COCOMPACTLY CUBULATED 2-DIMENSIONAL ARTIN GROUPS

JINGYIN HUANG, KASIA JANKIEWICZ, AND PIOTR PRZYTYCKI

Abstract. We give a necessary and sufficient condition for a 2-dimensional or a three-generator Artin group \( A \) to be (virtually) cocompactly cubulated, in terms of the defining graph of \( A \).

1. Introduction

We say that a group is \((\text{cocompactly})\) \textit{cubulated} if it acts properly (and compactly) by combinatorial automorphisms on a CAT(0) cube complex. We say that a group is \((\text{virtually cocompactly})\) \textit{cubulated}, if it has a finite index subgroup that is cocompactly cubulated. Such groups fail to have Kazhdan’s property (T) [NR97], are bi-automatic [´Swi06], satisfy the Tits Alternative [SW05] and, if cocompactly cubulated, they satisfy rank-rigidity [CS11]. For more background on CAT(0) cube complexes, see the survey article of Sageev [Sag14].

The Artin group with generators \( s_i \) and exponents \( m_{ij} = m_{ji} \geq 2, \) where \( i \neq j \), is presented by relations \( s_is_js_i\cdots = s_js_is_j\cdots \). Its defining graph has vertices corresponding to \( s_i \) and edges labeled \( m_{ij} \) between \( s_i \) and \( s_j \) whenever \( m_{ij} < \infty \).

Artin groups that are \((\text{right-angled})\) \textit{right-angled} (i.e. the ones with \( m_{ij} \in \{2, \infty\} \)) are cocompactly cubulated, and they play a prominent role in theory of special cube complexes of Haglund and Wise. However, much less is known about other Artin groups, in particular about braid groups. In [Wis11] Wise suggested an approach to cubulating Artin groups using cubical small cancellation. However, we failed to execute this approach: we were not able to establish the B(6) condition.

In this article we consider Artin groups that have three generators, or are 2-dimensional, that is, their corresponding Coxeter groups have finite special subgroups of maximal rank 2 (or, equivalently, 2-dimensional Davis complex). We characterise when such a group is virtually cocompactly cubulated. This happens only for very rare defining graphs. An \textit{interior} edge of a graph is an edge that is not a leaf.

Theorem 1.1. Let \( A \) be a 2-dimensional Artin group. Then the following are equivalent.

\( (i) \) \( A \) is cocompactly cubulated,
\( (ii) \) \( A \) is virtually cocompactly cubulated,
\( (iii) \) each connected component of the defining graph of \( A \) is either
\begin{itemize}
  \item a vertex, or an edge, or else
  \item all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.
\end{itemize}
Moreover, if \( A \) is an arbitrary Artin group, then (iii) implies (i).

**Theorem 1.2.** Let \( A \) be a three-generator Artin group. Then the following are equivalent.

(i) \( A \) is cocompactly cubulated,

(ii) \( A \) is virtually cocompactly cubulated,

(iii) the defining graph of \( A \) is as in Theorem 1.1(iii) or has two edges labelled by 2.

1.1. **Remarks.** From Theorem 1.2 it follows that the 4-strand braid group is not virtually cocompactly cubulated.

Note that, independently, Thomas Haettel has obtained a full classification of cocompactly cubulated Artin groups. We intend to bring with Haettel our results to common denominator and prove that an Artin group is virtually cocompactly cubulated only if it is cocompactly cubulated.

The equivalence of (i) and (ii) has no counterpart for Coxeter groups, where the group \( \tilde{A}_2 \) generated by reflections in the sides of an equilateral triangle in \( \mathbb{R}^2 \) is virtually cocompactly cubulated, but not cocompactly cubulated.

There are Artin groups that do not satisfy the equivalent conditions from Theorem 1.1 but are cubulated. Namely, it follows from [Bru92, HM99] that if the defining graph of \( A \) is a tree, then \( A \) is the fundamental group of a link complement that is a graph manifold with boundary. Hence by the work of Liu [Liu13] or Przytycki and Wise [PW14] the Artin group \( A \) is cubulated.

Artin groups of large type, that is, with all \( m_{ij} \geq 3 \) are 2-dimensional. For many of them Brady and McCammond constructed 2-dimensional CAT(0) complexes with proper and cocompact action [BM00]. However, these complexes are built of triangles, not squares.

1.2. **Some historical background.** Sageev invented a way of cubulating groups (i.e. showing that they are cubulated) using codimension 1-subgroups [Sag95], which was later also explained in the language of walls in the Cayley complex of the group [CN05, Nic04]. Here we give a brief account on some cubulation results, for a more complete one see [HW14].

Using the technology of walls,Niblo and Reeves cubulated Coxeter groups [NR97] and Caprace and M"uhlherr analysed when this cubulation is cocompact [CM05]. It is not known if all Coxeter groups are virtually cocompactly cubulated. Wise cocompactly cubulated small cancellation groups [Wis04], and Ollivier and Wise cocompactly cubulated random groups at density \( < \frac{1}{6} \) [OW11].

Furthermore, using the surfaces of Kahn and Markovic, Bergeron and Wise cocompactly cubulated the fundamental groups of closed hyperbolic 3-manifolds [KM12, BW12], and later Wise cocompactly cubulated the fundamental groups of compact hyperbolic 3-manifolds with boundary [Wis11]. Hagen and Wise cocompactly cubulated hyperbolic free-by-cyclic groups [HW15].

Groups that are not (relatively) hyperbolic are harder to cubulate cocompactly. Przytycki and Wise cubulated the fundamental groups of all compact 3-dimensional manifolds that are not graph manifolds, as well as graph manifolds with boundary [PW14, PW12]. In [Liu13] Liu gave a criterion for a graph manifold fundamental group to be virtually cubulated specially (meaning that the quotient of the action admits a local isometry into the Salvetti complex of a right-angled Artin group),
but we do not know if this is equivalent to just being cubulated. Hagen and Przytycki gave a criterion for a graph manifold fundamental group to be cocompactly cubulated [HP15]. In general, it is difficult to find obstructions for groups to be cubulated. Another result of this type is Wise’s characterization of tubular groups that are cocompactly cubulated [Wis14].

1.3. Proof outline for (i)⇒(iii) in Theorem 1.1. Given a 2-dimensional Artin group acting properly and cocompactly on a CAT(0) cube complex, we show that its two-generator special subgroups are convex cocompact. More precisely, each of them acts cocompactly on a convex subcomplex which naturally decomposes as a product of a vertical factor and a horizontal factor. Geometrically, the intersection of two such subgroups is either vertical or horizontal. However, if Theorem 1.1(iii) is not satisfied, then this intersection is neither vertical nor horizontal by algebraic considerations.

One of the ingredients of the proof is Theorem 3.8, which asserts that a top rank product of hyperbolic groups acting on a CAT(0) cube complex is always convex cocompact.

1.4. Organization. In Section 2 we give some background on CAT(0) spaces and CAT(0) cube complexes. Section 3 is devoted to the proof of Theorem 3.8. In Section 4 we give some background on Artin groups and discuss some algebraic properties of two-generator Artin groups. Finally, in Section 5 we prove Theorem 1.1 and in Section 6 we prove Theorem 1.2.

1.5. Acknowledgements. The authors would like to thank Daniel T. Wise for helpful discussions. The third author was partially supported by National Science Centre DEC-2012/06/A/ST1/00259 and NSERC.

2. Preliminaries

A group is a CAT(0) group if it acts properly and cocompactly on a CAT(0) space. We assume the reader is familiar with the basics of CAT(0) spaces and groups. For background, see [BH99]. In this section we collect some less classical results.

2.1. Asymptotic rank. The following definition was introduced in [Kle99].

Definition 2.1. Let $X$ be a CAT($\kappa$) space. For $x \in X$ we denote by $\Sigma_x X$ the CAT(1) space that is the completion of the space of directions at $x$ [BH99 Definition II.3.18]. The geometric dimension of $X$, denoted $\text{GeomDim}(X)$ is defined inductively as follows.

- $\text{GeomDim}(X) = 0$ if $X$ is discrete,
- $\text{GeomDim}(X) \leq n$ if $\text{GeomDim}(\Sigma_x X) \leq n - 1$ for any $x \in X$.

Definition 2.2. Let $X$ be a CAT(0) space. Then its asymptotic rank, denoted by $\text{asrk}(X)$, is the supremum of the geometric dimension of the asymptotic cones of $X$.

Theorem 2.3. Let $X$ and $Y$ be CAT(0) spaces. Then

1. $\text{asrk}(X \times Y) \geq \text{asrk}(X) + \text{asrk}(Y)$,
2. if $\text{asrk}(X) \leq 1$, then $X$ is hyperbolic.
The first assertion follows from Theorem A of [Kle99] and the second assertion follows from Corollary 1.3 of [Wen07].

