Bounds for the Clique Cover Width of Factors of the Apex Graph of the Planar Grid

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Abstract
The clique cover width of $G$, denoted by $ccw(G)$, is the minimum value of the bandwidth of all graphs that are obtained by contracting the cliques in a clique cover of $G$ into a single vertex. For $i = 1, 2, ..., d$, let $G_i$ be a graph with $V(G_i) = V$, and let $G$ be a graph with $V(G) = V$ and $E(G) = \bigcap_{i=1}^d(G_i)$, then we write $G = \bigcap_{i=1}^d G_i$ and call each $G_i$, $i = 1, 2, ..., d$ a factor of $G$. We are interested in the case where $G_1$ is chordal, and $ccw(G_i)$, $i = 2, 3, ..., d$ for each factor $G_i$ is “small”. Here we show a negative result. Specifically, let $\hat{G}(k, n)$ be the graph obtained by joining a set of $k$ apex vertices of degree $n^2$ to all vertices of an $n \times n$ grid, and then adding some possible edges among these $k$ vertices. We prove that if $\hat{G}(k, n) = \bigcap_{i=1}^d G_i$, with $G_1$ being chordal, then, $\max_{2 \leq i \leq d}\{ccw(G_i)\} \geq \frac{n^2}{2 \cdot \frac{2d}{c}}$, where $c$ is a constant. Furthermore, for $d = 2$, we construct a chordal graph $G_1$ and a graph $G_2$ with $ccw(G_2) \leq \frac{n^2}{2} + k$ so that $\hat{G}(k, n) = G_1 \cap G_2$. Finally, let $\hat{G}$ be the clique sum graph of $\hat{G}(k_i, n_i), i = 1, 2, ..., t$, where the underlying grid is $n_i \times n_i$ and the sum is taken at apex vertices. Then, we show $\hat{G} = G_1 \cap G_2$, where, $G_1$ is chordal and $ccw(G_2) \leq \sum_{i=1}^t(n_i + k_i)$. The implications and applications of the results are discussed, including addressing a recent question of David Wood.

1 Introduction
In this paper, $G = (V(G), E(G))$ denotes an undirected graph. A chordal graph is a graph with no chordless cycles [2]. An interval graph is the intersection graph of the intervals on the real line [16]. A comparability graph is a graph whose edges have a transitive orientation. An incomparability graph [4] is the complement of a comparability graph. A clique cover $C$ in $G$ is a partition of $V(G)$ into cliques. Throughout this paper, we will write
Let \( C = \{ C_0, C_1, ..., C_t \} \) to indicate that \( C \) is an ordered set of cliques in \( G \). Let \( L = \{ v_0, v_2, ..., v_{n-1} \} \) be a linear ordering of vertices in \( G \). The width of \( L \), denoted by \( w(L) \), is \( \max_{v_i, v_j \in E(G)} |j - i| \). The bandwidth of \( G \) denoted by \( bw(G) \), is the smallest width of all linear orderings of \( V(G) \) [3]. For a clique cover \( C = \{ C_0, C_1, ..., C_t \} \), in \( G \), let the width of \( C \), denoted by \( w(C) \), denote \( \max \{ |j - i| | xy \in E(G), x \in C_i, y \in C_j \} \). The clique cover width of \( G \) denoted by \( ccw(G) \), is the smallest width all ordered clique covers in \( G \) [11]. Note that \( ccw(G) \leq bw(G) \). It is easy to verify that any graph with \( ccw(G) = 1 \) is an incomparability graph.

Let \( d \geq 1 \), be an integer, and for \( i = 1, 2, ..., d \) let \( G_i \) be a graph with \( V(G_1) = V \), and let \( G \) be a graph with \( V(G) = V \) and \( E(G) = \cap_{i=1}^d (G_i) \).

Then we say \( G \) is the edge intersection graph of \( G_1, G_2, ..., G_d \), and write \( G = \cap_{i=1}^d G_i \). In this setting, each \( G_i, i = 1, 2, ..., d \) is a factor of \( G \). Roberts [8] introduced the 

\textit{breadth} and \textit{cubicity} of a graph \( G \) as the smallest integer \( d \) so that \( G \) is the edge intersection graph of \( d \) interval, or unit interval graphs, respectively. For more recent work see [15].

