Asymptotic results on the product of random probability matrices

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Abstract

I study the product of independent identically distributed $D \times D$ random probability matrices. Some exact asymptotic results are obtained. I find that both the left and the right products approach exponentially to a probability matrix (asymptotic matrix) in which any two rows are the same. A parameter $\lambda$ is introduced for the exponential coefficient which can be used to describe the convergent rate of the products. $\lambda$ depends on the distribution of individual random matrices. I find $\lambda = 3/2$ for $D = 2$ when each element of individual random probability matrices is uniformly distributed in $[0,1]$. In this case, each element of the asymptotic matrix follows a parabolic distribution function. The distribution function of the asymptotic matrix elements can be numerically shown to be non-universal. Numerical tests are carried out for a set of random probability matrices with a particular distribution function. I find that $\lambda$ increases monotonically from $\simeq 1.5$ to $\simeq 3$ as $D$ increases from 3 to 99, and the distribution of random elements in the asymptotic products can be described by a Gaussian function with its mean to be $\frac{1}{D}$.

Key words: random probability matrix, product, asymptotic behavior

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In recent years, there is an increasing interest in studying the properties of random matrices\textsuperscript{1}. Random matrices can be used to describe disordered systems, chaos, and biological problems, and in the statistical description of complex nuclei\textsuperscript{2,3}. There are two different kinds of problems related to random matrices. One of them is to study the statistical properties of a single random matrix. Wigner\textsuperscript{3} was the first to use a single large random matrix to explain the statistical behavior of levels in nuclear physics. The semicircular and the circular theorems\textsuperscript{1,3} had been obtained for the Gaussian unitary, Gaussian orthonormal, and Gaussian symplectic ensembles. The second problem is to study the statistical behavior of a product of random matrices. The product of random matrices has attracted much attention in most of recent works because many statistical problems in disordered systems and chaotic dynamical systems can be formulated as a study of such product. For the product of random matrices, there are beautiful Furstenberg\textsuperscript{4} and Oseledectheorems about the existence of Lyapunov characteristic exponents. However, a detailed analysis of the product of random matrices is very difficult due to the noncommutability of random matrices, and there are not so many general exact results about structures of the products of random matrices. Therefore, any exact results on the product of a particular type of random matrices should be interesting.

In this work, I present some exact results of the product of independent identically distributed random probability matrices. A probability matrix $T$ of $D \times D$ is defined as $T(ij) \geq 0$ and $\sum_{j=1}^{D} T(ij) = 1$, $i = 1, 2, ..., D$, where $T(ij)$ are the elements of $T$. Physically, such matrices can be used to describe the dynamics of a $D$ state classical system if $T_{t}(ij)$ is interpreted as the probability for the system in $i^{th}$ state jumping to $j^{th}$ state at time $t$. If the hopping process is stochastic, the evolution of the system is described by the product of the random probability matrices $T_{t}$. It is easy to show that the product of two probability matrices is still a probability matrix, i.e. $A = T_{1}T_{2}$ is a probability matrix provided $T_{1}$ and $T_{2}$ are two probability matrices of order $D \times D$. In this paper, I show that the product of independent identically distributed random probability matrices approaches exponentially, in terms of the number of matrices in the product, to a matrix in which any two rows are the
same. For 2 × 2 independent random probability matrices in which any element is uniformly distributed in [0, 1], I find that the asymptotic matrix elements have parabolic distribution. The main results are described by the following propositions.

