Some results from algebraic geometry over Henselian real valued fields

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Abstract

This paper develops algebraic geometry over Henselian real valued (i.e. of rank 1) fields $K$, being a sequel to our paper about that over Henselian discretely valued fields. Several results are given including: a certain concept of fiber shrinking (a relaxed version of curve selection) for definable sets; the canonical projection $K^n \times K \mathbb{P}^m \to K^n$ and blow-ups of the $K$-points of smooth $K$-varieties are definably closed maps; a descent property for blow-ups; a version of the Lojasiewicz inequality for continuous rational functions and the theorem on extending continuous hereditarily rational functions, established for the real and $p$-adic varieties in our joint paper with J. Kollár. The descent property enables application of desingularization and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. Our approach applies quantifier elimination due to Pas.

1. Introduction. We endeavour to carry over the results from algebraic geometry over Henselian discretely valued fields of equicharacteristic zero, given in our paper [21], to that over Henselian real valued (i.e. of rank 1) fields. The crucial result of paper [21] was the theorem that the projection $K^n \times K \mathbb{P}^m \to K^n$ is a definably closed map, which allowed us to attain

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the other results over the ground fields that are not locally compact. The approach developed here is different: the central idea is a certain concept of fiber shrinking (a relaxed version of curve selection) for sets definable in the language \( L \) of Denef–Pas. It is established by means of cell decomposition and the theory of \( L \)-definable functions of one variable. The line of reasoning in this paper has been converted: here these tools serve to achieve the closedness theorem. Having disposed of this theorem, we can prove the remaining results in the same manner as before in our paper \([21]\). In this paper, we deal with a Henselian real valued field with value group \( \Gamma \) and residue field \( k \). Throughout the paper we shall assume that \( \Gamma \) is without a minimal positive element (i.e. the valuation \( v \) is not discrete).

The organization of the paper is as follows. In Section 2, we set up the notation and terminology; in particular, the language \( L \) of Denef–Pas and the concept of a cell. We recall the theorems on quantifier elimination and on preparation cell decomposition, due to Pas \([22]\). Finally, we prove the theorem on cell decomposition for \( L \)-definable sets (Theorem 1).

In Section 3, we proceed with the study of \( L \)-definable functions of one variable. A result which will play an important role in the sequel is the theorem on existence of the limit (Theorem 2). Its proof makes use of Puiseux’s theorem for the local ring of convergent power series.

Section 4 is devoted to the proof of fiber shrinking (Theorem 3). In Section 5, we establish the closedness theorem (Theorem 4). Next several corollaries, including the descent property for blow-ups (Corollary 4), are stated. The descent property enables application of the desingularization and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. This allows us to achieve a version of the Lojasiewicz inequality for continuous rational functions (Theorem 5). Also presented is the theorem on extending continuous hereditarily rational functions (Theorem 6), established for the real and \( p \)-adic varieties in our joint paper with J. Kollár \([17]\). The proofs of the last two theorems from our paper \([21]\) can be repeated verbatim, because we now have at our disposal the descent property.

Finally, let us mention that the metric topology of a non-archimedean field \( K \) with a real valuation \( v \) is totally disconnected. Rigid analytic geometry (see e.g. \([3]\) for its comprehensive foundations), developed by Tate, compensates for this defect by introducing sheaves of analytic functions in a
Grothendieck topology. Another approach is due to Berkovich [1], who filled in the gaps between the points of $K^n$, producing a locally compact Hausdorff space (so-called analytification), which contains the metric space $K^n$ as a dense subspace if the ground field $K$ is algebraically closed. His construction consists in replacing each point $x$ of a given $K$-variety with the space of all real valuations on the residue field $\kappa(x)$ that extend $v$. The theory of stably dominated types, developed in paper [15], deals with non-archimedean fields with valuation of arbitrary rank and generalizes that of tame topology for Berkovich spaces. Currently, various analytic structures on Henselian real valued fields are intensively investigated (see e.g. [9] for more information).

2. Cell decomposition. This section recalls cell decomposition and quantifier elimination due to Pas [22] in the language $\mathcal{L}$ of Denef–Pas with three sorts: the valued field $K$-sort, the value group $\Gamma$-sort and the residue field $k$-sort. The language of the $K$-sort is the language of rings; that of the $\Gamma$-sort is any augmentation of the language of ordered abelian groups (and $\infty$); finally, that of the $k$-sort is any augmentation of the language of rings. Additionally, we add a map $v$ from the field sort to the value group (the valuation), and a map $ac$ from the field sort to the residue field (angular component map) which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring $R$ of $K$.

