Pulsar magnetospheres: a general relativistic treatment

Hongsu Kim,1* Hyung Mok Lee,1* Chul H. Lee2* and Hyun Kyu Lee2*

1Astronomy Program, SEES, Seoul National University, Seoul, 151-742, Korea
2Department of Physics, Hanyang University, Seoul, 133-791, Korea

Accepted 2005 January 17. Received 2005 January 10; in original form 2004 July 26

ABSTRACT
A fully general relativistic description of pulsar magnetospheres is provided. To be more specific, a study of pulsar magnetospheres is performed in the context of general relativistic magnetohydrodynamics (MHD) employing the so-called Grad–Shafranov approach. Not surprisingly, the resulting Grad–Shafranov equations and all the other related general relativistic MHD equations turn out to take essentially the same structures as those for a (rotating) black hole magnetosphere. Other different natures between the two cases including the structure of singular surfaces of MHD flows in each magnetosphere are essentially encoded in the different space–time (metric) contents. In this way, pulsar and black hole magnetospheres can be described in a unified fashion. Particularly, the direction of poloidal currents circulating in the neutron star magnetosphere turns out to be the same as that of currents circulating in a black hole magnetosphere, which in turn leads to pulsar and black hole spin-down via ‘magnetic braking’.

Key words: black hole physics – magnetic fields – MHD – stars: magnetic fields – pulsars: general.

1 INTRODUCTION
Radio pulsars are perhaps the oldest-known and the least energetic of all types of pulsar. The theoretical study of these radio pulsars can be traced back to Goldreich & Julian (1969). In their pioneering work, Goldreich and Julian argued that pulsars, which they thought to be rotating magnetized neutron stars, must have a magnetosphere with charge-separated plasma. They then demonstrated that an electric force much stronger than the gravitational force would be set up along the magnetic field, and as a result the surface charge layer could not be in dynamical equilibrium. Then there would appear steady current flows along the magnetic field lines which were taken to be uniformly rotating since they would be firmly rooted in the crystalline crust of the pulsar surface. Although it was rather implicit in their original work, this model of pulsar electrodynamics does indeed suggest that the luminosity of radio pulsars is due to the loss of their rotational energy (namely the spin-down) and its subsequent conversion into charged particle emission that eventually generates radiation in the far zone.

Since the pioneering proposals of Gold (1968) and by Pacini (1967) and (Pacini 1968), it has by now been widely accepted that indeed pulsars might be rotating magnetized neutron stars. Nevertheless, since then nearly all studies on pulsar electrodynamics have been performed by simply treating the region surrounding rotating neutron stars as flat. This simplification may be sufficient just to gain some insight into a rough understanding of the origin of pulsar radiation. However, more than thirty years study of pulsar electrodynamics have now passed and it seems time to treat the problem in a more careful and rigorous manner, namely in a fully relativistic fashion. Here we mean by relativistic treatment that we are dealing with highly self-gravitating compact objects and of course we shall have rotating neutron stars in mind. Thus we begin by providing the rationale for treating the vicinity of spinning compact objects such as rotating neutron stars relativistically as non-trivial curved space–time. Indeed, we now have a considerable amount of data observed for various pulsar types, from radio pulsars (Goldreich & Julian 1969) to (anomalous) X-ray pulsars (Mereghetti & Steeal 1995; Kouveliotou et al. 1998; Mereghetti 2000; Thompson 2001). Even if we take the oldest-known radio pulsars, for example, it is not hard to realize that these objects are compact enough to be treated in a general relativistic manner. To be more specific, note that the values of the parameters characterizing typical radio pulsars are: $r_0$ (radius) $\sim 10^6$ (cm); $M$ (mass) $\sim 1.4 M_\odot \sim 2 \times 10^{33}$ (g); $\tau$ (pulsation period) $\sim 10^{-3} - 1$ (s); $B$ (magnetic field strength) $\sim 10^{12}$ (G). Thus the Schwarzschild radius (gravitational radius) of a typical radio pulsar is estimated to be $r_{Sch} = 2G M_\odot/c^2 \sim 3 \times 10^5$ (cm) (where $G$ and $c$ denote Newton’s constant and the speed of light, respectively) and hence one ends up with the ratio $r_0/r_{Sch} \sim 10^6$ (cm)/$3 \times 10^5$ (cm) $\sim 3$. This simple argument indicates that indeed even the radio pulsar (which is perhaps the least energetic among all types of pulsar) is a highly self-gravitating compact object that needs to be treated relativistically. Once again, the ‘relativistic’ treatment here means that the

*E-mail: hongsu@astro.snu.ac.kr (HK); hmlee@astro.snu.ac.kr (HML); chlee@hepth.hanyang.ac.kr (CHL); hyunkyu@hanyang.ac.kr (HKL)

© 2005 RAS
region surrounding a pulsar, namely the pulsar magnetosphere, has to be described by a curved space–time rather than simply a flat one. The question now boils down to: ‘What would be the relevant metric to describe the vicinity of a rotating neutron star?’ Although it does not seem to be well-known, fortunately we have such a metric of the region exterior to slowly rotating relativistic stars such as neutron stars, white dwarfs and supermassive stars – it is the one constructed long ago by Hartle & Thorne (1968). Thus the Hartle–Thorne metric is a stationary axisymmetric solution to the vacuum Einstein equation and in the present work we shall take this Hartle–Thorne metric as the appropriate one to represent the space–time exterior to slowly rotating neutron stars.

Having provided the rationale for treating the region surrounding a magnetized rotating neutron star, namely a pulsar, fully relativistically as curved space–time, we now state our particular objective in this work. In the present work, we would like to study pulsar magnetospheres in the context of general relativistic magnetohydrodynamics (MHD) by employing the so-called Grad–Shafranov approach (Shafranov 1958; Grad 1960). We shall consider both the force-free and full MHD situations and accordingly derive the Grad–Shafranov equations for each case, namely the pulsar equation and the pulsar jet equation, respectively. Not surprisingly, then, the resulting Grad–Shafranov equations and all the other related force-free equations or general relativistic MHD equations turn out to take essentially the same structures as those for a (rotating) black hole magnetosphere (Macdonald & Thorne 1982; Thorne & Macdonald 1982; Okamoto 1992; Beskin & Pariev 1993; Beskin 1997). The essential distinction between the two cases, however, is the space–time (metric) content. For the pulsar magnetosphere case, one needs to choose the Hartle–Thorne metric mentioned above whereas for the black hole magnetosphere case, one has to select the Kerr black hole metric (Kerr 1963). As a consequence of this, the singular surfaces of MHD flows in pulsar and black hole magnetospheres exhibit substantially different natures. In this way, pulsar and black hole magnetospheres can be described in a unified fashion. This last point, namely a unified picture for both pulsar and black hole electrodynamics, is the key proposal of the present work. There is, however, as stronger motive for the present work – the uncomfortable current state of affairs that there is still no generally accepted stance on the structure of pulsar magnetospheres (Beskin 1999). To be more specific, there is as yet no complete model for the structure of longitudinal (or poloidal) currents circulating in neutron star magnetospheres that can provide a solution to the problem, say, of pulsar spin-down (namely, the ‘braking’ of rotating neutron stars). As we shall see shortly in the text, a partly satisfactory solution to this problem will be provided by treating the region outside a magnetized rotating neutron star as a curved space–time represented by the Hartle–Thorne metric. Namely it turns out that the structure of space-charge separation and the direction of poloidal current particularly in the force-free limit correctly lead to the braking torque that spins down the rotating neutron star. This implies that, particularly from a technical point of view, a Blandford–Znajek type mechanism can be adopted to provide a model that takes a magnetized rotating neutron star as the central engine for some radio, X-ray and even (soft) gamma-ray astrophysical phenomena (Mereghetti & Steelaer 1995; Kouveliotou et al. 1998; Mereghetti 2000; Thompson 2001), just as it has been employed to construct a model taking a rotating black hole as the central engine for active galactic nuclei (AGN)/quasars (Blandford & Znajek 1977; for some recent studies on its theoretical aspects, see Kim, Lee & Lee 2001a,b) or even gamma-ray bursts (GRBs) (Lee, Wijers & Brown 2000; Kim, Lee & Lee 2003). It seems fair, however, to say that this observation should not be taken as a new discovery but as something that has been expected to some extent. And it is due to the fact that historically the Blandford–Znajek mechanism has been strongly motivated by and thus constructed from the original pulsar model of Goldreich & Julian (1969) (and perhaps others) with the purpose of applying the main idea to the case of rotating black holes. We also note that there have been extensive critical examinations of the operational aspect of the Blandford–Znajek type mechanism carried out by Punsly (Punsly & Coroniti 1990; Punsly 1991, 2001).

2 ELECTRODYNAMICS AROUND SLOWLY ROTATING NEUTRON STARS

2.1 Hartle–Thorne metric for the region exterior to slowly rotating neutron stars

The Hartle–Thorne metric (Hartle & Thorne 1968) is given, in terms of the (3 + 1)-split form, by
\begin{equation}
\text{d}s^2 = -\alpha^2 \text{d}t^2 + g_{rr} \text{d}r^2 + g_{\theta\theta} \text{d}\theta^2 + g_{\phi\phi} (\text{d}\phi + \beta^\phi \text{d}t)^2,
\end{equation}
where the lapse \( \alpha \), (angular) shift \( \beta^\phi \) and the metric components are
\begin{equation}
\begin{aligned}
\alpha^2 &= R, \quad \beta^\phi = -\omega = -\frac{2J}{r^2 M}, \\
g_{rr} &= \frac{\Delta}{\rho^2 + \Delta}, \quad g_{\theta\theta} = r^2 A, \quad g_{\phi\phi} = r^2 A \sin^2 \theta,
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
\Delta &= \left(1 - \frac{2M}{r} + \frac{\beta^2}{r^2} \right), \\
R &= \left\{ 1 + 2 \left[ \frac{\beta^2}{r^2} (1 + \frac{\beta^2}{r^2}) \right] + \frac{3}{4} \frac{Q}{r^3 M} \left[ \frac{r}{2} - 1 \right] \right\} P_2(\cos \theta), \\
S &= \left\{ 1 - 2 \left[ \frac{\beta^2}{r^2} (1 - \frac{\beta^2}{r^2}) \right] + \frac{3}{4} \frac{Q}{r^3 M} \left[ \frac{r}{2} - 1 \right] \right\} P_2(\cos \theta), \\
A &= 1 + 2 \left[ -\frac{\beta^2}{r^2} \right] + \frac{3}{4} \frac{Q}{r^3 M} \left[ \frac{r}{2} - 1 \right] P_2(\cos \theta),
\end{aligned}
\end{equation}

© 2005 RAS, MNRAS 358, 998–1018
with $M$, $J$ and $Q$ being the mass, the angular momentum and the mass quadrupole moment of the (slowly) rotating neutron star, respectively, $P_2(\cos \theta) = (3 \cos^2 \theta - 1)/2$ being the Legendre polynomial, and $Q^n_\alpha$ being the associated Legendre polynomial, namely

$$\begin{align*}
Q^1_1(z) &= (z^2 - 1)^{1/2} \left[ \frac{z^2}{z^2 + 1} - \frac{1}{4} z \log \left( \frac{z + 1}{z - 1} \right) \right], \\
Q^2_2(z) &= \frac{3}{4}(z^2 - 1) \log \left( \frac{z + 1}{z - 1} \right) - \frac{3z^2 + 5z}{2z^2 + 1}
\end{align*}$$

and hence

$$\begin{align*}
Q^1_1 \left( \frac{M}{r} - 1 \right) &= \frac{r}{\sqrt{r}} \left[ 1 - \frac{2M}{r} \right]^{1/2} \left[ \frac{3Q/M^2(r - 2M)}{r} + \frac{3}{2} \log \left( 1 - \frac{2M}{r} \right) \right], \\
Q^2_2 \left( \frac{M}{r} - 1 \right) &= -\left[ \frac{3}{4} \log \left( 1 - \frac{2M}{r} \right) + \frac{M}{r} \left( 1 - \frac{2M}{r} \right) \right]
\end{align*}$$

As is well-known, the only known exact metric solution exterior to a rotating object is the Kerr metric (Kerr 1963). Thus it would be worth clarifying the relation of the Hartle–Thorne metric for slowly rotating relativistic stars given above to the Kerr metric. As Hartle & Thorne (1968) pointed out, by taking the Kerr metric given in Boyer–Lindquist coordinates and expanding it to second order in the angular velocity (namely, the angular shift $\beta^\phi$) followed by a coordinate transformation in the $(r, \theta)$-sector of

$$\begin{align*}
r \rightarrow r \left[ 1 - \frac{2M}{r} \left( 1 + \frac{Q}{M} \right) + \cos^2 \theta \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{Q}{M} \right) \right], \\
\theta \rightarrow \theta - a^2 \cos \theta \sin \theta \frac{r}{\sqrt{r}} \left( 1 + \frac{2M}{r} \right),
\end{align*}$$

where $a = J/M$, one can realize that the resulting expanded Kerr metric coincides with the particular case $Q = J^2/M$ (with $Q$ being the mass quadrupole moment of the rotating object) of the Hartle–Thorne metric. Therefore, in general this Hartle–Thorne metric is not a slow-rotation limit of the Kerr metric. Rather, the slow-rotation limit of Kerr metric is a special case of this more general Hartle–Thorne metric. As a result, the Hartle–Thorne metric with an arbitrary value of the mass quadrupole moment $Q$ can generally describe a (slowly rotating) neutron star of any shape (as long as it retains its axisymmetry).

