COSMOLOGICAL APPLICATIONS OF GRAVITATIONAL LENSING

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1. Introduction

Because gravitational lensing provides a unique tool to probe almost directly the dark matter, its use for cosmology is of considerable interest. The discovery of giant arc(let)s in clusters of galaxies (Soucail et al 1987, Lynds & Petrosian 1986, Fort et al 1988) or Einstein rings around galaxies (Hewitt et al 1988), and the spectroscopic proofs that they are produced by gravitational lensing effects (Soucail et al 1988) have revealed that gravitational distortion can probe with a remarkable amount of details the mass distribution of clusters (Tyson et al 1990, Kaiser & Squires 1993) and galaxies (Kochanek 1990).

The applications of gravitational lensing to cosmology are so important that one cannot ignore them in a course dedicated to observational cosmology. The most obvious applications are the determination of the mass of gravitational systems, because the total mass inferred from a simple gravitational lens model is remarkably robust. In this respect microlensing experiments as well as strong and weak lensing studies provide the most powerful techniques to probe the dark matter of the Universe from the Jupiter-like planets up to large-scale structures. That is more than 16 orders of magnitude in size, more than 19 orders of magnitude in mass, and more than 25 orders of magnitude in density contrast! These limits are only due to technical limitations of present-day instrumentation, and in principle gravitational lensing can probe a much broader range. Therefore, measuring the mass distribution with gravitational lensing can put important constraints on the gravitational history of our Universe and the formation of its structures and its virialized gravitational systems.
Gravitational lensing also can provide valuable constraints on the cosmological parameters. For examples, the fraction of quasars with multiple-image, or the magnification bias in lensing-clusters depend on the curvature of the Universe. Deep surveys devoted to these lenses have already put boundaries on the cosmological constant $\lambda$. Furthermore, the measurement of time delays of transient events observed in multiple images of lensed sources can potentially provide useful constraints on the Hubble constant, $H_0$, on cosmological scales where distortion of the measurements from any local perturbation is negligible. Of equal importance, gravitational lenses can also be used as gravitational telescopes in order to observe the deep Universe. When strongly magnified, detailed structures of extremely high redshift galaxies can be analyzed spectroscopically in order to understand the dynamical stage and the merging history of the young distant galaxies (Soifer et al 1998). Furthermore, the joint analysis of the dark matter distribution in clusters and the shape of the lensed galaxies can be used to recover their redshift distribution.

In this course, I will focus on some applications to cosmology. However, in order to avoid self-duplications, I will only address some aspects about the mass determination from gravitational lensing studies. The other topics more detailed presentations (namely, distant galaxies, cosmological parameters, lensing on the CMB) or more detailed presentations can be found in Bernardeau (this proceedings), Blandford & Narayan (1992), Fort & Mellier (1994), Mellier (1998, 1999), Refsdal & Surdej (1994) and obviously in the textbook written by Schneider, Ehlers & Falco (1992). The microlensing experiments will be also presented elsewhere (Rich, this proceedings). Since F. Bernardeau has already discussed the theoretical aspects in his lecture, I will only make some addenda on some specific quantities in order to clarify the order of magnitude, but I assume that all the basic concepts are know already.

2. Some important quantities and properties

2.1. IMAGE MULTIPLICITY AND EINSTEIN RADIUS

Let us assume the lens being a point mass of mass $M$ and a one-dimensional configuration. In that case, the solutions for $\theta_I$ are given by

$$\theta_S = \theta_I + \frac{D_{LS}}{D_{OS}D_{OL}} \frac{4GM}{c^2 \theta_I}$$

which in general has 2 solutions. This illustrates the fact that gravitational lenses can produce multiple images. In the special case of a point mass, the divergence of the deviation angle at the origin is meaningless. For a more realistic mass density, the divergence vanishes and this produces another
solution for the images close to the center. This reflects a more general theorem that the number of lensed images is odd for non-singular mass density (Burke 1981). A critical case occurs when \( \theta_S = 0 \), that is when there is a perfect alignment between the source, the lens and the observer. The positions of the images are degenerated and form a strongly magnified circle at the Einstein radius \( \theta_E \):

\[
\theta_E = \left( \frac{4GM}{c^2} \frac{D_{LS}}{D_{OL}D_{OS}} \right)^{1/2}.
\]  

For example,
- for a star of \( 1 M_\odot \) at distance \( D=1 \) kpc, \( \theta_E = 0.001 \) arcsec,
- for a galaxy of \( 10^{12} M_\odot \) at \( D = 1 \) Gpc, \( \theta_E = 1 \) arcsec,
- and for a clusters of \( 10^{14} M_\odot \) at \( z = 0.3 \) and sources at \( z_S = 1 \), \( \theta_E = 30 \) arcsec.

When the source is slightly off-centered with respect to the axis defines by the lens and the observed, a circular lens produces two elongated arcs diametrally opposite with respect to the center of the lens.

2.2. CRITICAL MASS

By definition, the magnification matrix is given by \( A = d\theta_S/d\theta_I \). This defines the convergence, \( \kappa \), and the shear, \((\gamma_1, \gamma_2)\),

\[
\begin{pmatrix}
1 - \kappa - \gamma_1 & -\gamma_2 \\
-\gamma_2 & 1 - \kappa + \gamma_1
\end{pmatrix} = \begin{pmatrix}
1 - \partial_{xx}\varphi & -\partial_{xy}\varphi \\
-\partial_{xy}\varphi & 1 - \partial_{yy}\varphi
\end{pmatrix},
\]

where “\( \partial_{ij} \)” denotes the partial derivative of the projected potential with respect to the coordinates \( ij \). The magnification is given by

\[
\mu = \frac{1}{|\det A|} = \frac{1}{|(1 - \kappa^2) - \gamma^2|},
\]  

where the shear amplitude \( \gamma = \sqrt{\gamma_1^2 + \gamma_2^2} \) expresses the amplitude of the anisotropic magnification. On the other hand, \( \kappa \) expresses the isotropic part of the magnification. In is worth noting that \( 2\kappa = \Delta \varphi = \Sigma/\Sigma_{crit} \), is directly related to the projected mass density. It may happen that the determinant vanishes. It that case, the magnification becomes infinite. From an observational point of view, these cases correspond to the formation of very extended images, like Einstein rings or giant arcs. The points of the
image plane where the magnification is infinite are called **critical lines**. To these critical lines correspond points in the source plane which are called **caustic lines**. The critical density is

$$\Sigma_{\text{crit}} = \frac{c^2}{4\pi G} \frac{D_{OS}}{D_{LS} D_{OL}}$$

expresses the capability of a gravitational system to produce strong lensing effect ($\Sigma_{\text{crit}} \geq 1$). If we scale to $2H_0/c^2$ in order to define the “normalized” angular distances $d_{ij}$, then

$$\Sigma_{\text{crit}} \approx 0.1 \left( \frac{H_0}{50 \text{km/sec/Mpc}} \right) \frac{d_{os}}{d_{ls} d_{ol}} \text{ g.cm}^{-2}$$

For example, for a lensing-cluster at redshift $z_L = 0.3$ and lensed sources at redshift $z_S = 1$, $d_{os}/(d_{ls} d_{ol}) \approx 3$. If the cluster can be modeled by an isothermal sphere with a core radius $R_c$ and with $M(R_c) = 2 \times 10^{14} \, M_\odot$, then

- For $R_c=250$ kpc, $\Sigma_{\text{crit}}=0.05 \, \text{g.cm}^{-2}$,
- For $R_c=50$ kpc, $\Sigma_{\text{crit}}=1. \, \text{g.cm}^{-2}$.

