Turing Impossibility Properties for Stack Machine Programming

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Abstract The strong, intermediate, and weak Turing impossibility properties are introduced. Some facts concerning Turing impossibility for stack machine programming are trivially adapted from previous work. Several intriguing questions are raised about the Turing impossibility properties concerning different method interfaces for stack machine programming.

Keywords instruction sequence processing, functional unit, halting problem, autosolvability

1 Introduction

The work presented in this paper constitutes a minor adaptation to a simplified setting, and a corresponding reformulation of the content of our [4]. We refer to that paper for further technical explanations of the formalism used below, for the justification of terminology, as well as for more information concerning connections with previous work [4]. By highlighting results from [4] from a different perspective their relevance for understanding the methodological impact of the well-known recursive unsolvability of the halting problem, in which we firmly believe, becomes more apparent. Like [4] this paper focuses on the off-line halting problem, which unlike the on-line halting problem analyzed in [6] need not always give way to a diagonal argument.

This paper concerns an investigation of issues relating to the halting problem perceived in terms of instruction sequences. Positioning Turing’s result of [8] regarding the recursive unsolvability of the halting problem as a result about pro-

1 In [4] the focus is on modeling Turing machine computation, while in this paper the focus is on stack machines. In addition [4] explains the semantics of instruction sequences via thread algebra (see [3]) and in this paper we will use an operational semantics instead.
grams rather than machines, and taking instruction sequences as programs, we
analyse the autosolvability requirement that a program of a certain kind must solve
the halting problem for all programs of that kind.

Below we will use the term execution both in connection with instructions
and in connection with instruction sequences. This is not entirely consistent with
[1] where execution is given a rather confined meaning, involving the use of real
computing devices. Here instruction sequences are mathematical objects and their
execution, by necessity is merely a mathematical or logical model for the (real)
putting into effect of (physical representations of) instruction sequences.

The paper follows the organization of [4], beginning with a survey of the in-
struction sequence notation that will be used used in this paper (Section 2). Next,
we introduce services and a composition operator for services families (Section 3).
In Section 4 an operational semantics is provided for instruction sequences under
execution in a context of service families. In Section 5 following [4], we add two
operators, named • and !, that are related to the processing of instructions by a
service family. Then, as in [4] we propose to comply with conventions that ex-
clude the use of terms that are not really intended to denote anything (Sections 6).
Thereafter, we introduce the concept of a functional unit and related concepts (Sec-
tion 7). Then, we define autosolvability and related notions in terms of functional
units related to stack machines (Section 8). In Section 9 we specify a familiar menu
of method names and operations on stacks. In Section 10 we introduce the strong,
intermediate, and weak Turing impossibility properties for programming environ-
ments. equipped with a given and fixed way to encode instruction sequences into
functional unit states. After that, we give positive and negative results concerning
the autosolvability of the halting problem (Section 11). In Section 12 we provide a
number of questions concerning Turing impossibility for stack machine program-
ing. Finally, we make some concluding remarks (Section 13).

2 PGLB with Boolean Termination

In this section, we introduce the program notation PGLB bt (PGLB with Boolean
termination). In [2], a hierarchy of program notations rooted in program algebra
is presented. One of the program notations that belong to this hierarchy is PGLB
(ProGramming Language B). This program notation is close to existing assembly
languages and has relative jump instructions. PGLB bt is PGLB extended with two
termination instructions that allow for the execution of an instruction sequence
to yield a Boolean value at termination. The extension makes it possible to deal
naturally with instruction sequences that implement some test, which is relevant
throughout the paper.

In PGLB bt, it is assumed that a fixed but arbitrary non-empty finite set A of
basic instructions has been given. The intuition is that the issuing of a basic in-
struction in most instances effects the modification of a state and in all instances

2 In fact execution as used in this paper corresponds to “directly putting into effect” as
used in [1]. Instructions are said to be executed as well, or alternatively instructions are said
to be issued, thus following a common terminology in computer architecture.
produces a reply at its completion. The possible replies are \( t \) (standing for true) and \( f \) (standing for false), and the actual reply is in most instances state-dependent. Therefore, successive executions of the same basic instruction may produce different replies.

\( \text{PGLB}_{bt} \) has the following primitive instructions:

- for each \( a \in \mathcal{A} \), a plain basic instruction \( a \);
- for each \( a \in \mathcal{A} \), a positive test instruction \(+a\);
- for each \( a \in \mathcal{A} \), a negative test instruction \(-a\);
- for each \( l \in \mathbb{N} \), a forward jump instruction \(#l\);
- for each \( l \in \mathbb{N} \), a backward jump instruction \( \backslash#l\);
- a plain termination instruction \(!\);
- a positive termination instruction \(!t\);
- a negative termination instruction \(!f\).

\( \text{PGLB}_{bt} \) instruction sequences have the form \( u_1; \ldots; u_k \), where \( u_1, \ldots, u_k \) are primitive instructions of \( \text{PGLB}_{bt} \).

