GAUSSIAN APPROACH FOR PHASE ORDERING IN NONCONSERVED SCALAR SYSTEMS WITH LONG-RANGE INTERACTIONS

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ABSTRACT

We have applied the gaussian auxiliary field method introduced by Mazenko to a non-conserved scalar system with attractive long-range interactions. This study provides a test-bed for the approach and shows some of the difficulties encountered in constructing a closed theory for the pair correlation function. The equation obtained for the equal-time two-point correlation function is studied in the limiting cases of small and large values of the scaling variable. A Porod regime at short distance and an asymptotic power-law decay at large distance are obtained. The theory, is not, however, consistent with the expected growth-law, and attempts to retrieve the correct growth lead to inconsistencies. These results indicate a failure of the gaussian assumption (at least in the form in which we use it) for this system.
I. INTRODUCTION

The phase ordering dynamics of systems quenched from the disordered phase to the ordered phase has been extensively studied [1]. There is a general consensus that at the late stages of domain coarsening these systems enter a scaling regime [2], in which the equal-time, two-point correlation function has the scaling form

\[ C(r, t) \equiv \langle \phi(x, t)\phi(x + r, t) \rangle = f(r/L(t)), \]

where \( \phi \) is the scalar order-parameter field, \( L(t) \) is the characteristic length scale at time \( t \) after the quench, \( f \) is a scaling function, and angled brackets indicate an average over initial conditions (and thermal noise, if present).

A first principles calculation of the scaling function has proved to be a most difficult task. Even for the simplest model dynamics, that of a nonconserved order parameter (model A) [3] with purely short-ranged (SR) interactions, exact results are rare and available only for cases of limited physical interest [4].

In the past few years closed-approximation schemes for the two-point correlation function of the SR model A (SRMA) have been proposed by a number of authors [5]-[11], based on a mapping \( \phi(r, t) = \phi(m(r, t)) \) between the order-parameter and an auxiliary field \( m(r, t) \) which has, near a domain wall, the physical interpretation of a coordinate normal to the wall. With this new variable the problem of describing the field at each instant of time is transformed into a problem of describing the evolution and statistics of the wall network. This approach enables the use of a physically plausible and mathematically convenient gaussian distribution for \( m \). Such a distribution is unacceptable for the order parameter field itself, since this is effectively discontinuous at the domain size scale.

The application of this sort of approach to both non-conserved and conserved (model B) dynamics, with purely SR interactions, has recently received a critical review by Yeung et al. [12]. Methods based on a description of the wall dynamics lead to an approximate linear equation for \( m(r, t) \), or for its correlator \( \langle m(x, t)m(x + r, t) \rangle \), [3]. A different and promising approach, due to Mazenko [7],
aims at deriving a closed non-linear equation for $C(r, t)$, built on the equation of motion for model A, using the single assumption that the field $m$ is gaussian distributed at all times. It has the advantage of yielding results with a non-trivial dependence on the spatial dimension $d$ and is also easily extensible to $O(n)$ component systems with topological defects, i.e. with $n \leq d$ [10].

The only uncontrolled feature of this approach is the gaussian assumption. Recent simulation tests have shown, however, that this assumption is not entirely satisfactory: Blundell et al. [13] have made an absolute test (free of adjustable parameters) of the relation between two different scaling functions, revealing a disappointing agreement with the theory. The discrepancy decreases, however, in higher dimensions, in agreement with an argument [11] that the gaussian approximation becomes exact in the limit $d \to \infty$. Yeung et al. [12], using data of Shinozaki and Oono [14] for $d = 3$, have checked the single-point probability distribution for $m$, finding it to be flatter at the origin than a gaussian. It is not difficult to derive an analytical expression for the two-point distribution $P(m(1), m(2))$, valid for $m(1)$, $m(2)$ and $|r|$ small compared to $L(t)$ [13]. It differs from a gaussian for fixed spatial dimension $d$, but is consistent with a gaussian in the limit $d \to \infty$.

Despite these reservations, the gaussian approach has been shown to give good results for the SRMA, displaying most of the expected physical properties [7, 10]. In this paper we try to extend the limits of this approach by applying it to model A dynamics with attractive long-range (LR) interactions. This application addresses a basic difficulty, not necessarily caused by the use of the gaussian assumption: the attempts to extend the approach beyond the simplicity of the SRMA produce equations for $C(r, t)$ which do not seem to respect the expected growth-law for the typical domain size $L(t)$. A lack of a proper scaling of the terms in the equation for $C(r, t)$, derived naively, is apparent for the case of a scalar order parameter, namely for the LR model A (LRMA) and for the SR model B (SRMB), although not for a vector order-parameter in which case a ‘naive’ dimensional analysis of the equation
agrees with the known growth law (with logarithmic corrections for $n = 2$) [17]. We shall see how this situation arises for the LRMA and present our understanding of it. In the case of the SRMB, a naive application of the method, however, omits the important bulk diffusion process which plays a vital role in the coarsening. Mazenko [8] has attempted to solve the problem by accounting explicitly for the bulk diffusion. It is not clear, however, that any analogous mechanism is present here.

