Effective Lagrangians and low energy photon-photon scattering

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Abstract

We use the behavior of the photon-photon scattering for photon energies $\omega$ less than the electron mass, $m_e$, to examine the implications of treating the Euler-Heisenberg Lagrangian as an effective field theory. Specifically, we determine the $\omega^2/m_e^2$ behavior of the scattering amplitude predicted by including one-loop corrections to the Euler-Heisenberg effective Lagrangian together with the counterterms required by renormalizability. This behavior is compared with the energy dependence obtained by expanding the exact QED photon-photon scattering amplitude. If the introduction of counterterms in the effective field theory is restricted to those determined by renormalizability, the $\omega^2/m_e^2$ dependences of the two expansions differ.

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I. INTRODUCTION

Low energy photon-photon scattering is a textbook example of a process which can be described using an effective interaction, the Euler-Heisenberg effective Lagrangian \([1]\). The precise form of this Lagrangian (given below) can be obtained by expanding the box diagrams for photon-photon scattering in powers of the photon energy over the electron mass, \(\omega/m_e\), and keeping the first non-vanishing term, which is of order \(\omega^4/m_e^4\). This leads to the cross section

\[
\frac{d\sigma}{d\Omega} = \frac{139\alpha^4}{(180\pi)^2} (3 + z^2)^2 \frac{\omega^6}{m_e^8},
\]

with its characteristic \(\omega^6\) dependence on the photon energy. Here, \(z\) is the cosine of the scattering angle. The energy scale in this case is \(m_e\), a result of ‘integrating out’ the heavy degree of freedom, the electron, to obtain an effective interaction.

There are compelling arguments suggesting that effective interactions can be used to define effective field theories which are adequate descriptions of Nature if one accepts a reasonable set of assumptions about the renormalization program \([2]\). With these assumptions, it is possible to obtain finite corrections to physical processes from the (ordinarily divergent) contributions calculated by using the effective interactions in higher orders of perturbation theory. The price for extracting these finite corrections is the introduction of local counterterms whose couplings are not known \textit{a priori}. It is, however, clear that the counterterms introduce corrections of higher order in the ratio of the energy to the energy scale.

The implications of this approach can be explored using the laboratory of photon-photon scattering as is evident from the plot of the total cross section given in Fig. 1 \([3,4]\). Here, it can be seen that the \(\omega^6\) behavior persists up to \(\omega \simeq 0.4 \, m_e\) or so, at which point the slope increases indicating the presence of higher powers of \(\omega\). Our purpose is to compute the one-loop corrections to low energy photon-photon scattering amplitude using the Euler-Heisenberg effective Lagrangian with the appropriate counterterms and to compare this result with that obtained by expanding the \textit{exact} amplitude in powers of \(\omega/m_e\). This will enable us to better understand how an effective field theory mimics a full field theory in a situation where explicit calculations in both approaches are feasible.

II. ONE-LOOP CORRECTIONS

The full Euler-Heisenberg Lagrangian describing photon self interactions is an expansion in powers of the electromagnetic field tensor \(F_{\mu\nu}\). Assuming parity conservation, it can be expressed in terms of the two independent, gauge invariant combinations \(F_{\mu\nu}F^{\mu\nu}\) and \(F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\). Due to the antisymmetry of \(F_{\mu\nu}\), scalars formed from an odd number of field tensors vanish. For a one-loop calculation, it suffices to use the terms

\[
\mathcal{L}_{E-H} = \frac{1}{180 \, m_e^4} \left[ 5 \left( F_{\mu\nu}F^{\mu\nu} \right)^2 - 14 F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu} \right] + \frac{1}{315 \, m_e^8} \left[ 9 \left( F_{\mu\nu}F^{\mu\nu} \right)^3 - 26 F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}F_{\alpha\beta}F^{\alpha\beta} \right].
\]
The one-loop contribution of the order $\alpha^3$ term in Eq. (2) to the photon-photon scattering amplitude, obtained by contracting a pair of $F_{\mu\nu}$’s, actually vanishes by dimensional regularization [9]. This leaves only the $\alpha^4$ one-loop contribution from the first term of Eq. (2) used in the second order of perturbation theory. Because the photons are massless, all the loop integrals, which take the form

$$B_{\alpha_1 \cdots \alpha_i} = \frac{1}{i \pi^2} \int d^4q \frac{q_{\alpha_1} \cdots q_{\alpha_i}}{q^2(q+k)^2},$$