In the process of executing a \( \text{PGLB}_{bt} \) instruction sequence, these primitive instructions have the following effects:

- the effect of a positive test instruction \(+a\) is that basic instruction \( a \) is executed and the execution proceeds with the next primitive instruction if \( t \) is produced and otherwise the next primitive instruction is skipped and the execution proceeds with the primitive instruction following the skipped one – if there is no primitive instruction to proceed with, deadlock occurs;
- the effect of a negative test instruction \(-a\) is the same as the effect of \(+a\), but with the role of the value produced reversed;
- the effect of a plain basic instruction \( a \) is the same as the effect of \(+a\), but a run always proceeds as if \( t \) is produced;
- the effect of a forward jump instruction \(#l\) is that the execution proceeds with the \( l \)th next primitive instruction – if \( l \) equals 0 or there is no primitive instructions to proceed with, deadlock occurs;
- the effect of a backward jump instruction \( \backslash#l\) is that the execution proceeds with the \( l \)th previous primitive instruction – if \( l \) equals 0 or there is no primitive instruction to proceed with, deadlock occurs;
- the effect of the plain termination instruction \(!\) is that the execution terminates and in doing so does not deliver a value;
- the effect of the positive termination instruction \(!t\) is that the execution terminates and in doing so delivers the Boolean value \( t \);
- the effect of the negative termination instruction \(!f\) is that the execution terminates and in doing so delivers the Boolean value \( f \).

A simple example of a \( \text{PGLB}_{bt} \) instruction sequence is

\[ +a; #2; \backslash#2; b; !t . \]

When executing this instruction sequence, first the basic instruction \( a \) is issued repeatedly until its execution produces the reply \( t \), next the basic instruction \( b \) is executed, and after that the run terminates with delivery of the value \( t \).
From Section [7], we will use a restricted version of PGLB<sub>bt</sub> called PGLB<sub>sbt</sub> (PGLB with strict Boolean termination). The primitive instructions of PGLB<sub>sbt</sub> are the primitive instructions of PGLB<sub>bt</sub> with the exception of the plain termination instruction. Thus, PGLB<sub>sbt</sub> instruction sequences are PGLB<sub>bt</sub> instruction sequences in which the plain termination instruction does not occur.

We will write IS to denote the set of PGLB<sub>bt</sub> instruction sequences below. We will view this set of instruction sequences as a sort in a many-sorted algebra for which the sort name IS will be used. Further each PGLB<sub>bt</sub> instruction sequence is used as a constant of sort IS denoting itself.

3 Services and Service Families

In this section, we introduce service families and a composition operator for service families. We start by introducing services.

It is assumed that a fixed but arbitrary non-empty finite set \( \mathcal{M} \) of methods has been given. A service is able to process certain methods. The processing of a method may involve a change of the service. At completion of the processing of a method, the service produces a reply value. The set \( \mathcal{R} \) of reply values is the set \( \{ t, f, d \} \). The reply value \( d \) stands for divergent.

For example, a service may be able to process methods for pushing a natural number on a stack (push \( n \)), testing whether the top of the stack equals a natural number (topop \( n \)), and popping the top element from the stack (pop). Execution of a pushing method or a popping method changes the service, because it changes the stack with which it deals, and produces the reply value \( t \) if no stack overflow or stack underflow occurs and \( f \) otherwise. Execution of a testing method does not change the service, because it does not change the stack with which it deals, and produces the reply value \( t \) if the test succeeds and \( f \) otherwise. Attempted processing of a method that the service is not able to process changes the service into one that is not able to process any method and produces the reply \( d \).

In SF, the algebraic theory of service families introduced below, the following is assumed with respect to services:

- a set \( \mathcal{S} \) of services has been given together with:
  - for each \( m \in \mathcal{M} \), a total function \( \delta_m : \mathcal{S} \to \mathcal{S} \);
  - for each \( m \in \mathcal{M} \), a total function \( \rho_m : \mathcal{S} \to \mathcal{R} \);
  satisfying the condition that there exists a unique \( S \in \mathcal{S} \) with \( \delta_m(S) = S \) and \( \rho_m(S) = d \) for all \( m \in \mathcal{M} \);
- a signature \( \Sigma_\mathcal{S} \) has been given that includes the following sort:
  - the sort \( S \) of services;

\[ ^3 \text{This treatment of the sort IS is a shortcut of the presentation of [4] and [2] where the sort of threads is used as a behavioral abstraction of instruction sequences.} \]
and the following constant and operators:

- the empty service constant \( \delta : S \);
- for each \( m \in M \), the derived service operator \( \frac{\partial}{\partial m} : S \to S \);
- \( S \) and \( \Sigma S \) are such that:
  - each service in \( S \) can be denoted by a closed term of sort \( S \);
  - the constant \( \delta \) denotes the unique \( S \in S \) such that \( \frac{\partial}{\partial m}(S) = S \) and \( \rho_m(S) = d \) for all \( m \in M \);
  - if closed term \( t \) denotes service \( S \), then \( \frac{\partial}{\partial m}(t) \) denotes service \( \frac{\partial}{\partial m}(S) \).

When a request is made to service \( S \) to process method \( m \):

- if \( \rho_m(S) \neq d \), then \( S \) processes \( m \), produces the reply \( \rho_m(S) \), and next proceeds as \( \frac{\partial}{\partial m}(S) \);
- if \( \rho_m(S) = d \), then \( S \) is not able to process method \( m \) and proceeds as \( \delta \).

The empty service \( \delta \) is the unique service that is unable to process any method.

It is also assumed that a fixed but arbitrary non-empty finite set \( \mathcal{F} \) of foci has been given. Foci play the role of names of services in the service family offered by an execution architecture. A service family is a set of named services where each name occurs only once.

