Combating Noisy-Labeled and Imbalanced Data by Two Stage Bi-Dimensional Sample Selection

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Abstract

Robust learning on noisy-labeled data has been an important task in real applications, because label noise directly leads to the poor generalization of deep learning models. Existing label-noise learning methods usually assume that the ground-truth classes of the training data are balanced. However, the real-world data is often imbalanced, leading to the inconsistency between observed and intrinsic class distribution due to label noises. Distribution inconsistency makes the problem of label-noise learning more challenging because it is hard to distinguish clean samples from noisy samples on the intrinsic tail classes. In this paper, we propose a learning framework for label-noise learning with intrinsically long-tailed data. Specifically, we propose a robust sample selection method called two-stage bi-dimensional sample selection (TBSS) to better separate clean samples from noisy samples, especially for the tail classes. TBSS consists of two new separation metrics to jointly separate samples in each class. Extensive experiments on multiple noisy-labeled datasets with intrinsically long-tailed class distribution demonstrate the effectiveness of our method.

1 Introduction

Under the support of large amount of high-quality labeled data, deep neural networks have achieved great success in various fields [Krizhevsky, Sutskever, and Hinton 2012] [Ren et al. 2015] [Devlin et al. 2019]. However, it is expensive and difficult to obtain a large amount of high-quality labeled data in many practical applications. Instead, the commonly used large-scale training data is usually obtained from the Internet or crowdsourcing platforms like Amazon Mechanical Turk, which is unreliable and may be mislabeled (Xiao et al. 2015) [Song, Kim, and Lee 2019]. The models trained on this kind of unreliable data, called noisy-labeled data, often produce poor generalization performance because deep neural networks tend to overfit noisy samples due to their large model capacity. In the literature, there have been a number of works to obtain a robust model trained on noisy-labeled data [Han et al. 2020b] [Song et al. 2020]. Among them, the most straightforward and effective way is to differentiate between clean and noisy samples based on their differences in specific metrics such as training loss [Han et al. 2018] [Li, Socher, and Hoi 2020] [Karim et al. 2022].

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Figure 1: Motivation. (a) The intrinsic class distribution of real labels is inconsistent with the observed class distribution with noisy labels. (b) The training loss of each sample under noisy-labeled and long-tailed data.

These label-noise learning methods generally assume that the intrinsic class distribution of the training data is balanced, that is, each class has almost the same number of samples in terms of their unknown ground-truth labels. However, the data in real-world applications is often imbalanced, e.g. the WebVision dataset (Li et al. 2017) and the iNaturalist dataset (Horn et al. 2018). Especially when the number of classes is large, the class imbalance usually exhibits in the form of a long-tail distribution, where a small portion of classes possess a large number of samples, and the other classes possess a small number of samples only [Reed 2001] [Horn and Perona 2017]. In this case, the model training tends to the head classes and ignores the tail classes [Zhang et al. 2021b] [Zhu et al. 2022]. When both noisy labels and long-tail distribution exist, training a robust model is even more challenging. There are two key challenges in this scenario. (1) Distribution inconsistency: The observed and intrinsic distributions are likely inconsistent due to noise labels, making the model more difficult to discover and focus on the intrinsic tail classes. As illustrated in Figure 1(a), the intrinsic class distribution of the dataset is long-tailed, while the existence of noisy labels makes the distribution more balanced. The intrinsic tail classes, e.g., class 6 and 9, are occupied by a large number of noisy data, making them no longer tail classes by observation. (2) Tail inseparability: Even if the tail class is identified, it is more difficult than ever to distinguish between clean and noisy samples in the tail class, because clean samples are overwhelmed by noises that make their values of the separation metric highly sim-
ilar. As illustrated in Figure 1b, the training loss of clean and noisy samples in the tail class are generally inseparable compared with the ones in the other classes. Several preliminary works have studied the joint problem of noisy label and long-tail distribution (Jiang et al. 2022; Wei et al. 2021b; Karthik, Revaud, and Boris 2021; Cao et al. 2021). These methods implicitly reduce the complexity of the problem by assuming an equal noise rate for each class. However, this assumption is too strong to make them applied in real applications because the noisy samples from the head classes may be the majority leading to a higher noise rate in the tail class than in other classes.

In this paper, we propose a learning framework to deal with the problem of label-noise learning with intrinsically long-tailed data. The framework consists of two steps in each training round. First, we propose a Two-stAge Bi-dimensionAl Sample seleCtiOn (TBSS) strategy to address the tail inseparability problem. Specifically, we propose two new separation metrics, i.e., weighted Jensen-Shannon divergence (WJSD) and adaptive centroid distance (ACD), which work corporately to separate clean samples from noisy samples in the tail classes. Specifically, the proposed metrics are complementary, where WJSD separates the samples from the output perspective while ACD does that from the feature perspective. Then, the clean and noisy samples are clustered by two-dimensional Gaussian mixture model (2D-GMM) on the bi-dimensional metrics calculated for each sample. Finally, we select the cluster with more clean samples as the labeled training data to update the model by semi-supervised learning. The key contributions of our work can be summarized as follows:

• We study a more general problem of label-noise learning with intrinsically long-tailed class distribution inconsistent with the observed class distribution.
• We propose a robust learning framework for label-noise learning with intrinsically long-tailed data learning to ensure that the model can focus on the intrinsic tail classes.
• In this framework, we propose two new separation metrics to overcome the difficulty of clean sample selection in the tail classes.