The scalar case, which is usually the more interesting one in the applications, is exceptional because an extra length, time independent at late stages, the domain-wall thickness, plays a role in the dynamics, and therefore power counting of lengths by dimensional analysis may not yield the right scaling (in terms of the characteristic length) for the different parts in the equation of motion. For the SRMB [16] and the LRMA [17, 18] dynamics the growth laws are $L(t) \sim t^{1/3}$ and $L(t) \sim t^{1/(1+\sigma)}$ (for $\sigma < 1$), respectively, for $n = 1$, and $L(t) \sim t^{1/4}$ and $L(t) \sim t^{1/\sigma}$, respectively, for $n > 2$ (with logarithmic corrections for $n = 2$ [17]), where $0 < \sigma < 2$ is the exponent describing the LR interactions, which decay as $1/r^{d+\sigma}$. For $n = 1$ and $1 < \sigma < 2$, the long-range interactions are irrelevant and the growth-law is the same as for the SR case [17, 18]. The SRMA, however, is exceptional since the predicted growth-law, $L(t) \sim t^{1/2}$, is the same for both the scalar and vector order-parameters, accidently allowing for a 'naive' dimensional analysis of the scalar equation of motion to agree with the growth-law. In this case the role of the extra length in the scalar equation can be ignored as the result of two canceling errors [18]. Therefore we wonder if the success of the Mazenko method with this scalar model might be somewhat fortuitous. In other words, we raise the question of whether this approach (or any other closed theory), naively applied, can succeed for those dynamical models where naive dimensional analysis gives the wrong growth law. In this respect it is interesting that a straightforward application of the method of Kawasaki, Yalabik and Gunton [8] to the LRMA [19] also gives the wrong growth law for $n = 1$, i.e. it gives the $t^{1/\sigma}$ growth suggested by naive dimensional analysis.
In this paper we have developed an extension of Mazenko's approach for the LRMA. Besides having interest by its own right, the study of this model provides a test-bed for the approach and shows some of the difficulties any approximate closed theory must resolve.

II. THE MODEL WITH LONG-RANGE INTERACTIONS

We consider a system with long-ranged attractive interactions, falling off with distance as $r^{-(d+\sigma)}$. A suitable Hamiltonian functional of the scalar field is

$$H[\phi] = \int d^dr [(\nabla \phi)^2/2 + V(\phi)] + (J_{LR}/2) \int d^dr \int d^dr' |\phi(r) - \phi(r')|^2 / |r - r'|^{d+\sigma} , \quad (2)$$

where as usual we have taken the short-range part to have the Ginzburg-Landau form, $J_{LR} > 0$, and $V(\phi)$ has a local maximum at $\phi = 0$ and global minima at $\phi = \pm 1$. The model is well defined for $0 < \sigma < 2$. The equation of motion for a non-conserved field reads $\partial \phi / \partial t = -\delta H / \delta \phi$, i.e.

$$\frac{\partial \phi(r,t)}{\partial t} = \nabla^2 \phi - V'(\phi) + V'_{LR}(\phi) , \quad (3)$$

where $V'(\phi) = dV/d\phi$ and the LR force is given, both in real and Fourier space, as

$$V'_{LR}(\phi) = J_{LR} \int d^dr' [\phi(r') - \phi(r)] / |r - r'|^{d+\sigma} \quad (4)$$

$$= J_{LR} h(d, \sigma) \int \frac{d^dk}{(2\pi)^d} \phi(k) k^\sigma e^{i r \cdot k} , \quad (5)$$

and

$$h(d, \sigma) = Q(d, \sigma) \sqrt{\pi} \frac{\Gamma(-\frac{\sigma}{2})}{2^\sigma \Gamma(1+\frac{\sigma}{2})} , \quad (6)$$

$$Q(d, \sigma) = \pi^{d+1} \frac{\Gamma(\frac{1+\sigma}{2})}{\Gamma(\frac{d+\sigma}{2})} . \quad (7)$$

In (3) noise is absent since temperature is an irrelevant variable [20]. From an analysis of (3), assuming the validity of the scaling hypothesis (1), the following growth-law has been predicted for a scalar order-parameter [17, 18]

$$L(t) \sim t^{1/(1+\sigma)} , \quad 0 < \sigma < 1$$

$$\sim t^{1/2} , \quad 1 < \sigma < 2 , \quad (8)$$
in which the crossover $\sigma = 1$ separates the regime where domain growth is faster due to the LR correlations from the regime where these become irrelevant \cite{17, 18, 21}.