**Definition 2.4.** If $G$ is a CAT(0) group acting properly and cocompactly on a CAT(0) space $X$, then the *asymptotic rank* of $G$ is the asymptotic rank of $X$. By [Kle99] Theorem C this is the maximal $n$ for which there is a quasi-isometric embedding $\mathbb{R}^n \to X$. Hence it does not depend on the choice of the CAT(0) space $X$.

**Lemma 2.5.** Suppose that $G$ is a CAT(0) group, and that $G$ acts properly and cocompactly on a contractible $n$-dimensional cell complex $X$ (not necessarily CAT(0)). Then the asymptotic rank of $G$ is $\leq n$.

**Proof.** Choose any $G$-equivariant length metric on $X$. We will prove that there does not exist a quasi-isometric embedding $f: \mathbb{R}^k \to X$ for $k>n$. Otherwise, since $X$ is contractible and admits a cocompact action of $G$, we can assume that $f$ is a continuous quasi-isometry: such $f$ can be defined by induction on consecutive skeleta of the standard cubical subdivision of $\mathbb{R}^k$.

Let $Y \subseteq X$ be the smallest subcomplex containing $f(\mathbb{R}^k)$. Then $f: \mathbb{R}^k \to Y$ is a quasi-isometry. Let $g: Y \to \mathbb{R}^k$ be a quasi-isometry inverse to $f$, we can again assume that $g$ is continuous. For any $x \in \mathbb{R}^k$ the distance $d(g \circ f(x),x)$ is uniformly bounded and consequently there is a proper geodesic homotopy between $g \circ f$ and the identity map.

Recall that for a topological space $X$ we can consider *locally finite chains* in $X$, which are formal sums $\sum_\lambda a_\lambda \sigma_\lambda$ where $a_\lambda$ are integers, $\sigma_\lambda$ are singular simplices, and any compact set in $X$ intersects the images of only finitely many $\sigma_\lambda$ with $a_\lambda \neq 0$. This gives rise to *locally finite homology* of $X$, denoted by $H^L_k(X)$. Moreover, proper maps induce homomorphisms on locally finite homology. See [BKS08a] Section 2.2 for more discussion.

Since there is a proper geodesic homotopy between $g \circ f$ and the identity map, $g \circ f$ induces the identity on $H^L_k(\mathbb{R}^k)$, and consequently $f_*: H^L_k(\mathbb{R}^k) \to H^L_k(Y)$ is injective. This leads to a contradiction, since $H^L_k(\mathbb{R}^k)$ contains the fundamental class $[\mathbb{R}^k]$ which is a nontrivial element, while $H^L_k(Y) = 0$ since $\dim(Y)<k$. \hfill $\Box$

### 2.2. Gate and parallel set

All CAT(0) cube complexes in our article are finite-dimensional. Throughout this paper the only metric that we consider on a CAT(0) cube complex $X$ is the CAT(0) metric $d$. The *convex hull* of a subspace $Y \subseteq X$ is the smallest convex subspace containing $Y$, and is not necessarily a subcomplex, while the *combinatorial convex hull* of $Y$ is the smallest convex subcomplex of $X$ containing $Y$. For a complete convex subspace $Y \subseteq X$ we denote by $\pi_Y: X \to Y$ the closest point projection onto $Y$.

The following lemma was proved in slightly different contexts by various authors [BHS14, Hua14b, BKS08a, AB08]:

**Lemma 2.6.** [Hua14b Lemma 2.10] Let $X$ be a CAT(0) cube complex of dimension $n$, and let $Y_1$, $Y_2$ be convex subcomplexes. Let $\Delta = d(Y_1,Y_2)$, $V_1 = \{ y \in Y_1 | d(y,Y_2) = \Delta \}$ and $V_2 = \{ y \in Y_2 | d(y,Y_1) = \Delta \}$. Then:

1. $V_1$ and $V_2$ are nonempty convex subcomplexes.
2. $\pi_{Y_1}$ maps $V_2$ isometrically onto $V_1$ and $\pi_{Y_2}$ maps $V_1$ isometrically onto $V_2$.

Moreover, the convex hull of $V_1 \cup V_2$ is isometric to $V_1 \times [0,\Delta]$.
Proof. The carrier $N_h$ of $h$, which is its neighbourhood, has form $N_h = h \times [0, 1]$. Thus if $F \notin h$, then $h \cap F$ is a codimension-1 submanifold of $F$. Moreover, the intersections $h \cap F$, $h^+ \cap F$, and $h^- \cap F$ are convex, thus the lemma follows. □
Lemma 3.2. Let \( h \) be a hyperplane in a CAT(0) cube complex \( X \). Suppose that \( l \) is a geodesic ray in \( X \) starting in \( h \). If \( l \not\subset h \), then there exists another hyperplane \( h' \) in \( X \) intersecting \( l \) and disjoint from \( h \).

Proof. Let \( N_h \) be the carrier of \( h \). Let \( B \) be the first cube outside \( N_h \) whose interior is intersected by \( l \). We claim that there is a hyperplane \( h' \) intersecting \( B \) and disjoint from \( h \). Indeed, pick a vertex \( v \in N_h \cap B \) and let \( e \) be an edge of \( B \) containing \( v \). If the hyperplane dual to \( e \) intersects \( h \), then \( e \subset N_h \). If this holds for any \( e \), then \( B \subset N_h \) by the convexity of \( N_h \), which yields a contradiction. This justifies the claim.

By the claim, there is a hyperplane \( h' \) intersecting \( B \) and disjoint from \( h \). It remains to prove that \( l \) intersects \( h' \). Otherwise, since \( l \) intersects the interior of the carrier of \( h' \), we have that \( l \) is contained in \( N_{h'} \). Since \( l \) starts at \( h \), we have that \( h \) intersects \( N_{h'} \) and hence it also intersects \( h' \), which is a contradiction.

We will also use a consequence of a result of Haglund \cite[Theorem 2.28]{Hag08}.

**Theorem 3.3.** Let \( X \) be a hyperbolic CAT(0) cube complex. Then any quasi-isometrically embedded subspace of \( X \) is at finite Hausdorff distance from its combinatorial convex hull.

In the following theorem we generalise our results from \cite[Section 3]{HP15}. Here \( d_{\text{Haus}} \) denotes the Hausdorff distance.

**Theorem 3.4.** Let \( X \) be a CAT(0) cube complex of asymptotic rank \( n \) and let \( F \subseteq X \) be an \( n \)-flat. Let \( Y \) be the combinatorial convex hull of \( F \). Then \( d_{\text{Haus}}(F, Y) < \infty \).

**Proof.** If \( F \) is contained in the carrier \( N_h = h \times [0,1] \) of a hyperplane \( h \), then we can replace \( X \) by \( h \) and \( F \) by its projection to \( h \). The combinatorial convex hull \( Y \) of \( F \) equals \( Y' \times [0,1], Y' \times \{0\}, \) or \( Y' \times \{1\}, \) where \( Y' \) is the combinatorial convex hull of the projection of \( F \) to \( h \). Henceforth we can and will assume that \( F \) is not contained in the carrier of any hyperplane.

Let \( \mathcal{H} \) be the collection of hyperplanes intersecting \( F \). We define a pencil of hyperplanes to be an infinite collection of mutually disjoint hyperplanes \( \{h_j\}_{j=-\infty}^{\infty} \) such that for each \( i \), \( \{h_{j+i}\}_{j=-\infty}^{\infty} \) and \( \{h_j\}_{j=i+1}^{\infty} \) are in different halfspaces of \( h_i \). It follows from Lemma 3.1 that every pencil of hyperplanes in \( \mathcal{H} \) intersects \( F \) in a collection of parallel family of codimension-1 flats. A collection of pencils of hyperplanes in \( \mathcal{H} \) is independent if their corresponding normal vectors are linearly independent in \( F = \mathbb{R}^n \).

Let \( \{P_i\}_{i=1}^m \) be a maximal collection of pairwise independent pencils in \( \mathcal{H} \). We claim that \( m = n \) and that \( \{P_i\} \) is independent. Suppose first \( m > n \). Note that if two pencils \( P, P' \subseteq \mathcal{H} \) are independent, then every hyperplane in \( P \) intersects every hyperplane in \( P' \). This gives rise to a quasi-isometric embedding of \( \mathbb{R}^m \) into \( X \), contradicting the bound on the asymptotic rank of \( X \). If \( m < n \) or if \( m = n \) but \( \{P_i\} \) is dependent, then there is a geodesic line \( l \) in \( F \) parallel to \( h \)\cap F \) for all hyperplanes \( h \) in \( \{P_i\} \). Using Lemma 3.2, we can then produce a new pencil \( P \) formed of some hyperplanes intersecting \( l \). Since \( P \) is independent from each \( P_i \), this contradicts the maximality of \( m \). This justifies the claim that \( m = n \) and \( \{P_i\} \) is independent.

