The Twistor Transform of a Verlinde formula

S. M. Salamon

Introduction

Let $\Sigma$ be a compact Riemann surface of genus $g$. The moduli space $\mathcal{M}_g = \mathcal{M}_g(2,1)$ of stable rank 2 holomorphic bundles over $\Sigma$ with fixed determinant bundle of degree 1 is a smooth complex $(3g-3)$-dimensional manifold \[25\]. The anticanonical bundle of $\mathcal{M}_g$ is the square of a holomorphic line bundle $L$, some power of which embeds $\mathcal{M}_g$ into a projective space. The dimensions of the vector spaces $H^0(\mathcal{M}_g, \mathcal{O}(L^{m-1}))$ of holomorphic sections of powers of $L$ are known to be independent of the choice of complex structure on $\Sigma$, and are given by the formula

$$h^0(\mathcal{M}_g, \mathcal{O}(L^{m-1})) = -m^g - \frac{2m-1}{2} \sum_{i=1}^{2m-1} (-1)^i \csc^2 \left( \frac{i\pi}{2m} \right)$$

(0.1)

predicted by Verlinde \[29\]. This is closely related to the structure of the cohomology ring of $\mathcal{M}_g$, and a number of independent proofs and generalizations of (0.1) are now known. Below we shall follow closely the approach of Szenes \[27\].

In the case in which $\Sigma$ is a hyperelliptic surface, and is therefore a 2-fold branched covering of $\mathbb{CP}^{g-1}$, Desale and Ramanan \[9\] exhibit $\mathcal{M}_g$ as a complex submanifold of the flag manifold $F_g = SO(2g+2)/(U(g-1) \times SO(4))$. As explained in \[27\] this reduces verification of (0.1) to certain $SO(2g)$-equivariant calculations. Our contribution is to observe that $F_g$ is the twistor space of the real oriented Grassmannian $G_g = SO(2g+2)/(SO(2g-2) \times SO(4))$ in the sense of \[7, 8\] for all $g \geq 3$. This enables us to relate the cohomology of the symmetric space $G_g$ directly to the cohomology of $\mathcal{M}_g$, and we obtain a set of generators for the latter which may be compared to the universal ones described in \[21, 28, 10\]. As a feasibility study, we illustrate the theory in the present paper for the case $g = 3$ which is worthy of special attention since the fibration $F_3 \to G_3$ encapsulates the quaternionic structure of the base space in a manner first identified by Wolf \[31\].

In the first section we investigate the cohomology of $G_3 = SO(8)/(SO(4) \times SO(4))$. Using its quaternionic spin structure, we prove that the odd Pontrjagin classes of $G_3$ vanish, and that its $A$ class simplifies remarkably. In the second section we recover Ramanan’s description \[24\] of the Chern ring of $M_3$ in the context of the natural mapping $\mathcal{M}_3 \to G_3$, enabling $h^0(\mathcal{M}_3, L^{m-1})$ to be computed rapidly. Whilst this provides only a particularly simple instance of (0.1), results of the third section identify $H^0(\mathcal{M}_3, L^k)$ with a virtual representation of $SO(8)$ that also arises from the kernels of coupled Dirac operators on $G_3$. Similar techniques can in theory be applied to higher genus cases, and formulae such as $p_1^g = 0$ on $\mathcal{M}_g$ \[28, 16\] may be expected to interact with properties of $G_g$ such as the constancy of the elliptic genera considered in \[20, 15\].

The material below was presented at the conference ‘Differential Geometry and Complex Analysis’ in Parma in May 1994, and the author is grateful to the organizers of that event.
1. Grassmannian cohomology

From now on we denote by \( \mathcal{G} \) the Grassmannian
\[
\mathcal{G}_3 = \frac{SO(8)}{SO(4) \times SO(4)}
\]
that parametrizes real oriented 4-dimensional subspaces of \( \mathbb{R}^8 \). Let \( W \) denote the tautological real rank 4 vector bundle over \( \mathcal{G} \), and \( W^\perp \) its orthogonal complement in the trivial bundle over \( \mathcal{G} \) with fibre \( \mathbb{R}^8 \). The bundles \( W \) and \( W^\perp \) arise from the standard representations of the two \( SO(4) \) factors constituting the isotropy subgroup in (0.1), and it follows that
\[
T\mathcal{G} \cong W \otimes W^\perp.
\]
(0.2)

The \( SO(8) \)-invariant Riemannian metric on \( \mathcal{G} \) determines an isomorphism \( W \cong W^* \) of vector bundles.

The decomposition (0.2) may be refined by lifting the \( SO(4) \) structure of \( W \) to \( Spin(4) \sim SU(2) \times SU(2) \) on a suitable open dense subset \( \mathcal{G}' \) of \( \mathcal{G} \). This procedure is one that is familiar from the study of Riemannian 4-manifolds, and
\[
W_C \cong U \otimes_C V,
\]
where \( U \) and \( V \) are each complex rank 2 vector bundles over \( \mathcal{G}' \). The resulting isomorphism
\[
(T\mathcal{G})_C \cong U \otimes (V \otimes W_C^\perp)
\]
reflects the fact that \( \mathcal{G} \) is a quaternion-Kähler manifold [31, 24]. In (0.3), \( U \) may be thought of as a quaternionic line bundle (usually called \( H \)), and its cofactor \( V \otimes W_C^\perp \) (usually called \( E \)) has structure group \( SU(2) \times SO(4) \) extending to \( Sp(4) \).