We now recall the necessary notation and terminology. Consider an $\mathcal{L}$-definable subset $D$ of $K^n \times k^m$, three $\mathcal{L}$-definable functions

$$a(x, \xi), b(x, \xi), c(x, \xi) : D \to K,$$

and a positive integer $\nu$. For each $\xi \in k^m$ set

$$C(\xi) := \{(x, y) \in K^n \times K : (x, \xi) \in D,$$

$$v(a(x, \xi)) <_1 v((y - c(x, \xi))^\nu) <_2 v(b(x, \xi)), \; ac(y - c(x, \xi)) = \xi_1\},$$

where $\nu$ is a positive integer and $<_1, <_2$ stand for $<$, $\leq$ or no condition in any occurrence. If the sets $C(\xi), \xi \in k^m$, are pairwise disjoint, the union

$$C := \bigcup_{\xi \in k^m} C(\xi)$$

is called a cell in $K^n \times K$ with parameters $\xi$ and center $c(x, \xi)$; $C(\xi)$ is called a fiber of the cell $C$. We need the following two theorems due to Pas ([22, Theorems 4.1 and 3.2]).
Quantifier Elimination. Let \((K, \Gamma, k)\) be a structure for the 3-sorted language \(\mathcal{L}\) of Denef–Pas. Assume that the valued field \(K\) is Henselian and of equicharacteristic zero. Then \((K, \Gamma, k)\) admits elimination of \(K\)-quantifiers in the language \(\mathcal{L}\).

Preparation Cell Decomposition. Let \(f_1(x, y), \ldots, f_r(x, y)\) be polynomials in one variable \(y\) with coefficients being \(\mathcal{L}\)-definable functions on \(\mathbb{K}_x^n\). Then \(\mathbb{K}^n \times \mathbb{K}\) admits a finite partition into cells such that for each cell \(A\) with parameters \(\xi\) and center \(c(x, \xi)\) and for all \(i = 1, \ldots, r\) we have:

\[v(f_i(x, y)) = v\left(\tilde{f}_i(x, \xi)(y - c(x, \xi))^{\nu_i}\right),\]

\[\overline{ac} f_i(x, y) = \xi_{\mu(i)},\]

where \(\tilde{f}_i(x, \xi)\) are \(\mathcal{L}\)-definable functions, \(\nu_i \in \mathbb{N}\) for all \(i = 1, \ldots, r\), and the map \(\mu : \{1, \ldots, r\} \to \{1, \ldots, m\}\) does not depend on \(x, y, \xi\).

Then we shall say that the the functions \(f_1(x, y), \ldots, f_r(x, y)\) are prepared with respect to the variable \(y\). The following result can be deduced.

Theorem 1 (Cell decomposition). Every \(\mathcal{L}\)-definable subset \(B\) of \(\mathbb{K}^n \times \mathbb{K}\) is a finite union of cells.

Proof. By elimination of \(K\)-quantifiers, \(B\) is a finite union of sets defined by conditions of the form

\[v(f_1(x, y), \ldots, f_r(x, y)) \in P,\]

\[(\overline{ac} g_1(x, y), \ldots, \overline{ac} g_s(x, y)) \in Q,\]

where \(f_i, g_j \in \mathbb{K}[x, y]\) are polynomials, and \(P\) and \(Q\) are \(\mathcal{L}\)-definable subsets of \(\Gamma^r\) and \(\mathbb{K}^s\), respectively (since \(a = 0\) iff \(\overline{ac} a = 0\)). Thus we may assume that \(B\) is such a set.

The sort \(\Gamma\) admits quantifier elimination in the language of ordered groups, because it is an ordered subgroup of the ordered additive group \((\mathbb{R}, +, <)\) without minimal positive element (see e.g. [7]). Therefore we can assume that the set \(P\) is defined by finitely many inequalities of the form

\[\{\alpha \in \Gamma^r : k_1\alpha_1 + \cdots + k_r\alpha_r + \beta < 0\},\]

where \(k_1, \ldots, k_r \in \mathbb{Z}, \beta \in \Gamma\) and \(<\) stands for \(<\) or \(=\).
But there exists a finite partition of $K^n \times K$ into cells, which prepares the polynomials

$$f_1(x, y), \ldots, f_r(x, y) \quad \text{and} \quad g_1(x, y), \ldots, g_s(x, y).$$

