One-parameter teleparallel limit of Poincaré gravity

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Poincaré gauge theories that, in the absence of spinning matter, reduce to the one-parameter teleparallel theory are investigated with respect to their mathematical consistency and experimental viability. It is argued that the theories can be consistently coupled to the known standard model particles. Moreover, we establish the result that in the classical limit, such theories share a large class of solutions with general relativity, containing, among others, the four classical black hole solutions (Schwarzschild, Reisner-Nordstrøm, Kerr and Kerr-Newman), as well as the complete class of Friedman-Robertson-Walker cosmological solutions, thereby extending older viability results that were restricted to the correct Newtonian limit and to the existence of the Schwarzschild solution.

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I. INTRODUCTION

In the quest of a Poincaré gauge theory with a suitable limit in the absence of spinning matter fields, we came to the conclusion that no consistent theory with teleparallel limit exists if we require the field equations to reduce, in the spinless limit, to the field equations of general relativity. The reason for the inconsistency of such theories was found in the frame invariance of the field equations in the classical limit, i.e., invariance under a local Lorentz transformation of the tetrad field with the connection held fixed, thereby generalizing the analysis of Kopczyński from purely teleparallel theories to the subclass of Poincaré gauge theories with teleparallel limit \( \mathbf{1} \). (Whenever we use the expressions spinless (or classical) limit, or simply limit, we mean in the limit of vanishing spin, i.e., in the absence of spinning matter fields.) On the other hand, we have already indicated in \( \mathbf{1} \) that one can modify the theories by adding a term of the form \( \lambda T_{ijkl} T_{ijkl} \) to the Lagrangian (\( \lambda \) means total antisymmetrization), without sensibly changing the classical limit of the theory. The limit of such theories corresponds to the so-called one-parameter teleparallel theory (sometimes called new general relativity \( \mathbf{3} \)). As pointed out in \( \mathbf{2} \), this reduces the frame invariance to a rather restricted class of transformations (special Lorentz transformations that leave the axial part of the torsion invariant), therefore taking care, to a certain extend, of the consistency problem. Moreover, it has already been established in \( \mathbf{2} \) that the resulting field equations are still in agreement with the experimental situation, independently of the value of \( \lambda \). This result was based on the consideration of the Newtonian limit on one hand and on the existence of the Schwarzschild solution on the other hand. This ensures agreement with the classical experiments (perihelion shift, light deflection, red shift).

Proceeding in the same fashion as in \( \mathbf{1} \), the above results are easily generalized from the framework of purely teleparallel theories to the class of Poincaré gauge theories with a teleparallel limit. In this paper, we deal with two questions: 1) Are the theories with a teleparallel limit corresponding to the one parameter theory, \( \lambda \neq 0 \), really consistent, or does the remaining frame invariance still represent a severe problem? 2) Can we extend the results on the experimental viability? Or otherwise stated, since general relativity seems to be in agreement with every experiment carried out so far, does the one-parameter theory admit more general relativity solutions apart from the Schwarzschild metric?

It should be noted that both questions can be dealt with within the framework of the purely teleparallel gravity. The actual problems are thus unrelated to the fact that the theory arises here as classical limit of a specific Poincaré gauge theory, whose main motivation is, in the words of Kopczyński, to grant the spin independent dynamical meaning \( \mathbf{2} \). In other words, the results are useful not only to those who, like us, consider the one-parameter theory as a starting point for the construction of viable Poincaré gauge theories with propagating curvature and torsion, but also to those who see in the theory merely an alternative to general relativity.

The paper is organized as follows. In section II, we introduce a concrete Lagrangian, exemplifying the theories we deal with. We derive the field equations and their classical limit. In section III we discuss the consistency question in presence of spinning matter fields, and finally, in section IV, we show that the classical limit of our theory, and more generally, of the one-parameter teleparallel theory, shares with general relativity an important class of metrics, containing, among many others, the known black hole solutions and the relevant cosmological solutions.

II. FIELD EQUATIONS AND CLASSICAL LIMIT

Our notations are identical to those of \( \mathbf{1} \). Especially, the torsion is given by \( T^a_{b,k} = \epsilon^a_{\ i,k} + \Gamma^a_{i,k} e^i_k - [i, k] \) and the curvature by \( R^a_{b,ik} = \Gamma^a_{b,k,i} + \Gamma^c_{b,k} \Gamma^a_{c,ik} - [i, k] \), where \( e^a_i \) and \( \Gamma^a_{bi} \) are the tetrad field and the Lorentz connection, respectively. As argued in \( \mathbf{1} \), the suitable Lagrangian for the construction of Poincaré gauge theories with propagating torsion and curvature presenting a teleparallel...
limit is given by
\[ L = R - \frac{1}{4} T^{ikl} T_{ikl} - \frac{1}{2} T^{ikl} T_{ikl} + T^k_{ik} T^{mi}_m + \lambda T^{ikl} T_{ikl}, \]
where we have added the term \( \lambda T^{ikl} T_{ikl} \), in order to break the frame invariance. Clearly, this term can also be written as combination of terms of the form of the second and third terms in \( L \), but we prefer here to write it separately because our study will focus on this term. (Note also that \( L \) is a scalar, while the Lagrangian density is given by \( \mathcal{L} = e L \), where \( e = \text{det} e^a_i \).

