A GENERALIZATION OF WILF’S CONJECTURE FOR GENERALIZED NUMERICAL SEMIGROUPS

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Abstract. A numerical semigroup is a submonoid of \(\mathbb{N}\) with finite complement in \(\mathbb{N}\). A generalized numerical semigroup is a submonoid of \(\mathbb{N}^d\) with finite complement in \(\mathbb{N}^d\). In the context of numerical semigroups, Wilf’s conjecture is a long standing open problem whose study has led to new mathematics and new ways of thinking about monoids. A natural extension of Wilf’s conjecture, to the class of \(C\)-semigroups, was proposed by García-García, Marín-Aragón, and Vigneron-Tenorio. In this paper, we propose a different generalization of Wilf’s conjecture, to the setting of generalized numerical semigroups, and prove the conjecture for several large families including the irreducible, symmetric, and monomial case. We also discuss the relationship of our conjecture to the extension proposed by García-García, Marín-Aragón, and Vigneron-Tenorio.

Introduction

Let \(S\) be a submonoid of \(\mathbb{N}\). The hole set of \(S\) is defined as \(H(S) = \mathbb{N} \setminus S\). If \(H(S)\) is a finite set then \(S\) is called a numerical semigroup. Every numerical semigroup \(S\) admits a unique, minimal, finite set of generators \(G(S)\). Thus, there is a unique finite set \(G(S)\) such that every element of \(S\) is an \(\mathbb{N}\)-linear combination of elements in \(G(S)\) while no proper subset of \(G(S)\) has the same property. Well-known invariants of a numerical semigroup \(S\) are \(e(S) = |G(S)|\), \(F(S) = \max\{k \mid k \in H(S)\}\), and \(n(S) = |\{s \in S \mid s < F(S)\}|\). For relationships between these and other invariants of numerical semigroups see [16]. An intriguing matter of study concerns the conjectured inequality \(e(S)n(S) \geq F(S) + 1\). This inequality is known as Wilf’s conjecture because of its first appearance in [20]. While Wilf’s conjecture has been proved for several classes of numerical semigroups (see for instance [7, 8, 14, 18]), it is still open in general. The survey [5] is a good reference for the state of the art on this long standing open problem.

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A monoid $S \subseteq \mathbb{N}^d$ is called a \textit{generalized numerical semigroup} (GNS) if the hole set $H(S) = \mathbb{N}^d \setminus S$ is finite. As in the case where $d = 1$, a generalized numerical semigroup, $S$, has a unique, minimal, finite set of generators, $G(S)$ and we let $e(S) = |G(S)|$. Additional properties and features of generalized numerical semigroups are provided in [3, 4, 9]. A problem posed in [9] was to formulate extensions of Wilf’s conjecture to the setting of generalized numerical semigroups. A first possible extension was given in [11] for a larger class of semigroups called $C$-semigroups. Their extension is quite natural and has the additional feature of depending on a monomial order. We propose an alternate generalization of Wilf’s conjecture to the setting of generalized numerical semigroups. Suppose $x, y \in \mathbb{N}^d$ with $x = (x^{(1)}, \ldots, x^{(d)})$ and $y = (y^{(1)}, \ldots, y^{(d)})$. There is a natural partial order, $\leq$ on $\mathbb{N}^d$, by setting $x \leq y$ if and only if $x^{(i)} \leq y^{(i)}$ for all $i = 1, \ldots, d$. Using this partial order, define $n(S) = |\{x \in S \mid x \leq h \text{ for some } h \in H(S)\}|$ and $c(S) = |\{x \in \mathbb{N}^d \mid x \leq h \text{ for some } h \in H(S)\}|$. With this notation in place, we propose the following generalization of Wilf’s conjecture to the setting of generalized numerical semigroups:

\textbf{Generalized Wilf Conjecture.} If $S \subseteq \mathbb{N}^d$ is a GNS then $e(S)n(S) \geq dc(S)$.

This paper is concerned with motivating the above conjecture, proving it for several large classes of generalized numerical semigroups, and contrasting it with the extension of Wilf’s conjecture proposed in [11]. Throughout this paper we will refer to the above conjecture as the \textit{Generalized Wilf Conjecture}.

In Section 1 we recall the most important properties about irreducible generalized numerical semigroups, a class studied in [4]. Furthermore, we explain why the Generalized Wilf Conjecture can be considered as a generalization of Wilf’s conjecture. In Section 2 we introduce an operation called \textit{thickening} and in Section 3 we use this operation to prove that irreducible generalized numerical semigroups satisfy the proposed conjecture. In Section 4 we consider the class of \textit{monomial semigroups}, i.e. semigroups satisfying the statistic $n(S) = 1$, and we show that elements in this class also satisfy the Generalized Wilf Conjecture. In Section 5 we compare our proposed extension of Wilf’s conjecture with the one given in [11]. The last two sections are devoted to providing additional computational evidence for the Generalized Wilf Conjecture and concluding remarks.

1. \textbf{Frobenius, irreducible, and symmetric semigroups and the Generalized Wilf Conjecture}

In this section, we discuss \textit{Frobenius} and \textit{irreducible} generalized numerical semigroups as introduced in [4]. We motivate our statement of the Generalized Wilf Conjecture by considering how Wilf’s conjecture can be extended to Frobenius semigroups, and we prove the Generalized Wilf Conjecture for a
certain class of Frobenius semigroups. We also fix the basic notation and vocabulary that will be used throughout the paper.

Throughout the paper, \( S \) refers to a generalized numerical semigroup. The set of pseudo-Frobenius elements of \( S \) is \( PF(S) = \{ h \in H(S) \mid h + S \subseteq S \} \) while the set of special gaps is \( EH(S) = \{ h \in PF(S) \mid 2h \in S \} \). \( S \) is called irreducible if it is not possible to express \( S \) as the intersection of two larger generalized numerical semigroups. We have the following theorems.

**Theorem 1.1** ([4], Theorem 2.9). Let \( S \subseteq \mathbb{N}^d \) be a GNS. Then the following statements are equivalent:

1. \( |PF(S)| = 1 \).
2. \( PF(S) = \{ f \} \) and \( f \) has at least one odd component.
3. There exists an \( f \in H(S) \) such that \( f - h \in S \) for all \( h \in H(S) \).

**Theorem 1.2** ([4], Theorem 2.10). Let \( S \subseteq \mathbb{N}^d \) be a GNS. Then the following statements are equivalent:

1. \( PF(S) = \{ f, f^2 \} \).
2. There exists an \( f \in H(S) \) with even components, such that \( f - h \in S \) for all \( h \in H(S) \setminus \{ f^2 \} \).

**Definition 1.3.** \( S \) is symmetric if it satisfies the conditions of Theorem 1.1. \( S \) is pseudo-symmetric if it satisfies the conditions of Theorem 1.2. In both cases \( EH(S) = \{ f \} \).

**Definition 1.4.** If there exists a unique maximal element \( f \in H(S) \), with respect to the natural partial order in \( \mathbb{N}^d \), then we say that \( S \) is Frobenius with Frobenius element \( f \). We say that \( (S, f) \) is a Frobenius GNS.

**Theorem 1.5** ([4]). Let \( S \subseteq \mathbb{N}^d \).

1) \( S \) is irreducible if and only if \( |EH(S)| = 1 \).
2) \( S \) is symmetric if and only if there exists \( f \in H(S) \) with \( 2|H(S)| = (f^{(1)} + 1)(f^{(2)} + 1) \cdots (f^{(d)} + 1) \).
3) \( S \) is pseudo-symmetric if and only if there exists \( f \in H(S) \) with \( 2|H(S)| - 1 = (f^{(1)} + 1)(f^{(2)} + 1) \cdots (f^{(d)} + 1) \).

In each case, \( (S, f) \) is a Frobenius GNS where in 1), we assume \( EH(S) = \{ f \} \).

Recall that \( S \subseteq \mathbb{N}^d \) has a unique finite set of minimal generators (see [3]). We will use the following notation throughout the paper:

**Definition 1.6.** Given a generalized numerical semigroup \( S \subseteq \mathbb{N}^d \) we define

- \( G(S) = \) the set of minimal generators of \( S \)
- \( H(S) = \mathbb{N}^d \setminus S \)
- \( C(h) = \{ x \in \mathbb{N}^d \mid x \leq h \} \)
- \( C(S) = \{ x \in \mathbb{N}^d \mid x \leq h \text{ for some } h \in H(S) \} \)
- \( N(h) = \{ x \in S \mid x \leq h \} \)
\[ N(S) = \{ x \in S \mid x \leq h \text{ for some } h \in H(S) \} \]
\[ H(h) = \{ x \in H(S) \mid x \leq h \} \]
\[ n(S) = |N(S)|, c(S) = |C(S)|, g(S) = |H(S)|, \text{ and } e(S) = |G(S)| \]

Consider the map:
\[ \Psi_h : N(h) \to H(h) \] defined by \[ \Psi_h(s) = h - s. \]

It is elementary to show that \( \Psi_h \) is well defined and injective. As a consequence, \( |N(h)| \leq |H(h)| \leq |H(S)|. \)

**Proposition 1.7.** Let \( S \subseteq \mathbb{N}^d \) be a symmetric GNS with Frobenius element \( f \). Then \( e(S)n(S) \geq d(f^{(1)} + 1) \cdots (f^{(d)} + 1) \).

