On invariant subspaces of dissipative operators in a space with indefinite metric

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Abstract. The theorem on the existence of maximal nonnegative invariant subspaces for a special class of dissipative operators in Hilbert space with indefinite inner product is proved in the paper. It is shown in addition that the spectra of the restrictions of these operators on the corresponding invariant subspaces lie in the closed upper half-plane. The obtained theorem is a generalization of well-known results of L. S. Pontrjagin, H. K. Langer, M. G. Krein and T. Ja. Azizov devoted to this subject.

1. Introduction

Let $H$ be a separable Hilbert space with usual scalar product $(x, y)$ and indefinite one $[x, y] = (Jx, y)$, where $J = P_+ - P_-$ and $P_+, P_-$ are the orthoprojectors such that $P_+P_- = P_-P_+ = 0$, $P_+ + P_- = I$ and $I$ is the identity operator. Obviously, $J$ admits such a representation if and only if $J = J^*$ and $J^2 = I$. The space $\{H, J\}$ is called the Pontrjagin space and is denoted by $\Pi_\kappa$, if either rank $P_+$ or rank $P_-$ is finite and equals $\kappa$. It is called the Krein space if both later numbers are infinite. A subspace $L$ in $\{H, J\}$ is called nonnegative (uniformly positive), if $[x, x] \geq 0$ ($\geq \varepsilon(x, x)$ with some $\varepsilon$ independent on $x$) for all $x \in L$. A nonnegative (uniformly positive) subspace $L$ is said to be maximal if there are no nontrivial nonnegative (uniformly positive) extensions of this subspace. Maximal nonpositive and uniformly negative subspaces are defined analogiously.

Let us represent the space $H$ in the form $H = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^\pm = P_\pm(H)$ are the ranges of the orthogonal projectors $P_\pm$. Consider a linear operator $A$ in $H$ with domain of definition $\mathcal{D}(A)$. The spectrum and the resolvent set of $A$ is denoted further by $\sigma(A)$ and $\rho(A)$. An operator $A$ is called dissipative in $H$ if $\text{Im}(Ax, x) \geq 0$ for all $x \in \mathcal{D}(A)$. A dissipative operator is called maximal dissipative if there are no nontrivial dissipative extensions of this operator. It is known [1] Ch. V, §3.10 that the later condition holds if and only if $\rho(A) \supset \mathbb{C}^+$ where $\mathbb{C}^+$ is the open upper half-plane. An operator $A$ is called dissipative (maximal dissipative) in the space $\{H, J\}$ if $JA$ is dissipative (maximal dissipative) in $H$. Analogously, $A$ is called symmetric (self-adjoint) in the space $\{H, J\}$ if $JA$ is symmetric (self-adjoint) in the space $H$.

In the sequel, we work only with operators $A$ for which the sum $\mathcal{D}^+ \oplus \mathcal{D}^-$ is dense in $H$, where $\mathcal{D}^\pm = \mathcal{D}(A) \cap \mathcal{H}^\pm$. We will always assume that $\mathcal{D}(A) = \mathcal{D}^+ \oplus \mathcal{D}^-$, otherwise we can consider the restriction of $A$ to this domain. In this case the operator $A$ can be represented as an operator matrix with respect to the decomposition $H = \mathcal{H}^+ \oplus \mathcal{H}^-:

\begin{equation}
A = \begin{pmatrix}
P_+AP_+ & P_+AP_-
p_+AP_+ & P_-AP_-
\end{pmatrix} = \begin{pmatrix}
A_{11} & A_{12}
A_{21} & A_{22}
\end{pmatrix}.
\end{equation}

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The vectors \( x = x_+ + x_- \in \mathcal{H} \) with \( x_\pm \in \mathcal{H}_\pm \) are identified in this representation with the columns \( x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \), and the action of \( A \) is determined by the formula

\[
Ax = A \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \begin{pmatrix} A_{11}x_+ + A_{12}x_- \\ A_{21}x_+ + A_{22}x_- \end{pmatrix}, \quad x_+ \in \mathcal{D}^+, \ x_- \in \mathcal{D}^-.
\]

Pontrjagin [2] proved in 1944 the following fundamental result.

Pontrjagin Theorem. Let \( A \) be a self-adjoint operator in the space \( \{\mathcal{H}, J\} \) and \( \text{rank} \, P_+ = \kappa < \infty \). Then there exists a maximal nonnegative \( A \)-invariant subspace \( \mathcal{L} \) (\( \dim \mathcal{L} = \kappa \)) such that the spectrum of the restriction \( A|_\mathcal{L} \) lies in the closed upper half-plane.

Starting from paper [2] the problem on the existence of maximal definite invariant subspaces has been a key-stone of the operator theory in Pontrjagin and Krein spaces. Krein [3] obtained an analogue of Pontrjagin theorem for unitary operators in \( \Pi^\kappa \) and developed a new approach to the problem in question. An important generalization of Pontrjagin theorem was obtained by Langer [4, 5] and Krein [6]. Let us present here the result [5].