Before we continue, let us remind that the consistency problems in the framework of the purely teleparallel theory (or \textit{new general relativity}) has been considered by Kopezyński [2] some twenty years ago, and in a series of follow up articles, notably in [3]. A more complete set of references can be found in [4]. Especially, in [3], the teleparallel theories were incorporated into the general framework of Poincaré gauge theory with the help of Lagrange multipliers. Recently, in [5], the consistency problem has once again been discussed under the aspects of Lagrange multipliers. Recently, in [6], the consistency problem has once again been discussed under the aspects of the symmetry properties of the stress-energy tensor. Such continuous interest in the subject underlines its importance and the need for a definite answer.

If we proceed adding any term of second order in the curvature tensor to \( L \), we find a theory with the required properties. To be concrete, we consider the following Lagrangian
\[ L = R - \frac{1}{4} T^{ikl} T_{ikl} - \frac{1}{2} T^{ikl} T_{ikl} + T^k_{ik} T^{mi}_m + \lambda T^{ikl} T_{ikl} + a R^ab_{ik} R_{ab}^{ik} + L_m. \]  
(1)

We have also included a matter Lagrangian \( L_m = e L_m \), (we write \( L = L_0 + L_m \)) and define the canonical stress-energy tensor \( T^a_{m} = (1/2e) \delta \mathcal{L}_m/\delta e^a_i \) as well as the spin density \( \sigma_{ab}^{m} = (1/\epsilon) \delta \mathcal{L}_m/\delta T^{ab}_m \). Variation with respect to the independent fields \( e^a_i \) and \( \Gamma_{ab}^m \) leads to the following set of field equations
\[ \tilde{G}_{ik} - \tau^{(1)}_{ik} - \tau^{(2)}_{ik} = T_{ik}, \]  
(2)
and
\[ D_m R^{ablm} + \frac{\lambda}{a} K^{[abl]} = \frac{1}{4a} \sigma^{abl}. \]  
(3)

Here, \( \tau^{(2)}_{ik} = -2a [R^ab_{ik} R_{ab}^l + \frac{1}{4} g_{ik} R^ab_{lm} R_{ab}^{lm}] \) is the symmetric stress-energy tensor of the (Lorentz)-Yang-Mills field, while \( \tau^{(1)}_{ik} \) is given by
\[ \tau^{(1)}_{ik} = \lambda \left[ -2(T_{[km]}^m - \frac{1}{2} T_{[lm]}^m)\Gamma^{l}_{ik} \right] + 2[T_{[km]}^i T^{ml}_i - \frac{1}{2} g_{ik} T_{[lm]}^i T^{lm}_i]. \]  
(4)

We refer to (2) and (3) as Einstein and Cartan equation, respectively. Recall that we took the convention to denote with a hat all quantities constructed from the torsionless connection and that \( D_m \) is defined to act with \( \Gamma^{ab}_m \) on tangent space indices \( a, b, \ldots \) and with \( T^a_{m} \) (the Christoffel symbols) on spacetime indices \( l, m, \ldots \), while the semicolon denotes the usual covariant differentiation with the Christoffel connection. (It should be obvious what is meant by the somewhat sloppy, but convenient notation involving the total antisymmetrization over indices of different nature in the second term of Eq. (3).)

The case \( \lambda = 0 \) has been discussed in [3]. It is an Einstein-Yang-Mills system for the Lorentz gauge field \( \Gamma_{ab}^m \). The Einstein equation is symmetric in that case, as a result of the invariance of \( L_0 \) under frame transformations, and thus, the coupling to spinning matter fields, which present an asymmetric stress-energy tensor, leads to inconsistencies.

The scope of this article is to analyze the question whether the introduction of the term \( \lambda T^{ikl} T_{ikl} \) resolves this problem (section III) and to what extend the classical limit is influenced by this modification (section IV).

In the classical, spinless limit, we see from Eq. (3) that the groundstate solution is given by \( R^ab_{lm} = 0, i.e., \), we have a teleparallel geometry. Further, we find the constraint on the torsion, \( T^{ikl} = 0 \) (no axial torsion). The remaining equations are thus
\[ \tilde{G}_{ik} = T_{ik}, \]  
(5)
\[ T^{ikl} = 0, \]  
(6)
under the condition \( R^ab_{lm} = 0 \). If we use the Poincaré gauge freedom (see [3]) to choose \( \Gamma_{ab}^m = 0, \) the set of equations (5,6) essentially determines the tetrad field \( e^a_i \). It is evident that the same classical limit will be achieved for Lagrangians containing additional terms quadratic in the curvature (e.g., \( R^2 \) or \( R^ab_{ik} R_{ik} \)).

It must be realized that our (classical) equations are stronger than those usually considered in teleparallel theories, where the Lagrangian is considered right from the start to depend only on \( e^a_i \) (tetrad gravity). In the latter approach, Eq. (6) is absent and consequently, Eq. (5) includes the contribution from \( \tau^{(1)} \). The statement \( \Gamma_{ab}^m \) is zero (as a consequence of the field equations) is quite different from the statement \( \Gamma_{ab}^m \) is absent (right from the start). Both approaches share the solutions with \( T^{ikl} = 0 \).

