INERTIAL PROXIMAL ADMM FOR SEPARABLE
MULTI-BLOCK CONVEX OPTIMIZATIONS AND
COMPRESSIVE AFFINE PHASE RETRIEVAL

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ABSTRACT. Separable multi-block convex optimization problem appears in many mathematical and engineering fields. In the first part of this paper, we propose an inertial proximal ADMM to solve a linearly constrained separable multi-block convex optimization problem, and we show that the proposed inertial proximal ADMM has global convergence under mild assumptions on the regularization matrices. Affine phase retrieval arises in holography, data separation and phaseless sampling, and it is also considered as a nonhomogeneous version of phase retrieval that has received considerable attention in recent years. Inspired by convex relaxation of vector sparsity and matrix rank in compressive sensing and by phase lifting in phase retrieval, in the second part of this paper, we introduce a compressive affine phase retrieval via lifting approach to connect affine phase retrieval with multi-block convex optimization, and then based on the proposed inertial proximal ADMM for multi-block convex optimization, we propose an algorithm to recover sparse real signals from their (noisy) affine quadratic measurements. Our numerical simulations show that the proposed algorithm has satisfactory performance for affine phase retrieval of sparse real signals.

1. Introduction

In the first part of this paper, we consider the following linearly constrained separable multi-block convex optimization,

\[
\min_{x_j \in \mathcal{X}_j, 1 \leq j \leq l} \sum_{j=1}^{l} f_j(x_j) \quad \text{subject to} \quad \sum_{j=1}^{l} A_j x_j = c,
\]

where \(A_j \in \mathbb{R}^{m \times n_j}\), \(c \in \mathbb{R}^m\), \(\mathcal{X}_j\) are closed convex sets in \(\mathbb{R}^{n_j}\) and \(f_j : \mathbb{R}^{n_j} \to (-\infty, \infty)\) are closed convex functions on \(\mathbb{R}^{n_j}\), \(1 \leq j \leq l\). The above minimization problem appears in machine learning, statistics, signal and image processing, and many more fields [44, 48, 55]. Denote the standard inner product and norm on the Euclidean space by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|_2\) respectively. A conventional approach to the convex optimization problem

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is the alternating direction method of multipliers (ADMM) with initial \((x_1^0, \ldots, x_l^0; z^0) \in W := X^1 \times \cdots \times X^l \times \mathbb{R}^m\) chosen appropriately or randomly, and with update in each iteration by

\[
\begin{align*}
(1.2a) & \quad x_1^{k+1} \in \arg \min_{x_1 \in X^1} \mathcal{L}_\beta(x_1, x_2^k, \ldots, x_l^k; z^k), \\
(1.2b) & \quad x_i^{k+1} \in \arg \min_{x_i \in X^i} \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i, x_{i+1}^k, \ldots, x_l^k; z^k), \quad i = 2, \ldots, l, \\
(1.2c) & \quad z^{k+1} = z^k - \beta \left( \sum_{j=1}^l A_j x_j^{k+1} - c \right),
\end{align*}
\]

where

\[
(1.3) \quad \mathcal{L}_\beta(x_1, \ldots, x_l; z) := \sum_{j=1}^l f_j(x_j) - \langle z, \sum_{j=1}^l A_j x_j - c \rangle + \frac{\beta}{2} \left\| \sum_{j=1}^l A_j x_j - c \right\|_2^2
\]

is the augmented Lagrange function with Lagrange multiplier \(z \in \mathbb{R}^m\) and penalty parameter \(\beta > 0\).

The ADMM algorithm with \(l = 2\) was introduced in the 1970s and its convergence has been well studied [26] [29]. For \(l \geq 3\), the multi-block ADMM [1.2] works very well for many concrete applications [4] [32] [48] [51], however it may not converge without additional information on the objective functions \(f_j\) and constraint matrices \(A_j, 1 \leq j \leq l\) [18]. For instance, Han and Yuan [31] showed that the scheme [1.2] is convergent if all the objective functions \(f_j, 1 \leq j \leq l\), are strongly convex and the penalty parameter \(\beta\) is chosen in a certain range. The above strongly convex condition on the objective functions is relaxed in [12] that not all functions in the objective are required to be strongly convex. For general multi-block convex problems, many convergent proximal variants of the multi-block ADMM [1.2] have been proposed to overcome the divergence issue, including the proximal parallel splitting method [30], the Jacobi-Proximal ADMM [22] and the twisted version of the proximal ADMM [54]. The reader may refer to the survey paper [28] for additional historical remarks and recent advances on the ADMM and its variations.

In this paper, we introduce an inertial proximal ADMM to solve the multi-block convex optimization problem (1.1). Prox-IADMM for abbreviation, with initial value \((x_1^0, \ldots, x_l^0; z^0)\) chosen appropriately or randomly in \(W\), and with update in each iteration given by the following:

\[
\begin{align*}
(1.4a) & \quad (\bar{x}_1^k, \ldots, \bar{x}_i^k; z^k) = (x_1^k, \ldots, x_l^k; z^k) + \alpha_k(x_1^k - x_1^{k-1}, \ldots, x_l^k - x_l^{k-1}; z^k - z^{k-1}), \\
(1.4b) & \quad x_1^{k+1} \in \arg \min_{x_1 \in X^1} \mathcal{L}_\beta(x_1, \bar{x}_2^k, \ldots, \bar{x}_l^k; z^k) + \frac{1}{2}(x_1 - \bar{x}_1^k)^T H_1 (x_1 - \bar{x}_1^k), \\
(1.4c) & \quad z^{k+1} = z^k - \beta \left( A_1 x_1^{k+1} + \sum_{j=2}^l A_j \bar{x}_j^k - c \right),
\end{align*}
\]
\( x_i^{k+1} \in \arg \min_{x_i \in X_i} (\mathcal{L}_\beta(x_i^{k+1}, x_2^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_l^k; z^{k+1}) + \frac{1}{2} (x_i - \bar{x}_i^k)^T H_i (x_i - \bar{x}_i^k)), \ 2 \leq i \leq l, \)

where \((x_1^{-1}, \ldots, x_l^{-1}; z^{-1}) = (x_1^0, \ldots, x_l^0; z^0), \alpha_k, \ k \geq 0\) are step sizes, \(\mathcal{L}_\beta\) is the augmented Lagrange function in (1.3), and \(H_j, 1 \leq j \leq l,\) are regularization matrices. Our illustrative examples of regularization matrices are prox-linear matrices
\[
H_j = \beta I/\eta_j - \beta A_j^T A_j, \quad 1 \leq j \leq l,
\]
and standard proximal matrices
\[
H_j = \beta I/\eta_j, \quad 1 \leq j \leq l,
\]
where \(\eta_j, 1 \leq j \leq l,\) are positive numbers, see [23] for additional regularization matrices of interest. The Prox-IADMM (1.4) extends the inertial proximal ADMM for two-block convex optimization in [16] nontrivially, it mixes the Jacobi method in [22, 34] and Gauss-Seidel method in [18, 33], and it also yields a new inertial variant of the ADMM for multi-block convex optimization when all regularization matrices are set to be zero. In the first part of this paper, we establish the convergence of its inertial version under the assumption that \(H_1\) and \(H_j - \beta(l - 2) A_j^T A_j, 2 \leq j \leq l,\) are positive definite, see Theorem 3.3. This is a nontrivial extension of the convergence result in [16] where \(l = 2\) and matrices \(H_1\) and \(H_2\) are under a strong assumption that \(H_1\) and \(H_2 - \beta A_2^T A_2\) are positive definite.

In the second part of this paper, we consider recovering sparse real vectors \(x \in \mathbb{R}^n\) from their affine quadratic measurements
\[
\bar{b} := |Ax + b|^2 = (|a_1^T x + b_1|^2, \ldots, |a_m^T x + b_m|^2)^T,
\]
where \(A = [a_1, \ldots, a_m]^T\) is a measurement matrix and \(b = (b_1, \ldots, b_m)^T\) is a reference vector. The above affine phase retrieval problem arises in holography [41], data separation [24, 43], phaseless sampling [20], phase retrieval with background information [25, 50], and phase retrieval with reference signal [3, 5, 6, 36, 37]. A sufficient and necessary condition on the pair \((A, b)\) of measurement matrix and reference vector is introduced in [19, 27] so that any (sparse) real vector \(x\) is uniquely determined by its affine quadratic measurements \(|Ax + b|^2\) in (1.7). However the reconstruction of the sparse real vector \(x \in \mathbb{R}^n\) from its affine quadratic measurements is highly nonlinear and notoriously difficult to solve numerically and stably. Observe that affine quadratic measurements in (1.7) is the same as the quadratic measurements of the vector \(\tilde{x} \in \mathbb{R}^{n+1}\) via the measurement matrix \(\tilde{A} = [\tilde{a}_1, \ldots, \tilde{a}_m]^T\),
\[
|Ax + b|^2 = |\tilde{A} \tilde{x}|^2,
\]
where \( \tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \) and \( \tilde{a}_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \ 1 \leq i \leq m \). Then a conventional approach for (sparse) affine phase retrieval is to recover the sparse real vector \( \tilde{x} \) from its quadratic measurements in (1.8) by applying available iterative reconstruction algorithms in phase retrieval, such as alternating minimization [45], semidefinite programming [14, 40, 46] and Wirtinger flow approach [9, 13] with additional normalization to the last component of the reconstructed vector in each iteration. We observe that there are some space for the improvement on the performance of those conventional approaches to sparse affine phase retrieval, see Subsections 5.3–5.6. In the second part of this paper, we apply the inertial Prox-ADMM scheme and propose the CAPReaL algorithm to reconstruct sparse real signals from their (noisy) affine quadratic measurements.

Define the \( \ell_0 \) norm \( \|x\|_0 \) (resp. \( \|X\|_0 \)) of a vector \( x \) (resp. a matrix \( X \)) by the number of its nonzero entries. Set \( X = xx^T \) for a real \( s \)-sparse vector \( x \), i.e., \( \|x\|_0 \leq s \). Then \( X \) is a positive semi-definite matrix with rank at most one and its \( \ell_0 \) norm \( \|X\|_0 \) is no larger than \( s^2 \). Moreover the affine quadratic measurements of \( x \) in (1.7) are affine measurements of \( x \) and \( X \),

\[
\bar{b} = \mathcal{A}(X) + Bx + |b|^2,
\]
where \( \mathcal{A} : \mathbb{R}^{n \times n} \ni X \mapsto (\langle a_1 a_1^T, X \rangle, \ldots, \langle a_m a_m^T, X \rangle)^T \in \mathbb{R}^m \) is a linear map, \( B = 2[b_1 a_1, \ldots, b_m a_m]^T \) and \( |b|^2 = (|b_1|^2, \ldots, |b_m|^2)^T \). Therefore our recovery problem reduces to finding a real signal \( x \) with minimal \( \ell_0 \) norm and a positive semi-definite matrix \( X \) with minimal rank and \( \ell_0 \) norm,

\[
\min_{x, X \succeq 0} \|x\|_0, \text{ rank}(X) \text{ and } \|X\|_0 \text{ subject to } \mathcal{A}(X) + Bx = c \text{ and } X = xx^T,
\]

where \( c = \bar{b} - |b|^2 \). Inspired by the lifting technique [11] for phase retrieval and the convex relaxation for rank of matrices and sparsity of matrices/vectors [12, 45, 49], we consider heuristically nuclear norm convex relaxation of matrix rank and \( \ell_1 \)-norm convex relaxation of vector/matrix sparsity in (1.10). This leads to the following multi-convex relaxation to solve the compressive affine phase retrieval problem (1.10):

\[
\min_{X \succeq 0, Y \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n} \text{tr}(X) + \tau \|Y\|_1 + \lambda \|x\|_1
\]

\[
\text{subject to } \frac{1}{2} \mathcal{A}(X) + \frac{1}{2} \mathcal{A}(Y) + Bx = c, \ X - Y = O \text{ and } Y = xx^T,
\]

where \( \tau > 0 \) and \( \lambda > 0 \) are balance parameters. We call the above model (1.11) as Compressive Affine Phase Retrieval via Lifting (CAPReaL).

Denote by \( I_n : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \) the identity operator on \( \mathbb{R}^{n \times n} \). Without imposing the constraint \( Y = xx^T \) in (1.11c), the proposed CAPReaL model
becomes

\[(1.12a) \quad \min_{X \succeq 0, Y \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n} \text{tr}(X) + \tau \|Y\|_1 + \lambda \|x\|_1\]

\[(1.12b) \quad \text{subject to } \frac{1}{2}A(X) + \frac{1}{2}A(Y) + Bx = c \quad \text{and} \quad X - Y = O,\]

which is a linearly constrained separable 3-block convex optimization problem \((1.1)\) with \(x_1 = x, x_2 = X, x_3 = Y, A_1 = [B; O], A_2 = [A/2; I_n] \) and \(A_3 = [A/2; -I_n].\) In Section 4, we apply the inertial proximal ADMM to solve \((1.12)\) and then take few more steps to compensate the relaxation of the constraints \((1.11c)\). Numerical simulations in Section 5 show that the proposed algorithm has a satisfactory performance to recover sparse real vectors from their (un)corrupted affine quadratic measurements.

1.1. Contributions. The inertial proximal ADMM for solving a two-block convex optimization has been proposed and well studied \([16]\). The first contribution of this paper is to extend the inertial proximal ADMM nontrivially for solving the multi-block convex optimization problem \((1.1)\). The proposed inertial proximal ADMM unifies and greatly extends the existing twisted version of the proximal ADMM \([54]\) and the proximal parallel splitting method \([30, \text{Algorithm 3.1}]\), with additional simpler iteration scheme. The second contribution is the global convergence of the proposed inertial proximal ADMM for a multi-block convex optimization with separable objective functions, see Theorem 3.3. The third contribution is to apply the inertial proximal ADMM to recover sparse real vectors from their (un)corrupted affine quadratic measurements. The numerical simulations show that in most cases, the proposed ADMM-based algorithm has better performance in retrieving sparse signals from their (un)corrupted affine quadratic measurements than conventional ADMM-based and phase-retrieval-based approaches do.