III. THE SCALING EQUATION

To obtain an equation for the two-point correlation function \footnote{\cite{11}} we multiply \footnote{\cite{3}}, evaluated at point \footnote{\cite{(1) \equiv (r_1, t_1),}} by $\phi$ evaluated at point \footnote{\cite{(2) \equiv (r_2, t_2) and average}} over the ensemble of initial conditions yielding, at equal-times,

$$\frac{1}{2} \frac{\partial C(1, 2)}{\partial t} = \nabla^2 C(1, 2) - \langle \phi(2) V'(\phi(1)) \rangle + \langle \phi(2) V_{LR}'(\phi(1)) \rangle \ . \quad (9)$$

We will call $\langle \phi(2) V'(\phi(1)) \rangle$ and $\langle \phi(2) V_{LR}'(\phi(1)) \rangle$ the ‘non-linear’ (NL) and the ‘long-range’ (LR) terms of the equation for $C(\mathbf{r}, t)$, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In \footnote{\cite{3}} the LR term reads, both in real and Fourier space,

$$\langle \phi(2) V_{LR}'(\phi(1)) \rangle = J_{LR} \int d^d r' \left[ C(\mathbf{r}', t) - C(\mathbf{r}, t) \right] / |\mathbf{r} - \mathbf{r}'|^{d+\sigma} \quad (10)$$

$$\quad = J_{LR} h(d, \sigma) \int \frac{d^d k}{(2\pi)^d} C(\mathbf{k}, t) k^\sigma e^{i\mathbf{r} \cdot \mathbf{k}} , \quad (11)$$

Assuming the existence of a late-time single-scaling regime, we expect $C(\mathbf{r}, t)$ to take the scaling form \footnote{\cite{11}}, in terms of which \footnote{\cite{9}} reads

$$- \frac{1}{2} \frac{\dot{L}}{L} x f' = \frac{1}{L(t)^2} \left( f'' + \frac{d-1}{x} f' \right) - \langle \phi(2) V'(\phi(1)) \rangle + \langle \phi(2) V_{LR}'(\phi(1)) \rangle \ , \quad (12)$$

where $x = \mathbf{r}/L(t)$ is the scaling variable and $f' = df/dx$, etc. In the equation above $\dot{L}/L \sim 1/t$, if $L(t)$ grows as a power-law. The LR term now reads

$$\langle \phi(2) V_{LR}'(\phi(1)) \rangle = \frac{J_{LR}}{L(t)^\sigma} \int d^d x' \left[ f(x') - f(x) \right] / |\mathbf{x} - \mathbf{x}'|^{d+\sigma} \quad (13)$$

$$\quad = \frac{J_{LR} h(d, \sigma)}{L(t)^\sigma} \int \frac{d^d y}{(2\pi)^d} g(y) y^\sigma e^{i\mathbf{x} \cdot \mathbf{y}} , \quad (14)$$

where $g(y)$ is the Fourier transform of $f(x)$, and $y = kL(t)$.

From an analysis of \footnote{\cite{3}} for an isolated, stationary, planar wall, we find that to leading order the equilibrium \textit{planar} wall profile saturates as

$$1 - \phi^2(r) \sim \frac{J_{LR}}{V_0 r^{\sigma}}, \quad (r \to \infty) \ , \quad (15)$$
where $V'' = (d^2V/d\phi^2)_{\phi^2=1}$ and $r$ is the distance from the wall. Hence we expect that throughout the bulk region $|\phi|$ will be below saturation by an amount $\sim 1/L(t)^{\sigma}$. Even with this power-law decay we still expect there to be well-defined walls, with a time-independent ‘thickness’ $w$, defined for example from (15) via $w^{\sigma} = J_{LR}/V''$. Therefore, domain walls may be regarded as ‘sharp’ at late-times, when $L(t) \gg w$.

