Torsion theories and coverings of preordered groups

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Abstract. We explore a non-abelian torsion theory in the category of preordered groups: the objects of its torsion-free subcategory are the partially ordered groups, whereas the objects of the torsion subcategory are groups (with the total order). The reflector from the category of preordered groups to this torsion-free subcategory has stable units, and we prove that it induces a monotone-light factorization system. We describe the coverings relative to the Galois structure naturally associated with this reflector, and explain how these coverings can be classified as internal actions of a Galois groupoid. Finally, we prove that in the category of preordered groups there is also a pretorsion theory, whose torsion subcategory can be identified with a category of internal groups. This latter is precisely the subcategory of protomodular objects in the category of preordered groups, as recently discovered by Clementino, Martins-Ferreira, and Montoli.

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1. Introduction

The category $\text{PreOrdGrp}$ of preordered groups is the category whose objects $(G, \leq)$ are groups $G$ endowed with a preorder relation $\leq$ on $G$ which is compatible with the group structure $+: a \leq c$ and $b \leq d$ implies $a + b \leq c + d$, for all $a, b, c, d \in G$. The morphisms in this category are preorder preserving group morphisms.
Alternatively, a preordered group \((G, \leq)\) can be seen in a different way. Indeed, consider the submonoid

\[ P_G = \{ g \in G \mid 0 \leq g \} \]

of \(G\), called the positive cone of \(G\), that has the property of being closed under conjugation in \(G\). It is well-known that the category of preordered groups is isomorphic to the category whose objects are pairs \((G, P_G)\), where \(G\) is a group and \(P_G\) is its positive cone, and whose arrows \((f, \bar{f}) : (G, P_G) \to (H, P_H)\) are pairs \((f, \bar{f})\) where \(f : G \to H\) is a group morphism and \(\bar{f} : P_G \to P_H\) is a monoid morphism, such that the following diagram commutes (the vertical morphisms are the inclusions):

\[
\begin{array}{ccc}
P_G & \xrightarrow{f} & P_H \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & H.
\end{array}
\]  

(1.1)

In this article we will always work with this latter equivalent presentation of the category \(\text{PreOrdGrp}\). In [11] Clementino, Martins-Ferreira and Montoli proved that \(\text{PreOrdGrp}\) has some remarkable exactness properties. First of all, \(\text{PreOrdGrp}\) is a normal category [27]: this means that \(\text{PreOrdGrp}\) has a zero-object, any arrow in it can be factorized as a normal epimorphism (i.e. a cokernel) followed by a monomorphism, and these factorizations are pullback-stable. Secondly, in this category normal epimorphisms and effective descent morphisms coincide, an observation which is fundamental in our study of the coverings in \(\text{PreOrdGrp}\).

Our first result is that \(\text{PreOrdGrp}\) contains two full (replete) subcategories, denoted by \(\text{Grp}\) and \(\text{ParOrdGrp}\), which form a (non-abelian) torsion theory \((\text{Grp}, \text{ParOrdGrp})\) (Proposition 3.1). Here the objects of the torsion subcategory \(\text{Grp}\) are those preordered groups \((G, G)\) such that the positive cone \(P_G\) is \(G\) itself, whereas the objects in the torsion-free subcategory \(\text{ParOrdGrp}\) have the property that the positive cone is a reduced monoid: \(x + y = 0\) implies \(x = y = 0\), for any \(x, y \in P_G\). Via the isomorphism of categories recalled above, preordered groups with a reduced monoid as positive cone exactly correspond to partially ordered groups. We then have a reflective subcategory

\[
\begin{array}{ccc}
\text{PreOrdGrp} & \xrightarrow{F} & \text{ParOrdGrp} \\
\xleftarrow{U} & & \text{ParOrdGrp},
\end{array}
\]  

(1.2)

where each component of the unit of the adjunction is a normal epimorphism. We prove that the reflector \(F : \text{PreOrdGrp} \to \text{ParOrdGrp}\) has stable units [9] in Proposition 3.7, and this implies that the adjunction can be studied from the point of view of Categorical Galois Theory [23]. By constructing, for any preordered group \((G, P_G)\), an effective descent morphism whose domain is a partially ordered group and whose codomain is \((G, P_G)\) (Proposition 3.10), we can show that this adjunction induces a monotone-light factorization system \((\mathcal{E}', \mathcal{M}^*)\) (Theorem 3.12). The class \(\mathcal{E}'\) consists of the morphisms in \(\text{PreOrdGrp}\).
which are stably in $\mathcal{E}$, this meaning that the pullback of a morphism in $\mathcal{E}'$ along any arrow is in $\mathcal{E}$, i.e. it is inverted by the reflector $F: \text{PreOrdGrp} \to \text{ParOrdGrp}$.

The class $\mathcal{M}^*$ is the important class of coverings, in the sense of Galois theory, with respect to the adjunction (1.2). In elementary terms, the coverings turn out to be the morphisms $(f, \overline{f}): (G, P_G) \to (H, P_H)$ as in (1.1) having a partially ordered kernel: $\text{Ker}(f, \overline{f}) \in \text{ParOrdGrp}$. In the fourth section we then compare our results with the ones on locally semisimple coverings from [22]. Categorical Galois Theory [23] then provides a classification theorem of the coverings in $\text{PreOrdGrp}$ in terms of the Galois groupoid of the effective descent morphism mentioned above. In our context this groupoid is actually an equivalence relation, and the above-mentioned description of the coverings in terms of actions (i.e. discrete fibrations) is explicitly given in Theorem 4.5.

It turns out that the adjunction (1.2) also induces a pretorsion theory (in the sense of [15,17]) in $\text{PreOrdGrp}$. This is given by the pair $(\text{ProtoPreOrdGrp}, \text{ParOrdGrp})$, where the torsion part is this time the category $\text{ProtoPreOrdGrp}$ whose objects $(G, P_G)$ are characterized by the fact that the positive cone $P_G$ is a group. As shown in [11] these objects are precisely the so-called protomodular objects [30] of $\text{PreOrdGrp}$. This interesting category, which can be also seen as the category of internal groups in the category $\text{PreOrd}$ of preordered sets (see [11]), is not only coreflective (as any torsion subcategory of a pretorsion theory is) but also reflective in $\text{PreOrdGrp}$: this is proved in Proposition 5.5, where an explicit description of the reflector is provided.

We conclude this introduction by mentioning the related work in [16,31], where similar results have been obtained in the context of internal preorders in an exact category. The results on preordered groups presented in this article are not special cases of the ones presented in those references, since a preordered group is not an internal preorder in the category $\text{Grp}$ of groups.

2. Preliminaries

2.1. Torsion theories in normal categories

In this part we briefly recall the notion of torsion theory in a normal category. There are several approaches to non-abelian torsion theories in various contexts, which can be found, for instance, in [4,10,12,26] (and in the references therein).

A finitely complete category $\mathcal{C}$ is normal [27] if

1. $\mathcal{C}$ has a zero object, denoted by 0;
2. any arrow $f: A \to B$ in $\mathcal{C}$ factors as a normal epimorphism (i.e. a cokernel) followed by a monomorphism;
3. normal epimorphisms are stable under pullbacks: in a pullback diagram

$$
\begin{array}{ccc}
E \times_B A & \xrightarrow{\pi_2} & A \\
\downarrow{\pi_1} & & \downarrow{f} \\
E & \xrightarrow{p} & B
\end{array}
$$

(2.1)
\[ \pi_2 \text{ is a normal epimorphism whenever } p \text{ is a normal epimorphism.} \]

Many familiar algebraic categories, such as groups, abelian groups, rings, Lie algebras, crossed modules of groups and of Lie algebras, are normal. For a variety whose theory has a unique constant 0, being normal is equivalent to being 0-regular (in the sense of [18]): each congruence is determined by the equivalence class of 0. Any semi-abelian category is normal, as well as any homological category [2]. The categories of topological groups [3], compact groups, Heyting semi-lattices [28] and cocommutative Hopf algebras over a field [20] are all examples of normal categories. It was recently proved that the category of preordered groups is also normal [11], and this observation will be important for our work.

In a normal category there is a natural notion of short exact sequence: two composable arrows \( \kappa \) and \( f \) form a short exact sequence

\[
0 \longrightarrow A \xrightarrow{\kappa} B \xrightarrow{f} C \longrightarrow 0
\]

if \( \kappa = \ker(f) \) and \( f = \coker(\kappa) \). Two useful properties of normal categories are the following (see [5]):

**Lemma 2.1.** Let \( C \) be a normal category.

1. A morphism \( f : A \to B \) in \( C \) is a monomorphism if and only if its kernel \( \ker(f) \) is trivial: \( \ker(f) \cong 0 \).
2. Given a commutative diagram of short exact sequences in \( C \)

\[
\begin{array}{ccc}
0 & \longrightarrow & A & \xrightarrow{\kappa} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \\
0 & \longrightarrow & A' & \xrightarrow{\kappa'} & B' & \xrightarrow{f'} & C' & \longrightarrow & 0
\end{array}
\]

the left-hand square is a pullback if and only if the arrow \( c \) is a monomorphism.

**Definition 2.2.** A torsion theory in a normal category \( C \) is given by a pair \((\mathcal{T}, \mathcal{F})\) of full (replete) subcategories of \( C \) such that:

(a) the only arrow from any \( T \in \mathcal{T} \) to any \( F \in \mathcal{F} \) is the zero arrow;
(b) for any object \( C \) of \( C \) there exists a short exact sequence

\[
0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0
\]

whith \( T \in \mathcal{T} \) and \( F \in \mathcal{F} \).

Given a torsion theory \((\mathcal{T}, \mathcal{F})\) in a normal category \( C \) the subcategory \( \mathcal{T} \) is called a torsion subcategory of \( C \) and the subcategory \( \mathcal{F} \) a torsion-free subcategory of \( C \), by analogy with the terminology used for the classical torsion theory \((\text{Ab}_t, \text{Ab}_f)\) in the category \( \text{Ab} \) of abelian groups, where \( \text{Ab}_t \) is the category of torsion abelian groups and \( \text{Ab}_f \) the category of torsion-free abelian groups.
Observe that the exact sequence in Definition 2.2(b) is unique, up to isomorphism. Indeed, assume that for an object \( C \) in \( \mathcal{C} \) we have two short exact sequences, with kernel in \( \mathcal{T} \) and cokernel in \( \mathcal{F} \):

\[
\begin{array}{c}
0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0 \\
\rotatebox{90}{$\cong$} \downarrow \quad \rotatebox{90}{$\cong$} \downarrow \quad \rotatebox{90}{$\cong$} \downarrow \\
0 \longrightarrow T' \xrightarrow{\epsilon_C'} C' \xrightarrow{\eta_C'} F' \longrightarrow 0.
\end{array}
\] (2.3)

Then, since \( \eta_C \) is the cokernel of \( \epsilon_C \) and \( \eta_C' \cdot \epsilon_C \) is the zero arrow (by (a)), there exists a unique morphism \( f: F \to F' \) such that \( f \cdot \eta_C = \eta_C' \). It is then easy to show that \( f \) is an isomorphism (its inverse is induced, symmetrically, by the universal property of the cokernel \( \eta_C' \)). Dually, by using the universal property of kernels, there is an isomorphism \( t: T \to T' \).

