Electromagnetic Field in de Sitter Expanding Universe: Majorana–Oppenheimer Formalism, Exact Solutions in non-Static Coordinates

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October 31, 2014

Abstract

Tetrad-based generalized complex formalism by Majorana–Oppenheimer is applied to treat electromagnetic field in extending de Sitter Universe in non-static spherically-symmetric coordinates. With the help of Wigner $D$-functions, we separate angular dependence in the complex vector field $E_j(t,r) + iB_j(t,r)$ from $(t,r)$-dependence. The separation parameter arising here instead of frequency $\omega$ in Minkowski space-time is quantized, non-static geometry of the de Sitter model leads to definite dependence of electromagnetic modes on the time variable.

Relation of 3-vector complex approach to 10-dimensional Duffin–Kemmer–Petiau formalism is considered. On this base, the electromagnetic waves of magnetic and electric type have been constructed in both approaches. In Duffin–Kemmer–Petiau approach, there are constructed gradient-type solutions in Lorentz gauge.

PACS numbers: 02.30.Gp, 02.40.Ky, 03.65Ge, 04.62.+v
MSC 2010: 33E30, 34B30.

Keywords: de Sitter Universe, electromagnetic field, Majorana–Oppenheimer formalism, non-static coordinates, electric and magnetic waves

1 Introduction

Special relativity arose from study of symmetry properties of the Maxwell equations with respect to a motion of a reference frame: Lorentz [1], Poincaré [2, 3], Einstein [4]. Indeed, the analysis of the Maxwell equations with respect to the Lorentz transformations was the first object of the special relativity: Minkowski [5], Silberstein [6, 7, 8], Marcelongo [9], Bateman [10], Lanczos [11], Gordon [12], Mandel’stam – Tamm [13, 14, 15]. After discovery of the relativistic equation for a particle with spin 1/2 – Dirac [16, 17] – much work was done to study spinor and vectors in the context of the Lorentz group theory: Möglich [18, 19], Ivanenko – Landau [20, 21], Neumann [22], van der Waerden [22, 23], Juvet [24]. As was shown, any quantity which transforms linearly under the Lorentz
transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations in particle physics should be written in a spinor form. For the first time, spinor formulation of the Maxwell equations was studied by Laporte and Uhlenbeck [24], also see Rumer [25]. In 1931, Majorana [26] and Oppenheimer [27] proposed to consider the Maxwell theory of electromagnetism as a basis for wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein [6], he showed the possibility to consider Maxwell equations in terms of complex 3-vector variables. In his second paper Silberstein [7] noted that the complex form of the Maxwell equations had been already known; he referred to the lecture notes on differential equations of mathematical physics given by B. Riemann that were edited and published by H. Weber in 1901 [29]. This is not widely used fact as noted by Bialynicki-Birula [30, 31].

And several general remarks. Maxwell equations in various forms were considered during long time by many authors. In this paper we did not discuss in detail did not classify these contributions – it should be a matter of special consideration. Instead we just try to give an historically organized list. They are: Luis de Broglie [32, 33, 35, 41, 44], Mercier [34], Petiau [35], Proca [36, 53], Duffin [37], Kemmer [39, 50, 79], Bhabha [40], Belinfante [41, 12], Taub [43], Sakata – Taketani [46], Schrödinger [45, 48, 49, 47], Heitler [51], Hoffmann [56], Utiyama [57], Mercier [59], Fujiwara [64], Gürsey [65], Gupta [66], Lichnerowicz [67], Ohmura [68], Borgardt [69, 76], Fedorov [70], Kuohsien [71], Bludman [72], Good [73], Moses [74, 75], Silveira [78], Lomont [75], Post [81], Bogush – Fedorov [82], Sachs – Schwebel [83], Newman – Penrose [84, 85], Ellis [86, 87], Beckers – Pirotte [88], Casanova [89], Carmeli [90], Bogush [90], Goldman – Tsai – Yildiz [91], Lord [92], Weingarten [93], Mignani – Recami – Baldo [94, 95], Stephani Stephani, Edmonds [98], Strazhev – Tomil'chik [99], Fushchych – Nikitin [100], Malin [102], Silveira [101], Frolov Frolov, Jena – Naik – Pradhan [103], Venuri [105], Chow [106], Fushchich – Vladimirov [107], Fushchich – Nikitin [7], [108], Cook [109, 110], Giannetto [111], Recami [112], Figueiredo, Oliveira and Rodrigues [130], Krivsky – Simulik [113], Vaz and Rodrigues [131, 132], Inagaki [114], Bialynicki-Birula [30, 31, 128], Sipe [115, 116], Dvoeglazov [118, 119, 120], Gersten [117], Gsponer [121], Varlamov [122], Khan [123, 124], Donev – Tashkova [125, 126, 127], Rodrigues et al [130, 131, 132], Bogush et al [133], Ovsiyuk and Red’kov [134, 135, 136].

The interest in the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years. In particular, elaboration of the complex formalism in electrodynamics to Riemannian space-time models has been performed in recent years – see [137-140, 141, 142, 143, 144].

In general, study fundamental particle fields on the background of expanding universe, in particular de Sitter and anti de Sitter models, has a long history [145-171]. Special value of these geometries consists in their simplicity and high symmetry groups underlying them which makes us to believe in existence of exact analytical treatment for some fundamental problems of classical and quantum field theory in curved spaces. In particular, there exist special representations for fundamental wave equations, Dirac’s and Maxwell’s, which are explicitly invariant under respective symmetry groups \(SO(4,1)\) and \(SO(3,2)\) for these models.

In the present paper, the Majorana–Oppenheimer approach and Duffin–Kemmer–Petiau formalism will be applied to treat electromagnetic field in extending de Sitter Universe in non-static spherically-symmetric coordinates. With the help of Wigner \(D\)-functions, we separate angular dependence from \((t, r)\)-variables. The separation parameter arising here instead of frequency \(\omega\) in Minkowski space-time is quantized, non-static geometry of the de Sitter model leads to definite dependence of electromagnetic modes on the time variable. Relation of 3-vector complex approach
to 10-dimensional Duffin–Kemmer–Petiau formalism is elaborated. On this base, the electromagnetic waves of magnetic and electric type have been constructed in both approaches. Besides, in Duffin–Kemmer–Petiau approach, there are constructed gradient-type solutions.

The popular group theoretical approach for spin one field (massive and massless cases) will be considered for treating the problem of spin one field in extending de Sitter Universe in a separate paper.

2 Electromagnetic field in Majorana–Oppenheimer formalism

The matrix form of Maxwell equations in Majorana–Oppenheimer formalism is

$$\alpha^c (e^\rho_{(c)} \partial_\rho + \frac{1}{2} j^{ab}_{\gamma_{abc}}) \Psi = 0,$$

$$\alpha^0 = -iI, \quad \Psi = \begin{bmatrix} 0 \\ E + iB \end{bmatrix}, \quad (1)$$

or in a more detailed form

$$-i (e^\rho_{(0)} \partial_\rho + \frac{1}{2} j^{ab}_{\gamma_{ab0}}) \Psi + \alpha^k (e^\rho_{(k)} \partial_\rho + \frac{1}{2} j^{ab}_{\gamma_{abk}}) \Psi = 0. \quad (2)$$

We will need expressions for matrices $\alpha^k$ and six generators of the 3-vector complex representation of the group $SO(3, C)$ (first specify them in Cartesian basis)

$$\alpha^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \alpha^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$S^1 = j^{23} = \begin{bmatrix} 0 & 0 \\ 0 & \tau_1 \end{bmatrix}, \quad N^2 = j^{01} = +i \begin{bmatrix} 0 & 0 \\ 0 & \tau_1 \end{bmatrix},$$

$$S^2 = j^{31} = \begin{bmatrix} 0 & 0 \\ 0 & \tau_2 \end{bmatrix}, \quad N^2 = j^{02} = +i \begin{bmatrix} 0 & 0 \\ 0 & \tau_2 \end{bmatrix},$$

$$S^3 = j^{12} = \begin{bmatrix} 0 & 0 \\ 0 & \tau_3 \end{bmatrix}, \quad N^3 = j^{03} = +i \begin{bmatrix} 0 & 0 \\ 0 & \tau_3 \end{bmatrix},$$

