Power Counting and Perturbative One Pion Exchange in Heavy Meson Molecules

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We discuss the possible power counting schemes that can be applied in the effective field theory description of heavy meson molecules, such as the \( X(3872) \) or the recently discovered \( Z_b(10610) \) and \( Z_b(10650) \) states. We argue that the effect of coupled channels is suppressed by at least two orders in the effective field theory expansion, meaning that they can be safely ignored at lowest order. The role of the one pion exchange potential between the heavy mesons, and in particular the tensor force, is also analyzed. By using techniques developed in atomic physics for handling power-law singular potentials, which have been also successfully employed in nuclear physics, we determine the range of center-of-mass momenta for which the tensor piece of the one pion exchange potential is perturbative. In this momentum range, the one pion exchange potential can be considered a subleading order correction, leaving at lowest order a very simple effective field theory consisting only on contact-range interactions.

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I. INTRODUCTION

Heavy meson molecules are a long established theoretical prediction of hadronic physics \[1,6\]. The discovery of the \( X(3872) \) by the Belle collaboration \[7\], and the subsequent confirmation by CDF \[8\], D0 \[9\] and BABAR \[10\], has provided so far the strongest candidate for a bound state of heavy mesons. Owing to the closeness of the \( X(3872) \) to the \( D^0\bar{D}^0 \) threshold, an interpretation in terms of a bound state of these two mesons is both appealing and natural. In turn, the recent finding of the \( Z_b(10610) \) and \( Z_b(10650) \) resonances by Belle \[11,12\], which are located just a few MeV above the \( B^+\bar{B} \) and \( B^+\bar{B}^* \) thresholds respectively, also calls for a molecular description. Moreover, from heavy quark spin symmetry \[13,14\] we should expect other low-lying states, the partners of the \( Z_b \)'s, in the bottom sector \[15,16\].

The shallow nature of the aforementioned candidates for molecular states, apart from being instrumental in their identification, also indicates the existence of a separation of scales between long and short range dynamics. The heavy mesons are far apart from each other and consequently are not able to resolve the details of the short range interaction that may be ultimately responsible for binding them. We therefore expect that heavy meson molecules will be amenable to an effective field theory (EFT) description of their properties and decays \[17,19\].

In the EFT formalism, the long range interaction between two heavy mesons (composed of a heavy and a light quark) is constrained by the low-energy symmetries of the system, in particular chiral symmetry, which determines the pion-exchange dynamics. In turn, the short range interaction is mimicked by local, contact-range operators or counterterms. A second ingredient of the EFT description is the existence of a power counting, that is, an organizational principle from which operators can be ordered from more to less relevant. From power counting we expect to be able to express operators and observables as a power series in term of a small expansion parameter \( x_0 \)

\[
O_{EFT} = \sum_{\nu=\nu_0}^{\nu_{\text{max}}} \mathcal{O}^{(\nu)} x_0^\nu + \mathcal{O}(x_0^{\nu_{\text{max}}+1}), \tag{1}
\]

where \( \nu_0 \) is the order at which the EFT expansion starts, \( \nu_{\text{max}} \) the order at which we perform the calculation, and \( x_0 \), the expansion parameter, can be written as the ratio

\[
x_0 = \frac{Q}{\Lambda_0}, \tag{2}
\]

with \( Q \) (\( \Lambda_0 \)) the generic energy scale associated with the long (short) range physics. A priori power counting entails a remarkable advantage: the relative error of a calculation of order \( \nu \) is known to be \( x_0^{\nu+1} \). Of course, the realization of this promise of accuracy depends on the implementation details and on the unambiguous identification of the expansion parameter. This manuscript will try to deal with this problem.

At this point a natural question arises: how important are the pion exchanges in the description of the heavy meson molecules? Of course, having a bound state requires the non-perturbative treatment of a certain subset of the meson-antimeson interaction, but not necessarily the one-pion exchange (OPE) potential. A nice illustration is provided by the \( X(3872) \), in which the distance between the heavy meson and antimeson is so large (\( \langle r^2 \rangle \sim 10 \text{ fm} \)) that even pions may not be clearly distinguished. This means that we can use a contact-range, pionless EFT in which pions are already considered to be short-ranged \[20,21\]. In such a case we are left with a much simpler theory. Even though the hard scale is the pion mass, \( \Lambda_0 = m_x \), which is an extremely light scale in
hadronic physics, the soft scale $Q$ can be even lighter if the
molecular state is close enough to threshold. In parti-
cular, $Q$ can be identified with the wave number of the
bound state, that is, $Q = \sqrt{2\mu_B}\langle B \rangle$, where $\mu_B$ is the
reduced mass of the two particle system and $B$ the binding
energy. For the $D^{*0}D^0$ mesons conforming the $X(3872)$,
$Q$ is of the order of mere tens of MeV. Thus the EFT
expansion is expected to converge fairly well, providing
a motivation and explaining the success of contact-range
descriptions of the $X(3872)$ [17, 22–24].

However, the applicability of pionless EFT is subjected
to limitations. In the $X(3872)$ we can list two: on the
one hand, the pion exchanged between the $D^{*0}$ and $D^0$
meson is almost on the mass shell, and consequently its
effects spread over a much larger range than expected.
That is, the irrelevance of pion exchanges is not so evi-
dent as it appears at first sight. Luckily, as shown in X-
EFT [19], pions are actually perturbative in the $X(3872)$.
On the other, if we consider the charged $D^{*+}D^-$ com-
ponent of the $X(3872)$, which is essential for explaining
certain branching ratios [25, 26], the associated soft
scale is $Q \sim 125$ MeV, of the order of $m_\pi$. Thus the
explicit inclusion of the charged channel may lie outside
the range of applicability of a pionless EFT. If we now
consider the bottom sector, the $Z_b(10610)$ and $Z_b(10650)$
resonances are a only few MeV away from the $B^* B$ and
$B^* B^*$ threshold [4]. However, the large reduced mass of
these systems imply that $Q \sim m_{Z_b}$ corresponds to a bind-
ing energy of merely 4 MeV at which the wave function
will start to probe the pions, requiring a non-perturbative
theory [28].

In all the previous examples the inclusion of pion ex-
changes may be required for a proper EFT description of the
molecular states. This can be rather cumbersome, owing

to the rich angular momentum coupled channel
structure triggered by tensor forces, especially in the
$D^* D^*$ and $B^* B^*$ cases. However, there exists a binding
energy range in which the OPE potential is subleading with
respect to the contact range interactions and hence
perturbative. In this energy window the pionfull EFT
will still be a contact-range theory at the lowest (or lead-
ing) order (LO), as pion exchanges will not enter until
next-to-leading order (NLO). In the two-nucleon system
the paradigmatic example of this kind of EFT is the
Kaplan, Savage and Wise (KSW) counting [29, 30], which
served as inspiration for X-EFT [19].

The question we want to answer is: where does the
limit between perturbative and non-perturbative pions
stand? If we take a second look to the deuteron in the two
nucleon system, which shares many similarities with the
heavy meson molecules, we see that the convergence of
the EFT with perturbative pions is numerically marginal.
As shown by Fleming, Stewart and Mehen (FMS) [31]
by a thorough next-to-next-to-leading order (N^2LO) cal-
culation, the convergence of the KSW counting in the
deuteron case is limited at most to $\Lambda_0 \sim 100$ MeV, a
rather low figure (indeed smaller than $m_{\pi}$). Taking
into account that in the deuteron the wave number is
$\gamma = 45$ MeV, the previous breakdown scale translates
into a rather slow convergence rate. In fact, the theory
would not converge at all had the deuteron be bound by
about 4 MeV or more. The purpose of this work is there-
fore to find the corresponding $B_{\text{max}}$ below which pions
are perturbative, as this limit will gives us essential in-
formation about the convergence of the EFT for heavy
meson molecules.

Of course, performing a N^2LO calculation is beyond
the scope of the present paper. Apart from that, the
scarce experimental input available on heavy meson-
antimeson systems makes the previous task impractical:
we do not have the information required to fix the coun-
terterms. Therefore, we need to resort to a more indirect
path. In the two-nucleon system the solution was pro-
vided by Birse [32]. The idea is to consider the OPE in
the chiral limit, in which it reduces to a pure power-law
potential of the type $1/r^3$. The interesting thing here
is that these kind of potentials have been studied and
analyzed in detail in the field of atomic physics. The
long-range solutions of OPE in the chiral limit can be
expressed in terms of a a particular type of series, as
shown by Cavagnero [33] and Gao [34]. Birse was able to
show that one can extract the breakdown scale of a the-
ory containing perturbative pions with the help of these
techniques, and the results help to explain remarkably
well the lengthier N^2LO calculations of FMS [31]. In this
work we will extend the observations of Birse to the pe-
culiarities of the tensor force in heavy meson molecules.

The article is structure as follows: in Sect. II we present
a brief overview of certain key EFT ideas that will help
to put the results of this work into proper context. In
Sect. III we will write the OPE and contact range po-
tentials between a heavy meson and antimeson, and in
Sect. IV we will discuss the different power counting
schemes that we can apply. In Sect. V we will extract
the breakdown scale of perturbative pion theories, and
finally, in Sect. VI we will discuss the results and their
implications on the EFT treatment of heavy molecular
states. We have also included an Appendix containing
the technical details involved in the derivation of the EFT
potential at lowest order.

II. GENERAL CONSIDERATIONS

The purpose of this section is to provide a quick re-
view of the EFT formalism for non-relativistic two-body
systems. The discussion is heavily based on the EFT for-
mulation of the two-nucleon system (see Refs. [35–39] for

1 It is interesting to notice that even though the original experi-
mental analysis by Belle [11, 12] locates the two states above the
$B^* B$ and $B^* B^*$ thresholds, this depends on the Breit-Wigner
parametrization employed for the $Z_b$’s as stressed in Ref. [24].
This work also indicates that the two $Z_b$’s may be located be-
low threshold and hence be bound states (instead of two-particle
resonances) after all.
reviews), which can be trivially translated and applied to heavy meson molecules with minor modifications, as demonstrated in Ref. 28. Thus, we begin with naive dimensional power counting, as originally proposed by Weinberg 40, 41, and then explain the modifications that have been required to successfully formulate a non-relativistic EFT to two-body systems forming shallow bound states 20, 29, 30, 42, 43. As we are dealing with heavy meson molecules, we will include notation related to heavy quark symmetry. In what follows $m_Q$ is the mass of the heavy quark conforming a heavy meson, and $H = P, P^*$ is used to denote a generic heavy meson with orbital angular momentum $l = 0$ between the heavy quark and the light quark. The heavy meson $P (P^*)$ has total spin $s = 0 (s = 1)$ and hence it is a pseudoscalar (vector) meson. If we are specifically dealing with the charm or bottom sector we will particularize the $P, P^*$ heavy meson notation by $D, D^*$ and $B, B^*$. We also use $D^{(s)}$ and $B^{(s)}$ as a generic for the pseudoscalar/vector cases. We will only consider the case in which the light quark is the $u$ or $d$ quark. The extension to the strange sector is straightforward.

A. Power Counting

The formulation of EFT depends on the existence of a separation of scales: we can distinguish between $Q$, the low energy scale that characterizes the physics we are interested in, and $\Lambda_0$, the high energy scale at which the effective description we are using stops to be applicable. In the EFT framework, the two-body potential can be expanded as a power series on the ratio of these scales, leading to

$$V_{\text{EFT}} = \sum_{\nu = -\nu_0}^{\nu_{\text{max}}} V^{(\nu)} + O\left(\left(\frac{Q}{\Lambda_0}\right)^{\nu_{\text{max}} + 1}\right),$$

(3)

where $\nu_0 \geq -1$ is the order at which the expansion begins and $\nu_{\text{max}}$ the order at which we perform the calculation. In chiral nuclear 40, 41 and heavy hadron 44 EFT the generic scale $Q$ usually includes the momenta of the two interacting particles and the mass of the pion, while $\Lambda_0$ refers to the mass of the rho meson or the momentum scale at which the internal structure of the two particles starts to be resolved.

As can be seen, we expect the theoretical error in the determination of the EFT potential to decrease, at least as long as $Q \lesssim \Lambda_0$. At this point we should take into account that the light scales include on the one hand the pion mass $m_\pi$, which does not change $4$ and on the other the momenta $Q \sim p, p'$ of the two heavy mesons, which can vary. This means that the expansion of the potential (and the scattering observables) is only valid for sufficiently small $p, p'$. To avoid the related divergences with taking $p, p' \gtrsim \Lambda_0$ in loops we usually include a cut-off $\Lambda$ that serves as an intermediate scale between $Q$ and $\Lambda_0$ (i.e. we take $Q \lesssim \Lambda \lesssim \Lambda_0$).

B. The Scaling of Operators

The power counting assignment of a certain operator tells us about its scaling properties. A contribution to the (momentum space) potential is assigned the order $\nu$ if it scales as

$$V^{(\nu)}(\lambda Q) = \lambda^\nu V^{(\nu)}(Q),$$

(4)

under the rescaling of all the light scales by a factor $\lambda$. This means that if we reduce the momentum or the pion mass by a factor $R = \frac{\lambda}{\lambda_0}$, with $R \leq 1$, thus increasing the separation of scales by a factor of $\lambda = \frac{1}{\lambda_0} \geq 1$, the size of the order $\nu$ contributions will decrease as $R^\nu$. That is, such a contribution becomes smaller the larger the scale separation, justifying their power counting assignment.

