Unstable metric pressure of partially hyperbolic diffeomorphisms with sub-additive potentials

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Abstract
In this paper, we define and study unstable measure theoretic pressure for $C^1$-smooth partially hyperbolic diffeomorphisms with sub-additive potentials. We show that this measure theoretic pressure for any ergodic measure equals the corresponding unstable measure theoretic entropy plus the Lyapunov exponent of the potentials with respect to the ergodic measure. On the other hand, we also give other definitions of unstable metric pressure, in terms of Bowen’s picture and the capacity picture. We show that all definitions of unstable metric pressure, including the one defined at the beginning, actually coincide for any ergodic measure.

Keywords: unstable measure theoretic pressure, sub-additive potential, variational principle
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1. Introduction
As a natural generalization of topological entropy, topological pressure for a given continuous function on the phase space roughly measures the orbit complexity of iterated maps on the potential functions. In [14], Ruelle first defined topological pressure for expansive maps. Under some assumptions, he also established a variational principle, which was generalized by Walters in full generality, see [16]. In [12], Pesin and Pitskel defined topological pressures for non-compact subsets and proved a variational principle under some supplementary conditions.

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Based on Katok’s work [11], He et al [6] introduced measure theoretic pressure for ergodic measures. All pressures mentioned are about additive potentials—the sequence of continuous functions consisting of summations over orbits of the dynamical map. On the other hand, sub-additive potentials for a dynamical system is a sequence of continuous functions satisfying sub-additivity condition involving the dynamical map. In [5], Falconer first introduced topological pressure for sub-additive potentials on mixing repellers. In [1], Barreira generalized Pesin and Pitskel’s work [12] to the case of general potentials. With some restrictions on potentials, he proved a variational principle. In [4], without any restriction, Cao et al obtained a variational principle of topological pressure for sub-additive potentials. Furthermore, Cheng, Cao, Hu, and Zhao investigated measure theoretic pressure for non-additive potentials, see [3, 8], etc.

In recent years, the theory of entropy and pressure for $C^1$-smooth partially hyperbolic diffeomorphisms are intensively investigated. In [9], Hu et al introduced the unstable topological and metric entropy, obtained the corresponding Shannon–McMillan–Breiman theorem, and established the corresponding variational principle. The main feature of these unstable entropies is to rule out the complexity on central directions and focus on that on unstable directions. In [15], Tian and Wu generalize the above result with additional consideration of an arbitrary subset (not necessarily compact or invariant). In [10], Hu et al investigated the unstable topological pressure for additive potentials, and obtained a variational principle.

It is a natural task to extend pressure theory to $C^1$-smooth partially hyperbolic diffeomorphisms with sub-additive potentials. In [18], we introduce unstable topological pressure for sub-additive potentials, and set up a corresponding variational principle.

In this paper, we define and study unstable measure theoretic pressure for sub-additive potentials. We show that for any ergodic measure this unstable metric pressure equals the corresponding unstable metric entropy plus the corresponding Lyapunov exponent of the potentials with respect to the measure. Moreover, we also formulate and study other definitions of unstable metric pressure, in terms of Bowen’s picture and the capacity picture. It turns out that all definitions of unstable metric pressure, including the one defined at the beginning, actually coincide for any ergodic measure. Although this work is inspired by the case of purely topological dynamical systems, we have to investigate dynamics on unstable manifolds in the setting of $C^1$-smooth partially hyperbolic systems, which causes new problems and difficulties. For example, for any Borel probability measure $\mu$ on a closed Riemannian manifold $M$ and any measurable partition $\eta$, we have to consider the measure disintegration of $\mu$ over $\eta$ with respect to unstable manifolds. Moreover, we also need to handle the sub-additive potentials, which causes additional work during estimation of pressures rather than the case of additive potentials.

**Theorem 1.1.** Suppose $M$ is a finite dimensional, smooth, connected, and compact Riemannian manifold without boundary; and $f: M \to M$ is a $C^1$-smooth partially hyperbolic diffeomorphism. Let there be given a sequence of sub-additive potentials $G = \{\log g_n\}_{n \geq 1}$ off on $M$. Then for any $\mu \in \mathcal{M}_f(M)$, one has

$$P^\mu(f, G) = h^\mu(f) + G_\mu(\mu).$$

Combing with theorem 1.1 in [18], we have the following variational principle for unstable topological pressure and unstable metric pressure.

**Corollary 1.2.** Under the same assumption as above, we have

$$P^\mu(f, G) = \sup \{ P^\mu(f, \tilde{G}) | \mu \in \mathcal{M}_f(M) \}.$$
Theorem 1.3. Suppose $M$ is a finite dimensional, smooth, connected, and compact Riemannian manifold without boundary; and $f: M \to M$ is a $C^1$-smooth partially hyperbolic diffeomorphism. Let there be given a sequence of sub-additive potentials $\mathcal{G} = \{\log g_n\}_{n \geq 1}$ off on $M$. Then for any $\mu \in \mathcal{M}_f(M)$, one has

$$P^\mu_n(f, \mathcal{G}) = CP^\mu_n(f, \mathcal{G}) = C\bar{P}^\mu_n(f, \mathcal{G}) = P^\mu_{\#N}(f, \mathcal{G}).$$

(All terms involved are defined in sections 2 and 4, see in particular definitions 2.5, 4.1 and 4.3–4.5. The sets $\mathcal{M}_f(M)$ and $\mathcal{M}_f(M)$ refer to the collection of $f$-invariant and ergodic probability measures on $M$ respectively.)

The paper is organized as follows. In section 2, we set up notation, and give the definition of the unstable measure theoretic pressure for sub-additive potentials. In section 3, we prove theorem 1.1 in two steps. In section 4, we formulate other definitions of unstable metric pressure, in terms of Bowen’s picture and the capacity picture. Moreover, we complete the proof of theorem 1.3.

2. Notation and definitions

Throughout the paper, we focus on the dynamical system $(M, f)$, where $M$ is a finite dimensional, smooth, connected, and compact Riemannian manifold without boundary; and $f: M \to M$ is a $C^1$-smooth partially hyperbolic diffeomorphism. We say $f$ is partially hyperbolic, if there exists a nontrivial $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle into stable, central, and unstable distributions, such that all unit vectors $v^\sigma \in E^\sigma_x (\sigma = s, c, u)$ with $x \in M$ satisfy

$$\|D_x f v^u\| < \|D_x f v^u\| < \|D_x f v^u\|,$$

and

$$\|D_x f \|_{E^c} < 1 \quad \text{and} \quad \|D_x f^{-1} \|_{E^c} < 1,$$

for some suitable Riemannian metric on $M$. The stable distribution $E^s$ and unstable distribution $E^u$ are integrable to the stable and unstable foliations $W^s$ and $W^u$ respectively such that $TW^s = E^s$ and $TW^u = E^u$ (cf [7]).

First we recall some basic facts about unstable entropy (see [9]).

Take $\epsilon_0 > 0$ small. Let $\mathcal{P} = \mathcal{P}_{\epsilon_0}$ denote the set of finite Borel partitions $\alpha$ of $M$ whose elements have diameters smaller than or equal to $\epsilon_0$, that is, $\text{diam } \alpha := \sup \{\text{diam } A : A \in \alpha\} \leq \epsilon_0$. For each $\beta \in \mathcal{P}$ we can define a finer partition $\eta$ such that $\eta(x) = \beta(x) \cap W^u_{\text{loc}}(x)$ for each $x \in M$, where $W^u_{\text{loc}}(x)$ denotes the local unstable manifold at $x$ whose size is greater than the diameter $\epsilon_0$ of $\beta$. Since $W^u$ is a continuous foliation, $\eta$ is a measurable partition with respect to any Borel probability measure on $M$.

Let $\mathcal{P}^{\eta}$ denote the set of partitions $\eta$ obtained in this way and subordinate to unstable manifolds. Here a partition $\eta$ of $M$ is said to be subordinate to unstable manifolds of $f$ with respect to a measure $\mu$ if for $\mu$-almost every $x$, $\eta(x) \subseteq W^u(x)$ and contains an open neighborhood of $x$ in $W^u(x)$. It is clear that if $\alpha \in \mathcal{P}$ satisfies $\mu(\partial \alpha) = 0$, where $\partial \alpha := \cup_{A \in \alpha} \partial A$, then the corresponding $\eta$ given by $\eta(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$ is a partition subordinate to unstable manifolds of $f$.

