On a Phase Separation Point for One - Dimensional Models

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Abstract: In the paper a one-dimensional model with nearest - neighbor interactions \(I_n, n \in \mathbb{Z}\) and spin values \(\pm 1\) is considered. It is known that under some conditions on parameters \(I_n\) the phase transition occurs for the model. We define a notion of "phase separation" point between two phases. We prove that the expectation value of the point is zero and its the mean square fluctuation is bounded by a constant \(C(\beta)\) which tends to \(\frac{1}{4}\) if \(\beta \to \infty\). Here \(\beta = \frac{1}{T}, T > 0\)-temperature.

1 Introduction

It is known that the curve of separation between two pure phases for two dimensional Ising model at low temperature is non rigid: Gallavotti [8] showed that the mean square fluctuation of the height of the interface, or phase separation line, diverges as \(\sqrt{L}\) (where \(L\) is side of square) in the thermodynamic limit. A different picture occurs in the three-dimensional case: there exists a value \(\beta_r > \beta_{cr}\) (where \(\beta_{cr}\) is the critical value of inverse temperature \(\beta\) of phase transition for the model) such that for \(\beta > \beta_r\) the phase separation membrane is at a finite distance from the plane \((x=0)\). For values of \(\beta\) between \(\beta_{cr}\) and \(\beta_r\) the membrane deviates from the plane \((x=0)\) at the distance \(\sim \log L\) (see [1], [4], [13]). These results were obtained for the model with short range, translation invariant interactions. In [18] Van Hove showed (see also [16, section 5.6.]) that a one-dimensional system could not exhibit a phase transition if the (translation-invariant) forces were of finite range. However, by breaking translation invariance we can obtain a phase transition in one-dimensional models with only nearest neighbor interactions [17], [10, p.95], [15]. In [3], [5-7], [10]-[12] other examples of phase transitions were considered for one-dimensional models with long range interactions.

In the paper we consider the Hamiltonian

\[ H(\sigma) = \sum_{l=(x-1,x): x \in \mathbb{Z}} I_x \mathbb{1}_{\sigma(x-1) \neq \sigma(x)}, \tag{1} \]

where \(Z = \ldots, -1, -2, 0, 1, 2, \ldots\), \(\sigma = \{\sigma(x) \in \{-1, 1\} : x \in \mathbb{Z}\} \in \mathcal{O} = \{-1, 1\}^Z\), and \(I_x \in R\) for any \(x \in Z\).

Note that [10, p.95] for the model (1) on \(N = \{1, 2, \ldots\}\) it was shown that there occurs a phase transition iff \(\sum_{n \geq 1} e^{-2I_n} < \infty\). In [15] using a contour argument it has been proven that for that model (1) the phase transition occurs if \(I_n + I_{n+k} > k\) for any \(n \in Z, k \in N\).
In two (resp. three) dimensional case the phase separation curve (resp. membrane) is defined as an "open" contour \[4,8\]. But that construction does not work for one dimensional case when interactions are only nearest neighbors. In this case the separation "line" is a point. To the best of our knowledge, there is no any paper devoted to PSP of one-dimensional models. One of the main reasons of this absence, we think, can be the fact that a one-dimensional model has phase transition if either it has long range (Dyson model) or non translational-invariant (Sullivan’s model) interactions. Therefore for such models definition and investigation of PSP is rather difficult problem.

In the present paper we are going to give a more natural definition of the phase separation point (PSP) between two phases in one dimensional setting. For the model (1) we show that the expectation value of the point is zero and its the mean square fluctuation (for point (PSP) between two phases in one dimensional setting. For Sullivan’s model) interactions. Therefore for such models definition and investigation of PSP is rather difficult problem.

