ON DYNAMICS IN A MEDIUM-TERM KEYNESIAN MODEL

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Abstract. This paper rigorously examines the (in)stability of limit cycles generated by Hopf bifurcations in a medium-term Keynesian model. The bifurcation equation of the model is derived and the conditions for stable and unstable limit cycles are presented. Numerical simulations are performed to illustrate the analytical results.

1. Introduction. In the theory of economic dynamics, the existence of periodic (or closed) orbits (including limit cycles) is viewed as the presence of persistent business cycles. The existence of periodic orbits is usually verified by the theory of dynamical systems (e.g., bifurcation theory).\(^1\) Taking this approach, a number of studies have demonstrated the presence of business cycles.\(^2\)

As is well known, limit cycles can be stable or unstable; stable limit cycles have inner and outer attracting domains, while unstable limit cycles have inner and outer repelling domains. In the theory of economic dynamics, both of them have important implications; it is indicated by the former that the state of an economy converges to persistent business cycles while by the latter that a stable region associated with “corridor stability” may exist.\(^3\) Thus, whether a limit cycle is stable

\(^1\)For detailed expositions on mathematical tools utilized in economic dynamics, see, for example, Lorenz [17] or Gandolfo [8]. In contrast to the methods expounded in Lorenz [17] or Gandolfo [8], which were used in many papers studying business cycles, we use more mathematically rigorous methods presented in Bibikov [4] or Wiggins [35].

\(^2\)The first application of the Hopf bifurcation theorem to economic theory can be found in Torre [34]. Recently, this theorem has been utilized for verifying the existence of business cycles in Keynesian models of, for example, Murakami [21, 22], Asada et al. [1, 2], Murakami and Asada [28] and Murakami and Zimka [30].

\(^3\)“Corridor stability” means a phenomenon in which the economy has a tendency to return to its equilibrium when perturbed by small shocks but to get away from it when perturbed by large shocks. In the theory of economic dynamics, the most inner unstable limit cycle surrounding an equilibrium point can be interpreted as the boundary of such a “corridor” because solution paths starting inside the limit cycle converge to the equilibrium while those starting outside do not.
or not is of great importance in economic dynamics. Nevertheless, it is not rigorously examined in many studies, especially those dealing with high-dimensional dynamical systems.\(^4\)

The purpose of this paper is to analyze rigorously the (in)stability of limit cycles in Murakami’s [21] medium-term Keynesian model (with dimensionality of four), which can be viewed as a hybrid one of Kaldor’s [11] and Tobin’s [32] (standard) Keynesian models. Murakami [21] verified the existence of limit cycles in three Keynesian models with dimensionality of two, three and four (the first two models are special cases of the four-dimensional one), but the question of their stability was only investigated in the two-dimensional model.\(^5\) This paper presents a rigorous investigation of the stability of periodic orbits generated by Hopf bifurcations in the four-dimensional model of Murakami [21] to evaluate the plausibility of business cycles in economies (with Keynesian features).\(^6\)

This paper is organized as follows. Section 2 sets up a medium-term Keynesian model. Section 3 analyzes the Keynesian model. In this section, we establish the existence and uniqueness of equilibrium and examine the existence and stability of limit cycles generated by Hopf bifurcations.\(^7\) Section 4 performs numerical simulations to evaluate the validity of Section 3. Section 5 concludes this paper.

2. The model. This section presents a medium-term Keynesian model based on Murakami [21]. The model consists of the four differential equations describing quantity adjustment, capital formation, price inflation and revisions of inflation expectations.

2.1. Quantity adjustment. Based on the Keynesian theory, it is assumed that aggregate *ex ante* (gross) investment \(I\) and aggregate *ex ante* (gross) saving \(S\) are represented as follows:

\[
I = I(Y, K, \rho^e),
\]

\[
S = S(Y),
\]

where \(Y, K\) and \(\rho^e\) stand for aggregate real output (or aggregate real income), aggregate stock of capital and the expected real rate of interest, respectively.\(^8\)

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\(^4\)For recent exceptional investigations of the stability of periodic orbits in high-dimensional dynamic models (with dimensionality of three or more) in economics, see Asada et al. [1, 2] and Murakami and Zimka [30].

\(^5\)Murakami [21] established the existence of stable limit cycles in the two-dimensional model but not in the three- or four-dimensional model. Certainly, this study aimed to prove the existence of periodic orbits, but it was more interested in the effects of price flexibility and of inflation expectations on economic stability (i.e., the local asymptotic stability of equilibrium) than in the stability of periodic orbits.

\(^6\)Murakami and Zimka [30] thoroughly studied a rigid-price two-sector Keynesian model (composed of the consumption-good and investment-good sectors) to verify that a stable limit cycle is generated, while we will examine a flexible-price one-sector Keynesian model to find that an unstable limit cycle appears.

\(^7\)Hopf bifurcations generating stable and unstable limit cycles are called *supercritical* and *subcritical* ones, respectively (cf. Marsden and McCracken [19]). This paper provides the conditions for supercritical and subcritical Hopf bifurcations. The stability of limit cycles can also be confirmed by proving (not numerically but analytically) the uniqueness, as well as existence, of a limit cycle. For recent studies on economic dynamics taking this approach, see Murakami [24, 25, 26, 27].

\(^8\)For mathematical derivations of the Keynesian (or non-Walrasian) investment and saving functions, see Murakami [22, 23].
It is also postulated that the nominal rate of interest $r$ is a function of aggregate real output $Y$ and the price level $P$ as follows:

$$r = R(Y, P).$$  \hspace{1cm} (3)$$

The function $R$ can be given as a solution of the following equilibrium condition of the money market:

$$\frac{M}{P} = L(r, Y),$$  \hspace{1cm} (4)$$

where $L$ is the liquidity preference (or real money demand) function and $M$ stands for the constant supply of money set by the monetary authority.\(^9\)

With the interest rate function (3), the (expected) real rate of interest $\rho^e$ can be expressed as follows:

$$\rho^e = R(Y, P) - \pi^e,$$  \hspace{1cm} (5)$$

where $\pi^e$ stands for the expected rate of inflation.

