WELL-POSEDNESS FOR THE DISPERSIVE HUNTER-SAXTON EQUATION

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Abstract. This article represents a first step towards understanding the well-posedness for the dispersive Hunter-Saxton equation. This problem arises in the study of nematic liquid crystals, and although the equation has formal similarities with the KdV equation, the lack of $L^2$ control gives it a quasilinear character, with only continuous dependence on initial data.

Here, we prove the local and global well-posedness of the Cauchy problem using a normal form approach to construct modified energies, and frequency envelopes in order to prove the continuous dependence with respect to the initial data.

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1. Introduction

In this article we consider the Cauchy problem for the dispersive Hunter-Saxton equation

\[
\begin{aligned}
\begin{cases}
    u_t + uu_x + u_{xxx} &= \frac{1}{2} \partial_x^{-1}(u_x^2) \\
    u(0) &= u_0,
\end{cases}
\end{aligned}
\]

(1.1)

where $u$ is a real-valued function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Due to the Galilean invariance of (1.1), we may fix a definition for $\partial_x^{-1}$:

\[
\partial_x^{-1} f(x) = \int_{-\infty}^x f(y) \, dy,
\]

where $f \in L^1_x(\mathbb{R})$. The dispersive Hunter-Saxton equation is a perturbation of the Hunter-Saxton equation

\[
\begin{aligned}
\begin{cases}
    u_t + uu_x &= \frac{1}{2} \partial_x^{-1}(u_x^2),
\end{cases}
\end{aligned}
\]

(1.2)

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which was introduced in [13] as an asymptotic model for the formation of nematic liquid crystals under a director field. The Hunter-Saxton equation (1.2) is completely integrable [14, 1] with a bi-Hamiltonian structure [18]. In the periodic case, the local well-posedness and blow up phenomena were studied in [13, 21], while global weak solutions were studied in [2, 3]. For the non-periodic case, the Cauchy problem local well-posedness and blow up were studied in [20].

The Hunter-Saxton equation is also the high frequency limit of the Camassa-Holm equation,

\[(1 - \partial_x^2)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}.\]

The local well-posedness and ill-posedness of the Camassa-Holm equation were studied in [7, 8, 10]. The global existence of strong solutions and blow up phenomena were investigated in [4, 6, 5, 7].

The dispersive Hunter-Saxton equation (1.1) first appeared in [15] as a dispersive regularization of (1.2). Complete integrability was later observed in [9].

In this paper, we initiate the study of the well-posedness for the dispersive Hunter-Saxton equation (1.1). For this purpose, we use the conserved quantities

\[E_1(t) = \int_R u_x(t)^2 \, dx,\]
\[E_2(t) = \int_R u_{xx}(t)^2 - u(t)u_x(t)^2 \, dx.\]

Throughout, we denote

\[X^s = L_x^\infty \cap \dot{H}^1_x \cap \dot{H}^{1+s}_x,\]

where \(s \in [0, 1]\). For brevity, we denote \(X = X^1\). Our first main result is the following local well-posedness statement:

**Theorem 1.1.** The dispersive Hunter-Saxton equation (1.1) is locally well-posed in \(X\). Precisely, for every \(R > 0\), there exists \(T = T(R) > 0\) such that for every \(u_0 \in X\) with \(\|u_0\|_X < R\), the Cauchy problem (1.1) has a unique solution \(u \in C([0, T], X)\). Moreover, the solution map \(u_0 \mapsto u\) from \(X\) to \(C([0, T], X)\) is continuous.

In both the dispersive and nondispersive cases of the Hunter-Saxton equation, the main difficulty is that the source term \(1/2 \partial_x^{-1}(u_x^2)\) is unbounded in any \(L^p\) space if \(p < \infty\), and in particular, in \(L^2\). As a result, it is necessary to consider the problem assuming only pointwise \(L^\infty\) control on \(u\), similar to the analysis in [20] for the nondispersive case (1.2). Further, the lack of spatial decay obstructs direct access to local smoothing estimates, so that (1.1) exhibits quasilinear behavior even in the presence of KdV-like dispersion. In particular, our solutions exhibit only continuous dependence on the initial data, instead of Lipschitz dependence.

Our proof follows a bounded iterative scheme which treats separately the high and low frequency components. To prove continuous dependence on the initial data in our quasilinear setting, we have used frequency envelopes, introduced by Tao in [19]. A systematic presentation of the use of frequency envelopes in the study of local well-posedness theory for quasilinear problems can be found in the expository paper [17], which we broadly follow in the present work.

Our second result is the following global well-posedness statement:
Theorem 1.2. The Cauchy problem \((1.1)\) is globally well-posed in \(X\). Moreover, for every \(t \geq 0\), we have the global in time bounds
\[
\|u(t)\|_{L^\infty_x} \lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}),
\]
\[
\|u(t)\|_{H^2_x}^2 \lesssim \|u_0\|_{H^2_x}^2 + \|u_0\|_{X^0} E_1 + t(E_1 + E_1^{1/2}) E_1.
\]
Its proof relies on Theorem 1.1 and on the conserved quantities \(E_1(t)\) and \(E_2(t)\). We remark that the \(L^\infty\) estimate holds even for solutions which are only in \(X^0 = L^\infty \cap \dot{H}^1\).

Using the \(X^1\) well-posedness as a starting point, our third and fourth results extend well-posedness to lower regularity data:

Theorem 1.3. For each \(s \in \left(\frac{1}{2}, 1\right)\), the Cauchy problem \((1.1)\) is locally well-posed in \(X^s\).

The local well-posedness of Theorem 1.3 is in the same sense as in Theorem 1.1. Here, we leverage Theorem 1.1 to construct \(X^s\) solutions as limits of sequences of smooth solutions, by proving an estimate for differences of two solutions in order to establish convergence. This in turn is a consequence of an estimate for the linearized equation associated to \((1.1)\),
\[
\frac{\partial w}{\partial t} + (uw)_x + w_{xxx} = \partial_x^{-1}(u_x w_x).
\]

Theorem 1.4. For each \(s \in \left(\frac{1}{2}, 1\right)\), the Cauchy problem \((1.1)\) is globally well-posed in \(X^s\). Moreover, for every \(t \geq 0\),
\[
\|u(t)\|_{L^\infty_x} \lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2})
\]
\[
\|u(t)\|_{\dot{H}^{1+s}_x}^2 \lesssim \langle t \rangle^4 E_1^2 \langle \|u_0\|_{X^0} + E_1 \rangle^2 + \|u_0\|_{\dot{H}^{1+s}_x}^2.
\]

To prove Theorem 1.4, we construct a modified energy functional for \(\dot{H}^{1+s}\) which is based on the quadratic normal form for \((1.1)\). The approach of constructing normal form inspired modified energies in the quasilinear setting was first introduced by Hunter-Ifrim-Tataru-Wong [12] which considered the Burgers-Hilbert equation. This approach was further developed in the gravity water wave setting by Hunter-Ifrim-Tataru in [11], which established almost-global well-posedness, and in the Benjamin-Ono setting by Ifrim-Tataru [16] which established dispersive decay.

Our paper is organized as follows. In Section 2 we present some existence results at various degrees of regularity for linear equations that arise throughout the proofs of the main results. In Section 3, using an iterative scheme, we prove the higher regularity local well-posedness result, while in Section 4, by using the conserved quantities \(E_1\) and \(E_2\), we show that the dispersive Hunter-Saxton equation \((1.1)\) is globally well-posed.

Section 5 analyzes a modified energy for the equation, which is based on the normal form associated to the Hunter-Saxton equation, in order to obtain bounds on the growth of the \(X^s\)-norm, whereas Section 6 discusses an estimate for the linearized equation \((1.3)\), as well as one for differences of solutions. These results are then used to prove the low regularity local well-posedness result in Section 7.

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## 2. Preliminaries

In this section we state and prove some results that will be used in the sequel. We begin by studying well-posedness for a linear equation which will be used in the iteration for the proof of Theorem 1.1.

We first prove well-posedness and energy estimates for initial data in $L^2$.

**Lemma 2.1.** Let $T > 0$, $a, b \in L^\infty_t([0, T], W^{1, \infty})$, $F \in L^1_t([0, T], L^2_x)$, $v_0 \in L^2_x(\mathbb{R})$. Then the Cauchy problem

$$\begin{cases}
v_t + av_x + bv + v_{xxx} = F \\
v(0) = v_0
\end{cases}$$

admits a unique solution $v \in L^\infty_t([0, T], L^2_x)$ which satisfies the energy estimate

$$\frac{d}{dt} \|v\|^2_{L^2_x} \lesssim \|F\|_{L^2_t} \|v\|_{L^2_x} + (\|a_x\|_{L^\infty_x} + \|b_x\|_{L^\infty_x}) \|v\|^2_{L^2_x}.$$ 

**Proof.** Let us assume that $v$ is a solution to the Cauchy problem. We have

$$\frac{d}{dt} \int_\mathbb{R} v^2(t) \, dx = 2 \int_\mathbb{R} v(t)v_t(t) \, dx$$

$$= 2 \int_\mathbb{R} v(t)(F(t) - a(t)v_x(t) - b_x(t)v(t) - v_{xxx}(t)) \, dx$$

$$= 2 \int_\mathbb{R} v(t)F(t) \, dx + \int_\mathbb{R} a_x(t)v^2(t) \, dx - 2 \int_\mathbb{R} b_x(t)v^2(t) \, dx$$

$$\lesssim \|v(t)\|_{L^2_x} \|F(t)\|_{L^2_x} + \|v(t)\|^2_{L^2_x} (\|a_x(t)\|_{L^\infty_x} + \|b_x(t)\|_{L^\infty_x}).$$

We obtain the desired energy estimate, which also establishes uniqueness.

It remains to show existence, for which we follow a standard duality argument. We first determine the adjoint problem. For an arbitrary $w$, a formal computation shows that

$$\int_0^T \int_\mathbb{R} (v_t + av_x + b_x v + v_{xxx})w \, dx \, dt = \int_\mathbb{R} v(T)w(T) \, dx - \int_\mathbb{R} v(0)w(0) \, dx$$

$$- \int_0^T \int_\mathbb{R} (w_t + aw_x + a_x w + b_x w + w_{xxx})v \, dx \, dt.$$ 

We write $w_t + aw_x + (a_x + b_x)w + w_{xxx} = G$ and $w(T) = w_T$. Thus,

$$\int_0^T \int_\mathbb{R} Fw \, dx \, dt + \int_\mathbb{R} v_0w(0) \, dx = \int_\mathbb{R} v(T)w_T \, dx - \int_0^T \int_\mathbb{R} Gv \, dx \, dt$$

and we have the adjoint problem

$$\begin{cases}
w_t + aw_x + (a_x + b_x)w + w_{xxx} = G \\
w(T) = w_T.
\end{cases}$$
Lemma 2.2. Let \( \alpha \) be the solution for the adjoint problem. It is also bounded, as
\[
\|v\|_{L^2} \lesssim \|F\|_{L^1_t L^2_x} + \|G\|_{L^1_t L^2_x}. 
\]
In particular, we conclude that if the adjoint problem has a solution, then it is unique.

