Full Counting Statistics in Quantum Contacts

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Abstract. Full counting statistics is a fundamentally new concept in quantum transport. After a review of basic statistics theory, we introduce the powerful Green’s function approach to full counting statistics. To illustrate the concept we consider a number of examples. For generic two-terminal contacts we show how counting statistics elucidates the common (and different) features of transport between normal and superconducting contacts. Finally, we demonstrate how correlations in multi-terminal structures are naturally included in the formalism.

1 Introduction

The probabilistic interpretation is a fundamental ingredient of quantum mechanics. While the wave function determines the full quantum state a system and its evolution in time, observable quantities are related to hermitian operators. Expectation values of these operators determine the average value of a large number of identical measurements. However, an individual measurement yields in general a different result. Applying this idea to a current measurement in a quantum conductor, leads directly to the concept of full counting statistics (FCS): during a given time interval a certain number of charges will pass the conductor. To predict the statistical properties of the number of transferred charges we need a probability distribution. The theoretical goal is to find this distribution.

Overview In this article we give an introduction to the field of full counting statistics in mesoscopic electron transport. We will concentrate on the powerful technique – using Keldysh-Green’s functions – which at the same time also is based on microscopic theory. To accomplish this goal we will first review concepts of basic statistics, which are relevant for counting statistics. In the next section we address the microscopic derivation of FCS using Keldysh-Green’s functions. In the rest of the article we demonstrate the use of counting statistics in a number of examples, like two-terminal contacts with normal and superconducting leads, diffusive metals and, finally, multi-terminal structures. But first we review briefly the development of the field.

History Full counting statistics has its roots in quantum optics [1], where the number statistics of photons is used, e. g., to characterize coherence properties of photon sources. The major step to adopt the concept to mesoscopic
electron transport has been undertaken by Levitov and Lesovik [2]. Since then the theory of FCS of charge transport in mesoscopic conductors has advanced substantially, see Refs. [3,4]. In Ref. [2] it was shown that scattering between uncorrelated Fermi leads with probability $T$ is described by a binomial statistics $P(N) = \binom{M}{N} (1-T)^{M-N}$. Here, $P(N)$ is the probability, that out of $M = 2e\theta V/h$ independent attempts $N$ charges are transferred. Furthermore, Levitov and coworkers studied the counting statistics of diffusive conductors [5], time-dependent problems [6] and of a tunnel junction [7]. A theory of full counting statistics based on the powerful Keldysh-Green’s function method was initiated by Nazarov [8]. This formulation allows a straightforward generalization to systems containing superconductors [9,10] and multi-terminal structures [11,12]. Classical approaches to FCS were recently put forward for Coulomb blockade systems [13,14], and, for chaotic cavities based on a stochastic path-integral approach [15]. The field of counting statistics in the quantum regime is closely related to the fundamental measuring problem of quantum mechanics, which has been addressed in a number of works [6,16,17,18,19]. Expressing the FCS of charge transport by the counting statistics of photons emitted from the conductor provides an interesting alternative to classical counting of electrons [20]. Counting statistics has been addressed by now for many different phenomena:

- Andreev contacts [21]
- generic quantum conductors [13,22,23,24]
- adiabatic quantum pumping [25,20,27,28]
- qubit-readout [10,29,30,31]
- superconducting contacts in equilibrium [9]
- proximity effect structures [10,32,33,43,44]
- cross-correlations with normal [35] or superconducting contacts [12,37]
- entangled electron pairs [38,39]
- phonon counting [40]
- relation between photon counting and electron counting [41]
- current biased conductors [42]
- interaction effects: weak and strong Coulomb blockade [14,43,44]
- multiple Andreev reflections in superconducting contacts [45,46]

Very recently, an important experimental step forward was achieved. Reulet, Senzier, and Prober measured for the first time the third cumulant of current fluctuations produced by a tunnel junction [47]. Surprisingly the measured voltage dependence deviated from the expected voltage-independent third cumulant of a simple tunnel contact [22,23]. A subsequent theoretical explanation is that the third cumulant is in fact susceptible to environmental effects [48]. This experiment has already triggered some theoretical activity [24,49,50].

2 Full Counting Statistics

The fundamental quantity of interest in quantum transport is the probability distribution

$$P_{t_0}(N_1, N_2, \ldots, N_M) \equiv P(N),$$

(1)
which denotes for a \textit{M}-terminal conductor the probability that during a certain period of time \( t_0 \), \( N_1 \) charges enter through terminal 1, \( N_2 \) charges enter through terminal 2, \ldots, and \( N_M \) charges enter through terminal \( M \) (negative \( N_i \) correspond to charges leaving the respective terminal). The same information is contained in the cumulant generating function (CGF), defined by

\[
S(\chi) = \ln \left[ \sum_N e^{iN\chi} P(N) \right], \tag{2}
\]

where we introduced the vector of counting fields \( \chi = (\chi_1, \chi_2, \ldots, \chi_N) \). The normalization condition requires \( \sum_N P(N) = 1 \leftrightarrow S(\chi = 0) = 0 \).

\textbf{Charge conservation} We are interested in the long-time limit of the charge counting statistics, which means that no extra charges remain inside the conductor after the counting interval. If we count only the total number of transferred charges, we simply have to consider \( P(N) = \sum_N \delta_{\sum N} P(N) \), or, equivalently, to put all counting fields equal \( S(\chi_1 = \chi, \chi_2 = \chi, \ldots, \chi_N = \chi) \). Charge conservation now means that \( S(\chi_1 = \chi, \chi_2 = \chi, \ldots, \chi_N = \chi) = 0 \). As a consequence the CGF depends only on differences between counting fields. This has the direct interpretation, that a difference \( \chi_\alpha - \chi_\beta \) is related to a charge transfer between terminal \( \alpha \) and \( \beta \). In general, this means that we need only \( M-1 \) counting fields to describe a \( M \)-terminal structure. If one of the counting fields, e. g. \( \chi_M \), has been eliminated, the charge transfer into terminal \( M \) can be restored from the CGF, in which all other \( \chi_\alpha \) are equal \( \chi_\alpha - \chi_M \). In the special case of a two-terminal device, the CGF depends only on \( \chi_1 = \chi_1 - \chi_2 \). We denote this below with \( S(\chi) \). Later we will see that the CGF’s are in general periodic functions of \( \chi \), i. e. \( S(\chi + 2\pi) = S(\chi) \). This ensures that the total charge transferred is an integer multiple of the electron charge \( e \), which makes sense, since we are talking about electron transport and want to neglect transient effects.

However, the interesting question, what the charge of an elementary event is, can be answered by FCS. Suppose the a CGF has the property \( S(\chi + 2\pi/n) = S(\chi) \). Direct calculation shows that

\[
P(Q) = \int \frac{d\chi}{2\pi} e^{-iN\chi+S(\chi)} = \begin{cases} P_n(Q/n), & (Q \text{ mod } n) = 0 \\ 0, & (Q \text{ mod } n) \neq 0 \end{cases}, \tag{3}
\]

where \( P_n(N) \) is the distribution \( S_n(\chi) = S(\chi/n) \). The probability distribution vanishes for all \( N \) which are not multiples of \( n \), thus the elementary charge transfer is in units of \( ne \), where \( e \) is the electron charge. This has interesting consequences in the context of superconductivity, in which multiple charge transfers can occur \cite{21,15,16}, or for fractional charge transfer \cite{23}.