Let us consider a sequence \( L_n = [-n, n], n = 0, 1, ... \) and denote \( L_n^c = Z \setminus L_n \). Consider a boundary condition \( \sigma_n^{(+)} = \sigma_{L_n^c} = \{ \sigma(x) = +1 : x \in L_n^c \} \). The energy \( H_n^+(\sigma) \) of the configuration \( \sigma \) in the presence of the boundary condition \( \sigma_n^{(+)} \) is expressed by the formula

\[
H_n^+(\sigma) = \sum_{l=(x-1,x): x \in L_n} I_x 1_{\sigma(x-1) \neq \sigma(x)} + I_{-x} 1_{\sigma(-x) \neq 1} + I_{n+1} 1_{\sigma(x) \neq 1}.
\]  

(2)

The Gibbs measure on \( \mathcal{O}_n = \{-1, 1\}^{L_n} \) with respect to the boundary condition \( \sigma_n^{(+)} \) is defined by the usual way

\[
\mu_{n,\beta}^+(\sigma) = Z^{-1}(n, \beta, +) \exp(-\beta H_n^+(\sigma)),
\]  

(3)

where \( \beta = T^{-1}, T > 0 \) temperature and \( Z(n, \beta, +) \) is the normalizing factor (statistical sum).

Denote by \( \sigma_n^+ \) the configuration on \( Z \) such that \( \sigma_n^+(x) = +1 \) for any \( x \in L_n^c \).

Put

\[
A(\sigma_n^+) = \{ x \in Z : \sigma_n^+(x) = -1 \}.
\]

Note that there is a one-to-one correspondence between the set of all configurations \( \sigma_n^+ \) and the set of all subsets of \( L_n \).

Let \( A'(\sigma_n^+) \) be the set of all maximal connected subsets of \( A(\sigma_n^+) \).

**Lemma 1.**\[15\] Let \( B \subset Z \) be a fixed connected set and \( p_{\beta}^+(B) = \mu_{n,\beta}^+(\sigma_n^+: B \in A'(\sigma_n^+)) \).

Then

\[
p_{\beta}^+(B) \leq \exp \left\{ -\beta \left[ I_{n_B} + I_{N_B+1} \right] \right\},
\]

where \( n_B \) (resp. \( N_B \)) is the left (resp. right) endpoint of \( B \).

Assume that the coupling interactions of the Hamiltonian (1) satisfy the following condition

\[
I_n + I_{n+r} \geq r \text{ for any } r \in \{1, 2, ...\} \text{ and } n \in Z
\]  

(4)
Theorem 2. [15] Assume the condition (4) is satisfied. For all sufficiently large \( \beta \) there are at least two Gibbs measures for the model (1).

Denote
\[ \mathcal{H} = \{ H : H \text{ (see (1)) satisfies the condition (4)} \} \]

The following example shows that the set \( \mathcal{H} \) is not empty.

Example. Consider Hamiltonian (1) with \( I_m \geq |m|, \ m \in \mathbb{Z} \). Then
\[ I_m + I_{m+k} \geq |m| + |m+k| \geq k \]
for all \( m \in \mathbb{Z} \) and \( k \geq 1 \). Thus the condition (4) is satisfied.

3 "±" -boundary condition

3.1 Statistical sum

Consider two type of statistical sums:

\[
Z^+ = \sum_{\sigma_n \in \mathbb{O}} \exp\{-\beta H^+_n(\sigma_n)\}, \tag{5}
\]

\[
Z^\pm = \sum_{\sigma_n \in \mathbb{O}} \exp\{-\beta H^\pm_n(\sigma_n)\}, \tag{6}
\]

where \( H^+_n \) is defined by (2) and
\[
H^\pm_n(\sigma_n) = H^+_n(\sigma_n) + I_{-n}\sigma(-n). \tag{7}
\]

In this paper for the simplicity assume
\[
I_n = I_{n+1}, \text{ for any } n \in \mathbb{Z}. \tag{8}
\]

Under the condition (8) we get
\[
Z^- = Z^+_n \text{ and } Z^\pm_n = Z^\mp_n. \tag{9}
\]