Hence, the existence of giant arcs in clusters implies that clusters should be strongly concentrated.

### 2.3. RELATION WITH OBSERVABLE QUANTITIES

Since, to first approximation, faint galaxies look like ellipses, their shapes can be expressed as function of their weighted second moments,

$$M_{ij} = \frac{\int S(\theta)(\theta_i - \theta_i^C)(\theta_j - \theta_j^C)d^2\theta}{\int S(\theta)d^2\theta},$$

where the subscripts $i \, j$ denote the coordinates $\theta$ in the source and the image planes, $S(\theta)$ is the surface brightness of the source and $\theta^C$ is the center of the source.

Since gravitational lensing effect does not change the surface brightness of the source (Etherington 1933), then, if one assumes that the magnification matrix is constant across the image, the relation between the shape of the source, $M^S$ and the lensed image, $M^I$ is

$$M^I = A^{-1} M^S A^{-1}$$

Therefore, the gravitational lensing effect transforms a circular source into an ellipse. Its axis ratio is given by the ratio of the two eigenvalues of the
magnification matrix. The shape of the lensed galaxies can then provide information about these quantities. However, though $\gamma_1$ and $\gamma_2$ describe the anisotropic distortion of the magnification, they are not directly related to observables (except in the weak shear regime). The reduced complex shear, $g$, and the complex polarization (or distortion), $\delta$,

$$g = \frac{\gamma}{(1 - \kappa)} ; \quad \delta = \frac{2g}{1 + |g|^2} = \frac{2\gamma(1 - \kappa)}{(1 - \kappa)^2 + |\gamma|^2},$$

are more relevant quantities because $\delta$ can be expressed in terms of the observed major and minor axes $a^I$ and $b^I$ of the image, $I$, of a circular source $S$:

$$\frac{a^2 - b^2}{a^2 + b^2} = |\delta|$$

In this case, the 2 components of the complex polarization write:

$$\delta_1 = \frac{M_{11} - M_{22}}{Tr(M)} ; \quad \delta_2 = \frac{2M_{12}}{Tr(M)},$$

where $Tr(M)$ is the trace of the magnification matrix. For non-circular sources, it is possible to relate the ellipticity of the image $\epsilon^I$ to the ellipticity of the lensed source, $\epsilon^S$. In the general case, it depends on the sign of $Det(A)$ (that is the position of the source with respect to the caustic lines) which expresses whether images are radially or tangentially elongated:

$$\epsilon^I = \frac{1 + b^I/a^I e^{2i\vartheta}}{1 + b^I/a^I} = \frac{\epsilon^S + g}{1 - g^* \epsilon^S}$$

for $Det(A) > 0$, and

$$\epsilon^I = \frac{1 + \epsilon^S + g^* \epsilon^S}{\epsilon^S + g^*}$$

for $Det(A) < 0$.

3. Academic examples

3.1. THE SINGULAR ISOTHERMAL SPHERE

The projected potential at radius $r$, $\varphi(r)$, of a singular isothermal sphere with 3-dimension velocity dispersion $\sigma$ is

$$\varphi = 4\pi \frac{\sigma^2}{c^2} \frac{D_{LS}}{D_{OS}} r$$

and the images are described by the lensing equation

$$\theta_S = \theta_I - 4\pi \frac{\sigma^2}{c^2} \frac{D_{LS}}{D_{OS}} \frac{\theta_I}{|\theta_I|}$$
and the magnification matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 - 4\pi \frac{\sigma^2}{c^2} \frac{D_{LS}}{D_{OS}} \frac{1}{|\theta_I|}
\end{pmatrix}
\]  

Hence, there is only one critical line which is given by the Einstein radius

\[
\theta_{SIS} = 4\pi \frac{\sigma^2}{c^2} \frac{D_{LS}}{D_{OS}} \approx 16'' \left( \frac{\sigma}{1000 \text{ km sec}^{-1}} \right)^2
\]

for an Einstein de Sitter universe, with \( z_L = 0.3 \) and \( z_S = 1 \). The total mass included within the radius \( \theta \) is then

\[
M (\theta) = 5.7 \times 10^{13} M_\odot h_{50}^{-1} \left( \frac{\theta}{16''} \right) \left( \frac{\sigma}{1000 \text{ km sec}^{-1}} \right)^2
\]

From Eq.(28) and (29), it is obvious that the magnification writes

\[
\mu(\theta_I) = \frac{\theta_I}{\theta_I - \theta_{SIS}}
\]

The singular isothermal sphere permits to keep in mind the various properties of gravitational lenses and order of magnitude estimates of the associated physical quantities. However, more complex lenses can produce somewhat different configurations of strong lensing. Some example are shown in Figure 2, where a series of arcs generated by an elliptical potential well is displayed. We see that various kind of multiple images can be produced, with some strange radial arcs for some configurations. Each panel can be found in the universe so this kind of template of lens features can be helpful for the understanding of the lens modeling. This has been extensively used, in particular on HST images (see Fort & Mellier 1994, Kneib et al 1996, Natarajan & Kneib 1997, Mellier 1998 and references therein).

### 3.2. THE GENERAL CASE OF AN AXIALLY SYMMETRIC LENS

Assuming the rescaled angle writes (see Bernardeau, this proceedings)

\[
\alpha = \nabla \varphi , \quad \Delta \varphi = 2\kappa (x)
\]
for an axially symmetric potential, we have
\[ \alpha = \frac{d\varphi}{dx} , \quad \frac{1}{x} \frac{d}{dx} \left( x \frac{d\varphi}{dx} \right) = 2\kappa(x) , \] (21)
and the mass at radius \( x \) is \( m(x) = m(|x|) = m(x) \). Hence
\[ \frac{dm}{dx} = 2x\kappa(x) \quad \Rightarrow \quad \frac{d\varphi}{dx} = \frac{2}{x} \int_{x_0}^{x} \kappa(x') x' dx' . \] (22)
Therefore, in general the magnification matrix writes
\[ A = I - \frac{m(x)}{x^2} \begin{pmatrix} x_1^2 - x_2^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} - \frac{1}{x^3} \left( \frac{dm(x)}{dx} \right) \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} , \] (23)
where \( I \) is the identity matrix. The expression of the shear terms is then immediate, and the total magnification writes:
\[ \text{Det} (A) = \left( 1 - \frac{m}{x^2} \right) \left( 1 - \frac{d}{dx} \left( \frac{m(x)}{x} \right) \right) \] (24)
where the first term on the right-hand part provides the position of the tangential arcs, whereas the second term provides the location of the radial arcs. These equations are useful in order to compute the lensing properties of mass distributions, like the universal profile discussed in Section 4.1.2.