For \( 1 \leq i \leq n \), choose \( h_i \in P_i \) and let \( F_i = h_i \cap F \). We will prove that for any hyperplane \( h \in \mathcal{H} \), there exists \( F_i \) such that \( h \cap F \) is parallel (possibly equal) to \( F_i \). Otherwise, choose a geodesic line \( l \) in \( F \) transverse to \( h \cap F \). By Lemma 3.2, \( h \) is
contained in a pencil \( P_h \) of hyperplanes intersecting \( l \). Note that \( P_h \) is independent from each \( P_i \) contradicting the maximality of \( m \).

Let \( \mathcal{H}_i \subseteq \mathcal{H} \) be the collection of hyperplanes whose intersection with \( F \) is parallel to \( F_i \). The above discussion implies \( \mathcal{H} = \bigcup_{i=1}^{m} \mathcal{H}_i \). Moreover, for \( i \neq j \), every hyperplane in \( \mathcal{H}_i \) intersects every hyperplane in \( \mathcal{H}_j \). Let \( Y \) be the combinatorial convex hull of \( F \). Since we assumed that \( F \) is not contained in the carrier of any hyperplane, the hyperplanes in \( Y \) are exactly the intersections with \( Y \) of the hyperplanes in \( \mathcal{H} \). Two hyperplanes of \( Y \) intersect if and only if the corresponding hyperplanes in \( \mathcal{H} \) intersect. Hence by Lemma 2.8 we have a product decomposition \( Y = Y_1 \times \cdots \times Y_n \).

Let \( \pi_i : Y \to Y_i \) be the coordinate projections. Let \( l_i = \bigcap_{j \neq i} F_j \), which is a geodesic line in \( F \). Note that for \( j \neq i \) we have \( l_i \subseteq F_j \subseteq h_j \) and hence the projection \( \pi_j(l_i) \) is a single point. Thus the restriction of \( \pi_i \) to \( l_i \) is an isometric embedding. It follows that \( F = \pi_1(l_1) \times \cdots \times \pi_n(l_n) \). Moreover, since \( \pi_i(l_i) = \pi_i(F) \), each \( Y_i \) is the combinatorial convex hull of \( \pi_i(l_i) \), since otherwise we could pass to a smaller convex subcomplex containing \( F \).

Since each of \( Y_i \) contains a line and their product has asymptotic rank \( \leq n \), by Theorem 2.3(1) each \( Y_i \) has asymptotic rank 1. By Theorem 2.3(2) each \( Y_i \) is hyperbolic. Thus by Theorem 3.3 we have \( d_{\text{Haus}}(\pi_i(l_i), Y_i) < \infty \), and consequently \( d_{\text{Haus}}(F, Y) < \infty \).

While we will not need it in the remaining part of the paper, from the proof above we can deduce the following interesting result which concerns flats that are not necessarily of top rank.

**Corollary 3.5.** Let \( X \) be a CAT(0) cube complex and let \( F \subseteq X \) be a flat. Let \( Y \subseteq X \) be the combinatorial convex hull of \( F \). Then \( Y \) has a natural decomposition \( Y = Y_1 \times \cdots \times Y_n \times K \) such that:

1. \( n \geq \dim(F) \) and \( K \) is a cube.
2. Each \( Y_i \) contains an isometrically embedded copy of \( \mathbb{R} \) that is the projection of a geodesic line in \( F \).
3. No \( Y_i \) contains a facing triple of hyperplanes, that is, a collection of three disjoint hyperplanes such that none of them separates the other two.

Roughly speaking, (3) means that \( Y_i \) do not “branch”.

### 3.2. Product of hyperbolic groups.

**Definition 3.6.** Let \( X \) be a CAT(0) cube complex. A group \( H \leq \text{Aut}(X) \) is convex cocompact if there is a convex subcomplex \( Y \subseteq X \) that is \( H \)-cocompact, meaning that \( H \) preserves \( Y \) and acts on it cocompactly.

**Lemma 3.7.** Let \( X \) be a CAT(0) cube complex and let \( H \leq \text{Aut}(X) \) be convex cocompact. Then there exists a minimal \( H \)-invariant convex subcomplex. Moreover, any minimal \( H \)-invariant convex subcomplex is \( H \)-cocompact and any two minimal \( H \)-invariant convex subcomplexes are parallel.

**Proof.** Let \( Y \subseteq X \) be an \( H \)-cocompact convex subcomplex. Let \( \mathcal{P} \) be the poset of \( H \)-invariant convex subcomplexes in \( Y \). For the first assertion, by the Kuratowski–Zorn Lemma, it suffices to show that every descending chain of elements \( \{Y_\lambda\}_{\lambda} \in \mathcal{P} \) has a lower bound, or equivalently that their intersection is nonempty. Let \( K \subseteq Y \)
be compact such that \( HK = Y \). Then each \( K \cap Y_{\lambda} \) is nonempty, and by compactness of \( K \) so is their intersection.

For the second and third assertion, let \( Y_{\min} \subseteq Y \) be a minimal element of \( P \) and let \( Y' \) be any other minimal \( H \)-invariant convex subcomplex. Let \( (V, V') = \mathcal{G}(Y_{\min}, Y') \). Then both \( V \) and \( V' \) are \( H \)-invariant. By Lemma 2.6(1) both \( V \) and \( V' \) are convex subcomplexes, hence from minimality of \( Y_{\min} \) and \( Y' \) we have \( V = Y_{\min} \) and \( V' = Y' \). Moreover, by Lemma 2.6(2) we have that \( Y' \) is \( H \)-equivariantly isometric to \( Y_{\min} \) and thus it is \( H \)-cocompact.

**Theorem 3.8.** Let \( X \) be a locally finite \( \text{CAT}(0) \) cube complex of asymptotic rank \( n \). Let \( H \leq \text{Aut}(X) \) be a subgroup satisfying

1. \( H = H_1 \times \cdots \times H_n \), where each \( H_i \) is an infinite hyperbolic group, and
2. for some (hence any) point \( x \in X \) the orbit map \( h \mapsto h \cdot x \) from \( H \) to \( X \) is a quasi-isometric embedding.

Then \( H \) is convex cocompact. More precisely, if among \( H_i \) exactly \( \{H_i\}_{i=1}^m \) are not virtually \( \mathbb{Z} \), then there is a convex subcomplex \( Y \subseteq X \) with a cubical product decomposition \( Y = Y_0 \times \prod_{i=1}^m Y_i \) such that

1. \( Y \) is \( H \)-cocompact, and the action \( H \acts Y \) respects the product decomposition, and
2. the induced action of \( \prod_{i=1}^m H_i \) on \( Y_0 \) is proper and cocompact, in particular \( Y_0 \) is quasi-isometric to \( \mathbb{R}^{n-m} \), and
3. for any pair \( i \neq j \) with \( 1 \leq j \leq m \) and \( 1 \leq i \leq n \), the induced action \( H_i \acts Y_j \) is almost trivial, i.e., by isometries at uniformly bounded distance from the identity.

In the proof we need the notion of coarse intersection. Let \( X \) be a metric space and let \( N_R(Y) \) be the \( R \)-neighbourhood of a subspace \( Y \subseteq X \). A subspace \( V \subseteq X \) is the coarse intersection of \( Y_1 \) and \( Y_2 \) if \( V \) is at finite Hausdorff distance from \( N_R(Y_1) \cap N_R(Y_2) \) for all sufficiently large \( R \). For example, in Lemma 2.6, in view of its part (3), the gates \( V_1, V_2 \) are the coarse intersections of \( Y_1 \) and \( Y_2 \). However, in general the coarse intersection of two subsets might not exist.

**Lemma 3.9 (MSW11 Lemma 2.2).** Let \( X \) be a finitely generated group with word metric. Then the coarse intersection of a pair of subgroups is well-defined and represented by their intersection.

See [MSW11] Chapter 2 for more discussion on coarse intersection.

**Proof of Theorem 3.8.** We first prove that \( H \) is convex cocompact, which we do by the induction on \( m \). Consider first the case \( m = 0 \). By [Hag07], \( H \) acts on \( X \) by semi-simple isometries. By the Flat Torus Theorem [BH99 Chapter II.7], \( H \) acts cocompactly on an \( n \)-flat \( F \subset X \). By Theorem 3.4, the combinatorial convex hull \( Y \) of \( F \) is at finite Hausdorff distance from \( F \). Since \( X \) is locally finite, \( Y \) is \( H \)-cocompact, as desired.

