THE GENUS ONE COMPLEX QUANTUM CHERN-SIMONS REPRESENTATION OF THE MAPPING CLASS GROUP

JØRGEN ELLEGAARD ANDERSEN AND SIMONE MARZIONI

Abstract. In this paper we compute explicitly, following Witten’s prescription, the quantum representation of the mapping class group in genus one for complex quantum Chern-Simons theory associated to any simple and simply connected complex gauge group $G_C$. We use a generalization of the Weil-Gel’fand-Zak transform to exhibit an explicit expression for the representation.

1. Introduction

In his seminal paper [Wi1], Witten introduce the study of quantum Chern-Simons theory with non-abelian gauge group $G$. For the compact gauge groups $G = SU(N)$, the so called Witten-Reshetikhin-Turaev TQFT’s was first constructed by Reshetikhin and Turaev for $G = SU(N)$ [RT1, RT2, T] and subsequently by Blanchet, Habegger, Masbaum and Vogel in [BHMV1, BHMV2, B1] using skein theory. Recently Ueno jointly with the first author of this paper showed in a series of four papers [AU1, AU2, AU3, AU4] that the Witten-Reshetikhin-Turaev TQFT is the same as the TQFT coming from Conformal Field Theory [TUY, BK] as was also proposed to be the case in Witten’s original paper [Wi1]. Witten further suggested in the same paper that the geometric quantization of the moduli space of flat connections should be related to this theory and he developed this approach further in his joint paper with Axelrod and Della Pietra [ADW]. Following shortly after, Hitchin gave a rigorous account of this work in [H]. For a purely differential geometric account of the construction of this connection see [A6, AG1, AG2]. By combining the work of Laszlo [La1] with the above mention work of Ueno and the first author, it has now been confirmed that one can use the geometric quantization of the moduli space of flat connections as an alternative construction of the Witten-Reshetikhin-Turaev TQFT. This has been exploited in, among others, the works [A3, AMU, A4, AH1, A5, AY, AHJMMc].

In the paper [Wi2] Witten also proposed a way to construct the mapping class groups representations of quantum Chern-Simons theory for complex gauge groups. This theory received much less attention, but the time now seems ripe to fully develop this theory. Recently Kashaev has, jointly with the first author of this paper, rigorously constructed the $n = 2$ theory in [AK1, AK2, AK3]. First the level $k = 1$ theory was examined in detail in [AK1, AK2] and recently has the picture for all levels become clear and in fact, a very general scheme has been proposed in [AK3] for construction of TQFT’s from Pontryagin self-dual Locally compact groups. It is further verified in that paper that the scheme for the LCA $\mathbb{R} \times \mathbb{Z}_k$ yields quantum Chern-Simons theory for $PSL(2, \mathbb{C})$ at level $k$. See also [AS] for
further discussion of the general level theory. This should be seen in parallel to
the developments on the index by Garoufalidis and Dimofte in [Ga1, DGa1], which
should be related to the level $k = 0$ theory and further their work on the quantum
modularity conjecture [DGa2]. In the physics literature, the complex quantum
Chern-Simons theory has been discussed from a path integral point of view in a
number of papers [Di1, BNW, DGG, DGLZ, DGu1, GM, Gu1, Hi1, Hi2, Wi3] (see
also references in these) and latest by Dimofte [Di2, Di3] using the more advanced
3d-3d correspondence.

In parallel to this Gammelgaard in collaboration with the first author of this
paper, has given [AG2] a finite dimensional differential geometric proof of the pro-
jective flatness of the connections proposed by Witten in [Wi2]. The basic setup
is here to consider the space of all sections of the Chern-Simons line bundle over
the moduli space of flat $G$ connections as the polarised Hilbert space for the quan-
tisation of the $G_C$ moduli space, where the polarisation on the $G_C$ moduli space
depends on the choice of a complex structure on the underlying surface, just as in
the case of the complex polarizations of the moduli space for the compact group $G$.
Witten shows that one also in this non-compact case gets a connection, which
he argues using partially physics arguments, should be flat. In the work [AG2]
the first author and Gammelgaard gave a differential geometric construction of this
connection, which they named the Hitchin-Witten connection. Further they gave
a proof that this Hitchin-Witten connection is projectively flat for higher genus
surfaces.

For the genus one case, it is immediately clear that Witten’s proposal produces a
flat connection. The purpose of this paper is to compute the resulting representation
of the mapping class group in this case. Let us now review the setup from Witten’s paper.

Suppose $G$ is a simple, simply connected compact Lie group and let $G_C$ be its
complexification. Let $\Sigma$ be a closed oriented surface of genus one. Then we get the
following description of the moduli space $M_G$ of flat $G$ connection on $\Sigma$

$$M_G \cong T_G \times T_G / W,$$

where $T_G$ is the maxim torus of $G$ and $W$ is the Weyl group of $G$. We thus have
the projection map

$$\pi : T_G \times T_G \to M_G,$$

whose fiber over a generic point in the target is a copy of $W$, but since the action
of $W$ on $T_G \times T_G$ has fixed points, this is not a global covering map. Using
the Chern-Simons action functional and the correct normalisation of the invariant
inner product on the Lie algebra of $G$ (see e.g. [Fr1, RSW]), one can construct a
complex line bundle $L_G$ over $M_G$ and from the construction we get the following
identification for all integers $k$

$$C^\infty(M_G, L_G^k) \cong C^\infty(T_G \times T_G, \pi^* L_G^k)^W.$$

Introducing $H^{(k)} \equiv C^\infty(M_G, L_G^k)$ and $\tilde{H}^{(k)} \equiv C^\infty(T_G \times T_G, \pi^* L_G^k)$, we of course
have that

$$H^{(k)} = (\tilde{H}^{(k)})^W.$$

At this point, we wish however, to remark that when considering the moduli space, the non-
stable bundles are ignored. As can be seen from [AG2], working with the stack of all bundles as
opposed to the moduli spaces of semi-stable bundles, might effect significantly the answer exactly
in genus one.
Consider \( \mathbb{H} \), the upper half plane, which is the Teichmüller space for \( \Sigma \), in the sense that it parametrises all marked complex structures on \( \Sigma \) up to natural equivalence (see also \[A2\] for the general torus case). The construction of Complex Quantum Chern-Simons Theory with gauge group \( G_C \) outlined in \[Wi2\] provides an identification of the Quantum Hilbert space of this theory with the pre-Quantum Hilbert space for the theory relative to the compact group \( G \), that is \( \mathcal{H}^{(k)} \). However, such an identification depends on a choice of complex structure on the \( \Sigma \). The complex structure on \( \Sigma \) induces a real polarization on the moduli space \( M_{G_C} \), the moduli space of flat \( G_C \)-connection, via the Hodge star operator on \( \Sigma \) as described in \[Wi2\]. Consequently we have a trivial vector bundle \( \mathcal{H}^{(k)} \times \mathbb{H} \to \mathbb{H} \) with a non-trivial flat Hitchin-Witten connection. For the genus one moduli spaces the Hitchin–Witten connection \( \nabla \) happens to have a simple description as follows. Fix a complex number \( t = k + is \), such that \( k \), the real part of \( t \), is an integer. Let \( d \) be the trivial connection on the bundle \( \mathcal{H}^{(k)} \times \mathbb{H} \). As Witten argues in \[Wi2\], if one chooses \( r \) such that 
\[
e^{-4r} = \frac{k - is}{k + is},
\]
then one finds that
\[
e^{-r\Delta} \circ d \circ e^{r\Delta} = \nabla,
\]
where \( \Delta = \Delta_\sigma \) is the Laplace operator on \( M_G \), which of course depends on \( \sigma \in \mathbb{H} \) parametrising the family of complex structures compatible with the Goldman symplectic form on \( M_G \), which are induced from the natural family of complex structures on \( \Sigma \). This means that parallel sections of \( \mathcal{H}^{(k)} \times \mathbb{H} \) are of the form
\[
s_\sigma = e^{-r\Delta_\sigma} s_0
\]
for any \( s_0 \in C^\infty(M_G, \mathcal{L}^k) \).

The mapping class group \( \Gamma = SL(2, \mathbb{Z}) \) acts on \( M_G \) and classical Chern-Simons theory allows one to lift this action to \( \mathcal{L}^k \), thus one gets an induced pre-quantum action
\[
\rho_k : \Gamma \to B(\mathcal{H}^{(k)}),
\]
where \( B(\mathcal{H}^{(k)}) \) refers to the continuous operators which respect to the Frechet topology on \( \mathcal{H}^{(k)} \). We call this action the pre-Quantum representation of \( \Gamma \). Note that we can also complete this to an action on the Hilbert space of \( L^2 \)-sections of \( L^k \) over \( M_G \), in which case \( \Gamma \) acts by bounded operators on this Hilbert space.

What we are seeking, however, are the quantum representations \( \eta_t \) which arise when we take into account the full action of \( \Gamma \) on the bundle \( \mathcal{H}^{(k)} \times \mathbb{H} \) composed with Hitchin-Witten parallel transport \( \mathcal{P}_{\sigma_0, \sigma_1} : (\mathcal{H}^{(k)}, \sigma_0) \to (\mathcal{H}^{(k)}, \sigma_1) \). That is, we first consider the action
\[
\tilde{\varphi} : (\mathcal{H}^{(k)}, \sigma) \to (\mathcal{H}^{(k)}, \varphi_\sigma)
\]
and then compose it with the parallel transport, to get a representation on \( (\mathcal{H}^{(k)}, \sigma) \)
\[
\eta_t, \sigma(\varphi) = \mathcal{P}_{\sigma, \varphi_\sigma} \circ \tilde{\varphi} : (\mathcal{H}^{(k)}, \sigma) \to (\mathcal{H}^{(k)}, \sigma).
\]
However, the description \( \mathbb{H} \) of \( \nabla \) leads to the following identification

**Proposition 1.** The densely defined transformation
\[
e^{r\Delta_\sigma} : \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}
\]
conjugates \( \rho_k \) to \( \eta_{t, \sigma} \) for all \( \sigma \in \mathbb{H} \). Thus for all \( \varphi \in \Gamma \), we have that

\[
\eta_{t, \sigma}(\varphi) = e^{-r \Delta_{\sigma}} \circ \rho_k(\varphi) \circ e^{r \Delta_{\sigma}}
\]
on the domain of \( e^{r \Delta_{\sigma}} \).

