ON THE THEORY OF GENERALIZED STIELTJES TRANSFORMS

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ABSTRACT. We identify measures arising in the representations of products of generalized Stieltjes transforms as generalized Stieltjes transforms, provide optimal estimates for the size of those measures, and address a similar issue for generalized Cauchy transforms. In the latter case, in two particular settings, we give criteria ensuring that the measures are positive. On this way, we also obtain new, applicable conditions for representability of functions as generalized Stieltjes transforms, thus providing a partial answer to a problem posed by Sokal and shedding a light at spectral multipliers emerged recently in probabilistic studies. As a byproduct of our approach, we improve several known results on Stieltjes and Hilbert transforms.

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Date: July 20, 2023.
2020 Mathematics Subject Classification. Primary 44A15, 26A33, Secondary 26A51, 60E10.
Key words and phrases. generalized Stieltjes transforms, generalized Cauchy transforms, representations, products, measures, norm estimates, Bernstein functions.
The authors were partially supported by the NCN grant 2017/27/B/ST1/00078.
1. Introduction

This paper is concerned with the theory of generalized Stieltjes functions and corresponding Stieltjes transforms, and it offers several insights on their structure and algebraic properties. To set the scene, for $\alpha \geq 0$ let $\mathcal{M}_\alpha^+(\mathbb{R}_+)$ be the set of positive Borel measures on $\mathbb{R}_+ = [0, \infty)$ such that $(1 + s)^{-\alpha} \in L^1(\mathbb{R}_+, \mu)$. The generalized Stieltjes transform of order $\alpha > 0$ is defined on $\mathcal{M}_\alpha^+(\mathbb{R}_+)$ by

$$S_\alpha[\mu](z) = \int_0^\infty \frac{\mu(dt)}{(z + t)^\alpha}, \quad \mu \in \mathcal{M}_\alpha^+(\mathbb{R}_+), \quad z > 0. \tag{1.1}$$

The set of $S_\alpha[\mu]$, where $\mu$ varies through $\mathcal{M}_\alpha^+(\mathbb{R}_+)$, coincides modulo constants with the so-called class of generalized Stieltjes functions, arising in various areas of analysis. Recall that a function $f : (0, \infty) \mapsto [0, \infty)$ is said to be generalized Stieltjes of order $\alpha > 0$ if

$$f(z) = a + S_\alpha[\mu](z), \quad z > 0, \tag{1.2}$$

where $a \geq 0$ and $\mu \in \mathcal{M}_\alpha^+(\mathbb{R}_+)$. In this case we will write $f \sim (a, \mu)_\alpha$, call $(a, \mu)_\alpha$ the Stieltjes representation of $f$, and denote the class of such $f$ by $S_\alpha$.

The studies on generalized Stieltjes functions go back to 1930’s and to well-known papers by Hirschman and Widder. Apart from classical analysis, these functions arise in many areas of mathematics including probability theory \cite{10, 18, 62, 71}, mathematical physics \cite{20, 21, 24, 56, 72}, complex analysis \cite{35, 36}, geometry \cite{53}, and operator theory \cite{5, 25, 29, 31, 34, 50, 51, 65}, even though they were outside mainstream research for a long while. (Here the division into separate topics is rather conditional, and the references are just samples.) Recently, the interest in generalized Stieltjes functions renewed, and several pertinent papers appeared on this subject, see e.g. \cite{3, 4, 9, 45, 46, 47}. Generalized Stieltjes functions also arise in a somewhat more general form of generalized Cauchy transforms, mostly in the setting of the unit circle, see \cite{35}, and numerous references given there. For some applications of generalized Cauchy transforms in the framework of the real line one may consult \cite{39}. Though apparently these two flows of research had very few points in common so far.

Most of studies on generalized Stieltjes functions centered around their subclass $S_1$, given by

$$S_1 := \{a + S_1[\mu] : \mu \in \mathcal{M}_1^+(\mathbb{R}_+), \ a \geq 0\}, \tag{1.3}$$

and usually called Stieltjes functions in the literature. This class along with the closely related class $\mathcal{CBF}$ of so-called complete Bernstein functions is fundamental in probability theory and spectral theory and their interactions. Stieltjes and complete Bernstein functions are thoroughly discussed, for instance, in \cite{62} Chapters 2-8. For their very recent and interesting generalizations see e.g. \cite{11, 48}.
The following geometric characterisation of Stieltjes functions is one of the most useful results of the theory. It is behind a number of important statements concerning Stieltjes functions and their properties.

**Theorem 1.1.** A function \( f : (0, \infty) \to [0, \infty) \) is Stieltjes if and only if it is the restriction to \((0, \infty)\) of an analytic function \( F \) on \( \mathbb{C} \setminus (-\infty, 0] \) satisfying \( \text{Im} \, F(z) \leq 0 \) for \( z \in \mathbb{C} \) with \( \text{Im} \, z > 0 \), or equivalently, \( \text{Im} \, z \, \text{Im} \, F(z) \leq 0 \) for \( z \in \mathbb{C} \setminus (-\infty, 0] \).

See e.g. [62] or [8] for a thorough analysis of this result and its relevance for the theory of Stieltjes functions.

However, in contrast to the case \( \alpha = 1 \), the classes \( S_\alpha \) of the generalized Stieltjes functions of arbitrary order \( \alpha > 0, \alpha \neq 1 \), seem to be far from being fully understood, and essential difficulties arise in putting a structure into these classes, at least when following intuition developed for the study of \( S_1 \). A number of important properties of \( S_1 \) fail for \( S_\alpha \) with \( \alpha \neq 1 \), even after natural reformulations. In particular, this concerns Theorem \( 1.1 \) where plausible variants for \( \alpha \neq 1 \) appear to be false, and the problem of finding an analogue of Theorem \( 1.1 \) for arbitrary \( \alpha > 0 \) was posed by Sokal in [66, p. 184-185]. So far the research on \( S_\alpha \) existed in the form of isolated results scattered over the literature. Several important properties of \( S_\alpha \) can be found in the survey paper [41]. It is instructive to note that, by e.g. [41, Theorem 3], one has \( S_\alpha \subset S_\beta, 0 < \alpha < \beta \), so that a function \( f \in S_\alpha \) admits different Stieltjes representations in \( S_\beta \) for \( \beta > \alpha \) with sometimes rather different properties. For instance if \( f(z) = z^{-1}, z > 0 \), then \( f = S_1[\delta_0] \) and \( f = S_2[\nu] \), where \( \nu \) stands for the Lebesgue measure on \( \mathbb{R}_+ \).

The lack of geometric or any other easy applicable characterisations of \( S_\alpha \) makes it, in particular, nontrivial to decide whether the product \( f_1 f_2 \) of \( f_1 \in S_{\alpha_1} \) and \( f_2 \in S_{\alpha_2} \) belongs to \( S_\gamma \) for some \( \gamma > 0 \) and to determine the optimal \( \gamma \) whenever such a \( \gamma \) exists. Nevertheless, Hirschman and Widder proved in [33, Chapter VII.7.4] (see also [32]) that if \( \mu_1 \in \mathcal{M}_{\alpha_1}^+(\mathbb{R}_+) \) and \( \mu_2 \in \mathcal{M}_{\alpha_2}^+(\mathbb{R}_+), \alpha_1, \alpha_2 > 0 \), then there exists \( \mu = [\mu_1, \mu_2] \in \mathcal{M}_{\alpha_1+\alpha_2}^+(\mathbb{R}_+) \) such that

\[
S_{\alpha_1} [\mu_1](z) S_{\alpha_2} [\mu_2](z) = S_{\alpha_1+\alpha_2} [\mu](z), \quad z > 0,
\]

i.e. \( S_{\alpha_1} S_{\alpha_2} \subset S_{\alpha_1+\alpha_2} \) for all \( \alpha_1, \alpha_2 > 0 \). Observe that \( \mu \) in (1.4) is determined uniquely. Moreover, simple examples show (see e.g. [31] below) that the order \( \alpha_1 + \alpha_2 \) in (1.4) cannot in general be improved.

The arguments of Hirschman and Widder, addressing in fact a more general situation, were somewhat involved. They relied on rescaling \( S_\alpha[\mu] \) as \( F(t) = e^{\alpha t/2} S_\alpha[\mu](e^{t/2}), t \in \mathbb{R} \), representing \( F \) in the form of a convolution transform

\[
F(t) = \int_{\mathbb{R}} G(t-s) \nu(ds), \quad t \in \mathbb{R},
\]
where $\Gamma$ is the Gamma function.

Despite the fact that (1.4) is one of the basic results in the theory of Stieltjes transforms, quite surprisingly, the measure $\mu$ in (1.4) and its fine properties remained implicit until now, and, apart from very specific cases, the form of $\mu$ seemed to be unknown. To remove this gap, in the general setting of complex Radon measures on $\mathbb{R}_+$, we prove (1.4) by providing an explicit formula for $\mu$. The argument is elementary. Thus derived $\mu$ is called the Stieltjes convolution of $\mu_1$ and $\mu_2$ to emphasize the similarity of the map $(\mu_1,\mu_2) \rightarrow \mu[\mu_1,\mu_2]$ to the usual convolution of measures. Our formula reveals several interesting properties of $\mu$ showing in particular that $\mu$ does not have singular continuous component. Moreover, representing $\mu$ via $\mu_1$ and $\mu_2$ we obtain a number of submultiplicative inequalities bounding $\|\mu\|_{\beta_1+\beta_2}$ in terms of the products $\|\mu_1\|_{\beta_1}\|\mu\|_{\beta_2}$ for $\beta_j \in [0,\alpha_j], j = 1, 2$. These estimates for the size of $\|\mu\|_{\beta_1+\beta_2}$ seem to be completely new.

Next, identifying absolutely continuous measures with their densities, we test our product formula in the classical framework of Stieltjes transforms of functions from $L^1(\mathbb{R}_+; (1+t)^{-1})$, and provide natural conditions on $g_1,g_2 \in L^1(\mathbb{R}_+; (1+t)^{-1})$ to ensure that the product $S_1[g_1]S_1[g_2]$ is again of the form $S_1[g]$ for some $g \in L^1(\mathbb{R}_+; (1+t)^{-1})$. This generalizes and extends similar results in the literature. Curiously, in the case of Hilbert transforms of $L^p$-functions, it appeared that our formula is essentially equivalent to the well-known Tricomi identity for Hilbert transforms in its most general formulation.

Although an easy applicable, e.g. geometrical, characterisation of $f \in S_\beta$ is still out reach, we provide a sufficient condition, which looks simple and easy to use. Namely, we show that if $f \in S_\alpha$, and if for a given $\beta \in (0,1]$ one defines $f_{(\beta)}(z) = f(z^{2\beta}), z > 0$, then $f_{(\beta)} \in S_\alpha$. So, in particular, $f_{(\beta)} \in S_\beta$ if $f \in S_1$. This implication can be considered as a partial answer to Sokal’s question mentioned above. Moreover, employing a different argument, we prove that if $f \sim (0,\nu)_1$, then $f_{(\beta)} \sim (0,\mu)_\beta$, where $\mu$ is an absolutely continuous measure given explicitly in terms of $\nu$. The case $\beta = 1/2$ is especially explicit and is a nice illustration of our considerations. These results stem from a general theorem ensuring that for an appropriate $f : (0,\infty) \rightarrow (0,\infty)$ and $\gamma = \gamma(f) > 0$ one has $f^\alpha \in S_\alpha$ for all $\alpha > 0$. Our version of the product formula (1.4) shows that measures in the Stieltjes representation for $f^\alpha$ do not have singular component and helps to refute several plausible conjectures appearing along the way. Moreover, we show that the theorem is close to be optimal.

Note that the set $S_{1/2}(\mathcal{M}_0^+ (\mathbb{R}_+))$ was crucial in the recent study [3] of spectral multipliers $\Phi$ on $L^p$-spaces. Without going into details of [3], which
rely on a probabilistic background, note that the main statements in [4] required \( \Phi \) to have a specific form \( S_{1/2}[\mu] \) with \( \mu \) being a (bounded) complex measure on \( \mathbb{R}_+ \). Our results allow one to produce such multipliers from a well-understood set of \( S_1[\mu], \mu \in \mathcal{M}_0^+ (\mathbb{R}_+) \), and moreover they help to identify \( \Phi \), dealing with functions themselves rather than with their Stieltjes representations. Recall that by Theorem (1.1) it is comparatively easy to check whether a given function \( f \) belong to \( S_1 \). Then \( f_{(1/2)} \in S_{1/2} \) yields a multiplier \( \Phi \) needed in [4] Theorems 1.1-1.3. In addition, we are able to identify the Stieltjes representation of \( f_{(1/2)} \), and thus to write \( f_{(1/2)} \) in the form used in [4].

Finally we study counterparts of (1.3) in the framework of generalized Cauchy transforms of complex Radon measures on \( \mathbb{R} \). To put our results into a proper context, we first recall a similar research done for generalized (or, in other terminology, fractional) Cauchy transforms of measures on the unit circle \( \mathbb{T} \). Let \( T_\alpha, \alpha > 0 \), be the space of the generalized Cauchy transforms \( C_\alpha[\mu] \) of order \( \alpha \) given by

\[
C_\alpha[\mu](z) := \int_{\mathbb{T}} \frac{\mu(d\zeta)}{(1 - \zeta z)^\alpha}, \quad |z| < 1, \tag{1.5}
\]

where \( \mu \) runs over the set of probability measures \( \mathcal{P}^+(\mathbb{T}) \) on \( \mathbb{T} \). By one of the fundamental results in the theory of generalized Cauchy transforms, see e.g. [14, Theorem 1] and [35, p. 22-24], if \( \mu_j \in \mathcal{P}^+(\mathbb{T}) \) and \( \alpha_j > 0, j = 1, 2 \), then there exists \( \mu \in \mathcal{P}^+(\mathbb{T}) \) such that

\[
C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) = C_{\alpha_1 + \alpha_2}[\mu](z), \quad |z| < 1. \tag{1.6}
\]

In other words, we have \( T_{\alpha_1}T_{\alpha_2} \subset T_{\alpha_1 + \alpha_2} \), which is an analogue of the corresponding inclusion for the generalized Stieltjes transforms discussed above. See [35] for many instances where this result played a role.

In view of (1.4) and (1.6), it is natural to ask whether there exists a relation similar to (1.6) for generalized Cauchy transforms of measures on \( \mathbb{R} \). Let \( \mathcal{M}_0^+ (\mathbb{R}), \alpha \geq 0 \), stand for the set of positive Borel measures \( \mu \) on \( \mathbb{R} \) satisfying \( (1 + |s|)^{-\alpha} \in L^1(\mathbb{R}, \mu) \). For \( \mu \in \mathcal{M}_0^+ (\mathbb{R}), \alpha > 0 \), define its generalized Cauchy transform \( C_\alpha[\mu] \) of order \( \alpha \) by

\[
C_\alpha[\mu](z) := \int_{\mathbb{R}} \frac{\mu(dt)}{(z + t)^\alpha}, \tag{1.7}
\]

for \( z \) from the upper half-plane \( \mathbb{C}^+ \). Using our technique developed for the generalized Stieltjes transforms, we write down explicitly a “product” measure \( \mu \) satisfying

\[
C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) = C_{\alpha_1 + \alpha_2}[\mu](z), \quad z \in \mathbb{C}^+, \tag{1.7}
\]

for \( \mu_1 \in \mathcal{M}_0^+ (\mathbb{R}) \) and \( \mu_2 \in \mathcal{M}_0^+ (\mathbb{R}), \alpha_1, \alpha_2 > 0 \), and even for \( \mu_1 \) and \( \mu_2 \) from more general classes of complex Radon measures on \( \mathbb{R} \). In this case, \( \mu \) is not unique, and we provide a \( \mu \) preserving good properties of \( \mu_1 \) and \( \mu_2 \) such as positivity, integrability or compact support, and given by a comparatively simple formula. As a byproduct, as in the case of Stieltjes transforms,
the resulting $\mu$ does not have singular continuous component. The task of expressing $\mu$ in terms of $\mu_1$ and $\mu_2$ appears to be rather involved, so we impose additional assumptions on the size of $\mu_1$ and $\mu_2$, which hold in several natural situations. If the size restrictions are dropped, then we also give a formula for $\mu$, but, in such generality, the formula becomes rather implicit. Moreover, it does not necessarily yield a positive $\mu$. Thus, given $\mu_1 \in \mathcal{M}_{\alpha_1}^+(\mathbb{R})$ and $\mu_2 \in \mathcal{M}_{\alpha_2}^+(\mathbb{R})$, we address the important problem of existence of $\mu \in \mathcal{M}_{\alpha_1+\alpha_2}^+(\mathbb{R})$ such that (L.7) holds. We prove the existence criteria when $\alpha_1 = \alpha_2 = 1$ or $\alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 > 0$, though the general case remains open. We also study a related problem of liftings of the generalized Cauchy transforms from $C_\alpha(\mathcal{M}_{\alpha}^+(\mathbb{R}))$ to $C_\beta(\mathcal{M}_{\beta}^+(\mathbb{R}))$, $0 < \alpha < \beta$, and describe several situations when such a lifting is possible. Despite the generalized Cauchy transforms on $\mathcal{P}^+(\mathbb{T})$ were studied intensively for some time, the product formulas and related matters for the generalized Cauchy transforms on $\mathcal{M}_{\alpha}^+(\mathbb{R})$ (and more general classes of measures) appear probably for the first time.

2. Notations, preliminaries and conventions

In this section we collect various notations used throughout the paper, and clarify parts of our terminology.

To be able to handle ”unbounded” complex measures properly, we use the notion of complex Radon measures. Recall that a bounded linear functional $\mu$ on the Frechét space $C_c(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support is said to be a complex Radon measure on $\mathbb{R}$. In other words, a complex Radon measure $\mu$ on $\mathbb{R}$ is a linear functional on $C_c(\mathbb{R})$ such that for every compact $K \subset \mathbb{R}$ the inequality $|\mu(f)| \leq C_K \max_K |f|$ holds for some $C_K > 0$ and for all $f \in C_c(\mathbb{R})$ with support in $K$. Usually, one uses a more intuitive notation for $\mu$ writing it as an integral: $\mu(f) = \int f \, d\mu$. If the context is clear, then the adjective ”complex” is usually omitted, and we will follow this rule in the sequel. Recall that a Radon measure $\mu$ is said to be real if $\mu(f) \in \mathbb{R}$ for all real-valued $f \in C_c(\mathbb{R})$, and $\mu$ is positive if $\mu(f) \geq 0$ for all $f \in C_c(\mathbb{R})$, $f \geq 0$. Note that the space of real Radon measures on $\mathbb{R}$ is a Riesz space with the natural order: $\mu_1 \geq \mu_2$ if $\mu_1 - \mu_2$ is a positive Radon measure. If $\mu$ is a Radon measure, then there exists the smallest positive Radon measure $|\mu|$ called the variation (or the absolute value) of $\mu$ satisfying $|\mu(f)| \leq |\mu|(f)$ for all $f \in C_c(\mathbb{R})$, $f \geq 0$. There is an analogue of Jordan’s decomposition for $\mu$:

$$\mu = \text{Re} \mu + i\text{Im} \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4),$$

where $\mu_i, 1 \leq i \leq 4$, are positive Radon measures, such that $|\mu_1 - \mu_2| = \mu_1 + \mu_2$ and $|\mu_3 - \mu_4| = \mu_3 + \mu_4$ (i.e. $\mu_1$ and $\mu_2$, as well as $\mu_3$ and $\mu_4$, are mutually singular). The Radon measures $\text{Re} \mu$ and $\text{Im} \mu$ satisfy $|\text{Re} \mu| \leq |\mu|$, $|\text{Im} \mu| \leq |\mu|$, and $|\mu| \leq |\text{Re} \mu| + |\text{Im} \mu|$. The space $L^1(\mathbb{R}, |\mu|)$ arises as a completion of $C_c(\mathbb{R})$ in an appropriate semi-norm constructed by means of $\mu$, and $\mu$ extends to this space accordingly. For such an extension of $\mu$, denoted
by the same symbol, and a locally integrable function \( g \) (with respect to \( \mu \)) the mapping \( C_{\alpha}(\mathbb{R}) \ni f \to (g\mu)(f) = \mu(gf) \) is a Radon measure again, and moreover \( |g\mu|(dt) = |g(t)||\mu|(dt) \).

Any Radon measure \( \mu \) on \( \mathbb{R} \) admits the Lebesgue decomposition \( \mu = \mu_a + \mu_d + \mu_s \), where \( \mu_a, \mu_d, \) and \( \mu_s \) stand for locally absolutely continuous (with respect the Lebesgue measure), purely discrete and singular continuous part of \( \mu \), respectively. Moreover, the Radon measures \( \mu_a, \mu_d \) and \( \mu_s \) are defined uniquely, mutually singular, and \( |\mu| = |\mu_a| + |\mu_d| + |\mu_s| \).

Observe that positive Radon measures on \( \mathbb{R} \) can be identified with locally finite Borel measures on \( \mathbb{R} \) by the Riesz representation theorem. It is also instructive to recall that the space of Radon measures on \( \mathbb{R} \) can be considered as the space of functions on \( \mathbb{R} \) with locally bounded variation, thus allowing one to use the Lebesgue-Stieltjes integration theory.

In general, using the measure-theoretic terminology, one may consider a Radon measure \( \mu \) on \( \mathbb{R} \) as a complex Borel measure \( \mu \) defined "locally" on \( \mathbb{R} \). In other words, if \( B(\mathbb{R}) \) is the Borel \( \sigma \)-algebra of \( \mathbb{R} \), then \( \mu \) is defined on \( B(\mathbb{R}) \cap K \) for all compact \( K \subset \mathbb{R} \), takes complex values, and is such that for any compact \( K \subset \mathbb{R} \) the function \( B \to \mu(B \cap K), B \in B(\mathbb{R}) \), is a complex measure. The need in such objects is clearly justified by a standard example of the Radon measure \( \text{sign}(t)dt \), which cannot be considered in the framework of complex (or signed) Borel measures on \( B(\mathbb{R}) \).

Finally, note that all of the above considerations are valid with \( \mathbb{R} \) replaced by \( \mathbb{R}_+ \), with obvious adjustments.

For the theory of Radon measures and their properties one may consult e.g. [12], [19, Chapter 7], [22, Chapter 4], and [64, Chapter 4], which sometimes complement each other. A compact treatment (though without using the Radon measures terminology) in [30, Chapter 4.16] is also relevant. Being unable to go into finer details of the theory of Radon measures, we just remark that with certain care, all of the basic facts from the integration theory of positive (Borel) measures can be adapted to this quite a general setting.

For \( \alpha \geq 0 \), let \( M_\alpha(\mathbb{R}_+) \) denote the set of complex Radon measures \( \mu \) on \( \mathbb{R}_+ \) such that

\[
\|\mu\|_\alpha := \int_{\mathbb{R}_+} \frac{|\mu|(dt)}{(1+t)^\alpha} < \infty.
\]

Note that \( (M(\mathbb{R}_+), \| \cdot \|_\alpha) \) is a linear normed space. If \( \mu \in M_\alpha(\mathbb{R}_+) \), then \( \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \), with \( \{ \mu_i : 1 \leq i \leq 4 \} \subset M_\alpha^+(\mathbb{R}_+) \). The normed space \( (M_\alpha(\mathbb{R}_+), \| \cdot \|_\alpha) \) can be identified with the dual to \( C_0(\mathbb{R}_+; (1+t)^\alpha) := \{ f \in C(\mathbb{R}_+) : (1+t)^\alpha f(t) \to 0, t \to \infty \} \) equipped with a natural sup-norm, so \( (M_\alpha(\mathbb{R}_+), \| \cdot \|_\alpha) \) is a Banach space. The positive part of \( M_\alpha(\mathbb{R}_+) \) will be denoted by \( M_\alpha^+(\mathbb{R}_+) \).

