Lorenz System Parameter Determination and Application to Break the Security of Two-channel Chaotic Cryptosystems

A. B. Orue, G. Alvarez, M. Romera, G. Pastor, F. Montoya and Shujun Li

Abstract—This paper describes how to determine the parameter values of the chaotic Lorenz system used in a two-channel crytosystem. The geometrical properties of the Lorenz system are used firstly to reduce the parameter search space, then the parameters are exactly determined, directly from the ciphertext, through the minimization of the average jamming noise power created by the encryption process.

Index Terms—Chaos, cryptography, cryptanalysis, nonlinear systems, security of data, Lorenz system.

I. INTRODUCTION

In recent years, a growing number of cryptosystems based on chaos synchronization have been proposed [1], many of them fundamentally flawed by a lack of robustness and security.

The first schemes of synchronization-related chaotic cryptography were based on the masking of a plaintext message by a system variable of a chaotic generator [2]–[4]. The receiver had to synchronize with the sender to generate the chaotic signal and then recover the message. This simple design is easily broken by elemental filtering of the ciphertext signal [5]–[7].

Recently, there appeared some chaotic cryptosystems with an enhanced plaintext concealment mechanism: the ciphertext consisted of a complicated non-linear combination of the plaintext and a variable of a chaotic transmitter generator, from which it was an unattainable goal to retrieve a clean plaintext. As it was impossible to synchronize a chaotic receiver with such ciphertext, a second channel was used for synchronization. The synchronization signal was a different sender chaotic variable, that was transmitted without modification. The same system parameters values were used at sender and receiver [8]–[10].

One of these cryptosystems, proposed by Jiang [8], made use of the Lorenz chaotic system [13], that is defined by the following equations:

\[ \begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= \rho x - y - x z, \\
\dot{z} &= x y - \beta z,
\end{align*} \]

where \( \sigma, \rho \) and \( \beta \) are fixed parameters.

The ciphertext \( s \) was defined as

\[ s = f_1(x, y, z) + f_2(x, y, z) m, \]

where \( m \) is the plaintext.

The receiver was designed as a reduced order nonlinear observer with a mechanism to achieve efficient partial synchronization, under the drive of \( x(t) \). It can generate two signals \( y_r(t) \) and \( z_r(t) \) that converge to the driver system variables \( y(t) \) and \( z(t) \), respectively, as \( t \to \infty \).

The recovered plaintext \( m^*(t) \) was retrieved with the function:

\[ m^* = \frac{s}{f_2(x, y_r, z_r)} - \frac{f_1(x, y_r, z_r)}{f_2(x, y_r, z_r)}. \]

It was given an example in [8, §III] with the following functions: \( f_1(x, y, z) = y^2 \) and \( f_2(x, y, z) = 1 + y^2 \); the following parameter values: \( \sigma = 10, \rho = 28 \) and \( \beta = 8/3 \); and with the following initial conditions: \((x(0), y(0), z(0)) = (0, 0.01, 0.01) \) and \((y_r(0), z_r(0)) = (0.05, 0.05) \). The plaintext was a small amplitude sinusoidal signal of 30 Hz, \( m(t) = 0.05 \sin(2\pi 30t) \). The author claimed that this cryptosystem guarantees higher security and privacy, showing that an error of 0.05 in the retrieval of \( y_r \) due to a poor parameter estimation, giving rise to a serious distortion in the retrieved plaintext.

In the vast majority of chaotic cryptosystems, the security relies on the secrecy of the system parameters, which play the role of secret key. Hence, the determination of the system parameters is equivalent to breaking the system. Recently, Solak [11] analyzed the cryptosystem [8] and showed how an eavesdropper could identify the value of the parameter \( \rho \), provided that it has the previous knowledge of the two other transmitter system parameters \( \beta \) and \( \sigma \). Solak’s approach was based on a novel expression of the Lorenz system. Formerly Stojanovski, Kocarev and Parlitz [12] described a generic method, to reveal simultaneously all the three parameters of a Lorenz system when one of the the variables \( x(t) \) or \( y(t) \) were known, that could be applied to break this cryptosystem.

