Experimental discovery of bulk-disclination correspondence

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Abstract

The past decade has witnessed the emergence of abundant topological phases protected by crystalline symmetries, such as topological crystalline insulators (TCIs) [1-3]. Recent discoveries of higher-order [4-7] and fragile [8-11] TCIs illustrate that gapless edge states may not be a universal feature of TCIs, and thus request alternative experimental signatures. Here, we demonstrate that disclinations, i.e., rotational topological defects in crystals, can serve as a novel probe of TCIs without invoking edge boundaries. Using photonic crystals, i.e., artificial periodic structures giving access to versatile manipulations of light [12], we observe topologically-protected photonic modes localized at disclinations in photonic TCIs. We create physical realizations of photonic TCIs and disclinations using macroscopic dielectric structures based on hexagonal lattices. By driving the photonic system across a topological transition through tuning the unit-cell geometry, we find that the topological disclination modes disappear in the trivial phase. The experimental discovery of the bulk-disclination correspondence unveils a new horizon in topological physics, while its photonic realization provides a pathway toward topological photonics [13-23] beyond the bulk-edge correspondence.
Symmetry plays a pivotal role in the classification of topological states of matter [1-3]. Recent attempts for the complete catalogue of topological phases in condensed matter systems reveal that the 230 space groups of three-dimensional crystals give rise to a vast number of symmetry-protected topological insulators (TIs) [24-26]. Underneath such a catalogue is the principle of adiabatic continuity: if an insulator can be transformed into a trivial atomic insulator (the atomic limit with vanishing inter-unit-cell coupling) without closing the band gap or breaking the protective symmetry, then it is a normal insulator (NI). If an insulator can be adiabatically transformed to a prototype TI, then it is a TI. Interestingly, there are a large number of insulators which are not adiabatically connected to NIs or conventional TIs (those having gapless edge states). For instance, higher-order TIs, which are adiabatically distinct from NIs and conventional TIs, have gapped edge states [4-7, 27-35]. Likewise, fragile TIs do not support stable gapless edge states [8-11]. In theory, fragile bands exhibit gapless Wannier bands---a feature of conventional TIs [36], yet their Wannier bands can become gapped by adding trivial energy bands to the system. However, such Wannier bands characteristics do not yield any physical consequence, making fragile bands intangible in experiments [37].

Topological defects, e.g., dislocations and disclinations, are stable crystalline defects which may also induce topological boundary states. For instance, dislocations can bind topological modes in certain conventional TIs [38-42]. It would be invaluable if experimental signatures of unconventional TIs, such as fragile and higher-order TIs, can be identified on topological defects. As suggested by recent theories [11, 35, 43-45], localized states bound to disclinations (denoted as “disclination states/modes”) can serve as observable signatures of those unconventional TIs [11, 35]. However, so far, such topological disclination states have not yet been observed.

Here, we use two-dimensional (2D) photonic crystals (PhCs) with six-fold rotation ($C_6$) symmetry to identify topological disclination states as the first experimental evidence of bulk-disclination correspondences. The underlying physics is illustrated in Fig. 1. The disclination is a rotational topological defect which is constructed by removing a $2\pi/6$ sector of the perfect crystal and gluing the remaining together (Fig. 1a). TCIs and NIs have distinct responses to disclinations. In the TCIs, the Wannier centers are located at the edges of the unit-cell (Fig. 1b). Thus, each Wannier center is shared by two neighboring unit-cells and
contributes $\frac{1}{2}$ electron to one of them. When going around the disclination core, the boundary of the core (the green lines in Fig. 1b) runs through five Wannier centers, which induces a separation of the charge associated with these Wannier centers. We hence expect the emergence of topological disclination states. In the NIs, the Wannier centers are at the unit-cell center and away from the core (Fig. 1c). Thus, there is no mechanism to support disclination states. This scenario is confirmed in Figs. 1d and 1e where the TCI and NI are realized by tight-binding models with six sites in each unit-cell. The TCI (NI) has inter-unit-cell couplings stronger (weaker) than the intra-unit-cell couplings. In the tight-binding picture, the disclination states in the TCI originate from the five weakly-coupled sites (purple dots in Fig. 1d) near the disclination core (see Supplementary Note 1 for the detailed properties of the two tight-binding models).

Figure 1 | Disclination as a probe of bulk topology. a, Illustration of a disclination in hexagonal crystals. b-c, Distributions of Wannier centers in disclinations for the topological (b) and trivial (c) cases. Green lines denote the boundary of the disclination core. d-e, Disclination spectrum for the TCI (d) and NI (e) realized by tight-binding models. Insets: Left, tight-binding models of disclinations (showing only a part
of the big structure; purple dots denote the weakly-coupled sites close to the disclination core); Right, tight-binding models of the unit-cell (black dots represent the sites; thick (thin) lines represent strong (weak) couplings; red dots represent the Wannier centers).