\textbf{Correlations} One commonly addressed question is, if two different events (say the charges transferred into terminals \( \alpha \) and \( \beta \)) are independent or not. For independent events the probability distributions are separable and we find that \( \langle N^\alpha \rangle^\beta_\beta = \langle N^\alpha \rangle \langle N^\beta \rangle \). In terms of the CGF this means that the CGF is the sum
of two terms: one which depends only on $\chi_\alpha$ and a second one, which depends only on $\chi_\beta$. On the contrary, if the CGF can not be written as such a sum, the charge transfers in terminals $\alpha$ and $\beta$ are correlated.

**Special distributions (two terminals)** If the elementary events are uncorrelated, the probability distribution is Poissonian. With the average number of events is $\bar{N}$ we have

$$P_{\text{Poisson}} = \frac{\bar{N}}{N!} e^{-\bar{N}} \leftrightarrow S(\chi) = \bar{N} (e^{i\chi} - 1) .$$

(4)

In the context of electron transport we encounter this distribution mostly for tunnel junctions with an almost negligible transmission probability at low temperatures. Here $\bar{N} = G_T V t_0 / e$ is simply related to the voltage bias and the tunnel conductance.

As second example we consider the binomial (or Bernoulli) distribution. This is obtained if an event occurs with a probability $T$ and the number of tries is fixed to $N_0$:

$$P_{\text{binomial}} = \binom{N_0}{N} T^N (1 - T)^{N_0 - N} \leftrightarrow S(\chi) = N_0 \ln \left[ 1 + T (e^{i\chi} - 1) \right] .$$

(5)

In some sense this is the most fundamental distribution in quantum transport: it gives the statistics of a voltage biased single channel quantum conductor if we identify $N_0 = e V t_0 / h$.

**Special distributions (many terminals)** For uncorrelated processes the CGF takes the simple form

$$S(\chi) = \sum_{\alpha,\beta} \bar{N}_{\alpha,\beta} \left( e^{i(\chi_\alpha - \chi_\beta)} - 1 \right) .$$

(6)

The resulting distribution is just the product of Poisson distributions, taking into account total charge conservation. An important example is a multinomial distribution for $N_0$ independent attempts, which can have different outcomes with probabilities $T_\alpha$. It has the form

$$S(\chi) = N_0 \ln \left[ 1 + \sum_{\alpha} T_\alpha (e^{i\chi_\alpha} - 1) \right] .$$

(7)

3 **Theoretical approach to full counting statistics**

**General theory** We will follow here the approach to FCS using the Green’s function technique [8]. Quantum-mechanically we define the cumulant generating function by

$$e^{S(\chi)} = \langle T_K e^{-i \int_{C_K} dx(t) J(t)} \rangle ,$$

(8)
Here $T_K$ denotes time ordering along the Keldysh-contour $C_K$, depicted in Fig. 1. The time-dependent field $\chi(t)$ is defined as $\pm \chi$ for $t \in C_1(2)$, i.e. $\chi(t)$ changes sign between the upper and the lower branch of $C_K$. $\hat{I}(t)$ is the usual operator of the current through a certain cross section. Expansion in the counting field yields the cumulants. In the second order we find the 2nd cumulant as

$$C_2(t_0) = \int_0^{t_0} dt \int_0^{t_0} dt' \langle \delta \hat{I}(t) \delta \hat{I}(t') \rangle.$$  \hfill (9)

Higher cumulants yield more complicated expressions.

**Current Correlation Functions** The cumulants $C_n(t_0)$ are directly related to experimentally accessible quantities like current noise or the third cumulant of the current fluctuations. Let us demonstrate the relation for the low-frequency current noise, defined by

$$S_I = 2 \Delta f \int_{-\infty}^{\infty} d\tau \langle \delta \hat{I}(\tau) \delta \hat{I}(0) \rangle,$$  \hfill (10)

where $\delta \hat{I}(\tau) = \hat{I}(\tau) - \langle \hat{I} \rangle$ and $\Delta f = f_{\text{max}} - f_{\text{min}}$ is the frequency band width, in which the noise is measured. The factor of 2 enters here to conform to the review article [3]. We now transform in (9) the integration variables from $t, t'$ to $T = (t + t')/2, \tau = t - t'$. In the limit $t_0 \equiv (\Delta f)^{-1}$ much larger than the correlation time of current-fluctuations, the integral over $T$ can be evaluated and we obtain from (9) the desired result $S_I/2$. Similar arguments hold for higher cumulants, for which the expression corresponding to (9) are less trivial, however. In Ref. [47] it was noted that $C_3$ depends in an quite unusual way on the frequency band measured, i.e. it is proportional to $2f_{\text{max}} - f_{\text{min}}$, which made it possible to prove experimentally that the third cumulant is actually measured.

**Keldysh-Green’s Functions** So far we have formally defined the CGF quantum mechanically. The relation to standard quantum-field theory methods is made in the following way. We introduce the standard Green’s function in the presence of a time-dependent Hamiltonian

$$H_c(t) = H_0 + \frac{1}{2e} \chi(t) \hat{I},$$  \hfill (11)

where the time-dependence is only in the ’counting’ field $\chi(t)$. The counting field couples to the operator $\hat{I}$ of the current through a cross section, which intersects...
the conductor entirely. The single-particle operators corresponding to \( H_0 \) and \( I \) are denoted by \( h_0 \) and \( j \).

Using the matrix notation for the Keldysh-Green’s functions, we arrive at the equation of motion

\[
\left[ \frac{i \partial}{\partial t} - \hat{h}_0 - \frac{\chi}{2e} \hat{\tau}_3 \hat{j}_c \right] \hat{G}(t, t'; \chi) = \delta(t - t').
\] (12)

Here \( \hat{\tau}_3 \) denotes the third Pauli matrix in the Keldysh space and is a result of the unusual time-dependence of the counting field. The relation of the Green’s function (12) to the CGF (8) is obtained from a diagrammatic expansion in \( \chi \) (the calculation is formally equivalent to the calculation of the thermodynamic potential in an external field, see e. g. [52]). One obtains the relation

\[
\frac{\partial S(\chi)}{\partial \chi} = \frac{i t_0}{e} \text{Tr} \left[ \hat{\tau}_3 \hat{\tau}_K \hat{G}(t, t; \chi) \right] = \frac{i t_0}{e} I(\chi),
\] (13)

where we have restricted us to a static situation, for which \( \hat{G}(t, t) \) is independent of time. Note, that the counting current \( I(\chi) \) should not be confused with the standard electrical current, which is actually given by \( I_{\text{el}} = I(0) \). Rather, \( I(\chi) \) contains (via an expansion in \( \chi \)) all current-correlators at once. It nevertheless resembles a current in the usual sense. E. g., it follows from Eq. (12) that the counting current is conserved.

