COUNTING PERVERSE COHERENT SYSTEMS
ON CALABI-YAU 4-FOLDS

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Abstract. Nagao-Nakajima introduced counting invariants of stable perverse coherent systems on small resolutions of Calabi-Yau 3-folds and determined them on the resolved conifold. Their invariants recover DT/PT invariants and Szendrői’s non-commutative invariants in some chambers of stability conditions. In this paper, we study an analogue of their work on Calabi-Yau 4-folds. We define counting invariants for stable perverse coherent systems using primary insertions and compute them in all chambers of stability conditions. We also study counting invariants of local resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1,-1,0)$ defined using torus localization and tautological insertions. We conjecture a wall-crossing formula for them, which upon dimensional reduction recovers Nagao-Nakajima’s wall-crossing formula on resolved conifold.

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0. Introduction

0.1. Background on CY 3-folds. For a contractible rational curve on a Calabi-Yau 3-fold $Z$, we have the flop

\[
\begin{array}{c}
Z \\
W \\
Z^+ \\
\end{array} \quad \begin{array}{c}
\xrightarrow{f} \\
\xleftarrow{f^+} \\
\end{array}
\]

where $f$ contracts the rational curve to a Gorenstein singularity and $f^+$ is a blow-up of $W$ in another way so that $Z \dashrightarrow Z^+$ is not an isomorphism. In [Bri02], Bridgeland introduced perverse coherent sheaves associated with 3-fold flopping contractions and used them to prove the equivalence of derived categories of $Z$ and $Z^+$ which was conjectured earlier by Bondal and Orlov [BO]. Shortly after that, Van den Bergh [VB] constructed non-commutative resolution of $W$ and realized Bridgeland’s equivalence through it.

Nagao-Nakajima [NN] introduced counting invariants for stable perverse coherent systems associated with flopping contractions of Calabi-Yau 3-folds. They determined their invariants for any chamber of stability conditions on resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$, which recover DT/PT
0.2. Perverse coherent systems on projective CY 4-folds. In this paper, we are interested in extending their work to Calabi-Yau 4-folds. Our setting is the following:

**Setting 0.1.** Let $X$ be a smooth projective Calabi-Yau 4-fold and $f: X \to Y$ be a projective birational contraction which contracts an irreducible surface $E \subset X$ to a curve $C \subset Y$. We assume that formal neighborhood at each point $p \in C \subset Y$ is of the form 

$$\hat{O}_{Y,p} \sim \mathbb{C}[x, y, z, w, u]/(xy - zw).$$

In the above setting, one can show that $Rf_*O_X = O_Y$, the singular locus $C \subset Y$ is a smooth connected curve, and the morphism $f|_E: E \to C$ is a ruled surface whose fibers have normal bundle $O_{P^1}(-1, -1, 0)$ in $X$. In this case, the abelian category $\text{Per}(X/Y)$ of perverse coherent sheaves still makes sense and Van den Bergh’s work [VB] applies. Following Nagao-Nakajima, we consider a pair (called perverse coherent system)

$$(0.1) \quad (F, s), \quad F \in \text{Per}(X/Y), \quad s: O_X \to F.$$

For $\Theta = (\theta_0, \theta_1) \in \mathbb{R}^2$, we will define $\Theta$-(semi)stability for perverse coherent systems (see Definition 1.5), and construct the coarse moduli space

$$P^\Theta_n(X, \beta) = \{(F, s) : \Theta\text{-semistable pairs } (0.1) \text{ with } [F] = \beta, \chi(F) = n\}/\sim$$

of $S$-equivalence classes of $\Theta$-semistable perverse coherent systems (see Theorem 1.6).

We are only interested in curve classes $\beta$ such that $f_*\beta = 0$, i.e. classes in fibers of $f$. We will classify walls for $\Theta$-stability, which turn out to consist of six types denoted by $L_\pm(k)$, $L_\pm(\infty)$ for $k \in \mathbb{Z}_{\geq 0}$ and $L_\pm(\infty)$ (see Proposition 1.12). This wall-chamber structure for $\Theta$-stability is described in Figure 1 which is the same as that of the resolved conifold in [NN] (see Lemma 1.18).

In special chambers, our moduli spaces recover moduli spaces of $Z_t$-stable pairs introduced in [CT19] (therefore also recover PT stable pairs [PT, CMT19]), Hilbert schemes of curves, and perverse Hilbert schemes (see Proposition 1.26, 1.14, 1.16 respectively).

When $\Theta$ lies outside walls, $P^\Theta_n(X, \beta)$ consists of only stable objects and admits a $(-2)$-shifted symplectic derived scheme structure in the sense of Pantev-Toën-Vaquie-Vezzosi [PTVV].

---

1 These are analogy of curve classes of resolved conifold considered by Nagao-Nakajima [NN].
Therefore there exists a virtual class
\[ [P_n^{\Theta}(X, \beta)]_{vir} \in H_{2n}(P_n^{\Theta}(X, \beta), \mathbb{Z}), \]
in the sense of Borisov-Joyce [BJ], which depends on the choice of orientation [CGJ] (see Proposition 1.19). In order to define counting invariants, we consider primary insertions:
\[ \tau: H^4(X, \mathbb{Z}) \to H^2(P_n^{\Theta}(X, \beta), \mathbb{Z}), \quad \tau(\gamma) := (\pi_P)_*(\pi_X^* \gamma \cup \text{ch}_3(\mathcal{F})), \]
where \( \pi_X, \pi_P \) are projections from \( X \times P_n^{\Theta}(X, \beta) \) onto corresponding factors, \( \mathcal{I} = (\pi_X^* \mathcal{O}_X \to \mathcal{F}) \) is the universal pair and \( \text{ch}_3(\mathcal{F}) \) is the Poincaré dual to the fundamental cycle of \( \mathcal{F} \).

The primary counting invariants of \( \Theta \)-stable perverse coherent systems are defined by
\[ P_{n, \beta}(\gamma) := \int_{[P_n^{\Theta}(X, \beta)]_{vir}} \tau(\gamma)^n. \]

The first purpose of this paper is to completely determine these invariants for all chambers of stability conditions:

**Theorem 0.2.** (Theorem 1.21) Let \( f: X \to Y \) be as in Setting 0.1. \( E \subset X \) be the exceptional surface and \( [P] \in H_2(X, \mathbb{Z}) \) be the fiber class of \( f|_E: E \to C \). Let \( \Theta = (\theta_0, \theta_1) \in \mathbb{R}^2 \) be outside walls defined in (1.13). Then for certain choice of orientation, we have
\[
\sum_{n \in \mathbb{Z}, f_\gamma = 0} \frac{P_{n, \beta}(\gamma)}{n!} q^n t^\beta = \begin{cases} 
\exp \left( q t^{[P]} \right) f_X \gamma_{\cup[E]} & \text{if } \theta_0 < 0, \theta_0 + 2\theta_1 > 0, \\
\exp \left( q t^{[P]} - q t^{-[P]} \right) f_X \gamma_{\cup[E]} & \text{if } \theta_0 < 0, \theta_0 + 2\theta_1 < 0, \\
\exp \left( -q t^{-[P]} \right) f_X \gamma_{\cup[E]} & \text{if } \theta_0 > 0, \theta_0 + 2\theta_1 < 0, \\
1 & \text{otherwise.}
\end{cases}
\]

We remark that the wall-chamber structure of primary counting invariants (see Figure 2) is different from that of corresponding moduli spaces as in Figure 1. Indeed most walls in Figure 1 (except those in Figure 2) do not contribute to the wall-crossing formula of primary invariants.

The result of this theorem in particular proves some of our previous conjectures:

**Corollary 0.3.** (Corollary 1.24) The LePotier-pair/GV conjecture [CT19, Conjecture 0.2], PT/GV conjecture [CMT19 §0.7] and DT/PT conjecture [CK19, Conjecture 0.3] hold for fiber classes in Setting 0.1.

### 0.3. Perverse coherent systems on local resolved conifold.

We also consider similar counting problem for the local resolved conifold
\[ X = \mathcal{O}_{P^1}(-1, -1, 0). \]

Since moduli spaces of \( \Theta \)-semistable perverse coherent systems on it are non-compact, we define counting invariants using torus localization. As there is no compact 4-cycle in \( X \), instead of primary insertions, we consider tautological insertions as in [CK18, CKM19, CT20b]. We take a CY torus
\[ T_0 = \{ t = (t_0, t_1, t_2, t_3) \in (\mathbb{C}^*)^4 : t_0 t_1 t_2 t_3 = 1 \}, \]
which acts on \( X \) preserving the CY 4-form. This action lifts to an action on \( P_n^{\Theta}(X, d) \) with finitely many reduced points as torus fixed loci (see Proposition 2.1). Therefore we can define equivariant tautological invariants (see Definition 2.3):
\[ P_{n, d}(e^m) := \sum_{I = (O_X \to F) \in P_n^{\Theta}(X, d) T_0} e_{T_0}(\chi_X(I, I)^X_0) \cdot e_{T_0 \times C^*}(\chi_X(F)^Y \otimes e^m) \in \Lambda. \]

Here we consider a trivial \( \mathbb{C}^* \)-action on moduli spaces and \( e^m \) is a trivial line bundle with \( \mathbb{C}^* \)-equivariant weight \( m \). \( \Lambda \) is the field of rational functions of equivariant parameters \( m \) and \( \lambda_i = e T_0(t_i) \). The above invariants depend on choice of sign for each torus fixed point.
Theorem 0.2.

For $\Theta_{PT} := (-1 + 0^+, 1)$, the corresponding invariants

$$P_{n,d}(e^m) := P_{n,d}^{\Theta_{PT}}(e^m)$$

enumerate PT stable pairs, which have a remarkable conjectural formula.

Conjecture 0.4. (Cao-Kool-Monavari [CKM19]) There exist choices of signs such that

$$\sum_{n,d} P_{n,d}(e^m) q^n t^d = \prod_{k \geq 1} \left( 1 - q^k t^k \right)^{\frac{\lambda_3}{k}} ,$$

where $-\lambda_3$ is the equivariant parameter of $\mathcal{O}_{P^1}$ in $X$.

Remark 0.5. In fact, the choices of signs such that the above formula holds are conjectured to be unique and checked for small $n, d$ (ref. [CKM19, Conjecture 0.16, Proposition B.1]).

The second purpose of this paper is to give an interpretation of Conjecture 0.4 in terms of wall-crossing of $\Theta$-stable perverse coherent systems. Suppose that $\Theta$ lies on one of the walls in Figure 1 except the DT/PT wall, and $\Theta_{\pm}$ lies on its adjacent chambers. We consider the flip type diagram of $T_0$-fixed loci of good moduli spaces:

$$0.2 \quad \bigcup_{n,d} P_n^{\Theta^-}(X, d)^{T_0} \xrightarrow{\pi} \bigcup_{n,d} P_n^{\Theta^+}(X, d)^{T_0} \xleftarrow{\pi^+} \bigcup_{n,d} P_n^{\Theta}(X, d)^{T_0}.$$  

Here $P_n^{\Theta}(X, d)^{T_0}$ consists of $\Theta$-polystable perverse coherent systems of type

$$I_0 \oplus S_{k-1}^{r_{T_0}}[-1], \quad r \geq 0,$$

where $I_0$ is a $T_0$-fixed $\Theta$-stable perverse coherent system, $S_{k-1}$ is a $T_0$-fixed $\Theta$-stable perverse coherent sheaf with $\Theta(S_{k-1}) = 0$, and $r$ can be computed from Chern character of $I_0$. $S_{k-1}$ is determined by the type of wall, e.g. $S_{k-1} = \mathcal{O}_{P^1}(k-1)$ if $\Theta \in L_{-}(k)$ (see [2.3] for details).

When $m = \lambda_3$, there exists a dimensional reduction which relates our invariants with Nagao-Nakajima’s invariants on the 3-fold $\mathcal{O}_{P^1}(-1, -1)$ (see Proposition 2.6). In [NN] Theorem 3.12 they proved a wall-crossing formula by stratifying $\pi$ into Grassmannian bundles and showed that the difference of invariants under wall-crossing is independent of the choice of $I_0$. Motivated by the idea of their wall-crossing formula, we conjecture a similar phenomenon holds for our 4-fold invariants on $\mathcal{O}_{P^1}(-1, -1, 0)$:

In this paper, we always use the numbering in the arxiv version of [NN].
Conjecture 0.6. (Conjecture 2.8) Let \( \Theta \) lie on one of the walls \( L_\pm(k), L_\mp(k) \) in \([1,13]\). For a \( T_0 \)-fixed \( \Theta \)-stable perverse coherent system \( I_0 \), we consider the following sequence of \( \Theta \)-polystable objects with \( r \geq 0 \):

\[
P_{k-1,r}^{I_0} := \{ I_0 \oplus S_{k-1,r}[-1] \} \in \bigcup_{n,d} P_n^\Theta(X,d)T_0.
\]

- If \( \Theta = (\theta_0, \theta_1) \in L_\mp(k) \) or \( L_\pm(k) \) (\( k \geq 1 \)) and \( \Theta_\pm = (\theta_0 \mp 0^+, \theta_1) \), then there exist choices of signs such that

\[
\sum_r t^r \sum_{\ell \in \pi_+^{-1}(P_{k-1,r}^{I_0})} \epsilon_{\ell_0}(\chi(X(I,J)_0^\pm) : \epsilon_{\ell_0} \times c_*(\chi(X(F)_0^\pm) \otimes e^m) = (1-t)^{k_\Theta}.
\]

- If \( \Theta = (\theta_0, \theta_1) \in L_\mp(k) \) or \( L_\pm(k) \) (\( k \geq 0 \)) and \( \Theta_\pm = (\theta_0 \mp 0^+, \theta_1) \), then there exist choices of signs such that

\[
\sum_r t^r \sum_{\ell \in \pi_+^{-1}(P_{k-1,r}^{I_0})} \epsilon_{\ell_0}(\chi(X(I,J)_0^\pm) : \epsilon_{\ell_0} \times c_*(\chi(X(F)_0^\pm) \otimes e^m) = (1-t^{-1})^{k_\Theta}.
\]

The point of those formulae in Conjecture 0.6 is that the quotient series in the LHS are independent of the choice of \( I_0 \). So by taking the summation for all \( T_0 \)-fixed \( \Theta \)-stable perverse coherent systems \( I_0 \), we obtain the following wall-crossing formula of tautological invariants:

Proposition 0.7. (Proposition 2.20) Assuming Conjecture 0.6 then we have the following:

- If \( \Theta = (\theta_0, \theta_1) \in L_\mp(k) \) or \( L_\pm(k) \) (\( k \geq 1 \)) and \( \Theta_\pm = (\theta_0 \mp 0^+, \theta_1) \), then there exist choices of signs such that

\[
\sum_{n,d} P_{n,d}^{\Theta_\pm}(e^m)q^nd = (1-q^kt)^{k_\Theta}.
\]

- If \( \Theta = (\theta_0, \theta_1) \in L_\mp(k) \) or \( L_\pm(k) \) (\( k \geq 0 \)) and \( \Theta_\pm = (\theta_0 \mp 0^+, \theta_1) \), then there exist choices of signs such that

\[
\sum_{n,d} P_{n,d}^{\Theta_\pm}(e^m)q^nd = (1-q^tk^{-1})^{k_\Theta}.
\]

In particular, by applying the above proposition from empty chamber to PT chamber, we obtain a wall-crossing interpretation of Conjecture [1,3] It also provides a conjectural formula for non-commutative tautological invariants of \( O_{\mathbb{P}^1}(-1,-1,0) \) (see Corollary 2.10).

When \( m = \lambda_3 \), Conjecture 0.6 reduces to Nagao-Nakajima’s wall-crossing formula (Proposition 2.6). Apart from this, we give several further evidence of our conjecture:

Theorem 0.8. (Theorem 2.10) Proposition 2.20, 2.22 Conjecture 0.6 holds for \( L_\mp(k) \) when

- \( I_0 = \mathcal{O}_X \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 1, \ k = 2, \ up to degree t^{16}, \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 2, \ k = 2, \ up to degree t^{10}, \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 3, 4, \ k = 2, \ up to degree t^9, \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 5, \ k = 2, \ up to degree t^8, \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 6, \ k = 2, \ up to degree t^7, \)
- \( I_0 = I_{\mathbb{P}^1}, \ l = 7, 8, 9, 10, \ k = 2, \ up to degree t^6, \)
- \( I_0 = I_{\mathbb{P}^1}, \ any \ l, \ k = 2, \ up to degree t^6, \)
- \( I_0 = I_{\mathbb{P}^1}, \ k = 3, \ up to degree t^5, \)
- \( I_0 = I_{\mathbb{P}^1}, \ k = 4, 5, \ up to degree t^3, \)
- \( I_0 = I_{\mathbb{P}^1}, \ k \leq 12, \ up to degree t^1. \)

Here \( I_{\mathbb{P}^1} := \big(O_X \to O_{\mathbb{P}^1} \oplus \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \mathcal{I}_{\mathbb{P}^1} \big) \) is the ideal sheaf of thickened \( \mathbb{P}^1 \) into \( O_{\mathbb{P}^1} \)-direction in \( X \).

The first case is proved using its compact analogue and Atiyah-Bott localization formula. Other cases are done with the help of a computer program, which usually involves checking very nontrivial combinatoric identities (see Example 2.19). We remark that we use consistent sign rule as discussed in Remark 2.4 to check our conjecture, and it is an interesting question to link our conjectural wall-crossing formula with the recent wall-crossing proposal of [GJT].
which restricts to an equivalence between \( \text{Per}(\ref{eq:1.3}) \) and a derived equivalence \( Y \) which exists as a vector bundle on \( \ref{eq:1.2} \). We assume that formal neighborhood at each point \( p \in C \subset Y \) is of the form

\[
\hat{O}_{Y,p} \cong \mathbb{C}[x,y,z,w,u]/(xy-zw).
\]

Under the above setting, one can show that \( Rf_\ast \mathcal{O}_X = \mathcal{O}_Y \), the singular locus \( C \subset Y \) is a smooth connected curve, the morphism \( f_\ast : E \to C \) is a ruled surface whose fibers have normal bundle \( \mathcal{O}_p(-1,-1,0) \) in \( X \).