The Keynesian principle of effective demand implies that aggregate real output is adjusted to aggregate effective demand.\(^10\) It is then postulated that aggregate real output is varied in response to its difference from aggregate effective demand, i.e., aggregate excess demand,\(^11\) in the following fashion:

$$\dot{Y} = \alpha(I - S),$$  \hspace{1cm} (6)$$

which can, by (1)-(5), be written as

$$\dot{Y} = \alpha[I(Y, K, R(Y, P) - \pi^e) - S(Y)],$$  \hspace{1cm} (6)$$

where $\alpha$ is a positive constant which is the coefficient of quantity (output) adjustment.

2.2. Capital formation. Aggregate stock of capital $K$ is changed by aggregate investment net of capital depreciation. It then follows from (1) that aggregate capital formation can be described by

$$\dot{K} = I(Y, K, R(Y, P) - \pi^e) - \delta K,$$  \hspace{1cm} (7)$$

where $\delta$ is a positive constant which represents the rate of capital depreciation.\(^13\)

\(^9\)For a sufficient condition for the (nominal) rate of interest, as a solution of (4), to be a well-defined function of the other relevant variables, see Murakami [21, Appx. A].

\(^10\)This is consistent with the dynamic interpretation of Keynes’ [12] principle of effective demand proposed by Leijonhufvud [14] and Tobin [33].

\(^11\)Aggregate effective demand $E$ is composed of aggregate consumption and aggregate investment:

$$E = C + I.$$ 

Also, aggregate real income is the sum of aggregate consumption and aggregate saving:

$$Y = C + S,$$

It then follows that aggregate excess demand is equal to the difference between aggregate investment and aggregate saving:

$$E - Y = I - S.$$ 

\(^12\)Throughout this paper, $\dot{x}$ stands for the time derivative of $x$, i.e., $\dot{x} \equiv dx/dt$.

\(^13\)This formalization identifies ex ante (planned) investment (net of capital depreciation) with ex post (realized) capital formation. Taking account of inventory adjustment, however, they are not necessarily equal to each other. For a more general formalization in the light of this issue, see Murakami [21]. Note that our analysis will remain almost unaltered even if this modification is made.
2.3. **Price inflation.** The natural rate hypothesis on the Phillips curve indicates that the rate of inflation, or the rate of change in the price level $P$, is assumed to be determined, in the following way, by the gap between aggregate real output and its natural rate level and the expected rate of inflation $\pi^e$: \[ \frac{\dot{P}}{P} = h(Y - Y^*) + \pi^e, \] (8) where $Y^*$ is a positive constant which stands for the level of aggregate real output corresponding to the natural rate of unemployment.

2.4. **Revisions of inflation expectations.** Following the adaptive expectations hypothesis, we assume that the expected rate of inflation $\pi^e$ is revised adjusting to the current rate of inflation $\frac{\dot{P}}{P}$ in the following fashion: \[ \dot{\pi}^e = \beta \left( \frac{\dot{P}}{P} - \pi^e \right), \] (9) where $\beta$ is a positive constant which is the adjustment coefficient in revisions of expectations. After equation (8) is substituted, this equation can be reduced to \[ \dot{\pi}^e = \beta h(Y - Y^*). \] (9)

2.5. **Full model: Model (K).** The full system of equations to be analyzed can be given as follows:

\[ \dot{Y} = \alpha[I(Y, K, R(Y, P) - \pi^e) - S(Y)], \] (6)
\[ \dot{K} = I(Y, K, R(Y, P) - \pi^e) - \delta K, \] (7)
\[ \dot{P} = h(Y - Y^*) P + P \pi^e, \] (8)
\[ \dot{\pi}^e = \beta h(Y - Y^*). \] (9)

The system of equations (6)-(9) is denoted by “Model (K)” (to signify “Keynesian”).

The following reasonable assumption is made on the functions $I$, $S$, $R$ and $h$.

**Assumption 1.** The real-valued functions $I$, $S$, $R$ and $h$ are of class $C^6$ with $I_Y > 0$, $I_K < 0$, $I_{\rho^e} < 0$, $S_Y > 0$, $R_Y > 0$, $R_P > 0$. $h(0) = 0$, $h' > 0$. 

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14Our formalization is consistent with Phelps’ [31] classic one.

15There are two major reasons for adopting the adaptive expectations hypothesis. The first one is that the rational expectations hypothesis, which is a representative one in modern macroeconomics, may not be supported in nonlinear dynamic models like ours; as Lorenz [17] and Gandolfo [8] argued, even if people fully understand the structure of the nonlinear model representing the state of an economy, they cannot form the rational expectations or precisely predict the future state of the economy because, in nonlinear dynamical systems, in general, the exact solution cannot be calculated and chaos may also occur. In this respect, the adaptive expectations hypothesis, which may seem naive from a viewpoint of rationality, is a proper one in our model. The second reason is that this hypothesis has been adopted as a realistic one in a lot of related works (cf. Tobin [32]). For details, see also Murakami [21, 24].

16Model (K) can be viewed as a synthesis of Kaldor’s [11] and Tobin's [32] models because Kaldor’s [11] consists of (6) and (7) while Tobin’s [32] of (6), (8) and (9).

17Throughout this paper, $F_x$ denotes the partial derivative of $F$ with respect to $x$, i.e., $F_x = \partial F/\partial x$. 
Conditions (10)-(13) are all standard in economic analysis. The assumption that all the relevant functions are of class $C^{18}$ (i.e., at least six-times continuously differentiable) is legitimate for the bifurcation analysis in the next section.\footnote{For periodic orbits generated by Hopf bifurcations to be of class $C^1$, it is only necessary to assume that the relevant functions are of class $C^2$ (cf. Marsden and McCracken [19, Sect. 5A]). Note that Murakami [21] only made the $C^2$ assumption for the four-dimensional model to verify the existence of periodic orbits, not the stability of them.}

3. Analysis. This section analyzes Model (K) to prove the existence and uniqueness of equilibrium and examine the existence and stability of a limit cycle generated by a Hopf bifurcation.

