Some remarks on the parametrized Borsuk-Ulam theorem

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Abstract. Given a locally trivial fibre bundle $E \to B$ (with fibres and base finite complexes), an orthogonal real line bundle $\lambda$ over $E$ and a real vector bundle $\xi$ over $B$, we consider a fibrewise map $f : S(\lambda) \to \xi$ over $B$ defined on the unit sphere bundle of $\lambda$. Following the fundamental work of Jaworowski and Dold on the parametrized Borsuk-Ulam theorem, we investigate lower bounds on the cohomological dimension of the set $\{v \in S(\lambda) \mid f(v) = f(-v)\}$.

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1. Introduction

Let $B$ be a connected compact ENR (Euclidean Neighbourhood Retract), $\pi : E \to B$ a locally trivial fibre bundle with compact ENR fibres, $\lambda$ a real line bundle over $E$ and $\xi$ a real vector bundle of dimension $n$ over $B$. We may assume that $\lambda$ and $\xi$ are equipped with inner products and write $S(\lambda)$ and $S(\xi)$ for their sphere bundles.

Suppose that $f : S(\lambda) \to \xi$ is a fibrewise map over $B$. We shall be concerned with the subset

$$\tilde{Z} = \{v \in S(\lambda) \mid f(v) = f(-v)\}$$

of $S(\lambda)$ and its image $Z$ in $E$ under the projection $S(\lambda) \to E$. In particular, if $f(-v) = -f(v)$ for all $v \in S(\lambda)$, then $\tilde{Z}$ is the zero-set $\{v \in S(\lambda) \mid f(v) = 0\}$ of $f$.

Throughout the paper we shall use representable cohomology $H^*$ (as in [4, Section 8]), usually with $\mathbb{F}_2$-coefficients. Our goal is to estimate the size of the space $\tilde{Z}$ by giving a lower bound on its cohomological dimension. Early estimates of this type were obtained by Jaworowski [8] and Dold [6].
We present here two theorems that generalize the result [6 Corollary 1.5] in Dold’s fundamental paper on the parametrized Borsuk-Ulam theorem.

The first is quite elementary. But it leads to easy proofs of a number of results in the recent literature (9 Corollary 1.5, 12 Theorem 1.3, 10 Theorem 1.5, 15 Theorem 1.3). In the statement below, $e(\lambda) = w_1(\lambda) \in H^1(E; \mathbb{F}_2)$ is the $\mathbb{F}_2$-Euler class of $\lambda$.

**Theorem 1.1.** Suppose that $H^i(B; \mathbb{F}_2) = 0$ for $i > d$ and that, for some $k \geq n$, there is a class $b \in H^k(B; \mathbb{F}_2)$ such that $\pi^*(b) \cdot e(\lambda)^k \neq 0$. Then the restriction of $\pi^*(b) \cdot e(\lambda)^k \in H^{d+k-n}(E; \mathbb{F}_2)$ to $Z$ is non-zero.

It follows that $H^j(Z; \mathbb{F}_2)$ is non-zero for some $j \geq d + k - n$.

The second is rather deeper and leads to stronger results in specific examples. Given a real vector bundle $\eta$, we shall write $P(\eta)$ for its projective bundle and $H$ for the Hopf line bundle over $P(\eta)$.

**Theorem 1.2.** Suppose that, for some $r \geq 1$, $\pi : E \to B$ admits a factorization $E = S(\zeta_r) \to S(\zeta_{r-1}) \to \cdots \to S(\zeta_1) \to P(\eta) \to B$, where $\eta$ is an $(m + 1)$-dimensional real vector bundle over $B$, and $\zeta_i$, for $i = 1, \ldots, r$, is a real vector bundle of dimension $l_i + 1$ over $P(\eta)$ if $i = 1$, over $S(\zeta_{i-1})$ if $i > 1$, with $\mathbb{F}_2$-Euler class $e(\zeta_i) = w_{l_i+1}(\zeta_i)$ equal to zero. The fibres of $\pi$ are thus manifolds of dimension $l + m$, where $l = l_1 + \ldots + l_r$. Suppose further that $\lambda$ is the pullback of $H$ over $P(\eta)$.

Let $d$ be maximal such that $H^d(B; \mathbb{F}_2) \neq 0$. Then, if $n \leq m$, the cohomology group $H^{d+l+m-n}(Z; \mathbb{F}_2)$ is non-zero, and hence $H^j(Z; \mathbb{F}_2)$ is non-zero for some $j \geq d + l + m - n$.

Two cases in which these conditions are satisfied are described in Propositions 3.15 and 3.17 which pursue ideas introduced in 15 and 12.

The proofs of Theorems 1.1 and 1.2 are given in Sections 2 and 3. An analogous theory for complex vector bundles is sketched in Section 4. In Section 5 we discuss the extension of the parametrized Borsuk-Ulam theorem for a sphere bundle $S(\xi)$ with the antipodal involution to the case of a spherical fibration with a fibre-preserving free involution. Necessary material on the Euler class of a spherical fibration is included as an appendix (Section 6).

2. An elementary condition

The basic Borsuk-Ulam Theorem as formulated, for example, in 4 Proposition 2.7 specializes to the following proposition, which will be fundamental to our discussion. To be precise we should write $\lambda \otimes \pi^*\xi$ in its statement, rather than $\lambda \otimes \xi$. In order to simplify notation we shall often, as here, use the same symbol for a bundle and its pullback to some space.

**Proposition 2.1.** Suppose that $a \in H^i(E; \mathbb{F}_2)$ is a cohomology class such that $a \cdot e(\lambda \otimes \xi) \neq 0 \in H^{i+n}(E; \mathbb{F}_2)$. Then $a$ restricts to a non-zero class in $H^i(Z; \mathbb{F}_2)$. 

