LOCAL CONFORMAL RIGIDITY IN CODIMENSION $\leq 5$

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Abstract. In this paper, for an immersion $f$ of an $n$-dimensional Riemannian manifold $M$ into $(n + d)$-Euclidean space we give a sufficient condition on $f$ so that, in case $d \leq 5$, any immersion $g$ of $M$ into $(n + d + 1)$-Euclidean space that induces on $M$ a metric that is conformal to the metric induced by $f$ is locally obtained, in a dense subset of $M$, by a composition of $f$ and a conformal immersion from an open subset of $(n + d)$-Euclidean space into an open subset of $(n + d + 1)$-Euclidean space. Our result extends a theorem for hypersurfaces due to M. Dajczer and E. Vergasta. The restriction on the codimension is related to a basic lemma in the theory of rigidity obtained by M. do Carmo and M. Dajczer.

1. Introduction

Let $M^n$ be an $n$-dimensional differentiable manifold and let $f: M^n \to \mathbb{R}^m$ and $g: M^n \to \mathbb{R}^l$ be two immersions into Euclidean spaces. We say that $g$ is conformal to $f$ when the metric induced on $M^n$ by $f$ and $g$ are conformal. That $f$ is conformally rigid means that for any other conformal immersion $h: M^n \to \mathbb{R}^m$, conformal to $f$, there exists a conformal diffeomorphism $\Upsilon$ from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^m$ such that $h = \Upsilon \circ f$.

In [2], do Carmo and Dajczer introduced a conformal invariant for an immersion $f: M^n \to \mathbb{R}^{n+d}$ into a Riemannian manifold $\mathbb{R}^{n+d}$, namely, the conformal $s$-nullity $\nu_s^c(p)$, $p \in M^n$, $1 \leq s \leq d$ (see Section 2 for definitions), and proved that is conformally rigid an immersion $f: M^n \to \mathbb{R}^{n+d}$ that satisfies $d \leq 4$, $n \geq 2d + 3$ and $\nu_s^c \leq n - 2s - 1$ for $1 \leq s \leq d$. Their result generalizes a result for hypersurfaces due to E. Cartan ([1]).

It was observed in Corollary 1.1 of [7] that the do Carmo-Dajczer’s conformal rigidity result also holds for $d = 5$. The restriction on the codimension is due to the following basic result in [2].

Theorem 1.1. Let $\sigma: V_1 \times V_1 \to W^{(r,r)}$ be a nonzero flat symmetric bilinear form. Assume $r \leq 5$ and $\dim N(\sigma) < 3 \dim V_1 - 2r$. Then $S(\sigma)$ is degenerate.

In the paper (see [4]), M. Dajczer and L. Florit proved that the above theorem can not be improved.

Denoting by $N_{cf}^{n+d}$ a conformally flat Riemannian manifold of dimension $n + d$, namely an $(n + d)$-dimensional Riemannian manifold $N^{n+d}$ which is locally conformally diffeomorphic to an open subset of Euclidean space $\mathbb{R}^{n+d}$ with the canonical metric, Dajczer and Vergasta ([6]) proved that if $f: M^n \to N_{cf}^{n+1}$ satisfies $n \geq 6$ and $\nu^c_s \leq n - 4$ along $M^n$, then any immersion $g: M^n \to N_{cf}^{n+2}$ conformal to $f$ is locally a composition in an open dense subset $U$ of $M$. This result, still in the context of hypersurfaces, was extended by Dajczer-Tojeiro in [5] for $g: M^n \to N_{cf}^{n+p}$. 2 $\leq p \leq n - 4$, assuming that $\nu^c_s \leq n - p - 2$ and, if $p \geq 6$, further that $M^n$ does not contain an open $n - p + 2$-conformally ruled subset for both $f$ and $g$. In this paper, we extend for codimension $\leq 5$ the result of Dajczer-Vergasta mentioned above in the following theorems.

Theorem 1.2. Let $f: M^n \to \mathbb{R}^{n+d}$ be an immersion with $d \leq 5$ and $n > 2d + 3$. Assume that everywhere $\nu^c_s \leq n - 2s - 2$, $1 \leq s \leq d$. If $g: M^n \to \mathbb{R}^{n+d+1}$ is an immersion conformal to $f$, then there exists an open
dense subset $\mathcal{U}$ of $M^n$ such that $g$ restricted to $\mathcal{U}$ is locally a composition $g = \Gamma \circ f$ for some local conformal immersion $\Gamma$ from an open subset of $\mathbb{R}^{n+d}$ into an open subset of $\mathbb{R}^{n+d+1}$.

Observing that Theorem 1.2 is local one, the following result is an immediate consequence of Theorem 1.2.

**Theorem 1.3.** Let $f : M^n \to N^{n+d}_{cf}$ be an immersion with $d \leq 5$ and $n > 2d + 3$. Assume that everywhere $\nu_s^c \leq n - 22 - 2$, $1 \leq s \leq d$. If $g : M^n \to N^{n+d+1}_{cf}$ is an immersion conformal to $f$, then there exists an open dense subset $\mathcal{U}$ of $M^n$ such that $g$ restricted to $\mathcal{U}$ is locally a composition $g = \Gamma \circ f$ for some local conformal immersion $\Gamma$ from an open subset of $N^{n+d}_{cf}$ into an open subset of $N^{n+d+1}_{cf}$.

2. **Proof of Theorem 1.2**

For a symmetric bilinear form $\beta : V \times V \to W$ we denote by $S(\beta)$ the subspace of $W$ given by

$$S(\beta) = \text{span} \{ \beta(X,Y) : X,Y \in V \},$$

and by $N(\beta)$ the nullity space of $\beta$ defined as

$$N(\beta) = \{ n \in V : \beta(X,n) = 0, \forall X \in V \}.$$

For an immersion $f : M^n \to \tilde{M}^{n+d}$ into a Riemannian manifold we denote by $\alpha^f : TM \times TM \to T^\perp M$ its vector valued second fundamental form and by $T_{f(p)}^\perp M$ the normal space of $f$ at $p \in M^n$. Given an $s$-dimensional subspace $U^s \subseteq T_{f(p)}^\perp M$, $1 \leq s \leq d$, consider the symmetric bilinear form

$$\alpha^f_{U^s} : T_p M \times T_p M \to U^s$$

defined as $\alpha^f_{U^s} = P \circ \alpha^f$, where $P$ denotes the orthogonal projection of $T_{f(p)}^\perp M$ onto $U^s$. The conformal $s$-nullity $\nu^c_s(p)$ of $f$ at $p$, $1 \leq s \leq d$, is the integer

$$\nu^c_s(p) = \max_{U^s \subseteq U^s} \dim N(\alpha^f_{U^s} - \langle \cdot, \cdot \rangle)$$

The Lorentz space $\mathbb{L}^k$, $k \geq 2$, is the Euclidean space $\mathbb{R}^k$ endowed with the metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle X, X \rangle = -x_1^2 + x_2^2 + \cdots + x_k^2$$

for $X = (x_1, x_2, \ldots, x_k)$. The light cone $\mathcal{V}^{k-1}$ is the degenerate totally umbilical hypersurface of $\mathbb{L}^k$ given by

$$\mathcal{V}^{k-1} = \{ X \in \mathbb{L}^k : \langle X, X \rangle = 0, X \neq 0 \}.$$

For $k \geq 3$ and $\zeta \in \mathcal{V}^{k-1}$ consider the hyperplane

$$H_\zeta = \{ X \in \mathbb{L}^k : \langle X, \zeta \rangle = 1 \}$$

and the $(k - 2)$-dimensional simply connected embedded submanifold $H_\zeta \cap \mathcal{V}^{k-1} \subset \mathbb{L}^k$. Note that $H_\zeta$ intersects only one of the two connected components of $\mathcal{V}^{k-1}$. More precisely, $H_\zeta$ is in the Euclidean sense parallel to $\zeta$ and it does not pass through the origin. Given $p \in H_\zeta \cap \mathcal{V}^{k-1}$ the normal space of this intersection in $\mathbb{L}^k$ is the Lorentzian plane $\mathbb{L}^2$ generated by $p$ and $\zeta$. Consequently, the metric induced by $\mathbb{L}^k$ on $H_\zeta \cap \mathcal{V}^{k-1}$ is Riemannian. Its second fundamental form is given by

$$\alpha = -\langle \cdot, \cdot \rangle_\zeta.$$

The Gauss equation for the inclusion $H_\zeta \cap \mathcal{V}^{k-1} \subset \mathbb{L}^k$ shows that $H_\zeta \cap \mathcal{V}^{k-1}$ is flat and, consequently, is the image of an isometric embedding $J_\zeta : \mathbb{R}^{k-2} \to H_\zeta \cap \mathcal{V}^{k-1}$.

Now suppose that $M^n$ is a Riemannian manifold and $h : M^n \to \mathbb{R}^{k-2}$ is a conformal immersion with conformal factor $\phi_h$, that is, $\langle h_* X, h_* Y \rangle = \phi_h^2 \langle X, Y \rangle$ over $M^n$, where $X, Y$ are vectors tangent to $M^n$ and
\( \phi_h \) is a positive differentiable real function on \( M^n \). We associate to \( h \) the isometric immersion \( H: M^n \to V^{k-1} \subset \mathbb{L}^k \) by setting
\[
H = \frac{1}{\phi_h} J_\zeta \circ h
\]
for a chosen \( \zeta \in V^{k-1} \).

Consider \( M^n \) with the metric induced by \( f \), a fixed \( \zeta \in V^{n+d+1} \) and an isometric embedding \( J_\zeta: \mathbb{R}^{n+d} \to H_\zeta \cap V^{n+d+1} \). The isometric immersion \( F: M^n \to V^{n+d+1} \subset \mathbb{L}^{n+d+2} \) associated to \( f \) is given by
\[
F = J_\zeta \circ f.
\]
Its second fundamental form in \( \mathbb{L}^{n+d+2} \) is the symmetric bilinear form
\[
\alpha^F = -\langle \cdot, \zeta \rangle + \alpha^f.
\]
Here, we are identifying the second fundamental form \( \alpha^f \) of \( f \) in \( \mathbb{R}^{n+d} \) with the symmetric bilinear form \( (J_\zeta)_* \alpha^f \).

