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Globally conformal invariant gauge field theory 
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Abstract

Operator product expansions (OPE) for the product of a scalar field 
with its conjugate are presented as infinite sums of bilocal fields 
\( V(\mathbf{x}_1, \mathbf{x}_2) \) of dimension \((\kappa, \kappa)\). For a globally conformal invariant (GCI) theory we write down the OPE of \( V_\kappa \) into a series of twist (dimension minus rank) 
\( 2\kappa \) symmetric traceless tensor fields with coefficients computed from the (rational) 4-point function of the scalar field.

We argue that the theory of a GCI hermitian scalar field \( \mathcal{L}(\mathbf{x}) \) of 
dimension 4 in \( D = 4 \) Minkowski space such that the 3-point functions of a 
pair of \( \mathcal{L} \)’s and a scalar field of dimension 2 or 4 vanish can be interpreted 
as the theory of local observables of a conformally invariant fixed point in 
a gauge theory with Lagrangian density \( \mathcal{L}(\mathbf{x}) \).

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contained in [21].
1 Introduction

The present paper offers a new step in the realization of the program set up in [19] and [20] of constructing a 4-dimensional conformal field theory (CFT) model with rational correlation functions of observable fields.

Global conformal invariance (GCI) allows to write down close form expressions for correlation functions involving (each) a finite number of free parameters. Thus, the truncated 4-point function $w_t^4$ of a neutral scalar field of (integer) dimension $d$ depends on $\left\lfloor \frac{d^2}{4} \right\rfloor$ real parameters ($[a]$ standing for the integer part of the positive number $a$). In addition to GCI it satisfies the constraints of locality and energy positivity. Operator product expansion (OPE) provides a method of taking the remaining condition of Wightman (i.e. Hilbert space) positivity into account. We organize systematically OPE into mutually orthogonal bilocal fields $V_{\cdot \cdot}(x_1; x_2)$ of dimension $(\cdot; \cdot)$ defined by the condition that $V_{\cdot \cdot}$ can be expanded into an (infinite) sum (of integrals) of twist 2 symmetric traceless tensor local fields. An algorithm is given for computing the full contribution of each $V_{\cdot \cdot}$ to the 4-point function. The effectiveness of this approach is enhanced by the fact that $V_1(x_1, x_2)$, which involves an infinite sum of conserved tensor fields, satisfies the d’Alembert equation in each argument and the 4-point function $h_0 j V_1(x_1; x_2) j h_0 i$ is rational by itself. For the model of a $d = 2$ neutral scalar field $\phi(x)$ the OPE of two $\phi$’s can be summed up simply as [20]

$$
\phi(x_1) \phi(x_2) = \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle + (12) V_1(x_1, x_2) + :\phi(x_1) \phi(x_2):,
$$

(1.1)

where (12) is the free 0-mass 2-point function,

$$(12) = \frac{1}{4\pi^2 \rho_{12}}, \quad \rho_{12} = x_1^2 + i 0 x_1^0, \quad x_{12} = x_1 - x_2, \quad x^2 = x^2 - x_0^2,$$

(1.2)

and the normal product $:\phi(x_1) \phi(x_2) :$ defined by (1.1) is non-singular for $x_1 = x_2$. The simplicity of $V_1$ in this model has been exploited in [20] to prove that it can be written as a sum of normal products of (mutually commuting) free 0-mass fields.

We focus in the present paper on the study of a model generated by a (neutral) scalar field which can be interpreted as a (gauge invariant) Lagrangian density $L(x)$. Taking into account the above cited triviality result for $d = 2$ we require that the OPE of $L(x_1) L(x_2) - \langle 0 | L(x_1) L(x_2) | 0 \rangle$ does not involve any field of dimension lower than 4 (which just amounts to excluding a possible scalar field of dimension 2). This requirement decreases the number of free parameters in $w_t^4$ from five to four. It already excludes most of the standard renormalizable interaction Lagrangians (like Yukawa and $\varphi^4$) but allows for the Lagrangian $L$ and for the pseudoscalar topological term $\bar{L}$ of a pure gauge theory

$$
L(x) = -\frac{1}{4} tr(F_{\mu\nu}(x) F^{\mu\nu}(x)),$$

(1.3)

$$
\bar{L}(x) = \frac{1}{4} \epsilon^{\kappa\lambda\mu\nu} tr(F_{\kappa\lambda}(x) F_{\mu\nu}(x)).
$$

(1.4)
The invariance of $L$ under conformal rescaling of the metric is made manifest by writing the action density in terms of the Yang-Mills curvature form
\[ F(x) := \frac{1}{2} F^\mu_{\nu}(x) \, dx^\mu \wedge dx^\nu \] (1.5)
and its Hodge dual, $^*F$:
\[ L(x) \sqrt{|g|} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \text{tr}(^*F(x) \wedge F(x)) \] (1.6)
(where $|g|$ is the absolute value of the determinant of the metric tensor, $|g| = 1$ for the Minkowski metric). (It does not seem superfluous to reiterate that the conformal invariance of the gauge field Lagrangian singles out $D = 4$ as the dimension of space time [6].)

It seems appropriate to demand further invariance of the theory under “electric-magnetic duality”, i.e., under the change $F \rightarrow ^*F$. Taking into account the fact that the Hodge star defines a complex structure in Minkowski space, we deduce that the Lagrangian (1.6) changes sign under $*$:
\[ ^* (^*F) = -F \Rightarrow L(^*F) = -L(F) \] (1.7)
It follows that odd point correlation functions of $L(x)$ should vanish. Requiring space reflection invariance we also find that the vacuum expectation values of an odd number of $L$’s (and any number of $L$’s) should vanish. This leads us to the assumption that no (pseudo) scalar field of dimension 4 appears in the OPE of two $L$’s thus eliminating one more parameter in the expression for $w_4$.

The case of a non-abelian gauge field is distinguished by the existence of a non-trivial 3-point function of $F$ which agrees with local commutativity of Bose fields. Consistency of the equations of motion satisfied by the 2- and 3-point functions of $F$ with OPE, however, requires the use of indecomposable representations of the conformal group $C$ which makes unpractical exploiting the compositeness of $L$ (Sec. 5.2).

The paper is organized as follows.

We outline basic ideas and review earlier work and notation in Sec. 2. In particular, we make precise the notation of an elementary positive energy representation of $C$, and indicate an extension of the present approach to any even space-time dimension $D$.

Sec. 3.1 provides a general treatment of the OPE of a pair of conjugate scalar fields of dimension $d \in \mathbb{N}$ in terms of bilocal fields $V_d(x_1, x_2)$. It also reproduces the formula for the expansion of $V_d$ into an infinite series of symmetric traceless tensor fields of twist 2$\kappa$. Sec. 3.2 displays the (crossing symmetrized) contribution of twist 2 ($\kappa$-conserved) tensors and their light cone expansions for $d = 2$ and $d = 4$.

In Sec. 4.1 we study systematically the truncated 4-point function of the $d = 4$ field $L(x)$ and discuss the special case of the Lagrangian $L_0$ (4.17) of the

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free Maxwell field. In Sec. 4. 2 we analyse the conformally invariant (rational) factor $f_1$ of the harmonic 4-point function

$$\langle 0 \vert V_1(x_1, x_2)V_1(x_3, x_4) \vert 0 \rangle = (13)(24)f_1(s, t) ,$$  \hspace{1cm} (1.8)

displaying a basis of solutions $j_\nu(s, t)$ of the “conformal Laplace equation” (4.7) which admits a natural crossing symmetrization. Sec 4.3 displays the operator content and the light cone expansion in twist 4 and twist 6 tensor fields in the case of general $\mathcal{L}$.

The restrictions on the parameters of the truncated 4-point function coming from Wightman positivity of the singular part of the the s-channel OPE are analysed in Sec. 5.1. Sec. 5.3 is devoted to a summary of results and concluding remarks.

Appendix A summarizes the derivation of the main result of [20] referred to in the text discussing on the way the possibility for a similar treatment of the $d = 4$ case of interest (and the difficulties lying ahead).

## 2 Consequences of GCI (a synopsis)

### 2.1 Elementary positive energy representations of the conformal group

The idea that quantum fields should be subdivided into local observables and “charged fields” (that are relatively local to the observables and act as intertwiners among different superselection sectors) emerged from several decades of work of Haag and collaborators reviewed in [16]. It has been implemented in 2-dimensional conformal field theory (CFT) models in which chiral currents (including the stress-energy tensor) play the part of local observables while primary (with respect to a given chiral algebra) “chiral vertex operators” correspond to “charged fields” intertwining between different superselection sectors. (Among the numerous reviews on 2D CFT we cite the textbook [7] and the earlier article [15] which refers to the axiomatic quantum field theory framework.) Local observable fields are assumed to satisfy Wightman axioms [24]. More specifically, we demand that their set contains both the stress energy tensor $T_{\mu\nu}(x)$ (cf. [18]) and the Lagrangian density $\mathcal{L}(x)$.

To set the stage we fix the class of local field representations of the quantum mechanical conformal group $\mathcal{C}(D) = \text{Spin}(D, 2)$ under consideration (cf. [9], [17], [25]). The forward tube,

$$\mathcal{T}_+ = \left\{ x + iy : y^0 > |y| := (y_1^2 + \ldots + y_{D-1}^2)^{\frac{1}{2}} \right\} ,$$  \hspace{1cm} (2.1)

the primitive analyticity domain of the vector valued function $\psi(x + iy) \vert 0 \rangle$ for any Wightman field $\psi$, appears as a homogeneous space of $\mathcal{C}(D)$,

$$\mathcal{T}_+ = \mathcal{C}(D) / K(D) \quad \text{for} \quad K(D) = \text{Spin}(D) \times U(1) ,$$  \hspace{1cm} (2.2)
$K(D)$ being the maximal compact subgroup of $C(D)$. An elementary positive energy representation of $C(D)$ is one induced by a (finite dimensional) irreducible representation of $K(D)$. Any such representation gives rise to a transformation law for local fields defined (as operator valued distributions) on the (conformally) compactified Minkowski space $\overline{M}_D$ which appears as a (part of the) boundary of $\mathbb{T}_D$. $\overline{M}_D$ also appears as a homogeneous space of $C(D)$ with respect to a $\parabolic subgroup$ $H(D)$ defined as a semidirect product $H(D) = N_D \rtimes (\text{Spin}(D - 1, 1) \times \text{SO}(1, 1))$ of a $D$ dimensional abelian subgroup $N_D$ of “special conformal transformations” with the direct product of the (quantum mechanical) Lorentz group and the 1-parameter subgroup of $C(D)$ of uniform dilations:

\[ \overline{M}_D \simeq C(D)/H(D) \simeq (S^1 \times S^{D-1})/\mathbb{Z}_2 . \] (2.3)