The present work describes an efficient determination method of the only two unknown parameters \( \rho \) and \( \beta \) needed to build up an intruder Lorenz system receiver, from the ciphertext alone, without partial knowledge of any transmitter parameters. Firstly, some geometrical properties of the Lorenz attractor are shown. Then, advantage is taken of them to minimize, as much as possible, the parameters search space.
Finally, the unknown receiver parameters are determined with high accuracy.

II. THE LORENZ ATTRACTOR’S GEOMETRICAL PROPERTIES

According to [13], the Lorenz system has three equilibrium points. The origin is an equilibrium point for all parameter values; for $0 < \rho < 1$ the origin is a globally attracting asymptotically stable sink; for $1 \leq \rho \leq \rho_H$ the origin becomes a non-stable saddle point, giving rise to two other stable twin equilibrium points $C^+$ and $C^-$, of coordinates $x_{C^\pm} = \pm \sqrt{\beta(\rho - 1)}$, $y_{C^\pm} = \pm \sqrt{\beta(\rho - 1)}$ and $z_{C^\pm} = \rho - 1$, being $\rho_H$ a critical value, corresponding to a Hopf bifurcation [14], whose value is:

$$\rho_H = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}. \quad (4)$$

When $\rho$ exceeds the critical value $\rho_H$ the equilibrium points $C^+$ and $C^-$ become non-stable saddle foci, by a Hopf bifurcation, and the strange Lorenz attractor appears. The flow, linearized around $C^+$ and $C^-$, has one negative real eigenvalue and a complex conjugate pair of eigenvalues with positive real part. As a consequence, the equilibrium points are linearly attracting and spirally repelling.

Figure I(a) shows the double scroll Lorenz attractor formed by the projection on the $x - y$ plane, in the phase space, of a trajectory portion extending along 12 s; the parameters are $\sigma = 16$, $\rho = 100$ and $\beta = 8/3$.

It is a well known fact that the Lorenz attractor trajectory draws two 3D loops, in the vicinity of the equilibrium points $C^+$ and $C^-$, with a spiral like shape of steadily growing amplitude, jumping from one of them to the other, at irregular intervals, in a random like manner though actually deterministic [13]. The trajectory may pass arbitrarily near to the equilibrium points, but never reach them while in chaotic regime.

The geometrical properties of Lorenz system allows for a previous reduction of the search space of the $\rho$ and $\beta$ parameters, taking advantage of the relation of them with the coordinates $x_{C^\pm} = \pm \sqrt{\beta(\rho - 1)}$ of the equilibrium points.

Let us call attractor *eyes* to the two neighborhood regions around the equilibrium points that are not filled with the spiral trajectory. The eye centres are the fixed points $C^+$ and $C^-$. The pending problem is to determine the eye centres when the inner turns are missing, as happens in normal chaotic regime. With the drive signal $x(t)$, we solved it by experimentally estimating the middle point value of the trajectory maxima and minima in the phase space projection on the $x - y$ plane. The best result was obtained by taking into account only the regular spiral cycle closest to the center, shown in Fig. II(b) as a thick continuous line. The $x$-coordinate of the eye center was calculated with the following empirical formula:

$$x_{C^\pm} = \frac{0.9 x_{m1} + 0.1 x_{m2} + x_{M1}}{2}, \quad (5)$$

where $x_{M1}$ is the minimum of all the maxima of $|x(t)|$ spiral trajectory, $x_{m1}$ and $x_{m2}$ are the two minima immediately preceding and following $x_{M1}$, respectively.