Let
\[
Y = \{(g, \tilde{G}) \in L^2_x \times L^2_x | \text{there exists } h \in L_t^\infty L^2_x \text{ solving the adjoint problem with } (w_T, G) = (g, \tilde{G}) \}. 
\]
We define the functional \( \alpha : Y \to \mathbb{R} \) by
\[
\alpha(g, \tilde{G}) = \int_0^T \int_x (\tilde{G} h dt) + \int_0 v_0 h(0) dt,
\]
which is well-defined by uniqueness for the adjoint problem. It is also bounded, as
\[
|\alpha(g, \tilde{G})| \lesssim \|v_0\|_{L^2} \|h(0)\|_{L^2} + \|F\|_{L^1_t L^2_x} \|h\|_{L^\infty_t L^2_x} 
\lesssim \|v_0\|_{L^2} (\|g\|_{L^2} + \|\tilde{G}\|_{L^2_x}) + \|F\|_{L^1_t L^2_x} (\|g\|_{L^2} + \|\tilde{G}\|_{L^1_t L^2_x}) 
\lesssim (\|v_0\|_{L^2} + \|F\|_{L^1_t L^2_x}) (\|g\|_{L^2} + \|\tilde{G}\|_{L^1_t L^2_x}). 
\]
Using the Hahn-Banach Theorem, we extend \( \alpha \) to a functional \( \beta \) defined on \( L^2_x \times L^1_t L^2_x \). This uniquely corresponds an element of \( L^2_x \times L^1_t L^2_x \), whose second component is the desired solution \( v \).

We extend the previous result to the case when the initial data is in \( H^1 \):

Lemma 2.2. Let \( T > 0 \), \( a, b \in L^\infty_t W^{1,\infty}_x \), \( b \in L^\infty_t \tilde{H}^2_x \), \( F \in L^1_t L^1_x \), and \( v_0 \in H^1_x \). Then the Cauchy problem (2.1) has a unique solution \( v \in L^\infty_t H^1_x \) which satisfies the energy estimate
\[
\frac{d}{dt}\|v\|^2_{H^1_x} \lesssim (\|F\|_{H^1_x} + \|b_{xx}\|_{L^2_x} \|v\|_{L^\infty_x}) \|v\|_{H^1_x} + (\|a_x\|_{L^2_x} + \|b_x\|_{L^\infty_x}) \|v\|^2_{H^1_x}.
\]
In particular, if \( u \) is a solution of the dispersive Hunter-Saxton equation (1.1), then
\[
\frac{d}{dt} \|u_x\|^2_{H^1_x} \lesssim \|u_x\|_{L^\infty_x} \|u_x\|^2_{H^1_x}.
\]

Proof. We first consider the regularized equation
\[
v_t + v_{xxx} + av_x + (b_{\leq m})_x v = F.
\]
By applying Lemma 2.1 we obtain a unique solution \( v^m \in L^\infty_t L^2_x \). We first observe that \( v^m \in L^\infty_t H^1_x \). Indeed, note that \( v^m_x \) formally satisfies
\[
\tilde{v}_t + \tilde{v}_{xxx} + (a_x + (b_{\leq m})_x) \tilde{v} + a \tilde{v}_x = F_x - (b_{xx})_{\leq m} v^m 
\]
where
\[
\|F_x - (b_{xx})_{\leq m} v^m\|_{L^\infty_t L^2_x} \leq \|F_x\|_{L^\infty_t L^2_x} + \|(b_{xx})_{\leq m} v^m\|_{L^\infty_t L^2_x} 
\leq \|F_x\|_{L^1_t L^2_x} + \|(b_{xx})_{\leq m} v^m\|_{L^\infty_t L^2_x} < \infty.
\]
By applying Lemma 2.1 once again, we obtain that (2.4) admits a unique solution \( \tilde{v}^m \in L^\infty_t L^2_x \) so that \( v^m_x = \tilde{v}^m \) and \( v^m \in L^\infty_t H^1_x \).
Using (2.2), we find
\[
\frac{d}{dt} \int_{\mathbb{R}} (v^m)^2 \, dx \lesssim \|v^m\|_{L^2_t}^2 \|F\|_{L^2_x} + (\|a_x\|_{L^\infty_x} + 2\|b_x\|_{L^\infty_x}) \|v^m\|_{L^2_x}^2
\]
and
\[
\frac{d}{dt} \int_{\mathbb{R}} (v^m_x)^2 \, dx \lesssim \|v^m_x\|_{L^2_t} \|F_x - (b_{xx})_{\leq m} v^m\|_{L^2_x} + \|a_x + 2(b_{\leq m} x)\|_{L^\infty_x} \|v^m_x\|_{L^2_x}^2
\]
\[
\lesssim \|v^m_x\|_{L^2_t} \|F_x\|_{L^2_x} + \|b_{xx}\|_{L^2_x} \|v^m\|_{L^\infty_x} \|v^m\|_{L^2_x}^2 + (\|a_x\|_{L^\infty_x} + \|b_x\|_{L^\infty_x}) \|v^m_x\|_{L^2_x}^2.
\]
Denoting
\[
E^m(t) = \int_{\mathbb{R}} (v^m(t))^2 \, dx + \int_{\mathbb{R}} (v^m_x(t))^2 \, dx,
\]
we have
\[
\frac{d}{dt} E^m(t) \lesssim (E^m(t))^{1/2} \|F(t)\|_{H^1_x} + (\|a_x(t)\|_{L^\infty_x} + \|b_x(t)\|_{L^\infty_x} + \|b_{xx}(t)\|_{L^2_x}) E^m(t).
\]
From Grönwall’s lemma, we infer that
\[
E^m(t) \leq e^{\frac{C}{2} \int_0^T \|a_x(s)\|_{L^\infty_x} + \|b_x(s)\|_{L^\infty_x \cap H^1_x} \, ds} \left( \|v_0\|_{H^1_x} + \int_0^T e^{-\frac{C}{2} \int_0^s \|a_x(\tau)\|_{L^\infty_x} + \|b_x(\tau)\|_{L^\infty_x \cap H^1_x} \, d\tau} \|F(s)\|_{H^1_x} \, ds \right),
\]
uniformly in $m$ and $t \in [0, T]$. Let $l \geq 0$ and $z = v^{m+l} - v^m \in L^\infty_t L^2_x$. We see that $z$ solves
\[
z_t + z_{xxx} + a z_x + (b_x)_{\leq m+l} z = -(b_x)_{m\leq x \leq m+l} v^m =: H.
\]
Let $\varepsilon := \sup_{m \geq 1} \sup_{t \in [0, T]} E^m(t) < \infty$. We estimate the source term:
\[
\|H\|_{L^\infty_t L^2_x} \lesssim \|(b_x)_{m \leq x \leq m+l}\|_{L^\infty_t L^2_x} \|v^m\|_{L^\infty_t L^2_x} \lesssim 2^{-m} \|(b_{m \leq x \leq m+l})_{xx}\|_{L^\infty_t L^2_x} \|v^m\|_{L^\infty_t L^2_x} \lesssim 2^{-m} \|b_{xx}\|_{L^\infty_t L^2_x} \varepsilon^{1/2}.
\]
By applying the energy estimate provided by Lemma 2.1 with Grönwall, we obtain
\[
\|z(t)\|_{L^2_x} \leq e^{\frac{C}{2} \int_0^t \|a_x(s)\|_{L^\infty_x} + 2\|b_x(s)\|_{L^\infty_x} \, ds} \left( \frac{C}{2} \int_0^t e^{-\frac{C}{2} \int_0^s \|a_x(\tau)\|_{L^\infty_x} + 2\|b_x(\tau)\|_{L^\infty_x} \, d\tau} \|H(s)\|_{L^2_x} \, ds \right)
\]
\[
\lesssim T 2^{-m} e^{1/2} \|b_{xx}\|_{L^\infty_t L^2_x}.
\]
Thus, $v^m$ is a Cauchy sequence in $L^\infty_t L^2_x$, which means that it converges to a solution $v$. As $v^m$ is bounded in $L^\infty_t H^1_x$, Lemma 2.3 implies $v \in L^\infty_t H^1_x$. The energy estimates of Lemma 2.1 also prove uniqueness. A similar computation to the one carried out for $v^m$ provides the desired energy estimate. In particular, if $u$ is a solution of (1.1), then $u_x$ is a solution of (2.1) with $a = u, \ b = -u_x/2$, and $F = 0$, so that the desired estimate follows. \hfill \Box

Using this, we establish persistence of regularity for (1.1):
Lemma 2.3. Let \( T > 0 \), and \( u \in C([0,T], X) \) a solution for the dispersive Hunter-Saxton equation \( \square \). If \( u(0) \in X \cap H^1_x(R) \), then \( u \in L^\infty_t([0,T], X \cap H^1_x) \). Furthermore, in the case \( n=2 \), we have the estimate

\[
\frac{d}{dt}\|u_{xx}(t)\|_{H^1_x} \lesssim \|u_x(t)\|_{L^\infty_x} \|u_{xx}(t)\|_{H^1_x}^2.
\]

Proof. Observe that \( u_{xx} \) formally satisfies

\[
v_t + u v_x + 2uv_x + v_{xxx} = 0.
\]

As \( u \in L^\infty_t X \), by applying Lemma 2.2 we infer that the problem admits a unique solution \( v \in L^\infty_t H^1_x \). In particular, \( v \) solves the problem in the sense of distributions, so that \( v = u_{xx} \) and \( u \in L^\infty_t (X \cap H^2_x) \), along with the energy estimate, as desired.

For \( n > 2 \), observe that \( \partial^n_u \) formally satisfies

\[
(\partial^n_u) v_t + u v_x + 2uv_x + v_{xxx} = P(u_{xx}, ..., \partial^n_{xx}u),
\]

where \( P \) is a quadratic polynomial. The result follows by induction and Lemma 2.2 \( \square \)

We now establish the following \( L^\infty \) estimate that will be used in the proof of several other results, including the iteration for the proof of Theorem \( \square \)

Lemma 2.4. Let \( T > 0 \), \( a \in L^\infty_t ([0,T], W^{1,\infty}), \) and \( w \in L^\infty_t ([0,T], L^\infty) \) satisfy

\[
w_t + aw_x + w_{xxx} = f.
\]

Then \( w \) satisfies

\[
\frac{d}{dt}\|w_{\leq 0}\|_{L^\infty} \lesssim \|w_{\leq 0}\|_{L^\infty} + \|f_{\leq 0}\|_{L^\infty} + \|a\|_{W^{1,\infty}}\|w\|_{L^\infty}
\]

Proof. By applying the frequency projection \( P_{\leq 0} \), we obtain

\[
(w_{\leq 0})_t + (aw_x)_{\leq 0} + (w_{\leq 0})_{xxx} = f_{\leq 0}
\]

and estimate

\[
\|(aw_x + w_{xxx})_{\leq 0}\|_{L^\infty} \lesssim \|(aw_x - (a_xw))_{\leq 0}\|_{L^\infty} + \|(w_{\leq 0})_{xxx}\|_{L^\infty}
\]

\[
\lesssim (\|a\|_{L^\infty} + \|a_x\|_{L^\infty})\|w\|_{L^\infty} + \|w_{\leq 0}\|_{L^\infty}.
\]

\( \square \)

Lastly, we observe a technical result which will be used in the proof of Theorem \( \square \) to show that the solution of \( \square \) has the desired regularity:

Lemma 2.5. Let \( T > 0 \) and \( \{v^n\}_{n \geq 0} \in L^\infty_t ([0,T], H^1_x) \) be a bounded sequence such that

\[
v^n \to v \in L^\infty_t ([0,T], L^2_x).
\]

Then \( v \in L^\infty_t ([0,T], H^1_x) \).

