MODEL-INDEPENDENT DETERMINATION OF CURVATURE PARAMETER USING $H(z)$ AND $D_A(z)$ DATA PAIRS FROM BAO MEASUREMENTS

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ABSTRACT

We present a model-independent determination of the curvature parameter $\Omega_k$ using the Hubble parameter $H(z)$ and the angular diameter distance $D_A(z)$ from recent baryon acoustic oscillation (BAO) measurements. Each $H(z)$ and $D_A(z)$ pair from a BAO measurement can constrain a curvature parameter. The accuracy of the curvature measurement improves with increased redshift of $H(z)$ and $D_A(z)$ data. By using the $H(z)$ and $D_A(z)$ pair derived from a BAO Lyman $\alpha$ forest measurement at $z = 2.36$, the $\Omega_k$ is confined to be $-0.05 \pm 0.06$, which is consistent with the curvature of $-0.037^{+0.044}_{-0.042}$ constrained by the nine year Wilkinson Microwave Anisotropy Probe data only. Considering future BAO measurements, at least one order of magnitude improvement of this curvature measurement can be expected.

Key words: cosmological parameters – cosmology: observations

1. INTRODUCTION

The strong degeneracy between the curvature of the universe and the dark energy equation of state causes difficulties when constraining the two parameters simultaneously. The curvature is commonly left out in dark energy analyses, or conversely, a dark energy constant is assumed in determinations of the curvature. However, a simple flatness assumption may result in erroneously reconstructing the dark energy equation of state even if the true curvature is very small (Clarkson et al. 2007), and a cosmological constant assumption may arise from confusion between a dynamical dark energy non-flat model and the flat $\Lambda$CDM model (Virey et al. 2008). In Clarkson et al. (2007), when arguing the defects of a zero curvature assumption, they proposed a direct curvature determination by combining measurements of the Hubble parameter $H(z)$ and the comoving angular diameter distance $D_A(z)$:

$$\Omega_k = \frac{[H(z)D'(z)']^2 - c^2}{[H_0 D(z)]^2}. \quad (1)$$

where $'$ denotes the derivative with respect to redshift $z$. This formula benefits from baryon acoustic oscillation (BAO) measurements which can provide $H(z)$ and $D_A(z)$ simultaneously at the same redshift. Here, $D_A(z)$ is the angular diameter distance, which is correlated to the comoving angular diameter distance by $D(z) = (1 + z)D_A(z)$. Since derivations of $H(z)$ and $D_A(z)$ pairs from BAO measurements are purely geometrical, Equation (1) can be evaluated without any assumption of dynamic evolution of the universe, and therefore breaks the degeneracy between curvature and the dark energy equation of state.

Several works have already derived $H(z)$ and $D_A(z)$ pairs from the data of the WiggleZ Dark Energy Survey at $z = 0.44, 0.6, 0.73$ (Blake et al. 2012), and the third generation Sloan Digital Sky Survey (SDSS-III) at $z = 0.35$ (Chuang & Wang 2012; Hemantha et al. 2013; Xu et al. 2013) and $z = 0.57$ (Reid et al. 2012; Kazin et al. 2013; Chuang et al. 2013; Anderson et al. 2014; Samushia et al. 2014). With the quasar-Lyman $\alpha$ forest in SDSS-III, measurements of $H(z)$ and $D_A(z)$ have extended to high redshifts such as $z = 2.36$ (Font-Ribera et al. 2014). These data can afford us the opportunity to directly determine the curvature parameter via Equation (1).

The remaining issue is to estimate reasonably the $D'(z)$. Mignone & Bartelmann (2008) has applied a novel method to obtain the derivative of the luminosity distance $D_L(a)$ with respect to the scale factor in their model-independent reconstruction of the Hubble parameter. They decomposed the observables into suitable basis functions, then recombinated the derivatives of the basis functions to yield $D'_L(a)$. The basis system was later optimized by Maturi & Mignone (2009) to be capable of describing cosmologies independently of their background physics and improve the quality of the estimation of $D'_L(a)$. This method is independent of any cosmology model and can be employed to estimate the $D'(z)$.