It follows that Porod’s law \[22\], $g(y) \sim A(d, \sigma)/y^{d+1}$ for $y \gg 1$, holds within the regime $kw \ll 1 \ll kL(t) \equiv y$ (corresponding to $w \ll r \ll L(t)$ in real space), in which case eq. (14) yields for the leading scaling behaviour of the LR term, as $x \to 0$,

\[
\langle \phi(2)V_{LR}(\phi(1)) \rangle = \frac{J_{LR}h(d, \sigma)}{L(t)^{\sigma}} \left( \int \frac{d^d y}{(2\pi)^d} g(y) y^\sigma + \frac{A(d, \sigma) h(d, 1-\sigma)}{(2\pi)^d} x^{1-\sigma} + \ldots \right), \quad 0 < \sigma < 1
\]

\[
= \frac{J_{LR}h(d, \sigma)}{L(t)^{\sigma}} \left( \frac{A(d, \sigma) h(d, 1-\sigma)}{(2\pi)^d} \frac{1}{x^{\sigma-1}} + O(1) \right), \quad 1 < \sigma < 2. \quad (16)
\]

This result will be exploited below to determine the amplitude $A(d, \sigma)$ of the Porod tail, within the gaussian approximation.

**IV. THE GAUSSIAN APPROXIMATION**

In order to transform (9) or (12) into a closed equation we need to express the NL term as some approximate non-linear function of $C(\mathbf{r}, t)$. A key idea, exploited by several authors [6-11] within SR model A dynamics, is to employ a non-linear mapping between the order parameter $\phi(\mathbf{r}, t)$, which at the scale of $L(t)$ is effectively discontinuous near walls, and an auxiliary ‘smooth’ field $m(\mathbf{r}, t)$, whose zeros define the wall network. This introduces the wall structure into the problem and allows the approximation to be implemented through the new field.

From the equation of motion (3) we can see that, just like in the SRMA, if the initial field satisfies $|\phi| \leq 1$ then this condition will hold at all times, assuring that a one-to-one mapping can be defined. For this model we have in mind, following Mazenko’s treatment for SR interactions, to identify the field $m(\mathbf{r}, t)$ at points $\mathbf{r}$ near domain walls as the \textit{(signed) distance to the nearest wall} (along its local normal), with the sign of $m$ being that of $\phi$. This determines $m$ uniquely when $m \ll L(t)$. 

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To specify $m$ everywhere in space, we define the function $\phi(r, t) = \phi(m(r, t))$ by extending Mazenko’s suggestion \cite{7} of using the equilibrium planar domain wall profile function for an isolated wall, with $m$ the coordinate normal to the wall, i.e. the function $\phi(m)$ is specified by the equation

$$0 = \frac{d^2 \phi(m)}{dm^2} - V'(\phi(m)) + J_{LR} \int d^{d-1} y \int_{-\infty}^{+\infty} dm' \frac{[\phi(m') - \phi(m)]}{[(m' - m)^2 + y^2]^{1/2}},$$

with boundary conditions $\phi(0) = 0$ and $\phi(m) \rightarrow \text{sign}(m)$ for $|m| \rightarrow \infty$. Using (17) we rewrite the NL term in (11)-(12) as

$$\langle \phi(2) V' (\phi(1)) \rangle = \langle \phi(2) \frac{d^2 \phi(m(1))}{dm(1)^2} \rangle +$$

$$Q(d, \sigma) \int_{-\infty}^{+\infty} ds \frac{J_{LR}}{|s|^{1+\sigma}} \langle \phi(m(2))[\phi(m(1) + s) - \phi(m(1))] \rangle \quad (18)$$

$$= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \tilde{\phi}(k) \bar{\phi}(k') \left[ J_{LR} h(d, \sigma) k^\sigma - k^2 \right] \langle e^{im(1)k + im(2)k'} \rangle \quad (19)$$

where $h(d, \sigma)$ and $Q(d, \sigma)$ are given by (3)- (7), and $\tilde{\phi}(k)$ is the Fourier transform of $\phi(m)$.

Following Mazenko \cite{7}, we now make the key assumption that $m(r, t)$ is a gaussian field (with zero mean) at all times, with a pair distribution function

$$P(m(1), m(2)) = N \exp \left[ -\frac{1}{2(1 - \gamma^2)} \left( \frac{m(1)^2}{S_0(1)} + \frac{m(2)^2}{S_0(2)} - \frac{2\gamma m(1)m(2)}{\sqrt{S_0(1)S_0(2)}} \right) \right], \quad (20)$$

$$S_0(1) = \langle m(1)^2 \rangle, \quad \gamma(1, 2) = \frac{\langle m(1)m(2) \rangle}{\sqrt{S_0(1)S_0(2)}}, \quad N = \frac{1}{2\pi \sqrt{(1 - \gamma^2)S_0(1)S_0(2)}}. \quad (21)$$

