On reducible partition of graphs and its application to Hadwiger conjecture

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Abstract

An undirected graph $H$ is called a minor of the graph $G$ if $H$ can be formed from $G$ by deleting edges and vertices and by contracting edges. If $G$ does not have a graph $H$ as a minor, then we say that $G$ is $H$-free. Hadwiger conjecture claim that the chromatic number of $G$ may be closely related to whether it contains $K_{n+1}$ minors. To study the coloring of a $K_{n+1}$-free $G$, we propose a new concept of reducible partition of vertex set $V_G$ of $G$. A reducible partition(RP) of a graph $G$ with $K_n$ minors and without $K_{n+1}$ minors is defined as a two-tuples \( \{S_1 \subseteq V_G, S_2 \subseteq V_G\} \) which satisfy the following conditions:

1. $S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset$
2. $S_2$ is dominated by $S_1$,
3. the induced subgraph $G[S_1]$ is a forest,
4. the induced subgraph $G[S_2]$ is $K_n$-free.

Further, one can obtain a special reducible partition(SRP) $\{S_1, S_2\}$ of $V_G$, which satisfy the following conditions:

1. $S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset$
2. $S_1$ is an independent set,
3. the induced subgraph $G[S_2]$ is $K_n$-free.

We will show that both SRP and RP are always exist for any graph. With the SRP of a $K_{n+1}$-free graph $G$, one can obtain some useful conclusion on
1. Exhaustive Reducible Partition(ERP) of graphs

Let \( G = (V_G, E_G) \) be a graph with vertex set \( V_G \) and edge set \( E_G \). A subset \( S \subset V_G \) is called a dominating set of \( G \) if each vertex in \( V_G - S \) is adjacent to at least one vertex of \( S \). When the subgraph \( G[S] \) induced by \( S \) is a forest \( F \), then the \( S \) is called a dominating forest (DF) of the graph \( G \). And when say a forest \( F \) is a DF, it refers that \( V_F \) is a DF. If \( G[S] \) is not a subgraph of any other dominating forests of \( G \), then \( G[S] \) is called a maximal DF.

If \( F \) is one of maximal DFs of \( G \), then we say \( V_G - V_F \) is dominated by \( V_F \) or \( F \), where \( V_F \) is the vertex set of the graph \( F \). The two-tuples \( \{V_F, V_G - V_F\} \) is called a dominating forest partition of \( G \).

**Theorem 1.** Any graph \( G \) possesses a DF.

**Proof.** The theorem can be proved by contradiction. Assume that there are no DFs in \( G \) and \( F \) is a maximal induced sub-forest of \( G \), namely \( F \) is not a subgraph of any other induced sub-forest of \( G \). As \( F \) is not a dominating forest, then there exists some vertex \( v \notin V_F \) has no neighbors in \( V_F \), then \( G[V_F \cup v] \) is a forest and \( F \) is a subgraph of \( G[V_F \cup v] \), which is a contradiction. \( \square \)

A minor of a graph \( G \) is any graph \( H \) that is isomorphic to a graph that can be obtained from a subgraph of \( G \) by contracting some edges. If \( G \) does not have a graph \( H \) as a minor, then we say that \( G \) is \( H \)-free.

**Definition 1.** For a \( K_{n+1} \)-free graph \( G \) with \( K_n \) minors and two subset \( S_1 \) and \( S_2 \) of \( V_G \), \( R = \{S_1, S_2\} \) is called a reducible partition of \( V_G \), if the following conditions are satisfied:

1. \( S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset \)
2. \( S_2 \) is dominated by \( S_1 \),

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Hadwiger conjecture.

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(3) the induced graph $G[S_1]$ is a forest,
(4) the induced graph $G[S_2]$ is $K_n$-free.

$S_1$ and $S_2$ are called the partitions of $\text{RP}$ of $G$.

Definition \[1\] implies that if $S_1$ is a maximal dominating forest of a graph $G$ with $K_n$ and without $K_{n+1}$ as minors, and $G[S_2]$ is $K_n$-free then $R = \{S_1, S_2\}$ is a reducible partition, where $S_2$ is the complement of $S_1$. It easy to see that if the dominationg forest $S_1$ happen to be a tree, then $R$ is a RP.

**Lemma 1.** $G$ is a given graph and $S \subseteq V_G$ is a subset of $V_G$. If the induced subgraph $G[S]$ is a maximal dominating tree, then there are at least 2 neighbors in $S$ for every vertex $v \in V_G - S$.

