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Dynamics of SIQR epidemic model with fractional order derivative

Subrata Paul a, Animesh Mahata b,*, Supriya Mukherjee c, Banamali Roy d

a Department of Mathematics, Arambagh Government Polytechnic, Arambagh, West Bengal, India
b Mahadevnagar High School, Maheshtala, Kolkata 700141, West Bengal, India
c Department of Mathematics, Gurudas College, 1/1, Suren Sarkar Road, Narkeldanga, Kolkata, Pin- 700054, West Bengal, India
d Department of Mathematics, Bangabasi Evening College, Kolkata 700009, West Bengal, India

A B S T R A C T

The dynamics of COVID-19 (Coronavirus Disease-2019) transmission are described using a fractional order SIQR model. The stability analysis of the model is performed. To obtain semi-analytic solutions to the model, the Iterative Laplace Transform Method [ILTM] is implemented. Real-time data from COVID-19 cases in India and Brazil is employed to estimate the parameters of the fractional order SIQR model. Numerical solutions obtained using Adam–Bashforth–Moulton predictor–corrector technique is compared with those obtained by ILTM. It is observed that the fractional order of the derivatives is more effective in studying the dynamics of the spread of COVID-19 in comparison to integral order of the SIQR model.

1. Introduction

The first cases of corona virus infection in human species with symptoms similar to common cold1 were reported in 1965 by Tyrrell and Bynoe. However, after numerous mutations, the virus has ultimately proved to be catastrophic in Wuhan, Hubei Province, China, in 2019. Henceforth, the virus was named SARS-CoV-2 and the disease was declared to be COVID-19. The devastation spread quickly throughout multiple nations, prompting the WHO to identify a pandemic on March 11, 2020.2 No sooner had the futile efforts of the doctors and the administrative heads of states succumbed to the wrath of the virus, researchers from all areas of science sprang to activity. Along with medical researches going on in full swing, the study of the transmission of the disease, effect of preventive measures, prediction of future outbreaks and potential control strategies were also investigated extensively. The simple epidemic model by Kermack and McKendrick in 1927 is among the initial works on mathematical modeling of infectious diseases.3–6 In order to study such mathematical models, the entire population is primarily divided into subclasses, namely, susceptible individuals S(t), infected individuals I(t), the quarantined individuals Q(t) and the recovered individuals R(t). Integral order ordinary differential equations are common among epidemiological modeling of biological systems.7,8 However, there are certain reservations regarding the integral order of the differential equations in such models. Such restrictions are taken into account by a comparatively new and emerging area of mathematical calculus, namely the Fractional Order Differential Equations.9–13 In fact, fractional-order differential equations (FODEs) and their applications have been intensively investigated and used in biology, physics, chemistry, biochemistry, hydrology, engineering and medicine.14–18

The stability results of the fractional order dengue model have been discussed in Ref. 19. Some interesting results are obtained in Ref. 20 on modeling the Hepatitis-C Virus replications by a fractional-order differential mathematical model. Particularly, fractional calculus has been applied in studying epidemic models related to different infectious diseases.21,22 Thus, extensive studies have evolved various techniques to construct real and approximate solutions to such fractional order differential equations.23–25

Research background and motivation

A mathematical modeling of infectious disease26 is critical for better understanding the transmission patterns of the disease and evaluating control strategies. It serves as motivation for mathematical and biological experts to investigate and evaluate the dynamical systems that regulate such diseases in order to anticipate their spread and control in the long term. The four epidemiological divisions that assess the susceptible part of individuals, the infected, the quarantine, and the recovered, have characterized basic definitions of infectious diseases. One of the several variations of the conventional SIR model is the SIQR model. It has been observed that quarantining the infected individuals is a better measure to control the spread of the disease. Over these years, several types of fractional calculus, for example, Riemann–Liouville, Caputo, Caputo–Fabrizio, Katugampola, Atangana–Baleanu, Hadamard etc. have been introduced to study the dynamics of the epidemic models, each displaying certain advantages and disadvantages.

* Corresponding author.
E-mail address: animeshmahata8@gmail.com (A. Mahata).

