Entropy of cosmological black holes and generalized second law in phantom energy-dominated universe

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Abstract

Adopting the thin-layer improved brick-wall method, we investigate the thermodynamics of a black hole embedded in a spatially flat Friedmann-Lemaître-Robertson-Walker universe. We calculate the temperature and the entropy at every apparent horizon for arbitrary solution of the scale factor. We show that the temperature and entropy display a non-trivial behavior as a functions of time. In the case of black holes immersed in universe driven by phantom energy, we show that for specific ranges of the equation-of-state parameter and apparent horizons the entropy is compatible with the D-bound conjecture, even the null, dominant and strong energy conditions are violated. In the case of accretion of phantom energy onto black hole with small Hawking-Hayward quasi-local mass, we obtain an equation-of-state parameter in the range $w \leq -5/3$, guaranteeing the validity of the generalized second law.

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1 Introduction

The discovery that the current universe is in an accelerating expansion phase, obtained from the observations of type Ia supernovae [1, 2], inaugurate an era of intense theoretical research to understand the mechanism driving this accelerating expansion. An immediate consequence is that gravity behaves differently on cosmological distance scales. A variety of possible solutions to the cosmic acceleration puzzle have been debated during this decade including the cosmological constant, exotic matter and energy, modified gravity, anthropic arguments, etc. The most favored ones are dark energy based models and modified gravity theories such $f(R)$ gravity and DGP

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gravity with an equation of state parameter $w = P/\rho < -1$ (where $\rho$ and $P$ are the energy density and pressure of the cosmic fluid, respectively).

The dark energy component is usually described by an equation-of-state parameter $w < -1/3$. The simplest and traditional explanation for dark energy is a cosmological constant, which is usually interpreted as the vacuum energy. However, the value required to explain the cosmic expansion, which is of the order of $10^{-120}$, cannot be explained by current particle physics. This is known as the cosmological constant problem. On the other hand, the first year WMAP data combined with the 2dF galaxy survey and the supernova Ia data favor the phantom energy equation-of-state of the cosmic fluid $w < -1$ over the cosmological constant and the quintessence field. A candidate for phantom energy is usually a scalar field with the wrong sign for kinetic energy term [3, 4]. One crucial fate of an expanding universe driven by phantom energy is that the phantom energy density and the scale factor diverge in finite time, ripping apart all bound systems of the universe (galaxies, stars, atoms, nuclei), before the universe approaches the Big-Rip singularity [3, 5].

In this work we are interested by the effect of cosmological expansion on local systems, namely a black hole embedded in a an expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) universe driven by phantom energy. The study of non-stationary event horizons and their thermodynamical parameters, relevant for a quantum theory of gravity, are currently attracting a great interest [6]-[19]. In the pure expanding flat FLRW universe dominated by phantom energy, the radius of the observer’s event horizon diminishes with time and consequently the horizon entropy, $\dot{S}_H < 0$. An other example is the quasi-de Sitter space, where the event horizon and apparent horizon are different [20, 21]. It was found that the first law and second law of thermodynamics cannot hold when both the horizons are considered. Other authors discussed the effect of the presence of black holes in expanding phantom energy-dominated universe on the validity of the generalized second law of gravitational thermodynamics (GSL) [22, 23].

In this paper, we present a detailed calculation of the entropy of the solution of Einstein’s equations recently found and representing a black hole embedded in an expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) universe [24, 25]. Our approach is based on the brick-wall method (BMW) invented by t’Hooft [26]. In this method a brick wall around the horizon prevents the divergence of the free energy (or entropy) of the black hole, which is identified with the canonical ensemble statistical-mechanical free energy (or entropy) due to quantum excitations in thermal equilibrium with the black hole. However, the BMW can not be used for non-equilibrium systems, like black holes with multi-horizons, where each horizon can be considered as an isolated thermodynamical system. The solution to the problem, known as the thin-layer improved method, consists in considering a thin layer near the horizon as a local equilibrium system and invoking thermodynamics of composite systems, such that the entropy of a multi-horizons black hole is the sum of the contributions arising from each horizon [27]. Then in the improved BWM, the global equilibrium has been replaced by a local equilibrium on microscopic scales. This method has been applied to a wide range of black holes with multi-horizons like Schwarzschild-de Sitter black hole,
Kerr-de Sitter black hole, Vaidya black hole, the 5D Ricci-flat black string and recently to the calculation of the entropy in the pure expanding FLRW universe \[28\].

The organization of the paper is as follows: in Section II we review the exact solution describing a black hole embedded in an expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. In section III, the statistical mechanical free energy, thermal energy and entropy due to a scalar field are computed using the improved thin-layer BWM. Then we discuss the conditions under which the D-bound conjecture \[29\] is protect in expanding universe in the presence of black holes. In section IV, our attention will be focused on the conditions under which the second law of thermodynamics and the GSL are satisfied when the universe is driven by phantom energy. Finally, we discuss and summarize our results in section V.