Given any probability measure $\nu$ and any measurable partition $\eta$ of $M$, and denote by $\eta(x)$ the element of $\eta$ containing $x$. The canonical system of conditional measures for $\nu$ and $\eta$ is a family of probability measures $\{\nu^\eta_x : x \in M\}$ with $\nu^\eta_x(\eta(x)) = 1$, such that for every measurable
set $B \subseteq M$, $x \mapsto \nu^B_\eta(x)$ is measurable and

$$
\nu(B) = \int_X \nu^B_\eta(x) dx.
$$

This is also called the measure disintegration of $\nu$ over $\eta$. A classical result of Rokhlin (cf [13]) says that if $\eta$ is a measurable partition, then there exists a system of conditional measures with respect to $\eta$. It is essentially unique in the sense that two such systems coincide for sets with full $\nu$-measure. For measurable partitions $\alpha$ and $\eta$, let

$$
H_\nu(\alpha|\eta) := -\int_M \log \nu^\alpha_\eta(x) d\nu(x).
$$

denote the conditional entropy of $\alpha$ for given $\eta$ with respect to $\nu$.

The unstable metric entropy in [9] is defined as follows.

**Definition 2.1.** For any $\mu \in M_f(M)$, any $\eta \in P^u$, and any measurable partition $\alpha$, define

$$
h_\mu(f, \alpha|\eta) := \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_f^n|\eta),
$$

and

$$
h_\mu(f|\eta) := \sup_{\alpha \in P} h_\mu(f, \alpha|\eta).
$$

The unstable metric entropy of $f$ is defined by

$$
h_n^u(f) := \sup_{\eta \in P^u} h_\mu(f|\eta).
$$

**Remark 2.2.** In [9], the authors proved that $h_\mu(f|\eta)$ is independent of $\eta \in P^u$. Moreover, for any ergodic measure $\mu$, any $\alpha \in P_\epsilon(\epsilon$ small enough), $\eta \in P^u$, one has $h_n^u(f) = h_\mu(f|\eta) = h_\mu(f, \alpha|\eta)$ (see lemma 2.8 and theorem A there). We shall use this fact frequently throughout the whole paper.

**Definition 2.3.** Given a sequence of continuous functions $G = \{\log g_n\}_{n=1}^\infty$ on $M$, $G$ is called a sequence of sub-additive potentials of $f$ if

$$
\log g_{m+n}(x) \leq \log g_m(x) + \log g_n(f^m x), \quad \text{for any } x \in M, m, n \in \mathbb{N}.
$$

**Remark 2.4.** For any $f$-invariant Borel probability measure $\mu$, set

$$
G_\mu(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \log g_n d\mu,
$$

and $G_\mu(\mu)$ is called the Lyapunov exponent of $G$ with respect to $\mu$. The existence of this limit follows from a sub-additive argument. It takes values in $[-\infty, +\infty)$. Moreover, the sub-additive Ergodic theorem (see [17], theorem 10.1) implies that for an ergodic measure $\mu$, one has

$$
G_\mu(\mu) = \lim_{n \to \infty} \frac{1}{n} \log g_n(x), \mu \text{-a.e.x.}
$$

We denote by $d^u$ the metric induced by the Riemannian structure on the unstable manifold and let

$$
d^u_\mu(x, y) = \max_{0 \leq j < n-1} d^u(f^j(x), f^j(y)).
$$
Let \( W^n(x, \delta) \) be the open ball inside \( W^n(x) \) centered at \( x \) with radius \( \delta \) with respect to the metric \( d^n \).

We define unstable metric pressure for sub-additive potentials as follows.

Take any \( \eta \in \mathcal{P}^n \), any \( x \in M \), any positive integer \( n \), any \( \epsilon > 0 \), and any \( 0 \leq \gamma < 1 \). A subset \( F \subseteq W^n(x) \) is called an \( (n, \epsilon, \gamma)u \)-spanning set of \( \eta(x) \), if

\[
\mu^n_\eta \left( \bigcup_{y \in F} B^n_\gamma(y, \epsilon) \right) \geq 1 - \gamma,
\]

where \( B^n_\gamma(y, \epsilon) = \{ z \in W^n(x) : d^n_\gamma(y, z) \leq \epsilon \} \) is the \((n, \epsilon)u\)-Bowen ball around \( y \).

**Definition 2.5.** For any \( x \in M \), any \( \eta \in \mathcal{P}^n \), any positive number \( \gamma \), any natural number \( n \), any sequence \( G = \{ \log g_n \}_{n=1}^\infty \) of sub-additive potentials of \( f \) on \( M \), and any \( \mu \in \mathcal{M}_f(M) \), set

\[
P^n_\mu(f, G, \epsilon, n, \eta(x), \gamma) := \inf \left\{ \sum_{y \in F} \sup_{x \in B^n_\gamma(y, \epsilon)} \log g_n(z) : F \text{ is an } (n, \epsilon, \gamma)u \text{-spanning set of } \eta(x) \right\},
\]

and

\[
P^n_\mu(f, G, \eta(x), \gamma) := \lim_{\epsilon \to 0} P^n_\mu(f, G, \epsilon, n, \eta(x), \gamma).
\]

(Note that \( P^n_\mu(f, G, \epsilon, n, \eta(x), \gamma) \) is non-decreasing when \( \epsilon \) gets smaller.) The unstable measure theoritic pressure of \( f \) with respect to \( G \) is defined by

\[
P^n_\mu(f, G) := \sup_{\eta \in \mathcal{P}^n} \int_M \lim_{\gamma \to 0} P^n_\mu(f, G, \eta(x), \gamma) \, d\mu(x).
\]

(Measurability follows from the fundamentals of partially hyperbolic systems, unstable leaves \( \{W^n(x) | x \in M \} \) form the unstable foliation.)

**Remark 2.6.** For any continuous function \( \varphi \in C(M) \), the corresponding sequence \( G = \{ S^n \varphi(x) = \sum_{j=0}^{n-1} \varphi(f^j(x)) \}_{n=1}^\infty \) is additive and hence sub-additive. We simply write \( P^n_\mu(f, G) \) as \( P^n_\mu(f, \varphi) \), which actually coincides with the classical definition.

### 3. Unstable metric pressure equals unstable metric entropy plus Lyapunov exponent

In this section, we prove theorem 1.1 in two steps. First we show the conclusion is true in the case of additive potentials. Second we prove theorem 1.1 for sub-additive potentials, with some help of the previous case.

#### 3.1. The case of additive potentials

**Theorem 3.1.** For any \( \varphi \in C(M, \mathbb{R}) \) and \( \mu \in \mathcal{M}_f(M) \), we have

\[
P^n_\mu(f, \varphi) = h^n_\mu(f) + \int_M \varphi \, d\mu.
\]

First we state lemma 3.3 of [2], which is needed in the following proof.
**Proposition 3.2.** Given \( \mu \in \mathcal{M}_1^s(M) \), \( \eta \in \mathcal{P}^s \), and \( \xi \in \mathcal{P} \) with \( H_\mu(\xi|\eta) < \infty \). Then for \( \mu\text{-a.e.} \, x \in M \), there exists a set \( G_x \subset \bigcap_{n=0}^\infty f^{-i}(f^i x) \) with \( \mu(G_x) = 1 \) such that

\[
\lim_{n \to \infty} -\frac{1}{n} \log \mu^s_i(\alpha_{n}^{i-1}(y)) = h_\mu(f, \xi|\eta)
\]

for each \( y \in G_x \), where \( \mu = \int \mu^s_i \, d\mu(x) \) is the measure disintegration of \( \mu \) over \( \eta \).

The following result will be used frequently in the rest of paper.

**Proposition 3.3.** Let there be given a probability measure space \((M, \mu)\), and let \( \eta \) be a measurable partition of \( M \). Suppose there is an increasing sequence \( \{K_n\}_{n \geq 1} \) of subsets of \( M \) with \( \mu(\bigcup_{n=1}^\infty K_n) = 1 \). Then for any \( \gamma > 0 \) small enough and any \( c_1, c_2 > 0 \), there is an \( N > 0 \) such that if \( n \geq N \), there is a \( M_n \subset M \) with \( \mu(M_n) \geq 1 - c_1 \gamma \), and for any \( x \in M_n \), one has \( \mu^s_i(K_n) \geq 1 - c_2 \gamma \), where \( \mu = \int \mu^s_i \, d\mu(x) \) is the measure disintegration of \( \mu \) over \( \eta \). Moreover, \( \{M_n\}_{n \geq 1} \) is increasing.

**Proof.** Since \( K_n \subset K_{n+1} \) and \( \mu(\bigcup_{n=1}^\infty K_n) = 1 \), for any \( \gamma > 0 \), there is an \( N > 0 \) such that if \( n \geq N \), then \( \mu(K_n) \geq 1 - \gamma^2(c_1c_2) \). Define

\[
M_n = \{x \in M \mid \mu^s_i(K_n) \geq 1 - c_2 \gamma \},
\]

then it is increasing and the complement

\[
M_n^C = \{x \in M \mid \mu^s_i(K_n^C) \geq c_2 \gamma \}.
\]

Note that

\[
\gamma^2(c_1c_2) \geq \mu(K_n^C) = \int_{M_n} \mu^s_i(K_n^C) \, d\mu(x)
\]

\[
\geq \int_{M_n} \mu^s_i(K_n) \, d\mu(x) \geq c_2 \mu(M_n^C),
\]

So \( \mu(M_n^C) \leq c_1 \), then \( \mu(M_n) \geq 1 - c_1 \gamma^2 \).

