DIFFUSION IN FLUID FLOW:
DISSIPATION ENHANCEMENT BY FLOWS IN 2D

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Abstract. We consider the advection-diffusion equation
\[ \phi_t^A + Au \cdot \nabla \phi^A = \Delta \phi^A, \quad \phi^A(0, x) = \phi_0(x) \]
on \( \mathbb{R}^2 \) (and on the strip \( \mathbb{R} \times \mathbb{T} \)), with \( u \) a periodic incompressible flow and \( A \gg 1 \) its amplitude. We provide a sharp characterization of all \( u \) that optimally enhance dissipation in the sense that for any initial datum \( \phi_0 \in L^p(\mathbb{R}^2), p < \infty \), and any \( \tau > 0 \),
\[ \| \phi^A(\cdot, \tau) \|_{L^\infty(\mathbb{R}^2)} \to 0 \quad \text{as} \quad A \to \infty. \]

Our characterization is expressed in terms of simple geometric and spectral conditions on the flow. Moreover, if the above convergence holds, it is uniform for \( \phi_0 \) in the unit ball of \( L^p(\mathbb{R}^2), p < \infty \), and \( \| \cdot \|_{L^\infty} \) can be replaced by any \( \| \cdot \|_q, q > p \). Extensions to higher dimensions and applications to reaction-advection-diffusion equations are also considered.

1. Introduction

In the present paper we study the influence of fast incompressible advection on diffusion. We will be mainly interested in the case of periodic flows on unbounded two-dimensional domains. More precisely, we will consider the passive scalar equation
\[ \phi_t^A + Au \cdot \nabla \phi^A = \Delta \phi^A, \quad \phi^A(x, 0) = \phi_0(x) \tag{1.1} \]
on the plane \( D = \mathbb{R}^2 \) or on the strip \( D = \mathbb{R} \times \mathbb{T} \), and with initial datum \( \phi_0 \in L^p(D) \) for some \( p < \infty \). Here \( u \) is a periodic divergence-free vector field and the parameter \( A \in \mathbb{R} \) accounts for its amplitude. We will be interested in the behavior of the solution \( \phi^A \) of (1.1) at fixed times \( \tau > 0 \) in the regime of large \( A \).

The problem of diffusion of passive scalars in the presence of a flow is one with a long history. It has been studied in both mathematical and physical literature, with applications to various areas of science and engineering. The long time behavior of the solutions of (1.1) for a fixed \( A \) is by now well-understood and, in particular, one has for each \( \phi_0 \),
\[ \| \phi^A(\cdot, t) \|_{L^\infty} \to 0 \quad \text{as} \quad t \to \infty. \tag{1.2} \]
The question of determining finer properties of the solutions has been addressed within the framework of homogenization theory, which identifies an effective diffusion equation as a suitable long time–large space limit of the solution of (1.1). The further question of the dependence of the corresponding effective diffusion matrix on the amplitude of the flow has been investigated by many authors, and we refer to [5, 17] and references therein for further details.
In contrast to these issues, the present work is interested in the strong flow limit $A \to \infty$ at finite times. This problem is one addressed by the Freidlin-Wentzell theory \cite{6,7,8,9} which applies to a class of Hamiltonian flows in $\mathbb{R}^2$. It shows the convergence of the solution of (1.1) to that of an effective diffusion equation on the Reeb graph of the hamiltonian, which is obtained by essentially collapsing each streamline of the flow to a point. However, the Freidlin-Wentzell method requires certain non-degeneracy and growth assumptions on the stream function and is not applicable to periodic flows.

Our main interest is in the question which flows enhance the dissipative effect of diffusion in the most efficient manner — so that they achieve the convergence in (1.2) on any fixed time scale as $A$ (rather than $t$) becomes large. That is, we want to identify those flows which satisfy

$$\|\phi^A(\cdot, \tau)\|_{L^\infty} \to 0 \quad \text{as } A \to \infty$$

for any $\phi_0 \in L^p(D)$ and any $\tau > 0$ (we will call them dissipation-enhancing). It turns out that this problem can be approached via methods very different from those above, pioneered in a recent work of Constantin, Kiselev, Ryzhik, and the author \cite{3}. That paper has studied the question of influence of advection on diffusion in the simpler setting of bounded domains and compact Riemannian manifolds (of any dimension) and we briefly review here the most relevant literature.

Long time behavior of solutions of (1.1) on bounded domains $D$ with Dirichlet boundary conditions at $\partial D$ has been investigated in many works. It is well known (see, e.g. \cite{10}) that the asymptotic rate of decay of the solution of (1.1) in this setting is given by the principal eigenvalue $\lambda^A_0$ of the corresponding elliptic operator $-\Delta + A u \cdot \nabla$. More precisely, we have

$$t^{-1} \log \|\phi^A(\cdot, t)\|_{L^2} \to -\lambda^A_0 \quad \text{as } t \to \infty.$$  

The question of dependence of $\lambda^A_0$ on $A$ has been addressed in the papers \cite{10,11,12,13} by Kifer (he actually considers $-\varepsilon \Delta + u \cdot \nabla$ and the related small diffusion problem). Using probabilistic methods, Kifer has obtained estimates on $\lambda^A_0$ for large $A$ under certain smoothness assumptions on $u$.

More recently and using PDE methods, Berestycki, Hamel, and Nadirashvili \cite{1} have characterized those flows $u$ for which $\lambda^A_0 \to \infty$ as $A \to \infty$ (the limit $\lim_{A \to \infty} \lambda^A_0$ in the opposite case is also determined via a variational principle). These are those that have no non-zero first integrals (i.e., solutions of $u \cdot \nabla \psi \equiv 0$) in $H^1_0(D)$. Moreover, \cite{1} shows that (1.3) with $L^2$ in place of $L^\infty$ holds precisely for these flows. This can be shown to imply (1.3) using Lemma 5.4 below, thus answering our basic question in the setting of bounded domains with Dirichlet boundary conditions.

However, the situation is quite different in the case of unbounded domains, when the equivalent of $\lambda^A_0$, the bottom of the spectrum of $-\Delta + A u \cdot \nabla$, is always zero. In the light of this fact, the problem on bounded domains with Neumann boundary conditions or on compact manifolds (when the principal eigenvalue is also always zero) seems more relevant to our investigation. This is precisely the focus of \cite{3}. In this setting the average of the solution of (1.1) over $D$ (which has a finite volume) stays constant and we are therefore interested in the enhancement of the speed of relaxation of $\phi^A$ to this average $\bar{\phi}_0$. That is,
one wants to characterize the flows for which

\[ \| \phi^A(\cdot, \tau) - \phi_0 \|_{L^\infty} \to 0 \quad \text{as } A \to \infty \]  

(1.5)

for each \( \phi_0 \in L^p(D) \) and \( \tau > 0 \). It has been proved in [3] that these relaxation-enhancing flows are precisely those for which the operator \( u \cdot \nabla \) has no non-constant eigenfunctions in \( H^1(D) \). The method is based on a functional-analytic approach and spectral techniques (the RAGE theorem, in particular), and the proof of an abstract result concerning evolution equations in a Hilbert space governed by the coupling of a dissipative evolution to a fast unitary evolution, with the latter having no “slowly dissipating” eigenfunctions (see Theorem 2.3 below). It is worth noting that [3] does not address the relation of the property (1.5) to the spectrum of \( -\Delta + Au \cdot \nabla \). It seems natural that the relevant quantity to look at should be the real part of the “second” eigenvalue (essentially the spectral gap), but it appears that currently very little is known about this problem.

As mentioned before, our main goal is to extend the above results to the non-compact setting of unbounded domains. Having [3] at hand, one may consider the following idea. Let \( \phi_0 \) have a compact support and assume that a periodic flow \( u \) on \( \mathbb{R}^n \) (let the period be 1 in all directions) is relaxation-enhancing on all scales. That is, \( u \) is relaxation-enhancing on each compact manifold \( M_k = (k\mathbb{T})^n \). Since the average of \( \phi_0 \) over \( M_k \) decays to zero as \( k \to \infty \), large \( k \) and \( A \) together with (1.5) will make \( \| \phi^A(\cdot, \tau) \|_{L^\infty} \) as small as desired when \( \phi^A \) solves (1.1) on \( M_k \). The comparison principle ensures that the solution on \( \mathbb{R}^n \) is dominated by that on \( M_k \) and so (1.3) follows. Moreover, one can show (see Lemma 5.3 below) that flows that are relaxation-enhancing on a single scale are also relaxation-enhancing on all other scales. Thus we obtain the result of Theorem 9.1 below that all flows which are relaxation-enhancing on their cell of periodicity are also dissipation-enhancing on \( \mathbb{R}^n \) (Theorem 9.1 also covers the more general case of space- and time-periodic flows).

It turns out, however, that these flows are only some of the periodic dissipation-enhancing ones on \( \mathbb{R}^n \). Indeed, it has been showed in [2, 15] that all generic shear (i.e., unidirectional) flows satisfy (1.3). But no shear flow, when considered on \( \mathbb{T}^n \), is relaxation-enhancing. The property (1.5) is quite strong and requires the flow to have certain mixing properties which are not possessed by shear flows whose streamlines preserve all but one coordinate. The issue here is that when \( k \) is large, the flow need not make \( \phi^A \) “evenly distributed” over all of \( M_k \) in order to make \( \| \phi^A \|_{L^\infty} \) small.

Nevertheless, we are able to provide a full characterization of the dissipation-enhancing flows on \( \mathbb{R}^2 \) and \( \mathbb{R} \times \mathbb{T} \) (Theorem 2.1). We do so by employing the above technique of periodization of the original problem and considering it on the large tori \( M_k \), along with other ideas and results. Most notably, we prove a generalization of the abovementioned abstract Hilbert space result from [3] that allows the existence of slowly dissipating eigenfunctions of the fast unitary evolution (see Theorem 2.4 and its time-periodic version Theorem 4.1). Somewhat surprisingly, it turns out that this characterization can still be stated in terms of a simple condition concerning \( H^1(C) \) eigenfunctions of the operator \( u \cdot \nabla \), with \( C \) the (compact) cell of periodicity of \( u \), plus the requirement that no open bounded set be invariant under \( u \). This time, however, the condition excludes only the existence of \( H^1(C) \) eigenfunctions of \( u \cdot \nabla \) with non-zero eigenvalues. Any \( H^1(C) \) first integrals are allowed, not only the constant ones.
We note that in more than two dimensions we only obtain the result of Theorem 9.1 mentioned above. In particular, a characterization of (periodic incompressible) dissipation-enhancing flows on \( \mathbb{R}^n, n \geq 3 \), remains open.

A part of our motivation to study dissipation enhancement by flows on fixed time scales comes from applications to quenching in reaction-diffusion equations (see, e.g., [2, 3, 4, 15, 20, 22]). In this case we consider (1.1) with an ignition-type non-negative non-linear reaction term added to the right-hand side (see (8.1)), and the question is which flows are able to extinguish (quench) any initially compactly supported reaction, provided their amplitude is large enough. Our main result here is Theorem 8.2 (and its extension to some strictly positive non-linearities, Theorem 8.3), which shows that outside of the class of flows that do have \( H^1(\mathcal{C}) \) eigenfunctions other than the first integrals but none of them belongs to \( C^{1,1}(\mathcal{C}) \), these strongly quenching flows are precisely the dissipation-enhancing ones.

The rest of the paper is organized as follows. In Section 2 we state our main result, Theorem 2.1 as well as the abstract Hilbert space result, Theorem 2.4, which is an important step in the proof. In Section 3 we prove Theorem 2.4 and in Section 4 we state and prove its time-periodic version, Theorem 4.1. Sections 5 and 6 contain the proof of Theorem 2.1, and Section 7 extends it to the case of the strip \( \mathbb{R} \times (0,1) \) with Dirichlet or Neumann boundary conditions, along with providing some examples. In Section 8 we prove an application of our main result to quenching in reaction-diffusion equations, and Section 9 extends some of our dissipation-enhancement and quenching results to space- and time-periodic flows in all dimensions.

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2. Statements of the Main Results

Let us start with some definitions. Let \( u \) be a periodic, incompressible, Lipschitz flow on the domain \( D = \mathbb{R}^n \times \mathbb{T}^m \), and let \( \mathcal{C} \) be its cell of periodicity with each couple of opposite \((n+m-1)\)-dimensional “faces” identified, so that \( \mathcal{C} \) is a blown-up \((m+n)\)-dimensional torus. Then \( u \) defines a unitary evolution \( \{U_t\}_{t \in \mathbb{R}} \) on \( L^2(\mathcal{C}) \) (and also on \( L^2(D) \)) in the following manner. For each \( x \in \mathcal{C} \) there is a unique solution \( X(x,t) \) to the ODE

\[
\frac{d}{dt}X(x,t) = u(X(x,0)), \quad X(x,0) = x.
\]  

(2.1)

We then let

\[
(U_t \psi)(x) = \psi(X(x,-t))
\]

for any \( \psi \in L^2(\mathcal{C}) \). Incompressibility of \( u \) implies that the group \( \{U_t\}_{t \in \mathbb{R}} \) is unitary, and its generator is the operator \(-iu \cdot \nabla\). It is self-adjoint on \( L^2(\mathcal{C}) \) and for each \( \psi \in H^1(\mathcal{C}) \) we have

\[
i \frac{d}{dt}(U_t \psi) = -iu \cdot \nabla(U_t \psi).
\]

(2.2)

If \( \psi \in L^2(\mathcal{C}) \) is an eigenfunction of the anti-self-adjoint operator \( u \cdot \nabla \) (i.e., \( u \cdot \nabla \psi = i\lambda \psi \) for some \( \lambda \in \mathbb{R} \)), and therefore also an eigenfunction of each \( U_t = e^{-i(u \cdot \nabla) t} \), we say that \( \psi \) is
an eigenfunction of the flow $u$ on $C$. The eigenfunctions $\psi$ of $u$ that correspond to eigenvalue zero (i.e., $u \cdot \nabla \psi \equiv 0$) are called the first integrals of $u$.

We also say that a set $V \subseteq D$ is invariant under the flow $u$ if and only if $X(x, t) \in V$ for all $t \in \mathbb{R}$ whenever $x \in V$.

Finally, if $v$ is an incompressible Lipschitz flow on $D$, we let $P_t(v)$ be the solution operator for the equation

$$
\psi_t + v \cdot \nabla \psi = \Delta \psi, \quad \psi(0) = \psi_0
$$

on $D$. That is, $P_t(v)\psi_0 = \psi(\cdot, t)$ when $\psi$ solves (2.3).

We can now state our main result.

**Theorem 2.1.** Let $u$ be a periodic, incompressible, Lipschitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R} \times \mathbb{T}$ with a cell of periodicity $C$, and let $\phi^A$ solve (1.1) on $D$. Then the following are equivalent.

(i) For some $1 \leq p \leq q \leq \infty$ and each $\tau > 0$, $\phi_0 \in L^p(D)$,

$$
\|\phi^A(\cdot, \tau)\|_{L^q(D)} \to 0 \quad \text{as} \quad A \to \infty.
$$

(ii) For any $1 \leq p \leq q \leq \infty$ such that $p < \infty$ and $q > 1$, and each $\tau > 0$, $\phi_0 \in L^p(D)$,

$$
\|\phi^A(\cdot, \tau)\|_{L^q(D)} \to 0 \quad \text{as} \quad A \to \infty.
$$

(iii) For any $1 \leq p < q \leq \infty$ and each $\tau > 0$,

$$
\|P_\tau(Au)\|_{L^p(D) \to L^q(D)} \to 0 \quad \text{as} \quad A \to \infty.
$$

(iv) No bounded open subset of $D$ is invariant under $u$ and any eigenfunction of $u$ on $C$ that belongs to $H^1(C)$ is a first integral of $u$.

**Remarks.** 1. The couples $p, q$ in the theorem are the only ones for which the corresponding claims can possibly hold. The conclusion of (ii) and (iii) cannot hold for $p > q$ because then $P_\tau$ does not map $L^p$ to $L^q$. As for $p = q$, note that since $\frac{d}{dt} \int_D \psi \, dx \equiv 0$ for solutions of (2.3) when $v$ is incompressible, the $L^1$ norm of non-negative solutions does not decay. This and the fact that $\psi \equiv 1$ is a constant solution mean that (ii) cannot hold for $p = q \in \{1, \infty\}$.

The above arguments and the maximum principle give

$$
\|P_\tau(v)\|_{L^p \to L^q} = 1
$$

for $p \in \{1, \infty\}$. Interpolation extends (2.7) to all $p$ and so (iii) cannot hold for any $p = q$.

2. The claim (iii) means that for $p < q$, the decay in (ii) is uniform for $\|\phi_0\|_{L^p} \leq 1$. In particular, taking $p = 1$ and $q = \infty$ yields a characterization of the (periodic incompressible Lipschitz) flows for which the corresponding heat kernel $k_{Au}(x, y, t)$ on $D$ satisfies

$$
\|k_{Au}(\cdot, \cdot, \tau)\|_{L^\infty} \to 0 \quad \text{as} \quad A \to \infty
$$

for each $\tau > 0$. Namely, these are the flows from Theorem 2.1 (iv). One direction of our proof — (iv) $\Rightarrow$ (i), (ii), (iii) in Section 5 — will actually only concentrate on the case $(p, q) = (1, \infty)$, since the others will follow by (2.7) and interpolation.

3. In the case of the strip $D = \mathbb{R} \times (0, 1)$ we only consider periodic boundary conditions here. The result remains unchanged (with $C$ the surface of a cylinder rather than a torus) if Dirichlet or Neumann boundary conditions are assumed and $u(x) \cdot (0, 1) = 0$ for $x \in \partial D$. 

See Section 7 below, which also provides examples demonstrating that the two conditions in (iv) are independent in general.

4. Notice that for some $u$ (e.g., vertical shear flows), the first condition in (iv) is satisfied when $D = \mathbb{R}^2$ but not when $D = \mathbb{R} \times \mathbb{T}$.

**Definition 2.2.** We will call the flows that satisfy (2.6) *dissipation-enhancing* on $D$.

It is natural to ask what makes the first integrals different from eigenfunctions corresponding to a non-zero eigenvalue. The answer is essentially the fact that the existence of a single $H^1$ eigenfunction corresponding to eigenvalue $\lambda \neq 0$ implies the existence of infinitely many eigenspaces of $u$ with $H^1$ eigenfunctions — those corresponding to all integer multiples of $\lambda$. This can be seen from the proof of Lemma 5.3 below.

