ORBITS OF TORI EXTENDED BY FINITE GROUPS AND THEIR POLYNOMIAL HullS: THE CASE OF CONNECTED COMPLEX OrBITS

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Abstract. Let $V$ be a complex linear space, $G \subset \text{GL}(V)$ be a compact group. We consider the problem of description of polynomial hulls $c^G_v$ for orbits $Gv$, $v \in V$, assuming that the identity component of $G$ is a torus $T$. The paper contains a universal construction for orbits which satisfy the inclusion $Gv \subset T^Cv$ and a characterization of pairs $(G, V)$ such that it is true for a generic $v \in V$. The hull of a finite union of $T$-orbits in $T^Cv$ can be distinguished in $	ext{clos}(T^Cv)$ by a finite collection of inequalities of the type $|z_1|^{s_1} \ldots |z_n|^{s_n} \leq c$. In particular, this is true for $Gv$. If powers in the monomials are independent of $v$, $Gv \subset T^Cv$ for a generic $v$, and either the center of $G$ is finite or $T^C$ has an open orbit, then the space $V$ and the group $G$ are products of standard ones; the latter means that $G = S_nT$, where $S_n$ is the group of all permutations of coordinates and $T$ is either $T^n$ or $SU(n) \cap T^n$, where $T^n$ is the torus of all diagonal matrices in $\text{U}(n)$. The paper also contains a description of polynomial hulls for orbits of isotropy groups of bounded symmetric domains. This result is already known, but we formulate it in a different form and supply with a shorter proof.

Introduction

Let $V$ be a finite-dimensional complex linear space and $G \subset \text{GL}(V)$ be a compact subgroup of $\text{GL}(V)$. We consider the problem of description of polynomially convex hulls for orbits $Gv$, $v \in V$. The polynomially convex hull (or polynomial hull) $\hat{Q}$ of a compact set $Q \subset V$ is defined as

$$\hat{Q} = \{ z \in V : |p(z)| \leq \sup_{\zeta \in Q} |p(\zeta)| \text{ for all } p \in \mathcal{P}(V) \},$$

where $\mathcal{P}(V)$ is the algebra of all holomorphic polynomials on $V$. It is usually difficult to find $\hat{Q}$. For $Q = Gv$, the answer is known if $G$ is an isotropy group of a bounded symmetric domain in $\mathbb{C}^n$. Paper [9] contains a description of $G$-invariant polynomially convex compact sets, including hulls of orbits ($Q \subset V$ is polynomially convex if $\hat{Q} = Q$); it continues paper [10] and uses results of [8]. On the other hand, it is known that an orbit of a compact linear group is polynomially convex if and only if the complex orbit $G^Cv$ is closed and $Gv$ is its real form (17). The cases $G = \text{U}(2), \text{SU}(2)$ were considered in [11], [12]. The problem of determination of polynomial hulls of orbits admits the following natural generalization: given a homogeneous space $M$ of a compact group $G$, describe maximal ideal spaces $\mathcal{M}_A$ of $G$-invariant closed subalgebras $A$ of $C(M)$, where $C(M)$ is the Banach algebra of all

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continuous complex-valued functions on $M$ with the sup-norm. If $A$ is generated by a finite-dimensional invariant subspace, then $\mathcal{M}_A$ can be realized as the polynomial hull of an orbit. Paper [6] contains a description of $\mathcal{M}_A$ for bi-invariant algebras on compact groups and partial results on spherical homogeneous spaces. Maximal ideal spaces for $U(n)$-invariant algebras on spheres in $\mathbb{C}^n$ are described in [11].

In this paper we consider orbits $Gv$ of groups $G = FT$, where $F \subseteq G$ is a finite subgroup and $T$ is a torus, such that $G^c v = T^c v$. Let $t \subseteq \mathfrak{gl}(V)$ be the Lie algebra of $T$ and set $t^R = it$, $T^R = \exp(t^R)$. Suppose that $v \in V$ has a trivial stable subgroup in $T$ and let $X \subset T^R v$ be finite. The hull of $Y = TX$ admits a simple description. If $X = \{v\}$, then $\hat{Y} = T \hat{v}$ is the closure of $T \exp(C_T v)$, where $C_T$ is a cone in $t^R$. If $T^C$ is closed, then $\hat{Y} = T \exp(Q_X) v$, where $Q_X \subseteq t^R$ is a convex polytope (the convex hull of the inverse image of $X$ for the mapping $\xi \rightarrow \exp(\xi)v$, $\xi \in t^R$). Any segment in $Q_X$ corresponds to an analytic strip or an annulus in $\hat{Y}$. In general, $\hat{Y}$ is the union of $\hat{T}u$, where $u$ runs over $\exp(Q_X)v$. Also, $\hat{Y}$ is distinguished in $\text{clos} T^C v$ by a finite family of monomial inequalities of the type

\[
|z_1|^{s_1} \ldots |z_n|^{s_n} \leq c,
\]

where $c \geq 0$ and $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ depend on $v$ and $X$. Vectors $s$ correspond to normals of faces of $C_T + Q_X$.

Thus, the problem of determination of $\hat{G}v$ is not difficult if $Gv \subset T^C v$. The latter is equivalent to the assumption that the complex orbit $G^c v$ is connected. In Example 3.4 we give a construction for orbits which satisfy this condition; here is a sketch. The group $G = FT$ acts on the space $V = C(K)$, where $K$ is a finite $F$-invariant subset of $t^*$: $F$ acts naturally on $C(K)$, $t = t^*$ is naturally embedded into $C(K)$, and $T = \exp(t)$ acts on $C(K)$ by multiplication. If $v \in C(K)$ is an $F$-invariant function, then $Gv \subset T^C v$. According to Theorem 3.5 each connected complex orbit can be realized in this way. Further, we describe pairs $(V, G)$ such that

\[
Gv \subset T^C v \quad \text{for a generic} \quad v \in V.
\]

By Theorem 4.3 under the additional assumption that the complex linear span of $T^C v$ coincides with $V$, this happens if and only if the group $G^c Z$, where $Z$ is the centralizer of $G$ in $\text{GL}(V)$, has an open orbit in $V$. There are two extreme cases: (A) $Z \subseteq G^c$; (B) $G$ has a finite center. An example for (A) is the group $G = S_n T^n$ acting in $\mathbb{C}^n$, where $T^n$ is the torus of all diagonal matrices in $\text{U}(n)$ and $S_n$ is the group of all permutations of coordinates. Replacing $T^n$ with $\text{SU}(n) \cap T^n$, we get an example for (B). Example 4.4 contains a construction for pairs $(V, G)$ that satisfy (0.3). Theorem 4.5 states that the construction is a universal one. In Theorem 4.10 we determine pairs which satisfy (0.3) and the following condition:

\[
\text{vectors } s \text{ in (0.3) are independent of } v.
\]

The paper also contains a description of hulls $\hat{G}v$ for $G = \text{Aut}_0(D)$, where $D$ is a bounded symmetric domain in the canonical realization and $\text{Aut}_0(D)$ is the stable subgroup of zero, which coincides with the group of all linear automorphisms of $D$. These hulls have already been described: the final step was done in paper [9], which essentially used [10], partial results appear in [14] and [8]. Most of them use the technique of Jordan triples and Jordan algebras. We use Lie theory, in particular, an explicit construction of paper [15] for a maximal abelian subspace $a$. A compact group acting in a Euclidean space is called polar if there exists a subspace
(a Cartan subspace) such that each orbit meets it orthogonally. The group $G$ is polar in the ambient linear space $\mathfrak{g}$, and $\mathfrak{a}$ is the Cartan subspace for $G$. Real polar representations are classified in paper [3]; they are orbit equivalent (i.e., have the same orbits) to isotropy representations of Riemannian symmetric spaces. If $D$ is a polydisc $\mathbb{D}^n \subset \mathbb{C}^n$, where $\mathbb{D}$ is the unit disc in $\mathbb{C}$, then $G = S_n \mathbb{T}^n$; the polynomial hulls $\hat{G}v$ are determined by the inequalities

\begin{equation}
\mu_k(z) \leq \mu_k(v),
\end{equation}

where $k = 1, \ldots, n$ and $\mu_k$ are defined by

\begin{equation}
\mu_k(z) = \max\{|z_{\sigma(1)} \cdots z_{\sigma(k)}| : \sigma \in S_n\}.
\end{equation}

The general case can be reduced to this one in the following way. Any bounded symmetric domain $D \subset \mathfrak{g}$ of rank $n$ admits an equivariant embedding of $\mathbb{C}^n$ to $\mathfrak{g}$, which induces an embedding of $\mathbb{D}^n$ to $D$, such that $\mathbb{R}^n \subset \mathbb{C}^n$ is the maximal abelian subspace $\mathfrak{a}$, and, for any $v \in \mathfrak{a}$, the hull of $\text{Aut}_0(D)v$ is the orbit of the hull of $\text{Aut}_0(\mathbb{D}^n)v$. Each $\mu_k(z)$ has a unique continuation to a $K$-invariant function on $\mathfrak{g}$. The extended functions determine hulls by the same inequalities. Moreover, they are plurisubharmonic and can be treated as products of singular values of $z \in \mathfrak{g}$ or as norms of exterior powers of adjoint operators in suitable spaces. The subsystem of long roots of the restricted root system (i.e., the root system for $\mathfrak{a}$) has type $n A_1$; this defines the above embedding $\mathbb{C}^n \to \mathfrak{g}$. Furthermore, this makes it possible to determine hulls in terms of the adjoint representation (Theorem [5.7]). Thus, there is no need to consider different types of domains separately.

The reduction to the case of a torus extended by a finite group, which is described above, is contained in Section 5 (in papers [9], [14], the problem is also reduced to this case by another method). It does not use essentially the results of the previous sections (only Proposition 3.2 in proof of Theorem 5.7). These extensions satisfy conditions (0.3) and (0.4); in addition, they possess the property that the complexified groups have open orbits. According to Theorem 5.10 any group with these properties is the product of groups $S_n \mathbb{T}^n$ acting in $\mathbb{C}^n$; it admits a natural realization as a group of automorphisms of a bounded symmetric domain (Corollary 5.3).

The following simple examples illustrate the case $Gv \not\subset T^Cv$ and show that condition (0.3) is essential. Let $G = S_n \mathbb{T}^n$, and let $\epsilon_1, \ldots, \epsilon_n$ be the standard base in $\mathbb{C}^n$. Then $G\epsilon_1$ is the closure of the union of discs $\mathbb{D}\epsilon_k$, $k = 1, \ldots, n$. Set $H = S_n \mathbb{T}$, where $\mathbb{T}$ acts by $z \to e^{it}z$, $t \in \mathbb{R}$, $z \in \mathbb{C}^n$. Then $H\epsilon_1 = \hat{G}\epsilon_1$. For $v = \epsilon_1 + \epsilon_2$, $\hat{G}v$ is the closure of the union of $\binom{n}{2}$ bidiscs but $\mathbb{T}^n$ contains no proper torus $T$ such that $\hat{G}v = \hat{H}v$ for $H = S_n T$. However, for any subgroup $F \subset S_n$ which acts transitively on 2-sets and $H = FT^\mathbb{T}$ we have $\hat{G}v = \hat{H}v$.