The resulting elementary (local field) representation [25] can be defined as a positive energy representation of $C(D)$ induced by a finite dimensional irreducible representation of $H(D)$ (energy positivity fixing the representation of the discrete centre $\mathbb{Z}_4$ for even $D$ – of $C(D)$ – see [17]). In the case $D = 4$ of interest (studied in [17]),

\[ C \equiv C(4) = SU(2, 2) (\equiv \text{Spin}(4, 2)) , \quad \overline{M}(\equiv \overline{M}_4) = U(2)(= (S^1 \times S^3)/\mathbb{Z}_2) , \] (2.4)

the elementary representations of $C$ are labeled by triples $(d, j_1, j_2)$ of non-negative half integers, which admit two equivalent to each other interpretations. Viewed as labels of a $K$–induced representation $d$ stands for the minimal eigenvalue of the conformal Hamiltonian (defined below) while $(j_1, j_2)$ label the irreducible representations of $SU(2) \times SU(2)$. Alternatively, referring to a $H$–induced representation, $d$ is the conformal dimension (which may take all positive values if we substitute $C$ by its universal covering) and $(j_1, j_2)$ stands for a $(2j_1 + 1)(2j_2 + 1)$ dimensional representation of the Lorentz group (a spin-tensor of $2j_1$ undotted and $2j_2$ dotted indices). The elementary positive energy representation of $C$ are, in general, neither unitary nor irreducible. They are proven [17] to be unitary (or to admit an irreducible unitary subrepresentation) iff

\[ (a) \quad d \geq 1 + j_1 + j_2 \quad \text{for } j_1 j_2 = 0 , \]
\[ (b) \quad d \geq 2 + j_1 + j_2 \quad \text{for } j_1 j_2 > 0 . \] (2.5)

For a symmetric traceless tensor with $\ell$ indices (i.e. for $2j_1 = 2j_2 = \ell$) the counterpart of (2.5) can be readily written for any space-time dimension $D$:

\[ d \geq d_0 := \frac{D - 2}{2} \quad \text{for } \ell = 0 ; \quad d \geq D - 2 + \ell \quad \text{for } \ell = 1, 2, \ldots , \] (2.6)

$d_0$ being the dimension of a free 0–mass scalar field. It is quite remarkable that conventional relativistic (quantum) fields fit precisely the above definition (rather than transforming under Wigner’s unitary irreducible representations of
the Poincaré group – cf. [5] or [24]). Furthermore, a free scalar, Weyl spinor
and Maxwell tensor correspond to the lower limit of the above series (a), for
weights (1, 0, 0), (\(\frac{3}{2}, \frac{1}{2}, 0\)) and (2, 1, 0) + (2, 0, 1), respectively, while conserved
symmetric traceless tensors (starting with the vector current (3, \(\frac{1}{2}, \frac{1}{2}\)) are twist
two fields (fitting the lower limit of the series (b) in (2.5) and of (2.6)).

In all the latter cases it is the subspace of solutions of the corresponding
field equations or conservation law that carries a unitary representation of \(\mathcal{C}\).

Note further that the conformal Hamiltonian, the generator of the centre,
\(U(1)\), of the maximal compact subgroup, \(\text{Spin}(D) \times U(1)/\mathbb{Z}_2\) of \(\mathcal{C}(D)\) (\(S(U(2) \times
U(2))\) for \(D = 4\)) is positive definite whenever the Minkowski space energy is
(cf. [23]) and has a discrete spectrum belonging to the set \(\{d + n \mid n = 0, 1, 2, \ldots\}\)
for the positive energy representation of \(\mathcal{C}\) of weight \((d, j_1, j_2)\).

2.2 Huygens principle and rationality of conformally in-
variant correlation functions

Global conformal invariance (GCI) and local commutativity in four dimensional
(4D) Minkowski space \(M\) imply the Huygens principle which can be stated in
the following strong form [19]. Let \(\psi(x)\) be a GCI local (Bose or Fermi) field of
weight \((d, j_1, j_2)\); then \(d + j_1 + j_2\) should be an integer and

\[
(x_1^2)^{d+j_1+j_2} \psi(x_1) \psi^*(x_2) - (-1)^{2j_1+2j_2} \psi^*(x_2) \psi(x_1) = 0
\]

for all \(x_1, x_2\) in \(M\). The Huygens principle and energy positivity (more precisely,
the relativistic spectral conditions) imply rationality of correlation functions
([19], Theorem 3.1).

Wightman positivity in 4D restricts the degree of singularities of truncated (\(\equiv\)
connected) \(n\)-point functions (which can only occur for coinciding arguments):
they must be strictly lower than the degree of the pole of the corresponding
2-point propagator. This allows to determine any correlation function up to
a finite number of (constant) parameters. In particular, the truncated 4-point
function of a hermitian scalar field \(\phi\) of (integer) dimension \(d \in \mathbb{N}\) can be written
in the form ([20] Sec. 1)

\[
w_4^t \equiv w^t(x_1, x_2, x_3, x_4) := \langle 1234 \rangle - \langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 24 \rangle - \langle 14 \rangle \langle 23 \rangle
\]

\[
= (12)(23)(34)(14) \left(\frac{\rho_{13}\rho_{24}}{\rho_{12}\rho_{23}\rho_{34}\rho_{14}}\right)^{d-2} P_d(s, t)
\]

where

\[
\langle 1 \ldots n \rangle := \langle 0 | \phi(x_1) \ldots \phi(x_n) | 0 \rangle,
\]

(12) is the free massless propagator (1.2), \(s\) and \(t\) are the conformally invariant
cross-ratios (so that the prefactor in (2.8) is a rational function of \(\rho_{ij}\)):

\[
s = \frac{\rho_{12}\rho_{24}}{\rho_{13}\rho_{24}}, \quad t = \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{24}},
\]

6
\( \rho_{ij} \) are defined in (1.2) and \( P_d(s, t) \) is a polynomial in \( s \) and \( t \) of total degree \( 2d - 3 \) (\( P_1(s, t) \equiv 0 \)):

\[
P_d(s, t) = \sum_{i, j \geq 0, \quad i + j \leq 2d - 3} c_{ij} s^i t^j. \tag{2.11}
\]

Furthermore, local commutativity implies the crossing symmetry conditions

\[
s_{12} P_d(s, t) = P_d(s, t) = s_{23} P_d(s, t) \quad (2.12)
\]

where \( s_{ij} \) is the substitution exchanging the arguments \( x_i \) and \( x_j \):

\[
s_{12} P_d(s, t) := t^{2d-3} P_d \left( \frac{s}{t}, \frac{1}{t} \right),
\]

\[
s_{23} P_d(s, t) := s^{2d-3} P_d \left( \frac{1}{s}, \frac{t}{s} \right). \tag{2.13}
\]

Invariance under (2.13) implies the symmetry property \( s_{13} P_d(s, t) := P_d(t, s) = P_d(s, t) \) where \( s_{13} = s_{12}s_{23}s_{12} = s_{23}s_{12}s_{23} \). (The Wightman function \( w_4 \) of a neutral scalar field is in fact symmetric – as a rational function – under the group \( S_4 \) of permutations of the four position variables \( x_a, \quad a = 1, 2, 3, 4 \). The normal subgroup \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) of \( S_4 \), however, generated by \( s_{12}s_{34} \) and \( s_{14}s_{23} \) leaves the conformal cross ratios \( s \) and \( t \) invariant, so that only the factor group \( S_4/\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq S_3 \) acts effectively on \( P_d(s, t) \).) The number of independent crossing symmetric polynomials is \( \left\lfloor \frac{d^2}{4} \right\rfloor \) (the integer part of \( \frac{d^2}{4} \), 1 for \( d = 2 \), 3 for \( d = 3 \), 5 for \( d = 4 \)).

3 OPE in terms of bilocal fields

3.1 Bilocal fields as infinite series of symmetric tensor fields of a given twist

It appears convenient, at least in analysing a (truncated) 4-point function, to organize the infinite series of integrals of local tensor fields in an OPE into a finite sum of bilocal fields. In order to avoid purely, technical complications we shall exhibit the basic idea in the simplest case of the product of a \( d \)-dimensional scalar field \( \psi(x) \) (in \( M = M_4 \)) with its conjugate. We look for an expansion of the form

\[
\psi^*(x_1) \psi(x_2) = \langle 12 \rangle + \sum_{\kappa=1}^{d-1} (12)^{d-\kappa} V_\kappa(x_1, x_2) + : \psi^*(x_1) \psi(x_2) :, \tag{3.1}
\]

where \( \langle 12 \rangle \) is the free massless propagator defined in (1.2), \( (12) \) is the 2-point function \( \langle 12 \rangle = \langle 0 \mid \psi^*(x_1) \psi(x_2) \mid 0 \rangle = N_\psi \langle 12 \rangle \), and \( V_\kappa(x_1, x_2) \) is a bilocal
conformal field of dimension \((\kappa, \kappa)\) which is assumed to have an expansion in a series of local (hermitian) symmetric traceless tensor fields

\[ O_{2\kappa,\ell}(x; \zeta) = O_{2\kappa,\mu_1 \ldots \mu_\ell}(x) \zeta^{\mu_1} \cdots \zeta^{\mu_\ell}, \quad \Box_{\zeta} O_{2\kappa,\ell}(x; \zeta) = 0 \]  

(3.2)

(for \(\Box_{\zeta} := \frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial \zeta^2}\)) of dimension \(2\kappa + \ell\) (i.e. of fixed twist \(2\kappa\)). We shall make use of the fact that the harmonic polynomial \(O_{2\kappa,\ell}(x, \zeta)\) in the auxiliary variable \(\zeta\) is uniquely determined by its values on the light cone \(\zeta^2 = 0\) \([2]\). Whenever such an expansion is valid, it can be written quite explicitly:

\[ V_\kappa(x_1, x_2) = \sum_{\ell=0}^{\infty} C_{\kappa \ell} \int_0^1 K_{\kappa \ell}(\alpha, \rho_{12} \Box_2) O_{2\kappa,\ell}(x_2 + \alpha x_{12}; x_{12}) d\alpha \]  

(3.3)

where

\[ K_{\kappa \ell}(\alpha, z) = \sum_{n=0}^{\infty} \frac{\alpha(1-\alpha)\ell+\kappa+n-1}{B(\ell+\kappa, \ell+\kappa)} \frac{(-\frac{z}{\ell})^n}{n!(\ell+2\kappa-d_0)n} \left(\frac{\lambda}{\Gamma(\lambda)}\right)^n, \]  

(3.4)

\(B\) is the Euler beta function, \(\Gamma(\mu + \nu) B(\mu, \nu) = \Gamma(\mu) \Gamma(\nu)\), and the d’Alembert operator \(\Box_2\) acts on \(x_2\) for fixed \(x_{12}\). The normal product in (3.1) has a similar expansion

\[ :\psi^*(x_1) \psi(x_2) : = \sum_{\kappa = d}^{\infty} \rho_{12}^{\kappa-d} V_\kappa(x_1, x_2) \]  

(3.5)

thus involving all higher (even) twists.

