Exotic Gauge Theories from Tensor Calculus

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c\textsuperscript{Abstract}

We construct non-standard interactions between exterior form gauge fields by gauging a particular global symmetry of the Einstein-Maxwell action for such fields. Furthermore we discuss generalizations of such interactions by adding couplings to gravitational Chern-Simons forms and to fields arising through dimensional reduction. The construction uses an appropriate tensor calculus.

\textbf{Introduction}

Exterior form gauge fields $A_p = (1/p!) dx^\mu_1 \cdots dx^\mu_p A_{\mu_1 \cdots \mu_p}$ generalize naturally the electromagnetic gauge potential and are therefore interesting on general grounds in the context of gauge theories. In particular they play a substantial rôle in supergravity models and string theory. It is therefore natural to study consistent interactions of such gauge fields, both the interactions between themselves and their couplings to other fields.

In this paper we construct non-standard interactions of exterior form gauge fields. This is done in curved spacetime, i.e. the coupling to the gravitational field is included as well. Our starting point is the Einstein-Maxwell action for a $p$-form gauge field $A_p$ and an $(n - p - 1)$-form gauge field $A_{n-p-1}$ in $n$-dimensional spacetime (for arbitrary $p$ and $n$). This action has a global symmetry which shifts $A_p$ by the hodge dual of the field strength of $A_{n-p-1}$, and $A_{n-p-1}$ by the hodge dual of the field strength of $A_p$ (with an appropriate sign factor, see below). We gauge this global symmetry, using an additional 1-form gauge field $V = dx^\mu V_\mu$. This yields inevitably interactions which are non-polynomial in the $V_\mu$. It would therefore be cumbersome to construct these interactions in a pedestrian way via the standard Noether procedure. Instead, we employ an appropriate tensor calculus which is analogous to the one introduced in [1] in flat four-dimensional spacetime. By formulating this tensor calculus in the differential form language we simplify the construction considerably as compared to the formulation in terms of components used in [1].
The resulting interactions can be generalized in several ways. We shall describe two such generalizations arising in gravitational theories: (a) additional couplings of $p$-form gauge fields to gravitational Chern-Simons forms; (b) couplings to additional vector gauge fields and scalar fields arising by standard Kaluza-Klein type dimensional reduction.

Our work is linked to recent progress in four-dimensional supersymmetric gauge theories, and theories with exterior form gauge fields in general. To our knowledge, gauge theories of the type constructed here appeared for the first time in [2]. There four-dimensional $N = 2$ supersymmetric gauge theories were constructed in which the central charge of the vector-tensor multiplet is gauged. This central charge is a global symmetry of the type described above and therefore the models found in [2] contain interactions of the same non-standard type as those we shall obtain. Further four-dimensional $N = 2$ supersymmetric gauge theories of the same type were constructed in [3]. The vector-tensor multiplet with gauged central charged is believed to arise in $N = 2$ heterotic string vacua [4]. Interactions of the type studied here may therefore be relevant in that context, among other things.

Independently of these developments related to the $N = 2$ vector-tensor multiplet, models of the type considered here, and generalizations thereof, were discovered in [5] (along with further new interactions of exterior form gauge fields) within a systematic classification of possible consistent interaction vertices of exterior form gauge fields. Four-dimensional $N = 1$ supersymmetric versions of some of these models were constructed in [6]. Our work thus adds to the list of new gauge theories in these references.

**The Basic Model**

The properly normalized Einstein-Maxwell action for $A_p$ and $A_{n-p-1}$ is

$$S_0 = \frac{1}{2} \int \left[ \frac{(-)^{(p+1)(n-1)}}{dA_p \star dA_p + \frac{(-)^p}{dA_{n-p-1} \star \star dA_{n-p-1}}} \right].$$  \hfill (1)

Here we use the following conventions for a $p$-form $\omega_p$ and its Hodge dual:

$$\omega_p = \frac{1}{p!} dx^{\mu_1} \cdots dx^{\mu_p} \omega_{\mu_1 \cdots \mu_p},$$

$$\star \omega_p = \frac{1}{p!(n-p)!} dx^{\mu_1} \cdots dx^{\mu_{n-p}} \varepsilon_{\mu_1 \cdots \mu_n} \omega^{\mu_{n-p+1} \cdots \mu_n},$$

where indices are raised with the inverse metric $G^{\mu\nu}$, and the curved Levi-Civita tensor is defined by $(x^1$ labels the time coordinate)

$$\varepsilon_{12 \cdots n} = -\sqrt{G}, \quad G = -\det G_{\mu\nu}.$$

One should of course add the Einstein action to $S_0$, which however we shall not write explicitly.

