Creep, Relaxation and Viscosity Properties for Basic Fractional Models in Rheology

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Abstract

The purpose of this paper is twofold: from one side we provide a general survey to the viscoelastic models constructed via fractional calculus and from the other side we intend to analyze the basic fractional models as far as their creep, relaxation and viscosity properties are considered. The basic models are those that generalize via derivatives of fractional order the classical mechanical models characterized by two, three and four parameters, that we refer to as Kelvin–Voigt, Maxwell, Zener, anti–Zener and Burgers. For each fractional model we provide plots of the creep compliance, relaxation modulus and effective viscosity in non dimensional form in terms of a suitable time scale for different values of the order of fractional derivative. We also discuss the role of the order of fractional derivative in modifying the properties of the classical models.

2010 Mathematics Subject Classification (MSC): 26A33, 33E12, 44A10
Physical and Astronomy Classification Subject (PACS):
Key Words and Phrases: Viscoelasticity, Rheology, Fractional derivatives, Mittag-Leffler function, Creep compliance, Relaxation modulus, Effective viscosity, Complex modulus, Hooke, Newton, Kelvin-Voigt, Maxwell, Zener, Anti-Zener, Burgers.
1 Introduction

A topic of continuum mechanics, where fractional calculus is suited to be applied, is without doubt the linear theory of viscoelasticity. In fact, an increasing number of authors have used fractional calculus as an empirical method of describing the properties of linear viscoelastic materials. A wide bibliography up to nowadays is contained in the recent book by Mainardi [27] including an historical perspective up to 1980’s.

The purpose of this paper is to provide, after a general survey to the linear theory of viscoelasticity, a (more) systematic discussion and a graphical representation of the main properties of the basic models described by stress–strain relationships of fractional order. The properties under discussion concern the standard creep and relaxation tests that have a relevance in experiments.

The plan of paper is as following. In Section 2 we recall the essential notions of linear viscoelasticity in order to present our notations for the analog mechanical models. We limit our attention to the basic mechanical models, characterized by two, three and four parameters, that we refer to as Kelvin–Voigt, Maxwell, Zener, anti–Zener and Burgers.

In Section 3 we consider our main topic concerning the creep, relaxation and viscosity properties of the previous basic models generalized by replacing in their differential constitutive equations the derivatives of integer order 1 and 2 with derivatives of fractional order $\nu$ and $1+\nu$ respectively, with $0 < \nu \leq 1$.

We provide the analytical expressions and the plots of the creep compliance, relaxation modulus and effective viscosity for all the considered fractional models.

We also enclose two Appendices for providing the readers with the essential notions of fractional derivative and with a discussion on initial conditions.

2 Essentials of linear viscoelasticity

In this section we present the fundamentals of linear viscoelasticity restricting our attention to the one–axial case and assuming that the viscoelastic body is quiescent for all times prior to some starting instant that we assume as $t = 0$. 
For the sake of convenience both stress $\sigma(t)$ and strain $\epsilon(t)$ are intended to be normalized, i.e. scaled with respect to a suitable reference state $\{\sigma_0, \epsilon_0\}$.

### 2.1 Generalities

According to the linear theory, the viscoelastic body can be considered as a linear system with the stress (or strain) as the excitation function (input) and the strain (or stress) as the response function (output). In this respect, the response functions to an excitation expressed by the Heaviside step function $\Theta(t)$ are known to play a fundamental role both from a mathematical and physical point of view. We denote by $J(t)$ the strain response to the unit step of stress, according to the creep test and by $G(t)$ the stress response to a unit step of strain, according to the relaxation test.

The functions $J(t)$ and $G(t)$ are usually referred to as the creep compliance and relaxation modulus respectively, or, simply, the material functions of the viscoelastic body. In view of the causality requirement, both functions are vanishing for $t < 0$.

The limiting values of the material functions for $t \to 0^+$ and $t \to +\infty$ are related to the instantaneous (or glass) and equilibrium behaviours of the viscoelastic body, respectively. As a consequence, it is usual to denote $J_g := J(0^+)$ the glass compliance, $J_e := J(+\infty)$ the equilibrium compliance, and $G_g := G(0^+)$ the glass modulus $G_e := G(+\infty)$ the equilibrium modulus. As a matter of fact, both the material functions are non-negative. Furthermore, for $0 < t < +\infty$, $J(t)$ is a non decreasing function and $G(t)$ is a non increasing function.

The monotonicity properties of $J(t)$ and $G(t)$ are related respectively to the physical phenomena of strain creep and stress relaxation. We also note that in some cases the material functions can contain terms represented by generalized functions (distributions) in the sense of Gel’fand–Shilov [12] or pseudo–functions in the sense of Doetsch [10].

Under the hypotheses of causal histories, we get the stress–strain relationships

\[
\begin{align*}
\epsilon(t) &= \int_{0^-}^{t} J(t - \tau) d\sigma(\tau) = \sigma(0^+) \cdot J(t) + \int_{0}^{t} J(t - \tau) \frac{d}{d\tau} \sigma(\tau) \, d\tau, \\
\sigma(t) &= \int_{0^-}^{t} G(t - \tau) d\epsilon(\tau) = \epsilon(0^+) \cdot G(t) + \int_{0}^{t} G(t - \tau) \frac{d}{d\tau} \epsilon(\tau) \, d\tau, \\
\end{align*}
\] (2.1)
where the passage to the RHS is justified if differentiability is assumed for the stress–strain histories, see also the excellent book by Pipkin [33]. Being of convolution type, equations (2.1) can be conveniently treated by the technique of Laplace transforms so they read in the Laplace domain

\[
\tilde{\varepsilon}(s) = s \tilde{J}(s) \tilde{\sigma}(s), \quad \tilde{\sigma}(s) = s \tilde{G}(s) \tilde{\varepsilon}(s),
\]

from which we derive the reciprocity relation

\[
s \tilde{J}(s) = \frac{1}{s \tilde{G}(s)}.
\]

Because of the limiting theorems for the Laplace transform, we deduce that \( J_g = 1/G_g, \) \( J_e = 1/G_e, \) with the convention that 0 and \(+\infty\) are reciprocal to each other. The above remarkable relations allow us to classify the viscoelastic bodies according to their instantaneous and equilibrium responses in four types as stated by Caputo and Mainardi in their 1971 review paper [8], see Table 2.1.

| Type | \( J_g \) | \( J_e \) | \( G_g \) | \( G_e \) |
|------|--------|--------|--------|--------|
| I    | > 0    | < \infty | < \infty | > 0    |
| II   | > 0    | = \infty | < \infty | = 0    |
| III  | = 0    | < \infty | = \infty | > 0    |
| IV   | = 0    | = \infty | = \infty | = 0    |

Table 2.1 The four types of viscoelasticity.

We note that the viscoelastic bodies of type I exhibit both instantaneous and equilibrium elasticity, so their behaviour appears close to the purely elastic one for sufficiently short and long times. The bodies of type II and IV exhibit a complete stress relaxation (at constant strain) since \( G_e = 0 \) and an infinite strain creep (at constant stress) since \( J_e = \infty \), so they do not present equilibrium elasticity. Finally, the bodies of type III and IV do not present instantaneous elasticity since \( J_g = 0 \) \((G_g = \infty)\). Other properties will be pointed out later on.

### 2.2 The mechanical models

To get some feeling for linear viscoelastic behaviour, it is useful to consider the simpler behaviour of analog mechanical models. They are constructed
from linear springs and dashpots, disposed singly and in branches of two (in series or in parallel). As analog of stress and strain, we use the total extending force and the total extension. We note that when two elements are combined in series [in parallel], their compliances [moduli] are additive. This can be stated as a combination rule: creep compliances add in series, while relaxation moduli add in parallel.

The mechanical models play an important role in the literature which is justified by the historical development. In fact, the early theories were established with the aid of these models, which are still helpful to visualize properties and laws of the general theory, using the combination rule.

Now, it is worthwhile to consider the simple models of Figs. 1, 2, 3 providing their governing stress–strain relations along with the related material functions.

**The Hooke and the Newton models.** The spring (Fig. 1a) is the elastic (or storage) element, as for it the force is proportional to the extension; it represents a perfect elastic body obeying the Hooke law (ideal solid). This model is thus referred to as the *Hooke* model. If we denote by \( m \) the pertinent elastic modulus we have

\[
\text{Hooke model} : \quad \sigma(t) = m \epsilon(t), \quad \begin{cases} J(t) &= 1/m, \\ G(t) &= m. \end{cases}
\] (2.4)

In this case we have no creep and no relaxation so the creep compliance and the relaxation modulus are constant functions: \( J(t) \equiv J_g \equiv J_e = 1/m; G(t) \equiv G_g \equiv G_e = m. \)

The dashpot (Fig. 1b) is the viscous (or dissipative) element, the force being proportional to rate of extension; it represents a perfectly viscous body obeying the Newton law (perfect liquid). This model is thus referred to as the *Newton* model. If we denote by \( b_1 \) the pertinent viscosity coefficient, we have

\[
\text{Newton model} : \quad \sigma(t) = b_1 \frac{d \epsilon}{dt}, \quad \begin{cases} J(t) &= t/b_1, \\ G(t) &= b_1 \delta(t). \end{cases}
\] (2.5)

In this case we have a linear creep \( J(t) = J_+ t \) and instantaneous relaxation \( G(t) = G_+ \delta(t) \) with \( G_+ = 1/J_+ = b_1. \)

We note that the *Hooke* and *Newton* models represent the limiting cases of viscoelastic bodies of type I and IV, respectively.
The Kelvin–Voigt and the Maxwell models. A branch constituted by a spring in parallel with a dashpot is known as the Kelvin–Voigt model (Fig. 1c). We have

\[ \text{Kelvin–Voigt model} : \sigma(t) = m \epsilon(t) + b_1 \frac{d\epsilon}{dt}, \] 

(2.6a)

and

\[
\begin{align*}
J(t) &= J_1 \left[ 1 - e^{-t/\tau_\epsilon} \right], \quad J_1 = \frac{1}{m}, \quad \tau_\epsilon = \frac{b_1}{m}, \\
G(t) &= G_e + G_- \delta(t), \quad G_e = m, \quad G_- = b_1,
\end{align*}
\]

(2.6b)

where \( \tau_\epsilon \) is referred to as the retardation time.