**Proof.** We prove the lemma by contradiction. As $S$ is a dominating tree, then $v$ has at least one neighbor in $S$. Assume that $v \in V_G - S$ only has one neighbor in one tree of $G[S]$, then $v$ can be added into $S$ and $S \cup v$ is also a dominating tree. Which contradicts that $S$ is maximal. \[ \square \]

**Theorem 2.** For a vertex partition $\{S_1, V_G - S_1\}$ of graph $G$, if $G[S_1]$ is a maximal tree and $V_G - S_1$ is dominated by $S_1$, then $\{S_1, V_G - S_1\}$ is a reducible partition of $G$.

**Proof.** The theorem can be proved by contradiction. Assume that a $K_{n+1}$-free graph $G$ contains $K_n$ minors and $\{S_1, S_2\}$ is not a reducible partition of $G$, then $G[S_2]$ has at least one $K_n$ minor, say $H$. From lemma \[1\] every vertex of $H$ has two neighbors in $S_1$. As $G[S_1]$ is connected then one can obtain a $K_{n+1}$ minor by contracting all edges of $G[S_1]$, which contradicts that $G$ is $K_{n+1}$-free. \[ \square \]

If the dominating forest $G[S_1]$ contains multiple disjoint trees, then $G[S_2]$ may contains a $K_n$ minor, one example is shown in Fig.1. But we will show that a reducible partition always exists by introduce the concept of minimal $K_{n+1}$-free minor.

**Definition 2.** If $H$ is a $K_{n+1}$ minor of graph $G$, and $H'$ obtained from $H$ by deleting any edge is $K_{n+1}$-free, then $H$ is called a minimal $K_{n+1}$ minor.
Figure 1: A $K_5$-free graph $G$. Where $S_1 = V_{T_1} \cup V_{T_2}$, $T_1$ and $T_2$ are two disjoint trees in the dominating forest $G[S_1]$, the red vertex set $S_2$ constitute a $K_4$. As both $G$ and $G[S_2]$ contain a $K_4$, then $\{S_1, S_2\}$ is not a reducible partition.

From definition 2, any subgraph obtained by deleting any vertex of a minimal $K_{n+1}$ minor is also $K_{n+1}$-free.

**Proposition 1.** If $G$ has no minimal $K_n$ minors, then $G$ has no $K_n$ minors.

**Theorem 3.** If $G$ contains only one minimal $K_{n+1}$ minor $H$, then $G'$ obtained from $G$ by deleting any edge $e$ of $H$ is $K_{n+1}$-free.

**Proof.** The theorem can be proved by contradiction. Assume that the theorem is not true, then $G'$ has at least one $K_{n+1}$ minor $H'$. Then it has at least another one minimal minor $H_1$ and $H_1 \neq H$ as $H_1$ does not contain all edges of $H$, which contradicts that $H$ is the unique minimal $K_{n+1}$ minor.

**Theorem 4.** If $G$ contains $m$ minimal $K_{n+1}$ minors $H_i, i = 1, \cdots, m$ and the intersection $H = \bigcap_{i=1}^{m} V_{H_i}$ of the vertex set of the $m$ $K_{n+1}$ minors is not null, then $G'$ obtained from $G$ by deleting any edge $e$ of $G[H]$ is $K_{n+1}$-free.

**Proof.** As each $H_i, i = 1, \cdots, m$ is a minimal $K_{n+1}$ minor, then any $K_{n+1}$ minor $H'$ contains at least one of $H_i$ as subgraph. Assume the theorem is not true, then $G'$ has one $K_{n+1}$ minor $H'$ that does not contain any $H_i, i = 1, \cdots, m$, which is a contradiction.

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Theorem 5. Let \( A_{ij} = V_{H_i} \cap V_{H_j}, i, j = 1, \ldots, m \) are intersections of every pair of minimal \( K_{n+1} \) minors \( H_i \) and \( H_j \) of \( G \), if \( A_{ij} = \emptyset \) is null then let \( A_{ij} = V_{H_i} \) and \( A_{ji} = V_{H_j} \). \( G' \) obtained from \( G \) by deleting \(|A|\) edge \( e_k \in G[A_k], k = 1, \ldots, |A| \) is \( K_{n+1} \)-free.

Proof. As \( A_{ij}, i, j = 1, \ldots, m \) is the intersection of every pair of \( K_{n+1} \) minors \( H_j \) and \( H_k \) of \( G \), then when \(|A|\) edges \( e_t \in E_{A_t}, t = 1, \ldots, |A| \) are deleted, there is no minimal \( K_{n+1} \) minors, \( G' \) is \( K_{n+1} \)-free by using proposition 1.

Definition 3. For a \( K_{n+1} \)-free graph \( G \) contains \( m_1 \) minimal \( K_n \) minors \( H_i, i = 1, \ldots, m_1 \). A vertex set \( F = \{v_i \in \bigcup_{j=1}^{m_1} V_{H_j}, i = 1, \ldots, m_2, m_2 \leq m_1\} \) is defined as a critical \( K_n \) set if \( G[F] \) is a forest.

