Plane drawings of the generalized Delaunay-graphs for pseudo-disks

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Abstract. We study general Delaunay-graphs, which are a natural generalizations of Delaunay triangulations to arbitrary families. We prove that for any finite pseudo-disk family and point set, there is a plane drawing of their Delaunay-graph such that every edge lies inside every pseudo-disk that contains its endpoints.

Keywords: Delaunay-graph · pseudo-disk · topological hypergraphs

1 Introduction

Delaunay triangulations play a central role in discrete and computational geometry. In many applications, however, one needs to deal with a different topology which requires to substitute disks in the definition with another family. If this other family consists of the homothets of some convex shape, then most properties generalize in a straight-forward manner. In this paper we study what happens when this is not the case, i.e., we only suppose that our underlying family is a family of (possibly non-convex) pseudo-disks. Now we make the exact definitions.

Definition 1 Given a finite set of points $S$ and a family of regions $F$, the vertices of the Delaunay-graph $D(S,F)$ of $S$ with respect to $F$ correspond to the points of $S$, and two vertices $p,q \in S$ are connected by an edge if there is an $F \in F$ such that $S \cap F = \{ p, q \}$.

Note that if $S \subset \mathbb{R}^2$ and $F$ is the family of disks, this gives back the usual definition of Delaunay triangulations. It is well-known that this graph with respect to disks is planar, moreover, drawing its edges as straight-line segments
we get a plane drawing in which the drawing of an edge \( pq \) lies inside every disk containing both \( p \) and \( q \). This is true also when the regions are the homothets of some convex region, or more generally, when \( \mathcal{F} \) is a pseudo-disk arrangement containing only convex regions (as we will soon see).

**Definition 2** A Jordan region is a (simply connected) closed bounded region whose boundary is a closed simple Jordan curve. A family of Jordan regions is called a family of pseudo-disks if the boundaries of every pair of the regions intersects in at most two points.

For points with respect to pseudo-disks if we draw the edges in an arbitrary way inside one of their defining pseudo-disks (that is, the edge connecting points \( p \) and \( q \) is drawn inside an \( F \in \mathcal{F} \) for which \( H_F = \{ p, q \} \)), we get a drawing in which non-adjacent edges intersect an even number of times, using the following simple lemma:

**Lemma 3** \([0]\) Let \( D_1 \) and \( D_2 \) be two pseudo-disks in the plane. Let \( x \) and \( y \) be two points in \( D_1 \setminus D_2 \). Let \( a \) and \( b \) be two points in \( D_2 \setminus D_1 \). Let \( e \) be any Jordan arc connecting \( x \) and \( y \) that is fully contained in \( D_1 \). Let \( f \) be any Jordan arc connecting \( a \) and \( b \) that is fully contained in \( D_2 \). Then \( e \) and \( f \) cross an even number of times.

The Hanani-Tutte theorem then implies that the Delaunay-graph of points with respect to pseudo-disks is planar.

If we additionally assume that the pseudo-disks in the family are all convex, then just like in the case of disks and homothets of a convex region, we can draw the edges as straight-line segments. As the regions are convex, the drawing of an edge \( pq \) indeed lies inside every pseudo-disk containing both \( p \) and \( q \). Furthermore, two adjacent edges do not intersect while non-adjacent edges intersect at most once and by Lemma 3 an even number of times, thus they also do not intersect. Thus, this is a plane drawing of the Delaunay-graph.

The aim of this paper is to prove that with additional effort we can also get such a plane drawing even when the pseudo-disks are not necessarily convex:

**Theorem 4** Given a finite pseudo-disk family \( \mathcal{F} \) and a finite point set \( S \), there is a plane drawing of the Delaunay-graph of \( S \) with respect to \( \mathcal{F} \) such that every edge \( pq \) lies inside every pseudo-disk containing both \( p \) and \( q \).

Note that self-intersecting edges can be always easily replaced by non-self-intersecting ones by removing loops, so in the rest of the paper we suppose that edges have no self-intersection, i.e., they are simple Jordan-curves.

