The confidence interval methods in quantum language

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Abstract

Recently we proposed quantum language (or, measurement theory), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. Also, we consider that this is a kind of system theory such that it is applicable to both classical and quantum systems. As far as classical systems, it should be noted that quantum language is similar to statistics. In this paper, we discuss the usual confidence interval methods in terms of quantum language. And we assert that three concepts (i.e., "estimator" and "quantity" and "semi-distance) are indispensable for the theoretical understanding of the confidence interval methods. Since our argument is quite elementary, we hope that the readers acquire a new viewpoint of statistics, and agree that our proposal is, from the pure theoretical point of view, the true confidence interval methods.

(Key words: Confidence interval, Chi-squared distribution, Student’s t-distribution)

1 Quantum language (Axioms and Interpretation)

In this section, we shall mention the overview of quantum language (or, measurement theory, in short, MT).

Quantum language is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics (cf. ref. [9]). Quantum language (or, measurement theory) has two simple rules (i.e. Axiom 1 (concerning measurement) and Axiom 2 (concerning causal relation)) and the linguistic interpretation (= how to use the Axioms 1 and 2). That is,

\[ \text{Quantum language} = \text{Axiom 1 (measurement)} + \text{Axiom 2 (causality)} + \text{linguistic interpretation (how to use Axioms)} \]  

(cf. refs. [2]- [8]).

This theory is formulated in a certain \( C^* \)-algebra \( A \) (cf. ref. [10]), and is classified as follows:

\[ \text{(A)} \quad \begin{cases} \text{quantum MT (when } A \text{ is non-commutative)} \\ \text{classical MT (when } A \text{ is commutative, i.e., } A = C_0(\Omega) \) \end{cases} \]

where \( C_0(\Omega) \) is the \( C^* \)-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space \( \Omega \).

Since our concern in this paper is concentrated to the usual confidence interval methods in statistics, we devote ourselves to the commutative \( C^* \)-algebra \( C_0(\Omega) \), which is quite elementary. Therefore, we believe that all statisticians can understand our assertion (i.e., a new viewpoint of the confidence interval methods).

Let \( \Omega \) is a locally compact Hausdorff space, which is also called a state space. And thus, an element \( \omega (\in \Omega) \) is said to be a state. Let \( C(\Omega) \) be the \( C^* \)-algebra composed of all bounded continuous complex-valued functions on a locally compact Hausdorff space \( \Omega \). The norm \( \| \cdot \|_{C(\Omega)} \) is usual, i.e., \( \| f \|_{C(\Omega)} = \sup_{\omega \in \Omega} | f(\omega) | \) (\( \forall f \in C(\Omega) \)).

Motivated by Davies’ idea (cf. ref. [1]) in quantum mechanics, an observable \( O = (X, \mathcal{F}, F) \) in \( C_0(\Omega) \) (or, precisely, in \( C(\Omega) \)) is defined as follows:
\((B_1)\) \(X\) is a topological space. \(\mathcal{F}(\subseteq 2^X\text{, i.e., the power set of } X)\) is a field, that is, it satisfies the following conditions (i)–(iii): (i): \(\emptyset \in \mathcal{F}\), (ii): \(\Xi \in \mathcal{F} \implies X \setminus \Xi \in \mathcal{F}\), (iii): \(\Xi_1, \Xi_2, \ldots, \Xi_n \in \mathcal{F} \implies \bigcup_{k=1}^n \Xi_k \in \mathcal{F}\).

\((B_2)\) The map \(F: \mathcal{F} \to C(\Omega)\) satisfies that

\[
0 \leq |F(\Xi)|(\omega) \leq 1, \quad |F(X)|(\omega) = 1 \quad (\forall \omega \in \Omega)
\]

and moreover, if

\[
\Xi_1, \Xi_2, \ldots, \Xi_n, \ldots \in \mathcal{F}, \quad \Xi_m \cap \Xi_n = \emptyset \quad (m \neq n), \quad \Xi = \bigcup_{k=1}^\infty \Xi_k \in \mathcal{F},
\]

then, it holds

\[
|F(\Xi)|(\omega) = \lim_{n \to \infty} \sum_{k=1}^n |F(\Xi_k)|(\omega) \quad (\forall \omega \in \Omega)
\]

Note that Hopf extension theorem (cf. ref. [11]) guarantees that \((X, \mathcal{F}, [F(\cdot)](\omega))\) is regarded as the mathematical probability space.

**Example 1** [Normal observable]. Let \(\mathbb{R}\) be the set of the real numbers. Consider the state space \(\Omega = \mathbb{R} \times \mathbb{R}_+\), where \(\mathbb{R}_+ = \{\sigma \in \mathbb{R}|\sigma > 0\}\). Define the normal observable \(O_N = (\mathbb{R}, \mathcal{B}_R, \mathcal{N})\) in \(C_0(\mathbb{R} \times \mathbb{R}_+)\) such that

\[
[N(\Xi)](\omega) = \frac{1}{\sqrt{2\pi}\sigma} \int_\Xi \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx
\]

\((\forall \Xi \in \mathcal{B}_\mathbb{R}(=\text{Borel field in } \mathbb{R}))\), \(\forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+\).

In this paper, we devote ourselves to the normal observable.

Now we shall briefly explain "quantum language (1)" in classical systems as follows: A measurement of an observable \(O = (X, \mathcal{F}, F)\) for a system with a state \(\omega (\in \Omega)\) is denoted by \(M_{C_0(\Omega)}(O, S|\omega)\). By the measurement, a measured value \(x(\in X)\) is obtained as follows:

**Axiom 1** (Measurement)

- The probability that a measured value \(x (\in X)\) obtained by the measurement \(M_{C_0(\Omega)}(O \equiv (X, \mathcal{F}, F), S|\omega_0)\) belongs to a set \(\Xi (\in \mathcal{F})\) is given by \([F(\Xi)](\omega_0)\).

**Axiom 2** (Causality)

- The causality is represented by a Markov operator \(\Phi_{21}: C_0(\Omega_2) \to C_0(\Omega_1)\). Particularly, the deterministic causality is represented by a continuous map \(\pi_{21}: \Omega_1 \to \Omega_2\)

**Interpretation** (Linguistic interpretation). Although there are several linguistic rules in quantum language, the following is the most important:

- **Only one measurement is permitted.**

In order to read this paper, it suffices to understand the above three.