generators obey the commutation relations

$$S^1 S^2 - S^2 S^1 = S^3,$$

$$N^1 N^2 - N^2 N^1 = -S^3,$$

$$S^1 N^2 - N^2 S^1 = +N^3,$$

and similar ones in accordance with cyclic symmetry. Below we will use notation

$$j^{23} = S^1, \quad j^{31} = S^2, \quad j^{12} = S^3,$$

$$j^{01} = iS^1, \quad j^{02} = iS^2, \quad j^{03} = iS^3.$$
Let us consider eq. (1) in non-static coordinates of the Sitter space-time
\[ x^\alpha = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi), \]
\[ ds^2 = dt^2 - \cosh^2 t [dr^2 + \sin^2 r(d\theta^2 + \sin^2 \theta d\phi^2)] \]
at the use of the following tetrad
\[ e^\alpha_{(0)} = (1, 0, 0, 0), \quad e^\alpha_{(1)} = (0, 0, \frac{1}{\cosh t \sin r}, 0), \]
\[ e^\alpha_{(2)} = (0, 0, 0, \frac{1}{\cosh t \sin r \sin \theta}), \quad e^\alpha_{(3)} = (0, \frac{1}{\cosh t}, 0, 0). \tag{3} \]
Non-vanishing Ricci rotation symbols are
\[ \gamma_{[01]} = \tanh t, \quad \gamma_{[02]} = \tanh t, \]
\[ \gamma_{[03]} = \tanh t, \quad \gamma_{[31]} = + \frac{\cot r}{\cosh t}, \]
\[ \gamma_{[32]} = + \frac{\cot r}{\cosh t}, \quad \gamma_{[12]} = \frac{1}{\cosh t \sin r}. \]
These permit us to find \( \frac{1}{2} j^{ab} \gamma_{abk} \):
\[ \frac{1}{2} j^{ab} \gamma_{ab1} = \left[ S^1 (\gamma_{231} + i \gamma_{011}) + S^2 (\gamma_{311} + i \gamma_{021}) + S^3 (\gamma_{121} + i \gamma_{031}) \right] = i S^1 \tanh t + S^2 \frac{\cot r}{\cosh t}, \]
\[ \frac{1}{2} j^{ab} \gamma_{ab2} = \left[ S^1 (\gamma_{232} + i \gamma_{012}) + S^2 (\gamma_{312} + i \gamma_{022}) + S^3 (\gamma_{122} + i \gamma_{032}) \right] = -S^1 \frac{\cot r}{\cosh t} + i S^2 \tanh t + S^3 \frac{1}{\cosh t \sin r}, \]
\[ \frac{1}{2} j^{ab} \gamma_{ab3} = \left[ S^1 (\gamma_{233} + i \gamma_{013}) + S^2 (\gamma_{313} + i \gamma_{023}) + S^3 (\gamma_{123} + i \gamma_{033}) \right] = i S^3 \tanh t. \]
Therefore, the matrix Maxwell equation
\[ -i (\partial_t + \frac{1}{2} j^{ab} \gamma_{ab0}) \Psi + \alpha^1 (e^2_{(1)} \partial_2 + \frac{1}{2} j^{ab} \gamma_{ab1}) \Psi \\
+ \alpha^2 (e^3_{(2)} \partial_3 + \frac{1}{2} j^{ab} \gamma_{ab2}) \Psi + \alpha^3 (e^1_{(3)} \partial_1 + \frac{1}{2} j^{ab} \gamma_{ab3}) \Psi = 0 \]
reads
\[ -i \partial_t \Psi + \alpha^1 \left( \frac{1}{\cosh t \sin r} \partial_\theta + \frac{1}{2} j^{ab} \gamma_{ab1} \right) \Psi + \alpha^2 \left( \frac{1}{\cosh t \sin r \sin \theta} \partial_\phi + \frac{1}{2} j^{ab} \gamma_{ab2} \right) \Psi = 0, \]
and then
\[
\left[ -i \partial_t + \alpha^3 \left( \frac{1}{\cosh t} \partial_r + iS^3 \tanh t \right) + i^2 \left( \frac{1}{\cosh t \sin r} \partial_\theta + iS^1 \tanh t + S^2 \frac{\cot \theta}{\cosh t} \right) \right] \Psi = 0.
\]

Let us compare it with the structure of DKP 10-component (massive case) equation for spin 1 field (see [172]) in the same coordinates and tetrad
\[
\left[ (i\beta^0 \frac{\partial}{\partial t} - m) + i \tanh t \left( \beta^1 j^{01} + \beta^2 j^{02} + \beta^3 j^{03} \right) \right. \\
\left. + \frac{i}{\cosh t} \left( \beta^3 \partial_r + \frac{\beta^1 j^{31} + \beta^2 j^{32}}{\tan r} \right) \right. \\
\left. + \frac{1}{\cosh t \sin r} \left( \frac{1}{\sin r} \frac{\alpha^1 \partial}{\partial \theta} + \frac{1}{\cosh t \sin r} \frac{\alpha^2 \partial_\phi + S^3 \cos \theta}{\sin \theta} \right) \right] \Psi = 0.
\]

Performing in (5) formal changes
\[i\beta^0 \rightarrow -i, \ i\beta^k \rightarrow \alpha^k, \ j^{0k} \rightarrow iS^k, \ j^{31} \rightarrow S^2, \ j^{32} \rightarrow -S^1, \ j^{12} \rightarrow S^3,\]
we get
\[
\left[ -i \partial_t + \alpha^3 \left( \frac{1}{\cosh t} \partial_r + \alpha^2 \right) + \frac{1}{\cosh t \sin r} \left( \frac{1}{\sin r} \frac{\alpha^1 \partial}{\partial \theta} + \frac{1}{\cosh t \sin r} \frac{\alpha^2 \partial_\phi + S^3 \cos \theta}{\sin \theta} \right) \right] \Psi = 0.
\]

Eq. (4) at \( m = 0 \) should coincide with (4) - it is indeed so. Below, we will represent eq. (4) in the form closed to DKP-form:
\[
\left\{ -i \partial_t + \tanh t \left( \alpha^1 \Delta^1 + \alpha^2 \Delta^2 + \alpha^3 \Delta^3 \right) + \frac{1}{\cosh t \sin r} \left( \alpha^1 \partial_r + \alpha^2 \partial_\phi + \frac{S^3 \cos \theta}{\sin \theta} \right) \right\} \Psi = 0, \quad \Sigma_{\theta \phi} = \left( \alpha^1 \frac{\partial}{\partial \theta} + \alpha^2 \frac{\partial_\phi}{\sin \theta} + \frac{S^3 \cos \theta}{\sin \theta} \right).
\]

To have the matrix \( j^{12} \) diagonal, we translate all description to the cyclic basis:
\[
\Psi' = U_4 \Psi, \quad U_4 = \begin{vmatrix} 1 & 0 & 0 \end{vmatrix}, \quad U = \begin{vmatrix} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ i/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix}, \quad U^{-1} = U_3^+ = \begin{vmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{vmatrix};
\]
\[
U \tau_1 U^{-1} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{vmatrix} = \tau_1', \quad j^{23} = s_1' = \begin{vmatrix} 0 & 0 \ 0 & \tau_1' \end{vmatrix};
\]
\[
U \tau_2 U^{-1} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \tau_2', \quad j^{31} = s_2' = \begin{vmatrix} 0 & 0 \ 0 & \tau_2' \end{vmatrix};
\]

5
\[ U \tau_3 U^{-1} = -i \begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \tau'_3, \quad j^{12} = s'_3 = \begin{vmatrix} 0 & 0 \\ 0 & \tau'_3 \end{vmatrix}; \]

\[ \alpha' = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ -1 & 0 & -i & 0 \end{vmatrix}, \quad \alpha'' = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 & -i \\ -i & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -i & 0 & 1 & 0 \end{vmatrix}, \quad \alpha''' = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \]

Equation (7) formally preserves its form

\[ \left\{ -i \frac{\partial}{\partial t} + i \tanh t (\alpha'^1 S'^1 + \alpha'^2 S'^2 + \alpha'^3 S'^3) \right. \]

\[ + \frac{1}{\cosh t} \left( \alpha'^3 \frac{\partial_r}{r} + \frac{\alpha'^1 S'^2 - \alpha'^2 S'^1}{\tan r} \right) + \frac{1}{\cosh t \sin r} \Sigma'_{\theta \phi} \left. \right\} \Psi' = 0, \]

\[ \Sigma'_{\theta \phi} = \left( \alpha'^1 \frac{\partial}{\partial \theta} + \alpha'^2 \frac{\partial}{\partial \phi} + S'^3 \cos \theta \right) \sin \theta. \] (8)

In the following, the primes will be omitted.