$S_0$ has among others a global symmetry generated by

$$\Delta A_p = \star dA_{n-p-1}, \quad \Delta A_{n-p-1} = \frac{(-)^{n(p+1)}}{\star dA_p}.$$  \hfill (2)
The corresponding Noether current, written in dual notation as an \((n-1)\)-form, is
\[
J = \star dA_p \star dA_{n-p-1} .
\] (3)

We remark that the global symmetry \(\Delta\) and its current \(J\) are non-trivial, see comments at the end of the paper.

We will now try to gauge this symmetry. To do this, we search for a modified generator \(\Delta'\), such that \(\Delta'A_p\) and \(\Delta'A_{n-p-1}\) transform covariantly under gauge transformations
\[
\delta_\epsilon A_p = g \epsilon \Delta'A_p , \quad \delta_\epsilon A_{n-p-1} = g \epsilon \Delta'A_{n-p-1} ,
\] (4)
where \(\epsilon\) is an arbitrary scalar field and \(g\) a coupling constant of mass dimension \(-1\). A reasonable ansatz is to replace the exterior derivative in (2) with a covariant one. Let us therefore introduce a connection 1-form \(V\) with the standard transformation law
\[
\delta_\epsilon V = d\epsilon
\] (5)
and a covariant derivative
\[
D = d - gV \Delta' ,
\] (6)
where \(\Delta'\) is the covariant version of (2),
\[
\Delta'A_p = \star DA_{n-p-1} , \quad \Delta'A_{n-p-1} = -(-)^{n(p+1)} \star DA_p .
\] (7)

These equations give the action of \(\Delta'\) only implicitly because \(D\) involves \(\Delta'\). We now determine the covariant derivatives \(DA_p\) and \(DA_{n-p-1}\). Starting with the former, we have
\[
DA_p = dA_p - gV \star DA_{n-p-1} = dA_p - gV \star [dA_{n-p-1} + (-)^{n(p+1)} g V \star DA_p] .
\]
Using the identity
\[
\star(V \star \omega_p) = (-)^{np} i_V \omega_p , \quad i_V = V^\mu \frac{\partial}{\partial(x^\mu)} ,
\] (8)
which holds for any \(p\)-form \(\omega_p\) (where \(V^\mu = G^{\mu\nu} V_\nu\)), this gives
\[
DA_p = dA_p - gV \star dA_{n-p-1} - g^2 V i_V DA_p .
\]
To solve for \(DA_p\), we have to invert the operator \(1 + g^2 V i_V\). With \((Vi_V)^2 = V^\mu V_\mu V_i V\), it is easily verified that
\[
(1 + g^2 V i_V)^{-1} = 1 - g^2 E^{-1} V i_V , \quad E = 1 + g^2 V^\mu V_\mu ,
\] (9)
and we obtain
\[
DA_p = dA_p - gE^{-1} V [\star dA_{n-p-1} + g i_V dA_p] .
\] (10)
Analogously, one finds
\[ DA_{n-p-1} = dA_{n-p-1} - g E^{-1} V \left[ -(-)^{n(p+1)} \star dA_p + g i_V dA_{n-p-1} \right]. \tag{11} \]

Note that due to the appearance of \( E^{-1} \) the covariant derivatives, and thus the gauge transformations, of \( A_p \) and \( A_{n-p-1} \) are non-polynomial in the connection \( V_\mu \) and the coupling constant \( g \).