3.1. Existence and uniqueness of equilibrium. An equilibrium point of Model (K) is defined as a point $(Y, K, P, \pi^e) \in \mathbb{R}^3_{++} \times \mathbb{R}$ at which $Y = K = P = \pi^e = 0.\footnote{Throughout this paper, $\mathbb{R}^n$, $\mathbb{R}_{++}^n$ and $\mathbb{R}_{+++}^n$ denote the $n$-dimensional Euclidean space, the subspace of $\mathbb{R}^n$ composed of all nonnegative vectors and the subspace of $\mathbb{R}^n$ composed of all strict positive vectors, respectively.}$

It then follows that an equilibrium point of Model (K), $(Y, K, P, \pi^e) \in \mathbb{R}^3_{++} \times \mathbb{R}$, can be defined as a solution of the following simultaneous equations:

\begin{align}
0 &= I(Y, K, R(Y, P) - \pi^e) - S(Y), \\
0 &= I(Y, K, R(Y, P) - \pi^e) - \delta K, \\
0 &= h(Y - Y^*)P + P\pi^e, \\
0 &= h(Y - Y^*).
\end{align}

\begin{align}
(14) & \quad (15) & \quad (16) & \quad (17)
\end{align}

It is seen from (16) and (17) that, under Assumption 1, the equilibrium values of $Y$ and $\pi^e$ are equal to $Y^*$ and 0, respectively. It then follows from (14) and (15) that equilibrium values of $K$ and $P$ are given by

\begin{align}
0 &= I(Y^*, K, R(Y^*, P)) - S(Y^*), \\
0 &= I(Y^*, K, R(Y^*, P)) - \delta K.
\end{align}

\begin{align}
(18) & \quad (19)
\end{align}

Thus, an equilibrium point of Model (K) exists if and only if simultaneous equations (18) and (19) have a solution $(K^*, P^*) \in \mathbb{R}^2_{++}$.

To ensure the existence of an equilibrium point of Model (K), we introduce two assumptions. The first assumption is as follows.

Assumption 2. The following condition is satisfied:

\begin{equation}
\lim_{K \to \infty} [I(Y^*, K, R(Y^*, 0)) - \delta K] < 0 < \lim_{P \to \infty} I(Y^*, 0, R(Y^*, P)).
\end{equation}

\begin{equation}
(20)
\end{equation}

Assumption 2, under Assumption 1, implies that for every $P > 0$,\footnote{Note that Murakami [21] only made the $C^2$ assumption for the four-dimensional model to verify the existence of periodic orbits, not the stability of them.}

\begin{equation}
\lim_{K \to \infty} [I(Y^*, K, R(Y^*, P)) - \delta K] \leq \lim_{R \to \infty} [I(Y^*, K, R(Y^*, 0)) - \delta K] < 0 < \lim_{P \to \infty} I(Y^*, 0, R(Y^*, P)) \leq I(Y^*, 0, R(Y^*, P)).
\end{equation}

\begin{equation}
(21)
\end{equation}

From an economic viewpoint, condition (21), implied by Assumptions 1 and 2, indicates that aggregate net investment (aggregate gross investment minus capital depreciation) is positive (resp. negative) when the level of aggregate capital stock is sufficiently low (resp. high), regardless of the price level (provided that aggregate income and the expected rate of inflation are both at the equilibrium values). Assumption 2 can then be interpreted to imply that shortage (resp. redundancy) of aggregate capital stock induces positive (resp. negative) aggregate capital formation.

Due to $I_K - \delta < 0$ (cf. Assumption 1), it follows from (21) (and the implicit function theorem) that there exists a continuously differentiable function $Q : \mathbb{R}^{++} \to \mathbb{R}^{++}$ such that

$$0 = I(Y^*, Q(P), R(Y^*, P)) - \delta Q(P),$$

(22)

$$Q'(P) = -\frac{I_{Iu}(Y^*, Q(P), R(Y^*, P))}{I_K(Y^*, Q(P), R(Y^*, P))} - \delta R_P(Y^*, P) < 0,$$

(23)

where the inequality holds under Assumption 1.

The second assumption is as follows.

**Assumption 3.** The following condition is satisfied:

$$\lim_{P \to \infty} I(Y^*, Q(P), R(Y^*, P)) < S(Y^*) < I(Y^*, Q(0), R(Y^*, 0)),$$

(24)

where $Q$ is defined by (22) and (23).

We can easily provide an economic interpretation for Assumption 3. Condition (24) means that aggregate investment falls short of (resp. exceeds) aggregate saving when the price level is sufficiently high (resp. low) (for the equilibrium levels of aggregate income and the expected rate of inflation). In this sense, Assumption 3 can be interpreted to imply that the Keynes effect is strong enough.

Assumption 3, combined with Assumption 1, implies that there exists a unique $P > 0$, denoted by $P^*$, that satisfies (18) because it is known from Assumption 1 and (23) that

$$I_K(Y^*, Q(P), R(Y^*, P))Q'(P) + I_{Iu}(Y^*, Q(P), R(Y^*, P))R_P(Y^*, P) = \delta I_{Iu}(Y^*, Q(P), R(Y^*, P)) - \delta R_P(Y^*, P) < 0.$$

Let $K^* = Q(P^*) > 0$. It then follows from (22) that

$$0 = I(Y^*, K^*, R(Y^*, P^*)) - S(Y^*),$$

$$0 = I(Y^*, K^*, R(Y^*, P^*)) - \delta K^*.$$

Thus, we can draw the following conclusion on the existence and uniqueness of equilibrium in Model (K).

**Proposition 1.** Let Assumptions 1-3 hold. Then, there exists a unique equilibrium point of Model (K), $(Y^*, K^*, P^*, 0) \in \mathbb{R}^3_{++} \times \mathbb{R}$.

3.2. **Existence of a periodic orbit.** For the analysis that follows, the following reasonable assumption is introduced.

**Assumption 4.** The following conditions are satisfied (at the unique equilibrium):\(^{21}\)

$$I_Y^* + I_{Iu}^* R_Y^* < S_Y^*,$$

(25)

$$\delta < R_P^* P^*.$$

(26)

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\(^{20}\)The “Keynes effect” indicates the negative effect of a rise in the price level on aggregate investment (due to the positive one of a rise in the price level on the rate of interest and the negative one of a rise in the rate of interest on aggregate investment). This effect is named after Keynes [12, Chap. 19].

