Dynamics of viscous dissipative gravitational collapse: A full causal approach

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Abstract

The Misner and Sharp approach to the study of gravitational collapse is extended to the viscous dissipative case in, both, the streaming out and the diffusion approximations. The dynamical equation is then coupled to causal transport equations for the heat flux, the shear and the bulk viscosity, in the context of Israel–Stewart theory, without excluding the thermodynamics viscous/heat coupling coefficients. The result is compared with previous works where these later coefficients were neglected and viscosity variables were not assumed to satisfy causal transport equations. Prospective applications of this result to some astrophysical scenarios are discussed.
1 Introduction

The gravitational collapse of massive stars represents one of the few observable phenomena where general relativity is expected to play a relevant role. This fact is at the origin of the great attraction that this problem exerts on the community of the relativists, since the seminal paper by Oppenheimer and Snyder [1].

Ever since that work, much was done by researchers trying to grasp essentials aspects of this phenomenon (see [2] and references therein). However this endeavour proved to be difficult and uncertain. Different kind of obstacles appear, depending on the approach adopted for the modelling and/or on the complexity of the physical description of the fluid, assumed to form the selfgravitating object. All these factors in turn are conditioned by the relevant time scales of different physical phenomena under consideration.

Thus, during their evolution, self–gravitating objects may pass through phases of intense dynamical activity, with time scales of the order of magnitude of (or even smaller than) the hydrostatic time scale, and for which the quasi–static approximation is clearly not reliable, e.g.,the collapse of very massive stars [3], and the quick collapse phase preceding neutron star formation, see for example [4] and references therein. In these cases, in which we are mainly concerned with here, it is mandatory to take into account terms which describe departure from equilibrium, i.e. a full dynamic description has to be used.

We shall assume that the process is dissipative. In fact, it is already an established fact, that gravitational collapse is a highly dissipative process (see [5], [6], [7] and references therein). This dissipation is required to account for the very large (negative) binding energy of the resulting compact object (of the order of $-10^{53}$erg).

Indeed, it appears that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole is neutrino emission [8].

In the diffusion approximation, it is assumed that the energy flux of radiation (and that of thermal conduction) is proportional to the gradient of temperature. This assumption is in general very sensible, since the mean free path of particles responsible for the transfer of energy in stellar interiors is usually very small as compared with the typical length of the object. Thus, for a main sequence star as the sun, the mean free path of photons at the
centre, is of the order of 2 cm. Also, the mean free path of trapped neutrinos in compact cores of densities about $10^{12} \, \text{g. cm}^{-3}$ becomes smaller than the size of the stellar core [9, 10].

Furthermore, the observational data collected from supernovae 1987A suggests that the regime of radiation transport prevailing during the emission process, is closer to the diffusion approximation than to the streaming out limit [11].

Dissipative effects in the diffusion approximation are further enhanced by very large values of thermal conductivity, which may be of the order of $\kappa \approx 10^{23} \, \text{erg s}^{-1} \, \text{cm}^{-1} \, \text{K}^{-1}$ for electron conductivity (see [12] and references therein) or even $\kappa \approx 10^{37} \, \text{erg s}^{-1} \, \text{cm}^{-1} \, \text{K}^{-1}$, for neutrino conductivity in a pre-supernovae event [13].

However, in many other circumstances, the mean free path of particles transporting energy may be large enough as to justify the free streaming approximation. Therefore we shall include simultaneously both limiting cases of radiative transport (diffusion and streaming out), allowing for describing a wide range situations.

Since we are mainly concerned with time scales that might be of the order of (or even smaller than) relaxation times, we have to appeal to a hyperbolic theory of dissipation in order to treat the transport equation for dissipative variables. The use of a hyperbolic theory of dissipation is further justified by the necessity of overcoming the difficulties inherent to parabolic theories (see references [5], [14]–[28] and references therein). Doing so we shall be able to give a description of processes occurring before thermal relaxation.

Some years ago, Misner and Sharp [29] and Misner [30] provided a full account of the dynamical equations governing the adiabatic [29], and the dissipative relativistic collapse in the streaming out approximation [30].

An extension of the Misner dynamical equations as to include dissipation in the form of a radial heat flow (besides pure radiation), was given in [5]. In that work the heat flux was assumed to satisfy a causal transport equation, but viscosity was absent and thereby the thermodynamics viscous/heat coupling coefficients were not taken under consideration. Furthermore, for simplification the fluid was assumed shearfree, in despite of the fact that the relevance of the shear tensor in the evolution of self-gravitating systems has been brought out by many authors (see [31] and references therein).

More recently [32], shear viscosity was introduced into the Misner approach, but again thermodynamics viscous/heat coupling coefficients were
neglected, and furthermore the assumed transport equation for the shear viscosity was the corresponding to the standard Eckart theory of relativistic irreversible thermodynamics [33, 34].