A branch constituted by a spring in series with a dashpot is known as the Maxwell model (Fig. 1d). We have

\[ \text{Maxwell model} : \sigma(t) + a_1 \frac{d\sigma}{dt} = b_1 \frac{d\epsilon}{dt}, \] 

(2.7a)

and

\[
\begin{align*}
J(t) &= J_g + J_+ t, \quad J_g = \frac{a_1}{b_1}, \quad J_+ = \frac{1}{b_1}, \\
G(t) &= G_1 e^{-t/\tau_\sigma}, \quad G_1 = \frac{b_1}{a_1}, \quad \tau_\sigma = a_1,
\end{align*}
\]

(2.7b)

where \( \tau_\sigma \) is referred to as the relaxation time.

The Voigt and the Maxwell models are thus the simplest viscoelastic bodies of type III and II, respectively. The Voigt model exhibits an exponential (reversible) strain creep but no stress relaxation; it is also referred to as the retardation element. The Maxwell model exhibits an exponential (reversible) stress relaxation and a linear (non reversible) strain creep; it is also referred to as the relaxation element.

The Zener and the anti–Zener models. Based on the combination rule, we can continue the previous procedure in order to construct the simplest models of type I and IV that require three parameters.

The simplest viscoelastic body of type I is obtained by adding a spring either in series to a Voigt model or in parallel to a Maxwell model (Fig. 2a and
Figure 1: The representations of the basic mechanical models: a) spring for Hooke, b) dashpot for Newton, c) spring and dashpot in parallel for Voigt, d) spring and dashpot in series for Maxwell.

So doing, according to the combination rule, we add a positive constant both to the Voigt–like creep compliance and to the Maxwell–like relaxation modulus so that we obtain $J_g > 0$ and $G_e > 0$. Such a model was introduced by Zener [42] with the denomination of Standard Linear Solid (S.L.S.). We have

\[
J(t) = J_g + J_1 \left[ 1 - e^{-t/\tau_e} \right], \quad J_g = \frac{a_1}{b_1}, \quad J_1 = \frac{1}{m} - \frac{a_1}{b_1}, \quad \tau_e = \frac{b_1}{m},
\]

\[
G(t) = G_e + G_1 e^{-t/\tau_\sigma}, \quad G_e = m, \quad G_1 = \frac{b_1}{a_1} - m, \quad \tau_\sigma = a_1.
\]

We point out the condition $0 < m < b_1/a_1$ in order $J_1, G_1$ be positive and hence $0 < J_g < J_e < \infty$ and $0 < G_e < G_g < \infty$. As a consequence, we note
that, for the S.L.S. model, the retardation time must be greater than the relaxation time, \( 0 < \tau_\sigma < \tau_\epsilon < \infty \).

\[ \begin{array}{cccc}
\text{a)} & \text{b)} & \text{c)} & \text{d)} \\
\end{array} \]

Figure 2: The mechanical representations of the Zener model, see a), b) and of the anti–Zener model, see c), d), where: a) spring in series with Voigt, b) spring in parallel with Maxwell, c) dashpot in series with Voigt, d) dashpot in parallel with Maxwell.

Also the simplest viscoelastic body of type IV requires three parameters, \( i.e. \ a_1 , b_1 , b_2 \); it is obtained adding a dashpot either in series to a Voigt model or in parallel to a Maxwell model (Fig. 2c and Fig 2d, respectively). According to the combination rule, we add a linear term to the Voigt–like creep compliance and a delta impulsive term to the Maxwell–like relaxation modulus so that we obtain \( J_\epsilon = \infty \) and \( G_g = \infty \). We may refer to this model as the anti–Zener model. We have

\[ \text{anti–Zener model : } \left[ 1 + a_1 \frac{d}{dt} \right] \sigma(t) = \left[ b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} \right] \epsilon(t), \quad (2.9a) \]

and

\[ \begin{cases} 
J(t) = J_+ t + J_1 \left[ 1 - e^{-t/\tau_\epsilon} \right], & J_+ = \frac{1}{b_1}, \quad J_1 = \frac{a_1}{b_1} - \frac{b_2}{b_1^2}, \quad \tau_\epsilon = \frac{b_2}{b_1}, \\
G(t) = G_- \delta(t) + G_1 e^{-t/\tau_\sigma}, & G_- = \frac{b_2}{a_1}, \quad G_1 = \frac{a_1}{a_1} - \frac{b_2}{a_1^2}, \quad \tau_\sigma = a_1. 
\end{cases} \quad (2.9b) \]
We point out the condition $0 < b_2/b_1 < a_1$ in order $J_1, G_1$ be positive. As a consequence, we note that, for the anti–Zener model, the relaxation time must be greater than the retardation time, i.e. $0 < \tau_e < \tau_\sigma < \infty$, on the contrary of the Zener (S.L.S.) model.

In Fig. 2 we exhibit the mechanical representations of the Zener model (2.8a)-(2.8b), see a), b), and of the anti–Zener model (2.9a)-(2.9b), see c), d).

**Remark:** We note that the constitutive equation of the anti–Zener model is formally obtained from that of the Zener model by replacing the strain $\epsilon(t)$ by the strain–rate $\dot{\epsilon}(t)$. However the Zener model, introduced by Zener in 1948 [42] for anelastic metals, was formerly introduced by Jeffreys with respect to bodily imperfection of elasticity in tidal friction, since from the first 1924 edition of his treatise on the Earth [17, 18], Sir Harold Jeffreys was usual to refer to the rheology of the Kelvin–Voigt model (that was suggested to him by Sir J. Larmor) to as firmoviscosity and to the rheology of the Maxwell model to as elastoviscosity. We observe that in the literature of rheology of viscoelastic fluids (including polymeric liquids) our anti–Zener model is (surprisingly for us) known as Jeffreys fluid, see e.g. the review paper by Bird and Wiest [2]. Presumably this is due to the replacement of the strain with the strain–rate (suitable for fluids) in the stress–strain relationship introduced by Jeffreys. As a matter of fact, in Earth rheology the Jeffreys model is known to be the creep model introduced by him in 1958, see [19], as generalization of the Lomnitz logarithmic creep law and well described in the subsequent editions of Jeffreys’ treatise on the Earth.

In view of above considerations, we are tempted to call our anti–Zener model as Standard Linear Fluid in analogy with the terminology Standard Linear Solid commonly adopted for the Zener model.

**The Burgers model.** In Rheology literature it is customary to consider the so–called Burgers model, which is obtained by adding a dashpot or a spring to the representations of the Zener or of the anti–Zener model, respectively. Assuming the creep representation the dashpot or the spring is added in series, so the Burgers model results in a series combination of a Maxwell element with a Voigt element. Assuming the relaxation representation, the dashpot or the spring is added in parallel, so the Burgers model results in two Maxwell elements disposed in parallel. We refer the reader to Fig. 3 for the two mechanical representations of the Burgers model.
According to our general classification, the Burgers model is thus a four–element model of type II, defined by the four parameters \( \{a_1, a_2, b_1, b_2\} \).

We have

\[
\text{Burgers model : } \left[ 1 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} \right] \sigma(t) = \left[ b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} \right] \epsilon(t), \tag{2.10a}
\]

so

\[
\begin{align*}
J(t) &= J_g + J_+ t + J_1 \left( 1 - e^{-t/\tau_e} \right), \\
G(t) &= G_1 e^{-t/\tau_{\sigma,1}} + G_2 e^{-t/\tau_{\sigma,2}}.
\end{align*} \tag{2.10b}
\]

We leave to the reader to express as an exercise the physical quantities \( J_g, J_+, \tau_e \) and \( G_1, \tau_{\sigma,1}, G_2, \tau_{\sigma,2} \), in terms of the four parameters \( \{a_1, a_2, b_1, b_2\} \) in the operator equation (2.10a).

**The operator equation for the mechanical models** Based on the combination rule, we can construct models whose material functions are of the following type

\[
\begin{align*}
J(t) &= J_g + \sum_n J_n \left[ 1 - e^{-t/\tau_{\epsilon,n}} \right] + J_+ t, \\
G(t) &= G_1 e^{-t/\tau_{\sigma,1}} + G_2 e^{-t/\tau_{\sigma,2}}.
\end{align*} \tag{2.11}
\]

where all the coefficient are non–negative, and interrelated because of the *reciprocity relation* (2.3) in the Laplace domain. We note that the four types of viscoelasticity of Table 2.1 are obtained from Eqs. (2.11) by taking into account that

\[
\begin{align*}
J_e < \infty \iff J_+ = 0, & \quad J_e = \infty \iff J_+ \neq 0, \\
G_g < \infty \iff G_- = 0, & \quad G_g = \infty \iff G_- \neq 0.
\end{align*} \tag{2.12}
\]

Appealing to the theory of Laplace transforms, we write

\[
\begin{align*}
s\tilde{J}(s) &= J_g + \sum_n \frac{J_n}{1 + s \tau_{\epsilon,n}} + \frac{J_+}{s}, \\
s\tilde{G}(s) &= (G_e + \beta) - \sum_n \frac{G_n}{1 + s \tau_{\sigma,n}} + G_- s.
\end{align*} \tag{2.13}
\]
Figure 3: The mechanical representations of the Burgers model: the creep representation (top) and the relaxation representation (bottom).

where we have put $\beta = \sum_n G_n$.

