Hawking Radiation Entropy and Horizon Divergences

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ABSTRACT

In this paper we review the problem of divergences in one–loop thermodynamical quantities for matter fields in thermal equilibrium on a black hole background. We discuss a number of results obtained for various thermodynamical quantities. Then we discuss the ansatz called “literal interpretation” of zeroth law of black hole mechanics and try to explain the diseases of the conical defect procedure in light of this ansatz. Finally, an analysis of the consequences implied by our ansatz on the calculation of the partition function is made.
1. Introduction

In this work we will discuss the following problem: the divergence of the 1 loop entropy for matter fields in equilibrium with a Schwarzschild black hole.

In fact, the underlying problem is the calculation of the 1 loop matter fields partition function on the Schwarzschild background.

The first calculations on this topic are due to G. ‘t Hooft [1]: using a WKB approximation for the eigenvalues of a scalar field hamiltonian on the Schwarzschild background, ‘t Hooft finds that thermodynamical quantities as free energy, internal energy and entropy have contributions divergent for the radial coordinate $r \to r_{bh}$, where $r_{bh}$ is the black hole radius. So one must introduce a short–distance cut–off $\epsilon$ representing a radial proper distance from the horizon. The divergences in the thermodynamical quantities behave as $\epsilon^{-2}$.

‘t Hooft proposal to face with these divergences is the so called brick wall model: his ansatz is to consider all the black hole entropy $S_{bh} = A/4$, where $A$ is the horizon area, as due to the radiation external to the black hole; then the cut–off is to be fixed to the value $\epsilon \sim l_{pl}$. This means that the “brick wall” proper distance from the horizon coordinate $r_{bh}$ is of the order of the Planck length; besides, the brick wall cut–off is shown to be an universal property: it is independent from the black hole mass.

An unsatisfactory result of this model, underlined by ‘t Hooft, is that the leading internal energy contribution of the radiation is a not negligible fraction of the black hole mass: it is about half of the black hole mass. This fact represents

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#1 The divergence problem for generic value of the space–time dimension is discussed in [2].

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a consistency problem for the approximation of fixed black hole background and
negligible backreaction of the linearized theory.

The most interesting hypothesis in the model is ‘t Hooft idea of a link between
the divergence problem and the unsolved problem of the explanation of black hole
entropy in terms of Statistical Mechanics. The proportionality of the leading di-
vergent term of the matter entropy to the black hole area $A$ is an indication that
‘t Hooft ansatz could be right.

Similar results about proportionality to the area and horizon divergences are
successively found by Bombelli et al. [3]. They study what they call “entanglement
entropy”: the entropy due to the ignorance of an observer external to the horizon
about the fields correlations existing between points internal and external to the
horizon. Recently, the divergence problem has been analyzed by many authors.
We will call “entanglement program” the research program identifying a strong
link between

(a) the 1 loop horizon divergence of the matter contribution to the entropy;

(b) the problem of a statistical mechanical explanation of black hole entropy. In
this program ‘t Hooft ideas and the concept of “entanglement entropy” are
often considered on the same foot, although they are not quite equivalent.

The major topic in the entanglement program has been the entropy divergence.
Between the current proposals of solution for this problem, we will take into account
essentially the following two:

1) to reabsorb (Susskind and Uglum [4]) the divergence in the gravitational cou-
pling constant $G$: studying the analogous divergence problem on the Rindler
background, Susskind and Uglum find that the same quadratic divergence of
the entropy enter into the renormalization of the coupling $G$ in the effective
action for matter fields and gravitational field; the renormalization of $G$ implies the renormalization of the entropy. This approach has been extensively pursued by Fursaev and Solodukhin [5,6,7]; the latter author discovered a logarithmic divergence in the entropy requiring to be renormalized the introduction in the gravitational action of terms quadratic in the curvature;

2] to find a physical mechanism to implement the brick wall model. Frolov et al. [8,9] identify this mechanism in the quantum fluctuations of the black hole horizon. The spreading of the radial coordinate $r_{bh}$ is of the order of the Planck length [8,10] (when expressed in radial proper distance from the horizon). The quantum fluctuation of the black hole radius acts as a physical thickness of the right size: used as a cut–off, it gives an horizon contribution for the matter entropy of the order of the black hole entropy.

We will present the main ideas and results underlying it and then we will explain our proposal. The plan of our work consists of the following paragraphs: in 2 we resume the divergence problem in its various aspects; in 3 a further discussion of the divergence problem is made in light of the discrepancy between the finiteness of the internal energy calculated by mean of the Hartle–Hawking tensor and its divergence in conical calculations; in 4 we will propose the ”intrinsic thermodynamics approach”; it is essentially a literal interpretation of the zeroth law of black hole mechanics and it is a candidate to solve the divergence problem. It represent an possibly coherent ansatz that allows us to include in a systematic picture various known results about Hawking radiation. In 5 we will analyze which consequences on the calculation of the partition function our main proposal have. Last paragraph involves a further discussion.
2. Divergence problems

In this paragraph we review the problem of the 1-loop divergence of the matter entropy contribution to the black hole entropy.

Let us consider preliminarily a mathematical problem: the 1-loop scalar field entropy calculation on the Rindler Wedge (RW).