Proof. Let \( M > 0 \) be such that \( \|v^n\|_{L^\infty_t H^1_x} \leq M \) for every \( n \geq 0 \). Fix \( t \in [0,T] \) such that \( v^n(t) \) converges to \( v(t) \) in \( L^2_x(R) \), and \( \|v^n(t)\|_{H^1_x} \leq M \). It suffices to show that \( \|v(t)\|_{H^1_x} \leq M \), independently of \( t \). We omit \( t \) in the notations below.

As \( v^n \) is bounded in \( H^1_x(R) \), which is a Hilbert space and hence reflexive, we infer that there exists a subsequence \( \{v^{n_k}\}_{k \geq 0} \) that converges weakly to some \( g \in H^1_x(R) \). In particular, \( v^{n_k} \) converges to \( g \) in the sense of distributions. On the other hand, \( v^n \) converges to \( v \) in \( L^2_x(R) \) and in the sense of distributions, so \( v = g \in H^1_x(R) \).
Let $w \in H_x^1(\mathbb{R})$ with $\|w\|_{H_x^1} = 1$, and observe that
\[ |\langle v, w \rangle| = \lim_{k \to \infty} |\langle v^{n_k}, w \rangle| \leq \lim_{k \to \infty} \|v^{n_k}\|_{H_x^1} \leq M. \]
We infer that $\|v\|_{H_x^1} \leq M$. This finishes the proof.

3. LOCAL WELL-POSEDNESS

In this section we prove Theorem 1.1.

Let $C > 0$ be a large absolute constant which may vary from line to line, and let small $T > 0$ be fixed later. Let $\|u_0\|_X < R$. We inductively define a sequence $\{u^n\}_{n \geq 0} \subset L_t^\infty([0, T] \times \mathbb{R})$. For $n = 0$ we set $u^0(t, x) = u_0(x)$. For $n > 0$, we will set $u^{n+1} \in L_t^\infty([0, T] \times \mathbb{R})$ as the unique solution of the Cauchy problem
\[
\begin{aligned}
&u^{n+1}_t + u^{n+1}_{xx} + u^n u^{n+1}_x = \frac{\partial^{-1}((u^n)^2)}{2}, \\
u^{n+1}(0) = u_0.
\end{aligned}
\]

3.1. Existence and uniform bounds for (3.1). Here we show existence and estimates for (3.1) in $L_t^\infty([0, T], X)$.

3.1.1. Existence for $u^{n+1}$ in $L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)$. We first show that (3.1) has a solution $u^{n+1} \in L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)$ with
\[
E^{n+1}(t) := \int_{\mathbb{R}} (u^{n+1}_{xx}(t))^2 + (u^{n+1}_x(t))^2 \, dx \leq K \|u_0\|_X^2 =: E,
\]
for $K > 0$ a large absolute constant. We assume by induction that this is true for $u^n$.

We consider the Cauchy problem
\[
\begin{aligned}
v_t + v_{xxx} + (u^n)_x v + u^n v_x &= \frac{(u^n)^2}{2}, \\
v(0) = (u_0)_x.
\end{aligned}
\]
By applying Lemma 2.2, we obtain that (3.2) admits a unique solution $v \in L_t^\infty\dot{H}_x^1$. By Sobolev embedding, we obtain that $v \in L_t^\infty$, which implies that for almost every $t \in [0, T]$, $v(t)$ is locally integrable. Then we may define
\[ u^{n+1}(t, x) = u_0(0, 0) + \int_0^x v(t, y) \, dy. \]

For the energy estimate, we apply the energy estimate of Lemma 2.2 to $(u^{n+1})_x$ with the induction hypothesis to obtain that for every $t \in [0, T]$, with $T$ chosen appropriately small depending on $C$ and $\|u_0\|_X$,
\[
\begin{aligned}
(E^{n+1}(t))^{1/2} &\leq e^{\frac{C}{2} \int_0^t (E^n(s))^{1/2} \, ds} \left( (E^{n+1}(0))^{1/2} + \frac{C}{2} \int_0^t e^{-\frac{C}{2} \int_0^s (E^n(r))^{1/2} \, dr} E^n(s) \, ds \right) \\
&\leq e^{\frac{C T E^{n+1/2}}{2}} \left( \frac{E^{1/2}}{2} + \frac{C T E}{2} \right) \lesssim E^{1/2}.
\end{aligned}
\]
In addition, the energy estimates for $u^{n+1}$ show that it is a unique solution, hence the iteration is well-defined.
3.1.2. $L^\infty_x$ control for $u^{n+1}$. Applying Lemma 2.4 and choosing $T$ appropriately small depending on $E$, we have
\[
\|u^{n+1}\|_{L^\infty_t L^2_x} \lesssim \|(u_0)\|_{L^\infty_x} + T(\|\partial_x^{-1}(u^n_x)^2\|_{L^\infty_t L^2_x} + \|u^n\|_{L^\infty_t W^{1,\infty}} \|u^{n+1}\|_{L^\infty_t H^1_x}) \\
\lesssim \|(u_0)\|_{L^\infty_x} + T(\|u^n_x\|_{L^\infty_t L^2_x}^2 + \|u^n\|_{L^\infty_t X} \|u^{n+1}\|_{L^\infty_t H^1_x}) \\
\lesssim \frac{1}{2} E^{1/2} + TE \lesssim E^{1/2}.
\]

Combined with Sobolev embedding for the high frequencies,
\[
\|u^{n+1}\|_{L^\infty_{t,x}} \lesssim \|u^{n+1}\|_{H_x^1 \cap H^1_x} \lesssim E^{1/2},
\]
we conclude that our iteration is well-defined with the uniform bound
\[
\|u^{n+1}\|_{L^\infty_t X} \leq E^{1/2}.
\]

3.2. Convergence for $u^n$. We shall now prove that $u^n$ is a Cauchy sequence in $L^\infty_t (L^\infty_x \cap \dot{H}^1_x)$. Let $m \geq 0$ be an arbitrary integer and $z = u^{n+2} - u^{n+1}$. In this case, $z$ satisfies
\[
z_t + u^{n+1} z_x + z_{xxxx} = \frac{\partial_x^{-1}( (u^n_x)^2 - (u^n_x)^2 )}{2} - (u^{n+1} - u^n) u^{n+1} =: H
\]
and thus $z_x$ satisfies
\[
(z_x)_t + u^{n+1} z_x + u^{n+1} z_{xx} + z_{xxxx} = H_x.
\]
We estimate the term:
\[
H_x \leq \|u^{n+1}_x\|^2 - \|u^n_x\|^2 + \|u^{n+1} - u^n\| u^{n+1}_x + \|u^{n+1} - u^n\| u^n_x \|u^{n+1}\|_{L^\infty_t L^2_x} \\
\lesssim \|u^{n+1}_x - u^n_x\|_{L^\infty_t L^2} \|u^{n+1}_x\|_{L^\infty_t X} + \|u^n_x\|_{L^\infty_t X} + \|u^{n+1} - u^n\|_{L^\infty_t X} \|u^{n+1}_x\|_{L^\infty_t L^2} \\
\lesssim E^{1/2} \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)}
\]
and
\[
H \lesssim \|\partial_x^{-1}( (u^n_x)^2 - (u^n_x)^2 )\|_{L^\infty_x} + \|u^{n+1} - u^n\|_{L^\infty_x} \|u^{n+1}_x\|_{L^\infty_x} \\
\lesssim \|u^{n+1}_x - u^n_x\|_{L^\infty_t L^2} \|u^n_x\|_{L^\infty_t L^2} + \|u^{n+1} - u^n\|_{L^\infty_t X} \|u^{n+1}_x\|_{L^\infty_t L^2} \\
\lesssim E^{1/2} \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)}.
\]

By applying the energy estimate provided by Lemma 2.1 and choosing $T$ sufficiently small, we have
\[
\|z_x(t)\|_{L^2_x} \leq e^{\frac{T}{2} \int_0^t \|u^{n+1}(s)\|_{L^\infty_x} ds} \left( C \int_0^t e^{-\frac{T}{2} \int_0^\tau \|u^{n+1}(s)\|_{L^\infty_x} d\tau} \|H_x(s)\|_{L^2_x} d\tau \right) \\
\lesssim e^{\frac{T}{2} \int_0^t \|H_x(s)\|_{L^\infty_t X} ds} T E^{1/2} \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)} \\
\lesssim \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)}.
\]

For the $L^\infty$ estimates, applying Lemma 2.4 and choosing $T$ appropriately small depending on $E$, we have
\[
\|z_{\leq 0}\|_{L^\infty_t X} \lesssim T(\|H_{\leq 0}\|_{L^\infty_t X} + \|u^{n+m}\|_{L^\infty_t W^{1,\infty}} \|z\|_{L^\infty_t H^1_x}) \lesssim \frac{1}{4} \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)}.
\]

For the high frequencies, we use Sobolev embedding:
\[
\|z_{> 0}\|_{L^\infty_t H^1_x} \lesssim \|z\|_{L^\infty_t \dot{H}^1_x} \lesssim \|u^{n+1} - u^n\|_{L^\infty_t (L^\infty_x \cap \dot{H}^1_x)}.
\]
Putting everything together, and choosing $T$ sufficiently small (depending on $R$), we get
\[
\|u^{n+2} - u^{n+1}\|_{L^\infty_t(L^\infty_x \cap H^1_x)} \leq \frac{1}{2}\|u^{n+1} - u^n\|_{L^\infty_t(L^\infty_x \cap H^1_x)}.
\]
By iterating, we get
\[
\|u^{n+2} - u^{n+1}\|_{L^\infty_t(L^\infty_x \cap H^1_x)} \leq 2^{-n-1}\|u^1 - u^0\|_{L^\infty_t(L^\infty_x \cap H^1_x)} \lesssim 2^{-n}E^{3/2},
\]
which shows that $u^n$ is a fundamental sequence in $L^\infty_t(H^1_x \cap L^\infty_x)$ converging to an element $u \in L^\infty_t(H^1_x \cap L^\infty_x)$. In particular, $u^n_x$ converges to $u_x$ in $L^\infty_tL^2_x$. As $u^n_x$ is bounded in $L^\infty_tH^1_x$ (because $u^n$ is bounded in $L^\infty_tX$), Lemma 2.5 implies that $u_x \in L^\infty_tH^1_x$. Therefore, $u \in L^\infty_tX$.

### 3.3. Uniqueness

Let $u$ and $v$ be two solutions to $(1.1)$ with initial data $u(0) = u_0$ and $v(0) = v_0$ such that $\|u_0\|_X < R$ and $\|v_0\|_X < R$. Let $w = u - v$. Recall that we have the bounds $\|u\|_{L^\infty_tX}, \|v\|_{L^\infty_tX} \leq E^{1/2}$.