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Zhang et al. developed the unique asymptotic stability criteria for fractional-order gene regulation systems with time delay. Wu et al. investigated global uniform asymptotical stability for fractional-order gene regulatory networks with time-varying delays and structured uncertainties. Based on the vector Lyapunov function, Zhang et al. presented a unique stability condition for fractional-order composite systems with time delay. We have used the Caputo–Fabrizio operator because of its possession of a nonlocal and nonsingular exponential kernel and is found to be best suited to study the dynamics of COVID-19. The dynamics of COVID-19 (Coronavirus Disease–2019) transmission are described using a fractional order SIQR model. We investigate the disease’s impact using an appropriate mathematical model (SIQR model) in context of the Caputo–Fabrizio fractional differential equation, motivated by early research.

The present research purpose is to:

1. To study local stability as well as global stability of the model system.
2. Determination of the Basic Reproduction number and Equilibrium points.
3. The Iterative Laplace Transform Method has been used to generate semi-analytic solutions to the model system.
4. To confirm the Iterative Laplace Transform Method’s predictions, the Adam–Bashforth scheme is applied to the SIQR model.
5. To confirm the findings and prevent the growth of COVID-19, a numerical simulation was used.

The following is an overview of the article’s structure: Section 2 discusses some fundamental concepts and findings for the fractional operator and the Laplace transform. Mathematical model with fractional order derivative is discussed in Section 3. Section 4 is focused on a description of the Model’s stability analysis and stability criterion. In Section 5, we perform Adam–Bashforth–Moulton predictor–corrector scheme for the SIQR model. In Section 6, numerical simulation and discussion are presented via MATLAB. Section 7 includes the validation of the model with real time data. Finally, Section 8 constitutes the conclusion of the paper.

2. Preliminaries

The definitions of fractional differential and integral operators are introduced in this part.

**Definition 2.1.** A function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) with fractional order \( 0 < a \leq 1 \) is defined as

\[
C^a I^a (f(t)) = \frac{1}{\Gamma(a)} \int_0^t (t-x)^{a-1} f(x) \, dx,
\]

where \( \Gamma() \) describes the Gamma function.

**Definition 2.2.** The Caputo fractional derivative operator of order \( 0 < a \leq 1 \) is defined as

\[
C^a D^a_t (f(t)) = I^{n-a} D^n_t (f(t)) = \frac{1}{\Gamma(n-a)} \int_0^t (t-x)^{n-a-1} \frac{d^n}{dx^n} f(x) \, dx,
\]

where \( n-1 < a < n \).

**Definition 2.3.** The Caputo–Fabrizio fractional derivative operator of order \( 0 < a \leq 1 \) is defined as

\[
C^{CF}_a D^a_t (f(t)) = \frac{M(a)}{(1-a)} \int_0^t \exp \left[ -\frac{a(1-x)}{1-a} \right] \frac{d}{dx} f(x) \, dx,
\]

where \( M(a) \) is a normalization function which depends on \( a \) and satisfies \( M(0) = M(1) = 1 \).

**Definition 2.4.** The Caputo–Fabrizio fractional integral operator of order \( 0 < a < 1 \) is defined as

\[
C^{CF}_a I^a_t (f(t)) = \frac{2(1-a)}{(2-a)} \frac{M(a)}{(1-a)} \int_0^t f(x) \, dx,
\]

where \( M(a) \) is a normalization function.

### Table 1

| Notation | Interpretations |
|----------|-----------------|
| \( \lambda \) | Natural birth rate |
| \( \beta \) | Contact rate between S and I |
| \( \gamma \) | Infection rate of Q class |
| \( \delta \) | Recovery rate |
| \( \mu \) | Natural death rate |

where \( M(a) \) is a normalization function.

**Definition 2.5.** The Laplace transform for the fractional operator of order \( 0 < a < 1 \) for \( m = 0, 1, 2, 3, \ldots \) is defined as

\[
L(C^{CF}_a D^a_t f(t)) = \left( \frac{p}{p+\alpha(1-p)} \right)^m L(f(t))^m \exp(-a \frac{a}{1-a} t),
\]

where \( m \) is a normalization function.

The parameters of the model and their descriptions are presented via MATLAB. Section 7 includes the validation of the model with real time data. Finally, Section 8 constitutes the conclusion of the paper.

3. Model formulation

Let us consider the case of a Susceptible–Infected–Quarantine–Removed (SIQR) epidemic. The SIQR disease transmission model is derived assuming several strong assumptions. The population (\( N \)) is divided into four classes: the susceptible individuals (\( S \)), the infected individuals (\( I \)), the quarantined individuals (\( Q \)) and the recovered individuals (\( R \)) at any time \( t \geq 0 \).

Now \( N(t) = S(t) + I(t) + Q(t) + R(t) \).