Scaling is very interesting in the sense that it determines the behaviour of the potential in coordinate space. If we consider the order $\nu$ contribution potential in momentum space, we simply have

$$V^{(\nu)}(\lambda \bar{q}, \lambda \lambda Q) = \lambda^\nu V^{(\nu)}(\bar{q}, Q),$$

(5)

where we have now explicitly considered the dependence on the momentum exchanges between the two particles, $\bar{q} = \bar{p} - \bar{p}'$. After Fourier-transforming into coordinate space, the previous scaling translates to

$$V^{(\nu)}(\frac{\bar{r}}{\lambda}, \lambda Q) = \lambda^{3+\nu} V^{(\nu)}(\bar{r}, Q),$$

(6)

which admits two kind of general solutions, contact range and finite range. The contact range solution is trivial to construct from the Dirac $\delta$-function and its derivatives, yielding

$$V_C^{(\nu)}(\bar{r}, Q) = C_\nu \partial^\nu \delta(\bar{r}),$$

(7)

where $\partial$ denotes a general derivative of the Dirac $\delta$. Of course, parity constraints imply that this kind of contribution only appears at $\nu = 2n$. On the other hand, the finite range solution must comply to the form

$$V^{(\nu)}(\bar{r}, Q) = \frac{F_{\nu}}{r^{3+\nu}} f^{(\nu)}(Q r),$$

(8)

with $f(x)$ an arbitrary (non exclusively power-law) function, which decays exponentially at large distances as it stems from meson exchanges.

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2 Unless we are considering chiral extrapolations.

3 Notice that we have restricted ourselves to the local potential case. Non-localities only appear at high orders in the chiral expansion ($\nu = 4$ in the two-nucleon case) and in addition they are suppressed by the mass of the heavy mesons.
In the previous equations $C_{0}$ and $F_{0}$ are constants with dimensions of $1/\text{[energy]}^{\nu+2}$. From power counting we expect the related energy scale to be $\Lambda_{0}$, that is,

$$C_{\nu} \sim \frac{1}{\Lambda_{0}^{\nu+2}} \quad \text{and} \quad F_{\nu} \sim \frac{1}{\Lambda_{0}^{\nu+2}}.$$ (9)

However, as we will see in the following paragraphs, sometimes there is an unexpected contamination of $C_{\nu}$ and $F_{\nu}$ by a light scale. This will trigger a change in the power counting of the potential, promoting a certain contribution from a higher to a lower order. An illustration is given by the scaling arguments discussed in the introduction of Ref. 43.

### C. Iteration

If we are considering scattering states, we should plug the potential into the Lippmann-Schwinger equation,

$$T = V + V G_{0} T,$$ (10)

where $V$ is the EFT potential, $T$ is the T-matrix and $G_{0} = 1/(E - H_{0})$ the resolvent operator. Analogously, if we are considering heavy meson molecules we should iterate the potential in the bound state equation

$$|\Psi_{B}\rangle = G_{0} V |\Psi_{B}\rangle,$$ (11)

to obtain the binding energy and the wave function $|\Psi_{B}\rangle$.

Within the EFT framework the previous equations are expected to be re-expanded according to the power counting of the potential. In this way, we are guaranteed to be able to estimate the error properly. For this, we need to take into account that the scaling of the resolvent operator is given by

$$G_{0} \sim \mu_{R} Q,$$ (12)

where $\mu_{R}$ is the reduced mass of the two body system, which for a non-relativistic theory can be considered a high energy scale, $\mu_{R} \sim \Lambda_{0}$. From the point of view of power counting, the scaling of $G_{0}$ means that only contributions to the EFT potential such that $O(G_{0} V) \sim 1$ should be iterated, i.e. the order $Q^{-1}$ piece of the potential.

The existence of shallow bound states between heavy mesons imply that there is a contribution to the potential of order $Q^{-1}$. However, when we compute the EFT potential from the chiral lagrangian we only find contributions of order $Q^{0}$ and higher. The OPE potential is order $Q^{0}$, as can be trivially checked from

$$V_{\text{OPE}}(\bar{q}) \propto \frac{q^{2}}{q^{2} + m_{\pi}^{2}},$$ (13)

and it is also clear that contact range potentials must be at least of order $Q^{0}$. This means that naive dimensional analysis is not enough to determine accurately the order of an operator if there is a bound state. What is missing is the unexpected contribution from a light energy scale, as has been discussed extensively in nuclear EFT 20, 29, 30, 42, 43. A way to see this is to consider the coupling $C_{0}$ of the lowest order contact operator: if there is a bound state, $C_{0}$ cannot be perturbative, that is, small. We can account for the fact that $C_{0}$ is big by multiplying the naive expectation for its size by a big number, say $\Lambda_{0}/Q$, yielding

$$C_{0}(Q) \sim \frac{1}{Q \Lambda_{0}},$$ (14)

from which it is obvious that $C_{0}(Q)$ is of order $Q^{-1}$, promoting the lowest order contact range potential from order $Q^{0}$ to $Q^{-1}$. A different way is to solve the bound state equation with a contact operator, in which case we obtain that $C_{0}$ should scale as

$$C_{0}(Q) \sim \frac{1}{\gamma \Lambda_{0}},$$ (15)

with $\gamma = \sqrt{2\mu_{R} B}$ the wave number of the heavy meson molecule, and $B$ the binding energy. For a shallow bound state we have $\gamma \sim Q$ and the wave number is the light scale contaminating $C_{0}$.

Moreover, there is a second mechanism that can lead to the promotion of the potential, in particular OPE, to order $Q^{-1}$. It is based on a well-known argument of Ericson and Karl 4 about the relative strength of OPE in heavy mesons molecules. The idea is that the intrinsic strength of the OPE potential, i.e. $F_{0}$ as defined in Eq. 9, is roughly independent on the heavy quark mass. As a consequence, the ratio

$$\frac{V^{(0)} G_{0} V^{(0)}}{V^{(0)}} \sim \frac{\mu_{R}}{\Lambda_{0}} \frac{Q}{\Lambda_{0}},$$ (16)

is only small for the naive identification $\mu_{R} \sim \Lambda_{0}$, for which the ratio above scales as expected if the OPE potential is a $Q^{0}$ operator. However, while the chiral hard scale $\Lambda_{0}$ is fixed, the reduced mass of the two heavy mesons systems scales as $m_{Q}$ in the heavy quark limit. Eventually we can have $\mu_{R} \gg \Lambda_{0}$, enhancing the pion loops to the point of making the previous ratio to be of order $Q^{0}$. In such a case the OPE potential will become non-perturbative, as expected from Ref. 42.

### D. Coupled Channels

If we consider heavy meson molecules, we expect from heavy quark symmetry that the energy gap between the thresholds of the three different combinations of pseudoscalar and vector mesons, that is

$$HH = PP, PP^{*}, P^{*}P^{*},$$ (17)

4 Actually, it is proportional to the axial pion coupling $g_{A}$, which contains a contribution of order $m_{Q}^{0}$ plus corrections of order $m_{Q}^{2}$, $m_{Q}^{3}$ and so on.
will shrink as $\Delta Q \sim 1/m_Q$ \cite{46,47}. This indicates, at first sight, the necessity of a coupled channel approach. In such a case, the Lippmann-Schwinger equation can be written as

$$T_{AB} = V_{AB} + V_{AC} G_0 T_{CB},$$

where $A, B, C = PP, PP^*, P^*P^*$. If we are considering the scattering in the vicinity of a given threshold $\alpha$, then we set $A = B = \alpha$ in the Lippmann-Schwinger equation. In addition, we set the center-of-mass energy to zero $E = 0$ at the $\alpha$ threshold. This means that we must include a proper energy shift for the resolvent operator if $G_0$ is acting on a channel $C \neq \alpha$, that is

$$G_0^{-1}(E) | \bar{q}, C) = (E - \frac{q^2}{2 \mu_C} - \Delta Q) | \bar{q}, C),$$

where we have considered the inverse of the $G_0$ operator for simplicity. In the expression above, $\mu_C$ is the reduced mass of channel $C$ and the energy shift is given by $\Delta Q = M_a - M_C$, with $M_a$ and $M_C$ the total mass of channels $\alpha$ and $C$ respectively.

However, the counting of the $G_0$ operator is not necessarily $Q$ on a channel $C \neq \alpha$. If we consider the ratio of the $G_0$ operator evaluated at the $C$ and $\alpha$ channels respectively, we find that

$$\frac{G_0^{-1}(E) | \bar{q}, C)}{G_0^{-1}(E) | \bar{q}, \alpha)} = \frac{2 \mu_A E - q^2}{2 \mu_C (E - \Delta Q) - q^2} \sim \left( \frac{Q}{\Lambda_C} \right)^2,$$

where $Q \sim \sqrt{2 \mu_R E} \sim q$ (we are assuming similar reduced masses), and $\Lambda_C = \sqrt{2 \mu_C \Delta Q}$ is the momentum scale related to the coupled channel effects. We can distinguish here two possible situations: (i) $\Lambda_C \ll \Lambda_0$ and (ii) $\Lambda_C \sim \Lambda_0$. If $\Lambda_C \ll \Lambda_0$, we can consider the coupled channel scale to be light, $\Lambda_C \sim Q$, and the resolvent operator will be of order $Q$, as expected (a more formal account can be consulted in Refs. \cite{18,49}). On the contrary, if $\Lambda_C \sim \Lambda_0$, then the resolvent operator for the channel $C \neq \alpha$ will be suppressed by two powers of $Q/\Lambda_C$, that is, by two orders in the EFT expansion.

In the particular case of heavy mesons, $\Lambda_C = \sqrt{2 \mu_{HH} \Delta Q}$, where $\mu_{HH}$ is the reduced mass of the system and $\Delta Q$ is the energy split between different pairs of heavy meson systems. For the charm sector, $\Lambda_C \simeq 520$ MeV for the DD – DD* and DD* – D* D* pairs and $\Lambda_C \simeq 740$ MeV for DD – D* D*. For the bottom sector, the coupled-channel scales are very similar, yielding $\Lambda_C \simeq 490$ MeV and $\Lambda_C \simeq 700$ MeV respectively. The previous estimations indicate that $\Lambda_C$ is similar to the hard scale of the theory, $\Lambda_0 \sim 0.5 - 1.0$ GeV, which indicates that coupled channel effects are suppressed.

A surprising aspect about the coupled channel effects is that the associated momentum scale is similar in the charm and bottom sectors. This can be easily understood if we consider the heavy quark limit, $m_Q \to \infty$, in which we expect $\mu_{HH} \sim m_Q$ and $\Delta Q \sim 1/m_Q$, meaning that $\Lambda_C \sim m_Q^0$. That is, the suppression of the coupled channel effects is basically independent of the heavy quark mass. Even though in the heavy quark limit the heavy mesons are degenerate, the energy split between them remains beyond the scope of a non-relativistic chiral EFT. The convergence of the EFT description for non-relativistic particles depends on momentum scales instead of energy scales. However, from the scaling of the center-of-mass energy with respect to the heavy quark mass, i.e. $E \sim p^2/m_Q$, we see that the energy window in which the EFT is valid decreases as the heavy quark mass grows.

## III. THE LEADING ORDER POTENTIAL

The lowest order potential in the effective field theory description of heavy meson molecules can be decomposed into a contact-range and a finite-range piece

$$V_{HH}^{(0)} = V_C^{(0)} + V_F^{(0)},$$

where the subscripts $C$ and $F$ are used to denote the contact and finite-range character of each contribution to the potential. In the equation above we have labelled the potential by the expected naive dimensional scaling, rather than the actual scaling that depends upon the particular power counting under consideration. Of course, the existence of a shallow molecular scale will imply that at least $V_C$ is of order $Q^{-1}$. However, we will not consider the actual order of the different contributions of the potential (and their consequences) until the next section.

### A. The Contact Range Potential

At LO we will naively expect a total of six counterterms in a given isospin channel. This number correspond to the number of s-wave channels for the HH system: a $0^+$ state for PP, two opposite C-parity states, $1^+$ and $1^-$ for the P$^*$$\bar{P}$/PP$^*$ system, and the three $0^+$, $1^+$ and $2^+$ P$^*$$\bar{P}$$^*$ states. However, the contact range is constrained by the requirements of heavy quark spin symmetry (HQSS) \cite{13,14}, which in turn implies that the six counterterm figure is reduced to only two independent counterterms \cite{18}. We will call these counterterms $C_{0a}$ and $C_{0b}$. Ignoring the coupled channel effects, the potential reads (see the Appendix for details)

$$V_{C,PP}^{(0)}(q, 0^{++}) = C_{0a},$$

$$V_{C,P^*P}^{(0)}(q, 1^{+-}) = C_{0a} - C_{0b},$$

$$V_{C,P^*P}^{(0)}(q, 1^{++}) = C_{0a} + C_{0b},$$

$$V_{C,P^*P}^{(0)}(q, 0^{++}) = C_{0a} - 2C_{0b},$$

$$V_{C,P^*P}^{(0)}(q, 1^{+-}) = C_{0a} - C_{0b},$$

$$V_{C,P^*P}^{(0)}(q, 2^{++}) = C_{0a} + C_{0b}.$$
depending on the $J^{PC}$ quantum number and the specific HH system under consideration. The explicit representation of these two counterterms in the coupled channel basis of HH states can be consulted in Ref. [16].