In this case, $w$ satisfies
\[
(3.5) \quad w_t + uw_x + w_{xxx} = -wv_x + \frac{\partial_x^{-1}(w_x(u_x + v_x))}{2} =: H
\]
so that $w_x$ satisfies
\[
(3.6) \quad (w_x)_t + uw_{xx} + \frac{1}{2}(u_x + v_x)w_x + w_{xxxx} = -wv_{xx}.
\]

By applying the energy estimate provided by Lemma 2.1 and choosing $T$ sufficiently small, we get that
\[
\|w_x\|_{L^\infty_tL^\infty_x} \leq C \left[\int_0^T \|u_x(s)\|_{L^\infty_x} + \|v_x(s)\|_{L^\infty_x} ds \right] \|u_0 - v_0\|_X L^\infty_x
\]
\[
+ C \int_0^T e^{-C\int_0^\tau \|u_x(\tau)\|_{L^\infty_x} + \|v_x(\tau)\|_{L^\infty_x} d\tau} \|wv_{xx}\|_{L^2_x} ds
\]
\[
\lesssim \|u_0 - v_0\|_X L^\infty_x + TE^{1/2}\|w\|_{L^\infty_tL^\infty_x}.
\]

For later use, we see that formally, we also have the energy estimate of Lemma 2.2
\[
\|w_x\|_{L^\infty_tH^2_x} \leq C \left[\int_0^T \|u_x(s)\|_{L^\infty_x \cap H^1_x} + \|v_x(s)\|_{L^\infty_x \cap H^1_x} ds \right] \|u_0 - v_0\|_X H^1_x
\]
\[
+ C \int_0^T e^{-C\int_0^\tau \|u_x(\tau)\|_{L^\infty_x \cap H^1_x} + \|v_x(\tau)\|_{L^\infty_x \cap H^1_x} d\tau} \|wv_{xx}\|_{H^1_x} ds
\]
\[
\lesssim \|u_0 - v_0\|_X H^1_x + T \|v_{xx}\|_{L^\infty_t(L^\infty_x \cap H^1_x)} \|w\|_{L^\infty_t(L^\infty_x \cap H^1_x)}.
\]

For $L^\infty$ estimates, we estimate the source term:
\[
\|H\|_{L^\infty_x} = \left\| -wv_x + \frac{\partial_x^{-1}(w_x(u_x + v_x))}{2} \right\|_{L^\infty_x} \lesssim \|w_x\|_{L^\infty_tL^\infty_x} \|u_x + v_x\|_{L^\infty_tL^2_x} + \|w\|_{L^\infty_tL^\infty_x} \|v_x\|_{L^\infty_tL^\infty_x}
\]
\[
\lesssim \|(w, v_x)\|_{L^\infty_tH^{1/2}_x} + \|w\|_{L^\infty_t(L^\infty_x \cap H^1_x)}.
\]

Then applying Lemma 2.4 and choosing $T$ appropriately small, we have
\[
\|w \leq 0\|_{L^\infty_tL^\infty_x} \lesssim \|(w(0)) \leq 0\|_{L^\infty_x} + T(\|H_{\leq 0}\|_{L^\infty_tL^\infty_x} + \|u\|_{L^\infty_tW^{1,\infty}} \|w\|_{L^\infty_tH^1_x})
\]
\[
\lesssim \|u_0 - v_0\|_{L^\infty_x} + T \|(w_x, v_x)\|_{L^\infty_tH^{1/2}_x} + \|w\|_{L^\infty_t(L^\infty_x \cap H^1_x)}.
\]

Moreover, by Sobolev embedding,
\[
\|w > 0\|_{L^\infty_tL^\infty_x} \lesssim \|w_x\|_{L^\infty_tL^2_x}.
\]
By adding this inequality, along with equations (3.7) and (3.9), we get that
\[ \|w\|_{L_t^\infty(L_x^\infty \cap H_x^1)} \lesssim \|(u_0)_x - (v_0)_x\|_{L_t^1} + \|u_0 - v_0\|_{L_t^\infty} + TE^{1/2}\|w\|_{L_t^1(L_x^\infty \cap H_x^1)}. \]
Choosing \( T \) sufficiently small, we find
\[ \|w\|_{L_t^\infty(L_x^\infty \cap H_x^1)} \lesssim \|u_0 - v_0\|_{L_x^\infty \cap H_x^1} \]
which establishes uniqueness.

3.4. Continuity with respect to the initial data. Consider a sequence of initial data
\[ u_{0j} \to u_0 \in X. \]
Here, since \( \|u_0\|_X < R \), we may assume that \( \|u_{0j}\|_X < R \) for every \( j \), and the existence part implies that \( u_j \) and \( u \) may be defined on a common time interval \([0, T]\), with uniform bounds in \( j \). Furthermore, by the Lipschitz estimate from the proof of uniqueness,
\[ u_j \to u \in L_t^\infty(L_x^\infty \cap \dot{H}_x^1). \]
By interpolation, it follows that
\[ u_j \to u \in L_t^\infty([0, T], L_x^\infty \cap \dot{H}_x^1 \cap \dot{H}_x^{2-\varepsilon}). \]
To obtain the endpoint, we take an approach similar to the one presented in [17].

We define \( u_{0j}^h = (u_{0j})_{\leq h} \) and \( u_0^h = (u_0)_{\leq h} \), and may assume that
\[ \|u_{0j}^h\|_X \lesssim \|u_0\|_X, \]
so that there exists \( T = T(\|u_0\|_X) > 0 \) and solutions \( u^h \) and \( u_{0j}^h \) that belong to \( L_t^\infty X \). Further, Lemma 2.3 shows that \( u^h \) and \( u_{0j}^h \) belong to \( L_t^\infty(X \cap \dot{H}_x^3) \). As
\[ \int_0^T \|u_x^h(s)\|_{L_x^\infty} \, ds \lesssim T \|u^h\|_{L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)}, \]
we have from the energy estimate of Lemma 2.3 that
\[ \|u^h\|_{L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)} \lesssim \|u_0^h\|_{\dot{H}_x^1 \cap \dot{H}_x^2}, \]
and likewise for \( u_{0j}^h \).

We consider \( \dot{H}_x^1 \) sharp frequency envelopes for \( (u_0)_x \) and \( (u_{0j})_x \), denoted by \( \{c_k\}_{k \in \mathbb{Z}} \) and \( \{c_k^j\}_{k \in \mathbb{Z}} \). As \( (u_{0j})_x \to (u_0)_x \) in \( \dot{H}_x^1 \), we can assume that \( c_i^j \to c_k \) in \( l^2 \). Moreover, as in [17], we can choose \( c_k \) having the following properties:

a) Uniform bounds:
\[ \|P_k(u_0^h)_x\|_{\dot{H}_x^1} \lesssim c_k \]
b) High frequency bounds:
\[ \|(u_0^h)_x\|_{\dot{H}_x^2} \lesssim 2^h c_h \]
c) Difference bounds:
\[ \|u_0^{h+1} - u_0^h\|_{\dot{H}_x^1} \lesssim 2^{-h} c_h \]
d) Limit as \( h \to \infty \):
\[ D_x u_0^h \to D_x u_0 \in H_x^1 \]
and likewise for $c^j_k$.

We first establish estimates for $(u - u^h)_{>0}$ and $(u_j - u^h_j)_{>0}$ in $L^\infty_t X$. We treat the low frequencies separately because the frequency envelopes that we are using are $\dot{H}^1 \cap \dot{H}^2$-based, and don’t allow us to control the $L^\infty$-component of the norm of $X$ at low frequencies. By applying the Lipschitz estimate from the proof of uniqueness, we can see that

$$
\|u^{h+1} - u^h\|_{L^\infty_t(\dot{H}^1_x \cap L^\infty_x)} \lesssim \|u_{>0}^{h+1} - u_0^h\|_{\dot{H}^1_x \cap L^\infty_x} \lesssim \|u_{>0}^{h+1} - u_0^h\|_{\dot{H}^1} \lesssim 2^{-h} c_h.
$$

Taking the high frequencies and interpolating with the estimate

$$
\|u^h\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)} \lesssim \|u^h\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)} \lesssim \|u^h\|_{\dot{H}^1_x \cap \dot{H}^2_x} \lesssim 2^h c_h,
$$

we get that

$$
\|u_{>0}^{h+1} - u_{>0}^h\|_{L^\infty_t X} \lesssim c_h.
$$

The analogous analysis and estimates hold for $u^h_j$. Moreover, as in [17], we get that

$$
\|u_{>0} - u_{>0}^h\|_{L^\infty_t X} \lesssim c_{>h} = \left(\sum_{k \geq h} c_k^2\right)^{1/2}, \quad \|(u_j)_{>0} - (u_j^h)_{>0}\|_{L^\infty_t X} \lesssim c_{>h}^j = \left(\sum_{k \geq h} (c_k^j)^2\right)^{1/2}.
$$

Next, we show that for fixed $h$, \(j \to \infty\), \(\lim u_j^h = u^h\) in $L^\infty_t X([0, T] \times \mathbb{R})$. Let us write \(w = u^h - u_j^h\), which by (3.8) satisfies

$$
\|w_x\|_{L^\infty_t H^1_x} \lesssim \|w_x(0)\|_{H^1_x} + T\|(u_j^h)_{xx}\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)}\|w\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)}.
$$

As $h$ is fixed, the previous discussion ensures that $\|(u_j^h)_{xx}\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)}$ is uniformly bounded with respect to $j$. Using as well (3.10), we conclude that

$$
\|w\|_{L^\infty_t X} \lesssim_h \|u_0 - u_0\| X
$$

as desired.

To complete the argument, we have

$$
\|u_{>0} - (u_j)_{>0}\|_{L^\infty_t X} \lesssim \|u^h - u_j^h\|_{L^\infty_t X} + \|u_{>0} - u_{>0}^h\|_{L^\infty_t X} + \|(u_j)_{>0} - (u_j^h)_{>0}\|_{L^\infty_t X}
\lesssim \|u^h - u_j^h\|_{L^\infty_t X} + c_{>h} + c_{>h}^j
$$

so that fixing $h$,

$$
\limsup_{j \to \infty} \|u_{>0} - (u_j)_{>0}\|_{L^\infty_t X} \lesssim c_{>h} + c_{>h}^j
$$

Then letting $h$ tend to $\infty$, we get that

$$
\lim_{j \to \infty} \|u_{>0} - (u_j)_{>0}\|_{L^\infty_t X} = 0.
$$

For the low frequencies, we directly estimate

$$
\|u_{\leq 0} - (u_j)_{\leq 0}\|_{L^\infty_t X} \lesssim \|u_{\leq 0} - (u_j)_{\leq 0}\|_{L^\infty_t(\dot{H}^1_x \cap L^\infty_x)} \lesssim \|u - u_j\|_{L^\infty_t(\dot{H}^1_x \cap \dot{H}^2_x)} \lesssim \|u_{\leq 0} - u_{\leq 0}\|_{\dot{H}^1_x \cap L^\infty_x}
$$

As $u_{j0} \to u_j$ in $X$, it follows that

$$
\lim_{j \to \infty} \|u_{\leq 0} - (u_j)_{\leq 0}\|_{L^\infty_t X} = 0.
$$

Combining the low and high frequencies, we obtain $u_j \to u$ in $L^\infty_t X$. 