Langer Theorem. Let \( A \) be a selfadjoint operator in Krein space \( \{\mathcal{H}, J\} \) and \( \mathcal{D}(A) \supset \mathcal{H}^+ \) (the later condition holds if and only if \( A \) admits representation (1) where \( A_{11} \) and \( A_{12} \) are bounded). If in addition the operator \( A_{12} = P_+AP_- \) is compact, then there exists a maximal \( A \)-invariant subspace \( \mathcal{L} \) such that the spectrum of the restriction \( A|_\mathcal{L} \) lies in the closed upper half-plane.

Later on the theorems on the existence of \( A \)-invariant subspaces have been obtained for other classes of operators. Krein brought into consideration and investigated the class of definite operators, and Langer [7, 8] proved the theorem on the existence of maximal definite invariant subspaces for a wider class of the so-called definitizable operators and obtained for these operators an analogue of the spectral theorem. Krein and Langer [9] and independently Azizov [10] showed that Pontrjagin theory remains to be valid (as before in Pontrjagin space \( \{\mathcal{H}, J\}, \text{rank} \, P_+ = \kappa < \infty \)) if the condition for \( A \) to be self-adjoint is replaced by the condition to be maximal dissipative. Later on, Azizov and Khoroshavin [11] proved an analogue of Langer theorem for a class of nonstretching operators in Krein space, and Azizov [12] Ch. 2 proved that Langer theorem [5] remains to be valid for maximal dissipative operators in Krein space. A direct and shorter proof of the later result was suggested by the author [13].

The Langer condition \( \mathcal{D}(A) \supset \mathcal{H}^+ \) (or equivalently the boundedness of the operators \( A_{11}, A_{21} \)) is rather restrictive. In particular, often in concrete problems (see [14, 15], for example) the operator \( A_{21} \) is unbounded.

2. Main result

The goal of the present paper is to obtain a generalization of Pontrjagin–Krein–Langer–Azizov theorem dropping out the Langer condition \( \mathcal{D}(A) \supset \mathcal{H}^+ \), i.e. the condition for the operators \( A_{11} \) and \( A_{21} \) to be bounded. The essence of the assumptions formulated below can be expressed as follows: the operator \( A_{22} \) is dominant with respect to the interlacing operators \( A_{21} \) and \( A_{12} \), and the so-called transfer-function of the operator matrix (1) is bounded. Let us formulate the main result.
Theorem. Let $A$ be a dissipative operator in Krein space $\{H, J\}$ and its domain $D(A) = D^+ \oplus D^-$ be dense in $H = H^+ \oplus H^-$. Let (1) be the matrix representation of $A$ in $H^+ \oplus H^-$ and the following conditions hold:

1. the operator $-A_{22}$ is maximal dissipative in the space $H^-$ (and hence the resolvent $(A_{22} - \mu)^{-1}$ exists for all $\mu \in \mathbb{C}^+$);
2. the operator $F(\mu) = (A_{22} - \mu)^{-1}A_{21}$ admits a bounded closure for some (and hence for all) $\mu \in \mathbb{C}^+$;
3. the operator $G(\mu) = A_{12}(A_{22} - \mu)^{-1}$ is compact for some (and hence for all) $\mu \in \mathbb{C}^+$;
4. the operator

$$S(\mu) = A_{11} - A_{12}(A_{22} - \mu)^{-1}A_{21}$$

admits a bounded closure for some (and hence for all) $\mu \in \mathbb{C}^+$.

Then the closure $\overline{A}$ of the operator $A$ is maximal dissipative in the space $\{H, J\}$, and there exists a maximal nonnegative $A$-invariant subspace $L$ such that the spectrum of the restriction $\overline{A}|_L$ lies in the closed upper half-plane. Moreover, $L \subset D(\overline{A})$, i.e. the operator $\overline{A}|_L$ is bounded.

First we shall make two remarks on the conditions of the above theorem. It is useful to view in mind that condition (ii) is valid if the operator $A_{21}$ is closable (hence the adjoint operator $A_{21}^*$ is densely defined) and $D(A_{21}^*) \supset D(A_{22}^*)$ (it is known [1] Ch. 5] that the adjoint to the dissipative operator $-A_{22}$ is densely defined). In fact, if the condition $D(A_{21}^*) \supset D(A_{22}^*)$ holds, then $F^*(\mu) = A_{21}^*(A_{22}^* - \mu)^{-1}$ is defined on the whole $H^-$ and the adjoint to this operator is the closure of the densely defined operator $F(\mu) = (A_{22} - \mu)^{-1}A_{21}$. Consequently, both operators $F^*(\mu)$ and $F(\mu)$ are bounded. The second remark concerns condition (i) which has not been met in the formulations of the previous theorems on this subject. However, it follows from [2] Ch. 2, Th. 2.9] that if $D(A) \supset H^+$ then $A$ is maximal dissipative in $\{H, J\}$ if and only if $-A_{22}$ is maximal dissipative in $H^-$. Hence conditions (i)-(iv) are weaker than those in theorems of Pontrjagin, Krein, Langer and Azizov.

Later on, if we meet no confusions, we shall write $\mu$ instead of $\mu I$ where $I$ is the identity operator in $H^+, H^-$ or in $H$.

