On Perfect Powers in $k$-Generalized Pell-Lucas Sequence

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Abstract

Let $k \geq 2$ and let $(Q_n^{(k)})_{n \geq 2-k}$ be the $k$-generalized Pell sequence defined by

$$Q_n^{(k)} = 2Q_{n-1}^{(k)} + Q_{n-2}^{(k)} + \cdots + Q_{n-k}^{(k)}$$

for $n \geq 2$ with initial conditions

$$Q_{-k}^{(k)} = Q_{-k-2}^{(k)} = \cdots = Q_{-2}^{(k)} = 0, \quad Q_0^{(k)} = 2, \quad Q_1^{(k)} = 2.$$

In this paper, we solve the Diophantine equation

$$Q_n^{(k)} = y^m$$

in positive integers $n, m, y, k$ with $m, y, k \geq 2$. We show that all solutions $(n, m, y)$ of this equation in positive integers $n, m, y, k$ such that $2 \leq y \leq 100$ are given by $(n, m, y) = (3, 2, 4), (3, 4, 2)$ for $k \geq 3$. Namely, $Q_3^{(k)} = 16 = 2^4 = 4^2$ for $k \geq 3$.

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1 Introduction

Let $k, r$ be an integer with $k \geq 2$ and $r \neq 0$. Let the linear recurrence sequence $(G_n^{(k)})_{n \geq 2-k}$ of order $k$ be defined by

$$G_n^{(k)} = rG_{n-1}^{(k)} + G_{n-2}^{(k)} + \cdots + G_{n-k}^{(k)}$$

(1)
for \( n \geq 2 \) with the initial conditions \( G_{-(k-2)}^{(k)} = G_{-(k-3)}^{(k)} = \cdots = G_{-1}^{(k)} = 0, \)
\( G_0^{(k)} = a, \) and \( G_1^{(k)} = b. \) For \((a, b, r) = (0, 1, 1)\), the sequence \( G_n^{(k)} \) is called \( k \)-generalized Fibonacci sequence \( \left( F_n^{(k)} \right)_{n \geq 2-k} \) (see [6]). For \((a, b, r) = (0, 1, 2)\) and \((a, b, r) = (2, 2, 2)\), the sequence \( G_n^{(k)} \) is called \( k \)-generalized Pell sequence \( \left( P_n^{(k)} \right)_{n \geq 2-k} \) and \( k \)-generalized Pell-Lucas sequence \( \left( Q_n^{(k)} \right)_{n \geq 2-k} \) respectively (see [17]). The terms of these sequences are called \( k \)-generalized Fibonacci numbers, \( k \)-generalized Pell numbers and \( k \)-generalized Pell-Lucas numbers, respectively. When \( k = 2 \), we have Fibonacci, Pell and Pell-Lucas sequences, \( (F_n)_{n \geq 0}, \ (P_n)_{n \geq 0}, \) and \( (Q_n)_{n \geq 0}, \) respectively.

There has been much interest in when the terms of linear recurrence sequences are perfect powers. For instance, in [13], Ljunggren showed that for \( n \geq 2, \) \( P_n \) is a perfect square precisely for \( P_7 = 13^2 \) and \( P_n = 2r^2 \) precisely for \( P_2 = 2. \) In [13], Cohn solved the same equations for Fibonacci numbers. Later, these problems are extended by Peth\'o for Pell numbers and by Bugeaud, Mignotte and Siksek for Fibonacci numbers. Peth\'o [21] and Cohn [14] independently found all perfect powers in the Pell sequence. They proved that the only positive integer solution \((n, y, m)\) with \( m \geq 2 \) and \( y \geq 2 \) of the Diophantine equation \( P_n = y^m \) is given by \((n, y, m) = (7, 13, 2). \) Bugeaud, Mignotte and Siksek [11] solved the Diophantine equation \( F_n = y^p \) for \( p \geq 2 \) using modular approach and classical linear forms in logarithms. Bravo and Luca showed in [6] that the Diophantine equation \( F_n^{(k)} = 2^m \) in positive integers \( n, k, m \) with \( k \geq 2 \) has the solutions \((n, k, m) = (6, 2, 3), (1, k, 0)\) and \((n, k, m) = (t, k, t - 2)\) for all \( 2 \leq t \leq k + 1. \) Except these, recently, for the studies related to \( k \)-generalized Fibonacci sequence, one can consult [11, 12, 10].

