On the conformal equivalence of meromorphic functions.

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Abstract

In 2002, Beardon, Carne, and Ng showed that given any sequence of complex numbers \( w_1, \ldots, w_n \), there is a polynomial with this list of critical values. In this paper we extend the notion of critical values of a function by defining the critical level curve configuration of a special class of meromorphic function elements, namely function elements \((f, G)\) where \(G\) is simply connected, \(|f|\) is constant on \(\partial G\), and \(f' \neq 0\) on \(\partial G\). This notion of critical level curve configuration preserves critical value information, but also incorporates information about the relative positions of the critical points of the function and, in particular, the topology of the level curves of the function. We then show that two function elements \((f_1, G_1)\) and \((f_2, G_2)\) are conformally equivalent (that is, \(f_1\) factors as the composition of \(f_2\) with a conformal map \(\phi: G_1 \to G_2\)) if and only if \(f_1\) and \(f_2\) have the same critical level curve configuration. Moreover, we extend the theorem of Beardon, Carne, and Ng by showing that for any of the critical level curve configurations which would correspond to an analytic function, there is a polynomial which has that critical level curve configuration. Finally, we conclude that for any function element \((f, G)\) such that \(f\) is analytic, \((f, G)\) is conformally equivalent to a polynomial function element.

Keywords: complex analysis; meromorphic functions; level curves; critical values; critical points

1 HISTORY AND OVERVIEW

Throughout this paper, we will use several preliminary results, largely developed in anticipation for this work, which may be found in [5]. Therein we also include a brief overview of the study of the level curves of a meromorphic function \(f\), culminating in the 1986 Level Curve Structure Theorem of Stephenson [7]. We will now pick up with some more recent work which directly bears on the subject of this paper.

The “fingerprint” \(k\) which a smooth curve imposes on the unit circle \(T\) was introduced by A. A. Kirillov [3][4] in 1987, and is defined as follows. Let \(\Gamma\) be a smooth simple closed curve in \(C\), with bounded face \(\Omega_-\) and unbounded face \(\Omega_+\). Let \(\phi_- , \phi_+\) denote Riemann maps from \(D,D_+\) to \(\Omega_-,\Omega_+\) respectively (here \(D_+\) is defined as \(C\setminus \text{cl}(D)\)). With certain normalizations on the Reimann maps, we define the fingerprint \(k\) of \(\Gamma\) by \(k := \phi_+^{-1} \circ \phi_-\). Since \(\Gamma\) is smooth it is easy to show that \(k\) is a diffeomorphism from \(T\) to \(T\). Moreover if \(\hat{\Gamma}\) equals the image of \(\Gamma\) under an affine transformation \(f(z) = az + b\), with corresponding fingerprint \(\hat{k}\), then \(k = \hat{k} \circ \phi\) for some automorphism \(\phi: D \to D\). Therefore we may define a function \(F\) which maps smooth simple closed curves (modulo composition with an automorphism of \(D\)) to the corresponding diffeomorphism of \(T\) which is its fingerprint (modulo pre-composition with an automorphism of \(D\)). (Note: This and more background may be found in [2].) Kirillov proved the following theorem [3][4].

**Theorem 1.1.** \(F\) is a bijection onto the diffeomorphisms of \(T\).

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If we restrict our attention to smooth curves which arise as level curves of polynomials, a similar result may be obtained. One first shows that if $\Gamma$ is a proper polynomial lemniscate (i.e., $\Gamma = \{z : |p(z)| = 1\}$ for some $n$-degree polynomial $p$ such that $\{z : |p(z)| = 1\}$ is smooth and connected) then the corresponding fingerprint may be shown to be of the form $k = B^\frac{1}{n}$ for some $n$-degree Blaschke product. If we let $\mathcal{F}_p$ denote the function $F$ viewed as having as its domain the equivalence classes of simple smooth closed curves which arise as proper polynomial lemniscates, and having codomain the equivalence classes of diffeomorphisms of $\mathbb{T}$ consisting of $n^{th}$ roots of $n$-degree Blaschke products ($n \in \mathbb{N}$), then one may prove the following theorem.

**Theorem 1.2.** $\mathcal{F}_p$ is a bijection.

This result was stated in [2] and follows rather directly from Theorem 1.1, though the authors proved it using other means. In Section 4 and Section 5, we will prove a somewhat stronger result which has Theorem 1.2 as a corollary, though in the following equivalent form.

**Theorem 1.3.** For every finite Blaschke product $B$ with degree $n$, there is some $n$-degree polynomial $p$ such that the set $G := \{z : |p(z)| < 1\}$ is connected, and some conformal map $\phi : \mathbb{D} \to G$ such that $B = p \cdot \phi$ on $\mathbb{D}$.

Our main goal in this paper, however, is to explore the way in which the configuration of level curves of a meromorphic function characterizes that function modulo conformal equivalence.

We begin in Section 2 by considering meromorphic functions $f$ with bounded simply connected domains $G$ such that the following hold.

- $f$ may be extended to a meromorphic function on $cl(G)$.
- $f' \neq 0$ on $\partial G$.
- $|f| = 1$ on $\partial G$.

We call such a pair $(f, G)$ a special type function element, and we rigorously construct a set $PC$ which represents all possible configurations of the critical level curves of $(f, G)$. We then define a function $\Pi$ which maps $(f, G)$ to the corresponding configuration in $PC$. Section 3 contains the main result (Theorem 3.1) of the paper, namely that $\Pi$ respects conformal equivalence.

**Theorem 1.4.** If $(f_1, G_1)$ and $(f_2, G_2)$ are two functions as described above, then $(f_1, G_1) \sim (f_2, G_2)$ if and only if $\Pi(f_1, G_1) = \Pi(f_2, G_2)$.

Here $(f_1, G_1) \sim (f_2, G_2)$ means that there is some conformal map $\phi : G_1 \to G_2$ such that $f_1 = f_2 \circ \phi$ on $G_1$ (clearly an equivalence relation on the set of special type function elements). This result implies that if we view $\Pi$ as having for its domain the set of equivalence classes of meromorphic functions modulo conformal equivalence, then first $\Pi$ is well defined and second $\Pi$ is injective.

In Section 4 and Section 5, we show that in a limited sense, $\Pi$ is surjective. That is, we define a subset $PC_a \subset PC$ of configurations which naturally correspond to the level curve configurations of analytic functions. Then, if we view $\Pi$ as having for its domain the equivalence classes of analytic $(f, G)$, and having codomain $PC_a$, then $\Pi$ is surjective. In Section 4 we prove the generic case, that the image of $\Pi$ contains each configuration in $PC_a$ whose critical values are non-zero and have different moduli. In Section 5 we extend this to all of $PC_a$ by approximating a general member of $PC_a$ by a generic one.

Finally, in the appendix we prove several lemmata used throughout the paper.

### 2 THE POSSIBLE LEVEL CURVE CONFIGURATIONS OF A MEROMORPHIC FUNCTION

We begin by re-defining a special type function element, and the relation $\sim$.

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1 Thanks to Prof. Dmitry Khavinson for pointing this fact out.
Definition: Let $G$ be an open bounded simply connected set in $\mathbb{C}$, and let $f : G \to \mathbb{C}$ be meromorphic, and such that $f$ can be extended to an meromorphic function on an open set containing the closure of $G$. Call such a pair $(f, G)$ a function element.

- Say $(f, G)$ is a special type function element if $|f| \equiv 1$ and $f' \neq 0$ on $\partial G$.
- If $(f_1, G_1)$ and $(f_2, G_2)$ are function elements, and there is some conformal map $\phi : G_1 \to G_2$ such that $f_1 \equiv f_2 \circ \phi$, then we say that $(f_1, G_1)$ and $(f_2, G_2)$ are conformally equivalent, and we write $(f_1, G_1) \sim (f_2, G_2)$.

It is easy to see that $\sim$ is an equivalence relation on the collection of all function elements, and we make the following definition.

Definition: Let $H'$ denote the set of all special type function elements, and define $H := H'/\sim$. Let $H'_a \subset H'$ denote the set of all special type function elements $(f, G)$ such that $f$ is analytic on $G$, and define $H_a := H'_a/\sim$.

In Section 3, we will show that two special type function elements are in the same member of $H$ if and only if they have the same level curve structure. We will see that to fully describe the configuration of level curves of a special type function element $(f, G)$, it suffices to consider only the configuration of the critical level curves of $f$. In order to rigorously define the configuration of critical level curves of $(f, G)$, in the next section we will define a mathematical object $PC$ (for “Possible Level Curve Configurations”) which will parameterize the different possible level curve configurations of a special type function element.

2.1 Construction of $PC$

We begin by defining a set $\hat{P}$ which will represent the different possible graphs one may obtain as a level curve of a special type function element. Members of $\hat{P}$ are certain connected finite graphs (that is, graphs with finitely many vertices and with finitely many edges), and may be viewed as sub-sets of $\mathbb{C}$, but are defined modulo orientation preserving homeomorphism (which will be defined shortly). We will now describe which finite graphs are contained in $\hat{P}$.

- Each edge of $\xi$ is incident to at least one bounded face of $\xi$.
- For each vertex $v$ of $\xi$, the number of edges of $\xi$ incident to $v$ is even and greater than 2 (where we count an edge twice if both endpoints of the edge are at $v$).

When modding out our graphs embedded in $\mathbb{C}$ by orientation preserving homeomorphisms, we mean that two graphs which meet the above restrictions are considered the same if there is an orientation preserving homeomorphism of $\mathbb{C}$ to itself which maps the one graph to the other. (In the setting of a special type level curve $(f, G)$, the single points will be used to represent zeros or poles of $f$, and the graphs will be used to represent the critical level curves of $f$.)

Note: Throughout when we refer to the single point element of $\hat{P}$, or constructions beginning with that element, we will just refer to it as $w$, though $\{w\}$ may be technically more accurate.

Given that the members of $\hat{P}$ will be used to help represent the critical level curves of a special type function element $(f, G)$, we now form another set $P$ by associating some auxiliary data to the members of $\hat{P}$. To each member $\xi$ of $\hat{P}$, we will associate auxiliary data to represent the following.

- The modulus of $f$ on the level curve being represented.
The number of zeros minus the number of poles of \( f \) in \( \xi \) (if \( \xi \) is the single point member of \( \check{P} \)) or in the bounded faces of \( \xi \) (if \( \xi \) is one of the graph members of \( \check{P} \)).

The points in the level curve being represented at which \( f \) takes non-negative real values. (These points we will call "distinguished points" of the graph.) If the member of \( \check{P} \) in question is the single point member, it will be used to represent a zero or pole of \( f \), and is thus automatically distinguished.

**Note:** We will view \( \infty \) as a non-negative real value.

To the members of \( \check{P} \) which are not single points, we will additionally associate auxiliary data to represent the following,

- The number of zeros minus the number of poles in each individual bounded face of the level curve being represented. (This will of course be equal to the number of distinguished points in the boundary of that face.)
- The argument of \( f \) at each vertex (critical point of \( f \)) of the level curve being represented.

We begin this process with the single point members of \( \check{P} \). Let \( w \) denote the single point member of \( \check{P} \).

From \( w \), we will construct a member \( \langle w \rangle_{P} \) of \( P \). We do this by associating the following pieces of data to \( w \).

- We define \( H(\langle w \rangle_{P}) \) to be a value in \( \{0, \infty\} \) (depending on whether \( \langle w \rangle_{P} \) will represent a zero of \( f \) or a pole of \( f \)).
- We write \( Z(\langle w \rangle_{P}) = n \) for some non-zero \( n \in \mathbb{Z} \). This represents the multiplicity of the point being represented as a zero or pole of \( f \) (positive if a zero, negative if a pole).
- We say that \( w \) is distinguished with multiplicity \( |n| \) to represent that \( f \) is non-negative real on \( w \), and the ramification of \( f \) at \( w \) is \( |n| + 1 \).

The resulting object we denote \( \langle w \rangle_{P} \).

If \( \lambda \) is a member of \( \check{P} \) which is not the single point, then we construct a member \( \langle \lambda \rangle_{P} \) of \( P \) from \( \lambda \) by associating the following pieces of data to \( \lambda \).

- We define \( H(\langle \lambda \rangle_{P}) = \epsilon \) for some value \( \epsilon \in (0, \infty) \) to denote the value of \( |f| \) on \( \lambda \).
- If \( D \) is a bounded face of \( \lambda \), we associate an integer \( z(D) \in \mathbb{Z} \setminus \{0\} \). (This represents the number of zeros of \( f \) in \( D \) minus the number of poles of \( f \) in \( D \).) If \( D_1, D_2, \ldots, D_k \) denote the bounded face of \( \lambda \), we define \( Z(\langle \lambda \rangle_{P}) = \sum_{i=1}^{k} z(D_i) \). This assignment of \( z(D_1), \ldots, z(D_k) \) must be done in such a way that \( Z(\langle \lambda \rangle_{P}) \neq 0 \) and if \( D_1 \) and \( D_2 \) are bounded faces of \( \lambda \) which share a common edge, then \( z(D_1) \) and \( z(D_2) \) are not both positive or both negative. (This is the case for level curves of \( f \) in view of the Open Mapping Theorem.)
- For each bounded face \( D \) of \( \lambda \), we distinguish \( z(D) \) points in \( \partial D \) (to represent the points in \( \lambda \) at which \( f \) takes non-negative real values).
- If \( x \in \lambda \) is a vertex of \( \lambda \), we designate a value \( a(x) \in [0, 2\pi) \). (This will represent the argument of \( f \) at \( x \).) We require that this assignment follows the following rules.
  - For a vertex \( x \) of \( \lambda \), \( a(x) = 0 \) if and only if \( x \) is a distinguished point.
  - If \( D \) is a face of \( \lambda \), and \( z(D) > 0 \), and \( x_1 \), \( x_2 \) are vertices of \( \lambda \) in \( \partial D \) such that \( a(x_1) \geq a(x_2) \), then there is some distinguished point \( z \in \partial D \) such that \( x_1, z, x_2 \) is written in increasing order as they appear in \( \partial D \). (This reflects the fact that if \( \lambda \) is a level curve of \( f \), and \( D \) contains more zeros of \( f \) than poles of \( f \), then the argument of \( f \) is increasing as \( \partial D \) is traversed with positive orientation.)
If \( D \) is a face of \( \lambda \), and \( z(D) < 0 \), and \( x_1, x_2 \) are vertices of \( \lambda \) in \( \partial D \) such that \( a(x_1) \geq a(x_2) \), then there is some distinguished point \( z \in \partial D \) such that \( x_2, z, x_1 \) is written in increasing order as they appear in \( \partial D \). (This reflects the fact that if \( \lambda \) is a level curve of \( f \), and \( D \) contains more poles of \( f \) than zeros of \( f \), then the argument of \( f \) is decreasing as \( \partial D \) is traversed with positive orientation.)

The resulting object with the above auxiliary data we denote \( \langle \lambda \rangle_P \), and we define \( P \) to be the set of all such \( \langle \lambda \rangle_P \) and \( \langle w \rangle_P \). We also define \( P_a \subset P \) by \( \langle w \rangle_P \in P_a \) if and only if \( Z(\langle w \rangle_P) > 0 \), and \( \langle \lambda \rangle_P \in P_a \) if and only if \( z(D) > 0 \) for each bounded face \( D \) of \( \lambda \).

Throughout this paper, \( \langle w \rangle_P \) will be used to refer to single point members of \( P \), \( \langle \lambda \rangle_P \) will be used for graph members of \( P \), and \( \langle \xi \rangle_P \) will be used when we do not wish to distinguish between the two types of members of \( P \).

We will now construct \( PC \). Each member of \( PC \) will be a collection of members of \( P \) arranged in different ways according to certain rules, with certain auxiliary data which will be described. As noted before, this will be used to represent the different ways in which the critical level curves of a special type function element may lie with respect to each other. There are two steps to this. First, determine which graphs lie in which bounded faces of which other graphs, and second, determine the orientations of each graph with respect to the others. We begin by describing the different ways in which the graphs may lie with respect to each other recursively.

**Note:** Each member of \( \langle \xi \rangle_P \) of \( P \) will give rise to possibly multiple different members of \( PC \) but, when it will not cause confusion, we will use \( \langle \xi \rangle_{PC} \) to denote a member of \( PC \) which arises from \( \langle \xi \rangle_P \). All members of \( PC \) will arise from some member of \( P \), so we use \( \langle \xi \rangle_{PC} \) to denote some member of \( PC \), then by \( \langle \xi \rangle_P \) we mean the member of \( P \) which gave rise to \( \langle \xi \rangle_{PC} \).

A level 0 member of \( PC \) will be a single point member of \( P \) viewed as a member of \( PC \), with no additional data. For \( n > 0 \), level \( n \) members of \( PC \) are constructed by taking \( \langle \lambda \rangle_P \) a graph member of \( P \), and assigning to each bounded face \( D \) of \( \lambda \) a level \( k \) member \( \langle \xi_D \rangle_{PC} \) of \( PC \) for some \( k < n \). This assignment must follow the following restrictions.

- For each bounded face \( D \) of \( \lambda \), \( Z(\langle \xi_D \rangle_P) = z(D) \).
- If \( z(D) > 0 \), then \( H(\langle \xi_D \rangle_P) < H(\langle \lambda \rangle_P) \).
- If \( z(D) < 0 \), then \( H(\langle \xi_D \rangle_P) > H(\langle \lambda \rangle_P) \).

Furthermore, let \( D \) be any bounded face of \( \lambda \). Then \( \langle \lambda \rangle_{PC} \) also comes equipped with a surjective map \( g_D \) from the distinguished points in \( \partial D \) to the distinguished points in \( \xi_D \). This map we call the gradient map, (since in the study of a special type function element \((f,G)\), the gradient maps will be determined by the gradient lines of \( f \)) and we require that \( g_D \) preserve the orientation of the distinguished points. That is, if \( \xi_D \) is a graph embedded in \( C \), and we read the distinguished points in \( \xi_D \) off as they appear around \( \xi_D \) when \( \xi_D \) is traversed one full time in positive order from the outside, let \( x_1, \ldots, x_{z(D)} \) be their enumeration as they appear in this way (if some vertex of \( \xi_D \) is distinguished, then it will appear more than one time in this list). Then there must be some point \( y_1 \in \partial D \) which is distinguished as a point in \( \lambda \), and when the distinguished points in \( \partial D \) are listed by their appearence in positive order starting with \( y_1 \), namely \( y_1, \ldots, y_{z(D)} \), then \( g_D(y_i) = x_i \) for each \( i \in \{1, \ldots, z(D)\} \). If \( \xi_D \) is just a single point with associated number 0, then \( g_D \) is just the map that takes every distinguished point in \( \partial D \) to that single point. (In the context of level curves of meromorphic functions, \( g_D(w) = z \) means that \( w \) and \( z \) are connected by a gradient line of \( f \). This keeps track of the "orientation" of a critical level curve of \( f \) with respect to the other critical level curves of \( f \).)

**Note:** If a member of \( PC \) has been formed at level \( n \) for some \( n \geq 0 \), we do not form it again at any later level, so we may say in a well defined way that a member of \( PC \) has a given level.
We call this assignment of members of $PC$ to the bounded faces of $\lambda$, along with the associated gradient maps, $(\lambda)_{PC}$. The set of all such $(\lambda)_{PC}$ and $\langle w \rangle_{PC}$ we denote $PC$, and we call $PC$ the set of possible level curve configurations. We define $PC_a \subset PC$ to be the collection of members of $PC$ which is constructed entirely using members of $P_a$. That is, $(\lambda)_{PC} \in PC_a$ if and only if $\langle \lambda \rangle_{P} \in P_a$, and each member of $PC$ which is assigned to a bounded face of $\lambda$ is in $PC_a$.

We adopt the same convention of $w, \lambda$ or $\xi$ for members of $PC$ as we did for members of $P$, namely that level 0 members of $PC$ we denote by $\langle w \rangle_{PC}$. Level $n > 0$ members of $PC$ we denote by $\langle \lambda \rangle_{PC}$, and if we do not wish to specify the level of a member of $PC$ we will denote it by $\langle \xi \rangle_{PC}$.

### 2.2 Construction of $\Pi$ 

We now make explicit the way in which $PC$ parameterizes the possible level curve configurations of a special type function element. We do this by defining a function $\Pi : H' \rightarrow PC$. In Section 3 we will show that $\Pi$ takes the same value on conformally equivalent members of $H'$, and therefore we may view $\Pi$ as acting on $H$. It is fairly easy to show that $\Pi$ acting on $H$ is injective, and that $\Pi(H_n) \subset PC_a$. The major goal of Sections 4 and 5 is to show that $\Pi : H_a \rightarrow PC_a$ is a bijection. We now define the action of $\Pi$ on $H'$. Let $(f, G)$ be some member of $H'$. Define $B := \{ w \in G : f(w) = 0 \text{ or } f(w) = 0 \text{ or } f(w) = \infty \}$. Proposition 2.1 from [5] shows that for each $w \in B$, $\Lambda_w$ is either a single point (if $w$ is a zero or pole of $f$), or is a planar graph of the type used to form members of $\tilde{P}$.

We begin by picking members of $P$ to represent $\Lambda_w$ for each $w \in B$. If $w$ is either a zero or a pole of $f$, then $\Lambda_w = \{ w \}$, and we define $Z(\langle w \rangle_{P}) := k$ where $k$ is the multiplicity of $w$ as a zero of $f$, $(k < 0 \text{ if } w \text{ is a pole of } f)$, $\Lambda_w$ is distinguished. It is also the case that if $k > 0$, and $z, z'$ are vertices of $\Lambda_w$ with $z \neq z'$ and $0 < a(z) \leq a(z')$, then there must be some $w' \in \partial D \subset \Lambda_w$ which is distinguished, and such that $z, w, z'$ are written as they appear in increasing order around $\partial D$. Similarly, if $k < 0$, and $z, z'$ are vertices of $\Lambda_w$ with $z \neq z'$ and $0 < a(z') \leq a(z)$, then there must be some $w' \in \partial D \subset \Lambda_w$ which is distinguished, and such that $z, w', z'$ are written as they appear in increasing order around $\partial D$. This can be shown by the interested reader by using the theorem of continuity of $f$ on the closure of $D$ and the Argument Principle. Finally, we define $H(\langle \Lambda_w \rangle_{P})$ to be the value that $|f|$ takes on $\Lambda_w$. Thus we obtain $\langle \Lambda_w \rangle_{P} \in P$.

Now we wish to stitch together the members of $P$ obtained from $\mathcal{B}$ above in such a way as to obtain a member of $PC$ (which we will call $\Pi(f, G)$). We will then identify a critical level curve of $(f, G)$ with the graph that it gives rise to in $\Pi(f, G)$. We will also identify the critical points of $f$ with the corresponding vertices, and the points in the critical level curves of $(f, G)$ at which $f$ takes non-negative real values with the corresponding distinguished points in $\Pi(f, G)$.

From each of the members $w$ of $\mathcal{B}$ which is a zero or pole of $f$, we form $\langle w \rangle_{PC}$ the level 0 member of $PC$ formed from $\langle w \rangle_{P}$. If $w \in \mathcal{B}$ is a zero of $f'$ which is not a zero of $f$, then we form $\langle \Lambda_w \rangle_{PC}$ as follows.

Let $D$ be a bounded face of $\Lambda_w$. Corollary 3.10 from [5] implies that one of the two following cases hold.

**Case 2.0.1.** There is a single distinct zero or pole of $f$ contained in $D$, and there is no zero of $f'$ contained in $D$ which is not a zero of $f$.

In this case, let $w'$ denote this single distinct zero or pole of $f$. $Z(\langle w' \rangle_{P}) = z(D)$, since $w'$ is the only zero or pole of $f$ in $D$. If $k > 0$ then $w'$ is a zero of $f$. Therefore $H(\langle w' \rangle_{P}) = 0 < H(\langle \Lambda_w \rangle_{P})$, so we may associate $(w')_{PC}$ to $D$. Similarly, if $k < 0$ then $w'$ is a pole of $f$. Therefore $H(\langle w' \rangle_{PC}) = \infty > H(\langle \Lambda_w \rangle_{P})$, so we may associate $\langle w' \rangle_{PC}$ to $D$. Finally, we define $g_D$ be the map which takes each distinguished point in $\partial D$ to $w'$.
Case 2.0.2. There is some critical point \( w' \) of \( f \) in \( G \) which is not a zero of \( f \), and such that each member of \( B \) which is in \( D \) is either in \( \Lambda_w \) or in one of the bounded faces of \( \Lambda_{w'} \).

Proceed recursively. Assume that \( \langle \Lambda_{w'} \rangle_{PC} \) has been already been formed. Since each zero and pole of \( f \) is in some bounded face of \( \Lambda_{w'} \), \( Z(\langle \Lambda_{w'} \rangle_{PC}) = z(D) \). Furthermore, if \( k > 0 \), then the value of \( |f| \) on \( \Lambda_{w'} \) is strictly less than the value of \( |f| \) on \( \Lambda_w \), and therefore \( H(\langle \Lambda_{w'} \rangle_{P}) = H(\langle \Lambda_{w} \rangle_{P}) \), so we may associate \( \langle \Lambda_{w'} \rangle_{PC} \) to \( D \). On the other hand, if \( k < 0 \), then \( H(\langle \Lambda_{w'} \rangle_{P}) < H(\langle \Lambda_{w} \rangle_{P}) \), so we may associate \( \langle \Lambda_{w'} \rangle_{PC} \) to \( D \). Now we wish to define \( g_D \).

Let \( z \in \partial D \) be distinguished (thus \( f(z) > 0 \)). The fact that no gradient lines may intersect in the region \( D \) which is exterior to \( \Lambda_{w'} \) (since there are no zeros of \( f' \) in that region) gives us that there is only a single distinguished point in \( \Lambda_{w'} \) which is connected to \( z \) by a portion of a gradient line of \( f \) which lies entirely in \( D \) but exterior to \( \Lambda_{w'} \). Call that distinguished point \( z' \in \Lambda_{w'} \). Then we define \( g_D(z) := z' \). Since the gradient lines of \( f \) do not cross in the region of \( D \) exterior to \( \Lambda_{w'} \), the map \( g_D \) so defined respects the order of the distinguished points as they appear in \( \partial D \).

Do this assignment process, and definition of the gradient map, for each bounded face of \( \Lambda_w \). The resulting object we call \( \langle \Lambda_w \rangle_{PC} \). Since \( B \) has finitely many members, this process terminates. Corollary 3.10 from [5] implies that there is some point \( w \in B \) such that each \( w' \in B \) is either in \( \Lambda_w \) or in one of the bounded faces of \( \Lambda_w \). Furthermore, if \( w, w' \) are any two such points, it is easy to see that \( \langle \Lambda_w \rangle_{PC} \) and \( \langle \Lambda_{w'} \rangle_{PC} \) as defined above are equal. Therefore we may define \( \Pi(f,G) := \langle \Lambda_w \rangle_{PC} \).

Thus we classify the ways in which the critical level curves of a function \( f \) may be configured in its domain \( G \) by \( \Pi(f,G) \).

### 3 \( \Pi \) RESPECTS CONFORMAL EQUIVALENCE

Our goal in this section is to show that conformal equivalence of special type function elements may be determined entirely by their respective level curve structures, and specifically by the data about a function element which is preserved by the map \( \Pi \). That is, we have the following theorem.

**Theorem 3.1.** If \( (f_1,G_1) \) and \( (f_2,G_2) \) are two special type function elements, then \( (f_1,G_1) \sim (f_2,G_2) \) if and only if \( \Pi(f_1,G_1) = \Pi(f_2,G_2) \).

**Proof.** The forward implication (if \( (f_1,G_1) \sim (f_2,G_2) \) then \( \Pi(f_1,G_1) = \Pi(f_2,G_2) \)) follows fairly straightforwardly from the definition of \( \sim \) and the definition of \( \Pi \). Let \( \phi : G_1 \to G_2 \) be a conformal map intertwining \( f_1 \) and \( f_2 \) (ie. \( f_1 \equiv f_2 \circ \phi \) on \( G_1 \)). Then if \( \lambda \) is a level curve of \( f_1 \) in \( G_1 \), then \( \phi(\lambda) \) is a level curve of \( f_2 \) in \( G_2 \). If \( w \) is a zero or pole or critical point of \( f_1 \) in \( G_1 \), then \( \phi(w) \) is a zero or pole or critical point respectively of \( f_2 \) in \( G_2 \) with the same multiplicity, and \( \phi \) carries gradient lines of \( f_1 \) to gradient lines of \( f_2 \). It follows immediately that the construction of \( \Pi(f_1,G_1) \) proceeds in exactly the same manner as the construction of \( \Pi(f_2,G_2) \), so we proceed to the more difficult backward implication.

Assume that \( \Pi(f_1,G_1) = \Pi(f_2,G_2) \). Let \( \langle \lambda \rangle_{PC} \) denote this common member of \( PC \).

For \( i \in \{1,2\} \), define \( B_i := \{ w \in G_i : f_i(w) = 0 \text{ or } f_i'(w) = 0 \text{ or } f_i(w) = \infty \} \). Define \( C_i \subset G_i \) by \( C_i := \bigcup_{w \in B_i} \Lambda_w \). Let \( \xi \) be some member of \( P \) used in the construction of \( \langle \lambda \rangle_{PC} \). For \( i \in \{1,2\} \), let \( \xi_i \) denote the level curve of \( f_i \) which gives rise to \( \xi \). Since \( \xi_1 \) and \( \xi_2 \) are the same when viewed as members of \( P \), there is an orientation preserving homeomorphism \( \phi : \xi_1 \to \xi_2 \). Furthermore, if \( E_1 \) is some edge in \( \xi_1 \), and \( E_2 \) is the corresponding edge in \( \xi_2 \), then \( E_1 \) and \( E_2 \) contain the same number of distinguished points. Therefore by reparameterizing \( \phi \), we may assume that \( \phi \) maps the distinguished points of \( \xi_1 \) to the distinguished points of \( \xi_2 \). Since we may form this orientation preserving homeomorphism for each \( \langle \xi \rangle_{PC} \) used to construct \( \langle \lambda \rangle_{PC} \), we may stitch these homeomorphisms together to obtain an orientation preserving homeomorphism \( \phi : \xi_1 \to \xi_2 \).