2 Related Work

2.1 Label-Noise Learning

A straightforward strategy to deal with noisy data is to reduce the proportion of noise in training samples by separating noisy samples from clean samples. Methods such as co-teaching (Han et al. 2018) and DivideMix (Li, Socher, and Hoi 2020) adopt the small loss trick, while Jo-SRC (Yao et al. 2021) and UNICON (Karim et al. 2022) use Jensen-Shannon divergence instead of the training loss for sample selection. In contrast to methods that separate at the label level, some methods (Li et al. 2022; Ortego et al. 2021) attempt to separate samples at the feature space. There are also some methods avoid over-fitting the model to noisy data by imposing regularization constraints on model parameters (Xia et al. 2021; Han et al. 2020a) or labels (Pereyra et al. 2017; Zhang et al. 2018; Lukasik et al. 2020). Other methods mitigate the influence of noisy data by adjusting the loss functions, such as backward and forward loss correction (Patrini et al. 2017), gold loss correction (Hendrycks et al. 2018), MW-Net (Shu et al. 2019) and Dual-T (Yao et al. 2020).

2.2 Long-tail Learning

Re-balancing the data for long-tail distributions is a classical strategy to solve the problem of long-tail learning, such as re-sampling (Chawla et al. 2002; Liu, Wu, and Zhou 2009; Estabrooks, Jo, and Japkowicz 2004; Han, Wang, and Mao 2005) and data augmentation (Zhong et al. 2021; Zhang, Huang, and Loy 2021; Chou et al. 2020). In addition, there are methods to improve the model generalization by introducing long-tail robust loss functions (Cui et al. 2019; Cao et al. 2019; Tan et al. 2020; Ren et al. 2020). Methods such as FTL (Yin et al. 2019), RIDE (Wang et al. 2021) and DiVE (He, Wu, and Wei 2021) try to solve the problem by using the idea of transfer learning. Recently, approaches based on decoupling (Kang et al. 2020; Zhang et al. 2021a) split the end-to-end learning into feature learning and classifier retraining such that the obtained feature extractor is less affected by the long-tail distribution. In contrast to the above methods with the supervised learning paradigm, CReST (Wei et al. 2021a), ABC (Lee, Shin, and Kim 2021) and DARP (Kim et al. 2020) attempt to solve the long-tail problem in the manner of semi-supervised learning.

2.3 Label-Noise Learning on Long-tailed Data

Research on this joint problem has just been explored. CNLCU (Xia et al. 2022) relaxes the constraint of the small loss trick by regarding a portion of large loss samples as clean samples to reduce the probability of misclassifying clean samples to noisy samples in the tail class. Based on MW-Net, CurveNet (Jiang et al. 2022) introduces sample labels and the addition of losses over a period of time so that the network can better weight clean and noisy samples. RoLT (Wei et al. 2021b) uses the distance from the samples to the centre of mass of the current class instead of the training loss for sample selection. HAR (Cao et al. 2021) uses an adaptive approach to regularize noise and tail class samples. Karthik et al. (Karthik, Revaud, and Boris 2021) use the idea of decoupling to fine-tune the loss function for better robustness after feature learning by the self-supervised method.

3 Problem Definition

Given a training set $D = \{x_i, \hat{y}_i\}_{i=1}^N$, where $\hat{y}_i \in [M]$ is the observed label of the sample $x_i$. The one-hot representation of $\hat{y}_i$ is denoted as $\mathbf{y}_i$. $N$ is the number of training samples, and $M$ is the number of classes. In our problem, $D$ has the following properties:

• Noisy-labeled. There exists a subset of samples $\tilde{D} \subset D$, where a sample $\{x, \tilde{y}\} \in \tilde{D}$ has an unknown ground-truth label $\tilde{y}$ different from its observed label $\tilde{y}$.
• Long-tailed. The ground-truth class distribution is long-tailed. Supposing $n_c$ is the number of class $c$ by counting the ground-truth label, we have $n_1 > n_2 > \cdots > n_M$. 
We call this kind of training data *intrinsically long-tailed* because the observed class distribution may not be long-tailed due to the existence of noisy labels. Therefore, our goal is to learn from noisy-labeled data with intrinsically long-tailed class distribution. The problem is challenging from two aspects. On the one hand, directly applying long-tail learning methods (Cui et al. 2019; Cao et al. 2019; Tan et al. 2020) is infeasible because the observed class distribution may be inconsistent with the intrinsic class distribution. On the other hand, label-noise learning methods poorly perform in the tail classes because they usually contain more noisy samples due to insufficient data, as shown in Figure 1(a). It results in high noise rates in the tail classes, which brings great challenges to separating clean samples and noisy samples for further training.