The RW represents the limit of infinite mass of the Schwarzschild space-time. The leading term of the 1-loop entropy is

\[ S_{\text{horizon}} = \frac{A}{48\pi \epsilon^2} \]

where \( \epsilon \equiv \) proper distance from the horizon; \( S_{\text{horizon}} \) is a term diverging on the horizon.

If the horizon is kept fixed and if one does not try to implement the renormalization approach, the divergence is not avoidable: it could be interpreted, in the standard statistical mechanics view, as a progressive growing dense [4] of the states of the Hilbert space available for the scalar field as far as one approach the horizon; on the horizon, they become infinite.

We begin our discussion about the black hole entropy divergence by revisiting 't Hooft brick-wall model [1]; we limit our considerations to a spin-zero massless scalar field in equilibrium with the black hole. The quantum field theory fluctuations at the Hawking temperature give the following leading 1-loop contributions to the entropy [1]:

\[
S_{\text{matter}} = S_{\text{horizon}} + S_{\text{volume}} \\
S_{\text{horizon}} = \frac{8\pi^3}{45} \frac{(2M)^4}{(\beta)^3} \frac{1}{h} \\
S_{\text{volume}} = \frac{8\pi^3}{135} \frac{V}{(\beta)^3}
\]
where $V$ is the box volume, $\beta$ is the inverse temperature and $h$ is a cut-off

$$ h \equiv \text{Inf}(r - 2M) \quad (2.3) $$

introduced in the calculation to avoid the divergence for $r \to 2M$: a strong growing dense of the states appears also near the horizon of the Schwarzschild space-time.

If we take *ad hoc*

$$ h = \frac{1}{720\pi M} \equiv h_{bw} \quad (2.4) $$

the horizon contribution is exactly the Bekenstein-Hawking value $S_{bh} = 4\pi M^2$; the interpretation of this ad hoc position is that the black hole entropy is entirely an *external* property, related with a hot brick-wall at a planckian proper distance from the horizon.

However, the entropy is not the only thermodynamical quantity divergent on the horizon: indeed, ‘t Hooft results for the free energy and the internal energy contributions near the horizon are

$$ F_{\text{horizon}} = -\frac{2\pi^3}{45} \frac{(2M)^4}{(\beta)^4} \frac{1}{h} \quad (2.5) $$

$$ E_{\text{horizon}} = \frac{2\pi^3}{15} \frac{(2M)^4}{(\beta)^4} \frac{1}{h}. $$

In [1] is also underlined the relevance of the neglected backreaction of the radiation mass: indeed the horizon contribution of the radiation internal energy in (2.5) evaluated for $h = h_{bw}$ and for $\beta = \beta_h$ is $3/8M$. For the free energy one gets the value $-1/8M$.

In [8,9] the brick-wall cut-off arises physically from the quantum fluctuations of the horizon. This position allows to implement a dynamical explanation of the
black hole entropy: as in the brick-wall model, the leading term represents not a new term to be summed to the black hole entropy, but again the black hole entropy itself. It is identified with the so called “dynamical entropy” [13]. The relevance of the quantum trembling of the horizon is obvious: it avoids the above infinite growing dense of the states; the same role plays also $h_{bw}$ in the brick-wall model.

Introducing the proper distance cut-off

$$\epsilon \sim 2\sqrt{r_{bh}} h$$

we can write the divergent terms as it follows:

$$S_{\text{horizon}} = \frac{A}{360\pi \epsilon^2} + \frac{1}{90\pi} \log\left(\frac{1}{\epsilon^2}\right)$$

$$= c_1\frac{A}{4 \epsilon^2} + c_2 \log\left(\frac{L^2}{\epsilon^2}\right);$$

(2.7)

$A$ is the horizon area, $L$ is an infrared cut-off.

In (2.7) it appears also the logarithmically divergent term, firstly found by Solodukhin [6].
2.1. QFT in Heisenberg Representation

In finite temperature quantum field theory on a fixed static curved background there are essentially two schemes for calculating the thermodynamical quantities: the Dowker-Kennedy [11] scheme and the Fursaev-Solodukhin [7] one.

The first is limited to conformally coupled scalar fields and it is based on the trick of passing to the optical manifold

$$\bar{g}_{\mu\nu} = \frac{g_{\mu\nu}}{g_{00}}.$$  \hspace{1cm} (2.9)

in order to implement the Matsubara sums on the periodic euclidean time and taking into account the difference w.r.t. the original static manifold by mean of a functional jacobian:

$$W[g_{\mu\nu}] = W[\bar{g}_{\mu\nu}] + \Delta W[g_{\mu\nu}, \omega]$$

$$\equiv \overline{W} + \Delta W$$  \hspace{1cm} (2.10)

$$\bar{g}_{\mu\nu} = e^{-2\omega}g_{\mu\nu}.$$  

For a manifold without boundary the jacobian is

$$\Delta W[g_{\mu\nu}, \omega] = \frac{1}{2880\pi^2} \int d^4x \sqrt{g}[(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - R_{\mu\nu}R^{\mu\nu} + \nabla^2 R)\omega$$

$$- 2R_{\mu\nu}\omega^\mu\omega^\nu - 4\omega^\mu\omega_{\mu}\nabla^2 \omega + 2(\omega^\mu\omega_{\mu})^2 + 3(\nabla^2 \omega)^2]$$  \hspace{1cm} (2.11)

$$\omega_{\mu} \equiv \nabla_{\mu}\omega.$$  

$\overline{W}$ is the optical manifold contribution. The boundary conditions are the Dirichelet ones. For further details see [11,12]. Given the relation between free energy and

As in ‘t Hooft pioneering calculations.