4. Astrophysical examples

4.1. MEASURING THE MASSES OF CLUSTERS OF GALAXIES

4.1.1. The case of MS2137-23
Strong lensing refers to lensing configurations where the source is located close to a caustic line and produces arcs or rings. In addition to the accurate estimate of the total mass, strong lensing features can probe some details on the mass density profile.
A nice example is the case of the lensing cluster MS2137-23 which contains a radial and a tangential arc. A simple investigation already permits to have some constraints on the lens configuration. First, since there is no counter-arc associated to the tangential arc on the diametrally opposite side of the cluster center, it is unlikely that the projected mass density is circularly distributed. Second, since the tangential arc and the radial arc are on the same side, it is unlikely that they are images of the same source. An interesting tentative to model the lens is to assume that the dark matter follows the light distribution of the central galaxy (see Figure 3). The isophotes then provide the center, the orientation and the ellipticity
of light as constraints to dark matter. Let us assume that the projected potential as the following geometry

\[
\varphi(r, \theta) = \varphi_0 \sqrt{1 + \left(\frac{r}{r_c}\right)^2 \left(1 - \epsilon \cos(2\theta)\right)}
\]  

(25)

where \(r_c\) is the core radius, \(\epsilon\) the ellipticity, and \(\varphi_0\) the depth of the potential, respectively. For an elliptical lens, the magnification matrix can be written

\[
\begin{pmatrix}
1 - 1 \frac{\partial \varphi}{r \partial r} - 1 \frac{\partial^2 \varphi}{r^2 \partial \theta^2} & 0 \\
0 & 1 - \frac{\partial^2 \varphi}{\partial r^2}
\end{pmatrix}
\]

(26)

which gives immediately the position or the radial and tangential arcs,

\[
\left[1 - \frac{\partial^2 \varphi}{\partial r^2}\right]_R = 0 ; \quad \left[1 - \frac{1}{r} \frac{\partial \varphi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2}\right]_T = 0
\]

(27)

From these two constraints, we can have a useful information of the core radius itself (Mellier et al 1993):

\[
\left(\frac{r_R}{r_c}\right) + 1 = \left(\frac{D_{OLR}}{D_{ORR}}\right) \left(\frac{D_{OS_T}}{D_{OLT}}\right) \left[\left(\frac{r_T}{r_c}\right)^2 + 1\right]^{3/2}
\]

(28)

where the subscripts \(R\) and \(T\) denote the radial and tangential arc respectively. Hence, provided the redshifts of the lens and the two sources are known, one can infer the core radius, or more generally the steepness of the dark matter density.

The modeling of the central regions of clusters of galaxies shows that the geometry of the dark matter follows the geometry of the diffuse light associated to the dominant galaxies. This is a remarkable success of the strong lensing analysis. However, there are still some uncertainties about the total mass which does not agree with the mass inferred from X-ray analysis. The mass-to-light ratio ranges in the same domain (\(\approx 100-300\)), but one can find frequently a factor of two difference.

4.1.2. The X-ray/lensing mass discrepancy

As it is illustrated in Figure 4, it turns out that the peaks of dark matter revealed by giant arcs in lensing-clusters correspond to the the peaks if X-ray emission as well as the location of the brightest components of
TABLE 1. Results obtained from strong lensing analyses of clusters. These are only few examples. $\sigma_{\text{obs}}$ is the velocity dispersion obtained from spectroscopy of cluster galaxies. $\sigma_{\text{DM}}$ is the velocity dispersion of the dark matter obtained from the modeling of the lens.

| Cluster     | z   | $\sigma_{\text{obs}}$ | $\sigma_{\text{DM}}$ | M/L | Scale     |
|-------------|-----|------------------------|-----------------------|-----|-----------|
|             |     | (kms$^{-1}$)           | (kms$^{-1}$)          |     | ($h_{100}$) ($h_{100}^{-1}$ Mpc) |
| MS2137-23   | 0.33| -                      | 900-1200              | 680-280 | 0.5       |
| A2218       | 0.18| 1370                   | ≈ 1000                | 200  | 0.9       |
| A2390       | 0.23| 1090                   | 1260                  | 250  | 0.4       |
| Cl0024+17   | 0.39| 1250                   | 1300                  | >200 | 0.5       |
| Cl0500-24   | 0.38| 1370                   | 850                   | >150 | 0.5       |
| Cl0500-24   | 0.316| -                      | <1200                 | <600 | 0.5       |

Optical images. On the other hand, Miralda-Escudé & Babul (1995) have pointed out an apparent contradiction between the mass estimated from X-ray data and the lensing mass ($M_{\text{lensing}} \approx 2 - 3M_X$). Further works done by many groups lead to somewhat inconclusive statements about this contradiction. Böhringer et al (1998) find an excellent agreement between X-ray and lensing masses in A2390 which confirms the view claimed by Pierre et al (1996); Gioia et al (1998) show that the disagreement reaches a factor of 2 at least in MS0440+0204; Schindler et al (1997) find a factor of 2-3 discrepancy for the massive cluster RXJ1347.4-1145, but Sahu et al (1998) claim that the disagreement is marginal and may not exist; Ota et al (1998) and Wu & Fang (1997) agree that there are important discrepancies in A370, Cl0500-24 and Cl2244-02.

There is still no definitive interpretation of these contradictory results. It could be that the modeling of the gravitational mass from the X-ray distribution is not as simple. By comparing the geometry of the X-ray isophotes of A2218 to the mass isodensity contours of the reconstruction, Kneib et al (1995) found significant discrepancies in the innermost parts. The numerous substructures visible in the X-ray image have orientations which do not follow the projected mass density. They interpret these features as shocks produced by the in-falling X-ray gas, which implies that the current description of the dynamical stage of the inner X-ray gas is oversimplified.

Recent ASCA observations of three lensing-clusters corroborate the view...
that substructures are the major source of uncertainties.

In order to study this possibility in more details, Smail et al (1997) and Allen (1998) have performed a detailed comparison between the lensing mass and X-ray mass for a significant number of lensing clusters. Both works conclude that the substructures have a significant impact on the estimate of X-ray mass. More remarkably, the X-ray clusters where cooling flows are present do not show a significant discrepancy with X-ray mass, whereas the others X-ray clusters do (Allen 1998). This confirms that the discrepancy is certainly due to wrong assumptions on the physical state of the gas. The interpretation of this dichotomy in cluster samples may be the following. Clusters with cooling flows are compact and rich systems which probably have probably virialised and have a well-defined relaxed core. Therefore, when removing the cooling flow contribution, the assumptions that the gas is in hydrostatic equilibrium is fully satisfied. Conversely, non-cooling flow clusters are generally poor, do have lot of substructures and no very dense core dominates the cluster yet. For these systems, the gas cannot be described simply (simple geometry, hydrostatic equilibrium) and the oversimplification of its dynamical stage produces a wrong mass estimator. This interpretation needs further confirmations. However, from these two studies we now have the feeling that we are now close to understand the origin of the X-ray and lensing discrepancy.

An alternative has been suggested by Navarro, Frenk & White (1997) who proposed that the analytical models currently used for modeling mass distributions may be inappropriate (hereafter NFW). They argue that the universal profile of the mass distribution produced in numerical simulations of hierarchical clustering may reconcile the lensing and X-ray masses. This kind of profile must be considered seriously because the universal profile is a natural outcome from the simulations which does not use external prescriptions.