The photonic TCI and NI are realized using the PhC illustrated in Fig. 2a. In each unit-cell, there are seven dielectric pillars with identical radius $r = 2$ mm. The lattice constant is $a = \sqrt{3}$ cm. The PhC can be configured by removing the central pillar or tuning the distance $d$, to switch between the TCI and NI phases. For instance, with no central pillar, the TCI can be realized with $d/a = 0.5$, while the NI can be realized with $d/a = 0.23$. Their photonic bands are presented in Figs. 2b and 2c, respectively. From the symmetry properties of the photonic bands, we find that the Wannier centers of the TCI (NI) are at the edges (center) of the unit-cell (see Supplementary Note 2 for detailed analysis).

Following Refs. [35, 45], the bulk topological index, as determined by the symmetry of the photonic bands, is given by $\chi = (\chi_M, \chi_K)$ with $\chi_M = \#M_1^{-2}) - \#M_1^{(2)}$ and $\chi_K = \#K_1^{(3)} - \#K_1^{(3)}$. Here, $\Pi_P^{(n)}$ is the number of bands below the band gap at a high-symmetry point $\Pi = \Gamma, M, K$ with the $C_n$ rotation eigenvalue $e^{i2\pi(p-1)/n}$ ($p = 1, ..., n$). We find that $\chi_M = -2$ or 0 for the TCI and NI phases, respectively, while $\chi_K = 0$ for both phases (see Supplementary Note 3 for details). A disclination index, $Q_{dis}$, is used to indicate the disclination anomaly [35, 45]. Remarkably, it is completely determined by the bulk topological indices and the Frank angle $\Omega$ (see Fig. 1a) as

$$ Q_{dis} = \frac{\Omega}{2\pi} \left( \frac{3}{2} \chi_M - \chi_K \right) \mod 1. \quad (1) $$

The above equation gives $Q_{dis} = \frac{1}{2}$ for the TCI and 0 for the NI, respectively. The nontrivial disclination index in the TCI indicates the emergence of the disclination states. In electronic systems, $Q_{dis} = \frac{1}{2}$ also yields a fractional disclination charge [35, 45], which, however, does not appear in photonics.

The ‘phase diagram’ in Fig. 2d demonstrates the transition between the TCI and NI phases at $d = a/3$, which agrees with previous studies on similar PhCs [21, 22]. We find that...
the TCI is a higher-order TI which supports gapped edge states and in-gap corner states (see Supplementary Note 3). Interestingly, the TCI can be adiabatically tuned to exhibit fragile bands [8-11]. This is achieved by adding the central pillars, as shown in Supplementary Note 4. Therefore, the same disclination physics also applies to fragile TIs, although fragile bands do not admit the construction of Wannier centers [11, 35, 45].

**Figure 2 | Photonic TCI and disclination.** a, Illustration of a 2D topological PhC. The center-to-center distance between the central pillar (dark-blue, permittivity $\varepsilon_2 = 11$) and the other pillars (light-blue, permittivity $\varepsilon_1 = 8.4$) is $d$. b-c, Photonic bands of PhCs with $d/a = 0.5$ (b) and 0.23 (c) (no central pillar). Little group representations are labeled at the high-symmetry points. Insets illustrate the Brillouin zone and the Wannier centers. d, Bulk topological ‘phase diagram’ for PhCs without the central pillar. Transition between the TCI and NI phases is indicated by the band-crossing between the $E_1$ (blue) and $E_2$ (red) representations at the $\Gamma$ point. In b-d, green/cyan zones indicate the band gaps of interest. e-f, Disclination spectra for the two PhCs studied in b-c.

The bulk-disclination correspondence is numerically tested by calculating the photonic spectrum of the disclination that contains 30 unit-cells and a hollow disclination core. Fig. 2e shows that there are five disclination states in the TCI phase, confirming the tight-binding
picture. For these disclination states, the electric field distributes mainly around the boundary of the disclination core (see Fig. 3). In contrast, in the NI phase, there is only one localized mode of which the electric field is strongly confined in the hollow disclination core (see Fig. 4). This mode is a defect mode which is common in PhCs with a hollow region [12]. The emergence of such a defect mode is due to the key difference between electrons and photons: unlike electrons described by the tight-binding models which are confined in the sites outside the disclination core, photons in PhCs can propagate into and get trapped in the hollow disclination core.