The path integral functional measure

$$D[\Phi] = \prod_x d\phi(x)g^{\frac{1}{2}}(x)$$

$$g = \text{det}g_{\mu\nu}$$  \hspace{1cm} (2.8)

is the usual covariant one.
effective action

\[ \beta F[\beta] = W \]  \hspace{1cm} (2.12)

we get

\[ F[\beta, g_{\mu\nu}] = F[\beta, \widetilde{g}_{\mu\nu}] + \Delta F[g_{\mu\nu}, \omega]. \]  \hspace{1cm} (2.13)

In (2.13) the second term does not depend on \( \beta \): the \( \beta \)-dependence is entirely in the optical term

\[ F[\beta, \widetilde{g}_{\mu\nu}] = -\frac{\pi^2 \, \overline{c}_0}{90 \, \beta^4} + \ldots \]  \hspace{1cm} (2.14)

Free energy and entropy are

\[ E[\beta, g_{\mu\nu}] = \partial_\beta (\beta F[\beta, \widetilde{g}_{\mu\nu}]) + \Delta F[g_{\mu\nu}, \omega] \]
\[ S[\beta, g_{\mu\nu}] = (\beta \partial_\beta - 1)(\beta F[\beta, \widetilde{g}_{\mu\nu}]). \]  \hspace{1cm} (2.15)

The term \( \Delta F \) in (2.15) for the internal energy is usually omitted in the high temperature expansions, whose leading terms are characterized by negative powers in \( \beta \).

The second one is based on a heat kernel expansion near the horizon, taking into account the peculiar nature of the manifold. It is in this case possible to implement the heat kernel periodicity in the euclidean time without passing to the optical manifold.\(^\text{#4}\)

In the second scheme it is to be noted that at the Hawking temperature the free energy is finite: it does not suffer of the divergence problem on the horizon characteristic of the entropy and of the internal energy. The claim of free energy

\(^\text{#4}\) The euclidean time is the angular variable of the cone and it is possible to obtain an heat kernel on the cone using periodic boundary conditions for the angular variable: see [5] and references therein.
finiteness appears in [13,5] (for the Rindler case it is evident from the inspection of formula (31) in ref. [14]).

The structure of the divergence for $\beta \neq \beta_h$ is very clear in [5, 6]: the effective action has a surface divergent term multiplied by a coefficient $\beta$-dependent having a simple zero for $\beta = \beta_h$ (in the following we will call “on shell” the thermodynamical quantities evaluated at $\beta = \beta_h$). As a consequence, we cannot have finite $\beta$-derivatives of the free energy, even if evaluated on shell.

In both these schemes, not only the entropy but also the internal energy is divergent on the horizon. Indeed, the leading optical contribution to the internal energy in the Dowker-Kennedy scheme gives

$$E_{rad} = \frac{\pi^2}{30 (\beta_h)^4} \int d^3x \sqrt{g} \frac{r^2}{(r - 2M)^2}.$$ (2.16)

We find the same divergence as in the entropy. Using as a cut-off the quantum trembling of the horizon or the brick-wall cut-off we get as leading term

$$E_{horizon} = \frac{3}{8} M$$ (2.17)

the same value than in [1].

In Fursaev-Solodukhin [7] renormalization approach, if we choose to renormalize the matter entropy leading term in such a way to get the black hole entropy value, we get straightforwardly for the leading term of the internal energy divergent at the horizon the following renormalized value

$$E_{horizon} = \frac{M}{2};$$ (2.18)

this is a value very similar to the one obtained by ’t Hooft and in (2.17).
The internal energy value obtained in both the approach is a very consistent fraction of the black hole mass. So, as noted firstly by ’t Hooft [1], there is a strong backreaction effect.

Analogous results characterize the approach of Barvinsky, Frolov, and Zelnikov’s work [9], quoted in the following as BFZ; it represents an implementation in Schroedinger representation of the idea of entropy as entanglement entropy and in a fixed background ansatz it is analogous to Thermofield Dynamics\(^5\) [15,16,17] that represents a natural framework in which thermodynamical entropy is equivalent to an entanglement entropy. For the entropy [9], for the free energy and the internal energy [18] one can get the same results than in ’t Hooft brick wall model.

Also the other physical quantities deduced by mean of \(\beta\)-derivatives are divergent on the horizon: in [18] the value for the divergent specific heat is calculated in the framework of the brick wall regularization and its problematic value discussed; an analysis about interpretative problems concerning the standard position

\[
S_{bh} \equiv S_{\text{radiation, leading}} \quad (2.19)
\]

and its consequences is also carried in [18].