Furthermore, since \( \Pi \) preserves information about how distinguished points are connected by gradient lines, we may assume that \( \phi \) respects gradient lines. That is, if \( D_1 \) is a bounded face of \( \xi_1 \), and \( D_2 \) is the corresponding bounded face of \( \xi_2 \) (that is, \( \partial D_2 = \phi(\partial D_1) \)), and \( w_1 \) is a distinguished point in \( \partial D_1 \), then \( \phi \) may be chosen so that \( \phi(g_{D_1}(w_1)) = g_{D_2}(\phi(w_1)) \).
We now show that we can assume that $f_1 = f_2 \circ \phi$ on $C_1$. Let $\xi_1$ be some component of $C_1$. If $\xi_1$ is just a single point, then $\phi(\xi_1)$ is a single point, and because $\xi_1$ and $\phi(\xi_1)$ give rise to the same member of $P$, $\xi_1$ and $\phi(\xi_1)$ are either both zeros or both poles of $f_1$ and $f_2$ respectively. Therefore $f_1 = f_2 \circ \phi$ on $\xi_1$.

Now assume that $\xi_1$ is a critical level curve of $f_1$ such that $H((\xi_1) P) \in (0, \infty)$ (where $(\xi_1) P$ refers to the member of $P$ used to construct $\Pi(f_1, G_1)$ which arises from $\xi_1$). Then let $D$ be a bounded face of $\xi_1$. Let $K$ denote the collection of distinguished points and vertices of $\xi_1$ which are contained in $\partial D$. That is, $K$ is the collection of points in $\partial D$ at which $f_1$ takes positive real values, or $f_1' = 0$. Then since $\phi$ is a homeomorphism which preserves the information recorded by $\Pi$, (ie. the points in $\xi_1$ at which $f_1$ takes positive real values, the argument that $f_1$ takes at each vertex of the $\xi_i$, and the modulus of $f_1$ on $\xi_i$ for $i \in \{1, 2\}$), we have that $f_1 = f_2 \circ \phi$ at each point in $K$. Let $T$ be a segment of $\partial D$ whose end points are in $K$ and such that no other point in $T$ is in $K$. So $T$ does not contain any vertex of $\xi_1$, except possibly at the endpoints. Let $w, w'$ be the endpoints of $T$ chosen so that $w, T, w'$ is the order in which they appear as $\partial D$ is traversed with positive orientation. If $w'$ is distinguished, view its argument as being $2\pi$ rather than 0 for the moment. Assume that $z(D) > 0$. Then $\arg(f_1(\cdot))$ is a continuous function on $T$ that is increasing as $T$ is traversed in the positive direction, and the same may be said of $\arg(f_2(\cdot))$ on $\phi(T)$, and it is a well known fact that under these circumstances, there is a homeomorphism $\hat{\phi}$ mapping $T$ to $\phi(T)$ such that $\arg(f_1(\cdot)) = \arg(f_2(\cdot))$.

Replace $\phi$ by $\hat{\phi}$ on $T$. In this way it may be seen that $f_1 = f_2 \circ \phi$ on $\xi_1$. If $z(D) < 0$, the same result holds by an analogous argument. So we may assume that $f_1 = f_2 \circ \phi$ on all of $\xi_1$. We now wish to extend $\phi$ to $G_1 \setminus C_1$ while maintaining the property $f_1 = f_2 \circ \phi$.

Let $F$ be a component of $G_1 \setminus C_1$ and let $\phi(F)$ denote the corresponding component of $G_2 \setminus C_2$ (that is, the component of $G_2 \setminus C_2$ whose boundary is $\phi(\partial F)$). By Theorem 3.11 in [5], $F$ and $\phi(F)$ are homeomorphic to annuli. Let $L_i$ and $L_e$ denote the interior and exterior components of $\partial F$ respectively (and thus $\phi(L_i)$ and $\phi(L_e)$ are the interior and exterior components of $\partial \phi(F)$). Let $\langle L_i \rangle_p, \langle \phi(L_i) \rangle_p$ denote the members of $P$ which correspond to $L_i$ and $\phi(L_i)$ as described in the construction of the map $\Pi$. Let $N_1, N_2$ denote the number of zeros of $f_1, f_2$ minus the number of poles of $f_1, f_2$ contained in the bounded components of $F^c, \phi(F)^c$ respectively. By the method of the construction of $\langle L_i \rangle_p$ and $\langle \phi(L_i) \rangle_p$, $N_1 = Z(\langle L_i \rangle_p)$ and $N_2 = Z(\langle \phi(L_i) \rangle_p)$. Since $\Pi(f_1, G_1) = \Pi(f_2, G_2)$, we have $Z(\langle L_i \rangle_p) = Z(\langle \phi(L_i) \rangle_p)$, and thus we define $N := N_1 = N_2$. Let $\epsilon_i, \epsilon_e$ denote the magnitude that $f_1$ takes on the interior and exterior components of $\partial F$ respectively (and thus by the definition of $\Pi$, $\|f_2\|$ equals $\epsilon_i, \epsilon_e$ on the interior and exterior components of $\partial \phi(F)$ respectively). By inspecting the proof of Theorem 3.11 in [5], we see that $\phi_1 := f_1^{1/N}$ and $\phi_2 := f_2^{1/N}$ conformally map $F$ and $\phi(F)$ respectively to $\text{ann}(0; \epsilon_i^{1/N}, \epsilon_e^{1/N})$ (for any choice of the $1/N^{th}$ root). We define $\phi : F \rightarrow \phi(F)$ by $\phi := \phi_2^{-1} \circ \phi_1$. We now need to specify which $1/N^{th}$ roots will be used in the definition of $\phi_1$ and $\phi_2$ in order to ensure that $\phi$ defined as above on $F$ extends continuously to our map $\phi$ which is already defined on $\partial F$. To this end, select some portion of a gradient line in $F$ one of whose endpoints is in the $L_i$, and the other of whose endpoints is in $L_e$, and on which $\arg(f_1) = 0$. Let $\gamma_1$ denote this path, and let $\gamma_2$ denote the corresponding portion of gradient line in $\phi(F)$. We make the choice of $1/N^{th}$ roots so that we have $\arg(\phi_j) = 0$ on $\gamma_j$ for $j = 1, 2$.

Since $\phi_1$ and $\phi_2$ are conformal maps with the same range, $\phi$ defined as above is clearly a conformal map from $F$ to $\phi(F)$ (which we may now call $\phi(F)$ in earnest because $\phi$ is now defined on $F$ itself, not just its boundary). Moreover, $f_1 = \phi_1^N$ on $F$ and $f_2 = \phi_2^N$ on $\phi(F)$, thus for $z \in F$,

$$f_2(\phi(z)) = \left[\phi_2 \left(\phi_2^{-1}(\phi_1(z))\right)\right]^N = f_1(z).$$

We will now show that $\phi$ has the property that for any $z_0 \in \partial F$

$$\lim_{z \to z_0, z \in F} \phi(z) = \phi(z_0).$$

As a consequence of Theorem 3.8 in [5], each level curve of $f_1$ in $F$ intersects $\gamma_1$. Therefore for $z \in F$, we may define $\lambda_z$ to be level curve of $f_1$ which contains $z$, and $z_{\gamma_1}$ to be the point at which $\lambda_z$ intersects $\gamma_1$. Let $\sigma_z$ denote the path from $z_{\gamma_1}$ to $z$ through $\lambda_z$ with positive orientation. Define $\Delta_{\arg(z)}$ to be the change in $\arg(f_1)$ along $\sigma_z$, and assume that $\sigma_z$ has domain $[0, \Delta_{\arg(z)}]$, and that $\sigma_z$ is parameterized so
that for \( t \in [0, \Delta_{\arg}(z)] \), \( \arg(f_1(\sigma_z(t))) = t \). Make the similar definition for any \( w \in \phi(F) \) of \( \lambda_w, w_{\gamma_2}, \sigma_w \), and \( \Delta_{\arg}(w) \).

For \( z \in F \), with \( w = \phi(z) \), since \( f_1 \equiv f_2 \circ \phi \) on \( F \), the path \( \phi \circ \sigma_z \) is contained in \( \lambda_w \), and for \( t \in [0, \Delta_{\arg}(z)] \), \( f_2(\phi(\sigma_z(t))) = f_1(\sigma_z(t)) \), so \( \arg(f_2) \) is increasing along \( \phi \circ \sigma_z \), and this path is equal to \( \sigma_w \), and we thus have \( \Delta_{\arg}(w) = \Delta_{\arg}(z) \).

Let \( z_0 \in L_i \) be given (if \( z_0 \) is in the exterior boundary of \( F \) the similar argument is made). Choose \( \alpha_1, \ldots, \alpha_m \in [0, 2\pi N) \) so that for each \( 1 \leq j \leq m \), if one begins at the endpoint of \( \gamma_1 \) in \( L_i \), and traverses \( L_i \) with a positive orientation so that the observed change in \( \arg(f_1) \) is \( \alpha_j \), then the point arrived at is \( z_0 \) (note that \( m \) is greater than 1 if and only if \( z_0 \) is a critical point of \( f_1 \)). Since \( \Pi(f_1, G_1) = \Pi(f_2, G_2) \), we have that the same list of numbers have the corresponding property for \( f_2 \) and \( \phi(z_0) \). That is, if one begins at the endpoint of \( \gamma_2 \) in \( \phi(L_i) \), and traverses \( \phi(L_i) \) with positive orientation so that the observed change in \( \arg(f_2) \) is \( \alpha_j \) for some \( 1 \leq j \leq m \), then the point arrived at is \( \phi(z_0) \).

We will now make a direct \( \epsilon \) and \( \delta \) argument that the desired limit is obtained. Let \( \epsilon > 0 \) be given. Choose \( \delta_{\arg}, \delta_{\mod} > 0 \) so that for \( w \in \phi(F) \), if \( w \) satisfies \( |f_2(w)| - \epsilon \) < \( \delta_{\mod} \) and \( \Delta_{\arg}(w) - \alpha_j < \delta_{\arg} \) for some \( 1 \leq j \leq m \), then \( |w - \phi(z_0)| < \epsilon \). Choose now a \( \delta > 0 \) so that for \( z \in F \), if \( z \) satisfies \( |z - z_0| < \delta \), then \( |f_1(z)| - \epsilon \) < \( \delta_{\mod} \) and \( \Delta_{\arg}(z) - \alpha_j < \delta_{\arg} \) for some \( 1 \leq j \leq m \). Therefore since \( |f_2(\phi(z))| = |f_1(z)| \) and \( \Delta_{\arg}(z) = \Delta_{\arg}(\phi(z)) \), we conclude that for \( z \in F \), if \( |z - z_0| < \delta \), \( |\phi(z) - \phi(z_0)| < \epsilon \). That is, \( \lim_{z \to z_0, z \in F} \phi(z) = \phi(z_0) \).

Extending \( \phi \) as described above to each component of \( G_1 \setminus C_1 \), we conclude that \( \phi \) is a continuous bijection from \( G_1 \) to \( G_2 \), analytic on \( G_1 \setminus C_1 \), and \( f_1 \equiv f_2 \circ \phi \) on \( G_1 \). If \( z \in C_1 \setminus B_1, C_1 \) is locally smooth at \( z \), so by the Schwartz Reflection Principle \( \phi \) is analytic at \( z \). Finally, \( B_1 \) consists of finitely many isolated points, so for \( z \in B_1, \phi \) is analytic in a punctured neighborhood of \( z \), continuous at \( z \), thus analytic at \( z \).

We recall again the fact observed at the beginning of this section that \( (f_1, G_1) \sim (f_2, G_2) \Rightarrow \Pi(f_1, G_1) = \Pi(f_2, G_2) \) gives us that we may view \( \Pi \) as acting on the equivalence classes of special type function elements. That is, defining \( \Pi : H \to PC \) by \( \Pi([f, G]) := \Pi(f, G) \) is well defined. The backwards implication of Theorem 3.11 (that \( \Pi(f_1, G_1) = \Pi(f_2, G_2) \Rightarrow (f_1, G_1) \sim (f_2, G_2) \)) gives us that \( \Pi : H \to PC \) is injective.

4 \( \Pi \) IS SURJECTIVE ONTO \( H_a \): THE GENERIC CASE

It is fairly easy to show from the definition of \( \Pi \) that \( \Pi(H_a) \subset PC_a \) (one need only use the Maximum Modulus Principle). We will now begin to prove the following theorem.

Theorem 4.1. \( \Pi(H_a) = PC_a \).

In order to prove this theorem, we will partition \( H_a \) by the critical values of its elements. We will then define a notion of critical value for a member of \( PC_a \), and partition \( PC_a \) by the critical values of its elements. Having done this, we will show that for a given list of critical values \( v_1, \ldots, v_n \), \( \Pi \) maps the partition set of members of \( H_a \) with critical values \( v_1, \ldots, v_n \) surjectively onto the partition set of members of \( PC_a \) with this same list of critical values. In the course of our proof we will in fact show a stronger result, that we may consider only the action of \( \Pi \) on the members of \( H_a \) which have representatives \( (f, G) \) such that \( f \) is a polynomial, and \( \Pi \) maps these members of \( H_a \) surjectively onto \( PC_a \). First several definitions.

Definition: For \( G \subset C \) an open simply connected set, and \( f : G \to C \) analytic on \( G \), and \( \epsilon > 0 \), define \( G_{f, \epsilon} := \{ z \in G : |f(z)| < \epsilon \} \). Because of the definition of a special type function element, we define \( G_{f, 1} := G_f \).

Definition: Let \( H_p' \) be the set of all special type function elements \( (f, G) \) such that \( (f, G) \sim (p_{G_p, G_p}) \) for some \( p \in \mathbb{C}[z] \). Henceforth we will write \( (p, G_p) \) for \( (p_{G_p, G_p}) \). We also define \( H_p := H_p' / \sim \).

Since \( H_p \subset H_a \), if we can show that \( \Pi(H_p) = PC_a \), then we are done. That is, we wish to show that for any \( (\lambda)_{PC} \in PC \), there is a polynomial \( p \in \mathbb{C} \) such that \( \Pi(p, G_p) = (\lambda)_{PC} \). Our method, broadly speaking,
will be to partition \( H_p \) by critical values. That is, a partition set will be the collection of members of \( H_p \) which have a given list of critical values. We then define a notion of critical values for members of \( PC_a \), and partition \( PC_a \) by these critical values. We then show that for any finite list of critical values, \( \{v_1, \ldots, v_{n-1}\} \), \( \Pi \) maps the partition set of \( H_p \) corresponding to this list of critical values bijectively to the partition set of \( PC_a \) corresponding to this list of critical values. Having shown this, we can conclude that \( \Pi \) maps \( H_p \) bijectively to \( PC_a \).

We begin by building up some notation for dealing with the critical values we will be working with.

**Definition:** For \( n \) a positive integer, define \( V_n \subset \mathbb{C}^n \) by \( V_n := \{v = (v^{(1)}, \ldots, v^{(n)}) \in \mathbb{C}^n : 0 \leq |v^{(1)}| \leq \cdots \leq |v^{(n)}| < 1\} \). Then define \( V := \bigcup_{n=1}^{\infty} V_n \).

**Definition:** For \( x, y \in \mathbb{C}^n \), we use the metric \( |x - y| := \max(|x^{(i)} - y^{(i)}| : 1 \leq i \leq n) \).

We now define the partition of \( H_p \) which we will use.

**Definition:** For \( n \geq 2 \) an integer, and \( v \in V_{n-1} \), let \( H_{p,v}' \) denote the subset of members \( (f,G) \in H_p' \) such that the critical values of \( f \) are exactly \( v^{(1)}, v^{(2)}, \ldots, v^{(n-1)} \). Further, define \( H_{p,v} := H_{p,v}'/\sim \).

We now will work out a notion of critical values for a member of \( PC_a \). In essence, the critical values of a member \( (\xi)_{PC} \) are the critical values of any member of \( \Pi^{-1}(\xi)_{PC} \). Since we do not yet know that this set is non-empty, we will have to define the critical values of \( (\xi)_{PC} \) directly from \( (\xi)_{PC} \). We begin with some definitions having to do with graphs.

**Definition:** For \( \lambda \) a member of \( \hat{\mathcal{P}} \), and \( w \) a vertex of \( \lambda \), then we let \( m(w) \) denote the number of edges of \( \lambda \) incident to \( z \), where we count an edge twice if both of its endpoints are at \( w \). Furthermore, we say that \( w \) is a vertex of \( \lambda \) with multiplicity \( m(w) = 2 - 1 = 0 \). Note also that if \( f, G \) is a special type function element, and \( w \in G \) is a zero of \( f' \) with multiplicity \( k \), then \( f \) is \( (k+1) \)-to-1 in a neighborhood of \( w \). Therefore there are \( 2(k+1) \) edges of \( \Lambda_w \) which are incident to \( w \). Thus the multiplicity of \( w \) as a vertex of \( \Lambda_w \) is \( \frac{2(k+1)}{2} - 1 = k \).

**Definition:** Let \( (\xi)_{PC} \in PC \) be given. If \( (w)_{PC} \) is one of the level 0 members of \( PC \) used to form \( (\xi)_{PC} \), then we say that \( 0 \) is a critical value of \( (\xi)_{PC} \) with multiplicity \( Z((w)_{PC}) - 1 \). Suppose that \( (\lambda')_{P} \) is a member of \( P \) used to build \( (\xi)_{PC} \). Then if \( \lambda \) is a vertex of \( \lambda' \), we say that \( H((\lambda')_{P}) e^{\text{int}(w)} \) is a critical value of \( (\lambda)_{PC} \) of multiplicity equal to the multiplicity of \( w \) as a vertex of \( \lambda' \).

With this notion of critical values of a member of \( PC_a \) built up, we may now partition \( PC_a \) as follows.

**Definition:** For \( u = (u^{(1)}, \ldots, u^{(n-1)}) \in V_{n-1} \), define \( PC_{a,u} \) and \( PC_{p,v} \) to be the collection of members of \( PC_a \) and \( PC_{p,v} \) respectively whose critical values listed according to multiplicity are \( v^{(1)}, \ldots, v^{(n-1)} \). Let \( |PC_{a,u}| \) denote the number of elements of \( PC_{a,u} \). (In the context of polynomials, \( n \) is the number of zeros of the polynomial, and thus there would be \( n - 1 \) critical values of the polynomial.)

From the definition of critical values of a member of \( PC_a \), it should be clear that \( \Pi(H_{p,v}) \subset PC_{a,v} \). Then to show that \( \Pi : H_a \rightarrow PC_a \) is surjective, we show that \( \Pi : H_{p,v} \rightarrow PC_{a,v} \) is surjective for each \( v \in V \). In this section we prove that \( \Pi : H_{p,v} \rightarrow PC_{a,v} \) is surjective for any \( v \) in a dense subset \( U \) of \( V \) about to be defined. We then extend this to all of \( V \) in Section \( 5 \).
Definition: For $n$ a positive integer, define $U_n \subset V_n$ to be the collection of $v = (v^{(1)}, \ldots, v^{(n)}) \in V_n$ such that $0 < |v^{(1)}| < \cdots < |v^{(n)}| < 1$. Then define $U := \bigcup_{n=1}^{\infty} U_n$.

Our goal in this section is to prove the following result.

Lemma 4.2. For any $v \in U$, $\Pi : H_{a,v} \to PC_{a,v}$ is surjective.

Proof. Fix some positive integer $n \geq 2$ and $v_0 = (v_0^{(1)}, \ldots, v_0^{(n-1)}) \in U_{n-1}$. Since $\Pi$ is injective, it suffices to show that $|H_{a,v_0}| \geq |PC_{a,v_0}|$. We will in fact show that $|H_{p,v_0}| = |PC_{a,v_0}|$, which immediately implies the desired result (since $H_{p,v_0} \subset H_{a,v_0}$).

In a paper by Beardon, Carne, and Ng, it was shown that if $n = 2$, and $v \in V_{n-1}$, then $|H_{p,v}| = 1$. If $n \in \mathbb{N}$ with $n \geq 3$, and $v \in V_{n-1}$, $H_{a,v}$ has exactly $n^{n-3}$ elements according to multiplicity, where multiplicity arises through a use of Bezout’s theorem. As an easy corollary to what was shown in this paper, one may prove that if $n = 2$, and $v \in U_{n-1}$, then $|H_{p,v}| = 1$, and if $n \geq 3$, then $H_{p,v}$ contains exactly $n^{n-3}$ distinct members.

Note: Since $0 < |v_0^{(1)}| < \cdots < |v_0^{(n-1)}|$, if $\langle \lambda \rangle_{PC} \in PC_{a,v_0}$, and $\langle \lambda' \rangle_P$ is a member of $P$ used in the construction of $\langle \lambda \rangle_{PC}$, then $\lambda'$ contains only a single vertex. (Since if there were two vertices in $\lambda'$, each would give rise to a critical value, and these critical values would have the same modulus.) If $w$ is the vertex of $\lambda'$, then $H(\langle \lambda' \rangle_P)_{\xi_a(w)}$ is a critical value of $\langle \lambda \rangle_{PC}$ with multiplicity 1, so by definition of multiplicity, the number of edges of $\lambda'$ which meet at $w$ is $2 \ast (1 + 1) = 4$. There is only one member of $P$ which has a single vertex at which exactly 4 edges meet, namely the “figure eight” graph. (Recall that an edge is counted twice if both ends meet at the vertex.) Thus each graph used while constructing $\langle \lambda \rangle_{PC}$ is this figure eight graph.

We will use an induction argument to count the number of members of $PC_{a,v_0}$. In order to do this, it will be helpful to have an ordering on the members of $P$ used to construct a given member of $PC_a$, which we define now.

Definition: Fix some $\langle \lambda \rangle_{PC} \in PC_a$, and let $\langle \xi_1 \rangle_{PC}, \ldots, \langle \xi_n \rangle_{PC}$ with $n \geq 2$ be the members of $PC_a$ which are used in constructing $\langle \lambda \rangle_{PC}$. Then we say $\xi_i \prec \lambda$ with respect to $\langle \lambda \rangle_{PC}$ for each $i \in \{1,2,\ldots,n\}$. Furthermore, if some $\langle \xi_i \rangle_{PC}$ has been associated to some bounded face of some $\xi_j$ while constructing $\langle \lambda \rangle_{PC}$ for some $i,j$, then we say $\xi_i \prec \xi_j$ with respect to $\langle \lambda \rangle_{PC}$ (this “with respect to $\langle \lambda \rangle_{PC}$” will usually be suppressed when the member $\langle \lambda \rangle_{PC}$ in question is clear). We extend this to be a transitive relation. That is, if $\xi_i \prec \xi_k$, then we say $\xi_i \prec \xi_k$. If $\langle \lambda \rangle_{PC}$ is a member of $PC_{a,v_0}$, since all critical values of $\langle \lambda \rangle_{PC}$ are non-zero, and thus come from a vertices of a graph used in constructing $\langle \lambda \rangle_{PC}$, and each planar graph used in constructing $\langle \lambda \rangle_{PC}$ contains a single vertex, and each of these vertices gives rise to a critical value of $\langle \lambda \rangle_{PC}$, there must be $n-1$ distinct planar graphs used to construct $\langle \lambda \rangle_{PC}$. Therefore we make the following definition in order to be able to refer to the vertex which gives rise to a given critical value of $\langle \lambda \rangle_{PC}$.

Definition: For $\langle \lambda \rangle_{PC} \in PC_{a,v_0}$, and $i \in \{1,\ldots,n-1\}$, let $z_i$ denote the point or vertex from which the critical value $v_0^{(i)}$ arose. Furthermore, let $\lambda_i$ denote the planar graph which contains the vertex $z_i$. (Note that since $v_0 \in U_{n-1}$, this is well defined.)

We now wish to show that $PC_{a,v_0}$ has exactly 1 member if $n = 2$, and $n^{n-3}$ distinct members if $n \geq 3$. We will have to handle the different possible values of $n$ separately up to $n = 6$. For $n \geq 6$ we will be able to make a general argument.

Case 4.2.1. $n = 2$. 

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Let $⟨\lambda⟩_{PC}$ be a member of $PC_{a,v_0}$. Since $⟨\lambda⟩_{PC}$ has a single critical value, $⟨\lambda⟩_{PC}$ is constructed from a single figure eight graph, namely $\lambda$. Let $D$ be either bounded face of $\lambda$, and let $⟨w⟩_{PC}$ denote the level 0 member of $PC$ associated with $D$. Since 0 is not a critical value of $⟨\lambda⟩_{PC}$, $Z(⟨w⟩_{PC}) = 1$, so there is only one distinguished point in $\partial D$, and thus there is only one possible choice of gradient map $g_D$, namely the one that takes the single distinguished point in $\partial D$ to $w$. Further if $z$ is the single vertex of $\lambda$, then $a(z) = \text{arg}(v_0^{(1)})$ and $H(⟨\lambda⟩_{P}) = |v_0^{(1)}|$ since $v_0^{(1)}$ is the only critical value of $⟨\lambda⟩_{PC}$. So all the data pertaining to $⟨\lambda⟩_{PC}$ is determined entirely by $v_0$. Thus $PC_{a,v_0}$ contains only a single element.

For the future cases we will need the following definition.

**Definition:** Let $⟨\lambda⟩_{PC}$ be a member of $PC_a$, and let $D$ denote one of the bounded faces of $\lambda$. For some $⟨\xi⟩_{P}$ used in constructing $⟨\lambda⟩_{PC}$, if $⟨\xi⟩_{PC}$ were associated to $D$, then we say $\xi < D$. We extend this as follows. If $⟨\xi_1⟩_{P}, \ldots, ⟨\xi_k⟩_{P}$ were used in the construction of $⟨\lambda⟩_{PC}$, and $\xi_1 < \cdots < \xi_k < D$, then we say $\xi_1 < D$.

**Note:** Let $⟨\lambda⟩_{PC}$ be any member of $PC$, let $⟨\lambda'⟩_{P}$ be any member of $P$ used in the construction of $⟨\lambda⟩_{PC}$, and let $D$ be any face of $\lambda'$. An easy induction argument gives that the number of single point elements $⟨w⟩_{P}$ of $P$ such that $w < D$ is exactly $z(D)$ (where these single point members of $P$ are counted according to multiplicity), and that the number of critical values of $⟨\lambda⟩_{PC}$ which come from members of $PC$ associated to $D$ is exactly $z(D) - 1$.

**Definition:** Let $⟨\lambda⟩_{PC}$ be any member of $PC_{a,v_0}$. Since $v_0$ was taken from $U_{n-1}$, $\lambda$ only contains one vertex and has only two bounded faces. Let $D_1$ denote the bounded face of $\lambda$ from which fewer critical values come. Let $D_2$ denote the other one. That is, the naming is done so that $z(D_1) \leq z(D_2)$. (If both bounded faces of $\lambda$ give rise to the same number of critical values, then this naming is arbitrary.)

**Note:** For any member $⟨\lambda⟩_{PC}$ of $PC_{a,v_0}$, $z(D_1) + z(D_2) = n$. There are $n - 1$ total critical values of $⟨\lambda⟩_{PC}$, and one of those critical values comes from the vertex of $\lambda$, so $n - 2$ of them come from the two regions $D_1$ and $D_2$. This together with the fact that $z(D_1) \leq z(D_2)$ immediately gives that the possible values for $z(D_1) - 1$ to take are exactly $\{k \in \mathbb{Z} : 0 \leq k \leq \frac{n-2}{2}\}$.

**Case 4.2.2.** $n = 3$.

Since $\lambda_1 < \lambda_2$, and $⟨\lambda_1⟩_{P}$ and $⟨\lambda_2⟩_{P}$ are the only planar graph members of $P$ used to construct $⟨\lambda_2⟩_{PC}$, we have that $⟨\lambda_1⟩_{PC}$ has been associated to $D_1$. Let $⟨w⟩_{PC}$ denote the level zero member of $PC_a$ associated to $D_1$. Then as in Case 4.2.1 there is only one possible choice of $g_{D_1}$, namely the one that maps the single distinguished point of $\partial D_1$ to $w$. We claim that there is only a single choice of $g_{D_2}$ as well. Since $\lambda_1$ has exactly two bounded faces, and each of these faces has only a single level zero member of $PC_a$ associated to it, we have that $Z(⟨\lambda_1⟩_{P}) = 2$, and thus $z(D_2) = 2$. And thus there are exactly two distinguished points in $\partial D_2$. The two distinguished points in $\lambda_1$ are either both at the vertex, or neither at the vertex, and in either case there is a orientation preserving homeomorphism of $\lambda_1$ which exchanges the distinguished points. Thus modulo orientation preserving homeomorphism of $\lambda_1$ (and recall that the members of $PC$ are formed modulo orientation preserving homeomorphisms) there is only one possible choice of $g_{D_2}$. And it is easy to see as in Case 4.2.1 that all the rest of the data (ie the values $H(\cdot)$ and $a(\cdot)$ take) for $⟨\lambda_2⟩_{PC}$ is fully determined by $v_0$ (up to orientation preserving homeomorphism), and thus $PC_{a,v_0}$ has exactly one member.

**Note:** Our general way of counting the number of elements in $PC_{a,v_0}$ when $n \geq 3$ will be to partition $PC_{a,v_0}$ by the value $z(D_1) - 1$ takes. For a given value of $z(D_1) - 1$, we find out how many ways the $z(D_1) - 1$ different critical values may be chosen from the $n - 2$ critical values available to come from $D_1$ and $D_2$ (which is of course $\binom{n-2}{z(D_1)-1}$). For that choice of critical values coming from $D_1$, we count the number of members of $PC_a$ which may be associated to $D_1$ and the number which may be associated to $D_2$ (a natural induction step). We then count the number of choices of $g_{D_1}$ and of $g_{D_2}$ (which are $z(D_1)$ and $z(D_2)$ respectively, except in the case where $z(D_1) - 1 = 1$ or $z(D_2) - 1 = 1$ as we will see). We then multiply these numbers to find the number of members of $PC_{a,v_0}$ with the given value of $z(D_1) - 1$.