Now let \( g: M^n \to \mathbb{R}^{n+d+1} \) be an immersion conformal to \( f \). Consider the isometric immersion \( G: M^n \to V^{n+d+2} \subset \mathbb{L}^{n+d+3} \) given by
\[
G = \frac{1}{\phi_g} J_\zeta \circ g
\]
for an fixed \( T \in V^{n+d+2} \), where \( J_\zeta: \mathbb{R}^{n+d+1} \to H_\zeta \cap V^{n+d+2} \) is an isometric embedding. Taking the derivative of \( \langle G, G \rangle = 0 \), we see that the null vector field \( G \) is normal to the immersion \( G \). The normal field \( G \) also satisfies \( A^G_\xi = -I \). The normal bundle of \( G \) is given by the orthogonal direct sum
\[
T_G^\perp M = T_g^\perp M \oplus \mathbb{L}^2
\]
where \( T_g^\perp M \) is identified with \( (J_\zeta)_* T_g^\perp M \) and \( \mathbb{L}^2 \) is a Lorentzian plane bundle that contains \( G \). There exists an unique orthogonal basis \( \{ \xi, \eta \} \) of \( \mathbb{L}^2 \), satisfying \( |\xi|^2 = -1 \), \( |\eta|^2 = 1 \) and
\[
G = \xi + \eta.
\]
Writing \( \alpha^G \) in terms of this orthogonal frame we obtain
\[
\alpha^G = -\langle \alpha^G, \xi \rangle \xi + \langle \alpha^G, \eta \rangle \eta + (\alpha^G)^*
\]
where \( (\alpha^G)^* = (1/\phi_g)(J_\zeta)_* \alpha^g \) is the \( T_g^\perp M \)-component of \( \alpha^G \).

Given an \( m \)-dimensional real vector space \( W \) endowed with a nondegenerate inner product \( \langle \cdot, \cdot \rangle \) of index \( r \), that is, the maximal dimension of a subspace of \( W \) where \( \langle \cdot, \cdot \rangle \) is negative definite, we say that \( W \) is of type \( (r,q) \) and we write \( W^{(r,q)} \) with \( q = m - r \).

At \( p \in M^n \), let
\[
W_p = T_{f(p)}^\perp M \oplus \text{span}\{ \xi(p) \} \oplus \text{span}\{ \eta(p) \} \oplus T_{g(p)}^\perp M
\]
be endowed with the natural metric of type \( (d+1,d+2) \) which is negative definite on \( T_{f(p)}^\perp M \oplus \text{span}\{ \xi(p) \} \).

We also define the symmetric tensor \( \beta: TM \times TM \to W \) setting \( \beta = \alpha^f + \alpha^G \), that is,
\[
\beta = \alpha^f - \langle \alpha^G, \xi \rangle \xi + \langle \alpha^G, \eta \rangle \eta + (\alpha^G)^*.
\]
The Gauss equations for \( f \) and \( G \) imply that \( \beta \) is flat, that is,
\[
\langle \beta(X,Y), \beta(Z,U) \rangle = \langle \beta(X,U), \beta(Y,Z) \rangle, \quad \forall X,Y,Z,U \in TM.
\]
Observe also that \( \beta(X,X) \neq 0 \) for all \( X \neq 0 \), because \( A^G_{\xi+\eta} = -I \).
With the aim of constructing locally the conformal immersion \( \Gamma \) we construct locally an isometric immersion \( \mathcal{T} \) of a neighborhood of \( L^{n+d+2} \) into a neighborhood of \( L^{n+d+3} \) such that \( G = \mathcal{T} \circ f \). This \( \mathcal{T} \) induces a conformal immersion \( \Gamma \) from an open subset of \( \mathbb{R}^{n+d} \) into an open subset of \( \mathbb{R}^{n+d+1} \) defined by

\[
J_{\mathcal{T}} \circ \Gamma = \frac{\mathcal{T} \circ J_{\mathcal{C}}}{\langle \mathcal{T} \circ J_{\mathcal{C}}, \zeta \rangle}
\]

which satisfies \( g = \Gamma \circ f \). Now we will construct locally the isometric immersion \( \mathcal{T} \).

**Lemma 2.1.** Given \( p \in M^n \), there exist an orthonormal basis \( \zeta_1, \ldots, \zeta_d \) of \( T^*_{f(p)}M \) and a basis \( G, \mu_o, \ldots, \mu_d, \mu_{d+1} \) of \( T^*_{f(p)}M \), with \( \langle G, \mu_o \rangle = 1, \langle \mu_o, \mu_o \rangle = 0, \langle \mu_o, \mu_j \rangle = 0 = \langle G, \mu_j \rangle, \langle \mu_i, \mu_j \rangle = \delta_{ij} \), \( 1 \leq i, j \leq d+1 \), such that

\[
\alpha^G = -\langle \cdot, \cdot \rangle_{\mu_o} + \sum_{j=1}^d \langle \alpha^f, \zeta_j \rangle \mu_j + \langle A_{\mu_{d+1}}^G (\cdot), \cdot \rangle \mu_{d+1},
\]

with \( \text{rank} A^G_{\mu_{d+1}} \leq 1 \). Here, \( \langle \alpha^f, \zeta_j \rangle \) is the inner product induced in \( T^*_{f(p)}M \) by \( \mathbb{R}^{n+d} \).

With respect to flat symmetric bilinear forms we need the following from [3]:

For \( p \in M^n \), set \( V := T_p M \) and for each \( X \in V \) define the linear map

\[
\beta(X) : V \to W
\]

by setting \( \beta(X)(v) = \beta(X, v) \) for all \( v \in V \). The kernel and image of \( \beta(X) \) are denoted by \( \ker \beta(X) \) and \( \beta(X,V) \), respectively. We say that \( X \) is a regular element of \( \beta \) if

\[
\dim \beta(X,V) = \max_{Z \subseteq V} \dim \beta(Z,V).
\]

The set of regular elements of \( \beta \) is denoted by \( \text{RE}^* (\beta) \). For each \( X \in V \), set \( U(X) = \beta(X,V) \cap \beta(X,V)^\perp \) and define

\[
\text{RE}^* (\beta) = \{ Y \in \text{RE} (\beta) : \dim U(Y) = d_0 \}
\]

where \( d_0 = \min \{ \dim U(Y) : Y \in \text{RE} (\beta) \} \).

**Lemma 2.2.** The set \( \text{RE}^* (\beta) \) is open and dense in \( V \) and

\[
\beta(\ker \beta(X), V) \subseteq U(X), \quad \forall X \in \text{RE} (\beta).
\]

Recall that a vector subspace \( L \) of \( W \) is said to be degenerate if \( L \cap L^\perp \neq \{0\} \) and isotropic when \( \langle L, L \rangle = 0 \). We also have that

\[
\dim L + \dim L^\perp = \dim W \quad \text{and} \quad L^\perp \perp = L.
\]

Notice that \( L \) is an isotropic subspace of \( W \) if and only if \( L \subseteq L^\perp \). In this case, it follows easily from (4) that \( L \) has dimension at most \( d+1 \). Recall that \( W_p \) is a \( (2d+3) \)-dimensional vector space. Since \( U(X) \) is isotropic by definition, we have that \( d_0 \leq d+1 \).

Before proving Lemma 2.1 we need several lemmas:

**Lemma 2.3.** In an arbitrary \( p \in M^n \) it holds that \( \dim S(\beta) \cap S(\beta)^\perp = d+1 \).

**Proof.** First we prove the following assertion

**Assertion.** There exist an orthogonal decomposition

\[
W_p = W_1^{(d,\ell)} \oplus W_2^{(d-\ell+1,d-\ell+2)}, \quad \ell \geq 1,
\]

and symmetric bilinear forms \( \omega_j : V \times V \to W_j, j = 1, 2 \), satisfying

\[
\beta = \omega_1 \oplus \omega_2
\]

such that:
Therefore, a consequence of the following result whose proof is part of the arguments for the Main Lemma 2.2 in ([2], p. 238),

\[ \langle \omega_1(X, Y), \omega_1(Z, U) \rangle = 0, \forall X, Y, Z, U \in V = T_p M, \]

and

\( \omega_2 \) is flat with \( \dim N(\omega_2) \geq n - \dim W_2 \).

Notice that \( \beta(Z, Z) \neq 0 \) for all \( Z \neq 0 \) in \( TM \). Given \( X \in RE(\beta) \), there exists \( Z \neq 0 \) such that \( Z \in \ker \beta(X) \) due to \( n > 2d + 3 \). Since \( \beta \) is flat, \( \beta(Z, \cdot) \subset U(X) \) by Lemma 2.2 and \( U(X) \) is isotropic, it holds that \( \beta(Z, Z) \in S(\beta) \cap S(\beta)^\perp \). Let \( v_i = \gamma_i + b_i \xi_i + c_i \eta_i + \delta_i, 1 \leq i \leq \ell \), with \( \gamma_i \in T_p^1 M \) and \( \delta_i \in T_p^2 M \), be a basis of the isotropic subspace \( S(\beta) \cap S(\beta)^\perp \). The vectors \( \gamma_i + b_i \xi_i, 1 \leq i \leq \ell \), are linearly independent. Otherwise, there are real numbers \( \rho_i \), with some of them different of zero, such that \( \sum_i \rho_i (\gamma_i + b_i \xi_i) = 0 \). Since the vectors \( v_i, 1 \leq i \leq \ell \), are linearly independent, we have \( \sum_i \rho_i (c_i \eta_i + \delta_i) = \sum_i \rho_i v_i \neq 0 \). The metric in \( \text{span} \{\eta\} \oplus T_{g_1} M \) being positive definite, we have obtained a contradiction because the vector \( \sum_i \rho_i (c_i \eta_i + \delta_i) \) is not isotropic. Analogously, we obtain that the vectors \( c_i \eta_i + \delta_i, 1 \leq i \leq \ell \), are linearly independent.