Using (8) and (9) from (5),(6) we obtain
\[
Z^+_n = (1 + e^{-2\beta I_{n+1}}) Z^+_n - 1 + 2e^{-\beta I_{n+1}} Z^+_n, \tag{10}
\]

\[
Z^\pm_n = (1 + e^{-2\beta I_{n+1}}) Z^\pm_n - 1 + 2e^{-\beta I_{n+1}} Z^\mp_n
\]

Putting \( X_n = Z^+_n - Z^-_n \) and \( Y_n = Z^+_n + Z^\pm_n \) from (10) one gets
\[
X_n = (1 - e^{-\beta I_{n+1}})^2 X_{n-1}, \tag{11}
\]

\[
Y_n = (1 + e^{-\beta I_{n+1}})^2 Y_{n-1}.
\]

The equalities \( X_0 = Z^+_0 - Z^-_0 = (1 - e^{-\beta I_1})^2 \), \( Y_0 = (1 + e^{-\beta I_1})^2 \) with (11) imply
\[
X_n = \prod_{i=0}^{n}(1 - e^{-\beta I_{i+1}})^2, \ Y_n = \prod_{i=0}^{n}(1 + e^{-\beta I_{i+1}})^2.
\]
Clearly, $H$ is invariant with respect to operator $S$, i.e. $S(U): \mathbb{O} \rightarrow \mathbb{O}$, where $U$ is a one-to-one map from $\mathbb{O}$ onto $\mathbb{O}$, such that $U(x) = \sigma(x, +)$ and $V(x) = \sigma(x, -)$. Hence

$$Z_n^+ = \frac{1}{2} \left( \prod_{i=0}^{n} (1 + e^{-\beta I_{i+1}}) + \prod_{i=0}^{n} (1 - e^{-\beta I_{i+1}}) \right),$$

$$Z_n^- = \frac{1}{2} \left( \prod_{i=0}^{n} (1 + e^{-\beta I_{i+1}}) - \prod_{i=0}^{n} (1 - e^{-\beta I_{i+1}}) \right).$$

For example, in a case of the usual Ising model, i.e. $I_n = I, \forall n$ from (12) denoting $\tau = \exp(-\beta I)$ we get

$$Z_n^+ = \frac{1}{2} \left( (1 + \tau)^{2(n+1)} + (1 - \tau)^{2(n+1)} \right),$$

$$Z_n^- = \frac{1}{2} \left( (1 + \tau)^{2(n+1)} - (1 - \tau)^{2(n+1)} \right).$$

Using these equalities (for usual Ising model) it is easy to see that

$$\frac{Z_n^+}{Z_n^-} \rightarrow 1, \text{ if } n \rightarrow \infty.$$  

### 3.2 Phase-separation point

Fix $n \in \{0, 1, 2, \ldots\}$. Denote by $\mathbb{O}_n$ the set of all configurations on $L_n = \{-n, ..., n\}$ i.e. $\mathbb{O}_n = \{-1, 1\}^{L_n}$. For every $\sigma_n \in \mathbb{O}_n$ define $\sigma_n^\pm \in \{-1, 1\}^Z$ as follows

$$\sigma_n^\pm(x) = \begin{cases} -1 & \text{if } x < -n \\ \sigma_n(x) & \text{if } x \in L_n \\ 1 & \text{if } x > n \end{cases}.$$  

Let $\mathbb{O}_n^\pm$ be the set of all configurations defined by (13). Denote

$$\mathbb{O}_n^{(+)} = \{\sigma_n \in \mathbb{O}_n^\pm : |\{x \in L_n : \sigma_n(x) = 1\}| \geq n + 1\};$$

$$\mathbb{O}_n^{(-)} = \{-\sigma_n : \sigma_n \in \mathbb{O}_n^{(+)}\}.$$  

Clearly $\mathbb{O}_n^{(+)} \cap \mathbb{O}_n^{(-)} = \emptyset$ and $\mathbb{O}_n^\pm = \mathbb{O}_n^{(+)} \cup \mathbb{O}_n^{(-)}$.