Lemma 2. For any \( K_{n+1} \)-free graph \( G \) contains 3 minimal \( K_n \) minors \( H_i, i = 1, 2, 3 \), critical \( K_n \) set always exist.

Proof. We will prove the lemma by contradiction. Assume that there is no critical \( K_n \) sets in \( G \), then for any vertex set \( \{v_i \in V_{H_i}, i = 1, 2, 3\} \), \( G \left[ \bigcup_{i=1}^{3} v_i \right] \) is a cycle \( C_3 \) with 3 vertices which is illustrateed in Fig. 2. If this is ture, then all vertices of \( H_k \) are adjacency to \( H_j \) for \( j \neq k \). That leads to multiple \( K_{n+2} \) minors, which is a contradiction as \( G \) is \( K_{n+1} \)-free.

![Figure 2: A graph with 3 K_n minors. Each black closed curve is used to denote a K_n minor.](image)

If there exists no critical \( K_n \) set, then \( w \) and \( v \) are the common neighbors all vertices of another \( K_n \) minor, this leads to a \( K_{n+2} \) minor.
**Corollary 1.** For any $K_{n+1}$-free graph $G$ contains 4 minimal $K_n$ minors $H_i$, $i = 1, \cdots, 4$, critical $K_n$ set always exist.

**Proof.** We will prove the theorem by contradiction. Based on lemma 2, one can pick a vertex set $F = \{v_i \in V_{H_i}, i = 1, 2, 3\}$ which satisfies that $G[F]$ is a forest. If $F$ is an independent set, then corollary 1 is clear. Suppose $F$ is not an independent set, than there are two cases

1. $G[F]$ contains a $K_2$ and one isolated vertex,
2. $G[F]$ is a tree with three vertices.

Assume that there is no critical $K_n$ set in $G$. For case (1), the $K_2$ of $G[F]$ must be the neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains the subgraphs $H_4 + K_2$ which is a $K_{n+2}$ minor, so it is a contradiction because $G$ is $K_{n+1}$-free. For case (2), there are two vertices of $G[F]$ must be neighbors of each vertex $v_4 \in H_4$. Then the $G$ contains a $K_{n+1}$ minor $G[F \cup V_{H_4}]$, which is a contradiction because $G$ is $K_{n+1}$-free. Hence, the corollary is always true. \(\square\)

![Schematic diagram of a graph with 4 minimal K_n minors. v_1, v_2 and v_3 are in three different minimal K_n minors, u_i and u_j are vertices of the fourth minimal K_n minor. If there exist no critical K_n sets, then there exist two of the three vertices (v_1, v_2 and v_3) which are neighbors of every vertex of the fourth minimal K_n minors. This leads to a K_{n+1} minor, which is a contradiction.](image)

**Lemma 3.** For any $K_{n+1}$-free graph $G$ contains $m$ minimal $K_n$ minors $H_i$, $i = \cdots$
1, · · · , m, if for each vertex \( v \in V_{H_i} \) has at least one neighbor in \( V_G - V_{H_i} \), then \( V_G - V_{H_i} \) is disconnected.

**Proof.** Assume the lemma is not true, then \( V_G - V_{H_i} \) is connected, this implies that one can contracts all edges in \( G[V_G - V_{H_i}] \) to obtain a \( K_{n+1} \) minor which is a contradiction. \( \square \)

**Lemma 4.** For any \( K_{n+1} \)-free connected graph \( G \) contains \( m \) minimal \( K_n \) minors \( H_i, i = 1, \ldots, m \). If \( G[V_G - V_{H_i}] \) is connected, then there always exist at least one vertex \( v \in V_{H_i} \) which has no neighbors in \( V_G - V_{H_i} \).

**Proof.** Assume the lemma is not true, namely every vertex \( v \in V_{H_i} \) has neighbors in \( V_G - V_{H_i} \). As \( G[V_G - V_{H_i}] \) is connected, one can contracts all edges in \( G[V_G - V_{H_i}] \) to obtain a \( K_{n+1} \) minor which is a contradiction. Hence there exists at least one vertex \( v \in V_{H_i} \) has no neighbors in \( V_G - V_{H_i} \). \( \square \)

**Corollary 2.** For any \( K_{n+1} \)-free graph \( G \) contains 5 minimal \( K_n \) minors \( H_i, i = 1, \ldots, 5 \), critical \( K_n \) forests always exist.

**Proof.** From Corollary 1, one can pick a vertex set \( F \) from any 4 of five minimal \( K_n \) minors satisfies that \( G[F] \) is a forest, without loss of generality, let \( F = \{v_i \in V_{H_i}, i = 1, 2, 3, 4 \} \). If \( F \) is an independent set, then Corollary 2 is clear. Suppose \( F \) is not an independent set, then \( G[F] \) may have 4 kinds of configurations.