The Betti numbers of a quaternion-Kähler 4\( n \)-manifold of positive scalar curvature satisfy \( b_{2k+1} = 0 \) for all \( k \) and \( b_{2k-4} \leq b_{2k} \) for \( k \leq n+1 \). They are also subject to the linear constraint of [17] which for \( n = 4 \) takes the form
\[
3(b_2 + b_4) = 1 + b_6 + 2b_8.
\]
This is well illustrated by \( \mathcal{G} \), which has Poincaré polynomial
\[
P_t(\mathcal{G}) = 1 + 3t^4 + 4t^8 + 3t^{12} + t^{16},
\]
and is the only real Grassmannian to have \( b_4 > 2 \). (These facts may be deduced from [12, chapter XI].) We shall in fact only be concerned with the subring generated by the Euler class \( e = e(W) \) and the first Pontrjagin class \( f = p_1(W) \).

Although the classes \( e \) and \( f \) are very natural, it will ultimately be more convenient to consider
\[
u = -c_2(U), \quad v = -c_2(V).
\]
Because of the \( \mathbb{Z}_2 \)-ambiguity in the definition of \( U, V \), the classes \( u, v \) are not integral, but the symmetric products \( \bigotimes^2 U, \bigotimes^2 V \) are globally defined so \( 4u, 4v \) belong to \( H^4(\mathcal{G}, \mathbb{Z}) \). If we write formally \( 4u = \ell^2 \) then
\[
\text{ch}(U) = e^{\ell/2} + e^{-\ell/2} = 2 + u + \frac{1}{12} u^2 + \frac{1}{300} u^3 + \frac{1}{20160} u^4.
\]
(0.4)
The class $\ell$ is given geometrical significance by the splitting (1.2). An analogous expression to (0.4) holds for $\text{ch}(V)$, and from $\text{ch}(W_C) = \text{ch}(U)\text{ch}(V)$, we obtain

\begin{align*}
e &= u - v, \\
f &= 2(u + v).
\end{align*}

(0.5)

We may add that $p_2(W) = c_4(W_C) = (u - v)^2$ confirming the well-known relation

\begin{equation*}
p_2(W) = e^2.
\end{equation*}

(0.6)

Moreover, the space $H^4(G, \mathbb{Z})$ is generated by $e, f$ together with $e(W^\perp)$ [19].

1.1 Proposition Evaluation on the fundamental cycle $[G]$ yields

\begin{align*}
e^4 &= 2 = e^2f^2, \quad e^3f = 0 = ef^3, \quad f^4 = 4; \\
u^4 &= \frac{21}{64} = v^4, \quad u^3v = -\frac{7}{64} = uv^3, \quad u^2v^2 = \frac{5}{64}.
\end{align*}

We shall deduce these Schubert-type relations from a description of the total Pontrjagin class and the $\hat{A}$ class

\begin{align*}
P(TG) &= 1 + P_1 + P_2 + P_3 + P_4, \\
\hat{A}(TG) &= 1 + \hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_4
\end{align*}

of the tangent bundle (1.2) of $G$. (Upper case $P_i$’s are used to prevent a future clash of notation.) The classes $\hat{A}_i$, $1 \leq i \leq 4$ are determined in terms of the $P_i$ in the usual way [14], and

1.2 Proposition $P_1 = 0 = P_3$ and $\hat{A}(G) = 1 - \frac{1}{240}f^2$.

Proof of both propositions. It is easy to check that, in the presence of (1.3), the two sets of equations of Proposition 1.1 are equivalent. The equalities $u^4 = v^4$ and $u^3v = uv^3$ are immediate from the symmetry between $U$ and $V$, and these are equivalent to $ef^3 = 0 = ef^3$. Using (0.6), we have

\begin{equation*}
\text{ch}(W_C) = 4 + f + \frac{1}{12}(-2e^2 + f^2) + \frac{1}{360}(-3e^2f + f^3) + \frac{1}{20160}(2e^4 - 4e^2f^2 + f^4).
\end{equation*}

(0.7)

From (1.2) and (1.7),

\begin{equation*}
\text{ch}(TG)_C = (\text{ch}(W_C))(8 - \text{ch}(W_C)) \\
= 16 - f^2 + \frac{1}{6}(2e^2f - f^3) + \frac{1}{720}(-20e^4 + 32e^2f^2 - 9f^4).
\end{equation*}

(0.8)

In particular $P_1 = 0$, and so we also have

\begin{equation*}
\text{ch}(TG)_C = 16 - \frac{1}{6}P_2 + \frac{1}{120}P_3 + \frac{1}{10080}(P_2^2 - 2P_4).
\end{equation*}

(0.9)