Let $S : \mathbb{O}_n^\pm \rightarrow \mathbb{O}_n^\pm$ be operator such that

$$S(\sigma_n)(x) = -\sigma_n(-x), \quad x \in Z.$$  

It is easy to see that

$$S(\mathbb{O}_n^{(\pm)}) = \mathbb{O}_n^{(\mp)},$$

i.e. the operator $S$ is one-to-one map from $\mathbb{O}_n^{(+)}$ (resp. $\mathbb{O}_n^{(-)}$) to $\mathbb{O}_n^{(-)}$ (resp. $\mathbb{O}_n^{(+)}$).

**Lemma 3.** The Hamiltonian (1) (under condition (8)) is invariant with respect to operator $S$ i.e. $H(S(\sigma)) = H(\sigma)$ for any $\sigma \in \mathbb{O}_n^\pm, n = 0, 1, ...$

**Proof.** Note that operator $S$ is the combination of the following two symmetry maps

$U : \mathbb{O}_n^{(+)} \rightarrow \mathbb{O}_n^{(\pm)}$ such that $U(\sigma_n)(x) = -\sigma_n(x)$, and $V : \mathbb{O}_n^{(-)} \rightarrow \mathbb{O}_n^{(\pm)}$ such that $V(\sigma_n)(x) = \sigma_n(-x)$. Clearly, $H$ is invariant with respect to $U$ and $V$ this completes the proof.

Denote $T_n = \{-n - \frac{1}{2}, -n + \frac{1}{2}, ..., -n - \frac{1}{2}, n + \frac{1}{2}\}$. Fix $\sigma_n \in \mathbb{O}_n^\pm$ and we say that $t \in T_n$ is an interface point for the configuration $\sigma_n$ if $\sigma_n(t - \frac{1}{2}) \neq \sigma_n(t + \frac{1}{2})$. For any interface point $t \in T_n$ denote

$$l_t^\pm(\sigma_n) = |\{x \in L_n : \sigma_n(x) = -1, x < t\}|,$$
\[
\begin{align*}
\sigma_n^+ &\equiv \sigma_n^+(x) = \{x \in \Omega_n : \sigma_n(x) = 1, x > t\}, \\
l_t^+ &= n + t + \frac{1}{2} - l_t^-,
\quad r_t^- = n - t + \frac{1}{2} - r_t^+.
\end{align*}
\]

Definition 4. We define PSP \( \gamma_n(\sigma_n) \in T_n \) as the following interface point

\[
\gamma_n(\sigma_n) = \begin{cases} 
\max \{t_0 \in T_n : \|\Delta t_0\| = \max_{t} \|\Delta t\|\}, & \text{if } \sigma_n \in \Omega^-_n, \\
\min \{t_0 \in T_n : \|\Delta t_0\| = \max_{t} \|\Delta t\|\}, & \text{if } \sigma_n \in \Omega^+_n.
\end{cases}
\] (16)

Lemma 5. For any \( \sigma_n \in \Omega^\pm_n \) we have

\[
\gamma_n(\sigma_n) = -\gamma_n(S(\sigma_n)).
\] (17)

Proof. Straightforward.

For \( \theta \in T_n \) denote

\[
P_n(\theta) = \mu^\pm_n \{\sigma_n : \gamma_n(\sigma_n) = \theta\},
\]

where \( \mu^\pm_n \) is the Gibbs measure with respect to \( \pm \)-boundary condition.

Lemma 6. For any \( \theta \) and \( n \in \mathbb{N} \) we have

\[
P_n(\theta) = P_n(-\theta).
\]

Proof. The proof follows from lemma 3, and equality (17).

As a corollary of lemmas 3 and 6 we have

Lemma 7. For any \( n \in \mathbb{N} \)

\[
E_{\mu^\pm_n}(\gamma_n) = 0,
\]

where \( E_{\mu^\pm_n} \) is the expectation value of the random variable \( \gamma_n \) with respect to the Gibbs measure \( \mu^\pm_n \).

