Small weight code words arising from the incidence of points and hyperplanes in $\text{PG}(n, q)$

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Abstract
Let $C_{n-1}(n, q)$ be the code arising from the incidence of points and hyperplanes in the Desarguesian projective space $\text{PG}(n, q)$. Recently, Polverino and Zullo (J Comb Theory Ser A 158:1–11, 2018) proved that within this code, all non-zero code words of weight at most $2q^{n-1}$ are scalar multiples of either the incidence vector of one hyperplane, or the difference of the incidence vectors of two distinct hyperplanes. We prove that all code words of weight at most $(4q - O(\sqrt{q}))q^{n-2}$ are linear combinations of incidence vectors of hyperplanes through a common $(n - 3)$-space. This extends previous results for large values of $q$.

Keywords Finite projective geometry · Coding Theory · Small weight code words

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1 Preliminaries

Let $n \in \mathbb{N}$ and $q = p^h$, with $p$ prime and $h \in \mathbb{N} \setminus \{0\}$. Let $\text{PG}(n, q)$ be the $n$-dimensional Desarguesian projective space over the finite field of order $q$. In line with other articles, we define

$$\theta_n = \frac{q^{n+1} - 1}{q - 1},$$

with the extension that $\theta_m = 0$ if $m \in \mathbb{Z} \setminus \mathbb{N}$. Denote the set of all points of $\text{PG}(n, q)$ by $\mathcal{P}(n, q)$ and the set of all hyperplanes of $\text{PG}(n, q)$ by $\mathcal{H}(n, q)$. Let $\mathbb{F}_p^{\mathcal{P}(n, q)}$ be the $p$-ary vector space of functions from $\mathcal{P}(n, q)$ to $\mathbb{F}_p$. Denote by $\mathbf{1}$ the function that maps all points to 1.
Lemma 2.1.3 Let $v \in \mathbb{F}_p^{P(n,q)}$. Define the support of $v$ as $\text{supp}(v) = \{ P \in P(n,q) : v(P) \neq 0 \}$ and the weight of $v$ as $\text{wt}(v) = |\text{supp}(v)|$. We will call all points of $P(n,q) \setminus \text{supp}(v)$ the holes of $v$.

We can identify each hyperplane $H \in \mathcal{H}(n,q)$ with the function $H \in \mathbb{F}_p^{P(n,q)}$ such that

$$H(P) = \begin{cases} 1 & \text{if } P \in H, \\ 0 & \text{otherwise}. \end{cases}$$

If a hyperplane $H$ is identified as a function, its representation as a vector will be called the incidence vector of the hyperplane $H$. It should be clear from the context whether we mean an actual hyperplane or such a function/vector.

Definition 1.2 The $p$-ary linear code $C_{n-1}(n,q)$ is the subspace of $\mathbb{F}_p^{P(n,q)}$ generated by $\mathcal{H}(n,q)$, where we interpret the elements of the latter as functions in $\mathbb{F}_p^{P(n,q)}$. The elements of $C_{n-1}(n,q)$ are called code words.

Definition 1.3 Let $v \in \mathbb{F}_p^{P(n,q)}$ and take a $k$-space $\kappa$ in $PG(n,q)$. If we identify $PG(k, q)$ with $\kappa$, we can naturally define the restriction of $v$ to the space $\kappa$ as the function $v|_\kappa \in \mathbb{F}_p^{P(k,q)}$ restricted to the point set $P(k,q) \subseteq P(n,q)$.

Definition 1.4 Let $s$ be a line in $PG(n,q)$ and $v \in \mathbb{F}_p^{P(n,q)}$. If $s$ intersects $\text{supp}(v)$ in $\alpha$ points ($0 \leq \alpha \leq q + 1$), we will call $s$ an $\alpha$-secant to $\text{supp}(v)$. Furthermore,

- if $\alpha < 3$, $s$ will be called a short secant to $\text{supp}(v)$,
- if $\alpha \geq q - 1$, $s$ will be called a long secant to $\text{supp}(v)$.

2 Known results

2.1 Results in general dimension

The minimum weight of the code $C_{n-1}(n,q)$ equals $\theta_{n-1}$. The code words corresponding to this weight are characterised.

Theorem 2.1.1 ([1,6]) The code words of $C_{n-1}(n,q)$ having minimum non-zero weight are the scalar multiples of the incidence vectors of hyperplanes.

Bagchi and Inamdar [4, Theorem 1] gave a geometrical proof of this theorem, using blocking sets. Recently, Polverino and Zullo [8] characterised all code words up to the second smallest non-zero weight:

Theorem 2.1.2 Let $q = p^h$ with $p$ prime.

1. [7] There are no code words of $C_{n-1}(n,q)$ with weight in the interval $]0, 2q^{n-1}[$.

2. [8] The code words of weight $2q^{n-1}$ in $C_{n-1}(n,q)$ are the scalar multiples of the difference of the incidence vectors of two distinct hyperplanes of $PG(n,q)$.

So far, Theorem 2.1.2 summarises the best results known concerning the characterisation of small weight code words in $C_{n-1}(n,q)$ when $n \geq 3$.

As a final note, we keep the following lemma in mind.

Lemma 2.1.3 ([8, Remark 3.1]) Let $c \in C_{n-1}(n,q)$ be a code word and $\kappa$ a $k$-space of $PG(n,q)$, $1 \leq k \leq n$. Then $c|_\kappa$ is a code word of $C_{k-1}(k,q)$.
2.2 Results in the plane

If $q$ is prime, the codes $C_1(2, q)$ look significantly different compared to the case $q$ not prime.

**Theorem 2.2.1** ([3, Theorem 1.1]) Let $p > 5$ be prime. Then the fourth smallest non-zero weight of $C_1(2, p)$ is $3p - 3$. The only code words of $C_1(2, p)$ of Hamming weight smaller than $3p - 3$ are the linear combinations of at most two lines in $PG(2, p)$.

Bagchi knew this bound was sharp, as he discovered a code word of weight $3p - 3$ which cannot be constructed as a linear combination of at most two lines when $p > 3$ [2, Theorem 5.2]. This code word was independently discovered by De Boeck and Vandendriessche [5, Example 10.3.4] as well and generalised by Sz¨ onyi and Weiner [9, Example 4.7].

**Example 2.2.2** ([2,5,9]) Let $p \neq 2$ be prime. Take three concurrent lines $m_1$, $m_2$ and $m_3$ in $PG(2, p)$. Choose coordinates $(X_0, X_1, X_2)$ for the plane $PG(2, p)$ in any way such that these lines have equations $X_0 = 0$, $X_1 = 0$, and $X_0 = X_1$, respectively. Choose for each line $m_i$ some scalar $\mu_i \in \mathbb{F}_p$. Define $c \in \mathbb{F}_p^{P(2,p)}$ as follows:

$$c(P) = \begin{cases} 
\lambda_1 + \mu_1 & \text{if } P = (0, 1, \lambda_1), \\
\lambda_2 + \mu_2 & \text{if } P = (1, 0, \lambda_2), \\
-\lambda_3 + \mu_3 & \text{if } P = (1, 1, \lambda_3), \\
\mu_1 + \mu_2 + \mu_3 & \text{if } P = (0, 0, 1), \\
0 & \text{otherwise.} 
\end{cases}$$

Remark that $\text{supp}(c) \subseteq m_1 \cup m_2 \cup m_3$, and that $\text{wt}(c) = 3p - 3$ if $\mu_1 + \mu_2 + \mu_3 = 0$ and $\text{wt}(c) = 3p - 2$ otherwise.

As each of the three lines $m_1$, $m_2$ and $m_3$ contains $p - 1$ points with pairwise different, non-zero values, it is easy to see that such a code word can never be written as a linear combination of less than $p - 1$ different lines.

For somewhat larger values of $p$, Sz¨ onyi and Weiner [9] improved Bagchi’s result:

**Theorem 2.2.3** ([9, Theorem 4.8 and Corollary 4.10]) Let $c$ be a code word of $C_1(2, p)$, $p > 17$ prime. If $\text{wt}(c) \leq \max\{3p + 1, 4p - 22\}$, then $c$ is either a linear combination of at most three lines or given by Example 2.2.2.

The same authors have proven the following results for $q$ not prime, proving for large values of $q$ that the code word described in Example 2.2.2 can only exist when $q$ is prime.

**Theorem 2.2.4** ([9, Theorem 4.3]) Let $c$ be a code word of $C_1(2, q)$, with $27 < q$, $q = p^h$, $p$ prime. If

- $\text{wt}(c) < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor)$, when $2 < h$, or
- $\text{wt}(c) < \frac{(p-1)(p-4)(p^2+1)}{2p-1}$, when $h = 2$,

then $c$ is a linear combination of exactly $\left\lceil \frac{\text{wt}(c)}{q+1} \right\rceil$ different lines.

We can now summarise these results concerning $C_1(2, q)$ in one corollary:

**Corollary 2.2.5** Let $c$ be a code word of $C_1(2, q)$, $q = p^h$, $p$ prime, $q \notin \{8, 9, 16, 25, 27, 49\}$.

- If $\text{wt}(c) \leq 3q - 4$, then $c$ is a linear combination of at most two lines.
If \( \text{wt}(c) \leq 3q + 1 \) and \( q = 121 \), then \( c \) is a linear combination of at most three lines.