\( SF \) has the sorts, constants and operators in \( \Sigma \mathcal{F} \) and in addition the following sort:

- the sort \( SF \) of service families;

and the following constant and operators:

- the empty service family constant \( 0 : SF \);
- for each \( f \in \mathcal{F} \), the unary singleton service family operator \( f : S \to SF \);
- the binary service family composition operator \( \oplus : SF \times SF \to SF \);
- for each \( F \subseteq \mathcal{F} \), the unary encapsulation operator \( \partial_F : SF \to SF \).

We assume that there is a countably infinite set of variables of sort \( SF \) which includes \( u, v, w \). Terms are built as usual in the many-sorted case (see e.g. \([9,7]\)). We use prefix notation for the singleton service family operators and infix notation for the service family composition operator.

The service family denoted by \( 0 \) is the empty service family. The service family denoted by a closed term of the form \( f.H \) consists of one named service only, the service concerned is the service denoted by \( H \), and the name of this service is \( f \). The service family denoted by a closed term of the form \( C \oplus D \) consists of all named services that belong to either the service family denoted by \( C \) or the service family denoted by \( D \). In the case where a named service from the service family denoted by \( C \) and a named service from the service family denoted by \( D \) have the same name, they collapse to an empty service with the name concerned. The service family denoted by a closed term of the form \( \partial_F(C) \) consists of all named services with a name not in \( F \) that belong to the service family denoted by \( C \). Thus, the service families denoted by closed terms of the forms \( f.H \) and \( \partial_{\{f\}}(C) \) do not collapse to an empty service in service family composition.
Using the singleton service family operators and the service family composition operator, any finite number of possibly identical services can be brought together in a service family provided that the services concerned are given different names.

The empty service family constant and the encapsulation operators are primarily meant to axiomatize the operators that are introduced in Section 4.

The axioms of SF are given in Table 1. In this table, \( f \) stands for an arbitrary focus from \( F \) and \( H \) and \( H' \) stand for arbitrary closed terms of sort \( S \). The axioms of SF simply formalize the informal explanation given above.

The \( \text{foci} \) operation \( \text{foci} \) defined by the equations in Table 2 (for \( \text{foci} \) \( f \in F \) and terms \( H \) of sort \( S \)) provides the collection of foci that occur within a service family. Knowledge of this collection plays a role when defining the operational semantics of instruction sequences acting on a service family. The operation \( \text{foci} \) gives, for each service family, the set of all foci that serve as names of named services belonging to the service family.

Given a service family \( C \), if \( f \not\in \text{foci}(C) \) then \( C \) can be written as \( \partial_f (\{f\}) \cdot C' \), and if \( f \in \text{foci}(C) \) then \( C \) can be written as \( f.H \cup \partial_f (\{f\}) \cdot C' \) for a suitable service \( H \) and an appropriate service family \( C' \).

### Table 1  Axioms of SF

| Axiom | Description |
|-------|-------------|
| \( u \oplus \emptyset = u \) | SFC1 | \( \partial_{\emptyset}(\emptyset) = \emptyset \) | SFE1 |
| \( u \oplus v = v \oplus u \) | SFC2 | \( \partial_F(f.H) = \emptyset \) if \( f \in F \) |
| \((u \oplus v) \oplus w = u \oplus (v \oplus w)\) | SFC3 | \( \partial_F(f.H) = f.H \) if \( f \not\in F \) |
| \( f.H \oplus f.H' = f.\delta \) | SFC4 | \( \partial_F(u \oplus v) = \partial_F(u) \oplus \partial_F(v) \) |

### Table 2  Defining equations for the foci operation

| Operation | Description |
|-----------|-------------|
| \( \text{foci}(\emptyset) = \emptyset \) | |
| \( \text{foci}(f.H) = \{f\} \) | |
| \( \text{foci}(u \oplus v) = \text{foci}(u) \cup \text{foci}(v) \) | |

4 Operational semantics

For the set \( \mathcal{A} \) of basic instructions, we take the set \( \{f.m \mid f \in \mathcal{F}, m \in \mathcal{M}\} \). Let \( 1 \leq i \leq k \), and let \( p = u_1; \ldots; u_k \) be a PGLBM instruction sequence, with basic instructions in \( \mathcal{A} \), and for that reason a closed \( IS \) term and let \( C \) denote a service family and at the same time a closed \( SF \) term. Then a triple \((i, p, C)\) can be read as the configuration consisting of \( p \) acting on service family \( C \) with program counter at value \( i \) when \( p \) is executed. Configurations are computational states but we will only use the term state for services and service families, and speak of a configuration if the instruction sequence is included as well as positional information about the instruction which is next to be issued, that is a program counter.

From a non-terminal configuration \((i, p, C)\), subsequent computational steps start with issuing the \( i^{th} \) primitive instruction, i.e. \( u_i \). By default, a run starts at the first primitive instruction. For technical reasons configurations with \( i = 0 \) or \( i > k \) will be considered as well.
The operational semantics describes how a configuration can develop step by step into other configurations. Terminal configurations are configurations that satisfy any of the following conditions:

- \( u_i = \,!, \) or \( u_i = \,!t, \) or \( u_i = \,!f, \) or
- \( i = 0 \) or \( i > 0, \) or
- \( u_i \equiv f.m, \) or \( u_i \equiv +f.m, \) or \( u_i \equiv -f.m \) for a focus \( f \) such that \( f \notin \text{foci}(C). \)

If \( u_i = \,!, \) or \( u_i = \,!t, \) or \( u_i = \,!f, \) then the configuration is correctly terminating. In all other cases the terminating configuration specifies an erroneous state indicating incorrect termination.

