Relic gravitational waves in cosmological models based on the modified gravity theories

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Abstract. We consider cosmological models based on the generalized scalar-tensor gravity, which correspond to the observational constraints on the parameters of cosmological perturbations for any model’s parameters. The estimates of the energy density of relic gravitational waves for such a cosmological models were made. The possibility of direct detection of such a gravitational waves using modern and prospective methods was discussed as well.

1. Introduction

To describe the physical processes occurring in the early universe, theoretical models of cosmological inflation based on the evolution of a scalar field and Einstein gravity theory \cite{1, 2, 3, 4} or its different modifications \cite{5, 6, 7, 8, 9} are usually used.

In accordance with the theory of cosmological perturbations, during the stage of inflation, quantum fluctuations of the scalar field induce perturbations of the space-time metric. In the context of inflationary cosmology the observed anisotropy and polarization of the CMB \cite{10} can be explained by the action of two types of perturbations, namely, scalar and tensor perturbations (relic gravitational waves). The third type of perturbations, namely vector perturbations, quickly decay in the process of accelerated expansion of the early universe \cite{2}. The observational constraints on the values of the cosmological perturbations parameters can be used for verification of different inflationary models based on GR and its modifications as well \cite{2, 3, 11, 12, 13, 14}.

For the case of inflationary models based on Einstein gravity the observational constraints on the values of the cosmological perturbations are satisfied for limited class of the physical potentials of a scalar field that restricts the possible relevant cosmological models \cite{2, 3, 4} However, in the case of modifications of general relativity, one can consider cosmological inflationary models with arbitrary parameters that correspond to the observational constraints (see, for example, in \cite{15, 16}). This approach makes it possible to obtain model-independent estimates of the characteristics of relic gravitational waves, the direct observation of which is an urgent task in the context of the confirmation of application of the inflationary paradigm to the description of physical processes in the early universe \cite{17}.

In this paper, we consider the models of cosmological inflation based on the generalized scalar-tensor gravity (STG) theory and the quadratic relationship between the Hubble parameter and coupling function that define the type of STG \cite{18, 19, 20} on the basis of a such approach.
Within the framework of the proposed approach, the relationship between the parameters of cosmological perturbations and the energy scale of inflation which doesn’t depend on the type of potential of a scalar field, the type of STG or the type of cosmological dynamics was defined. We also consider various possibilities of realizing transitions between the end of the inflationary stage and the stage of radiation domination, and estimate the parameters of relic gravitational waves for the proposed scenarios of the evolution of the early universe.

2. The dynamic equations for inflationary models based on STG

The inflationary models based on the generalized scalar-tensor theory are defined by the action [21, 22]

\[ S_{\text{GST}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R - \frac{\omega(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_m, \]

in the system of units \( 8\pi G = m_p^{-2} = c = 1 \), where \( g \) a determinant of the space-time metric \( g_{\mu\nu} \), \( \phi \) is a scalar field with the potential \( V = V(\phi) \), \( F(\phi) \) and \( \omega(\phi) \) are the coupling and kinetic functions that determine the type of STG, \( R \) is the Ricci scalar curvature of the space-time. The case of vacuum inflationary solutions implies the absence of matter \( S_m = 0 \).

The background dynamic equations in a spatially flat four-dimensional Friedmann-Robertson-Walker (FRW) space-time

\[ ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \]

corresponding to the action (1) in chosen system of units are

\[ 3FH^2 + 3H\dot{F} - \frac{\omega}{2} \dot{\phi}^2 - V(\phi) = 0, \]  
\[ 3FH^2 + 2H\dot{F} + 2FH + \ddot{F} + \frac{\omega}{2} \dot{\phi}^2 - V(\phi) = 0, \]  
\[ \omega\ddot{\phi} + 3\omega H\dot{\phi} + \frac{1}{2} \dot{\phi}^2 \omega' + V' - 6H^2 F' - 3\dot{H}F' = 0. \]

where a dot represents a derivative with respect to the cosmic time \( t \), \( H \equiv \dot{a}/a \) denotes the Hubble parameter and \( F' \equiv \partial F/\partial \phi \).

Also, we note, that the scalar field equation (5) can be derived from the equations (3)–(4). For this reason, equations (3)–(4) completely describe the cosmological dynamics, and can be represented as follow

\[ V(\phi) = 3FH^2 + \frac{5}{2} H\dot{F} + F\dot{H} + \frac{1}{2} \ddot{F}, \]
\[ \omega(\phi) \dot{\phi}^2 = H\dot{F} - 2F\dot{H} - \dddot{F}. \]

There are a lot of cosmological models were considered on the basis of the scalar-tensor gravity theories (see, for example, in [21, 22]). However, we are interested in the inflationary models, corresponding to observational constraints on the parameters of cosmological perturbations without restrictions on the type of the potential \( V(\phi) \), the parameters of a scalar-tensor gravity theory \( \{F(\phi), \omega(\phi)\} \) and the type of inflationary dynamics \( H(t) \).