The motivation to consider viscosity effects in the study of relativistic gravitational collapse is well founded. In fact, though they are often excluded in general relativistic models of stars, they are known to play a very important role in the structure and evolution of neutron stars. Indeed, depending on the dominant process, the coefficient of shear viscosity may be as large as \( \eta \approx 10^{20} \text{ g cm}^{-1} \text{ s}^{-1} \) (see [35] for a review on shear viscosity in neutron stars).

On the other hand the coefficient of bulk viscosity may be as large as \( \zeta \approx 10^{30} \text{ g cm}^{-1} \text{ s}^{-1} \) due to Urca processes in strange quark matter [36].

Similar and even larger values may be attained for two color superconducting quark matter phases [37, 38] and for hybrid stars [39] (see also [40, 41, 42] and references therein for a review on bulk viscosity in nuclear and quark matter).

The purpose of this work is to present a dynamical description of the gravitational collapse within the framework of the Misner approach, for the more general dissipative fluid distribution consistent with spherical symmetry. This includes the presence of both shear and bulk viscosity, with a full causal treatment for all dissipative variables as well as the inclusion of the thermodynamics viscous/heat coupling coefficients. These coefficients may be relevant in non-uniform stellar models [26].

The manuscript is organized as follows: in the next Section, besides the field equations, the conventions, and other useful formulae we obtain the resulting dynamical equation. In Section 3 transport equations in the context of the Müller–Israel–Stewart theory [15, 16, 17] are obtained. The coupling of these equations with the dynamical equation is performed in Section 4. After doing that we show how the effective inertial mass density of a fluid element reduces by a factor which depends on dissipative variables. This result was already known (see [43] and references therein), but for the case in which only the heat flux was assumed to satisfy a causal transport equation, without the presence of the thermodynamics viscous/heat coupling coefficients.

In Section 5 an expression is derived which relates the Weyl tensor with the density inhomogeneity and thermodynamical variables. This allows us to bring out the role of dissipative variables in a definition of the arrow of time.

Finally the results are discussed in the last section.
2 The basic equations

In this section we shall deploy the relevant equations for describing a viscous
dissipative self–gravitating fluid. This includes a full description of the matter
distribution, the line element, both, inside and outside the fluid boundary,
and the field equations this line element must satisfy.

2.1 Interior spacetime

We consider a spherically symmetric distribution of collapsing fluid, bounded
by a spherical surface Σ. We assume the fluid to undergo dissipation in the
form of heat flow, free streaming radiation and shearing and bulk viscosity.

Choosing comoving coordinates inside Σ, the general interior metric can
be written as

\[ ds^2 = -A^2 dt^2 + B^2 dr^2 + (Cr)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

where \( A, B \) and \( C \) are functions of \( t \) and \( r \) and are assumed positive. We
number the coordinates \( x^0 = t, x^1 = r, x^2 = \theta \) and \( x^3 = \phi \).

The assumed matter energy momentum \( T^{-\alpha\beta} \) inside Σ has the form

\[ T^{-\alpha\beta} = (\mu + p + \Pi) V_\alpha V_\beta + (p + \Pi) g_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha + \epsilon l_\alpha l_\beta + \pi_{\alpha\beta} \]  

where \( \mu \) is the energy density, \( p \) the pressure, \( \Pi \) the bulk viscosity, \( q^\alpha \) the
heat flux, \( \pi_{\alpha\beta} \) the shear viscosity tensor, \( \epsilon \) the radiation density, \( V^\alpha \) the four
velocity of the fluid, and \( l^\alpha \) a radial null four vector. These quantities satisfy

\[ \begin{align*}
V^\alpha V_\alpha &= -1, & V^\alpha q_\alpha &= 0, & l^\alpha V_\alpha &= -1, & l^\alpha l_\alpha &= 0, \\
\pi_{\mu\nu} V^\nu &= 0, & \pi_{[\mu\nu]} &= 0, & \pi^\alpha_\alpha &= 0. 
\end{align*} \]  

In the standard irreversible thermodynamics we have [26, 44]

\[ \pi_{\alpha\beta} = -2\eta \sigma_{\alpha\beta}, \quad \Pi = -\zeta \Theta \]  

where \( \eta \) and \( \zeta \) denote the coefficient of shear and bulk viscosity, respectively,
\( \sigma_{\alpha\beta} \) is the shear tensor and \( \Theta \) is the expansion. However, since we are inter-
ested in a full causal picture of dissipative variables we shall not assume (4).
Instead, we shall use the corresponding transport equation derived from the
Müller–Israel–Stewart theory.
The shear $\sigma_{\alpha \beta}$ is given by

$$\sigma_{\alpha \beta} = V_{(\alpha ; \beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha \beta}$$  \hspace{1cm} (5)$$

where the acceleration $a_{\alpha}$ and the expansion $\Theta$ are given by

$$a_{\alpha} = V_{\alpha ; \beta} V^{\beta}, \quad \Theta = V^{\alpha ; \alpha}$$  \hspace{1cm} (6)$$

and $h_{\alpha \beta} = g_{\alpha \beta} + V_{\alpha} V_{\beta}$ is the projector onto the hypersurface orthogonal to the four velocity.