In experiments, the disclination is surrounded by microwave absorption sponges (Fig. 3a). The whole sample is cladded by parallel metal plates above and below to form a 2D photonic system with transverse-magnetic modes (see Materials and Methods). We start with the disclination for the PhC with \( d/a = 0.5 \) (the TCI phase). The photonic spectrum (top panel of Fig. 3b) and eigenstates are calculated numerically with open boundary conditions (see Materials and Methods). The disclination states are well-localized around the disclination core (Fig. 3c) and are insensitive to the finite-size effect for the structure we adopted (see Supplementary Note 5 for details). Meanwhile, there are bulk states emerge in the photonic band gap due to the finite-size effect. We use the transmission measurements to detect the disclination modes. The positions of the source and the detector (labeled as A and B, respectively, in Fig. 3a) are designed to ensure visible resonances for the disclination modes, as indicated by the simulation in Fig. 3b (middle panel).

The consistent simulation and measurement results in Fig. 3b indicate that there are three resonances, denoted as \( \alpha, \beta \) and \( \gamma \), at the frequencies of 9.60 GHz, 10.35 GHz and 10.74 GHz, respectively. The \( \alpha \) resonance gives a non-degenerate mode which falls slightly into the bulk band region. The overlap between the \( \alpha \) mode and the bulk bands is mainly due to the hollow disclination core which acts as an attractive potential and reduces the frequency of the \( \alpha \) mode. The \( \beta \) and \( \gamma \) resonances give two sets of doubly-degenerate modes.

To further explore the disclination modes, we measure their photonic wavefunctions (i.e., the electric-field distributions) using the near-field scanning method (see Materials and Methods). The key challenge here is to detect the doubly degenerate modes. Within each resonance, the two degenerate states can hybridize with each other in arbitrary manners. A
single point-like source can excite only one hybridized state (the ‘bright-state’) and leave the other one (the ‘dark-state’) undetectable. We design two point-like sources, C and D, which locate symmetrically around the mirror symmetry line (white-dashed in Fig. 3a), to excite the two degenerate modes. With respect to the mirror symmetry, the doubly degenerate states form the symmetric and anti-symmetric states which can be excited, respectively, when C and D have phase difference 0 or π. Using this method, we selectively excite all five disclination states and measure their wavefunctions via the near-field scanning method (Fig. 3d). In the figures, $\psi_{\alpha}$ denotes the $\alpha$ state, while $\psi_{\beta,S}$ and $\psi_{\beta,A}$ ($\psi_{\gamma,S}$ and $\psi_{\gamma,A}$) represent, respectively, the symmetric and anti-symmetric states of the $\beta$ ($\gamma$) resonance. The measured disclination wavefunctions agree well with the calculated ones.
**Figure 3 | Topological disclination modes.** a, Illustration of the experimental system. A PhC disclination made of Al₂O₃ pillars (pink) is cladded by parallel metal plates (gray) above and below, and is surrounded by microwave absorption sponges (blue). Lower-right: top-down view. b, Eigenstates spectrum (top), and the simulated (middle) and measured (down) transmission between source A and detector B for the PhC with $d = a/2$ (no central pillar). Gray regions denote the bulk-band regions. c-d, Electric-field profiles of disclination states from eigenstates calculation (c) and measurements (d). The positive and negative signs indicate whether the sources C and D (green stars) have the same sign or opposite signs.

**Figure 4 | Bulk-disclination correspondence.** a, Eigenstates (top) and transmission between A and B (middle and down) for the PhC with $d/a = 0.23$ (no central pillar). Gray regions denote the bulk-band regions. b, Electric-field profiles of the disclination state $\alpha$ from eigenstates calculation and near-field-
scanning experiment (excited by the sources C and D). c, Eigenstates (top) and transmission between A and B (middle and down) for the PhC with \( d/a = 0.38 \) (no central pillar). d, ‘Phase diagram’ for the bulk bands (gray regions) and disclination states (lines). Dots with error-bars represent frequencies of disclination modes as extracted from transmission measurements. The black-dashed line denotes the topological transition at \( d/a = 1/3 \).

We then study the disclination for the PhC with \( d/a = 0.23 \) (the NI phase). Fig. 4a shows from both the eigenstates and transmission aspects that there is only one defect mode \( \delta \) at 9.93 GHz. The transmission experiments confirm the simulation results. The measured disclination wavefunction is also consistent with the calculated wavefunction (Fig. 4b). The minor differences between these wavefunctions originate from the fact that the experiments detect a superposition of the eigenstate wavefunction and the evanescent waves excited by the source antennas. The eigenstate wavefunction indicates that the electric-field of the defect mode is strongly confined in the disclination core---a feature of defect modes commonly seen in PhCs with a hollow region (see Supplementary Note 7).

To recheck the bulk-disclination correspondence, another PhC with \( d/a = 0.38 \) (the TCI phase) is measured. Eigenstates calculation indicates again that there are five disclination states, corresponding to the three resonance peaks, \( \alpha, \gamma \) and \( \beta \), at 8.65 GHz, 8.72 GHz, and 9.10 GHz in the simulated and measured transmission spectra (Fig. 4c). The disclination wavefunctions from simulations and measurements agree with each other (see Supplementary Note 6).