The sequence of steps from a configuration is called a computation. Each step involves either the execution of a jump or the application of a method to a service. The service involved in the processing of a method is the service whose name is the focus of the basic instruction in question. After proceeding 0 or more steps a computation can but need not end in a terminal configuration. If it ends in a terminal configuration the computation is said to converge, otherwise it proceeds forever and it is said to diverge. If the terminal configuration is correctly terminating, the computation is said to be successful, otherwise the terminal configuration is incorrectly terminating and the computation is said to be unsuccessful.

Computation steps for configurations are generated by the following four rules:

1. \( u_i \equiv \#k \)
   \[ (i, p, C) \xrightarrow{\text{fw-jmp}} (i + k, p, C) \]

2. \( u_i \equiv \\underline{\#k} \)
   \[ (i, p, C) \xrightarrow{\text{bw-jmp}} (i - k, p, C) \]

3. \( u_i \equiv f.m \lor (u_i \equiv +f.m \land \rho_m(H) = t) \lor (u_i \equiv -f.m \land \rho_m(H) = f) \)
   \[ (i, p, f.H \oplus \partial_{[f]}(C)) \xrightarrow{\text{b-act}} (i + 1, p, f.H \oplus \partial_{[f]}(C)) \]

4. \( u_i \equiv +f.m \land \rho_m(H) = f \lor (u_i \equiv -f.m \land \rho_m(H) = t) \)
   \[ (i, p, f.H \oplus \partial_{[f]}(C)) \xrightarrow{\text{b-act}} (i + 2, p, f.H \oplus \partial_{[f]}(C)) \]

An instruction sequence may interact with the named services from the service family offered by an execution architecture. That is, during its executed an instruction sequence may issue a basic instruction for the purpose of requesting a named service to process a method and to return a reply value at completion of the processing of the method.

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4 Incorrect termination can be understood to represent the occurrence of an error during a computation. For instance execution of the instruction sequence \#1;\#5;\#1 will lead to an error after the first instruction has been executed and for that reason the backward jump in the second instruction constitutes a fault in the instruction sequence.

5 As usual, we write \( i - j \) for the monus of \( i \) and \( j, \) i.e. \( i - j = i - j \) if \( i \geq j \) and \( i - j = 0 \) otherwise.
5 Apply and reply operators

In this section, we combine the sort $\text{IS}$ with the sort $\text{SF}$ and extend the combination with two operators, called apply operator and reply operator respectively, that relate to this kind of interaction between instruction sequences and services.

The reply operator is concerned with the effects of service families on the Boolean values that computations possibly deliver at their termination. The reply operator does not always produce Boolean values: it produces special values in cases where no Boolean value is delivered at termination or no termination takes place. The apply operator determines the successive effect that basic instructions issued during a terminating execution have on a service family. The apply operator is made total by stipulating that it produces the empty service family in the case of diverging computations.

Both operators mentioned above are concerned with the processing of methods by services from a service family in pursuance of basic instructions issued when an instruction sequence is executed.

We will use in addition the following sort:

- the sort $\text{R}$ of replies;

and the following constants and operators:

- the reply constants $t$, $f$, $d$, $m$ : $\text{R}$;
- the binary apply operator $\circledast : \text{IS} \times \text{SF} \rightarrow \text{SF}$;
- the binary reply operator $\circledcirc : \text{IS} \times \text{SF} \rightarrow \text{R}$.

We use infix notation for the apply and reply operators.

The service family denoted by a closed term of the form $p \circledast C$ is the service family that results from processing the method of each basic instruction issued by the instruction sequence $p$ by the service in the service family denoted by $C$ with the focus of the basic instruction as its name if such a service exists.

The value denoted by $p \circledcirc C$ is the Boolean value serving as the flag of the termination instruction at which computation starting from the initial configuration $(1, p, C)$ comes to a halt if that computation terminates correctly and in addition this termination instruction carries a Boolean value.

The value $m$ (standing for meaningless) is yielded if the computation terminates correctly ending with the program counter at a termination instruction not carrying a Boolean value, and the result is the value $d$ (standing for divergent) if the computation does not correctly terminate. Formally the connection between computations and the apply and reply operators is as follows (again assuming that $k$ is the number of instructions in $p = u_1; \ldots; u_k$):

- if $(1, p, C)$ produces a divergent computation then $p \circledast C = \emptyset$ and $p \circledcirc C = d$.
- if $(1, p, C)$ produces an incorrectly terminating computation, say in some configuration $(i, p, D)$ that satisfies one of these five conditions: $i = 0$, or $i > k$, or $u_i \equiv f.m$, or $u_i \equiv +f.m$, or $u_i \equiv -f.m$ for some basic instruction $f.m$ (for which $f \not\in \text{foci}(D)$ must necessarily hold), then $p \circledast C = \emptyset$ and $p \circledcirc C = d$.
- if $(1, p, C)$ produces a correctly terminating computation, say ending in a configuration $(i, p, D)$ such that $1 \leq i \leq k$, and either $u_i \equiv !$, or $u_i \equiv !t$, or $u_i \equiv !f$,
then \( p \cdot C = D \). Further in this case: if \( u_i \equiv ! \) then \( p \downarrow u = t \) or \( p \downarrow u = f \) or \( p \downarrow u = m \). We write \( p \downarrow u \) iff \( p \downarrow u = t \) or \( p \downarrow u = f \) or \( p \downarrow u = m \). We write \( p \downarrow B \) iff \( p \downarrow B = t \) or \( p \downarrow B = f \).