1.2. Organization. In Section 2, we first introduce a proximal ADMM and establish a mixed variational inequality. Then based on the proposed proximal ADMM, we introduce an inertial proximal ADMM to approximate KKT points of the multi-block convex optimization \((1.1)\). In Section 3, we establish the convergence of the proposed inertial proximal ADMM for the multi-block convex optimization \((1.1)\), which extends the corresponding conclusion in \([16, 54]\) where two-block convex optimizations are considered. In Section 4, based on the proposed inertial proximal ADMM, we introduce a compressive affine phase retrieval via lifting (CAPReaL) algorithm to recover a sparse real vector from its noiseless affine quadratic measurements, and also a compressive affine phase retrieval via lifting with \(\ell^p\)-constraints (\(p\)-CAPReaL) to reconstruct a real signal approximately from its affine quadratic measurements corrupted by Gaussian/Cauchy/bounded noises. The
performance of the CAPReaL and \( p \)-CAPReaL algorithms and the comparison with some conventional affine phase retrieval algorithms are presented in Section 5.

1.3. Notation. In this paper, we use boldfaced capital and small letters to denote a matrix and a vector, and denote the zero matrix and the zero vector by \( \mathbf{O} \) and \( \mathbf{0} \) respectively. For a real number \( t \), we denote its sign and positive part by \( \text{sgn}(t) \) and \( t_+ \) respectively. For a matrix \( \mathbf{X} \) (resp. a vector \( \mathbf{x} \)), we use \( \mathbf{X}^T \) (resp. \( \mathbf{x}^T \)) to denote its transpose, and \( \|\mathbf{X}\|_p, 0 < p \leq \infty \) (resp. \( \|\mathbf{x}\|_p \)) to denote its standard \( \ell_p \) (quasi-)norm. The matrix norm \( \|\mathbf{X}\|_p \) with \( p = 2 \) is the same as the Frobenius norm, denoted by \( \|\mathbf{X}\|_F \), of the matrix \( \mathbf{X} \). We use the notion \( \mathbf{A} \succeq \mathbf{O} \) (resp., \( \mathbf{A} \succ \mathbf{O} \)) to represent that the matrix \( \mathbf{A} \) is positive semidefinite (resp. positive definite) and denote the set of all positive semidefinite (resp. positive definite) matrices of size \( n \) by \( \mathbf{S}_+^n \) (resp. \( \mathbf{S}_+^n \)). Given \( \mathbf{A} \succeq \mathbf{O} \) of size \( n \), we define \( \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{A}} := \mathbf{u}^T \mathbf{A} \mathbf{v} \) and \( \|\mathbf{u}\|_{\mathbf{A}} := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{A}}} \) for vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \). For a positive definite matrix \( \mathbf{A} \), \( \langle \cdot, \cdot \rangle_{\mathbf{A}} \) and \( \|\cdot\|_{\mathbf{A}} \) define an inner product and norm on \( \mathbb{R}^n \) respectively, which become the standard inner product \( \langle \cdot, \cdot \rangle \) and Euclidean norm \( \|\cdot\|_2 \) respectively when \( \mathbf{A} \) is the identity matrix \( \mathbf{I} \). A matrix \( \mathbf{A} \) of size \( m \times n \) is also considered as a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), and its operator norm is denoted by \( \|\mathbf{A}\|_{p \rightarrow p} = \sup_{\mathbf{x} \neq \mathbf{0}} \|\mathbf{A}\mathbf{x}\|_p / \|\mathbf{x}\|_p, 0 < p \leq \infty \). Similarly for a linear map \( \mathcal{B} : \mathbb{R}^{m_1 \times n_1} \rightarrow \mathbb{R}^{m_2 \times n_2} \), we denote \( \|\mathcal{B}\|_{F \rightarrow F} = \sup_{\mathbf{X} \neq \mathbf{0}} \|\mathcal{B}(\mathbf{X})\|_F / \|\mathbf{X}\|_F \) as the induced norm of \( \mathcal{B} \).

2. Inertial Proximal ADMM

Let \( n = \sum_{i=1}^l n_i \). We define an affine function \( F \) on \( \mathcal{W} \subset \mathbb{R}^{m+n} \) by
\[
F(\mathbf{w}) := \begin{bmatrix}
\mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{A}_1^T \\
\mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{A}_2^T \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{A}_l^T \\
\mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_l & \mathbf{O}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_l \\
z
\end{bmatrix}
- \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
c
\end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_l \\
z
\end{bmatrix} \in \mathcal{W},
\]
and we say that \( \mathbf{w} \in \mathcal{W} \) is a Karush-Kuhn-Tucker (KKT) point of the convex optimization problem \((\mathbf{1.1})\) if
\[
\mathbf{A}_j^T \mathbf{z} \in \partial f_j(\mathbf{x}_j), \ 1 \leq j \leq l, \ \text{and} \ \sum_{j=1}^l \mathbf{A}_j \mathbf{x}_j = \mathbf{c}\]
\[\mathbf{7}\]. Then the convex optimization problem \((\mathbf{1.1})\) reduces to finding KKT points \( \mathbf{w}^* \) with the mixed variational property,
\[
\theta(\mathbf{w}) - \theta(\mathbf{w}^*) + \langle \mathbf{w} - \mathbf{w}^*, F(\mathbf{w}^*) \rangle \geq 0 \ \text{for all} \ \mathbf{w} \in \mathcal{W},
\]
where \( \theta(\mathbf{w}) := \sum_{j=1}^l f_j(\mathbf{x}_j), \ \mathbf{w} \in \mathcal{W} \). So in this paper we always assume the existence of KKT points for the convex optimization problem \((\mathbf{1.1})\).
Assumption 2.1. The set of KKT points of the convex optimization problem (1.1), denoted by $\mathcal{W}^*$, is nonempty.

In this section, we introduce an inertial proximal ADMM to approximate KKT points of the convex optimization problem (1.1).

2.1. Proximal ADMM and mixed variational inequality. For the proximal ADMM (1.4), we observe that in each iteration after updating the first variable $x_1$ and the multiplier $z$, variables $x_2, \ldots, x_l$ can be updated separately, and hence subproblems for $x_2, \ldots, x_l$ can be implemented in a parallel manner. In fact, we can minimize local versions of the augmented Lagrange function $L_\beta$ to update $x_{k+1}^i, 1 \leq i \leq l$, and $z_{k+1}$ in each iteration:

\begin{align}
(2.4a) \quad x_{k+1}^1 &\in \arg \min_{x_1 \in \mathcal{X}} f_1(x_1) - \langle z^k, A_1 x_1 \rangle + \frac{\beta}{2} \left\| A_1 x_1 + \sum_{j=2}^l A_j x_j^k - c \right\|_2^2 + \frac{1}{2} \left\| x_1 - x_1^k \right\|_{H_1}, \\
\end{align}

\begin{align}
(2.4b) \quad z_{k+1} & = z^k - \beta (A_1 x_1^k + \sum_{j=2}^l A_j x_j^k - c), \\
\end{align}

\begin{align}
(2.4c) \quad x_{k+1}^i &\in \arg \min_{x_i \in \mathcal{X}} \left\{ f_i(x_i) - \langle z^{k+1}, A_i x_i \rangle + \frac{1}{2} \left\| x_i - x_i^k \right\|_{H_i}^2 + \frac{\beta}{2} \left\| A_1 x_1^k + A_i x_i + \left( \sum_{j=2}^{i-1} + \sum_{j=i+1}^l \right) A_j x_j^k - c \right\|_2^2 \right\}, 2 \leq i \leq l.
\end{align}

Define a proximal regularization matrix $G$ by

\begin{equation}
G = \begin{bmatrix}
H_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \beta A_2^T A_2 + H_2 & 0 & \cdots & 0 & -A_2^T \\
0 & 0 & \beta A_3^T A_3 + H_3 & \cdots & 0 & -A_3^T \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \beta A_l^T A_l + H_l & -A_l^T \\
0 & -A_2 & -A_3 & \cdots & -A_l & \frac{1}{\beta} I_m
\end{bmatrix},
\end{equation}

which is introduced in [16] for $l = 2$. Following the argument used in [10] and [16], we can show that the proximal ADMM algorithm is a proximal-like scheme satisfying a mixed variational inequality, which is similar to the mixed variational inequality (2.3) to be satisfied for a KKT point $x^* \in \mathcal{W}^*$.

Theorem 2.2. Let $F$, $G$, and $w^k = (x_1^k)^T, \ldots, (x_l^k)^T, (z^k)^T$, $k \geq 0$, be as in (2.1), (2.5) and (2.4) respectively. Then

\begin{equation}
\theta(w) - \theta(w^{k+1}) + \langle w - w^{k+1}, F(w^{k+1}) + G(w^{k+1} - w^k) \rangle \geq 0, \quad w \in \mathcal{W},
\end{equation}

hold for all $k \geq 0$. 

2.2. Inertial proximal ADMM. To solve the separable multi-block convex optimization problem (1.1), we introduce an inertial proximal ADMM, Prox-IADMM for abbreviation, whose convergence analysis will be discussed in the next section.

Prox-IADMM Algorithm

Input: Given $H_1, \ldots, H_l \succeq O$, penalty parameter $\beta > 0$ and step sizes $\alpha_k, k \geq 0$.

Initials: Initial step $k = 0$, and initial vectors $(x_1^0, \ldots, x_l^0; z^0) \in \mathcal{W}$ with $(x_1^{-1}, \ldots, x_l^{-1}; z^{-1}) = (x_1^0, \ldots, x_l^0; z^0)$.

Circulate Step 1–Step 3 until “a stopping criterion is satisfied”:

Step 1 (Inertial Step)

\begin{equation}
(\bar{x}_1^k, \ldots, \bar{x}_l^k; \bar{z}^k) = (x_1^k, \ldots, x_l^k; z^k) + \alpha_k (x_1^k - x_1^{k-1}, \ldots, x_l^k - x_l^{k-1}; z^k - z^{k-1}).
\end{equation}

Step 2 (Prox-ADMM)

\begin{align}
& x_1^{k+1} \in \arg \min_{x_1 \in X_1} L_\beta(x_1, x_2^k, \ldots, x_l^k; z^k) + \frac{1}{2} \|x_1 - x_1^k\|_{H_1}^2, \\
& z^{k+1} = \bar{z}^k - \beta \left( A_1 x_1^{k+1} + \sum_{j=2}^l A_j \bar{x}_j^k - c \right), \\
& x_i^{k+1} \in \arg \min_{x_i \in X_i} L_\beta(x_1^{k+1}, x_2^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_l^k; z^{k+1}) + \frac{1}{2} \|x_i - \bar{x}_i^k\|_{H_i},
\end{align}

where $i = 2, \ldots, l$.

Step 3 Update $k$ to $k + 1$.

Output: $(\bar{x}_1, \ldots, \bar{x}_l; \bar{z})$.

The above inertial proximal ADMM is introduced in [16] for $l = 2$. It extrapolates at the current point in the direction of last movement, and then applies the proximal ADMM to the extrapolated point at each iteration. Taking $H_j = O, 1 \leq j \leq l$, in (2.8), we obtain a twisted version of the proximal ADMM in [54] where step sizes $\alpha_k, k \geq 0$, are also selected to be the same in each iteration.

Let $w^k = \left( (x_1^k)^T, \ldots, (x_l^k)^T, (z^k)^T \right)^T \in \mathcal{W}, k \geq 0$, be as in the above Prox-IADMM, and set

\begin{equation}
\bar{w}^k := w^k + \alpha_k (w^k - w^{k-1}), \quad k \geq 0.
\end{equation}

Similar to the conclusion in Theorem 2.2, we have the following mixed variational inequality for $w^k$ in the Prox-IADMM algorithm (2.7) and (2.8).

Theorem 2.3. Let $F$, $G$ and $w^k = \left( (x_1^k)^T, \ldots, (x_l^k)^T, (z^k)^T \right)^T$, $k \geq 0$, be as in (2.1), (2.5) and (1.4) respectively. Then

\begin{equation}
\theta(w) - \theta(w^{k+1}) + \langle w - w^{k+1}, F(w^{k+1}) + G(w^{k+1} - \bar{w}^k) \rangle \geq 0, \quad w \in \mathcal{W}
\end{equation}

hold for all $k \geq 0$. 
Remark 2.4. The stopping criterion in the Prox-IADMM algorithm (2.7) and (2.8) should be appropriately chosen. In this paper, we will use the following stopping criterion for any given accuracy $\epsilon$,
\begin{equation}
\|x_{k+1}^k - \bar{x}_k\|_H + 2 \sum_{j=2}^l \|x_{j+1}^k - \bar{x}_j\|_{A_j^T A_j + H_j} + \frac{l}{\beta} \|z_{k+1}^k - \bar{z}_k\| \leq \epsilon,
\end{equation}
see Subsection 5.2 for numerical demonstrations. Under the above stopping criterion, one may verify that $\|w_{k+1}^k - \bar{w}_k\| \leq \epsilon$.

3. Convergence Analysis of the Inertial Proximal ADMM

In this section, we establish convergence of the Prox-IADMM (2.7) and (2.8) for multi-block convex optimizations, and we discuss (non)asymptotic rates of convergence for the best primal function value and feasibility residues. This extends the corresponding conclusions in [16, 54] where two-block convex optimizations are considered.

The convergence of the Prox-IADMM (2.7) and (2.8) depends on adaptive selection of step sizes $\alpha_k, k \geq 0$, in (2.7), see [2, Proposition 2.1], [1, Proposition 2.5] and [16, Proposition 4.5]. In this paper, we always assume the following:

Assumption 3.1. Step sizes $\alpha_k, k \geq 0$, in (2.7) are nonnegative and bounded by some $\alpha \in (0, 1)$,
\begin{equation}
0 \leq \alpha_k \leq \alpha < 1, \ k \geq 0,
\end{equation}
and satisfy
\begin{equation}
\sum_{k=1}^{\infty} \alpha_k \|w^k - w^{k-1}\|_G < \infty.
\end{equation}

Assumption 3.1 has been used in [16] Assumption 1 for the convergence of the Prox-IADMM for the two-block convex optimization problem. In practice, we may select step sizes dynamically based on historical iterative information, for instance, $\alpha_k = \min \{1/3, (k\|w^k - w^{k-1}\|_G)^{-2}\}, k \geq 0$, see Section 5.1 for numerical demonstrations. Inspired by [17, Theorem 2], we can show that the monotonic family of step sizes in the following proposition satisfies Assumption 3.1, see Section 3.6 for the proof.

