Topological exact flat bands in two dimensional materials under periodic strain

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We study flat bands and their topology in 2D materials with quadratic band crossing points (QBCPs) under periodic strain. In contrast to Dirac points in graphene, where strain acts as a vector potential, strain for QBCPs serves as a director potential with angular momentum \( \ell = 2 \).

Introduction.—Electronic band structures of 2D materials can be controlled and designed by manipulating superlattice structures. One well-known example is twisted bilayer graphene (TBG), where interference between the two layers makes the band structure angle dependent. Remarkably, at some “magic” angles, isolated nearly flat topological bands arise \([1–3]\), which become exactly flat in the chiral limit \([1]\), and similar flat bands may arise in other twisted-bilayer systems as well, such as quadratic band crossing point (QBCP) bilayers \([4–6]\). This property of twisted bilayers makes them an exciting platform for studying strongly correlated phenomena such as unconventional superconductivity and correlated insulators \([7–9]\). In single layer moiré systems, exciting progress towards similar interference has been achieved via spatially varying electrostatic field, magnetic field, and elastic strain field \([10–13]\). However, for these systems, it was recently argued \([11]\) that (without a constant background magnetic field) exact flat bands cannot be achieved even in the chiral limit.

In this Letter, we study single layer systems with QBCPs under periodic strains. In contrast to Dirac points in graphene, where strain serves as a vector potential via minimal coupling \(i\partial \rightarrow i\partial + A\) \([44]\), for QBCPs at time-reversal invariant momenta, such gauge-field-like couplings are prohibited by the time-reversal symmetry. Because \(i\partial\) and strain fields have opposite parity under time reversal, the couplings allowed by symmetry for QBCPs take the form of \(\partial \partial \rightarrow \partial \partial + A\), where \(A\) is proportional to the strain field. In other words, the strain fields here provide a director potential with angular momentum \(\ell = 2\), instead of a vector with \(\ell = 1\).

Remarkably, we find that this strain-field coupling induces exact flat bands, in strong analogy to TBG. Here, instead of controlling the twisting angle, we vary the strength of the strain field. In the chiral limit, exact flat bands are obtained as the strength of the strain field hits certain “magic” values, and the flat bands carry Chern numbers \(C = \pm 1\). Away from the chiral limit, these magic flat bands survive in a wide range of phase space, although their bandwidth is no longer exactly zero. We prove analytically that these exact flat bands are protected by the same mathematical principles as magic flat bands in TBG, and thus in analogy to TBG, they are fragile topological bands and their quantum metric satisfy the trace condition. In addition, we find that their Berry curvature distributions are more uniform than the TBG flat bands, making them ideal for the realization of fractional Chern insulators \([45–51]\).

Despite these similarities, it is also worth highlighting that these strain-induced flat bands are due to a very different physical mechanism, and thus they exhibit some unique physical properties, sharply distinct from TBG. For example, these strain-induced flat bands can arise at the \(\Gamma\) valley and only need a single layer, while twisted-bilayer magic flat bands require a finite wave vector away from \(\Gamma\) \([52]\) and interference between two layers. Secondly, twisted bilayer magic flat bands arise for various dispersion, e.g., Dirac and QBCP \([5]\), while the strain-induced magic flat bands can only emerge from QBCPs.

QBCP, strain field and director potential.—Without requiring fine tuning, a stable QBCP can only arise at time-reversal \((T)\) invariant momenta \((Tk = -k)\) with proper rotational symmetry \([53]\), where \(T\) and rotational symmetries protect the QBCP from being gapped out or splitting into Dirac points \([53–54]\). Here we will focus on a QBCP with \(T\) and 3- or 6-fold rotational symmetry, e.g., the \(\Gamma\) point of a kagome metal. Up to a change of basis, the Hamiltonian near \(\Gamma\) takes the following form

\[
H_{\Gamma} = -t_0 a^2 \left( c_0 k^2 - (k_x^2 - k_y^2) \sigma_x - 2k_x k_y \sigma_y \right),
\]

where \(1, \sigma_x, \sigma_y\) represent the 2 \(\times\) 2 identity and Pauli matrices respectively, and \(k = |k|\). Without loss of generality, we set \(t_0 = 1\) and \(a = 1\). Here, \(c_0\) determines the average effective mass of the two bands. At \(c_0 = 0\), a chiral symmetry emerges, \(\{H, \sigma_z\} = 0\), and thus the...
FIG. 1: Exact flat bands from a QBCP under periodic strain. (a) Schematic of a kagome lattice, which has a QBCP at strain. (b) The band structure of a QBCP [Eq. (1)] at \(c_0 = 0, 0.5, 1, 1.5\), where \(c_0 = 0\) is the chiral limit. (c-e) Band structures in the chiral limit under strain \(\tilde{A}\) [Eq. (3)] at different critical values of \(\tilde{\alpha} = \alpha/|G_0| a\) along the high symmetry path in the Moiré Brillouin zone shown in (f). There are two exact flat bands at \(E = 0\) in each case. (g-h) Bandwidth \(E_w\) and bandgap \(E_g\) as a function of \(\tilde{\alpha}\), respectively. The bandwidth is exactly zero at the critical values of \(\tilde{\alpha}\). The minima of the bandwidth occur at the same values of \(\tilde{\alpha}\) as the maxima of the bandgap. (i) Table of the values of \(\Delta \tilde{\alpha} = \tilde{\alpha}_{i+1} - \tilde{\alpha}_i\).

 dispersion of the two bands is symmetric around \(E = 0\). A nonzero \(c_0\) breaks the chiral symmetry with dispersion similar to a kagome metal (Fig. 1(b)). The special case of \(c_0 = \pm 1\) generates a flat band, as in the nearest-neighbor-hopping kagome lattice (Fig. 1(a), also Supplemental material (SM) [56]).

Under a slowly varying strain field \(u_{ij}(r)\), the Hamiltonian becomes

\[
H(r) = H_\Gamma + A_I(r) \mathbb{1} + A_x(r) \sigma_x + A_y(r) \sigma_y,
\]

where \(A_I \propto u_{xx} + u_{yy}\) describes bulk deformation, and \(A_x \propto u_{xx} - u_{yy}\) and \(A_y \propto u_{xy}\) describe shear deformations. Here we introduce complex coordinates and strains \(z = x + iy\) and \(\hat{A} = A_x - iA_y\). Here, \(\hat{A}\) couples to \((k_x - ik_y)^2\), which is a director with angular momentum \(\ell = 2\). This director potential is the key reason why periodic strain can induce magic flat bands for QBCPs. In SM [56], we demonstrate a specific example using a kagome lattice tight-binding model to explicitly derive this Hamiltonian and the director potential coupling.

The two bands near a QBCP have opposite effective mass. This is worth commenting on how to achieve this limit. To make \(c_0\) close to zero, we need to use materials where the two bands near a QBCP have opposite effective mass (e.g., GaCu(OH)\(_6\)Cl\(_2\) [27]). To make \(A_I < < A_x\) or \(A_y\), it simply requires to use 2D materials with a bulk modulus \((B)\) larger than the shear modulus \((G)\), so that shear deformations \((A_x\) and \(A_y)\) have lower energy cost than bulk deformation \((A_I)\). In typical materials, this condition \(B > G\) is naturally satisfied, unless the material is auxetic, i.e., has negative Poisson ratio.

In the chiral limit, we apply a periodic strain with \(C_{3v}\) symmetry. Such strain fields can be categorized into two classes depending on whether the mirror plane is parallel or perpendicular to the reciprocal lattice vector \(G^m\). The first category (parallel) includes space groups \(p\bar{3}m1\) and \(p\bar{6}m\), while the latter (perpendicular) gives \(p\bar{3}1m\). Here we will focus on the first category. Within the first harmonic approximation, the strain can be written as

\[
\hat{A}(r) = t_0 \frac{\alpha^2}{2} \sum_{n=1}^{3} \omega^{n-1} \cos(G^m_n \cdot r + \phi),
\]

where \(\alpha^2 > 0\) is the strength of the strain, \(\phi\) is an arbitrary phase and \(\omega = \exp(2\pi i/3)\). \(G^m_0 = \frac{4\pi}{\sqrt{3} a} (0, 1, 0)\), \(G^m_1 = \frac{4\pi}{\sqrt{3} a} (\mp \sqrt{3}/2, -1/2, 1)\), the reciprocal lattice vectors. The second category has similar exact flat bands (see SM [56]) and its strain field is

\[
\hat{A}(r) = \pm t_0 \alpha^2 \sum_{n=1}^{3} \omega^{n-1} \exp(iG^m_n \cdot r).
\]

**Exact flat bands at \(\phi = 0\).**—We start by considering a special case with \(\phi = 0\) in Eq. (3), where the system exhibits a higher rotational symmetry \(C_{3v}\). As shown in Fig. 1(c-e), two exactly flat bands with \(E = 0\) arise at critical values of \(\alpha\) (the square root of the strain strength). We define a dimensionless parameter \(\tilde{\alpha} = \alpha/|G_0| a\), which fully controls the physics of the system. These critical \(\tilde{\alpha}\)’s are roughly equally spaced \(\Delta \tilde{\alpha} \approx 1.39\) [Fig. 1(g, i)], and they also maximize the band gap that separates these two
flat bands from others [Fig. 1(h)]. In analogy to TBG in the chiral limit [41], these critical strains and exact flat bands can be analytically proved, and their Bloch wavefunctions can be analytically constructed. We start from the Γ point. Because the strain preserves $T$ and $C_3$ symmetry, the two-fold degeneracy of the QBCP remains. In the chiral limit, the two of these degenerate states must remain at $E = 0$, and their wavefunctions can be obtained from the Hamiltonian Eq. (2):

$$\Psi_{\Gamma,1}(r) = \{\psi_{\Gamma}(r), 0\} \text{ and } \Psi_{\Gamma,2}(r) = \{0, \psi_{\Gamma}^*(r)\},$$

where $\psi_{\Gamma}(r)$ is a periodic function of the superlattice and obeys $(4\partial_z^2 + \hat{a}^*)\psi_{\Gamma}(r) = 0$. $\Psi_{\Gamma,1}$ and $\Psi_{\Gamma,2}$ are related with each other via time-reversal, $T = \sigma_x K$, where $K$ is the complex conjugation.