Now, consider a morphism \( \phi: C \to C' \) in \( \mathcal{C} \) as in the following diagram

\[
\begin{array}{c}
0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0 \\
\rotatebox{90}{$\cong$} \downarrow \phi \downarrow F(\phi) \downarrow \rotatebox{90}{$\cong$} \downarrow \\
0 \longrightarrow T' \xrightarrow{\epsilon_C'} C' \xrightarrow{\eta_C'} F' \longrightarrow 0.
\end{array}
\] (2.4)

where the two rows are the unique short exact sequences in Definition 2.2(b) associated with \( C \) and \( C' \), respectively. As above the dotted arrows \( F(\phi): F \to F' \) and \( T(\phi): T \to T' \) are induced by the universal properties of the cokernel \( \eta_C \) and of the kernel \( \epsilon_C' \), respectively. This construction then gives rise to two functors, \( T: \mathcal{C} \to \mathcal{T} \) and \( F: \mathcal{C} \to \mathcal{F} \), which are the right (respectively, the left) adjoint of the inclusion functor \( V: \mathcal{T} \to \mathcal{C} \) (respectively, \( U: \mathcal{F} \to \mathcal{C} \)) (see [4], for instance). The functor \( F: \mathcal{C} \to \mathcal{F} \) is a (normal epi)-reflector, i.e. a reflector with the property that each component \( \eta_C: C \to UF(C) \) of the unit \( \eta \) of the adjunction \( F \dashv U \) is a normal epimorphism. The dual statement is also true: the torsion subcategory \( \mathcal{T} \) is (regular mono)-coreflective in \( \mathcal{C} \). Note that the morphism \( \eta_C: C \to UF(C) \) is the arrow \( \eta_C: C \to F \) in Definition 2.2 above. Similarly the \( C \)-component of the counit of the adjunction \( V \dashv T \) is the arrow \( \epsilon_C: (T =) VT(C) \to C \) of Definition 2.2.

2.2. Effective descent morphisms

The notion of effective descent morphism can be defined in terms of discrete fibrations of internal equivalence relations, two concepts that we are now going to recall. For more details on the content of this section the interested reader can refer to [1, 25, 24]. Let \( \mathcal{C} \) be any category with pullbacks. An internal equivalence relation is a diagram

\[
\begin{array}{ccc}
R \times_X R & \xrightarrow{p_1} & R \\
\downarrow \sigma & \nearrow \tau_2 & \downarrow \tau_1 \\
\downarrow p_2 & & \downarrow \\
& X
\end{array}
\] (2.5)

in \( \mathcal{C} \), where \( (R \times_X R, p_1, p_2) \) is defined by the following pullback
the morphisms $r_1$ and $r_2$ are jointly monomorphic, and the following identities are satisfied:

1. $r_1 \cdot \Delta = 1_X = r_2 \cdot \Delta$ (reflexivity);
2. $r_1 \cdot \sigma = r_2$, and $r_2 \cdot \sigma = r_1$ (symmetry);
3. $r_1 \cdot p_1 = r_1 \cdot \tau$ and $r_2 \cdot p_2 = r_2 \cdot \tau$ (transitivity).

Of course, when $\mathcal{C}$ is the category $\text{Set}$ of sets and functions, $r_1$ and $r_2$ are the first and second projections of the relation $R$, $\Delta$ is the “diagonal map” yielding the reflexivity of the relation, $\sigma$ and $\tau$ are the “symmetry” and the “transitivity” maps, respectively. In other words, an internal equivalence relation in $\text{Set}$ is just an equivalence relation in the usual sense. More generally, an internal equivalence relation in any variety $\mathbb{V}$ of universal algebras is a congruence \cite{7}, i.e. an equivalence relation which is also compatible with the operations of the algebraic theory of $\mathbb{V}$.

**Example 2.3.** The kernel pair $(\text{Eq}(p), p_1, p_2)$ of a morphism $p: E \to B$ is always an internal equivalence relation in $\mathcal{C}$. Note that, in universal algebra, the kernel pair of a homomorphism is sometimes called its “kernel” \cite{7}.

From now on, to simplify the notations, an internal equivalence relation in a category $\mathcal{C}$ as in (2.5.) will be depicted as follows:

$$
\begin{array}{ccc}
R & \xrightarrow{r_1} & X \\
\downarrow{p_1} & & \downarrow{r_1} \\
R & \xrightarrow{r_2} & X,
\end{array}
$$

A discrete fibration of internal equivalence relations from $(R, r_1, r_2)$ to $(R', r'_1, r'_2)$ is given by a couple $(f_0, f_1)$ of arrows in $\mathcal{C}$ such that all the corresponding squares in the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{r_1} & X \\
\downarrow{f_1} & & \downarrow{f_0} \\
R' & \xrightarrow{r'_1} & X' \\
\downarrow{r'_2} & & \downarrow{r'_2}
\end{array}
$$

commute and such that the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{r_2} & X \\
\downarrow{f_1} & & \downarrow{f_0} \\
R' & \xrightarrow{r'_2} & X'
\end{array}
$$

is a pullback.

**Remark 2.4.** Note that, by the the symmetry of the relations, the conditions above imply that also the following diagram is a pullback:
Given a morphism $p: E \to B$, the discrete fibrations of equivalence relations with codomain $\text{Eq}(p)$

\[
\begin{array}{ccc}
R & \xrightarrow{r_1} & X \\
\downarrow f_1 & & \downarrow f_0 \\
R' & \xrightarrow{r'_1} & X'.
\end{array}
\]  

(2.8)

are the objects of a category, denoted by $\text{DiscFib}(\text{Eq}(p))$, where the morphisms are pairs $(\phi_0, \phi_1)$ of morphisms in $\mathcal{C}$ making the following diagram commute:

\[
\begin{array}{ccc}
R & \xrightarrow{\phi_1} & F \\
\downarrow f_1 & & \downarrow f_0 \\
\text{Eq}(p) & \xrightarrow{p_1} & E \\
\end{array}
\]

For a morphism $p: E \to B$ in $\mathcal{C}$, we write $p^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow E$ for the induced pullback functor along $p$, where $\mathcal{C} \downarrow B$ and $\mathcal{C} \downarrow E$ are the usual slice categories. A morphism $p: E \to B$ is called an effective descent morphism when the pullback functor $p^*: \mathcal{C} \downarrow B \to \mathcal{C} \downarrow E$ is monadic. Now, this property can also be expressed in terms of discrete fibrations, as follows: $p$ is an effective descent morphism if and only if the functor $K_p: \mathcal{C} \downarrow B \to \text{DiscFib}(\text{Eq}(p))$ sending an object $f: A \to B$ in $\mathcal{C} \downarrow B$ to the discrete fibration $(\pi_1, \bar{\pi}_1)$ of equivalence relations

\[
\begin{array}{ccc}
\text{Eq}(\pi_2) & \xrightarrow{\bar{\pi}_1} & E \times_B A \\
\downarrow \pi_1 & & \downarrow \pi_1 \\
\text{Eq}(p) & \xrightarrow{\pi_1} & E,
\end{array}
\]  

(2.9)

where $\bar{\pi}_1$ is the arrow induced by the universal property of $\text{Eq}(p)$ and by the commutativity of (2.1), is an equivalence of categories. In a regular category this is equivalent to the following properties [25]: $p$ is a regular epimorphism and, moreover, for any discrete fibration (2.8) of equivalence relations with codomain $\text{Eq}(p)$, the equivalence relation $R$ is effective (i.e. it is a kernel pair).

2.3. Factorization systems

We now recall the link between (reflective) factorization systems and (admissible) Galois structures. For this we mainly follow [9,8,12,23], where the reader will find more information about these topics. In this section we shall work in an arbitrary category $\mathcal{C}$. 
In order to define the notion of factorization system, some notations have to be introduced. For morphisms $e$ and $m$ in $C$, we write $e \downarrow m$ if there exists, for any commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow^a & \quad \xrightarrow{\phi} & \downarrow^b \\
C & \xleftarrow{m} & D,
\end{array}
\]

a unique arrow $\phi: B \to C$ such that $\phi \cdot e = a$ and $m \cdot \phi = b$. With respect to a given pair $(E, M)$ of classes of morphisms in $C$ one then defines:

- $E^\downarrow = \{ m \in C \mid e \downarrow m \ \forall \ e \in E \}$;
- $M^\uparrow = \{ e \in C \mid e \downarrow m \ \forall \ m \in M \}$.

**Definition 2.5.** A prefactorization system on the category $C$ is given by a pair $(E, M)$ of classes of morphisms in $C$ such that $E = M^\uparrow$ and $M = E^\downarrow$.

**Definition 2.6.** A factorization system on $C$ is a prefactorization system $(E, M)$ with the following additional property: for any morphism $f$ in $C$ there exist morphisms $e \in E$ and $m \in M$ such that $f = m \cdot e$.

Thanks to results from [9] we know that, given a full reflective subcategory $F$ of $C$

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{F} & \mathcal{F}, \quad (2.10)
\end{array}
\]

we then naturally get a prefactorization system $(E, M)$ defined as follows:

- $E = \{ f \in C \mid F(f) \text{ is an isomorphism} \}$;
- $M = \{ f \in C \mid \text{the following square (2.11) is a pullback} \}$:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & UF(A) \\
\downarrow^f & & \downarrow^{UF(f)} \\
B & \xleftarrow{\eta_B} & UF(B), \quad (2.11)
\end{array}
\]

where $\eta$ is the unit of the adjunction (2.10).