**Example 1.2.** Let \( g : Z \to W \) be a 3-fold flopping contraction of a \((-1,-1)\) curve on a smooth projective CY 3-fold \( Z \). For an elliptic curve \( E \), the \( E \)-copy of \( g \) gives a contraction above.

As in Bridgeland \( \text{\cite{Bri02}} \), for \( p \in \mathbb{Z} \) we consider the following heart of perverse t-structure on \( D^b \text{Coh}(X) \):

\[
\text{\( p \text{Per}(X/Y) := \left\{ E \in D^b \text{Coh}(X) : \begin{array}{c}
\text{Hom}(E,\mathcal{E}^{> p}) = 0 \\
\text{Hom}(\mathcal{E}^{< p}, E) = 0
\end{array} \right\}, \)
\]

where

\[
\mathcal{E}^{> p} := \{ F \in D^b \text{Coh}(X) : Rf_\ast F = 0, \mathcal{H}^{> p}(F) = 0 \},
\]

\[
\mathcal{E}^{< p} := \{ F \in D^b \text{Coh}(X) : Rf_\ast F = 0, \mathcal{H}^{< p}(F) = 0 \}.
\]

In this paper, we mainly use the \( p = -1 \) perversity

\[
\text{Per}(X/Y) := -^1 \text{Per}(X/Y).
\]

It is easy to see that \( \mathcal{O}_X \in \text{Per}(X/Y) \). By the renowned result of Van den Bergh \( \text{\cite{VB}} \), there exists a local projective generator of \( \text{Per}(X/Y) \)

\[
\mathcal{P} = \mathcal{O}_X \oplus \mathcal{P}_0,
\]

which exists as a vector bundle on \( X \), a sheaf \( \mathcal{A}_Y := f_\ast \mathcal{E}nd(\mathcal{P}) \) of non-commutative algebras on \( Y \) and a derived equivalence

\[
\Phi : D^b \text{Coh}(X) \simto D^b \text{Coh}(\mathcal{A}_Y), \quad (-) \mapsto R\text{Hom}(\mathcal{P}, -),
\]

which restricts to an equivalence between \( \text{Per}(X/Y) \) and \( \text{Coh}(\mathcal{A}_Y) \).

The morphism \( f : X \to Y \) is a flopping contraction, and we have the flop

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^+ \\
\downarrow f & \swarrow & \downarrow f^+ \\
Y & \xrightarrow{f} &
\end{array}
\]

The flopping contraction \( f^+ : X^+ \to Y \) also satisfies Setting \( \ref{setting:1.1} \) By Bridgeland \( \text{\cite{Bri02}} \) and Van den Bergh \( \text{\cite{VB}} \), there exists an equivalence

\[
\Upsilon : D^b \text{Coh}(X^+) \simto D^b \text{Coh}(X)
\]

which restricts to an equivalence between \( ^0 \text{Per}(X^+/Y) \) and \( -^1 \text{Per}(X/Y) \).
We are mainly interested in perverse coherent sheaves which are supported on fibers of $f$. We define the following categories:

$$\text{Coh}_{\leq 1}(X) := \{E \in \text{Coh}(X) : \dim \text{Supp}(E) \leq 1\},$$

$$\text{Coh}_{\leq 1}(X/Y) := \{E \in \text{Coh}_{\leq 1}(X) : \dim \text{Supp}(RF,E) = 0\},$$

$$D^b \text{Coh}_{\leq 1}(X/Y) := \{E \in D^b \text{Coh}(X) : H^*(E) \in \text{Coh}_{\leq 1}(X/Y)\},$$

$$\text{Per}_{\leq 1}(X/Y) := \text{Per}(X/Y) \cap D^b \text{Coh}_{\leq 1}(X/Y).$$

We will use the following lemma:

**Lemma 1.3.** We take $\beta \in H_2(X,\mathbb{Z})$ with $f_*\beta = 0$. Then any $F \in \text{Per}(X/Y)$ with $\text{ch}(F) = (0,0,0,\beta,n)$ is supported on fibers of $f$, and hence $F \in \text{Per}_{\leq 1}(X/Y)$.

**Proof.** Let $\Phi$ be the equivalence in (1.3). The object $\Phi(F)$ is given by

$$\Phi(F) = Rf_* (F) \oplus Rf_*(F \otimes P^\vee_0),$$

which is a coherent sheaf on $Y$. We claim that $\Phi(F)$ is zero dimensional. In fact, choose an ample divisor $H$ on $Y$. By the adjunction and Riemann-Roch formula, we have

$$\chi_Y(O_Y, Rf_* F \otimes O_Y(mH)) = \chi_X(O_X, F \otimes f^* O_Y(mH)) = n,$$

which is independent of $m$. Therefore $Rf_* F$ has a zero dimensional support. The same argument also shows that $Rf_*(F \otimes P^\vee_0)$ has a zero dimensional support.

Let $S = \{p_1, \cdots, p_n\} \subset Y$ be the set-theoretic support of $\Phi(F)$. Then we have $\Phi(F)|_{\gamma \setminus S} = 0$. Since the derived equivalence $\Phi$ in (1.3) is compatible with the base change to open subsets, we have

$$\Phi(F|_{X \setminus f^{-1}(S)}) = 0.$$

Therefore $F|_{X \setminus f^{-1}(S)} = 0$, i.e. $F$ is supported on fibers of $f$. \hfill \Box

Next we give another description of $\text{Per}_{\leq 1}(X/Y)$ using tilting theory of Happel-Reiten-Smalø. Below we fix a $\mathbb{Q}$-ample divisor $\omega$ on $X$ which is degree one on fibers of $f|_E: E \to C$. For $F \in D^b \text{Coh}_{\leq 1}(X/Y)$, we set

$$d(F) := [F] \cdot \omega \in \mathbb{Z}$$

where $[F]$ is the fundamental one cycle of $F$. In other word, $d(F)$ is determined by $[F] = d(F)[\mathbb{P}^1]$ in $H_2(X,\mathbb{Z})$, where $[\mathbb{P}^1]$ is the fiber class of $f|_E: E \to C$. For $F \in \text{Coh}_{\leq 1}(X/Y)$, its slope is defined to be

$$(1.5) \quad \mu_\omega(F) := \frac{\chi(F)}{d(F)} \in \mathbb{Q} \cup \{\infty\}$$

where we set $\mu_\omega(F) = \infty$ if $d(F) = 0$. The above slope function defines $\mu_\omega$-semistable sheaves on $\text{Coh}_{\leq 1}(X/Y)$ in the usual way. We define extension closed subcategories:

$$T_\omega := \{F \in \text{Coh}_{\leq 1}(X/Y) : F \text{ is } \mu_\omega\text{-semistable with } \mu_\omega(F) > 0\}_{\text{ex}},$$

$$F_\omega := \{F \in \text{Coh}_{\leq 1}(X/Y) : F \text{ is } \mu_\omega\text{-semistable with } \mu_\omega(F) \leq 0\}_{\text{ex}}.$$

Here $\langle - \rangle_{\text{ex}}$ is the extension closure. By the Harder-Narasimhan filtration, they form a torsion pair and we can define the tilting category in the sense of [HRS]:

$$\text{Coh}^+_\leq 1(X/Y) := \langle F_\omega[1], T_\omega \rangle_{\text{ex}}.$$

This is the heart of a bounded t-structure in $D^b \text{Coh}_{\leq 1}(X/Y)$, and is in particular an abelian category [BBD].

**Proposition 1.4.** As abelian subcategories of $D^b \text{Coh}_{\leq 1}(X/Y)$, we have

$$\text{Coh}^+_\leq 1(X/Y) = \text{Per}_{\leq 1}(X/Y).$$

**Proof.** Note that any object in $\text{Coh}_{\leq 1}(X/Y)$ is supported on points or fibers of $f|_E: E \to C$. By taking the Harder-Narasimhan and Jordan-Hölder filtrations, we have

$$T_\omega = \langle O_x, O_{f^{-1}(a)}(a) : x \in X, c \in C, a \geq 0 \rangle_{\text{ex}},$$

$$F_\omega = \langle O_{f^{-1}(a)}(a) : c \in C, a < 0 \rangle_{\text{ex}}.$$

It is straightforward to check $T_\omega, F_\omega[1] \subseteq \text{Per}_{\leq 1}(X/Y)$, therefore $\text{Coh}^+_\leq 1(X/Y) \subseteq \text{Per}_{\leq 1}(X/Y)$. Both sides are hearts of bounded t-structures on $D^b \text{Coh}_{\leq 1}(X/Y)$, they must be the same. \hfill \Box
For $\Theta = (\theta_0, \theta_1) \in \mathbb{R}^2$ and $F \in D^b \text{Coh}_{\leq 1}(X/Y)$, we set
\[
\Theta(F) := \theta_0 \cdot \chi(F) + \theta_1 \cdot (\chi(F) - d(F)) \in \mathbb{R}.
\]
Following Nagao-Nakajima [NN, Definition 2.1], we introduce the notion of (semi)stable perverse coherent systems. Here we always consider the rank one case.

**Definition 1.5.** A perverse coherent system is a pair
\[
(F, s), \quad F \in \text{Per}_{\leq 1}(X/Y), \quad s : O_X \to F.
\]
For $\Theta = (\theta_0, \theta_1) \in \mathbb{R}^2$, a perverse coherent system $(F, s)$ is $\Theta$-(semi)stable if
- for any non-zero subobject $0 \neq F' \subseteq F$ in $\text{Per}_{\leq 1}(X/Y)$, we have $\Theta(F') < (\leq) 0$. 
- for any proper subobject $F' \subseteq F$ in $\text{Per}_{\leq 1}(X/Y)$ such that $\text{Im}(s) \subseteq F'$, we have $\Theta(F') < (\leq) \Theta(F)$.

Following arguments from [NN, Theorem 1.10], [Yos, Proposition 1.6.1], one can construct moduli spaces of such semistable perverse coherent systems.

**Theorem 1.6.** ([NN, Theorem 1.10], [Yos, Proposition 1.6.1])
Let $f : X \to Y$ be as in Setting [1.1] and take $\beta \in H_2(X, \mathbb{Z})$ with $\ell \beta = 0$, $n \in \mathbb{Z}$, $\Theta \in \mathbb{R}^2$. Then there is a projective coarse moduli scheme $P_{[\Theta]}^b(X, \beta)$ which parametrizes $S$-equivalence classes of $\Theta$-semistable perverse coherent systems $(F, s)$ with $\chi(F) = (0, 0, 0, \beta, n)$.

### 1.2. Perverse coherent sheaves on local model.
Let $X_0 = O_{P_1}(-1, -1, 0)$ and take the affinization $f_0 : X_0 \to Y_0$, where $Y_0$ is given by
\[
Y_0 = \text{Spec } \mathbb{C}[x, y, z, w, u]/(xy - zw).
\]
By the assumption in Setting [1.1] the morphism $f : X \to Y$ is identified with $f_0 : X_0 \to Y_0$ formally locally on $Y$. By [VB], there is a projective generator for $\text{Per}(X_0/Y_0)$ given by
\[
P_0 = O_{X_0} \oplus O_{X_0}(1),
\]
and derived equivalences
\[
\Phi_0 : D^b \text{Coh}(X_0) \xrightarrow{\sim} D^b \text{mod}(A_{Y_0}), \quad (-) \mapsto \mathbf{R}\text{Hom}(P_0, -),
\]
\[
\Psi_0 : D^b \text{mod}(A_{Y_0}) \xrightarrow{\sim} D^b \text{Coh}(X_0), \quad (-) \mapsto (-) \otimes_{A_{Y_0}} P_0,
\]
which restrict to equivalences between $\text{Per}(X_0/Y_0)$ and $\text{mod}(A_{Y_0})$. Here $A_{Y_0} := \text{End}(P_0)$ is a non-commutative algebra.

The morphism $f_0 : X_0 \to Y_0$ is a flopping contraction, and we have the flop
\[
\begin{array}{ccc}
X_0 & \xrightarrow{\phi_0} & X_0^+ \\
\downarrow f_0 & & \downarrow f_0^+
\end{array}
\]
\[
\phi_0 : X_0 \to X_0^+\xrightarrow{\sim} X_0^+\xleftarrow{\sim} X_0.
\]
\[
f_0 : Y_0 \leftarrow Y_0^+\xrightarrow{\sim} Y_0^+\xleftarrow{\sim} Y_0.
\]

The flop $X_0^+$ is also isomorphic to $O_{P_1}(-1, -1, 0)$. By setting $P_0^+ = O_{X_0^+} \oplus O_{X_0^+}(-1)$, we have a derived equivalence
\[
\Phi_0^+ = \mathbf{R}\text{Hom}(P_0^+, -) : D^b \text{Coh}(X_0^+) \xrightarrow{\sim} D^b \text{mod}(A_{Y_0}).
\]
Here we have used the isomorphism induced by the strict transform 
\[
(\phi_0)_* : A_{Y_0} = \text{End}(P_0) \xrightarrow{\sim} \text{End}(P_0^+).
\]
By composing with the equivalence $\Phi_0$ in (1.7), we obtain the flop equivalence
\[
\Upsilon_0 := \Psi_0 \circ \Phi_0^+ : D^b \text{Coh}(X_0^+) \xrightarrow{\sim} D^b \text{Coh}(X_0),
\]
giving a local model of the equivalence (1.4).

The module category over the non-commutative algebra $A_{Y_0}$ is described in terms of representations of a quiver with relations as follows. Let $(Q, I)$ be the following quiver with relations:
\[
\begin{array}{cc}
& 0 & 1 \\
& b_2 & a_1 & a_2 & b_1 \\
& & a_1b_1 = 0, & a_2b_2 = 0, & a_1c = 0, & c = 0, & b_1d = 0, & i = 1, 2.
\end{array}
\]
Lemma 1.7. We have an equivalence

\[ \text{mod}(A_{\mathbb{C}^4}) \sim \text{mod}(\mathbb{C}Q/I). \]

Proof. We write \( \pi: \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0) \to \mathbb{P}^1 \) as the composition of projections

\[ \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0) \xrightarrow{\pi_1} \mathcal{O}_{\mathbb{P}^1}(-1, -1) \cong \mathbb{P}^1. \]

Since \( \mathcal{P}_0 = \pi_1^* \mathcal{E} \) for \( \mathcal{E} = \pi_2^* \mathcal{O}_{\mathbb{P}^1} \oplus \pi_3^* \mathcal{O}_{\mathbb{P}^1}(1) \), we have

\[ \text{Hom}(\mathcal{P}_0, \mathcal{P}_0) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \pi_1^* \mathcal{O}) \cong \text{Hom}(\mathcal{E}, \mathcal{E}) \otimes \mathbb{C}[t]. \]

We write \( B := \text{Hom}(\mathcal{E}, \mathcal{E}). \) By the above isomorphism, an \( A_{\mathbb{C}^4} \)-module \( M \) can be viewed as a \( B \)-module, which is an representation of the following quiver with relations (see [Sze, §2.1]):

\[ \begin{aligned}
 a_1 & \quad b_1 \\
 a_2 & \quad b_2 \\
 a_1 b_1 a_2 = a_1 b_2 a_2, & \quad b_2 a_1 b_1 = b_1 a_2 b_2.
\end{aligned} \]

Based on the action of idempotent elements at vertex 0 and 1, we can write

\[ M = M_0 \oplus M_1. \]

The \( \mathbb{C}[t] \)-module structure on \( M \) gives an action on \( M \)

\[ \times: M_i \to M_i, \quad i = 0, 1, \]

which we denote by loops \( c, d \) in \( I_0^2 \). This action commutes with \( B \)-module action, so we have commutative relations \( da_i = a_i c, c b_i = b_i d \).

For \( \Theta = (\theta_0, \theta_1) \in \mathbb{R}^2 \) and a finite dimensional representation \( V = (V_0, V_1) \) of the quiver \( I_0^2 \), we denote

\[ \Theta(V) := \theta_0 \dim V_0 + \theta_1 \dim V_1, \quad \mu_{\Theta}(V) := \frac{\theta_0 \dim V_0 + \theta_1 \dim V_1}{\dim V_0 + \dim V_1}. \]

The \( \Theta \)-stability for representations of the quiver \( I_0^2 \) is defined as follows:

Definition 1.8. A finite dimensional representation \( V \) of the quiver \( I_0^2 \) is \( \Theta \)-(semi)stable if for any subrepresentation \( 0 \neq V' \subset V \) we have \( \mu_{\Theta}(V') < (\leq) \mu_{\Theta}(V) \).

A simple extension of Lemma 1.7 (e.g. [NN, Proposition 3.3]) shows that perverse coherent systems on \( X_0 \) are in one-to-one correspondence with representations of the framed quivers \( \tilde{Q} \) with relations \( I \):

\[ \begin{aligned}
 &1 \\
 &b_1 \\
 &a_2 \\
 \infty &\quad 0 \\
 \cdots &\quad \cdots
\end{aligned} \]

\[ \begin{aligned}
 a_1 b_1 a_2 = a_1 b_2 a_2, & \quad b_2 a_1 b_1 = b_1 a_2 b_2, \\
 da_i = a_i c, & \quad cb_i = b_i d, \quad i = 1, 2.
\end{aligned} \]

More specifically, for a perverse coherent system \( (F, s) \), we have the associated vector spaces

\[ V_0 = \text{Hom}(\mathcal{O}_{X_0}, F), \quad V_1 = \text{Hom}(\mathcal{O}_{X_0}(1), F), \]

at the vertex 0 and 1. When \( F \) has compact support, we set \( d(F) := \text{rank}(\pi_* F) \) where \( \pi: X_0 \to \mathbb{P}^1 \) is the projection. In this case, we have

\[ (\dim V_0, \dim V_1) = (\chi(F), \chi(F) - d(F)). \]

Below when we consider a perverse coherent system \( (F, s) \) on \( X_0 \), we always assume that \( F \) is compactly supported. The stability of perverse coherent systems in Definition 1.8 for \( X_0 \) translates into the following King’s stability [King] of representations of \( (\tilde{Q}, I) \).

