The Indistinguishability of Quantum States is Independent of the Dimension of Quantum System

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(Dated: July 6, 2021)

The distinguishability of quantum states is important in quantum information theory and has been considered by authors. However, there were no general results considering whether a set of indistinguishable states become distinguishable by viewing them in a larger system without employing extra resources. In this paper, we consider this question for LOCC1, PPT and SEP distinguishabilities of states. We use mathematical methods to show that if a set of states is indistinguishable in \( \otimes_{k=1}^{K} C^{d_k} \), then it is indistinguishable even being viewed in \( \otimes_{k=1}^{K} C^{d_k+\rho_k} \), where \( K, d_k \geq 2, \rho_k \geq 0 \) are integers. This shows that LOCC1, PPT and SEP distinguishabilities of states are properties of states themselves and independent of the dimension of quantum system. With these results, we can give the maximal number of states which can be distinguished via LOCC and construct a LOCC indistinguishable basis of product states in a general system. Note that our results are also suitable for unambiguous discriminations. Also, we give a conjecture for other distinguishabilities and a framework by defining the Local-global indistinguishable property. Instead of considering such problems for special sets or special systems, we consider the problems for general states in general systems, which have not been considered yet, for our knowledge.

**Keywords:** LOCC; SEP; PPT; Local-global indistinguishability; Mixed states; Nonlocality; Multipartite system.

I. INTRODUCTION

In quantum information theory, the distinguishability of quantum states is of centrally important. If general POVMs are allowed, then states can be distinguished if and only if they are orthogonal[1]. However, in realistic tasks, multipartite states are often shared by separated owners, who can not use general POVMs. Fortunately, technologies of classical communications have been well-developed and are allowed for the owners of different partite. In spite of this, distinguishing a set of states by local operators and classical communications (LOCC) becomes available and significant. On the other hand, for a set of states, the distinguishability via LOCC POVMs implies the distinguishability via SEP POVMs, which implies the distinguishability via PPT POVMs. And sometimes, PPT POVMs or SEP POVMs have more simple properties than LOCC POVMs.

The problem of distinguishing a set of orthogonal states via LOCC can be described as follow. Alice, Bob, Charlie, et al. share a quantum state which is one of the states in a given set. The task for them is determining which state it is by LOCC protocols. A LOCC protocol is described as follow. Alice measures her partita firstly and publishes her result. Bob then provides a measurement on his partita depending on the information from Alice and publishes his result. And then Charlie provides his measurement depending on the informations from Alice and Bob, also publishes the result just like Bob did, and so do the others. A round is that every member measures the corresponding partita and publishes the result once.

If there is a protocol such that after \( r \)-round, they can discriminate which state they shared, then the set is said to be distinguishable via LOCC. In particular, when \( r=1 \), we say that the set is distinguishable via LOCC1.

In mathematics, a quantum system shared by K owners can be described as a Hilbert space \( \otimes_{k=1}^{K} C^{d_k} \). And a state they shared is described as a unit vector. The LOCC distinguishability of a set \( S \) of states is equivalent to that there is a LOCC POVM \( \{ M_j \}_{j=1,2,...,J} \) (note that each \( M_j \) is an operator) such that for any \( j \), \( \text{Tr}(M_j \rho) \neq 0 \) for at most one state \( \rho \) (written in the dense operator form) in \( S \). The mathematical statements for LOCC1, PPT and SEP distinguishabilities are similar, just replace the word "LOCC" by "LOCC1", "PPT" or "SEP".

The problem of distinguishabilities, especially the LOCC distinguishability, have been considered by authors. A result of Walgate et al. explains that two orthogonal pure states can always be distinguished by LOCC[1]. An innocent intuition might be that the more entanglements a set has, the harder the set can be distinguished by LOCC. However it is not true in general. Bennett et al. constructed a set with 9 elements in \( C^3 \otimes C^3 \) consists of orthogonal product states which is LOCC indistinguishable[2]. On the other hand, entanglements indeed give bounds to the LOCC distinguishability[3, 4]. Many authors considered the distinguishability of maximally entangled states. Nathanson proved that in \( C^3 \otimes C^3 \), every 3 orthogonal maximally entangled states are distinguishable via LOCC[5]. He also constructed three LOCC1 indistinguishable orthogonal maximally entangled states in \( C^d \otimes C^d \) when \( d \geq 4 \) be even or \( d = 3k + 2 \)[6]. The result was weakened but extended to all \( d \geq 4 \) by Wang et al. by constructing 4 orthogo-
nal maximally entangled states which can not be distinguished via LOCC when $d \geq 4$[7]. More results were given for Bell states and generalized Bell states. A paper by Ghosh et al. concludes that any 3 Bell states are LOCC indistinguishable[8]. However, in $C^d \otimes C^d$ with $d \geq 3$, any 3 generalized Bell states are LOCC distinguishable[9]. A result of Fan shows that if $d$ is a prime and $l(l - 1) \leq 2d$, then $l$ generalized Bell states can be distinguished by LOCC[10]. Note that any $d + 1$ maximally entangled states in $C^d \otimes C^d$ are LOCC indistinguishable[4]. A strange result is that if two copies of the state are allowed, then a set of generalized Bell states is LOCC distinguishable[11]. There were also some authors considering LOCC distinguishability of orthogonal product states. Bennett et al. showed that an unextendible product basis can not be distinguished by LOCC[12]. Others results include, for example, constructing indistinguishable orthogonal product states such as $[13–17]$. Let us mention that in $C^3 \otimes C^2$, there are 4 orthogonal product states LOCC1 indistinguishable when Alice goes firstly and there are 5 orthogonal product states LOCC1 indistinguishable in $C^3 \otimes C^3$ no matter who goes firstly[18]. An earlier result shows that if a completed product basis is LOCC distinguishable, then it is LPCC distinguishable[19]. Other methods include [20–23]. As for those LOCC indistinguishable sets, one might consider using auxiliary resources to make them distinguishable[24–28]. Another scheme is to accept distinguishing states with an inconclusive result, which is called the unambiguous discrimination[23, 29–35]. Also, one might concern about distinguishing states by infinite rounds of LOCC or asymptotically[36, 37].