Without loss of generality, we can suppose that the vectors \( \gamma_i = c_i \eta_i + \delta_i, 1 \leq i \leq \ell \), are orthonormal. In this case, due to \( \langle v_i, v_j \rangle = 0 \), \( 1 \leq i, j \leq \ell \), we have \( \langle \xi_i, \xi_j \rangle = -\delta_{ij} \) for \( \xi_i = \gamma_i + b_i \xi_i, 1 \leq i \leq \ell \). Extend \( \eta_1, \ldots, \eta_\ell \) to an orthonormal basis \( \eta_1, \ldots, \eta_{d+2} \) of \( \text{span} \{\eta\} \oplus T_{g_2} M \), and \( \xi_1, \ldots, \xi_\ell \) to a basis \( \xi_1, \ldots, \xi_{d+1} \) of \( \text{span} \{\xi\} \oplus T_{g_2}^2 M \) that satisfies \( \langle \xi_i, \xi_j \rangle = -\delta_{ij} \). Define \( \hat{v}_i = -\xi_i + \eta_i, 1 \leq i \leq \ell \),

\[ W_1 = \text{span}\{v_1, \ldots, v_\ell, \hat{v}_1, \ldots, \hat{v}_\ell\}, \quad W_2 = \text{span}\{\xi_{\ell+1}, \ldots, \xi_{d+1}, \eta_{\ell+1}, \ldots, \eta_{d+2}\} \]

and put

\[ \beta = \sum_{i=1}^\ell \phi_i v_i + \sum_{i=1}^{\ell} \psi_i \hat{v}_i + \sum_{i=\ell+1}^{d+1} \phi_i \xi_i + \sum_{i=\ell+1}^{d+2} \psi_i \eta_i. \]

Note that \( W = W_1 \oplus W_2 \) is an orthogonal decomposition of \( W \). For \( 1 \leq i \leq \ell \), we have \( \psi_i = \frac{1}{2} \langle \beta, v_i \rangle = 0 \).

Set

\[ \omega_1 = \sum_{i=1}^\ell \phi_i v_i \quad \text{and} \quad \omega_2 = \sum_{i=\ell+1}^{d+1} \phi_i \xi_i + \sum_{i=\ell+1}^{d+2} \psi_i \eta_i. \]

Since \( \ell = \dim S(\beta) \cap S(\beta)^\perp \geq 1 \), then \( \omega_1 \) is nonzero. It is easy to verify that \( \omega_1, \omega_2 \) are symmetric bilinear forms such that \( \omega_1 \) is null and \( \omega_2 \) is flat. In order to see that \( S(\omega_2) \) is nondegenerate, let \( \sum_i \omega_2(X_i, Y_i) \in W_2 \) be an arbitrary element in \( S(\omega_2) \cap S(\omega_2)^\perp \). For all \( v, w \in V \), we get

\[ \left( \sum_i \omega_2(X_i, Y_i), \beta(v, w) \right) = \left( \sum_i \omega_2(X_i, Y_i), \omega_2(v, w) \right) = 0. \]

Therefore, \( \sum_i \omega_2(X_i, Y_i) \in S(\beta) \cap S(\beta)^\perp \). Hence, \( \sum_i \omega_2(X_i, Y_i) \in W_1 \). Thus,

\[ \sum_i \omega_2(X_i, Y_i) \in W_1 \cap W_2 = \{0\}. \]

Since the subspace \( S(\omega_2) \) is nondegenerate and \( d - \ell + 1 \leq 5 \), the inequality \( \dim N(\omega_2) \geq n - \dim W_2 \) is a consequence of the following result whose proof is part of the arguments for the Main Lemma 2.2 in ([2], pp. 968-974).

**Lemma 2.4.** Let \( \sigma : V_1 \times V_1 \to W^{(r,t)} \) be a nonzero flat symmetric bilinear form. Assume \( r \leq 5 \) and \( \dim N(\sigma) < \dim V_1 - (r + t) \). Then \( S(\sigma) \) is degenerate.

Now to conclude Lemma 2.3, that is \( \ell = d + 1 \), we proceed exactly as in proof of Assertion 3 in ([2], p. 238).
Lemma 2.5. There exist an orthonormal basis $\zeta_1, \ldots, \zeta_d$ of $T^\perp_{f(p)} M$ and an orthonormal basis $\vartheta_1, \ldots, \vartheta_d$ of $T^\perp_{g(p)} M$ such that

$$\omega_1 = -\langle \alpha^G, \xi \rangle \cos \varphi + \vartheta_{d+1} \sin \theta \cos \varphi + \vartheta_1 \sin \varphi - \langle \alpha^f, \zeta_1 \rangle \cos \varphi - \vartheta_{d+1} \sin \theta \cos \varphi + \vartheta_1 \cos \varphi$$

$$- \sum_{j=2}^d \langle \alpha^f, \zeta_j \rangle (\vartheta_j + \vartheta_1)$$

(5)

$$\omega_2 = \langle \alpha^G, \eta \rangle \cos \theta + \vartheta_{d+1} \sin \theta \rangle (\eta \cos \theta + \vartheta_1 \cos \theta)) \cos \varphi + \vartheta_1 \cos \varphi).$$

Proof. Here our notations are as in proof of Assertion. We have $W_2 = \text{span}\{\eta_{d+2}\}$ due to $\ell = d + 1$. Put $\eta_{d+2} = \eta \cos \theta + \vartheta_{d+1} \sin \theta$. The vector $-\eta \sin \theta + \vartheta_{d+1} \cos \theta$ belongs to $\text{span}\{\eta_i \mid 1 \leq i \leq d + 1\}$ and its orthogonal complement in $\text{span}\{\eta_i \mid 1 \leq i \leq d + 1\}$ is the orthogonal complement of $\vartheta_{d+1}$ in $T^\perp_{g(p)} M$. The vectors $\xi_i = \gamma_i + b_i \xi$, $1 \leq i \leq d + 1$, being linearly independent, span the space $\text{span}\{\xi\} \oplus T^\perp_{f(p)} M$. Write $\xi = \sum_{i=1}^{d+1} \rho_i \xi_i$ and take the vectors in $S(\beta) \cap S(\beta)^\perp$ given by

$$\sum_{i=1}^{d+1} \rho_i v_i = \xi + \cos \varphi (-\eta \sin \theta + \vartheta_{d+1} \cos \theta) + \vartheta_1 \sin \varphi,$$

where $\vartheta_1 \in T^\perp_\varphi M$ is a unitary vector orthogonal to $\vartheta_{d+1}$. The vector $-\sin \varphi (-\eta \sin \theta + \vartheta_{d+1} \cos \theta) + \vartheta_1 \cos \varphi$ also belongs to $\text{span}\{\eta_i \mid 1 \leq i \leq d + 1\}$ since it is orthogonal to $\eta_{d+2}$. Take $\zeta_1$ as being the unitary vector in $T^\perp_\varphi M$ such that

$$\zeta_1 - \sin \varphi (-\eta \sin \theta + \vartheta_{d+1} \cos \theta) + \vartheta_1 \cos \varphi \in S(\beta) \cap S(\beta)^\perp.$$

Now extend $\zeta_1$ to an orthonormal basis $\zeta_1, \ldots, \zeta_d$ of $T^\perp_\varphi M$ and take the vectors $\vartheta_2, \ldots, \vartheta_d$ in $T^\perp_\varphi M$ such that $\zeta_i + \vartheta_1, 2 \leq i \leq d$, belong to $S(\beta) \cap S(\beta)^\perp$. Now it is not difficult to see that $\vartheta_1, \ldots, \vartheta_{d+1}$ is an orthonormal basis of $T^\perp_\varphi M$ and satisfies (5). \hfill $\square$

Note also that in (5) the form $\omega_1$ is a linear combination of vectors orthogonal to $\beta = \alpha^f \oplus \alpha^G$. These orthogonality give us

$$\langle \alpha^G, \zeta \rangle = \langle \alpha^G, \eta \rangle \sin \theta \cos \varphi - \langle \alpha^G, \vartheta_{d+1} \rangle \cos \theta \cos \varphi - \langle \alpha^G, \vartheta_1 \rangle \sin \varphi$$

$$\langle \alpha^f, \zeta_1 \rangle = -\langle \alpha^G, \eta \rangle \sin \theta \sin \varphi + \langle \alpha^G, \vartheta_{d+1} \rangle \cos \theta \sin \varphi - \langle \alpha^G, \vartheta_1 \rangle \cos \varphi$$

$$- \langle \alpha^f, \zeta_j \rangle = \langle \alpha^G, \vartheta_j \rangle, 2 \leq j \leq d.$$  

The above first two equations imply that

$$\langle \alpha^G, \vartheta_1 \rangle = -\langle \alpha^G, \xi \rangle \sin \varphi - \langle \alpha^f, \zeta_1 \rangle \cos \varphi.$$

of Lemma 2.1. First observe that in (5) we have $\sin \theta \cos \varphi \neq -1$. Otherwise, $G = \xi + \eta$ belongs to $S(\beta) \cap S(\beta)^\perp$ and, consequently,

$$0 = \langle G, \beta \rangle = \langle G, \alpha^G \rangle = -\langle \cdot, \rangle$$

which is a contradiction. Then, we can consider the orthonormal basis of $T^\perp_\varphi M$ given by

$$\mu_1 = \frac{1}{1 + \sin \theta \cos \varphi} \left[ \vartheta_1 (\cos \varphi + \sin \theta) - \vartheta_{d+1} \sin \varphi \cos \theta \right],$$

$$\mu_j = \vartheta_j, 2 \leq j \leq d,$$

$$\mu_{d+1} = \frac{1}{1 + \sin \theta \cos \varphi} \left[ \vartheta_1 \sin \varphi \cos \theta + \vartheta_{d+1} (\cos \varphi + \sin \theta) \right].$$

