ABELIAN SUBALGEBRAS IN $\mathbb{Z}_2$-GRADED LIE ALGEBRAS AND AFFINE WEYL GROUPS.

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Abstract. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple $\mathbb{Z}_2$-graded Lie algebra and let $\mathfrak{b}_0$ be a fixed Borel subalgebra of $\mathfrak{g}_0$. We describe and enumerate the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$.

§1 Introduction

In this paper we solve the following problem, which has been posed by D. Panyushev in [12, §3]. Suppose that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple $\mathbb{Z}_2$-graded Lie algebra and let $\mathfrak{b}_0$ be a fixed Borel subalgebra of $\mathfrak{g}_0$. Describe and enumerate the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$. We obtain uniform formulas (which will be displayed at the end of the Introduction) in terms of combinatorial data associated to the $\mathbb{Z}_2$-gradation. The interest in this question lies in a theorem by Kostant [7] (which has been generalized to the $\mathbb{Z}_2$ setting by Panyushev [11]) relating commutative subalgebras to the maximal eigenvalue of the Casimir element. More precisely, if $\mathfrak{a} \subseteq \mathfrak{g}_1$ is an abelian $\mathfrak{b}_0$-stable subalgebra of dimension $k$, the corresponding decomposable $k$-vector in $\Lambda^k \mathfrak{g}_1$ obtained by wedging the vectors of a basis of $\mathfrak{a}$ is an eigenvector of maximal eigenvalue for the Casimir operator $\Omega_0$ of $\mathfrak{g}_0$. Viceversa, any decomposable element in the “maximal” eigenspace of $\Omega_0$ which is a highest weight vector for the action of $\mathfrak{g}_0$ corresponds to an abelian $\mathfrak{b}_0$-stable subalgebra.

Panyushev has solved the previous problem in the very special case of the little adjoint module (i.e., when $\mathfrak{g}_1$ is the irreducible $\mathfrak{g}_0$-module of highest weight $\theta_s$, the highest short root of $\mathfrak{g}_0$). Panyushev’s strategy consists in identifying, in these cases, the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ with the abelian ideals consisting only of long roots of a Borel subalgebra of the Langlands dual $\mathfrak{g}_0^\vee$ of $\mathfrak{g}_0$. Then the enumerative result follows by providing a bijection between these ideals and the alcoves in the intersection of the fundamental chamber of the Weyl group of $\mathfrak{g}_0$ with the half-space $(\theta_s, x) < 1$.

Before describing our approach to the general case, let us discuss for a moment Peterson’s $2^{\text{rank}}$ abelian ideals theorem and its interpretations, since this result is crucial for our goals. In [8] Kostant attributed to D. Peterson the following result: the abelian ideals of a Borel subalgebra of a simple Lie algebra $\mathfrak{g}$ of rank $n$ are

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$2^n$ in number. Moreover they are parameterized by a special subset of the affine Weyl group $\hat{W}$ of $\mathfrak{g}$, the one consisting of the so-called minuscule elements. In [3] a geometric interpretation of the minuscule elements was proposed: they are exactly the elements which map the fundamental alcove $C_1$ of $\hat{W}$ into $2C_1$. In particular they are $2^n$ in number, since the ratio between the volumes of $2C_1$ and $C_1$ is $2^n$.

Very recently an alternative approach to Peterson’s result has been proposed in [9]: identify $\Lambda^k(\mathfrak{g})$ with $\Lambda^{(k,k)}\mathfrak{u}$, where $\mathfrak{u} = x\mathfrak{g}[x]$ is the space of $\mathfrak{g}$-valued polynomial maps without constant term and $\Lambda^{(k,k)}\mathfrak{u}$ is the subspace of elements in $\Lambda^k\mathfrak{u}$ having $x$-degree $k$. Then one can identify $\mathfrak{u}$ with the nilradical $\mathfrak{u}_F$ of the opposite parabolic subalgebra $\mathfrak{p}_F$ corresponding to $\mathfrak{g}$ in extended loop algebra of $\mathfrak{g}$. Now Garland-Lepowsky generalization of Kostant theorem on the cohomology of $\mathfrak{u}_F$ relates the abelian ideals of a Borel subalgebra of $\mathfrak{g}$ to the minuscule elements of $\hat{W}$.

Our approach to the description of the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$ is based on a suitable combination of the two ideas described above. Recall that $\mathbb{Z}_2$-gradings are in bijection with involutions. Given any involution $\sigma$ of $\mathfrak{g}$, we give the notion of $\sigma$-minuscule element in $\hat{W}$, and we prove that the set $\mathcal{W}_{ab}$ of $\sigma$-minuscule elements is a parameter space for the abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$. Then we describe a canonical polytope $D_\sigma$ such that the cardinality of $\sigma$-minuscule elements equals $\frac{Vol(D_\sigma)}{Vol(C_1)}$. Finally we calculate $Vol(D_\sigma)$. To describe more in detail the final outcome we have to fix some notation and to recall Kac’s classification of involutions of simple Lie algebras.

Let $\mathfrak{g}$ be a simple Lie algebra of type $X_N$, $\sigma$ be an involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding gradation. Let $k$ be the minimal integer such that $\sigma^k$ is of inner type and consider a realization $\hat{\mathfrak{h}}, \hat{\mathfrak{f}}, \hat{\mathfrak{f}}^\vee$ of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ of type $X_N^{(k)}$. Set $\hat{\Pi} = \{\alpha_0, \ldots, \alpha_n\}$, and let $a_0, \ldots, a_n$ be the labels of the affine Dynkin diagram of $\hat{\mathfrak{g}}$. Kac’s classification of involutions states that $\sigma$ is completely determined by a $(n + 2)$-tuple $(s_0, \ldots, s_n; k)$, where $n = rk(\mathfrak{g}_0)$, $k$ is as above, and $s_0, \ldots, s_n$ are coprime non negative integers satisfying $k \sum_{i=0}^{n} a_i s_i = 2$. We can define a $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \bigoplus \hat{\mathfrak{g}}_j$ of $\hat{\mathfrak{g}}$ (see §2.1), determined by $\sigma$, such that the degree zero space $\hat{\mathfrak{g}}_0$ is a reductive subalgebra of $\hat{\mathfrak{g}}$ containing $\hat{\mathfrak{h}}$. We denote by $\hat{\Delta}_0$ the set of roots of $\hat{\mathfrak{g}}_0$ relative to $\hat{\mathfrak{h}}$; if $\hat{\Delta}^+$ is the positive system of $\hat{\mathfrak{g}}$ corresponding to $\hat{\Pi}$ then we can choose $\hat{\Delta}^+_0 = \hat{\Delta}^+ \cap \hat{\Delta}_0$ as a positive system for $\hat{\mathfrak{g}}_0$. We can view $\mathfrak{g}_0$ and $\mathfrak{g}_1$ inside $\hat{\mathfrak{g}}$ as follows. Following [6, Ch. 8], we consider the extended loop algebra $\hat{L}(\mathfrak{g}, \sigma) = \bigoplus_{j \in \mathbb{Z}} (\mathfrak{g}_{j mod 2} \oplus t^j) \oplus \mathbb{C}c \oplus \mathbb{C}d$ (see §2.2), with its natural $\mathbb{Z}$-grading.

Fix a Cartan subalgebra $\mathfrak{h}_\sigma$ in $\mathfrak{g}_0$. There exists an isomorphism of graded algebras $\Psi : \hat{L}(\mathfrak{g}, \sigma) \to \hat{\mathfrak{g}}$ mapping $\mathfrak{h}_\sigma \otimes 1$ into $\hat{\mathfrak{h}}$. Indeed, we can identify the set of $\mathfrak{h}_\sigma$-roots of $\mathfrak{g}_0$ with $\hat{\Delta}_0$, so that $\hat{\Delta}^+_0$ defines a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$. Moreover, $\Psi$ maps $\mathfrak{g}_1 \otimes t^{-1}$ onto $\hat{\mathfrak{g}}_{-1}$, and the $\mathfrak{b}_0$-stable abelian subalgebras of $\mathfrak{g}_1$ correspond under $\Psi$ to the $\Psi(\mathfrak{b}_0)$-stable abelian subalgebras of $\hat{\mathfrak{g}}_{-1}$.

Now remark that for the $(n + 2)$-tuple $(s_0, \ldots, s_n; k)$ characterizing $\sigma$ there are only three kind of possibilities.

1. $k = 1$, and there exist two indices $p, q$ such that $a_p = a_q = s_p = s_q = 1$ and $s_i = 0$ for $i \neq p, q$.

2. $k = 1$, and there exists an index $p$ such that $s_p = 1$, $a_p = 2$ and $s_i = 0$ for $i \neq p$. 
(3) $k = 2$, and there exists an index $p$ such that $s_p = 1$, $a_p = 1$ and $s_i = 0$ for $i \neq p$.

Denote by $W_\sigma$ the Weyl group of $\widehat{\Delta}_0$. Let $W_f$ be the Weyl group of the root system generated by $\Pi_f = \{\alpha_1, \ldots, \alpha_n\}$.

Consider now case (1), the hermitian symmetric case. We can assume that $p = 0$, hence we may regard $W_\sigma$ as a subgroup of $W_f$. Denote by $\ell_\sigma$, $\ell_f$ the connection indices of $W_\sigma$, $W_f$, respectively. We have

$$|W_\sigma^{ab}| = \frac{|W_f|}{|W_\sigma|} \left(1 + \frac{\ell_\sigma}{\ell_f}\right).$$

This formula is proved by providing a decomposition $D_\sigma = D'_\sigma \cup D''_\sigma$ of the polytope $D_\sigma$ such that the volumes of $D'_\sigma$, $D''_\sigma$ are the two summands in the r.h.s of the previous formula. Indeed we prove directly that $D'_\sigma$ consists of $|W_f| \cdot |W_\sigma| \cdot \ell_f$ alcoves, whereas $D''_\sigma$ is shown to be isometric to a region whose alcoves are indexed by a set of coset representatives of $W_\sigma$ in $W_f$.

Now we deal with cases (2), (3). We prove that the polytope $D_\sigma$ is always contained in another polytope $P_\sigma$ which is the fundamental domain of a certain affine Weyl subgroup of $\widehat{W}$ associated to $g_0$. Hence we can compute the volume of $P_\sigma$ as the index of this subgroup in $\widehat{W}$.

Finally, it turns out that $P_\sigma \setminus D_\sigma$ is either empty or it consists of exactly one alcove, so the volume of $D_\sigma$ can be computed from that of $P_\sigma$, possibly correcting by $-1$. We finish the proof of our enumerative formulas by giving a simple criterion to decide whenever the correction occurs. We have

$$|W_\sigma^{ab}| = a_0 (\chi_\ell(\alpha_p) + 1)k^{n-L} \frac{|W_f|}{|W_\sigma|} - \chi_\ell(\alpha_p),$$

where $a_0$ is the label of the vertex 0 in the Dynkin diagram of type $X_N^{(k)}$, $\chi_\ell$ is the truth function which is 1 if $\alpha_p$ is long and 0 otherwise and $L$ is the number of long simple roots in $\Pi_f$.