**A simplification** In a typical mesoscopic transport problem we can access the full counting statistics based on the separation into terminals (or reservoirs) and a scattering region. Terminals provide boundary conditions to Green’s function far away from the scattering region. These are usually determined by external current or voltage sources and include material properties like superconductivity. Let us now take the following parameterization of the current operator in Eq. (12)

\[
\hat{j}(x) = (\nabla F(x)) \lim_{x \to x'} \frac{ie}{2m} (\nabla_x - \nabla_{x'}) \hat{\sigma}_3.
\] (14)

\( F(x) \) is chosen such that it changes from 0 to 1 across a cross section \( C \), which intersects the terminal, but is of arbitrary shape. Here we have introduced a matrix \( \hat{\sigma}_3 \) in the current operator, occurring e. g. in the context of superconductivity. We assume that the change from 0 to 1 should occur on a length scale \( L \), for which we assume \( \lambda_F \ll A \ll l_{\text{imp}}, \xi_0 \) (Fermi wave length \( \lambda_F \), impurity mean free path \( l_{\text{imp}} \), and coherence length \( \xi_0 = v_F/2\Delta \)). With this assumption we can reduce Eq. (12) inside the terminal to its quasiclassical version (see Ref. [51])

\[
\nu_F \nabla \hat{g}(x, \nu_F, t, t', \chi) = \left[ -i \frac{e}{2} (\nabla F(x)) \nu_F \hat{\tau}_K, \hat{g}(x, \nu_F, t, t', \chi) \right].
\] (15)

Here \( \hat{\tau}_K = \hat{\tau}_3 \hat{\sigma}_3 \) is the matrix of the current operator and \( \hat{g} \) obeys the normalization condition \( \hat{g}^2 = 1 \). Other terms can be neglected due to the assumptions.
we have made for \( \Lambda \). The counting field can then be eliminated by the gauge-like transformation

\[
\tilde{g}(x, v_F, t, t', \chi) = e^{-i \chi F(x) \tilde{\tau}_K/2} \tilde{g}(x, v_F, t, t', 0) e^{i \chi F(x) \tilde{\tau}_K/2}.
\]

(16)

We assume now that the terminal is a diffusive metal of negligible resistance. Then the Green’s functions are constant in space (except in the vicinity of the cross section \( C \)) and isotropic in momentum space. Applying the diffusive approximation \[51\] in the terminal leads to a transformed terminal Green’s function

\[
\tilde{G}(\chi) = e^{-i \chi \tilde{\tau}_K/2} \tilde{G}(0) e^{i \chi \tilde{\tau}_K/2},
\]

(17)

on the right of the cross section \( C \) (where \( F(x) = 1 \)) with respect to the case without counting field. Consequently, the counting field is entirely incorporated into a modified boundary condition imposed by the terminal onto the mesoscopic system.

Summary of Theoretical Approach
This concludes the theoretical approach to counting statistics of mesoscopic transport. Let us briefly summarize the scheme to follow. The FCS can be obtained by a slight extension of the usual Keldysh-Green’s function approach, which is widely employed to treat quantum transport problems. Making use of the separation of the mesoscopic structure into terminals and a scattering region, the formalism boils down to a very powerful, but nevertheless simple rule: we have to apply the counting rotation \[17\] to a terminal, thus providing new boundary conditions (now depending on the counting field \( \chi \)) to the scattering problem. We then proceed ‘as usual’ and calculated the current in the terminal, which again depends on \( \chi \). Finally the counting statistics is obtained from Eq. \[19\].

4 Two-Terminal contacts

Tunnel contact To illustrate the theoretical method we first calculate the counting statistics of a tunnel junction. As usual the system is described by a tunnel Hamiltonian \( H = H_1 + H_2 + H_T \), where \( H_1(2) \) describe the left(right) terminal and \( H_T \) describes the tunneling. The current is calculated in second order in the tunneling amplitudes and we obtain \( I(\chi) = \frac{G_T}{4 \pi} \int dE Tr \{ \tilde{\tau}_K [\tilde{G}_1(\chi), \tilde{G}_2] \} \), where \( G_T \) is the conductance of the tunnel junction and we have included the counting field in \( \tilde{G}_1 \). The CGF is (using \( (\partial/\partial \chi)G_1(\chi) = (i/2) [\tilde{\tau}_K, \tilde{G}_1(\chi)] \))

\[
S(\chi) = i \frac{t_0}{e} \int_0^X d\chi' I(\chi') = \frac{G_T t_0}{4 e^2} \int dE Tr \{ \tilde{G}_1(\chi), \tilde{G}_2 \},
\]

(18)

which is the general expression for the FCS of a tunnel junction. We use the pseudo-unitarity \( \tilde{\tau}_K^2 = \tilde{1} \) to write

\[
S(\chi) = N_{12}(e^{i \chi} - 1) + N_{21}(e^{-i \chi} - 1),
\]

(19)
where \( N_{ij} = \frac{t_0 G_T}{16 e^2} \int dE \text{Tr} \left( (1 + \tau_K) \hat{G}_i (1 - \tau_K) \hat{G}_j \right) \) denotes the average number of charges tunnel from \( i \) to \( j \). The statistics is therefore a bidirectional Poisson distribution\(^{23}\). It is easy to see that the cumulants are \( C_n = N_{12} + (-1)^n N_{21} \). If either \( N_{21} = 0 \) or \( N_{12} = 0 \) we obtain the Schottky limit.

Furthermore, in equilibrium \( N_{12} = N_{21} \) and the FCS is \( (2 G_T k_B T t_0/e^2)(\cos(\chi) - 1) \), which is non-Gaussian, remarkably.

**General CGF for quantum contacts** Using the method presented in the previous section, we can find the counting statistics for all conductors, which are characterized by a set of transmission coefficients \( \{ T_n \} \). Nazarov has shown that the transport properties of such a contact are described by a matrix current\(^{53}\)

\[
\hat{I}_{12} = -\frac{e^2}{\pi} \sum_n \frac{2T_n [\hat{G}_1, \hat{G}_2]}{4 + T_n (\{\hat{G}_1, \hat{G}_2\} - 2)}. \tag{20}
\]

Here \( \hat{G}_{1(2)} \) denote the matrix Green’s functions on the left and the right of the contact. We should emphasize that the matrix form of (20) is crucial to obtain the FCS, since it is valid for any matrix structure of the Green’s functions. The **scalar current** is obtained from the matrix current by

\[
I_{12} = \frac{1}{4e} \int dE \text{Tr} \tau_K \hat{I}_{12}. \tag{21}
\]

To find the FCS, we apply the counting rotation (17) to terminal 1, i.e. \( \hat{G}_1 \) becomes \( \chi \)-dependent. It turns out that the CGF can then be found generally from the relations (13), (20), and (21). To integrate (13) with respect to \( \chi \), we need the relations \( i(\partial/\partial \chi) G_1(\chi) = [\tau_K, G_1(\chi)] \) and \( \text{tr}[G_1(\chi), \{G_1(\chi), G_2\}^n] = 0 \). We find\(^9\)

\[
S(\chi) = \frac{t_0}{2\pi} \sum_n \int dE \text{Tr} \ln \left[ 1 + \frac{T_n}{4} \{\{G_1(\chi), G_2\} - 2]\right]. \tag{22}
\]

This is a very important result. It shows that the counting statistics of a large class of constrictions can be cast in a common form, independent of the contact types. Note, that Eq. (22) is just the sum over CGF’s of all eigenchannels. Thus, we can obtain the CGF’s of all constrictions from a known transmission eigenvalue density. These are known for a number of generic contacts (see e.g.\(^{54}\) and Table 1), can be determined numerically, or can be taken from experiment. Below we will discuss several illustrative examples for a single channel contacts.