3.5. **Continuity in time.** Let \( h > 0 \) be an arbitrary parameter, and \( u^h \) solve (1.1) with initial data \((u_0)_{\leq h}\). In particular,

\[
u^{h}_t = \frac{\partial_x^{-1}((u^h)^2)}{2} - u^h u^h_x - u^h_{xxx}.
\]

From Lemma 2.3, we know that \( u^h \in L_t^\infty(X \cap \dot{H}^5_x) \), so that the right hand side belongs to \( L_t^\infty X \). Thus, \( u^h \in C^0_t X \). From the previous section, we know that \( u^h \) converges to \( u \) in \( L_t^\infty X \), hence in \( C^0_t X \). This concludes the proof of Theorem 1.1.

### 4. Global well-posedness

In this section, we prove Theorem 1.2. Recall that the dispersive Hunter-Saxton (1.1) has the conserved quantities (see [9])

\[
E_1(t) = \int_X u_x^2 \, dx \\
E_2(t) = \int_X u_{xx}^2 - u u_x^2 \, dx.
\]

Throughout the proof, \( C > 0 \) shall denote a universal large constant. Consider a solution \( u \) of (1.1) on \([0, T)\) where \( T \) is finite. We shall determine a uniform bound for \( \|u(t)\|_X \).

We begin with the \( L^\infty \) estimate. The high frequencies can be controlled by the \( \dot{H}^1 \) norm, which is conserved via \( E_1 \), but the low frequencies need to be treated separately as follows. Projecting (1.1) onto frequencies less than or equal to 1, we consider

\[
(u_{\leq 0})_t + (uu_x)_{\leq 0} + (u_{\leq 0})_{xxx} = \frac{(\partial_x^{-1}(u_x^2))_{\leq 0}}{2}.
\]

For the transport term, write

\[
(uu_x)_{\leq 0} - u_{\leq 0}(u_{\leq 0})_x = (u_{> 0}u_x)_{\leq 0} + [P_{\leq 0}, u_{\leq 0}] u_x = (u_{> 0}u_x)_{\leq 0} + [P_{\leq 0}, P_0 u] u_x + [P_{\leq 0}, u_{< 0}] P_0 u_x
\]

and estimate

\[\|(u_{> 0}u_x)_{\leq 0}\|_{L_x^\infty} \lesssim \|u_{> 0}u_x\|_{L_x^2} \lesssim \|u_x\|_{L_x^2} \]

The same estimate holds for the first commutator directly, without using the commutator structure. For the second commutator,

\[\|[P_{\leq 0}, u_{< 0}] P_0 u_x\|_{L_x^\infty} \lesssim \|[P_{\leq 0}, u_{< 0}] P_0 u_x\|_{L_x^2} \lesssim \|\partial_x u_{< 0}\|_{L_x^\infty} \|P_0 u\|_{L_x^2} \lesssim \|u_x\|_{L_x^2}^2.\]

Besides this, we may estimate the dispersive and source terms by

\[\|(u_{\leq 0})_{xxx}\|_{L_x^\infty} \lesssim \|u_x\|_{L_x^2}, \quad \|(\partial_x^{-1}(u_x^2))_{\leq 0}\|_{L_x^\infty} \lesssim \|u_x\|_{L_x^2}^2.\]

Therefore, denoting

\[F = \frac{(\partial_x^{-1}(u_x^2))_{\leq 0}}{2} - ((uu_x)_{\leq 0} - u_{\leq 0}(u_{\leq 0})_x) - (u_{\leq 0})_{xxx},\]

we have

\[\|F\|_{L_x^\infty} \lesssim \|u_x\|_{L_x^2}^2 + \|u_x\|_{L_x^2} = E_1 + E_1^{1/2}.\]
As \( u \in C^0_tX([0,T] \times \mathbb{R}) \), we see that \( u \) is continuous with respect to \( t \) and \( x \), and Lipschitz with respect to \( x \), uniformly in \( t \). As in [20], let us consider the flow

\[
q_t = u_{t0}(t,q(t,x)), \quad q(0,x) = x.
\]

By standard ordinary differential equations theory, \( q \) exists, is unique, and is defined on the whole interval \([0,T]\) as a function in \( C^1([0,T])\). Moreover, it is not difficult to see that it is a \( C^1\)-diffeomorphism. We also note that \( q_{xt} = u_x q_x \), which means that \( q_x = e^{\int_0^t u_x(s,q(s,x))\,ds} > 0 \), hence \( q \) is strictly increasing in \( x \) for every \( t \). Further,

\[
\frac{d}{dt} u_{t0}(t,q(t,x)) = (u_{t0})_t + u_{t0}(u_{t0})_x = F.
\]

Then

\[
\|u_{t0}(t,q(t,x))\|_{L^\infty_x} \lesssim \|u_{t0}(0)\|_{L^\infty_x} + \int_0^t \|F\|_{L^\infty_x} \,ds \lesssim \|(u_{t0})_{t0}\|_{L^\infty_x} + \int_0^t E_1 + E_1^{1/2} \,ds.
\]

As \( q \) is a diffeomorphism, we now infer that

\[
\|(u(t))_{t0}\|_{L^\infty_x} \lesssim \|(u_{t0})_{t0}\|_{L^\infty_x} + t(E_1 + E_1^{1/2}).
\]

For the high frequencies, we apply Sobolev embeddings and Bernstein’s inequalities to estimate

\[
\|(u(t))_{t0}\|_{L^\infty_x} \lesssim E_1^{1/2}.
\]

Combining these estimates, we conclude that for every \( t \in [0,T) \),

\[
\|u(t)\|_{L^\infty_x} \lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}).
\]

Thus, for some constant \( C > 0 \), and for every \( t \in [0,T) \), we have

\[
\|u_{xx}(t)\|_{L^2_x}^2 \lesssim |E_2| + \|u(t)\|_{L^\infty_x}\|u_x(t)\|_{L^2_x}^2 \\
\lesssim \|u_0\|_{H^2_x}^2 + \|u_0\|_{X^0} E_1 + t(E_1 + E_1^{1/2}) E_1.
\]

We obtain the desired estimate for \( \|u(t)\|_X \), where \( t \in [0,T) \). In particular, the lifespan for \( u \) may be extended indefinitely.

5. A NORMAL FORM ANALYSIS

In this section, we use normal forms to construct an energy functional corresponding to \( \dot{H}^{1+s} \). Since [14] exhibits a quasilinear behavior at low frequencies, we use a modified energy approach as introduced in [12].

We may re-express the dispersive Hunter-Saxton (1.1) as

\[
u_t + u_{xxx} = \partial_x^{-2}(u_x u_{xx}) - uu_x =: Q_1 + Q_2 =: Q.
\]

Thus we see that the formal normal form variable, based on the normal form correction for the KdV equation, is

\[
\tilde{u} = u + B(u,u) = u - \frac{1}{6} \partial_x^{-2}(u^2) + \frac{1}{6}(\partial_x^{-1}u)^2.
\]

To construct a modified energy for \( \dot{H}^{1+s} \), write

\[
A(D) = D^s P_{>0}
\]
and consider
\[ \int Au_x \cdot A \left( u_x - \frac{1}{3} \partial_x^{-1} (u^2) + \frac{2}{3} (\partial_x^{-1} u) u \right) \, dx. \]
Integrating by parts on the last two terms and rearranging, we obtain
\[ \int (Au_x)^2 - \frac{1}{3} Au \cdot A (u^2 + 2\partial_x^{-1} u \cdot u_x) \, dx. \]
Then commuting \( A \) through the last term, we have
\[ \int (Au_x)^2 - \frac{1}{3} Au \cdot (A(u^2) + 2[A, \partial_x^{-1} u] u_x + 2\partial_x^{-1} u \cdot Au_x) \, dx. \]
Lastly, integrating by parts on the last term, we define the modified energy
\[ \tilde{E}(t) := \int (Au_x)^2 - \frac{1}{3} Au \cdot (A(u^2) + 2[A, \partial_x^{-1} u] u_x - Au \cdot u) \, dx. \]

**Lemma 5.1.** If \( u \in C_1^0 X^s([0, T) \times \mathbb{R}) \), then for every \( t \in [0, T) \), we have
\[ \|(u(t))_{>0}\|_{H^{1+s}}^2 = \tilde{E}(t) + O(E_1\|u(t)\|_{L^\infty}), \]
and
\[ \frac{d}{dt} \tilde{E}(t) \lesssim \|Au\|_{L^2_x}^2 (\|u_x\|_{L^2_x}^2 + \|u_x\|_{L^\infty} \|u\|_{L^\infty}). \]

**Proof.** We have
\[ \|A, \partial_x^{-1} v\|_{L^2_x} \lesssim \|Av\|_{L^2_x} \|v\|_{L^\infty} + \|Av\|_{L^2_x} \|w\|_{L^\infty}, \]
\[ \|A(vw)\|_{L^2_x} \lesssim \|Av\|_{L^2_x} \|w\|_{L^\infty} + \|Aw\|_{L^2_x} \|v\|_{L^\infty}. \]
Thus, the first bound is immediate.

We now prove the energy estimate. First observe that \( \frac{d}{dt} \tilde{E} \) consists only of quartic terms. Precisely, if we set
\[ L_A(v, w) := -\frac{1}{3} A(vw) - \frac{2}{3} [A, \partial_x^{-1} v] w_x + \frac{1}{3} Av \cdot w, \]
then a straightforward computation shows that
\[ \frac{d}{dt} \tilde{E} = \int AQ \cdot L_A(u, u) + Au \cdot L_A(Q, u) + Au \cdot L_A(u, Q) \, dx. \]