**Case 4.2.3.** $n = 4$.
As noted above, the only possible values of \( z(D_1) - 1 \) in this case are 0 and 1.

If \( z(D_1) - 1 = 0 \), then there is a single level 0 member of \( PC_a \) which could be associated to \( D_1 \). Let this member be called \( (w)_{PC} \). Then \( Z((w)_{PC}) = 1 \), so \( z(D) = 1 \). Thus there is a single choice of \( g_{D_1} \), namely the map that takes the single distinguished point in \( \partial D_1 \) to \( w \). On the other hand, since \( z(D_1) - 1 = 0, z(D_2) - 1 = 2 \), and from Case 4.2.2 we know that then there is only a single member of \( PC_a \) whose critical values are \( v_0^{(1)}, v_0^{(2)} \). However since \( z(D_2) - 1 = 2 \), \( \partial D_2 \) has 3 distinguished points, so there are three different choices of \( g_{D_2} \). Thus there are \( \binom{2}{0} \) * \( 1 \) * \( 1 \) * \( 1 \) = \( 3 \) members of \( PC_{a,v_0} \) for which \( z(D_1) - 1 = 0 \). If \( z(D_1) - 1 = 0 \), then there is a single level 0 member of \( PC_a \) which could be associated to \( D_1 \). Let this member be called \( (w)_{PC} \). Then \( Z((w)_{PC}) = 1 \), so \( z(D) = 1 \). Thus there is a single choice of \( g_{D_1} \), namely the map that takes the single distinguished point in \( \partial D_1 \) to \( w \). On the other hand, since \( z(D_1) - 1 = 0 \), \( z(D_2) - 1 = 2 \), and from Case 4.2.2 we know that then there is only a single member of \( PC_a \) whose critical values are \( v_0^{(1)}, v_0^{(2)} \). However since \( z(D_2) - 1 = 2 \), \( \partial D_2 \) has 3 distinguished points, so there are three different choices of \( g_{D_2} \). Thus there are \( \binom{2}{0} \) * \( 1 \) * \( 1 \) * \( 1 \) = \( 3 \) members of \( PC_{a,v_0} \) for which \( z(D_1) - 1 = 0 \).

Suppose \( z(D_1) - 1 = 1 \), and thus \( z(D_2) - 1 = 1 \). Hence there are \( \binom{2}{1} = 2 \) possible choices of the critical value which comes from \( D_1 \). By the work done for Case 4.2.1 there is a single possible member of \( PC_a \) which may be associated to \( D_1 \) and a single member of \( PC_a \) which may be associated to \( D_2 \). Also by the work done in Case 4.2.1 there is a single possible choice of \( g_{D_1} \) and a single possible choice of \( g_{D_2} \). And hence we count \( \binom{2}{1} \) = \( 2 \) members of \( PC_{a,v_0} \) for which \( z(D_1) - 1 = 1 \). But in this case, one critical value comes from \( D_1 \) and one comes from \( D_2 \), so we may switch the roles of \( D_1 \) and \( D_2 \) without breaking the restriction \( z(D_1) \leq z(D_2) \). That is, we are overcounting by a factor of 2. So finally we get that there is \( 2 * \frac{1}{2} = 1 \) member of \( PC_{a,v_0} \) for which \( z(D_1) - 1 = 1 \).

So \( |PC_{a,v_0}| = 3 + 1 = 4 = 4^{4-3} \).

**Case 4.2.4.** \( n = 5 \).

Here again the only possible values of \( z(D_1) - 1 \) are 0 and 1. For the sake of brevity we will leave out much of the explanation that can easily be translated from the previous case.

If \( z(D_1) - 1 = 0 \), there is only one member of \( PC_a \) which may be associated to \( D_1 \), and one choice of \( g_{D_1} \). From Case 4.2.3 above (prefiguring an inductive argument here), there are 4 members of \( PC_a \) that may be associated to \( D_2 \), and for each choice of the member of \( PC_a \) associated to \( D_2 \), there are 4 possible choices of \( g_{D_2} \). Hence there are \( \binom{2}{0} \) * \( 1 \) * \( 1 \) * \( 4 \) * \( 4 \) = \( 16 \) members for which \( z(D_1) - 1 = 0 \).

If \( z(D_1) - 1 = 1 \), there are \( \binom{3}{1} \) different choices of the critical value that comes from \( D_1 \). For that choice, there is a single member of \( PC_a \) which may be associated to \( D_1 \), and a single choice of \( g_{D_1} \). For that choice of the critical value in \( D_1 \) and thus the two critical values which come from \( D_2 \), there is a single member of \( PC_a \) which may be associated to \( D_2 \), and 3 choices of \( g_{D_2} \). Hence \( PC_{a,v_0} \) has \( \binom{3}{1} \) * \( 1 \) * \( 1 \) * \( 1 \) = \( 9 \) members for which \( z(D_1) - 1 = 1 \).

So \( |PC_{a,v_0}| = 16 + 9 = 25 = 5^{5-3} \).

**Case 4.2.5.** \( n \geq 6 \).

In the following calculations, the first number will be the number of ways of choosing the critical values which come from \( D_1 \). The second number will be the number of members of \( PC_a \) which may be associated to \( D_1 \) (induction step) for the given choice of critical values coming from \( D_1 \). The third number will be the number of possible choices of \( g_{D_1} \). The fourth number will be the number of members of \( PC_a \) which may be associated to \( D_2 \) (induction step) The fifth number will be the number of possible choices of \( g_{D_2} \).

Assume first that \( n \) is odd. Then \( z(D_1) - 1 \) can take any value in the set

\[
\left\{ 0, 1, \ldots, \frac{(n-2)-1}{2} = \frac{n-3}{2} \right\}.
\]

The number of members of \( PC_{a,v_0} \) for which \( z(D_1) - 1 = 0 \) is

\[
\binom{n-2}{0} * 1 * 1 * (n-2+1)^{(n-2+1)-3} * (n-2+1) = \binom{n-2}{0} (n-1)^{n-3}.
\]
The number of members of $PC_{a,v_0}$ for which $z(D_1) - 1 = 1$ is
\[
\binom{n-2}{1} * 1 * 1 * (n - 2 - 1 + 1)^{(n-2-1+1)-3} * (n - 2 - 1 + 1),
\]
which is equal to
\[
\binom{n-2}{1} (n - 2)^{n-4}.
\]
If $2 \leq i \leq \frac{n-3}{2}$, then the number of members of $PC_{a,v_0}$ for which $z(D_1) - 1 = i$ is
\[
\binom{n-2}{i} * (i + 1)^{i-3} * (i + 1) * (n - 2 - i + 1)^{(n-2-i+1)-3} * (n - 2 - i + 1).
\]
Simplifying this, we conclude that the number of members of $PC_{a,v_0}$ for which $z(D_1) - 1 = i$ is
\[
\binom{n-2}{i} (i+1)^{i-1} (n - i - 1)^{n-i-3}.
\]
Hence we get that
\[
|PC_{a,v_0}| = \binom{n-2}{0} (n - 1)^{n-3} + \binom{n-2}{1} (n - 2)^{n-4} + \sum_{i=2}^{\frac{n-3}{2}} \binom{n-2}{i} (i+1)^{i-1} (n - i - 1)^{n-i-3}.
\]
However,
\[
\binom{n-2}{0} (n - 1)^{n-3} = \binom{n-2}{0} (0 + 1)^{0-1} (n - 0 - 1)^{n-0-3},
\]
and
\[
\binom{n-2}{1} (n - 2)^{n-4} = \binom{n-2}{1} (1 + 1)^{1-1} (n - 1 - 1)^{n-1-3},
\]
so we may include these terms in the sum. That is,
\[
|PC_{a,v_0}| = \sum_{i=0}^{\frac{n-3}{2}} \binom{n-2}{i} (i+1)^{i-1} (n - i - 1)^{n-i-3}.
\]
By performing the substitution $m = n - 2$, we obtain
\[
|PC_{a,v_0}| = \sum_{i=0}^{\frac{m-3}{2}} \binom{m}{i} (i+1)^{i-1} (m - i - 1)^{m-i-1},
\]
where the length of $v_0$ is now $m + 1$. A brief examination of this sum should then convince the reader that due to the symmetric nature of the above sum, we have
\[
|PC_{a,v_0}| = \sum_{i=0}^{\frac{m-1}{2}} \frac{1}{2} \binom{m}{i} (i+1)^{i-1} (m - i + 1)^{m-i-1}.
\]
From here we may invoke a result in [6] which then gives that $|PC_{a,v_0}| = (m + 2)^{m-1} = n^{n-3}$. Thus we have the desired result when $n$ is odd.

Assume now that $n$ is even. Our calculations here are identical to the case where $n$ is odd except in calculating the last term of the sum. Since $n$ is even, $z(D_1) - 1$ can take any value in \{0, 1, \ldots, \frac{n-3}{2}\}. In
counting the number of members of $PC_{a,v_0}$ with a given value of $z(D_1) - 1$ we get the same results as when $n$ is odd if $0 \leq i = z(D_1) - 1 < \frac{m - 2}{2}$, namely that the number of members of $PC_{a,v_0}$ for which $z(D_1) - 1 = i$ is $\binom{n - 2}{i} (i + 1)^{i-1} (n - i - 1)^{n - i - 3}$. However if $z(D_1) - 1 = \frac{n - 2}{2}$, as described in Case 4.2.3 the roles of $D_1$ and $D_2$ may be reversed, so we are over counting by a factor of 2. Thus we need to include a factor of $\frac{1}{2}$ when we count the number of ways in which $\frac{n - 2}{2}$ of the $n - 2$ critical values may be chosen as the ones arising from vertices in $D_1$. Hence when $n$ is even, the number of members of $PC_{a,v_0}$ for which $z(D_1) - 1 = \frac{n - 2}{2}$ is exactly

$$\frac{1}{2} \left( \frac{n - 2}{2} \right) \left( \frac{n - 2}{2} + 1 \right)^{\frac{n - 2}{2} - 1} \left( n - \frac{n - 2}{2} - 1 \right)^{n - \frac{n - 2}{2} - 3},$$

and therefore $|PC_{a,v_0}|$ equals

$$\left( \sum_{i=0}^{\frac{n - 2}{2} - 1} \binom{n - 2}{i} (i + 1)^{i-1} (n - i - 1)^{n - i - 3} \right) + \left( \frac{1}{2} \left( \frac{n - 2}{2} \right) \left( \frac{n - 2}{2} + 1 \right)^{\frac{n - 2}{2} - 1} \left( n - \frac{n - 2}{2} - 1 \right)^{n - \frac{n - 2}{2} - 3} \right).$$

Again using the substitution $m = n - 2$, and taking advantage of the symmetry in the sum, we find that

$$|PC_{a,v_0}| = \sum_{i=0}^{m} \frac{1}{2} \binom{m}{i} (i + 1)^{i-1} (m - i + 1)^{m - i - 1}.$$

Again by [6] we conclude that $|PC_{a,v_0}| = (m + 2)^{m-1} = n^{n-3}$.

Thus under the assumption that $v_0 \in U_{n-1}$, we conclude that $PC_{a,v_0}$ has precisely $n^{n-3}$ members.

We now have that $|PC_{a,v_0}| = n^{n-3} = |H_{p,v_0}| \leq |H_{a,v_0}|$, and $\Pi : H_{a,v_0} \rightarrow PC_{a,v_0}$ is injective, so we conclude that $\Pi : H_{a,v_0} \rightarrow PC_{a,v_0}$ is also surjective.

So we have the desired result for each $v \in U_{n-1}$.

An example is in order here. Unfortunately it may be very difficult either to determine the critical level curve configuration of a given function element, or to find a polynomial with a given critical leve curve configuration. Therefore our example is quite simple.

**Example:** Consider the function

$$f(z) = \frac{1}{6} \left( z^2 + \frac{9}{25} \right) e^z.$$

The shaded region $G$ in Figure 1 is one of the components of the set $\{ w : |f(w)| < 1 \}$. The critical point of $f$ in $G$ is at $z = -2$ and the corresponding critical value is non-zero, so the vector $v = (f(-2))$ is in $U_1$. Therefore by Lemma 4.2 there is some polynomial $p$ such that $(f, G) \sim (p, G_p)$. Consider, for example, the polynomial

$$p(z) = z^2 + f(-2).$$

The shaded region $D$ in Figure 2 is the set $\{ w : |p(w)| < 1 \}$. The critical point of $p$ is at $z = 0$. It is easy to see that the critical value which arises from the critical point of $f$ in $G$ is equal to the critical value of $p$. Since there is only one member of $PC_{n}$ which has a given single critical value and no other critical values. It follows that $\Pi(f, G) = \Pi(p, D)$. Therefore by Theorem 5.1 there is some conformal function $\phi : G \rightarrow D$ such that $f \equiv p \circ \phi$ on $G$. 

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5 \( \Pi \) IS SURJECTIVE: THE GENERAL CASE

Our goal in this section is to complete the proof of the following theorem.

**Theorem 5.1.** For any \( v \in V \), \( \Pi : H_{a,v} \rightarrow PC_{a,v} \) is surjective.

**Proof.** We begin with several definitions.

**Definition:** For \( v \in V_{n-1} \) say \( v \) is typical if \( v \in U_{n-1} \), in which case we say \( v \) has atypical degree \( 0 \) (so \( 0 < |v^{(1)}| < \cdots < |v^{(n-1)}| \)). Say \( v \) has atypical degree \( 1 \) if \( 0 = |v^{(1)}| < |v^{(2)}| < |v^{(3)}| < \cdots < |v^{(n-1)}| \). Say \( v \) has atypical degree \( k \) for \( 2 \leq k \leq n-1 \) if \( 0 \leq |v^{(1)}| \leq \cdots \leq |v^{(k-1)}| = |v^{(k)}| < |v^{(k+1)}| < \cdots < |v^{(n-1)}| \).

**Definition:** For \( u = (u^{(1)}, \ldots, u^{(n-1)}) \in \mathbb{C}^{n-1} \), define the polynomial \( p_u \) by \( p_u(w) := \int_0^w \prod_{i=1}^{n-1} (z - u^{(i)})dz. \)

Then define \( \Theta : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1} \) by \( \Theta(u) = (p_u(u^{(1)}), \ldots, p_u(u^{(n-1)})) \).

Of course for any \( u \in \mathbb{C}^{n-1} \), the critical points of \( p_u \) are exactly \( u^{(1)}, \ldots, u^{(n-1)} \), so \( \Theta \) may be thought of as taking a prescribed list of critical points to the corresponding list of critical values via a normalized polynomial \( p_u \) is the polynomial with critical points \( u^{(1)}, \ldots, u^{(n-1)} \) normalized so that \( p_u(0) = 0 \) and \( p_u' \) is monic. This function \( \Theta \) was studied in \( \Pi \), in which it was shown that for any \( n > 0 \), and any \( v = (v^{(1)}, \ldots, v^{(n-1)}) \in V \), and any \( (p, G_p) \in H_p \) whose critical values are \( v^{(1)}, \ldots, v^{(n-1)} \), there is some \( u \in \Theta^{-1}(v) \) such that \((p, G_p) \sim (p_u, G_{p_u})\). It was shown moreover that for any \( v \in \mathbb{C}^{n-1} \), if \( n \) equals \( 1 \) or \( 2 \) then \( \Theta^{-1}(v) \) contains exactly \( 1 \) element. On the other hand, for \( n \geq 3 \), \( \Theta^{-1}(v) \) contains exactly \( n^{n-3} \) elements counted according to multiplicity.

**Definition:** For any \( i \geq 0 \), let \( \mathcal{J}(i) \) denote the statement "For any \( \langle \lambda \rangle_{PC} \in PC_a \) whose vector of critical values \( v \in V_{n-1} \) has atypicality degree less than or equal to \( i \), there is a \( u \in \Theta^{-1}(v) \) such that \((p_u, G_{p_u}) \in H_p \) and \( \Pi(p_u, G_{p_u}) = \langle \lambda \rangle_{PC} \).

We wish to show \( \mathcal{J}(i) \) holds for each \( i \geq 0 \). Lemma 4.2 may now be restated as saying that \( \mathcal{J}(0) \) holds. Fix some \( M \geq 1 \) and assume inductively that \( \mathcal{J}(i) \) holds for all \( i \in \{0, \ldots, M-1\} \). We now wish to show that \( \mathcal{J}(M) \) holds. Fix some \( N-1 \geq M \) and some \( v_1 = (v_1^{(1)}, \ldots, v_1^{(N-1)}) \in V_{N-1} \) with atypicality degree \( M \). Assume that \( v_1 \neq (0, \ldots, 0) \), since if this were the case then it is easy to show the desired result holds. Now fix some member \( \langle \Lambda \rangle_{PC} \) of \( PC_{a,v_1} \). Our plan is to choose another member \( \langle \hat{\Lambda} \rangle_{PC} \) of \( PC_{a} \) which is in some sense to be determined very close to \( \langle \Lambda \rangle_{PC} \), but whose list of critical values \( \hat{v}_1 \) has atypicality degree strictly less than \( M \). By the induction assumption, there is some \( \hat{u}_1 \in \Theta^{-1}(\hat{v}_1) \) such that \( \Pi(p_{\hat{u}_1}, G_{p_{\hat{u}_1}}) = \langle \hat{\Lambda} \rangle_{PC} \). If
we choose $(\widehat{\Lambda})_{PC}$ so that $\widehat{v}_1$ is sufficiently close to $v_1$, this will ensure that the members of $\Theta^{-1}(\widehat{v}_1)$ are close to the members of $\Theta^{-1}(v_1)$. If we let $u_1$ denote a member of $\Theta^{-1}(v_1)$ which $\widehat{u}_1$ is close to, then we will show that $\Pi(p_{u_1},G_{p_{u_1}}) = (\Lambda)_{PC}$. First a couple of definitions.

**Definition:** For non-zero $x^{(1)}, x^{(2)} \in \mathbb{C}$, define

$$d_{\arg}(x^{(1)}, x^{(2)}) := \begin{cases} |\arg(x^{(1)}) - \arg(x^{(2)})|, & \text{if } x^{(1)} \neq 0 \neq x^{(2)} \text{ and } \arg(x^{(1)}) \neq \arg(x^{(2)}) \\ 2\pi, & \text{if } x^{(1)} = 0 \text{ or } x^{(2)} = 0 \text{ or } \arg(x^{(1)}) = \arg(x^{(2)}) \end{cases},$$

where the choice of $\arg(x^{(1)})$ and $\arg(x^{(2)})$ in the definition above is made so as to minimize $d_{\arg}(x^{(1)}, x^{(2)})$. For $x = (x^{(1)}, \ldots, x^{(m)}) \in \mathbb{C}^m$ with $m \geq 2$, define

$$d_{\arg}(x) := \min(d_{\arg}(x^{(i)}, x^{(j)}): 1 \leq i, j \leq m).$$

**Definition:** For $a, b \in \mathbb{R}$ with $a < b$, and for $I : a = i_0 < i_1 < \cdots < i_n = b$ a partition on $[a, b]$, define $|I| := \min(i_k - i_{k-1}: 1 \leq k \leq n)$. Let $\gamma : [a, b] \to \mathbb{C}$ be a path and let $f$ be a function analytic and non-zero on the image of $\gamma$. Define $\Delta_{\arg}(f, \gamma, I) := \sum_{k=1}^n \arg(f(\gamma(i_k))) - \arg(f(\gamma(i_{k-1})))$, where the choice of the arguments in each summand is made so as to minimize the magnitude of the summand. Since $f$ is non-zero on $\gamma$, it is not hard to show that the limit as $|I| \to 0$ of $\Delta_{\arg}(f, \gamma, I)$ exists (and is finite). Let $\Delta_{\arg}(f, \gamma)$ denote this limit. We call $\Delta_{\arg}(f, \gamma)$ the winding number of $f$ along $\gamma$. Define $|\Delta_{\arg}|(f, \gamma, I) := \sum_{k=1}^n |\arg(f(\gamma(i_k))) - \arg(f(\gamma(i_{k-1})))|$, where again the choice of arguments is made so as to minimize the summands. Again it is not hard to show that the limit as $|I| \to 0$ of $|\Delta_{\arg}|(f, \gamma, I)$ exists (although possibly infinite). We let $|\Delta_{\arg}|(f, \gamma)$ denote this limit, and we call $|\Delta_{\arg}|(f, \gamma)$ the total variation of $\arg(f)$ along $\gamma$.

**Definition:** For $x = (x^{(1)}, \ldots, x^{(m)}) \in \mathbb{C}^m$, define

$$\minmod(x) := \begin{cases} 0, & \text{if } x^{(1)} = \cdots = x^{(m)} = 0 \\ \min(|x^{(i)}|: 1 \leq i \leq m, x^{(i)} \neq 0), & \text{otherwise} \end{cases}.$$ 

**Definition:** For any $m \geq 2$ and $r = (r^{(1)}, \ldots, r^{(m)}) \in \mathbb{C}^m$, define

$$\mindiff(r^{(1)}, \ldots, r^{(m)}) := \begin{cases} 0, & \text{if } r^{(1)} = \cdots = r^{(m)} \\ \min(|r^{(i)} - r^{(j)}|: r^{(i)} \neq r^{(j)}, 1 \leq i \neq j \leq m), & \text{otherwise} \end{cases}.$$ 

We may write this as $\mindiff(r)$ as well.

We will now determine how close $\widehat{v}_1$ must be to $v_1$. We will choose several constants along the way, culminating in a choice of $\nu_1 > 0$ which will be how close we require $\widehat{v}_1$ to be to $v_1$, and which will govern our construction of $(\widehat{\Lambda})_{PC}$.

We begin by choosing a $\delta_1 > 0$ small enough that the following hold.

1. Let $u = (u^{(1)}, \ldots, u^{(N-1)})$ be some point in $\Theta^{-1}(v_1)$. Fix some $i \in \{1, \ldots, N-1\}$ such that $v_1^{(i)} \neq 0$, and let $\widehat{u} \in B_{\delta_1}(u)$ be given. Let $L$ be a line segment contained in $B_{4\delta_1}(u^{(i)})$. Then $|\Delta_{\arg}|(p_{\widehat{u}}, L) < \frac{d_{\arg}(1,v_1)}{4}$. This follows from the finiteness of $\Theta^{-1}(v_1)$ and the compactness of $\text{cl}(B_{\delta_1}(u))$ and $\text{cl}(B_{4\delta_1}(u^{(i)}))$ for each $u \in \Theta^{-1}(v_1)$.
2. For any \( v = (v^{(1)}, \ldots, v^{(N-1)}) \in V_{N-1} \) such that \( |v - v_1| < 1 \) and \( d_{\arg}(v) \geq \frac{d_{\arg}(1,v^{(1)}, \ldots, v^{(N-1)})}{2} \), if \( u \in \Theta^{-1}(v) \), and \( \lambda \) is a critical level curve of \((p_u, G_{p_u})\) with \( |p_u| \equiv r \) on \( \lambda \) for some \( r \geq \minmod(v_1) \), then in each edge \( E \) of \( \lambda \) there is some point \( z \) which is greater than \( \delta_1 \) away from each critical point of \( p_u \), and greater than \( \delta_1 \) away from each edge of a critical level curve of \( p_u \) other than \( E \). We may do this by Lemma A.4.

3. For \( u \in \Theta^{-1}(v_1) \), let \( D \) denote either all of \( G_{p_u} \), or a bounded face of one of the critical level curves \( \lambda \) of \( p_u \) such that \( D \) contains a critical point of \( p_u \) whose corresponding critical value is non-zero. Let \( \lambda_D \) be the critical level curve of \( p_u \) in \( D \) which is maximal with respect to \( D \). Let \( m \) denote the number of distinct edges in \( \lambda \). Let \( E^{(1)}, \ldots, E^{(m)} \) be some enumeration of the edges of \( \lambda_D \). For each \( i \in \{1, \ldots, m \} \), choose some point \( z^{(i)} \) in \( E^{(i)} \) such that \( \arg(p_u(z^{(i)})) \) is greater than \( \frac{d_{\arg}(1,v^{(1)}, \ldots, v^{(N-1)})}{4} \) away from each of \( \{\arg(v^{(1)}), \ldots, \arg(v^{(N-1)})\} \). Let \( y^{(i)} \) be the point in \( \partial D \) which is connected to \( z^{(i)} \) by a section of a gradient line of \( p_u \), and let \( \sigma^{(i)} \) denote this section of gradient line which connects \( z^{(i)} \) and \( y^{(i)} \). Since \( \sigma^{(i)} \) is a portion of a gradient line of \( p_u \), \( \arg(p_u(y^{(i)})) = \arg(p_u(z^{(i)})) \), so \( y^{(i)} \) is not a critical point of \( p_u \). Since there are only finitely many such choices of \( u, \lambda, D \), we may construct such a collection of paths for each such choice of \( u, \lambda, D \), and choose \( \delta_1 \) so that for each such \( u, \lambda, D \), if \( i \in \{1, \ldots, m \} \) (here \( m \) depends on the choice of \( u, \lambda, D \), and \( j \in [0,1] \), there is no \( j \in [0,1] \) such that \( \sigma^{(j)}(s) \) is within \( 2\delta_1 \) of \( \sigma^{(i)}(t) \), and no critical point of \( p_u \) is within \( 2\delta_1 \) of \( \sigma^{(i)}(t) \), and there is no edge of any critical level curve of \( p_u \) other than the ones containing \( \sigma^{(i)}(0) \) and \( \sigma^{(i)}(1) \) within \( 2\delta_1 \) of \( \sigma^{(i)}(t) \).

4. For each \( u \in \Theta^{-1}(v_1) \), no critical level curve of \( p_u \) is within \( 2\delta_1 \) of \( \partial G_{p_u} \), and no critical level curve of \( p_u \) is within \( 2\delta_1 \) of any zero of \( p_u \), and no critical level curve of \( p_u \) is within \( 3\delta_1 \) of any other critical level curve of \( p_u \). This may be done by the finiteness of \( \Theta^{-1}(v_1) \).

5. For each \( u \in \Theta^{-1}(v_1) \), and each \( k \in \{1, \ldots, N-1 \} \), there is no point \( B_{2\delta_1}(u^{(k)}) \setminus \{u^{(k)}\} \) at which \( p_u \) takes the value \( v^{(k)} \).

6. For each \( u \in \Theta^{-1}(v_1) \), if \( |\hat{u} - u| < \delta_1 \), then for each \( k \in \{1, \ldots, N-1 \} \) such that \( v^{(k)} \neq 0 \), if \( |z - u^{(k)}| < 2\delta_1 \), then \( |p_u(z)| > \frac{\minmod(v_1)}{2} \). This follows because \( p_u \) depends continuously on \( u \).

7. \( \delta_1 < \frac{\minmod(v_1) \cdot d_{\arg}(1,v^{(1)})}{4\pi^2} \).

8. \( \delta_1 < \frac{\min\text{diff}(v_1)}{4} \).

9. Let \( u \in \Theta^{-1}(v_1) \) be given. If \( x_1, x_2 \in G_{p_u} \) are both in critical level curves of \( p_u \), and \( \arg(p_u(x_1)) = \arg(p_u(x_2)) = 0 \), then either \( x_1 = x_2 \) or \(|x_1 - x_2| > 2\delta_1 \).

We now choose \( \delta_2 > 0 \) small enough so that each of the following holds.

1. \( \delta_2 < \delta_1 \).

2. For each \( u \in \Theta^{-1}(v_1) \), for each \( k \in \{1, \ldots, N-1 \} \), for each \( z \in B_{3\delta_2}(u^{(k)}) \), we have \( |p_u(u^{(k)}) - p_u(z)| < \delta_1 \).

3. By Lemma A.10, we may choose \( \delta_2 > 0 \) and \( \rho_1 > 0 \) so that for any \( u \in \Theta^{-1}(v_1) \), let \( \hat{u} \) be any point in \( B_{\rho_1}(u) \) and let \( \hat{x}_1, \hat{x}_2 \in G_{\hat{p}_u} \) be given such that \( \arg(p_u(x_1)) = \arg(p_u(x_2)) = 0 \), and such that there is a path \( \hat{\sigma} : [0, 1] \to G_{\hat{p}_u} \) such that \( \hat{\sigma}(0) = \hat{x}_1 \) and \( \hat{\sigma}(1) = \hat{x}_2 \) and \( \arg(p_{\hat{u}}(\hat{\sigma}(r))) = 0 \) for all \( r \in [0, 1] \). Then if \( x_1, x_2 \in G_{p_u} \) are such that \( \arg(p_u(x_1)) = \arg(p_u(x_2)) = 0 \) and \( |x_1 - x_2| < \delta_2 \) and \( |\hat{x}_1 - x_1| < \delta_2 \) and \( |\hat{x}_2 - x_2| < \delta_2 \), then there is a path \( \sigma : [0, 1] \to G_{p_u} \) such that \( \sigma(0) = x_1, \sigma(1) = x_2 \), and for all \( r \in [0, 1] \), \( \arg(p_u(\sigma(r))) = 0 \) and \( |\sigma(r) - \sigma(r)| < \delta_1 \). Moreover, if \( |p_{\hat{u}}| \) is strictly increasing or strictly decreasing along \( \hat{\sigma} \), then \( \sigma \) may be chosen so that \( |p_u| \) is strictly increasing or strictly decreasing along \( \sigma \) respectively.

In Item 3 above we chose a \( \rho_1 > 0 \). We now require that \( \rho_1 > 0 \) be chosen smaller if necessary so that the following holds.
1. \( \rho_1 < \delta_2 \).

2. We will use this second item to refer to the restriction on \( \rho_1 \) described in Item 3 for the choice of \( \delta_2 \) above.