Next, because we shall employ the Hartle–Thorne metric in the present work to represent the space–time exterior of slowly rotating neutron stars, we would like to make a careful distinction between the Hartle–Thorne metric horizon and the actual radius of a neutron star. Indeed, one of the obvious differences between the black hole case and the neutron star case is the fact that a black hole is characterized by its event horizon while a neutron star has a hard surface. Since this solid neutron star surface (which we shall henceforth denote by $r_H$) lies outside its gravitational radius, which amounts to the Killing horizon radius of the Hartle–Thorne metric ($r_H$) at which $\dot{r} = (1 - 2M/r + 2J^2/r^3) = 0$ in equation (3), we have $r_H > r_H$. In the present work, however, we shall never speak of the Killing horizon of the Hartle–Thorne metric as it is an irrelevant quantity playing no physical role.

Now we turn to the choice of an orthonormal tetrad frame, and we shall choose in particular the zero-angular-momentum-observer (ZAMO) frame (Bardeen 1970; Bardeen, Press & Teukolsky 1972), which is a sort of fiducial-observer (FIDO) frame. Generally speaking, in order to represent a given background geometry, one needs first to choose a coordinate system in which the metric is to be given and next, in order to obtain the physical components of a tensor (such as the electric and magnetic field values), one has to select a tetrad frame (in a given coordinate system) on to which the tensor components are to be projected. As is well-known, the orthonormal tetrad is a set of four mutually orthogonal unit vectors at each point in a given space–time that give the directions of the four axes of a locally-Minkowskian coordinate system. Such an orthonormal tetrad associated with the Hartle–Thorne metric given above may be chosen as $e^\alpha = e^\alpha_i \ dx^i = (e^0, e^1, e^2, e^3)$,

$$\begin{align*}
e^0 &= \alpha \ dx = (R)^{1/2} \ dx, \\
e^1 &= g^1_{\phi \phi} d\phi = \left( \frac{1}{r} \right)^{1/2} \ dx, \\
e^2 &= g^2_{\theta \theta} d\theta = r A^{1/2} d\theta, \\
e^3 &= g^3_{\phi \phi} (d\phi + \beta^\phi \ dx) = r \sin \theta A^{1/2} \left( d\phi - \frac{2J}{r^2} \right)
\end{align*}$$

and its dual basis is given by $e^i = e^i_\alpha \partial_\alpha = (e_0 = e_{i0}, e_1 = e_{i1}, e_2 = e_{i2}, e_3 = e_{i3})$.

$$\begin{align*}
e_0 &= \frac{1}{2} \left( \partial_i - \beta^\phi \partial_\phi \right) = (R)^{-1/2} \left( \partial_i + \frac{2J}{r^2} \partial_\phi \right), \\
e_1 &= g^1_{\phi \phi} \partial_\phi = \left( \frac{1}{r} \right)^{1/2} \partial_i, \\
e_2 &= g^2_{\theta \theta} \partial_\theta = \frac{1}{r A^{1/2}} \partial_i, \\
e_3 &= g^3_{\phi \phi} \partial_\phi = \frac{1}{r A^{1/2} \sin \theta} \partial_i.
\end{align*}$$

The local, stationary observer at rest in this orthonormal tetrad frame $e^i$ has the worldline given by $\{dr = 0, \ d\theta = 0, (d\phi + \beta^\phi \ dx) = 0\}$ which is orthogonal to spacelike hypersurfaces and has orbital angular velocity given by

$$\omega = \frac{d\phi}{dr} = -\beta^\phi = -\frac{g_{\phi \phi}}{g_{\phi \phi}} = \frac{2J}{r^3}.$$ 

This is the long-known Lense–Thirring precession (Lense & Thirring 1918) angular velocity arising owing to the ‘inertial frame-dragging’ effect of a stationary axisymmetric space–time. Indeed, it is straightforward to demonstrate that this orthonormal tetrad observer can be
identified with a ZAMO carrying zero intrinsic angular momentum with it. To this end, recall that when a space–time metric possesses a rotational (azimuthal) isometry, there exists an associated rotational Killing field \( m^\alpha = (\partial / \partial \phi)\alpha = \delta^\alpha_\phi \) such that its inner product with the tangent (velocity) vector \( u^\mu = dx^\mu / d\tau \) (with \( \tau \) denoting the particle’s proper time) of the geodesic of a test particle is constant along the geodesic, i.e.

\[
\tilde{L} = g_{\alpha\beta} m^\alpha u^\beta = g_{\phi\phi} m^\phi u^\phi + g_{\phi\phi} m^\phi u^\phi = g_{\phi\phi} \frac{d\phi}{d\tau} + g_{\phi\phi} \frac{d\phi}{d\tau}.
\]

\( \tilde{L} = 0 \), its angular velocity becomes

\[
\omega = \frac{d\phi}{d\tau} = -\frac{g_{\phi\phi}}{g_{\phi\phi}} = -\beta^\phi
\]

and this confirms the identification of the local observer at rest in this orthonormal tetrad frame given above with a ZAMO.

2.2 Electrodynamics in curved space–time: the (3 + 1)-split formalism

Generally speaking, when dealing with electrodynamics in a curved space–time, relativists prefer a geometric, covariant, frame-independent approach, representing, say, the electromagnetic field by the field-strength tensor \( F_{\mu\nu} \). Astrophysicists, on the other hand, would prefer to split this tensor into a three-dimensional electric field \( E \) and magnetic field \( B \), sacrificing the general covariance of the theory in order to get some insight and achieve a comparison with the familiar flat space–time electrodynamics. As has been first developed by Thorne & Macdonald (1982) and Macdonald & Thorne (1982), fortunately in stationary curved space–times such as those outside rotating black holes (the Kerr metric) and rotating neutron stars (the Hartle–Thorne metric discussed above), one can actually reformulate electrodynamics in terms of an absolute three-dimensional space and a universal time – and the variables in this reformulation are the familiar electric and magnetic fields \( (E, B) \) and charge and current density \( (\rho, j) \). Indeed, this absolute-space/universal-time formulation of stationary curved space–times has deep roots in the so-called \( (3 + 1) \)-split formalism of general relativity, which had originally been employed in the canonical (or Hamiltonian) quantization of gravity in the 1950s and is nowadays being used in numerical relativity. Normally, however, such a \( (3 + 1) \)-split formalism has not been welcome by most relativists owing to the arbitrariness of the choice of fiducial reference frame. In the case of stationary black hole/neutron star electrodynamics, however, there is one set of fiducial observers preferred over all others: the zero-angular-momentum observer (ZAMO) or locally-non-rotating frame (LNRF) observer that we discussed in the above subsection. As is well-known and as we shall see in a moment, when one uses this ZAMO frames one realizes that the \( (3 + 1) \) equations of black hole/neutron star electrodynamics are nearly identical to their counterparts in flat space–time electrodynamics. For general formulations and more detailed discussions of this \( (3 + 1) \)-split formalism we refer the reader to (Macdonald & Thorne 1982) and (Thorne & Macdonald 1982). As such, throughout this work, we shall also employ this space-plus-time formalism in which all the physical quantities are represented by three-dimensional scalars and vectors as measured by ZAMO, the local observer. For instance, the physical electric and magnetic field components as measured by ZAMO can be read off as the projection of \( F_{\mu\nu} \) on to the ZAMO orthonormal tetrad frame (equation 8), \( F_{AB} = F_{\mu\nu} (e^A_\mu e^B_\nu) \) and \( F_{AB} = \{ F_{0i}, F_i \} \), where

\[
E_i = F_{i0}, \quad E = \{ E_i \}, \quad B_i = \frac{1}{2} \varepsilon_{ijk} F^{jk}, \quad B = \{ B_i \}. 
\]

3 PULSAR EQUATION – THE FORCE-FREE LIMIT OF THE GRAD–SHAFRANOV EQUATION

The so-called pulsar equation refers to the force-free limit of the more general Grad–Shafranov equation. The force-free condition essentially amounts to ignoring the (inertial) contributions of the plasma particles when the plasma energy density is assumed to be substantially smaller than that of the magnetic field.

3.1 Basic equations

3.1.1 Force-free condition

In order eventually to describe the force-free pulsar magnetosphere, we start with the Maxwell equations in the background of the stationary axisymmetric rotating neutron star space–time (Macdonald & Thorne 1982; Thorne & Macdonald 1982)

\[
\begin{align*}
\nabla \cdot E &= 4\pi \rho, \\
\n\nabla \times (\alpha E) &= (B \cdot \nabla) m, \\
\n\nabla \times (\alpha B) &= 4\pi \alpha j - (E \cdot \nabla) m,
\end{align*}
\]

where we dropped the terms \( \mathcal{L}_k(\ldots) = 0, \mathcal{L}_m(\ldots) = 0 \) owing to stationarity and axisymmetry. Additionally, here \( k = (\partial / \partial t) \) and \( m = (\partial / \partial \phi) \) denote the time-translational and the rotational Killing fields associated with the stationarity and the axisymmetry of the Hartle–Thorne metric, respectively, and hence \( m \cdot m = g_{\phi\phi} = \omega^2 \) and \( m = e_\phi = (g_{\phi\phi})^{1/2} e_\phi \). Note also that since all the measurements are made by ZAMO,
the lapse function $\alpha$ is introduced to convert the ZAMO’s proper time $d\tau$ over to the global time $dt$. Throughout this section, the force-free condition is assumed to hold, i.e.

$$\rho_e E + j \times B = 0, \quad B = B_T + B_p,$$  \hspace{1cm} (13)

which also implies the degeneracy condition $E \cdot B = 0$. Then this force-free condition indicates that the charged particles are flowing along the magnetic field lines and hence the toroidal (angular) velocity of magnetic field lines (which are frozen into plasma) relative to ZAMO is given by

$$v_\phi = \left( \frac{\Omega_F - \omega}{\alpha} \right) \sigma e_\phi = \left( \frac{\Omega_F - \omega}{\alpha} \right) m$$  \hspace{1cm} (14)

and then $j = j_T + j_p$ with $j_T = \rho_e v_\tau$, where $v_\tau$ consists of $v_T$ given above and the streaming velocity along the toroidal magnetic field lines. Then, from the force-free condition above, it follows that

$$E = E_p = -\frac{1}{\rho_e} j_T \times B_p.$$  \hspace{1cm} (15)

### 3.1.2 Poloidal field components

Consider a magnetic flux through an area $A$ whose boundary is an $m$-loop,

$$\Psi = \int_A B \cdot dS.$$  \hspace{1cm} (16)

Then from $d\Psi = \nabla \Psi \cdot d\mathbf{r}$ and alternatively $d\Psi = B \cdot (d\mathbf{r} \times 2\pi \sigma e_\phi) = (2\pi \sigma e_\phi \times B) \cdot d\mathbf{r}$, we get

$$B_p = \nabla \Psi \times e_\phi = \frac{\nabla \Psi \times m}{2\pi \sigma},$$  \hspace{1cm} (17)

where we used $m \cdot m = g_{\phi\phi} = \sigma^2$ and hence

$$E_p = -v_\tau \times B_p = -\frac{\Omega_F - \omega}{2\pi \sigma} \nabla \Psi.$$  \hspace{1cm} (18)

### 3.1.3 Poloidal current and toroidal magnetic field

Consider a poloidal current through the same area whose boundary is an $m$-loop,

$$I = -\int_A \alpha j \cdot dS.$$  \hspace{1cm} (19)

Then similarly to the case of a poloidal magnetic field, $\nabla I = -2\pi \sigma e_\phi \times (\alpha j_p)$, we get

$$\alpha j_p = -\frac{\nabla I \times e_\phi}{2\pi \sigma} = -\frac{\nabla I \times m}{2\pi \sigma^2}$$  \hspace{1cm} (20)

and then, from equations (17) and (20),

$$j_p = -\frac{1}{\alpha} \frac{dI}{d\Psi} B_p.$$  \hspace{1cm} (21)

Next, consider the Ampère’s law in Maxwell equations

$$\int_A \nabla \times (\alpha B) \cdot dS = 4\pi \int_A \alpha j \cdot dS - \int_A (E \cdot \nabla \omega) m \cdot dS,$$  \hspace{1cm} (22)

which, upon using Stokes’s theorem and $m \cdot dS = m \cdot (d\mathbf{r} \times 2\pi m) = 0$, becomes $2\pi \sigma \alpha | B_T | = -4\pi I$ and hence

$$B_T = \frac{2I}{\alpha \sigma} e_\phi = \frac{-2I}{\alpha \sigma^2} m.$$  \hspace{1cm} (23)

Next, using equations (12) and (18), one gets the Gauss law equation

$$\rho_e = \frac{1}{4\pi} \nabla \cdot E_p = -\frac{1}{8\pi^2} \nabla \cdot \left( \frac{\Omega_F - \omega}{\alpha} \nabla \Psi \right)$$  \hspace{1cm} (24)

while Ampère’s law in equation (12) yields

$$j_T = \frac{1}{4\pi \alpha} \left[ (\nabla \times (\alpha B))_T + \sigma (E \cdot \nabla \omega) \right]$$

$$= -\frac{\sigma}{8\pi^2 \alpha} \left[ \nabla \cdot \left( \frac{\alpha}{\sigma^2} \nabla \Psi \right) + \left( \frac{\Omega_F - \omega}{\alpha} \right) \nabla \Psi \cdot \nabla \omega \right].$$  \hspace{1cm} (25)
Alternatively, by combining equations (24) and (25), one gets (Okamoto 1992)

\[
\rho_e = \left( \frac{\Omega_e - \omega}{\alpha} \right) \sigma \left( j_T - \frac{1}{4\pi^2 \sigma} G \cdot \nabla \Psi \right),
\]

where \( G \equiv \frac{1}{2} \left[ \nabla \ln \left( \frac{(\Omega_e - \omega)^2 \sigma^2}{\alpha^2} \right) - \frac{(\Omega_e - \omega)\sigma^2}{\alpha^2} \nabla \omega \right] \).