If there are exact flat bands at $E = 0$, the eigen functions can be written as $\{\psi_k(r), 0\}$ and $\{0, \psi_k^*(r)\}$ with $(4\partial_z^2 + \hat{a}^*)\psi_k(r) = 0$. Since the kinetic part of $D^2 \psi_k$ is antiholomorphic, we can construct a trial wave function $\psi_k(r) = f_k(z) \psi_{\Gamma}(r)$. The $f_k(z)$ needs to satisfy Bloch periodicity (translation by a superlattice vector $a_m$ gives a phase shift of $e^{i k a_m}$). However, from Liouville’s theorem, such $f_k(z)$ must have poles, making $\psi_k(r)$ divergent. To avoid such singularity, $\psi_{\Gamma}(r)$ needs to have a zero that cancels the pole of $f_k(z)$. As shown in Fig. 2(a), at the magic values of $\alpha_T$, $\psi_{\Gamma}(0) = 0$. Hence, we have

$$\Psi_{k,1}(r) = \left[ \psi_k(r), 0 \right], \Psi_{k,2}(r) = \left[ 0, \psi_k^*(r) \right],$$

$$\psi_k(r) = \frac{\partial \psi_k^m}{\partial z} \left[ \begin{array}{c} x \cos \omega \tau - \frac{x}{\alpha_k^0} \frac{\partial \psi_k^m}{\partial \omega} \\ \frac{\partial \psi_k^m}{\partial z} \end{array} \right] \psi_{\Gamma}(r),$$

where $\partial_{\alpha \beta}(z, \tau)$ is the theta function of rational characteristic [35]. $a_m^0$ are lattice vectors $(a_m^0 = a_m^m(1, 0), a_m^2 = a_m^m(-1, \sqrt{3})/2)$ of the superlattice, $a_i = (a_m^0)_x + i(a_m^0)_y$. As shown in the SM [50], these wavefunctions give two exact flat bands with $E = 0$.

In analogy to the Chern basis in TBGs [59], because these two flat bands only involve holomorphic or antiholomorphic functions, they have Chern number ±1. Same as Landau levels, their Fubini-Study metric $g(k)$ satisfies the trace condition $tr(g(k)) = |F_{xy}(k)|$ [40, 51], where $F_{xy}(k)$ is the Berry curvature (see SM Sec. III [50]). In SM Fig. S1 [50], the distributions of the Berry curvature in $k$-space are shown for the first three critical $\alpha$, which we found to be quite uniform. To quantify the non-uniformity of Berry curvature, we measure the ratio between the root-mean-square deviation of the Berry curvature and its average value $\Delta F_{xy}/F_{xy}$ [35]. The smallest value (most uniform distribution) is found at the second critical $\alpha$ with $\Delta F_{xy}/F_{xy} \approx 0.027$, much smaller than the reported values in TBG flatbands [17]. Ideal quantum metric and very uniform Berry curvature make this system an ideal candidate for realizing fractional Chern insulators [15, 51].

Generic $\phi$, double zeros and 4-fold flat bands.- In addition to $\phi = 0$ (Eq. (3)), exact flat bands also arise at $\phi \neq 0$. At $\phi = n\pi/3$ ($\phi \neq n\pi/3$), the strain preserves 6-fold (3-fold) rotational symmetry with space group symmetry $p6mm$ ($p3m1$). In Fig. 3(a), we plot the bandwidth as a function of $\alpha^2$ and $\phi$ in polar coordinate, setting $\alpha^2$ and $\phi$ to be the radius and polar angle, respectively. The dark lines mark the exact flat bands and the dashed green line marks analytic solution from perturbation theory (See SM [50]). As we can see, for any value of $\phi$, exact flat bands can be reached at certain critical field strengths.

To further verify this conclusion, we show that $\psi_{\Gamma}(r)$ indeed contains zeros for all critical $\alpha$, and thus analytic wavefunctions can be constructed for these exact flat bands following the same procedure shown above. In Figs. 3(c-e), we plot in log scale $|\psi_{\Gamma}(r = 0)|$, $|\psi_{\Gamma}(r = (2a_m^m + a_m^2)/3)|$ and $|\psi_{\Gamma}(r = (2a_m^m + a_m^2)/3)|$, as well as $\min(|\psi_{\Gamma}(r = 0)|, |\psi_{\Gamma}(r = (2a_m^m + a_m^2)/3)|, |\psi_{\Gamma}(r = (2a_m^m + a_m^2)/3)|)$ in Fig. 3(b). These three real space points are the 3-fold rotation centers of the $p3m1$ lattice, and the dark lines in these figures mark $\psi_{\Gamma} = 0$, which perfectly match the exact flat bands shown in Fig. 3(a).

The reason, we only see $\psi_{\Gamma} = 0$ at these three high symmetry points in the unit cell, is that $\psi_{\Gamma}$ at other $r$ are in general complex. To make both the real and imaginary parts zero, it typically requires to adjust two real control parameters simultaneously. In contrast, $\psi_{\Gamma}$ at these high symmetry points must be real, due to the $T$ and $C_3$ symmetry (See SM [50]), and thus its value can be tuned to zero with only one control parameter, which is why exact flat bands can emerge as we scan $\alpha$. This is the reason why Figs. 3(a) and (b) are three-fold symmetric. At points highlighted by the red circles in Fig. 3(b), $\psi_{\Gamma}(r)$ has 2 zeros in a unit cell (two of the three real-space high-symmetry points). These double zeros are not accidental but due
we turn on the chiral symmetry breaking term survive, although a small bandwidth emerges. Here move away from the chiral limit, these magic flat bands filling Breaking chiral symmetry.

\[ \nu = 0 \] with Chern number \( \nu \) exactly (see SM Sec. XII \[56\]), similar to TBG \[60–62\].

The rotational symmetry of the system increases to 6-fold exactly zero, and the maximum bandgap and the minimum bandwidth slightly misalign. At \( \tilde{\alpha} = 0.79 \), the bandgap to bandwidth ratio is \( > 17 \) for \( c_0 = 0.1 \) and \( > 5 \) for \( c_0 = 0.3 \). Upon further increasing \( c_0 \), these two bands eventually mix with other bands. However, band hybridization and inversion result in an isolated highly flat band at \( c_0 = 0.9 \) and \( \tilde{\alpha} = 1.29 \) as shown in SM.

\[ \text{Fragile topology and exact solutions in interacting systems—} \text{As shown in SM Sec. IX-X} \[56\], in analogy to flat bands in TBG \[2, 3, 59\], these flat bands are fragile topological bands. In addition, for spin-1/2 fermions with charge repulsion, these flat bands exhibit an emergent \( U(2) \times U(2) \) symmetry and integer fillings can be solved exactly (see SM Sec. XII \[56\]), similar to TBG \[60–62\]. At charge neutrality (\( \nu = 0 \)), there are degenerate ground states with Chern number 0 or \( \pm 2 \), but thermal fluctuations stabilize the \( C = 0 \) state via order-by-disorder. At filling \( \nu = \pm 1 \), the exact ground state is a Chern insulator with \( C = \pm 1 \).

\[ \text{Breaking chiral symmetry—} \text{In analogy to TBG, as we move away from the chiral limit, these magic flat bands survive, although a small bandwidth emerges. Here we turn on the chiral symmetry breaking term} \ c_0 \text{ and} \ A_T \text{ in Eq. (2), setting} \ A \text{ as Eq. (3) with} \ \phi = 0 \text{ and} \ A_T(r) = -\frac{\alpha c_0 a}{4} \sum_{i=1}^{3} \cos(G_i \cdot r). \text{For a kagome lattice, this} A_T \text{ naturally arises from nearest-neighbor hoppings (See SM} \[56\]. \text{As shown in Fig. 4, magic flat bands survive even if} \ c_0 \text{ reaches 0.5, although the bandwidth is no longer exactly zero, and the maximum bandgap and the minimum bandwidth slightly misalign. At} \ \tilde{\alpha} = 0.79, \text{the bandgap to bandwidth ratio is} > 17 \text{ for} \ c_0 = 0.1 \text{ and} > 5 \text{ for} \ c_0 = 0.3. \text{Upon further increasing} \ c_0, \text{these two bands eventually mix with other bands. However, band hybridization and inversion result in an isolated highly flat band at} \ c_0 = 0.9 \text{ and} \ \tilde{\alpha} = 1.29 \text{ as shown in SM.}

\[ \text{FIG. 3: Exact flat bands in polar coordinate of} \ \tilde{\alpha}^2 \text{ (radius) and} \ \phi \text{ (polar angle). (a)} \text{ Bandwidth } \ln E_w \text{ as a function of} \ \phi \text{ and} \ \tilde{\alpha}. \text{The green dashed line shows the predicted value of principal critical} \ \tilde{\alpha} \text{ from 9th order perturbation theory. (b)} \min\{\ln |\psi(r = 0)|, \ln |\psi(r = (2a^m_2 + a^n_0)/3)|, \ln |\psi(r = (2a^m_2 + a^n_0)/3)|\} \text{ as a function of} \ \tilde{\alpha} \text{ and} \ \phi. \text{ (c-e) } \ln |\psi(r = (2a^m_2 + a^n_0)/3)|, \ln |\psi(r = (2a^m_2 + a^n_0)/3)| \text{ as a function of} \ \tilde{\alpha} \text{ and} \ \phi. \text{The dark lines in each plot show critical} \ \tilde{\alpha} \text{ with two exact flat bands. At the crossing point of two dark lines [red circles in (b)]}, \text{the strain induces four degenerate flat bands.}

\[ \text{FIG. 4: Effect of chiral symmetry breaking. (a)-(b) Bandwidth } E_w \text{ and bandgap } E_g \text{ as a function of} \ \tilde{\alpha} \text{ for different} \ c_0. \]
Fig. S4 [56]. Similarly, for $\tilde{A}$ with $\phi = \pi$ [Eq. (3)] and the same $A_I$, at $c_0 \approx 0.15$ and $\tilde{c} \approx 1.695$, the four degenerate flat bands around charge neutrality split into two pairs of isolated nearly flat bands as shown in SM. Sec. IX [58], and both of them remain fragile topological, protected by $C_2$ and $C_3$, respectively.

Discussions.—In this letter, we studied a QBCP near the $\Gamma$ point under periodic strain with $C_{3v}$ symmetry, and found exact topological flat bands in the chiral limit. Utilizing materials with nearly-chiral QBCPs (e.g., GaCu$_3$(OH)$_6$Cl$_2$ [57]) and a periodic strain field, which has already been achieved in experiments [32, 34], it is possible to access the vicinity of this chiral limit and explore these flat bands, offering a new platform to study topological flat bands, fragile topology, and correlated phases such as fractional Chern insulators and unconventional superconductivity. To further verify the feasibility of this proposal, in SM. Sec. XIV [56], we compared our model with existing experiments and the estimation indicates current strain-engineering technology would allow us to reach at least the first two magic flat bands at the temperature $T \sim 4$ K. In addition, the same principle also applies to other systems with quadratic band crossings, such as photonic/phononic crystals, magnons, and optical lattices [63–67].

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Note added: after the submission of this Letter, Ref. [68] showed that similar magic flat bands can arise in QBCP models with 4-fold rotational symmetry.