For this aim we consider the models with quadratic connection between the Hubble parameter and coupling function [18, 19, 20]

\[ H = \lambda \sqrt{F}. \]
Under condition (8) dynamic equations (6)–(7) are reduced to the following ones

\[ \tilde{F}(\phi) = \lambda^2 F(\phi) = H^2, \]
\[ \tilde{V}(\phi) = \lambda^2 V(\phi) = 3H^4 + 6H^2 \dot{H} + \dot{H}^2 + H \ddot{H}, \]
\[ \tilde{X}(\phi, \dot{\phi}) = \lambda^2 X(\phi, \dot{\phi}) = -\frac{\lambda^2}{2} \omega(\phi) \dot{\phi}^2 = \dot{H}^2 + H \ddot{H}, \]

where \( \lambda^2 \) is the constant parameter which defines the energy scale of inflation.

Based on the definitions of slow-roll parameters with corresponding slow-roll conditions

\[ \epsilon = -\frac{\dot{H}}{H^2} \ll 1, \quad \delta = -\frac{\ddot{H}}{2H \dot{H}} \ll 1, \]

from equations (10)–(11), one can obtain the following expressions

\[ \frac{\dot{V}}{F^2} = 3 - 6\epsilon + 2\epsilon\delta + \epsilon^2, \]
\[ \frac{\dot{X}}{F^2} = 2\epsilon\delta + \epsilon^2, \]

that connect the type of the scalar-tensor gravity, characteristics of a scalar field and the type of cosmological dynamics.

As one can see, the kinetic energy of a scalar field \( X \) is represented in the terms of a second order regarding its potential \( V \), therefore the inflationary slow-roll conditions implying \( X \ll V \) are satisfied for these models when \( \epsilon \ll 1 \) and \( \delta \ll 1 \).

3. The parameters of cosmological perturbations

The parameters of cosmological perturbations in inflationary models based on action (1) with quadratic connection \( H = \lambda \sqrt{F} \) were considered in [18, 19, 20].

The expressions of the parameters of cosmological perturbations on the crossing of Hubble radius \( (k = aH) \) were obtained in these papers in the following form

\[ P_S = A_S = \frac{\lambda^2}{16\pi^2 \epsilon (\epsilon - \delta)}, \]
\[ P_T = A_T = \frac{2\lambda^2}{\pi^2}, \]
\[ r = \frac{P_T}{P_S} = 32\epsilon (\epsilon - \delta), \]
\[ n_S = 1 - 4\epsilon + 2\delta + 2\epsilon\delta + (1 - \epsilon) \left( \frac{\epsilon\delta - \xi}{\epsilon - \delta} \right), \]
\[ n_T = 0, \]

where \( k \) is a wave number, \( \{A_S, A_T\} \) is the values of the power spectra of scalar and tensor perturbations \( \{P_S, P_T\} \) on the crossing of Hubble radius, \( \{n_S, n_T\} \) are the spectral indices of scalar and tensor perturbations and

\[ \xi = \epsilon\delta - \frac{1}{H} \delta \ll 1, \]

is the third slow-roll parameter.
In general case, the constraints on the values of the parameters of cosmological perturbations following from observation of anisotropy of CMB by PLANCK [10]

\[ A_S = 2.1 \times 10^{-9}, \quad n_S = 0.9663 \pm 0.0041, \quad r < 0.065, \]

restrict the possible inflationary model’s parameters. From expressions (15), (17) and (21) one has

\[ r = \frac{2 \lambda^2}{\pi^2 A_S} = 9.5 \times 10^8 \times \left( \frac{\lambda^2}{\pi^2} \right), \]

therefore, in these inflationary models tensor-to-scalar ratio \( r \) depends on the value of the parameter \( \lambda^2 \) only.

Thus, from constraint (23) one has following restriction

\[ \lambda^2 < 6.7 \times 10^{-10}, \]

on the energy scale of such a inflationary models.

Taking into account the quasi-de Sitter character of the expansion of the early universe \( H \simeq \lambda \), the energy scale of inflation can be estimated from (10) in terms of the Planck mass \( m_p^{-2} = 8\pi G = (2.4 \times 10^{18} \text{GeV})^{-2} \) as follows

\[ H < 2.6 \times 10^{-5} m_p, \quad V \simeq 3 \lambda^2 m_p^4, \quad V^{1/4} < 1.6 \times 10^{16} \text{GeV}, \]

(26)

corresponding to estimates given for standard inflationary models based on GR in order of magnitude [2, 23, 24].

