Abstract. In this paper, we consider the complex flows when all three regimes pre-Darcy, Darcy and post-Darcy may be present in different portions of a same domain. We unify all three flow regimes under mathematics formulation. We describe the flow of a single-phase fluid in $\mathbb{R}^d$, $d \geq 2$ by a nonlinear degenerate system of density and momentum. A mixed finite element method is proposed for the approximation of the solution of the above system. The stability of the approximations are proved; the error estimates are derived for the numerical approximations for both continuous and discrete time procedures. The continuous dependence of numerical solutions on physical parameters are demonstrated. Experimental studies are presented regarding convergence rates and showing the dependence of the solution on the physical parameters.

Key words. Porous media, pre-Darcy, Darcy, post-Darcy, error estimates, slightly compressible fluid, dependence on parameters, numerical analysis.

AMS subject classifications. 35Q35, 65M12, 65M15, 65M60, 76S05, 65N15, 65N30, 65N30.

1. Introduction. Fluid flow in porous media plays an important role in a wide range of science and engineering applications, e.g. water resources, geothermal system, chemical processes, gas and water purification, gas storage, oil extraction and petroleum engineering. Fluid flow in porous media is very complicated and are modeled by many different equations of various types due to the variety and complexity of the filtration matrix of the media in the path of the fluid. Based on the experimentally observed non-Darcy behavior of seepage flows by Basak [2], Soni et al. [39], Kundu et al. [30] classified flow regimes into three main flow regimes: pre-Darcy flow (i.e. pre-linear, non-Darcy), Darcy flow (linear), and post-Darcy flow (i.e. post-linear, non-Darcy). There is a general consensus that the Darcy regime is valid as long as the Reynolds number ($Re$) in the range of characteristic value between 1 and 10, see in [3]. When the Reynolds number is high ($Re > 10$), there is a deviation from the Darcy law and Forchheimer’s equations are usually used to account for it [12], see also in [3, 32, 33]. On the other end of the Reynolds number’s range, when it is very small ($Re \to 0$), the pre-Darcy regime is observed but not fully understood, although it contributes to unexpected oil extraction, and improved oil recovery in petroleum reservoirs, see in [4, 11, 37, 40] and references therein.

Most studies in porous media focus on the Darcy regime which is presented by the (linear) Darcy equation which is a linear relationship between the pressure gradient and the fluid velocity, see in [42]. Recently, the post-Darcy regime has been attracted attention with the (nonlinear) mathematical and numerical modeling of Forchheimer flows, see in [1, 8, 9, 15, 17, 18, 20, 21, 25, 26, 41] and references therein. In contrast, the (nonlinear) pre-Darcy regime has not received much attention among the researchers and engineers. Moreover, the three regimes are always treated separately. This is due to the different natures of the models and the ranges of their applicability. There are evidences in the literature that the flow of a fluid in porous media may present all three regimes in different unidentified portions of the confinement. The petroleum reservoir is an example of environment exhibiting all the three regimes: fast flow near wells and fractures is described by Forchheimer equation, very slow flow far away from wells is described by the Pre-Darcy equation and moderate flow in the between is described by Darcy equation. Therefore, there is a need to unify the three regimes into one formulation and study the fluid as a whole. This paper aims at deriving admissible models for this unification and analyze their properties mathematically.

We now start the investigation of different types of fluid flows in porous media. Consider fluid flows with velocity $v \in \mathbb{R}^d$, $d \geq 2$ pressure $p \in \mathbb{R}$, and density $\rho \in [0, \infty)$. Depending on the range of the Reynolds number, there are different groups of equations to describe their dynamics.

Darcy’s law is the most popular equation to describe flow in porous media and fractures at moderate flow rates, when the flow rate and the pressure gradient have a linear relationship.

$$v = -k(x,t)\nabla p(x,t), \text{ where } k(x,t) > 0. \tag{1.1}$$

However, as the flow rate (Reynolds’s number) decreases, there are Izbash-type equations that describe the pre-
For experimental values of \( \alpha \), see e.g. [37,40].

When the flow rate (Reynolds’s number) is high, the following Forchheimer equations are widely used in studying post-Darcy flows.

Forchheimer’s two-term law

\[
(a(x,t)v(x,t) + b(x,t)|v(x,t)|v(x,t)) = -\nabla p(x,t).
\] (1.3)

Forchheimer’s three-term law

\[
(a(x,t)v(x,t) + b(x,t)|v(x,t)|v(x,t) + c(x,t)|v(x,t)|^2v(x,t)) = -\nabla p(x,t).
\] (1.4)

Forchheimer’s power law

\[
(a(x,t)v(x,t) + d(x,t)|v(x,t)|^{m-1}v(x,t)) = -\nabla p(x,t).
\] (1.5)

Here, the \( a(x,t), b(x,t), c(x,t), d(x,t) \) are positive functions, and \( m \in (1,2) \) are derived from experiments for each case. The above three Forchheimer equations can be combined and generalized to the following form:

\[
g_F(|v(x,t)|)v(x,t) = -\nabla p(x,t),
\] (1.6)

where

\[
g_F(y) = a_0(x,t) + a_1(x,t)y^{\alpha_1} + \cdots + a_N(x,t)y^{\alpha_N},
\] (1.7)

with \( N \geq 1 \), \( \alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N \), \( a_0(x,t), a_N(x,t) > 0 \), \( a_1(x,t), a_2(x,t), \ldots, a_{N-1}(x,t) \geq 0 \). The generalized Forchheimer equation (1.6) was intensely used by the authors to model and study fast flows in the porous media, see in [1,8,14,16,19,21]. The techniques developed in those papers will be essential in our approach and analysis below.

In previous works, each regime pre-Darcy, Darcy, or post-Darcy was studied separately, even though they exist simultaneously in porous media. In particular cases, some models must consider multi-layer domains with each layer having a different regime of fluid flows, see for e.g. section 6.7.8 of [41]. The goal of this section is to model all regimes together in the same domain. We write a general equation of motion for all cases (1.1)–(1.6) as

\[
g(|v(x,t)|)v(x,t) = -\nabla p(x,t),
\] (1.8)

Different forms of \( g(y) \) give different models, for example,

\[
g(y) = k^{-1}(x,t)y^{-\alpha}, \quad k^{-1}(x,t), \quad a(x,t) + b(x,t)y, \quad a(x,t) + b(x,t)y + c(x,t)y^2, \quad a(x,t) + d(x,t)y^{m-1}, \quad g_F(y),
\]

for equations (1.2), (1.1), (1.3), (1.4), (1.5), (1.6), respectively.

We consider the following two main models. Below, \( \mathbf{1}_E \) denotes the characteristic (indicator) function of a set \( E \).

**Model 1.** Function \( g(y) \) is piecewise smooth on \((0, \infty)\). Based on (1.2), (1.1) and (1.6) and their validity in different ranges of Reynolds number, our first consideration is the following piecewise defined function

\[
g(y) = \tilde{g}(y) \overset{def}{=} av_{-\alpha}1_{(0,y_1)}(y) + a_{y_1,y_2}(s) + g_F(y)1_{(y_2,\infty)}(y) \quad \text{for} \quad s > 0,
\] (1.9)

where \( \alpha \in (0,1) \), and \( y_2 > y_1 > 0 \) are fixed threshold values.

Note that \( \tilde{g}(y) \) is not differentiable at \( y_{1,2} \).

**Model 2.** Function \( g(y) \) is smooth on \((0, \infty)\). Another generalization is to use a smooth interpolation between pre-Darcy (1.2) and generalized Forchheimer (1.6). Instead of (1.9), we propose the following

\[
g(y) = g_{\ell}(y) \overset{def}{=} a_{-\alpha}y^{-\alpha} + a_0 + a_1y^{\alpha_1} + \cdots + a_Ny^{\alpha_N} \quad \text{for} \quad y > 0,
\] (1.10)
where \( N \geq 1, \alpha \in (0, 1), \alpha_N > 0, \)

\[
a_{-1}, a_0, a_N > 0 \text{ and } a_i \geq 0 \quad \forall i = 1, \ldots, N - 1.
\]  

(1.11)

The main advantage of \( g_I \) over \( \bar{g} \) is that it is smooth on \((0, \infty)\). This allows further mathematical analysis of the flows. It also can be used as a framework for perspective interpretation of field data, i.e., matching the coefficients of non-negative functions with medium.

\[
N_i = \ldots
\]

We denote the vector of powers by \( \alpha \). In order to take into account the presence of density in a generalized Forchheimer equation, we modify (1.6) using dimension analysis by Muskat \[42\] and Ward \[43\]. They proposed the following equation for both laminar and turbulent flows in porous media:

\[
- \nabla p(x, t) = G(\nu^\frac{1}{2} \rho^{\frac{1}{2}} \mu^{\frac{1}{2}}),
\]  

(1.13)

where \( G \) is a function of one variable, \( \mu = \mu(x, t) \) is the viscosity of the fluid, \( \kappa = \kappa(x) \) is the permeability of the medium.

In particular, when \( i = 1 \), Ward \[43\] established the Darcy’s law to match the experimental data

\[
- \nabla p(x, t) = \frac{\mu(x, t)}{\kappa(x)} v(x, t),
\]  

(1.14)

and when \( i = 2 \) for Forchheimer’s law

\[
- \nabla p(x, t) = \frac{\mu(x, t)}{\kappa(x)} v(x, t) + c_F \frac{\rho(x, t)}{\sqrt{\kappa(x)}} |v(x, t)| v(x, t), \quad \text{where } c_F > 0.
\]  

(1.15)

Combining (1.8) with the suggestive form (1.13) for the dependence on \( \rho \) and \( v \), we propose the following equation

\[
- \nabla p(x, t) = \sum_{i=-1}^{N} a_i(x, t) \rho^{\alpha_i} |v(x, t)|^{\alpha_i} v(x, t),
\]  

(1.16)

where \( N \geq 1, \alpha_{-1} = -\alpha, \alpha = (0, 1), \alpha_N > 0, a_{-1}(x, t), a_N(x, t) > 0 \) and \( a_i(x, t) \geq 0 \quad \forall i = 1, \ldots, N - 1. \)

Multiplying both sides of the equation (1.16) to \( \rho \), we find that

\[
\left( \sum_{i=-1}^{N} a_i(x, t) \rho^{\alpha_i} |v(x, t)|^{\alpha_i} \right) \rho(x, t) v(x, t) = -\rho(x, t) \nabla p(x, t).
\]  

(1.17)

Denote the function \( F : \Omega \times [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) a generalized polynomial with non-negative coefficients by

\[
F(z) = a_{-1}(x, t) z^{-\alpha} + a_0(x, t) + a_1(x, t) z^{\alpha_1} + \cdots + a_N(x, t) z^{\alpha_N}, \quad z \geq 0.
\]  

(1.18)

We denote the vector of powers by \( \alpha = (\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_N) \) and vector coefficients by \( a(x, t) = (a_{-1}(x, t), a_0(x, t), a_1(x, t), \ldots, a_N(x, t)) \). The class of functions \( F(s) \) as in (1.18) is denoted by \( P(N, \alpha) \).
The equation (1.17) can be rewritten as
\[ F(|\rho(x,t)v(x,t)|)|\rho(x,t)v(x,t) = -\rho(x,t)\nabla p(x,t). \] (1.19)

Above, we have introduced several models which can be used to interpret experimental and field data. We now use them to investigate the fluid flow’s properties. They are used together with other basic equations of continuum mechanics which we recall here.

Continuity equation
\[ \phi(x)\rho(x,t) + \nabla \cdot (\rho(x,t)v(x,t)) = 0, \]
where \( \phi \in (0, 1) \) is the constant porosity.

Under isothermal condition the state equation relates the density \( \rho \) with the pressure \( p \) only, i.e., \( \rho = \rho(p) \). Therefore, the equation of state, for slightly compressible fluids, is
\[ \frac{d\rho}{dp} = \frac{\rho}{\kappa}, \]
where \( 1/\kappa > 0 \) is small compressibility.

