Interaction of global and local monopoles

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Abstract

We study the direct interaction between global and local monopoles. While in two previous papers, the coupling between the two sectors was only “indirect” through the coupling to gravity, we here introduce a new term in the potential that couples the Goldstone field and the Higgs field directly. We investigate the influence of this term in curved space and compare it to the results obtained previously.

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I. INTRODUCTION

Magnetic monopoles have raised a large interest since their first construction by Dirac [1]. While the Dirac-monopole has a singularity, the so-called Dirac string, ’t Hooft and Polyakov [2] came up with the construction of a particle-like magnetic monopole in SU(2) Yang-Mills-Higgs (YMH) theory with triplet Higgs scalar. The magnetic charge of this object results from the topological properties of the solution and is directly proportional to the degree of the map from space-time infinite to the vacuum manifold of the theory. Minimal coupling of the SU(2) YMH model to gravity leads (for suitable choice of the boundary conditions) to globally regular gravitating monopoles [3,4,5,6] which exist up to a maximal value of the gravitational coupling. For higher values of that coupling, the Schwarzschild radius of the solution becomes larger than the radius of the monopole core.

Considering only the theory with a scalar Goldstone field leads to a different type of topological defect [7], the so-called global monopole. Like all global defects this has infinite energy resulting from the $1/r^2$ fall-off of the energy density. Coupling to gravity [8,9] leads to the observation that the effective mass of the system becomes negative.

Recently, a self-gravitating magnetic monopole in the spacetime of a global monopole has been considered [10,11]. In both papers the potential is the sum of the Higgs potential and the analog Goldstone field potential. Thus, the interaction between the global Goldstone field and the local Higgs field is only indirect, namely through the coupling to gravity.

Considering the composite topological defect, the effective mass was found to be positive or negative, depending on the coupling constants of the model.

In this paper we continue the investigation of this system, allowing a direct interaction between the matter fields. This extra interaction is implemented by adding to the potential a gauge invariant term as follows:

$$V_3(\phi^a, \chi^a) = \frac{\lambda_3}{2} (\phi^a \phi^a - \eta^2_1)(\chi^a \chi^a - \eta^2_2).$$

Here, we are mainly interested in the analysis of the critical behaviour of the composite system considering now the most general, gauge invariant potential. Because of the high non-linearity of the set of coupled differential equation, an analytical analysis becomes impossible and only a numerical analysis can provide the results.

This paper is organized as follows. In Section II we describe our model and the Ansatz. In Section III, we give the equations of motion, the boundary conditions and the analysis of the asymptotic behaviour of the Goldstone and Higgs field functions. We present our numerical results in Section IV and describe how the extra term $[\text{II}]$ in the potential provides new results concerning the behaviour of the fields near the defect’s core as well as concerning the effective mass of the system. We observe e.g. a strong dependence of the mass on the coupling constant $\lambda_3$. We give our conclusions in Section V.

II. THE EXTENDED MODEL

This model is described by the following action which is composed of the action for the gravitating global monopole and the action of the gravitating local monopole:
\[ S = S_G + S_M = \int \mathcal{L}_G \sqrt{-g} \, d^4 x + \int \mathcal{L}_M \sqrt{-g} \, d^4 x , \]  
with the gravity Lagrangian \( \mathcal{L}_G \):

\[ \mathcal{L}_G = \frac{1}{16\pi G} R \]  
and \( G \) denotes Newton’s constant.

The matter Lagrangian \( \mathcal{L}_M \) of the extended model with extra direct interaction between the global Goldstone field \( \chi^a \) and the Higgs field \( \phi^a \) reads \((a = 1, 2, 3)\):

\[ \mathcal{L}_M = -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu, a} - \frac{1}{2} (D_\mu \phi^a)(D^\mu \phi^a) - \frac{1}{2} (\partial_\mu \chi^a)(\partial^\mu \chi^a) - V(\phi^a, \chi^a) , \]  
with covariant derivative of the Higgs field

\[ D_\mu \phi^a = \partial_\mu \phi^a - e \epsilon_{abc} A_\mu \phi^c , \]  
field strength tensor

\[ F_{\mu \nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e \epsilon_{abc} A^b_\mu A^c_\nu , \]  
and \( e \) being the gauge coupling constant. The potential \( V(\phi^a, \chi^a) \) is given by:

