PAPER

Quantum theory of Kerr nonlinearity with Rydberg slow light polaritons

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Abstract

We study the propagation of Rydberg slow light polaritons through an atomic medium in the regime where the dispersion relation for the polaritons is well described by the slow light velocity alone. In this regime, the quantum many-body problem can be solved analytically for arbitrary shape of the atomic cloud. We demonstrate the connection of Rydberg polaritons to the behavior of a conventional Kerr nonlinearity for weak interactions and determine the leading quantum corrections for increasing interactions. We propose an experimental setup which allows one to measure the effective two-body interaction potential between Rydberg polaritons as well as higher-body interactions. Our work shows that the locality and causality based no-go theorems for quantum gates do not apply to setups based on Rydberg polaritons.

A natural mechanism for an interaction between photons is provided by the Kerr nonlinearity of conventional materials and is well described within a classical theory for high intensities of the fields [1]. On the other hand, a strong interaction between individual photons would pave the way towards ultralow-power all-optical signal processing [2, 3], which in turn has important applications in quantum information processing and communication [4–7]. First attempts to quantize the phenomenological set of equations for a classical Kerr nonlinearity failed due to locality of the interaction and purely linear dispersion [8–10], while adding ad hoc a non-local response in time resolved this problems [11]. However, this approach also necessitates a strong noise term, and it was argued that the decoherence induced by this noise term leads to a no-go theorem for quantum two qubit gates based on propagating photons in a local nonlinear media [12], which motivated the research on several different approaches [13–17]. In turn, it has recently been proposed to use Rydberg slow light polaritons for the generation of a quantum two qubit gate [18, 19], which seem to be in contradiction to the above no-go theorem. In the present manuscript, we resolve this puzzle by demonstrating that the microscopic description of Rydberg polaritons also provides a consistent quantum theory of a Kerr nonlinearity without the necessity to introduce a strong noise term.

Rydberg slow light polaritons have emerged as a highly promising candidate to engineer strong interactions between optical photons with a tremendous recent experimental progress. A variety of applications were shown such as a deterministic single photon source [50], an atom–photon entanglement generation [51], as well as a single photon switch [27] and transistors [26, 28, 29]. Moreover, the regime of strong interaction between photons has been experimentally demonstrated leading to a medium transparent only to single photons [22], as well as the appearance of bound states for photons [25]. From theoretical point of view, the effective low energy theory is well understood from a microscopic approach [32, 52], but a full description of the propagation of photons through the medium is limited to extensive numerical simulations and low photon number [19, 22, 25, 34, 41–45].

In this manuscript, we provide the full input–output formalism of Rydberg polaritons for weak and intermediate interaction strengths, where the dispersion relation for the polaritons is well described by the slow light velocity alone, but arbitrary incoming photon number and shape of the atomic medium. The analysis is
performed in the regime with large detuning from the intermediate $p$-level, where losses are strongly suppressed and the effective low-energy theory for the polaritons is well described by an effective interaction potential \[52\]. We demonstrate that the quantum many-body problem can be solved analytically, and we find the connection of Rydberg polaritons to the behavior of a conventional Kerr nonlinearity for weak interactions. This allows us to determine the leading quantum corrections for such a Kerr nonlinearity. We demonstrate the possibility to experimentally probe the effective two-body as well as higher-body interaction potentials between the slow light polaritons.

Before we start with the analysis, we would like to summarize the implications of our findings on the potential application of Rydberg polaritons for quantum two-qubit gates. Previous approaches to describe the quantum propagation of photons in a nonlinear Kerr medium based on the quantization of the phenomenological nonlinear equations provide an inconsistent quantum field theory \[8–10\] as the interaction is local and the dispersion relation of the photons is linear. One can understand this observation as a contact interaction in combination with the linear dispersion leads to an infinite phase shift between two photons at the same position, while photons separated by a non-zero distance are non-interacting. This inconsistency was removed in an ad hoc approach by introducing a non-local response in time for the nonlinearity. As a consequence, also photons entering the media shortly after each other still acquire a phase shift due to the finite response time of the media. However, such a non-local response requires also the appearance of a noise term, which exhibits a similar strength as the Kerr nonlinearity \[11\], and leads to dephasing between the photons. As a consequence, the impossibility to generate a photonic quantum gate based on photons propagating through a medium with a large Kerr nonlinearity was concluded \[12\]; the latter has motivated researchers to focus on alternative approaches for the generation of two photon gates \[13–17\]. However, the microscopic analysis for Rydberg slow light polaritons shows that such a non-local response in time is absent, but naturally provides a mass term accounting for deviation from the slow light velocity as well as a finite range of the effective interaction potential accounting for the blockade phenomena. Here, we demonstrate that the finite range of the interaction potential is sufficient to derive a consistent quantum theory for a Kerr nonlinearity. As a consequence, we conclude that the proposed inability to generate a photonic phase gate by a large Kerr nonlinearity \[12\] does not apply to Rydberg slow light polaritons. However, it is important to point out that a co-propagating setup studied in the present manuscript is highly unsuitable for the realization of a quantum gate between photons, but rather a counter-propagating setup \[19\] should be pursued.