We also note that, as the walls become effectively sharp in the late-time regime, we can use the profile $\phi(m) = \text{sign}(m)$ to evaluate the leading contribution to the scaling functions. From (13) we expect the effect of ignoring the power tail in the profile is to neglect a quantity of relative order $\sim 1/L(t)^\sigma$ in the LR part of (13). The purely SR part of the NL term is then simply given, as a non-linear function of $C(r, t)$, by Mazenko’s result for the SRMA \cite{7}

$$\langle \phi(2) \frac{d^2 \phi(m(1))}{dm(1)^2} \rangle = -\frac{2}{\pi S_0(1)} \tan \left( \frac{\pi}{2} C \right). \quad (22)$$
Deriving a similar result for the LR part of the NL term is more tricky. There are three different ways to perform the calculation: we will outline the basic steps of each one. Representing \( \phi(m) \) in Fourier space and Taylor expanding \( \phi(m+s) \) in powers of \( s \), using the gaussian property and returning to real space, gives the formal expansion [23]

\[
F_{NL}(1,2) \equiv Q(d, \sigma) \int_{-\infty}^{+\infty} ds \frac{J_{LR}}{|s|^{1+\sigma}} \left\langle \phi(m(2))|\phi(m(1)+s)-\phi(m(1))| \right\rangle
\]

\[
= Q(d, \sigma) \int_{-\infty}^{+\infty} ds \frac{J_{LR}}{|s|^{1+\sigma}} \sum_{n=1}^{\infty} \frac{s^{2n}}{(2n)!} 2^n \partial^n C(1,2) \frac{\partial S_0(1)}{\partial \gamma} \cdot \tag{23}
\]

Using \( C(1,2) = \left\langle \text{sign}(m(1))\text{sign}(m(2)) \right\rangle \), the integral representation \( \text{sign}(m) = 1/(i\pi) \int_{-\infty}^{+\infty} dz \exp[izm]/z \), and the gaussian property, the series can be summed. Finally, differentiating with respect to \( C_0(1,2) = \gamma\sqrt{S_0(1)S_0(2)} \), performing the \( z \) and \( s \)-integrals and integrating back, yields the non-linear function

\[
F_{NL}(1,2) = \frac{J_{LR} a(d, \sigma)}{S_0(1)^{\sigma/2}} \int_0^{2\pi} d\theta \sec^\sigma(\theta) , \tag{24}
\]

\[
a(d, \sigma) = h(d, \sigma) \frac{2^{1+\sigma/2} \Gamma(\frac{1+\sigma}{2})}{\pi^{3/2}} . \tag{25}
\]

Alternatively, we can take \( \phi(m) = \text{sign}(m) \) from the start and do the \( s \)-integral, giving

\[
F_{NL}(1,2) = -\frac{2J_{LR}Q(d, \sigma)}{\sigma} \left\langle \frac{\text{sign}(m(1))\text{sign}(m(2))}{|m(1)|^{\sigma}} \right\rangle \cdot \tag{26}
\]

use integral representations for \( \text{sign} m \) and \( 1/|m|^\sigma \), do the gaussian integral, differentiate with respect to \( C_0(1,2) \), perform the remaining integrals, and finally integrate back over \( C_0(12) \), yielding the same result. Taking into account [13] and using a \( \text{sign} m \) profile, [24] can be recognized as the leading order result for a \( \phi^4 \)-potential NL term, i.e. \( <\phi(2)\phi(1)(1-\phi^2(1))> [24] \). Finally, the simplest derivation is to take the integral representation of \( \text{sign} m \) in [26], use the gaussian property, differentiate with respect to \( C_0(1,2) \) and do the gaussian integral, leading to the same point as the first calculation before its final integrations. This derivation, however, does not provide the appealing intermediate expressions [23] and [26].
According to our identification of $m(r, t)$ as a distance from the interface, we expect $S_0 \equiv \langle m^2 \rangle$ to have the scaling form $S_0 = L(t)^2$, which can be used along with (22) and (24) to rewrite equation (12) for the scaling function in the form

$$-\frac{1}{2} \frac{dL}{dt} x f' = \frac{1}{L(t)^2} \left( f'' + \frac{d-1}{x} f' + \frac{2}{\pi} \tan \left( \frac{\pi}{2} f \right) \right) + \frac{J_{LR}}{L(t)^{\sigma}} \left( \int d^d x' \frac{|f(x') - f(x)|}{|x - x'|^{d+\sigma}} - a(d, \sigma) \int_0^{\pi/2} d\theta \sec^\sigma(\theta) \right),$$  \hspace{1cm} (27)