(8)
By (6), for $2 \leq j \leq d$, it holds that
\[\langle \alpha^G, \mu_j \rangle = \langle \alpha^G, \vartheta_j \rangle = -\langle \alpha^f, \zeta_j \rangle.\]

Thus, we can write
\[\alpha^G = -\langle \alpha^G, \xi \rangle (\xi + \eta) - \langle , \rangle \eta + \langle \alpha^G, \varpi_1 \rangle \varpi_1 - \sum_{j=2}^{d} \langle \alpha^f, \zeta_j \rangle \mu_j + \langle \alpha^G, \varpi_{d+1} \rangle \varpi_{d+1}.\]

The equalities in (5) give that
\[\langle \alpha^G, \varpi_1 \rangle = \langle G, \alpha^G \rangle = \langle \xi + \eta, \beta \rangle = \langle \xi + \eta, \omega_1 + \omega_2 \rangle,\]

and, jointly with $-\langle , \rangle = \langle G, \alpha^G \rangle = \langle \xi + \eta, \beta \rangle = \langle \xi + \eta, \omega_1 + \omega_2 \rangle$, that
\[\langle \alpha^G, \eta \cos \theta + \vartheta_{d+1} \sin \theta \rangle \cos \theta = -\langle , \rangle - \langle \alpha^G, \xi \rangle (1 + \sin \theta \cos \varphi) + \langle \alpha^f, \zeta_1 \rangle \sin \theta \varphi.\]

Therefore,
\[\langle \alpha^G, \varpi_1 \rangle = \frac{\sin \theta \sin \varphi}{1 + \sin \theta \cos \varphi} \langle , \rangle - \langle \alpha^f, \zeta_1 \rangle.\]

From (6) and $A_{\xi + \eta}^G = -I$, we deduce that
\[
\cos \varphi \langle \alpha^G, \xi \rangle - \sin \varphi \langle \alpha^f, \zeta_1 \rangle = -\sin \theta \langle , \rangle - \cos \theta \langle \alpha^G, \vartheta_{d+1} \rangle - \sin \theta \langle \alpha^G, \xi \rangle.
\]

So
\[
\langle \cos \varphi + \sin \theta \rangle \langle \alpha^G, \xi \rangle = \sin \varphi \langle \alpha^f, \zeta_1 \rangle - \sin \theta \langle , \rangle - \cos \theta \langle \alpha^G, \vartheta_{d+1} \rangle.
\]

Multiplying the above equation by $\cos \varphi + \sin \theta$ and introducing $\varpi_{d+1}$ according to (8), it is a straightforward calculation to see that
\[
\langle \cos \varphi + \sin \theta \rangle^2 \langle \alpha^G, \xi \rangle = \sin \varphi (\cos \varphi + \sin \theta) \langle \alpha^f, \zeta_1 \rangle - \sin \theta (\cos \varphi + \sin \theta) \langle , \rangle \\
- \langle \alpha^G, \varpi_{d+1} \rangle (1 + \cos \varphi \sin \theta) \cos \theta + \langle \alpha^G, \vartheta_1 \rangle \cos^2 \theta \sin \varphi.
\]

This and (7) imply
\[
- \langle \alpha^G, \xi \rangle = -\frac{\sin \varphi \sin \theta}{1 + \cos \varphi \sin \theta} \langle \alpha^f, \zeta_1 \rangle + \frac{\sin \theta (\cos \varphi + \sin \theta)}{(1 + \cos \varphi \sin \theta)^2} \langle , \rangle \\
+ \frac{\cos \theta}{1 + \cos \varphi \sin \theta} \langle \alpha^G, \varpi_{d+1} \rangle.
\]

The formulae (9), (11) e (12) give us
\[
\alpha^G = \langle , \rangle \left[ \frac{\sin \theta (\cos \varphi + \sin \theta)}{(1 + \cos \varphi \sin \theta)^2} (\xi + \eta) - \eta + \frac{\sin \varphi \sin \theta}{1 + \cos \varphi \sin \theta} \varpi_1 \right] \\
+ \langle \alpha^f, \zeta_1 \rangle \left[ -\frac{\sin \varphi \sin \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta) - \varpi_1 \right] - \sum_{j=2}^{d} \langle \alpha^f, \zeta_j \rangle \mu_j \\
+ \langle \alpha^G, \varpi_{d+1} \rangle \varpi_{d+1} + \frac{\cos \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta).\]
From which it is not difficult to see that
\[
\alpha^G = \frac{1}{1 + \cos \varphi \sin \theta} [\xi - \eta \cos \varphi \sin \theta + \mu_1 \sin \varphi \sin \theta + \mu_{d+1} \cos \theta] \\
- \langle \alpha^f, \zeta_1 \rangle \left[ \frac{\sin \varphi \sin \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta) + \mu_1 \right] - \sum_{j=2}^{d} \langle \alpha^f, \zeta_j \rangle \mu_j \\
+ \left( \alpha^G, \mu_{d+1} + \frac{\cos \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta) \right) \left[ \mu_{d+1} + \frac{\cos \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta) \right].
\]

Now (3) follows if we define
\[
\mu_o = -\frac{1}{1 + \cos \varphi \sin \theta} [\xi - \eta \cos \varphi \sin \theta + \mu_1 \sin \varphi \sin \theta + \mu_{d+1} \cos \theta], \\
\mu_1 = \frac{\sin \varphi \sin \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta) + \mu_1, \\
\mu_{d+1} = \mu_{d+1} + \frac{\cos \theta}{1 + \cos \varphi \sin \theta} (\xi + \eta).
\]

Notice that if \( v \in V \) and \( w \in N(\omega_2) \), then
\[
\langle A^G_{\mu_{d+1}} v, w \rangle = \langle \alpha^G(v, w), \mu_{d+1} \rangle = \langle \beta(v, w), \mu_{d+1} \rangle = \langle \omega_1(v, w), \mu_{d+1} \rangle = 0
\]
due to (3), (8) and (10). Thus, \( \text{rank } A^G_{\mu_{d+1}} \leq 1 \). Recall that \( \text{dim } N(\omega_2) \geq n - 1 \) by Assertion in proof of Lemma 2.3 and \( \ell = d + 1 \).

We point out that \( p \) in Lemma 2.1 being arbitrary the decomposition (3) holds on \( M^n \).

**Lemma 2.6.** At \( p \in M^n \), it holds that \( \text{dim } S(\alpha^G) \geq d + 1 \). Furthermore, \( \text{dim } S(\alpha^G) = d + 1 \) at \( p \) if and only if \( A^G_{\mu_{d+1}} = 0 \) at \( p \) and \( \text{dim } S(\alpha^G) = d + 2 \) at \( p \) if and only if \( \text{rank } A^G_{\mu_{d+1}} = 1 \) at \( p \).

**Proof.** Consider \( S(\alpha^G)^\perp \), the orthogonal complement of \( S(\alpha^G) \) in the nondegenerate \( (d + 3) \)-dimensional vector space
\[
T^\perp_{G(p)} M = \text{span } \{ G, \mu_o, \mu_1, \ldots, \mu_{d+1} \}.
\]
Note that the coordinates of an arbitrary vector \( \mu \in T^\perp_{G(p)} M \) on the basis \( G, \mu_o, \mu_1, \ldots, \mu_{d+1} \) are given by
\[
(13) \quad \mu = \langle \mu, \mu_o \rangle G + \langle \mu, \mu_1 \rangle \mu_o + \sum_{i=1}^{d} \langle \mu, \mu_i \rangle \mu_i + \langle \mu, \mu_{d+1} \rangle \mu_{d+1}.
\]

Now take \( \mu \in S(\alpha^G)^\perp \). Then, for all \( v, w \in V \), we have
\[
0 = \langle \alpha^G(w, v), \mu \rangle = \left( \left( A^G_{\sum_{i=1}^{d} \mu_i} \zeta_i - \langle \mu_o, \mu \rangle I + \langle \mu_{d+1}, \mu \rangle A^G_{\mu_{d+1}} \right) (w), v \right)
\]
due to (3). For \( w \in \text{Ker } A^G_{\mu_{d+1}} \), we obtain
\[
0 = \left( \left( A^G_{\sum_{i=1}^{d} \mu_i} \zeta_i - \langle \mu_o, \mu \rangle I \right) (w), v \right), \quad \forall v \in T_p M.
\]
So
\[
\nu^o \geq \text{dim Ker } \left( A^G_{\sum_{i=1}^{d} \mu_i} \zeta_i - \langle \mu_o, \mu \rangle I \right) \geq \text{dim Ker } A^G_{\mu_{d+1}} \geq n - 1
\]
which contradicts the hypothesis on the 1-conformal nullity of \( f \) case \( \sum_{i=1}^{d} \langle \mu_i, \mu \rangle \zeta_i \neq 0 \). Then \( \langle \mu_i, \mu \rangle = 0, 1 \leq i \leq d \), and consequently \( \langle \mu_o, \mu \rangle \neq 0 \). Thus, \( \mu \) belongs to \( \text{span } \{ \mu_o, \mu_{d+1} \} \) due to (13). From (14) we deduce that \( \mu_{d+1} \in S(\alpha^G)^\perp \) if and only if \( A^G_{\mu_{d+1}} = 0 \). Since \( \mu_o \in S(\alpha^G)^\perp \) and \( \text{rank } A^G_{\mu_{d+1}} \leq 1 \) we have proved Lemma 2.6. \( \Box \)
As a consequence of Lemma 2.6, if we define
\[
(15) \quad \omega = -\langle \cdot, \cdot \rangle_{\mu_0} + \sum_{j=1}^{d} \langle \alpha^f, \zeta_j \rangle_{\mu_j},
\]
we have \( \dim S(\omega) = d + 1 \), that is,
\[
S(\omega) = \text{span} \{ \mu_0, \mu_1, \mu_2, \ldots, \mu_d \}.
\]
In relation to \( S(\omega) \) we also claim that
\[
(16) \quad S(\omega) = \text{span}\{ \omega(X, Y) : X, Y \in \text{Ker} A^G_{\mu_{d+1}} \}.
\]
In fact, if we put \( R = \text{span}\{ \omega(X, Y) : X, Y \in \text{Ker} A^G_{\mu_{d+1}} \} \), it suffices to verify that \( \dim R = d + 1 \). For to see this we show that the orthogonal complement of \( R \) in the nondegenerate \((d + 2)\)-dimensional vector space \( \text{span}\{G, \mu_0, \mu_1, \mu_2, \ldots, \mu_d \} \) has dimension one. Let \( \mu = aG + b\mu_0 + \sum_{j=1}^{d} a_j\mu_j \) be an arbitrary vector orthogonal to \( R \). Then, for all \( X, Y \in \text{Ker} A^G_{\mu_{d+1}} \), we can write
\[
0 = \langle \omega(X, Y), \mu \rangle = -a\langle X, Y \rangle + \sum_{j=1}^{d} \langle \alpha^f(X, Y), \zeta_j \rangle a_j = \left( \left( A^f_{\mu_{d+1}} \sum_{j=1}^{d} a_j\zeta_j - aI \right) X, Y \right).
\]
Consequently, if \( \gamma = \sum_{j=1}^{d} a_j\zeta_j \neq 0 \) then the nullity of \( \alpha^f_{\text{span}}(\gamma) = \langle \cdot, \cdot \rangle_{\mu_{d+1}} \) is at least \( n - 2 \). This implies that \( \nu^f_{1} \geq n - 2 \) and we have obtained a contradiction with our hypothesis. Thus, \( a = 0 = a_j, 1 \leq j \leq d, \) and the claim have been proved.