\(^{21}\)Throughout this paper, $F_x^*$ denotes $F_x$ evaluated at the unique equilibrium.
Condition (25) means that the marginal propensity to invest (including the indirect marginal effect through a change in the real rate of interest) evaluated at the unique equilibrium \( I^*_Y + I^*_p R^*_Y \) is less than that to save \( S^*_Y \) and implies that the Keynesian stability condition, which is a standard assumption in macroeconomics (cf. Marglin and Bhaduri [18]), holds at equilibrium. Condition (26) is, on the other hand, considered to be satisfied in practice because the rate of capital depreciation \( \delta \) is small enough (cf. Sect. 4). Assumption 4 can thus be justified from economic viewpoints.

The Jacobian matrix of Model (K) evaluated at the unique equilibrium, denoted by \( J^*_{(K)} \), is given as follows:

\[
J^*_{(K)} = \begin{pmatrix}
\alpha(I^*_Y + I^*_p R^*_Y - S^*_Y) & \alpha I^*_K & \alpha I^*_p R^*_p & -\alpha I^*_p \\
I^*_Y + I^*_p R^*_Y & I^*_K - \delta & I^*_p R^*_p & -I^*_p \\
\lambda^*_Y & 0 & 0 & I^*_p \\
\beta h^*_v & 0 & 0 & 0
\end{pmatrix},
\]

(27)

The characteristic equation associated with the Jacobian matrix \( J^*_{(K)} \) is then given by

\[
\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0,
\]

(28)

where

\[
a_1 = \alpha(S^*_Y - I^*_Y - I^*_p R^*_Y) + \delta - I^*_K > 0, \tag{29}
\]

\[
a_2 = \alpha[\beta I^*_p h^*_v + (\delta - I^*_K) S^*_Y - \delta I^*_Y - I^*_p (\delta R^*_Y + R^*_p h^*_v P^*)], \tag{30}
\]

\[
a_3 = \alpha[\beta I^*_p (\delta - R^*_p P^*) - \delta I^*_p (R^*_p P^*) h^*_v] > 0, \tag{31}
\]

\[
a_4 = -\alpha \beta \delta I^*_p R^*_p h^*_v P^* > 0. \tag{32}
\]

The inequalities hold under Assumptions 1 and 4.

It is seen from Liu [16, p. 252, Theorem] that the characteristic equation (28) has a pair of purely imaginary roots and two roots with negative real parts if

\[
a_1 > 0, \quad a_3 > 0, \quad a_4 > 0,
\]

\[
(a_1 a_2 - a_3) a_3 - a_1^2 a_4 = 0,
\]

(33)

which can, with (29), (31) and (32) taken into consideration, be reduced to

\[
(a_1 a_2 - a_3) a_3 - a_1^2 a_4 = b_0 \beta^2 + b_1 \beta + b_2 = 0,
\]

(34)

where \( b_j, \ i = 0, 1, 2 \), are independent from \( \beta \) and \( b_0 \) and \( b_2 \) are given by

\[
b_0 = \alpha^2 \left[ \alpha(S^*_Y - I^*_Y - I^*_p R^*_Y) + R^*_p P^* - I^*_K \right] (I^*_p h^*_v)^2 (\delta - R^*_p P^*) < 0,
\]

(35)

\[
b_2 = \alpha^2 \left[ \alpha(S^*_Y - I^*_Y - I^*_p R^*_Y) - I^*_K \right] (I^*_p h^*_v)^2 (\delta - R^*_p P^*)
\]

\[
- \alpha \left[ \alpha(S^*_Y - I^*_Y - I^*_p R^*_Y) + \delta - I^*_K \right] (\delta - I^*_K) S^*_Y
\]

\[
- \delta (I^*_Y + I^*_p R^*_Y) \right] (I^*_p h^*_v)^2 (\delta - R^*_p P^*) > 0.
\]

It can also be verified by the same theorem of Liu [16] that a Hopf bifurcation arises if the following condition holds for such a \( \beta > 0 \) that satisfies (33):

\[
\frac{d}{d \beta}(b_0 \beta^2 + b_1 \beta + b_2) = 2b_0 \beta + b_1 \neq 0.
\]

(36)

---

22 Throughout this paper, \( h^*_v \) stands for \( h'(0) \).

23 The value of \( b_1 \) is unnecessary for our analysis and omitted.
It follows from (34) and (35) that the quadratic equation of \( \beta \) (33) has positive and negative roots. Denoting the positive root by \( \beta^* \), it is seen that condition (33) is fulfilled for \( \beta = \beta^* > 0 \). Moreover, since \( \beta^* \) is a simple root of (33), condition (36) is satisfied for \( \beta = \beta^* > 0 \). Thus, Liu’s [16] criterion is satisfied for \( \beta = \beta^* \).

We have thus proved the following proposition on the existence of a periodic orbit by way of a Hopf bifurcation.

**Proposition 2.** Let Assumptions 1-4 hold. Then, there exists a \( \beta^* > 0 \) such that Model \((K)\) admits a Hopf bifurcation generating a periodic orbit, if \( \beta \) is sufficiently close to \( \beta^* \).

### 3.3. Stability and instability of a limit cycle.

Given the unique equilibrium \( (Y^*, K^*, P^*, 0) \) (cf. Proposition 1) and the bifurcation value of \( \beta \), or \( \beta^* \) (cf. Proposition 2), Model \((K)\) is transformed with the variables \( \tilde{Y} = Y - Y^* \), \( \tilde{K} = K - K^* \), \( \tilde{P} = P - P^* \), \( \tilde{\pi} = \pi^* \) and \( \mu = \beta - \beta^* \) as follows:

\[
\begin{align*}
\dot{\tilde{Y}} &= \alpha [I(\tilde{Y} + Y^*, \tilde{K} + K^*, R(\tilde{Y} + Y^*, \tilde{P} + P^*) - \tilde{\pi}^*) - S(\tilde{Y} + Y^*)], \\
\dot{\tilde{K}} &= I(\tilde{Y} + Y^*, \tilde{K} + K^*, R(\tilde{Y} + Y^*, \tilde{P} + P^*) - \tilde{\pi}^*) - \delta(\tilde{K} + K^*), \\
\dot{\tilde{P}} &= h(\tilde{Y})(\tilde{P} + P^*) + (\tilde{P} + P^*)\tilde{\pi}^*, \\
\dot{\tilde{\pi}} &= (\mu + \beta^*)h(\tilde{Y}).
\end{align*}
\]

(K1)

The transformed model is denoted by “Model \((K_1)\).”