Similarly, the set of complex Radon measures \( \mu \) on \( \mathbb{R} \) satisfying

\[
\|\mu\|_\alpha := \int_{\mathbb{R}} \frac{|\mu|(dt)}{(1+|t|)^\alpha} < \infty
\]
for some $\alpha \geq 0$ will be denoted by $\mathcal{M}_\alpha(\mathbb{R})$, and $\mathcal{M}_\alpha^+(\mathbb{R})$ will stand for positive Radon measures in $\mathcal{M}_\alpha(\mathbb{R})$. Note that $(\mathcal{M}_\alpha(\mathbb{R}), \|\cdot\|_\alpha)$ is a Banach space dual to $C_0(\mathbb{R}; (1 + |t|)^{\alpha})$. Moreover, $\mathcal{M}_\alpha(\mathbb{R}^+) \subset \mathcal{M}_\alpha(\mathbb{R})$ and $\mathcal{M}_\alpha^+(\mathbb{R}^+) \subset \mathcal{M}_\alpha^+(\mathbb{R})$ isometrically, so the norms in (2.1) and (2.2) agree in a natural sense.

For $\alpha \geq 0$ and $p \geq 1$ let $L_p^0(\mathbb{R}^+)$ stand for $L^p(\mathbb{R}^+; (1 + t)^{-\alpha})$. We identify $f \in L_1^1(\mathbb{R}^+)$ with $\mu = \mu_f \in \mathcal{M}_\alpha(\mathbb{R}^+)$ so that $\mu(dt) = f(t)dt$, and such an identification yields an isometric embedding of $L_1^1(\mathbb{R}^+)$ into $\mathcal{M}_\alpha(\mathbb{R}^+)$. Similarly, denoting $L^p(\mathbb{R}; (1 + |t|)^{-\alpha})$, with $L_p^0(\mathbb{R})$, observe that $L_1^1(\mathbb{R})$ embeds isometrically into $\mathcal{M}_\alpha(\mathbb{R})$. For the sequel, extending $g \in L_p^0(\mathbb{R}^+)$ by zero to $(-\infty, 0)$, it will be convenient to identify $g$ with its extension to $\mathbb{R}$, thus embedding $L_p^0(\mathbb{R}^+)$ into $L_p^0(\mathbb{R})$ isometrically.

If $\mu \in \mathcal{M}_\alpha^+(\mathbb{R}^+)$, then its generalized Stieltjes transform of order $\alpha > 0$ will be denoted by $S_\alpha[\mu]$, and for $\mu \in \mathcal{M}_\alpha(\mathbb{R})$ we denote its generalized Cauchy transform of order $\alpha > 0$ by $C_\alpha[\mu]$. If, moreover, $\mu(dt) = f(t)dt$, and $F$ is one of the two transformations above, then we will write $F[f]$ instead of $F[\mu]$.

Furthermore, we will let $S_\alpha$ stand for $\{a + S_\alpha[\mu] : a \geq 0, \mu \in \mathcal{M}_\alpha^+(\mathbb{R}^+)\}$, and if $f \in S_\alpha$, $f = a + S_\alpha[\mu]$, then we will write $f \sim (a, \mu)_\alpha$, and call $(a, \mu)_\alpha$ the Stieltjes representation of $f$. It is essential to note that $f$ admits an analytic extension to $\mathbb{C} \setminus (-\infty, 0]$, and the Stieltjes representation of $f$ is determined uniquely. The uniqueness is a direct consequence of a complex inversion theorem for the generalized Stieltjes transforms, see [67] or [15], and the property $S_\alpha[\mu](z) \to 0$ as $z \to \infty$. Alternatively, one may use a Laplace representation formula for $f$ ([11, Theorem 8]) and a uniqueness theorem for Laplace transforms. Though [15, 11, and 67] treat only positive measures, the corresponding reformulations for complex measures are straightforward.

As usual, $\delta_t, t \in \mathbb{R}$, denotes the Dirac delta measure at $t \in \mathbb{R}$, $\chi_E$ stands for the characteristic function of a set $E \subset \mathbb{R}$, and $\text{supp} \mu$ denotes the support of a measure $\mu$. In the sequel, for $-\infty \leq a < b \leq \infty$, the notation $\int_a^b$ is used for $\int_{[a,b]}$ if $a, b \in \mathbb{R}$, and for $\int_{(-\infty,b]}$ or $\int_{[a,\infty)}$ if $a = -\infty$ or $b = \infty$, respectively.

Throughout the paper, we will use the principal branch of the power function $z \to z^\alpha, \alpha \in \mathbb{R}$, given by

$$z^\alpha = |z|^\alpha e^{i\alpha \arg z}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \quad \arg(z) \in (-\pi, \pi).$$

Finally, we let $\mathbb{R}^+ := [0, \infty)$, $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im} \ z > 0\}$, $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Re} \ z > 0\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

3. MEASURES IN HIRSCHMAN-WIDDER’S REPRESENTATION

In this section, we study the structure and properties of measures $\mu \in \mathcal{M}_{\alpha_1 + \alpha_2}^+(\mathbb{R}^+)$ arising in Hirschman-Widder’s product formula (1.1).

The next simple but quite useful result will be essential for the sequel. For its proof, recall that for all $\alpha_1, \alpha_2 > 0$, $-\infty < s < t < \infty$, and $z \in$
where $B$ is the beta function, defined by

$$B(\alpha_1, \alpha_2) := \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1} \, dt, \quad \alpha_1 > 0, \alpha_2 > 0.$$ 

See e.g. [59, p. 301, 2.2.6(2)].

**Theorem 3.1.** Let $\mu_j \in M_{\alpha_j}(\mathbb{R}_+)$, $\alpha_j > 0$, $j = 1, 2$. Then

$$S_{\alpha_1}[\mu_2](z)S_{\alpha_2}[\mu_2](z) = S_{\alpha_1+\alpha_2}[\mu](z), \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

with the unique $\mu \in M_{\alpha_1+\alpha_2}(\mathbb{R}_+)$ defined by

$$\mu(\text{d}t) = u(\tau)\text{d}\tau + \mu_1(\{\tau\})\mu_2(\text{d}\tau),$$

and

$$B(\alpha_1, \alpha_2)u(\tau) = \int_{(0, \infty)} \int_{(0, \tau)} \frac{(\tau - \tau)^{\alpha_2-1}(1-\tau)^{\alpha_1-1}}{(1-\tau)^{\alpha_1+\alpha_2-1}} \mu_1(\text{d}s)\mu_2(\text{d}t)$$

$$+ \int_{(0, \infty)} \int_{(0, \tau)} \frac{(\tau - \tau)^{\alpha_1-1}(1-\tau)^{\alpha_2-1}}{(1-\tau)^{\alpha_1+\alpha_2-1}} \mu_2(\text{d}s)\mu_1(\text{d}t)$$

for all almost all $\tau \geq 0$. Moreover,

$$\|\mu\|_{\alpha_1+\alpha_2} \leq \|\mu_1\|_{\alpha_1} \|\mu_2\|_{\alpha_2},$$

with equality sign if $\mu_j \in M_{\alpha_j}^+(\mathbb{R}_+), j = 1, 2$.

**Proof.** First assume, in addition, that $\mu_j \in M_{\alpha_j}^+(\mathbb{R}_+), j = 1, 2$, and let $z > 0$ be fixed. Then using the integrability assumptions on $\mu_1$ and $\mu_2$, write

$$S[\mu_1](z)S[\mu_2](z) = \int_0^\infty \int_0^\infty \frac{\mu_1(\text{d}s)\mu_2(\text{d}t)}{(1+s)^{\alpha_1}(1+t)^{\alpha_2}} = g_1(z) + g_2(z) + g_0(z),$$

where

$$g_1(z) = \int_{(0, \infty)} \int_{(0, t)} \frac{\mu_1(\text{d}s)\mu_2(\text{d}t)}{(1+s)^{\alpha_1}(1+t)^{\alpha_2}},$$

$$g_2(z) = \int_0^\infty \int_{(t, \infty)} \frac{\mu_1(\text{d}s)\mu_2(\text{d}t)}{(1+s)^{\alpha_1}(1+t)^{\alpha_2}} = \int_{(0, \infty)} \int_{(0, t)} \frac{\mu_2(\text{d}s)\mu_1(\text{d}t)}{(1+s)^{\alpha_2}(1+t)^{\alpha_1}},$$

$$g_0(z) = \int_0^\infty \int_{(t, \infty)} \frac{\mu_2(\{t\})\mu_1(\text{d}t)}{(1+t)^{\alpha_1+\alpha_2}} = \int_0^\infty \int_{(0, \infty)} \frac{\mu_2(\{t\})\mu_1(\text{d}t)}{(1+s)^{\alpha_2+\alpha_2}}.$$

Clearly, all of the integrals above are finite, and the function $[0, \infty) \ni t \mapsto \mu_2(\{t\})$ is Borel since its support is countable. Next, by (3.1), positivity of...
integrands, and Fubini’s theorem,

\[ B(\alpha_1, \alpha_2)g_1(z) = \int_{(0,\infty)} \int_{(0,t)} \int_{(s,t)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1} d\tau}{(z + \tau)^{\alpha_1 + \alpha_2}} \mu_1(ds)\mu_2(dt) \]

\[ = \int_0^\infty \int_{(0,t)} \int_{(s,t)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2} - 1} \mu_1(ds)\mu_2(dt) \]

\[ = \int_0^\infty \int_{(\tau,\infty)} \int_{(0,\tau)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2} - 1} \mu_1(ds)\mu_2(dt) \frac{d\tau}{(z + \tau)^{\alpha_1 + \alpha_2} - 1}.
\]

Similarly,

\[ B(\alpha_1, \alpha_2)g_2(z) = \int_0^\infty \int_{(\tau,\infty)} \int_{(0,\tau)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2} - 1} \mu_2(ds)\mu_1(dt) \frac{d\tau}{(z + \tau)^{\alpha_1 + \alpha_2} - 1}.
\]

So, if \( z > 0 \), then

\[ (3.6) \quad S_{\alpha_1}[\mu_1](z)S_{\alpha_2}[\mu_2](z) = \int_0^\infty \frac{\mu(d\tau)}{(z + \tau)^{\alpha_1 + \alpha_2} - 1}, \]

where \( \mu = \mu[\mu_1, \mu_2] \in M_{\alpha_1 + \alpha_2}(R_+) \) is given by \((3.3)\) and \((3.4)\). (Note that the measurability of \( u \) is also guaranteed by Fubini’s theorem.) By analytic continuation, this formula extends to all \( z \in C \setminus (-\infty, 0) \).

To prove \((3.5)\) under our additional assumption note that

\[ (3.7) \quad \int_0^\infty \frac{\mu[\mu_1, \mu_2](dt)}{(z + t)^{\alpha_1 + \alpha_2}} = \int_0^\infty \frac{\mu_1(dt)}{(z + t)^{\alpha_1}} \int_0^\infty \frac{\mu_2(dt)}{(z + t)^{\alpha_2}}, \quad z > 0,
\]

so letting \( z = 1 \) in \((3.7)\) we obtain \((3.5)\).

In the general case, we apply the first part of the proof to positive measures \(|\mu_1|\) and \(|\mu_2|\) and, using Fubini’s theorem again, arrive at \((3.3)\) and \((3.4)\).

Furthermore, if \( \mu = \mu[\mu_1, \mu_2] \) is defined by \((3.3)\) and \((3.4)\), then

\[ (3.8) \quad |\mu[\mu_1, \mu_2]| \leq \mu(|\mu_1|, |\mu_2|).
\]

Hence, in view of \((3.7)\),

\[ (3.9) \quad \int_0^\infty \frac{|\mu[\mu_1, \mu_2]|(dt)}{(1 + t)^{\alpha_1 + \alpha_2}} \leq \int_0^\infty \frac{\mu[|\mu_1|, |\mu_2|](dt)}{(1 + t)^{\alpha_1 + \alpha_2}} \]

\[ = \int_0^\infty \frac{|\mu_1|(dt)}{(1 + t)^{\alpha_1}} \int_0^\infty \frac{|\mu_2|(dt)}{(1 + t)^{\alpha_2}},
\]

which is equivalent to \((3.5)\). \( \Box \)

**Remark 3.2.** Observe that by \((3.9)\) for almost every \( \tau > 0 \), the double integrals in \((3.3)\) converge absolutely and thus the integration limits in these integrals can be interchanged.
Given \( \mu_1 \in \mathcal{M}_{\alpha_1}(\mathbb{R}_+) \) and \( \mu_2 \in \mathcal{M}_{\alpha_2}(\mathbb{R}_+) \), \( \alpha_1, \alpha_2 > 0 \), the measure \( \mu \in \mathcal{M}_{\alpha_1+\alpha_2}(\mathbb{R}_+) \) defined by (3.3) and (3.4) will be called the Stieltjes convolution of \( \mu_1 \) and \( \mu_2 \) and denoted by \( \mu_1 \otimes_{\alpha_1,\alpha_2} \mu_2 \). Theorem 3.1 says that
\[
S_{\alpha_1}[\mu_1]S_{\alpha_2}[\mu_2] = S_{\alpha_1+\alpha_2}[\mu_1 \otimes_{\alpha_1,\alpha_2} \mu_2],
\]
so that with an appropriate choice of orders the Stieltjes transform maps the Stieltjes convolution of measures into the product of their Stieltjes transforms. This fact partially explains our terminology, see also Remark 4.11 below. There are other, rather simple algebraic properties of the Stieltjes convolution resembling the usual convolution of measures on \( \mathbb{R}_+ \), see e.g. [29], [54], [65], [69], and [73] for some of their variants in particular situations and a similar terminology (see also Remark 5.2 below). However, we decided to avoid a discussion of these properties, since they are not relevant for this paper.

At the same time, the Stieltjes convolution of measures has some unexpected features. The next direct corollary reveals a specific structure of representing measures \( \mu_1 \otimes_{\alpha_1,\alpha_2} \mu_2 \) arising from the product of the generalized Stieltjes transforms. It emphasizes, in particular, the fact that the Stieltjes convolution of measures does not have singular continuous part.

**Corollary 3.3.** For \( \alpha_j > 0, j = 1, 2 \) let \( \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}_+) \), and \( \mu = \mu_1 \otimes_{\alpha_1,\alpha_2} \mu_2 \). Then the following hold.

(i) If \( \mu_1 \) and \( \mu_2 \) are positive, then \( \mu \) is positive too.

(ii) If \( \text{supp} \mu_j \subset [a,b], j = 1, 2 \), then \( \text{supp} \mu \subset [a,b] \) as well.

(iii) \( \mu \) has no singular continuous part.

(iv) \( \mu \) is locally absolutely continuous if and only if \( \mu_1 \) and \( \mu_2 \) have disjoint sets of atoms.

(v) the discrete part \( \mu_d \) of \( \mu \) is of the form
\[
\mu_d(d\tau) = \mu_1(\{\tau\})\mu_2(d\tau) = \mu_2(\{\tau\})\mu_1(d\tau).
\]

As consequence, \( \mu \) is purely discrete if and only if \( \mu_1 = c_1\delta_t \) and \( \mu_2 = c_2\delta_t \) for some \( c_1, c_2 \in \mathbb{C} \) and \( t \in \mathbb{R}_+ \).

The proof of Corollary 3.3 is straightforward, and is therefore omitted.

**Remark 3.4.** Curiously, a similar effect of absence of singular continuous component appears for the so-called free multiplicative convolution of Borel probability measures on \( \mathbb{R}_+ \) and for the additive convolution of such measures on \( \mathbb{R} \) (when neither of measures is a point mass), see [37] and [6] respectively. Though the reasons for this phenomena lie probably much deeper.

To gain an intuition behind the Stieltjes convolution, consider the following illustrative example.
Example 3.5. Let $\alpha_j > 0$, $j = 1, 2$. If $\mu_1 \in \mathcal{M}_{\alpha_1}(\mathbb{R}_+)$ and $\mu_2 = \delta_a$ for some $a \geq 0$, then using (3.3) and (3.4), we have

$$\mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2 = B(\alpha_1, \alpha_2)u + \mu(\{a\})\delta_a,$$

where

$$u(\tau) = \begin{cases} (a - \tau)^{\alpha_1 - 1} \int_{[0, \tau]} \frac{(\tau - s)^{\alpha_2 - 1}}{(a - s)^{\alpha_1 + \alpha_2 - 1}} \mu_1(ds), & \tau \in (0, a), \\ (\tau - a)^{\alpha_1 - 1} \int_{(\tau, \infty)} \frac{(s - \tau)^{\alpha_2 - 1}}{(s - a)^{\alpha_1 + \alpha_2 - 1}} \mu_1(ds), & \tau > a, \end{cases}$$

if $a > 0$, and

$$u(\tau) = \tau^{\alpha_1 - 1} \int_{(\tau, \infty)} (s - \tau)^{\alpha_2 - 1}s^{1 - \alpha_1 - \alpha_2} \mu_1(ds), \quad \tau > 0,$$

if $a = 0$. In particular, setting $\alpha_1 = \alpha_2 = 1$, we have

$$\mu_1 \otimes_{1, 1} \mu_2 = u + \mu_1(\{a\})\delta_a$$

with

$$u(\tau) = \chi(a) - \tau \int_{[0, \tau]} \frac{\mu(ds)}{a - s} + \chi(\tau - a) \int_{\tau}^{\infty} \frac{\mu(ds)}{s - a}.$$ 

If $0 < a < b < \infty$, $\mu_1 = \delta_a$ and $\mu_2 = \delta_b$, then $\mu_1, \mu_2 \in \mathcal{M}_a(\mathbb{R}_+)$ for all $a > 0$, and we have

$$[\delta_b \otimes_{1, 1} \delta_a](dt) = \frac{1}{b - a} \chi(t - a)\chi(b - t) dt = \frac{1}{b - a} \chi_{[a, b]}(t) dt, \quad t > 0.$$ 

4. Inequalities for the Stieltjes convolution of measures

In this section, we shed a light on the Stieltjes convolution and generalize the estimate (3.5) by measuring the right-hand side of (3.5) in stronger norms.

Observe that if $\mu_j \in \mathcal{M}_0(\mathbb{R}_+)$, $j = 1, 2$, and $\mu = \mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2$, then by the dominated convergence theorem,

$$\|\mu\| \leq \lim_{s \to \infty} \int_0^\infty \frac{s^{\alpha_1} |\mu_1|(dt)}{(s + t)^{\alpha_1}} \int_0^\infty \frac{s^{\alpha_2} |\mu_2|(dt)}{(s + t)^{\alpha_2}},$$

$$= \int_0^\infty |\mu_1|(dt) \int_0^\infty |\mu_2|(dt) = \|\mu_1\|\|\mu_2\|_0.$$

The inequalities (3.5) and (4.1) suggest that the submultiplicative property $\|\mu\|_{\beta_1, \beta_2} \leq \|\mu_1\|_{\beta_1}\|\mu_2\|_{\beta_2}$ extrapolates to the whole of scale $\mathcal{M}_{\beta_1}(\mathbb{R}_+) \times \mathcal{M}_{\beta_2}(\mathbb{R}_+)$. Below we prove that it is so up to certain constants, apart from the boundary case when one of the numbers $\beta_1$ and $\beta_2$ equals zero. In the latter case, we show that a (necessarily) weaker inequality holds.

We will need the next simple lemma.

Lemma 4.1. Let $\alpha > 0$ and let

$$F_\alpha(t) := \int_1^\infty \frac{ds}{s(s + t)^\alpha}, \quad t > 0.$$
Then there exist \( \tilde{c}_\alpha, c_\alpha > 0 \) such that

\[
(4.3) \quad c_\alpha \frac{\log(t + e)}{(1 + t)^\alpha} \leq F_\alpha(t) \leq \tilde{c}_\alpha \frac{\log(t + e)}{(1 + t)^\alpha}, \quad t > 0.
\]

Proof. Fix \( t > 0 \). Integrating by parts, we obtain

\[
e^{\alpha} F_\alpha(t) = \int_{1/t}^\infty \frac{ds}{s(s + 1)^\alpha} = \log \frac{t}{(1/t + 1)^\alpha} + \alpha \int_{1/t}^\infty \frac{\log s ds}{(s + 1)^{\alpha + 1}},
\]

and the estimate (4.3) follows. \( \square \)

4.1. The numbers \( \beta_1 \) and \( \beta_2 \) are separated from zero. First we provide bounds for \( \|\mu\|_{\beta_1+\beta_2} \) in terms of \( \|\mu_1\|_{\beta_1} \) and \( \|\mu_2\|_{\beta_2} \), when \( \beta_j \in (0, \alpha_j] \), \( \alpha_j > 0 \), \( j = 1, 2 \).

Our arguments will rely on the following two identities. If \( \alpha_1, \alpha_2 > 0 \) and \( 0 < s < t \), then setting \( z = 1 \) in (3.1), we obtain

\[
(4.4) \quad \frac{1}{(t-s)^{\alpha_1+\alpha_2-1}} \int_s^t (\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1} d\tau = \frac{B(\alpha_1, \alpha_2)}{(1+s)^{\alpha_1}(1+t)^{\alpha_2}}.
\]

Moreover, multiplying both sides of (3.1) by \( (z + s)^{\alpha_1}(z + t)^{\alpha_2} \) and passing to the limit as \( z \to \infty \) we obtain

\[
(4.5) \quad \frac{1}{(t-s)^{\alpha_1+\alpha_2-1}} \int_s^t (\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1} d\tau = B(\alpha_1, \alpha_2),
\]

cf. [59, p. 298, 2.2.5(1)].

