PARAMETRIX CONSTRUCTIONS AND LOW FREQUENCY RESONANCES FOR THE CHARGED KLEIN-GORDON EQUATION ON THE DE SITTER-REISSNER-NORDSTRÖM METRIC

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Abstract. We show the index 0 property of several realizations of the spectral family of the charged Klein-Gordon operator on the De Sitter-Reissner-Nordström metric using a parametrix construction. We also deduce the absence of resonances in any compact neighborhood of the 0 energy for large masses or angular momenta of the field. The results are uniform in the cosmological constant. This work is a preliminary to the elaboration of a numerical scheme intended to compute low frequency resonances.

1. Introduction

1.1. The setting. Let $g$ denote the Euclidean metric on $\mathbb{S}^2$. In Boyer-Lindquist coordinates $(t, r, \theta, \phi) \in \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2$ with $r_\pm$ specified below, the (sub-extremal) De Sitter-Reissner-Nordström metric is given by

$$g := \mu(r)dt^2 - \mu(r)^{-1}dr^2 - r^2g$$

with the horizon function

$$\mu(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.$$ 

Above, $M > 0$ is the reduced mass of the black hole, $Q \in \mathbb{R}$ is its reduced electric charge and $\Lambda > 0$ is the cosmological constant. The metric $g$ is a solution to the coupled Einstein-Maxwell equation with the Coulomb potential

$$A(r) := \frac{Q}{r}dt.$$ 

The horizon function $\mu$ has four distinct real roots $r_n < 0 < r_c < r_- < r_+$ if and only if $Q = 0$ and $3\sqrt{\Lambda}M < 1$, or\footnote{Note that $Q \mapsto \frac{M + \sqrt{D}}{(3M + \sqrt{D})^3}$ is increasing so that $9\Lambda M^2 \geq 1$ is allowed for $Q \neq 0$.}

$$D := 9M^2 - 8Q^2 > 0, \quad \max \left(0, \frac{6(M - \sqrt{D})}{(3M - \sqrt{D})^3}\right) < \Lambda < \frac{6(M + \sqrt{D})}{(3M + \sqrt{D})^3}.$$ 

cf. [H1, Prop. 3.2]. In this setting, there exists a unique $r \in (r_-, r_+)$ such that

$$c^{-2} := \max_{[r_-, r_+]} \{\mu(r)\} = \mu(r)$$
(see \cite{H1}, Sect. 3) for the existence and uniqueness). The following constants, called surface gravities, will play an important role in the sequel:

\[ \kappa_{\alpha} := \frac{\mu'(r_{\alpha})}{2} = -\frac{1}{2} \frac{\Lambda}{3r^2} \prod_{\beta \in \{n,c,-,\} \setminus \{\alpha\}} (r_{\alpha} - r_{\beta}), \quad \alpha \in \{n,c,-,\}. \]

We also set:

\[ \kappa := \min\{\kappa_-|, |\kappa_+|\} > 0. \]

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{universe.png}
\caption{Graph of the horizon function $\mu$ of the sub-extremal De Sitter-Reissner-Nordström metric.}
\end{figure}

We set \( s := qQ \) (the charge product) and \( V(r) := \frac{1}{r} \). Following \cite{BeHa}, Sect. 2, we remove the coordinate singularities of the metric at \( \{r = r_{\pm}\} \) by introducing a new smooth coordinate. Let \( \nu(r) = \pm \sqrt{1 - \mu(r)c^2} \) for \( \pm (r - r) \geq 0 \) and define on \((r_-, r_+)\):

\[ \begin{cases} T(r) = -\int_r^\infty \frac{\nu(r')}{\mu(r')} dr', \\ R(r) = sV(r)T(r). \end{cases} \]

We then put \( t_* := t - T(r) \). Notice that \( |t_* - t| \leq C_K \) with \( C_K > 0 \) uniformly on any compact set \( K \subset (r_-, r_+) \times S^2 \), so \( t_* \) is like a "time" when \( r \in (r_-, r_+) \).

Consider a charged scalar field with electric charge \( q \in \mathbb{R} \) and mass \( m \geq 0 \). The charged wave operator on \((\mathcal{M}, g)\) is defined by

\[ \mathcal{E}_g := (\nabla_{\alpha} - iqA_{\alpha}) (\nabla^\alpha - iqA^\alpha) \]

and the corresponding charged Klein-Gordon equation, governing the evolution of the field, then reads:

\[ \mathcal{E}_g u + m^2 u = 0. \]

1.2. Resonances and decay of the local energy. Long time behaviour of waves in curved spacetimes is an important first step in the analysis of the stability of the underlying metric (cf. \cite{HV} for the non-linear stability of De Sitter-Kerr metric and \cite{HIV} for the linear stability of Kerr spacetime). It is also a prerequisite in proving asymptotic completeness for wave type equations (cf. \cite{HN} for massless Dirac fields on Kerr metric, \cite{GGH} for scalar waves on De Sitter-Kerr type metrics and \cite{B2} for charged Klein-Gordon fields on the De Sitter-Reissner-Nordström metric when the charged product is sufficiently small).

Theory of resonances gives a precise description of this long time behaviour. Similarly to solutions of the wave equation on a compact spatial domain expend as a sum over the (discrete set of) eigenvalues of
the spatial Laplacian, we know since the work of Bony-Häfner [BoHa] that scalar waves on cosmological uncharged spherical black hole metric with compactly supported initial data also have a local expansion of the form

$$\chi u(t, \bullet) = \sum_{\sigma_j \in \text{Res}, \text{Im}(\sigma_j) > -C} \sum_{k=0}^{m(\sigma_j)} t^k e^{-i\sigma_j t} u_{j,k} + \tilde{u}, \quad \|\tilde{u}(t, \bullet)\| \lesssim e^{-Ct}. \tag{3}$$

Above $C > 0$ and $\chi$ is a smooth spatial cut-off. The special frequencies appearing in the expansion are called resonances: they are complex frequencies, proper to the black hole, that describe late time oscillation and damping of waves. They are not the eigenvalues of the spatial part of the wave operator but rather those frequencies such that a related operator $P_\sigma$ (see Sect. 1.3 below) has a non trivial kernel in the appropriate functional space. Eigenvalues of $P_\sigma$ are called resonant states and those of the dual operator $P_\sigma^*$ are called dual states; the term $u_{jk}$ in (3) is the projection on the $(k + 1)$ dimensional vector space spanned by the resonant states associated to the resonance $\sigma_j$ with $k \leq m(\sigma_j)$, $m(\sigma_j)$ being the multiplicity of the resonance $\sigma_j$ (see [BoHa, eq. (1.10) & eq. (1.11)] for details for the wave equation on the De Sitter-Schwarzschild metric). Resonance expansions have first been obtained for (massive) waves on De Sitter-Kerr spacetimes by Dyatlov [D]; we refer to [H3, Sect. 1.1] for more references in this context, and to [B1] for charged Klein-Gordon fields on De Sitter-Reissner-Nordström spacetimes. Notice that (3) shows that the presence of resonances in the upper complex half plane entails exponential growth in time of the solution (this is the case for small charge product and sufficiently small mass of the field and rotation of the black hole, see [BeHa]).

Existence of resonances for (massive) scalar waves in perturbations of De Sitter-Kerr spacetimes has been proved by the seminal work of Vasy [V] by means of a robust Fredholm theory for non elliptic operators. Notice however that the operator $P_\sigma$ in [V] is defined on an enlarged spatial domain and a complex potential supported beyond the horizons is added, so that resonances are created beyond the horizons; however, in a cut-off sense, that is inside $(r_-, r_+)$, they do not depend on the choice of the extended domain nor on the absorbing potential. By the index 0 analytic Fredholm theory, they are thus characterized as the poles of the meromorphic extension from $\text{Im}(\sigma) > 0$ to some larger domain in $\mathbb{C}$ of the cut-off resolvent $\chi P_\sigma^{-1} \chi$. This definition coincide with that provided by Mazzeo-Melrose work [MM] (used by Sá Barreto-Zworski [SZ] then Bony-Häfner [BoHa] to define resonances).

In this paper, we propose another definition of the resonances based on the construction of parametrices in the context of charged Klein-Gordon fields on De Sitter-Reissner-Nordström spacetimes.

**Remark 1.** When $\Lambda = 0$, the manifold has a hyperbolic and an Euclidean ends. There is then an accumulation of resonances at the 0 energy (cf. [H3, Thm. 1.1]) so that 0 is not itself a resonance and the resolvent is not meromorphic in any neighborhood of 0. In this situation, only polynomial decay of waves is expected (cf. e.g. [H2] for a sharp rate of decay on Kerr metric).

### 1.3. Main results.

For all $\sigma \in \mathbb{C}$, we set

$$P_\sigma := e^{i(R(r) + \sigma t_*)} (\mathcal{L}_g + m^2) e^{-i(R(r) + \sigma t_*)}$$

---

2The physical coordinates are $(t, r, \theta, \phi)$ but mathematical computations will be carried out in the coordinates $(t_*, r, \theta, \phi)$. 
so that finding a solution to $P_\sigma u = 0$ is equivalent to finding a solution to $(\mathcal{L}_g + m^2)v(r, \omega) = 0$ with $v(r, \theta, \phi) = e^{-i(R(r) + \sigma t_\sigma)}u(r, \theta, \phi)$. The reason why we work with the coordinate $t_\sigma$ is that it allows us to use $\sigma$ independent Sobolev spaces below.

We fix $\Gamma \subset \mathbb{C}$ an open and simply connected set and consider $|s| < s_0$ for some fixed $s_0 > 0$. Let $\gamma \in \mathbb{R}$, $H_\gamma := H_\gamma((r_-, r_+), dr)$ and $\gamma\mathcal{H}$ be the functional space defined in Sect. 2.1 – $\gamma\mathcal{H}$ is locally equivalent to $H_\gamma$ i.e. if $\chi \in \mathcal{C}_c^\infty((r_-, r_+), \mathbb{C})$ and $u \in \gamma\mathcal{H}$, then $\chi u \in H_\gamma$. Using the parametrix construction of Sect. 3, we obtain the following result (the proof follows from Cor. 18 and the comment below):

**Theorem 2.** For all $\gamma \in \mathbb{R}$, $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -\kappa/2\} \ni \sigma \rightarrow P_\sigma \in \mathcal{B}(\gamma\mathcal{H}, \gamma\mathcal{H})$ are analytic families of index 0 Fredholm operators.

The critical strip $\{z \in \mathcal{C} \mid \text{Im}(z) > -\kappa/2\}$ can be lowered if we use standard Sobolev spaces $H_\gamma$. Let $X_\gamma := \{u \in H_\gamma \mid P_\sigma u \in H_\gamma^{-1}\}$. To simplify the expository, we will content ourselves with positive orders $\gamma$. The following result follows from Cor. 20 and the comment below:

**Theorem 3.** For all $\gamma \in \mathbb{N}\setminus\{0\}$, $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -(\gamma - 1/2)\kappa\} \ni \sigma \rightarrow P_\sigma \in \mathcal{B}(X_\gamma, H_\gamma^{-1})$ is an analytic family of index 0 Fredholm operators.

Vasy theory shows, in the context of rotating black holes and when $Q = q = 0$ (but the charged setting for a non rotating black hole enters Vasy framework [V, Sect. 2.2]), the index 0 Fredholm property of $(P_\sigma, X_\gamma)$ when $\text{Im}(\sigma) > -\kappa/2$ for all $\gamma \geq 1/2$ (cf. [V, eq. (1.2)]) using propagation of singularities on an enlarged spatial domain containing $[r_-, r_+]$. The approach of the present paper is different: we work on $(r_-, r_+)$ and consider $\{r = r_\pm\}$ as unreachable boundaries.

**Remark 4.** The analytic Fredholm theory (c.f. [DZ, Sect. C.3] for a reference using the Grushin problem approach) shows the existence of the meromorphic extension in $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -\kappa/2\}$ of each realization of $P_\sigma$ in Thm. 5 provided that $P_\sigma^{-1}$ exists for some $\sigma \in \mathbb{C}$ with $\text{Im}(\sigma) > 0$ (it is the case for $(P_\sigma, \mathcal{B}(\mathcal{H}^2, \mathcal{H}))$ as it follows from [B1, Prop. 3.4] and the comment below; it is also the case for $(P_\sigma, \mathcal{B}(X_\gamma, \gamma\mathcal{H}))$ by [V, Thm. 1.1]).

As $H_\gamma \rightarrow \mathcal{H}$, poles of $(P_\sigma^{-1}, \mathcal{B}(\gamma\mathcal{H}, X_\gamma))$ are also poles of $(P_\sigma^{-1}, \mathcal{B}(\gamma\mathcal{H}^{-1}, X_\gamma))$. Conversely, for all $u \in \gamma\mathcal{H}^{-1}$ and all cut-off $\chi \in \mathcal{C}_c^\infty((r_-, r_+), \mathbb{C})$, $\chi P_\sigma^{-1} \chi u \in X_\gamma$ with any realization $(P_\sigma, \mathcal{B}(\mathcal{H}^2, \mathcal{H}))$ or $(P_\sigma, \mathcal{B}(X_\gamma, \gamma\mathcal{H}^{-1}))$ since $H_\gamma$ and $\gamma\mathcal{H}$ are locally equivalent. Hence, poles in $\{\text{Im}(\sigma) > -\kappa/2\}$ of both the cut-off resolvents coincide. Notice that we can also work with the realization $(P_\sigma, \mathcal{B}(\mathcal{H}^2, \mathcal{H}))$ since inserting any element of $\gamma\mathcal{H}$ in expansion (3) makes sense independently of the meromorphic extension $P_\sigma^{-1} : \gamma\mathcal{H}^{-1} \rightarrow X_\gamma \subset \gamma\mathcal{H}$ or $P_\sigma^{-1} : \gamma\mathcal{H}^{-2} \rightarrow \gamma\mathcal{H}$. Hence, resonances are well defined as poles of any of the three meromorphic extensions above; moreover, resonances in $\{\text{Im}(\sigma) > -\kappa/2\}$ do not depend on $\gamma \geq 1$.

In particular, for all $\sigma \in \mathbb{C}$ with $\text{Im}(\sigma) > -\kappa/2$, $\chi(\mu(r)e^{-i(R(r) - \sigma T(r))}P_\sigma e^{i(R(r) - \sigma T(r))})^{-1} \chi$ is the cut-off (low frequency) resolvent in $\mathcal{B}[1].$

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3When $\gamma < 1/2$, the critical strip is $-\kappa/2$ max{1, $\kappa, \kappa_1$}.