1. \( G[F] \) contains one \( K_2 \) and two isolated vertices.
2. \( G[F] \) contains a tree \( T_3 \) with 3 vertices and an isolated vertex.
3. \( G[F] \) is a tree with 4 vertices.
4. \( G[F] \) contains two disjoint \( P_2 \).

The corollary is clearly for configurations (1), (2) and (3). Now we will show that the Corollary 2 is true for the configuration (4). For configuration (4), the two disjoint paths \( P_2 \) are denoted as \( T_1 \) and \( T_2 \) respectively and without loss of generality assume that the vertices of \( T_1 \) come from \( H_1 \) and \( H_2 \), and the vertices of \( T_2 \) come from \( H_3 \) and \( H_4 \). If the \( G[V_G - V_{H_5}] \) is connected, then there exists one vertex \( v_5 \in V_{H_5} \) that has no neighbor in \( G[V_G - V_{H_5}] \), hence the corollary is clearly true by using lemma 3. If \( G[V_G - V_{H_5}] \) is not connected,
then $H_1$ and $H_2$ will be in one component of $G[V_G - V_{H_5}]$, $H_3$ and $H_4$ will be in another component of $G[V_G - V_{H_5}]$. Assume the corollary is not true, then each vertex $u_j$ of $V_1 \subseteq V_{H_5}$ is adjacent to all vertices of $T_1$ and each vertex $w_j$ of $V_{H_5} - V_1$ is adjacent to all vertices of $T_2$. By using Lemma 4, one can re-peak two non-adjacent vertices as a set $Q = \{ v_1 \in V_{H_1}, v_2 \in V_{H_2}, u_j \in V_1 \}$ such that $G[Q]$ is a tree. Then the subgraph induced by the set $\{ v_1, v_2, u_j \} \cup V_{T_2}$ is a forest, as each vertex of $V_{T_2}$ are not adjacent with every vertex of $V_1$, which is a contradiction. Hence, a critical $K_n$ set always exists for a graph has 5 minimal $K_n$ minors.

**Theorem 6.** For any $K_{n+1}$-free graph $G$ contains $m$ minimal $K_n$ minors $H_i, i = 1, \cdots, m$, there exists a critical $K_n$ set $F = \{ v_j \in \bigcup_{i=1}^{m} V_{H_i}, j = 1, \cdots, k, k \leq m \}$ such that $G[F]$ is an independent set, and $k = m$ when the intersection of any pair $\{ H_i, H_j \}$ is null.

**Proof.** The situation that the intersection of every pair $\{ V_{H_i}, V_{H_j} \}$ is null will be consider firstly. For this case, we will produce an independent set by picking vertices from each $V_{H_i}$ in the following procedure. In the first round of the procedure, checking whether the vertices of $S_{H_i} = \{ V_{H_1}, \ldots, V_{H_{i-1}}, V_{H_{i+1}}, \ldots, V_{H_m} \}$, $i = 1, 2, \ldots, m$ are in one connected component $C_i$ of $G[V_G - V_{H_i}]$. If the answer is yes, then by using Lemma 4 there exists a vertex $v_i$ of $V_{H_i}$ has no neighbors in $S_{H_i}$. Hence, we will add $v_i$ into $F$ and update $G$ by deleting $v_i$ from $G$. If for each $i$, the minimal minors in $S_{H_i}$ are in one component of $G - v_i$, namely there exists one vertex $v_i$ in each $H_i$ can be added into $F$ and delete from $G$, then the procedure will halt and theorem is confirmed by using Lemma 4, where $v_i \in V_{H_i}$ has no neighbors in $V_{H_j}$ such that $j \neq i$ for each $j$. If the answer is no, i.e., the $K_n$ minors of $S_{H_i}$ are not in one component of $G[V_G - V_{H_i}]$, then keep $V_{H_i}$. After the first round of procedure, one will obtain an updated graph $G_1$, recursively execute the above procedure for $G_1$ to obtain a series $\{ G_1, G_2, \ldots, G_k \}$. The procedure will halt for some $k$ if all remained minimal minors are in one component of $G[G_k - V_{H_l}]$. Then by using Lemma 4 there exist a vertex $v_l \in V_{H_l}$ that only has neighbors in $H_l$ for every $l$. That is to
say, there exist a vertex set $F = \{v_i \in V_{H_i}, i = 1, \cdots, m\}$ which is an independent set.