Proposition 3.2. Let $w^* \in W^*$. If step sizes $\alpha_k, k \geq 0$, in (2.7) satisfy
\begin{equation}
0 \leq \alpha_k \leq \alpha_{k+1} \leq \alpha, \ k \geq 0,
\end{equation}
for some $\alpha < 1/3$, then
\begin{equation}
\sum_{k=1}^{\infty} \alpha_k \|w^k - w^{k-1}\|_G \leq \alpha \sum_{k=1}^{\infty} \|w^k - w^{k-1}\|_G \leq \frac{\alpha}{(1 - 3\alpha)(1 - \alpha)} \|w^0 - w^*\|_G.
\end{equation}
The main theoretical conclusion of this paper is the following theorem about feasibility and convergence of the Prox-IADMM scheme (2.7) and (2.8).

**Theorem 3.3.** Let $x_1^k, \ldots, x_l^k, w^k, k \geq 0$, be as in the Prox-IADMM (2.7) and (2.8), the family $\alpha_k, k \geq 0$, of step sizes satisfy Assumption 3.1 and the regularization matrices $H_j, 1 \leq j \leq l$, satisfy

\[
H_1 > 0 \quad \text{and} \quad H_j > \beta(l - 2)A_j^T A_j, \quad 2 \leq j \leq l.
\]

Then the following statements hold.

(i) The Prox-IADMM algorithm (2.7) and (2.8) is feasible,

\[
\lim_{k \to \infty} \sum_{j=1}^l A_j x_j^k = c.
\]

(ii) The objective function in the Prox-IADMM algorithm (2.7) and (2.8) converges to the optimal value,

\[
\lim_{k \to \infty} \sum_{j=1}^l f_j(x_j^k) = \min_{x_j \in X_j, 1 \leq j \leq l} \sum_{j=1}^l f_j(x_j).
\]

(iii) The sequence $w^k, k \geq 1$, in the Prox-IADMM algorithm (2.7) and (2.8) converges to a KKT point $w_* \in W^*$ of the convex optimization problem (1.1),

\[
\lim_{k \to \infty} w^k = w_*.
\]

We remark that the requirement (3.5) on regularization matrices $H_j, 1 \leq j \leq l$, are met for prox-linear matrices in (1.5) when

\[
\eta_j < (l - 1)^{-1} \|A_j\|^{-2}_{2 \to 2}, \quad 1 \leq j \leq l,
\]

and similarly for standard proximal matrices in (1.6) when

\[
\eta_j < (l - 2)^{-1} \|A_j\|^{-2}_{2 \to 2}, \quad 1 \leq j \leq l.
\]

The proximal regularization matrix $G$ in (2.5) plays an important role in our study of the convergence of the Prox-IADMM (2.7) and (2.8). In Section 3.1 we discuss its positive semi-definite assumption and the convergence of $\|w^k - w_*\|_G, k \geq 1$, for all $w_* \in W^*$. We divide the proof of Theorem 3.3 into several steps. In Sections 3.2 we consider feasibility of the Prox-IADMM (2.7) and (2.8), and we prove the first conclusion of Theorem 3.3. In Section 3.3 we discuss the convergence of objective functions and provide the proof of the second conclusion of Theorem 3.3. To prove the third conclusion of Theorem 3.3 we establish the boundedness of $w^k, k \geq 0$, in Section 3.4 first, and then in Sections 3.5 we give the proof of the third conclusion of Theorem 3.3.
3.1. Weak convergence. Let $G$ be the proximal regularization matrix in (2.5). In this paper, we always assume that proximal regularization matrix $G$ in (2.5) is positive semi-definite.

**Assumption 3.4.** The matrix $G$ in (2.5) is positive semi-definite, i.e., $G \succeq 0$.

**Remark 3.5.** The Assumption 3.4 is satisfied if $H_j, 1 \leq j \leq l$, are positive semi-definite matrices chosen appropriately. Assume that

\[(3.9) \quad H_j + \beta A^T_j A_j \succ 0, \quad 2 \leq j \leq l.\]

By (2.5) and the Schur complement [7, Section A.5.5], the positive semi-definite property $G \succeq 0$ reduces to

\[(3.10) \quad H_1 \succeq 0\]

and

\[(3.11) \quad \beta^{-1} I - \sum_{j=2}^{l} A_j (H_j + \beta A^T_j A_j)^{-1} A_j^T \succeq 0.\]

Clearly, the requirement (3.11) is met if

\[(3.12) \quad \sum_{j=2}^{l} \|A_j\|^2_{2 \rightarrow 2} \|(H_j + \beta A^T_j A_j)^{-1}\|_{2 \rightarrow 2} \leq \beta^{-1}.\]

From the above argument, we conclude that Assumption 3.4 is satisfied if (3.5) holds for regularization matrices $H_j, 1 \leq j \leq l$.

For the case that $H_j, 1 \leq j \leq l$, are prox-linear matrices in (1.5), we obtain from (3.10) and (3.12) that Assumption 3.4 is satisfied when

\[0 < \eta_1 \|A_1\|^2_{2 \rightarrow 2} \leq 1 \quad \text{and} \quad \sum_{j=2}^{l} \eta_j \|A_j\|^2_{2 \rightarrow 2} < 1.\]

We remark that the strictly positive definite property for the matrix $G$ is established in [22, Theorem 2.1] under a stronger assumption that

\[0 < \eta_j \|A_j\|^2_{2 \rightarrow 2} \leq l^{-1}, \quad 1 \leq j \leq l.\]

For standard proximal matrices $H_j, 1 \leq j \leq l$, in (1.6), we obtain from (3.10) and (3.11) that Assumption 3.4 is satisfied when

\[\sum_{j=2}^{l} \eta_j \|A_j\|^2_{2 \rightarrow 2} (1 + \eta_j \|A_j\|^2_{2 \rightarrow 2})^{-1} \leq 1.\]

We remark that the strictly positive definite property for the matrix $G$ is established in [22, Theorem 2.1] under a stronger assumption that

\[0 < \eta_j \|A_j\|^2_{2 \rightarrow 2} < 1/(l - 1), \quad 1 \leq j \leq l.\]

Motivated by [17, Theorem 1], we obtain that $\|w^k - w^*\|_G, k \geq 0$, converges for all $w^* \in \mathcal{W}^*$, in the following theorem.
Theorem 3.6. Let $w^* \in W^*$ and $w_k, k \geq 0$ be as in (2.9). If Assumption 3.4 is satisfied for the proximal regularization matrix $G$ in (2.5), then

\begin{equation}
\sum_{k=0}^{\infty} \|w^{k+1} - \bar{w}^k\|^2_G \leq \|w^0 - w^*\|^2_G + \frac{1 + \alpha}{1 - \alpha} \sum_{j=0}^{\infty} \alpha_j \|w^j - w^{j-1}\|^2_G < \infty,
\end{equation}

(3.13)

\begin{equation}
\lim_{k \to \infty} \|w^k - w^*\|^2_G \text{ exists,}
\end{equation}

(3.14)

and

\begin{equation}
\sup_{k \geq 0} \|w^k - w^*\|^2_G \leq \|w^0 - w^*\|^2_G + \frac{1 + \alpha}{1 - \alpha} \sum_{j=0}^{\infty} \alpha_j \|w^j - w^{j-1}\|^2_G.
\end{equation}

(3.15)

Proof. By (2.1), the function $F$ on $W$ satisfies

\begin{equation}
\langle w_1 - w_2, F(w_1) - F(w_2) \rangle = 0
\end{equation}

for all $w_1, w_2 \in W$. This together with the mixed variational inequalities (2.3) and (2.10) implies that

\begin{equation}
\langle w^{k+1} - w^*, w^{k+1} - \bar{w}^k \rangle_G \leq \theta(w^*) - \theta(w^{k+1}) - \langle w^{k+1} - w^*, F(w^{k+1}) \rangle
\end{equation}

(3.16)

\begin{equation}
\leq \theta(w^*) - \theta(w^{k+1}) - \langle w^{k+1} - w^*, F(w^*) \rangle \leq 0.
\end{equation}

By direct calculation, we have

\begin{equation}
2\langle w^{k+1} - w^*, w^{k+1} - w^k \rangle_G = \|w^{k+1} - w^k\|^2_G + \|w^{k+1} - w^*\|^2_G - \|w^{k+1} - w^*\|^2_G,
\end{equation}

(3.17)

\begin{equation}
2\langle w^{k+1} - w^*, w^k - w^{k-1} \rangle_G = 2\langle w^{k+1} - w^k, w^k - w^{k-1} \rangle_G + \|w^k - w^{k-1}\|^2_G + \|w^k - w^*\|^2_G - \|w^{k-1} - w^*\|^2_G,
\end{equation}

(3.18)

and

\begin{equation}
\|w^{k+1} - \bar{w}^k\|^2_G = \|w^{k+1} - w^k\|^2_G + \alpha^2 \|w^k - w^{k-1}\|^2_G - 2\alpha \langle w^{k+1} - w^k, w^k - w^{k-1} \rangle_G.
\end{equation}

(3.19)

Set $\nu_k = \|w^k - w^*\|^2_G - \|w^{k-1} - w^*\|^2_G, \ k \geq 0$. Then it follows from (3.16), (3.17), (3.18) and (3.19) that

\begin{equation}
\nu_{k+1} \leq \alpha \nu_k + (\alpha_k + \alpha^2) \|w^k - w^{k-1}\|^2_G - \|w^{k+1} - w^k\|^2_G, \ k \geq 0.
\end{equation}

(3.20)

This together with Assumption 3.1 implies that

\begin{equation}
\max(\nu_{k+1}, 0) \leq \alpha \max(\nu_k, 0) + (1 + \alpha) \alpha_k \|w^k - w^{k-1}\|^2_G
\end{equation}

(3.21)

\begin{equation}
\leq \cdots \leq (1 + \alpha) \sum_{j=0}^{k} \alpha^{k-j} \alpha_j \|w^j - w^{j-1}\|^2_G, \ k \geq 0.
\end{equation}

Therefore

\begin{equation}
\sum_{k=1}^{\infty} \max(\nu_k, 0) \leq \frac{1 + \alpha}{1 - \alpha} \sum_{j=0}^{\infty} \alpha_j \|w^j - w^{j-1}\|^2_G.
\end{equation}

(3.22)
\[ = \frac{1 + \alpha}{1 - \alpha} \sum_{j=1}^{\infty} \alpha_j \|w^j - w^{j-1}\|_G^2 < \infty, \]

where the last inequality holds by Assumption 3.1.

By (3.20), we obtain
\[
\|w^{k+1} - w^k\|_G^2 \leq -\nu_{k+1} + \alpha_k \nu_k + (\alpha_k + \alpha_k^2)\|w^k - w^{k-1}\|_G^2
\]
(3.23)
\[
\leq -\nu_{k+1} + \alpha \max(\nu_k, 0) + (1 + \alpha)\|w^k - w^{k-1}\|_G^2, \quad k \geq 0.
\]

Summing over all nonnegative \( k \geq 0 \) in the above inequality and applying (3.22) proves (3.13).

Set
\[ (3.24) \quad \gamma_k := \|w^k - w^*\|_G^2 - \sum_{j=0}^{k} \max(\nu_j, 0), \quad k \geq 0. \]

Then the sequence \( \{\gamma_k\}_{k=0}^{\infty} \) is bounded below by (3.22), and it is nonincreasing as \( \gamma_{k+1} - \gamma_k = \nu_{k+1} - \max(\nu_{k+1}, 0) \leq 0, \quad k \geq 0 \). Therefore the sequence \( \{\gamma_k\}_{k=0}^{\infty} \) converges. Hence the convergence in (3.14) follows from (3.22) and (3.24).

By (3.22) and the monotonicity of \( \gamma_k, k \geq 0 \), we have
\[
\|w^k - w^*\|_G^2 = \gamma_k + \sum_{j=0}^{k} \max(\nu_j, 0)
\]
\[
\leq \|w^0 - w^*\|_G^2 + \frac{1 + \alpha}{1 - \alpha} \sum_{j=1}^{\infty} \alpha_j \|w^j - w^{j-1}\|_G^2 < \infty.
\]

This proves (3.15). \( \square \)

**Remark 3.7.** By Theorem 3.6, we have that \( \min_{1 \leq i \leq k} \|w^{i+1} - w^i\|_G^2 = o(k^{-1/2}) \). For the case that the step sizes \( \alpha_k, k \geq 0 \) are chosen in (3.3), we can apply the argument used in the proof of Theorem 3.6 to show that
\[ (3.25) \quad \sum_{k=0}^{\infty} \|w^{k+1} - w^k\|_G^2 \leq \left(1 + \frac{\alpha(1 + \alpha)}{(1 - \alpha)^2(1 - 3\alpha)}\right) \|w^0 - w^*\|_G^2,
\]
which implies that
\[ (3.26) \quad \min_{1 \leq i \leq k} \|w^{i+1} - w^i\|_G \leq \sqrt{\left(1 + \frac{\alpha(1 + \alpha)}{(1 - \alpha)^2(1 - 3\alpha)}\right)} \|w^0 - w^*\|_G k^{-1/2}. \]

The above asymptotic/nonsymptotic convergence rates for \( \|w^{k+1} - w^k\|_G^2, \quad k \geq 1 \), are also given in [16, Theorems 4.4 and 4.6] and [17, Theorems 2, 4 and 7].
### 3.2. Feasibility of the Prox-IADMM

Set

\[
G_2 = \begin{bmatrix}
H_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & H_2 & -\beta A_T^T A_3 & \cdots & -\beta A_T^T A_l & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & -\beta(A_T^T A_l)^T & -\beta(A_T^T A_l)^T & \cdots & H_l & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 
\end{bmatrix}.
\]

(3.27)

In this section, we prove the first conclusion (3.6) of Theorem 3.3 under a weaker assumption that \( G \) and \( G_2 \) are positive semi-definite.

