Nonabelian Bundle Gerbes, Their Differential Geometry and Gauge Theory

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Abstract: Bundle gerbes are a higher version of line bundles, we present nonabelian bundle gerbes as a higher version of principal bundles. Connection, curving, curvature and gauge transformations are studied both in a global coordinate independent formalism and in local coordinates. These are the gauge fields needed for the construction of Yang-Mills theories with 2-form gauge potential.

1. Introduction

Fibre bundles, besides being a central subject in geometry and topology, provide the mathematical framework for describing global aspects of Yang-Mills theories. Higher abelian gauge theories, i.e. gauge theories with abelian 2-form gauge potential appear naturally in string theory and field theory, and here too we have a corresponding mathematical structure, that of the abelian gerbe (in algebraic geometry) and of the abelian bundle gerbe (in differential geometry). Thus abelian bundle gerbes are a higher version of line bundles. Complex line bundles are geometric realizations of the integral 2nd cohomology classes $H^2(M, \mathbb{Z})$ on a manifold, i.e. the first Chern classes (whose de Rham representative is the field strength). Similarly, abelian (bundle) gerbes are the next level in realizing integral cohomology classes on a manifold; they are geometric realizations of the 3rd cohomology classes $H^3(M, \mathbb{Z})$. Thus the curvature 3-form of a 2-form gauge potential is the de Rham representative of a class in $H^3(M, \mathbb{Z})$. This class is called the Dixmier-Douady class \cite{1,2}; it topologically characterizes the abelian bundle gerbe in the same way that the first Chern class characterizes complex line bundles.

One way of thinking about abelian gerbes is in terms of their local transition functions \cite{3,4}. Local “transition functions” of an abelian gerbe are complex line bundles on double overlaps of open sets satisfying cocycle conditions for tensor products over quadruple overlaps of open sets. The nice notion of abelian bundle gerbe \cite{5} is related to this picture. Abelian gerbes and bundle gerbes can be equipped with additional structures, that of a connection 1-form and of curving (the 2-form gauge potential), and that
of (bundle) gerbe modules (with or without connection and curving). Their holonomy can be introduced and studied [3, 4, 6–9]. The equivalence class of an abelian gerbe with connection and curving is the Deligne class on the base manifold. The top part of the Deligne class is the class of the curvature, the Dixmier-Douady class.

Abelian gerbes arise in a natural way in quantum field theory [10–12], where their appearance is due to the fact that one has to deal with abelian extensions of the group of gauge transformations; this is related to chiral anomalies. Gerbes and gerbe modules appear also very naturally in TQFT [13], in the WZW model [14] and in the description of D-brane anomalies in the nontrivial background 3-form $H$-field (identified with the Dixmier-Douady class) [15–17]. Coinciding (possibly infinitely many) D-branes are submanifolds “supporting” bundle gerbe modules [6] and can be classified by their (twisted) $K$-theory. The relation to the boundary conformal field theory description of D-branes is due to the identification of equivariant twisted $K$-theory with the Verlinde algebra [18, 19]. For the role of $K$-theory in D-brane physics see e.g. [20–22].

In this paper we study the nonabelian generalization of abelian bundle gerbes and their differential geometry, in other words we study higher Yang-Mills fields. Nonabelian gerbes arose in the context of nonabelian cohomology [23, 1] (see [24] for a concise introduction), see also ([25]). Their differential geometry –from the algebraic geometry point of view– is discussed thoroughly in the recent work of Breen and Messing [26] (and their combinatorics in [27]). Our study on the other hand is from the differential geometry viewpoint. We show that nonabelian bundle gerbes connections and curvings are very natural concepts also in classical differential geometry. We believe that it is primarily in this context that these structures can appear and can be recognized in physics. It is for example in this context that one would like to have a formulation of Yang-Mills theory with higher forms. These theories should be relevant in order to describe coinciding NS5-branes with D2-branes ending on them. They should be also relevant in the study of M5-brane anomaly. We refer to [28–30] for some attempts in constructing higher gauge fields.

Abelian bundle gerbes are constructed using line bundles and their products. One can also study $U(1)$ bundle gerbes; here line bundles are replaced by their corresponding principal $U(1)$ bundles. In the study of nonabelian bundle gerbes it is more convenient to work with nonabelian principal bundles than with vector bundles. Actually principal bundles with additional structures are needed. We call these objects (principal) bibundles and $D$-$H$ bundles ($D$ and $H$ being Lie groups). Bibundles are fibre bundles (with fiber $H$) which are at the same time left and right principal bundles (in a compatible way). They are the basic objects for constructing (principal) nonabelian bundle gerbes. The first part of this paper is therefore devoted to their description. In Sect. 2 we introduce bibundles, $D$-$H$ bundles (i.e. principal $D$ bundles with extra $H$ structure) and study their products. In Sect. 3 we study the differential geometry of bibundles, in particular we define connections, covariant exterior derivatives and curvatures. These structures are generalizations of the corresponding structures on usual principal bundles. We thus describe them using a language very close to that of the classical reference books [31] or [32]. In particular a connection on a bibundle needs to satisfy a relaxed equivariance property, this is the price to be paid in order to incorporate nontrivially the additional bibundle structure. We are thus lead to introduce the notion of a 2-connection $(a, A)$ on a bibundle. Products of bibundles with connections give a bibundle with connection only if the initial connections were compatible. We call this compatibility the summability conditions for 2-connections; a similar summability condition is established also for horizontal forms (e.g. 2-curvatures).