## 2 Cosmological expanding black hole

The first solution of Einstein’s theory of general relativity describing a black hole like object embedded in an expanding universe was introduced by McVittie in 1933 \[30\], and is given in isotropic coordinates by

\[
ds^2 = -\left(1 - \frac{M_0}{2a(t)r}\right)^2 dt^2 + a^2(t) \left(1 + \frac{M_0}{2a(t)r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right),
\]

where \(a(t)\) is the scale factor and \(M_0\) is the mass of the black hole in the static case. In fact, when \(a(t) = 1\), it reduces to the Schwarzschild solution. When the mass parameter is zero, the McVittie reduces to a spatially flat FLRW solution with the scale factor \(a(t)\). The global structure of (1) has been studied and particularly it has been shown that the solution possesses a spacelike singularity on the 2-sphere \(r = M_0/2\), and cannot describe an embedded black hole in an expanding spatially flat FLRW universe \[31\ \[32\].

In the following we adopt the new solution describing a black hole embedded in a spatially flat FLRW universe \[24\ \[25\]

\[
ds^2 = -\left(1 - \frac{M_0}{2r}\right)^2 dt^2 + a^2(t) \left(1 + \frac{M_0}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right),
\]

Using the areal radius

\[
\tilde{r} = r \left(1 + \frac{M_0}{2r}\right)^2, \quad R = a\tilde{r},
\]

the metric takes the following suitable Painlevé-Gullstrand form
\[
\begin{align*}
    ds^2 &= -\left[\left(1 - \frac{2M_0a}{R}\right) - \frac{R^2H^2}{(1 - \frac{2M_0a}{R})}\right]dt^2 + \left(1 - \frac{2M_0a}{R}\right)^{-1}dR^2 \\
    &\quad - 2RH\left(1 - \frac{2M_0a}{R}\right)^{-1}dtR + R^2d\Omega^2,
\end{align*}
\] (4)

where \( H = \dot{a}/a \) is the Hubble parameter and overdot stands for derivative with respect to the cosmic time. We observe that the term \( R^2H^2 \) plays the role of variable cosmological constant. In order to write the metric in the Nolan gauge, we introduce the time transformation \( t \rightarrow \bar{t} \) to remove the \( dtR \) term

\[
    d\bar{t} = F^{-1}(t, R) \left[ dt + \frac{HR}{(1 - \frac{2M_0a}{R})^2 - H^2R^2}dR \right],
\] (5)

where the integrating factor \( F(t, R) \) satisfy

\[
    \partial_RF^{-1} = \partial_t\left[\frac{F^{-1}HR}{(1 - \frac{2M_0a}{R})^2 - H^2R^2}\right].
\] (6)

Then substituting (5) into (4) and replacing \( \bar{t} \rightarrow t \), we obtain

\[
    ds^2 = dh^2 + R^2d\Omega^2,
\] (7)

where the two-dimensional metric is

\[
    dh^2 = -\left[\left(1 - \frac{2M_0a}{R}\right) - \frac{R^2H^2}{(1 - \frac{2M_0a}{R})}\right]F^2dt^2 + \left(1 - \frac{2M_0a}{R}\right)^{-1}\left[\frac{R^2H^2}{(1 - \frac{2M_0a}{R})}\right]dR^2.
\] (8)

The apparent horizons (AH) are solutions of the equation \( h^{ab}\partial_aR\partial_bR = 0 \), which leads to

\[
    \left(1 - \frac{2m_H(t)}{R} \mp RH\right) |_{R_A} = 0,
\] (9)

where we have used the Hawking-Hayward quasi-local mass \( m_H(t) = M_0a(t) \). This mass is always increasing in an expanding universe. Discarding the unphysical branch with the lower sign, we obtain

\[
    R_C = \frac{1}{2H} \left(1 + \sqrt{1 - 8\dot{m}_H(t)}\right),
\] (10)

\[
    R_B = \frac{1}{2H} \left(1 - \sqrt{1 - 8\dot{m}_H(t)}\right),
\] (11)
where $R_C$ and $R_B$ are the cosmic and the black hole AH, respectively. We observe that the AH coincides at a time $t_*$ for which $\dot{a}(t_*) = 1/8M_0$. This coincidence takes place in a future or past universe depending on the kind of matter accretion onto the black hole.

### 3 Statistical mechanical entropy

Now we consider the statistical mechanical entropy that arises from a minimally coupled quantum scalar field of mass $\mu$ in thermal equilibrium at temperature $\beta$ using the thin layer improved BWM method.