Moreover, we know that $(E, M)$ is a factorization system if the functor $F: C \to \mathcal{F}$ is semi-left-exact in the sense of [9]: it preserves all pullbacks of the form

\[
\begin{array}{ccc}
P & \xrightarrow{} & U(C) \\
\downarrow & & \downarrow^{U(f)} \\
B & \xleftarrow{} & UF(B),
\end{array}
\]

where $\eta_B: B \to UF(B)$ is the $B$-component of the unit of the adjunction (2.10) and $f: C \to F(B)$ is an arrow in the subcategory $\mathcal{F}$ of $C$.

In fact a reflection is semi-left-exact if and only if it is admissible in the sense of categorical Galois theory [23] (with respect to the classes of all morphisms, as explained in [8]). In this context the morphisms in $M$ defined above are called trivial coverings.
Note that, for a reflector \( F : C \to \mathcal{F} \), there exists a stronger property than being semi-left-exact:

**Definition 2.7** [9]. A reflector \( F : C \to \mathcal{F} \) as in (2.10) has stable units when it preserves pullbacks of the form

\[
\begin{array}{ccc}
P & \longrightarrow & C \\
\downarrow & & \downarrow f \\
B & \underset{\eta_B}{\longrightarrow} & UF(B)
\end{array}
\]

where \( \eta_B : B \to UF(B) \) is the \( B \)-component of the unit of the adjunction (2.10) and \( f : C \to UF(B) \) is any arrow in the category \( C \).

**Remark 2.8.** It is well known that, given a torsion theory \((\mathcal{T}, \mathcal{F})\) in a normal category \( C \), the reflector \( F : C \to \mathcal{F} \) to the torsion-free subcategory has stable units [13].

Given the reflection (2.10) we can define the following two subclasses of morphisms in \( C \):

\[ \mathcal{E}' = \{ f \in C \mid \text{the pullback of } f \text{ along any morphism in } C \text{ is in } \mathcal{E} \} \]

and

\[ \mathcal{M}^* = \{ f \in C \mid \text{there exists an effective descent morphism } p \text{ such that } p^*(f) \text{ is in } \mathcal{M} \} \]

Morphisms in \( \mathcal{M}^* \) are precisely the coverings which are the object of study in categorical Galois theory (whenever the reflection (2.10) is semi-left-exact). In particular, one of the goals of this paper is to describe these coverings in the category of preordered groups, and to show that the pair \((\mathcal{E}', \mathcal{M}^*)\) is a monotone-light factorization system in the following sense:

**Definition 2.9** [8]. A factorization system is said to be monotone-light when it is of the form \((\mathcal{E}', \mathcal{M}^*)\) for some factorization system \((\mathcal{E}, \mathcal{M})\).

We are now ready to state the main result of [12] (see also [8]), which will be useful later on:

**Theorem 2.10.** Let \( C \) be a normal category. Let \((\mathcal{T}, \mathcal{F})\) be a torsion theory in \( C \) such that, for any normal monomorphism \( k : K \to A \), the monomorphism \( k \cdot \epsilon_K : T(K) \to A \) is normal in \( C \), where \( \epsilon_K : T(K) \to K \) is the \( K \)-component of the counit \( \epsilon \) of the coreflection \( C \to \mathcal{T} \). We write \((\mathcal{E}, \mathcal{M})\) for the factorization system associated with the reflector \( F : C \to \mathcal{F} \), which has stable units. If for any object \( C \) in \( C \) there is an effective descent morphism \( p : F \to C \) with \( F \in \mathcal{F} \), then \((\mathcal{E}', \mathcal{M}^*)\) is a monotone-light factorization system and, moreover,

- \( \mathcal{E}' \) is the class of normal epimorphisms in \( C \) whose kernel is in \( \mathcal{T} \);
- \( \mathcal{M}^* \) is the class of morphisms in \( C \) whose kernel is in \( \mathcal{F} \).
2.4. Limits and short exact sequences in \textit{PreOrdGrp}

We recall the description of some limits and colimits, and of the short exact sequences in the category of preordered groups \cite{11}. The product of two preordered groups \((G, P_G)\) and \((H, P_H)\) is given by the direct product of groups \(G \times H\) with the positive cone \(P_{G \times H}\) defined by \(P_{G \times H} = P_G \times P_H\). Next, the equalizer of two arrows \((f, \bar{f}), (g, \bar{g})\): \((G, P_G) \Rightarrow (H, P_H)\)

is built by computing the equalizer \(e: E \to G\) of \(f\) and \(g\) in \textit{Grp}, and the positive cone \(P_E\) of \(E\) is then given by the intersection of \(P_G\) and \(E\), i.e. the pullback of the inclusion morphisms \(P_G \to G\) and \(E \to G\) in the category \textit{Mon} of monoids. It is then easily seen that the inclusion \(P_G \cap E \to P_G\) is the equalizer of \(\bar{f}\) and \(\bar{g}\) in the category \textit{Mon} of monoids. Pullbacks and kernels are computed in the same way, by considering pullbacks and kernels “componentwise” at each level (group and monoid). A description of colimits is also possible. The coequalizer of \((f, \bar{f})\) and \((g, \bar{g})\) is computed by taking the coequalizer \(q): H \to Q\) of \(f\) and \(g\) in \textit{Grp} and then the direct image of the submonoid \(P_H\) along \(q\): \(P_Q = q(P_H)\). The description of coproducts is more complicated, and it will not be needed for our work.

The description of normal epimorphisms and normal monomorphisms in the category \textit{PreOrdGrp} of preordered groups will also be useful. A morphism \((f, \bar{f}): (G, P_G) \to (H, P_H)\) in \textit{PreOrdGrp} is an epimorphism if and only if \(f\) is surjective. It is a normal epimorphism when, moreover, the morphism \(\bar{f}\) in \textit{PreOrdGrp} if and only if \(f\) is injective (which also implies that \(\bar{f}\) is injective). Such a morphism is a normal monomorphism if \(f\) is a normal monomorphism in \textit{Grp} and \(P_G = f^{-1}(P_H)\) (i.e. the square (1.1) is a pullback).

In the next proposition we gather the information from \cite{11} which is useful to describe short exact sequences in the category \textit{PreOrdGrp} of preordered groups:

**Proposition 2.11.** Consider, in \textit{PreOrdGrp}, a pair of composable arrows as in the following diagram

\[
\begin{array}{ccc}
P_A & \xrightarrow{k} & P_B & \xrightarrow{\bar{f}} & P_C \\
A & \xrightarrow{a} & (P) & \xrightarrow{b} & (P) \\
B & \xrightarrow{k} & f & \xrightarrow{c} & C.
\end{array}
\]  

Then:

1. the morphism \((k, \bar{k})\) is the kernel of \((f, \bar{f})\) if and only if \(k\) is the kernel of \(f\) in \textit{Grp} and the square \((P)\) is a pullback in \textit{Mon};
2. the morphism \((f, \bar{f})\) is the cokernel of \((k, \bar{k})\) if and only if \(f\) is the cokernel of \(k\) in \textit{Grp} and \(\bar{f}\) is surjective.
(3) the sequence (2.12) is a short exact sequence in $\text{PreOrdGrp}$ if and only if

$$0 \rightarrow A \xrightarrow{k} B \xrightarrow{f} C \rightarrow 0$$

is a short exact sequence in $\text{Grp}$, $(P)$ is a pullback in $\text{Mon}$, and $\bar{f}$ is surjective.

The category $\text{PreOrdGrp}$ of preordered groups is normal, as observed in [11], where it is also proved that a morphism in $\text{PreOrdGrp}$ is effective for descent if and only if it is a normal epimorphism (or, equivalently, if and only if it is a regular epimorphism).

2.5. Schreier points and special Schreier morphisms in monoids

As we saw in the introduction the category of preordered groups is equivalent to the one whose objects are pairs $(G, M)$ where $G$ is a group and $M$ is a submonoid of $G$ closed under conjugation. While the category of groups is protomodular, which means that the Split Short Five Lemma holds in $\text{Grp}$, this is not the case for the category of monoids [2]. Moreover, actions in monoids are not equivalent to split extensions of monoids (while this is the case for groups). Nevertheless, we can restrict our attention to a class of points (a “point” $(A, B, p, s)$ being a split epimorphism $p: A \rightarrow B$ with fixed section $s: B \rightarrow A$) in $\text{Mon}$, called Schreier points (or, equivalently, Schreier split epimorphisms) [6], which have a behavior which is quite similar to the ones in the category of groups. The class of Schreier points corresponds to monoid actions, and it was shown in [6] that the Split Short Five Lemma does hold for such points.

**Definition 2.12.** A Schreier point in the category $\text{Mon}$ of monoids is a point $(A, B, p, s)$ such that for any element $a$ in $A$ there exists a unique element $x$ in the kernel $\text{Ker}(p)$ of $p$ such that

$$a = x + (s \cdot p)(a).$$

This kind of points is useful to “locally” extend some classical properties of split extensions of groups to the context of monoids. We shall not develop these interesting aspects here, but we refer the reader to [6] for a thorough introduction to this subject. What will be of interest for the purpose of this paper is to briefly recall the properties of special Schreier morphisms in the category of monoids.

**Definition 2.13.**

- An internal reflexive relation in the category $\text{Mon}$ of monoids

$$\begin{array}{c}
R \xrightarrow{r_1} A \\
\downarrow s \quad \downarrow r_2 \\
\end{array}$$

is said to be a Schreier reflexive relation when the point $(R, A, r_1, s)$ is a Schreier one.

- A morphism $f: A \rightarrow B$ in the category $\text{Mon}$ of monoids is said to be a special Schreier morphism when its kernel pair
Eq\( (f) \)
\[ \begin{array}{ccc}
\text{Eq}(f) & \leftarrow (1,1) & \rightarrow A
\end{array} \]
\[ \begin{array}{ccc}
\text{Eq}(f) & \leftarrow (1,1) & \rightarrow A
\end{array} \]

is a Schreier reflexive relation, where \((1,1): A \rightarrow \text{Eq}(f)\) is such that \(f_1 \cdot (1,1) = 1 = f_2 \cdot (1,1)\).