As the spiral has a growing radius, it was necessary to take a weighted mean between the two minima $x_{m1}$ and $x_{m2}$, the optimal values of the two weights were determined experimentally. Instead of making two computations, one around $C^+$ and another around $C^-$, a unique computation was done on the absolute value waveform $|x(t)|$. It should be noted that all the first maxima after a change of sign of $x(t)$ and $y(t)$ must be discarded because they belong to the incoming trajectory portion attracted by the equilibrium points $C^\pm$ and do not belong to the spiral trajectory, one of them is shown in Fig. II(b) as a thick dashed line.

The result is illustrated in Fig. 2. It can be seen that the relative value of the error, taking the eye center coordinate...
and

\[ x_{C^+} \] instead of the true value of \( x_{C^+} \), is less than \( 2 \times 10^{-3} \). The system parameters were varied in the margins: \( \sigma \in (9.7, 37.4), \rho \in (25.6, 94.8) \) and \( \beta \in (2.6, 8.4) \). The system initial conditions were the same as the example of [8, §III]; the period of measurement was 20 s and the sampling frequency was 1200 Hz.

In this way, the search space of the unknown parameters \( \beta \) and \( \rho \) is reduced to a narrow margin defined as \( \beta^*(\rho^* - 1) \in \{0.996 x_{C^+}^2, 1.004 x_{C^+}^2\} \).

Applying this method to the proposed example of [8, §III], whose equilibrium point is \( x_{C^+} = \sqrt{\frac{\beta}{\rho}} \), the absolute determination error of \( x_{C^+}^2 \) was \( 7.5 \times 10^{-4} \), equivalent to a relative error of 0.0089%.

III. BREAKING OF THE PROPOSED ENCRYPTION SYSTEM

We designed an intruder receiver based on a homogeneous driving synchronization mechanism [15] between the transmitter drive Lorenz system and a receiver response subsystem, that was a partial duplicate of the drive system reduced to only two variables \( y_r(t) \) and \( z_r(t) \), driven by the drive variable \( x(t) \). The response system was defined by the following equations:

\[
\begin{align*}
\dot{y}_r &= \rho^* x - y_r - x z_r, \\
\dot{z}_r &= x y_r - \beta^* z_r.
\end{align*}
\]

(6)

Note that for breaking the system it is only necessary to get the knowledge of the parameters \( \rho \) and \( \beta \), i.e. the parameter \( \sigma \) may be ignored and need not be determined, unlike in the Solak method [11] which requires its previous knowledge, or in the Stojanovski et al method [12] which requires the simultaneous determination of all the three unknown parameters.

As it was shown in [15, §III], this drive-response configuration has two conditional Lyapunov exponents, both fairly negative, thus leading to a very stable system. The consequence is that, if the parameters of drive and response systems are moderately different, the drive and response variables will be alike, though not totally identical. This property may be exploited to search the right parameter values looking at the retrieved plaintext and applying an optimization procedure to find the parameters that provide the best retrieved plaintext quality.

When the synchronizing signal is fed to the response system described by Eq. (6) and the parameters of both systems agree, i.e. \( \rho^* = \rho \) and \( \beta^* = \beta \), the variables \( y \) and \( y_r \) of the drive and response systems are equal, hence the recovered text \( m^*(t) \) follows the plaintext \( m(t) \) exactly; being negligible the effect of different initial conditions after a very short transient. If the parameters of both systems do not agree, the recovered text will consist of a noisy distorted version of the original plaintext, growing the noise and distortion as the mismatch between drive and response systems parameters grows.