Suppose now that \( m \geq 1 \). Let \( H' = \prod_{i=m}^n H_i \). We first prove that the group \( H' \) is convex cocompact. Choose a subgroup \( Z \leq H_m \) isomorphic to \( \mathbb{Z} \) and choose \( h \in H_m \) such that the coarse intersection of \( hZ \) and \( Z \) is bounded. Let \( G = H' \times Z \subset H \). By induction assumption, there exists a \( G \)-cocompact convex subcomplex \( U \subset X \). Let \( V \subset U \) be the gate with respect to \( h \cdot U \). Note that both \( U \) and \( h \cdot U \) are \( H' \)-invariant, so \( V \) is \( H' \)-invariant. By Lemma 2.6(3), \( V \) is the coarse intersection of \( U \) and \( h \cdot U \). Hence, by Lemma 3.9 applied to \( G \) and \( hGh^{-1} \), the action \( H' \acts V \) is cocompact.
By Lemma 3.7 there exists a minimal $H'$-cocompact convex subcomplex, for which we keep the notation $V$. Then for any $h \in H_m$, the translate $h \cdot V$ is minimal $H'$-invariant, hence parallel to $V$ by Lemma 3.7. Let $P_V = V \times V^1$ be the combinatorial parallel set of $V$ (see Lemma 2.9). We have that $P_V$ is $H$-invariant. Moreover, since $V$ is $H'$-invariant, there are induced actions $H \sim V^1$ and $H_m \sim V^1$.

Choose a point $v \in V$. Let $\psi: H_m \to V^1$ be the composition of the orbit map $h \mapsto h \cdot v$ with the coordinate projection. We claim that $\psi$ is a quasi-isometric embedding. This follows from assumption (2) and the estimates below, where $\sim$ means equality up to a uniform multiplicative and additive constant. Namely, for any $h_1, h_2 \in H_m$ we have:

$$d_{H_m}(h_1, h_2) \sim d_H(h_1 H', h_2 H') \sim d_X(h_1 \cdot V, h_2 \cdot V) = d_{V^1}(\psi(h_1), \psi(h_2))$$

By Theorem 2.3 since $V$ contains an isometrically embedded copy of $\mathbb{R}_{n-1}$, the asymptotic rank of $V^1$ is $\leq 1$, and hence $V^1$ is hyperbolic. Let $V_m \subseteq V^1$ be the combinatorial convex hull of $\psi(H_m)$. Then $d_{\text{Haus}}(V_m, \psi(H_m)) < \infty$ by Theorem 3.3. Moreover, $V_m$ is $H$-invariant under the action $H \sim V^1$ since $\psi(H_m)$ is invariant under $H$. Thus $H$ acts cocompactly on the convex subcomplex $V \times V_m \subseteq P_V$. Notice that since $H' \sim \psi(H_m)$ is trivial, the action $H' \sim V_m$ is almost trivial.

By now we already know that $H$ is convex cocompact. As for properties (i)—(iii), if $m = 1$, then it suffices to take $Y_0 = V$ and $Y_1 = V_1$. If $m \geq 2$, to obtain the required decomposition, we consider $X' = V \times V_m$, $H'' = \prod_{i \in (m-1)} H_i$ and we repeat the previous argument. This gives rise to an $H$-cocompact convex subcomplex $V' \times V_{m-1} \subseteq V \times V_m$, where $V'$ is a minimal $H''$-cocompact convex subcomplex. Since $V_m$ is contained in some $R$-neighbourhood of a $V'$, the intersection $V_{m-1} \cap V_m$ is compact. Moreover, $V'$ and $V_{m-1}$ admit cubical product decompositions $V' = (V' \cap V) \times (V' \cap V_m)$ and $V_{m-1} = (V_{m-1} \cap V) \times (V_{m-1} \cap V_m)$, thus $V' \times V_{m-1} = (V' \cap V) \times (V' \cap V_m) \times (V_{m-1} \cap V) \times (V_{m-1} \cap V_m)$. The $H$-action respects the above decomposition. Moreover, the induced action $H' \sim (V' \cap V_m)$ is almost trivial and the induced action $H'' \sim (V_{m-1} \cap V)$ is almost trivial. If $m = 2$, then we take $Y_1 = V_1 \cap V$, $Y_2 = V' \cap V_2$, and $Y_0 = (V \cap V') \cup (V_1 \cap V_2)$. If $m \geq 3$, then we let $X'' = V' \times V_{m-1}$, $H''' = \prod_{i \in (m-2)} H_i$ and we repeat the previous process to obtain further product decomposition. We run this process $m$ times, obtaining the required decomposition as the result of the last step. In each step, we possibly get nontrivial compact factors similar to $V_{m-1} \cap V_m$. We absorb all these compact factors into the factor $Y_0$ (we can also discard them).

4. Artin groups

4.1. Background on Artin groups. Let $A$ be an Artin group with defining graph $\Gamma$, and generators $S$. Let $W$ be the Coxeter group defined by $\Gamma$. For any $T \subseteq S$ let $W_T$ (respectively $A_T$) be the special subgroup of $W$ (respectively $A$) generated by $T$. The special subgroup $W_T$ is naturally isomorphic to the Coxeter group defined by the subgraph $\Gamma_T$ induced on $T$ [Bou68]. Similarly, by [vdLS3] the special subgroup $A_T$ of $A$ is naturally isomorphic to the Artin group defined by $\Gamma_T$.

Lemma 4.1 ([CP14 Theorem 1.1]). Special subgroups of Artin groups are convex with respect to the word metric defined by standard generators.

A subset $T \subseteq S$ is spherical if the special subgroup $W_T$ is finite. The dimension of the Artin group $A$ is the maximal cardinality of a spherical subset of $S$.

The following is a consequence of [CD95b] and [CD95a Corollary 1.4.2].
Theorem 4.2. Let $A$ be an Artin group of dimension $n$. Suppose that
(A) $n \leq 2$, or
(B) every clique $T$ in $\Gamma$ is spherical.
Then there is a finite $n$-dimensional cell complex that is a $K(A,1)$.

4.2. Two-generator Artin groups. We start with the description of most two-generator Artin groups as virtually $F_k \times \mathbb{Z}$, where $F_k$ is the free group with $k$ generators.

Lemma 4.3. Let $A$ be an Artin group with defining graph $\Gamma$ a single edge labelled by $n > 2$. Then
(1) $A$ has a finite index subgroup of form $F_k \times \mathbb{Z}$ with $k \geq 2$, and
(2) no power of one of the two standard generators lies in the $\mathbb{Z}$ factor.

Proof. By [BM00] (or by our proof of Theorem 5.1) $A$ acts freely and cocompactly on a product of a tree and a line, where a central element acts as a translation in the line factor. By [BH99, Theorem II.6.12] $A$ virtually decomposes as $A' \times \mathbb{Z}$. The induced action of $A'$ on the tree factor has finite vertex stabilisers so by Bass-Serre theory $A'$ is a graph of finite groups, in particular $A'$ is virtually free, justifying (1). Part (2) follows from the fact that standard generators act hyperbolically on the tree factor. □

Throughout this section by $\bar{x}$ we denote the inverse of $x$. By $x^z$ we denote the conjugate $\bar{z}xz$.

Let $A_n = \langle a, b \mid aba\ldots = bab\ldots \rangle$. Denote $aba\ldots = bab\ldots$ by $\Delta$. Let $A'_n$ be the kernel of the homomorphism sending each generator to the generator of $\mathbb{Z}/2$ i.e. the subgroup consisting of all words of even length. The group $A'_n$ is generated by the elements: $r = ab, s = \bar{a}b, t = \bar{s}r, q = ba = \bar{s}r\bar{t}$. Since any word of even length can be written as a product of these elements and their inverses. If $\phi$ is a word in an alphabet $\Lambda$, and $x \in \Lambda$, then we denote by $\text{Exp}_x(\phi)$ the sum of all the exponents at $x$ in $\phi$.

By direct computation we immediately establish the following:

Lemma 4.4. If $n$ is odd, then the conjugation by $\Delta$ is an order two automorphism sending $s \mapsto \bar{s}, t \mapsto \bar{t}, r \mapsto q$, where $q = ba = \bar{s}r\bar{t}$. In particular, $\Delta^2$ is a central element.

If $n$ is even, then $\Delta$ is a central element.

Let $z$ be the element $\Delta^2$ for $n$ odd and the element $\Delta$ for $n$ even.

Lemma 4.5. If $n$ is odd, then we have
$$b^n = \phi(s, t, r)\Delta,$$
where $\text{Exp}_r(\phi) = 0$.

