The generalized porous medium equation on graphs: existence and uniqueness of solutions with $\ell^1$ data

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Abstract
We study solutions of the generalized porous medium equation on infinite graphs. For nonnegative or nonpositive integrable data, we prove the existence and uniqueness of mild solutions on any graph. For changing sign integrable data, we show existence and uniqueness under extra assumptions such as local finiteness or a uniform lower bound on the node measure.

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1 Introduction

Our model formal equation is the following:

$$\partial_t u(t, x) + \Delta \Phi(u(t, x)) = f(t, x)$$

for every $(t, x) \in (a, b) \times X$.

This equation is called the generalized porous medium equation (GPME) or filtration equation whenever $\Delta$ is the Laplace operator and $\Phi$ is the canonical extension to a function space of a map $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi$ is strictly monotone increasing, $\phi(\mathbb{R}) = \mathbb{R}$ and $\phi(0) = 0$. If $\phi(s) = s^m := s|s|^{m-1}$, then the above equation is known as the porous medium equation (PME) when $m > 1$ and the fast diffusion equation (FDE) when $0 < m < 1$. Clearly, when $m = 1$ and $f \equiv 0$ we recover the classic heat equation.
The GPME has a long story and we invite the interested reader to look at the seminal book by Vázquez [70] for a detailed and exhaustive account. In recent years, research interest about properties of solutions of the GPME has focused on the Riemannian setting as can be seen by the increasing number of related works, see, for example, [7, 8, 18, 26–34, 56, 57, 69] and references therein for an overview of the most significant developments.

In contrast, in the graph setting there are still relatively few results for the GPME. This is despite the fact that, on the one hand, the GPME is being used as a model equation for several real-world phenomena (e.g., the flow of gas through a porous medium, water infiltration or population dynamics) and, on the other hand, graphs are ubiquitous in many applied fields: in physics [62], biology [50, 51, 66], image and signal processing [19, 64, 68], engineering [17], etc.

To make our setting more precise, let us fix a graph $G = (X, w, \kappa, \mu)$ where $X$ is a countable node set, $w: X \times X \to [0, \infty)$ is a symmetric map with zero diagonal, $\kappa: X \to [0, \infty)$ is a possibly nontrivial killing term and $\mu: X \to (0, \infty)$ is a strictly positive node measure on $X$.

For notational convenience, let us fix $a = 0$ and $b = T \in (0, \infty]$. We will focus our attention on the following Cauchy problem posed on $G$:

\[
\begin{align*}
\text{Cauchy-GPME} \quad & \begin{cases} 
\partial_t u(t, x) + \Delta u(t, x) = f(t, x) & \text{for every } (t, x) \in (0, T) \times X, \\
\lim_{t \to 0^+} u(t, x) = u_0(x) & \text{for every } x \in X
\end{cases}
\end{align*}
\]

where $f: (0, T) \times X \to \mathbb{R}$ and $u_0: X \to \mathbb{R}$ are generic functions at the moment. In this setting, $\Delta$ represents the (formal) graph Laplacian operator defined by the formula

\[
\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \in X} w(x, y) (u(x) - u(y)) + \frac{\kappa(x)}{\mu(x)} u(x).
\]

The GPME on graphs belongs to the broader class of nonlinear diffusion equations with nonconstant diffusion since the edge-weight function $w(\cdot, \cdot)$ can be seen as a counterpart of the nonconstant diffusion coefficients $\{a_{i,j}(x)\}_{i,j=1}^d$ which characterize the second-order differential operator $\sum_{i=1}^d \partial_i \left( a_{i,j}(x) \sum_{j=1}^d \partial_j u(x) \right)$ acting on smooth functions on $\mathbb{R}^d$.

We now give a brief overview of some recent results concerning nonlinear equations in the graph setting. For the counterpart of the Kazdan-Warner equation, see [24, 45, 55]. Concerning the existence and uniqueness of solutions for reaction-diffusion type equations on the lattice $\mathbb{Z}$, see [65, 67]. For the Yamabe and other equations, see [23, 25, 54]. For parabolic equations involving the $p$-Laplacian, see [40, 61]. Finally, we mention some results concerning the existence and nonexistence of global nonnegative solutions of an abstract semilinear heat equation given in [52, 53, 73] which were recently extended to a general setting in [49].

With reference to the PME in the discrete setting, we highlight [20] where the authors study the (finite) discrete analogue of the Wasserstein gradient flow structure for the PME in $\mathbb{R}^n$. Concerning the existence and uniqueness of solutions of the Cauchy-GPME to the best of our knowledge there is an almost complete lack of a systematic treatment even in the case of finite graphs with one notable exception: In [61, Corollary 5.4], exploiting an interesting link between the PME and the $p$-heat equation (which is well-known in $\mathbb{R}$, see, e.g., [70, Section 3.4.3]), it is shown that if $G$ is an infinite tree, uniformly locally finite with $\mu \equiv 1$, then there exists a unique solution of the Cauchy-GPME for $\phi(s) = s|s|^{m-1}$ for any $u_0 \in \ell^2(X, \mu)$ and $f \equiv 0$. For more details and the definition of solutions in that setting we
invite the interested reader to look at the mentioned paper.

Our approach is different. The main goal of this article is to prove existence and uniqueness results for classes of solutions of the Cauchy-GPME problem under the weakest possible hypotheses on the graph \( G \), on the initial datum \( u_0 \) and on the forcing term \( f \). To achieve this, we will borrow techniques from the theory of semigroups on Banach spaces and, as it will become clear later, \( \ell^1(X, \mu) \) will turn out to be the ideal space for our considerations.

We will consider three classes of solutions: mild (Definition 3.3), strong (Definition 3.4) and classic (Definition 3.5) which are characterized by an increasing “regularity.” In particular, mild solutions \( u \) are limits of \( \epsilon \)-approximations \( u_\epsilon \) that satisfy the Cauchy-GPME for a time discretization well-adapted to \( f \). If the operator \( L := \Delta \Phi \) is \( m \)-accretive, then it is possible to immediately infer the existence and uniqueness of mild solutions for the Cauchy-GPME problem by appealing to well-known results, see [4, 6, 15]. Thus, the bulk of our work consists of establishing the \( m \)-accretivity of (a restriction of) the operator \( L \) on an appropriate Banach space.

Accretivity of an operator \( \mathcal{L} \) with respect to a norm \( \| \cdot \| \) on a real Banach space \( \mathfrak{E} = (E, \| \cdot \|) \) means that

\[
\|(u - v) + \lambda (\mathcal{L}u - \mathcal{L}v)\| \geq \|u - v\|
\]

for every \( u, v \in \text{dom}(\mathcal{L}) \subseteq E \) and for every \( \lambda > 0 \). Furthermore, \( m \)-accretivity means that \( \mathcal{L} \) is accretive and \( \text{id} + \lambda \mathcal{L} \) is surjective for every \( \lambda > 0 \). We note that accretivity implies that \( \text{id} + \lambda \mathcal{L} \) is injective, thus, \( m \)-accretivity gives that \( \text{id} + \lambda \mathcal{L} \) is bijective. For a more detailed introduction to the concepts of accretivity and \( m \)-accretivity, see Sect. 2.3.

As can be seen directly, the accretivity property depends on both the operator \( \mathcal{L} \) and on the underlying Banach space. For example, the graph Laplacian \( \Delta \) on \( \ell^p(X, \mu) \) is \( m \)-accretive on any finite graph for \( p \in [1, \infty) \), see Proposition 2.5. On the other hand, the nonlinear operator \( \mathcal{L} \) can fail to be accretive with respect to the \( \ell^2 \)-norm, see Example 2.3. What is crucial for our analysis is that the restriction of \( \mathcal{L} \) to a suitable dense subset of \( \ell^1(X, \mu) \) will be shown to be accretive for any graph. For \( m \)-accretivity to hold some additional hypothesis are required as will be discussed in what follows. We note that there is a parallel development concerning the surjectivity of the formal graph Laplacian \( \Delta \) which is always surjective on infinite, locally finite graphs but not necessarily surjective in the not locally finite case, see [11, 47].

For a complete introduction to the notation we refer to Sect. 2. We denote by \( \ell^{1,+}(X, \mu) \) and \( \ell^{1,-}(X, \mu) \) the cones of nonnegative and nonpositive integrable functions, respectively, and by \( \mathcal{L} \) the operator

\[
\mathcal{L} : \text{dom}(\mathcal{L}) \subseteq \ell^1(X, \mu) \rightarrow \ell^1(X, \mu),
\]

\[
\text{dom}(\mathcal{L}) := \{ u \in \ell^1(X, \mu) \mid \Phi u \in \text{dom}(\Delta), \Delta \Phi u \in \ell^1(X, \mu) \}
\]

whose action is given by

\[
\mathcal{L}u := \Delta \Phi u.
\]

For a subset \( \Omega \subseteq \text{dom}(\mathcal{L}) \), we write \( \mathcal{L}|_{\Omega} \) for the restriction of \( \mathcal{L} \) to \( \Omega \).

We now state the main results, whose proofs can be found in Sect. 4.1. The first main result discusses the accretivity and \( m \)-accretivity of \( \mathcal{L} \).
Theorem 1 Let G = (X, w, κ, μ) be a graph. Then, there exists a dense subset Ω ⊆ dom(ℒ) such that L|Ω is accretive. Moreover, for every λ > 0 and for every g ∈ ℓ1,±(X, μ) there exists a unique u ∈ ℓ1,±(X, μ) ∩ Ω such that

\[(\text{id} + λL)u = g.\]

If one of the following extra hypotheses holds:

(H1) G is locally finite;
(H2) \(\inf_{x \in X} \mu(x) > 0;\)
(H3) \(\sup_{x \in X} \frac{\sum_{y \in X} w(x, y)}{\mu(x)} < \infty\) and \(\Phi(\ell^1(X, \mu)) \subseteq \ell^1(X, \mu);\)

then \(\text{id} + λL\) restricted to Ω is also surjective. In particular, \(L|Ω\) is m-accretive. Moreover, in all cases, the solution u satisfies the contractivity estimate

\[||u|| \leq ||g||.\]

The second main result uses general theory along with the m-accretivity established in the first result to yield existence and uniqueness of mild solutions for the Cauchy-GPME. For this, we consider two cases, namely, when the initial data is nonnegative or nonpositive and when the initial data changes sign. In the second case, we need to add one of the extra hypotheses appearing in the first result above to guarantee existence. For the definitions of the various types of solutions for the Cauchy-GPME and the connections between them, see Sect. 3.

Theorem 2 Let G = (X, w, κ, μ) be a graph. Let

(i) \(u_0 \in \ell^1(X, \mu);\)
(ii) \(f \in L^1_{\text{loc}}([0, T]; \ell^1(X, \mu)).\)

If one of the following additional conditions holds:

(a) \(u_0, f(t) \geq 0\) (or \(\leq 0\)) for all \(t \geq 0;\)
(b) \(u_0\) or \(f(t)\) changes sign and at least one of (H1), (H2) or (H3) is satisfied;

then there exists a unique mild solution u of the Cauchy-GPME.

Furthermore, \(u(t) \in \ell^1(X, \mu)\) for all \(t \in [0, T]\) and for every \(\epsilon > 0\) there exists a continuous function \(δ: [0, \infty) \rightarrow [0, \infty)\) such that \(δ(0) = 0\) and if \(u_ε\) is an \(ε\)-approximate solution of the Cauchy-GPME, then

\[||u(t) - u_ε(t)|| \leq δ(ε)\quad \text{for } t \in [0, T - ε].\]  \hspace{1cm} (1.1)

Moreover, for any pair \((u_0, f), (\hat{u}_0, \hat{f})\), the corresponding mild solutions \(u, \hat{u}\) satisfy

\[||u(t_2) - \hat{u}(t_2)|| \leq ||u(t_1) - \hat{u}(t_1)|| + \int_{t_1}^{t_2} \|f(s) - \hat{f}(s)\| \, ds, \quad \forall 0 \leq t_1 < t_2 \leq T.\]  \hspace{1cm} (1.2)

Finally, under hypothesis (a), \(u(t) \geq 0\) (or \(\leq 0\)) for every \(t \geq 0.\)

The paper is organized in the following way:

- In Sect. 2 we present the main definitions and describe the tools that we will use.
- In Sect. 3 we introduce the abstract Cauchy problem along with a classification of types of solutions.
- Section 4 is the core of the paper: We present the proofs of Theorems 1 and 2 with an introductory part about the main issues to be addressed. As a concluding application, in Corollary 4.2 we prescribe some hypotheses on the graph that guarantee that a mild solution is indeed a classic solution.

Since the proofs involved in Sect. 4 are technical and rely on several auxiliary results, we collect them in Appendices A and B.
2 Preliminaries

In this section we collect background material for the graph setting and the main mathematical tools that we will use in our proofs.

2.1 Notation

Given a set $X$ and a real function space $\mathcal{F} \subseteq \{u: X \to \mathbb{R}\}$, we denote by $\text{id}: \mathcal{F} \to \mathcal{F}$ the identity operator. If $\phi: \text{dom}(\phi) \subseteq \mathbb{R} \to \mathbb{R}$ is a function, then we denote by the capital letter $\Phi$ the canonical extension of $\phi$ to $\mathcal{F}$, that is, the operator $\Phi: \text{dom}(\Phi) \subseteq \mathcal{F} \to \mathcal{F}$ given by
\[
\text{dom}(\Phi) := \{u \in \mathcal{F} \mid u(x) \in \text{dom}(\phi) \ \forall \ x \in X\},
\]
\[
\Phi u(x) := \phi(u(x)).
\]

Given a pair of real-valued functions $u$ and $v$ on $X$, we write $u \geq v$ if $u(x) \geq v(x)$ for every $x \in X$. All other ordering symbols are defined accordingly.

Given a real Banach space $\mathcal{E} = (E, \| \cdot \|)$, consider an $E$-valued function $f: [0, T) \subset \mathbb{R} \to E, t \mapsto f(t) \in E$. Such a function $f$ is called simple if $f$ is of the form
\[
f(t) = \sum_{k=1}^{n} e_k \mathbb{1}_{I_k}(t), \quad e_k \in E,
\]
where $I_k$ are Lebesgue measurable subsets of $[0, T]$ and $\mathbb{1}_{I_k}$ is the indicator function of $I_k$.

The integral of an $E$-valued simple function is defined by
\[
\int_{0}^{T} f(t) \, dt := \sum_{k=1}^{n} e_k m(I_k),
\]
where $m(\cdot)$ is the Lebesgue measure on $[0, T]$. A function $f$ is (strongly) measurable if there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of simple functions such that $f_n(t) \to f(t)$ in norm for almost every (a.e.) $t$ in $[0, T]$.

A strongly measurable function $f$ is Bochner integrable if there exists a sequence of simple functions such that $f_n \to f$ pointwise a.e. in $[0, T]$ and
\[
\lim_{n \to \infty} \int_{0}^{T} \|f_n(t) - f(t)\| \, dt = 0,
\]
or equivalently, by a theorem of Bochner, if and only if $\int_{0}^{T} \|f(t)\| \, dt < \infty$. The integral of $f$ is then defined by
\[
\int_{0}^{T} f(t) \, dt = \lim_{n \to \infty} \int_{0}^{T} f_n(t) \, dt.
\]

We denote the space of Bochner integrable functions from $[0, T]$ to $E$ by
\[
L^{1}([0, T]; E) := \left\{ f: [0, T] \to E \text{ measurable} \mid \int_{0}^{T} \|f(t)\| \, dt < \infty \right\}.
\]

In the same fashion, if $T = \infty$, we denote by $L^{1}_{\text{loc}}([0, T]; E)$ the space of $E$-valued functions that are locally Bochner integrable, that is, $f \in L^{1}_{\text{loc}}([0, T]; E)$ if and only if $f \in L^{1}([0, a]; E)$ for every $a \in (0, \infty)$. Clearly, if $T < \infty$, then $L^{1}_{\text{loc}}([0, T]; E) = L^{1}([0, T]; E)$. 

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A function \( f \in L^1_{\text{loc}}([0, T]; E) \) is weakly differentiable with weak derivative \( g \in L^1_{\text{loc}}([0, T]; E) \) if
\[
\int_0^T f(t)\eta'(t)\,dt = -\int_0^T g(t)\eta(t)\,dt, \quad \forall \eta \in C^\infty_c(0, T),
\]
where the integrals are understood in the Bochner sense. The first Sobolev space for locally Bochner integrable functions is defined as
\[
W^{1,1}_{\text{loc}}([0, T]; E) := \{ f \in L^1_{\text{loc}}([0, T]; E) \mid f \text{ is weakly differentiable} \}.
\]
Let us point out that \( f \in W^{1,1}_{\text{loc}}([0, T]; E) \) if and only if
\[
f(t) = e_0 + \int_0^t g(s)\,ds.
\]
Moreover, \( f \) is absolutely continuous and a.e. differentiable in \([0, T] \) with \( f'(t) = g(t) \).

For a review of integration and weak derivatives of vector-valued functions, see, for example, [74, Chapter 5, Sections 4 and 5] and [10, Chapter 1, Section 4.5].

Given an operator \( \mathcal{L} : \text{dom}(\mathcal{L}) \subseteq E \to E \) and a subset \( \Omega \subseteq \text{dom}(\mathcal{L}) \), then the restriction of \( \mathcal{L} \) to \( \Omega \) is the operator \( \mathcal{L}_{|\Omega} : \text{dom}(\mathcal{L}_{|\Omega}) \subseteq E \to E \) such that
\[
\text{dom}(\mathcal{L}_{|\Omega}) = \Omega, \quad \mathcal{L}_{|\Omega}u = \mathcal{L}u \quad \forall u \in \Omega.
\]

As a final piece of notation we mention that if \( E \subseteq \{ u : X \to \mathbb{R} \} \), then we write \( f(t, x) \) to indicate the value of \( f(t) \in E \) at \( x \in X \).

### 2.2 The graph setting

For a detailed introduction to the graph setting as presented here, see [44]. We begin with the definition of a graph.

**Definition 2.1** (Graph) A graph is a quadruple \( G = (X, w, \kappa, \mu) \) given by
- a countable set of nodes \( X \);
- a nonnegative edge-weight function \( w : X \times X \to [0, \infty) \);
- a nonnegative killing term \( \kappa : X \to [0, \infty) \);
- a positive node measure \( \mu : X \to (0, \infty) \)

where the edge-weight function \( w \) satisfies:

- (A1) Symmetry: \( w(x, y) = w(y, x) \) for every \( x, y \in X \);
- (A2) No loops: \( w(x, x) = 0 \) for every \( x \in X \);
- (A3) Finite sum: \( \sum_{y \in X} w(x, y) < \infty \) for every \( x \in X \).

If the cardinality of the node set is finite, i.e., \( |X| < \infty \), then \( G \) is called a finite graph, otherwise, \( G \) is called an infinite graph. The non-zero values \( w(x, y) \) of the edge-weight function \( w \) are called weights associated with the edge \([x, y]\). In this case we will write \( x \sim y \) meaning that \( x \) is connected to \( y \). On the other hand, if \( w(x, y) = 0 \), then we will write \( x \not\sim y \) meaning that \( x \) and \( y \) are not connected by an edge. A walk is a (possibly infinite) sequence of nodes \( \{x_i\}_{i \geq 0} \) such that \( x_i \sim x_{i+1} \). A path is a walk with no repeated nodes. A graph is connected if there is a finite walk connecting every pair of nodes, that is, for any pair of nodes \( x, y \) there is a finite walk such that \( x = x_0 \sim x_1 \sim \cdots \sim x_n = y \). Moreover, we will say that a subset \( A \subseteq X \) is connected if for every pair of nodes \( x, y \in A \) there exists
a finite walk connecting \( x \) and \( y \) all of whose nodes are in \( A \). A subset \( A \subseteq X \) is a connected component of \( X \) if \( A \) is maximal with respect to inclusion.