The paper is organized as follows. In section 2 we recall the facts we need about involutions, loop algebras, and Lie algebra cohomology. In section 3 we introduce the notion of $\sigma$-minuscule elements and we discuss the relationships of these elements with $b_0$-stable abelian subalgebras in $g_1$. Section 4 is devoted to the study of the polytope $D_\sigma$. In section 5 and 6 we prove our main results, in the semisimple and hermitian symmetric case respectively.

§2 Preliminaries

In this section we summarize the results on affine Kac-Moody Lie algebras that we shall need in the paper. Our main references are [6] and [10]. In particular we use the labeling of Dynkin diagrams as given in tables Aff $k$, $k = 1, 2, 3$ in [6, §4.8].

2.1. Given a diagram of type $X_N^{(k)}$, fix a realization $(\widehat{h}, \widehat{\Pi}, \widehat{\Pi'}^\vee)$ of the corresponding generalized Cartan matrix. Let $\widehat{g}$ denote the corresponding affine Kac-Moody Lie algebra and $g$ the finite dimensional Lie algebra of type $X_N$. If $n + 2 = dim \widehat{h}$ is the rank of $\widehat{g}$, then, in § 8.6 of [6], an automorphism of $g$ is associated to each $(n + 1)$-tuple $s = (s_0, \ldots, s_n)$ of non-negative coprime integers. We denote this automorphism by $\sigma_{s,k}$ and call it the automorphism of type $(s; k)$. The following theorem is Theorem 8.6 of [6].
Theorem A.
a) \( \sigma_{s;k} \) is of finite order and its order \( m \) is given by \( m = k(\sum_{i=0}^{n} a_i s_i) \) where \( a_i \)
are the labels of the diagram \( X_N^{(k)} \).
b) Up to conjugation by an automorphism of \( g \), the automorphisms \( \sigma_{s;k} \) exhaust all \( m \)-th order automorphisms of \( g \).
c) Two automorphisms \( \sigma_{s;k} \) and \( \sigma_{s';k'} \) are conjugate by an automorphism of \( g \) if and only if \( k = k' \) and the sequence \( s \) can be transformed in the sequence \( s' \) by an automorphism of
the diagram \( X_N^{(k)} \).

Let \( \sigma \) be an automorphism of \( g \) of order two and write \( g = g_0 \oplus g_1 \) for the corresponding gradation. By
the above Theorem, we can assume that \( \sigma \) is an automorphism of type \( (s_0, s_1, \ldots, s_n; k) \). Indeed we recover
the three possibilities described in the Introduction.

Let \( \hat{\Delta} \) to denote the set of roots of \( \hat{g} \). Let \( \hat{\Pi} = \{\alpha_0, \ldots, \alpha_n\} \) be the set of simple roots of \( \hat{g} \) and \( \hat{\Delta}^+ \) the corresponding set of positive roots. If \( \alpha \in \hat{\Delta} \) we let \( \hat{g}_\alpha \) be the

The numbers \( s_0, \ldots, s_n \) define a \( \mathbb{Z} \)-grading on \( \hat{g} \) as follows: if \( \alpha \in \hat{\Delta} \) write \( \alpha = \sum_{i=0}^{n} m_i \alpha_i \) and
\[
ht_{\sigma}(\alpha) = \sum_{i=0}^{n} s_i m_i.
\]

Then, if \( x \in \hat{g}_\alpha \), we set \( \text{deg}(x) = \text{ht}_{\sigma}(\alpha) \). We also set \( \text{deg}(h) = 0 \) for \( h \in \hat{h} \). We
denote by \( \hat{g}_i \) the span of all \( x \in \hat{g} \) such that \( \text{deg}(x) = i \). Set also \( \hat{\Delta}_i = \{\alpha \in \hat{\Delta} \mid \text{ht}_{\sigma}(\alpha) = i\} \). Notice that \( \hat{\Delta}_0 \) is the root system of \( \hat{g}_0 \).

2.2. Let \( L(g) \) be the loop algebra \( \mathbb{C}[t, t^{-1}] \otimes g \),
\[
\tilde{L}(g) = L(g) \oplus \mathbb{C}c
\]
its universal (one-dimensional) central extension, and
\[
\hat{L}(g) = \tilde{L}(g) \oplus \mathbb{C}d
\]
the algebra obtained by extending \( \tilde{L}(g) \) by the derivation defined by \( d(p(t) \otimes x) = tp'(t) \otimes x \) and \( d(c) = 0 \).

Let \( \hat{L}(g, \sigma) \) be the subalgebra \( \hat{L}(g) \) defined by
\[
\hat{L}(g, \sigma) = \sum_{j \in \mathbb{Z}} g_{\bar{j}} \otimes t^j + \mathbb{C}c + \mathbb{C}d
\]
where \( \bar{j} \in \{0, 1\} \) is defined by \( j \equiv \bar{j} \mod 2 \). The following is Theorem 8.5 of [6].

Theorem B. Let \( h_\sigma \) be a Cartan subalgebra of \( g_0 \). There exists an isomorphism
\[
\Phi : \hat{g} \rightarrow \hat{L}(g, \sigma)
\]
such that
i) \( \Phi \) maps \( \hat{g}_i \) onto \( t^i \otimes g_i \) for \( i \neq 0 \);
\[\Phi(\hat{h}) = h_{\sigma} \otimes 1 + Cc + Cd;\]

\[\Phi(\hat{g}_0) = g_0 \otimes 1 + Cc + Cd.\]

2.3. By means of the grading \(ht_{\sigma}\), we can define parabolic subalgebras

\[\hat{p}_{\sigma} = \hat{g}_0 \oplus \hat{u}_{\sigma}\] and \[\hat{p}^-_{\sigma} = \hat{g}_0 \oplus \hat{u}^-_{\sigma}\]

of \(\hat{g}\), where

\[\hat{u}_{\sigma} = \sum_{ht_{\sigma}(\alpha) > 0} \hat{g}_\alpha\] and \[\hat{u}^-_{\sigma} = \sum_{ht_{\sigma}(\alpha) < 0} \hat{g}_\alpha.\]

If we set \(Y = \{i \mid s_i = 0\}\), then the parabolic subalgebras \(\hat{p}_{\sigma}\) and \(\hat{p}^-_{\sigma}\) defined above are the subalgebras

\[\hat{p}_Y = \hat{g}_Y \oplus \hat{u}_Y\]

of \(\hat{g}\) as defined in § 1.2 of [10].

The grading on \(\hat{g}\) defines (by restriction) a grading on \(\hat{u}^-_{\sigma}\) and, henceforth, on \(\Lambda \hat{u}^-_{\sigma}\). Let \((\Lambda^p \hat{u}^-_{\sigma})_q\) denote the subspace of \((\Lambda^p \hat{u}^-_{\sigma})\) of degree \(q\). Notice also that \((\Lambda^p \hat{u}^-_{\sigma})_q = 0\) if \(q > -p\).

We set \(\partial_p : \Lambda^p \hat{u}^-_{\sigma} \to \Lambda^{p-1} \hat{u}^-_{\sigma}\) to be the standard boundary operator affording the Lie algebra homology \(H_i(\hat{u}^-_{\sigma})\). It is defined by setting

\[\partial_p(x_1 \wedge \cdots \wedge x_p) = \sum_{i < j} (-1)^{i+j+1}[x_i, x_j] \wedge \cdots \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_p.\]

if \(p > 1\) and \(\partial_1(x) = 0\).

We now recall Garland-Lepowsky generalization [4] of Kostant’s theorem. We need some more notation. Set \(\hat{\Delta}^+_0 = \hat{\Delta}^+ \cap \hat{\Delta}_0\) and \(\hat{\Pi}_0 = \hat{\Delta}_0 \cap \hat{\Pi}\). If \(\lambda\) is a dominant weight for this positive system, denote by \(V(\lambda)\) be the irreducible \(\hat{g}_0\)-module of highest weight \(\lambda\). Fix \(\hat{\rho} \in \hat{h}\) satisfying \(\hat{\rho}(\alpha_i) = 1\) for all \(i, 0 \leq i \leq 1\). If \(w \in \hat{W}\) set

\[N(w) = \{\beta \in \hat{\Delta}^+ \mid w^{-1}(\beta) \in -\hat{\Delta}^+\}.\]

Recall that \(W_\sigma\) is the Weyl group of \(\hat{\Delta}_0\). Denote by \(W'_\sigma\) the set of elements of minimal length in the cosets \(W_\sigma w, w \in \hat{W}\). The following is a special case of Theorem 3.2.7 from [10], which is an extended version of Garland-Lepowsky result.

**Theorem C.**

\[\hat{H}_p(\hat{u}^-_{\sigma}) = \bigoplus_{w \in W'_\sigma} V(w(\hat{\rho}) - \hat{\rho}).\]

Moreover a representative of the highest weight vector of \(V(w(\hat{\rho}) - \hat{\rho})\) is given by \(e_{-\beta_1} \wedge \cdots \wedge e_{-\beta_p}\), where \(N(w) = \{\beta_1, \ldots, \beta_p\}\) and the \(e_{-\beta_i}\) are root vectors.

2.4. Set

\[\hat{h}_R = \text{Span}_R(\alpha_1^\vee, \ldots, \alpha_n^\vee) + \mathbb{R}\Phi^{-1}(c) + \mathbb{R}\Phi^{-1}(d).\]
Remark 2.4.1. Notice that \( \partial^* H \) is an orthogonal sum. Therefore \( \partial^* H \) is conjugate to a root in \( \hat{W} \). We extend the form \( \{ \, , \} \) to a hermitian form on \( \Lambda^p \hat{u}_\sigma^- \) in the usual way, by determinants.

Let \( \partial^*_p : \Lambda^{p-1} \hat{u}_\sigma^- \to \Lambda^p \hat{u}_\sigma^- \) be the adjoint of \( \partial_p \) with respect to the hermitian form \( \{ \, , \} \) and consider the laplacian \( L_p : \Lambda^p \hat{u}_\sigma^- \to \Lambda^p \hat{u}_\sigma^- \)

\[
L_p = \partial_{p+1}\partial^*_p + \partial^*_p \partial_p.
\]

Set \( H_p = \text{Ker}(L_p) \). Then

1. \( H_p \subseteq \text{Ker}(\partial_p) \);
2. the natural map \( H_p \to (H_p \oplus \text{Im}(\partial_{p+1}))/\text{Im}(\partial_{p+1}) \) induces an isomorphism

\[
H_p \cong H_p(\hat{u}_\sigma^-).
\]

Remark 2.4.1. Notice that \( \partial_p((\Lambda^p \hat{u}_\sigma^-)_q) \subseteq (\Lambda^{p-1} \hat{u}_\sigma^-)_q \), and the decomposition

\[
\Lambda^p \hat{u}_\sigma^- = \bigoplus_{q \in \mathbb{Z}} (\Lambda^p \hat{u}_\sigma^-)_q
\]

is an orthogonal sum. Therefore \( \partial^*_p((\Lambda^{p-1} \hat{u}_\sigma^-)_q) \subseteq (\Lambda^p \hat{u}_\sigma^-)_q \). In particular, since \( (\Lambda^{p+1} \hat{u}_\sigma^-)_{-p} = 0 \), we have that \( \partial^*_{p+1} = 0 \) on \( (\Lambda^p \hat{u}_\sigma^-)_{-p} \), hence

\[
L_p|_{(\Lambda^p \hat{u}_\sigma^-)_{-p}} = \partial^*_p \partial_p|_{(\Lambda^p \hat{u}_\sigma^-)_{-p}}.
\]

2.5. We need some remarks on affine roots. Recall that a root is called real if it is \( \hat{W} \)-conjugate to a root in \( \hat{\Pi} \), imaginary otherwise. Imaginary roots are isotropic with respect to \( \{ \, , \} \) whereas real roots are not isotropic. Moreover there are only two possible roots lengths for real roots, except for the case \( \hat{\Delta} \cong A_n^{(2)} \), in which three lengths occur. We call long a real root of maximal length and short a real root of minimal length. If only one length occurs, we shall conventionally say that all roots are long. This convention will be relevant to the formulation of our results.