Individuals who are Susceptible in this model are those who are at risk of becoming infected. Quarantine is defined as an infected person who exhibits signs of the disease and is isolated. Quarantined individuals who recover from the disease are considered as Recovered (see Fig. 1, Table 1).

The differential equations governing the SIQR model are given as:

\[
D_t S(t) = \lambda - \beta S(t) \, I(t) - \mu S(t),
\]

\[
D_t I(t) = \beta S(t) \, I(t) - (\gamma + \mu) I(t),
\]

\[
D_t Q(t) = \gamma I(t) - (\delta + \mu) Q(t),
\]

\[
D_t R(t) = \delta Q(t) - \mu R(t),
\]

with initial condition

\[
S(0) = S_0 > 0, I(0) = I_0 \geq 0, Q(0) = Q_0 > 0 \text{ and } R(0) = R_0 \geq 0.
\]

The model (3.2) having fractional order derivatives with the Caputo–Fabrizio operator of order \( 0 < a \leq 1 \) have been proposed as follows:

\[
C^{CF}_a D^a_t S(t) = \lambda - \beta S(t) \, I(t) - \mu S(t),
\]

\[
C^{CF}_a D^a_t I(t) = \beta S(t) \, I(t) - (\gamma + \mu) I(t),
\]

\[
C^{CF}_a D^a_t Q(t) = \gamma I(t) - (\delta + \mu) Q(t),
\]

\[
C^{CF}_a D^a_t R(t) = \delta Q(t) - \mu R(t).
\]

3.1. Non-negativity and boundedness of solutions

**Proposition.** All the variables are positive for all \( t \geq 0 \). The closed region \( \Omega = \{(S,I,Q,R) \in \mathbb{R}^4 : 0 < N \leq \frac{2\mu}{\gamma}\mu \} \) is positive consistent for the model (3.4).
Thus \( l_i m C F \cdot C F \cdot C F \cdot \lambda = \beta SI - \mu S \geq -(\beta SI + \mu)S \).

Therefore, \( S(t) \geq S(0) \exp(-\int_0^t (\beta SI + \mu)dp) > 0 \). (3.5)

Now, \( C F \cdot D^{\lambda}_t I(t) = \beta SI - (\gamma + \mu)I \).

Therefore, \( I(t) \geq I(0) \exp(-\int_0^t (\gamma + \mu)dp) > 0 \). (3.6)

Also \( C F \cdot D^{\lambda}_t Q(t) = \gamma I - (\mu + \delta)Q \geq -(\mu + \delta)Q \).

We have, \( Q(t) \geq Q(0) \exp(-\int_0^t (\gamma + \mu)dp) > 0 \). (3.7)

Again \( C F \cdot D^{\lambda}_t (S + I + Q + R)(t) = \lambda - \mu(S + I + Q + T) \).

Therefore, \( C F \cdot D^{\lambda}_t N(t) = \lambda - \mu N \).

Thus \( \lim_{t \to \infty} \sup N(t) \leq \frac{\lambda}{\mu} \). (3.9)

Therefore, the model (3.4) is bounded by \( \frac{\lambda}{\mu} \).

Thus \( I, S, Q, R \) are positive functions and \( \Omega \) is positively consistent of the system (3.4).

### 3.2. Basic reproduction number, disease-free equilibrium, epidemic equilibrium state

The basic reproduction number, indicated by \( \Theta_0 \), is defined as the predicted number of secondary cases created by a single Susceptible individual’s infection. The disease-free equilibrium point is locally asymptotically stable when \( \Theta_0 < 1 \) and unstable when \( \Theta_0 > 1 \). So, the considered model has disease-free equilibrium at \((S, I, Q, R) = (0, 0, 0, 0, 0)\). An effective strategy should be developed, when a pandemic breaks out so that \( \Theta_0 \) reduces to less than 1 as soon as possible.