One important consequence of Theorem 4 is for pseudo-disk families that are shrinkable. To state it, first we need the following (restated) lemma from [0].

**Lemma 5** (Pinchasi [0]) If a pseudo-disk \( F \in \mathcal{F} \) contains exactly \( k \) points of \( S \), one of which is \( p \in S \), then for every \( 2 \leq l \leq k \) there exists a set \( F' \subset F \) such that \( p \in F' \) and \( |F' \cap S| = l \), and \( \mathcal{F} \cup \{ F' \} \) is again a family of pseudo-disks.
We say that a pseudo-disk family $\mathcal{F}$ is shrinkable over $S$ if for every $F \in \mathcal{F}$ the conclusion of Lemma 5 holds for some $F' \in \mathcal{F}$.

An example of a shrinkable pseudo-disk family is the collection of all disks in the plane over a finite $S$ that does not contain four points on a circle. More generally, instead of disks, we can take the family of all homothets of any convex set with a smooth boundary over a finite $S$ that does not contain four points on the boundary of a homothet.

**Corollary 6** Given a shrinkable pseudo-disk family $\mathcal{F}$ over a finite point set $S$, for every $F \in \mathcal{F}$ the subgraph of $\mathcal{D}(S, \mathcal{F})$ (the Delaunay-graph of $S$ with respect to $\mathcal{F}$) induced by $F \cap S$ is a connected graph.

**Proof.** Using 3 there is a point $p \in F \cap S$ such that there is an $F' \in \mathcal{F}$ for which $F' \cap S = F \cap S \setminus \{p\}$. By induction, the Delaunay-graph restricted to $F' \cap S$ is connected. The same holds for some other point $p' \in F \cap S$. Since $F \cap S \setminus \{p\}$ and $F \cap S \setminus \{p'\}$ both induce connected graphs, so does $F \cap S$ unless $F \cap S = \{p, p'\}$, but in this latter case $(p, p')$ is an edge of the Delaunay-graph because of $F$. This finishes the proof.

We expect that such a strong variant of planarity of the Delaunay-graph can be useful in several applications. As an example, in a new result of Ackerman and the authors about certain colorings of the edges of the Delaunay-graph [0] Theorem 4 was needed.

## 2 Proof of Theorem 4

We will need the following lemma:

**Lemma 7** Given a finite family of pseudo-disks such that all pseudo-disks contain a common point $p$, any point $q$ can be connected by a Jordan curve to $p$ such that this curve does not intersect the boundary of pseudo-disks containing $q$ (and $p$) and intersects once the boundary of pseudo-disks that do not contain $q$ (but contain $p$).

In [0] it is proved that for a finite family of pseudo-disks all containing a common point $p$ (and the regions are in general position, that is, no three of their boundaries intersecting in a common point), there exists a combinatorially equivalent family of pseudo-disks, all of which are star-shaped with respect to $p$. Clearly this also implies the above lemma, the only issue is that we do not want to assume that the regions are in general position. Alternately, appropriate application of the Sweeping theorem of Snoeyink and Hershberger [0] also implies Lemma 7 without assuming general position, yet the proof of their theorem is quite involved. Finally, in the manuscript [0] a relatively simple self-contained proof of Lemma 7 is shown, again without assuming general position.

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2 A region is star-shaped with respect to $p$ if for every line through $p$ intersects the region in a segment containing $p$. 
We need some further definitions and lemmas before we can prove our main result. From now on every point set $S$ we consider is finite, even if we do not always emphasize this.

**Definition 8** Given two pseudo-disks whose boundaries intersect, removing their boundaries the plane is split into three bounded and one unbounded region. We call the bounded regions the lens defined by these two pseudo-disks. If a set of points $S$ is given, we say that a lens is empty if it does not contain any point from $S$.