Proof. Consider the following word $\phi$ expressed as a product of terms indexed by decreasing $i$:
$$\phi(s, t, r) = \bar{s} \prod_{i=0}^{n-2} \bar{t}^i$$
Since $r^i$ appear in the expression defining $\phi$ only as elements that we conjugate by, we have $\text{Exp}_r(\phi) = 0$. 
To verify that $b^n = \phi \Delta$, note that

$$
\phi = \bar{s} \prod_{i=1}^{0} \bar{r}^i \bar{t}^i = \bar{s} (\bar{r}^{\frac{m_2}{2}} \bar{t} r \bar{t}^{\frac{m_2}{2}}) (\bar{r}^{\frac{m_2}{2}} - 1 \bar{t} r \bar{t}^{\frac{m_2}{2}} - 1) \ldots (\bar{r} \bar{t} r) \bar{t} = \bar{s} \bar{r}^{\frac{m_2}{2}} (\bar{r}^{\frac{m_2}{2}} - 1).
$$

Since $\bar{s} \bar{r}^{\frac{m_2}{2}} = b \bar{a}(\bar{b} \bar{a}) = b \Delta$ and $r \bar{t} \Delta = \Delta q t = \Delta b^2$, we have

$$
\phi(s, t, r) \Delta = \bar{s} \bar{r}^{\frac{m_2}{2}} \Delta b^{n-1} = b \Delta b^{n-1} = b^n.
$$

\[\square\]

**Corollary 4.6.** If $n$ is odd, we have $b^{2n} \bar{z} \in [A'_n, A'_n]$.

**Proof.** We have

$$
b^{2n} = \phi(s, t, r) \Delta \phi(s, t, r) \Delta = \phi(s, t, r) \phi(\bar{s}, \bar{t}, q) \bar{z}.
$$

Denote the word $\phi(s, t, r) \phi(\bar{s}, \bar{t}, q)$ by $\psi(s, t, r, q)$. By Lemma 4.5, we have $\text{Exp}_i(\psi) = \text{Exp}_i(\psi) = 0$. We also have $\text{Exp}_i(\psi) = \text{Exp}_i(\psi) = 0$ since the total exponents of $s$ and $t$ in $\phi(s, t, r)$ are equal to the total exponents of $s$ and $t$ in $\phi(\bar{s}, \bar{t}, q)$, respectively. Thus $\psi \in [A'_n, A'_n]$.

\[\square\]

Corollary 4.6 does not hold for $n$ even, since in that case $\Delta$ is a central element.

### 4.3. Surface lemma.

The following lemma will allow us to utilise the preceding result when discussing finite index subgroups of $A_n$.

**Lemma 4.7.** Let $G$ be a finitely generated group and let $z \in G$ be central. Let $H$ be a finite index normal subgroup of $G$, and let $h \in H \cap z[G, G]$. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho((z) \setminus H) \neq \{0\}$, there exist a positive integer $m$ and $g \in G$ with $\rho(h^m g^q) \neq 0$.

**Proof.** Let $X$ be a presentation complex for $G$. Let $S$ be an oriented surface with connected $\partial S$ and basepoint $s \in \partial S$, mapping to $X$, such that on the level of fundamental groups $\partial S \to \partial \hat{S}$. Let $\hat{X}$ be the finite cover of $X$ corresponding to $H$ and let $\hat{S}$ be a finite cover of $S$ such that $\hat{S} \to S$ lifts to $\hat{S} \to \hat{X}$. Choose a system $\Sigma$ of nonintersecting arcs that join the basepoint of $\hat{S}$ to the other preimages of $s$, one for each of the boundary components of $\hat{S}$. Consider the surface $S'$ obtained from $\hat{S}$ by cutting along the arcs of $\Sigma$, and the mapping $S' \to \hat{X}$ that factors through $\hat{S}$. Then, as the boundary of a surface, $\partial S'$ is mapped to an element $f \in H = \pi_1(\hat{X})$ contained in $[H, H]$. The arcs of $\Sigma$ map to paths in $\hat{X}$ that project to closed paths in $X$ corresponding to some $g_i \in G$. Thus we have $f = \prod_{i=1}^{m} (h^{m_i})^{g_i} \cdot z^M$, where $m_i \geq 1$ with $M = \sum m_i$.

Since $H$ is normal, each $(h^{m_i})^{g_i}$ lies in $H$. We have $\rho(\prod_{i=1}^{m} (h^{m_i})^{g_i}) = \rho(z^M) \neq 0$. That means that there is at least one element $(h^{m_i})^{g_i}$ such that $\rho((h^{m_i})^{g_i}) \neq 0$.

\[\square\]

**Corollary 4.8.** Let $n$ be odd and let $H$ be a finite index normal subgroup of $A'_n$. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho((z) \setminus H) \neq \{0\}$, there exist a positive integer $m$ and $g \in A'_n$ such that $b^m \in H$ and $\rho((b^m)^g) \neq 0$.

**Proof.** Let $k$ be large enough so that $b^{2nk} \in H$. By Corollary 4.6, we can apply Lemma 4.7 with $G = A'_n, h = b^{2nk}$, and $z^k$ in the role of $z$.

\[\square\]
Corollary 4.9. Let $n$ be even and let $H$ be a finite index normal subgroup of $A_n$. Then for any homomorphism $\rho : H \to \mathbb{Z}$ such that $\rho((z) \cap H) \neq \{0\}$, there exist a positive integer $m$ and $g \in A_n$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in $H$ and is not mapped to 0 under $\rho$.

Proof. Let $k = \frac{n}{2}k'$ be a nonzero integer such that $a^k, b^k \in H$. Since $z^{k'} = (ab)^k$, we have

$$a^k b^k \in z^{k'}[A_n, A_n].$$

By Lemma 4.7, we have $m > 0$ and $g \in A_n$ such that $\rho((a^k b^k)^m) \neq 0$. Let $f = (a^k)^g$ and $h = (b^k)^g$. We have $(fh)^m \in f^m h^m[H, H]$. Thus $\rho(f^m h^m) \neq 0$ and so at least one of $f^m = (a^k)^g$ and $h^m = (b^k)^g$ is not mapped to 0 under $\rho$. □

5. The main theorem

In this section we prove Theorem 1.1. The implication (i)⇒(ii) is obvious.

5.1. Implication (iii)⇒(i).

Theorem 5.1. Let $A$ be an Artin group with each connected component of the defining graph:

- a vertex, or an edge, or else
- all interior edges labeled by 2 and all leaves labelled by even numbers.

Then $A$ is the fundamental group of a nonpositively curved cube complex.

Proof. We assume without loss of generality that $\Gamma$ is connected, since if $\Gamma$ has more connected components, then $A$ is the fundamental group of the wedge of the complexes obtained for its connected components.

If $\Gamma$ is a single vertex, then $A$ is the fundamental group of a circle.

If $\Gamma$ is a single edge labelled by an odd $n$, let $x = ab$. The group $A$ is then presented as $(a, x | ax^{n/2} = x^{n/2}a)$. Let $K_{n,a}$ be the cube complex described in the figure below.

On the left side we see the 1-skeleton of $K_n$ consisting of three edges labelled by $a, b, t$, and the right side indicates how to attach the unique 2-cell (subdivided into $n$ squares) along its boundary path $ab\ldots a \bar{t}b\bar{a}\ldots \bar{b} \bar{t}$. It is easy to check that the link of each of the two vertices in $K_n$ is isomorphic to the spherical join of two points with $n$ points, hence $K_n$ is nonpositively curved. By collapsing the $t$-edge we obtain the presentation complex for the standard presentation of $A$, so $\pi_1(K_n) = A$. We learned this construction from Daniel Wise.

If $\Gamma$ is a single edge labelled by an even $n$, let $x = ab$. The group $A$ is then presented as $(a, x | ax^{n/2} = x^{n/2}a)$. Let $K_{n,a}$ be the cube complex described in the figure below.
One can check that the link of the unique vertex in $K_{n,a}$ is isomorphic to the spherical join of two points with $n$ points, hence $K_{n,a}$ is nonpositively curved. It is clear that $\pi_1(K_{n,a}) = A$.

Similarly if we let $y = ba$, then $A$ can be presented as $\langle b, y | by^{n/2} = y^{n/2}b \rangle$. We define $K_{n,b}$ in a similar way. Note that the $a$-circle in $K_{n,a}$ is a locally convex subcomplex, so is the $b$-circle in $K_{n,b}$.

If $\Gamma$ contains more than one edge, then let $\Gamma' \subseteq \Gamma$ be the nonempty subgraph induced on all the vertices that have at least two neighbours. Thus the edges of $\Gamma'$ are precisely the interior edges and by the hypothesis they are labelled by 2. Hence $A_{\Gamma'}$ is a right-angled Artin group. The Salvetti complex $S(\Gamma')$ is the nonpositively curved cube complex obtained from the presentation complex of $A_{\Gamma'}$ by adding the missing cubes of higher dimension (see [Cha07]). Let $\{(s_i, t_i)\}_{i=1}^k$ be the collection of leaves of $\Gamma$ with $s_i \in \Gamma'$. Let $n_i$ be the label of the edge $(s_i, t_i)$, which is even. Let $K$ be the amalgamation of $\{K_{n_i,s_i}\}_{i=1}^k$ and $S(\Gamma')$ along the $s_i$-circles. Then $\pi_1(K) = A$ and it follows from [BH99, Proposition II.11.6] that $K$ is nonpositively curved. □

5.2. Implication (ii)⇒(iii).

**Theorem 5.2.** Let $A$ be a 2-dimensional Artin group. If $A$ is virtually cocompactly cubulated, then each connected component of the defining graph of $A$ is either

- a vertex, or an edge, or else
- all its interior edges are labeled by 2 and all its leaves are labelled by even numbers.

