The shortest path problem for the distant graph of the projective line over \( \mathbb{Z} \)

by Andrzej Matraś and Artur Siemaszkowicz

**Keywords:** projective line, distant graph, shortest path problem

**AMS Subject Classification:** 51C05, 05C12

**Abstract**

The distant graph \( G = G(\mathbb{P}(R), \Delta) \) of the projective line over the ring of integers is considered. The shortest path problem in this graph is solved. In case the minimal path is non-unique all the possible splitting are described which allows to give necessary and sufficient conditions for existence of an unique shortest path.

**1 Introduction**

The projective line \( \mathbb{P}(R) \) over any ring \( R \) with 1 can be defined using admissible pairs([BHa00]). For some rings, for example for commutative ones, it is equivalent to using unimodular pairs. One of the basic concerning the projective line over ring \( \mathbb{P}(R) \) is the distant relation \( \Delta \) on the set of points. A pair \( (\bar{x}, \bar{y}) \in \mathbb{P}(R) \times \mathbb{P}(R) \) is in the distant relation if \( \det[\bar{x}, \bar{y}] \) is a unit in \( R \) for any choice \( \bar{x} \in \bar{x} \) and \( \bar{y} \in \bar{y} \). In that way one defines distant graph \( G = G(\mathbb{P}(R), \Delta) \) for any ring \( \mathbb{P}(R) \) ([BHe05]). The vertices of the distant graph on \( \mathbb{P}(R) \) are the points of \( \mathbb{P}(R) \), the edges of the graph are the undirected pairs of distant points. The distance function \( dist(\bar{x}, \bar{y}) \) is the minimal number of edges needed to walk from \( \bar{x} \) to \( \bar{y} \). Diameter of the \( G \) is the supremum of all distances between its points.

The notion of the distant relation was introduced in the nineties of the previous century (see e.g. [He95]). A. Blunck and H. Havlicek explored this...
notion later in [BHa01] and [BHa05]. They gave for instance the necessary and sufficient condition for the distant graph to be connected. Until today it is not known too much more about this class of graphs.

We consider the projective line over \( \mathbb{Z} \). In this case points of \( \mathbb{P}(\mathbb{Z}) \) are cyclic modules \( \mathbb{Z}(a, b) \), where \((a, b) \in \mathbb{Z} \times \mathbb{Z}\) is a unimodular pair. Because \( \mathbb{Z} \) is \( GE_2 \) ring ([C66]), the distant graph \( G \) is connected ([BHa01], [BHa05]) of the infinite diameter. In this paper we give a simple direct proof of this fact.

The aim of this paper is to analyse paths in the distant graph \( G(\mathbb{P}(\mathbb{R}), \Delta) \) and solve the shortest path problem for it.

The main tool in our considerations is the transition algorithm introduced in [SW11] (see also [SW14]) for bases of \( \mathbb{Z}^2 \) in order to investigate some kind of Markov partitions of the two dimensional torus. We adopt this algorithm and express it in terms of the projective line. Then we make use of it to find the shortest paths connecting given two non-adjacent vertices. It also allows us to provide the necessary and sufficient conditions for existence of the exactly one shortest path between them.

For any non-adjacent vertices \( \bar{x}, \bar{y} \in G(\mathbb{P}(\mathbb{R}), \Delta) \) we define so called cone relations on \( \mathbb{P}(\mathbb{Z}) \setminus \{\bar{x}, \bar{y}\} \) with precisely two equivalence classes. Any path entirely contained in one of this classes is called a consistent path. We prove that in each of equivalence classes of the cone relation there is precisely one shortest path connecting \( \bar{x} \) and \( \bar{y} \) (Theorem 3.1). All vertices of those two paths together with \( \bar{x} \) and \( \bar{y} \) form the set of vertices of some subgraph, called by us the Klein graph \( K(\bar{x}, \bar{y}) \).

If we abandon the assumption of consistency then the issue becomes more complicated. It turns out that in general there may be lots of shortest paths between two non-adjacent vertices. Nevertheless we prove that there is at least one shortest path which is contained in some induced subgraph of \( K(\bar{x}, \bar{y}) \) called by us a corner graph. This observation allows us to give a recipe to find one of the shortest path. Consequently we obtain the formula for the distance \( \text{dist}(\bar{x}, \bar{y}) \) between two vertices (Theorem 4.5).

Further, the careful insight into the corner graph enables to formulate the necessary and sufficient conditions for the existence of more than one shortest path (Theorem 5.3).

We would like to express our gratitude to professor Hans Havlicek for attracting our attention to the problem and fruitful discussion on the subject.

We also thank to Mariusz Bodzioch, a PhD student of our faculty, for help in preparing the figures.
2 Basic tools

Having two adjacent vertices \( \bar{e}, \bar{f} \in \mathbb{P}(\mathbb{Z}) \) we may create four bases in \( \mathbb{Z}^2 \):

\[
\{ e, f \}, \{ -e, f \}, \{ e, -f \}, \{ -e, -f \}.
\]

Every basis \( \{ e, f \} \) of \( \mathbb{Z}^2 \) defines two open cones in \( \mathbb{R}^2 \):

\[
C^+(e, f) = \{ \alpha e + \beta f : \alpha \beta > 0, \alpha, \beta \in \mathbb{R} \}
\]

\[
C^-(e, f) = \{ \alpha e + \beta f : \alpha \beta < 0, \alpha, \beta \in \mathbb{R} \}.
\]

The basic property of those bases is the following lemma proved in [SW11]:

**Lemma 2.1** (Lemma 2.1, [SW11]). Let \( \{ e, f \} \) and \( \{ e', f' \} \) be two bases in \( \mathbb{Z}^2 \). Then \( C^+(e, f) \) and \( C^+(e', f') \) do not overlap, i.e. if \( C^+(e, f) \cap C^+(e', f') \neq \emptyset \) then either \( C^+(e, f) \subset C^+(e', f') \) or \( C^+(e', f') \subset C^+(e, f) \).

The above lemma may be expressed in various ways. It is, for example, equivalent to the fact that the interior of the parallelogram spanned by members of a basis in \( \mathbb{Z}^2 \) contains no vectors from \( \mathbb{Z}^2 \). In other words the only vectors in the closed parallelogram are its vertices. By Pick’s formula this is equivalent for the parallelogram to have area equal to one. This is in turn equivalent to the equality

\[
det [e, f] = \pm 1,
\]

where \([e, f]\) stands for the matrix with columns equal to \( e \) and \( f \) respectively.