On each cell $C$ of this partition (of the form considered before), we thus have

$$v(f_i(x, y)) = v\left(\tilde{f}_i(x, \xi)(y - c(x, \xi))^\nu_i\right),$$

and

$$v(g_i(x, y)) = v\left(\tilde{g}_i(x, \xi)(y - c(x, \xi))^\eta_i\right),$$

Hence the intersection $B \cap C(\xi)$ is defined by finitely many conditions of the form

$$v\left(\prod_{i=1}^r \tilde{f}_i(x, \xi)^{k_i}(y - c(x, \xi))^{k_i\nu_i}\right) + \beta \ll 0$$

and

$$(\xi_{\eta(1)}, \ldots, \xi_{\eta(s)}) \in Q.$$

After renumbering the integers $k_i$, we may assume that $k_1, \ldots, k_p$, with $p \leq r$, are non-negative integers and $k_{p+1}, \ldots, k_r$ are negative integers. Take $b \in K$ such that $v(b) = \beta$. Then the conditions of the first form are equivalent to

$$v\left(b \prod_{i=1}^p \tilde{f}_i(x, \xi)^{k_i}(y - c(x, \xi))^{k_i\nu_i}\right) \ll v\left(\prod_{i=p+1}^r \tilde{f}_i(x, \xi)^{-k_i}(y - c(x, \xi))^{-k_i\nu_i}\right).$$

We can therefore partition the cell $C$ into a finite number of finer cells with the same center $c(x, \xi)$, so that the intersection $B \cap C(\xi)$ is the union of some of them. This finishes the proof. \square

**Remark.** A more careful analysis of cells in the above proof leads to the sharpening of Theorem 1 stated below:

*Every $\mathcal{L}$-definable subset $B$ of $K^n \times K$ can be partitioned into a finite union of cells.*
In Section 4, we shall apply the weaker version of cell decomposition (Theorem 1) to establish fiber shrinking over Henselian real valued fields (Theorem 3).

3. Definable functions of one variable. We begin with the following

**Proposition 1.** Let \( f : A \to K \) be an \( \mathcal{L} \)-definable function on a subset \( A \) of \( K^n \). Then there is a finite partition of \( A \) into \( \mathcal{L} \)-definable sets \( A_i \) and irreducible polynomials \( P_i(x, y), i = 1, \ldots, k \), such that for each \( a \in A_i \) the polynomial \( P_i(a, y) \) in \( y \) does not vanish and

\[
P_i(a, f(a)) = 0 \quad \text{for all } a \in A_i, \quad i = 1, \ldots, k.
\]

**Proof.** Observe again, as in the proof of Theorem 1, that the graph of \( f \) is a finite union of sets \( B_i, i = 1, \ldots, k \), defined by conditions of the form

\[
(v(f_1(x, y)), \ldots, v(f_r(x, y))) \in P,
\]

\[
(\overline{ac} g_1(x, y), \ldots, \overline{ac} g_s(x, y)) \in Q,
\]

where \( f_i, g_j \in K[x, y] \) are polynomials, and \( P \) and \( Q \) are \( \mathcal{L} \)-definable subsets of \( \Gamma^{r} \) and \( k^s \), respectively. Each set \( B_i \) is the graph of the restriction of \( f \) to an \( \mathcal{L} \)-definable subset \( A_i \). Since, for each point \( a \in A_i \), the fibre of \( B_i \) over \( a \) consists of one point, the above condition imposed on angular components includes one of the form \( \overline{ac} g_j(x, y) = 0 \) or, equivalently, \( g_j(x, y) = 0 \), for some \( j = 1, \ldots, s \), which may depend on \( a \), where the polynomial \( g_j(a, y) \) in \( y \) does not vanish. This means that the set

\[
\{ (\overline{ac} g_1(x, y), \ldots, \overline{ac} g_s(x, y)) : (x, y) \in B_i \}
\]

is contained in the union of hyperplanes \( \bigcup_{j=1}^{s} \{ \xi_j = 0 \} \) and, furthermore, that, for each point \( a \in A_i \), there is an index \( j = 1, \ldots, s \) such that the polynomial \( g_j(a, y) \) in \( y \) does not vanish and \( g_j(a, f(a)) = 0 \). Clearly, for any \( j = 1, \ldots, s \), this property of points \( a \in A_i \) is \( \mathcal{L} \)-definable. Therefore we can partition the set \( A_i \) into subsets each of which fulfils the condition required in the conclusion with some irreducible factors of the polynomial \( g_j(x, y) \). \( \square \)

Consider a complete real valued field \( L \). For every non-negative integer \( r \), let \( L\{x\}_r \) be the local ring of all formal power series

\[
\phi(x) = \sum_{k=0}^{\infty} a_k x^k \in L[[x]]
\]
in one variable $x$ such that $v(a_k) + kr$ tends to $\infty$ when $k \to \infty$; $L\{x\}_0$ coincides with the ring of restricted formal power series. Then the local ring

$$L\{x\} := \bigcup_{r=0}^{\infty} L\{x\}_r$$

is Henselian, which can be directly deduced by means of the implicit function theorem for restricted power series in several variables (see [4, Chap. III, § 4], [12] and also [13, Chap. I, § 5]).