Comparing (0.8) and (0.9) gives

\begin{equation*}
P_2 = 6f^2, \quad P_3 = 20(2e^2f - f^3), \quad P_4 = 140e^4 - 224e^2f^2 + 81f^4.
\end{equation*}

(0.10)

The remainder of the proof is based on the following less obvious facts.
(i) $\mathcal{G}$ is a spin manifold (see forward to (0.1)) carrying a metric of positive scalar curvature. Therefore its $\hat{A}$ genus

$$\hat{A}_4 = \frac{1}{2^{16}3^55^27} (762P_1^4 - 1808P_1^2P_2 + 416P_2^2 + 1024P_1P_3 - 384P_4)$$

vanishes. Thus

$$0 = 416(6f)^2 - 384(140e^4 - 224e^2f^2 + 81f^4) = 5376(-10e^4 + 16e^2f^2 - 3f^4).$$

(ii) The dimension $d$ of the isometry group of any quaternion-Kähler 16-manifold with positive scalar curvature is given by

$$d = 7 - \frac{8}{3}P_1u^3 + 64u^4$$

[24, page 170]. In the present case, $d = \dim SO(8) = 28$ and we obtain

$$21 = 64u^4 = \frac{1}{4}(16e^4 + 24e^2f^2 + f^4).$$

(iii) On any compact quaternion-Kähler 4n-manifold $M$ with positive scalar curvature and $n > 2$, the index

$$\hat{A}(M, \mathcal{O}^2U) = \langle \text{ch}(\mathcal{O}^2U) \hat{A}, [M] \rangle,$$

vanishes; this is a consequence of [24, Corollary 6.7] which is explained in [15]. Given that

$$\text{ch}(\mathcal{O}^2U) = 3 + 4u + \frac{4}{3}u^2 + \frac{8}{45}u^3 + \frac{4}{315}u^4,$$

$$\hat{A} = 1 - \frac{1}{24}P_1 - \frac{1}{2^53^25}P_2 - \frac{1}{2^63^45^7}P_3 = 1 - \frac{1}{240}f^2 + \frac{1}{1008}(2e^2f - f^3),$$

and $u = (2e + f)/4$, it follows that

$$24e^4 - 26e^2f^2 + f^4 = 0.$$ 

Proposition 1.1 now follows from (0.12), (0.13), (0.14), and it only remains to prove that $P_3 = 0$. Because of the symmetry between $W$ and $W^\perp$, it suffices to prove that $P_3e = 0 = P_3f$, but this follows from (0.10) and Proposition 1.1. QED

Remark. The vanishing of $\hat{A}_4$ and (0.12) above is in fact equivalent to the vanishing of the index $\hat{A}(M,T)$ of the Dirac operator coupled to the tangent bundle (see (0.2)), essentially the so-called Rarita-Schwinger operator. This index is known to be equivariantly constant on any spin manifold with $S^1$ action [30], and always vanishes in the homogeneous setting [15].

2. The flag manifold and moduli space

We denote by $\mathcal{F}$ the complex 9-dimensional homogeneous space

$$\mathcal{F}_3 = \frac{SO(8)}{U(2) \times SO(4)}$$

(0.1)
that parametrizes complex 2-dimensional subspaces \( \Pi \) of \( \mathbb{C}^8 \) that are isotropic with respect to a standard \( SO(8) \)-invariant bilinear form. It has a complex contact structure that was studied in [31] and exhibits it as the twistor space of \( \mathcal{G} \) in the sense of [24]. Projecting \( \Pi \) to a real 4-dimensional subspace of \( \mathbb{R}^8 \) determines an \( SO(8) \)-equivariant mapping \( \pi: \mathcal{F} \to \mathcal{G} \), and each fibre of \( \pi \) is isomorphic to \( SO(4)/U(2) \) and defines a rational curve in the complex manifold \( \mathcal{F} \).

From standard facts about twistor spaces [6, 24, 22], one knows that \( \text{Pic}(\mathcal{F}) \) is generated by a holomorphic line bundle \( L \) on \( \mathcal{F} \) such that

(i) the restriction of \( L \) to each fibre \( \pi^{-1}(x) \cong \mathbb{CP}^6 \) equals \( O(2) \);

(ii) \( L^5 \) is isomorphic to the anticanonical bundle \( K^{-1} \) of \( \mathcal{F} \).

The line bundle \( L \) admits a square root over an open set \( \mathcal{G}' \) of \( \mathcal{G} \) on which \( U \) and \( V \) are defined, there is a \( C^\infty \) isomorphism

\[
\pi^*U \cong L^{1/2} \otimes L^{-1/2},
\]

(0.2)

Let \( \ell \) denote the fundamental class \( c_2(L) \) in \( H^2(\mathcal{F}, \mathbb{Z}) \). From the Leray-Hirsch theorem, there is an identity \((\ell/2)^2 + \pi^*c_2(U) = 0\) of real cohomology classes. In terms of integral classes, and omitting \( \pi^* \),

\[
\ell^2 = 4u.
\]

(0.3)