$i = 1, \cdots, 4$, are proportional to the scalar integral

$$B_0(k^2) = \frac{1}{i \pi^2} \int d^4q \frac{1}{q^2(q+k)^2},$$

multiplied by a polynomial in the momentum $k$. Evaluating $B_0(k^2)$ using a cutoff $\Lambda$, we find

$$B_0(k^2) = \ln \left( \frac{\Lambda^2}{-k^2 - i\varepsilon} \right).$$

In second order, the $\alpha^2$ term in Eq. (2) has $s, t$ and $u$ channel contributions, illustrated in Fig. 2. As a consequence, the one-loop correction to the helicity amplitudes can be written

$$T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = \frac{\alpha^4}{(4\pi)^2} \frac{\omega^8}{m_e^8} \left( T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} B_0(s) + T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} B_0(t) + T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} B_0(u) \right),$$

with $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$ and $u = (k_1 - k_4)^2$. Since the vertex functions resulting from $\mathcal{L}_{E-H}$ are quite complicated [9], the programs SCHOONSCHIP [7] and FORM [8] were used to perform the algebraic manipulations. The explicit forms of the $T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}$ are given in Table I. In order to isolate the cutoff dependence, we introduce a renormalization scale $\mu$ and rewrite Eq. (3) as

$$T^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = \frac{\alpha^4}{(4\pi)^2} \frac{\omega^8}{m_e^8} \ln \left( \frac{\Lambda^2}{\mu^2} \right) \sum_{i=s,t,u} T^{i}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} + \tilde{T}^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4},$$

where $\tilde{T}^{(2)}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}$ is obtained from Eq. (3) by replacing $\Lambda^2$ with $\mu^2$ in $B_0$. The sums of the various helicity combinations are given in the last row of Table I.

To complete the interpretation of the cutoff-dependent terms in Eq. (7) as contributions to coupling constant renormalization, it is necessary to show that they can be obtained as matrix elements of local, gauge invariant operators. In addition to four $F_{\mu\nu}$’s, these operators must contain four derivatives in order to produce the $\omega^8$ behavior. Finding them is something of a trial and error process. There are many tensors which can be formed with the required properties, but most of them are not independent. The forms given below are sufficient to reproduce the helicity amplitudes in the last row of Table I.

$$\mathcal{L}^{(2)}_{E-H} = \frac{1}{10(90\pi^2)} \frac{\alpha^4}{m_e^8} \ln \left( \frac{\Lambda^2}{\mu^2} \right) \left[ 182 (\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu})(\partial_\beta F_{\lambda\rho})(\partial^\beta F^{\lambda\rho}) 
\quad - 639 (\partial_\alpha F_{\mu\nu})(\partial^\beta F^{\mu\nu})(\partial_\beta F_{\lambda\rho})(\partial^\rho F^{\lambda\rho}) + 1523 (\partial_\alpha \partial_\beta F_{\mu\lambda})(\partial^\mu F^{\rho\beta})(\partial^\rho F_{\alpha\rho}) - F^{\rho\beta} \right].$$