4Resonances showing up in (3) could depend on $\gamma$ in that the projection on the corresponding dual states (that is solutions of $P_\sigma^* u = 0$) could vanish for insufficiently regular initial data; however, this does not happen for solutions to (2) with initial data supported in $(r_-, r_+)$ because dual states are smooth therein, as $P_\sigma^* u \in \gamma\mathcal{H}^{-1}$ implies $u \in \gamma\mathcal{H}$ (cf. [V, eq. (2.21)]). Dual states can fail to be smooth at the horizons $\{r = r_\pm\}$, cf. e.g. [HX, App. B] for examples of such dual states for uncharged scalar fields in the De Sitter spacetime (including Dirac distributions supported at the horizon, whose associated resonant frequency does not show up in the long time behaviour of solution to the Klein-Gordon equation for initial data supported far away from the horizon).
The parametrix construction allows us to show the following result whose the proof is given in Sect. 4.2:

**Theorem 5.** There exists a constant $C > 0$ depending on $\sigma_0', \sigma_0'', s_0, r, \varepsilon, \chi_\pm$ such that, if $\ell + m > C$, then there is no resonance in $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -\kappa/2 + \varepsilon\}$.

**Remark 6.** Free resonance zones thus exist (at least) for large $\ell, m$ and $\text{Im}(\sigma)$. When $\ell \gg 0$, this is a well known fact when $s = 0$ (when $s$ is small, it is shown in [B1]) as $P_\sigma$ looks like a free semiclassical Laplacian (with semiclassical parameter $h := (\ell(\ell+1))^{-1/2}$) on the Euclidean space and resonances approach a lattice in the lower complex half plane called pseudo-poles (cf. [B1, Thm. 4.1] for a description in the context of charged Klein-Gordon fields). The mass also repels resonances from any compact neighborhoods of the 0 energy; it is not a generic phenomenon on black hole type spacetimes as when $\Lambda = 0$, masses create resonances in the upper complex half plane (cf. [SR]).

**1.4. Outlook.** The parametrix construction was initially motivated by the numerical computation of low frequency resonances and it is the natural continuation of the present work. We briefly explain how the parametrix can be used for numerics. We do not precise any functional space to stay simple.

Having constructed a (right) parametrix $Q_\sigma$ for $P_\sigma$, i.e.

$$P_\sigma Q_\sigma = 1 + K_\sigma$$

where $K_\sigma$ is compact and analytic in $\sigma$, we fix $\sigma_1 \in \mathbb{C}$ (in fact in the authorized domains of Thm. 2 and Thm. 3) and consider $\sigma \in D(\sigma_1, \varepsilon)$ with $\varepsilon > 0$ sufficiently small. Using compactness and analyticity of $K_\sigma$, we can write

$$K_\sigma = K_{\sigma_1} + \sum_{l=1}^{+\infty} K_l(\sigma - \sigma_1)^l = K'_{\sigma_1} + K''_{\sigma_1} + \sum_{l=1}^{+\infty} K_l(\sigma - \sigma_1)^l := K'_{\sigma_1} + K''_{\sigma_1}$$

where $K'_{\sigma_1}$ is of finite rank and

$$\|K''_{\sigma_1}\| + \left\|\sum_{l=1}^{+\infty} K_{\sigma_1}(\sigma - \sigma_1)^l\right\| < 1.$$ 

The smallness of the first norm above is due to the compactness of $K_{\sigma_1}$ while the smallness of the sum is due to the analyticity in $\sigma$ (using that $|\sigma - \sigma_1| < \varepsilon \ll 1$). Then:

$$1 + K_\sigma = (1 + K''_{\sigma_1})(1 + (1 + K''_{\sigma_1})^{-1}K'_{\sigma_1}).$$

As $(1 + K''_{\sigma_1})^{-1}K'_{\sigma_1}$ is of finite rank, the Fredholm determinant of $(1 + (1 + K''_{\sigma_1})^{-1}K'_{\sigma_1})$ is well-defined as the limit of the determinant of the projection of the latter operator on a finite-dimensional subspace of $\mathcal{H}$ as the dimension tends to $+\infty$ (cf. [DS, Lem. 22(f)]). By the index 0 Fredholm property, $\sigma$ is a resonance if and only if $P_{\sigma}$ has a non trivial cokernel which is equivalent to $\text{det}(1 + (1 + K''_{\sigma_1})^{-1}K'_{\sigma_1}) = 0$. The latter is an analytic function of $\sigma$ whose zeros can be localized and approximated by the argument principle and the residue theorem. The inverse $(1 + K''_{\sigma_1})^{-1}$ can be approximated by a truncated Neumann series.

It is expected that the parametrix construction presented in this paper applies to complex frequencies $\sigma$ such that $|\text{Re}(\sigma)| \gg 0$ or $\text{Im}(\sigma) \gg 0$ using the semiclassical parameter $h := |\text{Re}(\sigma)|^{-1}$ or $h := \text{Im}(\sigma)^{-1}$. Another possible continuation of the present work is the extension to the De Sitter-Kerr-Newman setting. In practice the approach is the same but the loss of spherical symmetry makes computations tougher.
Besides, the angular part of \( P_\sigma \) is not diagonalizable for non real \( \sigma \). The numerical scheme could however approximate its low frequency eigenvalues in order to apply the present work in this context. To our knowledge, localization of resonances in a strip \( \{ \text{Im}(\sigma) > -\varepsilon \} \) with \( \varepsilon > 0 \) for rotating black holes is totally unknown and numerical results would provide very important information in this context.

1.5. **Outline of the paper.** The paper is organized as follows. We first introduce in Sect. 2 the operators, functional spaces and quantizations used in the parametrix construction. In Sect. 3, we construct local (or boundary) parametrices then glue them to obtain global ones. Finally, we prove in Sect. 4 the main theorems presented in Sect. 1.3. The appendix, Sect. 5, contains norm estimates for Fourier type and Mellin type pseudo-differential operators with \( L^2 \) symbols.

2. **Functional framework and quantizations**

This section introduce the operators and functional spaces used throughout this paper. We also discuss some important properties of two quantizations used in the construction of parametrices for \( P_\sigma \).

2.1. **Functional framework.** We use the coordinates of Subsection 1.1. We henceforth fix \( \ell \in \mathbb{N} \) and restrict ourselves to \( \text{Ker}(\Delta_{S^2} + \ell(\ell + 1)) \). We also fix \( s_0, \sigma_0', \sigma_0'' > 0 \) and consider \( |s| < s_0 \) as well as a bounded open and simply connected set \( \Gamma \subset \mathbb{C} \) such that

\[
\max\{\text{Re}(\sigma) \mid \sigma \in \Gamma\} = \sigma_0', \quad \max\{\text{Im}(\sigma) \mid \sigma \in \Gamma\} = \sigma_0''.
\]

For all \( \sigma \in \Gamma \), we define the spectral family of operator pencils \( P_\sigma = P(\sigma, s, m, \ell) \) associated to the charged Klein-Gordon operator \( \mathcal{L}_g + m^2 \) by:

\[
P_\sigma := e^{i(R(r)+\sigma t_*)} (\mathcal{L}_g + m^2) e^{-i(R(r)+\sigma t_*)} = -c^2(\sigma + sV)^2 + \frac{i}{r^2} (\sigma + sV)r^2 \nu \partial_r + \frac{i}{r^2} \partial_r r^2 \nu (\sigma + sV) - \frac{1}{r^2} \partial_r r^2 \mu \partial_r + \frac{\ell(\ell + 1)}{r^2} + m^2.
\]

The conjugation by \( e^{i R(r)} \) is the appropriate gauge changing which smoothen the coefficients of the potential \( A = QV(r) \) in the new coordinates (cf. [BeHa, Sect. 2]). On the other hand, the conjugation by \( e^{i \sigma T(r)} \) ensures that the metric smoothly extends through the horizons.

Let \( \tilde{P}_\sigma := \Lambda^{-1} P_\sigma \). Note the following rescaling:

\[
\tilde{t} := \sqrt{\Lambda} t, \quad \tilde{r} := \sqrt{\Lambda} r, \quad \tilde{M} := \sqrt{\Lambda} M, \quad \tilde{Q} := \sqrt{\Lambda} Q, \quad \tilde{\Lambda} := 1,
\]

\[
\tilde{q} := \frac{q}{\sqrt{\Lambda}}, \quad \tilde{m} := \frac{m}{\sqrt{\Lambda}}, \quad \tilde{\sigma} := \frac{\sigma}{\sqrt{\Lambda}}.
\]

The last term above is of particular interest: resonance frequencies are in the new variables scattered by the scaling factor \( 1/\sqrt{\Lambda} \): in practice, this amounts to ”zooming” on resonances and changing the characteristic distance from \( \sqrt{\Lambda} \) to 1; this also implies that if there is no resonance in the upper complex half plane \( \mathbb{C}^+ \) for one positive value of \( \Lambda \) and for all \( M, m > 0 \) and \( Q, q \in \mathbb{R} \), then there is no resonance in \( \mathbb{C}^+ \) for all \( \Lambda > 0 \). Observe also that

\[
\tilde{s} := \frac{\tilde{q}}{\sqrt{\Lambda}} \tilde{Q} = q \sqrt{\Lambda} = s
\]

so that the charge product is invariant under the rescaling. Notice finally that

\[
\tilde{\mu}(\tilde{r}) := 1 - \frac{2 \tilde{M}}{\tilde{r}} + \frac{\tilde{Q}^2}{\tilde{r}^2} - \frac{\tilde{r}^2}{3} = 1 - \frac{2 M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{3} = \mu(r).
\]
In these rescaled variables, we have
\[ \tilde{P}_\sigma = -c^2(\tilde{\sigma} + sV(\tilde{r}))^2 + \frac{i}{\tilde{r}^2}(\tilde{\sigma} + sV(\tilde{r}))\tilde{\nu}(\tilde{r})\tilde{\partial}_r + \frac{i}{\tilde{r}^2}\tilde{\partial}_r\tilde{r}^2\tilde{\nu}(\tilde{r})(\tilde{\sigma} + sV(\tilde{r})) - \frac{1}{\tilde{r}^2}\tilde{\partial}_r\tilde{r}^2\tilde{\mu}(\tilde{r})\tilde{\partial}_r + \frac{\ell(\ell + 1)}{\tilde{r}^2} + \tilde{m}^2 \]
with \( \tilde{\nu}(\tilde{r}) := \nu(r) \). It is clear that \( \sigma \) is a pole of the meromorphic extension of \( P_\sigma \) if and only if \( \tilde{\sigma} \) is a pole of the meromorphic extension of \( \tilde{P}_\sigma \). In order to lighten notations, we henceforth drop the \( \tilde{\cdots} \) symbol.

We introduce:
\[ \begin{align*}
\mu(r) &:= r^2\mu(r), \\
\nu(r) &:= r\nu(r), \\
\kappa_\pm &:= r_\pm^2|\kappa_\pm|,
\end{align*} \]
\[ 
g_\sigma(r) = g(\sigma, s, r) := 2i(\sigma r + s)\nu(r) - (\partial_r\mu)(r),
\]
\[ 
f_\sigma(r) = f(\sigma, s, m, \ell, r) := -c^2(\sigma r + s)^2 + (\partial_r(\sigma r + s)\nu)(r) + \ell(\ell + 1) + m^2r^2.
\]

We multiply \( P_\sigma \) by \( r^2 \) which does not affect the spectral properties of \( P_\sigma \) far away from \( \{r = 0\} \) and \( \{r = +\infty\} \) but removes inverse powers\(^5\) of \( r \) in the coefficients of \( P_\sigma \):
\[ P_\sigma := r^2P_\sigma = -\mu(r)\partial_r^2 + g_\sigma(r)\partial_r + f_\sigma(r). \]

Because this operator is totally characteristic at the horizons \( \{r = r_\pm\} \), it is important to introduce new coordinates near these boundaries: on the intervals \((r_-,r)\) and \((r, r_+)\), we consider the boundary defining functions \( \rho_\pm := |r - r_\pm| \) and set
\[ 
\mu_\pm(\rho_\pm) := \frac{\mu(r_\pm \mp \rho_\pm)}{\rho_\pm}, \\
\nu_\pm(\rho_\pm) := \nu(r_\pm \mp \rho_\pm),
\]
\[ 
g_{\sigma,\pm}(\rho_\pm) := \mp 2i(\sigma r + s)\nu_\pm(\rho_\pm) \pm \rho_\pm(\partial_{\rho_\pm}\mu_\pm)(\rho_\pm),
\]
\[ 
f_{\sigma,\pm}(\rho_\pm) := f_\sigma(r_\pm \mp \rho_\pm).
\]

Note that \( \mu_\pm(0) = \mp 2r_\pm^2\kappa_\pm = 2r_\pm^2|\kappa_\pm| = 2\kappa_\pm \) and \( g_{\sigma,\pm}(0) = 2i(\sigma r_\pm + s)r_\pm \) (since \( \nu(r_\pm) = \mp 1 \)). Then \( \partial_r = \mp \partial_{\rho_\pm} \) and:
\[ P_\sigma = -\frac{\mu_\pm(\rho_\pm)}{\rho_\pm}(\rho_\pm \partial_{\rho_\pm})^2 + \frac{g_{\sigma,\pm}(\rho_\pm)}{\rho_\pm}(\rho_\pm \partial_{\rho_\pm}) + f_\sigma(\rho_\pm). \]

We turn to the functional spaces. Let \( \chi_\pm \in C^\infty([r_-, r_+[, [0, 1]) \) such that \( \chi_\pm(r_\pm) = 1 \) and \( \chi_- + \chi_+ = 1 \) (cf. Fig. 2).