**Remark 2.1.** A field \(V(x_1, x_2)\) is called bilocal if it commutes with any local field \(\phi(x_3)\) for spacelike \(x_{13}\) and \(x_{23}\). We note that GCI does not imply the Huygens principle for bilocal fields; hence, their correlation functions need not be rational. It is noteworthy that for any even space-time dimension \(D\), \(V_{d_0}\) does have rational correlation functions as will be demonstrated in Sec. 3.2 below using conservation of twist \(2d_0\) tensors. We shall also see (in Sec. 4.3) that this is not the case for \(V_\kappa\) if \(\kappa > d_0\).

In order to verify (3.3), (3.4) we need the general conformally invariant expression for the 2- and 3-point functions \([27]\)

\[ \langle 0 \mid O_{2\kappa,\ell}(x_1, \zeta_1) O_{2\kappa,\ell'}(x_2, \zeta_2) \mid 0 \rangle = \delta_{\ell \ell'} N_{\kappa \ell}(12)^{2\kappa} \zeta_1 R(x_{12}) \zeta_2 \]  

for \(\zeta_{1,2}^2 = 0\),

(3.6)

\[ \langle 0 \mid V_\kappa(x_1, x_2) O_{2\kappa,\ell}(x_3, \zeta) \mid 0 \rangle = A_{\kappa \ell}(13)^{\kappa} (23)^{\kappa} (X_{12}^3 \zeta)^{\ell} \]  

(3.7)

\[^2\text{Covariant conformal OPE were proposed in \([13]\), soon after the pioneer work of Wilson \([29]\). Further developments are reviewed in \([9], [27], [22],\) and \([14]\); we found useful the recent presentation \([10]\).} \]
where we have introduced the symmetric tensor
\[ \rho_{12} \zeta_1 R(x_{12}) \zeta_2 = \zeta_1 r(x_{12}) \zeta_2 := \zeta_1 \zeta_2 - \frac{2(\zeta_1 x_{12})(\zeta_2 x_{12})}{\rho_{12}} \quad (r(x)^2 = \Pi) \] (3.8)
and the vector
\[ X_{12}^3 = \frac{x_{13}}{\rho_{13}} - \frac{x_{23}}{\rho_{23}} \text{ of square } (X_{12}^3)^2 = \frac{\rho_{12}}{\rho_{13} \rho_{23}}. \] (3.9)

The integrodifferential operator in the right hand side of (3.3) is characterized by the property to transform a 2-point function of type (3.6) into a 3-point one, (3.7) (see [10]):
\[ \int_0^1 d\alpha K_{\alpha} (\alpha, \rho_{12} \square) (x_{12} R(y(\alpha)) \zeta)^\ell \rho_{y(\alpha)}^{-2\kappa} = \frac{(X_{12}^3)^\ell}{(\rho_{13} \rho_{23})^\kappa} \text{ for } \zeta^2 = 0, \] (3.10)
where \( y(\alpha) = x_{23} + \alpha x_{12} = \alpha x_{13} + (1 - \alpha)x_{23}, \rho_y = y^2 + i0y^0. \) The above relations are particularly easy to verify for \( \rho_{12} = 0 \) as the second argument of \( K_{\alpha} \) then vanishes and we obtain the light-cone expansion of \( V \).

Eqs. (3.3), (3.6) and (3.10) imply that the constants \( A_{\kappa \ell} \) in (3.7) are proportional to the expansion coefficients \( C_{\kappa \ell} \) in (3.3). Furthermore, a simple analysis shows that only their product is invariant under rescaling of \( O_{2\kappa, \ell} \) (for fixed \( V_\kappa \)).

We emphasize that the expansion (3.3) is universal: only the coefficients \( C_{\kappa \ell} \) depend on the field \( \psi \) in (3.1).

It follows from (3.1) that each \( V_\kappa \) satisfies the symmetry condition
\[ [V_\kappa(x_1, x_2)]^* = V_\kappa(x_2, x_1); \] (3.11)
taking into account the reality of \( O_{2\kappa, \ell} \) we deduce
\[ C_{\kappa \ell}^* = (-1)^\ell C_{\kappa \ell}. \] (3.12)
If \( \psi \) is hermitian then so is \( V_\kappa \), and \( C_{\kappa 2\ell + 1} \) vanish:
\[ C_{\kappa \ell} = 0 \text{ for odd } \ell \text{ if } V_\kappa(x_1, x_2) = V_\kappa(x_2, x_1) \quad (\psi^* = \psi). \] (3.13)

The tensor fields \( O_{2\kappa, \ell} \) transform under elementary representations of \( C \). It is convenient to work with such conformal fields, since they are mutually orthogonal under vacuum expectation values (see, e.g., [25] and references therein). It follows that each term in the expansion (3.1) is orthogonal to all others:
\[ \langle 0 | V_\kappa | 0 \rangle = 0 = \langle 0 | V_\kappa(x_1, x_2) V_\lambda^*(x_3, x_4) | 0 \rangle \text{ for } \kappa \neq \lambda, \]
\[ \langle 0 | V_\kappa(x_1, x_2) : \psi(x_3) \psi^*(x_4) : | 0 \rangle = 0, \] (3.14)
as the normal product (3.5) is expanded in higher twist fields.

From now on we restrict attention to the study of the OPE algebra of a hermitian GCI scalar field of dimension \( d \).
For a given (rational) truncated 4-point function we can compute the invariant under rescaling products

\[ B_{\kappa \ell} := A_{\kappa \ell} C_{\kappa \ell} \]

(for \( O_{2\kappa \ell} \rightarrow \lambda O_{2\kappa \ell}, \ A_{\kappa \ell} \rightarrow \lambda A_{\kappa \ell}, \ C_{\kappa \ell} \rightarrow \lambda^{-1} C_{\kappa \ell}, \ B_{\kappa \ell} \rightarrow B_{\kappa \ell} \) with \( A \) and \( C \) introduced in (3.7), (3.3)) using the light cone expansion of \( V_{\kappa}(x_3, x_4) \) in the 4-point function

\[ \langle 0 \mid V_{\kappa}(x_1, x_2) V_{\kappa}(x_3, x_4) \mid 0 \rangle = (13)^{\kappa} (24)^{\kappa} f_{\kappa}(s, t). \]  

As a consequence of (3.13) the conformally invariant amplitude \( f_{\kappa} \) is \( s_{12} \)-symmetric:

\[ s_{12} f_{\kappa}(s, t) := t^{-\kappa} f_{\kappa} \left( \frac{s}{t}, \frac{1}{t} \right) = f_{\kappa}(s, t). \]  

Combining (3.3), (3.7), (3.15) and using the standard integral representation for the hypergeometric function we find

\[ f_{\kappa}(0, t) = \sum_{\ell=0}^{\infty} B_{\kappa \ell} (1-t)^{2\ell} F(2\ell + \kappa, 2\ell + \kappa; 4\ell + 2\kappa; 1 - t). \]  

The symmetry condition (3.17) is reflected in a known transformation formula for the hypergeometric function:

\[ F \left( a, a; 2a; \frac{z}{z-1} \right) = (1-z)^{a} F(a, a; 2a; z) \text{ for } a = 2\ell + \kappa, \ z = 1 - t. \]  

### 3.2 Crossing symmetrized contribution of conserved tensors

The case \( \kappa = d_0 \) (2.6) is of particular interest since the expansion then comprises all conserved tensors \( O_{2d_0, \ell}(x; \zeta) := T_{\ell}(x; \zeta) \) (\( T_2 \) being the stress-energy tensor). The 3-point functions (3.7) are all harmonic in \( x_1 \) and \( x_2 \); hence, so is \( V_{d_0} \):

\[ \Box_1 V_{d_0}(x_1, x_2) = 0 = \Box_2 V_{d_0}(x_1, x_2). \]  

Applying \( \Box_1 \) to both sides of (3.16) for \( \kappa = d_0 \) we find

\[ \Box_1 \left\{ (13)^{d_0} (24)^{d_0} f_{d_0}(s, t) \right\} = \frac{4st}{\rho_{12} \rho_{14}} \rho_{24} (13)^{d_0} (24)^{d_0} \Delta f_{d_0}(s, t) = 0 \]  

where

\[ \Delta = s \frac{\partial^2}{\partial s^2} + t \frac{\partial^2}{\partial t^2} + (s + t - 1) \frac{\partial}{\partial s t} + \frac{D}{2} \left( \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right). \]  

For \( D = 4 \) the operator \( \Delta \) has been introduced in [11] (in the context of \( N = 4 \) supersymmetric Yang-Mills theory\(^3\)). The general solution of (3.21) can be

\(^3\)The authors thank Emery Sokatchev for this remark.
written in terms of the chiral variables \(u\) and \(v\) exploited in [10] (a similar procedure is used in Sec. 7 of [11]; see also Appendix A to [20]). It is given by

\[
f_{d_0}(s, t) = \frac{g(u) - g(v)}{(u - v)^{d_0}}\text{ for } s = uv, \ t = (1 - u)(1 - v). \tag{3.23}
\]

(For euclidean \(x\) the variables \(u\) and \(v\) are complex conjugate to each other.)

For a \(d\) dimensional field \((d > d_0)\) we are looking for a solution \(f = f_{d_0}\) of \(\Delta f = 0\) (3.21) such that the product of \(t^{d-1}f\) is a polynomial in \(s\) and \(t\) of overall degree not exceeding \(2d - 3\) (as a consequence of (2.11)). It can be obtained by viewing Eq. (3.21) as a Cauchy problem with initial condition (satisfying the symmetry property (3.17))

\[
f(0, t) = (1 + t^{-d_0}) \sum_{\nu=0}^{d_0-1} a\nu \frac{(1 - t)^{2\nu}}{t^\nu} \left( = t^{-d_0} f \left( 0, \frac{1}{t} \right) \right). \tag{3.24}
\]

It thus depends on \(d - d_0\) parameters \(a_0, a_1, \ldots, a_{d-d_0-1}\). Noting that for \(s = 0 = v\) we have \(t = 1 - u\), we can write the solution in the form (3.23) with

\[
u^{-d_0} g(u) = f(0, 1 - u) = \left[ 1 + (1 - u)^{-d_0} \right] \sum_{\nu=0}^{d_0-1} a\nu \frac{u^{2\nu}}{(1 - u)^\nu}. \tag{3.25}
\]

**Proposition 2.1.** The equation

\[
\Box I \int_0^1 K_{d_0, \ell}(\alpha, \rho_{12} \Box_y) O_{2d_0, \ell}(y; x_{12}) \, d\alpha = 0 \quad \text{for} \quad y = x_2 + \alpha x_{12} \tag{3.26}
\]

where \(K_{\alpha\ell}\) is given by (3.4) is necessary and sufficient for the conservation of the (traceless) twist \(2d_0\) tensor \(O_{2d_0, \ell}\):

\[
\frac{\partial^2}{\partial y \partial x_{12}} O_{2d_0, \ell}(y; x_{12}) = 0. \tag{3.27}
\]

**Sketch of proof.** We shall verify that (3.4) and (3.27) imply (3.26). (The proof of the sufficiency of (3.26) for the validity of (3.27) uses the same computation.) The statement follows from the differentiation formula

\[
\Box I \{ \rho_{12}^\alpha O_{2d_0, \ell}(y; x_{12}) \} = \rho_{12} \left\{ 4n \left( n + \ell + d_0 + \alpha \frac{\partial}{\partial \alpha} \right) + \right. \\
\left. + \alpha \rho_{12} \left( \frac{\partial^2}{\partial y \partial x_{12}} + \alpha \Box_y \right) \right\} O_{2d_0, \ell}(y; x_{12}) \tag{3.28}
\]

by integrating by parts the term involving \(\frac{\partial}{\partial \alpha}\). 