\(^5\) See also appendix 2, where for completeness a brief summary of the formalism is given.
3. Internal Energy and Hartle-Hawking Tensor

There is another interesting question, related to the previous discussion. Given the finite temperature renormalized action $W$, the tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g_{\mu\nu}} \quad (3.1)$$

for $\beta = \beta_h$ should be the Hartle-Hawking one:

$$T_{\mu\nu}|_{\beta=\beta_h} = T_{\mu\nu}^{HH} \quad (3.2)$$

The internal energy relative to the quantum field in thermal equilibrium can be calculated either by mean of

$$E = -\partial_\beta \log Z \quad (3.3)$$

or by mean of

$$E_{rad} = \int d^3 x \sqrt{g} \rho$$

where $\rho = -T_{0}^{0}$

$$E_{rad} = \frac{1}{1920 \pi R} y K(y)$$

$$K(y) = \frac{1}{3} y^3 + y^2 + 3y + 4logy - \frac{22}{3} - 5y^{-1} - 3y^{-2} + 11y^{-3} \quad (3.5)$$

$$y = \frac{R}{2M}$$

where a finite box radius $R$ is imposed. As previously shown, the conical strategy
gives a divergent result that requires a further renormalization w.r.t. the renormalization of the effective action or the introduction of the brick-wall cut-off:

\[ E_{ren} \sim \frac{3}{8} M \]  

We find a situation in which two a priori equivalent patterns for calculating the internal energy give rise to two different results. Particularly, for the second way the internal energy wants an ad hoc renormalization besides the free energy one. Moreover, given the finiteness of \((3.5)\) and of the effective action for \(\beta = \beta_h\), then also the entropy

\[ S[\beta]|_{\beta = \beta_h} = (\beta(E[\beta] - F[\beta]))|_{\beta = \beta_h} \]  

should be finite. Thermodynamics seems to be really different according to different (but a priori equivalent) schemes of calculation.

We can understand the cause of the above problem as it follows. The purely thermal part of the stress energy tensor is

\[ T_{\mu}^{\text{therm}} \nu = \frac{\pi^2}{90} \frac{1}{\beta^4} \frac{1}{(1 - \frac{2M}{r})^2} (\delta_{\mu}^\nu - 4\delta_{\mu}^0 \delta_{\nu}^0); \]  

if one performs the integral in \((3.4)\), one gets that the resulting internal energy is divergent and gives exactly the same result than the brick-wall model.

Besides, the following naive pattern (cf. Dowker [21]) is instructive: given the thermal stress energy tensor “off shell”, i.e. for \(\beta \neq \beta_h\) and given the internal energy

\[ E[\beta] = \int d^3x \sqrt{g}(-T_0^0[\beta]) \]  

\[ (3.9) \]
one can obtain by integration in beta the free energy and the entropy:

\[ F[\beta] = \frac{1}{\beta} \int_{\beta'}^{\beta} d\beta' E[\beta'] + \frac{C}{\beta} \]

\[ S[\beta] = \beta (E[\beta] - F[\beta]) \tag{3.10} \]

where \( C \) is an arbitrary constant. The brick wall results can be obtained in this pattern putting \( \beta = \beta_h \), \( C = 0 \) and choosing the brick-wall cut-off at the end of the calculation in (3.10).

The suggestion coming from the above results is that the conical strategy gives rise to the same disease one can obtain in actually treating the Hawking radiation as a purely thermal perfect gas. Due to vacuum polarization effects, Hawking radiation is not, at least near the horizon [32, 20], a purely thermal perfect gas. One can verify that a phenomenological study [22] of the Hawking radiation by mean of a purely thermal perfect gas approximation suffers of the same divergences than the brick-wall model.

4. The Relevance of an Intrinsic Thermodynamics

We propose a possibility to overcome the problems arising in the usual approach to the problem. We here further develop what we called “literal interpretation” of the zeroth law of black hole mechanics. The main ideas were presented in [23] and sketched in [18].

Hawking effect allows to identify the surface gravity with the physical temperature, as measured by a static observer at infinity, associated with a stationary black hole: black hole are sources of thermal particles with

\[ T = \frac{k}{2\pi}. \tag{4.1} \]
A free variation of the temperature, or equivalently, of its inverse $\beta$, as it is true in the canonical ensemble, implies that the geometrical parameters entering in the surface gravity are functions of the temperature by mean of (4.1). The equation defining the proper period of the manifold

$$\beta = \frac{2\pi}{k}$$

is interpreted as a constraint equation for the geometrical parameters appearing in (4.2); if the manifold is characterized by $n$ geometrical scales $L_j$, then the equation (4.2) becomes

$$\beta = \frac{2\pi}{k(L_1, \ldots, L_n)}.$$  

The above match gives rise to a $\beta$-dependence of the metric which represents the main difference w.r.t. the standard case: in standard manifolds, i.e. in manifolds not affected by Hawking effect, equilibrium thermodynamics of matter fields is usually calculated by mean of the canonical ensemble, in which the inverse temperature $\beta$ is an external parameter characterizing the equilibrium distribution of field microstates and there is no link between temperature and geometry. Instead in a black hole manifold, due to Hawking effect, thermal equilibrium is implemented only when matter fields and the black hole are at the same temperature given by (4.2).

