Separating families of convex sets

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Abstract

Two elements, \(x\) and \(y\), are separated by a set \(S\) if it contains exactly one of \(x\) and \(y\). We prove that any set of \(n\) points in general position in the plane can be separated by \(O(n \log \log n / \log n)\) convex sets, and for some point sets \(\Omega(n/ \log n)\) convex sets are necessary.

1 Introduction

We say that a set \(S\) separates elements \(x\) and \(y\), if exactly one of \(x\) and \(y\) is in \(S\). Given an underlying set \(X\), a family \(F\) of its subsets is called separating, if for any \(x, y \in X\), \(x \neq y\), there exists an \(F \in F\) which separates them.

Separating families are important tools in search theory. Suppose there is an unknown defective element in \(X\), and we can test subsets of \(X\) if they contain the defective element or not. We want to choose a family of sets in advance (non-adaptively), such that testing all of its members determines the defective element. It is not hard to see that the family satisfies this property if and only if it is separating. The usual goal is to test as few sets as possible, i.e. find a separating family of minimum cardinality.

It is clear that a separating family of \(X\), \(|X| = n\), contains at least \(\lceil \log n \rceil\) sets, since \(k\) subsets of \(X\) divide it into at most \(2^k\) parts. (All logarithms in this paper are of base 2.)

On the other hand, we can represent the elements of \(X\) by \(0−1\) sequences of length \(k = \lceil \log n \rceil\). Then, for \(1 \leq k \leq m\), let \(A_i\) denote the set of elements with 1 at coordinate \(i\). Then the sets \(A_i\) form a separating family.

Another well-known observation is the following, originally due to Bondy [B72], see also [W09].

Observation 1.1. Let \(F\) be a minimal separating family (in the sense that no proper subfamily of it is separating). Then \(|F| \leq n - 1\).

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There are several different versions of this simple concept, for a survey see [DH94]. An obvious idea is to consider adaptive algorithms, where we test only one set at a time, and choose the next set knowing the result of the previous test. Or one can have more defective elements, different types of tests, or errors in the results of the tests. One of the most studied generalizations is that not every subset can be asked. Instead, we are given a family \( \mathcal{A} \) of subsets of \( X \), and we can only test its members. A well-known example for this is to find the defective coins, or to sort a finite set, using only pairwise comparisons.

In this note, our underlying set \( X \) is a set of points in the plane, and we have certain geometric restrictions on the subsets we can ask.

For a family \( \mathcal{A} \) of planar sets and a point set \( X \) let \( \mathcal{A}_X = \{ A \cap X | A \in \mathcal{A} \} \). For simplicity, we will call \( \mathcal{A} \) separating (with respect to \( X \)) if \( \mathcal{A}_X \) is separating.

**Definition.** Let \( X \) be a set of \( n \) points in the plane, and let \( \mathcal{A} \) be a family of planar sets. Let \( s(X, \mathcal{A}) \) denote the size of the smallest subfamily \( \mathcal{A}' \subseteq \mathcal{A} \) with the property that the family \( \{ A \cap X | A \in \mathcal{A}' \} \) is separating. If there is no such subfamily, then let \( s(X, \mathcal{A}) = \infty \).

Let \( s(n, \mathcal{A}) \) be the maximum of \( s(X, \mathcal{A}) \) over all \( n \)-element point sets \( X \), and let \( s'(n, \mathcal{A}) \) be the maximum of \( s(X, \mathcal{A}) \) over all \( n \)-element point sets \( X \) in general position (that is, not three of its points are on a line).

By Observation 1.1, for any family \( \mathcal{A} \), \( s(n, \mathcal{A}) \leq n - 1 \) or \( s(n, \mathcal{A}) = \infty \), and similarly, \( s'(n, \mathcal{A}) \leq n - 1 \) or \( s'(n, \mathcal{A}) = \infty \).

For most of the natural families of planar sets \( \mathcal{A} \), it is not hard (or sometimes trivial) to give a linear lower bound for both \( s(n, \mathcal{A}) \) and \( s'(n, \mathcal{A}) \), and in many cases we can determine their exact values. We only give two examples here.