To interpret the above lemma in terms of projective line we need to define some equivalence relation on \( \mathbb{P}(\mathbb{Z}) \setminus \{ \bar{x}, \bar{y} \} \) for every distinct elements \( \bar{x}, \bar{y} \in \mathbb{P}(\mathbb{Z}) \):

\[
\bar{u} \sim_{\{\bar{x}, \bar{y}\}} \bar{v} \quad \text{if} \quad \alpha_{ux} \alpha_{uy} \alpha_{vx} \alpha_{vy} > 0,
\]

where \( \alpha_{ux}, \alpha_{uy}, \alpha_{vx}, \alpha_{vy} \in \mathbb{R} \) such that

\[
(1) \quad \bar{u} = \alpha_{ux} \bar{x} + \alpha_{uy} \bar{y}, \quad \bar{v} = \alpha_{vx} \bar{x} + \alpha_{vy} \bar{y}.
\]

Each such a relation call a cone relation.

Since each of vectors in the inequality appears twice, the relation is well defined. Every cone relation has two equivalence classes being images of

\[
C^+(x, y) \cap \mathbb{Z}^2 \quad \text{and} \quad C^-(x, y) \cap \mathbb{Z}^2
\]

by the canonical projection from the plane onto the projective line. Now Lemma 2.1 has the following interpretation.

**Lemma 2.2.** If \( \bar{x}, \bar{y}, \bar{x}', \bar{y}' \in \mathbb{P}(\mathbb{Z}) \) satisfy \( \bar{x} \triangle \bar{y} \) and \( \bar{x}' \triangle \bar{y}' \) then either \( \{\bar{x}, \bar{y}\} = \{\bar{x}', \bar{y}'\} \) or precisely one equivalence class of each cone relations \( \sim_{\{\bar{x}, \bar{y}\}} \) and \( \sim_{\{\bar{x}', \bar{y}'\}} \) is contained in the equivalence class of another cone relation.
We will need also the following obvious observation.

**Lemma 2.3.** If \( \bar{x}, \bar{y}, \bar{u} \) and \( \bar{v} \) are pairwise distinct elements of \( \mathbb{P}(Z) \) then

\[
\bar{u} \sim_{\{\bar{x}, \bar{y}\}} \bar{v} \iff \bar{x} \sim_{\{u, v\}} \bar{y}.
\]

In the graph theory the set of vertices is called a *clique* if its elements are pairwise adjacent. From Lemma 2.2 one can easily deduce the following corollary.

**Corollary 2.4.**

1. The maximal clique in \( \mathbb{P}(Z) \) has exactly three elements.
2. Every pair of adjacent vertices belongs to precisely two maximal cliques.
3. If \( \{\bar{x}, \bar{y}, \bar{u}\} \) and \( \{\bar{x}, \bar{y}, \bar{v}\} \) are two maximal cliques then

\[
\bar{u} \not\sim_{\{x, y\}} \bar{v}.
\]

It is not difficult to find maximal cliques for every pair \( \bar{x} \triangle \bar{y} \). Indeed

\[
\{\bar{x}, \bar{y}, \bar{x} + \bar{y}\} \quad \text{and} \quad \{\bar{x}, \bar{y}, \bar{x} - \bar{y}\}
\]

are the only maximal cliques containing \( \bar{x} \) and \( \bar{y} \).

Now we will express the algorithm from [SW11] in terms of the projective line and will use it in the two next section to find shortest paths connecting an arbitrary pair of vertices of the distant graph.

**TRANSITION ALGORITHM:**

Given two elements \( \bar{x}, \bar{y} \in \mathbb{P}(Z) \) fix one of them, say \( \bar{x} \).

Using the Extended Euclidean Algorithm find a sequence \( (\bar{c}_n)_{n \in \mathbb{Z}} \) of all vertices of the distant graph adjacent to \( \bar{x} \). The sequence \( (\bar{c}_n)_{n \in \mathbb{Z}} \) may be ordered in such a way that \( \{\bar{c}_n, \bar{c}_{n+1}, \bar{x}\} \) is a clique for every \( n \in \mathbb{Z} \). If \( \bar{y} \in (\bar{c}_n) \) then stop.

If not, find precisely one \( n_0 \in \mathbb{Z} \) such that \( \bar{c}_{n_0} \not\sim_{\{x, y\}} \bar{c}_{n_0+1} \). Denote \( \bar{c}_{n_0} \) and \( \bar{c}_{n_0+1} \) by \( \bar{e}_1 \) and \( \bar{f}_1 \) arbitrarily.

By 1. and 2. of Corollary 2.4 there is precisely one \( \bar{g}_2 \neq \bar{x} \) such that \( \{\bar{e}_1, \bar{f}_1, \bar{g}_2\} \) is a clique. Either
In the sequel of this paper we assume the case (TR). The other case is completely symmetric. It means that considering (TL) instead of (TR) we have to mutually exchange all notations. For instance we exchange symbols $\bar{e}_j$ and $\bar{f}_j$. Later on we introduce some numbers $r$, $a_j$, $A_j$ associated to vertices $\bar{e}_j$ and $l$, $b_j$, $B_j$ associated to vertices $\bar{f}_j$. We also must exchange them considering the case (TL) instead of (TR).

Denote $\bar{e}_2 := \bar{g}_2$. Among all adjacent to $\bar{f}_1$ vertices there are precisely two adjacent to each other that are not in the relation $\sim_{\{\bar{e}_1, \bar{y}\}}$. Moreover in the sequence of vertices adjacent to $\bar{f}_1$ there are finitely many in the relation $\sim_{\{\bar{e}_1, \bar{y}\}}$ with $\bar{e}_2$. We may naturally enumerate them: $(\bar{e}_2, \ldots, \bar{e}_{a_1+1})$ in such a way that $\bar{e}_k \Delta \bar{e}_{k+1}$. If the next vertex in this sequence and not in the relation $\sim_{\{\bar{e}_1, \bar{y}\}}$ with $\bar{e}_{a_1+1}$ is equal to $\bar{y}$ then stop.