We will diagonalize the square and third projection of the total angular moment, corresponding substitution for field function is \[172\]

\[ \psi = \begin{vmatrix} \varphi_1(t, r) D_{-1} \\ \varphi_2(t, r) D_0 \\ \varphi_3(t, r) D_{+1} \end{vmatrix}, \] (9)

where Wigner D-functions \[173\] are designated as \( D_{\sigma} = D_{j-m, \sigma}(\phi, \theta, 0), \sigma = -1, 0, +1; j, m \) stand for angular moment quantum numbers. With the help of recurrent formulas \[173\]

\[ \partial_\theta D_{-1} = \frac{1}{2} \left( a D_{-2} - \nu D_0 \right), \]

\[ \frac{m - \cos \theta}{\sin \theta} D_{-1} = \frac{1}{2} \left( a D_{-2} + \nu D_0 \right), \]

\[ \partial_\theta D_0 = \frac{1}{2} \left( \nu D_{-1} - \nu D_{+1} \right), \]

\[ \frac{m}{\sin \theta} D_0 = \frac{1}{2} \left( \nu D_{-1} + \nu D_{+1} \right), \]

\[ \partial_\theta D_{+1} = \frac{1}{2} \left( \nu D_0 - a D_{+2} \right), \]

\[ \frac{m + \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2} \left( \nu D_0 + a D_{+2} \right), \]

where

\[ \nu = \sqrt{j(j+1)}, \quad a = \sqrt{(j-1)(j+2)}. \]

we find action of the angular operator

\[ \Sigma'_{\theta \phi} \psi = \frac{\nu}{\sqrt{2}} \begin{vmatrix} (\varphi_1 + \varphi_3) D_0 \\ -i \varphi_2 D_{-1} \\ i (\varphi_1 - \varphi_3) D_0 \\ +i \varphi_2 D_{+1} \end{vmatrix}. \] (10)
With the use of intermediate formulas

\[ -i \frac{\partial}{\partial t} \Psi = -i \begin{vmatrix} 0 & \partial_t \varphi_1 D_{-1} - i \partial_t \varphi_2 D_0 \\ -i \partial_t \varphi_1 D_{-1} & 0 \end{vmatrix} i \tanh t (\alpha^1 S^1 + \alpha^2 S^2 + \alpha^3 S^3) \Psi = -2i \tanh t \begin{vmatrix} 0 & \varphi_1 D_{-1} \\ \varphi_2 D_0 & \varphi_3 D_{+1} \end{vmatrix} , \]

\[
\begin{vmatrix} \frac{1}{\cosh t} \alpha^3 \partial_r \Psi & 1 \\ -i \partial_r \varphi_2 D_0 \\ \frac{1}{\cosh t} \alpha^1 S^2 - \alpha^2 S^1 \cosh t \tan r \Psi = \frac{1}{\cosh t \tan r} \\ \frac{1}{\cosh t} \partial_r f_2 D_0 \\ -i \partial_r f_1 D_{-1} \\ 0 \end{vmatrix} 2 \varphi_2 D_0 - i \varphi_1 D_{-1} \\ 0 \\ i \varphi_3 D_{+1} \right) = 0. \] (11)

From whence it follows the system

\[ \left( \frac{\partial}{\partial r} + \frac{2}{\tan r} \right) \varphi_2 + \frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 + \varphi_3) = 0, \]

\[- \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) \varphi_1 - \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \varphi_1 - \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0, \]

\[- \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) \varphi_2 + \frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 - \varphi_3) = 0, \]

\[- \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) \varphi_3 + \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \varphi_3 + \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0. \] (13)

Separating in all functions a special factor

\[ \varphi_j = \frac{1}{\cosh^2 t} \frac{1}{\sin r} F_j, \]

we get more simple equations (for convenience on the left some extra numeration is added)

\[ (1') \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) F_2 + \frac{\nu/\sqrt{2}}{\sin r} (F_1 + F_3) = 0, \]

\[ (2') - \cosh t \frac{\partial}{\partial t} F_1 - \frac{\partial}{\partial r} F_1 - \frac{\nu/\sqrt{2}}{\sin r} F_2 = 0, \]
\[(3') - \cosh t \frac{\partial}{\partial t} F_2 + \frac{\nu/\sqrt{2}}{\sin r} (F_1 - F_3) = 0, \]
\[(4') - \cosh t \frac{\partial}{\partial t} F_3 + \frac{\partial}{\partial r} F_3 + \frac{\nu/\sqrt{2}}{\sin r} F_2 = 0. \]  

Let us sum and subtract equations (2') and (4') in (14) and allow equation (3') - this results (introducing intermediate parameter $\nu/\sqrt{2} = b$)

\[(2' + 4') - \cosh t \frac{\partial}{\partial t} (F_1 + F_3) - \frac{\partial}{\partial r} (F_1 - F_3) = 0, \]
\[-(2' + 4') \cosh t \frac{\partial}{\partial t} (F_1 - F_3) + \frac{\partial}{\partial r} (F_1 + F_3) + \frac{2b}{\sin r} F_2 = 0. \]

Let us prove that eq. (1') in (14) can be derived from three remaining equations. To this end, we should differentiate in time equation (1') in (14):

\[ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \frac{\partial}{\partial t} F_2 + \frac{b}{\cosh t \sin r} (F_1 - F_3) + \frac{b}{\sin r \partial t} (F_1 + F_3) = 0; \]

and then let us take into account equation (3') from (14), this gives

\[ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \frac{b}{\cosh t \sin r} (F_1 - F_3) + \frac{b}{\sin r \partial t} (F_1 + F_3) = 0, \]

which is equivalent to

\[ \frac{b}{\cosh t \sin r \partial r} (F_1 - F_3) + \frac{b}{\sin r \partial t} (F_1 + F_3) = 0. \]

It remains to allow for eq. (15)

\[ \frac{\partial}{\partial t} (F_1 + F_3) = - \frac{1}{\cosh t \partial r} (F_1 - F_3), \]

so we arrive at identity $0 \equiv 0$. Further, we may use only three independent equations

\[ \frac{\partial}{\partial t} F_2 = \frac{b}{\cosh t \sin r} (F_1 - F_3), \cosh t \frac{\partial}{\partial t} (F_1 + F_3) + \frac{\partial}{\partial r} (F_1 - F_3) = 0, \]
\[ \cosh t \frac{\partial}{\partial t} (F_1 - F_3) + \frac{\partial}{\partial r} (F_1 + F_3) + \frac{2b}{\sin r} F_2 = 0. \]  

One can exclude $F_2$ from the third equation in (17). To this end, it suffices to differentiate (17) in time and then to take into account expression for $\partial_t F_2$ from the first equation in (17):

\[ \cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} (F_1 - F_3) + \cosh t \frac{\partial}{\partial t} \frac{\partial}{\partial r} (F_1 + F_3) + \frac{2b^2}{\sin^2 r} (F_1 - F_3) = 0. \]

So, instead (17) one can use other (equivalent) system

\[ \frac{\partial}{\partial t} F_2 = \frac{b}{\cosh t \sin r} (F_1 - F_3), \]
\[ \cosh t \frac{\partial}{\partial t} (F_1 + F_3) = - \frac{\partial}{\partial r} (F_1 - F_3), \]
\[
cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} (F_1 - F_3) + \cosh t \frac{\partial}{\partial t} \frac{\partial}{\partial r} (F_1 + F_3) + \frac{2b^2}{\sin^2 r} (F_1 - F_3) = 0. \tag{18}
\]

With the use of the second equation in (18), one excludes the combination \((F_1 + F_3)\) from (18):

\[
\frac{\partial}{\partial t} F_2 = \frac{b}{\sin r} \cosh t \sin r (F_1 - F_3), \quad \cosh t \frac{\partial}{\partial t} (F_1 + F_3) = -\frac{\partial}{\partial r} (F_1 - F_3),
\]

\[
\left( \cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} + \frac{\partial^2}{\partial r^2} + \frac{2b^2}{\sin^2 r} \right) (F_1 - F_3) = 0. \tag{19}
\]

Let us simplify notation by introducing other functions:

\[
F = F_1 + F_3, \quad G = F_1 - F_3,
\]

then (19) reads

\[
\cosh t \frac{\partial}{\partial t} F_2 = \frac{b}{\sin r} G, \quad \cosh t \frac{\partial}{\partial t} F = -\frac{\partial}{\partial r} G, \quad \cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} + \frac{\partial^2}{\partial r^2} + \frac{2b^2}{\sin^2 r} \right) G = 0. \tag{20}
\]

It is convenient to introduce a new coordinate instead of \(t\) (\(\sinh t = x\) is an intermediate variable):

\[
cosh t \frac{d}{dt} = \frac{d}{d\tau}, \quad \frac{dt}{\cosh t} = \frac{dx}{1 + x^2}, \quad \tau = \arctan(\sinh t). \tag{21}
\]

Therefore, the system (19) can be presented in the form

\[
\left( -\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial r^2} + \frac{2b^2}{\sin^2 r} \right) G = 0, \quad \frac{\partial}{\partial \tau} F_2 = \frac{b}{\sin r} G, \quad \frac{\partial}{\partial \tau} F = -\frac{\partial}{\partial r} G. \tag{22}
\]