We will see from the proof of Theorem 2.1 that condition (iv) essentially tells us that the flow $Au$ quickly “stretches” any initial datum $\phi_0$ and exposes it to diffusion, thus enhancing the dissipation rate as much as desired when $A$ is large. (More specifically, the $H^1$ norm of $\phi^A$ becomes large.) One might therefore think that a sufficient condition for a flow to not be dissipation-enhancing would be the existence of a stable solution (not just a stable orbit!) of (2.1). It is not difficult to show using our methods that if (2.1) on $C$ has no dense orbits, then this is indeed the case. However, the claim is not true in general. We will not provide the details here, but a counterexample can be obtained in the following manner. One first constructs a 1-periodic flow $u$ whose spectrum is \( \{ n + m\alpha \mid n, m \in \mathbb{Z} \} \) for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and such that all the eigenfunctions of $u$ except of the constant function belong to $C(\mathbb{T}^2) \setminus H^1(\mathbb{T}^2)$. This can be done using Example 2 in Section 6 of \[3\], with the obtained flow smoothly isomorphic to a reparametrization of the constant flow $(\alpha, 1)$ (and therefore no bounded subset of $D$ is invariant under the flow). The construction goes back to Kolmogorov’s work \[16\] and is based on the existence of a smooth function $Q : \mathbb{T} \to \mathbb{T}$ with $\int_{\mathbb{T}} Q(\xi) d\xi = 1$ and $\alpha \in \mathbb{T}$ such that the homology equation (6.2) in \[3\]

\[
R(\xi + \alpha) - R(\xi) = Q(\xi) - 1,
\]

has a solution $R \in H^{1+\varepsilon}(\mathbb{T}) \setminus H^1(\mathbb{T}) \subseteq C(\mathbb{T}) \setminus H^1(\mathbb{T})$. This is possible when $\alpha$ is Liouville-lean, that is, very well approximated by rationals. The (continuous) eigenfunctions of the constructed flow have absolute value one and $u$ is non-zero everywhere. Our analysis in Section 6 below can then be used to show that all solutions of (2.1) are stable. Nevertheless, Theorem 2.1 shows that $u$ is dissipation-enhancing.

As we have mentioned in the Introduction, the proof of Theorem 2.1 crucially uses an abstract result concerning dissipative evolution in a Hilbert space. This is a generalization of Theorem 1.4 in \[3\] and we now state both results.

Let $\Gamma$ be a self-adjoint, non-negative, unbounded operator with a discrete spectrum on a separable Hilbert space $\mathcal{H}$. Let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $\Gamma$ (so that $\lambda_1 \geq 0$ and $\lambda_n \to \infty$) and let $\kappa_j$ be the corresponding orthonormal eigenvectors forming a basis in $\mathcal{H}$. The Sobolev space $H^m(\Gamma)$ associated with $\Gamma$ is formed by all vectors $\psi = \sum_j c_j \kappa_j$ such that

\[
\|\psi\|_{H^m(\Gamma)} \equiv \left( \sum_j \lambda_j^m |c_j|^2 \right)^{1/2} < \infty.
\]
This is the homogeneous Sobolev semi-norm (which is a norm if \( \lambda_1 > 0 \)), and the Sobolev norm is defined by \( \| \cdot \|_{H^m(\Gamma)} = \| \cdot \|^2_{H^m(\Gamma)} + \| \cdot \|^2_{L^2} \). Note that the domain of \( \Gamma \) is \( H^2(\Gamma) \).

Let \( L \) be a self-adjoint operator on \( \mathcal{H} \) such that, for any \( \psi \in H^1(\Gamma) \) and \( t > 0 \) we have
\[
\| L\psi \|_\mathcal{H} \leq C \| \psi \|_{H^1(\Gamma)} \quad \text{and} \quad \| e^{iLt} \psi \|_{H^1(\Gamma)} \leq B(t) \| \psi \|_{H^1(\Gamma)}
\]  
(2.9)
where the constant \( C < \infty \) and the function \( B(t) \in L^2_{\text{loc}}[0, \infty) \) are independent of \( \psi \). Here \( e^{iLt} \) is the unitary evolution group generated by the self-adjoint operator \( L \). It has been shown in [3] that the two conditions in (2.9) are independent in general.

Finally, let \( \phi^A(t) \) be a solution of the Bochner differential equation
\[
\frac{d}{dt} \phi^A(t) = iAL\phi^A(t) - \Gamma\phi^A(t), \quad \phi^A(0) = \phi_0.
\]  
(2.10)
Then we have the following result from [3].

**Theorem 2.3.** Let \( \Gamma \) be a self-adjoint, positive, unbounded operator with a discrete spectrum and let a self-adjoint operator \( L \) satisfy conditions (2.9). Then the following are equivalent.

(i) For any \( \tau, \delta > 0 \) and \( \phi_0 \in \mathcal{H} \) there exists \( A_0(\tau, \delta, \phi_0) \) such that for any \( A > A_0(\tau, \delta, \phi_0) \), the solution \( \phi^A(t) \) of (2.10) satisfies \( \| \phi^A(\tau) \|_\mathcal{H}^2 < \delta \).

(ii) For any \( \tau, \delta > 0 \) there exists \( A_0(\tau, \delta) \) such that for any \( A > A_0(\tau, \delta) \) and any \( \phi_0 \in \mathcal{H} \) with \( \| \phi_0 \|_\mathcal{H} \leq 1 \), the solution \( \phi^A(t) \) of (2.10) satisfies \( \| \phi^A(\tau) \|_\mathcal{H}^2 < \delta \).

(iii) The operator \( L \) has no eigenfunctions belonging to \( H^1(\Gamma) \).

**Remark.** Note that the theorem says that if \( A_0 \) above exists, it is independent of \( \phi_0 \) inside the unit ball in \( \mathcal{H} \). This is the same as the equivalence of Theorem 2.1(ii) and (iii).

If one takes \( \Gamma \equiv -\Delta \) and \( L \equiv iu \cdot \nabla \), both restricted to the space of mean-zero \( L^2 \) functions, then (2.10) is (1.1) and this result can be applied to the study of fast relaxation by flows on compact manifolds or on bounded domains \( D \) (where \( \Gamma > 0 \) has a discrete spectrum). It obviously only provides \( L^2 \rightarrow L^2 \) estimates, but after coupling these with Lemma 5.4 below, one can obtain the characterization of relaxation-enhancing flows on \( D \) from [3] mentioned in the Introduction.

In the light of our earlier observation that on unbounded domains not all periodic flows satisfying (1.3) are relaxation-enhancing on their cell of periodicity, a natural next question is what happens to the dissipative dynamics (2.10) when some eigenfunctions of \( L \) do lie in \( H^1(\Gamma) \). We denote by \( P_h \) the projection onto the closed subspace \( P_h \mathcal{H} \subseteq \mathcal{H} \) generated by all such eigenfunctions. That is, \( P_h \mathcal{H} \) is the closure of the set of all linear combinations of those eigenfunctions of \( L \) which lie in \( H^1(\Gamma) \). Notice that \( P_h \mathcal{H} \) need not be contained in \( H^1(\Gamma) \) since the latter is not closed in general. Now we can provide the following answer.

**Theorem 2.4.** Let \( \Gamma \) be a self-adjoint, non-negative, unbounded operator with a discrete spectrum and let a self-adjoint operator \( L \) satisfy conditions (2.9). Then for any \( \tau, \delta > 0 \) there exists \( A_0(\tau, \delta) \) such that for any \( A > A_0(\tau, \delta) \) and any \( \phi_0 \in \mathcal{H} \) with \( \| \phi_0 \|_\mathcal{H} \leq 1 \), the Lebesgue measure of the set of times \( t \) for which the solution \( \phi^A(t) \) of (2.10) satisfies
\[
(\| (I - P_h)\phi^A(t) \|_\mathcal{H}^2 \geq \delta)
\]  
(2.11)
is smaller than \( \tau \). Moreover, if \( \dim(P_h \mathcal{H}) < \infty \), then \( (\| (I - P_h)\phi^A(t) \|_\mathcal{H}^2 \geq \delta) \) for all \( t > \tau \).
Remarks. 1. That is, if $A$ is large, any solution starting in the unit ball in $\mathcal{H}$ will spend a lot of time $\sqrt{\delta}$-close to the subspace $P_h\mathcal{H}$. We will actually show that this is even true for some $(\tau, \delta)$-dependent finite-dimensional subspace of $P_h\mathcal{H} \cap H^1(\Gamma)$ (see the proof).

2. It follows from Lemma 3.3 below that $P_h\mathcal{H}$ is the smallest closed subspace (and its unit ball the smallest closed subset) of $\mathcal{H}$ for which a result like this holds. In this sense, Theorem 2.4 is not only natural but also optimal.

3. It remains an open problem whether the evolution stays close to $P_h\mathcal{H}$ for all $t > \tau$ when $\dim(P_h\mathcal{H}) = \infty$ and $A$ is large. We conjecture that this is the case. A related interesting problem is to find the $A \to \infty$ asymptotics of the evolution (2.10) and determine whether one recovers an effective evolution equation on the subspace $P_h\mathcal{H}$ in this way.

4. We allow here $\Gamma \geq 0$ rather than $\Gamma > 0$ (Theorem 2.3 can also be extended to this case). In the proof of Theorem 2.1 we will take $\Gamma \equiv -\Delta$ and $L \equiv iu \cdot \nabla$ on $\mathcal{H} \equiv L^2(\mathcal{M}_k)$ (with $\mathcal{M}_k$ from the Introduction), rather than just the mean-zero $L^2$ functions.

3. The Abstract Result

In this section we prove Theorem 2.4. Following [3], we reformulate (2.10) as a small diffusion–long time problem. By setting $\varepsilon = A^{-1}$ and rescaling time by the factor $1/\varepsilon$, we pass from considering (2.10) to considering

$$\frac{d}{dt}\phi^\varepsilon(t) = (iL - \varepsilon\Gamma)\phi^\varepsilon(t), \quad \phi^\varepsilon(0) = \phi_0.$$  \hspace{1cm} (3.1)

We now want to show that for all small enough $\varepsilon > 0$ the measure of times for which (2.11) holds (with $A$ replaced by $\varepsilon$) is smaller than $\tau/\varepsilon$.

We will be comparing this dissipative dynamics to the “free” one given by

$$\frac{d}{dt}\phi^0(t) = iL\phi^0(t), \quad \phi^0(0) = \phi_0,$$  \hspace{1cm} (3.2)

so that $\phi^0(t) = e^{itL}\phi_0$. Notice that if $L \equiv iu \cdot \nabla$, then this is just

$$\phi^0_t + u \cdot \nabla \phi^0 = 0, \quad \phi^0(x,0) = \phi_0(x),$$  \hspace{1cm} (3.3)

that is, (2.2) with $\phi_0 \equiv \psi$ and $\phi^0(x,t) \equiv (U_t\psi)(x) = \phi_0(X(x,-t))$.

For the sake of convenience, in the remainder of this section we will denote the norm $\| \cdot \|_{\mathcal{H}}$ by $\| \cdot \|$, the space $H^m(\Gamma)$ by $H^m$, and the semi-norm $\| \cdot \|_{H^m(\Gamma)}$ by $\| \cdot \|_m$.

We begin with collecting some preliminary results from [3].

Lemma 3.1. Assume that (2.9) holds.

(i) For $\varepsilon \geq 0$ and $\phi_0 \in H^1$ there exists a unique solution $\phi^\varepsilon(t)$ of (3.1) on $[0, \infty)$. If $\varepsilon > 0$, then for any $T < \infty$,

$$\phi^\varepsilon(t) \in L^2([0,T], H^2) \cap C([0, T], H^1), \quad \frac{d}{dt} \phi^\varepsilon(t) \in L^2([0,T], \mathcal{H}).$$  \hspace{1cm} (3.4)

If $\varepsilon = 0$, then for any $T < \infty$,

$$\phi^0(t) \in L^2([0,T], H^1) \cap C([0, T], \mathcal{H}), \quad \frac{d}{dt} \phi^0(t) \in L^2([0,T], \mathcal{H})$$
(ii) We have
\[ \frac{d}{dt} \| \phi^\varepsilon(t) \|^2 = -2\varepsilon \| \phi^\varepsilon(t) \|_1^2 \] (3.5)
for a.e. \( t \), and hence
\[ \| \phi^\varepsilon(t) \|^2 \leq \| \phi_0 \|^2 \quad \text{and} \quad \int_0^\infty \| \phi^\varepsilon(t) \|^2 dt \leq \frac{\| \phi_0 \|^2}{2\varepsilon}. \] (3.6)

(iii) If \( \phi^\varepsilon \) and \( \phi^0 \) solve (3.1) and (3.2), respectively, with \( \phi_0 \in H^1 \), then
\[ \frac{d}{dt} \| \phi^\varepsilon(t) - \phi^0(t) \|^2 \leq \frac{\varepsilon}{2} \| \phi^0(t) \|_1^2 \leq \frac{\varepsilon}{2} B(t)^2 (\| \phi_0 \|_1^2 + \| \phi_0 \|^2) \] (3.7)
for a.e. \( t \). In particular
\[ \| \phi^\varepsilon(t) - \phi^0(t) \|^2 \leq \frac{\varepsilon}{2} (\| \phi_0 \|_1^2 + \| \phi_0 \|^2) \int_0^{T_0} B(t)^2 dt \] (3.8)
for any \( t \leq T_0 \).

Remarks. 1. We consider here \( \phi^\varepsilon \) to be a solution of (3.1) if it is continuous and satisfies (3.1) for a.e. \( t \).

2. The solution \( \phi^\varepsilon \) also exists for any \( \phi_0 \in H \), but then it may be rougher on time intervals containing 0 when \( \varepsilon > 0 \), and everywhere when \( \varepsilon = 0 \). We will only need to consider \( \phi_0 \in H^1 \) in the proof of Theorem 2.4. This is because \( H^1 \) is dense in \( H \), and the norm of the difference of two solutions of (3.1) with the same \( \varepsilon \) cannot increase due to (3.6).

Notice that according to (3.5), the rate of decrease of \( \| \phi^\varepsilon \|^2 \) is proportional to \( \| \phi^\varepsilon \|_1^2 \). This illuminates the following result from which Theorem 2.4 will follow.

**Theorem 3.2.** Consider the setting of Theorem 2.4. Then for any \( \tau, \delta > 0 \) there exists \( \varepsilon_0(\tau, \delta) > 0 \) and \( T_0 = T_0(\tau, \delta) \) such that for any \( \varepsilon \in (0, \varepsilon_0(\tau, \delta)) \), any \( \phi_0 \in H^1 \) with \( \| \phi_0 \| \leq 1 \), and any \( t \geq 0 \), the solution \( \phi^\varepsilon \) of (3.1) satisfies at least one of the following:

(a) \[ \| \phi^\varepsilon(t) \|_1^2 > \frac{1}{\tau}; \] (3.9)

(b) \[ \int_{t_0}^{t_0 + t} \| \phi^\varepsilon(s) \|_1^2 ds \geq \frac{T_0}{\tau}; \] (3.10)

(c) \[ \| (I - P_h)\phi^\varepsilon(t) \|^2 < \delta \quad \text{and neither (a) nor (b) holds.} \] (3.11)

Remark. This result in the absence of \( H^1 \)-eigenfunctions was the cornerstone of the proof of Theorem 1.4 in [3].

**Proof of Theorem 2.4 given Theorem 3.2.** Let \( t_1 \geq 0 \) be the first time such that Theorem 3.2(b) holds for \( t = t_1 \), let \( t_2 \geq t_1 + T_0 \) be first such time after \( t_1 + T_0 \), etc. Thus we obtain a sequence of times \( t_j \) such that \( t_{j+1} \geq t_j + T_0 \) and (b) holds for \( t = t_j \). If \( J_1 = \bigcup_j [t_j, t_j + T_0] \), then (b) does not hold for any \( t \in \mathbb{R}^+ \setminus J_1 \) by construction. Let \( J_2 \) be the set of all \( t \in \mathbb{R}^+ \setminus J_1 \).
for which (a) holds and let \( J_0 \equiv J_1 \cup J_2 \), so that neither (a) nor (b) holds for \( t \in \mathbb{R}^+ \setminus J_0 \). Therefore (c) holds for these \( t \), and so (2.11) (with \( \varepsilon \) in place of \( A \)) can only hold for \( t \in J_0 \). From the definition of \( J_0 \) we have that

\[
\int_{J_0} \| \phi^c(t) \|^2 dt \geq \frac{|J_0|}{\tau}.
\]

From (3.6) we obtain \(|J_0| \leq \tau/2\varepsilon < \tau/\varepsilon\). This proves the first claim in Theorem 2.4 when \( \phi_0 \in H^1 \). As explained above, the case \( \phi_0 \in \mathcal{H} \) is immediate from the density of \( H^1 \) in \( \mathcal{H} \).

Let \( \{ \phi_n \}_{n \in \mathbb{N}} \) be an orthonormal basis of \( P_h \mathcal{H} \) with each \( \phi_n \) an \( H^1 \) eigenfunction of \( L \). Notice that each \( \phi^c(t) \) satisfying Theorem 3.2(c) belongs to

\[
K \equiv \left\{ \phi \big| \| \phi \| \leq 1, \| \phi \|_1 \leq \frac{1}{\tau}, \text{ and } \|(I - P_h)\phi\| \leq b \right\}.
\]

This set is compact and hence so is \( P_h K \subseteq P_h \mathcal{H} \). Each element of \( K \) is \( \sqrt{\delta} \)-close to \( P_h K \), and compactness shows that there is \( n_0 = n_0(\tau, \delta) < \infty \) such that each element of \( P_h K \) is \( \sqrt{\delta} \)-close to the subspace with basis \( \{ \phi_n \}_{n=1}^{n_0} \). Replacing \( \delta \) by \( \delta/4 \) proves the claim of Remark 1 after Theorem 2.4.

