Rényi and Tsallis entropies related to eigenfunctions of quantum graphs

Alexey E. Rastegin

Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia

For certain families of finite quantum graphs, we study the question of how eigenfunctions are distributed over the graph. To characterize properties of the distribution, generalized entropies of the Rényi and Tsallis types are considered. The presented approach is similar to entropic uncertainty relations of the Maassen–Uffink type. Using the Riesz theorem, we derive lower bounds on symmetrized generalized entropies of eigenfunctions. A quality of such estimates will depend on boundary conditions used at vertices of the given graph. Rényi and Tsallis entropies of eigenfunctions of star graphs are separately examined. Relations between generalized entropies and variances of eigenfunctions are considered as well. When such relations remain valid on average, they may be used in studies of quantum ergodicity.

Keywords: metric graph, Laplace operator, generalized entropy, Riesz theorem

I. INTRODUCTION

Using quantum networks of one-dimensional wires to model physical systems has a long history [1]. In the context of quantum chaos on graphs, Kottos and Smilansky [2] rediscovered the graph trace formula first discussed by Roth [3]. This result allows one to consider the connection between random matrix theory and chaotic classical dynamics. In such investigations, we try to understand the relationship between quantum mechanics and classical chaos (see [4–6] for graph models and [7] for other topics). Quantum graphs are often used as simplified models in mathematics, natural sciences, and engineering [8, 9]. Such models are naturally arisen in studies of nano- or meso-scale systems that are similar to a neighborhood of a graph. Studies of differential operators on metric graphs form an interesting branch of mathematical physics [10–15]. Such operators may be served as model systems arising in quantum chaos and related questions of statistical physics. They are typically treated as Hamiltonians including the negative Laplace operator [8]. Considering the Laplacian and other operators on a metric graph, suitable boundary conditions at the vertices should be assigned.

Entropies provide a powerful and flexible tool for investigation of distribution properties. Such functions can often be used as indicators of quantum chaos [10]. The authors of [17] used the standard Shannon entropy to characterize eigenfunctions of quantum graphs. These results are inspired by analogous studies of eigenfunctions on quantum maps [18, 19] and Riemannian manifolds [20, 21]. It is interesting that the notion of quantum graphs per se can be approached through considering one-dimensional piecewise linear maps [22]. The approach of [17] is to define entropies in terms of components of the corresponding eigenvectors. For other purposes, this idea was already realized in [16]. The estimates of [17] are essentially based on entropic uncertainty relations of the Maassen–Uffink type [23]. Although the Shannon entropy is fundamental, other entropic functions have found use in various disciplines [24, 25]. The Rényi entropy [26] and the Tsallis entropy [27] both give an important one-parameter extension of the Shannon entropy. Generalized entropies provide an additional tool in characterizing eigenfunctions on quantum graphs.

The aim of the present work is to characterize properties of eigenfunctions on quantum graphs by means of the Rényi and Tsallis entropies. The paper is organized as follows. In Section II, we recall basic definitions concerning quantum graphs. Section III introduces Rényi and Tsallis entropies corresponding to an eigenfunction of a quantum graph. Further, we obtain lower bounds on symmetrized generalized entropies of eigenvectors associated with eigenfunctions on quantum graphs. We also discuss circumstances under which the derived entropic bounds may be useful. In Section IV, generalized entropies of star graphs are considered. Due to a relative simplicity of such graphs, their properties can usually be described with more details. Section V is devoted to relations between entropies and the variance of a quantum graph. In particular, we address an asymptotic behavior of averaged entropies, when the number of bonds increases. In Section VI, we conclude the paper with a summary of results obtained.

II. PRELIMINARIES

In this section, we recall definitions and introduce the notation. The used notation closely follows chapter 1 of [8]. Let us consider a finite graph consisting of the set of vertices $V = \{v\}$ and the set of undirected edges $E = \{e\}$. By $V := |V|$ and $E := |E|$, we respectively mean the numbers of vertices and edges. In the following, we will assume the absence of loops and multiple edges. Two vertices $v$ and $v'$ are called adjacent, in symbols $v \sim v'$, when there exists an edge connecting them. For the given enumeration of vertices by numbers $i \in \{1, 2, \ldots, V\}$, each edge $e \in E$ is
labeled by pair $(ij)$ assumed to be symmetric. An undirected graph without loops and multiple edges is fully specified by its $V \times V$ adjacency matrix with entries equal to 1 for $(ij) \in \mathcal{E}$ and 0 for $(ij) \notin \mathcal{E}$. The degree $d_v$ of a vertex $v$ is the number of edges emanating from it. Graphs under consideration are all assumed to be connected.

In the following, we will mainly deal with directed graphs. The choice of orientation will be necessary for introducing coordinates along edges. Now, each edge has one origin vertex and one terminal vertex. Directed edges are referred to as bonds and comprise the set of bonds $\mathcal{B} = \{b\}$ of cardinality $B = |\mathcal{B}|$. Any undirected graph can be treated as a directed one by assigning two bonds $b$ and $\bar{b}$ with opposite directions to each edge $e$. In the following, we will deal only with such digraphs. It will be convenient to use an enumeration of vertices by numbers $i \in \{1, 2, \ldots, V\}$. For the given enumeration, each pair of adjacent vertices $i$ and $j$ is then linked by the two bonds $b = [ij]$ and $\bar{b} = [ji]$. According to [6], the notation $b = [ij]$ reads from the right to the left so that $j$ is its origin and $i$ is its terminus. In this case, we have $B = 2E$ since for all $i \in \{1, 2, \ldots, V\}$ the number of incoming bonds equals the number of outgoing ones. Up to now, graphs were discussed from the combinatorial perspective only as discrete structures. To approach quantum graphs, edges should be considered as one-dimensional segments sometimes called wires. Hence, digraphs will be equipped with an additional structure that will make them metric graphs [8].

A digraph becomes a metric graph, when each bond $b \in \mathcal{B}$ is assigned by a positive length $L_b \in (0, \infty)$ [8]. Thus, ordered points along $b$ are all identified with real numbers between 0 and $L_b$. On the bond $b = [ij]$, the coordinate $0 \leq x_{[ij]} \leq L_b$ is put by taking $x_{[ij]} = 0$ at the origin $j$ and $x_{[ij]} = L_b$ at the terminus $i$. The lengths of the bonds that are reversed to each other are treated as equal, namely $L_o = L_b$. Hence, the length $L_e$ of any edge $e \in \mathcal{E}$ is also defined. Between the coordinates along mutually reversed bonds, one has

$$x_{[ji]} = L_e - x_{[ij]},$$

where $e = (ij)$. The set of points of a metric graph include not only its vertices, but all intermediate points on the edges [8].