Hence
\[ \nabla \rho = \frac{1}{\kappa} \rho \nabla p, \quad \text{or} \quad \rho \nabla p = \kappa \nabla \rho. \] (1.20)

Combining (1.19) and (1.20) implies that
\[ F(|\rho(x,t)v(x,t)|)|\rho(x,t)v(x,t) = -\kappa(x)\nabla \rho(x,t). \] (1.21)

The continuity equation is
\[ \phi(x)\rho(x,t) + \text{div}(\rho(x,t)v(x,t)) = f(x,t), \] (1.22)
where \( \phi \) is the porosity, \( f \) is external mass flow rate.

By combining (1.21) and (1.22) we have
\[ F(|\mathbf{m}(x,t)|)|\mathbf{m}(x,t) = -\kappa(x)\nabla \rho(x,t), \]
\[ \phi(x)\rho(x,t) + \text{div} \mathbf{m}(x,t) = f(x,t), \]
where \( \mathbf{m}(x,t) = \rho(x,t)v(x,t) \).

By rescaling the variable \( \rho(x,t) \rightarrow \kappa(x)\rho(x,t) \), \( \phi(x) \rightarrow \kappa^{-1}(x)\phi(x) \), we obtain system of equations
\[ F(|\mathbf{m}(x,t)|)|\mathbf{m}(x,t) = -\nabla \rho(x,t), \]
\[ \phi(x)\rho(x,t) + \text{div} \mathbf{m}(x,t) = f(x,t). \] (1.23)

The outline of the paper is as follows. We introduce the notations and the relevant results in section 2. In section 3, we defined a numerical approximation using mixed finite element approximations to the initial boundary value problem (IVBP) (3.2). In section 4, we establish many estimates of the energy type norms for the approximate solution \( (\mathbf{m}_h, \rho_h) \) to the IVBP problem (3.10) in Lebesgue norms in terms of the boundary data and the initial data. In section 5, we focus on proving the continuous dependence of the solution on the coefficients of function \( g \). It is proved in Theorem 5.1 that the difference between the two solutions, which corresponds to two different coefficient vectors \( a_1 \) and \( a_2 \), is estimated in terms of \( |a_1 - a_2| \), see in (5.4). In section 6, we study in Theorem 6.1 the convergence and in Theorem 6.2 the dependence on coefficients of general polynomial \( g \) of the approximated solution to the problem (3.10). Furthermore, we can specify the convergent rate. In section 7, we study the fully discrete version of problem (3.10). In Lemma 7.1, the stability of the approximated solution is proved. Theorems 7.2 and 7.3 are for studying the error estimates and the continuous dependence on parameters of the numerical solution. In section 8, the numerical experiments in the two-dimensions using the standard finite elements \( P_1 \) are presented regarding the convergence rates and the dependence of the solution on the physical parameters.
2. Notations and preliminary results. Through out this paper, we assume that $\Omega$ is an open, bounded subset of $\mathbb{R}^d$, with $d = 2, 3, \ldots$, and has $C^1$-boundary $\partial \Omega$. For $s \in [0, \infty)$, we denote $L^s(\Omega)$ be the set of $s$-integrable functions on $\Omega$ and $(L^s(\Omega))^d$ the space of $d$-dimensional vectors which have all components in $L^s(\Omega)$. We denote $(\cdot, \cdot)$ the inner product in either $L^s(\Omega)$ or $(L^s(\Omega))^d$ that is $(\xi, \eta) = \int_{\Omega} \xi \eta dx$ or $(\xi, \eta) = \int_{\Omega} \xi \eta dx$ and $\|v\|_{L^s(\Omega)} = \left(\int_{\Omega} |v(x)|^s \, dx\right)^{1/s}$ for standard Lebesgue norm of the measurable function. For $m \geq 0, s \in [0, \infty)$, we denote the Sobolev spaces by $W^{m,s}(\Omega) = \{v \in L^s(\Omega) : D^\alpha v \in L^s(\Omega), |\alpha| \leq m\}$ and the norm of $W^{m,s}(\Omega)$ by $\|v\|_{W^{m,s}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha v|^s \, dx\right)^{1/s}$, and $\|v\|_{W^{m,-\infty}(\Omega)} = \sum_{|\alpha| \leq m} \operatorname{ess sup}_{\Omega} |D^\alpha v|$. Let $I = [0, T]$, we define $L^s(I; X)$ to be the space of all measurable functions $v : I \to X$ with the norm $\|v\|_{L^s(I; X)} = \left(\int_0^T \|v(t)\|_{X}^s \, dt\right)^{1/s}$, and $L^\infty(I; X)$ to be the space of all measurable functions $v : I \to X$ such that $v : t \to \|v(t)\|_X$ is essentially bounded on $I$ with the norm $\|v\|_{L^\infty(I; X)} = \operatorname{ess sup}_{t \in I} \|v(t)\|_X$. We use short hand notations, $\|\rho(t)\| = \|\rho(\cdot, t)\|_{L^s(I; X)}$.

Our estimates make use of coefficient-weighted norms. For some strictly positive, bounded function, we denote $\|f\| \|\| \leq \|f\|_{\omega}$ by $\|f\|_{\omega} = \int_{\Omega} \omega(x) |f|^2 \, dx$, and if $0 < \omega_x \leq \omega(x) \leq \omega^*$ throughout $\Omega$, we have the equivalent $\sqrt{\operatorname{det} \omega} \|f\|_{L^2(\Omega)} \leq \|f\|_{\omega} \leq \sqrt{\operatorname{det} \omega^*} \|f\|_{L^2(\Omega)}$. Our calculations frequently use the following exponents

\[ s = \alpha_N + 2, \quad s^* = \frac{s}{s - 1}. \]

The argument $C$ will represent for positive generic constants and their values depend on exponents, coefficients of polynomial $F$, the spatial dimension $d$ and domain $\Omega$, independent of the initial and boundary data and time step. These constants may be different place by place.

We will frequently use the following basic inequalities. By Young’s inequality, we have

\[ x^0 \leq x^\gamma + x^\beta \quad \text{for all } x > 0, \quad \gamma_1 \leq \beta \leq \gamma_2, \]
\[ x^\beta \leq 1 + x^\gamma \quad \text{for all } x \geq 0, \quad \gamma \geq \beta > 0. \]

For any $r \geq 1, x_1, x_2, \ldots, x_k \geq 0$,

\[ x_1^r + x_2^r + \cdots + x_k^r \leq (x_1 + x_2 + \cdots + x_k)^r \leq k^{r-1}(x_1^r + x_2^r + \cdots + x_k^r). \]

Lemma 2.1. For all $w > 0$,

(i) \[ -\alpha F(w) \leq wF'(w) \leq \alpha \delta F(w). \]

(ii) \[ a_s(w^{-\alpha} + 1 + w^{\alpha}) \leq F(w) \leq Na^s(w^{-\alpha} + w^{\alpha}). \]

Proof. (i) Since $-\alpha < \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_N$,

\[ -\alpha F(w) \leq wF'(w) = -\alpha a_1 w^{-\alpha} + \alpha_0 a_0 w^{\alpha} + \alpha_1 a_1 w^{\alpha} + \cdots + \alpha_N a_N w^{\alpha} \leq \alpha \delta F(w). \]

Thus, the estimate (2.5) follows.

(ii) It is easy to see that

\[ F(w) \leq \max_{i = 1, \ldots, N} a_i \cdot (w^{-\alpha} + w^{\alpha}) \leq Na^s(w^{-\alpha} + w^{\alpha}). \]

On the other hand,

\[ F(w) \geq a_s w^{-\alpha} + a_0 + a_N w^{\alpha} \geq a_s(w^{-\alpha} + 1 + w^{\alpha}). \]

Combining (2.7) and (2.8) we obtain the inequality in (2.6). \qed
LEMMA 2.2. (i) Assume \(-1 < p < 0\), then for all \(x, y \in \mathbb{R}^d\),
\[
\|x\|^p |x - y|^p \leq 2|x - y|^{1 + p}. \tag{2.9}
\]
(ii) Assume \(p > 0\), then for all \(x, y \in \mathbb{R}^d\), there is \(C > 0\) such that (see in [13, 38]).
\[
\|x\|^p |x - y|^p \leq C(|x| + |y|)^p |x - y|. \tag{2.10}
\]

Proof.
(i) Let \(\gamma(t) = \tau x + (1 - \tau)y\), \(\tau \in [0, 1]\) and \(h(t) = |\gamma(t)|^p \gamma(t)\). Then
\[
|\gamma(t)|^p |x - y| = |h(1) - h(0)| = \left| \int_0^1 h'(\tau)d\tau \right| = \left| \int_0^1 \left[ |\gamma(\tau)|^p (x - y) + p |\gamma(\tau)|^{p-1} \frac{\gamma(\tau) \cdot (x - y)}{|\gamma(\tau)|} \gamma(\tau) \right] d\tau \right|
\leq (1 + p)|x - y| \int_0^1 |\gamma(\tau)|^p d\tau.
\]
We claim
\[
\int_0^1 |\gamma(\tau)|^p d\tau \leq \frac{2}{p + 1} |x - y|^p. \tag{2.12}
\]

The inequality (2.9) follows by substituting (2.12) into (2.11).

Proof of claim (2.12)
- If \(|x| \geq |x - y|\). Since \(p < 0\),
\[
|\tau x + (1 - \tau)y|^p \leq |x| - (1 - \tau)|x - y|^p \leq |(1 - \tau)|x - y| - |x - y|^p = \tau^p |x - y|^p.
\]

This shows that
\[
\int_0^1 |\gamma(\tau)|^p d\tau \leq |x - y|^p \int_0^1 \tau^p d\tau \leq \frac{2}{p + 1} |x - y|^p.
\]
- If \(|x| < |x - y|\). Let \(\tau_* \in (0, 1)\) be defined by \((1 - \tau_*)|x - y| = |x|
\[
\int_0^1 |\gamma(\tau)|^p d\tau \leq \int_0^1 |x| - (1 - \tau)|x - y|^p d\tau
\leq |x - y|^p \int_0^1 |x - \tau_*|^p d\tau = \int_0^1 |x - \tau_*|^p d\tau = \frac{1}{p + 1} |x - y|^p (\tau_*^{p+1} + (1 - \tau_*)^{p+1})
\leq \frac{2}{p + 1} |x - y|^p.
\]

(ii) For the proof see Lemma 5.3 in [13, 38]. \(\square\)

For convenience, we use the following notations: let \(y = (y_1, y_2, \ldots, y_d)\) and \(y' = (y'_1, y'_2, \ldots, y'_d)\) be two arbitrary vectors of the same length, including possible length 1.

LEMMA 2.3. Let \(F(a, \cdot)\) and \(F(a', \cdot)\) belong to the class \(P(N, \alpha)\). Then for any \(y, y' \in \mathbb{R}^d\), one has
(i)
\[
|F(a, |y|)y - F(a', |y'|)y'| \leq C_1 (1 + |y| + |y'|)^{\frac{s-2+\alpha}{s-\alpha}} |y - y'|^{1-\alpha} + C_2 (1 + |y| + |y'|)^{s-1} |a - a'|. \tag{2.13}
\]
(ii)
\[
(F(a, |y|)y - F(a', |y'|)y') \cdot (y - y') \geq C_3 |y' - y'|^s - C_4 (1 + |y| + |y'|)^{s-1} |y - y'| |a - a'|, \tag{2.14}
\]
where \( C_1 = 2(s-1)/(1-\alpha), C_2 = 3N, C_3 = a_s(1-\alpha)/(2^{s-1}(s-1)) \).