A graph is said to be locally finite if for every \( x \in X \) there are at most a finite number of nodes \( y \) such that \( w(x, y) \neq 0 \). We define the degree \( \deg \) and weighted degree \( \text{Deg} \) of a node \( x \) as

\[
\deg(x) := \sum_{y \in X} w(x, y) + \kappa(x) \quad \text{and} \quad \text{Deg}(x) := \frac{\deg(x)}{\mu(x)}.
\]

Clearly, by \((A3)\), \( \deg(x) \) and \( \text{Deg}(x) \) are finite for every \( x \in X \). Observe that, if \( \kappa \equiv 0 \) and \( w(x, y) \in \{0, 1\} \), then \( \deg \) corresponds to the standard definition of degree in the literature on finite graphs (e.g., [21]).

The set of real-valued functions on \( X \) is denoted by \( C(X) \) and \( C_c(X) \) denotes the set of functions on \( X \) with finite support. As usual, for \( p \in [1, \infty) \) we define the \( \ell^p(X, \mu) \) subspaces as

\[
\ell^p(X, \mu) := \left\{ u \in C(X) \mid \sum_{x \in X} |u(x)|^p \mu(x) < \infty \right\} \quad \text{for } p \in [1, \infty),
\]

\[
\left\{ u \in C(X) \mid \sup_{x \in X} |u(x)| < \infty \right\} \quad \text{for } p = \infty
\]

with their norms

\[
\|u\|_p := \begin{cases} \left( \sum_{x \in X} |u(x)|^p \mu(x) \right)^{1/p} & \text{for } p \in [1, \infty), \\ \sup_{x \in X} |u(x)| & \text{for } p = \infty \end{cases}
\]

and with the standard remark that the \( \ell^2 \)-norm is induced by the inner product

\[
\langle u, v \rangle_{\ell^2} := \sum_{x \in X} u(x)v(x)\mu(x)
\]

making \( \ell^2(X, \mu) \) into a Hilbert space. In general, we will use the convention \( \| \cdot \| := \| \cdot \|_1 \) since we will work almost always with the \( \ell^1 \)-norm. However, in case of possible ambiguity in the text, we will specify the norm. In addition to the previous standard definitions, we introduce the following restrictions to the nonnegative/nonpositive cones:

\[
\ell^{1,+}(X, \mu) := \ell^1(X, \mu) \cap \{ u \in C(X) \mid u \geq 0 \}, \quad \ell^{1,-}(X, \mu) := -\ell^{1,+}(X, \mu).
\]

We now define the formal graph Laplacian \( \Delta : \text{dom}(\Delta) \subseteq C(X) \rightarrow C(X) \) associated to the graph \( G = (X, w, \kappa, \mu) \) by

\[
\text{dom}(\Delta) := \left\{ u \in C(X) \mid \sum_{y \in X} w(x, y)|u(y)| < \infty \quad \forall x \in X \right\}, \quad (2.1a)
\]

\[
\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \in X} w(x, y)(u(x) - u(y)) + \frac{\kappa(x)}{\mu(x)}u(x) = \text{Deg}(x)u(x) - \frac{1}{\mu(x)} \sum_{y \in X} w(x, y)u(y). \quad (2.1b)
\]
Remark 1 We observe that if \( u \geq 0 \), then \( \Delta u(x) \) can always be defined as an extended real-valued function taking values in \([-\infty, \infty)\). Furthermore, if \( u \geq 0 \), then \( u \in \text{dom}(\Delta) \) if and only if \( |\Delta u(x)| < \infty \) for every \( x \in X \) if and only if \( \Delta u(x) > -\infty \) for every \( x \in X \) if and only if \( \sum_{y \in X} w(x, y)u(y) > -\infty \) for every \( x \in X \).

As we will see in the upcoming sections, it will be useful to deal with subgraphs of a graph. We start by discussing the notion of the interior and two notions of boundary for a subset of the node set. Given a graph \( G = (X, w, \kappa, \mu) \) and a subset \( A \subset X \), then
\[
\hat{A} := \{ x \in A \mid x \sim y \text{ for every } y \in X \setminus A \}
\]
is called the interior of \( A \) and the elements of \( \hat{A} \) are called interior nodes of \( A \). On other hand, the sets of nodes
\[
\check{\partial} A := \{ x \in A \mid x \sim y \text{ for some } y \in X \setminus A \},
\]
\[
\cdot \partial A := \{ y \in X \setminus A \mid y \sim x \text{ for some } x \in A \}
\]
are called the interior boundary and the exterior boundary of \( A \), respectively. Although these notions are rather standard in the graph setting, we illustrate the definitions with an example in Fig. 1.

![Fig. 1 Example of a connected graph \( G = (X, w, \kappa, \mu) \) (in the left picture) with \( X = \{ x_i \mid i = 0, \ldots, 9 \} \) and a proper subset \( A = \{ x_4, x_5, x_6, x_7, x_8, x_9 \} \). A black line between two nodes \( x_i, x_j \in X \) means that \( x_i \sim x_j \). In the right picture, the interior \( \hat{A} = \{ x_5, x_6, x_8 \} \) is colored in green while the interior boundary \( \check{\partial} A = \{ x_4, x_7, x_9 \} \) is colored in light green. The nodes in the exterior boundary \( \cdot \partial A = \{ x_0, x_3 \} \subseteq X \setminus A \) are colored in light gray.](image)
We next introduce the concept of an induced subgraph.

**Definition 2.2** (Induced subgraph) We say that a graph \( F = (A, w', \kappa', \mu') \) is an *induced subgraph* of \( G \), and we write \( F \subset G \), if

- \( A \subset X \);
- \( w' \equiv w|_{A \times A} \);
- \( \kappa'(x) = \kappa(x) \) for every \( x \in \hat{A} \);
- \( \mu' \equiv \mu|_A \)

where \( w|_{A \times A} \) and \( \mu|_A \) denote the restrictions of \( w \) and \( \mu \) to the sets \( A \times A \) and \( A \), respectively. We call \( G \) the *host graph* or the *supergraph*. The corresponding formal graph Laplacian for a subgraph \( F \) is defined according to (2.1a) and (2.1b) where the quadruple \( (X, w, \kappa, \mu) \) is replaced by \( (A, w', \kappa', \mu') \). Observe that we do not require that \( \kappa' = \kappa \) on \( \hat{\partial} A \). Different choices of \( \kappa' \) on \( \hat{\partial} A \) will produce different subgraphs. We say that \( F \) is the *canonical induced subgraph* if \( \kappa' = \kappa|_A \).

**Remark 2** Our notion of induced subgraph is intrinsically related to the killing term \( \kappa' \). If we do not consider any \( \kappa \), then the definition of induced subgraph is equivalent to the classical one, see for example [21, Definition 2.2].

Of particular interest is the killing term \( \kappa_{dir} \) that arises from “Dirichlet boundary conditions.”

**Definition 2.3** (Dirichlet subgraph) An induced subgraph

\[
F_{dir} := (A, w|_{A \times A}, \kappa_{dir}, \mu|_A) \subset G
\]

is called a *Dirichlet subgraph* if

\[
\begin{align*}
\kappa_{dir}(x) &:= \kappa|_A(x) + b_{dir}(x), \\
b_{dir}(x) &:= \sum_{y \notin \partial A} w(x, y) = \sum_{y \notin A} w(x, y).
\end{align*}
\]

We note that \( b_{dir} : A \to \mathbb{R} \) is finite because of (A3). We call \( b_{dir} \) the *boundary (Dirichlet) weight function* and \( \kappa_{dir} \) the *Dirichlet killing term*. Clearly, \( b_{dir}(x) = 0 \) for every \( x \in \hat{A} \). We will denote by \( \Delta_{dir} \) the graph Laplacian of \( F_{dir} \) in order to distinguish it from the graph Laplacian of \( G \).

The Dirichlet killing term describes the edge deficiency of nodes in \( F_{dir} \) compared to the same nodes in \( G \), see Fig. 2. The name “Dirichlet” in the above definition comes from the following observation, see, for example, [43, pg. 197 and Proposition 2.4] and [44, Proposition 2.23]: Let \( i : C(A) \hookrightarrow C(X) \) be the canonical embedding and \( \pi : C(X) \to C(A) \) be the canonical projection, i.e.,

\[
i v(x) = \begin{cases} 
v(x) & \text{if } x \in A, \\
0 & \text{if } x \in X \setminus A, \end{cases} \quad \pi u = u|_A.
\]

Under some mild assumptions, it is almost straightforward to prove, see Lemma A.1, that

- \( \Delta_{dir} v(x) = \Delta i v(x) \) for \( v \in \text{dom}(\Delta_{dir}) \) and \( x \in A \);
- \( \Delta u(x) = \Delta_{dir} \pi u(x) \) for \( u \in \text{dom}(\Delta) \cap \{ u \in C(X) \mid u \equiv 0 \text{ on } X \setminus A \} \) and \( x \in A \).

Therefore, the Dirichlet graph Laplacian \( \Delta_{dir} \) can be viewed as the restriction of \( \Delta \) having imposed Dirichlet conditions on the exterior boundary of \( A \).
Fig. 2 Continuation of the example in Fig. 1. The left picture gives a canonical induced subgraph \( F \subset G \): In general, \( F \) is not “affected” by the complementary set \( X \setminus A \) since the weights of the edges lying in \( X \setminus A \) do not influence the graph \( F \). The right picture gives instead a Dirichlet subgraph \( F_{\text{dir}} \). Due to the presence of the Dirichlet weight function \( b_{\text{dir}} \) the subgraph \( F_{\text{dir}} \) is still affected by the complementary set \( X \setminus A \).

### 2.3 m-accretive operators

In preparation for Sect. 4, we introduce here the fundamental tools that will play a major role in the proofs of the existence and uniqueness of solutions to the Cauchy-GPME in the discrete setting. We begin by giving two equivalent definitions of accretivity. As a reference for this topic, see, for example [16].

**Definition 2.4** (Accretive and m-accretive operators) If \( E = (E, \| \cdot \|) \) is a real Banach space and \( \mathcal{L} : \text{dom}(\mathcal{L}) \subseteq E \rightarrow E \) is a (not necessarily linear) operator, then \( \mathcal{L} \) is said to be **accretive** if \( \mathcal{L} \) satisfies one of the following equivalent conditions:

(i) \[ \| (u - v) + \lambda (\mathcal{L}u - \mathcal{L}v) \| \geq \| u - v \| \] for every \( u, v \in \text{dom}(\mathcal{L}) \) and for every \( \lambda > 0 \).

(ii) \[ \langle \mathcal{L}u - \mathcal{L}v, u - v \rangle_{+} \geq 0 \] for every \( u, v \in \text{dom}(\mathcal{L}) \) where for \( z, k \in E \)

\[ \langle z, k \rangle_{+} := \| k \| \lim_{\lambda \rightarrow 0^{+}} \lambda^{-1} \left( \| k + \lambda z \| - \| k \| \right). \]

Concerning the well-posedness of condition (ii) and its application to \( \ell^{p} \)-spaces, see Remark 3 below. An accretive operator \( \mathcal{L} \) is called **m-accretive** if \( \text{id} + \lambda \mathcal{L} \) is surjective for every \( \lambda > 0 \).

Accretive operators are called monotone operators in the Hilbert space setting. Accretivity is a way to extend the property of monotonicity of real-valued functions of a real variable to spaces with a more complex structure. This follows by the trivial observation that a map \( f : \text{dom}(f) \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is monotone (increasing) if and only if \( (f(s_{1}) - f(s_{2}))(s_{1} - s_{2}) \geq 0 \) for all \( s_{1}, s_{2} \in \text{dom}(f) \).

Let us highlight that \( m \)-accretivity is related to the self-adjointness of linear operators in the Hilbert case setting. Indeed, a linear operator \( \mathcal{L} \) on a Hilbert space is self-adjoint and nonnegative if and only if \( \mathcal{L} \) is symmetric, closed and \( m \)-accretive (by the Minty theorem, \( m \)-accretive and maximal monotone are equivalent properties in Hilbert spaces), see
In this context, we note that there has been recent interest in the graph setting concerning the essential self-adjointness of the formal Laplacian and related operators restricted to finitely supported functions, see, for example, [35, 36, 41, 43, 58–60, 63, 72]. Concerning the \( m \)-accretivity of the graph Laplacian we also highlight a couple of recent results: The first is obtained in [59], where the authors establish the \( m \)-accretivity on \( \ell^p(X, \mu) \) for \( 1 \leq p < \infty \) in the more general setting of Hermitian vector bundles, under some hypotheses on the graph. The second result is obtained in [2] where the authors prove a criterion for the \( m \)-accretivity of a graph Laplacian (not necessarily self-adjoint) on directed graphs in the Hilbert case.

**Remark 3** Let us observe that condition (ii) in Definition 2.4 is well-posed. First of all, for \( \lambda > 0 \)

\[
-\|z\| \leq \frac{|k + \lambda z| - |k|}{\lambda} \leq \|z\|.
\]

Then we observe that, for every \( 0 < s < \lambda, \)

\[
\|k + sz\| - |k| = \left( 1 - \frac{s}{\lambda} \right) k + \frac{s}{\lambda} (k + \lambda z) \leq \|k\| \leq \left( 1 - \frac{s}{\lambda} \right) k + \frac{s}{\lambda} |k + \lambda z| - |k| = \frac{s}{\lambda} (|k + \lambda z| - |k|),
\]

i.e., the map \( \lambda \mapsto \lambda^{-1} \left( |k + \lambda z| - |k| \right) \) is monotone increasing in \( \lambda > 0 \). Therefore,

\[
\lim_{\lambda \to 0^+} \lambda^{-1} \left( |k + \lambda z| - |k| \right)
\]

exists and belongs to \([-\|z\|, \|z\|]\).

Now, let us fix \( E = \ell^p(X, \mu) \) with \( p \in [1, \infty) \). By the convexity of the map \( t \mapsto |t|^p \), for every \( \lambda \in (0, 1) \) we get

\[
\frac{|k \pm \lambda z|^p - |k|^p}{\lambda} = \frac{|(1 - \lambda) k + \lambda (k \pm z)|^p - |k|^p}{\lambda} \leq |k \pm z|^p - |k|^p
\]

and

\[
|k|^p - |k - \lambda z|^p \leq |k + \lambda z|^p - |k|^p
\]

where the second inequality can be easily derived from \( |f + g|^p \leq 2^{p-1}(|f|^p + |g|^p) \) with \( f := k - \lambda z \) and \( g := k + \lambda z \). Combining these inequalities gives

\[
|k|^p - |k - z|^p \leq \frac{|k|^p - |k - \lambda z|^p}{\lambda} \leq \frac{|k + \lambda z|^p - |k|^p}{\lambda} \leq |k + z|^p - |k|^p \quad \forall \lambda \in (0, 1],
\]

that is, \( \lambda^{-1} \left| k + \lambda z \right|^p - |k|^p \) is dominated by an integrable function. Then, by the mean value theorem and dominated convergence, for every \( p \geq 1 \) (and \( \|k\|_p \neq 0 \)) we get

\[
\lim_{\lambda \to 0^+} \lambda^{-1} \left( \|k + \lambda z\|_p - \|k\|_p \right)
\]

\[
= \lim_{\lambda \to 0^+} \lambda^{-1} \left( \|k + \lambda z\|_p^{1/p} - (\|k\|_p^{1/p}) \right)
\]

\[
= \frac{1}{p} \left( \|k\|_p^{1/p} \right)^{p-1} \lim_{\lambda \to 0^+} \sum_{x \in X} \frac{|k(x) + \lambda z(x)|^p - |k(x)|^p}{\lambda} \frac{\mu(x)}{\mu(x)}
\]

\[
= \left\{ \begin{array}{ll}
\|k\|_p^{1-p} \sum_{x \in X} z(x)|k(x)|^{p-1} \text{sgn}(k(x)) \frac{\mu(x)}{\mu(x)} & \text{for } p > 1,
\sum_{x \in X} |z(x)| \frac{\mu(x)}{\mu(x)} + \sum_{x \in X: k(x) \neq 0} z(x) \text{sgn}(k(x)) \frac{\mu(x)}{\mu(x)} & \text{for } p = 1
\end{array} \right.
\]

\( \text{sgn}(x) = \begin{cases} 1 & \text{if } k(x) > 0, \\ -1 & \text{if } k(x) < 0, \\ 0 & \text{if } k(x) = 0. \end{cases} \)
where

\[
\text{sgn}(s) := \begin{cases} 
1 & \text{if } s > 0, \\
0 & \text{if } s = 0, \\
-1 & \text{if } s < 0.
\end{cases} \tag{2.4}
\]

Summarizing, for \( E = \ell^p(X, \mu) \) with \( p \in [1, \infty) \),

\[
\langle z, k \rangle_+ = \begin{cases} 
\|k\|_p^{-p} \sum_{x \in X} z(x)k(x)|k(x)|^{p-2}\mu(x) & \text{if } p > 1, \\
\|k\|_1 \left( \sum_{x \in X} |z(x)|\mu(x) + \sum_{x \in X: k(x) \neq 0} z(x)\text{sgn}(k(x))\mu(x) \right) & \text{if } p = 1.
\end{cases} \tag{2.5}
\]

A simple example of an \( m \)-accretive operator is the graph Laplacian on finite graphs with respect to the \( \ell^p \)-norm. This result should be well-known but for completeness we give a proof in the following proposition.