The field equation on the background given by (4) is

\[
(\Box - \mu^2) \Phi (x^\mu) = 0, \tag{12}
\]

where $\Phi (x^\mu) = \Psi (R, t) Y_{lm} (\theta, \varphi)$.

Using the relation $2m_H(t) = R_A (1 - R_A H)$, the Klein-Gordon equation takes the form

\[
\left[ \partial_t + \frac{HR\partial_R}{1 - \frac{R_A}{R} (1 - R_A H)} \right] \Psi (R, t) = 0.
\]

In the semi-classical approximation, the wave-function is obtained by the ansatz

\[
\Psi (R, t) \sim e^{i \sum_n \hbar n S_n (R, t)}.
\]

Neglecting higher orders contributions and substituting in Eq. (13) with the ansatz $S_0 = -i\omega t + iB(R, t)$, we obtain

\[
\left[ \left( 1 - \frac{R_A}{R} (1 - R_A H) \right)^2 - R^2 H^2 \right] B'^2 + 2HR\omega B' + \omega^2 = 0.
\]

In our derivation we have assumed the constancy of the frequency $\omega$ near the AH and that $\dot{B}(R, t) \ll \omega$. Solving for $B'$ we obtain

\[
B'(R, t) \equiv p^\pm_R = -\frac{HR\omega}{f(R) \left( 1 - \frac{R_A}{R} (1 - R_A H) \right)} \pm \frac{1}{f(R)} \sqrt{\omega^2 - f(R) \left( \mu^2 + R^{-2}l(l + 1) \right)}, \tag{15}
\]

where we have set
\[ f(R) = 1 - \frac{R_A}{R} (1 - R_A H) - \frac{H^2 R^2}{1 - \frac{R_A}{R} (1 - R_A H)}, \quad (16) \]

which is just \( g_{tt} \). The sign ambiguity in Eq. (15) is related to outgoing \((\partial S_0/\partial R > 0)\) or ingoing \((\partial S_0/\partial R < 0)\) particles.

The number of radial modes with energy less than \( \omega \) is defined by

\[ n(\omega, l) = \frac{1}{2\pi} \int_{R_A} dR (p_R^+ - p_R^-), \quad (17) \]

where we used the average of the radial momentum. The second term in (17) is caused by a different direction such that the outgoing and ingoing particles are taken into account. Using Eq. (15) we get

\[ \pi n(\omega, l) = \int_{R_A} \frac{dR}{f(R)} \sqrt{\omega^2 - f(R) (\mu^2 + R^{-2} l(l + 1))}. \quad (18) \]

Now, according to the canonical ensemble theory the free energy is defined by

\[ \beta F = \int dN \ln (1 - e^{-\beta \omega}). \quad (19) \]

Integrating by parts we obtain

\[ F = - \int \frac{N(\omega)}{e^{\beta \omega} - 1} d\omega, \quad (20) \]

where \( N(\omega) \) is the total number of modes with energy less than \( \omega \) given by

\[ N(\omega) = \int dl (2l + 1) n(\omega, l). \quad (21) \]

Substituting the expression of \( n \) and restricting the integration over \( l \) to the range

\[ 0 \leq l \leq \frac{1}{2} \left[ -1 + \sqrt{1 + 4R^2 \left( \frac{\omega^2}{f(R)} - \mu^2 \right)} \right], \quad (22) \]

the free energy takes the form

\[ F = -\frac{2}{3\pi} \int_{\mu \sqrt{f(R)}}^{\infty} \frac{d\omega}{e^{\beta \omega} - 1} \int_{R} dR \frac{R^2}{f^2(R)} (\omega^2 - \mu^2 f(R))^{3/2}. \quad (23) \]

The integral over the radial variable is determined by the improved thin-layer BWM boundary conditions.
\[ \Phi (x^\mu) = 0 \quad \text{for} \quad R_B + h_B \leq R \leq R_B + h_B + \delta_B, \quad (24) \]
\[ \Phi (x^\mu) = 0 \quad \text{for} \quad R_C - h_C - \delta_C \leq R \leq R_C - h_C, \quad (25) \]

where \( h_B, h_C \ll R_B, R_C \) are ultraviolet cutoffs and \( \delta_B, \delta_C \) are thickness of the thin layer, near the BH horizon and cosmological horizon, respectively. Then, the integrals over \( R \) are performed by noting that \( f(R) \to 0 \) in the near vicinity of the AH. In fact \( f(R) \) can be approximated by

\[ f(R) \approx \frac{2}{R_A} (1 - 2R_A H) (R - R_A). \quad (26) \]