Let us consider the following generators of \( \Gamma \):

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

which satisfies the relations

\[
S^2 = (ST)^3, \quad S^4 = \text{Id}.
\]

In this paper we compute the representations \( \rho_k \) and \( \eta_t \) completely explicitly. We use standard notation from Lie theory, which we briefly recall in Appendix A where we further introduce a few quantities we will need in this paper. In particular we recall that \( t \subset g \) is the Lie algebra of \( T_G \) and that we denote by \( Z_{(k)} \) the finite set of \( \frac{1}{k} \)-scaled weights modulo the root lattice of \( g \).

In Section B we present a generalization of the Weil-Gel’fand-Zak transform which provides an isometry

\[
Z : \mathcal{H}_{(k)} \to \left( L^2(t) \otimes \mathbb{C} Z_{(k)} \right)^W.
\]

The \( W \)-invariance on the right is with respect to the following action of \( w \in W \)

\[
w \cdot f_{\gamma}(\theta) = f_{w\gamma}(w\theta) \quad \text{for} \quad \gamma \in Z_{(k)}, \quad \theta \in T.
\]

For technical reasons, we further conjugate with a discrete Fourier transform \( F_{Z_{(k)}} \) (see the definition from equation (35)). We use the following notation for the corresponding pre-quantum operators action on the square integrable functions

\[
\hat{\rho}_k(S) = (Z \circ F_Z)^{-1} \circ \hat{S} \circ (Z \circ F_Z), \\
\hat{\rho}_k(T) = (Z \circ F_Z)^{-1} \circ \hat{T} \circ (Z \circ F_Z)
\]

and correspondingly for the quantum representation

\[
\hat{\eta}_t(S) = (Z \circ F_Z)^{-1} \circ \eta_t(S) \circ (Z \circ F_Z), \\
\hat{\eta}_t(T) = (Z \circ F_Z)^{-1} \circ \eta_t(T) \circ (Z \circ F_Z).
\]

Considering the \( W \)-invariant subspace of \( L^2(t) \otimes \mathbb{C} Z_{(k)} \) we get the decomposition

\[
\mathcal{H}_{(k)} \simeq \left( L^2(t) \otimes \mathbb{C} Z_{(k)} \right)^W
\]

\[
= (L^2(F_0)) \otimes C_{(k)} \bigoplus (L^2(F_0) \otimes C_{(k)}),
\]

for which we refer again to Appendix A for definitions and notations.

**Theorem 1.** Let \( j \) and \( \omega \) be complex numbers satisfying

\[
j^4 = 1, \quad \omega^3 = i^{n/2} j^{-1}.
\]

The pre-Quantum representation \( \hat{\rho}_k \) splits into the direct sum of two representations, induced by the decomposition (35)

\[
\hat{\rho}_k = (\hat{\rho}_{k,0} \otimes \hat{\rho}_k''_{(0)}) \oplus (\hat{\rho}_{k,1} \otimes \hat{\rho}_k''_{(1)}).
\]
where, for $\theta \in F_0$, $\alpha, \beta, \alpha', \beta' \in I_k$

\[
\hat{\rho}'_{k,0}(S)(f)(\theta) = j \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} e^{2\pi i (w \theta, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

\[
\hat{\rho}'_{k,1}(S)(f)(\theta) = j \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} \det(w) e^{2\pi i (w \theta, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

\[
\hat{\rho}'_{k,0}(T)(f)(\theta) = \hat{\rho}'_{k,1}(T)(f)(\theta) = \omega e^{-\pi i (\theta, \tilde{\theta})} f(\theta),
\]

\[
\hat{\rho}''_{k,0}(S)_{\alpha,\beta} = \frac{j^{-1}}{\sqrt{|Z(k)|}} \sum_{w \in W} \det(w) e^{2\pi i (w, \alpha, \beta)},
\]

\[
\hat{\rho}''_{k,1}(S)_{\alpha',\beta'} = \frac{j^{-1}}{\sqrt{|Z(k)|}} \sum_{w \in W} e^{2\pi i (w, \alpha', \beta')},
\]

\[
\hat{\rho}''_{k,0}(T)_{\alpha,\beta} = \omega^{-1} e^{\pi i (\alpha, \beta)} \delta_{\alpha, \beta},
\]

\[
\hat{\rho}''_{k,1}(S)_{\alpha',\beta'} = \omega^{-1} e^{\pi i (\alpha', \beta')} \delta_{\alpha', \beta'}.
\]

Let $b \in \mathbb{C}$ such that $|b| = 1$, $\text{Re}(b) > 0$ and

\[
i s = k \frac{1 - b^2}{1 + b^2}.
\]

**Theorem 2.** The quantum representation $\hat{\eta}$ is obtained by conjugation with a unitary operator

\[
\hat{\eta}(\varphi) = e^{-r \Delta^\sigma} \circ \hat{\rho}_k(\varphi) \circ e^{r \Delta^\sigma}.
\]

It decomposes itself as

\[
\hat{\eta} = (\hat{\eta}'_{k,0} \otimes \hat{\rho}''_{k,0}) \oplus (\hat{\eta}'_{k,1} \otimes \hat{\rho}''_{k,1}).
\]

For the particular choice of complex structure $\sigma = ib$ the representations on $L^2(F_0)$ take the following explicit integral form

\[
\hat{\eta}'_{k,0}(S)(f)(\theta) =je^{\pi i (b - b\overline{\theta}) (\theta, \tilde{\theta})} \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} e^{2\pi i (w \theta, \tilde{\theta})} e^{-\pi i (b - b\overline{\theta})(\tilde{\theta}, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

\[
\hat{\eta}'_{k,1}(S)(f)(\theta) =je^{\pi i (b - b\overline{\theta}) (\theta, \tilde{\theta})} \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} \det(w) e^{2\pi i (w \theta, \tilde{\theta})} e^{-\pi i (b - b\overline{\theta})(\tilde{\theta}, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

\[
\hat{\eta}'_{k,0}(T)(f)(\theta) = \omega i^{-\frac{b}{2}} e^{\pi i (b - b\overline{\theta})(\theta, \tilde{\theta})} \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} e^{\pi i (w \theta - \tilde{\theta}, w \theta - \tilde{\theta})} e^{-\pi i (b - b\overline{\theta})(\tilde{\theta}, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

\[
\hat{\eta}'_{k,1}(T)(f)(\theta) = \omega i^{-\frac{b}{2}} e^{\pi i (b - b\overline{\theta})(\theta, \tilde{\theta})} \int_{F_0} f(\tilde{\theta}) \sum_{w \in W} \det(w) e^{\pi i (w \theta - \tilde{\theta}, w \theta - \tilde{\theta})} e^{-\pi i (b - b\overline{\theta})(\tilde{\theta}, \tilde{\theta})} d\text{vol}_k(\tilde{\theta}),
\]

In Section 4, Lemma 3 we provide the explicit expression for $e^{-r \Delta^\sigma}$ for generic $\sigma$. In Section 5 we provide comparisons to the known result for quantum Chern-Simons theory with compact gauge group $G$.

2. **Moduli space of flat connections and the pre-quantum line bundle**

In this section we briefly recall the construction of the pre-quantum space for Chern-Simons theory. We refer to [FT] and [ADW] for more details. See Appendix A for notations on Lie theory. Given a closed surface $\Sigma$ of genus $g = 1$, its
fundamental group is abelian, precisely
\[ \pi_1(\Sigma) = \mathbb{Z} \times \mathbb{Z}. \]

Let \( G \) be a compact, simple, connected and simply connected Lie group. The moduli space of flat connections is then
\[ M = \text{Hom}(\pi_1(\Sigma), G)/G = (\mathbb{T} \times \mathbb{T})/W, \]
where \( \mathbb{T} \) is the maximal torus of \( G \) and \( W \) is the Weyl group of \( G \). We can lift the description to quotient of its tangent space
\[ M = (t \oplus t)/((\mathbb{A}^R \times \mathbb{A}^R) \times W). \]

Since any principal \( G \)-bundle on \( \Sigma \) is trivial, we can fix a trivialization and so a \( G \)-connection \( A \in \Omega(\Sigma, g) \) is a one form on \( \Sigma \) with values in the lie algebra \( g \). In particular, we can identify \( t \oplus t \) with a subspace of flat connections with values in \( t \). The group \( (\mathbb{A}^R \times \mathbb{A}^R) \times W \) corresponds to the gauge transformations preserving this subspace of flat connections inside \( \Omega(\Sigma, t) \). Explicitly if \( x, y \) are local coordinates on \( \Sigma = \mathbb{R}^2/\mathbb{Z}^2 \), we can write \( A = \theta_1 dx + \theta_2 dy \), where \( \theta_1, \theta_2 \in t \) and identify \( A \) with \( \theta_1 \oplus \theta_2 \in t \oplus t \). Recall the preferred inner product \( (\cdot, \cdot) \) on \( t \) with the normalization specified in Appendix A. The Atiyah-Bott [46] symplectic form on \( t \oplus t \) is given by
\[ \omega(A, A') = 2\pi \int_{\Sigma} (A \wedge A')_1 = 2\pi ((\theta_1, \theta'_2)_1 - (\theta'_1, \theta_2)_1), \]
where \( A = \theta_1 \oplus \theta_2 \) and \( A' = \theta'_1 \oplus \theta'_2 \).

Let \( \{b_1, \ldots, b_n\} \) be a basis for \( t \). We can write coordinates for \( A = \theta_1 dx + \theta_2 dy \) as
\[ \theta_1 = \sum_{j=1}^{n} u_j b_j, \quad \theta_2 = \sum_{j=1}^{n} v_j b_j. \]

A complex structure on \( \Sigma \) is equivalent to a choice of holomorphic coordinate \( w = x + \tau y \) with \( \text{Im} \tau > 0 \). This complex structure induces a complex structure \( J = -\ast \) on \( M \), where \( \ast \) is the hodge star operator. \( J \) is compatible with \( \omega \) and we can write holomorphic coordinates on \( M \)
\[ z_j = u_j + \sigma v_j \quad \bar{z}_j = u_j + \bar{\sigma} v_j \]
where
\[ \sigma = -\frac{1}{\tau}. \]

The action of \( w \in W \) on \( \theta_1 \oplus \theta_2 \) is given by
\[ w(\theta_1, \theta_2) = \left( \sum_j u_j w(b_j), \sum_j v_j w(b_j) \right). \]

Let \( C^{(k)} = (C^{(k)}_{jl})_{j,l=1,\ldots,n} \) be the matrix defined as \( C^{(k)}_{jl} \equiv (b_j, b_l)_k \). The symplectic form \( \omega \) takes the following explicit form
\[ \omega = 2\pi \sum_{j,l} C^{(1)}_{jl} du_j \wedge dv_l. \]

The level 1 pre-Quantum line bundle \( \mathcal{L} \rightarrow M \) defined by classical Chern-Simons theory (see [46, [47, [48, [49]) can be constructed starting with the trivial bundle.
\( \bar{L} \equiv \mathfrak{t} \oplus \mathfrak{t} \times \mathbb{C} \). We lift the action of \( \Lambda^R \times \Lambda^R \) to \( \bar{L} \) as follows, for \( (A, \zeta) \in \mathfrak{t} \oplus \mathfrak{t} \times \mathbb{C} \) and \( \lambda = (\lambda_1, \lambda_2) \in \Lambda^R \times \Lambda^R \),

\[
(18) \quad \langle A, \zeta \rangle \cdot \lambda = (A + \lambda, e_\lambda(A) \zeta),
\]

\[
(19) \quad \text{where } e_\lambda(A) \equiv (-1)^{\langle \lambda_1, \lambda_2 \rangle} \exp - \frac{i}{2} (\omega(A, \lambda)).
\]

This defines a line bundle \( \pi^* L \rightarrow T \times T \), where \( \pi \) is the projection onto \( M \). The first Chern class of \( \pi^* L \) is \( c_1(\pi^* L) = [\frac{\sqrt{\pi}}{2\pi}] \in H^2(T \times T, \mathbb{Z}) \).