Finally, assume \( \dim(P_h \mathcal{H}) < \infty \). Then \( P_h \mathcal{H} \subseteq H^1 \) and there must be \( b < \infty \) such that \( \| \phi \|_1 \leq b \| \phi \| \) for all \( \phi \in P_h \mathcal{H} \). By Lemma 3.3 below, there is \( \tau_1 \equiv (\delta/4b)^2 \) such that for all \( \varepsilon > 0 \) and all \( \phi \in P_h \mathcal{H} \) with \( \| \phi \|_1 \leq 1 \), the solution of (3.1) with initial condition \( \phi_1 \) stays \( \sqrt{\delta} \)-close to \( P_h \mathcal{H} \) on the time interval \([0, \tau_1/\varepsilon]\).

Now change \( \tau \) to \( \min\{\tau, \tau_1\} \), and change \( \varepsilon_0(\tau, \delta) \) accordingly. The first claim of Theorem 2.4 says that for any \( \phi_0 \) with \( \| \phi_0 \|_1 \leq 1 \) and any \( t > \tau/\varepsilon \) there is \( t_0 \in [t - \tau/\varepsilon, \tau] \) and \( \phi_1 \in P_h \mathcal{H} \) with \( \| \phi_1 \|_1 \leq 1 \) such that \( \| \phi^c(t_0) - \phi_1 \| < \sqrt{\delta} \). But then from (3.6) and \( t - t_0 \leq \tau/\varepsilon \leq \tau_1/\varepsilon \),

\[
\text{dist}(\phi^c(t), P_h \mathcal{H}) \leq \| \phi^c(t) - \phi_1(t-t_0) \| + \text{dist}(\phi_1(t-t_0), P_h \mathcal{H}) < 2\sqrt{\delta},
\]

where \( \phi_1^c \) is the solution of (3.1) with initial condition \( \phi_1 \). Again, replacing \( \delta \) by \( \delta/4 \) gives the second claim of Theorem 2.4.

\[\square\]

**Lemma 3.3.** Let \( \phi^c \) and \( \phi^0 \) solve (3.1) and (3.2), respectively, and assume there is \( b < \infty \) such that \( \| \phi^0(t) \|_1 \leq b \| \phi_0 \| \) for all \( t \geq 0 \). Then for all \( t \),

\[
\| \phi^c(t) - \phi^0(t) \|^2 \leq 4b\sqrt{\varepsilon t} \| \phi_0 \|^2.
\]

**Proof.** With \( \langle \cdot, \cdot \rangle \) the inner product in \( \mathcal{H} \), we have using self-adjointness of \( L \),

\[
\left| \frac{d}{dt} \langle \phi^c, \phi^0 \rangle \right| = \left| \langle -\varepsilon \Gamma \phi^c, \phi^0 \rangle \right| \leq \varepsilon \| \phi^0 \|_1 \| \phi^c \|_1 \leq \varepsilon b \| \phi^c \|_1 \| \phi_0 \| \leq \frac{\varepsilon b}{c} \| \phi^c \|^2_1 + \varepsilon bc \| \phi_0 \|^2
\]

for any \( c > 0 \). Therefore by (3.6),

\[
\| \phi^c(t) - \phi^0(t) \| \geq \| \phi_0 \|^2 - \int_0^t \frac{\varepsilon b}{c} \| \phi^c(s) \|^2_1 + \varepsilon bc \| \phi_0 \|^2 ds \geq \| \phi_0 \|^2 - b \| \phi_0 \|^2 \left( \frac{1}{2c} + \varepsilon ct \right).
\]

Choosing \( c = (\varepsilon t)^{-1/2} \) gives

\[
\langle \phi^c(t), \phi^0(t) \rangle \geq \| \phi_0 \|^2 (1 - 2b\sqrt{\varepsilon t}),
\]

\[\square\]
and \( \| \phi^\varepsilon(t) \| \leq \| \phi^0(t) \| = \| \phi_0 \| \) then implies
\[
\| \phi^\varepsilon(t) - \phi^0(t) \|^2 \leq \| \phi_0 \|^2 4b\sqrt{\varepsilon}t
\]
whenever \( 2b\sqrt{\varepsilon}t \leq 2 \). But for \( 2b\sqrt{\varepsilon}t \geq 2 \) this is obvious from (3.6). \( \square \)

We devote the rest of this section to the proof of Theorem 3.2.

Proof of Theorem 3.2. We let \( P_c \) and \( P_p \) be the spectral projections in \( \mathcal{H} \) onto the continuous and pure point spectral subspaces of \( L \), respectively. We also let \( e_j \) be the eigenvalues of \( L \) and \( P_j \) the projection onto the eigenspace corresponding to \( e_j \). Finally, let \( Q_N \) be the projection onto the subspace generated by the eigenfunctions \( \kappa_1, \ldots, \kappa_N \) corresponding to the first \( N \) eigenvalues of \( \Gamma \). We note that [3] had the last notation exchanged (i.e., \( Q_j \) and \( P_N \)), but we feel that it is more transparent to reserve the letter \( P \) for those associated with \( L \), and \( Q \) for those associated with \( \Gamma \).

Take \( \varepsilon > 0 \) and let us assume that \( \| \phi^\varepsilon(t_0) \|_1^2 \leq 1/\tau \) and \( \| (I - P_h)\phi^\varepsilon(t_0) \|^2 \geq \delta \). We then need to show that (b) holds with \( t = t_0 \) provided \( \varepsilon \in (0, \varepsilon_0) \), with \( \varepsilon_0 = \varepsilon_0(\tau, \delta) \) and \( T_0 = T_0(\tau, \delta) \) to be determined later. To simplify notation we rename \( \phi^\varepsilon(t_0) \) to \( \phi_0 \) and \( \phi^\varepsilon(t_0 + t) \) to \( \phi^\varepsilon(t) \) so that \( \phi^\varepsilon(t) \) solves (3.11) and we have
\[
\| \phi_0 \|^2 \leq 1, \quad \| \phi_0 \|_1 \leq \frac{1}{\tau}, \quad \text{and} \quad \| (I - P_h)\phi_0 \|^2 \geq \delta.
\] (3.12)

We now need to show
\[
\int_0^{T_0} \| \phi^\varepsilon(t) \|^2 dt \geq \frac{T_0}{\tau}
\] (3.13)
in order to conclude (b), which is what we will do.

The idea, partially borrowed from [3], is as follows. We let \( \phi^0(t) \equiv e^{iLt}\phi_0 \) solve (3.2) and note that (3.8) guarantees \( \phi^\varepsilon(t) \) to be close to \( \phi^0(t) \) for all \( t \leq T_0 \) as long as \( \varepsilon \) is sufficiently small. As a result we will be left with studying the free dynamics \( \phi^0(t) \). We will show, in an averaged sense over \([0, T_0]\), that its pure point part \( P_p\phi^0(t) \) will “live” in low and intermediate modes of \( \Gamma \) (i.e., in \( Q_N\mathcal{H} \) for some \( N < \infty \)) with a large \( H^1 \) norm there if \( \| (P_p - P_h)\phi_0 \|^2 \geq \delta/2 \). On the other hand the continuous part \( P_c\phi^0(t) \) will live in high modes (i.e., in \( (I - Q_N)\mathcal{H} \)), and thus also have large \( H^1 \) norm if \( \| P_c\phi_0 \|^2 \geq \delta/2 \). Since \( I - P_h = P_c + (P_p - P_h) \), (3.12) will ensure (3.13) for both the free and dissipative dynamics. The key to these conclusions will be the compactness of the set of \( \phi_0 \) satisfying (3.12). The main point is that the pure point and continuous parts of the free dynamics effectively “decouple” into different modes of \( \Gamma \) and therefore do not cancel out each other’s contribution to the \( H^1 \) norm. The next three lemmas, the first and third of which are essentially from [3], make the above heuristic rigorous.

Lemma 3.4. Let \( K \subset \mathcal{H} \) be a compact set. For any \( N, \omega > 0 \), there exists \( T_c(N, K, \omega) \) such that for all \( T \geq T_c(N, K, \omega) \) and any \( \phi \in K \), we have
\[
\frac{1}{T} \int_0^T \| Q_N e^{iLt}P_c\phi \|^2 dt < \omega.
\] (3.14)
This is the “uniform RAGE theorem” for a compact set of vectors \([3]\). It says that if we wait long enough, the continuous part of the free dynamics starting in \(K\) escapes into the high modes of \(\Gamma\) in a time average. The next lemma shows that the pure point dynamics stays in low and intermediate modes.

**Lemma 3.5.** Let \(K \subset \mathcal{H}\) be a compact set. For any \(\omega > 0\), there exists \(N_p(K, \omega)\) such that for any \(N \geq N_p(K, \omega)\), \(\phi \in K\), and \(t \in \mathbb{R}\), we have

\[
\|(I - Q_N)e^{itL}P_p\phi\|^2 < \omega. \tag{3.15}
\]

**Proof.** Let \(\{\phi_n\}\) be an orthonormal basis of \(P_p\mathcal{H}\) such that each \(\phi_n\) is an eigenfunction of \(L\) with eigenvalue \(e_{j(n)}\). Since \(P_pK\) is compact, it has a finite \(1/k\) net for any \(k \in \mathbb{N}\). Moreover, this net can be chosen so that each its element is a finite linear combination of the \(\phi_n\), since these are dense in \(P_p\mathcal{H}\) (of course, it may happen that some elements of this net are not in \(P_pK\)). Let the net be \(\{\sum_{n=1}^{m_0} \alpha_{n,m,n} \phi_n\}^\infty_{m=1}\). Since \(e^{itL}\) is unitary, \(R \equiv \bigcup_{t \in \mathbb{R}} \{\sum_{n=1}^{m_0} e^{itj(n)\alpha_{n,m,n}} \phi_n\}^\infty_{m=1}\) is a 1/k net for \(K' \equiv \bigcup_{t \in \mathbb{R}} e^{itL}P_pK\). Let \(\alpha \geq \sup |\alpha_{n,m,n}|\) be an integer and \(S \equiv \frac{2\pi}{\max(1,2,\ldots,4n\alpha k)}\). Then

\[
\bigcup_{q_{m,n} \in S} \bigg\{ \sum_{n=1}^{m_0} e^{iq_{m,n}\alpha_{n,m,n}} \phi_n \bigg\}
\]

is a finite \(1/k\) net for \(R\), and thus a \(2/k\) net for \(K'\). Since \(k\) was arbitrary, \(K'\) must be compact. We have that \(I - Q_N\) converges strongly to zero as \(N \to \infty\), and so there must be \(N\) such that \(\|(I - Q_N)\phi\|^2 < \omega\) for all \(\phi \in K'\). \(\square\)

Finally, we show that the \(H^1\) norm of the pure point part of the free dynamics will become large provided \(P_p\phi_0\) is sufficiently “rough”. Recall that \(P_j\) are the projections onto the eigenspaces of \(L\).

**Lemma 3.6.** Let \(K \subset \mathcal{H}\) be a compact set and \(\Omega \subset \mathbb{R}\) be such that each \(\phi \in K\) satisfies

\[
\sum_j \|P_j\phi\|^2 \geq 3\Omega \quad \text{(the sum may be equal to } \infty) \text{. Then there exists } N_1(K, \Omega) \text{ and } T_1(K, \Omega) \text{ such that for all } N \geq N_1(K, \Omega) \text{ and } T \geq T_1(K, \Omega) \text{ we have}
\]

\[
\frac{1}{T} \int_0^T \|Q_N e^{itL} P_p \phi\|^2 dt > \Omega. \tag{3.16}
\]

This is almost identical to Lemma 3.3 in \([3]\) (which only treats the case \(\sum_j \|P_j\phi\|^2 = \infty\)) and the proof carries over. Namely, one first shows using compactness that there is \(N\) such that for all \(\phi \in K\) we have \(\sum_j \|Q_N P_j\phi\|^2 \geq 2\Omega\). Then one shows

\[
\left| \frac{1}{T} \int_0^T \|Q_N e^{itL} P_p \phi\|^2 dt - \sum_j \|Q_N P_j\phi\|^2 \right| \to 0
\]

as \(T \to \infty\), with the convergence being uniform on compacts. We refer to \([3]\) for details.
We are now ready to finish the proof of Theorem 3.2. Recall that we need to show (3.13), assuming (3.12). Let
\[
K_0 \equiv \left\{ \phi \mid \| \phi \|^2 \leq 1 \text{ and } \| \phi \|_1 \leq \frac{1}{\tau} \right\},
\]
\[
K_1 \equiv K_0 \cap \left\{ \phi \mid \| (P_p - P_h) \phi \|^2 \geq \frac{\delta}{2} \right\},
\]
(3.17)
\[
N_p = N_p \left( K_0, \frac{\delta}{12} \right),
\]
\[
N_1 = N_1 \left( K_1, \frac{10}{\tau} \right),
\]
(3.18)
\[
N_2 = \min \left\{ N \mid \lambda_N \geq \frac{48}{\tau \delta} \right\},
\]
\[
N_c = \max \{ N_p, N_1, N_2 \},
\]
\[
\omega = \min \left\{ \frac{\delta}{48}, \frac{1}{\tau \lambda_{N_1}} \right\},
\]
\[
T_0 = T_0(\tau, \delta) \geq \max \left\{ T_1 \left( K_1, \frac{10}{\tau} \right), T_c(N_c, K_0, \omega) \right\},
\]
(3.19)
\[
\varepsilon < \varepsilon_0(\tau, \delta) \equiv \frac{2\tau}{1 + \tau} \left( \int_0^{T_0} B(t)^2 \, dt \right)^{-1} \omega.
\]
Note that $K_0$ is compact and hence so is $K_1$. Also, $N_1$ and $T_1$ are well defined because if $\phi \in K_1$, then $P_n \phi \notin P_h \mathcal{H}$ for some $n$ by the definition of $K_1$. Since $P_n \phi$ is an eigenfunction of $L$, we have $P_n \phi \notin H^1$, and so $\sum_j \| P_j \phi \|^2_1 = \infty$ (note that we do not claim $P_p \phi \notin H^1$). This suggests that we could have used the version of Lemma 3.6 from [3] (with $3\Omega$ replaced by $\infty$). We shall see later that the current form will be necessary in the proof of Theorem 2.1.

From (3.12) we know that either $\| (P_p - P_h) \phi_0 \|^2 \geq \delta/2$ or $\| P_c \phi_0 \|^2 \geq \delta/2$. Assume the former. Then $\phi_0 \in K_1$ and so by Lemma 3.6,
\[
\frac{1}{T_0} \int_0^{T_0} \| Q_{N_1} e^{iLt} P_p \phi_0 \|^2 \, dt \geq \frac{10}{\tau}.
\]
We also know from Lemma 3.4 and 3.19 that
\[
\frac{1}{T_0} \int_0^{T_0} \| Q_{N_1} e^{iLt} P_c \phi_0 \|^2 \, dt \leq \frac{1}{T_0} \int_0^{T_0} \| Q_{N_1} e^{iLt} P_c \phi_0 \|^2 \, dt \leq \omega \leq \frac{1}{\tau \lambda_{N_1}}
\]
and so
\[
\frac{1}{T_0} \int_0^{T_0} \| Q_{N_1} e^{iLt} P_c \phi_0 \|^2 \, dt \leq \frac{1}{\tau}.
\]
It follows using the triangle inequality for $\| \cdot \|_1$ and $(a - b)^2 \geq \frac{1}{2} a^2 - b^2$ that
\[
\frac{1}{T_0} \int_0^{T_0} \| Q_{N_1} \phi_0(t) \|^2 \, dt \geq \frac{4}{\tau}.
\]
From (3.8) and (3.12) we know that
\[ \| \phi^\varepsilon(t) - \phi^0(t) \|_2^2 \leq \frac{\varepsilon}{2} \left( \frac{1}{\tau} + 1 \right) \int_0^{T_0} B(t)^2 \, dt \leq \omega \] (3.20)
for \( t \leq T_0 \), and so
\[ \frac{1}{T_0} \int_0^{T_0} \| Q_{N_1}(\phi^\varepsilon(t) - \phi^0(t)) \|_1^2 \, dt \leq \lambda_{N_1} \omega \leq \frac{1}{\tau}. \]
Using again \((a - b)^2 \geq \frac{1}{2}a^2 - b^2\) yields
\[ \frac{1}{T_0} \int_0^{T_0} \| Q_{N_1}(\phi^\varepsilon(t)) \|_1^2 \, dt \geq \frac{1}{\tau} \]
and (3.13) follows.

Next we assume (3.12) and \( \| P_c \phi_0 \|_2^2 \geq \delta/2 \). Since \( \phi_0 \in K_0 \), Lemma 3.4 gives
\[ \frac{1}{T_0} \int_0^{T_0} \| (I - Q_{N_c}) e^{itL} P_c \phi_0 \|_2^2 \, dt > \frac{\delta}{2} - \omega > \frac{\delta}{3}. \]
Lemma 3.5 and \( N_c \geq N_p \) give
\[ \frac{1}{T_0} \int_0^{T_0} \| (I - Q_{N_c}) e^{itL} P_p \phi_0 \|_2^2 \, dt < \frac{\delta}{12}, \]
so we obtain
\[ \frac{1}{T_0} \int_0^{T_0} \| (I - Q_{N_c}) \phi^0(t) \|_2^2 \, dt > \frac{\delta}{12}. \]
Applying (3.20) yields
\[ \frac{1}{T_0} \int_0^{T_0} \| (I - Q_{N_c}) \phi^\varepsilon(t) \|_2^2 \, dt > \frac{\delta}{24} - \omega \geq \frac{\delta}{48} \]
and so
\[ \frac{1}{T_0} \int_0^{T_0} \| (I - Q_{N_c}) \phi^\varepsilon(t) \|_1^2 \, dt > \frac{\delta}{48} \lambda_{N_c} \geq \frac{\delta}{48} \lambda_{N_2} \geq \frac{1}{\tau}. \]
Again (3.13) follows and the proof of Theorem 3.2 is complete. \( \square \)

4. THE TIME-PERIODIC CASE

Theorem 2.3 has a natural extension to the case of time-periodic family of operators \( L_t \) in place of \( L \) \[14\]. We provide here the corresponding extension of Theorem 2.4.