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See Supplemental Material ... for details, including the following: the construction of wave-function corresponding to exact flat bands, a description of symmetries required for exact flat band, example of exact flat band in p3lm space group, proof of fragile topology of the flat bands, construction of exact ground states of interacting Hamiltonian, and effect of chiral symmetry breaking, which includes refs. [69–75].
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KAGOME HAMILTONIAN UNDER STRAIN

Consider a buckled Kagome Hamiltonian,
\[
H = \begin{pmatrix}
0 & -2t_1 \cos k_1 & -2t_2 \cos k_2 \\
-2t_1 \cos k_1 & 0 & -2t_3 \cos k_3 \\
-2t_2 \cos k_2 & -2t_3 \cos k_3 & 0
\end{pmatrix}.
\] (S1)

Here \( k_1 = \mathbf{k} \cdot \mathbf{a}_1, \ k_2 = \mathbf{k} \cdot \mathbf{a}_2, \ k_3 = \mathbf{k} \cdot \mathbf{a}_3, \ \mathbf{a}_1 = a(1, 0), \ \mathbf{a}_2 = a(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}), \ \mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1. \)

To write the effective Hamiltonian near the quadratic band crossing point at \( \Gamma \) of this Hamiltonian, we write the Hamiltonian in the eigenbasis at \( \Gamma \):
\[
H_1 = U^\dagger HU \quad \text{where} \quad U = \begin{pmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}
\end{pmatrix}.
\] (S2)

The new Hamiltonian \( H_1 \) has the form
\[
H_1 = \begin{pmatrix}
B & C \\
C^\dagger & A
\end{pmatrix},
\] (S3)

where
\[
A = \begin{pmatrix}
2t_2 \cos k_2 & \frac{1}{\sqrt{3}}(2t_1 \cos k_1 - 2t_3 \cos k_3) \\
\frac{1}{\sqrt{3}}(2t_1 \cos k_1 - 2t_3 \cos k_3) & \frac{1}{3}(4t_1 \cos k_1 - 2t_2 \cos k_2 + 4t_3 \cos k_3)
\end{pmatrix},
\] (S4)
\[
B = -\frac{4}{3}(t_1 \cos k_1 + t_2 \cos k_2 + t_3 \cos k_3),
\] (S5)
\[
C = \begin{pmatrix}
\sqrt{\frac{2}{3}}(t_1 \cos k_1 - t_3 \cos k_3) \\
-\sqrt{\frac{2}{3}}(t_1 \cos k_1 - t_2 \cos k_2 + t_3 \cos k_3)
\end{pmatrix}.
\] (S6)

Then we integrate out the non-degenerate band to write an effective \( 2 \times 2 \) matrix for the two degenerate bands:
\[
H' = A - C^\dagger B^{-1} C
\] (S7)

Before applying any strain field to the Kagome lattice, \( t_1 = t_2 = t_3 = t_0 \), near the \( \Gamma \) point, the effective \( 2 \times 2 \) Hamiltonian is
\[
H_{\Gamma} = -\frac{1}{2}(-4 + k_x^2 + k_y^2)t_0 a^2 \mathbb{1} - k_x k_y t_0 a^2 \sigma_x + \frac{1}{2}(k_x^2 - k_y^2)t_0 a^2 \sigma_z.
\] (S8)

Here \( t_0 \) is the nearest-neighbor hopping strength, \( a \) is the bond length. When the strain field is nonzero, then \( t_1 \neq t_2 \neq t_3 \). Setting \( t_i = t_0 + \delta t_i \), and keeping only terms up to the lowest order in \( \delta t_i \), the Hamiltonian around the \( \Gamma \) point becomes (ignoring the constant part)
\[
H(\mathbf{r}) = -\frac{1}{2}(k_x^2 + k_y^2)t_0 a^2 \mathbb{1} + A_\| \mathbb{1} - k_x k_y t_0 a^2 \sigma_x + A_\| \sigma_x + \frac{1}{2}(k_x^2 - k_y^2)t_0 a^2 \sigma_z + A_\perp \sigma_z,
\] (S9)
\[
A_\| = \frac{2}{3}(\delta t_1 + \delta t_2 + \delta t_3),
\] (S10)
\[
A_\| = \frac{2}{\sqrt{3}}(\delta t_1 - \delta t_3),
\] (S11)
\[
A_\| = -\frac{2}{3}(\delta t_1 - 2\delta t_2 + \delta t_3).
\] (S12)

The above Hamiltonian can be mapped to the following one using the transformation \( H \rightarrow \hat{U}H\hat{U}^\dagger \) where \( \hat{U} = e^{i\pi \sigma_y/4} e^{-i\pi \sigma_z/4} \):
\[
H_{\Gamma} = -\frac{1}{2}(k_x^2 + k_y^2)t_0 a^2 \mathbb{1} + \frac{1}{2}(k_x^2 - k_y^2)t_0 a^2 \sigma_x - k_x k_y t_0 a^2 \sigma_y,
\] (S13)
\[
H(\mathbf{r}) = -\frac{1}{2}(k_x^2 + k_y^2)t_0 a^2 \mathbb{1} + A_\| \mathbb{1} - \frac{1}{2}(k_x^2 - k_y^2)t_0 a^2 \sigma_x - A_\| \sigma_x - k_x k_y t_0 a^2 \sigma_y + A_\perp \sigma_y.
\] (S14)
In this new basis, the representation of time reversal operator becomes $T = \sigma_x K$. To tune the coefficients of the terms proportional to $1$ in Eq. (S13) (S14), we multiply them by $c_0$ to obtain (and multiply the whole matrix by 2):

$$H_\Gamma = -c_0(k_x^2 + k_y^2)t_0a^2\mathbb{1} - (k_x^2 - k_y^2)t_0a^2\sigma_x - 2k_xk_yt_0a^2\sigma_y,$$

$$H(\mathbf{r}) = -c_0(k_x^2 + k_y^2)t_0a^2\mathbb{1} + 2c_0A_I\mathbb{1} - (k_x^2 - k_y^2)t_0a^2\sigma_x - 2A_z\sigma_y - 2k_xk_yt_0a^2\sigma_y + 2A_x\sigma_y.$$ (S15) (S16)

Now, if we set

$$\delta t_1 = -\frac{3}{16}t_0\alpha^2\cos(G_{m1} \cdot \mathbf{r}),$$
$$\delta t_2 = -\frac{3}{16}t_0\alpha^2\cos(G_{m2} \cdot \mathbf{r}),$$
$$\delta t_3 = -\frac{3}{16}t_0\alpha^2\cos(G_{m3} \cdot \mathbf{r}),$$ (S17) (S18) (S19)

we have

$$A_z = -\frac{1}{4}t_0\alpha^2[\cos(G_{m1} \cdot \mathbf{r}) - \frac{1}{2}\cos(G_{m2} \cdot \mathbf{r}) - \frac{1}{2}\cos(G_{m3} \cdot \mathbf{r})],$$
$$A_x = -\frac{1}{4}t_0\alpha^2[\sqrt{3}\cos(G_{m2} \cdot \mathbf{r}) - \sqrt{3}\cos(G_{m3} \cdot \mathbf{r})],$$
$$A_I = -\frac{1}{8}t_0\alpha^2[\cos(G_{m1} \cdot \mathbf{r}) + \cos(G_{m2} \cdot \mathbf{r}) + \cos(G_{m3} \cdot \mathbf{r})],$$ (S20) (S21) (S22)

where $G_{1m} = \frac{4\pi}{\sqrt{3}a^m}(0, 1)$, $G_{2,3m} = \frac{4\pi}{\sqrt{3}a^m}(\mp\sqrt{3}/2, -1/2)$ and $a^m$ is the lattice constant of the superlattice.
EXACT FLAT BANDS

Since the continuum model in Eq. (1) of the main text is obtained by expanding around the quadratic band crossing point protected by $C_3$ and $T$, as long as the $A$ field does not break any of these symmetries, the lowest two bands at the $\Gamma$ point are degenerate with energy $E = 0$ if the chiral symmetry is not broken. The two corresponding eigenfunctions are of the form $\Psi_{T1} = \{\psi_{T}(r), 0\}^T$ and $\Psi_{T2} = \{0, \psi_{T}^*(r)\}^T$ satisfying

\[
(4\partial_z^2 + \hat{A}^*)\psi_{T}(r) = 0. \tag{S23}
\]

To get an exact flat band for all wave vectors $k$ with $E = 0$, we can construct a trial wave function

\[
\psi_k(r) = f_k(z)\psi_{T}(r) \tag{S24}
\]

following [1]. Since $f_k(z)$ is a holomorphic function, $\psi_k(r)$ satisfies $(4\partial_z^2 + \hat{A}^*)\psi_k(r) = 0$. However, for $\psi_k(r)$ to be a wave function at the wave vector $k$, it must satisfy the Bloch periodicity:

\[
\psi_k(r + \mathbf{a}_m) = e^{ik\cdot\mathbf{a}_m}\psi_k(r), \tag{S25}
\]

where $\mathbf{a}_m$ are the lattice vectors of the superlattice. Hence, $f_k(z)$ must satisfy $f_k(z + a_i) = e^{ik\cdot\mathbf{a}_m}f_k(z)$, where $a_i = (\mathbf{a}_m)_x + i(\mathbf{a}_m)_y$, since $\psi_{T}(r)$ is a periodic function with $\psi_{T}(r + \mathbf{a}_m) = \psi_{T}(r)$. But from Liouville’s theorem, we know such $f_k(z)$ must have poles. This makes $\psi_k(r)$ divergent, unless $\psi_{T}(r)$ has a zero exactly at the same position as the pole of $f_k(z)$. We have shown in Fig. 2 of main text that at the critical value of $\alpha$, where the lowest two bands become exactly flat, $\psi_{T}(r)$ have zeros at the positions $r = r_0 + m\mathbf{a}_1 + n\mathbf{a}_2 = (r_{01} + m)a_1 + (r_{02} + n)a_2$ ($m, n \in \mathbb{Z}$). Then we can construct the following

\[
f_k(z) = \frac{\vartheta_{k,\mathbf{a}_m}(z - \frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}, \frac{2}{a_1} \omega + \frac{r_{02}}{a_2} \omega)}{\vartheta_{\frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}, \frac{2}{a_1} \omega}}, \tag{S26}
\]

where we choose $\mathbf{a}_1^m = a^m(1, 0)$, $\mathbf{a}_2^m = a^m(-1/2, \sqrt{3}/2)$ (hence $a_1 = a^m, a_2 = a^m\omega$ where $\omega = -1/2 + i\sqrt{3}/2$), and $\vartheta_{a,b}(z, \tau)$ is the theta function of rational characteristics $a$ and $b$ defined as [2]:

\[
\vartheta_{a,b}(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+a)^2} e^{2\pi i(n+a)(z+b)}. \tag{S27}
\]

Since $\vartheta_{a,b}(z, \tau)$ has zeros at $(n - 1/2 - a)\tau + (m + 1/2 - b)$ ($m, n \in \mathbb{Z}$) [2], the zeros of $\vartheta_{\frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}, \frac{2}{a_1} \omega}$ cancel with the zeros of $\psi_{T}(r)$. Moreover, since

\[
\vartheta_{a,b}(z + 1, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+a)^2} e^{2\pi i(n+a)(z+1+b)}
= \sum_{n=-\infty}^{\infty} e^{2\pi i(n+a)} e^{i\pi\tau(n+a)^2} e^{2\pi i(n+a)(z+b)}
= e^{2\pi i a} \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+a)^2} e^{2\pi i(n+a)(z+b)} = e^{2\pi i a} \vartheta_{a,b}(z, \tau), \tag{S28}
\]
where we used $e^{2\pi in} = 1$ and

$$
\vartheta_{a,b}(z + \tau, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau (n+a)^2} e^{2\pi i (n+a)(z+\tau+b)}
= \sum_{n=-\infty}^{\infty} e^{2\pi i \tau (n+a)} e^{i\pi \tau (n+a)^2} e^{2\pi i (n+a)(z+b)}
= e^{-i\pi \tau} \sum_{n=-\infty}^{\infty} e^{i\pi \tau (n+a+1)^2} e^{2\pi i (n+a)(z+b)}
= e^{-i\pi \tau} \sum_{n'=-\infty}^{\infty} e^{i\pi \tau (n'+a)^2} e^{2\pi i (n'\tau - 1+a)(z+b)}
= e^{-i\pi \tau - 2\pi i (z+b)} \sum_{n'=-\infty}^{\infty} e^{i\pi \tau (n'+a)^2} e^{2\pi i (n'+a)(z+b)}
= e^{-i\pi \tau - 2\pi i (z+b)} \sum_{n=-\infty}^{\infty} e^{i\pi \tau (n+a)^2} e^{2\pi i (n+a)(z+b)} = e^{-i\pi \tau - 2\pi i (z+b)} \vartheta_{a,b}(z, \tau),
$$

