Mean value properties of solutions to the modified Helmholtz equation and related topics (a survey)

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Recent results concerning solutions of the modified Helmholtz equation are reviewed; namely, various mean value properties and their corollaries, converse and inverse of these properties, and relations between these solutions and harmonic functions.

1 Introduction

In the present article, we consider real-valued $C^2$-solutions of the $m$-dimensional modified Helmholtz equation:

$$\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}; \quad (1.1)$$

$\nabla = (\partial_1, \ldots, \partial_m)$ is the gradient operator, $\partial_i = \partial / \partial x_i$. Unfortunately, it is not commonly known that these solutions are called panharmonic functions (or $\mu$-panharmonic functions) by analogy with solutions of the Laplace equation; this convenient abbreviation coined by Duffin [11] will be used in what follows. In the latter paper and elsewhere, Duffin refers to (1.1) as the Yukawa equation, surprisingly, without citing his original paper [49], in which the Nobel Prize winning theory of nuclear forces was proposed. For describing the force potential of a point charge that decays rapidly at infinity, Yukawa used the three-dimensional fundamental solution of (1.1), which has this property. Thus, it is quite reasonable to name equation (1.1) after Yukawa, even taking into account, that he did not use it in [49]. However, one can find still more confusing name of (1.1), namely, the Helmholtz equation; see [42] and [30], p. 231.

Undeservedly, panharmonic functions received much less attention than harmonic and subharmonic, despite the fact that studies of the three-dimensional equation (1.1) were initiated by C. Neumann in his monograph [34] published in 1896. Since then, ridiculously small number of papers treating rigorously solutions of equation (1.1) have been published and their content varies noticeably. Some consider particular boundary value problems (see, for example, [6], [7] and [42]), whereas others are concerned with the so-called $\mu$-regular class of pseudoanalytic functions satisfying the Cauchy–Riemann equations for the two-dimensional version of (1.1); see, for example, [11], [12] and [40]. Finally, it is worth mentioning the representation formulas for panharmonic functions obtained in [9].
Recently, the author published several notes [23], [24], [25], [26], [27], [28] and [29] dealing with various topics: mean value formulae for panharmonic functions, their corollaries, converse theorems and other related results such as characterization of balls via these functions. These studies were initiated after completing the survey [22], preparing which it was a surprise to discover that only mean value formulae for spheres and circumferences in three and two dimensions were derived earlier by C. Neumann and Duffin, respectively. Moreover, relations between harmonic, subharmonic and panharmonic functions also remained unnoticed.

The aim of this survey is to present the obtained results in a systematic, self-contained form similar, to some extent, to a comprehensive theory developed by Duffin for two-dimensional punharmonic functions; see [11] and [12]. The plan of the article is as follows.

Various mean value formulae (volume, spherical, asymptotic etc.) and their corollaries are presented in Sect. 2, whereas Sect. 3 deals with converse of these mean value properties. Two characterizations of balls via panharmonic functions are described in Sect. 4. Relations between harmonic and panharmonic functions are considered in Sect. 5.

2 Mean value formulae and their corollaries

The following analogue of the Gauss theorem on the arithmetic mean of a harmonic function over an $m-1$-dimensional sphere in $\mathbb{R}^m$ (the original memoir [16], Article 20, deals with the case $m = 3$) was derived for panharmonic functions in [23].

**Theorem 2.1** ([23]). Let $u$ be panharmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. Then for every $x \in D$ the identity

$$M^o(u, x, r) = a^o(\mu r) u(x), \quad a^o(\mu r) = \Gamma\left(\frac{m}{2}\right) \frac{I_{(m-2)/2}(\mu r)}{\left(\frac{\mu r}{2}\right)^{(m-2)/2}},$$

holds for each admissible sphere $S_r(x)$; $I_\nu$ denotes the modified Bessel function of order $\nu$.

Here and below the following notation and terminology are used. The open ball of radius $r$ centred at $x$ is denoted by $B_r(x) = \{ y : |y - x| < r \}$; the latter is called admissible with respect to a domain $D$ provided $\overline{B_r(x)} \subset D$, and $S_r(x) = \partial B_r(x)$ is the corresponding admissible sphere. If $u \in C^0(D)$, then its spherical mean value over $S_r(x) \subset D$ is

$$M^o(u, x, r) = \frac{1}{|S_r|} \int_{S_r(x)} u(y) dS_y = \frac{1}{\omega_m} \int_{S_m(0)} u(x + ry) dS_y,$$

where $|S_r| = \omega_m r^{m-1}$ and $\omega_m = 2 \pi^{m/2}/\Gamma(m/2)$ is the total area of the unit sphere (as usual $\Gamma$ stands for the Gamma function), and $dS$ is the surface area measure. It is clear that this function is continuous in $x$ and $r$; moreover, if $u \in C^k(D)$, then its mean value is in the same class in $x$ and $r$. By continuity we have that

$$M^o(u, x, 0) = u(x),$$

whereas further identities for $M^o$ can be found in [20], Chapter IV.
For \( m = 3 \) identity (2.1) (derived by C. Neumann [34] as early as 1896) is particularly simple because \( a^\circ(\mu r) = \sinh(\mu r)/(\mu r) \). Duffin independently rediscovered the proof (see [11], pp. 111-112), but in two dimensions with \( a^\circ(\mu r) = I_0(\mu r) \).

Our proof of Theorem 2.1 is based on the Euler–Poisson–Darboux equation for \( M^\circ \) (see [20], p. 88):

\[
M_{rr}^\circ + (m - 1) r^{-1} M_r^\circ = \nabla_x^2 M^\circ, \quad \text{where } r > 0, \ x \in D. \quad (2.3)
\]

It is valid provided \( u \in C^2(D) \) and follows from the obvious relation

\[
m \int_0^r t^{m-1} M^\circ(u, x, t) \, dt = r^m M^\bullet(u, x, r), \quad (2.4)
\]

where

\[
M^\bullet(u, x, r) = \frac{1}{|B_r|} \int_{B_r(x)} u(y) \, dy
\]

and \( |B_r| = \omega_m r^m/m \) is the volume of \( B_r \). Indeed, applying the Laplacian to both sides of (2.4), we obtain

\[
\omega_m \int_0^r t^{m-1} \nabla_x^2 M^\circ(u, x, t) \, dt = \int_{B_r(0)} \nabla_x^2 u(x + y) \, dy.
\]

By Green’s first formula the last integral is equal to

\[
\int_{|y|=r} \nabla_x u(x + y) \cdot \frac{y}{r} \, dS_y,
\]

and changing variables this can be written as follows:

\[
x r^{m-1} \frac{\partial}{\partial r} \int_{|y|=1} u(x + ry) \, dS_y = \omega_m r^{m-1} M_r^2(u, x, r).
\]

Thus we arrive at the equality

\[
r^{m-1} M_r^2(u, x, r) = \int_0^r t^{m-1} \nabla_x^2 M^\circ(u, x, t) \, dt.
\]

Differentiation of this relation with respect to \( r \) yields (2.3).

Proof of Theorem 2.1. It is straightforward to show that \( a(r) = a^\circ(\mu r) \) is a unique solution the following Cauchy problem:

\[
a_{rr} + (m - 1) r^{-1} a_r - \mu^2 a = 0, \ a(0) = 1, \ a_r(0) = 0. \quad (2.5)
\]
This follows by virtue of the relations (see [47], p. 79):

\[ zI_{\nu+1}(z) + 2\nu I_{\nu}(z) - zI_{\nu-1}(z) = 0, \quad [z^{-\nu}I_{\nu}(z)]' = z^{-\nu}I_{\nu+1}(z). \]  

(2.6)

In particular, the second one implies the second initial condition.