Definition 1.9. A representation \( (V_0, V_1, V_{\infty} = \mathbb{C}) \) of \( (\tilde{Q}, I) \) is \( \Theta \)-(semi)stable if

- for any non-zero subrepresentation \( (V'_0, V'_1, 0) \), we have \( \Theta(V'_0, V'_1) < (\leq) \Theta(V_0, V_1) \).
- for any proper subrepresentation \( (V'_0, V'_1, V'_{\infty} = \mathbb{C}) \), we have \( \Theta(V'_0, V'_1) < (\leq) \Theta(V_0, V_1) \).
1.3. Wall-chamber structures for local resolved conifold. In this section, we study wall-chamber structures for moduli spaces of (compactly supported) stable perverse coherent systems on the following local model:

\[ f_0 : X_0 = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0) \to Y_0 = \{ (x, y, z, w) \in \mathbb{C}^4 : xy = zw \} \times \mathbb{C}. \]

As mentioned above, we are reduced to study the wall-chamber structures for stability of finite dimensional representations of (1.10).

In order to classify all walls, we need to find out \( \Theta \) such that there exists a strictly \( \Theta \)-semistable 5 representation of the framed quiver \((\tilde{Q}, I)\) in (1.10). We have the Jordan-Hölder filtration:

\[ \tilde{V} = V_0 \supset V_1 \supset \cdots \supset \tilde{V}^l = 0, \quad l \geq 2, \]

such that \( \tilde{V}^i/\tilde{V}^{i+1} \)'s are \( \Theta \)-stable representations of \((\tilde{Q}, I)\). Since \( \dim V_\infty = 1 \), there must be some \( \tilde{V}^i/\tilde{V}^{i+1} \) which has zero dimension vector at the vertex \( \infty \) (the condition \( l \geq 2 \) is crucial here), i.e. \( \tilde{V}^i/\tilde{V}^{i+1} \) is a finite dimensional \( \Theta \)-stable representation of the unframed quiver \((Q, I)\) (1.9), satisfying \( \Theta(\tilde{V}^i/\tilde{V}^{i+1}) = 0 \). Therefore we see that a strictly \( \Theta \)-semistable representation of (1.9) produces a \( \Theta \)-stable representation of the unframed quiver (1.9). So in order to classify all walls, we are reduced to classify all \( \Theta \) such that there exists a non-zero finite dimensional \( \Theta \)-stable representations of \((Q, I)\). The following lemma is proved along with the similar argument of [NN Lemma 3.4]. Here we recall the key point to make us self-contained.

**Lemma 1.10.** Let \( V = (V_0, V_1) \) be a non-zero finite dimensional \( \Theta \)-stable representation of quiver (1.9). Then one of the following conditions hold:

1. \( \dim V_0 = \dim V_1 = 1 \),
2. \( a_1 = a_2 = 0 \),
3. \( b_1 = b_2 = 0 \).

**Proof.** By replacing \( \Theta = (\theta_0, \theta_1) \) with \((\theta_0 - \mu_{\Theta}(V), \theta_1 - \mu_{\Theta}(V))\), we may assume \( \Theta(V) = 0 \). We may also assume that \( V_0, V_1 \neq 0 \), as otherwise (2) or (3) holds trivially. For a fixed \((i, j)\), we set

\[ S_0 = \ker(b_ja_i), \quad S_1 = \ker(a jb_i), \quad T_0 = \text{Im}(b_ja_i), \quad T_1 = \text{Im}(a jb_i). \]

It is easy to check that \((S_0, S_1)\) and \((T_0, T_1)\) are subrepresentations of \( V \). Therefore the \( \Theta \)-stability of \( V \) implies

\[ \theta_0 \dim S_0 + \theta_1 \dim S_1 \leq 0, \quad \theta_0 \dim T_0 + \theta_1 \dim T_1 \leq 0. \]

These two inequalities must be equalities as we also have

\[ \theta_0 \dim V_0 + \theta_1 \dim V_1 = 0, \quad \dim S_i + \dim T_i = \dim V_i, \quad (i = 0, 1). \]

So either \((S_0, S_1) = (0, 0)\) or \((S_0, S_1) = (V_0, V_1)\) holds. Note that if the first case happens, then \( a_i \) and \( b_j \) are injective, hence they give isomorphisms \( V_0 \cong V_1 \). If the second case happens, then

\[ b_ia_j = a jb_i = 0. \]

From the above argument, we may assume that either one of the followings holds:

A) \( a_ib_j = b_ia_i = 0 \) for all \( i, j \), or
B) \( a_i, b_i \) are isomorphisms with \( \dim V_0 = \dim V_1 = 1 \).

In the case of (A), we first assume \( \theta_0 \geq 0 \). By taking the subrepresentation \((\ker(a_1) \cap \ker(a_2), 0)\) of \( V \), the \( \Theta \)-stability yields \( \ker(a_1) \cap \ker(a_2) = 0 \). Using \( a_1b_2 = 0 \) we have

\[ \text{Im}(b_1) \subseteq \ker(a_1) \cap \ker(a_2) = 0, \]

therefore \( b_1 = b_2 = 0 \). Similarly when \( \theta_0 \leq 0 \), i.e. \( \theta_1 \geq 0 \), we conclude that \( a_1 = a_2 = 0 \).

In the case of (B), note that \( b_1a_1, b_2a_1, b_1a_2, b_2a_2, c \) are pairwise commutating. Let us take a common eigenvector \( 0 \neq v_0 \in V_0 \). Then

\[ (S'_0, S'_1) = \langle (v_0), (a_1(v_0), a_2(v_0)) \rangle \]

is a subrepresentation of \( V \). Here \( \langle - \rangle \) is the linear span of \(-\). By stability, we obtain the inequality

\[ \theta_0 \dim S'_0 + \theta_1 \dim S'_1 \leq 0. \]

5Here \( V_\infty = \mathbb{C} \) as we are only interested in rank one perverse coherent systems (see Definition 1.5).

6Based on Definition 1.9, equivalently Definition 1.5, the slope function of \( \tilde{V} \) is assumed to be zero (compare with [NN §1.3]). The equality follows from the slope property of Jordan-Hölder filtration of \( \tilde{V} \).
Note that we have $\theta_0 + \theta_1 = 0$. If $\theta_0 < 0$, the above inequality is equivalent to $\dim S'_1 \leq \dim S'_0 = 1$. Since $a_1$ is an isomorphism, we have $a_1(v_0) \neq 0$. Therefore the equality holds and

$$(S'_0, S'_1) = (V_0, V_1), \quad \dim V_0 = \dim V_1 = 1.$$ If $\theta_0 > 0$ (i.e. $\theta_1 < 0$), by symmetry between vertex 0 and 1 and considering pairwise commuting paths $a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2, d$, we obtain the same conclusion. □

For $a \in \mathbb{C}$, we denote by $j_a$ the closed immersion

$$j_a : \mathbb{P}^1 \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1, -1) \times \{a\} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0),$$

where the first arrow is the zero section. We have the following classification of $\Theta$-stable representations.

**Proposition 1.11.** A non-zero finite dimensional representation $V$ of quiver (1.10) is $\Theta$-stable if it is either one of the following:

1. $\Phi_0(j_a \mathcal{O}_{\mathbb{P}^1}(m - 1))$ for $a \in \mathbb{C}$ and $m \geq 1$,
2. $\Phi_0(j_a \mathcal{O}_{\mathbb{P}^1}(-m - 1)[1])$ for $a \in \mathbb{C}$ and $m \geq 0$,
3. $\Phi_0(\mathcal{O}_x)$ for $x \in X_0$,
4. $\Phi_0(j_a \mathcal{O}_{\mathbb{P}^1}(-m - 1)[1])$ for $a \in \mathbb{C}$ and $m \geq 1$,
5. $\Phi_0(j_a \mathcal{O}_{\mathbb{P}^1}(m - 1))$ for $a \in \mathbb{C}$ and $m \geq 0$,
6. $\Phi_0(\mathcal{O}_x)$ for $x \in X_0^+$.

**Proof.** Suppose that $V$ satisfies (2) of Lemma 1.10. Then $V$ is a representation of the following quiver with relation:

$$(1.11) \quad \begin{array}{ccc} 0 & \xrightarrow{b_1} & 1 \\ & \xrightarrow{b_2} & \\ & \xrightarrow{c b_i = b_i d, \ i = 1, 2.} & d \end{array}$$

Geometrically, this quiver corresponds to $\mathbb{P}^1 \times \mathbb{C}$ from the tilting bundle. More precisely, let $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ be the tilting bundle of $\mathbb{P}^1$ whose endomorphism algebra $K := \text{End}(\mathcal{E})$ gives rise to the Kronecker quiver:

$$\begin{array}{ccc} 0 & \xrightarrow{b_1} & 1 \\ & \xrightarrow{b_2} & \\ & \xrightarrow{	ext{endomorphism algebra}} & \end{array}$$

Then the endomorphism algebra $L := \text{End}(\pi^* \mathcal{E})$ gives rise to the quiver (1.11), where $\pi : \mathbb{P}^1 \times \mathbb{C} \to \mathbb{P}^1$ denotes the projection. By the projection formula, we have

$$L \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \pi_* \mathcal{O}_{\mathbb{P}^1} \times \mathbb{C}) \cong K \otimes \mathbb{C}[t].$$

From the natural embedding $\mathbb{C}[t] \to L$, we can treat $L$-module $V$ as a $\mathbb{C}[t]$-module. Since $V$ is a stable $L$-module, we have $\text{End}(V) \cong \mathbb{C}$. Therefore as a $\mathbb{C}[t]$-module, we have

$$\text{End}(V) \cong \mathbb{C}[t]/(t - a),$$

for some $a \in \mathbb{C}$. In particular the $L$-module structure on $V$ descends to the $L/(t - a) \cong K$-module structure, so we can also treat $V$ as a stable $K$-module.

By [NN] Lemma 2.12, we can classify all $\Theta$-stable $K$-modules, which under the equivalence

$$\text{RHom}(\mathcal{E}, -) : D^b \text{Coh}(\mathbb{P}^1) \simto D^b \text{mod}(K)$$

correspond to $\mathcal{O}_{\mathbb{P}^1}(m - 1)$ for $m \geq 1$ or $\mathcal{O}_{\mathbb{P}^1}(-m - 1)[1]$ for $m \geq 0$. Now we view such $K$-modules as representations of the quiver (1.11). Then similarly to [NN] Remark 3.6, via the equivalence (1.17) they correspond to the following objects in $D^b \text{Coh}(X_0)$

$$(1.12) \quad j_{a*} \mathcal{O}_{\mathbb{P}^1}(m - 1) (m \geq 1), \quad j_{a*} \mathcal{O}_{\mathbb{P}^1}(-m - 1)[1] (m \geq 0).$$

Therefore $V$ is either of type (i) or (ii) in the proposition.

If $V$ satisfies (3) of Lemma 1.10 as in [NN] Remark 3.6 it corresponds to one of the geometric objects (1.12) in the flop side. So $V$ is either of type (iv) or (v) in the proposition. Finally if $V$ satisfies (1) of Lemma 1.10 a similar argument as above shows that we can treat $V$ as a stable representation of the quiver associated with the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$, unique up to a choice of $a \in \mathbb{C}$. Combining with [NN] Remark 3.6, we know it corresponds to a structure sheaf of a point in $X_0$ or that of a flop $X_0^+$.

To sum up, walls for $\Theta$-stability of representations of the framed quiver (1.10) can be classified as follows:
Proposition 1.12. For each $\Theta$-stable representation in Proposition \[NN, Proposition 1.11\] the corresponding wall is given as follows: \[L_{-}(m) := \{ (\theta_0, \theta_1) \in \mathbb{R}^2 : \theta_0 < \theta_1, \; m\theta_0 + (m - 1)\theta_1 = 0 \}, \quad (m \geq 1),\]
\[L_{-}(\infty) := \{ (\theta_0, \theta_1) \in \mathbb{R}^2 : \theta_0 < \theta_1, \; \theta_0 + \theta_1 = 0 \},\]
\[L_{+}(m) := \{ (\theta_0, \theta_1) \in \mathbb{R}^2 : \theta_0 > \theta_1, \; m\theta_0 + (m - 1)\theta_1 = 0 \}, \quad (m \geq 1),\]
\[L_{+}(\infty) := \{ (\theta_0, \theta_1) \in \mathbb{R}^2 : \theta_0 > \theta_1, \; \theta_0 + \theta_1 = 0 \}.\]

Proof. Let $V = (V_0, V_1)$ be a $\Theta$-stable representation (i) in Proposition \[NN, Proposition 1.11\] Then it has the dimension vector $(m, m - 1)$, so the condition $\Theta(V) = 0$ yields the equation of the wall $m\theta_0 + (m - 1)\theta_1 = 0$. Since it has a subrepresentation $(V_0, 0)$, when $m > 1$, it is $\Theta$-stable only if the inequality $m\theta_0 < 0$ holds. So $\theta_0 < 0$ and $\theta_1 > 0$ follows from the defining equation of the wall. This gives rise to the wall $L_{-}(m)$ when $m > 1$. When $m = 1$, the equation of wall is $\theta_0 = 0$ and we simply put $\theta_1 > 0$ to define $L_{-}(1)$. Similarly, the other stable representations in Proposition \[NN, Proposition 1.11\] give rise to other walls in (1.13).

A connected component of the complement of walls in $\mathbb{R}^2$ is called a chamber. Below we discuss some distinguished chambers. When $\theta_0, \theta_1 > 0$, we call this chamber the empty chamber:

Proposition 1.13. When $\theta_0, \theta_1 > 0$, there is no non-zero finite dimensional $\Theta$-stable representation of the framed quiver (1.10).

Proof. Assume that there is such a representation $V = (V_0, V_1, V_{\infty} = \mathbb{C})$. By considering the sub-representation $(V_0, V_1, 0)$, we obtain $\theta_0 \dim V_0 + \theta_1 \dim V_1 < 0$, which contradicts with the assumption.

The chambers adjacent to the wall $L_{-}(\infty)$ are the so-called DT/PT chambers. Following the proof [NN, Proposition 2.10, 2.11] in the resolved conifold case, it is easy to see:

Proposition 1.14. Let $\Theta^\pm = (-1 \mp 0^+, 1)$. Then under the derived equivalences in (1.7), we have the following:

- finite dimensional $\Theta^+$-stable representations of the framed quiver (1.10) correspond exactly to ideal sheaves of compactly supported subschemes in $X_0$.
- finite dimensional $\Theta^-$-stable representations of the framed quiver (1.10) correspond exactly to PT stable pairs $(\mathcal{O}_{X_0} \to F)$, i.e. $F$ is compactly supported pure one dimensional sheaf and Coker(s) is zero dimensional.

Remark 1.15. Similarly for the wall $L_{+}(\infty)$, finite dimensional $(-\Theta^\pm)$-stable representations correspond to those objects in the flop $X_0^{\infty}$ of $X_0$.

When $\theta_0, \theta_1 < 0$, we are in the non-commutative chambers, where stable representations correspond to perverse Hilbert schemes in the sense of Bridgeland [Bri02].

Proposition 1.16. When $\theta_0, \theta_1 < 0$, finite dimensional $\Theta$-stable representations of the framed quiver (1.10) are exactly those cyclic representations, i.e. representations generated by $V_{\infty} = \mathbb{C}$ as $\mathbb{C}[Q]/I$-modules.

Proof. Let $V = (V_0, V_1, V_{\infty} = \mathbb{C})$ be a $\Theta$-stable representation of $(\tilde{Q}, I)$. Suppose it has a subrepresentation $V' = (V'_0, V'_1, V'_{\infty} = \mathbb{C})$, the $\Theta$-stability yields $\theta_0 \dim V'_0 + \theta_1 \dim V'_1 \leq \theta_0 \dim V_0 + \theta_1 \dim V_1$.

However $\theta_0, \theta_1 < 0$, so the above inequality must be equality and $V'_i = V_i$ for $i = 0, 1$.

Conversely, let us take a cyclic representation $V = (V_0, V_1, V_{\infty} = \mathbb{C})$. Then a non-zero proper sub-representation of it must be of the form $(V'_0, V'_1, 0)$, and we have $\theta_0 \dim V'_0 + \theta_1 \dim V'_1 < 0$.

by the condition $\theta_0, \theta_1 < 0$, so $V$ is $\Theta$-stable.

In Section 1.6, we will discuss chambers in the region $\theta_0 < 0, \theta_1 > 0$ in details, and show that they are in one-to-one correspondence with chambers for $Z_1$-stable pairs introduced in [CT19, Definition 1.5] (see Proposition 1.26).
1.4. Classification of stable framed representations with $\dim V_0 = 1$. In this subsection, we classify $\Theta$-stable representations $V = (V_0, V_1, V_\infty = \mathbb{C})$ of the framed quiver (1.10) such that $V_\infty \to V_0$ is an isomorphism and $V_1 \neq \mathbb{C}$. This will be used in the proof of our main theorem 1.21. Note that by Proposition 1.13, we can assume $\theta_0 < 0$ or $\theta_1 < 0$.

**Proposition 1.17.** Let $V = (V_0, V_1, V_\infty = \mathbb{C})$ be a representation of the framed quiver (1.10) such that $V_\infty \to V_0$ is an isomorphism and $V_1 \neq \mathbb{C}$. Let $\Theta$ lie in one of the chambers in Figure 3. Then it is $\Theta$-stable if and only if $(V_0, V_1)$ is the following for some $a \in \mathbb{C}$:

$$
(V_0, V_1) = \begin{cases} 
\Phi_0(j_a \ast \mathcal{O}_{\mathbb{P}^1}), & \text{if } \Theta \in I, \\
\Phi_0(j_a \ast \mathcal{O}_{\mathbb{P}^1}) \text{ or } \Phi_0(j_a \ast \mathcal{O}_{\mathbb{P}^1}(-2)[1]), & \text{if } \Theta \in II \\
\Phi_0(j_a \ast \mathcal{O}_{\mathbb{P}^1}) \text{ or } \Phi_0^+(j_a \ast \mathcal{O}_{\mathbb{P}^1}), & \text{if } \Theta \in III, \\
\Phi_0^+(j_a \ast \mathcal{O}_{\mathbb{P}^1}), & \text{if } \Theta \in IV.
\end{cases}
$$

And there is no such $(V_0, V_1)$ if $\Theta \in V$.