### III. SPECIAL FRAME TRANSFORMATIONS

We now come to the problem that has been discussed extensively in [3] for the case \( \lambda = 0 \). The fact that the Lagrangian \( L_0 \) is invariant under frame transformations leads to problems in the coupling to spinning matter fields, which are described by a non-frame invariant Lagrangian. This is most easily seen in the fact that the Einstein equation in such cases has a symmetric left hand side, while the stress-energy tensor of the matter fields will be asymmetric. We also showed that, even in the case...
where $L_0$ is not frame invariant, but leads nevertheless to a frame invariant groundstate (by groundstate, we mean here the field configuration in the absence of spinning matter), problems arise when one considers spinning test particles entering the gravitational fields. It turns out that not all the measurable gravitational fields are determined by the field equations, and as a consequence, the behavior of spinning test particles is not entirely predictable.

Here, we wish to address the question whether those inconsistencies are remedied by the introduction of the term $\lambda T^{ikl}T_{ikl}$ in Eq. (1). The same problem, in the framework of the purely teleparallel theory, has been discussed in [2] and in the follow-up articles (see the references in [1]). Our specific Lagrangian is such that the (vacuum) field equations have the same symmetry properties (concerning the frame transformations) in both the general case [Eqs. (2,3)] and the classical limit [Eqs. (5,6)]. This allows us to make the discussion directly on the Lagrangian level. (If one adds terms like $R^2$ to $L_0$, the classical limit field configurations will have a larger symmetry than the Lagrangian itself, and the discussion will have to be transferred to test particles moving in classical field configurations, as in [1]. The conclusions, however, are the same in both cases.)

First, note that under a frame transformation $e^a_i \to \Lambda^a_i e^b_i$ (with $\Gamma^{ab}_{ikl}$ fixed), the metric $\eta_{ab}$ and the curvature $R^{ab}_{\;\;\;\;\;lm}$ are invariant, while the torsion transforms as

$$T^{a}_{\;\;ik} \to \Lambda^a_i T^{a}_{\;\;ik} - \epsilon^a_{[k} D_{l]} \Lambda^a_{\;\;b},$$

(7)

or, if we consider the spacetime form $T^l_{\;\;ik} = e^l_a T^a_{\;\;ik}$,

$$T^l_{\;\;ik} \to T^l_{\;\;ik} - \Lambda^a_i e^b_i \epsilon^a_{[k} D_{l]} \Lambda^a_{\;\;b}.$$  

(8)

Our notation is such that the transformation matrix $\Lambda^a_i$ can be dealt with as if it were a tensor, i.e., indices are raised and lowered with $\eta_{ab}$ and, in virtue of $\Lambda^a_i \Lambda^b_j \eta_{cd} = \eta_{ab}$, it holds $\Lambda^a_i = (\Lambda^{-1})^i_a$. (The meaning of the notation $D_l \Lambda^a_i$ is then also obvious.) We see that in the absence of the second terms in (7,8), $T^a_{\;\;ik}$ transforms as a tensor (precisely, as a vector valued two-form), while $T^l_{\;\;ik}$ remains invariant. Note that a sufficient condition for this is not, as is sometimes stated, that $\Lambda^a_i$ is a rigid frame transformation ($\Lambda^a_{\;\;b} = 0$), but rather that it is what could be called covariantly rigid, namely $D_i \Lambda^a_{\;\;b} = 0$. In the teleparallel limit $R^{ab}_{\;\;\;\;\;ik} = 0$, we can choose $\Gamma^{ab}_{ikl} = 0$ and both conditions coincide.

Apart from those covariantly rigid transformations, which do not represent a problem, since the matter fields too are invariant under those, it is clear that $L_0$ (i.e., Lagrangian (1) without the part $L_m$) is invariant under the specific subclass of frame transformations that leave invariant the totally antisymmetric part of the torsion, $T^{[ikl]}$. This is an immediate consequence of the fact that for $\lambda = 0$, $L_0$ is invariant under general frame transformations. That such restricted transformations exist has been established by Kopeczyński in [2], who has also given explicit examples.

Clearly, for this subclass of restricted frame transformations, the whole discussion of [1] applies, whenever the matter Lagrangian is not invariant under those specific transformations. We wish to point out in this article that no established theory exists that contains an elementary particle Lagrangian which is not invariant under the restricted frame transformations.

First, consider bosonic fields. It is commonly accepted (see the review article [3]) that scalar fields couple only to the term $L_0$. The same holds for the Maxwell field and for gauge fields in general, which are considered to be one-forms $A_i = A \wedge dx^i$, and not vector valued scalars $A^a$. Any other coupling to the gravitational fields leads to a violation of the internal gauge invariance (see [7]). Such fields, consequently, have a symmetric stress-energy tensor and a vanishing canonical spin density $\sigma_{ab} = (1/c) \delta L_m/\delta \psi_{ab}$. It is also clear that nothing will change on this situation if a symmetry is spontaneously broken, as in the case of the Weinberg-Salam model. Thus, although it is sometimes argued that massive vector fields (like the Proca field) can be coupled consistently to torsion (because they do not have the gauge invariance of their massless counterparts anyway), such models do not arise from the spontaneous breakdown of a gauge symmetry in a conventional Higgs model. Summarizing, the standard model spin zero and spin one fields are described by frame invariant Lagrangians.