Since we assumed the metric (1) comoving then

$$V^{\alpha} = A^{-1} \delta^{\alpha}_0, \quad q^{\alpha} = q B^{-1} \delta^{\alpha}_1, \quad l^{\alpha} = A^{-1} \delta^{\alpha}_0 + B^{-1} \delta^{\alpha}_1,$$  \hspace{1cm} (7)$$

where $q$ is a function of $t$ and $r$. Also it follows from (3) that

$$\pi_{00} = 0, \quad \pi_{11} = -2\pi_2^2 = -2\pi_3^3.$$  \hspace{1cm} (8)$$

In a more compact form we can write

$$\pi_{\alpha \beta} = \Omega (\chi_{\alpha} \chi_{\beta} - \frac{1}{3} h_{\alpha \beta})$$  \hspace{1cm} (9)$$

where $\chi^{\alpha}$ is a unit four vector along the radial direction, satisfying

$$\chi^{\alpha} \chi_{\alpha} = 1, \quad \chi^{\alpha} V_{\alpha} = 0, \quad \chi^{\alpha} = B^{-1} \delta^{\alpha}_1,$$  \hspace{1cm} (10)$$

and $\Omega = \frac{3}{2} \pi_1^1$.

With (7) we obtain for (5) its non null components

$$\sigma_{11} = \frac{2}{3} B^2 \sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\sin^2 \theta} = -\frac{1}{3} (Cr)^2 \sigma,$$  \hspace{1cm} (11)$$

where

$$\sigma = \frac{1}{A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right),$$  \hspace{1cm} (12)$$

and the dot stands for differentiation with respect to $t$, which gives the scalar quantity

$$\sigma_{\alpha \beta} \sigma^{\alpha \beta} = \frac{2}{3} \sigma^2.$$  \hspace{1cm} (13)$$

For (6) with (7) we have,

$$a_1 = \frac{A'}{A}, \quad \Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{2}{C} \right),$$  \hspace{1cm} (14)$$

where the prime stands for $r$ differentiation.
2.2 The Einstein equations

Einstein’s field equations for the interior spacetime (1) are given by

\[ G^-_{\alpha\beta} = 8\pi T^-_{\alpha\beta}. \]  \hspace{1cm} (15)

The non null components of (15) with (1), (2), (7), (8) and (9) become

\[ G^-_{00} = 8\pi T^-_{00} = 8\pi (\mu + \epsilon) A^2 = \left(2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \frac{\dot{C}}{C} + \left(\frac{A}{B}\right)^2 \left\{-2\frac{C''}{C} + \left(2\frac{B'}{B} - \frac{C'}{C}\right) \frac{C'}{C} + \frac{2}{r} \left(\frac{B'}{B} - 3\frac{C'}{C}\right) - \left[1 - \left(\frac{B}{C}\right)^2\right]\right\} \frac{1}{r^2} \]  \hspace{1cm} (16)

\[ G^-_{11} = 8\pi T^-_{11} = 8\pi \left[p + \Pi + \epsilon + \frac{2}{3} \Omega\right] B^2 = - \left(\frac{B}{A}\right)^2 \left[2\frac{\ddot{C}}{C} + \left(\frac{\dot{C}}{C}\right)^2 - 2\frac{\dot{A}\dot{C}}{AC}\right] + \left(\frac{C''}{C}\right)^2 + \frac{2A'C'}{AC} + \frac{2}{r} \left(\frac{A'}{A} + \frac{C'}{C}\right) + \left[1 - \left(\frac{B}{C}\right)^2\right]\frac{1}{r^2} \]  \hspace{1cm} (17)

\[ G^-_{22} = 8\pi T^-_{22} = 8\pi \left[p + \Pi - \frac{\Omega}{3}\right] (Cr)^2 = - \left(\frac{Cr}{A}\right)^2 \left[\frac{\dot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}}{A} \left(\frac{B}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{B}}{B} \frac{\dot{C}}{C}\right] + \left(\frac{Cr}{B}\right)^2 \left[\frac{A''}{A} + \frac{C''}{C} - \frac{A'}{A} \left(\frac{B'}{B} - \frac{C'}{C}\right) - \frac{B'C'}{BC} + \frac{1}{r} \left(\frac{A'}{A} - \frac{B'}{B} + 2\frac{C'}{C}\right)\right] \]  \hspace{1cm} (18)