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Proof. The map $f$ determines a section $s$ of $\lambda \otimes \xi$ over $E$:

$$s(x) = v \otimes (f(v) - f(-v))$$

for $v \in S(\lambda_x)$ with zero-set $\{ x \in E \mid s(x) = 0 \}$ equal to the subset $Z$. The conclusion follows at once from [4, Proposition 2.7] or its extension to spherical fibrations given as Proposition 6.4 in the Appendix.

Corollary 2.2. Under the conditions of Proposition 2.1, $H^j(\tilde{Z}; \mathbb{F}_2)$ is non-zero for some $j \geq i$.

Proof. Notice that $\tilde{Z} = S(\lambda|Z)$. There is, thus, a long exact sequence

$$\cdots \to H^j(Z; \mathbb{F}_2) \to H^j(\tilde{Z}; \mathbb{F}_2) \to H^j(Z; \mathbb{F}_2) \xrightarrow{e(\lambda)} H^{j+1}(Z; \mathbb{F}_2) \to \cdots$$

The result follows easily from the fact that $e(\lambda)$ is nilpotent.

Proof of Theorem 7.1. Recall that the mod 2 Euler class $e(\lambda \otimes \xi)$ is given by

$$w_n(\lambda \otimes \xi) = \sum_{i=0}^n e(\lambda)^i w_{n-i}(\xi).$$

Because $b \cdot w_{n-i}(\xi) = 0$ for $i < n$, we have $\pi^*(b) \cdot e(\lambda)^{k-n} \cdot e(\lambda \otimes \xi) = \sum_{i=0}^n \pi^*(b) \cdot e(\lambda)^{k-n} \cdot e(\lambda)^i w_{n-i}(\xi) = \pi^*(b) \cdot e(\lambda)^k$, which is non-zero. Take $a = \pi^*(b) \cdot e(\lambda)^{k-n}$ in Proposition 2.1

Remark 2.3. If we replace the condition $\pi^*(b) \cdot e(\lambda)^k \neq 0$ by $\pi^*(b) \cdot c \cdot e(\lambda)^n \neq 0$ for some class $c \in H^{k-n}(E; \mathbb{F}_2)$, the same argument shows that $\pi^*(b) \cdot c$ restricts to a non-zero class in $H^{d+k-n}(Z; \mathbb{F}_2)$.

Corollary 2.4. Suppose that $d \geq 0$ is maximal such that $H^d(B; \mathbb{F}_2)$ is non-zero and that, for any fibre $F$ of $\pi$, the restriction $H^*(E; \mathbb{F}_2) \to H^*(F; \mathbb{F}_2)$ is surjective and $e(\lambda|F)^k \in H^k(F; \mathbb{F}_2)$ is non-zero. Then $H^{d+k-n}(Z; \mathbb{F}_2)$ is non-zero.

Proof. In this case multiplication by $e(\lambda)^k : H^*(B) \to H^*(E)$ is injective, by the Leray-Hirsch Theorem, which says that the bundle $E \to B$ is ‘homologically trivial’.

Examples 2.5. We give some examples from [15, 9, 12, 10].

(i). ([15, Theorem 1.3]). Suppose that the restriction $S(\lambda|F) \to F$ of the bundle $S(\lambda) \to E$ to a fibre $F$ is homeomorphic to the projection

$$V_r(\mathbb{R}^{r+s}) \to V_r(\mathbb{R}^{r+s})/\{\pm 1\}$$

from the Stiefel manifold of orthogonal $r$-frames in $\mathbb{R}^{r+s}$ to the projective Stiefel manifold. Then we may take $k$ to be the smallest integer in the range $s \leq k < r + s$ such that $\binom{r+s}{k}$ is odd. (See [7, Theorem 1.6].)

(ii). ([9, Corollary 1.5], [12, Theorem 1.3]). Suppose that $S(\lambda|F) \to F$ is homeomorphic to the projection

$$S^{n_1} \times \cdots \times S^{n_r} \to (S^{n_1} \times \cdots \times S^{n_r})/\{\pm 1\}$$
We form the real projective bundles $P_{u, v, w}$ where $P$ may take $\lambda$ along each of the subspaces $P_i$. (See [10, Theorem 1.5].) Let $r, s > 1$ be integers. Suppose that the fibre $S(\lambda|F)$ has cohomology ring $H^*(S(\lambda|F); \mathbb{F}_2) = \mathbb{F}_21 \oplus \mathbb{F}_2u \oplus \mathbb{F}_2v \oplus \mathbb{F}_2w$, where $u, v, w$ have degrees $r, r+s, 2r+s$, respectively and $uv = 0$. Then we may take $k = 2r+s$. (See [13, Theorem 4.1], at least for the case $r = s$.)

We shall return to the examples (i) and (ii) in the next section.

**Remark 2.6.** Here is an illustration of the example (iii). Let $\alpha, \beta$ and $\gamma$ be real vector bundles over $B$ of dimension $r, s+1$ and $r+s+1$, respectively. We form the real projective bundles $P(\alpha \oplus \beta) \rightarrow B$ and $P(\alpha \oplus \gamma) \rightarrow B$. Both contain $P(\alpha)$ as a subbundle. We define $E \rightarrow B$ by gluing the two bundles along $P(\alpha)$:

$$E = P(\alpha \oplus \beta) \cup_{P(\alpha)} P(\alpha \oplus \gamma) \rightarrow B$$

and take $\lambda$ to be the line bundle that restricts to the Hopf line bundle on each of the subspaces $P(\alpha \oplus \beta)$ and $P(\alpha \oplus \gamma)$. The double cover $S(\lambda)$ may be described by a gluing construction or as a fibrewise join:

$$S(\lambda) = S(\alpha \oplus \beta) \cup_{S(\alpha)} S(\alpha \oplus \gamma) = S(\alpha) *_B (S(\beta) \cup S(\gamma)).$$

To be accurate, the papers [15, 9, 12, 10] work in a rather more general setting than that considered here: the base $B$ is not required to be a compact ENR and the involution on the vector bundle $\xi$ is not necessarily antipodal. We shall not deal with more general base spaces here, but we explain in Section 5 how to incorporate more general involutions into the theory.