### 4 Proposed Method

Under this circumstance, the existing sample selection methods (Li, Socher, and Hoi 2020; Karim et al. 2022) often fail to select clean samples in the tail class. The main reason is that it is difficult to distinguish the tail class data from the noisy label data by existing metrics like cross-entropy loss or Jensen-Shannon divergence, as shown in Figure 1(b). In this paper, we propose two-stage bi-dimensional sample selection (TBSS) to address the problem of label-noise learning with intrinsically long-tailed data. TBSS decouples the sample selection process into two phases: sample separation and cluster selection. Given an initial model \( \theta \) trained on the original training data \( D \), we propose to calculate two metrics for each sample in each observed class based on the model’s outputs. These two metrics complement each other for better separating clean and noisy samples into two clusters. A specific cluster selection method is adopted to select the cluster with more clean samples. The proposed sample selection method is fully self-adaptive without introducing any hyperparameter for thresholding. After sample selection, we adopt semi-supervised learning to update the model by regarding the selected clean cluster as labeled data. The overall framework to learn from noisy-labeled long-tailed data is shown in Figure 2.

#### 4.1 Bi-dimensional Sample Separation

Due to the deficiency of a single separation metric to distinguish clean samples from noisy samples in the complex situation of noisy labels with intrinsic long-tail distribution, we propose to jointly use two metrics from different perspectives: weighted Jensen-Shannon divergence (WJSD) and adaptive centroid distance (ACD). Both metrics are specifically designed for the joint problem, and they are complementary to cover the case when only one of them cannot separate samples well. WJSD fully utilizes the information of prediction confidence, while ACD relies on the distance in the feature space. Thus, using two-dimensional clustering on samples according to the values of their bi-dimensional metrics has the flexibility to separate samples with different imbalance ratios, noise ratios, and noise types.

To alleviate the interference from the head class to the tail class, we first reduce the separation granularity. Specifically, the separation according to the proposed bi-dimensional metrics is done within each observed class. We first divide training set \( D \) into subsets according to the observed labels \( D_\ell \), \( \ell \in \ell \). Given a sample \( (\mathbf{x}_i, \hat{y}_i) \in D_\ell \), the model predictions obtained from the model \( \theta \) is denoted as \( p^j_{\ell} = [p^1_{\ell}, p^2_{\ell}, ..., p^M_{\ell}] \), where \( p^j_{\ell} \) is the \( j \)-th dimension of vector \( p^j_{\ell} \). The one-hot representation of the observed label \( \hat{y}_i \) is denoted as \( \hat{y}_i \).

**Weighted Jensen-Shannon Divergence** Jensen-Shannon Divergence (JSD) is a commonly used metric to separate samples by assessing the variability of model predictions (Yao et al. 2021; Karim et al. 2022). It is defined as:

\[
JSD(x_i) = \frac{1}{2}KL\left(p_i \parallel \frac{p_i + \hat{y}_i}{2}\right) + \frac{1}{2}KL\left(\hat{y}_i \parallel \frac{p_i + \hat{y}_i}{2}\right),
\]

(1)

where \( KL(\cdot \parallel \cdot) \) is the Kullback-Leibler divergence. When the intrinsic distribution is balanced, the JSD values of clean samples are generally lower than the noisy samples so that the separation can be easily modeled. However, when the distribution is intrinsically long-tailed, the noisy and clean samples in a tail class tend to obtain similar prediction confidence on the observed class \( c \), because their predictions are both towards the head classes. In this case, using JSD as a
separation metric fails to separate them. The reason is given by Theorem 1, which shows the upper bound of the absolute value of the JSD difference between two samples. It is proved in Appendix B.

**Theorem 1 (Upper bound of JSD difference)** Suppose \( x_i \) and \( x_j \) are two samples in class \( c \), \( p_i^c \) and \( p_j^c \) are the \( c \)'s dimension of their prediction confidence \( p_i = [p_i^1, p_i^2, \ldots, p_i^M] \) and \( p_j = [p_j^1, p_j^2, \ldots, p_j^M] \), respectively. The upper bound of the absolute value of the difference between their JSD values is given by:

\[
|JSD(x_i) - JSD(x_j)| \leq \frac{1}{2} \log \left( \frac{p_i^j + 1}{p_i^c} \right) |p_i^c - p_j^c|. \tag{2}
\]

From Theorem 1, it can be known that the difference between values of JSD is only determined by the prediction confidence of the observed class, i.e., \( p_i^c \) and \( p_j^c \). In addition, as \( |p_i^c - p_j^c| \) gets smaller with a fixed value of \( p_i^c \), \( |JSD(x_i) - JSD(x_j)| \) is also smaller. This indicates that when two samples in the same class \( c \) with very close values of \( p_i^c \), their JSD values are also very close, which makes it difficult to separate them.