Let us now turn to fermions. Committing a common abuse of language, fermions were actually what we meant by spinning particles in the first place, namely, particles with a non-vanishing spin density $\sigma_{ab}$. As it seems, all the particles so far observed (leptons and quarks) are spin 1/2 fermions described by a Dirac type Lagrangian. In virtue of the well known result that such particles, when minimally coupled to the connection $\Gamma^{ab}_{lm}$, effectively couple only to the totally antisymmetric part of the torsion (see [2]), their Lagrangian is invariant under the restricted frame transformations which leave the axial part of the torsion unaffected. Consequently, all physical quantities will not change during such a restricted frame change. Especially, the parts of the torsion that are not completely determined by the field equations (due to the freedom in the choice of a frame) do not couple to spin 1/2 test particles entering the gravitational fields and are thus physically irrelevant.

This statement can already be found in [3] and also in [2]. It is based on the observation that the Lagrangian is constructed from a suitable contraction of the quantities $\psi, D_m \psi$ and $e^a_m$, which, in the absence of torsion, transform homogeneously (i.e., as spinors and tensors, respectively) under $e^a_m \to \Lambda^a_i e^b_i$, as we know from general relativity, and thus, the only non-homogeneously transforming quantity under the frame transformation is the torsion (appearing in $D_m \psi$). Restricting the transformations to those leaving the only part of the torsion, $T^{[ikl]}$, invariant, leads to the above conclusion.

We wish to point out here that, although the statement is correct in a certain sense, things are a little bit
more involved, and the above line of argumentation is not correct.

The Dirac Lagrangian, minimally coupled to the connection $\Gamma^{ab}_{m}$ is given by

$$L_{m} = \frac{i}{2} \bar{\psi} \gamma^{m} D_{m} \psi - \bar{D}_{m} \bar{\psi} \gamma^{m} \psi,$$

(9)

where $D_{m} \psi = (\partial_{m} \psi - \frac{1}{4} \Gamma^{ab}_{m} \sigma_{ab}) \psi$, $D_{m} \bar{\psi} = \partial_{m} \bar{\psi} + \frac{i}{4} \Gamma^{ab}_{m} \bar{\psi} \sigma_{ab}$, $\gamma^{m} = e^{m}_{a} \gamma^{a}$ and $\sigma_{ab} = \frac{i}{2} [\gamma_{a}, \gamma_{b}]$. The mass term $m \bar{\psi} \psi$ is unproblematic and has been omitted for simplicity.

The Lagrangian (9) is invariant under the following gauge transformation

$$\psi \rightarrow e^{-\frac{i}{2} \varepsilon^{cd} \sigma_{cd}} \psi,$$

$$\Gamma^{ab}_{m} \rightarrow \Lambda^{a}_{c} \Lambda^{b}_{d} \Gamma^{cd}_{m} - \Lambda^{a}_{b} \Lambda^{b}_{d} \Gamma^{cd}_{m},$$

$$e^{m}_{a} \rightarrow \Lambda^{a}_{b} e^{m}_{b},$$

(10)

under the condition that (infinitesimally) $\Lambda^{a}_{b} = \delta^{a}_{b} + e^{a}_{b}$. We have identified this as (Lorentz part of the) Poincaré gauge transformation in [1], in view of the relation of $e^{m}_{a}$ to the gauge potentials of the translational part of the Poincaré group (see [3]). The fact that (9) in invariant under (10) is not simply a result of suitable contractions on spinor and tensor indices, but in addition, a major role is played by the fact that the Lorentz generators $\sigma_{ab}$ do not commute with the Dirac matrix $\gamma^{m} = e^{m}_{a} \gamma^{a}$ that appears explicitly in (9). This is the result of the fact that, when dealing with the Lorentz group, as opposed to conventional gauge theories, inner space, where the gauge transformation takes place, coincides with Dirac (or spinor) space. Therefore, in contrast to other gauge theories, it is not necessary to add an additional index to $\psi$. It carries already a Lorentz representation.

Another, not unrelated remark concerns the transformation behavior of the Dirac matrices. Since the $\gamma^{a}$s, just as $\psi$, are defined in Dirac space, they will transform as $\gamma^{a} \rightarrow S \gamma^{a} S^{-1}$, with $S = \exp[-(i/4)\varepsilon^{cd} \sigma_{cd}]$. However, this transformation is exactly canceled by the transformation concerning the vector index, namely $\gamma^{a} \rightarrow \Lambda^{a}_{b} \gamma^{b}$, if, again, $\Lambda^{a}_{b} = \delta^{a}_{b} + e^{a}_{b}$. As a result, $\gamma^{a}$ and also $\sigma_{cd}$ are invariant under (10).