For the configuration that the intersections of multiple pairs of minimal $K_n$ minors are not null sets, one can also find an independent set $F$ by using a similar procedure. If $V_{H_i} \cap V_{H_j}$ is a null set for any $j \neq i$, then we will take the same actions as in the previous paragraph for $H_i$. Now consider $V_{H_i} \cap V_{H_j}$ is not a null set for some $j \neq i$. Let $P_j = V_{H_j} - V_{H_i} \cap V_{H_j}$ and $S_{H_i} = \bigcup_{j=1, j \neq i}^m P_j$. If all vertices of $S_{H_i}$ are in one connected component of $G[V_G - V_{H_i}]$, then there exists one vertex $v_i \in V_{H_i}$ has no neighbors in $S_{H_i}$. We will add $v_i$ into $F$ and update $G$ by deleting $v_i$. If the vertices of $S_{H_i}$ are in one connected component of $G[V_G - V_{H_i}]$, then keep $H_i$. After a vertex $v_i$ is added into $F$ and is deleted from $G$, one obtained an induced subgraph $G_1$, executes this procedure recursively to obtained a series of graphs $\{G_1, G_2, \cdots, G_k\}$. The procedure will halt when $G_k$ contains no minimal $K_n$ minors. It easy to see that $F$ is an independent set.

After $v_i$ is deleted from $G$, each minimal $K_n$ minor $H_j$ which contains $v_i$ also vanishes, hence the size of $F$ is less than or equals to $m$.

When the critical $K_n$ set $F$ is obtained, one can check each vertex $a_j$ in $V_G - F$, if $G[F \cup a_j]$ contains no cycle then update $F$ by adding $a_j$ to into $F$. By using theorem[1], after every $j$ is checked, one can obtain a maximal dominating forest $F$.

An instance of finding critical $K_n$ set is showed in Fig.4.
Figure 4: An instance graph $G$. There are four $K_4$ minors denoted as $H_i, i = 1, \cdots, 4$, where $H_i = G[A_i], i = 1, \cdots, 4$ and $A_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, A_2 = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, A_3 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8\}, A_4 = \{v_1, v_2, v_3, v_5, v_6, v_7, v_8\}$. It provides $S_{H_1} = v_8, S_{H_2} = v_1, S_{H_3} = v_7, S_{H_4} = v_4$. $S_{H_i}$ contains only single vertex, so $G[S_{H_i}]$ can be regard as a special connected graph. If $H_1$ be checked firstly, then one of $v_2, v_3$ and $v_7$ can be picked as all of them are non-adjacent to $v_8$. Hence, one can obtain $\{v_7, v_8\}, \{v_2, v_8\}$ or $\{v_3, v_8\}$ as a critical $K_4$ set. It distinctly to find that $G - v_7 - v_8$ is a $K_4$ free graph.

An schematic diagram of finding a critical $K_n$ set from a graph is showed in Fig.5 where the intersection between every pair of $K_4$ minors is a null set.
Figure 5: An instance of the procedure for finding a critical $K_n$ set, where the intersection of every pair of minimal $K_n$ minors is a null set. The block surrounded by a curve is used to denote a minimal $K_n$ minor, the line between two $K_n$ minors is used to denote that these two minors are connected. The figure 5(a) shows the original graphs. One can find that only the $K_n$ minors labelled by 1, 2 or 3 cannot be deleted, as if they are deleted then the remaining $K_n$ minors are not in one connected component of the resulting graph. Hence, one can find one vertex $u_j, j \neq 1, 2, 3$ which only has neighbors in the minimal $K_n$ minor labelled by $j$.

After each $u_j$ is deleted, the updated graph is showed in 5(b), where only $K_n$ minors are listed and each minimal $K_n$ minor cannot be deleted. This implies that there exists a vertex $v_j, j = 1, 2, 3$ which only has neighbors in the minimal $K_n$ minors labelled by $j$.

**Theorem 7.** For any connected graph $G$, there exists at least one reducible partition $\{S_1, S_2\}$.

**Proof.** This theorem is clear by using the theorem 6, one can let $S_1 = F$ and $S_2 = V_G - S_1$ to obtain the reducible partition $\{S_1, S_2\}$, where $F$ is a maximal dominating forest.
And then let $S_2 = V_G - S_1$, one can obtain a reducible partition $\{S_1, S_2\}$ of $G$.  

Now suppose $\{S_1, S_2\}$ is a reducible partition of $G$, if the induced subgraph $G[S_2]$ is a forest then $\{S_1, S_2\}$ is called an exhaustive reducible partition (ERP) of $G$. If $G[S_2]$ is not a forest, one can recursively perform the procedure showed in the proof of theorem 6 to obtain the reducible partition of $G[S_2]$.