**Proof.** If $(V_0, V_1)$ is of the form described in (1.13), then $\dim V_0 = 1$. By taking the $(\tilde{Q}, \tilde{I})$-representation $V = (V_0, V_1, V_\infty = \mathbb{C})$ so that $V_\infty \to V_0$ is an isomorphism, it is straightforward to check that $V$ is $\Theta$-stable.

In what follows, we show the converse direction, i.e. if $V = (V_0, V_1, V_\infty = \mathbb{C})$ satisfies the assumption of the proposition, then $(V_0, V_1)$ must be of the form described in (1.14). If $V_0 = 0$, then $(V_0, V_1) = \Phi_0(j_a \ast \mathcal{O}_{\mathbb{P}^1})$ for some $a \in \mathbb{C}$. In this case, $V$ is $\Theta$-stable if and only if $\Theta(V_0, V_1) = \theta_0 < 0$. Below we may assume that $\dim V_1 > 0$. Note that $(V_0, V_1, 0)$ is a subrepresentation of $V$, so the $\Theta$-stability yields

$$
\theta_0 + \theta_1 \dim V_1 < 0.
$$

**Case 1.** $\theta_0 < 0$, $\theta_1 > 0$.

In this case for the sub-representation $(0, \operatorname{Ker}(b_1) \cap \operatorname{Ker}(b_2), 0)$ of $V$, the $\Theta$-stability yields

$$
\dim(\operatorname{Ker}(b_1) \cap \operatorname{Ker}(b_2)) \cdot \theta_1 \leq 0.
$$

Since $\theta_1 > 0$, we obtain $\operatorname{Ker}(b_1) \cap \operatorname{Ker}(b_2) = 0$. If $\operatorname{Ker}(b_1) = 0$ or $\operatorname{Ker}(b_2) = 0$, then $\dim V_1 = 1$ so we can assume that $\operatorname{Ker}(b_1) \neq 0$ and $\operatorname{Ker}(b_2) \neq 0$. As $\operatorname{Ker}(b_1) \cap \operatorname{Ker}(b_2) = 0$, we have

$$
\operatorname{Ker}(b_1) \oplus \operatorname{Ker}(b_2) \subseteq V_1.
$$

Since $\operatorname{Im}(b_1) = V_0$ and it is one dimensional, we have

$$
\dim \operatorname{Ker}(b_1) = \dim V_1 - 1.
$$

Combined with (1.16), we obtain $\dim V_1 = 2$. If this happens, $\Theta$ must satisfy $\theta_0 + 2\theta_1 < 0$ by (1.14).

We show that $(V_0, V_1)$ is $\Theta$-stable. Since $\operatorname{Ker}(b_1) \cap \operatorname{Ker}(b_2) = 0$, the only possible non-zero proper sub-representations of $(V_0, V_1)$ are either $(V_0, 0)$ or $(V_0, \mathbb{C})$. By the inequalities

$$
\theta_0 < \frac{\theta_0 + 2\theta_1}{3}, \quad \frac{\theta_0 + \theta_1}{2} < \frac{\theta_0 + 2\theta_1}{3},
$$

Figure 3. Chambers in Proposition 1.17.
we conclude that \((V_0, V_1)\) is \(\Theta\)-stable. By Proposition \[1.11\] together with \((b_1, b_2) \neq (0, 0)\), we conclude that \((V_0, V_1) = \Phi_0(j_a, \mathcal{O}_D(-2)[1])\) for some \(a \in \mathbb{C}\). Therefore we proved the case of \(\Theta \in \mathrm{I}\) or \(\Theta \in \mathrm{II}\).

**Case 2.** \(\theta_1 < 0\).

In this case, we set

\[ V'_1 := (\text{Im } a_1, \text{Im } a_2) \subseteq V_1, \]

to be the spanned vector subspace. Using the relation \(da_i = a_i c\) of the quiver \[\[.95\]\], we have \(d(V'_1) \subseteq V'_1\). Therefore \((V_0, V'_1, V_\infty)\) is a subrepresentation of \(V\). The \(\Theta\)-stability yields

\[ \theta_0 + \theta_1 \dim V'_1 \leq \theta_0 + \theta_1 \dim V_1. \]

Therefore we have \(V'_1 = V_1\), hence \(\dim V_1 \leq 2\). So \(\dim V_1 = 2\) and by \[1.15\] this is possible when \(\theta_0 + 2\theta_1 < 0\). Note that \(V'_1 = V_1\) implies that \((a_1, a_2): V_0^\otimes 2 \rightarrow V_1\) is an isomorphism. So if we also take \(\theta_0 > 0\) then it is easy to see that \((V_0, V_1)\) is a \(\Theta\)-stable representation of the unframed quiver \((Q, I)\) in \[\[1.19\]\]. By Proposition \[1.11\] together with \((a_1, a_2) \neq (0, 0)\), we conclude that \((V_0, V_1) = \Phi_0(j_a, \mathcal{O}_D)\) for some \(a \in \mathbb{C}\). Therefore \[1.14\] holds when \(\Theta \in \mathrm{III}\), \(\Theta \in \mathrm{IV}\), and there is no such \((V_0, V_1)\) when \(\Theta \in \mathbb{V}\). □

1.5. Counting invariants. We go back to the compact setting where

\[ f: X \rightarrow Y \]

is the contraction as in Setting \[1.1\]. Consider the coarse moduli space \(P^\Theta_n(X, \beta)\) of \(\Theta\)-semistable perverse coherent systems \((F, s)\) with \(\text{ch}(F) = (0, 0, 0, \beta, n)\) and \(f, \beta = 0\). The set of walls for this moduli space coincides with that studied in Section \[1.3\].

**Lemma 1.18.** Suppose that \(\Theta\) lies outside all walls defined in \[1.13\]. Then \(P^\Theta_n(X, \beta)\) with \(f, \beta = 0\) depends only on the connected components where \(\Theta\) locates.

**Proof.** The argument is the same as in the beginning of Section \[1.3\]. A wall appears if there exists a strictly \(\Theta\)-semistable perverse coherent system. By taking the Jordan-Hölder filtration, there exists a \(\Theta\)-stable perverse coherent system \(V\) such that \(\Theta(V) = 0\). As \(\text{End}(V) = \mathbb{C}\), the support of \(V\) is connected. Therefore by Lemma \[1.3\] the support of \(V\) is contained in a fiber of \(E \rightarrow C\). By the assumption of Setting \[1.1\], it sits inside the local resolved conifold \(X_0 = \mathcal{O}_{P^2}(-1, -1, 0)\), i.e. reducing to the local case. □

For a general choice of \(\Theta \in \mathbb{R}^2\) such that it does not lie on a wall, the moduli space \(P^\Theta_n(X, \beta)\) has a universal family and consists of only stable objects. In this case, we have the following:

**Proposition 1.19.** When \(\Theta \in \mathbb{R}^2\) lies outside the walls in \[1.13\], the moduli space \(P^\Theta_n(X, \beta)\) can be given a \((-2)\)-shifted symplectic derived scheme structure in the sense of Pantev-Toën-Vaquie-Vezzosi \[PTVV\].

**Proof.** The \(\Theta\)-stability gives an open condition for any family of objects in \(D^b\text{ Coh}(X)\). Therefore as in \[CMT19\] Lemma 1.3, the existence of \((-2)\)-shifted symplectic structure is reduced to \[PTVV\] Theorem 0.1. □

In the above case, by \[CGJ Corollary 1.17\], we know \(P^\Theta_n(X, \beta)\) is orientable, hence it admits a Borisov-Joyce virtual class \[BJ\] :

\[ [P^\Theta_n(X, \beta)]^{\text{vir}} \in H_{2n}(P^\Theta_n(X, \beta), \mathbb{Z}), \]

which depends on the choice of orientation \[CGJ CL17\]. The virtual dimension of \(P^\Theta_n(X, \beta)\) is in general non-zero, and we need to involve some insertions to obtain enumerative invariants. As in \[CMT18\ CMJT19\ CT19\ CK19\], we consider primary insertions

\[ \tau: H^4(X, \mathbb{Z}) \rightarrow H^2(P^\Theta_n(X, \beta), \mathbb{Z}), \quad \tau(\gamma) := (\pi_F)_* (\pi_X^* \gamma \cup \text{ch}_3(F)), \]

where \(\pi_X, \pi_P\) are projections from \(X \times P^\Theta_n(X, \beta)\) onto corresponding factors, \(\mathbb{I} = (\pi_X^* \mathcal{O}_X \rightarrow \mathbb{F})\) is the universal pair and \(\text{ch}_3(F)\) is the Poincaré dual to the fundamental cycle of \(F\).

**Definition 1.20.** The primary counting invariants of \(\Theta\)-stable perverse coherent systems are

\[ P^\Theta_{n, \beta}(\gamma) := \int [P^\Theta_n(X, \beta)]^{\text{vir}} \tau(\gamma) \in \mathbb{Z}. \]

The following is the main result of this section.
Theorem 1.21. Let \( f: X \to Y \) be as in Setting 1.1. \( E \subset X \) be the exceptional surface and \([\mathbb{P}^1] \in H_2(X, \mathbb{Z})\) be the fiber class of \( f|_E: E \to C \). Let \( \Theta = (\theta_0, \theta_1) \in \mathbb{R}^2 \) be outside walls defined in (1.13). Then for certain choice of orientation, we have

\[
\sum_{n \in \mathbb{Z}, f, \beta = 0} \frac{P^\Theta_{n,d}(\gamma)}{n!} q^n t^\beta = \begin{cases} 
\exp \left( q^{[\mathbb{P}^1]} \right) f_X(\gamma)[E] & \text{if } \theta_0 < 0, \theta_0 + 2\theta_1 > 0, \\
\exp \left( q^{[\mathbb{P}^1]} - qt^{-[\mathbb{P}^1]} \right) f_X(\gamma)[E] & \text{if } \theta_0 < 0, \theta_0 + 2\theta_1 < 0, \\
\exp \left( -qt^{-[\mathbb{P}^1]} \right) f_X(\gamma)[E] & \text{if } \theta_0 > 0, \theta_0 + 2\theta_1 < 0, \\
1 & \text{otherwise.}
\end{cases}
\]

Here the first three cases correspond to \( \Theta \) lies in the chamber (a) I, (b) II and III, (c) IV in Figure 5 respectively.

Proof. We only need to consider curve classes \( \beta = d[\mathbb{P}^1] \) for \( d \in \mathbb{Z} \). Our aim is to evaluate

\[
(1.17) \quad P^\Theta_{n,d}(\gamma) = \int_{[P^\Theta_n(X,d[\mathbb{P}^1])]} \tau(\gamma)^n.
\]

We assume \( \gamma \cdot [E] \geq 0 \) (otherwise consider \( -\gamma \) instead). We take \( \{S_i\}_{i=1}^n \) to be \( n \)-different homological cycles which represent the class \( \gamma \in H^4(X, \mathbb{Z}) \) such that the intersections \( (S_i \cap E)'s \) are transverse, in general position and are disjoint for different choices of \( 1 \leq i \leq n \). For simplicity, we assume

\[
S_i \cap E = \{P_{i,1}, P_{i,2}, \ldots, P_{i,n_0(\gamma)}\}, \quad i = 1, 2, \ldots, n,
\]

where all points are with positive signs. Here \( n_0(\gamma) := \gamma \cdot [E] \) is the genus 0, degree 1 Gopakumar-Vafa type invariant defined by Klemm-Pandharipande [KP]. In the case when there is a point with negative sign, we can pair it with another point with positive sign, then it is easy to argue the pair will not contribute to (1.17).

For any \( (\mathcal{O}_X \to F) \in P^\Theta_n(X, \beta) \), by Lemma 1.3 \( F \) is supported on fibers of \( f|_E: E \to C \). We can decompose \( F \) into the direct sum

\[
(1.18) \quad F = \bigoplus_{i=1}^k F_i,
\]

such that \( \text{supp}(F_i) \)'s are connected. Then any \( (\mathcal{O}_X \to F_i) \) is supported on a formally local chart of \( X \to Y \), so can be regarded as a perverse coherent system for \( X_0 \to Y_0 \). Therefore we can present it as a finite dimensional \( \Theta \)-stable representation \( V = (V_0, V_1, V_\infty = \mathbb{C}) \) of the framed quiver (1.10).

Note that we have \( \chi(F_i) = \dim \text{Hom}(\mathcal{O}_X, F_i) \neq 0 \). Indeed otherwise \( (0 \to F_i) \) is a direct summand of \( (\mathcal{O}_X \to F) \) which violates the \( \Theta \)-stability. Therefore

\[
(1.19) \quad n = \chi(F) = \sum_{i=1}^k \chi(F_i) \geq k.
\]

In particular \( n = k \) if and only if \( \chi(F_i) = 1 \) for all \( i \), and \( n > k \) otherwise.

Since \( \gamma = [S_i] \), the class \( \tau(\gamma) \) is represented by a divisor of \( P^\Theta_n(X, \beta) \) supported on pairs \( (\mathcal{O}_X \to F) \) such that, under the decomposition (1.18), there is a unique \( 1 \leq j(i) \leq k \) satisfying

\[
(1.20) \quad \dim \text{Supp}(F_{j(i)}) = 1, \quad S_i \cap \text{Supp}(F_{j(i)}) \neq \emptyset.
\]

The multiplicity of this divisor at \( (\mathcal{O}_X \to F) \) is given by \( d(F_{j(i)}) \). Note that by our generic choice of \( S_i \), we have \( S_i \cap \text{Supp}(F_{j(i)}) = \emptyset \) for \( i' \neq i \). Therefore if the pair \( (\mathcal{O}_X \to F) \) satisfies the above condition for \( i \) and \( i' \), we have \( j(i) \neq j(i') \).

Now the cycle \( \tau(\gamma)^n \) imposes conditions (1.20) for each \( 1 \leq i \leq n \), so it is represented by a codimension \( n \) cycle supported on pairs \( (\mathcal{O}_X \to F) \) such that \( n \leq k \), hence \( n = k \), and \( d(F_i) \neq 0 \) for all \( i \). It follows that each \( F_i \) satisfies \( \chi(F_i) = 1 \) and \( d(F_i) \neq 0 \), so it corresponds to a \( \Theta \)-stable representation \( V \) of the form

\[
V = (V_0, V_1, V_\infty = \mathbb{C}), \quad V_i \neq \mathbb{C}, \quad V_\infty \cong V_0.
\]
Here the latter isomorphism follows since \( \dim V_0 = \chi(F_i) = 1 \) and \( V_\infty \to V_0 \) is non-zero. Therefore by Proposition \([1.17]\), \( F_i \) is either one of the following objects

\[
F_i = \begin{cases} 
  j_a \mathcal{O}_{\mathbb{P}^1}, & \text{if } \Theta \in \mathbb{I}, \\
  j_a \mathcal{O}_{\mathbb{P}^1} \text{ or } j_a \mathcal{O}_{\mathbb{P}^1}(-2)[1], & \text{if } \Theta \in \mathbb{II}, \\
  j_a \mathcal{O}_{\mathbb{P}^1} \text{ or } \Upsilon(j_a^+ \mathcal{O}_{\mathbb{P}^1}), & \text{if } \Theta \in \mathbb{III}, \\
  \Upsilon(j_a^+ \mathcal{O}_{\mathbb{P}^1}), & \text{if } \Theta \in \mathbb{IV}.
\end{cases}
\]

There is no such \( F_i \) in other cases and \( T \) is the flop equivalence \([1.14]\), \( j_a \) for \( a \in C \) is the composition

\[ j_a : \mathbb{P}^1 = (f|_E)^{-1}(a) \hookrightarrow E \hookrightarrow X, \]

and \( j_a^+ \) is similarly defined for the flop side.

Below we prove the desired formula in the case of \( \Theta \in \mathbb{II} \). Other cases are similarly obtained. We call an object of the form \( j_{a_i} \mathcal{O}_{\mathbb{P}^1}, j_{a_i} \mathcal{O}_{\mathbb{P}^1}(-2)[1] \) as type (i), (ii) respectively. By a computation of the numerical classes, the number of objects in \( \{ F_1, \ldots, F_n \} \) of type (i), (ii) is \( (n + d)/2 \), \( (n - d)/2 \) respectively. The number of such pairs is finite, so \( \tau(\gamma)^n \) is represented by a zero cycle. The total degree of the zero cycle \( \tau(\gamma)^n \) is calculated as follows. We first choose one of the points in \( S_i \cap E \) for each \( i = 1, 2, \ldots, n \) and then choose \( (n + d)/2 \) in \( n \) for a choice of type (i) objects. Since the type (ii) objects contribute to \(-1\), the total degree is

\[
(1.21) \quad (-1)^{(n-d)/2} \left( \frac{n+d}{2} \right) \cdot (n_{0,1}(\gamma))^n.
\]

The contribution from the virtual class is determined as follows: consider an open immersion

\[
U := (\text{Sym}^{(n+d)/2}(C) \times \text{Sym}^{(n-d)/2}(C)) \setminus \Delta_{\text{Big}} \hookrightarrow P_n^\Theta(X, \beta),
\]

where \( \Delta_{\text{Big}} \) is the big diagonal, sending \( (a_1, \ldots, a_{(n+d)/2}, b_1, \ldots, b_{(n-d)/2}) \) to the object

\[
\mathcal{O}_X \to \bigoplus_{i=1}^{(n+d)/2} j_{a_i} \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{i=1}^{(n-d)/2} j_{b_i} \mathcal{O}_{\mathbb{P}^1}(-2)[1].
\]

It is straightforward to check that \( \text{Ext}^2(-, -) \) of the above object is zero. Therefore by Lemma \([1.23]\) for some choice of orientation, the virtual class (up to invert 2 in the coefficient) is written as

\[
[P_n^\Theta(X, \beta)]^\text{vir} = [U] + \sum_{i \in I} c_i [C_i],
\]

where \( c_i \in \mathbb{Z}[1/2] \), each \( C_i \) is an irreducible closed subscheme of \( P_n^\Theta(X, \beta) \) with dimension \( n \) such that \( C_i \neq \emptyset \). Since the zero cycle which represents \( \tau(\gamma)^n \) is contained in \( U \), it follows that the integral \([1.17]\) coincides with \([1.21]\). Therefore we obtain the desired expression of the generating series in the case \( \Theta \in \mathbb{II} \).