3. Preliminary propositions

We shall premise several lemmas to the proof of Theorem. Lemmas [1] and [2] play the key role.

Lemma 1. A subspace $L$ is maximal nonnegative (uniformly positive) if and only if it can be represented in the form

$$(2)\quad L = \left\{ x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \quad x_+ \in H^+ \right\},$$

where $K : H^+ \to H^-$ is a linear operator with the norm $\|K\| \leq 1$ ($\|K\| < 1$). A nonnegative subspace $L$ is maximal if and only if there exists no nonzero element $y_+ \in H^+$ such that $[x, y_+] = (x, y^+) = 0$ for all $x \in L$.

Proof. (See [2].) Assuming that $L$ is nonnegative subspace in $\{H, J\}$ we have $\|x_+\| \geq \|x_-\|$ for all $x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in L$. Then the restriction $Q = P_+|_L : L \to P_+(L)$ is a
bijection, and \( \|Q^{-1}\| \leq 2 \). Therefore,

\[
\mathcal{L} = \left\{ x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \quad x_+ \in P_+(\mathcal{L}), \quad K = P_-Q^{-1} \right\}.
\]

Here \( \|K\| \leq 1 \) if \( \mathcal{L} \) is nonnegative and \( \|K\| < 1 \) if \( \mathcal{L} \) is uniformly positive. Obviously, \( \mathcal{L} \) is maximal if and only if \( P_+(\mathcal{L}) = \mathcal{H}^+ \). The second assertion of Lemma is also obvious. \( \square \)

The operator \( K \) participating in representation \([2]\) is said to be the angle operator of the subspace \( \mathcal{L} \).

**Lemma 2.** Let \( A \) be an operator with dense domain \( \mathcal{D}(A) = \mathcal{D}^+ \oplus \mathcal{D}^- \), the resolvent set \( \rho(A_{22}) \) be nonempty, and the operators

\[
G = A_{12}(A_{22} - \mu)^{-1}, \quad F = (A_{22} - \mu)^{-1}A_{21}, \quad S = A_{11} - A_{12}F
\]

be bounded for some \( \mu \in \rho(A_{22}) \). Then \( A \) is closable and its closure is given by the relation

\[
\overline{A} = \mu + \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S - \mu & 0 \\ 0 & A_{22} - \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix}.
\]

More precisely, the domain and the action of \( \overline{A} \) are defined by the relations

\[
\mathcal{D}(\overline{A}) = \left\{ \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in \mathcal{H}, \quad x_+ \in \mathcal{H}^+, \quad Fx_+ + x_- \in \mathcal{D}^- \subset \mathcal{D}(A_{22}) \right\},
\]

\[
\overline{A} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = \begin{pmatrix} Sx_+ + G(A_{22} - \mu)(Fx_+ + x_-) \\ (A_{22} - \mu)(Fx_+ + x_-) + \mu x_- \end{pmatrix}.
\]

**Proof.** (Cf. \([14]\)). One can easily check the validity of representation \([4]\) for \( x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in \mathcal{D}(A) \). Since the operators \( G, S, F \) are bounded, we conclude that the first and the third matrix in the right hand-side of \([4]\) are invertible, and the second one represents a closed operator. Therefore, \( A \) is closable and representation \([4]\) is valid. The description of \( \mathcal{D}(\overline{A}) \) and the formula for the action of \( \overline{A} \) follows from \([4]\). \( \square \)

**Lemma 3.** Suppose that \(-A_{22}\) is a maximal dissipative operator in \( \mathcal{H}^- \) and \( G(\mu) = A_{12}(A_{22} - \mu)^{-1} \) is compact for some \( \mu \in \mathbb{C}^+ \). Then \( \|G(\mu)\| \to \infty \) as \( \text{Im} \mu \to +\infty \).

**Proof.** It follows from the equation

\[
G(\lambda) = G(\mu) + (\lambda - \mu)G(\mu)(A_{22} - \lambda)^{-1}
\]

that \( G(\lambda) \) is compact for all \( \lambda \in \mathbb{C}^+ \). Further we make use from the relation

\[
G(\mu) = G(i)(A_{22} + i)(A_{22} - \mu)^{-1}.
\]

The norm of the operator function \((A_{22} + i)(A_{22} - \mu)^{-1}\) is uniformly bounded in the half-plane \( \text{Im} \mu \geq \varepsilon \). The compact operator \( G(i) : \mathcal{H}^- \to \mathcal{H}^+ \) can be approximated with arbitrary accuracy in the norm operator topology by a finite rank operator. Hence it suffices to prove that \( \|Q(A_{22} + i)(A_{22} - \mu)^{-1}\| \to 0 \) as \( \text{Im} \mu \to \infty \) for any operator \( Q \) of rank 1. Namely, for \( Q = (\cdot, \varphi_0)\varphi_+ \) where \( \varphi_\pm \in \mathcal{H}_\pm \). Observe that \( Q \) can be approximated with arbitrary accuracy in the norm operator topology by an operator of the form \( Q_0 = (\cdot, \varphi_0)\varphi_+ \) where \( \varphi_0 \in \mathcal{D}(A_{22}) \) (we already noted that the adjoint to a dissipative operator is densely defined). Now, the
operator $Q_0(A_{22} + i)$ is bounded, and $\|(A_{22} - \mu)^{-1}\| \leq 1/\text{Im} \mu$ for $\mu \in \mathbb{C}^+$. This gives the assertion of Lemma.