\[ G^-_{33} = \sin^2 \theta G^-_{22} \]  \hspace{1cm} (19)

\[ G^-_{01} = 8\pi T^-_{01} = -8\pi (q + \epsilon) AB = -2 \left(\frac{\dot{C'}}{C} - \frac{\dot{B}C'}{BC} - \frac{\dot{C} A'}{CA}\right) + \frac{2}{r} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \]  \hspace{1cm} (20)
observe that this last equation may be written as

\[ 4\pi(q + \epsilon)B = \frac{1}{3}((\Theta - \sigma)' - \sigma\frac{(Cr)'}{Cr}). \tag{21} \]

Next, the mass function \( m(t, r) \) introduced by Misner and Sharp [29] is defined by

\[ m = \frac{(Cr)^3}{2} - R_{23}^{23} = \frac{Cr}{2} \left\{ \left( \frac{rC'}{A} \right)^2 - \left[ \frac{(Cr)'}{B} \right]^2 + 1 \right\}. \tag{22} \]

2.3 The exterior spacetime and junction conditions

Outside \( \Sigma \) we assume we have the Vaidya spacetime (i.e. we assume all outgoing radiation is massless), described by

\[ ds^2 = - \left( 1 - \frac{2M(v)}{r} \right) dv^2 - 2drdv + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{23} \]

where \( M(v) \) denote the total mass, and \( v \) is the retarded time.

The matching of the full non-adiabatic sphere (including viscosity) to the Vaidya spacetime was discussed in [44].

From the continuity of the first and second differential forms it follows (see [44] for details),

\[ m(t, r) \equiv M(v), \tag{24} \]

and

\[ \begin{align*}
2\frac{C'}{C} &+ 2\frac{\dot{C}}{Cr} - 2\frac{\dot{B} C'}{B C} - 2\frac{\dot{B}}{Br} - 2\frac{A' \dot{C}}{AC} + \\
&+ \frac{B}{A} \left[ 2\frac{\ddot{C}}{C} - 2\frac{\dot{C}}{Cr} + \left( \frac{A}{Cr} \right)^2 + \left( \frac{C'}{C} \right)^2 - \left( \frac{A}{B} \right)^2 \left( \frac{C'}{C} + \frac{1}{r} \right) \left( \frac{C'}{C} + \frac{1}{r} + 2\frac{A'}{A} \right) \right] \\
\equiv 0, \tag{25} \end{align*} \]

where \( \Xi \) means that both sides of the equation are evaluated on \( \Sigma \) (observe a misprint in eq.(40) in [44] and a slight difference in notation).

Comparing (25) with (17) and (20) one obtains
Thus the matching of (1) and (23) on $\Sigma$ implies (24) and (26).

In the context of the standard irreversible thermodynamics where (4) is valid, we obtain

$$p + \Pi - \frac{4\eta\sigma}{3} \subseteq q,$$  \hspace{1cm} (27)

which reduces to eq.(41) in [44] with the appropriate change in notation. Observe a misprint in eq.(27) in [32] (the $\sigma$ appearing there is the one defined in [44], which is $-\frac{1}{3}$ of the one used here and in [32]).

### 2.4 Dynamical equations

The non trivial components of the Bianchi identities, $(T^{-\alpha\beta})_{;\beta} = 0$ yield

$$T_{;\nu}^{\nu\mu}V_{\mu} = \frac{1}{A} (\dot{\mu} + \dot{\epsilon}) - \frac{2}{B} (q' + \epsilon') - 2 (q + \epsilon) \frac{(ACr)' + \Omega}{ABCr} +$$

$$- \frac{2}{AC} \left( \mu + p + \Pi + \epsilon - \frac{\Omega}{3} \right) - \frac{1}{AB} \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3} \Omega \right) = 0$$

and

$$T_{;\nu}^{\nu\mu}X_{\mu} = \frac{1}{A} (\dot{q} + \dot{\epsilon}) + \frac{2}{A} \frac{(BCr)' + (q + \epsilon)}{BC}$$

$$+ \frac{1}{B} \left( \mu + p + \Pi + \epsilon + \frac{2}{3} \Omega' \right) + \frac{1}{B} \frac{A'}{A} \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3} \Omega \right)$$

$$+ \frac{2}{B} \frac{(Cr)' + \epsilon + \Omega}{Cr} = 0.$$  \hspace{1cm} (29)

To study the dynamical properties of the system, let us introduce, following Misner and Sharp [29], the proper time derivative $D_T$ given by

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}.$$  \hspace{1cm} (30)
and the proper radial derivative $D_R$,

$$D_R = \frac{1}{R^r} \frac{\partial}{\partial r}. \tag{31}$$

where

$$R = Cr \tag{32}$$

defines the proper radius of a spherical surface inside $\Sigma$, as measured from its area.