As can be checked, \( DA_p \) and \( DA_{n-p-1} \) are indeed covariant, i.e. their gauge transformations do not involve derivatives of \( \epsilon \), and one has
\[ \delta_\epsilon DA_p = g \epsilon D \Delta' A_p = g \epsilon D \star DA_{n-p-1} \]
\[ \delta_\epsilon DA_{n-p-1} = g \epsilon D \Delta' A_{n-p-1} = -(-)^n(-)^{(p+1)} g \epsilon D \star DA_p. \tag{12} \]

Now we can proceed to construct the gauge invariant action. To do this, we use the following fact: let \( X \) be a covariant volume form which transforms according to \( \delta_\epsilon X = g \epsilon D K \), with \( K \) a covariant \((n-1)\)-form. Then \( X + g V K \) transforms into a total derivative,
\[ \delta_\epsilon (X + g V K) = d(g \epsilon K). \tag{13} \]

In particular, thanks to (12) this applies to
\[ X = (-)^{(p+1)(n-1)} DA_p \star DA_p + (-)^{p(n-1)} DA_{n-p-1} \star DA_{n-p-1} \tag{14} \]
with
\[ K = -2(-)^n p \star DA_p \star DA_{n-p-1}. \tag{15} \]

Adding a kinetic term for the connection \( V \), the action thus reads
\[ S = \frac{1}{2} \int \left( X + g V K + dV \star dV \right), \tag{16} \]
with \( X \) and \( K \) as in (14) and (15). By construction, \( \delta_\epsilon S \) is a surface term, with the transformations as in (4), (5) and \( \delta_\epsilon G_{\mu \nu} = 0 \). Furthermore, the action is invariant under the standard spacetime diffeomorphisms and the abelian gauge transformations of \( A_p \) and \( A_{n-p-1} \),
\[ \delta A_p = d\Lambda_{p-1}, \quad \delta A_{n-p-1} = d\Lambda_{n-p-2}, \tag{17} \]
since these fields enter the covariant derivatives (10), (11) only via their exterior derivatives.

Alternatively, one may consider a first order formulation. Introducing auxiliary fields \( \beta_p \) and \( \beta_{n-p-1} \), with the form degree as indicated, the action is given by
\[
S = \frac{1}{2} \int \left[ dV \star dV + (-)^{(n-1)} \beta_p \star \beta_p + (-)^{(p+1)(n-1)} \beta_{n-p-1} \star \beta_{n-p-1} \\
+ 2 \beta_p dA_{n-p-1} - 2(-)^n(-)^{p+1} \beta_{n-p-1} dA_p - 2(-)^p g V \beta_p \beta_{n-p-1} \right]. \tag{18}
\]
while the gauge transformations read

\[
\delta_\epsilon V = d\epsilon, \quad \delta_\epsilon A_p = g\epsilon\beta_p, \quad \delta_\epsilon A_{n-p-1} = g\epsilon\beta_{n-p-1}
\]

\[
\delta_\epsilon\beta_p = \delta_\epsilon\beta_{n-p-1} = \delta_\epsilon G_{\mu\nu} = 0. \quad (19)
\]

The equations of motion for the auxiliary fields are coupled in exactly the same way as the equations that determine the covariant derivatives of \(A_p\) and \(A_{n-p-1}\),

\[
-(−)^{p(n-1)} \ast \beta_p = dA_{n-p-1} - gV\beta_{n-p-1}
\]

\[
-(−)^p \ast \beta_{n-p-1} = dA_p - gV\beta_p, \quad (20)
\]

and the solutions are thus

\[
\beta_p = \Delta' A_p, \quad \beta_{n-p-1} = \Delta' A_{n-p-1}. \quad (21)
\]

The action \((18)\) may also be used to derive a dual version of our model. To that end one solves the equations of motion for \(A_p\) and \(A_{n-p-1}\) through \(\beta_p = dA_{p-1}\) and \(\beta_{n-p-1} = dA_{n-p-2}\) and inserts the solution back into the action. The interaction vertex \(V\beta_p\beta_{n-p-1}\) then turns into a Chern-Simons term.

**Remark.** If \(n = 1 + 4k\) and \(p = 2k\), one may identify \(A_p\) with \(A_{n-p-1}\). For instance, in the case \(n = 5\), \(p = 2\) one then gets a 5-dimensional theory involving only one 2-form gauge field besides \(V\) and the gravitational field.