**Remark 3.4.** This proposition and its proof indicate that for a subset \( A \) whose \( \mu \)-measure is close to one, we can find another subset \( B \) whose \( \mu \)-measure is also close to one such that for any \( x \in B \), one has \( \mu^s_i(A) \) is large enough up to one. The parameters \( c_1, c_2 \) above are not essential, only for technical applications later. Such constructions appeared before in [2].

**Lemma 3.5.** For any \( \varphi \in C(M, \mathbb{R}) \) and \( \mu \in \mathcal{M}_1^s(M) \), we have

\[
P^s_\mu(f, \varphi) \leq h_\mu^s(f) + \int_M \varphi \, d\mu.
\]

**Proof.** Given any \( \epsilon > 0 \), any \( 0 < \gamma < 1 \), any large \( n \in \mathbb{N} \), any \( \rho > 0 \), and any \( \eta \in \mathcal{P}^s \). Let us choose a finite partition \( \alpha \) of \( M \) such that the diameter of \( \alpha \) is less than \( \epsilon / 2C \), where \( C > 1 \) satisfies

\[
d(y, z) \leq d^\rho(y, z) \leq Cd(y, z) \quad \text{whenever} \quad y, z \in \eta(x) \quad \text{for any} \quad x \in M.
\]

By proposition 3.2, for \( \mu\text{-a.e.} \, x \in M \), there is a set \( G_x \subset \bigcap_{i=0}^\infty f^{-i}(f^i x) \) with \( \mu(G_x) = 1 \) such that

\[
\lim_{n \to \infty} -\frac{1}{n} \log \mu^s_i(\alpha_{n}^{i-1}(y)) = h_\mu(f, \alpha|\eta)
\]
for each \( y \in G_x \). Then for each \( y \in G_x \) and for any \( \rho > 0 \), there exists an \( N(y, \rho) > 0 \) such that if \( n \geq N(y, \rho) \), then
\[
\mu_n^0(\alpha_0^{n-1}(y)) \geq e^{-n(h_n(f)+\rho)}.
\]
Set
\[
E_n = \{ y \in G_x | N(y, \rho) \leq n \},
\]
then
\[
\mu_n^0 \left( \bigcup_{n=1}^{\infty} E_n \right) = 1.
\]
So \( \mu_n^0(E_n) \geq 1 - \gamma/2 \) if \( n \) is large enough. Then one can see that \( E_n \) intersects at most \( e^{n(h_n(f)+\rho)} \) members of \( \alpha_0^{n-1} \). Furthermore,
\[
E_n \subseteq A \cap \left( \bigcap_{i=1}^{n-1} f^{-i} \eta(f^i x) \right),
\]
where \( A \in \alpha_0^{n-1} \) with \( \mu_n^0(A) \geq e^{-n(h_n(f)+\rho)} \). Moreover, each \( A \cap \left( \bigcap_{i=0}^{n-1} f^{-i} \eta(f^i x) \right) \) is contained in an \( (n, \epsilon) \)-\( u \)-Bowen ball. In fact, for any \( y, z \in A \cap \left( \bigcap_{i=0}^{n-1} f^{-i} \eta(f^i x) \right) \), since \( y, z \in \alpha_0^{n-1} \), one has \( d(y, z) \leq \epsilon/2C \) as before, while \( y, z \in \eta(f(x)) \), we have \( d^w(y, z) \leq Cd(y, z) \leq \epsilon/2 \). Similarly, \( f(y), f(z) \in \eta(f)(x) \), then we still have \( d^w(f(y), f(z)) \leq \epsilon/2 \) and so on. Note that we have at most \( e^{n(h_n(f)+\rho)} \) such \( A \), then \( E_n \) can be covered by the same number of \( (n, \epsilon) \)-\( u \)-Bowen balls. If we take a point from each member of \( \alpha_0^{n-1} \cap E_n \), then it is clear that they contribute to an \( (n, \epsilon) \)-\( u \)-spanning set \( F_n \) of \( E_n \). Moreover,
\[
|F_n| \leq e^{n(h_n(f)+\rho)} \quad (+).
\]
On the other hand, according to the Birkhoff’s ergodic theorem, one has
\[
\lim_{n \to \infty} \frac{1}{n} S_n \varphi(y) = \int_M \varphi \, d\mu, \quad \mu\text{-a.e.} y.
\]
By the Egoroff’s theorem, there is a measurable set \( B \) with \( \mu(B) \geq 1 - \gamma^2/2 \), and \( (1/n)S_n\varphi \) converges uniformly to \( \int_M \varphi \, d\mu \) on \( B \). By proposition 3.3 and its proof, there is a subset \( M(B) \) such that \( \mu_n^0(M(B)) \geq 1 - \gamma \), and for any \( x \in M(B) \), one has \( \mu_n^0(B) \geq 1 - \gamma/2 \).

So for each \( x \in M(B) \), if one can take \( n \) to be further large enough, and set \( E = B \cap E_n \), then \( \mu_n^0(E) > 1 - \gamma \); moreover,
\[
\frac{1}{n} S_n \varphi(y) \leq \int_M \varphi \, d\mu + \rho, \quad \text{for all } y \in E.
\]
Take \( F \) to be an \( (n, \epsilon) \)-\( u \)-spanning set of \( E \) with the smallest cardinality (without loss of generality one can assume \( E \) is compact, otherwise just pass to compact subsets of \( E \) with measure changed a little bit), then \( |F| \leq e^{n(h_n(f)+\rho)} \) based on (+). Then for any \( z \in F \), there is a \( y(z) \in E \) such that \( d_n^w(z, y(z)) < \epsilon \). Therefore,
\[ \sum_{z \in F} \sup_{w \in B_{\epsilon}(z)} \exp((S_{\mu}\varphi)(w)) \leq \sum_{z \in F} \exp((S_{\mu}\varphi)(z) + n\tau_{\epsilon}) \leq \sum_{z \in F} \exp((S_{\mu}\varphi)(y(z)) + 2n\tau_{\epsilon}) \leq \sum_{z \in F} \exp \left( n \left( \int_{M} \varphi \, d\mu + \rho \right) + 2n\tau_{\epsilon} \right) \leq \exp \left( h^{\mu}_{\epsilon}(f) + \int_{M} \varphi \, d\mu + 2\rho + 2\tau_{\epsilon} \right), \]

where \( \tau_{\epsilon} = \sup \{|\varphi(x) - \varphi(y)| : d(x, y) < \epsilon\} \). Then,

\[ P^{\mu}_{\epsilon}(f, \varphi, \epsilon, \eta(x), \gamma) \leq h^{\mu}_{\epsilon}(f) + \int_{M} \varphi \, d\mu + 2\rho + 2\tau_{\epsilon}. \]

Let \( \gamma \to 0 \) and \( \epsilon \to 0 \) (hence \( \tau_{\epsilon} \to 0 \)), since \( \rho > 0 \) is arbitrary, we obtain

\[ P^{\mu}_{\epsilon}(f, \varphi) \leq h^{\mu}_{\epsilon}(f) + \int_{M} \varphi \, d\mu. \]

**Lemma 3.6.** For any \( \varphi \in C(M, \mathbb{R}) \) and \( \mu \in \mathcal{M}^\gamma_{\mu}(M) \), we have

\[ P^{\mu}_{\epsilon}(f, \varphi) \geq h^{\mu}_{\epsilon}(f) + \int_{M} \varphi \, d\mu. \]

**Proof.** For any \( \epsilon > 0 \), any natural number \( n \), any \( \eta \in \mathcal{P}^{\mu} \), and any \( 0 < \gamma < 1 \), we first give a lower bound for the minimal cardinality \( S^{\mu}(f, \epsilon, n, \eta(x), \gamma) \) of \( (n, \epsilon, \gamma)\mu \)-spanning sets of \( \eta(x) \).

Let us recall some facts about the Hamming metric. For positive integers \( N \) and \( n \), we set

\[ \omega_{N,n} = \{ \omega = (\omega_0, \ldots, \omega_{n-1}) | \omega_i \in \{1, \ldots, N\}, 0 \leq i \leq n-1 \}. \]

The Hamming metric \( \rho^{H}_{\omega_{N,n}} \) on \( \omega_{N,n} \) is defined by

\[ \rho^{H}_{\omega_{N,n}}(\omega, \bar{\omega}) = \frac{1}{n} \sum_{i=0}^{n-1} (1 - \delta_{\omega_i, \bar{\omega}_i}), \]

where \( \delta_{ij} \) is the Kronecker symbol.