Also, under slow-roll conditions \( \epsilon \ll 1 \) and \( \delta \ll 1 \) from (18) one has

\[ n_S - 1 \simeq -4\epsilon + 2\delta + \left( \frac{\epsilon\delta - \xi}{\epsilon - \delta} \right). \]

(27)

Since, the deviations from Harrison-Zel’dovich spectrum (22) are

\[ |n_S - 1| \simeq 3 \times 10^{-2}, \]

(28)

for the following values of the slow-roll parameters on the crossing of the Hubble radius

\[ 0 < \epsilon \lesssim 10^{-2}, \quad -2 \times 10^{-1} \lesssim \delta \lesssim 10^{-2}, \quad -9 \times 10^{-2} \lesssim \xi \lesssim 10^{-4}, \]

(29)
inflationary models based on STG with connection \( H = \lambda \sqrt{F} \) and arbitrary type of accelerated expansion of the early universe can satisfy the observational constraint (22) when conditions (29) are fulfilled.

4. The spectrum of relic gravitational waves

As an additional tool for verifying cosmological models under consideration, we will consider the possibility of direct detection of relic gravitational waves through existing and promising methods [25, 26, 27, 28].

It should also be noted that relic gravitational waves have not been observed at the moment, and solving the problem of their direct detection is one of the most important problems of modern observational cosmology and gravitational-wave researching [25, 26, 27, 28].
The energy density of relic gravitational waves at the present time can be obtained from expression [25]

\[ \Omega_{GW}(f) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d \log f} \simeq \frac{P_T(f)}{24 z_{eq}}, \tag{30} \]

where \( \rho_c \) is the critical density, \( f \) is the frequency of relic gravitational waves and \( z_{eq} \simeq 3400 \) is the redshift at the matter-radiation equivalence.

The power-spectrum of tensor perturbations can be defined as [26, 27]

\[ P_T(f) \simeq A_T \left( \frac{f}{f_*} \right)^{\alpha(f)+n_T(f)}, \quad \alpha(f) = 2 \left( \frac{3w(f) - 1}{3w(f) + 1} \right), \tag{31} \]

where \( f_* \) is the characteristic frequency corresponding to the pivot scale \( k_* \) and \( w = w(f) \) is the value of the state parameter when transition from the end of inflation to the BBN epoch occurs. The dependence \( n_T = n_T(w) \) in explicit form was considered in [26].

For the models under consideration the spectral index of tensor perturbations \( n_T = 0 \) and for the case of standard transition between the end of inflation and the BBN epoch implying \( w = 1/3 \) from (16) and (31) we obtain following energy density of relic gravitational waves

\[ \Omega_{GW} \simeq \frac{\lambda^2}{12\pi^2 z_{eq}} < 1.7 \times 10^{-15}, \tag{32} \]

which is defined by the energy scale of inflation \( \lambda^2 \) only, and it doesn’t depend on the frequency, i.e. in this case one has exactly scale-invariant spectrum of relic gravitational waves.

Comparing this estimate of the energy density of the relic gravitational waves with the sensitivity of LIGO \( (\Omega_{GW} \simeq 10^{-9}) \) and LISA \( (\Omega_{GW} \simeq 10^{-13}) \) (for \( 10^2 \) Hz and \( 10^{-3} \) Hz respectively) [28], one can conclude that the relic gravitational waves predicted in such a models cannot be directly detected by these experiments.

Nevertheless, taking into account the additional stage of predominance of kinetic energy (kination) on the transition between the end of inflation and the BBN epoch for \( n_T = 0 \) has \( w \simeq 1/2 \) [26], and from (30)--(32) we obtain the following expression

\[ \Omega_{GW}(f) \simeq \frac{\lambda^2}{12\pi^2 z_{eq}} \left( \frac{f}{f_*} \right)^{2/5}, \tag{33} \]

where \( f_* \simeq 10^{-11} \) Hz since the gravitational waves with frequencies below this value reentered the Hubble horizon after BBN [26].

\textbf{Table 1.} The energy density of relic gravitational waves \( \Omega_{GW} \) at the modern era for different frequencies \( f \).

| \( f, \) Hz | \( 10^{-3} \) | \( 10^2 \) | \( 5 \times 10^6 \) | \( 10^8 \) | \( 10^{10} \) |
|---|---|---|---|---|---|
| \( \Omega_{GW} \) | \( 2.7 \times 10^{-12} \) | \( 2.7 \times 10^{-10} \) | \( 2.1 \times 10^{-8} \) | \( 6.8 \times 10^{-8} \) | \( 4.3 \times 10^{-7} \) |

The values of the energy density of relic gravitational waves at the modern era for different frequencies are presented in Table 1.