(S29)

$f_k(z)$ satisfies the Bloch periodicity. Therefore,

$$
\Psi_{k,1}(r) = \begin{pmatrix} f_k(z) \psi_\Gamma(r) \end{pmatrix} \quad \text{and} \quad \Psi_{k,2}(r) = \begin{pmatrix} 0 \\ f_{-k}^*(z) \psi_\Gamma^*(r) \end{pmatrix}
$$

(S30)

are the eigenfunctions at any $k$ with eigenvalues $E = 0$ as long as $\psi_\Gamma(r)$ has a zero in each superlattice unit cell.
proof of $\text{tr}(g) = |F_{xy}|$ for the wavefunction in Eq. (5) of the main text

Here we show that for exactly flat bands with eigenfunctions of the form Eq. (S30) (Eq. (5) of the main text) with Eq. (S26), the trace of the Fubini-Study metric $\text{tr}(g(k))$ is equal to the absolute value of the Berry curvature ($|F_{xy}(k)|$). To show this, we follow the arguments outlined in [3]. The Fubini-Study metric $g_{ab}(k)$ and the Berry curvature $F_{xy}(k)$ are given by:

$$g_{ab}(k) = \Re(\eta_{ab}(k)), \quad F_{xy}(k) = -2\Im(\eta_{xy}(k)),$$

where $\eta_{ab}(k) = \frac{\langle \partial_a u_k | \partial_b u_k \rangle \langle u_k | \partial_b u_k \rangle}{\| u_k \|^4} - \frac{\langle \partial_a u_k | u_k \rangle \langle u_k | \partial_b u_k \rangle}{\| u_k \|^2}$, (S31)

where $u_k(r)$ is the periodic part of the Bloch function, $\partial_a \equiv \partial_{k_a}$, and $\| u_k \|^2 = \langle u_k | u_k \rangle$.

Now, if $u_k$ is holomorphic in $k$, i.e., $\partial_a u_k = \frac{1}{2}(\partial_{k_a} + i \partial_{k_a})u_k = 0$, which we show below, then $\partial_{k_a} u_k = (\partial_k + \partial_{\bar{k}})u_k = \partial_k u_k$ and $\partial_{\bar{k}} u_k = i(\partial_k - \partial_{\bar{k}})u_k = i\partial_k u_k$, and hence

$$\begin{align*}
\eta_{xx}(k) &= \frac{\langle \partial_{k_a} u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^4} - \frac{\langle \partial_{k_a} u_k | u_k \rangle \langle u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^2}, \\
\eta_{xy}(k) &= i \frac{\langle \partial_{k_a} u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^4} - i \frac{\langle \partial_{k_a} u_k | u_k \rangle \langle u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^2} = i \eta_{xx}(k), \\
\eta_{yx}(k) &= -i \frac{\langle \partial_{k_a} u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^4} - i \frac{\langle \partial_{k_a} u_k | u_k \rangle \langle u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^2} = -i \eta_{xx}(k), \\
\eta_{yy}(k) &= \frac{\langle \partial_{k_a} u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^4} - \frac{\langle \partial_{k_a} u_k | u_k \rangle \langle u_k | \partial_{k_a} u_k \rangle}{\| u_k \|^2} = \eta_{xx}(k).
\end{align*}$$

(S32)

Hence we have,

$$\eta(k) = \eta_{xx}(k) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \text{and} \quad g(k) = \eta_{xx}(k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_{xy}(k) = -2\eta_{xx}(k),$$

(S33)

and consequently, $\text{tr}(g(k)) = |F_{xy}(k)|$.

To show that $\partial_{k_a} u_k = 0$, where $u_k = e^{-ik \cdot r} \psi_k(r)$ is the periodic part of the Bloch function in Eq. (S24) with $f_k(z)$ in Eq. (S26), we first do a $k$ dependent transformation $\tilde{\psi}_k(r) = \exp(-i\frac{a_k}{2}(\frac{1}{2} - r_0 - \frac{k_0^2}{8\pi} - \frac{k_0}{8\pi}))\psi_k(r) = \exp(-i\frac{a_k}{2}(\frac{1}{2} - r_0 - \frac{k_0^2}{8\pi} - \frac{k_0}{8\pi})) f_k(z) \psi_T(r)$. (Note that this function $\psi_k(r)$ is not periodic in $k$-space, but that is not required for a well-defined Berry curvature [3].) Then, we have

$$\begin{align*}
u_k(r) &= \exp(-ia_0 \frac{1}{2} \left( \frac{1}{2} - r_0 - \frac{k_0^2}{8\pi} - \frac{k_0}{8\pi} \right)) \exp(-i k \cdot r) f_k(z) \psi_T(r) \\
&= \exp(-ia_0 \frac{1}{2} \left( \frac{1}{2} - r_0 - \frac{k_0^2}{8\pi} - \frac{k_0}{8\pi} \right)) \exp(-i(k\bar{z} + k z)/2) f_k(z) \psi_T(r).
\end{align*}$$

(S34)
With this, we have

\[ \partial_k u_k(r) \]

\[ = \partial_k \left( \exp(-i \frac{a_1 \tilde{k}}{2} (\frac{1}{2} - r_{01} - \frac{k a_2}{4\pi} - \frac{\tilde{k} a_2}{8\pi})) \exp(-i (k \bar{z} + k z)/2) f_k(z) \psi(r) \right) \]

\[ = \frac{\psi(r)}{\partial_{\frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}}(\frac{z}{a_1}, \omega)} \partial_k \left( \exp(-i \frac{a_1 \tilde{k}}{2} (\frac{1}{2} - r_{01} - \frac{k a_2}{4\pi} - \frac{\tilde{k} a_2}{8\pi})) \exp(-i (k \bar{z} + k z)/2) \right) \]

\[ = \frac{a_1 \psi(r)}{\partial_{\frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}}(\frac{z}{a_1}, \omega)} \partial_{a_1} \left( \exp(-i \frac{a_1 \tilde{k}}{2} (\frac{1}{2} - r_{01} - \frac{k a_2}{4\pi} - \frac{\tilde{k} a_2}{8\pi})) \exp(-i (k \bar{z} + k z)/2) \right) \]

\[ = \frac{a_1 \psi(r)}{\partial_{\frac{1}{2} - r_{02}, \frac{1}{2} - r_{01}}(\frac{z}{a_1}, \omega)} \sum_{n=-\infty}^{\infty} \left( -i \frac{1}{2} (\frac{1}{2} - r_{01} - \frac{k a_2}{4\pi} - \frac{\tilde{k} a_2}{8\pi}) - i z/2 a_1 + 2i \pi \omega(n + \frac{k \cdot a_1}{2\pi} - \frac{1}{2} - \frac{r_0}{2}) \frac{1}{4\pi} \\
- 2\pi i(n + \frac{k \cdot a_1}{2\pi} - \frac{1}{2} - \frac{r_0}{2}) \frac{\omega}{4\pi} + 2\pi i \frac{1}{4\pi} (z/a_1 + \frac{1}{2} - r_{01} - \frac{k \cdot a_2}{2\pi}) \right) \times \]

\[ e^{i \pi \omega(n + \frac{k \cdot a_1}{2\pi} - \frac{1}{2} - \frac{r_0}{2})^2} e^{2\pi i(n + \frac{k \cdot a_1}{2\pi} - \frac{1}{2} - \frac{r_0}{2})(z/a_1 + \frac{1}{2} - r_{01} - \frac{k \cdot a_2}{2\pi})} \]

\[ = 0. \]

(S35)

This completes the proof.
FIG. S1: Berry Curvature distribution after adding $\sigma_z$ to separate the middle two bands at different critical values of $\tilde{\alpha}$ for the strain field defined in Eq. (3) in the main text with $\phi = 0$: (a) $\tilde{\alpha} = 0.79$ (b) $\tilde{\alpha} = 2.13$, and (c) $\tilde{\alpha} = 3.52$. The calculated values of the measure $\Delta F_{xy}/\bar{F}_{xy}$ of the fluctuation of the Berry curvature defined in the main text for these three cases are $\Delta F_{xy}/\bar{F}_{xy} = 0.287965$, $\Delta F_{xy}/\bar{F}_{xy} = 0.0270411$, and $\Delta F_{xy}/\bar{F}_{xy} = 0.136598$, respectively.
\[ \psi_T(r) \text{ IS REAL AT } C_{3v}, \text{ IN Variant POINTS IN THE SUPERLATTICE UNIT CELL} \]

We found earlier that if \( \psi_T(r = r_0) = 0 \), we can construct two exact flat bands with \( E = 0 \) and eigenfunctions are given in Eq. (S30). Generally, there is only one zero of \( \psi_T(r) \) in one superlattice unit cell, otherwise we could construct more flat bands; this is why the zeros of \( \psi_T(r) \) generally appear at isolated \( C_3 \) invariant points \( r_0 \) in the superlattice unit cell. However, it is generally not guaranteed that at a \( C_3 \) invariant point \( \psi_T(r) \) is real. Hence, if we want to set \( \psi_T(r = r_0) = 0 \) (where \( r_0 \) is a \( C_3 \) invariant point) to obtain the critical value of the strain amplitude \( \alpha^2 \) in the examples we show in the main text, we actually have two equations \( \Re(\psi_T(r = r_0)) = 0 = \Im(\psi_T(r = r_0)) \) and only one unknown \( \alpha \) to tune, hence there may not be any solution for the critical \( \alpha \). However, below we show that if there is extra mirror symmetry, \( \psi_T(r) \) has to be real at the \( C_{3v} \) symmetric points, hence there is only one equation \( \Re(\psi_T(r = r_0)) = 0 \) to satisfy to obtain the critical \( \alpha \).