**Definition 9** In a pseudo-disk family $\mathcal{F}$, replacing a pseudo-disk $F$ with some $F' \supseteq F$ such that the new family is still a pseudo-disk family, is called a shrinking of $F$ to $F'$. If such an $F'$ already exists in $\mathcal{F}\setminus\{F\}$, then simply deleting $F$ from $\mathcal{F}$ is also called a shrinking of $F$ to $F'$. Given a point set $S$, such a shrinking is hypergraph preserving on $S$ if $F' \cap S = F \cap S$.

Applying shrinking steps to multiple members of the family $\mathcal{F}$ after each other is called a shrinking of $\mathcal{F}$. A shrinking of $\mathcal{F}$ is hypergraph preserving if all shrinking steps are hypergraph preserving.

**Observation 10** If we do a hypergraph preserving shrinking on $\mathcal{F}$ to get $\mathcal{F}'$, then by definition in each step for the shrunk $F \in \mathcal{F}$ we have $H_F = H_{F'}$ and thus the geometric hypergraph of $S$ with respect to $\mathcal{F}$ is the same as of $S$ with respect to $\mathcal{F}'$. That is, a hypergraph preserving shrinking does indeed preserve the geometric hypergraph of $S$ with respect to $\mathcal{F}$.

**Lemma 11** Given a point set $S$ and a finite family $\mathcal{F}$ of pseudo-disks, if we take a containment-minimal empty lens $L$, and $L$ is defined by the pseudo-disks $F_1$ and $F_2$ such that $L \subset F_1$, then we can get rid of this lens in the following way: we shrink $F_1$ to some $F'_1$ such that $F'_1 \cap L = \emptyset$ while $F'_1 \cap S = F_1 \cap S$ (that is, shrinking $F_1$ to $F'_1$ is hypergraph preserving). Moreover, the number of intersections of the boundaries of pseudo-disks strictly decreases during this.

**Proof.** Let $l_1$ (resp. $l_2$) be the maximal curve which is on the boundary of both $F_1$ (resp. $F_2$) and $L$. We claim that every maximal curve inside $L$ which is part of a boundary of some pseudo-disk different from $F_1$ and $F_2$, has one endpoint on $l_1$ and another on $l_2$. Indeed, if such a maximal curve on the boundary of some $F_3$ would have both endpoints on $l_1$ (resp. $l_2$), then $F_1$ (resp. $F_2$) and $F_3$ would define a lens which lies inside $L$ contradicting its containment minimality.

Now we are ready to shrink $F_1$. Basically we want to delete $L$ from it, but we have to shrink it a bit more to avoid the introduction of common boundary parts which turns out to be a bit technical.\footnote{Deleting from $F_1$ an $\epsilon$-expansion of $L$ would not always work - we leave it to the interested reader why.} In order to do that we will define a curve inside $F_1 \setminus L$ which intersects the boundary of some pseudo-disk the same number of times as $l_2$ does. Then, we shrink $F_1 \setminus L$ by essentially replacing $l_2$ by this curve on the boundary. Next we give the details of how we do all of this.
Consider the arrangement determined by the boundaries of the pseudo-disks. The vertices of the arrangement are the intersection points of the boundaries of regions, the edges are the maximal connected parts of the boundaries of regions that do not contain a vertex and the cells are the maximal connected parts of the plane which are disjoint from the edges and the vertices of the arrangement.

Consider the vertices (that is, intersection points) of the arrangement that are on \(l_2\) and then for every such vertex and every edge incident to this vertex and lying inside \(F_1 \setminus L\), but not on \(l_2\), we take a small part of that edge ending in this vertex and call this a half-edge. These half-edges can be ordered naturally first according to the order of their endpoints on \(l_2\), second for two half-edges sharing an endpoint we order them according to their rotation order around this vertex. The same ordering defines uniquely for every consecutive such half-edges the cell of the arrangement ‘between’ them.