Consider measurements \(M_{C_0(\Omega)}(O_k \equiv (X_k, \mathcal{F}_k, F_k), S|\omega_0), (k = 1, 2, \ldots, n)\). However, the linguistic interpretation says that only one measurement is permitted. Thus we must consider a simultaneous measurement or a parallel measurement. The two are completely different, however in classical cases it suffices to consider only simultaneous measurement as follows.

**Definition 1** [(i):Simultaneous observable]. Let \(O_k \equiv (X_k, \mathcal{F}_k, F_k) (k = 1, 2, \ldots, n)\) be an observable in \(C_0(\Omega)\). The simultaneous observable \(\times_{k=1}^n O_k \equiv (\times_{k=1}^n X_k, \bigotimes_{k=1}^n \mathcal{F}_k, \hat{F}(\equiv \times_{k=1}^n F_k))\) in \(C_0(\Omega)\) is defined by

\[
[F(\Xi \times \cdots \times \Xi_n)](\omega) = \bigotimes_{k=1}^n [F_k(\Xi_k)](\omega) \quad (\forall \Xi_k \in \mathcal{F}_k \ (k = 1, \ldots, n), \forall \omega \in \Omega)
\]
Here, $\Xi_k$ is the smallest field including the family $\{X_k = \Xi_k : \Xi_k \in F_k, k = 1, 2, \ldots, n\}$. If $O \equiv (X, F, F)$ is equal to $O_k \equiv (X_k, F_k, F_k)$ ($k = 1, 2, \ldots, n$), then the simultaneous observable $X_k = \Xi_k$ $\equiv (X_k = \Xi_k, \Xi_k \in F_k, F(\equiv \chi_k F_k))$ is denoted by $O^n = (X^n, F^n).

[(ii):Parallel observable. Let $O_k \equiv (X_k, F_k, F_k)$ be an observable in $C_0(\Omega_k)$, ($k = 1, 2, \ldots, n$). The parallel observable $\otimes_{k=1}^n O_k = \otimes_{k=1}^n F_k$ ($\otimes_{k=1}^n F_k, F(\equiv \otimes_{k=1}^n F_k)$) in $C_0(\otimes_{k=1}^n \Omega_k)$ is defined by

$$\otimes_{k=1}^n F_k(\otimes_{k=1}^n \Omega_k)(\otimes_{k=1}^n \omega_k, \otimes_{k=1}^n \omega_k, \otimes_{k=1}^n \omega_k) = \otimes_{k=1}^n [F_k(\otimes_{k=1}^n \Omega_k)](\otimes_{k=1}^n \omega _k)$$

(\forall \otimes_{k=1}^n F_k, \forall \otimes_{k=1}^n \omega_k \in \Omega_k, \ (k = 1, \ldots, n))

Definition 2 [Image observable. Let $O \equiv (X, F, F)$ be observables in $C_0(\Omega)$. The observable $f(O)$ ($\equiv (Y, G, G(\equiv F \circ f^{-1}))$ in $C_0(\Omega)$ is called the image observable of $O$ by a map $f : X \rightarrow Y$, if it holds that

$$G(\Gamma) = F(f^{-1}(\Gamma)) \quad (\forall \Gamma \in G)$$

Example 2 [Simultaneous normal observable. Let $O_N = (R, B_R, N)$ be the normal observable in $C_0(R \times R_+)$ in Example 1. Let $n$ be a natural number. Then, we get the simultaneous normal observable $O^n_N = (R^n, B^n_R, N^n)$ in $C_0(R \times R_+)$. That is,

$$[N^n(\otimes_{k=1}^n \Xi_k)](\otimes_{k=1}^n \omega_k) = \frac{n}{(\sqrt{2})^n} \prod_{k=1}^n \exp\left[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n$$

(\forall \otimes_{k=1}^n F_k(\otimes_{k=1}^n \Xi_k), \forall \otimes_{k=1}^n \omega_k \in \Omega_k, \ (k = 1, \ldots, n))

Consider the maps $\overline{\mu} : R^n \rightarrow R$ and $\overline{\sigma} : R^n \rightarrow R$ such that

$$\overline{\mu}(x) = \overline{\mu}(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad (\forall x = (x_1, x_2, \ldots, x_n) \in R^n)$$

(7)

$$\overline{\sigma}(x) = \overline{\sigma}(x_1, x_2, \ldots, x_n) = \sum_{k=1}^n (x_k - \overline{\mu})^2 \quad (\forall x = (x_1, x_2, \ldots, x_n) \in R^n)$$

(8)

Thus, we have two image observables $\overline{\mu}(O^n_R) = (R, B_R, N^n \circ \overline{\mu}^{-1})$ and $\overline{\sigma}(O^n_R) = (R^+ B_R^+, N^n \circ \overline{\sigma}^{-1})$ in $C_0(R \times R_+)$. It is easy to see that

$$[(N^n \circ \overline{\mu}^{-1})(\Xi_1)](\otimes_{k=1}^n \omega_k) = \frac{1}{(\sqrt{2})^n \prod_{k=1}^n \exp[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}]} dx_1 dx_2 \cdots dx_n$$

(9)

and

$$[(N^n \circ \overline{\sigma}^{-1})(\Xi_2)](\otimes_{k=1}^n \omega_k) = \frac{1}{(\sqrt{2})^n \prod_{k=1}^n \exp[-\frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2}]} dx_1 dx_2 \cdots dx_n$$

(10)
Here, $p_{n-1}^2(x)$ is the chi-squared distribution with $n - 1$ degrees of freedom. That is,

$$p_{n-1}^2(x) = \frac{x^{(n-1)/2}e^{-x/2}}{2^{(n-1)/2}\Gamma((n-1)/2)} \quad (x > 0)$$

where $\Gamma$ is the gamma function.

### 2 Fisher’s maximum likelihood method

It is usual to consider that we do not know the pure state $\omega_0 \in \Omega$ when we take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$. That is because we usually take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ in order to know the state $\omega_0$. Thus, when we want to emphasize that we do not know the state $\omega_0$, $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ is denoted by $M_{C_0(\Omega)}(O, S_{[\ast]})$. Also, if we know that a state $\omega_0$ belongs to a certain set suitable $K \subseteq \Omega$, the $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ is denoted by $M_{C_0(\Omega)}(O, S_{[\ast]}(K))$.