**Normal contacts** Consider first a single channel with transmission \( T \) between two normal reservoirs. They are characterized by occupation factors \( f_{1(2)} = \{\exp((E - \mu_{1(2)})/k_B T_e) + 1\}^{-1} \) (\( T_e \) is the temperature). We obtain the result\(^{2,6}\) (see Appendix)

\[
S(\chi) = \frac{2t_0}{h} \int dE \ln \left[ 1 + T_{12}(E) \left(e^{i\chi} - 1\right) + T_{21}(E) \left(e^{-i\chi} - 1\right) \right]. \tag{23}
\]
\[ \rho(T) = \frac{G}{G_Q} \]

\[ \tilde{s}(A) = \ln(1 - T_a(A - 1)/2) \]

\[ \frac{1}{2} T_v \sqrt{1 - T} \frac{1}{4} \text{acosh}^2(A) \]

\[ \frac{1}{T^{1/2} \sqrt{1 - T}} \sqrt{2(1 + A)} \]

\[ \frac{1}{2} \pi \sqrt{T} \sqrt{1 - T} 4 \ln \left(2 + \sqrt{2(1 + A)}\right) \]

| Characteristic function | \[ \rho(T) \] | \[ \tilde{s}(A) \] |
|-------------------------|----------------|----------------|
| Single channel          | \[ \delta(T - T_1) \] | \[ \ln(1 - T_a(A - 1)/2) \] |
| Diffusive connector     | \[ \frac{1}{2} T_v \sqrt{1 - T} \] | \[ \frac{1}{4} \text{acosh}^2(A) \] |
| Dirty interface         | \[ \frac{1}{T^{1/2} \sqrt{1 - T}} \] | \[ \sqrt{2(1 + A)} \] |
| Chaotic cavity          | \[ \frac{1}{2} \pi \sqrt{T} \sqrt{1 - T} 4 \ln \left(2 + \sqrt{2(1 + A)}\right) \] |

**Table 1.** Characteristic functions of some generic conductors. The transmission eigenvalue densities are normalized to \( G/G_Q \), where \( G_Q = \frac{2e^2}{h} \) is the quantum conductance. The third column displays the CGF-density, which determines the CGF via \( S(\chi) = (t_0 G/4e) \int dE \text{tr} \tilde{s}(\{ G_1(\chi), \tilde{G}_2 \})/2 \).

Here we introduced the probabilities \( T_{12} = T_{f_1}(E) (1 - f_2(E)) \) for a tunneling event from 1 to 2 and \( T_{21}(E) \) for the reverse process. We see that the FCS (for each energy) is a trinomial of an electron going from left to right, from right to left, or no scattering at all. The counting factors \( e^{\pm i \chi} - 1 \) thus correspond to single charge transfers from 1 to 2 (2 to 1).

At zero temperature and \( \mu_1 - \mu_2 = eV \geq 0 \) the argument of the energy integral is constant in the interval \( \mu_1 < E < \mu_2 \) and we obtain the binomial form \( S(\chi) = \frac{2t_0 |V|}{h} \ln \left[1 + T \left(e^{i \chi} - 1\right)\right] \). Note that for reverse bias \( \mu_2 > \mu_1 \) the CGF has the same form, but with a counting factor \( e^{-i \chi} - 1 \). The prefactor denotes the number of attempts \( M = e t_0 V/h \) to transfer an electron. If the transmission probability is unity the FCS is non-zero only for \( N = M \), which therefore constitutes the maximal number of electrons occupying an energy strip \( eV \) that can be sent through one (spin-degenerate) channel in a time interval \( t_0 \).

In equilibrium it follows from Eq. (23) that the counting statistics is

\[ S(\chi) = \frac{2t_0 k_B T_{el}}{h} \text{asinh}^2 \left(\sqrt{T} \sin \frac{\chi}{2}\right). \]  

(24)

The fluctuations are non-Gaussian, except for \( T = 1 \), when \( S(\chi) = -\frac{t_0 k_B T_{el}}{h} \chi^2 \).

**SN-contact** The FCS of a contact between a superconductor and a normal metal also follows from the general expression Eq. (22). Using the Green’s functions given in the Appendix we find the result

\[ S(\chi) = \frac{t_0}{2\pi} \sum_n \int dE \ln \left[ 1 + \sum_{q=-2}^{2} A_{nq}(E) \left(e^{i q \chi} - 1\right)\right]. \]

(25)

The coefficients \( A_{nq}(E) \) are related to a charge transfer of \( q \times e \). For example, a term \( \exp(2i \chi) - 1 \) corresponds just to an Andreev reflection process, in which

\footnote{The noninteger values of \( M(t_0) \) occur due to the quasiclassical approximation. A more careful treatment reveals that \( M \) itself is described by a probability distribution. For large \( M \) the difference is negligible.}
two charges are transferred simultaneously. Explicit expressions for the various coefficients are given in Refs. [21,56]. The most interesting regime is that of pure Andreev reflection: \( eV, k_B T \ll \Delta \). Here we obtain

\[
S(\chi) = \frac{t_0}{\hbar} \int dE \ln \left[ 1 + R_A f_+ f_- (e^{i2\chi} - 1) + R_A (1 - f_+)(1 - f_-) (e^{-i2\chi} - 1) \right],
\]

(26)

where \( R_A = T^2/(2 - T)^2 \) is just the Andreev reflection probability and \( f_{\pm} = f(\pm E) \) denotes the occupation with electrons above(below) the chemical potential of the superconductor. For low temperatures \( k_B T_e \ll eV \ll \Delta \), the CGF becomes

\[
S(\chi) = \frac{2e t_0 |V|}{\hbar} \ln \left[ 1 + R_A (e^{i2\chi} - 1) \right].
\]

(27)

The CGF is now \( \pi \)-periodic, which according to Sec. 2 reflects that the charge transfer of an elementary event is now 2\( e \), a consequence of Andreev reflection. Quite remarkably, the statistics is again a simple binomial distribution. In equilibrium, we can adapt the result from Eq. 24 to find

\[
S(\chi) = -\frac{2e t_0 k_B T_e}{\hbar} \sin^2 \left( \sqrt{R_A} \sin \chi \right) \quad \text{for} \ \chi \in [-\pi/2, \pi/2].
\]

(28)

The counting statistics is also non-Gaussian, except for \( R_A = 1 \).

