On Non-Compact Compactifications with Brane Worlds

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Abstract
The possibility of neutral, brane-like solutions in a higher dimensional setting is discussed. In particular, we describe a supersymmetric solution in six dimensions which can be interpreted as a “three-brane” with a non-compact transverse space of finite volume. The construction can be generalized to $n+4$-dimensions and the result is a $n+3$-brane compactified on a $n-1$-dimensional Einstein manifold with a non-compact extra dimension. We find that there always exists a massless graviton trapped in four-dimensions while a bulk abelian gauge field gives rise to a unique four-dimensional massless photon. Moreover, all massless modes are accompanied by massive KK states and we show that it is possible in such a scenario the masses of the KK states to be at the TeV scale without hierarchically large extra dimensions.
1 Introduction

There is a renewal interest in higher-dimensional theories after the realization that string scale can be lowered even to few TeVs [1]. In this case, extra dimensions are expected to open up at this scale as was originally proposed in [2] motivated by the scale of supersymmetry breaking. It has also been realized [4], that the size of some dimensions can even macroscopic as long as the Standard Model (SM) sector lives solely in four-dimensions, in a three-brane for example. This proposal offers also a natural explanation to a long-standing problem in all efforts to extend the SM, namely, the hierarchy problem. The latter translates into the fact that the ratio $m_{EW}/M_{Pl}$ of the electroweak scale $m_{EW} \sim 10^3 \text{ GeV}$ to the Planck scale $M_{Pl} = G_N^{1/2} \sim 10^{18} \text{ GeV}$ is unnatural very small. According to this proposal, the hierarchy is due to the geometry of the higher-dimensional space-time. In particular, the higher $4 + n$ dimensional theory with $n \geq 2$ has a $4 + n$ dimensional TeV scale Planck mass $M_{Pl(4+n)}$ while the scale $R_c$ of the extra $n$ dimensions is less than a millimeter. The proposal has been designed in such a way as to generate the hierarchy $m_{EW}/M_{Pl}$.

These ideas have been pushed further by modeling four-dimensional space-time as a brane embedded in higher dimensions which is reminiscent of some old proposals that tried to model our world as a “domain wall” [3]. This is realized in modern terms by the brane world where our universe is viewed as a three-brane (or p-brane with appropriate compactified directions, or intersecting branes, etc.). An early example was the Horava-Witten picture for the non-perturbative heterotic $E_8 \times E_8$ string [6] and its relevance for the construction of realistic phenomenological models [7]. A general feature of all these models is that the gauge sector lives on the brane whereas gravity propagates in the bulk. Although a bulk SM (or partial bulk) has some nice properties like power-law running couplings [8],[9] and consequently a lower unification scale [9], by trying to put the gauge sector in the bulk in models with large extra dimensions, for example, one faces the problem that Kaluza-Klein (KK) states of the photons should have already been observed. Thus, a bulk SM constraints the size of the extra dimensions to be more than around 1 TeV depending on their number [10]. Thus, it seems that a bulk gauge sector is in conflict with large extra-dimensions.

Here, we will consider classical supergravity solutions which can be interpreted as “domain walls” in higher dimensions. The domain walls we will construct are formed in vacuum, contrary to the usual cases where other fields are present, usually scalars. So their formation is due to gravity. In particular, out of the domain wall, space-time is Ricci-flat and all the energy is concentrated at the position of the wall. Thus, our model differs essentially from other similar proposals. In the model of Randall and Sundrum (RS) for example [11], there is a bulk cosmological constant while in that of Cohen and Kaplan scalar fields [12]. In these models a massless four-dimensional graviton appears on the domain wall due to the finiteness of the transverse space as in the old KK literature [14]. In our case, there is no cosmological constant or matter fields and space-time is Ricci flat (or even completely flat) everywhere except at the position of the wall. Thus,
it can also be considered as a vacuum in heterotic theories. In the latter case there exist bulk gauge bosons and one may also ask if there exist massless gauge bosons trapped, like the graviton, in the wall. We find that indeed both bulk graviton and photons give rise to a four-dimensional massless graviton and photons with a discrete spectrum of KK modes. Gauge bosons in the RS case have been discussed in \[3\]. We also show that in this scenario it is possible for a TeV scale internal space.

