An analog of Matrix Tree Theorem for signless Laplacians

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\textbf{A B S T R A C T}
A spanning tree of a graph is a connected subgraph on all vertices with the minimum number of edges. The number of spanning trees in a graph $G$ is given by Matrix Tree Theorem in terms of principal minors of Laplacian matrix of $G$. We show a similar combinatorial interpretation for principal minors of signless Laplacian $Q$. We also prove that the number of odd cycles in $G$ is less than or equal to \[\frac{\det(Q)}{4},\] where the equality holds if and only if $G$ is a bipartite graph or an odd-unicyclic graph.

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1. Introduction

For a simple graph \( G \) on \( n \) vertices \( 1, 2, \ldots, n \) and \( m \) edges \( 1, 2, \ldots, m \) we define its degree matrix \( D \), adjacency matrix \( A \), and incidence matrix \( N \) as follows:

1. \( D = [d_{ij}] \) is an \( n \times n \) diagonal matrix where \( d_{ii} \) is the degree of the vertex \( i \) in \( G \) for \( i = 1, 2, \ldots, n \).
2. \( A = [a_{ij}] \) is an \( n \times n \) matrix with zero diagonals where \( a_{ij} = 1 \) if vertices \( i \) and \( j \) are adjacent in \( G \) and \( a_{ij} = 0 \) otherwise for \( i, j = 1, 2, \ldots, n \).
3. \( N = [n_{ij}] \) is an \( n \times m \) matrix whose rows are indexed by vertices and columns are indexed by edges of \( G \). The entry \( n_{ij} = 1 \) whenever vertex \( i \) is incident with edge \( j \) (i.e., vertex \( i \) is an endpoint of edge \( j \)) and \( n_{ij} = 0 \) otherwise.

We define the Laplacian matrix \( L \) and signless Laplacian matrix \( Q \) to be \( L = D - A \) and \( Q = D + A \), respectively. It is well-known that both \( L \) and \( Q \) have nonnegative real eigenvalues [1, Sec. 1.3]. Note the relation between the spectra of \( L \) and \( Q \):

**Theorem 1.1.** [1, Prop. 1.3.10] Let \( G \) be a simple graph on \( n \) vertices. Let \( L \) and \( Q \) be the Laplacian matrix and the signless Laplacian matrix of \( G \), respectively, with eigenvalues \( 0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \) for \( L \), and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) for \( Q \). Then \( G \) is bipartite if and only if \( \{\mu_1, \mu_2, \ldots, \mu_n\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \).

**Theorem 1.2.** [2, Prop. 2.1] The smallest eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.

We use the following notation for submatrices of an \( n \times m \) matrix \( M \): for sets \( I \subset \{1, 2, \ldots, n\} \) and \( J \subset \{1, 2, \ldots, m\} \),

- \( M[I; J] \) denotes the submatrix of \( M \) whose rows are indexed by \( I \) and columns are indexed by \( J \).
- \( M(I; J) \) denotes the submatrix of \( M \) obtained by removing the rows indexed by \( I \) and removing the columns indexed by \( J \).
- \( M(I; J) \) denotes the submatrix of \( M \) whose columns are indexed by \( J \), and obtained by removing rows indexed by \( I \).

We often list the elements of \( I \) and \( J \), separated by commas in this submatrix notation, rather than writing them as sets. For example, \( M(2; 3, 7, 8) \) is a \((n - 1) \times 3\) matrix whose rows are the same as the rows of \( M \) with the second row deleted and columns are respectively the third, seventh, and eighth columns of \( M \). Moreover, if \( I = J \), we abbreviate \( M(I; J) \) and \( M[I; J] \) as \( M(I) \) and \( M[I] \) respectively. Also we abbreviate \( M(\emptyset; J) \) and \( M(I; \emptyset) \) as \( M(I) \) and \( M(I) \) respectively.
A spanning tree of $G$ is a connected subgraph of $G$ on all $n$ vertices with minimum number of edges which is $n - 1$ edges. The number of spanning trees in a graph $G$ is denoted by $t(G)$ and is given by the Matrix Tree Theorem:

**Theorem 1.3 (Matrix Tree Theorem).** [1, Prop. 1.3.4] Let $G$ be a simple graph on $n$ vertices and $L$ be the Laplacian matrix of $G$ with eigenvalues $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Then the number $t(G)$ of spanning trees of $G$ is

$$t(G) = \det(L(i)) = \frac{\mu_2 \cdot \mu_3 \cdots \mu_n}{n},$$

for all $i = 1, 2, \ldots, n$.