**Theorem 3.8.** Let \( G, G_2 \) and \( \{x^k_j\}_{k=0}^\infty, 1 \leq j \leq l \), be as in (2.5), (3.27), and the Prox-IADMM (2.7) and (2.8) respectively, and \( w^* \in W^* \). If Assumption 3.4 is satisfied, and matrices \( G_2 \) and \( H_j, 1 \leq j \leq l \), are positive semi-definite, then

\[
\sum_{k=1}^\infty \left\| \sum_{j=1}^l A_j x_j^{k+1} - c \right\|^2_2 \leq \frac{1}{\beta} \| w^0 - w^* \|^2_G + \frac{1 + \alpha}{\beta(1 - \alpha)} \sum_{k=0}^\infty \alpha_k \| w^k - w^{k-1} \|^2_G < \infty.
\]

(3.28)

Proof. By (2.8b), we have

\[
\sum_{j=1}^l A_j x_j^{k+1} - c = \sum_{j=2}^n A_j (x_j^{k+1} - \bar{x}_j^k) - \frac{1}{\beta}(z^{k+1} - \bar{z}^k).
\]

Therefore

\[
\left\| \sum_{j=1}^l A_j x_j^{k+1} - c \right\|^2_2 \leq \beta^{-2} \| z^{k+1} - \bar{z}^k \|^2_2 \\
-2\beta^{-1} \sum_{j=2}^n (z^{k+1} - \bar{z}^k)^T A_j (x_j^{k+1} - \bar{x}_j^k) \\
+ \frac{1}{\beta} \sum_{j=2}^l (x_j^{k+1} - \bar{x}_j^k)^T (H_j + \beta A_j^T A_j)(x_j^{k+1} - \bar{x}_j^k) \\
\leq \frac{1}{\beta} \| w^{k+1} - \bar{w}^k \|^2_G;
\]

(3.29)

where the first inequality follows from (3.30) and positive semidefiniteness of the matrix \( G_2 \), the second equality holds by (2.5), and the last inequality is true as \( H_1 \succeq 0 \). The above estimate together with (3.13) in Theorem 3.6 proves (3.28). \( \square \)

**Remark 3.9.** The positive semi-definite requirement for the matrix \( G_2 \) is met if \( H_j \succeq 0, 1 \leq j \leq l \), are chosen appropriately. Clearly, \( G_2 \succeq 0 \) if and
Therefore the desired limit (3.6) follows from (3.28).

\[ \square \]

Proof of the first conclusion in Theorem 3.3. By Remarks 3.5 and 3.9, the positive semi-definite requirements for \( G \) which is given in [16, Theorem 4.6].

\[(3.34)\]

\[ \min_{1 \leq i \leq k} \left\| \sum_{j=1}^{l} A_j x_j^k - c \right\|_2 \leq \left( \frac{1}{\beta} \left( \frac{\alpha(1 + \alpha)}{(1 - 3\alpha)(1 - \alpha)^2} + 1 \right) \right) \| w^0 - w^* \|_G. \]

By Corollary 3.10, we have the following nonasymptotic convergence rate for the residual of constraint \( \| \sum_{j=1}^{l} A_j x_j - c \|_2 \),

\[ \min_{1 \leq i \leq k} \left\| \sum_{j=1}^{l} A_j x_j^i - c \right\|_2 \leq \sqrt{\frac{1}{\beta} \left( \frac{\alpha(1 + \alpha)}{(1 - 3\alpha)(1 - \alpha)^2} + 1 \right) \| w^0 - w^* \|_G k^{-1/2}}, \]

which is given in [16, Theorem 4.6].

We finish this section with the proof of the conclusion (i) in Theorem 3.3.

Proof of the first conclusion in Theorem 3.3. By Remarks 3.5 and 3.9, the positive semi-definite requirements for \( G \) and \( G_2 \) in Theorem 3.8 are met. Therefore the desired limit (3.6) follows from (3.28). \[ \square \]
3.3. Convergence of objective functions. In this section, we prove the following version of the second conclusion of Theorem 3.3 under a weak version that \( \mathbf{G} \) and \( \mathbf{G}_2 \) are positive semi-definite.

**Theorem 3.11.** Let \( x_j^k, 1 \leq j \leq l, k \geq 0 \) be as the inertial proximal ADMM (2.7) and (2.8). If matrices \( \mathbf{G}, \mathbf{G}_2, \mathbf{H}_j, 1 \leq j \leq l, \) and \( \alpha_k, k \geq 0 \) be as in Theorem 3.8, then

\[
|\theta(w^*) - \theta(w^{k+1})| \leq (\|w^* - w^{k+1}\|_G + \|z^*\|)\|w^{k+1} - \bar{w}^k\|_G
\]

and

\[
\sum_{k=1}^{\infty} \left| \sum_{j=1}^{l} f_j(x_j^k) - \min_{x_j \in X^j, 1 \leq j \leq l} \sum_{j=1}^{l} A_j x_j = c} \sum_{j=1}^{l} f_j(x_j) \right|^2 < \infty.
\]

**Proof.** We follow the argument used in [16, Theorem 4.3] where \( l = 2 \). Take \( w^* = ((x_1^T)^T, \ldots, (x_l^T)^T; (z^T)^T)^T \in W^* \). Then

\[
\sum_{j=1}^{l} Ax_j^k = c
\]

and

\[
\sum_{j=1}^{l} f_j(x_j^k) = \min_{x_j \in X^j, 1 \leq j \leq l} \sum_{j=1}^{l} A_j x_j = c} \sum_{j=1}^{l} f_j(x_j).
\]

Applying (2.3) with \( w = ((x_1^{k+1})^T, \ldots, (x_l^{k+1})^T; (z^*)^T)^T \) and using (3.29) and (3.37), we obtain

\[
\theta(w^{k+1}) - \theta(w^*) \geq \left\langle \sum_{j=1}^{l} Ax_j^{k+1} - c, z^* \right\rangle
\]

\[
\geq -\left\| \sum_{j=1}^{l} Ax_j^{k+1} - c \right\| z^* \geq -\|w^{k+1} - \bar{w}^k\|_G\|z^*\|.
\]

Applying (2.6) in Theorem 2.2 with \( w \) replaced by \( w^* \), and then using (3.29), we obtain

\[
\theta(w^*) - \theta(w^{k+1}) \geq -\langle w^*-w^{k+1}, \mathbf{G}(w^{k+1} - w^k) \rangle - \langle w^*-w^{k+1}, \mathbf{F}(w^{k+1}) \rangle
\]

\[
\geq -\|w^*-w^{k+1}\|_G\|w^{k+1} - w^k\|_G - \left\| \sum_{j=1}^{l} A_j x_j^{k+1} - c \right\| z^*\|
\]

\[
\geq -\|w^*-w^{k+1}\|_G\|w^{k+1} - w^k\|_G.
\]

Combining (3.39) and (3.40) gives

\[
|\theta(w^*) - \theta(w^{k+1})| \leq (\|w^* - w^{k+1}\|_G + \|z^*\|)\|w^{k+1} - w^k\|_G.
\]

This together with Theorem 3.6 completes the proof. \( \square \)
Therefore the desired limit (3.7) follows from (3.36).

Positive semi-definite requirements for 

By Remarks 3.5 and 3.9, the proof of the second conclusion in Theorem 3.3.

\[ \text{Theorem 4.6}. \]

Tortic convergence rate about objective functions has been given in [16, Theorem 3.4].

Let matrices \( G, G_2, H_j, 1 \leq j \leq l \), and \( \alpha_k, k \geq 0 \) be as in Theorem 3.8, \( x_j^k, 1 \leq j \leq l, k \geq 0 \) be as the inertial proximal ADMM (2.7) and (2.8), and let \( \alpha_k, k \geq 0 \) be chosen in (3.3). Then

\[
\sum_{k=1}^{\infty} \left( \sum_{j=1}^{I} f_j(x_j^k) - \min_{x_j \in X, 1 \leq j \leq l} \sum_{j=1}^{I} A_j x_j = c \right) \sum_{j=1}^{I} f_j(x_j) \right)^2 \leq \frac{2\alpha}{(1 - 3\alpha)^2 (1 - \alpha)^3} \left( \|w^0 - w^*\|_G + \|z^*\|^2 \right) \|w^0 - w^*\|_G k^{-1/2}
\]

For the case that \( \alpha_k, k \geq 0 \) are chosen in (3.3), we obtain from (3.42) that

\[
\min_{1 \leq i \leq k} \left( \sum_{j=1}^{I} f_j(x_j^i) - \min_{x_j \in X, 1 \leq j \leq l} \sum_{j=1}^{I} A_j x_j = c \right) \sum_{j=1}^{I} f_j(x_j) \right) \leq \sqrt{\frac{2\alpha}{(1 - 3\alpha)^2 (1 - \alpha)^3} \left( \|w^0 - w^*\|_G + \|z^*\|^2 \right) \|w^0 - w^*\|_G k^{-1/2}}
\]

hold for all \( k \geq 1 \). We remark that the above conclusion about nonasymptotic convergence rate about objective functions has been given in [16, Theorem 4.6].

Proof of the second conclusion in Theorem 3.3. By Remarks 3.5 and 3.9, the positive semi-definite requirements for \( G \) and \( G_2 \) in Theorem 3.8 are met. Therefore the desired limit (3.7) follows from (3.36). \( \square \)

3.4. Boundedness of the Prox-IADMM. In this section, we consider the boundedness of \( w^k, k \geq 0 \), in the Prox-IADMM (2.7) and (2.8).

**Theorem 3.14.** Let matrices \( G, G_2 \) and \( H_j, 1 \leq j \leq l \), and the family \( \alpha_k, k \geq 0 \) of step sizes be as in Theorem 3.8, and let \( x_j^k, 1 \leq j \leq l \) and \( z^k, k \geq 0 \), be as the Prox-IADMM (2.7) and (2.8). Then

\[
\sum_{j=1}^{I} \|x_j^k\|_{H_j + \beta A_j^T A_j}^2 + \frac{1}{\beta} \|z^k\|_2^2 \leq 30\beta \|A_1 x_1^k\|_2^2 + 12\beta \|x_1^k\|_{H_1}^2 + 36\beta \|c\|_2^2 + 30\beta^{-1} \|z^*\|_2^2 + 100 \|w^*\|_G^2 + 142 \|w^0 - w^*\|_G^2 + \frac{142(1 + \alpha)}{1 - \alpha} \sum_{j=0}^{\infty} \alpha_j \|w^j - w^{j-1}\|_G^2 < \infty.
\]

**Proof.** By the definition (2.5) of the matrix \( G \), we have

\[
\|w^k\|_G^2 = \sum_{j=2}^{I} \|x_j^k\|_{H_j + \beta A_j^T A_j}^2 + \|x_1^k\|_{H_1}^2 + \beta \left( \sum_{j=2}^{I} A_j x_j^k - \frac{z^k}{\beta} \right)^2 \right)_2^2 - \beta \left( \sum_{j=2}^{I} A_j x_j^k \right)_2^2.
\]
Therefore
\[
\sum_{j=1}^{l} \frac{\|x_j^k\|^2}{\|H_j + \beta A_j^T A_j\|} + \frac{1}{\beta} \|z^k\|^2 \leq \|w^k\|_G^2 + \beta \|A_1 x_1^k\|^2 + \frac{1}{\beta} \sum_{j=2}^{l} \|A_j x_j^k\|_2^2 + \frac{1}{\beta} \|z^k\|^2_2 - \beta \|\sum_{j=2}^{l} A_j x_j^k - \frac{z^k}{\beta}\|^2_2
\]

\[
\leq \beta \|\sum_{j=2}^{l} A_j x_j^k - \frac{z^k}{\beta}\|^2_2 + 6(\beta \|A_1(x_1^k - x_1^*)\|^2 + \beta^{-1}\|z^k - z^*\|^2_2) + \|w^k\|^2_G
\]

(3.46) \[+6\beta \|A_1 x_1^*\|^2_2 + 6\beta^{-1}\|z^*\|^2_2,
\]

where the first inequality follows from (3.45), and the second inequality is obtained by applying the elementary inequality \((a+b)^2 \leq 2(a^2 + b^2)\).

Next we estimate \(\beta \|\sum_{j=2}^{l} A_j x_j^k - \frac{z^k}{\beta}\|^2_2\). By (3.45) and the equivalent condition (3.30) for the positive semi-definiteness of \(G_2\), we have
\[
\beta \|\sum_{j=2}^{l} A_j x_j^k - \frac{z^k}{\beta}\|^2_2 \leq \|w^k\|^2_G.
\]

Hence by (3.46) and (3.47),
\[
\sum_{j=1}^{l} \frac{\|x_j^k\|^2}{\|H_j + \beta A_j^T A_j\|} + \frac{1}{\beta} \|z^k\|^2 \leq 6(\beta \|A_1(x_1^k - x_1^*)\|^2 + \beta^{-1}\|z^k - z^*\|^2_2)
\]

(3.48) \[+2\|w^k\|^2_G + 6\beta \|A_1 x_1^*\|^2_2 + 6\beta^{-1}\|z^*\|^2_2.
\]

Let’s turn our attention to estimate \(\beta \|A_1(x_1^k - x_1^*)\|^2 + \beta^{-1}\|z^k - z^*\|^2_2\). Applying (1.4c) and (2.8a), we obtain
\[
f_1(x_1^k) - f_1(x_1^{k+1}) + (x_1^* - x_1^{k+1})^T (-A_1^T z^{k+1} + H_1(x_1^{k+1} - x_1^k)) \geq 0.
\]

Applying (2.3) with \(w\) replacing by \((x_1^{k+1}, x_2^*, \ldots, x_l^*; z^*)\), we have
\[
f_1(x_1^{k+1}) - f_1(x_1^*) - (x_1^{k+1} - x_1^*)^T A_1^T z^* \geq 0.
\]

Summing up the estimates in (3.49) and (3.50) gives
\[
(x_1^{k+1} - x_1^*)^T A_1^T (z^{k+1} - z^*) \geq (x_1^{k+1} - x_1^*)^T H_1(x_1^{k+1} - x_1^k).
\]

Therefore
\[
\beta \|A_1(x_1^k - x_1^*)\|^2 + \beta^{-1}\|z^k - z^*\|^2_2
\]
\[
\leq \|(A_1 x_1^k + \beta^{-1}z^k) - (A_1 x_1^* + \beta^{-1}z^*)\|^2_2 - 2(x_1^k - x_1^*)^T H_1(x_1^k - x_1^k) - 2\beta\|A_1 x_1^k + \beta^{-1}z^*\|^2_2 + \|x_1^k - x_1^k - x_1^k - x_1^k\|^2_2
\]
\[
= 2\beta\|A_1 x_1^k + \beta^{-1}z^k\|^2_2 + 2\|w^k\|^2_G + \|w^k - w^{k-1}\|^2_G
\]