In particular, in the case \( a = a' \) then (2.13) and (2.14) become

\[
(iii) \quad |F(a, |y|)y - F(a, |y'|)y'| \leq C_1 (1 + |y| + |y'|)^{s-2+\alpha} |y - y'|^{1-\alpha}.
\]

\[
(iv) \quad (F(a, |y|)y - F(a, |y'|)y' \cdot (y - y') \geq C_3 |y' - y|^\alpha.
\]

**Proof.** (i). Let \( a, a' \in S \) and \( y, y', k \in \mathbb{R}^d \). We defined,

\[
\gamma(\tau) = \tau y + (1-\tau)y', \quad \mathbf{b}(\tau) = \tau a + (1-\tau)a' \quad \forall \tau \in [0,1],
\]

**Case 1:** The origin does not belong to the segment connecting \( y' \) and \( y \). Define

\[
z(\tau) = F(\mathbf{b}(\tau), |\gamma(\tau)|) \gamma(\tau) \cdot \mathbf{k}.
\]

By the Mean Value Theorem, we have

\[
I \equiv [F(a, |y|)y - F(a', |y'|)y'] \cdot k = z(1) - z(0) = \int_0^1 z'(\tau)d\tau.
\]

Elementary calculations give

\[
I = \int_0^1 f_1(\tau)d\tau + \int_0^1 f_2(\tau)d\tau \equiv I_1 + I_2,
\]

where

\[
f_1(\tau) = F(\mathbf{b}(\tau), |\gamma(\tau)|) (y - y') \cdot k + F'(\mathbf{b}(\tau), |\gamma(\tau)|) \frac{\gamma(\tau) \cdot (y - y')}{|\gamma(\tau)|} \gamma(\tau) \cdot k,
\]

\[
f_2(\tau) = F_a(\mathbf{b}(\tau), |\gamma(\tau)|)(a - a') \gamma(\tau) \cdot k.
\]

**Estimation of** \( I_1 \). Using triangle inequality, (2.5) and (2.6) we find that

\[
|f_1(\tau)| \leq |F(\mathbf{b}(\tau), |\gamma(\tau)|) (y - y') \cdot k| + F'(\mathbf{b}(\tau), |\gamma(\tau)|) \frac{|\gamma(\tau) \cdot (y - y')|}{|\gamma(\tau)|} |\gamma(\tau)| |k|
\]

\[
\leq (1 + \alpha_N) |F(\mathbf{b}(\tau), |\gamma(\tau)|) (y - y')| |k|
\]

\[
\leq (1 + \alpha_N) (|\gamma(\tau)|^{-\alpha} + |\gamma(\tau)|^{\alpha_N}) |y - y'| |k|
\]

\[
\leq (1 + \alpha_N) |\gamma(\tau)|^{-\alpha} (1 + |y| + |y'|)^{\alpha_N} |y - y'| |k|.
\]

Thus

\[
I_1 \leq \int_0^1 |f_1(\tau)|d\tau \leq (1 + \alpha_N) (1 + |y| + |y'|)^{\alpha_N} |y - y'| |k| \int_0^1 |\gamma(\tau)|^{-\alpha} d\tau.
\]

By using (2.12) yields

\[
I_1 \leq \frac{2(1 + \alpha_N)}{1-\alpha} (1 + |y| + |y'|)^{\alpha_N} |y - y'|^{1-\alpha} |k|,
\]

**Estimation of** \( I_2 \). We find the partial derivative of \( F(a, |\xi|) \) in \( a \). For \( i = -1,0,\ldots,N \), taking the partial derivative in \( a_i \), we find that

\[
\sum_{i=-1}^N F_{a_i}(a, |\xi|) = |\xi|^{-\alpha} + |\xi|^{\alpha_0} + \cdots + |\xi|^{\alpha_N} \leq N(|\xi|^{-\alpha} + 1 + |\xi|^{\alpha_N}) \quad \text{by (2.3)}.
\]
It is proved in Lemma 2.4 of [7] that,

\begin{align}
|f_2(\tau)| & \leq |F_a(b(\tau), |\gamma(\tau)|| |a - a'| |\gamma(\tau)||k| \\
& \leq N(|\gamma(\tau)|^{-\alpha} + 1 + |\gamma(\tau)|^{\alpha_N}) |a - a'| |\gamma(\tau)||k| \\
& \leq 3N(1 + |\gamma(\tau)|^{1+\alpha_N}) |a - a'| |k|. 
\end{align}

Using the estimate (2.6), we bound

\begin{align}
|f_2(\tau)| & \leq 3N(1 + |y| + |y'|)^{\alpha_N+1} |a - a'| |k|, 
\end{align}

and consequently,

\begin{align}
I_2 & \leq \int_0^1 |f_2(\tau)| d\tau \leq 3N(1 + |y| + |y'|)^{\alpha_N+1}|a - a'| |k|. 
\end{align}

Then (2.13) follows by combining (2.17), (2.18) and (2.22).

\textbf{Case 2:} The origin belongs to the segment connect \(y', y\). We replace \(y'\) by \(y' + y_\varepsilon\) so that \(0 \not\in [y' + y_\varepsilon, y]\) and \(y_\varepsilon \to 0\) as \(\varepsilon \to 0\). Apply the above inequality for \(y\) and \(y' + y_\varepsilon\), then let \(\varepsilon \to 0\).

(ii) If \(k = y - y'\) then

\begin{align}
f_1(\tau) & \geq F(b(\tau), |\gamma(\tau)||y - y'|^2 - \alpha F(b(\tau), |\gamma(\tau)|||\gamma(\tau)\cdot(y - y')|^2 |\gamma(\tau)|^2 \\
& \geq (1 - \alpha)F(b(\tau), |\gamma(\tau)||y - y'|^2. 
\end{align}

It follows (2.23) and (2.12) that

\begin{align}
I_1 & \geq (1 - \alpha)|y' - y|^2 \int_0^1 F(b(\tau), |\gamma(\tau)||d\tau \\
& \geq a_s(1 - \alpha)|y' - y|^2 \int_0^1 (|\gamma(\tau)|^{-\alpha} + 1 + |\gamma(\tau)|^{\alpha_N}) d\tau \\
& > a_s(1 - \alpha)|y' - y|^2 \int_0^1 |\gamma(\tau)|^{\alpha_N} d\tau. 
\end{align}

It is proved in Lemma 2.4 of [7] that,

\begin{align}
\int_0^1 |\gamma(\tau)|^{\alpha_N} d\tau \geq \frac{|y' - y|^{\alpha_N}}{2^{\alpha_N+1}(\alpha_N + 1)}. 
\end{align}

Substituting this into (2.24) gives

\begin{align}
I_1 & \geq \frac{a_s(1 - \alpha)}{2^{\alpha_N+1}(\alpha_N + 1)}|y' - y|^{\alpha_N+2}. 
\end{align}

On the other hand from (2.20), we see that

\begin{align}
I_2 & \geq - \int_0^1 |f_2(\tau)| d\tau \geq -3N(1 + |y| + |y'|)^{\alpha_N+1}|a - a'| |y - y'|. 
\end{align}

Thus, we obtain (2.14) by combining (2.17), (2.24) and (2.27). \(\square\)

We recall a discrete version of Gronwall Lemma in backward difference form, which is useful later.

\textbf{Lemma 2.4 (Lemma 2.4 in [24]):} Assume the nonnegative sequences \(\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty, \{g_n\}_{n=0}^\infty\) satisfying

\begin{align}
\frac{a_n - a_{n-1}}{\Delta t} - a_n + b_n \leq g_n, \quad n = 1, 2, 3 \ldots
\end{align}

then for sufficiently small \(\Delta t\),

\begin{align}
a_n + \Delta t \sum_{i=1}^n b_i \leq e^{\frac{\Delta t}{\Delta t}} \left( a_0 + \Delta t \sum_{i=1}^n g_i \right). 
\end{align}
3. **A mixed finite element method and mixed finite element approximation.** In this section we will represent the weak formulation and mixed element approximation of the mixed regime equation. We consider the following model problem

\[
F(|\mathbf{m}(x,t)|)\mathbf{m}(x,t) = -\nabla \rho(x,t) \quad (x,t) \in \Omega \times I,
\]

\[
\phi(x)\rho_t(x,t) + \nabla \cdot \mathbf{m}(x,t) = f(x,t) \quad (x,t) \in \Omega \times I,
\]

\[
\rho(x,t) = \psi(x,t) \quad (x,t) \in \partial\Omega \times I,
\]

\[
\rho(x,0) = \rho_0(x) \quad x \in \Omega.
\]

We make the following assumptions on the data and coefficients

(H1) The coefficients \(a_{-1}, \ldots, a_N \in W^{1,\infty}(I, L^\infty(\Omega))\) satisfy

\[
0 < a_s < a_{-1}(x,t), a_0(x,t), a_N(x,t) < a^* < \infty, 0 \leq a_i(x,t) \leq a^* < \infty, i = 1, \ldots, N - 1
\]

for almost every \((x,t) \in \tilde{\Omega} \times I\).

(H2) \(0 < \phi_s \leq \phi(x) \leq \phi^* < \infty; f \in L^\infty(I, L^2(\Omega)); \psi \in C^1(\tilde{\Omega} \times I)\).

Dealing with the boundary condition, let \(\Psi(x,t)\) be an extension of \(\psi\) from \(\partial\Omega \times I\) to \(\tilde{\Omega} \times I\) (see e.g \[14, 22\]). Let \(\bar{\rho} = \rho - \Psi\). Then

\[
F(|\mathbf{m}(x,t)|)\mathbf{m}(x,t) = -\nabla \bar{\rho}(x,t) - \nabla \Psi(x,t) \quad (x,t) \in \Omega \times I,
\]

\[
\phi(x)\bar{\rho}_t(x,t) + \nabla \cdot \mathbf{m}(x,t) = f(x,t) - \phi(x)\Psi_t(x,t) \quad (x,t) \in \Omega \times I,
\]

\[
\bar{\rho}(x,t) = 0 \quad (x,t) \in \partial\Omega \times I,
\]

\[
\bar{\rho}(x,0) = \rho_0(x) - \Psi(x,0) \quad x \in \Omega.
\]

Define \(Q = L^2(\Omega)\), and \(V = \{v \in (L^2(\Omega))^d; \nabla \cdot v \in L^2(\Omega)\}\) and equip it with the norm

\[
\|v\|_V = \|v\|_{L^2(\Omega)} + \|\nabla \cdot v\|_{L^2(\Omega)}.
\]

The mixed formulation of (3.1) is defined as follows: Find \((\mathbf{m}, \bar{\rho}) : I \rightarrow V \times Q\) such that

\[
(F(|\mathbf{m}|)\mathbf{m}, v) - (\bar{\rho}, \nabla \cdot v) = -(\nabla \Psi, v), \quad \text{for all } v \in V,
\]

\[
(\phi \bar{\rho}_t, q) + (\nabla \cdot \mathbf{m}, q) = (f, q) - (\phi \Psi_t, q), \quad \text{for all } q \in Q.
\]

Let \(\{\mathcal{R}_h\}_h\) be a family of globally quasi-uniform triangulation of \(\Omega\) with \(\max_{\tau \in \mathcal{R}_h} \text{diam } \tau \leq h\). Let \(k \geq 0\) be an integer we define

\[
Q_h = \{\rho_h \in C^0(\overline{\Omega}), \forall \tau \in \mathcal{R}_h; \rho_h|_\tau \in P_k(\tau)\},
\]

\[
V_h = \{\mathbf{m}_h \in (C^0(\overline{\Omega})))^d, \forall \tau \in \mathcal{R}_h; \mathbf{m}_h|_\tau \in P_k(\tau)\},
\]

with \(P_k(\tau)\) being the space of polynomial of degree at most \(k\) on the element \(\tau\). Let \(V_h \times Q_h\) be the mixed element spaces approximating the space \(V \times Q\).