By combining the data and method described above, we can determine the curvature parameter in a model-independent manner. This approach is different from previous works employing smoothing procedures in redshift bins or reconstruction of both the Hubble parameter and the comoving angular diameter distance (e.g., Mortess & Jonsson 2011). The property of Equation (1) has determined that the error on the measured curvature parameter decreases as redshift increases, and thus we can benefit from the BAO Lyman $\alpha$ forest measurement which can provide an $H(z)$ and $D_A(z)$ pair at high redshift.

The structure of this paper is as follows. In Section 2, we review the essential parts of the model-independent method. A description of the data and the application of the method is provided in Section 3. The discussions are presented in Section 4 and the conclusions are drawn in Section 5.

2. MODEL-INDEPENDENT METHOD

We follow the idea in Clarkson et al. (2007) and use the method proposed in Mignone & Bartelmann (2008), which was further developed by Maturi & Mignone (2009); Benitez-Herrera et al. (2012, 2013), to determine the curvature parameter in a model-independent manner.
2.1. Estimating the $D'(z)$

In the Friedmann–Robertson–Walker metric, the comoving angular diameter distance is written as

$$D(z) = (1+z)D_A(z) = \frac{c}{H_0 \sqrt{-\Omega_k}} \sin \left( \sqrt{-\Omega_k} \int_0^z \frac{dz'}{H(z')} \right),$$

where the Hubble parameter $H(z)$ at late time is given by

$$H(z) = \left[ \Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_r F(z) \right]^{1/2}.$$

The parameters $\Omega_m$, $\Omega_k$, and $\Omega_r$ are the current density of matter, the curvature, and dark energy in units of the critical density, respectively. The function

$$F(z) = \exp \left( 3 \int_0^z \frac{1+w(z')}{1+z'} dz' \right)$$

depends on the ratio $w(z)$ between the pressure and the density of the dark energy. By combining Equations (3) and (4), the degeneracy between the curvature and dark energy can easily be seen, as the non-zero curvature can be mimicked by the model with zero curvature and a dark energy component with $w(z) = -1/3$, which therefore results in difficulties in fitting Equation (2) to the observables. In fact, one can derive Equation (1) by solving the integral part from Equation (2) and taking the derivative with respect to redshift on both sides of the integral. The dynamic assumption of the universe thus has no place in the determination of the curvature.

The key issue here is to estimate $D'(z)$—the derivative of the comoving angular diameter distances with respect to the redshift in Equation (1). We suppose that there is an underlying function $D(z)$ describing the behaviors of the comoving angular diameter distances. $D(z)$ could be expanded into a series of suitable orthonormal functions $p_i(z)$:

$$D(z) = \sum_{i=1}^M c_i p_i(z).$$

By fitting Equation (5) to the observables $D_{\text{obs}}$, we determine the $M$ coefficients $c_i$ ($i = 1, 2, ..., M$). The number of terms to be included in the expansion depends on the choice of orthonormal basis and the quality of the data. The derivative of Equation (5) is then taken as the estimation of the derivative in Equation (1).

The basis $\{p_i\}$, in principle, could be arbitrary with ideal data, but it is not in practice. Benitez-Herrera et al. (2012) used an arbitrary orthonormal basis generated by Gram–Schmidt orthonormalization of $x^{i/2-1} (i = 0, 1, ...) \text{ to decompose the luminosity distances.}$ They observed a systematic trend on the slope of the reconstructed $H(z)$ at intermediate redshifts when compared with the predictions of a ΛCDM cosmology, although they were consistent within the error bars. This indicated that a randomly chosen system of orthonormal basis functions may not be well adapted to the behavior of the measured data. Maturi & Mignone (2009) have proposed an optimal basis system derived from principal component analysis (PCA). Under this optimal basis system, the number of coefficients $M$ in Equation (5) reaches a minimum and the possible bias introduced by the choice of the basis is removed (Benitez-Herrera et al. 2013).