Nevertheless, we can utilize the prediction confidence on other classes, i.e., \( p_i^d \) for \( d \neq c \), to distinguish two samples even if their values of \( p_i^c \) are close. Specifically, we propose WJSD by imposing an additional weight on JSD to further distinguish samples by inspecting their prediction confidence on other classes rather than the observed class. First, we take the maximum prediction confidence \( \max(p_i) \) into account because it may reflect the confidence of a noisy sample’s ground-truth class \( y_i \). To make it an additional weight whose value is greater or equal to 1, we calculate the ratio of the maximum prediction confidence to prediction confidence of the observed class \( \frac{\max(p_i)}{p_i^c} \). In this manner, the additional weight is greater than one only if the class with the maximum prediction confidence is not the observed class. The larger gap between \( \frac{\max(p_i)}{p_i^c} \) and \( p_i^c \) makes the weight higher. To avoid exceptionally large weights by the division during normalization over all samples in class \( c \) for later clustering, we set the upper bound according to the averaged prediction confidence over all samples in class \( c \). Finally, the additional weight can be calculated as:

\[
W(x_i) = \min(\max(p_i)/p_i^c, \max(p_c)/p_c^c), \tag{3}
\]

where \( p_c = [p_c^1, p_c^2, \ldots, p_c^M] = \frac{1}{|D_c|} \sum_{x_i \in D_c} p_i \) is the average prediction confidence of class \( c \). Thus, WJSD can be calculated by:

\[
WJSD(x_i) = W(x_i) \times JSD(x_i). \tag{4}
\]

**Remarks:** Compared with the value of JSD that is only related to the prediction confidence on the observed class, adopting the maximum prediction confidence in WJSD can better separate clean samples from noisy samples, especially in the tail classes (see Figure 7 and experiment discussion in Section 5.3).

**Adaptive Centroid Distance** Although the sample separation ability of WJSD is greatly improved compared with JSD, the separation metric still relies on model prediction. When all the noisy samples in a class belong to another ground-truth class, e.g., flip noise, the model prediction for both noisy and clean samples will be highly similar, because it is easy to learn a classifier that maps the features from two different classes into one. It results in low discrimination between the model predictions of clean samples and noisy samples. Thus, solely using WJSD is not enough to separate the clean and noisy samples when their model predictions (in terms of all classes, not only about \( p_i^c \)) are close.

Besides separating the samples in the output space, we can utilize another metric calculated in the feature space to eliminate the bias in the classifier. Specifically, for a given sample, we can calculate the distance between its feature and the class feature centroid to evaluate how the feature of a sample deviates from its class centroid. RoLT (Wei et al. [2021b]) adopts a similar idea to use class feature centroid for noisy sample detection. However, it directly calculates the feature centroid on the observed class, such that it only works when the number of noisy samples is far less than the number of clean samples in a class, which cannot be guaranteed in our problem apparently. In the tail classes, the class feature centroid will likely be overwhelmed by the noise samples.

In order to improve the purity of the class feature centroid for distance calculation between features, we propose ACD as the second separation metric to work with WJSD. Each class feature centroid is adaptively updated by involving samples with high confidence from other classes and eliminating samples with low confidence from the observed class. Therefore, we construct a high confidence sample set \( D^H_c \) and a low confidence sample set \( D^L_c \):

\[
D^H_c = \{x_i | x_i \in D_d, d \neq c, \max(p_i) = p_i^c, p_i^c > H_c\}, \tag{5}
\]

\[
D^L_c = \{x_i | x_i \in D_c, p_i^c < L_c\}, \tag{6}
\]

where \( H_c \) and \( L_c \) are two confidence thresholds defined as follows:

\[
H_c = \overline{p_c}, \quad L_c = \frac{1}{M - 1} \sum_{d \neq c} \overline{p_d}. \tag{7}
\]

The threshold \( H_c \) determines if a sample from other classes should be added to class \( c \) for centroid calculation when its prediction confidence of class \( c \) is higher than the prediction confidence of class \( c \) averaged over all samples in class \( c \). That indicates a high probability of being a clean sample in class \( c \). At the same time, the threshold \( L_c \) determines if a sample in the observed class should be removed from class \( c \) for centroid calculation when its prediction confidence in \( c \) is lower than the average prediction confidence class of \( c \) averaged over all the other classes. In this way, the samples with less probability in class \( c \) will not be involved in centroid calculation. Then, the adaptive centroid is calculated on the new sample set \( D'_c \) by removing \( D^L_c \) and adding \( D^H_c \):

\[
\alpha_c = \frac{1}{|D'_c|} \sum_{x_i \in D'_c} f_i, \quad D'_c = D_c \setminus D^L_c \cup D^H_c, \tag{8}
\]

where \( f_i \) is the feature of \( x_i \). In each round, the centroids are adaptively adjusted because the feature extractor in the
model is updated, as well as the two sets $D^L$ and $D^H$. Therefore, the proposed metric ACD can be calculated by:

$$ACD(x_i) = \cos(f_i, o_c).$$  (9)

Remarks: The proposed two metrics, WJSD and ACD, are complementary for sample separation in tail classes. When the ground-truth classes of the noisy samples are diverse, e.g. uniform noise, WJSD plays the dominant role because the predictions can hardly be unified due to the class diversity of noisy samples. They are more likely to be predicted towards their ground-truth class, which can be captured by $\max(p_i)$. In this case, ACD may not be effective in separating clean samples from noisy samples because the messy noisy samples may affect the purity of the calculated centroid. On the other hand, when the ground-truth classes of the noisy samples belong to only one class, e.g. flip noise, WJSD tends to produce similar predictions for noisy and clean samples, while the adaptive centroid in ACD can help distinguish them in the feature space.