In order to consider quantum graphs, metric graphs should be equipped with an additional operator called the Hamiltonian [8]. A function on the metric graph $\Gamma$ is defined as a collection of $E$ scalar functions such that $f_e : [0, L_e] \to \mathbb{C}$. In the studies of quantum graphs, the most frequently used operator is the negative second derivative acting on each edge. Other physically important forms of the Hamiltonian are discussed in [8]. The definition of the quantum graph Hamiltonian cannot be completed without adding smoothness conditions along the edges and junction conditions at the vertices. Junction conditions are similar to boundary ones used in the familiar case of differential operators on a single interval.

In the case of the negative Laplace operator, the eigenvalue problem is posed as [8]

$$- \frac{d^2 f_e}{dx_e^2} = \kappa^2 f_e(x_e).$$

(2.2)

We usually look for real (positive) values $\kappa \neq 0$. With the second derivative, we do not need to specify an orientation of coordinates along the edges. This is required in other cases such as the magnetic Schrödinger operator [8]. On each edge $e = (ij)$, an eigenfunction with eigenvalue $\kappa^2 \neq 0$ is written in the form

$$f_e(x_e) = a_{[ij]} \exp \left( i \kappa x_{[ij]} \right) + a_{[ji]} \exp \left( i \kappa x_{[ji]} \right).$$

(2.3)

To complete the formulation, one imposes suitable boundary conditions at the vertices where several edges meet. These conditions should guarantee self-adjointness of the Hamiltonian considered [8].

In general, boundary conditions can be described in two different forms. In the first approach, certain pair of $d_j \times d_j$ matrices is assigned to $j$-th vertex, with $j \in \{1, 2, \ldots, V\}$. The second approach is posed by prescribing how waves scatter at each vertex. This approach is typical in studies of quantum chaos on graphs. It is also more convenient for our purposes. The connections between the two approaches are considered in section 2.1 of [8]. Boundary conditions can be specified in terms of unitary scattering matrices assigned to graph vertices. For the given vertex $j$ with degree $d_j$, the corresponding matrix $\sigma^{(j)}$ has size $d_j \times d_j$ and entries $\sigma^{(j)}_{[ij][jk]}$. At the vertex $j$, the boundary conditions for eigenfunctions can be reformulated as

$$a_{[ij]} = \sum_{k \sim j} \sigma^{(j)}_{[ij][jk]} \exp \left( i \kappa L_{[jk]} \right) a_{[jk]},$$

(2.4)

where the sum is taken over those vertices $k \in \{1, 2, \ldots, V\}$ that are adjacent to $j$. The matrix $\sigma^{(j)}$ prescribes how the vertex $j$ scatters waves incoming into it from adjacent vertices. We will assume that vertex scattering matrices are $\kappa$-independent (for more details, see theorem 2.1.6 of [8]). The formula (2.4) provides the consistency requirement between the incoming and the outgoing coefficients and must be true simultaneously at all the vertices [4].
In general, the following two types of boundary conditions will be used. According to the so-called Neumann conditions, the function is continuous and the sum of its normal derivatives is zero at each vertex \([12, 13]\). These conditions are sometimes referred to as the Kirchhoff conditions \([11, 12]\) and the standard conditions \([8]\). Here, the scattering matrix at any vertex \(j\) reads \([28]\)

\[
\sigma^{(j)}_{[ij][jk]} = \frac{2}{d_j} - \delta_{ik},
\]

(2.5)

where \(\delta_{ik}\) is the Kronecker symbol. For graphs with large degrees of vertices, the Neumann conditions imply a dominance of back-scattering \([17, 28]\). For very large \(d_j\), the matrix \(-\sigma^{(j)}\) will approach the identity matrix of the corresponding size. The equi-transmitting boundary conditions were introduced in \([29]\). The corresponding matrix elements are characterized by the property \([29]\)

\[
\left|\sigma^{(j)}_{[ij][jk]}\right|^2 = \frac{1 - \delta_{ik}}{d_j - 1}.
\]

(2.6)

Thus, all the off-diagonal entries have equal amplitudes, and the diagonal ones are zero. Hence, back-scattering is forbidden so that an incoming wave is totally transmitted with equal weights to outgoing bonds. These boundary conditions cannot be realized with arbitrary \(d_j\) \([29]\). The authors of \([29]\) gave examples of an explicit construction of equi-transmitting scattering matrices. Their methods used skew-Hadamard matrices \([30–32]\) and properties of Dirichlet characters. In particular, the orthogonality of Dirichlet characters is important here (see, e.g., theorem 3.4 in chapter 5 of \([33]\)). The second construction provides an answer, in which \(d_j - 1\) is an odd prime. Using the Legendre symbol as a Dirichlet character, one can construct a symmetric equi-transmitting matrix with \(d_j - 1\) being a prime congruent to 1 modulo 4 \([29]\).

According to \([8]\), quantum graphs are defined as metric graphs equipped with a differential operator called the Hamiltonian and accompanied by vertex conditions. That is, the quantum graph \(\hat{\Gamma}\) is a triple of metric graph \(\Gamma\), the Hamiltonian and boundary conditions in the form of matrices \(\sigma^{(j)}\) assigned to vertices \(j \in \{1, 2, \ldots, V\}\). In the following, we restrict a consideration to the negative Laplace operator. For each quantum graph \(\hat{\Gamma}\), we write the unitary evolution \(B \times B\) matrix \(U_\hat{\Gamma}(x)\) with elements

\[
u_{[ij][k\ell]} = \delta_{jk} \sigma^{(j)}_{[ij][k\ell]} \exp(i x L_{[j\ell]}).
\]

(2.7)

Let \(a \in \mathbb{C}^B\) denote a column vector of coefficients \(a_{[ij]}\) that appear in \([23]\). These coefficients completely describe an eigenfunction of the problem \([2.2]\). The consistency requirement \([2.4]\) then reduces to

\[
U_\hat{\Gamma}(x) a = a.
\]

(2.8)

To each eigenfunction, we can herewith assign an eigenvector of \(U_\hat{\Gamma}(x)\) corresponding to eigenvalue 1. This vector specifies a distribution of the function \([2.3]\) over the graph. If \(x^2\) is an eigenvalue of the problem \([2.2]\), then \(x\) obeys the secular equation

\[
\det(\mathbb{I}_B - U_\hat{\Gamma}(x)) = 0,
\]

(2.9)

and vice versa \([8]\). Many results on quantum graphs hold under assumption that the eigenvalue is simple and the eigenfunction is non-vanishing on vertices. As was shown in \([34–36]\), these properties are generic with respect to small perturbations of the edge lengths. Of course, such perturbations have to break all symmetries of the graph.

There are various ways to characterize eigenvectors of the unitary evolution matrix. In the following, generalized entropies of the Rényi and Tsallis types will be utilized for such purposes. When we associate entropic measures with finite structures, usual vector norms in finite dimensions are convenient. For all \(p \geq 1\), the usual vector \(p\)-norm of \(B\)-tuple is defined as

\[
\|a\|_p = \left(\sum_{b=1}^B |a_b|^p\right)^{1/p}.
\]

(10.2)

The limiting value \(p = \infty\) is allowed and leads to \(\max\{|a_b| : 1 \leq b \leq B\}\).