Set \( \delta = \sum_{i=0}^n a_i \alpha_i \). The following statements hold.

1. The imaginary roots in \( \hat{\Delta} \) are \( \pm \mathbb{N}^+ \delta \).
2. Suppose that \( \hat{\mathfrak{g}} \) is of type \( X_N^{(k)} \). If \( \alpha \in \hat{\Delta} \) then \( k\delta + \alpha \in \hat{\Delta}^+ \cup \{0\} \).
3. If \( \alpha \) is not a long root then \( \delta + \alpha \in \hat{\Delta}^+ \cup \{0\} \).

All these properties follow from [6, Theorem 5.6, Proposition 6.3], where the relationships between the root system \( \hat{\Delta} \) and \( \Delta_f \) (the root system generated by \( \Pi_f \)) are described in detail. The explicit relation between \( \hat{W} \) and \( W_f \) is given in [6, Proposition 6.5]. Sometimes in the following we shall implicitly refer to these descriptions.
In the following we identify $\widehat{g}$ and $\widehat{L}(g, \sigma)$. We observe that, if $\alpha \in \widehat{\Delta}_0$, then $\alpha(d) = \alpha(c) = 0$. This allows us to identify the set of roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_\sigma$ with the set $\widehat{\Delta}_0$. Recall that $\widehat{\Delta}_0^+ = \widehat{\Delta}^+ \cap \widehat{\Delta}_0$ and let $\mathfrak{b}_0$ denote the corresponding Borel subalgebra of $\mathfrak{g}_0$. Notice also that

$$\mathfrak{g}_1 \simeq \widehat{\mathfrak{g}}_{-1} = (\widehat{\mathfrak{u}}_\sigma)_{-1}.$$  

**Definition.** We say that an element $w \in \widehat{W}$ is $\sigma$-minuscule if

$$N(w) \subset \{ \alpha \in \widehat{\Delta} \mid h_{\sigma}(\alpha) = 1 \}.$$  

Denote by $W^\sigma_{ab}$ the set of $\sigma$-minuscule elements of $\widehat{W}$. We can now state

**Theorem 3.1.** There is a bijection between $W^\sigma_{ab}$ and the set $I_{ab}^\sigma$ of abelian $\mathfrak{b}_0$-stable subalgebras of $\mathfrak{g}_1$.  

**Proof.** Let $i \subset \mathfrak{g}_1$ be a $\mathfrak{b}_0$-stable abelian subalgebra and fix a basis $\{x_1, \ldots, x_p\}$ of $i$. Set

$$v_i = t^{-1} \otimes x_1 \wedge \ldots \wedge t^{-1} \otimes x_p \in (\Lambda^p \widehat{\mathfrak{u}}_\sigma)_{-p}.$$  

By Remark 2.4.1 we have that $L_p|_{(\Lambda^p \widehat{\mathfrak{u}}_\sigma)_{-p}} = \partial_p^* \partial_p$. Clearly, since $i$ is abelian,

$$\partial_p^* \partial_p(v_i) = 0.$$  

It follows that $v_i$ is a cycle in $\Lambda^p \widehat{\mathfrak{u}}_\sigma$ and, since $i$ is $\mathfrak{b}_0$-stable and $v_i$ is $\widehat{\mathfrak{h}}$-stable, its homology class is an highest vector for an irreducible component $V_i$ of $H_p(\widehat{\mathfrak{u}}_\sigma)$. By Theorem C there exists an element $w \in \widehat{W}$ such that $\ell(w) = p$ and $V_i = V(w(\hat{\rho}) - \hat{\rho})$. We now check that $w$ is $\sigma$-minuscule. Suppose that $N(w) = \{\beta_1, \ldots, \beta_p\}$. Then there is a nonzero $c \in \mathbb{C}$ such that, fixing root vectors $e_{-\beta_1}$,

$$e_{-\beta_1} \wedge \ldots \wedge e_{-\beta_p} = c \cdot t^{-1} \otimes x_1 \wedge \ldots \wedge t^{-1} \otimes x_p.$$  

Hence $e_{-\beta_1} \wedge \ldots \wedge e_{-\beta_p}$ lies in the span of the vectors $t^{-1} \otimes x_1, \ldots, t^{-1} \otimes x_p$. This implies that $h_{\sigma}(\beta_i) = 1$.

Thus we have established a map $F : I_{ab}^\sigma \to W^\sigma_{ab}$. Suppose now conversely that $w \in W^\sigma_{ab}$ and set

$$N(w) = \{\beta_1, \ldots, \beta_p\}.$$  

Since $h_{\sigma}(\beta_i) = 1$ we have that $e_{-\beta_i} \in (\widehat{\mathfrak{u}}_\sigma)_{-1}$ hence we can write $e_{-\beta_i} = t^{-1} \otimes x_i$ with $x_i \in \mathfrak{g}_1$. It is well-known that $W_\sigma = \{w \in \widehat{W} \mid N(w) \cap \widehat{\Delta}_0^+ = \emptyset\}$. In particular, if $w$ is $\sigma$-minuscule, then $w \in W_\sigma$. Again by Theorem C, the element $v = e_{-\beta_1} \wedge \ldots \wedge e_{-\beta_p}$ represents a highest weight vector for $V(w(\hat{\rho}) - \hat{\rho})$ in $H_p(\widehat{\mathfrak{u}}_\sigma)$. By 2.4 (2) and the subsequent Remark, it follows that

$$L_p(v) = \partial_p^* \partial_p(v) = 0.$$  

It is a standard fact that $\partial_p^* \partial_p(v) = 0$ implies $\partial_p(v) = 0$. It easily follows that the space spanned by $\{x_1, \ldots, x_p\}$ is abelian. Since $v$ is $\mathfrak{b}_0$-stable, then $i$ is also $\mathfrak{b}_0$-stable. $\square$
§4 The polytope $D_\sigma$.

At this point we wish to count the elements in $I_{ab}^a$ by counting the elements of $\mathcal{W}_{ab}^\sigma$. This can be done by computing the volume of certain polytopes.

In the following we identify $\hat{h}_R$ with $\hat{h}_R^*$ via the standard invariant bilinear form, thus, for all real roots $\alpha, \alpha' = \frac{2\alpha}{(\alpha, \alpha)}$. Take $\omega_0$ in $\hat{h}_R^*$, such that $(\omega_0, \alpha_i^\vee) = 1$, $(\omega_0, \alpha_i^\vee) = 0$ for $i \in \{1, \ldots, n\}$ and $(\omega_0, \omega_0) = 0$.

Set
\[
\hat{h}_1^* = \{x \in \hat{h}_R^* | (x, \delta) = 1\}, \quad \hat{h}_0^* = \{x \in \hat{h}_R^* | (x, \delta) = 0\}.
\]
Let $\pi$ be the canonical projection $\mod \delta$ and set
\[
\hat{h}_1^* = \pi \hat{h}_1^*, \quad \hat{h}_0^* = \pi \hat{h}_0^*.
\]
For $x \in \hat{h}_R^*$ and $S \subseteq \hat{h}_R^*$, we set $\overline{x} = \pi(x)$, $\overline{S} = \pi(S)$. We define a $\hat{W}$-invariant nondegenerate pairing between $\hat{h}_0^*$ and $\hat{h}_R^*/\mathbb{R}\delta$ by setting
\[
(\alpha, \lambda + \mathbb{R}\delta) = (\alpha, \lambda).
\]
For $\alpha \in \Delta_+^+$ set
\[
H_\alpha = \{x \in \hat{h}_1^* | (\alpha, x) = 0\}
\]
and $H_\alpha^+ = \{x \in \hat{h}_1^* | (\alpha, x) \geq 0\}$. Set also
\[
C_1 = \{x \in \hat{h}_1^* | (\alpha, x) \geq 0 \forall \alpha \in \hat{H}\},
\]
the fundamental alcove of $\hat{W}$. It is well-known that there is a faithful action of $\hat{W}$ on $\hat{h}_1^*$. Set
\[
D_\sigma = \bigcup_{w \in \mathcal{W}_{ab}^\sigma} wC_1.
\]
Clearly the number of elements of $\mathcal{W}_{ab}^\sigma$ is equal to $\frac{\text{Vol}(D_\sigma)}{\text{Vol}(C_1)}$.

Given $w \in \hat{W}$, a root $\beta \in \hat{\Delta}^+$ belongs to $N(w)$ if and only if $H_\beta$ separates $wC_1$ and $C_1$. It follows that
\[
D_\sigma = \bigcap_{\alpha \in \hat{\Delta}^+, \beta w \alpha (\alpha) \neq 1} H_\alpha^+.
\]
If $\alpha \in \hat{\Delta}$ we set $\hat{H}_\alpha = \{x \in \hat{h}_R^*/\mathbb{R}\delta | (\alpha, x) = 0\}$ and $\hat{H}_\alpha^+ = \{x \in \hat{h}_R^*/\mathbb{R}\delta | (\alpha, x) \geq 0\}$.

Set also
\[
C_\sigma = \bigcap_{\alpha \in \hat{\Delta}^+, \beta w \alpha (\alpha) \neq 1} \hat{H}_\alpha^+.
\]
Obviously
\[
D_\sigma = C_\sigma \cap \hat{h}_1^*.
\]
For $i \in \{0, \ldots, n\}$ we define a number $\epsilon_i$ as follows:
\[
\epsilon_i = \begin{cases} 1 & \text{if } k = 1 \text{ or } \alpha_i \text{ is not a long root}, \\ 2 & \text{if } k = 2 \text{ and } \alpha_i \text{ is a long root}. \end{cases}
\]
Denote by $\hat{\Delta}_0^{max}$ the set of maximal roots in $\hat{\Delta}_0^+$. Set
\[
\Phi_\sigma = \{\alpha_i + \epsilon_i s_i \delta | i = 0, \ldots, n\} \cup \{k\delta - \gamma | \gamma \in \hat{\Delta}_0^{max}\}.
\]
Proposition 4.1. We have $\Phi_\sigma \subset \widehat{\Delta}^+$ and

$$C_\sigma = \bigcap_{\alpha \in \Phi_\sigma} \widehat{H}_\alpha^+.$$  

Proof. Set

$$P = \bigcap_{\alpha \in \Phi_\sigma} \widehat{H}_\alpha^+.$$  

First of all we prove that $\Phi_\sigma \subseteq \{ \alpha \in \widehat{\Delta}^+ \mid ht_\sigma(\alpha) \neq 1 \}$. This will imply obviously that $C_\sigma \subseteq P$.