Using (30) we can define the velocity $U$ of the collapsing fluid as the variation of the proper radius with respect to proper time, i.e.

$$U = r D_T C < 0 \quad \text{(in the case of collapse).} \tag{33}$$

Then (22) can be rewritten as

$$E \equiv \left( \frac{Cr}{B} \right)' = \left[ 1 + U^2 - \frac{2m(t,r)}{Cr} \right]^{1/2}. \tag{34}$$

With (31)-(32) we can express (21) as

$$4\pi(q + \epsilon) = E \left[ \frac{1}{3} D_R(\Theta - \sigma) - \frac{\sigma}{R} \right]. \tag{35}$$

Observe that in the non–dissipative, shearfree case, the equation above may be written, with the help of (12), (14) and (33) as

$$D_R(\frac{U}{R}) = 0 \tag{36}$$

implying $U \sim R$, which describes an homologous collapse [45].

Next, using (16-20) and (30-32) we obtain from (22)

$$D_T m = -4\pi R^2 \left[ \left( p + \Pi + \epsilon + \frac{2}{3} \Omega \right) U + (q + \epsilon)E \right] \tag{37}$$

and

$$D_R m = 4\pi R^2 \left[ \mu + \epsilon + (q + \epsilon)\frac{U}{E} \right]. \tag{38}$$
Expression (37) describes the rate of variation of the total energy inside a surface of radius \( R \). On the right hand side of (37), \( (p + \Pi + \epsilon + \frac{2}{3}\Omega)U \) (in the case of collapse \( U < 0 \)) increases the energy inside \( R \) through the rate of work being done by the “effective” radial pressure \( p + \Pi + \frac{2}{3}\Omega \) and the radiation pressure \( \epsilon \). In the stationary regime where we can use the standard thermodynamical relation \( \pi_{\alpha\beta} = -2\eta\sigma_{\alpha\beta} \), we recover the result obtained in [32]. The second term \( (q + \epsilon)E \) is the matter energy leaving the spherical surface.

Equation (38) shows how the total energy enclosed varies between neighboring spherical surfaces inside the fluid distribution. The first two terms on the right hand side of (38), \( \mu + \epsilon \), are due to the energy density of the fluid element plus the energy density of the null fluid describing dissipation in the free streaming approximation. The last term, \( (q + \epsilon)U/E \) is negative (in the case of collapse) and measures the outflow of heat and radiation.

The acceleration \( D_T U \) of an infalling particle inside \( \Sigma \) can be obtained by using (17), (22), (30) and (34), producing

\[
D_T U = -\frac{m}{R^2} - 4\pi R \left( p + \Pi + \epsilon + \frac{2}{3}\Omega \right) + \frac{EA'_A}{AB}, \tag{39}
\]

and then, substituting \( A'/A \) from (39) into (29), we obtain

\[
\left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) D_T U = \\
- \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) \left[ \frac{m}{R^2} + 4\pi R \left( p + \Pi + \epsilon + \frac{2}{3}\Omega \right) \right] \\
- E^2 \left[ D_R \left( p + \Pi + \epsilon + \frac{2}{3}\Omega \right) + \frac{2}{R} (\epsilon + \Omega) \right] \\
- E \left[ D_T q + D_T \epsilon + 4 (q + \epsilon) \frac{U}{R} + 2 (q + \epsilon) \sigma \right]. \tag{40}
\]

As it can be easily seen, the main difference between (40), and eq.( 40) in [32] (regarding the contributions from shear viscosity) stems from the \( \pi_{\alpha\beta} \) terms which now are not given by (4), but have to satisfy a transport equation obtained within the context of the causal dissipative theory (see next section).

Thus, the factor within the round bracket on the left (which equals the factor on the first round bracket on the right, as it should be) represents
the effective inertial mass (the passive gravitational mass according to the equivalence principle).

The first term on the right hand side of (40) represents the gravitational force. In this term, the factor within the square bracket shows how dissipation affects the “active” gravitational mass term.

There are two different contributions in the second square bracket. The first one is just the gradient of the total “effective” pressure (which includes the radiation pressure and the influence of shear and bulk viscosity). The second contribution comes from the local anisotropy of pressure induced by the radiation pressure and shear viscosity.

The last square bracket contains different contributions due to dissipative processes. The third term within this bracket is positive ($U < 0$) showing that the outflow of $q > 0$ and $\epsilon > 0$ diminish the total energy inside the collapsing sphere, thereby reducing the rate of collapse. The last term describes an effect resulting from the coupling of the dissipative flux with the shear of the fluid. The effects of $D_T \epsilon$ have been discussed in [30] and we shall not analyze them in detail here.