**Superconducting Contact** Now we turn to a slightly more involved problem: a contact between two superconductors biased at a finite voltage \( V \). For \( eV < 2\Delta \) the transport is dominated by multiple Andreev reflections (MAR). The microscopic analysis of the average current and the shot noise calculations suggest that the current at subgap energies proceeds in “giant” shots, with an effective charge \( q \sim e(1 + 2\Delta/|eV|) \). However, the question of size of the charge transferred in an elementary event can only be rigorously resolved by the FCS. The answer was given by Cuevas and the author [45] based on a microscopic Green’s function approach. Independently, Johansson, Samuelsson and Ingerman [46] arrived at the same conclusion using a different method.

Now, what would we like to have? In Sec. 2 we have discussed that one can speak of multiple charge transfers if the CGF allows an interpretation in terms of elementary events, which are described by counting factors \( e^{in\chi} - 1 \), where \( n \) denotes the charge transferred in the process. How can we ever hope to obtain this from the general formula (22)? We have to calculate the determinant of a \( 4 \times 4 \)-matrix, which can give only factors of the type \( e^{i2\chi} \) or even smaller charges. The answer to this puzzle is that we have to reinterpret the matrix structure in (22), since the Green’s functions of superconductors at a finite bias voltage are essentially nonlocal in energy. The general result for the CGF can be written as

\[
S(\chi) = \text{Tr} \ln \hat{Q}, \quad \text{where} \quad \text{Tr} = \int_0^{\infty} dt \text{tr} \quad \text{and} \quad \hat{Q}(t) = 1 + (T/4)(\{\hat{G}_1 \otimes \hat{G}_2\} - 2)(t,t).
\]

Here \( \hat{G}_1 \otimes \hat{G}_2(t,t') = \int dt'' \hat{G}_1(t, t'')\hat{G}_2(t'', t') \). Let us set the chemical potential of the right electrode to zero and represent the Green’s functions by \( \hat{G}_1(t,t') = e^{ieVt\tau_3} \hat{G}_S(t-t') e^{-ieVt'\tau_3} \) and \( \hat{G}_2(t,t') = \hat{G}_S(t-t') \). Here, we have
Full Counting Statistics

not included the dc part of the phase, since it can be shown that it drops from the expression of the dc FCS at finite bias. The Fourier transform leads to a representation of the form \( \tilde{G}(E, E') = \sum_n \tilde{G}_{0,n}(E) \delta(E - E' + n eV) \), where \( n = 0, \pm 2 \). Restricting the fundamental energy interval to \( E - E' \in [0, eV] \) we can represent the convolution as matrix product, i.e. \( (G_1 \otimes G_2)(E, E') \rightarrow (G_1 \tilde{G}_2)_{n,m}(E, E') = \sum_k (G_1)_{n,k}(E, E')(G_2)_{k,m}(E, E') \). The trace in this new representation is written as \( \int_0^{eV} dE \sum_n \text{Tr} \ln (\hat{Q})_{nn} \). In this way, the functional convolution is reduced to matrix algebra for the infinite-dimensional matrix \( \hat{Q} \). From these arguments it is clear that the statistics is a multinomial distribution of multiple charge transfers:

\[
S(\chi) = \frac{t_0}{\hbar} \int_0^{eV} dE \ln \left[ 1 + \sum_{n=-\infty}^{\infty} P_n(E, V) (e^{i\chi n} - 1) \right].
\]

(29)

General expressions for the probabilities \( P(E, V) \) have been derived in Ref. \[45\].

Here we will pursue a different path and study a toy model. Let us neglect all set \( f^{RA}(|E| < \Delta) = 1 \), \( g^{RA}(|E| > \Delta) = \pm 1 \), and equal to zero otherwise. Physically, this means that we neglect Andreev reflections above the gap and replace the quasiparticle density of states by a constant \( |E| > \Delta \). This simplifies the calculation a lot, since the matrix trace now becomes finite. Let us for example consider a voltage \( eV = 2\Delta/4 \). In that case, we consider the determinant of the matrix

\[
\det \left[ 1 - \frac{\sqrt{T}}{2} \begin{pmatrix} \hat{Q}_-(\chi) & 1 & e^{-i\chi \hat{\tau}_3} \\ 1 & 0 & e^{i\chi \hat{\tau}_3} \\ e^{i\chi \hat{\tau}_3} & 0 & 1 \end{pmatrix} \right] ,
\]

(30)

where \( \hat{Q}_\pm(\chi) \) describe quasiparticle emission (injection) and off-diagonal pairs \( e^{\pm \chi} \) are associated with Andreev reflection. Evaluating the determinant we find

\[
S(\chi) = \frac{2\Delta t_0}{(n-1)\hbar} \ln \left[ 1 + P_5 (e^{i\chi n} - 1) \right]
\]

(31)

where the probabilities are given by

\[
\begin{align*}
P_2 &= \frac{T^2}{(2-T)^2}, & P_3 &= \frac{T^3}{(1-3T)^2}, & P_4 &= \frac{T^4}{(1-8T+T^2)^2}, & P_5 &= \frac{T^5}{(1-20T+5T^2)^2}, \\
P_0 &= \frac{(2-T)^2(16-16T+T^2)^2}{T^2}, & P_1 &= \frac{64-112T+56T^2-7T^3}{T^2}.
\end{align*}
\]

(32)

Note the limiting cases of these probabilities \( P_n \sim T^n/4^{n-1} \) for \( T \ll 1 \) and \( P_n = 1 \) for \( T \rightarrow 1 \). We conclude this section by saying that the general results for the CGF \[45\] allow for a fast and efficient calculation of all dc-transport properties of contacts between superconductors (which may contain magnetic impurities, phonon broadening or other imperfections).
In this section, we illustrate a further advantage of the Keldysh-Green’s functions approach to counting statistics. We consider a normal metallic diffusive wire connected on one end to a normal metal reservoir and on the other side to a superconductor. The wire is supposed to have a mean free path \( l \gg \lambda_F \), a corresponding diffusion coefficient \( D = v_F l / 3 \), and a length \( L \). For \( eV, k_B T \ll \Delta \) the transport occurs through Andreev reflection at the interface to the superconductor. This system shows a quite remarkable property, which is the so-called reentrance effect of the conductance. The energy difference \( 2E \) of electron-hole pairs leads to a dephasing on a length scale \( \xi_E = \sqrt{D/2E} \). This has the consequence that the (otherwise) normal wire becomes partially superconducting and the conductance increases with decreasing energy. However, once the coherence length \( \xi_E \) reaches \( L \) the conductance decreases again. Finally for \( E = 0 \) the conductance is exactly equal to the conductance in the normal state. This is the reentrance effect occurring at an energy of the order of the Thouless energy \( E_c = \hbar D / L^2 \). In Fig. 2 (left panel, dotted curve) the resulting differential conductance at zero temperature is plotted.

The transport in this system is described by a matrix diffusion equation for the Keldysh Green’s functions, the so-called Usadel equation

\[
-\frac{D}{\sigma} \nabla \tilde{I} = \left[-i E \tilde{\tau}_3, \tilde{G} \right], \quad \tilde{I} = -\sigma \tilde{G} \nabla \tilde{G}.
\]

(33)

In these equations \( \sigma = 2e^2 N_0 D \) is the conductivity. The boundary conditions for this equation are that the Green’s functions in the terminal approach the bulk solution \( \tilde{G}_N \) or \( \tilde{G}_S \), respectively. This equation is in general difficult to solve, even if one is interested in the average current only. However, we can calculate the noise and the counting statistics using the recipe outlined in Sec. 3 and obtain the noise in the full parameter range of Eq. (33).