For momentum, let \(\Pi : V \rightarrow V_h\) be the Raviart-Thomas projection \[33\], which satisfies

\[
(\nabla \cdot (\Pi \mathbf{m} - \mathbf{m}), q) = 0, \quad \text{for all } \mathbf{m} \in V, q \in Q_h.
\]

For density, we use the standard \(L^2\)-projection operator, see in \[10\], \(\pi : Q \rightarrow Q_h\), satisfying

\[
(\pi \rho - \rho_0, q) = 0, \quad \text{for all } \rho \in Q, q \in Q_h,
\]

\[
(\pi \rho - \rho, \nabla \cdot \mathbf{m}_h) = 0, \quad \text{for all } \mathbf{m}_h \in V_h, \rho \in Q.
\]
This projection has well-known approximation properties, e.g. \cite{6,23}.

\begin{align}
\|\Pi m\|_{L^2(\Omega)} & \leq C \left( \|m\|_{L_p(\Omega)} + h \|\nabla \cdot m\|_{L^2(\Omega)} \right), \quad \forall m \in V \cap (W^{1,p}(\Omega))^d. \\
\|\Pi m - m\|_{L^q(\Omega)} & \leq C h^p \|m\|_{W^{p,q}(\Omega)}, \quad 1/q < p \leq k + 1, \forall m \in V \cap (W^{p,q}(\Omega))^d.
\end{align}

(3.5)

\begin{align}
\|\nabla \cdot (\Pi m - m)\|_{L^2(\Omega)} & \leq C h^p \|\nabla \cdot m\|_{W^{p-2}(\Omega)}, \quad 0 \leq p \leq k + 1, \forall m \in V \cap (H^p(\text{div}, \Omega))^d.
\end{align}

(3.6)

\begin{align}
\|\pi\rho - \rho\|_{L^q(\Omega)} & \leq C h^p \|\rho\|_{W^{p,q}(\Omega)}, \quad 0 \leq p \leq k + 1, q \in [1, \infty], \forall \rho \in W^{p,q}(\Omega).
\end{align}

(3.7)

The two projections \(\pi\) and \(\Pi\) preserve the commuting property \(\text{div} \circ \Pi = \pi \circ \text{div} : V \rightarrow Q_h\).

The semidiscrete formulation of (3.4) can read as follows: Find a pair \((m_h, \rho_h) : I \rightarrow V_h \times Q_h\) such that

\begin{align}
(F(m_h), v) - (\tilde{\rho}_h, \nabla \cdot v) = -(\nabla \Psi, v) & \quad \text{for all } v \in V_h, \\
(\phi \rho_h, q) + (\nabla \cdot m_h, q) = (f, q) - (\phi \Psi, q) & \quad \text{for all } q \in Q_h.
\end{align}

(3.10)

with initial data \(\tilde{\rho}_{h0} = \pi \tilde{\rho}(x, 0)\).

4. Stability. We study the equations (3.4), and (3.10) for the density with fixed functions \(F(s)\) in (1.17) and (1.18). Therefore, the exponents \(\alpha_i\) and coefficients \(d_i\) are all fixed, and so are the functions \(F(\xi)\) in (1.18). With the properties (2.5), (2.6), (2.15), the monotonicity (2.16), and by classical theory of monotone operators \cite{31,36,44}, the authors in \cite{25,27,29} proved the global existence and uniqueness of the weak solution of the problem (3.10).

4.1. A priori estimates for the solutions of the semi-discrete problems. For the \textit{a priori} estimates, we consider weak solutions with enough regularities in both \(x\) and \(t\) variables, but not necessarily classical, so that our calculations can be applied.

We will focus estimates for \(\tilde{\rho}_h\). The estimates for \(\rho_h\) can be obtained by simply using triangle inequality \(\|\rho_h(t)\| \leq \|\tilde{\rho}(t)\| + \|\Psi(t)\|\). Also our results are stated in term of \(\Psi(x, t)\). These can be written in term of \(\psi(x, t)\) by using specific extension. e.g. harmonic extension in \cite{14}.

**Lemma 4.1.** Let \((m_h, \rho_h)\) be a solution to the problem (3.10). There exists a positive constant \(C\) independence of \(h\) such that

\begin{align}
\|\tilde{\rho}_h\|_{L^2(I; L^2(\Omega))}^2 + \|m_h\|_{L^2(I; L^2(\Omega))}^2 & \leq \|\tilde{\rho}(0)\|_{L^2(\Omega)}^2 + C s\mathcal{A},
\end{align}

where

\begin{align}
\mathcal{A} = \|f\|_{L^2(I; L^2(\Omega))}^2 + \|\nabla \Psi\|_{L^1(I; L^2(\Omega))^d}^2 + \|\nabla \Psi\|_{L^2(I; L^1(\Omega))}^2.
\end{align}

(4.1)

(4.2)

**Proof.** Choosing \((v, q) = (m_h, \tilde{\rho}_h)\) in (3.10) and adding the resultant equations yield

\begin{align}
\frac{1}{2} \frac{d}{dt} \|\tilde{\rho}_h\|_\phi^2 + (F(m_h), m_h) = (f, \tilde{\rho}_h) - (\phi \Psi_t, \tilde{\rho}_h) - (\nabla \Psi, m_h).
\end{align}

(4.3)

By the monotonicity of the function \(F(\cdot)\) in (2.16), the second term of (4.3) is bounded from below by

\begin{align}
(F(m_h), m_h) \geq C_3 \|m_h\|_{L^2}^2.
\end{align}

(4.4)

We bound the right hand side of (4.3) by using Young’s inequality to obtain

\begin{align}
(f, \tilde{\rho}_h) - (\phi \Psi_t, \tilde{\rho}_h) - (\nabla \Psi, m_h) \leq \|f\|_{L^2}^2 + \|\Psi_t\|_\phi^2 + \frac{1}{2} \|\tilde{\rho}_h\|_\phi^2 + C_3 \|m_h\|_{L^2}^2 + \frac{(sC_3/2)^{-s'/s}}{s'} \|\nabla \Psi\|_{L^{s'}}^2.
\end{align}

(4.5)

Combining (4.3), (4.4) and (4.5) yields

\begin{align}
\frac{d}{dt} \|\tilde{\rho}_h\|_\phi^2 + C_3 \|m_h\|_{L^2}^2 \leq \|\tilde{\rho}_h\|_\phi^2 + C \left( \|f\|_{L^2}^2 + \|\Psi_t\|_\phi^2 + \|\nabla \Psi\|_{L^2}^2 \right).
\end{align}
By Gronwall’s inequality and using the boundedness of the function $\phi$, we find that
\[
\|\bar{\rho}_h\|_{L^2(I,L^2)}^2 + \|\mathbf{m}_h\|_{L^2(I,L^2)}^2 \leq \|\bar{\rho}_h(0)\|_{L^2}^2 + C \left( \|f\|_{L^2(I,L^2)}^2 + \|\nabla\Psi\|_{L^2(I,L^2)}^2 \right).
\]
Note that $\|\bar{\rho}_h(0)\| = \|\pi \bar{\rho}(0)\| \leq \|\bar{\rho}_0\|$. Thus the inequality (4.1) holds.

**Theorem 4.2.** There is a unique solution of the problem (3.10) satisfying (4.1).

**Proof.** The equation (3.10) can be interpreted as the finite system of ordinary differential equations in the coefficients of $(\mathbf{m}_h, \bar{\rho}_h)$ with respect to basis of $V_h \times Q_h$. The stability estimates (4.1) suffice to establish the local existence of $(\mathbf{m}_h(t), \bar{\rho}_h(t))$ for all $t \in I$. The proof of this statement is essential identical with that of [28, 34] for generalized Forchheimer flows. We will omit here.

Assume that $(\mathbf{m}_h^{(i)}, \bar{\rho}_h^{(i)}), i = 1, 2$ are two solutions of (3.10). Let $\mathbf{m}_h = \mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}$, $\bar{\rho}_h = \bar{\rho}_h^{(1)} - \bar{\rho}_h^{(2)}$. Then
\[
\left(F(\mathbf{m}_h^{(1)}), \mathbf{m}_h^{(1)} - F(\mathbf{m}_h^{(2)}), \mathbf{v}\right) - (\bar{\rho}_h, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in V_h,
\]
\[
(\bar{\rho}_h t, q) + (\nabla \cdot \mathbf{m}_h, q) = 0, \quad \forall q \in Q_h.
\]

It is easy to see that with $(\mathbf{v}, q) = (\mathbf{m}_h, \bar{\rho}_h)$ in (4.6)
\[
(\bar{\rho}_h t, q) + \left(F(\mathbf{m}_h^{(1)}), \mathbf{m}_h^{(1)} - F(\mathbf{m}_h^{(2)}), \mathbf{m}_h\right) = 0.
\]

Thanks to the monotonicity (2.16), we see that
\[
\frac{1}{2} \frac{d}{dt} \|\bar{\rho}_h\|^2_{\phi} + C_3 \|\mathbf{m}_h\|^2_{L^2} \leq (\bar{\rho}_h t, \bar{\rho}_h) + \left(F(\mathbf{m}_h^{(1)}), \mathbf{m}_h^{(1)} - F(\mathbf{m}_h^{(2)}), \mathbf{m}_h\right) = 0.
\]

This implies $\|\bar{\rho}_h\|^2_{\phi} + \|\mathbf{m}_h\|^2_{L^2(I,L^2)} \leq C \|\bar{\rho}_h(0)\|^2_{\phi} = 0$. Hence $\bar{\rho}_h = 0$ and $\mathbf{m}_h = 0$ a.e. 

**Lemma 4.3.** Let $(\mathbf{m}_h, \bar{\rho}_h)$ be a solution to the problem (3.10). Then, there exists a positive constant $C$ independence of $h$ such that
\[
\|\bar{\rho}_h\|^2_{L^2(I,L^2(\Omega))} + \|\mathbf{m}_h\|^2_{L^2(I,L^2(\Omega))} + \|\nabla \mathbf{m}_h\|^2_{L^2(I,L^2(\Omega))} \leq C(\mathcal{B}_0 + \mathcal{B}),
\]
where
\[
\mathcal{B}_0 = \|\bar{\rho}_0\|^2_{L^2(\Omega)} + \bar{\rho}_1(0)\|^2_{L^2(\Omega)}, \quad \text{and} \quad \mathcal{B} = \mathcal{A} + \|f\|^2_{L^2(\Omega)} + \|\Psi_t\|^2_{L^2(\Omega)} + \|\nabla\Psi_t\|^2_{L^2(\Omega)}.
\]

**Proof.** Differentiate (3.10) in time to see that
\[
\left(F(\mathbf{m}_h), \mathbf{m}_h + F'(\mathbf{m}_h)|\mathbf{m}_h|\mathbf{m}_h - \sum_{i=-1}^{N} F_0(|\mathbf{m}_h|)a_{i} \mathbf{m}_h, \mathbf{v}\right) - (\bar{\rho}_h, \nabla \cdot \mathbf{v}) = - (\nabla\Psi_t, \mathbf{v}),
\]
\[
(\bar{\rho}_h t, q) + (\nabla \cdot \mathbf{m}_h, q) = (f_t, q) - (\phi \Psi_t, q).
\]

Taking $(\mathbf{v}, q) = (\mathbf{m}_h, \bar{\rho}_h)$ and adding two resultant equations we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\bar{\rho}_h\|^2_{\phi} + \|F^{1/2}(\mathbf{m}_h)|\mathbf{m}_h\|^2_{L^2} = - \left(F'(\mathbf{m}_h)|\mathbf{m}_h|\mathbf{m}_h, \mathbf{m}_h\right)
\]
\[
- \sum_{i=-1}^{N} F_0(|\mathbf{m}_h|)a_{i} \mathbf{m}_h, \mathbf{m}_h\right) - (\nabla\Psi_t, \mathbf{m}_h) + (f_t, \bar{\rho}_h) - (\phi \Psi_t, \bar{\rho}_h).
\]