## 2 Novel vacua with non-compact dimensions

One way of constructing string theory vacua is to look for classical supergravity solutions. As supergravity is the low-energy limit of string theory, supergravity solutions describe accordingly low-energy string vacua. There will be $\alpha'$-corrections as well as string-loop corrections to these solutions but, nevertheless, these solutions will still be valid in some appropriate limits. These vacua are constructed by solving the classical field equations with appropriate fields turned on. Usually such fields are antisymmetric $p$-forms as well as scalars like the dilaton, axion etc., which appear in almost all supergravity theories. Some of these classical solutions are interpreted as the D-branes of string theory. There are also string vacua where there are no other fields, except the graviton, turned on. Such vacua are necessarily Ricci-flat

$$R_{MN} = 0,$$

i.e., they satisfy vacuum Einstein equations. Solutions to the above equations with four-dimensional Poincaré invariance are provided by $M^4 \times X$ where $M^4$ is ordinary Minkowski space-time and $X$ is a Ricci-flat manifold. Supersymmetry demands that $X$ should be a manifold of $U(1)^6$, $SU(2) \times U(1)^2$ or $SU(3)$ holonomy. Manifolds with such holonomies can be either compact or non-compact. We recall for example the case of the compact $K3$ surface and the non-compact Eguchi-Hanson gravitational instanton, both of $SU(2)$ holonomy. The four-dimensional Plank mass $M_P$ is proportional to the volume $V(X)$ of $X$

$$M_P^2 = M_s^8 V(X),$$

where $M_s^2 \sim 1/\alpha'$ is the string-mass scale. Propagating gravity therefore exists in four dimensions if the volume of $X$ is finite. In this case, $X$ must be compact so that $X$ is either $T^6$, $K3 \times T^2$ or $CY_3$ depending on the number of surviving supersymmetries. It should be stressed, however, that there are also non-compact spaces of finite volume which, according to eq.(2) will lead to a four-dimensional dynamical gravity. Such spaces have been considered in the Kaluza-Klein programme \[14\] and discussed \[15\] in connection to the chiral-fermion problem \[16\]. The drawback of non-compact spaces of finite volume is that they suffer from singularities. However, although singularities are considered in general to be disastrous, there exist singularities which are quite mild in the sense that they can be attributed to some form of matter. These are delta-function singularities
which may be interpreted as strings, domain walls or fundamental branes in general. Supergravity solutions with a bulk cosmological constant and such singularities have been constructed in [17]. Our aim here is to solve eq.(1) with a non-compact internal space $X$ of finite volume and delta-functions singularities.

In five dimensions the only solution to eq.(1) with four-dimensional Poincaré symmetry $ISO(1,3)$ is flat space i.e., $ISO(1,3)$-invariance and Ricci-flatness implies flat space in five dimensions. However, this is not true in higher dimensions. For example let us try to solve eq.(1) in six-dimensions with the $ISO(1,3) \times U(1)$-invariant metric

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + e^{2\Lambda(z)} \left( dz^2 + d\phi^2 \right),$$

where $0 < \phi \leq 2\pi a$ parametrize an $S^1$ with radius $a$ and $-\infty < z < \infty$. Eq.(1) is then written as

$$R_{zz} = R_{\phi\phi} = -\Lambda'' = 0.$$  

The solution of eq.(1) for $\Lambda$ is, up to an irrelevant constant,

$$\Lambda = \Lambda(z) = \epsilon \mu z, \quad \epsilon = \pm 1,$$

where $\mu > 0$ is a dimensionfull constant. The metric (3) turns out then to be

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\epsilon \mu z} \left( dz^2 + d\phi^2 \right), \quad \mu, \nu = 0, 1, 2, 3$$

with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ the Minkowski metric. One observes that the metric above is locally flat since it is an alternative way of writing the metric of a flat six-dimensional Minkowski space. Indeed, the transformation $\mu z = \epsilon \ln(\mu r)$ gives a manifestly flat-form of the metric (for $\mu = 1/a$). For $\mu \neq 1/a$, we get as solution the product of four-dimensional Minkowski space with a two-dimensional flat cone since in this case there exist a deficit angle $2\pi (1 - a\mu)$ for $\phi$.

In solving eq.(1), we have made the assumption that both $\Lambda$ and its first derivative are everywhere continuous. However, by relaxing the continuity conditions we may get other solutions as well. For example, there exist piecewise flat solutions in which $\Lambda$ is continuous but with discontinuous first derivatives. Such solutions are provided by the choice

$$\Lambda(z) = \epsilon \mu |z|, \quad \epsilon = \pm 1.$$  

As a result, we get

$$\Lambda'' = 2 \epsilon \mu \delta(z),$$

and the metric turns out to be

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\epsilon \mu |z|} \left( dz^2 + d\phi^2 \right).$$
The sing $\epsilon$ will be determined in a moment after we calculate the energy-momentum tensor. The Ricci tensor develops delta-function singularities. Indeed, from eq. (4) we find that

$$R_{zz} = R_{\phi\phi} = -2 \epsilon \mu \delta(z), \quad R = -4 \epsilon \mu \delta(z) e^{-2\mu |z|},$$