### 3.2 The Grad–Shafranov approach

#### 3.2.1 The Grad–Shafranov equation

Consider the force-free condition \( \rho_e E + j \times B = 0 \), and focus on its `poloidal component` equation,

\[-\rho_e (\nabla \times B)_p + j \times B|_p = 0,\]

which yields

\[-\rho_e \left( \frac{\Omega_e - \omega}{\alpha} \right) \sigma + j_T + \frac{1}{\alpha} \frac{dI}{d\Psi} B_T = 0. \tag{27}\]

Now by plugging equations (24) and (25) in (27) above and using equation (23), one arrives at (Macdonald & Thorne 1982; Thorne & Macdonald 1982; Okamoto 1992; Beskin & Pariev 1993; Beskin 1997)

\[
\nabla \cdot \left\{ \frac{\alpha}{\sigma^2} \left[ 1 - \frac{(\Omega_e - \omega)^2 \sigma^2}{\alpha^2} \right] \nabla \Psi \right\} + \frac{\Omega_e - \omega}{\alpha} \frac{d\Omega_e}{d\Psi} |\nabla \Psi|^2 + \frac{16\pi^2 I}{\alpha \sigma^2} \frac{dI}{d\Psi} = 0, \tag{28}\]

where we assumed a general situation in which the magnetic field lines are, although rooted in the pulsar surface, allowed to have differential rotation. This is the *stream equation* to determine the field structure of the force-free pulsar magnetosphere.

#### 3.2.2 Electric charge and toroidal current density

From equations (26) and (27), one gets the expressions for the charge and toroidal current density (Okamoto 1992)

\[
\rho_e = \left( \frac{\Omega_e - \omega}{4\pi^2 \sigma} \right) \frac{8\pi^2 I}{\alpha^2} \frac{dI}{d\Psi} G \cdot \nabla \Psi \tag{29}\]

\[
\frac{1}{4\pi^2 \sigma} \left\{ \frac{8\pi^2 I}{\alpha^2} \frac{dI}{d\Psi} \left[ \frac{\Omega_e - \omega}{\alpha} \right]^2 G \cdot \nabla \Psi \right\},
\]

\[
\frac{1}{4\pi^2 \sigma} \left\{ \frac{8\pi^2 I}{\alpha^2} \frac{dI}{d\Psi} \left[ \frac{\Omega_e - \omega}{\alpha} \right]^2 G \cdot \nabla \Psi \right\},
\]

\[
\frac{1}{4\pi^2 \sigma} \left\{ \frac{8\pi^2 I}{\alpha^2} \frac{dI}{d\Psi} \left[ \frac{\Omega_e - \omega}{\alpha} \right]^2 G \cdot \nabla \Psi \right\}. \tag{29}\]

#### 3.2.3 Singular surfaces

In order to distinguish the singular surfaces in a pulsar magnetosphere from those in a (rotating) black hole magnetosphere, we first note the generic distinction between a black hole magnetosphere and a pulsar magnetosphere.

(i) **Black hole magnetosphere.** In this case, the field angular velocity \( \Omega_e \) is generally in no way connected with the angular velocity of the black hole \( \omega(r_H) = \Omega_H \). Indeed, \( \Omega_e - \omega \) changes sign from minus to plus as one moves away from the symmetry axis (recall that \( \omega \) denotes the angular velocity of ZAMO).

(ii) **Pulsar magnetosphere.** In this case, all the magnetic field lines are firmly rooted in the crystalline crust of the pulsar surface, namely \( \Omega_e = \Omega_{NS} > \omega \). Thus \( \Omega_e - \omega > 0 \) namely, since \( \Omega_e = \Omega_{NS} \), the field angular velocity should be greater than that of ZAMO, \( \omega \), everywhere.

To summarize, a spinning black hole rotates the ‘surrounding’ magnetic field lines (generated, say, by currents in the accretion disc) essentially via the frame-dragging effect, whereas a neutron star rotates ‘its own dipole’ magnetic field lines via the spin motion itself.

We start with a light cylinder. When treating the region exterior to a rotating neutron star as flat space–time, there was a single light cylinder at \( \sigma = c/\Omega_e \). For the case at hand where the region outside of the rotating neutron star is described by a generic curved space–time, there still is a single light cylinder where the denominators of \( \rho_e \) and \( J_T \) vanish, namely at

\[
\sigma_L = \frac{\alpha c}{(\Omega_e - \omega)}. \tag{30}\]

The only change from the flat space–time treatment to the curved space–time one is the notion of relative angular velocity with respect to ZAMO since now all the measurements are made by a local fiducial observer, i.e. ZAMO. Particularly note that, in the present case of a rotating neutron star, there is only one zero for the denominators of \( \rho_e \) and \( J_T \) in equation (29) instead of two since the field angular velocity...
should be greater than that of ZAMO, \( \psi \) everywhere as we explained above. Certainly, this is in contrast to what happens in the case of a rotating black hole magnetosphere when there are two light cylinders for which the denominators of \( \rho_c \) and \( j_T \) vanish, namely

\[
\sigma_{OL} = \frac{ac}{(\psi - \Omega_f)}, \quad \sigma_{OL} = \frac{ac}{(\Omega_f - \psi)}.
\]

On the light cylinder, the numerators should vanish in order to have finite \( \rho_c \) and \( j_T \) there, i.e.

\[
\frac{4\pi^2}{\alpha^2} \left( \frac{dI^2}{d\psi} \right) = G \cdot \nabla \psi,
\]

(31)

which is called the ‘critical condition’.

We now point out the implication of this distinction between the single light cylinder in a pulsar magnetosphere and the double light cylinders in a black hole magnetosphere.

In the case of a rotating black hole magnetosphere, the source or origin of the plasma that will fill the magnetosphere is understood as follows. Note that as one moves from the inner light cylinder toward the outer one, the angular velocity of the magnetic field lines grows, namely \((\Omega_f < \psi) \rightarrow (\Omega_f > \psi)\). Thus somewhere between the two light cylinders, there should be a null surface where

\[
\Omega_f = \psi \quad \text{or} \quad \psi = \left[ \frac{\Omega_f - \psi}{\alpha} \right] m = 0
\]

(32)

and hence \( E_p = -\psi \times B_p = 0 \). Then on this null surface,

\[
\rho_c = -\frac{1}{8\pi^2} \cdot \left[ \frac{\Omega_f - \psi}{\alpha} \nabla \psi \right] = -\frac{1}{8\pi^2} \nabla \psi \cdot \nabla(\psi - \Omega_f),
\]

(33)

\[
J_T = \frac{2I}{\alpha^2} \frac{dI}{d\psi}
\]

and this null surface occurs at \( \sigma = \sigma_{OL} \) (in the context of the space-charge separation in the force-free limit). The presentation that will be given below for the present case of a pulsar (which is being treated in a fully general relativistic manner) essentially follows that for the case of a rotating black hole (Okamoto 1992) in the context of the Blandford–Znajek mechanism.

As advertized earlier in the introduction, we now address the issue of pulsar spin-down essentially due to the magnetic braking torque which is called the ‘null surface’ since there must be the spark gaps or the creation zone of nearly neutral plasma situated in its neighborhood. The velocity of the magnetic field lines relative to ZAMO

\[
|\psi| = \left[ \frac{\Omega_f - \psi}{\alpha} \right] m
\]

begins to increase from zero at \( \sigma = \sigma_N \) toward \( \sigma = \sigma_{OL} \) and then further to infinity as one moves far away from the outer light cylinder where \( \alpha \rightarrow 0, \psi \rightarrow 1 \). As one moves inwards, on the other hand, it begins to increase in magnitude (upon changing sign) from zero at \( \sigma = \sigma_N \) toward \(-\psi \) and then again further to (negative) infinity at the horizon \( r_H \) as \( \alpha = \alpha(r_H) = 0 \) there. Thus ZAMOs in the outer region of the magnetosphere \( \sigma > \sigma_N \) see the centrifugal ‘magnetic slingshot wind’ blowing outwards from the vicinity of the null surface to the acceleration zone and ZAMOs in the inner region of the magnetosphere \( \sigma < \sigma_N \) see the centrifugal magnetic slingshot wind blowing inwards to the horizon.

To summarize, the global structure of a pulsar magnetosphere associated with the singular surfaces is indeed quite different from that of a rotating black hole magnetosphere – and it indeed is related to the fact that a black hole is characterized by its event horizon while a neutron star has a hard surface.

### 3.3 Poloidal current in neutron star magnetospheres and pulsar spin-down

As advertized earlier in the introduction, we now address the issue of pulsar spin-down essentially due to the magnetic braking torque in terms of the structure of longitudinal (or poloidal) currents circulating in the neutron star magnetosphere. To this end, we should start with the space-charge separation in the force-free limit. The presentation that will be given below for the present case of a pulsar (which is being treated in a fully general relativistic manner) essentially follows that for the case of a rotating black hole (Okamoto 1992) in the context of the Blandford–Znajek mechanism.

Recall the definitions of poloidal magnetic flux (or stream function) \( \Psi \) and poloidal current \( I \) given, respectively, by

\[
\Psi = \int_A B \cdot dS, \quad I = -\int_A \alpha j \cdot dS.
\]

First, we consider the case when the angular momentum \( J_{NS} \) and the (asymptotic) direction of the magnetic field \( B \) are parallel. We begin by noting that the magnetic flux (and \( B \)) is defined to be positive when it is directed upwards while the poloidal current is defined to be positive when it is directed downwards. Normally one imposes the condition of no net loss of charge from a pulsar. This amounts to demanding that the net current flowing into and out of a neutron star surface \( A_{NS} \) vanishes, namely

\[
\int_{A_{NS}} \alpha j \cdot dS = \int_{A_{NS}} \left(-\frac{dI}{d\Psi} \right) B_p \cdot dS = -\left[I(\Psi_q) - I(\Psi_0)\right] = 0,
\]

(34)

where we used \( j = j_T + j_p \), \( j_T \cdot dS = 0 \) and \( d\Psi = B \cdot dS = B_p \cdot dS \). Additionally, for the expression for the poloidal current density \( j_p \), we used equation (21). Here, the surface integral is taken only over the (northern) hemisphere of the neutron star surface from the (north) pole...
where $\Psi = \Psi_0$ to the equator where $\Psi = \Psi_{eq}$ as the pulsar’s intrinsic dipole moment would generate dipole magnetic fields. This condition indeed implies the presence of some ‘critical’ magnetic surface $\Psi = \Psi_c$ such that

$$\frac{dI}{d\Psi} = \begin{cases} > 0 & (\text{for } \Psi_0 < \Psi < \Psi_c) : \text{Region I}, \\ 0 & (\text{for } \Psi = \Psi_c), \\ < 0 & (\text{for } \Psi_c < \Psi < \Psi_{eq}) : \text{Region II}. \end{cases} \quad (35)$$

Then one can realize from equations (35) and (21) that the electric current flows *inwards* along the magnetic field lines for $\Psi_0 < \Psi < \Psi_c$ from the acceleration region to the neutron star surface while it flows *outwards* for $\Psi_c < \Psi < \Psi_{eq}$ from the surface to the acceleration region. On the critical magnetic surface $\Psi = \Psi_c$, the net current is zero (i.e. inflowing current = outflowing current) but the charge density there may not be exactly zero and hence the critical magnetic surface may not serve as exactly a ‘charge-separating’ surface. This point can be envisaged from the expression for the charge density $\rho_c$ in equation (29) where one can notice that $dI/d\Psi$ is zero but $G \cdot \nabla \Psi$ is not necessarily zero along $\Psi = \Psi_c$. Indeed, this last quantity can roughly be identified with the vertical component of the magnetic field (i.e. $B_z$). In the Goldreich–Julian (GJ) model (Goldreich & Julian 1969) of a purely charge-separated magnetosphere, the critical magnetic surface has been defined as the one on which $B_z = 0$ all the way. In our model of a pulsar magnetosphere with infinite supply of quasi-neutral plasma, on the other hand, it is being defined as the one on which $dI/d\Psi = 0$. As such, generally the critical magnetic surfaces in the two models may not coincide and as a result, in our model the critical magnetic surface may not act as precisely a charge-separating surface. The origin of the different predictions of these two models will be discussed in detail later in Section 3.5.