**Proof.** Suppose that there exists a finite index subgroup $\hat{A} \leq A$ that acts properly and cocompactly by combinatorial automorphisms on a CAT(0) cube complex $X$. Without loss of generality, we assume that $\hat{A}$ is normal in $A$. It suffices to prove:

1. no edge of $\Gamma$ has an odd label, unless it is an entire connected component, and
2. no interior edge of $\Gamma$ has an even label $\geq 4$.

Let us first prove (1). Suppose to the contrary that $\Gamma$ has an edge $(a, b)$ with odd label and another edge $(b, c)$. Let $A_{ab}$ be the special subgroup generated by $a$ and $b$, and let $A'_{ab}$ be its index-two subgroup from the previous section. Let $A_{ab} = F_k \times Z$ be a finite index subgroup of $A'_{ab} \cap \hat{A}$ guaranteed by Lemma 4.3(1). We can also assume that $\hat{A}_{ab}$ is normal in $A'_{ab}$. Similarly, let $A_{bc}$ be the special subgroup generated by $b$ and $c$, and let $\hat{A}_{bc} = F_l \times Z$ be a finite index subgroup of $A_{bc} \cap \hat{A}$. Note that the edge $(b, c)$ might be labelled by 2 and then $l = 1$.

Since $\hat{A}$ is a CAT(0) group, we can speak of its asymptotic rank. By Theorem 4.2(A), there exists a finite 2-dimensional cell complex that is a $K(A, 1)$. Thus by Lemma 2.5 the asymptotic rank of $\hat{A}$ is $\leq 2$ and so is the asymptotic rank of $X$. The subgroup $A_{ab}$ is convex with respect to the standard generators of $A$ by Lemma 4.1 and so $\hat{A}_{ab}$ is quasi-isometrically embedded in $\hat{A}$. We can thus apply
Theorem 3.8 to find a convex subcomplex $Y_{ab}$ that is $\hat{A}_{ab}$-cocompact. Moreover, there is a cubical product decomposition $Y_{ab} = V_{ab} \times H_{ab}$ such that the action of $\hat{A}_{ab}$ respects this decomposition, the vertical factor $V_{ab}$ is quasi-isometric to $\mathbb{R}$, and the $Z$ factor $Z$ of $\hat{A}_{ab}$ acts almost trivially on $H_{ab}$.

Consider $\text{Min}(Z) = \mathbb{R} \times V_0 \subseteq V_{ab}$ for the induced action of $Z$, where $\mathbb{R}$ is an axis of $Z$. Since $Z$ is contained in the centre of $\hat{A}_{ab}$, we have an induced action of $\hat{A}_{ab}$ on $\mathbb{R} \times V_0$ respecting this decomposition. The factor $V_0$ is bounded, so $V_0$ contains a fixed-point of the action of $\hat{A}_{ab}$. Thus $\mathbb{R} \times V_0$ contains an $\hat{A}_{ab}$-invariant line $l$. Let $\rho : \hat{A}_{ab} \rightarrow \text{Isom}(l)$ be the induced map. Note that $\rho(\hat{A}_{ab})$ does not flip the ends of $l$. Moreover, since $V_{ab}$ is a cube complex, the translation lengths on $l$ are discrete. This gives rise to a homomorphism $\rho : \hat{A}_{ab} \rightarrow Z$ assigning to each element of $\hat{A}_{ab}$ its translation length on $l$. Note that $\rho(Z) \neq 0$. By Corollary 4.8 applied to $H = \hat{A}_{ab}$, there exists a nonzero integer $m$ and $g \in \hat{A}_{ab}$ such that $\rho((b^m)^g) \neq 0$.

By normality of $\hat{A}$, we have $\hat{A}_{bc} \subseteq \hat{A}$. Let $Y_{bc}$ be a convex $(\hat{A}_{bc})^g$-cocompact subcomplex guaranteed again by Theorem 3.8. By [vdLS3] we have $A_{ab} \cap A_{bc} = A_b$, and hence the groups $(b^m)^g$ and $\hat{A}_{ab} \cap (\hat{A}_{bc})^g$ have a common finite index subgroup $B$. Let $Y \subseteq Y_{bc}$ be the gate with respect to $Y_{bc}$. Then $Y$ is the coarse intersection of $Y_{ab}$ and $Y_{bc}$ by Lemma 2.6(3). By Lemma 3.9 $Y$ is $B$-cocompact.

Since $Y$ is a convex subcomplex, it has a product structure $Y = Y_V \times Y_H$ where $Y_V \subseteq V_{ab}$ and $Y_H \subseteq H_{ab}$. We have $\rho(B) \neq 0$, so $Y_V$ is unbounded. Since $Y$ is quasi-isometric to $\mathbb{R}$, the factor $Y_H$ is bounded. Since $Z$ acts almost trivially on $H_{ab}$, any of its orbits in $Y_{ab}$ is at a finite Hausdorff distance from $Y$. Hence $Z$ is commensurable with $B$. Thus there exists an integer $j \neq 0$ such that $(b^j)^g \in Z$, and hence $b^j \in Z$, contradicting Lemma 1.3(2).

Let us now prove (2). Suppose that $\Gamma$ has edges $(a, b), (b, c)$, and $(c', a)$ (here $c$ and $c'$ are possibly the same), where $(a, b)$ has an even label $\geq 4$. Let $\hat{A}_{ab}, \hat{A}_{bc}, \hat{A}_{c'a}$ be finite index subgroups of $A_{ab} \cap A, A_{bc} \cap A, A_{c'a} \cap A$, respectively, that are isomorphic to a product of a free group and $Z$. Assume moreover that $\hat{A}_{ab}$ is normal in $A_{ab}$. Let $Y_{ab} = V_{ab} \times H_{ab}$ be a convex $\hat{A}_{ab}$-cocompact subcomplex, and let $\rho : A_{ab} \rightarrow Z$ be defined as before. By Corollary 4.9 there exist a nonzero integer $m$ and $g \in A_{ab}$ such that at least one of $(a^m)^g$ and $(b^m)^g$ lies in $\hat{A}_{ab}$ and is not mapped to 0 under $\rho$. Without loss of generality we can assume $\rho((b^m)^g) \neq 0$. The rest of the argument is identical as in the proof of (1).

6. 3-generaTOR ARTIN GROUPS

This section is devoted to the proof of Theorem 1.2. Let $A$ be the three-generator Artin group with $m_{ab} = 3, m_{bc} = 2$, and $m_{ac} = 3, 4, 5$, and let $W$ be the Coxeter group with the same defining graph. Consider a longest word in $a, b, c$ which is a minimal length representative of the element it represents in $W$. This word represents also an element of $A$, which we call $\Delta$.

**Lemma 6.1.** (i) The centre $Z$ of $A$ is generated by $\Delta^2$ for $m_{ac} = 3$ and by $\Delta$ for $m_{ac} = 4$ or 5.

(ii) The intersections of $A_{ab}$ and $A_{bc}$ with $Z$ are trivial.

(iii) In $A$ we have $A_{ab} \times Z \cap A_{bc} \times Z = A_b \times Z$.

**Proof.** Assertion (i) follows from [Del72, Theorem 4.21].

For (ii), let $\Delta_{ab} = a b a$. By [Del72, Proposition 4.17], each element of $A_{ab}$ is represented by $\Delta_{ab}^k \phi(a, b)$, where $\phi$ is a positive word in $a, b$, and $k \geq 0$. If we had
\( \phi(a, b) = \Delta_{ab}^k \Delta^1 \) for some \( l > 0, k \geq 0 \), then by [Del72] Theorem 4.14] this equality would also hold in the Artin semigroup, contradicting the fact that \( \Delta \) is expressed as a positive word involving all \( a, b, c \). The same argument works for \( A_{bc} \).

For (iii) we need to show \( A_{ab} \times Z \cap A_{bc} \times Z \subseteq A_b \times Z \). Since \( b \) and \( c \) commute, it suffices to show that for each \( m \neq 0 \) we have \( c^m \notin A_{ab} \times Z \). If \( m_{ac} = 3 \), then this follows from a well known fact that \( A/Z \) is the mapping class group of the four punctured disc, where \( A_{ab} \) fixes a curve around the first three punctures and \( c \) is a half-Dehn twist in a curve around the third and the fourth.