Then using the near horizon approximation \( (\omega^2 - \mu^2 f(R))^{3/2} \to \omega^3 \), and the following integral

\[ \int_0^\infty \frac{\omega^3 d\omega}{e^{\beta \omega} - 1} = \frac{\pi^4}{15 \beta^4}, \quad (27) \]

the free energy is written as \( F = F_B + F_C \) with

\[ F_B = -\frac{2\pi^3}{45 \beta^4_B} \int_{R_B + h_B}^{R_B + h_B + \delta_B} dR \frac{R^2}{f^2(R)}, \quad (28) \]
\[ F_C = -\frac{2\pi^3}{45 \beta^4_C} \int_{R_C - h_C - \delta_C}^{R_C - h_C} dR \frac{R^2}{f^2(R)}, \quad (29) \]

where \( \beta_B \) and \( \beta_C \) are the Hawking temperatures evaluated at the BH and cosmic AH, respectively. Using (26) we obtain

\[ F \simeq \frac{\pi^3}{90} \left[ \frac{R^4_B}{\beta^4_B (1 - 2R_B H)^2 h_B (h_B + \delta_B)} + \frac{R^4_C}{\beta^4_C (1 - 2R_C H)^2 h_C (h_C + \delta_C)} \right]. \quad (30) \]

Let us now introduce the invariant distances \( \tilde{h}_B \) and \( \tilde{\delta}_B \) defined by

\[ \tilde{h}_B = \int_{R_B}^{R_B + h_B} \frac{dR}{\sqrt{f(R)}} = \sqrt{\frac{2R_B h_B}{1 - 2R_B H}}, \quad (31) \]
\[ \tilde{\delta}_B = \int_{R_B + h_B}^{R_B + h_B + \delta_B} \frac{dR}{\sqrt{f(R)}} = \sqrt{\frac{2R_B (h_B + \delta_B) - 2R_B h_B}{1 - 2R_B H}}, \quad (32) \]

The invariant distances near the cosmic AH, \( \tilde{h}_C \) and \( \tilde{\delta}_C \), are obtained by the substitutions \( (h_B, \delta_B) \to (h_C, \delta_C) \) and \( (1 - 2R_B H) \to (2R_C H - 1) \) in the above relations. Then, the free energy takes the form
\[ F \simeq -\frac{\pi^3}{45} \left[ \frac{R_B^5}{\beta_B^4 (1 - 2 R_B H)^3 h_B^2} + \frac{R_C^5}{\beta_C^3 (2 R_C H - 1)^3 h_C^2} \right]. \]  

The internal energy and entropy defined by \( U = \frac{\beta}{\partial \beta} (\beta F) \) and \( S = \beta (U - F) \), respectively, are then obtained as

\[
U = \frac{\pi^3}{15} \left[ \frac{R_B^5}{\beta_B^4 (1 - 2 R_B H)^3} h_B^2 + \frac{R_C^5}{\beta_C^4 (2 R_C H - 1)^3} h_C^2 \right],
\]

\[
S = \frac{2\pi^3}{45} \left[ \frac{R_B^5}{\beta_B^3 (1 - 2 R_B H)^3} h_B^2 + \frac{R_C^5}{\beta_C^3 (2 R_C H - 1)^3} h_C^2 \right].
\]

Using the area of the BH horizon and cosmic AH, \( A_B = 4\pi R_B^2 \) and \( A_C = 4\pi R_C^2 \), respectively, we rewrite the entropy as

\[
S = \frac{\pi}{90} \left[ \left( \frac{R_B}{\beta_B} \right)^3 \frac{A_B}{(1 - 2 R_B H)^3 h_B^2} + \left( \frac{R_C}{\beta_C} \right)^3 \frac{A_C}{(1 - 2 R_C H)^3 h_C^2} \right].
\]

Now the inverse temperature is calculated at the AH using \( \beta^{-1} = |\kappa_A|/2\pi \), where the dynamical surface gravity associated with the AH is \( \kappa_A = (2\sqrt{-h})^{-1} \partial_a (\sqrt{-h} h^{ab} \partial_b R) |_{R_A} \quad [16] \), and where the metric \( h_{ab} \) is defined by \( ds^2 = h_{ab} dx^a dx^b + R(x) d\Omega^2 \). The calculation yields

\[
\kappa_A = \frac{m_H}{R_A^2} - H - \frac{\dot{H}}{2H},
\]

where \( H = \dot{a}(t)/a(t) \), \( \dot{H} = dH/dt \). We also show that the Misner-Sharp energy defined by \( M (R, t) = \frac{R}{2} (1 - g^{\mu\nu} \partial_\mu R \partial_\nu R) \) is given by

\[
M (R, t) = m_H(t) + \frac{HR_A^2}{2}.
\]

Writing \( m_H \) is terms of the AH, the temperature and the Misner-Sharp energy become

\[
\beta_A^{-1} = \frac{1}{4\pi R_A} \left| 1 - \frac{R_A}{H} \left( 3H^2 + \dot{H} \right) \right|,
\]

\[
M (R, T) = \frac{R_A}{2}.
\]