For integers \( d \geq 2 \) and \( w \geq 1 \), let \( \mathcal{C}(d, w) \) be the class all graphs \( G \) so that \( G = \cap_{i=1}^d G_i \), where \( G_1 \) is chordal, and for \( i = 2, 3, ..., d \), \( ccw(G_i) \leq w \).

A function that assigns non-negative values to subgraphs of \( G \), is called a measure, if the following hold. (i) \( \mu(H_1) \leq \mu(H_2) \), if \( H_1 \subseteq H_2 \subseteq G \), (ii) \( \mu(G) = \mu(G) \cup H_2 \) \( \leq \mu(G) + \mu(H_2) \), if \( H_1, H_2 \subseteq G \), (iii) \( \mu(H_1 \cup H_2) = \mu(H_1) + \mu(H_2) \), if there are no edges between \( H_1 \) and \( H_2 \).

In [10], we have derived a separation theorem whose statement without a proof was announced in [11].

**Theorem 1.1 ([11, 10])**

Let \( \mu \) be a measure on \( G = (V(G), E(G)) \), and let \( G_1, G_2, ..., G_p \) be graphs with \( V(G_1) = V(G_2) = ..., V(G_p) = V(G) \), \( d \geq 2 \) and \( E(G) = \cap_{i=1}^p E(G_i) \) so that \( G_1 \) is chordal. Then there is a vertex separator \( S \) in \( G \) whose removal separates \( G \) into two subgraph so that each subgraph has a measure of at most \( 2\mu(G) / 3 \). In addition, the induced graph of \( G \) on \( S \) can be covered with at most \( O(2^{d^3} \frac{w}{\mu(G)} \frac{d}{\mu(G)}) \) many cliques from \( G \), where \( l^* = \max_{2 \leq i \leq d} ccw(G_i) \).

A number of geometric applications of Theorem 1.1 were announced in [11], however, in all these cases \( G_1 \) happened to be an interval graph, and hence the full power of the separation theorem was not utilized. We remark that a slightly stronger version of the above Theorem has been proved in [10], where \( l^* = \Pi_{2 \leq i \leq d} ccw(G_i) \).

Recall that a \textit{tree decomposition} [9, 1] of a graph \( G \) is a pair \( (X, T) \) where \( T \) is a tree, and \( X = \{ X_i | i \in V(T) \} \) is a family of subsets of \( V(G) \), each called a bag, so that the following hold:

- \( \cup_{i \in V(T)} X_i = V(G) \)
for any $uv \in E(G)$, there is an $i \in V(T)$ so that $v \in X_i$ and $u \in X_i$.

- for any $i, j, k \in V(T)$, if $j$ is on the path from $i$ to $k$ in $T$, then $X_i \cap X_k \subseteq X_j$.

Width of $(X, T)$, is the size of largest bag minus 1. Treewidth of $G$, or $tw(G)$ is the smallest width, overall tree decompositions of $G$. In an attempt to explore the structure of class $C(2, w)$ we proved Theorem 1.2 in [13].

**Theorem 1.2** (Universal Representation Theorem:[13]) Let G be a graph and let $L = \{L_1, L_2, \ldots, L_k\}$ be a partition of vertices, so that for any $xy \in E(G)$, either $x, y \in L_i$ where $1 \leq i \leq k$, or, $x \in L_i, y \in L_{i+1}$, where, $1 \leq i \leq k - 1$. Let $(X, T)$ be a tree decomposition of $G$. Let $t^* = \max_{i=1,2,\ldots,k} \{|L_i \cap X_j| : j \in V(T)\}$. (Thus, $t^*$ is the largest number of vertices in any element of $L$ that appears in any bag of $T$). Then, there is a chordal graph $H_1$ and a graph $H_2$ with $ccw(H_2) \leq 2t^* - 1$ so that $G = G_1 \cap G_2$.

Noting that for any planar graph $G$, the parameter $t^*$ is at most 4, it follows that any planar $G$ is in class $C(2, 7)$ [13] (with $G_1$ being a chordal graph). Consequently, the planar separation theorem [7] follows from Theorem 1.1. We further speculated in [13] that similar results would hold for graphs drawn of surfaces, and graphs excluding specific minors.