2.2. Building the Optimal Basis

The derivation of the optimal basis starts with writing the comoving angular diameter distances and their redshifts in the column vectors $D_{\text{obs}}$ and $z$, respectively. The data set $D_{\text{obs}}$ is therefore considered as a single point in $n$-dimensional space, where $n$ is the number of data points. Then, we select a group of models that are believed to span the viable cosmologies and calculate the comoving angular diameter distances $D(z)$ for each model at the redshifts in $z$ to generate a set of vectors $D_i$ ($i = 1, 2, ..., M$), where $M$ is the number of models. These new vectors correspond to a cluster of points in the $n$-dimensional space and should meet the condition that the data set $D_{\text{obs}}$ be tightly enclosed in the distribution of the cluster (Benitez-Herrera et al. 2013). This ensemble of models, referred to as the training set, samples the possible behaviors of the comoving angular diameter distances and initializes the PCA.

Once the training set models $T_{n \times M} = (D_1, D_2, ..., D_M)$ are defined, we build the so-called scatter matrix $S = \Delta \Delta^T$ with $\Delta = (D_1 - D_{\text{ref}}, D_2 - D_{\text{ref}}, ..., D_M - D_{\text{ref}})$. $S$ contains the differences between each training vector $D_i$. $D_{\text{ref}}$ is the reference model that defines the origin of the n-dimensional space. $\Delta$ is actually the deviation of the training set from the reference model. $D_{\text{ref}}$ could in principle be any combination of $\{D_i\}$, and could usually be set to the average of the training set $D_{\text{ref}} = (D_i)$. Any other choice of a reference model only affects the number of principal components (PCs) in decomposing the comoving angular diameter distances, e.g., $M$ in Equation (5). It does not affect the final reconstruction of $D(z)$.

The PCs are derived by solving the eigenvalue problem $\Sigma w_i = \lambda_i w_i$, where $\lambda_i$ and $w_i$ are the $i$th eigenvalue and eigenvector, respectively. The eigenvector of the largest eigenvalue is the first PC. It corresponds to the direction in which the projection of $\Delta$ has the largest variance, which means that the cluster of points mainly align in this direction. The second PC is the eigenvector of the second largest eigenvalue corresponding to the direction of the secondary largest variance, and so on. Since PCA aims to reduce the dimensionality of the training set substantially while retaining almost all of the variation, an important issue arises concerning how many PCs should be employed in the reconstruction of $D(z)$. The selection criterion is based on the cumulative percentage of total variation (Benitez-Herrera et al. 2013; Jolliffe 2002, Section 6.1.1), defined as

$$t_m = 100 \times \sum_{k=1}^{m} \frac{\lambda_k}{\sum_{k=1}^{n} \lambda_k},$$

where $\lambda_k (k = 1, 2, ..., n)$ are the eigenvalues corresponding to the PCs and sorted in a descending sequence $\lambda_k > \lambda_{k+1}$, $n$ is the number of total PCs, and $m$ is the number of PCs that we will employ in the reconstruction. The $t_m$, varying between 0 and 100, quantifies what percentage of variance in the training set is preserved in the first $m$ PCs. After setting a threshold (e.g., $t_m > 95$), $t_m$ returns the suitable number of PCs.

Then, the deviation of the $D(z)$ from the reference model is decomposed by the $m$ PCs

$$D(z)|_{S} = D_{\text{ref}} + \sum_{i=1}^{m} c_i w_i.$$

Equation (7) is similar to Equation (5), except that the reference model here is not zero and the basis $w_i$ is optimized via PCA. The coefficients $c_i$ are determined by fitting Equation (7) to $D_{\text{obs}}$ through $\chi^2$ minimization.

The optimal basis for the decomposition of $D(z)$ is also the optimal basis for the decomposition of the $H_0$-independent angular comoving distances $\bar{D}(z) = H_0 D(z)$. When build
the training set \( \tilde{T}_{n \times M} \) with \( H_0 \)-independent angular comoving distances \( \{ \tilde{D}_i \} \), Equation (7) can be rewritten as

\[
D(z) = \frac{1}{H_0} \left( \tilde{D}_{\text{ref}} + \sum_{i=1}^{m} \tilde{c}_i \tilde{w}_i \right),
\]

(8)

where the \( \tilde{D}_{\text{ref}} \) is the \( H_0 \)-independent reference model. In the \( \chi^2 \) minimization step, we use Equation (8) to leave the Hubble constant \( H_0 \) as a free parameter. The \( D'(z) \) in Equation (1) is derived by taking the derivative on both sides of Equation (8) with respect to redshift.