3. Let \( u = (u^{(1)}, \ldots, u^{(N-1)}) \in \Theta^{-1}(v_1) \) be chosen. For \( \hat{u} \in B_{\rho_1}(u) \) define \( \hat{v} = (v^{(1)}, \ldots, v^{(n-1)}) := \Theta(\hat{u}) \). Suppose that \( \arg(v^{(k)}) = \arg(u^{(k)}) \) for each \( k \in \{1, \ldots, N-1\} \). For some \( k \in \{1, \ldots, N-1\} \) with \( |v^{(k)}| \neq 0 \), let \( \hat{\lambda} \) denote the level curve of \( p_{\hat{u}} \) which contains \( u^{(k)} \). Then the following holds. Let \( \hat{E} \) denote some edge of \( \hat{\lambda} \) which is incident to \( u^{(k)} \), and let \( \hat{\gamma} \) denote a parameterization of \( \hat{E} \) beginning at \( u^{(k)} \) parameterized with respect to \( \arg(p_{\hat{u}}) \). That is, if \( \Delta \) is the total change in argument of \( \arg(p_{\hat{u}}) \) along \( \hat{E} \), and \( \alpha \in [0,2\pi) \) is the argument of \( \hat{v}^{(k)} \), then \( \hat{\gamma} : [\alpha, \alpha + \Delta] \rightarrow \hat{\lambda} \) and satisfies \( \hat{\gamma}(\alpha) = \hat{v}^{(k)} \) and \( \arg(p_{\hat{u}}(\hat{\gamma}(t))) = t \) for all \( t \in [\alpha, \alpha + \Delta] \). Then if we let \( \lambda \) denote the critical level curve of \( p_u \) containing \( u^{(k)} \), there is a path \( \gamma : [\alpha, \alpha + \Delta] \rightarrow \hat{\lambda} \) such that \( \gamma(\alpha) = \hat{v}^{(k)} \), and for each \( r \in [\alpha, \alpha + \Delta] \), \( \arg(p_u(\gamma(r))) = r \) and \( |\gamma(r) - \hat{\gamma}(r)| < \delta_2 \). This may be done by Lemma \( \text{A.9} \).

4. For each \( u \in \Theta^{-1}(v_1) \), for each \( i \in \{1, \ldots, N-1\} \), \( B_{\rho_1}(u^{(i)}) \subseteq G_{p_u} \).

5. For each \( u \in \Theta^{-1}(v_1) \), if \( |\hat{u} - u| < \rho_1 \), and \( z \in G_{p_u} \), and \( |z - z'| < \rho_1 \), then \( |p_{\hat{u}}(z') - p_u(z)| < \delta_2 \).

6. For each \( u \in \Theta^{-1}(v_1), \rho_1 < \text{mindiff}(0, u^{(1)}, \ldots, u^{(N-1)}) \).

Finally, choose some \( \nu_1 > 0 \) small so that the following holds.

1. \( \nu_1 < \rho_1 \).

2. If \( \hat{v}_1 \in V_{N-1} \) satisfies \( |v_1 - \hat{v}_1| < \nu_1 \), and \( \hat{u} \in \Theta^{-1}(\hat{v}_1) \), then there is some \( u \in \Theta^{-1}(v_1) \) such that \( |u - \hat{u}| < \frac{\nu_1}{4} \). This may be done by Lemma \( \text{A.5} \).

3. \( \nu_1 < \text{mindiff}(0, |v_1^{(1)}|, \ldots, |v_1^{(N-1)}|) \).

4. \( \nu_1 < 1 \).

This \( \nu_1 \) just found will be how close \( \hat{v}_1 \) must be to \( v_1 \) to make the argument described before we began picking constants and thus obtain the desired result. We now proceed to construct a critical level curve configuration \( (\hat{\lambda})_{PC} \in PC \) with critical values \( \hat{v}_1 \in V_{N-1} \) satisfying \( |v_1 - \hat{v}_1| < \nu_1 \), and such that \( \hat{v}_1 \) has atypicality degree strictly less than \( M \).

We will first introduce some notation. Recall that all single point members of \( P \) are identical except for the value that \( Z(\cdot) \) takes. Therefore we make the following definition.

**Definition:** For each non-zero integer \( k \), let \( \langle w_k \rangle_P \) denote the single point member of \( P \) such that \( Z(\langle w_k \rangle_P) = k \).

**Definition:** For \( \langle \xi \rangle_{PC} \in PC \), and \( \epsilon > 0 \) we define \( E(\langle \xi \rangle_{PC}, \epsilon) \) to be the collection of members \( \langle \psi \rangle_{P} \in P \) used to construct \( \langle \xi \rangle_{PC} \) such that \( H(\langle \psi \rangle_{P}) = \epsilon \).

We will construct \( (\hat{\lambda})_{PC} \in PC \) differently depending on which of the following three cases into which \( (\lambda)_{PC} \in PC \) falls.

- \( |v_1^{(M)}| = 0 \).
- \( |v_1^{(M)}| > 0 \) and for each \( \langle \lambda \rangle \in E(\langle \lambda \rangle_{PC}, |v_1^{(M)}|) \), \( \langle \lambda \rangle \) only contains a single vertex (counting multiplicity).
- \( |v_1^{(M)}| > 0 \) and there is some member of \( E(\langle \lambda \rangle_{PC}, |v_1^{(M)}|) \) which contains more than one vertex (counted with multiplicity).
Case 5.1.1. $|v_1^{(M)}| = 0$.

Since 0 is a critical value of $\langle \Lambda \rangle_{PC}$, there is some level 0 member $\langle w_k \rangle_{PC} \in PC$ used in the construction of $\langle \Lambda \rangle_{PC}$ such that $k \geq 2$. That is, in the construction of $\langle \Lambda \rangle_{PC}$, $\langle w_k \rangle_{PC}$ was associated to a face of some member of $P$. Let $\langle \psi \rangle_p$ denote this member of $P$, and let $D$ denote the face of $\psi$ to which $\langle w_k \rangle_{PC}$ was associated. Then $g_D$ mapped each distinguished point in $\partial D$ to $w_k$. We will define $\langle \hat{\lambda} \rangle_{PC}$, another member of $PC$, to replace $\langle w_k \rangle_{PC}$ as we construct $\langle \hat{\lambda} \rangle_{PC}$, and in every other respect we will construct $\langle \hat{\lambda} \rangle_{PC}$ in the same manner as $\langle \lambda \rangle_{PC}$.

Let $\hat{\lambda}$ denote the "figure eight" planar graph. Let $x$ denote the vertex of $\hat{\lambda}$. Define $H(\langle \hat{\lambda} \rangle_p) := \frac{\nu_1}{2}$, and $a(x) := 0$. Let $D^{(1)}$ denote one of the bounded faces of $\hat{\lambda}$, and $D^{(2)}$ the other. Distinguish $x$ and distinguish $k - 2$ distinct points other than $x$ in the boundry of $D^{(1)}$.

With this auxiliary data we have formed a member of $P$, namely $\langle \hat{\lambda} \rangle_p$. To $D^{(1)}$ we associate $\langle w_{k-1} \rangle_{PC}$, and define $g_{D^{(1)}}$ by mapping each distinguished point in $\partial D^{(1)}$ to $w_{k-1}$. We associate $\langle w_1 \rangle_{PC}$ to $D^{(2)}$, and define $g_{D^{(2)}}$ to map the single distinguished point in $\partial D^{(2)}$ (namely $x$) to $w_1$. The resulting object is a member of $PC$, namely $\langle \hat{\lambda} \rangle_{PC}$.

We wish to construct $\langle \hat{\hat{\lambda}} \rangle_{PC}$ in exactly the same manner as $\langle \lambda \rangle_{PC}$, except by replacing $\langle w_k \rangle_{PC}$ with $\langle \hat{\lambda} \rangle_{PC}$. We may do this because by construction, $Z(\langle \hat{\lambda} \rangle_p) = k = Z(\langle w_k \rangle_p)$. The only thing remaining to do in the construction of $\langle \hat{\hat{\lambda}} \rangle_{PC}$ is specify $g_D$. Let $w^{(1)}$ be any fixed distinguished point in $\partial D$. Then define $g_D(w^{(1)}) := x$, and if $w$ is the $i^{th}$ distinguished point in $\partial D$ (for some $i \in \{1, \ldots, k - 1 \}$) in the positive direction after $w^{(1)}$, define $g_D(w)$ to be the $i^{th}$ distinguished point in $\partial D^{(1)}$ in the positive direction after $x$ (where the $(k - 1)^{st}$ distinguished point in $\partial D^{(1)}$ after $x$ is interpreted as being $x$ itself). Then proceeding with the construction in every other way the same as with $\langle \lambda \rangle_{PC}$, we obtain a member of $PC$, namely $\langle \hat{\hat{\lambda}} \rangle_{PC}$.

Note that the critical values of $\langle \hat{\lambda} \rangle_{PC}$ will be exactly $\vec{v}_1 := (0, \ldots, 0, \frac{\nu_1}{2}, v_1^{(M+1)}, \ldots, v_1^{(N-1)})$ (with $M - 1$ copies of 0), while $v_1 = (0, \ldots, 0, v_1^{(M+1)}, \ldots, v_1^{(N-1)})$, so $|v_1 - \vec{v}_1| = \frac{\nu_1}{2} < v_1$. Note also that since $\frac{\nu_1}{2} < \min\text{mod}(v_1) < |v_1^{(M+1)}|$, $\vec{v}_1$ has atypicallity degree $M - 1 < M$.

Case 5.1.2. $|v_1^{(M)}| > 0$ and for each $\langle \lambda \rangle_p \in E_{\langle \Lambda \rangle_{PC}, |v_1^{(M)}|}$, $\langle \lambda \rangle_p$ only contains a single vertex (counting multiplicity).

Let $\langle \lambda \rangle_p$ be some fixed member of $E_{\langle \Lambda \rangle_{PC}, |v_1^{(M)}|}$. We construct $\langle \hat{\hat{\lambda}} \rangle_{PC}$ identically to the construction of $\langle \Lambda \rangle_{PC}$, except we replace $\langle \lambda \rangle_{PC}$ in the construction with $\langle \hat{\lambda} \rangle_{PC}$, where $\langle \hat{\lambda} \rangle_{PC}$ is identical to $\langle \lambda \rangle_{PC}$ except that $H(\langle \hat{\lambda} \rangle_p) := (1 + \frac{\nu_1}{2})H(\langle \lambda \rangle_p) = (1 + \frac{\nu_1}{2})|v_1^{(M)}|$. Note that with this construction, the critical values of $\langle \hat{\hat{\lambda}} \rangle_{PC}$ are exactly $\vec{v}_1 := (v_1^{(1)}, \ldots, v_1^{(M-1)}, 1 + \frac{\nu_1}{2})v_1^{(M)}, v_1^{(M+1)}, \ldots, v_1^{(N-1)}$, so $|v_1 - \vec{v}_1| = \frac{\nu_1}{2} |v_1^{(M)}| \leq \frac{\nu_1}{2} < v_1$. For each $k \in \{1, \ldots, N - 1 \}$, let $v_1^{(k)}$ denote the $k^{th}$ entry of $\vec{v}_1$. Then

$$|v_1^{(M-1)}| = |v_1^{(M-1)}| = |v_1^{M}| < |(1 + \frac{\nu_1}{2})v_1^{(M)}| = |\vec{v}_1^{(M)}|.$$

So we have

$$|v_1^{(M+1)}| - |\vec{v}_1^{(M)}| = |v_1^{(M+1)}| - (1 + \frac{\nu_1}{2})|v_1^{(M)}| = |v_1^{(M+1)}| - |v_1^{(M)}| - \frac{\nu_1}{2}|v_1^{(M)}|.$$

And $|v_1^{(M+1)}| - |v_1^{(M)}| \geq \nu_1 > \text{mindiff}(0, |v_1^{(1)}|, \ldots, |v_1^{(N-1)}|)$, so

$$|v_1^{(M+1)}| - |v_1^{(M)}| \geq \nu_1 - \frac{\nu_1}{2}|v_1^{(M)}| > \frac{\nu_1}{2} > 0.$$

Thus we conclude that $\vec{v}_1$ has atypicality degree less than $M$.

Case 5.1.3. $|v_1^{(M)}| > 0$ and some member of $E_{\langle \Lambda \rangle_{PC}, |v_1^{(M)}|}$ contains more than one vertex (counting multiplicity).
We again let \( \langle \lambda \rangle_{PC} \) denote one of the members of \( PC \) used in constructing \( \langle \lambda \rangle_{PC} \) such that \( \langle \lambda \rangle_p \in E_{\langle \lambda \rangle_{PC},m^{(0)}(\lambda)} \), and such that \( \lambda \) contains more than one vertex. (Possibly \( \langle \lambda \rangle_{PC} = \langle \lambda \rangle_{PC} \).) We now construct \( \langle \tilde{\lambda} \rangle_{PC} \) which will take the place of \( \langle \lambda \rangle_{PC} \) as we construct \( \langle \tilde{\lambda} \rangle_{PC} \). First a definition.

**Definition:** Let \( \langle \lambda \rangle_{PC} \in PC_a \) be given with the assumption that \( \lambda \) has more than two edges. By Lemma A.2, we may find some bounded face \( F \) of \( \lambda \) such that the boundary of \( F \) consists of a single edge \( E \) of \( \lambda \).

- We define \( \lambda \setminus E \) to be the member of \( \tilde{P} \) which arises from the graph \( \lambda \) when the edge \( E \) is removed.
- We define \( \langle \lambda \setminus E \rangle_p \) to be the member of \( \tilde{P} \) which arises from \( \lambda \setminus E \) and inherits all of its auxiliary data from \( \langle \lambda \rangle_p \). Note that if \( x \) is the vertex of \( \lambda \) which \( E \) has as its endpoints, if the multiplicity of \( x \) as a vertex of \( \lambda \) equals 1, then \( x \) is no longer a vertex of \( \lambda \setminus E \), and thus \( a(x) \) no longer has any meaning for \( \langle \lambda \setminus E \rangle_p \).
- We define \( \langle \lambda \setminus E \rangle_{PC} \) to be the member of \( \tilde{P} \) which arises from \( \lambda \setminus E \) and inherits all of its auxiliary data from \( \langle \lambda \rangle_{PC} \). For example, if \( D \) is a bounded face of \( \lambda \) other than \( F \), and \( \langle \xi \rangle_{PC} \) is the member of \( PC \) associated to \( D \) in \( \langle \lambda \rangle_{PC} \), then we associate \( \langle \xi \rangle_{PC} \) to \( D \) in the construction of \( \langle \lambda \setminus E \rangle_{PC} \), and we carry over \( g_D \) to \( \langle \lambda \setminus E \rangle_{PC} \) as well.

By Lemma A.2, there is some face of \( \lambda \) which has only one edge of \( \lambda \) in its boundary. Let \( F^{(1)}, \ldots, F^{(h)} \) be the bounded faces of \( \lambda \), ordered so that \( F^{(h)} \) has only one edge of \( \lambda \) in its boundary, and let \( E \) denote the edge that forms the boundary of \( F^{(h)} \). Let \( z \) denote the vertex of \( \lambda \) which is incident to \( F^{(h)} \). Let \( m \) denote the multiplicity of \( z \) as a vertex of \( \lambda \). As noted above, if \( m = 1 \), then \( z \) is not a vertex of \( \lambda \setminus E \) (or one might say that \( z \) is a vertex of \( \lambda \setminus E \) with multiplicity 0), while if \( m > 1 \) then \( z \) is a vertex of \( \lambda \setminus E \) with multiplicity \( m - 1 \). Note that \( Z(\langle \lambda \setminus E \rangle_p) = \sum_{k=1}^{h-1} z(F^{(k)}) \).

Let \( \tilde{\lambda} \) denote the "figure eight" graph, and let \( D^{(1)}, D^{(2)} \) denote its two faces. Let \( x \) denote the vertex in \( \tilde{\lambda} \). From \( \lambda \) we will form a member of \( \tilde{P} \), and eventually a member of \( PC \) which will replace \( \langle \lambda \rangle_{PC} \) in the construction of \( \langle \lambda \rangle_{PC} \). Define \( H(\tilde{\lambda})_p := H(\langle \lambda \rangle_p) \). Define \( z(D^{(1)}) := Z(\langle \lambda \setminus E \rangle_p) \), and \( z(D^{(2)}) := z(F^{(h)}) \) (where \( F^{(h)} \) is viewed as a face of \( \lambda ) \). Distinguish \( z(D^{(i)}) \) points in \( \partial D^{(i)} \) for \( i = 1, 2 \), distinguishing \( x \) if and only if \( z \) is distinguished as a vertex in \( \langle \lambda \rangle_p \). Define \( a(x) := a(z) \) where \( a(z) \) comes from \( \langle \lambda \rangle_p \). With this data, we have a member of \( \tilde{P} \), namely \( \langle \tilde{\lambda} \rangle_p \).

Let \( \langle \xi \rangle_{PC} \) be the member of \( PC \) which was associated to \( F^{(h)} \) in the construction of \( \langle \lambda \rangle_{PC} \). Then we associate \( \langle \lambda \setminus E \rangle_{PC} \) to \( D^{(1)} \), and \( \langle \xi \rangle_{PC} \) to \( D^{(2)} \). We now wish to define \( g_{D^{(1)}} \) and \( g_{D^{(2)}} \).

In order to define \( g_{D^{(i)}} \), we define an enumeration of the distinguished points in \( \lambda \setminus E \). Let \( E' \) denote the edge in \( \lambda \) which is directly after \( E \) as \( \lambda \) is traversed with a positive orientation. Define \( y^{(1)} := z \) if \( z \) is a distinguished point in \( \lambda \setminus E \). Otherwise define \( y^{(1)} \) to be the first distinguished point after \( z \) in \( \lambda \setminus E \) with a positive orientation beginning with \( E' \). Continue traversing \( \lambda \setminus E \) with a positive orientation, and let \( y^{(2)}, \ldots, y^{(z(D^{(1)})+1)} \) be the distinguished points after \( y^{(1)} \) of \( \lambda \setminus E \) as they appear as \( \lambda \setminus E \) is traversed one full time beginning with \( E' \). Note that a distinguished point will appear in this list exactly \( n \) times if it is a distinguished point of \( \lambda \setminus E \) with multiplicity \( n \). Now let \( x^{(i)} \) be an enumeration of the distinguished points in \( \partial D^{(1)} \) as they appear in increasing order beginning with \( x \) if \( x \) is a distinguished point, and beginning with the first distinguished point after \( x \) otherwise. Finally we define \( g_{D^{(1)}}(x^{(i)}) := y^{(i)} \) for each \( i \).

We will now define \( g_{D^{(2)}} \). Let \( y^{(1)}, \ldots, y^{(z(D^{(2)})+1)} \) be the distinguished points in \( \partial F^{(h)} \) listed in increasing order, with \( y^{(1)} = z \) if \( z \) is distinguished in \( \langle \lambda \rangle_p \), and \( y^{(1)} \) the first distinguished point after \( z \) in \( \partial F^{(h)} \) otherwise. Let \( x^{(1)}, \ldots, x^{(z(D^{(2)})+1)} \) be the distinguished points in \( \partial D^{(2)} \) listed in increasing order, with \( x^{(1)} = x \) if \( x \) is distinguished, and \( x^{(1)} \) the first distinguished point in the positive direction from \( x \) in \( \partial D^{(2)} \) otherwise. Then for \( 1 \leq i \leq z(D^{(2)}) \), we define \( g_{D^{(2)}}(x^{(i)}) := g_{F^{(h)}}(y^{(i)}) \). With this data we have a member of \( PC \), namely \( \langle \tilde{\lambda} \rangle_{PC} \).
We now wish to construct $\langle \Lambda \rangle_{PC}$ in every respect the same as $\langle \lambda \rangle_{PC}$, except that $\langle \lambda \rangle_{PC}$ will be replaced in this construction with $\langle \hat{\lambda} \rangle_{PC}$. If $\langle \lambda \rangle_{PC} = \langle \hat{\lambda} \rangle_{PC}$, then we are done, and we define $\langle \Lambda \rangle_{PC} := \langle \hat{\lambda} \rangle_{PC}$. If $\langle \lambda \rangle_{PC} \neq \langle \hat{\lambda} \rangle_{PC}$, then $\langle \lambda \rangle_{PC}$ was associated to some face $D$ of $\langle \psi \rangle_{P}$ a member of $P$ during the construction of $\langle \lambda \rangle_{PC}$. Then $\langle \hat{\lambda} \rangle_{PC}$ may inherit all of its data from $\langle \lambda \rangle_{PC}$ (other than $\langle \lambda \rangle_{PC}$, which we have exchanged for $\langle \hat{\lambda} \rangle_{PC}$) except the gradient function $g_D$. Let $g_D$ denote the gradient map for $D$ in $\langle \lambda \rangle_{PC}$ (which maps the distinguished points in $\partial D$ to the distinguished points in $\hat{\lambda}$), and let $\hat{g}_D$ denote the gradient map for $D$ in $\langle \hat{\lambda} \rangle_{PC}$ (which will map the distinguished points in $\partial D$ to the distinguished points in $\hat{\lambda}$). To construct $\hat{g}_D$, we have two possible cases, first that $z$ is a distinguished point in $\lambda$, and, in fact, the only distinct distinguished point in $\lambda$, and second that there are distinguished points in $\lambda$ which are distinct from $z$.

**Sub-case 5.1.3.1.** $z$ is a distinguished point in $\lambda$, and $z$ is the only distinct distinguished point in $\lambda$.

Let $x^{(1)}, \ldots, x^{(z(D))}$ be an enumeration of the distinguished points in $\partial D$ listed in increasing order. Let $y^{(1)}, \ldots, y^{(z(D))}$ be an enumeration of the distinguished points in $\lambda$ listed in the order in which they appear as $\lambda$ is traversed, beginning and ending with $x$. Then we define $\hat{g}_D(x^{(i)}) := y^{(i)}$ for each $i$.

**Sub-case 5.1.3.2.** There are distinguished points in $\lambda$ which are distinct from $z$.

We define an enumeration of the distinguished points of $\lambda$ which we will then use to define $\hat{g}_D$. Let $\overline{y}^{(1)}, \ldots, \overline{y}^{(z(D^{(1))})}$ be an enumeration of the distinguished points of $\hat{\Lambda}$ which are in $\partial D^{(1)}$, beginning at $x$ if $x$ is a distinguished point of $\hat{\lambda}$, and beginning at the first distinguished point past $x$ otherwise. For each $i \in \{1, \ldots, z(D^{(1)})\}$, $\overline{y}^{(i)}$ is a distinguished point in $\lambda \setminus E$. Define $y^{(i)}$ to be the distinguished point in $\lambda$ which corresponds to the distinguished point $\overline{y}^{(i)}$ in $\lambda \setminus E$. For $i \in \{1, \ldots, z(D^{(2)})\}$, let $y^{(z(D^{(1)}+i))}$ denote the $i^{th}$ distinguished point in $\partial D^{(2)}$, beginning with $z$ if $z$ is distinguished, and beginning with the first distinguished point past $z$ otherwise. Let $y^{(z(D^{(1)}+i))}$ denote the $i^{th}$ distinguished point in $\partial D^{(2)}$, beginning with $x$ if $x$ is a distinguished point of $\hat{\lambda}$, and with the first distinguished point past $x$ in increasing order around $\partial D^{(2)}$ otherwise. Then $\{y^{(1)}, \ldots, y^{(z(D^{(1)}+z(D^{(2)}))}\}$ is an enumeration of the distinguished points in $\lambda$ in increasing order, and $\{y^{(1)}, \ldots, y^{(z(D^{(1)}+z(D^{(2)}))}\}$ is an enumeration of the distinguished points in $\lambda$ in increasing order as they appear around $\lambda$. Let $z^{(1)}, \ldots, z^{(z(D^{(1)}+z(D^{(2)}))}$ be any enumeration of the distinguished points in $\partial D$ such that $g_D(z^{(i)}) = y^{(i)}$ for each $i \in \{1, \ldots, z(D^{(1)}+z(D^{(2)}))\}$. Now for $i \in \{1, \ldots, z(D^{(1)}+z(D^{(2)}))\}$, we define $\hat{g}_D(z^{(i)}) := y^{(i)}$. With this definition, we have all the data needed for a member of $PC$, namely $\langle \hat{\lambda} \rangle_{PC}$.

Notice, then, that by the construction of $\langle \hat{\lambda} \rangle_{PC}$, the critical values of $\langle \hat{\lambda} \rangle_{PC}$ are exactly

$$\hat{v}_1 := (v_1^{(1)}, \ldots, v_1^{(M-1)}, (1 + \frac{\nu_1}{2})v_1^{(M)}, v_1^{(M+1)}, \ldots, v_1^{(N-1)}) \in V_{N-1}.$$

Therefore by the same argument as in Case 5.1.2, $\hat{v}_1$ has atypicality degree less than $M$, and

$$|v_1 - \hat{v}_1| = \frac{\nu_1}{2}|v_1^{(M)}| < \frac{\nu_1}{2} < \nu_1.$$

We will call the method of construction of $\langle \hat{\lambda} \rangle_{PC}$ found in Case 5.1.3 the “scattering method”, as $\langle \hat{\lambda} \rangle_{PC}$ is constructed by “scattering” one of the vertices of one of the graphs used to construct $\langle \lambda \rangle_{PC}$.

As a result of Cases 5.1.1 5.1.2 and 5.1.3 we now have a member of $PC$, $\langle \hat{\lambda} \rangle_{PC}$, with critical values $\hat{v}_1$ such that $|v_1 - \hat{v}_1| < \nu_1$ and the atypicality degree of $\hat{v}_1$ is strictly less than $M$. Note also that by construction, for each $k \in \{1, \ldots, N-1\}$, $\text{arg}(v_1^{(k)}) = \text{arg}(v_1^{(k)})$ (where we adopt the convention that $\text{arg}(0) = 0$).

By the induction assumption there is some $\hat{u}_1 = (u_1^{(1)}, \ldots, u_1^{(N-1)}) \in \Theta^{-1}(\hat{v}_1)$ such that $\Pi(p_{\hat{v}_1}, G_{p_{\hat{v}_1}}) = \langle \hat{\lambda} \rangle_{PC}$. By Item 2 in the choice of $v_1$, there is a $u_1 = (u_1^{(1)}, \ldots, u_1^{(N-1)}) \in C_{N-1}$ such that $\Theta(u_1) = v_1$ and $|v_1 - \hat{u}_1| < \rho_1$. Define $\langle \hat{\lambda} \rangle_{PC} := \Pi(p_{\hat{u}_1}, G_{p_{\hat{u}_1}})$. Our goal is to show that $\langle \lambda \rangle_{PC} = \langle \hat{\lambda} \rangle_{PC}$. First two definitions.
Definition: Let $\langle \lambda \rangle_{PC}$ be some member of $PC$, and let $D$ denote some face of $\lambda$. Then we let $\langle \lambda_D \rangle_{PC}$ denote the member of $PC$ which was associated to $D$ in the construction of $\langle \lambda \rangle_{PC}$.

Definition: Let $\langle \lambda \rangle_P$ be a graph member of $P$, and let $n \geq 2$ denote the number of edges of $\lambda$. We say an enumeration $E^{(1)}, \ldots, E^{(n)}$ of these edges is in order with respect to $\lambda$ (or just "in order" when $\lambda$ is obvious) if the order in which the edges appear when $\lambda$ is traversed one full time with positive orientation beginning with $E^{(1)}$ is exactly $E^{(1)}, \ldots, E^{(n)}$. Let $D$ be a bounded face of $\lambda$, and let $k \geq 1$ denote the number of edges of $\lambda$ which are in $\partial D$. We say that an enumeration $E^{(1)}, \ldots, E^{(k)}$ of these edges is in order with respect to $D$ if $E^{(1)}, \ldots, E^{(k)}$ is the order in which these edges appear as $\partial D$ is traversed one full time with positive orientation beginning with $E^{(1)}$.

We will show that $\langle \Lambda \rangle_{PC} = \langle \bar{\Lambda} \rangle_{PC}$ recursively, working "outside in", by doing the following steps.

1. Show that $\langle \Lambda \rangle_P = \langle \bar{\Lambda} \rangle_P$, and establish a correspondence between the vertices of $\langle \Lambda \rangle_P$ and the vertices of $\langle \bar{\Lambda} \rangle_P$ which respects the data contained in a member of $P$. (That is, if $k \geq 1$ is the number of vertices in $\Lambda$ and $\bar{\Lambda}$, find enumerations $u^{(1)}, \ldots, u^{(k)}$ and $u^{(1)}, \ldots, u^{(k)}$ of the vertices of $\Lambda$ and $\bar{\Lambda}$ respectively such that the following holds. For each $i \in \{1, \ldots, k\}$, $a(u^{(i)}) = a(u^{(i)})$. For each $i, j \in \{1, \ldots, k\}$, $\{|u^{(i)}|_{\bar{\Lambda}}\}$ is an edge in $\bar{\Lambda}$ if and only if $\{|u^{(i)}|_{\Lambda}\}$ is an edge in $\Lambda$ and, moreover, if $\{|u^{(i)}|_{\Lambda}\}$ is an edge in $\Lambda$, then $\{|u^{(i)}|_{\bar{\Lambda}}\}$ and $\{|u^{(i)}|_{\bar{\Lambda}}\}$ contain the same number of distinguished points (as edges in $\langle \Lambda \rangle_P$ and $\langle \bar{\Lambda} \rangle_P$ respectively). Finally, if $n \geq 2$ is the number of edges in $\Lambda$, then $\{|u^{(i)}|_{\bar{\Lambda}}\}$ is a list of edges of $\bar{\Lambda}$ as they appear in order around $\bar{\Lambda}$, then $\{|u^{(i)}|_{\bar{\Lambda}}\}$ is the order in which the edges of $\bar{\Lambda}$ appear around $\bar{\Lambda}$. Note that this will immediately provide a well-defined correspondence between the bounded faces $D$ of $\Lambda$ and the bounded faces $\bar{D}$ of $\bar{\Lambda}$ and between the distinguished points $x$ of $\langle \Lambda \rangle_P$ and the distinguished points $\bar{x}$ of $\langle \bar{\Lambda} \rangle_P$.)

2. For each bounded face $D$ of $\Lambda$, let $\langle \lambda_D \rangle_{PC}$ and $\langle \bar{\lambda_D} \rangle_{PC}$ denote the members of $PC$ assigned to $D$ and $\bar{D}$ during the construction of $\langle \Lambda \rangle_{PC}$ and $\langle \bar{\Lambda} \rangle_{PC}$ respectively. Show that $\langle \lambda_D \rangle_P = \langle \bar{\lambda_D} \rangle_P$, and establish a correspondence between the vertices of $\lambda_D$ and the vertices of $\bar{\lambda_D}$ as described in Step 1. Then show that the correspondence between $\langle \lambda_D \rangle_P$ and $\langle \bar{\lambda_D} \rangle_P$ respects the gradient maps $g_D$ and $g_{\bar{D}}$. That is, if $x$ is one of the distinguished points of $\Lambda$ in $\partial D$, and $\bar{x}$ is the corresponding distinguished point of $\bar{\Lambda}$ in $\partial \bar{D}$, then show that $g_{\bar{D}}(\bar{x})$ is the distinguished point in $\bar{\lambda_D}$ which corresponds to the distinguished point $g_D(x)$ in $\lambda_D$.