11
In the case $D = d = 4$ ($d_0 = 1$) of interest we find a 3-parameter family of solutions:

\[ f_1(s, t) = a_0 i_0(s, t) + a_1 i_1(s, t) + a_2 i_2(s, t), \]

\[ i_0(s, t) := 1 + t^{-1}, \quad i_1(s, t) := \left(1 - \frac{t}{t_s}\right)^2 (1 + t - s) - 2 \frac{s}{t}, \]

\[ i_2(s, t) := \left(1 - \frac{t}{t_s}\right)^4 (1 + t - 2s) - 6(1 - \frac{t}{t_s})^2 + \frac{s^2}{t_s^2} (1 + t) \left(1 + \frac{(1 - t)^2}{t_s}\right). \] (3.29)

In both cases, $d = 2$ and $d = 4$ ($D = 4$), we can compute $B_{1\ell}(d)$ from (3.18).

The result is

\[ B_{1\ell}(2) = \frac{2a_0}{|t|} \] (3.30)

(we have used $c$ for $a_0$ in [20])

\[ B_{1\ell}(4) = \left[\frac{1}{|t|^{2\ell}}\right] \left[2a_0 + 2\ell(2\ell + 1)\left(2a_1 + (2\ell + 3)(\ell - 1)a_2\right)\right]. \] (3.31)

The condition for the absence of a $d = 2$ scalar field in the OPE is given by $a_0 = 0$ implying the vanishing of $V_1$ for $x_{12} \to 0$:

\[ a_0 = 0 \Rightarrow V_1(x_1, x_2) = C x_{12}^{\mu\nu} x_{12}^{\mu\nu} T_{\mu\nu}(x_1, x_2). \] (3.32)

The proportionality coefficient $C$ can be chosen in such a way that $T_{\mu\nu}(x) := T_{\mu\nu}(x, x)$ is the stress energy tensor of the theory, normalized by the standard Ward-Takahashi identity (cf. [18]).

We note that the system (3.18) is overdetermined: each $B_{1\ell}$ has to satisfy two conditions to fit the coefficients to $(1 - t)^{2\ell}$ and $(1 - t)^{2\ell+1}$. Thus the existence of a solution provides a non-trivial consistency check. The result (3.30) was proven analytically using the integral representation for the hypergeometric functions (see Appendix A of [20]); Eq. (3.31) was derived analytically for small values of $\ell$ and verified numerically for $2\ell \leq 300$.

Wightman (i.e. Hilbert space) positivity implies

\[ B_{\kappa\ell} = A_{\kappa2\ell} C_{\kappa2\ell} = N_{\kappa2\ell} C_{\kappa2\ell}^2 \geq 0 \] (3.33)

(since $C_{\kappa2\ell}$ is real, according to (3.12), while $N_{\kappa2\ell}$ is the normalization of the positive definite 2-point function (3.6) so it should be positive). The condition $B_{1\ell} > 0$ is indeed verified for non-negative $a_\nu$ with a positive sum. This is only a necessary condition for Wightman positivity. A necessary and sufficient positivity condition was established in the $d = 2$ case; it says that $c (= a_0)$ should be a positive integer (see [20] Theorem 5.1). As the ingredients in the derivation of this result have a more general significance we review them in Appendix A with an eye to a possible generalization to the theory of the Lagrangian field $\mathcal{L}(x)$ (of dimension $d = 4$).
Knowing the singular part of the OPE (3.1) allows, after crossing symmetrization to reconstruct the complete 4-point function. The result for the \( d = 2 \) case is:

\[
\langle 1234 \rangle = \langle 0 | \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle = (1 + s_{23} + s_{13}) \left\{ \langle 12 \rangle \langle 34 \rangle + \frac{1}{2} \langle 12 \rangle \langle 34 \rangle \langle 0 | V_1(x_1, x_2) V_1(x_3, x_4) | 0 \rangle \right\}
\]

\[
= \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 23 \rangle \langle 14 \rangle + w^f(x_1, \ldots, x_4)
\]

(3.34)

where, in the conventions of [20] (for \( d = 2 \)),

\[
h_{j'k} = c_2(j')^2, \quad \mu(j') = \frac{1}{4} \mu_{j'k} \left( j < j' \right),
\]

(3.35)

\[
w^f(x_1, x_2, x_3, x_4) = c \{(12)(23)(34)(14) + (13)(23)(24)(14) + (12)(13)(24)(34)\}
\]

(3.36)

for

\[
\langle 0 | V_1(x_1, x_2) V_1(x_3, x_4) | 0 \rangle = c \{(13)(24) + (14)(23)\}.
\]

(3.37)

Remark 2.2. The factor \( \frac{1}{2} \) in the second term in the braces in Eq. (3.34) reflects a special case of the following phenomenon. In general, we wish that the symmetrized contribution of \( V_\kappa(x_1, x_2) \) to the truncated 4-point function,

\[
F_\kappa(x_1, x_2, x_3, x_4) = \langle 12 \rangle \langle 34 \rangle \langle 0 | V_\kappa(x_1, x_2) V_\kappa(x_3, x_4) | 0 \rangle,
\]

(3.38)

where \( S \) stands for an yet unspecified symmetrization, \( \kappa = 1, \ldots, d-1 \), is not just symmetric under any permutation of its arguments but that the difference

\[
F_\kappa(x_1, x_2, x_3, x_4) - \langle 12 \rangle \langle 34 \rangle \langle 0 | V_\kappa(x_1, x_2) V_\kappa(x_3, x_4) | 0 \rangle
\]

is of order \( \langle 12 \rangle \langle 34 \rangle \langle 0 | V_\kappa(x_1, x_2) V_\kappa(x_3, x_4) | 0 \rangle \) (or smaller) for \( s \to 0 \). This second condition forces us to use the standard symmetrization \( \langle 12 \rangle \langle 34 \rangle \rightarrow (1 + s_{23} + s_{13}) \langle 12 \rangle \langle 34 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle \) for the disconnected term in the middle part of (3.34) while applying \( \frac{1}{2} \) to the second term. We shall see in Sec. 4.2 below that there exist a basis of rational harmonic functions whose symmetrized contribution satisfying the above condition is proportional to the standard symmetrization. We further note that the function \( F_1 \), which is rational by itself, can be viewed as providing a minimal model for the truncated 4-point function of \( \phi \).
4 General form of the 4-point function of $\mathcal{L}(x)$.
Operator content

4.1 s-channel contributions to the 4-point function for arbitrary twists

We shall now exploit a result of Dolan and Osborn [10] which permits to write down closed form expressions for $f_\kappa(s, t)$ (3.16) given the rational 4-point function for $d = 4 (= D)$. More precisely, keeping in mind the analysis of s-channel positivity we shall study the difference

$$w(x_1, x_2, x_3, x_4) - (12)(34) = (13)(24) + (14)(23) + w^t(x_1, x_2, x_3, x_4) =$$

$$= (2\pi)^{-3} (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-2} \left[ \frac{1}{8} P(s, t) + N^2 s^2 (t^2 + t^{-2}) \right].$$

(4.1)

with $P(s, t) = P_4(s, t))$, a polynomial in $s$ and $t$ of overall degree 5 satisfying the crossing symmetry conditions (2.12)(2.13) (see (4.28) below). The rational function in the brackets in the right hand side of (4.1) can be expressed as an infinite sum of even twist contributions

$$N^2 s^3 t (t^2 + t^{-2}) + P(s, t) = \sum_{\kappa = 4}^{\infty} s^{\kappa - 1} f_\kappa(s, t),$$

(4.2)

thus extrapolating (3.16) beyond the range $\kappa = 1, 2, 3$. Albeit $f_\kappa$ are, in general, not rational functions of $s$ and $t$ they are determined by the polynomial $P$ and the constant $N^2$. Indeed Eq. (3.10) of [10] allows to write down the following extension of (3.23) ($d_0 = 1$) to any $\kappa > 1$:

$$f_\kappa(s, t) = \frac{1}{u - v} \left\{ F(\kappa - 1, \kappa - 1; 2\kappa - 2; v) g_\kappa(u) - F(\kappa - 1, \kappa - 1; 2\kappa - 2; u) g_\kappa(v) \right\}. \quad (4.3)$$

We note that the normalization $N$ contributes to $\kappa \geq d (= 4)$ only. Then it has to be taken into account when verifying Wightman positivity condition (3.33). The hypergeometric functions $F(a; a; 2a; v), a = 1, 2, \ldots$, are expressed as linear combinations of $\log(1 - v)$ with rational in $v$ coefficients:

$$F(1; 1; 2; v) = \sum_{n = 1}^{\infty} \frac{v^{n-1}}{n} = \frac{1}{v} \ln \frac{1}{1 - v},$$

$$F(2; 2; 4; v) = \sum_{n = 1}^{\infty} \frac{6 n v^{n-1}}{(n + 1)(n + 2)} = \frac{6}{v^2} \left( \frac{2 - v}{v} \ln \frac{1}{1 - v} - 2 \right), \quad \text{etc.}$$

The OPE of the bilocal field $V_\kappa(x_1, x_2)$ in terms of twist $2\kappa$ rank $2\ell$ symmetric traceless tensors corresponds, according to (3.18), to an expansion of $g_\kappa(u)$ in terms of hypergeometric functions of the same type:

$$g_\kappa(u) = uf_\kappa(0, 1 - u) = \sum_{\ell = 0}^{\infty} B_{\kappa\ell} u^{2\ell+1} F(2\ell + \kappa, 2\ell + \kappa; 4\ell + \kappa; u) \quad (4.4)$$
(Eq. (4.4) is obtained from (4.3) just using $g_a(0) = 0, F(a, a; 2a; 0) = 1$.) The functions $f_a(0, t)$ can be determined recursively from the relations

$$f_a(0, t) = \lim_{s \to 0} \left\{ s^{1-a} \left[ t^{-3} P(s, t) - \sum_{\nu = 1}^{a-1} s^{\nu-1} f_{\nu}(s, t) \right] \right\},$$

$$f_1(0, t) = t^{-3} P(0, t).$$

and (4.3) which imply that the operator $	ilde{h}$,

$$\tilde{h} t^{-3} P(s, t) := \frac{1}{u-v} \left\{ \frac{u}{(1-u)^3} P(0, 1-u) - \frac{v}{(1-v)^3} P(0, 1-v) \right\} = f_1(s, t)$$

defines a (rational) harmonic projection of the product $t^{-3} P(s, t)$. In fact, the function $(13)(24)f_1(s, t)$ is harmonic in both $x_{12}$ and $x_{34}$ for fixed $x_{23}$ while its difference with $(13)(24)t^{-3} P(s, t)$ vanishes on either light cone ($\rho_{12} = 0$ and $\rho_{34} = 0$) since $t^{-3} P(0, t) = f_1(0, t)$ according to (4.5). Thus Eq. (4.3) can be viewed as an extension of the notion of harmonic projection to arbitrary twists.