For a given configuration \( \sigma_n \) denote by \( \theta_1 < \theta_2 < \ldots < \theta_k \) the interface points generated by \( \sigma_n \).

Theorem 8. 1. If an interface point \( t = \theta_1 \) (resp. \( t = \theta_k \)) is PSP then

\[
l_t^- \geq l_t^+ = 0, \quad r_t^+ > r_t^-.
\] (resp. \( l_t^- > l_t^+ \), \( r_t^+ \geq r_t^- = 0 \)).

2. If an interface point \( t \in T_n, t \neq \theta_1, \theta_k \) is PSP then

\[
l_t^- > l_t^+, \quad r_t^+ > r_t^-.
\] (19)

Proof. Consider case \( \sigma_n \in \Omega^{(+)}_n \) and \( t \neq \theta_1, \theta_k \) (all other cases can be proved similarly).

Assume \( l_t^- \leq l_t^+ \) then

\[
\|\Delta t\| = l_t^- + r_t^+ < l_t^+ + r_t^+ = r_{\theta_1}^+ \leq \|\Delta \theta_1\|.
\]
Thus by definition we get \( \gamma_n(\sigma_n) = \theta_1 \), which contradicts to \( t \neq \theta_1 \). This completes the proof.

**Remark 1.** In general for a given configuration \( \sigma_n \) a point \( t \) satisfying the condition (18), (19) is not unique. For example, take \( \sigma_2 = \{ \sigma_2(-2) = -1, \sigma_2(-1) = -1, \sigma_2(0) = 1, \sigma_2(1) = -1, \sigma_2(2) = 1 \} \), the interface points \( t = -0.5 \) and \( t = 1.5 \) satisfy the condition (19). Thus the conditions (18), (19) are necessary for \( t \) to be PSP but are not sufficient.

Summing over all configurations with a given \( \theta \) we obtain the probability \( P_n(\theta) \) of \( \theta \) which can be written by

\[
P_n(\theta) = \frac{e^{-\beta I_{\theta+1/2}} Y_{-n,\theta-1/2} Y_{\theta+1/2,n}}{Z_n},
\]

where \( Z_n^\pm \) is defined by (12) and \( Y_{-n,\theta-1/2} \) (resp. \( Y_{\theta+1/2,n} \)) is the ”crystal” partition function which contains only sum of terms \( \exp(-\beta H^+(\varphi)) \) with \( \varphi = \varphi' \in \{-1,1\}^{[-n,\theta-3/2]} \) (resp. \( \varphi = \varphi'' \in \{-1,1\}^{[\theta+3/2,n]} \)) such that the PSP of the total configuration \( \sigma = \sigma' \cup \{ \sigma(\theta - 1/2) = -1, \ \sigma(\theta + 1/2) = 1 \} \cup \sigma'' \) on \([-n, n]\) is \( \theta \).

**Remark 2.** In two-dimensional Ising model case an analog of the formula (20) is given in [2, formula (3.2)]. Comparing our formula (20) with the formula (3.2) we notice that the numerator of the formula (20) contains product of “full” (all possible terms) partition functions with pure ”+” boundary conditions (or ”−” boundary conditions which is equivalent by symmetry) in the different connected components of \( Z^2 \) which are separated by the phase separation curve. But in our setting the numerator of the formula (20) contains product of crystal partition functions which we have defined above. This is a remarkable difference between the notions of phase separation of one and two dimensional Ising models. In the sequel of this section we are going to estimate the crystal partition functions by ”rarefied” partition functions.