Furthermore, as a consequence of (2.13), $s\tilde{J}(s)$ and $s\tilde{G}(s)$ turn out to be rational functions in $\mathbb{C}$ with simple poles and zeros on the negative real axis and, possibly, with a simple pole or with a simple zero at $s = 0$, respectively.

In these cases the integral constitutive equations (2.1) can be written in differential form. Following Bland [3] with our notations, we obtain for these models

$$
\left[1 + \sum_{k=1}^{p} a_k \frac{d^k}{dt^k}\right] \sigma(t) = \left[m + \sum_{k=1}^{q} b_k \frac{d^k}{dt^k}\right] \epsilon(t),
$$

where $q$ and $p$ are integers with $q = p$ or $q = p + 1$ and $m, a_k, b_k$ are non-negative constants, subjected to proper restrictions in order to meet the
physical requirements of realizability. The general Eq. (2.14) is referred to as the operator equation of the mechanical models.

In the Laplace domain, we thus get

\[ s \tilde{J}(s) = \frac{1}{s \tilde{G}(s)} = \frac{P(s)}{Q(s)}, \]

where

\[
\begin{align*}
P(s) &= 1 + \sum_{k=1}^{p} a_k s^k, \\
Q(s) &= m + \sum_{k=1}^{q} b_k s^k.
\end{align*}
\]

(2.15)

with \( m \geq 0 \) and \( q = p \) or \( q = p + 1 \). The polynomials at the numerator and denominator turn out to be Hurwitz polynomials (since they have no zeros for \( \text{Re} \{s\} > 0 \)) whose zeros are alternating on the negative real axis (\( s \leq 0 \)). The least zero in absolute magnitude is a zero of \( Q(s) \). The four types of viscoelasticity then correspond to whether the least zero is \( (J_+ \neq 0) \) or is not \( (J_+ = 0) \) equal to zero and to whether the greatest zero in absolute magnitude is a zero of \( P(s) \) \( (J_g \neq 0) \) or a zero of \( Q(s) \) \( (J_g = 0) \).

In Table 2.2 we summarize the four cases, which are expected to occur in the operator equation (2.14), corresponding to the four types of viscoelasticity.

| Type | Order | \( m \) | \( J_g \) | \( G_e \) | \( J_+ \) | \( G_- \) |
|------|-------|--------|--------|--------|--------|--------|
| I    | \( q = p \) | \( > 0 \) | \( a_p/b_p \) | \( m \) | \( 0 \) | \( 0 \) |
| II   | \( q = p \) | \( = 0 \) | \( a_p/b_p \) | \( 0 \) | \( 1/b_1 \) | \( 0 \) |
| III  | \( q = p + 1 \) | \( > 0 \) | \( 0 \) | \( m \) | \( 0 \) | \( b_q/a_p \) |
| IV   | \( q = p + 1 \) | \( = 0 \) | \( 0 \) | \( 0 \) | \( 1/b_1 \) | \( b_q/a_p \) |

Table 2.2: The four cases of the operator equation.

We recognize that for \( p = 1 \), Eq. (2.14) includes the operator equations for the classical models with two parameters: Voigt and Maxwell, illustrated in Fig. 1, and with three parameters: Zener and anti-Zener, illustrated in Fig. 2. In fact we recover the Voigt model (type III) for \( m > 0 \) and \( p = 0, q = 1 \), the Maxwell model (type II) for \( m = 0 \) and \( p = q = 1 \), the Zener model (type I) for \( m > 0 \) and \( p = q = 1 \), and the anti-Zener model (type IV) for \( m = 0 \) and \( p = 1, q = 2 \). Finally, with four parameters we can construct two models, the former with \( m = 0 \) and \( p = q = 2 \), the latter with \( m > 0 \) and \( p = 1, q = 2 \), referred by Bland [3] to as four–element models of the first kind and of the second kind, respectively. According to our convention they
are of type II and III, respectively. We have restricted our attention to the former model, the Burgers model (type II), illustrated in Fig. 3, because it has found numerous applications, specially in geosciences, see e.g. the books by Klausner [21] and by Carcione [9].

2.3 Complex modulus, effective modulus and effective viscosity

In Earth rheology and seismology it is customary to write the one dimensional stress–strain relation in the Laplace domain in terms of a complex shear modulus \( \mu(s) \) as

\[
\tilde{\sigma}(s) = 2\tilde{\mu}(s)\tilde{\epsilon}(s),
\]

that is expected to generalize the relation for a perfect elastic solid (Hooke model)

\[
\sigma(t) = 2\mu_0 \epsilon(t),
\]

where \( \mu_0 \) denotes the shear modulus.

Adopting this notation we note comparing (2.16) with (2.2) that the functions \( \tilde{J}(s) \) and \( \tilde{G}(s) \) can be expressed in terms of the complex shear modulus \( \tilde{\mu}(s) \) as

\[
\tilde{J}(s) = \frac{1}{2s\tilde{\mu}(s)}, \quad \tilde{G}(s) = \frac{2\tilde{\mu}(s)}{s}.
\]

As a consequence we are led to introduce an effective modulus defined as

\[
\mu(t) := \frac{1}{2} \left[ \frac{d}{dt} G(t) + G_g \right],
\]

Recalling that for perfect viscous fluid (Newton model) we have

\[
\sigma(t) = 2\eta_0 \frac{d}{dt} \epsilon(t),
\]

where \( \eta_0 \) denotes the viscosity coefficient, similarly we are led to define, following Müller [31] an effective viscosity as

\[
\eta(t) := \frac{1}{2\tilde{J}(t)},
\]

where the dot denotes the derivative w.r.t. time \( t \).
We easily recognize that for the Hooke model (2.4) we recover \( \mu(t) \equiv \mu_0 = m/2 \), and for the Newton model (2.5) \( \eta(t) \equiv \eta_0 = b_1/2 \). Furthermore we recognize that for a spring (Hooke model) the corresponding complex modulus is a constant, that is
\[
\hat{\mu}_H(s) = \mu_0 ,
\]
whereas for a dashpot (Newton model)
\[
\hat{\mu}_N(s) = \eta_0 s = \mu_0 s \tau_0 ,
\]
where \( \tau_0 = \eta_0 / \mu_0 \) denotes a characteristic time related to viscosity.

In order to avoid possible misunderstanding, we explicitly note that the complex modulus is not the Laplace transform of the effective modulus but its Laplace transform multiplied by \( s \).

The appropriate form of the complex modulus \( \hat{\mu}(s) \) can be obtained recalling the combination rule for which creep compliances add in series while relaxation moduli add in parallel, as stated at the beginning of subsection 2.2. Accordingly, for a serial combination of two viscoelastic models with individual complex moduli \( \hat{\mu}_1(s) \) and \( \hat{\mu}_2(s) \), we have
\[
\frac{1}{\hat{\mu}(s)} = \frac{1}{\hat{\mu}_1(s)} + \frac{1}{\hat{\mu}_2(s)} ,
\]
whereas for a combination in parallel
\[
\hat{\mu}(s) = \hat{\mu}_1(s) + \hat{\mu}_2(s) .
\]

We close this section with a discussion about the definition of solid–like and fluid–like behaviour for viscoelastic materials. The matter is of course subjected to personal opinions.

Generally one may define a fluid if it can creep indefinitely under constant stress \( (\bar{J}_e = \infty) \), namely when it relaxes to zero under constant deformation \( (G_e = 0) \). According to this view, viscoelastic models of type II and IV are fluid–like whereas models of type I and III are solid–like. However, in his interesting book \[33\], Pipkin calls a solid if the integral of \( G(t) \) from zero to infinity diverges. This includes those cases in which the equilibrium modulus \( G_e \) is not zero. It also includes cases in which \( G_e = 0 \) but the approach to the limit is not fast enough for integrability, for example \( G(t) \approx t^{-\alpha} \) with \( 0 < \alpha \leq 1 \) as \( t \to \infty \).
3 Fractional viscoelasticity

The straightforward way to introduce fractional derivatives in linear viscoelasticity is to replace in the constitutive equation (2.5) of the Newton model the first derivative with a fractional derivative of order $\nu \in (0, 1)$, that, being $\epsilon(0^+) = 0$, may be intended both in the Riemann–Liouville or Caputo sense. We refer the reader to Appendix A and Appendix B for the essential notions of fractional calculus for causal systems and for a discussion on initial conditions.

Some people call the fractional model of the Newtonian dashpot (fractional dashpot) with the suggestive name pot: we prefer to refer such model to as Scott–Blair model, to honour to the scientist who already in the middle of the past century proposed such a constitutive equation to explain a material property that is intermediate between the elastic modulus (Hooke solid) and the coefficient of viscosity (Newton fluid), see e.g. [35]. Scott–Blair was surely a pioneer of the fractional calculus even if he did not provide a mathematical theory accepted by mathematicians of his time, as pointed out in [28].

It is known that the creep and relaxation power laws of the Scott–Blair model can be interpreted in terms of a continuous spectrum of retardation and relaxation times, respectively, see e.g. [27], p. 58. Starting from these continuous spectra, in [32] Papoulia et al. have interpreted the fractional dashpot by an infinite combination of Kelvin–Voigt or Maxwell elements in series or in parallel, respectively. We also note that Liu and Xu [24] have computed the relaxation and creep functions for higher–order fractional rheological models involving more parameters than in our analysis.

