Harmonic k-Uniformly Convex, k-Starlike Mappings and Pascal Distribution Series

Elif YAŞAR

Abstract
In this paper, connections between various subclasses of harmonic univalent functions by using a convolution operator involving the Pascal distribution series are investigated. Furthermore, an example is provided, illustrating graphically with the help of Maple, to illuminate the convolution operator.

Keywords: Harmonic functions, Univalent functions, Pascal distribution.

AMS Subject Classification (2020): Primary: 30C45; Secondary: 30C80.

1. Introduction
Let \( H \) denote the family of continuous complex valued harmonic functions of the form \( f = h + g \) defined in the open unit disk \( \mathfrak{U} = \{ z : |z| < 1 \} \), where

\[
   h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n
\]

are analytic in \( \mathfrak{U} \).

A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( \mathfrak{U} \) is that \( |h'(z)| > |g'(z)| \) in \( \mathfrak{U} \) (see [2],[3]).

Denote by \( SH \) the subclass of \( H \) consisting of functions \( f = h + g \) which are harmonic, univalent and sense-preserving in \( \mathfrak{U} \) and normalized by \( f(0) = f_z(0) = 0 \). One can easily show that the sense-preserving property implies that \( |b_1| < 1 \).

The subclass \( SH^0 \) of \( SH \) consist of all functions in \( SH \) which have the additional property \( b_1 = 0 \). Note that \( SH \) reduces to the class \( S \) of normalized analytic univalent functions in \( \mathfrak{U} \), if the co-analytic part of \( f \) is identically zero.

Define \( H UC_i \) \( (i = 1, 2) \) be the subclass of \( SH \) consisting of the functions \( f = h + g \) such that \( h(z) \) and \( g(z) \) are of the form

\[
   h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad \text{and} \quad g(z) = (-1)^i \sum_{n=1}^{\infty} |b_n| z^n.
\]

Let \( H UC(k, \alpha) \) be a subclass of the functions \( f = h + g \) in \( SH \) which satisfy the condition

\[
   \text{Re} \left\{ 1 + (1 + ke^{i\eta}) \frac{z^2 h''(z) + 2g'(z) + z^2 g''(z)}{zh'(z) - zg'(z)} \right\} \geq \alpha,
\]

for some \( k \geq 0 \), \( \alpha \geq 0 \), \( k \leq \alpha < 1 \) and \( z \in \mathfrak{U} \). Define \( H UC(k, \alpha) : = H UC(k, \alpha) \cap H UC^1 \). A mapping in \( H UC(k, \alpha) \) or \( H UC(k, \alpha) \) is called harmonic k-uniformly convex in \( \mathfrak{U} \). These classes were studied in [5]. For \( g \equiv 0 \), \( k = 1 \) and

Received : 02–02–2020, Accepted : 10–04–2020
\( \alpha = 0 \), the class \( \text{HUC}(k, \alpha) \) reduces to the class \( \text{UC} \) of analytic uniformly convex functions defined by [4]. Let \( \text{HS}^*(k, \alpha) \) be a subclass of the functions \( f = h + \overline{g} \) in \( \mathcal{S}\mathcal{H} \) which satisfy the condition

\[
\text{Re} \left\{ \frac{zf'(z)}{zf(z)} - \alpha \right\} \geq k \left| \frac{zf'(z)}{zf(z)} - 1 \right|
\]

for some \( k \geq 0 \), \( \alpha \) \((0 \leq \alpha < 1)\) and \( z \in \mathcal{U} \). Also define \( \text{HS}^*(k, \alpha) := \text{HS}^*(k, \alpha) \cap \mathbb{H} \). These mappings are called harmonic \( k \)-starlike in \( \mathcal{U} \). For \( \alpha = 0 \) these classes were studied in [7]. For \( g \equiv 0, k = 1 \) and \( \alpha = 0 \), the class \( \text{HS}^*(k, \alpha) \) reduces to the class \( \text{US}^* \) of analytic uniformly starlike functions defined by [6].