Using next generation matrix method, \( C_F(S, I, Q, R) \) the leading eigenvalue of the matrix \( FV^{-1} \) may be used to calculate the reproduction number \( \Theta_0 \) where,

\[
F = \begin{bmatrix}
\beta \lambda & 0 \\
\mu & 0 \\
0 & \gamma \\
0 & \mu + \delta
\end{bmatrix}
\]

And

\[
V = \begin{bmatrix}
\mu + \gamma & 0 \\
-\gamma & \mu + \delta
\end{bmatrix}.
\]

Therefore, the reproduction number \( \Theta_0 = \frac{\beta \lambda}{\mu (\gamma + \mu)} \). (3.10)

### 3.3. Iterative scheme

Consider the model (3.4) along with initial conditions (3.3). Applying the Laplace transform to both sides of model (3.4), we derive

\[
\frac{pLS(t) - S(0)}{p + a(1 - p)} = L(\lambda - \beta SI - \mu S),
\]

\[
\frac{pLI(t) - I(0)}{p + a(1 - p)} = L(\beta SI - (\gamma + \mu)I),
\]

\[
\frac{pLQ(t) - Q(0)}{p + a(1 - p)} = L(\gamma I - (\delta + \mu)Q),
\]

\[
\frac{pLR(t) - R(0)}{p + a(1 - p)} = L(\delta Q - \mu R).
\]

Rearranging, we get

\[
LS(t) = \frac{S(0)}{p} + \left(\frac{p + a(1 - p)}{p}\right)L(\lambda - \beta SI - \mu S),
\]

\[
LI(t) = \frac{I(0)}{p} + \left(\frac{p + a(1 - p)}{p}\right)L(\beta SI - (\gamma + \mu)I),
\]

\[
LQ(t) = \frac{Q(0)}{p} + \left(\frac{p + a(1 - p)}{p}\right)L(\gamma I - (\delta + \mu)Q),
\]

\[
LR(t) = \frac{R(0)}{p} + \left(\frac{p + a(1 - p)}{p}\right)L(\delta Q - \mu R).
\]

Using inverse Laplace transform of the equations (3.12), we get

\[
S(t) = S(0) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\lambda - \beta SI - \mu S)\right],
\]

\[
I(t) = I(0) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\beta SI - (\gamma + \mu)I)\right],
\]

\[
Q(t) = Q(0) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\gamma I - (\delta + \mu)Q)\right],
\]

\[
R(t) = R(0) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\delta Q - \mu R)\right].
\]

Using initial conditions, we get the recursive equations given by,

\[
S_{n+1}(t) = S_n(t) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\lambda - \beta SI_n - \mu S_n)\right],
\]

\[
I_{n+1}(t) = I_n(t) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\beta SI_n - (\gamma + \mu)I_n)\right],
\]

\[
Q_{n+1}(t) = Q_n(t) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\gamma I_n - (\delta + \mu)Q_n)\right],
\]

\[
R_{n+1}(t) = R_n(t) + L^{-1}\left[\left(\frac{p + a(1 - p)}{p}\right)L(\delta Q_n - \mu R_n)\right].
\]

### 4. Stability analysis

To obtain the equilibrium points of the system (3.4) we have, i.e., \( C_F D^{\lambda}_t S(t) = C_F D^{\lambda}_t I(t) = C_F D^{\lambda}_t Q(t) = C_F D^{\lambda}_t R(t) = 0 \). (4.1)
We have two equilibrium points given by $E_0 = (\frac{1}{\mu}, 0, 0, 0)$ and $E_1 = (S^*, I^*, Q^*, R^*)$, where $E_0$ is the disease-free equilibrium point and $E_1$ is the unique epidemic equilibrium point of the system (3.4), with $S^* = \frac{1}{\mu S}$, $I^* = \frac{1}{\mu S} - \frac{\beta S}{\mu + \gamma}$, $Q^* = \frac{\delta I^*}{\mu S}$, $R^* = \frac{\delta Q^*}{\mu}$.

For further analysis, the Jacobian matrix of the system (3.4) at any equilibrium point $(S, I, Q, R)$ is given by

$$J = \begin{bmatrix}
-\beta I - \mu & -\beta S & 0 & 0 \\
\beta I & \beta S - (\mu + \gamma) & 0 & 0 \\
0 & \gamma & - (\mu + \delta) & 0 \\
0 & 0 & \delta & -\mu
\end{bmatrix}.$$

### 4.1. Theorem

The disease-free equilibrium of the system (3.4) is locally stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

**Proof.** At the equilibrium point $E_0 = (\frac{1}{\mu}, 0, 0, 0)$, the Jacobian matrix becomes

$$J(E_0) = \begin{bmatrix}
-\mu & -\frac{\beta I}{\mu} & 0 & 0 \\
0 & - (\mu + \gamma) + \frac{\beta I}{\mu} & 0 & 0 \\
0 & \gamma & -(\mu + \delta) & 0 \\
0 & 0 & \gamma & -\mu - x
\end{bmatrix}.$$

Therefore, its characteristic equation is,

$$-\lambda_x - \frac{\beta I}{\mu} 0 0 0$$

$$0 - (\mu + \gamma) + \frac{\beta I}{\mu} 0 0$$

$$0 \gamma -(\mu + \delta) 0$$

$$0 0 \gamma -\mu - x = 0.$$

The characteristic roots are $-\mu$, $-\mu$, $-(\mu + \delta)$ and $(\mu + \gamma)$ ($\mathcal{R}_0 - 1$).