Equation for \(G(t, r)\) is solved by separation the variables:

\[
G = T(\tau)R(r), \quad \frac{1}{T(\tau)} \frac{\partial^2}{\partial \tau^2} T(\tau) = -\omega^2, \quad \frac{1}{R(r)} \left( \frac{\partial^2}{\partial r^2} + \omega^2 - \frac{j(j + 1)}{\sin^2 r} \right) R(r) = -\omega^2;
\]

from this it follows

\[
T(\tau) = e^{-i\omega \tau}, \quad \left( \frac{\partial^2}{\partial r^2} + \omega^2 - \frac{j(j + 1)}{\sin^2 r} \right) R(r) = 0. \tag{23}
\]

To treat the equation for \(R(r)\), let us introduce the new variable \(z = 1 - e^{-2ir}; \ z\) runs along the closed circle of unit length in complex plane. Allowing for relations

\[
\frac{d}{dr} = 2i(1 - z) \frac{dz}{dz}, \quad \frac{1}{\sin^2 r} = -\frac{4(1 - z)}{z^2},
\]
we derive
\[ 4(1-z)^2 \frac{d^2 f}{dz^2} - 4(1-z) \frac{df}{dz} - \omega^2 f - \frac{4(1-z)\nu^2}{z^2} f = 0, \]
and further through the substitution \( R = z^a (1-z)^b f(z) \) the problem reduces to
\[
z(1-z) \frac{d^2 f}{dz^2} + [2a - (2a + 2b + 1)z] \frac{df}{dz}
+ \left[ \frac{\omega^2}{4} - (a+b)^2 + (a(a-1) - \nu^2) \frac{1}{z} + (b^2 - \frac{\omega^2}{4}) \frac{1}{1-z} \right] f = 0.
\]
With requirements
\[ a = j + 1, \quad b = \pm \frac{\omega}{2}, \]
we get equation of hypergeometric type
\[ z(1-z) \frac{d^2 f}{dz^2} + [2a - (2a + 2b + 1)z] \frac{df}{dz} - [(a+b)^2 - \frac{\omega^2}{4}] f = 0 \quad (24) \]
with parameters
\[ \gamma = 2a, \quad \alpha = a + b - \frac{\omega}{2}, \quad \beta = a + b + \frac{\omega}{2}. \]
The function \( R \) is specified by
\[ R = z^a (1-z)^b f(z) = (2i \sin re^{-ir})^a \left(1 - 2i \sin re^{-ir}\right)^b f(z); \]
this solution is finite at the points \( r = 0 \) and \( r = \pi \) only at positive \( a \): \( a = j + 1 \), and at \( b = -\omega/2 \),
then hypergeometric series can be restricted to polynomials (by physical grounds one should assume \( \omega > 0 \)):
\[ \alpha = j + 1 - \omega = -n = \{0, -1, -2, \ldots\} \quad \Rightarrow \quad \omega = n + 1 + j. \]
Thus, the appropriate solutions are given by
\[ R = z^a (1-z)^b f(z) = (2i \sin re^{-ir})^a \left(1 - 2i \sin re^{-ir}\right)^b f(z), \]
\[ f(z) = F(-n, j + 1, 2j + 2; 2i \sin re^{-ir}); \quad (25) \]
quantization is determined by the formula
\[ \omega = n + 1 + j, \quad j = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots; \quad (26) \]
or in usual units (\( \rho \) is the curvature radius)
\[ \omega = \frac{c}{\rho} (n + 1 + j). \]

Let us turn to the first two equations in (20) – they can be presented as
\[ \frac{\partial}{\partial \tau} F_2 = \frac{b}{\sin r} e^{-i\omega \tau} R(r), \quad \frac{\partial}{\partial \tau} F = -\frac{\partial}{\partial r} e^{-i\omega \tau} R(r); \]
from whence it follows
\[ F_2(t, r) = -\frac{1}{i\omega} e^{-i\omega \tau} \frac{b}{\sin r} R(r), \quad F(t, r) = +\frac{1}{i\omega} e^{-i\omega \tau} \frac{d}{dr} R(r). \quad (27) \]
3 Relation between Majorana–Oppenheimer and Duffin–Kemmer–Petiau formalisms; electromagnetic waves of magnetic and electric types

In contrast to Majorana–Oppenheimer approach, the DKP formalisms permits us to follow gauge degrees of freedom of the electromagnetic field. Let us relate these descriptions. It is convenient to start with yet known structures for electromagnetic complex 3-vector and 10-dimensional DKP field function

\[
\Psi = e^{-i\omega t} \begin{vmatrix} 0 \\ \varphi_1 D_{-1} \\ \varphi_2 D_0 \\ \varphi_3 D_{+1} \end{vmatrix},
\]

\[
\Phi = e^{-i\omega t} \begin{bmatrix} f_1 D_0; f_2 D_{-1}, f_3 D_0, f_4 D_{+1}; \\ f_5 D_{-1}, f_6 D_0, f_7 D_{+1}; \\ f_8 D_{-1}, f_9 D_0, f_{10} D_{+1} \end{bmatrix}. \tag{28}
\]

In accordance with definitions, there must exist relationships

\[
E_1 = F_{01}, \quad E_2 = F_{02}, \quad E_3 = F_{03},
\]

\[
B_1 = -F_{23}, \quad B_2 = -F_{31}, \quad B_3 = -F_{12}; \tag{29}
\]

to avoid misunderstanding we denote \((t, r)-\)constituents in Majorana–Oppenheimer picture by \(\varphi_j(t, r)\):

\[
\varphi_1 D_{-1} = E_1 + iB_1 = f_5 D_{-1} - i f_8 D_{-1},
\]

\[
\varphi_2 D_0 = E_2 + iB_2 = f_6 D_0 - i f_9 D_0,
\]

\[
\varphi_3 D_{+1} = E_3 + iB_3 = f_7 D_{+1} - i f_{10} D_{+1}. \tag{30}
\]

From \(29\) it follows

\[
\varphi_2 = f_6 - i f_9, \quad \varphi_1 = f_5 - i f_8, \quad \varphi_3 = f_7 - i f_{10}. \tag{31}
\]

Allowing for restrictions, referred to spacial parity (see in [172]):

\[
P = (-1)^{j+1}, \quad f_6 = 0, \quad f_7 = -f_5, \quad f_{10} = +f_8;
\]

\[
P = (-1)^j, \quad f_9 = 0, \quad f_7 = +f_5, \quad f_{10} = -f_8,
\]

instead of \(31\) we introduce two classes of solutions:

\[
P = (-1)^{j+1}, \quad \varphi_2 = -i f_9,
\]

\[
\varphi_1 = f_5 - i f_8, \quad \varphi_3 = -f_5 - i f_8; \tag{32}
\]

\[
P = (-1)^j, \quad \varphi_2 = f_6,
\]

\[
\varphi_1 = f_5 - i f_8, \quad \varphi_3 = f_5 + i f_8; \tag{33}
\]
Inverse relations are

\[ P = (-1)^{j+1}, \quad f_9 = i \varphi_2, \]
\[ f_5 = \frac{1}{2} (\varphi_1 - \varphi_3), \quad f_8 = \frac{i}{2} (\varphi_1 + \varphi_3); \quad (34) \]
\[ P = (-1)^j, \quad f_6 = \varphi_2, \]
\[ f_5 = \frac{1}{2} (\varphi_1 + \varphi_3), \quad f_8 = \frac{i}{2} (\varphi_1 - \varphi_3). \quad (35) \]

It should be stressed that solutions of Majorana–Oppenheimer equation which differ in the imaginary unit \( i \), from physical standpoint represent substantially different electromagnetic fields (in fact, one should take in mind complex conjugated equation and its solutions).

After translating description to other functions

\[ \varphi_j = \frac{1}{\cosh^2 t \sin r} F_j, \]
\[ F = F_1 + F_3, \quad G = F_1 - F_3, \]

the formulas \((34)\) and \((35)\) will read

\[ P = (-1)^{j+1}, \quad f_9 = i \varphi_2, \]
\[ f_5 = \frac{1}{2} \frac{1}{\cosh^2 t \sin r} G, \quad (36) \]
\[ f_8 = \frac{i}{2} \frac{1}{\cosh^2 t \sin r} F; \]
\[ P = (-1)^j, \quad f_6 = \varphi_2, \]
\[ f_5 = \frac{1}{2} \frac{1}{\cosh^2 t \sin r} F, \quad (37) \]
\[ f_8 = \frac{i}{2} \frac{1}{\cosh^2 t \sin r} G. \]

Now we should relate the systems of \((t,r)\)-equations in these two formalism (we omit here all details of deriving such equations in DK formalism – it is separate and rather laborious task).

