Stability conditions for a decentralised medium access algorithm: single- and multi-hop networks

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Abstract

We consider a decentralised multi-access algorithm, motivated primarily by the control of transmissions in a wireless network. For the finite single-hop network with arbitrary interference constraints we prove stochastic stability under the natural conditions. For the infinite (as well as finite) single-hop network, we obtain broad rate-stability conditions. We also consider symmetric (in terms of both arrival intensities and routing) finite multi-hop networks and show that the natural condition is sufficient for stochastic stability.

1 Introduction

We consider a model motivated by wireless networks. A key feature of wireless transmissions is that they interfere with each other, especially if the receivers are in close proximity, and this interference may prevent some of the simultaneous transmissions from being received correctly. This creates the need for the design of algorithms regulating the behaviour of transmitters in wireless networks, so that simultaneous interfering transmissions do not occur at all, or occur rarely.

The transmitter-receiver pairs in a network are represented by vertices (we will refer to them as nodes) on a graph, and an edge between two vertices is present if the corresponding transmissions interfere with each other. Thus, the resulting interference graph represents the (interference) structure of the network.

We consider both single- and multi-hop networks. In a single-hop network jobs (or messages to be transmitted) arrive at nodes in a network and, upon a successful transmission, leave the network. The dependence between states of different nodes exists because the interference graph imposes constraints on simultaneous transmissions. In a multi-hop network, a message, upon successful transmission at one node, may leave the network or may move to another node, where it needs to be transmitted again. Thus, multi-hop networks add a further layer of complexity, as the states of the nodes are dependent not only due to interference constraints, but in this case also due to message movement between the nodes.

We are interested in stability of a network. In finite networks by stability we mean the stochastic stability of the nodes’ queues, where messages wait for transmission. One usually calls a (transmission scheduling) algorithm maximally stable if it guarantees stability if such is feasible at all, under at least one algorithm. The celebrated BackPressure (sometimes referred to as MaxWeight) algorithm introduced in [18] is maximally stable. It is however centralised, i.e. it requires the presence of a central entity that is aware of the state of the entire network. This is not practical in wireless networks that tend to be large and ever-changing.

There is thus the need for designing decentralised algorithms where each node regulates its behaviour on its own, without any global knowledge. In principle, decentralised behaviour may lead to conflicts, when several interfering transmissions will be attempted simultaneously, and

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the messages will not be received. We consider the so-called CSMA (Carrier Sense Multiple Access) networks where each transmitter can sense if a neighbouring node is transmitting and will never initiate an interfering transmission. Conflicts are thus avoided in CSMA networks.

In this paper, it is not our goal to design decentralised algorithms (or, protocols) that are maximally stable. We study a different question: what is the stability performance of some simple specific decentralised protocols? We consider the following protocol. Assume that the network is finite. Assume also that time is slotted, i.e. arrivals happen at discrete time instances denoted \(1, 2, \ldots\), all transmission times are equal to 1, and transmissions start at the beginning of a unit-long time slot and complete at its end. Assume that at the beginning of each time slot, each message is assigned a random number, drawn independently from some fixed absolutely continuous distribution; the lower this number, the higher the message transmission priority in the slot. Then, a given message is transmitted in a slot, if and only if its priority is the highest among all messages within its neighborhood, which includes its own node and all adjacent (neighbouring) nodes. We note that this protocol is different from that considered in [14]. First, in [14] the nodes compete for access, as opposed to individual messages. Second, the protocol in this paper is more conservative than that in [14] in that, a message will stay silent in a slot even if all higher priority messages in its neighbourhood do not actually transmit in the slot.

On the face of it, the protocol described in the previous paragraph is centralised, as priorities need to be assigned. It is, however, easily implemented in a decentralised fashion (with an arbitrarily small loss of efficiency), for example as follows. Let an arbitrarily small duration \(\varepsilon\) at the beginning of each time slot be fixed. Assume that each actual (payload) transmission lasts \(1 - \varepsilon\), while the initial \(\varepsilon\)-long interval within each time slot is devoted to medium-access competition. At the beginning of each slot there are no transmissions. Each message starts an “access transmission” at a random time uniformly distributed in \((0, \varepsilon)\), independent of everything else; the access transmission stops at (or before) time \(\varepsilon\). A message starts actual (payload) transmission at time \(\varepsilon\) if and only if its access transmission was the first within its neighbourhood.

In the rest of the paper, for simplicity we consider a “cleaner” version of the protocol, as described in the previous paragraph (and in Section 2) that ignores the loss of an \(\varepsilon\) proportion of the throughput.

The following example illustrates the conservative nature of the protocol. Consider 4 nodes, with interference graph being a “circle”, so that either nodes 1 and 3, or nodes 2 and 4 can transmit simultaneously. Assume that, in a given time slot, the message with the highest priority is located at node 1, the message with the second-highest priority is located at node 2 and the message with the third-highest priority is located at node 3. Under the algorithm considered here, only the first message will be transmitted in this slot, when in fact nodes 1 and 3 could successfully transmit simultaneously. Of course, under other priority orderings, transmission of two messages will occur.