**Definition 4.** For a $K_{n+1}$-free graph $G$ contains $K_n$ minors, an exhaustive reducible partition (ERP) $\{S_1, S_2, \ldots, S_m\}$ is defined as a collection of subsets of $V_G$ such that:

1. $\bigcup_{i=1}^{m} S_i = V_G$, if $i \neq j$, $S_i \cap S_j = \emptyset$,
2. $S_j$ is dominated by $S_i$ if $i \leq j$,
3. the induced subgraph $G[S_i]$ is a forest,
4. the induced subgraph $G\left[\bigcup_{j=k}^{m} S_j\right]$ is $K_{n-k+2}$-free.

And each $S_j$ is called as a partition of ERP of $G$, $m$ is defined as the depth of ERP.

**Theorem 8.** For a $K_{n+1}$-free graph $G$ contains $K_n$ minors, the depth of ERP of $G$ is at most $n - 1$.

**Proof.** As $G\left[\bigcup_{j=k}^{m} S_j\right]$ is $K_{n-k+2}$-free, then take $k = m$ and $n - k + 2 = 3$ to obtain

$$m = n - 1$$

at most.  

In the definition of RP and ERP, it requires that the first partition of ERP is a dominating forest. As a matter of fact, the restrictions of the first partition $S_1$ of RP can be more tighter. If $S_1$ is required as an independent set, then the $\{S_1, S_2\}$ still be reducible. This means that if $G$ has $K_n$ minors but has no $K_{n+1}$ minors, then $G[S_2]$ will be $K_n$ free. Hence we have the definition of the special reducible partition of $G$. 

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**Definition 5.** For a $K_{n+1}$-free graph $G$ with $K_n$ minors and two subset $S_1$ and $S_2$ of $V_G$, $R = \{S_1, S_2\}$ is called a special reducible partition (SRP) of $V_G$, if the following conditions are satisfied:

1. $S_1 \cup S_2 = V_G, S_1 \cap S_2 = \emptyset$
2. $S_1$ is an independent set,
3. the induced graph $G[S_2]$ is $K_n$-free.

$S_1$ and $S_2$ are called the partitions of the SRP of $G$.

**Theorem 9.** For any $K_{n+1}$-free graph $G$ contains $K_n$ minors, There always exist a SRP $\{S_1, S_2\}$.

*Proof.* See the proof of Theorem 6.

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### 2. Coloring of planar graphs

In the previous section, we have introduced the conception of RP and ERP of a graph. In this section, we will apply RP to the problem of colouring of planar graphs. A coloring of a graph is an assignment of colors to its points so that no two adjacent points have the same color, the chromatic number $\chi(G)$ is defined as the minimum $n$ for which $G$ has an $n$-coloring. A graph $G$ is $n$-colorable if $\chi(G) \leq n$ and is $n$-chromatic if $\chi(G) = n$. Four-color theorem states that if $G$ is a planar graph, then $\chi(G) \leq 4$. As a planar graph contains no $K_5$ minors, we will start from the RP or ERP of a planar graph $G$. The depth of ERG of any $K_{n+1}$-free graph is at most $n - 1$, then the depth of ERP is at most 3 for any planar graph. Hence for a planar graph $G$ with cycles, there exists a ERP $\{S_1, S_2, S_3\}$ or $\{S_1, S_2\}$ such that

1. $\bigcup_{i=1}^{m} S_i = V_G, m < 4, S_i \cap S_j = \emptyset$ if $i \neq j$
2. each $G[S_i], i = 1, ..., m.$ is a forest;
3. $S_j$ is dominated by $S_i$, if $i < j$;
4. When $S_3 \neq \emptyset$, $G[S_2 \cup S_3]$ is $K_4$-free and $G[S_3]$ is $K_3$-free. If $S_3 = \emptyset$, $G[S_2]$ is $K_3$-free.
When the depth of ERP of a planar graph is 2, the chromatic number \( \chi(G) \leq 4 \) is clear as \( G[S_1 \cup S_2] \) is a subgraph of the sum of two complete bipartite graphs. Therefore, only the planar graphs with depth 3 needs to be considered. By applying the RP to planar graph, the proof of four-color theorem, which is proved by Appel and Haken with computer\(^4\), without computer can be given.

**Theorem 10.** Any planar graph \( G \) is 4-colorable.

**Proof.** From Theorem\(^7\) and \(^8\) we know that the depth of ERP of any planar graph is 3. Assume the ERP of \( G \) is \( \{S_1, S_2, S_3\} \). It is clearly that \( G[S_2 \cup S_3] \) is 3-colorable as \( G[S_2 \cup S_3] \) has no \( K_4 \) minors. If every vertex of \( S_1 \) can be labelled within 4 colors, then the theorem is proved.

Let \( N(v; S_2 \cup S_3) \) denote the neighbors of \( v \) in set \( S_2 \cup S_3 \) and \( N(v_1, v_2; S_2 \cup S_3) \) denote the common neighbors of \( v_1 \) and \( v_2 \) in \( S_2 \cup S_3 \).