We consider first the contribution from \( Q_1 \). Since
\[ \|L_A(v, w)\|_{L^2_x} \lesssim \|Av\|_{L^2_x} \|w\|_{L^\infty} + \|Aw\|_{L^2_x} \|v\|_{L^\infty}, \]
we have
\[ \int AQ_1 \cdot L_A(u, u) + Au \cdot L_A(Q_1, u) + Au \cdot L_A(u, Q_1) \, dx \lesssim \|Au\|_{L^2_x} (\|AQ_1\|_{L^2_x} \|u\|_{L^\infty} + \|Au\|_{L^2_x} \|Q_1\|_{L^\infty}). \]
To bound \( Q_1 \), we have
\[ \|\partial_x^{-1} (u_x^2)\|_{L^\infty} \lesssim \|u_x\|_{L^2_x}^2, \]
\[ \|A\partial_x^{-1} (u_x^2)\|_{L^2_x} \lesssim \|u_x\|_{L^\infty} \|Au\|_{L^2_x} \]
which suffices.
For the contribution from $Q_2$, we consider each of the three terms in
\[ L_A(u, u) = -\frac{1}{3}A(u^2) - \frac{2}{3}[A, \partial_x^{-1}u]u_x + \frac{1}{3}Au \cdot u \]
successively. From the third term, and the $Q_2$ contribution arising from the case where the
time derivative falls on $u$,
\[
\int \frac{1}{3}Au \cdot Au \cdot uu_x \, dx \lesssim \|u\|_{L^\infty} \|u_x\|_{L^\infty} \|Au\|_{L^2}^2.
\]
On the other hand, when the derivative falls on $Au$, we write
\[
\int \frac{1}{6}Au \cdot A\partial_x(u^2) \cdot u \, dx = \int \frac{1}{3}Au \cdot [A, u]u_x \cdot u + \frac{1}{3}Au \cdot u \cdot Au_x \cdot u \, dx.
\]
The latter term is the same as the previous case after an integration by parts, while
\[
\int \frac{1}{3}Au \cdot [A, u]u_x \cdot u \, dx \lesssim \|Au\|_{L^2}^2 \|u_x\|_{L^\infty} \|u\|_{L^\infty}.
\]
From the first term in $L_A$, the case when the time derivative falls on $Au$ vanishes via an
integration by parts. Then from the remaining contribution,
\[
\int \frac{1}{3}Au \cdot A\partial_x(u^3) \, dx = \int Au \cdot [A, u^2]u_x + Au \cdot u^2 \cdot Au_x \, dx.
\]
The latter term has already appeared, while
\[
\int Au \cdot [A, u^2]u_x \, dx \lesssim \|Au\|_{L^2}^2 \|u_x\|_{L^\infty} \|u\|_{L^\infty}.
\]
Lastly, we have the commutator term from $L$. When the time derivative falls inside the
commutator, we have
\[
\int Au \cdot [A, \partial_x^{-1}(uu_x)]u_x \, dx \lesssim \|Au\|_{L^2} \|[A, \partial_x^{-1}(uu_x)]u_x\|_{L^2} \lesssim \|Au\|_{L^2}^2 \|u_x\|_{L^\infty} \|u\|_{L^\infty}.
\]
From the remaining contributions of $Q_2$, we are left with
\[
\int A(uu_x) \cdot [A, \partial_x^{-1}u]u_x + Au \cdot [A, \partial_x^{-1}u](uu_x)_x \, dx.
\]
Integrating by parts on the second term, and since
\[
\int Au \cdot [A, u](uu_x) \, dx \lesssim \|Au\|_{L^2} \|u_x\|_{L^\infty} \|A(u^2)\|_{L^2} \lesssim \|Au\|_{L^2}^2 \|u_x\|_{L^\infty} \|u\|_{L^\infty},
\]
remains to bound
\[
(5.1) \quad \int A(uu_x) \cdot [A, \partial_x^{-1}u]u_x - Au_x \cdot [A, \partial_x^{-1}u](uu_x) \, dx = - \int u_x \cdot [A[A, \partial_x^{-1}u], u]u_x \, dx.
\]
Before exploiting the full commutator structure, we first reduce to paraproducts. From the first integral on the left hand side of (5.1), we write
\[
\int A(uu_x) \cdot [A, \partial_x^{-1}u]u_x \, dx = \int A(uu_x) \cdot [A, T_{\partial_x^{-1}u}]u_x \, dx
\]
\[ - \frac{1}{2} \int A(u^2) \cdot \partial_x(A(T_{u_x} \partial_x^{-1}u) + A\Pi(u_x, \partial_x^{-1}u)) \, dx \]
\[ + \frac{1}{2} \int A(u^2) \cdot \partial_x(T_{Au_x} \partial_x^{-1}u + \Pi(Au_x, \partial_x^{-1}u)) \, dx. \]
The last two lines are perturbative and may be discarded. Precisely, we have
\[ \| A(u^2) \|_{L^2_x} \lesssim \| u \|_{L^\infty_x} \| Au \|_{L^2_x} \]
while
\[ \| \partial_x A(T_{u_x} \partial_x^{-1} u) \|_{L^2_x} \lesssim \| u_x \|_{L^\infty_x} \| Au \|_{L^2_x}, \]
\[ \| \partial_x (T_{Au_x} \partial_x^{-1} u) \|_{L^2_x} \lesssim \| u_x \|_{L^\infty_x} \| Au \|_{L^2_x}, \]
with the same estimate for the balanced frequency terms.

Next, we proceed further to write
\[ \int A(uu_x) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx = \int A(T_u u_x) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx \]
\[ + \int A(T_{u_x} u) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx + \int A \Pi(u_x, u) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx. \]

The second line is perturbative as before. Precisely,
\[ \int A(T_{u_x} u) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx \lesssim \| u_x \|_{L^\infty_x} \| Au \|_{L^2_x} \cdot \| u \|_{L^\infty_x} \| Au \|_{L^2_x} \]
with the same estimate for the balanced frequency term.

A similar analysis holds for the second term on the left hand side of (5.1), so we are only left to estimate
\[ \int A(T_u u_x) \cdot [A, T_{\partial_x^{-1} u}] u_x \, dx - \int Au_x \cdot [A, T_{\partial_x^{-1} u}] (T_u u_x) \, dx = - \int u_x \cdot [A[A, T_{\partial_x^{-1} u}], T_u] u_x \, dx. \]

Define
\[ L(u, v, w) = D^{-s} \partial_x [A[A, T_{\partial_x^{-1} u}], T_{\partial_x^{-1} v}] D^{-s} w_x \]
and let \( L_k \) denote the frequency \( k \) component.

Let \( a(\xi) = |\xi|^s(1 - \phi(\xi)) \) be the symbol of \( A \), where \( \phi \) is the symbol of the Littlewood-Paley projector \( P_{\leq 0} \), and
\[ \phi_k(\xi) = \phi \left( \frac{\xi}{2^{k-4}} \right), \quad \psi_k(\xi) = \phi \left( \frac{\xi}{2^{k}} \right) - \phi \left( \frac{\xi}{2^{k-1}} \right). \]
The symbol of \( L_k \) is
\[ L_k(\xi, \eta, \zeta) = \phi_k(\xi) \phi_k(\eta) \psi_k(\zeta)(\xi \eta + \eta + \zeta) |\xi|^s |\eta| |\zeta|^s \]
\[ \cdot \left( a(\xi + \eta + \zeta) a(\xi + \eta + \zeta) - a(\eta + \zeta) - a(\xi + \zeta) (a(\xi + \zeta) - a(\zeta)) \right). \]
This symbol is supported in the region \( \{ (\xi, \eta, \zeta) : |\xi|, |\eta|, |\zeta| \lesssim 2^k, \zeta \sim 2^k \} \), is smooth, and its associated kernel is bounded and integrable. Thus, we have the estimate:
\[ - \int (u_k)_x \cdot [A[A, \partial_x^{-1} u_{<k}], u_{<k}] (u_k)_x \, dx = \int Au_k \cdot L_k(u, u_x, Au_k) \, dx \]
\[ \lesssim \| u \|_{L^\infty_x} \| u_x \|_{L^\infty_x} \| Au_k \|_{L^2_x}^2. \]

Thus,
\[ \int u_x \cdot [A[A, T_{\partial_x^{-1} u}], T_u] u_x \, dx \lesssim \sum_k \| u \|_{L^\infty_x} \| u_x \|_{L^\infty_x} \| Au_k \|_{L^2_x}^2 \lesssim \| u \|_{L^\infty_x} \| u_x \|_{L^\infty_x} \| Au \|_{L^2_x}^2. \]

By putting everything together, we obtain the desired estimate.
Thus, we have the local result of Theorem 1.3, which we prove in the next two sections. □

Proposition 5.2. Let $T > 0$, $I = [0, T]$ or $I = [0, T)$, and $u \in C^0_t X^s(I \times \mathbb{R})$ solve (1.1). Then we have the bounds (1.4).

Proof. We have from Theorem 1.2 the pointwise estimates. It remains to establish the energy bounds.

Let $\tilde{E}$ be the modified energy functional of Lemma 5.1 so that for $t \in [0, T)$,

\[
\frac{d}{dt} \tilde{E}(t) \lesssim \|Au\|^2_{L^2} \left(\|u_x\|^2_{L^2} + \|u_x\|_{L^\infty} \|u\|_{L^\infty}\right) \\
\lesssim \|u_x(t)\|^4_{L^2} + \|u_x(t)\|^3_{L^2} \|u(t)\|_{L^\infty} + \|u_x(t)\|^2_{L^2} \|u(t)\|_{L^\infty} \|u(t)\|_{H^{1+s}} \langle t \rangle \lesssim \tilde{E}(t) + \tilde{E}(0).
\]

and since we have the energy equivalence

\[
\|(u(t))_{>0}\|^2_{H^{1+s}} = \tilde{E}(t) + O(E_1 \|u(t)\|_{L^\infty}),
\]

we find

\[
\frac{d}{dt} \tilde{E}(t) \lesssim E_1^2 + E_1^{3/2} \|u(t)\|_{L^\infty} + E_1 \|u(t)\|_{L^\infty} \left(\tilde{E}(t) + CE_1 \|u(t)\|_{L^\infty}\right)^{1/2} \\
\lesssim E_1^2 + E_1 \|u(t)\|_{L^\infty} \left(\tilde{E}(t) + E_1 + CE_1 \|u(t)\|_{L^\infty}\right)^{1/2} \\
\lesssim E_1 \|u(t)\|_{X^0} \left(\tilde{E}(t) + E_1 + CE_1 \|u(t)\|_{L^\infty}\right)^{1/2} \\
\lesssim E_1 \|u\|_{L^\infty} \|u\|_{X^0} \left(\tilde{E}(t) + E_1 + CE_1 \|u\|_{L^\infty}\right)^{1/2}.
\]

Integrating in $t$, we find that for every $t \in [0, T)$,

\[
\left(CE_1 \|u\|_{L^\infty_{t,x}} + E_1 + \tilde{E}(t)\right)^{1/2} \lesssim tE_1 \|u\|_{L^\infty_{t,x} X^0} + \left(CE_1 \|u\|_{L^\infty_{t,x}} + E_1 + \tilde{E}(0)\right)^{1/2}.
\]

Thus,

\[
\tilde{E}(t) \lesssim t^2 E_1^2 \|u\|^2_{L^\infty_{t,x} X^0} + E_1(\|u\|_{L^\infty_{t,x}} + 1) + \tilde{E}(0).
\]

Using the first inequality from Lemma 5.1 and the low frequency bound

\[
\|(u(t))_{<0}\|^2_{H^{1+s}} \lesssim E_1,
\]

we have

\[
\|u(t)\|^2_{H^{1+s}} \lesssim t^2 E_1^2 \|u\|^2_{L^\infty_{t,x} X^0} + E_1(\|u\|_{L^\infty_{t,x}} + 1) + \|u_0\|^2_{H^{1+s}}.
\]

Combined with the pointwise estimates, we obtain the stated bound. □

This establishes the bounds in Theorem 1.4. Global well-posedness now follows from the local result of Theorem 1.3, which we prove in the next two sections.
6. An estimate for the linearized equation

The linearized equation corresponding to (1.1) is

\[ w_t + (uw)_x + w_{xxx} = \partial_x^{-1}(u_x w_x), \]

which can be rewritten as

\[ w_t + \partial_x^{-1}(u_x w) + u w_x + w_{xxx} = f. \]

Applying \( D^s \) with \( s \in \left( \frac{1}{2}, 1 \right) \) to (6.1), and writing \( v = D^s w \), we have

\[ v_t + w_x + v_{xxx} = -D^s \partial_x^{-1}(T_{uxx} w + T_u u_{xx} + \Pi(w, u_{xx})) - [D^s, u]w_x + D_x f. \]