3.1 Fractional derivatives in mechanical models

The use of fractional calculus in linear viscoelasticity leads to generalizations of the classical mechanical models: the basic Newton element is substituted by the more general Scott–Blair element (of order $\nu$). In fact, we can construct the class of these generalized models from Hooke and Scott–Blair elements, disposed singly and in branches of two (in series or in parallel). Then, extending the procedures of the classical mechanical models (based on springs and dashpots), we will get the fractional operator equation (that is an operator equation with fractional derivatives) in the form which properly
generalizes (2.14), i.e.

\[
\left[ 1 + \sum_{k=1}^{p} a_k \frac{d^{\nu_k}}{dt^{\nu_k}} \right] \sigma(t) = \left[ m + \sum_{k=1}^{q} b_k \frac{d^{\nu_k}}{dt^{\nu_k}} \right] \epsilon(t), \quad \nu_k = k + \nu - 1. \tag{3.1}
\]

so, as a generalization of (2.11),

\[
\begin{align*}
J(t) &= J_g + \sum_n J_n \{ 1 - E_\nu [-(t/\tau,c_n)^\nu] \} + J_+ \frac{t^\nu}{\Gamma(1+\nu)}, \\
G(t) &= G_e + \sum_n G_n E_\nu [-(t/\tau_\sigma,n)^\nu] + G_- \frac{t^{-\nu}}{\Gamma(1-\nu)},
\end{align*}
\tag{3.2}
\]

where all the coefficient are non–negative. Here \( \Gamma \) denotes the well known Gamma function, and \( E_\nu \) denotes the Mittag–Leffler function of order \( \nu \) discussed hereafter along with its generalized form in two parameters \( E_{\nu,\mu} \). Of course, for the fractional operator equation (3.1) the four cases summarized in Table 2.2 are expected to occur in analogy with the operator equation (2.14). The definitions in the complex plane of the Mittag–Leffler functions in one and two parameters are provided by their Taylor powers series around \( z = 0 \), that is

\[
E_\nu(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}, \quad E_{\nu,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + \mu)}, \quad \nu, \mu > 0, \tag{3.3}
\]

related between them by the following expressions, see e.g. [27], Appendix E,

\[
E_\nu(z) = E_{\nu,1}(z) = 1 + z E_{\nu,1+\nu}(z), \quad \frac{d}{dz} E_\nu(z^\nu) = z^{\nu-1} E_{\nu,\nu}(z^\nu). \tag{3.4}
\]

In our case the Mittag–Leffler functions appearing in (3.2) are all of order \( \nu \in (0.1] \) and argument real and negative, namely, they are of the type

\[
E_\nu[-(t/\tau)^\nu] = \sum_{n=0}^{\infty} (-1)^n \frac{(t/\tau)^{\nu n}}{\Gamma(\nu n + 1)}, \quad 0 < \nu < 1, \quad \tau > 0, \tag{3.5}
\]

that for \( \nu = 1 \) reduce to \( \exp(-t/\tau) \).
Let us now outline some noteworthy properties of the Mittag–Leffler function (3.5) assuming for brevity $\tau = 1$. Since the asymptotic behaviours for small and large times are as following

\[
E_\nu(-t^\nu) \sim \begin{cases} 
1 - \frac{t^\nu}{\Gamma(1+\nu)}, & t \to 0^+, \\
\frac{t^{-\nu}}{\Gamma(1-\nu)}, & t \to +\infty,
\end{cases}
\]

we recognize that for $t \geq 0$ the Mittag–Leffler function decays for short times like a stretched exponential and for large times with a negative power law. Furthermore, it turns out to be completely monotonic in $0 < t < \infty$ (that is its derivatives of successive order exhibit alternating signs like $e^{-t}$), so it can be expressed in term of a continuous distribution of elementary relaxation processes:

\[
E_\nu(-t^\nu) = \int_0^\infty e^{-rt} K_\nu(r) \, dr, \quad K_\nu(r) = \frac{1}{\pi} \frac{\sin(\nu\pi)}{r^\nu + 2\cos(\nu\pi) + r^{-\nu}} \geq 0. \quad (3.7)
\]

For completeness we also recall its relations with the corresponding Mittag–Leffler function of two parameters as obtained from (3.4):

\[
E_\nu(-t^\nu) = E_{\nu,1}(-t^\nu) = 1 - t^\nu E_{\nu,1+\nu}(-t^\nu), \quad \frac{d}{dt} E_\nu(-t^\nu) = -t^{\nu-1} E_{\nu,\nu}(-t^\nu). \quad (3.8)
\]

We recognize that in fractional viscoelasticity governed by the operator equation (3.1) the corresponding material functions are obtained by using the combination rule valid for the classical mechanical models. Their determination is made easy if we take into account the following correspondence principle between the classical and fractional mechanical models, outlined in 1971 by Caputo and Mainardi [8]. Taking $0 < \nu \leq 1$ and denoting by $\tau > 0$ a characteristic time related to viscosity, such correspondence principle can be formally stated by the following three equations where the Laplace transform pairs are outlined as well:

\[
\delta(t/\tau) \div \tau \Rightarrow \frac{(t/\tau)^{-\nu}}{\Gamma(1-\nu)} \div \frac{1}{s} \frac{1}{(s\tau)^\nu}, \quad (3.9)
\]

\[
t/\tau \div \frac{1}{s} \frac{1}{(s\tau)} \Rightarrow \frac{(t/\tau)^\nu}{\Gamma(1+\nu)} \div \frac{1}{s} \frac{1}{(s\tau)^\nu}, \quad (3.10)
\]
\[
e^{-t/\tau} \div \frac{\tau}{1 + s\tau} \Rightarrow E_{\nu}[-(t/\tau)^{\nu}] \div \frac{1}{s} \frac{(s\tau)^{\nu}}{1 + (s\tau)^{\nu}}.
\]

(3.11)

In the following, we will provide the creep compliance \( J(t) \), the relaxation modulus \( G(t) \) and the effective viscosity \( \eta(t) \) for a set of fractional models which properly generalize with fractional derivatives the basic mechanical models discussed in Section 2, that is Kelvin–Voigt, Maxwell, Zener, anti–Zener, and Burgers, by using their connections with Hooke and Scott–Blair elements. Henceforth, for brevity we refer to all the basic models with their first letters, that is \( H, SB, KV, M, Z, AZ \) and \( B \). Our analysis will be carried out by using the Laplace transform and the complex shear modulus of the elementary Hooke and Scott–Blair models. For this purpose we recall the constitutive equations for the \( H \) and \( SB \) models

\[
\text{Hooke model : } \sigma(t) = m \epsilon(t), \quad \text{Scott – Blair model : } \sigma(t) = b_1 \frac{d^{\nu} \epsilon(t)}{dt^{\nu}},
\]

(3.12)

and, setting

\[
\mu := \frac{m}{2}, \quad \tau^{\nu} := \frac{b_1}{2\mu},
\]

(3.13)

we get the complex shear moduli for these elements

\[
\hat{\mu}_H(s) = \mu, \quad \hat{\mu}_{SB}(s) = \mu(s\tau)^{\nu},
\]

(3.14)

where \( \nu \in (0, 1] \) and \( \tau > 0 \) is a characteristic time of the \( SB \) element. So, whereas the \( H \) element is characterized by a unique parameter, its elastic modulus \( \mu \), the \( SB \) element is characterized by a triplet of parameters, that is \( \{\mu, \tau, \nu\} \).

### 3.2 Fractional Kelvin–Voigt model

The constitutive equation for the fractional Kelvin–Voigt model (referred to as \( KV \) body) is obtained from (2.6a) in the form

\[
\text{fractional Kelvin – Voigt model : } \sigma(t) = m \epsilon(t) + b_1 \frac{d^{\nu} \epsilon}{dt^{\nu}}.
\]

(3.15)

The mechanical analogue of the \( KV \) body is represented by a Hooke (\( H \)) element in parallel with a Scott–Blair (\( SB \)) element. The parallel
Figure 4: Normalized creep compliance (a), relaxation modulus (b) and effective viscosity (c) for the KV body, for some values of the fractional power $\nu$ in the range $(0 < \nu \leq 1)$, as a function of normalized time $\xi$. The thick lines ($\nu = 1$) represent material functions and effective viscosity for the traditional Kelvin–Voigt body.