For the covering dimension there is a simpler estimate.

**Corollary 2.7.** Suppose that $B$ is a closed smooth manifold of dimension $d$ and that, for some point $b \in B$, the restriction $e(\lambda|F)k \in H^k(F; \mathbb{F}_2)$ to the fibre $F$ of $\pi$ at $b$ is non-zero. Then the covering dimension of $Z$ and of $\tilde{Z}$ is at least $d + k - n$.

**Proof.** We choose an embedding $D(\mathbb{R}^d) \hookrightarrow B$ of a closed $d$-disc centred at $b$. The restriction of the bundle $E \rightarrow B$ to $D(\mathbb{R}^d)$ is trivial. Now we map the $d$-dimensional sphere $B' = S(\mathbb{R} \oplus \mathbb{R}^d)$ to $D(\mathbb{R}^d)$, and so to $B$, by projection $(t, v) \mapsto v$. The bundles $E \rightarrow B$, $\lambda$ over $E$ and $\xi$ over $B$ pullback to bundles $E' \rightarrow B'$, $\lambda$ over $E'$ and $\xi'$ over $B'$, and the map $f$ lifts to $f' : S(\lambda') \rightarrow \xi'$ with the associated subset $Z' \subseteq E'$ lifting $Z \subseteq E$.

We can apply Corollary 2.4 to $f'$ to deduce that $H^{d+k-n}(Z'; \mathbb{F}_2)$ is non-zero and so that the covering dimension of $Z'$ is at least $d + k - n$. But $Z'$ is the union of two closed subspaces each homeomorphic to $Z|D(\mathbb{R}^d)$. Hence $Z|D(\mathbb{R}^d)$ and so $Z$ have covering dimension greater than or equal to $d + k - n$. □

A more general result on the covering dimension is described in Proposition 6.7.
3. Sphere bundles

Consider an $m + 1$-dimensional real vector bundle $\eta$ over $B$ with projective bundle $P = P(\eta) \to B$.

It is convenient to introduce the polynomials

$$p(T) = T^n + w_1(\xi)T^{n-1} + \ldots + w_n(\xi)$$

and

$$q(T) = T^{m+1} + w_1(\eta)T^m + \ldots + w_{m+1}(\eta)$$

in $H^*(B; \mathbb{F}_2)[T]$.

Remark 3.1. Recall that

$$p(T + w_1(\lambda)) = \sum_{i=0}^{n} w_{n-i}(\lambda \otimes \xi)T^i \in H^*(E; \mathbb{F}_2)[T]$$

Remark 3.2. The Euler class $e(H \otimes \xi) \in H^n(P; \mathbb{F}_2)$ (of the tensor product of the Hopf line bundle $H$ and the pullback of $\xi$) is zero if and only if $q(T)$ divides $p(T)$ in $H^*(B; \mathbb{F}_2)$.

Suppose that $\rho : E \to P$ is a fibre bundle with structure group a compact Lie group $G$ acting smoothly on a closed connected manifold $M$ of dimension $l$. Thus $E = Q \times_G M$ for a principal $G$-bundle $Q \to P$. (More generally, we could just take $E \to P$ to be a fibrewise manifold with fibre $M$.) We take $\pi$ to be the composition $E \to P \to B$ and $\lambda = \rho^*H$ to be the pullback of the Hopf line bundle.

Lemma 3.3. Suppose that there is a class $\sigma \in H^l(E; \mathbb{F}_2)$ restricting to the generator of the cohomology $H^l(M; \mathbb{F}_2) = \mathbb{F}_2$ of a fibre. Then, for each $i \geq 0$, the homomorphism

$$x \mapsto \rho^*(x) \cdot \sigma : H^l(P; \mathbb{F}_2) \to H^{l+i}(E; \mathbb{F}_2)$$

is a split injection.

Proof. We have a fibrewise Umkehr map $\rho_! : H^{i+l}(E; \mathbb{F}_2) \to H^i(P; \mathbb{F}_2)$ and $\rho_!(\rho^*(x) \cdot \sigma) = x \cdot \rho_!(\sigma) = x$ for $x \in H^i(P; \mathbb{F}_2)$, because $\rho_!(\sigma) = 1 \in H^0(P; \mathbb{F}_2)$.

Remark 3.4. The class $\sigma$ should be thought of as a ‘homology section’ of $\rho : E \to P$. A genuine section $s : P \to E$ determines a class $\sigma = s_!(1) \in H^l(E; \mathbb{F}_2)$.

Proposition 3.5. Suppose that there is a class $\sigma \in H^l(E; \mathbb{F}_2)$ satisfying the condition of Lemma 3.3 and that $n \leq m$. Let $b \in H^d(B; \mathbb{F}_2)$ be a non-zero class of maximal degree. Then $\pi^*(b) \cdot e(\lambda)^{m-n} \cdot \sigma \in H^{d+m-n+l}(E; \mathbb{F}_2)$ restricts to a non-zero class in $H^{d+m-n+l}(Z; \mathbb{F}_2)$.

Proof. Write $a = \pi^*(b) \cdot e(\lambda)^{m-n} \cdot \sigma$. Then

$$a \cdot e(\lambda \otimes \xi) = \rho^*(b \cdot e(H)^{m-n} \cdot e(H \otimes \xi)) \cdot \sigma$$

But $b \cdot e(H)^{m-n} \cdot e(H \otimes \xi) \in H^{d+m}(P; \mathbb{F}_2)$ is non-zero. So the result follows from Proposition 2.1

□
Remark 3.6. When $\rho$ is the identity $E = P \to P$, this reduces to the original result of Dold [6, (1.7)].

Let $\zeta$ be a real vector bundle of dimension $l + 1$ over $P$. We consider the bundle $E = S(\zeta) \to P$.