Fix \( \alpha_j > 0 \) and \( \beta_j \in (0, \alpha_j], j = 1, 2 \), and define

\[
(4.6) \quad J(s, t) := \int_s^t (\tau - s)^{\alpha_1-1}(t - \tau)^{\alpha_2-1} d\tau, \quad 0 < s < t.
\]

Lemma 4.2. Let \( \beta_j \in (0, \alpha_j) \) and \( \gamma_j = \alpha_j - \beta_j, j = 1, 2 \). Then for all \( 0 < s < t \),

\[
(4.7) \quad J(s, t) \leq \frac{\beta_1^{\alpha_1} \beta_2^{\alpha_2} B(\gamma_1, \gamma_2)}{(\beta_1 + \beta_2)^{\beta_1+\beta_2}} \cdot \frac{(t-s)^{\alpha_1+\alpha_2-1}}{(1+s)^{\beta_1}(1+t)^{\beta_2}},
\]

and

\[
(4.8) \quad J(s, t) \leq \frac{\gamma_1^{\alpha_1} \gamma_2^{\alpha_2} B(\beta_1, \beta_2)}{(\gamma_1 + \gamma_2)^{\gamma_1+\gamma_2}} \cdot \frac{(t-s)^{\alpha_1+\alpha_2-1}}{(1+s)^{\beta_2}(1+t)^{\beta_1}}.
\]

Proof. Let \( \nu \in (0, 1) \) and \( 0 < s < t \) be fixed. Then by Hölder’s inequality,

\[
(4.9) \quad J(s, t) = \int_s^t (\tau - s)^{\beta_1-\nu}(t - \tau)^{\beta_2-\nu} (\tau - s)^{\gamma_1-(1-\nu)}(t - \tau)^{\gamma_2-(1-\nu)} d\tau
\]

\[
\leq \left( \int_s^t (\tau - s)^{\beta_1/\nu-\nu}(t - \tau)^{\beta_2/\nu-1} (\tau - s)^{\gamma_1/(1-\nu)-1}(t - \tau)^{\gamma_2/(1-\nu)-1} d\tau \right)^{1-\nu}
\]

\times \left( \int_s^t (\tau - s)^{\gamma_1/(1-\nu)-1}(t - \tau)^{\gamma_2/(1-\nu)-1} d\tau \right)^{1-\nu}.
\]
Using (4.4) and (4.5) to transform the last two terms in (4.9), we obtain

\[(4.10)\]
\[J(s, t) \leq B_\nu \frac{(t-s)^{\alpha_1+\alpha_2-1}}{(1+s)^{\beta_2}(1+t)^{\beta_1}},\]

where

\[B_\nu = \left( B\left(\frac{\beta_1}{\nu}, \frac{\beta_2}{\nu}\right) \right)^\nu \left( B\left(\frac{\gamma_1}{1-\nu}, \frac{\gamma_2}{1-\nu}\right) \right)^{1-\nu}.\]

Next, employing the relation \(B(s, t) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}, t, s > 0\), and Stirling’s formula, note that

\[(4.11)\]
\[\lim_{s, t \to \infty} \frac{(s+t)^{s+t-\frac{3}{2}}B(s, t)}{\sqrt{2\pi s^{s-\frac{1}{2}}t^{t-\frac{1}{2}}} = 1.}\]

Thus, in view of (4.11),

\[(4.12)\]
\[\lim_{\nu \to 0} B_\nu = \frac{\beta_1 \beta_2}{(\beta_1 + \beta_2)^{\beta_1+\beta_2}} B(\gamma_1, \gamma_2).\]

So letting \(\nu \to 0\) in (4.10), the estimate (4.7) follows.

On the other hand, since similarly to (4.12),

\[\lim_{\nu \to 1} B_\nu = \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2)^{\gamma_1+\gamma_2}} B(\beta_1, \beta_2),\]

we let \(\nu \to 1\) in (4.10) and obtain (4.8).

Next, for \(\alpha_1, \alpha_2 > 0\) and \(\beta_1 \in [0, \alpha_1], \beta_2 \in [0, \alpha_2]\), we estimate the size of \(\|\mu \odot_{\alpha_1, \alpha_2} \mu\|_{\beta_1+\beta_2}\). To simplify our presentation, we separate the cases when \(\beta_j \in (0, \alpha_j), j = 1, 2\), and when either \(\beta_1 = \alpha_1\) or \(\beta_2 = \alpha_2\). The first case is covered by the next result.

**Theorem 4.3.** For \(\alpha_j > 0\) let \(\beta_j \in (0, \alpha_j)\) and \(\gamma_j := \alpha_j - \beta_j, j = 1, 2\). If \(\mu_j \in \mathcal{M}_{\beta_j}(\mathbb{R}^+), j = 1, 2\), and \(\mu = \mu_1 \odot_{\alpha_1, \alpha_2} \mu_2\), then \(\mu \in \mathcal{M}_{\beta_1+\beta_2}(\mathbb{R}^+)\) and

\[(4.13)\]
\[\|\mu\|_{\beta_1+\beta_2} \leq \min(\alpha_1, \alpha_2)\|\mu\|_{\beta_1}\|\mu\|_{\beta_2},\]

where

\[(4.14)\]
\[A_1 := \frac{\beta_1 \beta_2}{(\beta_1 + \beta_2)^{\beta_1+\beta_2}} B(\gamma_1, \gamma_2) \quad \text{and} \quad A_2 := \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2)^{\gamma_1+\gamma_2}} B(\beta_1, \beta_2).\]

**Proof.** By Theorem 3.1 the Stieltjes convolution \(\mu\) is given by (3.3) and (3.4). Assume first that \(\mu_1\) and \(\mu_2\) are positive, so that \(\mu\) and \(u\) in (3.3) are...
positive. Then, by (3.4), Fubini’s theorem and Lemma 4.2, we have
\[
\int_0^\infty \frac{u(\tau)\,d\tau}{(1 + \tau)^{\beta_1 + \beta_2}} \leq \frac{1}{B(\alpha_1, \alpha_2)} \int_{(0, \infty)} \int_{[0, t)} J(s, t) \frac{\mu_1(ds) \mu_2(dt)}{(t - s)^{\alpha_1 + \alpha_2 - 1}} + \frac{1}{B(\alpha_1, \alpha_2)} \int_{(0, \infty)} \int_{[0, t)} J(s, t) \frac{\mu_2(ds) \mu_1(dt)}{(t - s)^{\alpha_1 + \alpha_2 - 1}} \\
\leq A_j \int_{(0, \infty)} \int_{[0, t)} \frac{\mu_1(ds) \mu_2(dt)}{(1 + s)^{\beta_1}(1 + t)^{\beta_2}} + A_j \int_{(0, \infty)} \int_{[0, t)} \frac{\mu_2(ds) \mu_1(dt)}{(1 + s)^{\beta_2}(1 + t)^{\beta_1}},
\]
where \(A_j\) stands for either \(A_1\) or \(A_2\). Furthermore, note that
\[
(4.15) \quad A_j \geq 1, \quad j = 1, 2.
\]
Indeed, setting \(t = s + 1\) in (4.10) we infer that
\[
\frac{B(\alpha_1, \alpha_2)}{(s + 2)^{\beta_1 + \beta_2}} \leq J(s, s + 1) \leq \frac{B_\nu}{(s + 1)^{\beta_1 + \beta_2}},
\]
so
\[
B_\nu \geq \frac{B(\alpha_1, \alpha_2)}{(s + 2)^{\beta_1 + \beta_2}}, \quad s > 0.
\]
Passing to the limit in the above inequality as \(s \to \infty\) and recalling the definition of \(A_j\), we deduce (4.15).
Therefore, taking into account (4.15) and (3.3), we conclude that
\[
\|\mu\|_{\beta_1 + \beta_2} \leq A_j \int_0^{\infty} \int_{[0, t)} \frac{\mu_1(ds) \mu_2(dt)}{(1 + s)^{\beta_1}(1 + t)^{\beta_2}} + A_j \int_0^{\infty} \int_{(t, \infty)} \frac{\mu_1(ds) \mu_2(dt)}{(1 + s)^{\beta_1}(1 + t)^{\beta_2}},
\]
and the statement follows under our positivity assumption. In the general case, it suffices to use (3.8) and to apply the first part of the proof. \(\square\)

Now we consider the second case where it is clearly enough to assume that \(\beta_2 = \alpha_2\) and \(\beta_1 \in (0, \alpha_1)\).

**Corollary 4.4.** For \(\alpha_1, \alpha_2 > 0\) let \(\beta_2 = \alpha_2\) and \(\beta_1 \in (0, \alpha_1)\). If \(\mu_j \in \mathcal{M}_{\beta_j}(\mathbb{R}^+)\), \(j = 1, 2\), and \(\mu = \mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2\), then \(\mu \in \mathcal{M}_{\beta_1 + \beta_2}(\mathbb{R}^+)\) and
\[
(4.16) \quad \|\mu\|_{\beta_1 + \beta_2} \leq \frac{B(\beta_1, \beta_2)}{B(\alpha_1, \alpha_2)} \|\mu_1\|_{\beta_1} \|\mu_2\|_{\beta_2}.
\]

**Proof.** Note that if \((\beta_1, \beta_2) \in (0, \alpha_1) \times (0, \alpha_2)\) and \(A_1\) and \(A_2\) are defined in (4.14), then
\[
\lim_{\beta_2 \to \alpha_2} A_1 = \infty \quad \text{and} \quad \lim_{\beta_2 \to \alpha_2} A_2 = \frac{B(\beta_1, \alpha_2)}{B(\alpha_1, \alpha_2)}.
\]
Moreover, $\|\mu\|_{\beta_1} \to \|\mu\|_{\alpha_2}$ and $\|\mu\|_{\beta_1+\beta_2} \to \|\mu\|_{\beta_1+\alpha_2}$ as $\beta_2 \to \alpha_2$. Thus, passing to the limit in (4.13), we obtain (4.16). □

Remark 4.5. Observe that

$$\lim_{(\beta_1,\beta_2) \to (0,0)} (\beta_1, \beta_2) = (0,0) \quad A_1 = 1$$

and

$$\lim_{(\beta_1,\beta_2) \to (\alpha_1,\alpha_2)} (\beta_1, \beta_2) = (\alpha_1,\alpha_2) \quad A_2 = 1,$$

so passing to the limit in (4.13) as $(\beta_1,\beta_2) \to (0,0)$ or as $(\beta_1,\beta_2) \to (\alpha_1,\alpha_2)$ we obtain either (4.1) or (3.5), respectively.

It is instructive to note that the constant $\min(A_1, A_2)$ in (4.13) cannot in general be replaced by 1, and in fact the optimal constant in (4.13) can be as large as one pleases. To see this consider the following example.

Example 4.6. Let $a > 0$, and let $\alpha_1 = \alpha_2 = 1$ and $\mu_1 = \delta_0, \mu_2 = \delta_a$. Then, in view of Example 3.5, one has

$$\mu = \mu_1 \otimes \delta_{0,1} \mu_2 = a^{-1} \chi_{[0,a]}.$$

Moreover,

$$\|\mu\|_1 = \frac{1}{a} \int_0^a \frac{dt}{1+t} = \frac{\log(1+a)}{a},$$

and for any $\alpha > 0$,

$$\|\mu_1\|_\alpha = 1 \quad \text{and} \quad \|\mu_2\|_\alpha = \frac{1}{(1+a)^\alpha}.$$

Thus, if $\delta \in (0,1)$, then there is $C(\delta) > 0$ such that

$$\|\mu\|_1 \leq C(\delta) \|\mu_1\|_{1-\delta} \|\mu_2\|_\delta,$$

hence

$$\frac{\log(1+a)}{a} = \frac{C(\delta)}{(1+a)^\delta}.$$

Letting now $\delta \to 1$, we infer that

$$\lim_{\delta \to 1} C(\delta) = \infty.$$

The constants given in Theorem 4.3 are not optimal and can be improved in several situations of interest. In particular, using (4.10), one can prove that if $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = \beta \in (0,1)$, and $\mu_1, \mu_2$ and $\mu$ are as in Theorem 4.3 then $\|\mu\|_{2\beta} \leq \|\mu_1\|_\beta \|\mu_2\|_\beta$ for every $\beta \in (0,1)$. Since, in this paper, we are not interested in the best constants, we omit a discussion of further details.

4.2. The boundary case: one of the numbers $\beta_1$ and $\beta_2$ equals zero.

In this subsection we show that in the boundary case when $\beta_1 \in (0, \alpha_1]$ and $\beta_2 = 0$ the inequality (4.13) holds up to small perturbations of $\alpha_1$. (The case when $\beta_2 \in (0, \alpha_2]$ and $\beta_1 = 0$ is clearly analogous.) On the other hand, we prove that our bounds are optimal with respect to $\beta_1$ and $\beta_2$, and in this case (4.13) does not, in general, hold. For $\mu \in \mathcal{M}_\alpha(\mathbb{R}_+), \alpha > 0$, define

$$\|\mu\|_{\alpha, \log} := \int_0^\infty \frac{\log(t+e)}{(1+t)^\alpha} |\mu|(dt),$$

(4.17)
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where the integral can be infinite.

**Theorem 4.7.** For $\alpha_1, \alpha_2 > 0$ let $\mu_1 \in \mathcal{M}_{\alpha_1}(\mathbb{R}_+)$, $\mu_2 \in \mathcal{M}_0(\mathbb{R}_+)$, and $\mu = \mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2$.

(i) If $\psi : [0, \infty) \to [1, \infty)$ is an increasing function satisfying

\[ B_{\psi} := \int_1^\infty \frac{dt}{t \psi(t)} < \infty, \]

then there is $C = C(\alpha_1, \alpha_2, \psi) > 0$ such that

\[ \int_0^\infty \frac{|\mu| (dt)}{\psi(t)(1 + t)^{\alpha_1}} \leq C \|\mu_1\|_{\alpha_1} \|\mu_2\|_0. \]

(ii) If $\|\mu_1\|_{\alpha_1, \log} < \infty$, then

\[ \|\mu\|_{\alpha_1} \leq \frac{\alpha_2^{-1} + \tilde{c}_{\alpha_1}}{B(\alpha_1, \alpha_2)} \|\mu_1\|_{\alpha_1, \log} \|\mu_2\|_0, \]

with $\tilde{c}_{\alpha_1}$ as in Lemma 4.1.

**Proof.** Taking into account (3.8), without loss of generality, we may suppose that $\mu_1$ and $\mu_2$ are positive, so that $\mu$ is positive as well.

To prove (i), note that if $\alpha > \nu > 0$ and $t \geq 2$, then

\[ \int_1^\infty \frac{s^{\nu-1}}{\psi(s)(s + t)^\alpha} ds \geq \frac{1}{\psi(t)} \int_t^{1/2} \frac{s^{\nu-1}}{(s + t)^\alpha} ds = \frac{1}{\psi(t)(1 + t)^{\alpha - \nu}} \int_1^{1/2} \frac{s^{\nu-1}}{(s + 1)^\alpha} ds. \]

So, there exists $D_{\nu, \alpha} > 0$ such that

\[ \int_1^\infty \frac{s^{\nu-1}}{\psi(s)(s + t)^\alpha} ds \geq \frac{D_{\nu, \alpha}}{\psi(t)(1 + t)^{\alpha - \nu}}, \quad t > 0. \]

So, using (4.21), (4.18) and Fubini’s theorem, we infer that

\[ D_{\alpha_2, \alpha_1 + \alpha_2} \int_0^\infty \frac{\mu(dt)}{\psi(t)(1 + t)^{\alpha_1}} \leq \int_0^\infty \int_1^\infty \frac{s^{\alpha_2-1}}{\psi(s)(s + t)^{\alpha_1 + \alpha_2}} \mu(dt) \]

\[ = \int_1^\infty \int_0^\infty \frac{\mu_1(dt)}{s \psi(s)(s + t)^{\alpha_1}} \int_0^\infty \frac{s^{\alpha_2-1} \mu_2(dt)}{(s + t)^{\alpha_2}} ds \]

\[ \leq \|\mu_2\|_0 \int_0^\infty \int_1^\infty \frac{ds}{s \psi(s)(1 + t)^{\alpha_1}} \mu_1(dt) \]

\[ = B_{\psi} \|\mu_1\|_{\alpha_1} \|\mu_2\|_0, \]

hence (4.19) is true.

Let us now prove (ii). Recall that

\[ \int_0^\infty \frac{s^{\alpha_2 - 1}}{(s + t + 1)^{\alpha_1 + \alpha_2}} ds = B(\alpha_1, \alpha_2) \frac{(t + 1)^{\alpha_2}}{(t + 1)^{\alpha_2}}, \quad t > 0, \]

see e.g. [59, p. 298, no.24].
If (4.17) holds, then by (4.22), (4.3) and Fubini’s theorem we have
\[ B(\alpha_1, \alpha_2)\|\mu\|_{\alpha_1} = \int_0^\infty \int_0^{\infty} \frac{s^{\alpha_2-1}ds}{(s + t + 1)^{\alpha_1 + \alpha_2}} \mu(dt) \]
\[ \leq \int_0^1 s^{\alpha_2-1}ds \int_0^\infty \frac{\mu(dt)}{(t + 1)^{\alpha_1 + \alpha_2}} \]
\[ + \int_1^\infty s^{-1} \int_0^\infty \frac{\mu_1(dt)}{(s + t + 1)^{\alpha_1}} \int_0^\infty \frac{s^{\alpha_2}\mu_2(dt)}{(s + t + 1)^{\alpha_2}} ds \]
\[ \leq \alpha_2^{-1}\|\mu_1\|_{\alpha_1}\|\mu_2\|_{\alpha_2} + \|\mu_2\|_0 \int_0^\infty F_{\alpha_1}(t)\mu_1(dt) \]
\[ \leq (\alpha_2^{-1} + c_{\alpha_1})\|\mu_1\|_{\alpha_1, \log}\|\mu_2\|_0, \]
where \(F_{\alpha_1}\) is given by (4.2), hence (4.20) follows.

Thus we arrive at the immediate consequence of Theorem 4.7. Since only one of the parameters \(\beta_1\) and \(\beta_2\) is non-zero, we discard the \((\beta_1, \beta_2)\)-notation and use instead a parameter \(\beta\).

**Corollary 4.8.** Let \(\alpha_j > 0, j = 1, 2\). If \(\mu_1 \in \mathcal{M}_{\alpha_1}(\mathbb{R}_+), \mu_2 \in \mathcal{M}_{0}(\mathbb{R}_+),\) and \(\mu = \mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2\), then there exists \(C = C(\alpha_1, \alpha_2) > 0\) such that
\[ (4.23) \quad \|\mu\|_{\beta} \leq \begin{cases} C\|\mu_1\|_{\alpha_1}\|\mu_2\|_0, & \text{if } \beta > \alpha_1, \\ C\|\mu_1\|_{\beta}\|\mu_2\|_0, & \text{if } 0 < \beta < \alpha_1 \text{ and } \|\mu_1\|_{\beta} < \infty. \end{cases} \]

It appears that (4.23) is close to be optimal with respect to the size of \(\mu_1\), and one cannot in general replace \(\|\mu_1\|_{\alpha_1, \log}\) by \(\|\mu\|_{\alpha_1}\).

**Theorem 4.9.** Let \(\mu_j \in \mathcal{M}_{\alpha_j}^+(\mathbb{R}_+), \alpha_j > 0, j = 1, 2,\) and \(\mu = \mu_1 \otimes_{\alpha_1, \alpha_2} \mu_2\). Then
\[ (4.24) \quad \|\mu_1\|_{\alpha_1, \log}\|\mu_2\|_{\alpha_2} \leq \frac{2^{\alpha_1 + \alpha_2}\|\mu\|_{\alpha_1}}{c_{\alpha_1} B(\alpha_1, \alpha_2)}, \]
where \(c_{\alpha_1}\) is as in Lemma 4.7.

**Proof.** By (4.3), noting that
\[ \frac{s}{s + t + 1} \geq \frac{1}{2(t + 1)}, \quad s \geq 1, \quad t > 0, \]
and using (4.22), we have
\[ B(\alpha_1, \alpha_2)\|\mu\|_{\alpha_1} \geq \int_0^\infty \int_0^{\infty} \frac{s^{\alpha_2-1}ds}{(s + t + 1)^{\alpha_1 + \alpha_2}} \mu(dt) \]
\[ \geq \int_1^\infty s^{-1} \int_0^\infty \frac{\mu_1(dt)}{(s + t + 1)^{\alpha_1}} \int_0^\infty \frac{s^{\alpha_2}\mu_2(dt)}{(s + t + 1)^{\alpha_2}} ds \]
\[ \geq \|\mu_2\|_{\alpha_2} \int_0^\infty F_{\alpha_1}(t)\mu_1(dt) \geq \frac{c_{\alpha_1}}{2^{\alpha_1 + \alpha_2}}\|\mu_1\|_{\alpha_1, \log}\|\mu_2\|_{\alpha_2}, \]
i.e. (4.24) holds. \(\square\)
The following corollary of Theorem 4.9 is straightforward.

**Corollary 4.10.** Under the assumptions of Theorem 4.9

\[(4.25)\]

\[\|\mu\|_{\alpha_1} < \infty \implies \|\mu_1\|_{\alpha_1, \log} < \infty,\]

and the implication in (4.25) becomes equivalence if \(\mu_2 \in \mathcal{M}_1^+(\mathbb{R}_+)\).

For a concrete example showing that (4.23) may fail if \(\beta > \alpha\), consider, for instance,

\[\mu_1(dt) = \frac{(1+t)^{\alpha_1-1}}{\log^2(t+e)} dt \quad \text{and} \quad \mu_2(dt) = \delta_0(dt).\]

**Remark 4.11.** In the context of Sections 3 and 4 let us consider the products \(f_1f_2\), where

\[f_j = a_j + S_{\alpha_j}[\mu_j], \quad a_j \in \mathbb{C}, \quad \alpha_j > 0, \quad \text{and} \quad \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}_+), \quad j = 1, 2.\]

Note that if \(a_j \geq 0\) and \(\mu_j \in \mathcal{M}_{\alpha_j}^+(\mathbb{R}_+), \quad j = 1, 2\), then \(f_1\) and \(f_2\) are generalized Stieltjes.

Recall that by e.g. Theorem 3 if \(\beta > \alpha > 0\) and \(\nu \in \mathcal{M}_\alpha^+(\mathbb{R}_+)\), then \(S_\alpha[\nu] = S_\beta[\mu]\) for \(\mu \in \mathcal{M}_\beta^+(\mathbb{R}_+)\) defined by

\[(4.26)\]

\[\mu(dt) = (B(\alpha, \beta - \alpha))^{-1} \int_{[0,t]} \frac{\nu(ds)}{(t-s)^{\alpha+1-\beta}} dt.\]

This fact generalizes to the case when \(\nu \in \mathcal{M}_\alpha(\mathbb{R}_+)\) and \(\mu \in \mathcal{M}_\beta(\mathbb{R}_+)\) by decomposing \(\nu\) and \(\mu\) into linear combinations of four summands from \(\mathcal{M}_\alpha^+(\mathbb{R}_+)\) and \(\mathcal{M}_\beta^+(\mathbb{R}_+)\), respectively. So, employing (4.26) in such a general setting, we have

\[(4.27)\]

\[B(\alpha_1, \alpha_2)(f_1(z)f_2(z) - S_{\alpha_1}[\mu_1](z)S_{\alpha_2}[\mu_2](z)) = a_1a_2B(\alpha_1, \alpha_2)\]

\[+ a_1 \int_0^\infty \left( \int_{[0,\tau]} (\tau - t)^{\alpha_1-1} \mu_2(dt) \right) \left( \frac{d\tau}{(z + \tau)^{\alpha_1+\alpha_2}} \right)\]

\[+ a_2 \int_0^\infty \left( \int_{[0,\tau]} (\tau - t)^{\alpha_2-1} \mu_1(dt) \right) \left( \frac{d\tau}{(z + \tau)^{\alpha_1+\alpha_2}} \right)\]

\[= S_{\alpha_1+\alpha_2}[u](z), \quad z > 0,\]

where \(u\) is given by

\[B(\alpha_1, \alpha_2)u(\tau) = a_1 \int_{[0,\tau]} (\tau - t)^{\alpha_1-1} \mu_2(dt) + a_2 \int_{[0,\tau]} (\tau - t)^{\alpha_2-1} \mu_1(dt),\]

and \(u \in L_{\alpha_1+\alpha_2}^1(\mathbb{R}_+)\) by Fubini’s theorem. By Theorem 3, \(S_{\alpha_1}[\mu_1]S_{\alpha_2}[\mu_2] = \mu_{\alpha_1+\alpha_2}[\mu]\) with \(\mu = \mu_{\alpha_1+\alpha_2}[\mu]\), so

\[(4.28)\]

\[f_1f_2 = a_1a_2 + S_{\alpha_1+\alpha_2}[u] + S_{\alpha_1+\alpha_2}[\mu] = a_1a_2 + S_{\alpha_1+\alpha_2}[u + \mu].\]

The term \(S_{\alpha_1+\alpha_2}[u]\) is much simpler than \(S_{\alpha_1+\alpha_2}[\mu]\). So the analogues of statements obtained in Sections 3 and 4 can be formulated for \(S_{\alpha_1+\alpha_2}[\mu+u]\) as well, covering, in particular, the situation when \(f_1\) and \(f_2\) are generalized
Stieltjes functions. However, to not obscure our presentation with technical
details, we have decided to consider only the case \( a_1 = a_2 = 0 \) and to deal
solely with the generalized Stieltjes transforms.