If \( \text{wt}(c) \leq \max\{3q + 1, 4q - 22\} \) and \( q > 17, q \neq 121 \), then \( c \) is a linear combination of at most three lines or given by Example 2.2.2.

**Proof** If \( q \leq 4 \), then \( 3q - 4 \leq 2q \) and we can use Theorem 2.1.2. If \( q > 4 \) and \( q \) is prime, the proof immediately follows from Theorem 2.2.1 and Theorem 2.2.3.

Suppose \( q > 4 \) is not prime. Then, by assumption, \( q > 27 \), which means that \( \max\{3q + 1, 4q - 22\} = 4q - 22 \). To apply Theorem 2.2.4, we only have to check the weight assumptions. One can verify that

- \( 4q - 22 < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor) \) if \( q \geq 10 \),
- \( 3p^2 + 1 < \frac{(p-1)(p-4)(p^2+1)}{2p-1} \) if \( q = p^2 \geq 121 \),
- \( 4p^2 - 22 < \frac{(p-1)(p-4)(p^2+1)}{2p-1} \) if \( q = p^2 \geq 144 \).

We conclude that \( c \) is a linear combination of at most \( \left\lceil \frac{4q-22}{q+1} \right\rceil = 4 \) lines. If \( c \) is a linear combination of precisely 4 lines, then its weight is at least \( 4 \cdot ((q + 1) - 3) = 4q - 8 \), a contradiction. \( \square \)

## 3 The main theorem

Throughout this section, let \( n \in \mathbb{N} \setminus \{0, 1\} \) and \( q = p^h \), with \( p \) prime and \( h \in \mathbb{N} \setminus \{0\} \). Let \( c \in C_{n-1}(n, q) \) be an arbitrary code word. Furthermore, define

\[
B_{n,q} = \begin{cases} 
2q^{n-1} & \text{if } q < 7 \text{ or } q \in \{8, 9, 16, 25, 27, 49\}, \\
(3q - \sqrt{6q} - \frac{1}{2})q^{n-2} & \text{if } q \in \{7, 11, 13, 17\}, \\
(3q - \sqrt{6q} + \frac{9}{2})q^{n-2} & \text{if } q \in \{19, 121\}, \\
(4q - 4\sqrt{q} - \frac{25}{2})q^{n-2} & \text{if } q \in \{29, 31, 32\}, \\
(4q - \sqrt{8q} - \frac{33}{2})q^{n-2} & \text{otherwise.}
\end{cases}
\]

This will be the assumed upper bound on the weight of \( c \). By Theorem 2.1.2, we can always assume that \( q \geq 7 \) and \( q \notin \{8, 9, 16, 25, 27, 49\} \).

### 3.1 Preliminaries

Using Corollary 2.2.5 and Lemma 2.1.3, we can distinguish several types of small weight code words:

**Definition 3.1.1** Let \( c \) be a code word, and \( \pi \) a plane. We will call \( c|_{|\pi} \)

- a code word of type \( T_w \) if \( c|_{|\pi} \) is a linear combination of at most two lines, with \( w \) the weight of \( c|_{|\pi} \).
- a code word of type \( T^{\text{odd}} \) if \( c|_{|\pi} \) is a code word as described in Example 2.2.2.
- a code word of type \( T^{\Delta} \) if \( c|_{|\pi} \) is a linear combination of precisely three nonconcurrent lines.
- a code word of type \( T^{\star} \) if \( c|_{|\pi} \) is a linear combination of precisely three concurrent lines.
- a code word of type \( T = \{T_0, T_{q+1}, T_{2q}, T_{2q+1}, T^{\text{odd}}, T^{\Delta}, T^{\star}\} \) if \( c|_{|\pi} \) is a code word of one of the types mentioned above.
• a code word of type $\mathcal{O}$ if $c|_{\pi}$ is not a code word of one of the types mentioned above.

We will often make no distinction between the code word $c|_{\pi}$ and the plane $\pi$; if $c|_{\pi}$ is a code word of a certain type $T$, we will call $\pi$ a plane of type $T$.

**Proposition 3.1.2** If $\pi$ is a plane of type $T$ in $\text{PG}(n, q)$, then all lines of $\pi$ are either short or long secants to $\text{supp}(c)$. If the type of $\pi$ is an element of $\{T_0, T_{q+1}, T_{2q}, T_{2q+1}\}$, then all lines intersect $\text{supp}(c)$ in at most 2 or in at least 9 points.

**Lemma 3.1.3** Take a code word $c \in C_{n-1}(n, q)$ and a $k$-space $\Pi$, with $3 \leq k \leq n$. Let $\kappa$ be a $(k-3)$-space in $\Pi$. Then $c|_{\Pi}$ can be written as a linear combination of $(k-1)$-spaces through $\kappa$ if and only if there exist a plane $\pi \subset \Pi \setminus \kappa$ and a scalar $\mu \in \mathbb{F}_p$ such that

$$c|_{\Pi}(P) = \begin{cases} \mu & \text{if } P \in \kappa, \\ c(\pi \cap \langle P, \kappa \rangle) & \text{if } P \in \Pi \setminus \kappa. \end{cases} \quad (1)$$

If this holds for some plane $\pi \subset \Pi \setminus \kappa$, then it holds for any plane in $\Pi \setminus \kappa$. Moreover, if $\pi$ is of type $T \in \mathcal{T}$, then all planes in $\Pi \setminus \kappa$ are of type $T$.

**Proof** Assume that $c|_{\Pi}$ is a linear combination of $(k-1)$-spaces through $\kappa$. Then all points of $\kappa$ lie in all these $(k-1)$-spaces, so they have the same value in $c|_{\Pi}$. A $(k-1)$-space through $\kappa$ contains a point $P \in \Pi \setminus \kappa$ if and only if it contains $\langle P, \kappa \rangle$. Therefore, all points in $\langle P, \kappa \rangle$ lie in the same $(k-1)$-spaces through $\kappa$ and thus have the same value in $c$. The fact that a plane $\pi \subset \Pi \setminus \kappa$ intersects every $(k-2)$-space through $\kappa$ in a point concludes this part of the proof.

Conversely, assume that there exist a plane $\pi \subset \Pi \setminus \kappa$ and a scalar $\mu \in \mathbb{F}_p$ such that all points $P \in \Pi$ satisfy (1). If $c|_{\pi} = \sum_i \alpha_i l_i$, with $\alpha_i \in \mathbb{F}_p$ and $l_i$ lines in $\pi$, then $c|_{\Pi}$ takes the same values as $\sum_i \alpha_i (l_i, \kappa)$ in all points of $\Pi \setminus \kappa$. Therefore, their Hamming distance is at most $\theta_{k-3} = \theta_{k-1}$, thus they must be equal. Hence, $c|_{\Pi}$ can be written as a linear combination of $(k-1)$-spaces through $\kappa$. Take another plane $\sigma \in \Pi \setminus \kappa$. Since $c|_{\Pi}$ is a linear combination of $(k-1)$-spaces through $\kappa$, the first part of the proof implies that (1) holds when replacing $\pi$ with $\sigma$. Moreover, $c|_{\sigma} = \sum_i \alpha_i (l_i, \kappa) \cap \sigma$. Note that mapping $l_i$ to $(l_i, \kappa) \cap \sigma$ induces a collineation from $\pi$ to $\sigma$. Thus, it is not hard to see that if $c|_{\pi}$ is of type $T$, $c|_{\sigma}$ must be of type $T$ as well.

**Definition 3.1.4** Let $\Pi$ be a $k$-space of $\text{PG}(n, q)$, with $3 \leq k \leq n$. We call $c|_{\Pi}$ a code word of type $T \in \mathcal{T}$ if and only if there exist a $(k-3)$-space $\kappa \subset \Pi$ and a plane $\pi \subset \Pi \setminus \kappa$ of type $T$ such that $c|_{\Pi}$ is a linear combination of $(k-1)$-spaces through $\kappa$, or, equivalently, if $\kappa$ and $\pi$ satisfy (1) for some scalar $\mu \in \mathbb{F}_p$.

**Lemma 3.1.5** Let $\Pi$ be a $k$-space of $\text{PG}(n, q)$, with $3 \leq k \leq n$, and assume that $\text{wt}(c|_{\Pi}) \leq B_{k,q}$. Then $c|_{\Pi}$ is of type $T$ if and only if $c|_{\Pi}$ is a linear combination of $(k-1)$-spaces through a common $(k-3)$-space.

**Proof** Assume that $c|_{\Pi}$ is a linear combination of $(k-1)$-spaces through the $(k-3)$-space $\kappa$ and take a plane $\pi \subset \Pi \setminus \kappa$. It suffices to prove that $\pi$ is of type $T$. Note that $\text{wt}(c|_{\Pi}) = \text{wt}(c|_{\pi})q^{k-2} + \delta \theta_{k-3}$ with $\delta = 0$ if $\mu = 0$, and $\delta = 1$ otherwise. Since $\text{wt}(c|_{\Pi}) \leq B_{k,q}$, this implies that $\text{wt}(c|_{\pi}) \leq B_{2,q}$. By Corollary 2.2.5, $\pi$ is of type $T$.