Let \( x = (\tilde{Y}, \tilde{K}, \tilde{P}, \tilde{\pi})^T \) and define \( u = (u_1, u_2, u_3, u_4)^T \) as follows:

\[
x = Mu,
\]

(37)

where \( M \) consists of the eigenvectors associated with the eigenvalues of \( J^*_{(K)} \) given in (27).

We denote all the eigenvalues of \( J^*_{(K)} \) by \( \lambda_j \), \( j = 1, 2, 3, 4 \). Since two of them are a pair of purely imaginary ones and the others have negative real parts for \( \beta = \beta^* \) or \( \mu = 0 \), we may have \( \lambda_1 = i\omega_0 \), \( \lambda_2 = -i\omega_0 \) and \( \text{Re}\lambda_j < 0 \), \( j = 3, 4 \), for \( \mu = 0 \), where \( \omega_0 \) is a positive constant determined by the relevant partial derivatives and parameters. It then follows from (37) that Model \((K_1)\) can be reduced to the following form with the Jordan linear approximation matrix:

\[
\begin{pmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3 \\
\dot{u}_4
\end{pmatrix} = \begin{pmatrix}
i\omega_0 & 0 & 0 & 0 \\
0 & -i\omega_0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix} + \begin{pmatrix}
G_1(u_1, u_2, u_3, u_4, \mu) \\
G_2(u_1, u_2, u_3, u_4, \mu) \\
G_3(u_1, u_2, u_3, u_4, \mu) \\
G_4(u_1, u_2, u_3, u_4, \mu)
\end{pmatrix},
\]

(K2)

where \( u_2 = \pi_1 \) and \( G_2 = \pi_1 \). This model is denoted by “Model \((K_2)\).”

For a transformation of Model \((K_2)\), the following vector \( z = (z_1, z_2, z_3, z_4)^T \) is introduced:

\[
u = z + f(z_1, z_2, \mu),
\]

(38)

where the vector function \( f = (f_1, f_2, f_3, f_4)^T \) is given by

\[
f_j(z_1, z_2, \mu) = \sum_{m_1 + m_2 + m \geq 2, m = 0,1}^{4-2m} f^{(m_1, m_2, m)}(z_1^m z_2^m \mu^m),
\]

(39)

---

24 Throughout this paper, the superscript \( T \) represents the transpose operator.

25 Throughout this paper, the over-line (\( \bar{\pi} \)) represents the complex conjugate.
for \( j = 1, 2, 3, 4 \).

Substituting (38) and (39), we can transform Model (K2) into its partial normal form on the center manifold, denoted by “Model (K3)”: 
\[
\begin{align*}
\dot{z}_1 &= (i\omega_0 + \epsilon_1\mu)z_1 + \epsilon_2 z_1^2 z_2 + U^0(z_1, z_2, z_3, z_4, \mu) + \tilde{U}(z_1, z_2, z_3, z_4, \mu), \\
\dot{z}_2 &= (-i\omega_0 + \epsilon_1\mu)z_2 + \epsilon_2 z_2 z_1^2 + \overline{U}^0(z_1, z_2, z_3, z_4, \mu) + \overline{U}(z_1, z_2, z_3, z_4, \mu), \\
\dot{z}_3 &= \lambda_3 z_3 + V_1^0(z_1, z_2, z_3, z_4, \mu) + \tilde{V}_1(z_1, z_2, z_3, z_4, \mu), \\
\dot{z}_4 &= \lambda_4 z_4 + V_2^0(z_1, z_2, z_3, z_4, \mu) + \tilde{V}_2(z_1, z_2, z_3, z_4, \mu),
\end{align*}
\]

\text{(K3)}

where \((\star)^0(z_1, z_2, 0, 0, \mu) = 0\) and \((\star)(\sqrt{|\mu|}z_1, \sqrt{|\mu|}z_2, \sqrt{|\mu|}z_3, \sqrt{|\mu|}z_4, \mu) = O((\sqrt{|\mu|})^5)\).

The resonant terms \(\epsilon_1\) and \(\epsilon_2\) in Model (K3) are given by
\[
\begin{align*}
\epsilon_1 &= \frac{\partial^2 G_1}{\partial u_1 \partial \mu}, \\
\epsilon_2 &= -\frac{1}{2i\omega_0} \frac{\partial^2 G_1}{\partial u_1^2} \frac{\partial^2 G_1}{\partial u_2 \partial \mu} + \frac{1}{i\omega_0} \frac{\partial^2 G_1}{\partial u_1 \partial u_2} \frac{\partial^2 G_2}{\partial u_1 \partial \mu} \\
&\quad + \frac{1}{6i\omega_0} \frac{\partial^2 G_1}{\partial u_2^2} \frac{\partial^2 G_2}{\partial u_1 \partial \mu} - \frac{1}{\lambda_3} \frac{\partial^2 G_1}{\partial u_1 \partial u_3} \frac{\partial^2 G_2}{\partial u_1 \partial \mu} \\
&\quad - \frac{1}{\lambda_4} \frac{\partial^2 G_1}{\partial u_1 \partial u_3} \frac{\partial^2 G_4}{\partial u_1 \partial \mu} + \frac{1}{2(2i\omega_0 - \lambda_3)} \frac{\partial^2 G_1}{\partial u_2 \partial u_4} \frac{\partial^2 G_3}{\partial u_1^2} \\
&\quad + \frac{1}{2(2i\omega_0 - \lambda_4)} \frac{\partial^2 G_1}{\partial u_2 \partial u_4} \frac{\partial^2 G_4}{\partial u_1^2} + \frac{1}{2} \frac{\partial^2 G_1}{\partial u_2^2}.
\end{align*}
\]

where all the derivatives are evaluated at \((u_1, u_2, u_3, u_4, \mu) = (0, 0, 0, 0, 0)\).