1. Preliminaries

We keep the notation of Introduction, in particular, (0.1) and (0.3). Linear spaces are supposed to be finite dimensional and complex unless the contrary is explicitly stated. "Generic" means "in some open dense subset". Throughout the paper, we use the following notation:

- $\mathbb{D}$ and $\mathbb{T}$ are the open unit disc and the unit circle in $\mathbb{C}$, respectively;
- $V$ denotes a complex linear space (except for Section 5);
Clearly, $\exp$ is bijective on $t$ is a lattice in the vector group $t$ defines an embedding of $\hat{\mathbb{V}}$ to $\mathbb{V}$ be the corresponding isotypical component of multiplicative group of all invertible functions in $\mathbb{F}$ where $T$ is the standard base in $\mathbb{C}^n$ and $\mathbb{R}^n$; $\mathbb{R}^+_n$ is the set of vectors in $\mathbb{R}^n$ with positive entries; $S_K$ denotes the group of all permutations of a finite set $K$; if $K = \{1, \ldots, n\}$, then $S_K = S_n$; $C(K)$ is the algebra of all complex-valued functions on $K$; $I$ is the identity of $C(K)$; $G \subset GL(V)$ is a compact group whose identity component is a torus $T$ (except for Section 5); $t \subset \mathfrak{gl}(V)$ is the Lie algebra of $T$, $t^R = i t$, $t^C = t + t^R$; $T^R = \exp(t^R)$, $T^C = \exp(t^C)$; $\mathbb{C}^* = T^C = \mathbb{C} \setminus \{0\}$; $\hat{T} = \text{Hom}(T, \mathbb{T})$ is the dual group to $T$; $\text{Aut}(D)$ is the group of all holomorphic automorphisms of a domain $D \subset V$, $\text{Aut}_0(D) = \text{Aut}(D) \cap GL(V)$; cone $X$ denotes the least convex cone which contains the set $X$; conv $X$ is the convex hull of $X$; $\text{cos}_{\mathbb{F}} X$ is the closure of $X$; span$_\mathbb{F} X$ is the linear span of $X$ over the field $\mathbb{F} = \mathbb{C}, \mathbb{R}, \mathbb{Q}$.

Clearly, $\exp$ is bijective on $t^R$ and $T^R \cong T^C/T$. The differentiating at the identity $e$ defines an embedding of $T$ into the dual space $t^*$: $\chi \to -i d_{\chi}$, where $\chi \in \hat{T}$. This is a lattice in the vector group $t^*$, moreover, $T \cong t/L$, where $L$ is the dual lattice to $T$ in $t$. For $\chi \in \hat{T}$, let

$$V_\chi = \{ v \in V : gv = \chi(g)v \text{ for all } g \in T \}$$

be the corresponding isotypical component of $V$. Then

$$(1.1) \quad V = \sum_{\chi \in \hat{T}} V_\chi.$$  

We assume that $V$ is equipped with a $G$-invariant inner product $\langle , \rangle$. Then decomposition $1.1$ is orthogonal. Let $\text{spec}(v)$ denote the spectrum of $v \in V$ (the set of $\chi \in \hat{T}$ such that the $\chi$-component of $v$ is nonzero); for $X \subseteq V$, $\text{spec}(X) = \cup_{x \in X} \text{spec}(x)$.

We say that $T$ has a simple spectrum if

$$(1.2) \quad \dim V_\chi \leq 1$$

for all $\chi \in \hat{T}$. If $1.2$ is true, then there exists a unique (up to scaling factors) orthogonal base in $V$ which agree with $1.1$ and a unique maximal torus $T^n$ in $GL(V)$ which contains $T$. In what follows, we assume that $1.2$ holds; we shall see in the next section that such assumption is not restrictive. Thus, we may fix an identification

$$(1.3) \quad V = \mathbb{C}^n = C(K),$$

where $K = \{1, \ldots, n\}$. If $F$ is a subgroup of $S_K$, then $C(K)^F$ denotes the set of all $F$-invariant functions on $K$; clearly, $1 \in C(K)^F$. Further, $(\mathbb{C}^*)^n$ is the multiplicative group of all invertible functions in $C(K)$, $T^n$ consists of functions with values in $\mathbb{T}$, and $(T^n)^C = (\mathbb{C}^*)^n$. The Lie algebra of $T^n$ is realized as $i\mathbb{R}^n \subset \mathbb{C}^n$. 

If $V$ is equipped with a linear base identifying it with $\mathbb{C}^n$, then $T^n$ is the group of all diagonal unitary transformations; $\mathbb{Z}_2^n$ consists of all transformations in $T^n$ with eigenvalues $\pm 1$; $e_1, \ldots, e_n$ is the standard base in $\mathbb{C}^n$ and $\mathbb{R}^n$; $\mathbb{R}^+_n$ is the set of vectors in $\mathbb{R}^n$ with positive entries; $S_K$ denotes the group of all permutations of a finite set $K$; if $K = \{1, \ldots, n\}$, then $S_K = S_n$; $C(K)$ is the algebra of all complex-valued functions on $K$; $I$ is the identity of $C(K)$; $G \subset GL(V)$ is a compact group whose identity component is a torus $T$ (except for Section 5); $t \subset \mathfrak{gl}(V)$ is the Lie algebra of $T$, $t^R = i t$, $t^C = t + t^R$; $T^R = \exp(t^R)$, $T^C = \exp(t^C)$; $\mathbb{C}^* = T^C = \mathbb{C} \setminus \{0\}$; $\hat{T} = \text{Hom}(T, \mathbb{T})$ is the dual group to $T$; $\text{Aut}(D)$ is the group of all holomorphic automorphisms of a domain $D \subset V$, $\text{Aut}_0(D) = \text{Aut}(D) \cap GL(V)$; cone $X$ denotes the least convex cone which contains the set $X$; conv $X$ is the convex hull of $X$; $\text{cos}_{\mathbb{F}} X$ is the closure of $X$; $\text{span}_{\mathbb{F}} X$ is the linear span of $X$ over the field $\mathbb{F} = \mathbb{C}, \mathbb{R}, \mathbb{Q}$.
The embedding $T \rightarrow \mathbb{T}^n$ induces embeddings of the Lie algebra and the fundamental group: $t \rightarrow i\mathbb{R}^n$, $\pi_t(T) \rightarrow i\mathbb{Z}^n \subset i\mathbb{R}^n$, respectively. Let $\Gamma$ be the image of $\pi_1(T)$. Then $\text{span}_\mathbb{R} \Gamma = t$; moreover, $t \cap i\mathbb{Z}^n = \Gamma$ and $t/\Gamma = T$. The dual mapping $\hat{T}^n \rightarrow \hat{T}$, which is defined by the restriction of characters $e^{-i(x,y)}$, where $x \in i\mathbb{Z}^n$, to $t$, is the orthogonal projection $\pi_t : i\mathbb{Z}^n \rightarrow t$. Thus, $\Gamma$ is a subgroup of finite index in $\hat{T} = \pi_t i\mathbb{Z}^n$. Vectors in $\text{span}_\mathbb{R} \hat{T}$ are called rational. The image of $t$ in $i\mathbb{R}^n$ can be distinguished by linear equations with integer coefficients. Hence, $\text{clos}(T^c_i)$, for a generic $v \in V$, is the set of all solutions to a finite number of equalities with holomorphic monomials. Thus, $Y \subset T^c$ implies $\bar{Y} \subset \text{clos}(T^c_i)$. Set

\begin{equation}
C_T = t^\mathbb{R} \cap \text{clos}(-\mathbb{R}^n_+),
\end{equation}

The cone $iC_T$ is dual to $\text{cone} \text{spec}(V) \subseteq t^* \subseteq i\mathbb{R}^n$. If $-\xi \in \text{clos}(\mathbb{R}^n_+)$, then $t = \lim_{t \rightarrow +\infty} \exp(t \xi)$ is an idempotent in $C(K)$ such that the multiplication by the complementary idempotent $1 - t$ is a projection onto $\text{span}_C \text{spec}(\xi)$. Set

\begin{equation}
I_T = \{ \lim_{t \rightarrow +\infty} \exp(t \xi) : \xi \in C_T \}.
\end{equation}

Clearly, $I_T$ is finite and contains $1$.

**Lemma 1.1.** The closure of $\exp(C_T)$ is equal to $I_T \exp(C_T)$.

**Proof.** Due to the evident inclusion $\text{clos}(\exp(C_T)) \supseteq I_T \exp(C_T)$, it is sufficient to prove that the set $S_T = I_T \exp(C_T)$ is closed. Clearly, $S_T$ is an abelian semigroup. The cone $C_T$ is polyhedral; hence, it is finitely generated:

$$C_T = \text{cone}\{\xi_1, \ldots, \xi_m\},$$

where $\mathbb{R}^+\xi_k$ are the extreme rays of $C_T$, $k = 1, \ldots, m$. Obviously, $I_T$ is a finite semigroup, which is generated by the idempotents $\lim_{t \rightarrow +\infty} \exp(t \xi_k)$. Thus, the correspondence $(e^{-t_1}, \ldots, e^{-t_m}) \rightarrow \exp(t_1 \xi_1 + \ldots + t_m \xi_m)$ defines a mapping of $[0, 1]^m$ onto $\exp(C_T)$, which continuously extends to $[0, 1]^m$. It follows that its image is closed and coincides with $S_T$. \hfill \square

Note that there is a natural one-to-one correspondence between $I_T$ and the set of faces of $C_T$.

