SOME REMARKS ON A MINKOWSKI SPACE \((R^n, F)\)

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Abstract. We consider a complete, totally umbilical hypersurface \(M\) of Riemannian space \((\hat{R}^n, \hat{g})\) induced by a Minkowski space \((R^n, F)\). Under certain conditions we prove that \(M\) is isometric to a "round" hypersphere of the \((n + 1)\)-dimensional Euclidean space. We also prove that the Minkowski norm \(F\) must be arised from an inner product if there exist a non-zero vector field, which is parallel according to Levi-Civita connection of the metric tensor \(\hat{g}\).

1. Introduction

Let \((N, F)\) be an \(n\)-dimensional Finsler space, then for any \(x \in N\) the restriction \(F_x\) of \(F\) to the tangent space \(T_xN\) is a Minkowski norm. That is, \(F_x : T_xN \rightarrow R\) is non-negative, smooth function on \(T_xN - \{0\}\), positive homogenous of degree one and strongly convex. The last condition means that the functions \(g_{ij}(x, y) := \frac{1}{2} F^2(x, y)\partial y_i \partial y_j\) give a positive definite quadratic form on \(T_xN - \{0\}\), that is \(g := g_{ij}(x, y)dy^i dy^j\) is a Riemannian metric on \(T_xN - \{0\}\).

If for any \(x \in N\), \(g_{ij}(x, y)\) doesn’t depend on \(y\), then \(g = g_{ij}(x, y)dy^i dy^j\) is an inner product on \(T_xN\), and \(G := g_{ij}(x, y)dx^i dx^j\) is a Riemannian metric on \(N\). Of course, any Riemannian space is also Finslerian, but the inverse is not true. It is easy to see that \(G := g_{ij}(x, y)dx^i dx^j\) is Riemannian metric on \(N\) iff the Cartan torsion tensor \(C_{ijk} := \frac{\partial g_{ij}}{\partial y^k}\) vanishes.

When does the Minkowski norm \(F_x\) arise from an inner product on \(T_xN\)? Or equivalently when does the Finslerian metric function \(F(x, y)\) on \(TN\) determine a Riemannian metric \(G := g_{ij}(x, y)dx^i dx^j\)? This is a very intersting and important question. Using the maximum principles, the famous Deicke’s theorem says that this occurs if the mean Cartan torsion \(A_k := g^{ij}C_{ijk}\) vanishes. Since for any \(x \in N\), the tangent space \(T_xN\) is an \(n\)-dimensional vector space, so for simplicity in this paper we consider the \(n\)-dimensional Euclidean space and a Minkowski norm \(F\) on it. So, let \((R^n, F)\) be a Minkowski space and \(g_{ij}(y) := (\frac{1}{2} F^2(y))_{y^i y^j}\). Then \(\hat{g} := g_{ij}(y)dy^i \otimes dy^j\) defines a Riemannian metric on \(\hat{R}^n := R^n - \{0\}\) and \((\hat{R}^n, \hat{g})\) becomes

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a Riemannian space. It is well known that when $F$ is induced by an inner product on $R^n$, $(\hat{R}^n, \hat{g})$ is flat (that is, the curvature tensor vanishes). Furthermore, (1) any proper totally umbilical hypersurface is a "round" hypersphere and (2) any constant vector field is parallel according to a canonical Levi-civita connection induced by $\hat{g}$.

In the present paper, concerning a proper totally umbilical hypersurface $M$ of a Riemannian space $(\hat{R}^n, \hat{g})$ we try to find the answer for the following two questions: 1. Under which conditions is $M$ isometric to a "round" sphere. 2. If there exists a non-zero constant, and parallel vector field according to $\hat{g}$, then under which conditions is the Minkowski norm $F$ arised from an Euclidean inner product? Our main results are:

**Theorem 1.** Let $M$ be a proper totally umbilical hypersurface of a Riemannian space $(\hat{R}^n, \hat{g})$ induced by a Minkowski space $(\mathbb{R}^n, F)$, and let $M$ be no level set $S(r) := \{y | F(y) = r\}$, $r > 0$. Suppose that the normal part of $R(X, Y)Z$, $X, Y, Z \in \mathcal{X}(M)$ vanishes, i.e. $(R(X, Y)Z)^\perp = 0$. Then $M$ must be isometric to a "round" hypersphere of the $(n + 1)$-dimensional Euclidean space.

**Theorem 2.** Let $(\hat{R}^n, \hat{g})$ be a Riemannian space induced by a Minkowski space $(\mathbb{R}^n, F)$. If $n \geq 3$, and $F$ is absolutely homogeneous, then $F$ is arised from an inner product if and only if there exists a non-zero constant vector field $b$ which is parallel according to the Levi-civita connection $\hat{\nabla}$ of $\hat{g}$, that is if $\hat{\nabla}_X b = 0$.