To lift the action of \( W \) to \( \pi^* L \) one takes the trivial lift to \( \bar{L} \)

\[
(20) \quad \langle A, \zeta \rangle \cdot w = (w(A), \zeta), \quad \text{for every } w \in W.
\]

It is simple to prove that such maps are equivariant with respect to the action of \( \Lambda^R \times \Lambda^R \) (see [6, Lemma A.2]). This completes the description of the Chern–Simons line bundle \( L \rightarrow M \).

**Remark 1.** The \( e_\lambda \) are called **multipliers** in the literature. The factor \((-1)^{\langle \lambda_1, \lambda_2 \rangle}\) is usually referred to as a **theta–characteristic** and it is not uniquely defined by the Chern–Simons action. In fact it is irrelevant in the definition of \( L \rightarrow M \) but a choice of it is needed to lift it to \( \pi^* L \) (see [6]).

The line bundle \( L \) support a preferred pre-Quantum connection, given explicitly on \( \bar{L} \) by

\[
(21) \quad \nabla = d + \alpha
\]

\[
(22) \quad \alpha_A(X) = \omega(A, X), \quad \text{for any } X \in T_A(\mathfrak{t} \oplus \mathfrak{t}).
\]

This connection has curvature \( F_\nabla = -i\omega \) and is compatible with the actions (18) and (20), therefore it descends to a well defined connection on \( L \).

The line bundle \( L^k \equiv \mathcal{L} \otimes k \) can be easily obtained from the description above substituting the inner product \( \langle \cdot, \cdot \rangle_1 \) with the \( k \)-**scaled** inner product \( \langle \cdot, \cdot \rangle_k \).

Consider the space \( \mathcal{H}^{(k)} = C^\infty(T \times T, \pi^* L^k) \). We are interested in the level \( k \) quantum space

\[
\mathcal{H}^{(k)} \equiv C^\infty(M, L^k) = C^\infty(T \times T, \pi^* L^k)^W
\]

of Weyl group invariant smooth sections of \( \pi^* L^2 \).

The inner product given by

\[
(23) \quad \varphi \cdot \psi(p) = \varphi(p)\bar{\psi}(p), \quad p \in T \times T, \quad \varphi, \psi \in \mathcal{H}^{(k)}
\]

is parallel with respect to our choice of \( \nabla \). This allows us to define an inner product on \( \mathcal{H}^{(k)} \) as

\[
(24) \quad \langle \varphi, \psi \rangle = \frac{1}{|\Lambda|} \int_{T \times T} \varphi \cdot \bar{\psi} \cdot d\text{vol}_k, \quad p \in T \times T, \quad \varphi, \psi \in \mathcal{H}^{(k)}.
\]

Here \( d\text{vol}_k \) is the volume form induced by the inner product \( \langle \cdot, \cdot \rangle_k \).

An element \( \gamma \in \Gamma \) acts by pull back on \( M \) mapping the equivalence class or representations \([\rho] \in M \) to \([\rho \circ \gamma]\). Classical Chern-Simons theory enables us to lift the action to the line bundle \( L^k \) to get an operator

\[
(25) \quad \bar{\gamma} : \mathcal{H}^{(k)} \rightarrow \mathcal{H}^{(k)},
\]
by defining, on \( \tilde{L} \)
\[
(A, \zeta) \cdot \gamma = (\gamma^*(A), \zeta).
\]

(26)

It can be easily shown that this lift is equivariant with respect to \( \Lambda^R \times \Lambda^R \) (see [Je]).

This is not sufficient to define a complex quantum representation of \( \Gamma \), indeed an identification of the \( SL(2, \mathbb{C}) \) quantum Hilbert space with \( \mathcal{H} \) depends on a choice of complex structure \( \tau \) on \( \Sigma \). Such a choice is not invariant under the action of \( \Gamma \), indeed \( \gamma \in \Gamma \) acts on \( \mathcal{H} \) via Mobius transformations. Nevertheless we can compute the action of \( \tilde{S} \) and \( \tilde{T} \) on \( t \oplus t \) as follows
\[
\tilde{S}\psi(\theta_1, \theta_2) = \psi(\theta_2, -\theta_1)
\]
(27)
\[
\tilde{T}\psi(\theta_1, \theta_2) = \psi(\theta_1, \theta_1 + \theta_2).
\]
(28)

We remark that these operators commute with the action of the Weyl group and are equivariant with respect to the action of \( \Lambda^R \times \Lambda^R \) (see proof of Lemma A.2 in [Je]).

3. Weil-Gel’fand-Zak Transform on lattices.

Let \((E, (\cdot, \cdot))\) be a fixed euclidean space of dimension \( \dim E = n < \infty \). Let \( \Lambda \subset E \) be a full-rank lattice. Denote \( T_E \) the quotient \( E/\Lambda \). Let \( L \rightarrow T_E \otimes T_E \) be the line bundle defined by the following multipliers
\[
e_{\lambda_1 \oplus \lambda_2}(x \oplus y) = (-1)^{(\lambda_1, \lambda_2)}e^{-\pi i((x, \lambda_2)-(\lambda_1, y))}
\]
(29)

Define the following notation for Fourier kernels
\[
\langle x, y \rangle \equiv e^{2\pi i(x, y)}
\]
(30)

and define the dual \( \Lambda^* \subset E \) as the lattice
\[
\Lambda^* \equiv \{ \gamma \in E : \langle \lambda, \gamma \rangle = 1, \forall \lambda \in \Lambda \}.
\]
(31)

We say that \( \Lambda \) is integral if \( \Lambda \subset \Lambda^* \). In such case \( Z \equiv \Lambda^*/\Lambda \) is a well defined finite abelian group.

Suppose now \( \Lambda \) is integral. The space \( L^2(E) \otimes \mathbb{C}^Z \) has an inner product defined as follows. Let \( f_\gamma, g_\gamma \in L^2(E) \) for \( \gamma \in Z \) be square integrable functions on \( E \), for \( f = (f_\gamma)_{\gamma \in Z} \) and \( g = (g_\gamma)_{\gamma \in Z} \in L^2(E) \otimes \mathbb{C}^Z \)

we define
\[
\langle f, g \rangle = \sum_{\gamma \in Z} \int_E f_\gamma \overline{g_\gamma} dvol,
\]
(32)

where the volume form \( dvol \) is specified by the inner product \((\cdot, \cdot)\). We write
\[
\text{Vol} \Lambda = \int_{F_\Lambda} dvol
\]

for the volume of a fundamental domain \( F_\Lambda \subset E \) for the action of \( \Lambda \).

On the space of sections \( L^2(T_E \times T_E, L) \) we consider the inner product
\[
\langle \psi, \varphi \rangle = \frac{1}{\text{Vol} \Lambda} \int_{F_\Lambda \times F_\Lambda} \psi(\theta_1, \theta_2) \overline{\varphi(\theta_1, \theta_2)} dvol(\theta_1, \theta_2)
\]
(33)

where the measure is the product measure on \( E \times E \).
Let $S(E)$ be the space of Schwartz functions on $E$. The following is the Weil-Gel'fand-Zak transform on the lattice $\Lambda$.

**Theorem 3.** We have an isomorphism

$$Z : S(E) \otimes \mathbb{C}^Z \rightarrow C^\infty(\mathbb{T}_E \times \mathbb{T}_E, L)$$

given by

$$Z(f)(\theta_1, \theta_2) = \frac{1}{\sqrt{|Z|}} (-\theta_1/2, \theta_2) \sum_{\gamma \in Z} \sum_{\lambda \in \Lambda^*} f_\gamma(\theta_1 + \lambda) \langle -\lambda, \theta_2 \rangle \langle -\lambda, \gamma \rangle,$$

with inverse

$$Z^{-1}(s)(\theta, \gamma) = \frac{1}{\sqrt{|Z|}} \sum_{\tilde{\gamma} \in Z} \sum_{\lambda \in \Lambda^*} s(\theta - \tilde{\gamma}, \tilde{\theta}) \langle \theta + \tilde{\gamma}, \tilde{\theta}/2 \rangle \text{dvol}(\tilde{\theta}).$$

It satisfies the following unitarity property

$$\langle Z(f), Z(g) \rangle = \langle f, g \rangle$$

thus $Z$ extends to an isometry between $L^2(E) \otimes \mathbb{C}^Z$ and $L^2(\mathbb{T}_E \times \mathbb{T}_E, L)$.