III. LOWER BOUNDS ON SYMMETRIZED ENTROPIES OF EIGENVECTORS

In this section, we derive lower bounds on symmetrized entropies defined for an eigenfunction of some quantum graph. Let \(0 \neq a \in \mathbb{C}^B\) and \(0 < \alpha \neq 1\); then Rényi’s \(\alpha\)-entropy of the column \(a\) with entries \(a_1, a_2, \ldots, a_B\) is defined
\[ R_\alpha(a) := \frac{1}{1 - \alpha} \ln \left( \sum_{b \in B} w_b^\alpha \right), \]  

(3.1)

where the weights are put as

\[ w_b = \|a\|^{-2} |a_b|^2. \]

(3.2)

Rényi considered this type of information measures in connection with formal postulates characterizing entropic functions \[26\]. The Rényi \( \alpha \)-entropy cannot increase with growth of \( \alpha \) (see, e.g., section 5.3 of \[24\]). It has many interesting properties summarized in section 2.7 of \[25\]. The maximal value \( \ln B \) of (3.1) is reached when \( w_b = 1/B \) for all \( b \in B \). The following limiting cases should be mentioned separately. For \( \alpha \to \infty \), we have the min-entropy defined as

\[ R_{\min}(a) := -\ln(\max w_b). \]

(3.3)

The limit \( \alpha \to 0 \) gives the max-entropy. By \( \text{rank}(a) \), we denote the number of non-zero elements of \( a \). Then the max-entropy is written as

\[ R_{\max}(a) := \ln \left\{ \text{rank}(a) \right\}. \]

(3.4)

As the Rényi \( \alpha \)-entropy is a non-increasing function of order \( \alpha \), we have

\[ R_{\min}(a) \leq R_\alpha(a) \leq R_{\max}(a). \]

(3.5)

For \( \alpha = 2 \), the definition (3.1) gives the so-called collision entropy. Tsallis entropies form another especially important family of generalized entropies. For \( 0 < \alpha \neq 1 \), the Tsallis \( \alpha \)-entropy of non-zero \( a \in \mathbb{C}^B \) is defined as

\[ H_\alpha(a) := \frac{1}{1 - \alpha} \left( \sum_{b \in B} w_b^\alpha - 1 \right). \]

(3.6)

With the factor \( (2^{1-\alpha} - 1)^{-1} \) instead of \( (1 - \alpha)^{-1} \), this entropic form was considered by Havrda and Charvát \[37\]. For \( \xi > 0 \), we define the \( \alpha \)-logarithm

\[ \ln_\alpha(\xi) := \begin{cases} \xi^{1-\alpha} - 1, & \text{for } 0 < \alpha \neq 1, \\ \ln \xi, & \text{for } \alpha = 1. \end{cases} \]

(3.7)

The maximal value of (3.6) is equal to \( \ln_\alpha(B) \) and reached when \( w_b = 1/B \) for all \( b \in B \). The choice \( \alpha = 2 \) gives the so-called linear entropy equal to 1 minus the sum of squared probabilities \[25\]. Conditional form of this entropy is directly related to the minimal error probability on checking a finite or countable number of hypotheses \[38\]. Due to non-additivity, the Tsallis entropy is well known in non-extensive thermostatistics \[27\]. Nevertheless, entropic functions of this type have found use far beyond the context of thermostatistics. For instance, such information measures were applied in formulation of Bell inequalities \[39\] and in studies of combinatorial problems \[40, 41\]. For Tsallis entropies, we do not consider the limit \( \alpha \to \infty \), as it leads to the same zero value for all vectors. In the limit \( \alpha \to 1 \), both the above entropies reduce to the Shannon entropy

\[ H_1(a) = -\sum_{b \in B} w_b \ln w_b. \]

(3.8)

This entropy was used in studying properties of graph eigenfunctions \[17\].

In the following, entropic bounds will be expressed in terms of the so-called symmetrized entropies. Such entropies were used in formulating quantum-mechanical uncertainty relations \[42, 43\]. Let positive orders \( \alpha \) and \( \beta \) satisfy \( 1/\alpha + 1/\beta = 2 \). It is convenient to parametrize them by means of \( s \in [0; 1) \),

\[ \max\{\alpha, \beta\} = \frac{1}{1 - s}, \quad \min\{\alpha, \beta\} = \frac{1}{1 + s}. \]

(3.9)

The symmetrized entropies Rényi and Tsallis entropies are respectively defined as

\[ \tilde{R}_\alpha(a) := \frac{1}{2} \left( R_\alpha(a) + R_\beta(a) \right), \]

(3.10)

\[ \tilde{H}_\alpha(a) := \frac{1}{2} \left( H_\alpha(a) + H_\beta(a) \right). \]

(3.11)
The above condition on $\alpha$ and $\beta$ in entropic relations will follow from the use of Riesz’s theorem \[44\]. We should also remember that the symmetrized entropies lead to the standard Shannon entropy in the limit $s \to 0$. For symmetrized entropies of the Rényi type, we will also use the limiting value $s = 1$, when

$$
\bar{R}_1(a) = \frac{1}{2} \left( R_{\min}(a) + R_{\max}(a) \right).
$$

(3.12)

For symmetrized entropies of the Tsallis type, the value $s = 1$ is not considered.

We shall now derive lower bounds on symmetrized Rényi and Tsallis entropies of eigenfunctions of quantum graphs. For finite-dimensional systems, the entropic uncertainty principle is most known in the formulation conjectured by Kraus \[45\] and later proved by Maassen and Uffink \[23\]. Their proof is based upon a deep mathematical result known as Riesz’s theorem \[44\]. Using the Maassen–Uffink result, the authors of \[17\] studied lower bounds on the Shannon entropy of eigenfunctions on some quantum graphs. To examine the quantized baker’s map, the authors of \[18\] obtained lower bounds on the entropies associated with semiclassical measures. In such questions, the formulation of Maassen and Uffink is widely used. So, we begin with recalling a simplified version of the Riesz theorem.