Most of results consider states in a fixed system, that is a fixed Hilbert space. Hence, we consider whether a set of indistinguishable states can become distinguishable by viewing them as states in a larger Hilbert space. Instead of considering special sets of states or in a special system, we consider problems for general states in general systems, which have not been considered, for our knowledge. Moreover, we consider such problems for both perfect and unambiguous discriminations. These problems are considerable for at least 3 reasons as follow.

Firstly, the local distinguishability of states are bounded by the dimension of the system. For example, in a bipartite system, a necessary condition of a local distinguishable (that is distinguishable via LOCC) set is that the total schmidt rank of the set is not larger than the dimension of the system[4]. However, if we view the states in a larger space, for example, viewing a set of states in $C^2 \otimes C^2$ as states in $C^3 \otimes C^3$ by the nature embedding, this restriction could become meaningless. But whether the states are still indistinguishable?

Secondly, if we can use extra resources such as entangled states, then a local indistinguishable set may become distinguishable[24–28]. And an universal resource may not exist in a system of the same dimension as the ordinary one. That is, to find an universal resource, we may have to enlarge the dimension of Hilbert space[27]. This gives a feeling that the local indistinguishability of states may not be remained if the states are viewed in a larger system.

Finally, one might describe the distinguishability of points in a Hilbert space by distances. As an example, we can define a distance of points in a plane by setting the distance of two different points be 1, and otherwise be 0. On the other hand, the distance of points may depend on the dimension of the space one chooses. For example, let us consider the usual distance of two diagonal points of a unit square. If the observer chooses the space of dimension one, says the space consists of the edges of the square, then the distance of the two diagonal points is 2. However, if the observer chooses the other space of dimension two, says the plane of the square, then the distance is different, it is $\sqrt{2}$. This gives a suspicion that the distinguishability of states may depend on the dimension of space one chooses.

The conclusion of this paper includes that if a set of states is indistinguishable via LOCC1, unambiguous LOCC, PPT or SEP POVMs in $\otimes_{k=1}^{K} C^{d_k}$, then it is in-distinguishable via the same kind of POVMs even being viewed in $\otimes_{k=1}^{K} C^{d_k+b_k}$, where $K, d_k \geq 2$, $b_k \geq 0$ are integers. These mean that LOCC1, unambiguous LOCC, PPT and SEP distinguishabilities are properties of states themselves and independent of the dimension of space one chooses.

The result has some immediately consequences. Firstly, it implies that in any system with at least two partite are of dimension at least 2 (Let us called such a system non-trivial), there exist 3 orthogonal pure states which are LOCC1 indistinguishable. Note that, any 2 orthogonal pure states are always LOCC (in fact LOCC1) distinguishable[1]. Secondly, the result with [18] imply that in $C^m \otimes C^n$, there exist 4 orthogonal product states which are LOCC1 indistinguishable if Alice goes firstly, when $m \geq 3$, $n \geq 2$ and 5 orthogonal product states which are LOCC1 indistinguishable when $m, n \geq 3$, no matter who goes firstly, and these can be extended to multipartite systems by fixing states of other partite. Therefore, our result solves the problem of finding minimum number of one-way local indistinguishable orthogonal product states in a general system once for all. Finally, together with [2], our result can be used to construct a LOCC indistinguishable product basis in $C^m \otimes C^n$, if $m, n$ are at least 3, and a similar construction can extend this to multipartite systems. Though we give corollaries for product states, we mention that our result is for general states, not only for product states.

The rest of the article is organized as follow. Section II is devoted to the main results and the proofs will be given in section III. Then in section IV, we give a framework by defining the Local-global indistinguishable property, restate the theorems and give a conjecture. Also, we give a sketch illustration of why the method may not work for LOCC and LPCC cases. Finally, in section V, there is a conclusion.
II. RESULTS

This section is devoted to main results.