The combination rule (2.25) provides the complex modulus

$$\hat{\mu}_{KV}(s) = \mu \left[ 1 + (s\tau)^\nu \right], \quad \mu := \frac{m}{2}, \quad \tau^\nu := \frac{b_1}{2\mu},$$

(3.16)
where our time constant $\tau$ reduces for $\nu = 1$ to the retardation time $\tau_e$ of the classical $KV$ body, see (2.6b). Hence, substitution of (3.16) in Eqs. (2.18) gives

$$\tilde{J}_{KV}(s) = \frac{1}{2\mu s} \left[ 1 - \frac{(s\tau)^\nu}{1 + (s\tau)^\nu} \right], \quad \tilde{G}_{KV}(s) = \frac{2\mu}{s} \left[ 1 + (s\tau)^\nu \right].$$  \hspace{1cm} (3.17)

By inverting the above Laplace transforms according to Eqs. (3.11) and (3.9), we get for $t \geq 0$,

$$J_{KV}(t) = \frac{1}{2\mu} \left[ 1 - E_\nu(-(t/\tau)^\nu) \right], \quad G_{KV}(t) = 2\mu \left[ 1 + \frac{(t/\tau)^{-\nu}}{\Gamma(1 - \nu)} \right].$$  \hspace{1cm} (3.18)

For plotting purposes, it is convenient to introduce a non–dimensional time $\xi := t/\tau$ and define normalized, non–dimensional material functions. With $J'_{KV}(\xi) = \mu J_{KV}(t)|_{t = \tau\xi}$ and $G'_{KV}(\xi) = (1/\mu)G_{KV}(t)|_{t = \tau\xi}$ these can be written in the non–dimensional form

$$J'_{KV}(\xi) = \frac{1}{2} \left[ 1 - E_\nu(-\xi^\nu) \right], \quad G'_{KV}(\xi) = 2 \left[ 1 + \frac{\xi^{-\nu}}{\Gamma(1 - \nu)} \right], \quad \xi = \frac{t}{\tau}. \hspace{1cm} (3.19)$$

Finally, by a straightforward application of Eq. (2.21), recalling the derivative rule in (3.8) for the Mittag–Leffler function, the effective viscosity of the $KV$ body turns out to be

$$\eta'_{KV}(\xi) = \frac{\xi^{1-\nu}}{E_\nu(\xi^\nu)}.$$  \hspace{1cm} (3.20)

Figure 4 shows plots of of $J'_{KV}(\xi)$, $G'_{KV}(\xi)$ and $\eta'_{KV}(\xi)$ as a function of non–dimensional time $\xi$. For $\nu \to 1$, the response of the fractional $KV$ body reduces to that of a classical $KV$ body. Taking into account that $E_1(-\xi) = e^{-\xi}$, the creep compliance reduces, in this limiting case, to $J'_{KV}(\xi) = (1 - e^{-\xi})/2$. Recalling the Dirac’s delta representation $\delta(t) = t^{-1}/\Gamma(0)$, the relaxation modulus reduces, for $\nu \to 1$, to $G'_{KV}(t) = 2[1 + \tau\delta(t)] = 2[1 + \delta(t/\tau)]$. Finally, for the effective viscosity we obtain the classical exponential law $\eta'_{KV}(\xi) = e^\xi$. We note the effective viscosity exceeds the classical value since the early stage of creep.
3.3 Fractional Maxwell model

The constitutive equation for the fractional Maxwell model (referred to as $M$ body) is obtained from (2.7a) in the form

\[ \text{fractional Maxwell model : } \sigma(t) + a_1 \frac{d^\nu \sigma}{dt^{\nu}} = b_1 \frac{d^\nu \epsilon}{dt^{\nu}}. \quad (3.21) \]

The mechanical analogue of the $M$ body is composed by a Hooke ($H$) element connected in series with a Scott–Blair ($SB$) element. From the series combination rule (2.24) we obtain the complex modulus as

\[ \tilde{\mu}_M(s) = \mu \frac{(s \tau)^\nu}{1 + (s \tau)^\nu}, \quad \mu = \frac{b_1}{2a_1}, \quad \tau^\nu := \frac{b_1}{2\mu}, \quad (3.22) \]

where now our time constant $\tau$ reduces for $\nu = 1$ to the relaxation time $\tau_\sigma$ of the classical $M$ body, see (2.7b). Hence, substitution of (3.22) into Eqs. (2.18) gives the Laplace transforms of the material functions

\[ \tilde{J}_M(s) = \frac{1}{2\mu s} \left[ 1 + \frac{1}{(s \tau)^\nu} \right], \quad \tilde{G}_M(s) = \frac{2\mu}{s} \frac{(s \tau)^\nu}{1 + (s \tau)^\nu}. \quad (3.23) \]

By inverting the Laplace transforms according to Eqs. (3.10) and (3.11), we get for $t \geq 0$,

\[ J_M(t) = \frac{1}{2\mu} \left[ 1 + \frac{(t/\tau)^\nu}{\Gamma(1 + \nu)} \right], \quad G_M(t) = 2\mu \, E_\nu \left( -(t/\tau)^\nu \right). \quad (3.24) \]

For plotting purposes, it is convenient to write non-dimensional forms of $J_M(t)$ and $G_M(t)$. These can be obtained, following the example of the fractional $KV$ body, by introducing a non-dimensional time with $\xi = t/\tau$ and defining normalized, non-dimensional material functions $J'_M(\xi) = \mu J_M(t)|_{t=\tau\xi}$ and $G'_M(\xi) = (1/\mu) G_M(t)|_{t=\tau\xi}$, which provides

\[ J'_M(\xi) = \frac{1}{2} \left[ 1 + \frac{\xi^\nu}{\Gamma(1 + \nu)} \right], \quad G'_M(\xi) = 2 \, E_\nu \left( -\xi^\nu \right), \quad \xi = \frac{t}{\tau}. \quad (3.25) \]

Following this normalization scheme, the effective viscosity (in non-dimensional form) can be readily obtained from (2.21) and (3.25) as

\[ \eta'_M(\xi) = \frac{\Gamma(1 + \nu)}{\nu} \xi^{1-\nu}, \quad (3.26) \]
Figure 5: Normalized creep compliance (a), relaxation modulus (b) and effective viscosity (c) for the M body, for some rational values of $\nu$ in the range $(0 < \nu \leq 1)$, as a function of normalized time $\xi$. The thick lines, corresponding to $\nu = 1$, show the classical Maxwell response.

Plots of $J'_M(\xi)$, $G'_M(\xi)$ and $\eta'_M(\xi)$ are shown in Figure 5. Thick lines show material functions and effective viscosity in the limit of $\nu \rightarrow 1$, i.e., when the response of the fractional $M$ body degenerates into that of a classical
In particular, from Eq. (3.25) we obtain $J'_M(\xi) = (1 + \xi)/2$ and $G'_M(\xi) = 2e^{-\xi}$ and the effective viscosity is constant ($\eta'_M(\xi) = 1$), which denotes the lack of transient effects. For $0 < \nu < 1$, the effective viscosity always increases with time and, except during the very early stages of creep ($\xi \approx 1$) it exceeds the value corresponding to the classical $M$ body ($\nu = 1$).

### 3.4 Fractional Zener model

The constitutive equation for the fractional Zener model (referred to as $Z$ body) is obtained from Eq. (2.8a) in the form

\[
\text{fractional Zener model : } \sigma(t) + a_1 \frac{d^\nu \sigma}{dt^\nu} = m \epsilon(t) + b_1 \frac{d^\nu \epsilon}{dt^\nu}. \tag{3.27}
\]

The mechanical analogue of the fractional $Z$ body is represented by a Hooke ($H$) element in series with a Kelvin–Voigt ($KV$) element. Here we indicate with $\mu_1$ the shear modulus of the $H$ body while with $\{\mu_2, \tau_2, \nu\}$ the triplet of parameters characterizing the $KV$ body.

The material functions for the fractional $Z$ body in the time domain can be derived following the same procedure outlined above for the fractional $KV$ and $M$. However, the algebraic complexity increases because of the increased number of independent rheological parameters involved. The use of the combination rule (2.24), which holds for connections in series, provides the complex modulus

\[
\frac{1}{\tilde{\mu}_Z(s)} = \frac{1}{\mu_1} + \frac{1}{\mu_2} \frac{1}{1 + (s\tau_2)^\nu}, \quad \mu_1 = \frac{b_1}{2a_1} \quad \mu_2 = \frac{m}{2} \quad \tau_2^\nu = \frac{b_1}{2\mu_2}, \tag{3.28}
\]

where now our time constant $\tau_2$ reduces for $\nu = 1$ to the retardation time $\tau_c$ of the classical $Z$ body, see (2.8b). Hence, substitution of (3.28) in Eqs. (2.18) provides the Laplace transform of the creep compliance and relaxation modulus. For the creep compliance we have

\[
\tilde{J}_Z(s) = \frac{1}{2s} \left[ \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) - \frac{1}{\mu_2} \frac{(s\tau_2)^\nu}{1 + (s\tau_2)^\nu} \right], \tag{3.29}
\]

that can be easily inverted to obtain

\[
J_Z(t) = \frac{1}{2} \left[ \frac{1}{\mu_1} + \frac{1}{\mu_2} \left( 1 - E_{\nu}(-(t/\tau_2)^\nu) \right) \right], \quad t \geq 0. \tag{3.30}
\]
Figure 6: Normalized creep compliance (a), relaxation modulus (b) and effective viscosity (c) for the Z body, for some values of the fractional power $\nu$ in the range $(0 < \nu \leq 1)$, as a function of normalized time $\xi$. Here we adopt the ratio $r_\mu = \mu_2/\mu_1 = 1$. The thick lines $(\nu = 1)$ represent material functions and effective viscosity for the traditional Zener body.