**Remark 1.22.** In the non-commutative chamber, our formula shares a similar shape as Szendrői’s formula \([Sze]\), which is a product of counting invariants on \( X \) and its flopping side \( X^+ \). See the RHS of the formula in Corollary \([2.10]\) (taking \( m = \lambda_\gamma \)) for an expression of Szendrői’s formula.

In the above theorem, we used the following technical lemma on Borisov-Joyce virtual classes, which we now prove using the recent work of Oh-Thomas \([OT]\) lifting the virtual classes in Chow groups (up to invert 2 in the coefficient).

**Lemma 1.23.** Let \( M \) be a projective fine moduli scheme of simple objects in \( D^b\text{-Coh}(X) \) of a Calabi-Yau 4-fold \( X \), which can be given a \((-2)\)-shifted symplectic derived scheme structure. Let \( [F] \in M \) be a point such that \( \text{Ext}^2(F, F) = 0 \), and take the unique irreducible component \( M' \subset M \) which contains \([F]\). Then for some choice of orientation, the Borisov-Joyce virtual class is written as

\[
[M]^\text{vir} = [M'] + \sum_{i \in I} c_i [C_i], \quad c_i \in \mathbb{Z}[1/2]
\]

in \( H_{2n}(M, \mathbb{Z}[1/2]) \). Here \( 2n \) is the (real) virtual dimension of \( M \), and each \( C_i \subset M \) is an irreducible \( n \)-dimensional subscheme such that \( C_i \neq M' \).

**Proof.** By \([OT]\), the BJ virtual class is lifted to an element of the Chow group with \( \mathbb{Z}[1/2] \)-coefficient (which we call Oh-Thomas virtual class below):

\[
[M]^\text{vir}_{OT} \in A_n(M, \mathbb{Z}[1/2]).
\]
We briefly review their construction. Let
\[(\mathcal{R}\pi_* \mathcal{R}\text{Hom}(\mathcal{E}, \mathcal{E})_o[1])^\vee \to \mathbb{L}_M\]
be the obstruction theory for \(M\), where \(\mathcal{E}\) is a universal object on \(X \times M\) and \(\pi_M: X \times M \to M\) is the projection. Let \(M \hookrightarrow A\) be a closed immersion into a smooth scheme \(A\) with defining ideal \(I \subset O_A\). It is proved in [OT] Proposition 4.1 that the above obstruction theory is represented by a map of complexes
\[(1.22) \quad (T \to E \to T') \to (0 \to I/I^2 \to \Omega_A|_M).\]
Here \(E, T\) are vector bundles on \(M\) such that \(E\) is equipped with a nondegenerate quadratic form, satisfying some compatibility with Serre duality pairing. The stupid truncation of the above map
\[(E \to T') \to (I/I^2 \to \Omega_A|_M)\]
is a Behrend-Fantechi perfect obstruction theory [BF], so we have the intrinsic normal cone \(\mathcal{C}_M \subset [E^\vee / T]\). By pulling it back to \(E^\vee \cong E\), we obtain the cone \(C_{E^\vee} \subset E\). Then Oh-Thomas virtual class
\[[M]_{\text{vir}} := \sqrt{0}_{E}[C_{E^\vee}] \in A_n(M, \mathbb{Z}[1/2])\]
is given by the square root Gysin pull-back ([OT] Definition 3.3) of the zero section \(0_E: M \to E\). Here an orientation is required in the definition.

The moduli space \(M\) is smooth at \([F] \in M\) by the assumption \(\text{Ext}^2(F, F) = 0\). Hence there is an irreducible smooth Zariski open subset \(U \subset M\) which contains \([F]\), so \(M' := \overline{U}\) is the unique irreducible component which contains \([F]\). Then obviously
\[[M]_{\text{vir}} = c'[M'] + \sum_{i \in I} c_i[C_i]\]
for some \(c', c_i \in \mathbb{Z}[1/2]\), where \(C_i \subset M\) is irreducible with dimension \(n\) and \(C_i \neq M'\).

We are left to show \(c' = 1\). From the construction of \(\sqrt{0}_{E}\), it is easy to see it commutes with pull-back by an open immersion \(U \hookrightarrow M\). So we have
\[(1.23) \quad [M]_{\text{vir}}|_U = \sqrt{0}_{E}|_U[C_{E^\vee}|_U] \in A_n(U, \mathbb{Z}[1/2]) = \mathbb{Z}[1/2]|U].\]
Here the last identity holds as \(U\) is an irreducible smooth scheme of dimension \(n\). By [OT] Equation (56) the class \((1.23)\) is independent of the choice of the 3-term complex \((1.22)\). So on \(U\), we can replace \((1.22)\) by \(T_U \to 0 \to T'_U\). The resulting virtual class on \(U\) is then \(\pm |U]\). By choosing a suitable orientation, we can take \(c' = 1\).

1.6. Comparison with \(Z_t\)-stable pairs. For a birational contraction \(f: X \to Y\) as in Setting [11], recall that we have fixed a Q-ample divisor \(\omega\) on \(X\) with degree one on the fibers of \(f|_E: E \to C\), and the associated slope function is defined by \((1.5)\). Here we recall the definition of \(Z_t\)-stability:

**Definition 1.24.** ([CT19] Lemma 1.7) Let \(F\) be a one dimensional coherent sheaf and \(s: O_X \to F\) be a section. We say \((F, s)\) is a \(Z_t\)-(semi)stable pair for \(t \in \mathbb{R}\) if
(i) for any subsheaf \(0 \neq F' \subset F\), we have \(\mu_\omega(F') < (\leq) t\),
(ii) for any subsheaf \(F'' \subset F\) such that \(s\) factors through \(F''\), we have \(\mu_\omega(F/F'') > (\geq) t\).

We only consider \(Z_t\)-stable pairs \((F, s)\) such that \([F] = \beta\) satisfies \(f_*\beta = 0\), i.e. \(F\) is supported on fibers of \(f: X \to Y\). Then the wall-chambers of \(Z_t\)-(semi)stable pairs are classified as follows.

**Lemma 1.25.** The set of walls for \(Z_t\)-stability of pairs \((F, s)\) on \(X\) is given by \(Z \subset \mathbb{R}\). Moreover, there exists a \(Z_t\)-stable pair \((F, s)\) with \([F] \neq 0\) only if the following inequalities hold:
\[(1.24) \quad t > \frac{\lambda(F)}{d(F)} \geq 1.\]

**Proof.** The first claim holds since any one dimensional stable sheaf on \(X\) supported on fibers of \(f: X \to Y\) is of the form \(f_* \mathcal{O}_{P_k}(k)\) for some \(k \in \mathbb{Z}\) and \(a \in C\), whose slopes are integers. We claim that if there is a \(Z_t\)-stable pair \((F, s)\), we have the inequalities \((1.24)\). Let \(Z \subset X\) be the closed subscheme such that \(\text{Im}(s) = \mathcal{O}_Z\). By the \(Z_t\)-stability, \(\mathcal{O}_Z\) is a non-zero subsheaf of \(F\). If \(\mathcal{O}_Z \neq F\), we have
\[\mu_\omega(F/\mathcal{O}_Z) = \frac{\lambda(F) - \lambda(\mathcal{O}_Z)}{d(F) - d(\mathcal{O}_Z)} > t > \frac{\lambda(F)}{d(F)}\]
which implies that
\[
\frac{\chi(\mathcal{O}_Z)}{d(\mathcal{O}_Z)} < \frac{\chi(F)}{d(F)}.
\]
This is an equality if \( F = \mathcal{O}_Z \). Finally, using the fact that any Cohen-Macaulay curve \( Z \) in \( X \)
supported on fibers of \( f \) satisfies \( \chi(\mathcal{O}_Z) \geq d(\mathcal{O}_Z) \), we are done.

The following proposition gives a comparison between stable perverse coherent systems and \( \mathbb{Z}_t \)-stable pairs:

**Proposition 1.26.** Let \( m \geq 2 \) and take \( \Theta = (-m + 1 + 0^+, m) \), i.e. \( \Theta \) lies in the chamber between walls \( L^- (m - 1) \) and \( L^- (m) \). Then a \( \Theta \)-stable perverse coherent system on \( X \) is a \( \mathbb{Z}_t \)-stable pair for \( t = m - 0^+ \), i.e. \( t \) lies in the chamber \( (m - 1, m) \subseteq \mathbb{R} \), and vice versa.

**Proof.** Let \((F, s)\) be a \( \Theta \)-stable perverse coherent system on \( X \) supported on fibers of \( f \). There is an exact sequence in \( \text{Per}(X/Y) \):

\[
0 \to \mathcal{H}^{-1}(F)[1] \to F \to \mathcal{H}^0(F) \to 0.
\]

Note that \( \chi(\mathcal{H}^{-1}(F)[1]) \geq 0 \) as \( R^1f_*(\mathcal{H}^{-1}(F)[1]) \) is a zero dimensional sheaf. Assume that \( F \) is not a sheaf (so \( \mathcal{H}^{-1}(F)[1] \neq 0 \)). Then we have

\[
0 \geq (\theta_0 + \theta_1) \cdot \chi(\mathcal{H}^{-1}(F)) > \theta_1 \cdot d(\mathcal{H}^{-1}(F)),
\]

where the second inequality uses the \( \Theta \)-stability of \( F \), i.e.

\[
\Theta(\mathcal{H}^{-1}(F)[1]) = \theta_0 \cdot \chi(\mathcal{H}^{-1}(F)[1]) + \theta_1 \cdot (\chi(\mathcal{H}^{-1}(F)[1]) - d(\mathcal{H}^{-1}(F)[1])) < 0.
\]

This implies that \( d(\mathcal{H}^{-1}(F)) \) is negative, a contradiction. Therefore \( F \) is a one dimensional sheaf, and it is easy to see that \( \Theta \)-stability is equivalent to \( \mathbb{Z}_t \)-stability by choosing \( t = \theta_1/(\theta_0 + \theta_1) \).

Conversely given a \( \mathbb{Z}_t \)-stable pair \((F, s)\) for \( t = m - 0^+ \), we show that it is a \( \Theta \)-stable perverse coherent system. Let \( \text{Im}(s) = \mathcal{O}_Z \subseteq F \) for a closed subscheme \( Z \subseteq X \). Then applying \( Rf_* \) to the exact sequence in \( \text{Coh}(X) \)

\[
0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0,
\]

we obtain \( R^1f_*\mathcal{O}_Z = 0 \). For any \( A \in \text{Coh}(X) \) such that \( Rf_*A = 0 \), we have an exact sequence

\[
0 \to \text{Hom}(\mathcal{O}_Z, A) \to \text{Hom}(\mathcal{O}_X, A) = 0.
\]

Therefore \( \text{Hom}(\mathcal{O}_Z, A) = 0 \), so by [Bri02, Lemma 3.2] we have \( \mathcal{O}_Z \in \text{Per}(X/Y) \). From the \( \mathbb{Z}_t \)-stability, we know that any Harder-Narasimhan factor of \( F/\mathcal{O}_Z \) satisfies \( \mu(F/\mathcal{O}_Z) \geq t > 0 \). By Proposition 1.4, \( F/\mathcal{O}_Z \in \text{Per}(X/Y) \). Therefore it follows that \( F \in \text{Per}(X/Y) \).

Next we verify the \( \Theta \)-stability of the pair \((F, s)\). Let us take an exact sequence in \( \text{Per}(X/Y) \)

\[
(1.25)
0 \to F_1 \to F \to F_2 \to 0.
\]

Since \( F \) is a sheaf, by taking the cohomology long exact sequence we see that \( F_1 \) is also a sheaf. We have an exact sequence in \( \text{Per}(X/Y) \):

\[
(1.26)
0 \to \mathcal{H}^{-1}(F_2)[1] \to F_2 \to \mathcal{H}^0(F_2) \to 0.
\]

By combining (1.25) with (1.26), we obtain a distinguished triangle

\[
(1.27)
F_1 \to F_2 \to \mathcal{H}^{-1}(F_2)[1],
\]

where \( F_3 \) fits into a distinguished triangle

\[
F_3 \to F \to \mathcal{H}^0(F_2).
\]

By (1.27), \( \mathcal{H}^1(F_3) = 0 \), so the above triangle is an exact sequence in \( \text{Coh}(X) \). Then the \( \mathbb{Z}_t \)-stability gives \( \mu(F_3) \leq t \). Note also (1.27) is equivalent to an exact sequence in \( \text{Coh}(X) \)

\[
0 \to \mathcal{H}^{-1}(F_2) \to F_1 \to F_3 \to 0.
\]

Since \( \mathcal{H}^{-1}(F_2)[1] \in \text{Per}(X/Y) \), then \( \mathcal{H}^{-1}(F_2) \) belongs to the category \( \mathcal{F}_\omega \) in Proposition 1.4 hence we know \( \chi(\mathcal{H}^{-1}(F_2)) \leq 0 \), so

\[
\chi(F_3) = \chi(F_1) - \chi(\mathcal{H}^{-1}(F_2)) \geq \chi(F_1),
\]

\[
d(F_3) = d(F_1) - d(\mathcal{H}^{-1}(F_2)) \leq d(F_1).
\]

Therefore \( \mu(F_3) \leq \mu(F_2) \). Together with \( \mu(F_2) \leq t \), we conclude that \( \mu(F_1) \leq t \) and it is easy to see it is a strict inequality if \( F_1 \neq F \). Choosing \( \Theta \) such that \( t = \theta_1/(\theta_0 + \theta_1) \), we have proved the first condition in Definition 1.3. Similar argument also shows that the second condition of Definition 1.3 and Definition 1.24 are equivalent. \( \square \)
Let $P^t_n(X, \beta)$ be the moduli space of $Z_t$-stable pairs $(F, s)$ with $\text{ch}(F) = (0, 0, 0, \beta, n)$. For a generic $t \in \mathbb{R}$, the moduli space $P^t_n(X, \beta)$ is a projective scheme, and the following invariant for $\gamma \in H^4(X, \mathbb{Z})$ was defined in [CT19]:

$$P^t_n,\beta(\gamma) := \int_{[P^t_n(X, \beta)]^{vir}} \tau(\gamma)^n \in \mathbb{Z}.$$ 

In the $t \to \infty$ limit, $P^t_n(X, \beta)$ recovers the moduli space $P_n(X, \beta)$ of PT stable pairs.

Let $I_n(X, \beta)$ be the moduli space of ideal sheaves $I_Z = (\mathcal{O}_X \rightarrow \mathcal{O}_Z)$ of one dimensional subschemes $Z$ such that $([Z], \chi(\mathcal{O}_Z)) = (\beta, n)$. We have (primary) DT/PT invariants

$$I_n,\beta(\gamma) := \int_{[I_n(X, \beta)]^{vir}} \tau(\gamma)^n, \quad P_n,\beta(\gamma) := \int_{[P_n(X, \beta)]^{vir}} \tau(\gamma)^n.$$ 

Combining Theorem 1.21 with Proposition 1.26, 1.14, we prove some of our previous conjectures, which give sheaf theoretic interpretations of Gopakumar-Vafa type invariants defined by Klemm-Pandharipande [KP] (see also [CMT18, CT20a] for other approaches).

**Corollary 1.27.** Let $f: X \rightarrow Y$ be as in Setting 1.4. $E \subset X$ be the exceptional surface and $[\mathbb{P}^1] \in H_2(X, \mathbb{Z})$ be the fiber class of $f|_E: E \rightarrow C$. For any $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$ with $f_*\beta = 0$, a generic $t > n/\text{deg}(\beta)$ and $\gamma \in H^4(X, \mathbb{Z})$, we have identities

$$I_n,\beta(\gamma) = P_n,\beta(\gamma) = P^t_n,\beta(\gamma),$$ 

for certain choice of orientation. Moreover, their generating series satisfies

$$\sum_{n \in \mathbb{Z}, t, \beta = 0} P_n,\beta(\gamma) q^n t^\beta = \exp \left( qt^{[\mathbb{P}^1]} \right) \int_X \gamma^{\vir}[E].$$ 

**Therefore the LePotier-pair/GV conjecture [CT19] Conjecture 0.2, PT/GV conjecture [CMT19] §0.7 and DT/PT conjecture [CT19] Conjecture 0.3 hold in this case.**

Here the first equality is the correspondence “DT=PT=LePotier-pair” and the second equality gives the “PT/GV” correspondence, where the power in the RHS is the only nontrivial GV invariant.

## 2. Perverse coherent systems on local resolved conifold

In the previous section, we studied counting invariants of perverse coherent systems on projective CY 4-folds. In this section, we focus on the local model $X := X_0 = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, with a contraction $f: X \rightarrow Y$ for $Y = Y_0$ in 1.3. We define counting invariants of perverse coherent systems on $X$ using tautological insertions as in [CK18, CKM19] and torus localization formulae as in [CK19, CMT19, CT19].

### 2.1. Moduli spaces.

Recall the framed quiver $\tilde{Q}$ with relation $I$ associated with $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$:

![Framed Quiver](image)

$$a_2 b_1 a_1 = a_1 b_1 a_2, \quad b_2 a_1 b_1 = b_1 a_1 b_2,$$

$$d a_i = a_i c, \quad c b_i = b_i d, \quad i = 1, 2.$$ 

For a dimension vector $d = (d_0, d_1) \in (\mathbb{Z}_{\geq 0})^2$, let $V_i$ be vector spaces with dim $V_i = d_i$. The space of representations of the framed quiver $\tilde{Q}$ is

$$R^d(\tilde{Q}) := \text{Hom}(V_0, V_1)^{\otimes 2} \oplus \text{Hom}(V_1, V_0)^{\otimes 2} \oplus \text{Hom}(V_0, V_0) \oplus \text{Hom}(V_1, V_1) \oplus V_0.$$ 

We have the closed subscheme

$$R^d(\tilde{Q}, I) \subset R^d(\tilde{Q}),$$

corresponding to representations which preserve the relation $I$. For $\Theta \in \mathbb{R}^2$, the $\Theta$-semistable $(\tilde{Q}, I)$-representations (see Definition 1.4) give an open subscheme of $R^d(\tilde{Q}, I)$, denoted by

$$R^d_+(\tilde{Q}, I) \subset R^d(\tilde{Q}, I).$$
The good moduli space of $\Theta$-semistable representations with dimension vector $d$ is given by the GIT quotient

$$M_0^{\Theta}(\bar{Q}, I) := R_0^{\Theta}(\bar{Q}, I)/(GL(V_0) \times GL(V_1)).$$

If $\Theta \in \mathbb{R}^2$ lies outside walls $\mathbf{1}_{\mathbf{13}}$, it is a fine moduli space consisting of $\Theta$-stable representations. The equivalence in $\mathbf{1}_{\mathbf{7}}$ induces an isomorphism

$$(2.1) \quad M_0^{\Theta}(\bar{Q}, I) \cong P_\Theta^\omega(X, \beta), \quad (\beta, n) = ((d_0 - d_1)[\mathbb{P}^1], d_0),$$

where $P_\Theta^\omega(X, \beta)$ is the moduli space of $\Theta$-stable (compactly supported) perverse coherent systems as in Theorem $\mathbf{1}_{\mathbf{6}}$.