For completeness, we point out, that an alternative approach to provide a consistent quantum theory for a Kerr nonlinearity was studied by requiring a finite bandwidth of the Kerr nonlinearity \[53\]. Introducing a cut-off in bandwidth transfers, via the linear propagation of the photons, into a finite length scale which the photons can resolve—the latter corresponds to introducing a finite range of the interaction. Then, a high fidelity gate requires the compression of the photonic wave packets on the range of this effective interaction, which leads to additional limitations for the gate. However, this argument holds only for co-propagating setups, and these restrictions are absent in a counter-propagating setup.

We consider a system of Rydberg slow light polaritons in the dispersive limit with large detuning $|\delta| \gg \gamma, \Omega$ from the intermediate $p$-level, see figure 1. Here, $\gamma$ describes the decay rate of the $p$-level, while $\Omega$ denotes the Rabi frequency of the coupling laser. Within this regime, losses are strongly suppressed and the intermediate $p$-level can be adiabatically eliminated \[52\]. Note, that the decay rates of highly excited Rydberg levels are negligible. We are interested in the propagation of photons along a one-dimensional mode through the medium with frequency close to the condition of electromagnetic induced transparency. In the regime with a low density of Rydberg polaritons, the system is well described by an effective low energy quantum theory \[52\]. The interaction potential between the polaritons is characterized by a blockade radius $\xi$ and the potential depth $2\hbar \Omega^2/\delta$ at short distances. For a microscopic van der Waals interaction with $C_6, \delta < 0$ the effective interaction potential reduces to $V(x) = -(2\hbar \Omega^2/\delta)[1 + (x/\xi)^2]^{-1}$ with the blockade radius $\xi = (|C_6, \delta|/2)^{1/6}$, see figure 1. It accounts for the fact that at large distances the van der Waals interaction is directly transferred onto the polaritons, while at short distances (due to the Rydberg blockade mechanism) only one photon is coupled to a Rydberg excitation and the other photon acquires an AC-Stark shift due to the coupling to the $p$-level. The latter describes exactly the saturation of the effective potential. The detailed derivation of the effective interaction potential is presented in \[52\] and requires the full resummation of all virtual scattering processes to the polaritons having high momentum, conveniently preformed within the $T$-matrix formalism. Note, that for increasing polariton densities additional many-body interactions are expected to appear \[46\]. In the following, we mainly focus on the two body interactions, and the extension to include many-body interactions is discussed at the end of the manuscript.

The kinetic energy for the polaritons at low energies is determined by the slow light velocity of the polaritons and an effective mass term accounting for the curvature in the dispersion relation. The important aspect for the present analysis is the possibility to drop the mass term for moderate interactions between polaritons. The precise condition for the validity of this approximation is discussed below. Then, the Hamiltonian describing the
propagation of photons through the spatially inhomogeneous medium with atomic density $n_a(x)$ is given by

$$H = \int dx \left[ \beta(x) \psi^\dagger(x) (-ic\hbar \partial_x) \beta(x) \psi(x) \right]$$

$$+ \frac{1}{2} \int dx dy \ n(x) n(y) \left( V(x - y) \psi^\dagger(x) \psi^\dagger(y) \psi(y) \psi(x) \right).$$

(1)

Here, $\psi$ and $\psi^\dagger$ denote the bosonic field operators annihilating and creating the Rydberg slow light polaritons and satisfy $[\psi(x), \psi^\dagger(x')] = \delta(x - x')$. Furthermore, $\beta(x)$ describes the amplitude of the polariton to be in a photonic state and is related to the slow light velocity $v_g = c/\beta(x)^2$, while $n(x) = 1 - \beta(x)^2$ is the probability for the polariton to be in the Rydberg state. These quantities are determined by the atomic density $n_a(x)$ via $\beta(x) = \Omega / \sqrt{\Omega^2 + g_0^2 n_a(x)}$ with $g_0$ the single atom coupling. Note that outside the atomic medium the operator $\psi$ describes non-interacting photons. The inclusion of higher many-body interactions into the Hamiltonian is straightforward and the influence of a three-body interaction is discussed at the end of the manuscript.