**Lemma 6.1.** Let \( T > 0 \) and \( I = [0, T] \). If \( u \in L^\infty_t X^s(I \times \mathbb{R}) \) is a solution of (1.1) in \( I \) and \( w \in \mathcal{C}_c^0(L^\infty_x \cap \dot{H}^s)(I \times \mathbb{R}) \) is a solution of (6.1), then by shrinking \( T \) enough depending on \( \| u \|_{L^\infty_t X^s(I \times \mathbb{R})} \), we have

\[ \| w \|_{L^\infty_t (L^\infty_x \cap \dot{H}^s_x)} \lesssim \| w_0 \|_{L^\infty_x \cap \dot{H}^s_x} + \| f \|_{L^1_t (L^\infty_x \cap \dot{H}^s_x)}. \]

**Proof.** We consider the homogeneous problem with \( f = 0 \), as the proof below easily generalizes. We first bound the source terms of (6.2) in \( L^2 \). For the first two source terms, we have

\[ \| D^s \partial_x^{-1}(T_{uxx} w) \|_{L^2_x} \lesssim \| w \|_{\dot{H}^s_x} \| u_x \|_{L^\infty_x} \lesssim \| w \|_{\dot{H}^s_x} \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x} \]

and

\[ \| D^s \partial_x^{-1}(T_u u_{xx}) \|_{L^2_x} \lesssim \| w \|_{L^\infty_x} \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x}. \]

For the balanced frequency case, we have

\[ \| D_x^s \partial_x^{-1}(\Pi(w, u_{xx})) \|_{L^2_x} \lesssim \| D^s \partial_x^{-1}(\Pi(w, u_{xx})) \|_{L^2_x} \lesssim \| D^s w \|_{L^2_x} \| D^{2-s} u \|_{L^{1+s}_x} \]

\[ \lesssim \| D_x^s w \|_{L^2_x} \| D_x^{2} u \|_{L^2_x} \lesssim \| w \|_{\dot{H}^s_x} \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x}. \]

Lastly, for the commutator term, we have

\[ \| \left[ D^s, u \right] w_x \|_{L^2_x} \lesssim \| u_x \|_{L^\infty_x} \| w \|_{\dot{H}^s_x}. \]

By applying the energy estimate from Lemma (2.1), we get that for every \( t \in [0, T] \),

\[ \| w(t) \|_{\dot{H}^s_x} \leq e^{\frac{C}{2} \int_0^t \| u_x(\tau) \|_{L^\infty_x} \, d\tau} \left( \| w_0 \|_{\dot{H}^s_x} + \int_0^t e^{-\frac{C}{2} \| u_x(\eta) \|_{L^\infty_x}} \| u(\tau) \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x} \| w(\tau) \|_{L^\infty_x \cap \dot{H}^s_x} \, d\tau \right) \]

\[ \lesssim \| w_0 \|_{\dot{H}^s_x} + T \| u \|_{L^\infty_t X^s} \| w \|_{L^\infty_x \cap \dot{H}^s_x}. \]

Next, to obtain an \( L^\infty \) estimate, it suffices to consider the low frequencies since by Sobolev embedding,

\[ \| w \|_{L^\infty_x} \lesssim \| w \|_{\dot{H}^s_x}. \]

For the first source term, we decompose into paraproducts as before to estimate

\[ \| P_{\leq 0} \partial_x^{-1}(T_{uxx} w) \|_{L^\infty_x} \lesssim \| P_{\leq 0} \partial_x^{-1}(T_{uxx} w) \|_{L^2_x} \lesssim \| D_x^{-\frac{s}{2}} u_x \|_{L^2_x} \| D_x^s w \|_{L^2_x} \]

\[ \lesssim \| u_x \|_{L^\infty_x} \| w \|_{\dot{H}^s_x} \lesssim \| w \|_{\dot{H}^s_x} \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x}, \]

\[ \| P_{\leq 0} \partial_x^{-1}(T_u u_{xx}) \|_{L^\infty_x} \lesssim \| \partial_x^{-1}(T_u u_{xx}) \|_{\dot{H}^s_x} \lesssim \| w \|_{L^\infty_x} \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x}, \]

and

\[ \| P_{\leq 0} \partial_x^{-1}(\Pi(u_{xx}, w)) \|_{L^\infty_x} \lesssim \| P_{\leq 0} \Pi(u_{xx}, w) \|_{L^1_x} \lesssim \| u_x \|_{\dot{H}^{1+s}_x} \| w \|_{\dot{H}^s_x} \lesssim \| u \|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x} \| w \|_{\dot{H}^s_x}. \]
Thus,
\[ \|P_{\leq 0} \partial_x^{-1}(wu_{xx})\|_{L^2} \lesssim \|w\|_{L^\infty} \|u\|_{\dot{H}^{1+s}_{\lambda}} + \|u\|_{\dot{H}^{1+s}_{\lambda}} \|w\|_{\dot{H}^{s}_{\lambda}}. \]
From Lemma 2.4 with \(T')\ sufficiently small, we have
\[ \|w_{\leq 0}\|_{L^\infty} \lesssim \|(w_0)_{\leq 0}\|_{L^\infty} + T'\|u\|_{L^\infty} \|w\|_{L^\infty_{\dot{H}^{s}_{\lambda}}}. \]
Putting everything together, for \(t \in [0, T')\) we get
\[ \|w(t)\|_{L^\infty \cap \dot{H}^{s}_{\lambda}} \lesssim \|w_0\|_{L^\infty \cap \dot{H}^{s}_{\lambda}} + T'\|u\|_{L^\infty} \|w\|_{L^\infty \cap \dot{H}^{s}_{\lambda}}. \]
By further shrinking \(T')\ depending on \(\|u\|_{L^\infty_{X^s}}\), we obtain the desired estimate. \(\square\)

We now prove a result regarding differences of solutions, that is going to be used in order to justify uniqueness of \(C^0_tX^s\)-solutions in the proof of Theorem 1.3.

**Lemma 6.2.** Let \(T > 0\) and \(I = [0, T]\). Let \(u, v \in C^0_tX^s(I \times \mathbb{R})\) solve (1.1) with \(u_0 - v_0 \in L^\infty \cap \dot{H}^{s}_{\lambda}\). Then \(u - v \in L^\infty_t(L^\infty_{\mathbb{R}} \cap \dot{H}^{s}_{\lambda})(I \times \mathbb{R})\), and for \(T\) sufficiently small depending on \(\|(u, v)\|_{L^\infty_{X^s}}\), we have
\[ \|u - v\|_{L^\infty(I \times \dot{H}^{s}_{\lambda})} \lesssim \|u_0 - v_0\|_{L^\infty \cap \dot{H}^{s}_{\lambda}}. \]

**Proof.** Let \(z = u - v\), which solves the equation
\[ z_t + uz_x + vz_x + z_{xxx} = \frac{\partial_x^{-1}(z_x(u_x + v_x))}{2}. \]
We apply \(D^s_x\) and rearrange to consider the Cauchy problem
\[ w_t + w_{xxx} = D^s_x \left( \frac{\partial_x^{-1}(z_x(u_x + v_x))}{2} \right) - D^s_x(uz)_x - D^s_x(zz)_x := H \]
with initial data \(w(0) = D^s_x(u_0 - v_0) \in L^2_x(\mathbb{R})\). For the first term in \(H\), we decompose into paraproducts and have the \(L^2\) bounds
\[
\|D_x^s \partial_x^{-1}(T_z(u_x + v_x))\|_{L^2} \lesssim \|z\|_{L^\infty} \|u + v\|_{\dot{H}^{1+s}} \]
\[
\|D_x^s \partial_x^{-1}(T_{ux + vz}z)\|_{L^2} \lesssim \|z\|_{\dot{H}^{1+s}} \|u + v\|_{L^\infty} \]
\[
\|D_x^s \partial_x^{-1}\Pi(u_x + v_x, z)\|_{L^2} \lesssim \|D_x \partial_x^{-1}\Pi(u_x + v_x, z)\|_{L^2} \lesssim \|z\|_{\dot{H}^{s}} \|D_x^{s+1/2} (u + v)\|_{L^2_x}. \]

The other terms are estimated directly using product estimates:
\[
\|D_x^s \partial_x(uz)_x\|_{L^2} \lesssim \|u\|_{L^\infty} \|z\|_{\dot{H}^{1+s}} + \|z\|_{L^\infty} \|u\|_{\dot{H}^{1+s}} \]
\[
\|D_x^s \partial_x(z^2)\|_{L^2} \lesssim \|z\|_{L^\infty} \|z\|_{\dot{H}^{1+s}}. \]

Thus, \(H \in L^\infty_tL^2_x([0, T] \times \mathbb{R})\). By applying Lemma 2.1, we infer that (6.3) has a unique solution in \(L^\infty_tL^2_x([0, T] \times \mathbb{R})\). However, both \(w\) and \(D^s_xz\) are solutions (in the sense of tempered distributions), hence \(w = D^s_xz\), and \(z = u - v \in L^\infty_t\dot{H}^{s}_{\lambda}([0, T] \times \mathbb{R})\). It is also clear that \(u - v \in L^\infty_t([0, T] \times \mathbb{R})\).

We now observe that \(z\) satisfies the linearized equation (6.1) with source,
\[ z_t + \partial_x^{-1}(u_{xxx}z) + uz_x + z_{xxx} = zz_x = \frac{\partial_x^{-1}(z_x^2)}{2} =: f. \]
After taking $T$ small enough (depending on $||u||_{L^\infty_tX^s}$), we can apply Lemma 6.1. But first, we have to estimate $f \in L^1_t(L^\infty_x \cap \dot{H}^s_x)([0, T] \times \mathbb{R})$. For the first term of $f$,
\[
||D^s_x \partial_x (z^2)||_{L^2_t} \lesssim ||z||_{L^\infty_t} (||u||_{X^s} + ||v||_{X^s}),
\]
\[
||\partial_x (z^2)||_{L^\infty_t} \lesssim ||z||_{L^\infty_t} (||u||_{X^s} + ||v||_{X^s}).
\]

For the second, we have
\[
||D^s_x \partial_x^{-1}(T_{z_x} z_x)||_{L^2_t} \lesssim ||z||_{H^s_{\dot{X}^s_t}} ||z_x||_{L^\infty_t} \lesssim ||z||_{H^s_{\dot{X}^s_t}} (||u||_{X^s} + ||v||_{X^s})
\]
\[
||D^s_x \partial_x^{-1} \Pi(z_x, z_x)||_{L^2_t} \lesssim ||\Pi(z_x, z_x)||_{L^\infty_t} \lesssim ||D^s_x z||_{L^2_t} ||D^{2-s}_x z||_{L^\infty_t} \lesssim ||z||_{H^s_{\dot{X}^s_t}} ||D^{3/2}_x z||_{L^2_t}
\]
\[
\lesssim ||z||_{H^s_{\dot{X}^s_t}} (||u||_{X^s} + ||v||_{X^s}).
\]

For the $L^\infty_t$ estimate, it suffices to consider the low frequencies since by Sobolev embedding,
\[
||\partial_x^{-1}(z^2) \geq 0||_{L^\infty_t} \lesssim ||\partial_x^{-1}(z^2) \geq 0||_{H^s_{\dot{X}^s_t}}.
\]