**Generalizations**

**Coupling to Gravitational Chern-Simons Forms**

We shall now discuss how the basic model introduced above can be generalized by including couplings of \(p\)-form gauge potentials to gravitational Chern-Simons forms. We denote a gravitational Chern-Simons \(p\)-form by \(q_p\), where \(p = 3, 7, \ldots\), or \(p = n-1\) in \(n = 2k\) dimensional spacetime (the latter case corresponds to the Euler density in even dimensional spacetime). So, for instance, in 5-dimensional spacetime there is only one gravitational Chern-Simons form, \(q_3\), which satisfies \(dq_3 = R^\sigma \ast R_\sigma\ast\), where \(R^\sigma = \frac{1}{2} dx^\mu dx^\nu R_{\mu\nu\sigma}\) (explicitly one has \(q_3 = \Gamma^\mu_\nu \ast R_\mu\ast - \frac{1}{3} \Gamma^\mu_\nu \ast \Gamma_\rho \ast \Gamma_\sigma \ast\), where \(\Gamma^\sigma_\rho = dx^b \Gamma^\sigma_{\mu\nu}\)).

In 4-dimensional spacetime there are two independent gravitational Chern-Simons 3-forms, \(q_3\) and \(q_3'\), satisfying \(dq_3 = R^\rho \ast R_\rho\ast\) and \(dq_3' = \varepsilon_{\mu\nu\rho\sigma} R^{\mu\nu} R^\rho\sigma\) respectively.

In a first order formulation, a gravitational Chern-Simons \((p+1)\)-form can be coupled to a \(p\)-form gauge potential through the following extension of the action \((18)\),

\[
S = \frac{1}{2} \int [dV \ast dV + (−)^{p(n-1)} \beta_p \ast \beta_p + (−)^{(p+1)(n-1)} \beta_{n-p-1} \ast \beta_{n-p-1}]
\]

\[
+ 2\beta_p dA_{n-p-1} - 2(−)^{n(p+1)} \beta_{n-p-1} (dA_p + q_{p+1}) - 2(−)^p gV\beta_p\beta_{n-p-1}]. \quad (22)
\]

This action is invariant under the gauge transformations \((17)\) and \((19)\), and the following modification of the spacetime diffeomorphisms,

\[
\delta_\zeta \Phi = \mathcal{L}_\zeta \Phi \quad \text{for} \quad \Phi \in \{G_{\mu\nu}, \beta_p, \beta_{n-p-1}, A_{n-p-1}\}
\]
\[ \delta_\xi A_p = \mathcal{L}_\xi A_p - r_p , \]

where \( \mathcal{L}_\xi \) is the standard Lie derivative along the vector field \( \xi^\mu \) (with \( \mathcal{L}_\xi \beta_\mu = (1/p!) \cdot dx^{\mu_1} \ldots dx^{\mu_p} \mathcal{L}_\xi \beta_{\mu_1 \ldots \mu_p} \) etc.), and \( r_p \) is the \( p \)-form whose exterior derivative is the non-covariant part of \( \delta_\xi q_{p+1} \),

\[ \delta_\xi q_{p+1} = \mathcal{L}_\xi q_{p+1} + dr_p . \]

So for \( q_3 \) as above, one gets \( r_2 = \partial_\mu \xi^\nu d\Gamma^\mu_{\nu\lambda} \), for instance. As usual, the combination \( dA_p + q_{p+1} \) transforms covariantly under \( \delta_\xi \), \( \delta_\xi (dA_p + q_{p+1}) = \mathcal{L}_\xi (dA_p + q_{p+1}) \), and therefore the action (22) is indeed invariant under \( \delta_\xi \).

From the above formulae one easily infers the second order version of the action and the gauge transformations in presence of Chern-Simons couplings. It is simply obtained from (18) and (19) by substituting \( dA_p + q_{p+1} \) for \( dA_p \) everywhere. Note that this results in the presence of the gravitational Chern-Simons form \( q_{p+1} \) both in \( \delta_\epsilon A_p \) and \( \delta_\epsilon A_{n-p-1} \).

**Dimensional Reduction**

We shall now perform a dimensional reduction from \( n \) to \( n - 1 \) dimensions. This gives rise to couplings to additional gauge fields and scalars. We shall discuss explicitly only the dimensional reduction of the basic model. The extension to more involved models, such as those with additional Chern-Simons couplings, is straightforward.

We denote the coordinates by

\[ x^\mu = (x^a, x^n) , \quad a = 1, \ldots, n - 1 , \]

and take all fields to be constant along the \( x^n \)-direction. Then the metric decomposes in the usual manner,

\[ G_{\mu\nu} dx^\mu \otimes dx^\nu = g_{ab} dx^a \otimes dx^b + e^{2\varphi} (W + dx^n) \otimes (W + dx^n) , \]

where \( W = dx^a W_a \) is a 1-form in \( n - 1 \) dimensions.