In the case of the Schwarzschild solution, that is the classical black hole solution we mainly are interested in, the relation $\beta = \frac{2\pi}{k} = 8\pi M$ for the unperturbed Schwarzschild black-hole implies

$$M = \frac{\beta}{8\pi}.$$  

The relation (4.4) between its period in euclidean time and the black hole mass
is coherent with the perturbative expansion of the path integral calculation of
the partition function: in general, the tree level value of (4.1) represents a link
between geometry and thermodynamics that is not modified by 1–loop matter
field contributions unless one does take into account their backreaction on the
geometry.

Hawking’s considerations [24,25], Frolov’s ideas in [13] and in particular Gross
et al. [26] statements support the literal interpretation of (4.1).

We get the following immediate consequences of the ansatz:

a) the horizon divergences are absent; they are due to a unphysical shift from
the physical temperature, as it follows from Fursaev and Solodukhin heat
kernel expansion;

b) quantum field theory at finite temperature in a fixed black hole manifold
becomes a theory in a background field (the metric) depending on the tree
level temperature. An expansion of the action around the background fields
up to quadratic terms gives [27]

\[
S[g_{\mu\nu}, \Phi] = S[g^0_{\mu\nu} + h_{\mu\nu}, \Phi^0 + \phi]
= S[g^0_{\mu\nu}, \Phi^0] + S_2[g^0_{\mu\nu}, \Phi^0; \phi] + S_2[g^0_{\mu\nu}, \Phi^0; h_{\mu\nu}];
\]

(4.5)

we choose for simplicity \( \Phi^0 = 0 \). The classical background term corresponds
to the gravitational term, that is to the Einstein–Hilbert action. The second
quadratic term in (4.5) depends only on the metric. In the perturbative
path–integral calculation of the partition function [27], taking into account
the \( \beta \)–dependence of the background metric (in the Schwarzschild case cf.
(4.4)) it follows for the 1–loop matter field partition function

\[
\log Z_{\text{matter}} \sim \log \int_{P(\beta)} [D\phi] e^{-S_2[g_{\mu\nu}(\beta);\phi]}.
\]

An analogous formula holds (cf. statements in [26]) for the 1–loop gravitational contribution. We stress again that the matter field backreaction on the geometry at this level is not taken into account. So the temperature has to be the one associated to the unperturbed background manifold;
c) the matter field Hamiltonian, being a function of the metric, becomes \(\beta\)-dependent:

\[
H = H[\Phi, g_{\mu\nu}(\beta)] = H(\beta).
\]

The meaning of (4.7) in relation with the canonical ensemble will be discussed in the next section.

Although our discussion is here limited to black hole manifold, the zeroth law ansatz is to be considered an approach valid in every manifold characterized by an Hawking effect and can straightforwardly generalized to a literal interpretation of the zeroth law of horizon’s thermodynamics [28], including the case of De Sitter spacetime.

4.1. \(H(\beta)\) and Canonical Ensemble

In general in the canonical ensemble the following formulas are valid:

\[
Z = Tr(e^{-\beta H})
\]

\[
\rho = \frac{e^{-\beta H}}{Z},
\]

\[
< O > = Tr(\rho O)
\]

where \(\rho\) is the density matrix and \(O\) is an observable operator. In (4.8) \(\beta\) plays
the double role of Lagrange multiplier and of physical equilibrium temperature furnished to the system by a thermostat. The well known relations

\[ E = <H> = -\partial_\beta \log Z \]
\[ S = -(\beta \partial_\beta - 1) \log Z \] (4.9)

and other similar relations allowing to calculate thermodynamical quantities by mean of \( \beta \)-derivatives of the partition function understand that the hamiltonian is not explicitly temperature dependent, that is

\[ \partial_\beta H = 0. \] (4.10)

If (4.10) is not true, than the machinery of \( \beta \)-derivatives generates spurious terms: e.g. one obtains

\[ -\partial_\beta \log Z = E + Tr(\rho \beta \partial_\beta H(\beta)). \] (4.11)

So, if the external parameter \( \beta \) appears explicitly in \( H \), as it should be for fields in equilibrium with a Schwarzschild black hole \(^6\) in our ansatz, then thermodynamical quantities of interest cannot be calculated by mean of \( \beta \)-derivatives unless one finds an algorithm to get rid of spurious terms \(^7\).

In the case of the Hawking radiation on the Schwarzschild background, a conical strategy could seem a possible solution to the problem: one could just make the following shift

\[ Z = Tr(e^{-\gamma H(\beta)}) \] (4.12)

where \( \gamma \) now is to be considered a Lagrange multiplier different from \( \beta \) appearing

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\(^6\) We mean to come back on this topic [29] for general stationary black hole geometries and to face with stringy black holes in a future work.

\(^7\) Note that (3.10) gives results that are equivalent to the standard canonical \( \beta \)-derivatives of the partition function only for a perfect gas in a “cool” geometry.
in $H$. (4.12) should allow to take $\gamma$–derivatives without generating spurious terms, and at the end of the calculation one could pose $\gamma = \beta$. In fact the strategy does not work: it does not reproduce the real physics of Hawking radiation, as seen. The reason we can guess for this fact is that (4.12) implies to relax the second property of the Lagrange multiplier $\gamma$ to be a physical equilibrium temperature, and it could be assumed as a posteriori proof corroborating our ansatz.