**Theorem 1** [Fisher’s maximum likelihood method (cf. refs. [3], [4], [8])]. Consider a measurement $M_{C_0(\Omega)}(O = (X, F), S_{\ast}(K))$. Assume that we know that the measured value $x \in X$ obtained by a measurement $M_{C_0(\Omega)}(O = (X, F), S_{\ast}(K))$ belongs to $\Xi \in \mathcal{F}$. Then, there is a reason to infer that the unknown state $[\ast]$ is equal to $\omega_0 \in K$ such that

$$F(\Xi)(\omega_0) = \max_{\omega \in K} F(\Xi)(\omega)$$

if the righthand side of this formula exists. Also, if $\Xi = \{x\}$, it suffices to calculate the $\omega_0 \in K$ such that

$$\lim_{\Xi \ni \{x\}, \Xi \to \{x\}} \max_{\omega \in K} F(\Xi)(\omega) = 1$$

**Example 3** [Fisher’s maximum likelihood method]. Consider the simultaneous normal observable $O_N^o = (\mathbb{R}^n, \mathcal{B}_\mathbb{R}^n, N^n)$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. Assume that $M_{C_0(\mathbb{R} \times \mathbb{R}_+)}(O_N^o = (\mathbb{R}^n, \mathcal{B}_\mathbb{R}^n, N^n), S_{[\ast]}(K))$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. Assume that a measured value $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is obtained by the measurement. Since the likelihood function $L_x(\mu, \sigma)$ is defined by

$$L_x(\mu, \sigma) = \frac{1}{(2\pi\sigma)^n} \exp \left[ - \frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2} \right]$$

($\forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+$).

it suffices to calculate the following equations:

$$\frac{\partial L_x(\mu, \sigma)}{\partial \mu} = 0, \quad \frac{\partial L_x(\mu, \sigma)}{\partial \sigma} = 0$$

Thus, Fisher’s maximum likelihood method says as follows.

(i): Assume that $K = \mathbb{R} \times \mathbb{R}_+$. Solving the equation (15), we can infer that $[\ast] = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$\mu = \bar{x}(x) = \frac{x_1 + x_2 + \ldots + x_n}{n}, \quad \sigma = \bar{\sigma}(x) = \sqrt{\frac{\sum_{k=1}^n (x_k - \mu)^2}{n}}$$

(ii): Assume that $K = \mathbb{R} \times \{\sigma_1\} \subseteq \mathbb{R} \times \mathbb{R}_+$. It is easy to see that there is a reason to infer that $[\ast] = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$\mu = \bar{x}(x) = \frac{x_1 + x_2 + \ldots + x_n}{n}, \quad \sigma = \sigma_1$$

(iii): Assume that $K = \{\mu_1\} \times \mathbb{R}_+ \subseteq \mathbb{R} \times \mathbb{R}_+$. There is a reason to consider that $[\ast] = (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$\mu = \mu_1, \quad \sigma = \sqrt{\frac{\sum_{k=1}^n (x_k - \mu_1)^2}{n}}$$
3 Confidence interval

Let \( \mathcal{O} = (X, \mathcal{F}, F) \) be an observable formulated in a commutative \( C^* \)-algebra \( C_0(\Omega) \). Let \( \Theta \) be a locally compact space with the semi-distance \( d_\Theta^O (\forall x \in X) \), that is, for each \( x \in X \), the map \( d_\Theta^O : \Theta^2 \to [0, \infty) \) satisfies that (i): \( d_\Theta^O (\theta, \theta) = 0 \), (ii): \( d_\Theta^O (\theta_1, \theta_2) = d_\Theta^O (\theta_2, \theta_1) \), (ii): \( d_\Theta^O (\theta_1, \theta_2) \leq d_\Theta^O (\theta_1, \theta_3) + d_\Theta^O (\theta_2, \theta_3) \).

Let \( \pi : \Omega \to \Theta \) be a continuous map, which is a kind of causal relation (in Axiom 2), and called “estimator”. Let \( E : X \to \Theta \) be a continuous (or more generally, measurable) map, which is called “quantity”. For any \( \omega \in \Omega \), define the positive number \( \eta^\omega (\equiv 0) \) such that:

\[
\eta^\omega = \inf \{ \eta > 0 : [F(\{ x \in X : d_\Theta^O (E(x), \pi(\omega)) \leq \eta \})](\omega) \geq \gamma \} \tag{19}
\]

For any \( x \in X \), put

\[
D^\gamma_{x_0} = \{ \pi(\omega)(\in \Theta) : \omega \in \Omega, \ d_\Theta^O (E(x), \pi(\omega)) \leq \eta^\omega \} \tag{20}
\]

The \( D^\gamma_{x_0} \) is called the \( (\gamma) \)-confidence interval of \( x \).

Note that,

(C) for any \( \omega_0 \in \Omega \), the probability, that the measured value \( x \) obtained by the measurement \( M_{C_0(\Omega)} (\mathcal{O} := (X, \mathcal{F}, F), S_{\omega_0}) \) satisfies the following condition (b), is larger than \( \gamma \) (e.g., \( \gamma = 0.95 \)).

(b) \( d_\Theta^O (E(x), \pi(\omega_0)) \leq \eta^\omega_0 \).

Assume that we get a measured value \( x_0 \) by the measurement \( M_{C_0(\Omega)} (\mathcal{O} := (X, \mathcal{F}, F), S_{\omega_0}) \). Then, we see the following equivalence:

\[
(b) \iff \ D^\gamma_{x_0} \ni \pi(\omega_0) . \tag{21}
\]

\[\text{Figure 1. Confidence interval } D^\gamma_{x_0}\]

Summing the above argument, we have the following proposition.

**Theorem 2** [Confidence interval]. Let \( \mathcal{O} = (X, \mathcal{F}, F) \) be an observable formulated in a commutative \( C^* \)-algebra \( C_0(\Omega) \). Let \( \omega_0 \) be any fixed state, i.e., \( \omega_0 \in \Omega \). Consider a measurement \( M_{C_0(\Omega)} (\mathcal{O} := (X, \mathcal{F}, F), S_{\omega_0}) \). Let \( \Theta \) be a locally compact space with the semi-distance \( d_\Theta^O (\forall x \in X) \). Let \( \pi : \Omega \to \Theta \) be a quantity. Let \( E : X \to \Theta \) be an estimator. Let \( \gamma \) be a real number such that \( 0 < \gamma < 1 \), for example, \( \gamma = 0.95 \). For any \( x \in X \), define \( D^\gamma_{x_0} \) as in (20). Then, we see,

(1) the probability that the measured value \( x_0 \in X \) obtained by the measurement \( M_{C_0(\Omega)} (\mathcal{O} := (X, \mathcal{F}, F), S_{\omega_0}) \) satisfies the condition that

\[
D^\gamma_{x_0} \ni \pi(\omega_0) , \tag{22}
\]

is larger than \( \gamma \).
This theorem is the generalization of our proposal in refs. [4] and [7].