In [4], the authors found all perfect powers in the Pell-Lucas sequence and proved the following result:

**Theorem 1** Let \( n, y, m \in \mathbb{N} \) with \( m \geq 2. \) Then the equation \( Q_n = y^m \) has no integer solutions and the equation \( Q_n = 2y^m \) has only the solution \( Q_1 = 2 \cdot 1^m. \)

In this paper, we will handle the Diophantine equation
\[
Q_n^{(k)} = y^m, \ k, n, m, y \in \mathbb{Z}^+ \text{ with } k, y, m \geq 2.
\] (2)

We will show that the solutions \((n, m, y)\) of Diophantine equation (2) with \( 2 \leq y \leq 100 \) are given by \((n, m, y) = (3, 2, 4), (3, 4, 2)\) for \( k \geq 3. \) Namely, \( Q_1^{(k)} = 16 = 2^4 = 4^2 \) for \( k \geq 3. \)

## 2 Preliminaries

In this section, we will mention some facts and properties of \( k \)-generalized Pell and Pell-Lucas sequences. It can be seen that the characteristic polynomial of
these sequences is
\[ \Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1. \quad (3) \]

Let us denote the roots of the polynomial in (3) by \( \alpha_j \) for \( j = 1, 2, \ldots, k \). We know from Lemma 1 given in [23] that the polynomial \( \Psi_k(x) \) has exactly one positive real root located between 2 and 3. Particularly, let \( \alpha = \alpha(k) = \alpha_1 \), be the positive real root of the polynomial \( \Psi_k(x) \). So,
\[ 2 < \alpha < 3. \quad (4) \]

The other roots are strictly inside the unit circle.

In [8], the Binet-like formula for the \( k \)-generalized Pell number is given by
\[ P_n^{(k)} = \sum_{j=1}^{k} \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n. \]

If we follow the method given in [8] to obtain the Binet-like formula for \( k \)-generalized Pell-Lucas numbers, then we get
\[ Q_n^{(k)} = \sum_{j=1}^{k} \frac{2(\alpha_j - 1)^2}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \alpha_j^n. \quad (5) \]

From [9], we can give the following lemma, which will be used in the proof of Theorem [11].

**Lemma 2** Let \( \alpha_j \)'s for \( j = 1, 2, \ldots, k \), be the roots of \( \Psi_k(x) \), and let \( \alpha = \alpha_1 \) be dominant root of \( \Psi_k(x) \). Then for \( j \geq 1 \) and \( k \geq 2 \), the inequality
\[ \left| \frac{(\alpha_j - 1)}{\alpha_j^2 - 1 + k(\alpha_j^2 - 3\alpha_j + 1)} \right| < 1 \quad (6) \]
holds.

Throughout this paper, \( \alpha \) denotes the positive root of the polynomial \( \Psi_k(x) \). The relation between \( \alpha \) and \( P_n^{(k)} \) is given by
\[ \alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1} \quad (7) \]
for all \( n \geq 1 \). For a proof of (7), see [8]. Also, Kılıç [17] proved that
\[ P_n^{(k)} = F_{2n-1} \quad (8) \]
for all \( 1 \leq n \leq k + 1 \).

Now, we prove a result concerning the relation between \( P_n^{(k)} \) and \( Q_n^{(k)} \).
Lemma 3 Let \( k \geq 2 \) be an integer. Then the relation

\[
Q_n^{(k)} = 2 \left( P_{n+1}^{(k)} - P_n^{(k)} \right)
\]

always holds.

Proof. Let

\[
G(x) = \sum_{i=0}^{\infty} G_i^{(k)} x^i
\]

be the generating function for \( G_n^{(k)} \) given by the relation (1). Using the relation (1) with initial conditions, it can be seen that

\[
G(x) = a + \frac{(b - 2a)x}{1 - 2x - x^2 - \ldots - x^k}
\]

for \( r = 2 \). So, the generating function \( P(x) \) for \( P_n^{(k)} \) is

\[
P(x) = \frac{x}{1 - 2x - x^2 - \ldots - x^k}
\]

and the generating function \( Q(x) \) for \( Q_n^{(k)} \) is

\[
Q(x) = \frac{2 - 2x}{1 - 2x - x^2 - \ldots - x^k}.
\]

From (10) and (11), the proof follows.

Now, we will give some theorems and lemmas from [8], which will be useful in the next section.

Theorem 4 ([8], Theorem 3.1) Let \( k \geq 2 \) be an integer. Then, for all \( n \geq 2 - k \), we have

\[
P_n^{(k)} = \sum_{i=1}^{k} g_k(\alpha_i)\alpha_i^n
\]

and

\[
\left| P_n^{(k)} - g_k(\alpha)\alpha^n \right| < \frac{1}{2},
\]

where \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_k \) are the roots of the characteristic equation \( \Psi_k(x) = 0 \) and

\[
g_k(z) = \frac{z - 1}{(k + 1)z^2 - 3kz + k - 1}.
\]

Thus, from (5), we can write

\[
Q_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 2)g_k(\alpha_i)\alpha_i^n.
\]
Lemma 5 ([8, Lemma 3.2]) Let $k, l \geq 2$ be integers. Then

(a) If $k > l$, then $\alpha(k) > \alpha(l)$, where $\alpha(k)$ and $\alpha(l)$ are the values of $\alpha$ relative to $k$ and $l$, respectively.

(b) $\varphi^2(1 - \varphi^{-k}) < \alpha < \varphi^2$, where $\varphi = \frac{1 + \sqrt{5}}{2}$ is the golden section.

(c) $g_k(\varphi^2) = \frac{1 + \sqrt{5}}{\varphi + 2}$.

(d) $0.276 < g_k(\alpha) < 0.5$.