For \( \omega \in \omega_{N,n}, r > 0 \), we denote by \( B^{H}(\omega, r) \) the closed \( r \)-ball in the metric \( \rho^{H}_{\omega_{N,n}} \) with the center at \( \omega \). The standard combinatorial arguments show that the number of points in \( B^{H}(\omega, r) \), say \( B(r, N, n) \), depends only on \( r, N, n \) (not on \( \omega \)), and equals

\[ B(r, N, n) = \sum_{m=0}^{\lfloor n \rfloor} (N - 1)^m C^m_n. \]

By the Stirling's formula, if \( 0 < r < (N - 1)/N \), then it is easy to see that

\[ \lim_{n \to \infty} \frac{\log B(r, N, n)}{n} = r \log(N - 1) - r \log r - (1 - r) \log(1 - r). \] (3.1)
For any $y \in M$ and $P_{0} \ni \alpha = \{A_{1}, \ldots, A_{N}\}$, set

$$\omega_{y, n} = \{\omega = (\omega_{0}, \ldots, \omega_{n-1})|\omega_{i} \in \{1, \ldots, N\}, f^{i}y \in A_{\omega_{i}}, 0 \leq i \leq n - 1\}.$$  

Now we define a semi-metric $d_{n}^{\alpha}$ on $M$ by

$$d_{n}^{\alpha}(y, z) := \rho_{\mu, \alpha}^{H}(\omega_{y, n}, \omega_{z, n}) = \frac{1}{n} \sum_{i=0}^{n-1} (1 - \delta_{\omega_{i} \omega_{i+1}}).$$

Now for every $\epsilon > 0$, set

$$\partial_{\epsilon}(\alpha) = \bigcup_{A \in \alpha} \partial_{\epsilon}(A),$$

where

$$\partial_{\epsilon}(A) = \{y \in A : \text{there exists a } z \in M \setminus A \text{ such that } d(y, z) < \epsilon\}.$$

Since $\cap_{\epsilon > 0} \partial_{\epsilon}(\alpha) = \partial_{0}(\alpha)$, one has $\lim_{\epsilon \rightarrow 0} \mu(\partial_{\epsilon}(\alpha)) = \mu(\partial_{0}(\alpha))$. (Moreover, we can assume that the measure $\mu$ is everywhere dense in $M$, i.e., the measure of any non-empty open subset of $M$ is positive.)

Let us focus on those partition $\alpha \in P_{0}$ with $\mu(\partial_{0}(\alpha)) = 0$. For any $s > 0$, if $\epsilon$ is small enough, then $\mu(\partial_{\epsilon}(\alpha)) < s^{3}/4$. If $y, z \in M$ and $d_{n}(y, z) < \epsilon$, then for every $0 \leq i \leq n - 1$ either $f^{i}y$ and $f^{i}z$ belong to the same member of $\alpha$, or both of them belong to $\partial_{\epsilon}(\alpha)$. Let us denote for brevity the characteristic function on $\partial_{\epsilon}(\alpha)$ by $\chi_{\epsilon}$ and set

$$B_{n, s} = \left\{y \in M : \sum_{i=0}^{n-1} \chi_{\epsilon}(f^{i}y) < \frac{ns}{2}\right\}.$$  

Since $\int_{M} \chi_{\epsilon} d\mu < s^{3}/4$ and $f$ preserves the measure $\mu$, we have

$$\frac{ns^{3}}{4} \geq \int_{M} \sum_{i=0}^{n-1} \chi_{\epsilon}(f^{i}y) d\mu \geq \int_{M \setminus B_{n, s}} \sum_{i=0}^{n-1} \chi_{\epsilon}(f^{i}y) d\mu \geq \frac{ns}{2} \mu(M \setminus B_{n, s}),$$

and so $\mu(B_{n, s}) > 1 - s^{2}/2$. By proposition 3.3 and its proof, there is a subset $M_{n, s}$ with $\mu(M_{n, s}) \geq 1 - s/2$, and for any $x \in M_{n, s}$, one has $\mu_{\eta}(B_{n, s}) \geq 1 - s$. If $y \in B_{n, s}$ and $d_{n}(y, z) < \epsilon$, then $d_{n}^{\alpha}(y, z) < s/2$. In other words, any intersection of an $\epsilon$-ball in the metric $d_{n}$ with the set $B_{n, s}$ is contained in some $s/2$-ball in the semi-metric $d_{n}^{\alpha}(y, z)$.

Since $\mu$ is ergodic, for the given $\eta$, according to theorem B in [9], one has

$$\lim_{n \rightarrow \infty} - \frac{1}{n} \log \mu_{\eta}(\alpha_{n}^{0, 1}(x)) = h_{\mu}(f|\eta), \mu\text{-a.e.x.}.$$  

Then there exists a subset $M_{1}$ of $M$ with $\mu(M_{1}) = 1$ such that for any $x \in M_{1}$, one has

$$\lim_{n \rightarrow \infty} - \frac{1}{n} \log \mu_{\eta}(\alpha_{n}^{0, 1}(y)) = h_{\mu}(f|\eta), \mu_{\eta}^{0}\text{-a.e.y } \in \eta(x)$$
since $\mu^n = \mu^0$. Therefore, for $\mu^n$ a.e. $y$, there exists an $N(y) = N(y, \rho) > 0$ such that if $n \geq N(y)$, then

$$
\mu^n((\alpha_0^{n-1}(y))) \leq e^{-n\rho(f(y) - \rho)}.
$$

Denote by $E_n = E_n(\rho) = \{ y \in \eta(x) | N(y, \rho) \leq n \}$, then $E_n \subseteq E_{n+1}$ and $\mu^n(\bigcup_{n=1}^{\infty} E_n) = 1$. So for each $\gamma > 0$, there exists an $N$, such that $\mu^n(E_N) \geq 1 - \gamma$.

Now for each $x \in M_1 \cap M_{2,3}$, consider a system $\Gamma$ of $a$-balls in the $d_n^\alpha$ metric, such that these balls cover the subset $E_n \subseteq \eta(x)$ with $\mu^n(E_n) \geq 1 - \gamma$ [note that $S = S^n(f, \epsilon, n, \eta(x), \gamma)$]. In other words,

$$
\Omega := \left\{ B_n^\alpha(y_i, \epsilon), 1 \leq i \leq S | E_n \subseteq \bigcup_{i=1}^{S} B_n^\alpha(y_i, \epsilon) \text{ and } \mu^n(E_n) \geq 1 - \gamma \right\}.
$$

Then

$$
\mu^n(E_n \cap B_{n,3}) \geq 1 - \gamma - s.
$$

Suppose that $s < (1 - \gamma)/2$, then $\mu^n(E_n \cap B_{n,3}) > (1 - \gamma)/2$. Since every ball $B_n^\alpha(y_i, \epsilon)$ is contained in $B_n(y_i, \epsilon)$, we claim that the intersection of every ball of $\Omega$ with $B_{n,3}$ is contained in some $s/2$-ball in $d_n^\alpha$. Then there exist $S^\alpha(f, \epsilon, \eta(x), \delta, \gamma)$ balls of radius $s/2$ in the metric $d_n^\alpha$, which cover the set $E_n \cap B_{n,3}$ whose $\mu^n$-measure is greater than $(1 - \gamma)/2$.

To be precise, set

$$
P(n, y) := (\alpha(y), \alpha(fy), \alpha(f^2y), \ldots, \alpha(f^{n-1}y)),
$$

we call $P(n, y)$ the $(\alpha, n)$-path of $y$. Suppose $V \in \alpha_0^{n-1}$, it is obvious that for any two points $y, z \in V$, $P(n, y) = P(n, z)$, denote it by $P(n, V)$. Set

$$
B_1^{\alpha}(y_i) := \left\{ V \in \alpha_0^{n-1}|d_n^\alpha(P(n, V), P(n, y)) < \frac{s}{2} \right\},
$$

where $y_i, i = 1, 2, \ldots, S^\alpha(f, \epsilon, n, \eta(x), \gamma)$ are the centers of the balls in $\Omega$. These are the $s/2$-balls we claimed.

While for sufficiently large $n$, some subset of the set $E_n \cap B_{n,3}$ with measure greater than $(1 - \gamma)/4$ consists of elements of $\alpha_0^{n-1} \cap \eta(x)$ and the measure of such an element is less than $e^{-n\rho(f(y) - \rho)}$ by the conclusion before. Consequently, the number of such elements is more than $(1 - \gamma)e^{-n\rho(f(y) - \rho)/4}$.