If not, denoted the above vertex by $\bar{f}_2$ and repeat the same construction with $\bar{e}_{a_1+1}$ and $\bar{f}_2$ playing role of $\bar{f}_1$ and $\bar{e}_2$, respectively. We get the sequence $(\bar{f}_2, \ldots, \bar{f}_{b_1+1})$. Note that $\bar{f}_k \Delta \bar{f}_{k+1}$, $k = 2, \ldots, b_1$, and $\bar{f}_j \sim_{\{\bar{f}_1, \bar{y}\}} \bar{f}_2$, $j = 2, \ldots, b_1 + 1$.

Then repeat the above construction with $\bar{e}_{a_1+1}$ and $\bar{f}_{b_1+1}$ playing role of $\bar{e}_1$ and $\bar{f}_1$, respectively. The transition algorithm is depicted in Fig. 1 for $x = [2, 3]$ and $y = [-3, 2]$.

Since all lines in $\mathbb{R}^2$ determined by members of $\mathbb{P}(Z)$ are rational our algorithm eventually stops after reaching $\bar{y}$. The proof of this is specially easy if we express the transition algorithm in terms of bases of $Z^2$. It will be presented in the forthcoming paper [MS15].
We have constructed two sequences of vertices:

$$(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{A_1}, \bar{e}_{A_1+1}, \ldots, \bar{e}_{A_2}, \bar{e}_{A_2+1}, \ldots, \bar{e}_{A_{r-1}}, \bar{e}_{A_{r-1}+1}, \ldots, \bar{e}_{A_r})$$

and

$$(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{B_1}, \bar{f}_{B_1+1}, \ldots, \bar{f}_{B_2}, \bar{f}_{B_2+1}, \ldots, \bar{f}_{B_{l-1}}, \bar{f}_{B_{l-1}+1}, \ldots, \bar{f}_{B_l})$$

where $A_k = \sum_{n=0}^{k} a_n$, $k = 0, \ldots, r + 1$, and $B_k = \sum_{n=0}^{k} b_n$, $k = 0, \ldots, l + 1$. Here $a_0 = a_{r+1} = b_0 = b_{l+1} = 1$. This is caused by the fact that $\bar{e}_{A_r}$ and $\bar{f}_{B_l}$ are both adjacent to $\bar{y}$ so we also have $\bar{e}_{A_{r+1}} = \bar{f}_{B_{l+1}} = \bar{y}$.

By construction both sequences are contained in the different equivalence classes of $\sim_{\{x, y\}}$ and two consecutive vertices in each of them are adjacent.
Vertices $\bar{e}_{A_k}$ and $\bar{f}_{B_k}$, excluded $\bar{y}$, will be called corner vertices. Note that by construction $r - l \in \{0, 1\}$.

Two sequences constructed in the above algorithm may be organized as a sequence of adjacent pairs of vertices.

$$
\bar{f}_{B_m} \triangle \bar{e}_k, \ k = A_m, \ldots, A_{m+1}, \ m = 0, 1, \ldots, r - 1;
$$

$$
\bar{e}_{A_{m+1}} \triangle \bar{f}_k, \ k = B_m, \ldots, B_{m+1}, \ m = 0, 1, \ldots, l - 1.
$$

To the set of vertices defined above add $\bar{x}$ and $\bar{y}$. The resulting set with appropriate edges is an induced subgraph of the distant graph and will be called a Klein graph associated to $\bar{x}$ and $\bar{y}$. Denote it by $K(\bar{x}, \bar{y})$. If we consider only corner vertices in this graph we get an induced subgraph which we will call a corner graph and denote by $\tilde{K}(\bar{x}, \bar{y})$.

The Klein graph is depicted in Fig. 2. All vertices of the corner graph are marked in black. The bolded edges are edges of the standard path to be defined in the beginning of Section 4.

## 3 Consistent paths

**Definition 1.** A path connecting $\bar{x}$ and $\bar{y}$ in the distant graph is called consistent if all its elements but $\bar{x}$ and $\bar{y}$ are contained in the same equivalence classes of $\sim_{\{\bar{x}, \bar{y}\}}$.

The length of the shortest consistent path between $\bar{x}$ and $\bar{y}$ is denoted by $d_c(\bar{x}, \bar{y})$.

**Theorem 3.1.** Let $\bar{x}, \bar{y} \in \mathbb{P}(Z)$.

1. The union of every equivalence classes of $\sim_{\{\bar{x}, \bar{y}\}}$ and $\{\bar{x}, \bar{y}\}$ contains a unique consistent path connecting $\bar{x}$ and $\bar{y}$. The lengths of those two paths are equal to

$$
d_a(\bar{x}, \bar{y}) = 1 + A_r \quad \text{and} \quad d_b(\bar{x}, \bar{y}) = 1 + B_l,
$$

where the numbers $A_r$ and $B_l$ are taken from the transition algorithm. In particular the distant graph of $\mathbb{P}(Z)$ is connected.