A. Viewing states in a larger system

There is a nature embedding from \( C^d \) to \( C^{d+h} \), for example, view \((x, y)\) in \( C^2 \) as \((x, y, 0)\) in \( C^3 \). As for states in \( \otimes_{k=1}^K C^{d_k} \), they can be viewed as in \( \otimes_{k=1}^K C^{d_k+h_k} \), by the nature embedding. That is, extending a basis of \( \otimes_{k=1}^K C^{d_k} \) to a basis of \( \otimes_{k=1}^K C^{d_k+h_k} \) and viewing the states in the larger space via the rest components are zeros. As for dense operators, that means viewing matrices (under the computational basis) \( \rho_i \) of \( \otimes_{k=1}^K C^{d_k} \) as \( \tilde{\rho}_i = \begin{pmatrix} \rho_i & 0 \\ 0 & 0 \end{pmatrix} \) (under the computational basis extending from the smaller space) of \( \otimes_{k=1}^K C^{d_k+h_k} \). In the rest of this paper, let us use this view.

B. The \( \text{LOCC}_1 \) case and its corollaries

The most important result of this paper is that the \( \text{LOCC}_1 \) indistinguishability of a set of orthogonal states is independent of the dimension of system. That is, when we view states of a lower dimensional space in a larger dimensional space, the \( \text{LOCC}_1 \) indistinguishability of them is still remained. This fact is stated as follows.

Theorem 1 Let \( \{ \rho_i | i = 1, 2, ..., N \} \) be a set of orthogonal states (pure or mixed), written in dense operators form, in \( \otimes_{k=1}^K C^{d_k} \), where \( N \) is a finite positive integer. If they are indistinguishable via \( \text{LOCC}_1 \) in \( \otimes_{k=1}^K C^{d_k} \), then they are indistinguishable via \( \text{LOCC}_1 \) in \( \otimes_{k=1}^K C^{d_k+h_k} \) (viewed as \( \tilde{\rho}_i = \begin{pmatrix} \rho_i & 0 \\ 0 & 0 \end{pmatrix} \), where \( h_k \) are non-negative integers).

The following corollaries somehow show the abilities of this result. These corollaries generalize results in some very special systems to very general systems.

Since there exist 3 orthogonal pure states in \( C^2 \otimes C^2 \) which are \( \text{LOCC}_1 \) indistinguishable (an example is 3 different Bell states), Theorem 1 together with the result in [1] imply that:

Corollary 1 In any non-trivial system, the maximal number \( T \) such that any \( T \) orthogonal pure states are \( \text{LOCC}_1 \) distinguishable is 2.

In [18], four \( \text{LOCC}_1 \) indistinguishable orthogonal product states are given in \( C^3 \otimes C^2 \) when Alice goes firstly and five \( \text{LOCC}_1 \) indistinguishable orthogonal product states are given, no matter who goes firstly, in \( C^3 \otimes C^3 \). On the other hand, results in [18] also show that any 3 orthogonal product states are \( \text{LOCC}_1 \) distinguishable no matter who goes firstly and any 4 orthogonal product states are \( \text{LOCC}_1 \) distinguishable if one can choose who goes firstly, in a bipartite system.

Thus, Theorem 1 implies the following corollary for product states:

Corollary 2 The maximal number \( P \) such that any \( P \) orthogonal product states are \( \text{LOCC}_1 \) distinguishable when Alice goes firstly is 3, in \( C^m \otimes C^n \), where \( m \geq 3 \), \( n \geq 2 \), and the maximal number \( Q \) such that any \( Q \) orthogonal product states are \( \text{LOCC}_1 \) distinguishable if one can choose who goes firstly is 4, in \( C^m \otimes C^n \), where \( m, n \geq 3 \).

In [2], an orthogonal product basis which is \( \text{LOCC} \) indistinguishable in \( C^3 \otimes C^3 \) is given. Using Theorem 1 together with that, it is easy to construct an orthogonal product basis which is \( \text{LOCC} \) indistinguishable in \( C^m \otimes C^n \), where \( m, n \) are at least 3, with the help of the theorem in [19].

Corollary 3 The 9 domino states in [2] together with \(|i,j\rangle, i=3,4,\ldots,m-1, j=3,4,\ldots,n-1\), form a \( \text{LOCC} \) indistinguishable completed orthogonal product basis in \( C^m \otimes C^n \), where \( m, n \) are at least 3.

The basis we find here is \( \text{LOCC} \) indistinguishable, not only \( \text{LOCC}_1 \) indistinguishable. Let us give details.

The 9 domino states (unnormalized) in [2] form an orthogonal product basis of \( C^3 \otimes C^3 \). They are \(|0\rangle|0\pm1\rangle, |0\pm1\rangle|0\rangle, |0\rangle|0\pm1\rangle, |0\pm1\rangle|0\rangle, |0\rangle|0\rangle, |1\rangle|1\rangle \). It is easy to see that in \( C^m \otimes C^n \), where \( m, n \) are at least 3, these states together with \(|i,j\rangle, i=3,4,\ldots,m-1, j=3,4,\ldots,n-1\), form a completed orthogonal product basis. One will see that the basis is \( \text{LOCC} \) indistinguishable.

Here we need a lemma which is an easy corollary of the theorem in [19].