It is then possible to prove [6] that any surjective special Schreier morphism \(f: A \rightarrow B\) is the cokernel of its kernel. Accordingly, we get an extension of monoids:

\[ 0 \rightarrow \text{Ker}(f) \rightarrow k \rightarrow A \rightarrow f \rightarrow B \rightarrow 0. \]

These \textit{special Schreier extensions} satisfy some remarkable properties. The following proposition states two of them, which will be needed for our future investigations:

\textbf{Proposition 2.14 [6].}

1. \textit{Special Schreier extensions are pullback stable in \text{Mon}.}
2. \textit{The Short Five Lemma holds for special Schreier extensions. This means that given any commutative diagram (2.2) of short exact sequences in \text{Mon}, where \(f\) and \(f'\) are special Schreier morphisms, and \(a\) and \(c\) are isomorphisms, then \(b\) is also an isomorphism.}

\section{Coverings in the category of preordered groups}

\textbf{Proposition 3.1.} The pair of full (replete) subcategories \((\text{Grp}, \text{ParOrdGrp})\) of \text{PreOrdGrp} is a torsion theory in the normal category \text{PreOrdGrp}.

\textit{Proof.} Let us first show that the only arrow in \text{PreOrdGrp} from an object of \text{Grp} to an object of \text{ParOrdGrp} is the zero morphism. Consider an arrow \((f, \bar{f}): (G,G) \rightarrow (H,P_H)\) in \text{PreOrdGrp}, with \((G,G)\) in \text{Grp} and \((H,P_H)\) in \text{ParOrdGrp}:

\[ G \xrightarrow{\bar{f}} P_H \]
\[ G \xrightarrow{f} H. \]

For any \(x \in G\), its opposite \(-x\) is also in \(G\), and

\[ 0 = \bar{f}(x-x) = \bar{f}(x) + \bar{f}(-x), \]

with \(\bar{f}(x), \bar{f}(-x) \in P_H\), and \(P_H\) is a reduced monoid. This implies that \(\bar{f}(x) = \bar{f}(-x) = 0\), \(\bar{f} = 0\), and then \(f = 0\).

Consider then an object \((G,P_G)\) of \text{PreOrdGrp}, and define

\[ N_G = \{ n \in G \mid n \in P_G \text{ and } -n \in P_G \}. \]

It is a normal subgroup of \(G\): indeed, if \(n \in N_G\) and \(x \in G\), then we have that \(x + n - x \in P_G\) and \(-(x + n - x) = x - n - x \in P_G\), since the submonoid \(P_G\) is closed under conjugation in \(G\). Accordingly, the sequence
\[ N_G \xrightarrow{k_G} G \xrightarrow{\eta_G} G/N_G \] is a short exact sequence in the category Grp of groups. Consider next the direct image factorization of the morphism \( \eta_G \cdot g \) in the category Mon of monoids, where \( P_G \xrightarrow{g} G \) is the inclusion:

\[
\begin{array}{ccc}
  P_G & \xrightarrow{\eta_G} & \eta_G(P_G) \\
  g & \downarrow & \downarrow \psi_G \\
  G & \xrightarrow{\eta_G} & G/N_G.
\end{array}
\]

Let us now prove that the sequence

\[
\begin{array}{ccc}
  N_G & \xrightarrow{k_G} & P_G \xrightarrow{\eta_G} \eta_G(P_G) \\
  \downarrow & & \downarrow \psi_G \\
  N_G & \xrightarrow{k_G} & G \xrightarrow{\eta_G} G/N_G
\end{array}
\]

(3.1)

is exact in the category PreOrdGrp of preordered groups. This follows from Proposition (2.11), since the left-hand square in (3.1) is clearly a pullback in Mon, the lower sequence is exact, and the morphism \( \bar{\eta}_G \) is surjective by construction.

It is obvious that \((N_G, N_G) \in \text{Grp}\), so that the proof will be complete if we show that \((G/N_G, \eta_G(P_G))\) is in ParOrdGrp. Now, if \(y+z = 0\) for \(y, z \in \eta_G(P_G)\), then there exist \(x, x' \in P_G\) such that \(\eta_G(x) = y\) and \(\eta_G(x') = z\), so that \(\eta_G(x + x') = y + z = 0\), that is \(x + x' \in N_G\). Since \(N_G\) is a group and \(P_G\) is a monoid it follows that

\[ -x = x' - x = x' - (x + x') \in P_G, \]

hence \(x \in N_G\), which implies that \(y = \eta_G(x) = 0\) and therefore \(z = 0\). Accordingly, the submonoid \(\eta_G(P_G)\) is reduced, and \((G/N_G, \eta_G(P_G))\) a partially ordered group. \(\square\)

**Remark 3.2.** Note that a similar result can be proved in the category of commutative monoids, as observed in [14] (Section 2.3): the pair \((\text{Ab}, \text{RedCMon})\) is a torsion theory in the category CMon of commutative monoids, where we write \(\text{Ab}\) for the category of abelian groups and \(\text{RedCMon}\) for the category of reduced commutative monoids.

As a consequence of Proposition 3.1 we get the following result:

**Corollary 3.3.**

- The category ParOrdGrp is reflective in PreOrdGrp

\[
\begin{array}{ccc}
  \text{PreOrdGrp} & \xrightarrow{F} & \text{ParOrdGrp} \\
  \downarrow U & \parallel & \downarrow \eta \\
  \text{PreOrdGrp} & \xleftarrow{\eta} & \text{ParOrdGrp}.
\end{array}
\]

(3.2)

and each component of the unit \(\eta\) of the adjunction (as in (3.1)) is a normal epimorphism.
The category \( \text{Grp} \) is coreflective in \( \text{PreOrdGrp} \) and each component of the counit \( \kappa \) of the adjunction (as in (3.1)) is a normal monomorphism.

**Proof.** This follows from the Proposition 3.1 and the (only) Proposition in [26] (see also [4], and [10]).

We now make some useful comments on the short exact sequence (3.1) constructed in the proof of Proposition 3.1:

**Lemma 3.4.** Consider the following commutative diagram in the category \( \text{Mon} \) of monoids, where the two rows are special Schreier extensions:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(f) & \xrightarrow{k} & A & \xrightarrow{f} & B & \rightarrow & 0 \\
& & a & \downarrow{b} & & c & & \downarrow & \\
0 & \rightarrow & \text{Ker}(f') & \xrightarrow{k'} & A' & \xrightarrow{f'} & B' & \rightarrow & 0.
\end{array}
\] (3.3)

If the morphism \( a \) is an isomorphism, then the right-hand square of diagram (3.3) is a pullback.

**Proof.** Let us consider the pullback \( (P,p_A',p_B) \) of \( f' \) and \( c \), and then the kernel \( (\text{Ker}(p_B),k'') \) of \( p_B \).

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(f) & \xrightarrow{k} & A & \xrightarrow{f} & B & \rightarrow & 0 \\
& & a & \downarrow{b} & & c & & \downarrow & \\
0 & \rightarrow & \text{Ker}(f') & \xrightarrow{k'} & A' & \xrightarrow{f'} & B' & \rightarrow & 0.
\end{array}
\]

We are going to show that the arrow \( \phi \) induced by the universal property of the pullback \( (P,p_A',p_B) \) is an isomorphism. By the universal property of the kernel \( k' = \ker(f') \) we first get the morphism \( \psi \) such that \( k' \cdot \psi = p_A' \cdot k'' \).

In the same way the universal property of the kernel \( k'' = \ker(p_B) \) gives a unique arrow \( \gamma \) such that \( k'' \cdot \gamma = \phi \cdot k \). It is easily seen that \( \psi \cdot \gamma = a \), with \( a \) an isomorphism by assumption. In addition, \( \psi \) is also an isomorphism since \( (P,p_A',p_B) \) is a pullback, so that \( \gamma \) is itself an isomorphism. We have that the bottom row is a special Schreier extension, hence by the pullback stability of special Schreier extensions (Proposition 2.14) we get that the middle row of the above diagram is a special Schreier extension. Again by Proposition 2.14 (second assertion) we apply the Short Five Lemma to the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker}(f) & \xrightarrow{k} & A & \xrightarrow{f} & B & \rightarrow & 0 \\
& & a & \downarrow{b} & & c & & \downarrow & \\
0 & \rightarrow & \text{Ker}(p_B) & \xrightarrow{k''} & P & \xrightarrow{p_B} & B & \rightarrow & 0.
\end{array}
\]

to conclude that \( \phi \) is an isomorphism, that is the right-hand square in the diagram (3.3) is a pullback. \( \square \)
**Corollary 3.5.** Consider a short exact sequence in $\text{PreOrdGrp}$ of the following form:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K & \overset{\tilde{k}}{\longrightarrow} & P_G & \overset{\tilde{f}}{\longrightarrow} & P_H & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \overset{k}{\longrightarrow} & G & \overset{f}{\longrightarrow} & H & \longrightarrow & 0
\end{array}
$$

(3.4)

Then the upper sequence is a special Schreier extension, and the right-hand square in (3.4) is a pullback in the category $\text{Mon}$ of monoids.

**Proof.** By Lemma 3.4, since any short exact sequence in the category $\text{Grp}$ of groups is a special Schreier extension, it suffices to show that the sequence

$$
K \overset{\tilde{k}}{\longrightarrow} P_G \overset{\tilde{f}}{\longrightarrow} P_H
$$

is a special Schreier extension in $\text{Mon}$. Consider the kernel pair $(\text{Eq}(\tilde{f}), r_1, r_2)$ of $\tilde{f}$ where the projections $r_1$ and $r_2$ are split by the diagonal morphism $s: P_G \rightarrow \text{Eq}(\tilde{f})$. Note that there is no restriction in assuming that $f: G \rightarrow H \cong G/K$ is the canonical quotient of $G$ by its normal subgroup $K$, and we write $f(g) = \bar{g}^K$.

For any $(a, b) \in \text{Eq}(\tilde{f})$, one has the equalities $f(a) = \tilde{f}(a) = \tilde{f}(b) = f(b)$, since $\tilde{f}$ is the restriction of $f$ to the positive cone $P_G$ of $G$. It follows that $\bar{a}^K = \bar{b}^K$, and there exists $x \in K$ such that $b = x + a$. As a consequence $(0, x) \in \text{Ker}(r_1)$ satisfies the equalities

$$(0, x) + (s \cdot r_1)(a, b) = (0, x) + (a, a) = (a, x + a) = (a, b),$$

and $(0, x)$ is the only element of $\text{Ker}(r_1)$ with this property. This shows that $(r_1, s)$ is a Schreier point, and the arrow $\tilde{f}: P_G \rightarrow P_H$ is a special Schreier morphism. Since any surjective special Schreier morphism is the cokernel of its kernel and since $\tilde{f}$ is surjective, this shows that $K \rightarrow P_G \rightarrow P_H$ is a special Schreier extension. $\square$

In particular the previous result implies the following

**Corollary 3.6.** Consider the short exact sequence (3.1) in $\text{PreOrdGrp}$. Then the square

$$
\begin{array}{ccc}
P_G & \overset{\bar{\eta}_G}{\longrightarrow} & \eta_G(P_G) \\
g \downarrow & & \downarrow \psi_G \\
G & \overset{\eta_G}{\longrightarrow} & G/N_G
\end{array}
$$

is a pullback in the category $\text{Mon}$ of monoids.