Since the first three roots are negative and other will be negative if $\mathcal{R}_0 < 1$ and positive if $\mathcal{R}_0 > 1$. Thus the equilibrium point $E_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

### 4.2. Theorem

If $\mathcal{R}_0 > 1$, the epidemic equilibrium $E_1 = (S^*, I^*, Q^*, R^*)$ is locally asymptotically stable.

**Proof.** At the equilibrium point $E_1 = (S^*, I^*, Q^*, R^*)$, the Jacobian matrix becomes

$$J(E_1) = \begin{bmatrix}
-\beta I^* - \mu & -\beta S^* & 0 & 0 \\
\beta I^* & \beta S^* - (\mu + \gamma) & 0 & 0 \\
0 & \gamma & -(\mu + \delta) & 0 \\
0 & 0 & \gamma & -\mu - x
\end{bmatrix}.$$

Therefore, its characteristic equation is,

$$-\lambda_x - \beta S^* 0 0 0$$

$$\beta I^* - (\mu + \gamma) - x 0$$

$$0 \gamma -(\mu + \delta) - x 0$$

$$0 0 \gamma -\mu - x = 0.$$

This gives, $(-\mu - x)(-\mu + \delta - x) (x^2 - ax + b) = 0$. Where, $a = \beta S^* - \beta I^* - (2\mu + \gamma)$ and $b = \beta I^* (\mu + \gamma) + \mu (\mu + \gamma) - \beta \mu S^*$.

The roots of the characteristic equations $J(E_1)$ are $x_1 = -\mu < 0, x_2 = (\mu + \delta) < 0$ and other two satisfies the following quadratic equation,

$$x^2 - ax + b = 0. \quad (4.2)$$

It is noted that the roots of (4.2) will be negative if $a < 0$ and $b > 0$.

### 4.3. Theorem

The disease-free equilibrium point of the system (3.4) is globally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.

**Proof.** Considering the suitable Lyapunov function as $F = I$ .

Calculate the time fractional derivative of the above function $CF D_t^\alpha F(t) = CF D_t^\alpha I(t)$. Utilizing system (3.4) we get, $CF D_t^\alpha F(t) = [\beta SI - (\mu + \gamma) I]$.

$$\leq [\beta S_0 - (\mu + \gamma)] I.$$  

Since, $S = \frac{1}{\mu} \leq N$. It follows that,

$$CF D_t^\alpha F(t) \leq I (\mu + \gamma) [\frac{\beta S_0}{(\mu + \gamma) + 1}] \leq I (\mu + \gamma) [R_0 - 1]. \quad (4.3)$$

Hence if $\mathcal{R}_0 < 1$, then $CF D_t^\alpha F(t) < 0$. By LaSalle’s extension to Lyapunov’s principle, the disease-free equilibrium point is globally asymptotically stable and unstable if $\mathcal{R}_0 > 1$.

### 4.4. Theorem

If $\mathcal{R}_0 > 1$, the epidemic equilibrium $E_1$ is globally asymptotically stable if $\{2I + \frac{I^*}{\mu} - \frac{I^*}{\mu S} \leq 0 \}$. 

**Proof.** Consider the model (3.4) and $\mathcal{R}_0 > 1$, so that the epidemic equilibrium $E_1$ of model exists.

We consider the following non-linear Lyapunov function of Goh-Volterra type:

$$V = \left( S - S^* \right) \log \left( \frac{S}{S^*} \right) + \left( I - I^* \right) \log \left( \frac{I}{I^*} \right) + \left( Q - Q^* \right) \log \left( \frac{Q}{Q^*} \right).$$

The time fractional derivative of the above function is, $CF D_t^\alpha V(t) = \left( 1 - \frac{S^*}{S} \right) CF D_t^\alpha I(t) + \left( 1 - \frac{I^*}{I} \right) CF D_t^\alpha Q(t). \quad (4.4)$