(10)

where $R$ is the scalar curvature. Thus, the metric (4) is again everywhere flat for $\mu = 1/a$ (or locally flat for $\mu \neq 1/a$ but now it develops a delta-function singularity at the point $z = 0$. To see if this singularity can be attributed to some form of matter, we have to calculate the energy-momentum tensor. The latter may be read off from

$$T_{MN} = \frac{1}{8\pi G_6} \left( R_{MN} - \frac{1}{2} G_{MN} R \right),$$

(11)

where $G_n$ is, in general, the $n$-dimensional Newton constant. By using eq. (10) we get that

$$T_{\mu\nu} = \frac{1}{4\pi G_6} \epsilon \mu e^{-2\mu |z|} \delta(z) \eta_{\mu\nu}$$

and

$$T_{zz} = T_{\phi\phi} = 0.$$  

(12)

Positivity now of the energy-density,

$$T_{00} = \rho = -\epsilon \frac{\mu}{4\pi G_6} e^{-2\mu |z|} \delta(z)$$

demands that $\epsilon = -1$. As a result, the metric turns out to be

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2\mu |z|} \left( dz^2 + d\phi^2 \right).$$

(13)

This metric describes a “string” (string if we count the codimension of the object, or a three-brane if we count its actual dimensions) with a four-dimensional world-volume in six dimensions as can be seen from the form of energy-momentum tensor

$$T_{MN} = \rho \text{ diag}(1, -1, -1, -1, 0, 0).$$

(14)

It should be stressed that this three-brane is not the one of type IIB theory since it is neutral and, in particular, it does not carry any RR charge to justify its name. However, it has a four-dimensional world-volume and for this reason we will call it three-brane. The transverse space of this three-brane is a non-compact surface $\Sigma_2$ with metric

$$ds_\perp = e^{-2\mu |z|} \left( dz^2 + d\phi^2 \right).$$

(15)

Remarkably, the area of the surface $\Sigma_2$ is

$$V_\perp = 2\pi a \int_{-\infty}^{\infty} e^{-2\mu |z|} dz = 2\pi \mu^{-1} a < \infty,$$

(16)

and thus, a massless graviton is expected in four-dimensions as will see later.
2.1 Supersymmetry

We will examine now if the background of eq. (13) is supersymmetric. In this case, the gravitino shifts

$$
\delta \psi_M = D_M \epsilon ,
$$

(17)

where $D_M = \partial_M + \omega_{MAB} \Gamma^{AB}/4$ is the spin connection, vanish for appropriate spinors $\epsilon$. We may split $\epsilon$ as

$$
\epsilon = \theta \otimes \eta ,
$$

(18)

where $\theta, \eta$ are four-dimensional and two-dimensional spinors, respectively. We choose for the gamma matrices the representation

$$
\Gamma^\alpha = \gamma^\alpha \otimes 1 , \quad \alpha = 0,1,2,3 ,
$$

$$
\Gamma^z = \gamma^5 \otimes \sigma^1 , \quad \Gamma^\phi = \gamma^5 \otimes \sigma^2 ,
$$

(19)

where $\gamma^\alpha$ are four-dimensional gamma matrices and $\sigma^{1,2}$ are Pauli matrices. The vanishing of the gravitino shifts is then equivalent to the existence of covariantly constant spinors in the transverse space $\Sigma_2$ with metric (15). In particular, the number of supersymmetries in four-dimensions is the number of independent Killing spinors in the transverse space $\Sigma_2$. The Killing spinor equation is

$$
D_i \eta = 0 ,
$$

(20)

where $D_i = \partial_i + \omega_{iab} \sigma^{ab}/4$ is the spin connection in the two-dimensional space $\Sigma_2$. For the metric (15) the Killing spinor equation splits as follows

$$
\partial_z \eta = 0 ,
$$

(21)

$$
\left( \partial_\phi + \frac{1}{2} \Lambda' \sigma^2 \sigma^1 \right) \eta = 0 .
$$

(22)

The solution to eq.(22) for $\eta$ is given by

$$
\eta = \left( \Theta(-z) \left( \sigma^2 \cos \frac{\mu}{2} \phi + \sigma^1 \sin \frac{\mu}{2} \phi \right) + \Theta(z) \left( \cos \frac{\mu}{2} \phi - \sigma^1 \sigma^2 \sin \frac{\mu}{2} \phi \right) \right) \eta_0 ,
$$