The function \( w(r, x) = a^\circ(\mu r)u(x) - M^\circ(u, x, r) \) is defined for all \( x \in D \) and all \( r \geq 0 \) such that \( S_r(x) \) is admissible and satisfies the initial conditions

\[ w(x, 0) = 0, \quad w_r(x, 0) = 0. \]

The first one is a consequence of the identities \( a^\circ(0) = 1 \) and \( M^\circ(u, x, 0) = u(x) \), whereas the second one follows from equation (2.3) multiplied by \( r \) in the limit as \( r \to 0 \). Moreover, equations (2.3) and (2.5) yield that

\[ w_{rr} + (m - 1)r^{-1}w_r - \mu^2w = 0 \quad \text{for } r > 0. \]

Since the latter Cauchy problem has only a trivial solution, we obtain (2.1). \( \square \)

It follows from Theorem 2.1 that a panharmonic function of fixed sign belongs to one of two well studied classes of functions; namely, subharmonic or superharmonic. In our context, it is sufficient to define these classes as follows.

**Definition 2.1** (Gilbarg and Trudinger [17], p. 23). A function \( u \in C^0(D) \) is called subharmonic (superharmonic) in \( D \) if for every admissible ball \( B \subset D \) and every function \( h \) harmonic in \( B \) and satisfying \( u \leq h \) (\( u \geq h \)) on \( \partial B \), the same inequality holds throughout \( B \).

**Corollary 2.1** ([29]). Let a panharmonic function \( u \) be nonnegative (nonpositive) in a domain \( D \). Then \( u \) is subharmonic (superharmonic) in \( D \).

**Proof.** The function \( a^\circ \) increases monotonically on \([0, \infty)\) from \( a^\circ(0) = 1 \) to infinity. Indeed, the second relation (2.6) implies the monotonicity, whereas the behavior at infinity is a consequence of the asymptotic formula

\[ I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[ 1 + O(|z|^{-1}) \right], \quad |\arg z| < \pi/2, \]  

(2.7)

whose principal term does not depend on \( \nu \), is valid as \( |z| \to \infty \); see [47], p. 80.

Since \( a^\circ(\mu r) > 1 \), identity (2.1) yields that \( u(x) \leq M^\circ(u, x, r) \) for every \( S_r(x) \subset D \) provided the \( \mu \)-panharmonic \( u \) is nonnegative in \( D \). The result immediately follows from this inequality and Definition 2.1. \( \square \)

The converse of Corollary 2.1 is not true, because any nonzero constant is subharmonic and superharmonic, but not panharmonic. Another consequence of Theorem 2.1 is the following.
Corollary 2.2 ([23]). Let \( D \) be a domain in \( \mathbb{R}^m \), \( m \geq 2 \). If \( u \) is panharmonic in \( D \), then
\[
M^\bullet(u, x, r) = a^\bullet(\mu r) u(x), \quad a^\bullet(\mu r) = \Gamma \left( \frac{m}{2} + 1 \right) \frac{I_{m/2}(\mu r)}{(\mu r/2)^{m/2}},
\]
and
\[
a^\circ(\mu r)M^\bullet(u, x, r) = a^\bullet(\mu r)M^\circ(u, x, r)
\]
for every admissible ball \( B_r(x) \).

Proof. Let us write formula (2.1) in the form
\[
\omega_m^{-1} \int_{S_1(0)} u(x + \rho y) \, dS_y = a^\circ(\mu \rho) u(x),
\]
multiply by \( \rho^{m-1} \), and integrate with respect to \( \rho \) over \((0, r)\), where \( r \) is such that \( B_r(x) \) is admissible. Thus we obtain \( M^\bullet(u, x, r) \) on the left-hand side after division by \( r^m \).

Applying formula 1.1.5. [38], namely,
\[
\int_0^x x^{1+\nu} I_\nu(x) \, dx = x^{1+\nu} I_{\nu+1}(x), \quad \Re \nu > -1.
\]
with \( \nu = (m-2)/2 \) while integrating the right-hand side, identity (2.8) follows. Combining (2.1) and (2.8), one arrives at (2.9).

Remark 2.1. It is clear from the proof of Corollary 2.2 that identities (2.1) and (2.8) are equivalent. Like \( a^\circ \), the function \( a^\bullet \) increases monotonically on \([0, \infty)\) from \( a^\bullet(0) = 1 \) to infinity. Moreover, \( a^\bullet(t)/a^\circ(t) < 1 \) for \( t > 0 \), which immediately follows from their definitions and the first formula (2.6).

Identity (2.9) couples the mean values over spheres and balls for a \( \mu \)-panharmonic \( u \). In this identity, the ratio of coefficients \( a^\circ(\mu r)/a^\bullet(\mu r) \) tends to unity in the limit as \( \mu \to 0 \), thus reducing (2.9) to the formula equating the mean values of a harmonic function over spheres and balls.

Another corollary of Theorem 2.1 deals with iterated spherical means introduced by John; see [23], p. 78, but the notation is different here. Let \( D_r \) be the subdomain of \( D \) with boundary ‘parallel’ to \( \partial D \) at the distance \( r > 0 \); namely, \( D_r = \{ x \in D : B_r(x) \subset D \} \).

Thus, \( D_r \) is nonempty only when \( r \) is less than the radius of the open ball inscribed into \( D \). Since \( M^\circ(u, \cdot, r) \) is defined on \( D_r \), it is clear that
\[
I(u, x, r', r) = M^\circ(M^\circ(u, \cdot, r), x, r') = \frac{1}{\omega_m} \int_{S_1(0)} M^\circ(u, x + \rho y, r) \, dS_y
\]
is defined on \( D_{r+r'} \); here the second equality is a consequence of (2.2). Substituting the expression for \( M^\circ \), we obtain
\[
I(u, x, r', r) = \frac{1}{\omega_m^2} \int_{S_1(0)} \int_{S_1(0)} u(x + \rho y + \rho z) \, dS_z \, dS_y,
\]
(2.11)
and so it is symmetric in $r'$ and $r$, that is, $I(u, x, r', r) = I(u, x, r, r')$. Moreover,  
\[ I(u, x, 0, r) = I(u, x, r, 0) = I^0(u, x, r) \quad \text{and} \quad I(u, x, 0, 0) = u(x). \]

Now we see that (2.10) implies the following assertion concerning the iterated mean value property.

**Corollary 2.3.** Let $u$ be $\mu$-panharmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. If the domain $D_r$ is nonempty for $r > 0$, then $M^0(u, \cdot , r)$ is $\mu$-panharmonic in it and  
\[ I(u, x, r', r) = a^0(\mu r') M^0(u, x, r) = a^0(\mu r') a^0(\mu r) u(x) \]
for every $x \in D_r$ and all $S_r'(x)$ admissible with respect to $D_r$.

Two more mean value properties of panharmonic functions are analogous to the classical theorems of Blaschke [5], Priwaloff [37] and Zaremba [50] concerning asymptotic mean values of harmonic functions (see [33], Sect. 9, for a discussion).