For the reader who want to look more deeply at the lensing properties of this profile, it is interesting to describe the NFW properties into more details. Let us assume that the cluster 3-dimension mass density has the
NFW shape:
\[ \rho(x) = \frac{\rho_s}{x (1 + x)^2}, \text{ with } x = \frac{r}{r_s}. \]  
(29)

The projected mass density of the NFW profile is (use equations of Sect. 3.2)
\[ \Sigma(x) = \int_{-\infty}^{+\infty} \rho(x) dz = 2 \rho_s r_s f(x), \]  
(30)

where
\[ f(x) = \begin{cases} 
1 - \frac{2}{\sqrt{x^2 - 1}} \arctg \left( \frac{x - 1}{x + 1} \right) & (x > 1) \\
0 & (x = 1) \\
1 - \frac{2}{\sqrt{1 - x^2}} \text{argth} \left( \frac{1 - x}{1 + 1} \right) & (x < 1) 
\end{cases} \]  
(31)

Then the convergence and the mass write as follows:
\[ \kappa(x) = \frac{2 \rho_s r_s f(x)}{\Sigma_{crit} x^2 - 1}, \quad m(x) = 2 \kappa g(x), \]  
(32)

where
\[ g(x) = \ln \left( \frac{x}{2} \right) + \begin{cases} 
\frac{2}{\sqrt{x^2 - 1}} \arctg \left( \frac{x - 1}{x + 1} \right) & (x > 1) \\
1 & (x = 1) \\
\frac{2}{\sqrt{1 - x^2}} \text{argth} \left( \frac{1 - x}{1 + 1} \right) & (x < 1) 
\end{cases}. \]  
(33)

Now, from the analytical shape of \( M(x) \), it is clear that \( (d/dx)(M/x) \to \infty \) when \( x \to 0 \) and that \( (d/dx)(M/x) \to 0 \) when \( x \to \infty \). Therefore, there is always a radial line, whatever \( \rho_s \) and \( r_s \).

In the fortunate cases of \( MS2137 - 23 \) and \( A370 \) which both have a tangential and a radial arc, one can infer \( r_s \) and \( \rho_s \) from the analysis of the tangential and radial arcs. As far as the redshifts of the two arcs are known, then the ratio of the critical mass density at these two positions is,
\[ \frac{\Sigma_{Crit,r}}{\Sigma_{Crit,t}} = \left( \frac{x_t^2}{g(x_t)} \right) \left[ \frac{1}{R_{rt}} \left( \frac{dg(R_{rt} x_t)}{dx} \right) - \left( \frac{1}{R_{rt} x_t} \right)^2 g(R_{rt} x_t) \right], \]  
(34)

where \( R_{rt} = x_r/x_t \). Once used jointly, the positions and redshifts of the two arcs and the two independent equations, permit to infer \( r_s \) and \( \rho_s \).

For example, for \( A370 \) (see the radial arc in Figure 8), we have
\[ z_{\text{lens}} = 0.375, \quad z_t = 0.724 \text{ (measured)}, \quad z_r \approx 1.5 \text{ (assumed)}, \]
\[ R_{rt} = 0.7 \]
\[ \Sigma_{\text{Crit},t} = 1.4 \, h_{100} \text{ g.cm}^{-2}, \quad \Sigma_s = \rho_s r_s = 0.28 \, h_{100} \text{ g.cm}^{-2} \]
\[ r_s \approx 250 \, h_{100}^{-1} \text{ kpc}, \]
\[ \text{and the overdensity } \delta_c = \frac{\rho(0)}{\rho_{\text{crit}}} = 2 \times 10^4, \text{ if } \Omega_0 = 1. \]

It is worth noting that the statement that NFW profiles do predict radial arcs contradicts the general view that their existence rules out mass profiles with singularity.

Despite this interesting prediction which makes the NFW profile even more attractive, Bartelmann (1996) has shown that the caustics produced by this profile predict that radial arcs should be thicker than observed in MS2137-23 and in A370, unless the sources are very thin (\approx 0.6 \text{ arcsecond for MS2137-23}). This is not a strong argument against the universal profile because this is possible in view of the shapes of some faint galaxies observed with HST that some distant galaxies are indeed very thin. But it is surprising that no radial arcs produced by “thick galaxies” have been detected so far. Even a selection bias would probably favor the observation of large sources rather than small thin and hardly visible ones.

4.1.3. Clusters from weak lensing analysis

Deep images of lensing-clusters show many weakly lensed galaxies having a correlated distribution of ellipticity/orientation which maps the projected mass density (Fort et al 1988 and Figure 5). The first attempt to use this distribution of arclets as a probe of dark matter has been done by Tyson et al (1990), but the rigorous inversion technique was first proposed by Kaiser & Squires (1993).

The weak lensing analysis starts from the following hypotheses:

- Assume that the orientation of the sources is isotropic.
- Assume that the orientation of the source is not correlated to their ellipticity.
- Assume that the redshift distribution of sources is known.

Then it proceeds as follows:

- Measure the averaged ellipticity and orientation of the galaxies inside all subareas of the field.
- Produce a (ellipticity, orientation) map (see Figure 6).
- Provide a relation between the (ellipticity, orientation) and the components of the shear.
- Provide a relation between the shear and the mass density.
- Provide a relation between the shape of the source and the shape of the image.
Figure 5. Distortion field generated by a lens. The top panel shows the grid of randomly distributed background sources as it would be seen in the absence of the lens. The projected number density corresponds to very deep exposure, similar to the HDF. The bottom panel shows the same population once they are distorted by a foreground (invisible) circular cluster with a typical velocity dispension of \(1300 \text{ km} \text{s}^{-1}\). The geometrical signature of the cluster is clearly visible. The potential can be recovered by using the formalism defined in part 4. In this simulation, the sources are at \(z = 1.3\), and the cluster at \(z = 0.15\).

The hypotheses can be rather well controled (in principle) from the observations of unlensed areas. Ellipticities of field galaxies provide control fields for the first and second assumptions. Spectroscopic surveys and photometric redshifts allows to model the redshift distribution of galaxies. The procedure itself requires technical analysis of the data (see Mellier 1998 for the technical issues) and theoretical relations provided by gravitational lensing theory. The crucial points are the relations between shear, mass density and geometry of the lensed galaxies. This is done by combining the following equations:

\[
\begin{aligned}
&\gamma = \gamma_1 + i\gamma_2 = \frac{1}{2} (\partial_{xx} - \partial_{yy}) \varphi + i\partial_{xy} \varphi \\
&\kappa = \frac{1}{2} (\partial_{xx} + \partial_{yy}) \varphi \\
&\varphi = \frac{1}{\pi} \int \kappa(\theta') \ln(|\theta - \theta'|) \, d\theta',
\end{aligned}
\]  

(35)

from which one can express the complex shear as a function of the convergence, \(\kappa\) (see Seitz & Schneider 1996 and references therein):

\[
\gamma(\theta) = \frac{1}{\pi} \int \mathcal{D}(\theta - \theta') \kappa(\theta') \, d^2 \theta',
\]  

(36)

where

\[
\mathcal{D}(\theta - \theta') = \frac{(\theta_2 - \theta'_2)^2 - (\theta_1 - \theta'_1)^2 - 2i(\theta_1 - \theta'_1)(\theta_2 - \theta'_2)}{|(\theta - \theta')|^4}.
\]  

(37)

This equation can be inverted in order to express the projected mass density, or equivalently \(\kappa\), as function of the shear:

\[
\kappa(\theta) = \frac{1}{\pi} \int \Re[\mathcal{D}^*(\theta - \theta') \gamma(\theta')] \, d^2 \theta' + \kappa_0,
\]  

(38)

where \(\Re\) denotes the real part. Finally, from Eq.(9-12) we can express the shear as a function of the complex ellipticity. Hence, if the background ellipticity distribution is randomly distributed, then \(<|\epsilon^S|> = 0\) and

\[
<|\epsilon^I|> = |\gamma| = \frac{|\gamma|}{1 - \kappa},
\]  

(39)
(Schramm & Kayser 1995). In the most extreme case, when $\kappa << 1$ (the linear regime), $<|\epsilon'|>-\approx |\gamma|$, and therefore, the projected mass density can be recovered directly from the measurement of the ellipticities of the lensed galaxies.