**Proof.** Since \( S \) is Frobenius, we have \( |H(f)| = |H(S)|, |N(f)| = |N(S)| \), and \( |C(f)| = |C(S)| \). Since \( S \) is symmetric, we have \( 2|H(S)| = (f^{(1)} + 1)(f^{(2)} + 1) \cdots (f^{(d)} + 1) = |C(f)| = |C(S)| = |N(S)| + |H(S)| \). This implies that \( n(S) = |N(S)| = |H(S)| \). Since \( e(S) \geq 2d \) by [11 Theorem 11], we have \( e(S)n(S) \geq 2d|H(S)| = d(f^{(1)} + 1) \cdots (f^{(d)} + 1) \) (by Theorem [11]). \( \square \)

Let \( S \) be a symmetric numerical semigroup with Frobenius number \( F(S) \), then by Proposition [11.7] with \( d = 1 \), \( S \) satisfies \( e(S)n(S) \geq F(S) + 1 \). This inequality is known as Wilf’s conjecture and has been shown to be satisfied by several classes of numerical semigroups. In general, the conjecture is wide open and is one of the long standing open problems in the study of numerical semigroups. The proposition above suggests a straightforward generalization for the Wilf conjecture for Frobenius generalized numerical semigroups.

**Conjecture 1.8.** (The Generalized Wilf Conjecture for Frobenius GNS) Let \( (S, f) \) be a Frobenius GNS in \( \mathbb{N}^d \). Then \( e(S)n(S) \geq d(f^{(1)} + 1) \cdots (f^{(d)} + 1) \).

Observe that, if \( S \subseteq \mathbb{N}^d \) is a GNS and \( h \in H(S) \) then
\[ |C(h)| = |N(h)| + |H(h)| = (h^{(1)} + 1)(h^{(2)} + 1) \cdots (h^{(d)} + 1). \]

The key idea in the previous conjecture is to substitute the value \( F(S) + 1 \), for numerical semigroups, with the cardinality of the set \( C(f) \) in the case of Frobenius generalized numerical semigroups \( (S, f) \), for which there exists a unique Frobenius element. However there are more general situations and \( F(S) + 1 \) may be replaced in a different way. Note that if the GNS is a \( (S, f) \) Frobenius GNS then
\[ |C(S)| = |C(f)| = (f^{(1)} + 1) \cdots (f^{(d)} + 1). \]

\( |C(S)| \) is known as the conductor if \( S \) is a numerical semigroup.

**Lemma 1.9.** Let \( S \subseteq \mathbb{N}^d \) be a GNS of genus \( g(S) \). Then
\[ (1) \ |C(S)| = |H(S)| + |N(S)|. \]
\[ (2) \ (h^{(1)} + 1) \cdots (h^{(d)} + 1) \leq |C(S)| \text{ for every } h \in H(S). \]

**Proof.** Trivial. \( \square \)
Example 1.10. In Figure 1, we consider the generalized numerical semigroup \( S = \mathbb{N}^2 \setminus \{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 0), (3, 2)\}. The minimal system of generators of \( S \) is the set \( G(S) = \{(2, 0), (5, 0), (0, 2), (0, 3), (1, 5), (1, 6), (3, 1), (4, 1)\}. The holes of \( S \), marked black in Figure 1, determine the red region \( C(S) \) in the figure. The elements of \( S \) lying in the red region are the elements of \( N(S) = \{(0, 0), (2, 0), (3, 1), (0, 2), (2, 2), (0, 3), (0, 4)\} \) (marked by a circle). We have \( n(S) = |N(S)| = 7 \), \( e(S) = |G(S)| = 8 \), \( C(S) \) is the disjoint union of \( H(S) \) and \( N(S) \), and \( c(S) = |C(S)| = 16 \). Note that for any \( h \in H(S) \) we have \((h^{(1)} + 1)(h^{(2)} + 1) \leq 16\).

If we let \( c(S) \), for generalized numerical semigroups, play the role of \( F(S) + 1 \) in numerical semigroups, we can extend Conjecture 1.8 to arbitrary generalized numerical semigroups as follows.

Generalized Wilf Conjecture. If \( S \subseteq \mathbb{N}^d \) then \( e(S)n(S) \geq d c(S) \).

Remark 1.11. The Generalized Wilf Conjecture can also be stated for the class of \( C \)-semigroups considered in [11]. For readability (and to aid intuition) we save discussion of this for a future paper.

2. Multiplicity and thickenings

A crucial concept for numerical semigroups is multiplicity. There is a natural way to define this notion for generalized numerical semigroups which we will use to verify the Generalized Wilf Conjecture for large classes of generalized numerical semigroups.
Definition 2.1. Let \( S \subset \mathbb{N}^d \) be a generalized numerical semigroup. Let \( M(S)^* = \{ h \in H(S) \mid C(h) \cap S = \{ 0 \} \} \). Equivalently, \( M(S)^* \) consists of all non-zero \( x \in \mathbb{N}^d \) satisfying that \( 0 \) is the only element of \( S \) less than or equal to \( x \) in the natural partial order on \( \mathbb{N}^d \). Following [12], we call the elements of \( M(S)^* \) the fundamental holes of \( S \). Let \( M(S) = M(S)^* \cup \{ 0 \} \). The multiplicity of \( S \) is defined as \( m(S) = |M(S)| \).

Lemma 2.2. The set \( M(S) \) is the minimal subset of \( \mathbb{N}^d \) satisfying that every \( x \in \mathbb{N}^d \) can be written as \( x = m + s \), where \( m \in M(S) \) and \( s \in S \).

Proof. We first prove that every \( x \in \mathbb{N}^d \) can be written as \( x = m + s \), where \( m \in M(S) \) and \( s \in S \). Suppose \( x \in \mathbb{N}^d \). Let \( s \) be a maximal element of \( S \) (under the natural partial order on \( \mathbb{N}^d \)) so that \( s \leq x \). Write \( x = s + (x - s) \); clearly \( x - s \in \mathbb{N}^d \) since \( s \leq x \). We prove that \( x - s \in M(S) \). If \( x - s \notin M(S) \) then there is some \( s' \in S \) with \( s' \neq 0 \) so that \( s' \leq x - s \). But then \( s < s + s' \leq x \), contradicting how \( s \) was chosen. So \( x - s \in M(S) \) and \( x = s + (x - s) \) gives a decomposition of the desired form.

Now suppose that \( T \subset \mathbb{N}^d \) satisfies that every \( x \in \mathbb{N}^d \) can be written as \( x = s + t \) for some \( s \in S \) and \( t \in T \). Suppose that \( m \in M(S) \). Then \( m = s + t \) for some \( s \in S \), \( t \in T \). Since the only element of \( S \) less than \( m \) is \( 0 \), we have \( m = 0 + t = t \). Thus \( M(S) \subseteq T \). \( \square \)

Remark 2.3. Even when \( S \) has infinitely many holes, the set of fundamental holes is finite (see [12]). From an algebraic perspective this is explained by the fact that the integral closure of a ring is module finite over the ring.

Lemma 2.4. Suppose \( S \subset \mathbb{N}^d \) is a generalized numerical semigroup. Then \( c(S) \leq m(S)n(S) \).

Proof. Let \( < \) be any total order on \( \mathbb{N}^d \) which refines the natural partial order. Define a map \( \psi_\prec : \mathbb{N}^d \to M(S) \times S \) as follows: for \( x \in \mathbb{N}^d \), select \( s = \max_{\prec} \{ t \in S \mid t \leq x \} \). As in the proof of Lemma 2.2, \( x - s \in M(S) \). Since \( \prec \) is a total order, the decomposition \( x = s + (x - s) \) chosen in this way is unique. Define \( \psi_\prec(x) = (x - s, s) \). Clearly this is a well-defined injection.