The temperature associated with dynamical horizons is not only related to the Misner-Sharp mass on the AH, but also to the matter content of the universe \[33\]. Here, we point that in a series of papers devoted to thermodynamics of expanding FLRW universe with dark energy it is assumed the usual law \( T_A = 1/2\pi R_A \) for stationary horizons \[34\], which certainly invalids the results obtained
The expression of temperature \(39\) is similar with the temperature in York theory of black holes confined in isothermal cavities \(36\). In this context, a local observer at rest will measure a local temperature which scale as \(1/\sqrt{-g_{tt}}\) for a self-gravitating mass in thermal equilibrium with the boundary of the cavity. The effect of the boundary is reflected in the relation \(38\), which state that the Hawking-Hayward quasi-local mass is the sum of the thermal energy and a gravitational self-energy due to the cavity. In our case, the black hole is embedded in an expanding universe, and the cosmic apparent horizon plays the role of the boundary of the cavity. On the other hand the behavior of temperature with time is contrasted with the naive expectation that the temperature is redshifted by the scale factor \(a(t)\) relative to its value in static background. The additional correction terms to the temperature become important when the density energy of the universe is negligible compared to the expansion rate. The same remark apply to the entropy which will show a behavior with time different from the expected one \(S \sim S_{\text{static}}a^{-3}\). Here some comments are appropriate about the assumption of local equilibrium used in the calculation, even we are considering dynamical horizons. The improved thin-layer BMW method is based on the fact that Hawking radiation is mainly due to the vacuum fluctuations near the black hole horizon(s), and then allows to identify the black hole entropy with the statistical-mechanical entropy associated with quantum fields outside the black hole and confined in small region(s) in the vicinity of the horizon(s). This cure the problem of maintaining thermal equilibrium at large scales of the original BWM of t’Hooft, without taking into account the back-reaction of the background. However, in small regions we are dealing with the assumption of local equilibrium and the validity of the statistical laws we used in the calculation, and therefore need a physical justification. The local equilibrium state is achieved if thermodynamics properties of the system are slowly varying. The last condition when applied to the case of a black hole immersed in an expanding FLRW universe is equivalent to the following statement: “local equilibrium is achieved if local thermalization of the system is faster than the expansion rate of the universe”. In fact assuming that \(H \ll 1\), and using Eq.(39) we obtain \(\beta_C^{-1} \sim \frac{\delta C}{H} - \frac{\delta H}{H} \ll 1\). This condition can be rewritten as \(\frac{\tau}{H} \gg 1\), which is an expression of the last statement.

In the pure flat FLRW case where only the cosmic AH survives, \(R_C = 1/H\), we obtain

\[
\beta_C^{-1} = \frac{H}{2\pi} \left| 1 + \frac{\dot{H}}{2H^2} \right|
\]

which is proportional to the de Sitter temperature, \(\beta_C^{-1} = H/2\pi\).

Now, substituting Eq.(39) in (36), the entropy takes the form

\[
S = \frac{1}{1440\pi} \left[ \frac{1 - \frac{R_B}{H} \left(3H^2 + \dot{H}\right)}{(1 - 2HR_B)^3} A_B^{3/2} + \frac{1 - \frac{R_C}{H} \left(3H^2 + \dot{H}\right)}{(2HR_c - 1)^3} A_C^{3/2} \right].
\]

In order to compare our results with the standard FLRW universe, we set the cutoffs as
\[ \tilde{h}_{B,C} = \sqrt{\frac{G}{180\pi}}, \] (43)

and introduce the following factors

\[ \mathcal{F}_B = \left| \frac{1}{2} - \frac{R_B}{2H} \left( 3H^2 + \dot{H} \right) \right|^3, \quad \mathcal{F}_C = \left| \frac{1}{2} - \frac{R_C}{2H} \left( 3H^2 + \dot{H} \right) \right|^3. \] (44)

Then the final expressions of thermal energy and entropy take the forms

\[ U = \frac{3}{8G} (R_B \mathcal{F}_B + R_C \mathcal{F}_C), \] (45)

\[ S = \frac{1}{4G} (\mathcal{F}_B \mathcal{F}_A + \mathcal{F}_C \mathcal{F}_C) \] (46)

We assume now that matter in the universe is in the form of an imperfect fluid with a radial heat flux, described by the stress-energy tensor

\[ T_{\mu \nu} = (P + \rho) u_\mu u_\nu + Pg_{\mu \nu} + q_\mu u_\nu + q_\nu u_\mu, \] (47)

where \( w^\mu = (\frac{A}{B}, 0, 0, 0) \) is the fluid four velocity, \( q^\mu = (0, q, 0, 0) \) a spacial vector describing radial heat current, and \( A = (1 + M_0/2r), \quad B = (1 - M_0/2r). \) The general solution of the Einstein equations are given by [24],

\[ 8\pi G \rho = 3 \left( \frac{A}{B} \right)^2 H^2, \] (48)

\[ 8\pi G p = - \left( \frac{A}{B} \right)^2 \left[ 3H^2 + 2\dot{H} \right]. \] (49)