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [8],[9],[10],[11],[12],[13]). Let us consider a non-negative discrete random variable \( X \) with a Pascal probability generating function

\[
P(X = n) = \binom{n + r - 1}{r - 1} p^n (1 - p)^r, \quad n \in \{0, 1, 2, 3, \ldots\}
\]

where \( p, r \) are called the parameters.

Now we introduce a power series whose coefficients are probabilities of the Pascal distribution, that is

\[
P^r_p (z) = z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r z^n, \quad (r \geq 1, \ 0 \leq p \leq 1, \ z \in \mathcal{U})
\]

(1.4)

Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity. Now, for \( r, s \geq 1 \) and \( 0 \leq p, q \leq 1 \), we introduce the operator

\[
P^r_s_{p,q} (f)(z) = P^r_p (z) * h(z) + \overline{P^q_s (z)} * g(z) = H(z) + \overline{G(z)}
\]

where

\[
H(z) = z + \sum_{n=2}^{\infty} \binom{n + r - 2}{r - 1} p^{n-1} (1 - p)^r a_n z^n
\]

\[
G(z) = b_1 z + \sum_{n=2}^{\infty} \binom{n + s - 2}{s - 1} q^{n-1} (1 - q)^s b_n z^n
\]

and "\(*\)" denotes the convolution (or Hadamard product) of power series.

**Example 1.1.** Consider the harmonic polynomial \( f_1(z) = z + \frac{1}{6} z^2 + \frac{1}{6} \pi^4 \). If we take \( r = 7, s = 7, p = 0.1 \) and \( q = 0.3 \) then from (1.5), we have

\[
P^r_s_{p,q} (f_1)(z) = z + 0.05 z^2 + 0.03 \pi^4.
\]

Images of concentric circles inside \( \mathcal{U} \) under the functions \( f_1 \) and \( P^r_s_{p,q} (f_1) \) are shown in Figure 1 and Figure 2.

In this paper, we deal mainly with connections between the classes harmonic starlike, harmonic convex, harmonic \( k \)-uniformly convex and harmonic \( k \)-starlike by using above convolution operator involving the Pascal distribution series.

### 2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

**Lemma 2.1.** [2] If \( f = h + \overline{g} \in \mathcal{KH}^0 \) where \( h \) and \( g \) are given by (1.1) with \( b_1 = 0 \), then

\[
|a_n| \leq \frac{n + 1}{2}, \quad |b_n| \leq \frac{n - 1}{2}.
\]

**Lemma 2.2.** [5] Let \( f = h + \overline{g} \) be given by (1.1). If \( k \geq 0, 0 \leq \alpha < 1 \) and

\[
\sum_{n=2}^{\infty} n (n (k + 1) - (k + \alpha)) |a_n| + \sum_{n=1}^{\infty} n (n (k + 1) + (k + \alpha)) |b_n| \leq 1 - \alpha,
\]

then \( f \) is harmonic, sense-preserving, univalent in \( \mathcal{U} \) and \( f \in \text{HUC}(k, \alpha) \).
Figure 2. Image of $P_{p,q}^r(f_1)(U)$

**Theorem 3.1.** If the inequality

$$
\frac{(k+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{(4k+5-\alpha)r(r+1)p^2}{(1-p)^2} + \frac{(2k+4-2\alpha)r p}{1-p} \\
+ \frac{(k+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{(6k+5+\alpha)s(s+1)q^2}{(1-q)^2} + \frac{(6k+4+2\alpha)s q}{1-q} \\
\leq 2(1-\alpha)(1-p)^r
$$

is hold, then $P_{p,q}^{r,s}(KH^0) \subset HUC(k, \alpha)$.