To show this we start from the \( 2 \times 2 \) Hamiltonian with chiral symmetry (S):

\[
\mathcal{H}_0 = \int d^2 r [c_1^\dagger(r) c_2(r) H(r) + c_1(r) c_2^\dagger(r)], \quad H(r) = \begin{bmatrix} 0 & 4\partial_z^2 + \hat{A}(r) \\ 4\partial_z^2 + \hat{A}^*(r) & 0 \end{bmatrix},
\]

where \( \hat{A} \) is the strain field, \( c_i^\dagger(r) \) are creation operators. In the following we prove that as long as the strain field \( \hat{A} \) has \( C_3 \), one mirror symmetry (\( M_x \) or \( M_y \), parallel or perpendicular to the reciprocal lattice vector \( \mathbf{G}^m \), respectively) and time reversal symmetry \( T \), \( \psi_T(r = r_0) \) is real at the \( C_{3v} \) invariant point \( r_0 \). For simplicity, we use \( M_y \) and \( C_3 \) to derive the result. Before adding any strain field, the matrix representations \( \rho \) of \( C_3, M_y, S \) and \( T \) of the Hamiltonian are:

\[
\begin{align*}
\rho(C_3) &= \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \\
\rho(M_{x/y}) &= \begin{pmatrix} \sigma_x & \rho(M_{x/y})H(r) \rho(C_3)^{-1} = H(M_{x/y}r), \\
\rho(S) &= \sigma_z, \\
\rho(T) &= \sigma_x, \\
\end{align*}
\]

where \( \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \), \( K \) is complex conjugate operator, and we have \( R c_i^\dagger(r) R^{-1} = c_j^\dagger(Rr) [\rho(R)]_{ji} \) for \( \forall R \in \{C_3, M_x, M_y, S, T\} \). We have the following (anti)commutation relations between the symmetries:

\[
[C_3, S] = 0, [C_3, T] = 0, [M_{x/y}, S] = 0, [M_{x/y}, T] = 0, \{T, S\} = 0.
\]

After adding a strain field which has \( M_y \) and \( C_3 \), the QBCP still remains at \( \Gamma \) at \( E = 0 \) because the system still has \( C_{3v}, S \) and \( T \). Let the corresponding eigenbasis be \( c_i^\dagger(r) = 0 \). The eigenbasis transforms under the symmetries of the \( \Gamma \) point in the following way:

\[
\begin{align*}
C_3 c_i^\dagger(r) C_3^{-1} &= c_i^\dagger(r), \\
M_{x/y} c_i^\dagger(r) M_{x/y}^{-1} &= c_i^\dagger(r), \\
S c_i^\dagger(r) S^{-1} &= c_i^\dagger(r), \\
T c_i^\dagger(r) T^{-1} &= c_i^\dagger(r),
\end{align*}
\]

where the matrices \([B_R]\) are called the sewing matrices. This implies that \( \Psi_{\Gamma,1}(r) \) and \( \Psi_{\Gamma,2}(r) \) satisfy the following equations:

\[
\begin{align*}
\rho(C_3)\Psi_{\Gamma,1}(r) &= \Psi_{\Gamma,2}(C_3 r), \\
\rho(M_{x/y})\Psi_{\Gamma,1}(r) &= \Psi_{\Gamma,2}(M_{x/y} r), \\
\rho(S)\Psi_{\Gamma,1}(r) &= \Psi_{\Gamma,2}(r), \\
\rho(T)\Psi_{\Gamma,2}(r) &= \Psi_{\Gamma,1}(r)
\end{align*}
\]

In general, wavefunctions have the form \( \Psi_{\Gamma,1}(r) = \{f_1(r), g_1(r)\} \) and \( \Psi_{\Gamma,2}(r) = \{f_2(r), g_2(r)\} \). However, we can gauge fix the sewing matrices in a certain way such that we have \( \Psi_{\Gamma,1}(r) = \{\psi_T(r), 0\}^T \) and \( \Psi_{\Gamma,2}(r) = \{0, \psi_T^*(r)\}^T \), and
ψ_T(r) is real at the \( C_3 \) symmetric points, which we choose to be the origin without loss of generality. To show this, we start by choosing \([B_{C_3}] = \text{Diag}\{\omega^2, \omega\}\) such that:

\[
\begin{pmatrix}
\omega^2 & 0 \\
0 & \omega
\end{pmatrix}
\Psi_{\Gamma,1}(r) = \omega^2 \Psi_{\Gamma,1}(C_3 r) \Rightarrow f_1(r) = f_1(C_3 r), \omega^2 g_1(r) = g_1(C_3 r), \quad (S41a)
\]

\[
\begin{pmatrix}
\omega^2 & 0 \\
0 & \omega
\end{pmatrix}
\Psi_{\Gamma,2}(r) = \omega \Psi_{\Gamma,2}(C_3 r) \Rightarrow \omega f_2(r) = f_2(C_3 r), g_2(r) = g_2(C_3 r). \quad (S41b)
\]

This implies that at the \( C_3 \) symmetric point (the origin) \( g_1(0) = f_2(0) = 0 \). Moreover, since \( C_3 \) and \( T \) commute with each other, we have

\[
C_3 T \tilde{c}^\dagger_{\Gamma,1}(C_3 T)^{-1} = TC_3 \Psi_{\Gamma,1}(r) \tilde{c}^\dagger_{\Gamma,1} C_3^{-1} T^{-1} = T \omega^2 \tilde{c}^\dagger_{\Gamma,1} T^{-1} = \omega T \tilde{c}^\dagger_{\Gamma,1} T^{-1}. \quad (S42)
\]

This implies

\[
T \tilde{c}^\dagger_{\Gamma,1} T^{-1} = e^{i\theta} \tilde{c}^\dagger_{\Gamma,2}. \quad (S43)
\]

We set \( \theta = 0 \) to get \( T \tilde{c}^\dagger_{\Gamma,1} T^{-1} = \tilde{c}^\dagger_{\Gamma,2} \). This implies

\[
\sigma_x \Psi^*_{\Gamma,1}(r) = \Psi_{\Gamma,2}(r) \Rightarrow f_2(r) = g_1^\ast(r), g_2(r) = f_1^\ast(r). \quad (S44)
\]

Furthermore, since \( C_3 M_{x/y} = M_{x/y} C_3^{-1} \), we require that \([B_{M_{x/y}}] = \sigma_x\) to obtain:

\[
\sigma_x \Psi_{\Gamma,1}(r) = \Psi_{\Gamma,2}(M_{x/y} r) \Rightarrow g_1(r) = g_1^\ast(M_{x/y} r), f_1(r) = f_1^\ast(M_{x/y} r). \quad (S45)
\]

Lastly, since \([S, C_3] = 0\), we can fix \([B_S] = \sigma_z\) implying \( g_1(0) = 0 \). Using all these and renaming \( f_1 \) as \( \psi_T \), we get

\[
\Psi_{\Gamma,1}(r) = \begin{pmatrix} \psi_T(r) \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_{\Gamma,2}(r) = \begin{pmatrix} 0 \\ \psi_T^\ast(r) \end{pmatrix} \quad \text{with} \quad \psi_T(r) = \psi_T^\ast(M_{x/y} r) \quad \text{and} \quad \psi_T(r) = \psi_T(C_3 r). \quad (S46)
\]

This means that on the mirror invariant lines (and hence on the \( C_{3\nu} \) invariant points) in real space \( \psi_T(r) \) is real as promised.
GENERAL FORM OF $C_{3v}$ SYMMETRIC STRAIN FIELD

In this section, we show that the most general form of $\tilde{A}(r)$ that satisfies $C_3$ rotation symmetry and mirror symmetry $M_x$ and translation symmetry is of the form:

$$\tilde{A}(r) = \sum_{\mathbf{G}^m} a_{\mathbf{G}^m} [e^{i\phi} (e^{i\mathbf{G}^m \cdot r} + \omega e^{iC_3 \mathbf{G}^m \cdot r} + \omega^2 e^{iC_3^2 \mathbf{G}^m \cdot r}) + e^{-i\phi} (e^{iM_3 \mathbf{G}^m \cdot r} + \omega e^{iC_3 M_3 \mathbf{G}^m \cdot r} + \omega^2 e^{iC_3^2 M_3 \mathbf{G}^m \cdot r})],$$

(S47)

where $\mathbf{G}^m$ is a reciprocal lattice vector of the superlattice, $a_{\mathbf{G}^m}$ is the amplitude at $\mathbf{G}^m$ and is real. Also notice that when we have mirror $M_x$, the amplitude of $\mathbf{G}^m$ is related to the amplitude at $M_y \mathbf{G}^m$. Any $\tilde{A}$ of this form has $C_{3v}$ symmetry, and consequently, the corresponding $\psi_T(r)$ is real at the $C_{3v}$ invariant points in the superlattice unit cell as shown in the previous section.

To this end, let us consider a strain field $\tilde{A}(r) = \sum_{\mathbf{G}^m} \tilde{A}_{\mathbf{G}^m} e^{i\mathbf{G}^m \cdot r}$, where $\tilde{A}_{\mathbf{G}^m}$ is the complex amplitude at the reciprocal lattice vector $\mathbf{G}^m$. We start with $C_3$. Since $C_3$ has representation $\rho(C_3) = \text{Diag}\{\omega^2, \omega\}$ (S37), if $\tilde{A}$ obeys this symmetry, we have

$$\rho(C_3) H(r) \rho(C_3)^{-1} = H(C_3 r) \Rightarrow \tilde{A}(C_3 r) = \omega \tilde{A}(r).$$

(S48)

To satisfy this condition, if $\tilde{A}$ must have amplitudes $\tilde{A}_{C_3 \mathbf{G}^m} = \omega \tilde{A}_{\mathbf{G}^m}$ and $\tilde{A}_{C_3^2 \mathbf{G}^m} = \omega^2 \tilde{A}_{\mathbf{G}^m}$ at reciprocal lattice vectors $C_3 \mathbf{G}^m$ and $C_3^2 \mathbf{G}^m$, respectively.

Next, we consider the mirror symmetry $M_x$. Since $M_x$ has the representation $\rho(M_x) = \sigma_x$ (S37), if $\tilde{A}$ obeys this symmetry, we have

$$\rho(M_x) H(r) \rho(M_x)^{-1} = H(M_x r) \Rightarrow \tilde{A}(M_x r) = \tilde{A}^*(r).$$

(S49)

Furthermore, we have the following:

$$\tilde{A}^*(r) = \sum_{\mathbf{G}^m} \tilde{A}_{\mathbf{G}^m} e^{-i\mathbf{G}^m \cdot r},$$

(S50)

and

$$\tilde{A}(M_x r) = \sum_{\mathbf{G}^m} \tilde{A}_{\mathbf{G}^m} e^{i\mathbf{G}^m \cdot M_x r}$$

$$= \sum_{\mathbf{G}^m} \tilde{A}_{\mathbf{G}^m} e^{i(M_x^{-1} \mathbf{G}^m) \cdot r}$$

$$= \sum_{\mathbf{G}^m} \tilde{A}_{\mathbf{G}^m} e^{i(M_x \mathbf{G}^m) \cdot r} \text{ since } M_x^{-1} = M_x$$

$$= \sum_{\mathbf{G}^m} \tilde{A}_{M_y \mathbf{G}^m} e^{i(M_x M_y \mathbf{G}^m) \cdot r} \text{ with } \mathbf{G}^m = M_y \mathbf{G}^m \text{ and } \mathbf{G}^m = M_y^{-1} \mathbf{G}^m = M_y \tilde{\mathbf{G}}^m$$

$$= \sum_{\mathbf{G}^m} \tilde{A}_{M_y \mathbf{G}^m} e^{i(-\mathbf{G}^m) \cdot r} \text{ where we used } M_x M_y \mathbf{G}^m = C_2 \mathbf{G}^m = -\mathbf{G}^m.$$