We estimate the right hand side term by term.
By \((2.5), (2.19)\), Hölder’s inequality and Young’s inequality we have

\[
- \left(F'(|m_h|) \frac{m_h \cdot m_{ht}}{|m_h|^2} m_h, m_{ht}\right) - \left(\sum_{\tau=1}^N F_{\tau}(|m_h|) a_{\tau h} m_h, m_{ht}\right) \leq \alpha \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 + C \left( F(|m_h|) m_h, |m_{ht}|\right) \\
\leq (\alpha + \epsilon) \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 + C \epsilon \left\| F^{1/2}(|m_h|) m_h\right\|_{L^2}^2 , \quad (4.11)
\]

and

\[
- (\nabla \Psi_t, m_{ht}) + (f_t, \hat{\rho}_{ht}) - (\phi \Psi_t, \hat{\rho}_{ht}) \leq \|f_t\|^2_{\phi} + \|\Psi_t\|^2_{\phi} + 1 \|\hat{\rho}_{ht}\|^2_{\phi} + \epsilon \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 + C \epsilon \left\| F^{1/2}(|m_h|) \nabla \Psi_t\right\|_{L^2}^2 . \quad (4.12)
\]

Comparing \((4.10), (4.11), (4.12)\) and taking \(\epsilon = (1 - \alpha)/4\) yield

\[
\frac{d}{dt} \|\hat{\rho}_{ht}\|^2_{\phi} + (1 - \alpha) \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 \\
\leq \|f_t\|^2_{\phi} + \|\Psi_t\|^2_{\phi} + 1 \|\hat{\rho}_{ht}\|^2_{\phi} + C \left( \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 + \left\| F^{1/2}(|m_h|) \nabla \Psi_t\right\|_{L^2}^2 \right) .
\]

By the virtue of \((2.6), (2.3)\), one has

\[
\left\| F^{1/2}(|m_h|) \nabla \Psi_t\right\|_{L^2}^2 + \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 \leq a^{-1}_t \left\| \nabla \Psi_t\right\|_{L^2}^2 + N a^* \int_{\Omega} (|m_h|^{-\alpha} + |m_h|^0) m_h^2 dx \\
\leq a^{-1}_t \left\| \nabla \Psi_t\right\|_{L^2}^2 + 2 N a^* \int_{\Omega} (1 + |m_h|^{0\alpha+2}) dx \\
\leq a^{-1}_t \left\| \nabla \Psi_t\right\|_{L^2}^2 + 2 N a^* (|\Omega| + 1) (1 + \|m_h\|_{L^2}) .
\]

It follows that

\[
\frac{d}{dt} \|\hat{\rho}_{ht}\|^2_{\phi} + \left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2}^2 \leq \frac{1}{2} \|\rho_{ht}\|^2_{\phi} + C \left( \|f_t\|^2_{\phi} + \|\Psi_t\|^2_{\phi} + \|m_h\|_{L^2}^2 + \left\| \nabla \Psi_t\right\|_{L^2}^2 \right) .
\]

By Gronwall’s inequality and the boundedness of \(\phi\) we find that

\[
\|\hat{\rho}_{ht}\|^2_{L^2(I,L^2)} + \left\| F^{1/2}(|m_h|) m\right\|_{L^2(I,L^2)}^2 \leq C \left( \|\hat{\rho}_{ht}(0)\|_{L^2}^2 + \|f_t\|_{L^2(I,L^2)}^2 + \|\Psi_t\|_{L^2(I,L^2)}^2 + \|m_h\|_{L^2(I,L^2)}^2 + \left\| \nabla \Psi_t\right\|_{L^2(I,L^2)}^2 \right) .
\]

Note that by \((2.6)\), one has

\[
\left\| F^{1/2}(|m_h|) m_{ht}\right\|_{L^2(I,L^2)}^2 \geq a_t \|m_{ht}\|_{L^2(I,L^2)}^2 .
\]

Combining this fact with estimate \((4.1)\), we obtain the first part of estimate \((4.8)\).

To verify the last part of \((4.8)\), we choose \(q = \nabla \cdot m_h\) in \((3.10)\) yielding

\[
\|\nabla \cdot m_h\|_{L^2}^2 = - (\phi \hat{\rho}_{ht}, \nabla \cdot m_h) + (f_t, \nabla \cdot m_h) - (\phi \Psi_t, \nabla \cdot m_h) .
\]

Then by Hölder’s inequality

\[
\|\nabla \cdot m_h\|_{L^2} \leq \|\phi \hat{\rho}_{ht}\|_{L^2} + \|f\|_{L^2} + \|\phi \Psi_t\|_{L^2} \leq (\phi^* + 1) (\|\hat{\rho}_{ht}\|_{L^2} + \|f\|_{L^2} + \|\Psi_t\|_{L^2}) ,
\]

which implies

\[
\|\nabla \cdot m_h\|_{L^2} \leq 2(\phi^* + 1)^2 (\|\hat{\rho}_{ht}\|_{L^2}^2 + \|f\|_{L^2}^2 + \|\Psi_t\|_{L^2}^2) .
\]
Using the first part of (4.8) to bound $\|\bar{\rho}_h\|_{L^2}^2$, we find that $\|\nabla \cdot \mathbf{m}_h\|_{L^2}^2$ holds (4.3). \[\square\]

In the same manner to the problem (3.34), we have as the following:

**Theorem 4.4.** Suppose $(\mathbf{m}, \bar{\rho})$ be a solution of the problem (3.34). Then, there exists a positive constant $C$ such that

$$
\|\bar{\rho}\|_{L^2(L^2(\Omega))}^2 + \|\mathbf{m}\|_{L^2(L^1(\Omega))}^2 \leq C \|\bar{\rho}_0\|_{L^2(\Omega)}^2 + C \mathcal{A},
$$

(4.13)

$$
\|\bar{\rho}_h\|_{L^2(L^2(\Omega))}^2 + \|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 \leq C \|\bar{\rho}_0\|_{L^2(\Omega)}^2 \leq C(\mathcal{B}_0 + \mathcal{B}),
$$

(4.14)

where $\mathcal{A}, \mathcal{B}_0, \mathcal{B}$ are defined in (4.2) and (4.9).

5. Dependence of solutions on parameters. In this section, we study the dependence of the solution on the coefficients of Forchheimer polynomial $F(s)$ in (1.18). Let $N \geq 1$, the exponent vector $\alpha = (\alpha_1, 0, \alpha_2, \ldots, \alpha_N)$ and the boundary data $\psi(x,t)$ be fixed. Let $D$ be a compact subset of $\{\mathbf{a} = (a_1, 0, a_2, \ldots, a_N) : 0 < a_1 \leq a_2, a_0, a_N \leq a^*, 0 \leq a_1, \ldots, a_{N-1} \leq a^*\}$. Let $F_1(y) = F(a_1, y)$ and $F_2(y) = F(a_2, y)$ be two functions of class $P(N, \alpha)$, where $a_1$ and $a_2$ belong to $D$. Let $\bar{\rho}_h^{(1)} = \bar{\rho}_h^{(1)}(x,t; a_1), \bar{\rho}_h^{(2)} = \bar{\rho}_h^{(2)}(x,t; a_2)$ be the two solutions of (3.10) respective to $F(a_1, y), F(a_2, y)$ with the same boundary data $\psi$ and initial data $\rho_0$. We will estimate $\|\bar{\rho}_h^{(1)} - \bar{\rho}_h^{(2)}\|_{L^2(L^2(\Omega))}^2, \|\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}\|_{L^2(L^2(\Omega))}^2$ in the term of $\|\mathbf{a}_1 - \mathbf{a}_2\|_{L^\infty(L^\infty(\Omega))}$.

Thus, there exists a positive constant $C$ such that

$$
\|\bar{\rho}_h^{(1)} - \bar{\rho}_h^{(2)}\|_{L^2(L^2(\Omega))}^2 + \|\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}\|_{L^2(L^2(\Omega))}^2 \leq C \|\mathbf{a}_1 - \mathbf{a}_2\|_{L^\infty(L^\infty(\Omega))}^2.
$$

Proof. Choosing $(v,q) = (\mathbf{m}_h, \bar{\rho}_h)$ in (5.1), adding the resultant equations, we find that

$$
\frac{1}{2} \frac{d}{dt} \|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 + \frac{\alpha_1 - \alpha_2}{2} \|\nabla \cdot \mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 = 0.
$$

(5.2)

According to (2.14),

$$
\frac{1}{2} \frac{d}{dt} \|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 \leq -C_3 \|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 C_4 \int_{\Omega} |\mathbf{a}_1 - \mathbf{a}_2| (1 + |\mathbf{m}_h^{(1)}| + |\mathbf{m}_h^{(2)}|)^{\alpha_2-1} |\mathbf{m}_h| dx.
$$

(5.3)

Then by the Young’s inequality

$$
\frac{d}{dt} \|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 \leq C \int_{\Omega} |\mathbf{a}_1 - \mathbf{a}_2|^{\alpha_2} (1 + |\mathbf{m}_h^{(1)}| + |\mathbf{m}_h^{(2)}|)^{\alpha_2-1} \mathbf{m}_h dx.
$$

Integrating in $t$ gives

$$
\|\mathbf{m}_h\|_{L^2(L^2(\Omega))}^2 \leq C \int_{\Omega} |\mathbf{a}_1 - \mathbf{a}_2|^{\alpha_2} (1 + |\mathbf{m}_h^{(1)}| + |\mathbf{m}_h^{(2)}|)^{\alpha_2-1} \mathbf{m}_h^2 \mathbf{m}_h^2 dx.
$$

Using (4.11), the result (5.2) follows. The proof is complete. \[\square\]

In the same manner to the problem (3.34), we have as the following:

**Theorem 5.2.** Let $(\mathbf{m}_h^{(i)}, \bar{\rho}_h^{(i)}), i = 1, 2$ be two solutions to problem (3.34) corresponding to vector coefficients $\mathbf{a}_i$ of polynomial $F(a_i, y)$ in (1.18). There exists a positive constant $C > 0$ such that

$$
\|\bar{\rho}_h^{(1)} - \bar{\rho}_h^{(2)}\|_{L^2(L^2(\Omega))}^2 + \|\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}\|_{L^2(L^2(\Omega))}^2 \leq C \|\mathbf{a}_1 - \mathbf{a}_2\|_{L^\infty(L^\infty(\Omega))}^2.
$$

(5.4)
6. Error estimates for semidiscrete approximation. In this section, we will give the error estimate between the analytical solution and approximate solution. We define the new variables:

\[ \mathbf{m} - \mathbf{m}_h = \mathbf{m} - \Pi \mathbf{m} - (\mathbf{m}_h - \Pi \mathbf{m}) = \eta - \zeta_h, \]
\[ \bar{\rho} - \bar{\rho}_h = \bar{\rho} - \pi \bar{\rho} - (\bar{\rho}_h - \pi \bar{\rho}) = \theta - \vartheta_h. \]

THEOREM 6.1. Let \((\mathbf{m}, \bar{\rho})\) be the solution of (3.4) and \((\mathbf{m}_h, \bar{\rho}_h)\) be the solution of (3.10). Suppose that \((\mathbf{m}, \bar{\rho}) \in V \times Q\), and \(\bar{\rho}_h \in L^2(I, L^2(\Omega))\). Then there exists a positive constant \(C\) independence of \(h\) such that

\[ \| \bar{\rho} - \bar{\rho}_h \|_{L^2(I, L^2(\Omega))} + \| \mathbf{m} - \mathbf{m}_h \|_{L^2(I, L^2(\Omega))} \leq C h^\nu. \]

Proof. Error equations

\[ (F(|\mathbf{m}|)\mathbf{m} - F(|\mathbf{m}_h|)\mathbf{m}_h, \mathbf{v}) - (\bar{\rho} - \bar{\rho}_h, \nabla \cdot \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in V_h, \]
\[ (\phi(\bar{\rho} - \bar{\rho}_h), q) + (\nabla \cdot (\mathbf{m} - \mathbf{m}_h), q) = 0 \quad \text{for all } q \in Q_h. \]

Using \(L^2\)-project and Raviar-Thomas projection we rewrite equation as form

\[ (F(|\mathbf{m}|)\mathbf{m} - F(|\Pi \mathbf{m}|)\Pi \mathbf{m}, \mathbf{v}) + (F(|\Pi \mathbf{m}|)\Pi \mathbf{m} - F(|\mathbf{m}_h|)\mathbf{m}_h, \mathbf{v}) + (\bar{\rho}_h, \nabla \cdot \mathbf{v}) = 0, \]
\[ (\phi(\bar{\rho}_h), q) + (\phi \bar{\theta}_h, q) - (\nabla \cdot \zeta_h, q) = 0. \]