First, let us consider the system of \((t,r)\)-equations in DK approach for states with parity (we omit all details concerning deriving these equations, they are part of another paper in preparation on treating the massive spin one particle in expending Universe)

\[ P = (-1)^{j+1}, \]

\[ - \cosh t \frac{\partial}{\partial t} f_5 - 2 \sinh tf_5 \]
\[ + i ((\frac{\partial}{\partial r} + \frac{1}{\tan r}) f_8 + \frac{i \nu/\sqrt{2}}{\sin r} f_9 = 0, \]
\[ \cosh t (\frac{\partial}{\partial t} f_2 - f_5) + \sinh tf_2 = 0, \]
\[ - \cosh tf_8 - i ((\frac{\partial}{\partial r} + \frac{1}{\tan r}) f_2 = 0, \]
\[ - \cosh tf_9 + \frac{i \sqrt{2} \nu}{\sin r} f_2 = 0. \quad (38) \]
Let us translate (38) to notation according to (34), which results in

\[
\frac{1}{2} \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) (\varphi_1 - \varphi_3) \\
+ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \cosh t (\varphi_1 + \varphi_3) + \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0,
\]

\[
(\cosh t \frac{\partial}{\partial t} + \sinh t) f_2 - \cosh t \frac{1}{2} (\varphi_1 - \varphi_3) = 0,
\]

\[
\cosh t \frac{1}{2} (\varphi_1 + \varphi_3) + \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_2 = 0,
\]

\[
- \cosh t \varphi_2 + \frac{\sqrt{2\nu}}{\sin r} f_2 = 0; \quad (39)
\]

The last equation permits us to exclude \( f_2 \) (it is referred to 4-potential of electromagnetic field)

\[
f_2 = \cosh t \sin r \frac{\varphi_2}{\sqrt{2\nu}}
\]

from second and third equations in (39):

\[
\left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) (\varphi_1 - \varphi_3) \\
+ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) (\varphi_1 + \varphi_3) + \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0,
\]

\[- \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) \varphi_2 + \frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 - \varphi_3) = 0,
\]

\[
\frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 + \varphi_3) + \left( \frac{\partial}{\partial r} + \frac{2}{\tan r} \right) \varphi_2 = 0. \quad (40)
\]

It should be noted that acting on the last equation in (40) by the operator \( (\cosh t \partial_t + 2 \sinh t) \):

\[
\frac{\nu/\sqrt{2}}{\sin r} \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) (\varphi_1 + \varphi_3) \\
+ \left( \frac{\partial}{\partial r} + \frac{2}{\tan r} \right) \left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) \varphi_2 = 0,
\]

and then allowing for second equation in (40), we arrive at the equation

\[
\left( \cosh t \frac{\partial}{\partial t} + 2 \sinh t \right) (\varphi_1 + \varphi_3) \\
+ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) (\varphi_1 - \varphi_3) = 0. \quad (41)
\]
This means that the system (40) is equivalent to the following one

\[
\left(cosh t \frac{\partial}{\partial t} + 2 sinh t\right) (\varphi_1 - \varphi_3) + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right)(\varphi_1 + \varphi_3) + \frac{\sqrt{2} \nu}{\sin r} \varphi_2 = 0,
\]

\[-(cosh t \frac{\partial}{\partial t} + 2 sinh t) \varphi_2 + \frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 - \varphi_3) = 0,
\]

\[
\left(cosh t \frac{\partial}{\partial t} + 2 sinh t\right) (\varphi_1 + \varphi_3) + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right)(\varphi_1 - \varphi_3) = 0.
\]

(42)

Now we are ready to demonstrate equivalence of equations derived in the frames of DKP approach (40), (42) to equations arising in Majorana–Oppenheimer approach (13). To this end, let us turn to the system (13) with excluded first equation (which is a consequence of these three):

\[
\left(cosh t \frac{\partial}{\partial t} + 2 sinh t\right) \varphi_1 + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right) \varphi_1 + \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0,
\]

\[-(cosh t \frac{\partial}{\partial t} + 2 sinh t) \varphi_2 + \frac{\nu/\sqrt{2}}{\sin r} (\varphi_1 - \varphi_3) = 0,
\]

\[-(cosh t \frac{\partial}{\partial t} + 2 sinh t) \varphi_3 + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right) \varphi_3 + \frac{\nu/\sqrt{2}}{\sin r} \varphi_2 = 0.
\]

(43)

Note that the second equations in (42) and (43) coincide. Next, let us sum equations one and three in (43):

\[
\left(cosh t \frac{\partial}{\partial t} + 2 sinh t\right) (\varphi_1 + \varphi_3) + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right)(\varphi_1 - \varphi_3) = 0,
\]

this one coincides with the first equation in (42).

Now, let us subtract the first equation from the third equation in (43):

\[
\left(cosh t \frac{\partial}{\partial t} + 2 sinh t\right) (\varphi_1 + \varphi_3) + \left(\frac{\partial}{\partial r} + \frac{1}{\tan r}\right)(\varphi_1 - \varphi_3) = 0,
\]

this one coincides with the third equation in (42).

Thus, for states of electromagnetic field with parity \((-1)^{j+1}\), DK- and MO-approaches give equivalent systems in \((t, r)\)-variables.

Now let us turn to the system of \((t, r)\)-equations derived in DK formalism for states with the opposite parity
\[ P = (-1)^j \]
\[
\left( \frac{\partial}{\partial r} + \frac{2}{\tan r} \right) f_6 + \frac{2\nu}{\sin r} f_5 = 0 ,
\]
\[
- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) f_5 + i \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_8 = 0 ,
\]
\[
- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) f_6 - \frac{2i\nu}{\sin r} f_8 = 0 ; \quad (44)
\]
\[
\cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_2 + \frac{\nu}{\sin r} f_1 - \cosh t f_5 = 0 ,
\]
\[
- \cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_3 + \frac{\partial}{\partial r} f_1 + \cosh t f_6 = 0 ,
\]
\[
i \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_2 + \frac{i\nu}{\sin r} f_3 + \cosh t f_8 = 0 ; \quad (45)
\]

where \( \nu = \sqrt{j(j+1)/2} \).

First let us consider the first three equations (44), in which only related to tensors functions enter. Translate them to new function according to (35):

\[
\left( \frac{\partial}{\partial r} + \frac{2}{\tan r} \right) \varphi_2 + \frac{\nu}{\sin r} (\varphi_1 + \varphi_3) = 0 ,
\]
\[
- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) (\varphi_1 + \varphi_3) - \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) (\varphi_1 - \varphi_3) = 0 ,
\]
\[
- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) \varphi_2 + \frac{\nu}{\sin r} (\varphi_1 - \varphi_3) = 0 .
\]

Next, let us translate them to \( F_j \):

\[
\varphi_j = \frac{1}{\cosh^2 t \sin r} F_j, \quad F = F_1 + F_3 , \quad G = F_1 - F_3 ;
\]

so we get

\[
\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) F_2 + \frac{\nu}{\sin r} F = 0 ,
\]
\[
- \cosh t \frac{\partial}{\partial t} F_2 + \frac{\nu}{\sin r} G = 0 ,
\]
\[
\cosh t \frac{\partial}{\partial t} F + \frac{\partial}{\partial r} G = 0 . \quad (46)
\]

Note that third equation is a consequence of the other. Indeed, differentiating the first one in time

\[
\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \frac{\partial}{\partial t} F_2 + \frac{\nu}{\sin r} \frac{\partial}{\partial t} F = 0 ,
\]

and allowing for the second, we derive the third

\[
\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \frac{\nu}{\cosh t \sin r} G + \frac{\nu}{\sin r} \frac{\partial}{\partial t} F = 0 \implies \frac{1}{\cosh t \frac{\partial}{\partial r} G + \frac{\partial}{\partial t} F = 0 .
Now, let us turn to the equations (14) derived in MO approach ($\nu = \sqrt{j(j+1)/2}$)

\[
\left( \frac{\partial}{\partial r} \tan r \right) F_2 + \frac{\nu}{\sin r} (F_1 + F_3) = 0,
\]

\[
(\cosh t \frac{\partial}{\partial t} F_2 + \frac{\nu}{\sin r} (F_1 - F_3) = 0,
\]

\[
- \cosh t \frac{\partial}{\partial t} (F_1 + F_3) - \frac{\partial}{\partial r} (F_1 - F_3) = 0,
\]

\[
\cosh t \frac{\partial}{\partial t} (F_1 - F_3)
+ \frac{\partial}{\partial r} (F_1 + F_3) + \frac{2\nu}{\sin r} F_2 = 0;
\] (47)

in the functions $F_2$, $F$, $G$ it reads

\[
\left( \frac{\partial}{\partial r} \tan r \right) F_2 + \frac{\nu}{\sin r} F = 0,
\]

\[
- \cosh t \frac{\partial}{\partial t} F_2 + \frac{\nu}{\sin r} G = 0,
\]

\[
\cosh t \frac{\partial}{\partial t} F + \frac{\partial}{\partial r} G = 0,
\]

\[
\cosh t \frac{\partial}{\partial t} G + \frac{\partial}{\partial r} F + \frac{2\nu}{\sin r} F_2 = 0.
\] (48)