As a possible way to overcome the above problems we propose the following one: let us calculate perturbatively the partition function according to the standard finite temperature quantum field theory

$$Z = Tr(e^{-\beta H(\beta)}) = e^{-\beta F}; \quad (4.13)$$

as for the gravitational tree level, the assumption is that the partition function is still related with (4.13). Being standard formulas (4.9) inhibited, our choice is to calculate the internal energy according to a general relativistic formula and to get the entropy as for the tree level by mean of a thermodynamic relation:

$$E = \int d^3x \sqrt{g}(-T^0_0)$$

$$S = \beta(E - F); \quad (4.14)$$

the stress–energy tensor appearing in (4.14) is the one for the Hartle–Hawking state.

All the calculations are made “on shell” and no $\beta$ derivative is involved.

Given the finiteness on the horizon of the effective action and of the internal energy also the matter field entropy is finite, because of the second formula in (4.14).

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#8 A similar conclusion can be deduced for Thermofield Dynamics: see appendix 2.
A remark is found in [13] about the finiteness of what the author calls the “thermodynamical” entropy. The main difference w.r.t. [13] is that we get a finite result for the statistical mechanical entropy of the radiation by mean of (4.14), whereas in [13] one faces with a infinite statistical mechanical entropy to be compared with a finite thermodynamical entropy.\#9

5. Matter Field Partition Function in Schwarzschild Manifold

What follows it appears to be relevant to the discussion of the calculation of thermal equilibrium quantities when the Hawking effect is involved in the physics one is treating. The $\beta$–dependence of the geometry modifies strongly the standard picture. To see why we first consider the one–loop finite temperature contribution for a scalar field on a fixed curved background. The standard periodicity condition in the imaginary time $\tau$ with period $\beta$ can be implemented by mean of the following imagine sum on a non–periodic (“zero–temperature”) heat kernel $K_\infty$:

$$K_\beta(x, y; s) = \sum_{n=-\infty}^{+\infty} K_\infty(x, y - nl\beta; s)$$  \hspace{1cm} (5.1)

where $s$ is the usual “fifth coordinate” and $l$ is a four vector in the same direction as the periodic coordinate. The $n = 0$ term in (5.1) is ordinarily a zero temperature–term [11] and it is the only divergent one:

$$K_\beta(x, y; s) = K_\infty(x, y; s) + \sum_{n\neq 0} K_\infty(x, y - nl\beta; s).$$  \hspace{1cm} (5.2)

If the zeroth law ansatz is right, then also the $n = 0$ term is $\beta$–dependent, because

\#9 For more details see [13].
of the $\beta$-dependence of the geometry. In the calculation of the partition function

$$Z(\beta) = \int_{P(\beta)} [D\Phi] e^{-S[\Phi]},$$

(5.3)

where $P(\beta)$ stays for the periodicity condition in imaginary time (with period $\beta$) required for a boson field, one cannot get rid of the $n = 0$ term in thermodynamical considerations. Note that

$$\log Z = \frac{1}{2} \int_0^\infty ds \frac{1}{s} \int dx K_\beta(x, x; s).$$

(5.4)

A way to compute the effective action is the one discussed in sec. 2.1: in order to factorize the heat kernel in a part dependent only on the euclidean time coordinate and in a part dependent on the spatial section, a conformal transformation to the optical manifold is made. For conformally coupled scalar fields in a static manifold it is known (see e.g. [33,34,35, 36,37]) that the effective action for the static manifold is given by the effective action $W$ in the related optical manifold plus a functional jacobian $\Delta W$ (sometimes called “Liouville action” [34,37]) representing the difference between trace anomaly contributions in the two manifolds:

$$\log Z_{\text{static}}^{\text{conf}} = \log Z_{\text{ultrastatic}}^{\text{conf}} + \log Z_{\text{jacobian}}^{\text{conf}}.$$

(5.5)

The generalization to the nonconformal coupling should be straightforward, and it consists simply in writing

$$\log Z_{\text{static}} = \log Z_{\text{ultrastatic}} + \log Z_{\text{jacobian}}$$

$$+ \log Z_{\text{non conformal jacobian}},$$

(5.6)

where the last term is due to the nonconformally invariant action terms (see [38]).
Note that in the case of "standard" thermodynamics (i.e. thermodynamics not involving Hawking effect) the jacobian contributions, being linear in $\beta$, don’t affect the thermodynamics involved in the problem. Instead, in the case of the Hawking effect, the Jacobians are not simply proportional to $\beta$, and so they cannot be a priori neglected in thermodynamical considerations.

This is also the reason why one can expect quite different results by the calculation of the partition function by mean of

$$\log Z_{\text{optical}}(\beta) = - \sum_k \log (1 - e^{-\beta E_k})$$

(5.7)

where $k$ is the index for single particle states, calculated in the optical manifold, whose energy is $E_k$. The point is that Allen’s theorem [30] allowing the thermodynamical equivalence between “thermal partition function” (5.7) and “quantum partition function” (5.3) does not hold, due to the $\beta$–dependence of the geometry. We will discuss further this topic in appendix 1.

5.1. THERMOFIELD DYNAMICS

The question that should be raised is: which is the place of Thermofield Dynamics in the above picture?