For $\sigma < 2$ the SR part in (27), scaling as $1/L^2$, is negligible compared to the LR part, scaling as $1/L^\sigma$, and can be ignored (but see the discussion in section VI!). Demanding that the left side of (27) balance the terms of order $1/L^\sigma$ on the right requires $\dot{L}/L \sim 1/L^\sigma$, i.e. that $L(t) \sim t^{1/\sigma}$. Note that this disagrees with the expected form (8)! In section V we will argue that a resolution of this discrepancy requires us to drop the left side of (27) in leading order. For the moment, however, we pursue the original (and a priori natural) assumption that the left side scales as $1/L^\sigma$ and write

$$L(t) = (J_{LR} \mu t)^{1/\sigma},$$  \hspace{1cm} (28)

where $\mu$ is to be determined. Dropping the SR terms from (27) gives the final equation for the scaling function $f(x)$:

$$0 = (\mu/2\sigma) x f' + \int d^d x' \frac{|f(x') - f(x)|}{|x - x'|^{d+\sigma}} - a(d, \sigma) \int_0^{\pi/2} d\theta \sec^\sigma(\theta).$$  \hspace{1cm} (29)

Equation (29) has to be solved numerically for general scaling variable $x$. However, it is straightforward to derive analytically the behaviour for small and large $x$. Using the Porod’s law form $f(x) = 1 - a(d, \sigma)x + ...$ for small $x$ (it is simple to show that this is only consistent short-distance behavior), we find that the LR part of the NL term, (24), has a leading scaling behaviour as $x \to 0$ which is similar to (16)

$$F_{NL}(1, 2) = \frac{J_{LR} a(d, \sigma)}{L(t)^{\sigma}} \left( \frac{B(1-\sigma, 1+\sigma)}{2^{d+\sigma}} - \left( \frac{\pi \alpha}{2} \right)^{1-\sigma} \frac{x^{1-\sigma}}{1-\sigma} + ... \right), \hspace{1cm} 0 < \sigma < 1$$

$$= \frac{J_{LR} a(d, \sigma)}{L(t)^{\sigma}} \left( \frac{\pi \alpha}{2} \right)^{1-\sigma} \frac{1}{x^{\sigma - 1}((\sigma - 1)} + O(1) \right), \hspace{1cm} 1 < \sigma < 2, \hspace{1cm} (30)$$
where $B(x,y)$ is the beta function. Performing a small-$x$ expansion of eq. (29), we find that the dominant terms for $x \to 0$ are obtained from the terms multiplying $J_{LR}$ in (27), whose small-$x$ expansions are given by (16) and (30). Matching powers of $x$ for general $0 < \sigma < 2$, and using $A(d,\sigma) = -\alpha(d,\sigma)(2\pi)^d/h(d,1)$ in (16) (which follows from Fourier transforming the Porod tail [25]), we find

$$
\alpha(d,\sigma) = \frac{\sqrt{2}}{\pi} \left( \frac{\Gamma\left(\frac{d+1}{2} - \frac{\sigma}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \right)^{1/\sigma} , \quad 0 < \sigma < 2 ,
$$

for the coefficient of $x$ in the small-$x$ expansion of $f(x)$ [26]. For $\sigma = 2$, this reduces to the SR result $\alpha(d,2) = 2/(\pi\sqrt{d-1})$.

For $\sigma < 1$, (31) was obtained by matching the terms of $O(x^{1-\sigma})$ in (16) and (30). The leading (constant terms) yield an interesting sum rule to be satisfied by the structure factor scaling function $g(y)$ for this range of $\sigma$:

$$
\int \frac{d^dy}{(2\pi)^d} g(y) y^\sigma = \frac{2^{\sigma/2}}{\Gamma\left(\frac{d-\sigma}{2}\right)} \sin\left(\frac{\pi}{2} + \frac{\sigma}{2}\right) , \quad 0 < \sigma < 1 .
$$

We now look at the large-$x$ asymptotic form of eq. (29), and discuss the large-$x$ behaviour of $f(x)$. In this limit, $f(x) \to 0$ and the final two terms in (29) become $g(0)/x^{d+\sigma}$ and $-a(d,\sigma)\pi f(x)/2$ respectively. In this regime, (29) can be integrated to give

$$
f(x) \to \frac{2\sigma g(0)}{[(d+\sigma)\mu - \pi|a|]} \frac{1}{x^{d+\sigma}} + \frac{A}{x^{|a|\pi\sigma/\mu}} ,
$$