Set

$$
B_2^{\alpha}(y_i) := \bigcup_{i=1}^{S^\alpha(f, \epsilon, n, \eta(x), \gamma)} B_1^{\alpha}(y_i),
$$

note that cardinality of each $B_2^{\alpha}(y_i)$ is at most $B(\frac{s}{2}, |\alpha|, n)$, then

$$
\text{Card} \left( B_2^{\alpha} \right) \leq S^\alpha(f, \epsilon, n, \eta(x), \gamma) \cdot B(\frac{s}{2}, |\alpha|, n).
$$

Thus we have

$$
S^\alpha(f, \epsilon, n, \eta(x), \gamma) \cdot B(\frac{s}{2}, |\alpha|, n) \geq \frac{(1 - \gamma)e^{-n\rho(f(y) - \rho)}}{4}.
$$
On the other hand, since $\mu$ is ergodic, according to the Birkhoff’s ergodic theorem, one has
\[
\lim_{n\to\infty} \frac{1}{n} S_n \varphi(y) = \int_M \varphi \, d\mu, \mu\text{-a.e.y.}
\]
Hence for any $\lambda > 0$ and $\mu - \text{a.e.y}$, there exists an $N(y) = N(y, \lambda) > 0$ such that if $n \geq N(y)$, then
\[
\frac{1}{n} S_n \varphi(y) \geq \int_M \varphi \, d\mu - \lambda.
\]
Set $H_n = H_n(\lambda) = \{ y \in M | N(y, \lambda) \leq n \}$, then $H_n \subseteq H_{n+1}$ and $\mu(\bigcup_{n=0}^{\infty} H_n) = 1$. By proposition 3.3, there is a subset $M_n(H)$ with $\mu(M_n(H)) \geq 1 - \gamma$, and for any $x \in M_n(H)$, one has $\mu^n(H_n) \geq 1 - \gamma/2$. For any $x \in M_n(H)$, let $A_x$ be a subset of $\eta(x)$ with $\mu^n(A_x) > 1 - \gamma/2$, set $A = A_1 \cap H_N$, then $\mu^n(A) > 1 - \gamma$. Let $F$ be an $(n, \epsilon)\mu$-spanning set of $A$ with smallest cardinality. Then for any $z \in F$, there exists $y(z) \in A$ such that $d^n_\mu(z, y(z)) < \epsilon$.
Therefore,
\[
\sum_{z \in F} \exp(S_n \varphi)(z) \geq \sum_{z \in F} \exp((S_n \varphi)(y(z)) - n\tau_x)
\]
\[
\geq \sum_{z \in F} \exp \left( n \left( \int_M \varphi \, d\mu - \lambda \right) - n\tau_x \right)
\]
\[
\geq \frac{(1 - \gamma)e^{\gamma n \frac{\rho}{2}}}{4B \left( \frac{\rho}{2}, |x|, n \right)} \exp \left( n \left( \int_M \varphi \, d\mu - \lambda \right) - n\tau_x \right),
\]
where $\tau_x := \sup \{|\varphi(x) - \varphi(y)| : d(x, y) < \epsilon\}$. Therefore,
\[
P^n_\mu(f, \varphi, \epsilon, \eta(x), \gamma) \geq h^n(f) + \int_M \varphi \, d\mu - \lambda - \rho - \tau_x - O(s),
\]
where $O(s) = \frac{s}{2} \log(|\alpha| - 1) - \frac{s}{2} \log \frac{s}{2} - (1 - \frac{s}{2}) \log(1 - \frac{s}{2})$. Since $\lambda, \rho, s, \epsilon$ are arbitrarily small, let them tend to 0 [and hence $\tau_x \to 0$ and $O(s) \to 0$], we obtain
\[
P^n_\mu(f, \varphi) \geq h^n(f) + \int_M \varphi \, d\mu.
\]

### 3.2. The case of sub-additive potentials–a proof of theorem 1.1

**Lemma 3.7.** Let $f : M \to M$ be a $C^1$-smooth partially hyperbolic diffeomorphism and $\mathcal{G} = \{ \log g_n \}_{n=1}^{\infty}$ be a sequence of sub-additive potentials of $f$. For any positive integer $l$ and small number $\rho > 0$, there exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, the following inequality holds:
\[
\sup_{z \in B(y, \epsilon)} \log g_n(z) \leq \sum_{i=0}^{n-1} \frac{1}{l} \log g(f^i y) + nl \rho + C, \quad \forall n, \quad \forall y \in M,
\]
where $B^n(y, \epsilon) = \{ z \in W^u(y) : d^n_\mu(y, z) \leq \epsilon \}$ is the $(n, \epsilon)u$-Bowen ball around $y$ and $C$ is a constant independent of $\rho$ and $\epsilon$.

**Proof.** Note that $d^n$ and $d$ are equivalent at local unstable neighborhood (see the observation in front of proposition 2.4 of [10]), so any unstable local neighborhood $W^u(x, \delta)$ is compact under $d^n$. Then one can get the desired result using a similar argument of lemma 2.2 of [8].

Now we proceed to prove theorem 1.1.
**Proof.** First, we prove \( P^\mu_\eta(f, G, \epsilon, n, \eta(x), \gamma) \).

For any positive integer \( n \) and any \( \rho > 0 \), by lemma 3.1, there is a constant \( C \) such that if \( \epsilon \) is small enough, one has

\[
P^\mu_\eta(f, G, \epsilon, n, \eta(x), \gamma) \leq e^{C+n\rho} \inf \left\{ \sum_{y \in F} \exp \left( \frac{1}{n} \sum_{i=1}^{n-1} \log g_i(f^i(y)) \right) \mid F \text{ is an } (n, \epsilon, \gamma) \text{ } \mu \text{-spanning set of } \eta(x) \right\}.
\]

Set

\[
M(n, \epsilon) = \inf \left\{ \sum_{y \in F} \exp \left( \frac{1}{n} \sum_{i=1}^{n-1} \log g_i(f^i(y)) \right) \mid F \text{ is an } (n, \epsilon, \gamma) \text{ } \mu \text{-spanning set of } \eta(x) \right\},
\]

then apply theorem 3.1 for the potential \( \varphi = \frac{1}{n} \log g_i \), one has

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log n M(n, \epsilon) = h^\mu_\eta(f) + \int \frac{1}{M} \log g_i \, d\mu.
\]

Therefore,

\[
P^\mu_\eta(f, G, \eta(x), \gamma) \leq h^\mu_\eta(f) + \int \frac{1}{M} \log g_i \, d\mu + \rho.
\]

Let \( l \to \infty \) and by the arbitrariness of \( \rho \), one has

\[
P^\mu_\eta(f, G) \leq h^\mu_\eta(f) + G_\eta(\mu).
\]

Second, we prove the inverse inequality

\[
P^\mu_\eta(f, G) \geq h^\mu_\eta(f) + G_\eta(\mu).
\]

For each \( s > 0 \), there exists \( 0 < \rho \leq s \), a measurable partition \( \mathcal{P} \ni \alpha = \{ A_1, \ldots, A_m \} \), and a finite open cover \( \mathcal{U} = \{ U_1, \ldots, U_k \} \) of \( M \) with \( k \geq m \), such that the following properties hold (using regularity of the measure \( \mu \)):

(a) \( \text{diam } \alpha := \sup \{ \text{diam } A_i \mid A_i \in \alpha \} \leq s \) and \( \text{diam } \mathcal{U} := \sup \{ \text{diam } U_j \mid U_j \in \mathcal{U} \} \leq s \);

(b) \( \overline{U_i} \subseteq A_i, \quad 1 \leq i \leq m \);

(c) \( \mu(A_i \setminus U_i) \leq \rho, \quad 1 \leq i \leq m \) and \( \mu(\bigcup_{i=m+1}^k U_i) \leq \rho \);

(d) \( 2\rho \log m \leq s \).

Set

\[
S_n(x) := \text{Card} \left\{ 0 \leq l \leq n - 1 \mid f^l(x) \in \bigcup_{i=m+1}^k U_i \right\},
\]
Since $\mu$ is ergodic, take $h$ to be the characteristic function on the set $\bigcup_{i=m+1}^k U_i$, then $S_n(x) = \sum_{i=0}^{n-1} h(f^i(x))$. According to the Birkhoff’s ergodic theorem, one has

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} h(f^i(x)) = \int_M h \, d\mu = \mu\left(\bigcup_{i=m+1}^k U_i\right) \leq \rho, \mu - \text{a.e.}$$

By the sub-additive Ergodic theorem, one has

$$\lim_{n \to \infty} \frac{1}{n} \log g_n(y) = G_*(\mu), \mu - \text{a.e.}$$

By theorem B in [9], one has

$$\lim_{n \to \infty} -\frac{1}{n} \log \mu^\eta(\alpha_0^{-1}(y)) = h_\mu(f|\eta), \mu - \text{a.e.}$$

Hence for $\mu - \text{a.e.}$, there exists an $N(y) = N(y, \rho) > 0$ such that if $n \geq N(y)$, then

$$S_n(y) \leq 2n\rho, \mu^\eta(\alpha_0^{-1}(y)) \leq e^{-\alpha h_\mu(f|\eta)\rho},$$

and

$$G_*(\mu) - \rho \leq \frac{1}{n} \log g_n(y) \leq G_*(\mu) + \rho.$$

Set $E_N = \{y \in M|N(y) = N(y, \rho) \leq n\}$, then $\mu(\bigcup_{n=1}^\infty E_N) = 1$. So for any $\gamma > 0$ there exists an $N > 0$ large enough with $\mu(E_N) > 1 - \gamma^2$, such that if $n > N$, then for any $y \in E_N$, one has

(a) $S_n(y) \leq 2\rho n$;
(b) $\mu^\eta(\alpha_0^{-1}(y)) \leq e^{-\alpha h_\mu(f|\eta)\rho}$;
(c) $G_*(\mu) - \rho \leq \frac{1}{n} \log g_n(y) \leq G_*(\mu) + \rho$.