It is clear that, if $s_i = 0$, then $\alpha_i = \alpha_i + \varepsilon_i s_i \delta \in \widehat{\Delta}^+$ and $ht_\sigma(\alpha_i) = 0 \neq 1$. By 2.5 we have that $\Phi_\sigma \subseteq \widehat{\Delta} \cup \{ 0 \}$. We observe that

$$ht_\sigma(\delta) = ht_\sigma(\sum_{i=0}^{n} a_i \alpha_i) = \sum_{i=0}^{n} a_i s_i = \frac{2}{k},$$

hence, if $s_i \neq 0$, $ht_\sigma(\alpha_i + \varepsilon_i s_i \delta) = s_i(\frac{2}{k} + 1) > 1$. It follows that $\alpha_i + \varepsilon_i s_i \delta \in \widehat{\Delta}^+$ and $ht_\sigma(\alpha_i + \varepsilon_i s_i \delta) \neq 1$. In the same manner, if $\gamma \in \widehat{\Delta}_0^{max}$, then $ht_\sigma(k\delta - \gamma) = 2 > 1$, hence $k\delta - \gamma \in \widehat{\Delta}^+$ and $ht_\sigma(k\delta - \gamma) \neq 1$.

We now check that $P \subseteq C_\sigma$. If $x \in P$ and $\alpha \in \widehat{\Delta}^+$, then, writing $\alpha = \sum_{i=0}^{n} n_i \alpha_i$, we obtain

$$0 \leq \sum_{i=0}^{n} n_i (\alpha_i + \varepsilon_i s_i \delta, x) = (\alpha, x) + (\sum_{i=0}^{n} n_i \varepsilon_i s_i)(\delta, x).$$

If $\alpha = \delta$, then $(1 + \sum_{i=0}^{n} a_i \varepsilon_i s_i)(\delta, x) \geq 0$ hence $(\delta, x) \geq 0$. If $k = 1$ then we can rewrite the above formula as

$$(\alpha, x) \geq - (\sum_{i=0}^{n} n_i s_i)(\delta, x) = -ht_\sigma(\alpha)(\delta, x).$$

If $k = 2$ then there is a unique index $p$ such that $s_p \neq 0$, so we can write

$$(\alpha, x) \geq - \varepsilon_p (\sum_{i=0}^{n} n_i s_i)(\delta, x) \geq -2ht_\sigma(\alpha)(\delta, x).$$

In any case we have

$$(\alpha, x) \geq -k ht_\sigma(\alpha)(\delta, x).$$

In particular, if $\alpha \in \widehat{\Delta}_0^{+}$, then $(\alpha, x) \geq 0$.

If $\alpha \in \widehat{\Delta}$ is a real root then there is $m \in \mathbb{Z}$ such that $\alpha = km\delta + \beta$, with $\beta \in \widehat{\Delta}_0 \cup \widehat{\Delta}_1$. Indeed, if $ht_\sigma(\alpha) = r$, we can write $\alpha = k[r/2] \delta + \alpha - k[r/2] \delta$, so we can choose $m = [r/2]$ and $\beta = \alpha - k[r/2] \delta$. Notice also that, if $\alpha \in \widehat{\Delta}^+$, then $m \geq 0$. Therefore, if $\alpha \in \widehat{\Delta}^+$ is such that $ht_\sigma(\alpha) \neq 1$, there are the following four possibilities.

A) $\alpha = m\delta$ with $m > 0$. If $x \in P$ then we have seen above that $(\delta, x) \geq 0$.

B) $\alpha = km\delta + \beta$ with $\beta \in \widehat{\Delta}_0^+$ and $m \geq 0$. If $x \in P$ then $(\alpha, x) = km(\delta, x) + (\beta, x) \geq 0.$
C) \( \alpha = km\delta - \beta \) with \( \beta \in \hat{\Delta}_0^+ \) and \( m \geq 1 \). There is a root \( \gamma \in \hat{\Delta}_0^{max} \) such that \( \gamma - \beta \) is a sum of roots in \( \hat{\Delta}_0^+ \). This implies that, if \( x \in P \), then \( (\alpha, x) = km(\delta, x) - (\beta, x) = k(m-1)(\delta, x) + (\gamma, x) - (\beta, x) + k(\delta, x) - (\gamma, x) \geq 0 \).

D) \( \alpha = km\delta + \beta \) with \( \beta \in \hat{\Delta}_1 \) and \( m \geq 1 \). If \( x \in P \) then \( km(\delta, x) + (\beta, x) \geq (km - k)(\delta, x) \geq 0 \). \( \square \)

Let \( \Gamma_\sigma \) be the Dynkin graph of \( \hat{\Pi}_0 \). \( \Gamma_\sigma \) is, in general, disconnected. Each of its connected components is of finite type. We write \( \Sigma|\Gamma_\sigma \) if \( \Sigma \) is a connected component of \( \Gamma_\sigma \). Assume that \( \Sigma|\Gamma_\sigma \), and denote by \( \Pi_\Sigma \) the simple roots in \( \Sigma \), by \( W_\Sigma \) the relative Weyl group, \( W_\Sigma = \langle s_\alpha \mid \alpha \in \Pi_\Sigma \rangle \), and by \( \Delta_\Sigma \) the relative root system, \( \Delta_\Sigma = W_\Sigma\Pi_\Sigma \). Moreover, let \( \theta_\Sigma \) be the highest root of \( \Delta_\Sigma \), \( \alpha_\Sigma = k\delta - \theta_\Sigma \), \( \hat{\Pi}_\Sigma = \Pi_\Sigma \cup \{\alpha_\Sigma\} \), and \( \hat{W}_\Sigma = \langle s_\alpha \mid \alpha \in \hat{\Pi}_\Sigma \rangle \). If \( X|\Sigma| \) is the type of the (finite) system \( \Delta_\Sigma \), then the root system generated by \( \hat{\Pi}_\Sigma \) is clearly of type \( X^{(1)} \). We set \( \hat{\Pi}_\sigma = \bigcup_{\Sigma|\Gamma_\sigma} \hat{\Pi}_\Sigma \). Since \( \hat{\Delta}_0^{max} = \{\theta_\Sigma \mid \Sigma|\Gamma_\sigma \} \), we have

\[
\Phi_\sigma = \hat{\Pi}_\sigma \cup \{\alpha_i + \epsilon_i s_i \delta \mid \alpha_i \notin \hat{\Pi}_0\}.
\]

We also set

\[
W_\sigma = \langle s_\alpha \mid \alpha \in \hat{\Pi}_0 \rangle, \quad \hat{W}_\sigma = \langle s_\alpha \mid \alpha \in \hat{\Pi}_\sigma \rangle.
\]

\( W_\sigma \) and \( \hat{W}_\sigma \) are the direct products \( \prod_{\Sigma|\Gamma_\sigma} W_\Sigma \) and \( \prod_{\Sigma|\Gamma_\sigma} \hat{W}_\Sigma \).

Observe that \( rk g_0 = rk \hat{g} - 2 \) while the rank of \([g_0, g_0]\) is given by the rank of the subsystem \( \hat{\Delta}_0 \). It follows that \( g_0 \) is semisimple if and only if there is a simple root \( \alpha_p \in \hat{\Pi} \) such that \( \hat{\Pi}_0 = \hat{\Pi} \setminus \{\alpha_p\} \). The calculation of the order of \( \mathcal{W}_{ab} \) is better performed by separating the case when \( g_0 \) is semisimple from the case when \( g_0 \) has a nontrivial center.

To simplify notation, henceforward we identify \( h_0^* \) with \( \sum_{j=1}^n \mathbb{R} \alpha_j \) and \( \pi(\omega_0) \) with \( \omega_0 \). Hence \( h_1^* = \omega_0 + h_0^* \). As usual, we set \( \theta = \delta - a_0 \omega_0 \).

\( \S 5 \) The semisimple case

We assume that \( \hat{\Pi}_0 = \hat{\Pi} \setminus \{\alpha_p\} \), hence \( \Phi_\sigma = \hat{\Pi}_\sigma \cup \{\alpha_p + s_p \epsilon_p \delta\} \). We define

\[
P_\sigma = \bigcap_{\alpha \in \hat{\Pi}_\sigma} H_\alpha^+.
\]

Then \( D_\sigma \subseteq P_\sigma \). We shall first compute \( Vol(P_\sigma)/Vol(C_1) \); then we shall see that the set difference \( P_\sigma \setminus D_\sigma \) is either empty or exactly one alcove of \( \hat{W} \). It is easily seen that \( P_\sigma \) is a fundamental domain for \( \hat{W}_\sigma \), hence by [1, VI.4, Lemma 1] we have that \( Vol(P_\sigma)/Vol(C_1) = [\hat{W} : \hat{W}_\sigma] \). This description is not directly useful to obtain an explicit result. Indeed, we shall compute the volume of a certain \( translate \) of \( P_\sigma \). We need some preliminaries about translations and the structure theory of \( \hat{W} \).

For \( \alpha \in h_0^* \), we define the \( translation \) by \( \alpha \), \( t_\alpha : \hat{h}_\mathbb{R}^* \rightarrow \hat{h}_\mathbb{R}^* \), setting

\[
t_\alpha(x) = x + (x, \delta)\alpha - ((x, \alpha) + \frac{1}{2} |\alpha|^2(x, \delta))\delta.
\]
This agrees with Kac’s definition [6, 6.5.2], according to our conventions. The translation \( t_\alpha \) preserves the standard invariant form for all \( \alpha \in h_0^* \). Since \( t_\alpha(\delta) = \delta \) for all \( \alpha \in h_0^* \), \( t_\alpha \) induces a map \( \pi(h_0^*) \rightarrow \pi(h_0^*) \) which we still denote by \( t_\alpha \). Notice that for \( x \in h_1^* \) and \( y \in h_0^* \) we have

\[
t_\alpha(x) = x + \alpha, \quad t_\alpha(y) = y - (y, \alpha)\delta.
\]

For any \( S \subseteq h_0^* \), we set \( T(S) = \{ t_\alpha \mid \alpha \in S \} \). Then \( \hat{W} = W_f \ltimes T(M) \), with \( M \) the lattice generated by \( W_f(\theta^\vee) \) [6, Proposition 6.5]. We explicitly describe the lattice \( M \). The first part of the next proposition is well known.