### 2. Preliminaries

In this section, we collect some facts of the Minkowski space $(\mathbb{R}^n, F)$. For more details see [1]. We use the Einstein convention. That is, repeated indexes with one upper and one lower indexes denote summation over their range throughout this paper.

Let $(\mathbb{R}^n, F)$ be a Minkowski space, and let $(y^1, y^2, ..., y^n)$ denote a canonical global coordinate system on $\mathbb{R}^n$. Then at every point $y \in T_y \mathbb{R}^n$ we have $\{\frac{\partial}{\partial y^i}\}$ and $\{dy^i\}$ which are the bases of the tangent space $T_y \mathbb{R}^n$ and the cotangent space $T^*_y \mathbb{R}^n$, respectively. The Minkowski norm on $\mathbb{R}^n$ is a continuous function $F : \mathbb{R}^n \to \mathbb{R}^+$ which has the properties:

1. $F$ is smooth on $\hat{R}^n := \mathbb{R}^n - \{0\}$
2. $F(y) > 0$ for all vector $y \in \hat{R}^n$
3. positive homogenous, i.e., $F(\lambda y) = \lambda F(y)$ for all $y \in \mathbb{R}^n$ and all $\lambda > 0$, and
(4) $F$ is strongly convex, i.e., the quantities $g_{ij}(y) := \frac{1}{2} \frac{F^2}{\partial y^i \partial y^j}$ form a positive definite matrix. Thus in this case $\hat{g}(y) := g_{ij}(y) dy^i dy^j$ defines a metric tensor for $\hat{R}^n$.

Now we recall some important facts related to the Riemannian space $(\hat{R}^n, \hat{g})$ ([1]): Let $\hat{\nabla}$ and $\hat{\gamma}^i_{jk}$ denote the Levi-Civita connection and the Christoffel symbols of the second kind of the metric tensor $\hat{g}$. Then

$$\hat{\nabla} \frac{\partial}{\partial y^k} := \hat{\gamma}^i_{jk} \frac{\partial}{\partial y^i}$$

(2.1)

where

$$\hat{\gamma}^i_{jk} = \frac{\hat{g}^{is}}{2} \left( \frac{\partial \hat{g}_{sj}}{\partial y^k} + \frac{\partial \hat{g}_{sk}}{\partial y^j} - \frac{\partial \hat{g}_{jk}}{\partial y^s} \right).$$

(2.2)

It is easy to see that

$$\hat{\gamma}^i_{jk} = \frac{1}{F} C^i_{jk},$$

(2.3)

where

$$C^i_{jk}(x, y) := \hat{g}^{ks} C_{ij}(x, y)$$

comes from the Cartan torsion

$$C_{ijk}(x, y) := \frac{F \partial^3 F^2(x, y)}{4 \partial y^i \partial y^j \partial y^k}.$$

The curvature tensor of $\hat{\nabla}$ is

$$\hat{R}(U, V)W := (\hat{\nabla}_U \hat{\nabla}_V - \hat{\nabla}_V \hat{\nabla}_U - \hat{\nabla}_{[U,V]})W,$$ $U, V, W \in \mathfrak{X}(M)$.

In a local coordinate system,

$$\hat{R}(U, V)W = W^i U^k V^l \hat{R}^i_{jkl} \frac{\partial}{\partial y^j},$$

where in our case, it can be shown that

$$\hat{R}^i_{jkl} = \frac{1}{F^2} (C^s_{jk} C^i_{sl} - C^s_{jl} C^i_{sk}).$$

For any two linearly independent tangent vectors $U, V$ in $T_y \hat{R}^n$ the sectional curvature corresponding to the 2-plane defined by $U$ and $V$ is given by

$$\hat{K}(U, V) := \frac{\hat{g}(\hat{R}(U,V)V,V)}{\hat{g}(U,U)\hat{g}(V,V) - (\hat{g}(U,V))^2}.$$

If the sectional curvature $K$ does not depend on the choice of the 2-plane, then we speak of constant curvature or the space is said to be a space form.

In the case of a level set $S(r) := \{ y \in R^n : F(y) = r > 0 \}$, we have
Theorem 3. ([1], Prop. 14.6.1, p. 401) The sectional curvature $K(U,V)$ of a level hypersurface $S(r)$ is related to the sectional curvature $\hat{K}(U,V)$ of $(\hat{R}^n, \hat{g})$ by

$$K(U,V) = \hat{K}(U,V) + \frac{1}{r^2}$$

and the following statements are equivalent:

(a) The Riemannian space $(\hat{R}^n, \hat{g})$ is flat.