(S51)

Comparing the expressions for $\tilde{A}^*(r)$ and $\tilde{A}(M_x r)$, we find $\tilde{A}_{M_y \mathbf{G}^m} = \tilde{A}_{\mathbf{G}^m}^*$. The constraints on the amplitudes from $C_3$ and $M_x$ together give the general form of $\tilde{A}(r)$ written in Eq. (S47).
ANALYTICAL EXPRESSION OF $\psi_T(0)$ CORRESPONDING TO EQ. (3) OF THE MAIN TEXT

For the strain field introduced in Eq. (3) of the main text, assuming $\alpha$ to be a small parameter, one can solve for the zero energy eigenfunction $\psi_T(r)$ satisfying $D^0(r)\psi_T(r) = 0$ perturbatively, plugging the ansatz

$$\psi_T(r) = 1 + \alpha^2 u_1(r) + \alpha^4 u_2(r) + \alpha^6 u_3(r) + \ldots,$$

(S52)
to get the following equation at $n$th order in perturbation:

$$4\alpha^2\partial_r^2 u_n(r) + \tilde{A}^ru_{n-1}(r) = 0$$

(S53)
with $u_0(r) = 1$. Note that since $\psi_T(r)$ is a periodic function, $u_i(r)$ is also a periodic function. It can be easily checked that

$$u_1(r) = -\frac{\alpha^2}{2|G|^a} (\cos(G^m_1 \cdot r + \phi) + \cos(G^m_2 \cdot r + \phi) + \cos(G^m_3 \cdot r + \phi)).$$

(S54)
The perturbation formula for $\psi_T(0)$ up to 9th order is:

$$\psi_T(0) = 1 - \frac{1}{2} (3 \cos(\phi) (\frac{\alpha}{|G|^m})^2 + \frac{1}{8} \frac{9}{32} \cos(2\phi)) \left(\frac{\alpha}{|G|^m}\right)^4 + \frac{17}{384} \cos(3\phi) - \frac{393 \cos(\phi)}{6272} \left(\frac{\alpha}{|G|^m}\right)^6$$

$$+ \frac{10741}{903168} \frac{-10741 \cos(2\phi)}{2119936} + \frac{3041 \cos(4\phi)}{401408} \left(\frac{\alpha}{|G|^m}\right)^8$$

$$+ \frac{-3521832913 \cos(2\phi) + 152170241 \cos(6\phi)}{19199785782400} + \frac{-87209741337 \cos(4\phi)}{7686650880} + \frac{-10846581621}{19439765334366} \left(\frac{\alpha}{|G|^m}\right)^{10}$$

$$+ \frac{-552829160357991777 \cos(\phi) + 37972267125602627 \cos(3\phi) + 849597600041 \cos(7\phi)}{27280719333558000806659600} + \frac{-35867003655610030320}{451860674799206400} \left(\frac{\alpha}{|G|^m}\right)^{12}$$

$$- \frac{63936431554809217 \cos(5\phi)}{81878057994965798098296} \left(\frac{\alpha}{|G|^m}\right)^{14}$$

$$+ \frac{4230598211819493221507 \cos(2\phi) + 2343534326486042059 \cos(6\phi)}{572573259558795826062950400} + \frac{31581536655201093550080}{34934638077852073852928000} \left(\frac{\alpha}{|G|^m}\right)^{16}$$

$$+ \frac{47219002709954377995457 \cos(4\phi) + 78531556662628367677020797}{2045860525835144573552164856400} + \frac{96666736498654970756936499200}{17255449642919799326000320817767 \cos(\phi)} + \frac{62291290190456796486924869 \cos(3\phi)}{3251299997078666218715857513600}$$

$$+ \frac{189635096110200866171238461 \cos(5\phi)}{1754706580415347153670413078647562240} + \frac{3251299997078666218715857513600}{11440641141751693454081 \cos(9\phi)}$$

$$+ \frac{17857437306477152204091576811520}{305606213005049942065414400} + \frac{119966518304046968049670002623 \cos(7\phi)}{465174475699948737845160475990425600} \left(\frac{\alpha}{|G|^m}\right)^{18} + O\left(\left(\frac{\alpha}{|G|^m}\right)^{20}\right).$$

(S55)
The expressions for $\psi_T((2a^m_1 + a^m_2)/3)$ and $\psi_T((2a^m_1 + a^m_2)/3)$ can be obtained by using the transformations $\phi \rightarrow \phi + 2\pi/3$ and $\phi \rightarrow \phi + 4\pi/3$, respectively.
EXAMPLE OF EXACT FLAT BAND IN A SYSTEM WHERE THE STRAIN FIELD HAS \( p31m \) SYMMETRY

This strain field has \( p31m \) space group symmetry. (a) Band structure corresponding to \( \bar{\alpha} = \alpha / (|G_m| a) \sim 1.42 \). (b) Plot of \( 2|\bar{A}(r)| / \alpha^2 \) as a function of \( r \). (c) Plot \( -\ln |\psi(0)| \) as a function of \( r \) at \( \alpha \sim 1.42 \). The singularity at the origin indicates the zero of \( \psi(0) \).
FOUR FLAT BANDS AT $\phi = \pi$ FOR THE STRAIN FIELD IN EQ. (3) OF THE MAIN TEXT

FIG. S3: Four exact flat bands for the strain field in Eq. (3) of the main text at $\phi = \pi$. (a) Band structure at $\tilde{\alpha} = 1.43$. (b) Band structure at $\tilde{\alpha} \approx 1.695$. (c) Band structure at $\tilde{\alpha} = 1.91$. (d) Plot of $-\ln |\psi_\Gamma(r)|$ at $\tilde{\alpha} \approx 1.695$. (e) Band structure at $\tilde{\alpha} \approx 1.695$, and $c_0 = 0.15$. The set of four flat bands decomposes into two sets of nearly flat bands.
FRAGILE TOPOLOGY OF THE LOWEST TWO BANDS

Here we show that in the chiral limit \((c_0 = 0)\), the two lowest energy bands obtained from the continuum model in Eq. (2) of the main text are always fragile topological as long as: (i) the system retains three-fold rotation symmetry \((C_3)\) with \([S, C_3] = 0\) and time reversal symmetry \((T)\), (ii) the two bands are isolated from other bands. Since the continuum model is obtained from expanding around the QBCP protected by \(C_3\) and \(T\), so long as the \(\tilde{A}\) field does not break any of these symmetries, the lowest two bands at the \(\Gamma\) point are degenerate with energy \(E = 0\) and have irrep labels \(\Gamma_2\Gamma_3, \Gamma_3, \Gamma_3\) for space groups \(p3, p3m1,\) and \(p31m\), respectively (here we use the notation of the Bilbao Crystallography Server (BCS) [4–6]). Below we show that the two lowest bands are fragile topological for all four wallpaper groups \((p3, p3m1, p31m\) and \(p6mm)\) with \(C_3\) symmetry.

Space group \(p3\)

In this case, the little co-group at the \(M\) point is trivial, and the only irrep is \(M_1\). At the \(K\) point, there are three 1d irreps \(K_1, K_2\) and \(K_3\). Due to \([S, C_3] = 0\), the two bands under consideration have the same irrep label at the \(K\) point since the two bands are chiral symmetry partners of each other. Therefore, the only choices for the irrep labels at the high symmetry points (HSPs) are:

1. \(\Gamma_2 \oplus \Gamma_3 - 2M_1 - 2K_1\),
2. \(\Gamma_2 \oplus \Gamma_3 - 2M_1 - 2K_2\),
3. \(\Gamma_2 \oplus \Gamma_3 - 2M_1 - 2K_3\).

All these cases are fragile topological as listed in the SM of [7]. This is because none of these irrep labels can be obtained from symmetric exponentially localized Wannier functions (SLWF). In fact, these irrep labels are consistent with the following formal differences between SLWFs:

1. \((1E^2E \uparrow G)_{1b} \oplus (A_1 \uparrow G)_{1a} \ominus (A_1 \uparrow G)_{1c}\),
2. \((1E^2E \uparrow G)_{1c} \oplus (A_1 \uparrow G)_{1b} \ominus (A_1 \uparrow G)_{1a}\),
3. \((1E^2E \uparrow G)_{1a} \oplus (A_1 \uparrow G)_{1c} \ominus (A_1 \uparrow G)_{1b}\),

where \(a, b\) and \(c\) are the maximal Wyckoff positions in the unit cell.

Space group \(p3m1\)

In this case, the little co-group at the \(M_1\) point is \(C_s\) with one mirror. But since the mirror symmetry \(s_x\) (Eq. [S37]) anticommutes with the chiral symmetry \(S = s_x\) (Eq. [S37]), if one of the two bands has irrep \(M_1\) (even parity), the other one has \(M_2\) (odd parity). At the \(K\) point, the story is the same as in the case of the space group \(p3\). Therefore, the only choices for the irrep labels at the HSPs are:

1. \(\Gamma_3 - M_1 \oplus M_2 - 2K_1\),
2. \(\Gamma_3 - M_1 \oplus M_2 - 2K_2\),
3. \(\Gamma_3 - M_1 \oplus M_2 - 2K_3\).

All these cases are fragile topological as listed in the SM of [7]. This is because none of these irrep labels can be obtained from SLWFs. In fact, these irrep labels are consistent with the following formal differences between SLWFs:

1. \((E \uparrow G)_{1b} \oplus (A_1 \uparrow G)_{1a} \ominus (A_1 \uparrow G)_{1c}\),
2. \((E \uparrow G)_{1c} \oplus (A_1 \uparrow G)_{1b} \ominus (A_1 \uparrow G)_{1a}\),
3. \((E \uparrow G)_{1a} \oplus (A_1 \uparrow G)_{1c} \ominus (A_1 \uparrow G)_{1b}\).
Space group $p31m$

In this case, the irreps at the $M$ point are the same as in $p3m1$ since the mirror operator anticommutes with chiral symmetry. At the $K$ point, we know that the two bands have the same $C_3$ eigenvalues since $[C_3, S] = 0$, but different $M_y$ eigenvalues $\{M_y, S\} = 0$. This means that the irrep labels of the two bands at $K$ are $K_1$ and $K_2$. Therefore, the only choice for the irrep labels at the HSPs is:

1. $\Gamma_3 - M_1 \oplus M_2 - K_1 \oplus K_2$.

This is fragile topological as listed in the SM of [7]. This is because these irrep labels cannot be obtained from SLWFs. In fact, these irrep labels are consistent with the following formal difference between SLWFs:

1. $(^1E^2E \uparrow G)_{2b} \oplus (E \uparrow G)_{1a}$.