6 Relevant Use Conventions

In the setting of service families, sets of foci play the role of interfaces. The set of all foci that serve as names of named services in a service family is regarded as the interface of that service family. There are cases in which processing does not terminate or, even worse (because it is statically detectable), interfaces of services families do not match. In the case of non-termination, there is nothing that we intend to denote by a term of the form \( p \cdot C \) or \( p \downarrow C \). In the case of non-matching services families, there is nothing that we intend to denote by a term of the form \( C \oplus D \). Moreover, in the case of termination without a Boolean reply, there is nothing that we intend to denote by a term of the form \( p \downarrow C \).

We propose to comply with the following relevant use conventions:

- \( p \cdot C \) is only used if it is known that \( p \downarrow C \);
- \( p \downarrow C \) is only used if it is known that \( p \downarrow B \);
- \( C \oplus D \) is only used if it is known that \( \text{foci}(C) \cap \text{foci}(D) = \emptyset \).

The condition found in the first convention is justified by the fact that \( x \cdot u = \emptyset \) if \( x \uparrow u \). We do not have \( x \cdot u = \emptyset \) only if \( x \uparrow u \). For instance, \( !t \cdot \emptyset = \emptyset \) whereas \( !t \downarrow \emptyset \). Similar remarks apply to the condition found in the second convention.

The idea of relevant use conventions is taken from \cite{5}, where it plays a central role in an account of the way in which mathematicians usually deal with division by zero in mathematical texts. In the sequel, we will comply with the relevant use conventions described above.

7 Functional Units

In this section, we introduce the concept of a functional unit and related concepts.

It is assumed that a non-empty finite or countably infinite set \( S \) of states has been given. As before, it is assumed that a non-empty finite set \( \mathcal{M} \) of methods has been given. However, in the setting of functional units, methods serve as names of operations on a state space. For that reason, the members of \( \mathcal{M} \) will henceforth be called method names.

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\footnote{If it turns out that in some case \( p \downarrow C = f \) a failure has occurred because by using \( p \downarrow C \) the belief is implicitly assumed that \( p \downarrow B \). A plausible cause for that state of affairs is an instruction sequencing fault. That is a mismatch between instruction sequencer intentions and the operational semantics of the instruction sequence that was constructed, for instance if at some place \( ! \) was written where \( !t \) was meant. Another plausible cause is that a mistake was made concerning the choice which instruction sequence from a library of given ones to execute.}
A method operation on \( S \) is a total function from \( S \) to \( \mathbb{B} \times S \). A partial method operation on \( S \) is a partial function from \( S \) to \( \mathbb{B} \times S \). We write \( \mathcal{M}O(S) \) for the set of all method operations on \( S \). We write \( M^r \) and \( M^e \), where \( M \in \mathcal{M}O(S) \), for the unique functions \( R : S \to \mathbb{B} \) and \( E : S \to S \), respectively, such that \( M(s) = (R(s), E(s)) \) for all \( s \in S \).

A functional unit for \( S \) is a finite subset \( \mathcal{H} \) of \( \mathcal{M} \times \mathcal{M}O(S) \) such that \((m,M) \in \mathcal{H} \) and \((m,M') \in \mathcal{H} \) implies \( M = M' \). We write \( \mathcal{F}U(S) \) for the set of all functional units for \( S \). We write \( \mathcal{I}(\mathcal{H}) \), where \( \mathcal{H} \in \mathcal{F}U(S) \), for the set \( \{ m \in \mathcal{M} | \exists M \in \mathcal{M}O(S) \cdot (m,M) \in \mathcal{H} \} \). We write \( m_{\mathcal{H}} \), where \( \mathcal{H} \in \mathcal{F}U(S) \) and \( m \in \mathcal{I}(\mathcal{H}) \), for the unique \( M \in \mathcal{M}O(S) \) such that \((m,M) \in \mathcal{H} \).

We look upon the set \( \mathcal{I}(\mathcal{H}) \), where \( \mathcal{H} \in \mathcal{F}U(S) \), as the interface of \( \mathcal{H} \). It looks to be convenient to have a notation for the restriction of a functional unit to a subset of its interface. We write \( (I,\mathcal{H}) \), where \( \mathcal{H} \in \mathcal{F}U(S) \) and \( I \subseteq \mathcal{I}(\mathcal{H}) \), for the functional unit \( \{(m,M) \in \mathcal{H} | m \in I \} \).

Let \( \mathcal{H} \in \mathcal{F}U(S) \). Then an extension of \( \mathcal{H} \) is an \( \mathcal{H}' \in \mathcal{F}U(S) \) such that \( \mathcal{H} \subseteq \mathcal{H}' \).

According to the definition of a functional unit, \( \emptyset \in \mathcal{F}U(S) \). By that we have a unique functional unit with an empty interface, which is not very interesting in itself. However, when considering services that behave according to functional units, \( \emptyset \) is exactly the functional unit according to which the empty service \( \mathcal{H} \) (the service that is not able to process any method) behaves.

We will use PGLB_{ab} instruction sequences to derive partial method operations from the method operations of a functional unit. We write \( \mathcal{L}(fI) \), where \( I \subseteq \mathcal{M} \), for the set of all PGLB_{ab} instruction sequences, taking the set \( \{ f.m \ | \ m \in I \} \) as the set \( \mathcal{M} \) of basic instructions.

