Operads for x-physics

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Abstract

The essential parts of the operad algebra are presented, which should be useful when confronting with the operadic physics. It is also clarified how the Gerstenhaber algebras can be associated with the linear pre-operads (comp algebras). Their relation to mechanics is concisely discussed. A hypothesis that the Feynman diagrams are observables is proposed.

1 Introduction and outline of the paper

Operads, in essence, were invented by Gerstenhaber [3, 4] and Stasheff [19]. The notion of an operad was formalized by May [12] as a tool for iterated loop spaces. In 1994/95 [6, 22], Gerstenhaber and Voronov published main principles of the operad calculus. Quite a remarkable research activity on operad theory and its applications can be observed in the last decade (e.g. [1, 18]). It may be said that operads are also becoming an interesting and important mathematical tool for QFT (e.g. [7, 8, 20, 21, 10]) and deformation quantization [9].

In this paper, the essential parts of the operad algebra are presented, which should be useful when confronting with the operadic physics. We start from simple algebraic axioms. Basic algebraic constructions associated with a linear pre-operad are introduced. Their properties and the first derivation deviations of the pre-coboundary operator are explicitly given. Under certain condition (formal associativity constraint), the Gerstenhaber algebra structure appears in the associated cohomology. At last, it isconcisely discussed how operads and Gerstenhaber algebras are related to mechanics. A hypothesis that the Feynman diagrams are observables is proposed.

2 Pre-operad (composition system)

Let $K$ be a unital associative commutative ring, and let $C^n$ ($n \in \mathbb{N}$) be unital $K$-modules. For homogeneous $f \in C^n$, we refer to $n$ as the degree of $f$ and often write (when it does not cause confusion) $f$ instead of $\deg f$. For example, $(-1)^f := (-1)^n$, $C^f := C^n$ and $\circ_f := o_n$. Also, it is convenient to use the reduced degree $|f| := n - 1$. Throughout this paper, we assume that $\otimes := \otimes_K$.

**Definition 2.1.** A linear pre-operad (composition system) with coefficients in $K$ is a sequence $C := \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions hold.
For }0 \leq i \leq m - 1\text{ there exist partial compositions }\circ_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0.

(2) For all }h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g,\text{ the composition (associativity) relations hold,}

\begin{align*}
(h \circ_i f) \circ_j g &= \begin{cases} 
-1)^{|f||g|} (h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\
h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i + |f|, \\
(1)^{|f||g|} (h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|. 
\end{cases}
\end{align*}

(3) There exists a unit }I \in C^1\text{ such that }I \circ_0 f = f = f \circ_1 I, \quad 0 \leq i \leq |f|.

In the 2nd item, the first and third parts of the defining relations turn out to be equivalent.

Example 2.2 (endomorphism pre-operad \([3, 4]\)). Let }A\text{ be a unital }K\text{-module and }E^n_A := \text{End}_A^n := \text{Hom}(A \otimes^n A, A).\text{ Define the partial compositions for }f \otimes g \in E^f_A \otimes E^g_A\text{ as}

\begin{align*}
f \circ_i g &= (-1)^{|g|} f \circ (\text{id}_A^i \otimes g \otimes \text{id}_A^{(|f| - i)}), \quad 0 \leq i \leq |f|.
\end{align*}

Then }E_A := \{E^n_A\}_{n \in \mathbb{N}}\text{ is a pre-operad (with the unit }\text{id}_A \in E^1_A)\text{ called the endomorphism pre-operad of }A.

Example 2.3 (associahedra). A geometrical example of a pre-operad is provided by the Stasheff associahedra \([19]\). Quite a surprising realization of the associahedra as truncated simplices was discovered and studied in \([17, 20, 13]\). Markl and Shnider \([14, 15, 16]\) revealed its relevance to the Drinfel’d algebra \([2]\) cohomology and deformations.

3 Associated operations

Throughout this paper fix }\mu \in C^2\text{.

Definition 3.1. The cup-multiplication }\cup : C^f \otimes C^g \to C^{f+g}\text{ is defined by}

\begin{align*}
f \cup g := (-1)^{|f|} (\mu \circ_0 f) \circ f \otimes g \in C^{f+g}, \quad |\cup| = 1.
\end{align*}

The pair }\text{Cup }C := \{C, \cup\}\text{ is called a }\cup\text{-algebra (cup-algebra) of }C.