By proposition 3.3, there is a subset $M_N$ with $\mu(M_N) \geq 1 - \gamma$, and for any $x \in M_N$, one has $\mu^\eta(E_N) \geq 1 - \gamma$. For each $x \in M_N$, one has $\mu^\eta(E_N) = \mu^\eta(E_N \cap \eta(x)) > 1 - \gamma$. Set $A = E_N \cap \eta(x)$, if $n > N$, then for every $y \in E_N$, one has

(a) $S_n(y) \leq 2\rho n$;
(b) $\mu^\eta(\alpha_0^{-1}(y)) \leq e^{-\alpha h_\mu(f|\eta)\rho}$;
(c) $G_*(\mu) - \rho \leq \frac{1}{n} \log g_n(y) \leq G_*(\mu) + \rho$.

Where $S_n(y) := \text{Card}\{0 \leq i \leq n - 1|f^i(y) \in \bigcup_{i=m+1}^k (U_i \cap \eta(x))\}$.

Set

$$(\alpha_0^{-1})^* := \{D \in \alpha_0^{-1}|D \cap A \neq \emptyset\}.$$

Then for any $n \geq N$, one has

$$\text{Card}\left((\alpha_0^{-1})^*\right) \geq \sum_{D \in (\alpha_0^{-1})^*} \mu^\eta(D) e^{\alpha h_\mu(f|\eta)\rho} \geq \mu^\eta(A) e^{\alpha h_\mu(f|\eta)\rho}. \quad (3.3)$$

On the other hand, choose $C > 1$ satisfies $d(y, z) \leq d^0(y, z) \leq Cd(y, z)$ for any $y, z \in \eta(x)$. Let $2C\epsilon$ be less than the Lebesgue number of the open cover $\mathcal{U}$. Let $F'$ be an $(n, \epsilon)n$-spanning set of $A$. Suppose $F \subseteq F'$ satisfies that for any $y \in F$, $B^\eta_n(y, \epsilon) \cap A \neq \emptyset$. For each $y \in F$ and $B = B^\eta_n(y, \epsilon)$, set

$$pB, \alpha_0^{-1}) = \text{Card}\{C \in \alpha_0^{-1}|C \cap B \cap A \neq \emptyset\}.$$
We now estimate the number \( p(B, \alpha_0^{n-1}) \). Note that \( B^n_y(B(f \cdot y, \epsilon) \subseteq U^y_i = U_i \cap \eta(x) \) for some \( U_i \in \mathcal{U} \). If \( 1 \leq i \leq m \), then \( f^{-1}U^y_i \subseteq f^{-1}A_i^n \), where \( A_i^n = A_i \cap \eta(x) \). If \( m + 1 \leq i \leq k \), then there are at most \( m \) sets of the form \( f^{-1}A_i^n \) which have non-empty intersection with \( f^{-1}U_i^n \). Since \( S^\alpha(y) \leq 2n \), one has \( p(B, \alpha_0^{n-1}) \leq m^{2n \rho} \). Then it follows that

\[
\text{Card} \left( \left( \alpha_0^{n-1} \right)^* \right) \leq \sum_{y \in F} p \left( B^n_y \left( y, \epsilon, \alpha_0^{n-1} \right) \right) \leq \text{Card} (F) m^{2n \rho} = \text{Card} (F) e^{2n \rho \log m}.
\]

Hence

\[
\text{Card}(F) \geq \mu^n(y) e^{n(h_y(f | \eta(x)) - \rho) - 2n \rho \log m},
\]

together with the fact that \( B^n_z(z, \epsilon) \cap A \neq \emptyset \) for each \( z \in F \), then

\[
\sum_{z \in F} \exp \left( \sup_{y \in B^n_z(z, \epsilon)} \log g_y(y) \right) \geq \sum_{z \in F} \exp \left( \sup_{y \in B^n_z(z, \epsilon)} \log g_y(y) \right) \geq \text{Card}(F) \exp \left( n(G_\gamma(\mu) - \rho) \right) \geq \mu^n(y) \exp \left( n(h_y(f | \eta) + G_\gamma(\mu) - 2n \rho - 2n \rho \log m) \right).
\]

This leads to

\[
\frac{1}{n} P_n^\mu(f, G, \epsilon, n, \eta(x), \gamma) \geq \frac{1}{n} \log \mu^n(y) + h_y(f | \eta) + G_\gamma(\mu) - 2n \rho - 2n \rho \log m.
\]

Let \( n \to \infty \), since \( s \) is arbitrary, \( \rho \leq s \), and \( 2n \rho \log m \leq s \), one has

\[
P_n^\mu(f, G) \geq h_y^\mu(f) + G_\mu(\mu).
\]

Remark 3.8. From the proof above, one can see that for any \( \mu \in \mathcal{M}_f(M) \) the quantity \( P_n^\mu(f, G, \eta(x), \gamma) \) in definition 2.5 actually doesn’t depend on \( \gamma \) and \( \eta \in \mathcal{P}^\mu \) for \( \mu \)-a.e. \( x \).

4. Other definitions of unstable measure theoretic pressure

In this section, we investigate other definitions of unstable pressure, in terms of Bowen’s picture and the capacity picture.

Let \( G = \{ \log g_y \}_Y \) be a sequence of sub-additive potentials of \( f \) on \( M \). Let \( Z \subseteq M \) be an arbitrary subset, and \( Z \) needn’t to be compact or \( f \)-invariant. Take \( \eta \in \mathcal{P}^\mu \). Take the \((n, \epsilon)\)u-
Bowen ball around \( x \):

\[
B^n_{y}(x, \epsilon) = \{ y \in W^n_{y}(x) | d^n_{y}(x, y) \leq \epsilon \}.
\]

For each open cover \( \Gamma = \{ B^n_{y}(x, \epsilon) \}_{i \in I} \) of \( Z \cap \overline{W^n(x, \delta)} \), set \( n(\Gamma) = \min \{ n_i | i \in I \} \).

\[
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\]
Definition 4.1. For \( s \in \mathbb{R}, \delta > 0, N \in \mathbb{N}, \epsilon > 0, x \in M, \) and \( Z \subseteq M, \) set

\[
M^w (G, s, N, \epsilon, Z, W^w (x, \delta)) := \inf_{\Gamma} \left\{ \sum_i \exp \left( -s n_i + \sup_{y \in B_{n_i}^w (x, \epsilon)} \log g_{n_i} (y) \right) \right\},
\]

where \( \Gamma \) runs over all countable open covers \( \Gamma = \{ B_{n_i}^w (x_i, \epsilon) \}_{i \in I} \) of \( Z \cap W^w (x, \delta) \) with \( n(\Gamma) \geq N. \)

Let

\[
m^w (G, s, \epsilon, Z, W^w (x, \delta)) := \lim_{N \to \infty} M^w (G, s, N, \epsilon, Z, W^w (x, \delta)),
\]

\[
P^w_B (f, G, \epsilon, Z, W^w (x, \delta)) := \inf \left\{ s | m^w (G, s, \epsilon, Z, W^w (x, \delta)) = 0 \right\},
\]

\[
:= \sup \left\{ s | m^w (G, s, \epsilon, Z, W^w (x, \delta)) = \infty \right\},
\]

and

\[
P^w_B (f, G, Z, W^w (x, \delta)) := \lim \inf_{\epsilon \to 0} P^w_B (f, G, \epsilon, Z, W^w (x, \delta)),
\]

then define

\[
P^w_B (f, G, Z) := \lim_{\delta \to 0} \sup_{x \in M} P^w_B (f, G, Z, W^w (x, \delta)).
\]

We call \( P^w_B (f, G, Z) \) the Bowen unstable topological pressure of \( f \) on the subset \( Z \) w. r. t. \( G. \)

Remark 4.2. 1. As a matter of fact, in definition 4.1, we do not have to take the limit with respect to \( \delta \to 0. \) This can be seen by a simple modification of the proof of proposition 3.1 in [18].