5. THE STIELTJES CONVOLUTION, HILBERT TRANSFORM AND TRICOMI’S
IDENTITY

While our studies in Sections 3 and 4 concerned the Stieltjes convolution
on \( \mathcal{M}_{\alpha_1}(\mathbb{R}^+) \times \mathcal{M}_{\alpha_2}(\mathbb{R}^+) \) for arbitrary \( \alpha_1, \alpha_2 > 0 \), they have several cu-
rious applications in the setting of classical Stieltjes transforms, i.e. when
\( \alpha_1 = \alpha_2 = 1 \) and \( \mu_j(dt) = f_j(t) \, dt \) with \( f_j \in L^1_1(\mathbb{R}^+) \), \( j = 1, 2 \). In this section,
we will investigate this situation in some more details and reveal its close
relations to Hilbert transforms and Tricomi’s identity. These considerations
will be based on Theorem 3.1 yielding the explicit expression (3.4) for the
Stieltjes convolution of \( \mu_1 \) and \( \mu_2 \). Note that there is an illuminating prob-
abilistic interpretation of \( \mu_1 \otimes_{1,1} \mu_2 \), \( \mu_1, \mu_2 \in \mathcal{M}_{1}^1(\mathbb{R}^+) \), as the distribution
of a random variable uniformly distributed between two independent ran-
dom variables, see [71]. See also [38] for a similar interpretation and [68] for
additional references.

Thus we deal with the classical Stieltjes transform \( S_1 \) defined on \( L^1_1(\mathbb{R}^+) \)
by
\[
S_1[g](z) := \int_0^\infty \frac{g(t) \, dt}{z + t}, \quad g \in L^1_1(\mathbb{R}^+), \quad z \in \mathbb{C} \setminus (-\infty, 0].
\]
Given \( g_1, g_2 \in L^1_1(\mathbb{R}^+) \), it is of interest to know when \( S_1[g_1]S_1[g_2] \) is again
of the form \( S_1[g] \) for some \( g \in L^1_1(\mathbb{R}^+) \). Using Theorem 3.1 and the Stieltjes
convolution \( g_1 \otimes_{1,1} g_2 \), we proceed with providing a partial answer to this
question in terms of Hilbert transforms of \( g_1 \) and \( g_2 \). To this aim, note that
if \( g_1, g_2 \in L^1_1(\mathbb{R}^+) \), then Theorem 3.1 and Remark 3.2 imply that \( g_1 \otimes_{1,1} g_2 \in L^1_2(\mathbb{R}^+) \) is given by

\[
(5.1) \quad [g_1 \otimes_{1,1} g_2](\tau) = \int_0^\tau \int_0^\infty \frac{g_2(s) \, ds}{s - t} \, g_1(t) \, dt + \int_0^\tau \int_0^\infty \frac{g_1(s) \, ds}{s - t} \, g_2(t) \, dt,
\]
for almost every \( \tau \in \mathbb{R}^+ \), and

\[
\|g_1 \otimes_{1,1} g_2\|_{L^1_2(\mathbb{R}^+)} \leq \|g_1\|_{L^1_1(\mathbb{R}^+)} \|g_2\|_{L^1_1(\mathbb{R}^+)}.\]

The identity (5.1) suggests to employ Hilbert transforms. Recall that for
g \( \in L^1_1(\mathbb{R}) \) its Hilbert transform \( H[g] \) is defined by

\[
(5.2) \quad H[g](\tau) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t) \, dt}{t - \tau} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|t - \tau| > \epsilon} \frac{g(t) \, dt}{t - \tau},
\]
where the limit exists for almost all \( \tau > 0 \) by e.g. [28], p. 348-349]. We will
need a classical boundedness property of the Hilbert transforms on weighted
\( L^p \)-spaces saying that for \( p \in (1, \infty) \) and \( \alpha \in [0, 1) \) there exists \( c_{p, \alpha} > 0 \) such that

\[
(5.3) \quad \|H[g]\|_{L^p_\alpha(\mathbb{R})} \leq c_{p, \alpha} \|g\|_{L^p_\alpha(\mathbb{R})}, \quad g \in L^p_\alpha(\mathbb{R}),
\]
see e.g. [26, Example 9.1.7 and Theorem 9.4.6]. (Alternatively, one may consult [44, Theorem 3] for a direct argument.)

**Theorem 5.1.** Let \( g_j \in L_{\beta_j}^{p_j}(\mathbb{R}_+) \), with

\[
(5.4) \quad p_j \in (1, \infty), \quad \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \beta_j \in [0, 1), \quad j = 1, 2.
\]

Then \( g_1, g_2, g_2H[g_1], \) and \( g_1H[g_2] \) belong to \( L^1_1(\mathbb{R}_+) \), and

\[
S_1[g_1]S_1[g_2] = S_1[g_2H[g_1] + g_1H[g_2]].
\]

**Proof.** By Hölder’s inequality, one has \( g_j \in L^1_1(\mathbb{R}_+) \), \( j = 1, 2 \). Moreover, in view of (5.3), we have \( g_1H[g_2], g_2H[g_1] \in L^1_1(\mathbb{R}_+) \) as well. Indeed, let \( \gamma \in (0, \infty) \) be such that

\[
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{\gamma} = 1.
\]

Then, by Hölder’s inequality and (5.3),

\[
\|g_1H[g_2]\|_{L^1_1(\mathbb{R}_+)} \leq C_{\gamma} \|g_1\|_{L_{\beta_1}^{p_1}(\mathbb{R}_+)} \|g_2\|_{L_{\beta_2}^{p_2}(\mathbb{R}_+)},
\]

where \( C_{\gamma} = 1 \) if \( \gamma = \infty \), and

\[
C_{\gamma} = \left( \int_0^{\infty} \frac{dt}{(1 + t)^{(1-\beta_1/p_1-\beta_2/p_2)\gamma}} \right)^{1/\gamma} < \infty \quad \text{if} \quad \gamma \in (0, \infty).
\]

The estimate for \( \|g_1H[g_2]\|_{L^1_1(\mathbb{R}_+)} \) is completely analogous, so that if

\[
g := g_2H[g_1] + g_1H[g_2],
\]

then \( g \in L^1_1(\mathbb{R}_+) \).

If \( \tau > 0 \), then taking into account (5.3) and applying Parseval’s identity for Hilbert transforms (see e.g. [43, Eq. (4.176)]) to

\[
f_1 := \chi(0, \tau)g_1 \in L^p(\mathbb{R}_+), \quad p := p_1,
\]

and

\[
f_2 := \chi(0, \tau)g_2 \in L^q(\mathbb{R}_+), \quad q := \frac{p_1}{p_1 - 1},
\]

we conclude that

\[
(5.6) \quad 0 = \pi \int_0^{\infty} f_1(t)H[f_2](t) \, dt + \pi \int_0^{\infty} H[f_1](t)f_2(t) \, dt
\]

\[
= \int_0^{\tau} \int_0^{\tau} \frac{g_2(s)}{s-t} g_1(t) \, dt + \int_0^{\tau} \int_0^{\tau} \frac{g_1(s)}{s-t} g_2(t) \, dt,
\]

where the inner integrals are understood in the principal value sense. Next, adding the right-hand sides of (5.1) and (5.6), we can write

\[
(5.7) \quad (g_1 \ast_{1,1} g_2)(\tau) = \int_0^{\tau} g(t) \, dt
\]

for almost every \( \tau > 0 \). Since \( g \in L^1_1(\mathbb{R}_+) \), we have

\[
(5.8) \quad \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} g(t) \, dt = 0.
\]
Thus, employing (5.7), integrating by parts and taking into account (5.8), we infer that
\[
S_1[g_1](z)S_1[g_2](z) = -\int_0^\infty (g_1 \otimes_{1,1} g_2)(\tau) d(\tau + \tau)^{-1} = S_1[g](z)
\]
for all \(z > 0\), i.e. (5.5) holds. \(\square\)

Observe that \(g_2 H[g_1] + g_1 H[g_2]\) may, in general, change its sign even if \(g_j \geq 0\), \(j = 1, 2\).

**Remark 5.2.** It seems that the identity (5.5) appeared for the first time in [27, §15] (see also [73]), where it was proved for a certain class of functions associated with a Mellin transform. Later the result was extended to \(L^p\)-spaces in [69], where it was assumed that
\[
g_j \in L^{p_j}(\mathbb{R}^+), \quad p_j \in (1, \infty), \quad j = 1, 2, \quad \frac{1}{p_1} + \frac{1}{p_2} < 1.
\]

Theorem 5.1 is a substantial generalization of this result based on a direct use of the convolution formula (3.3) and (3.4). Note that the term "Stieltjes convolution" was employed in the literature to denote \(g_1 \otimes_{1,1} g_2\) rather than \(g_1 H[g_2] + g_2 H[g_1]\). In view of the material presented in Sections 3 and 4, we believe that our terminology is more natural and revealing.

The appearance of Hilbert transforms in the right-hand side of (5.5) suggests to look for a version of (5.5) concerning functions defined on the whole real line. On this way, we show that the convolution formula (5.5) for \(g_j \in L^{p_j}(\mathbb{R}^+), \quad p_j \in (1, \infty), \quad j = 1, 2, \quad 1/p_1 + 1/p_2 \leq 1\), is equivalent to the next well-known version of Poincaré-Bertrand’s formula:
\[
(5.9) \quad H[f_1](t)H[f_2](t) - f_1(t)f_2(t) = H[f_1 H[f_2] + f_2 H[f_1]](t)
\]
for \(f_1 \in L^{p_1}(\mathbb{R})\) and \(f_2 \in L^{p_2}(\mathbb{R})\) and almost all \(t \in \mathbb{R}\). See [43] Chapters 2.13, 4.16 and 4.23 and [57] for a general discussion of (5.9) and related identities. Apparently, the identity (5.9) was first proved by Tricomi for the case \(1/p_1 + 1/p_2 < 1\), see [70, Theorem IV]. It was then extended in [16, Theorem 5.1] and [49, Theorem] to include a more general assumption \(1/p_1 + 1/p_2 \leq 1\). The approach in the second paper was rather technical, while the first paper offered a comparatively simple proof and gave several interesting consequences. As a byproduct of our technique, we prove a slightly more general version of (5.9) valid under the assumptions of Theorem 5.1.

To prove the equivalence of Theorem 5.1 and (5.9) for \(f_j \in L^{p_j}_{\beta_j}(\mathbb{R})\), with \(\beta_j\) and \(p_j\) satisfying (5.4), \(j = 1, 2\), we first note that one implication is rather direct. Indeed, assume that (5.9) holds for \(f_j \in L^{p_j}_{\beta_j}(\mathbb{R}), j = 1, 2\). By restricting (5.9) to \(t \in (-\infty, 0)\), replacing in this identity \(f_1\) with \(f_1 \chi_{\mathbb{R}^+}\), \(f_2\) with \(f_2 \chi_{\mathbb{R}^+}\), and setting \(t = -s\), we obtain (5.5). So, in fact, the result in [69] is an easy corollary of a version of Tricomi’s identity.
We proceed with deriving the other implication. To this aim, recall the classical Sokhotski-Plemelj jump formula for Hilbert transforms:

\( \lim_{\epsilon \to +0} \frac{1}{\pi} S_1[f](t \mp i\epsilon) = \pm i f(t) + H[f](t), \quad \text{a.e. } t > 0. \) \hfill (5.10)

for all \( f \in L^p(\mathbb{R}^+_+), p \in (1, \infty) \), see e.g. [28, p. 348-349] or [60, Theorem 5.30].

**Theorem 5.3.** Let \( f_j \in L^p_{\beta_j}(\mathbb{R}), j = 1, 2, \) be such that (5.4) holds. Then \( f_1, f_2, f_2 H[f_1], \) and \( f_1 H[f_2] \) belong to \( L^1_1(\mathbb{R}) \), and \( f_1 \) and \( f_2 \) satisfy the Tricomi identity (5.9).

**Proof.** As in the proof of Theorem 5.1, the fact that \( f_1, f_2 \) and \( f_2 H[f_1] + f_1 H[f_2] \) are in \( L^1_1(\mathbb{R}) \) is a direct corollary of (5.3) and Hölder’s inequality.

Thus, the main issue is to transfer Theorem 5.1 to the setting of the whole of \( \mathbb{R} \). This will be done by shifting \( f_1 \) and \( f_2 \) backwards and observing that via density arguments it suffices to prove (5.9) for \( f_1 \) and \( f_2 \) with compact support. Let \( a \in (0, \infty) \) be fixed and \( \text{supp} (f_j) \subset [-a, a], j = 1, 2. \) Define

\[ f_{j,a}(t) := f_j(t-a), \quad \text{a.e. } t > 0, \quad j = 1, 2. \]

Then, by Theorem 5.1 we have

\[ \frac{1}{\pi} S_1[f_{1,a}](z) \frac{1}{\pi} S_1[f_{2,a}](z) = \frac{1}{\pi} S_1\left[ f_{1,a} H[f_{2,a}] + f_{2,a} H[f_{1,a}] \right](z) \] \hfill (5.11)

for all \( z \in \mathbb{C} \setminus (-\infty, 0] \). In particular, setting \( z = t, t > 0 \), in (5.11), we obtain

\[ H[f_{1,a}](t) H[f_{2,a}](t) = H[ f_{1,a} H[f_{2,a}] + f_{2,a} H[f_{1,a}] ](-t). \]

This means that (5.9) holds for \( f_1 \) and \( f_2 \) and \( t < -a \).

Next, setting \( z = -t - i\epsilon \) with \( t, \epsilon > 0 \) in (5.11), letting \( \epsilon \to 0 \), and taking into account (5.11), we conclude that

\[ (i f_{1,a}(t) + H[f_{1,a}](t))(i f_{2,a}(t) + H[f_{2,a}](t)) \]
\[ = i f_{1,a}(t) H[f_{2,a}](t) + i f_{2,a}(t) H[f_{1,a}](t) \]
\[ + H[ f_{1,a} H[f_{2,a}] + f_{2,a} H[f_{1,a}] ](t) \] \hfill (5.12)

for almost all \( t > 0 \). Similarly, applying the same argument for \( z = -t + i\epsilon \), we have

\[ (-i f_{1,a}(t) + H[f_{1,a}](t))(-i f_{2,a}(t) + H[f_{2,a}](t)) \]
\[ = - i f_{1,a}(t) H[f_{2,a}](t) - i f_{2,a}(t) H[f_{1,a}](t) \]
\[ + H[ f_{1,a} H[f_{2,a}] + f_{2,a} H[f_{1,a}] ](t), \] \hfill (5.13)

for almost all \( t > 0 \).

So, summing up (5.12) and (5.13), and returning to \( f_1 \) and \( f_2 \), we infer that for almost every \( t > -a \) :

\[ H[f_1](t) H[f_2](t) - f_1(t) f_2(t) = H[ f_1 H[f_2] + f_2 H[f_1] ](t). \] \hfill (5.14)

Since the choice of \( a \) was arbitrary, this implies the claim. \( \square \)
Note that quite an interesting application of Tricomi’s identity to the study of nonlinear PDE was found recently in [23].

6. Representability problem for generalized Stieltjes transforms

As we discussed in the introduction, finding a convenient criterion for representability of a function as a generalized Stieltjes transform of order \( \alpha > 0, \alpha \neq 1 \), is a difficult and challenging problem. In this section, among other things, we show that if \( f \in S_\alpha \) and \( f_\beta := f(z^\beta), \beta \in (0, 1] \), then \( f_\beta \in S_{\alpha\beta} \), so that \( f_\beta \in S_\beta \) if \( f \in S_1 \). Since the class \( S_1 \) is well-understood, our results lead to a transparent description of a substantial subset of \( S_\alpha \), and clarify the representability problem to some extent.

Our construction will be realized in two ways. One of them will be based on an abstract argument and the other will rely on a series of explicit transformations, thus yielding a bit more eventually. On the other hand, the ways are not independent, and the second one uses an intuition provided by the first, soft approach. In these studies, Theorem 3.1 will play a substantial role.

Recall from (1.3) that \( S_1 = \mathbb{R}_+ + S_1(M_1^+_{1}(\mathbb{R}_+)) \), and that \( S_1 \) can be neatly characterized by Theorem 1.1. It will useful to note that if \( f \in S_1, f \sim (a, \mu)_1 \), then by the monotone convergence theorem,

\[
\exists \lim_{s \to \infty} sf(s) = \begin{cases} 
\mu(\mathbb{R}_+), & \text{if } a = 0 \text{ and } \mu \text{ is finite,} \\
\infty, & \text{otherwise.}
\end{cases}
\]

The class \( S_1 \) is closely related to the class of so-called complete Bernstein functions, denoted by \( CBF \). One may define \( f \in CBF \) as \( f : (0, \infty) \to [0, \infty) \) such that \( s^{-1}f \in S_1 \), so that

\[
CBF := \{as + sS_1[\mu](s) : a \geq 0, \mu \in M_1^+_{1}(\mathbb{R}_+)\}.
\]

Recall that \( f \in CBF, f \neq 0 \), if and only if \( 1/f \in S_1 \), see [62, Theorem 7.3]. If \( f \in CBF \) or \( f \in S_1 \), then \( f \) extends analytically to \( \mathbb{C} \setminus (-\infty, 0] \), and we then identify \( f \) with its analytic extension. Moreover, if \( f \in CBF \), then \( f \) extends continuously to \( \mathbb{C} \setminus (-\infty, 0) \).

Our arguments will rely on specific properties of the subclass of \( CBF \) consisting of Thorin-Bernstein functions \( TBF \). Recall that \( TBF \) can be described by the right-hand side of (6.2) with \( \mu \), being in addition, absolutely continuous on \( (0, \infty) \) such that \( \mu(dt) = t^{-1}w(t)dt \), and \( w \) is non-decreasing on \( (0, \infty) \), see [62, Theorem 8.2, (v)]. The next two statements on Thorin-Bernstein functions can be found in [62, Theorem 8.2, (iii)] and [62, Proposition 8.7]. For short-hand, if \( f : (0, \infty) \to (0, \infty) \) is differentiable, then we let

\[
\Psi[f](s) := -(\log f)^\prime(s) = \frac{f'(s)}{f(s)}, \quad s > 0.
\]
Note that if $\Psi[f](s) \geq 0$ for all $s > 0$, and $\Psi[f] \in C(0, \infty)$, then $f' \in C(0, \infty)$ and $f'(s) \leq 0$ for every $s > 0$, so that there exists (possibly infinite) $f(0+) = \lim_{s \to 0} f(s)$.

**Theorem 6.1.**

(i) If $g : (0, \infty) \to (0, \infty)$, then $g \in TBF$ if and only if $g$ is of the form

$$g(s) = a + bs + \int_{(0, \infty)} \log \left(1 + \frac{s}{t} \right) \nu(dt), \quad s > 0,$$

where $a, b \geq 0$ and $\nu$ is a unique positive Borel measure on $(0, \infty)$ satisfying

$$\int_{(0, 1)} |\log t| \nu(dt) + \int_{1}^{\infty} \frac{\nu(dt)}{t} < \infty.$$

(ii) If $f : (0, \infty) \to (0, \infty)$ is differentiable, then $f = e^{-g}$ with $g \in TBF$ if and only $\Psi[f] \in S_1$ and $f(0+) \leq 1$.

See [62, Chapter 8] for more information on the properties of $TBF$ and relevance of $TBF$ in probability theory.

We start with a general result describing a class of functions $f$ on $(0, \infty)$ such that for an appropriate $\gamma > 0$ one has $f^\alpha \in S_{\gamma \alpha}$ for all $\alpha > 0$. Its proof is based on an approximation trick involving the Krein-Milman theorem and used already in the literature, see e.g. the proofs of [24, Theorem 3.2] or [14, Theorem 2]. We will need the following classical fact, separated for ease of reference.

**Lemma 6.2.** Let $(f_n)_{n \geq 1} \subset S_\alpha$, $\alpha > 0$. If for every $s > 0$ there exists $f(s) := \lim_{n \to \infty} f_n(s)$, then $f \in S_\alpha$, and moreover $\lim_{n \to \infty} f_n^{(k)}(s) = f^{(k)}(s)$, $s > 0$, for every $k \in \mathbb{N}$.

The proof of $f \in S_\alpha$ is analogous to the proof of [62, Theorem 2.2, (iii)], where $\alpha = 1$, and is therefore omitted. The last claim follows either from [62, Corollary 1.7] or by adjusting the proof of [62, Theorem 2.2, (iii)] accordingly.

**Theorem 6.3.** Let $f : (0, \infty) \to (0, \infty)$ be differentiable. Assume that

$$\Psi[f] \in S_1$$

and

$$\gamma := \lim_{s \to \infty} s\Psi[f](s) < \infty.$$

Then for any $\alpha > 0$,

$$f^\alpha \in S_{\gamma \alpha}.$$

Moreover, if $f^\alpha \sim (a, \mu)_{\gamma \alpha}$, then $\mu$ does not have singular continuous part.

**Proof.** We first prove (6.8). Let $\alpha > 0$ and $\epsilon > 0$ be fixed, and

$$f_\epsilon(s) := f(s + \epsilon), \quad s > 0.$$
Then, by Theorem 6.1(ii), there exists $g = g_\varepsilon \in TBF$ such that

$$f_\varepsilon(s) = f_\varepsilon(0)e^{-g(s)}, \quad s \geq 0,$$

and, in view of Theorem 6.1(i), $g$ is of the form (6.4) with $a = a_\varepsilon = 0$, $b = b_\varepsilon \geq 0$, and $\nu = \nu_\varepsilon$ being a positive Borel measure on $(0, \infty)$ satisfying (6.5). So, $\Psi[f_\varepsilon] \in S_1$ and

$$\Psi[f_\varepsilon](s) = g'(s) = b + \int_{(0, \infty)} \frac{\nu(dt)}{s + t},$$

for every $s > 0$. Hence, by (6.1), $b = 0$ and

$$\int_{(0, \infty)} \nu(dt) = \gamma.$$

We have

$$f_\varepsilon(s) = f_\varepsilon(0) \exp \left( -\int_{[0, \infty)} \log \left( 1 + \frac{s}{t} \right) \nu(dt) \right), \quad s > 0.$$

Fix $n \in \mathbb{N}$, let

$$f_{\varepsilon,n}(s) := \exp \left( -\int_{[1/n, \infty)} \log \left( 1 + \frac{s}{t} \right) \nu(dt) \right), \quad s > 0,$$

and note that $\gamma > 0$ by (6.1). Using the Krein-Milman theorem (see e.g. [63, Theorem 8.14]) and (6.11), we can approximate the restriction of $\gamma^{-1} \nu$ to $[1/n, n]$ by finite convex combinations of delta measures in the weak* topology of $(C([1/n, n]))^*$, so that

$$\int_{1/n}^{n} \log \left( 1 + \frac{s}{t} \right) \nu(dt) = \lim_{j \to \infty} \sum_{i=1}^{N_j} a_{ij} \log \left( 1 + \frac{s}{t_{ij}} \right), \quad s > 0,$$

for some $t_{ij} \subset [1/n, n]$ and $a_{ij} \geq 0$ with $\sum_{i=1}^{N_j} a_{ij} = \gamma$ for all $j \in \mathbb{N}$. Thus, for every $s > 0$,

$$(f_{\varepsilon,n}(s))^\alpha = \lim_{j \to \infty} \prod_{i=1}^{N_j} \left( 1 + \frac{s}{t_{ij}} \right)^{-\alpha a_{ij}}.$$

Setting

$$G_j(s) = \prod_{i=1}^{N_j} \left( 1 + \frac{s}{t_{ij}} \right)^{-\alpha a_{ij}}, \quad j \in \mathbb{N}, \ s > 0,$$

and invoking Theorem 3.1 (or the less explicit result by Hirschmann-Widder [33], mentioned in the introduction), we have $G_j \in S_{\gamma^\alpha}$ for all $j \in \mathbb{N}$. Hence Lemma 6.2 implies that $(f_{\varepsilon,n})^\alpha \in S_{\gamma^\alpha}$ as well.