Denote by \(N_i\) the neighbourhood of node \(i\) in the interference graph (all neighbours of the node and the node itself). Denote by \(X_i\) the number of messages at node \(i\) at the beginning of a time slot. It is easy to see that node \(i\) will transmit a message with probability

\[
\varphi_i = \frac{X_i}{\sum_{j \in N_i} X_j}.
\]

A model closely related to ours has been considered in a recent paper [13] (see also [12]). It is a single-hop network with nodes located on a grid. The model is in continuous time, with each message having an exponentially distributed size with unit mean. All messages may transmit simultaneously, with the instantaneous transmission rate depending on the interference from the messages in the neighborhood. (The neighborhood in [13] is defined slightly more generally.) A standard assumption that a message transmission rate is proportional to its Signal-to-Noise-Ratio is adopted in [13], which leads to a node \(i\) transmission rate given exactly by (1). The
model in [13] is symmetric in that the message arrival rates at all nodes are equal. The authors focus on infinite-grid networks and are interested in their stability. The authors define this as the finiteness of the minimal stationary regime for the system starting with all queues being empty (see [13] for more details). They show that a network is stable in this sense under the natural condition on the message arrival rates. The main tool in their analysis is monotonicity, i.e. the property that if one network starts with an initial condition dominating that of another, there exists a coupling preserving this dominance at all times.

In the single-hop scenario, we consider arbitrary networks: finite or infinite, arbitrary interference graph, arbitrary arrival intensities. For finite single-hop networks we prove that the system is stochastically stable if the arrival rates belong to a certain set. For infinite single-hop networks we obtain a rather broad sufficient condition for the rate-stability which is the property that, starting from any fixed initial state, the growth rates of the queues are sub-linear in time.

We also consider finite multi-hop networks; here we additionally assume that a network is symmetric: it is a regular graph, with equal exogenous arrival rates at the nodes and with a message path through the network being the standard random walk (until the message leaves the network). In the multi-hop setting, a further complication arises from the fact that monotonicity does not hold. We prove directly that the network is stochastically stable under a natural condition. Our approach is not based on monotonicity, in either setting.

Our stability proofs use the fluid-limits technique. The discrete-time setting motivating our work and the continuous-time network motivating [13] share the same fluid limits, and thus our results are valid in the continuous-time setting too. Note also that the random variables representing the number of successful transmissions from all nodes in the same time slot in our model are not independent. However, our stability results also apply to a different discrete-time model, where (perhaps rather unrealistically) each station transmits a message with a probability given by [1]: again, this is due to the fact that this model has the same fluid-limit dynamics as ours.

Another important concept in wireless networks is utility maximisation. Utility-optimal algorithms are known to guarantee maximal stability for finite single-hop networks, under some assumptions on the utility functions. However, these algorithms are centralised as the average service rates \( \varphi_i \) assigned to nodes form a solution \( \varphi = \{ \varphi_i \} \) to a global optimisation problem.

An important example of such algorithms is presented by the well-known \( \alpha \)-fair algorithms (see [8, 9, 10] for introduction of the fair-allocation concepts and [1, 6] for stability proofs). In \( \alpha \)-fair algorithms the average rates \( \varphi_i \) are such that

\[
\varphi \in \arg \max_{\mu \in \mathcal{C}} \sum_i X_i \frac{1}{1 - \alpha} \left( \frac{\mu_i}{X_i} \right)^{1-\alpha}, \quad \text{when } \alpha > 0, \alpha \neq 1,
\]

or

\[
\varphi \in \arg \max_{\mu \in \mathcal{C}} \sum_i X_i \log(\mu_i/X_i), \quad \text{when } \alpha = 1.
\]

In the finite single-hop setting we provide two proofs of stochastic stability, for the arrival rates within a certain (natural) set \( \mathcal{C} \). The first proof follows from a much more general result for monotone 0-homogeneous service rates, which is of interest on its own and may have other applications. Our infinite-system rate stability proof is based on the same ideas. We provide a second proof for finite systems, as it is based on discovering an important property of the rates [1]: they in fact happen to be \( \alpha \)-fair in \( \mathcal{C} \), with \( \alpha = 2 \). We believe this property to be interesting in its own right too as it presents an example of a decentralised protocol which happens to be centrally optimal in a certain sense, and it is known to imply stochastic stability in finite single-hop networks.

It is known that utility-maximising algorithms, with the exception of proportionally-fair algorithms (i.e. \( \alpha \)-fair algorithms with \( \alpha = 1 \); see [19] for a treatment of some of the multi-hop scenarios), in general do not guarantee stability in multi-hop networks. Therefore our result
on the fairness of the rates [1] does not imply stability in the multi-hop setting. In Section 3.2 we
present our main result in the finite multi-hop setting stating that a natural stability property
does hold under our algorithm (which happens to be $\alpha$-fair) for a class of symmetric networks.
Specifically, these networks are such that the interference graph is regular, the exogenous arrival
rates to all nodes are equal, and upon a successful transmission a message may either leave the
network or move to a neighbour node chosen uniformly at random. We show stochastic stability
under a natural condition on the per-node exogenous arrival rate.

The paper is organised as follows. We define our model in Section 2 and then present our
main results in Section 3 (for single-hop networks in Section 3.1 and for multi-hop networks –
in Section 3.2). We then comment on the construction and main properties of Fluid Sample
Paths for our models in Section 4. The proofs of our main results are presented in Section 5.2
for the finite single-hop case, in Section 5.3 for the infinite single-hop case, and in Section 6 for
the multi-hop case. We discuss some open problems in Section 7.