**Proof.** Suppose $f = h + \overline{g} \in KH^0$ where $h$ and $g$ are given by (1.1) with $b_1 = 0$. We need to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in HUC(k, \alpha)$ where $H$ and $G$ are given by (1.5) with $b_1 = 0$. By Lemma 2.2, we need to establish that

**Lemma 2.3.** [1] Let $f = h + \overline{g} \in T^1$ be given by (1.2). Then $f \in HUC(k, \alpha)$ if and only if the coefficient condition (2.1) is satisfied. Also, if $f \in HUC(k, \alpha)$, then

$$
|a_n| \leq \frac{1-\alpha}{n(k+1)-(k+\alpha)}, \ n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n(k+1)+(k+\alpha)}, \ n \geq 1.
$$

**Lemma 2.4.** [1] Let $f = h + \overline{g}$ be given by (1.1). If $k \geq 0, 0 \leq \alpha < 1$ and

$$
\sum_{n=2}^{\infty} (n(k+1)-(k+\alpha)) |a_n| + \sum_{n=1}^{\infty} (n(k+1)+(k+\alpha)) |b_n| \leq 1-\alpha,
$$

then $f$ is harmonic, sense-preserving, univalent in $U$ and $f \in HS^*(k, \alpha)$.

**Lemma 2.5.** [1] Let $f = h + \overline{g} \in T^2$ be given by (1.2). Then $f \in HS^*(k, \alpha)$ if and only if the coefficient condition (2.2) is satisfied. Also, if $f \in HS^*(k, \alpha)$, then

$$
|a_n| \leq \frac{1-\alpha}{n(k+1)-(k+\alpha)}, \ n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n(k+1)+(k+\alpha)}, \ n \geq 1.
$$

**Lemma 2.6.** [2] If $f = h + \overline{g} \in SH^{1,0}$ where $h$ and $g$ are given by (1.1) with $b_1 = 0$, then

$$
|a_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |b_n| \leq \frac{(2n-1)(n-1)}{6}, \ n \geq 2.
$$

**3. Main Results**

From now, throughout the main results, we will consider $0 \leq \alpha < 1$, $k \geq 0$, $r, s \geq 1$, and $0 \leq p, q < 1$. 

**Theorem 3.1.** If the inequality

$$
\frac{(k+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{(4k+5-\alpha)r(r+1)p^2}{(1-p)^2} + \frac{(2k+4-2\alpha)r p}{1-p} \\
+ \frac{(k+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{(6k+5+\alpha)s(s+1)q^2}{(1-q)^2} + \frac{(6k+4+2\alpha)s q}{1-q} \\
\leq 2(1-\alpha)(1-p)^r
$$

is hold, then $P_{p,q}^{r,s}(KH^0) \subset HUC(k, \alpha)$.
Using Lemma 2.2, we obtain

\[
Q_1 \leq 1 - \alpha, \text{ where }
\]

\[
Q_1 = \sum_{n=2}^{\infty} n (n (k + 1) - (k + \alpha)) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} |a_n| \\
+ \sum_{n=2}^{\infty} n (n (k + 1) + (k + \alpha)) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} |b_n|.
\]

Using Lemma 2.2, we obtain

\[
Q_1 \leq \frac{1}{2} \left\{ \sum_{n=2}^{\infty} n (n + 1) (n (k + 1) - (k + \alpha)) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+ \sum_{n=2}^{\infty} n (n - 1) (n (k + 1) + (k + \alpha)) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \right\} \\
= \frac{1}{2} \left\{ (k + 1) \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+ (6k + 7 - \alpha) \sum_{n=2}^{\infty} (n - 1) (n - 2) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+ (6k + 10 - 4\alpha) \sum_{n=2}^{\infty} (n (k + 1) (n + 1) - (k + \alpha)) \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+ 2(1 - \alpha) \sum_{n=2}^{\infty} \left( \frac{n + r - 2}{r - 1} \right) (1 - p)^r p^{n-1} \\
+k (n + 1) \sum_{n=2}^{\infty} (n - 1) (n - 2) (n - 3) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \\
+ (6k + 5 + \alpha) \sum_{n=2}^{\infty} (n - 1) (n - 2) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \\
+ (6k + 2) \sum_{n=2}^{\infty} (n - 1) \left( \frac{n + s - 2}{s - 1} \right) (1 - q)^s q^{n-1} \right\} \\
= \frac{1}{2} \left\{ (k + 1) r (r + 1) (r + 2) p^3 (1 - p)^r \sum_{n=4}^{\infty} \left( \frac{n + r - 2}{r + 2} \right) p^{n-1} \\
+ (4k + 5 - \alpha) r (r + 1) p^2 (1 - p)^r \sum_{n=3}^{\infty} \left( \frac{n + r - 2}{r + 1} \right) p^{n-1} \\
+ (2k + 4 - 2\alpha) r (1 - p)^r \sum_{n=2}^{\infty} \left( \frac{n + r - 2}{r - 1} \right) p^{n-1} \\
+ (k + 1) s (s + 1) (s + 2) q^3 (1 - q)^s \sum_{n=4}^{\infty} \left( \frac{n + s - 2}{s + 2} \right) q^{n-1} \\
+ (6k + 5 + \alpha) s (s + 1) q^2 (1 - q)^s \sum_{n=3}^{\infty} \left( \frac{n + s - 2}{s + 1} \right) q^{n-1} \\
+ (6k + 4 + 2\alpha) s q (1 - q)^s \sum_{n=2}^{\infty} \left( \frac{n + s - 2}{s} \right) q^{n-1} \right\}.
\]
Theorem 3.2. If the inequality