Alternatively, one can measure the total mass within a circular radius using the Aperture densitometry technique (or the “$\zeta$-statistics”), which consists in computing the difference between the mean projected mass densities within a radius $r_1$ and within an annulus $(r_2 - r_1)$ (Fahlman et al 1994, Kaiser 1995) as function of the tangential shear, $\gamma_t = \gamma_1 \cos(2\theta) + \gamma_2 \sin(2\theta)$, averaged inside the ring. Let us denote $\bar{\kappa}$ the averaged value of $\kappa$ inside the loop of a circle with radius $r$ and $\langle \kappa \rangle_{\theta}$ the averaged value of $\kappa$ over the loop. We have

$$\bar{\kappa} = \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \kappa(r', \theta') r' dr' d\theta'$$

(40)

and

$$\frac{d\bar{\kappa}}{dr} = -\frac{2}{r} \frac{1}{\pi r^2} \int_0^{2\pi} \int_0^r \kappa(r', \theta') r' dr' d\theta' + \frac{1}{\pi r^2} \frac{d}{dr} \left( \int_0^{2\pi} \int_0^r \kappa(r', \theta') r' dr' d\theta' \right)$$

(41)

therefore,

$$\frac{d\bar{\kappa}}{dr} = -\frac{2}{r} \bar{\kappa} + 2 \frac{\langle \kappa \rangle_{\theta}}{r}$$

(42)

or equivalently

$$\frac{1}{2} \frac{d\kappa}{d\ln r} = \langle \kappa \rangle_{\theta} - \bar{\kappa}$$

(43)

Now, the mean tangential shear writes

$$\langle \gamma_t \rangle = \frac{1}{2\pi} \int_0^{2\pi} \gamma_t d\theta'$$

(44)

whereas in polar coordinates

$$\gamma_t = \frac{1}{2} (\partial_{rr} \varphi - \partial_{rl} \varphi) = \partial_{rr} - \kappa$$

(45)

Therefore

$$\langle \gamma_t \rangle = \frac{d}{dr} \left( \int_0^{2\pi} \partial_{\varphi} \frac{d\theta'}{2\pi} \right) - \langle \kappa \rangle_{\theta}$$

(46)

which implies that

$$\langle \gamma_t \rangle = -\frac{1}{r^2} \left( \int_0^{2\pi} \frac{d\varphi}{dr} \right)_{\theta} + \frac{1}{r} \frac{d}{dr} \left( \int_0^{2\pi} \frac{d\varphi}{dr} \right)_{\theta} - \langle \kappa \rangle_{\theta}$$

(47)

Therefore, from Eq.(42), $\langle \gamma_t \rangle$ is related to $\bar{\kappa}$ by this simple relation:

$$\langle \gamma_t \rangle = \frac{1}{2} \frac{d\bar{\kappa}}{d\ln r}$$

(48)
Figure 6. Detection of the shear field around Cl0024+1654. The figure is composed of two deep CCD images obtained at CFHT. The small field on the right is the central region of the cluster. The off-centered field on the left covers a much larger field and has been observed in order to detect the mass distribution at the periphery of the cluster. The thick full lines indicate the local average ellipticity. Each line displays the amplitude and the orientation of the distortion. The pattern is typical of a coherent gravitational shear produced by the mass of the cluster. Note also the perturbation of the shear field in the upper left. This effect is due to a secondary deflector which locally modifies the shear field.

which provides the $\zeta$ estimator:

$$\zeta(r_1, r_2) = <\kappa(r_1) > - <\kappa(r_1, r_2) > = \frac{2}{1 - r_1^2/r_2^2} \int_{r_1}^{r_2} <\gamma_t > d\ln r . \quad (49)$$

This quite robust mass estimator minimizes the contamination by foreground and cluster galaxies and permits a simple check that the signal is produced by shear, simply by changing $\gamma_1$ in $\gamma_2$ and $\gamma_2$ in $-\gamma_1$ which should cancel out the true shear signal.

The generalization to the non-linear regime (Kaiser 1995 and Seitz & Schneider 1996) can be done by solving the integral equation obtained from Eq.(38) where $\gamma$ is replaced by $(1-\kappa)g$. Alternatively one can use the fact that both $\kappa$ and $\gamma$ depend on second derivatives of the projected gravitational potential $\varphi$:

$$\begin{align*}
\frac{\partial \kappa}{\partial x_1} &= \frac{\partial \gamma_1}{\partial x_1} + \frac{\partial \gamma_2}{\partial x_2} \\
\frac{\partial \kappa}{\partial x_2} &= \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}
\end{align*} \quad (50)$$

which permits to recover the mass density by this relation:

$$\nabla \log(1-\kappa) = \frac{1}{1 - |g|^2} \cdot \begin{pmatrix} 1 & g_1 \\ -g_2 & 1 & g_2 \end{pmatrix} \begin{pmatrix} \partial_1 g_1 + \partial_2 g_2 \\ \partial_1 g_2 - \partial_2 g_1 \end{pmatrix} \quad (51)$$

Both Eq.(36) and Eq.(51) express the same relation between $\kappa$ and $\gamma$ and can be used to reconstruct the projected mass density.

Although mass reconstruction is now as a robust technique (see the comparison of various algorithms in Mellier 1998), the mass distribution recovered is not unique because the addition of a lens plane with constant mass density keeps the distortion of the galaxies unchanged. Furthermore, the inversion only uses the ellipticity of the galaxies regardless of their dimension, so that changing $(1-\kappa)$ in $\lambda(1-\kappa)$ and $\gamma$ in $\lambda \gamma$ keeps $g$ invariant. This is the so-called mass sheet degeneracy (Gorenstein et al 1988).

The degeneracy could in principle be broken if the magnification can be
measured independently, since it is not invariant under the linear transformation mentioned above, but instead it is reduced but a factor $1/\lambda^2$. The magnification can be measured directly by using the magnification bias (Broadhurst et al 1995), which changes the galaxy number-counts. The magnification bias expresses the effects of the gravitational magnification, which increases the flux received from lensed galaxies and magnifies by the same amount the area of the projected lensed sky and thus decreases the apparent galaxy number density. The total amplitude of the magnification bias depends on the slope of the galaxy counts as a function of magnitude and on the magnification factor of the lens. For a circular lens, the radial galaxy number density of background galaxies writes:

\[ N(< m, r) = N_0(< m) \mu(r)^{2.5\alpha-1} \approx N_0 \left(1 + 2\kappa\right)^{2.5\alpha-1} \] if $\kappa$ and $|\gamma| \ll 1$, \hspace{1cm} (52)

where $\mu(r)$ is the magnification, $N_0(< m)$ the intrinsic (unlensed) number density, obtained from galaxy counts in a nearby empty field, and $\alpha$ is the intrinsic count slope:

\[ \alpha = \frac{d \log N(< m)}{d m}. \] \hspace{1cm} (53)

A radial magnification bias $N(< m, r)$ shows up only when the slope $\alpha \neq 0.4$: otherwise, the increasing number of magnified sources is exactly compensated by the apparent field dilatation. For slopes larger than 0.4 the magnification bias increases the galaxy number density, whereas for slopes smaller than 0.4 the radial density will show a depletion. Hence, no change in the galaxy number density can be observed for $B(< 26)$ galaxies, since the slope is almost this critical value (Tyson 1988). But it can be detected in the $B > 26$, $R > 24$ or $I > 24$ bands when the slopes are close to 0.3 (Smail et al 1995). The change of the galaxy number density can be used as a direct measurement of the magnification and can be included in the maximum likelihood inversion as a direct observable in order to break the mass sheet degeneracy.