Note coincidence of the first three equations in (48) and (46)

\[
\left( \frac{\partial}{\partial r} \tan r \right) F_2 + \frac{\nu}{\sin r} F = 0,
\]

\[
- \cosh t \frac{\partial}{\partial t} F_2 + \frac{\nu}{\sin r} G = 0,
\]

\[
\cosh t \frac{\partial}{\partial t} F + \frac{\partial}{\partial r} G = 0.
\]

One can easily exclude from the fourth equation in MO system (48) the functions $F$ and $F_2$. To this end, it suffices to differentiate it in time

\[
\frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} G + \frac{\partial}{\partial r} \frac{\partial}{\partial t} F + \frac{2\nu}{\sin r} \frac{\partial}{\partial t} F_2 = 0,
\]

and then take into account 2-nd and 3-rd equations in (48): 

\[
\left( \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} - \frac{1}{\cosh t \partial r^2} + \frac{2\nu^2}{\sin^2 r \cosh t} \frac{1}{\sin^2 r} \right) G = 0.
\]

Thus, we derive a yet familiar equation for $G(t,r)$:

\[
\cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} G = \left( \frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{\sin^2 r} \right) G,
\] (49)
or
\[ \cosh t \frac{d}{dt} = \frac{d}{d\tau}, \quad \frac{d^2}{d\tau^2} G = \left( \frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{\sin^2 r} \right) G. \] (50)

Its solution was found after (23)
\[ G = e^{-i\omega \tau} R(r). \]

In turn, equation from (48) provides us with expressions for two remaining functions
\[ \frac{\partial}{\partial \tau} F_2 = \frac{\nu}{\sin r} G, \quad \frac{\partial}{\partial \tau} F = -\frac{\partial}{\partial r} G; \]

from these it follows
\[ F_2 = -\frac{1}{i\omega} e^{-i\omega \tau} \frac{\nu}{\sin r} R(r), \quad F = \frac{1}{i\omega} e^{-i\omega \tau} \frac{d}{dr} R(r). \] (51)

Now let us consider three remaining equations from DKP system (45)
\[ (\cosh t \frac{\partial}{\partial t} + \sinh t) f_2 + \frac{\nu}{\sin r} f_1 - \cosh tf_5 = 0, \]
\[ -(\cosh t \frac{\partial}{\partial t} + \sinh t) f_3 + \frac{\partial}{\partial r} f_1 + \cosh tf_6 = 0, \]
\[ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_2 + \frac{\nu}{\sin r} f_3 - i \cosh tf_8 = 0. \] (52)

One can exclude functions \( f_2, f_3 \) from third equation in (52); it suffices to act on it by operator
\[ (\cosh t \frac{\partial}{\partial t} + \sinh t), \]

and then take into account second equation from (52), which results in
\[ \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) \left( -\frac{\nu}{\sin r} f_1 + \cosh tf_5 \right) \]
\[ + \frac{\nu}{\sin r} \left( \frac{\partial}{\partial r} f_1 + \cosh tf_6 \right) \]
\[ -i(\cosh t \frac{\partial}{\partial t} + \sinh t) \cosh tf_8 = 0. \]

In more detailed form it reads
\[ -\frac{\nu}{\sin r} \frac{\partial}{\partial r} f_1 + \frac{\nu \cos r}{\sin^2 r} f_1 - \frac{\nu \cos r}{\sin^2 r} f_1 \]
\[ + \cosh t \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_5 \]
\[ + \frac{\nu}{\sin r} \frac{\partial}{\partial r} f_1 + \cosh t \frac{\nu}{\sin r} f_6 \]
\[ -i(\cosh t \frac{\partial}{\partial t} + \sinh t) \cosh tf_8 = 0. \]
Terms with \( f_1 \) cancel out each other and we get

\[
\cosh t \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_5 + \cosh t \frac{\nu}{\sin r} f_6 \\
- i (\cosh t \frac{\partial}{\partial t} + \sinh t) \cosh t f_8 = 0 .
\] (53)

Now let translate this equation to other notation

\[
P = (-1)^j, \quad f_6 = \varphi_2 , \\
f_5 = \frac{1}{2}(\varphi_1 + \varphi_3), \quad f_8 = \frac{i}{2}(\varphi_1 - \varphi_3);
\]

so we get

\[
\cosh t \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) (\varphi_1 + \varphi_3) + \cosh t \frac{2\nu}{\sin r} \varphi_2 \\
+ (\cosh t \frac{\partial}{\partial t} + \sinh t) \cosh t (\varphi_1 - \varphi_3) = 0 .
\] (54)

Next, we make substitutions

\[
\varphi_2 = \frac{F_2}{\cosh^2 t \sin r} , \quad (\varphi_1 + \varphi_3) = \frac{F}{\cosh^2 t \sin r} , \\
(\varphi_1 - \varphi_3) = \frac{G}{\cosh^2 t \sin r} ;
\]

which results in

\[
\frac{1}{\cosh t \sin r} \frac{\partial}{\partial r} F + \frac{1}{\cosh t \sin^2 r} F_2 + \frac{1}{\sin r} (\cosh t \frac{\partial}{\partial t} + \sinh t) \cosh t G = 0 ,
\]

which leads to

\[
\frac{\partial}{\partial r} F + \frac{2\nu}{\sin r} F_2 + \cosh t \frac{\partial}{\partial t} G = 0 .
\] (55)

In this equation, one can exclude functions \( F \) and \( F_2 \). To this end, it suffices to differentiate it in time

\[
\frac{\partial}{\partial r} \cosh t \frac{\partial}{\partial t} F + \frac{2\nu}{\sin r} \cosh t \frac{\partial}{\partial t} F_2 + \cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} G = 0
\]

and allow for two equations from (48):

\[
\cosh t \frac{\partial}{\partial t} F_2 = \frac{\nu}{\sin r} G , \quad \cosh t \frac{\partial}{\partial t} F = - \frac{\partial}{\partial r} G .
\]

In result, we arrive at yet known equation (49)

\[
\left( - \frac{\partial^2}{\partial r^2} + \frac{2\nu^2}{\sin^2 r} + \cosh t \frac{\partial}{\partial t} \cosh t \frac{\partial}{\partial t} \right) G = 0 .
\]
Let us turn back to the first two equations in (52):

\[
\begin{align*}
\cosh t \frac{\partial}{\partial t} \sinh t f_2 + \nu \frac{f_1}{\sin r} - \cosh t f_5 &= 0, \\
-(\cosh t \frac{\partial}{\partial t} + \sinh t) f_3 + \frac{\partial}{\partial r} f_1 + \cosh t f_6 &= 0,
\end{align*}
\]

translating them to other functions

\[
\begin{align*}
f_6 &= \varphi_2 = \frac{F_2}{\cosh^2 t \sin r}, \\
f_5 &= \frac{1}{2} (\varphi_1 + \varphi_3) = \frac{F}{2 \cosh^2 t \sin r},
\end{align*}
\]

we get

\[
\begin{align*}
\cosh t \frac{\partial}{\partial t} f_2 + \nu \frac{f_1}{\sin r} - \frac{F}{2 \cosh t \sin r} &= 0, \\
(\cosh t \frac{\partial}{\partial t} + \sinh t) f_3 - \frac{\partial}{\partial r} f_1 - \frac{F_2}{\cosh t \sin r} &= 0. 
\end{align*}
\]

(56)

After separating the factor

\[
\begin{align*}
f_1 &= \frac{g_1}{\cosh t}, \\
f_2 &= \frac{g_2}{\cosh t}, \\
f_3 &= \frac{g_3}{\cosh t},
\end{align*}
\]

eqs. (57) read

\[
\begin{align*}
\cosh t \frac{\partial}{\partial t} g_2 + \nu \frac{g_1}{\sin r} &= \frac{F/2}{\sin r}, \\
\cosh t \frac{\partial}{\partial t} g_3 - \frac{\partial}{\partial r} g_1 &= \frac{F_2}{\sin r}.
\end{align*}
\]

(58)

In the variable \( \tau \) eqs. (58) become simpler

\[
\begin{align*}
\frac{\partial}{\partial \tau} g_2 + \nu \frac{g_1}{\sin r} &= \frac{F/2}{\sin r}, \\
\frac{\partial}{\partial \tau} g_3 - \frac{\partial}{\partial r} g_1 &= \frac{F_2}{\sin r}.
\end{align*}
\]

(59)

However, these equations do not permit us to calculate the functions \( g_1, g_2, g_3 \) by known \( F_2, F \); this is what we should expect because of existence of the gauge freedom in electrodynamics.