**B. The Finite Range Potential**

The OPE potential between a heavy meson and antimeson is local, and hence it only depends on the exchanged momentum between the heavy mesons:

$$\langle \vec{p}'' | V_F^{(0)} | \vec{p} \rangle = V_F^{(0)}(\vec{q}),$$  

with $\vec{q} = \vec{p} - \vec{p}''$. The explicit consideration of the pseudoscalar / vector heavy meson channels allows us to write the potential as

$$V_{F,PP\rightarrow PP}^{(0)}(\vec{q}) = 0,$$  

$$V_{F,P^*P\rightarrow PP}^{(0)}(\vec{q}) = -\frac{g^2}{2f^2}\vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{c}_1 \cdot \vec{q} \vec{c}_2^* \cdot \vec{q}}{q^2 + \mu^2},$$  

$$V_{F,P^*P\rightarrow PP}^{(0)}(\vec{q}) = -\frac{g^2}{2f^2}\vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q}}{q^2 + m^2},$$  

depending on the particular heavy meson channel under consideration. In the equations above $g$ is the pion axial coupling, $f = 132$ MeV the pion decay constant, $m_{\pi} = 138$ MeV the pion mass, $\vec{\tau}_{1(2)}$ the isospin operators on the heavy meson $1(2)$, $\vec{c}_{1(2)}$ is the heavy vector meson polarization, and $\vec{S}_{1(2)}$ is the spin operator for intrinsic spin $S = 1$. For the $P^*P \rightarrow PP^*$ channel, we use $\mu$ instead of $m_{\pi}$ as a consequence of the different masses of the $P$ and $P^*$ mesons, the pion is emitted with the zero component of the momentum different to zero. In the static limit this is equivalent to changing the effective mass of the pion. We have that $\mu^2 = m^2_{\pi} - \Delta_Q^2$, with $\Delta_Q$ the mass splitting between the $P$ and $P^*$ mesons. In the heavy quark limit, $m_Q \rightarrow \infty$, we obtain $\mu = m_{\pi} + O(1/m_Q)$, and it would be practical to expand the potential in powers of $1/m_Q$. While this is the situation we find in the bottom sector ($BB^*$), the charm one (the $D\bar{D}^*$ potential) is more complicated as we have $\mu^2 < 0$. The OPE potential thus acquires a small imaginary piece that is related to the probability loss induced by the open decay channel $DD^*/DD^* \rightarrow DD\pi$. In this regard it is not clear up to what extent the static approximation to the potential holds. The rigorous theoretical treatment of this situation requires the inclusion of pions as dynamical degrees of freedom and the explicit consideration of the $DD\pi$ three body channel [50], which indicates a significant impact of the three body dynamics on the $X(3872) \rightarrow DD\pi$ decay rate, but only a mild effect on the $DD^*$ wave functions (that is, the residue of the $X(3872)$ pole in the language of Ref. [51]). This seems to indicate that, if we are only interested in the wave functions, we can simply ignore the imaginary piece of the OPE potential for $\mu^2 < 0$ and continue using the static limit.

Apart from the previous, there are additional pieces of the potential that mix the different heavy meson channels. However, as we have seen, coupled channel effects can be safely disregarded (even in the heavy quark limit) as their size is similar to the short range effects beyond chiral symmetry. Therefore we have decided to ignore the particle coupled-channel terms.

The previous potential has been computed for the meson-meson case. For obtaining the meson-antimeson potential in the isospin symmetric basis, we perform a $G$-parity transformation: this changes the sign of the $P^*P^*$ potential, but leaves the $P^*P/PP^*$ potential unchanged. However, the $P^*P/PP^*$ potential is better written in a definite C-parity basis, for which we employ

$$|P^*(\eta)| = \frac{1}{\sqrt{2}} \left[ |P^*\bar{P}| - \eta |P\bar{P}^*| \right],$$  

where $\eta$ is the intrinsic C-parity of the meson-antimeson system. This translates into a factor of $-\eta$ for the $P^*P/PP^*$ potential when expressed in this basis. The final form of the meson-antimeson potentials is thus

$$V_{F,P^*P\rightarrow PP}^{(0)}(\vec{q}) = \eta \frac{g^2}{2f^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{c}_1 \cdot \vec{q} \vec{c}_2^* \cdot \vec{q}}{q^2 + \mu^2},$$  

$$V_{F,P^*P\rightarrow PP}^{(0)}(\vec{q}) = \frac{g^2}{2f^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{S}_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q}}{q^2 + m^2_{\pi}}.$$  

The general form of the potential can be schematically written as

$$V_{F,HH}^{(0)}(\vec{q}) = \eta \frac{g^2}{2f^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{a}_1 \cdot \vec{q} \vec{b}_2 \cdot \vec{q}}{q^2 + \mu^2},$$  

where $\vec{a}_1$, $\vec{b}_2$ represent the particular spin operators and $\mu$ the particular value of the effective pion mass to be used in each case. In this form, the intrinsic C-parity $\eta$ is then to be taken $\eta = 1$ for the $P^*P^*$ channel.

The coordinate space potential can be obtained by Fourier-transforming the momentum space one, in which case we obtain

$$V_{F,HH}^{(0)}(\vec{r}) = \eta \vec{\tau}_1 \cdot \vec{\tau}_2 \vec{a}_1 \cdot \vec{b}_2 \frac{g^2}{6f^2 \mu} \delta(\vec{r}) - \eta \vec{\tau}_2 \cdot \vec{\tau}_1 \left[ \vec{a}_1 \cdot \vec{b}_2 W_C(r) + (3\vec{a}_1 \cdot \hat{r} \vec{b}_2 \cdot \hat{r} - \vec{a}_1 \cdot \vec{b}_2) W_T(r) \right],$$  

where $W_C(r)$ and $W_T(r)$ are the central and tensor pieces of the potential, which read

$$W_C(r) = \frac{g^2 \mu^3}{24\pi f^2} \frac{e^{-\mu r}}{\mu r},$$  

$$W_T(r) = \frac{g^2 \mu^3}{24\pi f^2} \frac{e^{-\mu r}}{\mu r} \left( 1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2} \right).$$

The interesting feature of the tensor force is that it can mix channels with different angular momentum.

\[5\] Or, more properly, the spin-spin part of the potential.
C. The Partial Wave Decomposition

In this subsection we will show the explicit partial wave decomposition of the potential. We will use the spectroscopic notation $^{2S+1}L_J$ for characterizing a certain partial wave with spin $S$, orbital angular momentum $L$ and total angular momentum $J$. If two or more partial waves are coupled we will indicate it with a dash, for example $^3S_1-^3D_1$, $^3P_2-^3F_2$ or $^1F_3-^5P_3-^5F_3-^5H_3$.

There are certain rules on how partial waves coupled to each other. On the one hand, the central operator, which we define as
\begin{align}
C_{12}^{P^*P^*}(\hat{r}) &= \hat{e}_1 \cdot \hat{e}_2^* , \\
C_{12}^{'P'P^*}(\hat{r}) &= \hat{S}_1 \cdot \hat{S}_2^* ,
\end{align}
(39) (40)
conserves parity, C-parity, spin, orbital and total angular momentum. Consequently it does not mix partial waves. On the other hand, the tensor operator, given by
\begin{align}
S_{12}^{P^*P^*}(\hat{r}) &= 3 \hat{e}_1 \cdot \hat{e}_2^* \cdot \hat{r} - \hat{e}_1 \cdot \hat{e}_2^* , \\
S_{12}^{'P'P^*}(\hat{r}) &= 3 \hat{S}_1 \cdot \hat{S}_2^* \cdot \hat{r} - \hat{S}_1 \cdot \hat{S}_2^* ,
\end{align}
(41) (42)
conserves parity, C-parity and total angular momentum, but neither spin nor orbital angular momentum. However, parity and C-parity conservation imply that the tensor force can only change the spin $S$ and orbital angular momentum $L$ by an even number of units. This in turn implies that we have four possibilities, which we define as follows
\begin{align}
^3U_J &\equiv ^3J_J , \\
^3C_J &\equiv ^3(J-1)_J - ^3(J+1)_J , \\
^5C_J &\equiv ^5(J-1)_J - ^5(J+1)_J , \\
^{1-5}C_J &\equiv ^1J_J - ^5(J-2)_J - ^5J_J - ^5(J+2)_J ,
\end{align}
(43) (44) (45) (46)
where $(U)C$ stands for (un)coupled. In this notation, $^3C_1$ is $^3S_1-^3D_1$, $^5C_2$ stands for $^3P_2-^3F_2$ and $^{1-5}C_3$ would be $^1F_3-^5P_3-^5F_3-^5H_3$. Of course, low $J$ coupled channels can be uncoupled or have less members than expected. Examples are the $^3F_0 (^5C_0)$ or the $^1S_0-^5D_0 (^1-5C_0)$ partial waves.

The matrix elements of the central and tensor operators between different $^{2S+1}L_J$ partial wave are schematically calculated as
\begin{align}
\langle b|\hat{O}_{12}|a\rangle &= \delta_{J_aJ_b} \delta_{M_aM_b} \sum_\{\mu\} \int d^2\hat{r} \langle \mu_\alpha|\hat{O}_{12}(\hat{r})|\mu_\alpha\rangle \\
&= \delta_{J_aJ_b} \delta_{M_aM_b} O_{J(S_aS_bL_aL_a)} ,
\end{align}
(47)
where $a$, $b$ are the initial and final partial wave, characterized by the vector $|S_a(a)L_a(a)J_a(a)M_a(a)\rangle$, and $\{\mu\}$ is whatever internal angular momentum quantum numbers $a$ and $b$ have. The two Kronecker $\delta$’s are a consequence of total angular momentum conservation. In the case of the central operator, which also conserves $S$ and $L$, we simply have
\begin{align}
C_{J(S_aS_bL_aL_a)} = C_{J(SL)} \delta_{S_aS_b} \delta_{L_aL_a} ,
\end{align}
(48)
with
\begin{align}
C_{J(1L)}^{P^*P^*} &= 1 , \\
C_{J(3L)}^{P^*P^*} &= \frac{1}{2} [S(S+1) - 4] ,
\end{align}
(49) (50)
in the $P^*P^*$ and $P^*P^*$ cases respectively.

We write the matrix elements of the tensor operator directly in one of the four coupled channel basis previously defined. For the uncoupled channel case we have
\begin{align}
S_{J=0}^{P^*P^*}(3U_J) &= -1 , \\
S_{J=0}^{P^*P^*}(3U_J) &= +1 .
\end{align}
(51) (52)
In the $^3C_J$ case we obtain
\begin{align}
S_{J=1}^{P^*P^*}(3C_J) &= -2 , \\
S_{J=0}^{P^*P^*}(3C_J) &= +2 ,
\end{align}
(53) (54)
\begin{align}
S_{J=2}^{P^*P^*}(3C_J) &= \frac{1}{2J+1} \times \begin{pmatrix}
J - 1 \\
-3\sqrt{J(J+1)} & J + 2
\end{pmatrix} ,
(55)
\end{align}
\begin{align}
S_{J=2}^{P^*P^*}(3C_J) &= \frac{1}{2J+1} \times \begin{pmatrix}
J - 1 \\
-3\sqrt{J(J+1)} & J + 2
\end{pmatrix} ,
(56)
\end{align}
The other two set of coupled channels, $^5C_J$ and $^{1-5}C_J$, only happen for the $P^*P^*$ system. In the $^5C_J$ case, we obtain the matrices
\begin{align}
S_{J=1}^{P^*P^*}(5C) &= -1 , \\
S_{J=2}^{P^*P^*}(5C) &= \frac{1}{2J+1} \times \begin{pmatrix}
J + 5 \\
3\sqrt{J(J+1)} - 2 & J - 4
\end{pmatrix} ,
(57)
(58)
\end{align}
where there is no $J = 0$ partial wave with fits into the $^5C_J$ scheme (the $^5P_0$ wave is unphysical). In the $^{1-5}C_J$ case, we obtain
\begin{align}
S_{J=0}^{P^*P^*}(1-5C) &= \begin{pmatrix}
0 & -\sqrt{2} \\
-\sqrt{2} & -2
\end{pmatrix} ,
(59)
S_{J=1}^{P^*P^*}(1-5C) &= \begin{pmatrix}
0 & 2 \sqrt{2} - \sqrt{\frac{5}{2}} \\
2 \sqrt{2} - \sqrt{\frac{5}{2}} & -\sqrt{\frac{5}{2}} - \sqrt{\frac{2}{5}}
\end{pmatrix} ,
(60)
\end{align}
\[ S_{J \geq 2}^{(1-5)C} = \begin{pmatrix}
0 & \sqrt{\frac{3J(J-1)}{(2J+1)(2J+3)}} & \sqrt{\frac{2J(J+1)}{(2J+3)(2J+1)}} & \sqrt{\frac{3(J+1)(J+2)}{(2J+3)(2J+1)}} \\
\frac{3J(J-1)}{(2J+1)(2J+3)} & -\frac{2J-1}{2J+1} & \frac{6J(J+1)(J+3)}{(2J+3)(2J+1)} & \frac{6(J+1)(J+2)}{(2J+3)(2J+1)} \\
\frac{2J(J+1)}{(2J+3)(2J+1)} & \frac{6J(J+1)(J+3)}{(2J+3)(2J+1)} & 0 & \frac{6(J+1)(J+2)}{(2J+3)(2J+1)} \\
\frac{3(J+1)(J+2)}{(2J+3)(2J+1)} & \frac{6(J+1)(J+2)}{(2J+3)(2J+1)} & \frac{6(J+1)(J+2)}{(2J+3)(2J+1)} & 0 \\
\end{pmatrix}. \]  

(61)

For \( J = 0, 1 \) the number of angular momentum channels is smaller than expected and the corresponding tensor matrices are written in the \( 5S_0-5D_0 \) basis for \( J = 0 \) and \( 1P_1-5P_1-5F_1 \) for \( J = 1 \).