**Remark 1** [The statistical meaning of Theorem 2]. Consider the simultaneous measurement $\mathcal{M}_{C_0(\Theta)}(O^j := (X^j, E^j, F^j), S_{(\omega, j)\in L})$, and assume that a measured value $x = (x_1, x_2, \ldots, x_J) \in X^J$ is obtained by the simultaneous measurement. Then, it surely holds that

\[
\lim_{J \to \infty} \frac{\text{Num} [ \{ j \mid D_{x_j}^J \ni \pi(\omega) \} ]}{J} \geq \gamma (= 0.95)
\]

(23)

where Num[A] is the number of the elements of the set $A$. Hence Theorem 2 can be tested by numerical analysis (with random number).

4 Examples

4.1 The case that $\Omega = \Theta$, and $d^*_\Theta$ does not depend on $x$

In this section, we assume that $\Omega = \Theta$, that is, we do not need $\Theta$ but $\Omega$. And moreover, we assume that $d^*_\Theta$ does not depend on $x$.

The arguments in this section are continued from Example 2. Consider the simultaneous measurement $\mathcal{M}_{C_0(\mathbb{R} \times \mathbb{R}^+)}(O^N_\Omega = (\mathbb{R}^n, \mathcal{B}^n_\mathbb{R}, \mathcal{N}^n), S_{(\mu, \sigma)\in L})$ in $C_0(\mathbb{R} \times \mathbb{R}^+)$. Thus, we consider that $\Omega = \mathbb{R} \times \mathbb{R}^+$, $X = \mathbb{R}^n$. The formulas (7) and (8) urge us to define the estimator $E : \mathbb{R}^n \to \Omega (\equiv \Theta \equiv \mathbb{R} \times \mathbb{R}^+)$ such that

\[
E(x) = E(x_1, x_2, \ldots, x_n) = (\overline{\pi}(x), (\overline{x})(x)) = \left( \frac{x_1 + x_2 + \cdots + x_n}{n}, \sqrt{\frac{\sum_{k=1}^n (x_k - \overline{x}(x))^2}{n}} \right)
\]

(24)

Let $\gamma$ be a real number such that $0 \ll \gamma < 1$, for example, $\gamma = 0.95$.

**Example 4** [Confidence interval for the semi-distance $d^{(1)}_{\Omega J}$]. Consider the following semi-distance $d^{(1)}_{\Omega J}$ in the state space $\mathbb{R} \times \mathbb{R}^+$:

\[
d^{(1)}_{\Omega J}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = |\mu_1 - \mu_2|
\]

(25)

For any $\omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}^+$, define the positive number $\eta^*_{\omega} (> 0)$ such that:

\[
\eta^*_{\omega} = \inf \{ \eta > 0 : [F^{-1}(\text{Ball}_{d^{(1)}_{\Omega J}}(\omega; \eta))] \geq \gamma \}
\]

where $\text{Ball}_{d^{(1)}_{\Omega J}}(\omega; \eta) = \{ \omega_1 \in \Omega : d^{(1)}_{\Omega J}(\omega, \omega_1) \leq \eta \} = [\mu - \eta, \mu + \eta] \times \mathbb{R}^+$.

Hence we see that

\[
E^{-1}(\text{Ball}_{d^{(1)}_{\Omega J}}(\omega; \eta)) = E^{-1}([\mu - \eta, \mu + \eta] \times \mathbb{R}^+) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \mu - \eta \leq \frac{x_1 + \cdots + x_n}{n} \leq \mu + \eta \}
\]

(26)

Thus,

\[
[N^n(E^{-1}(\text{Ball}_{d^{(1)}_{\Omega J}}(\omega; \eta)))](\omega) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mu - \eta}^{\mu + \eta} \cdots \int_{\mu - \eta}^{\mu + \eta} \exp \left[ - \frac{\sum_{k=1}^n (x_k - \mu)^2}{2\sigma^2} \right] dx_1 dx_2 \cdots dx_n
\]

\[
= \frac{1}{(\sqrt{2\pi})^n} \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \exp \left[ - \frac{\sum_{k=1}^n (x_k)^2}{2\sigma^2} \right] dx_1 dx_2 \cdots dx_n
\]

\[
= \frac{\sqrt{\pi}}{\sqrt{2\pi}} \int_{-\eta}^{\eta} \exp \left[ - \frac{n\eta^2}{2\sigma^2} \right] dx = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{n}\eta}^{\sqrt{n}\eta} \exp \left[ - \frac{x^2}{2} \right] dx
\]

(27)
Solving the following equation:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z((1-\gamma)/2)} \exp\left[-\frac{x^2}{2}\right] dx = \frac{1}{\sqrt{2\pi}} \int_{z((1-\gamma)/2)}^{\infty} \exp\left[-\frac{x^2}{2}\right] dx = \frac{1-\gamma}{2}
\] (28)
we define that

\[
\eta_\gamma^2 = \frac{\sigma}{\sqrt{n}} z \left(\frac{1-\gamma}{2}\right)
\] (29)

Therefore, for any \( x \in \mathbb{R}^n \), we get \( D_x^\gamma \) (the (\( \gamma \))-confidence interval of \( x \)) as follows:

\[
D_x^\gamma = \{ \omega (\in \Omega) : d_{11}(E(x), \omega) \leq \eta_\gamma^2 \}
\]
\[
= \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : |\mu - \mu(x)| = |\mu - \frac{x_1 + \ldots + x_n}{n}| \leq \frac{\sigma}{\sqrt{n}} z \left(\frac{1-\gamma}{2}\right) \}
\] (30)

![Figure 2. Confidence interval \( D_x^\gamma \) for the semi-distance \( d_{11}^{(1)} \)](image)

Thus, strictly speaking, the "confidence interval" should be said to be the "confidence domain" in quantum language.