As a consequence of Corollary 3.6 from now on we will write $P_G/N_G$ instead of $\eta_G(P_G)$ for the codomain of $\bar{\eta}_G$ in the short exact sequence (3.1). This means that in this sequence in $\text{PreOrdGrp}$ we have short exact sequences both at the group and at the monoid level.
As reminded in the previous section, if the reflector \( F : \text{PreOrdGrp} \to \text{ParOrdGrp} \) has stable units, then the adjunction (3.2) gives rise to a factorization system and is admissible in the sense of the categorical Galois theory [23]. The following proposition states that this is in fact the case for our adjunction (3.2).

**Proposition 3.7.** The reflector \( F : \text{PreOrdGrp} \to \text{ParOrdGrp} \) in the adjunction (3.2) has stable units.

**Proof.** We have to prove that the functor \( F \) preserves pullbacks of the form of the right-hand cube in the following commutative diagram

\[
\begin{array}{ccccccc}
P_G \times_{P_G/N_G} P_H & \xrightarrow{F(p_2)} & P_H & \xrightarrow{p_2} & H & \xrightarrow{f} & H \\
\downarrow \phi & & \downarrow h & & \downarrow p_2 & & \downarrow \psi \\
G \times_{G/N_G} H & \xrightarrow{f} & P_G/N_G & \xrightarrow{p_1} & G/N_G & \xrightarrow{\eta_G} & G/N_G \\
\downarrow \bar{\eta}_G & & \downarrow \bar{p}_1 & & \downarrow g & & \downarrow \bar{\eta}_G \\
N_G & \xrightarrow{\bar{k}_G} & \bar{P}_G & \xrightarrow{\bar{\bar{\eta}}_G} & G/N_G & \xrightarrow{\bar{\eta}_G} & G/N_G \\
\downarrow \bar{i} & & \downarrow \bar{i} & & \downarrow k_G & & \downarrow \bar{i} \\
N_G & \xrightarrow{i} & P_G & \xrightarrow{\eta_G} & G & \xrightarrow{\eta_G} & G \\
\end{array}
\]

in which \((k_G, \bar{k}_G)\) is the kernel of \((\eta_G, \bar{\eta}_G)\), and the induced arrow \((i, \bar{i})\) is the kernel of the arrow \((p_2, \bar{p}_2)\). If we apply the functor \( F \) to the diagram (3.5) we get the commutative diagram

\[
\begin{array}{ccccccc}
(P_G \times_{P_G/N_G} P_H)/N & \xrightarrow{F(p_2)} & P_H/N_H & \xrightarrow{\psi} & H/N_H & \xrightarrow{F(f)} & H/N_H \\
\downarrow \psi & & \downarrow F(p_2) & & \downarrow F(f) & & \downarrow F(f) \\
(G \times_{G/N_G} H)/N & \xrightarrow{F(p_1)} & P_G/N_G & \xrightarrow{\psi} & G/N_G & \xrightarrow{\psi} & G/N_G \\
\downarrow \psi & & \downarrow F(p_1) & & \downarrow \psi & & \downarrow \psi \\
0 & \xrightarrow{F(i)} & P_G/N_G & \xrightarrow{\psi} & G/N_G & \xrightarrow{\psi} & G/N_G \\
\end{array}
\]

where we write \( N \) for \( N_{G \times_{G/N_G} H} \) and \((F(a), F(\bar{a}))\) for the image by \( F \) of any arrow \((a, \bar{a})\) in \text{PreOrdGrp}. We observe that the arrows \( p_2 \) and \( p_2 \) are normal epimorphisms since \( \eta_G \) and \( \bar{\eta}_G \) are normal epimorphisms and the front and back squares of the cube in (3.5) are pullbacks in \text{Grp} and \text{Mon}, respectively. This means that \((p_2, \bar{p}_2)\) is the cokernel of its kernel: \((p_2, \bar{p}_2) = \text{coker}(i, \bar{i})\). It follows that \((F(p_2), F(\bar{p}_2))\) is the cokernel of \((F(i), F(\bar{i}))\), which is the zero arrow, and the arrow \((F(p_2), F(\bar{p}_2))\) is then an isomorphism. Accordingly, the front and the back squares of the cube in the diagram (3.6) are pullbacks in \text{Grp} and in \text{Mon}, respectively, i.e. the cube of this diagram is a pullback in \text{ParOrdGrp}, as desired. \( \square \)
Remark 3.8. By taking into account the fact that the pair \((\text{Grp}, \text{ParOrdGrp})\) is a torsion theory in the normal category \(\text{PreOrdGrp}\) (thanks to Proposition 3.1) the above result can be deduced from Theorem 1.6 in [13]. We have included a direct proof here in order to make the article more self-contained, and also to give an explicit description of the behavior of the reflector \(F : \text{PreOrdGrp} \to \text{ParOrdGrp}\).

Let us now characterize the two classes \(\mathcal{E}\) and \(\mathcal{M}\) of the factorization system induced by the reflector \(F : \text{PreOrdGrp} \to \text{ParOrdGrp}\) in the category \(\text{PreOrdGrp}\) of preordered groups:

Proposition 3.9. Given the adjunction \((3.2)\), we have a factorization system \((\mathcal{E}, \mathcal{M})\) in \(\text{PreOrdGrp}\) where:

- \((f, \tilde{f}) : (G, P_G) \to (H, P_H)\) is in the class \(\mathcal{E}\) if and only if the following conditions hold:
  - (a) \(f^{-1}(N_H) = N_G\),
  - (b) for any \(y \in H\) there exists \(x \in G\) such that \(\bar{f}(x)^{N_H} = \gamma^{N_H}\),
  - (c) for any \(y \in P_H\) there exists \(x \in P_G\) such that \(\tilde{f}(x)^{N_H} = \gamma^{N_H}\).

- \((f, \tilde{f}) : (G, P_G) \to (H, P_H)\) is in the class \(\mathcal{M}\) if and only if the morphism \(\phi : N_G \to N_H\) (which is the restriction of \(f : G \to H\) to \(N_G\)) is an isomorphism.

Proof. Consider the commutative diagram

\[
\begin{array}{ccccccccc}
N_G & \xrightarrow{\phi} & G & \xrightarrow{\eta_G} & P_G & \xrightarrow{\tilde{f}} & P_G/N_G & \xrightarrow{\eta_H} & P_H/N_H \\
\downarrow{\ker(\eta_G)} & & \downarrow{\eta_G} & & \downarrow{\bar{f}} & & \downarrow{\bar{\eta}_H} & & \downarrow{\tilde{\eta}_H} \\
\ker(\eta_H) & \xrightarrow{\phi} & H & \xrightarrow{\eta_H} & P_H & \xrightarrow{\tilde{\alpha}} & P_H/N_H \\
\end{array}
\]

where \((\alpha, \tilde{\alpha})\) stands for \(F(f, \tilde{f})\), and where the front and the back squares of the cube are the \((G, P_G)\)-component and the \((H, P_H)\)-component of the unit of the adjunction \((3.2)\), respectively.

- Assume that \((f, \tilde{f}) : (G, P_G) \to (H, P_H)\) is in the class \(\mathcal{E}\), so that \((\alpha, \tilde{\alpha})\) is an isomorphism in \(\text{PreOrdGrp}\). The fact that \(\alpha\) is a monomorphism implies that the square
is a pullback, i.e. \( f^{-1}(N_H) = N_G \) (by Lemma 2.1(2)). Furthermore, knowing that \( \alpha \) is surjective, for any \( y \) in \( H \) there exists an \( x \) in \( G \) such that \( \alpha(x^{N_G}) = y^{N_H} \), that is \( f(x)^{N_H} = y^{N_H} \). We can show the analogue assertion for \( \bar{f} \) in a similar way, since \( \bar{\alpha} \) is surjective.

Conversely, if \( f^{-1}(N_H) = N_G \), the square (3.8) is a pullback, and this implies that \( \alpha \) is a monomorphism (by Lemma 2.1(2) in the category \( \text{Grp} \)). Now, the assumption (b) guarantees that \( \eta_H \cdot f = \alpha \cdot \eta_G \) is surjective, hence \( \alpha \) is surjective. Similarly, \( \bar{\alpha} \) is surjective because the assumption (c) says that \( \bar{\eta}_H \cdot \bar{f} = \bar{\alpha} \cdot \bar{\eta}_G \) is surjective. It follows that \( (\alpha, \bar{\alpha}) \) is an isomorphism in \( \text{PreOrdGrp} \), i.e. that \( (f, \bar{f}) \) belongs to the class \( \mathcal{E}' \).

To prove the second point we observe that the arrow \( (f, \bar{f}) \) belongs to the class \( \mathcal{M} \) if and only if the cube in the diagram (3.7) is a pullback in \( \text{PreOrdGrp} \), and this is equivalent to the bottom and the top squares of this cube being pullbacks in the categories \( \text{Grp} \) and \( \text{Mon} \), respectively. If these two squares are pullbacks then the induced arrow \( \phi : N_G \rightarrow N_H \), the restriction of the morphism \( f : G \rightarrow H \) to \( N_G \), is obviously an isomorphism.

Conversely, if \( \phi \) is an isomorphism, then the bottom square of the cube is a pullback (since the Short Five Lemma holds in the category \( \text{Grp} \) of groups). The fact that the top square of the same cube is a pullback (in the category \( \text{Mon} \) of monoids) is a consequence of Corollary 3.6. Indeed, this latter states that the front and the back squares in the cube of diagram (3.7) are pullbacks, hence the top square is a pullback since the bottom one is a pullback. \( \square \)

Now that we have a description of the trivial coverings, i.e. the morphisms in the class \( \mathcal{M} \), we would like to have a description of the class \( \mathcal{M}^* \) of coverings. We shall actually prove that there is a monotone-light factorization system \( (\mathcal{E}', \mathcal{M}^*) \), by applying Theorem 2.10. In the following two propositions we verify the two fundamental assumptions needed to apply that theorem:

**Proposition 3.10.** For any object \((G, P_G)\) in the category \( \text{PreOrdGrp} \) of preordered groups, there exist an object \((H, P_H)\) in the subcategory \( \text{ParOrdGrp} \) of partially ordered groups and an effective descent morphism 
\[(f, \bar{f}) : (H, P_H) \rightarrow (G, P_G)\]
from \((H, P_H)\) to \((G, P_G)\).