\[ V(\phi^a, \chi^a) = \frac{\lambda_1}{4} \left( \phi^a \phi^a - \eta_1^2 \right)^2 + \frac{\lambda_2}{4} \left( \chi^a \chi^a - \eta_2^2 \right)^2 + \frac{\lambda_3}{2} \left( \phi^a \phi^a - \eta_1^2 \right) \left( \chi^a \chi^a - \eta_2^2 \right) , \]

where the third term on the rhs couples the two sectors directly to each other with coupling constant \( \lambda_3 \). \( \lambda_1, \lambda_2 \) denote the self-coupling constants of the Higgs and Goldstone field, respectively, while \( \eta_1, \eta_2 \) are the corresponding vacuum expectation values.

The potential \((7)\) has different properties according to the sign of \( \Delta \equiv \lambda_1 \lambda_2 - \lambda_3^2 \). For \( \Delta > 0 \) the potential has positive values and its minima are attained for \( \phi_1^2 = \eta_1^2, \chi_1^2 = \eta_2^2 \), for which \((7)\) is obviously zero. For \( \Delta < 0 \), these configurations become saddle points and two minima occur for

\[ \phi_1^2 = 0 , \ \chi_1^2 = \eta_2^2 + \frac{\lambda_1}{\lambda_3} \eta_1^2 \]  
and

\[ \phi_1^2 = 0 , \ \phi_1^2 = \eta_2^2 + \frac{\lambda_2}{\lambda_3} \eta_1^2 . \]  
The potential’s values for these extrema are, respectively,

\[ V_{\min} = \frac{\eta_1^4}{4\lambda_1} \Delta , \ V_{\min} = \frac{\eta_2^4}{4\lambda_2} \Delta \]

which are negative since \( \Delta < 0 \).
A. The Ansatz

The Ansatz for the metric tensor in Schwarzschild-like coordinates reads:

\[ ds^2 = -A^2(r)N(r)dt^2 + N^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (11) \]

where we define for later convenience the mass function \( m(r) \) as follows:

\[ N(r) = 1 - 2\alpha^2 q^2 - \frac{2m(r)}{r} . \quad (12) \]

The Ansatz for the global Goldstone field \( \chi^a \), the Higgs field \( \phi^a \) and the gauge field \( A^a_\mu \) in Cartesian coordinates reads:

\[ \phi^a(x) = \eta_1 h(r)\hat{x}^a , \quad (13) \]

\[ \chi^a(x) = \eta_1 f(r)\hat{x}^a , \quad (14) \]

\[ A^a_i(x) = \epsilon_{iaj}\hat{x}^j \frac{1 - u(r)}{er} , \quad (15) \]

and

\[ A^a_0(x) = 0 . \quad (16) \]

Substituting the above configurations into the matter Lagrangian density we obtain:

\[ \mathcal{L}_M = -4\pi \int_0^\infty dr \ r^2 \ A \left[ N K(f, h, u) + \mathcal{U}(f, h, u) \right] , \quad (17) \]

where

\[ K(f, h, u) = \frac{1}{2} \eta_1^2 (f')^2 + \frac{1}{2} \eta_1^2 (h')^2 + \frac{(u')^2}{e^2 r^2} , \quad (18) \]

and

\[ \mathcal{U}(f, h, u) = \frac{(u^2 - 1)^2}{2e^2 r^4} + \frac{\eta_1^2 u^2 h^2}{r^2} + \frac{\eta_1^2 f^2}{r^2} + \frac{\lambda_1 \eta_1^4}{4} (h^2 - 1)^2 + \frac{\lambda_2 \eta_1^4}{4} (f^2 - q^2)^2 \]

\[ + \frac{\lambda_3 \eta_1^4}{2} (h^2 - 1)(f^2 - q^2) . \quad (19) \]

The prime denotes the derivative with respect to \( r \).