**Lemma 4.** Let the conditions of Lemma 2 be preserved for an operator $\overline{A}$ as well as the notations (3) for the operators $G$, $F$, $S$, and $\overline{A}$ for the closure of $A$. Then a subspace

$$\mathcal{L} = \left\{ x = \begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix}, \quad x_+ \in \mathcal{H}^+ \right\}$$

with the angle operator $K : \mathcal{H}^+ \to \mathcal{H}^-$ lies in $\mathcal{D}(\overline{A})$ if and only if the operator $L = A_{21} + (A_{22} - \mu)K : \mathcal{H}^+ \to \mathcal{H}^-$ is well defined on $\mathcal{D}^+$ and admits a bounded closure. If the later condition holds, then the subspace $\mathcal{L}$ is $\overline{A}$-invariant and if and only if

$$L = K(S - \mu + GL),$$

and then the restriction $\overline{A}|_{\mathcal{L}}$ is represented in the form

$$\overline{A}|_{\mathcal{L}} = Q^{-1}(S + GL)Q,$$

where $Q = P^{|\mathcal{L}} : \mathcal{L} \to \mathcal{H}^+$, $\|Q^{-1}\| \leq 1 + \|K\|$.

**Proof.** Let $L = A_{21} + (A_{22} - \mu)K$ be defined on $\mathcal{D}^+$ and admit a bounded closure. Then

$$(A_{22} - \mu)^{-1}Lx_+ = (F + K)x_+ \in \mathcal{D}^+$$

for all $x_+ \in \mathcal{H}^+$. Recalling the description of $\mathcal{D}(\overline{A})$ obtained in Lemma 2, we find $\mathcal{L} \subset \mathcal{D}(\overline{A})$ and

$$(\overline{A} - \mu)\begin{pmatrix} x_+ \\ Kx_+ \end{pmatrix} = \begin{pmatrix} (S - \mu + GL)x_+ \\ Lx_+ \end{pmatrix}.$$  

Conversely, if $\mathcal{L} \subset \mathcal{D}(\overline{A})$, then $(F + K)x_+ \in \mathcal{D}^+$. Hence $L = (A_{22} - \mu)(F + K)$ is defined on the whole $\mathcal{H}^+$. Since the operator $\overline{A} : \mathcal{H} \to \mathcal{H}$ is closed, its restriction $\overline{A} : \mathcal{L} \to \mathcal{H}$ is also closed. The later operator is defined on the whole $\mathcal{L}$. Therefore, it is bounded by virtue of the closed graph theorem, and it follows from (7) that the operator $L$ is bounded. Now, suppose that the subspace $\mathcal{L}$ is $\overline{A}$-invariant. Then given $x_+ \in \mathcal{H}^+$ there exists an element $y_+ \in \mathcal{H}^+$ such that

$$(S - \mu + GL)x_+ = Lx_+$$

This implies equation (5). Conversely, suppose that $\mathcal{L} \subset \mathcal{D}(\overline{A})$ and equation (5) holds. Then relation (8) is valid, and it is equivalent to $\overline{A}$-invariance of the subspace $\mathcal{L}$. The last assertion of Lemma follows from (7). We remark only that the estimates $\|Q\| \leq 1$ and $\|Q^{-1}\| \leq 1 + \|K\|$ follow from the definition of $Q$. \hfill \Box

**Lemma 5.** Let $A$ be an uniformly dissipative operator in the space $\{\mathcal{H}, J\}$, i. e.

$$\text{Im}[Ax, x] \geq \varepsilon(x, x) \quad \text{for} \quad x \in \mathcal{D}(A),$$

where $\varepsilon > 0$. Let $\mathcal{D}(A) \supset \mathcal{H}^+$, and $-A_{22}$ be a maximal dissipative operator in $\mathcal{H}^-$. Then the operator $A$ is maximal dissipative in $\{\mathcal{H}, J\}$, the real axis belongs to the resolvent set $\rho(A)$ and its spectrum in $\mathbb{C}^+$ is bounded. If a Jordan contour $\Gamma_+$ surrounds the set $\sigma(A) \cap \mathbb{C}^+$ and

$$Q_+ = \frac{1}{2\pi i} \int_{\Gamma_+} (\lambda - A)^{-1} d\lambda$$
is the corresponding Riesz projector, then \( \mathcal{L} = Q_+(\mathcal{H}) \) is an \( A \)-invariant maximal uniformly positive subspace. Moreover,

\[
|x, x| \geq 2\varepsilon(\pi\|A_+\|^{-1})^2 \quad \text{for } x \in \mathcal{L}
\]

where \( A_+ = A|_\mathcal{L} \). If for some \( \mu \in \mathbb{C}^+ \) the estimate \( \|G(\mu)\| = \gamma < 1 \) holds, then

\[
\|A_+\| \leq 2(\|S\| + \gamma(1 - \gamma)^{-1}(\|S\| + |\mu|)) \quad \text{for } S = A_{11} - G(\mu)A_{21}.
\]

**Proof.** We already noted referring to [12] Ch. 2, §2 that \( A \) is maximal dissipative in \( \{\mathcal{H}, J\} \) under the assumptions of this Lemma (in fact, if \( JA \) admits a nontrivial dissipative extension in \( \mathcal{H} \), then the condition \( \mathcal{D}(A) \supset \mathcal{H}^+ \) implies that \( -A_{22} \) admits nontrivial dissipative extensions in \( \mathcal{H}^- \), and we come to a contradiction). The other assertions of Lemma but the last estimates were proved in the author’s paper [13]. To prove estimate (12) we have to repeat partially the arguments from [13]. We do this in several steps.