(e) If $k \geq 6$, then

$$c_k < \alpha < \varphi^2,$$

where

$$c_k = \frac{3k + \sqrt{5k^2 + 4}}{2k + 2}.$$

It is easy to observe that the inequality in Lemma 5 (e) holds for $k \geq 2$. If we consider the function $g_k(x)$ defined in (12) as a function of a real variable, then it can be easily seen that the function $g_k(x)$ is decreasing and continuous in the interval $(c_k, \infty)$ (also see Lemma 3.1 in [8]). Therefore, by Lemma 5, we have

$$g_k(\varphi^2) < g_k(t) < g_k(\alpha) < \frac{1}{2}$$

for every $t \in (\alpha, \varphi^2)$. The inequality (14) implies that

$$(k + 1)t^2 - 3kt + k - 1 > 2$$

for $t \in (\alpha, \varphi^2)$.

For solving the equation (2), we use linear forms in logarithms and Baker’s theory. For this, we will give some notions, theorem, and lemmas related to linear forms in logarithms and Baker’s Theory.

Let $\eta$ be an algebraic number of degree $d$ with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the $a_i’s$ are integers with $\gcd(a_0, \ldots, a_n) = 1$ and $a_0 > 0$ and the $\eta^{(i)}$’s are conjugates of $\eta$. Then

$$h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \left( \max \{ |\eta^{(i)}|, 1 \} \right) \right)$$

is called the logarithmic height of $\eta$. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b \geq 1$, then $h(\eta) = \log (\max \{ |a|, b \})$.

We give some properties of the logarithmic height whose proofs can be found in [12]:

$$h(\eta + \gamma) \leq h(\eta) + h(\gamma) + \log 2,$$

$$h(\eta^{\pm 1}) \leq h(\eta) + h(\gamma),$$

$$5$$
\[ h(\eta^m) = |m|h(\eta). \]  \hspace{1cm} (19)

Now, we can deduce the following estimation for \( h(g_k(\alpha)) \) from Lemma 6 given in [10].

**Lemma 6** Let \( k \geq 2 \). Then \( h(g_k(\alpha)) < 5 \log k \).

The following theorem is deduced from Corollary 2.3 of Matveev [20] and provides a large upper bound for the subscript \( n \) in the equation (2) (also see Theorem 9.4 in [11]).

**Theorem 7** Assume that \( \gamma_1, \gamma_2, \ldots, \gamma_t \) are positive real algebraic numbers in a real algebraic number field \( K \) of degree \( D \), \( b_1, b_2, \ldots, b_t \) are rational integers, and

\[ \Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \]

is not zero. Then

\[ |\Lambda| > \exp \left( -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1A_2\cdots A_t \right), \]

where

\[ B \geq \max \{|b_1|, \ldots, |b_t|\}, \]

and \( A_i \geq \max \{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \) for all \( i = 1, \ldots, t \).

Now we give a lemma which was proved in [5]. It is a variation of the lemma given by Dujella and Pethő [15]. The lemma given in [15] is a variation of a result of Baker and Davenport [3]. This lemma will be used to reduce the upper bound for the subscript \( n \) in the equation (2). For any real number \( x \), we let \( ||x|| = \min \{|x-n| : n \in \mathbb{Z}\} \) be the distance from \( x \) to the nearest integer.

**Lemma 8** Let \( M \) be a positive integer, let \( p/q \) be a convergent of the continued fraction of the irrational number \( \gamma \) such that \( q > 6M \), and let \( A, B, \mu \) be some real numbers with \( A > 0 \) and \( B > 1 \). Let \( \epsilon := ||\mu q|| - M||\gamma q|| \). If \( \epsilon > 0 \), then there exists no solution to the inequality

\[ 0 < |u\gamma - v + \mu| < AB^{-w}, \]

in positive integers \( u, v, \) and \( w \) with

\[ u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}. \]

The following lemma can be found in [22].

**Lemma 9** Let \( a, x \in \mathbb{R} \). If \( 0 < a < 1 \) and \( |x| < a \), then

\[ |\log(1 + x)| < \frac{-\log(1-a)}{a} \cdot |x| \]

and

\[ |x| < \frac{a}{1 - e^{-a}} \cdot |e^x - 1|. \]
3 Main Theorem

Now, we prove a lemma, which will be used in the next theorem.

Lemma 10 Let \( k \geq 2 \) be an integer. Then

(a) \( \alpha^{n-1} < Q_n^{(k)} < 2\alpha^n \) for all \( n \geq 1 \).

(b) \( \left| Q_n^{(k)} - (2\alpha - 2)g_k(\alpha)\alpha^n \right| < 2 \) for all \( n \geq 2 - k \).

(c) \( Q_n^{(k)} = 2F_{2n} \) for \( 1 \leq n \leq k \).