Let \( \Gamma \) be as before and let \( L_t \) be a periodic family of self-adjoint operators on \( \mathcal{H} \) such that for some \( C < \infty \), all \( \psi \in H^1(\Gamma) \), and all \( t \in \mathbb{R} \),
\[ \| L_t \psi \|_{\mathcal{H}} \leq C \| \psi \|_{H^1(\Gamma)}. \] (4.1)
Without loss of generality assume \( L_t \) has period 1. Let \( \{ U_t \}_{t \in \mathbb{R}} \) be a strongly continuous family of unitary operators on \( \mathcal{H} \) such that for any \( \phi_0 \in H^1(\Gamma) \), the function \( \phi^0(t) \equiv U_t \phi_0 \) satisfies
\[ \frac{d}{dt} \phi^0(t) = iL_t \phi^0(t) \] (4.2)
for almost every $t$. Notice that if $L_t \equiv L$ is constant, then $U_t = e^{iLt}$. Finally, assume there is a locally bounded function $B(t)$ such that for any $\psi \in H^1(\Gamma)$ and $t \in \mathbb{R}$,

$$
\|U_t \psi\|_{H^1(\Gamma)} \leq B(t)\|\psi\|_{H^1(\Gamma)}.
$$

(4.3)

We denote by $P_k$ the projection onto the closed subspace $P_k \mathcal{H} \subseteq \mathcal{H}$ generated by all $H^1(\Gamma)$ eigenfunctions of $U_1$ (these coincide with those of $L$ when $L_t \equiv L$) and we let $\phi_k(t)$ be the solution of

$$
\frac{d}{dt} \phi_k(t) = iAL \phi_k(t) - \Gamma \phi_k(t), \quad \phi_k(0) = \phi_0.
$$

(4.4)

Note that this is the right choice of the fast dissipative evolution to consider since the orbits of the fast free evolution $\frac{d}{dt}\phi(t) = iALt \phi(t)$ coincide with those of (4.2).

**Theorem 4.1.** Let $\Gamma$ be a self-adjoint, non-negative, unbounded operator with a discrete spectrum and let $L_t$ and $U_t$ satisfy conditions $\text{(4.1)} - \text{(4.3)}$. Then for any $\tau, \delta > 0$ there exists $A_0(\tau, \delta)$ such that for any $A > A_0(\tau, \delta)$ and any $\phi_0 \in \mathcal{H}$ with $\|\phi_0\|_{\mathcal{H}} \leq 1$, the Lebesgue measure of the set of times $t \geq 0$ for which the solution $\phi_k(t)$ of (4.4) satisfies

$$
\| (I - P_k)U_t^* \phi_k(t) \|^2_{\mathcal{H}} \geq \delta
$$

is smaller than $\tau$.

**Remark.** Let $U_{s,t} \equiv U_tU_t^*$ with $U_t^*$ the adjoint of $U_t$. Then $B(t), B(-t) < \infty$ and periodicity of $L_t$ guarantee that $U_t = U_{0,t}$ maps $H^1$ eigenfunctions of $U_1 = U_{0,1}$ onto those of $U_{t,t+1}$, and that $U_{t,0} = U_t^*$ maps $H^1$ eigenfunctions of $U_{t,t+1}$ onto those of $U_1$. Hence $P_{t,h} \equiv U_tP_h$ is the projection on the subspace of $\mathcal{H}$ generated by all $H^1$ eigenfunctions of $U_{t,t+1}$, and so

$$
\| (I - P_h)U_t^* \phi \| = \| (I - P_{t,h})\phi \|.
$$

This illuminates (4.3). Notice also that $P_{t+1,h} = P_{t,h}$ by definition.

We will now sketch the proof. It follows the lines of the proof of Theorem 2.4 and uses the method from [14] to deal with the time-dependence of $L_t$. The point is to obtain Theorem 3.2 with (c) replaced by

$$
\| (I - P_h)U_t^* \phi(t) \|^2 < \delta \quad \text{and neither (a) nor (b) holds}
$$

(4.6)

(from which Theorem 4.1 follows immediately). This is done in two steps.

First we fix any $\gamma \in [0, 1)$. We then obtain Theorem 3.2 with (b) replaced by

$$
\sum_{n=0}^{T_0-2} \| \phi^\gamma([t] + \gamma + n) \|^2 \geq \frac{2T_0}{\tau}
$$

(4.7)

and (c) by (4.8). Here $[t]$ is the least integer not smaller than $t$ (and we let $\beta \equiv [t] - t$), and the obtained $T_0, \epsilon_0$ additionally depend on $\gamma$. The proof extends directly with the following changes. After renaming $\phi^\gamma(t)$ to $\phi_0$, we replace all integrals $\int_0^{T_0} \ldots dt$ in the proof by the sums $\sum_{n=0}^{T_0-2}$, with the argument $t$ inside the integrals replaced by the argument $\beta + \gamma + n$.
inside the sums. The role of $\phi_0$ is then played by $\phi^0(\beta + \gamma)$, and that of $e^{itL} U_{\gamma\gamma+1}$ (since $\phi^0(\beta + \gamma + n) = U_{\gamma\gamma+1}^n \phi^0(\beta + \gamma)$). The assumption (3.12) now reads

$$\|\phi_0\|^2 \leq 1, \quad \|\phi_0\|^2_1 \leq \frac{1}{\tau}, \quad \text{and} \quad \|(I - P_h)U^*_t \phi_0\|^2 \geq \delta,$$

which together with

$$\|(I - P_{\gamma,h}) \phi^0(\beta + \gamma)\| = \|U^*_t(I - P_h)U^*_t \phi^0(\beta + \gamma)\| = \|(I - P_h)U^*_t \phi^0(\beta + \gamma)\| = \|(I - P_h)U^*_t \phi_0\|$$

guarantees

$$\|\phi^0(\beta + \gamma)\|^2 \leq 1, \quad \|\phi^0(\beta + \gamma)\|^2_1 \leq \frac{b}{\tau}, \quad \text{and} \quad \|(I - P_{\gamma,h}) \phi^0(\beta + \gamma)\|^2 \geq \delta,$$

where $b \equiv \sup_{t \in [0,1]} B(t)$. From this (4.7) follows as in Section 3, with the definitions of $K_0$ and $K_1$ involving $\|\phi_0\|^2 \leq b/\tau$ and $\|(P_{\gamma,h} - P_h)\|\phi_0\|^2 \geq \delta/2$, respectively, and with $\tau$ replaced by $\tau/2$ in order to account for the extra factor of two in (4.7).

Next we notice that we can actually pick $T_0, \varepsilon_0$ uniformly for all $\gamma$ inside a set $G$ of measure $\frac{1}{2}$. This is because the maximum in (3.19) is finite for each $\gamma$, and so the same $T_0$ (and hence the same $\varepsilon_0$) can be chosen for all $\gamma$ outside of a set of a small measure. Integrating (4.7) over $G$ now gives Theorem 3.2 with (a) and (b) the same as in (3.9) and (3.10), and (c) replaced by (4.6). This finishes the proof.

5. **Proof of Theorem 2.1** Part I

We devote the next two sections to the proof of Theorem 2.1. We will consider $D = \mathbb{R} \times \mathbb{T}$ and since the case $D = \mathbb{R}^2$ is almost identical, we will just indicate along the way where adjustments for this setting are required. We will also assume that $u$ has period one in each coordinate, that is, $C = \mathbb{T}^2$. The general case is again identical.

In this section we prove that if Theorem 2.1(iv) holds, then so do parts (i)–(iii). Let us therefore assume that the 1-periodic incompressible Lipschitz flow $u$ leaves no open bounded subset of $D$ invariant and has no $H^1(\mathbb{T}^2)$ eigenfunctions except possibly with eigenvalue zero (i.e., first integrals). We will then show that Theorem 2.1(i)–(iii) hold.

Let us start with a description of the main idea. Fix any $\tau, \delta > 0$ and let $\|\phi_0\|_{L^2} \leq 1$ (we will actually take $\|\phi_0\|_{L^1} \leq 1$ to obtain the desired $L^1 \rightarrow L^\infty$ bounds). As mentioned in the Introduction, we periodize the domain and consider the solution $\phi^A$ of (1.1) on $\mathcal{M} \equiv k\mathbb{T} \times \mathbb{T}$ with $k \gg 1$ depending on $\tau, \delta$ (we use $\mathcal{M} \equiv (k\mathbb{T})^2$ when $D = \mathbb{R}^2$). Here $k\mathbb{T}$ for $k \in \mathbb{N}$ is the interval $[0,k]$ with 0 and $k$ identified, and our $\phi^A$ on $\mathcal{M}$ will dominate the $\phi^A$ on $D$. We will show that on $\mathcal{M}$ the flow $u$ also cannot have $H^1$ eigenfunctions other than the first integrals (i.e., the operator $u \cdot \nabla$ on $\mathcal{M}$ can only have $H^1(\mathcal{M})$ eigenfunctions with eigenvalue zero). We will then show that if $k$ and $A$ are large enough, $\|\phi^A(\tau)\|_{L^\infty}$ will be small.

To this end, we notice that we are now in the setting of our main abstract result, because the Laplacian on $\mathcal{M}$ has a discrete spectrum. Theorem 3.2 shows that $\|\phi^A\|_{L^2}$ will decay quickly (when $A$ is large) as long as $\|\phi^A\|_{H^1}$ stays large. The theorem says that this can only be prevented by $\phi^A$ becoming close to an $H^1$ first integral $\psi$ of $u$. 
If \( \| \psi \|_{L^\infty} \) is small, we will be done after using Lemma 5.4 below to take care of \( \phi^A - \psi \) (which is small). If, on the other hand, \( \| \psi \|_{L^\infty} \) is large, then we will show that \( \psi \) has to be large on a long streamline of \( u \). More precisely, we will show using \( \dim(M) = 2 \) that under our hypotheses \( \psi \) has to be continuous, constant on the streamlines of \( u \) on \( M \), and that long streamlines must be dense.

As a result, we will obtain that \( \| \psi \|_{H^1} \) is large (again using \( \dim(M) = 2 \)). From \( \| \phi^A - \psi \|_{L^2} \) being small we will then show that \( \| \phi^A \|_{H^1} \) must also be large (which is obviously not true as stated and we will actually have to take an alternate route here). Thus the fast decay of \( \| \phi^A \|_{L^2} \) will continue until \( \| \phi^A \|_{L^\infty} \) is small. Since this fast decay can only be sustained for a short time due to \( \| \phi_0 \|_{L^2} \leq 1 \), we will indeed obtain that \( \| \phi^A(\tau) \|_{L^\infty} \) is small. Lemma 5.4 and interpolation will take care of the rest.

In what follows we make this heuristic rigorous. We will start with proving some of the above statements as auxiliary lemmas.

We will need to use a stream function for \( u \). This is a function \( U \in C^1(D) \) with values in \( \mathbb{R} \) if \( D = \mathbb{R}^2 \) and in \( a\mathbb{T} \) for some \( a > 0 \) if \( D = \mathbb{R} \times \mathbb{T} \) such that

\[
U(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2)) = \nabla^\perp U(x_1, x_2) \equiv \left( -\frac{\partial}{\partial x_2}U(x_1, x_2), \frac{\partial}{\partial x_1}U(x_1, x_2) \right).
\]

If \( D = \mathbb{R}^2 \), then we can take

\[
U(x_1, x_2) \equiv \int_0^{x_1} u_2(s, 0) \, ds - \int_0^{x_2} u_1(x_1, s) \, ds,
\]

which satisfies (5.1) because \( u \) is incompressible and so

\[
\int_0^{x_1} u_2(s, 0) \, ds - \int_0^{x_2} u_1(x_1, s) \, ds = -\int_0^{x_2} u_1(0, s) \, ds + \int_0^{x_1} u_2(s, x_2) \, ds.
\]

For the same reason and from periodicity of \( u \) we also have that \( \tilde{a} \equiv U(x_1, x_2 + 1) - U(x_1, x_2) \) is independent of \((x_1, x_2)\). Let \( a \equiv |\tilde{a}| \) if \( \tilde{a} \neq 0 \) and let \( a \) be any positive number otherwise. Changing \( U \) to \( (U \mod a\mathbb{Z}) \) gives a \( C^1 \) stream function with values in \( a\mathbb{T} \) which is 1-periodic in \( x_2 \), that is, a stream function on \( \mathbb{R} \times \mathbb{T} \). Without loss of generality we will assume \( a \equiv 1 \), as this can be achieved by changing \( u \) to \( a^{-1}u \). Note also that

\[
u \cdot \nabla U = 0
\]

by (5.1), so that \( U \) is constant on the streamlines of \( u \).

**Lemma 5.1.** Let \( u \) be a 1-periodic incompressible Lipschitz flow. Then \( u \) leaves no open bounded subset of \( D \) invariant if and only if the union of unbounded streamlines of \( u \) is a dense subset of \( D \).

**Proof.** If the unbounded streamlines of \( u \) are dense in \( D \), clearly no open bounded subsets of \( D \) are left invariant by the flow.

Assume now that the unbounded streamlines of \( u \) are not dense in \( D \) and let \( Y \subset D \) be open bounded and such that all streamlines intersecting \( Y \) are bounded. If \( u \equiv 0 \) on \( Y \), then \( Y \) is an open bounded set invariant under \( u \).
Lemma 5.2. Let $u(x_0) = 0$. This means that $0 \neq \nabla U(x_0) \perp u(x_0)$, and since $U \in C^1$, there is a neighborhood $V \subseteq Y$ of $x_0$ such that for each $y_0 \in V$ the set

$$\{ x \in V \mid U(x) = U(y_0) \}$$

is precisely the intersection of $V$ with the streamline passing through $y_0$. Pick $V$ small enough so that there is $t_0 > 0$ such that $X(V, t_0) \cap V = \emptyset$, with $X$ the solution of (2.1) on $D$. This is possible because $u$ is continuous. Finally, we let

$$W_0 \equiv \{ x \in V \mid |X(t, x)| \leq M \text{ for all } t \in \mathbb{R} \}$$

with $M$ large enough so that $|W_0| > 0$. This is possible because all streamlines intersecting $V$ are bounded.

Let $W_j \equiv X(W_0, jt_0)$, so that $W_j \subseteq B(x_0, M)$ and incompressibility of $u$ gives $|W_j| = |W_0| > 0$. Hence there must be $j < k$ with $W_j \cap W_k \neq \emptyset$, which in turn gives existence of $y_0 \in W_0 \cap W_m$ for $m \equiv k - j > 0$ (and then obviously we must have $m \geq 2$). So there is $y \in W_0$ such that $X(t, y) = y_0$ for some $t \in [(m - 1)t_0, (m + 1)t_0]$. But then $U(y) = U(y_0)$, and so $y$ must lie on the streamline through $y_0$. It follows that this non-trivial streamline $S$ is closed, that is, $X(y, \tau) = y$ for some $\tau \geq (m - 1)t_0 > 0$.

If $S$ is homotopic to a point, then it encloses an open bounded set invariant under $u$. If $S$ is not homotopic to a point (which can only happen if $D = \mathbb{R} \times \mathbb{T}$ and $S$ winds around it), we let $S' \equiv S + (1, 0)$. Since $S \neq S'$ due to the periodicity of $u$ and boundedness of $S$, the open bounded domain between $S$ and $S'$ is invariant under $u$. \hfill \square

Lemma 5.2. Let $u$ be an incompressible Lipschitz flow on $\mathcal{M} \equiv k\mathbb{T} \times l\mathbb{T}$ and let $\psi \in H^1(\mathcal{M})$ satisfy $u \cdot \nabla \psi \equiv 0$. Then $\psi$ is constant on each streamline of $u$ and continuous at each $x \in \mathcal{M}$ for which $u(x) \neq 0$. Moreover, if for some $\varepsilon > 0$ the union of streamlines of $u$ of diameter at least $\varepsilon$ is dense in $\mathcal{M}$, then $\psi$ is continuous.

Proof. Let $x_0 \in \mathcal{M}$ be such that $u(x_0) \neq 0$, let $v \perp u(x_0)$ have length 1, and set $x_s \equiv x_0 + sv$ for $s \in \mathbb{R}$. Define $g(t, s) \equiv X(x_s, t)$ with $X$ from (2.1). Since $u$ is Lipschitz, $g$ is a bilipschitz diffeomorphism between some neighborhoods of $0 \in \mathbb{R}^2$ and $x_0$. This means that the $H^1$ function $\omega(\cdot) \equiv \psi(g(\cdot))$ satisfies $(1, 0) \cdot \nabla \psi \equiv 0$. That is, $\omega(t, s) = \bar{\omega}(s)$ almost everywhere, with $\bar{\omega}$ an $H^1$ function of a single variable and so continuous on a neighborhood of 0. We conclude that $\psi$ is continuous on a neighborhood of $x_0$ (after possibly changing it on a measure-zero set). This means that $\psi$ is (equivalent to a function) continuous at each $x$ such that $u(x) \neq 0$. This and the dependency of $\omega$ on $s$ only means that $\psi$ is constant on all non-trivial streamlines. It is obviously constant on the trivial ones, too.

Next assume that the union of streamlines of diameter at least $\varepsilon > 0$ is dense in $\mathcal{M}$. It is sufficient to consider $\varepsilon = 1$, the general case is identical. The open set $R$ of all $x$ with $u(x) \neq 0$ is dense in $D$. We will now show that $\psi|_R$ can be continuously extended to $\mathcal{M}$. Assume the contrary, that is, there is $x_0 \in \mathcal{M}$ and $x_n, z_n \in R$ with $\lim x_n = \lim z_n = x_0$ such that either $\lim |\psi(x_n)| = \infty$ or $\lim \psi(x_n) \neq \lim \psi(z_n)$. We can assume without loss of generality that $x_n, z_n \in S$, the union of streamlines with diameter at least 1, because $S$ is dense in $R$ and $\psi$ is continuous on $R$. 

If \( \lim |\psi(x_n)| = \infty \), then for each \( M < \infty \) there is a curve joining the inner and outer perimeter of the annulus \( B_2 \equiv B(x_0, \frac{1}{2}) \setminus B(x_0, \frac{1}{4}) \) on which \( |\psi| \) is continuous and larger than \( M \). Namely, it is a part of the streamline going through \( x_n \in B(x_0, \frac{1}{4}) \) (which cannot be completely contained inside \( B(x_0, \frac{1}{2}) \), and on which \( \psi \) is constant). On the other hand, we have \( |J| \geq \frac{1}{8} \) where \( J \subseteq [\frac{1}{4}, \frac{1}{2}] \) is the set of all \( r \) for which the measure of all \( \theta \in [0, 2\pi] \) such that \( |\psi(x_0 + r\hat{e}^\theta)| \leq 4\|\psi\|_{L^2} \) is positive. But then

\[
\|\psi\|^2_{H^1} \geq \int_{B_2} |\nabla \psi|^2 \, dx \geq \int_{J} \int_0^{2\pi} \left( \frac{1}{r} \left| \frac{\partial \psi}{\partial \theta} \right| \right)^2 \, \rho \, d\theta \, dr \geq \int_{J} \frac{1}{2\pi r} \left( \int_0^{2\pi} \left| \frac{\partial \psi}{\partial \theta} \right| \, d\theta \right)^2 \, dr \geq \frac{(M - 4\|\psi\|_{L^2})^2}{8\pi}
\]

using the Schwartz inequality in the third step. Since the rightmost expression diverges as \( M \to \infty \), we have a contradiction.