We explore if there is an analog of the Matrix Tree Theorem for the signless Laplacian matrix $Q$. First note that unlike $\det(L(i))$, $\det(Q(i))$ is not necessarily the same for all $i$ as illustrated in the following example.

**Example 1.4.** For the paw graph $G$ with its signless Laplacian matrix $Q$ in Fig. 1, $\det(Q(1)) = 7 \neq 3 = \det(Q(2)) = \det(Q(3)) = \det(Q(4))$.

The Matrix Tree Theorem can be proved by the Cauchy–Binet formula:

**Theorem 1.5 (Cauchy–Binet).** [1, Prop. 1.3.5] Let $m \leq n$. For $m \times n$ matrices $A$ and $B$, we have

$$\det(AB^T) = \sum_S \det(A(;S)) \det(B(;S)),$$

where the summation runs over $\binom{n}{m}$ $m$-subsets $S$ of $\{1, 2, \ldots, n\}$.

The following observation provides a decomposition of the signless Laplacian matrix $Q$ which enables us to apply the Cauchy–Binet formula on it.

**Observation 1.6.** Let $G$ be a simple graph on $n \geq 2$ vertices with $m$ edges, and $m \geq n - 1$. Suppose $N$ and $Q$ are the incidence matrix and signless Laplacian matrix of $G$, respectively. Then
(a) \( Q = NN^T \),
(b) \( Q(i) = N(i; i)N(i; i)^T \), \( i = 1, 2, \ldots, n \), and
(c) \( \det(Q(i)) = \det(N(i; i)N(i; i)^T) = \sum_S \det(N(i; S))^2 \), where the summation runs over all \((n - 1)\)-subsets \( S \) of \( \{1, 2, \ldots, m\} \) (by Cauchy–Binet formula 1.5).

2. Principal minors of signless Laplacians

In this section we find a combinatorial formula for a principal minor \( \det(Q(i)) \) for the signless Laplacian matrix \( Q \) of a given graph \( G \). We mainly use Observation 1.6(c) given by Cauchy–Binet formula which involves determinant of submatrices of incidence matrices. This approach is completely different from the methods applied for related spectral results in [2]. But we borrow the definition of $TU$-subgraphs from [2] slightly modified as follows: A $TU$-graph is a graph whose connected components are trees or odd-unicyclic graphs. A $TU$-subgraph of \( G \) is a spanning subgraph of \( G \) that is a $TU$-graph. The following lemma finds the number of trees in a $TU$-graph.

**Lemma 2.1.** If \( G \) is a $TU$-graph on \( n \) vertices with \( n-k \) edges consisting of \( c \) odd-unicyclic graphs and \( s \) trees, then \( s = k \).

**Proof.** Suppose the number vertices of the cycles are \( n_1, n_2, \ldots, n_c \) and that of the trees are \( t_1, t_2, \ldots, t_s \). Then the total number of edges is

\[
 n - k = \sum_{i=1}^{c} n_i + \sum_{i=1}^{s} (t_i - 1) = n - s
\]

which implies \( s = k \). \( \square \)

Now we find the determinant of incidence matrices of some special graphs in the following lemmas.

**Lemma 2.2.** If \( G \) is an odd (resp. even) cycle, then the determinant of its incidence matrix is \( \pm 2 \) (resp. zero).