### 4.2 A basis of crossing symmetrized conformal harmonic functions

We shall now display a basis $j_\nu(s, t)$ of rational solutions of the conformal Laplace equation (3.21),

$$\Delta j_\nu(s, t) = 0, \ \nu = 0, 1, 2 \ (\Delta = s \frac{\partial^2}{\partial s^2} + t \frac{\partial^2}{\partial t^2} + (s+t-1) \frac{\partial^2}{\partial s \partial t} + 2 \frac{\partial}{\partial s} + 2 \frac{\partial}{\partial t}).$$

that are symmetric under $s_{12}$

$$t^{-1} j_\nu \left( \frac{s}{t}, \frac{1}{t} \right) = j_\nu(s, t)$$

and are eigenfunctions of the operator

$$\tilde{h} \ (1 + s_{23} + s_{13}).$$

We shall verify that such a basis is given by

$$j_\alpha(s, t) = i_\alpha(s, t), \ \ \alpha = 0, 1, \ \ j_2(s, t) = i_1(s, t) + i_2(s, t),$$

$$t^{\nu} j_2(s, t) = (1 + t^{\nu})[(1 + s - t)^2 - s] - 3 s (1-t),$$

where $i_\nu$ are defined in (3.29). It will become clear on the way that $i_2$ is not an eigenvector of the operator (4.9) as $j_1$ and $j_2$ correspond to different eigenvalues.

Set indeed

$$I_\nu = (1 + s_{23} + s_{13}) \left\{ t^{\nu} i_\nu(s, t) \right\}, \ \ \nu = 0, 1, 2$$
we find

\[ I_0(s, t) = s^2(1 + s) + t^2(1 + t) + s^2t^2(s + t), \]
\[ I_1(s, t) = s(1 + s)(1 - s)^2 + t(1 + t)(1 - t)^2 + st[(s - t)(s^2 - t^2) - 2Q_1], \quad Q_1 := 1 + s^2 + t^2 \]
\[ I_2(s, t) = (1 + s)(1 - s)^2(2 - 3s + 2s^2) + (1 + t)(1 - t)^2(3 - 3t + 2t^2) - 2 + st\left[4Q_1 - (s + t)(5s^2 - 8st + 5t^2)\right]. \]

(4.12)

The simplest way to compute the harmonic projection \( \hat{h}(t^{-3}I_\nu(s, t)) \) consists in looking at the “initial conditions” for \( s = 0 \). We see that while \( I_\alpha(0, t) = t^3i_\alpha(0, t) \) for \( \alpha = 0, 1, \) we have

\[ I_2(0, t) = 2(1 - t)^4(1 + t) + t(1 - t)^2(1 + t) = t^3(2i_2(0, t) + i_1(0, t)). \]

(4.13)

It follows that \( i_2 \) is not an eigenfunction of the operator (4.9) but \( j_2 \) (4.10) is one (with eigenvalue 2). This suggests introducing a new basis of symmetrized twist two polynomial factors in the truncated 4-point function:

\[ J_\alpha(s, t) := I_\alpha(s, t), \quad \alpha = 0, 1; \]
\[ J_2(s, t) = \frac{1}{2}(1 + s^2 + s^3)(t^3,j_2(s, t)) = \frac{1}{2}(I_1(s, t) + I_2(s, t)) = (1 + t^3)(1 + s)^3\left[1 - s - 3s(1 - t) + s^2(1 - s + t^2) - t\right]; \]

(4.14)

the \( J_\alpha \) are distinguished by the properties

\[ \hat{h}\left\{t^{-3}J_\nu(s, t)\right\} = j_\nu(s, t), \quad \nu = 0, 1, 2; \]
\[ J_0(s, t) = t^3j_0(s, t) = s^2(1 + t^3) + s^3(1 + t^2) \]
\[ J_1(s, t) = t^3j_1(s, t) = s(1 - t)(1 - t^3) - s^2(1 + t^3) - s^3(1 + t^2) + s^4(1 + t) \]
\[ J_2(s, t) = t^3j_2(s, t) = s^3(1 + t + t^2) - 2s^4(1 + t) + s^5. \]

(4.15)

Comparing (4.15) with (4.5) we deduce that the differences \( J_1 - t^3j_1, J_0 - t^3j_0, \) and \( J_2 - t^3j_2 \) involve twist 4 and higher, twist 6 and higher, and twist 8 and higher, respectively. The expressions (3.29) for \( f_1 \) and the corresponding crossing symmetric amplitude \( F_1 \) can be rewritten in the basis (4.10) (4.14) with the result

\[ f_1(s, t) = a_0j_0(s, t) + a_{12}j_1(s, t) + a_2j_2(s, t), \quad a_{12} = a_1 - a_2; \]
\[ F_1(x_1, x_2, x_3, x_4) = \left(\frac{1}{16}\right)^{\frac{5}{2}}\left(\frac{1}{16}\right)^2\left(\frac{1}{16}\right)^3\left(\frac{1}{16}\right)^2 \frac{1}{\pi t} \times \right. \]
\[ \times \left\{a_0j_0(s, t) + a_{12}j_1(s, t) + a_2j_2(s, t)\right\}. \]

(4.16)

We note that the truncated 4-point function of the (vacuum) Maxwell Lagrangian,

\[ \mathcal{L}_0(x) = -\frac{1}{4} : F_{\mu\nu}(x)F^{\mu\nu}(x) :, \]

(4.17)
(or a sum of mutually commuting expressions of this type) is a special case of (4.16). Here $F_{\mu\nu}$ is a free electromagnetic field with 2-point function
\[
(0 \mid F_{\mu_1\nu_1}(x_1) F_{\mu_2\nu_2}(x_2) \mid 0) = 4D_{\mu_1\nu_1\mu_2\nu_2}(x_{12})
\]
where
\[
4\pi^2 D_{\mu_1\nu_1\mu_2\nu_2}(x) = R_{\mu_1\mu_2}(x) R_{\nu_1\nu_2}(x) - R_{\mu_1\nu_2}(x) R_{\mu_2\nu_1}(x),
\]
the symmetric tensor $R(x)$ being defined in (3.8). The OPE of the product of two $\mathcal{L}_0$’s has the form
\[
\mathcal{L}_0(x_1) \mathcal{L}_0(x_2) = (12)_0 + \frac{8}{\pi^2 \rho_{12}^4} T(x_1, x_2; x_{12}) + \mathcal{O}(x_1, x_2) : \mathcal{L}_0(x_1) \mathcal{L}_0(x_2) :
\]
where $(12)_0 = 3(\pi \rho_{12})^{-4}$, and $T(x_1, x_2; x_{12})$ (a multiple of $V_1$) is the harmonic bilocal field
\[
T(x_1, x_2; x_{12}) = \frac{1}{4} : F^{\sigma\tau}(x_1) F_{\sigma\tau}(x_2) : x_{12}^2 - x_{12}^\nu : F_\nu(x_1) F_{\sigma\tau}(x_2) : x_{12}^\nu =
\]
\[
= \frac{1}{4} x_{12}^2 (r_{12})^\mu
\]
where $f(x) = 0 = d^* F(x)$,
\[
dF(x) = 0 = d^* F(x),
\]
\[
r(x_{12}) r(x_{13}) r(x_{13}) = r(X_{12}^1),
\]
\[
X_{12}^1 = \frac{x_{13}}{\rho_{13}} - \frac{x_{12}}{\rho_{12}}
\]
(see Appendix B of [20]). The contribution of $T$ to the 4-point function is given by the solution (3.29) of (3.21):
\[
\pi^4(0 \mid T(x_1, x_2; x_{12}) T(x_3, x_4; x_{34}) \mid 0) = (13)(24) f_1(s, t)
\]
for $a_0 = 0$, $a_1 = a_2$;
\[
\pi^4(0 \mid T(x_1, x_2; x_{12}) T(x_3, x_4; x_{34}) \mid 0) = (13)(24) f_1(s, t)
\]
(4.21)
\[
D_{12} := D_{\mu_1\nu_1\mu_2\nu_2}(x_{12}),
\]
\[
D^{23} := D^{\mu_2\nu_2\mu_3\nu_3}(x_{23})
\]
(see Appendix B of [20]). The contribution of $T$ to the 4-point function is given by the solution (3.29) of (3.21):
\[
\pi^4(0 \mid T(x_1, x_2; x_{12}) T(x_3, x_4; x_{34}) \mid 0) = (13)(24) f_1(s, t)
\]
for $a_0 = 0$, $a_1 = a_2$;
\[
\pi^4(0 \mid T(x_1, x_2; x_{12}) T(x_3, x_4; x_{34}) \mid 0) = (13)(24) f_1(s, t)
\]
(4.24)
\[
\text{corresponding to } a_1 = a_2 = 2^3 c_0, a_0 = a_{12} = 0.
\]
In fact, a finite sum of expressions of type (4.17) will give rise to a discrete subset of such values with a positive integer $c_0$. 