(3.52) \[+4\beta \|A_1 x_1^*\|^2_2 + 4\beta^{-1}\|z^*\|^2_2 + 2\beta\|x_1^*\|^2_G.
\]
where the first inequality holds by \(3.51\), and the third inequality follows as \(\|x_k^i\|_{H_1} \leq \|w_k\|_G\) and \(\|x_k^i - x_{k-1}^i\|_{H_1} \leq \|w_k - w_{k-1}\|_G\) by the definition \(2.5\) of the matrix \(G\) and Assumption \(3.4\). Combining \(3.48\) and \(3.52\), we obtain

\[
\sum_{j=1}^{l} \|x_j^i\|^2_{H_j + \beta A_j^T A_j} + \frac{1}{\beta} \|z^i\|^2_2 \\
\leq 12\beta \|A_1 x_1^i + \beta^{-1} z^i\|^2_2 + 14 \|w_k\|^2_G + 6 \|w_k - w_{k-1}\|^2_G \\
+ 30\beta \|A_1 x_1^i\|^2_2 + 12\beta \|x_1^i\|^2_{H_1} + 30\beta^{-1} \|z^*\|^2_2.
\]

(3.53)

Now we estimate \(\|A_1 x_1^i + \beta^{-1} z^i\|^2_2\). By \(3.29\) and \(3.47\), we have

\[
\beta \|A_1 x_1^i + \beta^{-1} z^i\|^2_2 \leq 3\beta \left\| \sum_{j=1}^{l} A_j x_j^i - c \right\|^2_2 + 3\beta \left\| \frac{z^i}{\beta} - \sum_{j=2}^{l} A_j x_j^i \right\|^2_2 + 3\beta \|c\|^2_2
\]

(3.54)

This together with (3.53) implies that

\[
\sum_{j=1}^{l} \|x_j^i\|^2_{H_j + \beta A_j^T A_j} + \frac{1}{\beta} \|z^i\|^2_2 \leq 50 \|w_k\|^2_G + 42 \|w_k - w_{k-1}\|^2_G + 30\beta \|A_1 x_1^i\|^2_2 \\
+ 12\beta \|x_1^i\|^2_{H_1} + 30\beta^{-1} \|z^*\|^2_2 + 36\beta \|c\|^2_2.
\]

(3.55)

Finally we estimate \(\|w_k\|_G\) and \(\|w_k - w_{k-1}\|_G\). By Theorem 3.6 we obtain

\[
\|w_k\|^2_G \leq 2 \|w^*\|^2_G + 2 \|w_k - w^*\|^2_G
\]

(3.56)

\[
\|w_k - w_{k-1}\|^2_G \leq 2 \|w^*\|^2_G + 2 \|w_0 - w^*\|^2_G + \frac{2 + 2\alpha}{1 - \alpha} \sum_{j=0}^{\infty} \alpha_j \|w^j - w^{j-1}\|^2_G,
\]

and

\[
\|w_k - w_{k-1}\|^2_G \leq \left( \|w^0 - w^*\|^2_G + \frac{1 + \alpha}{1 - \alpha} \sum_{k=0}^{\infty} \alpha_k \|w^k - w^{k-1}\|^2_G \right).
\]

(3.57)

Then the desired conclusion \(3.44\) follows from \(3.55\), \(3.56\) and \(3.57\). \(\Box\)

**Remark 3.15.** For the case that step sizes \(\alpha_k, k \geq 0\), are chosen to satisfy \(3.3\), then

\[
\sum_{k=1}^{\infty} \alpha_k \|w_k - w_{k-1}\|^2_G \leq \frac{\alpha}{(1 - 3\alpha)(1 - \alpha)} \|w^0 - w^*\|^2_G.
\]

by Proposition 3.2. This together with \(3.44\) leads to the following estimate

\[
\sum_{j=1}^{l} \|x_j^i\|^2_{H_j + \beta A_j^T A_j} + \frac{1}{\beta} \|z^i\|^2_2 \leq 30\beta \|A_1 x_1^i\|^2_2 + 12\beta \|x_1^i\|^2_{H_1} + 36\beta \|c\|^2_2
\]
0, is bounded and hence it has limit points.

Proof of Theorem 3.16. By (3.59) and Theorem 3.14, the sequence and Remarks 3.5 and 3.9. Then it remains to prove Theorem 3.16.

Let matrices \( G, G_2 \), and \( \alpha, k \geq 0 \) be as in Theorem 3.8 and let \( w^k, k \geq 0 \) be as the inertial proximal ADMM (2.7) and (2.8). If \( H_j, 1 \leq j \leq l \), are positive semi-definite and satisfy (3.59), then there exists a unique \( w^* \in W^* \) such that \( \lim_{k \to \infty} w^k = w^* \).

The third conclusion in Theorem 3.3 follows easily from Theorem 3.16 and Remarks 3.5 and 3.9. Then it remains to prove Theorem 3.16.

Proof of Theorem 3.16. By (3.59) and Theorem 3.14 the sequence \( w^k, k \geq 0 \), is bounded and hence it has limit points.

Take a limit point \( w^* \) of the sequence \( w^k, k \geq 0 \). As the sequence is contained in \( W \) and the set \( W \) is closed, we have that \( w^* \in W \). Let \( w_{j_1}, j_1 \geq 1 \) be a convergent subsequence which has limit \( w^* \). Taking the limit over \( k = k_j \) in (2.10) and applying the observation that \( \lim_{k \to \infty} \| w^k - w^{k-1} \|_G = 0 \), we obtain

\[
\theta(w) - \theta(w^*) + \langle w - w^*, F(w^*) \rangle \geq 0, \quad w \in W.
\]

This implies that \( w^* \in W^* \) and hence any limit point of the sequence \( w^k, k \geq 0 \) lie in \( W^* \).

Now we prove the uniqueness of the limit points. Let \( w_1^* \) and \( w_2^* \) be two limits points of the sequence \( w^k, k \geq 0 \). This together with the observation that

\[
\| w^k - w_1^* \|^2_G - \| w^k - w_2^* \|^2_G = \| w_1^* - w_2^* \|^2_G + 2\langle w_2^* - w_1^*, w^k - w_1^* \rangle_G,
\]

implies that the sequence \( \| w^k - w_1^* \|^2_G - \| w^k - w_2^* \|^2_G, k \geq 0 \) has two limit points \( \pm \| w_1^* - w_2^* \|^2_G \). On the other hand, it follows from Theorem 3.6 that the sequence \( \| w^k - w_1^* \|^2_G - \| w^k - w_2^* \|^2_G, k \geq 0 \) is convergent. Therefore two limit points \( w_1^* \) and \( w_2^* \) of the sequence \( w^k, k \geq 0 \) satisfy

\[
\| w_1^* - w_2^* \|^2_G = 0.
\]

This together with Assumption 3.4 on \( G \) implies that

\[
G(w_1^* - w_2^*) = 0.
\]
Write \( w^*_t = ((x^*_1)^T, \ldots, (x^*_t)^T ; (z^*_t)^T) \), \( t = 1, 2 \). Then it follows from (3.60) that
\[
H_1(x^*_1, x^*_2) = 0
\]
(3.61)
\[
\begin{cases}
H_j + \beta A_j A_j^T(x^*_j - x^*_j) - A_j^T(z^*_j - z^*_2) = 0, \quad j = 2, \ldots, l \\
- \sum_{j=2}^l A_j (x^*_j - x^*_j) + (z^*_j - z^*_2) = 0.
\end{cases}
\]

By \( w^* \in W^* \), we have that \( \sum_{j=1}^l A_j (x^*_j - x^*_j) = c - c = 0 \). This together with the third equality in (3.61) implies that
\[
A_1(x^*_1, x^*_2) + (z^*_1 - z^*_2) = 0.
\]
On the other hand, applying the mixed variational property, (2.3) with \( w^* \) replaced by \( w^*_1 \) and \( w^*_2 \) respectively, we obtain that \( f_1(x^*_1, x^*_2) + \langle x^*_1, x^*_2, -A^T_1 z^*_1 \rangle \geq 0 \) and \( f_1(x^*_1, x^*_2) + \langle x^*_1, x^*_2, -A^T_1 z^*_1 \rangle \geq 0 \). Taking the sum of the above two inequalities gives
\[
\langle A_1(x^*_1, x^*_2), z^*_1 - z^*_2 \rangle \geq 0.
\]
Combining (3.62) and (3.63) proves that
\[
A_1 x^*_1 + A_1 x^*_2 = z^*_1 + z^*_2.
\]
By (3.64) and the first two equations in (3.61), we have that \( (H_j + \beta A_j A_j^T)(x^*_j - x^*_j) = 0 \) for all \( 1 \leq j \leq l \). This together with (3.59) implies that
\[
x^*_j = x^*_j, \quad 1 \leq j \leq l.
\]
Combining (3.64) and (3.65) proves that \( w^*_t = w^*_t \). This completes the proof on uniqueness of the limit points of the sequence \( w^k, k \geq 0 \).

3.6. Proof of Proposition 3.2. Our proof is inspired by [17, Theorem 2]. The first inequality in (3.4) follows from (3.3). Therefore it suffices to prove
\[
\sum_{k=1}^\infty \| w^k - w^{k-1} \|_G^2 \leq \frac{\| w^0 - w^* \|_G^2}{(1-3\alpha)(1-\alpha)}.
\]
Take \( w^* \in W^* \). Recall that
\[
\hat{w}^k = w^k + \alpha_k (w^k - w^{k-1}), \quad k \geq 0.
\]
This together with (3.16), (3.17) and (3.18) implies that
\[
\begin{align*}
(3.67) \| w^{k+1} - w^* \|^2_G & = (1 + \alpha_k) \| w^k - w^* \|^2_G + \alpha_k \| w^{k-1} - w^* \|^2_G \\
& = \langle w^{k+1} - w^*, w^{k+1} - w^k \rangle_G + \| w^{k+1} - w^k \|^2_G + \alpha_k \| w^{k-1} - w^* \|^2_G \\
& \quad + 2\alpha_k \langle w^{k+1} - w^k, w^{k-1} - w^k \rangle_G \\
& \leq -\| w^{k+1} - w^k \|^2_G + \alpha_k \| w^{k-1} - w^* \|^2_G \\
& \quad + 2\alpha_k \langle w^{k+1} - w^k, w^{k-1} - w^k \rangle_G \\
& \leq -(1-\alpha_k) \| w^{k+1} - w^k \|^2_G + 2\alpha_k \| w^{k-1} - w^* \|^2_G.
\end{align*}
\]
Set \( \mu_k := \| w^k - w^* \|^2_G - \alpha_k \| w^{k-1} - w^* \|^2_G + 2\alpha_k \| w^{k-1} - w^* \|^2_G, \quad k \geq 0 \).
Then it follows from (3.67) and the assumption $0 \leq \alpha_k \leq \alpha_{k+1} \leq \alpha < \frac{1}{3}$ that
\[
\mu_{k+1} - \mu_k \leq -(1 - \alpha_k - 2\alpha_{k+1})\|w^{k+1} - w^k\|_G^2 + (\alpha_k - \alpha_{k+1})\|w^k - w^*\|_G^2 \leq -3\alpha\|w^{k+1} - w^k\|_G^2 \leq 0, \quad k \geq 0,
\]
which implies that $\mu_k, k \geq 0$ is an nonincreasing sequence bounded above by
\[
\mu_0 \leq (1 - \alpha_0)\|w^0 - w^*\|_G^2 \leq \|w^0 - w^*\|_G^2.
\]
Therefore
\[
\|w^k - w^*\|_G^2 - \alpha\|w^{k-1} - w^*\|_G^2 \leq \mu_k \leq \|w^0 - w^*\|_G^2, \quad k \geq 0.
\]
Applying the above upper estimate repeatedly gives
\[
\|w^k - w^*\|_G^2 \leq \|w^0 - w^*\|_G^2 + \alpha\|w^{k-1} - w^*\|_G^2 \leq \ldots \leq \sum_{j=0}^{k-1} \alpha^j\|w^0 - w^*\|_G^2 + \alpha^k\|w^0 - w^*\|_G^2 \leq \frac{\|w^0 - w^*\|_G^2}{1 - \alpha}.
\]

By (3.68), we have
\[
(1 - 3\alpha)\|w^{k+1} - w^k\|_G^2 \leq \mu_k - \mu_{k+1}, \quad k \geq 0.
\]
Taking sum over $k$ on above inequality and applying (3.69), we obtain
\[
(1 - 3\alpha)\sum_{j=0}^{k} \|w^{j+1} - w^j\|_G^2 \leq \mu_0 - \mu_{k+1}
\]
\[
\leq \|w^0 - w^*\|_G^2 + \alpha_{k+1}\|w^k - w^*\|_G^2 \leq \frac{1}{1 - \alpha}\|w^0 - w^*\|_G^2.
\]
This proves (3.66) and completes the proof.

4. Inertial Proximal ADMM and Compressive Affine Phase Retrieval

The problem to reconstruct of a (sparse) real signal $x$ from its affine quadratic measurements (1.9) is highly nonlinear. Based on the Prox-IADMM for separable multi-block convex optimizations, we propose a compressive affine phase retrieval via lifting (CAPReaL) approach (1.11) for the affine phase retrieval problem in Section 4.1. The affine quadratic measurements (1.9) could be corrupted in practice. In Section 4.2, we propose compressive affine phase retrieval via lifting with $\ell^p$-constraint ($p$-CAPReaL) to reconstruct a real signal approximately from its corrupted affine quadratic measurements. The demonstration of our proposed algorithms to recover sparse signals stably from their (un)corrupted affine quadratic measurements will be presented in Section 5.
4.1. **Compressive affine phase retrieval via lifting.** Define the soft thresholding operator $S(x, r), r \geq 0,$ for $x = (x_1, \ldots, x_n)^T$ by
\begin{align}
S(x, r) = (\text{sgn}(x_1)(|x_1| - r)_+, \ldots, \text{sgn}(x_n)(|x_n| - r)_+)^T,
\end{align}
and denote the projection onto the positive semi-definite cone $S^n_+$ by $P_+ : S^n \rightarrow S^n_+$. For the case that $X$ has the eigenvalue decomposition $X = U\Lambda U^T$, then $P_+(X) = U\Lambda_+ U^T$, where $U$ is an orthogonal matrix, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix and $\Lambda_+ = \text{diag}((\lambda_1)_+, \ldots, (\lambda_n)_+)$. Observe that the CAPReaL model \([1, 12]\) is a linearly constrained separable 3-block convex optimization problem \([1, 1]\) with $x_i$ and $A_i, i = 1, 2, 3$ given by $x_1 = x$, $x_2 = X$, $x_3 = Y$, and
\begin{align}
A_1 = \begin{bmatrix} B \\ O \end{bmatrix}, 
A_2 = \begin{bmatrix} A/2 \\ \mathcal{I}_n \end{bmatrix}, 
A_3 = \begin{bmatrix} A/2 \\ -\mathcal{I}_n \end{bmatrix}.
\end{align}
Therefore taking
\begin{align}
H_1 = \frac{\beta}{\eta_1} I_n - \beta B^T B 
H_i = \frac{\beta}{\eta_i} \mathcal{I}_n - \frac{\beta}{4} (A^* A + 4 \mathcal{I}_n) \text{ for } i = 2, 3,
\end{align}
with
\begin{align}
0 < \eta_i < (\|B^T B\|_2 → 2)^{-1} \text{ and } 0 < \eta_2, \eta_3 < 2(\|A^* A + 4 \mathcal{I}_n\|_{F → F})^{-1},
\end{align}
we obtain the following concrete form of the corresponding Prox-IADMM algorithm, where $A^* : \mathbb{R}^m \ni c = (c_1, \ldots, c_m)^T \mapsto \sum_{j=1}^m c_j a_j^T a_j \in \mathbb{R}^{n \times n}$ is the adjoint operator of $A$, and $\mathcal{I}_n^* : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is the adjoint operator of $\mathcal{I}_n$.