3. APPLICATION TO REAL DATA

3.1. The Data

The \( H(z) \) and \( D_A(z) \) pairs are collected from the current literature of BAO measurements. Since the SDSS and BOSS CMASS samples have been analyzed multiple times; we tend to use the most recent published results to avoid overlap. The \( (H,D_A) \) pairs and the BAO surveys are listed in Table 1. We consider the covariance between distances constrained by different BAO samples at different redshift ranges to have negligible effect on our determination of the derivative of distance with respect to redshift.

To lower the statistical errors in our \( D'(z) \) estimation, we enlarge the distance samples with the luminosity distances \( D_L(z) \) from the currently largest homogeneously reduced compilation of SN Ia, the Union 2.1 (Suzuki et al. 2012), which contains 580 SNe Ia. The data are in the form of a distance modulus \( \mu \) which should be converted to the luminosity distance by

\[
D_L = 10^{\frac{\mu}{5}} - 5.
\]

(9)

The corresponding comoving angular diameter distances are derived via Etherington’s relation, which holds for any space–time \( D(z) = D_L(z)/(1+z) \). For simplicity, we use the covariance matrix of distance modules of SNe Ia with statistical errors only, which is in diagonal form, during the \( \chi^2 \) minimization procedure.

3.2. Results

Figure 1 depicts the first four PCs for the comoving angular diameter distances. The training set \( \tilde{T}_{n \times M} \) is built from 1000 non-flat ΛCDM models with parameters uniformly sampled in the cubic parameter space with boundaries \( 0.1 < \Omega_m < 0.5, 0.5 < \Omega_\Lambda < 0.9, \) and \(-0.2 < \Omega_k < 0.2;\) the sum of \( \Omega_m, \Omega_\Lambda, \) and \( \Omega_k \) can deviate from 1. Here, the reference model takes the average of the training set \( \tilde{D}_{\text{ref}} = \langle \tilde{D}_i \rangle \). A different reference model would not change the reconstruction of \( D(z) \) as long as the distribution of the training set tightly enclosed the data (Benitez-Herrera et al. 2013, Section 3). The first PC retains 98.8% of the total variance in the sample, which means it has already considered the major properties in the expansion of the training set. We therefore use the first PC to decompose the \( D(z) \). Here, we emphasize that although the PCs are determined by the training set sampled from different non-flat ΛCDM models, they are able to constrain other cosmologies that are not explicitly contained in the training set (see Maturi & Mignone 2009, Section 4.2).

Fitting Equation (8) to the observables \( D_{\text{obs}} \) yields the coefficients \( \tilde{c}_1 = (-11.3 \pm 9.2) \times 10^4 \) and \( H_0 = 69.8 \pm 1.9 \) km s\(^{-1}\) Mpc\(^{-1}\). By substituting the coefficients and the BAO data into Equations (8) and (1), each \( (H,D_A) \) pair derives a curvature parameter. The upper panel of Figure 2 shows \( \Omega_k \) with 1σ errors. The curvature measurement using high redshift data at \( z = 2.36 \) has the best constraint, \( \Omega_k = -0.05 \pm 0.06 \), which is consistent with the curvature \( \Omega_k = -0.037^{+0.044}_{-0.043} \) constrained by the nine year Wilkinson Microwave Anisotropy Probe (WMAP) data only (Bennett et al. 2013). At low and medium redshifts, although the deviations of \( \Omega_k \) from zero are nearly of the order of unity, the measurements are still consistent with a flat universe.

The lower panel of Figure 2 shows a trend where the errors in \( \Omega_k \) decrease with increasing redshift. By comparing the errors of the input \( (H,D_A) \) data with the errors of the output \( \Omega_k \), we have

### Table 1

| \( z \) | \( H(z) \) (km s\(^{-1}\) Mpc\(^{-1}\)) | \( D_A(z) \) (Mpc) | Survey | Reference |
|---|---|---|---|---|
| 0.44 | 82.6 ± 7.8 | 1205 ± 114 | WiggleZ | Blake et al. (2012) |
| 0.6 | 87.9 ± 6.1 | 1380 ± 95 | | |
| 0.73 | 97.3 ± 7.0 | 1534 ± 107 | | |
| 0.35 | 84.4 ± 7.0 | 1050 ± 38 | SDSS DR7 | Xu et al. (2013) |
| 0.57 | 93.1 ± 3.0 | 1380 ± 23 | BOSS DR11 CMASS | Samushia et al. (2014) |
| 2.36 | 226 ± 8 | 1590 ± 60 | BOSS DR11 Ly-\( \alpha \) forest | Font-Ribera et al. (2014) |
demonstrated that even if the BAO data at higher redshift have less precision than the lower ones, the curvatures determined by the higher redshift BAO data still have smaller errors, such as 3.7% for $D_A(2.36)$ and 3.5% for $H(2.36)$, but 1.6% for $D_A(0.57)$ and 3.2% for $H(0.57)$.