Before considering Eq. (33) in its full generality, we consider the limiting cases of low and high energies (compared to \( E_c \)). For \( E = 0 \) the r.h.s. is absent and the system is completely analogous to a diffusive connector as discussed in \( \text{[4]} \). From Table 1 and using the eigenvalues \( \text{[12]} \) we find

\[
S(\chi) = \frac{t_0 G}{16 e^2} \int dE \text{acosh}^2 \left[ 2 \left(f_+(e^{2ix} - 1) + (1 - f_+)(1 - f_-)(e^{-2ix} - 1) \right) - 1 \right].
\]

(34)

This result shows, once again, that the charges are transferred in pairs. It is interesting to compare with the CGF for a diffusive wire between two normal metals, for which we obtain \( \text{[18]} \)

\[
S(\chi) = \frac{t_0 G}{4 e^2} \int dE \text{acosh}^2 \left[ 2 \left(f_1(1 - f_2)(e^{ix} - 1) + f_2(1 - f_1)(e^{-ix} - 1) \right) - 1 \right].
\]

(35)

We see that the only difference in the CGF between the SN- and the NN-case is in the counting factors, and a prefactor 1/4. Note, that this coincidence only
Fig. 2. Noise in diffusive SN-systems. Left panel a): the differential conductance and the noise show a reentrant behavior. The effective charge, defined as $q_{\text{eff}}(E) = (3/2) dS/dI$ reveals that the correlated Andreev pair transport suppresses the noise below the uncorrelated Boltzmann-Langevin result $2e$. Right panels b) and c): Effective charge of the Andreev interferometer shown in the inset (realized experimentally in Ref. [33]). The upper panel b) shows the theoretical predictions and the lower panel c) the experimental results. The theoretical results contain no fitting parameter (the Thouless energy $E_c = 30 \mu eV$ was extracted from the sample geometry and the experimental temperature of 43mK was included in the calculation). Therefore, it is reasonable that the deviations between experimental and theoretical results comes from possible heating effects in the experiment, which are not accounted for in the theoretical calculation.

occurs for the diffusive connector, but is by no means a general rule. At zero temperature the results simplify and we find

$$S^{SN}(\chi) = \frac{1}{2} S^{NN}(2\chi),$$

$$S^{NN}(\chi) = \frac{t_0 G V}{4e} \text{acosh}^2 \left(2e^{i\chi} - 1\right),$$

(36)
a surprising simple relation between the CGF for the Andreev wire and the normal diffusive wire. It is easy to see that the cumulants obey the general relation $C_n^{SN} = 2^{n-1} C_n^{NN}$. We observe that we can read off the effective charge from the ratio $C_n^{SN} / C_n^{NN} = (q_{\text{eff}}/e)^{n-1}$ and, indeed, find $q_{\text{eff}} = 2e$. This result for the effective charge is a special property of the diffusive connector.

At energies large compared to $E_c$ it is also possible to find the CGF for the Andreev wire in general. Then the proximity effect in the wire is absent and it turns out [34] that the wire can be effectively mapped on a normal circuit, consisting of two identical wires in series to which twice the voltage is applied and twice the counting field. Thus, for $E \gg E_c$ we obtain $S^{SN}(\chi)$ from $S^{NN}(\chi)$ by the replacement $\chi \rightarrow 2\chi$ and $G \rightarrow G/2$, which exactly brings us to Eq. [34] and shows that the counting statistics is again the same in the incoherent limit.

The full quantum-mechanical calculation of the energy-dependent shot noise can be performed on the basis of the approach of Sec. 3. We expand up to linear order in $\chi$, i.e. $\hat{G}(\chi) = \hat{G}_0 - i(\chi/2)\hat{G}_1$ and $\hat{I}(\chi) = \hat{I}_0 - i(\chi/2)\hat{I}_1$. Substituting in [38] we find

$$\frac{D}{\sigma} \frac{\partial}{\partial x} \hat{I}_1 = \left[-iE\tau_3, \hat{G}_1\right], \quad \hat{I}_1 = -\sigma \left(\hat{G}_0 \frac{\partial}{\partial x} \hat{G}_1 + \hat{G}_1 \frac{\partial}{\partial x} \hat{G}_0\right).$$

(37)
The boundary conditions at the reservoirs read $\hat{G}_1(0) = [\hat{\tau}_K, \hat{G}_L]$ at the left end and $\hat{G}_1(L) = 0$ at the right end. Finally the noise is $S_I = -e \int dE \text{Tr} \hat{\tau}_K I_1(x)$. By taking the trace of Eq. (37) multiplied with $\hat{\tau}_K$ it follows that it does not matter, where the noise is evaluated, as it should be. From these equations the generalization of the Boltzmann-Langevin equation to superconductors can be derived \cite{57}, which allows for a faster numerical solution. The results for the energy dependent noise is shown in the left panel Fig. 2. A direct comparison of the differential shot noise and the differential conductance (for zero temperature) shows the difference in the energy dependence. The effective charge defined as $q_{\text{eff}} = (3/2) dS/dI$ displays the clear deviation of the quantum noise from the Boltzmann-Langevin result of $2e$. At energies below the Thouless energy $E_c$ the effective charge is suppressed below $2e$. This shows that the correlated Andreev pair transport suppresses the noise below the uncorrelated Boltzmann-Langevin result.

To experimentally probe the pair correlations in diffusive superconductor-normal metal-heterostructures it is most convenient to use an Andreev interferometer. An example is shown in the left part of Fig. 2. A diffusive wire connected to a normal terminal is split into two parts, which are connected to two different points of a superconducting terminal. By passing a magnetic flux through the loop one can effectively vary the phase difference between the two connections to the superconductor. Such a structure has been experimentally realized by the Yale group \cite{33}. In Fig. 2 we present a direct comparison between our theoretical predictions and the experimentally obtained effective charge. Note, that we have included the experimental temperature in the theoretical modeling. The finite temperature explains the strong decrease of the effective charge in the regime $|eV| \leq k_B T$, where the noise is fixed by the fluctuation-dissipation theorem. The disagreement between theory and experiment in this regime stems solely from differences in the measured temperature-dependent conductance from the theoretical prediction. We attribute this to heating effects. The qualitative agreement in the shot-noise regime $|eV| \geq k_B T$ is satisfactory, if one takes into account, that we have no free parameters for the theoretical calculation. Both, experiment and theory show a suppression of the effective charge for some finite energy, which is of the order of the Thouless energy and depends on flux in a qualitative similar manner. Remarkably for half-integer flux the effective charge is completely flat, in contrast to what one would expect from circuit arguments based on the conductance distribution in the fork geometry. Currently we have no explanation for this behavior, and therefore more work is needed in this direction.