Next, in the case when the angular momentum $J_{\text{NS}}$ and the (asymptotic) direction of the magnetic field $B$ are *antiparallel*, all the quantities involved would carry flipped signs, namely

$$\frac{dI}{d\Psi} = \begin{cases} < 0 & (\text{for } \Psi_0 < \Psi < \Psi_c) : \text{Region I}, \\ 0 & (\text{for } \Psi = \Psi_c), \\ > 0 & (\text{for } \Psi_c < \Psi < \Psi_{eq}) : \text{Region II}. \end{cases} \quad (36)$$

This second case when $J_{\text{NS}}$ (or $J_{\text{BH}}$, which we shall also discuss shortly) and $B$ are antiparallel is indeed likely to happen. First for the pulsar case, its spin angular momentum $J_{\text{NS}}$ and its intrinsic magnetic dipole moment may well be aligned and pointing in opposite directions at the same time to yield this type of antiparallel configuration. For the rotating black hole case, on the other hand, consider, for instance, that the poloidal magnetic fields come from the toroidal currents in the accretion disc around the hole. Clearly, the hole and the disc would be corotating but if the excess charge is due to that of ions, the spin of the hole $J_{\text{BH}}$ and the magnetic field would be parallel whereas if it is due to that of electrons, the two would be antiparallel instead.

With this preparation, we now turn to the determination of the structure of space-charge separation and the direction of the poloidal current in a neutron star (and a rotating black hole for comparison) magnetosphere. Consider the charge density in a neutron star magnetosphere given earlier in equation (29) (note that its structure remains the same even in a rotating black hole magnetosphere except that the associated space–time metric content is distinct in the two cases),

$$\rho_c = \frac{\left( \Omega_F - \Omega_\alpha \right)}{4\pi a^2} \frac{dI}{d\Psi} - G \cdot \nabla \Psi \left[ 1 - \frac{\Omega_F - \Omega_\alpha}{\Omega_\alpha} \right] \left[ 1 + \frac{\Omega_F - \Omega_\alpha}{\Omega_\alpha} \right]. \quad (37)$$

Now using this expression for the charge density along with equations (35) and (36), one can, *in principle*, determine the sign of the separated charges in different domains of the neutron star magnetosphere. The directions of poloidal current densities are determined using the equation (21). The resulting charge separation and the poloidal current direction are depicted in Fig. 1. It is indeed quite instructive to compare the present case of pulsar magnetosphere structure with that of a rotating black hole magnetosphere structure. The latter had been studied in detail in the literature (Blandford & Znajek 1977; Macdonald & Thorpe 1982; Thorne & Macdonald 1982; Takahashi et al. 1990; Nitta et al. 1991; Okamoto 1992; Beskin & Pariev 1993; Beskin 1997; Kim et al. 2001b) and here in the present work, we have elaborated on it by further considering the case when the spin of the hole and the (asymptotic) direction of the magnetic field are antiparallel – see Fig. 2.

In practice, however, determining the sign of the separated charges in different domains of a neutron star/black hole magnetosphere using equations (35), (36) and (37) is by no means a straightforward job. Technically, the associated difficulty arises from the fact that we need to figure out which term, between $\left[ (8\pi^2 I/a^3 \Omega F) dI/d\Psi \right]$ and $\left[ G \cdot \nabla \Psi \right]$ in the numerator of equation (37), is greater than the other in different domains of the magnetosphere. By contrast, the determination of the structure of charge separation (i.e. the sign of separated charges) in the GI model (but only inside the light cylinder) that we shall discuss shortly and summarized in Fig. 3 using equation (38) below is rather straightforward. Fortunately, however, there is a guiding principle that allows us to determine the sign of separated charges in different domains with confidence. And that is just the insightful realization that the different domains of the magnetosphere around a compact object such as a rotating neutron star or black hole should rotate in the *same* direction as the compact object itself (‘corotation’) if they rotate faster than ZAMO, the local inertial observer carrying out all the observations, whereas they should rotate in the *opposite* direction (‘counter-rotation’) provided they rotate slower than ZAMO. In order to work out this guiding principle and determine the structure of charge separation as a result, we begin by defining the angular momentum of the magnetosphere. The pulsar magnetosphere consists of the poloidal magnetic field generated essentially by the intrinsic dipole moment of the neutron star and the poloidal electric field generated both by the toroidal current-poloidal magnetic field in the force-free limit (see equations 14 and 18) and by the separated space charge. And particularly these two sources of the poloidal electric field are closely related since the toroidal current arises as a result of the angular motion of the poloidal magnetic field lines (being dragged along by the spin of the neutron star) along which the plasma flows (in the force-free limit). Therefore if
Figure 1. The pulsar magnetosphere – a general relativistic treatment. The presence of an accretion disc (Aly 1980; Sibgatullin et al. 2004) here is not mandatory but is assumed for parallel comparison with the case of a black hole magnetosphere. The star’s intrinsic dipole moments are not drawn in the figures. (a) When \( J_{\text{NS}} \) and \( B \) are parallel, (b) when \( J_{\text{NS}} \) and \( B \) are antiparallel.

one can determine the direction of the poloidal electric field, the structure of charge separation can be determined accordingly. Besides, since the motion of the plasma eventually generates the poloidal electric field, the angular momentum of the magnetosphere can be defined in terms of its poloidal electromagnetic field. In general, the angular momentum of an electromagnetic field is defined by \( J_{\text{em}} = \int d^3x [r \times (E \times B)] \). Thus we only need to determine the direction of this poloidal electric field \( E \) (for a given poloidal dipole magnetic field \( B \)) that leads to either corotation or counter-rotation of the different domains of the magnetosphere (as measured by ZAMO, the local inertial observer) with a rotating neutron star/black hole, i.e. \( J_{\text{em}} \sim \pm J_{\text{NS,BH}} \). This is how the structure of charge separation in different domains of the pulsar magnetosphere summarized in Fig. 1 and that of a black hole magnetosphere summarized in Fig. 2, have been actually fixed. It is worthy of note that for the case of a pulsar magnetosphere, the sign of separated charges in different domains such determined is the same as that in the original GJ model that we shall turn to in a moment except that now for our force-free treatment, it can be extrapolated outside the light cylinder. Next, for the case of black hole magnetosphere, the structure of charge separation such determined turns out to be exactly the same again as the result derived by Okamoto (1992) some time ago via different reasoning. Then to summarize, it is not surprising (since it has been expected to some extent) but still interesting to realize that the structure of magnetosphere of a pulsar and a rotating black hole are not quite the same let alone the different structure of singular surfaces that we stressed earlier. If we emphasize it once again, this difference can be attributed to the fact that all the magnetic field lines are firmly rooted in the crystalline crust of the pulsar surface and hence \( \Omega_{\text{F}} > \omega \) namely, the field angular velocity is greater than that of ZAMO, \( \omega \), everywhere. Consequently, to the ZAMO of a rotating neutron star, the whole pulsar magnetosphere looks corotating and thus the resulting direction of the poloidal electric field eventually determines the structure of space-charge separation as given in Fig. 1. In the black hole case, however, the field angular velocity \( \Omega_{\text{F}} \) is generally in no way connected with the angular velocity of a black hole \( \omega(\Omega_{\text{F}} - \omega) = \Omega_{\text{H}} \). Indeed, \( \Omega_{\text{F}} - \omega \) changes sign from minus to plus as one moves away from the symmetry axis (recall that \( \omega \) denotes the angular velocity of ZAMO). Namely, inside the ‘null surface’ that we discussed earlier in Section 3.2, ZAMO rotates faster than the magnetic field lines (and hence than the inner part of the magnetosphere) while outside of it, ZAMO rotates slower than the magnetic field lines (and thus the outer part of the magnetosphere). As a result, to the ZAMO of a rotating black hole, the part of the black hole magnetosphere outside the null surface looks corotating whereas the part inside the null surface appears to counter-rotating. From this one can realize the directions of the poloidal electric fields which, in turn, determines the structure of space-charge separation as given in Fig. 2. And for both the pulsar and rotating black hole cases, it is rather straightforward to see that the structure of charge separation and the direction of longitudinal (or poloidal) current (denoted in the figures by \( I \)) circulating in the magnetospheres actually lead to the magnetic braking torques, namely the Lorentz torques \( N = \int_{\text{surface}} \left[ r \times (j \times B) \right] dS \), that spin down rotating neutron stars and black holes as it is always directed opposite to the spins regardless of whether the spin and the (asymptotic) direction of the magnetic field are parallel.

© 2005 RAS, MNRAS 358, 998–1018
Figure 2. The rotating black hole magnetosphere. Although the critical magnetic surfaces have been sketched more or less as straight lines in these figures, they would in fact be curved in actual situation. (a) When $\mathbf{J}_{BH}$ and $\mathbf{B}$ are parallel, (b) when $\mathbf{J}_{BH}$ and $\mathbf{B}$ are antiparallel.

Figure 3. The pulsar magnetosphere – Goldreich–Julian’s aligned rotator model.
or antiparallel. Here, the surface area \( A_{\text{NS}} \) over which the integral of the non-vanishing Lorentz torque density is to be taken might need some careful clarification. For the case of rotating neutron star, this area obviously should be its surface where the poloidal current crosses the magnetic field lines and hence generates non-vanishing braking torque. For the case of a black hole, however, the nature of this area might seem quite ambiguous but our suggestion here is to invoke the notion of ‘stretched horizon’ as an incarnation of the so-called ‘membrane paradigm’ (Thorne, Price & Macdonald 1996). Indeed, the philosophy that underlies the membrane paradigm is an attempt to have an intuitive picture of Blandford–Znajek mechanism by first assuming the appearance of stretched horizon (just outside the event horizon) and then introducing (fictitious) surface charge and current density on it. Then one of the most intriguing consequences of such assumption is that if we choose to do so, the (stretched) horizon behaves as if it is a conductor with finite resistivity. To be more specific, since there are now both current and resistivity on the horizon, one might naturally wonder what would happen to the Joule heat generated when those surface currents work against the resistance and how it would be related to the electromagnetic energy going down the hole through the horizon. Indeed, Znajek and independently Damour (Damour 1978; Znajek 1978) provided a simple and natural answer to this question. Namely, they showed in a consistent and elegant manner that the total electromagnetic energy flux (i.e. the Poynting flux) into the rotating Kerr hole through the horizon is indeed precisely the same as the amount of Joule heat produced by the surface currents when they work against the surface resistivity of \( 4\pi \). Therefore, motivated and encouraged by these ground works for the advent of the membrane paradigm, here we also assume that the surface area over which the integral of the non-vanishing Lorentz torque is to be taken is just this stretched horizon with surface current.

Then the real poloidal current in a black hole magnetosphere and this surface current on the horizon together are supposed to complete the circuit.

Now to summarize, this unified picture can be thought of as a satisfying solution to both the magnetized rotating neutron star interpretation of radio/X-ray pulsars (Mereghetti & Steelar 1995; Kouveliotou et al. 1998; Mereghetti 2000; Thompson 2001) and the rotating supermassive black hole interpretation of AGNs/quasars (Blandford & Znajek 1977; Kim et al. 2001a,b) or even GRBs (Lee et al. 2000; Kim et al. 2003).

It is, however, the following point that is of great interest and has been the strong motive for the present study. And it is the difference in the nature of the origin/source of charges which get separated in the domains of the magnetosphere and of the resulting poloidal currents between the two pulsar models – ours and that of Goldreich–Julian’s. It goes as follows.