**Proof.** Let \( G \) be a cycle with the incidence matrix \( N \). Then up to permutation we have

\[
 N = PN'Q = P \begin{bmatrix}
 1 & 0 & 0 & \cdots & 0 & 1 \\
 1 & 1 & 0 & \cdots & 0 & 0 \\
 0 & 1 & 1 & \cdots & 0 & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 0 & \cdots & \cdots & \cdots & 1 & 0 \\
 0 & \cdots & \cdots & 0 & 1 & 1
\end{bmatrix} Q,
\]
for some permutation matrices $P$ and $Q$. By a cofactor expansion across the first row we have

$$\det(N) = \det(P) \det(N') \det(Q) = (\pm 1)(1 + (-1)^{n+1})(\pm 1).$$

If $n$ is odd (resp. even), then $\det(N) = \pm 2$ (resp. zero). \qed

**Lemma 2.3.** If $G$ is an odd unicyclic (resp. even unicyclic) graph, then the determinant of its incidence matrix is $\pm 2$ (resp. 0).

**Proof.** Let $G$ be a unicyclic graph with incidence matrix $N$ and $t$ vertices not on the cycle. We prove the statement by induction on $t$. If $t = 0$, then $G$ is an odd (resp. even) cycle and then $\det(N_t) = \pm 2$ (resp. 0) by Lemma 2.2. Assume the statement holds for some $t \geq 0$. Let $G$ be a unicyclic graph with $t + 1$ vertices not on the cycle. Then $G$ has a pendant vertex, say vertex $i$. The vertex $i$ is incident with exactly one edge of $G$, say $e_i = \{i, j\}$. Then $i$th row of $N$ has only one nonzero entry which is the $(i, l)$th entry and it is equal to 1. To find $\det(N)$ we have a cofactor expansion across the $i$th row and get

$$\det(N) = \pm 1 \cdot (\pm \det(N(i; l))).$$

Note that $N(i; l)$ is the incident matrix of $G(i)$, which is a unicyclic graph with $t$ vertices not on the cycle. By induction hypothesis, $\det(N(i; l)) = \pm 2$ (resp. 0). Thus $\det(N) = \pm 1 \cdot (\pm \det(N(i; l))) = \pm 2$ (resp. 0). \qed

By a similar induction on the number of pendant vertices we get the following result.

**Lemma 2.4.** Let $H$ be a tree with at least one edge and $N$ be the incidence matrix of $H$. Then $\det(N(i;)) = \pm 1$ for all vertices $i$ of $H$.

**Lemma 2.5.** Let $H$ be a graph on $n$ vertices and $n - 1$ edges with incidence matrix $N$. If $H$ has a connected component which is a tree and an edge which is not on the tree, then $\det(N(i;)) = 0$ for all vertices $i$ not on the tree.

**Proof.** Let $H$ have a connected component $T$ which is a tree and an edge $e_j$ which is not on $T$. Suppose $i$ is a vertex of $G$ that is not on $T$. If $T$ consists of just one vertex, then the corresponding row in $N(i;)$ is a zero row giving $\det(N(i;)) = 0$. Suppose $T$ has at least two vertices. Now consider the square submatrix $N'$ of $N(i;)$ with rows corresponding to vertices of $T$ and columns corresponding to edges of $T$ together with $e_j$. Then the column of $N'$ corresponding to $e_j$ is a zero row giving $\det(N') = 0$. Since entries in rows of $N_i[S]$ corresponding to $T$ that are outside of $N'$ are zero, the rows of $N(i;)$ corresponding to $T$ are linearly dependent and consequently $\det(N(i;)) = 0$. \qed

Now we break down different scenarios that can happen to a graph with $n$ vertices and $m = n - 1$ edges.
Proposition 2.6. Let $H$ be a graph on $n$ vertices and $m = n - 1$ edges. Then one of the following is true for $H$.

1. $H$ is a tree.
2. $H$ has an even cycle and a vertex not on the cycle.
3. $H$ has no even cycles, but $H$ has a connected component with at least two odd cycles and at least two connected components which are trees.
4. $H$ is a disjoint union of odd unicyclic graphs and exactly one tree, i.e., $H$ is a TU-graph.