**Proposition 2.5** Let \( G \) be a finite graph. Then, the graph Laplacian \( \Delta \) on \( \ell^p(X, \mu) \) is \( m \)-accretive for \( p \geq 1 \). In particular,

\[
\| (\text{id} + \lambda \Delta)u \|_p \geq \| u \|_p \quad \text{for every } u \in C(X), \; \lambda > 0.
\]

**Proof** Fix \( p \in (1, \infty) \). Applying the linearity of \( \Delta \) and (2.5) of Remark 3, \( \Delta \) is accretive if and only if

\[
\| u \|_p^{2-p} \sum_{x \in X} \Delta u(x)u(x)|u(x)|^{p-2}\mu(x) \geq 0.
\]

Using the fact that the sum is finite, we have

\[
\sum_{x \in X} \Delta u(x)u(x)|u(x)|^{p-2}\mu(x) = \sum_{x \in X} u(x)|u(x)|^{p-2} \sum_{y \in X} w(x, y)(u(x) - u(y)) + \sum_{x \in X} \kappa(x)|u(x)|^p \\
\geq \sum_{x \in X} \sum_{y \in X} w(x, y)(|u(x)|^p - |u(y)|u(x)|u(x)|^{p-2}) \\
\geq \frac{1}{2} \sum_{x, y \in X} w(x, y)(|u(x)|^p + |u(y)|^p - |u(x)|^{p-1}|u(y)| - |u(x)||u(y)|^{p-1}).
\]

The conclusion now follows from the following inequality

\[
a^p + b^p - a^{p-1}b - ab^{p-1} \geq 0 \quad \forall \; a, b \geq 0, \; \forall \; p \in (1, \infty)
\]

which can be established by elementary calculus as we now show. The inequality holds for \( a = 0 \) or \( b = 0 \), so assuming that \( b > 0 \), dividing through by \( b^p \) and setting \( t := a/b \), it is equivalent to prove that

\[
\beta(t) := t^p + 1 - t^{p-1} - t \geq 0 \quad \text{for } t \geq 0.
\]

Note that \( \beta(0) = 1 \) and \( \beta(t) \to \infty \) as \( t \to \infty \). We have

\[
\beta'(t) = pt^{p-1} - (p - 1)t^{p-2} - 1
\]
and thus
\[ \beta'(t) < 0 \text{ for all } t \text{ small, } \beta'(1) = 0, \quad \beta'(t) \to \infty \text{ as } t \to \infty. \]

Moreover,
\[ \beta''(t) = p(p-1)t^{p-2} - (p-1)(p-2)t^{p-3} = p(p-1)t^{p-3}\left( t - \frac{p-2}{p}\right). \]

Thus, if \( 1 < p \leq 2 \), then \( \beta'' \geq 0 \) and \( \beta' \) is increasing. If \( p > 2 \), then
\[ \beta''(t) < 0 \text{ for } t < \frac{p-2}{p} < 1, \quad \beta''(t) \geq 0 \text{ for } t \geq \frac{p-2}{p}. \]

Hence, \( \beta' \) is decreasing until \( (p-2)/p \) and increasing afterwards. In any case, \( \beta'(t) < 0 \) for \( t < 1 \) and \( \beta'(t) > 0 \) for \( t > 1 \), so
\[ \min_{t>0} \beta(t) = \beta(1) = 0. \]

By the equivalence between (ii) and (i) of Definition 2.4, we then have
\[ \| (\text{id} + \lambda \Delta)u \|_p \geq \| u \|_p \quad \text{for every } u \in C(X), \lambda > 0. \]

We conclude the lemma for \( p \in (1, \infty) \) by noticing that \( \text{id} + \lambda \Delta \) injective and \( C(X) \) finite imply that \( \text{id} + \lambda \Delta \) surjective by linearity. The case \( p = 1 \) is addressed in the more general case of Corollary B.3 in Appendix B.

As a final comment, we observe that if two operators are accretive on a given Banach space, this will not automatically imply that the composition (if defined) of the two operators is accretive. See the following simple example.

Example Consider the finite birth-death chain \( G = (X, w, \kappa, \mu) \) (see Fig. 3) with

- \( X = \{x_1, x_2, x_3, x_4\} \);
- \( w(x_i, x_j) = 1 \) if and only if \(|i - j| = 1 \) and zero otherwise;
- \( \kappa \equiv 0; \)
- \( \mu \equiv 1. \)

Define now \( \phi : \mathbb{R} \to \mathbb{R} \) by \( \phi(s) := s|s|^3 \). Since \( \phi \) is monotone, thanks to Remark 3, \( \Phi \), the canonical extension of \( \phi \) to \( \ell^2(X, \mu) \), is accretive. Consider now the graph Laplacian \( \Delta : \ell^2(X, \mu) \to \ell^2(X, \mu) \) associated to \( G \) which in this case acts as
\[ \Delta u(x_i) = \begin{cases} u(x_1) - u(x_2) & \text{if } i = 1, \\ 2u(x_i) - u(x_{i-1}) - u(x_{i+1}) & \text{if } i = 2, 3, \\ u(x_4) - u(x_3) & \text{if } i = 4. \end{cases} \]

By Proposition 2.5, \( \Delta \) is accretive.

On the other hand, the operator \( \mathcal{L} := \Delta \Phi \) is not accretive. Indeed, a computation shows that if \( u \) and \( v \) are defined by
\[ u(x_i) = \begin{cases} 3, & \text{for } i = 1, \\ 4, & \text{for } i = 2, \\ 0 & \text{otherwise}, \end{cases} \quad v(x_i) = \begin{cases} 3, & \text{for } i = 2, \\ 0 & \text{otherwise}, \end{cases} \]
then
\[ \langle \mathcal{L}u - \mathcal{L}v, u - v \rangle_{\ell^2} = -13 < 0. \]
Problem: Cauchy problem for the generalized porous medium equation (GPME) (or filtration equation):

\[ L = \Delta \Phi \]

only to the action of the operator but to the norm on the underlying space as well.

Remark 4

Definition 3.1 is well-posed. If \( f \in L^1([0, T]; \ell^1(X, \mu)) \), then for every \( \epsilon > 0 \) there exists an \( \epsilon \)-discretization \( D_\epsilon \) of \((0, T]; f\), see [22, Lemma 4.1].

3 The Cauchy model problem

Let \( f : (0, T) \times X \rightarrow \mathbb{R} \) and \( u_0 : X \rightarrow \mathbb{R} \). Given a graph \( G \), let us consider the following Cauchy problem for the generalized porous medium equation (GPME) (or filtration equation):

**Problem:**

\[
\text{Cauchy-GPME} \left\{ \begin{aligned}
\partial_t u(t, x) + \Delta \Phi u(t, x) &= f(t, x) \quad \text{for every } (t, x) \in (0, T) \times X, \\
\lim_{t \to 0^+} u(t, x) &= u_0(x) \quad \text{for every } x \in X
\end{aligned} \right.
\]

where \( \Phi : C(X) \rightarrow C(X) \) is the canonical extension of a function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \phi \) is strictly monotone increasing, \( \phi(\mathbb{R}) = \mathbb{R} \) and \( \phi(0) = 0 \).

If \( \phi(s) = s^m := s|s|^{m-1} \), then we will call the above equation the porous medium equation (PME) when \( m > 1 \) and the fast diffusion equation (FDE) when \( 0 < m < 1 \). Clearly, when \( m = 1 \) and \( f \equiv 0 \) we recover the classic heat equation. The function \( f \) is called the *forcing term*. For notational convenience, we specify the time interval \((0, T)\) but everything we will say can be generalized to any (not necessarily bounded above) interval \((a, b) \subset \mathbb{R}\). The “+” sign in our equation comes from the fact that we are considering the formal graph Laplacian which corresponds to minus the second derivative in the Euclidean case.

We now introduce the various classes of solutions for the Cauchy-GPME problem in order of increasing regularity. The weakest notion of solution is obtained by means of a discretization and approximation procedure in the time variable. More precisely, we first need to discretize the time interval \((0, T)\) with respect to the forcing term \( f \) such that the corresponding time-discretization \( f_n \) of \( f \) is “close” to \( f \) in a way that will be made clear next.

**Definition 3.1** \((\epsilon\text{-discretization})\) Given a time interval \([0, T]\) with \( T < \infty \) and a forcing term \( f \in L^1([0, T]; \ell^1(X, \mu)) \), we define a partition of the time interval

\[
\mathcal{T}_n := \{ [t_k]_{k=0}^n | 0 = t_0 < t_1 < \ldots < t_n \leq T \}
\]

and a time-discretization \( f_n \) of \( f \)

\[
f_n := \{ [f_k]_{k=1}^n | f_k \in \ell^1(X, \mu), f_k(x) := f(t_k, x) \}.
\]

Having fixed \( \epsilon > 0 \), we call \( D_\epsilon := (\mathcal{T}_n, f_n) \) an \( \epsilon \)-**discretization** of \((0, T]; f\) if

\[ t_k - t_{k-1} \leq \epsilon \text{ for every } k = 1, \ldots, n \text{ and } T - t_n \leq \epsilon; \]
\[ \sum_{k=1}^n \| f(t_k) - f_k \| dt \leq \epsilon. \]

**Remark 4** Definition 3.1 is well-posed. If \( f \in L^1([0, T]; \ell^1(X, \mu)) \), then for every \( \epsilon > 0 \) there exists an \( \epsilon \)-discretization \( D_\epsilon \) of \((0, T]; f\), see [22, Lemma 4.1].
Now, given an \( \epsilon \)-discretization \( \mathcal{D}_\epsilon \), consider the following system of difference equations which arises from an implicit Euler-discretization of the Cauchy-GPME:

\[
\frac{u_k - u_{k-1}}{\lambda_k} + \Delta u_k = f_k, \quad \lambda_k := t_k - t_{k-1} \text{ and } k = 1, \ldots, n \tag{3.1}
\]

with \( u_0 \) given. Writing \( \mathcal{L} = \Delta \Phi \), we then require that every \( u_k \) belongs to

\[
\text{dom}(\mathcal{L}) = \left\{ u \in \ell^1(\mathbb{X}, \mu) \mid \Phi u \in \text{dom}(\Delta), \Delta \Phi u \in \ell^1(\mathbb{X}, \mu) \right\}.
\]

**Definition 3.2** (\( \epsilon \)-approximate solution) If the system (3.1) admits a solution \( u_\epsilon = \{u_k\}_{k=1}^n \) such that \( u_k \in \text{dom}(\mathcal{L}) \) for every \( k = 1, \ldots, n \), then we define \( u_\epsilon \) as the piecewise constant function

\[
u_\epsilon(t) := \begin{cases} \sum_{k=1}^n u_k \mathbb{I}_{(t_{k-1}, t_k]}(t) & \text{ for } t \in (0, t_n], \\ u_0 & \text{ for } t = 0 \end{cases} \tag{3.2}
\]

and we call \( u_\epsilon \) an \( \epsilon \)-approximate solution of the Cauchy-GPME (subordinate to \( D_\epsilon \)).

We then have the following definition of a mild solution which first appeared as a formal definition in [14]. It can be viewed as a uniform limit of “numerical approximations” to solutions of the Cauchy-GPME obtained by the system of difference equations (3.1).

**Definition 3.3** (Mild solution) If \( T < +\infty \), we say that \( u : [0, T] \to \ell^1(\mathbb{X}, \mu) \) is a mild solution of the Cauchy-GPME problem if \( u \in C([0, T]; \ell^1(\mathbb{X}, \mu)) \) and \( u \) is obtained as a uniform limit of \( \epsilon \)-approximate solutions. Namely, for every \( \epsilon > 0 \) there exists an \( \epsilon \)-discretization \( D_\epsilon \) of \(([0, T]; f)\), as in Definition 3.1, and an \( \epsilon \)-approximate solution \( u_\epsilon \) subordinate to \( D_\epsilon \), as in (3.2), such that

\[
\|u(t) - u_\epsilon(t)\| < \epsilon \quad \text{ for every } t \in [0, t_n] \subseteq [0, T].
\]

If \( T = +\infty \), then we say that \( u \) is a mild solution of the Cauchy-GPME if the restriction of \( u \) to each compact subinterval \([0, a] \subseteq [0, +\infty)\) is a mild solution of the Cauchy-GPME on \([0, a]\).

We next introduce two further classes of solutions, namely, strong and classic solutions. Following the definitions, we will discuss the relationship between these notions.

**Definition 3.4** (Strong solution) We say that \( u : [0, T] \to \ell^1(\mathbb{X}, \mu) \) is a strong solution of the Cauchy-GPME problem if

- \( u(t) \in \text{dom}(\mathcal{L}) \) for every \( t \in [0, T] \);
- \( u \in C([0, T]; \ell^1(\mathbb{X}, \mu)) \cap W^{1,1}_{\text{loc}}(0, T; \ell^1(\mathbb{X}, \mu)) \);
- \( \partial_t u(t, x) + \Delta \Phi u(t, x) = f(t, x) \) for almost every \( t \in (0, T) \);
- \( u(0) = u_0 \).

**Definition 3.5** (Classic solution) We say that \( u : [0, T] \to \ell^1(\mathbb{X}, \mu) \) is a classic solution of the Cauchy-GPME problem if

- \( u(t) \in \text{dom}(\mathcal{L}) \) for every \( t \in [0, T] \);
- \( u \in C([0, T]; \ell^1(\mathbb{X}, \mu)) \cap C([0, T]; \ell^1(\mathbb{X}, \mu)) \);
- \( \partial_t u(t, x) + \Delta \Phi u(t, x) = f(t, x) \) for every \( t \in (0, T) \);
- \( u(0) = u_0 \).
In the literature, mild solutions are also known by other names: they are called limit solutions in [48] and weak solutions in [46]. In [5], P. Bénilan and H. Brézis introduced the definition of faible (i.e., weak) solutions of the abstract Cauchy problem

\[
\begin{aligned}
\partial_t u(t) + Au(t) &= f(t) \quad \text{for } t \in (0, T), \\
u(0) &= u_0
\end{aligned}
\tag{3.3}
\]

as a uniform limit of strong solutions \(u_n\) of (3.3) with \(f\) replaced by \(f_n\) where \(f_n \to f\) in \(L^1([0, T]; E)\).

Clearly, a classic solution is a strong solution. The fact that a strong solution is a mild solution assuming that \(f \in L^1_{\text{loc}}([0, T]; \ell^1(X, \mu))\) is a standard result, e.g. [6, Theorem 1.4]. Therefore, assuming \(f\) is strongly measurable and locally Bochner integrable, we have compatibility of the three different definitions in the sense that classic solution \(\Rightarrow\) strong solution \(\Rightarrow\) mild solution in order of descending “regularity.” As a final remark, we observe that a mild solution may not be differentiable and does not necessarily satisfy the Cauchy-GPME in a pointwise sense. Nonetheless, this notion is known as the most natural one of the generalized notions of solutions of (3.3).

### 4 Existence and uniqueness of mild solutions

The theory of nonlinear operators on Banach spaces is well-established. We refer the interested reader to [48, Chapter 3], [3, Chapter 4] or [1, Appendix A] and references therein for well-organized summaries of all of the main results.

If the operator \(L := \Delta \Phi\) is accretive and such that for every \(\epsilon > 0\) there exists an \(\epsilon\)-approximate solution as in Definition 3.2, then it is possible to infer the existence and uniqueness of mild solutions for the Cauchy-GPME, relying on some consequences of a result due to P. Bénilan, see [4] and [6, Theorem 3.3], which is an extension of the famous Crandall-Liggett theorem, see [15]. The main idea is the following: Given \(u_0 \in \ell^1(X, \mu)\) and an \(\epsilon\)-discretization \(D_\epsilon\) as in Definition 3.1, then solving system (3.1) means solving the equation

\[(\text{id} + \lambda_k \Delta \Phi)u_k = u_{k-1} + \lambda_k f_k\]

with

\[u_k \in \ell^1(X, \mu), \quad \Phi u_k \in \text{dom}(\Delta), \quad \Delta \Phi u_k \in \ell^1(X, \mu)\]

for every \(k = 1, \ldots, n\) where \(\lambda_k > 0\) and \(f_k \in \ell^1(X, \mu)\). This is doable, in particular, if

\[(\text{id} + \lambda \Delta \Phi)u = g\quad (4.1)\]

is solvable for any \(g \in \ell^1(X, \mu)\), \(\lambda > 0\) and the solution \(u \in \ell^1(X, \mu)\) is such that \(\Phi u \in \text{dom}(\Delta)\) and \(\Delta \Phi u \in \ell^1(X, \mu)\). Therefore, if \(L\) is \(m\)-accretive, then we would get the existence and uniqueness of mild solutions in one step.

For example, in the Euclidean case, when \(X = \Omega\) is a bounded domain in \(\mathbb{R}^n\), then the accretivity property holds for \(L := \Delta \Phi\) defined on

\[
\text{dom}(L) := \left\{ u \in L^1(\Omega) \mid \Phi(u) \in W^{1,1}_0(\Omega), \Delta \Phi u \in L^1(\Omega) \right\}
\]

where the Laplace operator \(\Delta\) is understood in the sense of distributions. The difficult part is to prove the \(m\)-accretivity, i.e., to prove that for every \(g \in L^1(\Omega)\) there exists \(u \in \text{dom}(L)\) that is a solution for equation (4.1) for any \(\lambda > 0\). To circumvent the direct approach, it
is common to switch to an equivalent formulation, namely, having defined $v := \Phi u$ and $u = \Psi v = \Phi^{-1} v$, the question is whether the equation

$$(\Psi + \lambda \Delta) v = g$$

admits a solution $v$ in

$$\left\{ v \in W^{1,1}_0(\Omega) \mid \Delta v \in L^1(\Omega) \right\}.$$  

The positive answer to this question in the Euclidean case was given by H. Brézis and W. Strauss in [9]. In particular, the trick of this approach is to relate the $m$-accretive property of the nonlinear operator $L$ to suitable properties of the linear operator $\Delta$ which is easier to handle.

The main issue in the discrete setting is to prove existence of solutions under minimal assumptions on the underlying graph $G$. Indeed, while the accretivity of $L$ can be established for any finite graph, see Corollary B.3 in Appendix B, the accretivity of $L$ can be a tricky property to prove for more general graphs. Moreover, the hypothesis required to make use of the result in [9], which are satisfied by the Euclidean Laplacian $\Delta$ over bounded domains $\Omega$, are, in general, not satisfied by the graph Laplacian. Consequently, we are forced to step back and to prove “by hand” the existence of $\epsilon$-approximate solutions $u_\epsilon(t)$ for every $\epsilon > 0$ with the property that $u_\epsilon(t)$ belongs to a dense subset of $\text{dom}(L)$ where $L$ is accretive.

The idea is to find a solution $u$ to $(\text{id} + \lambda \Delta \Phi)u = g$ by building $u$ as a limit of a sequence $\{\Psi v_n\}$ where the $v_n$ are solutions of $(\Psi + \lambda \Delta_n)v_n = g_n$ on suitable restrictions of the graph $G$ to finite subgraphs. In particular, we will see that this can be achieved by decomposing $G$ as an infinite ascending chain $\{G_{\text{dir},n}\}_{n=1}^{\infty}$ of finite connected Dirichlet subgraphs.

After this introduction, we are now ready to prove our main results. In Theorem 1 we will prove the accretivity of the operator $L$ on a suitable dense subset of $\text{dom}(L)$ and the surjectivity of $\text{id} + \lambda L$ on the nonnegative/nonpositive cones of $\ell^1(X, \mu)$ and, under three different additional hypotheses, on the entire space $\ell^1(X, \mu)$. In Theorem 2 we will then establish the existence and uniqueness of mild solutions for the Cauchy-GPME problem. As a concluding application, in Corollary 4.2, we prescribe some hypotheses on the graph that guarantee that a mild solution is indeed a classic solution.