### 2.2. Torus action

We consider the torus $(\mathbb{C}^*)^6$ which acts on the six edges, $a_1, a_2, b_1, b_2, c, d$ diagonally by scaling. It induces an action on the path algebra $CQ$. In order to preserve the relation $I$, we need the actions on edge $c$ and $d$ are the same, so we consider the subtorus

$$(2.2) \quad (\mathbb{C}^*)^5 := \{(q_1, q_2, q_3, q_4, q_5) \in (\mathbb{C}^*)^6 : q_5 = q_6\}.$$

Note that $C^* = \{(q, q, q^{-1}, q^{-1}, 1, 1) \in (\mathbb{C}^*)^6\}$ acts trivially on isomorphism classes of representations of $(Q, I)$, so we will consider the action of the quotient torus $\bar{T} := (\mathbb{C}^*)^5/C^*$

on moduli spaces of representations. The above torus action does not preserve the CY4 structure, as in [Sze, §2.2], we consider the 3-dimensional subtorus

$$T_0 := \{t \in T : q_1q_2q_3q_4q_5 = 1\}.$$

Both $\bar{T}_0$ and $\bar{T}$ lift to actions on moduli spaces $M_0^{\Theta}(\bar{Q}, I)$. Their fixed loci are the same and consist of finite number of reduced points.

**Proposition 2.1.** Let $\Theta \in \mathbb{R}^2$ be outside walls $\mathbf{1}_{\mathbf{13}}$. Then we have

$$(2.3) \quad M_0^{\Theta}(\bar{Q}, I)^{\bar{T}_0} = M_0^{\Theta}(\bar{Q}, I)^{\bar{T}},$$

and it is a finite set. Moreover the Zariski tangent space of any element has no $\bar{T}_0$-fixed subspace.

**Proof.** The proof is an easy adaption of the resolved conifold case [Sze, NN]. Let $A = CQ/I$ be the quotient of the path algebra by the ideal of relations of the quiver $\mathbf{1}_{\mathbf{9}}$. We first consider the case that $\Theta$ lies in the non-commutative chamber, i.e. $\theta_0, \theta_1 < 0$. By Proposition $\mathbf{1}_{\mathbf{10}}$ any element in $M_0^{\Theta}(\bar{Q}, I)$ is a cyclic module $(M, m)$, where $m \in M$ is based at vertex 0. We consider the surjection

$$\pi : \tilde{A} \twoheadrightarrow A, \quad 1 \mapsto m,$$

whose kernel is denoted by $J := \ker(\pi)$. Let $(1)$ denote the idempotent element of $CQ$ at vertex 1. Then $A(1)$ consists of paths starting from vertex 1 which surely annihilates $m$. So we write

$$J = J_0 \oplus A(1).$$

We claim that if $(M, m)$ is $\bar{T}_0$-fixed, then $J_0$ is a monomial ideal. In fact, $J_0$ is a $\bar{T}_0$-fixed ideal whose generators are of the form $f(a_1, a_2, b_1, b_2, c, d) \cdot W$, where $f$ is a monomial in those variables and $W$ is a weight zero $\bar{T}_0$-eigenvector. Note that a weight zero $\bar{T}_0$-eigenvector should have the same start and end point, so $\mathbb{C}^* = \{(q, q, q^{-1}, q^{-1}, 1, 1) \in (\mathbb{C}^*)^5\}$ acts trivially. So weight zero $\bar{T}_0$-eigenvectors are the same as weight zero $\bar{T}_0$-eigenvectors, where $\bar{T}_0 \subset (\mathbb{C}^*)^5$ is the lift of $T_0$ to $(\mathbb{C}^*)^5$. The set of weight zero $\bar{T}_0$-eigenvectors is generated by

$$cb_2a_2b_1a_1, \quad da_1b_1ab_2 \in A.$$

Therefore, generators of $J_0$ are of the form $f(a_1, a_2, b_1, b_2, c, d) \cdot p(cb_2a_2b_1a_1)$, where $p$ is a polynomial with nonzero constant term.

Let $Z(A) \subset A$ be the center of $A$. It is easy to see that

$$Z(A) = \langle a_i b_j + b_j a_i, c + d \rangle.$$

Applying it to the idempotent element at the vertex 0, we have

$$Z(A)(0) = \langle b_j a_i, c \rangle \cong Z(A).$$

Let $K := J_0 \cap Z(A)(0)$, which is an ideal in $Z(A)(0) \cong Z(A)$. Since $J_0$ is $\bar{T}_0$-fixed, the zero set of $K$ is supported on the origin of $\text{Spec}(Z(A)(0)) \cong Y$ (e.g. [CK18 Lemma 3.1]). This is disjoint from the zero set of $p(cb_2a_2b_1a_1) \subset Z(A)(0)$. By the Nullstellensatz, $(p, K) = Z(A)(0)$, hence $f \in J_0$. Therefore $J_0$ is a monomial ideal, so it is $\bar{T}$-fixed. It follows that the identity $(2.3)$ holds and both sides are finite sets.
Next we study the $T_0$-fixed subspace of the Zariski tangent space of $(M, m) \in M_4^0(\hat{Q}, I)^{T_0}$. Under the derived equivalence in \((\ref{eq:derived-equivalence})\), a cyclic module $(M, m)$ (resp. $A(0)$) corresponds to a pair $I = (\mathcal{O}_X \to F)$ (resp. $\mathcal{O}_X$). We have canonical isomorphisms

$$\text{Ext}^1_X(I, J)_0 \cong \text{Hom}_X(I, F) \cong \text{Hom}_A(J_0, M),$$

where the first isomorphism can be proved as \cite[pp. 14]{CMT19}, and the second one follows from the exact sequence of $A$-modules

$$0 \to J_0 \to A(0) \to M \to 0.$$

We claim that $\text{Hom}_A(J_0, M)^{T_0} = 0$. It is enough to show that under the edge torus $(\mathbb{C}^*)^6$ on $\text{Hom}_A(J_0, m)$, no weight is a multiple of $(1, 1, 1, 1, 1, 1)$. Suppose that $\phi: J_0 \to M = A/J$ is an eigenvector of weight $w(1, 1, 1, 1, 1)$ with $w \in \mathbb{Z}$. If $w \geq 0$, as the $(1, 1, 1, 1, 1, 1)$-eigenspace of $A(0)$ is spanned by $cb_{2a}b_1a_1$, we have

$$\phi(a) \equiv (cb_{2a}b_1a_1)^w \cdot a \equiv 0 \pmod{J},$$

for any $a \in J_0$, i.e. $\phi = 0$.

Next suppose that $w < 0$. Note that the $A$-module $M$ is also a coherent $O_Y$-module, supported on the origin $0 \in Y$ as it is $T_0$-fixed. Therefore the actions of $b_1a_1$ and $cb_{2a}$ on $M$ are nilpotent. Let $\alpha$ be the smallest positive integer such that $(b_1a_1)^{\alpha} \in J_0$ and $\beta$ be the smallest positive integer such that $(cb_{2a})^{\beta}(b_1a_1)^{\alpha-1} \in J_0$. As $\phi$ has weight $w(1, 1, 1, 1, 1, 1)$, we have

$$\phi((cb_{2a})^{\beta}(b_1a_1)^{\alpha-1}) \equiv (cb_{2a})^{\beta+w}(b_1a_1)^{\alpha+w} \pmod{J}.$$

By the commutativity between $b_2a_2$ and $b_1a_1$, we have

$$\phi((cb_{2a})^{\beta}(b_1a_1)^{\alpha}) \equiv (b_1a_1)\phi((cb_{2a})^{\beta}(b_1a_1)^{\alpha-1}) \equiv (cb_{2a})^{\beta+w}(b_1a_1)^{\alpha+w} \pmod{J}.$$

Since there is no monomial in $A(0)$ with negative torus weights,

$$\phi((cb_{2a})^{\beta}(b_1a_1)^{\alpha}) \equiv (cb_{2a})^{\beta}\phi((b_1a_1)^{\alpha}) \equiv 0 \pmod{J}.$$

By combining the above two expressions, we conclude

$$(cb_{2a})^{\beta+w}(b_1a_1)^{\alpha+w} \in J.$$

Since $w < 0$, we have $(cb_{2a})^{\beta+w}(b_1a_1)^{\alpha-1} \in J$, which contradicts to the definition of $\beta$.

Since the wall-chamber structures of $O_{\mathbb{P}^1}(-1, -1, 0)$ and $O_{\mathbb{P}^1}(-1, -1)$ are the same, for other choices of $\Theta$, we can follow the approach of \cite[§4]{NN} and identify the moduli space $M_{\Theta}^0(\hat{Q}, I)$ with the moduli space of cyclic representations of some other quiver (as introduced in Chuang-Jafferis \cite{CJ}) and reduce to a similar argument as above.

In actual computations, we will first fix torus action on $X = O_{\mathbb{P}^1}(-1, -1, 0)$; let

$$(\ref{eq:torus-action}) \quad T_0 = \{ t = (t_0, t_1, t_2, t_3) \in (\mathbb{C}^*)^4 : t_0t_1t_2t_3 = 1 \},$$

which acts on $X$ in local coordinates such that the normal bundle of the zero section satisfies

$$N_{\mathbb{P}^1/X} = O_{\mathbb{P}^1}(-Z_{\infty}) \oplus t_{-1}^{-1} \oplus O_{\mathbb{P}^1}(-Z_{\infty}) \oplus t_{-2}^{-1} \oplus O_{\mathbb{P}^1} \oplus t_{-3}^{-1},$$

where $Z_0 := \{ 0 : 1 \}$, $Z_{\infty} := \{ 1 : 0 \} \in \mathbb{P}^1$ are torus fixed points. The torus lifts to an action on $\text{Per}_{\mathbb{C}^1}(X/Y)$ and moduli spaces $P_{\beta}^0(X, \beta)$. By Lemma \((\ref{lemma:torus-action})\) it also acts on representations of quiver \((\ref{eq:quiver})\) as described at the beginning of this section (up to reparametrizations), which preserves the equivalence \((\ref{eq:derived-equivalence})\) and the isomorphism \((\ref{eq:isomorphism})\).

2.3. Tautological invariants. By Proposition \((\ref{prop:tautological})\) we can define the tautological counting invariants of $P_{\beta}^0(X, \beta)$ using the isomorphism \((\ref{eq:isomorphism})\) and torus localization. We first recall the following notion of square roots.

**Definition 2.2.** Let $K^{T_0}(pt) \cong \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm]/(t_0t_1t_2t_3 - 1)$ denote the $T_0$-equivariant $K$-theory of one point. A square root $V^\pm$ of $V \in K^{T_0}(pt)$ is an element in $K^{T_0}(pt)$ such that

$$V^\pm + \overline{V}^\mp = V.$$

Here $\overline{\cdot}$ denotes the involution on $K^{T_0}(pt)$ induced by $\mathbb{Z}$-linearly extending the map

$$t_0^w t_1^{w_1} t_2^{w_2} t_3^{w_3} \mapsto t_0^{-w} t_1^{-w_1} t_2^{-w_2} t_3^{-w_3},$$

where $t_i$'s denote torus weights in notation \((\ref{eq:torus-action})\).
For a $T_0$-equivariant pair $I = (\mathcal{O}_X \rightarrow F)$ with compactly supported $F \in \text{Per}(X/Y)$, by Serre duality, the following square root exists:

$$\chi_X(I, I^\frac{2}{3}) := -\chi_X(F) + \chi_X(F, F^\frac{1}{2}) \in K^{T_0}(pt),$$

where $\chi_X(-, -)$ denotes the Euler pairing on $X$ and $\chi_0(-, -)$ denotes its trace-free part, $\chi_X(-) := \chi_X(\mathcal{O}_X, -)$. Here see Remark 2.4 for a choice of $\chi_X(F, F^\frac{1}{2})$, which is not unique, though its Euler class is unique up to a sign. If $(F, s)$ is $\Theta$-stable for a generic $\Theta \in \mathbb{R}^2$, then $\text{Ext}^1(I, I)$ has no $T_0$-fixed subspace by Proposition 2.1. Therefore its equivariant Euler class is non-zero, so the equivariant Euler class of $\chi_X(I, I^\frac{2}{3})$ is well-defined. For a different choice of square root, the corresponding Euler class may differ by a sign.

Let $\Lambda$ be the field of rational functions defined by

$$\Lambda := \frac{\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \mu)}{\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3}.\quad (2.5)$$

Here $\lambda_i = e_{T_0}(t_i)$’s are equivariant parameters of $T_0$ in (2.4). As in [CK18, CKM19], we use tautological insertions to define invariants.

**Definition 2.3.** Let $X = \mathcal{O}_{P^1}(-1, -1, 0)$ and $\Theta \in \mathbb{R}^2$ be outside walls (1.13). Consider a trivial $\mathbb{C}^*$-action on moduli spaces such that $\epsilon^m$ is an $\mathbb{C}^*$-equivariant line bundle with $\mathbb{C}^*$-equivariant weight $m$. We define the tautological invariant to be

$$P_{n,d}(\epsilon^m) := \sum_{I = (\mathcal{O}_X \rightarrow F) \in P^n_d(X, d)^{T_0}} e_{T_0}(\chi_X(I, I^\frac{2}{3}) : e_{T_0}(\chi_X(F, F^\frac{1}{2}) \in \Lambda).$$

The above invariant depends on the choice of sign for each torus fixed point.

When $\Theta = (-1 + 0^+, 1)$ (i.e., $\Theta$ lies in the PT chamber), the invariants in Definition 2.3 recover the cohomological invariants studied in [CKM19] §0.4.

**Remark 2.4.** In actual computations, we fix the Fano 3-fold $Y = \mathcal{O}_{P^1}(-1, 0)$ such that the normal bundle of the zero section satisfies

$$N_{P^1/Y} = \mathcal{O}_{P^1}(-Z_\infty) \oplus t_1^{-1} \oplus \mathcal{O}_{P^1} \oplus t_3^{-1}.\quad (2.5)$$

We take

$$\chi_X(F, F^\frac{1}{2}) := \chi_Y(F, F),$$

where RHS is defined by pushforward to $\mathbb{P}^1$ followed by taking inclusion to $Y$ via zero section.

We then put an extra sign as follows:

$$e_T(\chi_X(F, F^\frac{1}{2})) = (-1)^{\deg(F) + \text{sign}(F)} \cdot e_T(-\chi_X(F) + \chi_Y(F, F)),\quad (2.5)$$

where $\text{sign}(F) \in \mathbb{Z}$. When $F$ is scheme theoretically supported on $Y$, motivated by [Cao] Eqn. (0.1)], we take $\text{sign}(F) = 0$. In the thickened case, we will explain how to choose it in examples computed in Section 2.5.8.

**Remark 2.5.** In the computations of $\chi_Y(F, F), \chi_X(F)$ and their equivariant Euler classes, we use the adjunction formula

$$\chi_Y(F_i, F_j) = \chi_{P^1}(F_i, F_j) - \chi_{P^1}(F_i, F_j \otimes N_{P^1/Y}) + \chi_{P^1}(F_i, F_j \otimes \lambda^2 N_{P^1/Y}),$$

and equivariant Riemann-Roch formula

$$\text{ch} \left( \chi(\mathcal{O}_{P^1}(aZ_0 + bZ_\infty)) \right) = \frac{e^{-a\lambda_0}}{1 - e^{-\lambda_0}} + \frac{e^{b\lambda_0}}{1 - e^{-\lambda_0}} = \frac{e^{(b+1)\lambda_0} - e^{-a\lambda_0}}{e^{\lambda_0} - 1}.\quad (2.6)$$

Let $Z$ be the resolved conifold embedded into $X$:

$$\iota: Z := \mathcal{O}_{P^1}(-1, -1) \times \{0\} \hookrightarrow X.$$

We say that $I \in P^n_d(X, d)$ is scheme theoretically supported on $Z$ if it is of the form $(\mathcal{O}_X \rightarrow \iota_*F)$. Motivated by the dimensional reduction and cohomological limit in [CKM19], we show the following:

---

It is also an interesting question to link them with global orientations obtained in [Boj].
Proposition 2.6. Let us take \( \Theta \in \mathbb{R}^2 \) which lies outside walls in \((1.13)\). We have the following:

1. (Dimensional reduction) For each \( I \in \bar{P}^\Theta_n(X, d)^{T_0} \), we have
\[
eq 0, \quad \text{otherwise.}
\]

Here \((C^*)^3 = T_0|Z\) is the restricted torus.

2. (Insertion-free limit)
\[
\lim_{Q \to \infty} \left( \sum_{n, d} \bar{P}^\Theta_{n, d}(e^m) q^n e^d \right) = \sum_{n, d} Q^n e^d \sum_{I = (O_X \to F) \in \bar{P}^\Theta_n(X, d)^{T_0}} e_{T_0}(\chi_X(I, I)_0^\frac{1}{2}).
\]

\[\text{Proof.}\]
1. We first assume that \( I \) is scheme theoretically supported on \( Z \), so that it is written as \( I = (O_X \to t_{1,2}, F) \).