4.2 Cluster Selection

Once the values of the bi-dimensional metrics for all samples in class $c$ are calculated, each sample can be represented by a point in a 2D space. Two-dimensional Gaussian mixture model (2D-GMM) can be adopted to separate the samples in class $c$ into two clusters $G^*_1, G^*_2$. Usually, the cluster with a small average metric value can be regarded as the cluster with more clean samples. However, in our problem, the noisy samples may be the majority in the tail classes due to the influence of the long-tail distribution, which makes it challenging to identify which cluster contains more clean samples. Therefore, cluster selection is an additional important task in our problem where the case that noisy samples being the majority may usually happen. It will be harmful to the subsequent training if cluster selection fails, even if the clean samples are successfully separated from the noisy samples.

As an empirical observation, the main difficulty of cluster selection in the tail classes is in the early training stage, when the two clusters are hard to be identified as clean or noisy, although samples can be separated to some extent. This phenomenon is demonstrated in Figure 5 in Section 5.3. As a good selection criterion, it should be effective throughout the entire training process. Therefore, we utilize another model as the evaluation model for cluster selection. This model is a pre-trained model directly trained on the noisy-labeled and long-tailed dataset without any sample selection method. Here, we can use the pre-trained model, which is obtained by any common methods. Therefore, we simply use a supervised pre-trained model as the evaluation model to calculate the average WJSD for each cluster and select the one with a smaller value as the cluster with more clean samples, denoted as $D^\text{clean}_c$. Another cluster is thus denoted as $D^\text{noisy}_c$. After sample selection is conducted for each class, we adopt semi-supervised learning to train with all clean samples $D^\text{clean}$ as labeled data and all noisy samples $D^\text{noisy}$ as unlabeled data. The model is thus updated for the next round of training. The proposed method is summarized in Algorithm 1 in Appendix A.

5 Experiments

5.1 Experimental Setup

Datasets In our experiments, we perform experiments on CIFAR-10, and CIFAR-100 (Krizhevsky 2009) with various noise ratios and imbalance ratios. We also conduct experiments on the real-world dataset WebVision (Li et al. 2017) which is naturally noise-labeled and imbalanced. For CIFAR-10/100, we generate the training data by constructing the imbalance data distribution first and then adding noisy labels. As for the noisy labels, we adopt uniform noise and flip noise, which is commonly used in the area of label-noise learning (Shu et al. 2019; Li, Socher, and Hoi 2020). It should be noted that different from the previous methods (Shu et al. 2019; Li, Socher, and Hoi 2020; Wei et al. 2021b) to deal with the joint problem of noisy labeled and long-tailed data, the noise transition matrix is randomly generated that is only related to the noise ratio. Therefore, the observed and intrinsic distribution are likely inconsistent, especially when it is long-tail distributed. We set different imbalance factors, 0.1 and 0.01, in the experiment, which is calculated by the ratio between the size of the largest class and that of the smallest class. We also set different noise ratios, 0.4 and 0.6, in the experiment, which is calculated by the ratio between the number of noisy samples and the total number of samples.

Compared Methods We compare with the following two types of approaches: (1) Methods aiming at dealing with noisy label and long-tail distribution. They are MW-Net (Shu et al. 2019), CurveNet (Jiang et al. 2022), RoLT (Wei et al. 2021b) and HAR (Cao et al. 2021); (2) State-of-the-art label-noise learning methods. They are Iterative-CV (Chen et al. 2019), DivideMix (Li, Socher, and Hoi 2020) and UNICON (Karim et al. 2022).

Implementation Details During training, we use PreAct ResNet18 (He et al. 2016) as the backbone network and adopt SGD with an initial learning rate of 0.02, a momentum of 0.9, and a weight decay of $5 \times 10^{-4}$ and a batch size of 64. We train all models for 100 epochs on a single NVIDIA GTX 3090.

5.2 Comparative Results

Table 1 and 2 report the accuracy of different methods for intrinsically long-tailed CIFAR-10/100 with uniform and flip noise, respectively. It can be observed that the compared methods perform differently on different imbalance ratios, noise ratios and noise types. For example, methods aiming at dealing with noisy labels and long-tail distribution perform relatively well in the case of flip noise, but state-of-the-art label-noise learning methods perform relatively well in the case of uniform noise. However, the proposed method achieves the best performance in different cases compared with both types of methods. Therefore, the effectiveness of our learning framework can be validated in label-noise learning with intrinsically long-tailed data.