Space group $p6mm$

In this case, at the $\Gamma$ point, we have $C_{6v}$ symmetry. Before adding the periodic strain field, the 2d irrep at the $\Gamma$ point is $\Gamma_5$ since $\rho(C_2) = 1$ (i.e., $H(\mathbf{r}) = H(C_2\mathbf{r})$ when $\tilde{A} = 0$) along with Eq. (S37). However, as we argue in the main text, as long as the strain field does not break these symmetries, the degeneracy remains at $E = 0$ at $\Gamma$ due to the commutation relation $[C_3, S] = 0$. Therefore, the irrep label is still $\Gamma_5$ at $\Gamma$ even when $\tilde{A} \neq 0$. At the $M$ point, which has little co-group $C_{2v}$, due to the anti-commutation between the mirrors and the chiral symmetry, the irrep labels can only be $M_1 \oplus M_2$ or $M_3 \oplus M_4$. At the $K$ point, the story is the same as in the space group $p31m$. Therefore, the only choice for the irrep labels at the HSPs is:

1. $\Gamma_5 - M_1 \oplus M_2 - K_1 \oplus K_2$,

2. $\Gamma_5 - M_3 \oplus M_4 - K_1 \oplus K_2$.

All these cases are fragile topological as listed in the SM of [7]. This is because none of these irrep labels can be obtained from SLWFs. In fact, these irrep labels are consistent with the following formal differences between SLWFs:

1. $(E \uparrow G)_{2b} \oplus (E_1 \uparrow G)_{1a}$,

2. $(A_1 \uparrow G)_{3c} \oplus (B_1 \uparrow G)_{1a} \ominus (A_1 \uparrow G)_{2b}$.
FRAGILE TOPOLOGY OF THE LOWEST FOUR BANDS AT $\phi = (2n + 1)\pi/3$ FOR A SYSTEM UNDER STRAIN DEFINED IN EQ. (3) OF MAIN TEXT

At these values of $\phi$, the lowest 4 bands are connected as is shown in Fig. S3. Here we show that these 4 bands are fragile topological. The irrep labels at the HSPs are $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_5 - M_1 \oplus M_2 \oplus M_3 \oplus M_4 - 2K_3$. These irreps cannot be obtained from SLWFs. These irreps correspond to the following:

$$ (A_2 \uparrow G)_{1a} \oplus (A_1 \uparrow G)_{2b} \oplus (A_1 \uparrow G)_{3c} \oplus (A_1 \uparrow G)_{1a} \oplus (B_1 \uparrow G)_{1a}. $$

(S56)

These irrep labels of the 4 bands can also be decomposed into two sets of fragile roots [7] $\Gamma_1 \oplus \Gamma_2 - M_1 \oplus M_2 - K_3$ and $\Gamma_5 - M_3 \oplus M_4 - K_3$ (or $\Gamma_1 \oplus \Gamma_2 - M_3 \oplus M_4 - K_3$ and $\Gamma_5 - M_1 \oplus M_2 - K_3$). The first set corresponds to fragile bands protected by $C_3$ symmetry, and the second set corresponds to fragile bands protected by $C_2$ symmetry.
With this, we can rewrite the density operator
\[ \rho = \frac{1}{2 \Omega_{\text{tot}}} \sum_{q \in \text{MBZ}} \sum_{G} V(q + G) \delta_{\rho_{q-G}} \delta_{\rho_{q+G}}, \]  
(S57)

where \( \Omega_{\text{tot}} \) is the area of the superlattice, \( V(q) = \frac{2 \pi e^2}{q} \tan(\xi/2) \) is the screened Coulomb potential, where \( \xi \) is the screening length, \( G \) are the moiré reciprocal lattice vectors (we dropped the superscript \( m \) here from the notation for moiré reciprocal lattice vectors for simplicity), MBZ is abbreviation of moiré Brillouin zone, and \( \delta_{\rho_{q+G}} \) is the density operator at the wave vector \( q + G \) relative to the charge neutrality point:
\[ \delta_{\rho_{q+G}} = \sum_{k \in \text{MBZ}} \sum_{G'} \sum_{\alpha, s} \left( c_{k+q,G+G',\alpha,s}^\dagger c_{k,G,\alpha,s} - \frac{1}{2} \delta_{q,0} \delta_{G,0} \right), \quad (S58) \]

where \( c_{k,G,\alpha,s}^\dagger \) is the \( \alpha \)th (of the basis in which the QBCP Hamiltonian is written) creation operator at the wave vector \( k + G \) with spin \( s \). In this notation, the non-interacting part of the Hamiltonian (\( \mathcal{H} \) in Eq. (S35)) has the form:
\[ \mathcal{H}_0 = \sum_{k \in \text{MBZ}} \sum_{G' \alpha, \beta} \sum_{s} [h_{G,G'}(k)]_{\alpha,\beta} c_{k,G,\alpha,s}^\dagger c_{k,G',\beta,s}, \]
\[ [h_{G,G'}(k)] = \begin{bmatrix} 0 & -(k_+ + G_+)^2 \delta_{G,G'} + \tilde{A}(G - G') \\ -(k_- + G_-)^2 \delta_{G,G'} + \tilde{A}(G - G') & 0 \end{bmatrix}, \quad (S59) \]

where \( k_\pm = k_x \pm i k_y \) and \( G_\pm = G_x \pm i G_y \). To project the interacting part of the Hamiltonian to the lowest two bands, we first diagonalize the harmonic part:
\[ \sum_{G',\beta} [h_{G,G'}(k)]_{\alpha,\beta} u_{k,G',\beta} = \epsilon_n(k) u_{k,G,\alpha}, \quad (S60) \]

where \( \{ u_{k,n} \} \) is the eigenvector of the matrix \([ h(k) ]\) with eigenvalue \( \epsilon_n(k) \). We denote the eigenvalues as \( \epsilon_{-n}(k) \leq \cdots \leq \epsilon_{-1}(k) \leq \epsilon_1(k) \leq \cdots \epsilon_n(k) \leq \cdots \), where \( \epsilon_{-1}(k) \) and \( \epsilon_1(k) \) are the eigenvalues with lowest magnitude. With this, we can define the energy band basis \( c_{k,n,s}^\dagger = \sum_{G,\alpha} u_{k,G,\alpha,n} c_{k,G,\alpha,s} \) (also, \( c_{k,G,\alpha,s}^\dagger = \sum_{n} u_{k,G,\alpha,n} c_{k,n,s} \)). With this, we can rewrite the density operator
\[ \delta_{\rho_{q+G}} = \sum_{k \in \text{MBZ}} \sum_{G'} \sum_{\alpha, s} \sum_{mn} \left( u_{k+q,G+G',\alpha,m}^* u_{k,G',\alpha,n} c_{k+q,m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{G,0} \right) \]
\[ = \sum_{k \in \text{MBZ}} \sum_{G'} \sum_{\alpha, s} \sum_{mn} u_{k+q,G+G',\alpha,m}^* u_{k,G',\alpha,n} \left( c_{k+q,m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right) \]
\[ = \sum_{k \in \text{MBZ}} \sum_{s} \sum_{m} M_{m,n}(k, q + G) \left( c_{k+q,m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right), \quad (S61) \]

where \( M_{m,n}(k, q + G) = \sum_{G',\alpha} u_{k+q,G+G',\alpha,m}^* u_{k,G',\alpha,n} \) is the form factor matrix. To project the interacting Hamiltonian, we just need to project the density operator to the lowest two bands:
\[ \overline{\delta_{\rho_{q+G}}} = \sum_{k \in \text{MBZ}} \sum_{s} \sum_{m,n=\pm 1} M_{m,n}(k, q + G) \left( c_{k+q,m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right), \]
\[ \mathcal{O}_{q,G} = \sqrt{V(q + G)} \delta_{q,G}, \]
\[ \mathcal{H}_I = \frac{1}{2 \Omega_{\text{tot}}} \sum_{q \in \text{MBZ}} \sum_{G} V(q + G) \overline{\delta_{\rho_{q-G}}} \delta_{\rho_{q+G}} \]
\[ = \frac{1}{2 \Omega_{\text{tot}}} \sum_{q \in \text{MBZ}} \sum_{G} \mathcal{O}_{-q-G} \mathcal{O}_{q,G}. \quad \text{(S62)} \]
Note that $\mathcal{O}(-q, -G) = \mathcal{O}^\dagger(q, G)$ since $M^*_{mn}(k, q + G) = M_{mn}(k + q, -q - G)$. Hence, $\mathcal{H}_f$ is positive semidefinite. The form factor matrix $M$ that goes into this projected interacting Hamiltonian is a $2 \times 2$ matrix since we are considering only two bands. Therefore, it can be written in terms of Pauli matrices:

$$M(k, q + G) = \alpha_0(k, q + G) \mathbb{I} + \alpha_1(k, q + G) \zeta_1 + \alpha_2(k, q + G) \zeta_2 + \alpha_3(k, q + G) \zeta_3,$$

where $\zeta_i$ are Pauli matrices, $\alpha_i(k, q + G)$ are complex numbers. The symmetries of $\mathcal{H}_0$ constrain $\alpha_i(k, q + G)$. Here we focus on chiral symmetry $(S)$. This was already introduced in SM. Sec. S-V. In the notation of this section, the representation matrix is:

$$\rho(S)|m,n\rangle = \delta_G|\sigma_2\rangle_{m,n},$$

This implies that the eigenvectors satisfy $\{u_{k,n}\}$ the following:

$$[\rho(S)]\{u_{k,n}\} = \sum_m \{u_{k,m}\} [B_S(k)]_{mn},$$

where $[B_S(k)]$ is the sewing matrix. Specializing to the exact flat bands in the chiral limit, since we already know from Eq. (5) of the main text that we can choose a gauge where the eigen wavefunctions for the exact flat bands are eigenfunctions of the chiral symmetry operator, we can gauge-fix $[B_S(k)]$ to be $[B_S(k)] = \zeta_3$. Then, the form factor matrix satisfies the following:

$$M_{m,n}(k, q + G) = \sum_{G',\alpha} u_{k,G',\alpha,m}^* u_{k,G',\alpha,n}$$

$$= \sum_{G',\alpha} \{[\rho(S)]\{u_{k,m}\} \}^\dagger |G',\alpha\rangle \{[\rho(S)]\{u_{k,n}\} \} |G',\alpha\rangle$$

$$= \sum_{G',\alpha} \sum_{m' n'} \{u_{k,m'}(\zeta_3)_{m'n'} \}^\dagger |G',\alpha\rangle \{u_{k,n'}(\zeta_3)_{n'm'} \} |G',\alpha\rangle$$

$$= \sum_{m' n'} \sum_{G'} (\zeta_3)_{m'n'}^* u_{k,G',\alpha,m'} u_{k,G',\alpha,n'} (\zeta_3)_{n'm'}$$

$$\Rightarrow [M(k, q + G)] = [\zeta_3^\dagger M(k, q + G) \zeta_3].$$

This means that $\alpha_1(k, q + G) = 0 = \alpha_2(k, q + G)$, and $[M(k, q + G)]$ is diagonal. Moreover, we know from Eq. (5) of the main text that the corresponding eigen-wavefunctions have Chern numbers $\pm 1$. This means that for this gauge choice, the basis $\{c_{k,1,s}^\dagger, c_{k,-1,s}^\dagger\}$ is the Chern basis. Furthermore, since

$$SO(q, G) S^{-1} = \sum_{k,s} \sum_{m,n=\pm 1} \sqrt{V(q + G)} M_{m,n}(k, q + G) \left( S c_{k,q+m,s}^\dagger c_{k,n,s} S^{-1} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right)$$