Therefore it only remains to analyze the effects of transport equations when coupled to (40); we will proceed to carry on that task in the next section.

3 Transport equations

As stated in the Introduction, the main purpose of this work consists in providing a full causal description of viscous dissipative gravitational collapse. This implies that all dissipative variables (noy only $q$) have to satisfy the corresponding transport equation derived from causal thermodynamics. Furthermore the thermodynamics viscous/heat coupling coefficients will not be neglected, as they are expected to be relevant in non-uniform stellar models [26].

Accordingly, we shall use transport equations derived from the Müller-Israel-Stewart second order phenomenological theory for dissipative fluids [15, 16, 17].

This theory was proposed to overcome the pathologies [20] found in the approaches of Eckart [33] and Landau [34] for relativistic dissipative processes. The important point to retain is that this theory provides transport
equations for the dissipative variables, which are of Cattaneo type [14], leading thereby to hyperbolic equations for dissipative perturbations.

The starting point is the general expression for the entropy four–current, which in the context of the Müller-Israel-Stewart theory, reads (see [26] for details)

\[ S^\mu = S n V^\mu + \frac{q^\mu}{T} - \left( \beta_0 \Pi^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_{\alpha \nu} \pi^{\nu \kappa} \right) \frac{V^\mu}{2T} + \frac{\alpha_0 \Pi q^\mu}{T} + \frac{\alpha_1 \pi^{\mu \nu} q_\nu}{T} \]  

(41)

where \( n \) is particle number density, \( \beta_A(\rho, n) \) are thermodynamic coefficients for different contributions to the entropy density, and \( \alpha_A(\rho, n) \) are thermodynamics viscous/heat coupling coefficients.

Next, from the Gibbs equation and Bianchi identities, it follows that

\[ \begin{align*}
T S^\alpha_{;\alpha} &= -\Pi \left[ V^\alpha_{;\alpha} - \alpha_0 q^\alpha_{;\alpha} + \beta_0 \Pi V^\alpha + \frac{T}{2} \left( \frac{\beta_0}{T} V^\alpha \right)_{;\alpha} \right] \\
&\quad - q^\alpha \left[ h^\mu_{;\alpha}(\ln T)_{,\mu}(1 + \alpha_0 \Pi) + V_{\alpha ;\mu} V^\mu - \alpha_0 \Pi V^\alpha - \alpha_1 \pi^\mu_{;\alpha;\mu} + \alpha_1 \pi_{\alpha ;\mu} h^\beta_{;\mu}(\ln T)_{,\beta} \\
&\quad + \beta_1 q_{\alpha ;\mu} V^\mu + \frac{T}{2} \left( \frac{\beta_1}{T} V^\mu \right)_{;\mu} q_\alpha \right] \\
&\quad - \pi_{\alpha \mu} \left[ \sigma_{\alpha \mu} - \alpha_1 q_{\mu ;\alpha} + \beta_2 \pi_{\alpha \mu ;\nu} V^\nu + \frac{T}{2} \left( \frac{\beta_2}{T} V^\nu \right)_{;\nu} \pi_{\alpha \mu} \right].
\end{align*} \]

(42)

Finally, by the standard procedure, the constitutive transport equations follow from the requirement \( S^\alpha_{;\alpha} \geq 0 \)

\[ \tau_0 \Pi_{;\alpha} V^\alpha + \Pi = -\zeta \Theta + \alpha_0 \zeta q^\alpha_{;\alpha} - \frac{1}{2} \zeta T \left( \frac{\tau_0}{\zeta T} V^\alpha \right)_{;\alpha} \Pi, \]

(43)

\[ \tau_1 h^\beta_{;\alpha} q_{\beta ;\mu} V^\mu + q_\alpha = -\kappa \left[ h^\beta T_{,\beta}(1 + \alpha_0 \Pi) + \alpha_1 \pi^\mu_{;\alpha} h^\beta_{;\mu} T_{,\beta} + T(a_\alpha - \alpha_0 \Pi_{;\alpha} - \alpha_1 \pi^\mu_{;\alpha;\mu}) \right] \\
\quad - \frac{1}{2} \kappa T^2 \left( \frac{\tau_1}{\kappa T^2} V^\beta \right)_{;\beta} q_\alpha \]

(44)
\[ \tau_2 h_\alpha^\mu h_\beta^\nu \pi_{\mu\nu}\rho V^\rho + \pi_{\alpha\beta} = -2\eta \sigma_{\alpha\beta} + 2\eta \alpha_1 q_{<\beta;\alpha>} - \eta T \left( \frac{\tau_2}{2\eta T} V^\nu \right) \pi_{\alpha\beta} \]  

(45)

with

\[ q_{<\beta;\alpha>} = h_\beta^\mu h_\alpha^\nu \left( \frac{1}{2} (q_{\mu\nu} + q_{\nu\mu}) - \frac{1}{3} q_{\sigma\kappa} h_\sigma^\kappa h_{\mu\nu} \right) \]  

(46)

and where the relaxational times are given by

\[ \tau_0 = \zeta \beta_0, \quad \tau_1 = \kappa T \beta_1, \quad \tau_2 = 2\eta \beta_2. \]  

(47)

Equations (43)–(45) reduce to equations (2.21), (2.22) and (2.23) in [26] when thermodynamics coupling coefficients vanish.