Proof. If $H$ is connected then it is a tree. This implies Case 1. Now assume $H$ is not connected. If $H$ has no cycles, then it is a forest with at least two connected components. This would imply that $m < n - 2$, contradicting the assumption that $m = n - 1$. Thus $H$ has at least one cycle. Suppose $H$ has $t \geq 2$ connected components $H_i$ with $m_i$ edges and $n_i$ vertices, where the first $k$ of them have at least a cycle and the rest are trees. For $i = 1, \ldots, k$, $H_i$ has $m_i \geq n_i$. Note that

$$-1 = m - n = \sum_{i=1}^{k} (m_i - n_i) = \sum_{i=1}^{k} (m_i - n_i) + \sum_{i=k+1}^{t} (m_i - n_i) \quad (2.1)$$

Since $H_i$ has a cycle for $i = 1, \ldots, k$ and $H_i$ is a tree for $i = k + 1, \ldots, t$,

$$\ell := \sum_{i=1}^{k} (m_i - n_i) \geq 0,$$

and

$$\sum_{i=k+1}^{t} (m_i - n_i) = -(t - k).$$

Then $t - k = \ell + 1$ by (2.1). In other words, in order to make up for the extra edges in the connected components with cycles, $H$ has to have exactly $\ell + 1$ connected components which are trees.

If $H$ has an even cycle, then $\ell \geq 0$ and hence $t - k \geq 1$. This means there is at least one connected component which is tree and it contains a vertex which is not in the cycle. This implies Case 2. Otherwise, all of the cycles of $H$ are odd. If it has more than one cycle in a connected component, then $\ell \geq 1$ and thus $t - k \geq 2$. This implies Case 3. Otherwise, each $H_i$ with $i = 1, \ldots, k$ has exactly one cycle in it, which implies $\ell = 0$, and then $t - k = 1$. This implies Case 4. □

Theorem 2.7. Let $G$ be a simple connected graph on $n \geq 2$ vertices and $m$ edges with the incidence matrix $N$. Let $i$ be an integer from $\{1, 2, \ldots, n\}$. Let $S$ be an $(n - 1)$-subset of
\( \{1, 2, \ldots, m\} \) and \( H \) be a spanning subgraph of \( G \) with edges indexed by \( S \). Then one of the following holds for \( H \).

1. \( H \) is a tree. Then \( \det(N(i; S)) = \pm 1 \).
2. \( H \) has an even cycle and a vertex not on the cycle. Then \( \det(N(i; S)) = 0 \).
3. \( H \) has no even cycles, but it has a connected component with at least two odd cycles and at least two connected components which are trees. Then \( \det(N(i; S)) = 0 \).
4. \( H \) is a \( TU \)-subgraph of \( G \) consisting of \( c \) odd-unicyclic graphs \( U_1, U_2, \ldots, U_c \) and a unique tree \( T \). If \( i \) is a vertex of \( U_j \) for some \( j = 1, 2, \ldots, c \), then \( \det(N(i; S)) = 0 \). If \( i \) is a vertex of \( T \), then \( \det(N(i; S)) = \pm 2^c \).

**Proof.** Suppose vertices and edges of \( G \) are \( 1, 2, \ldots, n \) and \( e_1, e_2, \ldots, e_m \), respectively. Note that \( m \geq n - 1 \) since \( G \) is connected.

1. Suppose \( H \) is a tree. Since \( n \geq 2 \), \( H \) has an edge. Then by Lemma 2.4, \( \det(N(i; S)) = \pm 1 \).
2. Suppose \( H \) contains an even cycle \( C \) as a subgraph and a vertex \( j \) not on \( C \).

   **Case 1.** Vertex \( i \) is not in \( C \).
   Then the square submatrix \( N' \) of \( N(i; S) \) corresponding to \( C \) has determinant zero by Lemma 2.3. Since entries in columns of \( N(i; S) \) corresponding to \( C \) that are outside of \( N' \) are zero, the columns of \( N(i; S) \) corresponding to \( C \) are linearly dependent and consequently \( \det(N(i; S)) = 0 \).