**Proof of Theorem 1.2.** Let \( U \) be the subset of \( M^n \) constituted of the points \( q \) so that \( \dim S(\alpha^G) \) is constant in a neighborhood of \( q \). The subset \( U \) is open and dense in \( M^n \). In fact, clearly \( U \) is open in \( M^n \). For to see that \( U \) is dense in \( M^n \), consider an arbitrary point \( p \) in \( M^n \). By Lemma 2.6, in each point \( p \) of \( M^n \), the dimension of \( S(\alpha^G) \) is either \( d + 1 \) or \( d + 2 \). If \( \dim S(\alpha^G) = d + 2 \) at \( p \), then \( p \) belongs to \( U \) since \( \dim S(\alpha^G) \) does not decrease in a neighborhood of \( p \) by continuity. If \( \dim S(\alpha^G) = d + 1 \) at \( p \) and \( p \) does not belong to \( U \), there is a sequence of points where \( \dim S(\alpha^G) \) is \( d + 2 \), consequently a sequence of points in \( U \), that converges to \( p \). Thus, \( U \) is dense in \( M^n \). Now, for an arbitrary point \( p \in U \), we consider a connected neighborhood \( U \subset U \) of \( p \) where \( F \) is an embedding and \( \dim S(\alpha^G) \) is constant. We divide the proof of Theorem 1.2 in two cases.

**Case I.** \( \dim S(\alpha^G) = d + 2 \) on \( U \). In this case, \( \text{rank} A^G_{\mu_{d+1}} = 1 \) on \( U \) by Lemma 2.6 and
\[
S(\alpha^G) = \text{span} \{ \mu_0, \mu_1, \ldots, \mu_{d+1} \}
\]
due to \( \bullet \)

**Lemma 2.7.** In Case I we can choose \( \mu_{d+1} \) such that the unitary vector field \( \mu_{d+1} \in T^+_U \) be smooth on \( U \).

**Proof of Lemma 2.7.** In each point of \( U \) we choose \( \mu_{d+1} \) so that the unique nonzero eigenvalue of \( A^G_{\mu_{d+1}} \) be positive. We affirm that with this choice \( \mu_{d+1} \) is smooth. First we prove that for all continuous tangent vector fields \( X, Y \in TU \) the function \( \langle \alpha^G(X, Y), \mu_{d+1} \rangle \) is continuous. Since \( \alpha^G \) is bilinear and symmetric it suffices to prove that \( \langle \alpha^G(X, X), \mu_{d+1} \rangle \) is continuous for all continuous field \( X \) in \( TU \). All eigenvalues of \( A^G_{\mu_{d+1}} \) being nonnegative, it holds that
\[
(17) \quad \langle \alpha^G(X, X), \mu_{d+1} \rangle = \langle A^G_{\mu_{d+1}} X, X \rangle \geq 0.
\]
From \( \bullet \) it follows that
\[
\langle \alpha^G(Z, W), \mu_{d+1} \rangle^2 = |\alpha^G(Z, W)|^2 - |\alpha^f(Z, W)|^2, \forall Z, W \in TM.
\]
Recall that $\mu_o$ has zero length. In particular, $\langle \alpha^G(X, X), \mu_{d+1} \rangle^2$ is continuous. So $\langle \alpha^G(X, X), \mu_{d+1} \rangle$ is continuous due to \cite{17}. (Note that $\langle \alpha^G(Z, W), \mu_{d+1} \rangle^2$ is smooth when $Z$ and $W$ are smooth.) Now in a fixed $q_o \in U$ consider an orthonormal basis $e_1, \ldots, e_n$ of $T_{q_o}M$ of eigenvectors of $A^G_{\mu_{d+1}}$ so that $A^G_{\mu_{d+1}} e_1 = \rho e_1$, $\rho > 0$, and $A^G_{\mu_{d+1}} e_j = 0$, $2 \leq j \leq n$. Extend $e_1, \ldots, e_n$ locally to a smooth orthonormal frame $E_1, \ldots, E_n$ of tangent vectors and define the local smooth fields $Y_1 = E_1$ and $Y_j = E_1 + E_j$, $2 \leq j \leq n$. Observe that in each point $q$ where the vectors $Y_1(q), \ldots, Y_n(q)$ are defined they are linearly independent and so they are a basis of $T_qM$. Since $\dim S(\alpha^G) = d + 2$ on $U$ we can take vectors $\alpha^G(Y_{i_k}, Y_{j_k})$, $1 \leq k \leq d + 2$, that are a basis of $S(\alpha^G)$ in a neighborhood of $q_o$. Consider the locally defined continuous functions

$$\psi_k = \langle \alpha^G(Y_{i_k}, Y_{j_k}), \mu_{d+1} \rangle, 1 \leq k \leq d + 2.$$ 

As we have seen above the functions $\psi_k$ are continuous. We claim that each $\psi_k$ is smooth. In fact, at $q_o$ we have $\psi_k(q_o) = \rho > 0$. Then in a neighborhood of $q_o$ we can suppose that $\psi_k$ is positive since it is continuous.

We have observed previously that $\psi_k^2$ is smooth. Therefore, $\psi_k$ is smooth since it is positive. Let us denote $a_1, \ldots, a_{d+2}$ the coordinates functions of $\mu_{d+1}$ on the basis $\alpha^G(Y_{i_k}, Y_{j_k})$, $1 \leq k \leq d + 2$. Using \cite{3}, we can write

$$\mu_{d+1} = \sum_{k=1}^{d+2} a_k \alpha^G(Y_{i_k}, Y_{j_k}) = - \left( \sum_{k=1}^{d+2} a_k \langle Y_{i_k}, Y_{j_k} \rangle \right) \mu_o + \sum_{j=1}^d \left( \sum_{k=1}^{d+2} a_k \alpha^f(Y_{i_k}, Y_{j_k}), \xi_j \right) \mu_j + \left( \sum_{k=1}^{d+2} a_k \psi_k \right) \mu_{d+1}.$$ 

So the functions $a_k$ satisfy the equations

$$\sum_{k=1}^{d+2} a_k \langle Y_{i_k}, Y_{j_k} \rangle = 0, \quad \sum_{k=1}^{d+2} a_k \alpha^f(Y_{i_k}, Y_{j_k}) = 0, \quad \sum_{k=1}^{d+2} a_k \psi_k = 1.$$ 

Taking a smooth orthonormal frame $\xi_1, \ldots, \xi_d$ of $T_{q_o}^U M$ in a neighborhood of $q_o$, we have that the functions $a_k$ satisfy the following system of $d + 2$ linear equations

$$\sum_{k=1}^{d+2} a_k \langle Y_{i_k}, Y_{j_k} \rangle = 0, \quad \sum_{k=1}^{d+2} a_k \alpha^f(Y_{i_k}, Y_{j_k}), \xi_j = 0, 1 \leq j \leq d, \quad \sum_{k=1}^{d+2} a_k \psi_k = 1.$$ 

The order $d + 2$ matrix of the system is an invertible smooth matrix. That it is smooth follows from the smoothness of the functions $Y_i, \alpha^f$ and $\psi_k$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, d + 2$. If we consider a vector $(c_1, \ldots, c_{d+2}) \in \mathbb{R}^{d+2}$ in the kernel of the system, then the equations bellow are satisfied

$$\sum_{k=1}^{d+2} c_k \langle Y_{i_k}, Y_{j_k} \rangle = 0, \quad \sum_{k=1}^{d+2} c_k \alpha^f(Y_{i_k}, Y_{j_k}), \xi_j = 0, 1 \leq j \leq d, \quad \sum_{k=1}^{d+2} c_k \psi_k = 0.$$ 

By \cite{3}, we deduce that $\sum_{k=1}^{d+2} c_k \alpha^G(Y_{i_k}, Y_{j_k}) = 0$. Being the vectors $\alpha^G(Y_{i_k}, Y_{j_k})$, $1 \leq k \leq d + 2$, linearly independent, we have $c_k = 0$ for all $k$. So the matrix of the system is invertible. Then, the functions $a_k$ are smooth and, consequently, $\mu_{d+1}$ is an unitary smooth vector field on $T_{q_o}^U U$.