**Proposition 2.1.** Let $D$ be a domain in $\mathbb{R}^m$, $m \geq 2$. If $u$ is panharmonic in $D$, then  
\[ \lim_{r \to +0} \frac{M^*(u, x, r) - u(x)}{r^2} = \frac{\mu^2 u(x)}{2(m + 2)} \quad \text{for every } x \in D. \tag{2.12} \]
The assertion also holds with $M^*(u, x, r)$ changed to $M^0(u, x, r)$ and the right-hand side term in (2.12) changed to $\mu^2 u(x)/(2m)$.

**Proof.** The well-known relationships between the Laplacian and asymptotic mean values (see [8], Ch. 2, Sect. 2) follow from Taylor’s formula:
\[ u(x + y) - u(x) = y \cdot \nabla u(x) + 2^{-1} y \cdot [H_u(x)] y + o(r^2). \]
It is valid for $u \in C^2(D)$ at $x \in D$ as $r \to 0$ provided $B_r(x)$ is admissible and $|y| \leq r$; here $H_u(x)$ denotes the Hessian matrix of $u$ at $x$ and “$\cdot$” stands for the inner product in $\mathbb{R}^m$. Averaging each term of the equality with respect to $y \in B_r(0)$, one obtains
\[ M^*(u, x, r) - u(x) = \frac{1}{2|B_r|} \int_{B_r(0)} y \cdot [H_u(x)] y \, dy + o(r^2), \]
because the mean value of the first order term vanishes. It is straightforward to calculate that  
\[ \lim_{r \to +0} \frac{M^*(u, x, r) - u(x)}{r^2} = \frac{\nabla^2 u(x)}{2(m + 2)}, \quad x \in D. \tag{2.13} \]
Now (2.12) follows from panharmonicity of $u$.

Similarly, averaging Taylor’s formula with respect to $y \in S_r(0)$, we obtain
\[ \lim_{r \to +0} \frac{M^0(u, x, r) - u(x)}{r^2} = \frac{\nabla^2 u(x)}{2m}, \quad x \in D. \tag{2.14} \]
This and panharmonicity of $u$ yield the second assertion. \[ \square \]
Remark 2.2. Another proof of Proposition 2.1 is as follows. Identity (2.1) holds for a panharmonic $u$ provided $B_r(x)$ is admissible. It implies that (2.12) is equivalent to
\[
\lim_{r \to +0} \frac{a^*(\mu r) - 1}{(\mu r)^2} = \frac{1}{2(m + 2)},
\]
which follows from the definition of $I_{m/2}$.

In order to obtain the second assertion of Proposition 2.1 one has to use the equality
\[
\lim_{r \to +0} \frac{a^0(\mu r) - 1}{(\mu r)^2} = \frac{1}{2m},
\]
which is true by the definition of $I_{(m-2)/2}$.

2.1 Applications of Theorem 2.1

Turning to applications of the obtained mean value property, we recall that the most important consequences of the corresponding property in the case of harmonic functions are the strong maximum principle and Liouville’s theorem. The first asserts that a function harmonic in a domain $D$ cannot have local maxima or minima there; moreover, if it is continuous in $D$, which is bounded, then its maximum and minimum are attained on $\partial D$. The second theorem says that every harmonic on $\mathbb{R}^m$ function bounded below (or above) is constant.

It is clear that $u(x) = (\mu |x|)^{-1} \sinh(\mu |x|)$, which is panharmonic in $\mathbb{R}^3$, violates both these assertions; indeed, it has the local (and global) minimum at the origin. Since the maximum principle and Liouville’s theorem, as formulated above, are not true for panharmonic functions, some extra restrictions must be imposed in order to convert these theorems into true ones.

2.1.1. The weak maximum principle. We begin with the following assertion concerning the behaviour of $|u|$ for a nontrivial panharmonic function $u$.

**Proposition 2.2.** Let $u$ be a nonvanishing identically panharmonic function in a domain $D \subset \mathbb{R}^m$, $m \geq 2$. Then for every $x \in D$ there exists $y \in D$ such that $|u(y)| > |u(x)|$.

**Proof.** Without loss of generality, we assume that $u(x) \geq 0$; indeed, $-u$ should be considered otherwise. Then Theorem 2.1 implies that $M^0(u, x, r) \geq 0$ for every admissible sphere $S_r(x)$ and $u(x) < M^2(u, x, r)$ because $a^0(\mu r) > 1$. Therefore, there exists a point $y \in S_r(x) \subset D$ such that $u(y) > u(x)$. 

An immediate consequence of this proposition is the weak maximum principle for panharmonic functions.
Theorem 2.2. Let $D$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$. If $u \in C^0(D)$ is panharmonic in $D$, then
\[
\sup_{x \in D} |u(x)| = \max_{x \in \partial D} |u(x)|. \tag{2.15}
\]

Proof. In the case of $u$ nonvanishing identically, we take a sequence $\{x_k\}_{k=1}^\infty \subset D$ such that
\[
|u(x_k)| \to \sup_{x \in D} |u(x)| \quad \text{as} \quad k \to \infty.
\]
Since $D$ is bounded, $\{x_k\}_{k=1}^\infty$ has a limit point in $\overline{D}$, say $x_0$, and $|u(x_0)| = \sup_{x \in D} |u(x)|$ by continuity. Moreover, $x_0 \in \partial D$; indeed, if $x_0 \in D$, then there exists $y \in D$ such that $|u(y)| > |u(x_0)|$ by Proposition 2.2, but this is impossible. Now (2.15) follows from the equality $\sup_{x \in D} |u(x)| = \max_{x \in \partial D} |u(x)|$ valid because $u \in C^0(D)$.

Here, only the mean value property is used for proving this principle for panharmonic functions. Of course, the approach to proving this principle that is applicable to general elliptic equations (see [17], Sect. 3.1) is valid for these functions as well, but our aim was to use a minimal tool. An immediate consequence of Theorem 2.2 (see [34], p. 260, for the original formulation) is the uniqueness of a solution to the Dirichlet problem for equation (1.1) in a bounded domain as well as the continuous dependence of solutions to this problem on boundary data.

2.1.2. Liouville’s theorem. In the whole $\mathbb{R}^m$, self-similarity allows us to restrict ourselves to the equation:
\[
\nabla^2 u - u = 0; \tag{2.16}
\]
indeed, it follows from (1.1) by a change of variables.

Theorem 2.3. Let $u$ be a solution of (2.16) on $\mathbb{R}^m$. If the inequality
\[
|u(x)| \leq C(1 + |x|)^n \quad \text{holds for all} \quad x \in \mathbb{R}^m \tag{2.17}
\]
with some $C > 0$ and a nonnegative integer $n$, then $u$ vanishes identically.

Proof. Substituting the asymptotic formula (2.7) in the expression for $a^\circ(r)$, we obtain
\[
a^\circ(r) = \frac{\Gamma(m/2) 2^{(m-3)/2} e^r}{\sqrt{\pi} r^{(m-1)/2}} \left[1 + O(r^{-1})\right] \quad \text{as} \quad r \to \infty.
\]
Using this and (2.17) in identity (2.10) with $\mu = 1$, we see that the inequality
\[
|u(x)| \leq \tilde{C} (1 + |x| + r)^n \frac{r^{(m-1)/2}}{e^r}
\]
holds with some $\tilde{C} > 0$ for all $x \in \mathbb{R}^m$ and all $r > 0$. Letting $r \to \infty$, the required assertion follows.