![Figure 2. Graphs of the cut-offs \( \chi_- \) and \( \chi_+ \).](image)

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\(^5\)These terms worsen some estimates in the numerical scheme discussed in Sect. 1.4 since \( r_- \ll 1 \) for \( \Lambda = 1 \).
Let $\gamma \in \mathbb{R}$. We define b-Sobolev spaces adapted to the boundaries:

$$\mathcal{H}_\pm^\gamma := \left\{ u \in L^2\left( \mathbb{R}_+^*, \frac{d\sigma_{\pm}}{\rho_{\pm}} \right) \mid \| u \|_{\mathcal{H}_\pm^\gamma} < +\infty \right\}, \quad \| u \|_{\mathcal{H}_\pm^\gamma} := \| \rho_{\pm} \tilde{\sigma}_{\rho_{\pm}} \gamma u \|_{L^2(\mathbb{R}_+^*, \frac{d\rho_{\pm}}{\rho_{\pm}})}.$$ 

We use the Mellin transform in these definitions for non even integer $\gamma$, cf. Sect. 2.2 below. In order to view $\chi_{\pm} u$ as an element of the boundary space when $u \in L^2((r_-, r_+), dr)$, we need to introduce weighted b-Sobolev spaces:

$$\mathcal{H}_\pm^{\gamma, \beta} := \rho_{\pm}^\beta \mathcal{H}_\pm^\gamma = \{ \rho_{\pm}^\beta u \mid u \in \mathcal{H}_\pm^\gamma \}, \quad \| u \|_{\mathcal{H}_\pm^{\gamma, \beta}} := \| \rho_{\pm}^{-\beta} u \|_{\mathcal{H}_\pm^\gamma}.$$

We then define

$$\mathcal{H}^\gamma := \{ u \in L^2((r_-, r_+), dr) \mid \| u \|_{\mathcal{H}^\gamma} < +\infty \}, \quad \| u \|_{\mathcal{H}^\gamma} := \sum_\pm \| \chi_{\pm} u \|^2_{\mathcal{H}^{\gamma, -1/2}_\pm},$$

$\mathcal{H} := \mathcal{H}^0$ and set

$$\mathcal{X}^\gamma := \{ u \in \mathcal{H}^\gamma \mid P_\sigma u \in \mathcal{H}^{\gamma-1} \}, \quad \| u \|_{\mathcal{X}^\gamma} := \| u \|^2_{\mathcal{H}^\gamma} + \| P_\sigma u \|^2_{\mathcal{H}^{\gamma-1}}.$$

As we will see in Sect. 3.1, using the b-Sobolev spaces allows to realize $P_\sigma$ with elliptic estimates (i.e. $P_\sigma u \in \mathcal{H}^\gamma$ implies $u \in \mathcal{H}^{\gamma+2}$) but imposes the restriction $\text{Im}(\sigma) > -\kappa/2$ with $\kappa > 0$ defined in (1). This restriction is removable if we are willing to lose one derivative with respect to the elliptic case (that is, $P_\sigma u \in \mathcal{H}^{\gamma, \beta}$ only entails $u \in \mathcal{H}^{\gamma+1, \beta}$). To proceed, we need to work in standard Sobolev spaces $H^\gamma$. We will consider only integer orders to simplify the expository. For all $(\gamma, \beta) \in \mathbb{N} \times \mathbb{R}$, we define

$$H_\pm := H_0^0 := L^2\left( \mathbb{R}_+^*, \frac{d\rho_{\pm}}{\rho_{\pm}} \right)$$

equipped with its standard inner product and

$$H_\pm^{\gamma, \beta} := \{ u \in H \mid \rho_{\pm}^\beta \tilde{u} \in H_\pm \}, \quad \| u \|^2_{H_\pm^{\gamma, \beta}} := \| \rho_{\pm}^\gamma \tilde{u} \|^2_{H_\pm},$$

$H_\pm^{\gamma, \beta} := \{ u : \mathbb{R}_+^* \to \mathbb{C} \mid u, \rho_{\pm}^{-\beta} u, \rho_{\pm}^{-\beta} \tilde{u} \in H_\pm \}, \quad \| u \|^2_{H_\pm^{\gamma, \beta}} := \| \rho_{\pm}^{-\beta} u \|^2_{H_\pm} + \| \rho_{\pm}^{-\beta} \tilde{u} \|^2_{H_\pm}.$

Notice that $\mathcal{H}_\pm^{\gamma, \beta} \neq \mathcal{H}_\pm^\gamma$ but $H^\gamma(\mathbb{R}_+^*, dr) = H^{\gamma-1/2}_\pm$. Finally, we set $H := H^0 := L^2((r_-, r_+), dr)$ equipped with its standard scalar product,

$$H^\gamma := H^0((r_-, r_+), dr), \quad \| u \|^2_{H^\gamma} := \sum_\pm \| \chi_{\pm} u \|^2_{H^{\gamma, \beta}}.$$

then

$$\mathcal{X}^\gamma := \{ u \in H^\gamma \mid P_\sigma u \in \mathcal{H}^{\gamma-1} \}, \quad \| u \|^2_{\mathcal{X}^\gamma} := \| u \|^2_{H^\gamma} + \| P_\sigma u \|^2_{H^{\gamma-1}}.$$

The following density result will be useful (cf. [AF, Thm. 3.22]):

**Lemma 7.** Let $C_0^\infty(\mathbb{R}_+^*, \mathbb{C})$ be the space of smooth functions on $\mathbb{R}_+$ that vanish at infinity together with all its derivatives. Then for all $k \in \mathbb{N}$ and $1 \leq p < +\infty$, $C_0^\infty(\mathbb{R}_+^*, \mathbb{C})$ is dense in the Sobolev space $W^{k, p}(\mathbb{R}_+^*, d\rho_{\pm})$.

2.2. Quantization. We introduce in this section the Mellin type quantization. All the properties needed below directly follows from those associated to the Fourier type quantization. We thus start by recalling them.
2.2.1. Fourier type quantization. The Fourier transform \( \mathcal{F} \in \mathcal{B}(L^2(\mathbb{R}, dx)) \) and its inverse are defined by:

\[
\mathcal{F}[u](\xi) := \int_{\mathbb{R}} e^{-ix\xi} u(x) dx, \quad \mathcal{F}^{-1}[v](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} v(\xi) d\xi.
\]

Real order Sobolev spaces on \( \mathbb{R} \) are defined using the Fourier transform:

\[
\|u\|_{H^\gamma(\mathbb{R}, dx)} := \|\langle \xi \rangle^\gamma \mathcal{F}[u]\|_{L^2(\mathbb{R}, d\xi)}.
\]

We define the following symbol classes:

\[
S^k_{\mathcal{F}} := \left\{ a \in C^\infty(\mathbb{R}^2, \mathbb{C}) \mid \forall k', l' \in \mathbb{N}, \exists C_{k', l'} > 0; \forall (x, \xi) \in \mathbb{R}^2, \ |(\partial_x^k \partial_\xi^l a)(x, \xi)| \leq C_{k', l'} |\langle \xi \rangle^{k-k'}|ight\}.
\]

Then:

\[
S^k_{\mathcal{F}} S^{k'}_{\mathcal{F}} = S^{k+k'}_{\mathcal{F}}, \quad S^k_{\mathcal{F}} \subset S^{k'}_{\mathcal{F}} \quad \text{if} \ k \leq k'.
\]

Let \( \text{Op}_{\mathcal{F}} \) denote the standard Fourier type quantization on \( \mathbb{R} \): for all \( a \in S^k_{\mathcal{F}}, \)

\[
\text{Op}_{\mathcal{F}}[a]u := \mathcal{F}^{-1}[a(x, \bullet)\mathcal{F}[u]] \quad \forall u \in S(\mathbb{R}, \mathbb{C}).
\]

The classes of pseudo-differential operators are defined in the usual way:

\[
\Psi^k_{\mathcal{F}} := \text{Op}_{\mathcal{F}}[S^k_{\mathcal{F}}].
\]

Recall (cf. e.g. [Hö, Chap. XVIII]) that \( \Psi^{k}_{\mathcal{F}} \subset \mathcal{B}(H^\gamma(\mathbb{R}, dx), H^{\gamma-k}(\mathbb{R}, dx)) \) is a filtered algebra:

\[
\Psi^k_{\mathcal{F}} \Psi^{k'}_{\mathcal{F}} = \Psi^{k+k'}_{\mathcal{F}}, \quad \Psi^k_{\mathcal{F}} \subset \Psi^{k'}_{\mathcal{F}} \quad \text{if} \ k \leq k'.
\]

We also recall the following compactness condition:

**Lemma 8.** Let \( a \in S^{k}_{\mathcal{F}}. \) If \( |a(x, \xi)| \rightarrow 0 \) as \( \| (x, \xi) \| \rightarrow +\infty \) (where \( \| \bullet \| \) is any norm in \( \mathbb{R}^2 \)), then \( \text{Op}_{\mathcal{F}}[a] \in \mathcal{B}(H^\gamma(\mathbb{R}, dx)) \) is compact.

**Proof.** When \( \gamma = 0 \) this follows from e.g. [Z, Thm. 4.28]. For the general case, we observe that the symbol \( \langle \xi \rangle^\gamma a(\xi) \langle \xi \rangle^{-\gamma} \) is still decaying so that we can apply the case \( \gamma = 0 \). \( \square \)

2.2.2. Mellin type (or boundary) quantization. The Mellin transform \( \mathcal{M} \in \mathcal{B}\left(L^2\left(\mathbb{R}^*_+, \frac{d\rho}{\rho}\right), L^2(\mathbb{R}, d\xi)\right) \) and its inverse are defined by:

\[
\mathcal{M}[u](\xi) \equiv \tilde{u}(\xi) := \int_{\mathbb{R}^+_+} \rho^{-i\xi} u(\rho) \frac{d\rho}{\rho}, \quad \mathcal{M}^{-1}[v](\rho) = \frac{1}{2\pi} \int_{\mathbb{R}} \rho^{i\xi} v(\xi) d\xi.
\]

Real order Sobolev spaces on \( \mathbb{R}^*_+ \) are defined using the Mellin transform:

\[
\|u\|_{H^\gamma(\mathbb{R}^*_+, \frac{d\rho}{\rho})} := \|\langle \xi \rangle^\gamma \mathcal{M}[u]\|_{L^2(\mathbb{R}, d\xi)}.
\]

The change of variable \( \rho := e^x, x \in \mathbb{R}, \) allows us to transpose properties of Fourier type quantization to Mellin type quantization\(^6\). Setting \( v = u \circ \exp \), we have \( \mathcal{M}[u] = \mathcal{F}[v] \) and \( (\rho \partial_\rho u)(\rho) = (\partial_x v)(x) \); in particular,

\[
(4) \quad u \in H^\gamma_{\mathcal{F}} \iff v \in H^\gamma(\mathbb{R}, dx)
\]

\(^6\)In particular, that \( \mathcal{M} \in \mathcal{B}\left(L^2\left(\mathbb{R}^*_+, \frac{d\rho}{\rho}\right), L^2(\mathbb{R}, d\xi)\right) \) is simply a consequence of Plancherel formula.
Therefore, we define

\[ S^k_M := \{ a \in C^\infty(R_+^* \times R, \mathbb{C}) \mid \forall k', l' \in \mathbb{N}, \exists C_{k', l'} > 0; \forall (\rho, \xi) \in R_+^* \times R, |(\xi^{k'} \rho \rho') a(\rho, \xi)| \leq C_{k', l'} |\xi|^{k-k'} \} \]

so that

\[ S^k_M S^k_M = S^{k+k'}_M, \quad S^k_M \subset S^k_M \quad \text{if} \quad k \leq k', \]

and

\[ \Psi^k_M := Op_M[S^k_M] \]

where \( Op_M \) is the (standard) Mellin-type quantization

\[ Op_M[a] := \mathcal{M}^{-1}[a(\rho, \bullet) M[u]]. \]

The above oscillatory integral can be rewritten as

\[ Op_M[a]u = \mathcal{F}^{-1}[a(exp, \bullet) \mathcal{F}[v]], \]

hence using the standard mapping properties of Fourier type pseudo-differential operators, we see that

\[ \Psi^k_M \subset \mathcal{B}(\mathcal{H}^\gamma_+, \mathcal{H}^{\gamma-k}_+), \]

\[ \Psi^k_M \Psi^k_M = \Psi^{k+k'}_M, \quad \Psi^k_M \subset \Psi^k_M \quad \text{if} \quad k \leq k'. \]

and:

**Lemma 9.** Let \( a \in S^k_M \). If \( |a(\rho, \xi)| \to 0 \) as \( |(\rho + \rho^{-1}, \xi)| \to +\infty \) (where \( \| \cdot \| \) is any norm in \( \mathbb{R}^2 \)), then \( Op_M[a] \in \mathcal{B}(\mathcal{H}^\gamma_+) \) is compact.

On the weighted space \( \mathcal{H}^{\gamma, \beta}_\pm \), we have to use the symbol classes

\[ S^\beta_{k,M} := \{ a \in C^\infty(R_+^* \times R, \mathbb{C}) \mid R_+^* \times R \ni (\rho, \xi) \mapsto a(\rho, \xi - i\beta) \in S^k_M \} \]

so that for all \( u \in \mathcal{H}^{\gamma, \beta}_\pm \),

\[ R^*_+ \ni \rho \mapsto \left( Op^\beta_{M} [a] u \right)(\rho) := \frac{1}{2\pi} \int_{\{\text{Im}(\xi) = -\beta\}} \rho^k a(\rho, \xi) \left( \int_{R^*_+} y^{-i\xi} u(y) \frac{dy}{y} \right) d\xi \]

\[ = \frac{\rho^\beta}{2\pi} \int_{R} \rho^k a(\rho, \xi - i\beta) \left( \int_{R^*_+} y^{-i\xi} (y^{-i\xi} u(y)) \frac{dy}{y} \right) d\xi \]

defines an element in \( \mathcal{H}^{\gamma-k, \beta}_\pm \) when \( a \in S^\beta_{k,M} \), that is \( \Psi^\beta_{k,M} := Op^\beta_{M} [S^\beta_{k,M}] \subset \mathcal{B}(\mathcal{H}^{\gamma, \beta}_+, \mathcal{H}^{\gamma-k, \beta}_+) \). From Lem. 9, we deduce:

**Lemma 10.** Let \( a \in S^\beta_{k,M} \). If \( |a(\rho, \xi)| \to 0 \) as \( |(\rho + \rho^{-1}, \xi)| \to +\infty \) (where \( \| \cdot \| \) is any norm in \( \mathbb{R}^2 \)), then \( Op^\beta_{M} [a] \in \mathcal{B}(\mathcal{H}^{\gamma, \beta}_+) \) is compact.