Basic notation. We will use the following notation throughout: $\mathbb{R}$ and $\mathbb{R}_+$ are the sets
of real and real non-negative numbers, respectively; $\overrightarrow{y}$ means (finite- or infinite-dimensional)
vector ($y_i$); for a finite-dimensional vector $\overrightarrow{y}$, $\|y\| = \sum_i \|y_i\|$; for a set of functions ($f_i$) and
a vector ($y_i$), $\overrightarrow{f}(\overrightarrow{y})$ denotes the vector ($f_i(\overrightarrow{y})$); vector inequalities are understood component-
wise: $\frac{d^+}{dt}$ is the right derivative; $\frac{d^+}{dt} y(t_0) = \liminf_{t \downarrow t_0} y(t) - y(t_0)$ – the lower right Dini derivative;
$y(\cdot) = (y(t), t \geq 0)$; we also use the convention that $0/0 = 0$.

2 Model and notation
Denote by $\mathcal{V}$ the set of nodes represented by vertices on a graph, and by $\mathcal{E}$ the set of its edges.
The set $\mathcal{V}$ may be finite (in which case we will refer to the network as finite) or countable
(in which case we will refer to the network as infinite). Denote by $N$ the (finite or infinite)
cardinality of $\mathcal{V}$.

For a vertex $i$, denote by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\} \cup \{i\}$ its neighbourhood. We assume throughout that $\mathcal{N}_i$ is finite for each $i$, the graph $G = (\mathcal{V}, \mathcal{E})$ is connected, and that the
neighbourhood relationship is symmetric (or that the graph is undirected), i.e. if $i \in \mathcal{N}_j$, then
$j \in \mathcal{N}_i$.

Each node has an infinite buffer for storing messages but there is no queue. Time is slotted,
and at the beginning of each time slot first transmissions are initiated, and then arrivals happen.
Each transmission time is equal to 1.

At the beginning of each time slot, every message in the system is assigned a random number
which is drawn, independently of everything else, from a certain fixed absolutely continuous
distribution; the smaller this number, the higher the message transmission priority. A message is
transmitted if and only if it has the highest priority in its neighbourhood, i.e. node $i$ transmits a
message if that message’s priority is the maximal over all the messages in $\mathcal{N}_i$. We refer the reader
to the introduction for an explanation of how this may be implemented in a decentralised way,
by using Uniform distribution on a small interval, with an arbitrarily small loss of throughput.

At each time slot $k$, a random number $\xi_i(k)$ new messages arrive at node $i$. We assume that
$\xi_i(k)$ are i.i.d. with $\mathbb{E}(\xi_i(k)) = \lambda_i > 0$.

Throughout the paper we use notation

$$\varphi_i(\overrightarrow{p}) = \frac{p_i}{\sum_{j \in \mathcal{N}_i} p_j}, \quad \overrightarrow{\varphi}_i(\overrightarrow{p}) = (\varphi_i(\overrightarrow{p})),$$

where $\overrightarrow{p} = (p_i)$ is a vector with finite non-negative components. By convention, $\varphi_i(\overrightarrow{p}) = 0$ when
$\sum_{j \in \mathcal{N}_i} p_j = 0$. 

4
3 Main results

The results are split into two subsections covering single- and multi-hop networks. The subsection on single-hop networks contains results for both finite and infinite networks.

3.1 Single-hop network

For the single-hop network we make no extra assumptions on graph $G$ apart from those listed in Section 2. We assume that, upon a successful transmission, a message leaves the system. The evolution of the state of the queue of node $i$ may then be written as

$$X_i(k + 1) = X_i(k) + \xi_i(k) - \eta_i(k),$$

where by $X_i(k)$ we denoted the state of the queue of node $i$ at time $k$, and by $\eta_i(k)$ - the number of messages leaving node $i$ during the $k$-th time slot. The random variable $\eta_i(k)$ can only take values 0 and 1, and it is easy to see that, as priorities are chosen independently from the same fixed distribution,

$$P(\eta_i(k) = 1 | X(k) = X) = \varphi_i(X) = \frac{X_i}{\sum_{j \in N_i} X_j}.$$ 

Denote

$$C = \{ \lambda : \lambda \leq \varphi(p) \text{ for some } p \}.$$  \hspace{1cm} (3)

We will call a finite network stochastically stable if the countable Markov chain $X(\cdot)$ is positive recurrent.

**Theorem 1.** Consider a finite network. If $\lambda < \nu$ for some $\nu \in C$, then the system is stochastically stable.

We present a proof of Theorem 1 in Section 5.2.

**Corollary 2.** In a finite symmetric network, where the graph $G$ is $(m - 1)$-regular (so that each vertex has degree $(m - 1)$) and $\lambda_i = \lambda$ for each $i$, the condition of Theorem 1 is equivalent to the requirement that $\lambda < 1/m$. In particular, for a network of nodes located on a circle with the same arrival intensity $\lambda$ at each node, $\lambda < 1/3$ guarantees stability.

Indeed, if $\lambda < 1/m$, then the vector $(\lambda, \ldots, \lambda)$ is component-wise upper-bounded by the vector $(1/m, \ldots, 1/m)$ which belongs to the set $C$ trivially (one needs to take the vector $p = (1, \ldots, 1)$ to verify this). Assume now that for some $p$ the vector $(\lambda, \ldots, \lambda)$ is component-wise smaller than a vector $\varphi(p) \in C$. Then

$$\frac{1}{\lambda} > \sum_{j \in N_i} \frac{p_j}{p_i}$$

for each $i$, and if we add up these inequalities, we obtain

$$\frac{N}{\lambda} > \sum_i \sum_{j \in N_i} \frac{p_j}{p_i} = \frac{1}{2} \sum_i \sum_{j \in N_i} \left( \frac{p_j}{p_i} + \frac{p_i}{p_j} \right) \geq mN,$$

which implies $\lambda < 1/m$.