IV. THE POWER COUNTING MAP FOR HEAVY MESON MOLECULES

The presence of a bound state between a heavy meson and antimeson indicates the necessity of promoting a piece of the EFT potential from order \( Q^0 \) to \( Q^{-1} \). If the molecular state is shallow, as happens in the \( X(3872) \) or the \( Z_b(10610) \) and \( Z_b(10650) \), probably it is enough to promote a contact range interaction only. However, if the state is sufficiently deep (exactly how deep will be the subject of discussion of the next section), we are required to additionally promote the OPE potential as well. In this second case, the renormalizability of the theory will in turn generate important changes in the counterpart structure already at LO, as has been repeatedly discussed in the EFT description of nuclear forces \[32, 51–55\]. Therefore we distinguish two cases: power counting with perturbative and non-perturbative pions.

A. Counting with Perturbative Pions

We begin by considering a theory in which there is at least a \( Q^{-1} \) contact operator but where pions are perturbative. In this case, pions represent a NLO correction. As there are two contact operators at LO, \( C_{0a} \) and \( C_{0b} \), we have three options in what pertains to operator promotions, defining three power counting schemes:

(a) \( C_{0a} \) is of order \( Q^{-1} \),

(b) \( C_{0b} \) is of order \( Q^{-1} \),

(c) both \( C_{0a} \) and \( C_{0b} \) are of order \( Q^{-1} \).

Power countings (a), (b) and (c) are the EFT restatement of the observation made by Voloshin \[15\] about the heavy quark spin structure of the \( Z_b \) resonances: depending on which piece of the heavy quark spin symmetric interaction is responsible of the appearance of the two low-lying \( Z_b \) states, we should expect a total of four or six molecular \( s \)-wave states of \( B^{(*)}B^{(*)} \). The later situation, six states, correspond to the promotion (a), while the former, giving four states, with (b) \[5 \] and (c). It is worth commenting that the third possibility (c) is quite general and can also accommodate other situations. For example if \( C_{0a} < 0, C_{0b} > 0 \) and \( C_{0a} + C_{0b} \) is negative and attractive enough as to bind, then there will be six states, but at most only two of them will be shallow (the \( 1^{++} \) and \( 2^{++} \)). However, this specific situation does not necessarily correspond to what we encounter in the \( Z_b(10610) \) and \( Z_b(10650) \) resonances.

Power countings (a) and (b/c) also differ with respect to the treatment of coupled channel effects. The contact range interaction responsible for mixing different particle channels is \( C_{0a} \). In contrast, \( C_{0b} \) is diagonal in the HH space. This means that in the (b/c) counting the coupled channels effects are promoted by one order in the EFT expansion, and are therefore \( O(Q^2) \), instead of the naive estimate \( O(Q^3) \). In turn, the coupled channel effects dependent on the OPE interaction will still be of order \( Q^2 \).

In what regards the subleading contact operators, \( C_{2n} \), their scaling is affected by the corresponding scaling of \( C_{0} \). By this we mean that if \( C_{0} = Q^{-1} (Q^0) \), then \( C_{2n} = Q^{2n-2} (Q^{2n}) \), as can be deduced from renormalization group analysis arguments \[43\]. Of course, only the \( C_{2n} \) operators with a similar HQSS structure as their \( Q^{-1} \) counterparts will be promoted, depending on the whether we are considering the (a), (b) or (c) countings. However, the point is that, independently of the kind of promotion, we will have new free parameters at NLO. This also implies that without new data to fix the \( C_{0} \)’s we cannot proceed to NLO. Finally, the expected relative accuracy of a LO calculation is of order \( Q/A_0 \).

There is still one contribution that we have not addressed: the effect of heavy quarkonia states with the adequate quantum numbers in the vicinity of any of the HH thresholds. This situation may happen in the \( X(3872) \) if the \( \chi_{c1}(2S) \) lies nearby, as has been discussed in Ref. \[56\]. However, the coupling to heavy quarkonia is surely going to represent a subleading contribution only. The natural expectation is that the counterpart mixing these two states is of order \( Q^0 \). Therefore the coupled chan-

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\(^6\) Strictly speaking in (b) we should only expect three low-lying states, the two \( 1^{++} \) states and the \( PP^{*} 0^{++} \) state. However, if we recover the coupled channel effects, a fourth \( PP^{*} 0^{++} \) state may be expected. In this regard, it is more natural to use the promotion (c) to obtain the four states.
nel effects will be of order $Q^0$ and hence subleading if the thresholds of the $X(3872)$ and the $\chi_{c1}(2S)$ are very close, or order $Q^2$ otherwise. This type of contribution, being subleading, does not really affect the LO calculation and its accuracy.

### B. Counting with Non-Perturbative Pions

As we will see in the next section, pion exchanges can eventually become non-perturbative above a certain critical energy in heavy meson-antimeson systems. Interestingly, the non-perturbative treatment of pion exchange will be able to significantly alter the power counting scheme, as happens in the two-nucleon system $[32, 51, 55]$. We notice that Ref. $[28]$ already develops an EFT description of heavy meson molecules with non-perturbative OPE. However, the previous work only considers the particular case of the $P^*P/PP^*$ channels. In the next lines we will concentrate instead on the general HQSS structures relating the different $HH$ particle channels.

We must begin with distinguishing which piece of the OPE potential is to be iterated, the central or the tensor. If only the central pieces is non-perturbative, the changes in the power counting are minimal with respect to the theorems in which we already iterate the contact range operators, as demonstrated by Barford and Birse $[57]$. The reason is that the $1/r$ singularity of central OPE is not enough to change the power-law behaviour of the wave function at short distances, which is intimately related to the scaling of the counterterms (see, for example, the detailed discussion of Ref. $[53]$). In such a case, we simply refer to the results of the previous section.

On the contrary, the non-perturbative treatment of tensor OPE entails significant changes in the scaling of the contact range operators, as has been extensively discussed in the context of nuclear EFT $[32, 51, 55, 58, 59]$. The reason is that tensor OPE is a singular potential behaving as $1/r^3$ at distances below the pion Compton wavelength. The most evident effect of a singular potential is on the LO counterterms: if the singular potential is attractive, the renormalizability of the theory requires the inclusion of a contact interaction to stabilize the cut-off dependence $[58, 59]$. This can be appreciated if we consider the solution of the reduced Schrödinger equation for a $-1/r^3$ potential, which for short enough distances read $[58, 59]$

$$u_k(r) \rightarrow \left(\frac{r}{a_3}\right)^{3/4} \sin \left[\frac{a_3}{r} + \varphi\right]. \quad (62)$$

The problem with the wave function above is that it is always regular at the origin, no matter what the values of the semiclassical phase $\varphi$ is. If we try to regularize the potential by cutting it off at short distances, i.e. $V(r; r_c) = V(r) \theta(r - r_c)$, we do obtain a value of $\varphi(r_c)$. However, there is no well-defined $r_c \rightarrow 0$ limit for $\varphi(r_c)$: the semiclassical phase simply oscillates faster and faster on its way to the origin. The solution is to make the counterterm oscillate $[60]$,

$$\frac{\mu_R C_0(r_c)}{2\pi r_c^2} \rightarrow -\frac{2}{a_3} \left(\frac{r}{a_3}\right)^{3/4} \cot \left[\frac{a_3}{r} + \varphi\right], \quad (63)$$

so $\varphi$ remains a constant. In contrast, if the potential is repulsive, we have a unique and regular solution at short distances

$$u_k(r) \rightarrow \left(\frac{r}{a_3}\right)^{3/4} \exp \left[-\frac{a_3}{r}\right], \quad (64)$$

and no counterterm is required $[58, 59]$.

The (angular momentum) coupled channel case can be a bit more complicated however. In this case the important factor are the eigenvalues of the potential matrix in coupled channel space: if there are attractive (i.e. negative) eigenvalues, we will need $n(n + 1)/2$ counterterms $[58, 59]$. For the tensor matrices we have calculated in the previous section, the eigenvalues are

$$\lambda_{PP}(0^{++}) = 0, \quad (65)$$

$$\lambda_{PP}(1^{++}) = \tau \{1, -2\}, \quad (66)$$

$$\lambda_{PP}(1^{+-}) = \tau \{2, -1\}, \quad (67)$$

$$\lambda_{P^-P^-}(0^{++}) = \tau \{1 + \sqrt{3}, 1 - \sqrt{3}\}, \quad (68)$$

$$\lambda_{P^-P^-}(1^{--}) = \tau \{2, -1\}, \quad (69)$$

$$\lambda_{P^-P^-}(2^{++}) = \tau \{1, -2, 1 + \sqrt{3}, 1 - \sqrt{3}\}, \quad (70)$$

with $\tau = \vec{\tau}_1 \cdot \vec{\tau}_2$, that is, we have included the $-\eta \vec{\tau}_1 \cdot \vec{\tau}_2$ factor multiplying the potential of Eq. $[66]$ for obtaining the overall sign right. From the eigenvalues above it is apparent that the $C_{0a}$ and $C_{0b}$ counterterms are not enough as to renormalize the scattering amplitude if tensor OPE is non-perturbative.

Each distinct negative eigenvalue requires a different, independent counterterm. However, if two different channels share a negative eigenvalue, a common counterterm will be able to renormalize the two channels $[60]$. In principle we can count a total of eight counterterms for renormalizing the six $s$-wave states (one per channel, with the exception of the $2^{++}$ that requires three counterterms). This figure is then reduced by counting the shared eigenvalues, that is, (i) the $1^{++} P^*P$ and $P^*P^*$ molecules can be renormalized with the same counterterm, (ii) the $0^{++}$ and $2^{++}$ $P^*P^*$ share another eigenvalue and finally (iii) another common contact operator between the $1^{++}$ and $2^{++}$ channels. In total we end up with five independent counterterms.

All this makes the theory with non-perturbative pions rather cumbersome. In addition, with five contact interactions HQSS, although still there, is not so manifest as in the pionless theory $[4]$. It is interesting to notice

\footnote{What is actually happening is that certain $s$- to $d$-wave operators...}
though that the $1^{+-}$ $P^*\bar{P}$ and $P^*\bar{P}^*$ states are still expected to be degenerate: this situation corresponds to the quantum numbers of the $Z_0(10610)$ and $Z_0(10650)$ resonances. Luckily, as we will see in the next section, non-perturbative tensor forces are only expected in the isoscalar $B^{(*)}\bar{B}^{(*)}$ states.

However the non-perturbative OPE counting has certain advantages too. The momentum dependent $C_2$ operators are not expected until order $Q^{1/2}$ at least (or even $Q^2$: this is still an unresolved issue in nuclear EFT [53]). The lowest subleading order corrections are expected to happen at order $Q^1$: they are subleading contributions to the $C_0$ operators that are needed for absorbing the expected divergences in (particle) coupled channels, owing to the $1/r^3$ singularity of the tensor force. Thus the expected relative error of a LO calculation is now $(Q/\Lambda_0)^2$, representing an improvement over the $(Q/\Lambda_0)$ error in the perturbative pion case.

V. THE PERTURBATIVE TREATMENT OF ONE PION EXCHANGE

A. Central One Pion Exchange

\begin{verbatim}
  |   |   |
  |   |   |
(a) \quad |   |
  |   |
(b)
\end{verbatim}

FIG. 1. Feynman diagrams corresponding to (a) the OPE potential between two heavy mesons and (b) the first iteration of the OPE potential. The ratio of diagram (b) over (a) can be used to determine the energy range in which OPE is perturbative.