**Proof.** We write $d\theta$ instead of $\text{dvol}(\theta)$ in the computations.

$$Z \left( Z^{-1}(s) \right)(\theta_1, \theta_2) = \frac{1}{\sqrt{|Z|}} (-\theta_1/2, \theta_2) \sum_{\gamma \in Z} \sum_{\lambda \in \Lambda^*} Z^{-1}(s)(\theta_1 + \lambda, \gamma) \langle -\lambda, \theta_2 \rangle \langle -\lambda, \gamma \rangle$$

$$= \frac{1}{|Z|} (-\theta_1/2, \theta_2) \sum_{\gamma \in Z} \sum_{\lambda \in \Lambda^*} \frac{1}{\text{Vol} \Lambda} \langle -\lambda, \theta_2 \rangle \sum_{\gamma \in Z} \langle \gamma, \tilde{\gamma} - \lambda \rangle$$

$$\times \int_{F_x} s(\theta_1 + \lambda - \tilde{\gamma}, \tilde{\theta}) \langle \theta_1 + \lambda + \tilde{\gamma}, \tilde{\theta}/2 \rangle d\tilde{\theta}$$

$$= \langle -\theta_1/2, \theta_2 \rangle \sum_{\gamma \in Z} \sum_{\lambda \in \Lambda^*} \frac{1}{\text{Vol} \Lambda} \int_{F_x} s(\theta_1 + \lambda, \tilde{\theta}) \langle \theta_1 + \lambda + 2\tilde{\gamma}, \tilde{\theta}/2 \rangle d\tilde{\theta}$$

$$\times \langle \tilde{\lambda} - \tilde{\gamma}, \theta_2 \rangle$$

$$= \langle -\theta_1/2, \theta_2 \rangle \sum_{\gamma \in Z} \sum_{\lambda \in \Lambda^*} \frac{1}{\text{Vol} \Lambda} \int_{F_x} s(\theta_1 + \tilde{\theta}) \langle \theta_1 + 2\tilde{\gamma}, \tilde{\theta}/2 \rangle d\tilde{\theta}$$

$$\times \langle \tilde{\lambda} - \tilde{\gamma}, \theta_2 \rangle$$

$$= s(\theta_1, \theta_2),$$

where the last equation is given by the Fourier series properties of the $\Lambda$-periodic function $s(\theta_1, \tilde{\theta}) \langle \theta_1/2, \tilde{\theta} \rangle$ in the variable $\tilde{\theta}$. 
To establish the unitarity of the transform, we calculate

$$
\langle Z(f), Z(g) \rangle = \frac{1}{\text{Vol} \Lambda} \int_{F_\Lambda \times F_\Lambda} Z(f)(\theta_1, \theta_2) \overline{Z(g)(\theta_1, \theta_2)} d\theta_1 d\theta_2
$$

$$
= \frac{1}{\text{Vol} \Lambda |Z|} \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}, \lambda, \hat{\lambda} \in \Lambda^*} \langle -\lambda, \gamma \rangle \langle \hat{\lambda}, \hat{\gamma} \rangle \int_{F_\Lambda} f(\theta_1 + \lambda, \gamma) \overline{g(\theta_1 + \hat{\lambda}, \hat{\gamma})} d\theta_1
$$

$$
\times \int_{F_\Lambda} \langle \hat{\lambda} - \lambda, \theta_2 \rangle d\theta_2
$$

$$
= \frac{1}{|Z|} \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}, \lambda, \hat{\lambda} \in \Lambda^*} \langle -\lambda, \gamma \rangle \langle \hat{\lambda}, \hat{\gamma} \rangle \int_{F_\Lambda} f(\theta_1 + \lambda, \gamma) \overline{g(\theta_1 + \hat{\lambda}, \hat{\gamma})} d\theta_1
$$

$$
= \frac{1}{|Z|} \sum_{\gamma, \hat{\gamma}, \tilde{\gamma} \in \mathbb{Z}, \lambda \in \Lambda} \langle -\tilde{\gamma}, \gamma \rangle \langle \tilde{\gamma}, \hat{\gamma} \rangle \int_{F_\Lambda} f(\theta_1 + \tilde{\gamma}, \gamma) \overline{g(\theta_1 + \lambda + \tilde{\gamma}, \hat{\gamma})} d\theta_1
$$

$$
= \sum_{\tilde{\gamma} \in \mathbb{Z}} \int_{E} \frac{1}{|Z|} \sum_{\gamma \in \mathbb{Z}} f(\theta_1, \gamma) \langle -\tilde{\gamma}, \gamma \rangle \frac{1}{\sqrt{|Z|}} \sum_{\hat{\gamma} \in \mathbb{Z}} \overline{g(\theta_1, \hat{\gamma})} \langle \tilde{\gamma}, \hat{\gamma} \rangle d\theta_1
$$

We define the following auxiliary operators on $L^2(E) \otimes \mathbb{C}^Z$

$$
\mathcal{F}_Z(f)(\theta, \gamma) = \frac{1}{\sqrt{|Z|}} \sum_{\tilde{\gamma} \in \mathbb{Z}} f_\gamma(\theta, \tilde{\gamma}) \quad \mathcal{G}_Z(f)(\theta, \gamma) = \langle \gamma/2, \gamma \rangle f(\theta, \gamma)
$$

$$
\mathcal{F}_E(f)(\theta, \gamma) = \int_{E} f_\gamma(\tilde{\theta}) \langle \theta, \tilde{\theta} \rangle d\text{vol}(\tilde{\theta}) \quad \mathcal{G}_E(f)(\theta, \gamma) = \langle \theta/2, \theta \rangle (f)(\theta, \gamma).
$$

Let $\hat{S}$ and $\hat{T}$ be two operators acting on $L^2(T_E \times T_E; L)$ as expressed in (27-28). Define the following operators acting on $L^2(E) \otimes \mathbb{C}^Z$

$$
\hat{S} = (Z \circ \mathcal{F}_Z)^{-1} \circ \hat{S} \circ (Z \circ \mathcal{F}_Z), \quad \hat{T} = (Z \circ \mathcal{F}_Z)^{-1} \circ \hat{T} \circ (Z \circ \mathcal{F}_Z).
$$

They can be explicitly computed as

**Proposition 2.**

$$
\hat{S}(f)(\theta, \gamma) = \mathcal{F}_Z^{-1} \circ \mathcal{F}_E(f)(\theta, \gamma),
$$

$$
\hat{T}(f)(\theta, \gamma) = \mathcal{G}_Z \circ \mathcal{G}_E^{-1}(f)(\theta, \gamma).
$$
Proof:

Proof.  

\[ Z^{-1} \circ \tilde{S} \circ Z(f)(\theta, \gamma) = \frac{1}{\sqrt{|Z|}} \sum_{\gamma \in \mathbb{Z}} \langle \gamma, \tilde{\gamma} \rangle \frac{1}{\text{Vol } \Lambda} \int_{F_\Lambda} S(Z(f))(\theta - \tilde{\gamma}, \tilde{\theta}) \left\langle \theta + \tilde{\gamma}, \tilde{\theta} / 2 \right\rangle d\tilde{\theta} \]

\[ = \frac{1}{\sqrt{|Z|}} \sum_{\gamma \in \mathbb{Z}} \langle \gamma, \tilde{\gamma} \rangle \frac{1}{\text{Vol } \Lambda} \int_{F_\Lambda} Z(f)(\tilde{\theta}, -\theta + \tilde{\gamma}) \left\langle \theta + \tilde{\gamma}, \tilde{\theta} / 2 \right\rangle d\tilde{\theta} \]

\[ = \frac{1}{\text{Vol } \Lambda} \frac{1}{|Z|} \sum_{\gamma \in \mathbb{Z}} \sum_{\lambda \in \Lambda^*} \int_{F_\Lambda} f(\tilde{\theta} + \tilde{\lambda}, \tilde{\gamma}) \left\langle \theta, \tilde{\theta} \right\rangle \left\langle \lambda, \theta - \tilde{\gamma} \right\rangle d\tilde{\theta} \]

\[ \times \sum_{\gamma \in \mathbb{Z}} \langle \gamma, \gamma - \lambda \rangle \]

\[ = \frac{1}{\text{Vol } \Lambda} \sum_{\gamma \in \mathbb{Z}} \sum_{\lambda \in \Lambda^*} \int_{F_\Lambda} f(\tilde{\theta}, \tilde{\gamma}) \left\langle \theta, \tilde{\theta} \right\rangle d\tilde{\theta} \langle \gamma, -\gamma \rangle . \]

Noticing that \(|Z| = (\text{Vol } \Lambda)^2\) we get the first equation.

\[ Z^{-1} \circ \tilde{T} \circ Z(f)(\theta, \gamma) = \frac{1}{\sqrt{|Z|}} \sum_{\gamma \in \mathbb{Z}} \langle \gamma, \tilde{\gamma} \rangle \frac{1}{\text{Vol } \Lambda} \int_{F_\Lambda} T(Z(f))(\theta - \tilde{\gamma}, \tilde{\theta}) \left\langle \theta + \tilde{\gamma}, \tilde{\theta} / 2 \right\rangle d\tilde{\theta} \]

\[ = \frac{1}{\sqrt{|Z|}} \sum_{\gamma \in \mathbb{Z}} \langle \gamma, \tilde{\gamma} \rangle \frac{1}{\text{Vol } \Lambda} \int_{F_\Lambda} Z(f)(\theta - \tilde{\gamma}, \tilde{\theta} + \theta - \tilde{\gamma}) \left\langle \theta + \tilde{\gamma}, \tilde{\theta} / 2 \right\rangle d\tilde{\theta} \]

\[ = \frac{1}{|Z|} \text{Vol } \Lambda \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}} \langle \gamma, \hat{\gamma} \rangle \int_{F_\Lambda} \sum_{\lambda \in \Lambda^*} f(\theta - \hat{\gamma} + \lambda, \hat{\gamma}) \left\langle -\theta - \hat{\gamma}, \theta - \hat{\gamma} + \tilde{\theta} \right\rangle \]

\[ \times \langle -\lambda, \theta - \hat{\gamma} + \tilde{\theta} \rangle \langle -\lambda, \hat{\gamma} \rangle \left\langle \theta + \hat{\gamma}, \tilde{\theta} / 2 \right\rangle d\tilde{\theta} \]

\[ = \frac{1}{|Z|} \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}} \langle \gamma, \hat{\gamma} \rangle \sum_{\lambda \in \Lambda^*} f(\theta - \hat{\gamma} + \lambda, \hat{\gamma}) \left\langle -\theta - \hat{\gamma}, \theta - \hat{\gamma} \right\rangle \]

\[ \times \langle -\lambda, \theta - \hat{\gamma} \rangle \langle -\lambda, \hat{\gamma} \rangle \frac{1}{\text{Vol } \Lambda} \int_{F_\Lambda} \langle \tilde{\theta}, \gamma - \lambda \rangle d\tilde{\theta} \]

\[ = \frac{1}{|Z|} \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}} \langle \gamma, \hat{\gamma} \rangle f(\theta, \hat{\gamma}) \left\langle \frac{\theta - \hat{\gamma}}{2}, \theta - \hat{\gamma} \right\rangle \left\langle -\hat{\gamma}, \theta - \hat{\gamma} \right\rangle \langle -\hat{\gamma}, \gamma \rangle \]

\[ = \langle -\theta / 2, \theta \rangle \frac{1}{\sqrt{|Z|}} \sum_{\gamma, \hat{\gamma} \in \mathbb{Z}} \langle \gamma, \hat{\gamma} \rangle \int_{\gamma \in \mathbb{Z}} \frac{1}{\sqrt{|Z|}} \sum_{\hat{\gamma} \in \mathbb{Z}} f(\theta, \hat{\gamma}) \langle -\hat{\gamma}, \gamma \rangle \]

\[ = G_e^{-1} \circ F_Z \circ G_Z \circ F_Z^{-1}(f)(\theta, \gamma) . \]

\[ \square \]

It is evident that both \( \tilde{S} \) and \( \tilde{T} \) have a (unique up to a scalar) tensor product decomposition corresponding to \( L^2(E) \otimes \mathbb{C}^Z \). We write this precisely in the next proposition.