A disclination ‘phase diagram’ (Fig. 4d) is obtained through the evolution of the bulk bands and the disclination states with the distance \( d \). We find that the disclination states emerge only in the TCI phase and disappear in the NI phase. Besides, we numerically test the robustness of the disclination and defect modes against disorders and find that the topological disclination modes in the TCI phase are indeed more robust than the defect mode in the NI phase (see Supplementary Note 8).

This work explores the unprecedented regime of bulk-disclination physics in photonic experiments. Many interesting future studies in this regime could be inspired, especially in three-dimensional photonic [23] and electronic [1-3] TCIs, where rich bulk-disclination
physics emerges due to the versatile crystalline symmetry. From material perspectives, beside photonic and electronic systems, the study of bulk-disclination physics can be generalized to polaritonic, plasmonic, acoustic and phononic metamaterials to reveal novel topological phenomena and their applications.

**Materials and Methods**

**Sample fabrication**

All samples are constructed by carefully pasting dielectric cylinders into a pre-printed drawing on which the location of each cylinder is marked. As the basic element of the PhCs, the dielectric cylinders and half-cylinders are made of commercial alumina ceramics (Al₂O₃) doped with chromium dioxide. These cylinders and half-cylinders have the same height of 10 mm and the same radius of 2 mm. The relative permittivity of the alumina ceramics is measured by the transmission/reflection method using the Weir-Nicolson-Ross method. It changes little at the X-band and can be treated as a constant of $8.4 - 0.02i$ in the frequency range studied in this work. The disclination samples are constructed by removing $2\pi/6$ sector from the $C_6$-symmetric hexagonal lattices. Each disclination structure includes 30 unit-cells and a hollow disclination core as depicted in Fig. 3a. When the dielectric cylinders overlap with each other, we combine the overlapping cylinders together and there may be fractional cylinders in the boundary unit-cells. For instance, for the case with $d/a = 0.5$, half-cylinders are used in the unit-cells close to the boundary or the disclination core.

**Experimental setup**

The experimental systems are based on the quasi-2D electromagnetic systems with metallic cladding from the above and the below. The distance between these parallel metal plates is 11 mm. This setup ensures that the first three photonic bands are dominated by the transverse-magnetic harmonic modes and the system can be well approximated by the 2D simulation. Unlike electronic systems where the band filling effect is important, in photonic systems any band gap is experimentally accessible by choosing the right excitation frequency. A 1 mm air gap between the PhC and the upper metal plate is introduced to enable the moving of the lower
metal plate where the PhC is mounted on. In the scanning measurements, the sources are fed by two antennas (diameter 0.53 mm) with phase difference 0 or \( \pi \). They have a height of 9.9 mm and are inserted through and fixed on the lower plate. A vector network analyzer Agilent E8363A is used to record the magnitude and phase of the electromagnetic signal received by the detector cable. In the scanning measurements, the lower plate (and the source antennas on it) is mounted on a translational stage with the scan step of 2 mm, while the upper plate and the detector antenna (which is embedded in the upper metal plate and connected to the detector cable) is kept stationary. The transmission spectra are obtained in a stationary setup without the air gap. In the transmission measurements, the source and the detector (both have diameter 1.24 mm) are fixed. The frequency dependence of the transmission is quantified through the intensity of the detection signal, while the intensity of the input signal is fixed.

**Simulation**

Numerical simulations are performed using a commercial finite-element simulation software (COMSOL MULTIPHYSICS) via the RF module. Simulations are performed for 2D transverse-magnetic harmonic modes (i.e., the electric-field is perpendicular to the 2D plane, along the \( z \) direction). The bulk band regions (gray) in all figures are calculated for infinite PhCs, while the disclination states are calculated using the disclination structures with 30 unit-cells depicted in Figs. 3 and 4.

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**Author contributions**
J.H.J initiated the project. J.H.J and Y.P guided the research. J.H.J and Y.L established the theory. Y.L, Z.K.L and Y.P performed the numerical calculations and simulations. F.F.L, S.L, X.T, J.H.J and Y.P designed and achieved the experimental set-up and the measurements. All the authors contributed to the discussions of the results and the manuscript preparation. J.H.J, Y.P, Y.L, and Z.K.L wrote the manuscript and the Supplementary Information.

Competing Interests

The authors declare that they have no competing financial interests.

Data availability

All data are available in the manuscript and the Supplementary Information. Additional information is available from the corresponding authors through proper request.

Code availability

We use the commercial software COMSOL MULTIPHYSICS to perform the simulation and calculations. Request to the details can be addressed to the corresponding authors.

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