### 4.1 Proofs of the main theorems

Let us recall that $L$ is the operator

$$L: \text{dom}(L) \subseteq \ell^1(X, \mu) \to \ell^1(X, \mu),$$

$$\text{dom}(L) = \{ u \in \ell^1(X, \mu) \mid \Phi u \in \text{dom}(\Delta), \Delta \Phi u \in \ell^1(X, \mu) \}$$

whose action is given by

$$Lu = \Delta \Phi u,$$

and that for a subset $\Omega \subseteq \text{dom}(L)$, we write $L|_\Omega$ for the restriction of $L$ to $\Omega$. The extra hypotheses listed in Theorem 1 are

(H1) $G$ is locally finite;
(H2) $\inf_{x \in X} \mu(x) > 0$;
(H3) $\sup_{x \in X} \frac{\sum_{y \in X} w(x, y)}{\mu(x)} < \infty$ and $\Phi(\ell^1(X, \mu)) \subseteq \ell^1(X, \mu)$.
Proof of Theorem 1 We divide the proof into several steps. From Steps 0 to IV we will assume that $G$ is connected. To help orient the reader, we first give a brief outline of the structure of the proof: In Step 0, we will introduce a sequence of operators $L_n : C_c(X) \to \ell^1(X, \mu)$ and discuss that for every graph $G$ there exists a dense subset $\Omega \subseteq \text{dom}(L)$ where $L$ is accretive. Later, we will show that all solutions of the equation $(\text{id} + \lambda L)u = g$ that we construct along the way belong to $\Omega$. In Step I, assuming that $G$ is finite, we will prove that $\text{id} + \lambda L$ is surjective and preserves nonnegativity/nonpositivity and also give an upper bound of the norm of the solutions with respect to $g$. This step plays a crucial role in proving the surjectivity of $\text{id} + \lambda L$ in the infinite case where we will approximate the graph $G$ by an ascending chain of finite Dirichlet subgraphs. In Step II, given $\lambda > 0$ and $g \in \ell^1(X, \mu)$, we will show that there exists a sequence of compactly supported functions $u_n$ and $u \in \ell^1(X, \mu)$ such that $\lim_{n \to \infty} \|u_n - u\| = 0$ and $\lim_{n \to \infty} \|(i_n \circ \text{id}_n) \pi_n + \lambda L)u_n - g\| = 0$. Using this construction, in Step III we will show that $u$ solves $(\text{id} + \lambda L)u = g$ for every $g \in \ell^1(X, \mu)$ and that $u \in \Omega$. In Step IV-H1, H2, H3 we will prove that $u$ solves $(\text{id} + \lambda L)u = g$ for every $g \in \ell^1(X, \mu)$ and that $u \in \Omega$ under any one of the three different assumptions (H1), (H2) or (H3). Finally, in Step V we remove the assumption of connectedness that we used while proving Steps 0 to IV.

Step 0 (When $G$ is connected, there exists a dense subset $\Omega$ of $\text{dom}(L)$ where $L|_\Omega$ is accretive): This is exactly the content of Lemma B.8 in Appendix B. To help orient the reader, we recall here the notations involved and the definition of $\Omega$. If $G$ is finite, then $\Omega = \text{dom}(L) = C(X)$. If $G$ is infinite, we take an exhaustion $\{X_n\}_{n=1}^\infty$ of $X$, i.e., a sequence of subsets $X_n$ of $X$ such that $X_n \subseteq X_{n+1}$ and $X = \bigcup_{n=1}^\infty X_n$, where we assume that each $X_n$ is additionally finite, along with the canonical embedding $i_{n,\infty}$ and the canonical projection $\pi_n$ for each $X_n$:

$$i_{n,\infty} : C(X) \to C(X) \quad i_{n,\infty}u(x) := \begin{cases} u(x) & \text{if } x \in X_n, \\ 0 & \text{if } x \in X \setminus X_n; \end{cases}$$

$$\pi_n : C(X) \to C(X,\mu) \quad \pi_nu(x) := u(x) \text{ for every } x \in X_n.$$ 

At this point, the exhaustion $\{X_n\}_{n=1}^\infty$ can be arbitrary but should consist of finite sets. We then define the operators $L_n$ as in Definition B.1, namely,

$$L_n : \text{dom}(L_n) \subseteq \ell^1(X, \mu) \to \ell^1(X, \mu)$$

with

$$\text{dom}(L_n) := C_c(X), \quad L_nu := i_{n,\infty}\Delta_{\text{dir},n}\Phi \pi_nu$$

where $\Delta_{\text{dir},n}$ is the graph Laplacian associated to the Dirichlet subgraph $G_{\text{dir},n} \subseteq G$ on the node set $X_n$. Then, the set $\Omega$ is defined as

$$\Omega := \{ u \in \text{dom}(L) \mid \exists \{u_n\}_n \text{ s.t. supp } u_n \subseteq X_n, \lim_{n \to \infty} \|u_n - u\| = 0, \lim_{n \to \infty} \|L_nu_n - Lu\| = 0 \}$$

where $\text{supp } u_n$ denotes the support of the function $u_n$. We have $\overline{\Omega} = \text{dom}(L) = \ell^1(X, \mu)$ by Lemma B.7 and that $L|_\Omega$ is accretive by Lemma B.8.

Step I (When $G$ is finite and connected, id + $\lambda L$ is bijective and preserves nonnegativity/nonpositivity): Assume that $G$ is finite and connected, i.e., $|X| = n$. In this case,

$$\text{dom}(\Delta) = C(X) \simeq \mathbb{R}^n, \quad \ell^1(X, \mu) = (C(X, \| \cdot \|)) \quad \text{with } \|u\| = \sum_{x \in X} |u(x)|\mu(x).$$
Fix now \( \lambda > 0 \). Writing \( \psi := \phi^{-1} \) and \( v := \Phi u \), we can rewrite equation

\[
(id + \lambda \Delta \Phi)u = g
\]

(4.2)
in the equivalent form

\[
(\Psi + \lambda \Delta) v = g.
\]

(4.3)

Clearly, \( \psi \) is strictly monotone increasing, \( \psi(\mathbb{R}) = \mathbb{R} \) and \( \psi(0) = 0 \).

Let us enumerate the nodes of \( X \), that is, we write \( X = \{x_1, x_2, \ldots, x_n\} \). Owing to the isomorphism between \( C(X) \) and \( \mathbb{R}^n \), we identify \( n \)-dimensional vectors and real-valued functions on \( X \) in the standard way, that is, given \( v \in C(X) \) we associate to \( v \) the vector \( v = (v_1, \ldots, v_n) := (v(x_1), \ldots, v(x_n)) \) and vice-versa. Define \( M : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
M v := (\Psi + \lambda \Delta) v.
\]

Let us observe that:

(i) \( (\Psi v)_i = \psi(v_i) \) for every \( i = 1, \ldots, n \) where \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) is surjective and strictly monotone increasing;

(ii) For every \( \lambda > 0 \), \( \lambda \Delta \) is a diagonally dominant matrix (e.g., [39, Definition 6.1.9]), i.e.,

\[
(\lambda \Delta)_{i,i} = \frac{\lambda}{\mu(x_i)} \left( \kappa(x_i) + \sum_{j=1}^{n} w(x_i, x_j) \right)
\]

\[
\geq \frac{\lambda}{\mu(x_i)} \sum_{j=1}^{n} w(x_i, x_j) = \sum_{j=1}^{n} |(\lambda \Delta)_{i,j}|, \quad \forall i = 1, \ldots, n.
\]

Therefore, by [71, Theorem 1], for every \( g \in \mathbb{R}^n \) there exists a unique solution \( v \) to the equation

\[
M v = g.
\]

We now show that the norm of the solution is bounded above by the norm of \( g \). Let \( v \) be the solution of (4.3) with right-hand side \( g \). Since \( \psi \) is strictly monotone increasing and \( \psi(0) = 0 \), we get

\[
\text{sgn}(v(x)) = \text{sgn}(\psi(v(x))).
\]

By Proposition B.2, and recalling that \( v = \Phi u \), i.e., \( u = \Psi v \), it follows that

\[
\sum_{x \in X} \Delta v(x) \text{sgn}(\Psi v(x)) \mu(x) = \sum_{x \in X; \mu(x) \neq 0} \Delta \Phi u(x) \text{sgn}(u(x)) \mu(x) \geq 0.
\]

Therefore, we conclude

\[
\|u\| = \|\Psi v\| = \sum_{x \in X} |\Psi v(x)| \mu(x)
\]

\[
= \sum_{x \in X} \Psi v(x) \text{sgn}(\Psi v(x)) \mu(x)
\]

\[
= \sum_{x \in X} g(x) \text{sgn}(\Psi v(x)) \mu(x) - \lambda \sum_{x \in X} \Delta v(x) \text{sgn}(\Psi v(x)) \mu(x)
\]
\[
\leq \sum_{x \in X} g(x) \operatorname{sgn}(\Psi v(x)) \mu(x) \\
\leq \|g\|
\]

that is,

\[
\|u\| \leq \|g\|. \tag{4.4}
\]

By Case 2) of Theorem A.2, if \(g \geq 0\), then \(v \geq 0\), and if \(g \leq 0\), then \(v \leq 0\). Consequently, \(u = \Psi v\) has the same sign as \(g\). Therefore, if \(G\) is finite we can conclude that for every \(g \in \ell^{1,\pm}(X, \mu)\), the unique solution \(u\) of (4.2) belongs to \(\ell^{1,\pm}(X, \mu)\) and satisfies \(\|u\| \leq \|g\|\).

**Step II** (Constructing a solution when \(G\) is infinite and connected): We want to show that if \(G\) is infinite and connected, then for every fixed \(\lambda > 0\) and \(g \in \ell^{1}(X, \mu)\) there exists \(u \in \ell_{\pm}^{1}(X, \mu)\) and a sequence \(\{u_{n}\}_{n}\) such that

\[
\operatorname{supp} u_{n} \subseteq X_{n}; \tag{4.5a}
\]
\[
\lim_{n \to \infty} \|u_{n} - u\| = 0; \tag{4.5b}
\]
\[
\lim_{n \to \infty} \| (i_{n,\infty} \pi_{n} + \lambda L_{n}) u_{n} - g \| = 0 \tag{4.5c}
\]

where \(i_{n}\) is the identity operator on \(C(X_{n})\). We divide this step into two sub-steps consisting of the cases when \(g\) is nonnegative (or nonpositive) and then general \(g\). 

**Step II-1** \((g \in \ell^{1,\pm}(X, \mu))\): Assume that \(g \in \ell^{1,\pm}(X, \mu)\). By Lemma A.4, we can choose the exhaustion \(\{X_{n}\}_{n=1}^{\infty}\) with the following properties:

\[
X_{n} \subset X_{n+1}, \quad X = \bigcup_{n=1}^{\infty} X_{n} \tag{4.6}
\]

and such that the set

\[
\{x \in X_{n} \mid x \sim y \text{ for some } y \in X_{n+1} \setminus X_{n}\} \tag{4.7}
\]

is not empty for all \(n\). For each \(n\), we define the subgraph

\[
G_{\text{dir},n} = (X_{n}, w_{n}, \kappa_{\text{dir},n}, \mu_{n}) \subseteq G \tag{4.8}
\]

as a Dirichlet subgraph of \(G\), see Definition 2.3. That is,

- \(w_{n} \equiv w|_{X_{n} \times X_{n}}\);
- \(\mu_{n} \equiv \mu|_{X_{n}}\);
- for every \(x \in X_{n}\), \(b_{\text{dir},n}(x) = \sum_{y \in \partial X_{n}} w(x, y)\);
- for every \(x \in X_{n}\), \(\kappa_{\text{dir},n}(x) = \kappa|_{X_{n}}(x) + b_{\text{dir},n}(x)\).

If we define

\[
\partial X_{n,n+1} := \{x \in X_{n} \mid \exists y \in X_{n+1} \setminus X_{n} \text{ such that } x \sim y\},
\]
\[
\partial X_{n,n+1} := \{y \in X_{n+1} \setminus X_{n} \mid \exists x \in X_{n} \text{ such that } x \sim y\}
\]

which are not empty by construction and

\[
b'_{\text{dir},n}(x) = \sum_{y \in \partial X_{n,n+1}} w(x, y)
\]
then, for every \( x \in X_n \), it holds that
\[
\kappa_{\text{dir},n}(x) = \kappa_{|X_n}(x) + b_{\text{dir},n}(x)
\]
\[
= \kappa_{|X_n}(x) + \sum_{y \in \partial X_n} w(x, y)
\]
\[
= \kappa_{|X_n+1}(x) + \sum_{y \in X \setminus X_n} w(x, y)
\]
\[
= \kappa_{|X_n+1}(x) + \sum_{y \in X \setminus X_n} w(x, y) + \sum_{y \in X_{n+1} \setminus X_n} w(x, y)
\]
\[
= \kappa_{|X_n+1}(x) + \sum_{y \in \partial X_{n+1}} w(x, y) + \sum_{y \in \partial^* X_{n+1}} w(x, y)
\]
\[
= \kappa_{|X_n+1}(x) + b_{\text{dir},n+1}(x) + b'_{\text{dir},n}(x)
\]
\[
= \kappa_{\text{dir},n+1}(x) + b'_{\text{dir},n}(x).
\]

So, the collection \( \{G_{\text{dir},n} \mid n \in \mathbb{N}\} \) is a sequence of connected finite Dirichlet subgraphs such that each subgraph \( G_{\text{dir},n} \) is a Dirichlet subgraph of \( G_{\text{dir},n+1} \), that is,
\[
G_{\text{dir},1} \subset \ldots \subset G_{\text{dir},n} \subset G_{\text{dir},n+1} \subset \ldots \subset G.
\]

Denoting by
\[
i_n : C(X_n) \hookrightarrow C(X_n+1), \quad i_{n, \infty} : C(X_n) \hookrightarrow C(X), \quad \pi_n : C(X) \to C(X_n)
\]
the canonical embeddings and projections, respectively, define
\[
g_n := \pi_n g \quad \text{where} \quad g \in \ell^{1,+}(X, \mu).
\]

From **Step I**, for every \( n \in \mathbb{N} \) there exist \( \hat{v}_n \in C(X_n) \) such that \( \hat{v}_n \geq 0 \) and
\[
(\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n = g_n. \tag{4.9}
\]

Setting
\[
g_n(x) := (\Psi + \lambda \Delta_{\text{dir},n+1}) i_n \hat{v}_n(x) \in C(X_{n+1})
\]
by Lemma A.1, and the fact that every \( G_{\text{dir},n} \) is a Dirichlet subgraph of \( G_{\text{dir},n+1} \), we have
\[
(\Psi + \lambda \Delta_{\text{dir},n+1}) i_n \hat{v}_n(x) = (\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n(x) \quad \forall x \in X_n
\]
and
\[
g_n(x) = \begin{cases} 
  g_n(x) & \text{if } x \in X_n, \\
  -\frac{\lambda}{\mu(x)} \sum_{y \in X_n} w(x, y) \hat{v}_n(y) & \text{if } x \in \partial X_{n+1}, \\
  0 & \text{if } x \in X_{n+1} \setminus (X_n \cup \partial X_{n+1}).
\end{cases}
\]

Since \( 0 \leq \hat{v}_n \) and \( 0 \leq g_{n+1} \), it follows that \( g_n(x) \leq g_{n+1}(x) \) for every \( x \in X_{n+1} \setminus X_n \).

In particular, from the fact that \( g_{n+1}(x) = g_n(x) \) for every \( x \in X_n \), we get \( g_n \leq g_{n+1} \).

By Corollary A.3, we have \( i_n \hat{v}_n \leq \hat{v}_{n+1} \), and by the fact that \( \psi \) is monotone increasing and \( \psi(0) = 0 \), we have
\[
\Psi i_n \hat{v}_n \leq \Psi \hat{v}_{n+1} \quad \text{and} \quad \Psi i_n \hat{v}_n(x) = i_n \Psi \hat{v}_n(x) = \begin{cases} 
  \psi(\hat{v}_n(x)) & \text{if } x \in X_n, \\
  0 & \text{if } x \in X_{n+1} \setminus X_n.
\end{cases}
\]
Therefore, $i_n \Psi \hat{v}_n \leq \Psi \hat{v}_{n+1}$. In particular, we get
\begin{align}
i_{n, \infty} \hat{v}_n(x) &= i_{n+1, \infty} \hat{v}_n(x) \leq i_{n+1, \infty} \hat{v}_{n+1}(x), \\
i_{n, \infty} \Psi \hat{v}_n(x) &= i_{n+1, \infty} \Psi \hat{v}_n(x) \leq i_{n+1, \infty} \Psi \hat{v}_{n+1}(x)
\end{align}
(4.10) (4.11)
for every $x \in X$. Moreover, writing $\hat{u}_n(x) = \psi (\hat{v}_n(x)) \geq 0$ for every $x \in X_n$ and indicating by $\| \cdot \|_n$ the restriction of $\| \cdot \|$ to $C(X_n)$ from (4.4) we have
\[
0 \leq \hat{u}_n(x) \mu(x) \leq \sum_{x \in X_n} \hat{u}_n(x) \mu(x) = \| \hat{u}_n \|_n \leq \| g \|_n \leq \| g \|,
\]
that is, for every fixed $x$, $\hat{u}_n(x)$ is bounded uniformly in $n$. In particular, writing
\[
v_n := i_{n, \infty} \hat{v}_n \in C_c(X) \quad \text{and} \quad u_n := i_{n, \infty} \hat{u}_n \in C_c(X)
\]
it follows that
\[
\Psi v_n(x) = u_n(x) \in \left[0, \frac{\| g \|}{\mu(x)}\right] .
\]
Consequently, by (4.11) and (4.12), for every fixed $x \in X$ we have a sequence
\[
\{u_n(x)\}_n := \{\Psi v_n(x)\}_n
\]
that is monotonic and bounded. We can then define $u, v \in C(X)$ such that
\[
u(x) := \lim_{n \to \infty} u_n(x) \quad \text{for } x \in X,
\]
(4.14)
\[
v := \Phi u.
\]
(4.15)
Notice that, by construction, $x \in X_n$ eventually so that $u_n(x) = \hat{u}_n(x)$ eventually. In particular, by the continuity of $\phi$ and the fact that $\psi = \phi^{-1}$
\[
v(x) = \lim_{n \to \infty} v_n(x)
\]
and the limit is monotone. Moreover, $u_n$ satisfies (4.5a), i.e., $\supp u_n \subseteq X_n$ for every $n$ and
\[
(i_n + \lambda \Delta_{dir,n} \Phi) \pi_n u_n = g_n.
\]
Therefore,
\[
(i_n, \infty) i_n \pi_n + \lambda \mathcal{L}_n) u_n = (i_{n, \infty} i_n \pi_n + \lambda i_{n, \infty} \Delta_{dir,n} \Phi \pi_n) u_n = i_{n, \infty} (i_n + \lambda \Delta_{dir,n} \Phi) \pi_n u_n = i_{n, \infty} g_n.
\]
In particular, since $g \in \ell^1(X, \mu)$,
\[
\lim_{n \to \infty} \| (i_{n, \infty} i_n \pi_n + \lambda \mathcal{L}_n) u_n - g \| = 0,
\]
which is exactly (4.5c).
Let $\| \cdot \|_n$ be the restriction of $\| \cdot \|$ to $C(X_n)$. Since every $X_n$ is finite, from (4.4) in Step I we obtain
\[
\| \Psi \hat{v}_n \|_n \leq \| g_n \|_n
\]
and, consequently, by Fatou’s lemma
\[
\| u \| \leq \liminf_{n \to \infty} \| u_n \| = \liminf_{n \to \infty} \| \Psi v_n \| = \liminf_{n \to \infty} \| \Psi \hat{v}_n \|_n \leq \liminf_{n \to \infty} \| g_n \|_n = \| g \|.
\]
(4.16)
In particular, \( u = \Psi v \) is in \( \ell^{1,+}(X, \mu) \). Finally, by dominated convergence, we get (4.5b), i.e., \( \lim_{n \to \infty} \| u_n - u \| = 0 \).