Proof. We have the relation \( Q_n^{(k)} = 2 \left( P_{n+1}^{(k)} - P_n^{(k)} \right) \) by Lemma 3. We use (7), and obtain

\[
Q_n^{(k)} = 2 \left( P_{n+1}^{(k)} - P_n^{(k)} \right) \leq 2(\alpha^n - \alpha^{n-2}) \leq 2\alpha^{n-2}(\alpha^2 - 1) < 2\alpha^n,
\]

and also

\[
Q_n^{(k)} = 2 \left( P_{n+1}^{(k)} - P_n^{(k)} \right) \geq P_{n+1}^{(k)} + P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k+1}^{(k)} > \alpha^{n-1}.
\]

(b) By using Theorem 4, we obtain

\[
\left| Q_n^{(k)} - (2\alpha - 2)g_k(\alpha)\alpha^n \right| = \left| P_{n+1}^{(k)} - 2P_n^{(k)} - 2g_k(\alpha)\alpha^{n+1} + 2g_k(\alpha)\alpha^n \right|
\leq 2 \left| P_{n+1}^{(k)} - g_k(\alpha)\alpha^{n+1} \right| + 2 \left| P_n^{(k)} - g_k(\alpha)\alpha^n \right|
< 2.
\]

(c) From the equalities (8) and (9), the proof follows. ■

Theorem 11 All solutions of Diophantine equation (2) satisfies the inequality

\[
n < 1.64 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log y \cdot \log n.
\] (20)

Proof. Assume that the Diophantine equation (2) holds. If \( 1 \leq n \leq k \), then we have \( Q_n^{(k)} = 2F_{2n} = y^m \) by (8) and (9). From here, by Theorem 2 given in [19], \( F_{2n} = 2^{n-1}(y/2)^m \) implies that \( n \leq 6 \). Thus, the identity (20) is satisfied. Then we suppose that \( n \geq k + 1 \). In this case, \( n \geq 3 \). Let \( \alpha \) be positive real root of \( \Psi_k(x) \) given in [3]. Then \( 2 < \alpha < \varphi^2 < 3 \) by (4) and Lemma 5. Using Lemma 10 (a), we get

\[
\alpha^{n-1} < y^m < 2\alpha^n.
\]

Making necessary calculations, we obtain

\[
m < \frac{\log 2}{\log y} + n \frac{\log \varphi^2}{\log y} < 1 + n \frac{\log \varphi^2}{\log 2} < 1.73n
\] (21)

for \( n \geq 3 \). Now, let us rearrange the equation (2) by using Lemma 10 (b). Thus, we have

\[
|y^m - (2\alpha - 2)g_k(\alpha)\alpha^n| < 2.
\] (22)
If we divide both sides of the inequality (22) by $(2\alpha - 2)g_k(\alpha)\alpha^n$, we get
\[
|y^m\alpha^n((2\alpha - 2)g_k(\alpha))^{-1} - 1| < \frac{2}{(2\alpha - 2)g_k(\alpha)\alpha^n} \leq \frac{\alpha^{-n}}{2 \cdot 0.276} \leq \frac{1.82}{\alpha^n}
\] (23)

by Lemma (d). In order to use the result of Theorem we take
\[
(\gamma_1, b_1) := (y, m), \ (\gamma_2, b_2) := (\alpha, -n), \ (\gamma_3, b_3) := ((2\alpha - 2)g_k(\alpha), -1).
\]

The number field containing $\gamma_1, \gamma_2,$ and $\gamma_3$ are $K = \mathbb{Q}(\alpha)$, which has degree $D = k$. We show that the number
\[
\Lambda_1 := y^m\alpha^n((2\alpha - 2)g_k(\alpha))^{-1} - 1
\]
is nonzero. Contrast to this, assume that $\Lambda_1 = 0$. Then
\[
y^m = (2\alpha - 2)g_k(\alpha)\alpha^n = \frac{(2\alpha - 2)(\alpha - 1)}{(k + 1)\alpha^2 - 3k\alpha + k - 1}\alpha^n.
\]
Conjugating the above equality by some automorphism of the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and taking absolute values, we get
\[
y^m = \left| \frac{(2\alpha_i - 2)(\alpha_i - 1)}{(k + 1)\alpha_i^2 - 3k\alpha_i + k - 1}\alpha_i^n \right|
\]
for some $i > 1$, where $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $\Psi_k(x)$. As $|\alpha_i| < 1$, using the inequality (d), we can write the inequality
\[
y^m = |2\alpha_i - 2| \left| \frac{(\alpha_i - 1)}{(k + 1)\alpha_i^2 - 3k\alpha_i + k - 1}\alpha_i^n \right| |\alpha_i|^n < 4.
\]
Hence, we have $y_m < 4$, which is impossible as $m, y \geq 2$. Therefore $\Lambda_1 \neq 0$. Moreover, since $h(y) = \log y, h(\gamma_2) = \frac{\log \alpha}{k} < \frac{\log 3}{k}$ by (16), we can take $A_1 := k \log y, A_2 := \log 3$. Now, let find the approximate value of $h((2\alpha - 2)g_k(\alpha))$. We know that $h(g_k(\alpha)) < 5 \log k$ by Lemma (d). Besides, since the minimal polynomial of $\alpha - 1$ over the integers is
\[
(x + 1)^k - 2(x + 1)^{k-1} - (x + 1)^{k-1} - \cdots - (x + 1) - 1,
\]
it can be seen that
\[
h(\alpha - 1) = \frac{1}{k} \left( \log 1 + \sum_{i=1}^k \log \left( \max\{|\alpha_i - 1|, 1\} \right) \right) \leq \log 2.
\]
Consequently, we have
\[
h((2\alpha - 2)g_k(\alpha)) \leq h(2) + h(\alpha - 1) + h(g_k(\alpha)) < \log 2 + \log 2 + 5 \log k < 8 \log k.
\]
Hence, we can take $A_3 := 8k \log k$. Also, since $m \leq 1.73n$, it follows that $B := 1.73n$. Thus, taking into account the inequality (23) and using Theorem 7, we obtain