**Example 5** \([\text{Confidence interval for the semi-distance } d_{11}^{(2)}]\). Consider the following semi-distance \( d_{11}^{(2)} \) in \( \mathbb{R} \times \mathbb{R}_+ \):

\[
d_{11}^{(2)}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \int_{\sigma_2}^{\sigma_1} \frac{1}{\sigma} d\sigma = |\log \sigma_1 - \log \sigma_2|
\] (31)

For any \( \omega = (\mu, \sigma)(\in \Omega = \mathbb{R} \times \mathbb{R}_+) \), define the positive number \( \eta_\gamma^c \) (\( > 0 \)) such that:

\[
\eta_\gamma^c = \inf\{ \eta > 0 : |F(E^{-1}(\text{Ball}_{d_{11}^{(2)}}(\omega; \eta)))(\omega)| \geq \gamma \}
\] (32)

where \( \text{Ball}_{d_{11}^{(2)}}(\omega; \eta) = \{ \omega_1(\in \Omega) : d_{11}^{(2)}(\omega, \omega_1) \leq \eta \} \). Note that

\[
\text{Ball}_{d_{11}^{(2)}}(\omega; \eta) = \text{Ball}_{d_{11}^{(2)}}((\mu; \sigma), \eta) = \mathbb{R} \times \{ \sigma' \in \mathbb{R}_+ : |\log(\sigma'/\sigma)| \leq \eta \} = \mathbb{R} \times [\sigma e^{-\eta}, \sigma e^{\eta}]
\] (33)

Then,

\[
E^{-1}(\text{Ball}_{d_{11}^{(2)}}(\omega; \eta)) = E^{-1}(\mathbb{R} \times [\sigma e^{-\eta}, \sigma e^{\eta}])
\]
\[
= \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sigma e^{-\eta} \leq \left(\sum_{k=1}^{n} (x_k - \mu(x))^2\right)^{1/2} \leq \sigma e^{\eta} \}
\] (34)
Hence we see, by (10), that
\[
[N^n(E^{-1}(\text{Ball}_{d_{H^2}^{(2)}}(\omega; \eta)))](\omega) = \frac{1}{\sqrt{2\pi \sigma}} \int \cdots \int_{\sigma^2 \epsilon^{-2n} \leq \sum_{k=1}^{n}(x_k - \mu)^2 \leq \sigma^2 \epsilon^{2n}} \exp\left[-\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n
\]
\[
= \int_{\sigma^2 \epsilon^{-2n}}^{\sigma^2 \epsilon^{2n}} p_{n-1}^2(x) dx
\]
(35)

Using the chi-squared distribution \(p_{n-1}^2(x)\) (with \(n - 1\) degrees of freedom) in (11), define the \(\eta_n^\gamma\) such that
\[
\gamma = \int_{\sigma^2 \epsilon^{-2n}}^{\sigma^2 \epsilon^{2n}} p_{n-1}^2(x) dx
\]
(36)
where it should be noted that the \(\eta_n^\gamma\) depends on only \(\gamma\) and \(n\). Thus, put
\[
\eta_n^\gamma = \eta_n^\gamma
\]
(37)

Hence we get, for any \(x \in X\), the \(D^\gamma_x\) (the \((\gamma)\)-confidence interval of \(x\)) as follows:
\[
D^\gamma_x = \{\omega(\in \Omega) : d_{H^2}^{(2)}(E(x), \omega) \leq \eta_n^\gamma\} = \{\mu, \sigma \in \mathbb{R} \times \mathbb{R}_+ : \sigma e^{-\eta_n^\gamma} \leq \left(\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}\right)^{1/2} \leq \sigma e^{\eta_n^\gamma}\}
\]
(38)

Recalling (16), i.e., \(\overline{\sigma}(x) = \left(\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}\right)^{1/2} = \left(\frac{\sum_{k=1}^{n}(x_k - \overline{x})^2}{n}\right)^{1/2}\), we conclude that
\[
D^\gamma_x = \{\mu, \sigma \in \mathbb{R} \times \mathbb{R}_+ : \overline{\sigma}(x)e^{-\eta_n^\gamma} \leq \sigma \leq \overline{\sigma}(x)e^{\eta_n^\gamma}\}
\]
\[
= \{\mu, \sigma \in \mathbb{R} \times \mathbb{R}_+ : \frac{e^{-\eta_n^\gamma} \overline{\sigma}(x)}{n} \leq \sigma^2 \leq \frac{e^{2\eta_n^\gamma} \overline{\sigma}(x)}{n}\}
\]
(39)

![Figure 3. Confidence interval \(D^\gamma_x\) for the semi-distance \(d_{H^2}^{(2)}\)](image)