**Step II-2** \( g \in \ell^{1}(X, \mu) \): Using the same notation as in **Step II-1**, we define for \( g \in \ell^{1}(X, \mu) \)
\[
 g_n := \pi_n g, \quad g_n^+ := \max\{0, g_n\}, \quad g_n^- := \min\{0, g_n\}.
\]

From **Step I** there exist \( \hat{v}_n, \hat{v}_n^+, \hat{v}_n^- \in C(X_n) \) that satisfy
\[
\begin{cases}
(\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n = g_n, \\
(\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n^+ = g_n^+, \\
(\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n^- = g_n^-.
\end{cases}
\]

Define
\[
\hat{u}_n := \Psi \hat{v}_n, \quad u_n := i_n,\infty \hat{u}_n.
\]

Clearly, \( u_n \) satisfies (4.5a) and, by construction,
\[
(i_n + \lambda \Delta_{\text{dir},n} \Phi)\pi_n u_n = g_n.
\]

Therefore,
\[
(i_n,\infty \text{id}_n \pi_n + \lambda \mathcal{L}_n)u_n = (i_n,\infty \text{id}_n \pi_n + \lambda i_n,\infty \Delta_{\text{dir},n} \Phi \pi_n)u_n
\]
\[
= i_n,\infty (\text{id}_n + \lambda \Delta_{\text{dir},n} \Phi)\pi_n u_n
\]
\[
= i_n,\infty g_n,
\]

that is,
\[
\lim_{n \to \infty} \| (i_n,\infty \text{id}_n \pi_n + \lambda \mathcal{L}_n) u_n - g \| = 0,
\]

which is exactly (4.5c).

Define now
\[
\hat{u}_n^+ := \Psi \hat{v}_n^+, \quad u_n^+ := i_n,\infty \hat{u}_n^+, \quad u^+ := \lim_{n \to \infty} u_n^+.
\]

In particular, \( u^+ \in \ell^{1}(X, \mu) \) is the monotone limit solution of (4.2) obtained in (4.13) and (4.14) of **Step II-1**. In the same way, define \( \hat{u}_n^-, u_n^- \) and \( u^- \). Finally, define
\[
v^+ := \Phi u^+, \quad v^- := \Phi u^-\]

as in (4.15) of **Step II-1**. Let us observe that, by the definitions (4.17)-(4.18) and Corollary A.3, and monotone limits, it holds that
\[
u_n(x) = \hat{u}_n(x) \leq \hat{u}_n^+(x) \leq u^+(x) \quad \text{if} \ x \in X_n,
\]
\[
u_n(x) = 0 \leq u^+(x) \quad \text{if} \ x \notin X_n,
\]

and
\[
u_n^-(x) \leq \hat{u}_n^-(x) \leq u_n^-(x) \quad \text{if} \ x \in X_n,
\]
\[
u_n^-(x) \leq 0 = u_n^-(x) \quad \text{if} \ x \notin X_n.
\]

In particular,
\[
u_n^-(x) \leq u_n^-(x) \leq u^+(x), \quad \forall \ x \in X, \ n \in \mathbb{N},
\]
that is, for every fixed $x \in X$ the sequence $u_n(x)$ is uniformly bounded in $n$ with

$$|u_n(x)| \leq c_x := \max \{|u^{-}(x)|, |u^{+}(x)|\} < \infty \text{ } \forall n \in \mathbb{N}. \tag{4.19}$$

Therefore, by passing to a subsequence using a diagonal sequence argument, the limit functions

$$u(x) := \lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} \Psi v_n(x),$$

$$v := \Phi u$$

exist and are well-defined on $X$. By the same arguments in (4.16), it follows that

$$\|u\| \leq \|g\| \text{ and } u = \Psi v \in \ell^1(X, \mu). \tag{4.20}$$

Moreover, from the previous Step II-1 we know that $u^+, u^- \in \ell^1(X, \mu)$ and then from (4.19) it follows that $|u_n|$ is bounded above by an integrable function. By dominated convergence, we get (4.5b), i.e., $\lim_{n \to \infty} \|u_n - u\| = 0$.

**Step III** (When $G$ is infinite and connected, $id + \lambda L|_{\Omega}$ maps bijectively onto $\ell^1(\pm(X, \mu))$): We want to show that the function $u \in \ell^1(\pm(X, \mu)$ that we constructed in Step II-1 as limit of a sequence of finitely supported functions $\{u_n\}_n$ is a solution of (4.2) which belongs to $\Omega$. In order to do so, it remains to show that:

$$u \in \text{dom}(\Delta); \tag{4.21a}$$

$$(id + \lambda L)u = g; \tag{4.21b}$$

$$\lim_{n \to \infty} \|L_n u_n - Lu\| = 0. \tag{4.21c}$$

Let us now highlight that $v_n = \Phi u_n \in C_c(X) \subseteq \text{dom}(\Delta)$ for every $n$, that is, $\Delta v_n$ is well-defined. Since, for every $x \in X$,

$$\sum_{y \in X} \lim_{n \to \infty} w(x, y)v_n(y) = \lim_{n \to \infty} \sum_{y \in X} w(x, y)v_n(y)$$

by (4.10) and monotone convergence, and recalling that every $G_{\text{dir}, n}$ is a Dirichlet subgraph of $G$ for every $n \in \mathbb{N}$, by Lemma A.1 we get

$$\Psi v(x) + \frac{\lambda}{\mu(x)} \left[ \deg(x) v(x) - \sum_{y \in X} w(x, y)v(y) \right]$$

$$= \lim_{n \to \infty} \left( \Psi v_n(x) + \frac{\lambda}{\mu(x)} \left[ \deg(x) v_n(x) - \sum_{y \in X} w(x, y)v_n(y) \right] \right)$$

$$= \lim_{n \to \infty} (\Psi + \lambda \Delta)v_n(x)$$

$$= \lim_{n \to \infty} (\Psi + \lambda \Delta_{\text{dir}, n}) \hat{v}_n(x)$$

$$= \lim_{n \to \infty} g_n(x) = g(x). \tag{4.22}$$

Notice that along the way we used the fact that $\psi$ is continuous. Moreover, by Remark 1, we observe that $v \in \text{dom}(\Delta)$, that is, $\Phi u \in \text{dom}(\Delta)$ and

$$(\Psi + \lambda \Delta)v(x) = g(x),$$

namely, $v$ is a nonnegative solution of (4.3) and thus $u$ is a nonnegative solution of (4.21b).
By the fact that \( \lambda \Delta \Phi u = g - u \) and \( g, u \in \ell^1(X, \mu) \), we obtain \( \Delta \Phi u \in \ell^1(X, \mu) \). We can then conclude that \( u \in \text{dom}(\mathcal{L}) \), i.e., \((4.21a)\). Let us prove \((4.21c)\): By the fact that \( \lambda \mathcal{L} u = g - u \) we obtain

\[
\| \mathcal{L} u_n - \mathcal{L} u \| \leq \frac{1}{\lambda} \left( \| (i_{n,\infty} \text{id}_n \pi_n + \lambda \mathcal{L} u) u_n - g \| + \| i_{n,\infty} \text{id}_n \pi_n u_n - u \| \right)
\]

\[
= \frac{1}{\lambda} \left( \| (i_{n,\infty} \text{id}_n \pi_n + \lambda \mathcal{L} u) u_n - g \| + \| u_n - u \| \right)
\]

(4.23)

and we conclude \((4.21c)\) by using \((4.5b)\) and \((4.5c)\).

In particular, we have shown that for every \( \lambda > 0 \) and \( g \in \ell^{1,+}(X, \mu) \) there exists a unique \( u \in \Omega \cap \ell^{1,+}(X, \mu) \) such that \( (\text{id} + \lambda \mathcal{L})u = g \), and \( \| u \| \leq \| g \| \). If \( g \in \ell^{1,-}(X, \mu) \), then the arguments of the proof are completely symmetric and the (nonpositive) solution \( u \) can be built as monotone decreasing limit.

**Step IV** (When \( G \) is infinite, connected and satisfies assumptions (H1), (H2) or (H3), then \( \text{id} + \lambda \mathcal{L}_\Omega \) maps bijectively onto \( \ell^1(X, \mu) \)): The statement follows immediately by the same arguments in **Step III** if we can show that \( u \in \text{dom}(\mathcal{L}) \) and

\[
(\text{id} + \lambda \mathcal{L})u = g.
\]

We divide this step into three sub-steps.

**Step IV-H1** (\( G \) is locally finite): Let us highlight that \( \text{dom}(\Delta) = C(X) \) because of the local finiteness of \( G \) so that \( v_n, v \in \text{dom}(\Delta) \), that is, \( \Delta v_n \) and \( \Delta v = \Delta \Phi u \) are well-defined. Furthermore, by the local finiteness of \( G \), for every fixed \( x \) there exists a finite number of nodes \( y \in X \) such that \( w(x, y) \neq 0 \). By Lemma A.5, we can assume moreover that the sequence \( \{X_n\}_{n=1}^\infty \) in addition to \((4.6)\) and \((4.7)\) also satisfies

\[
\hat{X}_n \subset \hat{X}_{n+1} \quad \text{and} \quad \bigcup_{n=1}^\infty \hat{X}_n = X.
\]

Let \( N = N(x) \) be such that \( x \in \hat{X}_n \) for every \( n \geq N \). As a consequence, for every fixed \( x \in X \), the series are in fact finite sums and, passing to the limit, we get

\[
\lim_{n \to \infty} \sum_{y \in X} w(x, y)|v_n(y)| = \lim_{n \to \infty} \sum_{y \in X} w(x, y)|v_n(y)|
\]

\[
= \sum_{y \in X} \lim_{n \to \infty} w(x, y)|v_n(y)|.
\]

Therefore, by the above considerations, we have

\[
(\Psi + \lambda \Delta)v(x) = \lim_{n \to \infty} (\Psi + \lambda \Delta)v_n(x)
\]

\[
= \lim_{n \to \infty} (\Psi + \lambda \Delta_{\text{dir},n}) \hat{v}_n(x)
\]

\[
= \lim_{n \to \infty} g_n(x) = g(x)
\]

where the second equality follows from Lemma A.1 (and the fact that every \( G_{\text{dir},n} \) is a Dirichlet subgraph of \( G \) for every \( n \in \mathbb{N} \)).

Since \( \lambda \Delta \Phi u = g - u \), we get \( \Delta \Phi u \in \ell^1(X, \mu) \), i.e., \( u \in \text{dom}(\mathcal{L}) \). By the same arguments as in **Step III**, see \((4.23)\), we can check that \( u \in \Omega \). By the accretivity of \( \mathcal{L}_\Omega \) we obtain the
uniqueness of $u$. In particular, we have proven that for every $\lambda > 0$ and $g \in \ell^1(X, \mu)$ there exists a unique $u \in \Omega$ such that $(\id + \lambda L)u = g$, and $\|u\| \leq \|g\|$. 

**Step IV-H2** (inf$_{x \in X} h(x) > 0$): Once we show that for every fixed $x \in X$

$$
\sum_{y \in X} \lim_{n \to \infty} w(x, y)|v_n(y)| = \lim_{n \to \infty} \sum_{y \in X} w(x, y)|v_n(y)| < \infty, \tag{4.24}
$$

that is, $v \in \text{dom}(\Delta)$ and $\lim_{n \to \infty} \Delta v_n(x) = \Delta (\lim_{n \to \infty} v_n)(x)$ by dominated convergence, then we can conclude the proof as in the final part of **Step IV-H1**. Indeed, one of the main issues in the previous steps was to show that the solution $v$ of equation (4.3) belongs to $\text{dom}(\Delta)$ so that $(\Psi + \lambda \Delta) v$ is well-defined. If this is established, we can apply the same arguments as in (4.22). However, here (4.24) is immediate: By the uniformly lower boundedness of the measure $\mu$ it follows that $\ell^1(X, \mu) \subseteq \ell^\infty(X, \mu)$ and, since $\Psi v = u \in \ell^1(X, \mu)$, we get $v \in \ell^\infty(X, \mu)$ by the surjectivity of $\psi$. So, $v \in \text{dom}(\Delta)$ and (4.24) follows.

**Step IV-H3** (sup$_{x \in X} \frac{\sum_{y \in X} w(x, y)}{\mu(x)} \leq c < \infty$ and $\Phi(\ell^1) \subseteq \ell^1$): The reasoning of the previous step applies here as well. By (4.20) we have $u \in \ell^1(X, \mu)$ and from the hypothesis on $\Phi$ it follows that $v = \Phi u \in \ell^1(X, \mu)$. Therefore,

$$
\sum_{y \in X} w(x, y)|v(y)| = \sum_{y \in X} \frac{w(x, y)}{\mu(y)}|v(y)|\mu(y) \leq \sum_{y \in X} \sup_z \sum_{x \in X} \frac{w(x, z)}{\mu(z)}|v(y)|\mu(y) \leq c\|v\| < \infty.
$$

Thus, $v \in \text{dom}(\Delta)$ and (4.24) holds.

**Step V** (Constructing a solution when $G$ is not connected): Assume now that $G$ is not connected and write $X$ as a disjoint union of connected components, that is, $X = \bigcup_{k=1}^K Y_k$ where $Y_k$ are connected components of $X$ and $K \in \mathbb{N} \cup \{\infty\}$.

We first observe that, if $u \in \text{dom}(\Delta)$ and $x \in Y_k$, then $\Delta u(x) = \Delta_k \pi_k u(x)$ where $\Delta_k$ is the formal graph Laplacian associated to the canonical induced subgraph $G_k = (Y_k, w|_{Y_k \times Y_k}, \kappa|_{Y_k}, \mu|_{Y_k})$ and $\pi_k$ is the projection onto $C(Y_k)$. We then define $L_k : \text{dom}(L_k) \subseteq \ell^1(Y_k, \mu|_{Y_k}) \to \ell^1(Y_k, \mu|_{Y_k})$ where

$$
\text{dom}(L_k) := \{v \in \ell^1(Y_k, \mu|_{Y_k}) \mid \Phi v \in \text{dom}(\Delta_k), \Delta_k \Phi v \in \ell^1(Y_k, \mu|_{Y_k})\},
$$

$L_k v := \Delta_k \Phi v$.

Notice that if $u \in \text{dom}(L)$, then $\pi_k u \in \text{dom}(L_k)$ for every $k$ and $L u(x) = L_k \pi_k u(x)$ for every $x \in Y_k$.

Now, for every $Y_k$, we fix an exhaustion $\{Y_{k,n}\}_n$ as in Lemma A.4 and define the set $\Omega_k$ associated to the subgraph $G_k$ and $\{Y_{k,n}\}_n$ as in Definition B.5. As we already know from **Step 0**, $L_k$ is accretive on $\Omega_k$.

We next define

$$
\Omega := \{u \in \text{dom}(L) \mid \exists \{u_k\}_k \text{ s.t. } u_k \in \Omega_k \text{ for } k = 1, \ldots, K, \pi_k u = u_k\}.
$$

By Lemma B.6, $C_c(X) \subseteq \text{dom}(L)$. Furthermore, by Lemma B.7, for every $u \in C_c(X)$,

$$
u_k := \pi_k u \in C_c(Y_k) \subseteq \Omega_k.$$

It follows that $C_c(X) \subseteq \Omega$ and, in particular, $\Omega = \overline{\text{dom}(L)} = \ell^1(X, \mu)$. It is not difficult to show that $L$ is accretive on $\Omega$, as each $L_k$ is accretive on $\Omega_k$. 

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Indeed, for every \( u, v \in \Omega \)
\[
\| (u - v) + \lambda (\mathcal{L}u - \mathcal{L}v) \| = \sum_{k=1}^{K} \| (\pi_k u - \pi_k v) + \lambda (\mathcal{L}_k \pi_k u - \mathcal{L}_k \pi_k v) \|_{Y_k}
\]
\[
= \sum_{k=1}^{K} \| (u_k - v_k) + \lambda (\mathcal{L}_k u_k - \mathcal{L}_k v_k) \|_{Y_k}
\]
\[
\geq \sum_{k=1}^{K} \| u_k - v_k \|_{Y_k} = \| u - v \|.
\]

Finally, fix \( g \in \ell^1(X, \mu) \) and \( \lambda > 0 \). Clearly, if \( g \in \ell^1(X, \mu) \), then \( \pi_k g \in \ell^1(Y_k, \mu|_{Y_k}) \) for every \( k \) and, if \( G \) satisfies one of the assumptions \((H1), (H2) \) or \((H3)\) in the statement of Theorem 1 holds in each connected component \( Y_k \) of \( X \), not necessarily the same condition for different \( Y_k \).

Using the constructions carried out in the proof of Theorem 1, we can now prove the existence and uniqueness of solutions for the Cauchy-GPME.

**Proof of Theorem 2** From the definition of mild solutions, Definition 3.3, without loss of generality we can suppose that \( T < \infty \). Then, since \( f \in L^1_{\text{loc}}([0, T]; \ell^1(X, \mu)) = L^1([0, T]; \ell^1(X, \mu)) \), there exists an \( \epsilon \)-discretization \( D_{\epsilon} \) of \(([0, T]; f)\) for every \( \epsilon > 0 \). Let us observe that solving (3.1), i.e.,
\[
\frac{u_k - u_{k-1}}{\lambda_k} + \Delta \Phi u_k = f_k, \quad \lambda_k := t_k - t_{k-1}
\]
for \( k = 1, \ldots, n \) means to solve at each step the equation
\[
(id + \lambda_k \Delta \Phi)u_k = u_{k-1} + \lambda_k f_k
\]
in such a way that
\[ u_k \in \ell^1(X, \mu), \quad \Phi u_k \in \text{dom}(\Delta), \quad \Delta \Phi u_k \in \ell^1(X, \mu) \]
where \( \lambda_k > 0 \) and \( f_k \in \ell^1(X, \mu) \). Therefore, given \( u_0 \) and \( \{f_k\}_{k=1}^n \), the solution \( \{u_k\}_{k=1}^n \) (if any) of (3.1) is computed recursively starting from
\[ (id + \lambda_1 \Delta \Phi) u_1 = u_0 + \lambda_1 f_1. \]  
(4.25)

If \( u_0, f_1 \in \ell^1(X, \mu) \) are nonnegative (nonpositive), then \( g := u_0 + \lambda_1 f_1 \in \ell^{1, \pm}(X, \mu) \) and by Theorem 1 there exists a unique nonnegative (nonpositive) solution \( u_1 \in \Omega \) of (4.25).

Iterating the procedure, each \( u_{k-1} + \lambda_k f_k \in \ell^{1, \pm}(X, \mu) \). Therefore, for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-approximate solution \( u_\varepsilon \) of the Cauchy-GPME (see (3.2)), such that \( u_\varepsilon(t) \geq 0 \) and \( u_\varepsilon(t) \in \Omega \) for every \( t \in (0, T) \). In Theorem 1 we also proved that \( \mathcal{L}_{|\Omega} \) is accretive and \( \Omega = \text{dom}(\mathcal{L}) = \ell^1(X, \mu) \) by Lemma B.7.