In the following, it is convenient to introduce a coordinate transformation which removes the reduced velocity $v_g$ of the polaritons inside the media, i.e., we measure distances in the time $z/c$ which is required for the polaritons to reach the position $x$. The coordinate transformation takes the form $z = \zeta^{-1}(x) = \int_0^x dy (1/\beta(y)^2)$, and the Hamiltonian reduces to

$$H = -i\hbar \int dz \ \hat{\psi}^\dagger(z) \partial_z \hat{\psi}(z)$$

$$+ \frac{1}{2} \int dz dw \ \bar{n}(z) \bar{n}(w) \tilde{V}(z, w) \hat{\psi}^\dagger(z) \hat{\psi}^\dagger(w) \hat{\psi}(w) \hat{\psi}(z)$$

(2)

with $\bar{n}(z) = n(\zeta(z))$, $\tilde{V}(z, w) = V(\zeta(z) - \zeta(w))$, and $\hat{\psi}^\dagger(z) = \psi^\dagger(\zeta(z)) \beta(\zeta(z))$; the new operators $\hat{\psi}$ still satisfy the bosonic canonical commutation relations.

The quantum many-body theory in equation (2) is exactly solvable. Note that this property is not a specific feature of the Rydberg polaritons but occurs for any system with a linear dispersion relation and, moreover, is independent of the exact shape of the interaction. This remarkable property is most conveniently observed by analyzing the Heisenberg equations for the field operator $\hat{\psi}(z, t)$,

$$i\hbar \partial_t \hat{\psi}(z, t) = -i\hbar \partial_z \hat{\psi}(z, t) + K(z, t) \hat{\psi}(z, t)$$

(3)

with the operator $K(z, t)$ accounting for the interaction,

$$K(z, t) = \int dw \ \bar{n}(z) \bar{n}(w) \tilde{V}(z, w) \hat{\psi}^\dagger(w, t) \hat{\psi}(w, t).$$

(4)

In the following, we denote by $\hat{\psi}_0(z)$ the non-interacting field operator at time $t = 0$. Then, the interacting field operator $\hat{\psi}(z, t)$, satisfying the Heisenberg equation above, reduces to $\hat{\psi}(z, t) = e^{-i\tilde{H}(z, t)} \hat{\psi}_0(z - ct)$ with the operator 

Figure 1. Setup of Rydberg slow light polaritons: each atom consists of three relevant levels: ground state $|G\rangle$, intermediate $p$-level $|P\rangle$ and Rydberg state $|S\rangle$; the latter are coupled by a strong laser with the Rabi frequency $\Omega$. The single atom coupling is denoted by $g_0$ and the detuning from the intermediate $p$-level by $\delta$. Incoming photons within a single transverse channel enter the medium of length $L$ and are converted into slow light $v_g < c$ Rydberg polaritons. The interaction between the Rydberg states provides an effective interaction $V(x)$ for the polaritons which is characterized by a blockade radius $\xi$ and the potential depth $2\hbar \Omega^2/\delta$. New J. Phys. 18 (2016) 123026 P Bienias and H P Büchler
and the polariton density operator $\hat{\Psi}(z, t)$ describes the correlations built up between the photons during the propagation through the medium and takes the form

$$\varphi(u) = \frac{1}{\hbar c} \int_{-\infty}^{\infty} dw \, \hat{n}(w + u) \hat{n}(w) \hat{V}(w + u, w). \tag{8}$$

Note, that in the case of a local Kerr interaction the phase shift is proportional to a delta function $\delta(u)$, which leads to an ill defined phase shift. Therefore, we recover the problems to provide a consistent quantum theory for a local Kerr nonlinearity as discussed, e.g., in [9]. Next, it is instructive to analyze the phase factor $\varphi(u)$ for a specific homogeneous atomic density distribution $n_0(x) = n_0 \theta(L^2/4 - x^2)$ where $\theta$ is a Heaviside step function. The time delay simplifies to $\Delta t = L/(\tilde{v}_L - 1/c)$, where the slow light velocity $\tilde{v}_L = c\Omega^2/(g^2 + \Omega^2)$ and the collective coupling between photon and matter $g = g_0 \sqrt{n_0}$. In turn, the phase shift acquires the peak value