**Proof.** Let \((G, P_G) \in \text{PreOrdGrp} \). Define \((H, P_H)\) in the following way:

- \( H = \mathbb{Z} \times G \);
- \( P_H = (\mathbb{N} \times P_G) \setminus \{(0, g) \mid g \neq 0\} \).
It is easy to check that $P_H$ is a submonoid of the group $H$ (endowed with the natural group structure). To show that $P_H$ is closed in $H$ under conjugation consider $(z, g) \in H$ and $(n, h) \in P_H$. Then

$$(z, g) + (n, h) - (z, g) = (z + n - z, g + h - g) \in \mathbb{N} \times P_G$$

since $P_G$ is closed under conjugation in $G$, and if $z + n - z = 0$, i.e. if $n = 0$, then $(n, h) \in P_H$ implies $h = 0$, that is $g + h - g = g - g = 0$. This means that $(z, g) + (n, h) - (z, g) \in P_H$, and $(H, P_H)$ is a preordered group, which actually lies in $\text{ParOrdGrp}$, since by definition the submonoid $P_H$ is reduced (the only element having an inverse in $P_H$ is $(0, 0)$).

Let us next consider the function $f : H \to G$ defined, for any $(z, g) \in H$, by $f(z, g) = g$. It is a morphism in $\text{PreOrdGrp}$, since it is a group morphism and $f(z, g) = g \in P_G$, for any $(z, g) \in P_H$. In other words, the restriction $\bar{f}$ of $f$ to $P_H$ takes its values in $P_G$. The morphism $(f, \bar{f}) : (H, P_H) \to (G, P_G)$ is also a normal epimorphism in $\text{PreOrdGrp}$, since both $f$ and $\bar{f}$ are easily seen to be surjective. Since effective descent morphisms coincide with normal epimorphisms in $\text{PreOrdGrp}$ [11], the proof is complete.  

**Proposition 3.11.** For any normal monomorphism $(i, \bar{i}) : (K, P_K) \to (G, P_G)$ in $\text{PreOrdGrp}$, the monomorphism $(i, \bar{i}) \cdot (k, \bar{k}) : (N_K, N_K) \to (G, P_G)$ is normal, where $(k, \bar{k}) : (N_K, N_K) \to (K, P_K)$ is the $(K, P_K)$-component of the counit of the coreflection $T : \text{PreOrdGrp} \to \text{Grp}$.

**Proof.** Since $(i, \bar{i})$ is a normal monomorphism, there exists an arrow $(f, \bar{f}) : (G, P_G) \to (H, P_H)$ in $\text{PreOrdGrp}$ such that $(i, \bar{i}) = \ker(f, \bar{f})$. With that notation we then have that $K = \ker(f)$ and that $P_K = K \cap P_G$ since kernels in $\text{PreOrdGrp}$ are computed componentwise at the level of groups and monoids, respectively.

Let us first show that $N_K$ is normal in $G$, where

$$N_K = \{ x \in K \mid x \in P_K \text{ and } -x \in P_K \}$$

$$= \{ x \in K \mid x \in K \cap P_G \text{ and } -x \in K \cap P_G \}$$

$$= K \cap N_G.$$

Since $K$ and $N_G$ are two normal subgroups of $G, N_K$ is normal in $G$. In order to prove that the inclusion $(i, \bar{i}) \cdot (k, \bar{k}) : (N_K, N_K) \to (G, P_G)$ is a normal monomorphism in $\text{PreOrdGrp}$ one observes that the following rectangle is a pullback

$$\begin{array}{ccc}
N_K & \xrightarrow{k} & P_K & \xrightarrow{i} & P_G \\
\downarrow & & \downarrow & & \downarrow_{\psi_G} \\
N_K & \xrightarrow{k} & K & \xrightarrow{i} & G,
\end{array}$$

since it is made of two pullbacks, and the result then follows from Proposition 2.11.  

We are now ready to state the final result of this section.
Theorem 3.12. Let us consider in $\text{PreOrdGrp}$ the following classes of morphisms:

- $\mathcal{E}' = \{ (f, \bar{f}) \in \text{PreOrdGrp} \mid (f, \bar{f}) \text{ is a normal epimorphism such that} \text{Ker}(f, \bar{f}) \in \text{Grp} \}$;
- $\mathcal{M}^* = \{ (f, \bar{f}) \in \text{PreOrdGrp} \mid \text{Ker}(f, \bar{f}) \in \text{ParOrdGrp} \}$.

Then $\left( \mathcal{E}', \mathcal{M}^* \right)$ is a monotone-light factorization system.

Proof. This result follows from Theorem 2.10, which can be applied to the reflection 3.2 thanks to the two previous propositions. □

The coverings with respect to the adjunction (3.2) are then the morphisms $f: A \to B$ in $\text{PreOrdGrp}$ such that $\text{Ker}(f) \in \text{ParOrdGrp}$. This description is then similar to the one of the locally semisimple coverings relative to a generalized semisimple class, given by Janelidze, Márki and Tholen in [22]. We explain the link with this latter approach in the next section.

4. Classification of the coverings of preordered groups

Let us first recall the approach to locally semisimple coverings based on Galois theory developed in [22]. Here below we shall adapt the context in order to include the example of the category $\text{PreOrdGrp}$ of preordered groups.

Let $\mathcal{C}$ be any normal category in which normal epimorphisms and effective descent morphisms coincide. Let us consider a fixed class $\mathcal{X}$ of objects in $\mathcal{C}$, called a generalized semisimple class, having the property that the following two properties hold for any pullback

$$
\begin{array}{c}
E \times_B A \\
\downarrow \pi_1 \\
E \\
\alpha \downarrow \\
\pi_2 \\
A
\end{array}
$$

where $p$ is a normal epimorphism in $\mathcal{C}$:

1. $E \in \mathcal{X}$ and $A \in \mathcal{X}$ implies that $E \times_B A \in \mathcal{X}$;
2. $B \in \mathcal{X}$, $E \in \mathcal{X}$ and $E \times_B A \in \mathcal{X}$ implies that $A \in \mathcal{X}$.

The notion of locally semisimple covering is then defined relatively to a generalized semisimple class $\mathcal{X}$ in a category $\mathcal{C}$: a morphism $\alpha: A \to B$ is a locally semisimple covering in $\mathcal{C}$ if there is a normal epimorphism $p: E \to B$ such that the pullback $p^*(\alpha)$ of $\alpha$ along $p$ lies in the corresponding full subcategory $\mathcal{X}$ of $\mathcal{C}$.

For a fixed $B \in \mathcal{C}$, let $\text{LocSSimple}_{\mathcal{X}}(B)$ be the full subcategory of the slice category $\mathcal{C} \downarrow B$ over $B$ whose objects are pairs $(A, \alpha)$, where $\alpha: A \to B$ is a locally semisimple covering.

Under our assumptions, a normal epimorphism $p: E \to B$ in $\mathcal{C}$ induces a category equivalence $K_p: \mathcal{C} \downarrow B \to \text{DiscFib}(\text{Eq}(p))$, since $p$ is an effective descent morphism. When, moreover, $p: E \to B$ is such that $E$ belongs to $\mathcal{X}$, the functor $K_p$ restricted to the category of locally semisimple coverings gives an equivalence of categories

$$
\text{LocSSimple}_{\mathcal{X}}(B) \cong \text{DiscFib}_{\mathcal{X}}(\text{Eq}(p)),
$$
where $\text{DiscFib}_X(\text{Eq}(p))$ is the full subcategory of $\text{DiscFib}(\text{Eq}(p))$ whose objects are the discrete fibrations over $\text{Eq}(p)$ (as in (2.8)) with $F \in X$:

**Theorem 4.1** [22]. Consider a normal category $\mathcal{C}$ where normal epimorphisms are effective descent morphisms, and $\mathcal{X}$ a generalized semisimple class in $\mathcal{C}$. If $p: E \to B$ is a normal epimorphism in $\mathcal{C}$ such that $E \in \mathcal{X}$, there is an equivalence of categories

$$
\text{LocSSimple}_X(B) \cong \text{DiscFib}_X(\text{Eq}(p)).
$$

(4.1)

**Proof.** This essentially follows from the two properties of the generalized semisimple classes recalled above, that guarantee that a morphism $f: A \to B$ belongs to the subcategory $\text{LocSSimple}_X(B)$ if and only if the corresponding discrete fibration (2.9) is such that $E \times_B A \in \mathcal{X}$ (see [22] for the details). The equivalence $K_p: \mathcal{C} \downarrow B \to \text{DiscFib}(\text{Eq}(p))$ then (co)restricts to the full subcategories $\text{LocSSimple}_X(B)$ (and $\text{DiscFib}_X(\text{Eq}(p))$), yielding the announced equivalence (4.1). □

**Remark 4.2.** Observe that the category $\text{DiscFib}(\text{Eq}(p))$ is also called the category of internal $\text{Eq}(p)$-actions in the literature [25].

In particular we can consider $\mathcal{C} = \text{PreOrdGrp}$, and $\mathcal{X}$ the class of objects of $\text{ParOrdGrp}$, which is easily seen (by using Lemma 2.7 in [19], for instance) to be a generalized semisimple class. We are therefore in a situation where we can apply Theorem 4.1. We first of all state the following lemma:

**Lemma 4.3.** A morphism $(h, \bar{h}): (H, P_H) \to (G, P_G)$ in $\text{PreOrdGrp}$ is a locally semisimple covering (relatively to the subcategory $\text{ParOrdGrp}$) if and only if its kernel is a partially ordered group.