**Step 1.** It follows from the condition \( \mathcal{D}(A) \supset \mathcal{H}^+ \) that the operator \( AP_+ \) is bounded. Take a number \( a > 2\|AP_+\| \). Denote as before \( G(\lambda) = A_{12}(A_{22} - \lambda)^{-1} \) and show that

\[
\|G(\lambda)\| \leq 2 + 2a\varepsilon^{-1} \quad \text{for all } \lambda \in \mathbb{C}^+.
\]

Consider the operator

\[
T(\lambda, a) = \begin{pmatrix} ia & A_{12} \\ 0 & -A_{22} + \lambda \end{pmatrix} = \begin{pmatrix} JA + ia \end{pmatrix} + (\lambda - ia)P_- - AP_+.
\]

The operators \( JA + ia \) and \( T(\lambda, a) + AP_+ \) are maximal dissipative for \( \lambda \in \mathbb{C}^+ \). Moreover, we have the estimate

\[
\text{Im}(T(\lambda, a)x, x) \geq (a/2)\|x\|^2 \quad \text{for } x \in \mathcal{D}(T) = \mathcal{H}^+ \oplus \mathcal{D}(A_{22}),
\]

provided that \( \lambda \in \mathbb{C}_a^+ = \{\lambda \mid \text{Im } \lambda \geq a\} \). Therefore \( T(\lambda, a) \) is invertible for \( \lambda \in \mathbb{C}_a^+ \) and \( \|T^{-1}(\lambda, a)\| \leq 2a^{-1} \). Since \( G(\lambda) = aP_+T^{-1}(\lambda, a)P_- \), we have \( \|G(\lambda)\| \leq 2 \) for \( \lambda \in \mathbb{C}_a^+ \). Using the equation

\[
G(\lambda) = G(\lambda + ia) + iaG(\lambda + ia)(A_{22} - \lambda)^{-1}
\]

we get estimate (13). Here we view in mind that \( \text{Im}(A_{22}x, x) \leq -\varepsilon(x, x) \) and hence \( \|(A_{22} - \lambda)^{-1}\| \leq \varepsilon^{-1} \) for \( \lambda \in \mathbb{C}_a^+ \).

**Step 2.** It follows from representation (11) that \( \lambda \in \rho(A) \cap \mathbb{C}^+ \) if and only if the operator \( S(\lambda) - \lambda \) is invertible. Since the operator \( A_{11} \) is bounded, we have \( (A_{11} - \lambda)^{-1} = -\lambda^{-1} + O(\lambda^{-2}) \) as \( \lambda \to \infty \). Viewing in mind that \( A_{21} \) is bounded and \( G(\lambda) \) is subject to estimate (13) in \( \mathbb{C}^+ \) we find

\[
(S(\lambda) - \lambda)^{-1} = (A_{11} - \lambda)^{-1}(1 - G(\lambda)A_{21})(A_{11} - \lambda)^{-1} = -\lambda^{-1} + O(\lambda^{-2}),
\]

as \( \lambda \in \mathbb{C}_a^+ \) and \( \lambda \to \infty \). Hence the spectrum of \( A \) in \( \mathbb{C}^+ \) is bounded.

**Step 3.** Take a contour \( \Gamma_+ \) consisting of a segment \([-R, R]\) and the half of the circle \( C_R \) in \( \mathbb{C}^+ \) of the radius \( R \) and the center at zero. Taking \( R \) sufficiently large we may insulate that the spectrum of \( A \) in \( \mathbb{C}^+ \) lies inside \( \Gamma_+ \). Consider the Riesz projector \( (14) \). Obviously, the subspace \( \mathcal{L} = Q_+(\mathcal{H}) \) is \( A \)-invariant, and the restriction \( A_+ = A|_\mathcal{L} \) is a bounded operator. We can replace \( A \) by \( A_+ \) in (10). Then we have as \( R \to \infty \)

\[
\frac{1}{2\pi i} \int_{C_R} (\lambda - A_+)^{-1} d\lambda = \frac{1}{2\pi i} \int_{C_R} (\lambda^{-1} + O(\lambda^{-2}))^{-1} d\lambda = \frac{1}{2} I + O(R^{-1}).
\]
Let \( x = Q_+ x \in \mathcal{L} \) and \( y = (\lambda - A_+)^{-1} x \). Then

\[
[x, x] = \text{Re}[Q_+ x, x] = \frac{1}{2} [x, x] + \frac{1}{2\pi} \int_{-R}^{R} \text{Im}[y, (\lambda - A)y] d\lambda + O(R^{-1}).
\]