Upon dimensional reduction, \( A_p \) gives rise to a \( p \)-form \( \hat{A}_p \) and a \((p-1)\)-form \( \hat{A}_{p-1} \), while \( A_{n-p-1} \) decomposes into an \((n-p-1)\)-form \( \hat{A}_{n-p-1} \) and an \((n-p-2)\)-form \( \hat{A}_{n-p-2} \). The connection \( V \) introduces in addition to a 1-form \( \hat{V} \) a scalar field \( \phi \),

\[ A_p = \hat{A}_p + \hat{A}_{p-1} dx^n , \quad A_{n-p-1} = \hat{A}_{n-p-1} + \hat{A}_{n-p-2} dx^n , \quad V = \hat{V} + \phi dx^n . \]

In the following, one should keep in mind that the descendants \( \hat{A}_p, \hat{A}_{n-p-1} \) and \( \hat{V} \) all transform non-trivially under the abelian gauge transformation associated with \( W \), while \( \hat{A}_{p-1}, \hat{A}_{n-p-2} \) and \( \phi \) are invariant,

\[ \delta_\lambda W = d\lambda , \quad \delta_\lambda \hat{A}_p = \hat{A}_{p-1} d\lambda , \quad \delta_\lambda \hat{A}_{n-p-1} = \hat{A}_{n-p-2} d\lambda , \quad \delta_\lambda \hat{V} = \phi d\lambda \]

\[ \delta_\lambda \hat{A}_{p-1} = \delta_\lambda \hat{A}_{n-p-2} = \delta_\lambda \phi = 0 . \]
When decomposing the dual of a $p$-form $\omega_p$, we make use of the relation
\[ \star \omega_p = e^{-\varphi} \star \hat{\omega}_{p-1} + (-)^p e^{\varphi} \star (\hat{\omega}_p - \hat{\omega}_{p-1} W) \ (W + dx^n) , \]
where the Hodge star $\star$ in $n - 1$ dimensions involves the reduced Levi-Civita tensor $\varepsilon_{a_1 \ldots a_{n-1}} = e^{-\varphi} \varepsilon_{a_1 \ldots a_{n-1} n}$ and indices are raised with $g^{ab}$.

There are now two ways of determining the covariant derivatives and gauge transformations of the $\hat{A}$-fields, which turn out to give the same results. One may either start from the first order formulation and solve the dimensionally reduced equations of motion for the auxiliary fields, or one may directly decompose the equations (10) and (11). Let us follow the latter approach: we define a covariant derivative $\hat{D}$ by the relations
\[ DA_p = \hat{D} \hat{A}_p + \hat{D} \hat{A}_{p-1} dx^n , \quad DA_{n-p-1} = \hat{D} \hat{A}_{n-p-1} + \hat{D} \hat{A}_{n-p-2} dx^n . \quad (29) \]
The decomposition of the left-hand sides is straightforward, and by comparison of the terms with and without $dx^n$ one derives the action of $\hat{D}$ on the four gauge fields. Similarly to the gauge fields themselves, the descendants $\hat{D} \hat{A}_p$ and $\hat{D} \hat{A}_{n-p-1}$ transform non-covariantly under $\delta_\lambda$. It is convenient to use $\delta_\lambda$-invariant generalized field strengths instead, analogous to those appearing in Kaluza-Klein supergravity models (see e.g. [1]),
\[ F_{p+1} = \hat{D} \hat{A}_p - \hat{D} \hat{A}_{p-1} W , \quad F_{n-p} = \hat{D} \hat{A}_{n-p-1} - \hat{D} \hat{A}_{n-p-2} W . \quad (30) \]
Then also the connection $\hat{V}$ appears always in a $\delta_\lambda$-invariant combination, which we denote by
\[ U = \hat{V} - \phi W . \quad (31) \]
Since the original $n$-dimensional fields do not appear anymore, we shall omit all hats in the following. Explicitly, one finds
\[ F_{p+1} = dA_p - dA_{p-1} W - g E^{-1} U[e^{-\varphi} \star dA_{n-p-2} + g i_U(dA_p - dA_{p-1} W) \]
\[ + (-)^p g e^{-2\varphi} \phi dA_{p-1}] \]
\[ F_{n-p} = dA_{n-p-1} - dA_{n-p-2} W - g E^{-1} U[-(-)^n(p+1)e^{-\varphi} \star dA_{p-1} \]
\[ + g i_U(dA_{n-p-1} - dA_{n-p-2} W) - (-)^{n-p} g e^{-2\varphi} \phi dA_{n-p-2}] , \quad (32) \]
and
\[ DA_{p-1} = dA_{p-1} - g E^{-1} U[(-)^{n-p} e^{-\varphi} \star (dA_{n-p-1} - dA_{n-p-2} W) + g i_U dA_{p-1}] \]
\[ - g E^{-1} \phi[(-)^{n-p} e^{-\varphi} \star dA_{n-p-2} + (-)^p g i_U(dA_p - dA_{p-1} W) \]
\[ + g e^{-2\varphi} \phi dA_{p-1}] \]
\[ DA_{n-p-2} = dA_{n-p-2} - g E^{-1} U[-(-)^{(p+1)(n-1)} e^{-\varphi} \star (dA_p - dA_{p-1} W) + g i_U dA_{n-p-2}] \]
\[ - g E^{-1} \phi[(-)^{p(n-1)} e^{-\varphi} \star dA_{p-1} - (-)^{n-p} g i_U(dA_{n-p-1} - dA_{n-p-2} W) \]
\[ + g e^{-2\varphi} \phi dA_{n-p-2}] , \quad (33) \]
where now
\[ E = 1 + g^2 U^a U_a + g^2 e^{-2\phi} \phi^2 \, . \quad (34) \]