Take \(q = -\vartheta_h \in Q_h\) and \(\mathbf{v} = -\zeta_h \in V_h\). Add these two equations together

\[ (\phi \bar{\theta}_h, \vartheta_h) + (\phi(\Pi \mathbf{m})\Pi \mathbf{m} - F(|\mathbf{m}_h|)\mathbf{m}_h, \Pi \mathbf{m} - \mathbf{m}_h) = (F(|\mathbf{m}|)\mathbf{m} - F(|\Pi \mathbf{m}|)\Pi \mathbf{m}, \zeta_h) + (\phi \theta_h, \vartheta_h). \]

Then by (2.16),

\[ \frac{1}{2} \frac{d}{dt} \left\| \vartheta_h \right\|_\phi^2 + C_3 \left\| \zeta_h \right\|_{L^r}^r \leq (F(|\mathbf{m}|)\mathbf{m} - F(|\Pi \mathbf{m}|)\Pi \mathbf{m}, \zeta_h) + (\phi \theta_h, \vartheta_h). \]

Using Young’s and Hölder’s inequality we find that

\[ (\phi \theta_h, \vartheta_h) \leq (4\varepsilon)^{-1} \left\| \theta_h \right\|_\phi^2 + \varepsilon \left\| \vartheta_h \right\|_\phi^2, \]

and

\[ (F(|\mathbf{m}|)\mathbf{m} - F(|\Pi \mathbf{m}|)\Pi \mathbf{m}, \zeta_h) \leq (F(|\mathbf{m}|)\mathbf{m} - F(|\Pi \mathbf{m}|)\Pi \mathbf{m}, |\zeta_h|) \]
\[ \leq C_1 \left( (1 + |\Pi \mathbf{m}| + |\mathbf{m}|)^{r+2-\alpha} |\eta|^{1-\alpha} |\zeta_h| \right) \quad \text{by (2.15)} \]
\[ \leq \varepsilon \left\| \zeta_h \right\|_{L^r}^r + C\varepsilon \left( 1 + |\Pi \mathbf{m}| + |\mathbf{m}| \right)^{r+2-\alpha} \left( |\eta|^{1-\alpha} \right)^{\frac{1}{r+1}} \left\| \eta \right\|_{L^{r+1}}^\nu. \]

Choose \(\varepsilon = C_3/2\), we find that

\[ \frac{d}{dt} \left\| \vartheta_h \right\|_\phi^2 + C_3 \left\| \zeta_h \right\|_{L^r}^r \leq \varepsilon \left\| \vartheta_h \right\|_\phi^2 + C \left( 1 + |\Pi \mathbf{m}| + |\mathbf{m}| \right)^{\frac{r+2-\alpha}{r+1}} \left\| \eta \right\|_{L^{r+1}}^{\nu (1-\alpha)} + (4\varepsilon)^{-1} \left\| \theta_h \right\|_\phi^2. \]

Integrating in time from 0 to \(T\) and take sup-norm

\[ \sup_{t \in [0,T]} \left\| \vartheta_h \right\|_\phi^2 + \left\| \zeta_h \right\|_{L^2(I, L^2)} \leq \varepsilon T \sup_{t \in [0,T]} \left\| \vartheta_h \right\|_\phi^2 + (4\varepsilon)^{-1} \int_0^T \left\| \theta_h \right\|_\phi^2 dt \]
\[ + C \int_0^T \left( 1 + |\Pi \mathbf{m}| + |\mathbf{m}| \right)^{\frac{r+2-\alpha}{r+1}} \left\| \eta \right\|_{L^{r+1}}^{\nu (1-\alpha)} dt. \]
Now take $\varepsilon = 1/(2T)$, we find that
\[
\sup_{t \in [0,T]} \left\| \tilde{\theta}_h \right\|_{2, h}^2 + \left\| \tilde{\xi}_h \right\|_{2, h}^2 \leq T \frac{2}{3} \int_0^T \left\| \theta_h \right\|_{h}^2 dt + C \int_0^T \left(1 + \left\| \Pi m \right\|_{h}^2 + \left\| \Pi m \right\|_{h}^s \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt.
\]
Thus
\[
\left\| \tilde{\theta}_h \right\|_{L^2(I, L^2)} + \left\| \tilde{\xi}_h \right\|_{L^2(I, L^2)} \leq T \frac{2}{3} \int_0^T \left\| \theta_h \right\|_{h}^2 dt + C \int_0^T \left(1 + \left\| \Pi m \right\|_{h}^2 + \left\| \Pi m \right\|_{h}^s \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt.
\]
Using (2.4), (6.5) we obtain
\[
\left\| \tilde{\rho} - \tilde{\rho}_h \right\|_{L^2(I, L^2)} + \left\| m - m_h \right\|_{L^2(I, L^2)} \leq 2 \left( \left\| \theta \right\|_{L^2(I, L^2)} + \left\| \tilde{\theta}_h \right\|_{L^2(I, L^2)} \right) + 2 \left( \left\| \eta \right\|_{L^2(I, L^2)} + \left\| \tilde{\xi}_h \right\|_{L^2(I, L^2)} \right)
\]
\[
\leq 2 \left( \left\| \theta \right\|_{L^2(I, L^2)} + \left\| \eta \right\|_{L^2(I, L^2)} + \left\| \tilde{\theta}_h \right\|_{L^2(I, L^2)} + \left\| \tilde{\xi}_h \right\|_{L^2(I, L^2)} \right)
\]
\[
\leq 2 \left( \left\| \theta \right\|_{L^2(I, L^2)} + \left\| \eta \right\|_{L^2(I, L^2)} + T \int_0^T \left\| \theta_h \right\|_{h}^2 dt + C \int_0^T \left(1 + \left\| \Pi m \right\|_{h}^2 + \left\| \Pi m \right\|_{h}^s \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt \right).
\]
Applying estimates (3.6) and (3.9) implies that
\[
\left\| \tilde{\rho} - \tilde{\rho}_h \right\|_{L^2(I, L^2)} + \left\| m - m_h \right\|_{L^2(I, L^2)} \leq C h^2 \left( \left\| \tilde{\rho} \right\|_{L^2(I, L^2)} + \left\| \tilde{\rho}_h \right\|_{L^2(I, L^2)} \right)
\]
\[
+ C h^s \left\| \Pi m \right\|_{L^2(I, W^{1,s})} + C \left( \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{W^{1,s}} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt.
\]
Note that $\frac{s^2}{2} + s^2 (1 - \alpha) = s$,
\[
\left(1 + \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)}
\]
\[
\leq C \left(1 + \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} \quad \text{(Young's inequality)}
\]
\[
\leq C \left(1 + \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} \quad \text{(By (3.5))}
\]
\[
\leq C \left(1 + \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} \right). \quad \text{(By (3.5))}
\]

Thus
\[
\left\| \tilde{\rho} - \tilde{\rho}_h \right\|_{L^2(I, L^2)} + \left\| m - m_h \right\|_{L^2(I, L^2)} \leq C h^2 \left( \left\| \tilde{\rho} \right\|_{L^2(I, L^2)} + \left\| \tilde{\rho}_h \right\|_{L^2(I, L^2)} \right)
\]
\[
+ C h^s \left\| \Pi m \right\|_{L^2(I, W^{1,s})} + C \left( \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt.
\]
Since $s^2 (1 - \alpha) < 2 < s$,
\[
\left\| \tilde{\rho} - \tilde{\rho}_h \right\|_{L^2(I, L^2)} + \left\| m - m_h \right\|_{L^2(I, L^2)} \leq C \left(1 + \left\| \tilde{\rho} \right\|_{L^2(I, L^2)} + \left\| \tilde{\rho}_h \right\|_{L^2(I, L^2)} \right)
\]
\[
+ \left( \left\| \Pi m \right\|_{L^2(I, L^2)} + \left\| \Pi m \right\|_{L^2(I, W^{1,s})} + \left\| \Pi m \right\|_{L^2(I, L^2)} \right) \frac{s^2}{2} \left\| \eta \right\|_{L^2}^{s(1-\alpha)} dt.
\]

The proof is concluded. \( \square \)

**Theorem 6.2.** Let $(m^{(i)}_h, \rho^{(i)}_h)$, $i = 1, 2$ be two solutions to problems (3.10) corresponding to vector coefficients $a_i$ of generalized polynomials $F(a, s)$ in (1.18). Suppose that $(m^{(i)}_h, \tilde{\rho}^{(i)}_h) \in V \times Q$ and $\tilde{\rho}^{(i)}_h \in L^2(I, L^2(\Omega))$. Then, there exists a constant positive constant $C$ independent of $h$ and $\|a_1 - a_2\|_{L^2(I,L^2)}$ such that
\[
\left\| \rho^{(1)}_h - \rho^{(2)}_h \right\|_{L^2(I, L^2)} + \left\| m^{(1)}_h - m^{(2)}_h \right\|_{L^2(I, L^2)} \leq C \left( \left\| \rho^{(1)}_h \right\|_{L^2(I, L^2)} + \left\| \rho^{(2)}_h \right\|_{L^2(I, L^2)} \right).
\]
Proof. The triangle inequality shows that
\[
\|\hat{\rho}_h^{(1)} - \hat{\rho}_h^{(2)}\|^2_{L^2(I, L^2)} + \|m_h^{(1)} - m_h^{(2)}\|^2_{L^2(I, L^2)} \leq C \sum_{i=1}^{n} \left( \|\hat{\rho}_h^{(i)} - \bar{\rho}^{(i)}\|^2_{L^2(I, L^2)} + \|m_h^{(i)} - m_h^{(i)}\|^2_{L^2(I, L^2)} \right) + C \left( \|\hat{\rho}_h^{(1)} - \hat{\rho}_h^{(2)}\|^2_{L^2(I, L^2)} + \|m_h^{(1)} - m_h^{(2)}\|^2_{L^2(I, L^2)} \right).
\]

Then by using (6.2) to treat the sum-term and (5.4) to the last terms we obtain (6.6). □

7. Fully discrete method. Let \( \{t_n\}_{n=0}^{M} \) be the uniform partition of \([0, T]\) with \( t_n = n\Delta t \), for time step \( \Delta t > 0 \). We define \( \varphi_n = \varphi(\cdot, t_n) \). The discrete time mixed finite element approximation to (3.10) is defined as follows:

For given \( \bar{\rho}_{0, n}(x) = \pi \rho_{0}(x) \) and \( \{f_n\}_{n=1}^{M} \in L^2(\Omega) \), \( \{\Psi_n\}_{n=1}^{M} \in C(\Omega) \). Find a pair \( (m_{hn}, \rho_{hn}) \) in \( V_h \times Q_h \), \( n = 0, 1, 2, \ldots, M \) such that

\[
(F(|m_{hn}|)m_{hn}, v) - (\rho_{hn}, \nabla \cdot v) = - (\nabla \Psi_n, v) \quad \text{for all } v \in V_h,
\]

\[
(\phi \frac{\rho_{hn} - \rho_{hn-1}}{\Delta t}, q) + (\nabla \cdot m_{hn}, q) = (f_n, q) - (\phi \Psi_n, q) \quad \text{for all } q \in Q_h,
\]

for \( \bar{\rho}_{0, n}(x) = \pi \rho_{0}(x) \).