Therefore, summarizing, we have that:

1. By hypotheses a), (i), and (ii) we have \( u_0 \geq 0, u_0 \in \ell^1(X, \mu) = \text{dom}(\mathcal{L}_{|\Omega}) = \overline{\Omega} \), and \( f \in L^1([0, T]; \ell^1(X, \mu)) \), respectively;
2. \( \mathcal{L}_{|\Omega} \) is accretive;
3. For every \( u_0 \geq 0 \) and \( f(t) \geq 0 \), there exists an \( \varepsilon \)-approximate solution \( u_\varepsilon \) such that \( u_\varepsilon(t) \geq 0 \) and \( u_\varepsilon(t) \in \text{dom}(\mathcal{L}_{|\Omega}) = \Omega \) for every \( t \in (0, T) \).

Then, by standard results (see [6, Theorem 3.3] or [3, Theorem 4.1]), there exists a unique mild solution \( u \) of the Cauchy-GPME which satisfies (1.1). Since the limit is uniform and \( u_\varepsilon(t) \geq 0 \), then \( u(t) \geq 0 \) and \( u(t) \in \ell^1(X, \mu) \) for every \( t \in [0, T] \). The validity of (1.2) is again standard, see [3, Theorem 4.1]. If \( u_0 \leq 0 \) and \( f(t) \leq 0 \), then we get the same results in a completely analogous way.

Under the extra hypothesis (H1), (H2) or (H3) in Theorem 1 we have established the \( m \)-accretivity of \( \mathcal{L}_{|\Omega} \) which implies the existence of \( \varepsilon \)-approximate solutions for every \( \varepsilon \) as above. Therefore, under the hypotheses i), ii) and b), there exists a unique mild solution \( u \) of the Cauchy-GPME which satisfies (1.1) and (1.2), see [3, Corollary 4.1]. \( \square \)

We now recall the following general result see, e.g., [13, Proposition 3] or [6, Theorem 1.6].

**Proposition 4.1** Let \( f \in L^1_{\text{loc}}([0, T]; \ell^1(X, \mu)) \). Let \( \text{dom} (\mathcal{L}) \) be closed and let \( \mathcal{L} \) be continuous on \( \text{dom} (\mathcal{L}) \). If \( u \) is a mild solution on \( (0, T) \), then \( u \) is a strong solution and \( u \) satisfies for every \( 0 < t < T \)
\[ u(t) = u(0) - \int_0^t \mathcal{L} u(s) ds + \int_0^t f(s) ds. \]

Moreover, if \( f \in C([0, T]; \ell^1(X, \mu)) \), then \( u \) is a classic solution.

We now provide a direct application of the proposition above to the graph setting.

**Corollary 4.2** Let \( G = (X, w, \kappa, \mu) \) be a graph. If
1. \( \sup_{x \in X} \text{Deg}(x) < \infty \);
2. \( \Phi: \ell^1(X, \mu) \to \ell^1(X, \mu) \) is continuous;

then \( \text{dom}(\mathcal{L}) = \ell^1(X, \mu) \) and \( \mathcal{L} \) is continuous. In particular, the conclusions of Proposition 4.1 hold.
Let $u \in \ell^1(X, \mu)$. By (ii) we have that $\Phi u \in \ell^1(X, \mu)$ and then by (i)
\[ \sum_{y \in X} w(x, y) |\phi(u(y))| \leq c_1 \sum_{y \in X} |\phi(u(y))| \mu(y) < \infty \]
for some $c_1 > 0$, that is, $\Phi u \in \text{dom} (\Delta) \cap \ell^1(X, \mu)$. Let us recall from [38, Theorem 9.2] or [44, Theorem 2.15] that the formal graph Laplacian $\Delta$ is bounded on $\ell^1(X, \mu)$ (indeed, on $\ell^p(X, \mu)$ for all $p \in [1, \infty]$) if and only if (i) holds. Therefore,
\[ \|\Delta \Phi u\| \leq c_2 \|\Phi u\| < \infty \]
for some $c_2 > 0$, namely, $\Delta \Phi u \in \ell^1(X, \mu)$ and $\text{dom}(\mathcal{L}) = \ell^1(X, \mu)$. Therefore, by (ii), $\mathcal{L}$ is continuous as the composition of continuous operators is continuous. \qed

**Remark 6** Observe that, if $G$ is finite, then both hypotheses (i) and (ii) in Corollary 4.2 are trivially satisfied and if $f$ is continuous, then the Cauchy-GPME always has a unique classic solution for any $\phi$.

About hypothesis (ii), if $G$ is not finite, one condition that ensures the continuity of $\Phi$ is if $\phi$ is Lipschitz continuous with uniform Lipschitz constant. Another sufficient condition for the continuity of the operator $\Phi$ on $\ell^1(X, \mu)$ is that $\mu$ is bounded away from zero, i.e., assumption (H2) and that $\phi$ is uniformly Lipschitz on every interval $[-R, R]$. This is for instance the case of the PME where $\phi(s) = s|m|^{m-1}$ with $m > 1$.

To prove these statements, recall first that we are assuming $\Phi(\ell^1(X, \mu)) \subseteq \ell^1(X, \mu)$ and that if $\inf_{x \in X} \mu(x) \geq c > 0$, then $\ell^1(X, \mu) \subseteq \ell^\infty(X, \mu)$ and $\|u\|_\infty \leq c^{-1} \|u\|_1$ for every $u \in \ell^1(X, \mu)$. Therefore, if $\|u_n - u\|_1 \to 0$ then $|u_n - u|$ is uniformly bounded. In particular, there exists $R > 0$ such that $u_n(x), u(x) \in [-R, R]$ for every $x \in X$ and for every $n \in \mathbb{N}$. Consequently, $|\phi(u_n(x)) - \phi(u(x))| \leq L_R |u_n(x) - u(x)|$ where $L_R$ is the Lipschitz constant of $\phi$ on $[-R, R]$ and then $\|\Phi u_n - \Phi u\|_1 \to 0$.

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**Appendix A: Auxiliary results**

In this appendix we collect several results which are used in various parts of the paper. The first result concerns the relationship between the Dirichlet Laplacian and restrictions of the formal Laplacian. For related material, see [43, 44].

**Lemma A.1** Let $F = (X, w, k, \mu)$ be a graph, let $A$ be a subset of $X$ such that $\partial A \neq \emptyset$ and let $F_{\text{dir}} = (A, w|_{A \times A}, k_{\text{dir}}, \mu|_A)$ be the Dirichlet subgraph associated to $A$. Define
\[ \Delta|_A : \text{dom} \left( \Delta|_A \right) \subseteq C(X) \to C(A) \]
with
\[ \text{dom} \left( \Delta|_A \right) := \left\{ u \in C(X) \mid u \equiv 0 \text{ on } X \setminus A, \sum_{y \in A} w(x, y) |u(y)| < \infty \forall x \in A \right\} , \]
\[ \Delta|_A u(x) := \Delta u(x) \quad \text{for every } x \in A. \]
Then, we have the following commutative diagrams

\[ D_1: \quad \begin{array}{c}
\operatorname{dom}(\Delta_{\text{dir}}) \xrightarrow{i} \operatorname{dom}(\Delta_A) \\
\Delta_{\text{dir}} \downarrow \quad \downarrow \Delta_A \\
C(A) \end{array} \quad \begin{array}{c}
\operatorname{dom}(\Delta_A) \xrightarrow{\pi} \operatorname{dom}(\Delta_{\text{dir}}) \\
\downarrow \Delta_A \downarrow \Delta_{\text{dir}} \\
C(A) \end{array} \]

where \( i \) and \( \pi \) are the canonical embedding and the canonical projection, respectively, as defined in (2.3). In particular, we have \( \Delta_{\text{dir}} \equiv \Delta_A i \), \( \Delta_{\text{dir}} \pi \equiv \Delta_A \) and

\[
\Delta_{\text{dir}} v(x) = \Delta_A i v(x) \quad \forall v \in \operatorname{dom}(\Delta_{\text{dir}}) \subseteq C(A), \quad \forall x \in A,
\]

\[
\Delta_A \pi u(x) = \Delta_A u(x) \quad \forall u \in \operatorname{dom}(\Delta_A) \subseteq C(X), \quad \forall x \in A.
\]

If every node in \( \partial A \) is connected to a finite number of nodes in \( A \), then

\[
\operatorname{dom}(\Delta_A) = \operatorname{dom}(\Delta) \cap \{ u \in C(X) \mid u \equiv 0 \text{ on } X \setminus A \}
\]

and \( \Delta_A u \) can be uniquely extended to \( X \) for every \( u \in \operatorname{dom}(\Delta_A) \) in such a way that

\[
\Delta_A u(x) = \Delta u(x) \quad \forall x \in X,
\]

that is, \( \Delta_A \) is the restriction of \( \Delta \) to the set of functions which vanish on \( X \setminus A \).

**Proof** Clearly, \( \Delta_A \) is well-defined and

\[
\operatorname{dom}(\Delta_A) \supseteq \operatorname{dom}(\Delta) \cap \{ u \in C(X) \mid u \equiv 0 \text{ on } X \setminus A \}.
\]

If every node in \( \partial A \) is connected to a finite number of nodes in \( A \), then for every \( u \in \operatorname{dom}(\Delta_A) \) and \( x \in X \setminus A \)

\[
\sum_{y \in X} w(x, y) \lvert u(y) \rvert = \sum_{y \in \partial A} w(x, y) \lvert u(y) \rvert < \infty.
\]

Therefore,

\[
\operatorname{dom}(\Delta_A) \subseteq \operatorname{dom}(\Delta) \cap \{ u \in C(X) \mid u \equiv 0 \text{ on } X \setminus A \}
\]

so that the two domains are equal as claimed. Furthermore, for every \( u \in \operatorname{dom}(\Delta_A) \), we can uniquely extend \( \Delta_A u \) to \( X \setminus A \) so as to satisfy \( \Delta_A u = \Delta u \) by defining

\[
\Delta_A u(x) = -\frac{1}{\mu(x)} \sum_{y \in \partial A} w(x, y) u(y) \quad \text{for } x \in X \setminus A.
\]

Observe now that

\[
\operatorname{dom}(\Delta_A i) = \{ v \in C(A) \mid i v \in \operatorname{dom}(\Delta_A) \} \subseteq C(A)
\]

and if \( v \in C(A) \), then

\[
\sum_{y \in X} w(x, y) \lvert i v(y) \rvert = \sum_{y \in A} w(x, y) \lvert v(y) \rvert \quad \text{for every } x \in A.
\]

Therefore, it is immediate to check that \( \operatorname{dom}(\Delta_{\text{dir}}) = \operatorname{dom}(\Delta_A i) \).

Finally, if \( v \in \operatorname{dom}(\Delta_{\text{dir}}) \), then for every \( x \in A \) we have

\[
\Delta_{\text{dir}} v(x) = -\frac{1}{\mu_A(x)} \sum_{y \in A} w_{|A \times A}(x, y) (v(x) - v(y)) + \frac{\kappa_{\text{dir}}(x)}{\mu_A(x)} v(x)
\]
Combining the above inequalities (A.1) and (A.2), we get

\[ v(x) \leq \frac{1}{\mu(x)} \sum_{y \in A} w(x, y) (v(x) - v(y)) + \frac{\kappa(x)}{\mu(x)} v(x) \]

\[ = \frac{1}{\mu(x)} \sum_{y \in A} w(x, y) (v(x) - v(y)) + \frac{\sum_{y \in \bar{A}} w(x, y)}{\mu(x)} v(x) + \frac{\kappa(x)}{\mu(x)} v(x) \]

\[ = \frac{1}{\mu(x)} \sum_{y \in X} w(x, y) (iv(x) - iv(y)) + \frac{\kappa(x)}{\mu(x)} iv(x) \]

\[ = \Delta iv(x). \]

This concludes the proof of diagram \( D_1 \). The proof of diagram \( D_2 \) is basically the same following suitable modifications.

The next theorem is a comparison principle for a nonlinear operator. This result generalizes [12, Theorem 2, Section 23.1] and [43, Theorem 8 and Proposition 3.1], see also the proof of Theorem 1.3.1 in [72]. In particular, we relax the assumptions on the function \( u \) by letting it not attain a minimum or maximum on \( X \) if the graph \( G \) does not have any infinite paths. We recall that for us a path is a walk without any repeated nodes.

**Theorem A.2** (Comparison principle) Let \( G = (X, w, \kappa, \mu) \) be a connected graph. Let \( \lambda > 0 \) and \( v \in \text{dom} (\Delta) \). Assuming that \( \psi : \mathbb{R} \to \mathbb{R} \) is strictly monotone increasing and \( \psi(0) \leq 0 \) we consider three cases:

**Case 1** There exists \( x_0 \in X \) such that \( u \) attains a minimum at \( x_0 \), i.e.,

\[ v(x_0) = \inf_{x \in X} \{ v(x) \} > -\infty. \]

**Case 2** \( G \) does not contain any infinite path.

**Case 3** \( G \) has an infinite path and for every infinite path \( \{x_n\} \) we have \( \sum_{n} \mu(x_n) = \infty \) and there exists \( p > 0 \) such that \( \sum_{x_n} |v(x_n)|^p \mu(x_n) < \infty \).

In all the three cases, if \( (\Psi + \lambda \Delta) v \geq 0 \), then \( v \geq 0 \).

Assuming instead \( \psi(0) \geq 0 \) and substituting \( v(x_0) = \sup_{x \in X} \{ v(x) \} < \infty \) for \( v(x_0) = \inf_{x \in X} \{ v(x) \} > -\infty \) in Case 1, if \( (\Psi + \lambda \Delta) v \leq 0 \), then \( v \leq 0 \). Moreover, in any case, if \( v(x) = 0 \) for some \( x \in X \), then \( v = 0 \).

**Proof** Let \( v \in \text{dom} (\Delta) \) be such that \( (\Psi + \lambda \Delta) v \geq 0 \) for \( \psi(0) \leq 0 \) strictly monotone increasing. If \( v \geq 0 \), then there is nothing to prove. Hence, we assume that there exists \( x_0 \in X \) such that \( v(x_0) < 0 \). We will show that this leads to a contradiction in all three cases.

Since \( \psi(0) \leq 0 \) and \( \psi \) is strictly monotone increasing,

\[ \psi(v(x_0)) + \frac{\kappa(x_0)}{\mu(x_0)} v(x_0) < 0. \]  

(A.1)

Furthermore, as \( (\Psi + \lambda \Delta) v(x_0) \geq 0 \),

\[ 0 \leq \psi(v(x_0)) + \frac{\lambda}{\mu(x_0)} \sum_{y \in X} w(x_0, y) (v(x_0) - v(y)) + \frac{\kappa(x_0)}{\mu(x_0)} v(x_0). \]  

(A.2)

Combining the above inequalities (A.1) and (A.2), we get

\[ 0 < - \left[ \psi(v(x_0)) + \frac{\kappa(x_0)}{\mu(x_0)} v(x_0) \right] \leq \frac{\lambda}{\mu(x_0)} \sum_{y \in X} w(x_0, y) (v(x_0) - v(y)) \]
and because $w(\cdot, \cdot) \geq 0$ and $G$ is connected, there exists $y = x_1 \sim x_0$ such that $v(x_1) < v(x_0)$. In particular, $v(x_1) < 0$.

Hence, we see that every node where $v$ is negative is connected to a node where $v$ is strictly smaller. This is the basic observation that will be used in all three cases.

(Case 1) From the discussion above, it is clear that $v$ cannot achieve a negative minimum.

(Case 2) Iterating the procedure above, we find a sequence of distinct nodes $\{x_k\}_{k=0}^n$ such that $x_0 \sim x_1 \sim \ldots \sim x_n$ and

$$v(x_n) < v(x_{n-1}) < \ldots < v(x_0) < 0.$$ 

Since $G$ does not have any infinite path this sequence must end which leads to a contradiction.

(Case 3) In this case, we can obtain an infinite sequence $\{v(x_n)\}_n$ such that $\{x_n\}_n$ is an infinite path and

$$\ldots < v(x_n) < v(x_{n-1}) < \ldots < v(x_0) < 0.$$ 

It follows that $|v(x_n)| > |v(x_0)| > 0$, for every $n$, and therefore

$$\sum_n |v(x_n)|^p \mu(x_n) > |v(x_0)|^p \sum_n \mu(x_n) = \infty$$

for every $p > 0$ which gives a contradiction.

Hence, we have established that $v \geq 0$ in all three cases. Now, if there exists $x_0 \in X$ such that $v(x_0) = 0$, then

$$0 \leq \Psi v(x_0) + \lambda \Delta v(x_0) = -\frac{\lambda}{\mu(x_0)} \sum_{y \in X} w(x_0, y)v(y) \leq 0$$

and thus $v(y) = 0$ for all $y \sim x_0$. Using induction and the assumption that $G$ is connected we get $v \equiv 0$.

The proof that $(\Psi + \lambda \Delta) v \leq 0$ implies $v \leq 0$ when $\psi(0) \geq 0$ is completely analogous. $\square$

Remark 7 The above proof also shows that if $v$ satisfies $(\Psi + \lambda \Delta) v \geq 0$, on any connected graph, $\psi(0) \leq 0$ and $\min_{x \in X} v(x) \leq 0$, then $v \equiv 0$.

Remark 8 A closer look at the proof shows that the conclusion of the theorem also holds for the operator

$$\sigma \Psi + \lambda \Delta$$

where $\sigma \in C(X)$ is positive.

Corollary A.3 With the hypotheses of Theorem A.2, let $\psi(0) = 0$ for $\psi : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing. Let $v_1, v_2$ be solutions of

$$(\Psi + \lambda \Delta) v_k = g_k, \quad \lambda > 0 \text{ and } g_k \in C(X) \text{ for } k = 1, 2.$$ 

If $g_1 \geq g_2$, then $v_1 \geq v_2$.

Proof Notice that, since $\psi$ is strictly increasing and $\psi(0) = 0$, there exists a positive function $\sigma \in C(X)$ such that

$$\psi(v_1(x)) - \psi(v_2(x)) = \sigma(x)\psi(v_1(x) - v_2(x))$$
for all \( x \in X \). Indeed, we can define

\[
\sigma(x) = \begin{cases} 
1 & \text{if } v_1(x) = v_2(x) \\
\frac{\psi(v_1(x)) - \psi(v_2(x))}{\psi(v_2(x)) - \psi(v_2(x))} & \text{otherwise.}
\end{cases}
\]

It follows that \( v_1 - v_2 \) satisfies

\[
\sigma \Psi(v_1 - v_2) + \lambda \Delta(v_1 - v_2) = g_1 - g_2 \geq 0 \quad \text{on } X
\]

and the conclusion follows from Remark 8.

\[\square\]

**Remark 9** As in Remark 7, the proof shows that if \( v_k \) satisfy \((\Psi + \lambda \Delta) v_k = g_k, g_1 \geq g_2 \) and \( \min_{x \in X} \{ v_1(x) - v_2(x) \} \leq 0 \), then \( v_1 \equiv v_2 \) and \( g_1 \equiv g_2 \) on \( X \).

Finally, in the following two lemmas we discuss how to exhaust the graph via finite subgraphs which are nested and such that each subgraph is connected to the next subgraph. We note that as we do not assume local finiteness, we have to take a little bit of care in how we choose the exhaustion. Although this should certainly be well-known, for the convenience of the reader we include a short proof.