**Remark 3.** As the proof of Theorem 1 is based on the fluid limits, its results are also valid for a continuous version of the model similar to that of [13] (see also the introduction to this paper for an explanation of the connection between models).
A network (finite or infinite) is called rate-stable, if w.p.1,
\[ \lim_{k \to \infty} X_i(k)/k = 0, \quad \forall i, \]
for any initial state \( X(0) \) with all components being finite, \( X_i(0) < \infty \).

Rate-stability is a weaker property than the stochastic stability. The following result gives a sufficient condition for rate-stability.

**Theorem 4.** The infinite or finite system is rate-stable if \( \bar{\lambda} \leq \bar{\phi}(\bar{p}) \) for some \( \bar{p} \) such that \( 0 < c \leq p_i \leq 1 \) for all \( i \).

A proof of Theorem 4 is given in Section 5.3. (We actually prove a slightly more general fact there.)

### 3.2 Symmetric multi-hop networks with Geometric service requirements

In this section we restrict our attention to a finite \((m-1)\)-regular graph, \( m \geq 2 \). Assume that the access procedure is the same as before. Now however, upon service, a message leaves the system with probability \( 1/k \), and goes to a neighbouring node with probability \( (1 - 1/k)^{m-1} \).

Assume that arrival rate into each node is \( \lambda/k \), so that the total workload for each node is \( \lambda \) (this follows from standard rate-balance equations).

One can think of each message needing a Geometric(1/k) number of successful transmissions to leave the system and, conditionally on not leaving the system upon a successful transmission, performing a simple random walk on the graph (choosing a neighbouring node uniformly at random).

**Theorem 5.** Suppose the system graph is finite \((m-1)\)-regular, \( m \geq 2 \). Then, if \( \lambda < 1/m \), the system is stochastically stable.

We provide a proof of Theorem 5 in Section 6.

**Remark 6.** Similarly to Remark 3, our results in the multi-hop setting hold for a continuous-time version of the model.

### 4 Fluid limits

Our main results are based on the fluid-limit technique (see [11, 4, 16]). For the application of this technique to discrete time processes, see e.g. [3, 17]. In fact the proof of Lemma 7 below follows the exact same lines as that of [17, Theorem 2], for a model close to ours, and is omitted here.

We consider the Markov chain \( \{X(k)\}_{k \geq 0} \) and extend it to continuous time with the convention \( X(t) = X(\lfloor t \rfloor) \). Consider a sequence of processes \( X^{(r)}(\cdot) \), indexed by \( r \uparrow \infty \). Consider further their fluid-scaled versions
\[ \bar{X}^{(r)}(t) = \frac{X^{(r)}(rt)}{r}, \quad t \geq 0, \]
and assume that
\[ \bar{X}^{(r)}(0) \to \bar{\pi}(0), \]
as \( r \to \infty \), for some fixed \( \bar{\pi}(0) \) with all components being finite, \( x_i(0) < \infty \).

**Lemma 7.** For either the single- or the multi-hop setting, all processes (for all \( r \)) may be constructed on a common probability space, such that, with probability 1, any subsequence of realisations of \( \bar{X}^{(r)}(\cdot) \) contains a further subsequence such that each \( x_i^{(r)}(\cdot) \) converges, uniformly on compact sets, to a Lipschitz function \( x_i(t), \ t \geq 0 \). Moreover, the limiting functions \( x_i(\cdot) \) are uniformly Lipschitz. The corresponding limiting vector-function \( \bar{\pi}(t) = (x_i(t)), \ t \geq 0 \) is called a fluid sample path (FSP) with initial state \( \bar{\pi}(0) \).
FSPs are formally defined as possible limits of the realisations of $\overline{x}(\cdot)$, with common “driving” processes’ realisations, satisfying the functional strong law of large numbers. This definition/construction is quite standard (again, see e.g. the proof of [17, Theorem 2]) and we omit it here.

For a finite system, to establish stochastic stability (positive recurrence) of the Markov chain $\{X(k)\}_{k \geq 0}$, it suffices to prove that for some $T > 0$ and $\varepsilon > 0$ any sequence of processes $\overline{X}^{(r)}(\cdot)$, with $\|\overline{X}^{(r)}(0)\| = r$, is such that

$$\limsup_{r \to \infty} \mathbb{E} \left[ \frac{1}{r} \|\overline{X}^{(r)}(rT)\| \right] \leq 1 - \varepsilon.$$  

It is a standard result when applying fluid-limit techniques, that for the above to hold, it is sufficient to show that for some $\varepsilon > 0$ and $T > 0$, any FSP with $\|x(0)\| = 1$ is such that

$$||x(T)|| \leq 1 - \varepsilon. \quad (4)$$

For a finite or infinite system, to show rate-stability, it suffices to prove (see Lemma 13 below) that any FSP starting from zero initial state (all $x_i(0) = 0$), stays in zero state at all times, $x_i(t) = 0$, $t \geq 0$. This follows from the rate-stability definition and Lemma 7.