Now, let $L$ be the completion of the algebraic closure $\overline{K}$ of the ground field $K$. Clearly, the Henselian local ring $L\{x\}$ is closed under division by the coordinate and power substitution. Therefore it follows from our paper [20, Section 2] that Puiseux’s theorem holds for $L\{x\}$. We still need an auxiliary lemma.

**Lemma 1.** *The field $K$ is a closed subspace of its algebraic closure $\overline{K}$.***

Indeed, Denote by $\text{cl} (E, F)$ the closure of a subset $E$ in $F$, and let $\hat{K}$ be the completion of $K$. We have

$$\text{cl} (K, \overline{K}) = \text{cl} (K, L) \cap \overline{K} = \hat{K} \cap \overline{K}.$$ 

But, by the transfer principle of Ax-Kochen–Ershov (see e.g. [6]), $K$ is an elementary substructure of $\hat{K}$ and, a fortiori, is algebraically closed in $\hat{K}$. Hence

$$\text{cl} (K, \overline{K}) = \hat{K} \cap \overline{K} = K,$$

as asserted. \qed

Consider an irreducible polynomial

$$P(x, y) = \sum_{i=0}^{d} p_i(x)y^i \in K[x, y]$$

in two variables $x, y$ of $y$-degree $d \geq 1$. Let $Z$ be the Zariski closure of its zero locus in $\overline{K} \times \overline{K}^{\mathbb{P}^1}$. Performing a linear fractional transformation over the ground field $K$ of the variable $y$, we can assume that the fiber $\{w_1, \ldots, w_s\}, s \leq d,$ of $Z$ over $x = 0$ does not contain the point at infinity,
i.e. \(w_1, \ldots, w_s \in \overline{K}\). Then \(p_d(0) \neq 0\) and \(p_d(x)\) is a unit in \(L\{x\}\). Via Hensel’s lemma, we get the Hensel decomposition

\[
P(x, y) = p_d(x) \cdot \prod_{j=1}^{s} P_j(x, y)
\]

of \(P(x, y)\) into polynomials

\[
P_j(x, y) = (y - w_j)^{d_j} + p_{j1}(x)(y - w_j)^{d_j-1} + \cdots + p_{j\delta_j}(x) \in L\{x\}[y - w_j]
\]

which are Weierstrass with respect to \(y - w_j\), \(j = 1, \ldots, s\), respectively. By Puiseux’s theorem, there is a neighbourhood \(U\) of \(0 \in \overline{K}\) such that the trace of \(Z\) on \(U \times \overline{K}\) is a finite union of sets of the form

\[
Z_{\phi_j} = \{(x^q, \phi_j(x)) : x \in U\} \quad \text{with some } \phi_j \in L\{x\}, \ q \in \mathbb{N}, \ j = 1, \ldots, s.
\]

Obviously, for \(j = 1, \ldots, s\), the fiber of \(Z_{\phi_j}\) over \(x \in U\) tends to the point \(\phi_j(0) = w_j\) when \(x \to 0\).

If \(\phi_j(0) \in \overline{K} \setminus K\), it follows from Lemma 1 that

\[
Z_{\phi_j} \cap ((U \cap K) \times K) = \emptyset,
\]

after perhaps shrinking the neighbourhood \(U\).

Let us mention that if

\[
\phi_j(0) \in K \quad \text{and} \quad \phi_j \in L\{x\} \setminus \widehat{K}\{x\},
\]

then

\[
Z_{\phi_j} \cap ((U \cap K) \times K) = \{(0, \phi(0))\}
\]

after perhaps shrinking the neighbourhood \(U\). Indeed, let

\[
\phi_j(x) = \sum_{k=0}^{\infty} a_k x^k \in L[[x]]
\]

and \(p\) be the smallest positive integer with \(a_p \in L \setminus \widehat{K}\). Since \(\widehat{K}\) is a closed subspace of \(L\), we get

\[
\sum_{k=p}^{\infty} a_k x^k = x^p \left( a_p + x \cdot \sum_{k=p+1}^{\infty} a_k x^{k-(p+1)} \right) \notin \widehat{K}
\]
for $x$ close enough to 0, and thus the assertion follows.