\[
\frac{(k+1)r(r+1)}{1-p^2} + \frac{(3k+4-\alpha)}{1-p} + \frac{(k+1)s(s+1)}{1-q} + \frac{(3k+2+\alpha)}{1-q} \leq 2(1-\alpha)(1-p)^r
\]

(3.2)

is hold, then \(P^{p,s}_{p,q}(KH^0) \subset HS^*(k, \alpha)\).

Proof. Suppose that \(f = h + g \in KH^0\) where \(h\) and \(g\) are given by (1.1) with \(b_1 = 0\). It suffices to show that \(P^{p,s}_{p,q}(f) = H + G \in HS^*(k, \alpha)\) where \(H\) and \(G\) are given by (1.5) with \(b_1 = 0\) in \(I\). Using Lemma 2.4, we need to show that \(Q_2 \leq 1-\alpha\), where

\[
Q_2 = \sum_{n=2}^{\infty} (n(k+1) - (k+\alpha)) \binom{n+r-2}{r-1} (1-p)^{r-1} |a_n| \] 

\[
+ \sum_{n=2}^{\infty} (n(k+1) + (k+\alpha)) \binom{n+s-2}{s-1} (1-q)^{s-1} |b_n|.
\]

Using Lemma 2.2, we obtain

\[
Q_2 \leq \frac{1}{2} \left\{ \sum_{n=2}^{\infty} (n+1)(n(k+1) - (k+\alpha)) \binom{n+r-2}{r-1} (1-p)^{r-1} p^{n-1} \right. 
\]