3. For each bounded face $D$ of $\Lambda$, iterate Step 2 for each face of $\Lambda_D$, etc.

Since $\langle \Lambda \rangle_{PC}, \langle \bar{\Lambda} \rangle_{PC}$ are constructed with finitely many steps, this process will terminate after finitely many steps. When this process terminates, we will have shown that $\langle \Lambda \rangle_{PC}$ and $\langle \bar{\Lambda} \rangle_{PC}$ have all the same data, and are therefore equal. Notice that the base case (Step 1 and Step 2 as written) is just a simpler case of the recursive step (Step 2 and Step 2 with any $\langle \lambda \rangle_{PC}$ used in the construction of $\langle \Lambda \rangle_{PC}$ inserted in the place of $\langle \lambda \rangle_{PC}$). Therefore we just do the recursive step.

Now for any $\langle \lambda \rangle_{PC}$ used to construct $\langle \Lambda \rangle_{PC}$, we will see that in the process of establishing the correspondence described above between the vertices and edges of $\langle \lambda \rangle_{PC}$ and the vertices and edges of the corresponding $\langle \Lambda \rangle_{PC}$ used to construct $\langle \Lambda \rangle_{PC}$ we will do the following. Let $D$ be a face of $\lambda$. Let $\bar{D}$ be the corresponding face of $\bar{\lambda}$. We will select a face $\bar{D}$ of some graph member $\langle \bar{\xi} \rangle_{PC}$ used to construct $\langle \bar{\Lambda} \rangle_{PC}$, such that $\bar{D}$ corresponds in a natural way to $D$ in the construction of $\langle \bar{\Lambda} \rangle_{PC}$. We will view $\bar{\lambda}$ and $\bar{\xi}$ as embedded in $\mathbb{C}$ as critical level curves of $p_{u_1}$ and $p_{\bar{\xi}}$, respectively, and find paths $\sigma$ and $\bar{\sigma}$ which parameterize $\bar{D}$ and $\bar{D}$ respectively such that $|\sigma(r) - \bar{\sigma}(r)| < \delta_1$ for each $r$. This then implies that for any $i \in \{1, \ldots, N - 1\}$, if $u^{(i)}_1 \in \bar{D}$, then $u^{(i)}_1 \in \bar{D}$. To see this implication, observe that, by Item 3 in the choice of $\delta_1$, if $u^{(i)}_1 \in \bar{D}$, then $B_{2\delta_1}(u^{(i)}_1) \subset \bar{D}$. But $|u^{(i)}_1 - u^{(i)}_1| < \rho_1 < \delta_1$, so $B_{2\delta_1}(u^{(i)}_1) \subset \bar{D}$. Since $|\sigma(r) - \bar{\sigma}(r)| < \delta_1$ for all $r$, the
winding number of \( \bar{\sigma} \) around \( u_i^{(1)} \) is the same as the winding number of \( \sigma \) around \( u_1^{(1)} \). Thus we conclude that \( u_1^{(1)} = \bar{D} \). Now let us return to our induction argument.

Suppose that Step 2 has just been completed for some \( \langle \lambda \rangle_p \) used to construct \( \langle \Lambda \rangle_{PC} \), with corresponding \( \langle \tilde{\lambda} \rangle_p \) used to construct \( \langle \tilde{\Lambda} \rangle_{PC} \). Let \( D \) be one of the bounded faces of \( \lambda \), and let \( \bar{D} \) be the corresponding bounded face of \( \tilde{\lambda} \). We will now describe the selection of the face \( \bar{D} \) referred to above. Let \( \langle \tilde{\Lambda} \rangle_{PC} \) denote the member of \( PC \) which replaced \( \langle \lambda \rangle_{PC} \) in the construction of \( \langle \tilde{\Lambda} \rangle_{PC} \).

**Case 5.1.4.** \( \langle \tilde{\lambda} \rangle_p = \langle \lambda \rangle_p \).

Then let \( \bar{D} \) denote the face \( D \) of \( \lambda \) viewed as a face of \( \tilde{\lambda} \).

**Case 5.1.5.** \( \langle \tilde{\lambda} \rangle_p \neq \langle \lambda \rangle_p \) and \( \langle \tilde{\lambda} \rangle_{PC} \) was not constructed using the scattering method.

Then \( \langle \tilde{\lambda} \rangle_p \) was formed by merely changing the value of \( H(\cdot) \) (that is, \( \langle \tilde{\lambda} \rangle_p = \langle \lambda \rangle_p \) except that \( H(\langle \tilde{\lambda} \rangle_p) = (1 + \frac{\nu}{\Lambda})H(\langle \lambda \rangle_p) \)). In this case we choose \( \bar{D} \) to be the face \( D \) of \( \lambda \) viewed as a face of \( \tilde{\lambda} \) as in the previous case.

**Case 5.1.6.** \( \langle \tilde{\lambda} \rangle_p \neq \langle \lambda \rangle_p \) and \( \langle \tilde{\lambda} \rangle_{PC} \) was constructed using the scattering method.

Recall that in this case, \( \langle \tilde{\lambda} \rangle_{PC} \) was formed by selecting a bounded face \( F \) of \( \lambda \) such that \( \partial F \) consists of only a single edge \( E^{(1)} \) of \( \lambda \). We then set \( \tilde{\lambda} \) to be the figure eight graph, and assign \( \langle \lambda_F \rangle_{PC} \) to one face of \( \tilde{\lambda} \), and \( \langle \lambda \setminus E^{(1)} \rangle_{PC} \) to the other face of \( \tilde{\lambda} \). We define auxiliary data (values of \( a(\cdot), \) values of \( H(\cdot), \) distinguished points, gradient maps) as described earlier in this section, and the resulting member of \( PC \) we call \( \langle \tilde{\lambda} \rangle_{PC} \).

(From now on we will refer to \( \langle \lambda \setminus E^{(1)} \rangle_{PC} \) as \( \langle \lambda \setminus E^{(1)} \rangle_{PC} \) to remind us that we are viewing \( \langle \lambda \setminus E^{(1)} \rangle_{PC} \) as a member of \( PC \) used to construct \( \langle \tilde{\lambda} \rangle_{PC} \).)

With the above description, if \( D = F \) we define \( \bar{D} \) to be the face of \( \tilde{\lambda} \) to which \( \langle \lambda_F \rangle_{PC} \) was assigned. If \( D \neq F \), we define \( \bar{D} \) to be the face of \( \lambda \setminus E^{(1)} \) which corresponds to \( D \) in the construction of \( \langle \lambda \setminus E^{(1)} \rangle_{PC} \).

We now let \( \langle \lambda_D \rangle_{PC}, \langle \lambda_D \rangle_{PC} \), and \( \langle \lambda_D \rangle_{PC} \) be the members of \( PC \) which are assigned to \( D, \bar{D} \), and \( \bar{D} \) in the construction of \( \langle \lambda \rangle_{PC}, \langle \tilde{\lambda} \rangle_{PC} \), and \( \langle \lambda \rangle_{PC} \) respectively.

Before we dive into the next argument, let us step back for a moment and review our strategy. \( \langle \lambda \rangle_{PC} \) was given to us as a member of \( PC \) with the prescribed critical values \( v_1 \), and our goal is to find a polynomial \( p \) such that \( \Pi(p, G_p) = \langle \lambda \rangle_{PC} \). We constructed \( \langle \lambda \rangle_{PC} \) in such a way as to be in some sense very similar to \( \langle \lambda \rangle_{PC} \), have critical values \( \bar{v}_1 \) very close to \( v_1 \), and so that, by the induction assumption, there is some \( u_1 \in C^{N-1} \) such that \( \Theta(u_1) = \bar{v}_1 \) and \( \Pi(p_{u_1}, G_{p_{u_1}}) = \langle \lambda \rangle_{PC} \). We then used Lemma 5.5 to find a point \( u_1 \in C^{N-1} \) close to \( \bar{u}_1 \) and such that \( \Theta(u_1) = v_1 \). We define \( \langle \lambda \rangle_{PC} := \Pi(p_{u_1}, G_{p_{u_1}}) \). We view \( \langle \tilde{\lambda} \rangle_{PC} \) and \( \langle \tilde{\lambda} \rangle_{PC} \) as being embedded in \( C \) as the critical level curves of \( p_{\bar{u}_1} \) and \( p_{u_1} \), respectively, and we wish to use the fact that \( \bar{u}_1 \) and \( u_1 \) are so close, along with the fact that \( \langle \lambda \rangle_{PC} \) and \( \langle \tilde{\lambda} \rangle_{PC} \) are close (in some sense), to show that \( \langle \lambda \rangle_{PC} = \langle \tilde{\lambda} \rangle_{PC} = \Pi(p_{u_1}, G_{p_{u_1}}) \), which is our desired result.

Right now we wish to show that \( \langle \lambda_D \rangle_p = \langle \lambda_D \rangle_p \).

**Case 5.1.7.** \( \langle \lambda_D \rangle_p \neq \langle \lambda_D \rangle_p \), and \( \langle \lambda_D \rangle_{PC} \) was formed using the scattering method.

Let \( L \geq 2 \) denote the number of edges in \( \lambda_D \). We will again let \( F \) denote the bounded face of \( \lambda_D \) which was removed during the formation of \( \langle \lambda_D \rangle_{PC} \), and let \( E^{(1)} \) denote the edge of \( \lambda_D \) which forms \( \partial F \). Let \( \tilde{F} \) denote the face of \( \tilde{\lambda_D} \) to which we assigned \( \langle \lambda_F \rangle_{PC} \). Let \( \tilde{G} \) denote the other face of \( \tilde{\lambda_D} \), namely the one to which \( \langle \lambda_D \rangle_{PC} \) was assigned. Let \( E^{(2)}, \ldots, E^{(L)} \) be the enumeration of the other edges of \( \lambda_D \) so that the order in which the edges of \( \lambda_D \) appear in order around \( \lambda_D \) beginning with \( E^{(1)} \) is exactly \( E^{(1)}, \ldots, E^{(L)} \). Let \( K \geq 1 \) denote the number of distinct vertices of \( \lambda_D \), and let \( x^{(1)}, \ldots, x^{(K)} \) be any enumeration of these vertices such that \( E^{(1)} \) has both its endpoints at \( x^{(1)} \). Now, for each \( i \in \{1, \ldots, L\} \) we wish to choose an edge (or a portion of an edge) of a graph used to construct \( \langle \lambda_D \rangle_{PC} \) which will correspond to the edge \( E^{(i)} \) in \( \lambda_D \).
Sub-case 5.1.7.1. $i = 1$.

Recall that $E^{(1)}$ forms the boundary of the face $F$ of $\lambda_D$. In this case we define $\hat{E}^{(i)}$ to be the edge of $\hat{\lambda_D}$ which forms the boundary of $\hat{F}$.

Sub-case 5.1.7.2. $i \neq 1$, and $x^{(1)}$ is not an end point of $E^{(i)}$.

In this case we define $\hat{E}^{(i)}$ to be the edge $E^{(i)}$ viewed as an edge of $\lambda_D \setminus E^{(1)}$.

Sub-case 5.1.7.3. $i \neq 1$, and $x^{(1)}$ is an end point of $E^{(i)}$.

The difficulty in this case arises from the fact that if $x^{(1)}$ is incident to only two bounded faces of $\lambda_D$, then $x^{(1)}$ is no longer a vertex of $\lambda_D \setminus E^{(1)}$.

Sub-sub-case 5.1.7.3.1. There are more than two bounded faces of $\lambda_D$ which are incident to $x^{(1)}$.

Then $x^{(1)}$ is still a vertex of of $\lambda_D \setminus E^{(1)}$, so we may define $\hat{E}^{(i)}$ to be the edge $E^{(i)}$ viewed as an edge in the graph $\lambda_D \setminus E^{(1)}$.

Sub-sub-case 5.1.7.3.2. There are only two bounded faces of $\lambda_D$ which are incident to $x^{(1)}$.

First a definition.

**Definition:** Let $(\xi)_P$ be a graph member of $P$, and let $E$ be an edge of $\xi$. Let $x$ denote the initial point of $E$ and let $y$ denote the final point of $E$. We wish to define a quantity which we will call the change in argument along $E$ (with respect to $(\xi)_P$). Define $r_1 := a(x)$ and define $r_2 := a(y)$. If $a(y) = 0$, then we instead define $r_2 := 2\pi$. Then we define the change in argument along $E$ with respect to $(\xi)_P$ to be $r_2 - r_1 + 2\pi n$, where $n$ denotes the number of distinguished points in $E$ which are not end points of $E$. Note that if $(\xi)_P$ arises as a critical level curve of some analytic function $f$, then the change in argument along $E$ with respect to $(\xi)_P$ is the same as the change in $\arg(f)$ along $E$.

Let $j \in \{1, \ldots, L\}$ be the index of the other end point of $E^{(i)}$. Recall that in this sub-case $x^{(1)}$ is one end point of $E^{(i)}$. Recall that $F$ is one of the bounded faces of $\lambda_D$ which is incident to $x^{(1)}$. Let $G$ denote the other. If $\partial G$ is formed by a single edge of $\lambda_D$, then $\lambda_D$ is the figure eight graph. However since $\lambda_D$ was formed using the scattering method, $\lambda_D$ is not the figure eight graph, so $\partial G$ contains more than one edge of $\lambda_D$. Since $E^{(i)}$ does not form the boundary of $F$, we conclude that $E^{(i)}$ is contained in $\partial G$. Therefore $E^{(i)}$ does not have both ends at $x^{(1)}$, and therefore $j \neq 1$. Therefore also we may view $x^{(1)}$ as a vertex in $\lambda_D \setminus E^{(1)}$.

Let $\Delta > 0$ denote the change in argument along $E^{(i)}$ where $E^{(i)}$ is traversed from $x^{(j)}$ to $x^{(1)}$. Let $\hat{E}$ denote the edge of $\lambda_D \setminus E^{(1)}$ which contains the point $x^{(1)}$ (which is no longer a vertex of $\lambda_D \setminus E^{(1)}$). Then let $z$ denote the point in $\hat{E}$ such that the change in $\arg(p_{\xi})$ along the portion of $\hat{E}$ beginning at $x^{(j)}$ and ending at $z$ is exactly $\Delta$. Then we define $\hat{E}^{(i)}$ to be this portion of $\hat{E}$.

Note that in each case, by the construction of $(\hat{\Lambda})_{PC}$, the change in argument along $\hat{E}^{(i)}$ with respect to $(\hat{\lambda_D})_P$ is the same as the change in argument along $E^{(i)}$ with respect to $(\lambda_D)_P$.

We now wish, for each $i \in \{1, \ldots, K\}$, to choose a vertex $x^{(i)}$ of one of the graphs used to construct $(\hat{\lambda_D})_{PC}$ in such a way that $x^{(i)}$ gives rise to $\hat{x}^{(i)}$ during the construction of $(\hat{\Lambda})_{PC}$. Let $\hat{x}^{(1)}$ denote the vertex in $\hat{\lambda_D}$ and, for each $i \in \{2, \ldots, K\}$, let $\hat{x}^{(i)}$ denote the vertex $x^{(i)}$ viewed as a vertex of $\lambda_D \setminus E^{(1)}$.

We now have a name for the vertex in $\hat{\lambda_D}$ and for each vertex in $\lambda_D \setminus E^{(1)}$, unless $x^{(1)}$ is still a vertex of $\lambda_D \setminus E^{(1)}$. In this case, let $\hat{x}^{(0)}$ denote the vertex $x^{(1)}$ in $\lambda_D \setminus E^{(1)}$. Now view $(\hat{\Lambda})_{PC}$ as embedded in $\mathbb{C}$ as the critical level curves of $p_{\xi}$. For each $i \in \{1, \ldots, K\}$, fix some choice of $t_i \in \{1, \ldots, N - 1\}$ such that $\hat{u}^{(t_i)}$ is the critical point of $p_{\xi}$ which gives rise to $\hat{x}^{(i)}$ in the construction of $\Pi(p_{\xi}, G_{p_{\xi}})$. Then we define $\hat{x}^{(i)} := u^{(t_i)}$ (note that this implies that $t_1 = M$ by the construction of $(\hat{\Lambda})_{PC}$). If $x^{(0)}$ is defined, then let
Thus \( x_{(0)} \) is the critical point of \( p_{\lambda} \) which gives rise to \( \lambda_{\hat{D}} \) in the construction of \( \Pi(p_{\lambda}, G_{p_{\lambda}}) \). Then we define \( x(0) := u_{1(0)} \).

Define \( \epsilon := H((\lambda_{D})_{P}) \). We now wish to show that for each \( k \in \{1, \ldots, K\} \) (and for \( k = 0 \) if \( x(0) \) is defined), \( \lambda_{(k)} \in \lambda_{\hat{D}} \). If \( k \in \{2, \ldots, K\} \) (or \( k = 0 \) if \( x(0) \) is defined), then \( t_{k} \neq M \), so

\[
|p_{u_{1}(x_{(k)})}| = |p_{u_{1}(u_{1}(t_{k}))}| = |v_{1}(t_{k})| = |v_{1}(t_{k})| = H((\lambda_{D})_{E_{1}})_{P}) = H((\lambda_{D})_{P}) = \epsilon.
\]

For \( k = 1 \), \( t_{k} = M \), so

\[
|p_{u_{1}(x_{(k)})}| = |p_{u_{1}(u_{1}(M))}| = |v_{1}(M)| = \left| \frac{1}{1 + \frac{\epsilon}{\mu}} v_{1}(t_{1}) \right| = \frac{1}{1 + \frac{\epsilon}{\mu}} H((\lambda_{\hat{D}})_{P}) = H((\lambda_{D})_{P}) = \epsilon.
\]

Now suppose by way of contradiction that there is some critical point \( z \) of \( p_{u_{1}} \) in \( \hat{D} \) such that \( |p_{u_{1}}(z)| > \epsilon \). Then \( |p_{u_{1}}(z)| \geq \epsilon + \text{mindiff}(v_{1}) \) by definition of mindiff. Choose some \( t \in \{1, \ldots, N-1\} \), such that \( u_{1}(t) = z \). Therefore \( u_{1}(t) \in \hat{D} \) and, since \( t_{k} = M \), \( t \neq M \), so \( |v_{1}(t)| \geq |v_{1}(t)| > \epsilon + \text{mindiff}(v_{1}) > H((\lambda_{\hat{D}})_{P}) \) by Item 8 in the choice of \( v_{1} \). Thus \( u_{1}(t) \) is not in one of the bounded faces of \( \lambda_{\hat{D}} \), which is a contradiction by the definition of \( (\lambda_{\hat{D}})_{PC} \). Therefore the value that \( |p_{u_{1}}| \) takes at each point in \( \{x_{(1)}, \ldots, x_{(K)}\} \) (and \( x(0) \) if it is defined) is \( \epsilon \), and there is no critical point of \( p_{u_{1}} \) in \( \hat{D} \) at which \( |p_{u_{1}}| \) takes a value larger than \( \epsilon \), so we conclude by Theorem 3.8 in [5] that each point \( x_{(i)} \) is in the critical level curve in \( \hat{D} \) on which \( |p_{u_{1}}| \) takes its largest value, namely \( \lambda_{\hat{D}} \).

Suppose now that \( x_{(0)} \) is defined. We have already seen that \( |p_{u_{1}(x_{(1)})}| = |p_{u_{1}(x(0))}| \). Now for each \( k \in \{1, \ldots, N-1\} \), arg \( (v_{1}(k)) = \text{arg}(v_{1}(k)) \). Therefore by the construction of \( \lambda_{\hat{D}} \), we have

\[
\text{arg}(p_{u_{1}(x_{(1)})}) = \text{arg}(v_{1}(t_{1})) = \text{arg}(v_{1}(t_{1})) = a(x_{(1)}) = a(x_{(0)}) = \text{arg}(v_{1}(t_{0})) = \text{arg}(v_{1}(t_{0})) = \text{arg}(p_{u_{1}(x_{(0)})})
\]

Therefore \( p_{u_{1}}(x(1)) = p_{u_{1}}(x(0)) \). We now wish to show that \( x(1) = x(0) \). Define \( L_{1} \) to be the straight line path from \( x(1) \) to \( x(1) \). Let \( L_{D} \) denote the portion of the gradient line of \( p_{\lambda} \) which connects \( x(1) \) with \( x(0) \). Let \( L_{0} \) denote the straight line path from \( x(0) \) to \( x(0) \). By Item 1 in the choice of \( \lambda_{1} \), Item 5 in the choice of \( \delta_{1} \), and Item 5 in the choice of \( v_{1} \), it can be shown that this path from \( x(1) \) to \( x(0) \) does not intersect any critical level curve of \( p_{u_{1}} \), other than \( \lambda_{\hat{D}} \). Therefore we can project this path along gradient lines to a path \( \sigma : [0, 1] \rightarrow \lambda_{\hat{D}} \) from \( x(1) \) to \( x(0) \). Then it can easily be shown that either \( x(1) = x(0) \), or there is some \( r \in (0, 1) \) such that \( \sigma(r) \) is a critical point of \( p_{u_{1}} \) or \( \text{arg}(p_{u_{1}}(\sigma(r))) = \text{arg}(p_{u_{1}}(x(1))) + \pi \). However by Item 8 in the choice of \( \sigma_{1} \), Item 1 in the choice of \( \delta_{2} \), Item 5 in the choice of \( \rho_{1} \), and Item 5 in the choice of \( v_{1} \), no such \( r \) exists, so we conclude that \( x(1) = x(0) \).

Now choose some \( k_{0} \in \{1, \ldots, L\} \). We will now find an edge \( E_{(k_{0})} \) of \( \lambda_{\hat{D}} \) which corresponds to the edge \( E_{(k_{0})} \) in \( \lambda_{D} \).

**Sub-case 5.1.7.4.** \( x_{(1)} \) is not an end point of \( E_{(k_{0})} \).

Let \( i, j \in \{1, \ldots, L\} \) be the indices such that \( x_{(i)} \) is the initial point of \( E_{(k_{0})} \) and \( x_{(j)} \) is the final point of \( E_{(k_{0})} \). Recall that as we are viewing \( \lambda_{PC} \) as embedded in \( C \) as the critical level curves of \( p_{\lambda} \), we have \( x_{(i)} = u_{1}(t_{i}) \) and \( x_{(j)} = u_{1}(t_{j}) \). And \( a(x_{(i)}) = a(x(0)) \) and \( a(x_{(j)}) = a(x(0)) \) by the construction of \( \lambda_{PC} \). Thus

\[
a(x_{(i)}) = a(x_{(i)}) = \text{arg}(p_{\lambda}(u_{1}(t_{i}))),
\]

and similarly
\[a(x^{(j)}) = a(x^{(j)}) = \arg(p_{\Lambda_D}(\hat{u}_1^{(t_j)})].\]

Assume that \(a(x^{(i)}) = 0\) (otherwise make the appropriate minor changes throughout the argument). Let \(\Delta > 0\) denote the change in \(\arg(p_{\Lambda_D})\) along \(E^{(k_0)}\), and let \(\hat{\gamma}\) be a parameterization of \(E^{(k_0)}\) according to \(\arg(p_{\Lambda_D})\). That is, \(\hat{\gamma} : [0, \Delta] \to E^{(k_0)}\), and for each \(r \in [0, \Delta]\), \(\arg(p_{\Lambda_D}(\hat{\gamma}(r))) = r\). By Item 3 in the choice of \(\rho_1\), we may find a path \(\gamma : [0, \Delta] \to \tilde{\Lambda}_D\) such that \(\gamma(0) = u_1^{(t_j)}\), and for each \(r \in [0, \Delta]\), \(\arg(p_{\Lambda_D}(\gamma(r))) = r\), and \(|\gamma(r) - \hat{\gamma}(r)| < 2\delta_2\).

Now \(\hat{\gamma}(\Delta) = \hat{u}_1^{(t_j)}\), so \(\Delta = \arg(v_1^{(t_j)}) \mod 2\pi\). Therefore \(p_{\Lambda_D}(\gamma(\Delta)) = |v_1^{(t_j)}|e^{i\arg(v_1^{(t_j)})} = v_1^{(t_j)}\). Moreover,

\[|\gamma(\Delta) - u_1^{(t_j)}| \leq |\gamma(\Delta) - \hat{\gamma}(\Delta)| + |\hat{\gamma}(\Delta) - u_1^{(t_j)}| + |u_1^{(t_j)} - u_1^{(t_j)}|.
\]

However \(u_1^{(t_j)} = \hat{\gamma}(\Delta)\), so we have

\[|\gamma(\Delta) - u_1^{(t_j)}| \leq |\gamma(\Delta) - \hat{\gamma}(\Delta)| + |u_1^{(t_j)} - u_1^{(t_j)}| < \delta_2 + \rho_1 < 2\delta_2.
\]

By Item 4 in the choice of \(\delta_1\) and Item 4 in the choice of \(\delta_2\), there is no point in \(B_{2\delta_2}(u_1^{(t_j)}) \setminus \{u_1^{(t_j)}\}\) at which \(p_{\Lambda_D}\) takes the value \(v_1^{(t_j)}\), we conclude that \(\gamma(\Delta) = u_1^{(t_j)}\). Therefore we have that \(\gamma\) is a path from \(u_1^{(t_j)}\) to \(u_1^{(t_j)}\) through \(\tilde{\Lambda}_D\). We now wish to show that the image of \(\gamma\) consists of a single edge of \(\tilde{\Lambda}_D\). That is, we wish to show that for any \(r \in (0, \Delta]\), \(\gamma(r)\) is not a critical point of \(p_{\Lambda_D}\).

Suppose by way of contradiction that there is some \(r(0) \in (0, \Delta)\) such that \(\gamma(r(0))\) is a critical point of \(p_{\Lambda_D}\). Therefore we have \(r_0, 0, \Delta \in \{\arg(v_1), \ldots, \arg(v_{N-1})\}\), and thus both \(r_0\) and \(\Delta - r_0\) are greater than \(d_{\arg}(1, v_1^{(1)}, \ldots, v_1^{(N-1)})\). Since by Item 3 in the choice of \(\delta_1\) and Item 4 in the choice of \(\delta_2\), \(\delta_2 < \frac{\min_{\arg}}{2\pi} < \frac{\min_{\arg}}{2\pi}\), we conclude that both \(r(0)\) and \(\Delta - r(0)\) are greater than \(\frac{\min_{\arg}}{2\pi}\).

Choose some \(l_0 \in \{1, \ldots, N - 1\}\) so that \(\gamma(r(0)) = u_1^{(l_0)}\). We first wish to show that \(u_1^{(l_0)} \in \tilde{\Lambda}_D\). By construction of \(\tilde{\Lambda}_D_{\arg}\), there is only one critical level curve of \(p_{\Lambda_D}\) in \(\tilde{D}\) on which \(|p_{\Lambda_D}| = \epsilon\), namely \(\tilde{\Lambda}_D \setminus E^{(1)}\). If \(l_0 \neq M\), then \(v_1^{(l_0)} = v_1^{(l_0)}\), so \(|v_1^{(l_0)}| = |v_1^{(l_0)}| = \epsilon\). Thus we conclude that either \(l_0 = M\) or \(u_1^{(l_0)} \in \tilde{\Lambda}_D \setminus E^{(1)}\). Therefore there is some \(k \in \{1, \ldots, K\}\) such that \(u_1^{(l_0)} = x^{(k)}\), and therefore, as shown earlier, \(u_1^{(l_0)} \in \tilde{\Lambda}_D\).

Let \(L\) denote the straight line segment from \(u_1^{(l_0)}\) to \(\hat{\gamma}(r(0))\). Since \(\gamma(r(0)) = u_1^{(l_0)}\), we have

\[|u_1^{(l_0)} - \hat{\gamma}(r_0)| \leq |u_1^{(l_0)} - \gamma(r_0)| + |\gamma(r_0) - \hat{\gamma}(r_0)| = |u_1^{(l_0)} - u_1^{(l_0)}| + |\gamma(r_0) - \hat{\gamma}(r_0)|.
\]

So

\[|u_1^{(l_0)} - \hat{\gamma}(r_0)| < \rho_1 + \delta_2 < 2\delta_2.
\]

Therefore \(L \subset B_{2\delta_2}(u_1^{(l_0)})\). Now for all \(z \in L\),

\[|p_{\tilde{\Lambda}_D}(z) - u_1^{(l_0)}| \leq |p_{\tilde{\Lambda}_D}(z) - p_{\Lambda_D}(z)| + |p_{\Lambda_D}(z) - v_1^{(l_0)}| + |v_1^{(l_0)} - u_1^{(l_0)}|,
\]

and

\[|z - u_1^{(l_0)}| \leq |z - u_1^{(l_0)}| + |u_1^{(l_0)} - u_1^{(l_0)}| < 2\delta_2 + \rho_1 < 3\delta_2.
\]

So by Item 2 in the choice of \(\delta_2\) and Item 4 in the choice of \(\rho_1\), we have

\[|p_{\tilde{\Lambda}_D}(z) - v_1^{(l_0)}| < \delta_0 + \delta_1 + \frac{|v_1|}{2} < 3\delta_1.
\]
Therefore the path \( p_{\mathcal{G}_1}(L) \) is contained in \( B_{3\delta_1}(v_1^{(l_0)}) \), and \( |v_1^{(l_0)}| \geq \text{minmod}(v_1) \), so by Item \( \bigcirc \) in the choice of \( \delta_1 \), we conclude that the net change in argument of \( p_{\mathcal{G}_1} \) along \( L \) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \).