**Proposition 5.1.**

1. If \( \Delta \) is untwisted, or \( a_0 = 2 \), then \( M = Q^\vee \).
2. If \( k > 1 \) and \( a_0 = 1 \), then

\[
M = \sum_{i=1}^{n} r_i \mathbb{Z} \alpha_i^\vee,
\]

with \( r_i = 1 \) if \( \alpha_i \) is short, and \( r_i = k \) if \( \alpha_i \) is long.

**Proof.**

1. If \( \Delta \) is untwisted, or \( a_0 = 2 \), then \( \theta \) is long, so that \( W_f(\theta^\vee) \) is the set of short coroots. Since \( Q^\vee \) is generated by the short coroots, we obtain \( M = Q^\vee \).

2. If \( k > 1 \) and \( a_0 = 1 \), then \( \theta \) is a short root and since \( w(\theta^\vee) = w(\theta) \) for all \( w \in W_f \), \( M \) includes \( \alpha^\vee \) for all short \( \alpha \in \Delta_f \). Any long \( \beta \in \Delta_f \) is an integral linear combination of short roots, say \( \beta = c_1 \beta_1 + \cdots + c_s \beta_s \), and \( \beta^\vee = 2 \frac{\beta}{(\beta, \beta)} (c_1 \beta_1 + \cdots + c_s \beta_s) \). Since for any long \( \beta \) we have that \( \frac{\beta}{(\beta, \beta)} = k \), we obtain \( k \beta^\vee \in M \) for all long \( \beta \). Set \( M' = \sum_{i=1}^{n} r_i \mathbb{Z} \alpha_i^\vee \), with \( r_i = 1 \) if \( \alpha_i \) is short, and \( r_i = k \) if \( \alpha_i \) is long. Then we have that \( M' \subseteq M \). We shall prove that in fact \( M = M' \). It suffices to prove that for any short \( \alpha_i \) and \( w \in W_f \) we have \( w(\alpha_i^\vee) \in M' \), since \( W_f \alpha_i^\vee \) generates \( M \). It is clear that if \( \alpha_j \) and \( \alpha_i \) are short roots, then \( s_{\alpha_i}(\alpha_j^\vee) \in M' \). If \( \alpha_j \) is long and \( \alpha_i \) is short, then \( s_{\alpha_i}(\alpha_j^\vee) = \alpha_j^\vee - (\alpha_j^\vee, \alpha_i)\alpha_j^\vee \) and since \( k(\alpha_i^\vee, \alpha_j^\vee) \), \( s_{\alpha_i}(\alpha_j^\vee) \in M' \). Since moreover \( W_f(k \alpha_i^\vee) \subseteq M' \) for all \( \alpha \in \Delta_f \), we inductively obtain \( W_f \alpha_i^\vee \subseteq M' \). \( \square \)

The group \( \hat{W}_\Sigma \) is not in general a product of a subgroup of \( W_f \) and a subgroup of \( T(M) \). This motivates the following construction.

Assume that \( \Sigma \mid \Gamma_\sigma \). If \( \alpha_0 \in \Pi_\Sigma \), we set \( \Pi_{\Sigma,f} = \Pi_\Sigma \setminus \{ \alpha_0 \} \cup \{ -\theta \} \), and if \( \alpha_0 \notin \Pi_\Sigma \), we set \( \Pi_{\Sigma,f} = \Pi_\Sigma \). Moreover, we set \( \hat{\Pi}_{\Sigma,f} = \Pi_{\Sigma,f} \cup \{ k\delta - \overline{\theta}_\Sigma \} \) and denote by \( \Delta_{\Sigma,f} \), \( \Delta_{\Sigma,f} \) the root systems generated by \( \Pi_{\Sigma,f} \), \( \hat{\Pi}_{\Sigma,f} \), respectively.

If \( \hat{\Delta} \not\cong A_2^{(1)} \), it is clear that \( \overline{\theta}_\Sigma \in \Delta_f \) and \( k\delta - \overline{\theta}_\Sigma \in \hat{\Delta} \) (see [6], warning after 6.3.8). Moreover, \( \overline{\theta}_\Sigma \) is the highest root of \( \Delta_{\Sigma,f} \). Therefore \( \Delta_{\Sigma,f} \) and \( \hat{\Delta}_{\Sigma,f} \) are isomorphic to \( \Delta_{\Sigma} \) and \( \hat{\Delta}_{\Sigma} \), respectively. Moreover \( \hat{\Delta}_{\Sigma,f} \) is isomorphic to the untwisted affine system associated to \( \Delta_{\Sigma,f} \).

Assume that \( \hat{\Delta} \cong A_2^{(2)} \). Then \( p = n \), and there is just one connected component \( \Sigma = \Gamma_\sigma \). \( \Delta_{\Sigma} \) is of type \( B_n \), but \( \Delta_{\Sigma,f} \) is of type \( C_n \). In fact, \( \Pi_{\Sigma,f} = \{ -\theta, \alpha_1, \ldots, \alpha_{n-1} \} \), \( \alpha_1, \ldots, \alpha_{n-1} \) are short roots of \( \Delta_{\Sigma,f} \), and \( -\theta \) is long. Moreover, \( \theta_{\Sigma} = 2\alpha_0 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} \), and hence \( \overline{\theta}_\Sigma = -\theta + 2\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} \).
Thus $\theta_\Sigma$ is the highest short root of $\Delta_{\Sigma,f}$ and $\widehat{\Delta}_{\Sigma,f}$ is a twisted affine root system of type $A^{(2)}_{2n-1}$.

We define $W_{\Sigma,f} = \langle s_\alpha \mid \alpha \in \Pi_{\Sigma,f} \rangle$ and $\widehat{W}_{\Sigma,f} = \langle s_\alpha \mid \alpha \in \widehat{\Pi}_{\Sigma,f} \rangle$. Since $s_{-\theta} = s_{\theta_0}$, $W_{\Sigma,f} = \langle s_\alpha \mid \alpha \in \Pi_{\Sigma} \rangle$. Moreover, since for any non-isotropic $\alpha, \beta \in \overline{h_0}$, $s_\alpha s_\beta$ and $s_\theta$, have the same period, we obtain that $\widehat{W}_{\Sigma}$ and $\widehat{W}_{\Sigma,f}$, with the given sets of generators, are naturally isomorphic as Coxeter systems, though the respective root systems are not necessarily isomorphic.

Finally, we set $\Pi_{\sigma,f} = \bigcup_{\Sigma \mid \Gamma_\sigma} \Pi_{\Sigma,f}$, $\widehat{\Pi}_{\Sigma,f} = \bigcup_{\Sigma \mid \Gamma_\sigma} \widehat{\Pi}_{\Sigma,f}$, and

$$W_{\sigma,f} = \langle s_\alpha \mid \alpha \in \Pi_{\sigma,f} \rangle, \quad \widehat{W}_{\sigma,f} = \langle s_\alpha \mid \alpha \in \widehat{\Pi}_{\sigma,f} \rangle.$$ 

$W_{\sigma,f}$ is the direct product of the (finite) Weyl groups $W_{\Sigma,f}$, and $\widehat{W}_{\sigma,f}$ is the direct product of the $\widehat{W}_{\Sigma,f}$, for all $\Sigma \mid \Gamma_\sigma$.

**Proposition 5.2.** We have $\widehat{W}_{\sigma,f} = W_{\sigma,f} \rtimes T(M_\sigma)$, where

$$M_\sigma = \begin{cases} k\mathbb{Z}\alpha^\vee & \text{if } \widehat{\Delta} \not\cong A^{(2)}_{2n} \\ 4\mathbb{Z}\theta^\vee + 2\mathbb{Z}\alpha^\vee_1 + \cdots + 2\mathbb{Z}\alpha^\vee_{n-1} & \text{if } \widehat{\Delta} \cong A^{(2)}_{2n} \end{cases}$$

*Proof.* By [6, Proposition 6.5] $\widehat{W}_{\Sigma,f} = W_{\Sigma,f} \rtimes T(M_\Sigma)$, where $M_\Sigma$ is the lattice generated by $W_{\Sigma,f}(k\overline{\theta}_\Sigma)$. Hence the claim follows by the above discussion and Proposition 5.1. $\square$

The above structure results allows us to compute explicitly the index of $\widehat{W}_{\sigma,f}$ in $\widehat{W}$.

**Proposition 5.3.** We have

$$[\widehat{W} : \widehat{W}_{\sigma,f}] = r_p k^{n-L} [W_f : W_{\sigma,f}],$$

where $L$ is the number of long roots in $\Pi_f$ and $r_p$ is the ratio between the squared length of $\alpha_p$ and that of any short root.

*Proof.* By standard group theory we obtain $[\widehat{W} : \widehat{W}_{\sigma,f}] = [W_f : W_{\sigma,f}] [M : M_\sigma]$.

We have only to prove that, in all cases, $[M : M_\sigma] = r_p k^{n-L}$.

First assume that $p = 0$. This implies that $k = 2$, $a_0 = 1$. Then $W_f = W_{\sigma,f}$ and $M_\sigma = \sum_{\alpha \in \Pi_f} k\mathbb{Z}\alpha^\vee$. Moreover, $r_p = 1$. Hence the claim follows by Proposition 5.1 (2).

Next assume that $p \geq 1$. Set $\theta^\vee = b_1\alpha^\vee_1 + \cdots + b_n\alpha^\vee_n$. Suppose that $k = 1$, so that $a_p = 2$. If $\alpha_p$ is short, then $b_p = 1$, hence $\alpha_p^\vee \in \sum_{\alpha \in \Pi_f} \mathbb{Z}\alpha^\vee = M_\sigma$. It follows that $M_\sigma = Q^\vee = M$, hence the claimed equality holds. If $\alpha_p$ is long, then $b_p = 2$ and $2\alpha_p^\vee \in M_\sigma$, but $\alpha_p^\vee \notin M_\sigma$. It follows that $[Q^\vee : M_\sigma] = 2$, hence the claim is true in this case, too.
Now we assume that $k = 2$, so that $a_p = 1$. If $\alpha_p$ is short, then $b_p = 1$. Moreover, $\tilde{\Delta} \not\cong A^{(2)}_{2n}$. Thus we obtain $2\alpha_p^e \in M_\sigma$. It follows that $M_\sigma = 2Q^e$, hence $[M : M_\sigma] = 2^R$, where $R$ is the number of short roots in $\Pi_f$, and the claim holds. Then we assume that $\alpha_p$ is long. If $\tilde{\Delta} \cong A^{(2)}_{2n}$, then $b_p = 2$, hence $4\alpha_p^e \in M_\sigma$ and $2\alpha_p^e \notin M_\sigma$. It follows that $M_\sigma = 4\mathbb{Z}\alpha_p + \sum_{\alpha \in \Pi_f} 2\mathbb{Z}\alpha^e$, hence $[M : M_\sigma] = 2^{R+1}$, where $R$ is the number of short roots in $\Pi_f$, which is equivalent to our claim. Finally, if $\tilde{\Delta} \cong A^{(2)}_{2n}$, then $p = n$ and $\alpha_n$ is long. We have $b_n = 1$, hence $4\alpha_n^e \in M_\sigma$ and $2\alpha_n^e \notin M_\sigma$. It follows that $M_\sigma = 2\mathbb{Z}\alpha_1^e + \cdots + 2\mathbb{Z}\alpha_{n-1}^e + 4\mathbb{Z}\alpha_n^e$, hence $[M : M_\sigma] = 2^{n+1}$, which is the claim in this case. \hfill \Box

We then prove that $Vol(P_\sigma)$ is equal to $[\widehat{W} : \widehat{W}_{\sigma,f}]$. Set

$$A_{\sigma,f} = \bigcap_{\alpha \in \Pi_{\sigma,f}} H^+_\alpha.$$

It is clear that $H^+_{-\theta} = H^+_{\alpha_0}$, hence $A_{\sigma,f} = (\bigcap_{i \neq p} H^+_{\alpha_i}) \cap H^+_{k\delta - \tilde{\theta}}$.