**Proposition 1:** Let \( \{T_k\} \) be a set of \( D \times D \) independent identically distributed random probability matrices in which all elements are random and have the same distribution function. If \( A(n) = \prod_{k=1}^{n} T_k \equiv T_n T_{n-1} \cdots T_2 T_1 \) (the left product) and \( B(n) = \prod_{k=1}^{n} T_k \equiv T_1 T_2 \cdots T_{n-1} T_n \) (the right product), then

\[
\lim_{n \to \infty} A(n) = \begin{pmatrix}
a(1) & a(2) & \cdots & a(D) \\
a(1) & a(2) & \cdots & a(D) \\
\vdots & \vdots & \ddots & \vdots \\
a(1) & a(2) & \cdots & a(D)
\end{pmatrix}
\]

and

\[
\lim_{n \to \infty} B(n) = \begin{pmatrix}
a(1) & a(2) & \cdots & a(D) \\
a(1) & a(2) & \cdots & a(D) \\
\vdots & \vdots & \ddots & \vdots \\
a(1) & a(2) & \cdots & a(D)
\end{pmatrix},
\]

where \( a(i), (i = 1, 2, \ldots D) \), are positive random numbers with \( \sum_i a(i) = 1 \). However, the values of \( a's \) in the left product are fixed for a given sequence of random matrices while they keep changing in the right product.

Before we prove this proposition, let us look at a special case of \( D = 2 \). Let

\[
A(n) \equiv T_n \cdots T_2 T_1 = \begin{pmatrix}
y_n(1) & 1 - y_n(1) \\
y_n(2) & 1 - y_n(2)
\end{pmatrix},
\]

then, from \( A(n) = T_n A(n - 1) \), we obtain the following recursion relations for \( y_n(1) \) and \( y_n(2) \)

\[
y_n(1) = x(1)y_{n-1}(1) + (1 - x(1))y_{n-1}(2), \quad (4a)
\]

\[
y_n(2) = x(2)y_{n-1}(1) + (1 - x(2))y_{n-1}(2), \quad (4b)
\]
where $x(1)$ and $x(2)$ are the two independent random elements of $T_n$, i.e.,

$$T_n = \begin{pmatrix} x(1) & 1 - x(1) \\ x(2) & 1 - x(2) \end{pmatrix}.$$  

Therefore,

$$y_n(1) - y_n(2) = [x(1) - x(2)][y_{n-1}(1) - y_{n-1}(2)]$$

which gives

$$\left| \frac{y_n(1) - y_n(2)}{y_{n-1}(1) - y_{n-1}(2)} \right| = |x(1) - x(2)| \leq 1.$$  

(5)

Thus, we expect $|y_n(1) - y_n(2)|$ approaches to zero exponentially. If $x(1)$ and $x(2)$ are uniformly distributed in $[0, 1]$, then, for a large $n$,

$$y_n(1) - y_n(2) \sim e^{-\lambda n},$$  

(6)

with

$$\lambda = -<\ln|x(1) - x(2)|> = -\int_0^1 \int_0^1 \ln|x(1) - x(2)| dx(1) dx(2) = 3/2.$$  

Therefore, the left product of independent identically distributed $2 \times 2$ random probability matrices exponentially approaches to a matrix of the form

$$\begin{pmatrix} a_1 & 1 - a_1 \\ a_2 & 1 - a_2 \end{pmatrix},$$  

(7)

with $\lambda = 1.5$. Similarly, it is easy to show that the same conclusion can be made for a right product.

The above approach can be extended to the general cases. Without losing generality, we need only show that the values of elements in the first column of the product of random probability matrices approach to each other. Let

$$A_n \equiv \begin{pmatrix} y_n(1) \\ y_n(2) \\ \ldots \\ \ast \\ \ldots \\ y_n(D) \end{pmatrix} = T_n A(n-1).$$  

(8)
We want to show that $< |y_n(k) - y_n(1)| > \sim \exp^{-\lambda n}$, $\lambda > 0$. It is not hard to see that

$$y_n(i) - y_n(D) = \sum_{k=1}^{D-1} [x(i, k) - x(D, k)][y_{n-1}(k) - y_{n-1}(D)],$$

(9)

where $x(i, j)$ are matrix elements of $T_n$ which satisfies the conditions of a probability matrix. $\sum_j x(i, k) = 1$ is used in the above derivation. Define

$$z_n(i) = y_n(i) - y_n(D) \quad i = 1, \ldots, D - 1,$$

(10)
equation (9) can be written in the following matrix form

$$\begin{pmatrix}
  z_n(1) \\
  z_n(2) \\
  \vdots \\
  z_n(D - 1)
\end{pmatrix} = C_n
\begin{pmatrix}
  z_{n-1}(1) \\
  z_{n-1}(2) \\
  \vdots \\
  z_{n-1}(D - 1)
\end{pmatrix}$$