   **Case 2.** Vertex \( i \) is in \( C \).
   Since \( i \) is in \( C \), we have \( j \neq i \). Consider the square submatrix \( N' \) of \( N(i; S) \) that has rows corresponding to vertex \( j \) and vertices of \( C \) excluding \( i \) and columns corresponding to edges of \( C \). Since vertex \( j \) is not on \( C \), the row of \( N' \) corresponding to vertex \( j \) is a zero row and consequently \( \det(N') = 0 \). Since entries in columns of \( N_i[S] \) corresponding to \( C \) that are outside of \( N' \) are zero, the columns of \( N(i; S) \) corresponding to \( C \) are linearly dependent and consequently \( \det(N(i; S)) = 0 \).
3. Suppose \( H \) has no even cycles, but it has a connected component with at least two odd cycles and at least two connected components which are trees. Then vertex \( i \) is not in one of the trees. Then \( \det(N(i; S)) = 0 \) by Lemma 2.5.
4. Suppose \( H \) is a \( TU \)-subgraph of \( G \) consisting of \( c \) odd-unicyclic graphs \( U_1, U_2, \ldots, U_c \) and a unique tree \( T \). If \( i \) is a vertex of \( U_j \) for some \( j = 1, \ldots, c \), then \( \det(N(i; S)) = 0 \) by Lemma 2.5. If \( i \) is a vertex of the tree \( T \), then \( N(i; S) \) is a direct sum of incidence matrices of odd-unicyclic graphs \( U_1, U_2, \ldots, U_c \) and the incidence matrix of the tree \( T \) with one row deleted (which does not exist when \( T \) is a tree on the single vertex \( i \)). By Lemma 2.3 and 2.4, \( \det(N(i; S)) = (\pm 2)^c \cdot (\pm 1) = \pm 2^c \). □

The preceding results are summarized in the following theorem.
Theorem 2.8. Let $G$ be a simple connected graph on $n \geq 2$ vertices and $m$ edges with the incidence matrix $N$. Let $i$ be an integer from $\{1, 2, \ldots, n\}$. Let $S$ be an $(n-1)$-subset of $\{1, 2, \ldots, m\}$ and $H$ be a spanning subgraph of $G$ with edges indexed by $S$.

(a) If $H$ is not a TU-subgraph of $G$, then $\det(N(i; S)) = 0$.

(b) Suppose $H$ is a TU-subgraph of $G$ consisting of $c$ odd-unicyclic graphs $U_1, U_2, \ldots, U_c$ and a unique tree $T$. If $i$ is a vertex of $U_j$ for some $j = 1, 2, \ldots, c$, then $\det(N(i; S)) = 0$. If $i$ is a vertex of $T$, then $\det(N(i; S)) = \pm 2^c$.

For a TU-subgraph $H$ of $G$, the number of connected components that are odd-unicyclic graphs is denoted by $c(H)$. So a TU-subgraph $H$ on $n-1$ edges with $c(H) = 0$ is a spanning tree of $G$.

Theorem 2.9. Let $G$ be a simple connected graph on $n \geq 2$ vertices $1, 2, \ldots, n$ with the signless Laplacian matrix $Q$. Then

$$\det(Q(i)) = \sum_H 4^{c(H)},$$

where the summation runs over all TU-subgraphs $H$ of $G$ with $n-1$ edges consisting of a unique tree on vertex $i$ and $c(H)$ odd-unicyclic graphs.

Proof. By Observation 1.6, we have,

$$\det(Q(i)) = \sum_S \det(N(i; S))^2,$$

where the summation runs over all $(n-1)$-subsets $S$ of $\{1, 2, \ldots, m\}$. By Theorem 2.8, we have,

$$\det(Q(i)) = \sum_S \det(N(i; S))^2 = \sum_H (\pm 2^{c(H)})^2 = \sum_H 4^{c(H)},$$

where the summation runs over all TU-subgraphs $H$ of $G$ with $n-1$ edges consisting of a unique tree on vertex $i$ and $c(H)$ odd-unicyclic graphs. \QED

Example 2.10. Consider the Paw $G$ and its signless Laplacian matrix $Q$ in Fig. 1. To determine $\det(Q(1))$, consider the TU-subgraphs of $G$ with 3 edges consisting of a unique tree on vertex 1: $H_1, H_2, H_3, H_4$ in Fig. 2. Note $c(H_1) = c(H_2) = c(H_3) = 0$ and $c(H_4) = 1$. Then by Theorem 2.9,

$$\det(Q(1)) = \sum_H 4^{c(H)} = 4^{c(H_1)} + 4^{c(H_2)} + 4^{c(H_3)} + 4^{c(H_4)} = 4^0 + 4^0 + 4^0 + 4^1 = 7.$$
Fig. 2. TU-subgraphs of Paw $G$ with 3 edges consisting of a unique tree on vertex 1.