2. With the replacement of \( W^w (x, \delta) \) by \( \eta (x), \) then we have the following definitions.

Definition 4.3. For any \( \mu \in \mathcal{M}_f (M), \) any \( x \in M, \) and any \( \eta \in \mathcal{P}^\mu, \) we define

\[
P^\mu_{B, \eta} (f, G, \eta (x)) := \inf \left\{ P^\mu_B (f, G, Z, W^w (x, \delta)) | \mu_\eta (Z) = 1 \right\},
\]

\[
P^\mu_{B, \eta} (f, G) := \sup_{\eta \in \mathcal{P}^\mu} \int_M P^\mu_{B, \eta} (f, G, \eta (x)) \, d\mu (x),
\]

which is called the Bowen unstable metric pressure of \( f \) w. r. t. \( G, \) where \( \mu = \int \mu_\eta ^0 \, d\mu (x) \) is the measure disintegration of \( \mu \) over \( \eta. \)

Definition 4.4. For any positive integer \( n, \) any \( \epsilon > 0, \) any subset \( Z \subseteq M, \) any \( \delta > 0, \) and any \( x \in M, \) set

\[
\Lambda^w (G, n, \epsilon, Z, W^w (x, \delta)) := \inf_{\Gamma} \left\{ \sum_i \sup_{y \in B_{n_i}^w (x, \epsilon)} g_{n_i} (y) \right\},
\]

where \( \Gamma \) runs over all open covers \( \Gamma = \{ B_{n_i}^w (x_i, \epsilon) \}_{i \in I} \) of \( Z \cap W^w (x, \delta) \) with \( n_i = n \) for all \( i. \)

Then define
\[ CP_n^u(f, G, \epsilon, Z, W^u(x, \delta)) := \liminf_{n \to \infty} \frac{1}{n} \log \Lambda^n(G, n, \epsilon, Z, W^u(x, \delta)), \]
\[ CP^u(G, \epsilon, Z, \omega(x, \delta)) := \limsup_{n \to \infty} \frac{1}{n} \log \Lambda^n(G, n, \epsilon, Z, \omega(x, \delta)), \]
\[ CP_n^u(f, G, \epsilon, Z, \omega(x, \delta)) := \liminf_{\epsilon \to 0} CP_n^u(f, G, \epsilon, Z, \omega(x, \delta)), \]
\[ CP^u(f, G, \epsilon, Z, \omega(x, \delta)) := \liminf_{\epsilon \to 0} CP^u(f, G, \epsilon, Z, \omega(x, \delta)). \]

Then the lower and upper capacity unstable topological pressures of \( f \) on \( Z \) w. r. t. \( G \) are defined by
\[ CP^u(f, G, Z) := \limsup_{\delta \to 0} \int_{M} CP_n^u(f, G, Z, W^u(x, \delta)) \]
and
\[ CP^u(f, G, Z) := \limsup_{\delta \to 0} \int_{M} CP^u(f, G, Z, W^u(x, \delta)). \]

**Definition 4.5.** For any \( \mu \in \mathcal{M}_f(M) \), any \( x \in M \), and any \( \eta \in \mathcal{P}^u \), we define
\[ CP_n^u(f, G, \eta(x)) := \liminf_{\gamma \to 0} \left\{ CP_n^u(f, G, Z, \eta(x)) \mid \mu_{\gamma}(Z) \geq 1 - \gamma \right\}, \]
and
\[ CP^u(f, G) := \sup_{\eta \in \mathcal{P}^u} \int_{M} CP_n^u(f, G, \eta(x)) \, d\mu(x). \]

This is called the lower capacity metric pressure of \( f \) w. r. t. \( G \), and similarly the upper capacity metric pressure can be defined, where \( \mu = \int \mu_{\gamma}^1 \, d\mu(x) \) is the measure disintegration of \( \mu \) over \( \eta \).

Next we collect some basic properties of these pressures.

**Proposition 4.6.** For pressures defined above, the following properties hold.
\( (i) \) \( P(f, G, Z_1) \leq P(f, G, Z_2) \) if \( Z_1 \subseteq Z_2 \), where \( P \) can be chosen to be \( P_n^u, CP_n^u, \) or \( CP^u \).
\( (ii) \) \( P(f, G, \bigcup Z_i) = \sup P(f, G, Z_i) \) for a family \( \{Z_i\} \) of subsets of \( M \), where \( P \) can be chosen to be \( P_n^u, CP_n^u, \) or \( CP^u \).
\( (iii) \) \( P_n^u(f, G, Z) \leq CP_n^u(f, G, Z) \leq CP^u(f, G, Z) \) for any subset \( Z \subseteq M \).
\( (iv) \) For any \( \mu \in \mathcal{M}_f(M) \), one has
\[ P_{\mu}^u(f, G) \leq CP_{\mu}^u(f, G) \leq CP^u(f, G). \]

**Proof.** (i), (ii) Follow from the definitions. (iii) Can be proved by a quite similar argument as the proof of theorem 1.4 (a) in [1]. (iv) Follows immediately from (iii). \( \square \)

To prove theorem 1.3, we need the following two lemmas.

**Lemma 4.7.** For any \( \mu \in \mathcal{M}_f(M) \), one has
\[ CP_{\mu}^u(f, G) \leq h_{\mu}^U(f) + G_\mu(\mu). \]

**Proof.** For any positive integer \( k \), any \( \epsilon > 0 \), and any small number \( \rho > 0 \), take \( \eta \in \mathcal{P}^u \), by corollary 3.2 in [9] and the Birkhoff’s ergodic theorem, one has
\[ \lim_{n \to \infty} -\frac{1}{n} \log \mu^1_{\eta}(B^u_n(y, \epsilon/2)) = h_{\mu}(f|\eta) \]
for $\mu$-a.e. $y$, and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{k} \log g_k(f^i y) = \int \frac{1}{k} \log g_k \, d\mu
\]
for $\mu$-a.e. $y$.

Hence for $\mu$-a.e. $y$, there exists an $N(y, \rho, \epsilon) > 0$ such that if $n \geq N(y, \rho, \epsilon)$, then
\[
\mu^n_\rho(B^n_\rho(y, \epsilon/2)) \geq e^{-n(h(f|\eta) + \rho)},
\]
and
\[
\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{k} \log g_k(f^i y) - \int \frac{1}{k} \log g_k \, d\mu \leq \rho.
\]

Set $K_n(\rho, \epsilon) = \{ y \in M \mid N(y, \rho, \epsilon) \leq n \}$. Then $K_n(\rho, \epsilon) \subseteq K_{n+1}(\rho, \epsilon)$, and $\mu(\bigcup_{n=1}^{\infty} K_n(\rho, \epsilon)) = 1$. So for any $\gamma > 0$, by proposition 3.3, there is a subset $M_N(\rho, \epsilon)$ with $\mu(M_N(\rho, \epsilon)) \geq 1 - \gamma$, and for any $x \in M_N(\rho, \epsilon)$, one has $\mu^n_\rho(K_N(\rho, \epsilon)) \geq 1 - \gamma$.

Furthermore, for each $x \in M_N(\rho, \epsilon)$, let $G(x) = \eta(x) \cap K_N(\rho, \epsilon)$, then $\mu^n_\rho(G(x)) \geq 1 - \gamma$, and for each $y \in G(x)$ and $n > N$, one has
\[
\mu^n_\rho(B^n_\rho(y, \epsilon/2)) = \mu^n_\rho(B^n_\rho(y, \epsilon/2)) \geq e^{-n(h_f(\eta|\rho) + \rho)} \quad \text{(since $\mu^n_\rho = \mu^n_\rho$).} \quad (4.1)
\]

By lemma 3.7, one has
\[
\sup_{z \in B^n_\rho(y, \epsilon)} \log g_n(z) \leq n \int \frac{1}{k} \log g_k \, d\mu + 2n \rho + C.
\]

Let $E$ be an $(n, \epsilon)$-separated set of $\eta(x) \cap K_N(\rho, \epsilon)$ with the largest cardinality. Then
\[
\eta(x) \cap K_N(\rho, \epsilon) \subseteq \bigcup_{y \in E} B^n_\rho(y, \epsilon).
\]
Furthermore, the $\epsilon$-balls $\{ B^n_\rho(y, \epsilon/2) \mid y \in E \}$ are mutually disjoint, and by (4.1), the cardinality of $E$ is less than or equal to $e^{n(h_f(\eta|\rho) + \rho)}$.