**Proposition 3.** Let \( j \) and \( \omega \) be complex numbers satisfying

\[ (40) \quad j^4 = 1, \quad \omega^3 = i^{n/2} j^{-1} . \]
The operators \( \hat{S} \) and \( \hat{T} \) generate a representation of \( SL(2, \mathbb{Z}) \) as endomorphisms of \( L^2(E) \otimes \mathbb{C}^Z \). For any possible choice of \( j \) and \( \omega \) there is a tensor product decomposition

\[
\begin{align*}
\hat{S} &= \hat{S}' \otimes \hat{S}'' \\
\hat{T} &= \hat{T}' \otimes \hat{T}''
\end{align*}
\]

where \( \hat{S}' \) and \( \hat{T}' \) (resp. \( \hat{S}'' \) and \( \hat{T}'' \)) generates a representation on \( L^2(E) \) (resp. \( \mathbb{C}^Z \)).

**Proof.** The proof is just a direct verification of the relations of \( SL(2, \mathbb{Z}) \). \( \square \)

We now apply these computations to the genus 1 Chern–Simons theory presented in the previous section (see also Appendix A for further notation). The Euclidean space is \( (t, (\cdot, \cdot)_k) \) while the line bundle is \( \pi^* \mathcal{L}^k \rightarrow \mathbb{T} \times \mathbb{T} \). The integral lattice we consider is \( \Lambda^H \) and its dual will be \( \Lambda^H_\ast \). Their quotient is the finite abelian group \( \mathcal{Z}_k \). The operator \( Z \circ F_{\mathcal{Z}_k} \) defines an isometry between \( \mathcal{H}^{(k)} \) and \( L^2(t) \otimes \mathbb{C}^{\mathcal{Z}_k} \).

In this way, Proposition 3 provides an explicit description of a lift of the action of the mapping class group \( \Gamma \) to \( \mathcal{H}^{(k)} \).

Consider the following action of the Weyl group \( W \) on \( L^2(t) \otimes \mathbb{C}^{\mathcal{Z}_k} \)

\[
\hat{w} \cdot f(\theta, \gamma) = f(w(\theta), w(\gamma)), \quad \forall w \in W.
\]

The operator \( (Z \circ F_{\mathcal{Z}_k}) \) is equivariant with respect to this action and the one on \( \mathcal{H}^{(k)} \). Therefore it defines an isometry

\[
Z \circ F_{\mathcal{Z}_k} : \mathcal{H}^{(k)} \longrightarrow (L^2(t) \otimes \mathbb{C}^{\mathcal{Z}_k})^W.
\]

The action \( \hat{w} \) behaves well with respect to the tensor product decomposition \( L^2(t) \otimes \mathbb{C}^{\mathcal{Z}_k} \). Therefore it can be written as the tensor product \( \hat{w} = \hat{w}' \otimes \hat{w}'' \) of the two actions

\[
\begin{align*}
\hat{w}'(f)(\theta) &= f(w(\theta)) \\
\hat{w}''(x)(\gamma) &= x(w(\gamma)).
\end{align*}
\]

On a \( W \)-invariant vector \( f \otimes x \), i.e. \( \hat{w} \cdot f \otimes x = f \otimes x \), there will exist \( \lambda_w \in \mathbb{C}^* \) such that

\[
\begin{align*}
\hat{w}'(f)(\theta) &= \lambda_w f(\theta) \\
\hat{w}''(x)(\gamma) &= \lambda_w^{-1} x(\gamma).
\end{align*}
\]

Since \( W \) is generated by elements of order 2, \( \lambda_w \in \{1, -1\} \). In fact \( \lambda_w \) has to be either the identical character 1 or the alternating character \( \det(w) \). In the end, a vector \( f \otimes x \) is \( W \) invariant if and only if \( f \) and \( x \) are both invariant or both anti-invariant for the actions of \( \hat{w}' \) and \( \hat{w}'' \) respectively. With the notation from Appendix A we can write

\[
\mathcal{H}^{(k)} \simeq (L^2(t) \otimes \mathbb{C}^{\mathcal{Z}_k})^W = (L^2_{inv}(t) \otimes \mathbb{C}^{\mathcal{Z}_k}) \bigoplus (L^2_{anti}(t) \otimes \mathbb{C}^{\mathcal{Z}_k})
\]

Moreover, all the operators involved in the description of the action of the mapping class group from proposition 3 are invariant with respect to the action \( (44) \), so they preserves the whole decomposition \( (48) \).
On tensors $f \otimes x$, and $g \otimes y \in L^2(t) \otimes \mathbb{C}^Z(k)$ the inner product $\langle \cdot, \cdot \rangle$ from (32) factor into two inner products

\begin{equation}
\langle f, g \rangle_k = \int f(\theta)g(\bar{\theta})d\omega_k(\theta), \quad \langle x, y \rangle_{Z(k)} = \sum_{\gamma \in Z(k)} x(\gamma)\bar{y}(\gamma)
\end{equation}

We can use the second one, together with the basis from [SS] and [SSS], to compute the matrix elements of the action of SL(2, Z) expressed in Proposition [3] on the spaces $\mathcal{C}^Z(k)$ and $C^Z(k)$

**Lemma 1.**

\[ \langle e_{\alpha\beta} F_{Z(k)} e_{\beta} \rangle_{Z(k)} = \frac{j^{-1}}{|Z(k)|} \sum_{w \in W} \langle w\alpha, \beta \rangle, \quad \langle e_{\alpha\beta} \omega^{-1} G_{Z(k)} e_{\beta} \rangle_{Z(k)} = \omega^{-1} \left\langle -\frac{\alpha}{2}, \alpha \right\rangle \delta_{\alpha, \beta}, \]

\[ \langle \tilde{e}_{\alpha} F_{Z(k)} \tilde{e}_{\beta} \rangle_{Z(k)} = \frac{j^{-1}}{|Z(k)|} \sum_{w \in W} \det(w) \langle w\alpha, \beta \rangle, \quad \langle \tilde{e}_{\alpha} \omega^{-1} G_{Z(k)} \tilde{e}_{\beta} \rangle_{Z(k)} = \omega^{-1} \left\langle -\frac{\alpha}{2}, \alpha \right\rangle \delta_{\alpha, \beta}, \]

\[ \langle \tilde{e}_{\alpha} \omega^{-1} G_{Z(k)} e_{\beta} \rangle_{Z(k)} = \langle \tilde{e}_{\alpha} F_{Z(k)} e_{\beta} \rangle_{Z(k)} = 0. \]

**Proof.** This is a direct verification from the definitions. \hfill \square

**Proof of Theorem 4.** Lemma 4 together with Proposition 3 provides a proof for the decomposition and explicit formulas of $\hat{\rho}_k$ in Theorem 2. \hfill \square

Before proceeding with a review of the genus 1 Hitchin–Witten connection, we describe the pre-quantum connection $\nabla$ from equation (24), as an explicit differential operators acting on $L^2(t) \otimes \mathbb{C}^Z(k)$. Let namely $\nabla_X = (Z \circ F_{Z(k)})^{-1} \circ \nabla_X \circ (Z \circ F_{Z(k)})$ for any $X \in C^\infty(T \times T, T(T \times T))$. Define the following operator

\begin{equation}
D_{\sigma,j}: S(t) \otimes \mathbb{C}^Z(k) \rightarrow S(t) \otimes \mathbb{C}^Z(k)
\end{equation}

\[ f(\theta, \gamma) \mapsto df_{\gamma}[b_j](\theta) + 2\pi i \sigma^{-1}(b_j, \theta)k f_{\gamma}(\theta) \]

Notice that these operators depend on the basis $\{b_j\}_j$ of $t$. \hfill \square

From now on, given a connection $\nabla$ and a local coordinate function $u$ we will write $\nabla_u$ in place of $\nabla_\frac{\partial}{\partial u}$. Similarly we will write $\partial_u$ in place of $\frac{\partial}{\partial u}$. We have that

**Lemma 2.** Let $u_j$ and $v_j$, $\sigma \in \mathbb{H}$, $\zeta_j$ and $\bar{\zeta}_j$, for $j = 1, \ldots, n$ be defined as in (14) and (15). We have that

\begin{align}
\hat{\nabla}_{u_j} f_{\gamma}(\theta) &= df_{\gamma}[b_j](\theta) \\
\hat{\nabla}_{v_j} f_{\gamma}(\theta) &= -2\pi i (b_j, \theta)_k f_{\gamma}(\theta) \\
\hat{\nabla}_{\zeta_j} f_{\gamma}(\theta) &= \frac{\sigma}{\sigma - \overline{\sigma}} D_{\zeta_j} f_{\gamma}(\theta) \\
\hat{\nabla}_{\bar{\zeta}_j} f_{\gamma}(\theta) &= \frac{\sigma}{\sigma - \overline{\sigma}} D_{\bar{\zeta}_j} f_{\gamma}(\theta)
\end{align}

**Proof.** This is a simple verification using the explicit formulas for $\nabla$ and $Z$. \hfill \square

We remark that the pre-quantum connection $\hat{\nabla}$ decomposes under the tensor decomposition $L^2(t) \otimes \mathbb{C}^Z(k)$ and it is trivial in the second factor. In particular the Hitchin–Witten parallel transport does not affect the second factor.
4. The Hitchin-Witten Connection in genus 1

The quantum Hilbert space for Chern-Simons theory with gauge group $G_C$ can be identified with the pre-quantum Hilbert space for Chern Simons theory with gauge group $G$, as explained in [Wi3]. This identification, however, depends on a choice of a complex structure on the surface $\Sigma$. In genus 1 this means that there is a trivial bundle

\[ \mathbb{H} \times \mathcal{H}^{(k)} \rightarrow \mathbb{H} \]

and the identification between the fibers is obtained through parallel transport with the Hitchin-Witten connection $\nabla$ defined in [Wi3] for genus $g$ surfaces and in [AG2] in a more general setting. This connection is in this case flat, so the identifications between different quantum spaces will be well defined. The explicit formula for genus 1 is

\[ \nabla_{\partial_\sigma} = \partial_\sigma + \frac{1}{2t} \Delta_G \]
\[ \nabla_{\partial_\tau} = \partial_\tau - \frac{1}{2t} \Delta_G \]

where $\sigma$ is the holomorphic coordinate for $\mathbb{H}$ and $t \in \mathbb{C}^*$ with $t = k + is$. The operators $\Delta_G$ and $\Delta_{\tau}$ are second order differential operators on the line bundle $L^k$ defined in [AB] and [AG2]. Explicitly, for genus 1, we have that

\[ \Delta_G = \frac{i}{\pi} \sum_{p,q=1}^{n} C_{p,q}^{(1)} \nabla_{z_p} \nabla_{z_q} \]
\[ \Delta_{\tau} = -\frac{i}{\pi} \sum_{p,q=1}^{n} C_{p,q}^{(1)} \nabla_{\tau_p} \nabla_{\tau_q} \]

where $C_{p,q}^{(1)}$ is the $p,q$ entry of the inverse matrix of $C_{l,m}^{(k)} = (b_l b_m)_k$.