(23)

where $\Theta(z) = 0,1$ for $z < 0, z > 0$, respectively is the step function and $\eta_0$ is a two-component constant spinor. Substituting eq.(23) back in eq.(21), we get

$$
\partial_z \eta = \delta(z) \left( 1 - \sigma^2 \right) \left( \cos \frac{\mu}{2} \phi - \sigma^1 \sin \frac{\mu}{2} \phi \right) \eta_0 .
$$

(24)

The matrix $(1 - \sigma^2)$ has one zero eigenvalue and thus, there exist only one covariant constant spinor localized at $z = 0$. Therefore, the solution we found breaks half of the supersymmetries. The non-zero eigenvalue of $\partial_z \eta$ is the Golstone fermion and lives only inside the wall due to the delta function.
3 Domain Walls in higher dimensions

We have seen that the conditions of four-dimensional Poincaré invariance and Ricci-flatness lead, in six dimensions, to three-brane like solutions. This construction can be generalized to higher dimensions. Here we will consider an n+4-dimensional space-time of the form $M^{1,3} \times X^n$ with metric

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + e^{2\Lambda(z)}(dz^2 + d\sigma^2),$$

where

$$d\sigma = k_{ij}(y)dy^idy^j,$$

is the metric of an $n-1$-dimensional space $\Sigma$. The Ricci tensor for the metric (25) is

$$R_{\mu\nu} = 0,$$

$$R_{zz} = -(n-1)\Lambda'',$$

$$R_{ij} = R(k)_{ij} - \Lambda''k_{ij} - (n-2)\Lambda^2k_{ij},$$

where $R(k)_{ij}$ is the Ricci tensor of the space $\Sigma$. Eq.(1) is then satisfied by

$$\Lambda = \epsilon \mu z, \quad \epsilon = \pm 1,$$

$$R(k)_{ij} = (n-2)\mu^2k_{ij},$$

so that the space $\Sigma$ is a Einstein space of positive constant scalar curvature $(n-1)(n-2)\mu^2$. $\Sigma$ can be any compact Einstein space and, in particular, the solution is flat n+4-dimensional space-time if $\Sigma$ is the round sphere $S^{n-1}$ with radius $1/\mu$.

As in the six-dimensional case we discussed before, we may take $\Lambda$ to be continuous but with discontinuous first derivatives. In this case $\Lambda$ is given by eq.(7) and the metric turns out to be

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + e^{2\epsilon z}\left(dz^2 + k_{ij}dy^idy^j\right).$$

Here again, the space is everywhere Ricci-flat except at the point $z = 0$ where it develops a delta-function singularity such that

$$R_{\mu\nu} = 0, \quad R_{ij} = -2\epsilon \mu \delta(z)k_{ij}, \quad R_{zz} = -2(n-1)\epsilon \mu \delta(z).$$

The energy-momentum tensor can be calculated from eq.(11) and we find

$$T_{\mu\nu} = \epsilon(n-1)\frac{\mu}{4\pi G(n+4)}e^{-2\epsilon z}\delta(z)\eta_{\mu\nu},$$

$$T_{ij} = \epsilon(n-2)\frac{\mu}{4\pi G(n+4)}e^{-2\epsilon z}\delta(z)k_{ij},$$

$$T_{zz} = 0.$$
Positivity of the energy-density requires again $\epsilon = -1$ and from the form of the energy-momentum tensor we see that the solution

$$d s^2 = \eta_{\mu \nu} d x^\mu d x^\nu + e^{-2\mu|z|} \left( d z^2 + k_{ij} d y^i d y^j \right), \quad (32)$$

represents a domain wall at $z = 0$. This domain wall however, is not a flat $n + 3$-dimensional Minkowski space-time $M^{1,n+2}$ as in the usual case but rather is of $M^{1,3} \times \Sigma$ topology. In a sense it may be viewed as a compactified flat Minkowski space-time on a compact Einstein space $\Sigma$ as in the old Kaluza-Klein programme. The transverse space $X^n$ to the four-dimensional Minkowski space-time has metric

$$d s^2 = e^{-2\mu|z|} \left( d z^2 + k_{ij} d y^i d y^j \right), \quad (33)$$

and its volume is

$$V_\perp = V(\Sigma) \int_{-\infty}^{\infty} e^{-\eta \mu |z|} d z = \frac{2}{n} V(\Sigma) \mu^{-1} < \infty, \quad (34)$$

which is again finite.

### 4 The bosonic spectrum

We will examine now the spectrum of small fluctuations of the bulk fields. As usual, the bulk fields are the graviton, scalars (like dilaton or axions), gauge fields, antisymmetric tensor fields, fermions and gravitinos. We will discuss here the case of bulk graviton and gauge fields.