The perturbative character of the central piece of the OPE potential can be determined by a direct comparison of the diagrams of Fig. 1. This corresponds to computing the ratio of the matrix elements of the operators $V^{(0)}G_0V^{(0)}$ and $V^{(0)}$, which in terms of power counting is expected to scale as

$$\frac{\langle p|V^{(0)}G_0V^{(0)}|p\rangle}{\langle p|V^{(0)}|p\rangle} \sim \frac{Q}{\Lambda_C},$$

indicating the breakdown scale of a theory with central perturbative pions, where it is understood that we only consider the central piece of $V^{(0)}$. The evaluation is trivial for $s$-waves and $p = 0$, for which the generic low energy scale $Q$ can only be identified with the pion mass $m_\pi$, leading to [31]

$$\Lambda_C = \frac{1}{|\sigma|} \frac{24\pi f_\pi^2}{\mu_{HR} g^2}$$

where $\sigma = \frac{1}{2} S(S+1)-2$ and $\tau = 2 I(I+1)-3$ with $S$ and $I$ the total spin and isospin of the system. In general, this number is quite large for isovector channels ($\Lambda_C \gtrsim 1$ GeV), so the central piece of the OPE potential can always be treated as a perturbation in these cases. In contrast, for the $S = 0$ isoscalar case, corresponding to a $J^{PC} = 0^+(0^{++})$ $P^*\bar{P}^*$ molecule, the central piece can be quite important, but not so much as the tensor piece, as we will see later.

B. Tensor One Pion Exchange

The determination of the breakdown scale for a power counting in which tensor OPE is order $Q^0$ and hence perturbative is not trivial. The reason is that the argument of comparing the relative size of diagrams does not work when considering the tensor piece of OPE. In first place, tensor OPE does not directly act on $s$-waves, but only indirectly owing to transitions to intermediate $d$-wave states. This means that we should consider second and third order perturbation theory to obtain the breakdown scale

$$\frac{\langle V^{(0)}G_0V^{(0)}|V^{(0)}G_0V^{(0)}\rangle}{\langle V^{(0)}G_0V^{(0)}\rangle} \sim \frac{Q}{\Lambda_T},$$

where by $V^{(0)}$ we refer to the tensor piece only. However, there is a serious problem at this point: in the equation above the numerator and denominator are divergent. The reason lies in the $1/r^3$ divergent behaviour of the tensor force at distances below the Compton wavelength of the pion, which induces a linear (quadratic) divergence in the denominator (numerator), see for example Ref. [53] for the details of this kind of calculations. From the power counting point of view, $V^{(0)}G_0V^{(0)}$ requires the inclusion of the order $Q$ correction to the $C_0$ counterterm, while $V^{(0)}G_0V^{(0)}G_0V^{(0)}$ involves in addition two new order $Q^2$ counterterms, a purely $s$-wave one and another one connecting the $s$- and $d$-wave channels for the result to be properly renormalized [3]. The divergence spoils the evaluation of the previous ratio. If we

\footnote{Of course, this is assuming that the leading counterterm appears at order $Q^0$. If we are considering instead a two heavy meson
include counterterms, the ratio is again finite, but is also contaminated by whatever we assume to be the hard scale of the counterterms. In principle it is still possible to find the value of $\Lambda_T$ a posteriori by means of a full $Q^2$ calculation as in the nuclear case [31], but that requires plenty of experimental information about the low energy scattering of heavy mesons to fix the counterterms. Therefore we need to resort to other kind of arguments.

The failure of standard perturbation theory for inverse power-law potentials ($1/r^n$) is well-known in atomic physics, where techniques have been developed to handle this type of potentials. Of particular interest in this regard is the work of Cavagnero [32], who pointed out that the divergences of the perturbative series are analogous to the role of secular perturbations in classical mechanics, that is, small perturbations that however end up diverging at large enough time scales. The solution is to formulate a secular series in which we redefine (or renormalize) some quantity, in this case the angular momentum, in order to obtain finite results at arbitrary order. By this we specifically mean that the zeroth order perturbative approximation to the $l$-wave radial wave function is taken to be

$$\Psi_l(r; k) = \frac{J_{\nu}(ka)(kr)}{\sqrt{r}} + O(g),$$  \hspace{1cm} (74)

instead of the customary solution $r^{-1/2}J_{\nu}(kr)$, where $\nu(ka)$ is the renormalized angular momentum, $k$ the center-of-mass momentum, $a$ is the coupling constant and $J_{\nu}(x)$ the Bessel function of order $n$. The interesting feature is that at low energies the secular expansion can be reinterpreted as a particular resummation of the perturbative expansion, i.e. the renormalized angular momentum $\nu(ka)$ can be expanded as a power series in $a$. This observation, if translated to the EFT potential between heavy mesons, indicates the existence of an energy range for which the inverse power-law potential can be treated perturbatively at long distances 4.

For the particular inverse power-law potential of interest in this work, the $1/r^3$ potential (corresponding to the form of the tensor force for $m_A r < 1$), the secular series for the wave function have been studied and analyzed in detail by Gao [34] for the uncoupled channel case. These techniques were extended by Birse [32] to the coupled channel case, with a particular emphasis on the applications to nucleon-nucleon scattering. In the present work, we particularize the results of Birse [32] to the specific type of coupled channels appearing in two heavy meson systems.

In agreement with EFT expectations, the secular expansion of the wave function converges fast at long distances / low energies. Meanwhile, at high enough energies a very interesting thing happens: the secular expansion becomes a non-converging series in powers of the coupling constant above a certain critical value of the center-of-mass momentum [14]. That is, the secular expansion can no longer be considered as a perturbative series in disguise. From the EFT viewpoint this critical momentum is to be identified with the hard scale of the theory: if we map the secular series of the wave function at low energies onto the EFT expansion

$$\Psi_{\overline{HH}} = \sum_{\nu = -1}^{\infty} \hat{\Psi}_{\overline{HH}}^{(\nu)} \left( \frac{Q}{\Lambda_T} \right)^{\nu},$$  \hspace{1cm} (75)

it is apparent that the mapping will stop being valid at $Q = \Lambda_T$, where $Q$ is to be understood as the momentum (as the pion mass is fixed). This means that $\Lambda_T$ is the tensor breakdown scale we were looking for. Above this critical value of the momentum the EFT description of the two heavy meson system will require the full iteration of the $1/r^3$ tensor force.

In the following we will consider the OPE potential in the $\mu \to 0$ limit, for which it simplifies to a pure power-law potential of the type $1/r^3$:

$$2 \mu_{\overline{HH}} V^{(0)}_F(r) = \frac{G_s}{r} S_j,$$  \hspace{1cm} (76)

where $S_j$ is the tensor force matrix, $a_3$ is a length scale related to the strength of the tensor force, and the subscript $\overline{HH}$ denotes any combination of heavy meson and antimeson for which the OPE potential is non-zero. As can be appreciated, only the tensor piece of the OPE potential survives in this limit. Even though $\mu = 0$ can be physically interpreted as taking the chiral and heavy quark limit simultaneously ($m_\pi \to 0$ and $m_Q \to \infty$), here we simply consider the $\mu \to 0$ limit as a mathematical convenience that allows us to directly apply the results of Refs. [32, 34]. Deviations from $\mu = 0$ will in general rise the value of the critical momentum, as they will reduce the strength of the potential at distances $\mu r \geq 1$ and therefore increase the perturbative character of the interaction at long distances. Analogously, the finite size of the heavy mesons is expected to weaken the potential at short distances. In this regard, the $\mu = 0$ (plus $\Lambda_0 = \infty$) determination are merely a lower bound: the real critical momenta will be higher. We will estimate the finite pion mass corrections to the critical momenta at the end of this section.

Curiously there is a case for which the condition $\mu = 0$ is almost fulfilled: the $X(3872)$, in which $m_\pi$ and $\Delta Q$ are very similar. In such a case the critical momenta calculated in this work will really represent the actual boundary to the perturbative treatment of tensor OPE.

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4 In this statement we are implicitly assuming the presence of exponential suppression ($e^{-m_\pi r}$) for standard perturbation theory to make sense at long distances and of suitable counterterms to absorb the divergences we commented previously.

10 Specifically $\nu(x)$ encounters a non-analyticity in $x = k a$. 
1. The Uncoupled Channel Case

We start with the uncoupled channel case, for which the tensor operator \( S_j = S_j \) is simply a number (instead of a matrix in the coupled channel basis). In this case, the reduced Schrödinger equation reads

\[
-u''_{k,j} + \left( S_j \frac{a_3}{r^3} + \frac{l_j(l_j+1)}{r^2} \right) u_{k,j}(r) = k^2 u_{k,j}(r),
\]

(77)

where \( u_{k,j} \) is the reduced wave function, \( l_j \) is the angular momentum of the two body system, which for the uncoupled channel case coincides with the total angular momentum of the system \( (l_j = j) \), and \( k \) is the center-of-mass momentum. We have \( |S_j| = 1 \) in all the HH uncoupled channels, with the exception of the \( ^3P_0 \) partial wave for which \( |S_j| = 2 \).

The general solution of this equation (for \( k \neq 0 \)) can be written as the linear combination

\[
u_{k,j}(r) = \alpha \xi_j(r;k) + \beta \eta_j(r;k),
\]

(78)

where \( \xi_{k,j}(r) \) and \( \eta_{k,j}(r) \) are expressible as series in terms of Bessel functions

\[
\xi_j(r;k) = \sum_{m=-\infty}^{+\infty} b_m \sqrt{r} J_{m+\nu} (kr),
\]

(79)

\[
\eta_j(r;k) = \sum_{m=-\infty}^{+\infty} (-1)^m b_m \sqrt{r} J_{m-\nu} (kr).
\]

(80)

The shift \( \nu \) is a function of the dimensionless parameter \( \kappa = ka_3 \). The \( b_m \) coefficients can be expressed in terms of a recursion relation that admits an explicit solution \[34\].

The dependence of the renormalized angular momentum \( \nu \) in terms of \( \kappa \) is determined by the condition that \( \nu \) is the zero of a characteristic function

\[
F_1(\nu, \kappa) = 0,
\]

(81)

where the function \( F_1(\nu, \kappa) \) is in turn defined by

\[
F_1(\nu, \kappa) \equiv (\nu^2 - \nu_0^2) - \frac{\kappa^2}{\nu} \left[ R_0(\nu) - R_0(-\nu) \right],
\]

(82)

where \( \nu_0 = (l_j + \frac{1}{2}) \). In the equation above, \( R_0(\nu) \) can be expressed again in terms of a recursive relation

\[
R_n(\nu) = \frac{1}{(\nu+1)((\nu+1)^2 - \nu_0^2) - S_j^2 \kappa^2 R_{n+1}(\nu)}.
\]

(83)

Birse \[32\] notices that between 20 and 30 iterations are enough as to obtain \( \nu(\kappa) \) with six significant digits. As already commented, the analytical properties of the function \( \nu(\kappa) \) are the essential ingredient for determining the range of applicability of the perturbative reexpansion of the wave function defined in Eqs. (78), (79) and (80).

The behaviour of the solutions \( \nu = \nu(\kappa) \) can be described as follows: for \( \kappa = 0 \) (plus the condition \( k \neq 0 \)) we have the Schrödinger equation for free waves with angular momentum \( l_j \). In such a case, the solution of Eq. (77) are the Spherical Bessel functions, which is equivalent to having

\[
\nu(\kappa = 0) = \nu_0 = l_j + \frac{1}{2}.
\]

(84)

If \( \kappa \) is smaller than a critical value \( \kappa_c \), the shift can be written as

\[
\nu(\kappa < \kappa_c) = \nu_0 - \delta \nu,
\]

(85)

with \( 0 < \delta \nu < \frac{1}{2} \). For sufficiently small \( \kappa \) the \( \delta \nu \) correction is of order \( |\kappa| (\kappa^2) \) for \( l_j = 0 \) \((l_j \neq 0)\), where the exact expression can be consulted in Ref. \[32\]. For larger values of \( \kappa \), \( \delta \nu \) is still a power series in \( \kappa \) and therefore perturbative. However, this situation changes once \( |\kappa| \) is large enough for the shift to reach the value \( \nu = l_j \): above \( \kappa_c \) the shift \( \nu \) splits into two imaginary solutions of the type

\[
\nu(\kappa > \kappa_c) = l_j \pm i \rho(\kappa).
\]

(86)

Beyond this point, the renormalized angular momentum \( \nu(\kappa) \) cannot be expressed as a power series in terms of \( \kappa \), as a consequence of the non-analytical character of the split. Consequently, we expect the standard perturbative treatment to fail above this critical value \( \kappa_c \). The critical reduced momenta for the uncoupled channels can be consulted in Table I.

2. The Coupled Channel Case

The (angular momentum) coupled channel case was worked out in detail by Birse \[32\]. The extension is trivial and only requires the following definitions. In first place, the Schrödinger equation reads

\[
-u''_{k,j} + \left[ S_j \frac{a_3}{r^3} + \frac{L_j^2}{r^2} \right] u_{k,j}(r) = k^2 u_{k,j}(r),
\]

(87)

where there are \( N \) angular momentum channels, \( S_j \) is the tensor matrix, and \( L_j \) is a diagonal matrix containing the value of the angular momenta

\[
L_j = \text{diag}(l_1(l_1+1), \ldots, l_N(l_N+1)).
\]

(88)

As in the previous case, the general solution of the Schrödinger equation is a linear combination of the functions

\[
u_{k,j}(r) = \sum_{\{l_j\}} \left[ \alpha_{l_j} \xi_{l_j}(r;k) + \beta_{l_j} \eta_{l_j}(r;k) \right],
\]

(89)

For simplicity we are expressing \( u_{k,j} \) in terms of wave functions which do not have a well-defined \( k \rightarrow 0 \) limit. This limitation can be easily overcome by adding energy-dependent normalization coefficients, see Ref. \[32\] for the details of such a normalization.
where we sum over the possible values of the angular momenta. In this case $\xi$ and $\eta$ are $N$-component vectors that can be expressed as sums of Bessel functions

\begin{align}
\xi_{l_i}(r; k) &= \sum_{m=-\infty}^{\infty} b_m(\nu_{l_i}) \sqrt{\tau} J_{m+\nu_{l_i}}(kr), \\
\eta_{l_i}(r; k) &= \sum_{m=-\infty}^{\infty} (-1)^m b_m(\nu_{l_i}) \sqrt{\tau} J_{-m-\nu_{l_i}}(kr),
\end{align}

where we have labelled the different solutions by the subscript $l_i$, which indicates that for $\nu = 0$ the solutions will behave as free waves with angular momentum $l_i$. The $b_m(\nu_{l_i})$ coefficients are now vectors satisfying a certain recursive relation, see Ref. [32] for details.