By Lemma 6 it is enough to consider the case \( \theta \geq \frac{1}{2} \). For \( A \subset \mathbb{Z} \) we denote \( \emptyset_A = \{-1,1\}^A \) - the set of all configurations defined on \( A \). Denote

\[
H^-_{n,\theta}(\sigma) = \sum_{x=-n}^{\theta-\frac{3}{2}} I_x 1_{\sigma(x) \neq \sigma(x+1)} + I_{-n} 1_{\sigma(-n) \neq -1} + I_{\theta-\frac{3}{2}} 1_{\sigma(\theta-\frac{3}{2}) \neq -1}, \quad \sigma \in \emptyset_{\{-n,\ldots,\theta-\frac{3}{2}\}};
\]

\[
H^+_{n,\theta}(\sigma) = \sum_{x=\theta+\frac{3}{2}}^{n} I_x 1_{\sigma(x) \neq \sigma(x+1)} + I_{n+1} 1_{\sigma(n) \neq 1} + I_{\theta+\frac{3}{2}} 1_{\sigma(\theta+\frac{3}{2}) \neq 1}, \quad \sigma \in \emptyset_{\{\theta+\frac{3}{2},\ldots,n\}};
\]

\[
H^\pm_{n,\theta}(\sigma) = H^-_{n,\theta}(\sigma) - I_{-n} \sigma(-n);
\]

\[
H^\mp_{n,\theta}(\sigma) = H^+_{n,\theta}(\sigma) + I_{n+1} \sigma(n).
\]

Now we are ready to define the ”rarefied” partition functions i.e.

\[
\overline{Z}_{n,\theta} = \sum_{\sigma \in \emptyset_{\{\theta+\frac{3}{2},\ldots,n\}}} \exp(-\beta H^+_{n,\theta}(\sigma)); \quad \overline{Z}^-_{n,\theta} = \sum_{\sigma \in \emptyset_{\{-n,\ldots,\theta-\frac{3}{2}\}}} \exp(-\beta H^-_{n,\theta}(\sigma));
\]

\[
\overline{Z}^+_{n,\theta} = \sum_{\sigma \in \emptyset_{\{-n,\ldots,\theta-\frac{3}{2}\}}} \exp(-\beta H^-_{n,\theta}(\sigma)); \quad \overline{Z}^\pm_{n,\theta} = \sum_{\sigma \in \emptyset_{\{\theta+\frac{3}{2},\ldots,n\}}} \exp(-\beta H^\pm_{n,\theta}(\sigma)).
\]

Note that (see (20))

\[
Y_{-n,\theta-1/2} \leq \overline{Z}_{n,\theta}; \quad Y_{\theta+1/2,n} \leq \overline{Z}^-_{n,\theta}.
\]
It is easy to check that
\[
\begin{align*}
\mathcal{Z}_{n,\theta} &= \mathcal{Z}_{n-1,\theta} + e^{-\beta I_{n+1}} \mathcal{Z}_{n-1,\theta}, \quad n \geq \theta + \frac{3}{2} \\
\mathcal{Z}_{n,\theta}^- &= \mathcal{Z}_{n-1,\theta}^- + e^{-\beta I_{n+1}} \mathcal{Z}_{n-1,\theta}^-, \\
\mathcal{Z}_{\theta + \frac{1}{2},\theta}^- &= 1; \quad \mathcal{Z}_{\theta + \frac{1}{2},\theta} = e^{-\beta I_{\theta + \frac{1}{2}}}.
\end{align*}
\] (22)

Denote \( u_{n,\theta} = \mathcal{Z}_{n,\theta} - \mathcal{Z}_{n,\theta}^- \), \( v_{n,\theta} = \mathcal{Z}_{n,\theta} + \mathcal{Z}_{n,\theta}^- \). Then from (22) we get
\[
\begin{align*}
u_{n,\theta} &= (1 - e^{-\beta I_{n+1}}) u_{n-1,\theta}, \quad n \geq \theta + \frac{3}{2}, \\
u_{\theta + \frac{1}{2},\theta} &= 1 - e^{-\beta I_{\theta + \frac{1}{2}}},
\end{align*}
\]

i.e.
\[
u_{n,\theta} = \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 - e^{-\beta I_{i+1}}\right).
\]