To study the spectrum, we need certain Hodge-de Rham operators $\Delta_p$ in the $n$-dimensional space $X^n$. The action of the latter on scalars $Y$ and one-forms $Y_m$ is

$$\Delta_0 Y = -\nabla_m \nabla^m Y,$$

$$\Delta_1 Y_m = -\nabla_p \nabla^p Y_m + R_m^p Y_p, \quad m, k, p, q = 1, ..., n. \quad (35)$$

Another operator which is involved in the discussion is the Lichnerowitz operator $\Delta_L$ which acts on traceless transverse symmetric two-tensors as

$$\Delta_L h_{mk} = -\nabla_p \nabla^p h_{mk} + R_m^p h_{kp} + R_k^p h_{mp} - 2 R_{mpkq} h^{pq}. \quad (36)$$

Next we need the eigenvalues of the Hodge-de Rham and Lichnerowitz operators $\Delta_p$ and $\Delta_L$, respectively on the space $X$. We will work out explicitly the eigenvalue problem for the Laplace operator $\Delta_0$ which is directly involved in the discussion for massless four-dimensional gravitons, whereas for the rest, we will find bounds on their lowest eigenvalue. For simplicity, we will assume that the metric of $X$ is

$$d s_\perp^2 = e^{-2\mu|z|} \left( d z^2 + \frac{1}{\mu^2} d \Omega^2_{n-1} \right), \quad (37)$$
where $d\Omega_{n-1}^2$ is the metric on the unit $n-1$-sphere $S^{n-1}$. Thus, away from the $z = 0$ point, the transverse space is flat $n$-dimensional Euclidean space and all the curvature is at the $z = 0$ point.

We will first consider the scalar Laplacian $\Delta_0$ and its eigenvalue problem

$$\Delta_0 Y = M^2 Y.$$  \hspace{1cm} (38)

By writing

$$Y_\ell(y^i, z) = e^{(n-2)\mu|z|/2} Z(z) \Phi_\ell(y^i),$$  \hspace{1cm} (39)

where $\Phi_\ell(y^i)$ are the eigenfunctions of the scalar Laplacian $\Delta_0(S^{n-1})$ on the unit $S^{n-1}$ with eigenvalues $\ell(\ell + n - 1)$

$$\Delta_0(S^{n-1}) = \ell(\ell + n - 2) \Phi_\ell, \hspace{0.5cm} \ell = 0, 1, ...$$  \hspace{1cm} (40)

we get from eq. (38) that $Z(z)$ satisfies

$$-\frac{d^2}{dz^2} Z + \left(\frac{(n-2)^2}{4} \mu^2 + \ell(\ell + n - 2) \mu^2 - (n-2) \mu \delta(z)\right) Z = M^2 e^{-2\mu|z|} Z.$$  \hspace{1cm} (41)

The problem has been reduced to an one-dimensional Schrödinger equation with potential

$$V(z) = \left(\frac{(n-2)^2}{4} + \ell(\ell + n - 2)\right) \mu^2 - (n-2) \mu \delta(z).$$  \hspace{1cm} (42)

In general, an attractive potential $V(z) = -g^2/4$. We see that this bound state satisfies eq. (41) for $\ell = 0$ and $M = 0$. As a result, the four-dimensional massless graviton is just the unique bound state of the potential (42). To find the rest of the spectrum, we have to solve eq. (41) with appropriate boundary conditions. If we denote by $Z_+(z), Z_-(z)$ the solution for $z > 0, z < 0$, respectively, the boundary conditions are

$$Z_+(0) = Z_-(0) = Z(0),$$  \hspace{1cm} (43)

$$Z'_+(0) - Z'_-(0) = -\mu(n-2)Z(0),$$  \hspace{1cm} (44)

where prime denotes differentiation with respect to $z(=d/dz)$. In addition, there exist one more relation $Z$ has to satisfy, namely,

$$e^{-(n-2)\mu z} Z_+(z) Z'_+(z)|_{z=\infty} - e^{-(n-2)\mu z} Z_-(z) Z'_-(z)|_{z=-\infty} = 0,$$  \hspace{1cm} (45)

which is just the condition of conservation of the current $J^\mu = Z \partial^\mu Z$ on the transverse space, i.e.,

$$\int dz d^m y \partial_\mu \left( e^{-(n-2)\mu|z|} \sqrt{\hbar} Z \partial^\mu Z \right).$$  \hspace{1cm} (46)
Continuity of \( Z \) eq.\( (44) \) and the condition eq.\( (45) \), specify the solutions to be, up to a multiplicative constants,

\[
Z_+(z) = M^{(2-n)/2} J_\nu \left( \frac{M}{\mu} e^{-\mu z} \right),
\]
\[
Z_-(z) = M^{(2-n)/2} J_\nu \left( \frac{M}{\mu} e^{\mu z} \right),
\]