If on the other hand \( \lim \psi(x_n) = L_1 \neq L_2 = \lim \psi(z_n) \), then for each \( n \in \mathbb{N} \) there must be two curves joining the inner and outer perimeters of the annulus \( B_n \equiv B(x_0, \frac{1}{2}) \setminus B(x_0, 2^{-n}) \), on which \( \psi \) is continuous and has constant values \( a_n \) and \( b_n \), respectively, with \( |a_n - b_n| \geq \frac{1}{2}|L_1 - L_2| \). A similar argument as above gives

\[
\|\psi\|^2_{H^1} \geq \int_{B_n} |\nabla \psi|^2 \, dx \geq \int_{2^{-n}}^{1/2} \frac{1}{r} \int_0^{2\pi} \left| \frac{\partial \psi}{\partial \theta} \right|^2 \, d\theta \, dr \geq \frac{|L_1 - L_2|^2}{2} \int_{2^{-n}}^{1/2} \frac{1}{2\pi r} \, dr,
\]

with a contradiction when \( n \to \infty \).

Hence \( \psi|_R \) has a continuous extension \( \omega \) to \( M \) and it remains to show \( \psi = \omega \) almost everywhere. Assume this is not the case and let \( x_0 \in M \) be a Lebesgue point of the set \( P_\varepsilon \) of all \( x \in \mathbb{T}^2 \) such that \( |\psi(x) - \omega(x)| > 2\varepsilon \) (by the hypothesis, \( |P_\varepsilon| > 0 \) for some \( \varepsilon > 0 \)). Then for some \( r > 0 \) and all \( x \in B(x_0, r) \) and \( z \in B(x_0, r) \cap P_\varepsilon \) we have

\[
|\psi(z) - \omega(x)| > \varepsilon \quad (5.3)
\]

because \( \omega \) is continuous. Since \( x_0 \) is a Lebesgue point of \( P_\varepsilon \), we have

\[
|B(x_0, r) \cap P_\varepsilon| \cdot |B(x_0, r)|^{-1} \to 1
\]
as \( r \to 0 \). Hence for any \( \delta > 0 \) and a small enough \( \delta \)-dependent \( r_0 \), there is a set \( J \) with \( |J| \geq (1 - \delta)r_0 \) of \( r \in [0, r_0] \) such that \( |\{ \theta | x_0 + r\hat{e}^\theta \in P_\varepsilon \}| > 0 \). Again we can find a curve joining the inner and outer perimeter of the annulus \( B \equiv B(x_0, r_0) \setminus B(x_0, \delta r_0) \) on which \( \psi \) is continuous and equal to \( \omega \), and an argument as above together with (5.3) gives

\[
\|\psi\|^2_{H^1} \geq \int_{B} |\nabla \psi|^2 \, dx \geq \int_{J} \int_0^{2\pi} \left| \frac{\partial \psi}{\partial \theta} \right|^2 \, d\theta \, dr \geq \varepsilon^2 \int_{2\delta r_0}^{r_0} \frac{1}{2\pi r} \, dr = \frac{\varepsilon^2}{2\pi} \log(2\delta).
\]

Taking \( \delta \to 0 \) yields a contradiction, so \( \psi \) must be continuous. \( \square \)

**Lemma 5.3.** Let \( u \) be a 1-periodic incompressible Lipschitz flow on \( \mathbb{R}^n \) and let \( M \equiv \prod_{j=1}^n k_j \mathbb{T} \) for some \( k_j \in \mathbb{N} \).

(i) If \( \psi \) is an \( H^1 \) eigenfunction of \( u \) on \( M \), then \( |\psi| \) is an \( H^1 \) eigenfunction of \( u \) on \( M \) with eigenvalue 0.

(ii) The flow \( u \) on \( M \) has an \( H^1 \) eigenfunction with a non-zero eigenvalue if and only if the same is true for \( u \) on \( \mathbb{T}^n \).
(iii) The flow \( u \) on \( \mathcal{M} \) has a non-constant \( H^1 \) eigenfunction if and only if the same is true for \( u \) on \( \mathbb{T}^n \).

Remark. The exclusion of constants in (iii) is natural as these are always eigenfunctions of \( u \). We will use this part in Section 9.

Proof. (i) is an easy computation using

\[
(\nabla \psi)(x) = \begin{cases} \frac{\psi(x) \nabla \psi(x) + \psi(x) \nabla \psi(x)}{2 |\psi(x)|} & \text{if } \psi(x) \neq 0, \\ 0 & \text{if } \psi(x) = 0, \end{cases}
\]

the fact that \( u \) is real, and that all eigenvalues of \( u \cdot \nabla \) are purely imaginary.

(ii) Let us first consider the case \( \mathcal{M} = 2\mathbb{T} \times \mathbb{T}^{n-1} \). If \( \psi \) is an \( H^1 \) eigenfunction of \( u \) on \( \mathbb{T}^n \), then \( \phi(x_1, x') \equiv \psi(\{x_1\}, x') \) is obviously an \( H^1 \) eigenfunction on \( \mathcal{M} \) with the same eigenvalue (here \( \{x_1\} \) is the fractional part of \( x_1 \) and \( x' = (x_2, \ldots, x_n) \)). This proves one implication.

Let us now assume \( \psi \) is an \( H^1 \) eigenfunction of \( u \) on \( \mathcal{M} \) with eigenvalue \( i\lambda \in i\mathbb{R} \), and define \( \psi_e(x_1, x') \equiv \frac{1}{2} [\psi(x_1, x') + \psi(x_1 + 1, x')] \) and \( \psi_o(x_1, x') \equiv \frac{1}{2} [\psi(x_1, x') - \psi(x_1 + 1, x')] \). Periodicity of \( u \) shows that \( \psi_e, \psi_o \) are also \( H^1 \) eigenfunctions on \( \mathcal{M} \) with the same eigenvalue. If \( \psi_e \neq 0 \) then it is an \( H^1 \) eigenfunction on \( \mathbb{T}^n \) because it is 1-periodic. If \( \psi_e \equiv 0 \), then \( \psi_o \neq 0 \), and we let \( \phi \equiv \psi_o/|\psi_o| \). Again using (5.4) we find that \( \phi \) is an \( H^1 \) eigenfunction of \( u \) with eigenvalue \( 2i\lambda \). But \( \phi \) is 1-periodic (because \( \psi_o(x_1 + 1, x') = -\psi_o(x_1, x') \)), and so it is also an \( H^1 \) eigenfunction of \( u \) on \( \mathbb{T}^n \). Since \( i\lambda \) and \( 2i\lambda \) are either both zero or both non-zero, this proves (ii) for \( \mathcal{M} = 2\mathbb{T} \times \mathbb{T}^{n-1} \).

If now \( \mathcal{M} = k\mathbb{T} \times \mathbb{T}^{n-1} \), we use the same argument but with \( \psi_e, \psi_o \) replaced by

\[
\left\{ \psi_j(x_1, x') \equiv \frac{1}{k} \sum_{m=0}^{k-1} e^{\frac{2\pi i jm}{k}} \psi(x_1 + m, x') \right\}^{k-1}
\]

and \( \phi \equiv \psi_j/|\psi_j|^{k-1} \in H^1 \) associated to the eigenvalue \( k i \lambda \) (when \( \psi_j \neq 0 \)). The general case is treated by subsequently repeating this “unfolding” for each coordinate.

(iii) The proof is essentially identical to that of (ii) after noting that \( \psi_j/|\psi_j|^{k-1} \) cannot be a constant function when \( \psi_j \in H^1(\mathcal{M}) \) is non-constant. \( \square \)

The final lemma is based on [3], [4].

Lemma 5.4. For each \( p \in [1, \infty] \) and each integer \( d \geq 1 \) there exists \( C(d) \geq 1 \) such that for any \( D = \mathbb{R}^n \times \prod_{j=1}^m k_j \mathbb{T} \) with \( n + m = d \) and \( n, m \geq 0 \), any \( 1 \)-periodic incompressible flow \( v \in \text{Lip}(D) \), any \( \psi_0 \in L^1(D) \), and any \( t \leq 1 \) the solution of (2.3) on \( D \) satisfies

\[
\|\psi(\cdot, t)\|_{L^\infty(D)} \leq C(d) t^{-d/2p} \|\psi_0\|_{L^p(D)}. \tag{5.5}
\]

Proof. Interpolation and (2.7) for \( p = \infty \) imply that we only need to obtain (5.5) for \( p = 1 \).

Consider first \( d = 2 \). When \( D = \mathbb{T}^2 \) and \( \psi_0 \equiv |D|^{-1} \int_D \psi_0(x) \, dx = 0 \) (in which case \( \psi \) is mean zero at all times because the evolution given by (2.3) preserves its mean), then this is just Lemma 3.3 in [4]. That is,

\[
\|\psi(\cdot, t) - \tilde{\psi}_0\|_{L^\infty} \leq C t^{-d/2} \|\psi_0 - \tilde{\psi}_0\|_{L^1} \tag{5.6}
\]
for any \( \psi \). Using
\[
\max\{\|\tilde{\psi}_0\|_{L^\infty}, \|\tilde{\psi}_0\|_{L^1}\} \leq \|\psi_0\|_{L^1}
\]
and \( t \leq 1 \), we obtain \((5.5)\) for \( D = \mathbb{T}^2 \) and \( p = 1 \).

Take now any other \( D \) with \( n + m = d = 2 \) and let \( \tilde{\psi} \) solve \((2.3)\) on \( \mathbb{T}^2 \) with \( \tilde{\psi}(x_1, x_2) \equiv \sup_{j,m} |\psi(x_1 + j, x_2 + m)| \). Then by the comparison principle \((21)\), \( \tilde{\psi}(x_1, x_2, t) \geq \sup_{j,m} |\psi(x_1 + j, x_2 + m, t)| \), and so
\[
\|\tilde{\psi}(\cdot, t)\|_{L^\infty} \leq \|\tilde{\psi}(\cdot, t)\|_{L^\infty} \leq C t^{-d/2}\|\tilde{\psi}_0\|_{L^1} \leq C t^{-d/2}\|\psi_0\|_{L^1}
\]
with the same \( C \) (we then have \( C(2) = \max\{C, 1\} \)).

If \( d \geq 3 \), then the proof is identical, using Lemma 5.6 in \([3]\) in place of Lemma 3.3 in \([4]\) to obtain \((5.6)\). Finally, the case \( d = 1 \) is obvious since the only incompressible flows in one dimension are the constant ones, so \((2.3)\) is just the heat equation in a moving frame.

Next we show that given our assumptions on \( u \), we have for each fixed \( \tau > 0 \),
\[
\|P_{\tau}(Au)\|_{L^1(D) \to L^\infty(D)} \to 0 \quad \text{as } A \to \infty.
\] (5.7)
More precisely, we let \( \tilde{\phi}_0 \in L^1(D) \) be such that
\[
\|\tilde{\phi}_0\|_{L^1} \leq C^{-1/2} \tau^{1/2}
\] (5.8)
with \( C \equiv C(2) \), and we will show that for each \( \delta \in (0, 1) \) and \( A > A_1(\tau, \delta) \), the solution \( \tilde{\phi}^A \) of \((1.1)\) with initial condition \( \tilde{\phi}_0 \) satisfies
\[
\|\tilde{\phi}^A(\cdot, 3\tau)\|_{L^\infty} \leq 3\delta.
\] (5.9)
Equality \((2.7)\) with \( p = \infty \) shows that it is only necessary to consider \( \tau \leq 1 \).

We will actually replace the problem on \( D \) by the same problem on \( \mathcal{M} \equiv k\mathbb{T} \times \mathbb{T} \), with \( k > 270/\tau\delta^2 \) and with \( \tilde{\phi}_0 \) replaced by \( \sup_{j \in \mathbb{Z}} |\tilde{\phi}_0(x_1 + jk, x_2)| \). This new \( \tilde{\phi}_0 \) also satisfies \((5.5)\), and by the argument in the proof of Lemma 5.4, it is sufficient to show \((5.9)\) for the new \( \tilde{\phi}^A \). Note that if \( D = \mathbb{R}^2 \), then we consider the problem on \( \mathcal{M} \equiv (k\mathbb{T})^2 \) and change \( \tilde{\phi}_0 \) accordingly. In either case Lemma 5.3 shows that \( u \) can only have \( H^1 \) eigenfunctions with eigenvalue 0 on \( \mathcal{M} \).

From \((5.5)\) and \((5.8)\) we get \( \|\tilde{\phi}^A(\cdot, \tau)\|_{L^\infty} \leq C^{1/2} \tau^{-1/2} \) and \((2.7)\) gives \( \|\tilde{\phi}^A(\cdot, \tau)\|_{L^1} \leq \|\tilde{\phi}_0\|_{L^1} \leq C^{-1/2} \tau^{1/2} \), so \( \|\tilde{\phi}^A(\cdot, \tau)\|_{L^2} \leq 1 \). It is important here that \( C \) is independent of \( k \) and \( A \). Let us now define \( \phi_0(x) \equiv \tilde{\phi}^A(x, \tau) \) and \( \phi^A(x, t) \equiv \tilde{\phi}^A(x, \tau + t) \) so that \( \|\phi_0\|_{L^2} \leq 1 \) and \( \phi^A \) solves \((1.1)\).

We now use the abstract framework of Theorem 2.4 with \( \mathcal{H} \equiv L^2(\mathcal{M}) \), \( \Gamma \equiv -\Delta \), and \( L \equiv iu \cdot \nabla \). It is easy to see \([3]\) that the hypotheses of Theorem 2.4 are satisfied in this setting since \( \mathcal{M} \) is a compact manifold and \( u \) is Lipschitz. However, instead of directly applying the result we will need to alter the proof a little. Namely, we replace \((3.17)\) by
\[
K_1 \equiv K_0 \cap \left\{ \phi \mid \|(P_p - P_h)\phi\|_{L^2} \geq \frac{\delta}{2} \right\} \text{ or } |W_{\phi, \delta}| \geq \delta
\] (5.10)
where \( \tilde{\delta} \equiv \delta^2 \tau^2 / C^2 \) and
\[
W_{\phi, \gamma} \equiv \{ x \mid |(P_h \phi)(x)| \geq \gamma \}
\]
(the addition of $W_{\phi,\gamma}$ is the alternate route mentioned at the beginning of the present section). Note that $K_1$ is again compact because $\phi_n \to \phi_\infty$ implies $P_h \phi_n \to P_h \phi_\infty$, and so $(3.18)$ will be meaningful provided we show
\[ \sum_j \|P_j \phi\|^2_{L^1} \geq \frac{30}{\tau} \quad (5.11) \]
for all $\phi \in K_1$.

Since $u$ can only have $H^1$ eigenfunctions for eigenvalue zero, $P_h \mathcal{H}$ must be a subspace of the eigenspace of $u \cdot \nabla$ corresponding to eigenvalue zero. If now $\phi \in K_1$, then either $(P_p - P_h) \phi \neq 0$ in which case $\sum_j \|P_j \phi\|^2_{L^1} = \infty$, or $(P_p - P_h) \phi = 0$ and so $|W_{\phi,\delta}| \geq \delta$ (in particular, $P_p \phi = P_h \phi \neq 0$). In the latter case we have either $P_h \phi \notin H^1$ and so again $\sum_j \|P_j \phi\|^2_{L^1} = \|P_h \phi\|^2_{L^1} = \infty$, or $P_h \phi \in H^1$. If the latter happens, then $\psi \equiv P_h \phi$ is an $H^1$ eigenfunction of $u$ with eigenvalue zero. Lemma 5.2 shows that $\psi$ is continuous, and $|W_{\phi,\delta}| > 0$ together with the density of unbounded streamlines of $u$ (by Lemma 5.1) and the fact that $\psi$ is constant on them imply that there is a streamline of $u$ joining $\{0\} \times \mathbb{T}$ and $\{k\} \times \mathbb{T}$ inside $[0,k] \times \mathbb{T}$ on which $|\psi|$ is greater than $2\delta/3$. This is because any unbounded streamline must wind infinitely many times around $M$ in the first coordinate. Since obviously $\|\psi\|^2_{L^2} \leq 1$ and $k > 9\delta^{-2}$, the same reasoning shows that there must also be a streamline of $u$ joining $\{0\} \times \mathbb{T}$ and $\{k\} \times \mathbb{T}$ on which $|\psi|$ is smaller than $\delta/3$. Therefore
\[ \|\psi\|^2_{H^1} \geq \int_0^1 \int_0^1 \left| \frac{\partial \psi}{\partial x_2} \right|^2 dx_2 dx_1 \geq \int_0^1 \left( \int_0^1 \left| \frac{\partial \psi}{\partial x_2} \right| dx_2 \right)^2 dx_1 \geq \int_0^1 \left( \frac{\delta}{3} \right)^2 dx_1 \geq \frac{30}{\tau}. \]
In particular, $\sum_j \|P_j \phi\|^2_{L^1} = \|\psi\|^2_{H^1} \geq \frac{30}{\tau}$, and hence (5.11) holds for all $\phi \in K_1$.

We note that in the case $D = \mathbb{R}^2$ the last argument has to be changed slightly. Namely, we obtain that there must be a streamline of $u$ joining either $\{0\} \times k\mathbb{T}$ and $\{k\} \times k\mathbb{T}$ inside $[0,k] \times k\mathbb{T}$, or one joining $k\mathbb{T} \times \{0\}$ and $k\mathbb{T} \times \{k\}$ inside $k\mathbb{T} \times [0,k]$, on which $|\psi|$ is greater than $2\delta/3$. Assume the former. Then for each $x_1 \in [0,k]$ there is $x_2(x_1)$ such that $\psi(x_1, x_2(x_1)) > 2\delta$. But since $\|\psi\|^2_{L^2} \leq 1$ and $k > 18\delta^{-2}$, there must be at least measure $\frac{1}{2}$ set of $x_1 \in [0,k]$ such that $|\psi(x_1, x_2(x_1) + z(x_1))| < \delta/3$ for some $|z(x_1)| \leq \frac{\delta}{2}$. As above, $\|\psi\|^2_{H^1} \geq \frac{30}{\tau}$ follows.