2. **Hulls of finite unions of $T$-orbits in a $T^c$-orbit**

Let $v \in T^c$. If $v = \sum_{\chi \in \hat{T}} v_\chi$, where $v_\chi \in V_\chi$, $g \in T^c$, and $u = gv$, then $u = \sum_{\chi \in \hat{T}} \chi(g)v_\chi$. Since $\chi(g) \neq 0$ for all $g \in G$ and $\chi \in \hat{T}$, we get

\begin{equation}
(2.1) \quad u \in T^c v \quad \iff \quad \text{spec}(u) = \text{spec}(v);
\end{equation}

\begin{equation}
(2.2) \quad \dim(V_\chi \cap \text{span}_C T^c v) \leq 1 \quad \text{for all} \quad v \in V \quad \text{and} \quad \chi \in \hat{T}.
\end{equation}

Thus, the assumption that $T$ has a simple spectrum in $V$ is not restrictive in the problem of description of polynomial hulls of orbits $Gv$ such that $Gv \subset T^c_i$. Clearly, $\mathbb{D}^n = \hat{T}^n$ in $L(V)$. For each $x \in C_T$ and any polynomial $p$ on $L(V)$, the holomorphic function $f(\zeta) = p(\exp(\zeta x))$ is bounded in the halfplane $\Re \zeta \geq 0$. Hence, $\exp(\Pi)$ is contained in $\hat{T}$. On the other hand, if $z \in \mathbb{D}^n \cap T^c$, then $z = t \exp(x)$ for some $t \in T$ and $x \in C_T$ (the polar decomposition). By Lemma 1.1,

$$\hat{T} = \text{clos}(\mathbb{D}^n \cap T^c) = T \text{clos}(\exp(C_T)) = TI_T \exp(C_T).$$
If \( v \in (\mathbb{C}^*)^n \), then \((\mathbb{C}^*)^n v = (\mathbb{C}^*)^n \), and the mapping \( z \rightarrow zv \) is a linear nondegenerate transformation of \( \mathbb{C}^n \). Therefore,

\[
(2.3) \quad v \in (\mathbb{C}^*)^n \implies \hat{T}v = T I T \exp(C_T) v.
\]

For an arbitrary \( v \in V = \mathbb{C}^n \), set

\[
C_T^v = \{ \xi \in \mathbb{R}^\nu : \xi_k \leq 0 \text{ if } v_k \neq 0, \ k = 1, \ldots, n \}.
\]

Applying (2.3) to \( \text{span}_\mathbb{C}(\text{spec}(v)) = \mathbb{C}^n v \), we get

\[
(2.4) \quad \hat{T}v = T \text{ clos}(\exp(C_T^v)) v.
\]

Clearly, \( C_T^v \) depends only on \( \text{spec}(v) \). For \( s \in \mathbb{R}^n \) and \( v \in (\mathbb{C}^*)^n \), set

\[
\nu_s(z) = \prod_{k=1}^n |z_k|^{s_k}.
\]

If \( s_k \geq 0 \), then the \( k \)-th factor in \((\mathbb{C}^*)^n\) can be replaced with \( \mathbb{C} \) (i.e., \( \nu_s \) continuously extends to this product).

It is well known that for any holomorphically convex \( T \)-invariant set \( U \subseteq T^\mathbb{C} \), the set \( \log(U \cap T^\mathbb{R}) \subseteq \mathbb{R}^\nu \) is convex. In particular, this is true for sets of \( g \in T^\mathbb{C} \) such that \( gv \in TX \), where \( X \subset T^\mathbb{C} v, v \in V \). Nevertheless, it is convenient to have an explicit construction of an analytic strip (or an annulus, if it is periodic) in a \( T^\mathbb{C} \)-orbit, which corresponds to a segment that joins two points in \( t^\mathbb{R} \); it is contained in the following lemma. Set

\[
S = \{ z \in \mathbb{C} : 0 \leq \text{Im} z \leq 1 \}.
\]

**Lemma 2.1.** Let \( v \in \mathbb{C}^n \) and \( u \in T^\mathbb{R} v \). Then, there exists \( \xi \in \mathbb{R}^\nu \) such that

\[
(2.5) \quad \lambda(z) = \exp(z \xi) v
\]

is a holomorphic mapping \( \lambda : S \rightarrow T^\mathbb{C} v \) which satisfies conditions

\[
\lambda(\partial S) \subseteq Tv \cup Tu,
\]

\[
\lambda(0) = v, \quad \lambda(1) = u.
\]

If the stable subgroup of \( v \) in \( T^\mathbb{R} \) is trivial, then \( \xi \) is unique.

**Proof.** These properties hold for \( \xi \in \mathbb{R}^\nu \) such that \( \exp(\xi) v = u \); such a \( \xi \) exists, since \( \exp \) is a bijection \( t^\mathbb{R} \rightarrow T^\mathbb{R} \). The last assertion is clear. \( \square \)

If \( \xi \in C_T \), then (2.3) defines an analytic halfplane in \( \hat{T}v \); for \( \Gamma \)-rational \( \xi \), \( \lambda \) is periodic and defines an analytic disc in \( \hat{T}v \). Together with Lemma 2.1 this gives a characterization of hulls for finite unions of \( T \)-orbits in \( T^\mathbb{C} \). Suppose that \( X \subset T^\mathbb{R} v \) is finite and the stable subgroup of \( v \) in \( T \) is trivial. Then, the inverse to the mapping \( x \rightarrow \exp(x) v \) is well defined. Let us denote it by \( \log_v \) and set

\[
(2.6) \quad Q_X = \text{conv}(\log_v X),
\]

\[
(2.7) \quad P_X = Q_X + C_T.
\]

The set \( P_X \) is a convex polyhedron, which is unbounded if \( C_T \neq 0 \). Hence, there exists a finite set \( \mathfrak{N}_X \subset \mathbb{R}^n \) and, for each \( s \in \mathfrak{N}_X \), real numbers \( c_s \) such that

\[
(2.8) \quad P_X = \{ x \in \mathbb{R}^\nu : \langle x, s \rangle \leq c_s \text{ for all } s \in \mathfrak{N}_X \}.
\]

The set \( \mathfrak{N}_X \) consists of vectors orthogonal to faces of \( P_X \), whose projections into \( \text{span}_\mathbb{R} P_X \) look outside of it; clearly, it is not unique in general.
**Proposition 2.2.** Let $v \in (\mathbb{C}^*)^n$. Suppose that $Y \subset T^Cv$ is a finite union of $T$-orbits (including $Tv$), and set $X = T^Rv \cap Y$. Then $X$ is finite and

$$
(2.9) \quad \hat{Y} = \text{clos} (T \exp(P_X)v)
$$

$$
(2.10) \quad = \text{clos} \{ z \in (\mathbb{C}^*)^n : \nu_s(z) \leq e^{c_s} \nu_s(v), \ s \in \mathfrak{N}_X \}
$$

$$
(2.11) \quad = \bigcup_{u \in \exp(Q_X)v} \hat{T}u
$$

$$
(2.12) \quad = T \exp(P_X)I_T v,
$$

where $Q_X, P_X, \mathfrak{N}_X$ are as above and $I_T$ is defined in (1.10).

**Proof.** Due to the polar decomposition, the set $Tu \cap T^Rv$, for each $u \in T^Cv$, is nonvoid and consists of a single point. Hence, $X$ is finite and $Y = TX$. The inclusion $(\exp Q_X)v \subseteq \hat{Y}$ follows from Lemma 2.1 and Phragmén–Lindelöf Principle. The inclusion $\exp(C_T)u \subseteq \hat{T}u$ is true for any $u \in \mathbb{C}^n$. Since it holds for all $u \in T \exp(Q_X)$, the left-hand side of (2.9) includes the right-hand side. If $z = \exp(\xi)v$, where $\xi \in \mathfrak{t}^C$, then $z_k = e^{c_k} \nu_s(v)$, $k = 1, \ldots, n$; due to (2.8), this implies that the right-hand side of (2.9) coincides with (2.10). According to (2.3), the right-hand side of (2.9) and (2.11) intersect $T^Cv$ by the set

$$
T \exp(P_X)v = \exp(Q_X)v \exp(C_T)v;
$$

clearly, it is dense in (2.11). Since $Q_X$ is compact, the set (2.11) is closed. The compactness of $Q_X$, the above equality, and Lemma 1.1 imply that (2.12) is closed; hence, it is the same as the right-hand side of (2.9).

Each of the sets (2.9)–(2.12) includes $Y$. Thus, it remains to prove that (2.12) is polynomially convex. If $x \in T^R \setminus P_X$, then there exists $s \in \mathbb{R}^n$ such that

$$
(2.13) \quad \sup \{ \langle y, s \rangle : y \in P_X \} < \langle x, s \rangle.
$$

Since $Q_X$ is compact, the linear functional on $\mathfrak{t}$ in the right-hand side of (2.13) must be nonnegative on $C_T$. According to (1.14), we may assume that $s \in \text{clos} \mathbb{R}^n_+$. It follows that (2.13) holds in a neighborhood of $s$ in $\text{clos} \mathbb{R}^n_+$. Thus, $s$ can be assumed rational (hence, integer) with strictly positive entries. Then, $p(z) = z_1^{s_1} \ldots z_n^{s_n}$ is a holomorphic polynomial such that $|p|$ separates $\exp(x)v$ and $T \text{clos} (\exp(P_X)v)$. Therefore,

$$
\hat{Y} \cap T^Cv = T \exp(P_X)v.
$$

For any $\iota \in I_T$, the projection $z \rightarrow \iota z$ commutes with $T$. This makes it possible to apply the above arguments to the vector $\iota v$, the set $\iota X$, and to the restriction of $T$ to $\iota \mathbb{C}^n$. Consequently,

$$
(2.14) \quad \iota \hat{Y} \cap T^Cw = T \exp(P_X)\iota w = \iota T \exp(P_X)v
$$

(clearly, $\iota \exp(P_X)v = \exp(P_X)\iota v$). By (1.5), $\iota Y \subseteq \hat{Y}$, hence, $\iota \hat{Y} \subseteq \hat{Y}$; on the other hand, $\iota \hat{Y} \subseteq \iota \hat{Y}$ since $p \circ \iota$ is a polynomial on $\mathbb{C}^n$ for any polynomial $p$ on $\iota \mathbb{C}^n$. Thus, $\iota \hat{Y} = \hat{Y}$ and

$$
\hat{Y} = \hat{Y} \cap \iota \mathbb{C}^n.
$$

Together with (2.14), this implies the polynomial convexity of (2.12).

If $T = \mathbb{T}^n$, then Proposition 2.2 follows from the well-known characterization of polynomially convex Reinhardt domains.

**Corollary 2.3.** For any $v \in (\mathbb{C}^*)^n$, the orbit $T^Cv$ is closed in $\mathbb{C}^n$ if and only if $Tv$ is polynomially convex, and this is equivalent to $C_T = 0$. Then, $\hat{Y} = T \exp(Q_X)v$ for all $Y, X$ as above.
Proof. The orbit $T^C v$ is closed if and only if the convex hull of $\text{spec}(v) = \text{spec}(\mathbb{C}^n)$ contains 0 in its relative interior (see, for example, [13, Proposition 6.15]). Since $T \subseteq \text{GL}(n, \mathbb{C})$, the set $\text{spec}(\mathbb{C}^n)$ is generating in $t^*$. Hence, $T^C v$ is closed if and only if $C_T = 0$; by (2.1), this is equivalent to $T^v = T v$. Then, $\hat{Y} = T \exp(Q_X) v$ by (2.4) and (2.1).

There is a version of the first assertion for an arbitrary compact linear group $G$: a $G^C$-orbit is closed if and only if it contains a polynomially convex $G$-orbit ([17, Theorem 1 and Theorem 5]). For a torus $T$, all $T$-orbits in $T^C v$ are simultaneously polynomially convex or non-convex, but this is not true if $G$ is not abelian.

3. Finite extensions of $T$ that keep a $T^C$-orbit

In this section, we consider the case where the set $X$ defined in the previous section is an orbit of a finite group $F$ which normalizes $T$ and keeps the $T^C$-orbit. We assume that $T \subseteq G$, $T$ is a torus, $G$ is a subgroup of $\text{GL}(V)$, $F$ is a finite subgroup of $G$, and

\begin{align}
(3.1) & \quad G = FT = TF, \quad F \cong G/T, \\
(3.2) & \quad Gv \subseteq T^C v, \\
(3.3) & \quad v \in (\mathbb{C}^*)^n \subset \mathbb{C}^n = V.
\end{align}

By (3.1), $T$ is normal in $G$. Clearly, (3.2) is equivalent to $Fv \subseteq T^C v$ and to the connectedness of $G^C$. Here is an illustrating example.