In [1], we have split the transformation (10) in a pure Lorentz transformation, concerning only the Lorentz connection and leaving $e^{m}_{a}$ invariant, and a frame transformation, $e^{m}_{a} \rightarrow \Lambda^{a}_{b} e^{m}_{b}$, with $\Gamma_{m}^{ab}$ invariant. Here, we deal with the question, whether (9) is invariant under the restricted class of frame transformations that leave $T^{[kl]}$ invariant. Before we do this, it is necessary to define the behavior of the spinor field under those transformations. Following [1], we require again that the result of both transformations (Lorentz + frame) should be equivalent to a Poincaré transformation (10). This leaves us essentially with two possibilities. The first is to extend the Lorentz transformation to

$$\psi \rightarrow e^{-\frac{i}{2} \varepsilon^{cd} \sigma_{cd}} \psi,$$

and consequently, the frame transformation affects only $e^{m}_{a}$ and leaves connection and spinor field invariant. However, it is quite easy to see that (9) is neither invariant under (11), nor under the corresponding frame transformation, not even under the restricted class. Indeed, even if we eliminate the torsion completely, (9) would still not be invariant under (11).

Nevertheless, it turns out that (9) is invariant under a different kind of restricted frame transformations. Let us define, instead of (11), the following Lorentz transformation

$$\psi \rightarrow e^{-\frac{i}{2} \varepsilon^{cd} \sigma_{cd}} \psi,$$

$$\Gamma^{ab}_{m} \rightarrow \Lambda^{a}_{c} \Lambda^{b}_{d} \Gamma^{cd}_{m} - \Lambda^{a}_{b} \Lambda^{b}_{d} \Gamma^{cd}_{m},$$

$$e^{m}_{a} \rightarrow e^{m}_{a},$$

(12)

as well as the corresponding frame transformation

$$\psi \rightarrow e^{-\frac{i}{2} \varepsilon^{cd} \sigma_{cd}} \psi,$$

$$\Gamma^{ab}_{m} \rightarrow \Lambda^{a}_{c} \Lambda^{b}_{d} \Gamma^{cd}_{m},$$

$$e^{m}_{a} \rightarrow \Lambda^{a}_{b} e^{m}_{b}.$$

(13)

Clearly, the combination of (12) and (13) leads again to the Poincaré transformation (10), under which our Lagrangian is invariant. Therefore, invariance under (12) implies invariance under (13) and vice versa.

Applying an infinitesimal transformation (12), $\delta \Gamma^{ab}_{m} = -D_{m} \varepsilon^{ab}$, to the Lagrangian (9), we find for the variation

$$\delta L_{m} = -\frac{1}{8} \bar{\psi} (\gamma^{a} \sigma_{ab} + \sigma_{ab} \gamma^{c}) e^{m}_{c} D_{m} \varepsilon^{ab}.$$ 

(14)

For general $\varepsilon^{ab}$, this does certainly not vanish. However, for our restricted class of transformations, we require the totally antisymmetric part of the torsion to be invariant. Therefore, from Eq. (8), we find (for $\Lambda^{a}_{b} = \delta^{a}_{b} + e^{a}_{b}$) the constraint

$$e_{[a} e^{b]} D_{m} \varepsilon^{ab} = 0,$$

where the antisymmetrization is to be performed over $[kl]$. Since the antisymmetrization process is not troubled by changing spacetime indices into tangent space indices, we also have

$$e^{m}_{[a} e^{b]} D_{m} \varepsilon^{ab} = 0,$$

(15)

for the restricted class of transformations (antisymmetry over $[abc]$). It is well known that the factor $(-\gamma^{a} \sigma_{ab} + \sigma_{ab} \gamma^{c})$ appearing in (14) is totally antisymmetric and therefore, we can conclude that (9) is indeed invariant under (12) and (13) for the restricted class of transformations underlying the constraint (15).

Unfortunately, (12) has nothing to do with a Lorentz gauge transformation, and in (13), there are quite a few
things more involved than just a frame transformation. (Especially since the spinor space transformation on $\psi$ induces also a transformation of the Dirac matrices, as outlined above, which will have to be canceled by the corresponding Lorentz transformation.) We do not know whether the authors of [2] and [3] had the transformation (13) in mind, when they stated that the Dirac Lagrangian was invariant under a restricted class of frame transformations, but it is the only way to realize this.

Fortunately, for the derivation of the field equations and of conservation equations, it is of no importance whether $\psi$ transforms, under a certain transformation, in the way of (11) or of (12), since in the total variation $\delta L_m = (\delta L_m/\delta e^a_m)\delta e^a_m + (\delta L_m/\delta \Gamma_{ab}^c_m)\delta \Gamma_{ab}^c_m + (\delta L_m/\delta \psi)\delta \psi$, the last term does not contribute anyway, independently of the explicit form of $\delta \psi$, because on-shell, the Dirac equation $\delta L_m/\delta \psi = 0$ will be satisfied.

Although in a different way as naively expected, the spin 1/2 Lagrangian reveals itself to be invariant under those transformations that leave the axial part of the torsion tensor invariant, and can thus be consistently coupled to our theory.

Unfortunately, in the case of spin 3/2 particles, the situation is not so favorable. Those particles couple to each of the irreducible parts of the torsion (vector, axial and tensor), and a complete determination of the torsion by the field equations is thus required in order to get a predictable behavior of those particle. Otherwise stated, spin 3/2 particle Lagrangians are not invariant under the restricted frame invariance. However, it must be realized that such particles have not been observed in experiments so far and the reasons for introducing them are purely theoretical and confined to supersymmetric theories. More precisely, in supergravity theories, the supersymmetric partner of the spin two graviton is described by a spin 3/2 field. However, we wish to point out that supergravity theories in their present form are not suitable as constraint, we would not be able to solve it algebraically in generalized theories.