**Theorem 1.** Let \( H \) and \( D \) denote the family of the halfplanes and discs, respectively. Then we have

(i) \( s(n, H) = n - 1 \), \( s'(n, H) = \lceil n/2 \rceil \),

(ii) \( s(n, D) = s'(n, D) = \lceil n/2 \rceil \).

The case when \( \mathcal{A} \) is the family of convex sets seems to be the most interesting.

**Theorem 2.** Let \( \mathcal{A} \) denote the family of planar convex sets. Then we have

(i) \( s(n, \mathcal{A}) = \lceil n/2 \rceil \),

and (ii) \( n/2 \log n \leq s'(n, \mathcal{A}) \leq 20 \log \log n/ \log n \).

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

2 Some simple families of planar sets; Proof of Theorem 1

It is easy to see that both families are separating, for any point set, hence they contain separating subfamilies of cardinality at most \( n - 1 \). This implies by Observation 1.1 an upper bound of \( n - 1 \) in
each case. In fact, we can give a separating subfamily of size \( n - 1 \) directly for the hyperplanes. Assume without loss of generality that all points have different \( x \)-coordinates, (otherwise we slightly rotate the coordinate system) and let \( p_1, p_2, \ldots, p_n \) be the points, ordered according to their \( x \)-coordinates. Then let \( H_1, H_2, \ldots, H_{n-1} \) be halfplanes such that \( H_i \) contains exactly \( p_1, p_2, \ldots, p_i \) of the points. It is clearly a separating family.

Now let \( P \) be a set of \( n \) collinear points, \( p_1, p_2, \ldots, p_n \), and let \( \mathcal{H}(P) \) be a separating family of halfplanes. For any \( i, 1 \leq i < n \), there is a halfplane \( H_i \in \mathcal{H}(P) \) which separates \( p_i \) and \( p_{i+1} \), and these halfplanes are all different. Therefore, \( \mathcal{H}(P) \) contains at least \( n - 1 \) halfplanes, consequently, \( s(n, \mathcal{H}) = n - 1 \).

Now we show the lower bound for \( s'(n, \mathcal{H}) \). Let \( P \) be a set of \( n \) points, \( p_1, p_2, \ldots, p_n \), on a circle, ordered clockwise, and let \( \mathcal{H}(P) \) be a separating family of halfplanes. For any \( i, 1 \leq i < n \), there is a halfplane \( H_i \in \mathcal{H}(P) \) which separates \( p_i \) and \( p_{i+1} \), and there is a halfplane \( H_n \in \mathcal{H}(P) \) which separates \( p_n \) and \( p_1 \). These are \( n \) halfplanes, and we counted a halfplane at most twice. Therefore, \( \mathcal{H}(P) \) contains at least \( \lceil n/2 \rceil \) halfplanes.

Finally, we show that the upper bound holds for \( s'(n, \mathcal{H}) \). Suppose that \( P \) is a set of \( n \) points in general position. We obtain a separating family \( S \) of \( \lfloor n/2 \rfloor \) halfplanes by the following procedure. We can assume without loss of generality that \( n \) is even.

**Halfplane-Separate(\( P \))**

**Step 0.** Let \( \ell \) be a line which has exactly \( n/2 \) of the points on both sides. Let \( Q_0 \subset P \) and \( R_0 \subset P \) denote the points on the two sides of \( \ell \). Let \( H_0 \) be a halfplane whose bounding line is \( \ell \). Let \( S_0 = \{H_0\}, i = 1 \).

**Step i.** Take the convex hull of \( Q_i \cup R_i \). It has two edges that cross \( \ell \), let \( e = q_ir_i \) be one of them, \( q_i \in Q_i, r_i \in R_i \). Take a halfplane \( H_i \) that separates \( q_i \) and \( r_i \) from the rest of \( Q_i \) and \( R_i \).