For \( \lambda \in \mathbb{R} \)

\[
\text{Im}[y, (\lambda - A)y] = \text{Im}[Ay, y] \geq \varepsilon(y, y).
\]

Passing to the limit as \( R \to \infty \) and taking into account the inequality

\[
\|x\| \leq \|\lambda - A_+\| \|y\| \leq (|\lambda| + \|A_+\|) \|y\|,
\]

we get

\[
[x, x] \geq \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im}[Ay, y] d\lambda \geq \frac{\varepsilon}{\pi} \|x\|^2 \int_{-\infty}^{\infty} (\|A_+\| + |\lambda|)^{-2} d\lambda = \frac{2\varepsilon}{\pi \|A_+\|} \|x\|^2.
\]

This proves that \( \mathcal{L} \) is uniformly positive subspace, and the estimate (11) is valid.

**Step 4.** Let us prove that \( \mathcal{L} \) is a maximal uniformly positive subspace. It easily follows from (4) that

\[
((\lambda - A)^{-1} z, z) = ((\lambda - S(\lambda))^{-1} z, z),
\]

where by \(*\) we assign operators which representation is not used in the sequel. For \( z \in \mathbb{H}^+ \) we have

\[
((\lambda - A)^{-1} z, z) = ((\lambda - S(\lambda))^{-1} z, z).
\]

Integrating the function \((2\pi i)^{-1}(\lambda - A)^{-1} z, z)\) along the contour \( \Gamma_+ \), using relations (14) and (15) for the integrals along the half circle \( C_R \) and passing to the limit as \( R \to \infty \), we obtain

\[
(Q_+ z, z) = [Q_+ z, z] = \frac{1}{2}(z, z) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Im}[Ay, y] d\lambda,
\]

where \( y = (\lambda - A)^{-1} z \). Consequently, \( 2(Q_+ z, z) \geq (z, z) \) for all \( z \in \mathcal{H}^+ \). Hence there is no nonzero element \( z \in \mathcal{H}^+ \) such that \( z \perp \mathcal{L} = Q_+(\mathcal{H}) \). By lemma 4 the subspace \( \mathcal{L} \) is maximal positive.

**Step 5.** Finally, let us prove estimate (12) provided that \( \|G(\mu)\| = \gamma < 1 \) for some \( \mu \in \mathbb{C}^+ \). Since \( \mathcal{L} \subset \mathcal{D}(A) \) and \( \mathcal{L} \) is an \( A \)-invariant subspace, by Lemma 3 we have

\[
L = K(S - \mu + GL),
\]

where \( K \) is the angle operator of the subspace \( \mathcal{L} \), and \( L = A_{21} + (A_{22} - \mu)K \). Consequently,

\[
L = (1 - KG)^{-1} K(S - \mu), \quad \|L\| \leq (1 - \gamma)^{-1}(\|S\| + |\mu|).
\]

From (6) we get the inequality

\[
\|A_+\| \leq 2(\|S\| + \gamma\|L\|),
\]

which implies estimate (12). Lemma is proved. \( \square \)
Further, the operators $K(\lambda)$ are empty inside $\Omega$. Therefore, the Riesz projector of $T$ along the contour $\Gamma$ equals zero (since the corresponding Riesz projectors of $T_n$ equal zero). Consequently, the spectrum of $T$ is compact.

4. Proof of Theorem

Take a system of linear independent elements $\{\varphi_n\}_{n=1}^\infty$ belonging to $D^+ = \mathcal{H}^+ \cap D(A)$ such that the linear span of this system is dense in $\mathcal{H}$. Denote by $\mathcal{H}^+_n$ the finite dimensional subspaces with the bases $\{\varphi_1^n\}$ and by $P_n$ the orthoprojectors on these subspaces. Consider the operator

$$A_{n,\varepsilon} = \left( \begin{array}{cc} P_n A_{11} P_n & P_n A_{12} \\ A_{21} P_n & A_{22} \end{array} \right) + i\varepsilon J, \quad \varepsilon > 0,$$

acting in the space $\mathcal{H}_n = \mathcal{H}^+_n \oplus \mathcal{H}^-$ with the domain $D(A_{n,\varepsilon}) = \mathcal{H}^+_n \oplus D^- \subset D(A) = D^+ \oplus D^-$. Let us sketch the plan of the proof. The conditions of Lemma 5 are fulfilled for the operators $A_{n,\varepsilon}$, since

$$\text{Im}[A_{n,\varepsilon} x, x] = \text{Im}[A x, x] + \varepsilon(x, x) \quad \text{for } x \in D(A_{n,\varepsilon}).$$

By virtue of Lemmas 3 and 5 there exist maximal uniformly positive subspaces $L_n$ with the angle operators $K_n : \mathcal{H}^+_n \to \mathcal{H}^-$, $\|K_n\| < 1$, such that

$$\left( 1 - K_n G \right) L_n = K_n (S_n - \mu),$$

where $\mu \in \mathbb{C}^+$ and

$$G = G(\varepsilon + \mu), \quad S_n = P_n (i\varepsilon + S(i\varepsilon + \mu)) P_n, \quad L_n = A_{21} P_n + (A_{22} - i\varepsilon - \mu) K_n.$$