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Inequality (2.17) with any (arbitrarily large) \( n > 0 \) implies that a solution of equation (2.16) is trivial. On the other hand, if the same inequality is imposed on a harmonic function, then it is a (harmonic) polynomial, whose degree is less than or equal to \( n \); see [46], p. 290.

3 Converse of mean value properties

In the classical Kellogg’s monograph [21], the section, that follows the proof of the Gauss theorem on the spherical arithmetic means for harmonic functions, begins with the sentence.

The property of harmonic functions given by Gauss’ theorem is so simple and striking, that it is of interest to inquire what properties functions have which are, as we shall express it, their own arithmetic means on the surface of spheres.

Then Kellogg proves the converse of the arithmetic mean theorem due to Koebe (1906), and adds: “This theorem will be of repeated use to us.” Its analogue for panharmonic functions was obtained only in 2021.

**Theorem 3.1** ([23]). Let \( D \) be a bounded domain in \( \mathbb{R}^m \). If identity (2.1) with \( \mu > 0 \) is fulfilled for \( u \in C^0(D) \) at every \( x \in D \) and for all \( r \in (0, r(x)) \), where \( B_r(x) \) is admissible, then \( u \) is \( \mu \)-panharmonic in \( D \).

If instead of (2.1) identity (2.8) is fulfilled for \( u \in C^0(D) \) in the same way as above, then \( u \) is \( \mu \)-panharmonic in \( D \).

**Proof.** First, we have to show that \( u \) is smooth for which purpose a trick applied by Mikhlin in his proof of Koebe’s theorem is helpful; see [31], Ch. 11, Sect. 7. It is based on using the mollifier \( \omega_\epsilon(|y - x|) = \omega_\epsilon(r) \); see its properties in [31], Ch. 1, Sect. 1.

Let \( \epsilon > 0 \) be small and let \( D' = D_{2\epsilon} \) (the definition of this domain is given prior to Corollary 2.3). Assuming that \( x \in D' \), we multiply (2.1) by \( \omega_\epsilon \), thus obtaining

\[
\int_{B_\epsilon(x)} u(y) \omega_\epsilon(|y - x|) \omega_\epsilon(r) \int_{S_r(x)} u(y) \, dS_y.
\]

Now, integration with respect to \( r \) over \( (0, \epsilon) \) yields

\[
u(x) a^\circ(\mu r) |S_r| \omega_\epsilon(r) = \omega_\epsilon(r) \int_{S_r(x)} u(y) \, dS_y.
\]

Here the last equality is valid because \( x \in D' \), whereas \( \omega_\epsilon(|y - x|) \) vanishes outside \( B_\epsilon(x) \). Also,

\[
c(\mu, \epsilon) = \int_0^\epsilon a^\circ(\mu r) |S_r| \omega_\epsilon(r) \, dr > 0,
\]
because \( a^0(\mu r) > 1 \). Since \( \omega_r \) is infinitely differentiable, the obtained representation shows that \( u \in C^\infty(D') \). However, \( \epsilon \) is arbitrarily small, and so \( u \in C^\infty(D) \).

Now we are in a position to demonstrate that \( u \) is panharmonic in \( D \). Since \((2.1)\) holds for every \( x \in D \) and all \( r \in (0, r(x)) \) provided \( B_{r(x)}(x) \) is admissible, identity \((2.8)\) holds as well. Applying the Laplacian to the integral on the left-hand side of the latter identity, we obtain

\[
\int_{|y|<r} \nabla^2 u(x+y) \, dy = \int_{|y|=r} \nabla_x u(x+y) : \frac{y}{r} \, dS_y.
\]

Here the equality is a consequence of Green’s first formula. By changing variables this can be written as follows:

\[
r^{-m-1} \frac{\partial}{\partial r} \int_{|y|=1} u(x+ry) \, dS_y = |S_1(0)| \frac{m}{m-2} \frac{\partial}{\partial r} M^0(u(x,r)).
\]

However, \( M^0(u(x,r)) = a^0(\mu r) u(x) \), and so the second formula \((2.6)\) yields that

\[
\frac{\partial}{\partial r} M^0(u(x,r)) = - \frac{\mu I_{m/2}(\mu r)}{(\mu r)^{m-2/2}} u(x).
\]

Combining the above considerations and \((2.8)\), we find that for every \( x \in D \) the equality

\[
\int_{|y|<r} [\nabla^2 u - \mu^2 u](x+y) \, dy = 0 \quad \text{holds for all } r \in (0, r(x)).
\]

Thus, every ball \( B_r(x) \) contains a point \( y(r,x) \) such that \([\nabla^2 u - \mu^2 u](y(r,x)) = 0\). Since \( y(r,x) \to x \) as \( r \to 0 \), we conclude by continuity that \( u \) satisfies equation \((1.1)\) at every \( x \in D \), that is, \( u \) is panharmonic in \( D \).

Let identity \((2.8)\) hold for \( u \) instead of \((2.1)\). Since these two identities are equivalent (see Remark 2.1), \((2.1)\) holds for \( u \) as well. Then the previous considerations yield the assertion.

**Corollary 3.1.** Let \( D \subset \mathbb{R}^m, m \geq 2 \), be a bounded domain, and let \( u \in C^0(D) \). If for every \( x \in D \) there exists \( r(x) \) such that \( B_{r(x)}(x) \) is admissible and \( M^0(u(x,r))/a^0(\mu r) \) does not depend on \( r \in (0, r(x)) \), then \( u \) is \( \mu \)-panharmonic in \( D \).

**Proof.** According to the mean value theorem for integrals, for every \( x \in D \) and each \( r \in (0, r(x)) \) there exists \( x_0(r) \in \partial B_{r(x)}(x) \) such that \( M^0(u(x,r)) = u(x_0(r)) \), and so

\[
M^0(u(x,r),a^0(\mu r)) = u(x_0(r))/a^0(\mu r).
\]

Since this continuous function of \( r \) is constant on \((0, r(x))\), it is equal to its limit as \( r \to 0 \). Since \( a^0(\mu r) \to 1 \) and \( u(x_0(r)) \to u(x) \) as \( r \to 0 \), we obtain that \((2.1)\) holds for every \( x \in D \) and all \( r \in (0, r(x)) \). Then Theorem 3.1 yields the assertion. \( \square \)
Now, we prove the converse of identity (2.9), which generalizes the result of Beckenbach and Reade \[3\] for harmonic functions; it was announced in \[26\] without proof.

**Theorem 3.2.** Let \( D \subset \mathbb{R}^m, m \geq 2, \) be a bounded domain, and let \( u \in C^0(D). \) If identity (2.9) holds for every \( x \in D \) and all \( r \in (0, r(x)) \), where \( r(x) > 0 \) is such that the ball \( B_{r(x)}(x) \) is admissible, then \( u \) is \( \mu \)-panharmonic in \( D \).