(11)

where $C_n$ is a $(D - 1) \times (D - 1)$ matrix related to probability matrix $T_n$. It is not difficulty
to show the following relation between $T_n$ and $C_n$

$$||T_n - \mu I|| = (1 - \mu) ||C_n - \mu I||,$$

(12)
that is the eigenvalues of matrix $C_n$ are $D - 1$ of eigenvalues of matrix $T_n$ (one of the
eigenvalues of $T_n$ with value 1 is excluded)\[4. It is well known that the magnitudes of
eigenvalues $\mu_i$, $i = 1, \ldots, D$, of a $D \times D$ probability matrix are not greater than 1, i.e.
$|\mu_i| \leq 1$. Furthermore, matrices $C'$s do not have any common eigenvectors\[4. Therefore,

$$< \begin{pmatrix}
  z_n(1) \\
  z_n(2) \\
  \vdots \\
  z_n(1)
\end{pmatrix} > \sim \exp^{-\lambda N} \quad (for \ a \ large \ N)$$

(13)
with \( \lambda > 0 \), i.e., the values of elements in the first column of the left product \( A \) approach exponentially to each other. It is easy to show that the same result is true for any other column of \( A \). Thus, relation (1) holds. Similarly, it can be shown that relation (2) also holds.

This result is not really surprising. It is known that the magnitudes of eigenvalues of a probability matrix are equal to or smaller than 1. A probability matrix always has an eigenvalue 1 with the corresponding eigenvector (mode)

\[
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

(There may exist other eigenvectors with eigenvalue 1). The eigenvalues of the product will either approach to 0 or stay at 1 when such matrices are multiplied together. Because each matrix is random and independent, these matrices are not commutable among themselves, and they do not in general have the same eigenvector except

\[
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

which will keep unchanged since its eigenvalue is equal to 1. Therefore, all other modes are mixed together, and decay with the multiplication. The relation (1) is then expected.

**Proposition 2:** Let \( \{T_k\} \) be a set of \( 2 \times 2 \) independent identically distributed random probability matrices in which all elements are uniformly distributed in \([0,1]\), proposition 1 guarantees that

\[
\lim_{n \to \infty} A(n) \equiv \lim_{n \to \infty} T_nT_{n-1} \cdots T_2T_1 = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}
\]

and

\[
\lim_{n \to \infty} B(n) \equiv \lim_{n \to \infty} T_1T_2 \cdots T_{n-1}T_n = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}
\]

Then \( a \) is a random variable whose distribution function is

\[
f(a) = 6a(1-a).
\]

(14)
To prove proposition 2, we notice that \( a \) is a random variable which obeys recursion relation

\[
a_n = x(1)a_{n-1} + x(2)(1 - a_{n-1}),
\]

where \( x(1) \) and \( x(2) \) are two independent random numbers uniformly distributed in \([0, 1]\). Since \( f(a) \) is the asymptotic distribution function of \( a_n \), i.e., \( n \to \infty \), \( a_n \) and \( a_{n-1} \) should have the same distribution function \( f(a) \) when \( n \to \infty \). Therefore, we can obtain the following equation in the integral form for distribution function \( f(a) \)

\[
f(a) = \int_0^1 \int_0^1 \int_0^1 f(b) \delta(a - bx_1 + bx_2 - x_2) dx_1 dx_2 db. \tag{16}
\]

Substitute \( \delta(a - bx_1 + bx_2 - x_2) \) by

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqa-bx_1+bx_2-x_2} dq \tag{17}
\]

and integrate over \( x_1 \) and \( x_2 \), equation \((16)\) becomes

\[
f(a) = \frac{1}{2\pi} \int_0^1 db \frac{f(b)}{b(b-1)} \int_{-\infty}^{\infty} dq \frac{e^{iqa} + e^{iqa-1} - e^{i(a-b)} - e^{i(a+b-1)}}{q^2}. \tag{18}
\]