As one can see, the relic gravitational waves from inflation based on scalar-tensor gravity with connection \( H = \lambda \sqrt{F} \) can be in principle detected by LISA experiment. Since the energy density in this case is maximal in for high-frequency gravitational waves, the methods of their detection in the high-frequency range [28, 29, 30, 31, 32] can also be considered as relevant ones for verification of the proposed cosmological models on the basis of direct detection of relic gravitational waves.
5. Conclusion

In this paper, we considered the models of cosmological inflation based on generalized scalar-tensor theories of gravity, which satisfy observational constraints on the values of the parameters of cosmological perturbations for arbitrary model’s parameters and restricted by the energy scale of inflation only. The proposed approach allows one to give a model-independent estimate of the energy density of relic gravitational waves in the present era of the universe’s evolution. It was shown that the verification of the proposed models can be carried out on the basis of the possibility of direct detection of relic gravitational waves in both the low-frequency and high-frequency ranges of their spectrum.

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References

[1] Linde A 1990 Particle physics and inflationary cosmology (Boca Raton: CRC Press)
[2] Baumann D and McAllister L 2014 Inflation and String Theory (Cambridge: Cambridge University Press)
[3] Chervon S, Fomin I, Yurov V and Yurov A 2019 Scalar field cosmology (Singapore: World Scientific)
[4] Martin J, Ringeval C, Vennin V 2014 Phys. Dark Univ. 5-6 75
[5] Nojiri S and Odintsov S 2008 Phys. Lett. B 659 821
[6] Clifton T, Ferreira P, Padilla A and Skordis C 2012 Phys. Rept. 513 1
[7] Nojiri S, Odintsov S and Oikonomou V 2017 Phys. Rept. 692 1
[8] Ishak M 2019 Living Rev. Rel. 1 1
[9] Fomin I and Chervon S 2019 Phys. Rev. D 100 023511
[10] Ade P et al [PLANCK Collaboration] 2016 Astron. Astrophys. 594 A13
[11] Melchiorri A, Sazhin M, Shulga V and Vittorio N 1999 Astrophys. J. 518 562
[12] Kopeikin S, Ramirez J, Mashhoon B and Sazhin M 2001 Phys. Lett. A 292, 173
[13] Hwang J and Noh H 2005 Phys. Rev. D 71 063536
[14] Chervon S and Fomin I 2008 Grav. Cosmol. 14 163
[15] Fomin I and Chervon S 2020 J. Phys. Conf. Ser. 1557 012020
[16] Fomin I 2020 Eur. Phys. J. C 80 1145
[17] Baumann D et al [CMBPol Study Team] 2009 AIP Conf. Proc. 1141 10
[18] Fomin I and Chervon S 2018 Eur. Phys. J. C 78 918
[19] Fomin I, Chervon S and Tsyganov A 2020 Eur. Phys. J. C 80 350
[20] Fomin I and Chervon S 2020 Universe 6 199
[21] Fujii Y and Maeda K 2007 The Scalar-Tensor Theory of Gravitation (Cambridge: Cambridge University Press)
[22] Faraoni V 2004 Cosmology in Scalar-Tensor Gravity (New York: Springer)
[23] Liddle A 1994 Phys. Rev. D 49 739
[24] Guo Z, Schwarz D and Zhang Y 2011 Phys. Rev. D 83 083522
[25] Cabass G, Pagano L, Salvati L, Gerbino M, Giusarma E and Melchiorri A 2016 Phys. Rev. D 93 063508
[26] Boyle L and Buonanno A 2008 Phys. Rev. D 78, 043531
[27] Giovannini M 2020 Prog. Part. Nucl. Phys. 112 103774
[28] Aggarwal N, Aguilar O, Bauswein A, Cella G, Clesse S, Cruise A, Domcke V, Figueroa D, Geraci A and Goryachev M et al [arXiv:2011.12414 [gr-qc]].
[29] Gladyshev V and Morozov A 1993 Tech. Phys. Lett. 19 7 449
[30] Morozov A and Gladyshev V 2002 Russ. Phys. J. 45 113
[31] Esakov A et al. 2015 (in Russian) Herald of the Bauman Moscow State Tech. Univ. Nat. Sci. 1 26
[32] Fomin I and Morozov A 2017 J. Phys. Conf. Ser. 798 012008