$$\varphi(0) = \frac{g^4 V(0)L}{(g^2 + \Omega^2)\Omega^2 c \hbar} = -\frac{g^2}{g^2 + \Omega^2} \frac{\kappa^2}{\delta} \tag{9}$$

with $\kappa = 2g^2 L/c$ the optical depth of the medium. The width of the signal in the phase $\varphi(u)$ is enhanced from the blockade radius by the slow light velocity to $\xi_{\text{out}} = \xi (g^2 + \Omega^2)/\Omega^2$. The exact phase shapes for different medium lengths are shown in figure 2. The determination of $\varphi(u)$ for other physical distributions of atoms is straightforward.

Note, that the interaction provides a spatially dependent phase factor correlating the photons, but is unable to induce a modification in the intensity correlations. A bunching of photons as observed in the experiments by Firstenberg et al [25] requires the inclusion of the mass term. Here, we estimate the influence of this term, and determine the regime of validity for our approximation to drop it, for details see the appendix. First, the inclusion of the mass would lead to an additional phase shift estimated by $\varphi_{\text{in}} \sim \hbar \Delta t/(m\Omega^2)|\varphi(0)|^2 + i|\varphi(0)| = |\varphi(0)| + ig^2/(g^2 + \Omega^2)^2 L^2/\xi^2$, where we used the expression for the
polariton mass \( m = \hbar (g^2 + Q^2) / 2c^2 \Delta \Omega \) [52]. In order to drop phenomena like the bunching of photons we require \( g^2 / \Delta \Omega \ll 1 \). Secondly, we would like the phase shift induced by the interaction to dominate the behavior, i.e., \( \varphi(0) > \varphi_m \). The two conditions are either satisfied for weak coupling of the photons to the atomic media with \( g \ll \min(\Gamma, \Omega / \sqrt{\Delta}) \), or for a short medium \( L < \xi \) with a constraint on optical depth \( L/\xi^2 < \kappa \gamma / \delta < (\xi / L)^2 \).

The two-photon analysis can be generalized straightforwardly to an \( N \)-photon Fock state. Then, the wave function reduces to

\[
\psi_{\text{out}}(\tau_1, \ldots, \tau_N) = \exp \left[ -i \sum_{i<j}^N \varphi(\tau_i - \tau_j) \right],
\]

where \( \tau_i = x_i - c[t - \Delta t] \). This allows one to derive the outgoing wave function for an arbitrary incoming state. Of special experimental interest however is the behavior of coherent states. A general incoming coherent state is characterized by its incoming electric field expectation value \( \mathcal{E}(x - ct) = \lim_{t \rightarrow -\infty} \mathcal{E}(x, t) \) with \( \mathcal{E}(x, t) = \langle \phi|\psi(x, t)\beta(x)|\phi \rangle \). Then, the outgoing electric field (see appendix) behaves as

\[
\frac{E_{\text{out}}(\tau)}{E(\tau)} = \exp \left( \int \mathrm{d}u |\mathcal{E}(u)|^2 [e^{-i\varphi(u - \tau)} - 1] \right),
\]

where \( \tau = x - c[t - \Delta t] \). In the limit of a weak nonlinearity \( \varphi(u) \ll 1 \), we can recover the result of a classical Kerr nonlinearity. In this regime, the incoming wave packet has a size \( l_{\text{coh}} \) much larger than the characteristic size of the interaction \( l_{\text{coh}} \gg \xi_{\text{out}} \) and propagates through a long medium \( L \gg \xi \). Then, equation (10) reduces to

\[
\frac{E_{\text{out}}(\tau)}{E(\tau)} = \mathcal{E}(\tau) \exp \left( -i \sigma \|\mathcal{E}(\tau)\|^2 \right) \text{ with } \sigma = \text{the strength of the Kerr nonlinearity.}
\]

The latter follows from the fact that a coherent state is a superposition of different number states, where each number state picks up a slightly different phase factor. Note, that residual decay processes as well as dephasing can lead to an additional suppression. The observation of the quantum correction therefore requires a precise control of dephasing and a large detuning \( |\delta| \gg \gamma \). We would like to stress that these technical noises can be suppressed independently from the strength of the Kerr nonlinearity; in contrast to a system with contact interactions [12].