17
4.3 General form of $w_4'$. Higher twist contributions

The most general GCI truncated 4-point function of a $d = 4$ scalar field contains, in addition to $F_1$ (4.16), two more terms corresponding to crossing symmetrized twist 4 contributions. They can be written as $s t Q_i(s, t), i = 1, 2$, where $Q_i$ are crossing symmetric polynomials of degree 2:

$$Q_1 = 1 + s^2 + t^2, \quad Q_2 = s + t + st,$$

$$t^2 Q_j \left( \frac{s}{t}, \frac{1}{t} \right) = Q_j(s, t) = s^2 Q_j \left( \frac{1}{s}, \frac{t}{s} \right), \quad j = 1, 2. \quad (4.27)$$

Thus, the polynomial $P(s, t)$ of Eq. (4.1) can be presented in the form:

$$P(s, t) = a_0 J_0(s, t) + a_{12} J_1(s, t) + a_2 J_2(s, t) + s t [b_1 Q_1(s, t) + b_2 Q_2(s, t)]. \quad (4.28)$$

Remark 3.1. The above basis of crossing symmetric polynomials $J, Q_j$ and the $P, Q_j$ basis of [20], where

$$P_0(s, t) = 1 + s^5 + t^5, \quad P_1(s, t) = s(1 + s^3) + t(1 + t^3) + st(s^3 + t^3),$$

$$P_2(s, t) = s^2(1 + s) + t^2(1 + t) + s^2 t^2(s + t) \quad (= I_0(s, t)), \quad (4.29)$$

are related by:

$$J_0 = P_2, \quad J_1 = P_1 - P_2 - 2 s t Q_1, \quad J_2 = P_0 - 2 P_1 + P_2 + s t Q_1. \quad (4.30)$$

We shall now apply the procedure outlined in Sec.4.1 to compute the twist 4 and 6 contributions in (4.28). It follows from (4.4) (4.15) and (4.28) that

$$f_2(0, 1 - u) = a_{12} \left( 1-u + \frac{1}{(1-u)^3} \right) + (b_1 - a_{12}) \left( 1 + \frac{1}{(1-u)^3} \right) + b_2 \frac{1}{1-u} \equiv \frac{g_2(u)}{u} = \sum_{\ell = 0}^{\infty} B_{2\ell} u^{2\ell} F(2\ell + 2, 2\ell + 2; 4\ell + 4; u) \quad (4.31)$$

yielding

$$B_{2\ell} = \left( \frac{4\ell + 1}{2\ell} \right)^{1-1} \left[ \ell (\ell + 1) (2\ell + 1) (2\ell + 3) a_{12} + 2 (\ell + 1) (2\ell + 1) b_1 + b_2 \right]. \quad (4.32)$$

We see, in particular, that the absence of a scalar field of dimension 4 in the OPE of two $\mathcal{L}$’s implies

$$B_{20} = 2 b_1 + b_2 = 0 \quad (\Rightarrow \langle 123 \rangle = 0). \quad (4.33)$$

The difference between $P(s, t)$ (4.28) and the twist 2 (Eq. (4.16)) and twist 4 part, computed from (4.3), defines $t^3 f_3(0, t)$ as the coefficient to the $s^2$ term:

$$P(s, t) - t^3 [f_1(s, t) + s f_2(s, t)] = s^2 t^3 f_3(0, t) + O(s^3). \quad (4.34)$$
A computer aided calculation (using Maple) gives
\[
\frac{g_3(u)}{u} = \sum_{\ell=0}^\infty B_{3\ell} u^{2\ell} F(2\ell + 3, 2\ell + 3; 4\ell + 6; u) = \\
\frac{1}{2} a_{12} + a_0 + \frac{1}{2} \frac{b_2 - b_1}{1 - u} + \frac{1}{2} \frac{b_1 + 2b_2}{(1 - u)^2} + \frac{1}{2} \frac{a_{12} + 2a_0}{(1 - u)^3} - \\
- \frac{1}{2} (2b_1 + b_2) \left( \frac{1}{u} + \frac{2}{u^2} \right) - (2b_1 + b_2) \ln \left( \frac{1 - u}{u^3} \right)
\]
(reg. 4.35)

with
\[
B_{3\ell} = \frac{1}{2} \frac{(4\ell + 3)}{(2\ell + 1)} \left[ (\ell + 1) (\ell + 2) (2\ell + 1) (2\ell + 3) (2a_0 + a_{12}) + \\
+ 2 (\ell + 1) (2\ell + 3) (b_1 + 2b_2) - 2b_1 - b_2 \right].
\]

Remarkably, \( g_3(u) \) is rational precisely when the condition (4.33), reflecting the electric-magnetic duality, is satisfied.

5 Implications for the gauge field Lagrangian

5.1 Restrictions from OPE, Hodge duality, and Wightman positivity

As discussed in the Introduction the gauge field Lagrangian (1.3) is characterized by the absence of scalar fields of dimension 2 and 4 in the OPE of two \( L \)'s. According to (3.32) and (4.33) this implies
\[
a_0 = 0 = 2b_1 + b_2.
\]

This allows to write the polynomial \( P(s, t) \) in (4.1) as
\[
P(s, t) = a'J_1(s, t) + aJ_2(s, t) + b[Q_1(s, t) - 2Q_2(s, t)]
\]

where we have set \( a_2 = a, a_{12} = a', b_1 = b = \frac{b_2}{2} \); the difference
\[
Q_1(s, t) - 2Q_2(s, t) = (1 - s - t)^2 - 4st
\]

is characterized by having a (second order) zero at \( t = 1 \) for \( s = 0 \) (and being negative for euclidean \( x_{ij} \)).

The restrictions on the three remaining constants, \( a, a', \) and \( b, \) coming from the positivity condition (2.33) for \( B_{kl} \) given by (3.31) (4.32) and (4.36) are
\[
a + a' > 0, \quad a, a' \geq 0;
\]

\((\ell + 1)(2\ell + 1) a' + 2b \geq 0, \) for \( l > 0; (\ell + 2)(2\ell + 1) a' \geq 6b \) for \( \ell \geq 0. \) The last two inequalities imply
\[
-3a' \leq b \leq \frac{1}{3} a'.
\]
In particular, if $a' = 0$ it would follow that also $b = 0$ and we would end up with the truncated 4-point function that is a multiple of the one of the free electromagnetic Lagrangian (4.17).

Remark 4.1. If we allow for a positive $a_0$ in the 4-point function (including, say, a scalar field contribution in the Lagrangian) then, according to (4.36), the restriction (5.5) would be replaced by a more general one

$$-3 a' \leq b \leq \frac{1}{3} \left( 2 a_0 + a' \right)$$

(5.6)

which leaves room for a (non-zero) positive $b$ even for $a' = 0$. A numerical analysis indicates that the most general GCI 4-point function involves a non-trivial open domain in the 5-dimensional projective space of the parameters $a_\nu$, $b_\sigma$ and the 2-point function normalization, $N^2$, defined up to a common positive factor, in which Wightman positivity is verified for all twists.

5.2 Difficulties in exploiting the compositeness of $\mathcal{L}$

It is instructive to understand the restrictions, coming from (5.1), within an axiomatic treatment of non-abelian gauge field theory. At the same time we shall try to answer the question: can we extract more information from the expression (1.3) for $\mathcal{L}$ in terms of $F^{\mu\nu}(x)$ than just saying that the OPE of $\mathcal{L}(x_1)\mathcal{L}(x_2)$ contains neither $\mathcal{L}$ nor a scalar field of dimension 2?

In (perturbative) quantum electrodynamics amplitudes with an odd number of photon external legs (and no external charged particles) vanish because of charge conjugation invariance (Furry’s theorem). The general conformally invariant 3-point function of a Maxwell field $F^{\mu\nu}(x)$, on the other hand, violates local commutativity of Bose fields and should hence be also set equal to zero (without having to assume any discrete symmetry). This last argument fails for a non-abelian gauge field

$$F(x, \omega) := \frac{1}{2} \omega^{\mu\nu} F^{a}_{\mu\nu}(x) t_a, \quad [t_a, t_b] = i f_{abc} t_c, \quad (\omega^{\mu\nu} = -\omega^{\nu\mu})$$

(5.7)

where $t_a$ are orthonormal hermitian matrices generating the defining representation of a (compact, semi-simple) non-abelian gauge group $G$, $f_{abc}$ is the totally antisymmetric tensor of (real) structure constants of the Lie algebra $\mathcal{G}$ of $G$ ($f_{abc} = \varepsilon_{abc}$, the Levi-Civita tensor for $\mathcal{G} = su(2)$, $t_a = \frac{1}{2} \sigma_a$, $a, b, c = 1, 2, 3$), $\omega$ is a constant skew-symmetric tensor (or differential form) introduced for notational convenience. While the general (gauge and) conformally invariant 2-point function of $F^a(x, \omega) = \frac{1}{2} \omega^{\mu\nu} F^{a}_{\mu\nu}(x)$ coincides with the free Maxwell one, (4.18),

$$\langle 0 \mid F^a(x_1, \omega_1) F^b(x_2, \omega_2) \mid 0 \rangle = N_F \delta^{ab} D(x_{12}; \omega_1, \omega_2)$$

(5.8)

where we have introduced a contracted form of (4.19),

$$4\pi^2 D(x; \omega_1, \omega_2) = R_{\mu_1\mu_2}(x) R_{\nu_1\nu_2}(x) \omega^{\mu_1\nu_1} \omega^{\mu_2\nu_2}$$

(5.9)
there is a 2-parameter family of local gauge and conformally invariant 3-point functions:

\[ W^{abc}(x_1, \omega_1, x_2, \omega_2, x_3, \omega_3) = f^{abc}(N_1 W^{(1)} + N_2 W^{(2)}), \quad (5.10) \]

\[
\begin{align*}
W^{(1)}(x_1, \omega_1, x_2, \omega_2, x_3, \omega_3) &= (X_{12}^{13} \mu_1 R_{\mu_1 \mu_2}(x_{13}) X_{23}^{2} \mu_2 R_{\mu_2 \mu_3}(x_{23})) \\
&- X_{23}^{12} \mu_1 R_{\mu_1 \mu_2}(x_{12}) X_{13}^{1} \mu_3 R_{\mu_3 \mu_1}(x_{13}) \\
&+ X_{13}^{23} \mu_1 R_{\mu_1 \mu_2}(x_{12}) X_{23}^{12} \mu_2 R_{\mu_2 \mu_3}(x_{23}) \\
&\omega_1^{\mu_1 \nu_1} \omega_2^{\mu_2 \nu_2} \omega_3^{\mu_3 \nu_3},
\end{align*} \quad (5.11) \]

\[
W^{(2)}(x_1, \omega_1, x_2, \omega_2, x_3, \omega_3) = R_{\mu_1 \mu_2}(x_{12}) R_{\nu_1 \nu_3}(x_{13}) R_{\nu_2 \nu_3}(x_{23}) \\
\omega_1^{\mu_1 \nu_1} \omega_2^{\mu_2 \nu_2} \omega_3^{\mu_3 \nu_3},
\quad (5.12) \]

where \( R_{\mu \nu} \) and \( X_{12}^{13} \) are defined in (3.8) and (3.9).

The expression (5.10) can, sure, be used to produce a non-zero 3-point function of \( L \) (1.3). Thus, the second condition (5.1) is not, at first sight, an automatic consequence of (1.3), conformal invariance and locality, Hodge duality providing an independent restriction on correlation functions. It turns out, however, that the combination of (5.8) and (5.10) implies the existence of a field \( I \) in the OPE of \( F(x_1) \otimes F(x_2) \) transforming under an indecomposable representation of \( C \) (which would severely complicate the use of conformal invariance).

This statement follows from the observation that the 3-point function (5.10) does not satisfy the free Maxwell equation \( d^* F = 0 \) for any non-zero choice of \( N_1 \) and \( N_2 \) while the 2-point function does:

\[
\langle 0 \mid d^* F(x_1) \otimes F(x_2) \mid 0 \rangle = 0,
\]

\[
\langle 0 \mid d^* F(x_1) \otimes F(x_2) \otimes F(x_3) \mid 0 \rangle \neq 0.
\quad (5.13) \]

It follows that the OPE of \( F(x_2) \otimes F(x_3) \) should involve a local field \( I \) that is not orthogonal to \( d^* F \). \( I \) cannot be a derivative of \( F \) because of the first equation (5.13). On the other hand, \( I \) cannot be an elementary conformal field transforming under an inequivalent representation of \( C \) since then it should again be orthogonal to \( F \), and hence to \( d^* F \) thus violating the second equation (5.13).