Similarly
\[
v_{n,\theta} = \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 + e^{-\beta I_{i+1}}\right).
\]

Hence
\[
\begin{align*}
\mathcal{Z}_{n,\theta} &= \frac{1}{2} \left(\prod_{i=\theta + \frac{1}{2}}^{n} \left(1 + e^{-\beta I_{i+1}}\right) + \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 - e^{-\beta I_{i+1}}\right)\right); \\
\mathcal{Z}_{n,\theta}^- &= \frac{1}{2} \left(\prod_{i=\theta + \frac{1}{2}}^{n} \left(1 + e^{-\beta I_{i+1}}\right) - \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 - e^{-\beta I_{i+1}}\right)\right).
\end{align*}
\] (23)

Analogically, using condition (8) we get
\[
\begin{align*}
\mathcal{Z}_{n,\theta} &= \frac{1}{2} \left(\prod_{i=\theta + \frac{1}{2}}^{n} \left(1 + e^{-\beta I_{i+1}}\right) \prod_{i=1}^{\theta - \frac{1}{2}} \left(1 + e^{-\beta I_{i+1}}\right)^2 + \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 - e^{-\beta I_{i+1}}\right) \prod_{i=1}^{\theta - \frac{1}{2}} \left(1 - e^{-\beta I_{i+1}}\right)^2\right); \\
\mathcal{Z}_{n,\theta}^+ &= \frac{1}{2} \left(\prod_{i=\theta + \frac{1}{2}}^{n} \left(1 + e^{-\beta I_{i+1}}\right) \prod_{i=1}^{\theta - \frac{1}{2}} \left(1 + e^{-\beta I_{i+1}}\right)^2 - \prod_{i=\theta + \frac{1}{2}}^{n} \left(1 - e^{-\beta I_{i+1}}\right) \prod_{i=1}^{\theta - \frac{1}{2}} \left(1 - e^{-\beta I_{i+1}}\right)^2\right).
\end{align*}
\] (24)

Using formulas (12), (23), (24) and inequalities (21) from (20) one gets an upper bound of \( P_n(\theta) \).

4 Variation of the PSP

In this section, for simplicity, we consider the following case
\[
I_n = \begin{cases} 
  n, & \text{if } n > 0 \\
  \theta + 1, & \text{if } n \leq 0
\end{cases}
\] (25)

By lemmas 6 and 7 the variation of \( \gamma_n \) can be written as
\[
\text{Var}(\gamma_n) = 2 \sum_{\theta = \frac{1}{2}}^{n+\frac{1}{2}} \theta^2 P_n(\theta).
\] (26)
Theorem 9. If interactions $I_n$ satisfy (25) and $\beta$ large enough then

$$\frac{1}{4} \leq \text{Var}(\gamma_n) \leq \sim \frac{\tau A(\tau) \cosh \left( \frac{\tau^2}{1-\tau} \right) \left( 1 + \frac{3\tau(\tau + 3)}{(1-\tau)^2} \right)}{2 \sinh(2\tau)},$$

where $\tau = e^{-\beta}$, $A(\tau) = \cosh \left( \frac{\tau^2}{1-\tau} \right) \cosh (\tau(1+\tau)) - \sinh (\tau^2) \sinh (\tau(1+\tau))$.