For example, in the case that \(n = 3\), \(\gamma = 0.95\), the (36) says that
\[
0.95 = \gamma = \int_{3e^{-2\eta_n^\gamma}}^{3e^{2\eta_n^\gamma}} p_{2}^2(x) dx = \int_{3e^{-2\eta_n^\gamma}}^{3e^{2\eta_n^\gamma}} \frac{e^{-x/2}}{2^{2/2}\Gamma(1)} dx = \left[-e^{-x/2}\right]_{x=3e^{-2\eta_n^\gamma}}^{x=3e^{2\eta_n^\gamma}} = e^{-\frac{3}{2}e^{-2\eta_n^\gamma}} - e^{-\frac{3}{2}e^{2\eta_n^\gamma}}
\]
(40)
which implies that
\[
e^{-\eta_3^{0.95}} = 0.1849, \quad e^{\eta_3^{0.95}} = 5.4077,
\]
and,
\[
e^{-2\eta_3^{0.95}}/3 = 0.0114 \cdots, \quad e^{2\eta_3^{0.95}}/3 = 9.748 \cdots
\]
Thus, we see that
\[
D_x^{0.95} = \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : (0.0114 \cdots) \cdot \bar{S}(x) \leq \sigma^2 \leq (9.748 \cdots) \cdot \bar{S}(x) \}
\]
\[\text{Remark 2.} \ [\text{Other estimator}] \ \text{Instead of (24), we may consider the unbiased estimator} \ E' : \mathbb{R}^n \to \Omega(= \mathbb{R} \times \mathbb{R}_+) \ \text{such that}
\]
\[
E'(x) = E(x_1, x_2, \ldots, x_n) = (\bar{m}(x), (\bar{\sigma}(x)) = \left( \frac{x_1 + x_2 + \cdots + x_n}{n}, \sqrt{\frac{\sum_{k=1}^{n} (x_k - \bar{m}(x))^2}{n - 1}} \right)
\]
In this case, we see that
\[
(D_2')' = \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \bar{\sigma}(x) e^{-\eta_3^{0.95}'} \leq \sigma \leq \bar{\sigma}(x) e^{\eta_3^{0.95}'} \}
\]
\[= \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \frac{e^{-2\eta_3^{0.95}'}}{n - 1} \bar{S}(x) \leq \sigma^2 \leq \frac{e^{2\eta_3^{0.95}'}}{n - 1} \bar{S}(x) \}
\]
where the \((\eta_3^{0.95}')\) is defined by
\[
\gamma = \int_{(n-1)e^{-2\eta_3^{0.95}'}}^{(n-1)e^{2\eta_3^{0.95}'}} p_{n-1}^2(x) dx
\]
For example, in the case that \(n = 3\), \(\gamma = 0.95\), the (36) says that
\[
0.95 = \gamma = \int_{e^{-2\eta_3^{0.95}'}}^{2e^{2\eta_3^{0.95}'}} p_2^2(x) dx = \int_{e^{-2\eta_3^{0.95}'}}^{2e^{2\eta_3^{0.95}'}} \frac{e^{-x/2}}{2^{2/2} \Gamma(1)} dx = \left[ -e^{-x/2} \right]_{x=2e^{2\eta_3^{0.95}'}}^{x=2e^{-2\eta_3^{0.95}'}} = e^{2\eta_3^{0.95}'} - e^{-2\eta_3^{0.95}'}
\]
which implies that
\[
e^{\eta_3^{0.95}'} = 0.2265, \quad e^{-\eta_3^{0.95}'} = 4.4154
\]
Thus,
\[
e^{-2\eta_3^{0.95}'}/2 = 0.00256 \cdots, \quad e^{2\eta_3^{0.95}'}/2 = 9.748 \cdots
\]
Thus, we see that
\[
(D_x^{0.95})' = \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : (0.00256 \cdots) \cdot \bar{S}(x) \leq \sigma^2 \leq (9.748 \cdots) \cdot \bar{S}(x) \}
\]
Hence it should be noted that \(D_2' \neq (D_2')'\).

\[\text{Remark 3} \ [\text{Other semi-distance} \ d_{\Omega}^{(3)}] \ \text{We believe that the semi-distance} \ d_{\Omega}^{(2)} \ \text{is natural in Example 5, although we have no firm reason to believe in it. For example, consider a positive continuous function} \ h : \mathbb{R}_+ \to \mathbb{R}_+. \ \text{Then, we can define another semi-distance} \ d_{\Omega}^{(3)} \ \text{in the state space} \ \mathbb{R} \times \mathbb{R}_+:\]
\[
d_{\Omega}^{(3)}((\mu_1, \sigma_1), (\mu_2, \sigma_2)) = \left| \int_{\sigma_1}^{\sigma_2} h(\sigma) d\sigma \right|
\]
Thus, many \((\gamma)\)-confidence intervals exist, though the \(\eta_3^{n}\) may depend on \(\omega\). Now, we have the following problem:
• Is there a better $h(\sigma)$ than the $1/\sigma$?

whose answer we do not know.

**Remark 4** [So called $\alpha$-point method]. In many books, it conventionally is recommended as follows:

$$(D'_2)^{''} = \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \sqrt{\frac{\sum_{k=1}^{n} (x_k - \overline{X}(x))^2}{S_\infty^2}} \leq \sigma \leq \sqrt{\frac{\sum_{k=1}^{n} (x_k - \overline{X}(x))^2}{S_0^2}} \}$$

where

$$\int_0^{\chi_0^2} p_n^2(x) \, dx = \int_{\chi_0^2}^{\infty} p_n^2(x) \, dx = (1 - \gamma)/2$$

which may be an analogy of (19).

In the case that $n = 3, \gamma = 0.95$, we see

$$\int_0^{0.0506} p_2^3(x) \, dx = \int_{7.378}^{\infty} p_2^3(x) \, dx = 0.025$$

and thus,

$$\frac{1}{\chi_2^\infty} = \frac{1}{7.378} = 0.1355, \quad \frac{1}{\chi_0^2} = \frac{1}{0.0506} = 19.763.$$  

Thus, we see that

$$(D'_2^{0.95})^{''} = \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : (0.1355 \cdots) \cdot \overline{S}(x) \leq \sigma^2 \leq (19.763 \cdots) \cdot \overline{S}(x) \}$$

which should be compared to the estimations (43) and (50). It should be noted that both estimator and semi-distance are not declared in this $\alpha$-point method. Thus, we have the following problem:

(C) What is the $\alpha$-point method (52)?

This will be answered in the following remark.

**Remark 5** [What is the $\alpha$-point method (52)?]. Instead of (24) or (44), we consider the estimator $E'' : \mathbb{R}^n \to \Omega (\equiv \mathbb{R} \times \mathbb{R}_+)$ such that

$$E''(x) = E(x_1, x_2, \ldots, x_n) = \overline{X}(x), \quad (\overline{X}')(x) = (\frac{x_1 + x_2 + \cdots + x_n}{n}, \sqrt{\frac{\sum_{k=1}^{n} (x_k - \overline{X}(x))^2}{cn}})$$

where $c > 0$. In this case, by the same argument of (35), we see that Then,

$$(E'')^{-1}(\text{Ball}_{d_{\alpha}^{(2)}}(\omega; \eta)) = E^{-1}(\mathbb{R} \times [\sigma e^{-\eta}, \sigma e^\eta])$$

$$= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sigma e^{-\eta} \leq \left(\frac{\sum_{k=1}^{n} (x_k - \overline{X}(x))^2}{cn}\right)^{1/2} \leq \sigma e^\eta \}$$

Hence we see, by (10), that

$$[N^{(E''^{-1}(\text{Ball}_{d_{\alpha}^{(2)}}(\omega; \eta)))](\omega)$$

$$= \frac{1}{(\sqrt{2\pi} \sigma)^n} \int_{\sigma e^{-2\eta} \leq \sum_{k=1}^{n} (x_k - \overline{X}(x))^2 \leq \sigma e^{2\eta}} \exp\left[-\frac{\sum_{k=1}^{n} (x_k - \mu)^2}{2\sigma^2}\right] dx dx_2 \cdots dx_n$$

$$= \int_{\sigma e^{-2\eta}}^{\sigma e^{2\eta}} p_n^2(x) \, dx$$
Hence we get, for any \( x \in X \), the \( D^n_x \) (the (\( \gamma \))-confidence interval of \( x \)) as follows:

\[
D^n_x = \{ \omega(\in \Omega) : d^{(2)}_\Theta(E(x), \omega) \leq \eta^n \}
\]

\[
= \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \sigma e^{-\eta^n} \leq \left( \sum_{k=1}^{n}(x_k - \overline{x})^2 \right)^{1/2} \leq \sigma e^{\eta^n} \}
\]

\[
= \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : c \sigma e^{-2\eta^n} \leq S(x) \leq c \sigma e^{2\eta^n} \}
\]

\[
= \{ (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+ : \frac{S(x)}{c \sigma e^{2\eta^n}} \leq \sigma^2 \leq \frac{S(x)}{c \sigma e^{-2\eta^n}} \}
\]