The lens parallax method (Bartelmann & Narayan 1995) which compares the angular sizes of lensed galaxies with an unlensed sample can be also used as an alternative to break the degeneracy. Another approach consists in using wide field cameras with a field of view much larger than clusters of galaxies. In that case $\kappa$ should vanish at the boundaries of the field, so that the degeneracy could in principle be broken.

Since 1990, many clusters have been investigated using the weak lensing inversion, either using ground-based or HST data. They are summarized in Table 2, but the comparison of these results is not straightforward because of the different observing conditions which produced each set of data and the different mass reconstruction algorithms used by each author. Nevertheless, all these studies show that on scales of about 1 Mpc, the geometry
TABLE 2. Results obtained from weak lensing analyses of clusters. The scale is the typical radial distance with respect to the cluster center. The last cluster has two values for the M/L ratio. This corresponds to two extreme redshifts assumed for the lensed population, either \( z = 3 \) or \( z = 1.5 \). For this case, the two values given for the velocity dispersion are those inferred when \( z = 3 \) or \( z = 1.5 \) are used.

| Cluster  | \( z \) | \( \sigma_{\text{obs}} \) (kms\(^{-1}\)) | \( \sigma_{\text{wl}} \) (kms\(^{-1}\)) | M/L | Scale (\( h_{100} \) Mpc) |
|----------|--------|--------------------------------|--------------------------------|------|---------------------|
| A2218    | 0.17   | 1370                          | -                              | 310  | 0.1 (0.1)          |
| A1689    | 0.18   | 2400                          | 1200-1500                      | -    | 0.5                |
| A2163    | 0.20   | 1680                          | 740-1000                       | 300  | 0.5                |
| A2390    | 0.23   | 1090                          | \( \approx \) 1000            | 320  | 0.5                |
| Cl1455+22 | 0.26 | \( \approx \) 700             | -                              | 1080 | 0.4                |
| AC118    | 0.31   | 1950                          | -                              | 370  | 0.15               |
| Cl1358+62 | 0.33 | 910                           | 780                            | 180  | 0.75               |
| MS1224+20 | 0.33 | 770                           | -                              | \( \approx \) 800 | 1.0              |
| Q0957+56 | 0.36   | 715                           | -                              | -    | 0.5                |
| Cl0024+17 | 0.39 | 1250                          | -                              | 150  | 0.15               |
| Cl0049+78 | 0.41 | 1080                          | -                              | 120  | 0.2                |
| Cl0302+17 | 0.42 | 1080                          | -                              | 80   | 0.2                |
| RXJ1347-11 | 0.45 | -                             | 1500                           | 400  | 1.0                |
| 3C295    | 0.46   | 1670                          | 1100-1500                      | -    | 0.5                |
| Cl0412-65 | 0.51 | -                             | -                              | 70   | 0.2                |
| Cl1601+43 | 0.54 | 1170                          | -                              | 190  | 0.2                |
| Cl0016+16 | 0.55 | 1700                          | -                              | 180  | 0.2                |
| Cl0054-27 | 0.56 | -                             | -                              | 400  | 0.2                |
| MS1137+60 | 0.78 | 859                           | -                              | 270  | 0.5                |
| RXJ1716+67 | 0.81 | 1522                          | -                              | 190  | 0.5                |
| MS1054-03 | 0.83 | 1360                          | 1100-2200                      | 350-1600 | 0.5            |
of mass distributions, the X-ray distribution and the galaxy distribution are similar (see Figure 4), though the ratio of each component with respect to the others may vary with radius. The inferred median M/L value is about 300, with a trend to increase with radius. Contrary to the strong lensing cases, there is no evidence of discrepancies between the X-ray mass and the weak lensing mass. It is worth noting that the strong lensing mass and the weak lensing mass estimates are consistent in the region where the amplitude of two regimes are very close. This is an indication that the description of the X-ray gas, and its coupling with the dark matter on the scales corresponding to strong lensing studies is oversimplified, whereas on larger scales, described by weak lensing analysis, the detailed description of the gas has no strong impact.

The large range of M/L could partly be a result of one of the issues of the mass reconstruction from weak lensing. In particular, the deviation angle depends on the ratios of the three angular-diameter distances (see Bernardeau, these proceedings), which depends on the redshift we assume for the sources. For low-redshift lenses, the dependence with redshift of the background galaxies is not important, so the calibration of the mass can be provided with a reasonable confidence level. However, distant clusters are highly sensitive to the redshift of the sources, and it becomes very difficult to scale the total mass without this information.

From the investigation of about 20 clusters, it turns out that the median M/L is lower than 400. This implies that weak lensing analyses predict $\Omega < 0.3$ with a high significance level. These constraints on $\Omega$ are in good agreement with other observations.

Another strong statement results from the mass reconstruction obtained by Luppino & Kaiser (1997) and Clowe et al (1998) or from the detection of giant arcs in very distant clusters (Deltorn et al 1997): massive clusters do exist at redshift $\approx 1$! This is a strong but reliable statement, though the total mass and the M/L cannot be given with a high accuracy. Therefore, unless unknown important systematics have been disregarded, we now have the first direct observational evidences that high mass-density peaks have generated massive clusters of galaxies at redshift 1. These promising results are corroborated by weak lensing studies around radio sources and quasars (Mellier 1998).

4.2. MEASURING THE MASSES OF GALAXIES

Gravitational lensing can also provide valuable insight on the halos of galaxies. Since it works on all scales, in principle the halos of galactic dark matter could be probed from their gravitational lensing effects on background
TABLE 3. Results on Einstein ring analyses. This is not a complete survey of the rings detected. I only report on those for which enough data have been obtained and a model has been presented.\(^{(1)}\) Kochanek 1995, \(^{(2)}\) Impey et al 1998, \(^{(3)}\) Warren et al 1998.

| Lens          | \(z_{\text{lens}}\) | \(z_{\text{source}}\) | \(\sigma_{DM}\) (\(\text{kms}^{-1}\)) | \(\frac{M}{L} (h_{100})\) |
|--------------|---------------------|------------------------|--------------------------------------|-----------------------------|
| MG 1654+134\(^{(1)}\) | 0.25                | 1.74                   | \(\approx 220\)                      | \(\approx 20.4\) (in B)    |
| PG 1115+080\(^{(2)}\) | 0.310               | 1.722                  | \(\approx 240\)                      | \(\approx 8.2\) (in I)     |
| 0047-2808\(^{(3)}\) | 0.485               | 3.595                  | \(\approx 270\)                      | -                           |

4.2.1. Einstein rings

Rings occur when the alignment of the observer, the lens and the source is almost perfect, and if the source is covering the whole internal caustic, forming the so called "Einstein ring". The first rings were observed around galaxies in radio surveys (see Refsdal & Surdej 1994 for a recent review). They have provided unique targets to measure the mass-to-light ratios and to probe the mass profiles of galaxies (Kochanek 1991). In the case of rings, the mass of the lensing galaxies can be very well constrained (see for instance Kochanek 1995 and Table 3), so the properties of the halos inferred from modeling are reliable. The results are somewhat reasonable, with typical velocity dispersion and mass-to-light ratio in good agreement with other techniques. However, I would like to stress again that, due the simplicity of these lens configurations and the very strong constraints provided by the size of the ring, these results are very robust.