Now restrict \( \psi_\prec \) to the subset \( C(S) = \{ x \in \mathbb{N}^d : x \leq h \text{ for some } h \in H(S) \} \). The largest \( s \in S \) so that \( s \leq x \) also is in \( C(S) \). This gives an injection \( \psi_\prec : C(S) \to M(S) \times S \cap C(S) \). We observe that \( c(S) = |C(S)| \), \( m(S) = |M(S)| \), and \( |S \cap C(S)| = n(S) \), which concludes the proof. \( \square \)

Definition 2.5. We say a generalized numerical semigroup \( S \) has minimal multiplicity if \( c(S) = m(S)n(S) \). In this case every \( x \in C(S) \) can be written uniquely as \( x = m + s \), where \( m \in M(S) \) and \( s \in S \).

In the following, \( S^* \) denotes \( S \setminus \{ 0 \} \).

Definition 2.6. Write \( e_1, \ldots, e_{d+1} \) for the standard semigroup generators of \( \mathbb{N}^d \) and consider the semigroup isomorphic to \( \mathbb{N}^d \) inside \( \mathbb{N}^{d+1} \) generated by \( \{ e_1, \ldots, e_{d+1} \} \setminus e_i \) for some \( i \). By abuse of notation we refer to the latter
semigroup as \( \mathbb{N}^d \). Suppose \( S \subset \mathbb{N}^d \) is a semigroup. The \( k \)-thickening of \( S \subset \mathbb{N}^d \) along axis \( i \) in \( \mathbb{N}^{d+1} \) is the semigroup \( T_k(S,i) \subset \mathbb{N}^{d+1} \) defined as
\[
T_k(S,i) = S \cup (e_i + S) \cup \cdots \cup (ke_i + S) \cup ((k+1)e_i + \mathbb{N}^{d+1}).
\]

Remark 2.7. If \( S \subset \mathbb{N}^d \) is a GNS then \( T_0(S,i) \) corresponds to embedding \( S \) in \( \mathbb{N}^{d+1} \) in the coordinate hyperplane \( x_i = 0 \).

Proposition 2.8. Suppose \( S \subset \mathbb{N}^d \) is a semigroup with minimal generating set \( G(S) \), \( S \)-module generators \( M(S) \), and multiplicity \( m(S) \). The minimal generating set of \( T_k(S,i) \subset \mathbb{N}^{d+1} \) is
\[
\{e_i\} \cup G(S) \cup ((k+1)e_i + M(S)^*),
\]
where \( M(S)^* \) is the set of fundamental holes of \( S \).

Proof. We first show that any \( x \in T_k(S,i) \) can be written in terms of the prescribed generators. First, if \( x \in j e_i + S \) for some \( 0 \leq j \leq k \), then clearly \( x \) is the sum of \( j e_i \) and some number of elements of \( G(S) \). Now suppose \( x = (k+1)e_i + n \) for some \( n \in \mathbb{N}^{d+1} \). By Lemma 2.2 there is some \( m \in M(S) \), \( s \in S \) so that \( n = m + s \). Hence \( x \) can be written as a sum of \( (k+1)e_i, m \), and some number of generators of \( S \). For minimality, clearly \( e_i \) and \( G(S) \) cannot be removed from the generating set. If any element of \( (k+1)e_i + M(S)^* \) is removed from the generating set, then Lemma 2.2 guarantees that all of \( (k+1)e_i + \mathbb{N}^{d+1} \) will not be generated.

Corollary 2.9. Suppose \( S \subset \mathbb{N}^d \) is a semigroup. Then \( e(T_k(S,i)) = e(S) + m(S), n(T_k(S,i)) = (k+1)n(S), \) and \( c(T_k(S,i)) = (k+1)c(S) \).

Proposition 2.10. If \( S \) satisfies the Generalized Wilf Conjecture \( (d \cdot c(S) \leq n(S)e(S)) \) then so does \( T_k(S,i) \). Moreover, if \( S \) has minimal multiplicity and satisfies \( d \cdot c(S) = n(S)e(S) \), then \( (d+1)c(T_k(S,i)) = n(T_k(S,i))e(T_k(S,i)) \).

Proof. By Corollary 2.3 \( (d+1)c(T_k(S,i)) = (k+1)(d \cdot c(S) + c(S)) \) and \( n(T_k(S,i))e(T_k(S,i)) = (k+1)n(S)(e(S) + m(S)) \). Thus it suffices to show that \( d \cdot c(S) + c(S) \leq n(S)e(S) + n(S)m(S) \), with equality if \( d \cdot c(S) = n(S)e(S) \) and \( c(S) = n(S)m(S) \). This follows from Lemma 2.4 and our assumption that \( S \) satisfies \( d \cdot c(S) \leq n(S)e(S) \).

Remark 2.11. By Proposition 2.10 it suffices to prove the Generalized Wilf Conjecture for semigroups which are not of the form \( T_k(S,i) \) for a semigroup \( S \) of strictly smaller dimension.

Thickening is a process that can be iterated any number of times. We use the following notation.

Definition 2.12. Let \( S \subset \mathbb{N}^d \) be a GNS and suppose \( \mathbb{N}^d \) is embedded in \( \mathbb{N}^{d+t} = \text{Span}_\mathbb{N}\{e_1, \ldots, e_{d+t}\} \) along the axes \( e_i, \ldots, e_i \) and put \( \{e_j, \ldots, e_j\} = \{e_1, \ldots, e_{d+t}\} \setminus \{e_1, \ldots, e_i\} \). Consider the iterative sequence of thickenings \( S_1 = T_{k_1}(S, j_1), S_2 = T_{k_2}(S_1, j_2), \ldots, S_t = T_{k_t}(S_{t-1}, j_t) \). We write
$T_{k_1,\ldots,k_t}(S, j_1, \ldots, j_t)$ for $S_i$. If $k_1 = \cdots = k_t = k$, then we simply write $T_k(S, j_1, \ldots, j_t)$ for $S_i$.

**Remark 2.13.** In the sequence $S_1, \ldots, S_t$ constructed in Definition 2.12, order does not matter. Thus, once the axis directions $j_1, \ldots, j_t$ are chosen, there is a unique way to iteratively thicken $S$ along these axis directions.

Applying Proposition 2.14 repeatedly, we see that if $S$ is a semigroup which satisfies the Generalized Wilf Conjecture, then $T_{k_1,\ldots,k_t}(S, e_{j_1}, \ldots, e_{j_t})$ also satisfies the Generalized Wilf Conjecture. We now consider a special case of iterative thickening; this is the case when $k = 0$ for each step.

Let $A$ be a subset of $\mathbb{N}^d$, denote by $\text{Span}_\mathbb{R}(A)$ the $\mathbb{R}$-vector subspace of $\mathbb{R}^d$ spanned by the elements of $A$. Recall that a vector subspace of $\mathbb{R}^d$ is a coordinate linear space if it is spanned by a subset of the standard basis $\{e_1, e_2, \ldots, e_d\}$. The results in the second part of this section are inspired by the following proposition.

**Proposition 2.14** ([21], Proposition 5.2). Let $S \subseteq \mathbb{N}^d$ be a GNS and $H(S)$ the set of its holes. Then $\text{Span}_\mathbb{R}(H(S))$ is a coordinate linear space.

We will use the following notation:

- $S_{g,d}$ is the set of all GNS with genus $g$ in $\mathbb{N}^d$.
- $S_{g,d}^{(r)} = \{S \in S_{g,d} \mid \dim(\text{Span}_\mathbb{R}(H(S))) = r\}$.

**Definition 2.15.** Let $S \in S_{g,d}^{(r)}$.

1. Put $\text{Axes}(S) = \{k \in \{1, 2, \ldots, d\} \mid \text{for all } h \in H(S), h^{(k)} = 0\}$, where $h^{(k)}$ is the $k$-th coordinate of $h \in \mathbb{N}^d$.
2. Set $\{i_1, i_2, \ldots, i_r\} = \{1, 2, \ldots, d\} \setminus \text{Axes}(S)$ and put $\bar{e}_j = e_{i_j}$ for $j = 1, \ldots, r$. By abuse of notation we write $\mathbb{N}^r$ for the sub-monoid of $\mathbb{N}^d$ generated by $\bar{e}_1, \ldots, \bar{e}_r$.
3. We define $\bar{S} = \mathbb{N}^r \cap S$.

**Lemma 2.16.** The semigroup $\bar{S}$ in Definition 2.15 is a generalized numerical semigroup of $\text{Span}_\mathbb{R}(H(S)) \cap \mathbb{N}^d \cong \mathbb{N}^r$.

**Proof.** Proposition 2.11 shows that $\text{Span}_\mathbb{R}(H(S)) \cap \mathbb{N}^d \cong \mathbb{N}^r$ (by abuse of notation we refer to $\text{Span}_\mathbb{R}(H(S)) \cap \mathbb{N}^d$ as $\mathbb{N}^r$). Since $|\mathbb{N}^d \setminus S|$ is finite, so is $|\mathbb{N}^r \setminus \bar{S}|$. Hence $\bar{S}$ is a generalized numerical semigroup in $\mathbb{N}^r$. 