The gravity Lagrangian \( \mathcal{L}_G \) is given by

\[ \mathcal{L}_G = \frac{1}{2G} \int_0^\infty dr r(N - 1)A' . \quad (20) \]
III. EQUATIONS OF MOTION

Varying (14) with respect to the matter fields and gravitational fields and introducing the dimensionless variable \( x \) and dimensionless mass function \( \mu(x) \):

\[
x = e \eta_1 r , \quad \mu(x) = e \eta_1 m(r)
\]

we obtain the following set of differential equations:

\[
\frac{d}{dx} \left[ x^2 \frac{df}{dx} \right] = A \left[ 2f + x^2 \beta^2_2 (f^2 - q^2) + x^2 \beta^2_2 (h^2 - 1) \right], \quad (22)
\]

\[
\frac{d}{dx} \left[ x^2 \frac{dh}{dx} \right] = A \left[ 2u^2 h + x^2 \beta^2_1 (h^2 - 1) h + x^2 \beta^2_2 h (f^2 - q^2) \right], \quad (23)
\]

\[
\frac{d}{dx} \left[ \frac{du}{dx} \right] = A \left[ u \left( \frac{u^2 - 1}{x^2} \right) + uh^2 \right], \quad (24)
\]

\[
\frac{d}{dx} (xAN) = [1 - 2\alpha^2 x^2 \bar{U}] A, \quad (25)
\]

with

\[
\bar{U} = \frac{(u^2 - 1)^2}{2x^4} + \frac{u^2 h^2}{x^2} + \frac{f^2}{x^2} + \frac{\beta^2_1}{4} (h^2 - 1)^2 + \frac{\beta^2_2}{4} (f^2 - q^2)^2
\]

\[
+ \frac{\beta^2_3}{2} (h^2 - 1)(f^2 - q^2), \quad (26)
\]

and

\[
\frac{dA}{dx} = 2\alpha^2 Ax \bar{K}, \quad (27)
\]

with

\[
\bar{K} = \frac{1}{2} \left( \frac{df}{dx} \right)^2 + \frac{1}{2} \left( \frac{dh}{dx} \right)^2 + \frac{1}{x^2} \left( \frac{du}{dx} \right)^2. \quad (28)
\]

The equations only depend on the dimensionless coupling constants:

\[
\alpha^2 = 4\pi G \eta_1^2, \quad \beta^2_i = \lambda_i / e^2, \quad i = 1, 2, 3, \quad q = \eta_2 \eta_1. \quad (29)
\]

With the definition (12), Eq. (25) can be brought to the form

\[
\mu' = \alpha^2 x^2 \left( \bar{K} + (\bar{U} - \frac{q^2}{x^2}) \right). \quad (30)
\]

The finite energy of the solution can then be obtained by taking the value of \( \mu(x) \) at infinity.
A. Boundary conditions

The boundary conditions at the origin which follow from the requirement of regularity read:

\[ u(x = 0) = 1, \quad f(x = 0) = 0, \quad h(x = 0) = 0, \quad \mu(x = 0) = 0. \] \hspace{1cm} (31)

In fact the behaviour of the function \( \mu(x) \) near the origin is \( \mu(x) \approx -\alpha^2 q^2 x \). The requirement of finite energy solutions leads to:

\[ f(x = \infty) = q, \quad h(x = \infty) = 1, \quad u(x = \infty) = 0, \quad N(x = \infty) = 1 - 2\alpha^2 q^2. \] \hspace{1cm} (32)

B. Asymptotic behaviour

The integration of the equations for generic values of the parameters needs a better understanding of the asymptotic behavior of the functions \( f(x) \) and \( h(x) \). Two different types of behaviour for the functions \( f \) and \( h \) in flat space seem possible. Either:

\[ h(x) = 1 + \frac{A}{x^2} + O(\frac{1}{x^3}) \quad A = \frac{\beta_3^2}{\beta_2^2 \beta_1^2 - \beta_3^2} \]
\[ f(x) = q + \frac{B}{x^2} + O(\frac{1}{x^3}) \quad B = \frac{-\beta_1^2}{q(\beta_2^2 \beta_1^2 - \beta_3^2)} \] \hspace{1cm} (33)

or

\[ h(x) = 1 + C_1 \exp(\rho_1 x) + C_2 \exp(-\rho_1 x) + C_3 \exp(\rho_2 x) + C_4 \exp(-\rho_2 x), \]
\[ f(x) = q + \tilde{C}_1 \exp(\rho_1 x) + \tilde{C}_2 \exp(-\rho_1 x) + \tilde{C}_3 \exp(\rho_2 x) + \tilde{C}_4 \exp(-\rho_2 x), \] \hspace{1cm} (34)

where \( \rho_1^2, \rho_2^2 \) are the eigenvalues of the matrix

\[
\begin{pmatrix}
\beta_1^2 & q \beta_3^2 \\
q \beta_2^2 & q^2 \beta_3^2 \\
\end{pmatrix}.
\] \hspace{1cm} (35)