Suppose now that an $\mathcal{L}$-definable function $f : A \to K$ satisfies the equation

$$P(x, f(x)) = 0 \quad \text{for } x \in A$$

and 0 is an accumulation point of the set $A$. It follows immediately from the foregoing discussion that the set $A$ can be partitioned into a finite number of $\mathcal{L}$-definable sets $A_j, j = 1, \ldots, r$ with $r \leq s$, such that, after perhaps renumbering of the fiber $\{w_1, \ldots, w_s\}$ of the set $\{P(x, f(x)) = 0\}$ over $x = 0$, we have

$$\lim_{x \to 0} f|_{A_j}(x) = w_j \quad \text{for each } j = 1, \ldots, r.$$

Hence and by Proposition 1, we immediately obtain the following

**Theorem 2** (Existence of the limit). Let $f : A \to K$ be an $\mathcal{L}$-definable function on a subset $A$ of $K$ and suppose 0 is an accumulation point of $A$. Then there is a finite partition of $A$ into $\mathcal{L}$-definable sets $A_1, \ldots, A_r$ and points $w_1, \ldots, w_r \in K\mathbb{P}^1$ such that

$$\lim_{x \to 0} f|_{A_j}(x) = w_j \quad \text{for } j = 1, \ldots, r.$$

Moreover, there is a neighbourhood $U$ of 0 such that the $\mathcal{L}$-definable set

$$\{(v(x), v(f(x))) : x \in (E \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}) \subset \mathbb{R} \times (\mathbb{R} \cup \{\infty\})$$

is contained in a finite number of affine lines and $\mathbb{R} \times \{\infty\}$. □

**Remark.** In the above theorem, the existence of the limits could also be established through the lemma on the continuity of roots of a monic polynomial, which can be found in e.g. [3, Chap. 3, § 3]). The second conclusion, in turn, follows from Puiseux’s parametrization.

4. **Fiber shrinking for $\mathcal{L}$-definable sets.** Let $A$ be an $\mathcal{L}$-definable subset of $K^n$ with accumulation point $a \in K^n$, and $E$ an $\mathcal{L}$-definable subset of $K$ with accumulation point $a_1$. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$
an $L$-definable $x_1$-fiber shrinking for the set $A$ at $a$ if
\[
\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),
\]
i.e. for any neighbourhood $U$ of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood $V$ of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E, t \neq a_1$. When $n = 1$, $A$ is itself a fiber shrinking for the subset $A$ of $K$ at an accumulation point $a \in K$. This concept is a relaxed version of curve selection, available in the non-archimedean geometry treated in this paper. While the real case of the classical curve selection lemma goes back to papers \cite{5, 26} (see also \cite{18, 19}), the $p$-adic one was achieved in papers \cite{25, 14}.

**Theorem 3** (Fiber shrinking). *Every* $L$-definable subset $B$ of $K^n$ *with accumulation point* $b \in K^n$ *has*, after a permutation of the variables, an $L$-definable $x_1$-fiber shrinking at $a$.

**Proof.** We proceed with induction with respect to the dimension of the ambient affine space $n$. The case $n = 1$ is evident. So assuming the assertion to hold for $n$, we shall prove it for $n + 1$. The case $n = 1$ is trivial. We may, of course, assume that $b = 0$. Let $x = (x_1, \ldots, x_n), y$ be coordinates in $K^{n+1} = K^n \times K_y$.

If $0$ is an accumulation point of the intersections
\[ B \cap \{x_i = 0\}, \quad i = 1, \ldots, n, \text{ or } B \cap \{y = 0\}, \]
we are done by the induction hypothesis. Thus we can assume that the intersection
\[ B \cap \left( \bigcup_{i=1}^n \{x_i = 0\} \cup \{y = 0\} \right) = \emptyset \]
is empty. By Theorem 1 (on cell decomposition), we can assume that $B$ is a cell in $K^n \times K$ with parameters $\xi \in k^n$ and center $c(x, \xi)$, i.e. $B$ is the disjoint union of sets $C_\xi$ of the form (cf. Section 1):
\[ C(\xi) := \{(x, y) \in K^n_x \times K_y^y : (x, \xi) \in D, v(a(x, \xi))_1 \prec v(((y - c(x, \xi))_1) \prec v(b(x, \xi)), \overline{ac}(y - c(x, \xi)) = \xi_1) \}. \]
But the set
\[ \{(v(x_1), \ldots, v(x_n), v(y), \xi) \in \Gamma^{n+1} \times k^n : (x, y) \in C_\xi \} \]
is an $L$-definable subset of the product $\Gamma^{n+1} \times k^m$ of the two sorts, and thus is a finite union of the Cartesian products of definable subsets in $\Gamma^{n+1}$ and in $k^m$, respectively. Therefore, 0 is an accumulation point of the fiber $C(\xi')$ of the cell $B$ for a parameter $\xi' \in k^m$. Hence the definable (in the $\Gamma$-sort) set

$$P := \{(v(x_1), \ldots, v(x_n), v(y)) \in \Gamma^{n+1} : (x, y) \in C_{\xi'}\}$$