For instance, one can set \( g_1(t, r) = 0 \), then eqs. (59) will read

\[
\begin{align*}
\frac{\partial}{\partial \tau} g_2 &= \frac{F/2}{\sin r}, \\
\frac{\partial}{\partial \tau} g_3 &= \frac{F_2}{\sin r},
\end{align*}
\]

(60)

which leads to explicit functions \( g_2, g_3 \) without any gauge freedom. In this case, eqs. (60) determine, in fact, wave of electric type in Landau gauge (when the component \( \Psi_0 \) of the 4-potential vanishes).

When considering (59) and (60) we should remember on the known expressions for \( F_2, F \) (51):

\[
\begin{align*}
F_2 &= -\frac{1}{i \omega} e^{-i \omega \tau} \nu \frac{1}{\sin r} R(r), \\
F &= \frac{1}{i \omega} e^{-i \omega \tau} \frac{d}{dr} R(r).
\end{align*}
\]
To obtain description of the waves of electric type in Lorentz gauge eqs. (58) consistent with the Lorentz condition. The Lorentz condition looks as follows (see the proof of this formula in the next section)

\[- \frac{2\nu}{\sin r} g_2 + (\cosh t \partial_t + 2 \sinh t) g_1 - (\partial_r + \frac{2}{\tan r}) g_3 = 0. \quad (61)\]

Excluding here the functions $g_2$, $g_3$ with the help of (58)

\[
cosh t \frac{\partial}{\partial t} g_2 = -\frac{\nu}{\sin r} g_1 + \frac{F/2}{\sin r}, \quad \cosh t \frac{\partial}{\partial t} g_3 = \frac{\partial}{\partial r} g_1 + \frac{F_2}{\sin r};
\]

we get

\[- \frac{2\nu}{\sin r} \left( -\frac{\nu}{\sin r} g_1 + \frac{F/2}{\sin r} \right) + \cosh t \frac{\partial}{\partial t} (\cosh t \partial_t + 2 \sinh t) g_1

- (\partial_r + \frac{2}{\tan r}) \left( \frac{\partial}{\partial r} g_1 + \frac{F_2}{\sin r} \right) = 0,
\]

after transformation it looks

\[
\left[ \cosh t \frac{\partial}{\partial t} (\cosh t \partial_t + 2 \sinh t)

- \left( \frac{\partial^2}{\partial r^2} + \frac{2}{\tan r} \frac{\partial}{\partial r} + \frac{2\nu^2}{\sin^2 r} \right) \right] g_1

= \frac{\nu F}{\sin^2 r} + \frac{1}{\sin r} (\partial_r + \frac{1}{\tan r}) F_2.
\]

The right hand part of the equation turns to vanish due to the first relation in (46). Thus, we arrive at the equation

\[
\left[ \cosh t \frac{\partial}{\partial t} (\cosh t \partial_t + 2 \sinh t)

- \left( \frac{\partial^2}{\partial r^2} + \frac{2}{\tan r} \frac{\partial}{\partial r} + \frac{2\nu^2}{\sin^2 r} \right) \right] g_1 = 0, \quad (62)
\]

differently it reads as

\[
\left[ \frac{\partial^2}{\partial t^2} + 3 \tanh t \frac{\partial}{\partial t} - \frac{1}{\cosh^2 t} \frac{\partial^2}{\partial r^2}

+ \frac{2}{\tan r} \frac{\partial}{\partial r} - \frac{j(j + 1)}{\sin^2 r} \right] g_1 = 0. \quad (63)
\]

Evidently, it is $(t, r)$-part of the conformally invariant massless wave equation in de Sitter space

\[
\left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} + 2 \right) \Phi = 0. \quad (64)
\]
Equation (63) with the help of substitution
\[ g_1(t) = \frac{g_1(t)}{\cosh t} g_1(r) \sin r \]
leads to equations in separated variables
\[
\left( \frac{d^2}{dr^2} + \omega^2 - \frac{j(j + 1)}{\sin^2 r} \right) g_1(r) = 0, \quad (65)
\]
\[
\left( \cosh \frac{d}{dt} \cosh t \frac{d}{dt} + \omega^2 \right) g_1(t) = 0
\]
or
\[
\left( \frac{d^2}{d\tau^2} + \omega^2 \right) g_1(\tau) = 0. \quad (66)
\]

The functions \( g_1(t, r) \), determined by
\[
\frac{\partial}{\partial \tau} g_2 = -\frac{\nu}{\sin r} g_1 + \frac{F/2}{\sin r},
\]
\[
\frac{\partial}{\partial \tau} g_3 = \frac{\partial}{\partial r} g_1 + = \frac{F_2}{\sin r},
\]
provide us with the complete description of electromagnetic waves of electric type in Lorentz gauge.

4 Gradient type solutions

Among all electromagnetic solutions here exist solution of a pure gauge nature, that are constructed as gradient of a scalar function; therefore having trivial (vanishing) electromagnetic tensor:
\[ f_6 = 0, \quad f_7 = 0, \quad f_8 = 0, \quad f_9 = 0, \quad f_{10} = 0. \]

It is easily see that such solutions cannot have parity \( P = (-1)^{j+1} \) (see (38));
\[
- \cosh t \frac{\partial}{\partial t} f_5 - 2 \sinh t f_5
\]
\[
+i(\frac{\partial}{\partial r} + \frac{1}{\tan r}) f_8 + \frac{i\nu/\sqrt{2}}{\sin r} f_9 = 0,
\]
\[
\cosh t(\frac{\partial}{\partial t} f_2 - f_5) + \sinh t f_2 = 0,
\]
\[
- \cosh t f_8 - i(\frac{\partial}{\partial r} + \frac{1}{\tan r}) f_2 = 0,
\]
\[
- \cosh t f_9 + \frac{i\sqrt{2}\nu}{\sin r} f_2 = 0, \quad (68)
\]
as according to (60) components \( f_8, f_9 \) do not vanish.

For states with opposite parity (see (44) and (45))
\[ P = (-1)^j \]
\[
\frac{\partial}{\partial r} + \frac{2}{\tan r} f_6 + \frac{2\nu}{\sin r} f_5 = 0,
\]
\[- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) f_5 + i \left( \frac{\partial}{\partial r} - \frac{1}{\tan r} \right) f_8 = 0 , \]

\[- \cosh t \left( \frac{\partial}{\partial t} + 2 \tanh t \right) f_6 - \frac{2i\nu}{\sin r} f_8 = 0 , \]  

\[\cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_2 + \frac{\nu}{\sin r} f_1 - \cosh t f_5 = 0 , \]

\[- \cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_3 + \frac{\partial}{\partial r} f_1 + \cosh t f_6 = 0 , \]

\[i \left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_2 + \frac{i\nu}{\sin r} f_3 + \cosh t f_8 = 0 ; \]  

(69)

here one must set \( f_5 = f_6 = f_8 = 0 \); and get three identities \( 0 = 0 \); three reaming equations will take the form

\[\cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_2 + \frac{\nu}{\sin r} f_1 = 0 , \]

\[- \cosh t \left( \frac{\partial}{\partial t} + \tanh t \right) f_3 + \frac{\partial}{\partial r} f_1 = 0 , \]

\[\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) f_2 + \frac{\nu}{\sin r} f_3 = 0 . \]  

(70)

Translating the system to functions \( g_j \):

\[ f_1 = \frac{g_1}{\cosh t} , \quad f_2 = \frac{g_1}{\cosh t} , \quad f_3 = \frac{g_1}{\cosh t} , \]

we have

\[\cosh t \frac{\partial}{\partial t} g_2 + \frac{\nu}{\sin r} g_1 = 0 , \]

\[- \cosh t \frac{\partial}{\partial t} g_3 + \frac{\partial}{\partial r} g_1 = 0 , \]

\[\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) g_2 + \frac{\nu}{\sin r} g_3 = 0 . \]  

(71)

With the use of the variable \( \tau \), eqs. (72) are written as

\[\frac{\partial}{\partial \tau} g_2 + \frac{\nu}{\sin r} g_1 = 0 , \quad - \frac{\partial}{\partial \tau} g_3 + \frac{\partial}{\partial r} g_1 = 0 , \]

\[\left( \frac{\partial}{\partial r} + \frac{1}{\tan r} \right) g_2 + \frac{\nu}{\sin r} g_3 = 0 . \]  

(73)

Note that the third equation is a consequence of two first. As must be expected, these two first independent equations coincide with (67)

\[\frac{\partial}{\partial r} g_2 + \frac{\nu}{\sin r} g_1 = \frac{F/2}{\sin r} , \quad \frac{\partial}{\partial \tau} g_3 - \nu \frac{\partial}{\partial r} g_1 = F_2 , \]

22
at $F_2 = 0$, $F = 0$.