**Proposition 4.** Suppose $s \in \mathbb{R}$, $\varphi, \psi \in \mathcal{H}^{(k)}$ and $\langle \cdot, \cdot \rangle : \mathcal{H}^{(k)} \times \mathcal{H}^{(k)} \rightarrow \mathbb{C}$ as in [21]. We then have that

\[ d \langle \varphi, \psi \rangle = \langle \nabla \varphi, \psi \rangle + \langle \varphi, \nabla \psi \rangle. \]

**Proof.** Partial integration together with the hypothesis $s \in \mathbb{R}$ gives

\[ \langle \varphi, \frac{1}{2t} \Delta_G \psi \rangle = \langle \frac{1}{2t} \Delta_{\tau} \varphi, \psi \rangle. \]

A direct computation of the sum $\langle \nabla \varphi, \psi \rangle + \langle \varphi, \nabla \psi \rangle$ using the equation above gives the result. \qed

Let us define the Laplace operator operator

\[ \Delta_\sigma = \frac{i}{2\pi} (\sigma - \tau) \sum_{p,q=1}^{n} C_{p,q}^{(1)} \left( \nabla_{z_p} \nabla_{z_q} + \nabla_{\tau_p} \nabla_{\tau_q} \right) \]

which clearly dependent on the complex structure determined by $\sigma \in \mathbb{H}$.
Lemma 3. On $H^{(k)} \times \mathbb{H}$, the following hold true

\begin{align}
\nabla_{z_p} \nabla_{\tau q} &= \frac{2\pi i C^{(k)}_{p,q}}{\sigma - \bar{\sigma}} \\
[\partial_\sigma \nabla_{z_p}, \nabla_{\tau q}] &= \frac{-1}{\sigma - \bar{\sigma}} \nabla_{z_p}, \\
[\partial_\tau \nabla_{z_p}, \nabla_{\tau q}] &= \frac{-1}{\sigma - \bar{\sigma}} \nabla_{\tau q}, \\
[\partial_\sigma, \Delta_\sigma] &= \Delta_G, \\
[\partial_\tau, \Delta_\sigma] &= \Delta_{\bar{G}}, \\
[\Delta_\sigma, \Delta_G] &= 4k \Delta_G, \\
[\Delta_\sigma, \Delta_{\bar{G}}] &= -4k \Delta_{\bar{G}}
\end{align}

Proof. The first commutator follows from the explicit curvature of $\nabla$. The others follow from iterations of the first one or from the explicit dependence of $z_p$ on $\sigma$. \qed

Lemma 4. Let $x$, $y$ and $z$ be three elements of a Lie algebra satisfying the relations $[x,y] = z$ and $[y,z] = az$ for a central $a$. Suppose that the formal exponential $e^y = \sum_{k \geq 0} \frac{y^k}{k!}$ is well defined. Then we have

\begin{align}
[x,e^y] &= z a (e^y + a - e^y)
\end{align}

As was noted by Witten [Wi2], Lemma 4 together with Lemma 3 gives us the following conjugation rule

\begin{align}
e^{-r \Delta_\sigma} \circ d \circ e^{r \Delta_\sigma} = \nabla
\end{align}

where $r$ is chosen so that

\begin{align}
e^{-4kr} = \frac{k - is}{k + is}.
\end{align}

In particular, equation (61) implies

Proposition 5. For every $\psi \in H^{(k)}$ independent on the complex structure of $M$, the section $e^{-r \Delta} \psi$ of the vector bundle is parallel with respect to $\nabla$.

Given $\gamma \in \Gamma$, its pre-quantum action on $H^{(k)}$ was defined in (27), however when we look at the action on the whole bundle $H^{(k)} \times \mathbb{H} \to \mathbb{H}$, $\gamma$ acts on $\mathbb{H}$ by $\gamma_*$ as pull-back via the Mobius transformation $\gamma^{-1}$. We will then need to compose $\gamma$ with the parallel transport $\mathcal{P}_{\sigma, \gamma_*}$ of the pre-quantum action with the Hitchin-Witten connection from $\gamma_* \sigma$ back to $\sigma$. By the results in Proposition 5 we have that

\begin{align}
\mathcal{P}_{\sigma_0, \gamma_*} \psi_{\sigma_1} &= e^{-r \Delta_{\sigma_0}} e^{r \Delta_{\gamma_* \sigma}} \psi_{\sigma_1}, \\
\rho_k(\varphi) &= e^{-r \Delta_{\gamma_*} \nabla_{\rho_k(\varphi)}}.
\end{align}

If $w = x + \tau y$ are holomorphic coordinates for the surface $\Sigma$, then we have that

\begin{align}
S_* \tau = -\frac{1}{\tau} & \quad T_* \tau = \tau - 1,
\end{align}
and recalling that the holomorphic coordinate \( z_p = u_p + \sigma v_p \) on \( M \) are related with \( \tau \) by \( \sigma = -\frac{1}{\tau} \), we get that

\[
S_*\sigma = -\frac{1}{\sigma} \quad \quad T_*\sigma = \frac{\sigma}{1 + \sigma}.
\]

However the Laplace operator is itself is mapping class group invariant, as one can directly verify that

\[
\tilde{T} \circ \Delta_\sigma = \Delta_{T_*\sigma} \circ \tilde{T}, \quad \quad \tilde{S} \circ \Delta_\sigma = \Delta_{S_*\sigma} \circ \tilde{S}.
\]

For any \( \varphi \in \Gamma \) we obtain the equation

\[
e^{-r\Delta_\sigma} e^{r\Delta_{T_*\sigma}} \circ \varphi = e^{-r\Delta_\sigma} \circ \varphi \circ e^{r\Delta_{T_*}}
\]

and this proves Proposition 1.

\section*{Proposition 6} We have that

\[
\hat{\Delta}_\sigma = nk + \frac{i}{\pi} \sum_{p,q=1}^{n} \frac{\sigma \sigma}{\pi - \sigma} C_{(1)}^{p,q} D_{\sigma,p} D_{\sigma,q},
\]

For every multi-index \( l = (l_1, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n \), of length \( |l| = l_1 + \ldots + l_n \) we write

\[
D_{\sigma,l} \equiv D_{\sigma,1}^{l_1} \ldots D_{\sigma,n}^{l_n}
\]

Then the set

\[
\{ v_l(\theta, \sigma) \equiv D_{\sigma,l}(v)(\theta, \sigma) \in S(t) : l \in \mathbb{Z}_{\geq 0}^n, v(\theta, \sigma) := e^{-\pi i (\theta, \theta)/\sigma} \}
\]

is a complete set of eigenvectors for \( \hat{\Delta}_\sigma \) with corresponding eigenvalues

\[
\hat{\Delta}_\sigma v_l(\theta, \sigma) = 2k \left( |l| + \frac{n}{2} \right) v_l(\theta, \sigma).
\]

\section*{Remark 2} This Proposition follows from standard theory of multi-dimensional Hermite polynomials. In fact, the eigenfunctions of \( \hat{\Delta}_\sigma \) depend on the the choice of a basis \( \{ b_j \} \) of \( t \). If we choose the basis to be orthonormal with respect to \( (\cdot, \cdot)_k \) the eigenfunctions \( \{ v_l \} \) define a Hilbert basis of \( L^2(t) \) with respect to the inner product \( (\cdot, \cdot)_t \) from \([19]\). Indeed one can easily verify that, in such a basis, \( v_l(\theta, \sigma) = (-1)^{|l|} H_{l,\sigma}(\theta) v(\theta, \sigma) \) where

\[
H_{l,\sigma}(\theta) = (-1)^{|l|} e^{\alpha(\theta, \theta)_k} \partial^l e^{-\alpha(\theta, \theta)_k},
\]

\[
\alpha \equiv \pi i \frac{\sigma - \sigma}{\sigma \sigma}, \quad \partial^l = \partial_{l_1}^{l_1} \ldots \partial_{l_n}^{l_n}.
\]
which are orthogonal polynomials (multi-dimensional Hermite polynomials). Moreover \( v(\theta, \sigma)v(\theta, \sigma) = e^{-\alpha(\theta, \sigma)} \), so the \( v_l \) are orthogonal \( L^2(t) \) functions. Furthermore recall the following Mehler formula for every \( w \in \mathbb{C} \) such that \( \text{Re}(w) > 0 \)

\[
\sum_{l \in \mathbb{Z}_{\geq 0}} \frac{w^l}{\|v_l, v_l\}_t} H_t,\sigma(x)H_t,\sigma(y) =
\]

\[
\sqrt{\frac{\alpha}{\pi(1-w^2)}} \exp \left( \frac{\alpha}{1-w^2} (2w(x,y)_k - w^2((x,x)_k + (y,y)_k)) \right)
\]

(68) 

Choose \( b \in \mathbb{C} \) such that \( |b| = 1 \), \( \text{Re}(b) > 0 \) and

\[
is = \frac{k - b^2}{1 + b^2}.
\]

The choice of a square root of \( -b^2 \) gives

\[
e^{-4kr} = -b^2 \quad e^{-2kr} = ib
\]

From Proposition 6 we see that \( \hat{\Delta}_\sigma = 2k \left( \hat{N}_\sigma + \frac{\alpha}{b^2} \right) \) where \( \hat{N}_\sigma \) has spectrum equal to \( \mathbb{Z}_{\geq 0} \). So, its exponential can be written as

\[
e^{-r\hat{\Delta}_\sigma} = e^{-kr} e^{-2kr\hat{N}_\sigma} = (ib)^\frac{2}{\alpha} (ib)^\frac{\alpha}{2} \hat{N}_\sigma.
\]