5.3 Ablation Study and Discussions

In the following experiments, we analyze each component of the proposed TBSS on CIFAR-10 to verify its effectiveness.
### Table 1: Performance comparison with uniform noise and long-tail distribution. The best results are shown in bold.

| Dataset   | CIFAR-10 | CIFAR-100 |
|-----------|----------|-----------|
|            | 0.1      | 0.01      | 0.1      | 0.01     |
| Imbalance Factor | 0.4 | 0.6 | 0.4 | 0.6 | 0.4 | 0.6 |
| Noise Ratio  | 0.4       | 0.6       | 0.4       | 0.6       | 0.4       | 0.6       |
| Baseline   | 71.67     | 61.16     | 47.81     | 28.04     | 34.53     | 23.63     | 21.99     | 15.51     |
| MW-Net     | 70.90     | 59.85     | 46.62     | 39.33     | 32.03     | 21.71     | 19.65     | 13.72     |
| CurveNet   | 78.03     | 67.82     | 58.55     | 43.16     | 41.06     | 29.83     | 23.64     | 17.41     |
| RoLT       | 81.62     | 76.58     | 60.11     | 44.23     | 42.95     | 32.59     | 23.51     | 16.61     |
| HAR        | 77.44     | 63.75     | 51.54     | 38.28     | 38.17     | 26.09     | 20.21     | 14.89     |
| DivideMix  | 82.67     | 80.17     | 32.42     | 34.73     | 54.71     | 44.98     | 36.20     | 26.29     |
| UNICON     | 84.25     | 82.29     | 61.23     | 54.69     | 52.34     | 45.87     | 32.09     | 24.82     |
| Proposed   | **87.21** | **85.11** | **63.64** | **58.40** | **57.04** | **46.59** | **37.25** | **26.43** |

### Table 2: Performance comparison with flip noise and long-tail distribution. The best results are shown in bold.

| Dataset   | CIFAR-10 | CIFAR-100 |
|-----------|----------|-----------|
|            | 0.2      | 0.4       | 0.2      | 0.4      |
| Imbalance Factor | 0.1       |           |           |           |
| Noise Ratio  | 0.2       | 0.4       | 0.2       | 0.4       |
| Baseline   | 79.90     | 62.88     | 44.45     | 32.05     |
| MW-Net     | 79.34     | 65.49     | 42.52     | 30.42     |
| CurveNet   | 82.64     | 77.44     | 51.16     | 38.49     |
| RoLT       | 83.88     | 58.29     | 48.19     | 39.32     |
| HAR        | 82.85     | 69.19     | 48.50     | 33.20     |
| DivideMix  | 80.92     | 69.35     | 58.09     | 41.99     |
| UNICON     | 72.81     | 69.04     | 55.99     | 44.70     |
| Proposed   | **86.04** | **80.53** | **59.14** | **46.75** |

We select class 1 as the head class, class 4 as the medium class, and class 9 as the tail class. We use the imbalance factor of 0.1 and flip noise ratio of 0.4 in the case of flip noise, in which the noise ratios of the head class, the medium class and the tail class are 0.18, 0.15, and 0.84, respectively. Besides, we use the imbalance ratio of 0.01 and uniform noise ratio of 0.4 in the case of uniform noise, in which the noise ratios of head class, medium class and tail class are 0.16, 0.51, 0.94, respectively.

**Effectiveness of WJSD** To validate the advantage of the proposed WJSD, we compare it with JSD to show their separability. Figure 3 shows the distributions of samples in the head and tail class by JSD and WJSD, respectively. It can be observed in Figure 3(a) that the distributions have no significant difference between JSD and WJSD in the head class. Because the clean and noisy samples in the head class can be separated well by JSD, the additional weight in WJSD does not show further improvement. However, the prediction confidence of the samples in the tail class is very similar because the clean samples are overwhelmed by the large number of noise samples. As a result, their JSD values are highly overlapped, as shown in the left subfigure of Figure 3(b), which leads to the failure of sample separation. In this case, the noisy samples usually have higher prediction confidence in their ground-truth class. Thus, WJSD shows its advantage because it depends on the prediction confidence in other classes, which provides other information for sample separation. Therefore, the additional weights of noisy samples are higher than clean samples, so there is a significant difference in WJSD between them. As shown in the right subfigure of Figure 3(b), most of the noisy samples are concentrated in the interval of [0.8,1], which can be easily clustered by GMM.

**Effectiveness of Bi-dimensional Sample Separation** To validate how the proposed bi-dimensional metrics complement each other, we plot the value of bi-dimensional metrics of clean and noisy samples under different noise types in Figure 4. For the case of uniform noise, it can be observed in Figure 4(a) that both WJSD and ACD can well separate clean and noisy samples in the head and medium class. However, ACD cannot effectively distinguish clean samples from noisy samples in the tail class, as shown in the right subfigure of Figure 4(a). In this case, WJSD shows its advantage because the values of WJSD for most of the noisy samples are clustered in the top region, and the clean samples are scattered in the whole region. For the case of flip noise, it can be ob-
observed in Figure 4(b) that WJSD cannot distinguish clean samples from noisy samples well for all the three classes, while ACD can distinguish them well, although the clean and noisy samples are closer in the tail class. Therefore, the experimental observation is consistent with the previous discussion, which validates the complementarity of the bi-dimensional metrics, especially for the tail class.