First, the original GJ model can be thought of as the purely charge-separated solution in which only one charge species can be assigned at a given point in space. As a result, the poloidal current in this charge-separated solution is directly proportional to charge density. Now, the origin/source of charges in this GJ model is basically the surface of the pulsar itself and the poloidal current flows only \textit{until} these charges get separated (by the strong local electric field near the surface of the star) and then reach the equilibrium GJ charge density given by

\[
\nu_{e}^\text{GJ} = -\frac{1}{2\pi c} \frac{B \cdot \Omega}{1 - \left(\frac{\Omega}{\Omega_1}\right) \sin^2 \theta} = -\frac{\Omega}{2\pi c} \frac{B_z}{1 - \left(\frac{\Omega}{\Omega_1}\right) \sin^2 \theta},
\]

where we restored the speed of light \( c \) and \( \Omega \) denotes the angular velocity of the pulsar. Working with this expression for the pulsar charge density, the resulting charge separation can be determined as depicted in Fig. 3. By contrast, our force-free (and fully general relativistic) model may be referred to as the solution of quasi-neutral plasma in which two species of charges can coexist at a given point in space while allowing for still non-zero net space-charge density. And non-trivial poloidal current can still exist as it can be defined in terms of the difference in velocities between the two charge species. Then the origin/source of charges in this force-free treatment is mainly the pair creation due to strong electromagnetic field in space which provides infinite supply of ample plasma and hence the continuous flow of poloidal current without end. This difference in the nature of charges and the resulting poloidal currents between the two models is indeed the key to understanding why our force-free and general relativistic treatment of pulsar magnetospheres presents a more self-consistent and hence more satisfying view of the pulsar spin-down mechanism in the sense that it is consistent with the mechanism of Blandford & Znajek (1977), employing a rotating black hole magnetosphere. This essential difference between the two pulsar models, however, does not necessarily mean that our force-free and general relativistic treatment is able to provide a successful mechanism for pulsar spin-down in terms of the magnetic braking torque while Goldreich–Julian’s original but non-general relativistic (GR) one fails to do so. And obviously in order to be convinced that both of the two models can successfully provide the mechanism for the pulsar’s magnetic spin-down, one needs to demonstrate that the directions of poloidal currents particularly at the surface of the neutron star are indeed the \textit{same} correctly leading to the magnetic braking torque that we discussed earlier. Thus in the following, this last point shall be addressed. Indeed, the directions of the poloidal current (that closes globally in the magnetosphere) are defined \textit{differently} in the two models. First in our force-free treatment, the direction of the poloidal current and the structure of the charge densities given in equations (21) and (29), respectively, are determined \textit{simultaneously} via the behaviour of the quantity \( (dI/d\Psi) \) as given in equations (35) or (36) just as it was the case with a rotating black hole magnetosphere (Thorne & Macdonald 1982; Macdonald & Thorne 1982; Okamoto 1992). Namely, one is not determined as a result of the other. Of course, this is because the continuous supply of ample plasma is assumed in our treatment as discussed above. And as depicted in Fig. 1, the direction of poloidal current such determined particularly at the surface of the star correctly leads to the magnetic braking torque directed opposite to the star’s angular momentum.

In the original model of Goldreich–Julian’s (Goldreich & Julian 1969: Punsly & Coroniti 1990), on the other hand, the structure (i.e. the sign) of charge density is actually determined as a result of the direction of the poloidal current. That is, the strong local electric field near the surface of the neutron star first determines the direction of local poloidal current (i.e. the flow of local charge carriers) which, in turn, determines the sign of charge density in a given domain of space. To be more specific, Goldreich and Julian began their analysis by assuming
that the neutron star with an aligned dipole magnetic field is surrounded first by vacuum. Then the longitudinal electric field at the neutron star surface turns out to have component parallel to the poloidal magnetic field given by \( E_\parallel \sim -\cos^3 \theta \) and particularly at the equator the vacuum electric field is directed radially outwards. Thus in the polar region, the vacuum longitudinal electric field drives a current towards the star (by pulling space ions if present or by ripping electrons off the pulsar surface) while at the equator it drives current away from the star (by pulling space electrons if present or by ripping ions off the surface). In this way, the vacuum longitudinal electric field causes charge emission, i.e. the flow of the poloidal current, only until the magnetosphere is filled with plasma with the charge density being given by the Goldreich–Julian’s equilibrium value given in equation (38). Namely, the poloidal current cannot flow without end because once the charges get accumulated up to GJ’s equilibrium value in equation (38), the emission of charges becomes electrostatically unfavourable. And if, particularly in the particle acceleration region, there appears the difference between the local plasma charge density and the equilibrium Goldreich–Julian density \( \rho^{GJ} \) given in equation (38), the longitudinal electric field arises and as a result, the plasma in the magnetosphere would be streaming out along open magnetic field lines past the light cylinder as a centrifugally slung, relativistic wind leading eventually to the observed radio emission.

Note that if the current driven by the vacuum longitudinal electric field can close in a global current system, particularly the direction of the current flowing on the stellar surface again would exert the correct magnetic braking or spin-down torque on the neutron star. In this non-GR Goldreich–Julian model, therefore, the charge separation depicted in Fig. 3 does not really conflict with the spin-down process as the direction of the poloidal current is indeed the same as that of our force-free treatment discussed above correctly leading to essentially the same magnetic braking torque. Thus to summarize, regardless of the difference in the origin/source of the space charges and in the definition of the direction of poloidal current, both the Goldreich–Julian’s original model and our present force-free treatment provide the working mechanism for magnetic pulsar spin-down. Nevertheless, we would like to stress again that there indeed is a more desirable feature in our treatment that distinguishes it from the original model of Goldreich and Julian’s. And it is the fact that the force-free and fully general relativistic treatment of the problem of pulsar magnetospheres presented in this work appears to provide a much upgraded and closer view of the pulsar spin-down mechanism in the sense that it is consistent with the mechanism of Blandford and Znajek (Blandford & Znajek 1977; Kim et al. 2001a,b; Thorne & Macdonald 1982; Macdonald & Thorne 1982; Takahashi et al. 1990; Nitta et al. 1991; Okamoto 1992; Beskin & Pariev 1993; Beskin 1997) employing a rotating black hole magnetosphere.

Thus far, we have been interested in the comparison between our force-free and general relativistic model and the Goldreich–Julian’s non-GR model of pulsar magnetospheres. And it has been realized that the main differences between the two arise not from the GR effect but from the different nature and source of the charges. Now this realization leads us to turn to another relevant comparison. That is, it seems equally relevant to consider the comparison of our force-free and general relativistic treatment with a force-free but non-GR treatment of pulsar magnetospheres and see if there is actually something generic in fully general relativistic treatment of pulsar electrodynamics. Indeed such a force-free but non-GR study of pulsar magnetospheres has been performed some time ago by Okamoto (1974) and by Contopoulos, Kazanas & Fendt (1999). Later on in Section 3.5, the rigorous comparison of our present treatment with this second class of study shall be carried out and if we mention the essential result in advance, as far as pulsar electrodynamics goes, the GR treatment does not seem to have any generic effect other than the stereotypical complications and elaborations such as the large redshift near the neutron star’s surface and the frame-dragging effect and hence the quest for the introduction of ZAMO, the local inertial observer carrying out the actual observations. Indeed, the existence of and the observations by ZAMO is a non-trivial deviation from non-GR treatment of pulsar electrodynamics since it is the strong electric field felt by ZAMO that actually renders the pair creation of charges via the so-called Schwinger process work (see, for instance, Muslimov & Tsygan 1992). Recall that our model of pulsar magnetosphere depends, for the source of ample supply of quasi-neutral plasma, heavily on the pair production of charges in space.

To summarize, it has been quite difficult to accept that two relativistic spinning compact objects of nearly the same type, a neutron star (i.e. pulsar) (based on the Goldreich–Julian model) and a black hole (based on, say, the Blandford–Znajek model), have generically different magnetosphere structures. And in our force-free and general relativistic pulsar model, we realized that it shares the same structure of singular surfaces of flows with that of the original Goldreich–Julian model on the one hand and shares essentially the same structures of charge separation and poloidal current with those of a rotating black hole (Blandford & Znajek 1977; Kim et al. 2001a,b; Okamoto 1992) on the other.

### 3.4 The energy and the angular momentum flux

In the above, we showed, in terms of the space-charge separation structure of pulsar magnetospheres, that in the force-free case the longitudinal (or poloidal) currents circulating in the neutron star magnetosphere leads to the magnetic braking torque that actually spins it down in a similar manner to the case with the Blandford–Znajek mechanism for the extraction of rotational energy from Kerr black holes. In this subsection, we shall demonstrate, in terms of the energy and the angular momentum flux at the surface of the neutron star, that this argument does indeed hold true. The general expression for the redshifted energy flux \( S_\varepsilon \) and the angular momentum flux about the axis of rotation \( S_L \) are given, respectively, by (Macdonald & Thorne 1982) and (Macdonald & Thorne 1982)

\[
S_\varepsilon = \frac{1}{4\pi} \left[ \alpha (E \times B) - \omega B \cdot m E - \omega (E \cdot m) B + \frac{1}{2} \omega (E^2 + B^2) m \right],
\]

\[
S_L = \frac{1}{4\pi} \left[ -(E \cdot m) E - (B \cdot m) B + \frac{1}{2} (E^2 + B^2) m \right].
\]

(39)

© 2005 RAS, MNRAS 358, 998–1018
Since the toroidal component of the fluxes are irrelevant, we only need to consider the poloidal components

\[ S_L^\varpi = \frac{i}{4\pi} \mid B_T \mid B_P = \frac{I}{2\pi \alpha} B_P, \]

\[ S_E^\varpi = \frac{\alpha}{4\pi} E_P \times B_T + \omega S_L^\varpi = \frac{I}{2\pi} \left( \frac{\omega}{\alpha} B_P - \frac{1}{\varpi} E_P \times m \right). \]  

(40)

Thus, at the neutron star surface where \( \alpha = \alpha(r_o) \neq 0 \),

\[-S_L \cdot n \rightarrow \frac{dJ}{d\Sigma_o} \, dr = - \frac{I}{2\pi \alpha} B_\perp = - \frac{I}{4\pi \alpha \alpha} (\nabla \Psi \times e_\varpi) \cdot n, \]

\[-S_E \cdot n \rightarrow \frac{dM}{d\Sigma_o} \, dr = - \frac{I}{2\pi} \left[ \frac{\omega}{\alpha} B_\perp - \frac{1}{\varpi} (E_P \times e_\varpi) \cdot n \right] \]

\[ = - \frac{I}{2\pi \varpi} \Omega_B B_\perp = \Omega_B \frac{dJ}{d\Sigma_o} \, dr, \]  

(41)

where \( n \) denotes the unit vector outer normal to the neutron star surface. Now note that when the spin \( J \) of the rotating neutron star and the magnetic field \( B \) are parallel, \( B_\perp > 0, I > 0 \) whereas when \( J \) and \( B \) are antiparallel, \( B_\perp < 0, I < 0 \) due to their definitions equations (16) and (19). Namely, the magnetic flux (and \( B \)) is defined to be positive/negative when it directs upward/downward while the poloidal current is defined to be positive/negative when it directs downward/upward as we noted earlier. Thus one always has \( J B_\perp > 0 \), and hence from equation (41) above, we always have

\[-S_L \cdot n = - \frac{I}{2\pi \alpha} B_\perp < 0, \]

\[-S_E \cdot n = - \frac{I}{2\pi \varpi} \Omega_B B_\perp < 0. \]  

(42)

Since the angular momentum and the energy flux going into the neutron star surface are all negative, this means that a rotating neutron star (i.e. pulsar) experiences magnetic braking torque, namely spins-down, and as a result always loses part of its rotational energy (at the surface).

### 3.5 Limit of vanishing general relativistic effects

In earlier subsections, we have studied the detailed comparison between our force-free and general relativistic model and the Goldreich–Julian’s non-GR model of pulsar magnetospheres. It then has been argued that the main differences between the two arise not from the GR effect but from the different nature and source of the charges. This could be checked in a rigorous manner if we erase the GR content in our force-free treatment of the pulsar magnetosphere and see if these differences still remain. We also have turned to another equally relevant comparison. Namely, we have considered the comparison of our force-free and general relativistic treatment with a force-free but non-GR treatment of pulsar magnetospheres to see whether there is actually something generic in fully general relativistic treatment of pulsar electrodynamics. Again, such a comparison would be made explicit if the GR component in our treatment is washed out. Besides, such force-free but non-GR study of pulsar magnetospheres has been performed in the literature (Okamoto 1974; Contopoulos et al. 1999) and hence the result of the comparison can be directly tested. Therefore in this subsection, we shall reconsider our force-free and general relativistic model and take its particular limit of vanishing GR content for these purposes.

Evidently, taking the limit of vanishing GR content would amount to replacing the curved Hartle–Thorne space–time exterior to the rotating neutron star with the flat space–time while maintaining the force-free nature of pulsar electrodynamics. And technically, this is equivalent to setting all the parameters associated with the non-trivial curved space–time structure, i.e. the mass \( M \), angular momentum \( J \) and the mass quadrupole moment \( Q \) in the Hartle–Thorne metric for the neutron star, to zero in all the equations of pulsar electrodynamics presented in Sections 3.1 and 3.2 above.