**Lemma 2.8.** The null vector field $\mu_o$ is smooth on $U$.

**Proof.** The proof is identical to one in Lemma 2.7, noticing that the coordinates $a_1, \ldots, a_{d+2}$ of $\mu_o$ on the basis $\alpha^G(Y_{i_k}, Y_{j_k})$, $1 \leq k \leq d + 2$, are determined by

$$\sum_{k=1}^{d+2} a_k \langle Y_{i_k}, Y_{j_k} \rangle = -1, \quad \sum_{k=1}^{d+2} a_k \alpha^f(Y_{i_k}, Y_{j_k}), \xi_j = 0, 1 \leq j \leq d, \quad \sum_{k=1}^{d+2} a_k \psi_k = 0.$$ 

\square
Lemma 2.9. On $U$ the null vector field $\mu_o$ is parallel along $\text{Ker} A^G_{\mu_{d+1}}$.

Proof. We denote by $\nabla$ the connection of the Lorentz space $E^n_{d+1}$ and by $\nabla^\perp$ the normal connection of the immersion $G$. First note that the coordinate of $\nabla^\perp \mu_o$ in the direction $\mu_o$ is zero for all $X \in TU$. In fact, we have $\nabla^\perp G = 0$ since $\nabla X G = G \cdot X$ is tangent. Now if we take derivatives on $\langle G, \nabla^\perp \mu_o \rangle = 1$ in the direction $X$, we obtain that $\langle G, \nabla^\perp \mu_o \rangle = 0$ and the coordinate of $\nabla^\perp \mu_o$ in the direction $\mu_o$ is zero by (13). Being $\mu_o$ a vector field of zero length, the vector $\nabla^\perp \mu_o$ has not component in the direction $G$ by (13). Now we claim that $\nabla^\perp \mu_o$ also has not component on $L = \text{span} \{ \mu_1, \mu_2, \ldots, \mu_d \}$. In fact, consider $q \in U$ and the linear map

$$T_q M \xrightarrow{\Psi} \text{span} \{ \mu_1, \ldots, \mu_d, \mu_{d+1} \}$$

Using the Codazzi’s equations for $\alpha^G$, the compatibility of the normal connection of the immersion $G$ with the metric in $T_q^\perp M$ and that $\langle \alpha^G, \mu_o \rangle = 0$, it is a straightforward calculation to see that

$$\langle \alpha^G (X,Y), \nabla^\perp \mu_o \rangle = \langle \alpha^G (Z,Y), \nabla^\perp \mu_o \rangle, \forall X, Y, Z \in T_q M.$$  

This equation is equivalent to

$$A^G_{\nabla^\perp \mu_o} X = A^G_{\nabla^\perp \mu_o} Z, \forall X, Z \in T_q M.$$  

Let $\Psi_L$ be the component of $\Psi$ on $L = \text{span} \{ \mu_1, \mu_2, \ldots, \mu_d \}$, that is,

$$\Psi_L (Z) = \sum_{i=1}^{d} \langle \nabla^\perp \mu_o, \mu_i \rangle \mu_i, \forall Z \in T_q M.$$  

Suppose that $\dim (\text{Im} \Psi_L) = r$ and $r \geq 1$. Consider a basis $\Psi_L (Z_1), \ldots, \Psi_L (Z_r)$ of $(\text{Im} \Psi_L)$. Observe that $d \geq r \geq \dim (\text{Im} \Psi) - 1$. Taking $\xi_j = \sum_{i=1}^{d} \langle \Psi_L (Z_j), \mu_i \rangle \xi_i, 1 \leq j \leq r$, and using (3) and (18), we obtain that

$$A^f_{\xi_j} X = A^G_{\nabla^\perp \mu_o} Z_j - \rho (X, E) \langle \nabla^\perp \mu_o, \mu_{d+1} \rangle E,$$

being $E$ an unitary eigenvector such that $A^G_{\mu_{d+1}} E = \rho \cdot E, X \in T_q M$ and $j, 1 \leq j \leq r$. For each $j, 1 \leq j \leq r$, consider the linear map $\psi_j : T_q M \to \text{span} \{ E \}$ given by

$$\psi_j (X) = -\rho (X, E) \langle \nabla^\perp \mu_o, \mu_{d+1} \rangle E = A^f_{\xi_j} X - A^G_{\nabla^\perp \mu_o} Z_j.$$  

Notice that if $X \in \text{Ker} \Psi \cap \cap_{j=1}^{r} \text{Ker} \psi_j$ then $X \in \cap_{j=1}^{r} \text{Ker} A^f_{\xi_j}$ due to (20). We have $\dim \left( \cap_{j=1}^{r} \text{Ker} \psi_j \right) \geq n - r$. Then, for the $r$-dimensional space $L = \text{span} \{ \xi_1, \ldots, \xi_r \}$, using the formula

$$\dim (L_1 + L_2) = \dim L_1 + \dim L_2 - \dim (L_1 \cap L_2),$$

valid for any two finite dimensional vector subspaces of any vector space, we obtain

$$\nu^c_r \geq \dim N (\alpha^G_L) \geq \dim \left( \text{Ker} A^f_{\xi_j} \cap \cap_{j=1}^{r} \text{Ker} \psi_j \right) \geq (n - \dim (\text{Im} \Psi)) + (n - r) - n \geq n - 2r - 1$$

which is in contradiction with our hypothesis on the $r$-conformal nullity. Then, $r = 0$ and $\nabla^\perp \mu_o = \langle \nabla^\perp \mu_o, \mu_{d+1} \rangle \mu_{d+1}$ for all $Z \in T_q M$. Now from (19) we deduce that

$$\langle \nabla^\perp \mu_o, \mu_{d+1} \rangle A^G_{\mu_{d+1}} X = \langle \nabla^\perp \mu_o, \mu_{d+1} \rangle A^G_{\mu_{d+1}} Z.$$
for all $X, Z \in T_q M$. Thus, if we choose $X = E$ and $Z \in \text{Ker} A_{\mu_{d+1}}^G$, we have

$$\rho \left( \nabla^+_E \mu_\alpha, \mu_{d+1} \right) E = 0$$

for all $Z \in \text{Ker} A_{\mu_{d+1}}^G$. Hence $\nabla^+_E \mu_\alpha = 0$ for all $Z \in \text{Ker} A_{\mu_{d+1}}^G$. Recall that our choice for $\mu_{d+1}$ is such that the unique nonzero eigenvalue $\rho$ of the smooth linear map $A_{\mu_{d+1}}^G$ is positive. \hfill $\square$

**Lemma 2.10.** On $U$ the unitary vector field $\mu_{d+1}$ is parallel along $\text{Ker} A_{\mu_{d+1}}^G$.

**Proof.** Notice that the coordinate of $\nabla_X \mu_{d+1}$ in the direction $\mu_\alpha$ is zero due to

$$\langle G, \nabla_X \mu_{d+1} \rangle = -\langle \nabla_X G, \mu_{d+1} \rangle = -\langle G_* X, \mu_{d+1} \rangle = 0.$$ 

By Lemma 2.9, for $X \in \text{Ker} A_{\mu_{d+1}}^G$, $\nabla_X \mu_{d+1}$ also has not component in the direction $G$ since $\langle \mu_\alpha, \nabla_X \mu_{d+1} \rangle = -\langle \nabla_X \mu_\alpha, \mu_{d+1} \rangle = 0$. Being $\mu_{d+1}$ an unitary vector field, the vector $\nabla_X \mu_{d+1}$ has not component in the direction $\mu_{d+1}$. Thus, we only need proving that $\nabla_X \mu_{d+1}$ also has not component on $L = \text{span} \{ \mu_1, \mu_2, \ldots, \mu_d \}$ for all $X \in \text{Ker} A_{\mu_{d+1}}^G$. Consider the linear map

$$\text{Ker} A_{\mu_{d+1}}^G \xrightarrow{\Phi} \text{span} \{ \mu_1, \ldots, \mu_d \} \xrightarrow{\nabla_X \mu_{d+1}} \nabla_X \mu_{d+1}.$$ 

Our choice for $\mu_{d+1}$ so that the unique nonzero eigenvalue $\rho$ of the smooth linear map $A_{\mu_{d+1}}^G$ is positive, for a well known argument, implies that $\rho$ is smooth. Simple calculations shows that in an arbitrary point of $U$ the $(n-1)$-dimensional distribution $\text{Ker} A_{\mu_{d+1}}^G$ is given by

$$\text{Ker} A_{\mu_{d+1}}^G = \left\{ \left( A_{\mu_{d+1}}^G - \rho I \right) W \mid W \in TU \right\}.$$ 

So $\text{Ker} A_{\mu_{d+1}}^G$ is a differentiable distribution on $U$. Therefore, if in some point $q_0$ of $U$ we consider $X \in \text{Ker} A_{\mu_{d+1}}^G$, we can take in a neighborhood of $q_0$ in $U$ a differentiable extension of $X$ that lies on $\text{Ker} A_{\mu_{d+1}}^G$. With these observations, using the Codazzi’s equations for $\alpha^G$ and the compatibility of the normal connection of the immersion $G$ with the metric in $T_q M$, it is a straightforward calculation to see that

$$A_{\Phi(X)} Z = A_{\Phi(X)} Z + \rho \langle [X, Z], E \rangle E,$$

for all $X, Z \in \text{Ker} A_{\mu_{d+1}}^G$. Suppose that $\dim (\text{Im} \Phi) = r$ and $r \geq 1$. At this point we proceed like in proof of Lemma 2.9, with $\text{Ker} A_{\mu_{d+1}}^G$ instead of $T_q M$, for deduce that $\dim (\text{Im} \Phi) = 0$ and Lemma 2.10 has been proved. \hfill $\square$

Now we consider the linear isometry $\tau: T^*_F U \to (\text{span} \{ \mu_{d+1} \}^\perp \subset T^*_F U$ given by

$$\tau(F) = G, \quad \tau(\alpha) = \mu_\alpha \quad \text{and} \quad \tau(\zeta_i) = \mu_i, \quad 1 \leq i \leq d.$$ 

Notice that $\tau$ is smooth since

$$T^*_F M = \text{span} \{ F \} \oplus S (\alpha^F), \quad \text{span} \{ \mu_{d+1} \}^\perp \oplus \text{span} \{ G \} \oplus S(\omega)$$

and $\tau (\alpha^F(X, Y)) = \omega(X, Y), \forall X, Y \in T M$, due to (11) and (15). Observe also that our hypothesis on the 1-conformal nullity of $f$ implies that $S(\alpha^F)$ is a $(d+1)$-dimensional subspace of $T^*_F M$.