Effectiveness of Cluster Selection In Section 4.2, we propose to use a pre-trained model to select the cluster with more clean samples. In this experiment, we demonstrate the necessity of adopting a pre-trained model for cluster selection other than directly using the training model. Figure 5 compares the values of average WJSD over the entire training period by a plain model without sample selection and the training model with TBSS. For the head class, the gaps between clean and noisy samples are obvious for both methods, which indicates that cluster selection is not a problem for the head class. However, for the tail class, the average WJSD of the clean samples is very close to that of the noisy samples in the early stage for both methods. This is also why existing methods fail to correctly select clean samples of the tail class by directly referring to the smaller average metric value. However, it can be observed that at the later stage of training, the average WJSD of the clean and noise samples in the tail class has an obvious gap for both methods. Even though the plain model does not apply any sample selection method, it can still distinguish between clean and noisy samples, although the gap is much smaller than that of the training model with TBSS. Therefore, it is safe to use a pre-trained model on the original data without sample selection because it can select the cluster with more clean samples, although it cannot separate samples.

We also compare the performance of TBSS with a baseline that directly selects the cluster with a small value as the clean set in the tail classes. The results is shown in Figure 6. It can be observed that TBSS can effectively improve the accuracy of selecting clean samples in the tail classes compared with the baseline. Note that both of them achieve the same performance in class 7 because there is no noise in this class.

Consistency between Selected and Intrinsic Distribution Benefit from the proposed TBSS to select the clean samples for semi-supervised learning, the class distribution of the selected clean samples can be more consistent with the intrinsic distribution. It can be observed in Figure 7 that the proportion of the head class in the observed distribution is much lower, while the proportion of the tail class is much higher due to the influence of noise, such as classes 0, 1, 5, 7, and 9. After adopting the sample selection by TBSS, the clean samples we selected recover the proportion of the head class and reduce the proportion of the tail class. Therefore, this experiment validates the effectiveness of TBSS in addressing the problem of distribution inconsistency because it recovers the intrinsic class distribution to some extent.

6 Conclusion

In this paper, we have studied a more general and realistic problem of label-noise learning with intrinsically long-tailed data. The major challenge in this problem is that it is hard to distinguish clean samples from noisy samples on the intrinsic tail classes. Accordingly, we have proposed a learning framework for this problem, mainly consisting of a novel sample selection method called TBSS. In TBSS, two new metrics have been specifically proposed to address the problem of sample selection in the tail classes, such that it can be robust on different imbalance ratios, noise ratios and noise types. Extensive experiments on multiple noisy-labeled datasets with intrinsic long-tail distribution have demonstrated the effectiveness of our method.
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Algorithm 1: The training process of the proposed learning framework

**Input:** Noisy training data $\mathcal{D} = \{x_i, \hat{y}_i\}_{i=1}^N$

1. Initialize the model $\theta$ trained on $\mathcal{D}$;
2. while $\varepsilon < \text{MaxIterationNumber}$ do
   3. for $c = 1$ to $M$ do
      4. Obtain and store the feature $f_i$ and the prediction confidence $p_i$ for $x_i$ by model $\theta$;
      5. end
      6. Calculate the average prediction confidence $\bar{p}_c$ for class $c$;
   7. end
   8. for $i = 1$ to $|\mathcal{D}_c|$ do
      9. Calculate the confidence thresholds $H_c$ and $L_c$ for class $c$ by Equation (7);
      10. Calculate the adaptive centroid $o_c$ for class $c$ by Equation (8);
      11. for $i = 1$ to $|\mathcal{D}_c|$ do
            12. Calculate the additional weight $W(x_i)$ by Equation (3);
            13. Calculate $W \cdot \text{JSD}(x_i)$ by Equation (4);
            14. Calculate $ACD(x_i)$ by Equation (5);
      15. end
   16. end
   17. Apply 2D-GMM with values of bi-dimensional metrics to all the samples in class $c$;
   18. Adopt cluster selection to generate two sets $\mathcal{D}_c^{\text{clean}}$ and $\mathcal{D}_c^{\text{noisy}}$;
21. end

Update the model $\theta$ by SSL training with $\mathcal{D}_c^{\text{clean}}$ as labeled data and $\mathcal{D}_c^{\text{noisy}}$ as unlabeled data.

### Appendix

#### A Pseudocode of the Proposed Method

Algorithm 1 details the training procedure of the proposed framework for label-noise learning with intrinsically long-tailed data.