Now letting $n \to \infty$ and using the monotone convergence theorem, we infer that

$$(f_{\varepsilon}(s))^\alpha = \lim_{n \to \infty} (f_{\varepsilon,n}(s))^\alpha, \quad s > 0,$$
so by Lemma 6.2 again,

\[(f_\varepsilon)^\alpha \in S_{\gamma \alpha}.
\]

Finally, letting \(\varepsilon \to 0\), we note that

\[
(f(s))^\alpha = \lim_{\varepsilon \to 0} (f(s + \varepsilon))^\alpha = \lim_{\varepsilon \to 0} (f_\varepsilon(s))^\alpha,
\]

for all \(s > 0\). So (6.13) and Lemma 6.2 imply that \(f^\alpha \in S_{\gamma \alpha}\), and the proof of (6.8) is complete.

Let \(f^\alpha \sim (a, \mu)_{\gamma \alpha}\). To prove the last claim, note first that in view of (6.8) one clearly has \(f^{\alpha/2} \in S_{\alpha \gamma/2}\). Applying the representation (4.28) to the product \(f^\alpha\) of the generalized Stieltjes functions \(f^{\alpha/2}\) and \(f^{\alpha/2}\) and using Corollary 3.3 and (4.27), we conclude that \(\mu\) has no singular continuous part.

\[\square\]

Remark 6.4. Observe that by Theorem 6.3 if \(f\) satisfies (6.6) and (6.7) with \(\gamma = 1\), then \(f \in S_1\). However, the opposite implication does not hold, i.e. neither (6.6) nor (6.7) follow from \(f \in S_1\). Indeed, considering first (6.6), let \(f(s) := (\log(s + 1))^{-1}, s > 0\), and observe that \(f \in S_1\) by, for example, Theorem 1.1 (or [62, p.9]). Then

\[
\Psi[f](s) = ((s + 1) \log(s + 1))^{-1},
\]

and \(\Psi[f] \notin S_1\) by (6.1).

Turning now to (6.7), define

\[
f(s) := (s + 1)^{-1} + (s + 2)^{-1}, \quad s > 0.
\]

Then

\[
\Psi[f](s) = (s + 1)^{-1} + (s + 2)^{-1} - 2(2s + 3)^{-1}
\]

and \(\Psi[f] \notin S_1\). One may just note that \(\Psi[f] = S_1[\mu]\) for a signed measure \(\mu \in M_1(\mathbb{R}_+)\), and refer to a uniqueness theorem for Stieltjes transforms.

Thus, by Remark 6.4, the assumption \(f \in S_1\) is strictly weaker than the assumptions in Theorem 6.3 with \(\gamma = 1\), and it is natural to ask whether Theorem 6.3 holds if one only assumes that \(f \in S_1\). More precisely, one can ask whether the following implication is true:

\[(6.14) \quad f \in S_1 \implies f^\alpha \in S_\alpha, \quad \alpha \in (0, 1).\]

It is instructive to note the reformulation of (6.14) via analytic extensions of functions from \(S_1\):

\[g \in S_1, \quad g(\Sigma_\pi) \in \Sigma_{\alpha \gamma} \implies g \in S_\alpha, \quad \alpha \in (0, 1),\]

where \(\Sigma_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}, \theta \in (0, \pi]\). To see that (6.14) and (6.15) are equivalent, let first (6.14) hold. If \(\alpha \in (0, 1), g \in S_1\) and \(g(\Sigma_\pi) \in \Sigma_{\alpha \gamma}\), then by Theorem 1.1 we have \(g^{1/\alpha} \in S_1\), and in view of (6.14),

\[g = (g^{1/\alpha})^\alpha \in S_\alpha,\]
so that (6.15) is true. On the other hand, if (6.15) holds and \( f \in S_1 \), then using Theorem 1.1 we infer that
\[
f^\alpha \in S_1 \quad \text{and} \quad f^\alpha(\Sigma_{\pi}) \in \Sigma_{\pi\alpha},
\]
hence \( f^\alpha \in S_\alpha \) by (6.15), and (6.14) holds as well. Using Theorem 3.1, we show that (6.14), and thus (6.15), fail dramatically.

**Theorem 6.5.** Let \( f \in S_1, f \sim (a, \mu)_1 \), and let \( \mu \) satisfy one of the following two conditions:

(i) \( \mu \) has a nonzero continuous singular part;

(ii) \( \mu \) is purely discrete and its set of atoms contains at least two points.

Then for each \( n \geq 2, n \in \mathbb{N} \), there exists \( \epsilon_n \in (0, 1/n) \) such that
\[
f^\alpha \not\in S_\alpha, \quad |\alpha - 1/n| < \epsilon_n.
\]
In particular, \( f^{1/n} \not\in S_{1/n} \).

**Proof.** Fix \( n \in \mathbb{N}, n \geq 2 \), and, arguing as in the proof of Theorem 6.3, write \( f = (f^{1/n})^n \). If \( f \) satisfies either (i) or (ii), then using either Corollary 3.3 (iii) or Corollary 3.3 (v), respectively, along with (4.27), we conclude that \( f^{1/n} \not\in S_{1/n} \).

Next recall that by [66, Theorem 2], one has \( f \in S_\alpha \) if and only if
\[
(-1)^n \sum_{j=0}^{n} \binom{n+j}{j} \frac{\Gamma(n+k+\alpha)}{\Gamma(n+j+\alpha)} s^j f^{(n+j)}(s) \geq 0, \quad s > 0,
\]
for all integers \( n, k \geq 0 \). Hence, if \( f_m \in S_{\alpha_m}, \ m \geq 1 \), and there exist
\[
\alpha = \lim_{m \to \infty} \alpha_m \quad \text{and} \quad \gamma = \lim_{m \to \infty} f_m(s), \quad s > 0,
\]
then, taking into account Lemma 6.2, we infer that \( g \in S_\alpha \). In other words, if \( f \in S_1 \), then the set \( \{ \alpha \in [0, 1] : f^\alpha \in S_\alpha \} \) is closed. Since for \( n \in \mathbb{N}, n \geq 2 \), we have \( f^{1/n} \not\in S_1 \), it then follows that there exists \( \epsilon_n \in (0, 1/n) \) such that \( f^\alpha \not\in S_\alpha \) for all \( \alpha \) with \( |\alpha - 1/n| < \epsilon_n \), as required. \( \square \)

**Example 6.6.** For fixed \( t \geq 0 \) and \( \beta \in (0, 1] \), define
\[
(6.16) \quad K_{\beta,t}(s) := \frac{1}{s^\beta + t}, \quad s > 0.
\]
We have
\[
\Psi[K_{\beta,t}](s) = \frac{\beta s^{\beta-1}}{s^\beta + t},
\]
and (6.7) holds with \( \gamma = \beta \). Moreover,
\[
\frac{1}{\Psi[K_{\beta,t}]}(s) = \beta^{-1} (s + ts^{1-\beta}) \in \mathcal{CBF},
\]
hence $\Psi [K_{\beta,t}] \in S_1$. So, by Theorem 6.3 for every $\alpha > 0$,

$$(K_{\beta,t})^{\alpha} \in S_{\alpha\beta},$$

and, in particular, $K_{\beta,t} \in S_\beta$. If $K_{\beta,t} \sim (a,\nu)_\beta$, then using

$$\lim_{s \to \infty} s^\beta K_{\beta,t}(s) = 1,$$

and the monotone convergence theorem, we conclude that $a = 0$ and $\nu \in M_0^+(\mathbb{R}_+)$ with $\|\nu\|_0 = 1$.

For $\beta \in (0,1]$ and $f : (0,\infty) \to [0,\infty)$ let

$$(6.17) \quad f_\beta(s) := f(s^\beta), \quad s > 0.$$  

In view of Theorem 1.1 the next result can be considered as a geometric condition describing a substantial subset of $S_\alpha$, and thus filling partially a gap in the theory of generalized Stieltjes functions indicated by Sokal in [66, p. 184-185].

**Theorem 6.7.** Let $f \in S_\alpha$, $\alpha > 0$, and $\beta \in (0,1]$. Then

$$f(\beta) \in S_{\alpha\beta}.$$ 

In particular, $f(\beta) \in S_\beta$ if $f \in S_1$.

**Remark 6.8.** Note for the sequel that similarly to (6.1) if $f \in S_\beta$ and $\lim_{s \to \infty} s^\beta f(s)$ is finite, then by the monotone convergence theorem $f = S_\beta[\mu]$ with $\mu \in M_0^+(\mathbb{R}_+)$. Thus, Theorem 6.7 implies that if $f \in S_1$ is such that $\lim_{s \to \infty} sf(s) < \infty$, then $f(\beta) = S_\beta[\nu]$ with $\nu \in M_0^+(\mathbb{R}_+)$.  

**Proof.** By assumption, $f \sim (a,\nu)_\alpha$, so

$$f(\beta)(s) = a + \int_0^\infty \frac{\nu(dt)}{(s^\beta + t)^\alpha} = a + \int_0^\infty (K_{\beta,t}(s))^{\alpha} \nu(dt), \quad s > 0,$$

where $a \geq 0$ and $\nu$ is a positive Borel measure on $[0,\infty)$ satisfying

$$(6.18) \quad \int_0^\infty \frac{\nu(dt)}{(1+t)^\alpha} < \infty.$$ 

For $n \in \mathbb{N}$ let

$$f_{\beta,n}(s) := a + \int_0^n (K_{\beta,t}(s))^{\alpha} \nu(dt).$$

By Example 6.6 for every fixed $t > 0$ we have $(K_{\beta,t})^{\alpha} \in S_{\alpha\beta}$. So using (6.18) and the Krein-Milman theorem and arguing as in the proof of Theorem 6.3, we infer that $f_{\beta,n} \in S_{\alpha\beta}$. Since by the monotone convergence theorem,

$$f(\beta)(s) = \lim_{n \to \infty} f_{\beta,n}(s), \quad s > 0,$$

we conclude that $f(\beta) \in S_{\alpha\beta}$, and the statement follows. \qed

The next corollary of Theorem 6.7 concerning complex measures is direct.

**Corollary 6.9.** Let $f = a + S_1[\mu]$, where $\mu \in M_0(\mathbb{R}_+), a \in \mathbb{C}$. Then there exists $\nu \in M_0(\mathbb{R}_+)$ such that $f(1/2) = a + S_{1/2}[\nu]$.
To prove the statement it suffices to write \( \mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-) \) with \( \mu_j^\pm \in M_0^+(\mathbb{R}_+), j = 1, 2 \), and to apply Theorem 6.7 to each of the four transforms \( S_1[\mu_j^\pm], j = 1, 2 \).

Corollary 6.9 (along with Remark 6.8) can be used to identify spectral multipliers \( \Phi \) relevant for the main results in [4]. For more on that see Remark 6.16 and Example 6.17.

Remark 6.10. Theorem 6.7 is essentially sharp. If e.g. \( f(s) = s^{-1}, s > 0 \), then \( f(\beta) = s^{-1/\beta} \in S_3 \), and \( f(\beta) \notin S_\alpha \) for any \( \alpha < \beta \). (For the latter claim, see e.g. [41, Corollary 2, p. 63].)

Next, consider

\[
(6.19) \quad f(s) = 2 \int_0^2 \frac{dt}{(t+s)^3} + \int_1^2 \frac{dt}{(t+s)^3}, \quad s > 0,
\]

studied already in [41, Remark 10]. Since \( f \in S_3 \), Theorem 6.7 implies that \( f(1/2) \in S_{3/2} \). In fact, more is true. By a direct computation (see again [41, Example 10]), the analytic extension of \( f \) to \( \mathbb{C}^+ \), denoted by the same symbol, satisfies

\[
\text{Im} f(z) \in \mathbb{C} \setminus \mathbb{C}^+, \quad z \in \mathbb{C}^+.
\]

From this and Theorem 1.1 we infer that \( f(1/2) \in S_1 \subset S_{3/2} \). On the other hand,

\[
f(s) = \frac{1}{(s+1)^2} - \frac{1}{2(s+2)^2},
\]

so \( f \notin S_2 \) since the representing measure for \( f \) is signed.

Thus, in general, for \( \beta > 0 \) and \( \alpha \in (0, 1) \), the implication

\[
f \in S_\beta, \quad f(\alpha) \in S_\gamma, \quad \gamma < \beta \alpha \implies f \in S_{\gamma/\alpha},
\]

is not valid. (For \( f \) given by (6.19) we have \( \beta = 3, \alpha = 1/2, \gamma = 1, \) and \( \gamma/\alpha = 2 \).)

Let us illustrate Theorems 6.3 and 6.7 by the following example.

Example 6.11. Let

\[
f(s) := \frac{\log s}{s-1}, \quad s > 0.
\]

Then

\[
\Psi[f](s) = \frac{1}{s-1} - \frac{1}{s \log s},
\]

for all \( s > 0 \), so the assumption (6.7) holds for \( \gamma = 1 \). Moreover, we have

\[
\Psi[f](s) = \frac{q(1) - q(s)}{s-1}, \quad q(1) = 1,
\]

with \( q(s) = (sf(s))^{-1} = (s-1)(s \log s)^{-1} \). Since \( f \in S_1 \) by [25, Example 2.9], we also have \( 1/f \in CBF \), and then \( q \in S_1 \). Hence \( \Psi[f] \in S_1 \) by 6.2.
Theorem 7.17] (see also [55, Theorem 2]). So, Theorem 6.3 implies that for any \( \alpha > 0 \),

\[
\left( \frac{\log s}{s - 1} \right)^\alpha \in \mathcal{S}_\alpha.
\]

Moreover, by Theorem 6.7 for all \( \beta \in (0,1] \) and \( \alpha > 0 \),

\[
(6.20) \quad \left( \frac{\log s}{s^\beta - 1} \right)^\alpha \in \mathcal{S}_{\beta \alpha}.
\]

Observe that since

\[
\lim_{s \to \infty} f(s) = 0 \quad \text{and} \quad \lim_{s \to \infty} s f(s) = \infty,
\]

one has \( f \sim (0,\nu)_1 \), but the measure \( \nu \) is not finite on \([0,\infty)\) in contrast to the situation considered in Example 6.6.

Now we turn to our second approach to the Stieltjes representability problem leading to an explicit representation of \( f(\beta) \) and being alternative to Theorem 6.7. However, this approach uses an intuition provided by Theorem 6.7, and it is independent of Theorem 6.7 under a thin disguise. If \( f \in \mathcal{S}_1, f \sim (a,\mu)_1 \), then by Theorem 6.7 we have \( f(\beta) \in \mathcal{S}_\beta, f(\beta) \sim (a,\nu)_\beta \), for every \( \beta \in (0,1] \), and it is natural to try to determine \( \nu \) in terms of \( \mu \). This will be the content of Theorem 6.15 below.

We will need the following statement, which is a corollary of [41, Theorem 3], see also [61, Lemma 2.1].

**Lemma 6.12.** Let \( g \in \mathcal{S}_1 \) be such that

\[
(6.21) \quad g(s) = \int_0^\infty \frac{v(t) \, dt}{s + t}, \quad s > 0,
\]

for some positive \( v \in L_1^1(\mathbb{R}_+) \), absolutely continuous on \([0,a]\) for every \( a > 0 \). If moreover \( g \in \mathcal{S}_\beta, \beta \in (0,1) \), then \( v_\beta \) given by

\[
(6.22) \quad v_\beta(t) = \frac{d}{dt} \int_0^t \frac{v(r) \, dr}{(t - r)^{1-\beta}}, \quad a.e \ t > 0,
\]

is well-defined, \( v_\beta \in L_1^1(\mathbb{R}_+), v_\beta \geq 0, \) and

\[
(6.23) \quad g(s) = \int_0^\infty \frac{v_\beta(t) \, dt}{(s + t)^\beta}, \quad s > 0.
\]

**Lemma 6.13.** For \( \beta \in (0,1) \) let \( K_\beta := K_{\beta,1} \) be given by (6.16). Then \( K_\beta \in \mathcal{S}_\beta \), and

\[
(6.24) \quad K_\beta(s) = \int_0^\infty \frac{\varphi_\beta(t) \, dt}{(s + t)^\beta}, \quad s > 0,
\]

where a positive \( \varphi_\beta \in L_1^1(\mathbb{R}_+) \) is defined for all \( t > 0 \) by

\[
(6.25) \quad \varphi_\beta(t) := \frac{2\beta \sin(\pi \beta)}{\pi} \int_0^1 \frac{t^{2\beta-1}(1 + \cos(\pi \beta)t^\beta r^\beta)r^\beta \, dr}{(t^{2\beta}r^{2\beta} + 2\cos(\pi \beta)t^\beta r^\beta + 1)^2(1 - s)^{1-\beta}}.
\]
Proof. Let $\beta \in (0,1)$ be fixed. First, we infer by Example 6.6 that $K_\beta \in S_\beta$, so $K_\beta$ admits the representation (6.24) with a positive $\varphi_\beta \in L^1_1(\mathbb{R}_+)$, and we have only to determine $\varphi_\beta$ explicitly. To this aim, recall that $K_\beta \in S_1$, and

$$(6.26) \quad K_\beta(s) = \int_0^\infty \frac{\psi_\beta(r)}{s + r} dr, \quad s > 0,$$

where

$$\psi_\beta(r) := \frac{\sin(\pi \beta)}{\pi (r^{2\beta} + 2 \cos(\pi \beta) r^\beta + 1)}, \quad r > 0,$$

and $\psi_\beta \in L^1_1(\mathbb{R}_+)$. See e.g. [42, Theorem 2], where (6.26) was deduced in a more general operator context, or [51, Example 4.1.3]. Using now Lemma 6.12, for every $t > 0$, we have

$$\varphi_\beta(t) = \frac{\sin(\pi \beta)}{\pi} \int_0^t \frac{\psi_\beta(r) r^\beta}{(r^{2\beta} + 2 \cos(\pi \beta) r^\beta + 1)(t - r)^{1-\beta}} dr$$

and

$$(t^{2\beta} r^{2\beta} + 2 \cos(\pi \beta) t^{\beta} r^\beta + 1)^2 \frac{d}{dt} \left( \frac{t^{2\beta}}{(t^{2\beta} r^{2\beta} + 2 \cos(\pi \beta) t^{\beta} r^\beta + 1)} \right) = 2\beta t^{2\beta-1} (1 + \cos(\pi \beta) t^{\beta} r^\beta).$$

Thus

$$\varphi_\beta(t) = \frac{2\beta \sin(\pi \beta)}{\pi} \int_0^t \frac{t^{2\beta-1} (1 + \cos(\pi \beta) t^{\beta} r^\beta) s^\beta dr}{(t^{2\beta} r^{2\beta} + 2 \cos(\pi \beta) t^{\beta} r^\beta + 1)^2 (1 - r)^{1-\beta}}, \quad t > 0,$$

so (6.24) and (6.25) hold. \qed

It is crucial to note that to write down (6.24) we used the fact that $K_\beta \in S_\beta$ proved in Example 6.6. Thus, our argument is not independent of Example 6.6. However, once we have guessed (6.24), we can verify (6.24) directly and do not need to invoke any additional results. At the same time, we are not able to directly check the positivity of $\varphi_\beta$ for $\beta \in (1/2, 1)$, and this part of Lemma 6.12 depends on Example 6.6 and so on Theorem 6.3. See also Remark 6.18 below.

While the formulas (6.24) and (6.25) for $K_\beta$ are somewhat cumbersome, the case $\beta = 1/2$ is exceptional as the next example shows.

**Example 6.14.** If $\beta = 1/2$, then by (6.24), (6.25) and [59, p. 302, no.9], we have

$$\varphi_{1/2}(t) = \frac{1}{\pi t} \int_0^t \frac{r^{1/2} dr}{(r + 1)^2 (t - r)^{1/2}} = \frac{1}{2(t + 1)^{3/2}}$$

for all $t > 0$, hence

$$\frac{1}{s^{1/2} + 1} = \frac{1}{2} \int_0^\infty \frac{dt}{(s + t)^{1/2}(t + 1)^{3/2}}, \quad s > 0.$$
Now, given \( f \in S_1 \), we provide an explicit Stieltjes representation for \( f(\beta) \).

**Theorem 6.15.** Let \( f \in S_1 \), \( f \sim (\alpha, \mu) \), and let \( f(\beta) \) be defined by (6.17). Then \( f(\beta) \in S_\beta \) for every \( \beta \in (0, 1) \), and

\[
(6.27) \quad f(\beta)(s) = a + \mu(\{0\}) s^{-\beta} + \int_0^{\infty} \psi_\beta(r) \frac{dr}{(s+r)^\beta}, \quad s > 0,
\]

where \( \psi_\beta \in L_\beta^1(\mathbb{R}_+) \) is given by

\[
(6.28) \quad \psi_\beta(r) := \int_{(0, \infty)} \varphi_\beta \left( \frac{r}{\tau^{1/\beta}} \right) \frac{\mu(d\tau)}{\tau^{1/\beta}}, \quad r > 0,
\]

and \( \varphi_\beta \) is defined by (6.25).

**Proof.** It suffices to consider only the case when \( f \sim (0, \mu) \) and \( \mu(\{0\}) = 0 \).

By Lemma 6.13, for every \( \tau > 0 \), we have

\[
(6.29) \quad \frac{1}{s^{\beta} + \tau} = \frac{1}{\tau((s/\tau^{1/\beta})^{\beta} + 1)} = \int_0^{\infty} \varphi_\beta(t) \frac{dt}{(s + \tau^{1/\beta}t)^\beta},
\]

where \( \varphi_\beta \) is given by (6.24). Then, taking into account the positivity of integrands and using Fubini’s theorem,

\[
\begin{align*}
(6.27) \quad f(\beta)(s) &= \int_0^{\infty} \frac{\mu(d\tau)}{s^{\beta} + \tau} = \int_0^{\infty} \left( \int_0^{\infty} \frac{\varphi_\beta(t) \, dt}{(s + \tau^{1/\beta}t)^\beta} \right) \frac{d\tau}{\tau^{1/\beta}} \\
&= \int_0^{\infty} \left( \int_0^{\infty} \frac{\varphi_\beta(r/\tau^{1/\beta}) \, dr}{(s + r)^\beta} \right) \frac{d\tau}{\tau^{1/\beta}} \\
&= \int_0^{\infty} \left( \int_0^{\infty} \frac{\varphi_\beta \left( \frac{r}{\tau^{1/\beta}} \right) \, \mu(d\tau)}{\tau^{1/\beta}} \right) \frac{dr}{(s + r)^\beta}, \quad s > 0,
\end{align*}
\]

which is equivalent to (6.27) and (6.28). \( \square \)

Despite a fully expanded formula (6.27) looks quite “heavy”, it can be used successfully in particular situations, and the following example illustrates this point.