We remark that we can write $G$ in equation (17) instead of $G_n = P_n G$, since $K_n G_n = K_n G$. It will be shown that one can pass to the limit in the weak operator topology in equation (17), choosing a subsequence $n_k \to \infty$. The limit equation

$$\left( 1 - KG \right) L = K(S - \mu),$$

holds with $\|K\| < 1$, $L = A_{21} + (A_{22} - i\varepsilon - \mu) K$, $\|L\| \leq \text{const}$. By virtue of Lemmas 3 and 4 the subspace $L$ with the angle operator $K$ is $A + i\varepsilon J$-invariant and maximal uniformly positive. We remark that one can hardly realize a direct proof of the analogue of Lemma 5 for operators of the form $A + i\varepsilon J$, $\varepsilon > 0$, since there is no simple way to get representation (15) for $S(\lambda) - \lambda$. Further, the operators $K$, $G$, $L$, and $S$ in equation (18) depend on $\varepsilon$. Choosing a proper subsequence $\varepsilon_n \to 0$ one can pass again to the limit in the weak operator topology and obtain equation (18) with an operator $K$, $\|K\| < 1$, and the operators $L = A_{21} + (A_{22} - \mu) K$ and $S = S(\mu)$. Here $L$ is bounded and by Lemmas 3 and 4 the subspace $L$ with the angle operator
$K$ is $\overline{A}$-invariant and maximal nonpositive. On this way we have also to prove that the spectra of the restrictions $\overline{A} + i\varepsilon J$ onto the invariant subspaces $\mathcal{L}_\varepsilon$ lie in the upper half-plane $\mathbb{C}^+$ for $\varepsilon > 0$ and in $\overline{\mathbb{C}^+}$ for $\varepsilon = 0$. From now on we realize the above plan.

By virtue of Lemma 3 we can choose a number $\mu \in \mathbb{C}^+$ such that $\|G\| = \|G(i\varepsilon + \mu)\| < 1/2$ for all $0 \leq \varepsilon \leq 1$. The operator function $i\varepsilon + S(i\varepsilon + \mu)$ is continuous for $0 \leq \varepsilon \leq 1$ in the norm operator topology. Hence there is a constant $c$ such that

$$
\|i\varepsilon + S(i\varepsilon + \mu)\| \leq c \quad \text{for all } 0 \leq \varepsilon \leq 1.
$$

It follows from (17) that $\|L_n\| \leq 2(c + |\mu|)$. We remark that

$$
[x, x] \geq \delta(x, x) \quad \text{for all } x \in \mathcal{L}_n
$$

if and only if $\|K_n\| \leq 1 - \delta$, where $K_n$ is the angle operator of the subspace $\mathcal{L}_n$. By Lemma 5 there is a number $\delta > 0$ such that $\|K_n\| \leq 1 - \delta$. The operators $K_n$ and $L_n$ acting from $\mathcal{H}^+$ into $\mathcal{H}^+$ can be treated as operators from $\mathcal{H}^+$ into $\mathcal{H}^-$ after their zero extension on the orthogonal complement $\mathcal{H}^+ \ominus \mathcal{H}^-$. Certainly, the norms of these operators are preserved. Since $\mathcal{H}^+$ and $\mathcal{H}^-$ are separable spaces and $\|K_n\| < 1 - \delta$, one can choose a weakly convergent subsequence $K_{n_j} \rightarrow K$ (here we make use of the fact that the unit ball of a separable Hilbert space is a compact set in the weak topology). Since the norms of the operators $\{L_{n_j}\}$ are bounded by a constant $2(c + |\mu|)$, one can choose from the sequence $\{L_{n_j}\}$ a weakly convergent subsequence. Hence there are indices $m = n_k \rightarrow \infty$ such that $K_m \rightarrow K$, $L_m \rightarrow L$. Let us prove that

$$
L = A_{21} + (A_{22} - i\varepsilon - \mu)K.
$$

We have $(A_{22} - i\varepsilon - \mu)^{-1}L_m = F(\mu + i\varepsilon)P_m + K_m \rightarrow F(\mu + i\varepsilon) + K$. Consequently, $(A_{22} - i\varepsilon - \mu)^{-1}L = F(\mu + i\varepsilon) + K$, and this implies relation (20).