If \( i < n/2 - 1 \), then let \( S_{i+1} = S_i \cup \{H_i\}, \) let \( Q_{i+1} = Q_i \setminus \{q_i\}, R_{i+1} = R_i \setminus \{r_i\}. \) Increase \( i \) by one, and go to **Step i.**

Otherwise (for \( i = n/2 - 1 \)), let \( S = S_i \cup \{H_i\}, \) and **STOP.**

Clearly, \( S \) is a set of \( n/2 \) halfplanes. We claim that it separates \( P \). Let \( p, p' \in P \). If \( p = q_i \) and \( p' = r_j \) for some \( i, j \), then \( H_0 \) separates them. If \( p = q_i \) and \( p' = q_j, i < j \) (or if \( p = r_i \) and \( p' = r_j, i < j \)), then \( H_i \) separates them.

In the case of discs, the proofs are very similar. To show that \( s(n, \mathcal{D}), s'(n, \mathcal{D}) \geq \lfloor n/2 \rfloor \), let \( P \) be a set of \( n \) points on a circle. We can argue exactly as in the case of halfplanes, any disc can separate at most two consecutive pairs, therefore, we need at least \( \lfloor n/2 \rfloor \) discs.

To prove that \( \lceil n/2 \rceil \) discs are always enough, we can use a procedure very similar to **Halfplane-Separate(\( P \)).** Observe, that in the case of discs, it works for any point set, we do not have to assume that the points are in general position. Therefore, we have \( s(n, \mathcal{D}) = s'(n, \mathcal{D}) = \lceil n/2 \rceil \).
3 Convex sets; Proof of Theorem 2

Proof of part (i). The proof of the lower bound is again very similar to the previous proofs. Let \( P \) be a set of \( n \) points, \( p_1, p_2, \ldots, p_n \), on a line in this order, and let \( \mathcal{A}(P) \) be a separating family of convex sets. For any \( i, 1 \leq i < n \), there is a set \( A_i \in \mathcal{A}(P) \) which separates \( p_i \) and \( p_{i+1} \), and there is a set \( A_n \in \mathcal{A}(P) \) which separates \( p_n \) and \( p_1 \). These are \( n \) sets, and we counted a set at most twice. Therefore, \( \mathcal{A}(P) \) contains at least \( \lceil n/2 \rceil \) sets, so \( s(n, \mathcal{D}) \geq \lceil n/2 \rceil \). On the other hand, since discs are convex sets, \( s(n, \mathcal{A}) \leq s(n, \mathcal{D}) = \lceil n/2 \rceil \). Therefore, \( s(n, \mathcal{A}) = \lceil n/2 \rceil \).

Proof of part (ii). Let \( ES(k) \) denote the least integer such that among any \( ES(n) \) points in general position in the plane there are always \( k \) in convex position. In 1935, P. Erdős and G. Szekeres [ES35] showed that \( ES(k) \) exists for every \( k \), and \( ES(k) \leq \binom{2k-4}{k-2} + 1 \). The best known bounds for \( ES(k) \) are

\[ 2^{k-2} + 1 \leq ES(k) \leq \binom{2k-5}{k-2} + 1, \]

they were proved by P. Erdős and G. Szekeres [ES60], and by Tóth and Valtr [TV05], respectively.

It is easy to see that both the lower and upper bound holds for \( n \leq 16 \), hence we can assume that \( n > 16 \).

First we prove the lower bound for \( s'(n, \mathcal{A}) \). Assume without loss of generality that \( n \) is even. Using the construction of Erdős and Szekeres [ES60], or a subset of it, we can obtain a point set \( P_n \) of size \( n/2 \), in general position, such that it does not contain more than \( 2 \log n \) points in convex position. Take a line \( \ell \) which is not parallel to any line determined by the points of \( P_n \). Substitute each point \( p \) of \( P_n \) by two points, \( p' \) and \( p'' \), both very close to \( p \) such that the line \( p'p'' \) is parallel to \( \ell \). Points \( p' \) and \( p'' \) are called twins, and \( p \) is their parent. Let \( Q_n \) be the resulting set of \( n \) points, which is clearly in general position.

Suppose that \( S \) separates \( Q_n \). Clearly, for each pair of twins \((p', p'')\) in \( Q_n \), there is a set \( S \in S \) which separates them, that is, contains exactly one of \( p' \) and \( p'' \). Assign such a set \( S(p', p'') \) to each pair \((p', p'')\).