We have thus shown that $N_1, T_1$ are well defined, and so Theorem 3.2(b) must hold whenever $\phi^\varepsilon(\cdot, t) \in K_1$ (with $\varepsilon = A^{-1}$ and $\phi^\varepsilon$ as in Section 3). This allows us to strengthen the condition in Theorem 3.2(c) by adding the requirement $|W_{\phi^\varepsilon(\cdot, t), \delta}| < \delta$. Ultimately we obtain Theorem 2.4 on $L^2(M)$ with the conclusion that if $\|\phi_0\|_{L^2} \leq 1$ (which is our case) and $A > A_1(\tau, \delta)$ (with $A_1$ only dependent on $\tau, \delta$ because $k = k(\tau, \delta)$ and $C$ is universal), then the set of all times $t$ for which
\[ \|(I - P_h) \phi^A(\cdot, t)\|^2_{L^2} \geq \delta \quad \text{or} \quad |W_{\phi^A(\cdot, t), \delta}| \geq \delta \]
has measure less than $\tau$. Since $\tilde{\phi}^A(x, t) = \phi^A(x, t - \tau)$, this says that there must be a time $t_0 \in [\tau, 2\tau]$ such that
\[ \|(I - P_h) \tilde{\phi}^A(\cdot, t_0)\|^2_{L^2} < \delta \quad \text{and} \quad |W_{\tilde{\phi}^A(\cdot, t_0), \delta}| < \delta. \quad (5.12) \]
We now let \( \chi \) be the characteristic function of \( W_{\tilde{\phi}_A(\cdot,t_0),\delta} \), define
\[
\psi^1_0(\cdot) \equiv (I - P_h)\tilde{\phi}_A(\cdot,t_0),
\psi^2_0(\cdot) \equiv \chi(\cdot)P_h\tilde{\phi}_A(\cdot,t_0),
\psi^3_0(\cdot) \equiv (1 - \chi(\cdot))P_h\tilde{\phi}_A(\cdot,t_0),
\]
and let \( \psi^j \) solve (1.1) with initial condition \( \psi^j_0 \) so that \( \tilde{\phi}_A(x,t) = \sum_{j=1}^{\infty} \psi^j(x,t-t_0) \). Lemma 5.4 and \( \tau \leq 1 \) give \( \|\psi^1(\cdot,\tau)\|_{L^\infty} \leq C\tau^{-1/2}\delta \leq \delta \), and obviously \( \|\psi^2(\cdot,\tau)\|_{L^\infty} \leq \|\psi^3_0\|_{L^\infty} \leq \delta \). Finally, we have
\[
\|\psi^3_0\|_{L^1} \leq |\text{supp}(\psi^3_0)|^{1/2}\|\psi^3_0\|_{L^2} \leq \delta^{1/2}\|P_h\tilde{\phi}_A(\cdot,t_0)\|_{L^2} \leq \delta^{1/2}\|\tilde{\phi}_A(\cdot,t_0)\|_{L^2} \leq \tilde{\delta}^{1/2},
\]
and Lemma 5.4 again gives \( \|\psi^3(\cdot,\tau)\|_{L^\infty} \leq C\tau^{-1}\tilde{\delta}^{1/2} = \delta \). It follows using (2.7) that
\[
\|\tilde{\phi}_A(\cdot,3\tau)\|_{L^\infty} \leq \|\tilde{\phi}_A(\cdot,t_0+\tau)\|_{L^\infty} \leq 3\delta,
\]
that is, (5.9) holds and (5.7) follows.

Interpolation and (2.7) then give (2.6) for any \( 1 \leq p < q \leq \infty \), thus yielding Theorem 2.1(i)–(iii) for \( p < q \). The case \( p = q \in (1,\infty) \) in part (ii) follows by splitting \( \phi_0 = \phi''_0 + \phi'''_0 \) with \( \phi''_0 \in L^1 \) and \( \|\phi'''_0\|_{L^p} \) small. Using (2.6) for \( \phi''_0 \) and (2.7) for \( \phi'''_0 \) then gives the result.

6. Proof of Theorem 2.1 Part II

In the present section we complete the proof of Theorem 2.1. We now assume that \( u \) is a 1-periodic incompressible Lipschitz flow on \( D = \mathbb{R} \times \mathbb{T} \) that either leaves a bounded open subset of \( D \) invariant or has an \( H^1(\mathbb{T}^2) \) eigenfunction with a non-zero eigenvalue. We will then show that Theorem 2.1(i)–(iii) do not hold. Again the cases of \( D = \mathbb{R}^2 \) and/or of other periods are handled similarly.

The main point here is that flows with the above properties do not “stretch” compactly supported initial data in the way the flows considered in the previous section do, which means the exposure of the solution to the effects of diffusion is limited (at least for a short time), regardless of the flow strength. More precisely, we will show

Lemma 6.1. Under the above assumptions on \( u \), there is a bounded non-zero compactly supported \( \phi_0 \in H^1(D) \) and \( b < \infty \) such that the solution of (3.3) on \( D \) satisfies \( \|\phi^0(\cdot,t)\|_{H^1(D)} \leq b \) for all \( t \geq 0 \).

Assume for the moment that Lemma 6.1 holds. Then Lemma 3.3 with \( \Gamma \equiv -\Delta \) and \( L \equiv iu \cdot \nabla \) on \( \mathcal{H} \equiv L^2(D) \), and after setting \( A = \varepsilon^{-1} \) and rescaling time appropriately, shows that for each \( A \),
\[
\|\phi^A(\cdot,t) - \phi^0(\cdot,At)\|_{L^2} \leq 4b\sqrt{t}\|\phi_0\|_{L^2}.
\]
Note that Lemma 3.3 extends to the non-compact setting of \( D \), where \( \Gamma \) does not have a discrete spectrum. Since the measure of the set
\[
\{x \mid |\phi^0(x,t)| \geq \gamma\}
\]
is constant in $t$ for each $\gamma$, this means that $\phi^A(\cdot,t)$ cannot be small in any $L^p$ norm for $t$ sufficiently small, regardless of the choice of $A$. Thus none of Theorem 2.1(i)–(iii) cannot be valid and we are left with establishing Lemma 6.1.

Proof of Lemma 6.1. Let us first assume that $u$ leaves an open bounded domain $Y \subseteq D$ invariant. If $u \equiv 0$ on some such $Y$, then we only need to take $\phi_0$ to be any bounded $H^1$ function supported in $Y$.

If this is not the case, then we know from the proof of Lemma 5.1 that there is such a domain $Y$ with $\partial Y$ a union of one or two non-trivial streamlines of $u$. If $U$ is a stream function for $u$, then we have $U(\partial Y) = \{\beta, \gamma\}$ (with possibly $\beta = \gamma$). Since $U$ cannot be constant inside $Y$ (because $u(y) \neq 0$ on $\partial Y$), there is $y_0 \in Y$ with $U(y_0) \notin \{\beta, \gamma\}$. Then

$$\phi_0(x) \equiv \chi_Y(x)(U(x) - \beta)(U(x) - \gamma) \neq 0.$$ 

is a compactly supported Lipschitz (and therefore $H^1$) function that is constant on the streamlines of $u$ and thus $\phi^0(x,t) \equiv \phi_0(x)$ for all $t$. The claim of the lemma follows.

It remains to consider the case that $u$ on $\mathbb{T}^2$ has an eigenfunction $\psi \in H^1(\mathbb{T}^2)$ with eigenvalue $i\lambda \in i\mathbb{R} \setminus \{0\}$. Notice that the first paragraph of the proof of Lemma 5.2 again shows that $\psi$ has to be continuous at each $x$ for which $u(x) \neq 0$ (the only difference is that now we obtain $(1,0) \cdot \nabla \omega \equiv i\lambda \omega$ and so $\omega(t,s) = e^{i\lambda t} \tilde{\omega}(s)$).

Let $x_0 \in \mathbb{T}^2$ be such that $\psi(x_0) \neq 0 \neq u(x_0)$. Such $x_0$ exists because $\lambda \neq 0$ implies $u(x_0) \neq 0$ for almost all $x_0$ with $\psi(x_0) \neq 0$. Without loss of generality we can assume that $x_0 = 0$ and on a neighborhood $V$ of 0 we have $u(x) \equiv (1,0)$; otherwise a Lipschitz change of coordinates as in the proof of Lemma 5.2 will bring us to this situation. Then

$$\psi(x_1, x_2) = e^{i\lambda x_1} \tilde{\psi}(x_2) \quad (6.1)$$

(with $\tilde{\psi}$ continuous) for $|x_1|, |x_2| \leq 2\alpha$ and some small $\alpha \in (0, \pi/\lambda)$. Also, $\nabla U \equiv (0, -1)$ on $V$.

Choose a non-negative function $\omega : \mathbb{C} \to \mathbb{R}$ that is smooth as a function from $\mathbb{R}^2$ to $\mathbb{R}$ and is supported on a small ball around $\tilde{\psi}(0)$, so that for some small $\beta, \gamma > 0$ we have $\omega(\tilde{\psi}(x_1, x_2)) = 0$ for $(x_1, x_2) \in \{(\alpha, \alpha + \gamma) \cup [-\alpha - \gamma, -\alpha]) \times [-\beta, \beta]\). This is possible because of the continuity of $\tilde{\psi}$ and $\lambda \neq 0$ in (6.1), together with $\alpha < \pi/\lambda$ (this is where we crucially use $\lambda \neq 0$).

We also let $\theta$ with $\theta(0) \neq 0$ be a smooth non-negative function supported in $[-\alpha - \gamma, \alpha + \gamma] \times [-\beta, \beta]$ which only depends on $x_2$ in $R \equiv [-\alpha, \alpha] \times [-\beta, \beta] \subseteq V$. Since $U(x) = c - x_2$ on $V$ (for some $c$), we have $\theta(x) = \tilde{\theta}(U(x))$ for all $x \in R$ and a smooth compactly supported $\tilde{\theta}$.

Now extend $\psi$ periodically and $\theta$ by 0 onto $D$ and consider

$$\phi_0(x) \equiv \theta(x)\omega(\psi(x)) = \chi_R(x)\tilde{\theta}(U(x))\omega(\psi(x)) \in H^1(D).$$

Then $\phi^0(x,t) = \phi_0(X(x,-t))$ (with $X$ from (2.1)) is supported in $R_t \equiv X(R,t)$ and

$$\omega(\psi(X(x,-t))) = \omega(e^{-i\lambda t}\psi(x))$$

because $u \cdot \nabla \psi = i\lambda \psi$. Since constancy of $U$ on the streamlines of $u$ gives

$$\theta(X(x,-t)) = \tilde{\theta}(U(X(x,-t))) = \tilde{\theta}(U(x)) \quad (6.2)$$
for \( x \in R_t \), we have
\[
\phi^0(x, t) = \chi_{R_t}(x) \tilde{\theta}(U(x)) \omega(e^{-i\lambda t} \psi(x)).
\] (6.3)

Note that since \( R \subseteq \mathbb{T}^2 \), the domain \( R_t \subseteq D \) is simply connected and the natural map from \( D \) onto \( \mathbb{T}^2 \) is one-to-one when restricted to \( R_t \). Hence
\[
\int_{R_t} |\nabla [\omega(e^{-i\lambda t} \psi(x))]|^2 dx \leq \int_{\mathbb{T}^2} |\nabla [\omega(e^{-i\lambda t} \psi(x))]|^2 dx \leq \|\nabla \omega\|_{L^\infty(\mathbb{T}^2)} \|\psi\|_{H^1(\mathbb{T}^2)}^2.
\]
Since \( \tilde{\theta} \) and \( \omega \) are bounded and \( \phi^0 \) vanishes on \( \partial R_t \), to obtain the claim of the lemma, we only need to show that \( \int_{R_t} |\nabla(\tilde{\theta}(U(x)))|^2 dx \) is uniformly bounded in \( t \). But \( |R_t| \leq 1 \) and
\[
|\nabla(\tilde{\theta}(U(x)))| \leq \|\nabla \tilde{\theta}\|_{L^\infty(\mathbb{R})} \|U\|_{L^\infty(R_t)} \leq \|\nabla \tilde{\theta}\|_{L^\infty(\mathbb{R})} \|u\|_{L^\infty(\mathbb{T}^2)},
\]
for \( x \in R_t \), so this is obvious. \( \square \)

We note that \( \theta \) is only needed when \( |\tilde{\psi}(x_2)| \) is constant on an open interval containing zero. Otherwise \( \phi_0(x) \equiv \chi_R(x) \omega(\psi(x)) \) does the job. This finishes the proof of Theorem 2.1.

7. Other Boundary Conditions and Examples

In the case \( D = \mathbb{R} \times (0, 1) \) we have so far only considered periodic boundary conditions on \( \partial D \). It turns out that there is no change to Theorem 2.1 when we impose Dirichlet or Neumann boundary conditions, provided \( u(x) \cdot (0, 1) = 0 \) for \( x \in \partial D \).

**Corollary 7.1.** Assume that \( u \) is a periodic, incompressible, Lipschitz flow on \( D = \mathbb{R} \times (0, 1) \) with a cell of periodicity \( C = \alpha \mathbb{T} \times (0, 1) \) such that \( u(x) \cdot (0, 1) = 0 \) for \( x \in \partial D \). Let \( \phi^A \) be the solution of (1.1) on \( D \) with either Dirichlet or Neumann boundary conditions on \( \partial D \). Then Theorem 2.1(i)–(iv) are again equivalent.

**Remarks.** 1. The operator \( u \cdot \nabla \) is again anti-self-adjoint on \( L^2(C) \) due to \( u_2 \equiv 0 \) on \( \partial C \).

2. Notice that there is no distinction between dissipation-enhancing flows in the Dirichlet and Neumann boundary conditions cases. This is because \( u_2 \equiv 0 \) on \( \partial D \) means that boundary conditions do not considerably affect dissipation away from \( \partial D \) on short time scales.

**Proof.** Extend \( u \) to \( D' \equiv \mathbb{R} \times 2\mathbb{T} \) by letting
\[
(u_1(x_1, x_2), u_2(x_1, x_2)) \equiv (u_1(x_1, 2 - x_2), -u_2(x_1, 2 - x_2))
\]
for \( x_2 \in [1, 2] \). That is, \( u \) is periodic and symmetric across \( x_2 = 1 \). Consider the the Dirichlet boundary conditions case first. It is sufficient to show that each of Theorem 2.1(i)–(iv) holds on \( D \) if and only if \( u \) is dissipation-enhancing on \( D' \).

The “if” part of this claim is immediate. Indeed, if \( \phi^A \) is a solution on \( D \) with Dirichlet boundary conditions, then we can extend it to a solution on \( D' \) by letting \( \phi^A(x_1, x_2) \equiv -\phi^A(x_1, 2 - x_2) \). Hence any of Theorem 2.1(i)–(iii) on \( D' \) implies its counterpart on \( D \). The same is true in the case of part (iv) because if \( \psi \) is an eigenfunction of \( u \) in \( H^1(C) \), then by letting \( \psi(x_1, x_2) \equiv \psi(x_1, 2 - x_2) \) one extends \( \psi \) to an eigenfunction of \( u \) in \( H^1(\alpha \mathbb{T} \times 2\mathbb{T}) \).

As for the “only if” part, assume \( u \) on \( D' \) is not dissipation-enhancing. Take some \( \phi_0 \) that satisfies Lemma 6.1 for \( D' \) and that is supported inside \( D \). This can be done because the streamlines of \( u \) do not cross \( \partial D \). For the same reason \( \phi^0 \) from Lemma 6.1 stays inside \( D \),
and so if we extend \( \phi^0 \) to \( D \) by letting \( \phi^0(x_1, x_2) \equiv -\phi^0(x_1, 2 - x_2) \), then this \( \phi^0 \) satisfies all conditions of that lemma. The corresponding \( \phi^A \) vanishes on \( \partial D \) and as in Section 6 it follows that none of Theorem 2.1(i)–(iii) can hold on \( D \). The same is true for part (iv) after realizing that the restriction to \( D \) of a bounded open subset of \( D' \) invariant under \( u \) (or the restriction to \( C \) of an \( H^1(aT \times 2T) \) eigenfunction of \( u \)) has the same property on \( D \) (on \( C \)).

This finishes the case of Dirichlet boundary conditions. Neumann boundary conditions are treated identically, with \( \phi^A \) and \( \phi^0 \) extended evenly (rather than oddly) to \( D' \).

We will now present a simple example of flows on \( \mathbb{R}^2 \) that demonstrates the independence of the two conditions in Theorem 2.1(iv).

**Example 7.2.** Let \( p : \mathbb{T} \to \mathbb{T} \) and \( \tilde{U} : \mathbb{T} \to \mathbb{R} \) be \( C^1 \) with \( \int_0^1 p'(s)ds = 0 \). Define \( U(x_1, x_2) \equiv \tilde{U}(y(x_1, x_2)) \) with \( y(x_1, x_2) \equiv \{ p(x_1) - x_2 \} \) and consider the flow

\[
u(x_1, x_2) \equiv \nabla^{-1}U(x_1, x_2) = (\tilde{U}'(y), p'(x_1)\tilde{U}'(y))
\]  

(7.1)
on \( \mathbb{R} \times \mathbb{T} \) or on \( \mathbb{R}^2 \). In particular, we have \( u(x_1, x_2) = 0 \) if and only if \( \tilde{U}'(y(x_1, x_2)) = 0 \). If \( p \equiv 0 \) then this is a mean-zero shear flow. For general \( p \) (and \( \tilde{U}' \not\equiv 0 \)) these are examples of *percolating flows*.