6 Multi Terminal Circuits

In circuits with more than two terminals it is of particular interest to study non-local correlations of currents in different terminals. For that purpose we need a slight extension of the theoretical approach of Sec. 5 suitable for multi-terminal circuits. We will now introduce this method.
**Circuit Theory** To study transport in general mesoscopic multi-terminal structures the so-called circuit theory for quantum transport was developed by Nazarov [53,58]. Its main idea, borrowed from Kirchhoff’s classical circuit theory, is to represent a mesoscopic device by discrete elements, which resemble the known elements of electrical transport. We briefly repeat the essentials of the circuit theory. Topologically, one distinguishes three elements: terminals, nodes and connectors. Terminals are the connections to the external voltage or current sources and provide boundary conditions, specifying externally applied voltages, currents or phase differences in the case of superconductors. The actual circuit is represented by a network of nodes and connectors, the first determining the approximate layout and the second describing the connections between different nodes, respectively.

To describe quantum effects it is necessary to represent the variables describing a node by matrix Green’s function $\hat{G}$, which can be either Nambu or Keldysh matrices, or a combination thereof. Consequently, we describe the current through a connector by a matrix current $\hat{I}$, which relates the fluxes of all elements of $\hat{G}$ on neighboring nodes. The current has been derived by Nazarov [53] and is given by Eq. (20) for a connector, characterized by a set of transmission coefficients $\{T_n\}$. Note that the electrical current is obtained from $I_{12} = \frac{1}{i} \int dE \text{Tr}_K \hat{I}_{12}$. The boundary condition are given in terms of fixed matrix Green’s functions $\hat{G}_i$, which are determined by the applied potential, the temperature, the type of lead, and a counting field $\chi_i$.

Once the network is determined and all connectors are specified, the transport properties can be found by means of the following circuit rules. We associate an (unknown) Green’s function $\hat{G}_j$ to each node $j$. The two rules are

1. $\hat{G}_j^2 = \hat{I}$ for the Green’s functions of all internal nodes $j$.
2. the total matrix current in a node is conserved: $\sum_i \hat{I}_{ij} = 0$, where the sum goes over all nodes or terminals connected to node $j$ and each matrix current is given by (20).

Finally, the observable currents into the terminals are given by $I_i = \sum_j I_{ij}$, where the sum runs over all nodes connected to the terminal $i$. To obtain the counting statistics, we finally integrate all currents $I_i(\chi) = (\partial/\partial \chi_i)S(\chi)$ to find the CGF $S(\chi)$.

**Multi tunnel junction structure** A general expression of $S(\chi)$ can be obtained for a system of an arbitrary number of terminals connected to one common node by tunnel contacts, see Fig. [36,12]. At the same time it nicely demonstrates the application of the circuit theory rules, presented above. Let us denote the unknown Green’s function of the central node by $\hat{G}_c(\chi)$. The matrix current from a terminal $\alpha$ ($\alpha = 1, \ldots, K$) into the central node is given by the relation

$$\hat{I}_\alpha(\chi) = \frac{g_\alpha}{2} [\hat{G}_c(\chi), \hat{G}_\alpha(\chi_\alpha)] ,$$

where $g_\alpha = G_Q \sum_n T_n$ is the conductance of the respective tunnel junction junction, for which we have assumed that all $T_n \ll 1$ and $g_\alpha \gg G_Q$ to avoid Coulomb
blockade. The Green’s function of the central node is determined by matrix current conservation, reading $\sum_{\alpha=1}^{K} I_{\alpha} = [\sum_{\alpha=1}^{K} g_{\alpha} \tilde{G}_{\alpha}, \tilde{G}_{c}] / 2 = 0$. Employing the normalization condition $G_{c}^{2} = 1$, the solution is

$$
\tilde{G}_{c}(\chi) = \frac{\sum_{\alpha=1}^{K} g_{\alpha} \tilde{G}_{\alpha}(\chi_{\alpha})}{\sqrt{\sum_{\alpha,\beta=1}^{K} g_{\alpha} g_{\beta} \{ \tilde{G}_{\alpha}(\chi_{\alpha}), \tilde{G}_{\beta}(\chi_{\beta}) \}}}. 
$$

To find the cumulant-generating function (CGF) $S(\chi)$ we integrate the equations $\partial S(\chi)/\partial \chi_{\alpha} = (-it_{0}/4e^{2}) \int dE \text{Tr} \tilde{\tau}_{K} \tilde{I}_{\alpha}(\chi)$. We obtain

$$
S(\chi) = \frac{t_{0}}{2e^{2}} \int dE \text{Tr} \left[ \sqrt{\sum_{\alpha,\beta=1}^{M} g_{\alpha} g_{\beta} \{ \tilde{G}_{\alpha}(\chi_{\alpha}), \tilde{G}_{\beta}(\chi_{\beta}) \}} \right]. 
$$

This is the general result for an M-terminal geometry in which all terminals are tunnel-coupled to a common node. It is valid for arbitrary combinations of normal metal and superconductor, fully accounting for the proximity effect. Note, that we have dropped the normalization of $S(\chi)$ to write the expression more compact.

**Normal metals** If all terminals are normal metals, the matrices in Eq. (40) are all diagonal and trace is trivial. We obtain

$$
S(\chi) = \frac{t_{0}}{2e^{2}} \int dE \left[ g_{\Sigma}^{2} + \sum_{\alpha \neq \beta} g_{\alpha} g_{\beta} f_{\alpha}(E)(1 - f_{\beta}(E))(e^{i(\chi_{\alpha} - \chi_{\beta})} - 1) \right], 
$$

where $f_{\alpha}$ is the occupation function of terminal $\alpha$. Here, we introduced the abbreviation $g_{\Sigma} = \sum_{\alpha=1}^{N} g_{\alpha}$ for the sum of all conductances. We note, that the statistics is essentially non-Poissonian, despite the fact the we are considering tunnel junctions.

We now restrict us to two terminals (in which case we have to consider only one counting field $\chi = \chi_{1} - \chi_{2}$). For zero temperature and voltage bias $V$ the

---

**Fig. 3.** Multi tunnel junction structure: a) general setup with $K$ terminals connected to a common node. b) beam splitter setup in which terminal 3 is either a normal metal or a superconductor.
CGF reads then

\[ S(\chi) = \frac{t_0 V}{2e} \sqrt{g_2^2 + 4g_1g_2(e^{i\chi} - 1)}, \]  

(42)

the result for a double tunnel junction first obtained by de Jong \[13\] using a master equation approach. We obtain as limiting cases for an asymmetric junction (either \(g_1 \ll g_2\) or \(g_1 \gg g_2\)) Poisson statistics \(S(\chi) = (t_0 V g_1 g_2 / (g_1 + g_2)) \exp(i\chi) - 1\).