A. Parameter determination

In the particular case of the example in [8, §III], the encryption and decryption functions were:

\[
\begin{align*}
s &= y^2 + (1 + y^2) m, \\
m^* &= \frac{s}{1 + y_r^2} - \frac{y_r^2}{1 + y_r^2}.
\end{align*}
\]

(7)

(8)

Equation (8) of the recovered text can be rewritten as:

\[
m^* = m \frac{1 + y^2}{1 + y_r^2} + \frac{y^2 - y_r^2}{1 + y_r^2}.
\]

(9)

This equation has two terms, the first one is a function of the plaintext message \( m(t) \) and the variables \( y \) and \( y_r \). When \( y = y_r \), the term is reduced to the undistorted plaintext, but if \( y \neq y_r \), a distortion appears. The second term is a function of \( y \) and \( y_r \) and can be considered as a jamming noise. Figure 3 depicts the spectrum of the recovered text corresponding to the example, but with a wrong guessing of the response system parameters: \( \rho^* = 28.01 \) and \( \beta^* = 2.667 \). It can be seen that the spectrum has two main frequency bands: one around the plaintext \( m(t) \) frequency of 30 Hz, that corresponds to the distorted plaintext, and another near 0 Hz that corresponds to the jamming noise. Assuming that the plaintext will always consist of an a.c. band limited signal without d.c. component, as in the numerical example given in [8], it is clear from Fig. 3 that the second term of Eq. (9) may be isolated from the first by means of a suitable filter.

The most important band of the jamming noise \( \varepsilon \) was isolated by means of a finite impulse response low pass
filter with 2048 terms and a cutoff frequency of 0.2 Hz, that suppressed the contribution of the plaintext $m(t)$ and most of the frequency terms generated by the modulation with the chaotic signal $y^2(t)$. Figure 3 illustrates the mean value of the squared noise $\varepsilon^2$, i.e. the average noise power, as a function of $\rho^*$, with the eye center $x_{C^2}$ as parameter, with the same transmitter system parameters of the numerical example presented in [8] and the intruder receiver described by Eq. (6).

The mean of $\varepsilon^2$ was computed along the first 20 s, after a delay of 2 seconds, to let the initial transient finish. It is clearly seen that the noise grows monotonically with the mismatch between the transmitter and receiver parameters $|\rho^* - \rho|$ and that the minimum error corresponds to the receiver system parameter $\rho^*$ exactly matching the transmitter system parameter $\rho$, when $x_{C^2} = x_{C^2} = \sqrt{\beta(\rho - 1)} = \sqrt{2}$. The search of the correct parameter values $\beta^*$ and $\rho^*$ is carried out with the following procedure:

1. Determine the approximate value of the eye center $x_{C^2}$ as described in Section III from the $x(t)$ waveform.
2. Keeping the last value of $x_{C^2}$, vary the value of $\rho^*$ until a minimum of the average noise power is reached.
3. Keeping the last value of $\rho^*$, vary the value of eye center $x_{C^2}$ until a new minimum of the average noise power is reached.
4. Repeat the previous two steps until a stable result of average noise power will be reached and retain the last values of $\rho^*$ and $x_{C^2}$ as the ultimate ones.
5. Calculate the value of $\beta^*$ as $\beta^* = (x_{C^2})^2/(\rho^* - 1)$.

Table I shows the evolution of the relative eye center error, the relative $\rho^*$ parameter error and the average jamming noise power. It can be seen that the procedure converges rapidly to the exact values: $\rho^* = \rho = 28$ and $x_{C^2} = x_{C^2} = \sqrt{2}$. The value of the unknown parameter $\beta^*$ was deduced from Eq. (5) with the estimated values of $\rho^*$ and $x_{C^2}$ as $\beta^* = (x_{C^2})^2/((\rho^* - 1)/3)$.

Note that this method works as well for the general case described by Eqs. (2) and (3) that have similar structure to Eqs. (7) and (8) which describe the special case of the example in [8, §III], just selected here for experimental demonstration.

### B. Plaintext retrieving

As the system parameters are equivalent to the system key, once the exact values of $\beta^*$ and $\rho^*$ are known, the ciphertext can be efficiently decoded by the intruder receiver defined by Eq. (6). Figure 5 shows the three first seconds of the retrieved plaintext with the response system receiver described by Eq. (6), corresponding to the ciphertext example of [8, §III]. It can be seen that the plaintext is perfectly recovered after a short transient period of less than one second.