In Laflamme [16] and in BFZ euclidean path integral implementation of Thermofield Dynamics (however remember BFZ remarks about the difference of their approach w.r.t. the thermofield approach of Laflamme [16]) there lacks an explicit definition of the functional measure for the Euclidean Path–Integral; Barbon

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#10 In fact, in this case there are horizon divergences due to the explosion of the optical volume: the covariant volume density of the optical manifold is divergent on the horizon; but also the conformal jacobian is divergent. If, as it is expected, the effective action is finite on the horizon, then the horizon divergent contribution of the jacobian has to be canceled by the horizon divergences of the optical effective action.
and Emparan [34] underlined that Thermofield Dynamics results for the vacuum density matrix are compatible only with the choice of the non–covariant measure

\[ D[\Phi] = \prod_x d\phi(x) (\frac{\sqrt{g}}{g_{00}})^{\frac{1}{2}}(x). \] (5.8)

The measure in (5.8) is inferable by inspection in BFZ of the scalar product: the eigenfunctions are orthonormalized according to the scalar product of the optical manifold. But it is possible also to choose the covariant measure in the implementation of Thermofield Dynamics: indeed, BFZ and Laflamme results can be related to a covariant measure by understanding that a conformal transformation of the metric to the optical manifold is made in order to consider the scalar product in the optical manifold. Then one gets the same results than in Laflamme and in BFZ, with the only difference that one has to take into account also the above jacobian factors due to the transformation to the optical manifold.

We stress again that it is not a substantially irrelevant problem the presence or the absence of such jacobian factors; this discussion on Thermofield Dynamics is made in order to show that, with the appropriate choice of the functional measure, Thermofield Dynamics can give the same results than the standard finite temperature quantum field theory also in presence of the Hawking effect.
6. Conclusions and Summary

In order to underline the main points of this work, let us compare again the thermodynamics of a massless quantum field in thermal equilibrium at the temperature $\beta$ in a static manifold without horizons with the case of thermal radiation in equilibrium with a black hole.

In the first case, thermal and quantum partition function give the same thermodynamics (terms in $\log Z$ proportional to $\beta$ are in fact contributions to the cosmological constant [30], shifting by a constant the value of the free energy and of the internal energy, but leaving unaffected the entropy).

In the black hole case, if we take $\beta$ as a free parameter, and if we want to describe thermal equilibrium between Hawking radiation and the black hole, we have to give a $\beta$ dependence to the geometrical parameters involved in the expression of the black hole surface gravity. This is the fundamental difference with the standard case. Being the geometry temperature dependent, it is not possible a priori to neglect terms ordinarily linear in $\beta$: indeed, they are not in fact linear in $\beta$. So, we cannot neglect the jacobian factor $\Delta W$, because it is not simply proportional to $\beta$. The correct approach then appears to be the one of the quantum field theory partition function (5.3), and, allowing for a covariant choice of the functional measure, one could expect from Thermofield Dynamics the same results than from ordinary finite temperature quantum field theory.

A possible exception to the zeroth law ansatz is represented by the extremal black hole: a literal interpretation would give the value zero for the temperature associated with the event horizon; instead according to Hawking et al. point of view [39] the temperature of the manifold is arbitrary and would not be related to geometrical parameters. Anyway, the very peculiar geometry of the extremal
case can justify its exceptionality, and the problem of the temperature associated with an extremal black hole is still open. A recent result due to Moretti [40] is in the direction of the impossibility to make finite temperature quantum field theory on the extremal black hole manifold: from a QFT construction it results that the only quantum state consistent with a quantum generalized equivalence principle is the zero temperature one.

As a concluding remark we stress that the zeroth law ansatz, as formulated in this work, does not necessarily clash against the idea of black hole entropy as entanglement entropy, that still appears to be a major proposal [8].

Further investigations are necessary in order to understand full consistency of this approach [29] and if it can give some insights about the main problem of a statistical interpretation of black hole entropy [41].

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APPENDIX

1. Allen Theorem

Our aim in this appendix is first to generalize Allen’s theorem from ultrastatic manifolds to static ones, what we shall do, although not in fully rigorous form; and then to discuss the result in light of the zeroth law ansatz.

We start reviewing Allen’s distinction [30] between the thermal partition function

\[ Z^T(\beta) = \sum_n e^{-\beta E_n} \equiv \text{thermal } Z \]  \hspace{1cm} (A.1)

and the quantum partition function

\[ Z^Q(\beta) = \int_{P(\beta)} [D\Phi] e^{-S[\Phi]} \equiv \text{quantum } Z. \]  \hspace{1cm} (A.2)

In (A.2) \( P(\beta) \) stays for the periodicity condition in imaginary time (with period \( \beta \)) required for a scalar field.

In (A.1) the index \( n \) stays for all the multiparticle states whose energy is \( E_n \). For a free bosonic field (A.1) becomes:

\[ Z^T(\beta) = \prod_k (1 - e^{-\beta E_k}) \]  \hspace{1cm} (A.3)

where \( k \) is the index for single particle states whose energy is \( E_k \). From (A.3) it follows

\[ \log Z^T(\beta) = - \sum_k \log(1 - e^{-\beta E_k}) \]  \hspace{1cm} (A.4)

that is the usual formula for bosonic particles.
For quantum partition function (A.2) the one-loop calculation is exact and it can be expressed in terms of $\zeta$-function [32]:

$$\log Z^Q(\beta) = \frac{1}{2}(\zeta'(0, \beta) + \zeta(0, \beta) \log(\mu^2)). \quad (A.5)$$

$\mu$ is the renormalization scale usual in the $\zeta$-function regularization.