A single vertex \( v \in S_1 \) can be labelled by one of four colors, as \( G[V_G - S_1] \) contains no \( K_4 \) minors which can be labelled by using within 3 colors\(^5\).

As \( G[S_1] \) is a forest, then the non-adjacent vertices can be labelled by a same color. So, only the adjacent vertices of \( S_1 \) should be considered. Now we claim that a proper map \( \theta \) always exist:

\[
\theta : V_G - S_1 \rightarrow i, i \leq 3.
\]

such that the subset \( N(v_i, v_{i+1}; V_G - S_1) \subset V_G - S_1 \) can be labelled by 2 colors for any two adjacent vertices \( v_i \) and \( v_{i+1} \). We will prove this claim by the method of contradiction.

Assume that the claim is not true, then there exists two adjacent vertices \( v_i \) and \( v_{i+1} \) such that vertices in \( N(v_i, v_{i+1}; S_2 \cup S_3) \) must be labelled by 3 colors. However, if \( N(v_i, v_{i+1}; S_2 \cup S_3) \) must be labelled by 3 colors, then \( G[N(v_i, v_{i+1}; S_2 \cup S_3)] \) contains \( K_3 \) minors \( H_1 \) or there exists a path \( W = \{u_1, w_1, \cdots, w_m, u_2\} \) connecting \( u_1 \) and \( u_2 \), where \( u_1, u_2 \in N(v_i, v_{i+1}; S_2 \cup S_3) \), \( d_{12} \geq 2 \) is the distance of \( u_1 \) and \( u_2 \) in \( G[N(v_i, v_{i+1}; S_2 \cup S_3)] \) and \( w_j \notin N(v_i, v_{i+1}; S_2 \cup S_3), j = 1, \cdots, m, \) see Fig\(^6\) and \(^7\). Apparently, one can contracts edges of \( W \) to obtain a \( K_3 \) minor \( H_2 \) where every vertex \( u \in V_{H_2} \) also belongings
to \(N(v_i, v_{i+1}; S_2 \cup S_3)\). That is to say if \(N(v_i, v_{i+1}; S_2 \cup S_3)\) must be labelled by 3 colors, then all vertices of a \(K_3\) minor (\(H_1\) or \(H_2\)) are the common neighbors of \(v_i\) and \(v_{i+1}\). This implies that \(G\) has a \(K_5\) minor which is a contradiction. This provides that for any adjacent vertices \(v_i\) and \(v_{i+1}\), \(N(v_i, v_{i+1}; S_2 \cup S_3)\) has no \(K_3\) minors. And if \(N(v_i, v_{i+1}; S_2 \cup S_3)\) has a \(K_2\) minors, then there is no a path \(W = \{u_1, w_1, \ldots, w_m, u_2\}\) connecting \(u_1, u_2 \in N(v_i, v_{i+1}; S_2 \cup S_3)\), where \(d_{12} \geq 3\). So, there exists a color map such that \(N(v_i, v_{i+1}; S_2 \cup S_3)\) is 2 colorable for any two adjacent vertices \(v_i \in S_1\) and \(v_{i+1} \in S_1\). And then any vertex of \(S_1\) can be labelled by at most one extra color than \(G[V_G - S_1]\), i.e., any planar graph is 4-colorable.

Figure 6: A planar graph \(G\). The red vertices 1 and 2 are in \(S_1\) and the blue vertices 3 to 7 are in \(S_2\). These blue vertices are the common neighbors of the two red vertices. If there exists a path, say the \(\{3, 8, 9, 7\}\), which connecting two non-adjacent blue vertices. Then a \(K_5\) minor can be obtained by contracting edges of that path, which contradicts that \(G\) has no \(K_5\) minor.
Figure 7: A planar graph $G$. The red vertices belong to $S_1$, and the blue vertices belong to $S_2$. This figure is used to illustrate that if $1 \cup 2 \cup S_2$ has to be labelled by 4 colors, then $1, 2, 3; V_G - S_1$ contains no $K_2$ minors which contains 3 or more vertices. If this is not true, say there is one more edge connecting 3 and 6, then one can contracts this edge to obtain a $K_5$ minor.

3. Special Reducible Partition of Graph and It’s Application to Color of Graph Without $K_{n+1}$ Minors

In the previous section, we have introduced the conception of RP, ERP and SRP of a graph. In this section, we will apply SRP to the problem of colouring of planar graphs. A coloring of a graph is an assignment of colors to its points so that no two adjacent points have the same color, the chromatic number $\chi(G)$ is defined as the minimum $n$ for which $G$ has an $n$-coloring. A graph $G$ is $n$-colorable if $\chi(G) \leq n$ and is $n$-chromatic if $\chi(G) = n$ \cite{2}. Four-color theorem states that if $G$ is a planar graph, then $\chi(G) \leq 4$ \cite{4}. Scholars believe that four color theorem is a special case of Hadwiger conjecture which states that if $G$ is loopless and has no $K_{n+1}$ minor then its chromatic number satisfies $\chi(G) \leq n$. Hadwiger conjecture is known to be true for $1 \leq t \leq 5$\cite{5,6}.