It is sufficient to focus on unitary transformations in finite dimensions. Due to the unitarity of $B \times B$ matrix $U = [u_{bb'}]$, we have

$$
\|Ua\|_2 = \|a\|_2.
$$

(3.13)

Using this fact, we can apply the Riesz theorem \[44\] (see also theorem 297 of the book \[46\]). Let positive indices $p$ and $q$ be conjugated so that $1/p + 1/q = 1$, and let $\eta$ be the maximal modulus of matrix entry, namely

$$
\eta := \max |u_{bb'}|.
$$

(3.14)

It then holds that, for $1 \leq q \leq 2$ and arbitrary $a \in \mathbb{C}^B$,

$$
\|Ua\|_p \leq \eta^{(2-q)/q} \|a\|_q.
$$

(3.15)

In general, inequalities of the form (3.15) remain valid for those transformations that do not increase the vector 2-norm. However, we deal with the unitary transformation, which is invertible and the inversion is unitary as well. Under the same conditions $1/p + 1/q = 1$ and $1 \leq q \leq 2$, we write a “twin” inequality

$$
\|a\|_p \leq \eta^{(2-q)/q} \|Ua\|_q.
$$

(3.16)

The latter is obtained by application Riesz’s theorem to the transformation $U^\dagger$ acting on $Ua$. The above results hold irrespectively to the normalization of used vectors.

Let positive indices $\alpha$ and $\beta$ obey $1/\alpha + 1/\beta = 2$ and $\nu = \max \{\alpha, \beta\}$. For $a \in \mathbb{C}^B$ and unitary $B \times B$ matrix $U$, we have

$$
R_\alpha(a) + R_\beta(Ua) \geq -2 \ln \eta,
$$

(3.17)

$$
H_\alpha(a) + H_\beta(Ua) \geq \ln(\nu^{-2}).
$$

(3.18)

The relations (3.17) and (3.18) follow from (3.15) and (3.16). The derivation uses a method similar to that of \[47\]. A reformulation for rank-one resolutions of the identity in Hilbert space was later proposed in \[48\]. The only distinction is that the papers \[43, 47\] deal with probabilistic vectors calculated for a quantum state.

When $a$ is an eigenvector of $U$, the relations (3.17) and (3.18) involve two entropies of the same vector with different entropic parameters. It will be convenient here to use symmetrized entropies \[43\]. From (3.17) and (3.18), we immediately obtain

$$
\bar{R}_s(a) \geq -\ln \eta,
$$

(3.19)

$$
\bar{H}_s(a) \geq \frac{1}{2} \ln \nu(\eta^{-2}),
$$

(3.20)

where $\nu = (1-s)^{-1}$. The formulas (3.19) and (3.20) are one-parameter extensions of the inequality

$$
H_t(a) \geq -\ln \eta.
$$

(3.21)

The authors of \[17\] used (3.21) for studying the question of how eigenfunctions are distributed over a graph. Some results were shown to be related to geometric properties of the given graph. If $a$ is an eigenvector of $U^t$ for all $t \in \mathbb{Z}$, in addition, the power $U^t$ is unitary as well. Hence, we have the following.
Proposition 1 Let \( \mathbf{U} \) be a unitary \( B \times B \) matrix, and let \( \mathbf{a} \) be an eigenvector of \( \mathbf{U} \). Denoting entries of \( \mathbf{U}^t \) with natural power \( t \) by \( u_{bb'}^{(t)} \), we define
\[
\eta^{(t)} = \max |u_{bb'}^{(t)}|.
\] (3.22)

For all \( t \in \mathbb{N} \) and \( \nu = (1 - s)^{-1} \), we then have
\[
\tilde{R}_s(\mathbf{a}) \geq -\ln \eta^{(t)}
\] (0 \( s \leq 1 \)),
\[
\tilde{H}_s(\mathbf{a}) \geq \frac{1}{2} \ln \left\{ (\eta^{(t)})^{-2} \right\}
\] (0 \( s < 1 \)).
(3.23)
(3.24)

The maximal possible value of Rényi entropies is \( \ln B \). Rescaling to the latter, we obtain lower bound on normalized entropies,
\[
\frac{\tilde{R}_s(\mathbf{a})}{\ln B} \geq -\frac{\ln \eta^{(t)}}{\ln B}.
\] (3.25)

For \( s = 0 \), this formula reduces to the lower bound on the normalized Shannon entropy derived in [17]. The lower bounds (3.23) and (3.24) will be useful, when the quantity (3.22) is sufficiently far from 1. Since the squared absolute values of elements of the rows or columns of a unitary matrix sum to one, they should not deviate essentially from 1/\( B \). Thus, the above entropic relations may lead to a good estimate, when some powers of the corresponding unitary matrix are not too sparse. This condition can be treated in the context of stochastic classical dynamics in a Markov chain [17].

For quantum graphs with equi-transmitting boundary conditions, the notion of graph girth is useful [17]. Let us take a combinatorial graph without loops and multiple edges. Any sequence \((v_0, \ldots, v_0)\) of adjacent vertices is referred to as the path of length \( \tau \). If \((v_0, \ldots, v_0)\) is a path and \( \tau \geq 3 \), then the sequence \((v_0, v_1, \ldots, v_0)\) of adjacent vertices is a \( \tau \)-cycle. The minimum length of a cycle contained in the graph \( \Gamma \) is its girth \( g(\Gamma) \). For a quantum graph, we always refer to the girth of underlying combinatorial structure. Assigning two bonds to each edge, we will obtain two directed paths from any undirected one. Recall also that vertices of a regular graph are all of the same degree. Further, we consider \((d + 1)\)-regular graphs with equi-transmitting boundary conditions. The following statement holds.

Proposition 2 Let \( \hat{\Gamma} \) be a \((d + 1)\)-regular quantum graph with equi-transmitting boundary conditions and girth \( g(\Gamma) \). For each eigenvector \( \mathbf{a} \) of \( \mathbf{U}_\Gamma(\mathbf{x}) \) and \( \nu = (1 - s)^{-1} \), we then have
\[
\tilde{R}_s(\mathbf{a}) \geq \frac{g(\Gamma)}{4} \ln d
\] (0 \( s \leq 1 \)),
\[
\tilde{H}_s(\mathbf{a}) \geq \frac{1}{2} \ln \nu (d^{g(\Gamma)/2})
\] (0 \( s < 1 \)).
(3.26)
(3.27)

Proof. Following [17], we use a unitary matrix \( \mathbf{U}_\Gamma(\mathbf{x})^t \), where the power \( t \) obeys \( g(\Gamma)/2 \leq t < g(\Gamma)/2 + 1 \). By equi-transmitting boundary conditions, back-scattering is forbidden. To a non-zero entry \( u_{bb'}^{(t)} \), we can herewith assign a unique path of length \( \tau = t - 1 \) from the terminus of \( b' \) into the origin of \( b \). As our graph is \((d + 1)\)-regular, one finally gets [17]
\[
|u_{bb'}^{(t)}|^2 \leq d^{-g(\Gamma)/2}.
\] (3.28)

Combining the latter with (3.23) and (3.24) completes the proof. \( \square \)

