Fast symmetric matrix inversion using modified Gaussian elimination

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Abstract—In this paper we present two different variants of method for symmetric matrix inversion, based on modified Gaussian elimination. Both methods avoid computation of square roots and have a reduced machine time spending. Further, both of them can be used efficiently not only for positive (semi-) definite, but for any non-singular symmetric matrix inversion. We use simulation to verify results, which is not interesting for researcher. Re-solution of (1) could be done with formulae (2) below:

\[ A_{ij} \text{ is a required variable (indeed, it is a number of required variables in } x) \]

\[ f_{i}^{m+1} = f_{i}^{m} \text{ for } i < m + 1, m = 0, n - 1, \text{ and } x_{i} \text{ is an unrequired variable; } \]

\[ f_{m+1}^{m+1} = \frac{1}{f_{m+1}^{m+1}} \text{ for } m = 0, n - 1. \]

Let \( a_{i} \) be an \( i^{th} \) column of \( A \); let \( F^{m} \) be a matrix \( F \) after \( m^{th} \) change; let \( f_{i}^{m} \) be an \( i^{th} \) row of \( F^{m} \), \( m = 0, n \); let \( F^{0} = I \), where \( I \) is an identity matrix. Then solution of (1)

\[ \text{I. INTRODUCTION} \]

Symmetric matrix inversion is one of the most important problems for many practical tasks e.g., analysis of electrical circuits with inductance elements [1], synthesis of Kalman or Wiener filters [2], using of finite element method [3].

Existing symmetric matrix inversion methods are Cholesky decomposition, LDL decomposition [4], bordering method [5], and the most efficient Krishnamoorthy-Menon’s method (based on Cholesky decomposition, and requires \( \frac{n^3}{2} + \frac{n^2}{2} \) operations with \( n \) square roots computation) [6], [7].

The aim of this paper is to propose a symmetric matrix inversion method, which reduces machine time spending compared to Krishnamoorthy-Menon’s method by avoiding of square root computations. Moreover, this fact allow us to use the proposed method for efficient inverse of symmetric matrix not only with strict diagonal dominance, but without diagonal dominance as well.

II. MODIFIED GAUSSIAN ELIMINATION

In this section shows modified Gaussian elimination, which proposed method based on [8]-[12].

Let there is a system of linear equations (1),

\[ Ax = b, \quad (1) \]

where \( A \in \mathbb{C}^{n \times n} \), \( x \in \mathbb{C}^{n \times 1} \), \( b \in \mathbb{C}^{n \times 1} \). During modified Gaussian elimination an addition matrix \( F : F \in \mathbb{C}^{n \times n} \) changes instead of matrix \( A \), but addition memory is not necessary in this case [12].

Let vector \( x \) consist of two types of variables: required, which should be find during elimination, and unrequired, which is not interesting for researcher. Re-solution of (1) after some changes of \( A \) could be done with reduced number of multiplications and divisions, using formulae from [8]-[12].

A. First variant of the proposed method

The first variant of the proposed method consists of two different stages. On the first stage we use formulae (2) for \( p = 1 \), and only \( x_{n} \) is a required variable (indeed, it is a valid proposition for \( p = 0 \) as well). On the second stage we use addition formulae described below.

Let us introduce some notation.

Let \( f_{i}^{m+1} \) be an element from \( i^{th} \) row and \( j^{th} \) column of \( F^{m+1} \), let \( a_{ij} \) be an element from \( i^{th} \) row and \( j^{th} \) column of \( A \); let \( S_{j}^{m+1} \) be a submatrix of \( F^{m+1} \), such that \( S_{j}^{m+1} = (f_{ij}^{m+1}), i = 1, m + 1, j = 1, m + 1 \); let \( S_{a} \) be a submatrix of \( A \), such that \( S_{a} = (a_{ij}), i = 1, m + 1, j = 1, m + 1 \).