### 3. Parametrices construction

We use the cut-offs \( \chi_\pm \) introduced in Sect. 2.1 and choose \( \phi_\pm \in C^\infty([-r_-, r_+], [0, 1]) \) such that \( \chi_\pm \phi_\pm = \chi_\pm \).
3.1. **Boundary parametries in b-Sobolev spaces.** We construct local parametries for $\chi_\pm P_\sigma$. For the Mellin-type quantization, $P_\sigma$ has the following (elliptic near, but singular at $r_\pm$) symbols:

$$(\rho_\pm, \xi) \mapsto \frac{\mu_\pm(\rho_\pm)}{\rho_\pm} \xi^2 + i \frac{g_{\sigma,\pm}(\rho_\pm)}{\rho_\pm} \xi + f_{\sigma,\pm}(\rho_\pm).$$

We introduce:

$p_{\sigma,\pm}(\rho_\pm, \xi) := \mu_\pm(\rho_\pm)\xi^2 + i (g_{\sigma,\pm}(\rho_\pm) - 2\mu_\pm(\rho_\pm))\xi + g_{\sigma,\pm}(\rho_\pm) - \mu_\pm(\rho_\pm).$

We use below the quantization $\mathcal{O}_\mathrm{P, M}^{-1/2}$ of symbols involving $(p_{\sigma,\pm}(0, \xi))^{-1}$. We thus will need the following result:

**Lemma 11.** The polynomial function $\zeta \mapsto p_{\sigma,\pm}(0, \zeta + i/2)$ has no real root if and only if

$$\text{Im}(\sigma) \neq -\frac{\kappa_\pm}{2r_\pm} = -\frac{\kappa}{2}.$$

**Proof.** Set $a_\pm + ib_\pm := -(\sigma r_\pm + s)r_\pm - i\kappa_\pm$ with $a_\pm, b_\pm \in \mathbb{R}$. For all $\zeta \in \mathbb{R}$, we compute:

$$p_{\sigma,\pm}(0, \zeta + i/2) = 2\kappa_\pm \zeta^2 + 2(-\sigma r_\pm + s)r_\pm - i\kappa_\pm)\zeta - i(-\sigma r_\pm + s)r_\pm - i\kappa_\pm) + \frac{\kappa_\pm}{2}$$

$$= 2\kappa_\pm \zeta^2 + 2(a_\pm + ib_\pm)\zeta - i(a_\pm + ib_\pm) + \frac{\kappa_\pm}{2}.$$

If $\zeta \in \mathbb{R}$ satisfies $p_{\sigma,\pm}(0, \zeta + i/2) = 0$, then

$$0 = \text{Im}(p_{\sigma,\pm}(0, \zeta + i/2)) = 2b_\pm \zeta - a_\pm.$$

If $b_\pm = 0$ then $a_\pm = 0$ too, that is $\sigma = -sr_\pm^{-1} \pm i\kappa_\pm r_\pm^{-2}$, and we have

$$p_{\sigma,\pm}(0, \zeta + i/2) = 2\kappa_\pm \zeta^2 + \frac{\kappa_\pm}{2}$$

which never cancels. Otherwise, if $b_\pm \neq 0$ then plugging $\zeta = \frac{a_\pm}{2b_\pm}$ in the equation $p_{\sigma,\pm}(0, \zeta + i/2) = 0$ yields:

$$0 = (a_\pm^2 + b_\pm^2)(b_\pm + \kappa_\pm/2).$$

Therefore we find real roots if and only if $b_\pm = -\kappa_\pm/2$, which completes the proof. \qed

We now define

$$Q_{\sigma,\pm} := \mathcal{O}_{\mathrm{P, M}}^{-1/2}[q_{\sigma,\pm}], \quad q_{\sigma,\pm} := q_{\sigma,\pm}^f + q_{\sigma,\pm}^b$$

where

$$q_{\sigma,\pm}^f(\rho_\pm, \xi) := \frac{\rho_\pm}{\mu_\pm(\rho_\pm)(\xi - \frac{1}{2})^2 + \rho_\pm C_\pm}, \quad q_{\sigma,\pm}^b(\rho_\pm, \xi) := \frac{p_{\sigma,\pm}(\rho_\pm, \xi) - \mu_\pm(\xi - \frac{1}{2})^2}{p_{\sigma,\pm}(0, \xi)} q_{\sigma,\pm}^f(\rho_\pm, \xi).$$

Above $\sim$ symbolizes multiplication by $\rho_\pm^{-1}$ (so $q_{\sigma,\pm} := \rho_\pm^{-1} q_{\sigma,\pm}$ and so on) and $C_\pm \in \mathbb{C}\setminus(-\infty, 0]$ are constants that will be fixed in Sect. 4.1. The quantizations $\mathcal{O}_{\mathrm{P, M}}^{-1/2}$ of the symbols $q_{\sigma,\pm}^b$ are well-defined when $\sigma \in \Gamma\setminus(\Gamma_\gamma \cup \Gamma_\gamma)$ since $p_{\sigma,\pm}(0, \xi)$ does not cancel on $\text{Im}(\xi) = 1/2$ by Lem. 11. As we will see in the proof of Lem. 12 below, $q_{\sigma,\pm}^f$ produces decay in the fiber (with respect to $\xi$) while $q_{\sigma,\pm}^b$ brings a correction
to produce decay in both the basis (with respect to $\rho_{\pm}$) and the fiber, hence produces compactness near the boundaries.

We have $q_{\sigma, \pm} \in S_{\mathcal{M}}^{-1; -1/2}$ and thus, following the discussion below Lem. 9, $Q_{\sigma, \pm} := \text{Op}_{\mathcal{M}}^{-1/2}[q_{\sigma, \pm}] \in \mathcal{B}(\mathcal{H}_{\pm}^{-1; -1/2}, \mathcal{H}_{\pm}^{-1, -1/2})$. The operators $Q_{\sigma, \pm} \phi_{\pm}$ are local right parametric families for $P_{\sigma}$.

**Lemma 12.** Set

$$\Gamma_{\pm} := \left\{ \sigma \in \mathbb{C} \ \mid \ \text{Im}(\sigma) = -\frac{\kappa_{\pm}}{2r_{\pm}} \right\}.$$

For all $\gamma \in \mathbb{R}$,

$$K_{\sigma, \pm} := \chi_{\pm}(P_{\sigma}Q_{\sigma, \pm} - 1_{\mathcal{H}^{-1}})\phi_{\pm} \in \mathcal{B}(\mathcal{H}_{\gamma}^{-1})$$

is compact and analytic in $\sigma \in \Gamma \setminus (\Gamma_{-} \cup \Gamma_{+})$.

**Proof.** Let $\tilde{a}$ be a Mellin-type symbol and set $a(\rho_{\pm}, \xi) := \rho_{\pm} \tilde{a}(\rho_{\pm}, \xi)$. Then:

$$P_{\sigma}\text{Op}_{\mathcal{M}}^{-1/2}[a] = \text{Op}_{\mathcal{M}}^{-1/2}[b],$$

$$b = \frac{\mu_{\pm} \xi^{2} + ig_{\sigma, \pm} \xi + \rho_{\pm} f_{\sigma, \pm} a + \rho_{\pm}^{2}(\rho_{\pm} \tilde{c}_{\rho_{\pm}} a) - \frac{\mu_{\pm}}{\rho_{\pm}}((\rho_{\pm} \tilde{c}_{\rho_{\pm}} a)^{2} a)}{\rho_{\pm}^{2}}$$

$$= p_{\sigma, \pm}(\rho_{\pm}, \xi) \tilde{a} + \rho_{\pm} f_{\sigma, \pm} \tilde{a} + (-2i\tilde{M}_{\pm} \xi \rho_{\pm} + g_{\sigma, \pm} - 3\tilde{M}_{\pm})\rho_{\pm}(\tilde{c}_{\rho_{\pm}} \tilde{a}) - \tilde{M}_{\pm} \rho_{\pm}^{2}(\tilde{c}_{\rho_{\pm}}^{2} \tilde{a})$$

$$= p_{\sigma, \pm}(0, 0) \tilde{a} + \rho_{\pm} \left(\frac{p_{\sigma, \pm}(\rho_{\pm}, 0) - p_{\sigma, \pm}(0, 0)}{\rho_{\pm}} + f_{\sigma, \pm}\right) \tilde{a} + (-2i\tilde{M}_{\pm} \xi + g_{\sigma, \pm} - 3\tilde{M}_{\pm})\rho_{\pm}(\tilde{c}_{\rho_{\pm}} \tilde{a}) - \tilde{M}_{\pm} \rho_{\pm}^{2}(\tilde{c}_{\rho_{\pm}}^{2} \tilde{a}).$$

Setting now $\tilde{q}_{\sigma, \pm} := \rho_{\pm}^{-1} q_{\sigma, \pm}$, we compute:

$$P_{\sigma}\text{Op}_{\mathcal{M}}^{-1/2}[\tilde{q}_{\sigma, \pm}^{f}] = 1_{\mathcal{H}^{-1; -1/2}} + \text{Op}_{\mathcal{M}}^{-1/2}[k_{\sigma, \pm}^{f}],$$

$$k_{\sigma, \pm}^{f} = \left(\tilde{p}_{\sigma, \pm}(\rho_{\pm}, 0) - \tilde{M}_{\pm}(\xi - i \frac{1}{2})\right) \tilde{q}_{\sigma, \pm}^{f} + \rho_{\pm} \left(\frac{p_{\sigma, \pm}(\rho_{\pm}, 0) - p_{\sigma, \pm}(0, 0)}{\rho_{\pm}} + f_{\sigma, \pm}\right) \tilde{q}_{\sigma, \pm}^{f} + (-2i\tilde{M}_{\pm} \xi + g_{\sigma, \pm} - 3\tilde{M}_{\pm})\rho_{\pm}(\tilde{c}_{\rho_{\pm}} \tilde{q}_{\sigma, \pm}^{f}) - \tilde{M}_{\pm} \rho_{\pm}^{2}(\tilde{c}_{\rho_{\pm}}^{2} \tilde{q}_{\sigma, \pm}^{f}).$$

Next, setting $\tilde{q}_{\sigma, \pm}^{b} := \rho_{\pm}^{-1} \tilde{q}_{\sigma, \pm}^{b}$, we compute:

$$P_{\sigma}\text{Op}_{\mathcal{M}}^{-1/2}[\tilde{q}_{\sigma, \pm}^{b}] = \text{Op}_{\mathcal{M}}^{-1/2}[k_{\sigma, \pm}^{b}],$$

$$k_{\sigma, \pm}^{b} = -\left(\tilde{p}_{\sigma, \pm}(\rho_{\pm}, 0) - \tilde{M}_{\pm}(\xi - i \frac{1}{2})\right) \tilde{q}_{\sigma, \pm}^{b} + \rho_{\pm} \left(\frac{p_{\sigma, \pm}(\rho_{\pm}, 0) - p_{\sigma, \pm}(0, 0)}{\rho_{\pm}} + f_{\sigma, \pm}\right) \tilde{q}_{\sigma, \pm}^{b} + (-2i\tilde{M}_{\pm} \xi + g_{\sigma, \pm} - 3\tilde{M}_{\pm})\rho_{\pm}(\tilde{c}_{\rho_{\pm}} \tilde{q}_{\sigma, \pm}^{b}) - \tilde{M}_{\pm} \rho_{\pm}^{2}(\tilde{c}_{\rho_{\pm}}^{2} \tilde{q}_{\sigma, \pm}^{b}).$$

The symbol

$$k_{\sigma, \pm} := k_{\sigma, \pm}^{f} + k_{\sigma, \pm}^{b} = \rho_{\pm}(f_{\sigma, \pm} - C_{\pm}) \tilde{q}_{\sigma, \pm}^{f} + \rho_{\pm} \left(\frac{p_{\sigma, \pm}(\rho_{\pm}, 0) - p_{\sigma, \pm}(0, 0)}{\rho_{\pm}} + f_{\sigma, \pm}\right) \tilde{q}_{\sigma, \pm}^{b} + (-2i\tilde{M}_{\pm} \xi + g_{\sigma, \pm} - 3\tilde{M}_{\pm})\rho_{\pm}(\tilde{c}_{\rho_{\pm}} \tilde{q}_{\sigma, \pm}) - \tilde{M}_{\pm} \rho_{\pm}^{2}(\tilde{c}_{\rho_{\pm}}^{2} \tilde{q}_{\sigma, \pm}).$$
verifies $k_{\sigma, \pm} \in \rho_+ S_{\Psi M}^{-1/2}$ since $p_{\sigma, \pm}(0, \xi) \neq 0$ and is clearly analytic in $\sigma \in \Gamma \setminus (\Gamma_- \cup \Gamma_+)$. As a result, $K_{\sigma, \pm} = \chi_{\pm} \text{Op}_M^{-1/2}[k_{\sigma, \pm}] \phi_\pm \in \Psi_{\Psi M}^{-2-1/2}$ and thus, using that $\chi_\pm : \mathcal{H}_{\mp}^{-1-1/2} \hookrightarrow \mathcal{H}_{\mp}^{-1}$, we find $K_{\sigma, \pm} \in \mathcal{B}(\mathcal{H}^{-1})$. Finally, compactness is a direct consequence of Lem. 10. □.

Remark 13. Lem. 12 holds true with $\mathcal{B}(\mathcal{H}^{-1})$ replace by $\mathcal{B}(\mathcal{H}^{-2})$ if instead we set $q_{\sigma, \pm}^f := q_{\sigma, \pm}^{f, 0} + q_{\sigma, \pm}^{f, 1} + q_{\sigma, \pm}^{f, 2}$ where

$$q_{\sigma, \pm}^{f, 0}(\rho, \xi) := \frac{\rho_+}{\mu_+(\rho, \xi) - \xi^2 + \rho_+^2 C_\pm},$$

$$q_{\sigma, \pm}^{f, 1}(\rho, \xi) := -\frac{\rho_+^2 (\mu_+(\rho, \xi) - \xi^2 + \rho_+^2 C_\pm)}{\mu_+(\rho, \xi) - \frac{1}{2} \xi^2 + \rho_+^2 C_\pm} q_{\sigma, \pm}^{f, 0}(\rho, \xi),$$

$$q_{\sigma, \pm}^{f, 2}(\rho, \xi) := -\frac{2\rho_+ \mu_+(\rho, \xi)}{\mu_+(\rho, \xi) - \frac{1}{2} \xi^2 + \rho_+^2 C_\pm} \rho_+ (\xi - \frac{1}{2} \xi) q_{\sigma, \pm}^{f, 0}(\rho, \xi),$$

$$q_{\sigma, \pm}^{b, 0}(\rho, \xi) := \frac{\rho_+ (\mu_+(\rho, \xi) - \xi^2 + \rho_+^2 C_\pm)}{\rho_+ (\rho, \xi) - \frac{1}{2} \xi^2 + \rho_+^2 C_\pm} q_{\sigma, \pm}^{f, 1}(\rho, \xi),$$

with $0 \leq \alpha < 4/13$. Taking $\alpha < 4/13$ is necessary for Prop. 16 below.

3.2. Boundary parametrices in standard Sobolev spaces. We turn to the construction of local parametrices for $\chi_{\pm} P_{\sigma}$ in standard Sobolev spaces. This will allow us to go beyond the strip $\{\text{Im}(\sigma) = -\kappa\}$ at the cost of using more regular functions and losing one derivative.