$$\frac{1.82}{\alpha^m} > |A_1| > \exp\left(-C \cdot k^2 (1 + \log k)(1 + \log (1.73n))(k \log y)(\log 3)(8k \log k)\right)$$

and so

$$n \log \alpha - \log(1.82) < C \cdot k^2 \cdot 3 \log k \cdot 3 \log n \cdot (k \log y)(\log 3)(8k \log k) ,$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5}$ and we have used the fact that $1 + \log k < 3 \log k$ for $k \geq 2$ and $1 + \log (1.73n) < 3 \log n$ for $n \geq 3$. From the inequality (24), a quick computation with Mathematica yields

$$n < 1.64 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log y \cdot \log n.$$ 

Thus, the proof is completed.

**Theorem 12** Let $2 \leq y \leq 100$. Then all solutions $(n, m, y)$ of Diophantine equation (2) are given by $(n, m, y) = (3, 2, 4), (3, 4, 2)$ with $k \geq 3$.

**Proof.** Assume that Diophantine equation (2) is satisfied for $2 \leq y \leq 100$. If $1 \leq n \leq k$, then we have $Q^{(k)} = 2F_{2n} = y^m$ by Lemma 10 (c). From here, by Theorem 2 given in [19], $F_{2n} = 2^{m-1}(y/2)^m$ implies that $(n, m, y) = (3, 2, 4), (3, 4, 2)$. Now we assume that $n \geq k + 1$. If $k = 2$, then $n \geq 3$ and we have $Q_n = y^m$. By Theorem 1, the equation $Q_n = y^m$ has no solutions in positive integers $n \geq 3$ and $m \geq 2$. Therefore, assume that $k \geq 3$. In this case $n \geq 4$. Also, since $y \leq 100$, by (24), we get

$$n < 7.56 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log n.$$ 

Rearranging the last inequality as

$$\frac{n}{\log n} < 7.56 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2$$

and using the fact that

if $A \geq 3$ and $\frac{n}{\log n} < A$, then $n < 2A \log A,$

we obtain

$$n < 15.12 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot \log (7.56 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2) \quad (25)$$

$$< 15.12 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot (32 + 4 \log k + 2 \log(\log k))$$

$$< 15.12 \cdot 10^{13} \cdot k^4 \cdot (\log k)^2 \cdot 34 \log k$$

$$< 5.141 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3,$$
where we have used the fact that $32 + 4 \log k + 2 \log(\log k) < 34 \log k$ for all $k \geq 3$. Let $k \in [3, 510]$. Then, we get $n < 2.831 \cdot 10^{28}$ by (25). Now, let us try to reduce the upper bound on $n$ applying Lemma 8. Let $k \in [3, 510]$. Then, we get $n < 2.831 \cdot 10^{28}$ by (25). Now, let us try to reduce the upper bound on $n$ applying Lemma 8. Let $z_1 := m \log y - n \log \alpha + \log \left(\left((2\alpha - 2)g_k(\alpha)\right)^{-1}\right)$

and $x = e^{z_1} - 1$. Then, from (23), it is seen that

$$|x| = |e^{z_1} - 1| < \frac{1.82}{\alpha^n} < 0.12$$

for $n \geq 4$. Choosing $a := 0.12$, we get the inequality

$$|z_1| = |\log(x + 1)| < \frac{-\log(1 - 0.12)}{(0.12)} \cdot \frac{1.82}{\alpha^n} < \frac{1.94}{\alpha^n}$$

by Lemma 8. Thus, it follows that

$$0 < |m \log y - n \log \alpha + \log \left(\left((2\alpha - 2)g_k(\alpha)\right)^{-1}\right)| < \frac{1.94}{\alpha^n}.$$ 

Dividing this inequality by $\log \alpha$, we get

$$0 < |m \gamma - n + \mu| < A \cdot B^{-w}, \quad (26)$$

where

$$\gamma := \frac{\log y}{\log \alpha}, \quad \mu := \frac{\log \left(\left((2\alpha - 2)g_k(\alpha)\right)^{-1}\right)}{\log \alpha}, \quad A := 1.94, \quad B := \alpha, \quad w := n.$$