### IV. Simulations

All results were based on simulations with MATLAB 7.1, the Lorenz integration algorithm was a four-fifth order Runge-Kutta with an absolute error tolerance of $10^{-9}$, and a relative error tolerance of $10^{-6}$.

### V. Conclusions

A simple method was proposed to reduce the parameter search space of the Lorenz system, based on the determination of the system equilibrium points from the waveform analysis of one of its variables $x(t)$. Then the method was applied to the cryptanalysis of the cryptosystem [8], showing that it is rather weak since it can be broken without knowing its parameter values. The total lack of security discourages the use of this algorithm for secure applications.

This work was supported by Ministerio de Ciencia y Tecnologia of Spain, research grant SEG2004-02418, and by The Hong Kong Polytechnic University’s Postdoctoral Fellowships Scheme under grant no. G-YX63.

### References

[1] T. Yang, “A survey of chaotic secure communication systems,” Int. J. Comput. Cognit., vol. 2, pp. 81–130, June 2004.
[2] M. Boutayeb, M. Darouach, and H. Rafaralahy, “Generalized state-space observers for chaotic synchronization and secure communication,” IEEE Trans. Circuits Syst. I-Fundam. Theor. Appl., vol. 49, no. 3, pp. 345–349, March 2002.
[3] S. Bowong, “Stability analysis for the synchronization of chaotic systems with different order: application to secure communication,” Phys. Lett. A, vol. 326, no. 1-2, pp. 102–113, May 2004.
[4] Q. Memon, “Synchronized chaos for network security,” Comput. Commun., vol. 26, pp. 498–505, 2003.
[5] G. Álvarez, L. Hernández, J. Muñoz, F. Montoya, and S. Li, “Security analysis of a communication system based on the synchronization of different order chaotic systems,” Phys. Lett. A, vol. 345, no. 4, pp. 245–250, October 2005.

[6] G. Álvarez and S. Li, “Breaking network security based on synchronized chaos,” Comput. Communicat., vol. 27, pp. 1679–1681, 2004.

[7] G. Álvarez, F. Montoya, M. Romera, and G. Pastor, “Breaking two secure communication systems based on chaotic masking,” IEEE T. Circuits-II, vol. 51, no. 10, pp. 505–506, 2004.

[8] Z. P. Jiang, “A note on chaotic secure communication systems,” IEEE Trans. Circuits Syst. I-Fundam. Theor. Appl., vol. 49, no. 1, pp. 92–96, 2002.

[9] B.-H. Wang and S. Bu, “Controlling the ultimate state of projective synchronization in chaos: application to chaotic encryption,” Int. J. Mod. Phys. B, vol. 18, no. 17–19, pp. 2415–2421, 2004.

[10] D. X. Zhigang Li, “A secure communication scheme using projective chaos synchronization,” Chaos, Solitons and Fractals, vol. 22, pp. 477–481, 2004.

[11] E. Solak, “Partial identification of lorenz system and its application to key space reduction of chaotic cryptosystems,” IEEE Trans. Circuits Syst. II-Express briefs, vol. 51, no. 10, pp. 557–560, October 2004.

[12] T. Stojanovski, L. Kocarev, and U. Parlitz, “A simple method to reveal the parameters of the Lorenz system,” J. of Bifurcation adn Chaos, vol. 6, no. 12B, pp. 2645–2652, 1996.

[13] E. N. Lorenz, “Deterministic non periodic flow,” J. Atmos. Sci., vol. 20, no. 2, pp. 130–141, 1963.

[14] C. Sparrow, The Lorenz equations, ser. Applied mathematical sciences. Springer-Verlag, 1982.

[15] L. M. Pecora and T. L. Carroll, “Driving systems with chaotic signals,” Phys. Rev. A, vol. 44, pp. 2374–2383, 1991.