Example 3.1. Let $G = \text{Aut}_0(\mathbb{D}^2)$ be the group of linear automorphisms of the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$. Clearly, $G = FT$, where $F = S_2$ is generated by the transposition $\tau$ of the coordinates, $T = \mathbb{T}^2$, $T^C = (\mathbb{C}^*)^2$, and $T^C v = (\mathbb{C}^*)^2$ for any $v$ that lies outside the coordinate lines. Thus, (3.2) holds for all $v \in (\mathbb{C}^*)^2$ (however, (3.2) fails for any $v \neq 0$ in $\mathbb{C}^2 \setminus (\mathbb{C}^*)^2$). The hull $\hat{G}v$ can be distinguished by the inequalities

\begin{align}
(3.4) & \quad \max\{|z_1|, |z_2|\} \leq \max\{|v_1|, |v_2|\}, \\
(3.5) & \quad |z_1 z_2| \leq |v_1 v_2|.
\end{align}

Clearly, (3.4) and (3.5) define a polynomially convex set. Let $z_1, z_2 > 0$ (a generic $T$-orbit evidently contains such a point $z$). Then, $z$ and $\tau z$ can be joined by an analytic strip with the boundary in $Tz \cup \tau Tz$:

$$
\lambda_2(s) = (z_1^{1-s} z_2^s, z_1^s z_2^{1-s}), \quad s \in S.
$$

Set $q = \ln \frac{z_2}{z_1}$ and let $z_1 > z_2$. Then, the strip can be written in the form

$$
\lambda_2(s) = (e^{-s} z_1, e^s z_2), \quad 0 \leq \text{Re} \, s \leq q.
$$

It is periodic with the period $2\pi i$ and defines a $\tau$-invariant annulus in $\hat{G}v$ with $\tau$-fixed points $(\sqrt[4]{z_1 z_2}, \sqrt[4]{z_1 z_2})$ and $(-\sqrt[4]{z_1 z_2}, -\sqrt[4]{z_1 z_2})$. As $z_2 \to 0$, the annulus tends to a couple of discs: $\left( e^{-s} z_1, 0 \right)$ and $\left( 0, e^{-s} z_1 \right)$, where $\text{Re} \, s > 0$, $0 \leq \text{Im} \, s \leq 2\pi$ (the circle $\text{Re} \, s = \frac{q}{2}$, $0 \leq \text{Im} \, s \leq 2\pi$ collapses to zero). Let $z \in \hat{G}v \cap \mathbb{R}^2$. Then $\hat{G}v$ contains a bidisc $\mathbb{D}^2 z$. It intersects $\mathbb{R}^2$ by a rectangle, which is symmetric with respect to the coordinate axes. If $z$ lies on an axis, then the rectangle degenerates into a segment. Let $v_1 > v_2 > 0$. The union of these rectangles with vertices in the set $Q$ of real points of the annulus, which joins $v$ and $\tau v$, is a curvilinear octagon. It degenerates into a pair of segments if $v_2 = 0$ and into a square if $v_1 = v_2$ (see [10, Fig. 2] for the 3-dimensional case). In the logarithmic coordinates in the
first quadrant, $Q$ is a segment. Also, note that all nontrivial $T^C$-orbits are not closed.

In [2], Björk found a typical situation where analytic annuli appear in the maximal ideal space $M_A$ of a commutative Banach algebra $A$ which admits a nontrivial action of $T$ by automorphisms: this happens if $T$-invariant functions on $M_A$ do not separate distinct $T$-orbits. In [7], it was noted that analytic strips and/or annuli appear in $\tilde{G}v$ if the stable subgroup of $v$ in $G^C$ does not coincide with the complexification of the stable subgroup of $v$ in $G$.

**Proposition 3.2.** The hulls $\tilde{G}v$ for orbits of $G = Aut_0(\mathbb{D}^n) = S_nT^n$ are distinguished by inequalities (0.5), where $\mu_k$ are defined by (0.16).

**Proof.** The approximation by decreasing sequences of hulls makes it possible to reduce the proposition to the case of a generic $v$ in (0.5). Then, applying to $v = (v_1, \ldots, v_n)$ a suitable transformation in $T^n$, we may assume that

$$v_1 > v_2 > \cdots > v_n > 0.$$ 

Moreover, we may use Proposition 2.2 with $X = S_nv$, $C_T = -\text{clos} \mathbb{R}_n^+$ (we keep the notation of Proposition 2.2). Since $X$, $Q_X$, $C_T$, $P_X$, and $\mu_k$ are $S_n$-invariant, $S_n$ is transitive on $X$, by (0.5), and (2.10), it is sufficient to prove that the vectors $\xi_k = \sum_{r=1}^n \epsilon_r, k = 1, \ldots, n$, correspond to the faces of $P_X$ that meet at $v$, are orthogonal to them, and look outside of $P_X$.

Set $\eta_1 = \epsilon_2 - \epsilon_1, \ldots, \eta_{n-1} = \epsilon_n - \epsilon_{n-1}, \eta_n = -\epsilon_n$. Then $\{-\eta_k\}_{k=1}^n$ is a base in $\mathbb{R}_n$, which is dual to the base $\{\xi_k\}_{k=1}^n$. We claim that the cone of the polyhedron $P_X$ at the vertex $v$ is generated by $\{\eta_k\}_{k=1}^n$. This implies the assertion above (note that both cones are simplicial). If $\tau \in S_n$ is a transposition $(k, j)$, then $v - \tau v = (v_k - v_j)(\epsilon_k - \epsilon_j)$. If $\sigma, \kappa \in S_n$ then $v - \sigma \kappa v = (v - \kappa v) + (\kappa v - \sigma \kappa v)$. Furthermore, $S_n$ is generated by transpositions $(k, k+1)$, and $\epsilon_k - \epsilon_{k+1} > 0$ by (3.0), where $k = 1, \ldots, n - 1$. Therefore, vectors $\eta_1, \ldots, \eta_{n-1}$ generate the cone of $Q_X$ at $v$. Since $-\epsilon_k = \sum_{r=0}^{k-1} \eta_{n-r}$, and $C_T$ is generated by $-\epsilon_k, k = 1, \ldots, n$, this proves the proposition.

Property (3.2) implies $\text{span}_C Gv = \text{span}_C T^C v$. Hence, we may assume that (1.2) is valid. Then, a generic $\xi \in \mathfrak{t}$ has a simple spectrum. Any $f \in F$ permutes eigenvalues and eigenspaces. Thus, assuming (1.2) and identifying $V$ with $C(K)$ in accordance with (1.3), we get that each element of $F$ is a composition of a permutation of $K$ and a multiplication by a function on $K$. Further, (5.3) implies that the stable subgroup of $v$ in $T^C$ is trivial. Hence,

$$T^R v \cong T^C / T \cong \mathfrak{t}^R,$$

where the identification of $T^R v$ and $\mathfrak{t}^R$ is realized by $\xi \rightarrow \exp(\xi)v, \xi \in \mathfrak{t}^R$.

**Lemma 3.3.** Let $G \subset \text{GL}(V)$, a subgroup $F \subseteq G$, and $v \in V$ satisfy (2.1)–(6.3). Then $T^C v$ contains a $G$-invariant $T$-orbit. Moreover, there exists a mapping $f \rightarrow t_f, F \rightarrow T$, such that $\tilde{F} = \{t_f : f \in F\}$ is a subgroup of $G$ which has a fixed point in $T^C$ and satisfies (3.1)–(4.3).

**Proof.** The group $F$ naturally acts on $T^R v \cong T^C / T \cong \mathfrak{t}^R$. Any $g \in F$ is a composition of $\sigma \in S_K$ and a multiplication by a function in $C(K)$. Since $\mathfrak{t}$ acts on $C(K)$ by multiplication on linear functions and $\sigma$ induces a linear transformation in $\mathfrak{t}^C$, the induced action of $F$ on $\mathfrak{t}^R$ is affine. Since $F$ is finite, it has a fixed point in $\mathfrak{t}^R$. 


Hence, \( T^C v \) contains a \( G \)-invariant \( T \)-orbit. Let us fix a point \( u \) in it and define \( t_f \) by \( t_f u = u \); the choice is unique due to (3.3). Taken together with (3.1), this implies that \( \tilde{F} \) is a group, which obviously satisfies the lemma.

According to Lemma 3.3, we may assume without loss of generality that
\[
(3.7) \quad f v = v \quad \text{for all} \quad f \in F.
\]

In the following example we give a construction (associated with a given finite group \( F \)) for orbits with property (3.2).

Example 3.4. Let \( t \) be a real linear space, \( t^* \) be the dual space to \( t \), \( L \) be a lattice in \( t \), and \( L^* \subset t^* \) be the dual lattice to \( L \). Set
\[
\lambda_x(y) = y(x), \quad \text{where} \quad x \in t, \quad y \in t^*.
\]

Let \( K \) be a finite subset of \( L^* \) that generates \( L^* \) as a subgroup of the vector group \( t^* \). Then
\[
(3.8) \quad t^* = \text{span}_R K, \quad \text{and} \quad L = \{x \in t: \lambda_x(K) \subset \mathbb{Z}\}.
\]

Further, let \( F \) be a finite subgroup of \( \text{GL}(t) \) which keeps \( K \). Set \( V = C(K) \). The mapping
\[
(3.9) \quad \lambda: x \to i\lambda_x|_K
\]
is an embedding \( t \to V \), which has a natural extension to \( t^C \). Set
\[
(3.10) \quad \exp(x) = e^{2\pi i \lambda_x}.
\]

Clearly, \( L = \ker \exp \). Hence, \( \exp \) defines an embedding of \( T = t/L \) and \( T^C \) into the group \( (\mathbb{C}^*)^n \):
\[
T^C = \exp(t^C) \subset (\mathbb{C}^*)^n.
\]

The group \( T^C \) acts on \( C(K) \) by multiplication. The inclusion \( v \in C(K)^F \) is the same as (3.7); it implies (3.2). Furthermore, if \( v \in (\mathbb{C}^*)^n \), then
\[
(3.11) \quad \text{span}_C T v = V.
\]

Indeed, the space \( \text{span}_C T \) is a subalgebra of \( C(K) \), which separates points of the finite set \( K \). Hence, it coincides with \( C(K) \).

Theorem 3.5. Let a group \( G \subset \text{GL}(V) \), a finite subgroup \( F \subseteq G \), a torus \( T \), and a vector \( v \in V \) satisfy (3.1)–(3.3), (3.7), and (3.12). Then \( V, G, F, T, v \) can be realized as in Example 3.4, where
\[
(3.13) \quad v \in (\mathbb{C}^*)^n \cap C(K)^F.
\]

Conversely, if \( V, G, F, T, v \) are as in Example 3.4 and \( v \) satisfies (3.13), then (3.1)–(3.3), (3.7), and (3.12) are true.