In the notation of the Introduction, let \( \mathcal{M} = \mathcal{M}_3 \). Szenes exhibits the latter as the zero set of a non-degenerate holomorphic section \( s \in H^0(\mathcal{F}, \mathcal{O}(\sigma^*)) \), where \( \sigma = \mathcal{O}^2 \tau \) and \( \tau \) denotes the tautological rank 2 complex vector bundle acquired from the embedding \( \mathcal{F} \subset \mathcal{G} \mathcal{R}_2(\mathbb{C}^8) \). (Such a section \( s \) corresponds to a quadratic form on \( \mathbb{C}^8 \), but we shall not mention this again until the end of Section 3.) From the coset description (0.1), it follows that

\[
\tau \cong L^{-1/2} \otimes \pi^*V;
\]

(0.4)

the right-hand side is well defined on \( \mathcal{F} \), even though the individual factors only make sense locally (for example on \( \pi^{-1}(\mathcal{G}') \)). Since \( V \cong V^* \), we have \( \sigma^* \cong L \otimes \pi^* \mathcal{O}^2 V \). The resulting holomorphic structure on \( \pi^* \mathcal{O}^2 V \) coincides with that induced in a standard way from the fact that \( \mathcal{O}^2 V \) has a self-dual connection on the quaternion-Kähler manifold \( \mathcal{G} \), in the sense of [13]. In particular, \( \pi^* \mathcal{O}^2 V \) is trivial over each fibre \( \pi^{-1} \cong \mathbb{CP}^6 \). From now on we shall write \( \mathcal{O}^2 V \) in place of \( \pi^* \mathcal{O}^2 V \), and often omit tensor product signs.

The cohomology classes \( \ell, u, v \) may be pulled back from both \( \mathcal{G} \) and \( \mathcal{F} \) to \( \mathcal{M} \), and we shall denote the resulting elements of \( H^4(\mathcal{M}, \mathbb{R}) \) by the same symbols.

2.1 Proposition On \( \mathcal{M} \), \( 3u^2 + 10uv + 3v^2 = 0 \), and evaluation on \( [\mathcal{M}] \) yields

\[
u^3 = \frac{7}{2} = -\nu^3, \quad uv^2 = \frac{3}{2} = -u^2 v.
\]

Proof. The submanifold \( \mathcal{M} \) of \( \mathcal{F} \) is Poincaré dual to the Euler class \( c_3(\sigma^*) \), which is readily computed from the formula \( \text{ch}(\sigma^*) = e'\text{ch}(\mathcal{O}^2 V) \) (see (1.4)) and equals \( 4\ell(u - v) \). Then, for example,

\[
\langle u^3, [\mathcal{M}] \rangle = \langle u^3 c_3(\sigma^*), [\mathcal{F}] \rangle = \langle 4\ell(u^4 - u^3 v), [\mathcal{F}] \rangle = 8 \langle u^4 - u^3 v, [\mathcal{G}] \rangle = \frac{7}{2},
\]

the last equality from Proposition 1.1. The evaluation of \( u^2 v, uv^2 \) and \( v^3 \) follows in exactly the same way.

Since \( H^4(\mathcal{M}, \mathbb{R}) \cong H^4(\mathcal{M}, \mathbb{R}) \) is 2-dimensional [2], there must be a non-trivial linear relation \( au^2 + buv + cv^3 = 0 \). The solution \( (a = c)/b = 3/10 \) can be found by multiplying the left-hand side by \( u \) and \( v \) in turn.

QED
The next result gives an independent derivation of the characteristic ring in the context of the twistor fibration $F \to G$.

### 2.2 Proposition

The Chern and Pontrjagin classes of $M$ are given by

\[ c_1 = 2\ell, \quad c_2 = 4(3u+v), \quad c_3 = 8\ell u, \quad c_4 = -\frac{112}{3}uv, \quad c_5 = c_6 = 0; \]

\[ p_1 = -8(u+v), \quad p_2 = \frac{3}{5}p_1^2, \quad p_3 = 0. \]

**Proof.** It is known [24] that the fibration $\pi$ gives a $C^\infty$ splitting of the holomorphic tangent bundle of $F$:

\[ T^{1,0}F \cong L \oplus L^{1/2}(V \otimes W^c_\perp). \]

Combining this with the isomorphism

\[ T^{1,0}F|_M \cong T^{1,0}M \oplus (L \otimes^2 V)|_M, \]

we obtain

\[ \text{ch}(T^{1,0}M) = e^\ell + e^{\ell/2}\text{ch}(V W^c_\perp) - e^\ell \text{ch}(\otimes^2 V) \]

\[ = e^\ell (1 + e^{-\ell/2}\text{ch}V(8 - \text{ch}Wc) - \text{ch}(\otimes^2 V)). \]

This yields the required expressions for $c_1, c_2, c_3$. We also get $c_4 = 28(u+v)^2$ which reduces to $-112uv/3$ from Proposition 2.1. We next obtain $c_5 = -32\ell v(u+v)$, so that $c_5\ell = 0$ and the vanishing of $c_5$ follows from the fact that $H^2(M,\mathbb{R})$ is 1-dimensional [20]. Finally, all these equalities combine to yield

\[ c_6 = \frac{1}{3}(504u^3 + 2824u^2v + 1928uv^2 + 128v^3), \]

and Proposition 2.1 implies that $c_6 = 0$. The Pontrjagin classes $p_i$ of $M$ are now determined from the Chern classes by the usual relations. QED

**Remark.** The cohomology ring and Chern classes of $M$ were computed in [23, Theorem 4], and comparison with that shows that $h = \ell$, $\nu = \frac{1}{2}(3u+v)$.