It is easily shown that after the first stage \( F \) is a lower triangular matrix. After the second stage \( F \) became a matrix \( A^{-1} \), as if we use (2) for \( p = n \).
For symmetric matrix inversion using modified Gaussian elimination that is enough to use only lower triangular matrix. To prove this statement, we need a lemma 1.

Lemma 1: Let $F^{m+1}$ be a matrix from $\mathbb{K}$ for $p = n$, $m = 0, n - 1$. Then $S^{m+1}_j = (S^m_1)^{-1}$.

Proof: Using (2), let us consider 4 cases:

Case 1. Let $i = m + 1$. Then
$$f_{i+1}^{m+1}a_i = f_{m+1}^{m+1}a_{m+1},$$
$$f_{m+1}^{m+1}a_{m+1} = \frac{1}{f_{m+1}^{m+1}a_{m+1}}f_{m+1}^{m}a_{m+1} = 1,$$
$m = 0, n - 1$.

Case 2. Let $j = m + 1$. Then
$$f_{i+1}^{m+1}a_j = f_{i+1}^{m+1}a_{m+1},$$
$$f_{i+1}^{m+1}a_{m+1} = f_{i+1}^{m+1}a_{m+1} - f_{i+1}^{m}a_{m+1}f_{m+1}^{m+1}a_{m+1} = 0,$$
$m = 0, n - 1, i = m, m; n - 1$.

Case 3. Let $i = m$. Then
$$f_{i+1}^{m+1}a_i = f_{m+1}^{m+1}a_m,$$
$$f_{m+1}^{m+1}a_m = f_{m+1}^{m}a_m - f_{m+1}^{m}a_{m+1}f_{m+1}^{m+1}a_{m+1},$$
$$f_{m+1}^{m+1}a_m = f_{m+1}^{m+1}a_m - f_{m+1}^{m+1}a_m = 1;$$
It can be checked easily for $i < m + 1, m = 0, n - 1$.

Case 4. Let $j = m$. Then
$$f_{i+1}^{m+1}a_j = f_{i+1}^{m+1}a_m - f_{i+1}^{m+1}a_m = 0;$$
It can be checked easily for $j < m + 1, j \neq i, m = 0, n - 1$.

Since lemma 1 it follows that $S^{m+1}$ is a symmetric matrix.

This statement is needed for the second stage of the method. Let $F^k$ be a matrix $F$ after $k$th change, $k = \overline{0, n}$, where $F^0$ is a lower triangular matrix after the first stage, and $F^{m} = A^{-1}$. Let $f_{ij}^k$ is an element from $i$th row and $j$th column of $F^k$. Then the second stage describes with formulae below:

- $f_{ij}^{k+1} = f_{ij}^k$ for $i = k + 1, n, j = \overline{1, n}$, $k = 0, n - 1$,
- $f_{ij}^{k+1} = f_{ij}^k + f_{ij}^{k+1}$ for $i = \overline{1, k}, j = k + 1, n - 1$,
- $f_{ij}^{k+1} = f_{ij}^k + f_{ij}^{k+1}$ for $i = \overline{1, k}, j = \overline{1, i}, k = \overline{1, n - 1}$.

It is easily to prove that $A^{-1} = F^0 + (F^0 - D)^T$, where $D$ is a diagonal matrix, such that $d_{ii} = f_{ii}^0, i = \overline{1, n}$.

Number of multiplications and divisions for the first stage of the method describes with formula $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, as we note earlier.

It is easily shown that number of multiplications and divisions for the second stage describes with formula $\frac{n^3}{6} + \frac{n^2}{2} - \frac{n}{3}$.

Both stages for symmetric matrix inversion using the first variant of the proposed method requires $\frac{n^3}{2} + \frac{n^2}{2} - \frac{n}{2}$ multiplications and divisions. It is less than requirements of Cholesky decomposition or LDL decomposition (see table 1).

Let us remark that proposed method avoid square root computations; this considerably reduce machine time spending, and make it possible to use proposed method not only for positive determined, but for any invertible symmetric matrices as well.

B. Second variant of the proposed method

The second variant of the proposed method consist of only one stage.