Another point concerns the very base of the concept of supersymmetry. It is usually viewed as a gauge transformation that relates fermions to bosons and vice versa. This is indeed realized in the $N = 1$ model for instance, where the torsion is eliminated algebraically and the remaining dynamical fields are essentially the metric $g_{ik}$ (or the tetrad), and its superpartner, the spin 3/2 gravitino. However, if we start from a theory with propagating torsion and curvature, and try to construct its supersymmetric generalization, it can be expected that, apart from the gravitino, one will also need a fermionic superpartner for the connection $\Gamma_{ab}^c$, which then has the status of an independent dynamical field. A similar view has been expressed in [10] in the context of superconformal gravity.

As a result, there is no much sense in analyzing the transformation behavior of conventional supergravity theories under frame transformations, since they are not suitable for the generalization to Lagrangians of the form (1) in their present form. The first step would be to construct supersymmetric theories allowing for both propagating torsion and curvature, and then to analyze the question whether their are theories among them that reduce, in the spinless limit, to the one-parameter teleparallel field equations.

We conclude that, as far as the standard model particles (gauge bosons, leptons, quarks, scalar fields) are concerned, the restricted frame invariance of the gravitational Lagrangian (1) does not lead to inconsistencies, because those parts of the torsion that remain undetermined by the field equations, as a result of the frame invariance, do not couple to any of those particles and are thus unobservable. This is a result of the fact that the corresponding particle Lagrangians are invariant too under the restricted class of frame transformations.

**IV. THE CONFORMAL KERR-SCHILD CLASS**

This section is entirely devoted to the classical limit of the field equations and to the problem of the experimental viability of the theory. (Unfortunately, experiments involving spin effects of elementary particles in connection with gravity have not been carried out to date, because of the smallness of the expected effects.)

We take here the standpoint that general relativity is experimentally viable in all aspects, and therefore, instead of confronting our theory directly with the experiment, we simply compare it with general relativity.

As we have pointed out at the end of section II, our equations (5,6) are not exactly those of the purely teleparallel theory, because in the latter, the Cartan
equation is naturally absent. They share however the class of solutions with $T^{ikl} = 0$. Clearly, in virtue of Eq. (5), every solution of our classical equations is also a solution of general relativity.

Therefore, we have already solved our problem: Every solution of (5,6) is also a solution of general relativity (and a solution of the conventional one-parameter teleparallel theory). General relativity and our theory in the classical limit have in common the solutions that satisfy $T^{ikl} = 0$.

More precisely, one should say the solutions that can be brought into a form where $T^{ikl} = 0$, since for a given metric, there are many possible choices for the tetrad field (related by frame transformations), leading to different torsion tensors. (Recall that in the classical limit, we can always choose $\Gamma^a_{bc} = 0$.)

An example of such a class of metrics can already be found in [3]. It is the class of metrics that are diagonal and contains, according to the authors, many static solutions with high symmetry, especially the Schwarzschild and the Reisner-Nordstrøm solutions.

We have found yet another, larger class of metrics, namely the conformal Kerr-Schild class, given by metrics of the form

$$g_{ik} = \varphi^2 (\eta_{ik} - k_i k_k), \quad (16)$$

where $\varphi$ is a scalar field, $\eta_{ik}$ the Minkowski metric and $k_i$ a null-vector field, satisfying $\eta^{ik} k_i k_k = 0$.

For simplicity, we first investigate the case $\varphi = 1$. If we define $k^i = \eta^{ik} k_k$, then we have for the inverse of the metric $g^{ik} = \eta_{ik} + k^i k^k$ and consistently, $k^i = g^{ik} k_k$. Note also the relations $\eta_{ik} k^i k^k = \eta_{ik} k^i k^k = 0$. We choose the tetrad field as

$$e^a_i = \delta^a_i - \frac{1}{2} \delta^a_i k^k k_i. \quad (17)$$

Obviously, $e^a_i e^b_i \eta_{ab} = \eta_{ik} - k_i k_k$. The inverse is given by $e^a_i = \delta^a_i + \frac{1}{2} \delta^a_i k^k k_l$. For the torsion, we find

$$T^a_{ik} = e^a_{i,k} - e^a_i k^k = - \frac{1}{2} \delta^a_m (k^m k_k)_i + \frac{1}{2} \delta^a_m (k^m k_i)_k, \quad (18)$$

and its spacetime form reads

$$T_{mik} = - \frac{1}{2} (k^m k_k)_i + \frac{1}{2} (k^m k_i)_k. \quad (19)$$

In deriving those equations, it is helpful to note the relation $k^i k_i k_k = k_i k^i k_k = 0$. Obviously, the totally antisymmetric part of $T_{mik}$ vanishes.