5 Single-hop network

5.1 FSP properties

We start by establishing properties of the FSP dynamics in Lemma 8. The proof is, once again, standard and is omitted here.

Lemma 8. Any FSP in the single-hop case satisfies the following conditions:

$$x_i(t) > 0 \implies x'_i(t) = \lambda_i - \varphi_i(\overline{x}(t)), \text{ for almost all } t \geq 0,$$

$$[x_i(t) = 0 \text{ and } \sum_{j \in N_i} x_j(t) > 0] \implies \frac{d^+}{dt} x_i(t) = \lambda_i. \quad (6)$$

In particular, by property (6), any FSP is such that, for a fixed $i$, $\sum_{j \in N_i} x_i(t) > 0$ implies that $x_i(\tau) > 0$ for all $\tau > t$ sufficiently close to $t$. Furthermore, if the network is finite, any FSP is such that $\sum_i x_i(t) > 0$ implies that $x_i(\tau) > 0$, $\forall i$, for all $\tau > t$ sufficiently close to $t$.

A proof of Lemma 8 may be given following the exact same lines as that of [17, Theorem 2], for a model close to ours, and we omit it here.

5.2 Proof of Theorem 1

We present two different proofs of Theorem 1. One proof of (11), and then of Theorem 1, follows from the following much more general result.

For a function $\overline{\psi} = \overline{\psi}(\overline{p})$, mapping a finite-dimensional positive orthant $\mathbb{R}^N_+$, $N < \infty$, into itself, define

$$\mathcal{D} = \{ \overline{x} \in \mathbb{R}^N_+ : \overline{x} \leq \overline{\psi}(\overline{p}) \text{ for some } \overline{p} \}. \quad (7)$$

Lemma 9. Consider a family of Lipschitz trajectories $\overline{x}(t)$, $t \geq 0$, in $\mathbb{R}^N_+$, $N < \infty$, which satisfy the following conditions:

$$x_i(t) > 0 \implies x'_i(t) = \lambda_i - \psi_i(\overline{x}(t)), \text{ for almost all } t \geq 0, \text{ for any } i,$$

$$\sum_i x_i(t) > 0 \implies x_i(\tau) > 0, \text{ for all } i, \text{ for all } \tau > t \text{ sufficiently close to } t, \quad (9)$$
where the function $\psi$ is such that:

(A) each $\psi_i$ is non-decreasing in $x_j$ for all $j \neq i$;

(B) each $\psi_i$ is 0-homogeneous, i.e. $\psi_i(sx) = \psi_i(x)$ for all $s > 0$ and for all $x$.

Assume that $\lambda$ is such that $\lambda < \nu$ for some $\nu \in D$. Then for any constants $0 < \delta < K < \infty$, there exists $T > 0$ such that, for any such trajectory with $\|x(0)\| = K$,

$$\|x(T)\| \leq \delta.$$  

Proof of Lemma 9.

Fix a vector $\overline{p}$ such that $\nu \leq \psi_i(\overline{p})$ for every $i$ and consider the function

$$F(y) = \max_i \left( \frac{y_i}{p_i} \right).$$

For ease of notation, in the rest of the proof we drop the index $t$ and make the dependence of $\psi$ on $x(t)$ implicit.

Denote

$$K = \left\{ k : k \in \arg \max_i \left( \frac{x_i}{p_i} \right) \right\}.$$

The function $\max_i \{x_i/p_i\}$ is Lipschitz, because all $x_i(\cdot)$ are Lipschitz. The time points $t$, where the derivatives of all $x_i$ and of $\max_i \{x_i/p_i\}$ exist, are called regular. Almost all all points (with respect to Lebesgue measure) are regular. Then, due to [5, Lemma 2.8.6], $(\frac{x_k}{p_k})' = (\frac{x_l}{p_l})'$ at any regular point of $F$ for any $k, l \in K$. The derivative of the function is $F$ at a regular point is then

$$(F(\overline{x}))' = \frac{1}{p_k}(\lambda_k - \psi_k)$$

with an arbitrary $k \in K$. Note that, as $\frac{x_k}{p_k} \geq \frac{x_l}{p_l}$ for any $k \in K$ and for any $j$, due to property (A),

$$\psi_k = \psi_k(\overline{x}) \geq \psi_k \left( \frac{x_kp_k}{p_k}, \ldots, \frac{x_kp_{k-1}}{p_k}, \frac{x_kp_k}{p_k}, \ldots, \frac{x_kp_N}{p_k} \right)$$

$$= \psi_k \left( \frac{x_kp_k}{p_k}, \ldots, \frac{x_kp_{k-1}}{p_k}, \frac{x_kp_k}{p_k}, \ldots, \frac{x_kp_N}{p_k} \right)$$

$$= \psi_k(\overline{p}) = \nu_i$$

where in the last step we used property (B).

Noting that there exists $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for every $i$, we obtain

$$(F(\overline{x}))' = \frac{1}{p_k}(\lambda_k - \nu_k) + \frac{1}{p_k}(\nu_k - \psi_k) < -\frac{\varepsilon}{p_k}.$$  

This implies that $(F(\overline{x}))'$ is negative, bounded away from 0, as long as $F(\overline{x})$ is positive, bounded away from 0. In particular, if $\overline{x}$ “hits” 0, it stays at 0 thereafter. This concludes the proof of Lemma 9.