2. There are at most two shortest consistent paths connecting $\bar{x}$ and $\bar{y}$ in the distant graph.

3. There is a unique consistent path connecting $\bar{x}$ and $\bar{y}$ in the distant graph if and only if

$$
d_a(\bar{x}, \bar{y}) \neq d_b(\bar{x}, \bar{y}).$$
Proof. Take \( \bar{x}, \bar{y} \in P(Z) \). Consider the path
\[
p = p(\bar{x}, \bar{y}) = \{ \bar{x} = \bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{A_r}, \bar{y} \},
\]
where vertices \( \bar{e}_1, \ldots, \bar{e}_{A_r} \) are taken from the transition algorithm described in the previous section. It is by construction consistent. Take another consistent path
\[
\hat{p} = \hat{p}(\bar{x}, \bar{y}) = \{ \bar{x} = \bar{z}_0, \bar{z}_1, \ldots, \bar{y} \}
\]
contained in the same equivalence class \( C \) of \( \sim_{\{x,y\}} \) as \( p \). We will show that \( p \subset \hat{p} \). The set \( C \setminus \{ \bar{e}_1, \ldots, \bar{e}_{A_r} \} \) is a disjoint union of equivalence classes \( C_k \) of cone relations \( \sim_{\{x_k, x_{k+1}\}} \), \( k = 0, \ldots, A_r \) (here \( \bar{e}_0 = \bar{x} \)). If \( \bar{e}_1 = \bar{z}_1 \in \hat{p} \) we are done. Otherwise \( \bar{z}_1 \in C_0 \). It follows from the fact that \( \bar{e}_1 \) is the last vertex in the sequence of adjacent to \( \bar{x} \) vertices contained in \( C \) and moreover by Lemma 2.2 all other vertices in this sequence contained in \( C \) belong to \( C_0 \). We have either \( \bar{z}_2 = \bar{e}_1 \) or, again by Lemma 2.2, \( \bar{z}_2 \in C_0 \). Eventually we get \( \bar{e}_1 = \bar{z}_k \in \hat{p} \cap p \) because we have to go out of \( C_0 \) in order to get \( \bar{y} \). Assuming \( z_k = e_1 \) we are able to show, by the same arguments as above, that \( \bar{e}_{k+1} \in \hat{p} \) which shows that \( p \subset \hat{p} \). Therefore \( p \) is a unique shortest consistent path between \( \bar{x} \) and \( \bar{y} \) contained in \( C \). When computing the length of \( p \) we have to count steps between \( \bar{e}_1 \) and \( \bar{e}_{A_r} \) whose number is \( A_r - 1 \) and add one step between \( \bar{x} \) and \( \bar{e}_1 \) and another one between \( \bar{e}_{A_r} \) and \( \bar{y} \). We have finished the proof of 1.

Now 2. and 3. immediately follows from 1.

Example 1. Let \( \bar{x} = [1, 0] \) and \( \bar{y} = [37, 158] \). Then
\[
\begin{align*}
\bar{e}_1 &= [1, 1], \quad \bar{e}_2 = [1, 2], \quad \bar{e}_3 = [1, 3], \quad \bar{e}_4 = [1, 4], \quad \bar{e}_5 = [4, 17], \quad \bar{e}_6 = [15, 64], \\
\bar{e}_7 &= [26, 111]; \\
\bar{f}_1 &= [0, 1], \quad \bar{f}_2 = [1, 5], \quad \bar{f}_3 = [2, 9], \quad \bar{f}_4 = [3, 13], \quad \bar{f}_5 = [7, 30], \quad \bar{f}_6 = [11, 47].
\end{align*}
\]
Therefore \( d_a(\bar{x}, \bar{y}) = 8 \) and \( d_b(\bar{x}, \bar{y}) = d_c(\bar{x}, \bar{y}) = 7 \). We see that
\[
p(\bar{x}, \bar{y}) = \{ \bar{x}, \bar{f}_1, \ldots, \bar{f}_6, \bar{y} \}
\]
is the unique shortest consistent path between \( \bar{x} \) and \( \bar{y} \). Notice also that \( \bar{e}_1, \bar{e}_4, \bar{e}_5, \bar{e}_7, \bar{f}_1, \bar{f}_4, \bar{f}_6 \) are all corner vertices.

Example 2. Let \( \bar{x} = [1, 0] \) and \( \bar{y} = [26, 111] \). Then
\[
\begin{align*}
\bar{e}_1 &= [1, 1], \quad \bar{e}_2 = [1, 2], \quad \bar{e}_3 = [1, 3], \quad \bar{e}_4 = [1, 4], \quad \bar{e}_5 = [4, 17], \quad \bar{e}_6 = [15, 64]; \\
\bar{f}_1 &= [0, 1], \quad \bar{f}_2 = [1, 5], \quad \bar{f}_3 = [2, 9], \quad \bar{f}_4 = [3, 13], \quad \bar{f}_5 = [7, 30], \quad \bar{f}_6 = [11, 47].
\end{align*}
\]
Therefore \( d_a(\bar{x}, \bar{y}) = d_b(\bar{x}, \bar{y}) = d_c(\bar{x}, \bar{y}) = 7 \). We see that
\[
\{ \bar{x}, \bar{e}_1, \ldots, \bar{e}_6, \bar{y} \} \quad \text{and} \quad \{ \bar{x}, \bar{f}_1, \ldots, \bar{f}_6, \bar{y} \}
\]
are the only two shortest consistent paths between \( \bar{x} \) and \( \bar{y} \).
Definition 2. A Hamiltonian cycle \( p(\bar{x}) \) containing \( \bar{x} \in \mathbb{P}(\mathbb{Z}) \) is called consistent if there is \( \bar{y} \in \mathbb{P}(\mathbb{Z}) \) such that \( p(\bar{x}) \) is a union of two consistent paths connecting \( \bar{x} \) and \( \bar{y} \).

Formally to run the transition algorithm we have to fix both vertices \( \bar{x} \) and \( \bar{y} \). But actually we may fix only \( \bar{x} \) and arbitrary finite sequences of positive integers \((a_n)\) and \((b_m)\) whose lengths differ by one. Then we may run the transition algorithm starting from the longer sequence. After using all numbers \( a_n \) and \( b_m \) we get two last corner vertices \( \bar{e}_{A_r} \) and \( \bar{f}_{B_l} \). Then let \( \bar{y} \) be such that it forms a clique with these corner vertices and \( \bar{e}_{A_r} \not\sim \{\bar{x}, \bar{y}\} \bar{f}_{B_l} \).

The above consideration gives the following.

Corollary 3.2. For every \( \bar{x} \in \mathbb{P}(\mathbb{Z}) \) and every positive integers \( d > 2 \), \( d_a \) and \( d_b \) with \( d = d_a + d_b \) there exists a Hamiltonian consistent cycle containing \( \bar{x} \) whose length is equal to \( d \). Moreover there is \( \bar{y} \in \mathbb{P}(\mathbb{Z}) \) such that

\[
d_a = d_a(\bar{x}, \bar{y}) \quad \text{and} \quad d_b = d_b(\bar{x}, \bar{y}).
\]

In particular, \( G(\mathbb{P}(\mathbb{Z}), \triangle) \) is of the infinite diameter. ■

Note that one can eliminate \( \bar{y} \) from the cycle and get again a Hamiltonian one. In this way one is able to construct decreasing sequence of Hamiltonian cycles containing \( \bar{x} \) that lengths decrease by one. Of course the shortest such a cycle has length 3 and form a clique. In the forthcoming paper it will be shown how to construct all such sequences of Hamiltonian cycles.