Now we remark that the weak convergence $K_n \rightarrow K$ implies $K_nG \rightarrow KG$ and $GK_n \rightarrow GK$ for any bounded operator $G$. One can not guarantee the convergence $K_nGL_n \rightarrow KGL$ provided that the sequences $\{K_n\}$ and $\{L_n\}$ are weakly convergent. However, the convergence $K_nGL_n \rightarrow KGL$ does hold if $G$ is a compact operator (in this case the convergence holds even in the norm operator topology). In fact, a compact operator $G$ can be approximated with arbitrary accuracy in the norm operator topology by a finite rank operator, therefore it suffices to prove the convergence for an operator $G = (\cdot, v)u$ of rank 1. In the later case we have for all $x \in \mathcal{H}^+$ and $y \in \mathcal{H}^-$

$$
(K_nGL_n x, y) = (L_nx, v)(K_nu, y) \rightarrow (Lx, v)(Ju, y) = (KGLx, y).
$$

Hence, one can pass to the weak limit in (17) and obtain the relation

$$
(1 - KG)L = K(S(i\varepsilon + \mu) + i\varepsilon - \mu),
$$

where $L = A_{21} + (A_{22} - i\varepsilon - \mu)K$ is a bounded operator and $\|K\| \leq 1 - \delta$ with some $\delta > 0$. As we already mentioned, by Lemmas 11 and 14 the subspace $\mathcal{L}$ with the angle operator $K$ is $\overline{A} + i\varepsilon J$-invariant and maximal uniformly positive. The restriction $A_\varepsilon^+ = (A + i\varepsilon J)|_\mathcal{L}$ is a bounded uniformly dissipative operator on the subspace $\mathcal{L}$ with the inner product $\langle \cdot, \cdot \rangle$, which is equivalent to the usual inner product in $\mathcal{L}$, since the subspace $\mathcal{L}$ is uniformly positive. Consequently the spectrum of this restriction lies in the open upper half-plane $\mathbb{C}^+$. 
Now, we shall pass to the limit choosing a subsequence $\varepsilon_n \to 0$. Observe that
\begin{equation}
G(\mu + i\varepsilon) = G(\mu) + i\varepsilon G(\mu) (A_{22} - i\varepsilon - \mu)^{-1}
\end{equation}
\begin{equation}
S(\mu + i\varepsilon) = S(\mu) + i\varepsilon G(\mu + i\varepsilon) F\mu).
\end{equation}
Since $\|G(\mu + i\varepsilon)\| < 1/2$ for $\varepsilon \geq 0$, it follows from (21) that
\[\|L(\varepsilon)\| \leq 2(\|S(\mu)\| + \varepsilon(1 + \|G(\mu)\| \|F(\mu)\|) + \mu),\]
i. e. the norms of $L(\varepsilon)$ are uniformly bounded for $0 < \varepsilon \leq 1$. Take any sequence $K = K(\varepsilon_n)$ and choose a weakly convergent subsequence $K(\varepsilon_{n_j})$. Further, choose a weakly convergent subsequence from the sequence $L(\varepsilon_{n_j})$. On this way we find numbers $\varepsilon_m \to 0$ such that $K_n = K(\varepsilon_m) \rightharpoonup K$, $L_n = L(\varepsilon_n) \rightharpoonup L$. We can repeat the arguments applied while making the first limit procedure and obtain the relation $L(\mu) = A_{21} + (A_{22} - \mu)K$.

Taking into account relations (22) and recalling that the operator $G(\mu)$ is compact we can pass to the weak limit in relation (21) as $\varepsilon_m \to 0$. Thus we obtain that relation (21) holds with $\varepsilon = 0$ and the operators $K$, $L$, $\|K\| \leq 1$, $\|L\| \leq \text{const}$. By Lemma 4 the subspace $L$ with the angle operator $K$ is $\mathcal{A}$-invariant and maximal nonnegative. From Lemma 4 we also have
\[\mathcal{A}|_L = Q^{-1}(S(\mu) + G(\mu)L(\mu))Q.\]

It was already proved that the spectra of the operators
\[T(\varepsilon) = S(\mu + i\varepsilon) + G(\mu + i\varepsilon)L(\mu + i\varepsilon) + i\varepsilon\]
lie in $\mathbb{C}^+$ for each $\varepsilon > 0$. It follows from relations (22) that $T(\varepsilon) - i\varepsilon = T(0) + C$, where $C$ is a compact operator. Hence the spectrum of $T(0)$ in the half-plane $\text{Im}\lambda \geq -\varepsilon$ is discrete. Here $\varepsilon > 0$ is arbitrary number, therefore spectrum of $T(0)$ in the open lower-half plane is discrete. From (22) we obtain $T(\varepsilon) \to T(0)$ taking into account that $K_n G L_n \to KGL$ if $G$ is a compact operator. By Lemma 6 the spectrum of $T(0)$ (and hence the spectrum of $\mathcal{A}|_L$) lies in $\mathbb{C}^+$

It is left to prove that $\mathcal{A}$ is a maximal dissipative operator in the space $\{\mathcal{H}, J\}$. It follows from (4) that
\[\mathcal{A} - \mu + i\alpha P_+ = \begin{pmatrix} 1 & G(\mu) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S(\mu) - \mu + i\alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F(\mu) & 1 \end{pmatrix},\]
provided that $\mu \in \mathbb{C}^+$. Here the number $\alpha > \text{Im}\mu$ can be chosen sufficiently large to guarantee the invertibility of the operator $S(\mu) - \mu + i\alpha$. In this case the operator $J(\mathcal{A} - \mu + i\alpha)$ is dissipative in $\mathcal{H}$ and invertible. Therefore the dissipative operator $J\mathcal{A}$ is maximal dissipative. This ends the proof of Theorem.

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