Concurrent with our work in [13], in 2013, and independently from us, Dujmović, Morin, and Wood [6], formalized the notation of $t^*$, and introduced a parameter called the layered tree width, or $ltw(G)$, which is the minimum value of $t^*$, over all tree decompositions and layer partitions of $G$. Their work has significant applications in graph theory and graph drawing, under the requirement that $ltw(G)$ is bounded by a constant. Among other results, it was shown in [6] that $ltw(G) \leq 2g + 3$, for any genus $G$ graph, and as a byproduct some well known separation theorem followed with improved multiplicative constants. Dujmović, Morin, and Wood further classified those graphs $G$ with bounded $ltw(G)$ to be the class of $H$ minor free graphs, for a fixed apex graph $H$. ($H$ is an apex graph, if $H - \{x\}$ is a planar graph for some vertex $x$.) Combining this result and the Universal Representation Theorem, it can be concluded that for any fixed apex graph $H$, there is an integer $w(H)$, so that any $H$–minor free graph $G$, is in class $C(2, w(H))$.

Using the terminology in the abstract, let $G(1, n)$, be the graph obtained from $n \times n$ grid by adding a new vertex of degree $n^2$ which is adjacent to all vertices of the grid. Then, although this graph not have $K_6$ as a minor, it does not satisfy the requirement for the usage of the framework in [6], since $K_5$ is not planar. Specifically, as noted in [6], $ltw(G(1, n)) = \Omega(n)$, and hence the layered treewidth method is not applicable. Note that the Universal Representation Theorem also fails to show that $G(1, n)$ is in $C(2, w)$ for any constant $w$. David Wood [17] in private communications
raised the following question: Might it be true that for every \( H \)-minor-free graph \( G, G \in \mathcal{C}(2, w(H)) \), for some constant \( w(H) \) depending on \( H \)? Particularly, he inquired if this is true for \( G(1, n) \). Now, let \( G(k, n) \) be the graph obtained by joining a set \( X \) of \( n \) vertices of degree \( n^2 \) to all vertices of an \( n \times n \) grid, with possible addition of edges among these vertices in \( X \). In Section two we prove that if \( \hat{G}(k, n) = \bigcap_{i=1}^{d} G_i \), where \( G_i \) is chordal, then \( \max_{2 \leq i \leq d} \{ ccw(G_i) \} \geq \frac{n \sqrt{\pi} \Gamma}{2(2c)^{\pi + 1}} \), where \( c \) is a constant.

In the positive direction, for \( d = 2 \), we show \( \hat{G}(k, n) = G_1 \cap G_2 \), so that \( G_1 \) is chordal and \( ccw(G_2) \leq \frac{n}{2} + k \), where the upper bound of \( \frac{n}{2} + k \) is small enough for the effective application of the separation Theorem 1.1. We extend this result to clique sum graphs. Specifically, let \( \hat{G} \) be the clique sum graph of \( \hat{G}(k_i, n_i), i = 1, 2, \ldots, t \), where the underlying grid is \( n_i \times n_i \) and the sum is taken at apex vertices. Then, we show \( \hat{G} = G_1 \cap G_2 \), where \( G_1 \) is chordal and \( ccw(G_2) \leq \sum_{i=1}^{t} (n_i + k_i) \).

2 Main Results

**Theorem 2.1** Let \( \hat{G}(k, n) \) be the graph obtained by joining a set \( X \) of \( k \) vertices of degree \( n^2 \) to all vertices of an \( n \times n \) grid, with possible addition of edges among these vertices. Then, the following hold.

(i) If \( \hat{G}(k, n) = \bigcap_{i=1}^{d} G_i \), where \( G_1 \) is chordal, then, \( \max_{2 \leq i \leq d} \{ ccw(G_i) \} \geq \frac{n \sqrt{\pi} \Gamma}{2(2c)^{\pi + 1}} \), where \( c \) is a constant.

(ii) There is a chordal graph \( G_1 \) and a graph \( G_2 \) with \( ccw(G_2) \leq \frac{n}{2} + k \), so that \( \hat{G}(k, n) = G_1 \cap G_2 \).