Since the $(n-1)$-dimensional distribution $\text{Ker} A_{\mu_{d+1}}^G$ is smooth on $U$, if $E$ is a unitary differentiable vector field orthogonal to $\text{Ker} A_{\mu_{d+1}}^G$ then $E$ is in each point an eigenvector of $A_{\mu_{d+1}}^G$ corresponding to the eigenvalue $\rho$. Consequently, $E$ can be chosen smooth on $U$. Consider the vector bundle isometry $T: T^*_F U \oplus \text{span} \{ E \} \to (\text{span} \{ \mu_{d+1} \}^\perp \oplus \text{span} \{ E \}$ defined as $T(\delta + c E) = \tau(\delta) + c E$ for all $\delta + c E \in T^*_F U \oplus \text{span} \{ E \}$. Take the subbundle $\Lambda \subset (\text{span} \{ \mu_{d+1} \})^\perp \oplus \text{span} \{ E \}$ whose bundles are the orthogonal complement of the subspace generated by $\nabla_{E} \mu_{d+1} \perp - \rho E + \nabla_{E} \mu_{d+1}$. We observe that, being $(\text{span} \{ \mu_{d+1} \})^\perp \oplus \text{span} \{ E \}$ a $(d + 3)$-dimensional nondegenerate vector bundle, $\Lambda$ is a $(d + 2)$-dimensional nondegenerate subbundle and that $\Lambda$
is transversal to $TU$. Now define the $(d+2)$-dimensional nondegenerate subbundle $\Omega$ of $T^\perp U \oplus \text{span} \{E\}$, transversal to $TU$, by setting $T(\Omega) = \Lambda$. Let $F: \Omega \to \mathbb{L}^{n+d+3}$ be defined by $F(q, \vartheta) = F(q) + \vartheta$ for all $q \in U$ and all $\vartheta$ in the fibre $\Omega_q$. Since the fibres of $\Omega$ are transversal to $TU$ and $F$ is an embedding on $U$, $F$ is a diffeomorphism from a neighborhood $\Omega$ of the zero section in $\Omega$ into a neighborhood $W$ of $F(U)$ in $\mathbb{L}^{n+d+3}$.

Consider $\Omega$ endowed with the flat metric induced by $F$. Restrict to $\Omega_0$, $F$ become an isometry onto $W$. From now on $F$ will stand for this restriction. Now let $G: \Omega_0 \to \mathbb{L}^{n+d+3}$ be defined by $G(q, \vartheta) = G(q) + T(\vartheta)$ for all $q \in U$ and $\vartheta \in \Omega_q$. We claim that $G$ is an isometric immersion. In fact, for all differentiable curve $\theta(t) = (\gamma(t), \delta(\gamma(t)) + c(t)E(\gamma(t))) \in \Omega_0$ we have $G(\theta(t)) = G(\gamma(t)) + \tau(\delta(\gamma(t))) + c(t)E(\gamma(t))$. Consequently,

\begin{equation}
(G_\ast)_{\theta(0)} \theta'(0) = (G_\ast)_{\gamma(0)} \gamma'(0) + \nabla_{\gamma'(0)} \tau(\delta) + \tilde{\nabla}_{\gamma'(0)} cE
\end{equation}

Now if we put $\gamma'(0) = aE + v$, being $v$ the component of $\gamma'(0)$ in $\text{Ker} A^G_{\mu_{d+1}}$, we have

\begin{equation}
\left\langle \tilde{\nabla}_{\gamma'(0)} \tau(\delta) + a^G(\gamma'(0), cE, \mu_{d+1}) \right\rangle = \left\langle \nabla_{\gamma'(0)} \tau(\delta) + cE, \mu_{d+1} \right\rangle
\end{equation}

\begin{equation}
= - \left\langle \tau(\delta) + cE, \tilde{\nabla}_{\gamma'(0)} \mu_{d+1} \right\rangle = - \left\langle \tau(\delta) + cE, a\tilde{\nabla}_E \mu_{d+1} - A^G_{\mu_{d+1}} v + \tilde{\nabla}_v \mu_{d+1} \right\rangle = 0.
\end{equation}

Recall that $\tau(\delta) + cE \in \Lambda$ is orthogonal to $\tilde{\nabla}_E \mu_{d+1}$ and that, by Lemma 2.10, $\tilde{\nabla}_v \mu_{d+1} = 0$. For to finish the proof of the claim we will need of the following

**Lemma 2.11.** For $X \in TU$ and $\mu \in L = (\text{span} \{\mu_{d+1}\})^\perp$, let $\nabla^X_\mu$ be given by $\nabla^X_\mu = \tau(F^\ast \nabla^X_\tau^{-1}(\mu))$, being $F^\ast \nabla^X_\tau$ the normal connection of the immersion $F$, and let $\tilde{\nabla}^X_\mu$ be the component of $\tilde{\nabla}^X_\mu$ on $L$. Then, it holds that $\nabla^X_\mu = \tilde{\nabla}^X_\mu$.

**Proof.** For a fixed $X \in TU$, we define the linear map $K(X) : L \to L$ by $K(X)\mu = \nabla^X_\mu - \tilde{\nabla}^X_\mu$, $\forall \mu \in L$. Since $L$ is nondegenerate, for to prove that $K(X) \equiv 0$ it suffices to prove, for all $W, V, Y, Z \in \text{Ker} A^G_{\mu_{d+1}}$, the following relations

$\langle K(X)G, G \rangle = \langle K(X)G, \omega(Y, Z) \rangle = \langle \omega, K(X)\omega(Y, Z) \rangle = \langle K(X)\omega(Y, Z), \omega(W, V) \rangle = 0$.

due to \eqref{E16} and \eqref{E21}. First we note that $K(X)G = 0$ because $F^\ast \nabla^X_\tau F = \tilde{\nabla}^X_\tau G = 0$. Also, for all $Y, Z \in \text{Ker} A^G_{\mu_{d+1}}$, we have

$\langle G, K(X)\omega(Y, Z) \rangle = \langle G, \nabla^X_\omega \omega(Y, Z) \rangle - \langle G, \tilde{\nabla}^X_\omega \omega(Y, Z) \rangle$

$= \langle \tau(F), (F^\ast \nabla^X_\tau \alpha^F(Y, Z)) \rangle - \langle \tau(F), \tilde{\nabla}^X_\tau \omega(Y, Z) \rangle = \langle F, F^\ast \nabla^X_\tau \alpha^F(Y, Z) \rangle - \langle G, \tilde{\nabla}^X_\tau \alpha^G(Y, Z) \rangle$

$= X \langle F, \alpha^F(Y, Z) \rangle - X \langle G, \alpha^G(Y, Z) \rangle = -X \langle Y, Z \rangle + X \langle Y, Z \rangle = 0$.

Now we verify that holds the relation

$K(X)\omega(Y, Z) = K(Y)\omega(X, Z)$, for all $X \in TU$ and all $Y, Z \in \text{Ker} A^G_{\mu_{d+1}}$. 
In fact, for all \( \mu \in L \), we have
\[
\langle K(X)\omega(Y,Z),\mu \rangle = \langle \tau (F\nabla_{\mathring{X}}^G \alpha^F(Y,Z) - \nabla_{\mathring{X}}^G \alpha^G(Y,Z),\mu \rangle
\]
\[
= \langle \tau (F\nabla_{\mathring{X}}^G \alpha^F(Y,Z)) + \tau (\alpha^F(\nabla_X Y, Z) + \tau (\alpha^F(Y, \nabla_X Z)) - (\mathring{\nabla}_{\mathring{X}}^G)(Y,Z) - \alpha^G(\nabla_X Y, Z) - \alpha^G(Y, \nabla_X Z),\mu \rangle
\]
\[
= \langle \tau (F\nabla_{\mathring{Y}}^G \alpha^F(X,Z)) - \tau (\alpha^F(\nabla_Y X, Z)) - \tau (\alpha^F(X, \nabla_Y Z)) - \nabla_{\mathring{Y}}^G \alpha^G(X,Z) + \alpha^G(\nabla_Y X, Z) + \alpha^G(X, \nabla_Y Z),\mu \rangle
\]
\[
= \langle K(Y)\omega(X,Z),\mu \rangle.
\]
Above we have used that \( \omega = \tau (\alpha^F) \), \( \alpha^G(X,Z) = \omega(X,Z) \), for any \( X \in TM, Z \in \text{Ker} A^G_{\mu+1} \), and that \( \alpha^F \) and \( \alpha^G \) satisfy the Codazzi’s equations. It is a straightforward calculation to see that \( K(X) \) satisfies
\[
\langle K(X)\omega(Y,Z),\omega(W,V) \rangle = -\langle \omega(Y,Z), K(X)\omega(W,V) \rangle \text{ for all } W, V, X, Y, Z \in TM.
\]
Now if \( W, V, X, Y, Z \in \text{Ker} A^G_{\mu+1} \) and \( X \) is an arbitrary tangent vector, then
\[
\langle K(X)\omega(Y,Z),\omega(W,V) \rangle = \langle K(Y)\omega(X,Z),\omega(W,V) \rangle
\]
\[
= -\langle \omega(X,Z), K(Y)\omega(W,V) \rangle = -\langle \omega(X,Z), K(W)\omega(Y,V) \rangle
\]
\[
= \langle K(W)\omega(X,Z),\omega(Y,V) \rangle = \langle K(X)\omega(W,Z),\omega(Y,V) \rangle
\]
\[
= \langle K(Z)\omega(X,W),\omega(Y,V) \rangle = -\langle \omega(X,W), K(Z)\omega(Y,V) \rangle
\]
\[
= -\langle \omega(X,W), K(V)\omega(Y,Z) \rangle = \langle K(V)\omega(X,W),\omega(Y,Z) \rangle
\]
\[
= \langle K(X)\omega(W,V),\omega(Y,Z) \rangle = -\langle \omega(W,V), K(X)\omega(Y,Z) \rangle.
\]
Therefore, Lemma 2.11 has been proved.