(60)

Using \( \chi_0^2 \) and \( \chi_\infty^2 \) defined in (53), we obtain the following equation:

\[
c \sigma e^{-2\eta^n} = \chi_0^2, \quad c \sigma e^{2\eta^n} = \chi_\infty^2
\]

Thus, it suffices to put

\[
c = \sqrt{\frac{\chi_0^2 \cdot \chi_\infty^2}{n}}
\]

(61)

in the estimator \( E'' \) of (57). In this sense, the \( \alpha \)-point method (52) is true (cf. Remark 1), though it may be unnatural.

4.2 The case that \( \pi(\mu_1, \mu_2) = \mu_1 - \mu_2 \), and \( d^{(x)}_\Theta \) does not depend on \( x \)

The arguments in this section are continued from Example 2.

**Example 6** [Confidence interval the the case that "\( \pi(\mu_1, \mu_2) = \mu_1 - \mu_2 \)". Consider the parallel measurement \( M_{C_B(\mathbb{R} \times \mathbb{R}_+)} \) \( (\mathbb{O}^N_N \otimes \mathbb{O}_N^m = (\mathbb{R}^n \times \mathbb{R}^m, \mathbb{B}_N^\mathbb{R} \otimes \mathbb{B}_N^m, N^n \otimes N^m), S_{(\mu_1, \sigma_1, \mu_2, \sigma_2)}) \) in \( C_0((\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+)) \).

Assume that \( \sigma_1 \) and \( \sigma_2 \) are fixed and known. Thus, this parallel measurement is represented by \( M_{C_B(\mathbb{R} \times \mathbb{R})} \) \( (\mathbb{O}^N_{N_1} \otimes \mathbb{O}^m_{N_2}) = (\mathbb{R}^n \times \mathbb{R}^m, \mathbb{B}_N^\mathbb{R} \otimes \mathbb{B}_N^m, N_{n_1} \otimes N_{n_2}^m), S_{(\mu_1, \mu_2)}) \) in \( C_0(\mathbb{R} \times \mathbb{R}) \). Here, recall the (2), i.e.,

\[
[N_\sigma(\Xi)](\mu) = \frac{1}{\sqrt{2\pi \sigma}} \int_{\mathbb{R}} \exp[ -\frac{(x - \mu)^2}{2\sigma^2} ] dx \quad (\forall \Xi \in \mathbb{B}_\mathbb{R}(=\text{Borel field in } \mathbb{R})), \quad \forall \mu \in \mathbb{R}.
\]

(62)

Therefore, we have the state space \( \Omega = \mathbb{R}^2 = \{ \omega = (\mu_1, \mu_2) : \mu_1, \mu_2 \in \mathbb{R} \} \). Put \( \Theta = \mathbb{R} \) with the distance \( d_\Theta(\theta_1, \theta_2) = |\theta_1 - \theta_2| \) and consider the quantity \( \pi : \mathbb{R}^2 \to \mathbb{R} \) by

\[
\pi(\mu_1, \mu_2) = \mu_1 - \mu_2
\]

(63)

The estimator \( E : \tilde{X}(= X \times Y = \mathbb{R}^n \times \mathbb{R}^m) \to \Theta(= \mathbb{R}) \) is defined by

\[
E(x_1, \ldots, x_n, y_1, \ldots, y_m) = \frac{\sum_{k=1}^{n}x_k}{n} - \frac{\sum_{k=1}^{m}y_k}{m}
\]

(64)

For any \( \omega = (\mu_1, \mu_2)(\in \Omega = \mathbb{R} \times \mathbb{R}) \), define the positive number \( \eta_{\omega}^\gamma \) (\( > 0 \)) such that:

\[
\eta_{\omega}^\gamma = \inf\{ \eta > 0 : [F(E^{-1}(\text{Ball}_{d_\Theta}(\pi(\omega); \eta))](\omega) \geq \gamma \}
\]

where \( \text{Ball}_{d_\Theta}(\pi(\omega); \eta) = [\mu_1 - \mu_2 - \eta, \mu_1 - \mu_2 + \eta] \)

Now let us calculate the \( \eta_{\omega}^\gamma \) as follows:

\[
E^{-1}(\text{Ball}_{d_\Theta}(\pi(\omega); \eta)) = E^{-1}([\mu_1 - \mu_2 - \eta, \mu_1 - \mu_2 + \eta])
\]

\[
= \{ (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m : \mu_1 - \mu_2 - \eta \leq \frac{\sum_{k=1}^{n}x_k}{n} - \frac{\sum_{k=1}^{m}y_k}{m} \leq \mu_1 - \mu_2 + \eta \}
\]

\[
= \{ (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^n \times \mathbb{R}^m : -\eta \leq \frac{\sum_{k=1}^{n}(x_k - \mu_1)}{n} - \frac{\sum_{k=1}^{m}(y_k - \mu_2)}{m} \leq \eta \}
\]

(65)
Thus,

\[
[(N_{\sigma_1^n} \otimes N_{\sigma_2^m})(E^{-1}(\text{Ball}_{d_{\Theta}}(\pi(\omega); \eta)))](\omega)
\]

\[
= \frac{1}{(\sqrt{2\pi}\sigma_1^n)(\sqrt{2\pi}\sigma_2^m)} \times \int \cdots \int_{-\eta \leq \sum_{k=1}^n(x_k - \mu_1) \leq \eta} \cdots \int_{-\eta \leq \sum_{k=1}^m(y_k - \mu_2) \leq \eta} \exp\left[-\frac{\sum_{k=1}^n(x_k - \mu_1)^2}{2\sigma_1^2} - \frac{\sum_{k=1}^m(y_k - \mu_2)^2}{2\sigma_2^2}\right] dx_1 dx_2 \cdots dx_n dy_1 dy_2 \cdots dy_m
\]

\[
= \frac{1}{(\sqrt{2\pi}\sigma_1^n)(\sqrt{2\pi}\sigma_2^m)} \frac{1}{\sqrt{2\pi(\sigma_1^2/n + \sigma_2^2/m)^{1/2}}} \int_{-\eta}^{\eta} \exp\left[-\frac{x^2}{2(\sigma_1^2/n + \sigma_2^2/m)}\right] dx
\]