Lemma 3.7. There is a class $\sigma \in H^1(S(\zeta); \mathbb{F}_2)$ restricting to the generator of the cohomology $H^1(M; \mathbb{F}_2)$ of a fibre if and only if the Euler class $e(\zeta) \in H^{l+1}((P; \mathbb{F}_2)$ is zero.

Proof. We have an exact Gysin sequence of the pair $(D(\zeta), S(\zeta))$:

$$H^{-1}(P; \mathbb{F}_2) = 0 \to H^1(P; \mathbb{F}_2) \to H^1(S(\zeta); \mathbb{F}_2)$$

$$\to H^0(P; \mathbb{F}_2) = \mathbb{F}_2 \xrightarrow{\epsilon(\zeta)} H^{l+1}(P; \mathbb{F}_2),$$

where we have used the Thom isomorphism

$$H^{l+1+*}(D(\zeta), S(\zeta); \mathbb{F}_2) \cong H^*(P; \mathbb{F}_2),$$

so that the restriction $H^*(D(\zeta), S(\zeta); \mathbb{F}_2) \to H^*(D(\zeta))$ corresponds to multiplication by the Euler class $e(\zeta) = w_{l+1}(\zeta) : H^*(P; \mathbb{F}_2) \to H^{*+l+1}(P; \mathbb{F}_2)$.

Thus there is a lift $\sigma \in H^1(S(\zeta); \mathbb{F}_2)$ of $1 \in H^0(P; \mathbb{F}_2)$ if and only if $e(\zeta) = 0$. \hfill $\Box$

Remark 3.8. The class $\sigma$ satisfies $\text{Sq}(\sigma) - w(\zeta)\sigma \in H^*(P; \mathbb{F}_2)$, because the total Steenrod square $\text{Sq}$ acts on the Thom class of $\zeta$ as multiplication by the total Stiefel-Whitney class. In particular, we have the identity

$$\sigma^2 - w(\zeta)\sigma \in H^{2l}(P; \mathbb{F}_2).$$

Example 3.9. ([15]). Consider the bundle of Stiefel manifolds $E = O(\mathbb{R}^2, \eta)$, with fibre at $x \in B$ the space $V_2(\eta_x) = O(\mathbb{R}^2, \eta_x)$ of orthogonal 2-frames in $\eta_x$, equipped with the involution $-1$ on $\eta$. The quotient $E$ is a bundle of projective Stiefel manifolds. Restriction to the first factor $\mathbb{R}$ in $\mathbb{R}^2$ gives a map $O(\mathbb{R}^2, \eta) \to O(\mathbb{R}, \eta) = S(\eta)$. Then we can express $E$ as the sphere bundle $S(\eta) \to P(\eta)$, where $\zeta$ is the tensor product $H \otimes H^\perp$ of $H$ and its orthogonal complement $H^\perp$ in the pullback of $\eta$. Thus $l = m - 1$. The line bundle $\lambda$ is the pullback of $H$.

Now

$$w_{l+1}(\zeta) = (m + 1)w_1(H)^m + mw_1(\eta)w_1(H)^{m-1} + \ldots + w_m(\eta)$$

by the formula $w_m(H \otimes \eta) = q'(w_1(H))$. (See Remark 3.1) For $w_{l+1}(\zeta)$ to vanish it is necessary and sufficient that $m + 1$ be even and $w_i(\eta) = 0$ for $i$ odd. This is true if $\eta$ admits a complex structure, and in that special case $\zeta$ has a trivial 1-dimensional summand.

Example 3.10. Suppose that $\eta$ has a complex structure, so that $m + 1$ is even. Consider the complex Stiefel bundle $U(\mathbb{C}^2, \eta)$ with the involution $-1$ and quotient $E$. We can express $E$ as $S(\zeta) \to P = P(\eta)$, where $\zeta = H \otimes (\mathbb{C} \otimes H)^\perp$, so that $l = m - 2$. By Remark 3.1 again, we have

$$w_{l+1}(\zeta) = \binom{m+1}{2}w_1(H)^{m-1} + \binom{m}{2}w_1(\eta)w_1(H)^{m-2} + \ldots + \binom{2}{2}w_{m-1}(\eta),$$
which vanishes if and only if $m + 1$ is divisible by 4 and $w_{2i}(\eta) = 0$ for $i$ odd. This holds if $\eta$ admits a quaternionic structure.

Consider, more generally, $r$ vector bundles $\zeta_i$ of dimension $l_i + 1$, $i = 1, \ldots, r$, over $P$. Take $E = S(\zeta_1) \times_P \cdots \times_P S(\zeta_r)$ and let $\lambda$ be the pullback of $H$ over $P$. Write $l = l_1 + \ldots + l_r$.

**Lemma 3.11.** There is a class $\sigma \in H^l(E; \mathbb{F}_2)$ restricting to a generator of the cohomology $H^l(M; \mathbb{F}_2)$ of a fibre if and only if $e(\zeta_i) = 0 \in H^{l_i+1}(P; \mathbb{F}_2)$ for $i = 1, \ldots, r$.

**Proof.** If each $e(\zeta_i)$ is zero, choose $\sigma_i \in H^{l_i}(S(\zeta_i); \mathbb{F}_2)$ as in Lemma 3.7. Then take $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_r$.

Conversely, given $\sigma$, we can produce a class $\sigma_i$ as the image of $\sigma$ under the Umkehr homomorphism

$$(\pi_i)_* : H^l(S(\zeta_1) \times_P \cdots \times_P S(\zeta_r); \mathbb{F}_2) \to H^{l_i}(S(\zeta_i); \mathbb{F}_2)$$

of the fibrewise projection $\pi_i : S(\zeta_1) \times_P \cdots \times_P S(\zeta_r) \to S(\zeta_i)$ over $P$. \hfill $\Box$

A special case of our main result Theorem 1.2 follows immediately from Proposition 3.5.