The full characterization of the output state and relation to experimentally accessible quantities is most conveniently achieved by the normally ordered electric field correlations in the reduced coordinates \( \tau_i \),

\[
G_{n,m}(\tau_1, \ldots, \tau_{n+m}) = \left< \mathcal{E} \left| \prod_{j=1}^n \psi^\dagger(\tau_j) \prod_{j=n+1}^{n+m} \psi(\tau_j) \right| \mathcal{E} \right>,
\]

These correlation functions are experimentally accessible in a homodyne detection scheme. The full expression for the correlations of the outgoing fields for an incoming coherent state is presented in the appendix. In the following, we provide the result for the two point correlation function \( G_{0,1}^{\text{out}}(\tau, \tau') \), which reduces to

\[
G_{0,1}^{\text{out}}(\tau, \tau') = \mathcal{E}(\tau) \mathcal{E}(\tau') \exp \left[ -i\varphi(\tau - \tau') \right] \times \exp \left( \int \mathrm{d}u |\mathcal{E}(u)|^2 [e^{-i\varphi(u - \tau)} - 1] \right).
\]

We can distinguish two different contribution: first, we find a strong spatial correlation determined by the phase contribution \( \varphi(\tau - \tau') \), which provides direct information about the effective interaction potential between the polaritons. It is this contribution, which allows the access to the effective interaction potential within a homodyne detection scheme. The last factor describes additional phase shift and the suppression due to quantum fluctuations, which are small corrections for \( \xi_{\text{out}} \|\mathcal{E}(\tau)\|^2 \ll 1 \).

A full characterization of the outgoing field for an incoming field coherent field \( \mathcal{E} \) is provided by the Wigner function \( \mathcal{W}(q, p) \). In contrast to circuit and cavity QED experiments, where the photons within the resonator are characterized by a single photonic mode [54, 55], our system here corresponds to a multimode setup. Therefore, it is convenient to express the density matrix in terms of the Wigner function for a specific photonic mode. For this purpose, we define the annihilation operator for an arbitrary spatial mode \( u(x) \) as \( \hat{a}_u = \int \!\! \mathrm{d}x \, u(x) \psi(x) \) and the related quadrature operators as \( \hat{q} \equiv (\hat{a}_u + \hat{a}_u^\dagger) / 2, \hat{p} \equiv (\hat{a}_u - \hat{a}_u^\dagger) / 2i \). Then, the Wigner function derives directly from the analytical expression for the correlation functions \( G_{n,m}^{\text{out}} \) for the incoming coherent field, see appendix,
In order to characterize short range correlations between photons we consider a homodyne detection approach. For example, we present the results for a three-body interaction, which leads, in analogy to\cite{56–58}, to the three-body interaction potential observed in a homodyne detection scheme. Moreover, we have demonstrated that the contribution to the three photon wave-function established the connection to the conventional behavior of a Kerr nonlinearity. We have found a unique method to probe the two-body as well as higher-body interaction potentials between Rydberg polaritons using a homodyne detection scheme for a coherent input state. This approach demonstrates that no additional noise term is required. As a consequence, we conclude that the potential between the polaritons is sufficient to provide a consistent quantum description of a Kerr nonlinearity.

In contrast to previous descriptions expected to be valid for nonlinear media with purely local interactions, our results do not depend on the exact shape and the localization of\(q\) effects. Both plots are for\(|\xi(\gamma)\|^2 = 1^{2}/\xi_{\text{out}}^{2}\). The results are independent of the localization\(\tau_{0}\) of the probe mode and the shape of incoming photons. The Wigner function of the incoming state is a Gaussian function centered at the position of the probe mode, and the results do not depend on the exact shape and the localization of\(U\). The quasi-probability\(W(q, p)\) for different strengths of the interaction is shown in figure 3. For weak interactions\(\varphi(0) \ll 1\), the leading correction due to quantum fluctuations to the Gaussian coherent state is a small squeezing. However, for increasing interaction we obtain a strongly mixed state. This behavior is a result of the localized measurement tracing out all positions outside the localization of\(u(x)\). Such an operation, acting on our strongly spatially entangled state, leads to the mixed state.