Next we consider a three terminal structure, which is voltage biased such that the mean current \(\bar{I}_3\) in lead 3 vanishes (voltage probe) and a transport current \(\bar{I} = g_1 g_2 / (g_1 + g_2)\) flows between terminals 1 and 2. The CGF is \[59\]

\[ S(\chi) = \frac{t_0 |V|}{2e} \left( g_2 \sqrt{g_2^2 / 2 + 4g_3 g_1 (e^{-i\chi_1} - 1) + 4g_1 g_2 (e^{i\chi_2 - i\chi_1} - 1)} \right) + g_1 \sqrt{g_2^2 / 2 + 4g_3 g_2 (e^{i\chi_2} - 1) + 4g_1 g_2 (e^{i\chi_2 - i\chi_1} - 1)} \right). \]  

(43)

It is interesting to note that the presence of the voltage probe makes the CGF asymmetric under the transformation \(g_1 \leftrightarrow g_2\), whereas the current is symmetric.

In certain limits in which the square roots in Eq. 43 can be expanded one is able to find the counting statistics. E.g. in the strong-coupling limit \(g_3 \gg (g_1 + g_2)\) we find

\[ S(\chi) = \bar{N} \left[ e^{-i\chi_1} + e^{i\chi_2} - 2 \right]. \]  

(44)

The CGF is simply the sum of two Poisson distribution, demonstrating drastically the effect of the voltage probe. It completely suppresses the correlation between electrons entering and leaving the central node.

Another interesting geometry is a beam splitter configuration, in which a voltage bias is applied between one terminal and the other two. We find

\[ S_N(\chi_1, \chi_2) = \frac{t_0 |V|}{2e} \sqrt{g_2^2 / 2 + g_3 (e^{i\chi_1} - 1) + g_1 g_2 (e^{i\chi_2} - 1)}. \]  

(45)

In the limit that \(g_1 + g_2\) and \(g_3\) are very different, we can expand the CGF and find for the CGF \(S(\chi) = N_1 e^{i\chi_1} + N_2 e^{i\chi_2}\), i.e., the tunneling processes into the two terminals are uncorrelated. The corresponding probability distribution is simply the product of two Poisson distributions.

**SN-contact** We now consider the case of a double tunnel junction, in which one of the terminals is superconducting. From the general result \[10\] and \[52\] we find after some algebra

\[ S(\chi) = \frac{t_0 |V|}{e\sqrt{2}} \sqrt{g_1^2 + g_2^2 + \sqrt{(g_1^2 + g_2^2)^2 + 4g_1^2 g_2^2 (e^{i2\chi} - 1)}}. \]  

(46)

Remarkably, the statistics is fundamentally different from the corresponding normal case \[52\]. Still, the elementary events are transfers of pairs of electrons, which, however are correlated in a more complicated way than normal electrons.
If the junction is very asymmetric, the FCS reduces to Poissonian transfer of electron pairs. This is similar to the effect of decoherence between electrons and holes for energies of the order of the Thouless energy \[32\].

For the beam splitter configuration we are also able to find the FCS analytically. The CGF is \[12\]

\[
S(\chi_1, \chi_2) = \frac{Vt_0}{\sqrt{2e}} \times \sqrt{g_S^2 + 4g_3^2g_2^2(e^{i2\chi_1} - 1) + 4g_3^2g_2^2(e^{i2\chi_2} - 1) + 8g_3^2g_1g_2(e^{i(\chi_1+\chi_2)} - 1),}
\]

where we abbreviated \(g_S^2 = g_3^2 + (g_1 + g_2)^2\). From this result we see that the elementary processes are now double charge transfers to either terminal of a splitting of a Cooper pair among the two terminal. It is interesting to note, that, if we assume that \(g_1 + g_2\) and \(g_3\) are very different (but \(g_1 \approx g_2\)), we obtain non-separable statistics

\[
S(\chi) = N_{11}e^{i2\chi_1} + N_{22}e^{i2\chi_2} + N_{12}e^{i(\chi_1+\chi_2)}.
\]

This expression can not be written as a sum of two independent terms. Furthermore, the last term is positive, which implies that current crosscorrelation \(S_{12} = -(2e^2/t_0)(\partial^2/\partial\chi_1\partial\chi_2)S(\chi_1, \chi_2)|_{\chi_1, \chi_2 \to 0}\) are positive. Eq. \[48\] provides a simple explanation for this surprising effect: it is a consequence of independence of the different events, contributing to the current. This result, in fact, holds for a large class of superconducting beam splitters \[34,37,60,61\].

### 7 Conclusion

We have tried to give a pedagogical introduction to the field of counting statistics. Many technical details were left out, but we have tried to cover the essence of the derivation and concentrated on looking at concrete examples. For a more thorough study we recommend the recent book *Quantum Noise in Mesoscopic Physics* \[4\] or the original literature. While a number of aspects have already been explored, many open questions remain, e. g., experimental strategies to measure FCS, strongly interacting systems, or spin-dependent problems. For the future, we expect even more activity in the field and, consequently, even more interesting results will emerge.

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### 8 Appendix

We summarize here the matrix-Green’s function for superconducting and normal contact, as they were used in the text. The time-dependent Green’s functions are
expressed by their Fourier transforms $\hat{G}_0(t-t') = \int (dE/2\pi) e^{-iE(t-t')} \hat{G}_0(E)$.

The energy-dependent Green’s functions in the Keldysh×Nambu-space have the form

$$\hat{G}(E) = \begin{pmatrix} (\bar{A} - \bar{R}) \bar{f} + \bar{R} & (\bar{A} - \bar{R})(1 - \bar{f}) (\bar{R} - \bar{A}) \bar{f} + \bar{A} \end{pmatrix},$$

where the advanced, retarded and occupation Nambu matrices are

$$\bar{A}(\bar{R}) = \begin{pmatrix} g_{A(R)} & \bar{f}_{A(R)} \\ \bar{f}_{A(R)} & -g_{A(R)} \end{pmatrix}, \quad \bar{f}(E) = \begin{pmatrix} f(E) & 0 \\ 0 & f(-E) \end{pmatrix}.$$  \hfill (50)

The phase $\phi$ of the superconducting order parameter as well as the electrical potential $eV$ enter via the gauge transformation $\hat{G}(t, t') = \hat{U}(t) \hat{G}_0(t-t') \hat{U}^\dagger(t')$. Here $\hat{U}(t) = \exp \left[ i\phi(t) \tau_3/2 \right]$, where $\phi(t) = \varphi + eV t$.

In the calculation of the FCS of contacts between normal metals and superconductors we frequently need the eigenvalues of anticommutators of two Green’s functions. For two normal metals $\{\hat{G}_N(\chi), \hat{G}_N(\chi)\}/2$ is diagonal and the eigenvalue is

$$\left[ 1 + 2f_1(E) (1 - f_2(E)) (e^{i\chi} - 1) + 2f_2(E) (1 - f_1(E)) (e^{-i\chi} - 1) \right],$$

for the electron block and the same expression with $E \rightarrow -E$ for the 'hole'-block in Nambu space.

In the case of Andreev reflection, i.e. for $eV, k_B T_{el} \ll \Delta$, we find for $\{\hat{G}_N(\chi), \hat{G}_S\}/2$ the two eigenvalues

$$\pm \sqrt{f_N(E)f_N(-E)}(1 - e^{i2\chi}) + (1 - f_N(E))(1 - f_N(-E))(1 - e^{-i2\chi}).$$  \hfill (52)

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