In our case each of the equations (43)–(45) has only one independent component, they read

\[ \tau_0 \dot{\Pi} = - \left( \frac{A}{B} \alpha_0 \zeta \left[ q' + q \left( \frac{A'}{A} + \frac{2(rC)'rC}{rC} \right) \right] - \right. \]  

\[ \left. - \Pi \left[ \frac{\zeta T}{2} \left( \frac{\tau_0}{\zeta T} \right) + A \right] \right), \]  

(48)

\[ \tau_1 \dot{q} = - \frac{A}{B} \kappa \left\{ T' (1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega) + T \left[ \frac{A'}{A} - \alpha_0 \Pi' - \frac{2}{3} \alpha_1 (\Omega' + \left( \frac{A'}{A} + \frac{3}{rC} \right) \Omega) \right] \right\} \]  

\[ - q \left[ \frac{\kappa T_2^2}{2} \left( \frac{\tau_1}{\kappa T_2} \right) + \frac{\tau_1}{2} A \Theta + A \right] \]  

(49)

and

\[ \tau_2 \dot{\Omega} = -2\eta A \sigma + 2\eta \alpha_1 \frac{A}{B} \left( q' - q \left( \frac{(rC)'}{rC} \right) \right) - \Omega \left[ \eta T \left( \frac{\tau_2}{2\eta T} \right) + \frac{\tau_2}{2} A \Theta + A \right]. \]  

(50)

We shall now proceed to couple transport equations in the form above, to the dynamical equation (40), in order to bring out the effects of dissipation.
on the dynamics of the collapsing sphere. For that purpose, let us first substitute (49) into (40), then we obtain, after some rearrangements,

\[
\left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right)(1 - \Lambda)D_TU = (1 - \Lambda)F_{grav} + F_{hyd}
\]

\[
+ \frac{\kappa E^2}{\tau_1} \left\{ D_R T \left(1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega\right) - T \left[ \alpha_0 D_R \Pi + \frac{2}{3} \alpha_1 \left(D_R \Omega + \frac{3}{R} \Omega\right)\right] \right\}
\]

\[
+ E \left[ \frac{\kappa T^2 q}{2\tau_1} D_T \left(\frac{\tau_1}{\kappa T^2}\right) - D_T \epsilon \right] - E \left[ \left(\frac{3q}{2} + 2\epsilon\right) \Theta - \frac{q}{\tau_1} - 2(q + \epsilon) \frac{U}{R} \right],
\]

(51)

where \(F_{grav}\) and \(F_{hyd}\) are defined by

\[
F_{grav} = - \left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right)
\times \left[ m + 4\pi \left(p + \Pi + \epsilon + \frac{2}{3}\Omega\right) \frac{1}{R^3} \right],
\]

(52)

\[
F_{hyd} = -E^2 \left[ D_R \left(p + \Pi + \epsilon + \frac{2}{3}\Omega\right) + 2(\epsilon + \Omega) \frac{1}{R} \right],
\]

(53)

and \(\Lambda\) is given by

\[
\Lambda = \frac{\kappa T}{\tau_1} \left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right)^{-1} \left(1 - \frac{2}{3} \alpha_1 \Omega\right).
\]

(54)

Next we express \(\Theta\) by means of (48) and feed this back into (51), obtaining:

\[
\left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right)(1 - \Lambda + \Delta)D_TU = (1 - \Lambda + \Delta)F_{grav} + F_{hyd}
\]

\[
+ \frac{\kappa E^2}{\tau_1} \left\{ D_R T \left(1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega\right) - T \left[ \alpha_0 D_R \Pi + \frac{2}{3} \alpha_1 \left(D_R \Omega + \frac{3}{R} \Omega\right)\right] \right\}
\]

\[
- E^2 \left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right) \Delta \left(\frac{D_R q}{q} + \frac{2q}{R}\right)
\]

\[
+ E \left[ \frac{\kappa T^2 q}{2\tau_1} D_T \left(\frac{\tau_1}{\kappa T^2}\right) - D_T \epsilon \right] + E \left[ \frac{q}{\tau_1} + 2(q + \epsilon) \frac{U}{R} \right]
\]

\[
+ E \frac{\Delta}{\alpha_0 \zeta q} \left(\mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega\right) \left\{ 1 + \frac{\zeta T}{2} D_T \left(\frac{\tau_0}{\zeta T}\right) \right\} \Pi + \tau_0 D_T \Pi,
\]