First of all, we revisit Lem. 11:

Lemma 14. Let $\gamma \in \mathbb{R} \neq \{-1/2\}$. The polynomial function $\zeta \mapsto p_{\sigma, \pm}(0, \zeta - i(\gamma - 1/2))$ has no real root if and only if

$$\text{Im}(\sigma) \neq -(\gamma + 1/2) \frac{\kappa_+}{r_+} = -(\gamma + 1/2)\kappa.$$

Proof. Set again $a_\pm + ib_\pm := -\sigma r_\pm + s r_\pm = i\kappa_\pm$ with $a_\pm, b_\pm \in \mathbb{R}$. For all $\zeta \in \mathbb{R}$, we compute:

$$p_{\sigma, \pm}(0, \zeta + i(\gamma - 1/2)) = 2\kappa_\pm \zeta^2 + 2(a_\pm + ib_\pm - 2i\kappa_\pm \gamma) \zeta - 2i(a_\pm + ib_\pm)(\gamma + 1/2) - 2\kappa_\pm (\gamma + 1/2)(\gamma - 1/2).$$

If $\zeta \in \mathbb{R}$ satisfies $p_{\sigma, \pm}(0, \zeta + i(\gamma - 1/2)) = 0$, then

$$0 = \text{Im}(p_{\sigma, \pm}(0, \zeta + i(\gamma - 1/2))) = 2(b_\pm - 2\kappa_\pm \gamma) \zeta - 2a_\pm (\gamma + 1/2).$$

If $b_\pm = 2\kappa_\pm \gamma$ then $a_\pm (\gamma + 1/2) = 0$ i.e. $a_\pm = 0$ (as $\gamma \neq -1/2$), and we have:

$$p_{\sigma, \pm}(0, \zeta + i(\gamma - 1/2)) = 2\kappa_\pm (\zeta^2 + (\gamma + 1/2)^2) > 0.$$

Otherwise, if $b_\pm \neq 2\kappa_\pm \gamma$ then plugging $\zeta = a_\pm (\gamma + 1/2)/(b_\pm - 2\kappa_\pm \gamma)$ in the equation $p_{\sigma, \pm}(0, \zeta + i(\gamma - 1/2)) = 0$ yields:

$$0 = (a_\pm^2 + (b_\pm - 2\kappa_\pm \gamma)^2)(b_\pm - (\gamma - 1/2)2\kappa_\pm /2).$$

Therefore we find real roots if and only if $b_\pm = (\gamma - 1/2)\kappa_\pm$, which completes the proof. □.
Fix now $\gamma \in \mathbb{N}$ such that $\gamma > 1/2$ and denote by $(z)_{(w)} := z(z-1)\ldots(z-w+1)$ the falling factorial\(^7\). We define
\[
Q_{\sigma,\pm,\gamma} : H_{\pm}^{\gamma-1/2} \ni u \longmapsto \left( \rho_{\pm} \mapsto \text{Op}_M \left[ \rho_{\pm}^{-1/2} \frac{q_{\sigma,\pm,\gamma}(\rho_{\pm},\xi)}{(i\xi + \gamma + 1/2)(\gamma)} \right] (y \mapsto y^{1/2}(\partial_y^\gamma u)) \right).
\]
where $q_{\sigma,\pm,\gamma} := q_{\sigma,\pm,\gamma}^b + q_{\sigma,\pm,\gamma}^b$ and
\[
q_{\sigma,\pm,\gamma}^b (\rho_{\pm}, \xi) := \begin{cases}
\frac{\rho_{\pm}}{\rho_{\pm}^2 + \rho_{\pm} C_{\pm}^\gamma}, & \text{if } \gamma \neq 1/2, \\
\frac{\rho_{\pm}}{\rho_{\pm}(0, \xi - i(\gamma - 1/2)) - \tilde{\mu}_{\pm}(\rho_{\pm}, \xi)} (\partial_{\rho_{\pm}} q_{\sigma,\pm,\gamma}^f + \tilde{q}_{\sigma,\pm,\gamma}^f) (\rho_{\pm}, \xi) & \text{if } \gamma = 1/2,
\end{cases}
\]
\[
q_{\sigma,\pm,\gamma}^b (\rho_{\pm}, \xi) := -\frac{\rho_{\pm} (p_{\sigma,\pm}(\rho_{\pm}, \xi - i(\gamma - 1/2)) - p_{\sigma,\pm}(0, \xi - i(\gamma - 1/2))) + f_{\sigma,\pm})}{p_{\sigma,\pm}(0, \xi - i(\gamma - 1/2))}.
\]
\[
k_{\sigma,\pm,\gamma}^b := \rho_{\pm} (f_{\sigma,\pm} - C_{\pm}^\gamma \tilde{q}_{\sigma,\pm,\gamma}^f + \rho_{\pm} (p_{\sigma,\pm}(\rho_{\pm}, \xi - i(\gamma - 1/2)) - p_{\sigma,\pm}(0, \xi - i(\gamma - 1/2)) + f_{\sigma,\pm}) \tilde{q}_{\sigma,\pm,\gamma}^f + (-2i\tilde{\mu}_{\pm}(\xi - i(\gamma - 1/2)) + g_{\sigma,\pm} - 3\tilde{\mu}_{\pm}) \rho_{\pm} (\partial_{\rho_{\pm}} \tilde{q}_{\sigma,\pm,\gamma}^f + \tilde{\rho}_{\pm} \tilde{q}_{\sigma,\pm,\gamma}^{b,0}) - \tilde{\mu}_{\pm} \rho_{\pm}^2 (\partial_{\rho_{\pm}}^{2} \tilde{q}_{\sigma,\pm,\gamma}^f + \tilde{\rho}_{\pm}^{2} \tilde{q}_{\sigma,\pm,\gamma}^{b,0}).
\]
Above $\tilde{\gamma}$ symbolizes multiplication by $\rho_{\pm}^{-1}$ (so $q_{\sigma,\pm,\gamma}^{b,0} := \rho_{\pm} q_{\sigma,\pm,\gamma}^b$ and so on) and $C_{\pm} \in \mathbb{C} \setminus (-\infty, 0]$ are constants independent of those in Sect. 3.1; the symbols are defined as Sect. 3.1 but with $\xi - i(\gamma - 1/2)$ instead of $\xi + i/2$. The symbol $q_{\sigma,\pm,\gamma}^{b,0}$ is needed to increase by 1 the overall order of $\rho_{\pm}$ of the error term; this will be needed for compactness, see the end of the proof of Lem. 15 below. The operators $Q_{\sigma,\pm,\gamma} \phi_{\pm}$ are local right parametrices for $P_{\sigma}$.

**Lemma 15.** Let $\gamma \in \mathbb{N}\setminus\{0\}$. Set
\[
\Gamma_{\pm} := \left\{ \sigma \in \mathbb{C} \mid \text{Im}(\sigma) = -(\gamma - 1/2) \frac{\kappa_{\pm}}{i_{\pm}} \right\}.
\]
Then $Q_{\sigma,\pm,\gamma-1} \in B(H_{\pm}^{\gamma-1/2}, X_{\gamma})$ and
\[
K_{\sigma,\pm,\gamma-1} := \chi_{\pm} \left( P_{\sigma} Q_{\sigma,\pm,\gamma-1} - 1_{H_{\gamma-1}} \right) \phi_{\pm} \in B(H_{\gamma-1})
\]
is compact and analytic in $\sigma \in \Gamma_{\pm} \cup \bar{\Gamma}_{\pm}$.

**Proof.** That $Q_{\sigma,\pm,\gamma-1} \in B(H_{\pm}^{\gamma-1/2}, H_{\gamma-1})$ follows from the fact that taking $\gamma$ derivatives of $Q_{\sigma,\pm,\gamma-1} u$ with $u \in H_{\gamma-1/2}$ produces a term $\rho_{\pm}^{-1/2}(i\xi + \gamma - 1/2)(\gamma)$ in the Mellin quantization; we then observe that
\[
\left\| \text{Op}_M \left[ \rho_{\pm}^{-1/2}(i\xi + \gamma - 1/2)(\gamma) \frac{q_{\sigma,\pm,\gamma-1}(\rho_{\pm}, \xi)}{(i\xi + \gamma - 1/2)(\gamma-1)} \right] (y \mapsto y^{1/2}(\partial_y^{\gamma-1} u)) \right\|_{H_{\pm}^{\gamma-1/2}} \lesssim \| \rho_{\pm}^{1/2}(\partial_y^{\gamma-1} u) \|_{H_{\pm}^{\gamma-1/2}}.
\]

The above estimate is easily obtained using standard results of Fourier type pseudo-differential calculus for symbols of order $-1 \leq 0$ in $\xi$ (cf. Sect. 2.2.1) with the change of variable $\rho_{\pm} = e^x$, $x \in \mathbb{R}$, together with the relation $\| u \|_{H_{\pm}} = \| x \mapsto u(e^x) \|_{L^2(\mathbb{R}, dx)}$, as well as the fact that $q_{\sigma,\pm,\gamma-1}$ contains a factor $\rho_{\pm}$.

---

\(^7\)The falling factorial can be defined for complex $z$ and real $w$ such that $z, z - w$ are not integer using Euler’s $\Gamma$ function: $(z)_{(w)} := \frac{\Gamma(z+1)}{\Gamma(z-w+1)}$. 
Notice that we have to lose one derivative because of the weight $\rho_{\pm}$: indeed, taking $\gamma + 1$ derivatives would produce a symbol of order 0 in $\xi$ but of order $-1$ in $\rho_{\pm}$ and therefore would decrease by a factor 1 the weight of the target space.

Next, let $R_{\gamma-1/2}$ denote the restriction to $\{\text{Im}(\xi) = -i(\gamma - 1/2)\}$. Then:

$$P_{\sigma}Q_{\sigma,\pm,\gamma-1} u = \text{Op}_{\mathcal{M}}[k_{\sigma,\pm,\gamma-1}^{\prime\prime}](y \mapsto y^{1/2}(\partial_y^{-1}) u)$$

$$k_{\sigma,\pm,\gamma-1}^{\prime\prime} = \rho_{\pm}^{\gamma-1/2}$$

$$\times R_{\gamma-1}^{1/2} \left\{ \frac{\hat{\mu}_{\pm} \xi^2 + ig_{\sigma,\pm} \xi + \rho_{\pm} f_{\sigma,\pm}}{\rho_{\pm}} + \frac{-2i \hat{\mu}_{\pm} \xi + g_{\sigma,\pm}(\rho_{\pm} \partial_{\rho_{\pm}}) - \hat{\mu}_{\pm}(\rho_{\pm} \partial_{\rho_{\pm}})^2}{\rho_{\pm}} \right\} \frac{q_{\sigma,\pm,\gamma-1}}{(i \xi + \gamma - 1 + 1/2)(\gamma-1)}.$$  

The same computations as in the proof of Lem. 12 shows that $k_{\sigma,\pm,\gamma-1}^{\prime\prime} = k_{\sigma,\pm,\gamma-1}^{\prime} + k_{\sigma,\pm,\gamma-1}$ with ($\xi \in \mathbb{R}$):

$$k_{\sigma,\pm,\gamma-1}^{\prime} = \frac{\rho_{\pm}^{\gamma-1/2}}{(i \xi + (\gamma - 1 + 1/2)(\gamma-1))}$$

$$k_{\sigma,\pm,\gamma-1}^{\prime} = \frac{\rho_{\pm}^{\gamma-1/2}}{(i \xi + (\gamma - 1 + 1/2)(\gamma-1))} \left[ \rho_{\pm} \left( \frac{p_{\sigma,\pm}(\rho_{\pm} \xi - i(\gamma - 1/2)) - p_{\sigma,\pm}(0, \xi - i(\gamma - 1/2))}{\rho_{\pm}} + f_{\sigma,\pm} \right) q_{\sigma,\pm,\gamma-1}^{h,1} \right.$$  

$$+ (-2i \hat{\mu}_{\pm}(\xi - i(\gamma - 1/2)) + g_{\sigma,\pm} - 3 \hat{\mu}_{\pm}) \rho_{\pm} (\partial_{\rho_{\pm}} q_{\sigma,\pm,\gamma-1}^{h,1}) - \hat{\mu}_{\pm} \rho_{\pm}^2 (\partial_{\rho_{\pm}} q_{\sigma,\pm,\gamma-1}^{h,1}) \right].$$

If we momentarily consider $u \in C_0^\infty(\mathbb{R}_+, \mathbb{C})$, then notice that

$$\xi \mapsto \frac{1}{(i \xi + (\gamma - 1 + 1/2)(\gamma-1))} \left( \int_{\mathbb{R}_+^*} \frac{y^{-i \xi + 1} y^{1/2} \partial_y^{-1} u(y) \, dy}{y} \right)$$

is analytic in $\mathbb{R} + i[0, \gamma - 1]$ (smoothness and decay at $+\infty$ of $u$ allows us to verify Cauchy-Riemann equations under the integral). By Cauchy theorem, we can shift the integration strip to $\{\text{Im}(\xi) = \gamma - 1\}$ without changing the value of the integral. Integrating by parts in the integral in $y$, we thus have in the $L^2(\mathbb{R}, d\xi)$ sense:

$$\int_{\mathbb{R}} \rho_{\pm}^{i \xi + \gamma - 1 - 1/2} (i \xi + \gamma - 1 + 1/2)(\gamma-1) \left( \int_{\mathbb{R}_+^*} \frac{y^{-i \xi} y^{1/2} \partial_y^{-1} u(y) \, dy}{y} \right) d\xi$$

$$= \int_{\mathbb{R}} \rho_{\pm}^{i \xi - 1/2} (i \xi + 1/2)(\gamma-1) \left( \int_{\mathbb{R}_+^*} \frac{y^{-i \xi + \gamma - 1} y^{1/2} \partial_y^{-1} u(y) \, dy}{y} \right) d\xi$$

$$= \int_{\mathbb{R}} \rho_{\pm}^{i \xi - 1/2} (i \xi + 1/2)(\gamma-1) \left( \int_{\mathbb{R}_+^*} \frac{-1 \gamma^{-1} (\partial_y y^{-i \xi + \gamma - 1/2} u(y) \, dy}{y} \right) d\xi$$

$$= \int_{\mathbb{R}} \rho_{\pm}^{i \xi - 1/2} (i \xi + 1/2)(\gamma-1) \left( \int_{\mathbb{R}_+^*} \frac{(i \xi + 1/2)(\gamma-1) y^{-i \xi} y^{1/2} u(y) \, dy}{y} \right) d\xi$$

$$= 2\pi \text{Op}_{\mathcal{M}}^{-1/2} [1](u)$$

$$= 2\pi u.$$
By continuity in $\rho_\pm^{1/2} \partial_y^{-1} u$ of the above operator and density of $C_0^\infty(\mathbb{R}_+, \mathbb{C})$ in $H_{\pm}^{\gamma - 1}$ (cf. Lem. 7), we find that $\text{Op}_M[k_{\sigma, \pm, \gamma - 1}](y \mapsto y^{1/2}(\partial_y u)) = u$ for all $u \in H_{\pm}^{\gamma - 1, -1/2}$. We show that $\text{Op}_M[k_{\sigma, \pm, \gamma - 1}] \in \mathcal{B}(H_{\pm}^{\gamma - 1, -1/2}, H_{\pm}^{\gamma - 1/2})$ in the same way as we did above for $Q_{\sigma, \pm, \gamma - 1}$.

Finally, all the symbols above are analytic in $\sigma \in \Gamma \setminus (\tilde{\Gamma}_- \cup \tilde{\Gamma}_+)$ and the embeddings $\chi_\pm : H_{\pm}^{\gamma - 1, -1/2} \hookrightarrow H^{\gamma - 1}$ imply that $\Gamma \setminus (\tilde{\Gamma}_- \cup \tilde{\Gamma}_+) \ni \sigma \mapsto K_{\sigma, \pm, \gamma - 1} \in \mathcal{B}(H^{\gamma - 1})$ is analytic. As for compactness, it follows from Lem. 10 with $k = 0$: given a bounded sequence $(u_i)_{i \in \mathbb{N}}$ in $H_{\gamma - 1}$, we start by extracting a subsequence $(u_{i_0})_{i_0 \in \mathbb{N}}$ such that

$$K_{\sigma, \pm, \gamma - 1} u_{i_0} = \chi_\pm \text{Op}_M[k_{\sigma, \pm, \gamma - 1}](y \mapsto y^{1/2}(\partial_y^{-1}(\phi_\pm u_{i_0}))(y))$$

converges in $H$, then take $\gamma - 1$ derivatives and extract from $(u_{i_0})_{i_0 \in \mathbb{N}}$ a subsequence $(u_{i_1})_{i_1 \in \mathbb{N}}$ such that

$$\partial_y^{-1} K_{\sigma, \pm, \gamma - 1} u_{i_1} = \chi_\pm \text{Op}_M[\hat{k}_{\sigma, \pm, \gamma - 1}](y \mapsto y^{1/2}(\partial_y^{-1}(\phi_\pm u_{i_1}))(y))$$

converges in $H$ – this is possible because $\hat{k}$ is of order 1 in $\rho_\pm$ thanks to the correction of $q_{b, h, \gamma - 1}$. □

3.3. Global parametrix. We now construct a global right parametrix for the different realizations $(P_{\sigma}, B(\mathcal{H}^{\gamma - 1}, \mathcal{H}^{\gamma - 2}))$, $(P_{\sigma}, B(\mathcal{H}, \mathcal{H}^{\gamma - 2}))$ and $(P_{\sigma}, X^{\gamma})$ of the operator $P_{\sigma}$ by gluing together the local parametrices constructed above.