Using system (3.4) we get,

$$CF D_t^\alpha V(t) = \left( \lambda - \beta SI - \mu S - \frac{S^* (\lambda - \beta SI - \mu S)}{S} \right) \quad (4.5)$$

$$+ \left( \beta SI - (\mu + \gamma) I - \frac{I^* (\mu + \gamma) I}{I} \right) + \frac{(\gamma I - (\mu + \delta) Q) - \frac{Q^* (\gamma I - (\mu + \delta) Q)}{Q}}{Q^*}.$$  

At steady state from Eq. (3.4) we have

$$\lambda = \beta S^* I^* + \mu S^*.$$  

$$\mu + \gamma = \beta S^*.$$  

$$\mu + \delta = \frac{\gamma I^*}{Q^*}.$$  

Substituting Eq. (4.6) into (4.5) we have,

$$CF D_t^\alpha V(t) = \left( \beta S^* I^* + \mu S^* - \beta SI - \mu S - \frac{S^* (\beta S^* I^* + \mu S^* - \beta SI - \mu S)}{S} \right)$$

$$= 0.$$

4
\[ Q(t) = \begin{cases} (n + 1 - a)(n + 1), & \text{if } j = 0, \\ (n - j + 2)a^{n+1} - (n - j + 1)a^n, & \text{if } 0 \leq j \leq n, \\ 1, & \text{if } j = 1, \\ 0, & \text{if } j \leq n \text{ and } i = 1, 2, 3, 4. \end{cases} \]

Thus, \( C^T D^\alpha V(t) \leq 0 \) if \((2I - \frac{Q'\gamma I}{Q} - \frac{Q''\gamma I}{Q^2}) \leq 0\).

By LaSalle’s Invariance Principle, the epidemic equilibrium \( E_1 \) is globally asymptotically stable if \((2I - \frac{Q'\gamma I}{Q} - \frac{Q''\gamma I}{Q^2}) \leq 0\).

6. Numerical simulation and discussion

We will explore into numerical simulations of the Caputo–Fabrizio Coronavirus model (3.4) in this section. From the data given in Ref. 44, we use the total initial population of India \( N(0) = 1382339513 \). Using (3.1) at \( t = 0 \), we have \( N(0) = S(0) + I(0) + R(0) \). From the data given in Ref. 44, \( I(0) = 1401737, Q(0) = 350435, R(0) = 1184321 \) as on 1st August, 2020. Therefore, initial susceptible individuals are determined as \( S(0) = N(0) - I(0) - Q(0) - R(0) = 1379403021 \).

In the instance of COVID-19 in India, the estimated parametric values are as follows:

We can achieve an approximate solution to (3.13) by using ILTM up to four terms as given below:

\[ S(t) = 1.348475481619412 \times 10^9 + (2.0615739629214 \times 10^7 - 2.6015739629214 \times 10^7 i) a + (999727.478555693 - 999727.478555693) a^2 + (213892.138788847 - 42778.27757688) a^3 + (106427.947590277882) a^4 + (4908.8007228498) a^5 - 14726.42421685494 a^6 + 7861.062639637445 a^7 - 983.75314689445775 a^8 + (268.45068326930414) a^9 - 85.352049079124 a^{10} + 536.901365386838 a^{11} - 89.48356109876805 a^{12} + (11.68906244574971) a^{13} + (46.756249789916 - 13.321347928545 a^{14} - 7.36462505553762 a^{15} + 1.5207065071160833 a^{16} + 0.73356557223858 + 3.667827786164292 a^{17} - 5.125093830775163 a^{18} + 3.005236604151244 a^{19} - 0.7090605251997234 a^{20} + 0.0524455048194907 a^{21} + 0.00492636944044962 + 0.02955821664297716 a^{22} - 0.05415096498430903 a^{23} + 0.02459896675896235 a^{24} - 0.01532108782411106 a^{25} + 0.002442625047567154 a^{26} - 0.000135020691378284 a^{27} - 0.00003129266745992261 + 0.00009186076721914583 r + 0.00020996746792947618 r + 0.0002309043467506846 r^2 - 0.0001209286183459572 r^3 + 0.0000332448490883373 r^4 - 0.00000437432248530754 r^5 + 2.083010594538544 \times 10^{-7} r^6) a^7, \]

where \( \lambda > 0 \).
Fig. 2. Dynamical behavior of Susceptible individuals $S(t)$ with regard to time (days) for various values of $\alpha$.