Let \( d \) denote the combinatorial graph metric, that is, the least number of edges in a path connecting two nodes. Fix a node \( x_1 \in X \) and let \( S_r = S_r(x_1) \) denote the sphere of radius \( r = 0, 1, 2, \ldots \) about \( x_0 \), that is,

\[
S_r = \{ x \in X \mid d(x, x_0) = r \}.
\]

For \( x \in S_r \), we call \( y \in X \) a **forward neighbor** of \( x \) if \( y \sim x \) and \( y \in S_{r+1} \). We will denote the set of forward neighbors of \( x \) via \( N_+(x) \), i.e.,

\[
N_+(x) = \{ y \sim x \mid d(y, x_0) = d(x, x_0) + 1 \}.
\]

We now use the set of forward neighbors to inductively create our exhaustion sequence.

**Lemma A.4** Let \( G = (X, w, \kappa, \mu) \) be a connected and infinite graph. Then there exists a sequence of connected and finite induced subgraphs \( G_n = (X_n, w_n, \kappa_n, \mu_n) \) with \( X_n \subset X_{n+1}, \bigcup_{n=1}^{\infty} X_n = X \) and

\[
\{ x \in X_n \mid x \sim y \text{ for some } y \in X_{n+1} \setminus X_n \} \neq \emptyset
\]

for all \( n \in \mathbb{N}_0 \).

**Proof** We arrange the forward neighbors of each node in a sequence. Let \( X_1 = \{ x_0 \} \). For \( X_2 \) choose the first forward neighbor of \( x_0 \) and add it to \( X_0 \), that is, \( X_2 = \{ x_0, x_1 \} \) where \( N_+(x_0) = \{ x_1, x_2, \ldots \} \). We note that \( N_+(x_0) \neq \emptyset \) as the graph is infinite and connected.

Now, proceed inductively as follows: Given \( X_n \) let \( X_{n+1} \) consist of \( X_n \) and, for every node \( x \) in \( X_n \), we add to \( X_n \) the first forward neighbor in \( N_+(x) \) which is not included in \( X_n \) to get \( X_{n+1} \). Let \( G_n \) denote the induced subgraph.

As we only add at most a single forward neighbor for each node at every step, it follows that each \( X_n \) is finite with \( |X_n| \leq 2^n \). It is clear by construction that \( G_n \) is connected. Furthermore, as the graph is infinite and connected, it follows that \( \{ x \in X_n \mid x \sim y \text{ for some } y \in X_{n+1} \setminus X_n \} \neq \emptyset \) for each \( n \in \mathbb{N} \). Finally, to show that the union of the \( X_n \) is the entire node set, let \( x \in X \). Then, \( x \in S_r \) for some \( r \) which means that there exists a sequence \( \{ y_k \}_{k=0}^{\infty} \) with \( y_0 = x_0, y_r = x \) and \( y_k \in S_k \) such that \( y_{k+1} \in N_+(y_k) \). As each node \( y_k \) will then be included in some set of the exhaustion \( X_n \), it follows that \( x \in \bigcup_{n=1}^{\infty} X_n \). This completes the proof. \( \square \)
Fig. 4 Exhaustion of a locally finite graph $G$ by a chain of Dirichlet subgraphs $\{G_{\text{dir},n}\}_n$. The figure is read from left to right, from top to bottom. The chain $\{G_{\text{dir},n}\}_n$ is built starting from an inner node $x_0$ by applying the recursive procedure described in the proof of Lemma A.5. At each step $n = 1, 2, 3, \ldots$, the interior nodes in $\dot{X}_n$ are green while the inner boundary nodes in $\dot{\partial}X_n$ are light green. The nodes belonging to the exterior boundary $\partial X_n \subseteq X \setminus X_{n-1}$ are colored in light gray. We can visually see how every subgraph $G_{\text{dir},n}$ is still “chained” to the supergraph $G$ by the Dirichlet killing term $\kappa_{\text{dir},n}$ which is depicted by red rings and red dashed lines. In particular, each $G_{\text{dir},n}$ is a Dirichlet subgraph of $G_{\text{dir},n+1}$.

In the locally finite case, the above can be simplified by just using balls for our exhaustion sets. See Fig. 4 for a visual representation. In this case, it is also possible to exhaust in such a way that we have an inclusion between the interiors of the exhaustion sets.

**Lemma A.5** Let $G = (X, w, \kappa, \mu)$ be connected, infinite and locally finite. Then there exists a sequence of connected and finite induced subgraphs $G_n = (X_n, w_n, \kappa_n, \mu_n)$ with $\dot{X}_n \subset \dot{X}_{n-1}$.
\( \hat{X}_{n+1}, X_n \subset X_{n+1}, \bigcup_{n=1}^{\infty} \hat{X}_n = X \) and
\[
\{ x \in X_n \mid x \sim y \text{ for some } y \in X_{n+1} \setminus X_n \} \neq \emptyset
\]
for all \( n \in \mathbb{N} \).

**Proof** We modify the construction of the previous lemma: We take for all \( n \in \mathbb{N} \) constructed \( X_n \) of subsets projection for each \( X_n \).

Finally, since a node \( x_n \) carrying out the proof of Theorem 1 belongs to \( \hat{X}_n \). This subset is of particular importance because every solution that is constructed while following that \( \hat{\cdot} \) is required to satisfy any additional properties other than that each \( X_n \) is finite. We denote by \( i_{n,\infty} \) the canonical embedding and by \( \pi_n \) the canonical projection for each \( X_n \):

\[
i_{n,\infty}: C(X_n) \to C(X) \quad i_{n,\infty}u(x) := \begin{cases} u(x) & \text{if } x \in X_n, \\ 0 & \text{if } x \in X \setminus X_n; \end{cases}
\]

\[
\pi_n: C(X) \to C(X_n) \quad \pi_n u(x) := u(x) \text{ for every } x \in X_n.
\]

Appendix B: Accretivity

In this appendix we prove that there exists a dense subset \( \Omega \) of \( \text{dom}(\mathcal{L}) \) where \( \mathcal{L} \) is accretive. This subset is of particular importance because every solution that is constructed while carrying out the proof of Theorem 1 belongs to \( \Omega \).

From now on, if \( G \) is infinite, then we fix an exhaustion \( \{X_n\}_{n=1}^{\infty} \) of \( X \), i.e., a sequence of subsets \( X_n \) of \( X \) such that \( X_n \subseteq X_{n+1} \) and \( X = \bigcup_{n=1}^{\infty} X_n \), where we additionally assume that each \( X_n \) is finite. We denote by \( i_{n,\infty} \) the canonical embedding and by \( \pi_n \) the canonical projection for each \( X_n \):

\[
i_{n,\infty}: C(X_n) \to C(X) \quad i_{n,\infty}u(x) := \begin{cases} u(x) & \text{if } x \in X_n, \\ 0 & \text{if } x \in X \setminus X_n; \end{cases}
\]

\[
\pi_n: C(X) \to C(X_n) \quad \pi_n u(x) := u(x) \text{ for every } x \in X_n.
\]

We remark that, for the purpose of the results collected here, the exhaustion \( \{X_n\}_{n=1}^{\infty} \) is not required to satisfy any additional properties other than that each \( X_n \) is finite.

We recall that on a graph \( G = (X, w, \kappa, \mu) \) the operator \( \mathcal{L} : \text{dom}(\mathcal{L}) \subseteq \ell^1(X, \mu) \to \ell^1(X, \mu) \) is given by

\[
\text{dom}(\mathcal{L}) := \{ u \in \ell^1(X, \mu) \mid \Phi u \in \text{dom}(\Delta), \Delta \Phi u \in \ell^1(X, \mu) \}
\]

\[
\mathcal{L} u := \Delta \Phi u.
\]

For a subset \( \Omega \subseteq \text{dom}(\mathcal{L}) \), we write \( \mathcal{L}|_\Omega \) for the restriction of \( \mathcal{L} \) to \( \Omega \).

We first introduce a sequence of operators \( \mathcal{L}_n \) whose purpose is to ‘nicely approximate’ the operator \( \mathcal{L} \).

**Definition B.1** (The operators \( \mathcal{L}_n \)) Let \( G = (X, w, \kappa, \mu) \) be a graph. We define

\[
\mathcal{L}_n : \text{dom}(\mathcal{L}_n) \subseteq \ell^1(X, \mu) \to \ell^1(X, \mu)
\]

by

\[
\text{dom}(\mathcal{L}_n) := C_c(X), \quad \mathcal{L}_n u := i_{n,\infty} \Delta_{\text{dir},n} \Phi \pi_n u
\]

where \( \Delta_{\text{dir},n} \) is the graph Laplacian associated to the Dirichlet subgraph \( G_{\text{dir},n} \subseteq G \) on the node set \( X_n \).
We are going to prove that $\mathcal{L}_n$ is accretive for every $n$. This result will be a consequence of the next proposition for finite graphs.

**Proposition B.2** Let $G = (X, w, \kappa, \mu)$ be a finite graph. Then,

$$\sum_{x \in X: u(x) \neq v(x)} (\Delta \Phi u(x) - \Delta \Phi v(x)) \sgn(u(x) - v(x)) \mu(x) \geq 0 \ \forall \ u, v \in C(X).$$

**Proof** Define $h := \Phi u - \Phi v \in C(X)$. Since $\phi$ is strictly monotone increasing and $\phi(0) = 0$

\[ \sgn(u(x) - v(x)) = \sgn(\phi(u(x)) - \phi(v(x))) = \sgn(h(x)) \quad (B.1) \]

and, therefore,

\[ (\phi(u(x)) - \phi(v(x))) \sgn(u(x) - v(x)) = h(x) \sgn(h(x)) = |h(x)| \geq 0 \ \forall \ x \in X. \quad (B.2) \]

By the linearity of $\Delta$ and (B.1) we get

$$\sum_{x \in X: u(x) \neq v(x)} (\Delta \Phi u(x) - \Delta \Phi v(x)) \sgn(u(x) - v(x)) \mu(x) = \sum_{x \in X} \Delta h(x) \sgn(h(x)) \mu(x).$$

Since $G$ is finite, from the Green’s identity, see [37, 44], we get

$$\sum_{x \in X} \Delta h(x) \sgn(h(x)) \mu(x) = \frac{1}{2} \sum_{x, y \in X} w(x, y) (\sgn(h(x)) - \sgn(h(y))) (h(x) - h(y))$$

$$+ \sum_{x \in X} \kappa(x) h(x) \sgn(h(x)).$$

Combining the above identity with (B.2), we obtain

$$\sum_{x \in X} \Delta h(x) \sgn(h(x)) \mu(x) \geq \frac{1}{2} \sum_{x, y \in X} w(x, y) (\sgn(h(x)) - \sgn(h(y))) (h(x) - h(y)).$$

Setting, for ease of notation,

\[ \Gamma(x, y) := w(x, y) (\sgn(h(x)) - \sgn(h(y))) (h(x) - h(y)) \]

we have:

(i) If $\sgn(h(x)) = 0$, then $\Gamma(x, y) = w(x, y)|h(y)| \geq 0$;

(ii) If $\sgn(h(y)) = 0$, then $\Gamma(x, y) = w(x, y)|h(x)| \geq 0$;

(iii) If $\sgn(h(x)) = \sgn(h(y))$, then $\Gamma(x, y) = 0$;

(iv) If $\sgn(h(y)) = -\sgn(h(x))$, then $\Gamma(x, y) = 2w(x, y) (|h(x)| + |h(y)|) \geq 0$.

We obtain $\Gamma(x, y) \geq 0$ for every $x, y \in X$ and the required conclusion follows. \qed

We next show that $\mathcal{L}$ is accretive on finite graphs.

**Corollary B.3** Let $G = (X, w, \kappa, \mu)$ be a finite graph. Then, $\mathcal{L}$ is accretive.

**Proof** By condition (ii) in Definition 2.4, an operator $\mathcal{L}$ is accretive if $\langle \mathcal{L} u - \mathcal{L} v, u - v \rangle_+ \geq 0$ for every $u, v \in \text{dom}(\mathcal{L})$. From (2.5) in Remark 3, in the case of the $\ell^1$-norm we have

$$\langle z, k \rangle_+ = \|k\|_1 \left( \sum_{x \in X: k(x) = 0} |z(x)| \mu(x) + \sum_{x \in X: k(x) \neq 0} z(x) \sgn(k(x)) \mu(x) \right).$$
\[ \geq \|k\|_1 \sum_{x \in X: k(x) \neq 0} z(x)\text{sgn}(k(x))\mu(x) \quad \forall z, k \in \ell^1(X, \mu). \]

Therefore, to prove that \( \mathcal{L} \) is accretive on \( \ell^1(X, \mu) \), it is sufficient to prove that
\[
\sum_{x \in X: u(x) \neq v(x)} (\mathcal{L}u(x) - \mathcal{L}v(x)) \text{sgn}(u(x) - v(x))\mu(x) \geq 0 \quad \forall u, v \in \text{dom}(\mathcal{L}),
\]
that is,
\[
\sum_{x \in X: u(x) \neq v(x)} (\Delta \Phi u(x) - \Delta \Phi v(x)) \text{sgn}(u(x) - v(x))\mu(x) \geq 0 \quad \forall u, v \in C(X).
\]

Since \( G \) is finite, we conclude the proof by Proposition B.2. \( \square \)

We next establish that the operators \( \mathcal{L}_n \) are accretive.

**Corollary B.4** Let \( G = (X, w, \kappa, \mu) \) be a graph. Then, \( \mathcal{L}_n \) is accretive for every \( n \).

**Proof** By (B.3) in Corollary B.3, it suffices to show that
\[
\sum_{x \in X: u(x) \neq v(x)} (\mathcal{L}_n u(x) - \mathcal{L}_n v(x)) \text{sgn}(u(x) - v(x))\mu(x) \geq 0 \quad \forall u, v \in \text{dom}(\mathcal{L}_n),
\]
that is,
\[
\sum_{x \in X: u(x) \neq v(x)} (\Delta_{dir,n} \Phi \pi_n u(x) - \Delta_{dir,n} \Phi \pi_n v(x)) \text{sgn}(u(x) - v(x))\mu(x) \geq 0 \quad \forall u, v \in C_c(X).
\]

Let us observe that the left-hand side of the above is equal to
\[
\sum_{x \in X: \pi_n u(x) \neq \pi_n v(x)} (\Delta_{dir,n} \Phi \pi_n u(x) - \Delta_{dir,n} \Phi \pi_n v(x)) \text{sgn}(\pi_n u(x) - \pi_n v(x))\mu(x).
\]
Since \( \pi_n u, \pi_n v \in C(X_n) \) and \( \Delta_{dir,n} \) is the graph Laplacian associated to the finite graph \( G_{dir,n} \) with node set \( X_n \), by Proposition B.2 we conclude that \( \mathcal{L}_n \) is accretive. \( \square \)

The sequence of operators \( \mathcal{L}_n \) defines a subset \( \Omega \) of \( \text{dom}(\mathcal{L}) \). As we will see below, \( \Omega \) is dense in \( \text{dom}(\mathcal{L}) \) and \( \mathcal{L} \) restricted to \( \Omega \) is accretive. Let us introduce the following notation for the support of a function: Given \( u \in C(X) \) we let
\[ \text{supp } u := \{ x \in X \mid u(x) \neq 0 \}. \]

We start by defining the subset of the domain of interest.

**Definition B.5** (The set \( \Omega \)) Let \( G = (X, w, \kappa, \mu) \) be a graph. We define \( \Omega \subseteq \text{dom}(\mathcal{L}) \) by letting
\[ \Omega := \text{dom}(\mathcal{L}) = C(X) \]
if \( G \) is finite and
\[ \Omega := \{ u \in \text{dom}(\mathcal{L}) \mid \exists \{u_n\}_n \text{ s.t. supp } u_n \subseteq X_n, \lim_{n \to \infty} \|u_n - u\| = 0, \lim_{n \to \infty} \|\mathcal{L}_n u_n - \mathcal{L} u\| = 0 \} \]
if \( G \) is infinite.
While the definition of $\Omega$ depends on the choice of the exhaustion, this set always contains all finitely supported functions as will be shown in Lemma B.7. In order to establish this, we first prove that the finitely supported functions are contained in the domain of $\mathcal{L}$.

**Lemma B.6** Let $G = (X, w, \kappa, \mu)$ be a graph. Then, $C_c(X) \subseteq \text{dom}(\mathcal{L})$.

**Proof** Let $u \in C_c(X)$. Then, $u(x) = \sum_{j=1}^{n} \alpha_j \delta_{x_j}(x)$ where $\alpha_j \in \mathbb{R}$ and

$$\delta_{x_j}(x) = \begin{cases} 1 & \text{if } x = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by linearity, $\Delta(C_c(X)) \subseteq \ell^1(X, \mu)$ if and only if $\Delta \delta_z \in \ell^1(X, \mu)$ for every $z \in X$. Fix $z \in X$ and observe that

$$\sum_{x \in X} |\Delta \delta_z(x)| \mu(x) \leq \sum_{x \in X} \text{Deg}(x) \delta_z(x) \mu(x) + \sum_{x \in X} \sum_{y \in X} w(x, y) |\delta_z(y)|$$

$$= \text{Deg}(z) \mu(z) + \sum_{x \in X} w(x, z) < \infty$$

so that $\Delta(C_c(X)) \subseteq \ell^1(X, \mu)$. Since $\Phi u \in C_c(X)$ for every $u \in C_c(X)$, it follows that $\Phi u \in \text{dom}(\Delta)$ and $\Delta \Phi u \in \ell^1(X, \mu)$, that is, $u \in \text{dom}(\mathcal{L})$. \hfill $\square$

We now show that the set $\Omega$ contains the finitely supported functions.

**Lemma B.7** Let $G = (X, w, \kappa, \mu)$ be a graph. Then, $C_c(X) \subseteq \Omega$. In particular,

$$\overline{\Omega} = \text{dom}(\mathcal{L}) = \ell^1(X, \mu).$$

**Proof** Let us fix $u \in C_c(X)$. From Lemma B.6, we know that $u \in \text{dom}(\mathcal{L})$. Define

$$u_n(x) := \mathbf{i}_{n, \infty} \pi_n u(x) = \begin{cases} u(x) & \text{if } x \in X_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{supp } u_n \subseteq X_n$ and $\lim_{n \to \infty} \|u_n - u\| = 0$. Since, $u \in C_c(X)$, there exists an $N > 0$ such that $u(x) = 0$ for every $x \in X \setminus X_N$. In particular, $\Phi u_n(x) = 0$ for every $x \in X \setminus X_n$, $n \geq N$ and $\Phi u_n = \Phi u$ for every $n \geq N$.

By Lemma A.1, we have $\Phi u_n \in \text{dom}(\Delta|_{X_n})$ for every $n \geq N$ and then

$$\Delta_{\text{dir},n} \pi_n \Phi u_n(x) = \Delta \Phi u_n(x) = \Delta \Phi u(x) \quad \forall x \in X_n, \ n \geq N,$$

that is,

$$\Delta_{\text{dir},n} \pi_n \Phi u_n = \pi_n \Delta \Phi u \quad \forall n \geq N.$$

Therefore, using the trivial fact that $\Phi \pi_n = \pi_n \Phi$, $\mathcal{L} u_n = \mathbf{i}_{n, \infty} \Delta_{\text{dir},n} \Phi \pi_n u_n = \mathbf{i}_{n, \infty} \Delta_{\text{dir},n} \pi_n \Phi u_n = \mathbf{i}_{n, \infty} \pi_n \Delta \Phi u = \mathbf{i}_{n, \infty} \pi_n \mathcal{L} u, \ \forall n \geq N$

and, since $\mathcal{L} u \in \ell^1(X, \mu)$, it follows that $\lim_{n \to \infty} \|\mathcal{L} u_n - \mathcal{L} u\| = 0$ by dominated convergence. \hfill $\square$

To conclude this appendix, we prove that $\mathcal{L}|_{\Omega}$ is accretive.