If \( m_{ac} = 4 \) or \( 5 \), assume for contradiction that \( c^m = gz \), for some \( z \in Z \) and \( g \in A_{ab} \). Thus \( gc^m = g^2 z = g z g = c^m g \). Let \( g = \Delta_{ab}^k \phi(a, b) \), where \( \phi \) is a positive word in \( a, b, k \geq 0 \) is even. Thus \( \phi(a, b) c^m \Delta_{ab}^k = \Delta_{ab}^k c^m \phi(a, b) \).

By [Del72] Theorem 4.14] this equality also holds in the Artin semigroup. The relation \( acac = caca \) or \( acaca = cacac \) involves on each side 2 occurrences of \( c \) separated by an occurrence of \( a \). The word \( \phi(a, b) c^m \Delta_{ab}^k \) does not contain such a subword, and this property is invariant under the replacements \( bc = cb \), \( aba = bab \). Thus to pass from \( \phi(a, b) c^m \Delta_{ab}^k \) to \( \Delta_{ab}^k \phi(a, b) \) one can only use \( bc = cb \), and \( aba = bab \), which is the relation defining \( A_{ab} \). Thus there is \( l \) such that in \( A_{ab} \) we have \( \phi(a, b) b^l = \Delta_{ab}^k \).

Hence \( g = b^{-l} \). Thus \( c^m = b^{-l} z \), contradicting assertion (ii).

We also need the following consequence of rank-rigidity [CS11].

**Lemma 6.2.** Let \( G \) be a cocompactly cubulated group with centre containing \( Z \cong \mathbb{Z} \). Then \( G \) has a finite index subgroup \( G_0 \times Z \) with \( G_0 \) cocompactly cubulated.

**Proof.** Suppose that \( G \) acts properly and cocompactly by cubical automorphisms on a CAT(0) cube complex \( X \). By [CS11 Corollary 6.4(iii)], if we replace \( X \) with its essential core, and \( G \) with a finite-index subgroup, we obtain a cubical product decomposition of \( X \) respected by \( G \), such that for each factor there is an element of \( g \in G \) acting on it as a rank one isometry. Let \( X_V \) be a factor on which \( Z \) acts freely, and combine all other factors into \( X_H \), so that \( X = X_H \times X_V \).

Note that the generator \( z \) of \( Z \) acts on \( X_V \) as a rank one isometry. Otherwise an axis of \( g \) as above would not be parallel to an axis of \( z \). Hence \( g \) and \( z \) would generate \( \mathbb{Z}^2 \) acting properly on \( X_V \), contradicting the fact that \( g \) has rank one. Consider \( \text{Min}(Z) = \mathbb{R} \times Y \subseteq X_V \), where \( \mathbb{R} \) is an axis of \( Z \). Since \( Z \) is contained in the centre of \( G \), we have an induced action of \( G \) on \( \mathbb{R} \times Y \) respecting this decomposition. Since \( z \) has rank one, we have that \( Y \) does not contain a geodesic ray, and hence is bounded. Consequently, \( Y \) contains a fixed-point of the action of \( G \). Thus \( X_V \) contains a \( G \)-invariant line \( l \).

Let \( \rho : G \to \text{Isom}(l) \) be the induced map. Note that \( \rho(G) \) does not flip the ends of \( l \). Moreover, since \( X_V \) is a cube complex, the translation lengths on \( l \) are discrete. Thus the image of \( \rho \) can be identified with \( \mathbb{Z} \), which contains \( \rho(Z) \) as a finite index subgroup. Let \( G_0 = \ker(\rho) \). Thus \( Z \times G_0 \) is a finite index subgroup of \( G \). Moreover, \( G_0 \) acts properly by cubical automorphisms on \( X_H \subset X \). Since the action of \( Z \) on \( X_V \) is proper, the action of \( G_0 \) on \( X_H \) is cocompact.

We complement Lemma 6.2 with the following:

**Lemma 6.3.** Let \( G = G_0 \times Z \) be finitely generated, with \( Z \cong \mathbb{Z} \). Let \( H < G \) be a finite product of finitely generated free groups of rank \( \geq 2 \) that is quasi-isometrically embedded.

(i) The map \( H \to G/Z \) is a quasi-isometric embedding.
(ii) Let $G$ be cocompactly cubulated. If we require that $H \cap Z$ is trivial, then assertion (i) holds also if in the product we allow free groups of rank 1.

Proof. If $H$ is a free group of rank $\geq 2$, then we choose in $H$ a free generating set $S^\ast$. In $Z$ we consider the generating set $\{ \pm 1 \}$ and in $G_0$ any symmetric generating set. Let $|\cdot|_H$, $|\cdot|_Z$, $|\cdot|_{G_0}$ denote the corresponding word-lengths. Let $\pi_{G_0}, \pi_Z$ be the coordinate projections from $G$ to $G_0, Z$, respectively. By assumption, there exists a constant $c$ such that for any $h \in H$, we have $|h|_H \leq c(|\pi_{G_0}(h)|_{G_0} + |\pi_Z(h)|_Z)$. Viewing $h$ as a reduced word over $S^\ast$, choose $s \in S^\ast$ such that the word $w = hsh^{-1}s^{-1}$ is reduced. Then $|\pi_Z(w)|_Z = 0$, and applying the above inequality with $w$ in place of $h$ we obtain $2|h|_H + 2 \leq c|\pi_{G_0}(w)|_{G_0} \leq 2c(|\pi_{G_0}(h)|_{G_0} + |\pi_{G_0}(s)|_{G_0})$. Consequently $|h|_H \leq c|\pi_{G_0}(h)|_{G_0} + a$ for some uniform constant $a$, and thus the restriction of $\pi_{G_0}$ to $H$ is a quasi-isometric embedding, as desired.

Similarly, if $H$ is a product of free groups $H_i$ of rank $\geq 2$, then we choose generating sets $S_i^\ast$ in $H_i$. Let $h = \prod h_i$, with $h_i \in H_i$. To get an estimate on $|h|_H$, it suffices to use a product of reduced words $w = \prod h_i s_i h_i^{-1}s_i^{-1}$, with $s_i \in S_i^\ast$. This proves assertion (i).

If $G$ is cocompactly cubulated, then by Lemma 6.2 after passing to a finite index subgroup, the quotient $G/Z$ acts properly and cocompactly on a CAT(0) cube complex $X$. Let $H = \mathbb{Z}^n \times H_0 \leq G$, where $H_0$ is a finite product of finitely generated free groups of rank $\geq 2$. We keep the notation $H$ for the isomorphic image of $H$ in $G/Z$. Then $H$ preserves $\text{Min}(\mathbb{Z}^n) = \mathbb{R}^n \times Y \subseteq X$ and respects its product structure. We fix $v \in \mathbb{R}^n$ and $y \in Y$. From assertion (i), the orbit map $h_0 \mapsto (h_0 \cdot v, h_0 \cdot y)$ from $H_0$ to $\mathbb{R}^n \times Y$ is a quasi-isometric embedding. Since the commutator of $H_0$ acts trivially on the $\mathbb{R}^n$ factor, using the same argument as for assertion (i), we obtain $c$ satisfying $|h_0|_{H_0} \leq c \text{d}_Y (y, h_0 \cdot y)$. On the other hand, there is $c'$ such that for $f \in \mathbb{Z}^n$ we have $|f|_{\mathbb{Z}^n} \leq c' \text{d}_\mathbb{R}(v, f \cdot v)$. Let $d$ be the maximum of the displacements $d_\mathbb{R}(v, s \cdot v)$ over the generators $s$ of $H_0$. For $f h_0 \in H$ consider the supremum norm $\| fh_0 \| = \sup\{|f|_{\mathbb{Z}^n}, 2c d |h_0|_{H_0}\}$. If $|f|_{\mathbb{Z}^n} \geq 2c d |h_0|_{H_0}$, then

$$c'd_\mathbb{R}(v, fh_0 \cdot v) \geq |f|_{\mathbb{Z}^n} - c' |h_0|_{H_0} \geq \frac{1}{2} |f|_{\mathbb{Z}^n} \geq \frac{1}{2} \| fh_0 \|.$$ 

Otherwise, if $|f|_{\mathbb{Z}^n} < 2c' d |h_0|_{H_0}$, then

$$c' \text{d}_Y (y, fh_0 \cdot y) = c \text{d}_Y (y, h_0 \cdot y) \geq |h_0|_{H_0} > \frac{1}{2c' d} \| fh_0 \|.$$ 

This proves assertion (ii). \hspace{1cm} $\square$

Proof of Theorem 1.2. The implication (i)$\Rightarrow$(ii) is obvious. The implication (iii)$\Rightarrow$(i) follows from Theorem 5.1 unless the defining graph $\Gamma$ of $A$ has two edges $(a, c), (b, c)$ with label 2. By Theorem 5.1, $A_{ab}$ is the fundamental group of a nonpositively curved cube complex $K$. Then $K \times S^1$ is a nonpositively curved cube complex with fundamental group $A$.