We will further consider a sequence of \((d + 1)\)-regular combinatorial graphs \( \Gamma_n \) with natural \( n \) and suppose that the number of vertices \( V_n = |V_n| \) monotonically grows with \( n \). Such sequences of graphs are said to have large girth if there exists a constant \( C > 0 \) such that [48]
\[
g(\Gamma_n) = (C + o(1)) \frac{\ln V_n}{\ln d}
\] (3.29)
and \( o(1) \to 0 \) for \( n \to \infty \). In other words, the girth of a sequence element increases proportionally to the logarithm of the number of vertices. It can be shown that \( C \leq 2 \) with necessity [48]. Erdős and Sachs [49] gave a non-constructive proof of the existence of large-girth families of regular graphs with \( C = 1 \). At the same time, explicit examples are difficult to construct. The known explicit construction of such families is due to Margulis [50]. In the papers [49, 50], undirected graphs are considered. They can be transformed into digraphs as noticed above.
Let \( \{\Gamma_n\} \) be family of \((d+1)\)-regular quantum graphs with large girth and equi-transmitting boundary conditions, and let \( a_n \) be an eigenfunction of \( \Gamma_n(\varkappa) \). Combining (3.26) with (3.29) then gives
\[
\frac{R_\alpha(a_n)}{\ln V_n} \geq \frac{C + o(1)}{4}.
\]
This formula extends one of the results of [17] to symmetrized Rényi entropies. As was mentioned in [23], relations with a parametric dependence provide additional possibilities to analyze distributions under consideration.

IV. ENTROPIC CHARACTERIZATION OF EIGENFUNCTIONS OF STAR GRAPHS

In this section, eigenfunctions of star graphs will be characterized by means of entropies of associated eigenvectors. Properties of eigenvalues and eigenfunctions of such graphs were studied in detail [51, 52]. A star graph consists of a single central vertex together with outlying vertices, each of which is connected only to the central one. Here, the center has degree \( E \) and all other vertices have degree 1, so that \( V = E + 1 \). At the ends of the edges, Neumann boundary conditions are used. In each end, an incoming wave is merely reflected and will return into the center. Various boundary conditions at the central vertex may be assumed. The spectrum and eigenfunctions of a star graph do not behave typically for quantum chaotic systems, when it has Neumann like conditions at the central vertex [51, 52]. For other scattering matrices, e.g., equi-transmitting ones, the spectrum and eigenfunctions do behave generically. We will denote vertices by numbers \( i \in \{0, 1, \ldots, E\} \) with the central vertex 0. In this way, the edges are naturally labeled by elements of the set \( \{1, \ldots, E\} \). To \( j \)-th edge, with \( j \in \{1, \ldots, E\} \), one assigns the bonds \([j0]\) and \([0j]\). This notation mainly coincides with [54].

Due to the structure of star graphs, their properties can often be characterized in detail. Spectral determinants of various boundary conditions at the central vertex may be assumed. The spectrum and eigenfunctions of a star graph are then calculated in line with the definitions (3.1) and (3.6). The only point is that we now deal with \( E \) weights defined as
\[
\omega_e = \| A \|^2 |A_e|^2.
\]
Using (4.1), we can also construct an eigenfunction corresponding to the form (2.3). Along \( j \)-th edge, where \( j \in \{1, \ldots, E\} \), we write
\[
A_j \cos(\varkappa(x_{[j0]} - L_j)) = a_{[j0]} \exp(i \varkappa x_{[j0]}) + a_{[0j]} \exp(i \varkappa (L_j - x_{[j0]})),
\]
where the coefficients of incoming and outgoing waves are, respectively,
\[
a_{[j0]} = \frac{1}{2} A_j \exp(-i \varkappa L_j), \quad a_{[0j]} = \frac{1}{2} A_j.
\]
For \( j \in \{1, \ldots, E\} \), the matrix \( \sigma^{(j)} \) has size \( 1 \times 1 \) and entry 1. At the end of \( j \)-th edge, the condition (2.4) therefore reduces to multiplying \( a_{[j0]} \) by the factor \( \exp(i \varkappa L_j) \). The latter coincides with (4.4).

Thus, we assign to a star graph the two columns \( a \in \mathbb{C}^B \) and \( A \in \mathbb{C}^E \) with entries related by (4.3). Hence, the following connection between weights takes place. If the edge \( e \) generates the bonds \( b(e) \) and \( \bar{b}(e) \), then
\[
w_{b(e)} = w_{\bar{b}(e)} = \frac{\omega_e}{2}.
\]
In the case of star graphs, we deal with two entropic functions calculated on the base of \( \{w_e\} \) and \( \{\omega_e\} \), respectively. The link between theses functions is posed as follows.

**Proposition 3** Let columns \( a \in \mathbb{C}^B \) and \( A \in \mathbb{C}^E \) be assigned to the given eigenfunction of a star graph. For \( \alpha \in [0, \infty) \), the two Rényi \( \alpha \)-entropies are related as
\[
R_\alpha(a) = R_\alpha(A) + \ln 2.
\]
For \( \alpha \in (0, \infty) \), the two Tsallis \( \alpha \)-entropies are related as
\[
H_\alpha(a) = 2^{1-\alpha} H_\alpha(A) + \ln_\alpha(2).
\]
Proof. Due to \[\text{(4.15)},\] we first observe that
\[
\sum_{b \in B} w_b^\alpha = \sum_{e \in \mathcal{E}} \left\{ (w_{b(e)})^\alpha + (w_{b(e)})^\alpha \right\} = 2^{1-\alpha} \sum_{e \in \mathcal{E}} \sigma_e^\alpha.
\] (4.8)
Combining this with the definition \[\text{(3.1)},\] gives the claim \[\text{(4.6)}.\] Using \[\text{(4.8)},\] the entropy \(H_\alpha(a)\) is represented as
\[
\frac{1}{1-\alpha} \left( 2^{1-\alpha} \sum_{e \in \mathcal{E}} \sigma_e^\alpha - 1 \right) = \frac{2^{1-\alpha}}{1-\alpha} \left( \sum_{e \in \mathcal{E}} \sigma_e^\alpha - 1 \right) + \ln_\alpha(2).
\] (4.9)
The latter completes the proof of \[\text{4.7}.\] □
In the case \(\alpha = 1\), both the formulas \[\text{(4.6)}\] and \[\text{(4.7)}\] lead to the relation
\[
H_1(a) = H_1(A) + \ln 2.
\] (4.10)
The latter was mentioned in \[\text{[17]},\] but in the form with normalized entropies. Rescaling by maximal entropic values is not trivial, since the columns \(a\) and \(A\) have different numbers of entries. According to \[\text{(4.6)},\] the Rényi \(\alpha\)-entropies of \(a\) and \(A\) differ by additive term \(\ln 2\). The symmetrized entropies \(\tilde{R}_s(a)\) and \(\tilde{R}_s(A)\) are linked by the same relation.