**Lemma 5.4.** $A_{\sigma,f}$ is a fundamental domain for the action of $\widehat{W}_{\sigma,f}$ on $h_1^*$. Hence

$$\frac{Vol(A_{\sigma,f})}{Vol(C_1)} = [\widehat{W} : \widehat{W}_{\sigma,f}].$$

**Proof.** For any $\Sigma|\Gamma_\sigma$, set $h_{\Sigma,0}^* = \sum_{\alpha \in \Pi_{\Sigma,f}} \mathbb{R}\alpha$ and $A_{\Sigma,f} = \{ x \in h_{\Sigma,0}^* | (x, \alpha) \geq 0 \ \forall \alpha \in \Pi_{\Sigma,f}; (x, \tilde{\theta}_\Sigma) \leq k \}$. Then we have an orthogonal decomposition $h_0^* = \sum_{\Sigma|\Gamma_\sigma} h_{\Sigma,0}^*$, and since $h_1^* = \omega_0 + h_0^*$ we obtain $A_{\sigma,f} = \omega_0 + \sum_{\Sigma|\Gamma_\sigma} A_{\Sigma,f}$. Now $\widehat{W}_{\Sigma,f}$ acts faithfully onto $\omega_0 + h_{\Sigma,0}^*$, and $\omega_0 + A_{\Sigma,f}$ is its fundamental alcove, hence a fundamental domain for this action. Moreover $\widehat{W}_{\Sigma,f}$ fixes pointwise $h_{\Sigma,0}^*$, for all other $\Sigma|\Gamma_\sigma$. Since $\widehat{W}_{\sigma,f}$ is the direct product $\prod_{\Sigma|\Gamma_\sigma} \widehat{W}_{\Sigma,f}$, we obtain that $\omega_0 + \sum_{\Sigma|\Gamma_\sigma} A_{\Sigma,f}$ is a fundamental domain for the action of $\widehat{W}_{\sigma,f}$ on $h_1^*$. \hfill \Box

Let $\{ \omega_j^e | 1 \leq j \leq n \}$ be the dual basis of $\Pi_f$ in $h_0^*$ and $o_j = \frac{\omega_j^e}{a_j}$ for $1 \leq j \leq n$, where the $a_i$ are the labels of the Dynkin diagram. Thus we have $(\theta, o_j) = 1$. Set moreover $o_0 = 0$ and for $j \in \{0, 1, \ldots, n\} \setminus \{p\}$ define $\tilde{\omega}_j^e = a_j (o_j - o_p)$. Then $\{ \tilde{\omega}_j^e | j \in \{0, 1, \ldots, n\} \setminus \{p\} \}$ is the dual basis of $\{ \tilde{\pi} | \alpha \in \tilde{\Pi}_0 \}$ in $h_0^*$. In particular it is a basis of $h_0^*$. Indeed, $\{ \tilde{\omega}_j^e | \alpha_j \in \Pi_\Sigma \}$ is the dual basis of $\tilde{\Pi}_\Sigma$ in $h_{\Sigma,0}^*$, for all $\Sigma|\Gamma_\sigma$.

For all $\Sigma|\Gamma_\sigma$, let $\theta_\Sigma = \sum_{\alpha_j \in \Pi_\Sigma} a_j' o_j$: this defines integers $a_j'$ for all $j \in \{0, 1, \ldots, n\} \setminus \{p\}$. We set $\delta_j = k \frac{\tilde{\omega}_j^e}{a_j'}$, for all $j \in \{0, 1, \ldots, n\} \setminus \{p\}$, so that $(\delta_j, \theta_\Sigma) = (\delta_j, \tilde{\theta}_\Sigma) = 1$, for all $j$ such that $\alpha_j \in \Pi_\Sigma$. 


Proposition 5.5. We have \( P_\sigma = o_p + A_{\sigma, f} \), hence
\[
\text{Vol}(P_\sigma) = [\hat{W} : \hat{W}_{\sigma, f}].
\]
Moreover, \( t_{-o_p} \hat{W}_\sigma t_{o_p} = \hat{W}_{\sigma, f} \).

Proof. We have
\[
t_{-o_p}(\alpha_j) = \alpha_j + (o_p, \alpha_j)\delta = \begin{cases} \alpha_j & \text{if } j \notin \{0, p\}, \\ \alpha_0 - \frac{\delta}{a_0} = -\frac{\theta}{a_0} & \text{if } j = 0. \end{cases}
\]

Thus, in any case, \( t_{-o_p}(\alpha_j) = m_j \), and hence \( t_{-o_p}(k\delta - \theta_\Sigma) = k\delta - \bar{\theta}_\Sigma \). Since \( t_{-o_p} \) preserves the standard invariant form, this implies that \( t_{-o_p}(H^\pm_{\alpha_j}) = H^\pm_{\alpha_j} \) for all \( j \neq p \), and \( t_{-o_p}(H^+_{k\delta - \theta_\Sigma}) = H^+_{k\delta - \bar{\theta}_\Sigma} \), for all \( \Sigma | \Gamma_\sigma \). By the definitions of \( P_\sigma \) and \( A_{\sigma, f} \), it follows that \( t_{-o_p}(P_\sigma) = A_{\sigma, f} \), hence the claim.

Since \( t_{-o_p} \) preserves the standard invariant form, we also have that \( t_{-o_p} s_\alpha t_{o_p} = s_{t_{-o_p}(\alpha)} \). Since \( s_\theta = s_{\frac{\theta}{a_0}} \), we obtain that conjugation by \( t_{o_p} \) maps the generators of \( \hat{W}_\sigma \) onto those of \( \hat{W}_{\sigma, f} \), hence it maps \( \hat{W}_\sigma \) onto \( \hat{W}_{\sigma, f} \).

We are now going to study the difference set \( D_\sigma \setminus P_\sigma \).

Remark. (1). Notice that \( k = \frac{2}{a_p} \).

(2). If \( \alpha_p \) is long, then \( \epsilon_p s_p = k \), hence \( D_\sigma = P_\sigma \setminus H^\pm_{\alpha_p + k}\delta \).

Lemma 5.6. Assume that \( \alpha_p \) is long. Then, for each \( \Sigma | \Gamma_\sigma \), \( \alpha_p \) is connected to exactly one root \( \alpha_{j(\Sigma)} \) in \( \Sigma \). Moreover, \( a'_{j(\Sigma)} = 1 \), and \( -k\alpha^\vee_p = \sum_{\Sigma | \Gamma_\sigma} \delta_{j(\Sigma)} \).

Proof. Let \( \Sigma | \Gamma_\sigma \). Our assumptions imply that \( \hat{\Delta} \not\cong A^{(1)}_n \), \( n \geq 2 \), hence \( \alpha_p \) is connected to exactly one root in \( \Sigma \), say \( \alpha_{j(\Sigma)} \). Since \( \alpha_{j(\Sigma)} \) supports \( \theta_\Sigma \) and any other root in \( \text{Supp} \theta_\Sigma \) is orthogonal to \( \alpha_p^\vee \), we also obtain \( (\alpha_p^\vee, \theta_\Sigma) \neq 0 \). Since \( \alpha_p \) is a long root, we have that \( (\alpha_p^\vee, \alpha_{j(\Sigma)}) = -1 \) and \( |(\alpha_p^\vee, \theta_\Sigma)| \leq 1 \); hence \( (\alpha_p^\vee, \theta_\Sigma) = -1 \) and therefore \( a'_{j(\Sigma)} = 1 \). It follows that \( (\delta_{j(\Sigma)}, \alpha_{j(\Sigma)}) = k \) and that \( -k\alpha_p^\vee = \sum_{\Sigma | \Gamma_\sigma} \delta_{j(\Sigma)} \).

The above results imply in particular that if \( \alpha_p \) is long, then \( D_\sigma \) is properly included in \( P_\sigma \), since \( -k\alpha_p^\vee \in P_\sigma \) and \( (-k\alpha_p^\vee, \alpha_p) = -2k < -k \).

Proposition 5.7. We have \( D_\sigma = P_\sigma \) if and only if \( \alpha_p \) is short.

Proof. It suffices to prove that if \( \alpha_p \) is short then \( D_\sigma = P_\sigma \). Since \( \alpha_p \) is short, for at least one \( \Sigma | \Gamma_\sigma \) we have that \( \theta_\Sigma \) is a long root, and therefore \( (\theta_\Sigma, \alpha_p^\vee) < -1 \).
It follows that \( \theta_\Sigma + 2\alpha_p \) is a root. Hence \( k\delta - \theta_\Sigma - 2\alpha_p \in \hat{\Delta}_0 \). Now we have \( 2\alpha_p - k\delta = -(k\delta - \theta_\Sigma - 2\alpha_p) - \theta_\Sigma \) and since \( (x, \beta) < k \) for all positive roots \( \beta \in \hat{\Delta}_0 \) and \( x \in P_\sigma \), we obtain \( (x, 2\alpha_p - k\delta) > -2k \). It follows that \( (x, \alpha_p) \geq -\frac{1}{2}k \) for all \( x \in P_\sigma \), hence \( D_\sigma = P_\sigma \).
Finally we show that when $D_{\sigma} \neq P_{\sigma}$ then the difference set is an alcove of $\hat{W}$. In fact, we shall explicitly provide an element $w_{\sigma} \in \hat{W}$ such that $w_{\sigma}(C_1) = P_{\sigma} \setminus D_{\sigma}$.

For any $\Sigma|\Gamma_{\Sigma}$ set
\[ t_{\Sigma} = t_{\bar{\alpha}_j(\Sigma)}, \quad w_{\Sigma} = w_{\Sigma}^0 w_{\Sigma}^0 \]
where $w_{\Sigma}^0$ is the longest element in $W_{\Sigma}$ and $w_{\Sigma}^0$ is the longest element in $W(\Pi_{\Sigma} \setminus \{\alpha_j(\Sigma)\})$. Define then
\[ w_{\sigma} = \prod_{\Sigma|\Gamma_{\Sigma}} t_{\Sigma} w_{\Sigma}. \]

Next lemma implies in particular that $w_{\sigma} \in \hat{W}.$

**Lemma 5.8.** We have
\[ w_{\sigma}(P_{\sigma}) = P_{\sigma}. \]
Moreover
\[ w_{\sigma} = t_{-k\alpha_p} w_0^* w_0 \]
where $w_0$ is the longest element in $W_{\sigma}$ and $w_0^*$ is the longest element of $W(\hat{\Pi} \cap \{\alpha_p\})$.