The upcoming theorem is the main theorem of this article. It is an extension of Theorem 2.1.2 when $q \geq 7$ and $q \not\in \{8, 9, 16, 25, 27, 49\}$.

**Theorem 3.1.6** Let $c$ be a code word of $C_{n-1}(n, q)$, with $n \geq 3$, $q$ a prime power and $\text{wt}(c) \leq B_{n,q}$. Then $c$ is of type $T$, or, equivalently, $c$ can be written as a linear combination of incidence vectors of hyperplanes through a common $(n-3)$-space.
3.2 The proof

3.2.1 The weight spectrum concerning lines and planes

In this subsection, we will prove some intermediate results, the first one stating that all lines contain few or many points of supp(c).

**Lemma 3.2.1** Suppose that $c \in C_{n-1}(n, q)$ with $wt(c) \leq B_{n,q}$. Then

1. all lines are either short or long secants to supp(c). If $q \leq 17$, then all lines intersect supp(c) in at most 2 or in at least $q$ points.
2. all planes of type $\mathcal{O}$ contain at least $\frac{1}{2}q(q - 1)$ points of supp(c).

**Proof** First of all, define the values

$$A_q = \begin{cases} 3q - 3 & \text{if } q \in \{7, 11, 13, 17\}, \\ 3q + 2 & \text{if } q \in \{19, 121\}, \\ 4q - 21 & \text{otherwise}, \end{cases} \quad \text{and } i = \begin{cases} 3 & \text{if } q \in \{7, 11, 13, 17\}, \\ 4 & \text{otherwise}. \end{cases}$$

Suppose, on the contrary, that $s$ is an $m$-secant to supp(c), with $i \leq m \leq q + 2 - i$. By Proposition 3.1.2 and Corollary 2.2.5, all planes through $s$ contain at least $A_q$ points of supp(c). We get

$$wt(c) \geq A_q \theta_{n-2} - m(\theta_{n-2} - 1)$$

$$\Leftrightarrow m \geq \frac{A_q \theta_{n-2} - wt(c)}{\theta_{n-2} - 1} \quad (2)$$

$$\Rightarrow q + 2 - i \geq \frac{A_q \theta_{n-2} - wt(c)}{\theta_{n-2} - 1}. \quad (3)$$

From (2) and (3), we can conclude that all lines intersect supp(c) in at most $i - 1$ or in at least $\frac{A_q \theta_{n-2} - wt(c)}{\theta_{n-2} - 1}$ points. Let $\pi \supseteq s$ be an arbitrary plane, thus $wt(c|_\pi) \geq A_q$, and define $j = \min \{wt(c|_\ell) : l \subseteq \pi, wt(c|_\ell) \geq i\}$. Choose a point $P \in s \cap supp(c)$. If all other $q$ lines in $\pi$ through $P$ contain at most $i - 1$ points of supp(c), then $wt(c|_\pi) \leq (i - 2)q + m \leq (i - 1)q - 1 < A_q$, a contradiction. Thus, through each point of $s \cap supp(c)$, we find at least one line in $\pi$, other than $s$, containing at least $j$ points of supp(c). We find at least $m \geq j$ such lines, meaning that

$$wt(c|_\pi) \geq j + (j - 1) + \cdots + 1 = \frac{1}{2}j(j + 1). \quad (4)$$

This holds for all planes through an $m$-secant with $i \leq m \leq q + 2 - i$, in particular for all planes through a $j$-secant in $\pi$. As such, we get

$$wt(c) \geq \left(\frac{1}{2}j(j + 1) - j\right)\theta_{n-2} + j.$$

When combining this result with $j \geq \frac{A_q \theta_{n-2} - wt(c)}{\theta_{n-2} - 1}$, we get a condition on $wt(c)$, eventually leading to $wt(c) > B_{n,q}$, a contradiction. We refer to Appendix A for the arithmetic details.

Let $\pi$ be a plane of type $\mathcal{O}$. If no long secant is contained in this plane, $wt(c|_\pi) \leq 2q + 1 < A_q$, a contradiction.

Repeating the previous arguments, we get the same result as (4), for $j \geq q - 1$. This concludes the proof. \qed

Using this result, we can deduce the following.
Lemma 3.2.2 Suppose $\text{wt}(c) \leq B_{n,q}$. If there exists a $(q - 1)$-secant to $\text{supp}(c)$, then there exists a 3-secant to $\text{supp}(c)$ as well.

Proof Let $s$ be a $(q - 1)$-secant and suppose, on the contrary, that no 3-secants exist. Remark that planes of type $T$ containing a $(q - 1)$-secant always contain a 3-secant. Hence, by Lemma 3.2.1, all planes through $s$ contain at least $\frac{1}{2}q(q - 1)$ points of $\text{supp}(c)$. We get

\begin{align*}
B_{n,q} \geq \text{wt}(c) \geq \left(\frac{1}{2}q(q - 1) - (q - 1)\right)\theta_{n-2} + (q - 1),
\end{align*}

which is a contradiction for all values of $q$.

Lemma 3.2.3 Suppose $\text{wt}(c) \leq \min\{3q - 6\theta_{n-2} + 2, B_{n,q}\}$. Then all lines intersect $\text{supp}(c)$ in at most 2 or in at least $q$ points.

Proof. By Lemmas 3.2.1 and 3.2.2, it suffices to prove that 3-secants to $\text{supp}(c)$ cannot exist. Suppose there exists a 3-secant to $\text{supp}(c)$. By Corollary 2.2.5, all planes containing this 3-secant have at least $3q - 3$ points in common with $\text{supp}(c)$. This gives us the following contradiction:

\begin{align*}
(3q - 6)\theta_{n-2} + 2 \geq \text{wt}(c) \geq (3q - 3 - 3)\theta_{n-2} + 3.
\end{align*}

3.2.2 Code words of weight $2q^{n-1} + \theta_{n-2}$

In this section, we will prove Theorem 3.2.6, which essentially states that, if $\text{wt}(c) \leq \min\{3q - 6\theta_{n-2} + 2, B_{n,q}\}$, the code word $c$ corresponds to a linear combination of at most two hyperplanes.

Lemma 3.2.4 Assume that $S$ is a point set in $PG(n, q)$, $q \geq 7$, and every line intersects $S$ in at most 2 or in at least $q$ points. Then one of the following holds:

1. $|S| \leq 2q^{n-1} + \theta_{n-2}$.
2. The complement of $S$, denoted by $S^c$, is contained in a hyperplane.

Proof. We prove this by induction on $n$. Note that the statement is trivial for $n = 1$, so assume that $n \geq 2$. Furthermore, we can inductively assume that for every hyperplane $\Pi$, either $|S \cap \Pi| \leq 2q^{n-2} + \theta_{n-3}$, in which case we call $\Pi$ a small hyperplane, or $S^c \cap \Pi$ is contained in an $(n - 2)$-subspace of $\Pi$, in which case we call $\Pi$ a large hyperplane.

Case 1: There exist two large hyperplanes $\Pi_1$ and $\Pi_2$, and a point $P \in S \setminus (\Pi_1 \cup \Pi_2)$.

Consider the lines through $P$. At most $q^{n-2}$ of these lines intersect $\Pi_1 \setminus \Pi_3 - i$ in a point of $S^c$, and $\theta_{n-2}$ of these lines intersect $\Pi_1 \setminus \Pi_2$. Hence, at least $\theta_{n-1} - 2q^{n-2} - \theta_{n-2} = q^{n-1} - 2q^{n-2}$ of these lines intersect both $\Pi_1$ and $\Pi_2$ in distinct points of $S$. As $P \in S$, each of these lines contains at least three points of $S$. Therefore, they must contain at least $q$ points of $S$, thus at least $q - 3$ points of $S \setminus (\Pi_1 \cup \Pi_2 \cup \{P\})$. Since $\Pi_1 \cup \Pi_2$ contains at least $2q^{n-1} - q^{n-2}$ points of $S$, we know that

\begin{align*}
|S| \geq (q^{n-1} - 2q^{n-2})(q - 3) + (2q^{n-1} - q^{n-2}) + 1 = q^n - 3q^{n-1} + 5q^{n-2} + 1.
\end{align*}

Case 2: There exist a small hyperplane $\Pi$ and a point $P \in S^c \setminus \Pi$. 
The small hyperplane $\Pi$ must contain at least $\theta_{n-1} - (2q^{n-2} + \theta_{n-3}) = q^{n-1} - q^{n-2}$ points of $S'$.

Every line through $P$ and a point of $\Pi \cap S'$ intersects $S'$ in at least 2, thus in at least $q - 1$ points of $S'$. This yields that

$$|S'| \geq (q^{n-1} - q^{n-2})(q - 2) + 1 = q^n - 3q^{n-1} + 2q^{n-2} + 1.$$ 

If both cases would occur simultaneously, then

$$\theta_n = |S| + |S'| \geq (q^n - 3q^{n-1} + 5q^{n-2} + 1) + (q^n - 3q^{n-1} + 2q^{n-2} + 1)$$

$$= 2q^n - 6q^{n-1} + 7q^{n-2} + 2,$$

which is a contradiction if $q \geq 7$. Note that the existence of three large hyperplanes implies Case 1, and the existence of two small hyperplanes implies Case 2. Therefore, exactly one of these cases occurs.