Differentiate equation \((18)\) with respect to \( a \) twice and notice \( f(a) = f(1 - a) \), we can show that \( f(a) \) satisfies differential equation

\[
f'' = -\frac{2}{a(1-a)} f, \tag{19}
\]

with boundary conditions \( f(0) = f(1) = 0 \). Equation \((19)\) can be easily solved by power-series expansion method since \( a = 0 \) (or \( a = 1 \)) is a regular singular point. The solution of this equation (normalize to 1) is that of equation \((14)\).

I have carried out some numerical simulations to further confirm the results in the above propositions. Figure \( I \) is the distribution function of an element of the asymptotic matrix of the product of \( 2 \times 2 \) independent random probability matrices in which elements are uniformly distributed in \([0, 1]\). Solid line is the numerical result, and dashed line is the analytical expression \((14)\). They agree very well with each other. In order to check whether
the distribution function of the product depends on the distribution function of individual
random probability matrices, I also study the product of independent random probability
matrices in which elements are distributed in [0, 1] according to function

\[ f(x) = \int_0^1 \cdots \int_0^1 \delta(x - \frac{x_1}{\sum_1^D x_i}) dx_1 \cdots dx_D. \]  

Distribution (20) is chosen because it is easy to generate on a computer. Although I cannot
find the distribution function for the matrix elements of the asymptotic product analytically
in this case, numerical results can easily be obtained. Figure 2 is the single variable distri-
bution functions of random matrix elements in the right product of such random probability
matrices of order of 2 × 2 and 4 × 4. The dashed lines are the numerical results, and the
solid lines are the fits of Gaussian functions. The numerical results can be well described
by a Gaussian function with its mean to be $\frac{1}{D}$. Compare with that in figure 1, we can see
that the distribution function of the product of independent identically distributed random
probability matrices depends on the distribution of individual random matrices. In other
words, unlike the large number theorem for random numbers, the distribution function of
the product is not universal.

I check numerically the results in proposition 1 by using the random probability matrices
whose elements are independently, except the constrains of a probability matrix, distributed
in [0, 1] according to equation (20). I compute the decay of $<|A_n(1, 1) - A_n(2, 1)|>$ with
$n$, where $<\ldots>$ denotes ensemble average, and $A_n(i, j)$ are the elements of the product of
$n$ random matrices. Figure 3 is $\ln <|A_n(1, 1) - A_n(2, 1)|>$ vs. $\ln n$ for the left product of
3 × 3, 4 × 4, 8 × 8, 16 × 16, and 99 × 99 random matrices. 100 ensembles are used in the
numerical study. Figure 4 is a similar plot (as figure 1) for a right product. The exponential
decay of the quantity is clearly shown in these figures. Numerically, I find that $\lambda$ increases
monotonically from $\simeq 1.5$ to $\simeq 3$ as $D$ increases from 3 to 99.