contains an accumulation point $(\infty, \ldots, \infty)$. We work under the assumption that $\Gamma$ is an ordered subgroup of the ordered additive group $(\mathbb{R}, +, <)$ without minimal positive element. Thus $\Gamma$ admits quantifier elimination in the language of ordered groups and, moreover, every definable function is piece-wise $\mathbb{Q}$-linear (see e.g. [7]). Consequently, the set $P$, being a finite union of semi-linear subsets, contains an affine semi-line

$$L := \left\{ \frac{1}{s} \cdot (r_1k + \gamma_1, \ldots, r_nk + \gamma_n, r_0k + \gamma_0) \in \Gamma^{n+1} : k \in \Gamma, k > \beta \right\},$$

where $s, r_0, r_1, \ldots, r_n$ are positive integers, $\gamma_0, \gamma_1, \ldots, \gamma_n, \beta \in \Gamma$. Define an $L$-definable set $B_0$ of $C(\xi')$ by putting

$$B_0 := \{(x, y) \in K^n_x \times K_y : (x, y) \in C(\xi'), (v(x_1), \ldots, v(x_n), v(y)) \in L\},$$

and let $A := \pi(B_0)$ where $\pi : K^n_x \times K_y \rightarrow K^n_x$ is the canonical projection. Then

$$\{(v(x_1), \ldots, v(x_n)) \in \Gamma^n : x \in A\} =$$

$$= \left\{ \frac{1}{s} \cdot (r_1k + \gamma_1, \ldots, r_nk + \gamma_n) \in \Gamma^n : k \in \Gamma, k > \beta \right\}.$$

Hence 0 is an accumulation point of the set $A$. By the induction hypothesis, there exists, after a permutation of the variables, an $L$-definable $x_1$-fiber shrinking $\Phi$ for $A$ at 0:

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A, \quad \lim_{t \to 0} \Phi_t = 0;$$

here $E$ is the canonical projection of $A$ onto the $x_1$-axis.

Consider now the plane $L$-definable set

$$B^* := \{(t, y) \in K_t \times K_y : \exists u \in \Phi_t (t, u, y) \in B_0\}.$$
Then \((0, 0)\) is an accumulation point of \(B^*\). The theorem will be proven once we establish fiber shrinking for the set \(B^*\) at \((0, 0)\). Indeed, if

\[
\Psi = \bigcup_{t \in E} \{t\} \times \Psi_t \subset B^*, \quad \lim_{t \to 0} \Psi_t = 0,
\]

is an \(\mathcal{L}\)-definable \(x_1\)-fiber shrinking for \(B^*\) at 0, then it is easy to check that the \(\mathcal{L}\)-definable family of sets

\[
\{(t, u, y) \in K \times K^{n-1} \times K : (t, y) \in \Psi, (t, u) \in \Phi, (t, u, y) \in B_0\}
\]

is an \(\mathcal{L}\)-definable \(x_1\)-fiber shrinking for the set \(B_0\) at 0.

In this manner, we are reduced to the case where \(n = 2\) and \(b = (0, 0) \in K^2\). As before, we can assume that \(B\) is a cell in \(K^2\), and next that \(B\) is the fiber

\[
C(\xi') := \{(x, y) \in K_x \times K_y : (x, \xi') \in D, \quad v(a(x, \xi')) <_1 v((y - c(x, \xi'))^\nu) <_2 v(b(x, \xi')), \quad \overline{ac}(y - c(x, \xi')) = \xi'_1\}
\]

of the cell for a parameter \(\xi' \in k^m\); here \(<_1, <_2\) stand for <, \(\leq\) or no condition. For simplicity, we abbreviate \(c(x, \xi'), a(x, \xi'), b(x, \xi')\) to \(c(x), a(x), b(x)\).

By Theorem 2, we may assume that the limits, say \(c(0), a(0), b(0),\) of \(c(x), a(x), b(x)\) when \(x \to 0\) exist in \(K^P^1\). Performing a linear fractional transformation of the variable \(y\), we can assume that \(c(0), a(0), b(0) \in K\).

Put

\[
E := \{x \in K : (x, \xi') \in D\}.
\]

Let us recall that we have been working under the assumption that \((0, 0)\) is an accumulation point of \(B = C(\xi')\). Our proof falls into two major cases.

**Case I**: \(c(0) \neq 0\). When \(<_1\) occurs, there must be \(v(a(0)) \leq \nu \cdot v(c(0))\).

If \(v(a(0)) = \nu \cdot v(c(0))\), then \(<_1\) must be \(\leq\), and we can take \(\Phi = E \times \{0\}\).

When \(<_2\) occurs, there must be \(v(b(0)) \geq \nu \cdot v(c(0))\).