Assume that Case 1 occurs. Hence, Case 2 cannot occur, so if there exists a small hyperplane, it has to contain $S'$ completely and the proof is done. As such, we can assume that all hyperplanes are large. Take a hyperplane $\Pi$. If the points of $S' \cap \Pi$ span an $(n - 2)$-space $\Sigma$, then $S' \subseteq \Sigma \subseteq \Pi$. Otherwise, if a point $P \in S'$ lies outside of $\Sigma$, $\langle \Sigma, P \rangle$ would be a (necessarily large) hyperplane, spanned by elements of $S'$, a contradiction. In this way, we see that either some hyperplane contains all points of $S'$, or for every hyperplane $\Pi$, $S' \cap \Pi$ is contained in an $(n - 3)$-subspace of $\Pi$. We can now use the same reasoning to prove that either some hyperplane contains all points of $S'$, or for every hyperplane $\Pi$, $S' \cap \Pi$ is contained in an $(n - 4)$-space. Inductively repeating this process proves the theorem.

Assume that Case 2 occurs. Then there are at most two large hyperplanes, otherwise Case 1 would occur. If there are exactly two large hyperplanes $\Pi_1$ and $\Pi_2$, then they contain all points of $S$, else Case 1 would occur yet again. As a consequence, $|S| \leq |\Pi_1 \cup \Pi_2| = 2q^{n-1} + \theta_{n-2}$. So assume that there exists at most one large hyperplane. Consider the set $V = \{(P, \Pi) : P \text{ a point, } \Pi \text{ a hyperplane, } P \in S \cap \Pi\}$. Counting the elements of $V$ in two ways yields

$$|S|\theta_{n-1} = |V| \leq \theta_{n-1} + (\theta_n - 1)(2q^{n-2} + \theta_{n-3}) = \theta_{n-1} + q\theta_{n-1}(2q^{n-2} + \theta_{n-3}).$$

This yields

$$|S| \leq 1 + q(2q^{n-2} + \theta_{n-3}) = 2q^{n-1} + \theta_{n-2}. \hfill \Box$$

**Lemma 3.2.5** Suppose $\theta_{n-1} < wt(c) \leq \min\{(3q - 6)\theta_{n-2} + 2, B_{n,q}\}$. Then there exists a 2-secant to $\text{supp}(c)$.

**Proof** Suppose that no 2-secant to $\text{supp}(c)$ exists and suppose $t$ is a $q$-secant to $\text{supp}(c)$. By Corollary 2.2.5, all planes through $t$ containing at most $2q + 1$ points of $\text{supp}(c)$ correspond to planes of type $T_{2q}$. However, such planes contain several 2-sectants, contradicting the assumptions. Thus, by Lemmas 3.2.3 and 3.2.4, all planes through $t$ must contain at least $q^2$ points of $\text{supp}(c)$. In this way,

$$wt(c) \geq \theta_{n-2} \cdot (q^2 - q) + q = q^n,$$

which contradicts the weight assumptions. To conclude, all lines intersect $\text{supp}(c)$ in 0, 1 or $q + 1$ points, which is only possible if $\text{supp}(c)$ is a subspace. Once again, this contradicts the weight assumptions. \hfill \Box
Theorem 3.2.6 Suppose \( \text{wt}(c) \leq \min \{ (3q - 6)\theta_{n-2} + 2, B_{n,q} \} \). Then \( c \) is a linear combination of the incidence vectors of at most two distinct hyperplanes.

Proof By Theorem 2.1.2, we may assume that \( 2q^{n-1} < \text{wt}(c) \leq \min \{ (3q - 6)\theta_{n-2} + 2, B_{n,q} \} \). The proof will be done by induction on \( n \). If \( n = 2 \), Corollary 2.2.5 finishes the proof. Hence, let \( n \geq 3 \) and assume, for each hyperplane \( \Pi \), that if \( \text{wt}(c|\Pi) \leq \min \{ (3q - 6)\theta_{n-3} + 2, B_{n-1,q} \} \), \( c|\Pi \) is a linear combination of at most two distinct \((n - 2)\)-subspaces of \( \Pi \).

Suppose all hyperplanes contain at most \( 2q^{n-2} + \theta_{n-3} \) points of \( \text{supp}(c) \). Since \( \text{supp}(c) \neq \emptyset \), there must exist an \((n - 2)\)-space \( \Pi_{n-2} \) intersecting \( \text{supp}(c) \) in \( q^{n-2} \) or \( \theta_{n-2} \) points, such that all hyperplanes through \( \Pi_{n-2} \) contain either zero or \( q^{n-2} \) points of \( \text{supp}(c) \setminus \Pi_{n-2} \). This yields

\[
\text{wt}(c) \leq \theta_{n-2} + (q + 1)q^{n-2} = \theta_{n-1} + q^{n-2} < 2q^{n-1},
\]
a contradiction.

So consider a hyperplane \( \Pi_{n-1} \) containing more than \( 2q^{n-2} + \theta_{n-3} \) points of \( \text{supp}(c) \). Due to Lemma 3.2.4, \( \text{wt}(c|\Pi_{n-1}) \geq q^{n-1} \) and the holes of \( \Pi_{n-1} \) are contained in an \((n - 2)\)-space \( H_{n-2} \) of \( \Pi_{n-1} \). By Lemma 3.2.5, there exists a 2-secant \( l \) to \( \text{supp}(c) \). Let \( P \) and \( Q \) be the points in \( l \cap \text{supp}(c) \) and let \( \alpha = c(P) \).

Case 1: \( P, Q \notin \Pi_{n-1} \).

Suppose there is at most one hyperplane of type \( T_{2q} \) or \( T_{2q+1} \) through \( l \). Fix an \((n - 2)\)-space \( \Pi_{n-2} \) through \( l \). By Lemma 3.2.4, at least \( q \) hyperplanes through \( \Pi_{n-2} \) each contain at least \( q^{n-1} \) points of \( \text{supp}(c) \), thus

\[
\text{wt}(c) \geq q^{n-1} + (q - 1) \cdot (q^{n-1} - \theta_{n-2}) = q^n - q^{n-1} + 1,
\]
which exceeds the imposed upper bound on \( \text{wt}(c) \) for all prime powers \( q \), a contradiction.

Hence, we can choose a hyperplane \( \Sigma_{n-1} \) of type \( T_{2q} \) or \( T_{2q+1} \) through \( l \) and different from the hyperplane \( \langle H_{n-2}, l \rangle \). Therefore, all holes in \( \Sigma_{n-1} \cap \Pi_{n-1} \) are contained in the \((n - 3)\)-space \( \Sigma_{n-1} \cap H_{n-2} \). As \( \text{supp}(c|\Sigma_{n-1}) \) is the union or symmetric difference of precisely two \((n - 2)\)-subspaces and as \( \Sigma_{n-1} \cap \Pi_{n-1} \) must be one of these two, the latter contains either \( P \) or \( Q \), contrary to the assumption of this case.

Case 2: \( P \in \Pi_{n-1} \).

Remark that, due to Lemma 3.2.4, \( \text{wt}(c) \leq 2q^{n-1} + \theta_{n-2} \). From this, we get that there are at least \( q^{n-2} \) planes through \( l \) containing at most \( 2q + 1 \) points of \( \text{supp}(c) \). Otherwise, we would have

\[
2q^{n-1} + \theta_{n-2} \geq \text{wt}(c) > q^{n-2} \cdot (2q - 2) + (\theta_{n-2} - q^{n-2})q^2 + 2,
\]
a contradiction whenever \( q > 2 \).

The space \( \Pi_{n-1} \) contains \( \theta_{n-3} \) planes through a common line, so there exists a plane \( \pi \) through \( \Sigma \), not contained in \( \Pi_{n-1} \), having at most \( 2q + 1 \) points of \( \text{supp}(c) \). If \( Q \in \Pi_{n-1} \), we could choose another 2-secant lying in such an ‘external’ plane to \( \Pi_{n-1} \) and replace \( l \) (and \( Q \) correspondingly) with this 2-secant. In this way, we may assume that \( Q \in \pi \setminus \Pi_{n-1} \). Note that every line through \( P \) containing at least two holes of \( \Pi_{n-1} \) lies in \( H_{n-2} \). Therefore, there are at most \( \theta_{n-3} \) such lines through \( P \). Every plane through \( l \) intersects \( \Pi_{n-1} \) in a line through \( P \), hence there must be at least \( q^{n-2} - \theta_{n-3} \) planes through \( l \) of type \( T_{2q} \) or \( T_{2q+1} \), resulting in at least \( q^{n-2} - \theta_{n-3} \) lines in \( \Pi_{n-1} \), through \( P \), each containing at least \( q \) points all having the same non-zero value \( \alpha \) in \( c \). This yields at least

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\[(q^{n-2} - \theta_{n-3})(q - 1) + 1 > \frac{1}{2} \theta_{n-1}\]

points in \(\Pi_{n-1}\) with value \(\alpha\).