In general, it is known that the total Pontrjagin class of $M_g$ equals $(1 + \frac{1}{2g-2}p_1)^{2g-2}$ [21]. Moreover, $p_i^g = 0$ [16, 28] and $c_i = 0$ if $i > 2g-2$ [11].

The above enable the dimension $d_k$ of $H^0(M,\mathcal{O}(L^k))$ to be computed quickly. For this purpose it is convenient to set $k = m - 1$.

### 2.3 Theorem

\[ d_{m-1} = \frac{1}{45}m^2(11 + 20m^2 + 14m^4). \]

**Proof.** Given that $c_1(T^{1,0}F) = 2\ell$, the Todd class $\text{td}(T^{1,0}M)$ of $M$ equals

\[ e^\ell \hat{A}(TM) = e^\ell \left[ 1 - \frac{1}{27}p_1 + \frac{1}{243}p_1^2 - 4p_2 \right]. \]
Using Propositions 2.1, 2.2 and the Riemann-Roch theorem, we obtain
\[
d_{m-1} = \left\langle e^{mt} \left( 1 + \frac{1}{3}(u + v) - \frac{11}{155}w \right), [\mathcal{M}] \right\rangle = -\frac{22}{135} m^2 u^2 v + \frac{2}{9} m^4 (u^3 + u^2 v) + \frac{4}{45} m^6 u^3,
\]
and the result follows. QED

3. Equivariant indexes

In this section, we begin by considering the Dirac operator over the Grassmannian \(G\). Recall from (0.3) that the quaternionic structure of \(G\) is characterized by the vector bundles \(H = U\) and \(E \cong V W_C\) (juxtaposition denotes tensor product). For \(p \geq 4\), the exterior power \(\bigwedge^p E\) contains a proper subbundle \(\bigwedge^p_0 E\) with the property that \(\bigwedge^p E \cong \bigwedge^p_0 E \oplus \bigwedge^{p-2} E\) and, as described in [4], the total spin bundle \(\Delta\) of \(G\) decomposes as \(\Delta^+ \oplus \Delta^-\) where
\[
\Delta^+ \cong \bigotimes U \oplus \bigotimes U \bigwedge^2_0 E \oplus \bigwedge^4_0 E, \\
\Delta^- \cong \bigotimes^2 U E \oplus U \bigwedge^3_0 E. 
\]

The fact that all the summands on the right-hand side are globally defined confirms that \(G\) is spin, though we shall not in fact need the decompositions (0.1).

Now let \(X\) be any other complex vector bundle over \(G\). The choice of a connection on \(X\) allows one to extend the Dirac operator on \(G\) to an elliptic operator
\[D_X: \Gamma(\Delta^+, X) \to \Gamma(\Delta^-, X).\]

The index of this coupled Dirac operator is by definition \(\dim(\ker D_X) - \dim(\text{coker} D_X)\). This extends to a homomorphism \(\hat{A}(G) \to \mathbb{Z}\), so that the index of \(D_X\) is also defined when \(X\) is a virtual vector bundle. The Atiyah-Singer index theorem [3] asserts that the index of \(D_X\) equals
\[
\hat{A}(G, X) = \left\langle \text{ch}(X) \hat{A}(T \mathcal{G}), [\mathcal{G}] \right\rangle. 
\]

In our situation, this fact is closely related to the Riemann-Roch theorem on \(\mathcal{F}\) which provides the following interpretation of \(d_k\).

Theorem 3.1 Let \(X_k = \bigotimes^{2k+4} U - \bigotimes^{2k+2} U \bigotimes^2 V + \bigotimes^{2k} U \bigotimes^2 V - \bigotimes^{2k-2} U, k \geq 1\). Then \(d_k = \hat{A}(G, X_k)\).

Proof. Let \(\sigma\) denote the rank 3 vector bundle \(\bigotimes^2 \mathcal{F}\) as above, and let \((k)\) denote the operation of tensoring with \(L^k\). The description of \(\mathcal{M}\) as the zero set of a section of \(\sigma^* \cong \bigotimes^2 V(1)\) provides a Koszul complex
\[
0 \to \mathcal{O}_\mathcal{F}(\bigwedge^3 \sigma(k)) \to \mathcal{O}_\mathcal{F}(\bigwedge^2 \sigma(k)) \to \mathcal{O}_\mathcal{F}(\sigma(k)) \to \mathcal{O}_\mathcal{F}(k) \to \mathcal{O}_\mathcal{M}(k) \to 0,
\]
or equivalently,

$$0 \to \mathcal{O}_\mathcal{F}(k-3) \to \mathcal{O}_\mathcal{F}(\bigodot^2 V(k-2)) \to \mathcal{O}_\mathcal{F}(\bigodot^2 V(k-1)) \to \mathcal{O}_\mathcal{F}(k) \to \mathcal{O}_\mathcal{M}(k) \to 0.$$  