\[
+ \sum_{n=2}^{\infty} (n-1)(n(k+1) + (k+\alpha)) \binom{n+s-2}{s-1} (1-q)^{s-1} q^{n-1} \right\}
\]
\[
\begin{align*}
\frac{1}{2} \left\{ (k+1) \sum_{n=2}^{\infty} (n-1)(n-2) \left( \begin{array}{c} n + r - 2 \\ r - 1 \end{array} \right) (1-p)^r p^{n-1} \\
+(3k + 4 - \alpha) \sum_{n=2}^{\infty} (n-1) \left( \begin{array}{c} n + r - 2 \\ r - 1 \end{array} \right) (1-p)^r p^{n-1} \\
+2(1-\alpha) \sum_{n=2}^{\infty} (n+r-2) \left( \begin{array}{c} n + r - 2 \\ r - 1 \end{array} \right) (1-p)^r p^{n-1} \\
+(k+1) \sum_{n=2}^{\infty} (n-1)(n-2) \left( \begin{array}{c} n + s - 2 \\ s - 1 \end{array} \right) (1-q)^s q^{n-1} \\
+(3k + 2 + \alpha) \sum_{n=2}^{\infty} (n-1) \left( \begin{array}{c} n + s - 2 \\ s - 1 \end{array} \right) (1-q)^s q^{n-1} \right\} \\
= \frac{1}{2} \left\{ (k+1)r (r+1) p^2 (1-p)^r \sum_{n=3}^{\infty} \left( \begin{array}{c} n + r - 2 \\ r + 1 \end{array} \right) p^{n-3} \\
+(3k + 4 - \alpha)rp (1-p)^r \sum_{n=2}^{\infty} \left( \begin{array}{c} n + r - 2 \\ r \end{array} \right) p^{n-2} \\
+2(1-\alpha) (1-p)^r \sum_{n=2}^{\infty} \left( \begin{array}{c} n + r - 2 \\ r - 1 \end{array} \right) p^{n-1} \\
+(k+1)s (s+1) q^2 (1-q)^s \sum_{n=3}^{\infty} \left( \begin{array}{c} n + s - 2 \\ s + 1 \end{array} \right) q^{n-3} \\
+(3k + 2 + \alpha)sq (1-q)^s \sum_{n=2}^{\infty} \left( \begin{array}{c} n + s - 2 \\ s \end{array} \right) q^{n-2} \right\} \\
= \frac{1}{2} \left\{ (k+1)r (r+1) p^2 (1-p)^r \sum_{n=0}^{\infty} \left( \begin{array}{c} n + r + 1 \\ r + 1 \end{array} \right) p^n \\
+(3k + 4 - \alpha) rp (1-p)^r \sum_{n=0}^{\infty} \left( \begin{array}{c} n + r \\ r \end{array} \right) p^n \\
+2(1-\alpha) (1-p)^r \sum_{n=0}^{\infty} \left( \begin{array}{c} n + r - 1 \\ r - 1 \end{array} \right) p^n - 2(1-\alpha) (1-p)^r \\
+(k+1)s (s+1) q^2 (1-q)^s \sum_{n=0}^{\infty} \left( \begin{array}{c} n + s + 1 \\ s + 1 \end{array} \right) q^n \\
+(2k + 2 + \alpha) sq (1-q)^s \sum_{n=0}^{\infty} \left( \begin{array}{c} n + s \\ s \end{array} \right) q^n \right\} \\
= \frac{1}{2} \left\{ \frac{(k+1)r (r+1) p^2}{(1-p)^2} + \frac{(3k + 4 - \alpha) rp}{1-p} + 2(1-\alpha) - 2(1-\alpha) (1-p)^r \\
+ \frac{(k+1)s (s+1) q^2}{(1-q)^2} + \frac{(3k + 2 + \alpha) sq}{1-q} \right\}.
\end{align*}
\]

The last expression is bounded above by \((1-\alpha)\) by the condition (3.2). Thus the proof of Theorem 3.2 is complete. \(\square\)

**Theorem 3.3.** If the inequality
\[
(1-p)^r + (1-q)^s \geq 1 + \frac{(2k+1+\alpha)}{1-\alpha} |b_1|
\]  
(3.3)

is hold, then \(P_{p,q}^r \mathbb{HUC}(k,\alpha) \subset \mathbb{HUC}(k,\alpha)\).
Proof. Suppose \( f = h + \overline{g} \in HUC(k, \alpha) \) where \( h \) and \( g \) are given by (1.2) with \( i = 1 \). We need to establish that the operator

\[
P_{p,q}^{r,s}(f)(z) = z - \sum_{n=2}^{\infty} \left( \frac{n+r-2}{r-1} \right) p^{n-1} (1-p)^r a_n z^n - |b_1| z - \sum_{n=2}^{\infty} \left( \frac{n+s-2}{s-1} \right) q^{n-1} (1-q)^s |b_n| z^n
\]

is in \( HUC(k, \alpha) \) if and only if \( Q_3 \leq 1 - \alpha \), where

\[
Q_3 = \sum_{n=2}^{\infty} n (n(k+1)-(k+\alpha)) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} |a_n|
+ (2k + 1 + \alpha) |b_1| + \sum_{n=2}^{\infty} n (n(k+1)+(k+\alpha)) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} |b_n|.
\]