A crucial property of our analysis is that it demonstrates the possibility to probe the microscopic interaction potential between the Rydberg polaritons via a homodyne detection scheme for a coherent input state. This method can easily be extended to probe higher body interactions between the polaritons, which are expected to appear for higher polariton densities. Such a n-body interaction on the microscopic level takes the form

\[
H_n = \frac{1}{n!} \int dx U_n(x_1, \ldots, x_n) \prod_{i=1}^{n} n(x_i) \psi_i^\dagger(x_i) \psi(x_i)
\]

with the n-body interaction potential\(U_n\). This term can be straightforwardly included in the exact solution. As an example, we present the results for a three-body interaction, which leads, in analogy to\(\psi^\dagger\), to a phase factor\(\varphi^\dagger\) induced by the three-body interaction takes the form

\[
\frac{1}{c \hbar} \int -\infty dw \bar{n}(w + u) \bar{n}(w + v) \bar{n}(w) \bar{U}_3(w + u, w + v, w),
\]

with\(\bar{U}_3\) defined in analogy to \(\bar{V}\). The corresponding three-body interaction potential can then be experimentally observed in a homodyne detection of the correlations \(G_{0,3}\).

In conclusion, we have studied the quantum many-body theory of interacting Rydberg polaritons and have established the connection to the conventional behavior of a Kerr nonlinearity. We have found a unique method to probe the two-body as well as higher-body interaction potentials between Rydberg polaritons using a homodyne detection scheme. Moreover, we have demonstrated that the finite range of the effective interaction potential between the polaritons is sufficient to provide a consistent quantum description of a Kerr nonlinearity.

In contrast to previous descriptions expected to be valid for nonlinear media with purely local interactions, our approach demonstrates that no additional noise term is required. As a consequence, we conclude that the proposed inability to generate a photonic phase gate by propagating photons in a nonlinear media\cite{12} does not apply to Rydberg slow light polaritons. Therefore, a counter-propagating setup\cite{19} might be a viable approach.
for the realization of high fidelity quantum phase gates. Finally, it is important to point out that the spontaneous decay from the intermediate p-level as well as from the Rydberg level, also present in our approach, lead to a non-local response in time. However, in contrast to the previous approach, the strength induced by these decay rates is independent on the strength of the Kerr nonlinearity, and therefore can be controlled independently.

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Note added in proof. After writing this manuscript, we became aware of a recent related work by Gullans et al [59].

Appendix A. Regime of parameters in which mass term is negligible

In this section, we derive a regime of parameters in which we can drop the mass term in the polaritonic dispersion relation. For this purpose, we analyze two polaritons propagating through a cloud of atoms with a constant density. In the relative \( r \) and center of mass \( R \) coordinates, the Schrödinger equation for the two-body wave function \( \phi(R, r) \) takes the form

\[
\hbar \omega \phi(R, r) = \left( -i\hbar v_g \partial_R - \frac{\hbar^2}{m} \partial^2_r + \alpha V(r) \right) \phi(R, r),
\]

where

\[
m = \hbar \left( \frac{g^2 + \Omega^2}{2}\right) V(r) = \frac{2\Omega^2 \hbar}{\delta} \frac{1}{1 + r^2/\xi^2},
\]

\[
\alpha = \frac{g^4}{(g^2 + \Omega^2)^2}, \quad v_g = \frac{\Omega^2}{\Omega^2 + g^2}.
\]

Assuming that the mass term is negligible, we can find the analytical solution of (A.1) of the form

\[
\phi_0(R, r) = \phi(0, r) \exp \left( \frac{i\omega}{v_g} R - i\varphi(0) \frac{R}{L(1 + r^2/\xi^2)} \right),
\]

where we used the relation \( \varphi(0) = L\alpha V(0)/v_g \), see (9). For latter purposes let us define

\[
\varphi_{\text{int}}(R, r) = \varphi(0) R/L/(1 + (r/\xi)^2).
\]

Using the above solution, we calculate perturbatively the corrections due to the mass term to the phase shift. For this purpose we express the full solution \( \phi \) using \( \phi_0 \):

\[
\phi(R, r) = \phi_0(R, r) \exp \left( \frac{i\omega}{v_g} R - i\varphi(0) \frac{R}{L(1 + r^2/\xi^2)} \right),
\]

where \( \partial_m(R, r) \) takes into account the impact of the mass term. Next, we insert this ansatz into (A.1). Exploiting that \( \phi_0 \) is the solution for the unperturbed Hamiltonian, most of the terms cancel and we arrive with the equation for \( \partial_m \):

\[
0 = -i v_g \phi_0(R, r) \partial_R e^{-i\omega_a(R, r)} - \frac{\hbar^2}{m} \partial^2_r \phi_0(R, r) e^{-i\omega_a(R, r)}.
\]