4 Non-consistent paths

While, as has been shown in the previous section, there are at most two shortest consistent paths between two vertices in the distant graph of projective line over \( \mathbb{Z} \), it turns out that in general you can find many shortest paths if we abandon the assumption of consistence.

In order to compute the length of a shortest path between two vertices \( \bar{x}, \bar{y} \in \mathbb{P}(\mathbb{Z}) \) first run the transition algorithm described in Section 2 assuming the case (TR). Then it is convenient to consider the following path which exhaust all vertices of the corner path:

\[
(\bar{x}, \bar{f}_1, \bar{e}_{A_1}, \bar{f}_{B_1}, \ldots, \bar{f}_{B_{r-2}}, \bar{e}_{A_{r-1}}, \bar{f}_{B_{r-1}}, \bar{e}_{A_r}, \bar{y})
\]

or

\[
(\bar{x}, \bar{f}_1, \bar{e}_{A_1}, \bar{f}_{B_1}, \ldots, \bar{f}_{B_{r-2}}, \bar{e}_{A_{r-1}}, \bar{f}_{B_{r-1}}, \bar{y}).
\]
depending on whether \( r = l \) or \( r = l + 1 \). Denote this path by \( p_s(\bar{x}, \bar{y}) \) and call it a standard path between \( \bar{x} \) and \( \bar{y} \) (see Fig. 2). Let us agree that the vertices \( \bar{e}_{A_k} \) lie on the right side of the corner graph and the vertices of \( \bar{f}_{B_k} \) lie on its left side. The direct insight into the graph \( \tilde{K}(\bar{x}, \bar{y}) \) gives the following result.

**Lemma 4.1.** The length of the standard path between not adjacent \( \bar{x} \) and \( \bar{y} \) is equal to \( r + l + 1 \) provided \( l > 0 \). If \( l = 0 \), then it is equal to 2. □

The key observation of this section is the fact that if want to make a shortest walk from \( \bar{x} \) to \( \bar{y} \) then we have to use only vertices of \( K(\bar{x}, \bar{y}) \) but we may also use only vertices of \( \tilde{K}(\bar{x}, \bar{y}) \).

**Lemma 4.2.** Every shortest path between \( \bar{x} \) and \( \bar{y} \) is a subgraph of \( K(\bar{x}, \bar{y}) \). Moreover among all the shortest paths at least one is a subgraph of \( \tilde{K}(\bar{x}, \bar{y}) \).

**Proof.** We intend to show that having a path \( \tilde{p} \) from \( \bar{x} \) to \( \bar{y} \) one can find a path from \( \bar{x} \) to \( \bar{y} \) contained in \( K(\bar{x}, \bar{y}) \) which is not longer than \( \tilde{p} \).

Let then \( \tilde{z} \in \tilde{p} \setminus K(\bar{x}, \bar{y}) \). We have either \( \tilde{z} \sim_{\{\bar{x},\bar{y}\}} \bar{e}_1 \) or \( \tilde{z} \sim_{\{\bar{x},\bar{y}\}} \bar{f}_1 \). Assuming the former case we can find precisely one \( k \) with \( \tilde{z} \in C_k \), where \( C_k \) are taken from the proof of Theorem 3.1. By Lemma 2.1 all vertices adjacent to \( \tilde{z} \) belong to \( C_k \cup \{\bar{e}_k, \bar{e}_{k+1}\} \). Therefore the predecessor as well as the successor of \( \tilde{z} \) in \( \tilde{p} \) are contained in the same set, hence \( \{\bar{e}_k, \bar{e}_{k+1}\} \subset \tilde{p} \).

Since \( \bar{e}_k \triangle \bar{e}_{k+1} \), we may shorten \( \tilde{p} \) rejecting all vertices from \( \tilde{p} \cap C_k \). We argue analogically in the latter case.

Assume now that \( \tilde{p} \) is one of the shortest paths between \( \bar{x} \) and \( \bar{y} \). We already know that \( \tilde{p} \subset K(\bar{x}, \bar{y}) \). Let \( 1 \leq m \leq r \) and \( A_m < k < A_{m+1} \) be such that \( \bar{e}_k \in \tilde{p} \). In general we have always \( \bar{e}_{A_{m+1}} \in \tilde{p} \) since we can not omit \( \bar{e}_{A_{m+1}} \in \tilde{p} \). Walking from \( \bar{x} \) to \( \bar{e}_k \) we must go through \( \bar{f}_{B_m} \) or \( \bar{e}_{A_m} \) so \( \bar{f}_{B_m} \in \tilde{p} \) or \( \bar{e}_{A_m} \in \tilde{p} \). Since \( \bar{f}_{B_m} \triangle \bar{e}_{A_{m+1}} \) and \( \tilde{p} \) is one of the shortest paths, \( \bar{f}_{B_m} \not\subset \tilde{p} \) and \( \bar{e}_{A_{m+1}} \in \tilde{p} \). We know that \( \bar{e}_{A_m} \triangle \bar{f}_{B_m} \triangle \bar{e}_{A_{m+1}} \) is a path in \( \tilde{K}(\bar{x}, \bar{y}) \) of the length 2, hence there are no other vertices between \( \bar{e}_{A_m} \) and \( \bar{e}_{A_{m+1}} \) but \( \bar{e}_k \). We may replace \( \bar{e}_{A_m} \triangle \bar{e}_k \triangle \bar{e}_{A_{m+1}} \) by \( \bar{e}_{A_m} \triangle \bar{f}_{B_m} \triangle \bar{e}_{A_{m+1}} \subset \tilde{K}(\bar{x}, \bar{y}) \).