This way we found \( n/2 \) members of \( S \). Now we estimate how many times we could find the same set. Suppose e. g. that \( S(p_1', p_1'') = S(p_2', p_2'') = \cdots = S(p_k', p_k'') \). Then, since \( S \) is convex, and twins are very close to each other and to their parents, points \( p_1, p_2, \ldots, p_k \) are in convex position. Therefore, by the assumption, \( k \leq 2 \log n \). So, the number of different sets assigned to the twins is at least \( n/(2 \log n) \).

Now we prove the upper bound. Again, assume that \( n > 16 \).

By [TV05], any set of \( m \) points in general position contains \( \log m/2 \) in convex position. Let \( P \) be a set of \( n \) points in general position. We select a separating system \( S \) of convex sets such that its cardinality is at most \( 20n \log \log n / \log n \) by the following procedure.

**Convex-Separate**

Let \( P_1 = P \), \( S_1 = \emptyset \), \( i = 1 \).

**Step i.** Let \( Q_i \subset P_i \) be a subset of \( k = \lceil \log n \rceil \) points in convex position. Then there is a family \( A_i \) of cardinality \( \lceil \log k \rceil \) which separates \( Q_i \).
Let $S_i$ be the convex hull of all points of $Q_i$, and let $S'_i$ be slightly shrinked copy of $S_i$ (one which contains all the points of $P_i \cap S_i$, except for the points of $Q_i$).

Add $S_i$, $S'_i$, and $A_i$ to $S$.

Let $P_{i+1} = P_i \setminus Q_i$. If $|P_{i+1}| > \sqrt{n}$, then increase $i$ by one and go to Step $i$.

Otherwise, go to Final Step.

**Final Step.** For each point $p \in P$, add $S(p)$, a very small disc with center $p$, to $S$.

Stop.

When we execute Step $i$, $P_i$ contains more than $\sqrt{n}$ points, hence, by [TV05], we can select $k = \lceil \frac{\log n}{4} \rceil > \frac{\log n}{9}$ points among them in convex position. In each step, except for the final one, we delete $k$ points, hence we repeat Step $i$ at most $5n/\log n$ times. Each time we select at most $2 \log \log n + 2$ sets, and in the Final Step we select at most $\sqrt{n}$ sets to $S$. So, eventually, we have $|S| \leq 10n \log \log n / \log n + 10n / \log n + \sqrt{n} \leq 20n \log \log n / \log n$. We claim that $S$ separates $P$. We have to show that any two elements of $P$ can be separated by some member of $S$. If $p, p' \in Q_i$ for some $i$, then $A_i$ separates them.

Now suppose that $p \in Q_i$ and $p' \notin Q_i$ for some $i$. If $p'$ is in the convex hull of $Q_i$, then $S_0$ from Step $i$ separates $p$ and $p'$, if $p'$ is not in the convex hull of $Q_i$, then $S'_0$ from Step $i$ separates $p$ and $p'$.

Finally, suppose that there is no $i$ such that $p \in Q_i$. Then we selected set $S(p)$ in the Final Step, and it separates $p$ and $p'$.

This concludes the proof of Theorem 2.

**Remark.** Let $A$ be a family of connected planar sets, $\gamma$ a Jordan curve, and $k$ a constant. Suppose that each $A \in A$ is bounded by a closed Jordan curve, and intersects $\gamma$ in at most $k$ intervals. Then, it is not hard to see [P12] that $s(n, A) \geq (n - 1)/2k$. We believe that there is a linear bound for other “simple” families, in particular, for families of finite Vapnik-Chervonenkis dimension.

**Conjecture.** Suppose that $A$ is a family of planar sets, whose Vapnik-Chervonenkis dimension is finite. Then there is a $c = c(A) > 0$ such that $s(n, A) > cn$ for every $n$.

Note that this conjecture has nothing to do with geometry, it is a purely combinatorial statement. On the other hand, it might be easier to verify the conjecture if assume that the sets in $A$ are connected, or we add some geometric condition.

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