$$Q(t) = 1226844.644274208 - 232197.10714117886\alpha$$
$$+ (-306048.1919000001 + 306048.1919000001\alpha)$$
$$- 89257.51364467504\alpha^2 + (-231367.0251 + 462734.0502\alpha)$$
$$- 115683.51255\alpha^3 - 17632.36828945347a^4$$
$$+ 92.561795591256074a^4 + 7.713482965938004a^4$$
$$+ \frac{1}{6}t^3 \left( 17817.49187512786\alpha^3 - 555.370773475364a^4 \right)$$
$$+ t (232197.10714117886\alpha + 178515.02728935008\alpha^2)$$
$$+ 52897.10485183604a^3 - 370.2471823650243a^4$$
$$+ \frac{1}{2}t^2 \left( -89164.9518490838a^2 - 53082.228443018554a^3 \right)$$
$$+ 647.9325691387925a^4 \right)$$

Fig. 3. Dynamical behavior of Infected individuals $I(t)$ with regard to time (days) for various values of $\alpha$.

$$R(t) = 1624099.713711525 - 43416.78594852515\alpha$$
$$+ (-638557.57078218 + 638557.57078218\alpha)$$
$$- 2617.761263654674a^2 + (231354.70369109)$$
$$- 463109.40738218r + 115777.351845545r^2$$
Fig. 4. Dynamical behavior of Quarantine individuals $Q(t)$ with regard to time (days) for various values of $\alpha$.

Fig. 5. Dynamical behavior of Recovered individuals $R(t)$ with regard to time (days) for various values of $\alpha$.

We now study effect of the COVID–19 pandemic on the Brazilian population. From the data given in Ref. 45 the total initial population of Brazil $N(0) = 209500000$ as on 1st August 2020. Using (3.1) at $t = 0$ we have $N(0) = S(0) + I(0) + Q(0) + R(0)$. From the data given in Ref. 45, $I(0) = 2167100, Q(0) = 541775, R(0) = 2152361$ as on 1st August, 2020. Therefore, initial susceptible population is determined as $S(0) = N(0) - I(0) - Q(0) - R(0) = 204638764$.

The estimated parametric values are as follows in Brazil:

We achieve an approximate fractional order solution by applying ILTM to four terms in a succession, as shown below:

$$S(t) = 1.988324783119671 \times 10^6 + (5766154.964132981) \alpha + (42994.313931732555) \alpha^2 + (85988.62786346511t + 21575.945826256426t^2) \alpha^3$$

$$+ (2886.972994182714t + 8660.918982548143t^2 + 4384.476770316121t^3 + 498.774344475575t^4) \alpha^4$$

$$+ (1.17314911233352 - 16.692596449334072t - 16.70688294784533t^2 + 16.13046984651365t^3) \alpha^5$$. 

Fig. 2 shows the behavior between susceptible individuals versus time for different fractional order $\alpha$. We observe that number of susceptible individuals decrease with time because they are getting into infected class. Fig. 3 is indicating the relation between infected individuals versus time with different fractional order $\alpha$. An increase in the value of $\alpha$ leads to decrease in the infection rate in the infected individuals. We see in Fig. 4 that number of quarantine individuals increase with time for changes in the values of the fractional order $\alpha$. The number of recovered individuals grows exponentially, as shown in Fig. 5 when $\alpha$ decrease.

Fig. 6. shows the behavior of recovered individuals for different recovery rate $\delta$ versus time with $\alpha = 0.9$. A decrease in the recovery rate $\delta$ causes a decrease in the number of recovered individuals as expected.
Fig. 6. The dynamics of Recovered individuals for different values of $\delta$ with respect to time (days).

Fig. 7. The behavior of Susceptible individuals $S(t)$ with respect to time (days) with a change in the values of $\alpha$.

Fig. 8. The behavior of Infected individuals $I(t)$ with regard to time (days) when $\alpha$ changes.

\[
\begin{align*}
-2.3387206164314174t^4 + (1.2076687365300693t^4 &+ (0.0022060561968627744t^5 - 0.0796834788673288t^5) = 0 \\
-0.0132363371811765t + 0.0245116550820208t^2 - 0.0195702694327723t^3 + 0.00723995188278034t^4 &+ (0.001994028218474224t^5 + 0.0000698231972606121t^6) = 0 \\
-6.038343682650347t + 8.449330559097664t^2 &- 4.81440661545906t^3 + 1.0993855080902668t^4 \\
-0.0796834788673288t^5 + (0.0022060561968627744t^5 &+ (0.001994028218474224t^5 + 0.0000698231972606121t^6) = 0
\end{align*}
\]
Fig. 9. The behavior Quarantine individual $Q(t)$ with regard to time (days) when $\alpha$ changes.