Proof. The group \( F \) acts in \( t \) and \( t^* \) by the adjoint action. Let \( K \subset t^* \) be the collection of all weights for the representation of \( T \) in \( V \); clearly, \( K \) is \( F \)-invariant. It follows from (3.12) and (3.2) that the weights are multiplicity free. This defines an equivariant linear isomorphism between \( V \) and \( C(K) \), where the group \( T \) acts by multiplication. Thus, \( \lambda \) and \( \exp \) are well defined by (3.10) and (3.11). According to (3.7) and (3.3), (3.13) is true; (3.8) holds since \( T \subset \text{GL}(V) \) is compact and acts effectively on \( V \) (note that the annihilator of \( \text{span}_R K \) in \( t \) acts trivially due to...
Let us define $L$ by (3.9). Then $L = \ker \exp$ by (3.11). Hence, $L$ is a lattice in $\mathfrak{t}$ and the group $L^*$ generated by $K$ is the dual lattice in $\mathfrak{t}^*$.

The converse was proved in Example 3.4.

4. Finite extensions of $T$ which keep generic $T^C$-orbits

In what follows, we use the setting of Example 3.4. Let $Z$ denote the centralizer of $G$ in $\text{GL}(V)$. We assume that $(\mathbb{C}^*)^n$ acts in $V = C(K)$ by multiplication.

**Lemma 4.1.** $Z = C(K)^F \cap (\mathbb{C}^*)^n$.

*Proof.* Since $\lambda(\mathfrak{t})$ separates points of $K$, $Z \subseteq (\mathbb{C}^*)^n$. The multiplication by $u \in C(K)$ commutes with $F$ if and only if $u$ is $F$-invariant.

In general, condition (3.2) does not hold for a generic vector $v$. Hence, there is a natural problem: describe $V$ and $G$ such that generic orbits satisfy (3.2). The following proposition contains a simple criterion.

**Proposition 4.2.** Let $V, G$ be as in Example 3.4. Then $G$ satisfies (3.2) for a generic $v \in V$ if and only if

$$C(K) = \lambda(\mathfrak{t}^C) + C(K)^F.$$  

In this case, each $T^C$-orbit in $(\mathbb{C}^*)^n$ intersects $C(K)^F$.

*Proof.* It follows from Lemma 4.1 that the right-hand side of (3.1) is the tangent space at $1$ to the set $T^C Z$. Clearly, $\tilde{G}^C = ZG^C$ is a group, $T^C Z$ is the identity component of $\tilde{G}^C$, and the right-hand side of (3.1) is the tangent space to $\tilde{G}^C \mathbf{1}$. Hence, (3.1) holds if and only if $\tilde{G}^C \mathbf{1}$ is open. Moreover, this is equivalent to the equality $T^C Z = \exp(\lambda(\mathfrak{t}^C) + C(K)^F) = (\mathbb{C}^*)^n$. Therefore, each $T^C$-orbit in $(\mathbb{C}^*)^n$ intersects $C(K)^F$, i.e., contains an $F$-fixed point. Thus, (3.1) implies (3.2) for $v \in (\mathbb{C}^*)^n$.

Let (3.2) hold and let $W$ be an $F$-invariant neighborhood of $1$. If $W$ is sufficiently small, then the condition $\log 1 = 0$ defines a branch of $\log$ in $W$. We may assume that $\log W$ is convex and symmetric. This makes it possible to define roots in $W$:

$$w^{\lambda} = \exp \left( \frac{1}{r} \log w \right).$$

For $v \in W^{\frac{1}{r}}$ and $f \in F$, set $g_f = \left( \frac{I_v}{w} \right)^{\lambda}$, where $r = \text{card } F$, and $g = \prod_{f \in F} g_f$. Then $gv$ is $F$-fixed. If (3.2) holds for $v$, then $g_f \in T^C$ for all $f \in F$; hence, $gv \in T^C v$. Consequently, for all $v \in W$, $T^C v$ intersects $C(K)^F$. Since $Z$ keeps this property of orbits, it follows that $T^C Z$ has a nonempty interior. This implies (3.1).

**Theorem 4.3.** Let $G \subset \text{GL}(V)$ be a semidirect product of a torus $T$ and a finite subgroup $F$, and let $Z$ be the centralizer of $G$ in $\text{GL}(V)$. Suppose that $\text{span}_C T v = V$ for some $v \in V$. Then the following conditions are equivalent:

(i) $Gv \subseteq T^C v$ for a generic $v \in V$;

(ii) $G^C Z v$ is open in $V$ for a generic $v \in V$.

*Proof.* By Theorem 3.5, we may use the construction of Example 3.4. According to Lemma 4.1 (ii) is equivalent to (3.1), and the assertion follows from Proposition 4.2. □
We shall give a constructive description of these spaces and groups. Set
\[ C_0(K) = \left\{ u \in C(K) : \sum_{q \in K} u(q) = 0 \right\}. \]
Sometimes, we identify points in \( K \) with their characteristic functions.

**Example 4.4.** Let \( V = \mathbb{C}^n = C(K) \), where \( K = \{1, \ldots, n\} \), let \( F \) be a subgroup of \( S_n \), and
\begin{equation}
K = K_1 \cup \cdots \cup K_p
\end{equation}
be the partition of \( K \) into \( F \)-orbits. For \( k \in \{1, \ldots, p\} \), set \( V_k = C(K_k) \). Then \( V = V_1 \oplus \cdots \oplus V_p \). Set
\begin{align*}
\{0\}_k &= C_0(K_k) \cap i\mathbb{R}^n, \\
T_0 &= \exp(\{0\}_k) \subset C(K_k),
\end{align*}
where \( \exp \) is defined by (3.11). Set \( \{0\} = \{0\}_1 \oplus \cdots \oplus \{0\}_p \),
\[ T^0 = \exp(\{0\}) = T_1^0 \times \cdots \times T_p^0. \]

Let \( T \) be an \( F \)-invariant torus such that
\begin{equation}
T^0 \subseteq T \subseteq T^n
\end{equation}
and set \( G = FT \). Then generic \( G^C \)-orbits satisfy (3.12). The group \( G \) is irreducible if and only if \( F \) is transitive on \( K \); in general, \( F \)-orbits in \( K \) define \( G \)-irreducible components of \( V \). There are two extreme cases in (4.3).

(A) If \( T = T^n \), then there is one open orbit \((\mathbb{C}^*)^n \) of the group \( G^C = FT^C \), which evidently satisfies (3.2). If \( F \) is nontrivial, then there exist degenerate orbits that do not satisfy (3.2); moreover, if \( F \) is transitive on \( K \), then all non-open \( G^C \)-orbits, except for zero, are nontrivial finite unions of \( T^C \)-orbits.

(B) If \( T = T^0 \), then generic orbits are closed. They have codimension \( p \) and are distinguished by equations
\[ \prod_{r \in K_k} z_r = c_k, \]
where \( c_k \in \mathbb{C}^* \), \( k = 1, \ldots, p \).

Note that (A) and (B) are invariant under the Cartesian product (the group \( F \) need not be the product of groups \( F_k \) of irreducible components but must have the same orbits in \( K \) as \( F_1 \times \cdots \times F_p \)). In terms of Example 3.4 in (A), \( t = \mathbb{R}^n \), the mapping \( \lambda : t^C \to C(K) \) is surjective, \( K = \{\epsilon_1, \ldots, \epsilon_n\} \); in (B), \( t = i\mathbb{R}^n \cap C_0(K) \), \( \lambda(t^C) = C_0(K) \), and the set \( K \) is the projection of \( \{\epsilon_1, \ldots, \epsilon_n\} \) into \( t^0 = t \). In both cases, \( K \) is the set of all vertices of a regular simplex.

**Theorem 4.5.** Let \( V,G \) be as in Theorem 4.5 and let (i) hold. Then \( V,G \) can be realized as in Example 3.4. Furthermore,

1. \( V,G \) are of type (A) if and only if \( G^C \) has an open orbit,
2. (B) is equivalent to the assumption that the center of \( G \) is finite,
3. If \( G \) is irreducible, then either (A) or (B) holds.

Let \( C(K)^+_F \) be the cone of all nonnegative functions in \( C(K)^F \).

**Lemma 4.6.** Let \( G \) and \( V \) be as in Example 3.4. Then, the orbit \( G^C v \) is closed for a generic \( v \in V \) if and only if
\begin{equation}
t^R \cap C(K)^+_F = 0.
\end{equation}
Proof. Clearly, $G^C v$ is closed if and only if $T^C v$ is closed. Let $v \in (\mathbb{C}^*)^n$. By Proposition 2.2 and Corollary 2.3, $T^C v$ is not closed if and only if $C_T \neq 0$. Since $C_T$ is $F$-invariant by (1.4), it contains $\sum_{f \in F} fu$ for each $u \in C_T$. Thus, $C_T = 0$ is equivalent to (4.4).

Proof of Theorem 4.5. Suppose that $G$ is irreducible or, equivalently, $F$ is transitive. Then $Z = \mathbb{C}^* 1$ according to Lemma 4.1. If $1 \in \lambda(t^C)$, then $T^C \subseteq Z$ and $T^C v$ is open for a generic $v \in V$ by Theorem 4.3. If $1 \notin \lambda(t^C)$, then (4.4) is true; by Lemma 4.6, $T^C v$ is closed for a generic $v \in V$. By Proposition 4.2, a generic $T^C$-orbit intersects $C^* 1$. Consequently, we have

$$\text{codim } G^C v = 1.$$  

Let $1 \in T^C \cap Z$. The orthogonal projection of $1$ into the tangent space $T_1 T^C 1$ is $F$-fixed. Hence, it is proportional to $1$; since $1 \notin \lambda(t^C)$, this implies $1 \perp \lambda(t^C) 1$. Therefore, $T_1 T^C 1$ coincides with the tangent space to the hypersurface $z_1 \ldots z_n = 1$ at $1$; since the monomial on the left is an eigenfunction of $T^C$, this group keeps it. Due to (4.4), $T^C 1$ coincides with this hypersurface. Then, $T = T^n \cap SU(n)$, and any $T^C$-orbit that intersects $Z$ is a hypersurface $z_1 \ldots z_n = c$, for some $c \in \mathbb{C}^*$. This implies $t^C = C_0(K)$ and $T = T^0$.