(67)

Solving the equation (28), we get that

\[
\eta_\gamma = \left(\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)^{1/2} z\left(1 - \frac{\gamma}{2}\right)
\]

(68)

Therefore, for any \( \hat{x} = (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \) (\( \in \mathbb{R}^n \times \mathbb{R}^m \)), we get \( D^2_x \) (the (\( \gamma \))-confidence interval of \( \hat{x} \)) as follows:

\[
D^2_x = \{\omega(\in \Omega) : d_{\Theta}(E(\hat{x}), \pi(\omega)) \leq \eta_\gamma\}
\]

\[
= \{(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R} : \frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^m y_k}{m} - (\mu_1 - \mu_2) \leq \left(\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right)^{1/2} z\left(1 - \frac{\gamma}{2}\right)\}
\]

(69)

4.3 The case that \( d^2_{\Theta} \) depends on \( x \); Student’s t-distribution

The arguments in this section are continued from Example 2.

Example 7 [Student’s t-distribution]. Consider the simultaneous measurement \( M_{\mathbb{O}}(\mathbb{R} \times \mathbb{R}_+) \) (\( \mathbb{O}_N = (\mathbb{R}^n, \mathbb{B}_\mathbb{R}^n, \mathbb{N}^n \)), \( S_{(\mu, \sigma)} \)) in \( C_{\Theta}(\mathbb{R} \times \mathbb{R}_+) \). Thus, we consider that \( \Omega = \mathbb{R} \times \mathbb{R}_+, X = \mathbb{R}^n \). Put \( \Theta = \mathbb{R} \) with the semi-distance \( d^2_{\Theta}(\forall x \in X) \) such that

\[
d^2_{\Theta}(\theta_1, \theta_2) = \frac{|\theta_1 - \theta_2|}{\sigma'(x)/\sqrt{n}} \quad (\forall x \in X = \mathbb{R}^n, \forall \theta_1, \theta_2 \in \Theta = \mathbb{R})
\]

(70)

where \( \sigma'(x) = \sqrt{\frac{n}{n-1}} \sigma(x) \). The quantity \( \pi : \Omega(= \mathbb{R} \times \mathbb{R}_+) \to \Theta(= \mathbb{R}) \) is defined by

\[
\Omega(= \mathbb{R} \times \mathbb{R}_+) \ni \omega = (\mu, \sigma) \mapsto \pi(\mu, \sigma) = \mu \in \Theta(= \mathbb{R})
\]

(71)

Also, define the estimator \( E : X(= \mathbb{R}^n) \to \Theta(= \mathbb{R}) \) such that

\[
E(x) = E(x_1, x_2, \ldots, x_n) = \overline{p}(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}
\]

(72)

Let \( \gamma \) be a real number such that \( 0 \ll \gamma < 1 \), for example, \( \gamma = 0.95 \). Thus, for any \( \omega = (\mu, \sigma)(\in \Omega = \mathbb{R} \times \mathbb{R}_+) \),
we see that

\[
[N^n(\{x \in X : d^\Theta(x, \pi(\omega)) \leq \eta\}))/\omega ) = N^n(\{x \in X : \left| \frac{\overline{X}(x) - \mu}{\sigma(x)/\sqrt{n}} \right| \leq \eta\}))/\omega )
\]

\[
= \frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_{-\eta \leq \frac{|x| - \mu}{\sigma(x)/\sqrt{n}} \leq \eta} \exp\left[ -\sum_{k=1}^{n}(x_k - \mu)^2 \right] dx_1 dx_2 \cdots dx_n
\]

\[
= \frac{1}{(\sqrt{2\pi})^n} \int \cdots \int_{-\eta \leq \frac{|x| - \mu}{\sigma(x)/\sqrt{n}} \leq \eta} \exp\left[ -\frac{\sum_{k=1}^{n}(x_k)^2}{2} \right] dx_1 dx_2 \cdots dx_n
\]

\[
= \int_{-\eta}^{\eta} p^t_{n-1}(x) \, dx
\]

(73)

where \( p^t_{n-1} \) is the t-distribution with \( n - 1 \) degrees of freedom. Solving the equation \( \gamma = \int_{-\eta}^{\eta} p^t_{n-1}(x) \, dx \), we get \( \eta^t = t((1 - \gamma)/2) \).

Therefore, for any \( x \in X \), we get \( D^\gamma_2(x) \) (the \( (\gamma) \)-confidence interval of \( x \)) as follows:

\[
D^\gamma_2 = \{ \pi(\omega)(\in \Theta) : \omega \in \Omega, \ d^\Theta(x, \pi(\omega)) \leq \eta^t \}
\]

\[
= \{ \mu \in \Theta(= \mathbb{R}) : \overline{X}(x) - \frac{\overline{X}(x)}{\sqrt{n}} t((1 - \gamma)/2) \leq \mu \leq \overline{X}(x) + \frac{\overline{X}(x)}{\sqrt{n}} t((1 - \gamma)/2) \}
\]

(74)

5 Conclusions

It is sure that statistics and (classical) quantum language are similar. However, quantum language has the firm structure (1), i.e.,

\[
\text{Quantum language (MT(measurement theory))} = \text{Axiom 1 (measurement)} + \text{Axiom 2 (causality)} + \text{linguistic interpretation how to use Axioms}
\]

(75)

Hence, as seen in this paper, every argument cannot but become clear in quantum language. Thus, quantum language is suited for the theoretical arguments.

In fact, Theorem 2 (the confidence interval methods in quantum language) says that

- from the pure theoretical point of view, we can not understand the confidence interval methods without the three concepts, that is, "estimator \( E: X \rightarrow \Theta \)" and "quantity \( \pi: \Omega \rightarrow \Theta \)" and "semi-distance \( d^\Theta \)"

which is also shown throughout Remarks 1-5 and Examples 4-7. This is our new viewpoint of the confidence interval methods.

We hope that our approach will be examined from various points of view.

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