In the limit of vanishing GR content, apparently the exterior space–time is the flat Minkowski one and in the following we shall take the cylindrical coordinates \((r, \varphi, z)\) in which the Minkowski metric is given by

\[ ds^2 = -dt^2 + dR^2 + R^2 d\varphi^2 + dz^2. \]  

(43)

Then the role played by proper distance from the axis of (neutron star’s) rotation \( \varpi \) that has been employed thus far in the general relativistic treatment shall henceforth be taken over by the radial coordinate \( R \). Next, we start with the Maxwell equations in this flat space–time

\[ \nabla \cdot E = 4\pi \rho_e, \quad \nabla \cdot B = 0, \]

\[ \nabla \times E = 0, \quad \nabla \times B = 4\pi j, \]  

(44)

where we dropped the terms \( \partial (\ldots)/\partial t = 0, \partial (\ldots)/\partial \varphi = 0 \) due to stationarity and axisymmetry. Next, throughout, the force-free condition is still assumed to hold, i.e.

\[ \rho_e E + j \times B = 0, \quad B = B_T + B_P \]  

(45)

which also implies \( E \cdot B = 0 \). Then this force-free condition indicates that the charged particles are sliding along the magnetic field lines and hence the toroidal (angular) velocity of magnetic field lines (which are frozen into plasma) is given by

\[ v_{\varpi} = R \Omega_e e_\varpi = \Omega_e m \]  

(46)
and then \( j = j_T + j_p \) with \( j_T = \rho_e v_T \) where \( v_T \) consists of \( v_p \) given above and the streaming velocity along the toroidal magnetic field lines.

Here, again \( m = R e_\phi = (g_{\phi\phi})^{1/2} e_\phi \). Then from the force-free condition above, it follows that

\[
E = E_p = -\frac{1}{\rho_e} j_T \times B_p.
\]  (47)

We have established the force-free condition and based on this, we can now derive all the force-free pulsar electrodynamic equations. First we consider the poloidal field components.

As before, a magnetic flux through an area \( A \) whose boundary is a \( m \)-loop is given by

\[
\Phi = \int_A \mathbf{B} \cdot d\mathbf{S}.
\]  (48)

Then from \( d\Phi = \nabla \Phi \cdot d\mathbf{r} \) and alternatively \( d\Phi = \mathbf{B} \cdot (d\mathbf{r} \times 2\pi R e_\phi) = (2\pi R e_\phi \times \mathbf{B}) \cdot d\mathbf{r} \), we get

\[
\mathbf{B}_p = -\frac{\nabla \Psi \times e_\phi}{2\pi},
\]  (49)

where we used \( m \cdot m = g_{\phi\phi} = R^2 \) and hence

\[
\mathbf{E}_p = -\mathbf{v}_p \times \mathbf{B}_p = -\frac{\Omega^2}{2\pi} \hat{\mathbf{r}} \nabla \Phi.
\]  (50)

We are now ready to discuss the poloidal current and the associated toroidal magnetic field. Once again, a poloidal current through the same area whose boundary is a \( m \)-loop is given by

\[
I = -\int_A j \cdot d\mathbf{S}
\]  (51)

then similarly to the case of poloidal magnetic field, \( \nabla I = -2\pi R e_\phi \times j_p \), we get

\[
j_p = -\frac{\nabla I \times e_\phi}{2\pi R} = -\frac{\nabla I \times m}{2\pi R^2}
\]  (52)

and then from equations (49), (52),

\[
j_p = -\frac{\nabla I}{\nabla I} B_p.
\]  (53)

Next, we turn to the Ampère’s law in Maxwell equations

\[
\int_A (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = 4\pi \int_A j \cdot d\mathbf{S}
\]  (54)

which, upon using Stokes’s theorem, becomes \( 2\pi R |B_T| = -4\pi I \) and hence

\[
B_T = -\frac{2I}{R} e_\phi = \frac{2I}{R^2} m.
\]  (55)

Next, using equations (44) and (50), one gets the Gauss law equation

\[
\rho_e = \frac{1}{4\pi} \nabla \cdot \mathbf{E}_p = -\frac{\Omega^2}{8\pi^2} \nabla^2 \Phi
\]  (56)

while the Ampère’s law in equation (44) yields

\[
j_T = \frac{1}{4\pi} (\nabla \times \mathbf{B}_T) = -\frac{R}{8\pi^2} \nabla \cdot \left( \frac{1}{R^2} \nabla \Phi \right).
\]  (57)

Alternatively, by using \( B_T \) and \( j_T \), one gets

\[
\rho_e = R \Omega e j_T = \frac{\Omega^2}{2\pi} B_T.
\]  (58)

We are now in a position to write down the force-free limit of the Grad–Shafranov equation. Take the force-free condition \( \rho_e \mathbf{E} + j \times \mathbf{B} = 0 \), and focus on its ‘poloidal component’ equation,

\[-\rho_e (v_p \times \mathbf{B})_p + j \times \mathbf{B}_p = 0 \]

which gives,

\[-\rho_e R \Omega e j_T + j_T + \frac{\nabla I}{\nabla \Phi} B_T = 0.
\]  (59)

Now by plugging equations (56) and (57) in (59) above and using equation (55), one arrives at

\[
\nabla \cdot \left[ \frac{1}{R^2} \left\{ 1 - (R \Omega e)^2 \right\} \nabla \Phi \right] + \frac{16\pi^2 I}{R^2} \frac{dI}{d\Phi} = 0.
\]  (60)

This is the stream equation that, in principle, would allow us to determine the field structure of the force-free pulsar magnetosphere. Lastly, from equations (58) and (59), one gets the expressions for the charge and toroidal current density

\[
\rho_e = \frac{\Omega^2}{2\pi} \frac{1}{1 - \left( \frac{R}{R_1} \right)^2}, \quad j_T = \frac{c}{2\pi R} \frac{1}{1 - \left( \frac{R}{R_1} \right)^2} B_T.
\]  (61)
where we restored the speed of light $c$ for the sake of comparison with their counterparts in the Goldreich–Julian model and as usual $R_c = c/\Omega_c$ denotes the radius of the light cylinder.

We now test our force-free and general relativistic model by comparing its limit of vanishing GR content we have studied in this subsection first with the force-free but non-GR model of Okamoto (1974) and Contopoulos et al. (1999) in (I) and then next with non-GR model of Goldreich and Julian’s in (II) below.

(I) Among other things, it is noteworthy that all the pulsar electrodynamic equations and particularly these expressions in equation (61), except for the simplifications due to the absence of the GR content, remain essentially the same as their fully GR counterparts given earlier in the Subsections 1 and 2. This implies that the GR treatment does not seem to have any generic effect on pulsar electrodynamics other than the stereotypical complications and elaborations such as the large redshift near the neutron star’s surface and the frame-dragging effect and hence the quest for the introduction of ZAMO, the local inertial observer carrying out the actual observations. Thus in a sense, our present model can be thought of as a formal general relativistic generalization of the force-free but non-GR model of pulsar magnetospheres suggested in Okamoto (1974) and Contopoulos et al. (1999). Notice that the expressions in equation (61) above essentially coincide with the corresponding results constructed in Okamoto (1974) and Contopoulos et al. (1999).

(II) Next, one can immediately realize that the equilibrium GJ charge density given in equation (38) is just a special (vacuum-space-charge) case of equation (61) in which $I = 0$. Note also that the charge density in equation (61) above in our force-free treatment actually can be written as

$$\rho_{\text{FF}}^e = \frac{\Omega_e}{2\pi c} \left( \frac{4\alpha_d}{2\pi c} \right)^2 + \rho_{\text{GJ}}^e$$

with $\rho_{\text{GJ}}^e$ being the equilibrium GJ charge density given in equation (38). Recall here that $\rho_{\text{GJ}}^e$ represents the maximum amount of charge available in the Goldreich–Julian’s vacuum pulsar model and the origin/source of these charges is basically the surface of the star itself. In our force-free model, however, the origin/source of charges is mainly the pair creation due to strong field in space and hence it presumably guarantees the infinite supply of ample plasma and hence the continuous ample flow of poloidal current. Next, determining the structure of charge separation (i.e. the sign of separated charges) in our force-free pulsar model using equation (61) does not look so simple as we need to figure out which term, between $|d\Psi(\mathbf{r})| \cdot |B_z(\mathbf{r})|$ in the numerator of equation (61), is greater than the other in different domains of the magnetosphere. As we mentioned earlier, fortunately there is a guiding principle that allows us to determine the sign of separated charges in different domains with confidence and it is the insightful realization that with respect to ZAMO, the local inertial observer, the magnetosphere around the rotating neutron star should rotate in the same direction as the compact object itself. Namely, using the definition of the angular momentum of an electromagnetic field, $J_{\text{em}} = \int \mathbf{r} \times \mathbf{E} \times \mathbf{B} \, d^3x$ and demanding $J_{\text{em}} \sim J_{\text{NS}}$, one can determine in an unambiguous manner the structure of the charge separation in different domains of the pulsar magnetosphere as summarized in Fig. 1 and it turns out to be essentially the same as that in the GJ model (but only inside the light cylinder) given earlier in Fig. 3.

To summarize, it should now be clear that the main differences between the two models (i.e. ours versus GJ’s) arise not from the GR effect but from the different nature (such as the force-free assumption) and source of the charges. But the essential features such as the structure of charge separation and the direction of the poloidal current (particularly at the pulsar surface) leading to pulsar spin-down due to the magnetic breaking are shared by the two models. Next, it seems worth contrasting carefully the nature of the critical field lines in the two pulsar models. First, the critical magnetic surface in our model is the surface on which $(dI/d\Psi) = 0$. On the other hand, in GJ model of purely charge-separated pulsar magnetospheres, the critical field line has been defined as the one on which $\Omega_e B_z = 0$ (see equation (38)). Thus the critical magnetic surface $\Psi = \Psi_c$ in these two models generally may not coincide. Indeed, on the critical magnetic surface $\Psi = \Psi_c$ in our force-free treatment, the net current is zero (i.e. inflowing current = outflowing current) but the charge density there may not be exactly zero and hence the critical magnetic surface in Fig. 1 may not serve as exactly a ‘charge-separating’ surface. In the simpler model of Goldreich and Julian’s, however, the critical field line in Fig. 3 is indeed precisely a charge-separating boundary.

4 PULSAR JET EQUATION – THE GENERAL GRAD–SHAFFRANOV EQUATION

In this more general Grad–Shafranov equation, the role played by the plasma particles, i.e. their dynamics, has been taken into account.

4.1 Basic equations

First, the Maxwell equations in the background of the stationary axisymmetric rotating neutron star space–time given earlier in equation (12) should be supplemented by the charge conservation

$$\nabla \cdot J^e = 0 \quad \text{or} \quad \frac{\partial \rho_e}{\partial t} + \nabla \cdot (\alpha \mathbf{j}_e) = 0.$$ (63)

The remaining general relativistic magnetohydrodynamics (MHD) equations are as follows.
Particle (mass) conservation:
\[ \nabla u(\nu^a) = 0 \quad \text{or} \quad \frac{\partial}{\partial t}(\gamma n) + \nabla \cdot (\alpha \gamma nv) = 0, \]  
(64)

where \( u^a = dx^a/d\tau \) is the fluid 4-velocity and \( u_a u^a = g_{\alpha \beta} u^\alpha u^\beta = -1 \), \( \gamma = (1 - v^2/c^2)^{-1/2} \).

Energy–momentum conservation:
\[ \nabla \theta T^{\alpha \beta} = 0, \quad T^{\alpha \beta} = T_0^{\alpha \beta} + T_{\alpha n}^{\beta}. \]
\[ T_0^{\alpha \beta} = \left( \frac{n w}{c^2} \right) u^\alpha u^\beta + P g^{\alpha \beta}, \]
\[ T_{\alpha n}^{\beta} = \frac{1}{4\pi} \left[ F_\mu F_\beta^{\mu \alpha} - \frac{1}{4} \epsilon^{\alpha \beta \gamma} (F_\mu F_{\nu \gamma}^\mu) \right] \]
(65)
such that
\[ \nabla \theta T_{\alpha n}^{\beta} = -\frac{1}{c} F_\mu J_\beta^\mu, \]
where \( w = (\epsilon + P)/n \) is the specific enthalpy in which \( P \) denotes the proper pressure and \( \epsilon \) denotes the proper internal energy density given by \( \epsilon = nmc^2 + (n - 1)^{-1} P \) and hence \( w = mc^2 + \frac{P[n(\gamma - 1)]}{\gamma} \).

Infinite conductivity (Ideal MHD):
\[ F_\mu u^\mu = 0 \quad \text{or} \quad E + \frac{1}{c} v \times B = 0. \]  
(66)

Equation of state (Entropy conservation):
\[ s(w, P) = k_B(-1)^{-1} \ln(Pn^{-1}), \]  
(67)

where \( s = 5/3, 4/3 \) for non-relativistic motion and for ultrarelativistic motion, respectively. Then by contracting \( u_\alpha \) with equations (65) and (66) and using the 1st law of thermodynamics \( d\epsilon = TdS + \frac{P}{\gamma} dV \), one gets
\[ \nabla u(\nu^a) = 0 \quad \text{or} \quad \frac{\partial}{\partial t}(\gamma ns u^a) + \nabla \cdot (\alpha \gamma ns u v) = 0. \]  
(68)

Momentum conservation (Euler equation):

By contracting the energy–momentum conservation equation (65) with \( (g_{\alpha \beta} + u_\alpha u_\beta) \) and then employing the Maxwell equations, one gets
\[ n u^a \nabla_\alpha u_\beta = -\partial_\alpha P - u_\alpha(u^\beta \nabla_\beta P) + \frac{1}{c} F_\mu J^\mu_\beta. \]  
(69)

Particularly in the ‘cold limit’ \( p = 0, \epsilon = nmc^2, \) and \( w = \epsilon/n = mc^2 \), this reduces to
\[ nmc^2 (u^\beta \nabla_\beta u_\alpha) = \frac{1}{c} F_\mu J^\mu_\alpha. \]  
(70)

### 4.2 The Grad–Shafranov (GS) approach

In this section, we are mainly interested in the derivation of the Grad–Shafranov (GS) equation, which describes the dynamics of plasma particles. Additionally in the following, all the time derivative terms will be dropped, i.e. \( \partial (\ldots)/\partial t = 0 \), due to the stationarity of the background Hartle–Thorne metric for the region exterior to the rotating neutron star.