At considering electromagnetic field, as in Minkowski space it is possible to impose Lorentz conditions

$$\nabla_{\alpha} \Psi^\alpha = 0. \quad (74)$$

Let translate this relation to DKP basis. Instead of the 4-vector $\Psi^\alpha$ we should introduce its tetrad components

$$\Psi^\alpha = e^{(a)\alpha} \Psi_a,$$

where

$$e^{(0)\alpha} = (1, 0, 0, 0),$$
$$e^{(1)\alpha} = (0, 0, -1, \cosh t \sin r),$$
$$e^{(2)\alpha} = (0, 0, -1, \cosh t \sin r \sin \theta),$$
$$e^{(3)\alpha} = (0, 1, \cosh t, 0).$$

Correspondingly, (74) takes the form

$$e^{(b)\alpha} \Psi_b + e^{(b)\alpha} \partial_\alpha \Psi_b = 0. \quad (75)$$

We will need the formulas

$$e_{\alpha}^{(0)\alpha} = 3 \tanh t, \quad e_{\alpha}^{(3)\alpha} = -\frac{2}{\cosh t \tan r},$$
$$e_{\alpha}^{(1)\alpha} = -\frac{1}{\cosh t \sin r} \cos \theta, \quad e_{\alpha}^{(2)\alpha} = 0.$$  

Tetrad components of $\Psi_a$ are referred to the first four components of DKP function in cyclic basis by the formulas (let $W \equiv 1/\sqrt{2}$)

$$\begin{pmatrix}
\Phi_0 \\
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -W & 0 & +W \\
0 & -iW & 0 & -iW \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
f_1 D_0 \\
f_2 D_{-1} \\
f_3 D_0 \\
f_4 D_{+1}
\end{pmatrix},$$

so we have

$$\Psi_0 = f_1 D_0, \quad \Psi_3 = f_3 D_0,$$
$$\Psi_1 = \frac{1}{\sqrt{2}}(-f_2 D_{-1} + f_4 D_{+1}),$$
$$\Psi_2 = \frac{i}{\sqrt{2}}(-f_2 D_{-1} - f_4 D_{+1}). \quad (76)$$

Eq. (75) can be written as

$$e_{\alpha}^{(0)\alpha} \Psi_0 + e_{\alpha}^{(3)\alpha} \Psi_3 + e_{\alpha}^{(1)\alpha} \Psi_1 + e_{\alpha}^{(2)\alpha} \Psi_2$$
$$+ e^{(0)\alpha} \partial_\alpha \Psi_0 + e^{(3)\alpha} \partial_\alpha \Psi_3$$
$$+ e^{(1)\alpha} \partial_\alpha \Psi_1 + e^{(2)\alpha} \partial_\alpha \Psi_2 = 0,$$
which is equivalent to
\[
3 \tanh t \Psi_0 - \frac{2}{\cosh t \tan r} \Psi_3 \\
- \frac{1}{\cosh t \sin r \sin \theta} \Psi_1 + \partial_t \Psi_0 - \frac{1}{\cosh t} \partial_r \Psi_3 \\
- \frac{1}{\cosh t \sin r} \partial_{\theta} \Psi_1 - \frac{1}{\cosh t \sin \theta} \partial_{\phi} \Psi_2 = 0,
\]
and further with the use of (67) it reads
\[
3 \tanh t f_1 D_0 - \frac{2}{\cosh t \tan r} f_3 D_0 \\
- \frac{1}{\cosh t \sin r \sin \theta} \frac{1}{\sqrt{2}} (-f_2 D_{-1} + f_4 D_{+1}) \\
+ \partial_t f_1 D_0 - \frac{1}{\cosh t} \partial_r f_3 D_0 \\
- \frac{1}{\cosh t \sin r} \partial_{\theta} \frac{1}{\sqrt{2}} (-f_2 D_{-1} + f_4 D_{+1}) \\
- \frac{1}{\cosh t \sin r \sin \theta} \partial_{\phi} \frac{i}{\sqrt{2}} (-f_2 D_{-1} - f_4 D_{+1}) = 0.
\]
The last equation, after regrouping and some elementary manipulation takes the form
\[
\frac{1}{\sin r} \frac{1}{\sqrt{2}} \left[ \begin{array}{c} f_2 \left( \frac{\cos \theta}{\sin \theta} + \partial_\theta - \frac{m}{\sin \theta} \right) D_{-1} \\
- f_4 \left( \frac{\cos \theta}{\sin \theta} + \partial_\theta + \frac{m}{\sin \theta} \right) D_{+1} \end{array} \right] \\
+ \left[ \cosh t (\partial_\theta + 3 \tanh t) f_1 \right] \\
- \left( \partial_r + \frac{2}{\tan r} \right) f_3 \right] D_0 = 0. \tag{77}
\]
Finally, with the use of recurrent relations for Wigner functions
\[
\left( \partial_\theta - \frac{m - \cos \theta}{\sin \theta} \right) D_{-1} = - \sqrt{(j+1)j} D_0, \\
\left( \partial_\theta + \frac{m + \cos \theta}{\sin \theta} \right) D_{+1} = - \sqrt{(j+1)j} D_0,
\]
we derive the following equation (remember that \(\sqrt{j(j+1)/2} = \nu\))
\[
- \frac{\nu}{\sin r} (f_2 + f_4) + \cosh t (\partial_t + 3 \tanh t) f_1 \\
- \left( \partial_r + \frac{2}{\tan r} \right) f_3 = 0. \tag{78}
\]
This is the Lorentz condition after excluding the \((\theta, \phi)\) dependence.
For states with the parity
\[
P = (-1)^{j+1}, \quad f_1 = f_3 = 0, \quad f_4 = -f_2,
\]
24
it becomes identity $0 = 0$. For states with the opposite parity we have

$$P = (-1)^j, \quad f_4 = + f_2,$$

$$-\frac{2\nu}{\sin r} f_2 + \cosh t(\partial_t + 3 \tanh t)f_1$$

$$-(\partial_r + \frac{2}{\tan r}) f_3 = 0.$$  \hspace{1cm} (79)

Translating it to the functions

$$f_1 = \frac{g_1}{\cosh t}, \quad f_2 = \frac{g_1}{\cosh t}, \quad f_3 = \frac{g_1}{\cosh t},$$

we get

$$-\frac{2\nu}{\sin r} g_2 + (\cosh t\partial_t + 2 \sinh t)g_1$$

$$-(\partial_r + \frac{2}{\tan r}) g_3 = 0.$$  \hspace{1cm} (80)

Let the above gradient type solution

$$- \cosh t\frac{\partial}{\partial t} g_2 = \frac{\nu}{\sin r} g_1,$$

$$\cosh t\frac{\partial}{\partial r} g_3 = \frac{\partial}{\partial r} g_1,$$  \hspace{1cm} (81)

obey the Lorentz condition. To take into account (81) in Lorentz condition (80), one should act on eq. (80) by the operator $\cosh t\frac{\partial}{\partial t}$, after that we derive equation for $f_1$

$$\left[ \left( \frac{\partial^2}{\partial t^2} + 3 \tanh t \frac{\partial}{\partial t} + 2 \right) - \frac{1}{\cosh^2 t} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{\tan r} \frac{\partial}{\partial r} - \frac{2\nu^2}{\sin^2 r} \right) \right] g_1 = 0.$$  \hspace{1cm} (82)

Evidently, this is a $(t,r)$-part of Klein–Fock–Gordon equation with conformal term

$$\left( \frac{1}{\sqrt{-g}} \partial_\alpha \sqrt{-g} g^{\alpha\beta} \partial_2 + 2 \right) \Phi = 0.$$  \hspace{1cm} (83)

Eq. (82) is solved through separation of the variables (see (65), (66), and then one can obtain expressions for $g_2, g_3$ – see (76)).

5 Conclusion

Tetrad-based complex formalism by Majorana–Oppenheimer has been applied to treat electromagnetic field in extending de Sitter Universe in non-static spherically-symmetric coordinates. With the help of Wigner $D$-function we separate angular dependence in the complex vector field $E_j(t,r) + iB_j(t,r)$ from $(t,r)$-dependence. After that we separate variables $t$ and $r$. Non-static geometry of the de Sitter model lead to definite dependence of electromagnetic modes on the time.
variable. Relation of 3-vector complex approach to 10-dimensional Diffin–Kemmer-Petiau formalism is elaborated. On this base, the electromagnetic wave of magnetic and electric type have been constructed in both approaches, also in DK form, there are specified gradient-type solutions in Lorentz gauge.

This work was supported by the Fund for Basic Researches of Belarus, F 13K-079, within the cooperation framework between Belarus and Ukraine.

N. D. Vlasii and Yu. A. Sitenko were supported by the State Agency for Science, Innovations and Informatization of Ukraine under the SFFR-BRFFR grant F54.1/019.

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