Assuming \( r \gg m \) and radial energy inflow (\( q < 0 \)), the accretion rate can be written as

\[ \dot{m}_H \simeq G a A |q|, \]

where \( A = \int \int d\theta d\varphi \sqrt{g_{\Sigma}} = 4\pi r^2 a^2 A^4 \). In terms of the comoving AH, these equations reduce to

\[ H = \frac{8\pi G}{3} R_A \rho, \] (50)

\[ 3H + \frac{2H}{\dot{H}} = -8\pi G R_A \rho. \] (51)

At this stage we would like to comment about the law of temperature \( T_A = \frac{1}{2\pi R_A} \), widely used in
the literature. With the aid of Eqs. (50,51) we observe that the later law is only valid if the energy
density of the universe is significant relative to the expansion rate
\[ \rho \gg \dot{H}. \] (52)
Now, it is instructive to rewrite the dynamical surface gravity as an invariant. In fact using
the Misner-Sharp mass \( M(R_A, T) = R_A/2 \) and defining the projection of the (3+1)-dimensional
stress-energy tensor in the normal direction of the 2-sphere, \( T^{(2)}_{ab} = h^{ab}T_{ab} \) \((a, b = 1, 2)\), we obtain
\[ \kappa_A = \frac{M(R_A, t)}{R_A^2} + 2\pi R_AT^{(2)}_A, \] (53)
where we have set \( G = 1 \). The later invariant relation has been recently proved in different gauges [37]. On the other hand this equation leads immediately to the analogue of first law for dynamical
black holes. In fact using \( A_A = 4\pi R_A^2 \) and \( V_A = \frac{4}{3}\pi R_A^3 \) as the AH area and volume associated with
the AH, we obtain
\[ dM = \frac{\kappa_A}{8\pi} dA_A - \frac{1}{2} T^{(2)}_A dV_A. \] (54)
Assuming now an equation of state of the form \( p = w\rho \) and using Eqs. (50,51), we rewrite the
factors \( F_{B,C} \) as functions of the EoS parameter \( w \),
\[ F_B = \left| \frac{1}{2} - \frac{3}{4}x_B(1-w) \right|^3, \quad F_C = \left| \frac{1}{2} - \frac{3}{4}x_C(1-w) \right|^3, \] (55)
where we defined the reduced AH \( x_{B,C} = R_{B,C}H \). The factor in the standard FLRW universe,
already given in (41), follows by setting \( x_C = 1 \) in the second relation
\[ F_0 = \left| \frac{1 - 3w}{4} \right|^3, \] (56)
which is exactly the entropy factor obtained in [28]. In this case the temperature associated with
the cosmic AH becomes
\[ \beta_C^{-1} = F_0 \frac{H}{2\pi}. \] (57)
Since \( F_0 > 1 \) in a universe driven by phantom energy, this result put some doubts on the claim
that the GSL in the presence of a black hole in a phantom energy-dominated universe is protected
if the temperature is written as \( \beta_C^{-1} = bH/2\pi \), with \( 0 < b < 1 \) [38]. In this work, the backreaction
of the phantom fluid on the black hole has been ignored as the authors assumed the usual pure
flat FLRW metric.
In figure 1 we plotted the behavior of the factors \( F_B \) and \( F_C \) as functions of the EoS parameter
\( w \) for different values of the BH and cosmic AH, respectively. It is interesting to note that the
factors $F_B$ and $F_C$ do not depend on the AH when $w = -1/3$.

In the following let us check if the expression of the total entropy meets the D-bound conjecture \cite{29} generalized to dynamical black holes. The later conjecture asserts that the entropy of a system within a boundary is less than or equal to the gravitational entropy, $S \leq S_G = A/4G$. We consider the special ranges of values of $x_B$ and $x_C$ used in figure 1, and look for the possible values of the EoS parameter $w$ for which the D-bound conjecture is protected. In fact, from the relations giving the AH in Eqs.\cite{10,11}, and assuming that $M_0 \neq 0$, we have $0 \leq F_C \leq 1$ for $1 \leq x_C \lesssim 0.6$ and $-0.55 \lesssim w \lesssim 1/3$, and $0 \leq F_B \leq 1$ for $0 \leq x_B \lesssim 0.4$ and $-4/3 \lesssim w \leq 0$. Then, within these ranges of the EoS parameter and the AH, it is immediate to verify that the entropy associated with the BH and cosmic AH satisfy the D-bound conjecture,