#### 4.2.1 Constants of motion

(I) Substituting \( E = E_\beta = \left[ \frac{\partial \Omega_\eta}{\partial \Omega_\beta} \right] \nabla \Psi \) into the Maxwell equation (12) \( \nabla \times (\alpha E) = (B \cdot \nabla \omega) m \), one can readily realize that \( B \cdot \nabla \Omega_\beta = 0 \),
indicating that \( \Omega_\beta \) is constant on magnetic surfaces, i.e. \( \Omega_\beta = \Omega_\beta(\Psi) \) which represents the generalized Ferraro’s isorotation.

(II) Combining the freezing-in condition: \( E_T + \frac{1}{2}(v \times B)_T = 0 \),
the particle conservation: \( \nabla \cdot (\gamma n v) = 0 \),
and the Maxwell equation: \( \nabla \cdot B = 0 \)
one ends up with \( u_\beta = \gamma v_\beta = \eta(B_\beta/\alpha n) \) and hence from
\[ u_T = \gamma v_T = \eta \left( \frac{1}{\alpha n} B_\beta \right) + \gamma \left[ \frac{\Omega_\beta - \omega}{\alpha} \right] \sigma_\phi \]  
(72)

it follows that
\[ u = \gamma v = \frac{\eta}{\alpha n} B + \gamma \left[ \frac{\Omega_\beta - \omega}{\alpha} \right] \sigma_\phi. \]  
(73)
where the quantity $\eta$ represents the particle flow along the magnetic flux or the particle-to-magnetic field flux ratio.

Then plugging (73) back into the particle number conservation equation (64) yields

$$0 = \nabla \cdot (\alpha n u) = \nabla \cdot (\eta B)$$

$$= \eta (\nabla \cdot B) + B \cdot (\nabla \eta) = B \cdot (\nabla \eta),$$

which implies that $\eta$ must be constant on magnetic surfaces as well, i.e. $\eta = \eta(\Psi)$. (III),(IV)

Let $\chi^\mu$ be a Killing field associated with an isometry of the background space–time metric, then

$$\nabla \cdot T^{\mu\nu} = 0, \quad \nabla \cdot \chi_0 + \nabla \cdot \chi_0 = 0,$$

which yields

$$\nabla \cdot (T^{\mu\nu} \chi_\mu) = 0.$$  

Since the Hartle–Thorne metric possesses the time-translational isometry and the rotational isometry, there are corresponding Killing fields $k^\mu = (\partial/\partial t)^\mu$ and $m^\mu = (\partial/\partial \phi)^\mu$, respectively, such that the quantities $\epsilon^\mu = -T^{\mu\nu} k_\nu$ and $\mathcal{L}^\mu = T^{\mu\nu} m_\nu$ are covariantly conserved. To be a little more precise, $\epsilon^\mu = -T^{\mu\nu} k_\nu = -(T_{f0}^\mu + T_{em0}^\mu)$, $\mathcal{L}^\mu = T^{\mu\nu} m_\nu = (T_{f\phi}^\mu + T_{em\phi}^\mu)$.

Thus, using

$$T_{f0}^\mu + T_{em0}^\mu \left[ \left( \frac{ru}{c^2} \right) u^\mu u^\nu + P g^{\mu\nu} \right] + \frac{1}{4\pi} \left[ F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{4} g^{\mu\nu} (F_{\alpha\beta} F^{\alpha\beta}) \right],$$

and

$$\epsilon^\mu = (c\gamma, \gamma \nu) = \left[ c\gamma, \eta \frac{B}{n\alpha} + \gamma \left( \frac{\Omega - \omega}{\alpha} \right) \sigma \epsilon^\phi \right]$$

and

$$\epsilon' = -T_{f0}^\mu = nu^\mu E,$$

$$\mathcal{L}' = T_{f\phi}^\mu = nu^\mu L,$$

one gets two more integrals of motion (Beskin & Pariev 1993; Beskin 1997)

$$E = E(\Psi) = \frac{\Omega I}{2\pi} + w \eta (\alpha \gamma + \omega \sigma u_\phi),$$

$$L = L(\Psi) = \frac{I}{2\pi} + w \eta \sigma u_\phi$$

and the total loss of energy and angular momentum are given by

$$W_{\text{int}} = \int_0^{\nu_{\text{max}}} E(\Psi) d\Psi,$$

$$K_{\text{int}} = \int_0^{\nu_{\text{max}}} L(\Psi) d\Psi.$$  

(V) The entropy conservation $\nabla_\nu (nu^\nu) = 0$ reduces, for stationary axisymmetric case, to

$$\nabla \cdot (\alpha n u) = 0.$$  

Thus using

$$u = \frac{\eta}{\alpha n} B + \gamma \left[ \frac{\Omega - \omega}{\alpha} \right] \sigma \epsilon^\phi,$$

one gets

$$0 = \nabla \cdot (\alpha n u) = \nabla \cdot (\eta s B)$$

$$= s \nabla \cdot (\eta B) + \eta B \cdot (\nabla s) = \eta B \cdot (\nabla s)$$

which implies that the entropy per particle $s$ must be constant on magnetic surfaces as well

$$s = s(\Psi).$$

To summarize, for the stationary axisymmetric case, there are 5-integrals of motion (constants on magnetic surfaces)

$$\{ \Omega I(\Psi), \eta(\Psi), s(\Psi), E(\Psi), L(\Psi) \}.$$  

We shall now show that once the poloidal magnetic field $B_\phi$ and the 5-integrals of motion given above are known, the toroidal magnetic field $B_\phi$ and all the other plasma parameters characterizing a plasma flow can be determined.
To do so, we solve the two conservation laws in equation (80) and the toroidal component of equation (73)

\[ u_\phi = \frac{\eta}{\alpha n} B_\phi + \gamma \left[ \frac{\Omega_\ell - \omega}{\alpha} \right] \sigma = -\frac{2\eta I}{\alpha^2 n \sigma} + \gamma \left[ \frac{\Omega_\ell - \omega}{\alpha} \right] \sigma \]  

(87)

for \{I, \gamma, u_\phi\} to get (Beskin & Pariev 1993; Beskin 1997)

\[ I = \frac{\alpha^2 L - (\Omega_\ell - \omega) \sigma^2 (E - \omega L)}{\alpha^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2}, \]

\[ \gamma = \frac{\alpha^2 (E - \Omega_\ell L) - M^2 (E - \omega L)}{\alpha \eta \omega / \sigma^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2}, \]

\[ u_\phi = \frac{1}{\sigma \eta \omega / \sigma^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2}, \]

where \( M^2 = 4\eta^2 \omega / n = \alpha^2(u_p^2/u_A^2) \) is the square of the Mach number of the poloidal velocity \( u_p = \eta (B_\rho / na) \) with respect to the Alfvén velocity \( u_A = B_\rho (4\pi n \omega)^{-1/2}. \)

Now in order to determine this Mach number, consider

\[ \gamma^2 - u^2 = \gamma^2 - \gamma^2 w^2 = \gamma^2 (1 - v^2) = 1 \]

and into this relation, we substitute equations (73) and (88) to get (Beskin & Pariev 1993; Beskin 1997)

\[ \frac{K}{\sigma^2 A^2} \equiv \frac{1}{64\pi^4} \frac{M^2 (\nabla \Psi)^2}{\sigma^2} + \alpha^2 \eta^2 w^2, \]

(90)

where

\[ A = \alpha^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2 \equiv N^2 - M^2, \]

\[ K = \alpha^2 \sigma^2 (E - \Omega_\ell L) \left[ \alpha^2 - (\Omega_\ell - \omega)^2 \sigma^2 - 2M^2 \right] + M^2 [\sigma^2 (E - \omega L)^2 - \alpha^2 L^2] \]

which is the Bernoulli equation.

To summarize, once \( B_\rho, \Omega_\ell(\Psi), \Psi(\Psi), s(\Psi), E(\Psi), L(\Psi) \) are known, the characteristics of the plasma flow, \{I (or B_\phi), \gamma, u_\phi, u_p, M^2 (or u_A)\} can be determined by equations (88)–(90).

### 4.2.2 The Grad–Shafranov equation

The Grad–Shafranov equation is the ‘trans-field’ equation of magnetic field lines and it results from the poloidal component of the Euler equation (69). Further the Grad–Shafranov equation describes a ‘force-balance’ in the transfield (i.e. poloidal) directions. For the case at hand in which the content of plasma dynamics is taken into account, the Grad–Shafranov or the pulsar jet equation reads (Beskin & Pariev 1993; Beskin 1997)

\[ \frac{1}{\alpha} \nabla \cdot \left\{ \frac{1}{\alpha \sigma^2} \left[ \alpha^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2 \right] \nabla \Psi \right\} + \frac{(\Omega_\ell - \omega) d\Omega_\ell}{d\Psi} \left| \nabla \Psi \right|^2 + \frac{64\pi^4}{\alpha^2 \sigma^2} \frac{1}{2M^2} \frac{\partial}{\partial \Psi} \left( \frac{G}{A} \right) - 16\pi^2 n \omega \frac{d \eta}{\partial \Psi} - 16\pi^2 n T \frac{dx}{d\Psi} = 0, \]

(91)

where \( T \) denotes the temperature and \( G \equiv \alpha^2 \sigma^2 (E - \Omega_\ell L)^2 + \alpha^2 M^2 L^2 - M^2 \sigma^2 - (E - \omega L)^2 \). Note that this Grad–Shafranov equation contains only \( \Psi \) and 5-integrals of motion, position and physical constants. Thus the Grad–Shafranov equation is autonomous.

Also it is interesting to note that taking the limit, \( M^2 \to 0 \) and \( s \to 0 \), this pulsar jet equation given above reduces to the pulsar equation, i.e. the force-free limit of the Grad–Shafranov equation (neglecting the content of plasma dynamics) given earlier in equation (28) and at the same time the five integrals of motion also reduce to just 2-integrals of motion \( (\Omega_\ell(\Psi), I(\Psi)) \) which can be envisaged from equation (80).

### 4.2.3 Singular surfaces

The algebraic equations (88) and (90) allow for the determination of the locations of the singular surfaces of general relativistic MHD flows, as follows.

**Alfvén surfaces**

From equations (88) and (90), one realizes that there exists general relativistic version of the Alfvén points where \( A = \alpha^2 - (\Omega_\ell - \omega)^2 \sigma^2 - M^2 = 0 \) holds. Then using \( M^2 = \alpha^2 (u_p^2/u_A^2) \), one immediately sees that on the Alfvén surface (Beskin & Pariev 1993; Beskin 1997)

\[ u_p^2 = u_A^2 \left[ 1 - \frac{(\Omega_\ell - \omega)^2 \sigma^2}{\alpha^2} \right], \]

\[ \overline{\Omega_\ell} = \overline{\Omega_\ell} \]
must hold which, in the non-relativistic limit, coincides with the Alfvén velocity. On this Alfvén surface, in order to keep the value of \( \{ I, \gamma, u_\phi \} \) in equation (88) finite, one requires that numerators vanish there as well. This constraint amounts to a single relation (Takahashi et al. 1990; Nitta et al. 1991):

\[
\left[ \alpha^2 + \omega (\Omega_\nu - \omega) \sigma^2 \right] L - (\Omega_\nu - \omega) \sigma^2 E = 0
\]

or equivalently

\[
\Omega_\nu (L/E) = \frac{\sigma^2 \Omega_\nu (\Omega_\nu - \omega)}{[\alpha^2 + \omega (\Omega_\nu - \omega) \sigma^2]}.
\]

(93)

Note that it possesses essentially the same structure as its (rotating) black hole counterpart (Takahashi et al. 1990; Nitta et al. 1991). This is a general relativistic version of the Newtonian result that the angular momentum carried away by the wind is given by the position of the Alfvén point (Weber & Davis 1967; Camenzind 1986). Equations (92) and (93) also allows us to express the location of a single Alfvén point as

\[
\sigma_A = \left( \frac{\alpha^2 L}{(\Omega_\nu - \omega) (E - \omega L)} \right)^{1/2}.
\]

(94)

**Light cylinders**

Like in the force-free case we discussed earlier, the pulsar magnetosphere under consideration possesses a single light cylinder whose location is given by \( N^2 \equiv \alpha^2 - (\Omega_\nu - \omega)^2 \sigma^2 = 0 \), namely at

\[
\sigma_L = \frac{\alpha c}{(\Omega_\nu - \omega)}
\]

(95)
as \( \Omega_\nu > \omega \) everywhere for the case of rotating neutron star as we stressed earlier. And in the force-free limit, \( M^2 \rightarrow 0 \) and \( s \rightarrow 0 \) or equivalently \( E = \Omega_\nu L \), the Alfvén surface discussed above coincides with this light cylinder, i.e. \( \sigma_A = \sigma_L \). Next, the possible existence of the fast and the slow magnetosonic surfaces in this case of a pulsar magnetosphere can be checked following essentially the same procedure as that in the case of rotating (Kerr) black hole magnetosphere. Perhaps, the easiest way of defining these magnetosonic surfaces is to think of them as being singularities in the expression for the gradient of the Mach number \( M \). Here, however, we shall not go into any more detail and instead, we refer the interested reader to Beskin & Pariev (1993), Beskin (1997), Takahashi et al. (1990) and Nitta et al. (1991) for related discussions.