Thus we get a natural ordering of the half-edges, \(e_1, e_2, \ldots, e_l\). Notice that the first and last half-edge lies on the boundary of \(F_1 \setminus L\). Now for every \(e_i\) choose an arbitrary point \(c_i\) on it and connect for every \(1 \leq i \leq i - 1\) the points \(c_i\) and \(c_{i+1}\) by a curve lying inside the cell that lies between them. While we need to draw several curves inside one cell (see Figure 1), we can draw these curves such that no two of them intersects, as we can draw the curve connecting \(c_i\) and \(c_{i+1}\) close to their half-edges and (if they do not share an endpoint) the part of \(l_2\) separating their endpoints. The union of all the curves is a curve \(l'_1\) that connects \(c_1\) to \(c_l\).

Let \(F'_1\) be the region whose boundary consists of this curve and the boundary part of \(F_1 \setminus l\) from \(c_1\) to \(c_l\) which is disjoint from \(l_2\).

Having defined \(F'_1\), the shrinking of \(F_1\), we are left to prove that it has the properties we required.

Clearly, \(F'_1 \cap L = \emptyset\) and also \(F'_1 \cap S = F_1 \cap S\). So we need to show only that the new family is also a pseudo-disk family and has strictly less intersections between the boundaries of its members.

Consider now an intersection of \(l_1\) with the boundary of some \(F_3\). By definition, it must be \(c_i\) for some \(2 \leq i \leq i - 1\). The half-edge \(\gamma_i\), containing \(c_i\) has an endpoint on \(l_2\), which is an intersection point of the boundaries of \(F_3\) and \(F_2\).
Using our observation from the beginning of the proof, the maximal curve inside \( L \) whose starting point is this intersection ends on \( l_1 \), which was an intersection of the boundary of \( F_1 \) with the boundary of \( F_3 \). Arguing now in the opposite direction, for every intersection point of \( l_1 \) with a boundary of some \( F_3 \) there is a corresponding intersection point on \( l_2 \) and then also on \( l'_1 \) with the boundary of \( F_3 \). We conclude that the intersection points on the boundary of \( F_1 \) and \( F'_1 \) are in bijection except for the two intersection points of the boundaries of \( F_1 \) and \( F_2 \) as the boundaries of \( F'_1 \) and \( F_2 \) do not intersect. This implies that the family is still a pseudo-disk family and that the overall number of intersection points of boundaries decreased by 2, finishing the proof.

**Definition 12** Given a point set \( S \), we say that a pseudo-disk family \( F \) respects \( S \) if for every pair of pseudo-disks \( F_1, F_2 \in F \),

\[
\text{if } F_1 \cap S \subseteq F_2 \cap S, \text{ then } F_1 \subseteq F_2 \text{ as well and } \tag{1}
\]

\[
\text{if } (F_1 \cap S) \cap (F_2 \cap S) = \emptyset, \text{ then } F_1 \cap F_2 = \emptyset \text{ as well.} \tag{2}
\]

**Observation 13** If a pseudo-disk family \( F \) respects \( S \), then by definition for every subset \( S' \subseteq S \) there is at most one pseudo-disk \( F \) such that \( F \cap S = S' \).

Indeed, if \( F_1 \cap S = F_2 \cap S \), then by assumption (1) both \( F_1 \subseteq F_2 \) and \( F_2 \subseteq F_1 \), that is \( F_1 = F_2 \).

**Lemma 14** Given a point set \( S \) and a finite pseudo-disk family \( F \), we can shrink \( F \) to get a pseudo-disk family \( F' \) such that

(i) this shrinking is hypergraph preserving on \( S \) and

(ii) \( F' \) respects \( S \).

**Proof.** We keep applying Lemma 11 to a containment-minimal empty lens until there are no more empty lenses. This is a finite process as in each step the number of intersections between boundaries of pseudo-disks decreases. By Lemma 11 it follows that the new family is a pseudo-disk family and that this shrinking was hypergraph preserving on \( S \).

Next, if there is a pair of pseudo-disks which intersect \( S \) in the same subset \( S' \), then since there are no empty lenses, one of these must be contained in the other. The bigger one can be shrunk to the smaller, so we can delete it. We keep doing this until for every \( S' \) there is only at most one pseudo-disk which intersects \( S \) in \( S' \).