Now suppose, on the contrary, that \(c\) is a code word of minimal weight such that \(c\) cannot be written as a linear combination of at most two hyperplanes. Then \(\text{wt}(c - \alpha \Pi_{n-1}) < \text{wt}(c)\), thus the code word \(c - \alpha \Pi_{n-1}\) is a linear combination of exactly two hyperplanes. As a consequence, \(c\) must be a linear combination of precisely three hyperplanes, implying that \(\text{wt}(c) \geq 3(q^{n-1} - q^{n-2})\), contradicting the weight assumptions. \(\Box\)

### 3.2.3 Code words of weight approximately \(3q^{n-1}\)

It will turn out that we can go further than the code words of weight \(2q^{n-1} + \theta_{n-2}\). Moreover, we will be able to prove that a code word of weight at most \(B_{n,q}\) corresponds to a linear combination of hyperplanes through a common \((n - 3)\)-space (Theorem 3.1.6).

Due to Theorem 3.2.6, we can assume the following on the weight of the code word \(c\):

\[(3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}.\]

Remark that this inequality only holds in the case \(B_{n,q} = (4q - \sqrt{8q - \frac{33}{2}})q^{n-2}\), implying that \(q \geq 37\). On a side note, this is precisely the reason why the bound \(B_{n,q}\) differs in value for \(q \in \{29, 31, 32\}\): we needed to obtain \(q > 32\) for the remainder of this section.

Keep this lower bound on \(q\) in mind.

**Lemma 3.2.7** Suppose \((3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}\). Then there exists a 3-secant to \(\text{supp}(c)\).

**Proof** Suppose that there does not exist a 3-secant to \(\text{supp}(c)\). By Lemmas 3.2.1 and 3.2.2, all lines intersect \(\text{supp}(c)\) in at most 2 or in at least \(q\) points. Applying Lemma 3.2.4, we obtain that \(\text{wt}(c) \leq 2q^{n-1} + \theta_{n-2}\) or \(\text{wt}(c) \geq q^n\), contradicting the weight assumptions. \(\Box\)

**Lemma 3.2.8** Suppose \((3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}\). Then all planes containing a 3-secant are planes of type \(\mathcal{T}\).

**Proof** Suppose that \(\sigma\) is a plane of type \(\mathcal{O}\) containing a 3-secant \(t\) and suppose that \(\Sigma\) is a solid containing \(\sigma\). We claim that \(\text{wt}(c|\Sigma) \geq \frac{1}{4}q^3\).

In the first case, suppose that all planes in \(\Sigma\) through \(t\) are planes of type \(\mathcal{O}\). By Lemma 3.2.1,

\[
\text{wt}(c|\Sigma) \geq \left(\frac{1}{2}q^2 - \frac{1}{2}q - 3\right)(q + 1) + 3 \\
= \frac{1}{2}q^3 - \frac{7}{2}q \geq \frac{1}{4}q^3,
\]

the last inequality being valid whenever \(q > 3\).

In the second case, suppose there exists a plane \(\pi\) of type \(\mathcal{T}\) in \(\Sigma\) through \(t\). By Corollary 2.2.5, as \(\pi\) contains a 3-secant, \(\pi\) is either a plane of type \(T^{\text{odd}}\), type \(T^\Delta\) or type \(T^\star\). Regardless of this type, \(\pi\) always contains another 3-secant \(t'\) such that \(t \cap t' \notin \text{supp}(c)\).

Let \(y\) be the number of type-\(\mathcal{T}\) planes in \(\Sigma\) through \(t'\). Remark that such a plane intersects \(\sigma\) in at most three points of \(\text{supp}(c)\). Indeed, should a type-\(\mathcal{T}\) plane in \(\Sigma\) through \(t'\) intersect \(\sigma\) in at least 4, thus in at least \(q - 1\) points (Lemma 3.2.1), then one of the three points of
$t' \cap \text{supp}(c)$ must lie on this intersection line (as $\pi$ is a plane of type $T$). However, then $t' \cap \sigma \in \text{supp}(c)$, a contradiction to $t \cap t' \notin \text{supp}(c)$. In this way, we get

$$\frac{1}{2}q(q - 1) \leq \text{wt}(\sigma) \leq y \cdot 3 + (q + 1 - y)q = q^2 + q - y(q - 3),$$

which implies $y \leq \frac{1}{3}(q + 7)$, as $q \geq 37$.

Thus we get that $t'$ is contained in at least $q + 1 - \frac{1}{3}(q + 7) = \frac{1}{3}(q - 5)$ planes of type $\mathcal{O}$ (all lying in $\Sigma$). As each type-$T$ plane in $\Sigma$ through $t'$ contains at least $3q - 3$ points of $\text{supp}(c)$, we get

$$\text{wt}(c|_\Sigma) \geq \left[\frac{1}{2}(q - 5)\right] \cdot \left[\frac{1}{2}q(q - 1) - 3\right] + \left[\frac{1}{2}(q + 7)\right] \cdot (3q - 3) + 3$$

$$\geq \left[\frac{1}{2}(q - 5)\right] \cdot \left[\frac{1}{2}q(q - 1) - 3\right] + \left[\frac{1}{2}(q + 6)\right] \cdot (3q - 3) + 3$$

$$= \frac{1}{4}q^3 + \frac{23}{4}q - \frac{15}{2} \geq \frac{1}{4}q^3.$$

As the above claim holds for all solids containing $\sigma$, we get

$$\text{wt}(c) \geq \theta_{n-3}\left(\frac{1}{4}q^3\right) - (\theta_{n-3} - 1)(q^2 + q + 1).$$

One can easily check this implies $\text{wt}(c) \geq B_{n,q}$ for all prime powers $q$, a contradiction. \hfill \square

We can generalise the above lemma, which will prove its usefulness when using induction.

**Lemma 3.2.9** Suppose $(3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}$. Let $\psi$ be a $k$-space, $2 \leq k < n$, containing a 3-secant $s$. Then $\text{wt}(c|_\psi) \leq B_{k,q}$.

**Proof** By Lemma 3.2.8, we know that all planes in $\psi$ through $s$ contain at most $3q + 1$ points of $\text{supp}(c)$ (Corollary 2.2.5). This implies that $\text{wt}(c|_\psi) \leq \theta_{k-2}(3q + 1 - 3) + 3 \leq B_{k,q}$, for all $q \geq 37$. \hfill \square

We can now present some properties about certain types of subspaces sharing a common 3-secant.

**Lemma 3.2.10** Suppose $(3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}$. Let $\Pi_1$ and $\Pi_2$ be two $k$-spaces, $2 \leq k < n$, of type $T_1, T_2 \in \{T^{odd}, T^{A}, T^{\star}\}$, respectively, having a 3-secant $s$ in common. Then at least one of the following holds:

1. $T_1 = T^{\star}$.
2. $T_2 = T^{\star}$.
3. $T_1 = T_2$.

Furthermore, if $T_1 = T_2$, then $\text{wt}(c|_{\Pi_1}) = \text{wt}(c|_{\Pi_2})$.

**Proof** In each subspace $\Pi_i$, choose a plane $\pi_i$ through $s$, disjoint to the vertex corresponding with the cone $\text{supp}(c|_{\Pi_i})$. By definition, $\pi_i$ is a plane of type $T_i$. Define $\Sigma = \langle \pi_1, \pi_2 \rangle$.

Furthermore, let $P_\alpha, P_\beta$ and $P_\gamma$ be the points in $s \cap \text{supp}(c)$ with corresponding non-zero values $\alpha, \beta$ and $\gamma$ in $c$. Let $l^{(i)}_\alpha, l^{(i)}_\beta$ and $l^{(i)}_\gamma$ be the unique long secants in $\pi_i$ through $P_\alpha, P_\beta$ and $P_\gamma$, respectively ($i = 1, 2$).

Case 1: $T_1 \neq T_2$. 

\begin{align*}
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Assume, to the contrary, that \( T_1 \neq T^\star \neq T_2 \). W.l.o.g. we can assume that \( T_1 = T^{\text{odd}} \) and \( T_2 = T^\Delta \).

Suppose that \( \pi \) is a plane in \( \Sigma \) going through \( l_\alpha^{(2)} \). Remark that \( l_\alpha^{(2)} \) is a long secant, containing \( q - 1 \) points having non-zero value \( \alpha \), one point having value \( \alpha + \beta \) and one point having value \( \alpha + \gamma \). From this, we know that the plane \( \pi \) cannot be

- a plane of type \( T_0 \), as \( \alpha \neq 0 \).
- a plane of type \( T_{\alpha+1}, T_{2\alpha}, T_{2\alpha+1} \) or \( T^\star \), else \( \alpha + \beta = \alpha \) or \( \alpha + \gamma = \alpha \).
- a plane of type \( T^{\text{odd}} \), as \( l_\alpha^{(2)} \) contains at least three points with the same value \( \alpha \).
- a plane of type \( T^\Delta \), unless \( \text{wt}(c_{|\pi}) = \text{wt}(c_{|\pi_2}) \). Indeed, \( l_\alpha^{(2)} \) contains two points \( l_\alpha^{(2)} \cap l_\beta^{(2)} \) and \( l_\alpha^{(2)} \cap l_\gamma^{(2)} \) with corresponding values \( \alpha + \beta \) and \( \alpha + \gamma \), respectively, unambiguously fixing the weight of \( \text{wt}(c_{|\pi}) \).