**Proof.** Let \( \rho > 0 \) be sufficiently small. If \( r \in (0, \rho) \), then \( M^\bullet(x, r, u) \) is defined for every \( x \), which belongs to an open subset of \( D \) depending on the smallness of \( \rho \). Moreover, \( M^\bullet(x, r, u) \) is differentiable with respect to \( r \) and

\[
\frac{\partial M^\bullet(x, r, u)}{\partial r} = m r^{-1} [M^\circ(x, r, u) - M^\bullet(x, r, u)] \quad \text{for } r \in (0, \rho).
\]

Since

\[
\frac{a^\bullet(\mu r)}{a^\circ(\mu r)} = \frac{m I_{m/2}(\mu r)}{\mu r I_{(m-2)/2}(\mu r)},
\]

the previous relation takes the form

\[
\frac{\partial M^\bullet}{M^\bullet} = \frac{I_{(m-2)/2}(\mu r)}{I_{m/2}(\mu r)} - \frac{m}{r} = \frac{I_{m/2}'(\mu r)}{I_{m/2}(\mu r)} - \frac{m}{2r},
\]

where the last equality is a consequence of the recurrence formula (47), p. 79):

\[
I_{\nu-1}(z) = I_{\nu}'(z) + \frac{\nu}{z} I_{\nu}(z).
\]

The equation for \( M^\bullet \) has logarithmic derivatives on both sides. Therefore, integrating with respect to \( r \) over the interval \( (\epsilon, \rho) \), we obtain, after letting \( \epsilon \to 0 \), relation (2.8) with \( r \) changed to \( \rho \). Indeed, shrinking \( B_{\epsilon}(x) \) to its centre on the left-hand side, we see that \( M^\bullet(x, \epsilon, u) \to u(x) \) because \( u \in C^0(D) \), and this takes place for every \( x \) in an arbitrary closed subset of \( D \). By letting \( \epsilon \to 0 \) on the right-hand side, the factor \( \Gamma \left( \frac{m}{2} + 1 \right) \) arises due to the leading term of the power expansion of \( I_{m/2} \). Thus we have

\[
M^\bullet(x, \rho, u) = a^\bullet(\mu \rho) u(x)
\]

for every \( x \in D \) and all admissible \( \rho \). Hence \( u \) is \( \mu \)-panharmonic in \( D \) by the second assertion of Theorem 3.1. \( \square \)

### 3.1 Restricted mean value property

There is a long series of publications dealing with the so-called restricted mean value properties that characterize harmonicity; see the survey article \[33\], Sections 5 and 6. The following definition is accommodated for panharmonic functions.
Definition 3.1. A real-valued function $f$ defined on an open set $G \subset \mathbb{R}^m$ is said to have the restricted mean value property with respect to spheres if for each $x \in G$ there exists a single sphere centred at $x$ of radius $r(x)$ such that $B_{r(x)}(x) \subset G$ and identity (2.1) holds for $f$ with $r = r(x)$.

Theorem 3.3. Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain such that the Dirichlet problem for equation (1.1) is soluble in $C^2(D) \cap C^0(\overline{D})$ for every continuous function given on $\partial D$. If $u \in C^0(\overline{D})$ has the restricted mean value property in $D$ with respect to spheres, then $u$ is $\mu$-panharmonic in $D$.

Proof. First, let us show that the theorem’s assumptions yield that
\[
\max_{x \in D} |u(x)| = \max_{x \in \partial D} |u(x)|. \tag{3.1}
\]
Reasoning by analogy with the proof of Proposition 2.2, we see that the restricted mean value property implies that for every $x \in D$ there exists $y \in D$ such that $|u(y)| > |u(x)|$. Then the considerations used in the proof of Theorem 2.2 yield (3.1).

Let $f$ denote the trace of $v$ on $\partial D$; then there exists $u_0 \in C^0(\overline{D})$ solving the Dirichlet problem for equation (1.1) in $D$ with $f$ as the boundary data. Hence $u_0$ satisfies identity (2.1) for all $x \in D$ and all admissible $S_r(x)$, and so the restricted mean value property is valid for $u - u_0$. Then the weak maximum principle (3.1) holds for $u - u_0$, thus implying that $u \equiv u_0$ in $D$ because $u \equiv u_0$ on $\partial D$. Then $u$ also satisfies (2.1) for every $x \in D$ and all admissible $S_r(x)$.

The question about domains in which the Dirichlet problem for an elliptic equation is soluble has a long history going back to George Green’s Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism published in 1828, where this problem for the Laplace equation was posed for the first time. The final answer when the Dirichlet problem for harmonic functions has a solution was given by Wiener [48] in 1924; the notion of capacity was introduced for this purpose.

The class of bounded domains such that the Dirichlet problem is soluble is the same for the modified Helmholtz equation and for the Laplace equation. This follows from the results of Oleinik [36] and Tautz [43]; they demonstrated independently and published in 1949 that this fact about the solubility of the Dirichlet problem is a common characteristic which is true for elliptic equations of rather general form (see the monograph [32], Ch. IV, Sect. 28, for a review of related papers).

3.2 A function with panharmonic means is panharmonic itself

Theorem 3.3 allows us to prove the following converse of Corollary 2.3.

Theorem 3.4. Let $D \subset \mathbb{R}^m$, be a bounded domain in which the Dirichlet problem for equation (1.1) is soluble. If $u \in C^2(D) \cap C^0(\overline{D})$ has $\mu$-panharmonic $M^\mu(u, \cdot, r)$ in $D_r$ for all $r \in (0, r_*)$, where $r_* > 0$ is such that $D_{r_*} \neq \emptyset$, then $u$ is $\mu$-panharmonic in $D$. 

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Proof. It is clear that every \(x \in D\) belongs to each \(D_r\) provided \(r < \text{dist}(x, \partial D)/2\), where \(\text{dist}(x, \partial D)\) is the distance from \(x\) to \(\partial D\). Let us fix some \(r(x) \in (0, \text{dist}(x, \partial D)/2)\); hence \(B_{r(x)}(x) \subset D_r\) for all described values of \(r\). Since the mean \(M^\circ(u, \cdot, r)\) is \(\mu\)-panharmonic in \(D_r\) for every such \(r\), Theorem 2.1 yields that

\[
M^\circ(M^\circ(u, \cdot, r), x, r(x)) = a^\circ(\mu r(x)) M^\circ(u, x, r)
\]

In view of (2.11) and (2.2), this can be written as follows:

\[
\frac{1}{\omega_m^2} \int_{S_1(0)} \int_{S_1(0)} u(x + r(x)y + rz) \, dS_z \, dS_y = \frac{a^\circ(\mu r(x))}{\omega_m} \int_{S_1(0)} u(x + ry) \, dS_y.
\]

Letting \(r \to 0\) in this equality, we obtain that the identity

\[
M^\circ(u, x, r(x)) = a^\circ(\mu r(x)) u(x)
\]

holds for every \(x \in D\) with some \(r(x)\) such that \(B_{r(x)}(x) \subset D\). Now, Theorem 3.3 yields that \(u\) is \(\mu\)-panharmonic in \(D\). \(\square\)

3.3 Converse of the asymptotic mean value property

The following converse of Proposition 2.1 generalizes the classical result obtained by Blaschke [5], Priwaloff [37], and Zaremba [50] for harmonic functions.

**Theorem 3.5.** Let \(D\) be a domain in \(\mathbb{R}^m\), \(m \geq 2\), and let \(u \in C^2(D)\). If identity (2.12) holds for every \(x \in D\), then \(u\) is \(\mu\)-panharmonic in \(D\).

The assertion also holds with \(M^\bullet(u, x, r)\) changed to \(M^\circ(u, x, r)\) in (2.12), provided the right-hand side term is changed to \(\mu^2 u(x)/(2m)\).