For the relaxation modulus the following expression can be obtained

$$\tilde{G}_Z(s) = 2\mu^* \left( \frac{\tau_2}{\tau_a} \right)^{\nu} \frac{1}{s^{\nu}} \frac{1}{s^{\nu} + 1/\tau_a^{\nu}},$$

(3.31)
where
\[ \mu^* = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}, \] (3.32)
and we have introduced an additional characteristic time
\[ \tau_a^\nu = \frac{1}{1 + r_\mu} \tau_2^\nu, \quad r_\mu = \frac{\mu_1}{\mu_2}. \] (3.33)
Applying a partial fraction expansion to Eq. (3.31) and using Eqs. (3.8), the relaxation modulus in the time domain can be cast in the form
\[ G_Z(t) = 2 \mu^* \left( \frac{\tau_2}{\tau_a} \right)^\nu \left[ E_\nu \left( -\left( t/\tau_a \right)^\nu \right) + \left( t/\tau_a \right)^\nu E_{\nu,\nu+1} \left( -\left( t/\tau_a \right)^\nu \right) \right], \] (3.34)
hence, after some rearrangement, we finally obtain
\[ G_Z(t) = 2 \mu^* \left[ 1 + r_\mu E_\nu \left( -(t/\tau_a)^\nu \right) \right]. \] (3.35)
We now recognize that for \( \nu = 1 \) the constant \( \tau_a \) reduces to the relaxation time \( \tau_\sigma \) (0 < \( \tau_\sigma \) < \( \tau_\epsilon \) < \( \infty \)) for the classical Z model, see (2.8b).
Writing \( J'_Z(\xi) = \mu^* J_Z(t)|_{t=\tau_2}\xi \) and \( G'_Z(\xi) = (1/\mu^*)G_Z(t)|_{t=\tau_2}\xi \) the material functions for the Zener model can be written in the non–dimensional form
\[ J'_Z(\xi) = \frac{1}{2} \left[ 1 - \frac{h_1}{h_2} E_\nu \left( -\xi^\nu \right) \right], \quad G'_Z(\xi) = 2 \left[ 1 + h_1 E_\nu \left( -h_2 \xi^\nu \right) \right], \quad \xi = \frac{t}{\tau_2}, \] (3.36)
with \( h_1 = r_\mu, \quad h_2 = 1 + r_\mu. \)
As a consequence the effective viscosity of the Z body turns out to be
\[ \eta'_Z(\xi) = \frac{h_1}{h_2} \frac{\xi^{1-\nu}}{E_{\nu,\nu} \left( -\xi^\nu \right)}. \] (3.37)

Figure 6 shows plots of \( J'_Z(\xi), \ G'_Z(\xi) \) and \( \eta'_Z(\xi) \) as a function of \( \xi \), for various values of \( \nu \) in the range 0 < \( \nu \) ≤ 1. In the limiting case of \( \nu = 1 \) (thick curves), which corresponds to the classical Zener model, the material functions and the effective viscosity can be written in terms of exponential functions, in agreement with Eqs. (2.8b) above. In particular, since \( r_\mu = 1 \), we obtain \( J'_Z(\xi) = 1/2 \left[ 1 - \frac{1}{2} e^{-\xi} \right] \) and \( G'_Z(\xi) = 2 \left[ 1 + e^{-2\xi} \right] \). Furthermore, the effective viscosity grows exponentially, since \( \eta'_Z(\xi) = 1/2 e^\xi \). We note that, for sufficiently long times (\( \xi > 0.2 \) in Figure 6), the value of \( \eta'_Z(\xi) \) for a fractional Zener model (0 < \( \nu \) < 1) always exceeds the classical Zener viscosity.

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3.5 Fractional anti–Zener model

The constitutive equation for fractional anti–Zener model (referred to as AZ body) is obtained from (2.9a) in the form

\[
\text{fractional anti–Zener model} : \left[1 + a_1 \frac{d^\nu}{dt^\nu}\right] \sigma(t) = \left[b_1 \frac{d^\nu}{dt^\nu} + b_2 \frac{d^{1+\nu}}{dt^{1+\nu}}\right] \epsilon(t). \tag{3.38}
\]

The fractional anti–Zener model results from the combination in series of a SB element (with material parameters \{\mu, \tau_1, \nu\}) and a fractional KV element (\{\mu, \tau_2, \nu\}).

Using the series combination rule (2.24) we obtain

\[
\frac{1}{\mu_{AZ}(s)} = \frac{1}{\mu(s\tau_1)^\nu} + \frac{1}{\mu} \left[1 - \frac{(s\tau_2)^\nu}{1 + (s\tau_2)^\nu}\right], \quad \frac{1}{\mu} = 2 \left(\frac{a_1}{b_1} - \frac{b_2}{b_1^2}\right), \quad \tau_1^\nu = \frac{1}{2\mu b_1}, \quad \tau_2^\nu = \frac{b_2}{b_1}, \tag{3.39}
\]

where now our time constant \tau_2 reduces for \nu = 1 to the retardation time \tau_\epsilon of the classical AZ body, see (2.9b). Using (2.18) the above equation easily provides the Laplace transform of the creep compliance

\[
\tilde{J}_{AZ}(s) = \frac{1}{2\mu s} \left[\frac{1}{(s\tau_1)^\nu} + 1 - \frac{(s\tau_2)^\nu}{1 + (s\tau_2)^\nu}\right], \tag{3.40}
\]

whose inverse is

\[
J_{AZ}(t) = \frac{1}{2\mu} \left[\frac{(t/\tau_1)^\nu}{\Gamma(1 + \nu)} + (1 - E_\nu(-(t/\tau_2)^\nu))\right], \quad t \geq 0. \tag{3.41}
\]

The derivation of the relaxation modulus is more complicated but straightforward. Starting from the general relationship (2.18) and using (3.40), by a partial fraction expansion we obtain

\[
\tilde{G}_{AZ}(s) = \frac{2\mu}{s} \left[(s\tau_0)^\nu + \left(\frac{\tau_1}{\bar{\tau}}\right)^{2\nu} \frac{(s\bar{\tau})^\nu}{1 + (s\bar{\tau})^\nu}\right], \tag{3.42}
\]

where we have defined the new time constants

\[
\bar{\tau} = (\tau_1^{\nu} + \tau_2^{\nu})^{\frac{1}{\nu}}, \quad \tau_0 = \frac{\tau_1\tau_2}{\bar{\tau}}. \tag{3.43}
\]

Hence, using (3.9) and (3.11), the inverse Laplace transform of (3.43) is

\[
G_{AZ}(t) = 2\mu \left[\frac{(t/\tau_0)^{-\nu}}{\Gamma(1 - \nu)} + \left(\frac{\tau_1}{\bar{\tau}}\right)^{2\nu} E_\nu(-(t/\bar{\tau})^\nu)\right], \quad t \geq 0. \tag{3.44}
\]
Figure 7: Normalized creep compliance (a), relaxation modulus (b) and effective viscosity (c) for the AZ body, for some values of the fractional power $\nu$ in the range ($0 < \nu \leq 1$), as a function of normalized time $\xi$. The particular case $\tau_1 = \tau_2$ is considered. The thick lines, pertaining to $\nu = 1$, show results for the traditional AZ body.

We now recognize that for $\nu = 1$ the constant $\bar{\tau}$ reduces to the relaxation time $\tau_\sigma$ ($0 < \tau_e < \tau_\sigma < \infty$) for the classical AZ model, see (2.9b).
From (3.41) and (3.44) it is easy to derive non dimensional expressions for the material functions in the time domain. The choice of the scaling for the time variable is of course arbitrary. Here we define

\[ \xi = \frac{t}{\tau^*}, \quad \tau^* = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \]  

(3.45)

With this choice, defining \( J'_{AZ}(\xi) = \mu J_{AZ}(t)|_{t=\tau^*\xi} \) and manipulating (3.41) we obtain

\[ J'_{AZ}(\xi) = \frac{1}{2} \left[ (c_2 \xi)^\nu \Gamma(1 + \nu) + (1 - E_\nu (- (c_1 \xi)^\nu)) \right], \]  

(3.46)

where

\[ c_1 = \frac{r_\tau}{1 + r_\tau}, \quad c_2 = \frac{1}{1 + r_\tau}, \quad r_\tau \equiv \frac{\tau_1}{\tau_2}. \]  

(3.47)

Similarly, from (3.44), the relaxation modulus has the non dimensional form

\[ G'_{AZ}(\xi) = 2 \left[ \frac{(e_1 \xi)^{-\nu}}{\Gamma(1 - \nu)} + e_2^2 E_\nu((-e_3 \xi)^{\nu^2}) \right], \]  

(3.48)

where \( G'_{AZ}(\xi) = (1/\mu) G_{AZ}(t)|_{t=\tau^*\xi} \) and we have introduced the constants

\[ e_1 = \frac{(1 + r_\nu^\nu)^{1/2}}{1 + r_\nu}, \quad e_2 = \frac{r_\nu}{(1 + r_\nu)^{1/2}}, \quad e_3 = \frac{r_\tau}{(1 + r_\tau)(1 + r_\nu)^{1/2}}. \]  

(3.49)

Using the normalization scheme above, the expression for the effective viscosity of the \( AZ \) body can be obtained from Eqs. (3.45) and (3.46) as

\[ \eta'_{AZ}(\xi) = \frac{1}{\Gamma(1 + \nu)} \frac{c_2 (c_2 \xi)^{\nu-1} + c_1 (c_1 \xi)^{\nu-1} E_\nu((-c_1 \xi)^\nu)}{c_2 (c_2 \xi)^{\nu-1} + c_1 (c_1 \xi)^{\nu-1} E_\nu((-c_1 \xi)^\nu)}. \]  

(3.50)