Example 3.2. For the endomorphism pre-operad (Example 2.2) }E_A\text{ one has

\begin{align*}
f \cup g &= (-1)^{|g|} \mu \circ (f \otimes g), \quad \mu \otimes f \otimes g \in E^2_A \otimes E_A \otimes E_A.
\end{align*}

Definition 3.3. The total composition }\bullet : C^f \otimes C^g \to C^{f+|g|}\text{ is defined by

\begin{align*}
f \bullet g &= \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0.
\end{align*}

The pair }\text{Com }C := \{C, \bullet\}\text{ is called the composition algebra of }C.
Definition 3.4 (tribraces and tetrabraces). The Gerstenhaber braces \{ \cdot, \cdot, \cdot \} are defined as a double sum
\[
\{h, f, g\} := \sum_{i=0}^{\lfloor |h| - 1 \rfloor} \sum_{i+f}^{\lfloor |h| + |g| \rfloor} (h \circ_i f) \circ_j g \in C^{h+|f|+|g|}, \quad |\{\cdot, \cdot, \cdot\}| = 0.
\]
The tetrabraces \{ \cdot, \cdot, \cdot, \cdot \} are defined by
\[
\{h, f, g, b\} := \sum_{i=0}^{\lfloor |h| - 2 \rfloor} \sum_{j=i+f}^{\lfloor |h| + |f| - 1 \rfloor} \sum_{k=j+g}^{\lfloor |h| + |f| + |g| \rfloor} ((h \circ_i f) \circ_j g) \circ_k b \in C^{h+|f|+|g|+|b|}.
\]

It turns out that \( f \circ g = (-1)^{|f|}\{\mu, f, g\} \). In general, Cup \( C \) is a non-associative algebra. By denoting \( \mu^2 := \mu \cdot \mu \) it turns out that the associator in Cup \( C \) reads
\[
(f \circ g) \circ h - f \circ (g \circ h) = \{\mu^2, f, g, h\}.
\]

Thus the formal associator \( \mu^2 \) is an obstruction to associativity of Cup \( C \). For the endomorphism pre-operad \( E_A \), \( \mu^2 \) reads as an associator as well:
\[
\mu^2 = \mu \circ (\mu \otimes \text{id}_A - \text{id}_A \otimes \mu), \quad \mu \in E_A^2.
\]

4 Identities

In a pre-operad \( C \), the Getzler identity
\[
(h, f, g) := (h \bullet f) \bullet g - h \bullet (f \bullet g) = \{h, f, g\} + (-1)^{|f||g|}\{h, g, f\}
\]
holds, which easily implies the Gerstenhaber identity
\[
(h, f, g) = (-1)^{|f||g|}(h, g, f).
\]
The commutator \([ \cdot , \cdot ]\) is defined in Com \( C \) by
\[
[f, g] := f \circ g - (-1)^{|f||g|} g \circ f = -(-1)^{|f||g|}[g, f], \quad [\cdot, \cdot] = 0.
\]
The commutator algebra of Com \( C \) is denoted as Com\(^-\) \( C := \{ C, [\cdot, \cdot] \} \). By using the Gerstenhaber identity, one can prove that Com\(^-\) \( C \) is a graded Lie algebra. The Jacobi identity reads
\[
(-1)^{|f||h|}[\{f, g\}, h] + (-1)^{|g||f|}[\{g, h\}, f] + (-1)^{|h||f|}[h, f, g] = 0.
\]

5 Pre-coboundary operator

In a pre-operad \( C \), define a pre-coboundary operator \( \delta := \delta_\mu \) by
\[
-\delta f := \text{ad}^\text{right}_\mu f := [f, \mu] := f \bullet \mu = (-1)^{|f|}\mu \bullet f = f \circ I + f \bullet \mu + (-1)^{|f|} I \circ f, \quad \deg \delta = +1 = |\delta|.
\]

It turns out that \( \delta^2 = -\delta \mu^2 \). It follows from the Jacobi identity in Com\(^-\) \( C \) that \( \delta \) is a (right) derivation of Com\(^-\) \( C \),
\[
\delta[f, g] = (-1)^{|g|}[\delta f, g] + [f, \delta g].
\]

But \( \delta \) need not be a derivation of Cup \( C \), and \( \mu^2 \) again appears as an obstruction:
\[
\delta(f \circ g) - f \circ \delta g - (-1)^{\mu} \delta f \circ g = (-1)^{|g|}\{\mu^2, f, g\}.
\]
6 Derivation deviations

The derivation deviation of \( \delta \) over \( \bullet \) is defined by

\[
\text{dev}_{\bullet} \delta (f \otimes g) := \delta (f \bullet g) - f \bullet \delta g - (-1)^{|\delta|} \delta f \bullet g.
\]

**Theorem 6.1.** In a pre-operad \( C \), one has

\[
(-1)^{|\delta|} \text{dev}_{\bullet} \delta (f \otimes g) = f \leadsto g - (-1)^{|f|} g \leadsto f.
\]

The derivation deviation of \( \delta \) over \{\ldots, \} is defined by

\[
\text{dev}_{\{\ldots, \}} \delta (h \otimes f \otimes g) := \delta \{h, f, g\} - \{h, f, \delta g\} - (-1)^{|\delta|} \{h, \delta f, g\} - (-1)^{|\delta|+|f|} \{\delta h, f, g\}.
\]