It can be easily seen that $\frac{\log y}{\log \alpha}$ is irrational. If it were not, then we could write $\frac{\log y}{\log \alpha} = \frac{b}{a}$ for some positive integers $a$ and $b$. This implies that $y^a = \alpha^b$. Conjugating this equality by some automorphism belonging to the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and taking absolute values, we get $y^a = |\alpha_i|^b$ for any $i > 1$. This is impossible since $|\alpha_i| < 1$ and $y \geq 2$. If we take $M := 4.9 \cdot 10^{28}$, which is an upper bound on $m$ since $m \leq 1.73n < 4.9 \cdot 10^{28}$, we found that $q_{81}$, the denominator of the 81 th convergent of $\gamma$ exceeds $6M$. Furthermore, a quick computation with Mathematica gives us that the value

$$\frac{\log(Aq_{81}/\epsilon)}{\log B}$$

is less than 144.6 for all $k \in [3, 510]$. So, if the inequality (26) has a solution, then

$$n < \frac{\log(Aq_{81}/\epsilon)}{\log B} < 144.6,$$
which shows that \( n \leq 144 \). In this case, \( m \leq 249 \) by (21). A quick computation with Mathematica gives us that the equation \( Q_n^{(k)} = y^m \) with \( 2 \leq y \leq 100 \) has no solutions for \( n \in [4, 144] \), \( m \in [2, 249] \) and \( k \in [3, 510] \). Thus, this completes the analysis in the case \( k \in [3, 510] \). From now on, we can assume that \( k > 510 \). Then we can see from (25) that the inequality
\[
n < 5.141 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3 < \varphi^{k/2-2} < \varphi^{k/2}
\] holds for \( k > 510 \).

Now, let \( \lambda > 0 \) be such that \( \alpha + \lambda = \varphi^2 \). By Lemma 5 (b), we obtain
\[
\lambda = \varphi^2 - \alpha < \varphi^2 - \varphi^2 (1 - \varphi^{-k}) = \varphi^{-k+2},
\]
that is,
\[
\lambda < \frac{1}{\varphi^{k-2}}. \tag{28}
\]
Also \( 2\alpha - 2 = 2\varphi^2 - 2\lambda - 2 = 2\varphi - 2\lambda \), and so
\[
2\varphi > 2\alpha - 2 > 2\varphi - \frac{2}{\varphi^{k-2}}.
\]
Moreover,
\[
\alpha^n = (\varphi^2 - \lambda)^n = \varphi^{2n}(1 - \frac{\lambda}{\varphi^2})^n
\]
\[
= \varphi^{2n} e^{n \log (1 - \frac{\lambda}{\varphi^2})} \geq \varphi^{2n} e^{-n\lambda} \geq \varphi^{2n} (1 - n\lambda)
\]
\[
> \varphi^{2n} \left(1 - \frac{n}{\varphi^{k-2}}\right),
\]
where we have used the facts that \( \log (1 - x) \geq -\varphi^2 x \) for \( 0 < x < 0.906 \), and \( e^{-x} > 1 - x \) for all \( x \in \mathbb{R} \setminus \{0\} \). Thus,
\[
\alpha^n > \varphi^{2n} - \frac{n\varphi^{2n}}{\varphi^{k-2}} > \varphi^{2n} - \frac{\varphi^{2n}}{\varphi^{k/2}}
\]
by (27). In this case,
\[
(2\alpha - 2)\alpha^n > \left(2\varphi - \frac{2}{\varphi^{k-2}}\right) \left(\varphi^{2n} - \frac{\varphi^{2n}}{\varphi^{k/2}}\right) \tag{29}
\]
\[
= 2\varphi^{2n+1} - 2\varphi^{2n} \left(\frac{\varphi}{\varphi^{k/2}} + \frac{1}{\varphi^{k-2}} - \frac{1}{\varphi^{3k/2-2}}\right)
\]
\[
> 2\varphi^{2n+1} - \frac{4\varphi^{2n}}{\varphi^{k/2}},
\]
where we have used the fact that
\[
\frac{\varphi}{\varphi^{k/2}} + \frac{1}{\varphi^{k-2}} - \frac{1}{\varphi^{3k/2-2}} < \frac{2}{\varphi^{k/2}}
\]
for $k > 510$. Since $\alpha < \varphi^2$, it follows that

$$(2\alpha - 2)\alpha^n < 2\varphi \cdot \varphi^{2n} < 2\varphi^{2n+1} + \frac{4\varphi^{2n}}{\varphi^{k/2}}$$

and so we have

$$|(2\alpha - 2)\alpha^n - 2\varphi^{2n+1}| < \frac{4\varphi^{2n}}{\varphi^{k/2}}.$$  \hspace{1cm} (30)

Let us consider

$$g_k(x) = \frac{x - 1}{(k+1)x^2 - 3kx + k - 1},$$

defined in (12) as a function of a real variable. By the Mean-Value Theorem, we can say that there exist some $\theta \in (\alpha, \varphi^2)$ such that

$$g_k'(\theta) = g_k(\alpha) - g_k(\varphi^2).$$ \hspace{1cm} (31)