Let $\gamma \in \mathbb{R}$ and $\sigma \in \Gamma \setminus (\Gamma_- \cup \Gamma_+)$, $\Gamma_\pm$ as in Lem. 12, and define

$$Q_{\sigma} := \sum_\pm \chi_\pm Q_{\sigma, \pm} \phi_\pm \in \mathcal{B}(\mathcal{H}^{\gamma - 1}, X^{\gamma}).$$

Then:

$$P_{\sigma} Q_{\sigma} = \sum_\pm [P_{\sigma}, \chi_\pm] Q_{\sigma, \pm} \phi_\pm + \sum_\pm \chi_\pm P_{\sigma} Q_{\sigma, \pm} \phi_\pm$$

$$= \sum_\pm [P_{\sigma}, \chi_\pm] Q_{\sigma, \pm} \phi_\pm + \sum_\pm (\chi_\pm \phi_\pm + K_{\sigma, \pm})$$

$$= 1_{\mathcal{H}^{\gamma - 1}} + \sum_\pm \left((P_{\sigma}, \chi_\pm) Q_{\sigma, \pm} \phi_\pm + K_{\sigma, \pm}\right)$$

$$=: 1_{\mathcal{H}^{\gamma - 1}} + K_{\sigma}.$$ 

Since $\text{Supp}([P_{\sigma}, \chi_\pm]) \subseteq (r_-, r_+)$ and $q_{\sigma, \pm} \in S_{\mathcal{M}}^{-2}$, the operator $[P_{\sigma}, \chi_\pm] Q_{\sigma, \pm} \phi_\pm \in \Psi_{\mathcal{M}}^{-1}$ is compact; furthermore, we have $k_{\sigma, \pm} \in S_{\mathcal{M}}^{-1}$. As a result, the operator $K_{\sigma} \in \mathcal{B}(\mathcal{H}^{\gamma - 1})$ is compact and analytic in $\sigma \in \Gamma \setminus (\Gamma_- \cup \Gamma_+)$ as the sum of compact operators which are analytic. For the realization $(P_{\sigma}, B(\mathcal{H}^{\gamma - 1}, \mathcal{H}^{\gamma - 2}))$, we can use Rmk. 13.

In the same way, for $\gamma \in \mathbb{N} \setminus \{0\}$, we check that

$$Q_{\sigma, \gamma - 1} := \sum_\pm \chi_\pm Q_{\sigma, \pm, \gamma - 1} \phi_\pm \in \mathcal{B}(H^{\gamma - 1}, X^{\gamma})$$

is such that

$$P_{\sigma} Q_{\sigma} = 1_{H^{\gamma - 1}} + \sum_\pm \left((P_{\sigma}, \chi_\pm) Q_{\sigma, \pm, \gamma - 1} \phi_\pm + K_{\sigma, \pm, \gamma - 1}\right) =: 1_{H^{\gamma - 1}} + K_{\sigma, \gamma - 1}$$

where $K_{\sigma, \gamma - 1} \in \mathcal{B}(H^{\gamma - 1})$ is compact and analytic in $\sigma \in \Gamma \setminus (\tilde{\Gamma}_- \cup \tilde{\Gamma}_+)$, $\tilde{\Gamma}_\pm$ as in Lem. 15.
4. Proof of the main results

In this section, we use the parametrices $Q_\sigma$ and $Q_{\sigma,\gamma-1}$ from Sect. 3.3 to localize resonances in $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -\kappa/2\}$ and $\Gamma \cap \{z \in \mathbb{C} \mid \text{Im}(z) > -(\gamma - 1/2)\kappa\}$.

4.1. The Fredholm property. Let $\gamma \in \mathbb{R}$. We first show that $P_\sigma : \mathcal{H}_+ \to \mathcal{H}_-^{-1}$ is a family of index 0 Fredholm operators by inverting $Q_\sigma : \mathcal{H}_-^{-1} \to \mathcal{H}_+$. Recall the constants $C_\pm$ from Sect. 3.1 and the sets $\Gamma_\pm$ introduced in Lem. 12.

Proposition 16. Let $\sigma \in \Gamma \cap (\Gamma_+ \cup \Gamma_-^c)$ where $\Gamma_\pm^c$ is a $\varepsilon$-neighborhood of $\Gamma_\pm$. There exists $C > 0$ depending on $\gamma, \sigma,\alpha, \beta, \alpha', \beta'$ such that, if $C_- = C_+ > C$, then $Q_\sigma \in \mathcal{B}(\mathcal{H}_-^{-1}, \mathcal{H}_+)$ is invertible.

Proof. We first go back to the proof of Lem. 12: we can check that

$$
\chi_\pm([P_\sigma + \rho_\pm C_\pm] Q_{\sigma,\pm}) \phi_\pm = \chi_\pm + \chi_\pm \text{Op}_{\mathcal{M}}^{-1/2}[k'_\sigma,\pm] \phi_\pm
$$

where (recall that $\tilde{q}_{\sigma,\pm} := \rho_\pm^{-1} q_{\sigma,\pm}$)

$$
k'_\sigma,\pm = \rho_\pm f_{\sigma,\pm} \tilde{q}_{\sigma,\pm} + \rho_\pm \left( \frac{\rho_\pm}{p_{\sigma,\pm}(0,\xi)} \right) + f_{\sigma,\pm} \tilde{q}_{\sigma,\pm} + (g_{\sigma,\pm} - 3\tilde{\mu}_\pm) \rho_\pm (\tilde{\rho}_\pm \tilde{q}_{\sigma,\pm})
$$

$$
- 2i\tilde{\mu}_\pm \rho_\pm (\tilde{\rho}_\pm \tilde{q}_{\sigma,\pm} + \tilde{\rho}_\pm \tilde{q}_{\sigma,\pm} + \tilde{\rho}_\pm \tilde{q}_{\sigma,\pm}) - \tilde{\mu}_\pm \rho_\pm^2 (\tilde{\rho}_\pm \tilde{q}_{\sigma,\pm}).
$$

As

$$
\|\chi_\pm u\|^2_{\mathcal{H}_-^{-1},-1/2} \leq \sum \|\chi_\pm u\|^2_{\mathcal{H}_-^{-1},-1/2} = \|u\|^2_{\mathcal{H}_-^{-1}},
$$

we can use Lem. 23 and find a bound for $\|\text{Op}_{\mathcal{M}}^{-1/2}[k''_{\sigma,\pm}]\|_{\mathcal{B}(\mathcal{H}_-^{-1},-1/2)}$ in terms of $\|\tilde{q}_{\sigma,\pm}\|_{L^2(\mathbb{R} \times \mathbb{R}, \text{d}x \text{d}x)}$ with

$$
0 \leq |\gamma - 1| \quad \text{and} \quad k''_{\sigma,\pm}(x,\xi) := k'_\sigma,\pm(e^x, \xi + i/2).\quad \text{The latter norm can be bounded by sums and products of } L^\infty(\text{Supp}(\chi_\pm)) \quad \text{norms of } \tilde{\mu}_\pm, f_{\sigma,\pm}, g_{\sigma,\pm} \quad \text{and their derivatives, as well as by terms of the form}
$$

$$
\int_{\mathbb{R}} \left( \rho_\pm \tilde{\rho}_\pm \right)^j \left( \frac{\rho_\pm}{(\tilde{\mu}_\pm \xi^2 + \rho_\pm C_\pm)} \right) \frac{1}{p_{\sigma,\pm}(0,\xi + i/2)} ^{2j} \text{d}x,
$$

with

$$
(\rho_\pm \tilde{\rho}_\pm)^j \left( \frac{\rho_\pm}{(\tilde{\mu}_\pm \xi^2 + \rho_\pm C_\pm)} \right) = \frac{\rho_\pm}{(\tilde{\mu}_\pm \xi^2 + \rho_\pm C_\pm)} + \sum \frac{\rho_\pm^{1+k}(c_{j,k}(\rho_\pm) C_k + c_{j,k}'(\rho_\pm) C_k)}{(\tilde{\mu}_\pm \xi^2 + \rho_\pm C_\pm)^{1+k}}
$$

where $c_{j,k}, c_{j,k}' = \mathcal{O}(1)$ are smooth functions of derivatives of $\tilde{\mu}_\pm$ and of $\rho_\pm$. Let us write:

$$
p_{\sigma,\pm}(0,\xi + i/2) = a_2 \xi^2 + a_1 \xi + a_0, \quad a_0, a_1, a_2 \in \mathbb{C}, \quad a_0 = 2\kappa > 0,
$$

$$
\Delta := a_2^2 - 4a_0a_2 = A + iB, \quad A, B \in \mathbb{R}.
$$

A direct computation shows that

$$
\Delta = (\alpha + i\beta)^2, \quad \alpha = \sqrt{\frac{A + |\Delta|}{2}}, \quad \beta = \text{sign}(B) \sqrt{-\frac{A + |\Delta|}{2}}.
$$

If we set $a_1 := a_1 + ia_2'$ with $a_1', a_2' \in \mathbb{R}$, then:

$$
p_{\sigma,\pm}(0,\xi + i/2) = 2\kappa \xi (\xi - c_2)
$$
We similarly check that
\[
\begin{align*}
    c_1 &:= -\frac{a_1'}{4\kappa_\pm} - i\frac{a_1''}{4\kappa_\pm} + \alpha, \\
    c_2 &:= -\frac{a_1'}{4\kappa_\pm} - i\frac{a_1''}{4\kappa_\pm} - \beta.
\end{align*}
\]

We are led to the following computation: for all \(0 \leq k' \leq k\),
\[
(8)
\]
\[
\int \mathbb{R} \left( \frac{\rho^2 \xi^{4(k-k')}}{\tilde{\mu}_\pm(\rho)\xi^2 + \rho \pm C^{2(k+1)}_\pm} \right) |p_{\sigma,\pm}(0, \xi + i/2)|^{2l} d\xi
\]
\[
= \frac{2\pi i \rho^{2(k+1)}}{\tilde{\mu}_\pm(\rho)\xi^2 + \rho \pm C^{2(k+1)}_\pm} \xi^{4(k-k')} \left( \xi \mapsto \frac{\xi^{4(k-k')}}{\left( \xi + i\sqrt{\frac{\rho \pm C^{2(k+1)}}{\tilde{\mu}_\pm(\rho)}} \right) |p_{\sigma,\pm}(0, \xi + i/2)|^{2l} \right)_{\xi = c}
\]
\[
+ \frac{2\pi i \rho^{2(k+1)} \delta_{l,1}}{2\kappa_\pm} \sum_{c \in \{c_1, c_2, c_3, c_4\}} \prod_{\text{Im}(c') > 0} \left( \xi \mapsto \frac{\xi^{4(k-k')}}{(\tilde{\mu}_\pm(\rho)\xi^2 + \rho \pm C^{2(k+1)}_\pm)} \prod_{c' \in \{c_1, c_2, c_3, c_4\}} (\xi - c') \right)_{\xi = c}.
\]

The total order in \(\xi\) of the first term on the right hand side is \((4(k-k') - 2(k+1)) - (2(k+1) - 1) = -3 - 4k'\) so that evaluation at \(i\sqrt{\frac{\rho \pm C^{2(k+1)}_\pm}{\tilde{\mu}_\pm(\rho)}}\) yields a factor \(\rho^{3/2 - 2k'} C^{-3/2 - 2k'}\), which is in \(L^1\left(\mathbb{R}_+, \frac{d\rho}{\rho}\right)\) after multiplication by \(\rho^{2(k+1)}\); the second term on the right hand side above is harmless as it does not produce any powers of \(\rho^{-1}\). We get:
\[
(9)
\]
\[
\|\text{Op}_x[k'_{\sigma,\pm}]\|_{B(\mathcal{H}^{-1}, -1/2)} = O(C^{-3/2}_\pm).
\]

We similarly check that \(\|[P_{\sigma}, \chi_\pm]Q_{\sigma,\pm} \phi_\pm\|_{B(\mathcal{H}^{-1})}\) is of order \(C^{-3/2}_\pm\).

We now put \(C_- = C_+\). Reproducing the computation of Sect. 3.3, we find
\[
(10)
\]
\[
(P_{\sigma} + \rho_- C_-)Q_{\sigma} = \mathbb{1}_{\mathcal{H}^{-1}} + \sum_{\pm} \left( [P_{\sigma}, \chi_\pm]Q_{\sigma,\pm} \phi_\pm + K'_{\sigma,\pm} \right).
\]
where
\[
K'_{\sigma,\pm} = \chi_\pm \text{Op}_M^{-1/2}[k'_{\sigma,\pm}] \phi_\pm.
\]

Combining (9) together with the remark below this equations, we see that the right hand side of (10) is invertible if \(C^{-1/2}_\pm\) is sufficiently large, depending on all the parameters \(\gamma, s, m, \ell, \sigma_0, \sigma'', \chi_\pm\) as well as the distance to \(\mathbb{R}\) of the closest root of \(\mathbb{R} \ni \xi \mapsto p_{\sigma,\pm}(0, \xi + i/2)\) (depending itself on the distance to \(\Gamma_\pm\) of \(\sigma\)). This shows that \(Q_{\sigma} : \mathcal{H}^{-1} \rightarrow \mathcal{H}\) has a trivial kernel.