We analogically consider \( \bar{f}_k \in \tilde{p} \) instead of \( \bar{e}_k \) which ends the proof. □

Now let us ponder the question when the standard path is one of the shortest between \( \bar{x} \) and \( \bar{y} \). Let

\[(a_1, b_1, \ldots) = (c_1, c_2, \ldots, c_{r+1}).\]
The first obvious case is \( c_2 = 0 \). Indeed, if \( c_2 = b_1 = 0 \) then \( l = 0 \) and \( \bar{x} \triangle \bar{f}_1 \triangle \bar{y} \) is the standard path of the length 2. If additionally \( c_1 = a_1 = 0 \) then \( r = 0 \) and \( \bar{x} \triangle \bar{e}_1 \triangle \bar{y} \) is the other shortest path. If \( c_2 = 0 \) and \( c_1 \neq 0 \) then \( r = 1 \) and the standard path is the only shortest one. The other case is if all numbers \( c_k, k = 1, \ldots, r + l, \) are distinct from 0 (which is equivalent to \( c_2 \neq 0 \)) and all but \( c_1 \) or \( c_{r+l} \) are greater then 1. In such a situation no pairs \((\bar{e}_{A_k}, \bar{e}_{A_{k+1}}), k = 1, \ldots, l-1, (\bar{f}_{B_k}, \bar{f}_{B_{k+1}}), k = 0, \ldots, r-2,\) are adjacent. It means that the shortest walk from \( \bar{e}_{A_k} \) to \( \bar{e}_{A_{k+1}} \) is \( \bar{e}_{A_k} \triangle \bar{f}_{B_k} \triangle \bar{e}_{A_{k+1}} \). Similarly the shortest walk between \( \bar{f}_{B_k} \) and \( \bar{f}_{B_{k+1}} \) is \( \bar{f}_{B_k} \triangle \bar{e}_{A_{k+1}} \triangle \bar{f}_{B_{k+1}} \). We have shown the following.

**Fact 4.3.** The standard path is one of the shortest ones iff

- \( c_2 = 0 \) or
- \( c_2 > 0 \) and \( c_k > 1 \) for all \( k \in \{2, \ldots, r+l-1\} \).

Note that even \( c_1 = 1 \) or \( c_{r+l} = 1 \) the standard path remains one of the shortest ones. Indeed, if \( c_1 = a_1 = 1 \) then \( \bar{e}_1 \triangle \bar{e}_2 \), hence there are two shortest paths from \( \bar{x} \) to \( \bar{e}_2 \), namely \( \bar{x} \triangle \bar{f}_1 \triangle \bar{e}_2 \) and \( \bar{x} \triangle \bar{e}_1 \triangle \bar{e}_2 \). Similarly if \( c_{r+l-2} = b_{r-1} = 1 \) then \( \bar{f}_{B_{r-1}} \triangle \bar{f}_{B_{r}} \triangle \bar{y} \) and \( \bar{f}_{B_{r-1}} \triangle \bar{e}_{A_{r}} \triangle \bar{y} \) are two shortest walks from \( \bar{f}_{B_{r-1}} \) to \( \bar{y} \) and if \( c_{r+l-2} = a_{r-1} = 1 \) then \( \bar{e}_{A_{r-1}} \triangle \bar{e}_{A_{r}} \triangle \bar{y} \) and \( \bar{e}_{A_{r-1}} \triangle \bar{f}_{B_{r}} \triangle \bar{y} \) are two shortest walks from \( \bar{e}_{A_{r-1}} \) to \( \bar{y} \).

One can ask what is the necessary and sufficient condition for the Klein graph \( K(\bar{x}, \bar{y}) \) that the standard path is the only shortest one. The above considerations immediately convince us that first of all we must have \( c_1 > 1 \) and \( c_{r+l} > 1 \). We must also have \( c_k > 1 \) for \( k = 2, \ldots, r+l-1 \) to assure that the standard path is one of the shortest ones. Assume now that \( c_k = 2 \) for some \( k \in \{2, \ldots, r+l-1\} \). If \( c_k = a_j \) then \( \bar{e}_{A_{j-1}} \triangle \bar{f}_{B_{j-1}} \triangle \bar{e}_{A_j} \) (the standard one) and \( \bar{e}_{A_{j-1}} \triangle \bar{e}_{A_{j-1}+1} \triangle \bar{e}_{A_j} \) are two shortest path between \( \bar{e}_{A_{j-1}} \) and \( \bar{e}_{A_j} \). Similarly if \( c_k = b_j \) then \( \bar{f}_{B_{j-1}} \triangle \bar{e}_{A_j} \triangle \bar{f}_{B_j} \) (the standard one) and \( \bar{f}_{B_{j-1}} \triangle \bar{f}_{B_{j-1}+1} \triangle \bar{f}_{B_j} \) are two shortest path between \( \bar{f}_{B_{j-1}} \) and \( \bar{f}_{B_j} \). We have just shown that the necessary and sufficient condition is the following.

**Fact 4.4.** The standard path is the only shortest one iff

- \( c_1 > 0, \ c_2 = 0 \ or \)
- \( c_1 > 1, \ c_{r+l} > 1 \ and \ c_k > 2 \ for \ all \ k \in \{2, \ldots, r+l-1\} \).

Let us now consider a case when the standard path is not the shortest one. We already know that this is equivalent to

\[
c_2 = b_1 \neq 0 \ and \ c_k = 1 \ for \ some \ k \in \{2, \ldots, r+l-1\}.
\]
On the other hand, by Lemma 4.2 to compute the length of the shortest path between $\bar{x}$ and $\bar{y}$ it is enough to find the shortest one contained in the corner graph associated with those two vertices.

The considerations similar to those above allow us to give a recipe for finding one of the shortest paths in $\tilde{K}(\bar{x}, \bar{y})$. First assume the following convention: walking along the standard path we say that we meet the number $a_{k+1}$ or $b_{k+1}$ if we are in the vertex $\bar{e}_A$ or $\bar{f}_B$, respectively. Then the recipe is the following: start from $\bar{x}$ and walk along the standard path until you meet the number one. Then walk along the consecutive vertices on the same side of the corner graph. Once you meet the number different from one you come back to the standard path. Perform this procedure till you reach $\bar{e}_A$ or $\bar{f}_B$. Then make one step more to end at $\bar{y}$. In the sequel the path we have just described is called a standard shortest path.

To compute the distance between $\bar{x}$ and $\bar{y}$ it is left to determine by how many steps we shortened the standard path. For the first observe that if $r = l$ then we never meet $a_1$ nor $b_l$ and if $r = l + 1$ then we never meet $a_1$ nor $a_r$. Every other appearance of the number one in the "proper" place results in shortening of the standard path by precisely one step. It possible to shorten the standard path by one step in precisely two cases:

1. $b_1 = 1$; 2. $b_k > 1$ and $a_k = 1$.