Calculating $g_k'(\theta)$ and using Lemma 5, the inequalities (14) and (15), we get

$$|g_k'(\theta)| = \left| \frac{(k+1)\theta^2 - 2k\theta - 2\theta + 2k + 1}{((k+1)\theta^2 - 3k\theta + k - 1)^2} \right|$$

$$< 1 + \frac{(k-2)\theta + k + 2}{(k+1)\theta^2 - 3k\theta + k - 1}$$

$$= 1 + g_k(\theta)\frac{(k-2)\theta + k + 2}{\theta - 1}$$

$$= 1 + g_k(\alpha)\left( k - 2 + \frac{2k}{\theta - 1} \right)$$

$$< 1 + \left( \frac{k-2}{2} + \frac{k}{\theta - 1} \right)$$

$$< \frac{3k}{2}.$$  

Hence, from (31), it follows that

$$|g_k(\alpha) - g_k(\varphi^2)| = |\alpha - \varphi^2| \left| g_k'(\theta) \right| = \lambda \left| g_k'(\theta) \right|,$$

which implies that

$$|g_k(\alpha) - g_k(\varphi^2)| < \frac{3k/2}{\varphi^{k-2}} < \frac{4k}{\varphi^k}$$ \hspace{1cm} (32)

by (28).

Now let us record what we made.

**Lemma 13** Let $k > 510$ and let $\alpha$ be dominant root of $\Psi_k(x)$. Let consider $g_k(x)$ defined in (12) as a function of a real variable. Then

$$g_k(\alpha) = g_k(\varphi^2) + \eta,$$

where $|\eta| < \frac{4k}{\varphi^k}$.
Taking into account the inequality \((30)\) and Lemma 13, we can write
\[
(2\alpha - 2)\alpha^n = 2\varphi^{2n+1} + \delta \quad \text{and} \quad g_k(\alpha) = g_k(\varphi^2) + \eta
\] (33)
such that
\[
|\delta| < \frac{4\varphi^{2n}}{\varphi^{k/2}} \quad \text{and} \quad |\eta| < \frac{4k}{\varphi^k}.
\] (34)
Thus, since \(g_k(\varphi^2) = \frac{1}{\varphi + 2}\) by Lemma 15 (c), it is seen that
\[
(2\alpha - 2)g_k(\alpha)\alpha^n = \frac{2\varphi^{2n+1}}{\varphi + 2} + \frac{\delta}{\varphi + 2} + 2\varphi^{2n+1}\eta + \eta\delta.
\] (35)
So, using (22), (34), and (35), we obtain
\[
|y^m - \frac{2\varphi^{2n+1}}{\varphi + 2}| = |y^m - (2\alpha - 2)g_k(\alpha)\alpha^n + \frac{\delta}{\varphi + 2} + 2\varphi^{2n+1}\eta + \eta\delta|
\leq |y^m - (2\alpha - 2)g_k(\alpha)\alpha^n| + |\delta| + 2\varphi^{2n+1}|\eta| + |\eta||\delta|
< 2 + \frac{4\varphi^{2n}}{\varphi^{k/2}} \left(\frac{\varphi + 2}{\varphi^{k/2}}\right) + \frac{8k\varphi^{2n+1}}{\varphi^{k}} + \frac{16k\varphi^{2n}}{\varphi^{3k/2}}.
\] (36)
Dividing both sides of the above inequality by \(\frac{2\varphi^{2n+1}}{\varphi + 2}\), we get
\[
|y^m \varphi^{-(2n+1)} \left(\frac{\varphi + 2}{2}\right) - 1| < \frac{\varphi + 2}{\varphi^{2n+1} \varphi^{k/2}} + \frac{2/\varphi}{\varphi^{k/2}} + \frac{4k(\varphi + 2)}{\varphi^{k}} + \frac{8k(\varphi + 2)}{\varphi^{3k/2}}
< \frac{0.001}{\varphi^{k/2}} + \frac{1.24}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} + \frac{0.005}{\varphi^{k/2}} = \frac{1.251}{\varphi^{k/2}},
\] (37)
where we have used the facts that
\[
\frac{\varphi + 2}{\varphi^{2n+1}} < \frac{0.001}{\varphi^{k/2}} < \frac{0.005}{\varphi^{k/2}}
\]
and
\[
\frac{(8k/\varphi)(\varphi + 2)}{\varphi^{3k/2}} < \frac{0.005}{\varphi^{k/2}}
\]
for \(k > 510\). In order to use Theorem 7, we take
\[
(\gamma_1, b_1) := (y, m), \quad (\gamma_2, b_2) := (\varphi, -2n), \quad (\gamma_3, b_3) := \left(\frac{\varphi + 2}{2\varphi}, 1\right).
\]
The number field containing \(\gamma_1, \gamma_2\) and \(\gamma_3\) are \(K = \mathbb{Q}(\sqrt{5})\), which has degree \(D = 2\). We show that the number
\[
\Lambda_1 := y^m \varphi^{-(2n+1)} \left(\frac{\varphi + 2}{2}\right) - 1
\]
is nonzero. Contrast to this, assume that $\Lambda_1 = 0$. Then $y^m \left( \frac{\varphi + 2}{\varphi} \right) = \varphi^{2n+1}$ and conjugating this relation in $\mathbb{Q}(\sqrt{5})$, we get $y^m \left( \frac{\beta + 2}{\beta} \right) = \beta^{2n+1}$, where $\beta = \frac{1 - \sqrt{5}}{2} = \varphi$. From this, it is seen that