It is not the case for symmetrized Tsallis entropies due to specific factors of the form \(2\ln e\). Rescaling by maximal entropic values, we have
\[
\frac{\tilde{R}_s(a)}{\ln B} = \frac{\ln E}{\ln E + \ln 2} \frac{\tilde{R}_s(A)}{\ln E} + \frac{\ln 2}{\ln E + \ln 2}.
\] (4.11)
To reach a stronger estimate, we should rather focus on normalized Rényi entropies of \(A\). In the case of star graphs, such entropies are of primary interest. For normalized Shannon entropies, this conclusion was formulated in \[\text{[17]}\].

The Riesz theorem leads to lower bounds on various entropies calculated with \(A\). The center is the only vertex with complicated picture. The junction condition at the central vertex is written in line with \[\text{(2.4)}\]. Substituting \[\text{(4.11)},\] into this condition gives
\[
A_j \exp(-i\kappa L_j) = \sum_{k=1}^{E} \sigma_j^{(0)} \exp(i\kappa L_k) A_k,
\] (4.12)
where we accordingly shorten the notation of matrix entries, \(\sigma_j^{(0)} \equiv \sigma_j^{(0)}_{[a][b]}. In matrix form, we write
\[
A = \exp(i\kappa L) \sigma^{(0)} \exp(i\kappa L) A,
\] (4.13)
with the diagonal \(E \times E\) matrix \(L = \text{diag}(L_1, L_2, \ldots, L_E)\). In other words, the column \(A\) is an eigenvector of certain unitary matrix corresponding to eigenvalue 1. Similarly to the formulas \[\text{(3.19)}\] and \[\text{(3.20)},\] we have arrived at a conclusion.

Proposition 4 Let column \(A \in \mathbb{C}^E\) be assigned to the given eigenfunction of a star graph. Then we have
\[
\tilde{R}_s(A) \geq -\ln \eta_0 \quad (0 \leq s \leq 1),
\] (4.14)
\[
\tilde{H}_s(A) \geq \frac{1}{2} \ln \nu (\eta_0^{-2}) \quad (0 \leq s < 1),
\] (4.15)
where \(\nu = (1-s)^{-1}\) and \(\eta_0 := \max |\sigma_e^{(0)}|.\)

To obtain further results, we should specify explicit conditions at the central vertex. We first consider a star graph with the Neumann boundary conditions at the central vertex. Suppose also that \(E \geq 4\), whence \(1 - 2/E \geq 2/E\). In terms of symmetrized entropies, one gives the relations
\[
\tilde{R}_s(A) \geq -\ln \left(1 - \frac{2}{E}\right) \quad (0 \leq s \leq 1),
\] (4.16)
\[
\tilde{H}_s(A) \geq \frac{1}{2} \ln \nu \left(\frac{E^2}{(E-2)^2}\right) \quad (0 \leq s < 1),
\] (4.17)
where \(\nu = (1-s)^{-1}\). For sufficiently large \(E\), the symmetrized Rényi entropy obeys
\[
\tilde{R}_s(A) \geq \frac{2}{E} + O\left(\frac{1}{E^2}\right).
\] (4.18)
Thus, we have extended one of the results of [17] to two families of generalized entropies. With the Neumann boundary conditions, an eigenfunction cannot be gathered on a single edge, but may be concentrated on two edges only. The above lower bounds coincide with the related discussion in subsection 4.1 of [17]. For large $E$, these bounds become vanishing. As was noted in [17], the entropic uncertainty principle cannot lead to a good bound for star graphs with Neumann like conditions at the center.

Let us proceed to the equi-transmitting boundary conditions. For equi-transmitting conditions at the central vertex, lower entropic bounds turn out to be almost optimal. In this case, diagonal entries of $\sigma^{(0)}$ are all zero. According to (2.6), for $e \neq e'$ we have

$$|\sigma^{(0)}_{ee'}|^2 = \frac{1}{E - 1}.$$  

Combining this with (4.14) and (4.15) gives the lower bounds

$$\tilde{R}_s(A) \geq \frac{1}{2} \ln(E - 1) \quad (0 \leq s \leq 1),$$

$$\tilde{H}_s(A) \geq \frac{1}{2} \ln\nu(E - 1) \quad (0 \leq s < 1).$$

Of course, we assume here that $E > 1$. For $s = 0$, both the relations reduce to the lower bound on the Shannon entropy. The latter was proved in [17], but in terms of normalized entropies. Combining (4.16) with (4.20), we also obtain a good lower bound on Rényi entropies of $a$. For a star graph with $E = B/2$ edges and equi-transmitting conditions at the central vertex, it holds that

$$\frac{\tilde{R}_s(a)}{\ln B} \geq \frac{\ln(B - 2) + \ln 2}{2 \ln B}.$$  

Asymptotically, this lower bound is close to $1/2$. At the same time, the left-hand side of (4.22) cannot exceed 1. So, the ratio of $\tilde{R}_s(a)$ to its maximal possible value may range in sufficiently limited interval only. In the case of equi-transmitting conditions at the central vertex, we have obtained a good estimate of symmetrized entropies from below.

V. RELATIONS BETWEEN GENERALIZED ENTROPIES AND THE VARIANCE

In this section, we will consider relations between entropies of an eigenfunction and its variance. In this case, we do not need in symmetrization of entropies with respect to the entropic parameter. The variance is widely used to characterize how some vector deviates from the equi-distributed one [17]. The variance is defined as follows. Using the weights (3.2), for the given $a \in \mathbb{C}^B$ one has

$$D(a) := \frac{1}{B} \sum_{b \in B} (Bw_b - 1)^2.$$  

When the given vector is equi-distributed, $w_b = 1/B$ for all $b \in B$ and $D(a) = 0$. When the variance is small, the vector is close to equi-distribution. It will be more convenient to represent the variance as

$$D(a) = -1 + B \sum_{b \in B} w_b^2.$$  

The maximal value of (5.1) is therefore equal to $B - 1$. The authors of [17] showed how the variance is connected with the concept of quantum ergodicity. In general, quantum ergodicity is not realized on finite quantum graphs. It should be replaced with weaker notion, which the authors of [57] called the asymptotic quantum ergodicity. They further developed a Gaussian random wave model on quantum graphs. A version of this model was previously introduced in [58]. For ergodic systems the behavior of almost all eigenfunctions in the semiclassical limit is described by the quantum ergodicity theorem [59, 60]. One of difficult problems here is to estimate the rate by which the expectation values approach the classical mean [61–64]. Quantum graphs can be used as a relatively simple model for studies such questions. Various aspects of ergodicity on quantum graphs were addressed in [53, 63, 67].

First of all, the variance is directly connected with the collision entropy. Combining (5.2) with (5.1) immediately gives

$$R_2(a) = \ln B - \ln(1 + D(a)).$$
We also recall that the Rényi $\alpha$-entropy cannot increase with growth of $\alpha$. Normalizing Rényi entropies by the denominator equal to $\ln B$, for $\alpha \in [0; 2]$ we obtain
\[
\frac{R_\alpha(a)}{\ln B} \geq 1 - \frac{\ln(1 + D(a))}{\ln B}.
\] (5.4)

In particular, this lower bound is valid for the normalized Shannon entropy. Since $\ln(1 + D(a)) \leq D(a)$, the result (5.4) has improved the statement of lemma 5 of [17].