#### B Proof of Theorem 1

**Proof.** Suppose $x_i$ and $x_j$ are two samples in class $c$, $p_i^c$ and $q_j^c$ are the $c$’s dimension of their prediction confidence $p_i = [p_i^1, p_i^2, ..., p_i^M]$ and $p_j = [p_j^1, p_j^2, ..., p_j^M]$, respectively. Their common observed class label $\hat{y} = [\hat{y}_i^1, \hat{y}_i^2, ..., \hat{y}_i^M]$ is in the one-hot form where only the value of the $c$’s dimension is 1 ($\hat{y}_i^c = 1$) and the values on other dimensions are all 0 ($\hat{y}_i^d = 0$ for all $d \neq c$). Jensen-Shannon Divergence (JSD) for sample $x_i$ is defined as:

\[
JSD(x_i) = \frac{1}{2} KL(p_i \| p_i + \frac{\hat{y}_i}{2}) + \frac{1}{2} KL(p_i \| p_i + \frac{\hat{y}_i}{2})
\]

\[
= \frac{1}{2} \left( \sum_{d=1}^{M} p_i^d \log \frac{2p_i^d}{p_i^d + \frac{\hat{y}_i^d}{2}} + \sum_{d=1}^{M} \hat{y}_i^d \log \frac{2\hat{y}_i^d}{p_i^d + \frac{\hat{y}_i^d}{2}} \right)
\]

\[
= \frac{1}{2} \left( \sum_{d \neq c} p_i^d \log \frac{2p_i^d}{p_i^c + \frac{\hat{y}_i^c}{2}} + p_i^c \log \frac{2p_i^c}{p_i^c + \frac{\hat{y}_i^c}{2}} + \hat{y}_i^c \log \frac{2\hat{y}_i^c}{p_i^c + \frac{\hat{y}_i^c}{2}} \right)
\]

\[
= \frac{1}{2} \left( \sum_{d \neq c} p_i^d + p_i^c \log \frac{2p_i^c}{p_i^c + \frac{\hat{y}_i^c}{2}} + \log \frac{2}{p_i^c + 1} \right)
\]

\[
= \frac{1}{2} \left( 1 - p_i^c + p_i^c \log \frac{2p_i^c}{p_i^c + 1} + \log \frac{2}{p_i^c + 1} \right)
\]

\[
= \frac{1}{2} (1 - p_i^c + p_i^c \log p_i^c - p_i^c \log (p_i^c + 1) + 1 - \log (p_i^c + 1))
\]

\[
= \frac{1}{2} (2 + p_i^c \log p_i^c - (p_i^c + 1) \log (p_i^c + 1)).
\]

Proof.
The base of logarithm is 2 for the above derivation. Then
\[
|JSD(x_i) - JSD(x_j)| = \frac{1}{2} (2 + p_i^c \log p_i^c - (p_i^c + 1) \log (p_i^c + 1)) - \frac{1}{2} (2 + p_j^c \log p_j^c - (p_j^c + 1) \log (p_j^c + 1))
\]
\[= \frac{1}{2} |(p_i^c \log p_i^c - (p_i^c + 1) \log (p_i^c + 1)) - (p_j^c \log p_j^c - (p_j^c + 1) \log (p_j^c + 1))|
\]
(17)

Now, for \( u \in (0, 1) \), let
\[
f(u) = u \log u - (u + 1) \log(u + 1).
\]
(19)
The first- and second-order derivative of \( f(u) \) can be obtained by:
\[
f'(u) = \left( \log u + u \cdot \frac{1}{u} \cdot \frac{1}{\ln 2} \right) - \left( \log(u + 1) + (u + 1) \cdot \frac{1}{u + 1} \cdot \frac{1}{\ln 2} \right)
\]
\[= \log u - \log(u + 1).
\]
(20)
\[
f''(u) = \left( \frac{1}{u} - \frac{1}{u + 1} \right) \frac{1}{\ln 2}.
\]
(22)

It can be observed that \( f'(u) < 0 \) and \( f''(u) > 0 \) that makes \( f'(u) \) monotonically increase and \( |f'(u)| \) monotonically decrease. By Lagrange mean value theorem, if function \( f(u) \) is continuous and differentiable on the interval \((0, 1)\), then there is at least one point \( \xi \) between two real numbers \( a, b \in (0, 1) \):
\[
|f(a) - f(b)| = |f'(\xi)(a - b)| = |f'(\xi)| \cdot |a - b|.
\]
(23)

As \( \xi > a \) and \( |f'(u)| \) monotonically decrease, we have:
\[
|f'(\xi)| \cdot |a - b| \leq |f'(a)| \cdot |a - b|
\]
\[= |\log a - \log(a + 1)| \cdot |a - b|
\]
\[= \left| \log \frac{a}{a + 1} \right| \cdot |a - b|
\]
\[= \log \frac{a + 1}{a} \cdot |a - b|
\]
(25)
(26)
(27)

By replacing \( f(u) \) by \( JSD(x) \), \( a \) by \( p_i^c \) and \( b \) by \( p_j^c \), we have:
\[
|JSD(x_i) - JSD(x_j)| \leq \frac{1}{2} \log \left( \frac{p_i^c + 1}{p_i^c} \right) |p_i^c - p_j^c|.
\]
(28)