(47)

where \( J_\nu \) are the Bessel functions with

\[
\nu = \frac{2\ell + n - 2}{2}.
\]

(48)

The factor \( M^{(2-n)/2} \) in eq.\( (47) \) has been inserted in order the limit \( M \to 0 \) to give the zero-mode eigenfunction

\[
Z_0(z) \sim \exp^{-(n-2)\mu|z|}.
\]

(49)

The other solution to eq.\( (41) \), the second Bessel function \( Y_\nu \) fails to satisfy eq.\( (45) \). Finally, from the last condition eq.\( (44) \), we get

\[
\frac{M}{\mu} J'_\nu \left( \frac{M}{\mu} \right) = \frac{n-2}{2} J_\nu \left( \frac{M}{\mu} \right),
\]

(50)

which can be written, after using Bessel-function identities, as

\[
\frac{M}{\mu} J_{\ell+n/2} \left( \frac{M}{\mu} \right) = \ell J_{\ell-1+n/2} \left( \frac{M}{\mu} \right).
\]

(51)

Thus, the spectrum is \( M_{k,\ell} \) where \( M_{k,\ell} \) satisfies eq.\( (51) \). In particular, for \( \ell = 0 \), we find that \( M_{k,0} \) is

\[
M_{k,0} = \mu j_0^{(k)}, \quad M_{0,0} = 0, \quad k = 1, 2, \ldots,
\]

(52)

where \( j_0^{(k)} \) are the zeros of \( J_{n/2} \). Note that \( M_{0,0} = 0 \) corresponds to the bound state we found before. It should be noted that for \( \ell \neq 0 \), the value \( M = 0 \) which solves eq.\( (51) \), gives \( Z(z) = 0 \) and thus, there exist only zero eigenvalue \( M_{0,0} \) with corresponding eigenfunction eq.\( (43) \). The spectrum of \( \Delta_0 \) is given in table \( (53) \)

| eigenvalues of \( \Delta_0 \) | dim of \( SO(n-1) \) |
|-----------------------------|-------------------|
| \( j_0^{(k)} \)            | 1                 |
| \( M_{k,\ell}, \ell \neq 0 \) | \( \frac{(2\ell+n-2)(\ell+n-3)!}{(n-2)!\ell!} \) |

(53)

In particular, for the \( n = 2 \) case, where the transverse space is \( \Sigma_2 \) with metric eq.\( (13) \), eq.\( (11) \) is written as

\[
-\frac{d^2}{dz^2} Z + \ell^2 \mu^2 Z = M^2 e^{-2\mu|z|} Z.
\]

(54)
The solution is then

\[ Z_+(z) = J_\ell \left( \frac{M}{\mu} e^{-\mu z} \right), \quad Z_- (z) = J_\ell \left( \frac{M}{\mu} e^{\mu z} \right), \]  

(55)

while the eigenvalues (38) are specified by

\[ \frac{M}{\mu} J_{\ell+1} \left( \frac{M}{\mu} \right) = \ell J_\ell \left( \frac{M}{\mu} \right). \]  

(56)

The \( \ell = 0 \) tower consists of the zeroes of \( J_1 \) and the massless mode is the \( x = 0 \) of the equation \( J_1(x) = 0 \). The rest of the spectrum is obtained by solving eq. (56).

Concerning the operator \( \Delta_1 \), it is not difficult to verify that its eigenvalues \( M_1^2 \) are strictly positive, i.e., \( M_1^2 > 0 \) for \( n \geq 2 \). Indeed, from the eigenvalue problem

\[ -\nabla_p \nabla^p Y_m + R_{mp} Y^p = M_1^2 Y_m, \]  

(57)

we see, by multiplying both sides with \( Y^*_m \) and integrating over \( X \) that

\[ M_1^2 \geq \frac{1}{|Y^*_m Y_m|^2} \int R_{mp} Y^*_p Y^m, \quad |Y^*_m Y^m|^2 = \int Y^*_m Y^m. \]  

(58)

Since now \( R_{mp} \) is a strictly positive matrix, we see that there is no zero eigenvalue.