3
The effective Lagrangian
\[ \mathcal{L} = \mathcal{L}_{E-H} + \frac{1}{m_e^8} \left[ a_1 \left( \partial_\alpha F_{\mu\nu} \right) \left( \partial^\alpha F^{\mu\nu} \right) \left( \partial_\beta F_{\lambda\rho} \right) \left( \partial^\beta F^{\lambda\rho} \right) \right. 
+ \left. a_2 \left( \partial_\alpha F_{\mu\nu} \right) \left( \partial^\beta F^{\mu\nu} \right) \left( \partial_\beta F_{\lambda\rho} \right) \right. 
+ \left. a_3 \left( \partial_\alpha \partial_\beta F_{\mu\lambda} \right) F^{\lambda\rho} \left( \partial_\mu \partial_\nu F_{\alpha\rho} \right) F^{\alpha\beta} \right] \]

then gives a finite matrix element for elastic scattering at the one-loop level provided the
cutoff dependent terms represented by Eq. (8) are interpreted as renormalizations of the
coupling constants \( a_1, a_2 \) and \( a_3 \).

### III. CORRECTIONS TO THE QED AMPLITUDE

The leading \( \omega^6 \) dependence of the low energy cross section follows from the fact that the
terms in Eq. (2) involving four field tensors are of dimension 8. Thus, the amplitudes go
as \( \omega^4 \) and, from QED or using Eq. (2), one can obtain the helicity amplitudes [3] given in
the first row of Table II. The remaining amplitudes are equal to these by parity and time
reversal. Equation (3) follows by taking the sum of the squares of the \( T_{\lambda_1\lambda_2\lambda_3\lambda_4} \).

Table III compares the total cross section, as calculated approximately by integrating
Eq. (1) over \( z \),
\[ \sigma = \frac{1}{(45)^2} \frac{973}{5} \frac{\alpha^4}{\pi} \frac{\omega^6}{m_e^8}, \]
with the full one loop QED expression calculated by using a rapidly converging series for
the Spence functions [3] and performing a numerical integration over \( z \). As can be seen,
Eq. (10) provides a reasonable approximation for \( \omega/m_e \lesssim 0.50 \).

Again, corrections to Eqs. (11) or (10) can only involve effective interactions with deriva-
tives acting on four factors of \( F_{\mu\nu} \) and, by Lorentz invariance, must involve an even number
of derivatives. Thus the helicity amplitudes and therefore the cross section are expansions
in \( \omega^2/m_e^2 \). For comparison purposes, we have calculated the terms of order \( \omega^6 \) and \( \omega^8 \) by
expanding the electron box diagrams of QED. The results are given in the second and third
rows of Table II.

These amplitudes add terms to Eq. (11) which becomes
\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^4}{(180\pi)^2} \frac{\omega^6}{m_e^8} \left[ 139 (3 + z^2)^2 + 160 \frac{\omega^2}{m_e^2} (1 - z^2)(3 + z^2) \right. 
+ \left. \frac{4}{245} \frac{\omega^4}{m_e^2} \left( 48533 + 35885 z^2 + 18995 z^4 + 1641 z^6 \right) \right], \]
where the first term is the contribution from the square of the \( \omega^4 \) amplitude, the second term
is the cross term between the amplitudes of order \( \omega^4 \) with those of order \( \omega^6 \), the third term
consists of the square of the \( \omega^6 \) amplitudes and the cross terms between the \( \omega^4 \) amplitudes
and those of order \( \omega^8 \). The corresponding total cross section is
\[ \sigma = \frac{1}{(45)^2} \frac{\alpha^4 \omega^6}{\pi m_e^8} \left[ \frac{973}{5} + \frac{128}{3} \frac{\omega^2}{m_e^2} + \frac{409792}{1715} \frac{\omega^4}{m_e^8} \right]. \]
The last column of Table III gives the ratio of Eq. (12) to the full one loop cross section. Inclusion of the second and third terms in Eq. (12) provides a very good approximation, all the way to \( \omega/m_e \approx 0.90 \).