Of course, what is important are the analytical properties of $\nu(\nu)$. The functional dependence $\nu = \nu(\nu)$ is determined by finding the zeros of

$$
\det \left( \mathbf{F}_j(\nu, \nu) \right) = 0,
$$

where the characteristic function $\mathbf{F}_j(\nu, \nu)$ is a $N \times N$ matrix. The equation above admits $N$ solutions, one for each value of the angular momentum, which can be labelled by their value for $\nu = 0$, that is

$$
\nu_{l_i}(\nu = 0) = l_i + \frac{1}{2},
$$

with $i = 1, \ldots, N$. We define $\mathbf{F}_j(\nu, \nu)$ as follows

$$
\mathbf{F}_j(\nu, \nu) \equiv \mathbf{f}_j(\nu) - \frac{\kappa_j^2}{\nu} [\mathbf{R}_0(\nu) - \mathbf{R}_0(-\nu)],
$$

where $\mathbf{f}_j(\nu)$ is a diagonal matrix defined by

$$
\mathbf{f}_j(\nu) = \text{diag}(\nu^2 - (l_1 + \frac{1}{2})^2, \ldots, \nu^2 - (l_N + \frac{1}{2})^2).
$$

In turn, $\mathbf{R}_0(\nu)$ can be expressed in terms of a recursive relation

$$
\mathbf{R}_n(\nu) = [\mathbf{f}_j(\nu) - \kappa_j^2 \mathbf{S}_j \mathbf{R}_{n+1}(\nu)]^{-1}.
$$

As in the uncoupled channel case, between 20 and 30 iterations are more than enough to obtain a sufficiently accurate result.

The analytical properties of the shift $\nu_1(\kappa)$ are analogous to what happened in the uncoupled channel case: the shift begins its trajectory at $\nu_1(l_i = 1/2)$ for $\kappa = 0$ and moves slowly downwards. Once the shift reaches the value $\nu_1 = l_i$ at the critical value of the coupling $\kappa = \kappa_c$, the shift splits into two complex conjugate solutions of the type $\nu_1(\kappa) = l_i \pm i\nu_1(\kappa)$, signalling the breakdown of a long range perturbative expansion of the wave function.

In general, the first shift to split into the complex plane is the one corresponding to the smallest angular momentum. For that reason we will simply study the critical value of $\kappa$ corresponding to the lowest angular momentum in the coupled channel. The results for the $3C_J$, $5C_J$ and $1^-C_J$ families of coupled channels with $j \leq 3$ are listed in Table I. While the critical momenta for the $3C_J$ type of coupled channels was already determined by Birse in Ref. [32], the results for the $5C_J$ and $1^-C_J$ coupled channels are new.

In general, for the $3C_J$ channels we obtain values of $\kappa_c$ with are half the values obtained in Ref. [32]. The reason is simple: the tensor operator matrix elements in the $3C_J$ channels for heavy meson molecules is half of the value of the corresponding matrix in the two nucleon system. In general, the critical reduced momenta $\kappa_c$ grows rapidly with the angular momentum of the system, with the exception of the $3P_0$ partial wave for which a relatively low value of $\kappa_c$ is obtained. If we assume similar values of $a_3$, the tensor length scale, the channel less likely to be perturbative is the $1^-C_J$ partial wave, corresponding to a $J^{PC} = 2^{++}$ $P^* P^*$ meson molecule.

### C. Finite Pion Mass Effects

In the previous paragraphs we have calculated the critical reduced momenta $\kappa_c$ for a pure $1/r^3$ potential, corresponding to the form of OPE for $\mu = 0$. For $\mu > 0$ we expect the values of $\kappa_c$ to rise by a certain amount, as the finite pion mass $e^{-\mu r}$ will decrease the effective strength of the potential at large distances. Curiously, we will see that the dependence of $\kappa_c$ on $\mu$ is actually triggered by the far from perfect separation of scales in the chiral EFT, rather than from the finite pion mass alone.

The argument is the following: we begin by solving the Schrödinger equation stepwise, that is, we divide the possible radii $0 < r < \infty$ into several regions of a given size $\Delta r$. In particular, if we consider the region defined

| $3U_J$ | $^3C_J$ | $\kappa_c$ | $^5C_J$ | $\kappa_c$ | $^1^-C_J$ | $\kappa_c$ |
|-------|---------|------------|---------|------------|---------|------------|
| $3P_0$ | 1.259   |            | $3P_0$  | 1.328      | $^1S_{0}-^3D_0$ | 4.314    |
| $3P_1$ | 2.518   | $^3S_{1}-^3D_1$ | 1.367   | $^5D_1$   | 8.333   | $^1P_{1}-^3P_{1}$ | 4.314    |
| $^3P_2$ | 8.333   | $^3P_{2}-^3F_2$ | 3.237   | $^5D_2$   | 1.644   | $^1D_{2}-^3D_{2}$ | 1.089    |
| $^3P_3$ | 19.70   | $^3D_{3}-^3G_3$ | 7.913   | $^5D_3$   | 5.594   | $^1F_{3}-^3F_{3}$ | 2.615    |

**Table I.** Reduced critical momenta for the different types of coupled channels appearing in the HH system ($j \leq 3$).
by the condition
\[ R - \frac{1}{2} \Delta r < r < R + \frac{1}{2} \Delta r , \] (97)
then, for \( \mu \Delta r < 1 \) we can approximate the finite pion effects by the substitution
\[ \frac{a_3}{r^3} e^{-\mu r} \simeq \frac{a_3}{r^3} e^{-\mu R} , \] (98)
where we have ignored the \( 1/r \) and \( 1/r^2 \) components of the tensor force for simplicity. At this point it is worth noticing that the explicit solutions of the Schrödinger equation we have written in the preceding subsections can also be applied in a specific subset of the real axis \( 0 < r < \infty \). That is, if the \( 1/r^3 \) potential is only valid in the region defined by Eq. (97):
\[ V(r) = \frac{a_3}{r^3} \theta(|R + \frac{\Delta r}{2} - r|) \theta(r - |R - \frac{\Delta r}{2}|) , \] (99)
then, Eqs. (82), (79) and (80) represent the full solution of the Schrödinger equation in this region. Moreover, the critical reduced momenta \( \kappa_c \) calculated for \( 0 < r < \infty \) are still valid in \( R - \Delta r/2 < r < R + \Delta r/2 \).

The consequence for a decaying \( e^{-\mu r/r^3} \) potential is that, by making \( \Delta r \) sufficiently small, we can define a reduced critical momentum
\[ \kappa_c(\mu, R) = \kappa_c e^{\mu R} , \] (100)
which applies in the vicinity of \( R \). Therefore, the real \( \kappa_c(\mu) \) setting the actual limits of perturbative tensor OPE is the minimum of \( \kappa_c(\mu, R) \), which is in turn determined by the smallest radius for which the OPE potential is valid. In chiral EFT we expect this radius to be proportional to the inverse of the breakdown scale, \( R_0 \propto 1/\Lambda_0 \). Naively, we expect \( R_0 \) to lie around 0.5 fm, but we cannot discard larger (or smaller) values. In particular, a recent analysis of the convergence of the two-nucleon potential in nuclear EFT suggests values around 0.8 fm (and even higher). Of course, the extrapolation of this latter value to the heavy meson-antimeson potential is to be taken with a grain of salt, but we will nevertheless use the 0.8 fm figure only as an upper bound for \( R_0 \). For \( \mu = m_\pi \), the \( R_0 = 0.5 - 0.8 \) fm window yields
\[ \kappa(m_\pi) = \kappa_c(m_\pi, R_0) \simeq (\sqrt{2} - \sqrt{3}) \kappa_c , \] (101)
where we have approximated \( e^{m_\pi R_0} \) by \( \sqrt{2} (\sqrt{3}) \) for \( R_0 = 0.5 - 0.8 \) fm. In the nucleon-nucleon case the \( \sqrt{2} \) factor approximately corresponds to the observed mismatch between the critical momentum obtained by Birse for the \( ^3S_1-^3D_1 \) channel (\( p_c = 66 \text{ MeV} \)) and the momentum at which the FMS calculations fail (\( p_c \approx 100 \text{ MeV} \)). However, a more recent formulation of nuclear EFT with perturbative OPE suggests a larger \( p_c \geq 150 \text{ MeV} \), requiring \( R_0 \geq 1.0 \) fm at least.

VI. DISCUSSION AND CONCLUSIONS

The critical values \( \kappa_c \) can be converted into critical momenta by dividing by the tensor length scale \( a_3 \)
\[ a_3 = |\tau| \frac{\mu \Lambda_{\text{HI}} g^2}{4\pi g^2} . \] (102)
In the equation above, \( \tau = 2I(I + 1) - 3 \) is the eigenvalue of the isospin operator \( \hat{T}_1 \cdot \hat{T}_2 \) and \( \mu \Lambda_{\text{HI}} \) is the reduced mass of the heavy meson molecule under consideration. For the axial coupling constant \( g \), we take \( g = 0.6 \pm 0.1 \) in the charm sector (\( D^{(*)}D^{(*)} \)) and \( g = 0.5 \pm 0.1 \) in the bottom one (\( B^{(*)}B^{(*)} \)). For the charm mesons, the value of \( g \) is known from the \( D^* \rightarrow D \pi \) and \( D^* \rightarrow D \gamma \) decays [66, 67]. This yields \( g = 0.59 \pm 0.01 \pm 0.07 \), which we round to \( g = 0.6 \pm 0.1 \). For the bottom mesons, \( g \) is experimentally unknown, even though from heavy quark symmetry we should expect a value similar to the charm mesons one. There are however theoretical determinations, for example from lattice QCD [68, 71] or from the QCD Dyson-Schwinger equations [72] to name just a few of them, usually ranging from 0.3 to 0.7 [73]. The value we have chosen, \( g = 0.5 \pm 0.1 \), represent a compromise between the previous determinations, but it should be kept in mind that the uncertainties for the bottom sector may be much larger.

The critical momentum (including finite pion mass corrections) is defined as
\[ p_{\text{crit}}(\mu) = \frac{\kappa_c}{a_3} e^{\mu R_0} , \] (103)
with \( R_0 = 0.65 \pm 0.15 \) fm (i.e. \( R_0 = 0.5 - 0.8 \) fm), and \( \kappa_c \) to be taken from Table II. We can also translate \( p_{\text{crit}}(\mu) \) into critical binding energies by
\[ B_{\text{crit}}(\mu) = \frac{p_{\text{crit}}^2(\mu)}{2\mu \Lambda_{\text{HI}}} , \] (104)
where we can make the approximation \( B_{\text{crit}}(m_\pi) \simeq (2.5 \pm 0.5) B_{\text{crit}}(0) \). It is also interesting that the critical binding energy scales as \( 1/g^4 \). This dependence will generate remarkable uncertainties on the value of \( B_{\text{crit}} \) for bottom meson molecules.

In the charm isovector sector, that is, the \( D^{(*)}D^{(*)} \) molecules with \( I = 1 \) and \( \tau = 1 \), the critical momenta are of the order of 0.8 – 1.1 GeV, similar to the chiral breakdown scale \( \Lambda_0 \sim 0.5 - 1.0 \text{ GeV} \). In this case, central and tensor OPE are expected to be a perturbation in all the range of validity of the EFT. On the other extreme we have the bottom isoscalar sector, for which the critical momenta are of the order of \( m_\pi \) (\( \sim 200 \text{ MeV} \) for \( \mu = m_\pi \)). For these systems the onset of non-perturbative OPE start at around 4 MeV in the chiral limit, or around 10 MeV if we take into account the finite pion mass effects. It should be noted that the previous limits could be more stringent if \( g \) is larger than the expected 0.5, as the critical binding depends on \( 1/g^4 \). However, the bottomline is that in these heavy meson molecules OPE is expected to be non-perturbative unless the bound states are

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12 These components can be implicitly taken into account by a redefinition of \( a_3 \). However, provided \( R \) is sufficiently small, they are suppressed by a factor of \( \mu R \).
genuinely shallow. The corresponding EFT treatment is the one employed in Ref. [28] for exploring prospective B⁺B⁻ states.