$$S_B \leq \frac{\pi}{4H^2} , \quad S_C \leq \frac{\pi}{H^2},$$

respectively. When $M_0 = 0$, the BH horizon disappears and we obtain $S_C \leq \frac{\pi}{H^2}$ for $-1 \leq w \leq 1/3$. To verify the D-bound conjecture for the total entropy $S_B + S_C$, we have to verify that the following condition

$$F_B R_B^2 + F_C R_C^2 \leq R_0^2$$

holds, where $R_0 = 1/H$ corresponds to the AH in the pure flat FLRW universe. Using $R_0 =
$R_B + R_C$, the condition (59) is rewritten as

$$\mathcal{F}_C \leq 1 + (1 - \mathcal{F}_B) \frac{x_B^2}{x_C} + 2 \frac{x_B}{x_C},$$

where we have used the reduced AH. Since we have $x_B/x_C < 1$, the condition (60) becomes

$$\mathcal{F}_C + \mathcal{F}_B \leq 4.$$  

We easily verify that the above condition is verified for an EoS parameter in the interval $-0.8 < w < 0.6$, and $0 \leq x_B \lesssim 0.4$, and $1 \leq x_C \lesssim 0.6$. On the other hand, if we restrict the values of the reduced cosmic AH to $1 \leq x_C \lesssim 0.8$, the condition (61) is satisfied for $-1.35 \lesssim w < 1.7$. The later values of $x_C$ corresponds to a small Hawking-Hayward quasi local mass. Finally, we point that even the phantom field is unstable and violate the null, strong and dominant energy conditions (Null, SEC, DEC) \cite{39}, the entropy of dynamical black holes immersed in phantom energy-dominated universe, meets the D-bound conjecture.

In what follows, we would like to study the time evolution of the thermodynamics parameters when the black hole is embedded in an FLRW universe driven by phantom energy and accreting this cosmic fluid. In this situation, the scale factor is given by

$$a_{ph}(t) = a_0 (t_{\text{rip}} - t)^{\frac{2}{3(w+1)}},$$

where $t_{\text{rip}}$ in (62) is the big rip time. In that case, as time increases, the black hole horizon increases monotonically while the cosmic AH decreases monotonically. The AH coincide when

$$t_* = t_{\text{rip}} - \left[ \frac{16M_{\text{max}}}{3(w+1)} \right]^{\frac{3(w+1)}{2}},$$

and after which the black hole singularity will become naked in a finite time, violating the cosmic censorship conjecture \cite{25}. In figure 2 we show the evolution of entropy and free energy with time for $w = -1.2$, and different initial mass of the black hole. We observe, that entropy is decreasing and increasing at early times and late times when approaching $t_*$, respectively, and diverges at the critical AH radius $R_C = 1/2H$. This indicates that at times approaching $t_*$, the second law of thermodynamics is protected, $\dot{S} \geq 0$, while at early times the second lwa of thermodynamics is violated. The important result to note here is that, taking into account the effect of backreaction of the phantom fluid on the black hole in an expanding universe, the second law of thermodynamics becomes valid when approaching the coincidence time, while it is always violated in a purely phantom-energy dominated universe.
Figure 2: Variation of entropy and free energy as functions of time with \( w = -1.2 \). From right to left we have \( M_0 = 0 \) (solid), 0.01 (dot), 0.1 (dash), 1 (dash-dot).

4 Generalized second law

It has been advanced that when a black hole is embedded in an expanding universe driven by phantom energy, the GSL is verified under some restrictive conditions [22, 23]. Let us now proceed to discuss the GSL, \( \dot{S} + \dot{S}_f \geq 0 \), where \( S \) is the geometric entropy associated with the AH and \( S_f \) the entropy of the cosmic fluid confined between the BH and the cosmic AH. The entropy of the fluid can be obtained by using the Gibbs equation [40],

\[
T dS_f = V d\rho + (\rho + p) dV,
\]  

where \( T \) is the temperature of the energy in the vicinity of the AH. Using the continuity equation, which is given in our model by

\[
\dot{\rho} + \frac{\dot{R_A}}{R_A} \rho + \frac{3}{2} H (\rho + p) = 0,
\]

and assuming that the cosmic fluid is in thermal equilibrium with the boundary, the evolution of the fluid entropy is then given by

\[
T_A \dot{S}_f = \frac{R_A^2 \dot{H}}{2} \left( 1 - \frac{\dot{R_A} H}{R_A H} - \frac{2 \dot{R_A}}{R_A H} \right).
\]
Using this relation and since the temperature is meaningful only in the near horizon region, we assume that the fluid entropy is the sum of the contributions near the BH and cosmic AH, respectively,

\[ \dot{S}_f = \sum_{A=B,C} \frac{R_A^2 \dot{H}}{2T_A} \left( 1 - \frac{\dot{R}_A H}{R_A H} \left[ 1 + 2 \frac{\dot{H}}{H^2} \right] \right). \]  