where we note from (25) and (6) that $a$ is negative. In general, both terms in (33) will be present in the large-$x$ solution. On physical grounds, however, we do not expect $f(x)$ to fall off with distance more slowly (in a power-law sense) than the underlying interactions, which decay as $r^{-(d+\sigma)}$. (An exception is when sufficiently long-range power-law spatial correlations are present in the $t = 0$ state. This power-law can then persist for general times [27]. Here, however, we consider only short-range correlations in the initial state.) We infer that either $\mu < |a|\pi\sigma/(d+\sigma)$, so that the second term in (33) is subdominant for large $x$, or $A = 0$. The first possibility, however, implies that the coefficient of the (dominant) first term in (33) is negative (since $g(0) > 0$ by definition), i.e. $f(x)$ approaches zero from below, which also seems
unphysical (and disagrees with numerical simulations \[28\]). We conclude that the only physically sensible possibility is that $A$ vanishes in (33). This can, presumably, only happen for a special choice of $\mu$, so the condition $A = 0$ determines $\mu$. This mechanism is very similar to that which determines $\mu$ for short-range interactions [7, 10]. Note that, if $f(x)$ is to approach zero from above for $x \to \infty$, (33) gives the inequality

$$\mu > \pi |a|/(d + \sigma) .$$

(34)

A sum rule for $\mu$ can be obtained by integrating (29) over space:

$$\mu = \frac{2\sigma |a|}{dg(0)} \int d^d x \int_0^{\pi f/2} d\theta \sec^\sigma \theta .$$

Finally, it should be noted that the above analysis implicitly assumes that $g(0)$ is finite, i.e. that $f(x)$ decays faster than $x^{-d}$. In fact, the mathematical structure allows for $f(x) \sim x^{-p}$ with $p < d$ [29], but we reject this possibility on the physical grounds that we appealed to before, namely that, at least for initial states with only short-ranged spatial correlations, the scaling function should not decay with a smaller power than the underlying interactions.

V. TWO-TIME CORRELATIONS

The gaussian approach can also be used to evaluate the two-time correlation function $C(r, t_1, t_2) = \langle \phi(x, t_1)\phi(x + r, t_2) \rangle$ and, in particular, the autocorrelation function $A(t_1, t_2) = C(0, t_1, t_2)$. The calculation is simplest in the limit $t_2 \gg t_1$, when $C \to 0$ and the full nonlinear equation can be linearized, Fourier transformed, and explicitly integrated. In this regime the analog of (9) for two-time correlations reads, in Fourier space (dropping the SR term on the right),

$$\frac{\partial C_k}{\partial t_2} = -J_{LR}|h|k^\sigma C_k + \frac{\pi |a|}{2\mu t} C_k ,$$

(35)

where (28) has been used for $L(t)$. We integrate (35) forward from time $\alpha t_1$, where $\alpha \gg 1$ ensures that the condition $t_2 \gg t_1$, required for the validity of (35), holds at all times. This gives

$$C_k(t_1, t_2) = C_k(t_1, \alpha t_1) \left( \frac{t_2}{\alpha t_1} \right)^{\pi |a|/2\mu} \exp\{-J_{LR}|h|k^\sigma(t_2 - \alpha t_1)\} .$$

(36)
Using the scaling form \( C_k(t_1, \alpha t_1) = L_1^d g_\alpha(kL_1) \), where \( L_1 = L(t_1) \), and summing over \( k \) for \( t_2 \gg \alpha t_1 \) gives the autocorrelation function

\[
A(t_1, t_2) = \text{const} \left( \frac{L_1}{L_2} \right)^{d-\pi|\alpha|\sigma/2\mu},
\]

where ‘const’ is clearly independent of \( \alpha \). The physical requirement that \( A \) decrease with increasing \( t_2 \) gives the inequality \( \mu > \pi|\alpha|\sigma/2d \), which is guaranteed by (34) for \( d > \sigma \).

The connection between the parameter \( \mu \) and the exponent describing the decay of the autocorrelation function is similar to that obtained for purely short-ranged interactions [10, 30].

VI. DISCUSSION AND SUMMARY

We have extended the original Mazenko gaussian approach [7] to the LR model and evaluated the late-time leading contribution to the NL term of eq. (27), yielding a dominant LR part given by (24), which is of order \( 1/L^\sigma \). An infinitely sharp wall profile has been used which amounts to neglecting a quantity of relative order \( 1/L^\sigma \). The LR term in the equation, (11)-(14), is of the same order and has an amplitude which is a function of \( x, d \) and \( \sigma \), but its non-local nature (i.e. its dependence on the values of \( f(x) \) everywhere) makes the problem particularly hard to handle.

Despite the profile power-law decay (15) induced by the LR interactions, the scaling function exhibits Porod’s law, i.e. a linear short-distance behaviour in real space with coefficient given by (31). This is consistent with the assumption that at late-times there are well-defined walls with a constant ‘width’ independent of \( L(t) \). This is an important point of principle, on which the identification of the field \( m(r, t) \) and the mapping (17) rely, and also a key ingredient in the first-principles derivation [17, 18] of the growth-law (8).