**Lemma 2.17.** The following are equivalent:

1. $S \in S_{g,d}^{(r)}$
2. There is some $S' \in S_{g,r}^{(r)} \subseteq \mathbb{N}^r$ so that $S = T_0(S', \text{Axes}(S))$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $\text{Span}_\mathbb{R}(H(S)) = \text{Span}\{e_{j_1}, \ldots, e_{j_t}\}$ and $\text{Axes}(S) = \{j_1, \ldots, j_{d-r}\}$. Then $\bar{S} \in S_{g,r}^{(r)}$ and $S = \bar{S} \cup (e_{j_1} + \mathbb{N}^{r+1}) \cup (e_{j_2} + \mathbb{N}^{r+2}) \cup \cdots \cup (e_{j_{d-r}} + \mathbb{N}^d) = T_0(\bar{S}, \text{Axes}(S))$. 

Lemma 3.1. For all irreducible GNS, the proof of the conjecture for pseudo-symmetric GNS satisfy the Generalized Wilf Conjecture. Now we show that actually this occurs. If \( S = S^{(r)}_{\text{gen}} \), then \( \dim(\text{Span}_G(H(S))) = \dim(\text{Span}_G(H(S^*))) = r \). Since 0-thickenings do not effect genus, \( S \in S^{(r)}_{\text{gen}} \).

Example 2.18. Let \( S = \mathbb{N}^5 \setminus \{(0, 0, 0, 1, 0), (0, 0, 0, 2, 0), (0, 1, 0, 0, 0), (0, 1, 0, 3, 0)\} \). The set of minimal generators of \( S \) is \( G(S) = \{(1, 0, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 1), (1, 0, 0, 1, 0), (0, 1, 1, 0), (0, 0, 0, 1, 1), (0, 0, 0, 3, 0), (1, 0, 0, 2, 0), (0, 1, 0, 2, 0), (0, 0, 1, 2, 0), (0, 0, 2, 1), (0, 0, 0, 5, 0), (0, 0, 0, 4, 0), (1, 1, 0, 0, 0), (0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 2, 0, 0, 0), (0, 2, 0, 1, 0), (0, 3, 0, 0, 0)\} \). Furthermore \( e(S) = 20 \) and \( g(S) = 4 \). In this case Axes(\( S \)) = \{1, 3, 5\} and \( i_1 = 2, i_2 = 4 \). With the previous construction we have \( \mathcal{S} = \mathbb{N}^2 \setminus \{(0, 1), (0, 2), (1, 0), (1, 3)\} \). The set of minimal generators of \( \mathcal{S} \) is \( G(\mathcal{S}) = \{(1, 1), (0, 3), (1, 2), (0, 5), (0, 4), (2, 1), (2, 0), (3, 0)\} \). So \( e(\mathcal{S}) = 8 \). Notice that \( M(\mathcal{S}) = \{(0, 0), (0, 1), (0, 2), (1, 0)\} \) and \( m(S) = 4 \). If we iterate Proposition 2.8 three times, we see how \( G(S) \) is obtained from \( G(\mathcal{S}) \).

Corollary 2.19. Let \( S \in S^{(r)}_{\text{gen}} \) and suppose that \( \mathcal{S} \in S^{(r)}_{\text{gen}} \) satisfies the Generalized Wilf Conjecture. Then \( S \) satisfies the Generalized Wilf Conjecture. Moreover, if \( \mathcal{S} \) has minimal multiplicity and satisfies the Generalized Wilf Conjecture with equality, then so does \( S \).

Proof. This is immediate from Lemma 2.17 and Proposition 2.10.

3. The Generalized Wilf Conjecture for Irreducible GNS

Proposition 1.7 shows that all symmetric generalized numerical semigroups satisfy the Generalized Wilf Conjecture. Now we show that actually this occurs for all irreducible GNS. The proof of the conjecture for pseudo-symmetric GNS requires some preliminary results and Corollary 2.19.

Lemma 3.1. Let \( S \subseteq \mathbb{N}^d \) be an irreducible GNS such that \( e(S) = 2d \). Then \( S \) is symmetric.

Proof. If \( e(S) = 2d \) then by [3, Theorem 2.8] it follows that \( S = \langle A \rangle \) with \( A = \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d, a\mathbf{e}_1 + b\mathbf{e}_i \mid i \in \{1, \ldots, d\}, 1 < a < b \in \mathbb{N} \setminus \{0\}, \gcd(a, b) = 1 \} \cup \{e_i + h^{(j)}\mathbf{e}_j \mid j \in \{1, \ldots, d\} \setminus \{i\}, h^{(j)} \in \mathbb{N} \setminus \{0\}\}. \) Observe that \( a \) and \( b \) generate a numerical semigroup in the \( i \)-th axis. We distinguish two cases:

1) \( a = 2 \). In such a case \( H(\langle 2, b \rangle) = \{1, 3, 5, \ldots, b - 2\} \) and by a simple argument we see that \( H(S) \) is the set:

\[
\left\{ h\mathbf{e}_i + \sum_{j \neq i} l_j\mathbf{e}_j \mid h \in H(\langle 2, b \rangle), l_j \in \{0, \ldots, h^{(j)} - 1\}, j \in \{1, \ldots, d\} \setminus \{i\} \right\}.
\]

Moreover \( S \) is a Frobenius GNS with Frobenius element \( f = (b - 2)\mathbf{e}_i + \sum_{j \neq i}(h^{(j)} - 1)\mathbf{e}_j \) and genus \( g(S) = \frac{b - 1}{2} \prod_{j \neq i} h^{(j)}. \) By Theorem 1.5, \( S \) is...
symmetric.
2) $a > 2$. In such a case we show that $S$ is not a Frobenius GNS, so it is not irreducible. This will prove the claim of this lemma. Let $F = ab - a - b$ be the Frobenius number of $\langle a, b \rangle$ and consider the element $h = Fe_i + \sum_{j \neq i} (h^{(j)} - 1)e_j$. We show that $h$ is a maximal element in $H(S)$ with respect to the natural partial order in $\mathbb{N}^d$. First we prove that $h \in H(S)$. If not, $h \notin \langle A \rangle$ and since $h - (e_i + h^{(j)}e_j) \notin \mathbb{N}^d$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$, then $h = \lambda_1 ae_i + \lambda_2 be_i + \sum_{j \neq i} \mu_j e_j$, with $\lambda_1, \lambda_2, \mu_j \in \mathbb{N}$. But this implies $F = \lambda_1 a + \lambda_2 b$ that is a contradiction. So $h \in H(S)$, in order to prove that it is a maximal hole it suffices to prove that $h + e_k \in S$ for any $k \in \{1, \ldots, d\}$. It is obvious that $h + e_1 \in S$. So let $k \neq i$, then $h + e_k = (F - 1)e_i + \sum_{j \neq i, k} (h^{(j)} - 1)e_k + e_i + h^{(k)}e_k$. Since $\langle a, b \rangle$ is a symmetric numerical semigroup, $F - 1 \in \langle a, b \rangle$, hence $(F - 1)e_i \in S$. Therefore $h + e_k \in S$ and $h$ is maximal in $H(S)$. It remains to prove that there exists an element in $H(S)$ not comparable with $h$. Consider $x = 2e_i + h^{(k)}e_k$ with $k \neq i$. Obviously $x \nleq h$, moreover one can see by a simple argument that $x \in H(S)$. This concludes the proof. □

**Remark 3.2.** The proof of the previous Lemma shows actually a stronger result: If $S \subseteq \mathbb{N}^d$ is a GNS with $e(S) = 2d$, then $S$ is Frobenius if and only if $S$ is symmetric.

For the claim and the proof of the following Lemma we use the same notation of the previous section.

**Lemma 3.3.** Let $S \in S_{g,d}^{(r)}$ and $\overline{S} \in S_{g,d}^{(r)}$ be as in Definition 2.15. Then the following hold:

1) If $S$ is symmetric then $\overline{S}$ is symmetric.
2) If $S$ is pseudo-symmetric then $\overline{S}$ is pseudo-symmetric.