**Proof.** If $(h, \bar{h})$ is a locally semisimple covering there exists a normal epimorphism $(p, \bar{p}): (E, P_E) \to (G, P_G)$ such that $(p, \bar{p})^*(h, \bar{h}) \in \text{ParOrdGrp}$. It follows that $\text{Ker}((p, \bar{p})^*(h, \bar{h})) \in \text{ParOrdGrp}$. Now since the diagram

$$
\begin{array}{ccc}
(E, P_E) \times (G, P_G) & \to & (H, P_H) \\
(p, \bar{p})^*(h, \bar{h}) \downarrow & & \downarrow (h, \bar{h}) \\
(E, P_E) & \to & (G, P_G)
\end{array}
$$

(4.2)

is a pullback we have that $\text{Ker}(h, \bar{h}) \cong \text{Ker}((p, \bar{p})^*(h, \bar{h}))$, so that $\text{Ker}(h, \bar{h}) \in \text{ParOrdGrp}$.

Conversely, by Proposition 3.10, there exists an effective descent morphism (i.e. a normal epimorphism) $(p, \bar{p}): (E, P_E) \to (G, P_G)$ with $(E, P_E) \in \text{ParOrdGrp}$. Since the diagram (4.2) is a pullback, we have that

$$
\text{Ker}((p, \bar{p})^*(h, \bar{h})) \cong \text{Ker}(h, \bar{h})
$$

with $\text{Ker}(h, \bar{h}) \in \text{ParOrdGrp}$ by assumption. Knowing that any torsion-free subcategory is stable by extensions (see [26] for instance) and that $\text{ParOrdGrp}$ is a torsion-free subcategory of $\text{PreOrdGrp}$ (by Proposition 3.1), it follows that $(p, \bar{p})^*(h, \bar{h})$ is in $\text{ParOrdGrp}$. □
Remark 4.4. The previous lemma is a particular case of a more general fact observed in [22] (Proposition 2.3) where, more generally, the role of the kernel of an arrow was played by the “fibers” (as defined in [22]).

Theorem 4.5. Let \((G, P_G) \in \text{PreOrdGrp}\). Consider the effective descent morphism 
\[
(f, \bar{f}) : (\mathbb{Z} \times G, (\mathbb{N} \times P_G) \{ (0, g) \mid g \neq 0 \}) \to (G, P_G)
\] from Proposition 3.10. Then there exists an equivalence of categories 
\[
\mathcal{M}^* \downarrow (G, P_G) \cong \text{DiscFib}_{\text{ParOrdGrp}}(\text{Eq}(f, \bar{f}))
\] where \(\mathcal{M}^* \downarrow (G, P_G)\) is the category of coverings over \((G, P_G)\).

Proof. Since the morphism 
\[
(f, \bar{f}) : (\mathbb{Z} \times G, (\mathbb{N} \times P_G) \{ (0, g) \mid g \neq 0 \}) \to (G, P_G)
\] from Proposition 3.10 is an effective descent morphism in \(\text{PreOrdGrp}\) such that 
\[
(\mathbb{Z} \times G, (\mathbb{N} \times P_G) \{ (0, g) \mid g \neq 0 \}) \in \text{ParOrdGrp}
\] we are allowed to apply Theorem 4.1: there exists then an equivalence of categories 
\[
\text{LocSSimple}_{\text{ParOrdGrp}}(G, P_G) \cong \text{DiscFib}_{\text{ParOrdGrp}}(\text{Eq}(f, \bar{f})).
\]

Thanks to the previous lemma and Theorem 3.12 the proof is complete since both the coverings and the locally semisimple coverings (over \((G, P_G)\)) are described as the arrows \((h, \bar{h}) : (H, P_H) \to (G, P_G)\) with \(\text{Ker}(h, \bar{h}) \in \text{ParOrdGrp}\).

Note that the internal equivalence relation \(\text{Eq}(f, \bar{f})\) from Theorem 4.5 is in fact the Galois groupoid \(\text{Gal}(f, \bar{f})\) associated with the effective descent morphism \((f, \bar{f})\) [23]. By definition the Galois groupoid associated with \((f, \bar{f})\) is indeed the image of \(\text{Eq}(f, \bar{f})\) by the reflector \(F : \text{PreOrdGrp} \to \text{ParOrdGrp}\). But since the diagram 
\[
(\text{Eq}(f), \text{Eq}(\bar{f})) \leftrightarrow (E, P_E)
\] lies in \(\text{ParOrdGrp}\) (where we write \((E, P_E)\) for 
\[
(\mathbb{Z} \times G, (\mathbb{N} \times P_G) \{ (0, g) \mid g \neq 0 \}))
\] the image of \(\text{Eq}(f, \bar{f})\) by the reflector \(F\) is \(\text{Eq}(f, \bar{f})\) itself. In other words \(\text{Eq}(f, \bar{f}) = \text{Gal}(f, \bar{f})\), and 
\[
\mathcal{M}^* \downarrow (G, P_G) \cong \text{DiscFib}_{\text{ParOrdGrp}}(\text{Gal}(f, \bar{f})).
\] This equivalence is the classification of the coverings as internal \(\text{Gal}(f, \bar{f})\)-actions.

Remark 4.6. Besides its interest for the classification of coverings in the category of preordered groups, the above result also provides an example of application of Theorem 3.1 in [22] in a non-exact setting (see Remark 3.2 (e) in [22]).
5. The torsion subcategory of protomodular objects

In this last section we show that the reflection 3.2 gives also rise to a pretorsion theory in the category $\text{PreOrdGrp}$ of preordered groups.

5.1. Pretorsion theories in general categories

The concept of pretorsion theory [15] allows one to extend the notion of torsion theory to a non-pointed category. Here we only recall the basic results which will be useful for this work, and we refer to [17] for the fundamental aspects of the theory. To adapt Definition 2.2 to a general category (not necessarily pointed) a (non-empty) class $\mathcal{Z}$ of objects of $\mathcal{C}$ is introduced, which somehow plays the role of the zero object, and we denote by $\mathcal{N}$ the class of morphisms in $\mathcal{C}$ that factorize through an object of $\mathcal{Z}$. These special morphisms are called $\mathcal{Z}$-trivial. One can then extend the notions of kernel, cokernel and short exact sequence to get the notions of $\mathcal{Z}$-prekernel, $\mathcal{Z}$-precokernel and short $\mathcal{Z}$-preexact sequence.

From now on we assume $\mathcal{C}$ to be an arbitrary category. Given an arrow $f: A \to B$ in $\mathcal{C}$, one says that $k: K \to A$ is a $\mathcal{Z}$-prekernel of $f$ when

- $f \cdot k \in \mathcal{N}$;
- for any morphism $\alpha: X \to A$ such that $f \cdot \alpha \in \mathcal{N}$, there exists a unique arrow $\phi: X \to K$ such that $k \cdot \phi = \alpha$.

Dually, an arrow $c: B \to C$ is a $\mathcal{Z}$-precokernel of $f: A \to B$ when

- $c \cdot f \in \mathcal{N}$;
- for any morphism $\alpha: B \to X$ such that $\alpha \cdot f \in \mathcal{N}$, there exists a unique arrow $\phi: C \to X$ such that $\phi \cdot c = \alpha$.

Any $\mathcal{Z}$-prekernel is a monomorphism and, dually, any $\mathcal{Z}$-precokernel is an epimorphism.

**Definition 5.1.** Let $f: A \to B$ and $g: B \to C$ be two arrows in $\mathcal{C}$. The sequence

$$
0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
$$

is a short $\mathcal{Z}$-preexact sequence when $f$ is a $\mathcal{Z}$-prekernel of $g$ and $g$ is a $\mathcal{Z}$-precokernel of $f$.

We are now ready to recall the definition of pretorsion theory [15,17] (see also [29,21] for an interesting and closely related approach based on the notion of ideal of morphisms):

**Definition 5.2.** A $\mathcal{Z}$-pretorsion theory in the category $\mathcal{C}$ is given by a pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{C}$, with $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$, such that:

- any morphism in $\mathcal{C}$ from $T \in \mathcal{T}$ to $F \in \mathcal{F}$ belongs to $\mathcal{N}$;
- for any object $C$ of $\mathcal{C}$ there exists a short $\mathcal{Z}$-preexact sequence

$$
0 \to T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \to 0
$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
In a similar way as for classical torsion theories, the torsion-free subcategory $\mathcal{F}$ of a pretorsion theory $(\mathcal{F}, \mathcal{F})$ is epireflective in $\mathcal{C}$ and, dually, the torsion subcategory $\mathcal{T}$ is monocoreflective in $\mathcal{C}$ [17].

Also in this more general situation the $C$-component of the unit $\eta$ of the adjunction relative to the reflector $F: \mathcal{C} \rightarrow \mathcal{F}$ is given by the arrow $\eta_C: C \rightarrow F = UF(C)$ of Definition 5.2 where $U: \mathcal{F} \rightarrow \mathcal{C}$ is the inclusion functor, and the $C$-component of the counit of the adjunction $V \dashv T$ is given by the arrow $\epsilon_C: T = VT(C) \rightarrow C$, where $V: \mathcal{T} \rightarrow \mathcal{C}$ stands for the inclusion functor.

5.2. A pretorsion theory in the category $\text{PreOrdGrp}$ of preordered groups

Proposition 5.3. The pair $(\text{ProtoPreOrdGrp}, \text{ParOrdGrp})$ of full replete subcategories of $\text{PreOrdGrp}$ is a $\mathcal{Z}$-pretorsion theory in $\text{PreOrdGrp}$, where $\mathcal{Z} = \text{ProtoPreOrdGrp} \cap \text{ParOrdGrp}$ is given by

$$\mathcal{Z} = \{ (G, P_G) \mid P_G = 0 \}.$$

Observe that the preordered groups in $\mathcal{Z}$ are the ones endowed with the discrete order.

Proof. To prove that any arrow $(f, \bar{f}): (G, P_G) \rightarrow (H, P_H)$ in $\text{PreOrdGrp}$, with $(G, P_G) \in \text{ProtoPreOrdGrp}$ and $(H, P_H) \in \text{ParOrdGrp}$ factorizes through an object in $\mathcal{Z}$, first observe that the following diagram commutes

$$\begin{array}{ccc}
P_G & \xrightarrow{f} & P_H \\
\downarrow & & \downarrow \\
G & \xrightarrow{f} & H,
\end{array}$$

(5.1)

where $f(G)$ is the image of the group morphism $f: G \rightarrow H$. Indeed, as explained in the first part of the proof of Proposition 3.1, any monoid morphism from a group to a reduced monoid is the 0-arrow. Since $f(G)$ is a group and since any part of the diagram 5.1 commutes, we conclude that any arrow in $\text{PreOrdGrp}$ from an object of $\text{ProtoPreOrdGrp}$ to an object of $\text{ParOrdGrp}$ factorizes through an object of $\mathcal{Z}$, i.e it belongs to $\mathcal{N}$.