**Corollary 2.11.** Let $G$ be a simple connected graph on $n \geq 2$ vertices $1, 2, \ldots, n$. Let $Q$ be the signless Laplacian matrix of $G$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

(a) $\det(Q(i)) \geq t(G)$, the number of spanning trees of $G$, where the equality holds if and only if all odd cycles of $G$ contain vertex $i$.

(b) $\frac{1}{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}} = \frac{1}{n} \sum_{i=1}^{n} \det(Q(i)) \geq t(G)$,

where the equality holds if and only if $G$ is an odd cycle or a bipartite graph.

**Proof.** (a) First note that a TU-subgraph $H$ on $n - 1$ edges with $c(H) = 0$ is a spanning tree of $G$. Then $\det(Q(i)) = \sum_H 4^{c(H)} \geq \sum_T 4^0$, where the sum runs over all spanning trees $T$ of $G$ containing vertex $i$. So $\det(Q(i))$ is greater than or equal to the number of spanning trees of $G$ containing vertex $i$. Since each spanning tree contains vertex $i$, $\det(Q(i)) \geq t(G)$ where the equality holds if and only if all odd-unicyclic subgraphs of $G$ contain vertex $i$ by Theorem 2.9. Finally note that all odd-unicyclic subgraphs of $G$ contain vertex $i$ if and only if all odd cycles of $G$ contain vertex $i$.

(b) The first equality follows from the well-known linear algebraic result

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}} = \sum_{i=1}^{n} \det(Q(i)).$$

Now by (a) $\det(Q(i)) \geq t(G)$ where the equality holds if and only if all odd cycles of $G$ contain vertex $i$. Then

$$\frac{1}{n} \sum_{i=1}^{n} \det(Q(i)) \geq t(G)$$

where the equality holds if and only if $\det(Q(i)) = t(G)$ for all $i = 1, 2, \ldots, n$. So the equality holds if and only if all odd cycles of $G$ contain every vertex of $G$ which means $G$ is an odd cycle or a bipartite graph ($G$ has no odd cycles).
3. Number of odd cycles in a graph

In this section we find a combinatorial formula for \( \det(Q) \) for the signless Laplacian matrix \( Q \) of a given graph \( G \). As a corollary we show that the number of odd cycles in \( G \) is less than or equal to \( \frac{\det(Q)}{4} \).

**Proposition 3.1.** Let \( H \) be a graph on \( n \) vertices and \( m = n \) edges. Then one of the following is true for \( H \).

1. \( H \) has a connected component which is a tree.
2. All connected components of \( H \) are unicyclic and at least one of them is even-unicyclic.
3. All connected components of \( H \) are odd-unicyclic.

**Proof.** Suppose \( H \) has \( t \geq 2 \) connected components \( H_i \) with \( m_i \) edges and \( n_i \) vertices, where the first \( k \) of them have at least one cycle and the rest are trees. For \( i = 1, \ldots, k \), \( H_i \) has \( m_i \geq n_i \). Note that

\[
0 = m - n = \sum_{i=1}^{t} (m_i - n_i) = \sum_{i=1}^{k} (m_i - n_i) + \sum_{i=k+1}^{t} (m_i - n_i) \quad (3.1)
\]

Since \( H_i \) has a cycle for \( i = 1, \ldots, k \) and \( H_i \) is a tree for \( i = k + 1, \ldots, t \),

\[
\ell := \sum_{i=1}^{k} (m_i - n_i) \geq 0,
\]

and

\[
\sum_{i=k+1}^{t} (m_i - n_i) = -(t - k).
\]