**Proof.** Let $\{1, 2, \ldots, d\} \setminus \text{Axes}(S) = \{i_1, i_2, \ldots, i_r\}$. Suppose $S$ is symmetric or pseudo-symmetric and let $f = (f^{(1)}, \ldots, f^{(d)})$ be the Frobenius element of $S$. Then $\prod_{i=1}^d (f^{(i)} + 1) = \prod_{k=1}^r (f^{(ik)} + 1)$ since for $j \in \text{Axes}(S)$ we have $f^{(j)} + 1 = 1$. But $\overline{f} = (f^{(i_1)}, \ldots, f^{(i_r)})$ is the Frobenius element of $\overline{S}$. So both the statements follow easily from Theorem 1.5 and the fact that $g(S) = g(\overline{S})$. □

**Lemma 3.4.** Let $S \subseteq \mathbb{N}^d$ be a GNS. Then the following hold:

1) If $g(S) < d$ then $S \in S_{g,d}^{(r)}$ for some $r < d$. In particular $g(S) \geq r$.
2) If $g(S) = d$ and $S \in S_{g,d}^{(d)}$ then $S$ is not pseudo-symmetric.

**Proof.** The first statement is quite easy, considering that a vector space of dimension $r$ is spanned by exactly $r$ independent vectors. To prove the second statement, suppose that $(S, f)$ is a pseudo-symmetric GNS. Then $f/2, f \in
$H(S)$, so $S$ must have at least $d+1$ holes to have $d$ linearly independent holes. It follows that if $g(S) = d$ then $S$ is not pseudo-symmetric.

**Theorem 3.5.** Let $S \subseteq \mathbb{N}^d$ be a pseudo-symmetric GNS. Then $S$ satisfies the Generalized Wilf Conjecture.

**Proof.** Let $g = g(S)$. We know that $S$ has Frobenius element $f = (f^{(1)}, \ldots, f^{(d)})$ and, by Theorem 1.5 (2), we have $2g - 1 = (f^{(1)} + 1) \cdots (f^{(d)} + 1) = c(S)$. So it suffices to prove that $c(S)n(S) \geq d(2g - 1)$. If $S$ is pseudo-symmetric then, by the map $\Psi$, $g - 1 = |H(f)| - 1 = |N(f)| = n(S)$. Furthermore $e(S) \geq 2d + 1$ by Lemma 3.1. So $c(S)n(S) \geq (2d + 1)(g - 1) = d(2g - 1) + g - (d + 1)$, in particular if $g \geq d + 1$ we conclude. Now consider that $S \in S^{(r)}_{g,d}$ with $r \leq d$. If $r = d$ we have $S \in S^{(d)}_{g,d}$, then by Lemma 3.4 we have that $S$ is not pseudo-symmetric, a contradiction. If $r < d$ then we can consider $S \in S^{(r)}_{g,r}$ and by Lemma 3.3 it is pseudo symmetric. Moreover $g(S) \geq r$ hence by a similar argument we have that $S$ satisfies the Generalized Wilf Conjecture. By Corollary 2.19 the same holds for $S$. \qed

Combining the previous theorem with Proposition 1.7 we can state the following general result:

**Theorem 3.6.** Let $S \subseteq \mathbb{N}^d$ be an irreducible GNS. Then $S$ satisfies the Generalized Wilf Conjecture.

4. **Monomial Semigroups**

In this section we prove that generalized numerical semigroups satisfying $n(S) = 1$ satisfy the Generalized Wilf Conjecture. We do this by exploiting a connection between generalized numerical semigroups with $n(S) = 1$ and monomial ideals. We assume some familiarity with commutative algebra.

**Definition 4.1.** Let $R = k[x_1, \ldots, x_d]$ and suppose $M$ is a graded $R$-module. If $M$ has finite dimension as a $k$-vector space then we say that $M$ is zero-dimensional and set $t(R/I) = \dim_k M$. If $I$ is a homogeneous ideal of $R$ and $R/I$ is zero-dimensional, then we simply say $I$ is zero-dimensional.

**Remark 4.2.** The disconnect between calling $M$ zero-dimensional while the dimension of $M$ as a $k$-vector space is positive is an unfortunate side effect of using dimension to refer to both the Krull dimension of $M$ (which is zero) and the dimension of $M$ as a $k$-vector space (which is positive).

Throughout this section, if $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(d)}) \in \mathbb{N}^d$, then $x^\alpha$ means $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_d^{\alpha_d}$.

**Proposition 4.3.** Suppose $S \subseteq \mathbb{N}^d$ is a set containing $0$, and $S^* = S \setminus \{0\}$. The following are equivalent.

1. $S$ is a generalized numerical semigroup with $n(S) = 1$. 
2. $S$ is a monomial semigroup. 
3. $S$ is an irreducible GNS. 
4. $S$ satisfies the Generalized Wilf Conjecture.
Proof. (1)⇒(2): In the polynomial ring \( k[x_1, \ldots, x_d] \), consider the ideal \( I = \langle x^\alpha \mid \alpha \in S^* \rangle \). We claim that if \( x^\beta \) is a monomial and \( x^\beta \in I \), then \( \beta \in S^* \). To see this, notice that \( x^\beta \in I \) means \( x^\alpha \mid x^\beta \) for some \( \alpha \in S^* \). In other words \( \alpha \leq \beta \) in the natural partial order. If \( \beta \notin S^* \), then \( n(S) \geq 2 \) since \( \alpha \) would contribute to the set \( N(S) = \{ s \in S : s \leq h \text{ for some } h \in H(S) \} \) whose cardinality is \( n(S) \). Hence \( \beta \in S^* \). It follows that \( S^* \) is exactly the set of the exponent vectors of the monomials in the ideal \( I \). Since \( \mathbb{N}^d \setminus S^* \) is finite, \( I \) is zero-dimensional.

(2)⇒(1): Suppose \( I \) is a zero-dimensional monomial ideal, and \( S^* = \{ \alpha \in \mathbb{N}^d \mid x^\alpha \in I \} \). Since \( I \) is an ideal, if \( x^\alpha, x^\beta \in I \), then \( x^{\alpha + \beta} \in I \) and hence \( \alpha + \beta \in S^* \). Thus if we take \( S = S^* \cup \{ 0 \} \), \( S \) is a semigroup. Moreover, if \( x^\alpha \notin I \), then \( x^\alpha \) is not divisible by any monomial \( x^\beta \in I \), hence the set \( \{ n \in \mathbb{N}^d \mid n \leq \alpha \} \) does not contain any elements of \( S^* \). It follows that \( n(S) = 1 \). Also, \( \mathbb{N}^d \setminus S \) is finite since \( I \) is zero-dimensional. \( \square \)

In view of Proposition 4.3, we make the following definition.

**Definition 4.4.** If \( S \) is a generalized numerical semigroup satisfying \( n(S) = 1 \) then we call \( S \) a monomial semigroup and we call the ideal \( I = \langle x^\alpha \mid \alpha \in S^* \rangle \) the ideal corresponding to \( S \).

**Lemma 4.5.** If \( S \subset \mathbb{N}^d \) is a monomial semigroup and \( I \subset R = k[x_1, \ldots, x_d] \) is the ideal corresponding to \( S \), then \( e(S) = \ell(I/I^2) \) and \( c(S) = \ell(R/I) \).

**Proof.** It is well-known that a minimal generating set for \( S \) is provided by \( S^* \setminus (S^* + S^*) \). We can identify \( S^* \) with the monomials in \( I \) and \( S^* + S^* \) with the monomials in \( I^2 \). It follows that \( S^* \setminus (S^* + S^*) \) can be identified with the monomials in \( I \) but not in \( I^2 \). These form a \( k \)-vector space basis for \( I/I^2 \). Hence \( e(S) = |S^* \setminus (S^* + S^*)| = \ell(I/I^2) \).

Recall \( c(S) = |C(S)| \), and \( C(S) = \{ n \in \mathbb{N}^d \mid n \leq h \text{ for some } h \in H(S) \} \). Clearly the set \( C(S) \) can be identified with monomials not in \( I \). These form a basis for \( R/I \), hence \( c(S) = \ell(R/I) \). \( \square \)

**Example 4.6.** Let \( g = 3 \) and \( d = 2 \). Consider the GNS \( S = \mathbb{N}^2 \setminus \{(0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (3,0)\} \), which corresponds to the monomial ideal \( I \subset k[x,y] \) generated by all monomials in two variables of degree at least four. The minimal set of generators is \( G(S) = \{(0,4), (0,5), (0,6), (0,7), (4,0), (5,0), (6,0), (7,0), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (5,1), (5,2), (6,1)\} \); these are the exponent vectors of monomials in \( I \) but not in \( I^2 \). Figure 2 provides a graphical view of this GNS – black points are the holes of the GNS, while the red points are the minimal generators.

**Theorem 4.7** (The Generalized Wilf Conjecture for Monomial Semigroups).
If \( S \) is a monomial semigroup, then \( dc(S) \leq e(S) \). Equivalently (by
Lemma 4.5), if \( I \subset R = k[x_1, \ldots, x_d] \) is a zero-dimensional monomial ideal, then \( d\ell(R/I) \leq \ell(I/I^2) \).