Fig. 10. The behavior Recovered individuals $R(t)$ with regard to time (days) when $\alpha$ changes.

$I(t) = 2076914.236797711 + (111128.28717407388 - 111128.28717407388t)\alpha + (-20702.38871722496 + 41404.77743444992t - 10396.275037363199t^2)\alpha^2 + (-302.1686554503082 + 906.5059663509248t - 466.65206741227087t^2 + 55.221424917565585t^3)\alpha^3 + (63.243214166524844 - 252.9728566660994t + 250.6279597707888t^2 - 83.54683331257115t^3 + 7.956750889669616t^4 + (-1.207683765300693 + 6.038343682650347t - 8.449330559097664t^2 + 4.81440661545906t^2 - 1.09938550800902668t^4 + 0.07968347888673289t^3 + (-0.0022060561968627744 + 0.01323633718117665t - 0.02445116550820208t^2 + 0.0195702694327723t^3 - 0.00723995188278034t^4 + 0.001199402818474224t^2 - 0.0000698231972606111t^4 + (0.00006151578090385235 - 0.0004306104663269664t + 0.0009842524944616376t^2 - 0.00104576827536549t^3 + 0.0005638946582531334t - 0.0001558399782897593t^4)\alpha^5 + 0.0000205262031028412t^6 - 9.764409667278152 \times 10^{-7}t^7)\alpha^7$, $Q(t) = 2556591.201558373 + (-1936032.837279405t\alpha + (-81404.66029879282t + 162809.32059758564t - 40736.007990638325t^2)\alpha^2 + (2688.651702308761 - 8065.9551069262825t + 4100.332359469699t^2 - 470.5065112127235t^3)\alpha^3 + (-67.35568284838279^2 + 269.422799353116t - 235.7448886933976t^2 + 67.3556824838279t^3 - 5.6129735403189915t^4)\alpha^4$, $R(t) = 2259922.883515276 + (-189988.44328005833 + 189988.44328005833t\alpha + (81882.89643428754t + 163765.792868675508t - 40941.44821714377t^2)\alpha^2 + (543.6633304944202 - 1630.9899914832604t + 815.4949957416302t^2 - 90.6105508240335t^3)\alpha^3$.

Fig. 7 depicts the behavior of susceptible individuals over time and Fig. 8 indicates the relation between infected individuals and time for different values of $\alpha$. Fig. 9 depicts that the number of quarantined individuals grows with time for changes in the values of $\alpha$. Fig. 10...
depicts that the number of recovered individuals will increase with time and a decrease in the fractional order $\alpha$ leads to a drop in the number of recovered individuals in the Brazilian population (see Fig. 11).

7. Data fitting and model validation

The data fitting and model validation of the system (3.4) for Infected and Recovered cases are described in this section. The parametric values are given in Table 2. Figs. 12 and 13 depicts the graphical representation of the infected and the recovered individuals respectively of the system (3.4) and the real time data of the same reported in India from 1st July to 15th July 2020. The results of model (3.4) thus obtained are in complete agreement with the real time data.

8. Conclusion

The fractional order derivatives using the Caputo–Fabrizio of order $0 < \alpha \leq 1$ of the SIQR model were explored in this article. Based on the COVID-19 cases data in India and Brazil, collected up to 1st August,
2020, we estimated the basic reproduction number $R_0$ to be 1.7824 and 2.767 respectively. Modeling using fractional-order derivatives is often more efficient than modeling with integer-order derivatives because the option of derivative order gives one more degree of freedom, resulting in a better fit to real-time data with less inaccuracy than the integer-order model. We used the Iterated Laplace Transform Method to solve our proposed model and compared the results to numerical solutions obtained utilizing the Adams–Bashforth–Moulton predictor corrector technique. The parameter values in (3.4) have been estimated using the real time data given in Refs. 44, 45 and is presented in Tables 2 and 3. In Section 4, $E_0$ and $E_1$ are determined, along with their stability analysis. In Section 7 we observe that the proposed SIQR model with fractional order derivatives comply with the real time data in case of infected and recovered individuals. Thus, it may be considered to be an effective model to study contagious diseases. According to our understanding of the challenges, the incidence of disease transmission must be reduced, or a large part of the population would be afflicted in a very short span of time. For successful isolation and control of the disease’s transmission, common preventative methods include lockdown, curfews, and the designation of containment zones.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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