If \(v(b(0)) = \nu \cdot v(c(0))\), then \(<_2\) must be \(\leq\), and we can take \(\Phi = E \times \{0\}\).

Thus it remains to consider only the case where \(v(a(0)) < \nu \cdot v(c(0))\) (when \(<_1\) occurs) and \(v(b(0)) > \nu \cdot v(c(0))\) (when \(<_2\) occurs). Again, we can then take \(\Phi = E \times \{0\}\), being immaterial which \(<_1\) or \(<_2\) occur.

**Case II**: \(c(0) = 0\). There may be \(a(0) = 0\) or \(a(0) \neq 0\) or \(<_1\) is no condition; the last two subcases impose no real condition. Further, there must be \(b(0) = 0\) or \(<_2\) is no condition; the latter subcase is trivial.
Under the circumstances, it is sufficient to analyse the three subcases: only $<_1$ occurs and $a(0) = 0$; only $<_2$ occurs and $b(0) = 0$; $<_1$, $<_2$ occur and $a(0) = b(0) = 0$. In the first and third subcase, however, the set $B$ itself is an $x$-fiber shrinking at $(0, 0)$. Consider now the second one.

By elimination of $K$-quantifiers, for any positive integer $\mu$, the dense subset $v(K^\mu)$ of $\Gamma \cup \{\infty\}$ contains $\Gamma$ except a finite number of points. In particular, it contains a right hand side semi-line. Moreover, the set $K^\mu$ contains a set of the form

$$\{x \in K : \overline{ac}(x) = 1, \ v(x) > \alpha\}$$

with some $\alpha \in \Gamma$. According to Theorem 2, we can assume, after shrinking the set $E$, that the set

$$\{(v(x), v(b(x))) : x \in E\}$$

is contained in an affine semi-line

$$k \mapsto \frac{rk + \gamma}{s} \quad \text{for} \quad k \in \Gamma, \ \gamma > \beta,$$

where $r, s$ are positive integers and $\beta, \gamma \in \Gamma$. On the other hand, the $\mathcal{L}$-definable set $E$ contains the set of the form

$$\{x \in K : v(x) > \delta, \ \overline{ac}(x) = \lambda\}$$

with some $\delta \in \Gamma$ and $\lambda \in K$. Take elements $w, z \in K$ such that

$$v(w) < \gamma, \ v(z) = 0, \ \overline{ac}(w) = 1 \ \text{and} \ \overline{ac}(z) = \lambda.$$

Then it is easy to check that the set

$$\left\{ (x, y) : x \in E, \ (y - c(x))^\mu = w \left( \frac{x}{z} \right) \right\},$$

where $\mu = \nu \cdot s$, is the $x$-shrinking we are looking for. This completes the proof of Theorem 3.

5. Further conclusions. This section provides several results which are counterparts of those given in our previous paper \cite{21} over Henselian discretely valued fields of equicharacteristic zero. While the crucial theorem
of that paper upon which all other results relied was the closedness theorem, stated below, here it is established by means of fiber shrinking given in Section 4.

**Theorem 4** (Closedness theorem). Let $D$ be an $L$-definable subset of $K^n$. Then the canonical projection $\pi : D \times R^m \to D$ is definably closed in the $K$-topology, i.e. if $B \subset D \times R^m$ is an $L$-definable closed subset, so is its image $\pi(B) \subset D$.

**Proof.** The proof reduces easily to the case $m = 1$. We shall make use of fiber shrinking (Theorem 3), cell decomposition (Theorem 1) and existence of the limit (Theorem 2).

We must show that if a point $a$ lies in the closure of $A := \pi(B)$, then there is a point $b$ in the closure of $B$ such that $\pi(b) = a$. We may obviously assume that $a \notin A$. By Theorem 3, there exists, after a permutation of the variables, an $L$-definable $x_1$-fiber shrinking $\Phi$ for $A$ at $a$:

$$
\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A, \quad \lim_{t \to 0} \Phi_t = 0;
$$

here $E$ is the canonical projection of $A$ onto the $x_1$-axis. Put

$$
B^* := \{(t, y) \in K \times R : \exists u \in \Phi_t (t, u, y) \in B\};
$$

it is easy to check that if a point $(a_1, w)$ lies in the closure of $B^*$, then the point $(a, w)$ lies in the closure of $B$. The problem is thus reduced to the case $n = 1$. We may, of course, assume that $a = 0 \in K$.