Finding an explicit expression for \( e^{-r\hat{\Delta}_\sigma} \psi = f \) can be obtained via the kernel \( k_{\sigma, b} \) as follows

\[
f(\theta, \sigma, b) = \int k_{\sigma, b}(\theta, \hat{\theta})\psi(\hat{\theta})d\text{vol}_k(\hat{\theta}).
\]

Recall the Hilbert bases \( \{v_l\}_t \) for \( L^2(t) \) diagonalizing \( \hat{\Delta}_\sigma \). We can rewrite the kernel as

\[
k_{\sigma, b}(\theta, \hat{\theta}) = (ib)^\frac{\alpha}{2} \sum_{l \in \mathbb{Z}_{\geq 0}} \frac{(ib)^l}{\|v_l, v_l\}_t} v_l(\theta, \sigma)v_l(\theta, \sigma)
\]

(72) 

Since \( \text{Re}(-b^2) = 1 - 2(\text{Re}b)^2 < 0 \), Mehler’s Formula (68) gives the following explicit kernel

**Lemma 5.** For \( ib = e^{-2kr} \) we have

\[
e^{-r\hat{\Delta}_\sigma} \psi(\theta) =
\]

\[
(ib)^\frac{\alpha}{2} \sqrt{\frac{\alpha}{\pi(1+b^2)}} \int \exp \left( \frac{\alpha}{1+b^2} \left( 2ib(\theta, \hat{\theta})_k + b^2((\theta, \theta)_k + (\hat{\theta}, \hat{\theta})_k) \right) \right) v(\theta, \sigma)v(\theta, \sigma)\psi(\hat{\theta})d\text{vol}_k(\hat{\theta})
\]

where

\[
\alpha = \pi i \frac{\sigma - \sigma}{\sigma \sigma}.
\]

**Proof of Theorem 2** The conjugation relation stated in proposition 1 was already established on equation (60). In the case \( \sigma = ib \) the operator from Lemma 5 takes the form

\[
e^{-r\hat{\Delta}_{ib}} \psi(\theta) = e^{\frac{\pi}{ib}} e^{\pi(b-\bar{b})(\theta, \theta)_k} \circ F_1(\psi)(\theta)
\]

and the explicit expressions follow easily.

\[\square\]
5. Remark on Compact Quantum Chern–Simons Theory

In this last section we want to make precise contact with the results known for Chern–Simons theory with gauge group $G$. With reference to [Je], Proposition 4.2, consider $\mathfrak{g}$ simply laced. The matrix elements $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ presented there for the level-$k$ theory, corresponds to our matrix elements

$$\hat{\rho}_{k+h,1}(\varphi)_{(\alpha+\varphi) \in \frac{(\beta+\varphi)}{2}}$$

for $\varphi = S$ or $T$,

with the specific choice of $j = e^{-|\Delta_+|}$ and $\omega = e^{\pi i (\rho,\rho) / h}$. Here $h$ and $\rho$ have the same meaning as the ones in [Je]. The indexes $(\alpha+\rho)/k$ and $(\beta+\rho)/k$ run in the set $I_{(k+h)},$ that corresponds to the weights in the interior of the $(k+h)$-alcove, exactly as in [Je] for the level $k + h$ representation. The relation between the shift on the level $k \mapsto k + h$ and the corresponding shift on weights $\alpha \mapsto \alpha + \rho$ is explained in [Je], section A.5.

We remark that for $G = SU(N)$ without the shift $k \mapsto k + h$, we find similar formulae in [We], but there for compact Chern–Simons theory.

Appendices

A. Notation from Lie Theory

In this appendix we recall some notation and facts from Lie theory that is useful to formulate Chern–Simons theory in genus 1. We follow the presentations in [ADW] [Je] quit closely.

Let $G$ be a compact, simple, connected, simply connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $\mathbb{T} \subset G$ be a maximal torus in $G$ and let $\mathfrak{t} \subseteq \mathfrak{g}$ its Lie algebra, which is a Cartan subalgebra of $\mathfrak{g}$. The rank of $G$ is defined as the dimension of $\mathfrak{t}$ and is denoted by $n$.

There is a preferred inner product $\langle \cdot, \cdot \rangle_1$ in $\mathfrak{t}$ which is determined, up to a positive scalar $K$, by the trace as $(X,Y)_1 = -K \text{Tr}(XY)$.

Associated to $\mathfrak{g}$ we have the root system thought of as a certain finite set of $\alpha \in \mathfrak{t}^*$ in the dual of the Cartan subalgebra. Via $\langle \cdot, \cdot \rangle_1$ we can identify $\mathfrak{t}$ and $\mathfrak{t}^*$. In this way the roots can have only two possible lengths.

We fix the normalization factor $K > 0$ so that $(\alpha,\alpha)_1 = 2$ for $\alpha$ a longest root. For a short root $\beta$ one has $(\beta,\beta)_1 = 2/p$, where $p \in \mathbb{Z}$ depends only on $\mathfrak{g}$. A Lie algebra is called simply laced if all its roots have the same length.

A fixed set of positive roots is denoted $\Delta_+$, while the corresponding set of simple roots $\alpha_i$, $i = 1, \ldots, n$, is denoted $\Delta \subset \Delta_+$.

To each root $\alpha$ we can associate a coroot $h_\alpha \in \mathfrak{t}$ such that $h_\alpha(\alpha) = 2$. Via the preferred inner product we can identify $h_\alpha = \frac{2\alpha}{(\alpha,\alpha)_1}$. In particular the norm of the coroots of a long (resp. short) root $\alpha_l$ (resp. $\alpha_s$) are

$$\langle h_{\alpha_l},h_{\alpha_l} \rangle_1 = 2 \quad \langle h_{\alpha_s},h_{\alpha_s} \rangle_1 = 2p,$$

for some $p \in \mathbb{Z}_{>0}$.

The coroot lattice $\Lambda^R \subset \mathfrak{t}$ is the $\mathbb{Z}$-span of the coroots. This gives the identification

$$\mathbb{T} = \mathfrak{t}/\Lambda^R.$$
For every integer $k \geq 1$ we define a $k$-scaled inner product $(\cdot, \cdot)_k \equiv k(\cdot, \cdot)_1$. We use it to define the dual of $\Lambda^R$ with respect to $(\cdot, \cdot)_k$

$$
\Lambda^w_k \equiv \{ \gamma \in t \text{ such that } (\gamma, \lambda)_k \in \mathbb{Z}, \text{ for every } \lambda \in \Lambda^R \}.
$$

$\Lambda^R$ is integral with respect to $(\cdot, \cdot)_k$, meaning that $\Lambda^R \subset \Lambda^w_k$ for all $k$. In the special case $k = 1$, $\Lambda^w = \Lambda^w_{(1)}$ is called the *weights lattice*.

We will denote by $Z_{(k)}$ the finite abelian group

$$
Z_{(k)} \equiv \Lambda^w_{(k)}/\Lambda^R.
$$

For any full rank lattice $\Lambda$ we write $\text{Vol}_k \Lambda$ for the volume of a fundamental domain of the action of $\Lambda$ on $t$ with respect to the inner product $(\cdot, \cdot)_k$. For a finite set $C$, $|C|$ is its cardinality. We have

$$
\text{Vol}_1 \Lambda^w_{(k)} = \frac{\text{Vol}_1 \Lambda^w}{k^n} = \left(\frac{\text{Vol}_1 \Lambda^R}{k^n}\right)^{-1}, \quad |Z_{(k)}| = \frac{\text{Vol}_1 \Lambda^R}{\text{Vol}_1 \Lambda^w_{(k)}} = |\Lambda^R|^2 k^n
$$

The Weyl group $W$ is the group generated by the reflections $s_\alpha$ through the hyperplane orthogonal to $\alpha$, i.e.

$$
s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}.
$$

A fundamental domain for the action of $W$ on $t$ is called a *Weyl Chamber*. After a choice of a set of simple roots $\Delta$ is fixed, we can identify a preferred chamber $F_0$ as

$$
F_0 \equiv \{ x \in t : (x, \alpha)_1 > 0, \text{ for all } \alpha \in \Delta \}.
$$

In this way $t$ is decomposed into $|W|$ Weyl chambers and $W$ acts simply transitively on the set of them, i.e.

$$
t = \bigcup_{w \in W} \overline{w(F_0)}.
$$

The space $L^2(t, \langle \cdot, \cdot \rangle_t)$ is defined with respect to the following inner product

$$
\langle f, g \rangle_t \equiv \int_t f(x)g(x)d\text{vol}_k(x),
$$

where $d\text{vol}_k$ is the measure on $t$ induced by $(\cdot, \cdot)_k$.

An $f \in L^2(t)$ can be projected orthogonally into the subspace of $W$–*invariant* (resp. $W$–*anti-invariant*) functions $L^2_+(t)$ (resp. $L^2_-(t)$) as follows

$$
P_+(f)(x) = \frac{1}{|W|} \sum_{w \in W} f(w(x)) \quad \text{(invariant)}
$$

$$
P_-(f)(x) = \frac{1}{|W|} \sum_{w \in W} \text{det}(w)f(w(x)) \quad \text{(anti–invariant)}
$$

where $\text{det}(w)$ is the same as the sign character. In particular, both $L^2_+(t)$ and $L^2_-(t)$ are isomorphic to $L^2(F_0)$ by restriction to a Weyl chamber.

The *affine Weyl group* $W_a$ is the semidirect product $\Lambda^R \rtimes W$. Its *fundamental alcove* $A \subset t$ is the set

$$
A \equiv \{ x \in t : 0 < (x, \alpha)_1 < 1, \text{ for all } \alpha \in \Delta_+ \},
$$
\[ \text{Vol}_1 \Lambda^R = \text{Vol}_1(\Omega^R) = |W| \text{Vol}_1(A). \]

We define the following two sets of indexes
\[ I_k \equiv \Lambda^w_{(k)} \cap A, \quad \overline{I}_k \equiv \Lambda^w_{(k)} \cap \overline{A}. \]

The set \( \overline{I}_k \) can be thought as a fundamental domain for the action of \( W \) on \( Z_{(k)} \) or the action of \( W_\alpha \) on \( \Lambda^w_{(k)} \). By scaling by \( k \) we get a bijection between \( \overline{I}_k \) and the weights in the fundamental \( k \)-alcove (compare with [Je] equation A.16 and section A.5).