The two interesting cases are the isovector charm and isoscalar bottom meson molecules: in these two systems the values of the critical momenta lie between mₜ and λ₀, defining a specific window in which OPE is perturbative. The critical momenta for the charm and bottom sectors can be found in Table II. For the charm isoscalar the critical binding energy lies around 25–40 MeV (60–90 MeV) for μ = 0 (μ = mₜ). In this regard, the D⁺D⁻/DD⁺ systems are specially interesting in the sense that they naturally fulfill the condition |μ| ~ 0 \textsuperscript{13}. This means that the X(3872) is well within the energy range for which the LO EFT description consists only on contact range interactions. In particular, the critical momentum of 290 MeV implies that the charged D⁺D⁻/D⁻⁺D⁻ component of the X(3872) can be accounted for without the explicit inclusion of pion exchanges at the lowest order. This represents a remarkable simplification that allows, for example, to reinterpret the explanation of Gamermann et al. [25, 26] for the branching ratio

\[ \frac{\Gamma [X(3872) \rightarrow J/\psi \pi^+\pi^-\pi^0]}{\Gamma [X(3872) \rightarrow J/\psi \pi^+\pi^-]} \]  

\textsuperscript{13} Strictly speaking |μ| = 31 MeV ≪ mₜ, which is small enough as to consider μ ~ 0 as a pretty good approximation.

TABLE II. Critical value of the momenta for the tensor piece of OPE for the s-wave isoscalar DD⁺ and D⁺D⁻ and isovector BB⁺ and B⁺B⁻ systems. We compute the critical value both in the μ = 0 limit and in the finite μ limit. In the second case, we usually have μ = mₜ, with the exception of the DD⁺ (BB⁺) case in which |μ| = 31 MeV (μ = 131 MeV), generating no significant deviation from μ = 0 (μ = mₜ). The respective value of the critical momenta and maximum binding energies for the isovector DD⁺ and D⁺D⁻ molecules is three (nine) times larger than for their isoscalar counterparts. The contrary happens with the isoscalar BB⁺ and B⁺B⁻ molecules, for which the critical momenta are 1/3 of the isovector case and the maximum binding energies 1/9.

| I^G(J^{PC}) | 2S+1L_J | HH | \( p_{\text{crit}}(0) \) | \( B_{\text{crit}}(0) \) | \( p_{\text{crit}}(\mu) \) | \( B_{\text{crit}}(\mu) \) |
|----------------|----------------|-------|----------------|----------------|----------------|----------------|
| 0⁺(1⁺⁺)       | ¹S₁/2 D₁      | DD⁺   | 290\(\pm80\)   | 42\(\pm19\)   | 290\(\pm80\)   | 42\(\pm19\)   |
| 0⁺(0⁺⁺)       | ¹S₀/2 D₀      | D⁺D⁻  | 270\(\pm70\)   | 36\(\pm17\)   | 420\(\pm120\)  | 89\(\pm41\)   |
| 0⁺(1⁺⁺)       | ³S₃/2 D₃      | D⁺D⁻  | 280\(\pm80\)   | 38\(\pm18\)   | 440\(\pm200\)  | 90\(\pm100\)  |
| 0⁺(2⁺⁺)       | ¹D₂⁺S₂⁺D₂⁺G₂  | D⁺D⁻  | 220\(\pm100\)  | 24\(\pm14\)   | 350\(\pm160\)  | 60\(\pm65\)   |
| 1⁺(1⁺⁺)       | ³S₁/2 D₁      | B⁺B⁻  | 450\(\pm260\)  | 38\(\pm56\)   | 690\(\pm420\)  | 90\(\pm120\)  |
| 1⁺(0⁺++)      | ¹S₀/2 D₀      | B⁺B⁻  | 440\(\pm240\)  | 36\(\pm31\)   | 690\(\pm390\)  | 90\(\pm120\)  |
| 1⁺(1⁺⁺)       | ³S₃/2 D₃      | B⁺B⁻  | 450\(\pm250\)  | 38\(\pm40\)   | 710\(\pm410\)  | 90\(\pm140\)  |
| 1⁺(2⁺⁺)       | ¹D₂⁺S₂⁺D₂⁺G₂  | B⁺B⁻  | 360\(\pm200\)  | 24\(\pm35\)   | 560\(\pm320\)  | 60\(\pm87\)   |

Of course, the previous results are to be held in proper context: they refer to the EFT framework rather than to a more fundamental or phenomenological level of description of heavy meson molecules. In phenomenological approaches non-perturbative pion exchanges and coupled channel effects may be important in suggesting the possibility or explaining the location of heavy molecular states [61, 74], in particular if the states are shallow, in which case a fine tuning takes place between the different long and short range effects that are explicitly included. In contrast, in the EFT approach these fine tunings are absorbed in the counterterms, which are set to reproduce the already known position of a certain state. This in turn stabilizes the calculations and reduces the impact of new physical effects we may subsequently incorporate in the EFT description as they are expected to be constrained by power counting. The price to pay is the inability of the EFT framework to predict the positions of these states from first principles.

To summarize, we have found that pion exchange effects are perturbative in most types of heavy meson molecules over the expected range of applicability of heavy meson EFT. The exception is the isoscalar bottom meson molecules, for which OPE is particularly strong and presumably non-perturbative. To a lesser extent, in
the isoscalar charm sector non-perturbative OPE may be required if the molecular states are deep enough (around 80 MeV or more). At lowest order, below the critical binding energies we have computed, the EFT description is a contact-range theory. Surprisingly, this description holds for momenta much larger than naively expected.

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Appendix A: Derivation of The Leading Order Potential in Heavy Hadron Chiral Perturbation Theory

In this appendix, we derive the LO potential between a heavy mesons and anti-meson within heavy hadron chiral perturbation theory (HHChPT) [14]. The LO potential can be decomposed into a contact-range and a finite-range piece. The later can be identified with the well-known one pion exchange (OPE) potential. We also compute the complete partial wave projection of the LO potential.

1. The Effective Lagrangian at Leading Order

We define the heavy meson (antimeson) fields in terms of the pseudoscalar and vector meson (antimeson) fields in the following way.

\[ H^Q = \frac{1 + \gamma^5}{2} [P_v^{(*)\mu} \gamma^\mu - P_v \gamma^5] , \]
\[ \bar{H}^Q = \gamma^0 H^Q \dagger \gamma^0 , \]
\[ H^Q = \gamma^0 H^Q \dagger \gamma^0 , \]
\[ \bar{H}^Q = \gamma^0 H^Q \dagger \gamma^0 . \]

where \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \) is the usual Dirac 4-fermion gamma matrix. The subscript \( v \) indicates that the normalization of the one-particle states is given by

\[ \langle P_v(\vec{k}) | P_v(\vec{k}') \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta(3) (\vec{k} - \vec{k}') , \]
\[ \langle P_v^{(*)}(\vec{k}) | P_v^{(*)}(\vec{k}') \rangle = 2v^0 \delta_{vv'} (2\pi)^3 \delta(3) (\vec{k} - \vec{k}') \delta_{\mu\nu} . \]

depending on whether we are considering the pseudoscalar or vector meson case.

The HHChPT Lagrangian at lowest order \( (Q^0) \) can be decomposed as the sum of two different contributions:

\[ \mathcal{L}^{(0)} = \mathcal{L}^{(0)}_{4H} + \mathcal{L}^{(0)}_{\pi HH} , \]

where the first term represents a 4-meson interaction vertex and the second the meson-pion vertex. The contact interaction term reads [15]

\[ \mathcal{L}^{(0)}_{4H} = D_{0a} \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \right] \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \right] + D_{0b} \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \gamma^5 \right] \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \gamma^5 \right] \]
\[ + E_{0a} \mathrm{Tr} \left[ H^Q \tau_i H^Q \gamma^\mu \gamma^5 \right] \mathrm{Tr} \left[ \bar{H}^Q \tau_i \bar{H}^{Q*} \gamma^\mu \right] + E_{0b} \mathrm{Tr} \left[ H^Q \tau_i H^Q \gamma^\mu \gamma^5 \right] \mathrm{Tr} \left[ \bar{H}^Q \tau_i \bar{H}^{Q*} \gamma^\mu \right] , \]

where \( \tau_i \) are the isospin matrices (i.e. the Pauli matrices). That is, in the previous Lagrangian \( D_{0a} \) and \( D_{0b} \) are isospin independent, while \( E_{0a} \) and \( E_{0b} \) are isospin dependent. In general we will be only considering a specific isospin channel, in which case we will generically define

\[ C_{0a} = D_{0a} + \tau_1 \cdot \tau_2 E_{0a} , \]
\[ C_{0b} = D_{0b} + \tau_1 \cdot \tau_2 E_{0b} . \]

For the Lagrangian describing the pion-meson vertex we have [14]

\[ \mathcal{L}^{(0)}_{\pi HH} = i \frac{g}{2} \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \gamma^5 (\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger) \right] - i \frac{g}{2} \mathrm{Tr} \left[ H^Q H^{Q*} \gamma^\mu \gamma^5 (\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger) \right] , \]

where \( \xi \) is defined as

\[ \xi = e^{\frac{\pi}{f_\pi} \mathcal{M}} , \]
\[ M = \left( \begin{array}{c} \pi^0 \\pi^+ \pi^0 \\pi^0 \\ \pi^- \pi^- \frac{\pi^+}{\sqrt{2}} \frac{\pi^-}{\sqrt{2}} \end{array} \right) = \frac{1}{\sqrt{2}} \vec{\pi} \cdot \vec{\pi} , \]

with \( \pi \) the pion fields and \( f_\pi = 132 \text{MeV} \). The isospin indices convention for the heavy meson and antimeson fields is

\[ P_v^{(*)} = \begin{pmatrix} P_v^{(\mu)} \\ P_v^{(*)} \end{pmatrix} \quad \text{and} \quad -P_v^{(*)} = \begin{pmatrix} -P_v^{(\mu)} \\ P_v^{(*)} \end{pmatrix} . \]

This convention explains the minus sign in the second line of Eq. (A11). That is, we have performed a G-parity transformation to obtain the chiral Lagrangian for the heavy anti-meson fields from the heavy meson one.
2. The Non-Relativistic Normalization

In the heavy quark limit, $m_Q \to \infty$, the static potential between two heavy mesons is a well-defined object. Relativistic effects are suppressed, and the two heavy meson system can be effectively described in terms of non-relativistic quantum mechanics. For making the non-relativistic transition we specify the velocity parameter to be $v = (1, 0)$ and employ a new normalization of the heavy meson fields defined by

$$P^{(s)} = \sqrt{2}P^{(s)}_v,$$

(A15)

where we have substituted the Greek index $\mu$ by the latin index $j$, as the $\mu = 0$ polarization component is completely irrelevant for $v = (1, 0)$. In this heavy meson field normalization, the one-particle states have the usual non-relativistic normalization

$$\langle P(\vec{k})|P(\vec{k}')\rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}'),$$

(A16)

$$\langle P^i(\vec{k})|P^{*j}(\vec{k}')\rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}') \delta_{ij},$$

(A17)

which is more convenient for the definition of the quantum mechanical potential.

$$V_C(\vec{q}) = C_{0a} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \vec{e}_{1+}^* \cdot \vec{e}_1 & 0 & 0 \\ 0 & 0 & \vec{e}_{2+}^* \cdot \vec{e}_2 & 0 \\ 0 & 0 & 0 & \vec{e}_{1+}^* \cdot \vec{e}_{2+} \end{array} \right),$$

(A18)

In the equation above, $\vec{e}_{1(2)}/\vec{e}_{1(2)}'$ represents the polarization of the incoming particles $1(2)$ or the outgoing particles $1'(2')$. In turn, the matrix elements of the spin-1 operator $\vec{S}_{1(2)}$ are equivalent to the vector product of the polarization wave functions, that is

$$i \langle 1\lambda' | \vec{S} | 1\lambda \rangle = (\vec{e}_{\lambda'}^* \times \vec{e}_\lambda),$$

(A21)

where $|1\lambda\rangle$ is the vector corresponding to the polarization wave function $\vec{e}_\lambda$. 