Theorem 1 follows from Lemma 9 as the rates $\overline{\varphi}$ clearly satisfy conditions (A) and (B).

Remark 10. Lemma 9 is rather general and relates to the so-called cooperative dynamical systems (see, e.g. [7] [13]). We believe that this result is interesting on its own and may have other applications. This result also allows, in an obvious fashion, to obtain stability conditions for networks with a more general notion of neighbourhood considered in [13].
We also present a different proof of Theorem 1, which is specific to our model and is based on a global optimality of the rates $\overline{\varphi}$. We think that this optimality is interesting on its own as it is an important structural property of the rates, and as it provides an example of a decentralised algorithm which maximises a global utility function. We also present a simple proof showing stability of algorithms maximising utility functions over a set, without requiring convexity of the set.

The remainder of this second proof consists of the following steps, which correspond to two lemmas below:

1. We show that the FSPs of this model are such that the “service rates” the nodes receive are utility maximising (in fact 2-fair) in the set $C$. This property is proved in Lemma 11.

2. The property (1) of the FSPs (which is sometimes referred to as stability of FSPs), and hence the stability of the underlying Markov chain, follows from the utility-maximisation property of the “service rates.” We only need to note that this fact is usually proved for convex sets of possible rates, whereas our set $C$ is not convex. However, convexity is in fact not needed in the stability proof, and we provide a proof for any sets, based on the proof of [17, Theorem 2]; this is done in Lemma 12.

**Lemma 11.** For any $\overline{x}$ with $x_i > 0$ for all $i$, the rates $\overline{\varphi} = \overline{\varphi}(\overline{x})$ are 2-fair in the set $C$ (see the Introduction of this paper or, e.g. [7] for definition of $\alpha$-fairness).

Indeed, due to the definition of the set $C$, for any $\overline{x} \in C$,

$$\sum_i x_i \left( \frac{\mu_i}{x_i} \right)^{-1} \geq \sum_i x_i \left( \frac{p_i}{(\sum_{j \in \mathcal{N}_i} p_j) x_i} \right)^{-1}$$

for the corresponding vector $\overline{p}$. Hence, it is sufficient to show that

$$\sum_i x_i \left( \frac{\varphi_i}{x_i} \right)^{-1} \leq \sum_i x_i \left( \frac{p_i}{(\sum_{j \in \mathcal{N}_i} p_j) x_i} \right)^{-1}$$

for all vectors $\overline{p}$.

Note that the LHS of the above is equal to $\sum_i x_i \sum_{j \in \mathcal{N}_i} x_j = \sum_i x_i^2 + \sum_i \sum_{j \in \mathcal{N}_i, j \neq i} x_i x_j$. Consider now

$$\sum_i x_i \left( \frac{p_i}{(\sum_{j \in \mathcal{N}_i} p_j) x_i} \right)^{-1} = \sum_i x_i^2 \left( 1 + \sum_{j \in \mathcal{N}_i, j \neq i} \frac{p_j}{p_i} \right) = \sum_i x_i^2 + \sum_i \sum_{j \in \mathcal{N}_i, j \neq i} \left( x_i^2 \frac{p_j}{p_i} + x_j^2 \frac{p_i}{p_j} \right).$$

For any $i$ and $j$,

$$x_i^2 \frac{p_j}{p_i} + x_j^2 \frac{p_i}{p_j} \geq 2x_i x_j,$$

and the equality is possible if and only if $x_i^2 \frac{p_j}{p_i} = x_j^2 \frac{p_i}{p_j}$, which is equivalent to $\frac{p_i}{p_j} = \frac{x_i}{x_j}$. Therefore we obtain

$$\sum_i x_i \left( \frac{\sum_{j \in \mathcal{N}_i} p_j}{p_i x_i} \right)^{-1} \geq \sum_i x_i^2 + \sum_i \sum_{j \in \mathcal{N}_i, j \neq i} x_i x_j,$$

and the equality is possible if and only if $\frac{p_i}{x_i} = \frac{p_j}{x_j}$ for all $i$ and $j$. This implies that $\frac{p_i}{x_i}$ has to be a constant for each $i$, as the graph is connected. This concludes the proof of Lemma 11.

The result of Theorem 1 now follows from stability of FSPs under $\alpha$-fair algorithms. One only needs to note that such proofs are usually given for convex sets, but convexity is not in fact needed, and stability may be proved following the lines of the proof of Theorem 2 in [17], Section 8, where it was given in the case $\alpha = 1$. The proof for far more general rate allocations is essentially the same and we provide it here for completeness.
Lemma 12. Let $C$ be a compact coordinate-convex subset of $\mathbb{R}_+^N$. (Coordinate-convex means that $\vec{x} \in \mathbb{R}_+^N$ and $\vec{x} \leq \vec{y} \in C$ imply $\vec{x} \in C$.) Let $h_i : [0, \infty) \to \mathbb{R}$, for each $i$, be an increasing differentiable concave function (the case when $h_i(y) \downarrow -\infty$ as $y \downarrow 0$ is allowed). Let $g_i : [0, \infty) \to [0, \infty)$, for each $i$, be a continuous non-decreasing function such that $g_i(0) \geq 0$ and $g_i(y) > 0$ for $y > 0$.