Then \( t - k = \ell \) by (3.1). If \( H \) has a connected component which is a tree, we have Case 1. Otherwise \( t - k = 0 \) which implies \( \ell = \sum_{i=1}^{k} (m_i - n_i) = 0 \). Then \( m_i = n_i \), for \( i = 1, 2, \ldots, k \), i.e., all connected components of \( H \) are unicyclic. If one of the unicyclic components is even-unicyclic, we get Case 2. Otherwise all connected components of \( H \) are odd-unicyclic which is Case 3. Finally if \( H \) is connected, it is unicyclic and consequently it is Case 2 or 3. \( \square \)

**Lemma 3.2.** Let \( H \) be a graph on \( n \) vertices and \( n \) edges with incidence matrix \( N \). If \( H \) has a connected component which is a tree and an edge which is not on the tree, then \( \det(N) = 0 \).
Proof. Let $H$ have a connected component $T$ which is a tree and an edge $e_j$ which is not on $T$. If $T$ consists of just one vertex, say $i$, then row $i$ of $N$ is a zero row giving $\det(N) = 0$. Suppose $T$ has at least two vertices. Now consider the square submatrix $N'$ of $N$ with rows corresponding to vertices of $T$ and columns corresponding to edges of $T$ together with $e_j$. Then the column of $N'$ corresponding to $e_j$ is a zero row giving $\det(N') = 0$. Since entries in rows of $N$ corresponding to $T$ that are outside of $N'$ are zero, the rows of $N$ corresponding to $T$ are linearly dependent and consequently $\det(N) = 0$. \hfill \Box

Theorem 3.3. Let $G$ be a simple graph on $n$ vertices and $m \geq n$ edges with the incidence matrix $N$. Let $S$ be a $n$-subset of $\{1,2,\ldots,m\}$ and $H$ be a spanning subgraph of $G$ with edges indexed by $S$. Then one of the following is true for $H$:

1. $H$ has a connected component which is a tree. Then $\det(N[S]) = 0$.
2. All connected components of $H$ are unicyclic and at least one of them is even-unicyclic. Then $\det(N[S]) = 0$.
3. $H$ has $k$ connected components which are all odd-unicyclic. Then $\det(N[S]) = \pm 2^k$.

Proof. 1. Suppose $H$ has a connected component which is a tree. Since $H$ has $n$ edges, $H$ has an edge not on the tree. Then $\det(N[S]) = 0$ by Lemma 3.2.
2. Suppose all connected components of $H$ are unicyclic and at least one of them is even-unicyclic. Since $N[S]$ is a direct sum of incidence matrices of unicyclic graphs where at least one of them is even-unicyclic, then $\det(N[S]) = 0$ by Lemma 2.2.
3. Suppose $H$ has $k$ connected components which are all odd-unicyclic. Since $N[S]$ is a direct sum of incidence matrices of $k$ odd-unicyclic graphs, then $\det(N[S]) = (\pm 2)^k = \pm 2^k$ by Lemma 2.2. \hfill \Box

By Theorem 1.5 and 3.3, we have the following theorem.

Theorem 3.4. Let $G$ be a simple graph on $n$ vertices with signless Laplacian matrix $Q$. Then

$$\det(Q) = \sum_H 4^{e(H)},$$

where the summation runs over all spanning subgraphs $H$ of $G$ on $n$ edges whose all connected components are odd-unicyclic.

Proof. By Theorem 1.5 and Observation 1.6,

$$\det(Q) = \det(NN^T) = \sum_S \det(N(S[S]))^2,$$

where the summation runs over all $n$-subsets $S$ of $\{1,2,\ldots,m\}$. By Theorem 3.3, we have
\[ \det(Q) = \sum_S \det(N(S))^2 = \sum_H (\pm 2^{c(H)})^2 = \sum_H 4^{c(H)}, \]

where the summation runs over all spanning subgraphs \( H \) of \( G \) whose all connected components are odd-unicyclic. \( \square \)

Let \( ous(G) \) denote the number of spanning subgraphs \( H \) of a graph \( G \) where each connected component of \( H \) is an odd-unicyclic graph. So \( ous(G) \) is the number of \( TU \)-subgraphs of \( G \) whose all connected components are odd-unicyclic. Note that \( c(H) \geq 1 \) for all spanning subgraphs \( H \) of \( G \) whose all connected components are odd-unicyclic. By Theorem 3.4, we have an upper bound for \( ous(G) \).