It follows that 

$$\chi(\mathcal{M}, \mathcal{O}(k)) = a_k - b_{k-1} + b_{k-2} - a_{k-3}, \quad (0.3)$$

where

$$a_k = \chi(\mathcal{F}, \mathcal{O}(k)), \quad b_k = \chi(\mathcal{F}, \mathcal{O}(\bigodot^2 V(k))). \quad (0.4)$$

These holomorphic Euler characteristics may be computed using the Riemann-Roch theorem and the cohomological version [24, 7.2] of the twistor transform; the result is

$$a_k = \hat{A}(\mathcal{G}, \bigodot^{2k+4}U), \quad b_k = \hat{A}(\mathcal{G}, \bigodot^{2k+4}U \bigodot^2 V). \quad (0.5)$$

Finally, Proposition 2.2 implies that the canonical bundle $\mathcal{K}(\mathcal{M})$ is isomorphic to $L^{-2}$, so by Serre duality and Kodaira vanishing, $H^i(\mathcal{M}, \mathcal{O}(k)) = 0$ for all $i \geq 1$ and $k \geq -1$. In particular, $\chi(\mathcal{M}, \mathcal{O}(k)) = \dim H^0(\mathcal{M}, \mathcal{O}(k))$ for all $k \geq -1$, and the theorem now follows from (0.3). QED

The isometry group $SO(8)$ of $\mathcal{G}$ acts naturally on the cohomology groups over $\mathcal{F}$ of the sheaves $\mathcal{O}(k), \mathcal{O}(\bigodot^2 V(k))$ considered above. The integers $a_k, b_k$ and

$$d_k = a_k - b_{k-1} + b_{k-2} - a_{k-3}$$

are therefore the dimensions of certain virtual $SO(8)$-modules, and we identify these shortly.

Let $V(\gamma)$ denote the complex irreducible representation of $SO(8)$ with dominant weight $\gamma$, where $\gamma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$. We adopt standard coordinates so that $V(1,0,0,0) = \mathbb{C}^8$ is the fundamental representation, and $V(1,1,0,0) = \mathfrak{so}(8,\mathbb{C})$ is the complexified adjoint representation.

### 3.2 Proposition

Let $A_k = V(k,k,0,0)$ and $B_k = V(k+1,k-1,0,0)$. Then $a_k = \dim A_k$ and $b_k = \dim B_k$.

**Proof.** The Weyl dimension formula states that

$$\dim(V(\gamma)) = \prod_{\alpha \in R_+} \frac{\langle \alpha, d + \gamma \rangle}{\langle \alpha, d \rangle},$$

where $R_+$ denotes the set of positive roots and $d$ is half of their sum. With the above coordinates,

$$R_+ = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1),$$

$$(1,-1,0,0), (1,0,-1,0), (1,0,0,-1), (0,1,-1,0), (0,1,0,-1), (0,0,1,-1)\},$$

$d = (3,2,1,0)$ and we obtain 

$$\dim A_k = \frac{1}{4320} (k+1)(k+2)^3(2k+5)(k+3)(k+4),$$

$$\dim B_k = \frac{1}{1440} k(k+1)^2(k+2)(2k+5)(k+3)(k+4)^2(k+5).$$
We claim that the right-hand sides are equal to $a_k$ and $b_k$ respectively. It follows from (0.4) that $a_k$ and $b_k$ are polynomials in $k$ of degree 9, and by Serre duality,

$$a_{-k} = -a_{k-5}, \quad b_{-k} = -b_{k-5}, \quad k \in \mathbb{Z}. \quad (0.6)$$

By (0.4) and suitable vanishing theorems [5], $a_k = 0 = b_k$ for $k = -4, -3, -\frac{5}{2}, -2, -1$. In addition, $\mathcal{F}$ has Todd genus $a_0 = 1 = -a_{-5}$, and $b_0 = 0 = b_{-5}$. Accordingly,

$$a_k = \frac{1}{4320}(k+1)(k+2)(2k+5)(k+3)(k+4)\tilde{a}_k,$$
$$b_k = \frac{1}{1440}(k+1)(k+2)(2k+5)(k+3)(k+4)(k+5)\tilde{b}_k.$$

where $\tilde{a}_k$ is a quartic polynomial in $k$ with $\tilde{a}_0 = 36$ and $\tilde{b}_k$ is quadratic in $k$.

Let $n = 2k + 4$. The formulae (0.5) involve $\text{ch}(\bigodot^n U) = f(n)$, where

$$f(x) = \frac{e^{(x+1)\ell/2} - e^{-(x+1)\ell/2}}{e^{\ell/2} - e^{-\ell/2}} \quad (0.7)$$

(see (0.2)). To evade an explicit calculation of $\text{ch}(\bigodot^n U)$, we exploit the following formulae which are easily deduced from (0.7).