The derivation of partial method operations from the method operations of a functional unit involves services whose processing of methods amounts to replies and service changes according to corresponding method operations of the functional unit concerned. These services can be viewed as the behaviours of a machine, on which the processing in question takes place, in its different states. We take the set \( \mathcal{F}U(S) \times S \) as the set \( \mathcal{I} \) of services. We write \( \mathcal{H}(s) \), where \( \mathcal{H} \in \mathcal{F}U(S) \) and \( s \in S \), for the service \( (\mathcal{H},s) \). The functions \( \partial_{dm} \) and \( \rho_m \) are defined as follows:

\[
\partial_{dm}(\mathcal{H}(s)) = \begin{cases} 
\mathcal{H}(m'_{\mathcal{H}}(s)) & \text{if } m \in \mathcal{I}(\mathcal{H}) \\
\emptyset(s') & \text{if } m \notin \mathcal{I}(\mathcal{H}) 
\end{cases},
\]

\[
\rho_m(\mathcal{H}(s)) = \begin{cases} 
m'_{\mathcal{H}}(s) & \text{if } m \in \mathcal{I}(\mathcal{H}) \\
d & \text{if } m \notin \mathcal{I}(\mathcal{H}) 
\end{cases},
\]

where \( s' \) is a fixed but arbitrary state in \( S \). We assume that each \( \mathcal{H}(s) \in \mathcal{I} \) can be denoted by a closed term of sort \( S \). In this connection, we use the following notational convention: for each \( \mathcal{H}(s) \in \mathcal{I} \), we write \( \widehat{\mathcal{H}}(s) \) for an arbitrary closed term of sort \( S \) that denotes \( \mathcal{H}(s) \). The ambiguity thus introduced could be obviated by decorating \( \mathcal{H}(s) \) wherever it stands for a closed term. However, in this paper, it is always immediately clear from the context whether it stands for a closed
term. Moreover, we believe that the decorations are more often than not distracting. Therefore, we leave it to the reader to make the decorations mentally wherever appropriate.

Let $\mathcal{H} \in \mathcal{F} \mathcal{U}(S)$, and let $I \subseteq \mathcal{I}(\mathcal{H})$. Then an instruction sequence $x \in \mathcal{L}(f, I)$ produces a partial method operation $|x|_\mathcal{H}$ as follows:

$$
|x|_\mathcal{H}(s) = (|x|'_\mathcal{H}(s), |x|''_\mathcal{H}(s)) \text{ if } |x|'_\mathcal{H}(s) = t \lor |x|''_\mathcal{H}(s) = f,
$$

$$
|x|'_\mathcal{H}(s) \text{ is undefined} \quad \text{if } |x|''_\mathcal{H}(s) = d,
$$

where

$$
|x|'_\mathcal{H}(s) = x \ f \ H(s),
$$

$$
|x|''_\mathcal{H}(s) = \text{the unique } s' \in S \text{ such that } x \bullet f \ H(s) = f \ H(s').
$$

If $|x|_\mathcal{H}$ is total, then it is called a derived method operation of $\mathcal{H}$.

8 Functional Units for Stack Machines

In this section, we define some notions that have a bearing on the halting problem in the setting of $\text{PGLB}_{\text{str}}$ and functional units. The notions in question are defined in terms of functional units for the following state space:

$$
T_s = \{0, 1, :\}^*.
$$

The elements of $T_s$ can be understood as the possible contents of the tape of a stack whose alphabet is $\{0, 1, :\}$. It is assumed that the top is the left-most element.

The colon serves as a separator of bit sequences. This is for instance useful if the input of a program consists of another program and an input to the latter program, both encoded as a bit sequences. We could have taken any other tape alphabet whose cardinality is greater than one, but $\{0, 1, :\}$ is quite handy when dealing with issues relating to the halting problem.

Below, we will use a computable injective function $\alpha: T_s \rightarrow \mathbb{N}$ to encode the members of $T_s$ as natural numbers. Because $T_s$ is a countably infinite set, we assume that it is understood what is a computable function from $T_s$ to $\mathbb{N}$. An obvious instance of a computable injective function $\alpha: T_s \rightarrow \mathbb{N}$ is the one where $\alpha(a_1, \ldots, a_n)$ is the natural number represented in the quaternary number-system by $a_1 \ldots a_n$ if the symbols 0, 1, and : are taken as digits representing the numbers 1, 2, and 3, respectively.

A method operation $M \in \mathcal{M}(T_s)$ is computable if there exist computable functions $F, G: \mathbb{N} \rightarrow \mathbb{N}$ such that $M(v) = (\beta(F(\alpha(v)))), \alpha^{-1}(G(\alpha(v))))$ for all $v \in T_s$, where $\alpha: T_s \rightarrow \mathbb{N}$ is a computable injection and $\beta: \mathbb{N} \rightarrow \mathbb{B}$ is inductively defined by $\beta(0) = t$ and $\beta(n + 1) = f$. A functional unit $\mathcal{H} \in \mathcal{F} \mathcal{U}(T_s)$ is computable if, for each $(m, M) \in \mathcal{H}$, $M$ is computable.