The central question we want to address in this paper is whether the gaussian theory is able to yield the correct growth-law for this model. We have seen that the ‘naive’ application of the gaussian approach present in section IV ostensibly gives the wrong growth law: (28) instead of (8). A related problem is the SRMB to which
Mazenko has attempted to apply the gaussian approach, yet the correct growth-law does not come out of the theory as cleanly as in the SRMA. In this system local conservation imposes a bulk diffusion process which controls the interface motion and delays domain coarsening relative to the purely relaxational dynamics of model A. There are some common features between the dynamics of a conserved and a LR interacting field, namely the existence of a bulk profile which relaxes rapidly to a non-saturating value as the walls move. One key difference, though, is that the true growth law for SRMB \( t^{1/3} \) is faster than that obtained by a naive application of the gaussian approach (without allowing for bulk diffusion), which gives \( t^{1/4} \).

Before implementing any approximation we focus the analysis on the exact equation (12). If the growth-law holds, the time-derivative term must be negligible compared to the LR term (14), which scales as \( L^{-\sigma} \), and therefore the NL term must have a leading contribution of order \( 1/L^\sigma \) which exactly cancels the LR term in the scaling limit. In fact, this condition determines the late-time leading contribution to the scaling function. Within the gaussian approximation, it amounts to neglecting the first term in (29) (which came from the left-hand side of (27)), to give

\[
0 = \int d^d x \left[ f(x) - f(x') \right] \frac{\sigma}{|x-x'|^{d+\sigma}} - a(d, \sigma) \int_0^{\pi/2} f d\theta \sec^{\sigma}(\theta) .
\]  

(38)

Solving this equation gives the scaling function \( f(x) \), within the gaussian approach, provided the growth law is slower than \( t^{1/\sigma} \). However, there seems to be no way to determine the growth law within this scheme. Moreover, (38) has a serious shortcoming. If we integrate the equation over all space, the first term drops out, giving the sum rule

\[
\int_0^\infty dx \, x^{d-1} \int_0^{\pi f(x)/2} d\theta \sec^{\sigma}(\theta) = 0 .
\]  

(39)

Since the integrand is positive definite, the only way this sum rule can be satisfied is for \( f(x) \) to be negative for some range (or ranges) of \( x \), with sufficient negative weight to satisfy (39). This seems a priori improbable for a nonconserved order parameter, and indeed numerical simulations show no hint of it.
We emphasize, however, that since our fundamental equation (12) is exact, the analogue of (36) obtained without making the gaussian approximation must be exactly true. Because the true growth is slower than $t^{1/\sigma}$, the left side of (12) is negligible in the scaling limit. Taking the Fourier transform of the equation, and setting $k = 0$, the ‘long-range’ term vanishes. This leaves the identity

$$\int d^d x \langle \phi(2) V(\phi(1)) \rangle = 0,$$

(40)
of which (39) is the special case obtained within the gaussian approximation.

Our results seem to indicate that the gaussian approach, applied to the bulk equation of motion, is unable to account for the qualitative feature of coarsening in systems with long-range interactions. However, we cannot make a definitive statement as we have not exhausted all the possible choices for the gaussian field. There may exist a mapping definition which is physically more appropriate and works better than (17). What seem to be clear is that, beyond the simple nonconserved system with short-range interactions, one cannot apply the gaussian approach in a straightforward and ‘naive’ manner to construct a closed equation for the scaling function. Just as for the conserved scalar system, a deeper understanding of the underlying physics may be required in order to implement a more controlled approximate scheme. It is possible that this might be achieved by means of an interfacial approach [31].

To summarize, the gaussian approach, naively applied, is unable to yield a growth-law different from that obtained from dimensional analysis of the linear terms in the equation of motion. As mentioned above, the failure of the gaussian approach in this context could be due to the particular choice employed for the mapping between $\phi$ and the gaussian field. For example, one can in principle use the same mapping as Mazenko [7], $\phi''(m) = V''(\phi(m))$, appropriate to a purely short-range interactions. However, this leads to an inconsistent scaling analysis of the equation for $C(\mathbf{r}, t)$ (e.g. at short-distances there is no LR part in the NL term to match the LR term (16)), and $m(\mathbf{r}, t)$ can no longer be regarded as a distance.
from a wall. By contrast, the mapping employed here, defined by eq. (17), seems far more natural and physically suitable for a system with LR interactions.

Finally we note that the present methods can also be used for a vector order parameter with long-range interactions. In that case the $t^{1/\sigma}$ growth obtained within the gaussian approach is correct (apart from logarithmic corrections for $n = 2$ [17]). The purpose of the present paper, however, is to test the method on those systems which provide the greatest challenge, i.e. scalar systems, in the hope that the difficulties identified here may stimulate the development of more robust approximation schemes.

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