Thus, the theorem is proved for all irreducible $G$. The projection onto each irreducible component keeps the property (3.2) for generic orbits since it commutes with $G$. Hence, (i) holds for all irreducible components. They correspond to $F$-orbits $K_k$ in the partition (4.2). Let $t^0_k$, $k = 1, \ldots, p$, be defined as in Example 4.4. According to the arguments above, $\lambda(t| K_k) \supseteq \lambda(t^0_k)$ for all $k$. If $x \in t$, then the averaging

$$Ax = \frac{1}{r} \sum_{f \in F} fx, \quad r = \text{card } F,$$

distinguishes the $F$-fixed component of $x$ (i.e., $Ax \in C(K)^F \cap t$ and $x - Ax \in t^0$); since $t$ is $F$-invariant, it contains both components. By Lemma 4.1 if $G$ has a finite center, then $\lambda(t| K_k) = \lambda(t^0_k)$ for all $k$. It follows that

$$t \subseteq t^0 = t^0_1 \oplus \cdots \oplus t^0_p.$$

On the other hand, (ii) and Lemma 4.1 imply $\text{codim } t \leq \dim C(K)^F = p$. Hence, the inclusion above is in fact the equality. Thus, we get (B) assuming that $G$ has a finite center. The converse is true since $t^0$ does not contain a nontrivial $F$-fixed element. The same arguments show that any $F$-invariant torus $T$ includes $T^0$ if (i) is true. This proves that $V, G$ admit the realization of Example 4.4 (1) and (2) are clear.

Corollary 4.7. Let $G$ be as in Theorems 4.5 and 4.3. Then $G$ contains a closed subgroup $G^0$ such that

1. each connected component of $G$ contains a connected component of $G^0$,
2. $G^0$ has a finite center,
3. generic orbits of $(G^0)^C$ are closed,
4. $Gv \cap T^0 v = G^0 v \cap (T^0)^2 v$ for a generic $v \in V$.

Proof. By Theorem 4.5 and (4.3), $G \supseteq T^0$, where $T^0$ is as in (B). Clearly, $F$ normalizes $T^0$. Hence, $G^0 = FT^0$ is a group, which satisfies the corollary.
Proposition 2.2 makes it possible to find $\hat{G}v$ for $G$ as above. If $T = T^n$, then $T \supseteq \mathbb{Z}_2^n$, and generic $T$-orbits intersect $\mathbb{R}_+^n$; hence, we may assume $v \in \mathbb{R}_+^n$. Then $\overline{T}v \cap \mathbb{R}^n$ is a parallelepiped $\Pi_v = \text{conv}\{ (\pm v_1, \ldots, \pm v_n) \}$. Clearly, $\Pi_v = \mathbb{Z}_2\Pi_v^+$, where $\Pi_v^+ = \Pi_v \cap \text{clos}\mathbb{R}_+^n$. Since $\mathbb{R}_+^n = T\mathbb{R}^n v$, we may use Proposition 2.2 with $X = Fv$, $C_T = -\text{clos}\mathbb{R}_+^n$, $P_X = \text{conv}(Fv) - \mathbb{R}_+^n$.

$$\hat{G}v = \bigcup_{u \in \exp(Q_v)} \mathbb{R}^n u = T \bigcup_{u \in \exp(Q_v)} \Pi_u = T \bigcup_{u \in \exp(Q_v)} \Pi_u^+,$$

where $Q_v = \text{conv} Fv$. For the description in the form (2.10), one has to know normal vectors to faces of $\text{conv} Fv$. Since $F$ may be an arbitrary subgroup of $S_n$, they need not be proportional to rational vectors (for example, this is true for the cyclic subgroup of order 3 in $S_3$). We shall describe the situation where they are locally independent of $v$; since they depend on $v$ continuously, this is equivalent to the condition that they are rational. Note that the vector which joins two points in $t^i$ as in Lemma 2.1 is rational if and only if the strip reduces to an annulus. In Example 4.3, $F$ need not be the product of groups corresponding to the irreducible components; we shall see that $F$ possesses this property in the case under consideration.

Let $U$ be a real vector space and $F \subseteq \text{GL}(U)$ be a finite group. Set

$$C_u = \text{cone}(u - Fu);$$

this is the cone at the vertex $u$ of the polytope $\text{conv}(Fu)$ (which may be degenerate). We say that $C_u$ is locally independent of $u$ if, for a generic $u \in U$, $C_u = C_w$ for all $w$ that are sufficiently close to $u$.

**Lemma 4.8.** Let $U$ be a real vector space and $F$ be a finite subgroup of $\text{GL}(U)$. Suppose that $C_u$ is locally independent of $u$. Then $F$ is generated by reflections in hyperplanes in $U$.

**Proof.** We may assume without loss of generality that $U$ is equipped with an inner product and that $F \subseteq \text{O}(U)$. Let $\mathbb{R}_+(u - fu)$, $f \in F$, be an extreme ray of $C_u$. The equality $C_u = C_w$ for $w$ in a neighborhood of $u$ implies that this ray does not change near $u$. Hence, dim$(1 - f)U = 1$. Since $f$ is orthogonal and nontrivial, it is a reflection in a hyperplane. The stable subgroup of a generic $u \in U$ is trivial (hence, $F$ acts freely on a generic orbit) and each vertex of $\text{conv}(Fu)$ can be joined with $u$ by a chain of edges. Applying the above arguments repeatedly to $u, fu, \text{etc.}$, we get that $F$ is generated by reflections in hyperplanes. □

For any $g \in \mathbb{Z}_2^n S_n$ and $k = 1, \ldots, n$, $g \epsilon_k = \pm \epsilon_{\sigma(k)}$ for some $\sigma \in S_n$. The mapping $f \rightarrow \sigma$ is a natural homomorphism $\mathbb{Z}_2^n S_n \rightarrow S_n$, which we denote by $\phi$.

**Lemma 4.9.** Let $F$ be a transitive subgroup of $S_n$ acting in $\mathbb{R}^n$ by permutations of coordinates and let a group $H \subseteq \mathbb{Z}_2^n S_n$ be generated by reflections in hyperplanes in $\mathbb{R}^n$. If $\phi(H) = F$, then $F = S_n$.

**Proof.** Let $\rho$ be a reflection in a hyperplane in $\mathbb{R}^n$. If $\rho \in \mathbb{Z}_2^n S_n = BC_n$, then it is conjugate to a reflection in a wall of the Weyl chamber that is distinguished by the inequalities $x_1 > \cdots > x_n > 0$. Hence, $\phi(\rho)$ is a transposition if it is nontrivial. Since $F = \phi(H)$, $F$ is generated by transpositions. It remains to note that any subgroup of $S_n$, which is generated by transpositions, coincides with $S_n$ if it is transitive on $\{1, \ldots, n\}$ (consider the graph with the vertices $\{1, \ldots, n\}$ and edges corresponding to transpositions and note that inclusions $(k, l) \in F$, $(l, m) \in F$ imply $(k, m) \in F$; this makes it possible to use the induction). □
We say that a pair \((V, G)\) is standard if it is isomorphic to (A) or (B) in Example 4.4 with \(F = S_k\). The product of pairs \((V_k, G_k), k = 1, \ldots, m\), is the pair \((\sum_{k=1}^m V_k, \prod_{k=1}^m G_k)\).

**Theorem 4.10.** Let \(G = FT\) be a compact subgroup of \(GL(n, \mathbb{C})\), where \(T \subseteq T^n\) is a torus and \(F\) is a subgroup of \(S_n\). Suppose that \(Gv \subset T^Cv\) for a generic \(v \in V\) and

1. either \(T = T^n\) or the center of \(G\) is finite,
2. for a generic \(v \in C^n\), \(Gv\) can be distinguished in \(\text{clos}T^Cv\) by a family of inequalities

\[
|z_1|^{s_1} \cdots |z_n|^{s_n} \leq \rho_s(v),
\]

where \(\rho_s(v) \geq 0\) and vector \(s = (s_1, \ldots, s_n)\) runs over a certain finite subset of \(\mathbb{R}^n\) which is independent of \(v\).

Then \((V, G)\) is isomorphic to the product of standard pairs. Moreover, if \(G\) is irreducible, then \((V, G)\) is standard.

**Proof.** Let \(G\) be irreducible. Then \(F\) is transitive and \((V, G)\) are as in (A) or as in (B) by Theorem 4.5. Suppose that (B) is the case. It follows from (2) and Proposition 2.2 that the polytope \(Q_X \subset t^R\), where \(X = Gv \cap T^Rv\), for a generic \(v\), satisfies the assumption of Lemma 4.8. Therefore, \(F\) is generated by reflections (we may assume that \(F \subset O(t^R)\)). They extend to reflections in hyperplanes in \(t^R + \mathbb{R} \cdot 1 = \mathbb{R}^n\) if we assume that they fix 1. Then, Lemma 4.9 implies \(F = S_n\). The case (A) can be reduced to (B): it is sufficient to replace \(T^n\) with \(T = SU(n) \cap T^n\) since \(F\) evidently keeps \(T\) and to note that (2) remains true due to Proposition 2.2.

Thus, \((V, G)\) is standard.

Let the center of \(G\) be finite. According to Theorem 4.5, \(T\) may be identified with the group \(T^0\) in Example 4.4. In particular, \(G^Cv\) is closed for a generic \(v\) and \(C_T = 0\) due to Proposition 2.2. By Proposition 4.2, generic orbits contain \(F\)-fixed points. Applying the arguments above (which did not use the assumption that \(G\) is irreducible), we get that the cones at the vertices of the convex polytope \(Q_X\), \(X = Gv \cap T^R\subset t^R\), are locally independent of \(v\). Clearly, the same is true for its projection into each space \(t_k^0\) corresponding to an irreducible component \(V_k\) of \(V = C^n\). This implies that all irreducible components are standard. Thus, \(F_k = S(K_k)\), where \(k = 1, \ldots, p\) and \(K = K_1 \cup \cdots \cup K_p\) is the partition of \(K\) into \(F\)-orbits. Due to Theorem 4.5, it is sufficient to prove that

\[
F = F_1 \times \cdots \times F_p.
\]

By Lemma 4.8, \(F \big|_v\) is generated by reflections in hyperplanes in \(t^0\); the condition that they keep real \(F\)-invariant functions on \(K\) uniquely defines their extension to \(\mathbb{R}^n\). Hence, \(F\) is generated by reflections in \(\mathbb{R}^n\). A permutation which induces a reflection in a hyperplane in \(\mathbb{R}^n\) is a transposition of a pair of coordinates; this pair is necessarily contained in only one of the sets \(K_k\), \(k = 1, \ldots, p\). This proves 4.10.

If \(T = T^n\), then \(T\) is a product of tori in irreducible components. Thus, the case \(T = T^n\) follows from the above case, since the assumptions of the theorem hold true for the group \(T^0\) if they hold for \(T\) in 4.3 in Example 4.4.