Therefore,
\[
\Lambda^n (G, n, \epsilon, K_N(\rho, \epsilon), \eta(x)) \leq \sum_{y \in E} \sup_{z \in B^n_\rho(y, \epsilon)} g_n(z) \leq e^{n(h_f(\eta|\rho) + \rho) + \frac{1}{k} \log g_k \, d\mu + 3\rho + C}
\]
Hence
\[
\overline{CP}^\rho(f, G, x) \leq h_f(\eta) + \int \frac{1}{k} \log g_k \, d\mu + 3\rho.
\]
and so
\[
\overline{CP}^\rho(f, G, x) \leq h_f(\eta) + \frac{1}{k} \log g_k \, d\mu + 3\rho.
\]
Let $k \to \infty$, by the arbitrariness of $\rho$ and theorem A in [9], one gets that
\[
\overline{CP}^\rho(f, G, \eta(x)) \leq h_f(\eta) + G_\epsilon(\mu).
\]
Therefore,
\[ \mathcal{CP}_\mu(f, G) \leq h^\mu_n(f) + \mathcal{G}_n(\mu). \]

\[ \square \]

**Lemma 4.8.** For any \( \mu \in \mathcal{M}_f(M) \), one has
\[ P^{\mu}_{G_n}(f, G) \geq h^\mu_n(f) + \mathcal{G}_n(\mu). \]

**Proof.** For any \( \rho > 0 \), set \( \lambda = h^\mu_n(f) + \mathcal{G}_n(\mu) - 2\rho \). Given \( \eta \in \mathcal{P}^\mu \) and \( \epsilon > 0 \), by corollary 3.2 in [9] and the sub-additive ergodic theorem, for \( \mu \)-a.e. \( y \), there exists an \( N(y, \rho, \epsilon) > 0 \) such that if \( n > N(y, \rho, \epsilon) \), then
\[ \mu_n^\eta(B^\mu_n(y, \epsilon)) \leq e^{-\mu_n^\eta(f(y) - \rho)} \quad \text{and} \quad \frac{1}{n} \log g_n(y) \geq \mathcal{G}_n(\mu) - \rho. \]

Set
\[ K_n(\rho, \epsilon) = \{ y \in M \mid N(y, \rho, \epsilon) \leq n \}. \]

Then \( K_n(\rho, \epsilon) \subseteq K_{n+1}(\rho, \epsilon) \), and \( \mu\left( \bigcup_{n=1}^{\infty} K_n(\rho, \epsilon) \right) = 1 \). By proposition 3.3, there is a subset \( M_N(\rho, \epsilon) \) with \( \mu(M_N(\rho, \epsilon)) \geq 1 - (\rho + \epsilon) \), and for any \( x \in M_N(\rho, \epsilon) \), one has \( \mu_n^\eta(K_N(\rho, \epsilon)) \geq 1 - (\rho + \epsilon) \).

For any \( x \in M_N(\rho, \epsilon) \) and any \( Z \subseteq M \) with \( \mu_n^\eta(Z) = 1 \), then \( \mu_n^\eta(K_N(\rho, \epsilon) \cap \eta(x) \cap Z) > 1 - (\rho + \epsilon) \), denote this intersection by \( G_N(x) \), then for each \( y \in G_N(x) \) and \( n > N \), one has
\[ \mu_n^\eta(B^\mu_n(y, \epsilon)) \leq e^{-\mu_n^\eta(f(y) - \rho)} \quad (\text{since } \mu_n^\eta = \mu_N^\eta). \]

(4.2)

Take any countable open cover \( \Gamma = \{ B^\mu_n(y, \epsilon/2) \} \) of \( G_N(x) \) with \( n(\Gamma) \geq N \). We can assume \( G_N(x) \) is compact, otherwise approximate it by a compact subset within an error. Then we may assume this cover is finite, say \( \{ B^\mu_n(y_1, \epsilon/2), \ldots, B^\mu_n(y_l, \epsilon/2) \} \). For each \( i = 1, \ldots, l \), we can choose \( z_i \in G_N(x) \cap B^\mu_n(y_i, \epsilon/2) \), then \( B^\mu_n(y_i, \epsilon/2) \subseteq B^\mu_n(z_i, \epsilon) \), and \( \{ B^\mu_n(z_i, \epsilon) \} \), forms an open cover of \( G_N(x) \). Then
\[
\sum_{i=1}^l \exp \left( -n_i \lambda + \sup_{y \in B^\mu_n(z_i, \epsilon)} \log g_n(y) \right)
\geq \sum_{i=1}^l \exp \left( -n_i \lambda + n_i (\mathcal{G}_n(\mu) - \rho) \right)
= \sum_{i=1}^l \exp \left( -n_i (h^\mu_n(f) - \rho) \right)
\geq \sum_{i=1}^l \mu_n^\eta(B^\mu_n(z_i, \epsilon)) \geq \mu_n^\eta(K_N) \geq 1 - (\rho + \epsilon) > \frac{1}{2}. \]

Hence
\[ M^\mu \left( \mathcal{G}, \lambda, n, \epsilon, K_N \cap Z, \eta(x) \right) > \frac{1}{2}. \]
Thus
\[ m^\mu (\mathcal{G}, \lambda, \varepsilon, K_N \cap Z, \eta(x)) > \frac{1}{2}, \]
\[ P^\mu_B (f, \mathcal{G}, \lambda, \varepsilon, Z, \eta(x)) \geq \lambda = h^\mu_\mu (f) + \mathcal{G}_\varepsilon (\mu) - 2\rho. \]
so
\[ P^\mu_B (f, \mathcal{G}, \lambda, Z, \eta(x)) \geq \lambda = h^\mu_\mu (f) + \mathcal{G}_\varepsilon (\mu) - 2\rho. \]
Let \( \rho \) go to zero, one has
\[ P^\mu_B (f, \mathcal{G}, \varepsilon, Z, \eta(x)) \geq h^\mu_\mu (f) + \mathcal{G}_\varepsilon (\mu), \]
then
\[ P^\mu_B (f, \mathcal{G}, Z, \eta(x)) \geq h^\mu_\mu (f) + \mathcal{G}_\varepsilon (\mu). \]
Therefore,
\[ P^\mu_{B, \mu} (f, \mathcal{G}) \geq h^\mu_\mu (f) + \mathcal{G}_\varepsilon (\mu). \]
\[ \square \]

Now we proceed to prove theorem 1.3:

**Proof.** It follows from theorem 1.1, lemmas 4.7 and 4.8, and proposition 4.6. \( \square \)

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**References**

[1] Barreira L 1996 A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems *Ergod. Theor. Dynam. Syst.* 16 871–927

[2] Huang P, Chen E and Wang C 2018 Katok’s entropy formula of unstable metric entropy for partially hyperbolic diffeomorphisms (arXiv:1811.05278v1)
[3] Cheng W, Zhao Y and Cao Y 2012 Pressures for asymptotically subadditive potentials under a mistake function *Discrete Contin. Dynam. Syst. Ser. A* **32** 487–97

[4] Cao Y, Feng D and Huang W 2008 The thermodynamic formalism for sub-multiplicative potentials *Discrete Contin. Dynam. Syst.* **20** 639–57

[5] Falconer K 1988 A subadditive thermodynamic formalism for mixing repellers *J. Phys. A: Math. Gen.* **21** 737–42

[6] He L, Lv J and Zhou L 2004 Definition of measure theoretic pressure using spanning sets *Acta Math. Sin.* **20** 709–18

[7] Hirsch M W, Pugh C C and Shah M 1970 Invariant manifolds *Bull. Am. Math. Soc.* **76** 1015–9

[8] Cao Y, Hu H and Zhao Y 2013 Nonadditive measure theoretic pressure and applications to dimensions of an ergodic measure *Ergod. Theor. Dynam. Syst.* **33** 831–50

[9] Hu H, Hua Y and Wu W 2017 Unstable entropies and variational principle for partially hyperbolic diffeomorphisms *Adv. Math.* **321** 31–68

[10] Hu H, Wu W and Zhu Y 2016 Unstable pressure and $u$-equilibrium states for partially hyperbolic diffeomorphisms (arXiv:1601.05504)

[11] Katok A 1980 Lyapunov exponents, entropy and periodic orbits for diffeomorphisms *Inst. Hautes Études Sci. Publ. Math.* **51** 137–73

[12] Pesin Y and Pitskel B 1984 Topological pressure and the variational principle for noncompact sets *Funct. Anal. Appl.* **18** 307–18

[13] Rohlin V A 1952 On the fundamental ideas of measure theory *J. Am. Math. Soc. Translation* **71** 55

[14] Ruelle D 1973 Statistical mechanics on a compact set with $Z^n$ action satisfying expansiveness and specification *Trans. Am. Math. Soc.* **187** 237–51

[15] Tian X and Wu W 2018 Unstable entropies and dimension theory of partially hyperbolic systems (arXiv:1811.03797)

[16] Walters P 1975 A variational principle for the pressure of continuous transformations *Amer. J. Math.* **97** 937–71

[17] Walters P 1982 *An Introduction to Ergodic Theory* (Graduate Texts in Mathematics vol 79) (Berlin: Springer)

[18] Zhang W, Li Z and Zhou Y Unstable topological pressures of partially hyperbolic diffeomorphisms with sub-additive potentials