Consider a family of Lipschitz trajectories $\vec{\tau}(t), \ t \geq 0$, in $\mathbb{R}_+^N$, which satisfy the following conditions:

$$x_i(t) > 0 \implies x_i'(t) = \lambda_i - \psi_i(\vec{\tau}(t)), \ \text{for almost all } t \geq 0,$$

$$\sum_i x_i(t) > 0 \implies x_i(\tau) > 0, \ \text{for all } i, \text{for all } \tau > t \text{ sufficiently close to } t,$$

where the rates $\vec{\psi}$ satisfy

$$\vec{\psi} \in \arg \max_{\vec{\tau} \in C} \sum_i g_i(x_i)h_i(\mu_i).$$

Assume that $\vec{\lambda}$ is such that $\vec{\lambda} < \vec{\psi}$ for some $\vec{\psi} \in C$. Then for any constants $0 < \delta < K < \infty$, there exists $T > 0$ such that, for any such trajectory with $\|x(0)\| = K$,

$$\|x(T)\| \leq \delta.$$

Proof of Lemma 12. In this proof, we will drop the index $t$ for ease of notation. We will also write simply $\psi_i$, with its dependence on $\vec{\tau}(t)$ being implicit.

Property (11) implies that $x_i(t) > 0$, for all $i$, for $t \in (0, \theta)$, where $\theta$ is the first time (if any) when all $x_i(t)$ “hit” 0 simultaneously. Consider a trajectory $\vec{\tau}(\cdot)$ in this interval $(0, \theta)$.

Note that there exists $\varepsilon > 0$ such that $\lambda_i < \nu_i - \varepsilon$ for each $i$. Denote by

$$F(\vec{\psi}) = \sum_{i=1}^N G_i(y_i)h_i'(\nu_i)$$

with $G_i(\varepsilon) = \int_0^\varepsilon g_i(s)ds$, and note that

$$(F(\vec{\tau}))' = \sum_{i=1}^N h_i'(\nu_i)g_i(x_i)(\lambda_i - \psi_i) = \sum_{i=1}^N h_i'(\nu_i)g_i(x_i)(\lambda_i - \nu_i) + \sum_{i=1}^N h_i'(\nu_i)g_i(x_i)(\nu_i - \psi_i)$$

$$< -\varepsilon \sum_{i=1}^N h_i'(\nu_i)g_i(x_i) + \sum_{i=1}^N h_i'(\nu_i)g_i(x_i)(\nu_i - \psi_i).$$

(13)

As $\vec{\psi} \in C$, due to $[12]$,

$$\sum_{i=1}^N g_i(x_i)h_i(\psi_i) \geq \sum_{i=1}^N g_i(x_i)h_i(\nu_i).$$

Using this and the concavity of the functions $h_i$, we have

$$0 \leq \sum_{i=1}^N g_i(x_i)(h_i(\psi_i) - h_i(\nu_i)) \leq \sum_{i=1}^N g_i(x_i)h_i'(\nu_i)(\psi_i - \nu_i).$$

This, together, with (13), implies that

$$(F(\vec{\tau}))' < -\varepsilon \sum_{i=1}^N h_i'(\nu_i)g_i(x_i).$$

We see that $(F(\vec{\tau}))'$ is negative, bounded away from 0, as long as $F(\vec{\tau})$ is positive, bounded away from 0. In particular, if $\vec{\tau}$ “hits” 0, it stays at 0 thereafter. This concludes the proof.

It is easy to see that the condition (6) implies (11) as the graph $G$ is connected. The result of Theorem 1 now follows if we take $g_i(y) = y^2$ and $h_i(y) = -y^{-1}$. 

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5.3 Proof of Theorem 4

Recall that Theorem 4 is for the infinite, as well as finite, system. It suffices to prove the following.

Lemma 13. Any FSP starting from zero initial state (all $x_i(0) = 0$), stays in zero state at all times, $x_i(t) = 0$, $t \geq 0$.

**Proof of Lemma 13.** Consider any FSP with zero initial state. Denote $s(t) = \sup_i x_i(t)/p_i$. This function is Lipschitz, because all $x_i(\cdot)$ are uniformly Lipschitz and all $0 < c \leq p_i \leq 1$. Time points $t$ where all derivatives $x_i'(t)$ and $s'(t)$ exist, are called regular. Almost all points $t$ (with respect to Lebesgue measure) are regular. We will show that at any regular point $t$, $s'(t) \leq 0$. (This will imply that $s(\cdot)$ cannot escape from 0.) Suppose not, and at some regular point $t$, $s'(t) = \eta > 0$. If this is true, then there exists a positive function $\delta_1 = \delta_1(\delta) \downarrow 0$ as $\delta \downarrow 0$, such that the following holds for any sufficiently small $\delta > 0$: (a) there exists $i$ such that the increment

$$x_i(t + \delta)/p_i - x_i(t)/p_i \geq (\eta/2)\delta,$$

(b) $|x_i(\xi)/p_i - s(t)| < \delta_1$ for all $\xi \in [t, t + \delta]$, (c) $x_j(\xi)/p_j - s(t) < \delta_1$ for all $j \in N_i$ and all $\xi \in [t, t + \delta]$. If we consider such an $i$, we observe that for any regular $\xi \in [t, t + \delta]$, $x'_j(\xi)/p_j \leq \epsilon = \epsilon(\delta_1)$, where $\epsilon(\delta_1) \downarrow 0$ as $\delta_1 \downarrow 0$. Therefore, for a sufficiently small $\delta$ and a corresponding $i$, $x_i'(\xi)/p_i \leq \eta/3$ for all regular $\xi \in [t, t + \delta]$. This contradicts (13). \qed

6 Symmetric multi-hop networks with Geometric service requirements: Proof of Theorem 5

As in the single-hop case, we first present a Lemma on the conditions any FSP satisfies.