A relation between the variance and Rényi entropies of order $\alpha > 2$ is more complicated. Let us begin with the min-entropy (3.5). For the probability distribution with $B$ weights $w_b$, we have
\[
\max\{w_b : b \in B\} \leq \frac{1}{B} \left(1 + \sqrt{(B-1)D(a)}\right). \quad (5.5)
\]

This result is obtained by combining (5.2) with lemma 3 of [43]. As the function $\xi \mapsto -\ln \xi$ decreases, the min-entropy obeys
\[
R_{\min}(a) \geq \ln B - \ln \left(1 + \sqrt{(B-1)D(a)}\right). \quad (5.6)
\]

Using the lower bounds (5.3) and (5.6), we have arrived at a conclusion.

**Proposition 5** For $\alpha \in [2, \infty]$, the Rényi $\alpha$-entropy of an eigenfunction is bounded from below as
\[
R_\alpha(a) \geq \ln B - \frac{\ln(1 + D(a))}{\alpha - 1} - \frac{\alpha - 2}{\alpha - 1} \ln \left(1 + \sqrt{(B-1)D(a)}\right). \quad (5.7)
\]

**Proof.** It was proved in proposition 1 of [68] that the Rényi $\alpha$-entropy of order $\alpha \geq 2$ obeys
\[
R_\alpha(a) \geq \frac{1}{\alpha - 1} R_2(a) + \frac{\alpha - 2}{\alpha - 1} R_{\min}(a). \quad (5.8)
\]

Combining the latter with (5.3) and (5.6) completes the proof. \(\square\)

The relations (5.4) and (5.7) may be used, when we focus on mean values of some quantities. For a family of graphs with finite spectral gap, a quantum ergodicity statement has been formulated in [57]. Using this result, the authors of [17] described a model, in which the variance of an eigenfunction obeys on average $\langle D(a) \rangle = O(1)$. Let us consider an ensemble of eigenfunctions, over which a mean value should be taken. Averaging (5.4), due to convexity of the function $\xi \mapsto -\ln(1 + \xi)$ we obtain
\[
\frac{\langle R_\alpha(a) \rangle}{\ln B} \geq 1 - \frac{\ln(1 + \langle D(a) \rangle)}{\ln B}, \quad (5.9)
\]

where $\alpha \in [0, 2]$. Combining concavity of the square root function with decreasing of $\xi \mapsto -\ln(1 + \xi)$, for $\alpha \geq 2$ we write
\[
\frac{\langle R_\alpha(a) \rangle}{\ln B} \geq 1 - \frac{\ln(1 + \langle D(a) \rangle)}{(\alpha - 1) \ln B} - \frac{\alpha - 2}{(\alpha - 1) \ln B} \ln \left(1 + \sqrt{(B-1)\langle D(a) \rangle}\right). \quad (5.10)
\]

In both the formulas (5.9) and (5.10), the left-hand side cannot exceed 1. We now suppose that $\langle D(a) \rangle = O(1)$ as $B \to \infty$. If the average of variances goes to a constant, then the Rényi entropy will tend at a logarithmic rate to 1. In [17], this claim was formulated and numerically supported for the Shannon entropy. Our results show that similar reasons are applicable to averaged Rényi entropies.

Many findings of the above discussion can be reformulated with entropies of the Tsallis type. Like (5.3), the linear entropy is immediately connected with the variance. Substituting $\alpha = 2$ into (3.6) and using (5.2), we get
\[
H_2(a) = 1 - \frac{1 + D(a)}{B}. \quad (5.11)
\]

After averaging, we will still have the exact equality instead of inequalities such as (5.9) and (5.10). Assuming $\langle D(a) \rangle = O(1)$, the averaged linear entropy will obviously tend to the limit 1. For other entropic parameters, we should separately consider the intervals $\alpha \in (0, 2]$ and $\alpha \in [2, \infty)$. The following statement holds.
Proposition 6 For $\alpha \in (0, 2]$, the Tsallis $\alpha$-entropy of an eigenfunction is bounded from below as

$$H_\alpha(a) \geq \ln_\alpha \left( \frac{B}{1 + D(a)} \right).$$

Proof. According to (3.6) and (5.7), the Tsallis $\alpha$-entropy can be rewritten as

$$H_\alpha(a) = \sum_{b \in B} w_b \ln_\alpha \left( \frac{1}{w_b} \right).$$

By inspection of the second derivative, the function $\xi \mapsto \ln_\alpha(1/\xi)$ is convex for $\alpha \in (0, 2]$. Due to Jensen’s inequality, we then obtain

$$\sum_{b \in B} w_b \ln_\alpha \left( \frac{1}{w_b} \right) \geq \ln_\alpha \left\{ \left( \sum_{b \in B} w_b^2 \right)^{-1} \right\}.$$  

(5.14)

Combining the latter with (5.2) finally gives the claim (5.12). □

In a structure, the right-hand side of (5.12) is similar to (5.4) multiplied by $\ln B$. Similarly to the Rényi case, we will consider averages over an ensemble of eigenfunctions. Due to the mentioned concavity, averaging results in

$$\langle H_\alpha(a) \rangle \geq \ln_\alpha \left( \frac{B}{1 + \langle D(a) \rangle} \right),$$

(5.15)

where $0 < \alpha \leq 2$. Further, we recall the identity

$$\ln_\alpha(\xi z) = \ln_\alpha(\xi) + \xi^{1-\alpha} \ln_\alpha(z),$$

which follows from (3.7) immediately. The maximal value of Tsallis’ $\alpha$-entropy with $B$ weights is equal to $\ln_\alpha(B)$. Applying (5.16) to (5.15) and rescaling it by the denominator $\ln_\alpha(B)$, for $\alpha \in (0, 2]$ we have

$$\frac{\langle H_\alpha(a) \rangle}{\ln_\alpha(B)} \geq 1 - \frac{1}{\ln_\alpha(1/B)} \ln_\alpha \left( \frac{1}{1 + \langle D(a) \rangle} \right).$$

(5.17)

For $\alpha \geq 1$, the modulus of $\ln_\alpha(1/B)$ tends to infinity as $B \to \infty$. When an ensemble is such that $\langle D(a) \rangle = O(1)$ in this limit, the ratio of averaged Tsallis’ $\alpha$-entropy to $\ln_\alpha(B)$ is bounded by 1 from below. Also, this ratio cannot exceed 1. We may conclude that, on average, the Tsallis $\alpha$-entropies with $\alpha \in (1, 2)$ will reveal a behavior similar to both the Shannon entropy and the linear entropy (5.11). These are obtained from (3.6) for $\alpha = 1$ and $\alpha = 2$, respectively.