**Remark 6.16.** Under the assumptions of Theorem 6.15 if \( \beta = 1/2 \), then

\[
\varphi_{1/2}(t) = \frac{1}{2(t+1)^{3/2}}, \quad t > 0,
\]

so that the corresponding density \( \psi_{1/2} \) in the Stieltjes representation of \( f_{1/2} \) is given by

\[
\psi_{1/2}(r) = \int_0^{\infty} \varphi_{1/2} \left( \frac{r}{\tau^2} \right) \frac{\mu(d\tau)}{\tau^2} = \frac{1}{2} \int_0^{\infty} \varphi_{1/2} \left( \frac{\tau \mu(d\tau)}{(r+\tau^2)^{3/2}} \right), \quad r > 0,
\]

cf. Example 6.14. Thus, if \( \beta = 1/2 \), then Theorem 6.15 takes a transparent form and, being combined with Remark 6.8, yields explicit formulas for multipliers \( \Phi \) in [4, Theorems 1.1-1.3] given there in the form \( S_{1/2}[\nu] \) with \( \nu \in M_0(\mathbb{R}_+) \) (or an equivalent form).

The next example illustrates this a bit further.
Example 6.17. Let
\[ f(s) = \int_0^1 \frac{dt}{s + t} = \log(1 + 1/s), \quad s > 0. \]
Then \( f \in \mathcal{S}_1 \), and by (6.28) we have
\[ f_{(1/2)}(s) = \log(1 + 1/\sqrt{s}) = \int_0^\infty \psi_{1/2}(r) \frac{dr}{(s + r)^{1/2}}, \quad s > 0, \]
where
\[ \psi_{1/2}(r) = \int_0^\infty \varphi_{1/2}(r) \frac{\mu(\tau)}{\tau^2} = \frac{1}{2} \int_0^1 \left( \frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{1 + r}} \right) \frac{dr}{(s + r)^{1/2}}, \quad s > 0. \]

So,
\[ \log(1 + 1/\sqrt{s}) = \frac{1}{2} \int_0^\infty \left( \frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{1 + r}} \right) \frac{dr}{(s + r)^{1/2}}, \quad s > 0. \]

Alternatively, using Theorem 1.1 it is easy to verify that \( f \in \mathcal{S}_1 \). Then \( f_{(1/2)} \in \mathcal{S}_{1/2} \) by Theorem 6.15 and its representing measure is finite by Remark 6.8. This kind of arguments can be used to produce spectral multipliers \( \Phi \) in [4] starting from a given function from \( \mathcal{S}_1 \) rather than from its representation as a generalized Stieltjes transform.

Remark 6.18. Once Theorem 6.15 is obtained, one may replace the assumption \( f \in \mathcal{S}_1, f \sim (0, \mu)_1 \), by the assumption that \( f = \mathcal{S}_1[\mu] \) for, in general, a complex Radon measure \( \mu \in \mathcal{M}_1(\mathbb{R}_+) \). A direct verification using Fubini’s theorem shows that the formulas (6.27) and (6.28) remain valid. However, in this case, the formulas appear as a black box, without a natural explanation on how they could be obtained. Moreover, Theorem 6.15 relies on Theorem 6.7 as far as positivity of the representing measures is concerned.

7. PRODUCT FORMULAS FOR GENERALIZED CAUCHY TRANSFORMS AND RELATED MATTERS

7.1. General product formulas. In this section, being motivated by the property (1.6), we investigate product formulas in the setting of the generalized Cauchy transforms on \( \mathbb{R} \). Recall that for \( \mu \in \mathcal{M}_\alpha(\mathbb{R}) \) its generalized Cauchy transform of order \( \alpha > 0 \) is defined as
\[ C_\alpha[\mu](z) := \int_\mathbb{R} \frac{\mu(dt)}{(z + t)^\alpha}, \quad z \in \mathbb{C}^+, \]
where \( \mathbb{C}^+ \) stands for the upper half-plane \( \{ z \in \mathbb{C} : \text{Im} z > 0 \} \).

Given \( \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j > 0, j = 1, 2 \), we use our ideas from the preceding sections and obtain product formulas of the form
\[ C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) = C_{\alpha_1+\alpha_2}[\mu](z), \quad z \in \mathbb{C}^+, \]
with an emphasis on the case when \( \mu_j \in \mathcal{M}_{\alpha_j}^+(\mathbb{R}) \), \( j = 1, 2 \). It is essential to observe that given the right-hand side of (7.1) its representation as a generalized Cauchy transform is not unique. For example, we have

\[
\int_{\mathbb{R}} \frac{t^k}{d + t^2} \, dt \equiv 0, \quad z \in \mathbb{C}^+, \quad k \in \mathbb{N}, \quad \beta > k + 1.
\]

This fact leads to non-uniqueness of \( \mu \) in (7.2). In Theorems 7.1 and 7.4, we provide explicit, comparatively simple formulas for \( \mu \) resembling (8.3) and (8.4) and having similar useful features. These formulas are obtained under additional assumptions on the size of \( \mu_1 \) and \( \mu_2 \), which include several situations of interest. (As far as \( \mu \) is not unique, here we avoid using the convolution notation from the preceding sections.) At the same time, we produce a \( \mu \in \mathcal{M}_{\alpha_1+\alpha_2}(\mathbb{R}) \) satisfying (7.1) without any additional restrictions on \( \mu_1 \in \mathcal{M}_{\alpha_1}(\mathbb{R}) \) and \( \mu_2 \in \mathcal{M}_{\alpha_2}(\mathbb{R}) \). However, in this case, our formula for \( \mu \) appears to be rather complicated. In such a general framework, it is not always possible to choose a positive \( \mu \) in (7.2) even if \( \mu_j \in \mathcal{M}_{\alpha_j}^+(\mathbb{R}), j = 1, 2 \), and we explore this issue thoroughly. For \( \alpha_1 = \alpha_2 = 1 \), we characterize the existence of such a \( \mu \) by a simple integrability condition on \( \mu_1 \) and \( \mu_2 \). Moreover, we prove that the choice of positive \( \mu \) is possible if \( \alpha_1 + \alpha_2 = 1 \).

To formulate our first result, we define the functional \( J_{\alpha_1,\alpha_2} : \mathcal{M}_{\alpha_1}(\mathbb{R}) \times \mathcal{M}_{\alpha_2}(\mathbb{R}) \rightarrow \mathbb{R}_+ \cup \{\infty\} \) by

\[
J_{\alpha_1,\alpha_2}[\mu_1,\mu_2] := \int_{\mathbb{R}} \int_{(s,t)} \frac{(\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1}}{(1 + |\tau|)^{\alpha_1+\alpha_2}} \left| \mu_1([ds]) \mu_2([dt]) \right| \frac{1}{(t - s)^{\alpha_1+\alpha_2-1}}.
\]

**Theorem 7.1.** Let \( \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j > 0, j = 1, 2 \), and suppose that

\[
J_{\alpha_1,\alpha_2}[\mu_1,\mu_2] < \infty \quad \text{and} \quad J_{\alpha_2,\alpha_1}[\mu_2,\mu_1] < \infty.
\]

Then (7.2) holds with \( \mu \in \mathcal{M}_{\alpha_1+\alpha_2}(\mathbb{R}) \) given by

\[
\mu(d\tau) := u(\tau)d\tau + \mu_1(\{\tau\})\mu_2(d\tau)
\]

and

\[
B(\alpha_1,\alpha_2)u(\tau) := \int_{(\tau,\infty)} \int_{(-\infty,\tau)} \frac{(\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1}}{(t - s)^{\alpha_1+\alpha_2-1}} \mu_1(ds) \mu_2(dt)
\]

\[
+ \int_{[\tau,\infty)} \int_{(-\infty,\tau)} \frac{(\tau - s)^{\alpha_1-1}(t - \tau)^{\alpha_2-1}}{(t - s)^{\alpha_1+\alpha_2-1}} \mu_2(ds) \mu_1(dt)
\]

for almost all \( \tau \in \mathbb{R} \). Moreover, such a \( \mu \) is positive if \( \mu_1 \) and \( \mu_2 \) are positive.

**Proof.** The proof is similar to the proof of Theorem 3.1. Let \( z \in \mathbb{C}^+ \) be fixed. Write

\[
C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mu_1(ds) \mu_2(dt)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}} := g_1(z) + g_2(z) + g_3(z),
\]

where

\[
g_1(z) = \int_{\mathbb{R}} \frac{\mu_1(ds)}{(z + s)^{\alpha_1}}, \quad g_2(z) = \int_{\mathbb{R}} \frac{\mu_2(dt)}{(z + t)^{\alpha_2}}, \quad g_3(z) = \int_{\mathbb{R}} \frac{\mu_1(ds) \mu_2(dt)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}}.
\]
for some $c(7.6)$

Then, using (7.3), the elementary inequality

where $u$

where $u$

where $u$

By (3.1),

$$B(\alpha_1, \alpha_2)g_1(z) = \int_{\mathbb{R}} \int_{(-\infty, t)} \frac{g_1(t) \mu_2(dt)}{z + t} \frac{\alpha_1 - 1}{\alpha_1 + \alpha_2} dt \frac{\mu_1(ds) \mu_2(dt)}{(t-s)^{\alpha_1 + \alpha_2}}.$$ 

Then, using (7.3), the elementary inequality

(7.6) $|z + \tau| \geq c(1 + |\tau|), \quad \tau \in \mathbb{R},$

for some $c = c(z) > 0$, and Fubini’s theorem, we infer that

$B(\alpha_1, \alpha_2)g_1(z)$

$$= \int_{\mathbb{R}} \int_{t, \tau} \frac{\alpha_1 - 1}{\alpha_1 + \alpha_2} \frac{\mu_1(ds) \mu_2(dt)}{(t-s)^{\alpha_1 + \alpha_2}} dt \frac{\mu_1(ds) \mu_2(dt)}{(t-s)^{\alpha_1 + \alpha_2}}.$$ 

where $u_1 \in L^1_{\alpha_1 + \alpha_2}$. Similarly,

$$B(\alpha_1, \alpha_2)g_2(z) = \int_{\mathbb{R}} \int_{\tau, \sigma} \frac{\alpha_2 - 1}{\alpha_1 + \alpha_2} \frac{\mu_2(ds) \mu_1(dt)}{(t-s)^{\alpha_1 + \alpha_2}} dt \frac{\mu_2(ds) \mu_1(dt)}{(t-s)^{\alpha_1 + \alpha_2}}.$$ 

with $u_2 \in L^1_{\alpha_1 + \alpha_2}$. Hence (7.2) holds with $\mu$ given by (7.4) and (7.5), where $u = u_1 + u_2$ belongs to $L^1_{\alpha_1 + \alpha_2}$. Moreover, it is direct to check that for $\mu_2(dt) = \mu_1\{t\}\mu_2(dt)$ one has $\mu_2 \in M_{\alpha_1 + \alpha_2}$ and $||\mu_2||_{\alpha_1 + \alpha_2} \leq ||\mu_1||_{\alpha_1} ||\mu_2||_{\alpha_2}$. The claim on positivity of $\mu$ for $\mu_j \in M^{+}_{\alpha_j}$ is straightforward.

Remark 7.2. Note that, as in the situation of Corollary 3.3, $\mu$ does not have singular continuous component. In general, under the assumptions of Theorem 7.1, one may formulate an analogue of Corollary 3.3 but we omit easy details.

Now we consider several natural cases where one can get rid of the assumption (7.3). In particular, we include Theorem 3.1 into a more general context of Cauchy transforms, see Corollary 7.3, b) below.
Corollary 7.3. Let \( \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j > 0, j = 1, 2 \), be such that one of the following conditions holds:

a) One has \( \mu_j \in \mathcal{M}_0(\mathbb{R}), j = 1, 2 \);

b) There exists \( a \in \mathbb{R} \) such that \( \text{supp } \mu_j \subset [a, \infty), j = 1, 2 \);

c) There exists \( a \in \mathbb{R} \) such that \( \text{supp } \mu_j \subset (-\infty, a], j = 1, 2 \);

d) One of the measures \( \mu_1 \) or \( \mu_2 \) has compact support.

If \( \mu \) is defined by (7.4) and (7.5), then (7.2) holds.

Proof. To prove the assertion, we verify the assumption (7.3) of Theorem 7.1 in each of the cases a)-d).

a) Using (4.5), we obtain

\[
B(\alpha_1, \alpha_2)J_{\alpha_1, \alpha_2}[\mu_1, \mu_2] \\
\leq B(\alpha_1, \alpha_2) \int_\mathbb{R} \int_{(-\infty,t)} \int_{(s,t)} (\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1} \, d\tau \, |\mu_1|(d\sigma)|\mu_2|(dT) \\
= \int_\mathbb{R} \int_{(-\infty,t)} |\mu_1|(d\sigma)|\mu_2|(dT) = \|\mu_1\|_0\|\mu_2\|_0.
\]

Similarly,

\[
B(\alpha_1, \alpha_2)J_{\alpha_2, \alpha_1}[\mu_2, \mu_1] \leq \|\mu_1\|_0\|\mu_2\|_0.
\]

b) By considering appropriate shifts of measures \( \mu_1 \) and \( \mu_2 \), we may assume that \( a = 0 \). Using (3.4), we infer that

\[
B(\alpha_1, \alpha_2)J_{\alpha_1, \alpha_2}[\mu_1, \mu_2] \\
\leq B(\alpha_1, \alpha_2) \int_0^\infty \int_{[0,t)} \int_{(s,t)} (\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1} \, d\tau \, |\mu_1|(d\sigma)|\mu_2|(dT) \\
= \int_0^\infty \int_{[0,t)} |\mu_1|(d\sigma)|\mu_2|(dT) = \|\mu_1\|_{\alpha_1}\|\mu_2\|_{\alpha_2} < \infty.
\]

By symmetry, we also have

\[
B(\alpha_1, \alpha_2)J_{\alpha_2, \alpha_1}[\mu_2, \mu_1] \leq \|\mu_1\|_{\alpha_1}\|\mu_2\|_{\alpha_2} < \infty.
\]

c) We proceed along the lines of the proof of b). Assuming \( a = 0 \), from (3.1) it follows that

\[
\frac{1}{(t-s)^{\alpha_1+\alpha_2-1}} \int_s^t (\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1} \, d\tau \\
= \frac{1}{(t-s)^{\alpha_1+\alpha_2-1}} \int_{-s}^{-s} (\tau + s)^{\alpha_2 - 1}(\tau + t)^{\alpha_1 - 1} \, d\tau \\
= \frac{B(\alpha_1, \alpha_2)}{(1-s)^{\alpha_1}(1-t)^{\alpha_2}}, \quad s < t < 0.
\]
Then
\[B(\alpha_1, \alpha_2) J_{\alpha_1, \alpha_2}[\mu_1, \mu_2]\]
\[\leq B(\alpha_1, \alpha_2) \int_{(-\infty, 0)} \int_{(-\infty, t)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(1 - \tau)^{\alpha_1 + \alpha_2}} \frac{|\mu_1|(ds)|\mu_2|(dt)}{(t - s)^{\alpha_1 + \alpha_2 - 1}}\]
\[= \int_{(-\infty, 0]} \int_{(-\infty, t]} \frac{|\mu_1|(ds)|\mu_2|(dt)}{(1 - s)^{\alpha_1}(1 - t)^{\alpha_2}} \leq \|\mu_1\|_{\alpha_1} \|\mu_2\|_{\alpha_2} < \infty.\]

By symmetry, as above,
\[B(\alpha_1, \alpha_2) J_{\alpha_2, \alpha_1}[\mu_2, \mu_1] \leq \|\mu_1\|_{\alpha_1} \|\mu_2\|_{\alpha_2} < \infty.\]

d) Assume that supp\( \mu_1 \subset [-a, a] \) for some \( a > 0 \). Let \( \mu_2 = \mu_{2,1} + \mu_{2,2} \), where
\[\text{supp}\, \mu_{2,1} \subset (-\infty, 0] \subset (-\infty, a] \quad \text{and} \quad \text{supp}\, \mu_{2,2} \subset [0, \infty) \subset [-a, \infty).\]

Then, by the arguments in b) and c),
\[J_{\alpha_1, \alpha_2}[\mu_1, \mu_2] \leq J_{\alpha_1, \alpha_2}[\mu_1, \mu_{2,1}] + J_{\alpha_1, \alpha_2}[\mu_1, \mu_{2,2}] < \infty,\]
and
\[J_{\alpha_2, \alpha_1}[\mu_2, \mu_1] \leq J_{\alpha_2, \alpha_1}[\mu_{2,1}, \mu_1] + J_{\alpha_2, \alpha_1}[\mu_{2,2}, \mu_1] < \infty.\]

If \( \text{supp}\, \mu_2 \subset [-a, a], \ a > 0 \), then the considerations are completely analogous. \( \square \)

The next result is similar in spirit to Corollary 7.3. On the other hand, to make the relation (7.2) valid for \( \mu \) defined by a concrete formula, it offers a compromise between our assumptions on the supports on \( \mu_1 \) and \( \mu_2 \) and on their size at zero and at infinity.

**Theorem 7.4.** Let \( \mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j \geq 1, j = 1, 2, \) satisfy supp\( \mu_1 \subset (-\infty, a) \) and supp\( \mu_2 \subset (a, \infty) \) for some \( a \in \mathbb{R} \). Then
\[C_{\alpha_1}[\mu_1](z) C_{\alpha_2}[\mu_2](z) = C_{\alpha_1 + \alpha_2}[u](z), \quad z \in \mathbb{C}^+,\]
where \( u \in L^1_{\alpha_1 + \alpha_2}(\mathbb{R}) \) is defined by
\[B(\alpha_1, \alpha_2) u(\tau) := \left\{ \begin{array}{ll}
- \int_{(a, \infty)} \int_{(\tau, a)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2 - 1}} \mu_1(ds) \mu_2(dt), & \tau < a, \\
- \int_{(a, \tau)} \int_{(-\infty, a)} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2 - 1}} \mu_1(ds) \mu_2(dt), & \tau > a,
\end{array} \right.\]
with
\[(\tau - s)^{\alpha_2 - 1} = e^{i\pi(\alpha_2 - 1)}(s - \tau)^{\alpha_2 - 1}, \quad \tau < s,\]
\[(t - \tau)^{\alpha_1 - 1} = e^{i\pi(\alpha_1 - 1)}(\tau - t)^{\alpha_1 - 1}, \quad \tau > t.\]

**Proof.** Let \( z \in \mathbb{C}^+ \) be fixed. Without loss of generality, we may assume that \( a = 0 \). We have
\[C_{\alpha_1}[\mu_1](z) C_{\alpha_2}[\mu_2](z) = \int_{(0, \infty)} \int_{(-\infty, 0]} \frac{\mu_1(ds) \mu_2(dt)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}}.\]
Note that
\[ 0 = \lim_{\epsilon \to 0^+} \int_{\mathbb{R} + i\epsilon} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(z + \tau)^{\alpha_1 + \alpha_2}} \, d\tau = \int_{\mathbb{R}} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(z + \tau)^{\alpha_1 + \alpha_2}} \, d\tau. \]

Then, by \((3.1)\), for any \( s < t \),
\[
\frac{B(\alpha_1, \alpha_2)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}} = \frac{1}{(t - s)^{\alpha_1 + \alpha_2 - 1}} \left\{ \int_{-\infty}^{s} + \int_{t}^{\infty} \right\} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(z + \tau)^{\alpha_1 + \alpha_2}} \, d\tau.
\]
So, we can formally write
\[ B(\alpha_1, \alpha_2)C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) := I_1(z) + I_2(z), \]
where
\[
I_1(z) := -\int_{(0, \infty)} \int_{(-\infty, 0)} \int_{-\infty}^{s} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2 - 1}(z + \tau)^{\alpha_1 + \alpha_2}} \mu_1(ds) \mu_2(dt),
\]
\[
I_2(z) := -\int_{(0, \infty)} \int_{(-\infty, 0)} \int_{t}^{\infty} \frac{(\tau - s)^{\alpha_2 - 1}(t - \tau)^{\alpha_1 - 1}}{(t - s)^{\alpha_1 + \alpha_2 - 1}(z + \tau)^{\alpha_1 + \alpha_2}} \mu_1(ds) \mu_2(dt).
\]
To prove the theorem it suffices to show that the (triple) integrals \( I_1 \) and \( I_2 \) in \((7.8)\) are absolutely convergent. Taking into account \((7.6)\), we have
\[
\begin{align*}
&\int_{(0, \infty)} \int_{(-\infty, 0)} \int_{-\infty}^{s} \frac{|\tau - s|^{\alpha_2 - 1}|t - \tau|^{\alpha_1 - 1}}{|t - s|^{\alpha_1 + \alpha_2 - 1}|z + \tau|^{\alpha_1 + \alpha_2}} |\mu_1|(ds) |\mu_2|(dt) \\
&\leq \frac{1}{c^{\alpha_1 + \alpha_2}} \int_{(0, \infty)} \int_{(-\infty, 0)} \int_{-\infty}^{s} \frac{|\tau - s|^{\alpha_2 - 1}|t - \tau|^{\alpha_1 - 1}}{|t - s|^{\alpha_1 + \alpha_2 - 1}|z + \tau|^{\alpha_1 + \alpha_2}} |\mu_1|(ds) |\mu_2|(dt) \\
&\leq \frac{1}{c^{\alpha_1 + \alpha_2}} \int_{0}^{\infty} \int_{-\infty}^{0} \int_{|s|}^{\infty} \frac{|\tau - s|^{\alpha_2 - 1}|t + \tau|^{\alpha_1 - 1}}{|z + \tau|^{\alpha_1 + \alpha_2 - 1}} |\mu_1|(ds) |\mu_2|(dt).
\end{align*}
\]
To estimate the latter integral, observe that
\[
\begin{align*}
&2^{1 - \alpha_1} \int_{|s|}^{\infty} \frac{(\tau - |s|)^{\alpha_2 - 1}(t + \tau)^{\alpha_1 - 1}}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau \\
&\leq \int_{|s|}^{\infty} \frac{(\tau - |s|)^{\alpha_2 - 1} \tau\alpha_1 - 1}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau + \int_{|s|}^{\infty} \frac{(\tau - |s|)^{\alpha_2 - 1} \tau\alpha_2 - 1}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau \\
&= t^{\alpha_1 - 1} \int_{|s|}^{\infty} \frac{\tau^{\alpha_2 - 1} \alpha_1 + \alpha_2}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau + \int_{|s|}^{\infty} \frac{\tau^{\alpha_2 - 1} \alpha_2 + 1}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau \\
&= t^{\alpha_1 - 1} \int_{|s|}^{\infty} \frac{\tau^{\alpha_2 - 1} \alpha_1 + \alpha_2}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau + \int_{|s|}^{\infty} \frac{\tau^{\alpha_2 - 1} \alpha_2 + 1}{(\tau + |s|)^{\alpha_1 + \alpha_2}} \, d\tau \\
&\leq \left( \frac{(\alpha_1 - 1)}{|s|^{\alpha_1}} + \frac{1}{|s|} \right) \int_{0}^{\infty} \frac{(\tau + 1)^{\alpha_2 - 1} \alpha_2 + 1}{(\tau + 1)^{\alpha_2 + 1}} \, d\tau \leq \left( \frac{(\alpha_1 - 1)}{|s|^{\alpha_1}} + \frac{1}{|s|} \right),
\end{align*}
\]
where we used that
\[ \int_{0}^{\infty} \frac{\tau^{\alpha_2 - 1} \alpha_2 + 1}{(\tau + 1)^{\alpha_2 + 1}} \, d\tau = \frac{1}{\alpha_2} \leq 1. \]
satisfied simultaneously these results yield, in general, different measures. Thus, by Fubini’s theorem,

\[
\frac{1}{c^{\alpha_1+\alpha_2}} \int_{(0,\infty)} \int_{(-\infty,0)} \int_{-\infty}^{\infty} \left| \tau - s \right|^{\alpha_2-1} \left| t - \tau \right|^{\alpha_1-1} \frac{dt}{|s|^\alpha_1 + |s|^\alpha_2} |\mu_1|(ds) |\mu_2|(dt) \leq \frac{2}{c^{\alpha_1+\alpha_2}} \int_{(0,\infty)} \int_{(-\infty,0)} \left| |\mu_1|(ds) |\mu_2|(dt) \right| < \infty.
\]