By adjunction, we get
\[
\chi_X(I, I)_0^\frac{1}{2} := -\chi_X(t_{1,2}F) + \chi_X(t_{1,2}F, t_{1,2}F)^\frac{1}{2} = -\chi_Z(F) + \chi_Z(F, F).
\]

By the Serre duality for \( Z \), we have
\[
\chi_Z(I, I)_0 = \chi_Z(F, F) - \chi_Z(F) + \chi_Z(F)^\vee \otimes t_{3,2}.
\]

Then we have identities
\[
eq e_{T_0}(\chi_Z(F, F)) \cdot e_{T_0}(\chi_Z(F)^\vee \otimes t_{3,2})
\]
\[
= e_{T_0}(\chi_Z(I, I)_0)
\]
\[
= e_{T_0}(\chi_Z(I, I)_0).
\]

Therefore (1) holds when \( I \) is scheme theoretically supported on \( Z \).

Next, we consider pairs \( I = (O_X \to F) \in \bar{P}^\Theta_n(X, d)^{T_0} \) thickened into normal direction of \( Z \) inside \( X \). We first deal with the case that \( F \) is a sheaf, e.g. when \( \Theta \) lies in a \( Z_{1,2} \)-stable pair chamber (see Proposition 1.28). By Proposition 2.1, \( I \) is also fixed by the full torus \((C^*)^4\), hence fixed by the subtorus \((C^*)^3 \subset (C^*)^4\) acting on the fibers of the projection \( \pi : X \to \mathbb{P}^1 \). We have decompositions into \((C^*)^3\)-weight spaces
\[
\pi_* F = \bigoplus_{(i_1, i_2, i_3) \in \Delta \subset \mathbb{Z}^2} F_{i_1, i_2, i_3}, \quad \pi_* O_X = \bigoplus_{(i_1, i_2, i_3) \in \mathbb{Z}^2_{\geq 0}} L_{1}^{i_1} \otimes L_{2}^{-i_2} \otimes L_{3}^{-i_3}.
\]

Here
\[
L_{1} = O_{\mathbb{P}^1}(-Z_{\infty}) \otimes t_{1}^{-1}, \quad L_{2} = O_{\mathbb{P}^1}(-Z_{\infty}) \otimes t_{2}^{-1}, \quad L_{3} = O_{\mathbb{P}^1} \otimes t_{3}^{-1}
\]
are equivariant line bundles on \( \mathbb{P}^1 \) and \( \Delta \) is defined by
\[
\Delta := \left\{ (i_1, i_2, i_3) \in \mathbb{Z}^3 : F^{i_1, i_2, i_3} \neq 0 \right\}.
\]

The torus invariant section \( s \) is determined by a collection of morphisms
\[
s^{-i_1, -i_2, -i_3} : L_{1}^{-i_1} \otimes L_{2}^{-i_2} \otimes L_{3}^{-i_3} \to F^{-i_1, -i_2, -i_3}, \quad i_1, i_2, i_3 \geq 0.
\]

Since \( s \) is an \( O_X \)-module homomorphism, the above morphisms fit into a commutative diagram
\[
L_{1}^{i_1} \otimes L_{2}^{-i_2} \otimes L_{3}^{-i_3} \otimes L_{1}^{-1} \xrightarrow{\phi} L_{1}^{-i_1-1} \otimes L_{2}^{-i_2} \otimes L_{3}^{-i_3}.
\]

We have similar commutative diagrams by replacing the role of \( L_{1} \) with \( L_{2} \) or \( L_{3} \).

By the stability, \( s \) is not identically zero, so there exists \((i_1, i_2, i_3)\) such that \( s^{-i_1, -i_2, -i_3} \neq 0 \).

Therefore, \( F \) is thickened into the normal direction of \( Z \) inside \( X \), we may assume \( i_3 \geq 3 \). From the above commutative diagrams, we obtain
\[
s^{-i_1+1, -i_2, -i_3}, \quad s^{-i_1, -i_2+1, -i_3}, \quad s^{-i_1, -i_2, -i_3+1} \neq 0.
\]

By inductions, we know \( F^{0, 0, -1} \neq 0 \) and \( s^{1, 0, -1} \neq 0 \). So we have
\[
F^{0, 0, -1} = O_{\mathbb{P}^1}(aZ_0 + bZ_{\infty}) \otimes t_{3},
\]
for some \(a, b \geq 0\). Therefore we have identities

\[
e^T_0(\chi(F)^\vee \otimes t_3) = \sum_{(i_1, i_2) \in \Delta \subset \mathbb{Z}^3} e^T_0(\chi(F^{i_1, i_2})^\vee \otimes t_3)
\]

\[
= e^T_0(\chi(F^{0,0,-1})^\vee \otimes t_3 + \cdots)
\]

\[
= e^T_0(1 + \cdots) = 0.
\]

Here we have used the fact that \(\chi(F) = H^0(F)\) so that \(\chi(F)\) does not contain elements with negative signs (e.g. \(-1^{\mu_0} t_1^{\mu_1} t_2^{\mu_2} t_3^{\mu_3}\)) in the weight space decomposition.

A similar argument works if \(F = F'[1] \in \text{Per}(X/Y)\) for a one dimensional sheaf \(F'\). In general, we have a short exact sequence in \(\text{Per}(X/Y)\):

\[
0 \to H^{-1}(F)[1] \to F \to H^0(F) \to 0,
\]

where \(H^*(F)\) are one dimensional sheaves. Applying \(\text{Hom}(\mathcal{O}_X, -)\), we obtain the exact sequence

\[
0 \to \text{Hom}(\mathcal{O}_X, H^{-1}(F)[1]) \to \text{Hom}(\mathcal{O}_X, F) \to \text{Hom}(\mathcal{O}_X, H^0(F)) \to 0
\]

together with the vanishings

\[
\text{Hom}^{\neq 0}(\mathcal{O}_X, H^{-1}(F)[1]) = 0, \quad \text{Hom}^{\neq 0}(\mathcal{O}_X, H^0(F)) = 0.
\]

Since \((F, s)\) is \((C^*)^4\)-fixed, we pushforward \(F\) and \((2.7)\) to \(\mathbb{P}^1\) and do weight space decomposition as before. It is easy to see the above argument applies.

(2) We have

\[
\lim_{m \to \infty} \left( \sum_{n,d} P_{n,d}(e^m) q^n t^d |_{Q=q^m} \right)
\]

\[
= \lim_{m \to \infty} \left( \sum_{n,d} Q^n t^d \sum_{I \in P_n^d(X,d)_{T_0}} e^T_0(\chi_X(I, I)_0^{1/2}) \cdot e^T_0(\chi_X(F)^\vee \otimes e^m) \right)
\]

\[
= \lim_{m \to \infty} \left( \sum_{n,d} Q^n t^d \sum_{I \in P_n^d(X,d)_{T_0}} e^T_0(\chi_X(I, I)_0^{1/2}) \cdot e^T_0(\chi_X(H^0(X,F)^\vee \otimes e^m)) \right)
\]

\[
= \lim_{m \to \infty} \left( \sum_{n,d} Q^n t^d \sum_{I \in P_n^d(X,d)_{T_0}} e^T_0(\chi_X(I, I)_0^{1/2}) \cdot \left( \frac{m^n + \text{l.o.t.}}{m^n} \right) \right)
\]

\[
= \sum_{n,d} Q^n t^d \sum_{I \in P_n^d(X,d)_{T_0}} e^T_0(\chi_X(I, I)_0^{1/2}),
\]

where ‘l.o.t.’ means lower order terms of \(m\) and we use \(\chi(F) = n\) in the fourth identity. \(\square\)

### 2.4. Wall-crossing formula

Let \(\Theta_{\text{PT}} := (-1+0^+, 1)\) and consider tautological PT invariants

\[
P_{n,d}(e^m) := P_{n,d}^\Theta(e^m) \in \Lambda.
\]

In \cite{CKM19} Appendix B, the following closed formula is conjectured.

**Conjecture 2.7** \((\text{CKM19})\). There exist choices of signs such that

\[
\sum_{n,d} P_{n,d}(e^m) q^n t^d = \prod_{k \geq 1} (1 - q^k t)^{\frac{m k}{\lambda_k}},
\]

where \(-\lambda_0\) is the equivariant parameter of \(\mathcal{O}_{\mathbb{P}^1}\) in \(X\).

The aim of this section is to give an interpretation of the above conjecture in terms of wall-crossing of \(\Theta\)-stable perverse coherent systems. Suppose that \(\Theta\) lies on one of the walls in \(\mathbb{P}^4\) except the DT/PT wall \(L_\pm(\infty)\), and \(\Theta_\pm\) lie in its adjacent chambers. We consider the flip type

\footnote{As in \cite{NN}, we exclude the DT/PT wall here. A reason is that, at the DT/PT wall, the simple object in \(\mathbb{P}^4\) is not a line bundle on \(\mathbb{P}^1\) nor its Fourier-Mukai transform, but a skyscraper sheaf of a point, so a separate treatment is required.}
\[
\begin{align*}
(2.8) \quad \bigcup_{n,d} P_n^\Theta (X,d) T_0 & \quad \bigcup_{n,d} P_n^\Theta + (X,d) T_0 \\
& \bigcup_{n,d} P_n^\Theta (X,d) T_0.
\end{align*}
\]

Here \( P_n^\Theta (X,d) T_0 \) consists of \( \Theta \)-polystable perverse coherent systems of type

\[
I_0 \oplus S_{k-1}^{\Theta \bullet} [-1], \quad r \geq 0,
\]

where \( I_0 \) is a \( T_0 \)-fixed \( \Theta \)-stable perverse coherent system, \( S_{k-1} \) is a \( T_0 \)-fixed \( \Theta \)-stable perverse coherent sheaf with \( \Theta(S_{k-1}) = 0 \), and \( r \) can be computed from the Chern character of \( I_0 \).

By Proposition 2.6, when \( m \) is the zero section \( O \) by their work, we conjecture this phenomenon extends to the following sequence of \( \Theta \)-polystable objects for all \( r \geq 0 \):

\[
(2.9)
\]

\[
S_{k-1} = \begin{cases} \mathcal{O}_\mathbb{P} \cdot (k - 1), & \Theta \in L_-(k), \\ \mathcal{O}_\mathbb{P} \cdot (-k - 1)[1], & \Theta \in L_+(k), \\ S_0 \mathcal{O}_\mathbb{P} \cdot (-k - 1)[1], & \Theta \in L_-(k), \\ S_0 \mathcal{O}_\mathbb{P} \cdot (k - 1), & \Theta \in L_+(k). 
\end{cases}
\]

Here \( T_0 \) is the derived equivalence under flop \( LL \) and \( \mathcal{O}_\mathbb{P} \) is scheme theoretically supported on the zero section \( \mathbb{P} \times \{0\} \subset X \). For a \( T_0 \)-fixed \( \Theta \)-stable perverse coherent system \( I_0 \), we consider the following sequence of \( \Theta \)-polystable objects for all \( r \geq 0 \):

\[
P_{k-1,r}^I := \{ I_0 \oplus S_{k-1}^{\Theta \bullet} [-1] \} \in \bigcup_{n,d} P_n^\Theta (X,d) T_0.
\]

By Proposition 2.8 when \( m = \lambda_3 \) (also taking specialization \( \lambda_0 + \lambda_1 + \lambda_2 = 0 \)), the invariants in Definition 2.3 recover Nagao-Nakajima’s counting invariants of perverse coherent systems on the resolved conifold \( \mathcal{O}_\mathbb{P} \cdot (-1, -1) \) \([NN]\). In \([NN]\) Theorem 3.12, they proved a wall-crossing formula of their invariants by stratifying \( \pi^+ \) into Grassmannian bundles and showed that the difference of their invariants under wall-crossing is independent of the choice of \( I_0 \). Motivated by their work, we conjecture this phenomenon extends to \( \mathcal{O}_\mathbb{P} \cdot (-1, -1, 0) \):

**Conjecture 2.8.** Let \( \Theta \) lie on one of the walls \( L_-(k), L_+(k) \) in \([LL]\).

- If \( \Theta = (\theta_0, \theta_1) \in L_-(k) \) or \( L_+(k) \) \( (k \geq 1) \) and \( \Theta \pm = (\theta_0 \mp \theta_1) \), then there exist choices of signs such that

\[
\sum_r l^r \sum_{I \in \pi^+ \cdot (P_{k-1,r}^I)} e_{T_0} \cdot e_{\mathcal{O}_\mathbb{P}} \cdot (\chi(F) \cdot e^{-}) \otimes e^{+} = (1 - t)^{k \frac{\theta_1}{\lambda_1}}.
\]

- If \( \Theta = (\theta_0, \theta_1) \) \( L_-(k) \) or \( L_+(k) \) \( (k \geq 0) \) and \( \Theta \pm = (\theta_0 \mp \theta_1) \), then there exist choices of signs such that

\[
\sum_r l^r \sum_{I \in \pi^+ \cdot (P_{k-1,r}^I)} e_{T_0} \cdot e_{\mathcal{O}_\mathbb{P}} \cdot (\chi(F) \cdot e^{-}) \otimes e^{+} = (1 - t^{-1})^{k \frac{\theta_1}{\lambda_1}}.
\]

The formulae in Conjecture 2.8 in particular imply that the quotient series in the LHS are independent of the choice of \( I_0 \). The above conjecture implies the following wall-crossing formulae of tautological invariants given in Definition 2.3:

**Proposition 2.9.** Suppose that Conjecture 2.8 is true. Then we have the following:

- If \( \Theta = (\theta_0, \theta_1) \) \( L_-(k) \) or \( L_+(k) \) \( (k \geq 1) \) and \( \Theta \pm = (\theta_0 \mp \theta_1) \), then there exist choices of signs such that

\[
\sum_{n,d} P_{n,d}^\Theta (e^{-}) q^{n} t^{d} = (1 - q^k t)^{k \frac{\theta_1}{\lambda_1}}.
\]

- If \( \Theta = (\theta_0, \theta_1) \) \( L_-(k) \) or \( L_+(k) \) \( (k \geq 0) \) and \( \Theta \pm = (\theta_0 \mp \theta_1) \), then there exist choices of signs such that

\[
\sum_{n,d} P_{n,d}^\Theta (e^{-}) q^{n} t^{d} = (1 - q^{k} t^{-1})^{k \frac{\theta_1}{\lambda_1}}.
\]
Here in our setting, the JS stability for $(F, s)$, we have
\[
\sum_{n,d} P_{\Theta(n,d)}^\Theta(e^m)q^n t^d
\]
\[
= \sum_{n_0,d_0} q^{n_0} t^{d_0} \sum_{I_0 \in P^\Theta_{n_0}(X,d_0)^{\Theta}} \left( \sum_{r_0 \geq 0} \sum_{I \in \pi_{\pm}^{-1}(P^\Theta_{n_0}(X,d_0))} e_{T_0}(\chi_X(I,I)_{0})^\pm \right) \cdot \left( e_{T_0 \times \mathbb{C}^*}(\chi_X(F)^\vee \otimes e^m)(q^2 t)^r \right),
\]
where $P^\Theta_{n_0}(X,d_0)^{\Theta}$ denotes the set of $T_0$-fixed $\Theta$-stable perverse coherent systems with numerical class $(n_0, d_0)$. Applying Conjecture 2.8, we prove the proposition in this case. The other cases can be similarly obtained.

In particular, this gives a wall-crossing interpretation of Conjecture 2.7 and a conjectural formula for non-commutative tautological invariants.

**Corollary 2.10.** Conjecture 2.8 implies Conjecture 2.7. Moreover, if we further assume the DT/PT conjecture [CKM19, §0.4], then there exist choices of signs such that
\[
\sum_{n,d} P_{\Theta(n,d)}^\Theta(e^m)q^n t^d = M(q) \prod_{k \geq 1} (1 - q^k t)^{\frac{k}{\lambda_1}} \prod_{k \geq 1} (1 - q^k t^{-1})^{\frac{k}{\lambda_2}},
\]
where $\Theta_{NC} = (\theta_0 < 0, \theta_1 < 0)$ lies in the non-commutative chamber and
\[
M(q) := \prod_{k \geq 1} (1 - q^k)^{-k}
\]
is the MacMahon function.

**Remark 2.11.** By Proposition 2.6, the substitution $m = \lambda_3$ and specialization $\lambda_0 + \lambda_1 + \lambda_2 = 0$ allow us to recover the formula of non-commutative DT invariants of resolved conifold [Sze, Young] from Corollary 2.11.