Suppose, that
$$F^{m+1} = F^m + \Delta F^m, \quad (4)$$
where $F^m = F^m$, but all elements from $(m + 1)^{th}$ row of $F^m$ are zeros.

Let $f_{ij}$ be an element of $i$th row and $j$th column of $F^m$; let $f_{i*}$ be a $j$th column of $F^m$.

For explanation of the following formulae, we need a lemma 2.

Lemma 2: Let $F^{m+1}$ is a matrix from $\mathbb{K}$ for $p = n$, $m = 0, n - 1$. Then $\Delta F^{m+1}$ be $\Delta F^m = (f_{m+1}^{m+1}f_{m+1}^m)$.

Proof: From (4) $\Delta F^{m+1} = F^{m+1} - F^m$.

Using (2), we get 1.

- $f_{m+1}^{m+1} = \frac{1}{f_{m+1}^{m+1}a_{m+1}}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$.

If we combine lemma 1 with lemma 2 we get formulae below:

- $f_{m+1}^{m+1} = \frac{1}{f_{m+1}^{m+1}a_{m+1}}f_{m+1}^m$ for $m = 0, n - 1$,
- $f_{i+1}^{m+1} = f_{i+1}^{m+1}f_{i+1}^m$ for $i = m, n - 1$,
- $f_{i+1}^{m+1} = f_{i+1}^{m+1}f_{i+1}^m$ for $i = m + 2, n$,
- $f_{i+1}^{m+1} = f_{i+1}^{m+1}f_{i+1}^m$ for $i = \overline{1, m}, i = \overline{1, m}, i = \overline{1, m}$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$ for $m = 1, n - 1, j = m, n - 1$,
- $f_{m+1}^{m+1} = f_{m+1}^{m+1}f_{m+1}^m$ for $m = 1, n - 1, j = m, n - 1$, $j = 1, m$,
\[ f_{ij}^{m+1} = f_{ij}^m \quad \text{for } m = 2, n - 1, \quad j = 2, m, \quad i = 1, j - 1, \]
\[ f_{ij}^{m+1} = f_{ij}^m \quad \text{for } m = 0, n - 2, \quad i = 1, n, \quad j = m + 2, n. \]

The formulae (5) describe an idea of the method in detail, but for practical tasks it is better to use different formulae:

\[ f_{m+1}^{m+1} = \frac{1}{f_{m+1}^m} f_{m+1}^m f_{m+1}^m \quad \text{for } m = 0, n - 1, \quad j = 1, m + 1, \]
\[ f_{m+1}^{m+1} = -f_{m+1}^2 \quad \text{for } m = 0, n - 2, \quad i = m + 2, n, \]
\[ f_{ij}^{m+1} = f_{ij}^m + f_{m+1}^{m+1} f_{m+1}^m \quad \text{for } m = 1, n - 2, \quad i = m + 2, n, \quad j = 1, n, \]
\[ f_{ij}^{m+1} = f_{ij}^m \quad \text{for } m = 1, n - 1, \quad i = 1, n, \quad j = m + 1, n. \]

(6)

It can easily be shown that (5) and (6) are equivalent.

Number of multiplications and divisions for the second variant of the proposed method describes with formulae

\[ \frac{n^3}{2} + \frac{n^2}{2}. \] It is less then requirements of Cholesky decomposition, LDL decomposition or Krishnamoorthy-Mennon method. It is the same requirements as for bordering method, but it should be noted that bordering method could not be use for inversion of matrix with \( M_{ii} \neq 0, \quad i = 1, n, \) where \( M_{ij} \) is a minor of \( A, \quad i = 1, n, \quad j = 1, n. \)

At the same time, proposed method could be use for inversion of matrix with \( M_{ii} = 0, \) if \( A \) is not a singular matrix.

Let us remark that the second variant of the proposed method avoid square root computations as well.