This equation can be simplified once we take into account that in the perturbative limit \( |\partial_\omega \varphi_{\text{int}}(R, r)| \ll |\partial_\omega \varphi_{\text{int}}(R, r)| \ll |\partial^2_\omega \varphi_{\text{int}}(R, r)| \). Moreover, considered photons are much longer than \( \xi_{\text{int}} \) and, therefore, we drop \( \partial_\omega \phi(0, r) \) and \( \partial^2_\omega \phi(0, r) \) terms. Finally, (A.6) simplifies to

\[
0 = -i v_g \phi_0(R, r) \partial_R e^{-i\omega_a(R, r)} - \frac{\hbar^2}{m} \phi(0, r) e^{-i\omega_a(R, r)} e^{i\varphi(0)} \partial^2_r \exp \left( -i\varphi(0) \frac{R}{L(1 + r^2/\xi^2)} \right).
\]

This equation leads to the solution for \( \partial_m(L, r) \) of the form

\[
\partial_m(L, r) = -\frac{1}{v_g} \int_0^L dR \frac{\hbar^2}{m} \partial^2_r \exp \left( -i\varphi(0) \frac{R}{L(1 + r^2/\xi^2)} \right) \exp \left( -i\varphi(0) \frac{R}{L(1 + r^2/\xi^2)} \right).
\]
In order to estimate \( \vartheta_m(R = L, r) \) we consider its value at \( r = \xi \), which is equal to

\[
\vartheta_m(L, \xi) = 3(\varphi(0) + i) \frac{L^2 \xi^6}{(\xi^2 + \Omega^2)^3},
\]

and corresponds to the result for \( \varphi_m \) from the main text. We see that the mass term can be dropped if two conditions \( \varphi_m \ll 1 \) and \( \varphi_m/\varphi(0) \ll 1 \) are satisfied.

**Appendix B. Correlations of the outgoing fields for an incoming coherent state**

Here, we derive the general expression for the correlations \( G_{m,n}^{\text{out}} \) of the outgoing fields for an incoming coherent state \( |\mathcal{E}\rangle \). We start by inserting the exact solution for bosonic field operators \( \hat{\psi}(z, t) = e^{-i\hat{J}(z,t)} \hat{\psi}_0(z - ct) \) into the definition of \( G_{m,n}^{\text{out}} \) from the main text. This leads to

\[
G_{m,n}^{\text{out}}(\tau_1, \ldots, \tau_{n+m}) = \left\langle \mathcal{E} \prod_{i=1}^n e^{i\hat{J}(z_i, t_i)} \hat{\psi}_0^\dagger(z_i - ct_i) \prod_{j=n+1}^{n+m} e^{-i\hat{J}(z_j, t_j)} \hat{\psi}_0(z_j - ct_j) |\mathcal{E}\rangle \right\rangle,
\]

where \( \tau_i = z_i - ct_i \). Our goal is to transform the product of operators to the normally ordered expression, of which expectation value in a coherent state is trivial. For this purpose, we first use the relation

\[
\hat{\psi}_0(z_i - ct) e^{-i\hat{J}(z_i, t_i)} = e^{-i\hat{J}(z_i, t_i)} e^{-i\varphi(z_i - z')} \hat{\psi}_0(z_i - ct)
\]

to normally order the \( \hat{\psi}_0 \) operators in the (B.1),

\[
G_{m,n}^{\text{out}}(\tau_1, \ldots, \tau_{n+m}) = \left\langle \mathcal{E} \prod_{k=1}^n \hat{\psi}_0^\dagger(\tau_k) \prod_{i=1}^n e^{i\hat{J}(z_i, t_i)} \prod_{j=n+1}^{n+m} e^{-i\hat{J}(z_j, t_j)} \hat{\psi}_0(\tau_j) |\mathcal{E}\rangle \right\rangle \times \exp \left[ i \sum_{k>l=1}^{n+m} \varphi(\tau_k - \tau_l) \right] \exp \left[ -i \sum_{k>l=n+1}^{n+m} \varphi(\tau_k - \tau_l) \right].
\]

Next, we use the fact that in the limit of \( t \to \infty \) the expression for \( \hat{J}(z_i, t) \) can be written as \( \hat{J}(z_i, t) = \int_{-\infty}^\infty du \hat{I}(u) \varphi(u - z_i + ct_i) \), and that \( \hat{J}(z_i, t) \) commutes with \( \hat{J}(z_j, t) \), in order to rewrite the product of exponentials appearing in (B.3) in the following way:

\[
\prod_{i=1}^n e^{i\hat{J}(z_i, t_i)} \prod_{j=n+1}^{n+m} e^{-i\hat{J}(z_j, t_j)} = \exp \left[ i \sum_{i=1}^n \hat{J}(z_i, t_i) - i \sum_{j=n+1}^{n+m} \hat{J}(z_j, t_j) \right]
= \exp \left[ \int_{-\infty}^\infty du \hat{I}(u) \left[ i \sum_{i=1}^n \varphi(u - \tau_i) - i \sum_{j=n+1}^{n+m} \varphi(u - \tau_j) \right] \right].
\]