(66)

Let us now consider explicitly the case where the Hawking-Hayward mass is enough small so that the quantities associated with the BH horizon can be neglected. Then, we can set \( R_B \approx 0 \) and the horizon entropy is due essentially to the contribution near the cosmic AH

\[ S_C \approx F_0 \pi R_C^2, \]  

(67)

where \( F_0 \) is given by (56). The time derivative of the phantom fluid entropy becomes

\[ \dot{S}_f \approx \frac{R_C^2 \dot{H}}{2T_C} \left( 1 - \frac{\dot{R}_C H}{R_C H} \left[ 1 + 2 \frac{\dot{H}}{H^2} \right] \right). \]  

(68)

Now taking the time derivative of \( S_C \), we obtain

\[ \dot{S}_C + \dot{S}_f = \frac{R_C^2 \dot{H}}{2T_C} \left( 1 - \frac{\dot{R}_C H}{R_C H} \left[ 1 + 2 \frac{\dot{H}}{H^2} \right] \right) + 2F_0 \pi R_C \dot{R}_C. \]  

(69)

The GSL states that \( \dot{S}_C + \dot{S}_f \geq 0 \). Note that in the phantom era \( \dot{H} > 0, \dot{R}_C < 0 \), and since \( T_C > 0 \), a necessary condition for the GSL to be satisfied is

\[ 1 - \frac{\dot{R}_C H}{R_C H} \left[ 1 + 2 \frac{\dot{H}}{H^2} \right] \geq 0. \]  

(70)

This condition can be integrated and leads to

\[ R_C(t) \leq a(t) \frac{3 [1 + w \frac{1 + 3w}{1 + w}]}{1}. \]  

(71)

Using the expression of the cosmic AH, we get the following condition on the derivative of the Hawking-Hayward quasi-local mass

\[ \dot{m}_H \geq \frac{1}{8} \left[ 1 - \left[ 2H(t)a(t) \frac{3 [1 + w \frac{1 + 3w}{1 + w}]}{1} - 1 \right]^2 \right]. \]  

(72)

Now substituting the scale factor given by (62), and the expression of the Hubble parameter we finally obtain
\[ \dot{m}_H \geq \dot{m}_{H, \text{crit}} = \frac{1}{8} \left[ 1 - \left( \frac{4a^2_0}{3(1+w)} (t_{\text{rip}} - t)^{-\frac{3w}{1+3w}} + 1 \right)^2 \right]. \]  \hspace{1cm} (73)

In figure 3, we plotted the variation of \( \dot{m}_{H, \text{crit}} \) with time for different values of the EoS parameter. Knowing that the Hawking-Hayward mass is an increasing function of time in an expanding universe, we observe that the GSL remains protected from the past to the present time if \( w \leq -5/3 \).

![Figure 3: Variation of the derivative of the Hawking-Hayward quasi-local mass as a function of time.](image)

Using \( \dot{m}_H = M_0 \dot{a}(t) \) and considering the present time, we get for the mass of the black hole the following constraint

\[ M_0 \geq -\frac{1}{9} \frac{5 + 3w}{(1+w)^2}. \]

Assuming the positivity of the mass, we obtain again \( w \leq -5/3 \), in order for the GSL to be satisfied. This value of the EoS parameter is compatible with the analysis performed on the validity of the D-bound conjecture in the case where the cosmic AH is close to the critical value \( 1/2H \).

### 5 Conclusion

In this paper we have studied the thermodynamical properties of black holes immersed in an expanding spatially flat FLRW universe. We have particularly calculated the entropy and temperature associated with the apparent horizons using the improved thin-layer brick wall method and
the dynamical surface gravity, respectively. The temperature and entropy at the apparent horizons (AH) display a non trivial behavior as a function of time, and are not scaled by the expected factors \(a(t)\) and \(a^{-3}(t)\), respectively. The correction terms become relevant when the expansion rate is significant relative to the density energy of the universe. On the other hand, we found that the sum of entropies associated with the AH meets the D-bound conjecture for an EoS parameter in the interval, \(-1.35 < w < 1.7\), although for \(w < -1\), the null, strong and dominant energy conditions are violated. We have also discussed the validity of the second law of thermodynamics and the generalized second law for a black hole embedded in phantom energy-dominated FLRW universe. The analysis showed that for arbitrary Hawking-Hayward quasi-local mass, the second law of thermodynamics is protected when approaching the coincidence time, at which the apparent horizons coincide, \(R_B = R_C = 1/2H\). On the other, in the case of small Hawking-Hayward quasi-local mass, the GSL is only satisfied if \(w \leq -5/3\).

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