The first case always works. To decide whether 2. works or not we must check if it satisfies some additional condition. Put

$$s_k = \begin{cases} \max\{j : b_{k-1} = \ldots = b_{k-j} = 1\} : b_{k-1} = 1, \\ 0 : b_{k-1} > 1; \end{cases}$$

and

$$t_k = \begin{cases} \max\{j : a_{k-1} = \ldots = a_{k-j} = 1\} : a_{k-1} = 1, \\ 0 : a_{k-1} > 1. \end{cases}$$

Then 2. result in shortening iff $s_k \leq t_k$.

Now we are in position to determine by how many steps we shorten the standard path. To end this define

$$\tilde{a}_k = \begin{cases} a_k : k = 2, \ldots, l; \\ \max(a_k, 2) : k = 1 \end{cases}$$

and

$$\tilde{b}_k = \begin{cases} b_k : k = 1, \ldots, l - 1; \\ \max(b_k, 2) : k = l, \end{cases}$$

12
if \( r = l \). If \( r = l + 1 \) the former condition remains unchanged and the latter one becomes
\[
\tilde{b}_k = b_k \quad \text{for} \quad k = 1, \ldots, l.
\]
Now remove from \( \{1, \ldots, l\} \) those \( k \)'s that satisfy 2. with \( s_k > t_k \) and denote received set by \( D = D_{x,y} \). By the above consideration we see that we shorten the standard path by
\[
\sum_{k \in D} \left\lfloor \frac{1}{2} \left( \left\lfloor \frac{1}{a_k} \right\rfloor + \left\lfloor \frac{1}{b_k} \right\rfloor \right) + \frac{1}{2} \right\rfloor
\]
steps. The symbol \( \lfloor \cdot \rfloor \) stands for the floor function.

**Theorem 4.5.** Assume that \( \bar{x}, \bar{y} \in \mathbb{P}(Z) \) are not adjacent. Then the length of the shortest path connecting \( \bar{x} \) and \( \bar{y} \) is equal to
\[
\text{dist}(\bar{x}, \bar{y}) = \begin{cases} 
  r + l + 1 - \sum_{k \in D_{x,y}} \left\lfloor \frac{1}{2} \left( \left\lfloor \frac{1}{a_k} \right\rfloor + \left\lfloor \frac{1}{b_k} \right\rfloor \right) + \frac{1}{2} \right\rfloor : & l > 0 \\
  2 : & l = 0.
\end{cases}
\]

### 5 Uniqueness of the shortest paths

In this section we formulate necessary and sufficient conditions for existence of more than one shortest path between two vertices. In the previous section we described the construction of one of the shortest paths between two not adjacent vertices. This path may split, but need not do, into more shortest paths because of appearance of 1 or 2 in the sequence \((c_1, \ldots, c_{r+l})\). Two next lemmas explain the situation.

First we deal with ones. We say the two blocks of 1’s **overlap** if there are natural numbers \( 0 \leq l_1 \leq l_2 \leq l + 1 \) and \( 0 \leq r_1 \leq r_2 \leq r + 1 \) satisfying

(o1) \( l_1 < r_1 \leq l_2 + 1 \leq r_2 \) with \( l_2 < l + 1 \) iff \( r = l + 1 \)

or

(o2) \( l_1 < r_1 \leq l_2 = r_2 = r + 1 = l + 1 \)

or

(o3) \( 0 = r_1 = l_1 \leq l_2 < r_2 \)

or

(o4) \( r_1 \leq l_1 \leq r_2 \leq l_2 \), where \( l_1 \neq 0 \) with an additional condition \( r_2 < r + 1 \) iff \( r = l \)

or
such that
\[ b_{l_1} = b_{l_1+1} = \ldots = b_{l_2} = a_{r_1} = a_{r_1+1} = \ldots = a_{r_2} = 1. \]

**Lemma 5.1.** Appearance of 1 in the sequence \((c_1, \ldots, c_{r+l})\) results in splitting the shortest path iff two blocks of 1’s overlap or \(r = l\) and \(c_i = 1\) for all \(i\)’s.

**Proof.** First note that 1 in \((c_1, \ldots, c_{r+l})\) causes splitting of any path \(p\) contained in the Klein graph iff it appears in one of the following configurations depicted in Fig. 3:

1. \((s1)\) \(b_k = a_k = 1, 1 \leq k \leq l;\)
2. \((s2)\) \(b_k = 1, a_{k+1} = 1, \tilde{f}_k \in p.\)

In particular to split the shortest path 1 has to appear in both sides of the Klein graph. Then fix some block \(B\) of 1’s in the left side of the Klein graph: \(b_{l_1} = \ldots = b_{l_2} = 1\), where \(0 < l_1 \leq l_2 < l\). Then imagine that we “shift” some block \(A\) of 1’s in the right side: \(a_{r_1} = \ldots = a_{r_2} = 1\), where \(0 < r_1 \leq r_2 < r\). We want to assure that meeting of those two blocs results in splitting the shortest path.

Assume that \(|A| = r_2 - r_1 + 1 \leq |B| = l_1 - l_2 + 1\). Then in order to gain the configuration \((s1)\) or \((s2)\) that causes splitting we must have

\[ r_1 \leq l_2 + 1 \leq r_2 \text{ or } r_1 \leq l_1 \leq r_2. \]

Indeed, if neither of the above conditions holds then there are three possibilities. Two of them are \(l_2 + 1 < r_1\) or \(l_1 > r_2\). In this cases neither \((s1)\) nor
(s2) holds. The last case is \( r_1 > l_1 \) and \( l_2 \geq r_2 \). This forces \(|A| < |B|\) and appearance of (s1) and (s2). Nevertheless this does not cause splitting because the standard shortest path has as a part the path \( f_{B_{l_1-1}} \triangle f_{B_{l_2}} \triangle \ldots \triangle f_{B_{l_2}} \). Walking along this part we do not ”jump” onto the right side of the Klein graph although we meet 1’s in configuration (s1) and (s2). Therefore we are not able to gain splitting. Together with \(|A| \leq |B|\) the former condition gives (o1) and the latter one gives (o4).

Similarly, if \(|A| \geq |B|\) then we must have

\[ l_1 < r_1 \leq l_2 + 1 \quad \text{or} \quad l_1 \leq r_2 \leq l_2. \]

Together with \(|A| \geq |B|\) the former condition gives (o1) and the latter one gives (o4).

Until now we have assumed that our blocks neither start nor end at \( \bar{x} \) and \( \bar{y} \). Now we assume that.