**Lemma B.8** Let $G = (X, w, \kappa, \mu)$ be a graph. Then, $\mathcal{L}|_{\Omega}$ is accretive.
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**Proof** If $G$ is finite, then $\Omega = \text{dom}(\mathcal{L})$ and $\mathcal{L}$ is accretive by Corollary B.3. If $G$ is infinite, let $u, v \in \Omega$. Then, by the definition of $\Omega$, there exists $\{u_n\}_n, \{v_n\}_n$ such that

$$\lim_{n \to \infty} \|u_n - u\| = \lim_{n \to \infty} \|v_n - v\| = 0, \quad \lim_{n \to \infty} \|\mathcal{L}_n u_n - \mathcal{L} u\| = \lim_{n \to \infty} \|\mathcal{L}_n v_n - \mathcal{L} v\| = 0.$$

By the accretivity of $\mathcal{L}_n$ established in Corollary B.4 above, it follows readily that

$$\|(u - v) + \lambda (\mathcal{L} u - \mathcal{L} v)\| = \lim_{n \to \infty} \|(u_n - v_n) + \lambda (\mathcal{L}_n u_n - \mathcal{L}_n v_n)\| \geq \lim_{n \to \infty} \|u_n - v_n\| = \|u - v\|.$$

This completes the proof. \(\square\)

**Remark 10** One might be tempted to identify $\Omega$ with

$$\Omega' = \{u \in \text{dom}(\mathcal{L}) \mid \exists \{u_n\}_n \text{ s.t. } u_n \in C_c(X), \lim_{n \to \infty} \|u_n - u\| = 0, \lim_{n \to \infty} \|\mathcal{L}_{\min} u_n - \mathcal{L} u\| = 0\}$$

where $\mathcal{L}_{\min}$ is the minimal operator, that is, $\mathcal{L}_{\min} := \mathcal{L}|_{C_c(X)}$. It is possible to show that $\mathcal{L}$ is accretive on $\Omega'$ but unfortunately $\Omega$ is not equal to $\Omega'$. In particular, the solutions that are constructed in the proof of Theorem 1 may not belong to $\Omega'$. The main problem is that

$$|(id + \lambda \mathcal{L}_{\min}) u_n(x) - g(x)| = \begin{cases} 0 & \text{if } x \in X_n, \\ \sum_{y \in X_n} w(x, y) u_n(y) + g(x) & \text{if } x \in X \setminus X_n \end{cases}$$

with

$$|(id + \lambda \mathcal{L}_{\min}) u_n(x) - g(x)| = \begin{cases} 0 & \text{if } x \in X_n, \\ |\lambda \sum_{y \in X_n} w(x, y) u_n(y) + g(x)| & \text{if } x \in X \setminus X_n \end{cases}$$

and then $|(id + \lambda \mathcal{L}_{\min}) u_n - g|$ does not necessarily tend to 0. As a consequence, we cannot infer that $\lim_{n \to \infty} \|\mathcal{L}_{\min} u_n - \mathcal{L} u\| = 0$.

**References**

1. Andreu-Vaillo, F., Caselles, V., Mazón, J.M.: Parabolic Quasilinear Equations Minimizing Linear Growth Functionals. Vol. 223. Birkhäuser Basel, (2004). [https://doi.org/10.1007/978-3-0348-7928-6](https://doi.org/10.1007/978-3-0348-7928-6)
2. Anné, C., Balti, M., Torki-Hamza, N.: m-accretive Laplacian on a non symmetric graph. Indag. Math. 31, 277–293 (2020). [https://doi.org/10.1016/j.indag.2020.01.005](https://doi.org/10.1016/j.indag.2020.01.005)
3. Barbu, V.: Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer Monographs in Mathematics. Springer, New York (2010). [https://doi.org/10.1007/978-1-4419-5542-5](https://doi.org/10.1007/978-1-4419-5542-5)
4. Bénilan, P.: “Equations d’évolution dans un espace de Banach quelconque et applications”. Thesis (Ph.D.-Orsay, France: Université Paris-Sud, (1972)
5. Bénilan, P., Brézis, H.: Solutions faibles d’équations d’évolution dans les espaces de Hilbert. Ann. Inst. Fourier 22(2), 311–329 (1972). [https://doi.org/10.5802/afif.421](https://doi.org/10.5802/afif.421)
6. Bénilan, P., Crandall, M.G., Pazy, A.: “Nonlinear evolution equations in Banach spaces”. Preprint. (1988)
7. Bianchi, D., Setti, A.G.: Laplacian cut-offs, porous and fast diffusion on manifolds and other applications. Calc. Var. Partial Differ. Equ. 57(4), 1–33 (2018). [https://doi.org/10.1007/s00526-017-1267-9](https://doi.org/10.1007/s00526-017-1267-9)
8. Bonforte, M., Grillo, G., Vazquez, J.L.: Fast diffusion flow on manifolds of nonpositive curvature. J. Evol. Equ. 8, 99–128 (2008). [https://doi.org/10.1007/s00028-007-0345-4](https://doi.org/10.1007/s00028-007-0345-4)
9. Brézis, H., Strauss, W.A.: Semi-linear second-order elliptic equations in $L^1$. J. Math. Soc. Japan 25(4), 565–590 (1973). [https://doi.org/10.2969/JMSJ/02540565](https://doi.org/10.2969/JMSJ/02540565)
10. Cazenave, T., Haraux, A.: An Introduction to Semilinear Evolution Equations. Revised Edition. Vol. 13. Oxford Lecture Series in Mathematics and its Applications. Clarendon Press (1998)
11. Ceccherini-Silberstein, T., Coornaert, M., Dodziuk, J.: The surjectivity of the combinatorial Laplacian on infinite graphs. Enseign. Math. 58(2), 125–130 (2012)
12. Collatz, L.: Functional Analysis and Numerical Mathematics. Academic Press, New York (1966)
13. Crandall, M.G.: Nonlinear semigroups and evolution governed by accretive operators. In: Nonlinear Functional Analysis and its Applications (Berkeley, California, 1983). Browder, F.E. (ed.), Vol. 45. Proceedings of the Symposium in Pure Mathematics I. American Mathematical Society. pp. 305–337 (1986). https://doi.org/10.1090/pspum/045.1
14. Crandall, M.G., Bénilan, P.: Regularizing effects of homogeneous evolution equations. Tech. rep. 2076. Wisconsin Univ–Madison Mathematics Research Center, pp. 1–23 (1980)
15. Crandall, M.G., Liggett, T.M.: Generation of semi-groups of nonlinear transformations on general banach spaces. Amer. J. Math. 93(2), 265–298 (1971). https://doi.org/10.2307/2373376
16. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985). https://doi.org/10.1007/978-3-662-00547-7
17. Deo, N.: Graph Theory with Applications to Engineering & Computer Science. Dover Publications, New York (2016)
18. Dipierro, S., Gao, Z., Valdinoci, E.: Global gradient estimates for nonlinear parabolic operators. ESAIM Control Optim. Calc. Var. 27(21), 1–37 (2021). https://doi.org/10.1051/cocv/2021016
19. Elmoataz, A., Toutain, M., Tenbrinck, D.: On the p-Laplacian and ∞-Laplacian on graphs with applications in image and data processing. SIAM J. Imag. Sci. 8, 2412–2451 (2015). https://doi.org/10.1137/15M1022793
20. Erbar, M., Maas, J.: Gradient flow structures for discrete porous medium equations. Discrete Contin. Dyn. Syst. 34(4), 1355–1374 (2014). https://doi.org/10.3934/dcds.2014.34.1355
21. Estrada, E., Knight, P.A.: A First Course in Network Theory. Oxford University Press, New York (2015)
22. Evans, L.C.: Nonlinear evolution equations in an arbitrary Banach space. Israel J. Math. 26, 1–42 (1977). https://doi.org/10.1007/BF03007654
23. Grigor’yan, A., Lin, Y., Yang, Y.: Existence of positive solutions to some nonlinear equations on locally finite graphs. Sci. China Math. 60, 1311–1324 (2017). https://doi.org/10.1007/s11425-016-0422-y
24. Grigor’yan, A., Lin, Y., Yang, Y.: Kazdan-Warner equation on graph. Calc. Var. Partial Differ. Equ. 55(92), 1–13 (2016). https://doi.org/10.1007/s00526-016-1042-3
25. Grigor’yan, A., Lin, Y., Yang, Y.: Yamabe type equations on graphs. J. Differ. Equ. 261, 4924–4943 (2016). https://doi.org/10.1016/j.jde.2016.07.011
26. Grillo, G., Ishige, K., Muratori, M.: Nonlinear characterizations of stochastic completeness. J. Math. Pures Appl. 139, 63–82 (2020). https://doi.org/10.1016/j.matpur.2020.05.008
27. Grillo, G., Merlini, G., Punzo, F.: Global existence of solutions and smoothing effects for classes of reaction-diffusion equations on manifolds. J. Evol. Equ. 21, 2339–2375 (2021). https://doi.org/10.1007/s00028-021-00685-3
28. Grillo, G., Muratori, M.: Radial fast diffusion on the hyperbolic space. Proc. Lond. Math. Soc. 109, 283–317 (2014). https://doi.org/10.1112/plms/pdt071
29. Grillo, G., Muratori, M.: Smoothing effects for the porous medium equation on Cartan-Hadamard manifolds. Nonlinear Anal. 131, 346–362 (2016). https://doi.org/10.1016/j.na.2015.07.029
30. Grillo, G., Muratori, M., Punzo, F.: Blow-up and global existence for the porous medium equation with reaction on a class of Cartan-Hadamard manifolds. J. Differ. Equ. 266, 4305–4336 (2019). https://doi.org/10.1016/j.jde.2018.09.037
31. Grillo, G., Muratori, M., Punzo, F.: Fast diffusion on noncompact manifolds: Wellposedness theory and connections with semilinear elliptic equations. Trans. Amer. Math. Soc. 374, 6367–6396 (2021). https://doi.org/10.1090/tran/8431
32. Grillo, G., Muratori, M., Punzo, F.: The porous medium equation with large initial data on negatively curved Riemannian manifolds. J. Math. Pures Appl. 113, 195–226 (2018). https://doi.org/10.1016/j.matpur.2017.07.021
33. Güneysu, B., Keller, M., Schmidt, M.: A Feynman-Kac-Itô formula for magnetic Schrödinger operators on graphs. Probab. Theory Related Fields 165, 365–399 (2016). https://doi.org/10.1007/s00440-015-0633-9
34. Güneysu, B., Milatovic, O., Truc, F.: Generalized Schrödinger semigroups on infinite graphs. Potent. Anal. 41, 517–541 (2014). https://doi.org/10.1007/s11118-013-9381-6
37. Haeseler, S., Keller, M.: Generalized solutions and spectrum for Dirichlet forms on graphs. In: Lenz, D., Sobieczky, F.W.W. (eds.), Random Walks, Boundaries and Spectra. Vol. 64. Progress in Probability. Springer, Basel, pp. 181–199 (2011). https://doi.org/10.1007/978-3-0346-0244-0_10
38. Haeseler, S., Keller, M., Lenz, D., Wojciechowski, R.K.: Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. J. Spectr. Theory 2, 397–432 (2012). https://doi.org/10.4171/JST/35
39. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2012). https://doi.org/10.1017/CBO97805118111800187
40. Hua, B., Mugnolo, D.: Time regularity and long-time behavior of parabolic p-Laplace equations on infinite graphs. J. Differ. Equ. 259, 6162–6190 (2015). https://doi.org/10.1016/j.jde.2015.07.018
41. Huang, X., Keller, M., Masamune, J., Wojciechowski, R.K.: A note on self-adjoint extensions of the Laplacian on weighted graphs. J. Funct. Anal. 265, 1556–1578 (2013). https://doi.org/10.1016/j.jfa.2013.06.004
42. Kato, T.: Perturbation Theory for Linear Operators, 2nd edn. Springer, Berlin (1995). https://doi.org/10.1007/978-3-642-66282-9
43. Keller, M., Lenz, D.: Dirichlet forms and stochastic completeness of graphs and subgraphs. J. Reine Angew. Math. 666, 189–223 (2012). https://doi.org/10.1515/CRELLE.2011.122
44. Keller, M., Lenz, D., Wojciechowski, R.K.: Graphs and Discrete Dirichlet Spaces. Grundlehren der mathematischen Wissenschaften. Springer, Cham (2021). https://doi.org/10.1007/978-3-0348-0738-7
45. Keller, M., Schwarz, M.: The Kazdan-Warner equation on canonically compactifiable graphs. Calc. Var. Partial Differ. Equ. 57(70), 1–18 (2018). https://doi.org/10.1007/s00526-018-1329-7
46. Kobayashi, K., Kobayashi, Y., Oharu, S.: Nonlinear evolution operators in Banach spaces. Osaka J. Math. 21, 281–310 (1984)
47. Koberstein, J., Schmidt, M.: A note on the surjectivity of operators on vector bundles over discrete spaces. Arch. Math. 114, 313–329 (2020). https://doi.org/10.1007/s00013-019-01412-8
48. Lakshmikantham, V., Leela, S.: Nonlinear differential equations in abstract spaces. Vol. 2. International Series in Nonlinear Mathematics: Theory, Methods and Applications. Pergamon International Library, (1981). https://doi.org/10.1016/C2013-0-11031-8
49. Lenz, D., Schmidt, M., Zimmermann, I.: Blow-up of nonnegative solutions of an abstract semilinear heat equation with convex source. (2021). arXiv:2108.11291 [math.AP]
50. Lesne, A.: Complex networks: from graph theory to biology. Lett. Math. Phys. 78, 235–262 (2006). https://doi.org/10.1007/s11005-006-0123-1
51. Lieberman, E., Hauert, C., Nowak, M.A.: Evolutionary dynamics on graphs. Nature 433, 312–316 (2005). https://doi.org/10.1038/nature03204
52. Lin, Y., Wu, Y.: Blow-up problems for nonlinear parabolic equations on locally finite graphs. Acta Math. Sci. Ser. B (Engl. Ed.) 38, 833–856 (2018). https://doi.org/10.1016/S0252-9602(18)30788-4
53. Lin, Y., Wu, Y.: The existence and nonexistence of global solutions for a semilinear heat equation on graphs. Calc. Var. Partial Differ. Equ. 56(102), 1–22 (2017). https://doi.org/10.1007/s00526-017-1204-y
54. Lin, Y., Yang, Y.: A heat flow for the mean field equation on a finite graph. Calc. Var. Partial Differ. Equ. 60(206), 1–15 (2021). https://doi.org/10.1007/s00526-021-02086-3
55. Liu, S., Yang, Y.: Multiple solutions of Kazdan-Warner equation on graphs in the negative case. Calc. Var. Partial Differ. Equ. 59(164), 1–15 (2020). https://doi.org/10.1007/s00526-020-01840-3
56. Lu, P., Ni, L., Vázquez, J.L., Villani, C.: Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds. J. Math. Pures Appl. 91, 1–19 (2009). https://doi.org/10.1016/j.matpur.2008.09.001
57. Meglioli, G., Punzo, F.: Blow-up and global existence for solutions to the porous medium equation with reaction and fast decaying density. Nonlinear Anal. 203, 112187 (2021). https://doi.org/10.1016/j.na.2020.112187
58. Milatovic, O.: Essential self-adjointness of magnetic Schrödinger operators on locally finite graphs. Integral Equ. Oper. Theory 71(13), 13–27 (2011). https://doi.org/10.1007/s00020-011-1882-3
59. Milatovic, O., Truc, F.: Maximal accretive extensions of Schrödinger operators on vector bundles over infinite graphs. Integral Equ. Oper. Theory 81, 35–52 (2015). https://doi.org/10.1007/s00020-014-2196-z
60. Milatovic, O., Truc, F.: Self-Adjoint Extensions of Discrete Magnetic Schrödinger Operators. Ann. Henri Poincaré 15, 917–936 (2014). https://doi.org/10.1007/s00023-013-0261-9
61. Mugnolo, D.: Parabolic theory of the discrete p-Laplace operator. Nonlinear Anal. 87, 33–60 (2013). https://doi.org/10.1016/j.na.2013.04.002
62. Nakanishi, N.: Graph Theory and Feynman Integrals. Vol. 11. Mathematics and Its Applications. Gordon and Breach (1971)
63. Schmidt, M.: On the existence and uniqueness of self-adjoint realizations of discrete (magnetic) Schrödinger operators. In: Analysis and geometry on graphs and manifolds. Vol. 461. London Math. Soc. Lecture Note Ser. Cambridge University Press, Cambridge, pp. 250–327 (2020)
64. Shuman, D.I., Narang, S.K., Frossard, P., Ortega, A., Vandergheynst, P.: The emerging field of signal processing on graphs: extending high-dimensional data analysis to networks and other irregular domains. IEEE Signal Process. Mag. 30, 83–98 (2013). https://doi.org/10.1109/Msp.2012.2235192
65. Slavík, A., Stehlík, P., Volek, J.: Well-posedness and maximum principles for lattice reaction-diffusion equations. Adv. Nonlinear Anal. 8, 303–322 (2019). https://doi.org/10.1515/anona-2016-0116
66. Stam, C.J., Reijneveld, J.C.: Graph theoretical analysis of complex networks in the brain. Nonlinear Biomed. Phys. 1(3), 1–19 (2007). https://doi.org/10.1186/1753-4631-1-3
67. Stehlík, P.: Exponential number of stationary solutions for Nagumo equations on graphs. J. Math. Anal. Appl. 455, 1749–1764 (2017). https://doi.org/10.1016/j.jmaa.2017.06.075
68. Ta, V.-T., Elmoataz, A., Lézoray, O.: Nonlocal PDEs-Based Morphology on Weighted Graphs for Image and Data Processing. IEEE Trans. Image Process. 20, 1504–1516 (2011). https://doi.org/10.1109/TIP. 2010.2101610
69. Vázquez, J.L.: Fundamental solution and long time behavior of the Porous Medium Equation in hyperbolic space. J. Math. Pures Appl. 104, 454–484 (2015). https://doi.org/10.1016/j.matpur.2015.03.005
70. Vázquez, J.L.: The Porous Medium Equation: Mathematical Theory. Oxford Scholarship Online (2006). https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
71. Willson, A.N., Jr.: On the solutions of equations for nonlinear resistive networks. AT&T Bell Labs. Tech. J. 47, 1755–1773 (1968). https://doi.org/10.1002/j.1538-7305.1968.tb00101.x
72. Wojciechowski, R.K.: Stochastic completeness of graphs. Thesis (Ph.D.)-City University of New York. ProQuest LLC, Ann Arbor, MI (2008). ISBN: 978-0549-58579-4
73. Wu, Y.: Blow-up for a semilinear heat equation with Fujita’s critical exponent on locally finite graphs. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 115(133), 1–16 (2021). https://doi.org/10.1007/s13398-021-01075-7
74. Yosida, K.: Functional Analysis, 6th edn. Springer, Berlin (1995). https://doi.org/10.1007/978-3-642-61859-8

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