Finally, it is easy to see that if there was a pair of pseudo-disks in \( F' \) for which (1) or (2) did not hold, then either they would intersect \( S \) in the same subset or one of the lenses they form would be empty, contradicting the fact that there were no more empty lenses in \( F' \).

**Definition 15** Given a pseudo-disk family \( F \), the depth of a point is the number of pseudo-disks containing this point.
Lemma 16 We are given a point set $S$ and a family of pseudo-disks $F$ which respects $S$ such that every pseudo-disk contains exactly two points from $S$. Given a pseudo-disk $F_{x,y} \in F$ containing only the two points $x, y \in S$ from $S$, we can draw a curve connecting $x$ and $y$ which lies completely inside $F_{x,y}$ and intersects the boundary of every other pseudo-disk at most once.

Proof. For any pair $p, q$ of points of $S$, denote by $F_{p,q}$ the unique pseudo-disk containing exactly these two points from $S$, if it exists (uniqueness follows from Observation 13 as $F$ respects $S$).

We will draw the curve connecting $x$ and $y$ inside $F_{x,y}$. Thus, it cannot intersect any pseudo-disk which lies outside $F_{x,y}$, that is, as $F$ respects $S$, any pseudo-disk that contains two points of $S$, both different from $x$ and $y$. Thus when drawing the arc, we only need to care about pseudo-disks of two types, of type $F_{x,*}$, which is an $F_{x,p}$ for some $p$ different from $y$ and of type $F_{y,*}$, which is an $F_{y,q}$ for some $q$ different from $x$. Note that $F_{x,y}$ has no type.

If $p \neq q$, then $F_{x,p} \cap F_{y,q} = \emptyset$ as $F$ respects $S$. This implies the following:

Proposition 17 Inside $F_{x,y}$ if a point (not from S!) is contained in at least 3 pseudo-disks besides $F_{x,y}$ (that is, has depth at least 4), then all these pseudo-disks must be of the same type.

Now we can continue with the proof of Lemma 16 — for illustrations see Figure 2.
– Case 1. There is a point \( z \) (not from \( S \)) in \( F_{x,y} \) contained in pseudo-disks of both types.

Then by Proposition 17 it must be a 3-deep point which besides \( F_{x,y} \) is contained only in \( F_{x,p} \) and \( F_{y,p} \) for some \( p \). Also, in the arrangement of the pseudo-disks the cell containing \( p \) must be bounded only by boundary parts of the three regions otherwise a point in a neighboring cell would contradict Proposition 17. In fact, it is easy to see that it must be bounded by parts of the boundaries of all of these three regions, otherwise we would have an empty lense.

Take now the arrangement defined by \( F_{x,y} \) and the regions of type \( F_{x,*} \). Let \( C_x \) be the cell containing \( z \) in this arrangement. Take the first intersection point \( z_x \) of the ray guaranteed by Lemma 7 going from \( z \) to \( x \) with the boundary of \( C_x \). Note that \( C_x \) is disjoint from all pseudo-disks of type \( F_{y,*} \) except for \( F_{y,p} \). Moreover, again by Proposition 17, the boundary of \( F_{y,p} \) intersects the boundary of \( F_{x,y} \) and the boundary of \( F_{x,p} \) in two points but does not intersect the boundaries of other pseudo-disks of type \( F_{x,y} \). Thus, \( F_{y,*} \) subdivides \( C_x \) into at most three parts, to \( C' \), containing \( p \), to \( C'_x \), having \( z_x \) on its boundary, and to a possible third cell. It is easy to see that \( C' \) and \( C'_x \) must share boundary parts, and so we can choose a point \( z'_x \) on their common boundary. Note that \( C' \) is actually the cell containing \( z \) in the arrangement of all pseudo-disks.

Now take the arrangement defined by \( F_{x,y} \) and the regions of type \( F_{y,*} \). A symmetric argument gives the cells \( C' \) and \( C'_y \) and the points \( z_y \) and \( z'_y \) (note that we get the same \( C' \) as it is again the cell containing \( z \) in the arrangement of all pseudo-disks).