As noted above, \( r^{(0)} \) and \( \Delta - r^{(0)} \) are both greater than \( d_{\text{arg}(1,v_1)} \). Therefore, we can choose some \( s_1 \in (0,r_0) \), and some \( s_2 \in (r_0,\Delta) \), each of which is greater than \( \frac{d_{\text{arg}(1,v_1)}}{2} \) away from each member of \( \{ \text{arg}(v_1^{(1)}),\ldots,\text{arg}(v_1^{(N-1)}) \} \). Therefore \( \hat{\gamma}(s_1),\hat{\gamma}(s_2) \notin L \). We now consider the set \( \lambda_D \{ \hat{E}(1) \cup L \} \) as a graph whose vertices are the vertices of \( \lambda_D \{ \hat{E}(1) \} \) along with any intersections of \( \lambda_D \{ \hat{E}(1) \} \) and \( L \). (Note that the smoothness of the edges of \( \lambda_D \{ \hat{E}(1) \} \) and the smoothness of \( L \) imply that \( \lambda_D \{ \hat{E}(1) \} \) and \( L \) intersect at only finitely many places.) Let \( K_1 \) and \( K_2 \) denote the edges of \( \lambda_D \{ \hat{E}(1) \} \cup L \) which contain \( \hat{\gamma}(s_1) \) and \( \hat{\gamma}(s_2) \) respectively. Since \( L \) intersects \( E^{(k_0)} \) at \( \hat{\gamma}(r_0) \), we conclude that \( \hat{\gamma}(s_1) \) and \( \hat{\gamma}(s_2) \) are in different edges of \( \lambda_D \{ \hat{E}(1) \} \cup L \), so \( K_1 \neq K_2 \).

Let \( G_0 \) denote the bounded face of \( \lambda_D \{ \hat{E}(1) \} \) such that \( E^{(k_0)} \) is contained in \( \partial G_0 \).

**Sub-sub-case 5.1.7.4.1. Both \( K_1 \) and \( K_2 \) are adjacent to the unbounded face of \( \lambda_D \{ \hat{E}(1) \} \cup L \).**

Let \( D_1 \) and \( D_2 \) denote the bounded faces of \( \lambda_D \{ \hat{E}(1) \} \cup L \) which are adjacent to \( K_1 \) and \( K_2 \) respectively. Then in this case, \( L \) must intersect some portion of \( \partial G_0 \setminus \hat{\gamma}(s_1,s_2) \), and \( D_1 \neq D_2 \). By choice of \( s_j \), the change in argument of \( p_{\mathcal{G}_1} \) along \( K_j \) is greater than \( \frac{d_{\text{arg}(1,v_1)}}{2} \) for \( j = 1,2 \). Let \( \Gamma_{\lambda_D \{ \hat{E}(1) \}} \) denote the portion of \( \partial D_1 \) which is contained in \( \lambda_D \{ \hat{E}(1) \} \). Let \( \Gamma_L \) denote the portion of \( \partial D_1 \) which is contained in \( L \). Since the argument of \( p_{\mathcal{G}_1} \) is strictly increasing on \( \lambda_D \{ \hat{E}(1) \} \), and \( K \subset \lambda_D \{ \hat{E}(1) \} \), the net change in the argument of \( p_{\mathcal{G}_1} \) along \( \Gamma_{\lambda_D \{ \hat{E}(1) \}} \) is greater than \( \frac{d_{\text{arg}(1,v_1)}}{2} \). And since the total variation of \( \text{arg}(p_{\mathcal{G}_1}) \) on \( L \) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \), the magnitude of the net change in \( \text{arg}(p_{\mathcal{G}_1}) \) along \( \Gamma_L \) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \). Therefore we conclude that the net change in \( \text{arg}(p_{\mathcal{G}_1}) \) along \( \partial D_1 \) is greater than zero. We conclude that \( D_1 \) contains a zero of \( p_{\mathcal{G}_1} \).

The exactly similar argument implies that \( D_2 \) contains a zero of \( p_{\mathcal{G}_1} \). Let \( z_1 \) denote one of the zeros of \( p_{\mathcal{G}_1} \) in \( D_1 \), and let \( z_2 \) denote one of the zeros of \( p_{\mathcal{G}_1} \) in \( D_2 \). Then Theorem 3.8 from \( \Box \) implies that there is a critical level curve \( \hat{\xi} \) of \( p_{\mathcal{G}_1} \) in \( G_0 \), such that each of \( z_1 \) and \( z_2 \) is in a bounded face of \( \hat{\xi} \). Then \( \hat{\xi} \) intersects \( D_1 \) and \( D_1^c \), so \( \hat{\xi} \) intersects \( \partial D_1 \). However \( \hat{\xi} \) does not intersect \( \lambda_D \{ \hat{E}(1) \} \), so therefore we conclude that \( \hat{\xi} \) intersects \( L \). We have already shown that for all \( z \in L \), \( |p_{\mathcal{G}_1}(z) - v_1^{(l_0)}| < 3\delta_1 \). Therefore the value that \( |p_{\mathcal{G}_1}| \) takes on \( \hat{\xi} \) is contained in \( (\epsilon - 3\delta_1, \epsilon + 3\delta_1) \). By the Maximum Modulus Principle, since \( \hat{\xi} \subset G_0 \), the value that \( |p_{\mathcal{G}_1}| \) takes on \( \hat{\xi} \) is contained in \( (\epsilon - 3\delta_1, \epsilon) \). However by the construction of \( (\lambda) \{ \hat{PC} \} \) and Item \( \Box \) in the choice of \( \delta_1 \), no critical values of \( p_{\mathcal{G}_1} \) have moduli in \( (\epsilon - 3\delta_1, \epsilon) \), which supplies us with our contradiction.

**Sub-sub-case 5.1.7.4.2. One of \( K_1 \) or \( K_2 \) is not adjacent to the unbounded face of \( \lambda_D \{ \hat{E}(1) \} \cup L \).**

Assume that \( K_1 \) is not adjacent to the unbounded face of \( \lambda_D \{ \hat{E}(1) \} \cup L \). Since \( K_1 \) is adjacent to the unbounded face of \( \lambda_D \{ \hat{E}(1) \} \), one of the faces of \( \lambda_D \{ \hat{E}(1) \} \cup L \) which is adjacent to \( K_1 \) is contained in the unbounded face of \( \lambda_D \{ \hat{E}(1) \} \). Let \( D_1 \) denote this face. Let \( \Gamma_{\lambda_D \{ \hat{E}(1) \}} \) denote the portion of \( \partial D_1 \) which is contained in \( \lambda_D \{ \hat{E}(1) \} \). Let \( \Gamma_L \) denote the portion of \( \partial D_1 \) which is contained in \( L \). Then since \( D_1 \) is contained on the unbounded portion of \( \hat{\Lambda} \), \( \text{arg}(p_{\mathcal{G}_1}) \) is strictly decreasing as \( \Gamma_{\lambda_D \{ \hat{E}(1) \}} \) is traversed with positive orientation, and the change in \( \text{arg}(p_{\mathcal{G}_1}) \) as \( K_1 \) is traversed as a portion of \( \partial D_1 \) (thus with the opposite orientation as before) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \). Therefore the net change in \( \text{arg}(p_{\mathcal{G}_1}) \) as \( \Gamma_{\lambda_D \{ \hat{E}(1) \}} \) is traversed is less than \( -\frac{d_{\text{arg}(1,v_1)}}{2} \). And since the total variation of \( \text{arg}(p_{\mathcal{G}_1}) \) along \( L \) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \), the net change in \( \text{arg}(p_{\mathcal{G}_1}) \) along \( \Gamma_L \) is less than \( \frac{d_{\text{arg}(1,v_1)}}{2} \) in magnitude. Therefore the net change in \( \text{arg}(p_{\mathcal{G}_1}) \) along \( \partial D_1 \) is strictly less than 0. Therefore there is a pole of \( p_{\mathcal{G}_1} \) in \( D_1 \), which is obviously a contradiction as \( p_{\mathcal{G}_1} \) is a polynomial.
Therefore by the previous two subcases, we conclude that there is no $r \in (0, \Delta)$ such that $\gamma(r)$ is a critical point of $p_{u_1}$. Therefore $\gamma(0, \Delta)$ is an edge of $\lambda_{\beta}$. Let $E^{(k_0)}$ denote this edge.

**Sub-case 5.1.7.5.** $x^{(1)}$ is an end point of $E^{(k_0)}$.

In this case, there are three subcases to consider, two of which are essentially identical to the previous case, and which we list first.

**Sub-sub-case 5.1.7.5.1.** $k_0 = 1$ (so $E^{(k_0)} = \partial F$).

**Sub-sub-case 5.1.7.5.2.** $k_0 \neq 1$, but $x^{(1)}$ is still a vertex of $\lambda_{\beta} \setminus E^{(1)}$.

For the previous two cases, both end points of $E^{(k_0)}$ are critical points of $p_{\gamma_1}$, and $E^{(k_0)}$ is completely contained in a single critical level curve of $p_{\gamma_1}$. Therefore the method of Case 5.1.7.3 may be applied with minor changes, and the conclusion is that if $i, j \in \{0, \ldots, K\}$ are the indices such that $x^{(i)}$ is the initial vertex of $E^{(k_0)}$ and $x^{(j)}$ is the final point of $E^{(k_0)}$, then there is a path $\hat{\gamma}$ which parameterizes $E^{(k_0)}$ with respect to $\arg(p_{\gamma_1})$ and an edge $E^{(k_0)}$ of $\lambda_{\beta}$ from $x^{(i)}$ to $x^{(j)}$ with a parameterization $\hat{\gamma}$ which parameterizes $E^{(k_0)}$ with respect to $\arg(p_{u_1})$ such that for each $r$, $|\hat{\gamma}(r) - \gamma(r)| < \delta_2$.

**Sub-sub-case 5.1.7.5.3.** $k_0 \neq 1$ and $x^{(1)}$ is not a vertex of $\lambda_{\beta} \setminus E^{(1)}$.

Since $k_0 \neq 1$, $E^{(k_0)}$ does not form $\partial F$. As noted earlier, the fact that $(\lambda_{\beta})_{PC}$ was formed using the scattering method implies that both end points of $E^{(k_0)}$ are not at $x^{(1)}$. Let $i \in \{2, \ldots, K\}$ be the index so that $x^{(i)}$ is the other end point of $E^{(k_0)}$. Assume during the following argument that $x^{(i)}$ is the initial point of $E^{(k_0)}$ (otherwise make the appropriate minor changes, such as reversing orientations of paths, etc.). Let $\Delta \in \mathbb{R} \setminus \{0\}$ denote change in argument along $E^{(k_0)}$ from $x^{(i)}$ to $x^{(1)}$. Assume that $a(x^{(i)}) = 0$ (otherwise make the appropriate minor changes). Let $\bar{E}$ denote the edge of $\lambda_{\beta} \setminus E^{(1)}$ which contains $x^{(1)}$ (which is no longer a vertex of $\lambda_{\beta} \setminus E^{(1)}$). Recall that $E^{(k_0)}$ is the portion of $\bar{E}$ formed in the following way. Let $\Delta^{(1)}$ denote the change in argument of $p_{\gamma_1}$ along $\bar{E}$ beginning at $x^{(i)}$. Let $\gamma^{(1)}$ be a parameterization of $\bar{E}$ according to $\arg(p_{\gamma_1})$ and beginning at $x^{(i)}$. So $\gamma^{(1)} : [0, \Delta^{(1)}]$, and $\arg(p_{\gamma_1}(\gamma^{(1)}(r))) = r$ for each $r \in [0, \Delta^{(1)}]$. Then we define $E^{(k_0)}$ to be $\gamma^{(1)}([0, \Delta])$. Let $\hat{\gamma}$ denote the path $\gamma^{(1)}$ restricted to $[0, \Delta]$.

Again by Item [3] in the choice of $\rho_1$, there is a path $\gamma : [0, \Delta] \to \lambda_{\beta}$ such that $\gamma(0) = u_1(t_1)$ and for each $r \in [0, \Delta]$, $\arg(p_{u_1}(\gamma(r))) = r$ and $|\gamma(r) - \gamma^{(1)}(r)| \leq \delta_1$. The argument for this subcase is very similar to the argument for Case 5.1.7.3. The major difference is in the method by which we show that $\gamma(\Delta) = u_1(t_1)$. Our method here is similar to the way in which we showed that $\bar{x}(0) = x^{(1)}$ (when $\bar{x}(0)$ is defined).

Since the image of $\gamma$ is contained in $\lambda_{\beta}$, we conclude that $|p_{u_1}(\gamma(\Delta))| = |p_{u_1}(x^{(1)})|$. By definition of $\gamma$, $\arg(p_{u_1}(\gamma(\Delta))) = \Delta$. And by definition of $\Delta$, $\arg(p_{u_1}(x^{(1)})) = a(x^{(1)}) = \Delta$ (since we are assuming that $\arg(p_{u_1}(x^{(i)})) = 0$). Therefore $p_{u_1}(\gamma(\Delta)) = p_{u_1}(x^{(1)})$. Let $L_1$ denote the straight line path from $x^{(i)}$ to $x^{(1)}$. Let $L$ denote the portion of the gradient line of $p_{\gamma_1}$ which connects $x^{(i)}$ to $\gamma(\Delta)$. Let $L_2$ denote the straight line path from $\gamma(\Delta)$ to $\gamma(\Delta)$. Let $L_1, L, L_2$ denote denote the path obtained by concatenating the three paths $L_1$, $L$, and $L_2$. By Item [4] in the choice of $\delta_1$, Item [5] in the choice of $\delta_2$, Item [6] in the choice of $\rho_1$, and Item [7] in the choice of $\rho_1$, it can be shown (by considering the value of $|p_{u_1}|$ on $L_1, L, L_2$) that the path $L_1, L, L_2$ does not intersect any critical level curve of $p_{u_1}$ other than $\lambda_{\beta}$. Therefore we can project this path along gradient lines to a path $\sigma : [0, 1] \to \lambda_{\beta}$ from $x^{(i)}$ to $\gamma(\Delta)$. Then it can easily be shown that either $x^{(1)} = \gamma(\Delta)$, or there is some $r \in (0, 1)$ such that $\sigma(r)$ is a critical point of $p_{u_1}$ or $\arg(p_{u_1}(\sigma(r))) = \arg(p_{u_1}(x^{(1)})) + \pi$. However by Item [8] in the choice of $\delta_1$, Item [9] in the choice of $\delta_2$, Item [10] in the choice of $\rho_1$, and Item [11] no such $r$ can exist, so we conclude that $u_1(t_1) = x^{(1)} = \gamma(\Delta)$. The rest of the argument for this subcase is essentially the same as for Case 5.1.7.2 so we omit it.
The conclusion which we draw is as before. Namely, there is a path \( \gamma \) which parameterizes \( \tilde{E}(k_0) \) according to \( \arg(p_{\gamma_0}) \) and an edge \( E(k_0) \) of \( \lambda_{\tilde{D}} \) from \( x^{(i)} \) to \( x^{(1)} \) with a parameterization \( \gamma \) which parameterizes \( E(k_0) \) according to \( \arg(p_{\gamma_1}) \) such that for each \( r, |\gamma(r) - \gamma(r)| < \delta_2 \). Now for each \( k \in \{1, \ldots, L\} \), let \( \tilde{\gamma}^{(k)} \) be the path which parameterizes \( E^{(k)} \). Let \( \tilde{\gamma}^{(k)} \) be the path which parameterizes \( E^{(k)} \).

We now wish to show that the order in which the edges \( E^{(1)}, \ldots, E^{(L)} \) appear around \( \tilde{\lambda}_{\tilde{D}} \) is the same as the order of their corresponding edges around \( \lambda_{\tilde{D}} \).

Now it does not quite make sense to speak of the order in which the edges \( E^{(1)}, \ldots, E^{(L)} \) appear, because these edges are not all contained in a single level curve of \( p_{\gamma_1} \). However there is some way to make sense of the order of appearance of \( E^{(1)}, \ldots, E^{(L)} \). Begin with \( E^{(1)} \). Now for each \( k \in \{2, \ldots, L\} \), \( E^{(k)} \) is contained in \( \lambda_{\tilde{D}} \setminus E^{(1)} \). Select one point \( z^{(k)} \) in \( E^{(k)} \) which is not an end point of \( E^{(k)} \). Then the orientation of the edges \( E^{(1)}, \ldots, E^{(L)} \) appear around \( \tilde{\lambda}_{\tilde{D}} \) is the same as the order of appearance of \( E^{(1)}, \ldots, E^{(L)} \).

Define for the moment \( y^{(k)} \) to be the point in \( \partial \tilde{G} \) which connects to \( z^{(k)} \) by a gradient line of \( p_{\gamma_1} \). Then by definition of the edges \( E^{(2)}, \ldots, E^{(L)} \), the order in which the points \( y^{(2)}, \ldots, y^{(L)} \) appear around \( \partial \tilde{G} \) is exactly \( y^{(2)}, \ldots, y^{(L)} \). Therefore, by the construction of \( \tilde{p}_{C} \), if we begin at \( x^{(1)} \), and we travel down the gradient line containing \( x^{(1)} \) into \( \tilde{G} \) until we reach \( \lambda_{\tilde{D}} \setminus E^{(1)} \), and begin traversing \( \lambda_{\tilde{D}} \setminus E^{(1)} \), the order in which we traverse the edges of \( \lambda_{\tilde{D}} \setminus E^{(1)} \) is exactly \( E^{(2)}, \ldots, E^{(L)} \). This is the sense in which we will say that the edges \( E^{(1)}, \ldots, E^{(L)} \) appear in the order \( E^{(1)}, \ldots, E^{(L)} \).

We will make further use of the process just described of ”parameterizing” the order in which the edges of a member of \( \tilde{P} \) appear by points contained in the boundary of a region which contains this member of \( \tilde{P} \). Let us first describe this process more precisely. Let \( \xi \) denote a member of \( \tilde{P} \), and let \( n \) denote the number of edges \( \xi^{(1)}, \ldots, \xi^{(n)} \) of \( \xi \). Choose some simple closed path such that the bounded face \( G \) of the path contains \( \xi \) in its bounded face, and choose \( n \) distinct points in \( \partial G \). For each edge \( e \in \xi \), draw a path from a point in \( e \) which is not an end point of \( e \) to one of the chosen points in \( \partial \tilde{G} \). Do this in such a way that the paths are contained in the portion of \( G \) exterior to \( \xi \) except at the endpoints, so that each edge connects to a different point in \( \partial \tilde{G} \), and so that the paths do not intersect. For each \( i \in \{1, \ldots, n\} \), let \( y^{(i)} \) denote the point in \( \partial \tilde{G} \) which is connected to \( e^{(i)} \). Then the orientation of the edges \( e^{(1)}, \ldots, e^{(n)} \) in \( \xi \) is the same as the orientation of the points \( y^{(1)}, \ldots, y^{(n)} \) in \( \partial \tilde{G} \).

Recall that by Item 4 in the choice of \( \delta_1 \), there is a choice of points \( z^{(1)}, \ldots, z^{(L)} \) such that for each \( k \in \{1, \ldots, L\} \), the following holds.

- \( z^{(k)} \) is in \( E^{(k)} \) but is not an end point of \( E^{(k)} \).
- \( \arg(p_{\gamma_1}(z^{(k)})) \) is more than \( \frac{d_{\arg(1:v_1)}}{4} \) away from each of \( \{\arg(v_1^{(1)}), \ldots, \arg(v_1^{(N-1)})\} \).
- \( z^{(k)} \) is more than \( 2\delta_1 \) away from each critical point of \( p_{\gamma_1} \).

Now for each \( k \in \{1, \ldots, L\} \), let \( \sigma^{(k)} : [0, 1] \to \mathbb{C} \) be a parameterization of the portion of the gradient line of \( p_{\gamma_1} \) which connects \( z^{(k)} \) to a point in \( \partial \tilde{D} \). Let \( y^{(k)} \) denote this point in \( \partial \tilde{D} \). Then recall that \( \delta_1 \) is chosen so that for any \( j, k \in \{1, \ldots, L\} \) with \( j \neq k \), and for any \( s, t \in [0, 1] \), \( |\sigma^{(j)}(s) - \sigma^{(k)}(t)| > 2\delta_1 \), and there is no edge of a critical level curve of \( p_{\gamma_1} \) other than the edges which contain \( \sigma^{(k)}(0) \) and \( \sigma^{(k)}(1) \) within \( 2\delta_1 \) of \( \sigma^{(k)}(s) \). Let \( \sigma^{(k)} \) parameterize this gradient line so that \( \sigma^{(k)} : [0, 1] \to \mathbb{C} \) with \( \sigma^{(k)}(0) = y^{(k)} \) and \( \sigma^{(k)}(1) = z^{(k)} \).

Define \( i_1 := 1 \) and choose distinct indices \( i_2, \ldots, i_L \in \{2, \ldots, L\} \) so that the order in which the edges of \( \tilde{\lambda}_{\tilde{D}} \) appear around \( \tilde{\lambda}_{\tilde{D}} \) is \( E^{(i_1)}, \ldots, E^{(i_L)} \). Now by Item 5 in the choice of \( \delta_1 \) and Item 4 in the choice of \( \delta_2 \), \( \tilde{\lambda}_{\tilde{D}} \subset \tilde{D} \). We are now going to alter each \( \sigma^{(k)} \) so that it is a path from \( y^{(k)} \) to a point in \( \tilde{E}^{(k)} \). By Item 4 in the choice of \( \delta_2 \), there is no point in the path \( \sigma^{(k)} \) which intersects \( \tilde{E}^{(l)} \) for any \( l \neq k \). If there is any \( s \in [0, 1] \) such that \( \sigma^{(k)}(s) \in \tilde{E}^{(k)} \), then let \( s_0 \) denote the smallest such \( s \). Then define \( \sigma^{(k)} \) to be the restriction of \( \sigma^{(k)} \) to \( [0, s_0] \). If there is no such \( s \), let \( \Delta^{(k)} \) denote the change in \( \arg(p_{\gamma_1}) \) along \( \tilde{E}^{(k)} \) and let \( i \in \{1, \ldots, K\} \).
denote the index such that \( \tilde{x}^{(i)} \) is the initial point of \( \tilde{E}^{(k)} \). Let \( r_1 \in \{ a(\tilde{x}^{(i)}), a(\tilde{x}^{(i)}) + \Delta^{(k)} \} \) be chosen so that \( \gamma^{(k)}(r_1) = \tilde{x}^{(k)} \). Let \( L^{(k)} \) denote the straight line path from \( \gamma^{(k)}(r_1) \) to \( \gamma^{(k)}(r_1) \). Let \( \sigma^{(k)}L^{(k)} \) denote the path obtained by first traversing \( \sigma^{(k)} \), and then traversing \( L^{(k)} \) from \( \tilde{y}^{(k)} \) to \( \gamma^{(k)}(r_1) \). Then \( \sigma^{(k)}L^{(k)} \) is a path from \( \tilde{y}^{(k)} \) to \( \tilde{y}^{(k)} \). Let \( s_0 \) be the smallest number in the domain of the path \( \sigma^{(k)}L^{(k)} \) such that \( \sigma^{(k)}L^{(k)}(s_0) \in \tilde{E}^{(k)} \). Then define \( \sigma^{(k)}L^{(k)} \) to be this path \( \sigma^{(k)}L^{(k)} \) restricted to \([0, s_0]\). Now since \( L^{(k)} \subset B_{\delta_1}(\tilde{z}^{(k)}) \), by Item [3] in the choice of \( \delta_1 \), \( \tilde{y}^{(k)} \) does not intersect \( \tilde{y}^{(i)} \) for any \( i \neq k \), and it can be shown that \( \tilde{L}^{(k)} \) does not intersect the gradient line which connects \( x^{(1)} \) to \( \lambda^{(1)} \). Define \( \tilde{z}^{(k)} := \sigma^{(k)}L^{(k)}(s_0) \).

By definition of the \( \sigma^{(k)} \), the order in which the points \( y^{(1)}, \ldots, y^{(L)} \) appear around \( \partial \tilde{D} \) beginning with \( y^{(1)} \) is \( y^{(1)} = y^{(i_1)}, y^{(i_2)}, \ldots, y^{(i_L)} \). We wish to show that for each \( k \in \{2, \ldots, L\} \), \( i_k = k \). Recall that if one begins at \( x^{(k)} \), traverses the gradient line down to \( \lambda^{(1)} \), and begins traversing \( \lambda^{(1)} \) with positive orientation, the first edge one traverses is \( \tilde{E}^{(2)} \). Therefore the first point of the set \( \tilde{z}^{(2)}, \ldots, \tilde{z}^{(L)} \) which one encounters while traversing \( \lambda^{(1)} \) is \( \tilde{z}^{(2)} \). Now consider the path one obtains by traversing \( \sigma^{(1)} \) from \( y^{(1)} \) to \( \tilde{z}^{(1)} \), traversing \( \tilde{E}^{(1)} \) with positive orientation from \( \tilde{z}^{(1)} \) to \( x^{(1)} \), traversing the gradient line of \( \tilde{p}_{u_1}^{(1)} \) from \( x^{(1)} \) to the point where it intersects \( \lambda^{(1)} \) (let us call that point \( z \) for the moment), traversing \( \tilde{E}^{(2)} \) from \( z \) to \( \tilde{z}^{(2)} \), and finally traversing \( \tilde{E}^{(2)} \) from \( \tilde{z}^{(2)} \) to \( y^{(2)} \). Let \( \sigma \) denote this path. Since no \( \tilde{y}^{(i)} \) is in this path for \( l \notin \{1, 2\} \), and no \( \tilde{y}^{(1)} \) for \( l \notin \{1, 2\} \) can intersect this path, we conclude that if one traverses \( \partial \tilde{D} \) from \( y^{(1)} \) to \( y^{(2)} \) with positive orientation, then one does not encounter any \( y^{(i)} \) for \( l \notin \{1, 2\} \). Therefore since \( i_2 = 2 \) is the index such that \( y^{(i_2)} \) is the next point in \( \partial \tilde{D} \) after \( y^{(1)} \) (with respect to a positive orientation), therefore we conclude that \( i_2 = 2 \). The same argument gives us that for each \( k \in \{3, \ldots, L\} \), \( i_k = k \). Therefore the order in which the edges \( \tilde{E}^{(1)}, \ldots, \tilde{E}^{(L)} \) appear around \( \lambda^{(1)} \) is exactly \( \tilde{E}^{(1)}, \ldots, \tilde{E}^{(L)} \). Therefore the edges \( \tilde{E}^{(k)} \) appear in the same order around \( \lambda^{(k)} \) as the corresponding edges \( E^{(k)} \) appear around \( \lambda^{(k)} \).

Let \( G \) denote for the moment either the face \( \tilde{F} \) of \( \lambda^{(k)} \) or one of the bounded faces of \( \lambda^{(1)} \). Let \( n \geq 1 \) be the number of edges of \( \tilde{L} \) which are contained in \( \partial G \). Let \( i_1, \ldots, i_n \in \{1, \ldots, L\} \) be the indices such that \( \tilde{E}^{(i_1)}, \ldots, \tilde{E}^{(i_n)} \) are the edges which form \( \partial G \) listed in order of their appearance around \( \partial G \). Then \( \tilde{E}^{(i_1)}, \ldots, \tilde{E}^{(i_n)} \) forms a simple closed path and, by the Maximum Modulus Principle, no edge of \( \lambda^{(1)} \) is contained in the bounded face of this path. Let \( \tilde{G} \) denote the face of \( \lambda^{(1)} \) which has this path as its boundary. By definition of the paths \( \gamma^{(i)} \) defined earlier, the change of \( \arg(p_{u_1}) \) along \( \partial \tilde{G} \) is the same as the change in \( \arg(p_{u_1}) \) along \( \partial G \). Therefore the number of zeros of \( p_{u_1} \) in \( \tilde{G} \) is the same as the number of zeros \( p_{u_1} \) in \( G \). Let \( m \geq 2 \) denote the number of bounded faces of \( \lambda^{(1)} \). Then \( \lambda^{(1)} \) has \( m - 1 \) bounded faces. Define \( \tilde{D}^{(1)} := \tilde{F} \) and let \( \tilde{D}^{(2)}, \ldots, \tilde{D}^{(m)} \) be an enumeration of the bounded faces of \( \lambda^{(1)} \). Then we have \( \sum_{k=1}^{m} z(\tilde{D}^{(k)}) = \sum_{k=1}^{m} z(\tilde{D}^{(k)}) \) where \( z(\tilde{D}^{(k)}) \) denotes the number of zeros of \( p_{u_1} \) in \( \tilde{D}^{(k)} \) and \( z(\tilde{D}^{(k)}) \) denotes the number of zeros of \( p_{u_1} \) in \( \tilde{D}^{(k)} \). Therefore the number of zeros of \( p_{u_1} \) contained in the bounded faces of \( \lambda^{(1)} \) is greater than or equal to the number of zeros of \( p_{u_1} \) in the bounded faces of \( \lambda^{(1)} \). However by definition of \( \lambda^{(1)} \) and the map \( \Pi \), each zero of \( p_{u_1} \) which is contained in \( \tilde{D} \) is contained in one of \( \tilde{D}^{(1)}, \ldots, \tilde{D}^{(m)} \). Moreover, by the same argument as above it may easily be shown that there are the same number of zeros of \( p_{u_1} \) in \( \tilde{D} \) as the number of zeros of \( p_{u_1} \) in \( D \). These two facts together imply that for each bounded face \( \tilde{G} \) of \( \lambda^{(1)} \), \( \tilde{G} = D^{(k)} \) for some \( k \in \{1, \ldots, m\} \). Therefore \( \lambda^{(1)} \) contains no edges other than \( E^{(1)}, \ldots, E^{(L)} \).

We have already seen that for each \( k \in \{1, \ldots, K\} \),

\[
a(x^{(k)}) = a(x^{(k)}) = \arg(p_{u_1}(x^{(k)})) = \arg(p_{u_1}(x^{(k)})) = a(x^{(k)})
\]

and, by the definition of the paths \( \gamma^{(k)} \) and \( \tilde{y}^{(k)} \), each \( \tilde{E}^{(k)} \) contains the same number of distinguished points as \( E^{(k)} \), which contains the same number of distinguished points as \( E^{(k)} \) by construction of \( \tilde{A} \).
have also already seen that $H(\langle \lambda_D \rangle_P)$ equals $|v_1^{(k)}|$ for each $k \in \{1, \ldots, K\}$ and this value is in turn equal to $H(\langle \lambda_B \rangle_P)$. Therefore $\langle \lambda_D \rangle_P$ and $\langle \lambda_B \rangle_P$ share all their auxiliary data, and we conclude that $\langle \lambda_D \rangle_P = \langle \lambda_B \rangle_P$.