Lemma 1 An orthogonal product basis of a multipartite system is distinguishable via \( \text{LOCC} \) if and only if it is distinguishable via \( \text{LOCC}_1 \).[19]

Using Theorem 1, one can get that the above orthogonal basis is \( \text{LOCC}_1 \) indistinguishable, since the domino states are \( \text{LOCC} \) (and so \( \text{LOCC}_1 \)) indistinguishable in \( C^3 \otimes C^3 \). Then Lemma 1 tells us that since we are considering an orthogonal product basis, it is equivalent to that the basis is \( \text{LOCC} \) indistinguishable. Thus, the basis we constructed is \( \text{LOCC} \) indistinguishable.

An easy argument can generalize the construction to multipartite systems by a tensor a normalized orthogonal basis of other partite and then normalize.

C. The \( \text{PPT} \) and \( \text{SEP} \) cases

A similar statement for \( \text{SEP} \) or \( \text{PPT} \) indistinguishability can be proven.

Theorem 2 Let \( \{ \rho_i | i = 1, 2, ..., N \} \) be a set of orthogonal states (pure or mixed), written in dense operators
form, in $\otimes_{k=1}^{K}C_{d_k}$, where $N$ is a finite positive integer. If they are indistinguishable via PPT POVMs, then they are indistinguishable via SEP POVMs in $\otimes_{k=1}^{K}C_{d_k+h_k}$.

**Theorem 3** Let $\{\rho_i|i=1,2,...,N\}$ be a set of orthogonal states (pure or mixed), written in dense operators form, in $\otimes_{k=1}^{K}C_{d_k}$, where $N$ is a finite positive integer. If they are indistinguishable via SEP POVMs, then they are indistinguishable via SEP POVMs in $\otimes_{k=1}^{K}C_{d_k+h_k}$.

**D. The unambiguous discriminations**

The results are also suitable for unambiguous discriminations (in such cases, the states are only assumed to be linear independent, not needed to be orthogonal). We just need to modify the proofs a bit and the calculations are almost the same. Note that for unambiguous discriminations, the LOCC distinguishability of states is equal to the SEP distinguishability of them [23] and so Theorem 1 can be extended to the LOCC case.

**Theorem 4** Let $\{\rho_i|i=1,2,...,N\}$ be a set of states (pure or mixed), written in dense operators form, in $\otimes_{k=1}^{K}C_{d_k}$, where $N$ is a finite positive integer. If they are indistinguishable via LOCC(SEP, PPT) POVMs, then they are indistinguishable unambiguously via LOCC(SEP, PPT) POVMs in $\otimes_{k=1}^{K}C_{d_k+h_k}$.

**III. PROOF OF THE RESULTS**

In this section, we will prove the theorems. We only prove for perfect discriminations. The proofs for unambiguous discriminations are similar.

For convenience, we use the computational basis of $\otimes_{k=1}^{K}C_{d_k}$ and extend it to the computational basis of $\otimes_{k=1}^{K}C_{d_k+h_k}$, so that we can write all operators in a POVM in matrix forms. On one hand, we show that taking the up-left block of a POVM of $\otimes_{k=1}^{K}C_{d_k+h_k}$ is a POVM of $\otimes_{k=1}^{K}C_{d_k}$ of the same kind for a LOCC1, PPT or SEP POVM. And this also states why this method may not work for LOCC or LPCC case. Note that the up-left block of the LOCC(LPCC) POVM of $\otimes_{k=1}^{K}C_{d_k+h_k}$ may not be a LOCC(LPCC) POVM of $\otimes_{k=1}^{K}C_{d_k}$. And on the other hand, we prove the theorems by showing that the conditions of distinguishability in a lower dimensional system are the same as in the larger dimensional system, since the low-right blocks of the matrices are of traces 0.

**A. The up-left blocks of the matrices in a LOCC1 POVM form a LOCC1 POVM**

The lemma and the proof below is typical.

**Lemma 2** Let $\{M_j\}_{j=1,2,...,J}$ be a (LOCC1) POVM of $\otimes_{k=1}^{K}C_{d_k+h_k}$ such that $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ written in the block form (under the computational basis extended from the computational basis of $\otimes_{k=1}^{K}C_{d_k}$), where $M_{j1}$ is a $(\prod_{k=1}^{K}d_k) \times (\prod_{k=1}^{K}d_k)$ matrix, then $\{M_{j1}\}_{j=1,2,...,J}$ is a (LOCC1) POVM of $\otimes_{k=1}^{K}C_{d_k}$ (as matrices under the computational basis).

**Proof of Lemma 2** for bipartite system: A LOCC1 POVM in $\otimes_{k=1}^{2}C_{d_k+h_k}$ is given as follow. Alice provides a POVM $\{A_j\}_{j=1,2,...,J}$ on her partita and gets an outcome $j_A$. Bob provides a POVM $\{B_{j_A,j}\}_{j=1,2,...,J_A}$ on his partita depending on the outcome $j_A$, and gets an outcome $j_B$. Thus, the LOCC1 POVM is $\{M_{j_A,j_B} = A_{j_A} \otimes B_{j_A,j_B}\}_{j_A=1,2,...,J_A, j_B=1,2,...,J_A}$.