It is easy to see that the flow preserves \( y \), and its first coordinate \( \tilde{U}'(y) \) is therefore constant on the streamlines. The unbounded streamlines are those corresponding to \( y \)'s for which \( \tilde{U}'(y) \neq 0 \) (they are then 1-periodic functions of \( x_1 \) due to \( \int_0^1 p'(s)ds = 0 \)). This means that there is an open bounded domain invariant under the flow if and only if \( \tilde{U}' \) has a plateau (a non-trivial interval where it is constant) with \( \tilde{U}' = 0 \). Note that (7.1) on \( \mathbb{T} \times \mathbb{R} \) always has invariant bounded open domains.

There are always many \( H^1 \) eigenfunctions of such \( u \), since each \( \tilde{\psi}(y) \) is a first integral. On the other hand, it turns out that \( u \) has \( H^1 \) eigenfunctions other than the first integrals if and only if \( \tilde{U}' \) has plateaus with \( \tilde{U}' \not\equiv 0 \).

Indeed, if \( \tilde{U}'(y) \equiv C \neq 0 \) for \( y \in [a, b] \) and \( \theta \) is a smooth function supported on \( [a, b] \), then \( \psi(x_1, x_2) \equiv e^{2\pi i x_1}\theta(y) \) is an \( H^1 \) eigenfunction of \( u \) with eigenvalue \( 2\pi i C \). On the other hand, any \( H^1 \) eigenfunction with an eigenvalue \( i\lambda \neq 0 \) must be continuous a.e. where \( u \not\equiv 0 \) (i.e., \( \tilde{U}' \not\equiv 0 \)) and zero a.e. where \( u = 0 \) (i.e., \( \tilde{U}' = 0 \)). This means that it must be of the form \( \psi(x_1, x_2) \equiv e^{i\lambda x_1/\tilde{U}'(y)}\theta(y) \) with \( \theta \) continuous where \( \tilde{U}' \not\equiv 0 \) and zero where \( \tilde{U}' = 0 \). But for \( \psi \) to be well defined as a function on \( \mathbb{T}^2 \), \( 2\pi \lambda/\tilde{U}'(y) \) must be an integer where \( \theta(y) \not\equiv 0 \). Since \( \psi \not\equiv 0 \) and so \( \theta \not\equiv 0 \), this means that there must be a plateau of \( \tilde{U}' \) with \( \tilde{U}' \not\equiv 0 \).

Finally, since the existence of a plateau of \( \tilde{U}' \) with \( \tilde{U}' = 0 \) and the existence of a plateau of \( \tilde{U}' \) with \( \tilde{U}' \not\equiv 0 \) are “independent”, we can construct flows \( u \) given by (7.1) that demonstrate all four possibilities of the conditions of Theorem 2.1(iv) either being satisfied or not.

Notice that if \( \tilde{U}'(y) \equiv C \) for \( y \in [a, b] \), then the solutions of (2.1) starting inside the “channel” given by \( y(x_1, x_2) \in [a, b] \) move along this channel with the same (horizontal) velocity \( U'(y) \). This shows that any initial datum supported inside the channel will not get stretched too much regardless of the amplitude \( A \) of the flow, as was mentioned at the beginning of Section 6. On the other hand, \( \tilde{U}'(y) \) not locally constant means any compactly
supported initial datum will be stretched quickly when $A$ is large because “neighboring”
streamlines move at different horizontal speeds and this difference is magnified by $A$.

We also mention that in the case of shear flows (i.e., $p \equiv 0$) Theorem 2.1 follows from
the results of [15] (the earlier paper [2] also considers shear flows and can treat all $\bar{U}$ except
of those that have no plateaus but all their derivatives vanish at some $y_0$). The above
stretching argument was made rigorous there using probabilistic methods (Malliavin calculus
in particular), but unlike our functional-analytic method, the approach does not seem to be
applicable to general non-shear flows.

Finally, notice that if $\bar{U}' \neq 0$ only on a dense set of a small measure and $\bar{U}'$ has no plateaus,
then $u$ vanishes on a large set but it is dissipation-enhancing nevertheless.

8. Applications to Reaction-Diffusion Equations

We now turn to applications of Theorem 2.1 to quenching in reaction-advection-diffusion
equations. We consider the equation

$$T^A_t(x,t) + Au \cdot \nabla T^A(x,t) = \Delta T^A(x,t) + f(T^A(x,t)), \quad T^A(x,0) = T_0(x)$$

(8.1)

for $x \in \mathbb{R} \times \mathbb{T}$ or $x \in \mathbb{R}^2$. Here $T^A(x,t) \in [0,1]$ is the (normalized) temperature of a premixed
combustible gas that is advected by the periodic incompressible flow $Au(x)$. The nonlinear
reaction term $f(T^A)$ accounts for temperature increase due to burning and will be considered
to be of the \textit{ignition type}, that is,

(i) $f(0) = f(1) = 0$ and $f(T)$ is Lipschitz continuous on $[0,1]$,

(ii) $\exists \eta_0 \in (0,1)$ such that $f(T) = 0$ for $T \in [0,\eta_0]$ and $f(T) > 0$ for $T \in (\eta_0,1)$.

(8.2)

The value $\eta_0$ is called the (normalized) \textit{ignition temperature}. We also take $T_0(x)$ to be
compactly supported with values in $[0,1]$, so that $T^A(x,t) \in [0,1]$ for all $x,t$ by the maximum
principle.

\textbf{Definition 8.1.} We say that the initial “flame” $T_0$ is \textit{quenched} by the flow $Au$ if

$$\|T^A(\cdot,t)\|_{L^\infty} \to 0 \quad \text{as } t \to \infty.$$  \hfill (8.3)

A flow $u$ is said to be \textit{strongly quenching} if for each compactly supported $T_0$ and each ignition-
type reaction $f$ there exists $A_0$ such that $Au$ quenches $T_0$ for each $A > A_0$.

That is, strongly quenching flows are those that have the ability to extinguish any initially
localized reaction, provided their amplitude is large enough. Notice also that due to the
compactness of $\text{supp}(T_0)$ and $\eta_0 > 0$, (8.3) is equivalent to $\|T^A(\cdot,t_0)\|_{L^\infty} \leq \eta_0$ for some
$t_0 < \infty$.

We can now state

\textbf{Theorem 8.2.} Assume that $u$ is a periodic, incompressible, Lipschitz flow on $D = \mathbb{R}^2$ or
$D = \mathbb{R} \times \mathbb{T}$ with a cell of periodicity $C$.

(i) If $u$ is dissipation-enhancing, then $u$ is strongly quenching.

(ii) If either $u$ leaves an open bounded subset of $D$ invariant or $u$ has an eigenfunction
$\psi \in C^{1,1}(C)$ that is not a first integral of $u$, then $u$ is not strongly quenching.
Remarks. 1. $C^{1,1}(C)$ is the set of all $\psi \in C^1(C)$ with $\nabla \psi \in \text{Lip}(C)$.

2. This of course leaves open the case when no open bounded sets are invariant under $u$, the flow does have $H^1(C)$ eigenfunctions with non-zero eigenvalues, but none of them belongs to $C^{1,1}(C)$. Such flows can again be constructed using Example 2 in Section 6 of [3], this time with a smooth $Q : \mathbb{T} \rightarrow \mathbb{T}$ and a Liouvillean $\alpha$ such that (2.8) has a solution $R \in H^1(\mathbb{T}) \setminus H^2(\mathbb{T})$. We conjecture that $u$ is not strongly quenching in such cases, and hence that the strongly quenching periodic flows in two dimensions are precisely the dissipation-enhancing ones.

Proof. (i) Let $c$ be the Lipschitz constant for $f$ so that $f(T) \leq cT$. If $\phi^A$ solves (1.1) with initial condition $\bar{\phi}_0 \equiv T_0 \in L^1(D)$, then $T^A(x,t) \leq c^A(x,t)$. The result follows by choosing $A$ large enough so that $\|\phi^A(\cdot,1)\|_{L^\infty} \leq e^{-c_0}$.

(ii) Assume first there is an open bounded domain $Y \subseteq D$ invariant under $u$. From the proof of Lemma 6.1 we know that then there is such a $Y$ so that either $u \equiv 0$ on $Y$, or $\partial Y$ consists of one or two closed streamlines of $u$ (one if $Y$ is simply connected, two otherwise).

In either case we will construct a stationary subsolution $T_0$ of (8.1) for some $f$ and any $A$. From this the claim will follow, because then $T^A(x,t) \geq T_0(x)$ for all $A, x, t$ and so $u$ cannot be quenching.

Assume the first case (i.e., $u \equiv 0$ on $Y$) and choose a smooth function $T_0$ supported in $Y$ and bounded above by $\frac{2}{3}$ such that $\Delta T_0(x) \geq 0$ when $T_0(x) < \frac{1}{3}$. We then have

$$\Delta T_0 + f(T_0) \geq 0$$

whenever $f$ is larger than $\|\Delta T_0\|_{L^\infty}$ on $[\frac{1}{3}, \frac{2}{3}]$. Hence $T_0(x)$ is a subsolution of (8.1) for such $f$ and any $A$.

Next assume the second case above and assume $Y$ is bounded and simply connected (the other alternative can be handled by a simple modification of the following argument). Notice that we have that $u \neq 0$ on $\partial Y$ by construction (see Section 6) and so $|\nabla U| \geq c$ for some $c > 0$ on some open neighborhood $\bar{V}$ of $\partial Y$. This, the fact that we are in two dimensions, and $u$ Lipschitz ensure that all streamlines that are close enough to $\partial Y$ must also be closed. It follows that there is a domain $V \subseteq \bar{V} \cap Y$ with $\partial V$ consisting of two streamlines of $u$, one of which is $\partial Y$. Since $|\nabla U|$ is strictly positive on $V$ and continuous, $V$ can be chosen so that $U(\partial V) = \partial U(V)$.

Let $\tilde{\phi}_0$ be a smooth function on the interval $U(V)$ with $\tilde{\phi}_0(U(\partial Y)) = 0$ and $\tilde{\phi}_0(U(\partial V \setminus \partial Y)) = \frac{2}{3}$, with the first and second derivatives of $\tilde{\phi}_0$ vanishing on $\partial U(V)$, and with

$$\tilde{\phi}_0''(s) \geq c^{-2}\|\Delta U\|_{L^\infty}|\tilde{\phi}_0'(s)|$$

(8.4)

when $\tilde{\phi}_0(s) < \frac{1}{3}$. This is possible because $U \in C^{1,1}(D)$ and so $\|\Delta U\|_{L^\infty} < \infty$. We then let

$$M \equiv \|\tilde{\phi}_0''\|_{L^\infty}\|\nabla U\|_{L^\infty}^2 + \|\tilde{\phi}_0'\|_{L^\infty}\|\Delta U\|_{L^\infty}$$

(8.5)
and pick \( f \) that is larger than \( M \) on \([\frac{1}{3}, \frac{2}{3}]\). We define

\[
T_0(x) \equiv \begin{cases} 
\phi_0(U(x)) & x \in V, \\
\frac{2}{3} & x \in Y \setminus V, \\
0 & x \notin Y,
\end{cases}
\]

so that \( \Delta T_0 + f(T_0) = f(T_0) \geq 0 \) outside \( V \) and

\[
\Delta T_0(x) + f(T_0(x)) = \tilde{\phi}''_0(U(x))|\nabla U(x)|^2 + \tilde{\phi}'_0(U(x))\Delta U(x) + f(T_0(x)) \geq 0
\]

in \( V \) (using (8.4) when \( T_0(x) < \frac{1}{3} \) and (8.5) otherwise). This and the fact that \( T_0 \) is constant on the streamlines of \( u \) means that \( T_0 \) is a subsolution of (8.1) for any \( A \).

Let us now assume that \( u \) has an eigenfunction \( \psi \in C^{1,1}(\mathcal{C}) \) with eigenvalue \( i\lambda \in i\mathbb{R} \setminus \{0\} \). We will show that if we choose \( f \) and the functions \( \omega \) and \( \theta \) from the corresponding part of Section 6 appropriately, then the (time-dependent) solution of the fast free linear dynamics \( \psi^0(x,At) \) from (6.3) will be a subsolution of (8.1) for each \( A \).

Take \( x_0 \) such that \( \psi(x_0) \neq 0 = u(x_0) \). Without loss of generality we can assume that \( x_0 = 0, \psi(0) = 1, \) and \( U(0) = 0 \), as this can be achieved by a translation of the problem, multiplication of \( \psi \) by a constant, and additon of a constant to \( U \). In what follows we will call \( C^2 \) functions smooth.

Assume first that the flow \( u(x) \equiv (1,0) \) in a neighborhood of \( 0 \). Repeat the construction from Section 6 to obtain smooth non-negative \( \omega, \theta \), and a small rectangle \( R \equiv [-\alpha, \alpha] \times [-\beta, \beta] \) such the following hold with \( \psi \) extended periodically onto \( D \). The product \( \theta(x)\omega(\psi(x)) \) is supported in \( R \) (by slightly enlarging \( R \) we can actually assume that \( \theta(x)\omega(\psi(x)) \) is supported on a compact subset of the interior of \( R \)) and on \( R \) we have \( \theta(x) = \tilde{\theta}(U(x)) \) for some smooth non-negative compactly supported \( \tilde{\theta} \). Moreover, we will also pick \( \omega \) so that \( \omega(z) = \tilde{\omega}(3z) \) on \( \psi(R) \) for some compactly supported smooth \( \tilde{\omega} \) and \( 3z \) the imaginary part of \( z \). This can be achieved thanks to \( \psi(0) = 1 \), the continuity of \( \psi \) on \( R \), and \( \lambda \neq 0 \) in (6.1), provided \( R \) is small (recall that so far \( \omega \) was only required to be supported on a small ball around \( \psi(0) = 1 \)). The picture we are establishing here is that \( U(x) \) and \( 3\psi(x) \) determine a coordinate system on \( R \), while inside \( R \) each of the functions \( \theta \) and \( \omega \circ \psi \) depends on one of these coordinates only (and their product is supported in the interior of \( R \)). The main point is that, as we shall see, this setup will be preserved by the free evolution and hold on \( R_t \equiv X(R,t) \).

This time, however, we need to impose additional conditions on \( R_t, \tilde{\omega}, \) and \( \tilde{\theta} \). This will be necessary because we will deal with second derivatives here, and possible because these will not clash with the conditions we imposed so far — that \( R_t \) be small and \( \tilde{\omega}, \tilde{\theta} \) be non-negative, nonzero, smooth, and have small supports containing zero (since \( 3\psi(0) = U(0) = 0 \)).

We first ask that \( R \) is small enough so that

\[
|\psi(x) - 1| \leq \frac{1}{2}
\]

(8.6)
for \( x \in R \). Since the flow preserves \(|\psi|\), we have \(|\psi(x)| \geq \frac{1}{2} \) for \( x \in R_t \). This and \( u \cdot \nabla \psi = i\lambda \psi \) mean that if
\[
C \equiv \max\{\|u\|_{L^\infty}, \|\nabla \psi\|_{L^\infty}, \|\Delta U\|_{L^\infty}, \|\Delta \psi\|_{L^\infty}, \sqrt{\lambda}, 1\} < \infty,
\]
\[
c \equiv \min\left\{\frac{\lambda}{2C}, \frac{1}{2} \left(1 - \sqrt{1 - \frac{\lambda^2}{4C^4}}\right)\right\} > 0,
\]
then
\[
|\nabla U(x)| = |u(x)| \geq \frac{\lambda}{2C} \geq c
\]
for \( x \in R_t \). We let \( \kappa_t(x) \equiv \Im(e^{-i\lambda t}\psi(x)) \) so that
\[
u \cdot \nabla \kappa_t(x) = \lambda \Re(e^{-i\lambda t}\psi(x))
\]
together with
\[
e^{-i\lambda t}\psi(x) = \psi(X(x, -t)) \in \psi(R)
\]
for \( x \in R_t \) and with (8.6) gives
\[
|\nabla \kappa_t(x)| \geq \frac{\lambda}{2C} \geq c
\]
for \( x \in R_t \). Finally, we note that \( \nabla U \perp u \) and \( |\nabla U| = |u| \) give for \( x \in R_t \),
\[
|\nabla U(x) \cdot \nabla \kappa_t(x)| = (|\nabla U(x)|^2|\nabla \kappa_t(x)|^2 - |u(x) \cdot \nabla \kappa_t(x)|^2)^{1/2}
\]
\[
= |\nabla U(x)| |\nabla \kappa_t(x)| \sqrt{1 - \frac{|\lambda \Re(e^{-i\lambda t}\psi(x))|^2}{|\nabla U(x)|^2|\nabla \kappa_t(x)|^2}}
\]
\[
\leq |\nabla U(x)| |\nabla \kappa_t(x)| \sqrt{1 - \frac{\lambda^2}{4C^4}}
\]
\[
\leq (1 - 2c)|\nabla U(x)| |\nabla \kappa_t(x)|,
\]
where we again used (8.8) and (8.6) in the third step.

As for \( \tilde{\theta} \) and \( \tilde{\omega} \), we ask that they be smooth, bounded above by \( \frac{2}{3} \), and satisfy
\[
|\tilde{\omega}'(s)| = kK\tilde{\omega}(s)^{1-1/k} \quad \text{and} \quad \tilde{\omega}''(s) = k(k - 1)K^2\tilde{\omega}(s)^{1-2/k} \quad \text{when} \quad \tilde{\omega}(s) \leq \frac{1}{2},
\]
\[
|\tilde{\theta}'(s)| = kK\tilde{\theta}(s)^{1-1/k} \quad \text{and} \quad \tilde{\theta}''(s) = k(k - 1)K^2\tilde{\theta}(s)^{1-2/k} \quad \text{when} \quad \tilde{\theta}(s) \leq \frac{1}{2},
\]
for some \( K > 1 \) and
\[
k \equiv 1 + Cc^{-3}.
\]
This can be achieved by making \( \tilde{\omega}, \tilde{\theta} \) equal to translations of \((K|s|)^k\) close to the edges of their respective supports (with \( K \) large to ensure the supports are as small as needed) and taking values from \( [\frac{1}{2}, \frac{2}{3}] \) on the remainders of their supports. We then let
\[
L \equiv \max \left\{ \max \left\{ \frac{|\tilde{\omega}'(s)|}{\tilde{\omega}(s)}, \frac{|\tilde{\omega}''(s)|}{\tilde{\omega}(s)} \right| \tilde{\omega}(s) \geq \frac{1}{2} \right\}, \max \left\{ \frac{|\tilde{\theta}'(s)|}{\tilde{\theta}(s)}, \frac{|\tilde{\theta}''(s)|}{\tilde{\theta}(s)} \right| \tilde{\theta}(s) \geq \frac{1}{2} \right\}, 1 \right\}.
\]
From now on \( \tilde{\omega}, \tilde{\theta} \) will be fixed.