### 3.3 Lemma

$$f'(0) = \frac{\ell/2}{\tanh(\ell/2)}, \quad f''(0) = u.$$

The right-hand side of the first equation is the series used in the definition of Hirzebruch’s L-genus, and using (0.3) can be rewritten as

$$\frac{d}{dn} \bigg|_{n=0} \text{ch}(\bigodot^n U) = 1 - \sum_{j \geq 1} (-1)^j \frac{2^{2j}B_j}{(2j)!} u^{2j}$$
$$= \frac{1}{2} \left(1 - \frac{3}{4}u^2 + \frac{2}{945}u^3 - \frac{1}{4725}u^4\right),$$

where $B_j$ are the Bernoulli numbers [14]. From above, we obtain

$$\frac{d}{dk} \bigg|_{k=-2} a_k = \frac{1}{270} (u^4 + 2u^3v + u^2v^2) - \frac{1}{4725} u^4 = 0 = \frac{d^2}{dk^2} \bigg|_{k=-2} a_k.$$

It follows that $\tilde{a}_k$ is divisible by $(k+2)^2$, and by Serre duality by $(k+3)^2$. We obtain $\tilde{a}_k = (k+2)^2(k+3)^2$. The identification $\tilde{b}_k = (k+1)(k+4)$ is similar, and proceeds using a less-enlightening version of Lemma 3.3; we omit the details. QED

The following table displays some of the above dimension functions in terms of $k$.

| $k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|----|
| $a_k$ | 1  | 28 | 300| 1925| 8918| 32928| 102816| 282150| 698775|
| $b_k$ | 0  | 35 | 567| 4312| 21840| 85050| 274890| 772464| 1945944|
| $d_k$ | 1  | 28 | 265| 1392| 5145| 15100| 37681 | 83392 | 168273|


Applying Serre duality and Kodaira vanishing over \( F \), recalling that \( K(F) \cong L^{-5} \), shows that there is in fact an SO(8)-equivariant isomorphism \( A_k \cong H^0(F, O(k)) \). In particular, \( A_1 \) may be identified with both the space of holomorphic sections of \( L \) and the Lie algebra \( \mathfrak{so}(8, \mathbb{C}) \) of infinitesimal automorphisms of the contact structure of \( F \). There is an associated moment mapping \( F \to \mathbb{P}(\mathfrak{so}(8, \mathbb{C})^*) \cong \mathbb{CP}^{27} \) that identifies \( F \) with the projectivization of the nilpotent orbit of minimal dimension \( [27] \). Accordingly, the SO(8)-equivariant linear mapping

\[
\phi_k : \bigotimes^k H^0(F, O(1)) \to H^0(F, O(k))
\]

is onto for all \( k \geq 1 \). Indeed, \( A_k \) is the irreducible summand of \( \bigotimes^k A_1 \) of highest weight, and it suffices to show that the restriction of \( \phi_k \) to \( A_k \) is an isomorphism. Observe that \( A_k \) contains a decomposable tensor product \( \xi \otimes \xi \) for some non-zero \( \xi \in A_1 \) and \( \phi_k(\xi \otimes \xi) \), being the \( k \)th power of \( \xi \) regarded as a section of \( L \), is also non-zero. The irreducibility of \( A_k \) and Schur’s lemma establishes the claim.

A similar argument can be given to establish an SO(8)-equivariant isomorphism \( B_k \cong H^0(F, O(\bigotimes^2 V(k))) \), given that \( H^i(F, O(\bigotimes^2 V(k))) \) vanishes for all \( i > 0 \) and \( k \geq 0 \). One considers the mapping

\[
\psi_k : H^0(F, O(\bigotimes^2 V(1))) \otimes H^0(F, O(k - 1)) \to H^0(F, O(\bigotimes^2 V(k))),
\]

in which \( H^0(F, O(\bigotimes^2 V(1))) \) is isomorphic to the irreducible 35-dimensional SO(8)-module \( \bigotimes^2 \mathbb{C}^8 \) with highest weight \((2, 0, 0, 0)\). The irreducible summand of highest weight in the tensor product is isomorphic to \( B_k \) and the restriction of \( \psi_k \) to this is an isomorphism.

The above arguments can be streamlined by applying more sophisticated twistor transform machinery contained, for example, in [14]. In particular, \( A_k \) and \( B_k \) are known to be isomorphic to the respective kernels of natural twistor operators

\[
\alpha_k : \bigotimes^{2k} U \to E \bigotimes^{2k+1} U, \\
\beta_k : \bigotimes^{2k} U \bigotimes^2 V \to E \bigotimes^{2k+1} U \bigotimes^2 V.
\]

Recall that \( \mathcal{M} \) is the zero set of an element \( s \) of the space \( B_1 \cong \bigotimes^2 \mathbb{C}^8 \). For suitable hyperelliptic surfaces \( \Sigma \), the section \( s \) will be a real element; at each point of \( \mathcal{G} \) it then defines a section of \( W \oplus W^\perp \), which is a trivial bundle with fibre \( \mathbb{R}^8 \) (see (14.1)). In these terms the element \( \tilde{s} \in \ker \beta_1 \) determined by \( s \) is essentially the image of \( s \) by the homomorphism

\[
\bigotimes^2 (W \oplus W^\perp)_C \to \bigotimes^2 W_C \to \bigotimes^2 U \bigotimes^2 V \cong \text{Hom}(\bigotimes^2 V, \bigotimes^2 U).
\]

This may be used to describe \( \mathcal{M} \) as a ‘branched cover’ of a real subvariety of \( \mathcal{G} \).