**Injection surfaces**

Lastly, we introduce the injection surfaces, \( r = r_I(\theta, \Omega_\nu (\Psi)) \) for both plasma inflow and outflow where a poloidal flow starts with a sub-Alfvénic velocity. And the plasma inflow or outflow which starts from this injection point must pass through the Alfvénic point to reach the neutron star surface or the far region. In order to determine these surfaces, however, we need some concrete physical model which is beyond the scope of the present work.

### 4.3 Problems with the Grad–Shafranov approach

We now discuss the difficulties when treating a (rotating) black hole or pulsar magnetosphere in terms of the so-called Grad–Shafranov approach. As has been pointed out thus far, the central role is played by the Grad–Shafranov equation in determining the structure of electromagnetic field and the characteristics of the plasma flow in a black hole or pulsar magnetosphere. Thus we begin with the algorithm to solve the Grad–Shafranov equation.

**Step 1.** The physical constants \( \{ n, w, T, B_I \} \) are known and the 5-integrals of motion \( \{ \Omega_\nu (\Psi), \eta (\Psi), \sigma (\Psi), E (\Psi), L (\Psi) \} \) are given.

**Step 2.** One might be able to solve the Grad–Shafranov equation in equation (91) for the poloidal magnetic flux or the stream function \( \Psi = \Psi(r, \theta) \) as a function of the poloidal coordinates \( (r, \theta) \).

**Step 3.** Then from this \( \Psi = \Psi(r, \theta) \) and using

\[
B_r = \frac{\nabla \Psi \times e_\theta}{2 \pi \sigma}, \quad B_\theta = -\frac{2 I}{\alpha \sigma} e_\phi, \quad E_r = -\left( \frac{\Omega_\nu - \omega}{2 \pi \alpha} \right) \nabla \Psi, \quad E_\theta = 0
\]

(96)
one in principle determines the structure of the electromagnetic fields and then next using equations (88)–(90), one obtains the characteristics of the plasma flow \( \{ I, \gamma, u_\phi, u_\rho, M^2 \} \).

In this way, in principle, one can determine the structure of pulsar/black hole magnetospheres. In practice, however, this Grad–Shafranov approach does not appear to be so tractable since in the step (1), there is no known systematic way of evaluating the ‘physical constants’ and giving the ‘5-integrals of motion’ in terms of the stream function \( \Psi \).

In the force-free case we discussed earlier, however, the plasma content is now absent and the whole task of dealing with the Grad–Shafranov approach reduces to the attempt at finding the solution (i.e. the stream function \( \Psi(r, \theta) \)) of the stream equation (28). Even in this
The authors would like to thank Dr V. S. Beskin and Dr S. J. Park for interesting discussions during the winter school to describe the geometries of millisecond pulsars. They also thank the anonymous referee for valuable criticism and advice that much improved Section 3 of the manuscript. HK was

5 SUMMARY AND DISCUSSION

In the present work, we performed a study of pulsar magnetospheres in the context of general relativistic magnetohydrodynamics (MHD) by employing the so-called Grad–Shafranov approach. We considered both the force-free and full MHD situations and accordingly derived the pulsar equation and the pulsar jet equation, respectively. The resulting Grad–Shafranov equations and all the other related force-free equations or general relativistic MHD equations turn out to take essentially the same structures as those for a (rotating) black hole magnetosphere.

The essential distinction between the two cases, however, is the space–time (metric) contents. For the pulsar magnetosphere case, one needs to choose the Hartle–Thorne metric mentioned above whereas for the black hole magnetosphere case, one has to select the Kerr black hole metric. In this way, we demonstrated that pulsar and black hole magnetospheres can be described in a unified and consistent manner.

Also there is quite an uncomfortable state of affair that there has been no complete model for the structure of longitudinal (or poloidal) currents circulating in the neutron star magnetosphere that can provide the solution to the problem, say, of pulsar spin-down. To this problem, we have provided a partly satisfying solution again by treating the region outside a magnetized rotating neutron star as a curved space–time represented by the Hartle–Thorne metric. Namely, we have demonstrated that both for the pulsar and rotating black hole cases the structure of charge separation and the direction of longitudinal (or poloidal) current circulating in the magnetospheres actually lead to the magnetic braking torques that spin down rotating neutron stars and black holes regardless of whether the spin and the (asymptotic) direction of the magnetic field are parallel or antiparallel. We also remarked that the structure of charge separation that resulted from our force-free treatment of pulsar magnetospheres turns out to be the same as that in the original model of Goldreich & Julian (1969). And this unified picture can be thought of as a more satisfying solution to both the magnetized rotating neutron star interpretation of radio/X-ray pulsars (Mereghetti & Steelar 1995; Kouveliotou et al. 1998; Mereghetti 2000; Thompson 2001) and the rotating supermassive black hole interpretation of AGNs/quasars (Blandford & Znajek 1977; Kim et al. 2001a,b) or even GRBs (Lee et al. 2000; Kim et al. 2003).

Next, one might be worried about the validity of the Hartle–Thorne metric for the region surrounding the slowly rotating neutron stars employed in this work to describe the magnetosphere of pulsars which seem rapidly-rotating having typically millisecond pulsation periods. Thus in the following, we shall defend this point in a careful manner. Here, ‘slowly rotating’ means that a neutron star rotates relatively slowly compared to an equal mass Kerr black hole, which can rotate arbitrarily rapidly up to the maximal rotational $J = M^2$. Thus this does not necessarily mean that the Hartle–Thorne metric for slowly rotating neutron stars cannot properly describe the millisecond pulsars. To see this, note that according to the Hartle–Thorne metric, the angular speed of a rotating neutron star is given by the Lense–Thirring precession angular velocity in equation (9) at the surface of the neutron star, which, restoring the fundamental constants to get back to the gaussian unit, is

$$\omega = \frac{2J}{r_0} \left(\frac{G}{c^2}\right), \quad \text{with} \quad J = \tilde{a}M^2 \left(\frac{G}{c}\right) \quad (0 < \tilde{a} < 1).$$

As we mentioned earlier, one of the obvious differences between the black hole case and the neutron star case is the fact that a black hole is characterized by its event horizon while the neutron star has a hard surface. As such, in terms of the space–time metric generated by each of them, just as the Lense–Thirring precession angular velocity (due to frame-dragging) at the horizon represents the black hole angular velocity, the Lense–Thirring precession angular velocity at the location of neutron star’s surface should give the angular velocity of the rotating neutron star.

Thus the Hartle–Thorne metric gives the angular speed of a rotating neutron star, having the data of a typical radio pulsar, $M \sim 2 \times 10^{33}$ (g), $r_0 \sim 10^6$ (cm), as $\omega = 2\tilde{a}(M^2/r_0^4)(G/c)(G/c^2) \sim 10^4$ (1/s) which, in turn, yields the rotation period of $\tau = 2\pi/\omega \sim 10^{-2}$ (s). And here we used, $(G/c^2) = 0.7425 \times 10^{-28}$(cm$^{-1}$) and $(G/c) = 2.226 \times 10^{-18}$ (cm$^2$ g s$^{-1}$). Indeed, this is impressively comparable to the observed pulsation periods of radio pulsars $\tau \sim 10^{-3} - 1$ (s) we discussed earlier. As a result, we expect that the Hartle–Thorne metric is well-qualified to describe the geometries of millisecond pulsars.

Lastly, although the Grad–Shafranov approach toward the study of pulsar magnetospheres is not fully satisfying for reasons stated earlier, it nevertheless is our hope that at least here we have taken one step closer toward the systematic general relativistic study of the electrodynamics in the region close to the rotating neutron stars in association with their pulsar interpretation.
financially supported by the BK21 Project of the Korean Government and HML was supported by Korean Research Foundation Grant No. 2002-041-C20123. CHL and HKL were supported in part by grant No. R01-1999-00020 from the Korea Science and Engineering Foundation.

REFERENCES

Aly J. J., 1980, A&A, 86, 192
Bardeen J. M., 1970, ApJ, 162, 71
Bardeen J. M., Press W. H., Teukolsky S. A., 1972, ApJ, 178, 347
Beskin V. S., 1997, Phys. Uspek., 40, 659
Beskin V. S., 1999, Phys. Uspek., 42, 1071
Beskin V. S., Pariev V. I., 1993, Phys. Uspek., 36, 529
Beskin V. S., Gurevich A. V., Istonin Ya. N., 1983, Sov. Phys. JETP, 58, 235
Blandford R. D., Znajek R. L., 1977, MNRAS, 179, 433
Camenzind M., 1986, A&A, 156, 137
Contopoulos I., Kazanas D., Fendt C., 1999, ApJ, 511, 351
Dannour T., 1978, Phys. Rev. D, 18, 359b
Gold T., 1968, Nat, 218, 731
Goldrich P., Julian W. H., 1969, ApJ, 157, 869
Grad H., 1960, Rev. Mod. Phys., 32, 830
Hartle J. B., Thorne K. S., 1968, ApJ, 153, 807
Kerr R., 1963, Phys. Rev. Lett., 11, 552
Kim H., Lee C. H., Lee H. K., 2001a, Phys. Rev. D, 63, 064037
Kim H., Lee H. K., Lee C. H., 2001b, Phys. Rev. D, 63, 104024
Kim H., Lee H. K., Lee C. H., 2003, J. Cosmol. Astropart. Phys., 0309, 001
Kouveliotou C. et al., 1998, Nat, 393, 235
Lee C. H., Lee H. K., Kim H., 2003, J. Korean Phys. Soc., 43, 24
Lee H. K., Wijers R. A. M., Brown G. E., 2000, Phys. Rep., 325, 83
Lense J., Thirring H., 1918, Phys. Zh., 19, 156
Macdonald D. A., Thorne K. S., 1982, MNRAS, 198, 345
Mereghetti S., 2000, in Connaughton V., Kouveliotou C., van Paradijs J., Ventura J., eds, The Neutron Star–Black Hole Connection. Reidel, Dordrecht, in press (astro-ph/9911252)
Mereghetti S., Stee1 L., 1995, ApJ, 444, L17
Mestel L., Wang Y.-M., 1979, MNRAS, 188, 799
Michel F. C., 1973a, ApJ, 180, 207
Michel F. C., 1973b, ApJ, 180, L133
Muslimov A., Tsygan A. I., 1992, MNRAS, 255, 61
Nitta S., Takahashi M., Tomimatsu A., 1991, Phys. Rev. D, 44, 2295
Okamoto L., 1974, MNRAS, 167, 457
Okamoto L., 1992, MNRAS, 254, 192
Pacini F., 1967, Nat, 216, 567
Pacini F., 1968, Nat, 219, 145
Punsly B., 1991, ApJ, 372, 424
Punsly B., 2001, Black Hole Gravitohydromagnetics. Springer, Berlin
Punsly B., Coroniti F. V., 1990, ApJ, 350, 518
Shafranov V. D., 1958, Sov. Phys. JETP, 6, 545
Sibatunlin N. R., Sibatunlin I. N., Garcia A. A., Manko V. S., 2004, A&A, 422, 587
Takahashi M., Nitta S., Tomimatsu Y., Tomimatsu A., 1990, ApJ, 363, 206
Thompson C., 2001, in Feroci M., Mereghetti S., Stee1 L., eds, Soft Gamma Repeaters: the Rome 2000 Mini-Workshop, in press (astro-ph/0110679)
Thorne K. S., Macdonald D. A., 1982, MMRAS, 198, 339
Thorne K. S., Price R. H., Macdonald D. A., 1996, Black Holes: the Membrane Paradigm. Yale Univ. Press, New Haven
Weber E. J., Davis L., Jr, 1967, ApJ, 148, 217
Znajek R. L., 1978, MNRAS, 185, 833

This paper has been typeset from a TEX/LATEX file prepared by the author.