Via cell decomposition, we can assume that $B$ is a cell

$$C \subset K_x \times R \subset K_x \times K_y$$

and next, as before, that $B$ is the fiber

$$C(\xi') := \{(x, y) \in K_x \times K_y : (x, \xi') \in D, \quad v(a(x, \xi')) <_1 v((y - c(x, \xi'))^{\nu_1}) <_2 v(b(x, \xi')), \quad \overline{v(c(x, \xi'))} = \xi_1\}$$

of the cell for a parameter $\xi'$. Again, we abbreviate $c(x, \xi'), a(x, \xi'), b(x, \xi')$ to $c(x), a(x), b(x)$ for simplicity.

In the statement of Theorem 4, we may equivalently replace $R$ with the projective line $K\mathbb{P}^1$, because the latter is the union of two open and closed
charts biregular to \( R \). Hence and by Theorem 2, we can assume without
loss of generality that the limits, say \( c(0), a(0), b(0), \) of \( c(x), a(x), b(x) \) when
\( x \to 0 \) exist in \( K\mathbb{P}^1 \). Performing a linear fractional transformation of the
variable \( y \), we can assume that \( c(0), a(0), b(0) \in K \).

Now, our reasoning will be simpler than the discussion of Cases I and II
from the proof of Theorem 3. The role of the center \( c(x) \) is immaterial. We
can assume that it vanishes, \( c(x) \equiv 0 \), for if a point \( b = (0, w) \in K^2 \) lies
in the closure of the cell with zero center, the point \( (0, w + c(0)) \) lies in the
closure of the cell with center \( c(x) \).

It is straightforward to verify the case where only one \( \triangleleft_2 \) occurs, and the case
where \( \triangleleft_1 \) occurs and \( a(0) \neq 0 \). It remains to consider the case where \( \triangleleft_1 \)
occurs and \( a(0) = 0 \). But then the set \( B \) is itself an \( x \)-fiber shrinking at
\((0,0)\). The point \( b = (0,0) \) is thus an accumulation point of \( B \) lying over
\( a = 0 \), as desired. \( \square \)

Below we state without proofs several corollaries to Theorem 4, which
can be deduced in the same manner as before in our paper \[21\].

**Corollary 1.** Let \( K\mathbb{P}^m \) be the projective space of dimension \( m \) over \( K \)
and \( D \) an \( \mathcal{L} \)-definable subset of \( K^n \). Then the canonical projection
\[
\pi : D \times K\mathbb{P}^m \to D
\]
is definably closed. \( \square \)

**Corollary 2.** Let \( \phi_i, i = 0, \ldots, m, \) be regular functions on \( K^n \), \( D \) be an
\( \mathcal{L} \)-definable subset of \( K^n \) and \( \sigma : Y \to K\mathbb{A}^n \) the blow-up of the affine space
\( K\mathbb{A}^n \) with respect to the ideal \( (\phi_0, \ldots, \phi_m) \). Then the restriction
\[
\sigma : Y(K) \cap \sigma^{-1}(D) \to D
\]
is a definably closed quotient map. \( \square \)

**Corollary 3.** Let \( X \) be a smooth \( K \)-variety, \( \phi_i, i = 0, \ldots, m, \) regular
functions on \( X \), \( D \) be an \( \mathcal{L} \)-definable subset of \( X(K) \) and \( \sigma : Y \to X \) the
blow-up of the ideal \( (\phi_0, \ldots, \phi_m) \). Then the restriction
\[
\sigma : Y(K) \cap \sigma^{-1}(D) \to D
\]
is a definably closed quotient map. \( \square \)
Corollary 4 (Descent property). Under the assumptions of Corollary 3, every continuous $L$-definable function $g : \sigma^{-1}(D) \to K$ that is constant on the fibers of the blow-up $\sigma$ descends to a (unique) continuous $L$-definable function $f : D \to K$.

Also, the proofs of the following two theorems from our paper [21] carry over verbatim to Henselian real valued fields of equicharacteristic zero.

Theorem 5 (Lojasiewicz inequality). Let $f, g$ be two continuous rational functions on $X(K)$ and $U$ be an $L$-definable open (in the $K$-topology) subset of $X(K)$. If

$$\{ x \in U : g(x) = 0 \} \subset \{ x \in U : f(x) = 0 \},$$

then there exist a positive integer $s$ and a continuous rational function $h$ on $U$ such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in U$.

Theorem 6 (Extending continuous hereditarily rational functions). Additionally assume that the ground field $K$ is not algebraically closed. Let $X$ be a smooth $K$-variety and $W \subset Z \subset X$ closed subvarieties. Let $f$ be a continuous hereditarily rational function on $Z(K)$ that is regular at all $K$-points of $Z(K) \setminus W(K)$. Then $f$ extends to a continuous hereditarily rational function $F$ on $X(K)$ that is regular at all $K$-points of $X(K) \setminus W(K)$.

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