Finite-temperature transition in a quasi-2D Bose gas trapped in the harmonic potential

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Abstract. We study the finite-temperature transition of the quasi-2D Bose gas in an uniaxially-compressed harmonic trap by numerically solving the projected Gross-Pitaevskii equation. Gradual emergence of superfluidity is confirmed by calculating the moment of inertia when a temperature decreases. By investigating the long-distance behavior of a phase correlation function, superfluid density gradually increases reflecting the development of the phase correlation around the center of the system. From these results, we obtain the evidence for the emergence of superfluidity in this system directly.

1. Introduction

Superfluidity is one of the most characteristic quantum phenomenon and it has been intensively studied for a long time. While two dimensional (2D) system is prohibited from showing a true long range order by Mermin-Wagner-Hohenberg theorem [1, 2], the system exhibits a finite-temperature transition of the Kosterlitz-Thouless (KT) type and a quasi-long range order appears below the critical temperature $T_c$. The occurrence of such a finite-temperature transition is explained by binding/unbinding of vortex-antivortex pairs [3, 4, 5]. Indeed, this superfluid transition has been experimentally observed in several systems such as liquid helium thin films [6], superconducting Josephson-junction arrays [7] and spin-polarized atomic hydrogen [8]. Theoretically, the superfluid transition in the 2D $^4$He films has been studied using path-integral Monte Carlo methods and has been shown that this transition is of the KT type [9].

In 2D homogeneous systems, it is well known that Bose-Einstein condensation (BEC) does not exist at any finite temperature. On the other hand, it was theoretically predicted that a 2D ideal Bose gas trapped in the harmonic potential shows BEC at a finite temperature [10] and this phenomenon interprets that the long-wavelength phase fluctuations destroy the global phase coherence for BEC and the condensate only exists as the quasi-condensate [11, 12, 13]. In recent works by the ENS group, an interesting behavior of the ultracold bosonic atoms trapped in a harmonic potential has been observed; the phase defects derived from free vortices are measured by the interference patterns of multiple quasi-2D gases trapped in the valleys of an optical lattice [14]. They also captured the cross-over behavior from the quasicondensate under the KT theorem to the normal state by observing the spatial phase correlations [15]. The existence of the phase defects was confirmed by using classical field method [16] and also anomalous temperature dependence of the scissors-mode oscillation frequencies was observed [17]. However, they were fail to estimate the superfluidity qualitatively. In this paper, we study the thermal dynamics...
of a quasi-2D Bose gas in the uniaxially-compressed harmonic trap by numerically solving the projected Gross-Pitaevskii equation. We focus on the emergence of superfluidity in this system and discuss the finite-temperature transition qualitatively.

2. Model and Method
In order to treat the dynamics of a dilute Bose gas at a finite temperature, we consider the projected Gross-Pitaevskii equation described by [18, 19, 20]

\[ i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) \right) \psi(\mathbf{r}, t) + \mathcal{P}\{g|\psi(\mathbf{r}, t)|^2\psi(\mathbf{r}, t)\}, \tag{1} \]

where \(\psi(\mathbf{r}, t)\) means the field in the classical region \(C\) described below. In order to estimate the classical region \(C\), we introduce the energy cutoff \(E_{\text{cut}}(T)\) which is defined from the eigenvalue of a single-particle state with the mean occupation number \(n = 3\). The classical region \(C\) is defined by the subspace of the lower energy modes than \(E_{\text{cut}}(T)\). The subspace is projected by the operator \(\mathcal{P}\{F(\mathbf{r})\} = \sum_{n \in C} \phi_n(\mathbf{r}) \int d\mathbf{r}' \phi_n^*(\mathbf{r}') F(\mathbf{r}')\), where \(\phi_n(\mathbf{r})\) are single-particle eigenstates. In Eq. (1), \(g\) means the strength of the particle interaction and we set \(g = 4\pi\hbar^2a_s/m\) by using the s-wave scattering length \(a_s\) and the atomic mass \(m\). The external potential \(V_{\text{ext}}(\mathbf{r})\) is defined by \(V_{\text{ext}}(\mathbf{r}) = m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)/2\), where \(\omega_i\) (\(i = x, y, z\)) denotes the harmonic trap frequency along the \(i\)-axis direction. In our calculation, we consider \(^{87}\)Rb a dilute Bose gas of the total particle number \(N_{\text{tot}} = 10^4\). Following ref. [20], we estimate \(E_{\text{cut}}(T)\), the number of particles \(N_{\text{cl}}(T)\) and the energy \(E_{\text{cl}}(T)\) in the classical region \(C\) at \(N_{\text{tot}} = 10^4\) and then simulate the thermal dynamics in Eq. (1). From obtained results, we can also estimate the equilibrium temperature as is defined by ref. [21]. In our calculation, we apply the ergodic theorem to the estimation of quantity at equilibrium; we substitute the time average over the propagation of a single initial state at equilibrium for ensemble average over many different initial state.