**Case 5.1.8.** $(\lambda_B)_P$ was not formed using the scattering method.

In this case the graph $\hat{\lambda}_B$ is actually equal to $\lambda_D$ as members of $\hat{P}$, which removes many of the difficulties encountered in Case 5.1.7. Therefore the argument needed to show that $\langle \lambda_D \rangle_P = \langle \lambda_B \rangle_P$ and establish the correspondence between the vertices, edges, and distinguished points of $\langle \lambda_D \rangle_P$ and $\langle \lambda_B \rangle_P$ is a much simplified version of that found in Case 5.1.7, so we omit it here.

Note that Case 5.1.7 and Case 5.1.8 assume that $\langle \lambda_D \rangle_P$ is a graph member of $P$. If $\langle \lambda_D \rangle_P$ is a single point member of $P$ it is easy to show that $\langle \lambda_B \rangle_P$ must be the same single point member of $P$ by simply considering the different values that $|v_1^{(k)}|$ can take, and using the fact that by the construction of $\langle \hat{\lambda} \rangle_{PC}$, $\hat{z}(D) = z(D)$, and, as described above, $z(\hat{D}) = z(\hat{D})$.

We now wish to show that the correspondence we have established between the graphs, vertices, and distinguished points of $\langle \hat{\lambda} \rangle_{PC}$ and the graphs, vertices, and distinguished points of $\langle \hat{\lambda} \rangle_{PC}$ respects the gradient maps of $\langle \hat{\lambda} \rangle_{PC}$ and $\langle \hat{\lambda} \rangle_{PC}$.

As before, let $\langle \hat{\lambda} \rangle_{PC}$ be a member of $PC$ used to construct $\langle \hat{\lambda} \rangle_{PC}$. Let $D$ be a bounded face of $\lambda$, and let $\langle \hat{\lambda} \rangle_{PC}$, $\hat{D}$, and $\langle \hat{\lambda} \rangle_{PC}$, $\hat{D}$, and $\langle \hat{\lambda} \rangle_{PC}$ be the objects for $\langle \hat{\lambda} \rangle_{PC}$ and $\langle \hat{\lambda} \rangle_{PC}$ which correspond to $\langle \hat{\lambda} \rangle_{PC}$, $D$, and $\langle \hat{\lambda} \rangle_{PC}$ respectively. Choose some edge $E_1$ of $\partial D$ and let $\hat{E}_1$ and $\hat{E}_1$ be the corresponding edges in $\partial \hat{D}$ and $\partial \hat{D}$. Let $y_1$ and $y_1$ be the initial and final points of $E_1$, and let $\hat{y}_1, \hat{y}_2, \hat{y}_1$, and $\hat{y}_2$ be the corresponding points for $\langle \hat{\lambda} \rangle_{PC}$ and $\langle \hat{\lambda} \rangle_{PC}$. Let $\Delta_1$ denote the change in argument along $E_1$. Assume that $a(y_1) = 0$, (otherwise make the appropriate minor changes). Then let $\gamma^{(1)} : [0, \Delta_1] \to \mathbb{C}$ be the path which parameterizes $\hat{E}_1$ according to $\arg(p_{\hat{\gamma}})$. Let $\gamma^{(1)} : [0, \Delta_1] \to \mathbb{C}$ be the path which parameterizes $\hat{E}_1$ according to $\arg(p_{\hat{\gamma}})$. Let $y$ be a distinguished point in $E_1$. Let $\hat{y}$ and $\hat{y}$ be the corresponding points in $\hat{E}_1$ and $\hat{E}_1$. Then by choice of $\gamma^{(1)}$ and $\gamma^{(1)}$, $|\hat{y} - \hat{y}| < \delta_2$. Define $z$ to be the distinguished point in $\lambda_D$ such that $g_D(y) = z$. Let $\hat{z}$ and $\hat{z}$ be the distinguished points corresponding to $z$ for $\langle \hat{\lambda} \rangle_{PC}$ and $\langle \hat{\lambda} \rangle_{PC}$. Then since $g_D(y) = z$, the goal is to show that $g_{\hat{D}}(\hat{y}) = \hat{z}$. Let $E_2$ denote one of the edges of $\lambda_D$ which contains $z$ (if $z$ is a vertex of $\lambda_D$ then it will be contained in more than one edge of $\lambda_D$). Let $z_1$ and $z_2$ be the initial and final points of $E_2$. Let $\Delta_2$ denote the change in argument along $E_2$. Let $\gamma^{(2)} : [a(z_1), a(z_1) + \Delta_2] \to \mathbb{C}$ be the paths which parameterize $\hat{E}_2$ and $\hat{E}_2$ with respect to $\arg(p_{\hat{\gamma}})$ and $\arg(p_{\hat{\gamma}})$ respectively. Then by choice of $\gamma^{(2)}$ and $\gamma^{(2)}$, $|\hat{z} - \hat{z}| < \delta_2$. We will show the desired result in the case where $\langle \hat{\lambda}_B \rangle_{PC}$ was formed using the scattering method. As before, the other cases are just simpler versions of this case.

**Case 5.1.9.** $(\hat{\lambda}_B)_P$ was formed using the scattering method.

Recall that $\hat{F}$ denotes the face of $\hat{\lambda}_B$ to which $\langle \lambda_F \rangle_{PC}$ was assigned, and $\hat{G}$ denotes the other face of $\hat{\lambda}_B$. We now will define a path $\hat{\sigma}$ from $\hat{y}$ to $\hat{z}$.

**Sub-case 5.1.9.1.** $z \in \partial F$.

In this case $\hat{z}$ is a distinguished point in $\partial \hat{F}$. By definition of $\hat{y}$ and $\hat{z}$, $g_{\hat{D}}(\hat{y}) = \hat{z}$. Therefore there is a portion of a gradient line $\hat{\sigma} : [0, 1] \to \mathbb{C}$ of $p_{\hat{\gamma}}$ which connects $\hat{y}$ and $\hat{z}$ and such that $\hat{\sigma}((0, 1))$ is contained in the portion of $\hat{D}$ which is exterior to $\hat{\lambda}_B$.

**Sub-case 5.1.9.2.** $z \notin \partial F$.

In this case by the definition of the correspondence already established, $\hat{z}$ is a point in an edge of $\lambda_D \setminus E^{(1)}$. Recall that $\langle \lambda_D \setminus E^{(1)} \rangle_{PC}$ has been assigned to $\hat{G}$ during the construction of $\langle \hat{\lambda}_B \rangle_{PC}$, and by this construction, $g_{\hat{D}}(\hat{y})$ is a point in $\partial \hat{G}$, and one can show that $g_{\hat{D}}(\hat{y}) = \hat{z}$. Therefore there is a portion of a gradient line $\hat{\sigma}_1 : [0, 1] \to \mathbb{C}$ of $p_{\hat{\gamma}}$ which connects $\hat{y}$ to $g_{\hat{E}}(\hat{y})$, and another portion of a gradient line $\hat{\sigma}_2 : [0, 1] \to \mathbb{C}$ of $p_{\hat{\gamma}}$ which connects $g_{\hat{D}}(\hat{y})$ to $\hat{z}$. Let $\hat{\sigma}$ denote the concatenation of these two paths.
Therefore we have the desired path $\tilde{\sigma}$. By Item 3 in the choice of $\delta_2$ and Item 2 in the choice of $\rho_1$, we conclude that there is a path $\sigma : [0,1] \to \mathbb{C}$ such that $\sigma(0) = \tilde{y}$ and $\sigma(1) = \tilde{z}$ and, for all $r \in [0,1]$, $\arg(p_u(\sigma(r))) = 0$ and $|\sigma(r) - \tilde{\sigma}(r)| < \delta_1$. Moreover, since $|p_u|_f$ is strictly decreasing on $\tilde{\sigma}$, we may assume that $|p_u|$ is strictly decreasing on $\sigma$. Therefore for each $r \in (0,1)$, $|p_u(\sigma(r))| \in ([p_u(\tilde{z})], |p_u(\tilde{y})|)$. Therefore for each $r \in (0,1)$, $\sigma(r)$ is in the portion of $\tilde{D}$ which is in the unbounded face of $\lambda_D$. Therefore by definition of $g_D$, we conclude that $g_D(\tilde{y}) = \tilde{z}$.

Therefore the correspondence established above between the graphs, vertices, and distinguished points of $\langle \Lambda \rangle_{PC}$ and those of $\langle \tilde{\Lambda} \rangle_{PC}$ respects the gradient maps of $\langle \Lambda \rangle_{PC}$ and $\langle \tilde{\Lambda} \rangle_{PC}$. Finally we conclude that $\langle \Lambda \rangle_{PC}$ and $\langle \tilde{\Lambda} \rangle_{PC}$ share all auxiliary data, and are thus equal.

By inspecting this proof we immediately have the following corollary.

**Corollary 5.2.** For any $(f,G) \in H_a$ there is a polynomial $(p,G_p)$ such that $(p,G_p) \sim (f,G)$.

## A SEVERAL LEMMATA

**Lemma A.1.** For any disjoint closed sets $X,Y \subset \text{cl}(\mathbb{C})$, and $x,y \in \text{cl}(\mathbb{C}) \setminus (X \cup Y)$, if $x$ and $y$ are in the same component of $X^c$ and the same component of $Y^c$, then $x$ and $y$ are in the same component of $(X \cup Y)^c$.

**Proof.** Suppose by contradiction that $x$ and $y$ are in different components of $(X \cup Y)^c$. Assume without loss of generality that $y = \infty$. Let $A_1$ denote the component of $(X \cup Y)^c$ which contains $x$, and let $B_1$ denote the component of $(X \cup Y)^c$ which contains $y$. Let $Z$ denote the union of all bounded components of $A_1^c$. Define $A_2 := A_1 \cup Z$. Since $B_1$ is open and contains $\infty$, and $A_2$ does not intersect $B_1$, we may conclude that $A_2$ is bounded. Therefore $A_2$ has only a single unbounded component, so $A_2$ is simply connected. And the boundary of a simply connected set is connected, so $\partial A_2 \subset X \cup Y$ is connected. Because $X$ and $Y$ are disjoint and compact, this implies that $\partial A_2$ is either contained in $X$ or contained in $Y$. However this is a contradiction because $x$ and $y$ are in the same component of $X^c$ and in the same component of $Y^c$.

It may easily be seen that this lemma implies that if $A$ is an open connected set, and $X \subset A$ is compact, such that $X^c$ has a single component, then $A \setminus X$ is connected. This fact will be used several times in this paper, and we will cite the above lemma when it is needed.

**Lemma A.2.** Let $\lambda$ be a finite connected graph embedded in the plane. If $\lambda$ has the property that each edge of $\lambda$ is in the boundary both of a bounded and the unbounded face of $\lambda$, some bounded face of $\lambda$ has a single edge of $\lambda$ as its boundary.

**Proof.** We begin by constructing a graph $\mathcal{T}$ from $\lambda$. $\mathcal{T}$ will have two kinds of vertices. We place a $V$-type vertex for $\mathcal{T}$ at each vertex of $\lambda$, and we place one $F$-type vertex for $\mathcal{T}$ in each bounded face of $\lambda$. Let $u$ be an $F$-type vertex of $\mathcal{T}$. Let $D$ denote the bounded face of $\lambda$ which contains $u$. Then we draw an edge from $u$ to each $V$-type vertex of $\mathcal{T}$ which arises from a vertex of $\lambda$ which is contained in $\partial D$. We draw these edges in such a way that they are contained in $D$ (except at the end points) and do not intersect (except at $u$). Having done this for each $F$ type vertex in $\mathcal{T}$, the resulting connected graph is $\mathcal{T}$.

We now wish to show that $\mathcal{T}$ is a tree. Suppose by way of contradiction that $\mathcal{T}$ contains some cycle $C : u_1E_1u_2 \cdots u_nE_nv_1$. Consider this cycle as a path in $C$. Reduce this cycle if necessary so that it forms a simple closed path. Let $D_1$ denote the face of $\lambda$ which contains $u_1$. Since $C$ is a simple path, the only edges in $C$ which have $u_1$ as an end point are $E_1$ and $E_n$. Therefore $C$ bisects $D_1$. Let $E$ be one of the edges of $\lambda$ which is in $\partial D_1$ and which is contained in the bounded face of the path $C$. Then since $\mathcal{T}$ is contained in the closure of the bounded faces of $\lambda$, $E$ is not adjacent to the unbounded face of $\lambda$, which is a contradiction of the definition of $\lambda$.

Now that we have shown that $\mathcal{T}$ is a tree, and $\mathcal{T}$ is certainly finite, let $u$ denote one of the leaves of $\mathcal{T}$. Since each vertex of $\lambda$ is incident to more than one bounded face of $\lambda$, $u$ must be an $F$-type vertex of $\mathcal{T}$. Let $D$ now denote this face of $\lambda$. Since $u$ is a leaf of $\mathcal{T}$, $\partial D$ only contains one vertex of $\lambda$, and thus $\partial D$ consists of a single edge of $\lambda$.  

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Lemma A.3. Given any special type function element \((f,G)\) and \(\eta > 0\), and any compact set \(G' \subset G\) which does not contain any critical points of \(f\), there exists \(\tau > 0\) such that if \(g\) is analytic on \(G\), and \(|f(z) - g(z)| < \tau\) for all \(z \in G\), then the following hold:

1. If \(z(0) \in G'\), and \(w^{(1)} \in B_\tau(f(z(0)))\), then there is a point \(z(1) \in B_\eta(z(0))\) such that \(g(z(1)) = w^{(1)}\). (In particular we may put \(w^{(1)} = f(z(0))\).)

2. If \(z(0) \in G'\), and \(w^{(1)} \in B_\tau(g(z(0)))\), then there is a point \(z(1) \in B_\eta(z(0))\) such that \(f(z(1)) = w^{(1)}\). (In particular we may put \(w^{(1)} = g(z(0))\).)

Proof. For \(z \in G'\), since \(f\) is analytic, there is a \(\kappa \in (0,\eta)\) such that \(f\) is injective on \(B_\kappa(z)\). Let \(h^{(1)}(z)\) denote the supremum over \(G'\) of all such \(\kappa\). Since \(G'\) does not contain any critical points of \(f\), \(h^{(1)}\) is continuous on \(G'\). Therefore if we define \(\kappa^{(1)} := \frac{\inf(h^{(1)}(z) : z \in G'')}{2}\), we conclude that \(\kappa^{(1)} \in (0,\eta)\). Now define \(h^{(2)}(z) := \inf(|f(z') - f(z)| : |z' - z| = \kappa^{(1)})\). By definition of \(\kappa^{(1)}\), \(h^{(2)}\) is non-zero on \(G'\), and \(h^{(2)}\) is continuous, so if we define \(\kappa^{(2)} := \inf(h^{(2)}(z) : z \in G'\), we may conclude that \(\kappa^{(2)} > 0\). Now define \(\tau := \frac{\kappa^{(2)}}{100}\).

Let \(g\) be analytic on \(G\), with \(|f - g| < \tau\) on \(G\), and let \(z(0) \in G'\) and \(w^{(1)} \in B_\tau(f(z(0)))\) be given. Consider the function \(h^{(3)}(z) := f(z(0)) - g(z)\). For any \(z \in \partial B_{\kappa^{(2)}(z(0))}\),

\[|h^{(3)}(z)| = |f(z(0)) - g(z)| = |f(z(0)) - f(z) + f(z) - g(z)| \geq |f(z(0)) - f(z)| - |f(z) - g(z)|.\]

Thus we have

\[|h^{(3)}(z)| \geq \kappa^{(2)} - \frac{\kappa^{(2)}}{100} = \frac{99\kappa^{(2)}}{100}.\]

However \(|h^{(3)}(z(0))| = |f(z(0)) - g(z(0))| \leq \tau = \frac{\kappa^{(2)}}{100}\). Then by the Maximum Modulus Principle, \(h^{(3)}\) has a zero in \(B_{\frac{\kappa^{(2)}}{100}}(z(0))\), which proves that Item 1 holds for this \(\tau\).

Let us now define \(h^{(4)}(z) := g(z(0)) - f(z)\). For any \(z \in \partial B_{\frac{\kappa^{(2)}}{100}}(z(0))\),

\[|h^{(4)}(z)| = |g(z(0)) - f(z)| = |g(z(0)) - f(z(0)) + f(z(0)) - f(z)|.\]

By using the reverse triangle inequality, we obtain

\[|h^{(4)}(z)| \geq |f(z(0)) - f(z)| - |g(z(0)) - f(z(0))| \geq \frac{99\kappa^{(2)}}{100}.\]

However \(|h^{(4)}(z(0))| = |g(z(0)) - f(z(0))| \leq \tau = \frac{\kappa^{(2)}}{100}\). Thus by the Maximum Modulus Principle, \(h^{(4)}\) has a zero in \(B_{\frac{\kappa^{(2)}}{100}}(z(0))\), which proves that Item 2 holds for the chosen \(\tau\).

Lemma A.4. Let \(K \subset \mathbb{C}^{n-1}\) be a compact set, and let \(\nu^{(1)} \in [0,\pi)\) and \(r^{(1)} > 0\) be given. There is a \(\delta^{(1)} > 0\) such that the following holds. Let \(v \in K\) be chosen, such that \(d_{\arg}(v) > \nu^{(1)}\). Let \(u \in \Theta^{-1}(v)\) be chosen. Then if \(r \geq r^{(1)}\), and \(\lambda\) is any component of \(E_{p_u,r}\), and \(E\) is any edge in \(\lambda\), then there is some point \(z\) in \(E\) which is greater than \(\delta^{(1)}\) away from each critical point of \(p_u\).

Proof. By definition of \(p_u\), there are polynomials \(P^{(1)}, \ldots, P^{(n-1)}\) in \(n-1\) variables such that for \(u \in \mathbb{C}^{n-1}\), \(p_u(z) = \frac{1}{n}z^n + \sum_{k=1}^{n-1} P^{(k)}(u)z^k\). Therefore for \(z \in \mathbb{C}\),

\[|p_u(z)| \geq \frac{1}{n}|z|^n - \left(\sum_{i=k}^{n-1} |P^{(k)}(u)||z|^k\right) .

(1)
Since $\Theta$ is proper (by \ref{item:Theta-proper}), $\Theta^{-1}(K)$ is compact. By inspecting Equation \ref{item:Theta-proper} we conclude that there is some constant $S > 0$ such that if $u \in \Theta^{-1}(K)$, and $|z| > S$, then $|p_u(z)| \geq 2$. Therefore $G_{p_u} \subset B_S(0)$ for each $u \in \Theta^{-1}(K)$. Increase $S$ further if necessary so that $\Theta^{-1}(K) \subset B_S(0)$. Finally we set

$$T := \sup(\{p_u'(z) : u \in B_S(0) \text{ and } z \in B_S(0)\}).$$

By a similar argument as above, this $T$ is finite by the compactness of the sets involved. We now define

$$\delta^{(1)} := \frac{r^{(1)} \sin(\frac{\pi}{2})}{T}.$$

Now choose any $v \in K$ such that $d_{\arg}(v) > \delta^{(1)}$, and any $u \in \Theta^{-1}(v)$. Let $r \in [r^{(1)}, 1)$ be chosen, and let $\lambda$ be some component $E_{p_u,r}$, and let $E$ be any edge of $\lambda$. Since the endpoints of $E$ are critical points of $p_u$, the change in argument of $p_u$ along $E$ is greater than or equal to $\delta^{(1)}$. Therefore there is some point $z^{(1)}$ in $E$ such that $d_{\arg}(p_u(z), v^{(k)}) > \frac{\delta^{(1)}}{T}$ for each $k \in \{1, \ldots, n-1\}$. Fix some $i \in \{1, \ldots, n-1\}$. The angle between $p_u(z^{(1)})$ and $p_u(u^{(i)})$ is greater than or equal to $\frac{\delta^{(1)}}{T}$, and $|p_u(z^{(1)})| = r > r^{(1)}$. Therefore by geometry, $|p_u(z^{(1)}) - p_u(u^{(i)})| \geq r^{(1)} \sin(\frac{\pi}{2})$. Let $L$ denote the straight line path from $z^{(1)}$ to $u^{(i)}$. Then

$$\frac{|p_u(z^{(1)}) - p_u(u^{(i)})|}{|z^{(1)} - u^{(i)}|} \leq \max(|p_u'(z)| : z \in L) \leq T.$$

Therefore

$$|z^{(1)} - u^{(i)}| \geq \frac{|p_u(z^{(1)}) - p_u(u^{(i)})|}{T} \geq r^{(1)} \frac{\sin(\frac{\pi}{2})}{T} = \delta^{(1)}.$$

Since this holds for each $i \in \{1, \ldots, n-1\}$, we are done. \hfill $\square$

**Lemma A.5.** Let $v \in \mathbb{C}^{n-1}$ and $\rho > 0$ be given. Then there exists a $\nu > 0$ such that if $\widehat{v} \in \mathbb{C}^{n-1}$ and $|v - \widehat{v}| < \nu$, and $\widehat{u} \in \Theta^{-1}(\widehat{v})$, there is then a $u \in \Theta^{-1}(v)$ such that $|u - \widehat{u}| < \rho$.

**Proof.** It was shown in \ref{item:Theta-proper} that $\Theta$ is continuous, open, and proper, and that $\Theta^{-1}(v)$ is finite. Since $\Theta$ is open and $\Theta^{-1}(v)$ is finite, there is some $\nu > 0$ small enough that $B_{\rho}(v) \subset \bigcap_{u \in \Theta^{-1}(v)} \Theta(B_{\rho}(u))$. Since $\Theta$ is proper, $\Theta^{-1}(cl(B_{\rho}(v))$ is compact. Suppose by way of contradiction that there is a sequence of $\{v_k\}_{k=0}^\infty \subset B_{\rho}(v)$ such that $v_k \to v$, and for each $k \geq 0$ there is some $u_k \in \Theta^{-1}(v_k)$ such that $|u_k - u| > \rho$ for each $u \in \Theta^{-1}(v)$. Since $\Theta^{-1}(cl(B_{\rho}(v))$ is compact, there is some subsequence $\{u_{k_i}\}_{i=0}^\infty$ which converges to some point $u$. Since $\Theta$ is continuous, $\Theta(u) = v$, so $u \in \Theta^{-1}(v)$, which gives us our contradiction. \hfill $\square$

**Definition:** If $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, and $f$ is a function which is analytic and non-zero on the image of $\gamma$, then we say that $\gamma$ is parameterized according to $\arg(f)$ if for each $r \in [\alpha, \beta]$, $\arg(f(\gamma(r))) = r$.

**Lemma A.6.** Let $v \in V_{n-1}$, and $\delta^{(1)} > 0$ be given. There exists some $\delta^{(2)} \in (0, \delta^{(1)})$ such that if $u \in \Theta^{-1}(v)$, and $\lambda$ is a critical level curve of $(p_u, G_{p_u})$ (with $|f| \equiv \epsilon > 0$ on $\lambda$), and $x \in \lambda$, then if $y \in B_{\delta^{(2)}}(x)$ satisfies $|f(y)| = \epsilon$, then there is a path $\sigma$ from $y$ to $x$ which is contained in $\lambda \cap B_{\delta^{(2)}}(x)$. Moreover, we may choose $\sigma$ so that $\arg(p_u)$ is strictly increasing or strictly decreasing along $\sigma$, and parameterized according to $\arg(p_u)$.

**Proof.** Since $\Theta^{-1}(v)$ is finite (\ref{item:Theta-proper}), we need only show the result for some fixed $u \in \Theta^{-1}(v)$. Let $u \in \Theta^{-1}(v)$, and let $\lambda$ be one of the critical level curves of $(p_u, G_{p_u})$, (with $|f| \equiv \epsilon > 0$ on $\lambda$). Let $x \in \lambda$ be given. Let $k \in \mathbb{N}$ denote the multiplicity of $x$ as a zero of $p_u$ (possibly $k = 0$). Then there is some neighborhood $D \subset B_{\delta^{(2)}}(x)$ of $x$ and $S > 0$ and conformal map $\phi : D \to B_{\delta^{(2)}}(p_u(0))$ such that $p_u(z) = \phi(z)^{k+1} + p_u(x)$ for all $z \in D$. Define $f(w) := w^{k+1} + p_u(x)$. The level curves of $f$ are well understood. Let $L$ denote the level curve of $f$ which contains $0$. Then if $w \in L$, there is a path in $L$ from $w$ to $0$ which is contained in $B_{|w|}(0)$, along which $\arg(f)$ is either strictly increasing or strictly decreasing. Choose some $r > 0$ such that
Let $y \in B_r(x)$ be any point such that $|p_u(y)| = \epsilon$. Let $\sigma^{(1)}$ denote the path in $B_{|\phi(y)|}(0)$ from $\phi(y)$ to 0 along which $\arg(f)$ is strictly increasing or strictly decreasing. Then if we define $\sigma := \phi^{-1} \circ \sigma^{(1)}$, $\sigma \subset \phi^{-1}(B_{|\phi(y)|}(0)) \subset D \subset B_{\delta^{(1)}}(x)$, and for each $t \in [0, 1]$, $p_u(\sigma(t)) = f(\sigma^{(1)}(t))$, so $\arg(p_u)$ is either strictly increasing or strictly decreasing along $\sigma$.

Now for $x \in \lambda$, let $h(x)$ denote the supremum over all $r > 0$ such that for $y \in B_r(x)$ with $|p_u(y)| = \epsilon$, a path $\sigma$ with the desired properties may be found. We have just shown that $h(x) > 0$ for all $x \in \lambda$, and it is easy to see that $h$ is continuous, so if we define $h(\lambda) := \inf(h(x) : x \in \lambda)$, the compactness of $\lambda$ implies that $h(\lambda) > 0$. Then choosing $\delta^{(2)} > 0$ to be less than $h(\lambda)$ for each critical level curve $\lambda$ of $p_u$ on which $|p_u| \neq 0$, it is clear that $\delta^{(2)}$ has the desired property.

Lemma A.7. Let $v \in V_{n-1}$, and $\delta^{(1)} > 0$ be given. There exists some $\delta^{(2)} \in (0, \delta^{(1)})$ such that if $u \in \Theta^{-1}(v)$, and $\lambda$ is a critical level curve of $(p_u, G_{p_u})$ (with $|p_u| \equiv \epsilon > 0$ on $\lambda$), and $x \in \lambda$, then if $y \in B_{\delta^{(2)}}(x)$ satisfies $\arg(p_u(y)) = \arg(p_u(x))$, then there is a path $\sigma$ from $y$ to $x$ which is contained in $B_{\delta^{(1)}}(x)$ and such that $\arg(p_u(\sigma(r))) = \arg(p_u(x))$ for all $r$. Moreover we may choose $\sigma$ so that $|p_u|$ is strictly increasing or strictly decreasing along $\sigma$, and parameterized according to $|p_u|$.

Proof. Essentially the same argument for Lemma A.6 works here.

Lemma A.8. Let $v \in V_{n-1}$, and $\tau > 0$ and be given. Then there exists a $\rho > 0$ such that if $u \in \Theta^{-1}(v)$, and $\lambda \subset \Theta^{-1}(V_{n-1})$ such that $|u - \lambda| < \rho$, then the following holds. $G_{p_u,1} \subset G_{p_u,2}$, and $|p_u(z) - p_u(z')| < \tau$ for all $z, z' \in G_{p_u,2}$ satisfying $|z - z'| < \rho$.

Proof. This follows from the fact that the coefficients of $p_u$ are polynomials in the components of $u$. Therefore if $u^{(n)} \to u$ in $\Theta^{-1}(V_{n-1})$, then $p_u^{(n)} \to p_u$ uniformly on any compact set.

Definition: For $u \in \mathbb{C}^{n-1}$, if $\gamma : [0, 1] \to \mathbb{C}$ is a path, and $0 < a < b < 1$, then for $0 < \epsilon^{(1)} < \epsilon^{(2)}$, we say that $\gamma$ takes an $(\epsilon^{(1)}, \epsilon^{(2)})$ trip on $[a, b]$ if the following hold.

- There is some $\epsilon > 0$ such that for all $r \in (a - \epsilon, a) \cup (b, b + \epsilon)$, $\gamma(r)$ is less than $\epsilon^{(1)}$ away from any critical point of $p_u$.
- For each $r \in (a, b)$, $\gamma(r)$ is greater than or equal to $\epsilon^{(1)}$ away from every critical point of $p_u$.
- There is some $r \in (a, b)$ such that $\gamma(r)$ is greater than $\epsilon^{(2)}$ away from every critical point of $p_u$.

Definition: Let $u \in \mathbb{C}^{n-1}$ be given. Let $\gamma$ be a path in $E_{p_u,\epsilon}$ for some $\epsilon > 0$, such that $\gamma(0)$ and $\gamma(1)$ are critical points of $p_u$. For $0 \leq a < b \leq 1$, we say $\gamma$ has a $p_u$-edge on $[a, b]$ if $\gamma(a)$ and $\gamma(b)$ are critical points of $p_u$, and for all $r \in (a, b)$, $\gamma(r)$ is not a critical point of $p_u$.

Lemma A.9. Fix some $v = (v^{(1)}, \ldots, v^{(n-1)}) \in V_{n-1}$ not the zero vector, and $\delta^{(1)} > 0$. Then there exists a constant $\rho > 0$ such that the following hold. Let $u \in \Theta^{-1}(v)$ be chosen, and fix some $\tilde{u} \in B_{\rho}(u)$ such that if we define $\tilde{v} = (\tilde{v}^{(1)}, \ldots, \tilde{v}^{(n-1)}) := \Theta(\tilde{u})$, then $\arg(\tilde{v}^{(k)}) = \arg(\tilde{v}^{(k)})$ for each $k \in \{1, \ldots, n-1\}$. For some $k \in \{1, \ldots, n-1\}$ with $|v^{(k)}| \neq 0$, let $\lambda$ denote the level curve of $p_u$ which contains $u^{(k)}$. Let $\tilde{E}$ denote some edge of $\tilde{\lambda}$ which is incident to $u^{(k)}$, and let $\tilde{\gamma}$ denote a parameterization of $\tilde{E}$ such that $\tilde{\gamma} : [\alpha, \beta] \to \tilde{\lambda}$ (for some $\alpha, \beta \in \mathbb{R}$) satisfies $\tilde{\gamma}(\alpha) = u^{(k)}$ and $\arg(p_u(\tilde{\gamma}(t))) = t$ for all $t \in [\alpha, \beta]$. (Note: if $\arg(p_u)$ is increasing as this portion of $\tilde{E}$ is traversed, then $\alpha < \beta$, otherwise $\alpha > \beta$.) Then if we let $\lambda$ denote the critical level curve of $p_u$ containing $u^{(k)}$, there is a path $\gamma : [\alpha, \beta] \to \lambda$ such that $\gamma(\alpha) = u^{(k)}$, and for each $t \in [\alpha, \beta]$, $\arg(p_u(\gamma(t))) = t$ and $|\gamma(t) - \tilde{\gamma}(t)| < \delta^{(1)}$.