Applying the insertion-free limit in Proposition 2.6, we obtain a wall-crossing formula for cohomological invariants without insertions:

**Proposition 2.12.** Suppose that Conjecture 2.8 is true. Then we have the following:
- If $\Theta = (\theta_0, \theta_1) \in L^-_\pm(k)$ or $L^+_\pm(k) (k \geq 1)$ and $\Theta_{\pm} = (\theta_0 \mp \mp, \theta_1)$, then there exist choices of signs such that
  \[
  \frac{\sum_{n,d} q^n t^d \sum_{I \in P^\Theta_{n_0}(X,d_0)^{\Theta}} e_{T_0}(\chi_X(I,I)_{0})}{\sum_{n,d} q^n t^d \sum_{I \in P^\Theta_{n_0}(X,d_0)^{\Theta}} e_{T_0}(\chi_X(I,I)_{0})^\pm} = \begin{cases} \exp \left( -\frac{Q}{r^k} \right), & \text{if } k = 1, \\ 1, & \text{otherwise}. \end{cases}
  \]
- If $\Theta = (\theta_0, \theta_1) \in L^-_\pm(k)$ or $L^+_\pm(k) (k \geq 0)$ and $\Theta_{\pm} = (\theta_0 \mp \mp, \theta_1)$, then there exist choices of signs such that
  \[
  \frac{\sum_{n,d} q^n t^d \sum_{I \in P^\Theta_{n_0}(X,d_0)^{\Theta}} e_{T_0}(\chi_X(I,I)_{0})^\pm}{\sum_{n,d} q^n t^d \sum_{I \in P^\Theta_{n_0}(X,d_0)^{\Theta}} e_{T_0}(\chi_X(I,I)_{0})} = \begin{cases} \exp \left( -\frac{Q}{r^k} \right), & \text{if } k = 1, \\ 1, & \text{otherwise}. \end{cases}
  \]

**Proof.** Applying the insertion-free limit in Proposition 2.6 to the LHS of Proposition 2.9, we obtain the formula of the above formula. The RHS is obtained as
\[
\lim_{Q \text{ fixed} \atop m \to \infty} (1 - q^k t)^{\frac{k}{\lambda_1}} \left|_{Q = q^m} \right. = \lim_{m \to \infty} \left( 1 - \frac{Q^k t}{m^k} \right)^{\frac{k}{\lambda_1} \cdot \frac{1}{m^k}} = \begin{cases} \exp \left( -\frac{Q}{r^k} \right), & \text{if } k = 1, \\ 1, & \text{otherwise}. \end{cases}
\]

### 2.5. Computations

**Case 2.5.** In this section, we compute examples to support Conjecture 2.8.

- $I_0 = \mathcal{O}_X$. By Proposition 1.2, when $\Theta = (-n + 0^+, n + d)$, the moduli space $P^\Theta_n(X,d)$ parametrizes Joyce-Song type stable pairs introduced in [CT19, Definition 1.10]:
  \[
P^\Theta_n(X,d) = \{ JS \text{ stable pairs } (F, s) \text{ with } (d(F), \chi(F)) = (d, n) \}.
  \]

Here in our setting, the JS stability for $(F, s)$ is defined by:
- $F$ is a compactly supported one dimensional semistable sheaf,
- $s \neq 0$, and for any subsheaf $\text{Im}(s) \subset F^\prime \subsetneq F$ we have $\chi(F^\prime)/d(F^\prime) < \chi(F)/d(F)$.
Lemma 2.13. Suppose that $\Theta = (\theta_0, \theta_1) \in L^{-}(k)$ ($k \geq 1$) and take $I_0 = O_X$, we consider
\[ P^{O_X}_{k-1,d} := \{ O_X \oplus O_{\mathbb{P}^1}(k-1)^{e_{|d|}}[-1] \} \subset P^{O}_{k-1,d}(X, d)^{T_0}. \]
Then fibers of maps $\pi_\pm$ in (2.8) at the above points satisfy
\[ \pi_+^{-1}(P^{O_X}_{k-1,d}) = P^{JS}_{k,d}(X, d)^{T_0}, \quad \pi_-^{-1}(P^{O_X}_{k-1,d}) = \emptyset. \]

Proof. By Proposition 1.26, we compute directly using Lemma 2.14.

By Proposition 1.26, we compute directly using Lemma 2.14.

Remark 2.17. Therefore to prove Conjecture 2.8 for $L^{-}(k)$, it is enough to show the quotient series in the LHS is independent of the choice of $I_0$.

• $I_0 = I_{p^1}, k = 2$ case. We denote
\[ I_{p^1} := \left( O_X \longrightarrow O_{p^1} \otimes \sum_{j=0}^{l-1} t_j^3 \right), \quad l \geq 1 \]
where $O_{p^1}$ is scheme theoretically supported on the zero section $\mathbb{P}^1 \hookrightarrow X$ of the projection $\pi : X \rightarrow \mathbb{P}^1$. Namely $I_{p^1}$ is the ideal sheaf of the $l$-th thickening of $\mathbb{P}^1$ in the normal direction of $Z$ inside $X$ (2.6).
Lemma 2.18. Suppose that $\Theta = (\theta_0, \theta_1) \in L^-_\pi(2)$ and take $I_0 = I_{P^1}$, we consider

$$P^+_{1,d} := \{ I_{P^1} \oplus \mathcal{O}_{P^1}(1)^{\oplus d}[1] \} \in P^{\Theta}_{2d+1}(X, d + 1)^{T_0}.$$  

(1) An element in $\pi_+^{-1}(P^+_{1,d})$ is precisely of the form

$$s_1: I_{P^1} \to \bigoplus_{i=1}^4 F_i \oplus \sum_{j=0}^{d-1} t_j^{13},$$

where $d_1, \ldots, d_4 \geq 0$ with $\sum_{i=1}^4 d_i = d$, $F_i$ are the following $T_0$-equivariant sheaves and $s$ is the canonical $T_0$-equivariant morphism

$$F_i = \begin{cases} \mathcal{O}_{P^1}(Z_{\infty}) \otimes t_1, & i = 1, \\ \mathcal{O}_{P^1}(Z_{\infty}) \otimes t_2, & i = 2, \\ \mathcal{O}_{P^1}(Z_{\infty}) \otimes t_3, & i = 3, \\ \mathcal{O}_{P^1}(Z_0) \otimes t_3, & i = 4. \end{cases}$$

(2) The set $\pi_+^{-1}(P^+_{1,d})$ is empty for $d > 0$, and consists of $I_{P^1}$ for $d = 0$.

Proof. (1) The fiber of $\pi_+$ consists of isomorphism classes of $T_0$-fixed pairs

$$(s: I_{P^1} \to F), \quad F \in \langle \mathcal{O}_{P^1}(1) \rangle_{\text{ex}},$$

with no morphism to $(0 \to \mathcal{O}_{P^1}(1))$ by the $Z_{2+0}$-stability. From the $T_0$-equivariant Koszul resolution

$$\cdots \to \mathcal{O}_X(Z_{\infty}) \otimes t_1 \oplus \mathcal{O}_X(Z_{\infty}) \otimes t_2 \oplus \mathcal{O}_X \otimes t_3 \to I_{P^1} \to 0,$$

we see that $\mathbb{P}(\text{Hom}(I_{P^1}, \mathcal{O}_{P^1}(1)))$ consists of four $T_0$-fixed points, namely canonical morphisms $s_i: \mathcal{O}_X \to F_i$ where $F_i$ is one of (2.13). Let us consider the composition

$$I_{P^1} \to F \to F|_Z,$$

where $Z \subset X$ is given by (2.6). The above composition is $T_0$-equivariant and $F|_Z$ is a direct sum of $\mathcal{O}_{P^1}(1)$, so it is of the form

$$(s^{[k_i]}_l: I_{P^1} \to \bigoplus_{i=1}^4 F_i^{[k_i]}, \quad k_i \in \mathbb{Z}_{\geq 0}).$$

Then $Z_{2+0}$-stability forces $k_i \leq 1$. Hence $F$ is a direct sum of thickenings of $F_i$ for $k_i = 1$ in the normal direction of $Z$ inside $X$, so we obtain the desired description for fiber of $\pi_+$.

(2) By Proposition [1.20] a pair $(F, s)$ in the fiber of $\pi_-$ is a $Z_{2-0}$-stable pair. By the wall-chamber structures of $Z_{l}$-stable pairs in Lemma 1.25, it is also a $Z_{1+l}$-stable pair. Since we have the inequality

$$1 \leq \chi(F) \cdot \frac{d}{d(F)} = 1 + \frac{d}{d+l},$$

which is strict for $d > 0$, we obtain the desired description of the fiber of $\pi_-$ by Lemma 1.25. □

By the above lemma, we can explicitly compute the LHS of the formula in Conjecture 2.8 when $I_0 = I_{P^1}$ and $k = 2$. Once we know the relevant classification of torus fixed loci as in Lemma 2.18 the computations are similar to [CIT9] Theorem 6.9 which are direct applications of those formulae in Remark 2.4. Here we omit details and give one example.

Example 2.19. Let $I_0 = I_{P^1}$ and $\Theta \in L^-_\pi(2)$. The degree d-term of LHS in Conjecture 2.8 is

$$(-1)^d \sum_{d_1 + d_2 + d_3 + d_4 = d, \ d_i \geq 0} \prod_{1 \leq i,j \leq 4} (\frac{d - d_j}{d_i + d_j})^{-1} \cdot \prod_{0 \leq k \leq d_i - 1} ((k - 1) \lambda_3 + \lambda_i - \lambda_j) \cdot \prod_{1 \leq j \leq 2} (m - k \lambda_3 - \lambda_i) \cdot \prod_{0 \leq k \leq d_i - 1} (m - k \lambda_3 - \lambda_i + \lambda_1 + \lambda_2 + \lambda_3),$$

where $\lambda_4 := \lambda_1 + \lambda_2 + 2 \lambda_3$. Conjecture 2.8 predicts that this expression is equal to $(-1)^d \left( \frac{m}{d} \right)$. In the following Proposition 2.20, we verify this non-trivial identity up to $d \leq 16$.

A computer program\footnote{Our use of computer program is simply a brute force checking of whether two rational functions are equal. We list all cases that our computers can do.} enables us to check Conjecture 2.8 in the following cases.
Proposition 2.20. Let $\Theta = (\theta_0, \theta_1) \in L_-(k)$, and take $I_0 = I_{P_1}$. Then Conjecture 2.8 holds in the following cases:

- $l = 1$, up to degree $t_1^{16}$,
- $l = 2$, up to degree $t_2^{16}$,
- $l = 3, 4$, up to degree $t_5^7$,
- $l = 5$, up to degree $t_3^3$,
- $l = 6$, up to degree $t_7^5$,
- $l = 7, 8, 9, 10$, up to degree $t_6^6$,
- any $l$, up to degree $t_5^9$.

Here the sign rule is as follows: we take $\text{sign}(F) = 1$ for fibers of $\pi_+$ in (2.10) with $d_2 > 0$, and $\text{sign}(F) = 0$ otherwise.

- $I_0 = I_{P_1}$, $k \geq 3$ case.

Lemma 2.21. Suppose that $\Theta = (\theta_0, \theta_1) \in L_-(k)$ and take $I_0 = I_{P_1}$, we consider

$$ I_{P_1} : = \{ I_{P_1} \otimes \mathcal{O}_{\mathbb{P}}(k - 1)^{[d]}[-1] \} \in P_{k+1}^d(X, d + 1)^T_0. $$

(1) An element in $\pi_1^{-1}(I_{P_1, k-1, d})$ is precisely of form:

$$ \bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}_1}((k - 1 - i)Z_0 + iZ_\infty) \otimes t_1 \sum_{j=0}^{d_i - 1} t_3^j $$

(2.15)

$$ s : I_{P_1} \rightarrow \bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}_1}((k - 1 - i)Z_0 + iZ_\infty) \otimes t_2 \sum_{j=0}^{e_i - 1} t_3^j $$

where $d_i, e_i, f_i \geq 0$ with $\sum_{i=1}^{k-1} d_i + \sum_{i=1}^{k-1} e_i + \sum_{i=0}^{k-1} f_i = d$ and $s$ is given by the canonical map.

(2) An element in $\pi_1^{-1}(I_{P_1, k-1, d})$ is precisely of the form $(s : \mathcal{O}_X \rightarrow \mathcal{E})$, where $\mathcal{E}$ fits into the canonical $T_0$-equivariant extension

$$ 0 \rightarrow \mathcal{O}_{P_1} \rightarrow \mathcal{E} \rightarrow \bigoplus_{1 \leq s \leq k-2, \sum_{d_i \in \{0, 1\}} d_i + \sum_{d_i \geq 2} d_i = d} \mathcal{O}_{P_1}((k - 1 - i)Z_0 + iZ_\infty) \cdot d_i \rightarrow 0, $$

and $s$ is given by the composition $\mathcal{O}_X \rightarrow \mathcal{O}_{P_1} \hookrightarrow \mathcal{E}$.

Proof. (1) The fiber of $\pi_+$ can be described similarly as in Lemma 2.19 so we omit details.

(2) The fiber of $\pi_-$ consists of $T_0$-equivariant exact sequences of the form

$$ 0 \rightarrow I_{P_1} \rightarrow \star \rightarrow F[-1] \rightarrow 0, $$

where $F \in (\mathcal{O}_P(k - 1))_{ex}$ satisfies the $Z_{k+0}$-stability. Note that we have $\text{Hom}(F[-2], \mathcal{O}_X) = 0$ by the Serre duality. Therefore the map $F[-2] \rightarrow I_{P_1}$ in (2.16) factors through as

$$ F[-2] \rightarrow \mathcal{O}_{P_1}[-1] \rightarrow I_{P_1}. $$

By taking cones and a diagram chasing, we obtain an extension

$$ 0 \rightarrow \mathcal{O}_{P_1} \rightarrow \mathcal{E} \rightarrow F \rightarrow 0, $$

and $\star$ is isomorphic to a pair $(s : \mathcal{O}_X \rightarrow \mathcal{E})$, where $s$ is the composition $\mathcal{O}_X \rightarrow \mathcal{O}_{P_1} \hookrightarrow \mathcal{E}$. The $Z_{k+0}$-stability is equivalent to the condition

$$ \text{Hom}(\mathcal{O}_{P_1}(k - 1), \mathcal{E}) = 0. $$

The sheaf $\mathcal{E}$ is obtained as a $T_0$-equivariant extension of $F$ by $\mathcal{O}_{P_1}$. Using Serre duality together with Koszul resolution (2.14), one calculates

$$ \text{Ext}^1_X(\mathcal{O}_{P_1}(k - 1), \mathcal{O}_{P_1}) \cong \text{Hom}_{\mathcal{O}_{P_1}}(\mathcal{O}_{P_1}(Z_0 + Z_\infty), \mathcal{O}_{P_1}(k - 1))^\vee. $$

In the RHS, the $T_0$-fixed morphisms are given by canonical morphisms

$$ \mathcal{O}_{P_1}(Z_0 + Z_\infty) \rightarrow \mathcal{O}_{P_1}((k - 1 - i)Z_0 + iZ_\infty), \quad 1 \leq i \leq k - 2. $$
Therefore $F$ is of form

$$F = \bigoplus_{1 \leq i \leq k - 2} \mathcal{O}_{P_i}((k - 1 - i)Z_0 + iZ_\infty) \cdot (1 + t_3 + \cdots + t_{d_i}^{d_i - 1}).$$

We claim that the condition (2.18) forces $d_i \leq 1$. Suppose that $d_i \geq 2$ for some $i$. By writing $F_i := \mathcal{O}_{P_i}((k - 1 - i)Z_0 + iZ_\infty)$ and $F^d_i := F_i \otimes (\sum_{j = 0}^{d_i - 1} t_j)$, we have the exact sequence

$$0 \to F_i \to F^d_i \to F^{d_i - 1} \to 0.$$

By applying $\text{Hom}(-, \mathcal{O}_{P_i})$ to the above exact sequence, we obtain the long exact sequence

$$\cdots \to \text{Ext}^1_X(F^d_i, \mathcal{O}_{P_i}) \to \text{Ext}^1_X(F_i, \mathcal{O}_{P_i}) \to \text{Ext}^2_X(F^{d_i - 1}, \mathcal{O}_{P_i}) \to \cdots.$$

If $\lambda$ is the zero map, then the composition

$$F_i \to F^d_i \to F$$

factors through a map $F_i \to \mathcal{E}$, hence violating the condition (2.18).

We are left to show that $\lambda = 0$. It is enough to show $\nu$ is injective. By the local-to-global spectral sequence, we have $\text{Ext}^2_X(F_i, \mathcal{O}_{P_i}) \cong H^1(X, \mathcal{O}_{P_i}(-k))$ and

$$\text{Ext}^2_X(F^{d_i - 1}, \mathcal{O}_{P_i}) \cong H^0(X, \mathcal{E} \otimes^L (F^{d_i - 1}, \mathcal{O}_{P_i})) \oplus H^1(X, \mathcal{E}t^{1}(F^{d_i - 1}, \mathcal{O}_{P_i}))$$

$$\cong H^0(X, \mathcal{E} \otimes^L t^{1}(\mathcal{O}_{P_i})) \oplus H^1\left(X, \mathcal{E}t^{1}(\mathcal{O}_{P_i} \otimes (\sum_{j = 0}^{d_i - 1} t_j))\right)$$

$$\cong H^0(X, \mathcal{E} \otimes^L (\mathcal{O}_{P_i}(-1)) \oplus \mathcal{O}_{P_i}) \oplus H^1\left(X, (\mathcal{O}_{P_i}(-1)) \oplus \mathcal{O}_{P_i}(-k)\right),$$

where we have used (2.13) in the last isomorphism. So $\text{Ext}^2_X(F^{d_i - 1}, \mathcal{O}_{P_i})$ contains $H^1(X, \mathcal{O}_{P_i}(-k))$ as a direct summand and one can show $\nu$ is the inclusion of this summand.

As before, we explicitly compute invariants using Remark 2.4, 2.5. A computer program enables us to check Conjecture 2.8 in the following cases.

**Proposition 2.22.** Suppose that $\Theta = (\theta_0, \theta_1) \in L_-(k)$ ($k \geq 3$). Then Conjecture 2.8 holds for $I_0 = I_{P_i}$ in the following cases:

- $k = 3$, up to degree $t^5$,
- $k = 4, 5$, up to degree $t^2$,
- $k \leq 12$, up to degree $t^1$.

Here we use the following sign rule in (2.5): for fibers of $\pi_+$ in (2.15), we take $\text{sign}(F)$ to be the number of $e_i$’s which are positive; for fibers of $\pi_-$, we take $\text{sign}(F) = 0$.

As a corollary of the above computations, we can prove Conjecture 2.8 for the ‘first wall’ and provide several checks for the ‘second wall’ of (1.13):

**Corollary 2.23.** Let $\Theta = (\theta_0, \theta_1) \in L_-(k)$. Then Conjecture 2.8 holds in the following cases:

- $k = 1$ and any $I_0$,
- $k = 2$ and any $I_0$ up to degree $t^5$.

**Proof.** By Lemma 1.25 the only $Z_1$-stable pair is $\mathcal{O}_X$. When $k = 1$, the only possible choice of $I_0$ is $\mathcal{O}_X$. Using Lemma 2.13 we are reduced to Theorem 2.16.

By the openness of stability and wall-chamber structures of $Z_1$-stable pairs (see Lemma 1.25), $Z_2$-stable pairs are $Z_{1+\chi}$-stable pairs, which are JS type stable pairs $(F, s)$ with $\chi(F) = d(F)$. By Lemma 2.14 they are of the form (2.11), which are $Z_1$-stable for any $t > 1$. Therefore, the $k = 2$ case is reduced to Proposition 2.20.

**Remark 2.24.** Finally we remark that one can also study $K$-theoretical generalization of tautological invariants considered in this paper, following [CKM19 Definition 0.2], and lift the formula in [MMNS] to CY 4-folds. It may be interesting to pursue this direction in the future.
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