**Conjecture 1.** \cite{5} *Every connected graph $G$ without $K_{n+1}$ minors is at most $n$-colorable.*

Before studying the color of a graph without $K_{n+1}$, we will firstly discuss the ERP and SRP of a graph without $K_{n+1}$ minor. Theorem 8 tells us that the vertex set $V_G$ of a graph $G$ without $K_{n+1}$ minor can always be divided
into at most \( n - 1 \) subsets, and the subgraph induced by vertices in each of the \( n - 1 \) subsets is a collection of disjoint trees, see Fig.8. Further more, any

\[
\begin{align*}
S_1 & \quad \cdots \quad \cdots \\
S_2 & \quad \cdots \quad \cdots \\
S_{n-2} & \quad \cdots \quad \cdots \\
S_{n-1} & \quad \cdots \quad \cdots
\end{align*}
\]

Figure 8: Schematic diagram of ERP of a graph \( G \) has no \( K_{n+1} \) minor. The subgraph \( G[S_j] \) induced by \( S_k \) only contains disadjoint trees and \( G[S_j] \) is dominated by \( G[S_k] \) for \( j > k \)

graph \( G \) has a SRP \( \{S_1, S_2\} \). The most significant feature of SRP is that \( S_1 \) is an independent set. As \( S_1 \) is a independent set of \( G \) and \( G[S_2] \) is \( K_n \)-free, the method of induction for learning Hadwiger conjecture is feasible.

**Theorem 11.** *Any graph \( G \) has no \( K_{n+1} \) minor is \( n \)-colorable.*

**Proof.** We will prove the theorem by induction. Firstly, this theorem is ture for \( n \leq 4 \) [4, 5, 6]. Secondly, assume the theorem is ture for all \( n \leq t \). Thirdly, we will check whether the theorem is ture for \( n = t + 1 \), i.e., whether \( G \) is \((t+1)\)-colorable if \( G \) has no \( K_{t+2} \). For \( n = t \), \( G \) has no \( K_{t+1} \) minors and if \( G \) is \( t \) colorable it must contains \( K_t \) minors based the assumption. For \( n = t + 1 \), namely \( G \) has no \( K_{t+2} \) minors. Based on Theorem[9] \( G \) has a SRP which is denoted as \( \{S_1, S_2\} \) where \( S_1 \) is an independent set and \( G[S_2] \) has no \( K_{t+1} \) minors. For any vertex \( v \in S_1 \), let \( N(v) \) denote the neighbors of \( v \) and \( N(v; S_2) = N(v) \cap S_2 \) denote the neighbors of \( v \) in \( S_2 \). As \( S_1 \) is an independent set, then \( N(v) = N(v; S_2) \). So, the color of \( v \) completely depend on the coloring of \( N(v; S_2) \). From the assumption, \( G[S_2] \) is \( n \) colorable and has to be labelled by \( n \) colors if and only if it contains a \( K_t \) minor which has to be labelled by \( t \) colors. Clearly, \( N(v; S_2) \) is
t colorable for any \( v \). Therefore, at most \( t+1 \) colors will be used for labelling \( S_1 \).

This implies that if Hadwiger conjecture is true for \( n = t \), then it is true for \( n = t + 1 \). Hence, Hadwiger conjecture is true for any positive integer \( n \). \( \square \)

4. Conclusions and Discussions

In this theme, we propose the concepts of RP of graphs. We showed that any graph \( G \) possess a RP, an ERP and a SRP. And the depth of ERP of \( G \) is at most \( n − 1 \) if \( G \) does not contain \( K_{n+1} \) as minor. By applying the SRP to any graph one can obtain that Hadwiger conjecture is true. The method of induction is used. For the first procedure, we assume that the Hadwiger conjecture is true for all \( n \leq t \), and the we prove that the conclusion is also true for \( n = t + 1 \). The reason why the method of induction is used can be used is that any graph \( G \) with \( K_n \) minors and without \( K_{n+1} \) minors has a SRP \( R = \{ S_1, S_2 \} \) such that \( S_1 \) is an independent set and \( G[S_2] \) is \( K_n \)-free. The procedure showed that if the vertices of \( S_1 \) must be labelled by using \( t + 2 \) colors than \( G \) must have a \( K_{n+2} \) minor which is a contradiction.

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