Our numerical analysis strongly suggests the following results:

- for \( \beta_1^2 \beta_2^2 - \beta_3^4 > 0 \) the solutions obey the asymptotic behaviour (33) for \( f \) and \( h \),
- for \( \beta_1^2 \beta_2^2 - \beta_3^4 < 0 \) the functions \( f \) and \( h \) have the asymptotic behaviour (34). In this case, however, one of the eigenvalues of (35) becomes negative, consequently its square root is complex and the functions \( f, h \) thus oscillate. This behaviour is clearly observed when we set \( \beta_1 = 0, \beta_2 \neq 0 \) and increase \( \beta_3 \) from 0. For \( \beta_3 = 0 \), the solutions exist and \( f, h \) increase monotonically, as soon as \( \beta_3 \neq 0 \), however, oscillations occur.
IV. NUMERICAL RESULTS

Because of the reasons given previously, we restrict our analysis in the following to the case $\beta_1^2\beta_2^2 - \beta_3^4 > 0$.

The limit $\beta_3 = 0$ was studied in detail in [10], [11]. Here we discuss how the new term influences the behaviour of the solutions. One of the main features is that the Higgs function $h(x)$ does not reach its asymptotic value $h(x = \infty) = 1$ monotonically. It first reaches a maximum $h_{\text{max}} > 1$ for $x < \infty$ and then decreases to 1. This phenomenon, which can be expected from the inspection of (33) since $A > 0$ for $\beta_3 \neq 0$, is illustrated in Fig. 1. This is different from the phenomena observed in [10], [11]. There, for all values of the coupling constants, the function $h(x)$ was observed to be monotonically increasing from 0 to 1 as indicated in Fig. 1 for $\beta_3 = 0$. The Goldstone field functions $f(x)$ reaches its asymptotic value $q$ for increasing values of the coordinate $x$ when $\beta_3$ is increasing. This again can be explained by (33) since the value $B$ is a decreasing function of $\beta_3$ with a sharp drop at $\beta_3 \approx 1$. Thus, the function $f(x)$ decays less strong for higher values of $\beta_3$.

At the same time, the minimum of the metric function $N$ decreases with increasing $\beta_3$, while the matter function $\mu$ reaches a maximum at roughly the same $x$ at which the function $N$ attains its minimum and than drops down to its asymptotic value which determines the mass of the solution. Clearly, for $\beta_3 = 0$, this asymptotic value is positive, while for increasing $\beta_3$, it decreases and becomes negative for large enough $\beta_3$.

Another feature of the new term is that, for all parameters but $\beta_3$ fixed, the classical mass of the solution decreases when $\beta_3$ increases. This is illustrated by means of Fig. 2 where we have plotted the evolution of the mass as a function of $\beta_3$ for three different combinations of the coupling constants $\alpha, q$. For $q = 1.0, \alpha = 0.4$, the mass of the solution is already negative for $\beta_3 = 0$ indicating that the influence of the global monopole is already dominating in the limit of vanishing direct interaction of the Higgs field and Goldstone field. For smaller $q$, the mass becomes negative at some finite value $\beta_3 = \beta_3^0$. For fixed $q$, this value is increasing for decreasing $\alpha$. Since the local and global monopole are still “indirectly” coupled over gravity, a stronger gravitational coupling, of course, couples the two objects in a stronger way. For small gravitational coupling, $\beta_3$ thus has to be raised further to make the influence of the global monopole dominating. When the combination $\beta_1^2\beta_2^2 - \beta_3^4$ becomes negative, the mass of the solution reaches $-\infty$, independent of the combination of $q$ and $\alpha$. This again can be related to the fact that we observe the solutions to become oscillating for $\beta_1^2\beta_2^2 - \beta_3^4 < 0$.