**CAPReaL Algorithm**

**Input:** Given $(x^0, X^0, Y^0; z^0, Z^0)$, $\tau$, $\lambda$, $\beta > 0$, parameters $\eta_i, 1 \leq i \leq 3$, satisfying \([1, 3]\), and step sizes $\alpha_k, k \geq 0$.

**Initials:** Let $(x^{-1}, X^{-1}, Y^{-1}; z^{-1}, Z^{-1}) = (x^0, X^0, Y^0; z^0, Z^0)$ and $k = 0$.

**Circulate** Step 1–Step 6 until “some stopping criterion is satisfied”:

**Step 1** Iterate as
\begin{align}
(x^k, X^k, Y^k, z^k, Z^k) = (X^k, Y^k, x^k, z^k, Z^k) + \alpha_k (x^k - x^{k-1}, X^k - X^{k-1}, Y^k - Y^{k-1}; z^k - z^{k-1}, Z^k - Z^{k-1}),
\end{align}

**Step 2** Compute $x^{k+1}$ by
\begin{align}
x^{k+1} = S(x^k - \eta_1 B^T \left( \frac{1}{2} A(X^k) + \frac{1}{2} A(Y^k) + Bx^k - c - \frac{\bar{z}^k}{\beta} \right), \frac{\lambda \eta_1}{\beta}).
\end{align}

**Step 3** Update multiplier $z^{k+1}, Z^{k+1}$ via
\begin{align}
z^{k+1} &= \bar{z}^k - \beta \left( \frac{1}{2} A(X^k) + \frac{1}{2} A(Y^k) + Bx^{k+1} - c \right), \\
Z^{k+1} &= \bar{Z}^k - \beta (X^k - Y^k).
\end{align}
Step 4 Compute $X^{k+1}$ by

$$X^{k+1} = \mathcal{P}_S \left( \bar{X}^k - \frac{\eta_2}{\beta} I_n - \frac{\eta_2}{2} A^* \left( \frac{1}{2} A(\bar{X}^k) + \frac{1}{2} A(\bar{Y}^k) + Bx^k + c - \frac{z^{k+1}}{\beta} \right) \right)$$

$$- \eta_2 \left( \bar{X}^k - \bar{Y}^k - \frac{Z^{k+1}}{\beta} \right).$$

Step 5 Compute $Y^{k+1}$ by

$$Y^{k+1} = S \left( \bar{Y}^k - \frac{\eta_3}{2} A^* \left( \frac{1}{2} A(\bar{X}^k) + \frac{1}{2} A(\bar{Y}^k) + Bx^k + c - \frac{1}{\beta} z^{k+1} \right) \right)$$

$$+ \eta_3 \left( \bar{X}^k - \bar{Y}^k - \frac{Z^{k+1}}{\beta} \right), \frac{\tau \eta_2}{\beta}.$$ 

Step 6 Update $k$ to $k + 1$.

Output: $(\hat{x}, \hat{X}, \hat{Y})$.

By (4.2), (4.3) and Theorem 3.3, the above CAPReaL algorithm converges. However, the solution $(\hat{X}, \hat{Y}, \hat{x})$ of the above algorithm may not satisfy the constrained condition $\hat{Y} = \hat{x}\hat{x}^H$. In order to compensate for these relaxations, we take two additional steps:

(a) In addition to the stopping criterion (2.11) to the Prox-IADMM, we select the additional stopping criteria,

$$\|Y^k - x^k(x^k)^T\|_F \leq \tilde{\varepsilon}$$

in the implementation of the CAPReaL algorithm.

(b) Add the following steps after the implementation of the CAPReaL algorithm.

(b1) Find the best rank-one approximation $\hat{X}_{\text{rank}(1)} = \sigma_1 u_1 u_1^H$ of $\hat{X}$, and take $\hat{x} = \alpha \sqrt{\sigma_1} u_1$, where $\sigma_1$ is the maximal singular value of the matrix $\hat{X}$ and the sign $\alpha = \pm 1$ is chosen so that $\langle \hat{x}, \hat{x} \rangle \geq 0$.

(b2) Find the best $s^2$-sparse approximation $\hat{Y}_{\text{max}(s^2)}$ of $\hat{Y}$ in the norm $\|\cdot\|_1$, and compute the full rank decomposition of $\hat{Y}_{\text{max}(s^2)} = UV^T$, and then take $\bar{y} = \tilde{\alpha} \bar{u}_1$, where $u_1$ is the first column of $U$ and the sign $\tilde{\alpha} = \pm 1$ is chosen so that $\langle \bar{y}, \hat{x} \rangle \geq 0$.

(b3) Compute $x^* = (\hat{x} + \bar{x} + \bar{y})/3$.

4.2. Compressive affine phase retrieval via lifting with penalty. In this section, we consider compressive affine phase retrieval problem with corrupted measurements,

$$\bar{b} = (|\langle a_1, x \rangle + b_1|^2, \ldots, |\langle a_m, x \rangle + b_m|^2)^T + b$$

$$= A(xx^H) + Bx + |b|^2 + e,$$
where $e = (e_1, \ldots, e_m)^T \in \mathbb{R}^m$ is the noise. Similar to the bi-convex relaxation model (1.11), we propose the following approach:

\[
\min_{x \geq 0, Y \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n, y \in \mathbb{R}^m} \operatorname{tr}(X) + \tau \|Y\|_1 + \lambda \|x\|_1 + \rho \|y\|_p^p,
\]

subject to

\[
\frac{1}{2}A(x) + \frac{1}{2}A(Y) + Bx - y = c, \quad X - Y = O,
\]

where $c = b - |b|^2$, $\tau, \lambda, \rho > 0$ are balance parameters, and $h_p(y) = \begin{cases} \|y\|_p^p & \text{if } 0 < p < \infty \\ \|y\|_\infty & \text{if } p = \infty. \end{cases}$

We call the above approach as the Compressive Affine Phase Retrieval via Lifting with $p$-Constraint, and use the abbreviation $p$-CAPReaL. Holding the constraint in (4.6c) about $Y$, the approach in (4.6) becomes a separable 4-block convex optimization problem (1.1) with linearly constraint, where $x_1 = y$, $x_2 = X$, $x_3 = Y$, $x_4 = x$, and

\[
A_1 = \begin{bmatrix} -I_m & 0 \\ 0 & I_n \end{bmatrix}, \quad A_2 = \begin{bmatrix} A/2 & 0 \\ 0 & I_n \end{bmatrix}, \quad A_3 = \begin{bmatrix} A/2 \\ -I_n \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} B \\ 0 \end{bmatrix}.
\]

Thus we can use the Prox-IADMM to solve the above separable 4-block convex optimization problem with the regularization matrices (4.7)

\[
H_1 = \frac{\beta}{\eta_1}I_m - \beta I_m, \quad H_4 = \frac{\beta}{\eta_4}I_n - \beta B^T B, \quad \text{and} \quad H_i = \frac{\beta}{\eta_i}I_n - \frac{\beta}{4}A^* A \quad \text{for } i = 2, 3,
\]

where $\eta_i > 0, 1 \leq i \leq 4$, satisfy

\[
0 < \eta_1 < 1, \quad 0 < \eta_2, \eta_3 < \frac{4}{3\|A^* A + 4I_n\|_{F \rightarrow F}} \quad \text{and} \quad 0 < \eta_4 < \frac{1}{3\|B^T B\|_{2 \rightarrow 2}}.
\]

For the above selection of regularization matrices, the Prox-IADMM for the special cases that $p = 1, 2, \infty$ has the following concise formulation, where $S^*$ is the proximal operator of $\ell_\infty$ norm [21, 47, 57],

\[
S^*(b, \lambda) = \operatorname{Prox}_{\lambda\|\|_\infty}(b) := \arg \min_{x} \frac{1}{2}\|x - b\|_2^2 + \lambda\|x\|_\infty.
\]

For $p = 1, 2, \infty$, the $p$-CAPReaL scheme can be formulated as follows.

---

$p$-CAPReaL Algorithm

**Input:** $(y^0, X^0, Y^0, x^0, z^0, Z^0)$, $\tau > 0$, $\lambda > 0$, $\rho > 0$, $\beta > 0$, nonnegative step sizes $\alpha_k, k \geq 0$, and parameters $\eta_1, \eta_2, \eta_3, \eta_4$ in (4.8).

**Initials:** Set $(y^{-1}, X^{-1}, Y^{-1}, x^{-1}, z^{-1}) = (y^0, X^0, Y^0, x^0, z^0, Z^0)$ and $k = 0$.

**Circulate** Step 1-Step 8 until “some stopping criterion is satisfied”:

**Step 1** Iterate as

\[
(y^k, \bar{X}^k, \bar{Y}^k, \bar{x}^k, \bar{z}^k, \bar{Z}^k) = (y^k, X^k, Y^k, x^k, z^k, Z^k) + \alpha_k(y^k - y^{k-1}),
\]

where $\alpha_k$ is a constant parameter.
\[ X^k - X^{k-1}, Y^k - Y^{k-1}, x^k - x^{k-1}, z^k - z^{k-1}, Z^k - Z^{k-1}. \]

**Step 2** Compute \( y^{k+1} \) by

\[
y^{k+1} = S\left( \bar{y}^k + \eta_1 \left( \frac{1}{2} \mathcal{A}(\bar{x}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) + B\bar{x}^k - \bar{y}^k - c - \frac{\bar{z}^k}{\beta} \right), \frac{\rho \eta_1}{\beta} \right)
\]

if \( p = 1 \), and

\[
y^{k+1} = \frac{\beta}{\beta + 2 \rho \eta_1} \bar{y}^k + \frac{\beta \eta_1}{\beta + 2 \rho \eta_1} \left( \frac{1}{2} \mathcal{A}(\bar{X}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) + B\bar{x}^k - \bar{y}^k - c - \frac{\bar{z}^k}{\beta} \right)
\]

if \( p = 2 \), and

\[
y^{k+1} = S^s\left( \bar{y}^k + \eta_1 \left( \frac{1}{2} \mathcal{A}(\bar{x}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) + B\bar{x}^k - \bar{y}^k - c - \frac{\bar{z}^k}{\beta} \right), \frac{\rho \eta_1}{\beta} \right)
\]

if \( p = \infty \).

**Step 3** Update multipliers \( z^{k+1} \) and \( Z^{k+1} \) via

\[
z^{k+1} = z^k - \beta \left( \frac{1}{2} \mathcal{A}(\bar{X}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) + B\bar{x}^k - y^{k+1} - c \right),
\]

\[
Z^{k+1} = Z^k - \beta (\bar{X}^k - \bar{Y}^k).
\]

**Step 4** Compute \( X^{k+1} \) by

\[
X^{k+1} = \mathcal{P}_\Sigma \left( \bar{X}^k - \frac{\eta_2}{\beta} I_n - \frac{\eta_2}{\beta} A^* \left( \frac{1}{2} \mathcal{A}(\bar{X}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) \right) + B\bar{x}^k - y^{k+1} - c - \frac{z^{k+1}}{\beta} \right) - \eta_2 \left( \bar{X}^k - \bar{Y}^k - \frac{Z^{k+1}}{\beta} \right).
\]

**Step 5** Compute \( Y^{k+1} \) by

\[
Y^{k+1} = S \left( \left( \bar{Y}^k - \frac{\eta_3}{\beta} A^* \left( \frac{1}{2} \mathcal{A}(\bar{X}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) \right) + B\bar{x}^k - y^{k+1} - c - \frac{z^{k+1}}{\beta} \right) + \eta_3 \left( \bar{X}^k - \bar{Y}^k - \frac{Z^{k+1}}{\beta} \right), \frac{\tau \eta_3}{\beta} \right).
\]

**Step 6** Compute \( x^{k+1} \) by

\[
x^{k+1} = S \left( \bar{x}^k - \eta_4 B^T \left( \frac{1}{2} \mathcal{A}(\bar{X}^k) + \frac{1}{2} \mathcal{A}(\bar{Y}^k) + B\bar{x}^k - y^{k+1} - c - \frac{z^{k+1}}{\beta} \right), \frac{\eta_4 \lambda}{\beta} \right).
\]

**Step 7** Update \( k \) to \( k + 1 \).

**Output:** \( (\bar{y}, \bar{X}, \bar{Y}, \bar{x}) \).

By \((4.2), (4.3)\) and Theorem 3.3 the \( p \)-CAPReA algorithm converges for \( 1 \leq p \leq \infty \). However, the solution \( \bar{Y}, \bar{x}, \bar{y} \) of the above algorithm may not
satisfy the constrained condition $\hat{Y} = \hat{x}\hat{x}^T$. In order to compensate for that relaxation, we take same additional steps as those in Subsection 4.1.