(b) For any $r > 0$, the level hypersurface $S(r)$ has constant sectional curvature $\frac{1}{r^2}$.

(c) For some $r_0 > 0$, the level hypersurface $S(r_0)$ has constant sectional curvature $\frac{1}{r_0^2}$.

In the proof of our results we will also need the following important theorems.

Theorem 4. ([1], Theorem 14.9.2 (Brickell-Theorem), p.415) Let $(R^n, F)$ be a Minkowski space. If $n \geq 3$, $F$ is absolutely homogeneous of degree 1 and the Riemannian space $(\hat{R}^n, \hat{g})$ is flat, then $F$ must be the norm of an inner product on $R^n$.

From Theorem 3 and Theorem 4 we have

Corollary 1. ([1]) If $n \geq 3$, $F$ is absolutely homogeneous of degree 1 and for some $r > 0$, $(S(r), \hat{g})$ has constant sectional curvature $K = \frac{1}{r^2}$, then the Minkowski norm is induced by an inner product.

Theorem 5. (Obata’s Theorem, [10]) In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function $\varphi$ with $\nabla_X d\varphi + c^2 \varphi X = 0$ for any vector field $X$, it is necessary and sufficient that the manifold be isometric with a sphere $S^n(c)$ of radius $\frac{1}{c}$ in the $(n+1)$-Euclidean space.

Obata’s theorem is very useful to get important results in geometry of submanifolds, firstly based on the works of Bang-Yen Chen ([2]) and its version for Finsler geometry is given by B. Badabad ([3]). We can find the studies of submanifolds of Minkowski spaces in work of X. Cheng and J. Yan ([4]), and the studies of umbilical hypersurfaces of Minkowski spaces in the sense of Finsler geometry can be found in the paper of J. Li ([9]).

3. Proof of Theorem 1.

For the totally umbilical hypersurface $M$ of Riemannian space $(\hat{R}^n, \hat{g})$ the Gauss equation is

$$\hat{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

where $h$ is the second fundamental form. In our case $(M,g)$ is a proper totally umbilical hypersurface. So we have

$$\hat{\nabla}_X Y = \nabla_X Y + Hg(X,Y)\nu,$$
where $H$ and $\nu$ denote the mean curvature and the unit normal vector field of the hypersurface $M$ and $H \neq 0$. We choose the unit normal vector $\nu$ in such a way that the mean curvature $H$ takes positive value somewhere on $M$. From the identity $g_{ij}(y)y^iy^j = F^2(y)$ it follows that $\frac{\nu}{F(y)}$ is the unit normal vector field of the level hypersurface $S(r)$. Since $F^2(y)$ is positive homogenous of degree 2, the functions $g_{ij}(y)$ are positive homogenous of degree 1, from (2.2) we have

$$\hat{g}_{ij}^k(y)y^j = C_{ij}^k(y)y^j = 0,$$

from which we obtain

$$\hat{\nabla}_X y = X, \quad \forall X \in \mathcal{X}(\mathbb{R}^n).$$

Let us define a function $f(y) := \hat{g}(y, \nu)$ on $M$. Since $M$ is not a level set, the function $f$ is non-constant. For, if $f(y)$ is constant, then for any vector field $X$ on $M$, we have

$$0 = X(\hat{g}(y, \nu)) = \hat{g}(\hat{\nabla}_X y, \nu) + \hat{g}(\hat{\nabla}_X \nu, y) = \hat{g}(X, \nu) + \hat{g}(y, -HX) = -H\hat{g}(y, X),$$

from which $X(F^2(y)) = X(\hat{g}(y, y)) = 2\hat{g}(\hat{\nabla}_X y, y) = \hat{g}(X, y) = 0$. This means $F(y) = \text{constans}$. Its contradiction to the fact that $M$ is not a level set.