Utilizing the polar coordinate transformations \(z_1 = \rho \exp(i\phi)\) and \(z_2 = \rho \exp(-i\phi)\), Model (K3) can further be reduced to the following “Model (K*)”: 
\[
\begin{align*}
\dot{\rho} &= \rho(\zeta \rho^2 + \theta \mu) + \Psi^0(\rho, \phi, z_3, z_4, \mu) + \tilde{\Psi}(\rho, \phi, z_3, z_4, \mu), \\
\dot{\phi} &= \omega_0 + \kappa \mu + \iota \rho^2 + \frac{1}{\rho} \{\Phi^0(\rho, \phi, z_3, z_4, \mu) + \tilde{\Phi}(\rho, \phi, z_3, z_4, \mu)\}, \\
\dot{z}_3 &= \lambda_3 z_3 + W_1^0(\rho, \phi, z_3, z_4, \mu) + \tilde{W}_1(\rho, \phi, z_3, z_4, \mu), \\
\dot{z}_4 &= \lambda_4 z_4 + W_2^0(\rho, \phi, z_3, z_4, \mu) + \tilde{W}_2(\rho, \phi, z_3, z_4, \mu),
\end{align*}
\]

\text{(K*)}

where \(\zeta = \text{Re} \epsilon_2, \theta = \text{Re} \epsilon_1, \kappa = \text{Im} \epsilon_1\) and \(\iota = \text{Im} \epsilon_2\) for (40) and (41).

It is seen from Model (K*) that the bifurcation equation of Model (K) (cf. Bibikov [4, Chap. 1, Sect. 7]) is given by
\[
\zeta \rho^2 + \theta \mu = 0.
\]

We can then establish the following theorem on the stability of a limit cycle due to a Hopf bifurcation in Model (K).

**Theorem.** Let Assumptions 1-4 hold and let \(\zeta\) and \(\theta\) be given by (42).

1. If \(\zeta < 0\), then a supercritical Hopf bifurcation occurs, with an attracting limit cycle existing for \(\beta > \beta^*\) (resp. \(\beta < \beta^*\)) when \(\theta > 0\) (resp. \(\theta < 0\)).

2. If \(\zeta > 0\), then a subcritical Hopf bifurcation occurs, with a repelling limit cycle existing for \(\beta > \beta^*\) (resp. \(\beta < \beta^*\)) when \(\theta < 0\) (resp. \(\theta > 0\)).
4. Numerical analysis. This section provides numerical analysis for our medium-term Keynesian model. We first specify the relevant parameter values and functional forms and then perform numerical simulations.

4.1. Specifications. For the numerical analysis, we specify the relevant parameter values and functional forms in our model. To begin, we set the coefficient of quantity adjustment \( \alpha \) and the rate of capital depreciation \( \delta \) to the following:

\[
\alpha = 1.25, \tag{43}
\]
\[
\delta = 0.1. \tag{44}
\]

These values fit well with the empirical data.

Next, we turn to the investment, saving and inflation functions. We assume that they are represented in the following forms:

\[
I(Y, K, \rho^e) = I_0 + i_p \left( \frac{Y}{K} - \rho^e \right) K + i_u (\kappa Y - K), \tag{45}
\]
\[
S(Y) = sY - S_0, \tag{46}
\]
\[
h(Y - Y^*) = h_y \left( \frac{Y}{Y^*} - 1 \right), \tag{47}
\]

where \( i_p \), \( i_u \), \( s \) and \( h_y \) are positive coefficients; \( I_0 \) and \( S_0 \) are positive constants (cf. Assumption 1); \( \gamma \) and \( \kappa \) are positive constants which represent aggregate share of capital and the capital coefficient, which is the ratio of aggregate capital to aggregate potential output (identified with \( Y^* \)), respectively.

We determine the coefficients in (45)-(47). Based on Jones [9] and Flaschel et al. [6], we may first set aggregate share of capital \( \gamma \) and the capital coefficient \( \kappa \) to the following:

\[
\gamma = 0.35, \tag{48}
\]
\[
\kappa = 2. \tag{49}
\]

Following Flaschel et al. [6], the values of \( i_p \), \( i_u \) and \( s \) are set to

\[
i_p = 0.15, \quad i_u = 0.035 \tag{50}
\]
\[
s = 0.25. \tag{51}
\]

---

26 For the specifications, we make use of the results of several preceding works, but none of them discussed the existence or stability of limit cycles because their empirical studies were conducted for purposes different from ours.

27 Flaschel et al. [6] estimated the value of \( \alpha \) for the U.S. economy at 1.2610, and Juillard and Villemot [10] suggested that the (annual) value of \( \delta \) be set to 0.1 in calibration.

28 Functional forms of (45)-(47) are consistent with Flaschel et al. [6] under the assumption that aggregate share of capital and the capital coefficient are approximated to constants. This assumption reflects the fact that the U.S. share of capital and capital coefficient have been almost constant in the postwar period (cf. Jones [9]; Flaschel et al. [6]).

29 According to Jones [9], the U.S. share of capital was almost constant around 0.342 in the postwar period up to 2000. The ratio of aggregate potential output to capital, which is the reciprocal of the capital coefficient, was estimated by Flaschel et al. [6] to 0.5091 for the U.S. economy.