**Proof.** For (i), let \( H \) be the \( n \times n \) planar grid. Robertson and Seymour have shown that any chordalization of \( H \) has a clique size \( n/c \), for a constant \( b \). Since \( G_1 \) restricted to \( H \) is chordal, it follows that, \( G_1 \) induced to \( H \), must have a clique \( S \) of size \( n/2c \). It follows that the subgraph of \( \hat{G}(k, n) \) induced on \( S \) has a independent set \( S' \) of size at least \( n/2c \). Now let \( G' = \bigcap_{i=2}^{d} G_i \), and observe that \( S' \) must also be an independent set in \( G' \), since \( \hat{G}(k) = G_1 \cap G' \), and \( S' \) is a clique in \( G_1 \). Let \( \hat{S} = S \cup \{ x \} \) for some \( x \in X \). Next for \( i = 2, 3, \ldots, d \), assume that \( C_i \) is a clique cover in \( G_i \) with \( w(C_i) = ccw(G_i) \) and let \( B_i \) be the restriction of \( C_i \) to \( \hat{S} \), and let \( 2 \leq j \leq d \) so that \( |B_j| = \max_{2 \leq i \leq d} \{|B_i|\} \). Note that \( |(B_j)| \leq 2w(B_j) \), since \( x \) is adjacent to all vertices in \( S' \). Now if \( |(B_j)| \geq \frac{n \sqrt{\pi} \Gamma}{2(2c)^{\pi + 1}} \), then the claim follows. So assume that \( |(B_j)| < \frac{n \sqrt{\pi} \Gamma}{2(2c)^{\pi + 1}} \), then \( S' \) can be covered...
with strictly less than $|B_j|^{d-1} = \frac{n}{d}$ cliques in $G'$ which is a contradiction, since $S'$ is independent in $G'$ and $|S'| \geq \frac{n}{2d}$. For (ii), take the vertices of the grid $H$ in every two consecutive rows, and make them into one single clique, by the addition of edges. This way we get a graph $H'$ which is a unit interval graph. Now make the set $X$ a clique, and add this clique and all $k \cdot n^2$ edges between $X$ and vertices of $H$ to obtain a graph $G_1$ which can be shown to be chordal. To construct $G_2$, take any column of $H$, and make all vertices into one single clique. This way, we obtain a graph $I$ with a clique cover, $O = \{C_1, C_2, ..., C_n\}$, where the vertices in each clique in $O$ are the vertices in a column of $H$. It is easily seen that $ccw(I) = 1$. To obtain $G_2$, add to $I$ all vertices in $X$ and edges incident to them. To complete the proof place each vertex $x \in X$ as a clique between $C_2^x$, and $C_2^{x+1}$. Observe that $G(k, n) = G_1 \cap G_2$, and $ccw(G_2) \leq \frac{n}{2} + k$. □

Remark 2.1 Let $G$, $|V(G) = N|$ be an $H$ minor free graph, where $H$ is a fixed graph. It is known that $tw(G) = O(\sqrt{N})$ [5], consequently by the Universal Representation Theorem, $G = G_1 \cap G_2$, where $G_1$ is chordal and $ccw(G_2) = O(\sqrt{N})$. Moreover, the upper bound of $O(\sqrt{N})$ is tight, since $\hat{G}(1, n)$ has $N = n^2 + 1$ vertices, is $k_6$ minor free and, by Part (i) in Theorem 2.1, if $\hat{G}(1, n) = G_1' \cap G_2'$, where $G_1'$ is chordal, then $ccw(G_2') = \Omega(n)$.

Remark 2.2 By Part (i), $\hat{G}(k, n) \notin C(d, w)$, for any constants $w$ and $d$. Nonetheless, the upper bound for $ccw(G_2)$ in Part (ii) of Theorem 2.1 is sufficient to use Theorem 1.1 and show that $\hat{G}(k, n), |V(\hat{G}(k))| = N$ has a vertex separator of size $O(N^{1-\epsilon})$, with the splitting ratio $1/3 - 2/3$, where $N$, is the number of vertices in $G$, as long as $k = O(n)$. The bound on the separator size is sufficient for many algorithmic purposes, although it may not be the best possible. Thus, Theorem 1.1 may be effectively used, even when layered tree width is unbounded.

Let $G_1$ and $G_2$ be graphs so that $V(G_1) \cap V(G_2)$ is a clique in both $G_1$ and $G_2$. Then, the clique sum of $G_1$ and $G_2$, denoted by $G_1 \oplus G_2$ is a graph $G$ with $V(G) = V(G_1) \cup V(G_2)$, and $E(G) = E(G_1) \cup E(G_2) - E'$, where $E'$ is a subset of edges (possibly empty) in the clique induced by $V(G_1) \cap V(G_2)$. The clique sum of more than two graphs is defined iteratively, using the definition for two graphs. Let $G = G_1 \oplus ... \oplus G_2 \oplus ... \oplus G_k$ then we write $G = \oplus_{i=1}^k G_i$. Clique sums are intimately related to the concept of the tree width and tree decomposition; specifically, it is known that if the tree widths of $G_1$ and $G_2$ are at most $k$, then, so is the tree width of $G_1 \oplus G_2$.