Thus the vanishing of odd point correlation functions of \( L \) is a natural property of the Lagrangian (1.3) if the local observable algebra is spanned by elementary conformal fields (and their derivatives).

**Remark 4.2.** Let \( A^a_\mu(x) \), \( a = 1, 2, 3 \) be three commuting purely longitudinal gauge potentials, – i.e., generalized free fields such that

\[
\langle 0 \mid A^a_\mu(x_1) A^b_\nu(x_2) \mid 0 \rangle = \frac{1}{2} \delta^{ab} \delta_{\mu \nu} (x_{12}) \frac{(12)}{8\pi^2} R_{\mu \nu}(x_{12}),
\]

\[
\partial_\mu A^a_\nu(x) = \partial_\nu A^a_\mu(x).
\quad (5.14) \]
Then both the 2-point function (5.8) and the 3-point function (5.12) are reproduced by the corresponding $su(2)$ Yang-Mills curvature tensor

$$\begin{align*}
F_{\mu\nu}(x) &= \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - g\varepsilon^{abc} : A_\mu^a(x)A_\nu^b(x) : \\
&= -g\varepsilon^{abc} : A_\mu^a(x)A_\nu^b(x) : .
\end{align*}$$

(5.15)

This example is already excluded, however, by our requirement that no $d = 2$ scalar field appears in the OPE of two $L'$s (neither does $F$ (5.15) satisfy the Yang-Mills equation with connection $A$). Even for the more general $d = 4$ composite field

$$\mathcal{L}_{\xi\eta}(x) = \xi : (A_\mu^a(x)A_\mu^a(x))^2 : -\eta : A_\mu^a(x)A_\nu^b(x)A_\sigma^c(x)A_\sigma^c(x) :$$

(5.16)

(which includes (1.3) with $F$ given by (5.15) for $\xi = \eta = \frac{s^2}{4}$) we find that the leading term in the OPE of two $L'$s,

$$\mathcal{L}_{\xi\eta}(x_1)\mathcal{L}_{\xi\eta}(x_2) \approx 4(14\xi^2 - 16\xi\eta + 11\eta^2)(12)^3 (A_\mu^a(x_1)r^{\mu\nu}(x_{12})A_\nu^a(x_2)),$$

(5.17)

involves a scalar field of dimension $d = 2$ (with the same coefficient as the 2-point function of $\mathcal{L}_{\xi\eta}$ -- which is non-zero for any not simultaneously zero real $\xi, \eta$).

### 5.3 Concluding remarks

We have presented in the preceding sections two intertwined developments: (i) a systematic study of the theory of a GCI local scalar field $\psi$ of any (integer) dimension $d$ in terms of bilocal fields $V_\kappa(x_1, x_2)$ of dimension $(\kappa, \kappa)$ appearing in the OPE $\psi^*(x_1)\psi(x_2)$ (3.1) (3.5); (ii) first steps in an attempt to construct in a non-perturbative, axiomatic approach a conformally invariant fixed point of a gauge field theory, formulated entirely in terms of gauge invariant local observables of dimension 4: the Lagrangian density $\mathcal{L}(x)$ and the stress-energy tensor $T^{\mu\nu}(x)$ or, rather, its polarized bilocal counterpart that determines $V_1(x_1, x_2)$ according to (3.32).

(i) The use of bilocal fields simplifies substantially the analysis of the operator content of GCI correlation functions. In particular, $V_1(x_1, x_2)$, defined as the (infinite) sum of twist 2 conserved symmetric tensor fields, satisfies as a consequence the $d$'Alembert equation (3.20). This allows to compute the contribution of the correlator $\langle 0 | V_1(x_1, x_2)V_1(x_3, x_4) | 0 \rangle$ to the (truncated) 4-point function $w^4_1$ of $\mathcal{L}$. (Its expression for the $d = 2$ neutral scalar field $\phi$ has been computed in [20].) A minimal model for the connected part of the 4-point function of a general neutral scalar field of dimension 4 is given by $F_1$ (4.16) which is determined by its harmonic projection. A general procedure is outlined - based on the Dolan-Osborn formula (4.3) - for computing the expectation values (3.16) of $V_\kappa$. The case $\kappa = 1$ is distinguished by the fact that the conformal harmonic function $f_1(s,t)$ (and hence, the corresponding symmetrized contribution $F_1$)
is rational. For \( \kappa > 1 \) \( f_\kappa(s, t) \) are linear combinations (with rational function coefficients) of
\[
\log(1 - u) \text{ and } \log(1 - v) \text{ for } s = uv, \ s + 1 - t = u + v.
\] (5.18)

We have displayed these functions (taking into account also the additional term \( b_1Q_1 + b_2Q_2 \) in (4.28)) and the associated (infinite) OPE expansions in terms of symmetric traceless tensor fields of twist \( 2\kappa \) for \( \kappa = 1, 2, 3 \) (see Eqs. (3.31), (4.31)(4.32), (4.35)(4.36), respectively). The same procedure applies to higher \( \kappa \) as well, when \( f_\kappa \) also depend on the sum of products of 2-point functions
\[
(13)(24) + (14)(23) \quad (\text{which introduces one more parameter, the normalization } N \text{ of the 2-point function}).
\]
The resulting expressions (together with other computer aided results - concerning 6-point functions) will be presented in a forthcoming publication, in collaboration with K.-H. Rehren.

(ii) Conditions (5.1) exclude contributions of scalar fields of dimension 2 and 4 in the OPE of \( \mathcal{L}(x_1)\mathcal{L}(x_2) \) leaving us with a 3-parameter family of truncated 4-point functions. We assert that \( \mathcal{L}(x) \) then has the properties of the Lagrangian density of a gauge field curvature (without matter fields). The study of Wightman positivity for twists 4 and 6 contributions leads to a rather strong constraint (5.5) for the remaining parameters. Should, in particular, the analysis of the 2n-point function of \( \mathcal{L} \) for \( n \geq 3 \) yield the constraint \( a' = 0 \) Eq.(5.5) would also imply \( b = 0 \) and leave us with a multiple of the 4-point function of the Lagrangian of a free abelian gauge field. This would confirm the general belief (see, e.g. [12] [30]) that a (pure) non-abelian gauge theory necessarily involves a mass gap, thus violating conformal invariance. By contrast, if we do not impose (5.1) – i.e., if we allow for the presence of (at least) a scalar field in the Lagrangian, a full account of Wightman positivity for the 4-point function appears to allow for an open set of the 5-dimensional (projective) parameter space.

The evidence that we are displaying a gauge invariant 4-point function in a (non-abelian) gauge theory is rather indirect. One verifies on a case by case basis that any other renormalizable Lagrangian would involve fields of \( d < 4 \) in the OPE and that is excluded by condition (5.1). The difficulty in identifying the theory in terms of basic (gauge dependent) fields like \( F^\mu_{\nu} \) (briefly reviewed in Sec. 5.2) lies in the fact that the model we are trying to construct is necessarily non-perturbative (if it exists at all). From this point of view our model is not incompatible with the \( N = 4 \) supersymmetric Yang-Mills theory (see e.g. [1], [3], [4], [11] and references therein) for sufficiently large value of the coupling constant \( g \), such that the anomalous dimension of the Konishi field (which appears also in the OPE of two fields of the supermultiplet of the stress-energy tensor) is a positive even integer.

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Appendix A. Vacuum representation of the algebra generated by the harmonic bilocal field $V_1$. The case $d = 2$

There are two main ingredients in the proof of the central result, Theorem 5.1, of [20] to be reviewed in the two sections of this Appendix. The first is a (computer aided) study of the (5- and) 6-point function of the basic field $\phi(x)$ of dimension 2, combined with the expansion (3.3) (for $\kappa = 1$) and the conservation law for the twist two fields $T_\ell(x, \zeta)$ ($\ell = 2, 4, \ldots$). Here we sum up a modified version of the argument which uses Proposition 2.1. The (technical) difficulty of such an analysis increases drastically with increasing the dimension of the underlying scalar field and it has not been completed for $d = 4$. The second ingredient is quite general: it uses the discrete mode expansion of $V_1$ with respect to the conformal Hamiltonian which can be carried out for any $d$.

A.1 Analysis of 5- and 6-point functions for $d = 2$

The general GCI and crossing symmetric 5-point function of $\phi(x)$ (for $d_\phi = 2$) involves two independent terms: the sum of twelve 1-loop graphs

$$w^{(1)} = \frac{c}{2} \sum_{\sigma \in \text{Perm}(2, \ldots, n)}^{(n)} (1\sigma_2) \sigma_2 \sigma_3 \cdots \sigma_{n-1} \sigma_n (1\sigma_n) \text{ for } n = 5 \quad (A.1)$$

where $(ij)$ is defined in (2.9) and

$$\sigma_i \sigma_j = (\min(\sigma_i, \sigma_j), \max(\sigma_i, \sigma_j)), \quad (A.2)$$

and a sum, $w^{(2)}$, of 10 products of 7 factors each:

$$w^{(2)} = \lambda \sum_{1 \leq i < j \leq 5} p_{ij} \prod_{1 \leq k \leq \frac{1}{2} \notin \frac{5}{2}}^{(i, k, j)} i k j. \quad (A.3)$$

A rather nasty (computer aided) calculation shows that the 5-point function of three $\phi$'s and a $V_1$ satisfies the d’Alembert equation in the last two arguments

$$\Box_j \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) V_1(x_4, x_5) | 0 \rangle = 0 \quad \text{for } j = 4, 5 \quad (A.4)$$

iff $\lambda = 0$. In view of Proposition 2.1 (A.4) is equivalent to demanding the infinite set of conservation laws

$$\frac{\partial^2}{\partial x_\mu^2} \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) T_{2\ell}(x_4, \zeta) | 0 \rangle = 0. \quad (A.5)$$

Similarly, only the crossing symmetric sum of $\left(\frac{1}{2} \times 5! = 60\right)$ 1-loop graph contributions to the truncated 6-point function is consistent with $T_{2\ell}$ conservation. The 1-loop expression for the 6-point function allows to prove that the
limit
\[ V(x_1, x_2) = \lim_{\rho_{123} \to 0} \{ (2\pi)^4 \rho_{123} (\phi(x_1)\phi(x_2)\phi(x_3) \]
\[ - (13)\phi(x_2) - (23)\phi(x_1) - \langle 123 \rangle \} \] (A.6)
exists, does not depend on \( x_3 \) and defines a harmonic in each argument bilocal field \( V_1(x_1, x_2) \); moreover, the truncated \( n \)-point functions of \( \phi \) will be given by (A.1) for all \( n \) (see [20], Proposition 2.3).