Similarly,

\[
\frac{1}{c^{\alpha_1+\alpha_2}} \int_{(0,\infty)} \int_{(-\infty,0)} \int_{t}^{\infty} \left| \tau - s \right|^{\alpha_2-1} \left| t - \tau \right|^{\alpha_1-1} \frac{dt}{|s|^\alpha_1 + |s|^\alpha_2} |\mu_1|(ds) |\mu_2|(dt) \leq \frac{2}{c^{\alpha_1+\alpha_2}} \int_{(0,\infty)} \int_{(-\infty,0)} \left| |\mu_1|(ds) |\mu_2|(dt) \right| < \infty.
\]

Thus, by Fubini’s theorem,

\[
I_1(z) = -\int_{-\infty}^{0} \int_{(0,\infty)} \int_{(\tau,0)} u_1(\tau) d\tau = \int_{E \setminus (z + \tau)^{\alpha_1+\alpha_2}} \frac{(\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1}}{(t - s)^{\alpha_1+\alpha_2}} \frac{\mu_1(ds) \mu_2(dt) d\tau}{(z + \tau)^{\alpha_1+\alpha_2}},
\]

where \( u_1 \in L^{1}_{\alpha_1+\alpha_2}(E) \), and similarly

\[
I_2(z) = \int_{0}^{\infty} \int_{(0,\tau)} \int_{(\tau,0)} u_2(\tau) d\tau = \int_{E \setminus (z + \tau)^{\alpha_1+\alpha_2}} \frac{(\tau - s)^{\alpha_2-1}(t - \tau)^{\alpha_1-1}}{(t - s)^{\alpha_1+\alpha_2}} \frac{\mu_1(ds) \mu_2(dt) d\tau}{(z + \tau)^{\alpha_1+\alpha_2}},
\]

with \( u_2 \in L^{1}_{\alpha_1+\alpha_2}(E) \). Setting \( u = u_1 + u_2 \), the statement follows from (7.8).

Remark that even when the assumptions of Theorems (7.1) and (7.4) are satisfied simultaneously these results yield, in general, different measures \( \mu \), and the one originating from Theorem (7.4) may not be positive for positive \( \mu_1 \) and \( \mu_2 \). In general, if \( \mu_1 \) and \( \mu_2 \) are positive, then under the assumptions of Theorem (7.4) one cannot choose a positive \( \mu \) satisfying (7.2). See Theorem (7.5) and Example (7.9) below.

Finally, we prove a version of (7.2) without any additional assumptions on \( \mu_j \in M_{\alpha_j}(E) \), \( j = 1, 2 \). We will rely on an idea from (14) used to show (11.6). While (3.1) was convenient to deal with the products of the generalized Stieltjes transforms, a direct application of (3.1) to the products of the generalized Cauchy transforms leads to convergence problems. So we recast (3.1) into a form convenient for our current purposes. Such an approach seems to be more transparent and revealing than transforming (1.6) to (7.2) via an appropriate conformal mapping. On the other hand, we are not able to avoid the use of conformal mappings completely.
We proceed with several preparations. Setting \( z = 1 \) and \( \tau = (s-t)r + rt \), \( r \in (0,1) \), in (3.11), we conclude that for all \( \alpha_1, \alpha_2 > 0 \), \( a > 0 \) and \( t > s > -1 \),
\[
(7.10) \quad \frac{1}{(a(1+s))^{\alpha_1}(a(1+t))^{\alpha_2}} = (B(\alpha_1, \alpha_2))^{-1} \int_0^1 \frac{r^\alpha_1(1-r)^{\alpha_2} dr}{(a(1+c(r;s,t)))^{\alpha_1+\alpha_2}} = \int_0^1 \frac{\nu(dr)}{(a(1+c(r;s,t)))^{\alpha_1+\alpha_2}},
\]
where
\[
c(r;s,t) = rs + (1-r)t, \quad r \in (0,1),
\]
denotes the convex combination of \( s \) and \( t \), and the probability measure \( \nu = \nu_{\alpha_1, \alpha_2} \) on \([0,1]\) is given by
\[
\nu(dr) := (B(\alpha_1, \alpha_2))^{-1}r^\alpha_1(1-r)^{\alpha_2} dr.
\]
Note that, moreover, by analytic continuation, (7.10) holds for \( a \in \mathbb{C}_+ \) and \( s,t \) from the unit disc \( \mathbb{D} \).

We will also need the Cayley transform \( \omega(z) := \frac{z-i}{z+i} \) mapping \( \mathbb{C}^+ \) onto \( \mathbb{D} \setminus \{1\} \) homeomorphically, so that \( \omega(\mathbb{R}) = \mathbb{T} \setminus \{1\} \), with \( \omega^{-1} : \mathbb{D} \setminus \{1\} \mapsto \mathbb{C}^+ \), \( \omega^{-1}(z) = i\frac{1+z}{1-z} \). Recall that we deal with the principal branch of \( z \to z^\alpha \) with the cut along \(( -\infty, 0 ] \).

**Lemma 7.5.** Let \( \alpha_1, \alpha_2 > 0 \). If \( s, t \in \mathbb{R}, s \neq t \), and \( z \in \mathbb{C}^+ \), then
\[
(7.11) \quad \frac{(i+s)^{\alpha_1}(i+t)^{\alpha_2}}{(z+s)^{\alpha_1}(z+t)^{\alpha_2}} = \int_{\mathbb{R}} \frac{G(\tau; s, t) d\tau}{(z+\tau)^{\alpha_1+\alpha_2}},
\]
where
\[
(7.12) \quad G(\tau; s, t) := \frac{(i+\tau)^{\alpha_1+\alpha_2}}{\pi} \int_0^1 \mathrm{Im} \left( \frac{1}{\tau - \omega^{-1}(c(r; \omega(s), \omega(t)))} \right) \, d\nu(r),
\]
for all \( \tau \in \mathbb{R} \), and \( G(\cdot; s, t) \in L^1_{\alpha_1+\alpha_2}(\mathbb{R}) \) with \( \|G(\cdot; s, t)\|_{\alpha_1+\alpha_2} \leq 1 \).

**Proof.** Let \( s, t \in \mathbb{R}, s \neq t \), and \( z \in \mathbb{C}^+ \) be fixed. Note that by a simple calculation, for all \( \lambda \in \mathbb{C}^+ \),
\[
(7.13) \quad \frac{i+\lambda}{z+\lambda} = \frac{1}{a(z)(1-\omega(z)\omega(\lambda))},
\]
where for short-hand \( a(z) := (2i)^{-1}(i+z) \), so that \( a(z) \in \mathbb{C}_+ \).

Therefore,
\[
\frac{(i+s)^{\alpha_1}}{(z+s)^{\alpha_1}} \cdot \frac{(i+t)^{\alpha_2}}{(z+t)^{\alpha_2}} = \frac{1}{(a(z)(1-\omega(z)\omega(s)))^{\alpha_1}} \cdot \frac{1}{(a(z)(1-\omega(z)\omega(t)))^{\alpha_2}},
\]
and in view of (7.10),
\[
(7.14) \quad \frac{(i+s)^{\alpha_1}(i+t)^{\alpha_2}}{(z+s)^{\alpha_1}(z+t)^{\alpha_2}} = \int_0^1 \frac{\nu(dr)}{(a(z)(1-\omega(z)c(r; \omega(s), \omega(t)))^{\alpha_1+\alpha_2}).}
\]
On the other hand, by (7.13) again,
\[
\frac{1}{a(z)(1 - \omega(z)c(r;\omega(s),\omega(t)))} = \frac{\omega^{-1}(c(r;\omega(s),\omega(t)) + i}{\omega^{-1}(c(r;\omega(s),\omega(t)) + z}
\]
Hence, taking into account that \(s \neq t, s, t \in \mathbb{R}\), and using Poisson’s integral formula for \(\mathbb{C}^+\), we infer that
\[
(a(z)(1 - \omega(z)c(r;\omega(s),\omega(t))))^{-(\alpha_1 + \alpha_2)}
\]
\[
= \frac{1}{\pi} \int_{\mathbb{R}} \text{Im} \left( \frac{1}{\tau - \omega^{-1}(c(r;\omega(s),\omega(t)))} \right) (i + \tau)^{\alpha_1 + \alpha_2} d\tau.
\]
By combining (7.14) and the above formula, we obtain (7.11).
Moreover,
\[
\int_{\mathbb{R}} \frac{|G(\tau; s, t)| d\tau}{(1 + |\tau|)^{\alpha_1 + \alpha_2}} \leq \frac{1}{\pi} \int_{\mathbb{R}} \text{Im} \left( \frac{1}{\tau - \omega^{-1}(c(r;\omega(s),\omega(t)))} \right) d\tau \nu(dr) = 1,
\]
which finishes the proof.

Now we can state our product formula for the generalized Cauchy transforms of arbitrary \(\mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j > 0, j = 1, 2\).

**Theorem 7.6.** Let \(\mu_j \in \mathcal{M}_{\alpha_j}(\mathbb{R}), \alpha_j > 0, j = 1, 2\). Then (7.2) holds with \(\mu \in \mathcal{M}_{\alpha_1 + \alpha_2}(\mathbb{R})\) defined by
\[
(7.15) \quad \mu(d\tau) = u(\tau) d\tau + \mu_1(\{\tau\})\mu_2(d\tau)
\]
where
\[
u(\tau) := \int_{\mathbb{R}} \int_{\mathbb{R}, s \neq t} G(\tau; t, s) \frac{\mu_1(ds)\mu_2(dt)}{(i + s)^{\alpha_1}(i + t)^{\alpha_2}}.
\]
and \(G\) is given by Lemma 7.5.

**Proof.** First note that by (7.12), (7.6) and Fubini’s theorem,
\[
\int_{\mathbb{R}} \frac{|u(\tau)| d\tau}{(1 + |\tau|)^{\alpha_1 + \alpha_2}} \leq (\sqrt{2})^{\alpha_1 + \alpha_2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}, s \neq t} \frac{|G(\tau; s, t)||\mu_1(ds)||\mu_2(dt)| d\tau}{(1 + |s|)^{\alpha_1}(1 + |t|)^{\alpha_2}(1 + |\tau|)^{\alpha_1 + \alpha_2}}
\]
\[
\leq (\sqrt{2})^{\alpha_1 + \alpha_2} \|\mu_1\|_{\alpha_1} \|\mu_2\|_{\alpha_2},
\]
so that \(u \in L^1_{\alpha_1 + \alpha_2}(\mathbb{R})\). Let \(z \in \mathbb{C}^+\) be fixed. Write formally
\[
C_{\alpha_1}[\mu_1](z)C_{\alpha_2}[\mu_2](z) = \int_{\mathbb{R}} \int_{\mathbb{R}, s \neq t} \frac{\mu_1(ds)\mu_2(dt)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}} + \int_{\mathbb{R}} \frac{\mu_1(\{\tau\})\mu_2(d\tau)}{(z + \tau)^{\alpha_1 + \alpha_2}}.
\]
By Lemma 7.5 and Fubini’s theorem again,
\[
\int_{\mathbb{R}} \int_{\mathbb{R}, s \neq t} \frac{\mu_1(ds)\mu_2(dt)}{(z + s)^{\alpha_1}(z + t)^{\alpha_2}}
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}, s \neq t} G(\tau; s, t)\frac{\mu_1(ds)\mu_2(dt)}{(s + t)^{\alpha_1}(i + t)^{\alpha_2}(z + \tau)^{\alpha_1 + \alpha_2}} d\tau
\]
\[
= \int_{\mathbb{R}} \frac{u(\tau) d\tau}{(z + \tau)^{\alpha_1 + \alpha_2}}.
\]
Moreover, for \( \mu_d(d\tau) = \mu_1(\{\tau\})\mu_2(d\tau) \) it is easy to check that \( \mu_d \in M_{\alpha_1+\alpha_2}(\mathbb{R}) \), and \( \|\mu_d\|_{\alpha_1+\alpha_2} \leq \|\mu_1\|_{\alpha_1}\|\mu_2\|_{\alpha_2} \). \( \square \)

Note that as in Theorems 7.1 and 7.4, \( \mu \) does not have singular continuous component. However, in contrast to Theorem 3.1 dealing with Stieltjes transforms, the nature of \( \mu \) in (7.15) is rather implicit. Moreover, as we show below, the positivity of \( \mu_1 \) and \( \mu_2 \) is not, in general, inherited by \( \mu \).

As far as Theorem 7.1 yields positive \( \mu \) for positive \( \mu_1 \) and \( \mu_2 \), this shows that the assumptions (7.3) are, in general, necessary for its conclusion, and in particular for the validity of the formula for \( \mu \) given by (7.4) and (7.5).

### 7.2. Positivity of representing measures for products of Cauchy transforms, and related matters.

We proceed with deriving a criterion for \( \mu \) in (7.2) to be positive if \( \alpha_1 = \alpha_2 = 1 \) and \( \mu_j \in M_+^1(\mathbb{R}), j = 1, 2 \). The statement will help us to produce various examples of \( \mu_1 \) and \( \mu_2 \) from \( M_+^1(\mathbb{R}) \) not allowing for positive \( \mu \) in (7.2). First, we need to separate a simple lemma.

**Lemma 7.7.** Let \( \mu \in M_+^2(\mathbb{R}) \). If \( C_2[\mu](z) = 0 \) for all \( z \in \mathbb{C}^+ \), then \( \mu(dt) = cd\check{t} \) for some constant \( c \geq 0 \).

**Proof.** By assumption,

\[
\int_{\mathbb{R}} \frac{\mu(dt)}{(t+z)^2} = 0, \quad z \in \mathbb{C}^+.
\]

Passing to conjugates and using the positivity of \( \mu \), we have

\[
\int_{\mathbb{R}} \frac{\mu(dt)}{(t-z)^2} = 0, \quad z \in \mathbb{C}^+.
\]

Let

\[
F(z) := \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) \mu(dt) = \int_{\mathbb{R}} \frac{(1+tz)\mu(dt)}{(t-z)(1+t^2)}, \quad z \in \mathbb{C}^+,
\]

be the Nevanlinna-Pick function associated to \( \mu \). (See e.g. [60, p. 84] for the notion of Nevanlinna-Pick functions.) Then \( F \) is analytic on \( \mathbb{C}^+ \), and \( \text{Im} F(z) \geq 0 \) for all \( z \in \mathbb{C}^+ \). Differentiating \( F \) and using (7.16), we infer that \( F'(z) = 0 \) for all \( z \in \mathbb{C}^+ \). Hence there exists \( a \in \mathbb{C} \) with \( \text{Im} a \geq 0 \) such that \( F(z) = a \). Then, by the inversion formula for Nevanlinna-Pick functions, see e.g. [60, Theorem 5.4], \( \mu(dt) = (\text{Im} a/\pi) dt = c dt \). \( \square \)

**Theorem 7.8.** Let \( \mu_j \in M_1^+(\mathbb{R}), j = 1, 2 \), satisfy

\[
\text{supp} \mu_1 \subset (-\infty, a) \quad \text{and} \quad \text{supp} \mu_2 \subset (a, \infty)
\]

for some \( a \in \mathbb{R} \). Then there exists \( \mu \in M_2^+(\mathbb{R}) \) such that (7.2) holds if and only if

\[
\int_{(a,\infty)} \int_{(-\infty,a)} \frac{\mu_1(ds)\mu_2(dt)}{t-s} < \infty.
\]
Proof. Without loss of generality we may assume that $a = 0$. By Theorem 7.4, we have

$$C_1[\mu_1](z)C_1[\mu_2](z) = C_2[u](z), \quad z \in \mathbb{C}^+,$$

where the negative function $u \in L^1_2(\mathbb{R})$ is given by

$$(7.18) \quad u(\tau) = \begin{cases} - \int_{(0,\infty)} \int_{(\tau,0)} \frac{\mu_1(ds)\mu_2(dt)}{t-s}, & \tau < 0, \\ - \int_{(0,\tau)} \int_{(-\infty,0)} \frac{\mu_1(ds)\mu_2(dt)}{t-s}, & \tau > 0. \end{cases}$$

Suppose that there exists $\mu \in M^+_2(\mathbb{R})$ such that (7.2) holds. Letting then $\nu(\tau) := |u(\tau)|d\tau + \mu(\tau)$, we conclude that $\nu \in M^+_2(\mathbb{R})$ and $C_2[\nu](z) = 0$, $z \in \mathbb{C}^+$. Hence by Lemma 7.7 there exists $c \geq 0$ such that $\nu(\tau) = cd\tau$, for almost all $\tau \in \mathbb{R}$, and in view of (7.18),

$$\int_{(0,\tau)} \int_{(-\infty,0)} \frac{\mu_1(ds)\mu_2(dt)}{t-s} \leq c \quad \text{for a.e. } \tau > 0.$$

Thus, by applying Fatou’s Lemma, we infer that (7.17) (with $a = 0$) is true.

Conversely, let now (7.17) hold. Denote by $R(\mu_1, \mu_2)$ the left-hand side of (7.17) and define $\mu(\tau) := |u(\tau)|d\tau + \mu(\tau)$, where $u$ is given by (7.18). Then, taking into account the elementary equality

$$\int_{\mathbb{R}} \frac{d\tau}{(z + \tau)^2} = 0, \quad z \in \mathbb{C}^+,$$

we conclude that $\mu \in M^+_2(\mathbb{R})$, and (7.2) holds as well. (Alternatively, to arrive at (7.2), one may note that (7.17) implies (7.3) and refer to Theorem 7.1.)

We illustrate Theorem 7.8 with a simple concrete example, though other more general examples can be constructed easily.

**Example 7.9.** Let $\mu_1$ and $\mu_2$ from $M^+_1(\mathbb{R})$ be defined by

$$\mu_1(dt) = \chi_{(-\infty,-1]}(t) \frac{dt}{|t|^{1/2}} \quad \text{and} \quad \mu_2(dt) = \chi_{[1,\infty)}(t) \frac{dt}{|t|^{1/2}}.$$

Then we have

$$\int_{(0,\infty)} \int_{(-\infty,0)} \frac{\mu_1(ds)\mu_2(dt)}{t-s} = \int_1^\infty \int_1^\infty \frac{ds dt}{t^{1/2} s^{1/2} (t+s)} = \infty,$$

and therefore Theorem 7.8 with $a = 0$ implies that there is no $\mu \in M^+_2(\mathbb{R})$ satisfying (7.2).
On the other hand, if $\alpha_1 + \alpha_2 = 1, \alpha_j > 0$, and $\mu_j \in \mathcal{M}^+_{\alpha_j}(\mathbb{R}), j = 1, 2$, then under mild additional assumptions on $\mu_1$ and $\mu_2$, one can always find a $\mu \in \mathcal{M}^+_{\alpha}(\mathbb{R})$ satisfying $C_{\alpha_1}[\mu_1] C_{\alpha_2}[\mu_2] = C_1[\mu]$. To show this, recall that by the well-known Kac (Katz) criterion (see e.g. [39], [40], or [2, Section 3]), a function $f$ holomorphic on $\mathbb{C}^+$ admits the representation

$$f(z) = C_1[\mu](z), \quad z \in \mathbb{C}^+,$$

with $\mu \in \mathcal{M}^+_1(\mathbb{R})$ if and only if

$$(7.19) \quad \text{Im} \, f(z) \leq 0, \quad z \in \mathbb{C}^+,$$

and

$$(7.20) \quad \int_1^\infty \frac{\text{Im} \, f(iy)}{y} \, dy < \infty.$$

Such a $\mu$ is necessarily unique.

For $\alpha \in (0, 1)$, define

$$(7.21) \quad Q_{\alpha}(x, y) := \int_0^1 \frac{d\tau}{(x\tau + 1)^{\alpha}(y\tau + 1)^{1-\alpha}}, \quad x, y \geq 0.$$

**Theorem 7.10.** For $\alpha \in (0, 1)$, let $\mu_1 \in \mathcal{M}^+_{\alpha}(\mathbb{R})$ and $\mu_2 \in \mathcal{M}^+_{1-\alpha}(\mathbb{R})$. Then there exists $\mu \in \mathcal{M}^+_1(\mathbb{R})$ such that

$$(7.22) \quad C_\alpha[\mu_1](z) C_{1-\alpha}[\mu_2](z) = C_1[\mu](z), \quad z \in \mathbb{C}^+,$$

if and only if there exists $a \geq 0$ satisfying

$$(7.23) \quad \int_a^\infty \int_{-\infty}^{-a} Q_{\alpha}(|s|, |t|) \mu_1(ds) \mu_2(dt) < \infty,$$

and

$$(7.24) \quad \int_{-\infty}^{-a} \int_a^\infty Q_{\alpha}(s, |t|) \mu_1(ds) \mu_2(dt) < \infty.$$

**Proof.** Setting $f = C_\alpha[\mu_1] C_{1-\alpha}[\mu_2]$, we will show that $f$ satisfies the conditions $(7.19)$ and $(7.20)$ of Kac’s criterion if and only if $(7.23)$ and $(7.24)$ hold.