4. DISCUSSIONS

Consider the error propagation formula of Equation (1)

$$(\Delta \Omega_k)^2 = \sum_{\alpha \in \{H, D, D', \Delta H, \Delta D, \Delta D', \Delta H_0, \Delta D_0\}} \left( \frac{\partial \Omega_k}{\partial \alpha} \right)^2 \Delta \alpha^2$$

$$= 4 \left[ \Omega_k + \frac{c^2}{(H_0 D)^2} \left( \frac{\Delta H}{H} \right)^2 + \left( \frac{\Delta D}{D} \right)^2 \right]$$

$$+ 4 \Omega_k^2 \left[ \left( \frac{\Delta D}{D} \right)^2 + \left( \frac{\Delta H_0}{H_0} \right)^2 \right]. \quad (10)$$

We find that the behavior of $\sigma_{\Omega_k}^2$ is dominated by $D(z)^{-4}$ at low redshift. This feature results in large errors in the determination of the curvature when using low redshift data. Nevertheless, Equation (10) tends to be a constant which only depends on the errors of $H$, $H_0$, $D$, $D'$ when using very high redshift data, revealing that Equation (1) could tightly constrain the curvature parameter via precise measurements of $H(z)$ and $D_A(z)$ at high redshifts.

We repeat the procedure from previous sections for a synthetic sample simulated by a standard flat $\Lambda$CDM model with parameters $\Omega_m = 0.315$, $\Omega_\Lambda = 0.685$, and $h = 0.673$ from Planck Collaboration (2013). The uncertainties of the $H(z)$ and $D(z)$ measurements resemble the theoretical BAO cosmic variance forecast for a full-sky BAO survey in Table 2 of Weinberg et al. (2013), in which case the $H(z)$ and $D_A(z)$ measurements can reach a precision of 0.2% at $z > 1$.

Figure 3 shows the results of the curvature measurements on the mock data. The tendency of $\sigma_{\Omega_k}$ to change with $z$ is in agreement with those derived from real BAO data. With precise BAO measurements in the future, we could expect this method to constrain the curvature parameter within $10^{-3}$ error limit or better.

5. CONCLUSIONS

Based on the work of Clarkson et al. (2007), Mignone & Bartelmann (2008), Maturi & Mignone (2009), and Benitez-Herrera et al. (2013), we present a model-independent method to determine the curvature parameter. The $H(z)$ and $D_A(z)$ pairs involved in this method are derived by BAO measurements that only depend on the space-time geometry, and thus allow us to measure the curvature without any assumptions of the dynamic evolution of the universe. The luminosity distances $D_L(z)$ from the Union 2.1 SN Ia compilation are included to better constrain the estimation of the derivative of the comoving angular diameter distance with respect to redshift. The curvature parameters measured in this work are in agreement with a flat universe within the error limits.

The feature of Equation (1) leads to the fact that the accuracy of the curvature measurement improves with increasing redshift for $H$ and $D_A$ and will reach a limit primarily determined by the data quality. In this work, the errors of curvature measurements at low redshift are nearly of the order of unity, while the curvature measurement at high redshift $z = 2.36$ have derived a much better constraint: $\Omega_k = -0.05 \pm 0.06$, which is consistent with the nine year WMAP-only results.
We use the density parameters from Planck Collaboration (2013) and the theoretical BAO cosmic variance forecast from Weinberg et al. (2013) to generate a small synthetic sample to test the curvature measurement. At least one order of magnitude improvement of this curvature measurement could be expected.

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Note added in proof. We are aware of Sapone et al. (2014), who have accomplished similar work during the preparation of this paper.

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