5. Hulls of isotropy orbits of bounded symmetric domains

We start with a preliminary material on hermitian symmetric spaces following [15] but adapting the exposition to our purpose in order to be as self contained as
noncompact type in complex structure in where
possible. For a subset $X$ of a Lie algebra $\mathfrak{g}$, $\mathfrak{z}(X) = \{z \in \mathfrak{g} : [z, X] = 0\}$ is the centralizer of $X$. Let $G$ be a simple real noncompact Lie group with a finite center, $K$ be its maximal compact subgroup, and $\mathfrak{g}, \mathfrak{k}$ be their Lie algebras, respectively. If the center $\mathfrak{z} = \mathfrak{z}(\mathfrak{t})$ of $\mathfrak{t}$ is nontrivial, then $\mathfrak{g}$ is called hermitian. Then $\mathfrak{t} = \mathfrak{z}(\mathfrak{g})$ and $\dim \mathfrak{z} = 1$ (note that $K$ is irreducible in $\mathfrak{g}/\mathfrak{t}$). Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{t}$. Then $\mathfrak{c}$ is also a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{z} \subseteq \mathfrak{c}$. There exists $k \in \mathfrak{z}$ such that $\text{ad}(k)$ has eigenvalues $0, \pm i$ (it is unique up to a sign; $\text{ker ad}(k) = \mathfrak{t}$). Then $\kappa = e^{\pi \text{ad}(k)}$ is the Cartan involution which defines the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{d},$$

where $\mathfrak{t}, \mathfrak{d}$ are eigenspaces for $1, -1$, respectively. Furthermore, $j = \text{ad}(k)$ is a complex structure in $\mathfrak{d}$. This defines the structure of a hermitian symmetric space of noncompact type in $D = G/K$. These spaces can be realized as bounded symmetric domains in $\mathbb{C}^n$ with $K = \text{Aut}_\mathbb{C}(D)$. Any irreducible bounded symmetric domain admits such a realization. Let $\Delta \subseteq i\mathfrak{c}$ be the root system of $\mathfrak{g}^\mathbb{C}$. Each $\alpha \in \Delta$ corresponds to an $\mathfrak{sl}_2$-triple $h_\alpha, e_\alpha, f_\alpha$ such that $ih_\alpha \in \mathfrak{c}$. Thus, $\alpha(h_\alpha) = 2, [e_\alpha, f_\alpha] = h_\alpha$, and

$$[h, e_\alpha] = \alpha(h)e_\alpha, \quad [h, f_\alpha] = -\alpha(h)f_\alpha$$

for all $h \in \mathfrak{c}^\mathbb{C}$. We identify $\mathfrak{c}^\mathbb{C}$ and $(\mathfrak{c}^\ast)^\mathbb{C}$ equipping $\mathfrak{g}$ with an $\text{Ad}(K)$-invariant sesquilinear inner product and normalize it by the condition

$$\max\{ |\alpha| : \alpha \in \Delta \} = \sqrt{2}.$$ 

Then short roots must have length 1 (note that $G_2$ has no real hermitian form). The set $\Delta^\vee = \{h_\alpha : \alpha \in \Delta\}$ is the dual root system. The above normalization implies $h_\alpha = \alpha$ for long roots and $h_\alpha = 2\alpha$ for short ones. Since $\text{ad}(h), h \in \mathfrak{c}$, has eigenvalues $0$ and $\alpha(h)$, where $\alpha \in \Delta$, we get $\alpha(ik) = 0, \pm 1$, i.e., $ik$ is a microweight (of $\Delta^\vee$). For $s = 0, \pm 1$, set

$$\Delta_s = \{\alpha \in \Delta : \alpha(ik) = s\}.$$ 

Since $\mathfrak{t} \oplus i\mathfrak{d}$ is a compact real form of $\mathfrak{g}^\mathbb{C}$ and $\text{span}_\mathbb{R}\{ih_\alpha, e_\alpha - f_\alpha, i(e_\alpha + f_\alpha)\}$ is the $\mathfrak{su}(2)$-subalgebra corresponding to a root $\alpha \in \Delta$, we have

$$\mathfrak{d} = \text{span}_\mathbb{R}\{e_\alpha + f_\alpha, i(e_\alpha - f_\alpha) : \alpha \in \Delta_1\}.$$ 

Set $\mathfrak{s}_\alpha = \text{span}_\mathbb{R}\{ih_\alpha, e_\alpha + f_\alpha, i(e_\alpha - f_\alpha)\}$. Then $\mathfrak{s}_\alpha$ is an $\mathfrak{sl}(2, \mathbb{R})$-subalgebra of $\mathfrak{g}^\mathbb{C}$ and

$$\alpha \in \Delta_{\pm 1} \iff \mathfrak{s}_\alpha \subseteq \mathfrak{g}.$$ 

Let $E$ be a maximal subset of pairwise orthogonal long roots in $\Delta_1$. Set

$$h = \sum_{\alpha \in E} h_\alpha, \quad e = \sum_{\alpha \in E} e_\alpha, \quad f = \sum_{\alpha \in E} f_\alpha; \quad s = \sum_{\alpha \in E} \mathfrak{s}_\alpha.$$

Let $\alpha, \beta \in E$, $\alpha \neq \beta$. Since $\alpha, \beta$ are long and orthogonal, $\pm \alpha \pm \beta \notin \Delta$. Hence,

$$\alpha, \beta \in E, \quad \alpha \neq \beta \implies [\mathfrak{s}_\alpha, \mathfrak{s}_\beta] = 0.$$ 

It follows that $h, e, f$ is an $\mathfrak{sl}_2$-triple and $s$ is a subalgebra of $\mathfrak{g}$. Set

$$\theta = e^{\frac{\pi}{2} \text{ad}(e - f)},$$

$$a = \text{span}_\mathbb{R} \theta E.$$
Here is the standard realization of root systems $B_n$ and $C_n$:

\[
B_n = \{ \pm \epsilon_k \pm \epsilon_l : k, l = 1, \ldots, n, k < l \}; \\
C_n = \{ \pm \epsilon_k \pm 2\epsilon_l : k, l = 1, \ldots, n, k < l \}.
\]

Then $C_n = B_n'$, but $C_n$ does not satisfy (5.3). These systems have microweights; up to the action of the Weyl group, they are:

\[
B_n: \quad \epsilon_1; \\
C_n: \quad \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n).
\]

There are no other irreducible root systems which have microweights and contain roots of different lengths. Also, $B_n$ and $C_n$ have the same Weyl group $BC_n = \mathbb{Z}_2^n S_n$.

**Lemma 5.1.** The space $a$ is a maximal abelian subspace of $\mathfrak{d}$.

**Proof.** A straightforward calculation with 2-matrices shows that $\theta h = e + f$. By (5.7),

\[
(5.10) \quad \theta h_\alpha = e_\alpha + f_\alpha \quad \text{for all } \alpha \in E.
\]

It follows from (5.5) that $a \subseteq \mathfrak{d}$. Moreover, $a$ is abelian due to (5.7). Set $\Xi = \Delta \cap E^\perp$. We claim that

\[
(5.11) \quad \Xi \subseteq \Delta_0.
\]

Indeed, a root in $\Delta_1 \cap \Xi$ must be short. This may happen only in $B_n$ or $C_n$, since $G_2$ and $F_4$ have no microweights and other irreducible root systems have no roots of different lengths. In $B_n$, $k$ is a short root and all other short roots are orthogonal to $k$. Hence, they do not belong to $\Delta_1$. In $C_n$, $E = \{2\epsilon_1, \ldots, 2\epsilon_n\}$; then $\Xi = \emptyset$. Since $\Delta_1 = -\Delta_1$, this proves (5.11).

Set $b = E^\perp \cap c$ and $m = \text{span}_\mathbb{C}\{e_\alpha, f_\alpha : \alpha \in \Xi\}$. It follows from (5.11) that $m \subseteq \mathfrak{t}^\mathbb{C}$. Clearly, $\mathfrak{s}(E) = \mathfrak{c}^\mathbb{C} \oplus m$. The space $m$ is $\theta$-invariant, because $\theta$ fixes roots in $\Xi$. Due to (5.9), we get

\[
\mathfrak{s}(a) = \theta \mathfrak{s}(E) = b^\mathbb{C} \oplus a^\mathbb{C} \oplus m
\]

Since $b^\mathbb{C} \oplus m \subseteq \mathfrak{t}^\mathbb{C}$, this implies $\mathfrak{s}(a) \cap \mathfrak{d} = a$. \qed

The projection of $\theta \Delta$ into $a$ is the restricted root system $\Delta_a$ (it is also the set of roots for $\text{Ad}(a)$ in $\mathfrak{g}$). The group

\[
W = \{ \text{Ad}(g) : g \in K, \text{Ad}(g)a = a \}|_a,
\]

acting in $a$, is the Weyl group of $a$.

In what follows, we denote by $\mathfrak{v}$ the complexification of $a$ with respect to the complex structure $j$ (thus, $\mathfrak{v} \subset \mathfrak{d}$). The set $\theta E$ is a base in $\mathfrak{v}$; enumerating it, we identify $\mathfrak{v}$ with $\mathbb{C}^n$. Set $t = \text{span}_\mathbb{R} iE$, $T = \exp t$, $H = WT$. The torus $T = T^n$ is a maximal compact subgroup in the group $\text{exp} \mathfrak{s} \subseteq G$.

**Proposition 5.2.** The following assertions hold:

1. $\Delta_a$ is a root system of type $BC_n$ or $C_n$;
2. the pair $(\mathfrak{v}, H)$ is standard with $T = T^n$.

**Proof.** (1). Let $\Delta_a \setminus \theta E$ contain a long root $\alpha$. Then $\alpha = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$ for some $\alpha_1, \ldots, \alpha_4 \in \theta E$, since $|\alpha|^2 = 2$ and $\langle \alpha, \beta \rangle = 0, \pm 1$ for all $\beta \in E$ due to the normalization (5.3) (note that $\alpha, \beta$ generate $A_2$ if $\langle \alpha, \beta \rangle \neq 0$). Roots $\alpha, \alpha_1, \ldots, \alpha_4$ generate $D_4$, since only $A_4$ and $D_4$ among irreducible systems of rank
4 consist of roots of equal length, but $A_4$ does not contain an orthogonal base. Since $\langle i k, \beta \rangle = 1$ for all $\beta \in E$, the projection of $i \theta k$ into $\text{span}_H D_4$ is a microweight $\omega$ such that $\langle \omega, \alpha_k \rangle = 1$, $k = 1, \ldots, 4$, but $D_4$ has no microweight with this property (in the realization above, $D_4 = B_4 \cap C_4$ and the microweights are either $\pm \epsilon_4$ or $\frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$). Thus, $E \cup (-E)$ is the set of all long roots in $\Delta_4$. According to the classification of irreducible root systems, only $C_n$ and $BC_n = B_n \cup C_n$ has the property that linearly independent long roots are mutually orthogonal.

(2). The maximal compact subgroup of the group corresponding to $\mathfrak{s}$ is $T^n$. Hence, $T = T^n \supset \mathbb{Z}_2^n$. Systems $C_n$ and $BC_n$ have the same Weyl group $W = BC_n$.

Therefore, $H = WT = S_n T^n$. □

Let $D$ be a bounded symmetric domain in a complex linear space $\mathfrak{d}$ (may be, reducible) and $v \subseteq \mathfrak{d}$ be the complex linear span of a maximal abelian subspace in $\mathfrak{d}$ (thus, we identify $\mathfrak{d}$ with the corresponding space in the Cartan decomposition [5.1], which is induced by the Cartan involutions in irreducible components). Let $\text{Aut}_{00}(v, D)$ denote the subgroup of all linear transformations in $\text{Aut}(D)$ which keep $v$ and each irreducible component of $D$.