**Proof.** Let equality (2.12) hold; combining it and formula (2.13) one obtains that \(u\) is \(\mu\)-panharmonic in \(D\). In the same way, (2.12) and (2.14) yield the second assertion when \(M^\circ(u, x, r)\) stands in (2.12) instead of \(M^\bullet(u, x, r)\), whereas the right-hand side term is \(\mu^2 u(x)/(2m)\). \(\square\)

4 Characterizations of balls via panharmonic functions

It is worth mentioning first that analytic characterization of balls in the Euclidean space \(\mathbb{R}^m\) by means of harmonic functions has a long history; it started in the 1960s, in the pioneering notes [13], [14], and shortly afterwards the following general result was obtained.

**Theorem 4.1** (Kuran [21]). Let \(D\) be a domain (= connected open set) of finite (Lebesgue) measure in the Euclidean space \(\mathbb{R}^m\) where \(m \geq 2\). Suppose that there exists a point \(P_0\) in \(D\) such that, for every function \(h\) harmonic in \(D\) and integrable over \(D\), the volume mean of \(h\) over \(D\) equals \(h(P_0)\). Then \(D\) is an open ball (disk when \(m = 2\)) centred at \(P_0\).
Presumably, the paper \cite{18} was the first one in which this theorem was referred to as the property of harmonic functions inverse to the mean value identity for balls. The term became widely accepted. A slight modification of Kuran’s considerations shows that his theorem is valid even if $D$ is disconnected; see the survey article \cite{33}, p. 377, which also contains some improvements of Kuran’s theorem, and a discussion of its applications and of possible similar results involving certain averages over $\partial D$, when $D$ is a bounded domain. It occurs that panharmonic functions yield an analogous characterization of balls.

4.1 Inverse mean value property: volume means

The following result was recently proved in \cite{27}; see also the brief note \cite{25}. Before giving its precise formulation, we give two definitions. If $D$ is a bounded domain and a function $f$ is integrable over $D$, then

$$M^\bullet(f, D) = \frac{1}{|D|} \int_D f(x) \, dx$$

is the volume mean value of $f$ over $D$. Here and below $|D|$ is the domain’s volume (area if $D \subset \mathbb{R}^2$). Also, we define a dilated copy of $D$: $D^r = D \cup \{x \in \partial D : B_r(x) \not\subset D\}$. Thus, the distance from $\partial D^r$ to $D$ is equal to $r$.

**Theorem 4.2.** Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain, whose complement is connected, and let $r$ be a positive number such that $|B_r| \leq |D|$. Suppose that there exists a point $x_0 \in D$ such that for some $\mu > 0$ the mean value identity $u(x_0) a^\bullet(\mu r) = M^\bullet(u, D)$ holds for every positive function $u$, which is panharmonic in $D^r$, and $a^\bullet$ is defined in (2.8). If also $|D| = |B_r|$ provided $B_r(x_0) \setminus \overline{D} \neq \emptyset$, then $D = B_r(x_0)$.

Prior to proving this theorem, let us consider some properties of the function

$$U(x) = a^\circ(\mu|x|), \quad x \in \mathbb{R}^m,$$

The properties of $a^\circ$ defined in (2.1) show that this spherically symmetric function monotonically increases from unity to infinity as $|x|$ goes from zero to infinity.

Moreover, Poisson’s integral for $I_\nu$ (see \cite{35}, p. 223) implies that:

$$U(x) = \int_0^1 (1 - s^2)^{(m-3)/2} \cosh(\mu|x|s) \, ds.$$  \hspace{1cm} (4.1)

This representation is easy to differentiate, thus obtaining that $U$ is panharmonic in $\mathbb{R}^m$. Since the formulae for $a^\circ$ and $a^\bullet$ are similar, Poisson’s integral allows us to compare these functions. In that way, the inequality

$$[U(x)]_{|x|=r} > a^\bullet(\mu r)$$  \hspace{1cm} (4.2)

immediately follows.
Proof of Theorem 4.2. Without loss of generality, we suppose that the domain $D$ is located so that $x_0$ coincides with the origin. Let us show that the assumption that $D \neq B_r(0)$ leads to a contradiction.

It is clear that either $B_r(0) \subset D$ or $B_r(0) \setminus \overline{D} \neq \emptyset$ (the equality $|B_r| = |D|$ is assumed in the latter case), and we treat these two cases separately. Let us consider the second case first, for which purpose we introduce the bounded open sets

$$G_i = D \setminus B_r(0) \quad \text{and} \quad G_e = B_r(0) \setminus \overline{D},$$

whose nonzero volumes are equal in view of the assumptions about $D$ and $r$. The volume mean identity for $U$ over $D$ can be written as follows:

$$|D| a^*(\mu r) = \int_D U(y) \, dy; \quad (4.3)$$

here the condition $U(0) = 1$ is taken into account. Since formula (2.8) is valid for $U$ over $B_r(0)$, we write it in the same way:

$$|B_r| a^*(\mu r) = \int_{B_r(0)} U(y) \, dy. \quad (4.4)$$

Subtracting (4.4) from (4.3), we obtain

$$0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy > 0.$$

Indeed, the difference is positive since $U(y)$ (positive and monotonically increasing with $|y|$) is greater than $[U(y)]_{|y|=r}$ in $G_i$ and less than $[U(y)]_{|y|=r}$ in $G_e$, whereas $|G_i| = |G_e|$. This contradiction proves the result in this case.

In the case when $B_r(0) \subset D$, a contradiction must be deduced when $B_r(0) \neq D$, that is, $|G_i| = |D| - |B_r| > 0$. Now, subtracting (4.4) from (4.3), we obtain

$$\left(|D| - |B_r|\right) a^*(\mu r) = \int_{G_i} U(y) \, dy > |G_i| [U(y)]_{|y|=r},$$

where the last inequality is again a consequence of positivity of $U(y)$ and its monotonicity. This yields that $a^*(\mu r) > [U(y)]_{|y|=r}$, which contradicts (4.2). The proof is complete.

Remark 4.1. In Theorem 4.2, the domain $D$ is supposed to be bounded because it is easy to construct an unbounded domain of finite volume in which $U$ is not integrable. Thus, the boundedness of $D$ allows us to avoid imposing rather complicated restrictions on the domain.

In the limit $\mu \to 0$, one obtains Laplace’s equation from (1.1), whereas the assumption about $r$ becomes superfluous in this case. Hence, Theorem 4.2 turns into an improved version of Kuran’s theorem because only positive harmonic functions are involved.
Furthermore, the integral $\int_D u(y) \, dy$ can be replaced by the flux $\int_{\partial D} \partial u / \partial n_y \, dS_y$ in the formulation of Theorem 4.2 provided $\partial D$ is sufficiently smooth; here $n$ is the exterior unit normal. Indeed, we have

$$\int_D u(y) \, dy = \mu^{-2} \int_D \nabla^2 u(y) \, dy = \mu^{-2} \int_{\partial D} \partial u / \partial n_y \, dS_y.$$ 

These relations are used in [26]; see comments to Theorem 9 of that paper.

4.2 Characterization of balls via fundamental solutions of equation (1.1)

A different approach to harmonic characterization of balls was developed by Aharonov, Schiffer and Zalcman [1]. The origin of a rather unusual title of their paper (potato kugel is a traditional dish of Jewish cuisine commonly served for Shabbat) is explained in Zalcman’s comment; see [41], p. 497. Namely, these authors proved the following.

**Theorem 4.3** (ASZ, [1]). Let $D \subset \mathbb{R}^3$ be a bounded open set. If the equality

$$\int_D \frac{dy}{|y-x|} = \frac{a}{|x|} + b$$

holds with suitable real constants $a$ and $b$ for every $x \in \mathbb{R}^3 \setminus D$, then $D$ is an open ball centred at the origin, $a = |D|$ and $b = 0$.