### 4.1 Graviton

In order to find the spectrum of small fluctuations around the background metric \( g_{MN} \) of (32) we write \( \hat{g}_{MN} = g_{MN} + \delta g_{MN} \) and we keep only linear terms in \( \delta g_{MN} = h_{MN} \) in the equation

\[ \hat{R}_{MN}(g + h) = 0. \]  

(59)

Then, we get that \( h_{MN}(x^\mu, y^i, z) \) satisfies the equation

\[ \delta R_{MN} = -\nabla_K \nabla^K h_{MN} + \frac{1}{2} R_{MA} h^A_N + \frac{1}{2} R_{NA} h^A_M - R^A_{MKN} h^K_A \]

\[ + \frac{1}{2} \nabla_M \nabla^A h_{NA} + \frac{1}{2} \nabla_N \nabla^A h_{MA} - \frac{1}{2} \nabla_M \nabla^M h^A_A, \]

(60)

where \( \nabla_M \) is the covariant derivative with respect to the background metric (32). We may express the components of \( h_{MN} \) as

\[ h_{\mu\nu}(x, y) = h_{\mu\nu}(x) Y(y), \]

\[ h_{\mu n}(x, y) = B_\mu(x) Y_n(y), \]

\[ h_{mn}(x, y) = V(x) Y_{mn}(y) + \frac{1}{n} g_{mn} U(x) Y(y), \]

(61)

where \( Y(y), Y_m(y) \) have been defined in eqs (38, 57), and \( Y_{mn}(y) \) is transverse traceless. We see from the expansion (61) that we get in four dimensions a symmetric tensor field
\[ h_{\mu \nu} \text{ which contains the graviton, a vector } B_\mu \text{ and the scalars } U, V. \] We are particular interested for massless four dimensional fields and we will examine if there are such massless modes. Due to the invariance

\[ \delta g_{MN} = \nabla_M \xi_N + \nabla_N \xi_M \tag{62} \]

we may impose \( n + 4 \) conditions on \( h_{MN} \) which we choose to be

\[ \nabla^m h_{m\nu} = 0, \quad \nabla^m h_{mn} = \frac{1}{n} g_{mn} \nabla^m h^k_k. \tag{63} \]

By using the expansion (61), we get from the \((\mu\nu)\) component of eq.(60)

\[ \nabla_\rho \nabla_\rho h_{\mu\nu} - \nabla_\nu \nabla_\rho h^{\rho \mu} - \nabla_\nu \nabla_\rho h^{\rho \nu} + \nabla_\mu \nabla_\nu h^{\rho \rho} = M^2 h_{\mu\nu}, \tag{64} \]

while from the \((\mu, n)\) components we get, among others,

\[ \nabla_\mu \nabla_\nu B_\nu = M_1^2 B_\nu, \tag{65} \]

Eq.(64) is the equation for the four-dimensional graviton while eq.(65) is the equation for the KK vector \( B_\mu \). Since \( M^2 \) is the eigenvalues of the scalar Laplacian in \( X^n \) which, as we have seen has a zero mode, a massless graviton always exists and it is the unique bound state in the attractive delta-function potential of eq.(42). The KK modes of the four-dimensional graviton have masses given in table (53). On the other hand, the KK vector \( B_\mu \) is massive since the operator \( \Delta_1 \) does not have a zero eigenvalue. Proceeding as above for the scalars, we find that they are massive. Thus, the only massless mode in four-dimensions of the higher-dimensional bulk graviton is the four-dimensional graviton.

The massive KK modes of the graviton will affect the Newton law as usual generating Yukawa-type corrections [4, 18, 19]. The gravitational potential where also the massive KK modes of the graviton are taken into account is given by [19]

\[ V(r) = -\frac{1}{r} \sum_k d_k e^{-M_k r}, \tag{66} \]

where \( d_k \) is the degeneracy of the \( k \)-th massive KK state. Note that the degeneracy is due to the \( SO(n - 1) \) invariance and the range is set by the mass of the first KK state which is proportional to \( \mu \).

### 4.2 Gauge fields

Let us now consider a \( U(1) \) gauge field \( A_M(x, y^i, z) = (A_\mu, A_i, A_z) \) in the bulk geometry. We will examine the spectrum which appears on the brane at \( z = 0 \) due to the bulk gauge field. The field equations for the gauge field is just the Maxwell equations

\[ \nabla^M F_{MN} = 0, \tag{67} \]
where $\nabla_M$ is the covariant derivative in the $n+4$-dimensional space-time and $F_{MN} = \nabla_M A_N - \nabla_N A_M$ is the field strength. Eq.(67) follows from the action

$$S_{n+1} = -\frac{1}{4g^2} \int d^{n+4}x F_{MN} F^{MN}.$$  \hspace{1cm} (68)

In terms of the gauge field $A_M$ and in the covariant gauge, eq.(67) is written as

$$-\nabla_M \nabla^M A_N + R^M N A_M = 0, \quad \nabla^M A_M = 0.$$  \hspace{1cm} (69)

