Short-range dependent processes subordinated to the Gaussian may not be strong mixing

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| Version          | First author draft                                                                                                                                                                                                 |
|------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Citation (published version): | Shuyang Bai, Murad S Taqqu. 2016. "Short-range dependent processes subordinated to the Gaussian may not be strong mixing." STATISTICS & PROBABILITY LETTERS, Volume 110, pp. 198 - 200 (3). https://doi.org/10.1016/j.spl.2015.12.010 |

https://hdl.handle.net/2144/37411

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Short-range dependent processes subordinated to the Gaussian may not be strong mixing

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August 22, 2018

Abstract

There are all kinds of weak dependence. For example, strong mixing. Short-range dependence (SRD) is also a form of weak dependence. It occurs in the context of processes that are subordinated to the Gaussian. Is a SRD process strong mixing if the underlying Gaussian process is long-range dependent? We show that this is not necessarily the case.

Let \( \{Z_i\} \) be a standardized Gaussian process with covariance function \( \gamma(n) = n^{2H-2}L(n) \), where \( 1/2 < H < 1 \) and \( L(n) \) is slowly varying. We will consider instantaneous transformations \( X_i = P(Z_i) \), where \( \mathbb{E}P(Z_i)^2 < \infty \).

The sequence \( \{X_i\} \) is said to be LRD if the sum of its covariances diverges and SRD if the sum converges. Note that the sequence \( \{Z_i\} \) is LRD because \( \sum_{n=-\infty}^{+\infty} \gamma(n) = \infty \). The sequence \( \{X_i\} \), however, may be LRD or SRD depending on \( P(\cdot) \).

Suppose now that \( P(\cdot) \) is a finite-order polynomial. It can then be expressed as

\[
P(x) = c_0 + \sum_{k=m}^{n} c_k H_k(x), \quad 1 \leq m \leq n,
\]

with \( c_m \neq 0 \), where \( H_k(x) \) is the \( k \)-th order Hermite polynomial. The bottom index \( m \) is called the Hermite rank of \( P(x) \) and/or of the process \( \{P(X_i)\} \).

It is known from Breuer and Major [1] that when

\[
(2H - 2)m + 1 < 0,
\]

which can only happen when \( m \geq 2 \), then \( \{X_i\} \) is SRD and as \( N \to \infty \),

\[
N^{-1/2} \sum_{i=1}^{[N]} [P(Z_i) - \mathbb{E}P(Z_i)] \stackrel{f.d.d.}{\to} \sigma B(t),
\]

where \( \sigma^2 = \sum \gamma(n) \), \( B(t) \) is the standard Brownian motion and \( f.d.d. \) denotes convergence of finite-dimensional distributions. This seems to suggest that \( \{P(Z_i)\} \) has weak dependence. It is natural to ask whether \( \{P(Z_i)\} \) is strong mixing. We will show that this may not be the case.

Key words long-range dependence, short-range dependence, Hermite rank, strong mixing.

2010 AMS Classification: 60G18

1 A stationary process \( \{X_i\} \) is said to be strong mixing if

\[
\lim_{k \to \infty} \sup_{A \in \mathcal{F}_{-\infty}^0} \{P(A)P(B) - P(A \cap B), \ A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_k^\infty \} = 0,
\]

where \( \mathcal{F}_a^b \) is the \( \sigma \)-field generated by \( X_a, \ldots, X_b \).
**Theorem 1.** Suppose that \( \{Z_i\} \) is LRD with covariance \( \gamma(n) = n^{2H-2}L(n) \), where \( H \) satisfies (1). The SRD process \( \{X_i = P(Z_i)\} \) is not strong mixing if there exists a polynomial \( Q(x) \) such that the Hermite rank \( m' \) of \( Q(P(x)) \) satisfies

\[
(2H - 2)m' + 1 > 0.
\]

**Remark 2.** The process \( \{X_i = P(Z_i)\} \) in the theorem is SRD. The theorem states that this process is not strong mixing if there is a polynomial \( Q(x) \) such that the new process \( \{Q(P(Z_i))\} \) is LRD. Note that (2) implies, in view of (1), that \( m' < m \).

**Proof.** We argue by contradiction. Suppose that \( \{X_i\} \) is strong mixing. Then by the definition of strong mixing, \( \{Q(X_i)\} \) is also strong mixing. But (2) implies that (Taqqu [4])

\[
s_N^2 := \text{Var} \left[ \sum_{i=1}^{N} Q(X_i) \right] \sim c_H L(N)^{m'} N^{(2H-2)m'+2} \quad (2H - 2)m' + 2 > 1.
\]

On the other hand, \( S_N := \sum_{i=1}^{N} [Q(X_i) - \mathbb{E}Q(X_i)] \) is an element living on Wiener chaos of a finite order (see Janson [2], Chapter 2). By Janson [2], Theorem 5.10, for any \( p > 1 \), there exists a constant \( c_p > 0 \) depending only on \( p \), such that

\[
\mathbb{E} \left| s_N^{-1} S_N \right|^p \leq c_p \left( \mathbb{E} \left| s_N^{-1} S_N^2 \right|^{p/2} \right)^{p/2} = c_p.
\]

Therefore \( s_N^{-2} s_N^2, N \geq 2 \) is uniformly integrable. By Theorem 1.3 of Peligrad [3], strong mixing and uniform integrability imply that

\[
S_N^2 = l(N)N
\]

for some slowly varying function \( l(N) \). This contradicts (3).