Since $|y-x|^{-1}$ is a fundamental solution of the Laplace equation for $m = 3$, this theorem answers in the affirmative the following question posed to the authors (see [1], p. 331):

Let $D$ be a solid, homogeneous, compact, connected “potato” in space, which gravitationally attracts each point outside it as if all its mass were concentrated at a single point [. . .] Must $D$ be spherical, i.e. a ball?

There are various generalizations and improvements of this result. In particular, the following one was obtained in the recent article [10].

**Theorem 4.4** (Cupini, Lanconelli, [10]). Let $D \subset \mathbb{R}^m$, $m \geq 3$, be an open set such that $|D| < \infty$. If for some $x_0 \in D$ the identity

$$|D|^{-1} \int_D |y-x|^{2-m} \, dy = |x_0 - x|^{2-m}$$

holds for every $x \in \mathbb{R}^m \setminus D$, then $D$ is an open ball centred at $x_0$.

A similar result, to which we now turn, is valid for the potential

$$E_\mu(x, y) = \frac{\exp\{-\mu|x-y|\}}{|x-y|}, \quad \mu > 0, \quad x \in \mathbb{R}^3 \setminus \{y\}, \quad (4.5)$$
rapidly decaying with the distance; it was proposed by Yukawa \cite{49} to describe a source of nuclear force located at \( y \in \mathbb{R}^3 \). Following the paper \cite{1}, we restrict our considerations to three dimensions, thus answering the quoted question for the nuclear setting. Since there is another linearly independent fundamental solution of (1.1), namely,

\[
E^\pm_\mu(x,y) = \exp\left\{ \mu |x-y| \right\}, \quad \mu > 0, \quad x \in \mathbb{R}^3 \setminus \{y\},
\]

(4.6)
it must also be taken into account.

For every \( r > 0 \) and arbitrary \( x_0 \in \mathbb{R}^3 \), these fundamental solutions define two families of integrable panharmonic functions

\[
B_r(x_0) \ni y \mapsto E^\pm_\mu(y, x) \text{ parametrised by } x \in \mathbb{R}^3 \setminus B_r(x_0).
\]

For every element of these families the mean value property (2.8) yields that

\[
a_3(\mu r) E^\pm_\mu(x, x_0) = M^*(E^\pm_\mu(\cdot, x), x_0, r), \quad a_3(t) = \frac{\sqrt{2\pi} I_{3/2}(t)}{t^{3/2}};
\]

(4.7)
the latter function is \( a^*(t) \) for \( m = 3 \). In view of Theorem 4.4 and identity (4.7), we prove the following.

**Theorem 4.5.** Let \( D \subset \mathbb{R}^3 \) be a bounded domain, whose complement is connected, and let \( r > 0 \) be such that \( |B_r| = |D| \). If the fundamental solutions (4.5) and (4.6) satisfy the mean value identity

\[
a_3(\mu r) E^\pm_\mu(x, x_0) = M(E^\pm_\mu(\cdot, x), D)
\]

(4.8)
for some \( x_0 \in D \) and every \( x \notin D \), then \( D = B_r(x_0) \).

**Proof.** Since \( E^+_\mu(x, x_0) \) and \( E^-_\mu(x, x_0) \) satisfy (4.8) for every \( x \notin D \), the same is true for every linear combination of these fundamental solutions. In particular,

\[
|D| a_3(\mu r) \frac{\sinh(\mu |x-x_0|)}{\mu |x-x_0|} = \int_D \frac{\sinh(\mu |x-y|)}{\mu |x-y|} \, dy \quad \text{for every } x \notin D.
\]

Moreover, this identity is valid throughout \( \mathbb{R}^3 \), because real-analytic functions of \( x \) stand on both sides (a consequence of the fact that \( z^{-1} \sinh z \) is an entire function). Substituting \( x = x_0 \), we obtain

\[
|D| a_3(\mu r) = \int_D \frac{\sinh(\mu |x_0-y|)}{\mu |x_0-y|} \, dy.
\]

Let us relocate, without loss of generality, the domain \( D \) so that \( x_0 \) coincides with the origin, which simplifies the identity to

\[
|D| a_3(\mu r) = \int_D U(y) \, dy, \quad \text{where } U(y) = \frac{\sinh(\mu |y|)}{\mu |y|},
\]

(4.9)
because this is the function \((4.1)\) with \(m = 3\).

On the other hand, the mean value property \((2.8)\) is valid for \(U\) over \(B_r:\)

\[ |B_r| a_3(\mu r) = \int_{B_r} U(y) \, dy. \quad (4.10) \]

If we assume that \(D \neq B_r\), then \(G_i = D \setminus \overline{B_r}\) and \(G_e = B_r \setminus \overline{D}\) are bounded open sets such that \(|G_e| = |G_i| \neq 0\), which follows from the assumptions made about \(D\) and \(r\). Then, subtracting \((4.10)\) from \((4.9)\), we obtain

\[ 0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy > 0. \]

Indeed, the difference is positive since \(U(y)\) (positive and monotonically increasing with \(|y|\)) is greater than \([U(y)]_{|y|=r}\) in \(G_i\) and less than \([U(y)]_{|y|=r}\) in \(G_e\), whereas \(|G_i| = |G_e|\). The obtained contradiction proves the result.

**Remark 4.2.** The final part of this proof repeats literally the argument used in the proof of Theorem 4.2.

### 5 Relations between harmonic and panharmonic functions

The motivation to consider relations between harmonic and panharmonic functions comes from the theorem on subharmonic functions published by F. Riesz [39] in 1930. It establishes the decomposition of such a function into the sum of a harmonic function and a Newtonian potential. (The result was proved by Riesz for functions of two variables, whereas the general case can be found in [19], Section 3.5.) It occurs that panharmonic and subharmonic functions have a lot in common; see Corollary 2.1. Therefore, it is reasonable to apply methods developed for subharmonic functions in studies of panharmonic ones.

#### 5.1 Properties of positive panharmonic functions

It is worth to recall the Riesz decomposition theorem for subharmonic functions (see, for example, [19], Theorem 3.9).

**Theorem 5.1.** If \(u\) is subharmonic in a domain \(D \subset \mathbb{R}^m, m \geq 2\), then there exists a unique Borel measure \(m\) in \(D\) such that for any compact set \(K \subset D\)

\[ u(x) = \int_K E_m(x - y) \, dm(y) + h(x), \quad x \in \text{int}K, \quad (5.1) \]

where \(\text{int}K\) is the interior of \(K\) and \(h\) is harmonic there.
Here $E_m(x - y)$ is the fundamental solution of the Laplace equation:

$$E_m(x - y) = \left[ (2 - m) \omega_m |x - y|^{(m-2)} \right]^{-1} \quad \text{when } m \geq 3,$$

whereas $E(x - y) = (2\pi)^{-1} \log |x - y|$.

**Remark 5.1.** It follows from Treves’ considerations (see [44], pp. 288–289) that if $u \geq 0$ is subharmonic in a bounded domain $D$, then formula (5.1) holds with $K$ changed to $D$, whereas $d\mathbf{m}(y) = \nabla^2 u(y) \, dy$ and $h$ is the positive least harmonic majorant of $u$ in $D$; for its definition see also [2], p. 79.

Now we are in a position to formulate the following.