30 Flaschel et al. [6] estimated the values of \( i_p \) and \( i_u \) as 0.1353 and 0.034, respectively. They also estimated the marginal propensities to save of capitalists and workers at 0.6230 and 0.0510. It then follows from the value of aggregate share of capital in (48) that the marginal propensity to save can be calculated as 0.6230 · 0.35 + 0.0510(1 − 0.35) ≈ 0.25.
According to Murakami and Asada [28], the coefficient of the inflation function can be set to
\[ h_y = 0.25. \]  
To determine the values of \( I_0 \) and \( S_0 \), we have a closer look at the unique equilibrium of Model (K) under (45)-(47). As is indicated by the definition of the capital coefficient, it follows from (49) that the equilibrium values of \( Y \) and \( K \) satisfy
\[ \frac{K^*}{Y^*} = 2. \]  
According to Laubach and Williams [13], we can set the equilibrium rate of interest, denoted by \( r^* \), to
\[ r^* = 0.03. \]  
It is seen from (18) and (19) that \( Y^* \) and \( K^* \) are given by
\[ 0 = I_0 + i_p \left( \gamma \frac{Y^*}{K^*} - r^* \right) K^* + i_u (\kappa Y^* - K^*) - (s Y^* - S_0), \]  
\[ 0 = S_0 + i_p \left( \gamma \frac{Y^*}{K^*} - r^* \right) K^* + i_u (\kappa Y^* - K^*) - \delta K^*. \]  
These equations can, with (44)-(51), (53) and (54) substituted, be reduced to
\[ I_0 = 0.1565 Y^*, \]  
\[ S_0 = 0.05 Y^*. \]  
We find that \( I_0 \) and \( S_0 \) are proportional to \( Y^* \). We can then set \( I_0 \) and \( S_0 \) to
\[ I_0 = 0.1565, \]  
\[ S_0 = 0.05, \]  
under the following normalization of \( Y^* \):
\[ Y^* = 1. \]  
Thus, substituting (48)-(57) in (45)-(47), we can determine the investment, saving and inflation functions as follows:
\[ I(Y, K, \rho^e) = 0.1565 + 0.1225 Y - (0.15 \rho^e + 0.035) K, \]  
\[ S(Y) = 0.25 Y - 0.05, \]  
\[ h(Y - Y^*) = 0.3(Y - 1). \]  
Finally, we specify the interest rate function. Since it is derived from (4) as explained in Sect. 2, we take a look at the liquidity preference function. It is assumed for simplicity that this function can be represented in the form of
\[ L(r, Y) = L_0 r^{-\eta} Y^{\eta_s}, \]
where \( \eta_r \) and \( \eta_y \) are positive constants; \( L_0 \) is a positive constant. This implies from (4) that the interest rate function is given as

\[
R(Y, P) = Y^{\eta_y/\eta_r} \left( \frac{L_0}{M} P \right)^{1/\eta_r}.
\]  

(61)

According to Bahnamu-Oskooee and Chomsisengphet [3], the values of \( \eta_r \) and \( \eta_y \) can be approximated to

\[
\eta_r = 0.2, \quad \eta_y = 1.2.
\]

(62)

We can find from (54), (57), (61) and (62) that the equilibrium value of \( P \) is given by

\[
0.03 = \left( \frac{L_0}{M} P^* \right)^5.
\]

We can then assume that

\[
\frac{L_0}{M} = 0.03^{0.2},
\]

(63)

under the normalization of \( P^* \) to unity:

\[
P^* = 1.
\]

(64)

Thus, we can, substituting (62) and (63) in (61), obtain the interest rate function in the following form:

\[
R(Y, P) = 0.03Y^6 P^5.
\]

(65)

Substituting (43), (44), (58)-(60) and (65) in Model (K), we have the following model for numerical simulations:

\[
\dot{Y} = 1.25[0.2065 - 0.1275Y - (0.0045Y^6 P^5 - 0.15\pi^e + 0.035)K],
\]

(66)

\[
\dot{K} = 0.1565 + 0.1225Y - (0.0045Y^6 P^5 - 0.15\pi^e + 0.135)K,
\]

(67)

\[
\dot{P} = 0.3(Y - 1)P + P\pi^e,
\]

(68)

\[
\dot{\pi}^e = 0.3\beta(Y - 1).
\]

(69)

Model (K) is redefined as the system of (66)-(69) in this section. Note that Assumptions 1-3 are all satisfied and that the unique equilibrium of Model (K) is, due to (53), (57) and (64), given by

\[
(Y^*, K^*, P^*, \pi^e) = (1, 2, 1, 0).
\]

(70)

Also, Assumption 4 is satisfied for this equilibrium.

4.2. Existence and stability of a limit cycle. We can obtain the (Hopf) bifurcation value of \( \beta > 0 \) in Proposition 2 as follows:

\[
\beta^* = 0.121766.
\]

(71)

If \( \beta = \beta^* \), the eigenvalues of \( J^*_{(K)} \) are then calculated as follows:

\[
\lambda_1 = 0.08047i, \quad \lambda_2 = -0.08047i, \quad \lambda_3 = -0.225913, \quad \lambda_4 = -0.140462.
\]

The resonant terms, defined by (40) and (41), are obtained as follows:

\[
\epsilon_1 = 0.246528 + 0.306592i, \quad \epsilon_2 = 0.0125023 - 0.394415i.
\]

\[^{34}\text{Bahnamu-Oskooee and Chomsisengphet [3] estimated the values of } \eta_r \text{ and } \eta_y \text{ at 0.1753 and 1.1549, respectively, for the U.S. economy.} \]
Hence, the coefficients in (42) are given by
\[ \zeta = 0.0125023 > 0, \quad \theta = 0.246528 > 0. \]

It then follows from Theorem that Model (K) admits a subcritical Hopf bifurcation generating an unstable limit cycle if \( \beta \) is less than \( \beta^* \) and close to it.

We perform numerical simulations of Model (K). To see that an unstable limit cycle exists if \( \beta \) is less than \( \beta^* \) (given in (71)) and sufficiently close to it, we set the values of \( \beta \) as
\[ \beta = 0.12. \]

To check the existence and instability of a limit cycle, we examine the solution paths of Model (K) with the following two initial conditions:
\[
\begin{align*}
(Y(0), K(0), P(0), \pi^e(0)) &= (1, 1.95, 1.25, 0), \\
(Y(0), K(0), P(0), \pi^e(0)) &= (1, 1.95, 1.05, 0).
\end{align*}
\]

Figures 1 and 2 illustrate the differences of \( Y, K, P \) and \( \pi^e \) from their equilibrium values (given in (70) along the solution paths with the initial conditions (72) and (73), respectively. In Figure 3, on the other hand, the blue and red curves describe the projection on the \( K-P \) plane of the solution paths with the initial conditions (72) and (73), respectively. It is seen from these figures that the two solution paths diverge from the unstable limit cycle if \( \beta \) is less than \( \beta^* \), and it is indicated that there exists an unstable limit cycle. These simulation results are consistent with the conclusion of Theorem. In terms of macroeconomics, the unstable limit cycle can be interpreted as the boundary of a stable region associated with corridor stability (cf. Sect. 1). In this respect, the unstable limit cycle may matter in practice because the stable region implied by it can be a guidance for stabilization policy.\(^{35}\)

\(^{35}\)The existence of an unstable limit cycle does not necessarily imply the nonexistence of a stable limit cycle. Indeed, it is possible that stable and unstable limit cycles coexist. For studies on such a coexistence in economic models, see Matsumoto [20] or Dohtani [5].
Figure 2. Solution path with (73)

(—: $Y - Y^*$, ——: $K - K^*$, ---: $P - P^*$, ----: $\pi^*$)

Figure 3. Solution paths projected on $K$-$P$ plane

5. Conclusion. This paper has studied mathematical details of the medium-term Keynesian model proposed by Murakami [21]. We have presented the conditions for the (in)stability of limit cycles generated by Hopf bifurcations and confirmed the validity of them by numerical simulations. Although the (in)stability of limit cycles (or business cycles) is both theoretically and empirically important, it has seldom
been examined rigorously. In this respect, our analysis may contribute to deeper understanding of real economies.