**Theorem 6.2.** In a pre-operad \( C \), one has

\[
(-1)^{|\delta|} \text{dev}_{\{\ldots, \}} \delta (h \otimes f \otimes g) = (h \bullet f) \leadsto g + (-1)^{|h|} f \leadsto (h \bullet g) - h \bullet (f \leadsto g).
\]

Thus the left translations in \( \text{Com} C \) are not derivations of \( \text{Cup} C \), the corresponding deviations are related to \( \text{dev}_{\{\ldots, \}} \delta \). It turns out that the right translations in \( \text{Com} C \) are derivations of \( \text{Cup} C \),

\[
(f \leadsto g) \bullet h = f \leadsto (g \bullet h) + (-1)^{|h|} f \leadsto (h \bullet g) - h \bullet (f \leadsto g).
\]

By combining this formula with the one from Theorem 6.2, we obtain

**Theorem 6.3.** In a pre-operad \( C \), one has

\[
(-1)^{|\delta|} \text{dev}_{\{\ldots, \}} \delta (h \otimes f \otimes g) = [h, f] \leadsto g + (-1)^{|h|} f \leadsto [h, g] - [h, f] \leadsto g.
\]

7 Associated cohomology and Gerstenhaber algebra

Now, it can be clarified how the Gerstenhaber algebra can be associated with a linear pre-operad. If (formal associativity) \( \mu^2 = 0 \) holds, then \( \delta^2 = 0 \), which in turn implies \( \text{Im} \delta \subseteq \text{Ker} \delta \). Then one can form an associated cohomology (\( \mathbb{N} \)-graded module) \( H(C) := \text{Ker} \delta / \text{Im} \delta \) with homogeneous components

\[
H^n(C) := \text{Ker}(C^n \xrightarrow{\delta} C^{n+1}) / \text{Im}(C^{n-1} \xrightarrow{\delta} C^n),
\]

where, by convention, \( \text{Im}(C^{-1} \xrightarrow{\delta} C^0) := 0 \). Also, in this (\( \mu^2 = 0 \)) case, \( \text{Cup} C \) is associative and \( \delta \) is a derivation of \( \text{Cup} C \). Recall from above that \( \text{Com}^{-1} C \) is a graded Lie algebra and \( \delta \) is a derivation of \( \text{Com}^{-1} C \). Due to the derivation properties of \( \delta \), the multiplications \([\cdot, \cdot]\) and \( \leadsto \) induce corresponding (factor) multiplications on \( H(C) \), which we denote by the same symbols. Then \( \{H(C), [\cdot, \cdot]\} \) is a graded Lie algebra. It follows from Theorem 6.1 that the induced \( \leadsto \)-multiplication on \( H(C) \) is graded commutative,

\[
f \leadsto g = (-1)^{|f|} g \leadsto f
\]

for all \( f \otimes g \in H^f(C) \otimes H^g(C) \), hence \( \{H(C), \leadsto\} \) is an associative graded commutative algebra. It follows from Theorem 6.3 that the graded Leibniz rule holds,

\[
[h, f \leadsto g] = [h, f] \leadsto g + (-1)^{|h|} f \leadsto [h, g]
\]
for all \( h \otimes f \otimes g \in H^h(C) \otimes H^f(C) \otimes H^g(C) \). At last, it is also relevant to note that
\[
0 = |[\cdot, \cdot]| \neq |\sim| = 1.
\]
In this way, the triple \( \{H(C), \sim, [\cdot, \cdot]\} \) turns out to be a Gerstenhaber algebra. This fact is a known from [4, 5, 1].

In the case of an endomorphism pre-operad, the Gerstenhaber algebra structure appears on the Hochschild cohomology of an associative algebra [3].

8 Discussion: x-mechanics

Some people like commutative diagrams. Consider the following one:

\[
\begin{array}{ccc}
\text{Poisson algebras} & \overset{\text{algebraic abstraction}}{\leftarrow} & \text{mechanics} \\
\{ & & \\
\text{Gerstenhaber algebras} & \overset{\text{algebraic abstraction}}{\leftarrow} & \text{x-mechanics} \\
\end{array}
\]

_Poisson algebras_ can be seen as an algebraic abstraction of mechanics. Here \( \sim \) means _similarity_: Gerstenhaber algebras are graded analogs of the Poisson algebras.

It may be expected that there exists a kind of mechanics (x-mechanics) associated with operads and Gerstenhaber algebras. According to the diagram, x-mechanics is a graded analogue of mechanics and _observables_ of an x-mechanical model must satisfy the (homotopy [6, 22]) Gerstenhaber algebra identities.

Recently Kreimer [10] explained how the _insertion operad of Feynman graphs_ is present in renormalization in QFT because insertion operations are used in the Hopf algebra of Feynman graphs. Thus from the operad theoretical point of view it may be expected that the _Feynman diagrams can be seen as observables_.

Acknowledgement

Research supported by the ESF grant 3654.

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