4.2.2. Perturbations near giant arcs

Perturbations of caustics by intervening masses can locally change the length and shape of arcs or locally increase the intensity of unresolved arc substructures. Dramatic perturbations could even be responsible for the complete vanishing of an arc segment. The perturbation of caustics by a smaller interloping lens can be understood by considering that the magnification matrix degenerates to a single eigenvector tangent to the critical curve. Indeed the distortion of the images of objects close to the critical line corresponds mainly to a stretching along the direction of merging. There-
Figure 7. Gravitational distortion induced by a perturbation close to a giant fold arc. The top left panel shows the formation of two elongated images by a fold catastrophe. The vertical segment (A,B) is the length of the source in the source plane. The images are given by the antecedent of the parabola (fold caustic) and therefore two images are formed. In the next panel we introduce a perturbation represented as the dashed line which co-adds to the parabola. This perturbation roughly represents the deviation angle expected from an isothermal sphere with soft core, assuming its influence is zero beyond a given radius. The difference between the three configurations is the intensity of the perturbation. When the intensity is large enough (top right panel) it can break the nearest image and form multiple small images. When the intensity decreases (bottom left) the perturbation can break the images but can also form sub-ellipses of merging sub-images.

Figure 8. Perturbations close to giant arcs produced by galaxies. The top panel is the giant arc in A370. The galaxies with a number are those reported in Table IV. On this HST images the effects of these galaxies is clear, in particular for the galaxy #22. The middle panel shows similar perturbation in Cl0024+17. In particular, the effect of #158 is important. Though the central arc should be twice as long as the others (prediction of cups arcs), it is clearly smaller. The contraction is produced by the two galaxies which are located at the top and the bottom of the central arc. The bottom panel shows, the best lens model (left) of MS2137-23 and an example of the perturbation of the galaxy #7. When two much mass is put in this galaxy one can see that the giant arc is broken in three sub-arcs. This effect permits to put upper limits on the mass of this galaxy.

Therefore, the angular coordinates of the source can be developed in polynomial form along the direction of merging (Kassiola, Kovner & Fort 1992). Figure 7 gives a short description of the effect of perturbations on two merging images of an extended object near a fold. In this case the functional form of the unperturbed fold is approximated by a second-order polynomial. If a large perturbation from a nearby galaxy is added, the image can be split into many components (Figures 7 and 8).

Large perturbations of caustics have been used to constrain the galaxies located close to the giant arcs in A370, Cl0024+17, Cl2244 or MS2137-23. In general, the absence of breaks along a well-defined arc provides robust upper limits to the mass of perturbing galaxies. In summary, the results, as those shown in Table 4 are not surprising. The masses found for these cluster galaxies range between $10^{10} \, \text{M}_\odot$ and $2 \times 10^{11} \, \text{M}_\odot$, with typical mass-to-light ratios between 5 and 30.

4.2.3. Galaxy-galaxy lensing on field galaxies
A more promising approach consists in a statistical study of the deformation of distant galaxies by foreground galactic halos. The galaxy-galaxy lensing analysis uses the correlation between the position of foreground galaxies and the orientation of background population. If the correlation is produced by the gravitational shear of the foreground halos, then it is possible to probe their mass, if the redshift distributions of the foregrounds and the
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TABLE 4. Results cluster galaxies near arcs (perturbations). The table gives the constrains on the masses of the halos of galaxies around giant arcs which are shown in Figure 8.

| Lens   | $z_{\text{lens}}$ | Galaxy | $\sigma_{DM}$ (kms$^{-1}$) | M/L $(h_{100})$ |
|--------|------------------|--------|---------------------------|----------------|
| A370   | 0.38             | 201    | --                        | $2.4 < M/L < 8$ (in B) |
| A370   | 0.38             | 22     | --                        | $4.4 < M/L < 12.6$ (in B) |
| A370   | 0.38             | 63     | --                        | $9 < M/L < 30$ (in B) |
| MS2137-23 | 0.33       | 7      | $\leq 190$               | --              |
| Cl0024+17 | 0.39          | m1     | $\approx 180$            | --              |
| Cl0024+17 | 0.39          | 186    | $\approx 250$            | --              |
| Cl0024+17 | 0.39          | m2     | $\approx 120$            | --              |

backgrounds are known.

Let us define the shape of a galaxy by the vector $\epsilon$ as defined in Eq.(12). In the weak lensing regime, the shear simply translates in the $\epsilon$ plane at the new position $\epsilon = \epsilon_0 + \delta$, where $\epsilon_0$ is the intrinsic shape and $\delta$ the distortion produced by the shear. If we assume that the translation is done in the $x$-direction, then the shape distribution is simply translated:

$$f_{\epsilon}(\epsilon_x, \epsilon_y) = f^0_{\epsilon}(\epsilon_0, \epsilon_y) = f^0_{\epsilon}(\epsilon_x - \delta, \epsilon_y), \quad (54)$$

where $f^0_{\epsilon}$ is the intrinsic shape distribution. In that case, the orientations of the galaxies with respect to the $x$-axis, $\phi$, is modified and the final distribution, averaged over the ellipticity of the galaxies, is

$$P_\phi(\phi) = \int f^0_{\epsilon} \epsilon d\epsilon + \delta \cos(2\phi) \int \epsilon \frac{d f^0_{\epsilon}}{d\epsilon} d\epsilon \quad (55)$$

that is

$$P_\phi(\phi) = \frac{2}{\pi} \left(1 - \langle \delta \rangle \cos(2\phi) \langle \epsilon^{-1} \rangle \right) \quad (56)$$

Therefore, we expect a deficit of radially-oriented galaxies and conversely an excess of tangentially-oriented images.

A statistical analysis is then possible, if one assumes that all the foreground galaxies have similar halos, which can be scaled from observations. The procedure is the following. Assume an analytical shape for the projected
potential \( \varphi \) (or for the projected mass density \( \Sigma \), or the 3-dimension mass density \( \rho \)). Then, assuming the potential of halos are circular, the deflection angle is

\[
\alpha(r) = \frac{2}{c^2} \frac{D_{LS}}{D_{OS}} \frac{d\varphi}{dr} \tag{57}
\]

and the polarization writes

\[
p(r) = D_{OL} r \frac{d}{dr} \left( \frac{\alpha(r)}{r} \right) \tag{58}
\]

which depends on the typical scale, say \( r_c \) and the depth of the potential well (i.e. for an elliptical galaxy or a cluster of galaxies the velocity dispersion of the lens, \( \sigma_{los} \), and for a spiral galaxy its rotation velocity).

In the case of spiral galaxies, the depth of the potential can be related to the circular velocity of the galaxy, which can be obtained either directly or from the apparent magnitude and the redshift of the galaxy. Let us assume for simplicity that the mass-to-light ratio of the galaxy is independent of its luminosity. Then the typical scale

\[
r_c \propto \sqrt{M} , \text{ that is, from Tully – Fisher : } r_c \propto \sqrt{L} \propto V_c^2 . \tag{59}
\]

Therefore, the photometry of the foreground galaxies can, to first approximation, scale their mass, from the calibrated Tully-Fisher relation, as well as their redshift, from a magnitude-redshift relation. The main objective of the galaxy-galaxy lensing is then to calibrate the Tully-Fisher relation by measuring the physical scales \( r^* \) and \( L^* \) from the averaged polarization produced by the foreground galaxies onto the background lensed sources.