IV. SIMULATION SETUP

In order to demonstrate advantages of the proposed algorithms, we use MATLAB based simulation via different CPUs. We give results for Intel Core i5-3230M 2.60 GHz below (Intel Pentium Dual Core T 2390 1.86 GHz and Intel Atom N450 1.67 GHz gives familiar results). We compare proposed algorithms with the most efficient notable algorithms for symmetric matrix inversion: Cholesky decomposition [13], LDL decomposition [4], and Krishnamoorthy-Mennon method [6], [7], [14]. We generate table with full equations, which describes number of multiplications, divisions and square roots computation for every noted method.

The first row of the table 1 describes Cholesky decomposition and solving of systems of linear equations \( LB = I, \) and \( L^T A^{-1} = B. \) The second row describes LDL decomposition and solving of SLE \( LX = I, \) \( DB = X, \) and \( L^T A^{-1} = B. \) The third row describes matrix inversion using Krishnamoorthy-Mennon method, based on Cholesky decomposition [6], [7]. The fourth row describes the first variant, and the fifth row describes the second variant of the proposed method.

**Experiment 1.** Inversion of a real symmetric matrix with strict diagonal dominance. Let \( q_{\text{theor}} \) be a number of multiplications and divisions, and \( s_{\text{theor}} \) be a number of square root computations, determined with formulae from the table 1. Let \( q_{\text{pract}} \) and \( s_{\text{pract}} \) be numbers of operations, determined with counter variables from MATLAB scripts. Results of the experiment are given in tables 2 and 3.

**Experiment 2.** Inversion of real symmetric matrices of order \( n \) with strict diagonal dominance. Let \( t \) be a time for matrix inversion; let \( \text{norm} \) be a second \( \text{norm} \) \( : \) \( \text{norm} = \|A_m^{-1} - A_{\text{inv}}\|_2, \) where \( A_m^{-1} \) is a matrix, inverted one of described methods, and \( A_{\text{inv}}^{-1} \) is a matrix, inverted via MATLAB function \( \text{inv}(A). \) Results of the experiment are given in tables 4 and 5.

**Experiment 3.** Inversion of real symmetric matrices of order \( n \) without diagonal dominance. Results of the experiment are given in the table 6.

V. SIMULATION RESULTS

From tables 2 and 3 we can conclude that formulae from the table 1 are correct.

From tables 4 and 5 we can conclude that both variants of the proposed method provide notable reduction of machine time spending and has a good accuracy.

From the table 6 we can conclude that both variants of the proposed method increase advantages for matrices without diagonal dominance. Let us remark that it is especially important for inductance matrix inversion.

VI. CONCLUSIONS

We propose a new method for symmetric matrix inversion based on modified Gaussian elimination with avoiding of square root computations. Proposed method could be useful for any scientific and technical problem with symmetric matrix inversion, especially if matrix has not a diagonal dominance.
| METHOD OF MATRIX INVERSION | NUMBER OF MULTIPLICATIONS AND DIVISIONS | NUMBER OF SQUARE ROOT COMPUTATIONS |
|---------------------------|----------------------------------------|-----------------------------------|
| Cholesky decomposition    | $\frac{n^2}{2} + \frac{n^2}{2}$       | $n$                               |
| LDL decomposition         | $2n^3 + n^4 - \frac{n}{2}$            | 0                                 |
| Krishnamoorthy - Mennon’s method (based on Cholesky decomposition) | $\frac{n^3}{2} + \frac{n^2}{2}$ | $n$                               |
| The first variant of the proposed method | $\frac{n^3}{2} + n^2 - \frac{n}{2}$ | 0                                 |
| The second variant of the proposed method | $\frac{n^3}{2} + n^2 - \frac{n}{2}$ | 0                                 |

Table 1. Table summarizing the number of operations for inversion of a matrix with strict diagonal dominance via different methods.

| METHOD OF MATRIX INVERSION | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ |
|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Cholesky decomposition    | 315000                   | 315000                   | 100                      | 100                      |
| LDL decomposition         | 671650                   | 671650                   | 0                        | 0                        |
| Krishnamoorthy - Mennon’s method (based on Cholesky decomposition) | 505000                  | 505000                   | 100                      | 100                      |
| The first variant of the proposed method | 309950                  | 309950                   | 0                        | 0                        |
| The second variant of the proposed method | 505000                  | 505000                   | 0                        | 0                        |