However, \( \pi \) can only be a plane of type \( T^\Delta \) in some cases. Suppose that \( \pi \) is a plane of type \( T^\Delta \) and suppose that \( \pi \) intersects \( \pi_1 \) in a 3-secant \( \triangle \). One of the points of \( \triangle \) is obviously \( P_\alpha \), as this point belongs to both \( l_\alpha^{(2)} \) and \( \pi_1 \). The other two points of \( \triangle \) lie on \( l_\beta^{(1)} \) and \( l_\gamma^{(1)} \) and must have corresponding values \( \beta \) and \( \gamma \), as \( \text{wt}(c_{|\pi}) = \text{wt}(c_{|\pi_2}) \). As \( \pi_1 \) is a plane of type \( T^{\text{odd}} \), there are only two possibilities for \( \pi \) to intersect \( \pi_1 \), namely when the \( \beta \)-valued point of \( \triangle \) lies on \( l_\beta^{(1)} \) (then \( \pi = \pi_2 \)), or when the \( \beta \)-valued point of \( \triangle \) lies on \( l_\gamma^{(1)} \).

In conclusion, of the at least \( q - 2 \) planes through \( l_\alpha^{(2)} \) in \( \Sigma \), intersecting \( \pi_1 \) in a 3-secant, at least \( q - 4 \) of them cannot be a plane of type \( T^\Delta \), and thus must be planes of type \( \mathcal{O} \). In addition, the plane \( (l_\alpha^{(1)}, l_\alpha^{(2)}) \) can never be a plane of type \( T^\Delta \) as well, as \( l_\alpha^{(1)} \) contains many differentially valued points. Thus, we find at least \( q - 3 \) planes of type \( \mathcal{O} \) in \( \Sigma \) through \( l_\alpha^{(2)} \), each containing at least \( \frac{1}{2}q(q - 1) \) points of \( \text{supp}(c) \) (Lemma 3.2.1). The other planes in \( \Sigma \) through \( l_\alpha^{(2)} \), of which there are at most four, contain at least \( 3q - 3 \) points of \( \text{supp}(c) \). We get

\[
\text{wt}(c_{|\pi}) \geq \left( \frac{1}{2}q(q - 1) \right)(q - 3) + 4 \cdot (3q - 3) - q \cdot (q + 1)
\]

\[
= \frac{1}{2}q^3 - 3q^2 + \frac{25}{2}q - 12 > B_{3,q},
\]

which is, if \( n = 3 \), a direct contradiction or, if \( n > 3 \), a contradiction with Lemma 3.2.9, as \( \Sigma \) contains the 3-secant \( \triangle \).

Case 2 : \( T_1 = T_2 \).

Suppose, on the contrary, that \( \text{wt}(c_{|\pi_1}) \neq \text{wt}(c_{|\pi_2}) \). W.l.o.g. we can assume that \( \text{wt}(c_{|\pi_1}) \neq \text{wt}(c_{|\pi_2}) \) as well. Assume, in the first case, that \( T_1 = T_2 \in \{ T^{\text{odd}}, T^\star \} \). By observing the types of these planes and by Example 2.2.2, \( \text{wt}(c_{|\pi_1}) \neq \text{wt}(c_{|\pi_2}) \) implies that both \( \alpha + \beta + \gamma = 0 \) and \( \alpha + \beta + \gamma \neq 0 \), a contradiction.

Now assume \( T_i = T^\Delta \). Considering the plane \( \pi_1 \), we know that the lines \( l_\alpha^{(i)}, l_\beta^{(i)} \) and \( l_\gamma^{(i)} \) are not concurrent; in this way, we can consider the pairs of values of the points \( l_\alpha^{(1)} \cap l_\beta^{(1)} \) and \( l_\gamma^{(1)} \) and \( l_\alpha^{(1)} \cap l_\gamma^{(1)} \) and \( l_\beta^{(1)} \cap l_\gamma^{(1)} \), and \( l_\alpha^{(1)} \cap l_\beta^{(1)} \) and \( l_\gamma^{(1)} \). As \( \text{wt}(c_{|\pi_1}) \neq \text{wt}(c_{|\pi_2}) \), at least one of these pairs of values consists of a zero value and a non-zero value, implying conflicting conditions on the corresponding values \( \{ \alpha, \beta, \gamma \} \).

\[\square\]

**Lemma 3.2.11** Let \( \pi \) be a plane of type \( T \in \{ T^{\text{odd}}, T^\Delta \} \). Then all planes \( \sigma \) of type \( T \) intersecting \( \pi \) in a long secant are planes of type \( T \) as well. Moreover, \( \text{wt}(c_{|\sigma}) = \text{wt}(c_{|\pi}) \).

\( \square \) Springer
Proof Suppose the plane $\sigma$ is a plane of type $T^\sigma \in \mathcal{T}$; let $l$ be the long secant $\pi \cap \sigma$. As $T \in \{T^{odd}, T^\Delta\}$, no $q$ points on $l$ have the same non-zero value in $c$. As a consequence, $T^\sigma \notin \{T_0, T_{q-1}, T_{2q}, T_{2q+1}, T^\star\}$. If $T = T^{odd}$, we find at least $q$ points on $l$ having pairwise different values in $c$. If $T = T^\Delta$, we find at most 3 points on $l$ having pairwise different values in $c$. Hence, if $T^\sigma \neq T$, then $q \leq 3$, a contradiction. Furthermore, it is not hard to check that the set of values of points on $l$ fixes the weight of $c|_\sigma$.

Lemma 3.2.12 Suppose that $n = 3$ and $3q^2 - 3q - 3 \leq \text{wt}(c) \leq B_3.q$. Then a 3-secant is never contained in $q + 1$ planes of the same type $T \in \{T^{odd}, T^\Delta\}$.

Proof Suppose, on the contrary, that $t$ is such a 3-secant. Fix a plane $\pi$ through $t$. By Lemma 3.2.10, the weight of the code word $c$ is known, as we can count: $\text{wt}(c) = (q + 1)(\text{wt}(c|_\pi) - 3) + 3 = (q + 1)\text{wt}(c|_\pi) - 3q$.

Remark that, as $\pi$ is a plane of either type $T^{odd}$ or $T^\Delta$, we can always find a 1- or 2-secant $r$ in $\pi$ such that $t$ and $r$ intersect in a point $Q$ of supp$(c)$. Indeed,

- if $\pi$ is a plane of type $T^{odd}$, we can simply connect two points: a hole lying on a long secant in $\pi$, different from the intersection point of the three long secants in $\pi$, and a point of $t \cap$ supp$(c)$ on another long secant in $\pi$.
- if $\pi$ is plane of type $T^\Delta$, we can connect a point lying on two long secants with the unique point of $t$ lying on the third long secant.

Let $\sigma$ be a plane through $r$, not equal to $\pi$. Choose a long secant $s$ in $\sigma$ through $Q$. This is possible since every plane of type $\mathcal{T} \setminus \{T_0\}$ obviously contains a long secant, and planes of type $\mathcal{O}$ contain long secants as well (cfr. Lemma 3.2.1). The plane $\langle t, s \rangle$ contains the 3-secant $t$, thus this plane has to be of the same type as $\pi$. In particular, this means that $\langle t, s \rangle$ is a plane of type $T^{odd}$ or $T^\Delta$. However, by Lemma 3.2.11, the plane $\langle t, s \rangle$ then has to be of the same type as $\sigma$ as well, as they share the long secant $s$, unless $\sigma$ is a plane of type $\mathcal{O}$.

Therefore, all planes $\sigma$ through $r$ satisfy either $\text{wt}(c|_\sigma) = \text{wt}(c|_\pi)$ (if $\sigma$ is a plane of type $\mathcal{T}$), or $\text{wt}(c|_\sigma) \geq \frac{1}{2}q(q - 1) > \text{wt}(c|_\pi)$ (if $\sigma$ is a plane of type $\mathcal{O}$, by Lemma 3.2.1). In both cases, this yields the following lower bound on $\text{wt}(c)$:

$$ (q + 1)\text{wt}(c|_\pi) - 3q = \text{wt}(c) \geq (q + 1)(\text{wt}(c|_\pi) - 2) + 2, $$

which results in a contradiction. To clarify, on the left-hand side we count the weight of $\text{wt}(c)$ by considering all $q + 1$ planes through the 3-secant $t$, on the right-hand side we estimate $\text{wt}(c)$ by considering all $q + 1$ planes through the 1- or 2-secant $r$.

The following proposition is a consequence of the way code words of type $\mathcal{T}$ are defined (Lemma 3.1.3, Definition 3.1.4).

Proposition 3.2.13 Suppose that $\Pi$ is a hyperplane of type $T \in \mathcal{T}$, with $\kappa$ the $(n - 4)$-dimensional vertex of supp$(c|_\Pi)$. Suppose that $t$ is a 3-secant contained in $\Pi$. Then $t$ is disjoint to $\kappa$ and all $q^{n-3}$ planes in $\Pi$ that contain $t$ but that are disjoint to $\kappa$ are planes of type $T$. The other $\theta_{n-4}$ planes in $\Pi$ through $t$ intersect $\kappa$ in a point and are all planes of type $T^\star$.