We similarly prove that \(Q_{\sigma} : \mathcal{H}^{-1} \rightarrow \mathcal{H}\) is onto from the equation
\[
Q_{\sigma}(P_{\sigma} + \rho_- C_-) = \mathbb{1}_{\mathcal{H}} + K_{\sigma}.
\]
for some $\tilde{K}_\sigma \in \Psi^{-1}_{\lambda_\mathcal{M}}$ with norm of order $C^{3/2}_\lambda$. This is achieved by carrying out the same computations as above, using the appropriate modifications in the proof of Lem. 12, we omit the details here. □

**Remark 17.** The same proof show that the parametrix $Q_\sigma$ in Rmk. 13 is invertible as an element of $\mathcal{B}(\mathcal{H}^{-2}, \mathcal{H}^\gamma)$. In (8), we have to replace $\rho_\pm$ by $\rho_\pm^b$ and have to consider in plus integrals in $\xi$ of the suqared moduli of terms of the form

\[
\frac{\rho_\pm^2 \xi^3}{(\bar{\mu}_\pm \xi^2 + \rho_\pm^2 C_\pm^2)^3}, \quad \frac{\rho_\pm^{1-\alpha} \xi C_\pm}{(\bar{\mu}_\pm \xi^2 + \rho_\pm^2 C_\pm^2)^3}
\]

as well as of their b-derivatives $\rho_\pm \bar{\mu}_\pm$ (they come from $q_{\sigma, \pm}^{f_1}$ and $q_{\sigma, \pm}^{f_2}$ in Rmk. 13). The overall bound now is of order $\rho_\pm^{2-13\alpha/2} C_\pm^{-3/2}$ and is thus in $L^1 \left( \mathbb{R}^+; \frac{d\rho}{\rho_\pm} \right)$ if $\alpha < 4/13$.

**Corollary 18.** For all $\gamma \in \mathbb{R}$,

\[
\Gamma \setminus (\Gamma_- \cup \Gamma_+) \ni \sigma \mapsto P_\sigma \in \mathcal{B}(X^\gamma, \mathcal{H}^{\gamma-1}), \quad \Gamma \setminus (\Gamma_- \cup \Gamma_+) \ni \sigma \mapsto P_\sigma \in \mathcal{B}(\mathcal{H}_\gamma, \mathcal{H}^{\gamma-2})
\]

are analytic families of index 0 Fredholm operators.

**Proof.** We showed in Sect. 3.3 that $P_\sigma Q_\sigma = 1_{\mathcal{H}^{-2}} + K_\sigma$ where $K_\sigma \in \mathcal{B}(\mathcal{H}^{-2})$ is compact and analytic in $\sigma \in \Gamma \setminus (\Gamma_- \cup \Gamma_+)$. Then $\Gamma \setminus (\Gamma_- \cup \Gamma_+) \ni \sigma \mapsto 1_{\mathcal{H}^{-2}} + K_\sigma \in \mathcal{B}(\mathcal{H}^{-2})$ is an analytic family of index 0 Fredholm operators. Since $Q_\sigma \in \mathcal{B}(\mathcal{H}^{\gamma-2}, \mathcal{H}^\gamma)$ is invertible by Prop. 16, the corollary follows for this realization of $P_\sigma$. The proof is similar for $(P_\sigma, \mathcal{B}(\mathcal{H}_\gamma, \mathcal{H}^{\gamma-2}))$ using Rmk. 13 and Rmk. 17. □

Recall now the constants $C_\pm$ from Sect. 3.2 and the sets $\Gamma_\pm$ introduced in Lem. 15. Applying verbatim the arguments above to the case of standard Sobolev spaces, we get:

**Proposition 19.** Let $\gamma \in \mathbb{N} \setminus \{0\}$ and $\sigma \in \Gamma \setminus (\tilde{\Gamma}_- \cup \tilde{\Gamma}_+) \ni \sigma$ is a $\varepsilon$-neighborhood of $\tilde{\Gamma}_\pm$. There exists $C > 0$ depending on $\gamma, s, m, \ell, \sigma'_0, \sigma''_0, \chi_\pm, \varepsilon$ such that, if $C_\pm = C_\pm > C$, then $Q_{\sigma, \gamma-1} \in \mathcal{B}(\mathcal{H}_\gamma, X^\gamma)$ is invertible.

**Corollary 20.** For all $\gamma \in \mathbb{N} \setminus \{0\}$,

\[
\Gamma \setminus (\tilde{\Gamma}_- \cup \tilde{\Gamma}_+) \ni \sigma \mapsto P_\sigma \in \mathcal{B}(X^\gamma, \mathcal{H}^{\gamma-1})
\]

is an analytic family of index 0 Fredholm operators.

4.2. Large mass or angular momentum regime. We finally show that resonances are expelled from any neighborhood of 0 in $\Gamma \cap \{ z \in \mathbb{C} \mid \text{Im}(z) > -\kappa/2 \}$ when $\ell + m \gg 0$. Using Rmk. 4, it is sufficient to work with the realization $(P_\sigma, \mathcal{B}(X^\lambda, \mathcal{H}))$.

**Proposition 21.** Let $\Gamma^\varepsilon_\pm$ be a $\varepsilon$-neighborhood of $\Gamma_\pm$ with $\varepsilon > 0$. There exist two constants $\mathcal{C} > 0$ depending on $\sigma'_0, \sigma''_0, s_0, \Gamma, \varepsilon, \chi_\pm$ and $\mathcal{C}' > 0$ depending on $s_0, m, \ell, \Gamma, \varepsilon, \chi_\pm$ such that, if $\ell + m > C$ or if $|\text{Im}\sigma| > \mathcal{C}'(1 + \max_{r \in [r_-, r_+]} |\text{Re}(\sigma) + s/r|)$, then $\text{Ker}_\mathcal{B}(X^\lambda, \mathcal{H}) (P_\sigma) = \{ 0 \}$ for all $\sigma \in \Gamma \setminus (\Gamma^\varepsilon_- \cup \Gamma^\varepsilon_+)$.  

**Proof.** Recall from the proof of Lem. 12 that

\[
k_{\sigma, \pm} := k_{\sigma, \pm}^b + k_{\sigma, \pm}^a = \rho_\pm(f_{\sigma, \pm} - C_\pm)q_{\sigma, \pm}^f + \rho_\pm \left( \frac{p_{\sigma, \pm}(\rho_\pm, \xi) - p_{\sigma, \pm}(0, \xi)}{\rho_\pm} + f_{\sigma, \pm} \right) q_{\sigma, \pm}^b + (-2\bar{\mu}_\pm \xi + g_{\sigma, \pm} - 3\bar{\mu}_\pm) \rho_\pm(\bar{\mu}_\pm q_{\sigma, \pm} - \bar{\mu}_\pm^2 q_{\sigma, \pm}^2) \bar{q}_{\sigma, \pm}
\]
with \( \text{Im}(\xi) = 1/2 \). Since \( q_{\sigma,\pm}^b \) is proportional to \( q_{\sigma,\pm}^f \), we can factor out:

\[
\rho_{\pm}(f_{\sigma,\pm} - C_{\pm})\tilde{q}_{\sigma,\pm}^f + \rho_{\pm} f_{\sigma,\pm} \tilde{q}_{\sigma,\pm}^b = \rho_{\pm} \left( f_{\sigma,\pm} \left( 1 - \frac{p_{\sigma,\pm}(\rho_{\pm}, \xi) - \tilde{\mu}_{\pm}(\xi - \frac{i}{2})^2}{p_{\sigma,\pm}(0, \xi)} \right) - C_{\pm} \right) \tilde{q}_{\sigma,\pm}^f.
\]

In order to simplify notations, we set for \( \text{Im}(\xi) = 1/2 \):

\[
f_{\sigma,\pm} =: f' + if'', \quad C_{\pm} =: C' + iC'', \quad p_{\sigma,\pm}(\rho_{\pm}, \xi) - \tilde{\mu}_{\pm}(\xi - \frac{i}{2})^2 =: p_1 + ip_1'', \quad p_{\sigma,\pm}(0, \xi) =: p_2 =: p_2' + ip_2''.
\]

Then:

\[
\left| f_{\sigma,\pm} \left( 1 - \frac{p_{\sigma,\pm}(\rho_{\pm}, \xi) - \tilde{\mu}_{\pm}(\xi - \frac{i}{2})^2}{p_{\sigma,\pm}(0, \xi)} \right) - C_{\pm} \right|^2 = \left( f'(p_2' - p_1') - f''(p_2'' - p_1'') - C'p_2' + C''p_2'' \right)^2 \frac{1}{|p_2|^2}
\]

\[
+ \left( f''(p_2' - p_1') + f'(p_2'' - p_1'') - C''p_2' - C'p_2'' \right)^2 \frac{1}{|p_2|^2}
\]

\[
=: \frac{F(\rho_{\pm}, \xi - \frac{i}{2})^2}{|p_2|^2}.
\]

Recall from (7) that \( c_1, c_2 \) are the (non real) roots of \( p_2 \), and

\[
d_1 := \frac{1}{2} \sqrt{\frac{p_{\pm}}{\tilde{\mu}_{\pm}} \left( -\text{sign}(C'') \sqrt{-C' + |C_{\pm}|} + i \sqrt{C' + |C_{\pm}|} \right)}, \quad d_2 := -d_1
\]

are the (non real) roots of \( \mathbb{R} + \frac{1}{2} \ni \xi \mapsto \tilde{\mu}_{\pm}(\xi - \frac{i}{2})^2 + \rho_{\pm}C_{\pm} \). Using the residue theorem, we get:

\[
\int_{\{\text{Im}(\xi) = 1/2\}} \left| f_{\sigma,\pm} \left( 1 - \frac{p_{\sigma,\pm}(\rho_{\pm}, \xi) - \tilde{\mu}_{\pm}(\xi - \frac{i}{2})^2}{p_{\sigma,\pm}(0, \xi)} \right) - C_{\pm} \right|^2 |\tilde{q}_{\sigma,\pm}^f| d\xi = \frac{2\pi i}{4\kappa_0^2 \tilde{\mu}_{\pm}^2} \sum_{c \in \{c_1, c_1', c_2, c_2'\}} \frac{F(\rho_{\pm}, c)^2}{(c - d_1)(c - d_1')(c - d_2)(c - d_2')} \prod_{c' \in \{c_1, c_1', c_2, c_2'\}} \text{Im}(c') > 0
\]

\[
+ \frac{2\pi i}{4\kappa_0^2 \tilde{\mu}_{\pm}^2} \sum_{d \in \{d_1, d_1', d_2, d_2'\}} \frac{F(\rho_{\pm}, d)^2}{(d - c_1)(d - c_1')(d - c_2)(d - c_2')} \prod_{d' \in \{d_1, d_1', d_2, d_2'\}} \text{Im}(d') < 0.
\]

In terms of integrability with respect to \( \frac{d\rho_{\pm}}{p_{\pm}} \), the first term on the right hand side of (12) is of order \( \rho_0^{\pm} \) (since \( c \) is independent of \( \rho_\pm \)) while the second term is of order \( \rho_\pm^{3/2} \); multiplying with \( \rho_\pm^2 \) from the squared modulus of (11), we see that there is no integration issue.
Notice next that $(\xi \in \mathbb{R}$ below):

\[
(f'(p'_2 - p'_{1}) - C''p'_2)|_{\text{Im}(\xi) = 1/2} = \tilde{\mu}_\pm(0)\xi^2(f' - C')
+ f'(0) - \text{Im}(g_{\sigma,0}(0) - g_{\sigma,\pm})\xi + \text{Re}(g_{\sigma,\pm}(0) - g_{\sigma,0}) - \frac{5}{4}(\tilde{\mu}_\pm(0) - \tilde{\mu})
- C'(0) - \text{Im}(g_{\sigma,0}(0))\xi + \text{Re}(g_{\sigma,0}(0)) - \frac{5}{4}(\tilde{\mu}_\pm(0)),
\]

\[
(f'(p''_2 - p''_{1}))|_{\text{Im}(\xi) = 1/2} = f'((\text{Re}(g_{\sigma,0}(0) - g_{\sigma,\pm}) - (\tilde{\mu}_\pm(0) - \tilde{\mu}))\xi + \text{Im}(g_{\sigma,0}(0) - g_{\sigma,\pm})).
\]

Since $d_j = \mathcal{O}(|C_\pm|^{-1/2})$, we obtain:

\[
(12) \leq \left( |f' - C'|^2\mathcal{O}(|C_\pm|^2) + (|f'|^2 + |C_\pm|^2)\mathcal{O}(|C_\pm|)\right)\mathcal{O}(|C_\pm|^{-7/2}).
\]

As only $f'$ depends on $\ell$ and $m$, we can choose (say) $C'' = 1$ and $1 + (\sigma_0')^2 + (\sigma_{0}'')^2 + s_0^2 < C' < \ell^2 + m^2 < C' + C^{3/4}$ so that $\|\text{Op}_M^{-1/2}[k'_{\sigma,\pm}]\|_{\mathcal{B}(\mathcal{H}_\pm^{0,-1/2})}$ is as small as wished. Observe that the norm $\|\text{Op}_M^{-1/2}[q_{\sigma,\pm}]\|_{\mathcal{B}(\mathcal{H}_\pm^{0,-1/2})}$ is obviously small as $C' \to +\infty$.

On the other hand, $f_{\sigma,\pm}$ is of order 2 and $g_{\sigma,\pm}$ of order 1 in $(\sigma r + s)$, and thus $q_{\sigma,\pm}^t$ is of order 0 and $q_{\sigma,\pm}^b$ as well as its derivatives with respect to $\rho_\pm$ are of order 1 in $(\sigma r + s)$. As a result, if $\max_{\sigma \in [\tau_r, \tau_b]} |\text{Re}(\sigma) + s/r| \ll |\text{Im}(\sigma)|^{-1}$, then we can choose $C'' = 1$ and $1 + \ell^2 + m^2 + s_0^2 < C' \ll \text{Im}(\sigma)^2 < C' + C^{3/4}$ to get the same smallness of the norms as above.

In both cases, we obtain

\[
(13) \quad P_\sigma Q_\sigma = 1 + \sum_{\pm} \left( [P_\sigma, \chi_\pm]Q_{\sigma,\pm}\phi_\pm + K_{\sigma,\pm} \right),
\]

with

\[
\left\| \sum_{\pm} \left( [P_\sigma, \chi_\pm]Q_{\sigma,\pm}\phi_\pm + K_{\sigma,\pm} \right) \right\|_{\mathcal{H}^{1-2}} < 1.
\]

The right-hand side in (13) is thus invertible and $P_\sigma \in \mathcal{B}(\mathcal{H}^1, \mathcal{H})$ is onto, which means by the index 0 Fredholm property (cf. Cor. 18) that $P_\sigma \in \mathcal{B}(\mathcal{X}^1, \mathcal{H})$ is invertible. The proof is complete. \(\square\)

5. APPENDIX: NORM ESTIMATE FOR PSEUDO-DIFFERENTIAL OPERATORS WITH $L^2$ SYMBOL

This appendix is devoted to the estimation of the norm of $A$ between appropriate Sobolev spaces when $A$ is the Mellin quantization of a symbol in $L^2(\mathbb{R}_+^* \times \mathbb{R}, \frac{d\rho_\pm}{\rho_\pm}d\xi)$.

We start with an estimate for the Fourier type quantization.