5. Numerical Simulations

In this section, we demonstrate performance of the proposed $(p)$-CAPReaL algorithm to recover $s$-sparse real vectors $\mathbf{x}_o \in \mathbb{R}^n$ from either the noiseless quadratic measurement $\mathbf{c} = |\mathbf{A}\mathbf{x}_o + \mathbf{b}|^2$ or the noisy quadratic measurement $\mathbf{c} = |\mathbf{A}\mathbf{x}_o + \mathbf{b}|^2 + \mathbf{e}$, and compare it with the conventional phase retrieval algorithms [9, 39, 40, 46]. In our simulations, the measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the real standard Gaussian matrix of size $m \times n$, the true $s$-sparse signal $\mathbf{x}_o \in \mathbb{R}^n$ has each nonzero components randomly i.i.d. drawn according to the continuous uniform distribution $U(-1, 1)$ on $[-1, 1]$, and the reference vector $\mathbf{b} \in \mathbb{R}^m$ has its components $b_j = \xi_j y_j, j = 1, \ldots, m$, with $\xi_j$ and $y_j$ randomly i.i.d. drawn according to the continuous uniform distribution $U(-1, 1)$ and standard normal distribution $N(0, 1)$ respectively [8, 57]. In our simulations, we consider a Gaussian white noise $\mathbf{e}$ with variance $\sigma^2 > 0$, i.e., $\mathbf{e} \sim \sigma \mathcal{N}(0, \mathbf{I}_m)$, or a Cauchy noise $\mathbf{e} \sim \mathcal{C}(0, \gamma)$ with its probability density function given by $(\pi \gamma (1 + |x/\gamma|^2))^{-1}$, where $\gamma$ is the scale parameter to specify the noise half-width at half-maximum [50, 52, 53], and we also test for a uniformly distributed noise $\mathbf{e} \sim \mathcal{U}(-\delta, \delta)$ (i.e., the uniform distribution on the interval $(-\delta, \delta)$), where $\delta > 0$ is a noise range parameter [8, 57]. All experiments were performed under Windows Vista Premium and MATLAB v7.8 (R2016b) running on a Huawei laptop with an Intel(R) Core(TM)i5-8250U CPU at 1.8 GHz and 8195MB RAM of memory.

5.1. CAPReaL algorithm with different selection of step sizes. In this subsection, we demonstrate the performance of the CAPReaL algorithm with different selection of step sizes to recover sparse signals from their affine quadratic affine measurements. Shown in Table 1 are average success percentages of the CAPReaL algorithm for different selection of step sizes over 100 independent realizations to recover sparse signals from their noiseless quadratic measurements of size $m$, where the original sparsity signal $\mathbf{x}_o$ has sparsity $s = 4$ and length $n = 64$, and step sizes $\alpha_k = 1/8, 1/4, 1/3, 1/2$ are independent of $k$ for the first three simulations and $\alpha_k = 1/3 - (1/3)^{\lfloor k/5 \rfloor}$ and $(1/3)^{\lfloor k/5 \rfloor}, k \geq 0$ for the last two simulations. Here $\lfloor a \rfloor$ denotes the nearest integer less than or equal to $a$. In the simulation, the recovery is regarded as successful if $\|\mathbf{x}^* - \mathbf{x}_o\|_2/\|\mathbf{x}_o\|_2 \leq 0.01$, where $\mathbf{x}^*$ is the reconstructed signal via the CAPReaL algorithm. This indicates that step sizes in the CAPReaL algorithm should be chosen appropriately and the CAPReaL algorithm with step size $\alpha_k = 1/4$ for all $k \geq 0$ has highest success percentage to recover sparse signals from their phaseless affine measurements. Due to the above observation, in the following simulations, we always choose $\alpha_k = 1/4, k \geq 0$, as step sizes in the CAPReaL algorithm and also in the $p$-CAPReaL algorithm.
Table 1. Success percentage of the CAPReaL algorithm to recover sparse real signals over 100 repeated trials for different ratios $m/n$ between the number $m$ of measurements and the length of original signals, and for different selections of step sizes $\alpha_k$, $k \geq 0$.

| $\alpha_k$ | $m/n$ | 0.5 | 0.75 | 0.875 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|------------|-------|-----|------|--------|---|------|-----|------|---|
| 1/8        | 1/8   | 1   | 11   | 30     | 36| 82   | 98  | 100  | 100|
| 1/4        | 1/4   | 2   | 19   | 40     | 61| 95   | 98  | 100  | 100|
| 1/3        | 1/3   | 0   | 6    | 38     | 56| 88   | 96  | 100  | 100|
| 1/2        | 1/2   | 0   | 0    | 0      | 0 | 0    | 0   | 0    | 0  |
| $1/3 - 3^{-[k/5]}$ | 1/3 - 3^{-[k/5]} | 1   | 16   | 31     | 53| 86   | 97  | 100  | 100|
| $3^{-[k/5]}$ | 3^{-[k/5]} | 1   | 15   | 27     | 45| 90   | 96  | 100  | 100|

5.2. Comparison between CAPReaL and Jacobian/twisted ADMM-based algorithms. The proposed CAPReaL algorithm to recover sparse real vectors from their affine quadratic measurements is based on the Prox-IADMM. In our simulations, we always select step sizes $\alpha_k = 1/4, k \geq 0$, in the CAPReaL algorithm, see Subsection 5.1. As the Prox-IADMM (1.4) with zero step sizes becomes the classical ADMM (1.2), we may use the corresponding CAPReal algorithm based on the classical ADMM, CAPReaL-Zero for abbreviation, to solve (1.12). Based on the Jacobi-Proximal ADMM [22], we propose the following iterative algorithm, CAPReaL-Jacobi for abbreviation, to solve (1.12), where $\eta_1, \eta_2, \eta_3$ are proximal parameters, each iteration is modified from the proximal Jacobian ADMM [22],

$$
\begin{align*}
\mathbf{x}^{k+1} &= S\left(\mathbf{x}^k - \eta_1 \mathbf{B}^T \left(\frac{1}{2} \mathbf{A}(\mathbf{x}^k) + \frac{1}{2} \mathbf{A}(\mathbf{y}^k) + \mathbf{b}x^k - c - \frac{z^k}{\beta}\right), \lambda \eta_1\right) , \\
\mathbf{X}^{k+1} &= \mathcal{P}_\varepsilon \left(\mathbf{x}^k - \frac{\eta_2}{\beta} \mathbf{I}_n - \frac{\eta_2}{2} \mathbf{A}\left(\frac{1}{2} \mathbf{A}(\mathbf{x}^k) + \frac{1}{2} \mathbf{A}(\mathbf{y}^k) + \mathbf{b}x^{k+1} - c + \frac{z^k}{\beta}\right) \right. \\
&\quad - \left. \eta_2 \left(\mathbf{x}^k - \mathbf{y}^k - \frac{z^k}{\beta}\right) \right) , \\
\mathbf{Y}^{k+1} &= S\left(\mathbf{y}^k - \frac{\eta_3}{2} \mathbf{A}^\ast \left(\frac{1}{2} \mathbf{A}(\mathbf{x}^k) + \frac{1}{2} \mathbf{A}(\mathbf{y}^k) + \mathbf{b}x^{k+1} - c - \frac{1}{\beta}z^{k+1}\right) \right. \\
&\quad + \left. \eta_3 \left(\mathbf{x}^k - \mathbf{y}^k - \frac{z^{k+1}}{\beta}\right), \tau \eta_2\right) , \\
\mathbf{z}^{k+1} &= \mathbf{z}^k - \beta \left(\frac{1}{2} \mathbf{A}(\mathbf{x}^{k+1}) + \frac{1}{2} \mathbf{A}(\mathbf{y}^{k+1}) + \mathbf{b}x^{k+1} - \mathbf{y}^{k+1} - c\right) , \\
\mathbf{Z}^{k+1} &= \mathbf{z}^k - \beta \left(\mathbf{x}^{k+1} - \mathbf{y}^k\right) ,
\end{align*}
$$

and the compensation step is the same as the one in Subsection 4.1 being used to design the CAPReaL algorithm. Similarly, based on twisted version
of the proximal ADMM \cite{54} and following the same compensation step as the one in the CAPReal algorithm, we propose the following iterative algorithm, CAPReaL-Twisted for abbreviation, to solve \eqref{1.12}, where $\alpha \in (0, 2)$, $0 < \eta_2 < (2\|B^TB\|_{2+\alpha})^{-1}$, $0 < \eta_3 < 2(\|A^*A + 4I_n\|_{F \to F})^{-1}$ are proximal parameters, and each iteration is essentially the proximal twisted ADMM \cite{54},

$$
\hat{X}^k = P_{\gamma} \left( (\beta A^*A/4 + \beta I_n)^{-1} \left( \beta (Y^k + \frac{Z^k}{\beta}) - I_n \right.ight.
\left. - \frac{\beta}{2} B^*(\frac{1}{2} A(Y^k + Bx^k - c) - \frac{zk}{\beta} \right),
\hat{z}^k = z^k - \beta \left( \frac{1}{2} A(\hat{x}^k) + \frac{1}{2} A(Y^k) + Bx^k - c \right),
\hat{Z}^k = Z^k - \beta (\hat{x}^k - Y^k),
\hat{x}^k = S \left( x^k - \eta_1 B^T \left( \frac{1}{2} A(\hat{x}^k) + \frac{1}{2} A(Y^k) + Bx^k - c - \frac{zk}{\beta} \right), \lambda \eta_1 \right),
\hat{Y}^k = S \left( y^k - \frac{\eta_3}{2} A^* \left( \frac{1}{2} A(\hat{x}^k) + \frac{1}{2} A(Y^k) + Bx^k - c - \frac{1}{\beta} \hat{z}^k \right) \right)
\left. + \eta_3 (\hat{x}^k - Y^k - \hat{Z}^k), \frac{\tau \eta_2}{\beta} \right),
X^{k+1} = \hat{X}^k,
(Y^{k+1}; x^{k+1}; z^{k+1}; Z^{k+1}) = (1 - \alpha)(Y^k; x^k; z^k; Z^k) + \alpha(\hat{Y}^k; \hat{x}^k; \hat{z}^k; \hat{Z}^k).
$$

In this subsection, we present some numerical results to compare the performance of CAPReaL, CAPReaL-Zero, CAPReaL-Jacobi and CAPReaL-Twisted algorithms to recover $s$-sparse real vectors $x_o \in \mathbb{R}^n$ from their quadratic measurement $c = |Ax_o + b|^2$.

Shown in Table 2 are the average of the iteration number $Iter$ and the time consumption $Time$ in seconds to reach the stopping criterion, and the relative reconstruction error $\|x^* - x_o\|_2/\|x_o\|_2$ between the recovered sparse signal $x^*$ and the original sparse signal $x_o$ over 100 trials for different ratio $m/n$ between the number $n$ of measurements and the length $n$ of the original vector, where the original sparsity signal $x_o$ has sparsity $s = 4$ and length $n = 64$, and the stopping criteria in the compensation step are the same for all algorithms,

$$
\|Y^k - x^k(x^k)^T\|_F/\|x^k(x^k)^T\|_F \leq \varepsilon := 10^{-5},
$$

and the stopping criteria for the ADMM step are

$$
\|x^{k+1} - \hat{x}^k\|_{\beta/\eta_1 - \beta B^T B}^2 + \frac{2\beta}{\eta_2} \|X^{k+1} - X^k\|_2^2 + \frac{2\beta}{\eta_3} \|Y^{k+1} - Y^k\|_2^2
+ \frac{3}{\beta} \|z^{k+1} - \hat{z}^k\|_2^2 + \frac{3}{\beta} \|X^{k+1} - \hat{Z}^k\|_2^2 \leq \epsilon := 10^{-2}
$$

cf. \cite{14}, and the stopping criteria for the ADMM step are
Table 2. Average iteration number and time consumption over 100 trials to implement the proposed algorithms for different ratio \(m/n\) between the number \(m\) of measurements and the length \(n\) of the original signal.

| \(m/n\) | Algorithm       | Iter | Time  | \(\|x^*-x_0\|_2\)/\(\|x_0\|_2\) |
|---------|-----------------|------|-------|---------------------------------|
| 1       | CAPReaL-Jacobi  | 994.6| 2.7496| 2.70e-1                         |
|         | CAPReaL-Twisted | 1000 | 3.1777| 1.12e-1                         |
|         | CAPReaL-Zero    | 964.4| 2.6862| 1.05e-2                         |
|         | CAPReaL        | 960  | 2.6824| 6.12e-3                         |
| 1.5     | CAPReaL-Jacobi  | 866.6| 2.7674| 6.60e-3                         |
|         | CAPReaL-Twisted | 998.2| 3.5418| 3.90e-5                         |
|         | CAPReaL-Zero    | 905.5| 2.8921| 5.01e-5                         |
|         | CAPReaL        | 863.1| 2.5233| 3.51e-5                         |
| 2       | CAPReaL-Jacobi  | 834.3| 2.3517| 3.99e-4                         |
|         | CAPReaL-Twisted | 986.5| 3.5778| 9.56e-6                         |
|         | CAPReaL-Zero    | 846.4| 2.8398| 2.91e-5                         |
|         | CAPReaL        | 823.8| 2.2863| 2.95e-6                         |

for the CAPReaL and CAPReaL algorithms (cf. \((2.11))\),

\[
\max \left\{ \frac{\|x^k - \tilde{x}^k\|_2 \|Y^k - \tilde{Y}^k\|_2}{1 + \|x^k\|_2}, \frac{\|z^k - \tilde{z}^k\|_2}{1 + \|z^k\|_2}, \frac{\|Z^k - \tilde{Z}^k\|_2}{1 + \|Z^k\|_2} \right\} \leq \epsilon := 10^{-2}
\]

for the CAPReaL-Twisted algorithm (cf. \([54]\) Eqn. 51)), and

\[
\frac{2\beta\|x_j^{k+1} - x_j^k\|_2^2}{\eta_1} + \frac{2\beta\|X_j^{k+1} - X_j^k\|_2^2}{\eta_2} + \frac{2\beta\|Y_j^{k+1} - Y_j^k\|_2^2}{\eta_3} + \frac{3 - \gamma}{\beta\gamma^2} \|z^{k+1} - z^k\|_2^2 + \frac{3 - \gamma}{\beta\gamma^2} \|Z^{k+1} - Z^k\|_2^2 \leq \varsigma := 10^{-2}
\]

for the CAPReaL-Jacobi algorithm (cf. \([22]\) Lemma 2.1, Eqn 2.2)). Plotted in Figure 1 is the average of the relative error \(\|x^k - x_0\|_2/\|x_0\|_2\), \(1 \leq k \leq 1000\), between the reconstructed signal \(x^k\) in the \(k\)-th iteration and the original sparse signal \(x_0\) over 100 trials. From Table 2 and Figure 1, we observe that the proposed CAPReaL algorithm has more favorable performance on the recovery of sparse real vectors from their quadratic measurements than the CAPReaL-Twisted, CAPReaL-Jacobi, and CAPReaL-Zero algorithms do.