Now we compute the gradient, $\nabla f$ of a function $f$ in $M$. We have

$$X(f) = X\hat{g}(y, \nu) = \hat{g}(\hat{\nabla}_X y, \nu) + g(y, \hat{\nabla}_Y \nu)$$

$$= \hat{g}(X, \nu) + \hat{g}(y, -HX) = -H\hat{g}(y, X) = -H\hat{g}(y^T, X),$$

where $y^T := y - \hat{g}(y, \nu)\nu$ is the tangential part to $M$ of $y$. From $X(f) = g(X, \nabla f)$, it follows

$$\nabla f = -Hy^T.$$

Now, we prove that under our assumptions the mean curvature $H$ is constant. Recall the Ricci equation from the geometry of submanifold

$$(\hat{R}(X, Y)Z)^\perp = (\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z),$$

where

$$(\nabla h)(Y, X, Z) := \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

At any $y \in M$ we can choose a local normal coordinate system such that $\nabla_X Y(y) = \nabla_Y X(y) = 0$. Then we obtain

$$\nabla^\perp_X \hat{g}(X, Z)\nu = \nabla^\perp_X \hat{h}(Y, Z)\nu.$$

Choose the unit vector fields $X, Y, Z$ such that $X = Z$ and orthogonal to $Y$. Then we have $\nabla^\perp_X H\nu = 0$, from which it follows that $Y(H) = 0$, for any tangent vector $Y \in T_y M$. This means the mean curvature $H$ is constant.

Now,

$$\nabla_X \nabla f = (\nabla_X \nabla f)^\perp = -H(\hat{\nabla}_X y^T)^\perp =$$

$$= -H\hat{\nabla}_X (y - \hat{g}(y, \nu)\nu) = -H\{X + \hat{g}(y, \nu)HX\} = -H^2\left(\frac{1}{H} + f(y)\right)X.$$
Let now \( \tilde{f}(y) := f(y) + \frac{1}{H} \). Then we have
\[
\nabla_X \nabla \tilde{f} + H^2 \tilde{f} X = 0.
\]
From the theorem of Obata it follows that the totally umbilical hypersurface \( M \) is isometric to a round sphere \( S^n(c) \) of radius \( \frac{1}{H} \) in the \((n + 1)\)-Euclidean space. \( \square \)

**Remark 1.** We know that the level hypersphere \( S^n(r) \) is totally umbilical with unit normal vector field \( \frac{y}{F(y)} \) at any \( y \in S^n(r) \). It is easy to show that the converse is also true. For, if \( \nu = \frac{y}{F(y)} \) is a unit normal vector field at a point \( y \) of a totally umbilical hypersurface \( M \), then for any tangent vector field \( X \), we have
\[
0 = X\hat{g}(\nu, \nu) = X\hat{g}\left(\frac{y}{F(y)}, \frac{y}{F(y)}\right) = 2\hat{g}(X\left(\frac{1}{F}\right)y + \frac{y}{F(y)}X, y) = 2\left(\frac{1}{F}\right)F^2(y).
\]
Since \( M \) is a submanifold of \( \hat{R}^n \), \( F(y) > 0 \) for any \( y \in M \), thus \( X\left(\frac{1}{F}\right) = 0 \), and we obtain \( F(y) = r = \) constant, \( M \) is nothing but the level set \( S(r) \).

4. **Proof of Theorem 2.**

We suppose that \( b \) is a non-constant vector filed, which is parallel according to the Levi-Civita connection \( \hat{\nabla} \) of a Riemannian space \((\hat{R}^n, \hat{g})\). e. \( \hat{\nabla} b = 0 \). We consider the level set \( S(1) := \{ y \in \hat{R}^n : F(y) = 1 \} \), and we show that \( S(1) \) is a hypersurface of \((\hat{R}^n, \hat{g})\) whose sectional curvature is constant 1. Then, by Theorem 3. the Riemannian space \((\hat{R}^n, \hat{g})\) is flat and then according to Theorem 4. \( F \) must be a norm induced by an inner product on \( R^n \).

Let us define \( f(y) := \hat{g}(y, b) \). For any tangent vector field \( X \) on \( S(1) \) we have
\[
X(f) = X\hat{g}(y, b) = \hat{g}(\hat{\nabla}_X y, b) - \hat{g}(\nabla_X b) =
\]
\[
= \hat{g}(X, b) = \hat{g}(X, b^\top).
\]
Thus, we obtain the gradient of the function \( f \) on \( S(1) \),
\[
\nabla f = b^\top = b - \hat{g}(b, y)y.
\]
Now,
\[
\nabla_X \nabla f = (\hat{\nabla}_Y \nabla f)^\top = (\hat{\nabla}_Y (b - \hat{g}(b, y)y))^\top =
\]
\[
= \{-X\hat{g}(b, y)y - \hat{g}(b, y)\hat{\nabla}_Y y\}^\top = -f(y)X.
\]
Since \( b \) is constant vector field its easy to see that \( f(y) \) is non-constant function on \( S(1) \). From the Obata’s theorem (Theorem 5), it follows that \( S(1) \) is isometric to the round sphere of radius 1, and thus its sectional curvature is also constant 1. This completes the proof of Theorem 2. \( \square \)
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