Lemma 3.2.14 Suppose that $(3q - 6)\theta_{n-2} + 3 \leq \text{wt}(c) \leq B_{n,q}$. Then a 3-secant is never contained in $\theta_{n-2}$ hyperplanes of the same type $T \in \{T^{odd}, T^\Delta\}$.

Proof By Lemma 3.2.12, we can assume that $n > 3$. Suppose that $t$ is a 3-secant with the property that all hyperplanes through $t$ are of the same type $T \in \{T^{odd}, T^\Delta\}$. Now define

$$ S = \{(\pi, \Pi) : t \subseteq \pi \subseteq \Pi, \pi \text{ a plane, } \Pi \text{ a hyperplane, both of type } T\}. $$
Fix an arbitrary plane $\pi_0 \supseteq t$ of type $T$. As all hyperplanes through $t$ are of the same type $T$, all hyperplanes through $\pi_0$ have this property as well. Thus, the number of elements in $S$ with a fixed first argument $\pi_0$ equals $\theta_{n-3}$.

Fix an arbitrary hyperplane $\Pi_0 \supseteq t$ of type $T$. By Proposition 3.2.13, the number of elements in $S$ with a fixed second argument $\Pi_0$ equals $q^{n-3}$ (the number of planes in $\Pi_0$ through $t$, disjoint to an $(n-4)$-subspace not intersecting $t$).

Let $x_\pi$ be the number of type-$T$ planes through $t$. By double counting, we get:

$$x_\pi \cdot \theta_{n-3} = |S| = \theta_{n-2} \cdot q^{n-3} \iff x_\pi = \frac{q^{n-1} - 1}{q^{n-2} - 1}q^{n-3} = q^{n-2} + 1 - \frac{q^{n-3} - 1}{q^{n-2} - 1}$$

As $x_\pi$ is known to be an integer, this is only valid when the fraction on the right-hand side is an integer. As $n > 3$, this is never the case. □

**Lemma 3.2.15** Suppose that $(3q - 6)\theta_{n-2} + 3 \leq wt(c) \leq B_{n,q}$. Let $\Pi_1$ and $\Pi_2$ be two hyperplanes of type $T^*$ and let $C_i$ be the union of the three $(n-2)$-subspaces present in the linear combination $c|_{\Pi_i}$, thus intersecting in a common $(n-3)$-space $\kappa_i$ ($i = 1, 2$). Suppose that $C_1$ and $C_2$ have an $(n-2)$-subspace in common. Then $\kappa_1 = \kappa_2$.

**Proof** Let $\Sigma$ be the $(n-2)$-space that $C_1$ and $C_2$ have in common. As $q > 3$, $\Sigma$ must be one of the three subspaces present in the linear combination of both $c|_{\Pi_1}$ and $c|_{\Pi_2}$.

Suppose that $\kappa_1 \neq \kappa_2$. As these are spaces of the same dimension, we can find a point $P_1 \in \kappa_1 \setminus \kappa_2$ and a point $P_2 \in \kappa_2 \setminus \kappa_1$; define $l = \langle P_1, P_2 \rangle$. Remark that $l$ must be a $(q+1)$-secant to $\text{supp}(c)$. This follows from the fact that every point of $l$ lies in $\Sigma \setminus \kappa_i$, for at least one choice of $i$. Looking in $C_i$, we see that all points of $\Sigma \setminus \kappa_i$ lie in $\text{supp}(c)$. Now take planes $\pi_i$ in $\Pi_i$, for $i = 1, 2$, through $l$, not contained in $\Sigma$. Due to this choice, it is clear that the plane $\pi_i$ will intersect each $(n-2)$-subspace of $C_i$ in a line (through $P_i$). Define $S = \langle \pi_1, \pi_2 \rangle$.

Choose a $(q+1)$-secant $s$ in $\pi_1$, different from $l$. As $P_1 \neq P_2$, all planes in $S$ through $s$ (not equal to $\pi_1$) intersect $\pi_2$ in a 3-secant and thus, by Lemma 3.2.8, are planes of type $T$. As $\pi_1$ is a plane of type $T$ as well, we know that all planes in $S$ through $s$ are planes of type $T$. This means that $wt(c|_S) \leq q \cdot (2q) + (3q + 1) = 2q^2 + 3q + 1 \leq \min \left\{ (3q - 6)(q + 1) + 2, B_{3,q} \right\}$. By Theorem 3.2.6, $c|_S$ is a linear combination of the incidence vectors of at most two planes. As $s$ necessarily lies in one of such planes, we see that not all planes through $s$ in $S$ can be of type $T$, resulting in a contradiction. □

The following theorem connects all previous results and proves Theorem 3.1.6.

**Theorem 3.2.16** Each code word $c \in C_{n-1}(n, q)$ with $wt(c) \leq B_{n,q}$ is a code word of type $T$.

**Proof** If $wt(c) \leq \min \left\{ (3q - 6)\theta_{n-2} + 2, B_{n,q} \right\}$, Theorem 3.2.6 proves the statement, so let us assume that $(3q - 6)\theta_{n-2} + 3 \leq wt(c) \leq B_{n,q}$. We will prove that there exist a $(k-3)$-space $\kappa$ and a plane $\pi$ satisfying (1) of Lemma 3.1.3; this will be done by induction on $n$.

If $n = 2$, we can choose $\kappa = \emptyset$ and refer to Corollary 2.2.5. Now assume $n \geq 3$ and suppose the statement is true for $c$ restricted to any $k$-space, $2 \leq k < n$, given that the weight of this restricted code word is at most $B_{k,q}$. By Lemma 3.2.7, we can choose a 3-secant $t$ with corresponding non-zero values $\alpha, \beta$ and $\gamma$. By the induction hypothesis and Lemma 3.2.9, each hyperplane through $t$ is a hyperplane of type $T$ and by Lemma 3.2.10, we know that there exist two specific types $T_A = T^*$ and $T_B \in \{T^{\text{odd}}, T^\Delta, T^*\}$ such that all hyperplanes through $t$ are either of type $T_A$ or type $T_B$. Furthermore, by Lemma 3.2.14, we know that

\[ \text{ Springer} \]
there exists at least one hyperplane through \( t \) of type \( T_A \); consider such a hyperplane \( \Pi \). Remark that, by Proposition 3.2.13, all planes through \( t \) are planes of type \( T_A \) or \( T_B \) as well. We can now fix a certain plane \( \pi \) as follows: if all planes through \( t \) are planes of type \( T_A \), choose \( \pi \) to be an arbitrary plane through \( t \), not contained in \( \Pi \). Else, choose \( \pi \) to be a plane through \( t \) of type \( T_B \). By Proposition 3.2.13, \( \pi \) cannot be contained in \( \Pi \).

By the choice of \( \Pi \), we know that \( c|\Pi \) is a linear combination of three distinct \((n-2)\)-subspaces of \( \Pi \) through a common \((n-3)\)-space. Choose \( \kappa \) to be this \((n-3)\)-space. In this way, it’s clear that \( c|\kappa = (\alpha + \beta + \gamma) \cdot 1 \). Moreover, as all lines in \( \Pi \) not disjoint to \( \kappa \) are either 0-, 1-, or \((q+1)\)-secants, we know that \( \kappa \) must be disjoint to the 3-secant \( t \) and, as a consequence, disjoint to the plane \( \pi \supseteq t \), as that plane is not contained in \( \Pi \).

The only statement left to prove is that, for every point \( P \notin \kappa \),

\[
c(P) = c(\langle \kappa, P \rangle \cap \pi).
\]  

As \( c|\Pi \) is a linear combination of three different \((n-2)\)-spaces of \( \Pi \) having the space \( \kappa \) in common, we can choose one of those \((n-2)\)-spaces \( \Psi_1 \); w.l.o.g., this space corresponds to the value \( \alpha \). Choose an arbitrary 3-secant \( t_1 \) in \( \pi \) through the point \( \Psi_1 \cap \pi \), thus having corresponding non-zero values \( \alpha, \beta_1 \) and \( \gamma_1 \). By Lemma 3.2.9, the induction hypothesis implies that \( \Pi_1 = \langle \Psi_1, t_1 \rangle \) is a hyperplane of type \( \mathcal{T} \). We claim that \( \Pi_1 \) is a hyperplane of type \( T_A \). Indeed, let \( \pi_1 \) be a plane in \( \Pi_1 \) through \( t_1 \), thus intersecting \( \Pi \) in a line \( \sigma \) of \( \Psi_1 \). Then this intersection line \( \sigma \) must be a \((q+1)\)-secant. By Lemma 3.2.8, \( \pi_1 \) has to be a plane of type \( \mathcal{T} \), and, more specifically, a plane of type \( T_A \). Indeed, if this would not be the case, then the plane \( \pi_1 \) would be of type \( T_A^{odd} \) or \( T_A \), as it contains the 3-secant \( t_1 \). By Lemma 3.2.11, this would imply that all planes in \( \Pi \) through \( \sigma \) of type \( \mathcal{T} \) are of a type that is not \( T^* \), which is impossible. By the arbitrary choice of \( \pi_1 \), all planes in \( \Pi_1 \) through \( t_1 \) must be planes of type \( T_A \), thus \( \Pi_1 \) contains at least \( \theta_{n-3} \) planes of type \( T_A \) through a fixed 3-secant \( (t_1) \). By Proposition 3.2.13, at least one of these planes is of the same type as \( \Pi_1 \), thus this hyperplane must be of type \( T_A \).