The expected gravitational distortion is very weak: for foregrounds at redshift \( < z_f > = 0.1 \), backgrounds at \( < z_s > = 0.5 \), and typical halos with velocity dispersion of 200 kms\(^{-1}\) and radius of 100 kpc, \( \vert \gamma \vert \approx 1\% \) at about 20 kpc from the center. But if the observations go to very faint magnitudes there is a huge number of background lensed galaxies, so that the weakness of the signal is compensated by the large statistics.

The first reliable results came from deep sub-arcsecond seeing CCD observations (Brainerd et al 1996). The distortion was compared with simulations, based on analytical profiles assumed for the dark matter halos as well as the Tully-Fisher relation and a magnitude-redshift relation, in order to relate mass models to observations. They detected a significant polarization...
of about 1%, averaged over separation between 5" and 34" (see Figure 9). They concluded that halos smaller that $10h^{-1}$ kpc are excluded at a 2σ level, but the data are compatible with halos of size larger that $100h^{-1}$ kpc and circular velocities of 200 kms$^{-1}$.

The HST data look perfectly suited for this kind of program which demands high image quality and the observation of many field galaxies. Griffiths et al (1996) used the Medium Deep Survey (MDS) and measured the distortion produced by foreground elliptical and spiral galaxies. They found similar results as Brainerd et al (1996) but with a more significant signal for foreground elliptical than spiral galaxies. The comparison with shear signals expected from various analytical models seems to rule out de Vaucouleur’s law as mass density profile of ellipticals. Dell Antonio & Tyson (1996) and Hudson et al (1998) analyzed the galaxy-galaxy lensing signal in the HDF. As compared with the ground-based images or the MDS, the field is small but the depth permits to use many background galaxies even on scale smaller than 5 arcseconds. Furthermore, the UBRI data of the HDF permit to infer accurate photometric redshifts for the complete sample of galaxies. By comparing the lensing signal with predictions from an analytical model for the halo, Dell’Antonio & Tyson found a significant distortion of about 7% at 2" from the halo center which corresponds to halos with typical circular velocities of less than 200 km.sec$^{-1}$. All these results seem consistent with those of Brainerd et al.

4.2.4. Galaxy-galaxy lensing on cluster galaxies

As it has been shown from the observation of perturbations along giant arcs, the clumpiness of dark matter on small-scale can be probed by anomalies in lensing configurations (see Sect. 4.2.3). With the details visible on the HST images of arclets in A2218, AC114 or A2390, the sample of halos which can be constrained by this method is much larger and can provide more significant results by extending the method to perturbations along arclets. The number of details permits also to use more sophisticated methods of investigation.

The simplest strategy is to start with an analytical potential which reproduces the general features of the shear pattern of HST images, and in a second step, to include in the model analytical halos around the brightest cluster members. In practice, additional mass components are put in the model in order to interpret the arc(let)s which cannot be easily explained by the simple mass distribution. Some guesses are done in order to pair unexplained multiple images. The colors of the arc(let)s as well as their morphology help a lot to make these associations. This approach has been proposed by Natarajan & Kneib (1997), and Natarajan et al (1998). The detailed study done in AC114 by Natarajan et al (see Figure 10) indicates that
about 10% of the dark matter is associated with halos of cluster galaxies. These halos have truncation radii smaller than field galaxies ($r_t \approx 15\text{kpc}$) with a general trend of S0-galaxies to be even more truncated than the other galaxies. If this result is confirmed it would be a direct evidence that truncation by tidal stripping is really efficient in rich clusters of galaxies. This result is somewhat contradictory with the absence a clear decrease of rotation curves of spiral galaxies in nearby clusters (Amram et al. 1993) which is interpreted as a proof that massive halos of galaxies still exist in cluster galaxies. However, it could be explained if the spirals which have been analyzed appear to be in the cluster center only by projection effects but are not really located in the very dense region of the clusters where stripping is efficient.

Geiger & Schneider (1997, 1998) used a maximum likelihood analysis which explores simultaneously the distortions induced by the cluster as a whole and by its individual galaxies. They applied this analysis to the HST data of Cl0939+47 and reached similar conclusions as Natarajan et al. (1998). Several issues limit the reliability of their analyses and of the other methods as well (Geiger & Schneider 1998). First, depending on the slope of the mass profile of the cluster, the contributions of the cluster mass density and of the cluster galaxies may be difficult to separate. Second, it is necessary to have a realistic model for the redshift distribution of the background and foreground galaxies. Finally, the mass sheet degeneracy is also an additional source of uncertainties. Regarding these limitations, Geiger & Schneider discuss the capability of the galaxy-galaxy lensing in clusters to provide valuable constrains on the galactic halos from the data they have in hands. Indeed, some of the issues they raised can be solved, like for instance the redshift distribution of the galaxies. It would be interesting to look into more details how the analysis could be improved with more and better data.

5. Conclusion and future prospects

The measurement of mass of galaxies and cluster of galaxies made a major step after the discovery of gravitational lenses like rings, arcs and arclets. The results on clusters of galaxies seem quite robust. In summary, the total
masses and the mass-to-light ratio recovered from lensing theory are similar to those found by other techniques, which confirms that investigation of dark matter in clusters favors $\Omega < 0.4$. The new result is the evidence that dark matter is strongly concentrated at the cluster center. More interesting, it seems that the geometry of the light distribution of the brightest cluster members traces the dark matter with a good accuracy. Furthermore, despite the discrepancies found between the X-ray mass and the lensing mass on small scales, the agreement is good on Megaparsec scales. This is still a matter of debates, but the dichotomies between large scale and small scale as well as between non-cooling flows and cooling flow clusters are strong arguments that the discrepancy is produced by over simplifications on the dynamical stage of the gas.

The investigation of galaxies are more debated. Though Einstein rings provide very accurate mass, the galaxy-galaxy lensing approach is still at its infancy and needs much more attention before providing something reliable.

In the future, the investigation of gravitational systems will be extended to nearby clusters. For those systems, the shear amplitude is small but each angular scale probes a much smaller physical scale, so that one can use much more galaxies than in distant systems in order to map the same physical scale. Groups of galaxies will be soon analyzed extensively as well. In parallel, we are still looking for dark clusters that gravitational lensing could reveal easily. If these systems do exist, then their discovery would be a major breakthrough.

Indeed, the wide field CCD cameras and the future NGST observations will permit to increase considerably the scientific return from lensing studies. The two key technical points are the understanding of systematics and the statistics. Both wide field CCD cameras and HST/NGST will provide important insights and intrinsically much better data that present-day observations. In parallel, since the masses are scaled by angular distances, the redshifts of the lenses and the sources are needed. The new giant telescopes equipped with visible and infrared spectrographs, as well as the joint use of visible and near infrared photometric redshifts open a new area. We expect that the number of Einstein rings and giant arcs with redshift will increase significantly in the next decade, putting much better constrains on the mass and mass-to-light ratios of galaxies and clusters.

Acknowledgments

I am grateful to M. Lachièze-Rey for his invitation to give this lecture in Cargèse. I thank F. Bernardeau, D. Elbaz, B. Fort, J.-P. Kneib, J. Rich, P. Schneider, L. van Waerbeke for the numerous stimulating discussions.
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