(55)
where $\Delta$ is given by

$$\Delta = \alpha_0 \zeta q \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3} \Omega \right)^{-1} \frac{3q + 4\epsilon}{2\zeta + \tau_0 \Pi}.$$  \hspace{1cm} (56)

Thus, once transport equations have been taken into account, then the inertial energy density and the “passive gravitational mass density”, appear diminished by the factor $1 - \Lambda + \Delta$. This result generalizes the one obtained in [32], by means of a complete causal treatment of all dissipative variables and the inclusion of thermodynamics viscous/heat coupling coefficients.

4 The Weyl tensor

In this section we shall find some interesting relationships linking the Weyl tensor with matter variables, from which we shall extract some conclusions about the arrow of time.

From the Weyl tensor we may construct the Weyl scalar $C^2 = C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta}$ which can be given in terms of the Kretchman scalar $\mathcal{R} = R^{\alpha \beta \gamma \delta} R_{\alpha \beta \gamma \delta}$, the Ricci tensor $R_{\alpha \beta}$ and the curvature scalar $R$ by

$$C^2 = \mathcal{R} - 2R^{\alpha \beta} R_{\alpha \beta} + \frac{1}{3} R^2.$$ \hspace{1cm} (57)

With the help of the formulae given in the Appendix of [32] and the field equations, we may write (57) as

$$\mathcal{E} = m - \frac{4\pi}{3} R^3 (\mu - \Omega),$$ \hspace{1cm} (58)

where $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{C}{48^{1/2}} R^3.$$ \hspace{1cm} (59)

From (58) with (37) and (38) we have

$$D_R \mathcal{E} = 4\pi R^2 \left[ (q + \epsilon) \frac{U}{E} + \epsilon + \Omega - D_R (\mu - \Omega) \frac{R^3}{3} \right].$$ \hspace{1cm} (60)

From (60) we obtain at once for the non-dissipative, perfect fluid case

$$D_R \mathcal{E} + \frac{4\pi}{3} R^3 D_R \mu = 0,$$ \hspace{1cm} (61)
implying that \( D_R \mu = 0 \) produces \( C = 0 \) (using the regular axis condition), and conversely the conformally flat condition implies homogeneity in the energy density.

Since tidal forces tend to make the gravitating fluid more inhomogeneous as the evolution proceeds, a relationship like (61) led Penrose to propose a gravitational arrow of time in terms of the Weyl tensor [46].

However the fact that such a relationship is no longer valid in the presence of local anisotropy of the pressure and/or dissipative processes, already discussed in [6], explains its failure in scenarios where the above-mentioned factors are present [47].

Here we see how shear viscosity and dissipative fluxes affect the link between the Weyl tensor and density inhomogeneity. From the above it is evident that density inhomogeneities may appear in a conformally flat spacetime, if dissipative processes occur. Examples of this kind have been presented in [48].

5 Conclusions

We have established the set of equations governing the structure and evolution of self–gravitating spherically symmetric dissipative viscous fluids.

Dissipative variables have been assumed to satisfy transport equations derived from causal thermodynamics, and viscous/heat coupling coefficients have been included.

As a result of this approach we obtain a dynamic equation (55) which shows the influence of dissipative variables and viscous/heat coupling coefficients on the value of the “effective” inertial mass ( “passive” gravitational mass).

In a presupernovae event, dissipative parameters (in particular \( \kappa \)) may be large enough as to produce a significant decreasing of the gravitational force term, resulting in a reversal of the collapse. A numerical model exhibiting this kind of “bouncing” has been presented in [49].

Nevertheless, the role that this effect might play in the outcome of gravitational collapse of massive stars will critically depend on specific numerical values of those quantities. Such estimations are, however, well beyond the scope of this work.

Here we just want to display the way those quantities enter into the dynamic equation and stress the fact that they should not be excluded a
priori, particularly during the most rapid phases of the collapse.

From (60) it is apparent that the production of density inhomogeneities is related to a quantity involving dissipative fluxes and shear viscosity. Thus if following Penrose we adopt the point of view that self-gravitating systems evolve in the sense of increasing of density inhomogeneity, then any alternative definition for an arrow of time should include those variables.

Finally, it is worth mentioning that we have considered the fluid to be neutral. The reason for this is that, as it can be easily verified, there is not terms coupling electromagnetic and dissipative variables, in the relevant equations. Therefore the role of electric charge in the dynamics of collapse is the same already discussed in [32].

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