Let \( \kappa_1 \) be the \((n-3)\)-subspace of \( \Pi_1 \) in which the three hyperplanes of \( c|\Pi_1 \) intersect. By Lemma 3.2.15, we know that \( \kappa = \kappa_1 \). In this way, it is easy to see that all points in \( \Pi_1 \setminus \kappa \) fulfil property (5).

We can now repeat the above process by choosing another \((n-2)\)-space \( \Psi_2 \) in one of the linear combinations of \( c|\Pi \) or \( c|\Pi_1 \) and considering the span \( \Pi_2 = \langle \Psi_2, t_2 \rangle \), with \( t_2 \) an arbitrary 3-secant in \( \pi \) through the point \( \Psi_2 \cap \pi \). All points in \( \Pi_2 \setminus \kappa \) will fulfil property (5) as well.

To conclude, if, for each point \( P \) in \( \pi \), there exists a sequence of 3-secants \( t_1, t_2, \ldots, t_m \supseteq P \) in \( \pi \) such that \( t \cap t_1 \in \text{supp}(c) \) and \( t_i \cap t_{i+1} \in \text{supp}(c) \) for all \( i \in \{1, 2, \ldots, m-1\} \), then this theorem is proven by consecutively repeating the above arguments. Unfortunately, not all points in \( \pi \) satisfy this property. However, if a point \( P \in \pi \) does not lie on such a (sequence of) 3-secant(s), we can easily prove that this point lies on a 0-, 1- or 2-secant \( r \) in \( \pi \) of which the other \( q \) points are reached by such a (sequence of) 3-secant(s). Thus, we already know the value of many points in the hyperplane \( \langle \kappa, r \rangle \). As \( \text{wt}(c|\kappa, r) \leq 2q^{n-2} + \theta_{n-3} + \text{wt}(c|\kappa, \Pi) - \text{wt}(c|\kappa) \leq 3q^{n-2} + \theta_{n-3} \leq B_{n-1,q} (q \geq 37) \), this hyperplane is a hyperplane of type \( \mathcal{T} \) by the induction hypothesis. It is easy to see that all points in \( \langle \kappa, r \rangle \setminus \kappa \) must satisfy property (5).

\[ \square \]

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A Further details to Lemma 3.2.1

Suppose \( c \in C_{n-1}(n, q) \), with \( q \geq 7, q \notin \{8, 9, 16, 25, 27, 49\} \), and assume that \( wt(c) \leq E_{n,q} \), with

\[
E_{n,q} = \begin{cases} 
3q - \sqrt{3q} - \frac{1}{2}q^{n-2} & \text{if } q \in \{7, 11, 13, 17\}, \\
3q - \sqrt{3q} + \frac{9}{2}q^{n-2} & \text{if } q \in \{19, 121\}, \\
4q - \sqrt{8q} - \frac{33}{2}q^{n-2} & \text{otherwise}; 
\end{cases}
\]

\[
A_q = \begin{cases} 
3q - 3 & \text{if } q \in \{7, 11, 13, 17\}, \\
3q + 2 & \text{if } q \in \{19, 121\}, \\
4q - 21 & \text{otherwise.}
\end{cases}
\]

Remark that \( B_{n,q} < E_{n,q} \) if \( q \in \{29, 31, 32\} \) and \( B_{n,q} = E_{n,q} \) for all other considered values of \( q \), so it suffices to check the details of the lemma for this bound \( E_{n,q} \). We will derive a contradiction using the following two inequalities:

\[
wt(c) \geq \left( \frac{1}{2} j(j+1) - j \right) \theta_{n-2} + j \quad \text{and} \quad j \geq \frac{A_q \theta_{n-2} - wt(c)}{\theta_{n-2} - 1}. \tag{6}
\]

Define \( W := wt(c) \). Below, we will sketch the details when \( q > 19 \), and \( q \neq 121 \). The other two cases are completely analogous.

Combining the two equations in (6), together with \( A_q = 4q - 21 \), gives rise to the following inequality:

\[
0 \geq (q^{n+1} - 2q^n + q^{n-1} - q^2 + 2q - 1)W^2 \\
- (8q^{2n} - 49q^{2n-2} + 14q^{2n-2} - 17q^{n+1} + 100q^n - 83q^{n-1} + 9q^2 - 51q + 42)W \\
+ 16q^{3n-1} - 172q^{3n-2} + 462q^{3n-3} - 36q^2n + 441q^{2n-1} - 1323q^{2n-2} \\
- 8q^{n+2} + 82q^{n+1} - 458q^n + 1302q^{n-1} + 8q^3 - 62q^2 + 189q - 441.
\]

The above inequality is of the form \( 0 \geq aW^2 + bW + c \), with \( a \geq 0 \), implying that \( W \geq \frac{-b - \sqrt{D}}{2a} \) with \( D = b^2 - 4ac \). One can check that

\[
D = 32q^{4n-1} - 231q^{4n-2} + 366q^{4n-3} - 167q^{4n-4} \\
- 64q^{3n+1} + 398q^{3n} - 270q^{3n-1} - 398q^{3n-2} + 334q^{3n-3} \\
+ 32q^{2n+3} - 103q^{2n+2} - 526q^{2n+1} + 1066q^{2n} - 302q^{2n-1} - 167q^{2n-2} \\
- 64q^{n+4} + 398q^{n+3} - 270q^{n+2} - 398q^{n+1} + 334q^n \\
+ 32q^5 - 231q^4 + 366q^3 - 167q^2.
\]

Since \( q \geq 23 \), we can find the following upper bound on the right-hand side:

\[
D \leq 32q^{4n-1} - 231q^{4n-2} + 398q^{4n-3} - 46q^{3n+1}. \tag{7}
\]

On the other hand, we have that \( D \geq \left( - b - 2a(4q - \sqrt{8q} - \frac{33}{2}) \right)^2 \), which implies

\[
D \geq 32q^{4n-1} - 128q^{4n-2} - 264\sqrt{2q} \cdot q^{4n-3} + 192q^{4n-3} + 792\sqrt{2q} \cdot q^{4n-4} + 961q^{4n-4} \\
- 792\sqrt{2q} \cdot q^{4n-5} - 216q^{4n-5} + 264\sqrt{2q} \cdot q^{4n-6} + 1089q^{4n-6} \\
- 72\sqrt{2q} \cdot q^3 - 64q^3 + 552\sqrt{2q} \cdot q^{3n-1} + 850q^{3n-1} - 696\sqrt{2q} \cdot q^{3n-2}
\]
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\[ - 4344q^{3n-2} - 504\sqrt{2q} \cdot q^{3n-3} + 4216q^{3n-3} + 1248\sqrt{2q} \cdot q^{3n-4} + 1520q^{3n-4} \\
- 528\sqrt{2q} \cdot q^{3n-5} - 2178q^{3n-5} + 81q^{2n+2} + 144\sqrt{2q} \cdot q^{2n+1} - 886q^{2n+1} \\
- 1104\sqrt{2q} \cdot q^{2n} + 2041q^{2n} + 2184\sqrt{2q} \cdot q^{2n-1} + 3828q^{2n-1} - 1368\sqrt{2q} \cdot q^{2n-2} \\
- 951q^{2n-2} - 120\sqrt{2q} \cdot q^{2n-3} + 3398q^{2n-3} + 264\sqrt{2q} \cdot q^{2n-4} + 1089q^{2n-4} \\
- 162q^{n+3} - 72\sqrt{2q} \cdot q^{n+2} + 1836q^{n+2} + 552\sqrt{2q} \cdot q^{n+1} - 6120q^{n+1} \\
- 1224\sqrt{2q} \cdot q^n + 4608q^n + 1080\sqrt{2q} \cdot q^{n-1} + 2610q^{n-1} - 336\sqrt{2q} \cdot q^{n-2} \\
- 2772q^{n-2} + 81q^4 - 918q^3 + 3357q^2 - 4284q + 1764. \]

Since \( q \geq 23 \), we can find the following lower bound on the right-hand side:

\[ D \geq 32q^{4n-1} - 206q^{4n-2} - 72\sqrt{2q} \cdot q^{3n} - 64q^{3n}. \]  

(8)

Combining (7) and (8) yields

\[
32q^{4n-1} - 231q^{4n-2} + 398q^{4n-3} - 46q^{3n+1} \geq D \geq 32q^{4n-1} - 206q^{4n-2} \\
- 72\sqrt{2q} \cdot q^{3n} - 64q^{3n},
\]

resulting in

\[
0 \geq 25q^{4n-2} - 398q^{4n-3} + 46q^{3n+1} - 72\sqrt{2q} \cdot q^{3n} - 64q^{3n} \\
\implies 0 \geq 25q^{4n-2} - 398q^{4n-3} \\
\implies 398 \geq 25 \implies q,
\]

a contradiction.

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