The last expression can be transformed to the normally ordered version using the relation [11]:

\[
\exp \left( \int_{-\infty}^\infty du \, g(u) \hat{I}(u) \right) =: \exp \left( \int_{-\infty}^\infty du \, (e^{g(u)} - 1) \hat{I}(u) \right).
\]

In our case \( g(u) = i \sum_{i=1}^n \varphi(u - \tau_i) - i \sum_{j=n+1}^{n+m} \varphi(u - \tau_j) \), what leads to

\[
\prod_{i=1}^n e^{i\hat{J}(z_i, t_i)} \prod_{j=n+1}^{n+m} e^{-i\hat{J}(z_j, t_j)}
=: \exp \left( \int_{-\infty}^\infty du \hat{I}(u) \left[ i \sum_{i=1}^n \varphi(u - \tau_i) - i \sum_{j=n+1}^{n+m} \varphi(u - \tau_j) \right] - 1 \right).
\]

The last equation inserted into the (B.3) provides the final result,

\[
G_{m,n}^{\text{out}}(\tau_1, \ldots, \tau_{n+m})
= \prod_{i=1}^n \hat{\mathcal{E}}^\dagger(\tau_i) \prod_{j=n+1}^{n+m} \hat{\mathcal{E}}(\tau_j) \exp \left[ i \sum_{k>l=1}^{n+m} \varphi(\tau_k - \tau_l) \right] \exp \left[ -i \sum_{k>l=n+1}^{n+m} \varphi(\tau_k - \tau_l) \right]
\times \exp \left( \int_{-\infty}^\infty du \left| \mathcal{E}(u) \right|^2 \left[ i \sum_{i=1}^n \varphi(u - \tau_i) - i \sum_{j=n+1}^{n+m} \varphi(u - \tau_j) \right] - 1 \right).
\]

Two special cases of these correlations, i.e., \( G_{0,1}^{\text{out}} \) and \( G_{0,2}^{\text{out}} \) are presented in the main text, see (10) and (12), respectively.
Appendix C. Wigner function from correlation functions

Here, we show how Wigner function $W(q, p)$ can be calculate using the correlation functions $G_{nm}$. Our starting point is symmetrically ordered characteristic function $\chi(\eta)$ defined as

$$\chi(\eta) = \text{Tr} \left[ \rho \exp \{ \eta \hat{a}_n^\dagger - \eta^* \hat{a}_n \} \right].$$  

The function $\chi(\eta)$ can be expressed using correlation function $G_{nn} = \langle \hat{a}_n^\dagger \hat{a}_n^m \rangle$ as

$$\chi(\eta) = \sum_{nm} \frac{\eta^n (-\eta)^m}{n!m!} \text{e}^{-|\eta|^2/2} \text{Tr} \left[ \rho \hat{a}_n^\dagger \hat{a}_n^m \right] = \sum_{nm} \frac{\eta^n (-\eta)^m}{n!m!} \text{e}^{-|\eta|^2/2} G_{nn}.$$  

The Wigner function is defined as the Fourier transform of the characteristic function $\chi(\eta)$ [60],

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 \eta \text{ e}^{i \alpha^* - i \alpha} \chi(\eta).$$  

Finally, we insert $\chi(\eta)$ from (C.2) into the definition (C.3) and afterwards transform $W(\alpha)$ to a more concise expression:

$$W(\alpha) = \frac{1}{\pi^2} \sum_{nm} \frac{(-1)^n}{n!m!} \int d^2 \eta \text{ e}^{i \alpha^* - i \alpha} \eta^n \eta^m \text{e}^{-|\eta|^2/2} G_{nn}$$

$$= \frac{1}{\pi^2} \sum_{nm} \frac{(-1)^n + m}{n!m!} G_{nm} \partial_n \partial_m \int d^2 \eta \text{ e}^{i \alpha^* - i \alpha} \text{e}^{-|\eta|^2/2}$$

$$= \frac{2}{\pi^2} \sum_{nm} \frac{(-1)^n + m}{n!m!} G_{nm} \partial_n \partial_m \text{e}^{-2|\alpha|^2},$$

which is the formula presented in the main text.

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