In order to generalize this result to the metrics of the form (16), we show the transformation behavior of $T^a_{ik}$ under a conformal transformation $e^a_m \to \varphi e^a_m$, i.e., $g_{ik} \to \varphi^2 g_{ik}$. We find for the transformed torsion

$$\tilde{T}^a_{ik} = \varphi T^a_{ik} + \varphi_i e^a_k - \varphi^2 k_i e^a_i, \quad (20)$$

leading to

$$\tilde{T}_{mik} = \varphi^2 T_{mik} + \varphi [\varphi_i g_{mk} - \varphi, k g_{mi}]. \quad (21)$$

Clearly, if the axial part of $T_{mik}$ vanishes, then so does the axial part of $T_{mik}$, which completes our argument.

Let us point out that the above transformation of $T^a_{ik}$ follows directly from the specific form of the torsion tensor in the teleparallel limit $(\Gamma^a_{bc} = 0)$ of our theory, as given by the first line in (18), and the transformation behavior of $e^a_m$. This is not to be confused with what is usually called conformal transformation in the framework of Poincaré gauge theory, which requires the specification of the transformation behavior of both tetrad and connection. Such transformations have been considered in, e.g., [11]-[13] in relation with Riemann-Cartan-Weyl geometries. A complete classification of conformal, projective and dilational (or scale) transformations in the more general framework of metric affine theory can be found in [7].

It is important to remark that, although the explicit form of the metric (16) is highly coordinate dependent, the final relation, $T^{ikl} = 0$, is a tensor relation, and therefore holds independently of the coordinate choice. This means that, whenever a solution of general relativity can be brought into the form (16), there will always be a choice for the tetrad, unique up to the restricted frame transformations underlying the constraint (15), such that $T^{ikl}$ vanishes. This particular tetrad is then a solution to our classical field equations (5,6), and more generally to the field equations of the one-parameter teleparallel theory.

In the case where such a general relativity solution is given in a different coordinate system, we do not know a priori what will be the corresponding tetrad field satisfying $T^{ikl} = 0$, but we certainly know that it exists. For instance, to the Schwarzschild solution in its conventional, diagonal, form corresponds a diagonal tetrad field (see [3]).

The class (16) corresponds to the metrics conformal to the Kerr-Schild class and is well known in the context of general relativity (see, e.g., [14]). In the simple Kerr-Schild class ($\varphi = 1$), we find, among others, the known black hole solutions (Schwarzschild, Reisner-Nordstrøm, Kerr and Kerr-Newman) as well as exact wave solutions (see [14]). On the other hand, for $k_i = 0$, we are dealing with conformally flat metrics. This class contains, for instance, the important cosmological Friedman-Robertson-Walker solutions, when transformed into a suitable coordinate system. Examples of solutions containing both $\varphi$ and $k_i$ can be found in [15], where inner Schwarzschild solutions have been derived in the form (16).

Consequently, our classical limit field equations (5,6), although stronger than the equations of general relativity, nevertheless admit a large class of solutions, containing all those solutions that have been related to experiments so far. The theory described by the Lagrangian (1) can therefore be considered to be experimentally viable.

Let us note once again that in the conventional one-parameter teleparallel theory, as conceived by [3], the constraint $T^{ikl}$ is missing, and the Einstein equations therefore contain additional contributions. This allows
It is interesting to note that a teleparallel form of the Kerr metric has also been found for the case \( \lambda = 0 \) (i.e., the teleparallel equivalent of general relativity) in \[17\], where the torsion is, in complete contrast to our case, purely axial. The authors then conclude, correctly, on a spin precession effect for a Dirac electron in this field. On the other hand, using the Kerr solution in the form (17), we will find no such effect. This is a concrete example for the inconsistency of the theory with \( \lambda = 0 \), which allows for both, physically inequivalent solutions, illustrating very clearly the arguments given in \[1\]. Once we choose \( \lambda \neq 0 \), the only allowed solutions (for a classical source) are constraint to \( T[ikl] = 0 \), and the remaining freedom in the choice of the tetrad field (restricted frame invariance) does not affect the spin precession.

It is also worthwhile to note that, in view of Eq. (6) and the results of sec. III, all test particles in a classical field configuration will move on geodesics, since the only part of the torsion they eventually couple to is zero. More generally, the evolution of a spinning particle (momentum propagation, spin precession, . . . ) does not differ from that of a spinless particle. Consequently, in order to study intrinsic spin effects, one has to consider fields created by spinning sources.

Finally, we wish to point out that in the framework of the purely teleparallel theory, the Kerr black hole counterpart of the one-parameter theory has been found (see \[17\] and \[18\]) and later on generalized to the charged case \[19\]. The solutions contain an additional parameter (apart from the angular momentum parameter of the Kerr metric and the charge parameter in the Kerr-Newman case), and coincide with the Kerr and Kerr-Newman solutions only for a specific value of that parameter. It is concluded that no experimental distinction of those solutions to the corresponding general relativity solutions is possible in the presence of scalar, spin \( 1/2 \) and Yang-Mills fields. This is consistent with our results of section III. Indeed, the solutions corresponding to different values of the parameter only differ by their axial torsion parts, which do, however, not couple to the mentioned fields.