Note that since for all $s, t \in \mathbb{R}$ and $z \in \mathbb{C}^+$:

$$\text{Im} \, \frac{1}{(z + s)^{\alpha}(z + t)^{1-\alpha}} \leq 0,$$

we have

$$\text{Im} \, f(z) = \text{Im} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mu_1(ds) \mu_2(dt)}{(z + s)^{\alpha}(z + t)^{1-\alpha}} \leq 0, \quad z \in \mathbb{C}^+,$$

so that $f$ satisfies $(7.19)$, and it is necessary to check that $(7.20)$ is then equivalent to $(7.23)$ and $(7.24)$. To prove this we proceed with two reductions. First, since $Q_{\alpha}(x, y) \leq 1$ for all $x, y \geq 0$, we may assume without loss
of generality that \( a = 0 \). Second, note that it suffices to consider the case when

\[
(7.26) \quad \text{supp } \mu_1 \subset (-\infty, 0] \quad \text{and} \quad \text{supp } \mu_2 \subset [0, \infty).
\]

Indeed, for \( j = 1, 2 \), and \( k = 1, 2 \), define \( \mu_{j,k} \in \mathcal{M}_\alpha^+ (\mathbb{R}) \) as follows:

\[
\mu_{1,k} := \chi(-\infty,0)\mu_1 + \chi(0,\infty)\mu_1,
\]

\[
\mu_{2,k} := \chi(-\infty,0)\mu_2 + \chi(0,\infty)\mu_2.
\]

If \( f_{1,k} = C_\alpha[\mu_{1,k}] \) and \( f_{2,k} = C_{1-\alpha}[\mu_{2,k}] \), \( k = 1, 2 \), then write

\[
f = (f_{1,1} + f_{1,2})(f_{2,1} + f_{2,2}) = F_1 + F_2,
\]

where

\[
f_1 = f_{1,1} \cdot f_{2,1} + f_{1,2} \cdot f_{2,2} \quad \text{and} \quad f_2 = f_{1,1} \cdot f_{2,2} + f_{1,2} \cdot f_{2,1},
\]

and \( F_1 \) and \( F_2 \) satisfy \((7.19)\) in view of \((7.25)\) applied to each of the four summands above. Furthermore, \( f_1 \in C_1(\mathcal{M}_1^+ (\mathbb{R})) \) by Corollary \((7.3)\) b) and c). Hence, using Kac’s criterion, we infer that \( f \in C_1(\mathcal{M}_1^+ (\mathbb{R})) \) if and only if \( f_2 \in C_1(\mathcal{M}_1^+ (\mathbb{R})) \), and so may restrict our attention to \( f_2 \). Taking into account that \( Q_{1-\alpha}(x,y) = Q_\alpha(y,x) \), \( x,y \geq 0 \), it remains to note that \((7.26)\) holds for \( \mu_{1,1} \) and \( \mu_{2,2} \), as well as for \( \mu_{1,2} \) and \( \mu_{2,1} \), up to relabeling.

Thus, it is sufficient to consider the case when \( \mu_1 = \mu_{1,1} \), \( \mu_2 = \mu_{2,2} \),

\[
f = f_2 = f_{1,1} \cdot f_{2,2} = C_\alpha[\mu_1]C_{1-\alpha}[\mu_2],
\]

and to show that \((7.23)\) is then equivalent to \((7.20)\). To this aim, observe that for all \( y > 0 \) and \( t \leq 0 \), \( s \geq 0 \),

\[
\arg ((iy + s)^\alpha) \in [\pi\alpha/2, \pi\alpha], \quad \arg ((iy + t)^{1-\alpha}) \in [0, \pi(1 - \alpha)/2],
\]

so that

\[
- \arg \left( \frac{1}{(iy + s)^\alpha} \right) - \arg \left( \frac{1}{(iy + t)^{1-\alpha}} \right) \in [\pi\alpha/2, \pi/2 + \pi\alpha/2].
\]

Hence, if \( \varphi := \min(\pi\alpha/2, \pi/2 - \pi\alpha/2) \), then \( \varphi \in (0, \pi/4) \), and

\[
\sin \varphi \frac{1}{|iy + s|^\alpha|iy + t|^{1-\alpha}} \leq - \text{Im} \frac{1}{(iy + s)^\alpha(iy + t)^{1-\alpha}} \leq \frac{1}{|iy + s|^\alpha|iy + t|^{1-\alpha}}, \quad y > 0.
\]

Therefore, if \( y > 0 \), then

\[
(7.27) \quad \sin \varphi \int_0^\infty \int_{-\infty}^0 \frac{\mu_1(ds)\mu_2(dt)}{|iy + s|^\alpha|iy + t|^{1-\alpha}} \leq - \text{Im} f(iy)
\]

\[
\leq \int_0^\infty \int_{-\infty}^0 \frac{\mu_1(ds)\mu_2(dt)}{|iy + s|^\alpha|iy + t|^{1-\alpha}}.
\]
Using (7.27) and Fubini’s theorem, we infer that (7.20) for \( f \) is equivalent to

\[
\int_0^\infty \int_{-\infty}^0 \int_1^\infty \frac{dy}{y|iy + s|iy + t|^{1-\alpha}} \mu_1(ds) \mu_2(dt) < \infty.
\]

In turn, since for all \( s, t \in \mathbb{R} \),

\[
\int_1^\infty \frac{dy}{y|iy + s|iy + t|^{1-\alpha}} = \int_0^1 \frac{d\tau}{(s^2\tau^2 + 1)^{\alpha/2}(t^2\tau^2 + 1)^{(1-\alpha)/2}},
\]

and so

\[
Q_\alpha(|s|, t) \leq \int_1^\infty \frac{dy}{y|iy + s|iy + t|^{1-\alpha}} \leq \sqrt{2} Q_\alpha(|s|, t),
\]

the property (7.28) is equivalent to (7.23), and the statement follows. \( \square \)

It is useful to note that (7.23) and (7.24) can be replaced with a stronger but more transparent condition on the size of \( \mu_1 \in \mathcal{M}_1^+(\mathbb{R}) \) and \( \mu_2 \in \mathcal{M}_1^+(\mathbb{R}) \). On this way, the next simple estimate of \( Q_\alpha \) will be relevant.

**Lemma 7.11.** If \( Q_\alpha \) is given by (7.27), then for every \( \beta \in [0, 1] \),

\[
Q_\alpha(x, y) \leq 2 \log^\beta x \log^{1-\beta} y \frac{1}{x^\alpha y^{1-\alpha}}, \quad x, y \geq \exp(\pi/\sin(\pi \alpha)).
\]

**Proof.** Note that for every \( x \geq \exp(\pi/\sin(\pi \alpha)) \),

\[
Q_\alpha(x, y) \leq \frac{1}{y^{1-\alpha}} \int_0^1 \frac{d\tau}{\tau^{1-\alpha}(\tau + 1)^\alpha} = \frac{1}{x^\alpha y^{1-\alpha}} \int_0^x \frac{d\tau}{\tau^{1-\alpha}(\tau + 1)^\alpha}
\]

\[
\leq \frac{1}{x^\alpha y^{1-\alpha}} \left( \int_0^1 \frac{d\tau}{\tau^{1-\alpha}(\tau + 1)^\alpha} + \int_1^x \frac{d\tau}{\tau} \right) = \frac{1}{x^\alpha y^{1-\alpha}} \left( \frac{\pi}{\sin(\pi \alpha)} + \log x \right),
\]

hence

\[
(7.30) \quad Q_\alpha(x, y) \leq 2 \frac{\log x}{x^\alpha y^{1-\alpha}}, \quad x \geq \exp(\pi/\sin(\pi \alpha)).
\]

Similarly,

\[
(7.31) \quad Q_\alpha(x, y) \leq 2 \frac{\log y}{x^\alpha y^{1-\alpha}}, \quad y \geq \exp(\pi/\sin(\pi \alpha)).
\]

Now writing \( Q_\alpha = Q_\alpha^\beta Q_\alpha^{1-\beta} \) and estimating \( Q_\alpha^\beta \) by (7.30) and \( Q_\alpha^{1-\beta} \) via (7.31), we obtain (7.29). \( \square \)

**Corollary 7.12.** Let \( \alpha \in (0, 1) \), and assume that \( \mu_1 \in \mathcal{M}_0^+(\mathbb{R}) \) and \( \mu_2 \in \mathcal{M}_1^+(\mathbb{R}) \). If \( \beta_j \geq 0, j = 1, 2, \beta_1 + \beta_2 = 1 \), and

\[
A := \int_{|t| \geq a_0} \frac{\log^\beta |t| \mu_1(dt)}{|t|^\alpha} + \int_{|t| \geq a_0} \frac{\log^{\beta_2} |t| \mu_2(dt)}{|t|^{1-\alpha}} < \infty,
\]

for some \( a_0 \geq 1 \), then there exists \( \mu \in \mathcal{M}_1^+(\mathbb{R}) \) satisfying (7.22).
Example 7.13. Let us illustrate the sharpness of Corollary 7.12. Let $\alpha \in (0,1)$ be fixed and for $\delta_j > 0$, $j = 1, 2$, define $\mu_1 \in \mathcal{M}_\alpha^+(\mathbb{R})$ and $\mu_2 \in \mathcal{M}_{1-\alpha}(\mathbb{R})$ by
\begin{equation}
\mu_1(dt) := \frac{\chi(-\infty,-e) (t) dt}{|t|^{1-\alpha} \log^{1+\delta_1} |t|} \quad \text{and} \quad \mu_2(dt) := \frac{\chi(e,\infty) (t) dt}{t^\alpha \log^{1+\delta_2} t}.
\end{equation}
Then by Fubini’s theorem
\begin{align*}
&\int_0^1 \int_{s \geq e} \int_{t \leq -e} \frac{\mu_1(dt) \mu_2(ds)}{|t|^{1-\alpha} (\tau t + 1)^\alpha (s \tau + 1)^{1-\alpha}} d\tau \\
&\geq \int_0^{1/e} \left( \frac{1}{2^{1-\alpha} \pi^\alpha} \int_{1/\tau}^\infty \frac{1}{t^{1-\alpha} \log^{1+\delta_1} t} \frac{dt}{\log^{1+\delta_2} s} \right) d\tau \\
&\geq \frac{1}{2^{1-\alpha} \pi^\alpha} \int_0^{1/e} \frac{1}{\tau \log \tau |\delta_1 + \delta_2|} d\tau.
\end{align*}
If $\delta_1 + \delta_2 \leq 1$, then the latter inequality implies that for $\mu_1$ and $\mu_2$ given by (7.32) the property (7.2) does not hold. In fact, taking into account Corollary 7.12 (7.2) is satisfied for $\mu_1$ and $\mu_2$ from (7.32) if and only if $\delta_1 + \delta_2 > 1$.

7.3. Liftings of Cauchy transforms via positive measures. Recall from Remark 4.11 that if $\beta > \alpha > 0$, $\nu \in \mathcal{M}_\beta^+(\mathbb{R}_+)$, and $\mu \in \mathcal{M}_{1-\beta}(\mathbb{R}_+)$ is given by (4.26), then $S_\alpha[\nu] = S_\beta[\mu]$, so that $S_\alpha[\nu]$ can be ’lifted” to $S_\beta$. In particular, this fact leads sometimes to useful alternative versions of (3.1), where for $\mu_1 \in \mathcal{M}_{\alpha_1}^+(\mathbb{R})$ and $\mu_2 \in \mathcal{M}_{\alpha_2}^+(\mathbb{R})$ a measure $\mu$ belongs to the class $\mathcal{M}_\beta^+(\mathbb{R}_+), \beta > \alpha_1 + \alpha_2$, wider than $\mathcal{M}_{\alpha_1+\alpha_2}(\mathbb{R})$, and at the same time $\mu$ has better properties than the choice provided by (3.1). (For instance, one may pass from a singular $\mu$ to a locally absolutely continuous one.)

The considerations in the preceding subsections raise a natural question whether a similar property holds for the generalized Cauchy transforms, so that $C_\alpha(\mathcal{M}_\alpha^+(\mathbb{R})) \subset C_\beta(\mathcal{M}_\beta^+(\mathbb{R}))$. Though Theorem 7.6 may suggest a positive answer, the answer is in general negative, see Example 7.17 below. At the same time, we identify several natural situations, when such a choice is indeed possible. Our arguments will depend on the next auxiliary statement.
Proposition 7.14. Let $\nu_1 \in \mathcal{M}_\alpha^+(\mathbb{R})$, $\alpha > 0$, be such that $\text{supp} \; \nu_1 \in (-\infty, 0]$. Then for every $\beta > 0$, 

$$C_\alpha[\nu_1](z) = e^{i\pi \beta} C_{\alpha+\beta}[\nu_2](z), \quad z \in \mathbb{C}^+, \tag{7.33}$$

where the locally absolutely continuous $\nu_2 \in \mathcal{M}_{\alpha+\beta}^+(\mathbb{R})$ is given by

$$\nu_2(\tau) := \frac{1}{B(\alpha, \beta)} \left( \int_{[\tau, 0]} (t - \tau)^{\beta - 1} \nu_1(dt) \right) d\tau. \tag{7.34}$$

Proof. By Fubini’s theorem, for all $z \in \mathbb{C}^+$,

$$B(\alpha, \beta)C_{\alpha+\beta}[\nu_2](z) = \int_{-\infty}^{0} \left( \int_{-\infty}^{t} \frac{(t - \tau)^{\beta - 1} \nu_1(dt)}{(z + \tau)^{\alpha+\beta}} \right) \nu_1(dt)$$

$$= \int_{-\infty}^{0} \left( \int_{0}^{\infty} \frac{s^{\beta - 1} ds}{(z + t - s)^{\alpha+\beta}} \right) \nu_1(dt). \tag{7.35}$$

Next, let $t \leq 0$ be fixed. Extending the function $z \rightarrow z^\beta$ by continuity to the upper side of the cut along $(-\infty, 0]$, we have

$$\int_{0}^{\infty} \frac{s^{\beta - 1} ds}{(z + t - s)^{\alpha+\beta}} = e^{-i(\alpha+\beta)\pi} \int_{0}^{\infty} \frac{s^{\beta - 1} ds}{(|z + t| + s)^{\alpha+\beta}}$$

$$= e^{i(\alpha+\beta)\pi} |z + t|^\alpha \int_{0}^{\infty} \frac{s^{\beta - 1} ds}{(1 + s)^{\alpha+\beta}} = e^{i\pi \beta} B(\alpha, \beta) \frac{(z + t)^{\alpha}}{(z + t)^{\alpha}}$$

for all $z < 0$. Then, by the boundary uniqueness for functions bounded and analytic in $\mathbb{C}^+$,

$$\int_{0}^{\infty} \frac{s^{\beta - 1} ds}{(z + t - s)^{\alpha+\beta}} = e^{i\pi \beta} B(\alpha, \beta) \frac{(z + t)^{\alpha}}{(z + t)^{\alpha}}, \quad z \in \mathbb{C}^+. \tag{7.36}$$

Now (7.33) follows from (7.35) and (7.36). \qed

The following corollary from Proposition \ref{7.14} showing that $C_\alpha(\mathcal{M}_\alpha^+(\mathbb{R})) \subset C_{\alpha+2}(\mathcal{M}_{\alpha+1}^+(\mathbb{R}))$ is straightforward.

Corollary 7.15. Let $\nu \in \mathcal{M}_\alpha^+(\mathbb{R})$, $\alpha > 0$. Then there exists $\mu \in \mathcal{M}_{\alpha+2}^+(\mathbb{R})$ such that

$$C_\alpha[\nu](z) = C_{\alpha+2}[\mu](z), \quad z \in \mathbb{C}^+. \tag{7.37}$$

To deduce the result from Proposition 7.14, it suffices to write $\nu = \chi_{(-\infty, 0)\nu} + \chi_{[0, \infty)\nu}$ and to push-forward $\chi_{[0, \infty)\nu}$ to $(-\infty, 0]$.

However, the statement similar to Corollary 7.15 may not hold if $\mathcal{M}_{\alpha+2}^+(\mathbb{R})$ replaced by the smaller class $\mathcal{M}_{\alpha+1}^+(\mathbb{R})$. We proceed with a criterion addressing the problem when $\alpha = 1$ and relying on Proposition 7.14.

Theorem 7.16. Let $\nu \in \mathcal{M}_1^+(\mathbb{R})$. Then there exists $\mu \in \mathcal{M}_2^+(\mathbb{R})$ satisfying

$$C_1[\nu](z) = C_2[\mu](z), \quad z \in \mathbb{C}^+. \tag{7.37}$$
if and only if

\[(7.38) \quad \int_{-\infty}^{0} \nu(dt) < \infty.\]

Proof. Let us first assume that, in addition, supp \(\nu \in (-\infty, 0]\). Then the proof is similar to the proof of the Theorem 7.8. Suppose that \((7.37)\) holds, and let \(\nu_2\) be given \((7.34)\) with \(\nu_1 = \nu\). Then, by Proposition 7.14, we obtain that

\[C_2[\mu](z) + C_2[\nu_2](z) = 0, \quad z \in \mathbb{C}^+.\]

So, in view of Lemma 7.7 there exists \(c \geq 0\) such that

\[\mu(dt) + \nu_2(dt) = c dt,\]

and then

\[\int_{(\tau, 0]} \nu(dt) \leq c, \quad \tau \leq 0,\]

which is equivalent to \((7.38)\).

Conversely, if \((7.38)\) holds, then

\[C_1[\nu](z) = C_2[\mu](z), \quad z \in \mathbb{C}^+,\]

with the locally absolutely continuous measure \(\mu \in \mathcal{M}_2^+(\mathbb{R})\) defined by

\[
\mu(dt) := \begin{cases} 
\left(\int_{(-\infty, t]} \nu(d\tau)\right) dt, & t \leq 0, \\
\left(\int_{(-\infty, 0]} \nu(d\tau)\right) dt, & t > 0.
\end{cases}
\]

In the general case, write \(\nu = \nu_1 + \nu_2\) with \(\nu_1, \nu_2 \in \mathcal{M}_1^+(\mathbb{R})\) such that supp \(\nu_1 \subset (-\infty, 0]\) and supp \(\nu_2 \subset [0, \infty)\). By applying the preceding case to \(\nu_1\) and \((7.28)\) to \(\nu_2\), we get the sufficiency. For the proof of necessity one may simply restrict \(\nu\) to \((\tau, 0]\) and refer to the first part of the proof. \(\square\)

Example 7.17. Let \(\alpha \in (0, 1)\) and

\[\nu(dt) = \chi_{(-\infty,-1]}(t) \frac{dt}{|t|^\alpha},\]

so that \(\nu \in \mathcal{M}_1^+(\mathbb{R})\) but \((7.38)\) does not hold. Hence, there is no \(\mu \in \mathcal{M}_2^+(\mathbb{R})\) satisfying \((7.37)\). Note that, by a direct computation, for every \(z \in \mathbb{C}^+\),

\[C_1[\nu](z) = \int_{-\infty}^{-1} \frac{dt}{(z + t)|t|^\alpha} = - \int_{0}^{1} \frac{\tau^{\alpha-1} d\tau}{1 - z\tau} = -\alpha \, _2F_1(1, \alpha, \alpha + 1; z),\]

where \(_2F_1\) is a hypergeometric function, see [1, p. 65].

Invoking geometrical arguments and Kac’s theorem formulated above, we finish this section with a lifting criterion from \(\mathcal{M}_1^+(\mathbb{R}), \alpha \in (0, 1)\), to \(\mathcal{M}_2^+(\mathbb{R})\). It looks similar to Theorem 7.16 although its proof is based on a different idea.
Theorem 7.18. Let \( \nu \in \mathcal{M}_+^+(\mathbb{R}) \), \( \alpha \in (0, 1) \). Then there exists \( \mu \in \mathcal{M}_+^+(\mathbb{R}) \) such that

\[
C_\alpha[\nu](z) = C_1[\mu](z), \quad z \in \mathbb{C}^+,
\]

if and only if

\[
\int_{-\infty}^{0} \frac{\log(|t| + 1) \, \nu(dt)}{(1 + |t|)^\alpha} < \infty.
\]

Proof. As in the proof of Theorem 7.16, we may assume without loss of generality that \( \text{supp} \nu \subset (-\infty, 0] \). First, observe that

\[
\text{Im} \, C_\alpha[\nu](z) \leq 0, \quad z \in \mathbb{C}^+.
\]

Since

\[
\arg \left( (t + iy)^\alpha \right) \in [\pi\alpha/2, \pi\alpha], \quad y > 0, \quad t < 0,
\]

then, similarly to the proof of Theorem 7.10,

\[
c_1 \int_{-\infty}^{0} \frac{\nu(dt)}{|iy + t|^\alpha} \leq |\text{Im} \, C_\alpha[\nu](iy)| \leq c_2 \int_{-\infty}^{0} \frac{\nu(dt)}{|iy + t|^\alpha}, \quad y > 0,
\]

for some \( c_1, c_2 > 0 \). Thus,

\[
c_1 \int_{\mathbb{R}} \int_{1}^{\infty} \frac{dy}{y |iy + t|^\alpha} \nu(dt) \leq \int_{1}^{\infty} \frac{|\text{Im} \, C_\alpha[\nu](iy)| \, dy}{y}
\]

\[
\leq c_2 \int_{\mathbb{R}} \int_{1}^{\infty} \frac{dy}{y |iy + t|^\alpha} \nu(dt).
\]

So, using Lemma 4.1, we infer that

\[
\int_{-\infty}^{0} \int_{1}^{\infty} \frac{dy}{y |iy + t|^\alpha} \nu(dt) < \infty \quad \iff \quad \int_{-\infty}^{0} \frac{\log(|t| + 1) \, \nu(dt)}{(1 + |t|)^\alpha} < \infty,
\]

and obtain the assertion by the Kac criterion (7.19), (7.20).

\[\square\]

8. Final remarks

We finish the paper with two remarks addressing the scope of the paper and some relevant problems left open.

8.1. The emphasis in this paper is put on the study of the generalized Stieltjes and Cauchy transforms of measures from \( \mathcal{M}_+^+ \)-spaces or of functions from \( L_1^+ \)-spaces. As far as our considerations fit into a more general framework of complex Radon measures, which help us to define the resulting product measures properly and to not distinguish between functions and measures, we have chosen it to simplify and unify our presentation. However, our choice looks more like a matter of convenience than a substantial technical advance, and we are not aware of any of its applications of a separate interest.
8.2. Unfortunately, we were not able to answer the next two questions, which we believe might deserve further research. First, in the framework of Theorem 7.6, given \( \mu_1 \in M_{\alpha_1}^+ (\mathbb{R}) \) and \( \mu_2 \in M_{\alpha_2}^+ (\mathbb{R}) \), \( \alpha_1, \alpha_2 > 0 \), it is not, in general, clear whether there always exists a \( \mu \in M_{\alpha_1 + \alpha_2}^+ (\mathbb{R}) \) such that \( C_{\alpha_1} [\mu_1] C_{\alpha_2} [\mu_2] = C_{\alpha_1 + \alpha_2} [\mu] \). Theorems 7.8 and 7.10 do not clarify this issue, and Theorem 7.14 produces such a \( \mu \) under somewhat stringent additional assumptions. Second, given \( \alpha_2 > \alpha_1 > 0 \), it is not quite clear how to characterize \( \mu_1 \in M_{\alpha_1}^+ (\mathbb{R}) \), satisfying \( C_{\alpha_1} [\mu_1] = C_{\alpha_2} [\mu_2] \) for some \( \mu_2 \in M_{\alpha_2}^+ (\mathbb{R}) \), and thus liftable to \( C_{\alpha_2} (M_{\alpha_2}^+ (\mathbb{R})) \). (Recall that by Corollary 7.15 \( \mu_1 \) is liftable to \( C_{\alpha_1 + 2} (M_{\alpha_1 + 2}^+ (\mathbb{R})) \).) While we showed that such a lifting is not always possible and described in Theorems 7.16 and 7.18 the two particular situations allowing for a complete answer, the question is, in general, widely open.

9. Acknowledgments

We would like to thank the referee for careful reading the manuscript and useful remarks and suggestions.

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