We then have for the low frequencies
\[
||P_{\leq 0} \partial_x^{-1}(\Pi(z_x, z_x))||_{L^\infty_t} \lesssim ||P_{\leq 0} \Pi(z_x, z_x)||_{L^1_t} \lesssim ||z||_{H^s_{\dot{X}^s_t}} ||z||_{H^s_{\dot{X}^s_t}} \lesssim ||z||_{H^s_{\dot{X}^s_t}} (||u||_{X^s} + ||v||_{X^s})
\]
\[
||P_{\leq 0} \partial_x^{-1}(T_{z_x} z_x)||_{L^\infty_t} \lesssim ||\partial_x^{-1}(T_{z_x} z_x)||_{H^s_{\dot{X}^s_t}} \lesssim ||z||_{L^\infty_t} (||u||_{X^s} + ||v||_{X^s}).
\]

Thus,
\[
||f||_{L^1_t(L^\infty_t \cap H^s_{\dot{X}^s_t})} \lesssim T ||z||_{L^\infty_t(L^\infty_t \cap H^s_{\dot{X}^s_t})}(||u||_{L^\infty_tX^s} + ||v||_{L^\infty_tX^s}).
\]

Thus, we get that
\[
||w||_{L^\infty_t(L^\infty_t \cap H^s_{\dot{X}^s_t})} \lesssim ||w||_{L^\infty_tX^s} ||w_0||_{L^\infty_t \cap \dot{H}^s_x} + ||f||_{L^1_t(L^\infty_t \cap H^s_{\dot{X}^s_t})}
\]
\[
\lesssim ||w||_{L^\infty_tX^s} ||w_0||_{L^\infty_t \cap \dot{H}^s_x} + T ||z||_{L^\infty_t(L^\infty_t \cap H^s_{\dot{X}^s_t})}(||u||_{L^\infty_tX^s} + ||v||_{L^\infty_tX^s}).
\]

After further shrinking $T$ (depending on $||(u, v)||_{L^\infty_tX^s}$, Lemma 6.1 implies the desired conclusion.

\[
\square
\]

7. Local well-posedness at low regularity

In this section, we prove Theorem 1.3. As we have already noticed at the end of Section 4, this will also imply Theorem 1.4.

Let $R > 0$ be arbitrary. Given data $u_0$ satisfying $||u_0||_{X^s} < R$, we consider the corresponding regularized data
\[
u_0^h = P_{< h} u_0.
\]
Since $u_0^h \rightarrow u_0$ in $X^s$, we may assume that $||u_0^h||_{X^s} < R$ for all $h$.

We construct a uniform $\dot{H}^1_t \cap \dot{H}^{1+s}_x$ frequency envelope $\{c_k\}_{k \geq 0}$ for $u_0$ having the following properties:

a) Uniform bounds:
\[
||P_k(u_0^h)||_{\dot{H}^1_t \cap \dot{H}^{1+s}_x} \lesssim c_k
\]

b) High frequency bounds:
\[
||u_0^h||_{\dot{H}^1_t \cap \dot{H}^{2+s}_x} \lesssim 2^h c_h
\]

c) Difference bounds:
\[
||u_0^{h+1} - u_0^h||_{\dot{H}^s} \lesssim 2^{-h} c_h
\]
Moreover, we also obtain uniform bounds for such solutions, including the family \((u^h)_{h \in \mathbb{Z}}\). We now get that
\[
\|u^h\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x) \cap \dot{H}^1_x} \lesssim 2^h c_h,
\]
and
\[
\|u^{h+1} - u^h\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x)} \lesssim 2^{-h} c_h.
\]
By interpolation, we infer that
\[
\|u^{h+1} - u^h\|_{C^0_t(\dot{H}^{s+1}_x)} \lesssim c_h.
\]
Thus, for \(h \geq 0\),
\[
\|u^{h+1} - u^h\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x)} \lesssim c_h.
\]
As in \([17]\), we get that
\[
\|P_k u^h\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x)} \lesssim c_k.
\]
and that
\[
\|u^{h+k} - u^h\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x)} \lesssim c_{h \leq h+k} = \left(\sum_{n=h}^{h+k-1} c_n^2\right)^{\frac{1}{2}}
\]
for every \(k \geq 1\). Thus, \(u^h\) converges to an element \(u\) belonging to \(C^0_t(\dot{H}^1_x \cap \dot{H}^{1+1}_x)([0, T] \times \mathbb{R})\).

Moreover, we also obtain
\[
\|u^h - u\|_{C^0_t(\dot{H}^s_x \cap \dot{H}^{s+1}_x)} \lesssim c_{h \geq h} = \left(\sum_{n=h}^{\infty} c_n^2\right)^{\frac{1}{2}}.
\]
For pointwise convergence, we use Sobolev embedding for the high frequencies,
\[
\|(u^{h+k})_{>0} - (u^h)_{>0}\|_{C^0_t L^\infty_x} \lesssim \|u^{h+k} - u^h\|_{C^0_t(\dot{H}^1_x)}.
\]
and the estimate \((3.30)\) for the low frequencies:
\[
\|(u^{h+k})_{\leq 0} - (u^h)_{\leq 0}\|_{C^0_t L^\infty_x} \lesssim \|u^h_{0} - u^h_{0}\|_{L^\infty_x} + TR\|u^{h+k} - u^h\|_{C^0_t(\dot{H}^1_x)}.
\]
We conclude that \(u^h \rightarrow u \in C^0_t(X^s([0, T] \times \mathbb{R}))\).

Lemma \([6.2]\) also implies uniqueness for \((1.1)\). For continuity with respect to the initial data, consider a sequence
\[
u_{0j} \rightarrow u_0 \in X^s
\]
and an associated sequence of \(\dot{H}^1_x \cap \dot{H}^{1+1}_x\)-frequency envelopes \(\{c^j_k\}_{k \geq 0}\), each satisfying the analogous properties enumerated above for \(c_k\), and further such that \(c^j \rightarrow c\) in \(l^2(\mathbb{Z})\).

We may assume that \(\|u_{0j}\|_{X^s} < R\) for every \(j \geq 0\). As before, we get uniform bounds for \((u^h_{0j})_{(j,h) \in \mathbb{N} \times \mathbb{Z}}\), and we can interpolate to conclude
\[
\|u^{h+1}_{j} - u^h\|_{C^0_t(\dot{H}^1_x \cap \dot{H}^{1+1}_x)} \lesssim c^j_h.
\]
and

\[ \|P_k u_j^h\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} \lesssim C_k^j, \]

\[ \|u_{j+k}^h - u_j^h\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} \lesssim C_{h \leq h+k}(\sum_{n=h}^{h+k-1} (C_n^j)^2)^{1/2}, \]

\[ \|u_j^h - u_j\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} \lesssim C_{h \geq h} = \left(\sum_{n=h}^{\infty} (C_n^j)^2\right)^{1/2}. \]

Using the triangle inequality, we write

\[ \|u_j - u\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} \lesssim \|u_j^h - u\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} + \|u_j^h - u_j\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} + \|u_j^h - u^h\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)} \]

\[ \lesssim C_{h+k}^j + C_{h+1}^j \]

For every fixed \( h \), Theorem 1.2 tells us that \( u_j^h \to u^h \) in \( X \). This implies that \( u_j \to u \) in \( C^0_T(\dot{H}^1_x \cap \dot{H}^s_x)([0,T] \times \mathbb{R}) \). For pointwise estimates, by applying Sobolev embeddings and using Bernstein’s inequalities, we get that

\[ \|(u_j)_{>0} - u_{>0}\|_{C^0_T L^\infty_x} \lesssim \|u_j - u\|_{C^j_T(\dot{H}^1_x \cap \dot{H}^s_x)}. \]

Besides this, (3.39) implies that

\[ \|(u_j)_{\leq 0} - u_{\leq 0}\|_{C^0_T L^\infty_x} \lesssim \|(u_j)_{\leq 0} - u_{\leq 0}\|_{C^0_T L^\infty_x} + CT\|u_j - u\|_{C^0_T \dot{H}^1_x}. \]

Therefore, \( u_j \to u \) in \( C^0_T X^*[0,T] \times \mathbb{R} \). This finishes the proof.

References

[1] Richard Beals, David H. Sattinger, and Jacek Szmigielski. Inverse scattering solutions of the Hunter-Saxton equation. Appl. Anal., 78(3-4):255–269, 2001.

[2] Alberto Bressan and Adrian Constantin. Global solutions of the Hunter-Saxton equation. SIAM J. Math. Anal., 37(3):996–1026, 2005.

[3] Alberto Bressan, Helge Holden, and Xavier Raynaud. Lipschitz metric for the Hunter-Saxton equation. J. Math. Pures Appl. (9), 94(1):68–92, 2010.

[4] Adrian Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. Ann. Inst. Fourier (Grenoble), 50(2):321–362, 2000.

[5] Adrian Constantin and Joachim Escher. Global existence and blow-up for a shallow water equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 26(2):303–328, 1998.

[6] Adrian Constantin and Joachim Escher. Wave breaking for nonlinear nonlocal shallow water equations. Acta Math., 181(2):229–243, 1998.

[7] Adrian Constantin and Joachim Escher. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. Comm. Pure Appl. Math., 51(5):475–504, 1998.

[8] Raphaël Danchin. A note on well-posedness for Camassa-Holm equation. J. Differential Equations, 192(2):429–444, 2003.

[9] Mingwen Fei. The integrability of dispersive Hunter-Saxton equation. J. Partial Differ. Equ., 25(4):330–334, 2012.

[10] Zihua Guo, Xingxing Liu, Luc Molinet, and Zhaoyang Yin. Ill-posedness of the Camassa-Holm and related equations in the critical space. J. Differential Equations, 266(2-3):1698–1707, 2019.

[11] John K. Hunter, Mihaela Ifrim, and Daniel Tataru. Two dimensional water waves in holomorphic coordinates. Comm. Math. Phys., 346(2):483–552, 2016.

[12] John K. Hunter, Mihaela Ifrim, Daniel Tataru, and Tak Kwong Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. Proc. Amer. Math. Soc., 143(8):3407–3412, 2015.
[13] John K. Hunter and Ralph Saxton. Dynamics of director fields. *SIAM J. Appl. Math.*, 51(6):1498–1521, 1991.
[14] John K. Hunter and Yu Xi Zheng. On a completely integrable nonlinear hyperbolic variational equation. *Phys. D*, 79(2-4):361–386, 1994.
[15] John K. Hunter and Yu Xi Zheng. On a nonlinear hyperbolic variational equation. II. The zero-viscosity and dispersion limits. *Arch. Rational Mech. Anal.*, 129(4):355–383, 1995.
[16] Mihaela Ifrim and Daniel Tataru. Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(2):297–335, 2019.
[17] Mihaela Ifrim and Daniel Tataru. Local well-posedness for quasilinear problems: a primer. *arXiv e-prints*, page arXiv:2008.05684, August 2020.
[18] Peter J. Olver and Philip Rosenau. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E (3)*, 53(2):1900–1906, 1996.
[19] Terence Tao. Global regularity of wave maps. I. Small critical Sobolev norm in high dimension. *Internat. Math. Res. Notices*, (6):299–328, 2001.
[20] Weikui Ye and Zhaoyang Yin. On the Cauchy problem for the Hunter-Saxton equation on the line. *arXiv e-prints*, page arXiv:2012.15429, December 2020.
[21] Zhaoyang Yin. On the structure of solutions to the periodic Hunter-Saxton equation. *SIAM J. Math. Anal.*, 36(1):272–283, 2004.

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