The implication (ii)$\Rightarrow$(iii) follows from Theorem 5.2 if $A$ is 2-dimensional. Suppose now that $A$ is not 2-dimensional. Then the labels of $\Gamma$ are $m_{ab} = 3, m_{bc} = 2$, and $m_{ac} = 3, 4, 5, 6$. Let $Z$ be the centre of $A$ described in Lemma 6.1(i).

Suppose that there exists a normal finite index subgroup $\hat{A} \leq A$ that is cocompactly cubulated. Let $\hat{Z} = \hat{A} \cap Z$. By Lemma 6.2 up to replacing $\hat{A}$ with a further
finite index subgroup, we have we have $\hat{A} = \hat{A}_0 \times \hat{Z}$, where $\hat{A}_0$ is cocompactly cubulated. We keep the notation $\hat{A}_0$ for its isomorphic image in the quotient $A/Z$. Note that $\hat{A}_0 \leq A/Z$ is a normal finite index subgroup.

By Theorem 4.2(B), the Artin group $A$ is the fundamental group of a 3-dimensional cell complex which is a $K(A, 1)$. Thus, by Lemma 2.5 the asymptotic rank of $\hat{A}$ is $\leq 3$. Hence the asymptotic rank of $\hat{A}_0$ is $\leq 2$.

By Lemma 6.1(iii), the intersections of $A_{ab}$ and $A_{bc}$ with $Z$ are trivial. Thus $A_{ab}$ and $A_{bc}$ embed into $A/Z$ under the quotient map, and we keep the notation $\hat{A}_{ab}$ and $\hat{A}_{bc}$ for their images in $A/Z$. By Lemma 6.1(iii) in $A/Z$ we have $\hat{A}_{ab} \cap \hat{A}_{bc} = \hat{A}_0$.

Let $\hat{A}_{ab} = F_k \times Z$ be a finite index subgroup of $A'_{ab} \cap \hat{A}_0$ guaranteed by Lemma 4.3(1). We can assume that $\hat{A}_{ab}$ is normal in $A'_{ab}$. Let $\hat{A}_{bc} = A_{bc} \cap \hat{A}_0 = Z^2$. By Lemmas 4.1 and 6.3(ii), $\hat{A}_{ab}, \hat{A}_{bc} < A/Z$ are quasi-isometric embeddings.

From this point we argue to reach a contradiction exactly as in part (1) of the proof of Theorem 5.2. □

References

[AB08] Peter Abramenko and Kenneth S Brown. Buildings: theory and applications. Springer Science & Business Media, 2008.

[BH99] Martin R. Bridson and André Haefliger. Metric spaces of non-positive curvature, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999.

[BHS14] Jason Behrstock, Mark F Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces I: curve complexes for cubical groups. arXiv:1412.2171, 2014.

[BM00] Thomas Brady and Jonathan P. McCammond. Three-generator Artin groups of large type are biautomatic. J. Pure Appl. Algebra, 151(1):1–9, 2000.

[Bou68] N. Bourbaki. Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.Actualités scientifiques et industrielles, No. 1337. Hermann, Paris, 1968.

[Bru92] AM Brunner. Geometric quotients of link groups. Topology and its Applications, 48(3):245–262, 1992.

[BW12] Nicolas Bergeron and Daniel T. Wise. A boundary criterion for cubulation. Amer. J. Math., 134(3):843–859, 2012.

[CD95a] Ruth Charney and Michael W. Davis. Finite $K(\pi, 1)$s for Artin groups. In Prospects in topology (Princeton, NJ, 1994), volume 138 of Ann. of Math. Stud., pages 110–124. Princeton Univ. Press, Princeton, NJ, 1995.

[CD95b] Ruth Charney and Michael W. Davis. The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups. J. Amer. Math. Soc., 8(3):597–627, 1995.

[Cha07] Ruth Charney. An introduction to right-angled Artin groups. Geometriae Dedicata, 125(1):141–158, 2007.

[CM05] Pierre-Emmanuel Caprace and Bernhard Mühlherr. Reflection triangles in Coxeter groups and biautomaticity. J. Group Theory, 8(4):467–489, 2005.

[CN05] Indira Chatterji and Graham Niblo. From wall spaces to CAT(0) cube complexes. Internat. J. Algebra Comput., 15(5-6):875–885, 2005.

[CP14] Ruth Charney and Luis Paris. Convexity of parabolic subgroups in Artin groups. Bull. Lond. Math. Soc., 46(6):1248–1255, 2014.

[CS11] Pierre-Emmanuel Caprace and Michah Sageev. Rank rigidity for CAT(0) cube complexes. Geometric and functional analysis, 21(4):851–891, 2011.

[Del72] Pierre Deligne. Les immeubles des groupes de tresses généralisés. Invent. Math., 17:273–302, 1972.
Frédéric Haglund. Isometries of cat(0) cube complexes are semisimple. arXiv:0705.3386, 2007.

Frédéric Haglund. Finite index subgroups of graph products. Geometriae Dedicata, 135(1):167–209, 2008.

Susan M. Hermiller and John Meier. Artin groups, rewriting systems and three-manifolds. Journal of Pure and Applied Algebra, 130(2):141–156, 1999.

Mark F. Hagen and Piotr Przytycki. Cocompactly cubulated graph manifolds. Israel J. Math., 207(1):377–394, 2015.

Jingyin Huang. Quasi-isometry rigidity of right-angled Artin groups I: the finite out case. arXiv:1410.8512, 2014.

Jingyin Huang. Top dimensional quasiflats in CAT(0) cube complexes. arXiv:1410.8195, 2014.

G. C. Hruska and Daniel T. Wise. Finiteness properties of cubulated groups. Compos. Math., 150(3):453–506, 2014.

Mark F. Hagen and Daniel T. Wise. Cubulating hyperbolic free-by-cyclic groups: the general case. Geom. Funct. Anal., 25(1):134–179, 2015.

Bruce Kleiner. The local structure of length spaces with curvature bounded above. Mathematische Zeitschrift, 231(3):409–456, 1999.

Jeremy Kahn and Vladimir Markovic. Immersing almost geodesic surfaces in a closed hyperbolic three manifold. Ann. of Math. (2), 175(3):1127–1190, 2012.

Yi Liu. Virtual cubulation of nonpositively curved graph manifolds. J. Topol., 6(4):793–822, 2013.

Lee Mosher, Michah Sageev, and Kevin Whyte. Quasi-actions on trees II: Finite depth Bass-Serre trees. Mem. Amer. Math. Soc., 214(1008):vi+105, 2011.

Bogdan Nica. Cubulating spaces with walls. Algebr. Geom. Topol., 4:297–309 (electronic), 2004.

Graham Niblo and Lawrence Reeves. Groups acting on CAT(0) cube complexes. Geom. Topol., 1:approx. 7 pp. (electronic), 1997.

Yann Ollivier and Daniel T. Wise. Cubulating random groups at density less than 1/6. Trans. Amer. Math. Soc., 363(9):4701–4733, 2011.

Piotr Przytycki and Daniel T. Wise. Mixed 3-manifolds are virtually special. arXiv:1205.6742v2, 2012.

Piotr Przytycki and Daniel T. Wise. Graph manifolds with boundary are virtually special. J. Topol., 7(2):419–435, 2014.

Michah Sageev. Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3):585–617, 1995.

Michah Sageev. CAT(0) cube complexes and groups. In Geometric group theory, volume 21 of IAS/Park City Math. Ser., pages 7–54. Amer. Math. Soc., Providence, RI, 2014.

Michah Sageev and Daniel T. Wise. The Tits alternative for CAT(0) cubical complexes. Bull. London Math. Soc., 37(5):706–710, 2005.

Jacek Świątkowski. Regular path systems and (bi)automatic groups. Geom. Dedicata, 118:23–48, 2006.

H. van der Lek. The homotopy type of complex hyperplane complements. Katholieke Universiteit te Nijmegen, 1983.

Stefan Wenger. The asymptotic rank of metric spaces. arXiv:0701212, 2007.

D. T. Wise. Cubulating small cancellation groups. Geom. Funct. Anal., 14(1):150–214, 2004.

Daniel T. Wise. The structure of groups with a quasiconvex hierarchy. Available at http://www.math.mcgill.ca/wise/papers.html, 2011.

Daniel T. Wise. Cubular tubular groups. Trans. Amer. Math. Soc., 366(10):5503–5521, 2014.

Daniel T. Wise and Daniel J. Woodhouse. A cubical flat torus theorem and the bounded packing property. arXiv:1510.00965, 2015.