Lemma 14. Any FSP for a multi-hop symmetric network satisfies the following conditions:

$$x_i(t) > 0 \implies x'_i(t) = \lambda_i - \varphi_i(\tau(t)) + (1 - 1/k) \sum_{j \in N_i, j \neq i} \varphi_j(\tau(t)), \text{ for almost all } t \geq 0,$$

$$[x_i(t) = 0 \text{ and } \sum_{j \in N_i} x_j(t) > 0] \implies \frac{dt}{dt} x_i(t) \geq \lambda_i.$$

In particular, by property (10), any FSP for a finite network is such that $\sum_i x_i(t) > 0$ implies that $x_i(\tau) > 0$, $\forall i$, for all $\tau > t$ sufficiently close to $t$.

Once again, a proof of Lemma 14 may be given following the exact same lines as that of [17, Theorem 2], for a model close to ours, and we omit it here.

We will omit the index $t$ in the remainder of the proof. Fix $\epsilon > 0$ such that $\lambda + \epsilon < 1/m$ and consider the Lyapunov function $\frac{1}{2} \sum_i x_i^2$, whose drift is

$$\sum_i x_i \left( \frac{\lambda}{k} - \varphi_i + \sum_{j \in N_i, j \neq i} (1 - 1/k) \frac{1}{m - 1} \varphi_j \right)$$

$$= \frac{\lambda}{k} \sum_i x_i - \frac{1}{k} \sum_i x_i \varphi_i + \sum_i \left( -(1 - 1/k) \varphi_i + \sum_{j \in N_i, j \neq i} (1 - 1/k) \frac{1}{m - 1} \varphi_j \right)$$

$$< -\frac{\epsilon}{k} \sum_i x_i + \frac{1}{k} \left( \frac{1}{m} \sum_i x_i - \sum_i x_i \varphi_i \right) + (1 - 1/k) \sum_i \varphi_i \left( -x_i + \frac{1}{m - 1} \sum_{j \in N_i, j \neq i} x_j \right)$$

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\[ -\varepsilon k \sum x_i + \frac{1}{k} \left( \frac{1}{m} \sum x_i - \sum x_i \varphi_i \right) + (1 - 1/k) \sum \varphi_i \left( -x_i + \frac{1}{m - 1} \left( \frac{x_i}{\varphi_i} - x_i \right) \right) \]

\[ = -\varepsilon k \sum x_i + \frac{1}{k} \left( \frac{1}{m} \sum x_i - \sum x_i \varphi_i \right) + (1 - 1/k) \left( -\frac{m}{m - 1} \sum x_i \varphi_i + \frac{1}{m - 1} \sum x_i \right) \]

\[ = -\varepsilon k \sum x_i + \left( \frac{1}{k} + (1 - 1/k) \frac{m}{m - 1} \right) \left( \frac{1}{m} \sum x_i - \sum x_i \varphi_i \right) \]

\[ = -\varepsilon k \sum x_i + \frac{1}{m - 1} \left( m - \frac{1}{k} \right) \left( \frac{1}{m} \sum x_i - \sum x_i \varphi_i \right) . \]

Note now that

\[ \frac{1}{\sum_i x_i} \sum_i x_i \varphi_i = \sum_i \frac{x_i}{\sum_{j \in N_i} x_j} \geq \frac{\sum_i x_i}{\sum_i \sum_{j \in N_i} x_j} = \frac{1}{m} , \]

due to convexity of the function $1/x$. It now follows that

\[ \left( \frac{1}{2} \sum_i x_i^2 \right) > -\varepsilon k \sum x_i . \]

From here the FSP property (4), and then Theorem 5 follows.

7 Open problems

Our main result in the multi-hop setting only concerns symmetric (in terms of arrival intensities as well as routing) networks. We expect similar results to hold in greater generality. An interesting example is the following: assume that the graph is a circle, arrival intensities into each node are constant and equal to $\lambda/k$ but the routing is not symmetric. Upon successful transmission, a message leaves the system with probability $1/k$ or goes to its neighbour on the right with probability $1 - 1/k$. Rate-balance equations imply that the total workload of each node is $\lambda$ and we expect that the condition $\lambda < 1/3$ guarantees stability in this model, as well as in the model with symmetric routing covered by Theorem 5. In fact, we conjecture that the same Lyapunov function as the one used in the proof of Theorem 5 has a negative drift in this scenario as well. Simple calculus shows that to prove this, one needs to show that

\[ \sum_{i=1}^{N} \frac{x_i (x_i - x_{i+1})}{x_{i-1} + x_i + x_{i+1}} \geq 0 \]

for all vectors $\pi$, with the conventions that $x_0 = x_N$ and $x_{N+1} = x_1$. We have ample numerical evidence in support of this hypothesis but currently lack a proof. If the inequality above is proved, it will imply, furthermore, that condition $\lambda < 1/3$ guarantees stability on the circle topology with arrival intensities equal to $\lambda/k$ for each node, every message leaving the system upon a successful transmission with probability $1/k$ and for arbitrary (but same for all nodes) routing to neighbours if the message does not leave the system.

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