$$= \sum_{k,s} \sum_{m,n=\pm 1} \sqrt{V(q + G)} [\zeta_3^\dagger M(k, q + G) \zeta_3]_{mn} \left( c_{k,q+m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right)$$

$$= \sum_{k,s} \sum_{m,n=\pm 1} \sqrt{V(q + G)} M_{m,n}(k, q + G) \left( c_{k,q+m,s}^\dagger c_{k,n,s} - \frac{1}{2} \delta_{q,0} \delta_{m,n} \right)$$

$$= \mathcal{O}(q, G),$$

we have $S \mathcal{H}_f S^{-1} = \mathcal{H}_f$. However $\mathcal{H}_f$ already has $U(2)$ spin rotation symmetry (the generators of the spin rotation symmetry are $\Sigma^a = \sum_k \sum_{s,s'} \Sigma^a_{m,n}(s,s')_{k,s} c_{k,m,s}^\dagger c_{k,n,s'}$, where $a = 0, 1, 2, 3$, and $s^0$ is $2 \times 2$ identity matrix, $s^1$, $s^2$ and $s^3$ are the Pauli matrices). Adding the chiral symmetry enlarges the symmetry group to $U(2) \times U(2)$ with generators

$$\Sigma^{ab} = \sum_k \sum_{m,n=\pm 1} (\Sigma^a)_{m,n}(s^b)_{s,s'} c_{k,m,s}^\dagger c_{k,n,s'},$$

where $a \in \{0, 3\}$, $b \in \{0, 1, 2, 3\}$. (S68)
Furthermore, since \([M(k, q + G)]\) is diagonal, with
\[ M_{\pm 1}(k, q + G) = \alpha_0(k, q + G) + \alpha_3(k, q + G) = M_{\pm 1}(k, q + G) \]
and \(M_{1, -1}(k, q + G) = \alpha_0(k, q + G) - \alpha_3(k, q + G) \equiv M_{-1}(k, q + G)\) and \(M_{1, -1} = M_{-1, 1} = 0\), we can write the operator \(O_{q, G}\) in the following way
\[
O(q, G) = \sum_{k, s} \sum_{\epsilon = \pm 1} \sqrt{V(q + G)} M_\epsilon(k, q + G) \left( c^\dagger_{k+q, c, s} c_{k, c, s} - \frac{1}{2} \delta_{q, 0} \right),
\]
where \(\epsilon\) goes over the Chern band basis. Since the number of electrons \(N\) is conserved, we can introduce a Lagrangian multiplier \(A_G\) to rewrite the projected interacting part of the Hamiltonian as:
\[
\mathcal{H}_I = \frac{1}{2\Omega_{\text{tot}}} \sum_G \left[ \left( \sum_q \left( O_{q, G} - A_G N \delta_{q, 0} \right) (O_{-q, G} - A_G N \delta_{q, 0}) \right) + 2A_G N \Omega_{0, G} - A_{-G} A_G N^2 \right],
\]
(see [9] for details). At integer filling \(\nu\) w.r.t. the charge neutrality point \((-2 \leq \nu \leq 2\)), we define spin polarized Fock state
\[
|\Psi_{\nu^+, \nu^-}^\nu\rangle = \prod_k \prod_{j_1 = 1}^{\nu^+} c^\dagger_{k+1, j_1} \prod_{j_2 = 1}^{\nu^-} c^\dagger_{k-1, j_2} |0\rangle,
\]
where \(\nu^+, \nu^- \in \{0, 1, 2\}\) with \(\nu^+ + \nu^- = \nu + 2\) and \(s_{j_1}\) and \(s_{j_2}\) can be chosen arbitrarily. Defining \(A_G = \sqrt{V(G)} \sum_k \left[ \frac{\nu^+-1}{N} \right] M_{1}(k, G) + \frac{\nu^-1}{N} M_{-1}(k, G)\), it can be shown that
\[
(O_{q, G} - A_G N \delta_{q, 0}) |\Psi_{\nu^+, \nu^-}^\nu\rangle = 0.
\]
At the flat metric limit \(M_{m,n}(k, G) = \xi(G) \delta_{m,n} [9]\) or at the charge neutrality point \(\nu = 0\), the last two terms are constants; hence \(|\Psi_{\nu^+, \nu^-}^\nu\rangle\) is a ground state. Away from the flat metric limit, the Fock state \(|\Psi_{\nu^+, \nu^-}^\nu\rangle\) remains a ground state unless the gap between it and the excited states closes. In general, the flat metric condition is not satisfied except at \(G = 0\), but since the wavefunction decreases exponentially as \(G\) increases [10], the flat metric condition is not largely violated.

Clearly, \(|\Psi_{\nu^+, \nu^-}^\nu\rangle\) has the Chern number \((\nu^+ - \nu^-)\). At the charge neutrality point \(\nu = 0\), there are three choices: \((\nu^+ = 2, \nu^- = 0)\), \((\nu^+ = 0, \nu^- = 2)\) and \((\nu^+ = 1, \nu^- = 1)\). The first two cases have Chern numbers \(C = \pm 2\), whereas the last case has Chern number \(C = 0\). However, the last case is entropically more favorable since there are more ways of filling this than the other two. At other integer fillings, \(\nu = \pm 1\), \(|\Psi_{\nu^+, \nu^-}^\nu\rangle\) always carries a Chern number since in case of \(\nu = -1\), either \(\nu^+\) or \(\nu^-\) is zero, whereas in case of \(\nu = +1\), one of \(\nu^+\) and \(\nu^-\) is 1, the other is 2.
EFFECT OF BREAKING CHIRAL SYMMETRY

In Fig. S4 we show two example band structures for \( c_0 \neq 0 \) in Eq. (2) of the main text along with the specific choice \( A_I = -\frac{\alpha^2 \Omega_{Jc}}{4} \sum_{i=1}^{3} \cos(G_i \cdot \vec{r}) \).

(a) Band Structure when \( c_0 = 0.9 \) (defined in Eq. (2)) and \( \tilde{\alpha} = 1.29 \). The red band is the \( n \)th band among the total \( 2n \) bands. (b) Band structure when \( c_0 = 0.1 \) and \( \tilde{\alpha} = 0.79 \), which is the first critical strength. The red bands are the \( n \)th and \((n + 1)\)th bands.

FIG. S4: Effect of breaking chiral symmetry on the band structure. (a) Band Structure when \( c_0 = 0.9 \) (defined in Eq. (2)) and \( \tilde{\alpha} = 1.29 \). The red band is the \( n \)th band among the total \( 2n \) bands. (b) Band structure when \( c_0 = 0.1 \) and \( \tilde{\alpha} = 0.79 \), which is the first critical strength. The red bands are the \( n \)th and \((n + 1)\)th bands.
EXPERIMENTAL FEASIBILITY

In this section, we give an estimate of the experimentally achievable value of the nondimensional parameter $\alpha$ (defined in Eq. (3) in the main text) to argue that the critical strain strengths for exact flat bands can, in principle, be achieved in real experiments. The change in the hopping strength under applied strain in the long-wavelength limit can be quantified to be \[11\]

\[\delta t_i = t_i - t_0 = - \frac{\kappa \beta t_0}{a^2} \mathbf{a}_i \cdot (\mathbf{a}_i \cdot \nabla) \mathbf{u}, \quad (S73)\]

where $\kappa$ is the so-called reduction factor \[11\], $\beta = - \frac{\partial t_i}{\partial u_{xx}}$ and $\mathbf{a}_i$ is the vector corresponding to the bond. Plugging $\mathbf{a}_1 = a(1, 0)$, $\mathbf{a}_2 = a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\mathbf{a}_3 = \mathbf{a}_2 - \mathbf{a}_1$ for nearest neighbor kagome lattice, we get the following relations between strain tensor and modulation of hopping strength along different bond directions:

\[\delta t_1 = - \kappa \beta t_0 u_{xx},\]
\[\delta t_2 = - \kappa \beta t_0 \left(\frac{u_{xx}}{4} + \frac{3u_{yy}}{4} + \frac{\sqrt{3}u_{xy}}{2}\right),\]
\[\delta t_3 = - \kappa \beta t_0 \left(\frac{u_{xx}}{4} + \frac{3u_{yy}}{4} - \frac{\sqrt{3}u_{xy}}{2}\right). \quad (S74)\]

To apply a strain field, we can either employ in-plane displacement field or out-of-plane displacement field, the latter is easier to achieve in experiments. Following \[12\], where a graphene monolayer was placed on a substrate to undergo a buckling transition, we consider periodic out-of-plane displacement field $h(\mathbf{r})$ to estimate experimentally accessible $\alpha$. For only out-of-plane displacements, we have

\[u_{ij} = \frac{1}{2} \partial_i h \partial_j h \propto h_0^2 \beta G^2 = h_0^2 \left(\frac{4\pi}{\sqrt{3} \alpha_m}\right)^2, \quad (S75)\]

where $h_0$ is the amplitude of out-of-plane displacement, $G$ is the length of the reciprocal lattice vectors of the superlattice, $\alpha_m$ is the length of the lattice vectors of the superlattice. Hence, the magnitude of $\delta t_i$ is of order $\sim \kappa \beta h_0^2 \left(\frac{\pi}{\sqrt{3} \alpha_m}\right)^2$. We take $\kappa \approx \frac{1}{3}$ and $\beta \approx 2$ from \[11\], then $\delta t_i \approx - \frac{2}{3} t_0 h_0^2 \left(\frac{4\pi}{\sqrt{3} \alpha_m}\right)^2$. We use the values $h_0 = 0.17$ nm and $\alpha_m = (14.4 \pm 0.5)$ nm from the experiments in \[12\] to get the estimate $\delta t_i \approx -0.0048908 t_0$. On the other hand, from Eq. (S17)-(S19), we know that the magnitude of $\delta t_i$ is $\frac{3}{4} t_0 \alpha^2$. Therefore, $\delta t_i \approx -0.0048908 t_0$ corresponds to $\alpha \approx 0.161506$ and dimensionless $\tilde{\alpha} \approx 2.257$, which is larger than the first 2 critical $\tilde{\alpha}$ values shown in Fig. 1(c-e) of the main text.

Now we estimate the temperature scale below which the upper bands are not important. Due to band folding in the Moiré system, the band gap is of the order $t_0 \left(\frac{4\pi}{\sqrt{3} \alpha_m}\right)^2$, where $t_0$ is the nearest-neighbor hopping strength, $a$ is the bond length and $\alpha_m$ is the length of the lattice vectors of the superlattice. Here, we take the DFT calculation of $t_0 \approx 0.3 eV$ for GaCu$_3$(OH)$_6$Cl$_2$(a material with nearly-chiral quadratic band crossing point) in \[13\]. Using the $\alpha_m = (14.4 \pm 0.5)$ nm and $a = \frac{0.246}{\sqrt{2}}$ nm from the experiments in \[12\], we get the the energy gap $\sim 1.54$ meV, which corresponds to $T \approx 4.46$ K. In principle, as long as we keep the temperature below 4.46 K, thermal fluctuation will not destroy the flat band.

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