It is assumed that, for each $\mathcal{H} \in \mathcal{F} \mathcal{U}(T_s)$, a computable injective function from $\mathcal{L}(f, \mathcal{I}(\mathcal{H}))$ to $\{0, 1\}^*$ with a computable image has been given that yields, for each $x \in \mathcal{L}(f, \mathcal{I}(\mathcal{H}))$, an encoding of $x$ as a bit sequence. If we consider the case where the jump lengths in jump instructions are character strings representing
proof
This follows immediately from the definition of a Turing machine with tape alphabet \( \{0, 1, \ldots\} \) and the axioms for \(!\).

The method operation \( Dup \) is a derived method operation of the above-mentioned functional unit whose method operations correspond to the basic steps that a Turing machine can perform on its tape. This follows immediately from the computability of \( Dup \) and the universality of this functional unit.
Below we will make use of two simple transformations of PGLB_{sbt} instruction sequences that affect only their termination behaviour on and in particular the Boolean value yielded at termination in the case of termination. Here, we introduce notations for those transformations.

Let \( x \) be a PGLB_{sbt} instruction sequence. Then we write \( \text{swap}(x) \) for \( x \) with each occurrence of \(!t\) replaced by \(!f\) and each occurrence of \(!f\) replaced by \(!t\), and we write \( \text{f2d}(x) \) for \( x \) with each occurrence of \(!f\) replaced by \( #0 \). In the following proposition, the most important properties relating to these transformations are stated.

**Proposition 2** Let \( x \) be a PGLB_{sbt} instruction sequence. Then:

1. if \( x ! u = t \) then \( \text{swap}(x) ! u = f \) and \( \text{f2d}(x) ! u = t \);
2. if \( x ! u = f \) then \( \text{swap}(x) ! u = t \) and \( \text{f2d}(x) ! u = d \).

The proof is a trivial adaptation of the elementary proof of the corresponding statement in the case of Turing Machine tapes, instead of Stack Machine data.

### 9 Method names and method operations for a stack

At this stage it is useful to lay down the names and meaning of the common methods for stack manipulation. This can be done in many ways, and any choice will do. The interface \( I_s \) consists of the following ten method names. These eight methods are taken together in a functional unit \( H_s \) that represents a stack with this particular three symbol alphabet as a functional unit over \( T_s \).

- empty leaves the state of the functional unit unchanged and returns \( t \) if the state represent an empty stack and \( f \) otherwise.
- pop deletes the leftmost symbol, and returns reply \( t \), if the stack is non-empty, otherwise it leaves the stack empty and returns \( f \).
- push:0, push:1 and push:c insert respectively 0, 1 and \( : \) on the left most position and each return \( f \).
- topeq:0, topeq:1, and topeq:c each test for the presence of a specific character at the top of the stack. If the stack is empty or its top differs from the symbol mentioned in the basic instruction name the reply is \( f \), otherwise it is \( t \). In all cases the stack is left unchanged.

As mentioned above Dup is a method on stacks as well, but it is not included in the methods on \( H_s \).

### 10 Turing Impossibility Properties

The recursive unsolvability theorem by Turing is an impossibility result which may be found in many different circumstances. Looking at its proof that proof establishes the negation of potential autosolvability. Subsequently by combining it with the Church–Turing thesis that fact can be phrased in terms of recursive solvability in general.
As an impossibility result we take Turing’s theorem to establish the impossibility of a reflexive solution of the halting problem in any functional unit in $\mathcal{F}U(T_u)$ extending $\mathcal{H}$. That state of affairs concerning a programming environment will be termed the (strong) Turing impossibility property. We formulate this only for functional units in $\mathcal{F}U(T_u)$ but it should be clear that these definitions can be adapted to many contexts that allow an encoding of programs (instruction sequences) into the state space upon which a program is acting when executed. Consider a functional unit $\mathcal{H} \in \mathcal{F}U(T_u)$, and let $I = I(\mathcal{H})$. The pair $(L(f.I), \mathcal{H})$ constitutes an instruction sequence programming environment. For programming environments of this kind we introduce the following notions.

- The programming environment has the **strong Turing impossibility property** if its halting problem is not potentially autosolvable. By default Turing Impossibility refers to strong Turing impossibility if no further qualification is provided.
- The programming environment has the **intermediate Turing Impossibility Property** if its halting problem is not potentially recursively autosolvable.
- The programming environment has the **weak Turing impossibility property** if its halting problem is not autosolvable.

It has been established in [4] and implicitly in [6] that the strong Turing impossibility property holds for some programming environments where the halting problem is recursively solvable. This is an interesting situation because it combines the intuitions of two seemingly incompatible worlds: general computability on machines with an unbounded state space where Turing impossibility is taken for granted, and the computing devices that emerge from digitalized electrical engineering where everything is finite state and where for that reason all problems have computable solutions, however inefficient these solutions may be.

We have no information about the existence of programming environments that have the intermediate Turing impossibility property but not the strong one and also not about the existence of programming environments that satisfy the weak Turing impossibility property and not the intermediate one. At this stage we have no indication that such examples will be of methodological importance for the theory of computer programming.

### 11 Strong Turing impossibility in the presence of dup

The following theorem tells us essentially that potential autosolvability of the halting problem is precluded in the presence of the method operation $\text{Dup}$.

**Theorem 1** Let $\mathcal{H} \in \mathcal{F}U(T_u)$ be such that $(\text{dup}, \text{Dup}) \in \mathcal{H}$, and let $I \subseteq I(\mathcal{H})$ be such that $\text{dup} \in I$. Then there does not exist an $x \in L(f.I(\mathcal{H}))$ such that $x$ produces a reflexive solution of the halting problem for $L(f.I)$ with respect to $\mathcal{H}$.