Remark A.1. The above sketched analysis for the 5-point function has been carried out in the \( d = 4 \) case, too, with the following results. There are 37 independent GCI and crossing symmetric 5-point functions of \( L \) (compared to 2 for \( d = 2 \)). After imposing conservation of the twist 2 tensors \( T_{2\tau}(x, \zeta) \) there remain only 8, just 1 among them actually contributing to the twist 2 part of the OPE. The latter is non-zero only if the condition (5.1) is violated and then it yields \( a_2 = 0 \).

The analysis of the general 6-point function has to deal with 31990 (instead of 8 for \( d = 2 \)) independent structures (most of them consisting of \( 6! = 720 \) terms each).

A.2 Analytic compact picture fields. Conformal Hamiltonian. Mode expansions

Compactified Minkowski space
\[ \mathcal{M} = (S^1 \times S^3)/\mathbb{Z}_2 \] (A.7)
(see [8]) has a convenient realization in terms of (euclidean) complex 4-vectors ([26]):
\[ \mathcal{M} = \{ z_\mu = e^{2\pi i \zeta} u_\mu, \mu = 1, 2, 3, 4; \zeta \in \mathbb{R}, u \in S^3 \text{, i.e. } u^2 := u^2 + u_4^2 = 1 \}. \] (A.8)

Real Minkowski space \( M \) is mapped on an open dense subset of \( \mathcal{M} \):
\[ M \ni (x^0, x) \rightarrow z = \omega^{-1}(x)x, \ z_4 = \frac{1 - x^2}{2\omega(x)}, \ \omega(x) = \frac{1 + x^2}{2} - ix_0. \] (A.9)

\( \mathcal{M} \) can be obtained by adding to the image of \( M \) the 3-cone at infinity:
\[ K_\infty = \{ z \in \mathcal{M} : 1 + 2z_4 + z^2 (= 2(u_4 + \cos 2\pi \zeta)e^{2\pi i \zeta}) = 0 \}. \] (A.10)

Note that Eq. (A.9) can be interpreted as the Cayley map from the Lie algebra \( u(2)(\cong \mathbb{R}^4) \) to the group \( U(2) \) – cf. [28].

We note that the map (A.9) extends to complex arguments \( x \mapsto x + iy \) and is regular in the forward tube (2.1) which is mapped (for any \( D \)) on the domain
\[ T_+ = \{ z \in \mathbb{C}^D : |z|^2 < 1, \ 2 |z|^2 = \sum_{\mu=1}^D |z_\mu|^2 < 1 + |z|^2 \}. \] (A.11)
where $z^2 = \left[(1 - y^0 + ix^0)^2 + (x + iy)^2\right] \left[(1 + y^0 - ix^0)^2 + (x + iy)^2\right]^{-1}$. The maximal compact subgroup $K = K(4)$ acts by linear homogeneous transformations on $z_\mu$: $SU(2) \times SU(2)$ is represented by real $SO(4)$ rotations of the vector $(z_\mu)$ while the $U(1)$ factor acts by phase transformations: $z_\mu \mapsto e^{i\alpha}z_\mu$.

Thus the point $z = 0 \in T_+$ (the image of $(x + iy) = (i, 0) \in \mathbb{R}^+$) is left invariant by $K$.

A scalar $M$-space field $\phi_M(x)$ of dimension $d$ is related to its $z$-picture counterpart, $\phi(z)$ (for $z = z(x)$ given in (A.9)) by:

$$\phi_M(x) = [2\pi \omega(x)]^{-d/2} \phi(z(x)).$$ \hspace{1cm} (A.12)

The numerical factor $2\pi$ in (A.12) is chosen for convenience so that the free massless scalar propagator assumes a simple $z$-picture form

$$\langle 12 \rangle = \frac{1}{z_{12}^2}. \hspace{1cm} (A.13)$$

A $z$-picture scalar field $\phi(z)$ of dimension $d$ transforms in such a way that the form $\phi(z) (dz^2)^{d/2}$ remains invariant. (The transformation properties of the Lagrangian $\mathcal{L}(z)$, for $d = 4$, can be read off, alternatively, from the invariance of the volume form $\mathcal{L}(z) dz^4 \wedge dz^2 \wedge dz^1 \wedge dz^0$ - cf. (1.6).) This implies, in particular, that correlation functions are invariant under complex euclidean transformations. The conformal Hamiltonian $H$ (that generates translations in the conformal time $\xi$) acts on $\phi(z)$ and on the bilocal fields $V_\kappa(z_1, z_2)$ according to the law

$$[H, \phi(z)] = \left(d + z \frac{\partial}{\partial z}\right) \phi(z),$$

$$[H, V_\kappa(z_1, z_2)] = \left(2\kappa + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\right) V_\kappa(z_1, z_2). \hspace{1cm} (A.14)$$

Here $V_\kappa$ are again related to $\phi$ by an expansion of type (3.1) with (12) substituted by its $z$-picture counterpart (A.13).

The mode expansion of $V_1(z_1, z_2)$ is, in particular, an expansion in homogeneous harmonic polynomials:

$$V_1(z_1, z_2) = \sum_{m, n \in \mathbb{Z}} V_{m-1n-1}(z_1, z_2),$$

$$V_{m-1n-1}(\lambda_1 z_1, \lambda_2 z_2) = \frac{V_{m-1n-1}(z_1, z_2)}{\lambda_1^m \lambda_2^n} \text{ for } \lambda_1, \lambda_2 \in \mathbb{C}^* \hspace{1cm} (A.15)$$

$$\Delta_1 V_{m,n}(z_1, z_2) = 0 = \Delta_2 V_{m,n}(z_1, z_2) \text{ for } \Delta_n = \sum_{\mu=1}^{4} \left(\frac{\partial}{\partial z_{\mu}}\right)^2. \hspace{1cm} (A.16)$$

For a hermitian scalar field $\phi$ the modes $V_{m,n}$ have the following simple conjugation property. For $m, n \geq 0$

$$V_{-m-1, -n-1}(z, w) = V_{m, n} z_{\mu_1} \cdots z_{\mu_m} w_{\nu_1} \cdots w_{\nu_n} \hspace{1cm} (A.17)$$
where \( V^{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n} \) are symmetric traceless tensors in \( \mu_1, \ldots, \mu_m \) and in \( \nu_1, \ldots, \nu_n \) (separately) which are mapped under hermitian conjugation into tensors with the same properties,

\[
(V^{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n})^* = V_{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n}^{* \mu_1 \ldots *\mu_m *\nu_1 \ldots *\nu_n}
\]

so that

\[
V_{m+1n+1}(z, w) = \frac{1}{z^2 w^2} V^{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_n} z_{\mu_1} \cdots z_{\mu_m} w_{\nu_1} \cdots w_{\nu_n}.
\]

We are interested in the vacuum representation of the algebra of \( V_1 \) modes for which

\[
V_{mn} | 0 \rangle = 0 = \langle 0 | V_{-n \ldots -n} \text{ if either } m \geq 0 \text{ or } n \geq 0.
\]

For \( d = 2 \) the algebra generated by \( V_{nm}(z, w) \) coincides with a central extension \( \check{\mathfrak{sp}}(\infty, \mathbb{R}) \) of the infinite symplectic Lie algebra. This is particularly simple to see when the (invariant under rescaling) structure constant is a natural number:

\[
e := 8 \frac{(12)(13)(23)}{((123))^2} = N \in \mathbb{N}
\]

and

\[
V_1(z_1, z_2) = \sum_{j=1}^{N} : \varphi_j(z_1) \varphi_j(z_2) : \quad (V_1(z, z) = 2\phi(z)),
\]

where \( \varphi_j(z) \) are free (commuting for different \( j \)) massless scalar fields. The modes \( \varphi_n(z) (n \in \mathbb{Z}) \) of each \( \varphi(z) \) generate the infinite Heisenberg algebra:

\[
\varphi_n(z) = e^{-2\pi i(n+1)} \varphi_n(u) \quad [\varphi_n(u), \varphi_m(v)] = \delta_{n-m} \epsilon(n) C_n^1(vu) \quad (A.23)
\]

where \( \epsilon(n) = \text{sign } n \left( \frac{n}{|n|}, \epsilon(0) = 0 \right) \) \( C_n^1 \) are the Gegenbauer polynomials generated by the 2-point function of \( \varphi \):

\[
C_n^1(\cos 2\pi \alpha) = \frac{\sin 2\pi \alpha}{\sin 2\pi \alpha}, \quad \frac{1}{z_1^2} = \frac{1}{z_1^2} (1 - 2xz + z^2) = \frac{1}{z_1^2} \sum_{n=0}^{\infty} z^n C_n^1(x),
\]

\[
z = \sqrt{\frac{z_2^2}{z_1^2}}, \quad x = uw.
\]

It is well known that the quadratic combination of the generators of a Heisenberg algebra give rise to a symplectic Lie algebra. The central extension comes, as usual, from the normal products.

The algebra \( \check{\mathfrak{sp}}(\infty, \mathbb{R}) \) of \( V_{nm}(u, v) \) has a simple diagonal subalgebra, generated by \( v_{nm} := V_{nm}(u, u) \), \( u \in \mathbb{S}^3 \),

\[
[v_{n_1 m_1}, v_{n_2 m_2}] = cn_1 m_1 (\delta_{n_1, -n_2} \delta_{m_1, -m_2} + \delta_{n_1, -m_2} \delta_{m_1, -n_2}) + m_1 (\delta_{n_1, -n_2} v_{m_1 m_2} + \delta_{n_1, -m_2} v_{m_1 n_2}) + m_1 (\delta_{m_1, -n_2} v_{n_1 m_2} + \delta_{m_1, -m_2} v_{n_1 n_2}).
\]

(28)
The proof of Theorem 5.1 of [20] is now based on the construction of a sequence 
\( \langle \Delta_n \mid, n = 1, 2, \ldots \) of vectors (Lemma 5.2)

\[
\langle \Delta_n \mid = \frac{1}{n!} \langle 0 \mid \begin{vmatrix}
v_{11} & v_{12} & \cdots & v_{1n} \\
v_{21} & v_{22} & \cdots & v_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
v_{n1} & v_{n2} & \cdots & v_{nn}
\end{vmatrix},
\]

such that \( \langle \Delta_n \mid \Delta_n \rangle = (n+1)!c(c-1)\cdots (c-n+1), \) (A.26)

implying \( c \in \mathbb{N} \) for unitary vacuum representations of \( \widehat{sp}(\infty, \mathbb{R}) \). Then one deduces that Wightman positivity implies that \( V_1 \) has the form (A.22) for some \( N \).

The difficulty in extending this analysis to the case \( d = 4 \) again lies in the necessity of having the 6-point function of \( \mathcal{L} \) in order to be able to compute the structure constants of the infinite Lie algebra generated by the modes of \( V_1(z_1, z_2) \).
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