Using Lemma 2.4, we have

\[
Q_3 \leq (1-\alpha) \left\{ \sum_{n=2}^{\infty} \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \right\}
+ \sum_{n=2}^{\infty} \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} + (2k + 1 + \alpha) |b_1|
= (1-\alpha) \left\{ (1-p)^r \sum_{n=0}^{\infty} \left( \frac{n+r-1}{r-1} \right) p^n - (1-p)^r \right\}
+ (1-q)^s \sum_{n=0}^{\infty} \left( \frac{n+s-1}{s-1} \right) q^n - (1-q)^s \right\} + (2k + 1 + \alpha) |b_1|
= (1-\alpha) \left\{ 2 - (1-p)^r - (1-q)^s \right\} + (2k + 1 + \alpha) |b_1| \leq 1 - \alpha.
\]

Then inequality (3.3) completes the proof. \( \square \)

Theorem 3.4. If the inequality

\[
\frac{2(k+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{13k+15-2\alpha}{(1-p)^2} \frac{r(r+1)p^2}{1-p} + \frac{15k+24-9\alpha}{1-p} \frac{rp}{1-p}
+ \frac{2(k+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{11k+9+2\alpha}{(1-q)^2} \frac{s(s+1)q^2}{1-q} + \frac{9k+6+3\alpha}{1-q} \frac{sq}{1-q}
\leq 6(1-\alpha) (1-p)^r
\]

is hold, then \( P_{p,q}^{r,s}(SH^{*,0}) \subset HS^*(k, \alpha) \).

Proof. Suppose \( f = h + \overline{g} \in SH^{*,0} \) where \( h \) and \( g \) are given by (1.1) with \( b_1 = 0 \). We need to prove that \( P_{p,q}^{r,s}(f) = H + \overline{G} \in HS^*(k, \alpha) \). In view of Lemma 2.4, we need to prove that \( Q_4 \leq 1 - \alpha \), where

\[
Q_4 = \sum_{n=2}^{\infty} n (n(k+1)-(k+\alpha)) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} |a_n|
+ \sum_{n=2}^{\infty} n (n(k+1)+(k+\alpha)) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} |b_n|.
\]
Referring Lemma 2.6, we observe

\[
Q_1 \leq \frac{1}{6} \left\{ \sum_{n=2}^{\infty} (2n+1)(n+1)(n(k+1)-(k+\alpha)) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \\
+ \sum_{n=2}^{\infty} (2n-1)(n-1)(n(k+1)+(k+\alpha)) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} \right\}
\]

\[
= \frac{1}{6} \left\{ 2(k+1) \sum_{n=2}^{\infty} (n-1)(n-2)(n-3) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \\
+ (13k+15-2\alpha) \sum_{n=2}^{\infty} (n-1)(n-2) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \\
+ (15k+24-9\alpha) \sum_{n=2}^{\infty} (n-1) \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \\
+ 6(1-\alpha) \sum_{n=2}^{\infty} \left( \frac{n+r-2}{r-1} \right) (1-p)^r p^{n-1} \\
+ 2(k+1) \sum_{n=2}^{\infty} (n-1)(n-2)(n-3) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} \\
+ (11k+9+2\alpha) \sum_{n=2}^{\infty} (n-1)(n-2) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} \\
+ (9k+6+3\alpha) \sum_{n=2}^{\infty} (n-1) \left( \frac{n+s-2}{s-1} \right) (1-q)^s q^{n-1} \right\}
\]

\[
= \frac{1}{6} \left\{ 2(k+1)r(r+1)(r+2)p^3 (1-p)^r \sum_{n=0}^{\infty} \left( \frac{n+r+2}{r+2} \right) p^n \\
+ (13k+15-2\alpha)r(r+1)p^2 (1-p)^r \sum_{n=0}^{\infty} \left( \frac{n+r+1}{r+1} \right) p^n \\
+ (15k+24-9\alpha)rp (1-p)^r \sum_{n=0}^{\infty} \left( \frac{n+r}{r} \right) p^n \\
+ 6(1-\alpha) \sum_{n=0}^{\infty} \left( \frac{n+r-1}{r-1} \right) (1-p)^r p^n - 6(1-\alpha) (1-p)^r \\
+ 2(k+1)s(s+1)(s+2)q^3 (1-q)^s \sum_{n=0}^{\infty} \left( \frac{n+s+2}{s+2} \right) q^n \\
+ (11k+9+2\alpha)s(s+1)q^2 (1-q)^s \sum_{n=0}^{\infty} \left( \frac{n+s+1}{s+1} \right) q^n \\
+ (9k+6+3\alpha)sq (1-q)^s \sum_{n=2}^{\infty} \left( \frac{n+s}{s} \right) q^n \right\}
\]