Write $A_{j_A} = \begin{pmatrix} A_{j_A1} & A_{j_A2} \\ A_{j_A3} & A_{j_A4} \end{pmatrix}$ in the block form, where $A_{j_A1}$ is a $d_1 \times d_1$ matrix. Since $\{A_j\}_{j=1,2,...,J}$ is a POVM, $A_{j_A1} = A_{j1}^d$ and $\sum_j A_{j_Aj} = I_{d_1+h_1}$. Of course every $A_{j_A1}$ is positive semi-defined. Now:

$$\sum_j A_{j_Aj} = I_{d_1+h_1} \implies \sum_j A_{j_Aj} = I_{d_1} \implies A_{j_A1} \text{ is positive semi-defined,}$$

$$A_{j_A1} = A_{j1}^d \implies A_{j_A1} = A_{j1}^d.$$

Thus, $\{A_{j_A1}\}_{j_A=1,2,...,J_A}$ is a POVM of Alice’s partita of $\otimes_{k=1}^{2}C_{d_k}$.

Similarly, write $B_{j_A,j_B} = \begin{pmatrix} B_{j_A,j_B1} & B_{j_A,j_B2} \\ B_{j_A,j_B3} & B_{j_A,j_B4} \end{pmatrix}$ in the block form, where $B_{j_A,j_B1}$ is a $d_2 \times d_2$ matrix. Since $\{B_{j_A,j}\}_{j=1,2,...,J_A}$ is a POVM, $\{B_{j_A,j_B1}\}_{j_B=1,2,...,J_A}$ is a POVM of Bob’s partita of $\otimes_{k=1}^{2}C_{d_k}$.

Write $M_{j_A,j_B} = \begin{pmatrix} M_{j_A,j_B1} & M_{j_A,j_B2} \\ M_{j_A,j_B3} & M_{j_A,j_B4} \end{pmatrix}$ in the block form, where $M_{j_A,j_B1}$ is a $d_1d_2 \times d_1d_2$ matrix. Now, $M_{j_A,j_B1} = A_{j_A1} \otimes B_{j_A,j_B1}$.

In $\otimes_{k=1}^{2}C_{d_k}$, let Alice measure her partita by POVM $\{A_{j_A1}\}_{j_A=1,2,...,J_A}$ and gets an outcome, says $j_A$. Then Bob measures by POVM $\{B_{j_A,j_B1}\}_{j_B=1,2,...,J_A}$ and gets an outcome $j_B$. Now the final LOCC1 POVM of $\otimes_{k=1}^{2}C_{d_k}$ will be $\{A_{j_A1} \otimes B_{j_A,j_B1} = M_{j_A,j_B1}\}_{1 \leq j_A, j_B = 1 \leq J_A}$.

**B. The conditions of the LOCC1 distinguishability in lower dimensional systems are the same as in the larger dimensional systems**

Let us prove Theorem 1 for a bipartite system. The proof is also typical. The multipartite case is similar.

**Proof of Theorem 1 for bipartite systems:** Let $\{\rho_i|i=1,2,...,N\}$ be a set of orthogonal states
in $\otimes_{s=1}^{2} C^{d_s}$. If it is distinguishable via LOCC in $\otimes_{s=1}^{2} C^{d_s}+h_s$, where $h_s$ are non-negative integers, let us show that it is distinguishable in $\otimes_{s=1}^{2} C^{d_s}$ via LOCC 1.

Let the states in $\otimes_{s=1}^{2} C^{d_s}+h_s$ be LOCC 1 distinguishable via LOCC 1 POVM $\{M_j\}_{j=1,2,...,J}$ that means that for every $j$, $Tr(M_j \rho_i) \neq 0$ for at most one $i$.

Write $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $d_1 d_2 \times d_1 d_2$ matrix. Since $\{M_j\}_{j=1,2,...,J}$ is a LOCC 1 POVM, $\{M_{j1}\}_{j=1,2,...,J}$ is a LOCC 1 POVM by Lemma 2.

However, if for $j = 1$, $Tr(M_{j1}\rho_i) \neq 0$, then $Tr(M_j \rho_i) \neq 0$, since after an easy computation, they are the same. Thus, at most one $\rho_i$ satisfies that $Tr(M_{j1}\rho_i) \neq 0$, which means that they are LOCC 1 distinguishable via LOCC 1 POVM $\{M_{j1}\}_{j=1,2,...,J}$.

\[ \sum_{j} A_{j1}^{(s)} + A_{j2}^{(s)} + A_{j3}^{(s)} + A_{j4}^{(s)} = I_{d_s+h_s} \]

Thus, $\{A_{j1}^{(s)}\}_{j=1,2,...,J}$ is a POVM of partita $A^{(s)}$ of $\otimes_{s=1}^{K} C^{d_{s}^t}$.