Lemma 7.1 (Stability). Let \( (m_{hn}, \rho_{hn}) \) solve the fully discrete finite element approximation (7.1) for each time step \( n = 1, 2, \ldots, M \). There exists a positive constant \( C \) independent of \( n, \Delta t \) such that for \( \Delta t \) sufficiently small

\[
\|\rho_{hn}\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \Delta t \|m_{hn}\|^2_{L^2(\Omega)} \leq C \left( \|\bar{\rho}_{0}\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \Delta t (\|f_i\|^2_{L^2(\Omega)} + \|\Psi_i\|^2_{L^2(\Omega)} + \|\nabla \Psi_n\|^2_{L^2(\Omega)}) \right).
\]

Proof. Selecting \( (v, q) = 2(m_{hn}, \rho_{hn}) \) in (7.1), we find that

\[
2 (F(|m_{hn}|)m_{hn}, m_{hn}) - 2 (\rho_{hn}, \nabla \cdot m_{hn}) = -2 (\nabla \Psi_n, m_{hn}),
\]

\[
2 \left( \phi \frac{\rho_{hn} - \rho_{hn-1}}{\Delta t}, \rho_{hn} \right) + 2 (\nabla \cdot m_{hn}, \rho_{hn}) = 2 (f_n, \rho_{hn}) - 2 (\phi \Psi_n, \rho_{hn}).
\]

Adding the two above equations, and using the identity

\[
2 \left( \phi (\rho_{hn} - \rho_{hn-1}), \rho_{hn} \right) = \|\rho_{hn}\|^2_{\phi} - \|\rho_{hn-1}\|^2_{\phi} + \|\rho_{hn} - \rho_{hn-1}\|^2_{\phi} ,
\]

we obtain

\[
\|\rho_{hn}\|^2_{\phi} - \|\rho_{hn-1}\|^2_{\phi} + \|\rho_{hn} - \rho_{hn-1}\|^2_{\phi} + 2 \Delta t (F(|m_{hn}|)m_{hn}, m_{hn}) = 2 \Delta t \left( (f_n, \rho_{hn}) - (\phi \Psi_n, \rho_{hn}) - (\nabla \Psi_n, m_{hn}) \right).
\]

It follows from (2.16) that

\[
(F(|m_{hn}|)m_{hn}, m_{hn}) \geq C_3 \|m_{hn}\|^2_{L^2}.
\]

Using Hölder’s inequality to the RHS of (7.4) shows that

\[
(F(|f_n|)m_{hn}, m_{hn}) \leq \|f_n\|^2_{\phi} + \|\Psi_n\|^2_{\phi} + \frac{1}{2} \|\rho_{hn}\|^2_{\phi} + \frac{C_1}{2} \|m_{hn}\|^2_{L^2} + \frac{(sC_3/2)^{s'}/s}{s'^{s'}} \|\nabla \Psi_n\|^s_{L^{s'}}.
\]
Combining (7.4) and (7.6) yields
\[ \frac{\| \bar{\rho}_n \|_{\phi}^2 - \| \bar{\rho}_{n-1} \|_{\phi}^2 + C_3 \Delta t \| \mathbf{m}_{hn} \|_{L^2}^2}{\Delta t} \leq \Delta t \left( \frac{\| f_n \|_{\phi}^2 + \| \mathbf{\Psi}_n \|_{L^2}^2 + \| \nabla \mathbf{\Psi}_n \|_{L^2}^2}{\| \bar{\rho}_n \|_{\phi}^2 + \| \mathbf{m}_{hn} \|_{L^2}^2} \right), \]
which shows that
\[ \frac{\| \bar{\rho}_n \|_{\phi}^2 - \| \bar{\rho}_{n-1} \|_{\phi}^2 + C_3 \| \mathbf{m}_{hn} \|_{L^2}^2}{\Delta t} \leq C \left( \frac{\| f_n \|_{\phi}^2 + \| \mathbf{\Psi}_n \|_{L^2}^2 + \| \nabla \mathbf{\Psi}_n \|_{L^2}^2}{\| \bar{\rho}_n \|_{\phi}^2 + \| \mathbf{m}_{hn} \|_{L^2}^2} \right). \]

By discrete Gronwall’s inequality in Lemma 2.4
\[ \| \bar{\rho}_{hn} \|_{\phi}^2 + C_3 \sum_{i=1}^{n} \Delta t \| \mathbf{m}_{ih} \|_{L^2}^2 \leq C e^{\frac{n}{h_{\Delta}}} \| \bar{\rho}_{h0} \|_{\phi}^2 + C e^{\frac{n}{h_{\Delta}}} \sum_{i=1}^{n} \Delta t \left( \frac{\| f_i \|_{\phi}^2 + \| \mathbf{\Psi}_i \|_{L^2}^2 + \| \nabla \mathbf{\Psi}_i \|_{L^2}^2}{\| \bar{\rho}_i \|_{\phi}^2 + \| \mathbf{m}_{ih} \|_{L^2}^2} \right). \] (7.7)

Note that \( \| \bar{\rho}_{h0} \|_{\phi}^2 \leq \| \bar{\rho}_0 \|_{\phi}^2 \) and \( e^{\frac{n}{h_{\Delta}}} \leq e^{\frac{n}{h_{\Delta}}} = e^{\frac{n}{NT}} \) imply (7.2). The proof is complete. \( \square \)

**7.1. Error analysis.** As in the semidiscrete case, we use \( \eta = \mathbf{m} - \Pi \mathbf{m}, \zeta_t = \mathbf{m}_h - \Pi \mathbf{m}, \theta = \bar{\rho} - \bar{\rho}, \bar{\rho}_h = \bar{\rho}_h - \pi \bar{\rho} \) and \( \eta_{t}, \theta_{t}, \zeta_{th}, \theta_{th} \) be evaluating \( \eta, \theta, \zeta_t, \theta_t \) at the discrete time levels. We also define
\[ \partial \varphi_n = \frac{\varphi_n - \varphi_{n-1}}{\Delta t}. \]

**THEOREM 7.2.** Let \( (\mathbf{m}_n, \bar{\rho}_n) \) solve problem (3.10) and \( (\mathbf{m}_{hn}, \bar{\rho}_{hn}) \) solve the fully discrete finite element approximation (7.1) for each time step \( n, n = 1, \ldots, M \). Suppose that \( (\mathbf{m}, \bar{\rho}) \in V \times Q \) and \( \bar{\rho}_n \in L^2(I, L^2(\Omega)) \). Then, there exists a positive constant \( C \) independent of \( h \) and \( \Delta t \) such that for \( \Delta t \) sufficiently small
\[ \| \bar{\rho}_n - \bar{\rho}_{hn} \|_{L^2(\Omega)}^2 + \| \mathbf{m}_n - \mathbf{m}_{hn} \|_{L^2(\Omega)}^2 \leq C (h^{\alpha} + \Delta t^2). \] (7.8)

**Proof.** Evaluating equation (3.4) at \( t = t_n \) gives
\[ (F(\mathbf{m}_n), \mathbf{m}_n, v) - (\bar{\rho}_n, \nabla \cdot v) = -(\nabla \mathbf{\Psi}_n, v) \quad \text{for all } v \in V_h, \]
\[ (\phi_{\eta_{t}}, q) + (\nabla \cdot \mathbf{m}_n, q) = (f_{t}, q) - (\phi \mathbf{\Psi}_{t}, q) \quad \text{for all } q \in Q_h. \] (7.9)

Subtracting (7.1) from (7.9), we obtain
\[ (F(\mathbf{m}_n) - F(\mathbf{m}_{hn}), \mathbf{m}_n) - (\mathbf{m}_n, \nabla \cdot \mathbf{m}_n) = 0, \quad \text{for all } v \in V_h, \] (7.10)
\[ \left( \phi(\bar{\rho}_n - \bar{\rho}_{hn}, q) \right) + \left( \nabla \cdot (\Pi \mathbf{m}_n - \mathbf{m}_{hn}), q \right) = 0 \quad \text{for all } q \in Q_h. \] (7.11)

Choosing \( v = -\zeta_t, q = -\theta_{th} \), and adding the two equations shows that
\[ (\phi(\bar{\rho}_n - \bar{\rho}_{hn}, \theta_{th}) - (F(\mathbf{m}_n) - F(\mathbf{m}_{hn})) \mathbf{m}_n, \Pi \mathbf{m}_n - \mathbf{m}_{hn}) = 0. \] (7.12)

Since \( \bar{\rho}_n - \partial_\mathbf{m}_n = \bar{\rho}_n - \partial_\mathbf{m}_n + \partial_\mathbf{m}_n - \partial_\mathbf{m}_n \), we rewrite (7.12) in the form
\[ (\phi(\partial_\mathbf{m}_n, \theta_{th}) + (F(\Pi \mathbf{m}_n) - F(\mathbf{m}_{hn})) \mathbf{m}_n, \Pi \mathbf{m}_n - \mathbf{m}_{hn}) \]
\[ = (F(\mathbf{m}_n) - F(\Pi \mathbf{m}_n) \Pi \mathbf{m}_n, \zeta_{th}) + (\phi(\bar{\rho}_n - \bar{\rho}_{hn}), \theta_{th}) + (\phi \partial_\mathbf{m}_n, \theta_{th}). \] (7.13)

We will evaluate (7.13) term by term.
- For the first term, we use the identity
\[ (\phi \partial_\mathbf{m}_n, \theta_{th}) = \frac{1}{2 \Delta t} \left( \| \theta_{th} \|_{\phi}^2 - \| \theta_{th-1} \|_{\phi}^2 \right) \]
[1.2 \Delta t \| \theta_{th} \|_{\phi}^2]. \] (7.14)
For the second term, the monotonicity of $F(\cdot)$ in (2.16) yields
\begin{equation}
(F(\Pi m_n) \Pi m_n - F(m_{hn}) m_{hn}, \Pi m_n - m_{hn}) \geq C_3 \| \xi_{hn} \|^s_{L^2}. \tag{7.15}
\end{equation}

For the third term, using (6.4) gives
\begin{equation}
(F(\Pi m_n) m_n - F(\Pi m_n) \Pi m_n, \xi_{hn}) \leq C \frac{\alpha}{2} \| \xi_{hn} \|^s_{L^2} + C (1 + \| \Pi m_n \|_{L^2})^{s + \frac{1}{s-1}} \| \eta_i \|_{L^2}^{s(1-\alpha)}. \tag{7.16}
\end{equation}

Using Cauchy-Schwarz’s inequality, Poincare’s and Young’s inequalities, we obtain
\begin{equation}
(\phi (\tilde{\rho}_n - \tilde{\rho}_n), \tilde{\theta}_{hn}) + (\phi \tilde{\rho}_n, \tilde{\theta}_{hn}) \leq C\Delta t \int_{t_{n-1}}^{t_n} \| \tilde{\rho}_n (\tau) \|^2_\phi \, d\tau + \frac{C}{\Delta t} \int_{t_{n-1}}^{t_n} \| \theta (\tau) \|^2_\phi \, d\tau + \frac{1}{2} \| \tilde{\theta}_{hn} \|^2_\phi. \tag{7.17}
\end{equation}

In view of (7.14)-(7.17), (7.15) yields
\begin{equation}
\left\| \tilde{\theta}_{hn} \right\|^2_\phi - \left\| \tilde{\theta}_{hn-1} \right\|^2_\phi - \left\| \tilde{\rho}_n \right\|^2_\phi + C_3 \| \xi_{hn} \|^s_{L^2} \leq C\Delta t \int_{t_{n-1}}^{t_n} \| \tilde{\rho}_n (\tau) \|^2_\phi \, d\tau + \frac{C}{\Delta t} \int_{t_{n-1}}^{t_n} \| \theta (\tau) \|^2_\phi \, d\tau
\end{equation}
\begin{equation}
+ (1 + \| \Pi m_n \|_{L^2} + \| m_i \|_{L^2}) \frac{1}{s-1} \| \eta_i \|_{L^2}^{s(1-\alpha)}. \tag{7.18}
\end{equation}

By mean of discrete Gronwall’s inequality in Lemma 2.4 and the fact $\tilde{\theta}_{h0} = 0$, we find that
\begin{equation}
\left\| \tilde{\theta}_{hn} \right\|^2_\phi + C_3 \sum_{i=1}^{n} \Delta t \| \xi_{hn} \|^s_{L^2} \leq C e^{\frac{N\alpha}{\Delta t}} \sum_{i=1}^{n} \Delta t \int_{t_{i-1}}^{t_i} \| \tilde{\rho}_n (\tau) \|^2_\phi \, d\tau + \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} \| \theta (\tau) \|^2_\phi \, d\tau
\end{equation}
\begin{equation}
+ C e^{\frac{N\alpha}{\Delta t}} \sum_{i=1}^{n} (1 + \| \Pi m_n \|_{L^2} + \| m_i \|_{L^2}) \frac{1}{s-1} \| \eta_i \|_{L^2}^{s(1-\alpha)}.
\end{equation}