We also studied the way the solution bifurcates into a black hole when the parameter $\alpha$ increases while the others are fixed. Fixing $q = 0.4$ we have analyzed the critical behaviour of the solution and checked that, like for the case $\beta_3 = 0$, the solution bifurcates into a black hole for a finite value of $\alpha$, say $\alpha = \alpha_c$. As demonstrated in Fig. 3 for $\beta_1 = \beta_2 = 1, q = 0.4$ and $\beta_3 = 0.8$, the function $N$ develops a minimum which becomes deeper while $\alpha$ increases and becomes zero for $\alpha = \alpha_c$. The limiting solution thus represents an extremal black hole solution with horizon $x_h$. In Fig. 3, we also show the evolution of the mass with $\alpha$. Finally, the critical solution corresponding to $\beta_3 = 0.8$ and $\alpha_c \approx 1.085$ is displayed in Fig. 4. This figure clearly suggests that the limiting solution is not an abelian black hole for $x > x_h$ like e.g. in the case of the pure local monopole [3], where the solutions bifurcate with the branch of Reissner-Nordström (RN) solutions and consequently the functions reach their RN values for $x > x_h$. Here, the function $f$ is equal to zero for $0 \leq x \leq x_h$ and non-trivial
for $x_h \leq x \leq x_0$. Similarly, the functions $A$ and $h$ are non-trivial for $x \leq x_0$ and are equal to their asymptotic values for $x > x_0$, while the gauge field function $u$ reaches its asymptotic value for $x \approx x_h$. This solution thus represents a “black hole inside a global monopole” as was observed previously for the $\beta_3 = 0$ limit \[11\].

V. CONCLUSIONS

In this paper we have analyzed the composite system of a global and a local gravitating monopole considering the most general gauge-invariant potential. This potential contains a direct interaction between the Goldstone and the Higgs field. This term leads to important consequences concerning the local behaviour of the fields themselves as well as concerning the global properties of the system. One of the most relevant consequences is related to the effective mass associated with the composite topological defect. The numerical results show a strong dependence of this mass on the coupling constant $\lambda_3$. Although the Goldstone and Higgs fields are indirectly coupled through gravity, the extra direct interaction is more effective. Increasing the parameter $\lambda_3$, the mass becomes negative, indicating the dominance of the global sector over the local one. Compared to the results of the $\lambda_3 = 0$ case \[10,11\], we observe the modulus of the negative mass to become very large in our system. Another point which deserves to be mentioned is that the extra direct interaction term is not positive definite. Denoting by $x_{h=1}$ the value of $x$ for which the Higgs field function $h$ is equal to one, $h$ becomes bigger than one for $x > x_{h=1}$ and the new term in the potential becomes negative. However, for a particular choice of the self-coupling constants such that they fulfill $\Delta \equiv \lambda_1 \lambda_2 - \lambda_3^2 > 0$, the total potential is positive, vanishing only at the minima $\phi_a^2 = \eta_1^2$, $\chi_a^2 = \eta_2^2$.

As possible extensions of the model studied here, let us mention the coupling to a scalar dilaton which arises naturally in low energy effective actions of string theory. The gravitating local monopole was studied recently coupled to a dilaton \[12\] and it was found that in the limit of critical gravitational coupling, the solutions bifurcate with the branch of extremal Einstein-Maxwell-dilaton (EMD) solutions which are associated with naked singularities. It would be interesting to see what sort of critical solution the composite system of a global and local monopole reaches since our analysis indicates that the behaviour of the functions close to the core of the local monopole is strongly influenced by the global monopole.
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FIG. 1. The metric functions $N$, $\mu$, the Higgs field function $h$ and the Goldstone field function $f$ are shown as functions of the dimensionless variable $x$ for $q = 0.4$, $\alpha = 0.6$, $\beta_1 = \beta_2 = 1$ and three different values of $\beta_3$. 
FIG. 2. The mass in units of $\frac{4\pi m}{e}$ is shown as function of $\beta_3$ for three different combinations of the coupling constants $\alpha$ and $q$ and for $\beta_1 = \beta_2 = 1$. 
FIG. 3. The mass in units of $\frac{4\pi m}{e}$ and the minimum $N_m$ of the metric function $N(x)$ are shown as function of $\alpha$ for $\beta_3 = 0$, respectively $\beta_3 = 0.8$, and $q = 0.4$, $\beta_1 = \beta_2 = 1$. 
FIG. 4. The profiles of the functions $N, A, u, f, h$ are shown for $\alpha \approx \alpha_c = 1.08$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0.8$ and $q = 0.4$. 