In $\otimes_{s=1}^{K} C^{d_{s}^t}$, let $A^{(s)}$ measure its partita by POVM $\{A_{j1}^{(s)}\}_{j=1,2,...,J}$, and gets an outcome, $j$. Now the final LOCC 1 measurement will be $\{M_{j1}\}_{j=1,2,...,J}$, and note that $M_{j1} \rho_j = \otimes_{s=1}^{K} A_{j1}^{(s)}$.

\[ \text{Proof of Theorem 1:} \text{ Let } \{ \rho_i | i = 1, 2, ..., N \} \text{ be a set of orthogonal states in } \otimes_{s=1}^{K} C^{d_{s}^t}. \text{ If it is distinguishable via LOCC in } \otimes_{s=1}^{K} C^{d_{s}^t}+h_s, \text{ where } h_s \text{ are non-negative integers, let us see that it is distinguishable in } \otimes_{s=1}^{K} C^{d_{s}^t} \text{ via LOCC 1.}

In $\otimes_{s=1}^{K} C^{d_{s}^t}$, the states are LOCC 1 distinguishable via LOCC 1 POVM $\{M_{j1}\}_{j=1,2,...,J}$ if and only if for every $j$, $Tr(M_{j1}\rho_i) \neq 0$ for at most one $i$.

Write $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $d_1 \times d_1$ matrix. Since $\{M_{j1}\}_{j=1,2,...,J}$ is a LOCC 1 POVM, $\{M_{j1}\}_{j=1,2,...,J}$ is a LOCC 1 POVM of $\otimes_{s=1}^{K} C^{d_{s}^t}$, by Lemma 2.

In $\otimes_{s=1}^{K} C^{d_{s}^t}$, let use measure by POVM $\{M_{j1}\}_{j=1,2,...,J}$. However, if for $j = 1$, $Tr(M_{j1}\rho_i) \neq 0$, then $Tr(M_{j1}\rho_i) \neq 0$, since after an easy computation, they are the same. Thus, at most one $\rho_i$ satisfies that $Tr(M_{j1}\rho_i) \neq 0$, which means that they are LOCC 1 distinguishable via LOCC 1 POVM $\{M_{j1}\}_{j=1,2,...,J}$.

\[ \text{D. Proofs of Theorem 2 and Theorem 3} \]

To prove Theorem 2 and Theorem 3, just note that one can prove a similar lemma for PPT POVMs or SEP POVMs as Lemma 2.

\[ \text{Lemma 3 Let } \{M_{j1}\}_{j=1,2,...,J} \text{ be a PPT (SEP) POVM of } \otimes_{s=1}^{K} C^{d_{s}^t} \text{ such that } M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix} \text{ written in the block form (under the computational basis extended from the computational basis of } \otimes_{s=1}^{K} C^{d_{s}^t}, \text{ where } M_{j1} \text{ is a } (\prod_{s=1}^{K} d_{s}) \times (\prod_{s=1}^{K} d_{s}) \text{ matrix, then } \{M_{j1}\}_{j=1,2,...,J} \text{ is a PPT (SEP) POVM of } \otimes_{s=1}^{K} C^{d_{s}^t} \text{ (as matrices under the computational basis).} \]
Proof of PPT case for Lemma 3: Without loss generality, assume that the partie are held by $A(s)$, $s=1,2,...,K$. \{ $M_j$ \}$_{j=1,2,...,J}$ is a PPT POVM means that for every $j$, $M_j$ can be written as $M_j = \sum_{i = 1}^{K} \otimes_{s=1}^{K} (A_i^{(s)})$, where the sum is finite, and the operators are positive semi-defined after a partial transposition. Without loss generality, for $j$, assume that the partial transposition is on $A_1^{(s)}$. Thus, $\sum_{i} [(A_i^{(s)})^T \otimes \otimes_{s=2}^{K} (A_i^{(s)})]$ is positive semi-defined.

Write $A_j^{(s)} = \begin{pmatrix} A_j^{(s)} & A_j^{(s)} \\ A_j^{(s)} & A_j^{(s)} \end{pmatrix}$ in the block form, where $A_j^{(s)}$ is a $d_s \times d_s$ matrix. $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $\prod_{s=1}^{K} d_s \times \prod_{s=1}^{K} d_s$ matrix.

Thus, $M_{j1} = \otimes_{s=1}^{K} A_{j1}^{(s)}$.

Since \{ $M_j$ \}$_{j=1,2,...,J}$ is a POVM, $M_j = M_j^\dagger$ and $\sum_{j} M_j = I_{\prod_{s=1}^{K} (d_s+h_s)}$. Of course every $M_j$ is positive semi-defined. Now:

$\sum_{j} M_j = I_{\prod_{s=1}^{K} (d_s+h_s)}$ implies that $\sum_{j} M_{j1} = I_{\prod_{s=1}^{K} d_s}$. $M_j$ is positive semi-defined implies that $M_{j1}$ is positive semi-defined.

$M_j = M_j^\dagger$ implies that $M_{j1} = M_{j1}^\dagger$.

$\sum_{i} [(A_i^{(s)})^T \otimes \otimes_{s=2}^{K} (A_i^{(s)})]$ is positive semi-defined implies that $\sum_{i} [(A_i^{(s)})^T \otimes \otimes_{s=2}^{K} (A_i^{(s)})]$ is positive semi-defined, which means that $M_{j1}$ is a PPT operator.

Thus, \{ $M_{j1}$ \}$_{j=1,2,...,J}$ is a PPT POVM of $\otimes_{s=1}^{K} C_{d_s}$.

Proof of Theorem 2: Let \{ $\rho_i$ \}$_{i=1,2,...,N}$ be a set of orthogonal states in $\otimes_{s=1}^{K} C_{d_s}$. If it is distinguishable via PPT in $\otimes_{s=1}^{K} C_{d_s+h_s}$, where $h_s$ be non-negative integers. Let us see that it is distinguishable in $\otimes_{s=1}^{K} C_{d_s}$ via PPT.