Theorem 2.2 For $i = 1, 2, ..., t$, let $\hat{G}(k_i, n_i)$ denote the apex graph of an $n_i \times n_i$ grid, with an apex set $X_i$, $|X_i| = k_i$, so that every vertex in $X_i$ is adjacent to all vertices of the $n_i \times n_i$ grid. Let $G = \oplus_{i=1}^t \hat{G}_{k_i, n_i}$, where the
clique sum is only taken at apex sets. Then, \( \hat{G} = G_1 \times G_2 \), where, \( G_1 \) is chordal and \( ccw(G_2) \leq \sum_{i=1}^{t}(n_i + k_i) \).

**Proof.** We use the construction in part (ii) of Theorem 2.1. Thus, for \( i = 1, 2, ..., t \), there is a chordal graph \( G_1^i \) and a graph \( G_2^i \) with \( ccw(G_2^i) \leq \frac{n_i}{2} + k_i \), so that \( \hat{G}(k_i, n_i) = G_1^i \cap G_2^i \). Let \( G_1 = \oplus_{i=1}^{t} G_1^i \), and \( G_2 = \oplus_{i=1}^{t} G_2^i \). Then, \( \hat{G} = G_1 \cap G_2 \). It is easy to verify that \( G_1 \) is chordal. To verify the claim concerning \( ccw(G_2) \), we use the construction for general graphs, in [14], in a simpler setting. Particularly, let \( O_i \) be the clique cover for \( G_2^i, i = 1, 2, ..., t \), where each vertex \( x_i \) in \( X_i \) is represented as a clique and has already been placed in the “middle” of \( O_i \). When taking the sum of \( \hat{G}_k, n_i \) and \( \hat{G}_{k+1, n_{i+1}}, i = 1, 2, ..., t-1 \), we identify the vertices of the clique in \( X_i \) first, and then place the cliques in \( O_{i+1} \) (in the same order that they appear in \( O_{i+1} \)) so that each clique in \( O_i \) appears immediately to the left of a clique in \( O_{i+1} \). \( \square \)

We finish this section by exhibiting graphs in \( C(2, 1) \) with arbitrary large layered tree width. For these graphs, Theorem 1.1 is applicable, directly, but the Universal Representation Theorem (and layered treewidth approach) would fail to be effective.

**Example 2.1** (i) For positive integers \( n, k \), there is an incomparability graph \( G \in C(2, 1) \) of diameter \( \Theta(k) \) on \( \Theta(nk) \) vertices with \( ltw(G) = \Omega(n/k) \).

(ii) For positive integers \( n, k \), there is a graph \( G \in (2, 1) \) with \( n^2k \) vertices of diameter \( \Theta(n) \) so that \( ltw(G) = \Omega(k/n) \).

For (i), let \( G \) be a graph with \( ccw(G) = 1 \), or a unit incomparability graph on \( N = n.k \) vertices, of diameter \( k+2 \), where each maximal clique has \( n \) vertices. So there is a clique cover \( \{C_1, C_2, ..., C_k\} \) of \( G \) so that any edge is either in the same clique or joins two consecutive cliques. Let \( G_2 = G \). Now let \( G_1 \) be a graph which is obtained by adding all possible edges between two consecutive cliques of \( G \), then \( G_1 \) is an interval graph. Note that, \( G = G_1 \cap G_2 \), and \( n \leq tw(G) \leq ltw(G)k \).

For (ii), let \( H \) be \( n \times n \) grid embedded in the plane. To obtain \( G \) replace any vertex \( x \in V(G) \) by a clique \( c_x \) of \( k \) vertices. Now for any \( x, y \in V(H) \) with \( xy \in E(H) \) place \( k^2 \) edges joining every vertex in \( c_x \) to very vertex in \( c_y \) in \( G \). To obtain \( G_1 \), take every two consecutive rows in \( H \), and make all vertices of \( G \) in them into one single clique in \( G_1 \). It is easy to verify that \( G_1 \) is an interval graph. To construct \( G_2 \), take any column of \( H \), and make all vertices of \( G \) in them into one single clique in \( G_2 \). It can be verified that \( ccw(G_2) = 1 \), and that \( G = G_1 \cap G_2 \). \( \square \).
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