Finally, we note that if \( u \neq (1,0) \) around 0, we can map \( u \) onto \((1,0)\) via a bilipschitz mapping \( J \), construct \( R, \omega, \theta, \tilde{\omega}, \tilde{\theta} \) as above (using \( \psi \circ J \)), and then map \( R, \theta \) back via \( J^{-1} \), keeping \( \omega, \tilde{\omega}, \tilde{\theta} \) unchanged. This gives us \( R \) that is not necessarily a rectangle but has the properties we are interested in. Namely, \( \phi_0(x) \equiv \theta(x)\omega(\psi(x)) \) is supported in the interior of \( R \), and \( \theta(x) = \tilde{\theta}(U(x)) \) and \( \omega(\psi(x)) = \tilde{\omega}(3\psi(x)) \) for \( x \in R \). Therefore \( \phi_0(x) = \tilde{\theta}(U(x))\tilde{\omega}(3\psi(x)) \) on its support, and so \( \phi_0 \in C^{1,1} \) because \( \psi, U \in C^{1,1} \) and \( \tilde{\theta}, \tilde{\omega} \) are smooth.

Once again the solution \( \phi^0(x,t) = \phi_0(X(x,-t)) \) of the free linear dynamics (3.3) is supported in the interior of \( R \). The introduction of \( \tilde{\omega} \) turns (6.3) into

\[
\phi^0(x,t) = \theta(X(x,-t))\omega(\psi(X(x,-t))) = \chi_{R_t}(x)\tilde{\theta}(U(x))\tilde{\omega}(\kappa_t(x)).
\]

This is because the flow preserves \( U \), and for \( x \in R_t \) we have \( X(x,-t) \in R \) so that

\[
\omega(\psi(X(x,-t))) = \tilde{\omega}(3[\psi(X(x,-t))]) = \tilde{\omega}(3[e^{-iM}\psi(x)]) = \tilde{\omega}(\kappa_t(x)).
\]

We also have

\[
\frac{d}{dt} \phi^0(x,At) + Au \cdot \nabla \phi^0(x,At) = 0, \tag{8.12}
\]

and we will show that \( \phi^0(x,At) \) is a subsolution of (8.1) with an appropriate \( f \).

Obviously \( \Delta \phi^0(x,t) = 0 \) for \( x \notin R_t \), and for \( x \in R_t \),

\[
\Delta \phi^0(x,t) = \tilde{\theta}''(U(x))\tilde{\omega}'(\kappa_t(x))|\nabla U(x)|^2 + \tilde{\theta}'(U(x))\tilde{\omega}''(\kappa_t(x))|\nabla \kappa_t(x)|^2 + 2\tilde{\theta}'(U(x))\tilde{\omega}'(\kappa_t(x))\nabla U(x) \cdot \nabla \kappa_t(x)
\]

\[
+ \tilde{\theta}''(U(x))\tilde{\omega}'(\kappa_t(x))\Delta U(x) + \tilde{\theta}'(U(x))\tilde{\omega}'(\kappa_t(x))\Delta \kappa_t(x). \tag{8.13}
\]

Note that from \( \psi, U \in C^{1,1} \) and \( \tilde{\theta}, \tilde{\omega} \) smooth it follows that

\[
\Delta \phi^0(x,t) \geq -M
\]

for some large \( M \) independent of \( x, t \). Let us now assume that \( x \in R_t \) is such that \( \tilde{\omega}(\kappa_t(x)) \leq \frac{1}{2} \) and \( \tilde{\theta}(U(x)) \leq \frac{1}{2} \). Then we have (after dropping the arguments)

\[
\tilde{\omega}'' \tilde{\omega} = \frac{k-1}{k}(\tilde{\omega}')^2,
\]

\[
\tilde{\theta}'' \tilde{\theta} = \frac{k-1}{k}(\tilde{\theta}')^2,
\]

and so \( a^2 + b^2 \geq 2ab, \tag{8.10} \), and \( k > c^{-1} \) give

\[
(1-c)(\tilde{\theta}'' \tilde{\omega} |\nabla U|^2 + \tilde{\theta}'' \tilde{\omega}' |\nabla \kappa_t|^2) \geq 2(1-c)\frac{k-1}{k} |\tilde{\theta}' \tilde{\omega}'||\nabla U||\nabla \kappa_t|
\]

\[
\geq 2\frac{1-c}{1-2c} \frac{k-1}{k} |\tilde{\theta}' \tilde{\omega}'||\nabla U \cdot \nabla \kappa_t|
\]

\[
\geq 2\tilde{\theta}' \tilde{\omega}' |\nabla U \cdot \nabla \kappa_t|.
\]

On the other hand, (8.7) and (8.11) show that for \( x \in R_t \),

\[
c \tilde{\theta}'' \tilde{\omega} |\nabla U|^2 \geq c^3(k-1)|\tilde{\theta}' \tilde{\omega} |\nabla U | \geq c^3 C^{-1}(k-1) \tilde{\theta}' \tilde{\omega} \Delta U = \tilde{\theta}' \tilde{\omega} \Delta U,
\]

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and the same is true for $\tilde{\theta}$ and $\tilde{\omega}$ exchanged and $\kappa_t$ in place of $U$. Therefore $\Delta \phi^0(x, t) \geq 0$.

Next let $x \in R_t$ be such that $\tilde{\omega}(\kappa_t(x)) \leq \frac{1}{2}$ and $\tilde{\theta}(U(x)) \leq \frac{1}{2}$. Then

$$\Delta \phi^0(x, t) \geq \tilde{\theta}^0 \tilde{\omega}^2 - \tilde{\theta} \tilde{\omega} LC^2 - |\tilde{\theta}| \tilde{\omega} 2 LC^2 - |\tilde{\theta}| \tilde{\omega} C + \tilde{\theta} \tilde{\omega} LC \geq \tilde{\theta}^0 \tilde{\omega}^2 - (|\tilde{\theta}| + \tilde{\theta}) \tilde{\omega} 3 LC^2$$

by the definition of $L$. But then (8.11) gives

$$\Delta \phi^0(x, t) \geq \tilde{\theta}^1 \tilde{\omega}^2 k(k - 1) K^2 - \tilde{\theta}^1 \tilde{\omega} 6 k K L C^2 = \tilde{\theta}^1 \tilde{\omega} k K (\tilde{\theta}^1 \tilde{\omega} c^2 (k - 1) K - 6 L C^2).$$

This is greater than zero provided $\tilde{\theta} \leq \varepsilon \equiv \min\{(6 L C^2 c^{-2} (k - 1) K^{-1})^{-k}, \frac{1}{2}\}$. We get the same conclusion if $\tilde{\omega}(\kappa_t(x)) \leq \frac{1}{2}$ and $\tilde{\theta}(U(x)) \leq \frac{1}{2}$.

This all means that $\Delta \phi^0(x, t) \geq 0$ when $x \in R_t$ and either $\tilde{\omega}(\kappa_t(x)) \leq \varepsilon$ or $\tilde{\theta}(U(x)) \leq \varepsilon$. But then

$$\Delta \phi^0(x, t) + f(\phi^0(x, t)) \geq 0$$

for all $x \in R_t$ (and so for all $x \in D$), provided $f$ is such that $f(T) \geq M$ for $T \in [\varepsilon^2, \frac{4}{9}]$ (recall that $\tilde{\omega}, \tilde{\theta} \leq \frac{1}{2}$). Combining this with (8.12), we find that $\phi^0(x, At)$ is indeed a subsolution of (8.1), so that $u$ is not strongly quenching.

We note that the above method of construction of a subsolution to (8.1) does not extend to the case when $u$ only has $H^1 \setminus C^{1,1}$ eigenfunctions with non-zero eigenvalues.

It turns out that dissipation-enhancing flows quench some reactions without an ignition temperature cutoff, in particular, Arrhenius-type reactions with $f(T) \equiv e^{-c/T}(1 - T)$ and $c > 0$.

**Theorem 8.3.** Assume that $u$ is a periodic incompressible Lipschitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R} \times \mathbb{T}$, and that the reaction function $f$ satisfies (8.2)(i) and $f(T) \leq cT^\alpha$ for some $c > 0$ and $\alpha > 2$ (if $D = \mathbb{R}^2$) resp. $\alpha > 3$ (if $D = \mathbb{R} \times \mathbb{T}$). If $u$ is dissipation-enhancing, then for each $M$ there is $A_0(M)$ such that when $\|T_0\|_{L^1(D)} \leq M, T_0 \in [0, 1]$, and $A > A_0(M)$, the solution of (8.1) satisfies $\|T^A(\cdot, t)\|_{L^\infty(D)} \to 0$ as $t \to \infty$.

**Remarks.** 1. It follows from (18) (see also [22]) that if $f(T) \geq cT^\alpha$ for some $c > 0, \alpha < 2$ (if $D = \mathbb{R}^2$) resp. $\alpha < 3$ (if $D = \mathbb{R} \times \mathbb{T}$), and all small $T$, then the conclusion of the theorem does not hold for any $A$ and $u$.

2. Theorem (8.2)(ii) trivially extends to this setting since by the comparison principle, solution of (8.1) with $\tilde{f} \geq f$ dominates that of (8.1) with $f$.

**Proof.** Let $D = \mathbb{R} \times \mathbb{T}$ and define $I_A \equiv \int_0^\infty \|\phi^A(\cdot, t)\|_{L^\infty(D)}^{\alpha - 1} dt$ where $\phi^A$ is the solution of (1.1) with $\phi_0 \equiv T_0$. It follows from (18) (see also [22, Lemma 2.1]) that the conclusion of the theorem is valid whenever $c(\alpha - 1)I_A < 1$.

Lemma 3.1 in [14] shows that there exists $C < \infty$ such that for each incompressible Lipschitz flow $v$ on $D$ and $t \geq 1$ we have

$$\|\psi(\cdot, t)\|_{L^\infty(D)} \leq C t^{-1/2} \|\psi_0\|_{L^1(D)},$$

with $\psi$ the solution of (2.3). We pick $\tau_0 > 1$ so that

$$c(\alpha - 1)(CM)^{\alpha - 1} \frac{2}{\alpha - 3} \tau_0^{-(\alpha - 3)/2} < \frac{1}{3},$$

(8.15)
\( \delta > 0 \) so that \( c(\alpha - 1)\tau_0\delta^{\alpha - 1} < \frac{1}{3} \), and \( \tau \in (0, \tau_0) \) so that \( c(\alpha - 1)\tau < \frac{1}{3} \). If now \( A_0(M) \) is such that
\[
\| P_\tau(Au) \|_{L^1(D) \to L^\infty(D)} \leq \delta M^{-1}
\]
for all \( A > A_0(M) \), then \( c(\alpha - 1)I_A < 1 \) for such \( A \). This is obtained by estimating \( \| \phi^A(\cdot, t) \|_{L^\infty} \) by 1 for \( t \in [0, \tau] \), by \( \delta \) for \( t \in [\tau, \tau_0) \), and by (8.14) for \( t \geq \tau_0 \).

The case \( D = \mathbb{R}^2 \) is identical (with \( \tau \) in (8.15)) provided we show
\[
\| \psi(\cdot, t) \|_{L^\infty(D)} \leq Ct^{-1}\| \psi_0 \|_{L^1(D)}
\]
for some \( C \), any \( t \geq 1 \), any incompressible Lipschitz flow \( v \), and any solution \( \psi \) of (2.3) on \( D \). We provide the proof of this claim below, essentially following [4].

Solutions of (2.3) satisfy
\[
\frac{d}{dt}\| \psi \|_2^2 = -2\| \nabla \psi \|_2^2 \leq -C\| \psi \|_2\| \psi \|_1^2 \leq -C\| \psi \|_2^2\| \psi_0 \|_1^{-2},
\]
where we used the Nash inequality \( \| \psi \|_2 \leq C\| \nabla \psi \|_2\| \psi \|_1 \) [19] and (2.7) with \( p = 1 \). Dividing by \( \| \psi \|_2^2 \) and integrating in time gives
\[
\| \psi(\cdot, t) \|_{L^2} \leq Ct^{-1/2}\| \psi_0 \|_{L^1}.
\]
This shows that \( \| P_t(v) \|_{L^1 \to L^2} \leq Ct^{-1/2} \). But \( P_t(v) \) is the adjoint of \( P_t(-v) \) which satisfies the same bound, so we obtain
\[
\| P_{2t}(v) \|_{L^1 \to L^\infty} \leq \| P_t(v) \|_{L^1 \to L^2}\| P_t(v) \|_{L^2 \to L^\infty} = \| P_t(v) \|_{L^1 \to L^2}\| P_t(-v) \|_{L^1 \to L^2} \leq C^2t^{-1},
\]
which gives (8.16).

Note that the same proof with the inequality \( \| \psi \|_2^{1+2/n} \leq C\| \nabla \psi \|_2\| \psi \|_1^{2/n} \) in \( \mathbb{R}^n \) [19] gives (8.16) with \( t^{-n/2} \) when \( D = \mathbb{R}^n \). The claim of the theorem can be extended to this case with \( \alpha > 1 + \frac{2}{n} \).

9. DISSIPATION-ENHANCEING FLOWS IN MORE DIMENSIONS

Most of Sections 5 and 6 does not extend to higher dimensions or time-periodic flows. An exception are Lemmas 5.3 and 5.4 which have both been stated in any dimension. They also extend to time-periodic flows. In that case Lemma 5.3 deals with \( H^1 \) eigenfunctions of the unitary evolution operator \( U_{\tau_0} \) generated by the flow (with \( \tau_0 \) the time-period) rather than those of \( u \), and the proof stays the same. Notice that the two sets of eigenfunctions coincide when \( u \) is time-independent. The statement of Lemma 5.4 is unchanged in this case, and the proof uses [14] to obtain (5.6).

We call a time-dependent flow \( u \) on \( D = \mathbb{R}^n \times \mathbb{T}^m \) dissipation-enhancing if for any \( 1 \leq p < q \leq \infty \) and \( \tau > 0 \),
\[
\| P_{\tau}(Au) \|_{L^p(D) \to L^q(D)} \to 0 \quad \text{as} \quad A \to \infty,
\]
where \( u^A(x, t) \equiv u(x, At) \). This is the natural choice for \( u^A \) as it ensures that the solutions of \( X'(t) = u^A(X(t), t) \) with \( X(0) = x_0 \) have the same orbits for different \( A \). The definition of strongly quenching time-dependent flows is changed analogously.
Theorem 9.1. Assume that $u$ is a space- and time-periodic incompressible Lipschitz flow on
$D = \mathbb{R}^n \times \mathbb{T}^m$ with $n \geq 1$, $m \geq 0$, a cell of spatial periodicity $C \subseteq D$, and time-period $\tau_0$. If
the unitary evolution operator $U_{\tau_0}$ on $C$ has no non-constant eigenfunctions in $H^1(C)$, then
$u$ is dissipation-enhancing and strongly quenching.

Proof. The proof essentially follows Section 5, but is simpler due to the absence of non-
constant first integrals. Choose any $\tau, \delta > 0$ and let $k \in \mathbb{Z}$ be larger than $\delta^{-2/n}$. Let
$\|\bar{\phi}_0\|_{L^1(D)} \leq 1$ and periodize the problem and $\bar{\phi}_0$ onto $\mathcal{M} \equiv (k\mathbb{T})^n \times \mathbb{T}^m$ as we did in
Section 5. We define $\phi_0(x) \equiv \tilde{\phi}^A(x, \tau)$ so that by Lemma 5.4 in $d = n + m$ dimensions,
$\|\phi_0\|_{L^1(\mathcal{M})} \leq 1$ and $\|\phi_0\|_{L^\infty(\mathcal{M})} \leq C\tau^{-d/2}$
with $C = C(d)$. This then gives
$\|\phi_0\|_{L^2(\mathcal{M})} \leq C^{1/2}\tau^{-d/4}$ and $|\bar{\phi}_0| \leq k^{-n} \leq \delta k^{-n/2}$
where $\bar{\phi}_0$ is the average of $\phi_0$ over $\mathcal{M}$. Consider the operators $\Gamma \equiv -\Delta$ and $L_t \equiv iu(\cdot, t) \cdot \nabla$
on the space $\mathcal{H} \equiv L^2(\mathcal{M})$. From Lemma 5.3(iii) for time-periodic flows we know that $U_{\tau_0}$, now as an operator on $\mathcal{H}$, has no non-constant eigenfunctions in $H^1(\Gamma)$. It follows from
Theorem 4.1 that for each $A > A_1(\tau, \delta)$ (with $A_1$ independent of $\phi_0$), there is $t \leq \tau$ such that
the solution $\phi^A$ of (4.1) satisfies
$\|\phi^A(\cdot, t)\|_{L^2(\mathcal{M})} \leq \|\phi^A(\cdot, t) - \bar{\phi}_0\|_{L^2(\mathcal{M})} + \|\bar{\phi}_0\|_{L^2(\mathcal{M})} \leq \delta + (k^n (\delta k^{-n/2})^2)^{1/2} = 2\delta$.
This is because the average of $\phi^A$ stays constant and so $P_h \phi^A(\cdot, t) = |\mathcal{M}|^{-1} \int_{\mathcal{M}} \phi^A(x, t) \, dx = \bar{\phi}_0$. Another application of Lemma 5.4 gives
$\|\tilde{\phi}^A(\cdot, 3\tau)\|_{L^\infty(\mathcal{M})} = \|\phi^A(\cdot, 2\tau)\|_{L^\infty(\mathcal{M})} \leq \|\phi^A(\cdot, t + \tau)\|_{L^\infty(\mathcal{M})} \leq 2C\tau^{-d/2}\delta$
and so the same is true for the original problem on $D$. Since $\delta$ was arbitrary and $C$ only
depends on $d$, (9.1) follows with $p = 1$ and $q = \infty$ for each $\tau > 0$. As in Section 5
interpolation provides the other cases, so $u$ is dissipation-enhancing. Strong quenching is
then immediate as in Theorem 8.2(i). \qed

The complete characterization of (periodic incompressible) dissipation-enhancing flows in
more than two dimensions, even in the time-independent case, remains an open problem.

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