**Proposition 3.12.** Suppose that $e(\zeta_i) = 0$ for $i = 1, \ldots, r$, that $d$ is maximal such that $H^d(B; \mathbb{F}_2) \neq 0$ and that $n \leq m$. Then $H^{d+m-n+l}(Z; \mathbb{F}_2)$ is non-zero. \hfill $\Box$

**Remark 3.13.** By Leray-Hirsch, $H^*(E; \mathbb{F}_2)$ is a free $H^*(P; \mathbb{F}_2)$-module on the classes $\sigma_1 \cdot \ldots \cdot \sigma_k$, $1 \leq i_1 < \ldots < i_k \leq r$, where $0 \leq k \leq r$. Remark 3.8 allows us to describe $H^*(E; \mathbb{F}_2)$ as

$$H^*(P; \mathbb{F}_2)[V_1, \ldots, V_r]/(V_i^2 - w_i(\zeta_i)V_i - s_i | i = 1, \ldots, r),$$

where $V_i$ has dimension $l_i$ and $s_i = \sigma_i^2 - w_i(\zeta_i)\sigma_i \in H^{2l_i}(P; \mathbb{F}_2)$. (Compare [12] Section 4.)

This decomposition is a homological version of the stable splitting in the next Remark 3.14.

**Remark 3.14.** The condition of Lemma 3.11 is satisfied if each $S(\zeta_i)$ has a section over $P$ and we can then split $\zeta_i$ as $\mathbb{R} \oplus \nu_i$ and identify $S(\zeta_i)$ with the fibrewise one-point compactification $(\nu_i)_+^\perp$ over $P$. Now the fibrewise suspension $\Sigma_P(S(\zeta)_+^P)$ splits as a fibrewise wedge $\Sigma_P(P \times S^0) \vee_P \Sigma_P(\nu_i)_+^\perp$. It follows that we have a fibrewise homotopy equivalence

$$\Sigma_P(E_+^P) \simeq \Sigma_P(((P \times S^0) \vee_P (\nu_1)_+^\perp) \wedge_P \cdots \wedge_P ((P \times S^0) \vee_P (\nu_r)_+^\perp)).$$

This then permits us, using the notation $P^n$ for the Thom space of a vector bundle $\eta$ over $P$, to describe $\Sigma(E_+)$ as a wedge of $2^r$ suspensions of Thom spaces over $P$:

$$\Sigma(E_+) \simeq \bigvee_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \Sigma(P^{\nu_{i_1} \oplus \cdots \oplus \nu_{i_k}}).$$
where $0 \leq k \leq r$. There is thus a homotopy equivalence

$$\Sigma^r E_+ \simeq \bigvee_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \Sigma^{r-k}(P\xi_{i_1} \oplus \cdots \oplus \xi_{i_k}).$$

**Proof of Theorem 1.2.** Choose a real line bundle.  

2 not divisible by $l$ and is satisfied if $\mu$ is divisible by $l$ and $q = 0$.

There is a class $\sigma \in H^l(S(H \otimes \mu); \mathbb{F}_2)$ restricting to the generator of the cohomology $H^l(M; \mathbb{F}_2)$ of a fibre if and only if the polynomial $r(T)$ is divisible by $q(T)$. In that case, $\sigma^2 - r'(w_1(H))\sigma \in H^{2l}(P; \mathbb{F}_2)$.

Notice that this condition requires the dimensional restriction $l \geq m$ and is satisfied if $\mu$ admits $\eta$ as a subbundle.

**Proof.** This is contained in Remark 3.2. □
Proposition 3.17. (See [9,12].) Let \( \eta \) be a real vector bundle over \( B \) of dimension \( m + 1 \) and \( \mu_1, \ldots, \mu_r, r \geq 1 \), real vector bundles over \( B \) with \( \dim \mu_i = l_i + 1 \geq m + 1 \). Consider the bundle \( \hat{E} = S(\eta) \times_B S(\mu_1) \times_B \cdots \times_B S(\mu_r) \) with the free diagonal antipodal involution. Let \( \pi : E \to B \) be the quotient by the action of \( O(1) \) and let \( \lambda \) be the associated real line bundle, so that \( \hat{E} = S(\lambda) \).

Suppose that

\[
\rho_i(T) = T^{l_i+1} + w_1(\mu_i)T^{l_i-1} + \ldots + w_{l_i+1}(\mu_i) \in H^*(B; \mathbb{F}_2)[T]
\]

is divisible by \( q(T) = T^{m+1} + w_1(\eta)T^m + \ldots + w_{m+1}(\eta) \) for each \( i = 1, \ldots, r \).

Then \( \pi : E \to B \) and \( \lambda \) satisfy the conditions of Theorem 1.2.

Proof. The bundle \( E \to B \) can be identified with

\[
E = S(H \otimes \mu_1) \times_P \cdots \times_P S(H \otimes \mu_r) \to P \to B
\]

and \( \lambda \) with the pullback of the Hopf bundle \( H \).

Thus Theorem 1.2 or Proposition 3.12 extends [12, Theorem 1.4] to bundles.

Remark 3.18. The condition of Proposition 3.17 is satisfied if each \( \mu_i \) admits \( \eta \) as a subbundle, or equivalently if the sphere bundle \( S(\zeta_i) \) of \( \zeta_i = H \otimes \mu_i \) admits a section. In that case, Remark 3.14 allows us to describe \( \Sigma(E_+) \) as a wedge of \( 2^n \) Thom spaces over \( P \):

\[
\Sigma(E_+) \cong \bigvee_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \Sigma P^{\nu_{i_1} \otimes \cdots \otimes \nu_{i_k}}
\]

where \( 0 \leq k \leq r \). There is thus a homotopy equivalence

\[
\Sigma^r E_+ \cong \bigvee_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \Sigma^{r-k} P^{H \otimes (\mu_{i_1} \otimes \cdots \otimes \mu_{i_k})}.
\]

(Compare the argument of Davis in [5].)