The Horrocks instanton bundle over \( \mathbb{CP}^{27} \) discussed at the end of [13] provides an analogous situation in which a geometric object is defined by a non-degenerate solution of a twistor equation over a homogeneous space. Such situations are worthy of more systematic investigation.
References

1. M.F. Atiyah, R. Bott: Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. London 308 (1982), 523–615.
2. M.F. Atiyah, N.J. Hitchin, I.M. Singer: Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. Lond. A362 (1978), 425–461.
3. M.F. Atiyah, I.M. Singer: The index theory of elliptic operators III, Ann. Math. 87 (1968), 546–604.
4. R. Barker, S.M. Salamon: Analysis on a generalized Heisenberg group, J. London Math. Soc. 28 (1983), 184–192.
5. R.J. Baston and M.G. Eastwood: The Penrose transform: its interaction with representation theory, Oxford University Press (1989).
6. A. Besse: Einstein manifolds, Springer (1987).
7. R.L. Bryant: Lie groups and twistor spaces, Duke Math J. 52 (1985), 223–261.
8. F.E. Burstall, J.H. Rawnsley: Twistor theory for Riemannian symmetric spaces, Lect. Notes Math. 1424, Springer (1990).
9. U.V. Desale, S. Ramanan: Classification of vector bundles of rank 2 over hyperelliptic curves, Invent. Math. 38 (1976), 161–185.
10. S.K. Donaldson: Gluing techniques in the cohomology of moduli spaces, in Topological Methods in Modern Mathematics, Publish or Perish (1993), 137–170.
11. D. Gieseker: A degeneration of the moduli space of stable bundles, J. Diff. Geometry 19 (1984), 173–206.
12. W. Greub, S. Halperin, R. Vanstone: Curvature, connections and characteristic classes, Volume 3, Academic Press (1976).
13. G. Harder, M.S. Narasimhan: On the cohomology groups of moduli spaces of vector bundles on curves, Math. Ann. 212 (1978), 215-248.
14. F. Hirzebruch: Topological methods in algebraic geometry, 3rd edition, Springer (1966).
15. F. Hirzebruch, P. Sladowski: Elliptic genera, involutions, and homogeneous spin manifolds, Geom. Dedicata 35 (1990) 309–343.
16. F.C. Kirwan: The cohomology rings of moduli spaces of bundles over Riemann surfaces, J. Amer. Math. Soc. 5 (1992), 853–906.
17. C.R. LeBrun, S.M. Salamon: Strong rigidity of positive quaternion-Kähler manifolds, Invent. Math. 118 (1994), 109–132.
18. M. Mamone Capria, S. Salamon: Yang-Mills fields on quaternionic spaces, Nonlinearity 1 (1988), 517–530.
19. J.W. Milnor, J.D. Stasheff: Characteristic classes, Annals of Math. Studies 76, Princeton University Press (1974).
20. P.E. Newstead: Topological properties of some spaces of stable bundles, Topology 6 (1967), 241–262.
21. P.E. Newstead: Characteristic classes of stable bundles over an algebraic curve, Trans. Am. Math. Soc. 169 (1972), 337–345.
22. Y.S. Poon, S.M. Salamon: Eight-dimensional quaternion-Kähler manifolds with positive scalar curvature, J. Diff. Geometry 33 (1991), 363–378.
23. S. Ramanan: The moduli space of vector bundles over an algebraic curve, Math. Ann. 200 (1973), 69–84.
24. S. Salamon: Quaternionic Kähler manifolds, Invent. Math. 67 (1982), 143–171.
25. C.S. Seshadri: Space of unitary vector bundles on a compact Riemann surface, Ann. of Math. 85 (1967), 303–336.
26. A.F. Swann: Hyperkähler and quaternionic Kähler geometry, Math. Ann. 289 (1991), 421–450.
27. A. Szenes: Hilbert polynomials of moduli spaces of rank 2 vector bundles I, Topology 32 (1993), 587–597.
28. M. Thaddeus: Conformal field theory and the moduli space of stable bundles, J. Diff. Geometry 35 (1992), 131–149.
29. E. Verlinde: Fusion rules and modular transformations in 2d conformal field theory, Nucl. Phys. B 300 (1988), 360–376.
30. E. Witten: The index of the Dirac operator in loop space, in Elliptic curves and modular forms in algebraic topology, Lect. Notes Math. 1326, Springer (1988), 161–181.
31. J.A. Wolf: Complex homogeneous contact structures and quaternionic symmetric spaces, J. Math. Mech. 14 (1965) 1033–1047.