As mentioned above, the helicity amplitudes in the first row of Table II can be obtained from the effective Lagrangian

\[
\mathcal{L}_{\text{eff}} = \frac{1}{180} \frac{\alpha^2}{m_e^4} \left[ 5 (F_{\mu\nu} F^{\mu\nu})^2 - 14 F_{\mu\nu} F^{\mu\rho \lambda} F_{\lambda\rho} \right].
\]  

(13)

In order to generalize this expression to reproduce the \( \omega^6 \) and \( \omega^8 \) corrections, it is necessary to construct terms containing two or four additional derivatives acting on a combination of four \( F_{\mu\nu} \)’s to produce a scalar. Terms with four derivatives were encountered in the discussion of the one-loop corrections and suffice to describe the \( \omega^8 \) terms found here. The enumeration of the terms with two derivatives can be done similarly and we find that

\[
\mathcal{L}'_{\text{eff}} = \frac{1}{945} \frac{\alpha^2}{m_e^6} \left[ (\partial^\alpha \partial_\beta F_{\mu\nu}) F^{\mu\nu} F_{\alpha\lambda} F^{\lambda\rho} + 3 (\partial^\alpha F_{\mu\nu}) (\partial_\alpha F^{\mu\nu}) F_{\lambda\rho} F^{\lambda\rho} + 11 (\partial^\alpha F_{\mu\nu}) F^{\nu\lambda} (\partial_\alpha F_{\lambda\rho}) F^{\rho\mu} \right].
\]  

(14)

gives the \( \omega^6 \) correction and

\[
\mathcal{L}''_{\text{eff}} = -\frac{1}{9450} \frac{\alpha^2}{m_e^8} \left[ 33 (\partial_\alpha F_{\mu\nu}) (\partial^\alpha F^{\mu\nu}) (\partial_\beta F_{\lambda\rho}) (\partial^\beta F^{\lambda\rho}) - 106 (\partial_\alpha F_{\mu\nu}) (\partial^\beta F^{\mu\nu}) (\partial_\alpha F_{\lambda\rho}) (\partial^\beta F^{\lambda\rho}) + 262 (\partial^\alpha \partial_\beta F_{\mu\lambda}) F^{\lambda\nu} (\partial_\mu \partial_\nu F_{\alpha\rho}) F^{\alpha\beta} \right].
\]  

(15)

gives the \( \omega^8 \) correction.

The expansion of the photon-photon scattering amplitude can thus be obtained by computing the matrix element of the local effective Lagrangian \( \mathcal{L}_{\text{eff}} + \mathcal{L}'_{\text{eff}} + \mathcal{L}''_{\text{eff}} \), and it is accurate for photon energies very close to the energy scale \( m_e \).

IV. DISCUSSION

By using the Euler-Heisenberg Lagrangian, Eq. (2), to the one-loop level in perturbation theory we obtained an expansion of the low energy photon-photon scattering amplitude containing terms of order \( \omega^4/m_e^4 \) and \( \omega^8/m_e^8 \). Since there is no contribution of order \( \omega^6/m_e^6 \), it is not necessary to introduce a counterterm to ensure renormalizability in this order. This situation is not peculiar to the one-loop approximation. An \( \omega^6/m_e^6 \) correction cannot be generated by retaining more terms in the Euler-Heisenberg expansion or by including perturbative corrections with more than one loop. While we do obtain finite one-loop terms of order \( \omega^8/m_e^8 \), these are higher order in \( \alpha \) and thus should be negligible numerically. In fact, the discussion of the order \( \omega^8/m_e^8 \) is essentially complete. The only other potential \( \omega^8 \) correction to the elastic amplitude is associated with term in the Euler-Heisenberg expansion which consists of products of eight field tensors, but this contribution also vanishes by dimensional regulation.
This situation should be compared with the result of expanding the exact QED amplitude in powers of $\omega^2/m_e^2$. Here, one obtains, as expected, the same $\omega^4/m_e^4$ term, as well as an $\omega^6/m_e^6$ term and an $\omega^8/m_e^8$ correction which is not suppressed by additional powers of $\alpha$.