By writing the components of the gauge field as

$$A_\mu(x, y, z) = a_\mu(x) Y(y, z), \quad A_m(x, y, z) = a(x) Y_m(y, z),$$  \hspace{1cm} (70)

and recalling eq.(27) for the Ricci tensor in the background (32), we get that $a_\mu, a$ satisfy

$$\nabla^2 a_\mu = M_0^2 a_\mu, \quad \nabla^2 a = M_1^2 a.$$  \hspace{1cm} (71)

where $M_0^2, M_1^2$ are the eigenvalues of the $\Delta_0, \Delta_1$ operators, respectively, i.e.,

$$-\nabla^2_{(n)} Y = M_0^2 Y.$$  \hspace{1cm} (72)

$$-\nabla^2_{(n)} Y_i + R_{ij}^{} Y_j = M_1^2 Y_i.$$  \hspace{1cm} (73)

Thus a massless four-dimensional photon exists if there exist a zero mode of the Laplace operator in eq.(72) while a massless four-dimensional scalar appears if the operator in eq.(73) has a zero modes. From the analysis of the previous sections we know that indeed the operator (72) has a unique zero eigenvalue. This eigenvalue corresponds to a massless photon in four-dimensions which is the unique bound state of the attractive delta-function potential in eq.(42). On top of this, there exist a tower of massive KK states given in table (53). On the other hand, the operator of eq.(73) does not have a zero mode. As a result, there is no bound state, no massless scalars thereof and all four-dimensional scalars coming from the components of $n+4$-dimensional vector appear massive.

5 Exponentially large extra dimensions

It has recently be proposed that the hierarchy problem, the unnatural smallness of the ration $m_{EW}/M_P$ of the electroweak scale $m_{EW} \sim 10^3$ GeV to the four-dimensional Planck scale $M_P \sim 10^{18}$ GeV can be resolved in a higher $n+4$-dimensional setting. In such a framework, $M_P$ is related to the $n+4$-dimensional Planck scale $M_{P(n+4)}$ by

$$M_P^2 = M_{P(n+4)}^2 V(X^n),$$  \hspace{1cm} (74)

where $V(X^n)$ is the volume of the internal space $X^n$. Usually, for a more or less isotropic space $X^n$ of characteristic scale $R$ we have

$$V(X^n) = \alpha R^n,$$  \hspace{1cm} (75)

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where $\alpha$ is an $R$-independent constant so that,

$$M_P^2 = M_{P(n+4)}^{n+2} R^n. \quad (76)$$

Thus, as have been pointed out in [4], taking $M_{P(n+4)} \sim m_{EW}$, the hierarchy problem is solved if the scale of the internal space $X^n$ is large. For two extra dimensions for example with $m_{EW} \sim 10^3$ GeV we get $R \sim 1$ mm. However, in such a scenario one expects that $R \sim M_{P(n+4)}^{-1}$ as this is the scale in the higher dimensional theory. Thus, the hierarchy $m_{EW} R \sim 10^{15}$ for two extra dimensions) has still to be explained. This hierarchy can be traced back to eq.(75), and as we will see here this is not in general the case. Namely, we will construct a vacuum configuration in which the volume of the internal space $V(X^n)$ although by dimensional reasons satisfy eq.(75), the constant $\alpha$ is an exponential function of $R$. In this case, even $R \sim M_{P(n+4)}^{-1}$, an exponentially large volume emerges so that no hierarchy $m_{EW} R$ appears.

We will consider again the metric (3) where now $\Lambda(z)$ is

$$\Lambda(z) = -\mu_1 |z + L| + \mu_2 |z - L|, \quad (77)$$

where $\mu_1, \mu_2$ are, as before, dimensionful constants and $2L$ is the distance between the two branes sited at $-L, L$. It is natural to assume that all scales, namely, $\mu_1, \mu_2, 1/a, 1/L$ are of the order of the six-dimensional Planck scale $M_{P(6)}$. The volume of the transverse space is finite if $\mu_1 > \mu_2$ and in this case we find

$$V = \frac{2\pi a \mu_1}{\mu_1^2 - \mu_2^2} e^{4\mu_2 L} + \frac{2\pi a \mu_2}{\mu_1^2 - \mu_2^2} e^{-4\mu_1 L}. \quad (78)$$

The first term in the expression above dominates and the four-dimensional Planck scale $M_P$ is then

$$M_P^2 = M_{P(6)}^4 V = M_{P(6)}^4 \frac{2\pi a \mu_1}{\mu_1^2 - \mu_2^2} e^{4\mu_2 L}. \quad (79)$$

For $\mu_1 = 2$ TeV $, \mu_2 = 1$ TeV, $1/a = 1$ TeV, the value $M_P \sim 10^{18}$ GeV for the four-dimensional Planck scale is obtained for $1/L = 15$ TeV. The masses of the KK states are now at the TeV scale as was originally proposed in [2].

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