Assume first \( r_1 = l_1 = 0 \). Then in order to have splitting we must have \( r_2 > l_2 \) which gives (o3). If \( l_1 = 0 \neq r_1 \) then \( l_1 < r_1 \) and necessarily \( r_1 \leq l_2 + 1 \leq r_2 \). This gives (o1). If \( r_1 = 0 \neq l_1 \) then \( l_1 > r_1 \) and necessarily \( l_1 \leq r_2 \leq l_2 \) which is (o4).

Next let \( r = l \) and \( r_2 = l_2 = r \). Then in order to have splitting we must have \( l_1 < r_1 \) which is (o2). If \( l_2 = l = r \neq r_2 \) then \( r_2 < r \) which is equivalent to \( r_2 < r + 1 \). Moreover we must have \( r_1 \leq l_1 \leq r_2 \). This gives the additional condition in (o4). If \( l_2 \neq l = r = r_2 \) then \( l_2 + 1 \leq r_2 \) and we must have \( l_1 < r_1 \leq l_2 + 1 \) which is (o4).

Now assume that \( r = l + 1 \). First assume that \( r_2 < r \). Then \( r_2 < r = l + 1 = l_2 \) and in order to guarantee splitting we must have \( r_1 \leq l_1 \leq r_2 \). This gives (o4). If \( r_2 = r + 1 \) then to obtain splitting the standard shortest path has to end in the right side of the Klein graph. Therefore if \( r_1 \leq l_1 \) then \( l_2 \leq l + 1 < r_2 = l + 2 \) and we get (o5). On the other hand if \( r_1 > l_1 \) then we must have \( r_1 \leq l_2 + 1 \) and we have to ”jump” into the right side of the Klein graph which forces \( l_2 < l + 1 \). This is the additional condition in (o1).

There is left one case. Namely, \( c_1 = \ldots = c_{r+l} = 1 \). In this case it obvious that one has splitting iff \( r = l \).

We have considered all possible configurations of two blocks of 1’ that result in splitting. Since we get all conditions (o1)-(o5) and each of them obviously forces splitting, we have finished the proof.

Another reason of splitting is appearance of 2 in this sequence. By direct insight into the graph \( K(\bar{x}, \bar{y}) \) we get the following result.

**Lemma 5.2.** Appearance of 2 in the sequence \((c_1, \ldots, c_{r+l})\) results in splitting the standard shortest path iff one of the following conditions hold
(st1) there exists $k \in \{2, \ldots, l\}$ such that $b_k = 2$, $a_k > 1$, $a_{k+1} > 1$ or $b_1 = 2$ and $a_2 > 1$.

(st2) there exists $k \in \{2, \ldots, l\}$ such that $a_k = 2$, $b_k > 1$ and $s_k \leq t_k$,

where the numbers $s_k$ and $t_k$ in the above are defined in (3) and (4).

**Proof.** Note that if $b_k = 2$ then necessary and sufficient condition to split the standard shortest path is that it contains vertices $\bar{f}_{B_k-1}$, $\bar{f}_{B_k}$ and $\bar{e}_{A_k}$ (see the left side of Fig. 4). The standard shortest path contains $\bar{f}_{B_k-1}$ iff $k > 1$ and $a_k > 1$ or $k = 1$ even if $a_1 = 1$. It necessarily contains $\bar{e}_{A_k}$ since $b_k = 2 > 1$. The standard shortest path contains $\bar{f}_{B_k}$ iff $a_{k+1} > 1$.

Now assume that $a_k = 2$. Then necessary and sufficient condition to split the standard shortest path is appearance of vertices $\bar{e}_{A_k-1}$, $\bar{f}_{B_k-1}$ and $\bar{e}_{A_k}$ (see the right side of Fig. 4). The vertex $\bar{e}_{A_0} = \bar{e}_1$ is never contained in the standard shortest path so $k$ must be greater than 1. The vertex $\bar{e}_{A_k-1}$ appears in it iff $s_k \leq t_k$. This path necessarily contains $\bar{f}_{B_k-1}$ since $a_k = 2 > 1$. Then we see that $\bar{e}_{A_k}$ appears in the standard shortest path iff $b_k > 1$.

Observe that in the both cases $k$ must be less then $r + 1$. This gives range of $k$ in the thesis of the lemma.
All discussion of this section and direct insight into the Klein graph convince us that the following statement is true.

**Theorem 5.3.** Assume that \( \bar{x}, \bar{y} \in \mathbb{P}(Z) \) are not adjacent. There are more than one shortest path between \( \bar{x} \) and \( \bar{y} \) if and only if

- two blocks of 1’s overlap or
- \( r = l \) and all numbers \( a_k \) and \( b_k \) are equal to one or
- one of the conditions of Lemma 5.2 holds.

References

[BHa00] A. Blunck, H. Havlicek, *Projective representations I: Projective lines over a ring*. Abh. Math. Sem. Univ. Hamburg 70 (2000), 287 – 299.

[BHe05] A. Blunck, A. Herzer, *Kettengeometrien-Eine Einführung*. Shaker Verlag, Aachen, 2005.

[C66] P.H. Cohn, *On the structure of the GL₂ of a ring*. Inst. Hautes Etudes Sci. Publ. Math. 30 (1966), 5 – 53.

[BHa01] A. Blunck, H. Havlicek, *The connected components of the projective line over a ring*. Adv. Geom. 1 (2001), 107 – 117.

[BHa05] ——— , *On distant-isomorphisms of projective lines*. Aequationes Math. 69 (2005), 146 – 163.

[He95] A. Herzer, Chapter 14: *Chain Geometries* in *Handbook of Incidence Geometry*, edited by F. Buekenhout. Elsevier 1995.

[SW11] A. Siemaszko, M.P. Wojtkowski, *Counting Berg partitions*. Nonlinearity 24 (2011), 2383 – 2403.

[SW14] ——— , *Counting Berg partitions via Sturmian words and substitution tilings*. European Congress of Mathematics Kraków, 2 – 7 July, 2012. European Mathematical Society Publishing House 2014, 779 – 790.

[MS15] A. Matraš, A. Siemaszko, *The shortest path problem for the distant graph of projective line over Z*. Algorithms. Preprint
Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn
Słoneczna 54, 10-710 Olsztyn, POLAND

email: matras@uwm.edu.pl, artur@uwm.edu.pl