In some cases, no polynomial \( Q(x) \) satisfies the requirement of Theorem 1. For example, when \( P(x) = x^2 \), then the Hermite rank \( m = 2 \), and one always has

\[
\mathbb{E}Q(Z^2)H_1(Z) = \mathbb{E}Q(Z^2)Z = 0
\]

for arbitrary polynomials \( Q(x) \) (in fact for arbitrary \( L^2(\Omega) \) functions). This is because \( Q(Z^2) \) is an even function of \( Z \). So the Hermite rank of \( Q(P(x)) \) is at least 2, and hence we don’t have \( m' < m \).

In the simple case where \( P(x) \) is a Hermite polynomial, we have the following result:

**Proposition 3.** Suppose \( P(x) = H_m(x), m \in \mathbb{Z}_+ \). The polynomial \( Q(x) \) required in Theorem 1 exists in either of the following cases:

(a) \( m \geq 4 \) is even and \( H > 3/4 \).

(b) \( m \geq 3 \) is odd.

**Proof.** Using the product formula ((3.13) of Janson [2]) for Hermite polynomial, one has

\[
H_m(x)^2 = \sum_{k=0}^{m} \frac{m!}{k!} H_{2m-2k}(x),
\]

\[
H_m(x)^3 = \sum_{k_1=0}^{m} \sum_{k_2=0}^{(2m-2k_1) \wedge m} \frac{m!}{k_1!k_2!} \left( \frac{m}{k_1} \right)^2 \left( \frac{2m - 2k_1}{k_2} \right) H_{3m-2k_1-2k_2}(x).
\]

For case (a), choose \( 3/4 < H < 1 \), but not too big such that \( \{P(X_i) = H_m(X_i)\} \) is SRD. This will happen by constraining \( H \) to satisfy (1). Now choose \( Q(x) = x^2 \). Then by (1),

\[
Q(P(x)) = H_m(x)^2 = m! + (m - 1)! m^2 H_2(x) + \ldots,
\]
so \(\{Q(P(Z_i))\}\) has Hermite rank \(m' = 2\), which is less than \(m \geq 4\). Since \(m' = 2\), and \(H > 3/4\), we conclude that \(\{Q(P(Z_i))\}\) is LRD and satisfies (2).

For case (b), choose \(Q(x) = x^3\). Then

\[
Q(P(x)) = H_m(x)^3 = a_1 H_1(x) + \ldots
\]

for some \(a_1 > 0\). The term \(H_1(x)\) appears when \(3m - 2k_1 - 2k_2 = 1\), e.g., when \(k_1 = (m - 1)/2, k_2 = m\). The coefficient \(a_1 > 0\) because all the coefficients before the Hermite polynomials in (3) are positive. It is then clear that the Hermite rank of \(H_m(x)^3\) is \(m' = 1\). Hence the polynomial \(Q(x)\) satisfies (2). \(\square\)

Remark 4. In Proposition 3 case (b), we do not need a restriction on \(H\). We require \(m \geq 3\) since \(m = 1\) is incompatible with (1).

Remark 5. What about the converse? Can a strong mixing process not be subordinated to a Gaussian LRD process? The answer is clearly “yes”. Suppose for example \(\{X_i\}\) i.i.d. Gaussian. Then there is no \(\{X'_i\} \overset{d.d.}{=} \{X_i\}\) so that \(X'_i = G(Z'_i)\), where \(\{Z'_i\}\) is LRD Gaussian, because the covariance \(\text{Cov}[X'_i, X'_0] \neq 0\) for large \(i\).

Acknowledgments. This work was partially supported by the NSF grant DMS-1309009 at Boston University.

References

[1] P. Breuer and P. Major. Central limit theorems for non-linear functionals of Gaussian fields. Journal of Multivariate Analysis, 13(3):425–441, 1983.

[2] S. Janson. Gaussian Hilbert Spaces, volume 129. Cambridge University Press, 1997.

[3] M. Peligrad. Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables (a survey). In E. Eberlein and M. S. Taqqu, editors, Dependence in Probability and Statistics, pages 193–223. Birkhäuser, 1986.

[4] M.S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 31:287–302, 1975.

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