Now, due to (23), the equality (22) becomes
\[
(G_*)_{\gamma(0)} \theta'(0) = (G_*)_{\gamma(0)} \left[ \gamma'(0) - A^G_{\tau(\delta)} \gamma'(0) + \nabla_{\gamma'(0)} cE \right] + \mathring{\nabla}^G_{\gamma'(0)} \tau(\delta) + \alpha^G(\gamma'(0),cE)
\]
\[
= (G_*)_{\gamma(0)} \left[ \gamma'(0) - A^G_{\tau(\delta)} \gamma'(0) + \nabla_{\gamma'(0)} cE \right] + \mathring{\nabla}^G_{\gamma'(0)} \tau(\delta) + \omega(\gamma'(0),cE)
\]
\[
= \left[ (G_*)_{\gamma(0)} \circ (F_*)_{\gamma(0)} \right]^{-1} \circ (F_*)_{\gamma(0)} \left[ \gamma'(0) - A^F \gamma'(0) + \nabla_{\gamma'(0)} cE \right]
\]
\[
+ \tau \left[ F \nabla^F_{\gamma'(0)} \delta + \alpha^F(\gamma'(0),cE) \right].
\]
Observe that \( A^G_{\tau(\delta)} = A^F_{\delta} \). Then, \( (G_*)_{\gamma(0)} \theta'(0) \) for all differentiable curve \( \theta(t) \) in \( \Omega \). This finish the proof of the claim.

Finally, taking \( T = G \circ F^{-1} \), we obtain an isometric immersion from a neighborhood of \( F(U) \) in \( \mathbb{L}^{n+d+2} \) into a neighborhood of \( G(U) \) in \( \mathbb{L}^{n+d+3} \) such that
\[
T(F(q)) = T(F(q,0)) = G(q,0) = G(q), \forall q \in U.
\]
According to previous observations \( T \) induces a conformal immersion \( \Gamma \) from a neighborhood of \( f(U) \) in \( \mathbb{R}^{n+d} \) into a neighborhood of \( g(U) \) in \( \mathbb{R}^{n+d+1} \) such that \( g = \Gamma \circ f \). This prove Theorem 1.2 in Case I.
Case II. \( \dim S (\alpha^G) = d + 1 \) on \( U \). In this case, by Lemma 2.6, rank \( A^G_{\mu + 1} = 0 \) on \( U \) and consequently,

\[
S (\alpha^G) = \text{span} \{ \mu_0, \mu_1, \ldots, \mu_d \}.
\]

by (3).

The linear isometry \( \tau: T^*_F U \to (\text{span} \{ \mu_{d+1} \})^\perp \) given by

\[
\tau(F) = G, \quad \tau(\zeta) = \mu_o \quad \text{and} \quad \tau(\xi_i) = \mu_i, \quad 1 \leq i \leq d,
\]

now satisfies \( \tau(\alpha^F(X,Y)) = \alpha^G(X,Y), \forall X, Y \in TU \). Thus, \( \tau \) is smooth. Then \( \mu_o \) also is smooth. Since \( \text{span} \{ \mu_{d+1} \} \) is the orthogonal complement of \( \text{span} \{ G \} \oplus S (\alpha^G) \) in \( T^*_G U \), the unitary vector field \( \mu_{d+1} \) can be choose smooth. Define

\[
\begin{align*}
TU & \xrightarrow{\Psi} \text{span} \{ G, \mu_1, \ldots, \mu_d \} \\
X & \quad \quad \rightarrow \nabla^G_X \mu_{d+1},
\end{align*}
\]

Using the Codazzi’s equations for \( \alpha^G \), the compatibility of the normal connection of the immersion \( G \) with the metric in \( T^*_G M \) and that \( \langle \alpha^G, \mu_{d+1} \rangle \equiv 0 \), it is not difficult to see that

\[
\langle \alpha^G(X,Y), \nabla^G_Z \mu_{d+1} \rangle = \langle \alpha^G(Z,Y), \nabla^G_X \mu_{d+1} \rangle, \quad \forall q \in U, \forall X, Y, Z \in T_q M,
\]

that is,

\[
A^G_{\nabla^G_X \mu_{d+1}} X = A^G_{\nabla^G_X \mu_{d+1}} Z, \quad \forall q \in U, \forall X, Z \in T_q M.
\]

At \( q \in U \), let us denote \( S \) the subspace of \( T^*_f(q) M \) given by

\[
S = \left\{ \sum_{i=1}^{d} \langle \nabla^G_Z \mu_{d+1}, \mu_i \rangle \zeta_i : Z \in T_q M \right\}.
\]

Suppose that \( \dim S = r \) and \( r \geq 1 \). Take a basis of \( S \) given by \( \xi_j = \sum_{i=1}^{d} \langle \nabla^G_Z \mu_{d+1}, \mu_i \rangle \zeta_i, 1 \leq j \leq r \). It holds that \( d \geq r \geq \dim (\text{Im } \Psi) - 1 \). From (3) and (24) we deduce that

\[
A^G_{\nabla^G_X \mu_{d+1}} X = A^G_{\nabla^G_X \mu_{d+1}} Z_j,
\]

for all \( q \in U, X \in T_q M \) and \( j, 1 \leq j \leq r \). Define \( \gamma \in S \) by \( \langle \gamma, \xi_j \rangle = \langle \mu_o, \nabla^G_Z \mu_{d+1} \rangle, 1 \leq j \leq r \). If \( X \in (\text{Ker } \Psi) \) then \( X \in N (\alpha_S - \langle \cdot, \gamma \rangle) \) due to (26).

Thus,

\[
\nu^c \geq \dim N (\alpha_S - \langle \cdot, \gamma \rangle) \geq \dim (\text{Ker } \Psi) = n - \dim (\text{Im } \Psi) \geq n - r - 1
\]

which is in contradiction with our hypothesis on the \( r \)-conformal nullity. Then \( r = 0 \) and \( \nabla^G_{\nabla^G_X \mu_{d+1}} G \) for all \( q \in U \) and \( Z \in T_q M \) by (3). From (25) and \( A^G_{\nabla^G_X \mu_{d+1}} = -I \), we deduce that

\[
\langle \nabla^G_{\nabla^G_X \mu_{d+1}, \mu_o} \rangle X = \langle \nabla^G_{\nabla^G_X \mu_{d+1}, \mu_o} \rangle Z, \quad \forall q \in U, \forall X, Z \in T_q M.
\]

Since \( n \geq 2 \), this implies that \( \langle \nabla^G_{\nabla^G_X \mu_{d+1}, \mu_o} \rangle = 0 \) for all \( q \in U \) and \( Z \in T_q M \). Hence \( \nabla^G_{\nabla^G_X \mu_{d+1}} = 0 \) and the unitary vector field \( \mu_{d+1} \) is parallel along \( U \). Then \( L \), the orthogonal complement of \( \text{span} \{ \mu_{d+1} \} \) in \( T^*_G U \), is a \( (d + 2) \)-dimensional nondegenerate parallel subbundle of \( T^*_G U \), that is, \( \nabla^G_X \mu \in L \) for all \( X \in TU \) and \( \mu \in L \). We claim that \( \mu_{d+1} \) is constant along \( U \). In fact, for all \( X \in TU \) we have

\[
\nabla^G_X \mu_{d+1} = -A^G_{\mu_{d+1}} X + \nabla^G_X \mu_{d+1} = 0.
\]
We fix \( q \in U \) and consider a differentiable curve \( \gamma(t) \) in \( U \) with \( \gamma(0) = q \). Then, putting \( \mu_{d+1}(q) = \mu \), we have

\[
\frac{d}{dt} \langle G(\gamma(t)) - G(q), \mu \rangle = \frac{d}{dt} \langle G(\gamma(t)) - G(q), \mu_{d+1}(t) \rangle = \left( \langle (G_\ast)_{\gamma(t)} \gamma'(t), \mu_{d+1}(t) \rangle \right) + \left( \langle G(\gamma(t)) - G(q), \tilde{\nabla}_{\gamma(t)} \mu_{d+1}(t) \rangle \right) = 0.
\]

Therefore, \( \langle G(\gamma(t)) - G(q), \mu \rangle \) is constant and equal to zero. So \( G(\gamma(t)) \) lies on the \( (n+d+2) \)-dimensional affine Lorentz space \( G(q) + T_q M \oplus L(q) \). Since \( \gamma(t) \) is arbitrary, it follows that \( G(U) \) lies on \( G(q) + T_q M \oplus L(q) \). Notice that \( G(q) \) belongs to the vector space \( L(q) \) and, consequently, \( -G(q) \) also belongs to \( L(q) \). Thus, \( G(q) + T_q M \oplus L(q) \) pass through the origin. Hence, \( G(U) \subset (G(q) + T_q M \oplus L(q)) \cap V^{n+d+2} = V^{n+d+1} \). By \( 2 \), \( J_\gamma(g(U)) \subset H_{\gamma} \cap V^{n+d+1} = \mathbb{R}^{n+d} \). This implies that \( g \) restricts to \( U \) reduces codimension to \( n + d \). So we can apply Theorem 1.2 in \( 2 \) and Corollary 1.1 in \( 7 \) for conclude that there is a conformal diffeomorphism \( \Gamma \) from an open subset of \( \mathbb{R}^{n+d} \) containing \( f(U) \) to an open subset of \( \mathbb{R}^{n+d} \) containing \( g(U) \) such that \( g = \Gamma \circ f \). This finish proof of Case II and of Theorem 1.2. \( \square \)

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