The above expressions enter the gauge transformations \( \delta \) of the forms \( A_p \) etc., which are found to read
\[
\begin{align*}
\delta \epsilon A_p &= g \epsilon [e^{-\epsilon} \star DA_{n-p-2} + (-)^{n-p} e^\phi \star F_{n-p} W] \\
\delta \epsilon A_{n-p-1} &= (-)^{n-p} g \epsilon e^\phi \star F_{n-p} \\
\delta \epsilon A_{n-p-1} &= (-)^{n(p+1)} g \epsilon [e^{-\epsilon} \star DA_{n-p-1} - (-)^p e^\phi \star F_{p+1} W] \\
\delta \epsilon A_{n-p-2} &= (-)^{(p+1)(n-1)} g \epsilon e^\phi \star F_{p+1} \, ,
\end{align*}
\]

while those of the remaining fields are simply
\[ \delta \epsilon U = d\epsilon \, , \quad \delta \epsilon \phi = \delta \epsilon W = \delta \epsilon g_{ab} = 0 \, . \quad (36) \]

We observe that \( DA_{n-p-1} \) and \( DA_{n-p-2} \) contain terms which are accompanied by the scalar \( \phi \) rather than the connection \( U \). One may introduce “minimal” covariant derivatives \( D \) of \( A_{n-p-1} \) and \( A_{n-p-2} \) by substracting appropriate covariant terms from \( DA_{n-p-1} \) and \( DA_{n-p-2} \),
\[
\begin{align*}
DA_{p-1} &= DA_{p-1} + (-)^p g e^{-\phi} \phi \star DA_{n-p-2} \\
DA_{n-p-2} &= DA_{n-p-2} + (-)^{(p-1)(n-1)} g e^{-\phi} \phi \star DA_{p-1} \, .
\end{align*}
\]

This results in the following simpler expressions:
\[
\begin{align*}
DA_{p-1} &= dA_{p-1} - g E^{-1} U [(-)^{n-p} E e^\phi \star (dA_{n-p-1} - dA_{n-p-2} W) + g i_U dA_{p-1} \\
&\quad - (-)^p g^2 e^{-\phi} \phi i_U \star dA_{n-p-2}] \\
DA_{n-p-2} &= dA_{n-p-2} - g E^{-1} U [(-)^{(p+1)(n-1)} E e^\phi \star (dA_p - dA_{p-1} W) \\
&\quad + g i_U dA_{n-p-2} - (-)^{p(n-1)} g^2 \phi i_U \star dA_{p-1}] \, ,
\end{align*}
\]

where \( E \) is a function of the scalar fields only,
\[ E = 1 + g^2 e^{-2\phi} \phi^2 \, . \quad (39) \]