4. Complex versions

The methods extend readily to the complex theory. We suppose now that \( \lambda \) and \( \xi \) are complex vector bundles over \( B \) of dimension 1 and \( n \). From a map \( f : S(\lambda) \to \xi \) we construct a section \( s \) of the complex tensor product \( \lambda^* \otimes \xi \):

\[
s(x) = v^* \otimes \int_{\mathbb{T}} z f(z^{-1}v) \text{ for } v \in S(\lambda_x),
\]

where \( v^* \in S(\lambda^*_z) \) is the dual generator and the integral is over the circle group \( \mathbb{T} = \{ z \in \mathbb{C} | |z| = 1 \} \) (with Haar measure). We write \( Z = \text{Zero}(s) \), and then \( \hat{Z} = S(\lambda|Z) \) is the set of points \( v \) where \( \int z f(z^{-1}v) \) is zero.

We use cohomology with \( \mathbb{Z} \)-coefficients. The Euler classes are \( e(\lambda) = c_1(\lambda) \) and \( e(\lambda^* \otimes \xi) = \sum (-1)^i e(\lambda)^i c_{n-i}(\xi) \).
Theorem 4.1. Suppose that $H^i(B; \mathbb{Z}) = 0$ for $i > d$ and that, for some $k \geq n$, there is a class $b \in H^d(B; \mathbb{Z})$ such that $\pi^*(b) \cdot e(\lambda)^k \neq 0$. Then the restriction of $\pi^*(b) \cdot e(\lambda)^{2k-2n} \in H^{d+2k-2n}(E; \mathbb{Z})$ to $Z$ is non-zero.

It follows that $H^j(\tilde{Z}; \mathbb{Z})$ is non-zero for some $j \geq 1 + d + 2(k - n)$. □

There are corresponding results for cohomology with coefficients in $\mathbb{F}_p$ ($p$ a prime) or $\mathbb{Q}$. We state a corollary for $\mathbb{F}_p$-cohomology.

Corollary 4.2. Suppose that $d \geq 0$ is maximal such that $H^d(B; \mathbb{F}_p)$ is non-zero and that, for any fibre $F$ of $\pi$, the restriction $H^*(E; \mathbb{F}_p) \to H^*(F; \mathbb{F}_p)$ is surjective and $e(\lambda|F)^k \in H^{2k}(F; \mathbb{F}_p)$ is non-zero. Then $H^{d+2k-2n}(Z; \mathbb{F}_p)$ is non-zero, and $H^j(\tilde{Z}; \mathbb{F}_p)$ is non-zero for some $j \geq 1 + d + 2(k - n)$.

Example 4.3. ([15, Theorem 1.4]). Suppose that the restriction $S(\lambda|F) \to F$ of the bundle $S(\lambda) \to E$ to a fibre $F$ is homeomorphic to the projection $V^C_r(\mathbb{C}^{r+s}) \to V^C_r(\mathbb{C}^{r+s})/\mathbb{T}$ from the complex Stiefel manifold of unitary $r$-frames in $\mathbb{C}^{r+s}$ to the projective complex Stiefel manifold. We can take $k$ to be the least integer in the range $s \leq k \leq r + s$ such that $(\frac{r+s}{k+1})$ is not divisible by $p$. (See [1] Theorems 1.1 and 1.2.)

Theorem 4.4. Suppose that $\eta$ is a complex vector bundle of dimension $m+1$ over $B$ and that $\zeta_i$, for $i = 1, \ldots, r$, is a complex vector bundle of dimension $l_i + 1$ over $\mathbb{C}P(\eta)$ for $i = 1$ and over $S(\zeta_{i-1})$ for $i > 1$, with each $\mathbb{Z}$-Euler class $e(\zeta_i) = c_{l_i+1}(\zeta_i)$ zero. Let $E \to B$ be the bundle

$$E = S(\zeta_r) \to S(\zeta_{r-1}) \to \cdots \to S(\zeta_1) \to \mathbb{C}P(\eta) \to B$$

with $\lambda$ the pullback of the complex Hopf bundle $H$ over $\mathbb{C}P(\eta)$.

Let $d$ be maximal such that $H^d(B; \mathbb{Z}) \neq 0$. Then, if $n \leq m$, the group $H^{d+2m-2n+2l}(Z; \mathbb{Z})$ is non-zero.

It follows that $H^j(\tilde{Z}; \mathbb{Z})$ is non-zero for some $j \geq 1 + d + 2(m-n+l)$. □

Example 4.5. Let $E = U(\mathbb{C}^2, \eta)$, where $\eta$ is a complex bundle of even dimension $m+1$ such that $c_i(\eta) = 0$ for all odd $i$. We take $\zeta$ to be $H^* \otimes H^\perp$ over $\mathbb{C}P(\eta)$, where $H^\perp$ is the orthogonal complement of $H \subseteq \eta$, so that $l = m-1$. The condition on Chern classes is satisfied if the complex structure on $\eta$ extends to a quaternionic structure.

5. Spherical fibrations

We sketch a generalization of the theory in which the sphere bundle $S(\xi)$ with the antipodal involution is replaced by a fibrewise $O(1)$-space $\Xi \to B$, with the total space $\Xi$ compact Hausdorff and the trivial action of $O(1)$ on the base $B$, which is locally $O(1)$-equivariantly fibre homotopy equivalent to a trivial bundle with fibre $S(\mathbb{R}^n)$ equipped with the antipodal involution. We assume that the $O(1)$-space $\Xi$ admits an $O(1)$-equivariant embedding as a subspace of some finite dimensional real $O(1)$-module.
From a real line bundle $\lambda$ over $E$ we can form a spherical fibration $\Xi_\lambda \to E$:

$$\Xi_\lambda = (S(\lambda) \times_B \Xi)/\text{O}(1),$$

where $\text{O}(1)$ acts as $\pm 1$ on $S(\lambda)$.