**Proof.** By Lemma 5.6 $a_j'(\Sigma) = 1,$ hence (see [5, Section 1]) $w_{\Sigma}(\Pi_{\Sigma} \cup \{-\theta_{\Sigma}\}) = \Pi_{\Sigma} \cup \{-\theta_{\Sigma}\}$. Moreover, $w_{\Sigma}(-\theta_{\Sigma}) = a_j(\Sigma)$. Since $\bar{\alpha}_j(\Sigma) \perp \{\alpha \in \Pi_{\Sigma} \mid \alpha \neq \alpha_j(\Sigma)\}$ and $(\bar{\alpha}_j(\Sigma), \alpha_j(\Sigma)) = (\bar{\alpha}_j(\Sigma), \theta_{\Sigma}) = k,$ it follows that $t_{\Sigma} w_{\Sigma}(\hat{\Pi}_{\Sigma}) = \hat{\Pi}_{\Sigma}$. Moreover, $t_{\Sigma} w_{\Sigma}$ fixes pointwise $\hat{\Sigma}_{\hat{\Sigma}}$ for $\Sigma' \mid \Gamma_{\sigma}, \Sigma' \neq \Sigma$, hence $t_{\Sigma} w_{\Sigma}(\hat{\Pi}_{\sigma}) = \hat{\Pi}_{\sigma}$. It follows that $w_{\sigma}(\hat{\Pi}_{\sigma}) = \hat{\Pi}_{\sigma}$, and hence that $w_{\sigma}(P_{\sigma}) = P_{\sigma}$.

Since $w_{\Sigma} t_{\Sigma'} = t_{\Sigma'} w_{\Sigma}$ for all $\Sigma, \Sigma' \mid \Gamma_{\sigma}$ with $\Sigma \neq \Sigma'$, it is clear that $w_{\sigma} = \prod_{\Sigma|\Gamma_{\Sigma}} t_{\Sigma} w_0^* w_0$, hence by Lemma 5.6 $w_{\sigma} = t_{-k\alpha_p} w_0^* w_0$. □

It is well known that $-w_0$ induces a permutation of $\hat{\Pi}_0$. Since $\hat{\Pi}_0 = \hat{\Pi} \setminus \{\alpha_p\}$ we can define $\alpha_i' = -w_0(\alpha_i)$ for $i \neq p$. For calculating the action of $w_{\sigma}$ on $C_1$ we need the following lemma.

**Lemma 5.9.**
\[ \alpha_i = \alpha_i'. \]

**Proof.** We first prove that $g_1$ is irreducible as $g_0$-module. Remark that, as $g_0$-modules, $g_1 \cong (\Lambda^1 \hat{u}_-)^{-1}$. Since $L_1 = \partial_1^* \partial_1 = 0$, we see that $(\Lambda^1 \hat{u}_-)^{-1}$ is a nontrivial submodule of $H_1(\hat{u}_-)$, but, by Theorem C,
\[ H_1(\hat{u}_-) = \bigoplus_{w \in W_{\Sigma}} V(w(\hat{\rho}) - \hat{\rho}) = V(s_{\alpha_p}(\hat{\rho}) - \hat{\rho}) = V(-\alpha_p). \]

Moreover, the highest weight of $g_1$ as a $g_0$-module is $-\alpha_p$ restricted to $Span(\alpha_i' \mid i \neq p)$, which is equal to $\frac{1}{\alpha_p} \sum_{i \neq p} a_i \alpha_i$. Since $g_1^* \cong g_1$ as $g_0$-modules, applying $w_0$ gives the desired result. □
Lemma 5.10. We have

\[ w_0 t_{k\alpha_p}(\alpha_p) = -k\delta - \alpha_p \]

hence, in particular, \( w_\sigma(C_1) = P_\sigma \setminus D_\sigma \) and

\[ \text{Vol}(P_\sigma \setminus D_\sigma) = \text{Vol}(C_1). \]

Proof. We have \( t_{k\alpha_p}(\alpha_p) = \alpha_p - (k\alpha_p^\vee, \alpha_p)\delta = \alpha_p - 2k\delta \). To prove the Lemma we have to check that \( w_0(\alpha_p) = k\delta - \alpha_p \). Let \( \{\widehat{\omega}_0, \ldots, \widehat{\omega}_n, \delta\} \) be the dual basis of \( \{\alpha_0, \ldots, \alpha_n, \omega_0\} \). It suffices to check that \( (w_0(\alpha_p), \widehat{\omega}_i) = ka_i - \delta_{ip} \) (\( \delta_{ip} \) is a Kronecker \( \delta \)). For \( i = p \) this is obvious. Assume that \( i \neq p \). We have \( (w_0(\alpha_j), \widehat{\omega}_i) = (\alpha_j, w_0(\widehat{\omega}_i)) \) for \( 0 \leq j \leq n \). This implies that \( w_0(\widehat{\omega}_i) = -\widehat{\omega}_i' + m\widehat{\omega}_p + r\delta \) for some \( m, r \in \mathbb{Z} \). By applying \( \delta \) to both sides of the previous equation we find that \( a_i = -a_i' + a_pm \), hence, by Lemma 5.9, that \( m = \frac{2a_i}{a_p} = ka_i \). But clearly \( (w_0(\alpha_p), \widehat{\omega}_i) = m \), hence we get the claim. \( \square \)

Putting together 5.3, 5.5, and 5.10 we obtain the main result of this section.

Theorem 5.11. Assume that \( \mathfrak{g}_0 \) is semisimple. If \( \chi_\ell(\alpha_p) \) is the truth function which is 1 if \( \alpha_p \) is long and 0 otherwise, then

\[ |\mathcal{W}^\sigma_{ab}| = a_0(\chi_\ell(\alpha_p) + 1)k^{n-L}|W_f| - \chi_\ell(\alpha_p) \]

where \( L \) is the number of long roots in \( \Pi_f \).

The uniform formula established in the previous theorem can be made completely explicit in each case. If \( k = 1 \) and \( a_p = 2 \), we have

| type of \( \widehat{\mathfrak{g}} \) | \( p \) | type of \( \Delta_f \) | type of \( \mathfrak{g}_0 \) | \( |\mathcal{W}^\sigma_{ab}| \) |
|-----------------|-----|-----------------|-----------------|------------------|
| \( B_n^{(1)} \) | \( 2 \leq p \leq n-1 \) | \( B_n \) | \( D_p \times B_{n-p} \) | \( 4 \binom{n}{p} - 1 \) |
| \( p = n \) | \( B_n \) | \( D_n \) | 2 |
| \( C_n^{(1)} \) | \( 1 \leq p \leq n-1 \) | \( C_n \) | \( C_p \times C_{n-p} \) | \( \binom{n}{p} \) |
| \( D_n^{(1)} \) | \( 2 \leq p \leq n-2 \) | \( D_n \) | \( D_p \times D_{n-p} \) | \( 4 \binom{n}{p} - 1 \) |
| \( G_2^{(1)} \) | \( p = 1 \) | \( G_2 \) | \( A_1 \times A_1 \) | 5 |
| \( F_4^{(1)} \) | \( p = 1 \) | \( F_4 \) | \( A_1 \times C_3 \) | 23 |
| \( p = 4 \) | \( F_4 \) | \( B_4 \) | 3 |
| \( E_6^{(1)} \) | \( p = 2, 4, 6 \) | \( E_6 \) | \( A_1 \times A_5 \) | 71 |
| \( E_7^{(1)} \) | \( p = 1, 5 \) | \( E_7 \) | \( A_1 \times D_6 \) | 125 |
| \( p = 7 \) | \( E_7 \) | \( A_7 \) | 143 |
| \( E_8^{(1)} \) | \( p = 1 \) | \( E_8 \) | \( A_1 \times E_7 \) | 239 |
| \( p = 7 \) | \( E_8 \) | \( D_8 \) | 269 |
If \( k = 2 \) and \( a_p = 1 \), we have

| type of \( \hat{g} \) | type of \( \Delta_f \) | type of \( g_0 \) | \(|W_{ab}^\sigma|\) |
|-----------------|-----------------|-----------------|-----------------|
| \( A_{2n}^{(2)} \) | \( p = n \) | \( C_n \) | \( B_n \) | \( 2^{n+1} - 1 \) |
| \( A_{2n-1}^{(2)} \) | \( p = 0, 1 \) | \( C_n \) | \( C_n \) | \( 2^{n-1} \) |
| \( p = n \) | \( C_n \) | \( D_n \) | \( 2^{n+1} - 1 \) |
| \( D_{n+1}^{(2)} \) | \( 1 \leq p \leq n - 1 \) | \( B_n \) | \( B_p \times B_{n-p} \) | \( 4 \left( \begin{array}{c} n \\ p \end{array} \right) - 1 \) |
| \( p = 0, n \) | \( B_n \) | \( B_n \) | \( 2 \) |
| \( p = 0 \) | \( F_4 \) | \( F_4 \) | \( 4 \) |
| \( p = 4 \) | \( F_4 \) | \( B_4 \) | \( 23 \) |

\section{The hermitian symmetric case.}

In this section we assume that \( g_0 \) is not semisimple. Since \( k \sum_{i=0}^{n} a_i s_i = 2 \) this happens if and only if \( k = 1 \) and there are two indices \( p, q \) such that \( a_p = a_q = s_p = s_q = 1 \) and \( s_i = 0 \) for \( i \neq p, q \). By Theorem A (3) we can and do choose \( p = 0 \).

Set \( D'_\sigma = D_\sigma \cap \{(x, \alpha_\sigma) < 0\} \), \( D''_\sigma = D_\sigma \cap \{(x, \alpha_\sigma) \geq 0\} \).

Clearly, \( \text{Vol}(D_\sigma)/\text{Vol}(C_1) = \text{Vol}(D'_\sigma)/\text{Vol}(C_1) + \text{Vol}(D''_\sigma)/\text{Vol}(C_1) \). We first compute \( \text{Vol}(D'_\sigma)/\text{Vol}(C_1) \).

Denote by \( P'_f, Q'_f \), the coweight and the coroot lattices of \( \Delta_f \), and by \( \ell_f \) its connection index, \( \ell_f = [P_f : Q_f] \).

As in Section 5, let \( \omega_1^\vee, \ldots, \omega_n^\vee \) be the fundamental coweights of \( \Delta_f \). Moreover let \( \mathfrak{h}_0^\ast \) be the real span of \( \hat{\Delta}_0 \). Then we have an orthogonal decomposition

\[ \mathfrak{h}_0^\ast = \mathfrak{h}_\sigma^\ast \oplus \mathbb{R}\omega_q^\vee; \]

we denote by \( \pi_\sigma \) to the corresponding projection onto \( \mathfrak{h}_\sigma^\ast \).