Lemma 22. Let $a$ be a Fourier type symbol such that $\hat{a}^j \in L^2(\mathbb{R} \times \mathbb{R}, dx d\xi)$ for all $0 \leq j \leq |\gamma|$. For $\gamma \in 2\mathbb{N}$ and all $0 \leq j \leq k \leq \gamma/2$, set

\[
c_{j,k}(\gamma)^2 := \begin{cases} 1 & \text{if } 2k - j \in \{0, 2\gamma\}, \\ \left( \frac{\gamma - |2k-j|}{2\gamma - |2k-j|} \right)^{(2k-j)/2} & \text{if } 0 < 2k - j < 2\gamma, \end{cases}
\]

\[
c_{j,k}(\gamma)^2 := \begin{cases} 1 & \text{if } 2k - j \in \{0, 2\gamma\}, \\ \left( \frac{\gamma - |2k-j|}{2\gamma - |2k-j|} \right)^{(2k-j)/2} & \text{if } 0 < 2k - j < 2\gamma, \end{cases}
\]
and define inductively the coefficients \(d_{k,n} \in \mathbb{Z}\) by \(d_{0,0} = 1, d_{0,1} = -1, d_{1,1} = 1, d_{k,n+1} = d_{k-1,n} - d_{k,n}\) with the convention that \(d_{-1,n} = d_{n+1,n} = 0\). Set also:

\[
\begin{align*}
\hat{c}_{\gamma}(a) &= 2\pi \sum_{k=0}^{\gamma/2} \binom{\gamma/2}{k} \sum_{j=0}^{2k} \binom{2k}{j} c_{j,k}(\gamma) \|\hat{\partial}_x^j a\|_{L^2(\mathbb{R} \times \mathbb{R}, dxd\xi)}, \\
\hat{c}_{-\gamma}(a) &= 2\pi \sum_{k=0}^{\lceil \gamma/2 \rceil} \binom{\lceil \gamma/2 \rceil}{k} \sum_{j=0}^{2k} c_{j,k}(\gamma) \|\hat{\partial}_x^j a\|_{L^2(\mathbb{R} \times \mathbb{R}, dxd\xi)}.
\end{align*}
\]

Then

\[
\|\text{Op}_\mathcal{F}[a]\|_{\mathcal{B}(H^\gamma(\mathbb{R}, dx))} \leq \begin{cases} 
\hat{c}_{\gamma}(a) & \text{if } \gamma \in 2\mathbb{N}, \\
\max \left\{ \hat{c}_{\gamma,2}[\gamma](a), \hat{c}_{\gamma,2}[\gamma+2](a) \right\} & \text{if } \gamma \in (0, +\infty) \setminus 2\mathbb{N}, \\
\hat{c}_{-\gamma}(a) & \text{if } \gamma \in -2\mathbb{N}, \\
\max \left\{ \hat{c}_{-\gamma,2}[\gamma](a), \hat{c}_{-\gamma,2}[\gamma+2](a) \right\} & \text{if } \gamma \in (-\infty, 0) \setminus (-2\mathbb{N})
\end{cases}
\]

with \(\hat{c}_{-0} := \hat{c}_{+0}\).

**Proof.** We start with the case \(\gamma = 2\gamma'\) with \(\gamma' \in \mathbb{N}\). For all \(u \in \mathcal{H}^\gamma(\mathbb{R}, dx)\), we compute:

\[
\|\langle \xi \rangle^\gamma \mathcal{F}[\text{Op}_\mathcal{F}[a]u]\|_{L^2(\mathbb{R}, dx)} = \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} \xi^{2k} \mathcal{F}[\text{Op}_\mathcal{F}[a]u]_{L^2(\mathbb{R}, dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} \xi^{2k} \mathcal{F}[(i\partial_x)^{2k} \text{Op}_\mathcal{F}[a]u]_{L^2(\mathbb{R}, dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} (-1)^k \mathcal{F} \left( \sum_{j=0}^{2k} \binom{2k}{j} \text{Op}_\mathcal{F}[(i\xi)^{2k-j}(\partial_x^j a)]u \right)_{L^2(\mathbb{R}, dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} (-1)^{k-j} \mathcal{F} \left( \sum_{j=0}^{2k} \binom{2k}{j} \text{Op}_\mathcal{F}[\partial_x^j a](\partial_x^{2k-j} u) \right)_{L^2(\mathbb{R}, dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} \sum_{j=0}^{2k} \binom{2k}{j} \|\text{Op}_\mathcal{F}[\partial_x^j a](\partial_x^{2k-j} u)\|_{L^2(\mathbb{R}, dx)}
\]

\[
= 2\pi \sum_{k=0}^{\gamma'} \binom{\gamma'}{k} \sum_{j=0}^{2k} \binom{2k}{j} \text{Op}_\mathcal{F}[\partial_x^j a](\partial_x^{2k-j} u)_{L^2(\mathbb{R}, dx)}.
\]
The last line follows from Plancherel equality. Now observe that

\[
|\text{Op}_\mathcal{F}[\hat{\gamma}_1^j]v|^2 \leq \frac{1}{(2\pi)^2} \left( \int_{\mathbb{R}} |(\hat{\gamma}_1^j)(x,\xi)||\hat{v}(\xi)|d\xi \right)^2 \leq \frac{1}{(2\pi)^2} \left( \int_{\mathbb{R}} |(\hat{\gamma}_1^j)(x,\xi)|^2d\xi \right) \left( \int_{\mathbb{R}} |\hat{v}(\xi)|^2d\xi \right)
\]

\[
= \| (\hat{\gamma}_1^j)(x,\bullet) \|^2_{L^2(\mathbb{R},d\xi)} \| v \|^2_{L^2(\mathbb{R},dx)}
\]

by Cauchy-Schwarz inequality and again Plancherel equality. It follows:

\[
\| \langle \xi \rangle^\gamma \mathcal{F}[\text{Op}_\mathcal{F}[a]u] \|_{L^2(\mathbb{R},d\xi)} \leq 2\pi \sum_{k=0}^{\gamma'} \left( \frac{\gamma'}{k} \right) \sum_{j=0}^{2k} \| \hat{\gamma}_1^j \|_{L^2(\mathbb{R},d\xi)} \| \hat{\gamma}_1^{2k-j} u \|_{L^2(\mathbb{R},dx)}.
\]

Since\(^8\)

\[
\| \hat{\gamma}_1^{2k-j} u \|_{L^2(\mathbb{R},dx)} = \| \xi^{2k-j} \mathcal{F}[u] \|_{L^2(\mathbb{R},d\xi)} \leq c_{j,k}(\gamma) \| \langle \xi \rangle^\gamma \mathcal{F}[u] \|_{L^2(\mathbb{R},dx)} = c_{j,k}(\gamma) \| u \|_{H^\gamma(\mathbb{R},dx)},
\]

the estimate follows in the case \(\gamma \in 2\mathbb{N}\).

If \(\gamma \in (2k,2(k+1))\) for some \(k \in \mathbb{N}\), then we use that \(A_\gamma := \langle D_x \rangle^\gamma \text{Op}_\mathcal{F}[a] \langle D_x \rangle^{-\gamma} \in \mathcal{B}(L^2(\mathbb{R},dx))\) verifies

\[
\| A_\gamma \|_{L^2(\mathbb{R},dx)} = \| \text{Op}_\mathcal{F}[a] \|_{\mathcal{B}(H^\gamma(\mathbb{R},dx))}
\]

so that the maximum principle applied to the operator \(\langle D_x \rangle^\theta \text{Op}_\mathcal{F}[a] \langle D_x \rangle^{-\theta} \in \mathcal{B}(L^2(\mathbb{R},dx))\) where \(\theta \in [2k,2(k+1)] + i\mathbb{R}\) provides the estimate.

Finally, assume that \(\gamma = 2\gamma'\) with \(\gamma' < 0\). Note that for all \(n\) times differentiable functions \(f\) and \(g\), we have

\[
\hat{\gamma}_1^n (fg) = \sum_{k=0}^{n} d_{k,n}(\hat{\gamma}_1^k f(\hat{\gamma}_1^k g)).
\]

This formula is easily checked by induction. Then, for all \((u,v) \in L^2(\mathbb{R},dx) \times H^\gamma(\mathbb{R},dx)\), we compute:

\[
\langle \text{Op}_\mathcal{F}[a] \langle D_x \rangle^\gamma u, v \rangle_{L^2(\mathbb{R},dx)} = \sum_{k=0}^{\gamma'} \left( \frac{\gamma'}{k} \right) \langle \text{Op}_\mathcal{F}[\hat{\gamma}_1^k] u, v \rangle_{L^2(\mathbb{R},dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \left( \frac{\gamma'}{k} \right) \sum_{j=0}^{2k} (-1)^k d_{j,2k} \langle \hat{\gamma}_1^{2k-j} \text{Op}_\mathcal{F}[\hat{\gamma}_1^j] u, v \rangle_{L^2(\mathbb{R},dx)}
\]

\[
= \sum_{k=0}^{\gamma'} \left( \frac{\gamma'}{k} \right) \sum_{j=0}^{2k} (-1)^j d_{j,2k} \langle \text{Op}_\mathcal{F}[\hat{\gamma}_1^j] u, \hat{\gamma}_1^{2k-j} v \rangle_{L^2(\mathbb{R},dx)}.
\]

---

\(^8\)We find the critical values of the function \(f_{\gamma',\alpha} : \mathbb{R} \ni \xi \mapsto \frac{\xi^{\alpha}}{(1+\xi^2)^{\gamma'/2}}\) by noticing that \(f_{\gamma',\alpha}'(\xi) = -\frac{\alpha \xi^{\alpha-1}}{(1+\xi^2)^{\gamma'/2+1}}((4\gamma'-\alpha)\xi^2-\alpha)\) cancels at \(\pm \sqrt{\frac{\alpha}{4\gamma'-\alpha}}\) if \(\alpha < 4\gamma'\) and at 0 if \(\alpha > 1\); this provides the constants \(c_{j,k}(\gamma)\) – note that \(f_{\gamma',\alpha} \leq 1\) so it holds \(c_{j,k}(\gamma) \leq 1\).
It follows with the computations carried out in the positive $\gamma$ case:

\[
\left| \langle \text{Op}_\mathcal{F}[a] \hat{D}_x^\gamma, v \rangle_{L^2(\mathbb{R}, dx)} \right| \leq \sum_{k=0}^{\left\lfloor \frac{1}{2}\gamma \right\rfloor} \frac{|\gamma|!}{k!} \sum_{j=0}^{2k} d_{j,2k} \left\| \text{Op}_\mathcal{F}[\hat{\rho}^j_\mathcal{F}[a]] u \right\|_{L^2(\mathbb{R}, dx)} \left\| \hat{\rho}^{2k+j}_\mathcal{F}[v] \right\|_{L^2(\mathbb{R}, dx)}
\]

\[
\leq 2\pi \sum_{k=0}^{\left\lfloor \frac{1}{2}\gamma \right\rfloor} \frac{|\gamma|!}{k!} \sum_{j=0}^{2k} c_{j,k} (|\gamma|) d_{j,2k} \left\| \hat{\rho}^j u \right\|_{L^2(\mathbb{R} \times \mathbb{R}, dxd\xi)} \left\| v \right\|_{H^{\gamma}(\mathbb{R}, dx)}.
\]

This gives the announced estimate. The general case $\gamma < 0$ follows again by complex interpolation. □

We now turn to the Mellin type quantization. To lighten notations, we will simply write $\rho$ instead of $\rho_\pm$.

**Lemma 23.** Let $a$ be a Mellin type symbol such that $(\rho, \xi) \mapsto ((\rho \hat{\rho})^j a)(\rho, \xi - i\beta) \in L^2((\mathbb{R}_+)^* \times \mathbb{R}, \frac{d\rho}{\rho})$ for all $0 \leq j \leq |\gamma|$, and set $b(x, \xi) := a(e^x, \xi - i\beta)$. Then

\[
\left\| \text{Op}_\mathcal{M}^\beta[a] \right\|_{B(\mathcal{H}^\gamma, \hat{\mathcal{M}}^\beta)} \leq \begin{cases} 
\hat{c}_{+,\gamma}(b) & \text{if } \gamma \in 2\mathbb{N}, \\
\max \left\{ \hat{c}_{+,2 \left\lfloor \frac{1}{2}\gamma \right\rfloor}(b), \hat{c}_{+,2 \left\lfloor \frac{1}{2}\gamma \right\rfloor + 1}(b) \right\} & \text{if } \gamma \in (0, +\infty) \setminus 2\mathbb{N}, \\
\hat{c}_{-,\gamma}(b) & \text{if } \gamma \in -2\mathbb{N}, \\
\max \left\{ \hat{c}_{-,2 \left\lfloor \frac{1}{2}\gamma \right\rfloor}(b), \hat{c}_{-,2 \left\lfloor \frac{1}{2}\gamma \right\rfloor + 1}(b) \right\} & \text{if } \gamma \in (-\infty, 0) \setminus (-2\mathbb{N})
\end{cases}
\]

where the constants $\hat{c}_{\pm, \bullet}(\bullet)$ are as in Lem. 22.

**Proof.** Let $u \in \mathcal{H}^\gamma_{\pm, \beta}$ and set $\rho := e^x$ with $x \in \mathbb{R}$. Then:

\[
\text{Op}_\mathcal{M}[a] u(\rho) = \frac{\rho^\beta}{2\pi} \int_{\mathbb{R}_+} \rho^{i\xi} a(\rho, \xi - i\beta) \left( \int_{\mathbb{R}^+} y^{-i\xi} (y^{-\beta} u(y)) \frac{dy}{y} \right) d\xi
\]

\[
= \frac{e^{\beta x}}{2\pi} \int_{\mathbb{R}} e^{ix\xi} a(e^x, \xi - i\beta) \left( \int_{\mathbb{R}} e^{-i\xi z} (e^{-\beta z} u(e^z)) dz \right) d\xi
\]

Setting $v(x) := e^{-\beta x} u(e^x)$, it comes:

\[
\text{Op}_\mathcal{M}[a] u(\rho) = e^{\beta x} (\text{Op}_\mathcal{F}[b] v)(x).
\]

As already observed in (4), $\|v\|_{H^{\gamma}(\mathbb{R}, dx)} = \|u\|_{\mathcal{H}^\gamma_{\pm, \beta}}$; moreover, $b \in S_{-2}^{-2; \beta}$ since $a \in S_{-2}^{-2; \beta}$. As a result, we have

\[
\|\text{Op}_\mathcal{M}[a] u\|_{\mathcal{H}^\gamma_{\pm, \beta}} = \|\rho^{-\beta} \text{Op}_\mathcal{M}[a] u\|_{\mathcal{H}^\gamma_{\pm}} = \|\text{Op}_\mathcal{F}[b] v\|_{H^{\gamma}(\mathbb{R}, dx)},
\]

whence

\[
\|\text{Op}_\mathcal{M}[a]\|_{B(\mathcal{H}^\gamma_{\pm, \beta})} = \|\text{Op}_\mathcal{F}[b]\|_{B(\mathcal{H}^\gamma_{\pm, \beta})}.
\]

It remains to apply Lem. 22 to the symbol $b$ which verifies $\partial_1^j b \in L^2(\mathbb{R} \times \mathbb{R}, dxd\xi)$ for all $0 \leq j \leq |\gamma|$. □
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