Allen shows that for an ultrastatic manifold (with compact spatial sections) it holds

$$\log Z^Q(\beta) - \log Z^T(\beta) = -\frac{\beta}{2}\left(\frac{d}{dz} + \frac{1}{2}\log \mu^2\right)z\zeta^E(z - 1)|_{z=0} \quad (A.6)$$

where

$$\zeta^E(2z) \Gamma(z) = \int_0^\infty dt t^{z-1} Q(t) \quad (A.7)$$

$$Q(t) = \sum_k e^{-tE_k^2}.$$  

If $\zeta^E(z)$ is regular in $z = -1$ then the contribution in (A.2) not present in (A.1) is equivalent to a vacuum energy contribution. The difference between (A.1) and (A.2) does not affect standard thermodynamics: it represents a shift independent from the temperature for the free energy and the internal energy and it does not modify the entropy.

For the case of static manifolds it does not exist an analogous rigorous demonstration about a difference between the two partition functions. Anyway, a first natural extension of Allen theorem is possible for conformally coupled scalar fields in a static manifold: (5.5) allows to write

$$\log Z_{\text{static}}^{Q,\text{conf}} - \log Z_T = \log Z_{\text{ultrastatic}}^{Q,\text{conf}} - \log Z_T + \log Z_{\text{jacobian}}. \quad (A.8)$$

The jacobian is in fact the new contribution arising in the static case.
From (5.5) and (A.8) the generalization to the nonconformal coupling is

\[
\log Z^Q_{\text{static}} - \log Z_T = \log Z^Q_{\text{ultrastatic}} - \log Z_T \\
+ \log Z_{\text{jacobian}} + \log Z_{\text{non \ conformal \ jacobian}}.
\]  

The dependence of the hamiltonian eigenvalues on \(\beta\) in the case in which Hawking effect is involved (they depend on the geometry) and the use Allen does of a Mellin transform of the \(\zeta\)-function w.r.t. \(\beta\) in order to demonstrate (A.6) don’t allow to extend automatically Allen’s theorem to manifolds characterized by an intrinsic thermodynamics.

2. Thermofield Dynamics

The formalism [15], [16] is introduced in order to express a statistical mean of an observable \(A\) in terms of a expectation value on a “thermal vacuum state”:

\[
< A > = \text{Tr}(\rho A) \equiv < O(\beta) | A | O(\beta) > .
\]  

(A.10)

The equivalence in (A.10) is implementable only in an extended Hilbert space. Given a physical system described by an hamiltonian \(H\) in a Hilbert space \(\mathcal{H}\), one has to introduce a fictitious Hilbert space \(\tilde{\mathcal{H}}\) and an hamiltonian \(\tilde{H}\) describing an identical physical system. Total Hilbert space and total hamiltonian are

\[
\mathcal{H}_{\text{tot}} = \mathcal{H} \otimes \tilde{\mathcal{H}}
\]

\[
H_{\text{tot}} = H - \tilde{H}.
\]  

(A.11)

A physical (fictitious) \(n\)-particle state whose energy is \(E_n\) is denoted by \(|n > (|\tilde{n} >)\) and \(|n, \tilde{n} > = |n > |\tilde{n} >\) is the \(n\)-particle tensor product state.
Vacuum state in the extended Hilbert state is

\[ |O(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n, \tilde{n}\rangle; \quad (A.12) \]

\(Z(\beta)\) is a normalization.

In the calculation of (A.10) one has to sum over all the fictitious states \(\tilde{n}\); the result is

\[ < A > = < O(\beta)|A|O(\beta) > = \frac{1}{Z(\beta)} \sum_n e^{-\beta E_n} < n|A|n >. \quad (A.13) \]

Between Thermofield Dynamics and black hole thermodynamics, as discussed by Israel [42], the following identifications are possible [42]

\[ |O(k)\rangle \equiv |HH - vacuum\rangle = |O(\beta)\rangle \]

\[ < O(k)|A|O(k) > = Tr_{II}(\rho A) \]

\[ = \frac{1}{Z(k)} \sum_n e^{-\frac{2\pi}{k}E_n} < n|A|n > \quad (A.14) \]

\[ \mathcal{H}_{II} = \mathcal{H}; \]

\(k\) is the surface gravity, and \(|n\rangle\) are \(n\)-particle (Boulware) states. Physical Hilbert space available to a static observer (region I) is related to the physical Hilbert space of Thermofield Dynamics; statistical means in (A.14) are relative to the region I, and the fictitious space is identified with the space of states in region II (time reversed of region I).

We note that the above correspondence was derived and in fact holds “on shell”: the temperature \(\beta\) is proportional to the surface gravity \(k\).

If one allows the introduction of a conical defect in treating Hawking radiation in Thermofield Dynamics, in the sense that the mean w.r.t. HH–vacuum is taken but in the sum \(\sum_n e^{-\beta E_n} \ < n|A|n >\) the parameter \(\beta\) is off–shell, it happens that thermal equilibrium condition is broken.
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