Now the curve connecting \( x \) and \( y \), and intersecting the boundary of every pseudo-disk at most once consists of the following parts: a curve from \( x \) to \( z_x \) along a ray guaranteed by Lemma 7 (applied for the pseudo-disks containing \( x \)), a curve inside \( C' \) from \( z_x \) to \( z'_x \), a curve from \( z'_x \) to \( z'_y \) inside \( C' \) (which can go through \( z \) if we wish to), a curve from \( z'_y \) to \( z_y \) inside \( C'_y \) and finally a curve from \( z_y \) to \( y \) along a ray guaranteed by Lemma 7 (applied for the pseudo-disks containing \( y \)). By Lemma 7, every point of the curve connecting \( x \) and \( z_x \) is inside at least two pseudo-disks of type \( F_{x,*} \) and thus by Proposition 17 it cannot intersect any pseudo-disks of type \( F_{y,*} \). Similarly, the curve connecting \( y \) and \( z_y \) cannot intersect any pseudo-disks of type \( F_{x,*} \). Altogether, using again Lemma 7, we get that the union of these curves defines a curve intersecting every pseudo-disk boundary at most once, as required.

– Case 2. Every point in \( F_{x,y} \) is contained only in pseudo-disks of one type.

In this case going along an arbitrary curve \( \gamma \) from \( x \) to \( y \), the last point \( z_x \) which is contained in a pseudo-disk of type \( F_{x,*} \) must be at least 2-deep and contained in \( F_{x,y} \) and in pseudo-disks only of type \( F_{x,*} \) (if there are no pseudo-disks of type \( F_{x,*} \), let \( z_x = x \)). Going further along this curve towards \( y \), there are 1-deep points and then the first at least 2-deep point \( z_y \) must be contained in \( F_{x,y} \) and in pseudo-disks only of type \( F_{y,*} \) (or if there are none, let \( z_y = y \)). Now we can apply Lemma 7 to get a curve from \( x \)
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to $z_x$ (applied for the pseudo-disks containing $x$). This will be disjoint from each $F_{y,z}$, as there no points in $F_{x,y}$ contained in pseudo-disks of both types.

We similarly connect $y$ to $z_y$. Together with the part of $\gamma$ connecting $z_x$ to $z_y$ we get again a curve, which using again Lemma 7 is intersecting every pseudo-disk boundary at most once, as required.

Proof (of Theorem 4). Given a finite pseudo-disk family $\mathcal{F}$ and a point set $S$, we want to find a plane drawing of the Delaunay-graph of $S$ with respect to $\mathcal{F}$ such that every edge $pq$ lies inside every pseudo-disk containing both $p$ and $q$.

First we shrink $\mathcal{F}$ using Lemma 14 to get a family which respects $S$. As we did a hypergraph preserving shrinking on $S$, the new family has the same (possibly empty!) Delaunay-graph as $\mathcal{F}$. Next we remove all the pseudo-disks containing at least 3 points from $S$, by which the Delaunay-graph is again left intact. We get the pseudo-disk family $\mathcal{F}'$. Notice that as $\mathcal{F}'$ respects $S$, for every pair of points $p, q$ which are connected by an edge in the Delaunay-graph, in $\mathcal{F}'$ there is exactly one pseudo-disk $F_{pq}'$ which contains these two points (and no other point of $S$).

We claim that a plane drawing of the Delaunay-graph of $\mathcal{F}'$ with respect to $S$ such that for each edge $pq$ its drawing lies inside $F_{pq}'$ is as required for $\mathcal{F}$ with respect to $S$. Indeed, take an arbitrary edge $pq$ of the Delaunay-graph of $\mathcal{F}$ with respect to $S$ and let $F$ be an arbitrary pseudo-disk such that $p, q \in F \in \mathcal{F}$. After shrinking $\mathcal{F}$ to $\mathcal{F}'$, $F$ was shrunk to some $F'$ (possibly in multiple steps) which must contain $F_{pq}'$ as $F' \cap S = F \cap S \supseteq \{p, q\} = F_{pq}' \cap S$ and $\mathcal{F}'$ respects $S$. Thus, the drawing of the edge $pq$ lies inside $F_{pq}' \subset F' \subset F$, as claimed.