We consider an $\text{O}(1)$-map $f : S(\lambda) \to C_B(\Xi)$ to the fibrewise cone on $\Xi$ and write $\tilde{Z} = \{ v \in S(\lambda) | f(v) = 0 \}$ (where 0 is the vertex of the cone in the fibre). It determines a section $s$ of $\text{C}_E(\Xi_\lambda)$ with zero-set $\tilde{Z}$. (If $\Xi = S(\xi)$ with the antipodal involution, then $C_B(\Xi)$ is the disc bundle $D(\xi)$, $\Xi_\lambda = S(\lambda \otimes \xi)$ and $\text{C}_E(\Xi_\lambda) = D(\lambda \otimes \xi)$.)

The mod 2 Euler class $e(\Xi_\lambda) \in H^n(E; \mathbb{F}_2)$ can be written as

$$e(\Xi_\lambda) = \sum_{i=0}^{n} e(\lambda)^i w_{n-i}(\Xi),$$

where the classes $w_j(\Xi) \in H^j(B; \mathbb{Z})$ are defined by the universal example in which $E = B \times P(\mathbb{R}^N)$ and $\lambda$ is the pullback of the Hopf line bundle over $P(\mathbb{R}^N)$ for $N$ sufficiently large.

Then Proposition 2.1 generalizes as follows.

**Proposition 5.1.** Suppose that $a \in H^i(E; \mathbb{F}_2)$ is a cohomology class such that $a \cdot e(\Xi_\lambda) \neq 0 \in H^{i+n}(E; \mathbb{F}_2)$. Then $a$ restricts to a non-zero class in $H^i(\tilde{Z}; \mathbb{F}_2)$.

**Proof.** This follows from the generalization, included as Proposition 6.4 in the Appendix, of [4, Proposition 2.7] in which $S(\xi)$ is replaced by an arbitrary spherical fibration. □

In the complex case we replace $\text{O}(1)$ by $\text{U}(1)$. The spherical fibration $\Xi_\lambda$ is oriented and we have classes $c_j(\Xi) \in H^{2j}(B; \mathbb{Z})$.

### 6. Appendix: Euler classes for spherical fibrations

We follow closely the account in [4, Section 2] to construct Euler classes with integral coefficients. Corresponding definitions may be made, and theorems proved, for $\mathbb{F}_p$-coefficients by purely notational changes.

Let $X$ be a compact ENR and let $\Xi \to X$ be a fibrewise space, with $\Xi$ compact Hausdorff, which is locally fibre homotopy equivalent to a trivial sphere bundle with fibre $S^{n-1} = S(\mathbb{R}^n)$. From the spherical fibration $\Xi \to X$ we form the fibrewise cone $C_X(\Xi) \to X$, which contains $\Xi$ as a subspace. We shall refer to the vertex of the cone in any fibre as zero, written as ‘0’.

The non-zero elements of $C_X(\Xi)$ will be written as $[t, v]$, where $t \in (0, 1]$ and $v \in \Xi$.

We begin with the definition of the Euler class $e(\Xi)$ in the cohomology group $H^n(X; \mathbb{Z}(\Xi))$ with integral coefficients $\mathbb{Z}(\Xi)$ twisted by the orientation bundle of $\Xi$. There is a unique class $u \in H^n(C_X(\Xi); \mathbb{Z}(\Xi))$ restricting to the canonical generator of each fibre $H^n(C(\Xi_x); \mathbb{Z}(\Xi_x)) = \mathbb{Z}$. We define $e(\Xi)$ to be the pullback $z^*u$ by the zero-section $z : (X, \emptyset) \to (C_X(\Xi), \Xi)$. 

Let $s : X \to C_X(\Xi)$ be a section. We use the notation
$$\text{Zero}(s) = \{ x \in X \mid s(x) = 0 \}$$
for its zero-set. It is a closed, so compact, subspace of $X$.

**Definition 6.1.** Suppose that $U \subseteq X$ is an open subset and that $s$ is a section of $C_X(\Xi)$ such that $\text{Zero}(s) \cap U$ is compact. Choose an open neighbourhood $W$ of $\text{Zero}(s) \cap U$ such that $\overline{W}$ is compact and contained in $U$. There is an $\epsilon \in (0, 1)$ such that for each $x \in \overline{W} - W$ we have $s(x) = [t, v]$, where $t \in [\epsilon, 1]$, $v \in \Xi_x$. For $x \in \overline{W}$, put
$$s'(x) = \begin{cases} 0 & \text{if } s(x) = 0, \\ [t/\epsilon, v] & \text{if } s(x) = [t, v] \text{ with } t \in (0, \epsilon], v \in \Xi_x, \\ [1, v] & \text{if } s(x) = [t, v] \text{ with } t \in [\epsilon, 1], v \in \Xi_x, \end{cases}$$
so that $s'$ gives a map $(\overline{W}, \overline{W} - W) \to (C_X(\Xi), \Xi)$. The pullback of the Thom class $u \in H^n(C_X(\Xi), \Xi; \mathbb{Z}(\Xi))$ by this map $s'$ gives a class in the group $H^n(\overline{W}, \overline{W} - W; \mathbb{Z}(\Xi))$ which we may identify with the cohomology $H^n_c(W; \mathbb{Z}(\Xi))$ of $W$ with compact supports.

The image of this class in $H^n_c(U; \mathbb{Z}(\Xi))$ under the homomorphism induced by the inclusion $W \hookrightarrow U$ is independent of the choices made and will be called the **Euler class with compact supports**
$$e(s \mid U) \in H^n_c(U; \mathbb{Z}(\Xi))$$
of the section $s$.