**Proof** Assume the contrary. Let $x \in L(f.I(\mathcal{H}))$ be such that $x$ produces a reflexive solution of the halting problem for $L(f.I)$ with respect to $\mathcal{H}$, and let $y = f.\text{dup} : f.2d(\text{swap}(x))$. Then $x \downarrow f.\mathcal{H}(\mathcal{T}y)$. By Proposition [2] it follows that
swap(x) ↓ f. sóc (y, y) and either swap(x) ! f. sóc (y, y) = t or swap(x) ! f. sóc (y, y) = f.

In the case where swap(x) ! f. sóc (y, y) = t, we have by Proposition 2 that (i) f2d(swap(x)) ! f. sóc (y, y) = t and (ii) x ! f. sóc (y, y) = f. By Proposition 1 it follows from (i) that (f. dup; f2d(swap(x))) ! f. sóc (y) = t. Since y = f. dup; f2d(swap(x)), we have y ! f. sóc (y) = t. On the other hand, because x produces a reflexive solution, it follows from (ii) that y ↑ f. sóc (y). This contradicts with y ! f. sóc (y) = t.

In the case where swap(x) ! f. sóc (y, y) = f, we have by Proposition 2 that (i) f2d(swap(x)) ! f. sóc (y, y) = d and (ii) x ! f. sóc (y, y) = t. By Proposition 1 it follows from (i) that (f. dup; f2d(swap(x))) ! f. sóc (y) = d. Since y = f. dup; f2d(swap(x)), we have y ! f. sóc (y) = d. On the other hand, because x produces a reflexive solution, it follows from (ii) that y ↓ f. sóc (y). This contradicts with y ! f. sóc (y) = d. □

It is easy to see that Theorem 1 goes through for all functional units for T, of which Dup is a derived method operation.

Now, let sóc = \{(dup, Dup)\}. By Theorem 1 the halting problem for L(f, \{dup\}) with respect to sóc is not (potentially) autosolvable. However, it is recursively solvable.

**Theorem 2** Let sóc = \{(dup, Dup)\}. Then the halting problem for L(f, \{dup\}) with respect to sóc is decidable.

**Proof** Let x ∈ L(f, \{dup\}), and let x’ be x with each occurrence of f. dup and +f. dup replaced by #1 and each occurrence of −f. dup replaced by #2. For all v ∈ T, Dup’(v) = t. Therefore, x ↓ f. sóc (v) ⇔ x’ ↓ t for all v ∈ T. Because x’ is finite, x’ ↓ t is decidable. □

### 12 Open issues on Turing Impossibility properties for stack machine programming

About Turing impossibility properties for stack machine programming we know in fact almost nothing except the result just proven that presence of dup implies the strong Turing impossibility property.

Let sóc’ result from sóc by removing the method push: c. It follows from the results in 14 that this functional unit yields a programming system for which the halting problem is potentially recursively autosolvable. The difference made by the presence of this one method is quite remarkable.

It is now easy to formulate several plausible questions which are open to the best of our knowledge. Indeed the objective of this lengthy paper is no more than to introduce the terminology of Turing impossibility properties and to state these problems in full detail. Let sóc,dup denote the extension of sóc with the method dup. Is, dup is its interface.

1. Is the halting problem for L(f, Is, dup) w.r.t. sóc, dup recursively solvable?7
2. If so, can $\mathcal{H}_{s,\text{dup}}$ be extended with methods that are not derivable from $\mathcal{H}_{s,\text{dup}}$ without destroying recursive solvability of the halting problem?

3. Does the programming system $\mathcal{L}(f, I)$ with $\mathcal{H}_e$ feature the weak Turing impossibility property?

4. If so, what about the intermediate and strong Turing impossibility properties?

Some remarks concerning the motivation of these questions is in order. To begin with, the virtue of separating Turing impossibility from recursive unsolvability is that the technical content of the recursive unsolvability proof for the Halting problem is made independent from the Church-Turing thesis. However convincing that thesis may be, unquestionably it is strongly connected with general computability theory on the infinite set of natural numbers. As a conceptual toolkit for understanding the practice of computation recursion theory on the natural numbers can be questioned, however.

Phrasing the halting problem in terms of program machine interaction, rather than exclusively in terms of machines correlates with the fact that the intuition of computing on an unbounded platform has been so successful for the development and deployment of high level program notations. Much more so than for the area computer architecture which always keeps the underlying electric circuitry in mind, and for which the digital perspective means that an abstraction can be made from infinite state machines in need of a probabilistic analysis to finite state machines that can be understood, at least in principle, without the use of probabilities.

13 Concluding Remarks

We have put forward three flavors of the Turing impossibility property: strong, intermediate and weak. These notions have been applied to stack machine programming. Some results concerning that case have been translated from the work on Turing machines in [4], and several open questions have been formulated.

Programming environments which satisfy the strong Turing impossibility property and for which the halting problem is recursively solvable at the same time constitute an interesting bridge between the two worlds of computer science: general computation without bounds on memory and time, and finite state computation in bounded time. The existence of these combined circumstances depends on being specific on how the encoding of instruction sequences into data is achieved. The classical Turing impossibility property for a Turing complete programming environment is not dependent on the specific way in which that encoding is done, in that sense the classical approach is more general.

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