The space of functions \( x : Z_{(k)} \rightarrow \mathbb{C} \) is a finite dimensional vector space of dimension \( |Z_{(k)}| \). A basis is given by the functions \( \delta_\gamma \), \( \gamma \in Z_{(k)} \) defined as
\[ \delta_\gamma(\gamma^0) = \delta(\gamma - \gamma^0 \mod \Lambda^R). \]

The subspace \( \overline{C}_{(k)} \) and \( C_{(k)} \) of, respectively, \( W \)-invariant and \( W \)-anti-invariant functions are then spanned, respectively, by the following two basis
\[ \overline{C}_{(k)} = \text{Span}_\mathbb{C}\langle e_\gamma \rangle_{\gamma \in \overline{I}_k}, \quad C_{(k)} = \text{Span}_\mathbb{C}\langle \tilde{e}_\gamma \rangle_{\gamma \in I_k}. \]

We use the sets \( \overline{I}_k \) and \( I_k \) to index the basises \( \{e_\gamma\} \) and \( \{\tilde{e}_\gamma\} \) respectively. These basises are orthonormal with respect to the product
\[ \langle x,y \rangle_{Z_{(k)}} = \sum_{\gamma \in Z} x(\gamma)\overline{y(\gamma)}. \]

**References**

[A1] J.E. Andersen, *Geometric Quantization of Symplectic Manifolds with respect to reducible non-negative polarizations*, Commun. in Math. Phys. **183** (1997) 401–421.

[A2] J.E. Andersen, *Deformation quantization and geometric quantization of abelian moduli spaces*, Comm. of Math. Phys. **255**, (2005) 727-745.

[A3] J.E. Andersen, *Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups*, Annals of Mathematics, **163** (2006) 347–368.

[AMU] J.E. Andersen, G. Masbaum & K. Ueno, *Topological quantum field theory and the Nielsen-Thurston classification of \( M(0,4) \)*, Math. Proc. Cambridge Philos. Soc. **141** (2006) No. 3, 477–488.

[AU1] J.E. Andersen & K. Ueno, *Geometric construction of modular functors from conformal field theory*, Journal of Knot theory and its Ramifications. **16** 2 (2007), 127-202.

[AU2] J.E. Andersen & K. Ueno, *Abelian Conformal Field theories and Determinant Bundles*, International Journal of Mathematics. **18** (2007) 919–993.

[A4] J.E. Andersen, *The Nielsen-Thurston classification of mapping classes is determined by TQFT*, J. Math. Kyoto Univ. **48** (2008) No. 2, 323–338.

[A5] J.E. Andersen, *Toeplitz Operators and Hitchin’s projectively flat connection*, in *The many facets of geometry: A tribute to Nigel Hitchin*, 177–209, Oxford Univ. Press, Oxford, (2010).
[AG1] J.E. Andersen & N.L. Gammelgaard, Hitchin’s Projectively Flat Connection, Toeplitz Operators and the Asymptotic Expansion of TQFT Curve Operators, Grassmannians, Moduli Spaces and Vector Bundles, 1–24, Clay Math. Proc., 14, Amer. Math. Soc., Providence, RI, (2011).

[AB] J.E. Andersen & J. Blaavand, Asymptotics of Toeplitz operators and applications in TQFT, Travaux Mathématiques, 19 (2011) 167–201.

[AU3] J.E. Andersen & K. Ueno, Modular functors are determined by their genus zero data, Quantum Topology 3 (2012) No. 3/4, 255–291.

[A6] J.E. Andersen, Hitchin’s connection, Toeplitz operators and symmetry invariant deformation quantization, Quantum Topology 3 (2012) No. 3/4, 293–325.

[AGL] J.E. Andersen, N.L. Gammelgaard & M.R. Lauridsen, Hitchin’s Connection in Metaplectic Quantization, Quantum Topology 3 (2012) No. 3/4, 327–357.

[AHi] J.E. Andersen & B. Himpel, The Witten-Reshetikhin-Turaev invariant of finite order mapping tori II, Journal für Reine und Angewandte Mathematik 681 (2013) 1–36.

[A7] J.E. Andersen, The Witten-Reshetikhin-Turaev invariant of finite order mapping tori I, Journal für Reine und Angewandte Mathematik 681 (2013) 1–36.

[AU4] J.E. Andersen & K. Ueno, Construction of the Witten-Reshetikhin-Turaev TQFT from conformal field theory, Invent. Math. 201 (2015) No. 2, 519–559.

[AK1] J. E. Andersen and R. M. Kashaev, A TQFT from Quantum Teichmüller Theory, Commun. in Math. Phys., Published online June 14, (2014). Digital Object Identifier (DOI) 10.1007/s00220-014-2073-2.

[AK2] J. E. Andersen and R. M. Kashaev, A new formulation of the Teichmüller TQFT, ArXiv: 1305.4291, (2013).

[AJHMMc] J. E. Andersen, B. Himpel, S. F. Jørgensen, J. Martens, and B. McLellan, The Witten-Reshetikhin-Turaev invariant for links in finite order mapping tori i. Advances in Mathematics 304 (2017), 131–178.

[AK3] J. E. Andersen and R. M. Kashaev, Complex Quantum Chern Simons, ArXiv: 1409.1208, (2014).

[AG2] J. E. Andersen and Niels Leth Gammelgaard, The Hitchin-Witten-connection and Complex Quantum Chern Simons Theory, ArXiv: 1409.1035, (2014).

[AGP] J. E. Andersen, S. Gukov and D. Pei, The Verlinde formula for Higgs bundles, arXiv:1608.01761 (2016).

[AS] J. E. Andersen and S. Marzioni, Level N Teichmüller TQFT and Complex Chern–Simons Theory. To appear in Travaux Mathématiques (2017).

[AtB] M. Atiyah & R. Bott, The Yang-Mills equations over Riemann surfaces. Phil. Trans. R. Soc. Lond., Vol. A308 (1982) 523–615.

[B1] C. Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants, Topology 39 (2000), no. 1, 193–223.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, Three-manifold invariants derived from the Kauffman Bracket. Topology 31 (1992) 685–699.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum & P. Vogel, Topological Quantum Field Theories derived from the Kauffman bracket. Topology 34 (1995) 883–927.

[Cha1] K. Chandrasekharan, Elliptic Functions , Grundlehrender math.Wissenschaften, vol. 281. Berlin Heidelberg New York (1985).

[Dil1] T. Dimofte. Quantum Riemann surfaces in Chern-Simons theory Adv. Theor. Math. Phys., 17 (2013) No. 3, 479–599.

[Dil2] T. Dimofte. Complex Chern-Simons theory at level k via the 3d-3d correspondence. arXiv:1409.0857 (2014).

[Di3] Tudor Dimofte, Perturbative and nonperturbative aspects of complex Chern-Simons Theory, arXiv:1608.02961 (2016).

[DGG] T. Dimofte, D. Gaiotto, and S. Gukov. Gauge theories labelled by three-manifolds, Comm. Math. Phys., 325 (2014) No.2, 367–419.

[DGa1] T. Dimofte and S. Garoufalidis. The quantum content of the gluing equations. Geom. Topol. 17 (2013) No. 3, 1253–1315.
[DGa2] Tudor Dimofte, Stavros Garoufalidis, Quantum modularity and complex Chern-Simons theory, arXiv:1511.05628 (2016).
[DGu1] T. Dimofte and S. Gukov. Chern-Simons theory and S-duality. J. High Energy Phys. 109 (2013) No. 5., front matter+65.
[DGLZ] T. Dimofte, S. Gukov, J. Lenells, and D. Zagier. Exact results for perturbative Chern-Simons theory with complex gauge group. Commun. Number Theory Phys. 3 (2009) No. 2, 363–443.
[BNW] Dror Bar-Natan and Edward Witten. Perturbative expansion of Chern-Simons theory with noncompact gauge group, Comm. Math. Phys. 141 (1991) no. 2, 423–440.
[Fri1] D.S. Freed, Classical Chern-Simons Theory, Part I, Adv. Math. 113 (1995) 237–303.
[Gal] S. Garoufalidis. The 3d index of an ideal triangulation and angle structures, arXiv:1208.1663 (2012).
[Gu1] S. Gukov. Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial. Comm. Math. Phys., 255 (2005) No. 3, 577–627.
[GM] S. Gukov and H. Murakami. SL(2,C) Chern-Simons theory and the asymptotic behavior of the colored Jones polynomial. Lett. Math. Phys. 86 (2008) No. 2-3, 79–98.
[Hik1] K. Hikami. Hyperbolicity of partition function and quantum gravity. Nuclear Phys. B, 616 (2001) No. 3, 537–548.
[Hik2] K. Hikami. Generalized volume conjecture and the A-polynomials: the Neumann-Zagier potential function as a classical limit of the partition function. J. Geom. Phys. 57 (2007) No. 9, 1895–1940.
[H] N. Hitchin, Flat connections and geometric quantization, Comm.Math.Phys. 131 (1990) 347–380.
[Je] L. C. Jeffrey, Chern-Simons-Witten invariants of lens spaces and torus bundles, and the semiclassical approximation, Comm. Math. Phys. 147 (1992) no. 3, 563–604.
[La1] V. Laszlo, Hitchin's and WZW connections are the same, J. Diff. Geom. 49 (1998) no. 3, 547–576.
[RSW] T.R. Ramadas, I.M. Singer and J. Weitsman, Some Comments on Chern – Simons Gauge Theory, Comm. Math. Phys. 126 (1989) 409-420.
[RT1] N. Reshetikhin & V. Turaev, Ribbon graphs and their invariants derived from quantum groups, Comm. Math. Phys. 127 (1990) 1–26.
[RT2] N. Reshetikhin & V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991) 547–597.
[TUY] A. Tsuchiya, K. Ueno & Y. Yamada, Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries, Advanced Studies in Pure Mathematics, 19 (1989) 459–566.
[T] V. G. Turaev, Quantum invariants of knots and 3-manifolds, de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994. x+588 pp. ISBN: 3-11-013704-6
[We] J. Weitsman, Quantization via real polarization of the moduli space of flat connections and Chern-Simons gauge theory in genus one, Comm. Math. Phys. 137 (1991) 175–190.
[Wil] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (1989) 351–98.
[Wi2] E. Witten. Quantization of Chern-Simons gauge theory with complex gauge group. Comm. Math. Phys. 137 (1991) No. 1, 29–66.
[Wi3] E. Witten. Analytic continuation of Chern-Simons theory. In Chern-Simons gauge theory: 20 years after, volume 50 of AMS/IP Stud. Adv. Math., pages 347–446. Amer. Math. Soc., Providence, RI, 2011.

Center for Quantum Geometry of Moduli Spaces, University of Aarhus, DK-8000, Denmark
E-mail address: andersen@qgm.au.dk

Center for Quantum Geometry of Moduli Spaces, University of Aarhus, DK-8000, Denmark
E-mail address: marzioni@qgm.au.dk