Consider now any preordered group $(G, P_G)$. We will work as before with the normal subgroup $N_G$ of $G$. We then consider the (regular epimorphism, monomorphism)-factorization of $\eta_G \cdot g$ in the category $\text{Mon}$ of monoids, where $G \xrightarrow{\eta_G} G/N_G$ is the quotient morphism and $P_G \xrightarrow{g} G$ is the inclusion arrow:

$$\begin{array}{ccc}
P_G & \xrightarrow{\bar{\eta}_G} & P_G/N_G \\
\downarrow & & \downarrow \psi_G \\
G & \xrightarrow{\eta_G} & G/N_G.
\end{array}$$
Let us prove that the sequence

\[
\begin{array}{c}
N_G \xrightarrow{i} P_G \xrightarrow{\bar{\eta}_G} P_G/N_G \\
\phi_G \downarrow \quad g \downarrow \quad \eta_G \downarrow \\
G \quad G \quad G \quad G/\bar{\eta}_G \\
\end{array}
\]

in which \((G, N_G) \in \text{ProtoPreOrdGrp}\) and \((G/N_G, P_G/N_G) \in \text{ParOrdGrp}\) is a short \(\mathcal{Z}\)-preexact sequence in \text{PreOrdGrp}.

We begin by showing that \((\eta_G, \bar{\eta}_G)\) is the \(\mathcal{Z}\)-precokernel of the arrow \((1_G, i)\). Let us consider a morphism \((f, \bar{f}) : (G, P_G) \to (H, P_H)\) in \text{PreOrdGrp} such that \((f, \bar{f}) \cdot (1_G, i) \in \mathcal{N}'\), i.e. such that \((f, \bar{f}) \cdot (1_G, i)\) factorizes through an object \((A, 0)\) of \(\mathcal{Z}\): \((f, \bar{f}) \cdot (1_G, i) = (b, \bar{b}) \cdot (a, \bar{a})\).

In particular \(f \cdot \phi_G = b \cdot a \cdot \phi_G = 0\). Since \(\eta_G\) is the cokernel of \(\phi_G\) in the category \text{Grp} of groups, by the universal property of the cokernel, there exists a unique arrow \(\alpha : G/N_G \to H\) in \text{Grp} such that \(\alpha \cdot \eta_G = f\). Now, seeing that \(\bar{\eta}_G\) is a regular epimorphism in \text{Mon} and that \(h\) is a monomorphism, the universal property of strong epimorphisms yields a unique arrow \(\bar{\alpha} : P_G/N_G \to P_H\) such that \(\bar{\alpha} \cdot \bar{\eta}_G = \bar{f}\) and \(h \cdot \bar{\alpha} = \alpha \cdot \psi_G\). In other words there exists a unique arrow \((\alpha, \bar{\alpha}) : (G/N_G, P_G/N_G) \to (H, P_H)\) in \text{PreOrdGrp} such that \((\alpha, \bar{\alpha}) \cdot (\eta_G, \bar{\eta}_G) = (f, \bar{f})\), i.e. \((\eta_G, \bar{\eta}_G)\) is the \(\mathcal{Z}\)-precokernel of \((1_G, i)\).

Next we show that \((1_G, i)\) is the \(\mathcal{Z}\)-prekernel of \((\eta_G, \bar{\eta}_G)\). Consider \((f, \bar{f}) : (H, P_H) \to (G, P_G)\) in \text{PreOrdGrp} such that \((\eta_G, \bar{\eta}_G) \cdot (f, \bar{f}) \in \mathcal{N}'\), i.e. \((\eta_G, \bar{\eta}_G) \cdot (f, \bar{f})\) factorizes through an object \((A, 0)\) of \(\mathcal{Z}\): \((\eta_G, \bar{\eta}_G) \cdot (f, \bar{f}) = (b, \bar{b}) \cdot (a, \bar{a})\).
Let us take $\alpha = f$, since this is the only possible arrow such that $1_G \cdot \alpha = f$. Now we have that $\bar{\eta}_G \cdot \bar{f} = 0$, hence $\eta_G \cdot g \cdot \bar{f} = 0$. Since $\phi_G$ is the kernel of $\eta_G$ in $\text{Mon}$ (and in $\text{Grp}$) there is a unique arrow $\bar{\alpha} : P_H \to N_G$ such that $\phi_G \cdot \bar{\alpha} = g \cdot \bar{f}$. Then
\[ g \cdot i \cdot \bar{\alpha} = \phi_G \cdot \bar{\alpha} = g \cdot \bar{f} \]
and since $g$ is a monomorphism it follows that $i \cdot \bar{\alpha} = \bar{f}$. The morphism $\bar{\alpha}$ is moreover unique with this property, because $i$ is a monomorphism. As a conclusion $(\alpha, \bar{\alpha}) : (H, P_H) \to (G, N_G)$ is the unique arrow such that $(1_G, i) \cdot (\alpha, \bar{\alpha}) = (f, \bar{f})$, and $(1_G, i)$ is the $\mathcal{E}$-prekernel of $(\eta_G, \bar{\eta}_G)$. \hfill \Box

From this Proposition we deduce in particular that the subcategory $\text{ProtoPreOrdGrp}$ of protomodular objects is monocoreflective in $\text{PreOrdGrp}$:

**Corollary 5.4.** The subcategory $\text{ProtoPreOrdGrp}$ of protomodular objects is monocoreflective in the category $\text{PreOrdGrp}$ of preordered groups:

\[
\text{PreOrdGrp} \xleftarrow{V} \text{ProtoPreOrdGrp}. \quad (5.2)
\]

It turns out that the inclusion functor $V : \text{ProtoPreOrdGrp} \hookrightarrow \text{PreOrdGrp}$ is not only a left adjoint but also a right adjoint:

**Proposition 5.5.** The functor $V : \text{ProtoPreOrdGrp} \hookrightarrow \text{PreOrdGrp}$ has a left adjoint functor $E : \text{PreOrdGrp} \rightarrow \text{ProtoPreOrdGrp}$:

\[
\text{PreOrdGrp} \xleftrightarrow{E} \text{ProtoPreOrdGrp}. \quad (5.3)
\]

**Proof.** We begin with the construction of the functor $E$. Let $(G, P_G)$ be any preordered group. Consider the subgroup $M_G$ of $G$ generated by all elements in $P_G \cup (-P_G)$, where we write $-P_G$ for the submonoid
\[ -P_G = \{ x \in G \mid \exists g \in P_G \text{ such that } x = -g \}. \]
Any element $m$ of $M_G$ is of the form $m = g_1 - g_2 + g_3 - \cdots + g_{n-1} - g_n$ for $g_1, \ldots, g_n \in P_G$. Since both $P_G$ and $-P_G$ are submonoids of $G$ it is clear that $M_G$ is a subgroup of $G$. And this subgroup is in addition normal in $G$. Indeed, let $g \in G$ and let $m = g_1 - g_2 + \cdots + g_{n-1} - g_n$ be an element in $M_G$ (where $g_1, \ldots, g_n \in P_G$). Then
\[
g + m - g = g + g_1 - g_2 + \cdots + g_{n-1} - g_n - g = (g + g_1 - g) + (g - g_2 - g) + \cdots + (g + g_{n-1} - g) + (g - g_n - g) = (g + g_1 - g) - (g + g_2 - g) + \cdots + (g + g_{n-1} - g) - (g + g_n - g)
\]
with $g + g_i - g \in P_G$ for any $i \in \{1, \ldots, n\}$ since $P_G$ is closed under conjugation in $G$, and $g + m - g \in M_G$. Accordingly $(G, M_G)$ in an object of the subcategory $\text{ProtoPreOrdGrp}$. This construction is obviously functorial, and we write $E : \text{PreOrdGrp} \rightarrow \text{ProtoPreOrdGrp}$ for the functor defined on objects by $E(G, P_G) = (G, M_G)$, for any $(G, P_G) \in \text{PreOrdGrp}$. 

\[
\text{PreOrdGrp} \xleftarrow{V} \text{ProtoPreOrdGrp}. \quad (5.2)
\]
Let us now prove that this functor $E$ is a left adjoint of the functor $V: \text{ProtoPreOrdGrp} \rightarrow \text{PreOrdGrp}$. Let $(G, P_G) \in \text{PreOrdGrp}$, and let us check that the $(G, P_G)$-component of the unit of the adjunction is given by the arrow $(1_G, j)$

\[
\begin{array}{ccc}
P_G & \xleftarrow{j} & MG \\
\downarrow{g} & & \downarrow{n} \\
G & \cong & G
\end{array}
\]

where $P_G \xleftarrow{j} MG$ is the inclusion morphism. Let $(H, P_H)$ be an object in $\text{ProtoPreOrdGrp}$ and consider any morphism $(f, \tilde{f}): (G, P_G) \rightarrow V(H, P_H) = (H, P_H)$.

There exists a unique morphism $\phi = f: G \rightarrow H$ with the property $\phi \cdot 1_G = f$. We then observe that, for any $m = g_1 - g_2 + \cdots + g_{n-1} - g_n \in MG$ (with $g_1, \ldots, g_n \in P_G$),

\[
f(m) = f(g_1 - g_2 + \cdots + g_{n-1} - g_n)
    = f(g_1) - f(g_2) + \cdots + f(g_{n-1}) - f(g_n)
\]

with $f(g_i) \in P_H$ since $g_i \in P_G$ for any $i \in \{1, \ldots, n\}$. Hence $f(m) \in P_H$ since $P_H$ is a group by assumption. This means that the restriction $f|_{MG}$ of $f$ to $MG$ takes its values in $P_H$. Let us then define $\tilde{\phi} = f|_{MG}: MG \rightarrow P_H$. We can then check that $\tilde{\phi} \cdot j = \tilde{f}$, and we observe that, for any $m \in MG$,

\[
(\phi \cdot n)(m) = (f \cdot n)(m) = f(m) = \tilde{\phi}(m) = (h \cdot \tilde{\phi})(m),
\]

so that $\phi \cdot n = h \cdot \tilde{\phi}$. Accordingly there exists a unique morphism $(\phi, \tilde{\phi}) = (f, f|_{MG}): (G, MG) \rightarrow (H, P_H)$ such that $(\phi, \tilde{\phi}) \cdot (1_G, j) = (f, \tilde{f})$, and the proof is complete. \qed

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