Proof. We proceed by induction on the ambient dimension \( d \) and \( \ell(R/I) = c(S) \). If \( d = 1 \), then \( I = \langle x^k \rangle \) and \( I^2 = \langle x^{2k} \rangle \), so \( d\ell(R/I) = k = \ell(I^2/I) \).

Now suppose \( d > 1 \) and \( R = k[x_1, \ldots, x_d] \). If \( I \subset R \) is a monomial ideal so that \( \ell(R/I) = 1 \) then \( I = \langle x_1, \ldots, x_d \rangle \) and \( I/I^2 \cong \text{Span}_k \{x_1, \ldots, x_d\} \), so \( \ell(I/I^2) = d = d \cdot \ell(R/I) \). Now suppose that \( I \) is a monomial ideal in \( R \) and that the proposed inequality holds for all monomial ideals in less than \( d \) variables and all monomial quotients of length less than \( \ell(R/I) \). For simplicity we write \( y = x_d \) and put \( \overline{R} = R/\langle y \rangle \cong k[x_1, \ldots, x_{d-1}] \), \( \overline{I} = (I + \langle y \rangle)/\langle y \rangle \subset \overline{R} \).

We consider the short exact sequence
\[
0 \to \frac{R}{I : y} \xrightarrow{y} R/I \to \frac{R}{I + \langle y \rangle} \cong \frac{\overline{R}}{\overline{I}} \to 0.
\]
(1)

From (1) we get \( \ell(R/I) = \ell(R/(I : y)) + \ell(\overline{R}/\overline{I}) \). We always have the containment \( I \subset I : y \). Furthermore the containment is always proper since \( I \) is zero-dimensional, so \( \ell(R/(I : y)) < \ell(R/I) \). By the induction hypothesis, we thus have \( d\ell(R/(I : y)) \leq \ell((I : y)/(I : y)^2) \). Since \( \overline{R} \) involves one less variable, we also have \( (d-1)\ell(R/I) \leq \ell(\overline{R}/\overline{I}) \) by induction. Hence
\[
d\ell(R/I) = d\ell(R/(I : y)) + d\ell(\overline{R}/\overline{I}) \leq \ell((I : y)/(I : y)^2) + \ell(\overline{R}/\overline{I})^2 + \ell(\overline{R}/\overline{I}).
\]
To complete the induction, it suffices to prove

\[
\ell((I : y)|/y)^2) + \ell(T/T^2) + \ell(R/R) \leq \ell(I/I^2).
\]

We now simplify (2). Using the identity \( y(I : y) = I \cap \langle y \rangle \), there is a short exact sequence

\[
0 \to I : y \to I/I^2 \to I/I^2 + I \cap \langle y \rangle \cong \frac{I + \langle y \rangle}{I^2 + \langle y \rangle} \cong \frac{\mathbb{T}}{\mathbb{T}^2} \to 0,
\]

which yields \( \ell(I/I^2) = \ell((I : y)/(I^2 : y)) + \ell(R/R) \). Plugging this into the right hand side of (2) and simplifying, we rewrite (2) as:

\[
\ell((I : y)/(I : y)^2) + \ell(R/R) \leq \ell((I : y)/(I^2 : y)).
\]

Finally, notice that \( (I^2 : y) \subset (I : y)^2 \). From the short exact sequence

\[
0 \to (I : y)^2 \to I : y \to I : y/(I : y)^2 \to 0
\]

we get \( \ell((I : y)/(I^2 : y)) = \ell((I : y)/(I : y)^2) = \ell((I : y)^2/(I^2 : y)) \). So we can rewrite (3), hence also (2), as:

\[
\ell \left( \frac{R}{T} \right) \leq \ell \left( \frac{(I : y)^2}{I^2 : y} \right).
\]

We prove (4) in Lemma 4.8, completing the induction and the proof. □

**Lemma 4.8.** Suppose \( I \subset R = k[x_1, \ldots, x_{d-1}, y] \) is a zero-dimensional monomial ideal and put \( \overline{R} = R/\langle y \rangle \cong k[x_1, \ldots, x_{d-1}] \) and \( \overline{T} = (I + \langle y \rangle)/\langle y \rangle \subset \overline{R} \).

Then

\[
\ell \left( \frac{\overline{R}}{\overline{T}} \right) \leq \ell \left( \frac{(I : y)^2}{I^2 : y} \right).
\]

**Proof.** We first prove that \( I^2 : y = I(I : y) \). If \( m \) is a monomial in \( I^2 : y \) then \( ym = fg \) for some monomials \( f, g \in I \). Hence \( y \mid f \) or \( y \mid g \); without loss assume \( f = yf' \). Then \( m = f'g \in I(I : y) \). Conversely if \( m \in I(I : y) \) then \( m = fg \) where \( f \in I \) and \( g \in I : y \). Then \( my = f(gy) \in I^2 \).

Now we prove the inequality by producing an injective map \( \phi \) from the canonical basis of \( \overline{R}/\overline{T} \) into \( (I : y)^2/(I^2 : y) \). The canonical basis for \( \overline{R}/\overline{T} \) consists of monomials in the variables \( x_1, \ldots, x_{d-1} \) which are not in \( I \), and the canonical basis for \( (I : y)^2/(I^2 : y) \) consists of monomials in the variables \( x_1, \ldots, x_{d-1} \) which are in \( (I : y)^2 \) but not in \( (I^2 : y) \).

Let \( m \) be a monomial in \( \overline{R}/\overline{T} \), which we view as a monomial in \( R \) without the \( y \) variable. Since \( I^2 : y \) has finite colength, \( y^km \in (I^2 : y) \) for all \( k \gg 0 \). Let \( t = \min \{ k : y^km \in (I^2 : y) \} \). We claim that \( y^{k-1}m \in (I : y)^2 \), as follows. Since \( y^km \in (I^2 : y) = I(I : y) \), \( y^km = ab \) where \( a \in I \) and \( b \in (I : y) \). Suppose \( y \mid a \). Then \( y^k \mid b \), so \( b = y^k b' \) and \( m = ab' \in I \), a contradiction since \( m \notin I \). Hence \( y \mid a \) so \( a = ya' \) for some \( a' \in (I : y) \). Thus \( y^{k-1}m = a'b \in (I : y)^2 \).
Now we define the map \( \phi : R/I \to (I : y)^2/(I^2 : y) \). Given \( m \in R/I \) (regarded as a monomial in \( R \) not including the variable \( y \)), let \( \phi(m) = y^{t-1}m \), where \( t \) is the smallest integer such that \( y^tm \in (I^2 : y) \). By the above argument, \( \phi(m) \in (I : y)^2 \) but not in \( (I^2 : y) \). The map \( \phi \) is clearly injective, so we are done.

\[ \phi: \frac{R}{I} \to \frac{(I : y)^2}{(I^2 : y)} \]

Remark 4.9. There is a geometric interpretation of the inequality \( d\ell(R/I) \leq \ell(I/I^2) \) if \( I \) is a zero-dimensional monomial ideal which is the initial ideal of a radical ideal. Suppose that \( m \in R \) with corresponding ideals \( I \) and \( J \). We claim that \( d\ell(R/I) = \ell(I/I^2) \). Since \( \ell(R/I) = n \) (the number of points), it suffices to show that \( \ell(I/I^2) = nd \). Since \( I/I^2 \) has finite length, we have

\[ I/I^2 \cong \bigoplus_{i=1}^{n} (I/I^2)_{m_i}, \]

where \( (I/I^2)_{m_i} \) is the localization at \( m_i \). Since localization is exact and \( I_{m_i} \cong m_i, (I/I^2)_{m_i} \cong (m_i/m_i^2) \). It is straightforward to show that \( \ell(m_i/m_i^2) = d \) (it suffices to consider the case \( m_i = \langle x_1, \ldots, x_d \rangle \) ), hence \( \ell(I/I^2)_{m_i} = d \) for each summand above and \( \ell(I/I^2) = nd \). So \( d\ell(R/I) = \ell(I/I^2) \) for zero-dimensional radical ideals.

Now consider what happens under deformation to the initial ideal \( J = \text{in}(I) \). Since the deformation is flat, \( \ell(R/I) = \ell(R/J) \). However, \( \text{in}(I^2) \supset J^2 \), and these are not necessarily equal. Thus \( d\ell(R/J) = d\ell(R/I) = \ell(I/I^2) \leq \ell(J/J^2) \), verifying Theorem 4.7 if \( J \) is the initial ideal of a zero-dimensional radical ideal. Unfortunately, not all zero-dimensional monomial ideals are initial ideals of radical ideals (in technical terms, the Hilbert scheme of points is not necessarily smoothable), so the above argument does not work for all monomial ideals.