Thus, we are left to prove that there exists a plane drawing of the Delaunay-graph of $\mathcal{F}'$ with respect to $S$ such that for each edge $pq$ its drawing lies inside $F_{pq}'$. We will prove this by drawing the edges one-by-one using Lemma 16.

We take the edges one-by-one in an arbitrary order. We draw the first edge using Lemma 16. Now suppose that a subset of the edges is already drawn such that no pair of the drawn edges intersects and moreover every edge intersects the boundary of a pseudo-disk at most once (which implies that it intersects a boundary only when it is necessary, that is, when exactly one of its endpoints is inside this pseudo-disk). Note that this additional requirement holds for the first drawn edge by Lemma 16.

Suppose that the next edge we want to draw connects $x$ and $y$ and we want to draw it inside $F_{x,y}$. For an illustration of the rest of the proof see Figure 3. Draw it first using Lemma 16, then $f$ intersects the boundary of any other pseudo-disk at most once. Although $f$ may intersect previously drawn edges, but only edges connecting $x$ or $y$ to some other points of $S$ as all other edges lie outside $F_{x,y}$. Take the intersection $z_x$ of $f$ with a drawing of an edge $xp$ which is farthest from $x$ along $f$ among edges of this type (or $z_x = x$ if there is no such intersection). Take also the intersection $z_y$ of $f$ with a drawing of an edge $yq$ which is farther from $x$ along $f$ and is the closest to $x$ among edges of this type (or $z_y = y$ if there is no such intersection). Now take the curve $f'$ which goes from $x$ to $z_x$ along the already drawn $xp$ (very close to and on the appropriate side) on which $z_x$ lies, then goes along $f$ to $z_y$ and then goes from $z_y$ to $y$ along
the already drawn \( yq \) (again very close to and on the appropriate side) on which \( z_y \) lies. By appropriate sides we mean that we choose sides such that \( f' \) does not intersect the edges \( xp \) and \( yq \). This gives a drawing \( f' \) of the edge \( xy \) which does not intersect any earlier edge.

We need to prove that \( f' \) lies inside \( F_{x,y} \). As \( f \) lies inside \( F_{x,y} \), we only need to care about the two parts that are drawn along the drawings of the edges \( xp \) and \( yq \). Yet by induction these edges were drawn such that they intersect \( \partial F_{x,y} \) once, and as \( z_x, z_y \in F_{x,y} \), this intersection cannot lie on the parts that connect \( x \) to \( z_x \) and \( y \) to \( z_y \). That is, all three parts of \( f' \) lie inside \( F_{x,y} \), as required.

![Fig. 3. Adding the drawing of the edge \( xy \).](image)

Now we are left to prove that \( f' \) intersects the boundary of every pseudo-disk at most once. For this we prove that for an arbitrary pseudo-disk \( F \) its boundary \( \partial F \) intersects \( f' \) the same number of times as it intersects \( f \). Denote by \( f'_x \) the part of \( f' \) between \( x \) and \( z_x \) and by \( f_x \) the part of \( f \) between \( x \) and \( z_x \). As both \( f_x \) and \( f'_x \) can intersect \( \partial F \) at most once, we get that \( f'_x \) intersects \( \partial F \) if and only if \( f_x \) does. The similar statement holds for the part of \( f' \) between \( y \) and \( z_y \).

As the remaining (middle) parts of \( f \) and \( f' \) coincide, we can conclude that they intersect \( \partial F \) the same number of times. As \( f \) intersected \( \partial F \) at most once, the same holds for \( f' \) as well.

We have seen that we can add an arbitrary edge. Repeating this process for all edges we get that the whole Delaunay-graph can be drawn in the plane as required.

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