Figure 7 shows functions \( J'_{AZ}(\xi), G'_{AZ}(\xi) \) and \( \eta'_{AZ}(\xi) \) as a function of \( \xi \), in the particular case \( \tau_1 = \tau_2 \) (\( r_\tau = 1 \) according to Eq. (3.46)). The behaviour for \( \nu = 1 \) (thick curves), which corresponds to the classical anti–Zener model, can be expressed in terms of elementary functions. Since \( r_\tau = 1, c_1 = c_2 = 1/2 \), see Eq. (3.47). Hence, from (3.46), in this particular case the normalized creep function is \( J'_{AZ}(\xi) = 1/2 (1 + \xi/2 + e^{-\xi}) \), which, in the range of \( \xi \) values considered in Figure 7 appears to be dominated by the linear term. In the same limit, since from Eqs. (3.49) we have \( e_1 = 1, e_2 = 1/2, \) and \( e_3 = 1/4 \), omitting an additive \( \delta(\xi) \) term, the normalized relaxation modulus decays exponentially with \( \xi \) according to \( G'_{AZ}(\xi) = e^{-\xi/4}/2 \). The effective viscosity, according to (3.50), turns out to be \( \eta'_{AZ}(\xi) = 2/(1 + e^{-\xi/2}) \), which approximately is increasing linearly in the range of \( \xi \) values considered.
3.6 Fractional Burgers model

The constitutive equation for the fractional Burgers model (referred to as $B$ body) is obtained from (2.10a) as

\[
\text{fractional Burgers model : }\left[1 + a_1 \frac{d^\nu}{dt^\nu} + a_2 \frac{d^{1+\nu}}{dt^{1+\nu}}\right] \sigma(t) = \left[b_1 \frac{d^\nu}{dt^\nu} + b_2 \frac{d^{1+\nu}}{dt^{1+\nu}}\right] \epsilon(t).
\]

The mechanical analogue of the fractional Burgers ($B$) model can be represented as the combination in series of a fractional $KV$ element (with material parameters $\{\mu_1, \tau_1, \nu\}$) and a fractional $M$ fractional element ($\{\mu_2, \tau_2, \nu\}$).

Using the expressions (3.16) and (3.22) for complex moduli of fractional $KV$ and $M$ bodies into the series combination rule (2.24) provides the complex modulus

\[
\frac{1}{\hat{\mu}_B(s)} = \frac{1}{\mu_1} \left[1 + \frac{1}{(s\tau_1)^\nu}\right] + \frac{1}{\mu_2} \left[1 - \frac{(s\tau_2)^\nu}{1 + (s\tau_2)^\nu}\right],
\]

where the four constants $\mu_i$ and $\tau_i$ are related in some way to the four coefficients $a_i, b_i$ of the constitutive equation with $i = 1, 2$.

Using (2.18), from (3.52) it is possible to obtain immediately the Laplace transformed creep compliance

\[
\tilde{J}_B(s) = \frac{1}{2s} \left[\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) + \frac{1}{\mu_1} \frac{1}{(s\tau_1)^\nu} - \frac{1}{\mu_2} \frac{1}{(s\tau_2)^\nu}\right],
\]

which can be easily inverted with the aid of (3.10) and (3.11), obtaining

\[
J_B(t) = \frac{1}{2} \left[\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) + \frac{1}{\mu_1} (t/\tau_1)^\nu - \frac{1}{\mu_2} E_\nu\left(-(t/\tau_2)^\nu\right)\right], \quad t \geq 0.
\]

The computation of the Laplace–transformed relaxation modulus for the fractional $B$ body requires more cumbersome algebra compared to $J_B$. Using (3.52) into (2.18) provides, after some algebra:

\[
\tilde{G}_B(s) = 2\mu^* \tau_0^\nu s^{\nu-1} \frac{z + \gamma_0}{z^2 \delta_2 + z \delta_1 + \delta_0}.
\]

Above, for the sake of convenience, we have introduced the variable

\[
z \equiv (s\tau_0)^\nu,
\]

\[
(3.56)
\]
Figure 8: Normalized creep compliance (a), relaxation modulus (b) and effective viscosity (c) for the B body, for values of the fractional power $\nu = 1, 0.75, 0.50$ and $0.25$ as a function of normalized time $\xi$. The thick lines ($\nu = 1$), show results for the traditional B body.

where $\tau_0$ is defined as in the fractional AZ model (3.43), and several constants as follows:

$$\mu^* = \frac{\mu_1\mu_2}{\mu_1 + \mu_2}, \quad (3.57)$$
\[ \gamma_0 = \frac{r^\nu_\tau}{1 + r^\nu_\tau}, \quad \delta_2 = \frac{1}{1 + r^\mu}, \quad \delta_1 = \frac{1}{1 + r^\mu} + \frac{r^\nu_\mu \cdot r^\nu_\tau}{1 + r^\mu}, \quad \delta_0 = \frac{1}{1 + r^\mu} \cdot \left(1 + r^\nu_\tau\right)^2, \]  

(3.58)

where \( r_\tau = \tau_1 / \tau_2 \) and \( r^\mu = \mu_1 / \mu_2 \).

Denoting by \( z_1 \) and \( z_2 \) the roots of algebraic equation

\[ z^2 \delta_2 + z \delta_1 + \delta_0 = 0, \]  

(3.59)

we obtain

\[ \tilde{G}_B(s) = 2 \mu^* \tau^\nu_0 s^{\nu-1} \sum_{i=1}^{2} \frac{G_i}{z_i}, \]  

(3.60)

where the evaluation of constants \( G_i = G_i(r^\mu, r^\tau) \) is straightforward (the details are left to the reader). Since from (3.58) it can be easily recognized that the discriminant of Eq. (3.59) is positive, the roots are real. Hence, observing that \( \delta_i > 0 \) (\( i = 0, 1, 2 \)), Descartes’ rule of signs ensures that \( z_i < 0 \) (\( i = 1, 2 \)). Setting \( z_i = -1/\rho^i_0 \) with \( \rho^i_0 > 0 \) and recalling (3.56), we obtain a convenient expression of the relaxation modulus in the Laplace domain:

\[ \tilde{G}_B(s) = 2 \mu^* \sum_{i=1}^{2} \frac{G_i}{s} \cdot \frac{(s\tau_0 \rho_i)^\nu}{1 + (s\tau_0 \rho_i)^\nu}, \]  

(3.61)

which can be easily inverted with the aid of (3.11), obtaining the relaxation modulus in the time domain as a combination of two Mittag–Leffler functions with different arguments:

\[ G_B(t) = 2 \mu^* \sum_{i=1}^{2} G_i E^\nu_{\nu} \left( - \left( \frac{t}{\tau_0 \rho_i} \right)^\nu \right), \]  

(3.62)

Useful non–dimensional forms for the material functions in the time domain can be easily obtained. For the creep compliance (3.54) we have

\[ J'_B(\xi) = \frac{1}{2} \left[ 1 + c_2' \left( \frac{c_2 \xi}{\Gamma(1 + \nu)} \right)^\nu - c_1' E^\nu_{\nu} \left( (c_1 \xi)^\nu \right) \right], \quad \xi = \frac{t}{\tau^*}, \quad \tau^* = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2}, \]  

(3.63)

where \( J'_B(\xi) = \mu^* J_B(t)|_{t=\tau^* \xi} \), and

\[ c_2' = \frac{1}{1 + r^\mu}, \quad c_1' = \frac{r^\mu}{1 + r^\mu}, \quad r^\mu = \frac{\mu_1}{\mu_2}, \]  

(3.64)
For the non–dimensional form for the relaxation modulus we obtain:

\[
G'_B(\xi) = 2 \sum_{i=1}^{2} G_i E_\nu \left( - \left( \frac{\xi}{\rho_i} \right)^\nu \right), \tag{3.65}
\]

where \( G'_B(\xi) = (1/\mu^*) G_B(t)|_{t=\tau^*\xi}, \xi = t/\tau^* \).

By the above normalization scheme, it is possible to find the expression for the effective viscosity of the \( B \) body by using Eqs. (2.21) and (3.63) as

\[
\eta'_B(\xi) = \frac{\nu}{\Gamma(1+\nu)} \frac{1}{c_2 c'_2 (c_2 \xi)^{\nu-1} + c_1 c'_1 (c_1 \xi)^{\nu-1}} E_{\nu,\nu} \left( - (c_1 \xi)^\nu \right). \tag{3.66}
\]

As an illustration of the behaviour of the material functions for the \( B \) body, in Figure 8 we consider, for \( \tau_1 = 2\tau_2 \) and \( \mu_1 = \mu_2 \), \( J'_B(\xi), G'_B(\xi) \) and \( \eta'_B(\xi) \) as a function of variable \( \xi \). While the results for the creep compliance \( (a) \) and the effective viscosity \( (c) \) are qualitatively similar to those obtained for the \( AZ \) model in Figure 7, those pertaining to the relaxation modulus \( (c) \) differ significantly, being now removed the singularity for \( \xi \to 0 \) displayed in Figure 7c for \( \nu \neq 1 \). Curves obtained for \( \nu = 1 \) (thick), which corresponds to the classical \( B \) model, can be expressed in terms of elementary functions. Since \( r_\tau = 2 \) and \( r_\mu = 1 \), in this case we have \( c_1 = 2/3, c_2 = 1/3, c'_1 = c'_2 = 1/2 \). Hence, with these numerical values for the rheological parameters the creep function varies with time as

\[
J'_{AZ}(\xi) = \frac{1}{2} \left( 1 + \frac{\xi}{6} - \frac{1}{2} e^{-2\xi/3} \right). \tag{3.67}
\]

In the same limit, since \( E_1(-x) = e^{-x} \), the relaxation modulus \( G'_B(\xi) \) reduces to the sum of two exponentially decaying functions, and its initial value is only determined by the numerical value of the elastic parameters \( \mu_1 \) and \( \mu_2 \). The effective viscosity, according to (3.66), for \( \nu = 1 \) asymptotically approaches a constant value for \( \xi \to \infty \) as in the case of the \( AZ \) body; for all the other values of the fractional order \( \eta'_B(\xi) \) increases indefinitely, as in the case of the \( M \) body.