4. The Finite Range OPE Potential

For obtaining the finite range piece of the potential, we start by considering the HH$\pi$ vertex in the non-relativistic normalization, for which the Lagrangian reads

$$\mathcal{L}_{\pi HH} = \frac{g}{\sqrt{2}f_\pi} \left[ (P^* P^\dagger + P^\dagger P) + i (P^* \times P^\dagger) \right] \vec{\tau} \cdot \vec{\partial} \vec{\pi}$$

$$+ \frac{g}{\sqrt{2}f_\pi} \left[ (\vec{P}^* \vec{P}^\dagger + \vec{P}^\dagger \vec{P}) - i (\vec{P}^* \times \vec{P}^\dagger) \right] \vec{\tau} \cdot \vec{\partial} \vec{\pi},$$

(A22)

3. The Contact Range Potential

In the non-relativistic normalization, after expanding the $H^{(Q)}$ and $\bar{H}^{(Q)}$ fields, the contact range Lagrangian can be rewritten as

$$\mathcal{L}_{4H} = -C_{0a} \left( P^1 P + P^* P^\dagger \right) \right( \bar{P} \bar{P}^\dagger + \bar{P}^* \bar{P}^\dagger \right)$$

$$+ C_{0b} \left( P^* P^\dagger + P^\dagger P \right) \left( \bar{P} \bar{P}^\dagger + \bar{P}^* \bar{P}^\dagger \right)$$

$$- i C_{0a} \left[ \left( P^1 P + P P^* \right) \left( \bar{P}^\dagger \bar{P} + \bar{P} \bar{P}^\dagger \right) \right]$$

$$- \left( P^* \times P^\dagger \right) \left( \bar{P}^\dagger \bar{P} + \bar{P} \bar{P}^\dagger \right) \right)$$

(A19)

In the expressions above, the polarization of the vector mesons has been made implicit to simplify the notation. The potential corresponding to the Lagrangian above can mix different particle channels. Thus, it is convenient to write the potential in the coupled channel basis

$$\mathcal{B}_{HH} = \{ |P P\rangle, |P^* \bar{P}\rangle, |P P^*\rangle, |P^* \bar{P}^*\rangle \},$$

(A19)

for which we have (in momentum space)

$$+ C_{0b} \left( \begin{array}{cccc} 0 & 0 & 0 & -\vec{e}_{1+}^* \cdot \vec{e}_{2+}^* \\ 0 & 0 & -\vec{e}_1 \cdot \vec{e}_{2+}^* + \vec{S}_1 \cdot \vec{e}_{2+}^* & 0 \\ 0 & -\vec{e}_{1+}^* \cdot \vec{e}_2 & 0 & -\vec{e}_1 \cdot \vec{S}_2 \\ -\vec{e}_1 \cdot \vec{e}_2 + \vec{S}_1 \cdot \vec{e}_2 & -\vec{e}_1 \cdot \vec{S}_2 & 0 & \vec{S}_1 \cdot \vec{S}_2 \end{array} \right),$$

(A20)

where the polarization indices are implicit. If we define the non-relativistic amplitude as follows

$$A(H \to H \pi^a) = i \langle H | \mathcal{L}_{\pi HH} | H \pi^a \rangle,$$

(A23)

we find

$$A(P \to P \pi^a) = 0,$$

(A24)

$$A(P^* \to P \pi^a) = \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{e}_1 \cdot \vec{q},$$

(A25)

$$A(P^* \to P^* \pi^a) = -i \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{S} \cdot \vec{q},$$

(A26)

where $\vec{q}$ is the momentum of the outgoing pion, $a$ the isospin index of the pion and $\vec{e}_1$ the polarization wave function of the heavy vector meson $P^{*i}$. The amplitudes with heavy anti-mesons can be obtained either from the Lagrangian or from a $G$-parity transformation of the expressions above, yielding

$$A(\bar{P} \to \bar{P} \pi^a) = 0,$$

(A27)

$$A(\bar{P}^* \to \bar{P} \pi^a) = \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{e}_1 \cdot \vec{q},$$

(A28)

$$A(\bar{P}^* \to \bar{P}^* \pi^a) = i \frac{g}{f_\pi} \frac{\tau^a}{\sqrt{2}} \vec{S} \cdot \vec{q},$$

(A29)
where only the $\bar{P}P^*\pi$ vertex changes sign.

\[ V_{F,HH}^{(0)}(\vec{q}) = \frac{g^2}{2f_\pi^2} \frac{1}{\vec{q}^2 + m_\pi^2} \left( \begin{array}{cccc} 0 & 0 & 0 & -\epsilon_1^* \cdot \vec{q} \epsilon_2^* \cdot \vec{q} \\ 0 & 0 & -\epsilon_1^* \cdot \vec{q} \epsilon_2^* \cdot \vec{q} + S_1 \cdot \vec{q} \epsilon_2^* \cdot \vec{q} \\ -\epsilon_1^* \cdot \vec{q} \epsilon_2 \cdot \vec{q} + S_1 \cdot \vec{q} \epsilon_2 \cdot \vec{q} & 0 & -\epsilon_1^* \cdot \vec{q} \vec{S}_2 \cdot \vec{q} + S_1 \cdot \vec{q} \vec{S}_2 \cdot \vec{q} \\ \end{array} \right) + O\left(\frac{1}{m_Q^2}\right) \] (A30)

From the non-relativistic amplitudes above, the OPE potential in the heavy quark limit is given by

\[ V_{F,HH}^{(0)}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} V_{F,HH}^{(0)}(\vec{q}) e^{-i\vec{q} \cdot \vec{r}}, \] (A32)

in which the possible off-shell effects are strongly suppressed.

We can calculate the coordinate space representation of the OPE potential by Fourier transforming the momentum space potential

\[ V_{F,HH}^{(0)}(\vec{r}) = \vec{r} \cdot \vec{r} \cdot \frac{g^2}{6f_\pi^2} \delta(\vec{r}) - \vec{r} \cdot \vec{r} \left[ C_{12}(\vec{r}) W_C(\vec{r}) + S_{12}(\vec{r}) W_T(\vec{r}) \right] + O\left(\frac{1}{m_Q^2}\right), \] (A33)

with $W_C(\vec{r})$ and $W_T(\vec{r})$ defined in Eqs. (37) and (37). The central and tensor operators $C_{12}(\vec{r})$ and $S_{12}(\vec{r})$ are $4 \times 4$ matrices in the particle channel basis $B_{HH}$. Their explicit representation is

\[ C_{12}(\vec{r}) = \left( \begin{array}{cccc} 0 & 0 & 0 & -\epsilon_1^* \cdot \vec{q} \epsilon_2^* \cdot \vec{q} \\ 0 & 0 & -\epsilon_1^* \cdot \vec{q} \epsilon_2^* \cdot \vec{q} + S_1 \cdot \vec{q} \epsilon_2^* \cdot \vec{q} \\ -\epsilon_1 \cdot \epsilon_2 + S_1 \cdot \vec{q} \epsilon_2 \cdot \vec{q} & 0 & -\epsilon_1 \cdot \epsilon_2 - S_1 \cdot \vec{q} \epsilon_2 \cdot \vec{q} \\ \end{array} \right), \] (A34)

\[ S_{12}(\vec{r}) = \left( \begin{array}{cccc} 0 & 0 & 0 & -S_1(\epsilon_1^* \cdot \epsilon_2^* \cdot \hat{r}) \\ 0 & 0 & -S_1(\epsilon_1^* \cdot \epsilon_2^* \cdot \hat{r}) + S_1(\vec{S}_1 \cdot \vec{q} \epsilon_2 \cdot \vec{q}, \hat{r}) \\ -S_1(\epsilon_1^* \cdot \epsilon_2^* \cdot \hat{r}) + S_1(\vec{S}_1 \cdot \vec{q} \epsilon_2 \cdot \vec{q}, \hat{r}) & 0 & -S_1(\epsilon_1^* \cdot \vec{S}_2 \cdot \vec{q}, \hat{r}) \\ \end{array} \right), \] (A35)

where $S_{12}$ is defined as

\[ S_{12}(\vec{a}_1, \vec{b}_2, \hat{r}) = 3 \vec{a}_1 \cdot \hat{r} \vec{b}_2 \cdot \hat{r} - \vec{a}_1 \cdot \vec{b}_2. \] (A36)

5. Partial Wave Projection

We can simplify the representation of the OPE potential by projecting into HH states with well-defined $J^{PC}$ quantum numbers. This task is greatly simplified in coordinate space, for which only the operators $C_{12}(\vec{r})$ and $S_{12}(\vec{r})$ are to be projected. In the PP case, we consider the partial waves

\[ |\vec{P}\vec{P}(ljm)\rangle = \delta_{jl} Y_{lm}(\hat{r}), \] (A37)
The partial waves with well-defined orbital and total angular momentum \( l \) and \( j \) are defined as

\[
|P^* P^*(ljm)\rangle = \sum_{m_{l},m_{j}} Y_{lm_{l}}(\hat{r})|1\nu\rangle\langle lm_{l}1\nu|jm\rangle, \tag{A38}
\]

\[
|P^* \bar{P}(ljm)\rangle = \sum_{m_{l},\mu_{l}} Y_{lm_{l}}(\hat{r})|1\mu\rangle\langle lm_{l}1\mu|jm\rangle, \tag{A39}
\]

where \(|1\mu\rangle\), \(|1\nu\rangle\) are the polarization vectors of particle 1 and 2 and \(<l_{1}m_{1}2m_{2}|jm\rangle\) a Clebsch-Gordan coefficient. The polarization vectors are to be understood as giving matrix elements of the type

\[
\langle 1\mu'|\hat{e}_{1}\cdot\hat{e}_{1}|1\mu\rangle = \delta_{\mu_{\mu}}, \tag{A40}
\]

\[
\langle 1\nu'|\hat{e}_{2}\cdot\hat{e}_{1}|1\mu\rangle = \delta_{\nu_{\nu}}\delta_{\mu_{\mu}}, \tag{A41}
\]

and so on, where we have added primas to denote final states and with \( \hat{e}_{1} \) the unit vector in the spherical basis.

A problem with the two partial waves we have written above is that they do not have a well-defined C-parity. Thus, we define the linear combination

\[
|P^* P^*(\eta)(ljm)\rangle = \frac{1}{\sqrt{2}} [|P^* P^*(ljm)\rangle - \eta|P^* \bar{P}(ljm)\rangle], \tag{A42}
\]

with intrinsic C-parity \( \eta \) and total C-parity \( C = (-1)^l \eta \).

Finally, the \( P^*P^* \) partial wave with intrinsic, orbital and total angular momentum \( s, l \) and \( j \) is given by

\[
|P^* P^*(s|jm)\rangle = \sum_{m_{s},m_{l}} Y_{lm_{l}}(\hat{r})|s \rangle_{s}|m_{s}\rangle_{m_{s}}\langle lm_{l}sm_{s}|jm\rangle, \tag{A43}
\]

where the spin state \( |s\rangle_{s}\rangle_{m_{s}} \) is

\[
|s\rangle_{s}\rangle_{m_{s}} = \sum_{\mu_{s}} |1\mu\rangle\langle 1\mu|1\nu\rangle\langle 1\nu|sm_{s}\rangle. \tag{A44}
\]

The C-parity of these states is \( C = (-1)^{l+s} \). Alternatively we can also use the spectroscopic notation \( 2s+1l_{j} \) instead of \( (s)|lj \) for denoting the partial waves.

From the previous definitions we can compute the matrix elements of the central and tensor operators as

\[
\langle (s')l'j'm'|\hat{O}_{12}(s)|lj\rangle = \int d^{2}\hat{r}\langle (s')l'j'm'|\hat{O}_{12}(\hat{r})|lj\rangle = \delta_{jj'}\delta_{mm'}O_{lj}(s'), \tag{A45}
\]

where \( \hat{O}_{12}(\hat{r}) \) either represents \( \hat{C}_{12}(\hat{r}) \) or \( \hat{S}_{12}(\hat{r}) \). The particle channels have been made implicit owing to the matrix notation. The central and tensor operators preserve the \( J^{PC} \) quantum numbers, which for the particular case of the tensor operator means that the orbital angular momentum can only change by an even number of units. In addition, if we only consider the \( P^*P^* \) particle channel, the intrinsic angular momentum is also subjected to the restriction of even \( |s - s'| \). However, terms mixing the \( P^*P^* \) and \( PP^*/PP^* \) channels can change \( s \) by one unit.

Thus the most compact way to write the matrix elements of the \( C_{12}(\hat{r}) \) or \( S_{12}(\hat{r}) \) operators is to consider a particle/partial wave basis with well-defined \( J^{PC} \). Examples are the following

\[
B_{HH}(0^{++}) = \left\{ |1^{+}S_{0}(P\bar{P})\rangle, |1^{+}S_{0}(P^*\bar{P}^*)\rangle, \right\}, \tag{A46}
\]

\[
B_{HHi}(1^{-+}) = \left\{ |3^{+}S_{1}(P\bar{P}^* + P^*\bar{P})\rangle, \right\}, \tag{A47}
\]

\[
B_{HHi}(2^{++}) = \left\{ |3^{+}D_{1}(P\bar{P}^* + P^*\bar{P})\rangle, \right\}, \tag{A48}
\]

for which we obtain the central matrices

\[
C_{12}(0^{++}) = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A49}
\]

\[
C_{12}(1^{-+}) = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \tag{A50}
\]

\[
C_{12}(2^{++}) = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{3} & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{A51}
\]

and the tensor matrices

\[
S_{12}(0^{++}) = \begin{pmatrix} 0 & 0 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{2} & -2 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \end{pmatrix}, \tag{A52}
\]

\[
S_{12}(1^{-+}) = \begin{pmatrix} \sqrt{2} & -1 & \sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & -1 & \sqrt{2} & -1 \end{pmatrix}, \tag{A53}
\]
Other $J^{PC}$ possibilities are to be computed as these three
elements. As can be seen, the resulting coupled channel
structure can be quite complex: in certain cases we can
have up to six coupled channels. The simplifying feature
is that particle coupled channel effects are suppressed by
two orders in the chiral expansion of the potential, which
means that in low order calculations we can concentrate
in a specific particle channel and ignore the others. In
this case, we only need to consider the diagonal terms of
the central operator and, in what regard the tensor op-
erator, we end up with the matrices shown from Eq. (51)
to (61) with at most four coupled channels.

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