**Theorem 5.2.** Let $u \geq 0$ be $\mu$-panharmonic in a domain $D \subset \mathbb{R}^m$, $m \geq 2$, then (5.1) takes the following form:

$$h(x) = u(x) - \mu^2 \int_K E_m(x - y) \, u(y) \, dy, \quad x \in \text{int}K. \quad (5.2)$$

Here $K \subset D$ is a compact set and $h$ is harmonic in $\text{int}K$.

If $D$ is bounded and, besides, $u \in C^0(\overline{D})$, then (5.2) is valid in the whole $D$ with the integral over $D$, whereas $h \geq 0$ is the least harmonic majorant of $u$ in $D$.

**Proof.** According to Corollary 2.1, $u$ is subharmonic in $D$, and so it has the Riesz decomposition (5.1). Applying the Laplacian to both sides of (5.1) and taking into account equation (1.1) on the left-hand side and using the harmonicity of $h$ and the definition of $E_m$ on the right, we see that $\mathbf{m}$ is proportional to the Lebesgue measure with the coefficient $\mu^2 u$ (cf. Remark 5.1). Now (5.2) follows by rearranging.

The second assertion is obvious, whereas the last one is a consequence of the considerations mentioned in Remark 5.1.

Our next result involves mean values over a domain as well as over its boundary. In this case, one can hardly expect an identity analogous to (2.9) to be valid for panharmonic functions in a domain distinct from a ball. Indeed, Bennett [4] proved the following.

**Theorem 5.3.** Let $D \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary. If

$$\left| D \right|^{-1} \int_D h(y) \, dy = \left| \partial D \right|^{-1} \int_{\partial D} h(y) \, dS_y$$

for every $h \in C^2(\overline{D})$ harmonic in $D$, then $D$ is an open ball.

A similar conjecture for panharmonic functions based on identity (2.9) is made in [27]. At the same time, an inequality holds between the mean values of nonnegative panharmonic functions in a bounded domain under a suitable assumption about its boundary.
Proposition 5.1. Let $D \subset \mathbb{R}^m$ be a bounded domain satisfying the exterior sphere condition uniformly on $\partial D$. Then there exists a constant $c \in [1, \infty)$, depending on $D$ and $\mu$, and such that
\[
|D|^{-1} \int_D u(y) \, dy \leq c |\partial D|^{-1} \int_{\partial D} u(y) \, dS_y
\]
for every nonnegative panharmonic function $u \in C^0(D)$.

In view of Corollary 2.1, inequality (5.3), like Theorem 5.2, is a consequence of the corresponding theorem proved for subharmonic functions; see [15], p. 195. Moreover, if $D$ is a ball $B_r$ (there is no need to specify its center), then equality takes place in (5.3) with
\[
c = a^*(\mu r) = \frac{m I_{m/2}(\mu r)}{\mu r I_{(m-2)/2}(\mu r)} < \frac{m}{\mu r}.
\]
Here the equalities follow from identity (2.9) and formulae (2.1) and (2.8), whereas the inequality is a consequence of the definition of $I_\nu$. This not only demonstrates that $c$ depends on $\mu$, but also improves Proposition 5.1 for balls provided $\mu r > m$. It occurs that $c$ can be arbitrarily small when either $r$ ($\mu$ fixed) or $\mu$ ($r$ fixed) is sufficiently large (or both are sufficiently large).

5.2 Characterization of panharmonic functions

Let $D$ be a bounded domain in $\mathbb{R}^m$, $m \geq 3$; for $u \in L^2(D)$ we define the operator:
\[
(Tu)(x) = \int_D E_m(x - y) u(y) \, dy, \quad x \in D.
\]
Its symmetric kernel is positive after dropping the negative coefficient and it has well-known properties; for example, $T$ is compact in the Banach space $C^0(D)$ (see, for example, [31], Chapter 7).

In terms of this operator, the second assertion of Theorem 5.2 admits the following interpretation: $I - \mu^2 T$ (as usual, $I$ stands for the identity operator) maps the cone of nonnegative $\mu$-panharmonic functions into the cone of nonnegative harmonic functions within the Banach space $C^0(D)$.

Let us consider whether there exists an inverse mapping: harmonic $\mapsto \mu$-panharmonic functions. To this end we introduce the integral equation
\[
u(x) - \lambda(Tu)(x) = h(x), \quad x \in D, \quad \lambda \in \mathbb{R},
\]
where $u, h \in L^2(D)$. This is a natural setting because the operator $T$ has a weakly singular kernel, and so is compact and self-adjoint, whereas $-T$ is a positive operator in this space.
We recall that these properties of $T$ imply that it has a sequence $\{\lambda_n\}_{1}^{\infty}$ of characteristic values each having a finite multiplicity; moreover, these values are real negative numbers such that $|\lambda_n| \to \infty$ as $n \to \infty$. Finally, if $\lambda \neq \lambda_n$ for $n = 1, 2, \ldots$ (in particular, if $\lambda > 0$), then for any $h \in L^2(D)$ equation (5.4) has a unique solution $u \in L^2(D)$, which can be represented by virtue of the resolvent kernel. Taking into account these facts, we formulate and prove the following assertion, in which $C^{0,\alpha}(\overline{D})$ stands for the Banach space of functions that are Hölder continuous with exponent $\alpha \in (0, 1)$.

**Theorem 5.4.** Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^m$, $m \geq 3$, and let $\lambda = \mu^2 > 0$ in equation (5.4). If $h \in C^{0,\alpha}(\overline{D})$ is harmonic in $D$, then a unique solution $u$ of this equation belongs to $C^{0,\alpha}(\overline{D})$ and is $\mu$-panharmonic in $D$.

**Proof.** It is a classical result (see, for example, [31], Theorem 8.6.1) that an $L^2$-solution of a weakly singular integral equation (it exists in our case) is in $C^{0}(\overline{D})$ provided the right-hand side term has this property. However, the continuity of $u$ does not guarantee the existence of second derivatives of the Newtonian potential $T u$. Let us establish their existence under the assumptions made in the theorem.

Since $h \in C^{0,\alpha}(\overline{D})$, the solution $u$ has the same property. Indeed, writing the equation as follows

$$u = \mu^2 T u + h, \quad (5.5)$$

we see that both terms on the right are in $C^{0,\alpha}(\overline{D})$, because this is a consequence of the following result (see [15], Lemma 2.3). If $u \in L^\infty(D)$, then $T u \in C^{0,1}(\overline{D})$, that is, $T u$ is Lipschitz continuous. Now, another classical result (see [17], Lemma 4.2) yields that $T u \in C^2(D)$ and $\nabla^2(T u) = u$. Furthermore, relation (5.5) implies that $u \in C^2(D)$ since $h$ is harmonic in $D$. Then, applying the Laplacian to both sides of (5.5), we obtain that $u$ is $\mu$-panharmonic in $D$. $\square$

In other words, for every $\mu^2 > 0$ there exists the bounded operator

$$(I - \mu^2 T)^{-1} : L^2(D) \to L^2(D),$$

which maps each harmonic in $D$ function from $C^{0,\alpha}(\overline{D})$ to a $\mu$-panharmonic function belonging to the same Hölder space. It is not clear whether the range of this operator comprises the whole set of $\mu$-panharmonic functions. At the same time, Theorem 5.2 yields that every nonnegative function belonging to this set has a pre-image in the cone of nonnegative harmonic functions. Hence, all nonnegative $\mu$-panharmonic functions are in the operator’s range.

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