Proposition 4.10. If \( I \subset R[x_1, \ldots, x_d] \) is a zero-dimensional monomial ideal, then \( d\ell(R/I) = \ell(I/I^2) \) if and only if \( I \) is a complete intersection.

Proof. First, assume \( I \) is a complete intersection, so \( I = \langle x_1^{a_1}, \ldots, x_d^{a_d} \rangle \). The monomials in \( I/I^2 \) can be described as those which are divisible by precisely one of \( x_i^{a_i} \). Thus the monomials in \( I/I^2 \) are in bijection with the set \( X = \{ x_i^{a_i}m : m \notin I \text{ and } 0 \leq i \leq d \} \). Clearly \( |X| = d\ell(R/I) \), so we are done.

Now we prove by induction on \( d \) and \( \ell(R/I) \) that if \( d\ell(R/I) = \ell(I/I^2) \) then \( I \) is a complete intersection. The cases \( d = 1 \) and \( \ell(R/I) = 1 \) are clear. So suppose \( d > 1 \) and \( \ell(R/I) > 1 \). Using the same notation as in the proof of Theorem 4.7, we have the short exact sequence

\[ 0 \to \frac{R}{I : y} \to \frac{R}{I} \to \frac{R}{I} \to 0. \]
From the proof of Theorem 4.7
\[ \ell(R/I) = \ell(R/(I : y)) + \ell(R/I) \leq \ell((I : y)/(I : y)^2) + \ell(R/I) \leq \ell(I/I^2), \]
so if \( \ell(R/I) = \ell(I/I^2) \) then \( \ell(R/(I : y)) = \ell((I : y)/(I : y)^2) \) and \( (d - 1)\ell(R/I) = \ell(I/I^2) \). By induction on \( d \) and \( \ell(R/I) \), both \( I : y \) and \( \overline{T} \) must be complete intersections.

Now we prove that if \( I : y \) and \( \overline{T} \) are both complete intersections and \( I \) is not a complete intersection, then \( \ell(R/I) < \ell(I/I^2) \) (by the proof of Theorem 4.7). Examining the proof of Lemma 4.8, it suffices to prove that there is a monomial \( m \in \overline{T} \) so that \( m \in (I : y)^2 \) but not in \( I^2 : y \). This is what we show.

Since both \( T \) and \( I : y \) are complete intersections, \( I = T + yM \) where \( T \) and \( M \) are minimally generated as \( T = \langle x_1^{a_1}, \ldots, x_{d-1}^{a_{d-1}} \rangle \) and \( M = \langle x_1^{b_1}, \ldots, x_{d-1}^{b_{d-1}}, y^B \rangle \), and \( a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}, B \) are all positive integers with the exception that we allow \( b_i = -\infty \) with the convention that \( x_i^{-\infty} = 0 \). We stipulate that \( b_i < a_i \) (otherwise \( yx_i^{b_i} \) would be a redundant generator). Since \( I \) is not a complete intersection \( b_i \geq 1 \) for some \( 1 \leq i \leq d - 1 \). Thus \( x_i^{2b_i} \in (I : y)^2 \) but not in \( I^2 : y \). If \( x_i^{2b_i} \in \overline{T} \), then we are done. Otherwise \( 2b_i < a_i \) and we claim that \( x_i^{a_i} \in (I : y)^2 \) but not in \( I^2 : y \). \( \square \)

**Definition 4.11.** A generalized numerical semigroup \( S \subset \mathbb{N}^d \) is an ordinary GNS if there is some \( f \in \mathbb{N}^d \) so that \( S = \{0\} \cup (\mathbb{N}^d \setminus C(f)) \).

**Remark 4.12.** In [1] a numerical semigroup \( S \) is called ordinary if \( S = \mathbb{N} \setminus \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \). The corresponding monomial ideal is \( I = \langle x^{a+1} \rangle \). Definition 4.11 is a natural extension to generalized numerical semigroups of the notion of an ordinary numerical semigroup.

**Proposition 4.13.** The following conditions on a generalized numerical semigroup \( S \) are equivalent:

1. \( S \) is ordinary.
2. \( S \) is a monomial semigroup and its corresponding ideal is a complete intersection.
3. \( S \) satisfies \( n(S) = 1 \) and \( dc(S) = c(S) \).

**Proof.** This is immediate from Propositions 4.3 and 4.10. \( \square \)

The following example illustrates the equality \( dc(S) = c(S) \) satisfied by ordinary generalized numerical semigroups.

**Example 4.14.** Let \( f = (2, 3) \in \mathbb{N}^2 \). Then \( C((2, 3)) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (2, 3)\} \). The ordinary semigroup \( S = (\mathbb{N}^2 \setminus C(f)) \cup \{(0, 0)\} \) corresponds to the monomial complete intersection ideal \( I = \langle x^3, y^4 \rangle \). Visually, the minimal generators of \( S \) breaks
into two copies of $C((2, 3))$, namely $(3, 0) + C((2, 3))$ and $(0, 4) + C((2, 3))$. These are displayed as the red dots in Figure 3 (Notice that these red dots correspond to the exponent vectors of monomials in $I$ but not in $I^2$, where $I = \langle x^3, y^4 \rangle$). The holes of $S$ are marked in black (they are all the elements of $C((2, 3))$ except for $(0, 0)$), while all points which are red or not marked belong to $S$.

![Figure 3. The ordinary GNS in Example 4.14.](image)

5. Comparison with a different extension of Wilf’s conjecture

In [11] another generalization of Wilf’s conjecture is given. That generalization involves a larger class of affine semigroups, called C-semigroups. We will only consider the work of [11] in the case of generalized numerical semigroups. We will need some additional notation from [11] (and also [9]). Let $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. The maximum of $H(S)$ with respect to $\prec$ is the Frobenius element of $S$, denoted by $Fb(S)$. By convention, $Fb(\mathbb{N}^d)$ is the vector $(-1, \ldots, -1)$ with $d$ coordinates. Denote by $n_{\prec}(S)$ the cardinality of the finite set $\{x \in S \mid x \prec Fb(S)\}$. The Frobenius number of $S$ is defined as $n_{\prec}(S) + g(S)$ and denoted by $N(Fb(S))$. 
Conjecture 5.1 (Extended Wilf Conjecture, [11], Conjecture 14). Let \( S \subseteq \mathbb{N}^d \) be a GNS. Then \( n_<(S)e(S) \geq N(Fb(S)) + 1 \), for every monomial order \( \prec \) satisfying that every monomial is preceded only by a finite number of monomials.

We would like to compare the Generalized Wilf Conjecture and the Extended Wilf Conjecture [5.1]. First of all, we can remark that the Generalized Wilf Conjecture does not depend on the choice of a monomial order. In order to provide a simple link between the two conjectures, we recall the following property, stated in [4] in a more general case:

Proposition 5.2 ([4], Proposition 3.4). Every monomial order in \( \mathbb{N}^d \) extends the natural partial order in \( \mathbb{N}^d \).

Proposition 5.3. If \( S \subseteq \mathbb{N}^d \) is a GNS that satisfies the Generalized Wilf Conjecture then \( S \) satisfies the Extended Wilf Conjecture [5.1].

Proof. If \( d = 1 \) it is clear that the two inequalities are the same, so we suppose that \( d > 1 \) and assume that \( S \) satisfies the Generalized Wilf Conjecture. Fix a monomial order \( \prec \) in \( \mathbb{N}^d \). Let \( s \in N(S) \), then \( s \preceq h \) for some \( h \in H(S) \), with respect to the natural partial order in \( \mathbb{N}^d \). By Proposition 5.2, \( s \prec h \prec Fb(S) \), so \( s \in \{ x \in S \mid x \prec Fb(S) \} \). Therefore \( n(S) \leq n(S) \). Consider the Generalized Wilf Conjecture in the form \( n(S)(e(S) - d) \geq dg(S) \). Hence \( n(S)(e(S) - 1) \geq n(S)(e(S) - 1) \geq n(S)(e(S) - d) \geq dg(S) \geq g(S) + 1 \), in particular \( n(S)e(S) \geq n(S) + g(S) + 1 = N(Fb(S)) + 1 \).

In [11] other classes of generalized numerical semigroups are given for which the Extended Wilf Conjecture [5.1] is satisfied. Now we study the behaviour of those classes with respect to the Generalized Wilf Conjecture. The first class ([11], Lemma 15) provides another example, different from ordinary GNS, in which the Generalized Wilf Conjecture holds as an equality:

Proposition 5.4. Let \( h > 1 \) be a positive integer, \( i \in \{1, 2, \ldots, d\} \), \( k \in \{1, 2, \ldots, d\} \setminus \{i\} \). Consider the GNS \( S \subseteq \mathbb{N}^d \) generated by:

\[
\{ e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_d, 2e_i, 3e_i \} \\
\cup \{ e_i + he_k \} \cup \{ e_i + e_j \mid j \in \{1, 2, \ldots, d\} \setminus \{k, i\} \}
\]

Then \( S \) satisfies the Generalized Wilf Conjecture.