5.3. Noiseless quadratic measurements. (Sparse) phase retrieval plays an influential role in signal/image/speech processing and it has received considerable attention in recent years, see \([13, 14, 35]\) and references therein. A fundamental problem is whether and how a (sparse) vector \(x \in \mathbb{R}^n\) (or \(\mathbb{C}^n\)) can be reconstructed from its quadratic measurements \(c = |Ax|^2 = [|a_1^T x|^2, \ldots, |a_m^T x|^2]^T\), where \(A = [a_1, \ldots, a_m]^T\) is the measurement matrix.
Figure 1. The average relative error of $\|x^k - x_0\|_2 / \|x_0\|_2$, $1 \leq k \leq 1000$, in $k$-th iteration over 100 trials in the implementation of the proposed algorithms to reconstruct sparse signals from their quadratic measurements.

Various algorithms have been proposed to recover an (sparse) original signal, up to a trivial ambiguity, from its quadratic measurements, see the survey paper [38] and references therein. By (1.8), the recovery of a signal $x \in \mathbb{R}^n$ with sparsity $s$ from its affine quadratic measurement $|Ax + b|^2$ reduces to finding a signal $\tilde{x}$ with sparsity $s + 1$ and last component 1 from its quadratic measurement $|\tilde{A}\tilde{x}|^2 = |Ax + b|^2$. Therefore we may adjust the CPRL algorithm [46, 40], Thresholded Wirtinger flow method (TWF) [9], CoPRAM approach [39] by normalizing the last component to 1 in each iteration through dividing the last component and we denote the adjusted algorithms as CPRL$_r$, TWF$_r$ and CoPRAM$_r$ respectively. Shown in Table 3 is the success percentage of the proposed CAPReaL algorithm and the adjusted algorithms CPRL$_r$, TWF$_r$ and CoPRAM$_r$ to recover $s$-sparse vectors in $\mathbb{R}^n$ from their quadratic affine measurements of size $m$, over 100 trials, where $s = 4$, $n = 64$ and $1/2 \leq m/n \leq 2$. This indicates that the proposal CAPReaL method has the best performance to recover sparse signals from their noiseless affine quadratic measurements, followed close behind by CPRL$_r$ and then by TWF$_r$ and CoPRAM$_r$. On the other hand, our simulations indicates that TWF$_r$ and CoPRAM$_r$ consume much less time in the implementation than CAPReaL and CPRL$_r$ do.

5.4. Quadratic measurements corrupted by Gaussian noises. In this subsection, we demonstrate the performance of 2-CAPReaL algorithm to recover sparse signals $x_o \in \mathbb{R}^n$ from their quadratic measurements corrupted by Gaussian white noises. For the comparison, we compare the proposed 2-CAPReaL algorithm with adjusted CPRL-QC$_r$, TWF$_r$ and CoPRAM$_r$. Here CPRL-QC$_r$ is adjusted from the CPRL-QC algorithm [46],

$$\min_{X \succeq O} \text{tr}(X) + \tau \|X\|_1 \quad \text{subject to} \quad \|A(X) - c\|_2 \leq \varepsilon,$$

by normalizing the last entries of the matrix $X$ to 1 in each iteration by dividing $X_{n+1,n+1}$, where $\tau > 0$ is balancing parameter and $\varepsilon = \|e\|_2$ is the
Table 3. Success percentage of the CPRL\(_r\), TWF\(_r\), CoPRAM\(_r\) and CAPReaL algorithms to recover sparse signals with 100 repeated trials for different ratios \(m/n\) between the number \(m\) of measurements and length \(n\) of the original sparse signal.

| Alg.      | \(m/n\)  | 0.5 | 0.75 | 1    | 1.25 | 1.5  | 1.75 | 2    |
|-----------|-----------|-----|------|------|------|------|------|------|
| CPRL\(_r\) |           | 1   | 10   | 62   | 87   | 93   | 96   | 100  |
| TWF\(_r\)  |           | 0   | 3    | 14   | 33   | 51   | 62   | 70   |
| CoPRAM\(_r\)|         | 0   | 1    | 2    | 2    | 3    | 4    | 5    |
| CAPReaL    |           | 2   | 19   | 61   | 95   | 98   | 100  | 100  |

Table 4. The average SNR of the CPRL-QC\(_r\), TWF\(_r\), CoPRAM\(_r\) and 2-CAPReaL algorithms to recover sparse solutions over 100 trials for different ratios \(m/n\) between the number \(m\) of measurements and the length \(n\) of original signals and for two different Gaussian noise levels \(\sigma\).

| \(\sigma\) | Alg.      | \(m/n\)  | 0.5 | 0.75 | 1    | 1.25 | 1.5  | 1.75 | 2    |
|------------|-----------|-----------|-----|------|------|------|------|------|------|
| 10\(^{-3}\) | CPRL-QC\(_r\) |           | 5.68| 16.59| 31.03| 38.92| 40.24| 41.28| 41.93|
|            | TWF\(_r\)  |           | -8.72| 6.25| 23.42| 35.25| 45.65| 55.82| 57.63|
|            | CoPRAM\(_r\)|         | 2.31| 4.37| 6.33| 7.13| 8.73| 8.77| 9.81|
|            | 2-CAPReaL  |           | 4.58| 14.01| 40.44| 56.38| 63.27| 66.66| 67.97|
| 10\(^{-1}\) | CPRL-QC\(_r\) |           | 4.32| 8.19| 14.98| 24.54| 26.87| 30.43| 31.67|
|            | TWF\(_r\)  |           | -9.27| 6.89| 16.59| 25.29| 29.96| 31.05| 32.70|
|            | CoPRAM\(_r\)|         | 1.98| 3.71| 5.87| 7.36| 7.40| 8.71| 9.17|
|            | 2-CAPReaL  |           | 2.21| 5.92| 13.59| 24.66| 29.97| 32.96| 34.09|

noise bound. We use the average of the signal-to-noise ratio (SNR) in dB,

\[
\text{SNR}(x^*, x_o) = 20 \log_{10} \frac{\|x_o\|_2}{\|x^* - x_o\|_2},
\]

over 100 independent trials as our performance measure, where \(x^*\) is the reconstructed signal. Shown in Table 4 is the result of our proposed 2-CAPReaL algorithm to recover sparse signals from their quadratic affine measurements and the performance comparison with the CPRL-QC\(_r\), TWF\(_r\) and CoPRAM\(_r\), where the Gaussian white noise level \(\sigma = 10^{-3}, 10^{-1}\). This shows that for \(m/n \geq 1\), the proposal 2-CAPReaL is more robust against Gaussian white noises than the CPRL-QC\(_r\), TWF\(_r\) and CoPRAM\(_r\) does especially when the noise level is low, while for \(m/n < 1\) the CPRL-QC\(_r\) has best performance followed by proposal 2-CAPReaL.
Table 5. The average SNR of the CPRL-LADC<sub>r</sub>, TWF<sub>r</sub>, CoPRAM<sub>r</sub> and 1-CAPReaL algorithms to recover sparse signals from their quadratic affine measurements corrupted by Cauchy noises over 100 trials for different ratios m/n and for two different Cauchy noise levels γ.

| γ       | Alg.    | m/n | 0.5 | 0.75 | 1    | 1.25 | 1.5 | 1.75 | 2    |
|---------|---------|-----|-----|------|------|------|-----|------|-----|
| 10^-4   | CPRL-LADC<sub>r</sub> | | 3.61 | 12.17 | 34.61 | 47.51 | 53.25 | 55.03 | 55.56 |
|         | TWF<sub>r</sub>       | -30.65 | -21.83 | -8.91 | 8.64 | 23.28 | 32.91 | 48.53 |
|         | CoPRAM<sub>r</sub>    | 1.82 | 3.68 | 6.61 | 7.15 | 8.05 | 9.26 | 10.07 |
|         | 1-CAPReaL             | 5.15 | 21.60 | 55.27 | 70.84 | 75.66 | 77.63 | 77.80 |
| 10^-2   | CPRL-LADC<sub>r</sub> | 3.44 | 10.90 | 21.18 | 29.75 | 30.74 | 32.14 | 32.37 |
|         | TWF<sub>r</sub>       | -30.64 | -22.85 | -11.65 | 1.91 | 12.27 | 16.83 | 22.96 |
|         | CoPRAM<sub>r</sub>    | 1.47 | 3.65 | 5.12 | 5.43 | 6.23 | 7.01 | 8.25 |
|         | 1-CAPReaL             | 1.42 | 4.10 | 10.59 | 23.11 | 32.34 | 40.29 | 42.64 |

5.5. Quadratic measurements corrupted by impulsive noises. For the case that quadratic measurements are corrupted by the impulsive Cauchy noise, we will use the p-CAPReaL algorithm with p = 1 to recover sparse signals from their corrupted quadratic measurements. Presented in Table 5 are performances of CPRL-LADC<sub>r</sub>, TWF<sub>r</sub>, CoPRAM<sub>r</sub> and 1-CAPReaL algorithms to recover sparse solutions for different ratios m/n between the number m of measurements and the length n of original signals, and for two different Cauchy noise levels γ, where CPRL-LADC<sub>r</sub> is modified from the CPRL-LADC algorithm,

\[
\min_{X \succeq \mathbf{0}} \operatorname{tr}(X) + \tau \|X\|_1 \quad \text{subject to } \|A(X) - c\|_1 \leq \varepsilon,
\]

by adjusting the last entries of the matrix X to one in each iteration by dividing X_{n+1,n+1}, where \(\tau > 0\) is balancing parameter and \(\varepsilon = \|e\|_1\) is noise bound. Therefore for the recovery of sparse signals from their affine quadratic measurements corrupted by the impulsive noise of Cauchy type, the CPRL-LADC<sub>r</sub> and the proposed 1-CAPReaL have much better performance than TWF<sub>r</sub> and CoPRAM<sub>r</sub> do, the CPRL-LADC<sub>r</sub> achieves higher SNR than the 1-CAPReaL does when we have less measurements and the 1-CAPReaL does better job than CPRL-LADC<sub>r</sub> does when we have more measurements.

5.6. Quadratic measurements corrupted by bounded noises. In this subsection, we approximate the true sparse signal \(x_o\) when its quadratic measurements are corrupted by a uniformly distributed noise with different noise bound \(\delta\). Shown in Table 6 is the performances of CPRL-IC<sub>r</sub>, TWF<sub>r</sub>,

Table 6. The average SNR of the CPRL-IC<sub>r</sub>, TWF<sub>r</sub>, CoPRAM<sub>r</sub> and ∞-CAPReaL algorithms to recover sparse signals from their quadratic affine measurements corrupted by bounded noises over 100 trials for different ratios \( m/n \) and for four different bounded noise levels \( \delta \), where the sparsity is \( s = 4 \) and vector length is \( n = 64 \), and the quadratic affine measurements \( b = |Ax_0|^2 + e \) with \( e \) is the uniformly distribution noise with noise bound \( \delta \).

| \( \delta \) | \( m/n \) | Alg. | 0.5 | 0.75 | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|---|---|---|---|
| \( 10^{-3} \) | CPRL-IC<sub>r</sub> | 3.43 | 18.92 | 35.54 | 47.67 | 52.69 | 54.81 | 55.22 |
| &nbsp;&nbsp; | TWF<sub>r</sub> | -15.53 | 1.51 | 23.88 | 38.12 | 47.26 | 54.16 | 57.52 |
| &nbsp;&nbsp; | CoPRAM<sub>r</sub> | 1.86 | 4.37 | 5.37 | 7.39 | 7.45 | 9.54 | 10.02 |
| &nbsp;&nbsp; | ∞-CAPReaL | 5.19 | 15.10 | 36.55 | 55.98 | 63.89 | 65.94 | 69.03 |
| \( 10^{-2} \) | CPRL-IC<sub>r</sub> | 3.76 | 12.91 | 32.04 | 40.97 | 47.15 | 48.82 | 48.83 |
| &nbsp;&nbsp; | TWF<sub>r</sub> | -17.75 | 1.42 | 22.35 | 33.28 | 40.45 | 45.35 | 50.03 |
| &nbsp;&nbsp; | CoPRAM<sub>r</sub> | 1.75 | 4.13 | 6.38 | 7.12 | 7.69 | 9.21 | 9.68 |
| &nbsp;&nbsp; | ∞-CAPReaL | 4.66 | 13.43 | 28.45 | 41.38 | 45.79 | 48.39 | 51.95 |
| \( 10^{-1} \) | CPRL-IC<sub>r</sub> | 2.72 | 9.55 | 16.70 | 24.55 | 27.37 | 28.40 | 30.46 |
| &nbsp;&nbsp; | TWF<sub>r</sub> | -17.41 | -1.12 | 18.12 | 28.42 | 32.65 | 33.32 | 35.65 |
| &nbsp;&nbsp; | CoPRAM<sub>r</sub> | 2.33 | 3.69 | 5.03 | 6.48 | 7.46 | 8.54 | 9.02 |
| &nbsp;&nbsp; | ∞-CAPReaL | 3.93 | 13.33 | 19.31 | 28.68 | 30.80 | 33.35 | 34.97 |
| 1 | CPRL-IC<sub>r</sub> | 1.82 | 3.04 | 4.11 | 6.07 | 7.80 | 8.60 | 10.43 |
| &nbsp;&nbsp; | TWF<sub>r</sub> | -17.19 | -3.47 | 9.29 | 15.07 | 17.25 | 17.99 | 20.32 |
| &nbsp;&nbsp; | CoPRAM<sub>r</sub> | 0.02 | 1.46 | 2.44 | 3.25 | 4.57 | 4.38 | 5.87 |
| &nbsp;&nbsp; | ∞-CAPReaL | 2.05 | 4.14 | 6.19 | 8.87 | 10.68 | 11.88 | 14.87 |

CoPRAM<sub>r</sub> and ∞-CAPReaL, where CPRL-IC<sub>r</sub> is modified from the CPRL-IC algorithm,

\[
\begin{align*}
\text{(5.5a)} & \quad \min_{X \succeq 0} \text{tr}(X) + \tau\|X\|_1 \\
\text{(5.5b)} & \quad \text{subject to } \|A(X) - c\|_\infty \leq \delta,
\end{align*}
\]

by adjusting the last component of the matrix \( X \) to one in each iteration by dividing \( X_{n+1,n+1} \), where \( \delta = \|e\|_\infty \) is the noise bound. These results indicate that the proposed ∞-CAPReaL has much better performance than CPRL-IC<sub>r</sub>, TWF<sub>r</sub> and CoPRAM<sub>r</sub> do when the noise level is low, while the TWF<sub>r</sub> achieves higher SNR than the ∞-CAPReaL does when the noise level is high.
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