The existence of a leading order (in $\alpha$) $\omega^8/m_e^8$ correction in the QED expansion can be accommodated in the effective field theory approach by an appropriate choice of the renormalized couplings $a_i$ of Eq. (14). There is still a difference between the energy dependence derived from QED and that implied by simply requiring the renormalizability of the effective Lagrangian approach. This fact does not seem to be in conflict with the spirit of effective field theory, since it is always possible to add interactions such as those in Eq. (14) to recover the $\omega^6$ dependence. One cannot, however, restrict the number of such terms by appealing to renormalizability.

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TABLE I. The contributions to the helicity amplitudes of Eq. (6) from the Euler-Heisenberg Lagrangian used to one-loop are given in the first three rows. In addition, $T^s_{i_1 i_2 i_3 i_4}$ while $T^u_{i_1 i_2 i_3 i_4}$ and $T^w_{i_1 i_2 i_3 i_4}$ are the same cross section as in Fig. 1. The third column is the ratio of the approximate cross section (3) to the full cross section in column 2. The fourth column is the ratio of the improved cross section (7) to the full cross section.

| $T^s_{i_1 i_2 i_3 i_4}$ | $T^u_{i_1 i_2 i_3 i_4}$ | $T^w_{i_1 i_2 i_3 i_4}$ | $\sum_{s,t,u} T_{i_1 i_2 i_3 i_4}$ |
|------------------------|------------------------|------------------------|----------------------------------|
| $\frac{8}{5} (159 + 3z^2)$ | $-1760$ | $\frac{96}{5} (1 + z)^2$ | $\frac{8}{5} (1701 + 245z^2)$ |
| $\frac{96}{5} (1 - z)^2$ | $-110 (1 - z)^4$ | $\frac{24}{5} (1 - z)^2$ |
| $\frac{96}{5} (1 + z)^2$ | $-110 (1 + z)^4$ | $\frac{1}{5} (1 + z)^2 (743 + 1450z + 731z^2)$ |

TABLE II. The angular dependence of the helicity amplitude expansion in powers of $\omega$ is shown. In all cases, $T^s_{i_1 i_2 i_3 i_4}$ while $T^u_{i_1 i_2 i_3 i_4}$ and $T^w_{i_1 i_2 i_3 i_4}$ is the same cross section as in Fig. 1. The third column is the ratio of the approximate cross section (3) to the full cross section in column 2. The fourth column is the ratio of the improved cross section (7) to the full cross section.

| $\omega$ | $\sigma$ | $R_1$ | $R_2$ |
|----------|----------|-------|-------|
| 0.25     | $3.20 \times 10^{-35}$ | 0.983 | 1.001 |
| 0.35     | $2.46 \times 10^{-34}$ | 0.966 | 1.010 |
| 0.45     | $1.15 \times 10^{-34}$ | 0.931 | 1.020 |
| 0.55     | $4.08 \times 10^{-33}$ | 0.876 | 1.036 |
| 0.65     | $1.22 \times 10^{-32}$ | 0.799 | 1.054 |
| 0.75     | $3.32 \times 10^{-32}$ | 0.692 | 1.054 |
| 0.85     | $9.07 \times 10^{-32}$ | 0.537 | 0.977 |
| 0.95     | $3.02 \times 10^{-31}$ | 0.314 | 0.701 |
| 0.99     | $6.74 \times 10^{-31}$ | 0.180 | 0.438 |
| 1.00     | $1.26 \times 10^{-30}$ | 0.102 | 0.255 |

TABLE III. The first column gives the photon center-of-mass energy in units of the electron mass. The second column is the full one loop cross section, as calculated numerically, in cm$^2$. This is the same cross section as in Fig. 1. The third column is the ratio of the approximate cross section (3) to the full cross section in column 2. The fourth column is the ratio of the improved cross section (7) to the full cross section.
FIG. 1. The exact cross section for $\gamma\gamma \rightarrow \gamma\gamma$ is shown.

FIG. 2. One-loop diagrams for the $s$, $t$ and $u$ channel two photon exchanges are shown.