**Corollary 5.3.** Let $F$ be a subgroup of $S_n$, $G = FT^n \subset \text{GL}(n, \mathbb{C})$. Then $G$ satisfies condition (2) of Theorem 4.10 if and only if $(V, G)$ is isomorphic to a pair $(v, \text{Aut}_{00}(v, D))$ for a bounded symmetric domain $D$.

**Proof.** All pairs $(\mathbb{C}^n, S_n T^n)$ appear as $(v, \text{Aut}_{00}(v, D))$ for matrix balls $D$. It remains to combine Theorem 4.10 and Proposition 5.2. □

It is possible now to describe hulls of $K$-orbits in $\mathfrak{d}$ (with respect to the complex structure $j$) in terms of Proposition 5.2. The key point is that $K$ is polar in $\mathfrak{d}$: each $K$-orbit meets $\mathfrak{a}$ orthogonally (i.e., $\mathfrak{a}$ is a Cartan subspace). This is true, since all maximal abelian subspaces are conjugate in $\mathfrak{d}$ by $K$, $\text{ad}(a)$ is symmetric if $a \in \mathfrak{d}$ and, for a generic $a \in \mathfrak{a}$, $\text{ker ad}(a) = \mathfrak{a}$; hence,

$$ (5.12) \quad [a, \mathfrak{g}] = a^\perp. $$

We may include the linear base in $v$ into a base in $\mathfrak{d}$ as the first $n$ vectors of the latter. Then $z_1, \ldots, z_n$ are coordinates in $v$ and linear functions in $\mathfrak{d}$. The functions $\mu_k$ in (5.6) admit a $K$-invariant extension to $\mathfrak{d}$:

$$ (5.13) \quad \mu_k(z) = \sup\{(g z)_1 \ldots (g z)_k : g \in K\}, $$

where $k = 1, \ldots, n$. The following lemma shows that (5.13) is an extension indeed.

**Lemma 5.4.** For $z \in v$, (5.6) and (5.13) coincide.

**Proof.** It follows from (5.12) that any critical point of the linear function $\text{Re} z_1$ on the orbit $Kz$ belongs to $\mathfrak{a}$. If the lemma is not true, then there exist $z \in v$ and $k \in \{1, \ldots, n\}$ such that $|(g z)_k| > |z_k|$. Transformations in $S_n$ and $T$ reduce the problem to the case $z_1 > \cdots > z_n > 0$ and $k = 1$, but then the assumption implies that $\text{Re} z_1$ attains its maximal value on $Kz$ outside of $\mathfrak{a}$. □

**Proposition 5.5.** For any $v \in \mathfrak{d}$, $\overline{Kv} = \{z \in \mathfrak{d} : \mu_k(z) \leq \mu_k(v), k = 1, \ldots, n\}$.

**Proof.** Due to (5.13), each $\mu_k$ is a supremum of absolute values of holomorphic polynomials. Hence, the right-hand side is polynomially convex. Thus, it includes $\overline{Kv}$. The inverse inclusion holds, since each $K$-orbit intersects $v$ by an $H$-orbit and hulls of $H$-orbits are distinguished in $v$ by the same inequalities according to Proposition 5.2 and Lemma 5.4. □
The functions $\mu_k$ can be written in more invariant terms. To do it, note that the Weyl group of $\Delta_\alpha$ has the form $\mathbb{Z}_2^n S_n$ in the base $\theta E$ by (5.3); thus, $z_k = \alpha_k(z)$, $k = 1, \ldots, n$, where $\alpha_k \in \theta E$ and $z \in \mathfrak{a}$. Therefore, $z_k$ are eigenvalues of $\text{ad}(z)$ in the subspace generated by the corresponding root vectors. The problem is to distinguish this subspace (in fact, we use a slightly different version). After that, functions $\mu_k$ can be defined as norms of some operators according to the following lemma (this observation was used in [12] in another context).

Lemma 5.6. Let $V$ be a Euclidean space and $A$ be a symmetric nonnegative operator in $V$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$, where $m = \dim V$. Let $A^{\wedge k}$ be its natural extension to the $k$-th exterior power $V^{\wedge k} = \wedge^k V$. Then

$$\|A^{\wedge k}\|_{V^{\wedge k}} = \lambda_1 \ldots \lambda_k,$$

where $\| \|_k$ is the operator norm with respect to the inner product in $V^{\wedge k}$.

Proof. The norm of a nonnegative symmetric operator is equal to its maximal eigenvalue. $\square$

Let $v \in \mathfrak{g}$ be semisimple and $\pi(v)$ denote the projection onto $\ker \text{ad}(v)$ along other eigenspaces of $\text{ad}(v)$ (note that $\pi(v)$ is a function of $\text{ad}(v)$, since it is the residue at zero of the resolvent of $\text{ad}(v)$). Set

$$a(v) = \text{ad}([v, [v, k]])\pi(v)\text{ad}(k),$$

$$p_k(v) = \|a(v)^{\wedge k}\|_{\mathfrak{g}^{\wedge k}}, \quad k = 1, \ldots, n.$$ 

The space $\mathfrak{d}$ is $a(v)$-invariant and $\ker a(v) \supseteq \mathfrak{t}$. We assume that $\mathfrak{g}$ is equipped with some $K$-invariant inner product, which extends the inner product in $\mathfrak{d}$. It follows from the calculation below that $a(v)$ is symmetric and has range $\text{ad}(k)\mathfrak{a}$. Let $n = \dim \mathfrak{a}$ be the rank of the symmetric space $D$. It is equal to the codimension of a generic $K$-orbit in $\mathfrak{d}$.

Theorem 5.7. For any $v \in \mathfrak{d}$,

$$\hat{K}v = \{z \in \mathfrak{d} : p_k(z) \leq p_k(v), \quad k = 1, \ldots, n\}.$$ 

Proof. It is sufficient to prove the assertion for a generic $v \in \mathfrak{d}$. Clearly, $p_k$ are $K$-invariant. Hence, we may assume $v \in \mathfrak{a}$. Then, by (5.9) and (5.10),

$$v = \sum_{\alpha \in E} v_\alpha (e_\alpha + f_\alpha),$$

where $v_\alpha \in \mathbb{R}$. According to (5.2) and (5.3), $[k, v] = \sum_{\alpha \in E} iv_\alpha (e_\alpha - f_\alpha)$. Thus,

$$[v, [v, k]] = \sum_{\alpha \in E} 2i^2 \alpha h_\alpha$$

due to (5.7). Also, (5.7) implies that $\text{ad}(ih_\alpha)$ keeps $v$ and has eigenvalues $0, \pm 2i$ in it for each $\alpha \in E$. Therefore, $v$ is $\text{ad}([v, [v, k]])$-invariant and its eigenvalues are $\pm 4v_\alpha^2$, $\alpha \in E$. Since $\text{ad}(k)\mathfrak{g} = \mathfrak{d}$ and $\pi(v)\mathfrak{d} = \mathfrak{a}$ for a generic $v \in \mathfrak{a}$, the space $v$ is $a(v)$-invariant; moreover, $a(v)\mathfrak{g} = \text{ad}(k)\mathfrak{a} \subseteq v$. Thus, $a(v)$ has eigenvalues $0, \pm 4v_\alpha^2$ in $\mathfrak{g}$. According to Lemma 5.6 and (0.9),

$$(5.14) \quad p_k(v) = 4k^2 v_\alpha^2 (v)$$

for $v \in \mathfrak{v}$ and $k = 1, \ldots, n$. Since $p_k$ and $\mu_k$ are $K$-invariant, (5.14) holds for all $v \in \mathfrak{d}$. The theorem follows from Proposition 5.5. $\square$
Corollary 5.8. Functions $p_k$, $k = 1, \ldots, n$, are plurisubharmonic in $\mathfrak{d}$ with respect to the complex structure $j = \text{ad}(k)$.

Proof. By (5.14) and (5.13),

$$p_k(z) = 4^k \sup \{|(gz_1^2 \ldots (gz_k^2)| : g \in K\}.$$

The right-hand side is plurisubharmonic, since the functions $z_k^2$ are $j$-holomorphic and $j$ is $K$-invariant. □

One can get the same functions $p_k$ by replacing $g$ with $\mathfrak{d}$, endowed with the complex structure $j$, and $a(v)$ with $\text{ad}([v, jv])(\pi(v) + \pi(jv))$.

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References

[1] J.T. Anderson, Polynomial hulls of sets invariant under an action of the special unitary group, Can. J. Math., v. XL, (1988) no. 5, 1256-1271.
[2] J.-E. Björk, Compact groups operating on Banach algebras, Math. Ann., v. 205 (1973), no. 4, 281–297.
[3] J. Dadok, Polar coordinates induced by actions of compact Lie groups, Transactions Amer. Math. Soc., 288 (1985), 125-137.
[4] A. Debiard, B. Gaveau, Equations de Cauchy-Riemann sur SU(2) et leurs enveloppes dholomorphie, Can. J. Math., v. 38 (1986), 1009-1024.
[5] T. Gamelin, Uniform Algebra, Prentice-Hall, Englewood Cliffs, N. J., 1969.
[6] V.M. Gichev, Maximal ideal spaces of invariant function algebras on compact groups, preprint, 46 pp., available at http://arXiv.org/abs/math/0603449
[7] V.M. Gichev, I.A. Latypov, Polynomially Convex Orbits of Compact Lie Groups, Transformation Groups, v. 6 (2001) no. 4, 321-331.
[8] J. Faraut, L. Bouattour, Enveloppes polynomiales densembles compacts invariants, Math. Nachrichten 266 (2004), 20-26.
[9] W. Kaup, Bounded symmetric domains and polynomial convexity, Manuscr. Math. 114 (2004), No.3, 391-398.
[10] W. Kaup, D. Zaitsev, On the CR-structure of compact group orbits associated with bounded symmetric domains, Invent. Math., 153 (2003), 45-104.
[11] J. Kane, Maximal ideal spaces of $U$-algebras, Illinois J. Math., v.27 (1983), No 1, 1-13.
[12] B. Kostant, On convexity, the Weil group and the Iwasawa decomposition, Ann. Sc. Ecole. Norm. Super., 6 (1973), 413–455.
[13] V.L. Popov, E.B. Vinberg, Invariant theory, Itogi Nauki i Tekhniki, Sovr. Probl. Mat. Fund. Napravl., v. 55, VINITI, Moscow 1989, pp. 137–309 (in Russian); English transl.: Algebraic Geometry IV, Encyclopaedia of Math. Sciences, v. 55, Springer-Verlag, Berlin 1994, pp. 123-278.
[14] C. Sacre, Enveloppes polynomiales de compacts, Bull. Sci. math. 116 (1992), 129-144.
[15] J.A. Wolf, Fine Structure of Hermitian Symmetric Spaces, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969-1970), Pure Appl. Math., Vol. 8. New York: Dekker 1972, pp. 271-357.

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