In order to consider a quasi-2D Bose gas in the uniaxially-compressed harmonic trap under the experimental condition, we treat the 3D harmonic potential which is compressed toward \(z\)-axis under the condition of \(\hbar\omega_z > k_BT\). This condition provides the 2D system because of

\[ \text{Figure 1. Two dimensional slice of the instantaneous density profiles } n(x, y, z = 0) \text{ (upper) and phase profiles } \theta(x, y, z = 0) \text{ (lower) at (a) } T = 137[\text{nK}] \text{ and (b) } T = 178[\text{nK}]. \]
Irot 0
0.0001
180 190
Irot 
T [nK]
-1.2
-0.8
-0.4
0
-1 0 1 2
154nK
168nK
178nK
184nK
189nK
195nK
log(r) [mm]
log(g(0, r))
~Trot
*
Figure 2. (a) Moment of inertia $I_{\text{rot}}$ at $\Omega/\omega_x = 0.1$ in rotating frame against temperature $T$. The dotted-line denotes the characteristic temperature $T_{\text{rot}}$ at which superfluidity begins to emerge. (b) Phase correlation function $g(0, r)$ at $z = 0$ in the radius direction at various temperatures. The dotted-line denotes the power-law decay of the exponent $-1/4$.

the no excited states from ground state for the z-axis direction. Therefore, we set the harmonic trap frequencies $\{\omega_x, \omega_y, \omega_z\} = 2\pi \times \{50, 50, 4000\} [\text{Hz}]$. In the range of a temperature of our numerical calculation, $\hbar \omega_z/k_B T > 6.3$ and then the 2D system is realized.

3. Results

We show the results of numerical calculation for a quasi-2D Bose gas trapped in the uniaxially-compressed harmonic potential at a finite temperature. In Figure 1, we show the results of the two dimensional slice of the instantaneous density profiles $n(x, y, z = 0)$ and phase profiles $\theta(x, y, z = 0)$ at $T = 137 [\text{nK}]$ and $T = 178 [\text{nK}]$. It is obvious from the figure that the central region is phase coherent, while the size of the phase coherence region depends on the temperature. At a high temperature $T = 178 [\text{nK}]$, the phase coherence only survives for $r < 5 [\mu\text{m}]$ and many vortices and antivortices are activated by thermal fluctuations for $5 [\mu\text{m}] < r < 15 [\mu\text{m}]$, where $r$ is the radial distance at $z = 0$. On the other hand, at a low temperature $T = 137 [\text{nK}]$, the phase coherence is developed for $r < 8 [\mu\text{m}]$ and vortices and antivortices are activated for $8 [\mu\text{m}] < r < 15 [\mu\text{m}]$.

In the experiment of superfluid helium, the superfluid density has been measured by the moment of inertia using torsion oscillator technique. For the direct observation of the emergence of superfluidity, we calculate the moment of inertia $I_{\text{rot}}$ in the rotating frame in association with the experimental observation of the superfluid density. In the present calculation, $I_{\text{rot}}$ is represented by $I_{\text{rot}} = \langle L_z \rangle / \Omega$, where $\langle L_z \rangle$ is the angular momentum and $\Omega$ is the angular velocity. In the rotating frame, the emergence of superfluidity is signified by non-zero $I_{\text{rot}}$ because the superfluid component does not move with the frame and then generate a finite angular momentum. Since the temperature dependence of $I_{\text{rot}}$ hardly shows the $\Omega$ dependence, we only present the results at $\Omega/\omega_x = 0.1$ in Figure 2 (a). As shown in Figure 2 (a), we find that $I_{\text{rot}}$ increases gradually below the characteristic temperature $T_{\text{rot}} \sim 189 [\text{nK}]$. In the 2D $^4\text{He}$ films, the jump of the superfluid density at the KT transition temperature was reported [16]. However, in the case of the system with a harmonic potential, the transition temperature strongly depends on the positions due to the nonuniform density of particles and then the superfluid density hardly show a clear jump.

In order to discuss a property of a finite temperature transition, we also investigate the
long-distance decay of the phase correlation function in the radius direction described by 
\[ g(0, r) = \langle e^{i\theta(0) - \theta(r)} \rangle - \langle e^{i\theta(0)} \rangle \langle e^{i\theta(r)} \rangle. \]
Based on the KT theorem, the correlation function in 2D homogenous system shows the power-law decay with the distance, 
\[ g(0, r) \sim r^{-\eta(T)} \]
below the critical temperature \( T_c \). The exponent \( \eta(T) = T/4T_c \) and it takes a value \( \eta = 1/4 \) at \( T_c \). In Figure 2 (b), we plot the distance dependence of \( g(0, r) \) at various temperatures. \( I_{rot} \) gradually increases for \( T < T_{rot} \), reflecting the development of the phase coherence region. Therefore, the region where \( g(0, r) \) shows almost power-law decay having the exponent \(-1/4 \leq -\eta < 0\), note that the value of the exponent probably depends on the position, exists around the center of the system. On the other hand, the thermal fluctuation well developed for \( T > T_{rot} \) and the \( g(0, r) \) shows the exponential decay as expected. As with the crossovers behavior of \( I_{rot} \), \( g(0, r) \) dose not shows \( r^{-1/4} \) at \( T \sim T_{rot} \). Finally, we comment that an universal behavior of the boundary of the phase coherence region has been found from our calculation [22].

4. Summary
In summary, we have studied the finite-temperature transition of a quasi-2D Bose gas in the uniaxially-compressed harmonic trap. By rotating the system and investigating the long-distance behavior of the phase correlation function, the obtained results suggest that the superfluid state exists for \( T < T_{rot} \).

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