Equations (37) can be inverted to express \( DA_{p-1} \) and \( DA_{n-p-2} \) in terms of the minimal covariant derivatives,
\[
\begin{align*}
DA_{p-1} &= E^{-1} [DA_{p-1} - (-)^p g e^{-\phi} \phi \star DA_{n-p-2}] \\
DA_{n-p-2} &= E^{-1} [DA_{n-p-2} - (-)^{(p-1)(n-1)} g e^{-\phi} \phi \star DA_{p-1}] \, .
\end{align*}
\]

Finally, we obtain the gauge invariant action in \( (n-1) \)-dimensional spacetime by reduction of equation (16). Using the equations (34), (35) and (36), it can be written entirely in terms of \( \delta_\lambda \)-invariant as well as \( \delta_\epsilon \)-covariant expressions,
\[ S = \frac{1}{2} \int [e^\phi (dU + \phi dW) \star (dU + \phi dW) + (-)^n e^{-\phi} d\phi \star d\phi \]
As compared to (16), the new feature of the action (41) is the coupling of \( W \). Indeed, if we set these fields to zero, (41) reduces to an action of the form (16) in zero scalar and vector fields coming from \( G \) procedure to derive more involved models of the above type. Again, after setting to zero scalar and vector fields, plus kinetic terms for the additional fields in the models obtained by dimensional reduction. The non-polynomial structure and the non-triviality of the interactions can be traced to the properties of the global symmetry (2) that is gauged.

Namely, to first order in the coupling constant \( g \), the deformed Einstein-Maxwell action (14) reads

\[
S_1 = (-)^{n-p} g \int V \mathcal{J},
\]

where \( \mathcal{J} \) is the Noether current (3). Hence, \( S_1 \) is nothing but the standard Noether coupling of the gauge field \( V \) to the conserved current of the global symmetry (2). The corresponding first order deformation of the gauge symmetry involves the global symmetry (2) itself: it is just \( \delta_1 = g \epsilon \Delta \) (i.e. \( \delta_1 A_p = g \epsilon \star dA_{n-p-1} \), \( \delta_1 A_{n-p-1} = -(-)^{n(p+1)} g \epsilon \star dA_p \)). It can be readily checked that \( \delta_1 S_1 \) does not vanish, due to the fact that \( J \) is not invariant under \( \epsilon \Delta \). Rather, one has \( \delta_1 S_1 \approx -\delta_0 S_2 \), where \( \approx \) denotes weak equality ("on-shell equality"), i.e. equality up to terms that involve the left hand sides of the equations of motion. As a consequence, consistency of the interactions at second order makes it necessary to introduce second order deformations both of the action and of the gauge symmetry (this can be rigorously proved by BRST cohomological arguments along the lines of [8]). This extends to all higher orders and leads to the non-polynomial structure of the interactions and gauge transformations.

In fact the situation is somewhat similar to the coupling of the gravitational field to matter in standard gravity. Indeed, viewed as a deformation using \( G_{\mu\nu} = \eta_{\mu\nu} + m_{pl}^{-1} h_{\mu\nu} \), the first order deformation couples \( h_{\mu\nu} \) to the energy momentum tensor \( T^{\mu\nu} \) in flat spacetime. \( T^{\mu\nu} \) is the Noether current of translations and not invariant under the generators of translations given by the spacetime derivatives. This leads to interactions

\[
+ (-)^{n(p+1)} e^\varphi \left( \mathcal{F}_{p+1} \star \mathcal{F}_{p+1} + \mathcal{F}_{n-p} \star \mathcal{F}_{n-p} \right)
+ (-)^{np} \varepsilon^{-\varphi} \left( \mathcal{D} A_{p-1} \star \mathcal{D} A_{p-1} + \mathcal{D} A_{n-p-2} \star \mathcal{D} A_{n-p-2} \right)
- 2g \varepsilon^{-1} U \left( \mathcal{D} A_{p-1} \star \mathcal{F}_{n-p} - (-)^{n(p+1)} \mathcal{D} A_{n-p-2} \star \mathcal{F}_{p+1} \right)
- 2(-)^{p(n-1)} g^2 \varepsilon^{-2} \phi U \left( \mathcal{D} A_{n-p-2} \star \mathcal{F}_{n-p} - (-)^{n} \mathcal{D} A_{p-1} \star \mathcal{F}_{p+1} \right)
+ 2(-)^{p} g \varepsilon^{-1} e^{-2\varphi} \phi \mathcal{D} A_{p-1} \mathcal{D} A_{n-p-2} \right].
\]
that are non-polynomial in $h_{\mu\nu}$ and in the coupling constant $m_{\text{pl}}^{-1}$ which are compactly constructed by means of the familiar tensor calculus.