It is clear that \( \{\pi_\sigma(\omega_i^\vee) \mid i \neq q\} \) is the dual basis of \( \hat{\Pi}_0 \) in \( \mathfrak{h}_\sigma^\ast \). We denote by \( P'_\sigma, Q'_\sigma, \ell_\sigma \) the coweight lattice, the coroot lattice and the connection index of \( \hat{\Delta}_0 \):

\[ P'_\sigma = \sum_{i \neq 0, q} \mathbb{Z}\pi_\sigma(\omega_i^\vee), Q'_\sigma = \sum_{i \neq 0, q} \mathbb{Z}\alpha_i^\vee, \text{ and } \ell_\sigma = [P'_\sigma : Q'_\sigma]. \]

**Lemma 6.1.**

\[ \frac{\text{Vol}(D'_\sigma)}{\text{Vol}(C_1)} = \frac{\ell_\sigma|W_f|}{\ell_f|W_\sigma'|}. \]

**Proof.** Set

\[ \hat{I}_\sigma = \left\{ \sum_{i \neq q} x_i \omega_i^\vee \mid 0 \leq x_i \leq 1 \right\}, \quad I_q = \left\{ x \omega_q^\vee \mid 0 \leq x \leq 1 \right\}, \quad I = \hat{I}_\sigma + I_q, \]
and
\[ I' = \left\{ \sum_{i \neq 0} x_i \alpha_i^\vee \mid 0 \leq x_i \leq 1 \right\}. \]

Then \( \omega_0 + I \) and \( \omega_0 + I' \) are fundamental domains for the action of \( T(P_f^\vee) \) and \( T(Q_f^\vee) \) on \( \mathfrak{h}_1^+ \), hence by [1, VI.4, Lemma 1] we have that
\[
\frac{\text{Vol}(I)}{\text{Vol}(C_1)} = \frac{\text{Vol}(I)}{\text{Vol}(I')} \frac{\text{Vol}(I')}{\text{Vol}(C_1)} = \frac{1}{\ell_f}|W_f|.
\]

\( \widehat{W}_\sigma \) acts faithfully on \( \omega_0 + \mathfrak{h}_\sigma^+ \). Set
\[
I_\sigma = \pi_\sigma(\widehat{I}_\sigma); \quad A_\sigma = (\bigcap_{\alpha \in \Pi_\sigma} H_\alpha^+) \cap (\omega_0 + \mathfrak{h}_\sigma^+); \quad \widehat{A}_\sigma = (\bigcap_{\alpha \in \Pi_\sigma} H_\alpha^+) \cap (\omega_0 + \sum_{i \neq q} \mathbb{R} \omega_i^\vee).
\]

We notice that \( A_\sigma = \pi_\sigma(\widehat{A}_\sigma) \), and that \( D'_\sigma = \widehat{A}_\sigma - I_q \).

Arguing as in the proof of Lemma 5.4 we obtain that \( A_\sigma \) is a fundamental domain for the action of \( \widehat{W}_\sigma \) on \( \omega_0 + \mathfrak{h}_\sigma^+ \). Moreover, \( \omega_0 + I_\sigma \) is a fundamental domain for the action of \( T(P_\sigma^\vee) \) onto \( \omega_0 + \mathfrak{h}_\sigma^+ \), hence we also have that
\[
\frac{\text{Vol}_{n-1}(I_\sigma)}{\text{Vol}_{n-1}(A_\sigma)} = \frac{1}{\ell_\sigma}|W_\sigma|.
\]

Now we observe that \( \text{Vol}(I) = \text{Vol}(I_\sigma + I_q) = \text{Vol}_{n-1}(I_\sigma) \), and similarly \( \text{Vol}(D'_\sigma) = \text{Vol}(A_\sigma - I_q) = \text{Vol}_{n-1}(A_\sigma) \). It follows that
\[
\frac{\text{Vol}(D'_\sigma)}{\text{Vol}(C_1)} = \frac{\text{Vol}(D'_\sigma)}{\text{Vol}(I)} \frac{\text{Vol}(I)}{\text{Vol}(C_1)} = \frac{\text{Vol}_{n-1}(A_\sigma)}{\text{Vol}_{n-1}(I_\sigma)} \frac{\text{Vol}_{n-1}(I_\sigma)}{\text{Vol}(C_1)} = \frac{\ell_\sigma|W_f|}{\ell_f|W_\sigma|}.
\]

\( \square \)

We now compute \( \frac{\text{Vol}(D'_\sigma)}{\text{Vol}(C_1)} \).

**Lemma 6.2.**
\[
\frac{\text{Vol}(D'_\sigma)}{\text{Vol}(C_1)} = \frac{|W_f|}{|W_\sigma|}.
\]

**Proof.** For \( w \in W_f \) we denote by \( \text{Des}(w) \) the descent set of \( w \), i.e. \( \text{Des}(w) = N(w) \cap \Pi_f \). For \( I \subset \Pi_f \), set \( X^I = \{ w \in W_f \mid \text{Des}(w) \cap I = \emptyset \} \). Note that \( W_\sigma = \langle s_\alpha \mid \alpha \in I \rangle \) with \( I = \Pi_f \setminus \{ \alpha_q \} \). It is well known that \( X^I \) is the set of minimal length representatives of the right cosets \( W_\sigma \backslash W_f \). In particular \( |X^I| = |W_f : W_\sigma| \).

We shall prove that \( D'_\sigma - o_q = X_q C_1 \), where \( X_q = \{ w \in W_f \mid \text{Des}(w) \subseteq \{ \alpha_q \} \} \), and \( o_q = \frac{\omega_q^\vee}{a_q} \); by the above discussion this implies the claim.

It is well known (see [2, Lemma 1.2]) that \( W_f C_1 = \{ x \in \mathfrak{h}_1^+ \mid -1 \leq (\beta, x) \leq 1 \text{ for all } \beta \in \Delta_f^+ \} \). Moreover \( \text{Des}(w) \subseteq \{ \alpha_q \} \) if and only if \( (\alpha_i, w(C_1)) \geq 0 \text{ for } i \neq q \).

Hence \( X_q C_1 = \{ x \in \mathfrak{h}_1^+ \mid -1 \leq (\beta, x) \leq 1 \text{ for all } \beta \in \Delta_f^+, (\alpha_i, x) \geq 0 \text{ for } i \neq q \} \).

We first prove that if \( x \in D''_\sigma \), then \( x - o_q \in X_q C_1 \). We notice that \( o_q = \omega_q^\vee \), that \( (\beta, o_q) \leq 1 \text{ for all } \beta \in \Delta_f^+ \), and \( (\beta, o_q) = 0 \) if and only if \( \beta \in \widehat{\Delta}_0 \). Moreover, \( x \) is a dominant element, hence for positive \( \beta, \alpha \in \Delta_f \) such that \( \beta \leq \alpha \) (in the standard
partial order on roots) we have $0 \leq (\beta, x) \leq (\alpha, x)$. Now assume that $\beta \in \Delta^+_f$. If $\beta \in \Delta^+_0$, then $\beta \in \Delta$ for some $\Sigma|\Pi$, hence $0 \leq (x, \beta) = (x - o_q, \beta) \leq (x, \theta_\Sigma) \leq 1$. If $\beta \not\in \Delta^+_0$, then $-1 \leq (x, \beta) - 1 = (x - o_q, \beta) \leq (x, \theta) - 1 \leq 2 - 1 = 1$.

Next we prove the reverse inclusion. We consider $y \in X_q C_1$ and prove that $y + o_q \in D''_{\sigma}$. We have $(y + o_q, \alpha_q) = (y, \alpha_q) + 1 \geq 0$, and if $i \neq q$ we have $(y + o_q, \alpha_i) = (y, \alpha_i) \geq 0$. Moreover, $(y + o_q, \theta_\Sigma) = (y, \theta_\Sigma) \leq 1$ for all $\Sigma|\Pi$. This concludes the proof. □

Combining the two lemmas we find

**Theorem 6.3.** If $g_0$ is not semisimple then

$$|W_{ab}^\sigma| = \left|\frac{W_f}{W_\sigma}\right| \left(1 + \frac{\ell_\sigma}{\ell_f}\right).$$

We summarize the explicit results in the following table:

| type of $\hat{g}$ | $q$ | type of $\Delta_f$ | type of $[g_0, g_0]$ | $|W_{ab}^\sigma|$ |
|-------------------|-----|--------------------|----------------------|------------------|
| $A_n^{(1)}$       | $1 \leq q \leq n$ | $A_n$ | $A_{q-1} \times A_{n-q}$ | $(n+1) + q(n)$ |
| $B_n^{(1)}$       | $q = 1$ | $B_n$ | $B_{n-1}$ | $4n$ |
| $C_n^{(1)}$       | $q = n$ | $C_n$ | $A_{n-1}$ | $2^{n-1}(n+2)$ |
| $D_n^{(1)}$       | $q = 1$ | $D_n$ | $D_{n-1}$ | $4n$ |
|                   | $q = n - 1, n$ | $D_n$ | $A_{n-1}$ | $2^{n-3}(n+4)$ |
| $E_6^{(1)}$       | $q = 1, 6$ | $E_6$ | $D_5$ | $63$ |
| $E_7^{(1)}$       | $q = 7$ | $E_7$ | $E_6$ | $140$ |

**References**

[1] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4–6*, Hermann, Paris, 1968.

[2] P. Cellini, P. Möseneder Frajria and P. Papi, ‘Compatible discrete series’, Pacific J. of Math. 212 (2) (2003), 201–230.

[3] P. Cellini and P. Papi, ‘$\text{ad}$-nilpotent ideals of a Borel subalgebra’, J. Algebra 225 (2000), 130–141.

[4] H. Garland, J. Lepowsky, ‘Lie algebra homology and the Macdonald-Kac formulas’, Invent. Math. 34 (1976), 37–76.

[5] N. Iwahori, H. Matsumoto, ‘On some Bruhat decomposition and the structure of the Hecke rings of $p$-adic Chevalley groups’, Inst. Hautes Études Sci. Publ. Math. 25 (1965), 5–48.

[6] V.Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, Cambridge, 1985.

[7] B. Kostant, ‘Eigenvalues of a Laplacian and commutative Lie subalgebras’, Topology 3, suppl. 2 (1965), 147–159.

[8] B. Kostant, ‘The Set of Abelian ideals of a Borel Subalgebra, Cartan Decompositions, and Discrete Series Representations’, Internat. Math. Res. Notices 5 (1998), 225–252.
[9] B. Kostant, ‘Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra’ (2003), Math. GR/0309232.

[10] S. Kumar, Kac-Moody groups, their flag varieties and representation theory, Birkhäuser, Boston, 2002.

[11] D. Panyushev, ‘Isotropy representations, eigenvalues of a Casimir element, and commutative Lie subalgebras’, J. London Math. Soc. 61 (2001), 61–80.

[12] D. Panyushev, ‘Long abelian ideals’, Adv. Math. (to appear) (2003), Math. RT/0303222.