Proof. The lower bound easily follows from (26). We shall prove upper bound. It follows from (26), (20) and (21) that

$$\text{Var}(\gamma_n) = 2 \sum_{\theta = \frac{1}{2}}^{n+\frac{1}{2}} \theta^2 e^{-\beta I_n} \frac{Y_{-n,\theta-1/2} Y_{\theta+1/2,n}}{Z_n^{\pm}} \leq$$

$$2 \sum_{\theta = \frac{1}{2}}^{n+\frac{1}{2}} \theta^2 e^{-\beta I_n} \frac{Z_{n,\theta} Z_{n,\theta}}{Z_n^+}.$$

By (25) from (12) we get

$$Z_n^+ = \frac{1}{2} \left( \exp \left( 2 \sum_{i=0}^{n} \ln(1 + \tau^{i+1}) \right) - \exp \left( 2 \sum_{i=0}^{n} \ln(1 - \tau^{i+1}) \right) \right) \sim \frac{1}{2} \left( \exp \left( 2 \sum_{i=0}^{n} \tau^{i+1} \right) - \exp \left( -2 \sum_{i=0}^{n} \tau^{i+1} \right) \right) = \sinh \left( \frac{2\tau(1 - \tau^{n+1})}{1-\tau} \right) \geq \sinh(2\tau). \quad (27)$$

Here we used $\ln(1 + \tau^i) \sim \tau^i$ for small $\tau$ (i.e. large $\beta$).

Similarly from (23) and (24) for $\theta \geq \frac{1}{2}$ we get

$$\frac{\tau^{\theta+\frac{1}{2}} \left( 1 - \tau^{n-\theta+\frac{1}{2}} \right)}{1-\tau} \leq \cosh \left( \frac{\tau^{\theta+\frac{1}{2}}}{1-\tau} \right) \leq \cosh \left( \frac{\tau^2}{1-\tau} \right); \quad (28)$$

$$\frac{\tau^2 (1 - \tau^n)}{1-\tau} - \tau^{\theta+\frac{1}{2}} (1+\tau) \leq A(\tau). \quad (29)$$

Hence

$$\text{Var}(\gamma_n) \leq 2\tau \left( \frac{1}{4} + \sum_{m=1}^{\infty} (m + \frac{1}{2})^2 \tau^m \right) \frac{\cosh \left( \frac{\tau^2}{1-\tau} \right) A(\tau)}{\sinh(2\tau)}. \quad (30)$$

One can check that

$$\sum_{m=1}^{\infty} (m + \frac{1}{2})^2 \tau^m = \frac{3\tau(\tau+3)}{4(1-\tau)^2}. \quad (31)$$

Thus from (30) and (31) one gets the assertion of the Theorem.

Remark 3. The estimation $\frac{1}{4} \leq \text{Var}(\gamma_n)$ is true for any interactions $I_n$, i.e. the condition (25) is not necessary.

Corollary. For any $n$ the following holds

$$\lim_{\beta \to \infty} \text{Var}(\gamma_n) = \frac{1}{4}.$$
5 Conclusions

In usual one-dimensional case there can be no phase transition. But it is known that phase transition occurs in the following cases:

a) The set of spin values is \{-1, 1\} and interactions are long range (Dyson’s model).

b) The set of spin values is \{-1, 1\} and interactions are nearest neighbors, but they are spatially inhomogeneous (Sullivan’s model).

c) The set of spin values is a countably infinite set and interactions are nearest neighbors (Spitzer’s model).

For a detailed description of history of phase separation properties of lattice models see [9].

We have considered here a one-dimensional model of type b). As mentioned above for such kind of model a phase transition occurs (see Theorem 2). In such a case of the existence of the phase transition, it would be interesting to know certain properties of a phase separation point (PSP) (curve (membrane) in two (three) dimensional case).

In two (resp. three) dimensional case a phase separation curve (resp. membrane) is defined as an “open” contour [4],[8]. But this construction does not work for one dimensional case with interactions of only nearest neighbors. A notion of PSP, to our knowledge, have not yet been introduced for one-dimensional models. In one-dimensional case the separation "line" is a point.

We have introduced here a natural definition of the PSP between two phases in one-dimensional case. We studied asymptotical properties of the PSP. Our definition of PSP is rather natural and properties of the PSP more special than ones in two and three dimensional cases. Namely, from Theorem 9 it follows that with probability 1 the PSP should be $-\frac{1}{2}$ or $\frac{1}{2}$.

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