The non-standard coupling of the $p$-form gauge potentials to scalar fields present in the dimensionally reduced models can be understood analogously from the point of view of consistent deformations. If we specialize these models by setting $W$ and $\varphi$ to zero, the first order coupling of the scalar field $\phi$ is just $g \int \phi dA_{p-1} dA_{n-p-2}$ (up to a factor), i.e. it couples $\phi$ to the topological density $dA_{p-1} dA_{n-p-2}$. Again, the latter is not invariant under $\delta_1$. This enforces further interaction terms of higher order, with higher powers of $\phi$, and eventually non-polynomial interactions of $\phi$. Analogous statements apply in presence of $W$ and $\varphi$, where further interactions are present. One may disentangle the various couplings in these interactions by introducing further coupling constants $g_1$ and $g_2$ through the rescalings $W \to g_1 W$, $\varphi \to g_2 \varphi$.

Hence, though one might have suspected that the non-polynomial interactions of the scalar fields (and of $W$) in the above models are just an artefact of the dimensional reduction, such interactions are actually to be expected in gauge theories of the type studied here, whether or not the models can be obtained by dimensional reduction. In fact, the reader may check that non-polynomial interactions of scalar fields similar to those found here are present in all supersymmetric models constructed in [2, 3, 6].

Finally we comment briefly on the non-triviality of the deformations constructed here. Deformations of actions and gauge transformations are called trivial if they can be removed by mere field redefinitions. The non-triviality of the deformations constructed here traces to the non-triviality of the global symmetry $\Delta$ and its current $J$ (again, this can be shown by BRST cohomological arguments).

In general, the non-triviality of conserved currents is concisely formulated in the so-called characteristic cohomology (see e.g. [11]) related to the equations of motion. This cohomology is defined through the cocycle condition $d\omega_p \approx 0$ and the coboundary condition $\omega_p \approx d\omega_{p-1}$, where $\omega_p$ and $\omega_{p-1}$ are local forms constructed of the fields and their derivatives (of first or higher order) and $\approx$ denotes weak equality as described above. In our case $J$ is indeed a non-trivial cocycle of the characteristic cohomology related to the Einstein-Maxwell equations: it is conserved, $dJ \approx 0$, and non-trivial since there is no local $(n-2)$-form $\omega_{n-2}$ such that $J \approx d\omega_{n-2}$.

This implies already that $\Delta$ is non-trivial too, because a global symmetry is trivial (= weakly equal to a gauge transformation) if and only if the corresponding Noether current is trivial in the characteristic cohomology [14] (when global subtleties are absent or irrelevant). In fact, in our case the non-triviality of $\Delta$ traces also directly to the characteristic cohomology: $\star dA_p$ and $\star dA_{n-p-1}$ are non-trivial in that cohomology (cf. [11]) and thus $\Delta A_p$ and $\Delta A_{n-p-1}$ are not weakly equal to a gauge transformation.

Remark. The non-triviality of $J$, $\star dA_p$ and $\star dA_{n-p-1}$ in the characteristic cohomology does not contradict the fact that these quantities are exact (at least locally) when one evaluates them for a specific solution to the equations of motion. In fact, by the ordinary Poincaré lemma, they are locally exact for any specific solution to the equations of motion since they are weakly closed. The latter statement applies of course to every cocycle of the characteristic cohomology (except for the constant 0-forms), and thus in particular to every Noether current, whether or not it is trivial. The point is that a form $\omega_p$ which is trivial in the characteristic cohomology is weakly equal to $d\omega_{p-1}$ with
a *definite* $\omega_{p-1}$, that involves the fields and their derivatives and does not depend upon which solution to the equations of motion one considers.

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