In conclusion, I have shown that both left and right products of a sequence of independent
identically distributed random probability matrices exponentially approach to a probability
matrix in which all elements in any column vector are the same. An exponential exponent
is used to describe this approach rate. I also find that $\lambda$ increases monotonically from $\simeq 1.5$ to $\simeq 3$. as $D$ increases from 3 to 99 when the distribution function of individual random matrices is described by equation (20). I also find that $\lambda = 3/2$ for $D = 2$ when random matrix elements are uniformly distributed in $[0, 1]$. It is well known that at least one of the eigenvalues of a probability matrix is equal to 1 while the rest of them are distributed in $[0, 1]$. A large $D$ means that the product has more channel to decay to the stable structure (1) or (2). Thus it is expected that the decay is faster for large $D$, i.e., $\lambda$ increases with $D$. In order to understand the meaning of the results, let us look at a physical model system of $D$ states. Assume the system can move randomly from $i'$th state to $j'$th state with probability $T_t(ij)$ at time step $t$. If the system starts from an initial distribution, one might want to know the probability of the system in $i'$ state, i.e., the distribution function, after a long time. The questions may be whether there is a stable distribution (equilibrium state), and/or what it is if there is one. Obviously, the long time distribution(s) are the non-trivial left eigenstate(s) of the right product of the random matrices $T_t$. In equation (2), although the structure for a product will not change when $n$ is larger than certain value, elements $a(i)$ do change as another independent random probability matrix is multiplied to the product. Therefore, the system does not have a stable distribution as one expected since the dynamics of the system is a stochastic process, and transition probabilities keep change with time. Both of the right and left products, however, have a non-trivial unique right eigenstate with eigenvalue 1. The eigenstate takes the same value in each of its $D$ components. Unfortunately, I am not able to obtain any meaningful non-trivial results by applying the propositions to this simple model system. It will be interesting to find some interesting physical systems in which the propositions can be used to extract useful information. In contrast to the sum of independent random variables whose distribution is Gaussian no matter what is the distribution function of individual random variables, the distribution function of a element in the product of independent identically distributed random matrices depends on the distribution of individual random matrices. Therefore, the distribution function of the product of independent identically distributed random matrices is not universal.
The author would like to point out that proposition 2 results from the fruitful discussions with Prof. B. Derrida. Prof. Derrida made the major contribution to the result. The author thanks Prof. Derrida for useful suggestions and for reading the manuscript. The stimulating discussions with Prof. Y. Shapir, Dr. W. K. Ge, and Prof. E. Domany are also acknowledged. This work was supported by UPGC, Hong Kong, through RGC Grant.
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6 Add first $D - 1$ columns to the last column of $\|T_n - \mu I\|$. All elements of the last column are equal to $1 - \mu$ because $\sum_j x(i, j) = 1$. Take the common factor $1 - \mu$ out, and subtract the last row from the first $D - 1$ rows, then the left-top $(D - 1) \times (D - 1)$ block is exactly $\|C_n - \mu I\|$.

7 A probability matrix $T_i$ always has an eigenstate $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ with eigenvalue 1. The rest of eigenvalues and eigenstates are given by matrix $C_n$ which depends on the detail structure of $T_i$. In general, $C_n$ do not have common eigenstates.

8 See e.g., Feller W., \textit{An Introduction to Probability Theory and its Applications} (Wiley, New York, N.Y.) 1967.
FIGURES

FIG. 1. Distribution function of a random matrix elements in the right product of independent identically distributed $2 \times 2$ random probability matrices in which all elements are uniformly distributed in $[0, 1]$. The dash line is the numerical result, and the solid line is $f(a) = 6a(1 - a)$.

FIG. 2. Distribution function of an arbitrary random matrix element in the right product of independent identically distributed $2 \times 2$, $4 \times 4$ random probability matrices in which all elements are distributed in $[0, 1]$ according to equation (20). The dashed lines are the numerical results, and the solid lines are the fits of Gaussian functions.

FIG. 3. $\ln < |A_n(1, 1) - A_n(2, 1)| >$ vs $n$ of the left product of independent identically distributed random matrices of $3 \times 3$, $4 \times 4$, $8 \times 8$, $16 \times 16$, $99 \times 99$, with $100$ ensembles. $\lambda$'s can be obtained from the slopes. The slopes increase monotonically from $\lambda \simeq 1.5$ to $\lambda \simeq 3$, as $D$ changes from $D = 3$ to $D = 99$.

FIG. 4. $\ln < |A_n(1, 1) - A_n(2, 1)| >$ vs $n$ of the right product of independent identically distributed random matrices of $3 \times 3$, $4 \times 4$, $8 \times 8$, $16 \times 16$, $99 \times 99$. $\lambda$'s can be obtained from the slopes. The slopes increase monotonically from $\lambda \simeq 1.5$ to $\lambda \simeq 3$, as $D$ changes from $D = 3$ to $D = 99$. 

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