Table 2. Table summarizing the number of operations for inversion of a matrix with strict diagonal dominance via different methods ($n = 100$).

| METHOD OF MATRIX INVERSION | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ |
|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Cholesky decomposition    | 62875000                 | 62875000                 | 500                      | 500                      |
| LDL decomposition         | 83458250                 | 83458250                 | 0                        | 0                        |
| Krishnamoorthy - Mennon’s method (based on Cholesky decomposition) | 62625000                | 62625000                 | 500                      | 500                      |
| The first variant of the proposed method | 62749750                | 62749750                 | 0                        | 0                        |
| The second variant of the proposed method | 62625000                | 62625000                 | 0                        | 0                        |

Table 3. Table summarizing the number of operations for inversion of a symmetric matrix with strict diagonal dominance via different methods ($n = 500$).

| METHOD OF MATRIX INVERSION | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ |
|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Based on Cholesky decomposition | 0.384 s.                | 9.499 s.                 | 41.96 s.                 | 329.7 s.                 |
| LDL decomposition         | 0.332 s.                 | 7.950 s.                 | 33.23 s.                 | 261.0 s.                 |
| Krishnamoorthy Mennon’s method (program by A. Krishnamoorthy [14]) | 0.125 s.                | 2.950 s.                 | 12.12 s.                 | 96.84 s.                 |
| The first variant of the proposed method | 0.113 s.                | 2.793 s.                 | 12.44 s.                 | 103.3 s.                 |
| The second variant of the proposed method | 0.046 s.                | 0.743 s.                 | 3.086 s.                 | 27.13 s.                 |

Table 4. Table summarizing times of numerical computations for inversion of a symmetric matrix with strict diagonal dominance via different methods.

| METHOD OF MATRIX INVERSION | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ |
|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Based on Cholesky decomposition | 9.4E-19 s.               | 1.3E-18 s.               | 1.9E-18 s.               | 3.0E-18 s.               |
| LDL decomposition         | 1.0E-18 s.               | 1.5E-18 s.               | 2.0E-18 s.               | 3.0E-18 s.               |
| Krishnamoorthy Mennon’s method (program by A. Krishnamoorthy [14]) | 9.4E-19 s.               | 1.3E-18 s.               | 1.9E-18 s.               | 3.0E-18 s.               |
| The first variant of the proposed method | 9.4E-19 s.               | 1.3E-18 s.               | 1.9E-18 s.               | 3.0E-18 s.               |
| The second variant of the proposed method | 1.5E-18 s.               | 2.5E-18 s.               | 4.1E-18 s.               | 6.7E-18 s.               |

Table 5. Table summarizing $\|A_m^{-1} - A_s^{-1}\|_2$ of numerical computations for inversion of a symmetric matrix with strict diagonal dominance via different methods.

| METHOD OF MATRIX INVERSION | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ | $\theta_{\text{theor}}$ | $\theta_{\text{pract}}$ |
|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Based on Cholesky decomposition | 0.448 s.                | 11.16 s.                 | 50.47 s.                 | 401.2 s.                 |
| LDL decomposition         | 0.328 s.                 | 7.329 s.                 | 33.22 s.                 | 264.3 s.                 |
| Krishnamoorthy Mennon’s method (program by A. Krishnamoorthy [14]) | 0.153 s.                | 6.185 s.                 | 54.42 s.                 | 861.7 s.                 |
| The first variant of the proposed method | 0.318 s.                | 10.66 s.                 | 73.34 s.                 | 1021 s.                  |
| The second variant of the proposed method | 0.045 s.                | 0.742 s.                 | 3.093 s.                 | 27.05 s.                 |

Table 6. Table summarizing times for inversion of a symmetric matrix without diagonal dominance via different methods.
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