4 Conclusions

Starting from the mechanical analog of the basic fractional models of linear viscoelasticity we have verified the consistency of the correspondence.
principle. This principle allows one to formally get the material functions and the effective viscosity of the fractional models starting from the known expressions of the corresponding classical models by using the corresponding rules (3.9), (3.10), (3.11).

In particular, in the present paper, we were able to plot all these functions versus a suitable time scale in order to visually show the effect of the order $\nu \in (0,1]$ entering in our basic fractional models. This can help the researchers to guess which fractional model can better fit the experimental results. Presentation of the results has greatly benefited from a recently published Fortran code for computing the Mittag–Leffler function of complex argument [39, 40], coupled with an open source program for manipulating and visualizing data sets [41].

In Earth rheology, the concept of effective viscosity is often introduced to describe the behaviour of composite materials that exhibit both linear and a non–linear (i.e., non–Newtonian) stress–strain components of deformation), see e.g., Giunchi and Spada [13]). However, according to Müller [31] and to the results outlined in this paper, it is clear that this concept has a role also within the context of linear viscoelasticity, both for describing the transient creep of classical models and for characterizing the mechanical behavior of fractional models.

Transient rheological effects are currently the subject of investigation in the field of post–seismic deformations [4, 29, 30], global isostatic deformations [36, 37] and regional sea level variations [38]. The fractional mechanical models described in this paper have the potential of better characterizing the time–dependence of these processes, also in view of the increased quality of the available geophysical and geodetic observations.

5 Acknowledgments

We thank Davide Verotta for kindly providing a Fortran 90 function (mlfv.f90) for the evaluation of the two–parameters Mittag–Leffler function $E_{\alpha,\beta}(z)$, which has been essential for the preparation of this manuscript. The figures have been drawn using the Generic Mapping Tools (GMT) public domain software [11]. This work was supported by funding the ice2sea project from the European Union 7th Framework Programme through grant number 226375 (ice2sea contribution number 022).
Appendix A. The two fractional derivatives in $\mathbb{R}^+$

For a sufficiently well-behaved function $f(t)$ ($t \in \mathbb{R}^+$) we may define the fractional derivative of order $\beta$ ($m - 1 < \beta \leq m$, $m \in \mathbb{N}$), see e.g. the lectures notes by Gorenflo and Mainardi [15] and the text by Podlubny [34] in two different senses, that we refer here as to Riemann-Liouville derivative and Caputo derivative, respectively. Both derivatives are related to the so-called Riemann-Liouville fractional integral of order $\alpha > 0$, defined as

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, \quad \alpha > 0. \quad (A.1)$$

We note the convention $I^0 = \text{Id}$ (Identity operator) and the semi-group property

$$I^\alpha I^\beta = I^\beta I^\alpha = I^{\alpha+\beta}, \quad \alpha \geq 0, \beta \geq 0. \quad (A.2)$$

The fractional derivative of order $\beta > 0$ in the Riemann-Liouville sense is defined as the operator $D^\beta$ which is the left inverse of the Riemann–Liouville integral of order $\beta$ (in analogy with the ordinary derivative), that is

$$D^\beta I^\beta = \text{Id}, \quad \beta > 0. \quad (A.3)$$

If $m$ denotes the positive integer such that $m - 1 < \beta \leq m$, we recognize from Eqs. (A.2) and (A.3) $D^\beta f(t) := D^m I^{m-\beta} f(t)$, hence

$$D^\beta f(t) = \begin{cases} \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta+1-m}} \, d\tau \right], & m-1 < \beta < m, \\
\frac{df(t)}{dt}, & \beta = m. \end{cases} \quad (A.4)$$

For completion we define $D^0 = \text{Id}$.

On the other hand, the fractional derivative of order $\beta > 0$ in the Caputo sense is defined as the operator $D^\beta_*$ such that $D^\beta_* f(t) := I^{m-\beta} D^m f(t)$, hence

$$D^\beta_* f(t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} \, d\tau, & m-1 < \beta < m, \\
\frac{df(t)}{dt}, & \beta = m. \end{cases} \quad (A.5)$$
where $f^{(m)}$ denotes the derivative of order $m$ of the function $f$. We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the Laplace transformation, according to which

\[
L\{D_0^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k} f^{(k)}(0^+) , \quad m - 1 < \beta \leq m , \quad (A.6)
\]

where $\tilde{f}(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) \, dt$, $s \in \mathbb{C}$ and $f^{(k)}(0^+) := \lim_{t \to 0^+} f^{(k)}(t)$. The corresponding rule for the Riemann-Liouville derivative is more cumbersome: for $m - 1 < \beta \leq m$ it reads

\[
L\{D_0^\beta f(t); s\} = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} [D^k f^{(m-\beta)}] f(0^+) s^{m-1-k} , \quad (A.7)
\]

where, in analogy with (A.6), the limit for $t \to 0^+$ is understood to be taken after the operations of fractional integration and derivation. As soon as all the limiting values $f^{(k)}(0^+)$ are finite and $m - 1 < \beta < m$, the formula (A.7) simplifies into

\[
L\{D_0^\beta f(t); s\} = s^\beta \tilde{f}(s) . \quad (A.8)
\]

In the special case $f^{(k)}(0^+) = 0$ for $k = 0, 1, m - 1$, we recover the identity between the two fractional derivatives. The Laplace transform rule (A.6) was practically the starting point of Caputo, see [5, 6], in defining his generalized derivative in the late sixties of the last century.

For further reading on the theory and applications of fractional calculus we recommend the more recent treatise by Kilbas, Srivastava and Trujillo [20]. For the basic results on fractional integrability and differentiability we refer the interested reader to the survey paper by Li and Zhao [23].

**Appendix B: Remark on initial conditions in classical and fractional operator equations**

We note that the initial conditions at $t = 0^+$, $\sigma^{(h)}(0^+)$ with $h = 0, 1, \ldots p - 1$ and $\epsilon^{(k)}(0^+)$ with $k = 0, 1, \ldots q - 1$, do not appear in the operator equation
but they are required to be compatible with the integral equations (2.1). In fact, since Eqs. (2.1) do not contain the initial conditions, some compatibility conditions at \( t = 0^+ \) must be implicitly required both for stress and strain. In other words, the equivalence between the integral Eqs. (2.1) and the differential operator Eq. (2.14) implies that when we apply the Laplace transform to both sides of Eq. (2.14) the contributions from the initial conditions are vanishing or cancel in pair-balance. This can be easily checked for the simplest classical models described by Eqs. (2.6)–(2.9). It turns out that the Laplace transform of the corresponding constitutive equations does not contain any initial conditions: they are all hidden being zero or balanced between the RHS and LHS of the transformed equation. As simple examples let us consider the Kelvin–Voigt model for which \( p = 0, q = 1 \) and \( m > 0 \), see Eq. (2.6), and the Maxwell model for which \( p = q = 1 \) and \( m = 0 \), see Eq. (2.7).

For the Kelvin–Voigt model we get \( s \bar{\sigma}(s) = m \bar{\epsilon}(s) + b \bar{s}(s) - \epsilon(0^+) \), so, for any causal stress and strain histories, it would be

\[
    s \tilde{J}(s) = \frac{1}{m + bs} \iff \epsilon(0^+) = 0.
\]

We note that the condition \( \epsilon(0^+) = 0 \) is surely satisfied for any reasonable stress history since \( J_g = 0 \), but is not valid for any reasonable strain history; in fact, if we consider the relaxation test for which \( \epsilon(t) = \Theta(t) \) we have \( \epsilon(0^+) = 1 \). This fact may be understood recalling that for the Voigt model we have \( J_g = 0 \) and \( G_g = \infty \) (due to the delta contribution in the relaxation modulus).

For the Maxwell model we get \( \bar{\sigma}(s) + a [s \bar{\sigma}(s) - \sigma(0^+)] = b [s \bar{\epsilon}(s) - \epsilon(0^+)] \), so, for any causal stress and strain histories it would be

\[
    s \tilde{J}(s) = \frac{a}{b} + \frac{1}{bs} \iff a \sigma(0^+) = b \epsilon(0^+).
\]

We now note that the condition \( a \sigma(0^+) = b \epsilon(0^+) \) is surely satisfied for any causal history both in stress and in strain. This fact may be understood recalling that for the Maxwell model we have \( J_g > 0 \) and \( G_g = 1/J_g > 0 \).

Then we can generalize the above considerations stating that the compatibility relations of the initial conditions are valid for all the four types of viscoelasticity, as far as the creep representation is considered. When the relaxation representation is considered, caution is required for the types III.
and IV, for which, for correctness, we would use the generalized theory of integral transforms suitable just for dealing with generalized functions.

Similarly with the operator equations of integer order we note that the initial conditions at \( t = 0^+ \) for the stress and strain do not explicitly enter into the fractional operator equation (3.1). This means that the approach with the Caputo derivative, which requires in the Laplace domain the same initial conditions as the classical models is quite correct. However, if we assume the same initial conditions, the approach with the Riemann–Liouville derivative provides the same results since, in view of the corresponding Laplace transform rule (A.8), the initial conditions do not appear in the Laplace domain. The equivalence of the two approaches was formerly outlined by Mainardi \[26\] and more recently noted by Bagley \[1\] for the fractional Zener model.

We refer the reader to the paper by Heymans and Podubny \[16\] for the physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville derivatives, especially in viscoelasticity. In such field, however, we prefer to adopt the Caputo derivative since it requires the same initial conditions as in the classical cases. By the way, for a physical viewpoint, these initial conditions are more accessible than those required in the more general Riemann-Liouville approach, see (A.7).

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