\[
= \frac{1}{6} \left\{ \frac{2(k+1)r(r+1)(r+2)p^3}{(1-p)^3} + \frac{(13k+15-2\alpha)r(r+1)p^2}{(1-p)^2} \\
+ \frac{(15k+24-9\alpha)rp}{1-p} + 6(1-\alpha) - 6(1-\alpha) (1-p)^r \\
+ \frac{2(k+1)s(s+1)(s+2)q^3}{(1-q)^3} + \frac{(11k+9+2\alpha)s(s+1)q^2}{(1-q)^2} \\
+ \frac{(9k+6+3\alpha)sq}{1-q} \right\}.
\]
The last expression bounded above by \((1 - \alpha)\) by the given condition (3.4).

The proof of the following theorem is similar to those of the previous theorems so we state only the result.

**Theorem 3.5.** If the inequality \((1 - p)^r + (1 - q)^s \geq 1 + |b_1|\) is hold, then \(P^r,s_{p,q}(HS^*(k, \alpha)) \subset HS^*(k, \alpha)\).

---

**References**

1. Ahuja, O.P., Aghalary, R., Joshi, S.B.: *Harmonic univalent functions associated with k-uniformly starlike functions*. Math. Sci. Res. J. 9 (1), 9-17 (2005).
2. Clunie, J., Sheil-Small, T.: *Harmonic univalent functions*. Ann. Acad. Sci. Fenn. Ser. A I Math. 9, 3-25 (1984).
3. Duren, P.: Harmonic Mappings in the Plane. Cambridge University Press. Cambridge (2004).
4. Goodman, A.W.: *On uniformly convex functions*. Ann. Polon. Math. 56 (1), 87-92 (1991).
5. Kim, Y.C., Jahangiri, J.M., Choi, J.H.: *Certain convex harmonic functions*. Int. J. Math. Sci. 29 (8), 459-465 (2002).
6. Ronning, F.: *Uniformly convex functions and a corresponding class of starlike functions*. Proc. Amer. Math. Soc. 118, 189-196 (1993).
7. Rosy, T., Stephen, B.A., Subramanian, K.G., Jahangiri, J.M.: *Goodman-Rønning-type harmonic univalent functions*. Kyungpook Math. J. 41(1) 45-54 (2001).
8. Altunkaya, Ş., Yalçın, S.: *Poisson Distribution Series for Certain Subclasses of Starlike Functions with Negative Coefficients*. Annals of Oradea University Mathematics Fascicola 2. 24 (2), 5-8 (2017).
9. Porwal, S., Kumar, M.: *A unified study on starlike and convex functions associated with Poisson distribution series*. Afrika Matematika. 27 (5-6), 1021-1027 (2016).
10. Porwal, S.: *An application of a Poisson distribution series on certain analytic functions*. J. Complex Anal. Article ID 984135, 1-3 (2014).
11. Nazeer, W., Mehmood, Q., Kang, S.M., Haq, A.U.: *An application of Binomial distribution series on certain analytic functions*. Journal of Computational Analysis and Applications. 26 (1), 11-17 (2019).
12. Çakmak, S., Yalçın, S., Altunkaya, Ş.: *Some connections between various classes of analytic functions associated with the power series distribution*. Sakarya Univ. Jour. Science. 23 (5), 982-985 (2019).
13. El-Deeb, M.S., Bulboaca, T., Dziok, J.: *Pascal Distribution Series connected with certain subclasses of univalent functions*. Kyunpook Math. J. 59, 301-314 (2019).

---

**Affiliations**

**ELIF YAŞAR**  
**ADDRESS:** Bursa Uludag University, Faculty of Arts and Science, Dept. of Mathematics, 16059, Bursa-Turkey.  
**E-MAIL:** elifyasar@yahoo.com  
**ORCID ID:** 0000-0003-0176-4961