**Corollary 3.5.** Let \( G \) be a simple graph with signless Laplacian matrix \( Q \). Then \( \det(Q) \geq 4ous(G) \).

For example, if \( G \) is bipartite graph, then \( \frac{\det(Q)}{4} = 0 = ous(G) \). If \( G \) is an odd-unicyclic graph, then \( \frac{\det(Q)}{4} = 1 = ous(G) \).

Note that by appending edges to an odd cycle in \( G \) we get at least one \( TU \)-subgraph of \( G \) with a unique odd-unicyclic connected component. Let \( oc(G) \) denote the number of odd cycles in a graph \( G \). Then \( oc(G) \leq ous(G) \), where the equality holds if and only if \( G \) is a bipartite graph or an odd-unicyclic graph. Then we have the following corollary.

**Corollary 3.6.** Let \( G \) be a simple graph with signless Laplacian matrix \( Q \). Then \( \frac{\det(Q)}{4} \geq oc(G) \), the number of odd cycles in \( G \), where the equality holds if and only if \( G \) is a bipartite graph or an odd-unicyclic graph.

4. Open problems

In this section we pose some problems related to results in Sections 2 and 3. First recall Corollary 3.6 which gives a linear algebraic sharp upper bound for the number of odd cycles in a graph. So an immediate question would be the following:

**Question 4.1.** Find a linear algebraic (sharp) upper bound of the number of even cycles in a simple graph.

To answer this one may like to apply Cauchy–Binet Theorem as done in Sections 2 and 3. Then a special \( n \times m \) matrix \( R \) will be required with the following properties:

1. \( RR^T \) is a decomposition of a fixed matrix for a given graph \( G \).
2. If \( G \) is an even (resp. odd) cycle, then \( \det(R) \) is \( \pm c \) (resp. zero) for some fixed nonzero number \( c \).

For other open questions consider a simple connected graph \( G \) on \( n \) vertices and \( m \geq n \) edges with signless Laplacian matrix \( Q \). The characteristic polynomial of \( Q \) is
\[ P_Q(x) = \det(xI_n - Q) = x^n + \sum_{i=1}^{n} a_i x^{n-i}. \]

It is not hard to see that \( a_1 = -2m \) and \( a_2 = 2m^2 - m - \frac{1}{2} \sum_{i=1}^{n} d_i^2 \) where \((d_1, d_2, \ldots, d_n)\) is the degree-sequence of \( G \). Theorem 4.4 in [2] provides a broad combinatorial interpretation for \( a_i, \ i = 1, 2, \ldots, n. \) A combinatorial expression for \( a_3 \) is obtained in [3, Thm. 2.6] by using mainly Theorem 4.4 in [2]. Note that

\[ a_3 = (-1)^3 \sum_{1\leq i_1 < i_2 < i_3 \leq n} \det(Q[i_1, i_2, i_3]). \]

So it may not be difficult to find corresponding combinatorial interpretation of \( \det(Q[i_1, i_2, i_3]) \) in terms of subgraphs on three edges. Similarly we can investigate other coefficients and corresponding minors which we essentially did for \( a_n \) and \( a_{n-1} \) in Sections 3 and 2 respectively. So the next coefficient to study is \( a_{n-2} \) which entails the following question:

**Question 4.2.** Find a combinatorial expression or a lower bound for \( \det(Q(i_1, i_2)) \).

By Cauchy–Binet Theorem,

\[ \det(Q(i_1, i_2)) = \sum_{S} \det(N(i_1, i_2; S))^2, \]

where the summation runs over all \((n-2)\)-subsets \( S \) of the edge set \( \{1, 2, \ldots, m\} \). So it comes down to finding a combinatorial interpretation of \( \det(N(i_1, i_2; S)) \).

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