V. FINAL REMARKS

Hoping to resolve the inconsistency of the Poincaré gauge theories with a teleparallel limit, we followed the step performed by Hayashi and Shirafuji in the framework of the purely teleparallel theory and introduced the term \( \lambda T[ikl]T[ikl] \) that breaks the frame invariance of the theory. We showed that this improved theory can indeed be coupled consistently to all the standard model elementary particles, using the fact that the remaining, restricted, frame invariance of the gravitational Lagrangian is also a symmetry of the matter Lagrangians (Yang-Mills, Higgs, Dirac . . . ). It turns out that in the spin \( 1/2 \) case, in order to assure this invariance, a spinor transformation has to be performed simultaneously with the frame change.

In a second part, we concentrated on the experimental aspects of the theory. Although the field equations, in the classical limit, now differ from those of general relativity (in contrast to the theories discussed in \[1\]), we were able to show that they share a very important class of solutions with the latter. More precisely, every solution of general relativity that can be brought into a form conformal to the Kerr-Schild metric is also a solution to our theory. This class contains every solution so far involved in any experimental procedure, and many other solutions that have been subject to theoretical investigations in the past, notably the black hole solutions (Schwarzschild, Kerr, Reisner-Nordstrøm, Kerr-Newman) and many cosmological solutions. Thus, we can conclude that, as far as the classical limit is concerned, the theory is experimentally indistinguishable from general relativity. Experiments involving the gravitational interaction of particles with intrinsic spin are needed to conclude further.

In spite of these positive results, we would like to express some doubts that the introduction of the term \( \lambda T[ikl]T[ikl] \) is really the way to proceed. Concerning the free gravitational fields, it seems quite arbitrary to break the frame invariance as far as possible, leading to constraints in the field equations that leave just enough freedom to allow for the most important general relativity solutions. The fact that this (explicit) symmetry breaking reveals itself as, again, just enough to resolve the inconsistencies concerning the coupling to elementary particles could be seen as a coincidence. Or, otherwise stated, does the fact that, so far, the only observed half-integer spin particles are of spin \( 1/2 \), justify the use of a theory that excludes the coupling to higher spin particles a priori?

On the other hand, apart from the inconsistencies arising from the restricted frame invariance, there are also algebraic consistency problems. Indeed, it is well known that in a Riemannian geometry, the presence of matter fields with spin higher than 2 leads to severe restrictions (the so-called consistency conditions) on the curvature tensor which are so strong that it is actually almost impossible to construct reasonable theories involving such fields in curved spacetime. Extending the analysis to the case of Riemann-Cartan geometry, it was found that strong consistency conditions on curvature and torsion arise already for fields with spin larger than \( 1/2 \) (except for spin one gauge fields, which are considered to be one-forms and do not couple to torsion.) For details, see \[20\]-\[23\] and references therein. Thus, in view of these algebraic inconsistencies concerning higher spin fermions on a Riemann-Cartan background geometry, it does not seem much of a drawback to have additional dynamical inconsistencies arising as a result of the specific choice of the gravitational Lagrangian. It simply means that we cannot allow for higher spin fermions. (Note that supergravity, as a specific combination of spin 2 and spin 3/2 fields, is not subject to the above problem, but as men-
tioned earlier, the torsion is non-dynamical and directly related to the spin density of the gravitino. Therefore, in that framework, it makes hardly sense to talk of a gravitino in a torsion background.

Another point concerns the interpretation of the restricted frame invariance, especially in view of the spinor transformation involved [see Eqs. (12,13)]. Even in the unrestricted form, the transformations cannot be identified neither as gauge, nor as frame transformations in the strict sense. The role of the additional constraint [Eq. (15)] is even less clearer. One might prefer to work with theories that are free of symmetries to which no physical interpretation can be asserted to, as is the case for Einstein-Cartan theory or other Poincaré gauge models with a Riemannian (instead of teleparallel) limit.

There, however, one has to pay the price of either putting some limits on the involved coupling constants (in order to reduce the deviation from general relativity to an experimentally acceptable level) or of having only non-propagating torsion fields (Einstein-Cartan).

Finally, and most importantly, there is the question of how we deal with compound particles, or macroscopic, spin polarized bodies (e.g., neutron stars). It has been argued in [24] that in the limit of large spin (>> 1), particles will couple to the complete torsion tensor, and that this will be the case for macroscopic spin polarized matter too (see [25] for a detailed review concerning the spin-torsion couplings and the resulting equations of motion). A concrete example is given by the model of the Weyssenhoff fluid with intrinsic spin, where the spin density is considered to be of the form \( \int e \sigma_{a'b'}d^4x \sim \sigma_{a'b'}u^m \), where \( \sigma_{a'b'} \) is the spin tensor and \( u^m \) the four-velocity (see [26] for the detailed treatment of the Weyssenhoff fluid in a Riemann-Cartan framework). Especially, since \( \sigma_{a'b'} \) is not totally antisymmetric, it will couple to non-axial torsion contributions. Thus, if we do not want to run into trouble, we have to reconsider those semi-classical models of intrinsic spinning matter distributions.

We conclude that the theories presented in this article are consistent in the presence of the standard model matter fields and are experimentally viable. The coupling to higher half-integer spin fields is not possible and there are doubts on the consistency of the macroscopic spin-torsion coupling. Further, the restricted frame invariance of the theories lacks a physical interpretation and the introduction of the term \( \lambda T^{[ikl]}T_{[ikl]} \) seems arbitrary.

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