$$y^m \left( \varphi^2 + \frac{2}{\varphi} \right) = \varphi^{2n+1},$$

and conjugating this relation in $\mathbb{Q}(\sqrt{5})$, we get

$$y^m \left( \beta^2 + \frac{2}{\beta} \right) = \beta^{2n+1},$$

where $\beta = 1 - \frac{1 - \sqrt{5}}{2} = \frac{2}{\varphi}$. From this, it is seen that

$$\frac{\varphi^{2n+1}}{\varphi} + \frac{2}{\varphi} = \frac{\beta^{2n+1}}{\beta} + 2 < 0,$$

which is impossible since $\varphi > 0$. Therefore $\Lambda_1 \neq 0$. Moreover, since

$$h(\gamma_1) = h(y) = \log y, h(\gamma_2) = h(\varphi) \leq \frac{\log \varphi}{2},$$

and

$$h(\gamma_3) = h \left( \frac{\varphi + 2}{2\varphi} \right) = h \left( \frac{\sqrt{5}}{2} \right) \leq \frac{\log 5}{2},$$

by (18), we can take $A_1 := 2 \log y$, $A_2 := \log \varphi$ and $A_3 := \log 5$. Also, since $m < 1.73n$, we can take $B := 2n$. Thus, taking into account the inequality (37) and using Theorem 7, we obtain

$$(1.251) \cdot \varphi^{-k/2} > |\Lambda_1| > \exp \left( C \cdot (1 + \log 2n) (2 \log y) (\log \varphi) (\log 5) \right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. This implies that

$$\frac{k}{2} \log \varphi - \log(1.251) < 6.92 \cdot 10^{12} \cdot (1 + \log 2n)$$

or

$$k < 3.31 \cdot 10^{13} \cdot \log n,$$

(38)

where we have used the fact that $(1 + \log 2n) < (2.3) \log n$ for $n \geq 4$. On the other hand, from (25), we get

$$\log n < \log (5.141 \cdot 10^{15} \cdot k^4 \cdot (\log k)^3)$$

$$< 36.2 + 4 \log k + 3 \log(\log k)$$

$$< 38 \log k$$

for $k \geq 3$. So, from (38), we obtain

$$k < 3.31 \cdot 10^{13} \cdot 38 \log k,$$

which implies that

$$k < 4.84 \cdot 10^{16},$$

(39)

To reduce this bound on $k$, we use Lemma 8. Substituting this bound of $k$ into (25), we get $n < 1.6 \cdot 10^{87}$, which shows that $m < 2.77 \cdot 10^{87}$.

Now, let

$$z_2 := m \log y - (2n + 1) \log \varphi + \log \left( \frac{\varphi + 2}{2} \right)$$
and \( x := 1 - e^{z^2} \). Then
\[
|x| = |1 - e^{z^2}| < \frac{1.251}{\phi^{k/2}}
\]
by (57) and so \( |x| < 0.1 \) for \( k > 510 \). Choosing \( a := 0.1 \), we get the inequality
\[
|z^2| = |\log(x + 1)| < \frac{\log(100/90)}{0.1} \cdot \frac{1.251}{\phi^{k/2}} < \frac{1.32}{\phi^{k/2}}
\]
by Lemma 9. That is,
\[
0 < \left| m \log y - (2n + 1) \log \phi + \log \left( \frac{\phi + 2}{\phi} \right) \right| < \frac{1.32}{\phi^{k/2}}.
\]
Dividing both sides of the above inequality by \( \log \phi \), we obtain
\[
0 < |m\gamma - (2n + 1) + \mu| < A \cdot B^{-w},
\]
where
\[
\gamma := \frac{\log y}{\log \phi} \notin \mathbb{Q}, \quad \mu := \frac{\log \left( \frac{\phi + 2}{\phi} \right)}{\log \phi}, \quad A := 2.75, \quad B := \phi, \quad \text{and} \quad w := k/2.
\]
If we take \( M := 2.77 \cdot 10^{87} \), which is an upper bound on \( m \), we found that \( q_{207} \), the denominator of the 207 th convergent of \( \gamma \) exceeds 6M. Furthermore, a quick computation with Mathematica gives us that the value
\[
\frac{\log (Aq_{207}/\epsilon)}{\log B}
\]
is less than 585.91. So, if the inequality (40) has a solution, then
\[
\frac{k}{2} < \frac{\log (Aq_{207}/\epsilon)}{\log B} \leq 585.91,
\]
which implies that \( k \leq 1171 \). Hence, from (25), we get \( n < 3.41 \cdot 10^{30} \), which shows that \( m < 5.9 \cdot 10^{30} \). If we apply the inequality (40) to Lemma 5 again with \( M := 5.9 \cdot 10^{30} \), we found that \( q_{69} \), the denominator of the 69 th convergent of \( \gamma \) exceeds 6M. After doing this, then a quick computation with Mathematica show that in case the inequality (40) has a solution, we get \( k < 505 \). This contradicts the fact that \( k > 510 \). This completes the proof. \( \blacksquare \)

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