Proof. First we have \( e(S) = 2d \). The set of holes of \( S \) is \( H(S) = \{ e_i, e_i + e_k, e_i + 2e_k, \ldots, e_i + (h - 1)e_k \} \), so \( S \) is a Frobenius GNS ([4]) with Frobenius element \( f = e_i + (h - 1)e_k \). Therefore \( c(S) = |C(f)| = 2h \). Furthermore \( \bigcup_{h \in H(S)} N(h) = \{ 0, e_k, 2e_k, \ldots, (h - 1)e_k \} \), so \( n(S) = h \). Finally \( dc(S) = 2dh = n(S)e(S) \).

\[\square\]
Remark 5.5. Observe that the GNS in the previous proposition can be expressed using thickenings. In particular if we consider the numerical semigroup \( \langle 2, 3 \rangle \) in the \( i \)-th axis and the GNS \( \hat{S} = T_{h-1}(\langle 2, 3 \rangle, k) \), then \( S = T_0(\hat{S}, \{1, \ldots, d\} \setminus \{i, k\}) \). So \( S \) satisfies the Generalized Wilf Conjecture also by Proposition 2.10 and the fact that \( \langle 2, 3 \rangle \) satisfies Wilf’s conjecture.

The second class contains semigroups \( S = \mathbb{N}^d \setminus \{e_i, 2e_i, \ldots, (q - 1)e_i\} \), with \( i \in \{1, 2, \ldots, d\} \) and \( k \in \mathbb{N} \setminus \{0\} \) ([11], Lemma 16). In this case \( S = (\mathbb{N}^d \setminus C((q - 1)e_i)) \) and \( S \) satisfies the Generalized Wilf Conjecture by Theorem 4.7 or Proposition 2.10.

For the third class ([11], Lemma 17) we prove the following more general result

**Proposition 5.6.** Let \( Q \subseteq \mathbb{N} \) be a numerical semigroup satisfying Wilf’s conjecture, \( j \in \{1, 2, \ldots, d\} \) and a set \( \{q_i \in \mathbb{N} \mid i \in \{1, 2, \ldots, d\} \setminus \{j\}\} \). Then \( S = \mathbb{N}^d \setminus \{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_j \notin Q, x_i \leq q_i, i \in \{1, 2, \ldots, d\} \setminus \{j\}\} \) is a GNS and it satisfies the Generalized Wilf Conjecture.

**Proof.** Consider the numerical semigroup \( Q \) on the axis \( j \). Observe that \( S \) is obtained by the following sequence of thickenings:

\[
S = T_{q_1, q_2, \ldots, q_{j-1}, q_j}(Q, \{1, \ldots, d\} \setminus \{j\}).
\]

Then \( S \) satisfies the Generalized Wilf Conjecture by Proposition 2.10.

6. Some computational tests

The GAP [10] package *numericalsgps* [6] offers tools to deal with numerical and affine semigroups. In [2], in particular, some procedures for generalized numerical semigroups are described and these algorithms are implemented in the development version site of the package. Such tools allow to compute all generalized numerical semigroups of a given genus and to test the Generalized Wilf Conjecture for them. Using this technique we verified that the Generalized Wilf Conjecture is satisfied by all generalized numerical semigroups in \( \mathbb{N}^2 \) up to genus \( g = 13 \), and in \( \mathbb{N}^3 \) up to genus \( g = 10 \). Moreover the function *RandomAffineSemigroupWithGenusAndDimension* allows to produce a random GNS in \( \mathbb{N}^d \) of genus \( g \), so it is possible to make a random test of the Generalized Wilf Conjecture. Considering a random GNS of genus \( g \), from \( g = 1 \) up to \( g = 500 \) we checked that different random tests give a positive answer for the Generalized Wilf Conjecture in \( \mathbb{N}^d \) from \( d = 2 \) to \( d = 5 \).

We summarize the computational positive answers to the Generalized Wilf Conjecture in the following table:
inequality is negative. Such a measure could expose additional terms either improving the monomial ideal $I$.

The test confirms a positive answer to the Generalized Wilf Conjecture for a wide number of generalized numerical semigroups.

7. Concluding Remarks

If $S \subset \mathbb{N}^d$ is a GNS, it is natural to ask for a measure of the size of $n(S)e(S) - dc(S)$, which the Generalized Wilf Conjecture postulates is non-negative. Such a measure could expose additional terms either improving the inequality $n(S)e(S) \geq dc(S)$ or indicating where one could look for a counterexample. We briefly consider this question for the case where $n(S) = 1$, so $S^*$ is the set of exponent vectors of monomials inside a zero-dimensional monomial ideal $I \subset k[x_1, \ldots, x_d]$. Let $a_1, \ldots, a_d$ be the smallest integers such that $x_i^{a_i} \in I$, and put $J = \langle x_1^{a_1}, \ldots, x_d^{a_d} \rangle$. Suppose that $I^2 = JI$. In the language of integral closures, this means that $J$ is a reduction of $I$ and the reduction number of $I$ with respect to $J$ is one (see [19, Chapter 8]). This is a very special situation - in general the ideal $J$ need not even be a reduction of $I$, let alone with reduction number one. See for instance the recent preprint [13] which bounds the reduction number of monomial ideals in two variables which have $J = \langle x^a, y^b \rangle$ as a minimal reduction.

**Proposition 7.1.** Suppose $I \subset R = k[x_1, \ldots, x_d]$ is a zero-dimensional monomial ideal, $J = \langle x_1^{a_1}, \ldots, x_d^{a_d} \rangle$ where $a_1, \ldots, a_d$ are the smallest integers such that $x_i^{a_i} \in I$, and $I^2 = JI$. Then $\ell(I/I^2) - d\ell(R/I) = \prod_i a_i - \ell(R/I) = \ell(R/J) - \ell(R/I)$.

**Proof.** Consider the set $X = \{x_i^{a_i} \cdot m : m \notin I\}$. We first show that $X \subset I \setminus I^2$. Suppose for a contradiction that $x_i^{a_i} \cdot m \in I^2$ where $m \notin I$. Since $x_i^{a_i} \cdot m \in I^2$ and $I^2 = JI$, it follows that $x_i^{a_i} \cdot m = x_j^{a_j} \cdot n$ for some $1 \leq j \leq d$ and $n \in I$. As $m \notin I$, $x_j^{a_j} \nmid m$. Thus the only way the equation can be satisfied is if $j = i$ and $m = n$, a contradiction since $m \notin I$. So $X \subset I \setminus I^2$. We can just as easily see that monomials of the form $x_i^{a_i} \cdot m$, $x_j^{a_j} \cdot n$ where $m, n \notin I$ must be distinct. Hence $|X| = d\ell(R/I)$. Now suppose $n$ is a monomial satisfying $n \notin X$ and $n \in I \setminus I^2$. Then $n = \prod_i x_i^{b_i}$ where $0 \leq b_i \leq a_i - 1$. It follows that the exponent vector of $n$ is in the box $B = [0, a_1 - 1] \times [0, a_2 - 1] \times \cdots \times [0, a_d - 1]$. By
our assumption that $I^2 = IJ$, every monomial whose exponent vector is in $B$ is not in $I^2$. Notice also that any monomial not in $I$ has an exponent vector which is also in $B$. Hence the number of monomials which are in $I \setminus I^2$ but not in $X$ is exactly $\prod_i a_i - \ell(R/I)$. This completes the proof.

**Example 7.2.** Suppose $I = \langle x^5, x^3y^3, y^5 \rangle \subset R = k[x,y]$ and $J = \langle x^5, y^5 \rangle$. Then $I^2 = IJ$, so this is an example of the situation in Proposition 7.1. Here $\ell(I/I^2) = 46$, $\ell(R/I) = 21$, and $\ell(R/J) = 25$. We can check that $46 = 2 \cdot 21 + (25 - 21)$. In contrast, $L = \langle x^5, xy^4, y^5 \rangle$ does not satisfy $L^2 = LJ$; the reduction number of $L$ with respect to $J$ is 4 so we only have equality in the equation $L^{k+1} = JL^k$ when $k \geq 4$.

Proposition 7.1 raises the question of whether the Generalized Wilf Conjecture is just the largest term of an inequality incorporating other invariants of the semigroup $S$ and its integral closure.

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