For $\alpha \geq 2$, lower bounds on the Tsallis $\alpha$-entropy are more complicated in formulation. For such values of $\alpha$, we first write the inequality

$$\sum_{b \in B} w_b^\alpha \leq (\max w_b)^{\alpha-2} \sum_{b \in B} w_b^2 \leq B^{1-\alpha} \left( 1 + D(a) \right) \left( 1 + \sqrt{(B-1)D(a)} \right)^{\alpha-2},$$

(5.18)

which is based on (5.5). For $\alpha \geq 2$, the Tsallis $\alpha$-entropy of an eigenfunction is bounded from below as

$$H_\alpha(a) \geq \frac{1}{\alpha - 1} \left\{ 1 - B^{1-\alpha} \left( 1 + D(a) \right) \left( 1 + \sqrt{(B-1)D(a)} \right)^{\alpha-2} \right\}. $$

(5.19)

However, converting the latter into a relation with averaged values seems to be difficult. This question may deserve further investigations.

To sum up, we see the following. Lower bounds on suitably rescaled Rényi and Tsallis entropies are expressed as 1 minus a term depending on the variance. Such normalized entropies are also bounded from above by 1. Many of the derived relations remain valid after averaging over an ensemble of eigenfunctions. Suppose that we deal with some eigenfunction ensemble and consider growing number of bonds. When $\langle D(a) \rangle = O(1)$ as $B \to \infty$, averaged Rényi’s $\alpha$-entropy will tend at a logarithmic rate to 1 for all $\alpha \in [0, \infty]$. In the same case, the ratio of averaged Tsallis $\alpha$-entropy to $\ln_\alpha(B)$ will tend to 1 for all $\alpha \in [1, 2]$. These findings suggested that, on average, an eigenfunction will be distributed over a graph with equal weights.
VI. CONCLUSIONS

We have studied generalized entropies of eigenfunctions of finite quantum graphs. The derivation is based on Riesz’s theorem using an approach similar to that of Maassen and Uffink [23]. Lower bounds on the symmetrized Rényi and Tsallis entropies were given in terms of the maximal modulus among entries of the unitary evolution matrix. Such estimates can sometimes be related to certain geometric characteristics of a graph. Due to relative simplicity of star graphs, they give a good example to test derived entropic bounds. The central vertex of a star graph is the only vertex, at which complicated redistribution of incoming waves takes place. For star graphs with Neumann conditions at the center, entropic lower bounds do not provide a good estimate. For the case of equi-transmitting conditions at the center, our approach leads to good lower bounds on symmetrized entropies.

Lower bounds for the Rényi and Tsallis entropies without symmetrization were expressed in terms of the variance of a graph eigenfunction. The variance is a natural measure of the deviation of corresponding eigenvector from the equi-distributed one. The obtained bounds are shown to be useful, when we average quantities over an ensemble of eigenfunctions and assume growing number of bonds. If the considered ensemble is of small variation, normalized entropies will tend to 1. For the normalized Shannon entropy, this fact was proved and illustrated in [17]. We have extended the above conclusion to the Rényi $\alpha$-entropy for $\alpha \geq 0$ and to the Tsallis $\alpha$-entropy for $1 \leq \alpha \leq 2$. In models with growing number of bonds and a limited average variance, we will see the equi-distribution of an eigenfunction at the microscopic scale.

[1] Kottos T and Smilansky U 1999 Ann. Phys. 274 76
[2] Kottos T and Smilansky U 1997 Phys. Rev. Lett. 79 4794
[3] Roth J-P 1983 C. R. Acad. Sci. Paris, Sér. I Math. 296 793
[4] Kottos T and Schanz H 1997 Physica E 9 523
[5] Gnutzmann S and Smilansky U 2006 Adv. Phys. 55 527
[6] Smilansky U 2007 J. Phys. A: Math. Theor. 40 F621
[7] Stöckmann H-J 1999 Quantum Chaos: An Introduction (Cambridge: Cambridge University Press)
[8] Berkolaiko G and Kuchment P 2012 Introduction to Quantum Graphs (Providence, RI: American Mathematical Society)
[9] Berkolaiko G 2016 An elementary introduction to quantum graphs arXiv 1603.07356 [math-ph]
[10] Exner P 1997 Ann. Inst. H. Poincaré: Phys. Théor. 66 359
[11] Kostrykin V and Schrader R 1999 J. Phys. A: Math. Gen. 32 595
[12] Kuchment P 2004 Waves Random Media 14 S107
[13] Kuchment P 2005 J. Phys. A: Math. Gen. 38 4887
[14] Gnutzmann S and Altland A 2005 Phys. Rev. E 72 056215
[15] Kurasov P, Malenová G, and Naboko S 2013 J. Phys. A: Math. Theor. 46 275309
[16] Życzkowski K 1990 J. Phys. A: Math. Gen. 23 4427
[17] Kameni L and Schubert R 2014 Entropy of eigenfunctions on quantum graphs arXiv:1405.5871 [math-ph]
[18] Anantharaman N and Nonnenmacher S 2007 Ann. Henri Poincaré 8 37
[19] Gutkin B 2010 Commun. Math. Phys. 294 303
[20] Anantharaman N and Nonnenmacher S 2007 Ann. Inst. Fourier, Grenoble 57 2465
[21] Anantharaman N 2008 Ann. Math. 168 435
[22] Pakoński P, Życzkowski K, and Kuś M 2001 J. Phys. A: Math. Gen. 34 9303
[23] Maassen H and Uffink J B M 1988 Phys. Rev. Lett. 60 1103
[24] Beck C and Schlögl F 1993 Thermodynamics of Chaotic Systems: An Introduction (Cambridge: Cambridge University Press)
[25] Bengtsson I and Życzkowski K 2006 Geometry of Quantum States: An Introduction to Quantum Entanglement (Cambridge: Cambridge University Press)
[26] Rényi A 1961 On measures of entropy and information Proc. 4th Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, CA: University of California Press) 547–61
[27] Tsallis C 1988 J. Stat. Phys. 52 479
[28] Tanner G J 2001 J. Phys. A: Math. Gen. 34 8485
[29] Harrison J M, Smilansky U, and Winn B 2007 J. Phys. A: Math. Theor. 40 14181
[30] Whitteman A L 1971 Pacific J. Math. 38 817
[31] Doković D Z 1992 J. Comb. Theory, Ser. A 61 319
[32] Georgiou S, Koukouvinos C, and Stylianou S 2002 Comput. Stat. Data Anal. 41 171
[33] Rose H E 1994 A Course in Number Theory (Oxford: Oxford University Press)
[34] Friedlander L 2005 Israel J. Math. 146 149
[35] Colin de Verdière Y 2015 Annales Henri Poincaré 16 347