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REFERENCES

[1] T. Asada, M. Demetrian and R. Zimka, On dynamics in a Keynesian model of monetary stabilization policy with debt effect, Communications in Nonlinear Science and Simulation, 58 (2018), 131–146.
[2] T. Asada, M. Demetrian and R. Zimka, On dynamics in a Keynesian model of monetary and fiscal stabilization policy mix with twin debt accumulation, Metroeconomica, 70 (2019), 365–383.
[3] M. Bahmanu-Oskooee, S. Chomsisengphet, Stability of M2 money demand function in industrial countries, Applied Economics, 34 (2010), 2075–2083.
[4] Y. N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations, Lecture Notes in Mathematics, 702. Springer-Verlag, Berlin-New York, 1979.
[5] A. Dohtani, A growth-cycle model of Solow-Swan type, I, Journal of Economic Behavior and Organization, 76 (2010), 428–444.
[6] P. Flaschel, G. Gong and W. Semmer, A Keynesian macroeconomic framework for the analysis of monetary policy rules, Journal of Economic Behavior and Organization, 46 (2001), 101–136.
[7] P. Flaschel and H. M. Kroizig, Wage-price Phillips curves and macroeconomic stability: Basic structural form, estimation and analysis, Quantitative and Empirical Analysis, 277 (2006), 7–47.
[8] G. Gandolfo, Economic Dynamics, 4th ed., Springer-Verlag, Berlin, Heidelberg, 2009.
[9] C. I. Gouruc, The facts of economic growth, Handbook of Macroeconomics, 2, (2016), 3–69.
[10] M. Juillard and S. Villemot, Multi-country real business cycle models: Accuracy tests and test bench, Journal of Economic Dynamics and Control, 35 (2010), 178–185.
[11] N. Kaldor, A model of the trade cycle, Economic Journal, 50 (1940), 78–92.
[12] J. M. Keynes, The General Theory of Employment, Interest and Money, Macmillan, Cham, 1936.
[13] T. Laubach and J. C. Williams, Measuring the natural rate of interest, Review of Economics and Statistics, 85 (2003), 1063–1070.
[14] A. Leijonhufvud, On Keynesian Economics and the Economics of Keynes: A Study in Monetary Theory, Oxford University Press, Oxford, 1968.
[15] A. Leijonhufvud, Effective demand failures, Scandinavian Journal of Economics 75 (1973), 27–48.
[16] W. Liu, Criterion of Hopf bifurcations without using eigenvalues, Journal of Mathematical Analysis and Applications, 182 (1994), 250–256.
[17] H.-W. Lorenz, Nonlinear Dynamical Economics and Chaotic Motion, 2nd ed., Springer, Berlin, 1993.
[18] S. A. Marglin and A. Bhaduri, Profit squeeze and Keynesian theory, The Golden Age of Capitalism: Reinterpreting the Postwar Experience, Clarendon Press, Oxford, (1990), 153–186.
[19] J. E. Marsden and M. McCracken, The Hopf Bifurcation and its Applications Springer-Verlag, New York, 1976.
[20] A. Matsumoto, Note on Goodwin’s 1951 nonlinear accelerator model with an investment delay, Journal of Economic Dynamics and Control, 33 (2009), 832–842.
[21] H. Murakami, Keynesian systems with rigidity and flexibility of prices and inflation-deflation expectations, Structural Change and Economic Dynamics, 30 (2014), 68–85.
[22] H. Murakami, Wage flexibility and economic stability in a non-Walrasian model of economic growth, *Structural Change and Economic Dynamics*, **32** (2015), 25–41.
[23] H. Murakami, A non-Walrasian microeconomic foundation of the “profit principle” of investment, *Essays in Economic Dynamics: Theory, Simulation Analysis, and Methodological Study*, Springer, Singapore, (2016), 123–141.
[24] H. Murakami, Existence and uniqueness of growth cycles in post Keynesian systems, *Economic Modelling*, **75** (2018), 293–304.
[25] H. Murakami, A note on the “unique” business cycle in the Keynesian theory, *Metroeconomica*, **70** (2019), 384–404.
[26] H. Murakami, Monetary policy in the unique growth cycle of post Keynesian systems, *Structural Change and Economic Dynamics*, **52** (2020), 39–49.
[27] H. Murakami, The unique limit cycle in post Keynesian systems, IERCU Discussion Paper, Chuo University, **334** (2020), 1–26.
[28] H. Murakami and T. Asada, Inflation-deflation expectations and economic stability in a Kaleckian system, *Journal of Economic Dynamics and Control*, **92** (2018), 183–201.
[29] H. Murakami and H. Sasaki, Economic development with accumulation of public capital: The crucial role of wage flexibility on business cycles, *Economic Modelling*, **93** (2020), 299–309.
[30] H. Murakami and R. Zimka, On dynamics in a two-sector Keynesian model of business cycles, *Chaos, Solitons and Fractals*, **130** (2020), 109419, 8 pp.
[31] E. S. Phelps, Phillips curves, expectations of inflation and optimal unemployment over time, *Economica*, **34** (1967), 254–281.
[32] J. Tobin, Keynesian models of recession and depression, *American Economic Review*, **65** (1975), 195–202.
[33] J. Tobin, Price flexibility and output stability: An old Keynesian view, *Journal of Economic Perspectives*, **7** (1993), 45–65.
[34] V. Torre, Existence of limit cycles and control in complete Keynesian system by theory of bifurcations, *Econometrica*, **45** (1977), 1457–1466.
[35] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Texts in Applied Mathematics, Vol. 2, Springer-Verlag, New York, 1990.