Proof. We assume that $\arg(p_u)$ is increasing as $\gamma$ is traversed. Otherwise make the appropriate changes. We will show that the result of the lemma holds for any fixed $u \in \Theta^{-1}(v)$, which will suffice because $\Theta^{-1}(v)$ is finite by [I]. Reduce $\delta^{(1)} > 0$ if necessary so that for each $k \in \{1, \ldots, n-1\}$ with $|v^{(k)}| \neq 0$, if
\[ |z - u(k)| < \delta(1), \text{ then } |p_u(z) - u(k)| < \min_{\theta(0, |v^{(1)}|, \ldots, |v^{(n-1)}|)} \frac{\text{min_diff}(0, |v^{(1)}|, \ldots, |v^{(n-1)}|)}{4}. \] Of course \( \text{min_diff}(0, |v^{(1)}|, \ldots, |v^{(n-1)}|) \leq |v(k)| \), so by geometry, \( |\arg(p_u(z)) - \arg(u(k))| < \frac{\pi}{2} \).

By Lemma A.6, we may choose \( \delta(2) \in (0, \frac{\delta(1)}{3}) \) such that the following holds. If \( y \in B_{\delta(2)}(u(k)) \) for some \( k \in \{1, \ldots, n-1\} \) such that \( |p_u(y)| = |v(k)| \), then there is a path \( \sigma \) from \( y \) to \( u(k) \) contained in \( B_{\delta(2)}(u(k)) \) such that \( \arg(p_u) \) is strictly increasing or strictly decreasing along \( \sigma \).

Since \( p_u \) is an open mapping, we may choose some \( M > 0 \) small enough so that for each \( k \in \{1, \ldots, n-1\} \), \( B_{2M}(v(k)) \subset p_u(B_{\delta(2)}(u(k))) \). By Lemma A.8, we may choose a \( \rho(1) > 0 \) so that \( \rho(1) < \frac{\delta(2)}{2} \), and if \( \hat{u} \in B_{\rho(1)}(v(k)) \), then \( |p_u(z) - p_u(\hat{u})| < M \) for all \( z, \hat{u} \in G_{p_u} \) such that \( |z - \hat{u}| < \rho(1) \).

Let \( K \) denote the set of all points \( x \in G_{p_u} \) such that the following hold.

- \( x \in E_{p_u,|v(k)|} \) for some \( k \in \{1, \ldots, n-1\} \).
- \( |x - u(k)| > \frac{\delta(2)}{2} \) for each \( k \in \{1, \ldots, n-1\} \).

Then by the compactness of \( K \), we may choose an \( \eta > 0 \) such that for each \( x \in K \) the following holds. Let \( l \in \{1, \ldots, n-1\} \) be chosen so that \( |p_u(x) - |u(l)|| \).

- \( \eta < \min(\{\eta, \partial G_{p_u} : z \in K\}) \).
- \( p_u \) is injective on \( B_\eta(x) \).
- \( \eta < \rho(1) \).
- \( |x - u(k)| > \eta \) for each \( k \in \{1, \ldots, n-1\} \).

Define \( G' := \{ x \in G_{p_u} : d(x, \partial G_{p_u}) > \eta, d(x, u(k)) > \eta \text{ for each } k \} \). By Lemma A.3, we may choose \( \tau > 0 \) so that \( \tau < \min(M, \frac{\text{min_mod}(x)}{4}) \), \( \tau < \min_{\theta(0, |v^{(1)}|, \ldots, |v^{(n-1)}|)} \), and if \( f \) is analytic on \( G' \) with \( |f - p_u| < \tau \) of \( G' \), then for all \( x \in G' \), the following hold.
- For any \( w \in B_\tau(p_u(x)) \), there is a \( y \in B_\eta(x) \) with \( f(y) = w \).
- For any \( w \in B_\tau(f(x)) \), there is a \( y \in B_\eta(x) \) with \( p_u(y) = w \).

By Lemma A.8 and the continuity of \( \Theta \), we may choose \( \rho \in (0, \delta(1)) \) so that if \( \hat{u} \in B_{\rho}(u) \), then \( |p_u(z) - p_u(\hat{u})| < \tau \) for all \( z, \hat{u} \in G_{p_u} \) such that \( |z - \hat{u}| < \rho \), and for \( \hat{v} = (u^{(1)}, \ldots, u^{(n-1)}) := \Theta(\hat{u}) \), \( |\hat{v} - v| < \tau \). We now show that the statement of the lemma holds for the chosen \( \rho \).

Let \( \hat{u} \in B_{\rho}(u) \) be chosen. Fix some \( k \in \{1, \ldots, n-1\} \) such that \( |v(k)| \neq 0 \). Then \( |v(k)| \geq \min \text{mod}(v) \).

Note that since \( \tau > \frac{\text{min_mod}(x)}{4} \), and \( |v - \hat{v}| < \frac{\tau}{2} \), we have \( |\hat{v}(k)| > \frac{\text{min_mod}(x)}{2} \). Let \( \hat{\lambda} \) denote the level curve of \( p_{\hat{u}} \) which contains \( \hat{u} \). Let \( \hat{E} \) denote some edge of \( \hat{\lambda} \) which is incident to \( \hat{\lambda} \). Let \( \alpha \) denote some choice of the argument of \( p_{\hat{u}}(u(k)) \), and let \( \hat{\gamma} : [\alpha, \beta] \rightarrow \hat{\lambda} \) (for some \( \beta > \alpha \) because \( \arg(p_{\hat{u}}) \) is increasing as \( \hat{\gamma} \) is traversed) be a path which parameterizes \( \hat{E} \) according to the argument of \( p_{\hat{u}} \) (that is, \( \arg(p_{\hat{u}}(\hat{\gamma}(t))) = t \) for all \( t \in [\alpha, \beta] \)).

Note that by the definition of a \( (\rho(1), \delta(1)) \) trip over an interval, if \( \hat{\gamma} \) takes \( (\rho(1), \delta(1)) \) trips over two intervals \( I^{(1)}, I^{(2)} \subset [0, 1] \), then either \( I^{(1)} = I^{(2)} \), or \( I^{(1)} \) and \( I^{(2)} \) are disjoint. Therefore since \( \hat{\gamma} \) is a rectifiable path, \( \hat{\gamma} \) takes at most finitely many distinct \( (\rho(1), \delta(1)) \) trips.

**Case A.9.1.** \( \hat{\gamma} \) takes a \( (\rho(1), \delta(1)) \) trip on some sub-interval of \( [\alpha, \beta] \).

Let \( [s^{(1)}, s^{(1)}], \ldots, [s^{(N)}, s^{(N)}] \subset [\alpha, \beta] \) be the disjoint subintervals of \( [\alpha, \beta] \) over which \( \gamma \) takes \( (\rho(1), \delta(1)) \) trips, ordered so that \( s^{(k)} < s^{(k+1)} \) for each \( k \in \{1, \ldots, N-1\} \). We begin by defining \( \gamma \) on \( \bigcup_{i=1}^{N}[s^{(i)}, s^{(i)}] \). Fix for the moment some \( j \in \{1, \ldots, N\} \).
For all \( t \in [r^{(j)}, s^{(j)}) \), define \( w(t) := |v^{(k)}| e^{it} \). Then by choice of \( \tau \)
\[
|w(t) - p_{u}(\gamma(t))| = |v^{(k)}| e^{it} - |v^{(k)}| e^{it} = ||v^{(k)}| - |v^{(k)}|| < \tau
\]

Thus there is some \( y \in B_{\eta}(\gamma(r)) \) such that \( p_{u}(y) = w(r) \). Moreover, since \( p_{u} \) is injective in \( B_{\eta}(\gamma(r)) \), this choice of \( y \) is unique. Define \( \gamma(r) = y \).

Since \( p_{u} \) is injective on \( B_{\eta}(\gamma(r)) \) for each \( r \in [r^{(j)}, s^{(j)}) \), and \( p_{u} \) is an open mapping, it is easy to show that \( \gamma \) is a continuous function, and thus a path from \( \gamma(r^{(j)}) \) to \( \gamma(s^{(j)}) \). Further, if \( r \in [r^{(j)}, s^{(j)}] \), \( |p_{u}(\gamma(r))| = |w(r)| = |v^{(k)}| \). Therefore we conclude that \( \gamma|_{[r^{(j)}, s^{(j)}]} \) is a path in \( E_{p_{u},|v^{(k)}|} \), and by construction, for each \( r \in [r^{(j)}, s^{(j)}], \left| \gamma(r) - \gamma(r) \right| < \eta \) and \( \arg(p_{u}(\gamma(r))) = r \). Having done this for each \( j \in \{1, \ldots, N\} \), we now wish to define \( \gamma \) on \( (s^{(j)}, r^{(j+1)}) \) for each \( j \in \{1, \ldots, N - 1\} \).

Again fix for the moment some new \( j \in \{0, \ldots, N\} \).

Since there is no sub-interval of \( (s^{(j)}, r^{(j+1)}) \) on which \( \gamma \) takes a \((\rho^{(1)}, \delta^{(1)})\) trip, \( \gamma(r) \) is within \( \frac{\delta^{(1)}}{4} \) of some critical point of \( p_{u} \) for each \( r \in (s^{(j)}, r^{(j+1)}) \). However \( \delta^{(1)} < \frac{\min\text{diff}(v)}{2} \), thus there is some unique \( l \in \{1, \ldots, n - 1\} \) such that for each \( j \in (s^{(j)}, r^{(j+1)}), |\gamma(r) - u^{(l)}| \leq \delta^{(1)} \). Since \( \gamma \) takes a \((\rho^{(1)}, \delta^{(1)})\) trip over \( [r^{(j)}, s^{(j)}], \left| \gamma(s^{(j)}) - u^{(l)} \right| = \rho^{(1)} \). Therefore
\[
|\gamma(s^{(j)}) - u^{(l)}| \leq |\gamma(s^{(j)}) - \gamma(s^{(j)})| + |\gamma(s^{(j)}) - u^{(l)}| < \eta + \rho^{(1)} < \delta^{(2)}.
\]

In addition to this, \( |p_{u}(\gamma(s^{(j)}))| = |v^{(k)}| \), but by choice of \( \delta^{(1)} \),
\[
||v^{(k)}| - |v^{(l)}| = |p_{u}(\gamma(s^{(j)}))| - |u^{(l)}|| < \delta^{(2)}
\]

and
\[
\delta^{(2)} < \min\text{diff}(0, |v^{(1)}|, \ldots, |v^{(n-1)}|).
\]

Therefore we conclude that \( |p_{u}(\gamma(s^{(j)}))| = |v^{(l)}| = |v^{(k)}| \). Then by choice of \( \delta^{(2)} \), there is some path \( \sigma^{(1)} \) from \( \gamma(s^{(j)}) \) to \( u^{(l)} \) contained in \( B_{\frac{\delta^{(2)}}{\min\text{diff}(u)}}(u^{(l)}) \cap E_{p_{u},|v^{(k)}|} \) such that \( \arg(p_{u}) \) is strictly monotonic on \( \sigma^{(1)} \), and \( \sigma^{(1)} \) is parameterized according to \( \arg(p_{u}) \). Since \( \arg(p_{u}) \) is increasing along \( \gamma \), \( \arg(p_{u}) \) is increasing along the portions of \( \gamma \) which have already been defined. Let \( D \) denote an open region containing \( \gamma(s^{(j)}) \) on which \( p_{u} \) is injective. Choose some \( t^{(0)} \in (r^{(j)}, s^{(j)}) \) such that \( \gamma(t^{(0)}, s^{(j)}) \subset D \). If \( \arg(p_{u}) \) is decreasing on \( \sigma^{(1)} \), then since \( p_{u} \) is injective on \( D \), for each \( r \in (s^{(j)}, t^{(0)}), \sigma^{(1)}(r) = \gamma(r) \). Furthermore, since \( p_{u} \) is injective in a neighborhood of each point of \( \gamma((r^{(j)}, s^{(j)})), \sigma^{(1)} \) must continue to trace back along the entire length of \( \gamma([r^{(j)}, s^{(j)}]) \). This is because both \( \sigma^{(1)} \) and \( \gamma \) are parameterized according to \( \arg(p_{u}) \), so any branching off of \( \sigma^{(1)} \) from \( \gamma \) would have to be a critical point of \( p_{u} \). However \( \sigma^{(1)} \) may not trace back along \( \gamma([r^{(j)}, s^{(j)}]) \) because the image of \( \sigma^{(1)} \) is contained in \( B_{\frac{\delta^{(2)}}{2}}(u^{(l)}) \). Therefore we conclude that \( \arg(p_{u}) \) is increasing on \( \sigma^{(1)} \).

By very similar reasoning we may obtain a path \( \sigma^{(2)} \) from \( u^{(l)} \) to \( \gamma(r^{(j+1)}) \) contained in \( B_{\frac{\delta^{(2)}}{2}}(u^{(l)}) \cap E_{p_{u},|v^{(k)}|} \) parameterized according to \( \arg(p_{u}) \), and along which \( \arg(p_{u}) \) is increasing.

Let \( s^{(j)} \) be the choice of \( \arg(p_{u}(\gamma(s^{(j)}))) \) which is the starting point for the domain of \( \sigma^{(1)} \), and choose some \( t^{(1)} > 0 \) so that the domain of \( \sigma^{(1)} \) is \([s^{(j)}, s^{(j)} + t^{(1)}]\). Now let \( s^{(j)} + t^{(1)} \) be the choice of \( \arg(u^{(l)}) \) which is the starting point for the domain of \( \sigma^{(2)} \), and choose \( t^{(2)} > 0 \) so that the domain of \( \sigma^{(2)} \) is \([s^{(j)} + t^{(1)}, s^{(j)} + t^{(1)} + t^{(2)}]\). Let \( \sigma : (s^{(j)} + t^{(1)}, s^{(j)} + t^{(1)} + t^{(2)}) \rightarrow B_{\frac{\delta^{(2)}}{2}}(u^{(l)}) \) denote the concatenation of \( \sigma^{(1)} \) and \( \sigma^{(2)} \). Then \( s^{(j)} + t^{(1)} + t^{(2)} = r^{(j+1)} \) (mod 2\pi), so \( t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)} \) (mod 2\pi). However by choice of \( \delta^{(1)} \), the total change in argument of \( p_{u} \) along \( \sigma^{(1)} \) must be less than \( \pi \). And since \( \gamma(s^{(j)}, r^{(j+1)}) \subset B_{\frac{\delta^{(2)}}{2}}(u^{(l)}) \), the total change in argument of \( p_{u} \) along \( \gamma([s^{(j)}, r^{(j+1)}) \) is less than \( \pi \), and thus the total change in argument of \( p_{u} \) along \( \gamma([s^{(j)}, r^{(j+1)}) \) (which of course equals \( r^{(j+1)} - s^{(j)} \)) is less than \( 2\pi \) in magnitude (since \( |p_{u} - p_{u}| < \tau \) on the image of \( \gamma \)). Thus we have that \( t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)} \) (mod 2\pi) and both sides are contained in \([0, 2\pi]\), so \( t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)} \). Therefore we may define \( \gamma(r) := \sigma(r) \) for each \( r \in (s^{(j)}, r^{(j+1)}) \). With this definition, we have that for each \( r \in (s^{(j)}, r^{(j+1)}) \), \( \arg(p_{u}(\gamma(r))) = \tau \), and
\[|\gamma(r) - \hat{\gamma}(r)| \leq |\gamma(r) - u^{(0)}| + |u^{(0)} - \hat{u}^{(0)}| + |\hat{u}^{(0)} - \gamma(r)| < \delta(1).\]

We extend \(\gamma\) in this manner to \((\sigma^{(j)}, r^{(j+1)})\) for each \(j \in \{1, \ldots, N - 1\}\). Moreover, we may extend \(\gamma\) in using the exactly similar construction to \([\alpha, r^{(1)})\) and \((\sigma^{(N)}, \beta]\), and this extended \(\gamma\) has all of the desired properties.

**Case A.9.2.** There is no sub-interval of \([\alpha, \beta]\) along which \(\hat{\gamma}\) takes a \((\rho^{(1)}, \delta^{(1)})\) trip.

Then either \(|\hat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}\) for all \(r \in [\alpha, \beta]\), or there is some \(r^{(0)} \in (\alpha, \beta)\) such that for all \(r \in [\alpha, r^{(0)}]\), \(|\hat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}\), and for all \(r \in (r^{(0)}, \beta]\), \(\hat{\gamma}\) is greater than \(\rho^{(1)}\) from any critical point of \(p_u\).

**Sub-case A.9.2.1.** \(|\gamma(r) - u^{(k)}| \leq \delta^{(1)}\) for all \(r \in [\alpha, \beta]\).

In this case, we construct \(\gamma\) using the same method as in the second part of Case A.9.1.

**Sub-case A.9.2.2.** There is some \(r^{(0)} \in (\alpha, \beta)\) such that for all \(r \in [\alpha, r^{(0)}]\), \(|\hat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}\), and for all \(r \in (r^{(0)}, \beta]\), \(\hat{\gamma}\) is greater than \(\rho^{(1)}\) from any critical point of \(p_u\).

In this case, we construct \(\gamma\) on \([\alpha, r^{(0)})\) using the same method as in the second part of Case A.9.1 and we construct \(\gamma\) on \(r^{(0)}, \beta\) using the same method as in the first part of Case A.9.1.

**Lemma A.10.** Fix some \(v = (v^{(1)}, \ldots, v^{(n-1)}) \in V_{n-1} \) not the zero vector, and \(\delta^{(1)} > 0\). Then there exists constants \(\rho, \delta^{(2)} > 0\) such that the following hold. Let \(u \in \Theta^{-1}(v)\) be chosen, and fix some \(\hat{u} \in B_{\rho}(u)\). Let \(\hat{x}_1, \hat{x}_2 \in G_{\rho \hat{u}}\) be given such that \(\arg(p_u(x_1)) = \arg(p_u(x_2)) = 0\), and such that there is a path \(\hat{\sigma} : [0, 1] \rightarrow G_{\rho \hat{u}}\) such that \(\sigma(0) = \hat{x}_1\) and \(\sigma(1) = \hat{x}_2\) and \(\arg(p_u(\hat{\sigma}(r))) = 0\) for all \(r \in [0, 1]\). Then if \(x_1, x_2 \in G_{\rho \hat{u}}\) are such that \(\arg(p_u(x_1)) = \arg(p_u(x_2)) = 0\) and \(|\hat{x}_1 - x_1| < \delta^{(2)}\) and \(|\hat{x}_2 - x_2| < \delta^{(2)}\), then there is a path \(\sigma : [0, 1] \rightarrow G_{\rho \hat{u}}\) such that \(\sigma(0) = x_1\), \(\sigma(1) = x_2\), and for all \(r \in [0, 1]\), \(\arg(p_u(\sigma(r))) = 0\) and \(|\sigma(r) - \sigma(r)| < \delta^{(1)}\). Moreover, if \(|p_u|\) is strictly increasing or strictly decreasing on \(\sigma\), then we may assume that \(|p_u|\) is strictly increasing or strictly decreasing on \(\sigma\) respectively.

**Proof.** The exact same method of proof used for Lemma A.9 works here except that instead of invoking Lemma 4.8 we would invoke the gradient line version Lemma A.7.

**Lemma A.11.** Fix some \(v = (v^{(1)}, \ldots, v^{(n-1)}) \in V_{n-1}\) and some \(u \in \Theta^{-1}(v)\). Let \(\delta > 0\) be given, and choose some point \(x \in G_{\rho \hat{u}}\). There are constants \(\rho, \nu > 0\) small enough so that for any \(\hat{u} \in B_{\rho}(u)\), if \(\hat{y} \in B_{\nu}(p_u(x))\), then there is some \(\hat{x} \in B_{\nu}(x)\) such that \(p_u(\hat{x}) = \hat{y}\).

**Proof.** Note that the statement of the lemma is similar to the statement of Lemma A.3 but more general in that the point \(x\) which is chosen may be a critical point of \(p_u\). The proof is similar to a portion of the proof of Lemma A.3 but we will reproduce it here.

Reduce \(\delta\) if necessary so that \(B_{\delta}(x) \subset G_{\rho \hat{u}}\), and there is no point \(w\) such that \(|w - x| = \delta\) and \(p_u(w) = p_u(x)\). Then define \(\eta > 0\) to be the minimum that \(|p_u(w) - p_u(x)|\) takes on the set \(\{w \in \mathbb{C} : |w - x| = \delta\}\). Now choose \(\rho > 0\) so that if \(\hat{u} \in B_{\rho}(u)\), then for all \(w \in G_{\rho \hat{u}}\), \(|p_u(w) - p_u(\hat{u})| < \frac{\eta}{2}\). Define \(h(z) := p_u(x) - p_u(\hat{z})\). On the set \(\{w \in \mathbb{C} : |w - x| = \delta\}\) by using the reverse triangle inequality we have

\[|h(z)| = |p_u(x) - p_u(\hat{z})| = |p_u(x) - p_u(z) + p_u(z) - p_u(\hat{z})| \geq |p_u(x) - p_u(z)| + |p_u(z) - p_u(\hat{z})|,\]

and thus

\[|h(z)| \geq \eta - \frac{\eta}{2} = \frac{\eta}{2}.\]

However \(|h(x)| = |p_u(x) - p_u(\hat{z})| \leq \frac{\eta}{2}\), so by the Maximum Modulus Theorem, we conclude that \(h\) contains a zero in the set \(\{w : |w - x| < \delta\}\), which is the desired result.
Lemma A.12. Fix some compact set $K \subset \mathbb{C}^{n-1}$ and some $\tau > 0$ and $R^{(0)} \in (0, 1)$. There exists some $\delta > 0$ so that the following holds. Let $u \in K$, and let $z \in \mathbb{C}$ be such that $|p_u(z)| := R \in [R^{(0)}, 1]$. For all $w \in B_\delta(z)$, $|p_u(w)| \in (R - \tau, R + \tau)$. Fix some $w \in B_\delta(z)$, and let $L$ denote the straight line path from $z$ to $w$, then $|\Delta_{\arg}(p_u, L)| < \tau$.

Proof. Reduce $\tau$ if necessary so that $\tau \in (0, \pi)$. Since $K$ is compact, there is some $S > 0$ such that for each $u \in K$, $G_{p_u} \subset \mathbb{B}_{\frac{\tau}{S}}(0)$. Again since $K$ is compact, and $cl(B_\delta(0))$ is compact, there is some $M > 0$ such that for all $u \in K$, for all $z \in B_S(0)$, $|p_u(z)| < M$. Choose $\delta > 0$ so that the following hold.

- $\delta < \frac{S}{\tau}$.
- $\delta < \frac{R^{(0)}}{2\pi M}$.

Fix some $u \in K$, and let $z \in \mathbb{C}$ be chosen so that $R := |p_u(z)| \in [R^{(0)}, 1]$. Then $B_\delta(z) \subset B_S(0)$, so if $w$ is any point in $B_\delta(z)$, then $|p_u(w) - p_u(z)| \leq |w - z| < \delta$, for all $z \in B_\delta(z)$ and $w \in \mathbb{B}_\frac{\delta}{S}(0)$. Fix some $K \in S \subset \mathbb{B}_\frac{\delta}{S}(0),$, and let $\tau, R \in (R - \tau, R + \tau)$.

Now fix some $w \in B_\delta(z)$, and let $L$ denote the straight line path from $z$ to $w$. Let $P : 0 = x^{(0)} < x^{(1)} < \cdots < x^{(N)} = 1$ be any fixed partition of $[0, 1]$. Then for each $i \in \{0, \ldots, N\}$, $|L(x^{(i)}) - z| < \delta$, so $|f(L(x^{(i)})) - f(z)| \leq M \delta < \frac{R^{(0)}}{2\pi}$. Therefore $|p_u(L(x^{(i)}))| \geq \frac{R^{(0)}}{\pi M}$. Therefore, by geometry, for each $i \in \{1, \ldots, N\}$,

$$|\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)}))))| < \frac{2\pi}{R^{(0)}|f(L(x^{(i)})) - f(L(x^{(i-1)}))|}.$$

Then since $L$ is contained in $B_S(0)$, we have

$$|\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)}))))| \leq \frac{2\pi M}{R^{(0)}|L(x^{(i)}) - L(x^{(i-1)})|}.$$

Since $L$ is the straight line path from $z$ to $w$, $\sum_{i=1}^{N} |L(x^{(i)}) - L(x^{(i-1)})| = |z - w| < \delta$. Therefore

$$\sum_{i=1}^{N} |\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)}))))| < \frac{2\pi M \delta}{R^{(0)}} < \tau,$$

which gives us the desired result.

Lemma A.13. Let $(f, G)$ be a special type function element, and let $\lambda$ be a level curve of $f$ in $G$. Let $D$ be some face of $\lambda$, and let $x, y \in \partial D$ be given such that the line segment $(x, y)$ is contained entirely in $D$. Define $R^{(0)} := \min(|f(z)| : z \in [x, y])$. Define $R^{(1)} := \max(|f'(z)| : z \in [x, y])$. Let $\sigma^+$ denote the path in $\lambda$ obtained by traversing $\partial D$ from $y$ to $x$ with a positive orientation, and let $\sigma^-$ denote the path through $\lambda$ obtained by traversing $\partial D$ from $y$ to $x$ with a negative orientation. Define $\delta^+ := \Delta_{\arg}(f, \sigma^+)$, and $\delta^- := \Delta_{\arg}(f, \sigma^-)$.

Then either there are zeros of $f$ in $D$ on both sides of $[x, y]$, or $|x - y| \geq \frac{2R^{(0)}\min(|\delta^+|, |\delta^-|)}{\pi R^{(1)}}$.

Proof. Assume that all the zeros of $f$ in $D$ are on one side of $(x, y)$ or the other. Let $\sigma$ denote the path obtained by concatenating $[x, y]$ with $\sigma^+$. Then $\sigma$ is a simple closed path. Let $D$ denote the bounded face of $\sigma$. Assume that $D$ does not contain any zeros of $f$. (Otherwise, we make the exactly similar argument with $\sigma^+$ replaced by $\sigma^-$.) Since $D \subset \pi \delta$ does not contain any zero of $f$, $\Delta_{\arg}(f, \sigma) = 0$, and thus $\Delta_{\arg}(f, [x, y]) = -\Delta_{\arg}(f, \sigma^+)$, and thus $|\Delta_{\arg}(f, [x, y])| = |\delta^+|$.

Let $\gamma$ denote the standard parameterization of $[x, y]$. Let $0 = t^{(0)} < \cdots < t^{(N)} = 1$ be a partition of $[0, 1]$ such that $\Delta_{\arg}(f, \gamma|[t^{(i-1)}, t^{(i)})] < \frac{\pi}{\delta}$ for each $i \in \{1, \ldots, N\}$, and define $\delta^+_{(i)} := \Delta_{\arg}(f, \gamma|[t^{(i-1)}, t^{(i)})]$. For each $i \in \{1, \ldots, N\}$, define $z^{(i)} := \gamma(t^{(i)})$. Since $[x, y]$ is a straight line, $|x - y| = \sum_{k=1}^{N} |z^{(i)} - z^{(i-1)}|$. Now since the $|\delta^+_{(i)}| < \frac{\pi}{2}$, we may use elementary trigonometry to show that
\[ |f(x^{(i)}) - f(x^{(i-1)})| < 2R^{(0)} \sin\left(\frac{\delta^+}{2}\right). \]

Moreover, for \( \alpha \in [0, \frac{\pi}{2}] \), \( \sin(\alpha) \geq \frac{2\alpha}{\pi} \). Therefore we have
\[
R^{(1)} \geq \frac{f(z^{(i)}) - f(z^{(i-1)})}{z^{(i)} - z^{(i-1)}} \geq \frac{2R^{(0)}|\delta^+|}{\pi |z^{(i)} - z^{(i-1)}|}.
\]

Solving for \( |z^{(i)} - z^{(i-1)}| \), we obtain
\[
|z^{(i)} - z^{(i-1)}| \geq \frac{2R^{(0)}|\delta^+|}{\pi R^{(1)}}.
\]

And if we sum over all \( i \), we obtain
\[
|x - y| \geq \frac{2R^{(0)}}{\pi R^{(1)}} \sum_{i=1}^{N} |\delta^+| \geq \frac{2R^{(0)}|\delta^+|}{\pi R^{(1)}} \geq \frac{2R^{(0)} \min(|\delta^-|)}{\pi R^{(1)}},
\]
which is the desired result.

\[ \square \]

**Lemma A.14.** Let \((f, G)\) be a special type function element, and let \( \lambda \) be a level curve of \( f \) in \( G \). Let \( x, y \in \lambda \) be given such that the line segment \((x, y)\) is contained entirely in the unbounded face of \( \lambda \). Let \( \hat{D} \) denote the bounded face of \( \lambda \cup [x, y] \) which is not a bounded face of \( \lambda \). Let \( \sigma^{(0)} \) be a parameterization of the portion of \( \partial \hat{D} \) which is in \( \lambda \) from \( y \) to \( x \), and define \( \delta := \Delta_{\arg f}(f, \sigma^{(0)}) \). Define \( R^{(0)} := \min(|f(z)| : z \in [x, y]) \).

Define \( R^{(1)} := \max(|f'(z)| : z \in [x, y]) \). Then either \( \hat{D} \) contains zeros of \( f \), or \( |x - y| \geq \frac{2R^{(0)}|\delta|}{\pi R^{(1)}} \).

\[ \square \]

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