There is a PPT POVM \{ $M_j$ \}$_{j=1,2,...,J}$ of $\otimes_{s=1}^{K} C_{d_s+h_s}$ such that for every $j$, at most one $\rho_i$ satisfies $Tr(M_j \rho_i) \neq 0$.

Write $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $(\prod_{s=1}^{K} d_s) \times (\prod_{s=1}^{K} d_s)$ matrix. Since \{ $M_j$ \}$_{j=1,2,...,J}$ is a PPT POVM, Lemma 3 concludes that \{ $M_{j1}$ \}$_{j=1,2,...,J}$ is a PPT POVM of $\otimes_{s=1}^{K} C_{d_s}$.

Thus, $M_{j1} = \otimes_{s=1}^{K} A_{j1}^{(s)}$, measure by the PPT POVM \{ $M_{j1}$ \}$_{j=1,2,...,J}$. If for an outcome $j$, $Tr(M_{j1} \rho_i) \neq 0$, then $Tr(M_j \rho_i) \neq 0$, since after an easy computation, they are the same. Thus, at most one $\rho_i$ satisfies that $Tr(M_{j1} \rho_i) \neq 0$, which means that they are PPT distinguishable via this PPT POVM.

Proof of SEP case for Lemma 3: Without loss generality, assume that the partie are held by $A(s)$, $s=1,2,...,K$. \{ $M_j$ \}$_{j=1,2,...,J}$ is a SEP POVM means that $M_j$ can be written as $M_j = \otimes_{s=1}^{K} A_j^{(s)}$, for every $j$.

Write $A_j^{(s)} = \begin{pmatrix} A_j^{(s)} & A_j^{(s)} \\ A_j^{(s)} & A_j^{(s)} \end{pmatrix}$ in the block form, where $A_j^{(s)}$ is a $d_s \times d_s$ matrix. $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $\prod_{s=1}^{K} d_s \times \prod_{s=1}^{K} d_s$ matrix.

Thus, $M_{j1} = \otimes_{s=1}^{K} A_{j1}^{(s)}$.

Since \{ $M_j$ \}$_{j=1,2,...,J}$ is a POVM, $M_j = M_j^\dagger$ and $\sum_{j} M_j = I_{\prod_{s=1}^{K} (d_s+h_s)}$. Of course every $M_j$ is positive semi-defined. Now:

$\sum_{j} M_j = I_{\prod_{s=1}^{K} (d_s+h_s)}$ implies that $\sum_{j} M_{j1} = I_{\prod_{s=1}^{K} d_s}$. $M_j$ is positive semi-defined implies that $M_{j1}$ is positive semi-defined.

$M_j = M_j^\dagger$ implies that $M_{j1} = M_{j1}^\dagger$.

$M_{j1} = \otimes_{s=1}^{K} A_{j1}^{(s)}$ means that it is a separable operator.

Thus, \{ $M_{j1}$ \}$_{j=1,2,...,J}$ is a SEP POVM of $\otimes_{s=1}^{K} C_{d_s}$.

Proof of Theorem 3: Let \{ $\rho_i$ \}$_{i=1,2,...,N}$ be a set of orthogonal states in $\otimes_{s=1}^{K} C_{d_s}$. If it is distinguishable via SEP in $\otimes_{s=1}^{K} C_{d_s+h_s}$, where $h_s$ be non-negative integers. Let us see that it is distinguishable in $\otimes_{s=1}^{K} C_{d_s}$ via SEP.

There is a SEP POVM \{ $M_j$ \}$_{j=1,2,...,J}$ of $\otimes_{s=1}^{K} C_{d_s+h_s}$ such that for every $j$, at most one $\rho_i$ satisfies $Tr(M_j \rho_i) \neq 0$.

Write $M_j = \begin{pmatrix} M_{j1} & M_{j2} \\ M_{j3} & M_{j4} \end{pmatrix}$ in the block form, where $M_{j1}$ is a $(\prod_{s=1}^{K} d_s) \times (\prod_{s=1}^{K} d_s)$ matrix. Since \{ $M_{j1}$ \}$_{j=1,2,...,J}$ is a SEP POVM, Lemma 3 concludes that \{ $M_{j1}$ \}$_{j=1,2,...,J}$ is a SEP POVM of $\otimes_{s=1}^{K} C_{d_s}$.

In $\otimes_{s=1}^{K} C_{d_s}$, measure by SEP POVM \{ $M_{j1}$ \}$_{j=1,2,...,J}$. If for an outcome $j$, $Tr(M_{j1} \rho_i) \neq 0$, then $Tr(M_{j1} \rho_i) \neq 0$, since after an easy computation, they are the same. Thus, at most one $\rho_i$ satisfies that $Tr(M_{j1} \rho_i) \neq 0$, which means that they are SEP distinguishable via this SEP POVM.