Then it follows
\begin{equation}
\left\| \tilde{\theta}_{hn} \right\|^2_\phi + C_3 \sum_{i=1}^{n} \Delta t \| \xi_{hn} \|^s_{L^2} \leq C\Delta t \int_{0}^{T} \| \tilde{\rho}_n (\tau) \|^2_\phi \, d\tau + \int_{0}^{T} \| \theta (\tau) \|^2_\phi \, d\tau
\end{equation}
\begin{equation}
+ (1 + \| \Pi m_n \|_{L^2} + \| m_i \|_{L^2}) \frac{1}{s-1} \| \eta_i \|_{L^2}^{s(1-\alpha)}. \tag{7.19}
\end{equation}

Using (2.4) and (7.19) we find that
\begin{equation}
\| \tilde{\rho}_n - \tilde{\rho}_{hn} \|^2_{L^2} + \sum_{i=1}^{N} \Delta t \| m_i - m_{hn} \|^s_{L^2} \leq 2 \left( \| \theta_n \|^2_{L^2} + \| \tilde{\theta}_{hn} \|^2_{L^2} \right) + 2^{-1} \sum_{i=1}^{n} \Delta t \left( \| \eta_i \|_{L^2} + \| \tilde{\xi}_{hn} \|^s_{L^2} \right)
\end{equation}
\begin{equation}
\leq 2^{-1} \left( \| \theta_n \|^2_{L^2} + \sum_{i=1}^{N} \Delta t \| \eta_i \|_{L^2} + \| \tilde{\theta}_{hn} \|^2_{L^2} + \sum_{i=1}^{n} \Delta t \| \tilde{\xi}_{hn} \|^s_{L^2} \right)
\leq C \left( \| \theta_n \|^2_{L^2} + \sum_{i=1}^{N} \Delta t \| \eta_i \|_{L^2} + \Delta t \int_{0}^{T} \| \tilde{\rho}_n (\tau) \|^2_\phi \, d\tau + \int_{0}^{T} \| \theta (\tau) \|^2_\phi \, d\tau \right)
\end{equation}
\begin{equation}
+ C \sum_{i=1}^{n} \Delta t \left( 1 + \| \Pi m_i \|^s_{L^2} + \| m_i \|^s_{L^2} \right) \frac{1}{s-1} \| \eta_i \|_{L^2}^{s(1-\alpha)}. \tag{7.20}
\end{equation}
Applying (3.6) and (3.9), (2.4) and Young’s inequality gives
\[
\|\tilde{\rho}_n - \tilde{\rho}_{hn}\|_{L^2_\Omega}^2 + \sum_{i=1}^n \Delta t \|m_i - m_{ih}\|_{L^2_\Omega}^2 \leq C(h^2 \|\tilde{\rho}_n\|_{L^2_\Omega}^2 + h^2 \sum_{i=1}^n \Delta t \|m_i\|_{L^2 \Omega}^2 + \Delta t^2 \|\tilde{\rho}_n(\tau)\|^2_{L^2(T,L^2)}) + C h^{r(1-\alpha)} \sum_{i=1}^n \Delta t \left(1 + \|\Pi m_i\|_{L^r}^r + \|m_i\|_{W^{1,r}}^r\right).
\]

The proof is complete. \(\square\)

The following theorem about an error estimate for \((m_{hn}, \tilde{\rho}_{hn})\) is obtained by using the same manner as in the proof of Theorem 5.1.

**Theorem 7.3.** Let \((m^{(i)}_{hn}, \tilde{\rho}^{(i)}_{hn})\), \(i=1,2\) solve problem (7.1) corresponding to vector coefficients \(a_i\) of generalized polynomials \(F(a_i,s)\) in (1.18) for each time step \(n, n = 1, \ldots, M\). Suppose that each \((m^{(i)}_{hn}, \tilde{\rho}^{(i)}_{hn}) \in V \times Q\) and \(\tilde{\rho}^{(i)}_{hn} \in L^2(I,L^2(\Omega))\). There exists a positive constant \(C\) independent of \(h\), \(\Delta t\) and \(\|a_1 - a_2\|_{L^2(I,L^2(\Omega))}\) such that
\[
\|m^{(1)}_{hn} - \tilde{\rho}^{(2)}_{hn}\|^2_{L^2(\Omega)} + \|m^{(1)}_{hn} - m^{(2)}_{hn}\|^2_{L^2(\Omega)} \leq C \left(h^{r(1-\alpha)} + \Delta t^2 + \|a_1 - a_2\|^2_{L^2(I,L^2(\Omega))}\right).
\]

### 8. Numerical results.
In this section we carry out numerical simulations using mixed finite element approximation to solve problem (7.1) in two dimensions to validate our theoretical estimates. For simplicity, the region of examples are unit square \(\Omega = [0, 1] \times [0, 1]\). We use the piecewise-linear elements for both density and momentum variables. We divided the unit square into an \(\mathcal{N} \times \mathcal{N}\) mesh of squares, each of them subdivided into two right triangles. For each mesh, we solved the solve problem (7.1) numerically. Our problem is solved at each time level starting at \(t = 0\) until the given final time \(T = 1\). A Newton iteration was used to solve the nonlinear equation generated at each time step. The error control in each nonlinear solve is \(\text{tol} = 10^{-6}\). At time \(T\), we measured the error in \(L^2\)-norm for density and \(L^2\)-norm for the vector momentum. We obtain the convergence rates \(r = \log(h_{(c_{\alphaN})})\) of finite approximation at 7 levels with the discretization parameters \(h = 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256\) respectively and time step \(\Delta t = h/2\). The numerical examples in this section are constructed in two categories:

- Example 1 is used to study the convergence rates of the method proposed in the paper.
- Example 2 is used to study the dependence of solution on physical parameters.

**Example 1.** We test the convergence of our method with the function \(F(y) = y^{-1/2} + 1 + y\). In this case, \(\alpha = 1/2\), \(\alpha_N = 1\), \(s = 3\), \(s^* = 3/2\), \(s^*(1-\alpha) = 3/4\). To test the convergence rates, we choose the analytical solution
\[
\rho(x,t) = \frac{1}{\sqrt{2}} \left(e^{-t} + e^{-2t} + e^{-4t}\right)(x_1 + x_2), \quad \text{and} \quad m(x,t) = -e^{-2t} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \quad \forall (x,t) \in \Omega \times I.
\]

For simplicity, we take \(\phi(x) \equiv 1\) on \(\Omega\). The forcing term \(f\) is determined from equation \(\rho_t + \nabla \cdot m = f\). Explicitly,
\[
f(x,t) = -\frac{1}{\sqrt{2}} \left(e^{-t} + 2e^{-2t} + 4e^{-4t}\right)(x_1 + x_2).
\]

The initial condition and boundary condition are determined according to the analytical solution as follows:
\[
\rho_0(x) = \frac{3}{\sqrt{2}}(x_1 + x_2), \quad \psi(x,t) = \frac{1}{\sqrt{2}} \left(e^{-t} + e^{-2t} + e^{-4t}\right) \begin{cases} x_2 & \text{on } \{0\} \times [0,1], \\ 1 + x_2 & \text{on } \{1\} \times [0,1], \\ x_1 & \text{on } [0,1] \times \{0\}, \\ 1 + x_1 & \text{on } [0,1] \times \{1\}. \end{cases}
\]

Then \(\Psi(x,t) = \rho(x,t)\). Thus \(\tilde{\rho}(x,t) = 0\).

We report the approximate errors in Table 1.
We observe that by refining the time step and mesh size, the convergence of our method with the functions $F_1(y) = y^{-1/2} + 1 + y$ and $F_2(y) = 0.95 y^{-1/2} + 1 + 0.95 y$. The analytical solution is computed according to $F_2(y)$ given as below

$$\rho(x,t) = \frac{1}{\sqrt{2}} \left( 0.95 e^{-t} + e^{-2t} + 0.95 e^{-4t} \right) (x_1 + x_2)$$

and

$$\mathbf{m}(x,t) = -e^{-2t} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \forall (x,t) \in \Omega \times I.$$

The forcing term $f$, initial condition and boundary condition accordingly are

$$f(x,t) = \frac{1}{\sqrt{2}} \left( 0.95 e^{-t} + 2 e^{-2t} + 3.80 e^{-4t} \right) (x_1 + x_2), \quad \rho_0(x) = 1.45 \sqrt{2} (x_1 + x_2),$$

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left( 0.95 e^{-t} + e^{-2t} + 0.95 e^{-4t} \right) \begin{cases} x_2 & \text{on } \{0\} \times [0,1], \\ 1 + x_2 & \text{on } \{1\} \times [0,1], \\ x_1 & \text{on } [0,1] \times \{0\}, \\ 1 + x_1 & \text{on } [0,1] \times \{1\}. \end{cases}$$

We use $\|\rho_h^{(1)} - \rho_h^{(2)}\|_{L^2}$ and $\|\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}\|_{L^3}$ as the criterion to measure the dependence of solutions on the coefficients of $g$. The numerical results are listed in Table 2.

| $N$ | $\|\rho_h^{(1)} - \rho_h^{(2)}\|_{L^2(\Omega)}$ | Rates | $\|\mathbf{m}_h^{(1)} - \mathbf{m}_h^{(2)}\|_{L^3(\Omega)}$ | Rates |
|-----|------------------------------------------|-------|------------------------------------------|-------|
| 4   | $1.328 e - 2$                           | –     | $1.126 e - 2$                           | –     |
| 8   | $7.946 e - 3$                           | 0.741 | $7.225 e - 3$                           | 0.640 |
| 16  | $4.423 e - 3$                           | 0.845 | $4.595 e - 3$                           | 0.653 |
| 32  | $2.365 e - 3$                           | 0.903 | $2.912 e - 3$                           | 0.658 |
| 64  | $1.236 e - 3$                           | 0.937 | $1.840 e - 3$                           | 0.662 |
| 128 | $6.364 e - 4$                           | 0.957 | $1.160 e - 3$                           | 0.665 |
| 256 | $3.247 e - 4$                           | 0.971 | $7.313 e - 4$                           | 0.666 |

Table 2. Study the dependence of the solution of mixed regime flows using mixed FEM in 2D.

We observe that by refining mesh size and the time step, the $L^2$-errors for the density and the $L^3$-error for momentum decrease with a rate higher the theoretical rates of convergence. This seems to indicate that the practical rate of convergences are better than the theories ones from Theorem 7.2.

REFERENCES

[1] E. Aulisa, L. Bioshanskaya, L. Hoang, and A. Ibragimov. Analysis of generalized Forchheimer flows of compressible fluids in porous media. *J. Math. Phys.*, 50(10):103102, 44, 2009.
[38] J. Simon. Caractérisation d’espaces fonctionnels. Bollettino della Unione Matematica Italiana. Series V. B, 01 1978.
[39] J. Soni, N. Islam, and P. Basak. An experimental evaluation of non-Darcian flow in porous media. Journal of Hydrology, 38(3):231 – 241, 1978.
[40] J. Soni, N. Islam, and P. Basak. An experimental evaluation of non-Darcian flow in porous media. Journal of Hydrology, 38(3-4):231–241, 1978.
[41] B. Straughan. Stability and wave motion in porous media, volume 165 of Applied Mathematical Sciences. Springer, New York, 2008.
[42] J. L. Vázquez. The porous medium equation. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.
[43] J. C. Ward. Turbulent flow in porous media. Journal of the Hydraulics Division, Proc. Am. Soc. Civ. Eng., 90(HY5):1–12, 1964.
[44] E. Zeidler. Nonlinear functional analysis and its applications. II/B. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.