**Lemma 6.2.** (Properties of the Euler class with compact supports).
(i) Suppose that $U' \subseteq U$ is an open subspace of $U$ such that $\text{Zero}(s) \cap U \subseteq U'$. Then $e(s \mid U) = i_1^* e(s \mid U')$, where $i_1 : H^n_c(U'; \mathbb{Z}(\Xi)) \to H^n_c(U; \mathbb{Z}(\Xi))$ is induced by the inclusion $i : U' \hookrightarrow U$.
(ii) Suppose that $U = U_1 \sqcup U_2$ is a disjoint union of two open subspaces $U_1$ and $U_2$. Then $e(s \mid U) = (i_1)_! e(s \mid U_1) + (i_2)_! e(s \mid U_2)$, where $i_1 : U_1 \hookrightarrow U$ and $i_2 : U_2 \hookrightarrow U$ are the inclusion maps.

We shall use the Euler class with compact supports to localize the Euler class of a spherical fibration to an arbitrarily small neighbourhood of the zero-set of a section. The basic result follows directly from the definition (and is, indeed, a special case of the property (i) above).

**Lemma 6.3.** Let $s$ be a section of $C_X(\Xi)$ with zero-set $\text{Zero}(s) \subseteq U$, where $U \subseteq X$ is open. Then the Euler class $e(s \mid U) \in H^n_c(U; \mathbb{Z}(\Xi))$ with compact supports maps under the homomorphism $j_!$ induced by the inclusion $j : U \hookrightarrow X$ to $e(\Xi) \in H^n(X; \mathbb{Z}(\Xi))$.

This yields the basic result on the cohomology of the zero-set.

**Proposition 6.4.** Let $s$ be a section of $C_X(\Xi)$ with zero-set $\text{Zero}(s)$. Suppose that $a \in H^i(X; \mathbb{Z})$ is a cohomology class that restricts to $0 \in H^i(\text{Zero}(s); \mathbb{Z})$. Then $a \cdot e(\Xi) = 0 \in H^{i+n}(X; \mathbb{Z}(\Xi))$. 
Proof. Choose an open neighbourhood $U$ of $\text{Zero}(s)$ in $X$ such that $a$ restricts to zero in $H^i(U; \mathbb{Z})$. We have an Euler class with compact supports $e(s|U) \in H^n_a(U; \mathbb{Z}(\Xi))$ mapping to $e(\Xi) \in H^n(X; \mathbb{Z}(\Xi))$. So $a \cdot e(s|U)$ maps to $a \cdot e(\Xi)$. But $a$ restricts to 0 in $H^i(U; \mathbb{Z})$ and hence $a \cdot e(s|U) = 0$. □

Corollary 6.5. Suppose that the kernel of multiplication by $e(\Xi) : H^i(X; \mathbb{Z}) \to H^{i+n}(X; \mathbb{Z}(\Xi))$ is zero. Then the restriction map $H^i(X; \mathbb{Z}) \to H^i(\text{Zero}(s); \mathbb{Z})$ is injective. □

Corollary 6.6. Suppose that $s$ is a section of $C_X(\Xi)$ with zero-set $\text{Zero}(s)$. If there is a class $x \in H^i(X; \mathbb{Z})$ such that $a \cdot e(\xi) \neq 0$, then the covering dimension of $\text{Zero}(s)$ is greater than or equal to $i$. □

Proposition 6.7. Let $X$ be a closed smooth manifold and let $Y \subseteq X$ be a closed submanifold of codimension $d$. Suppose that $s$ is a section of $C_X(\Xi)$. If there is a class $a \in H^i(Y; \mathbb{Z})$ such that $a \cdot e(\Xi|Y) \neq 0 \in H^{i+n}(Y; \mathbb{Z}(\Xi))$, then the covering dimension of $\text{Zero}(s) \subseteq X$ is at least $d + i$.

Proof. Writing $\nu$ for the normal bundle of $Y$ in $X$, choose a tubular neighbourhood $D(\nu) \to X$. Now we have a map $f : X' = S(\mathbb{R} \oplus \nu) \to D(\nu) \to X$, given by the projection $(t, v) \mapsto v$. The section $s$ lifts to a section $s'$ of $C_{X'}(f^*\Xi)$. And its zero-set $\text{Zero}(s')$ is the pullback of $\text{Zero}(s) \cap D(\nu)$. Let $u \in H^d(S(\mathbb{R} \oplus \nu); \mathbb{Z}(\nu))$ be the Thom class of $\nu$. Then $a' = u \cdot f^*a \in H^{d+i}(X'; \mathbb{Z}(\nu))$ is non-zero. Applying Corollary 6.6 (with twisted coefficients) to $s'$, we see that the covering dimension of $\text{Zero}(s')$ is at least $d + i$.

But $\text{Zero}(s')$ is the union of two closed subspaces homeomorphic to $\text{Zero}(s) \cap D(\nu)$. It follows that $\text{Zero}(s) \cap D(\nu)$ and a fortiori $\text{Zero}(s)$ have covering dimension greater than or equal to $d + i$. □

Remark 6.8. If $n$ is large, then $\Xi$ is canonically equivalent to a free $O(1)$-spherical fibration. See [2, Theorem 3.10]. For this canonical structure, the classes $w_j(\Xi)$ defined in Section 5 satisfy $w_j(\Xi) \cdot u = \text{Sq}^j u$, where $u \in H^n(C_X(\Xi), \Xi; \mathbb{F}_2)$ is the mod 2 Thom class.
Footnote. This paper originated in the observation by the first author that two remarks (Remark 6.1 and Remark 6.2) in [14] were incompatible with a result of Stolz [16] (see, also, [3]) on the level of real projective spaces. The error lies in the proof of Corollaries 4.2 and 4.4, which is modelled on the somewhat misleading approach in [9] to the result stated there as Corollary 1.5. We trust that the argument used in [9, Corollary 1.5], and also in [12, Theorem 1.3], [10, Theorem 1.5] and [15, Theorem 1.3], will now be superseded by our proof of those theorems as applications (Examples 2.5) of Corollary 2.4.

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