IV. DISCUSSION

The result of other indistinguishabilities may also hold. However, the method in this paper may not work. For example, the method may not be used to prove the (perfect) LOCC case. The reason may relate to that measuring a state in a lower dimensional system via LOCC measurement of a larger dimensional system, the collapsing state may not be in the lower dimensional system. Another example is that the method in this paper may not work for LPCC case. We construct a projective POVM of $C^4 \otimes C^4$ which is not projective when only looking at the left-up 3 $\times$ 3 blocks under computational basis as follow.
However, it will not be surprised if the result can be extended to LOCC case. We note that for a completed product basis or an unextendible product basis, the result holds for LOCC\cite{19, 36}. Here, we make a conjecture:

**Conjecture 1** If a set of states in $\otimes_{K=1}^{K} C^{d_k}$ is indistinguishable (perfectly or unambiguously) via LOCC (LPCC, LOCC$_r$, LPCC$_r$, PPT, SEP), then it is indistinguishable (perfectly or unambiguously) via LOCC (LPCC, LOCC$_r$, LPCC$_r$, PPT, SEP) in $\otimes_{K=1}^{K} C^{d_k+h_k}$, where $h_k$ be non-negative integers.

The cases of LOCC$_1$, unambiguous LOCC, (perfect or unambiguous) PPT and SEP have been discussed in the paper.

The above discussions lead to the following definition:

**Definition 1** Let $\{d_k\}_{k=1,2,\ldots,N}$ be integers at least 2.

1. Given a set of states $S$ in $\otimes_{K=1}^{K} C^{d_k}$ and kinds of indistinguishabilities $M$, $M'$. We say that $S$ satisfies the $M \rightarrow M'$ Local-global indistinguishable property, if $S$ is indistinguishable via $M$ in $\otimes_{K=1}^{K} C^{d_k}$ implies that $S$ is indistinguishable via $M'$ in $\otimes_{K=1}^{K} C^{d_k+h_k}$, for any $h_k$ be non-negative integers.

2. If for any states set $S$ in $\otimes_{K=1}^{K} C^{d_k}$, $S$ satisfies the $M \rightarrow M'$ Local-global indistinguishable property, then $\otimes_{K=1}^{K} C^{d_k}$ is said to be satisfying the $M \rightarrow M'$ Local-global indistinguishable property.

3. If for any set $S$ of states in any system, $S$ satisfies the $M \rightarrow M'$ local-global indistinguishable property, then $M \rightarrow M'$ is said to be satisfying the Local-global indistinguishable property.

The definition gives a framework of Local-global indistinguishability, which somehow states the independence of indistinguishability of states and the dimension of space one chooses. Using the definition, Theorem 1, Theorem 2, Theorem 3 and Theorem 4 can be restated as

- **LOCC$_1$** (unambiguous LOCC, (perfect or unambiguous) PPT, SEP)
- **LOCC$_1$** (unambiguous LOCC, (perfect or unambiguous) PPT, SEP)
- satisfies the Local-global indistinguishable property.

The conjecture is restated to be if $M=LOCC$ (LPCC, LOCC$_r$, LPCC$_r$, PPT, SEP), perfect or unambiguous, then $M \rightarrow M$ satisfies the Local-global indistinguishable property.

**V. CONCLUSION**

In this paper, we consider a problem that whether the indistinguishability of states is depend on the dimension of space one chooses. We prove that the LOCC$_1$, PPT and SEP indistinguishabilities, both perfect and unambiguous, are properties of states themselves and independent of the dimensional choice of the space. More exactly, we show that if a set of states is LOCC$_1$ (or unambiguous LOCC, (perfect or unambiguous) PPT, SEP) indistinguishable in a lower dimensional system, then it is LOCC$_1$ (or unambiguous LOCC, (perfect or unambiguous) PPT, SEP) indistinguishable in a dimensional extended system. The result is true for both bipartite and multipartite systems and for both pure and mixed states.

Using the result in [1], Theorem 1 implies that the maximal number $T$ such that any $T$ orthogonal pure states are LOCC$_1$ distinguishable is 2, in a non-trivial system. Also, Theorem 1 together with results in [18] give that the maximal number $P$ such that any $P$ orthogonal product states are LOCC$_1$ distinguishable is 3, if Alice goes firstly for a bipartite system such that Alice’s partita is of dimension at least 3 and Bob’s partita is of dimension at least 2. And that the maximal number $Q$ such that any $Q$ orthogonal product states are LOCC$_1$ distinguishable is 4, if one can choose who goes firstly, in a bipartite system with partite are of dimension at least 3. And using Theorem 1 together with the domino states in [2], an orthogonal product basis which is LOCC indistinguishable in any system with at least two partite are of dimension at least 3 can be constructed. And note that all the corollaries can be extended to multipartite systems.

Finally, we discuss other distinguishabilities and give a definition of the Local-global indistinguishable property. We also restate the theorems above and give a conjecture using the statement of the definition.

The remaining problems include whether the conjecture is true for those kinds of distinguishabilities. Another question is to find counter examples for which the Local-global indistinguishable property is not satisfied. Also, one can discuss such a problem for other kinds of distinguishabilities such as asymptotically LOCC distinguishability\cite{37}, or for special kinds of states.
We wish to acknowledge professor Zhu-Jun Zheng who gave many advices and helped to check the paper.

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