A NOTE ON THE $\mathbb{Z}_2$-EQUIVARIANT MONTGOMERY-YANG CORRESPONDENCE

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Abstract. In this paper, a classification of free involutions on 3-dimensional homotopy complex projective spaces is given. By the $\mathbb{Z}_2$-equivariant Montgomery-Yang correspondence, we obtain all smooth involutions on $S^6$ with fixed-point set an embedded $S^3$.

1. Introduction

In \cite{5}, Montgomery and Yang established a 1 : 1 correspondence between the set of isotopy classes of smooth embeddings $S^3 \hookrightarrow S^6, C_3^3$, and the set of diffeomorphism classes of smooth manifolds homotopy equivalent to the 3-dimensional complex projective space $\mathbb{C}P^3$ (these manifolds are called homotopy $\mathbb{C}P^3$). It is known that the latter set is identified with the set of integers by the first Pontrjagin class of the manifold. Therefore so is the set $C_3^3$.

In a recent paper \cite{4}, Bang-he Li and Zhi Lu established a $\mathbb{Z}_2$-equivariant version of the Montgomery-Yang correspondence. Namely, they proved that there is a 1 : 1 correspondence between the set of smooth involutions on $S^6$ with fixed-point set an embedded $S^3$ and the set of smooth free involutions on homotopy $\mathbb{C}P^3$. This correspondence gives a way of studying involutions on $S^6$ with fixed-point set an embedded $S^3$ by looking at free involutions on homotopy $\mathbb{C}P^3$. As an application, by combining this correspondence and a result of Petrie \cite{7}, saying that there are infinitely many homotopy $\mathbb{C}P^3$’s which admit free involutions, the authors constructed infinitely many counter examples for the Smith conjecture, which says that only the unknotted $S^3$ in $S^6$ can be the fixed-point set of an involution on $S^6$.

In this note we study the classification of the orbit spaces of free involutions on homotopy $\mathbb{C}P^3$. As a consequence, we get the classification of free involutions on homotopy $\mathbb{C}P^3$, and further by the $\mathbb{Z}_2$-equivariant Montgomery-Yang correspondence, the classification of involutions on $S^6$ with fixed-point set an embedded $S^3$.

The manifolds $X^6$ homotopy equivalent to $\mathbb{C}P^3$ are classified up to diffeomorphism by the first Pontrjagin class $p_1(X) = (24j + 4)x^2, j \in \mathbb{Z}$, where $x^2$ is the canonical generator of $H^4(X; \mathbb{Z})$ (c. f. \cite{5}, \cite{10}). We denote the manifold with $p_1 = (24j + 4)x^2$ by $H\mathbb{C}P^3_j$.

**Theorem 1.1.** The manifold $H\mathbb{C}P^3_j$ admits a (orientation reversing) smooth free involution if and only if $j$ is even. On every $H\mathbb{C}P^3_{2k}$, there are exactly two free involutions up to conjugation\(^1\).

\(^1\)The same result was also obtained independently by Bang-he Li (unpublished).
Corollary 1.2. An embedded $S^3$ in $S^6$ is the fixed-point set of an involution on $S^6$ if and only if its Montgomery-Yang correspondence is $H\mathbb{C}P_3^{2k}$. For each embedding satisfying the condition, there are exactly two such involutions up to conjugation.

Theorem 1.1 is a consequence of a classification theorem (Theorem 3.5) of the orbit spaces. Theorem 3.5 will be shown in Section 3 by the classical surgery exact sequence of Browder-Novikov-Sullivan-Wall. In Section 2 we show some topological properties of the orbit spaces, which will be needed in the solution of the classification problem.

2. Topology of the Orbit Space

In this section we summarize the topological properties of the orbit space of a smooth free involution on a homotopy $\mathbb{C}P^3$. Some of the properties are also given in [4]. Here we give shorter proofs from a different point of view and in a different logical order.

Let $\tau$ be a smooth free involution on $H\mathbb{C}P^3$, a homotopy $\mathbb{C}P^3$. Denote the orbit manifold by $M$.

Example 2.1. The 3-dimensional complex projective space $\mathbb{C}P^3$ can be viewed as the sphere bundle of a 3-dimensional real vector bundle over $S^4$. The fiberwise antipodal map $\tau_0$ is a free involution on $\mathbb{C}P^3$ (c. f. [7, A.1]). Denote the orbit space by $M_0$.

As a consequence of the Lefschetz fixed-point theorem and the multiplicative structure of $H^*(H\mathbb{C}P^3)$, we have

Lemma 2.2. [4, Theorem 1.4] $\tau$ must be orientation reversing.

Lemma 2.3. The cohomology ring of $M$ with $\mathbb{Z}_2$-coefficients is

$$H^*(M; \mathbb{Z}_2) = \mathbb{Z}_2[t, q]/(t^3 = 0, q^2 = 0),$$

where $|t| = 1$, $|q| = 4$.

Proof. Note the the fundamental group of $M$ is $\mathbb{Z}_2$. There is a fibration $\tilde{M} \to M \to \mathbb{R}P^\infty$, where $M \to \mathbb{R}P^\infty$ is the classifying map of the covering. We apply the Leray-Serre spectral sequence. Since $\tilde{M}$ is homotopy equivalent to $\mathbb{C}P^3$, the nontrivial $E_2$-terms are $E_2^{p,q} = H^p(\mathbb{R}P^\infty; \mathbb{Z}_2)$ for $q = 0, 2, 4, 6$. Therefore all differentials $d_2$ are trivial, and henceforth $E_2 = E_3$. Now the differential $d_3: E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism. For otherwise the multiplicative structure of the spectral sequence implies that the spectral sequence collapses at the $E_3$-page, which implies that $H^*(M; \mathbb{Z}_2)$ is nontrivial for $* > 6$, a contradiction. Then it is easy to see that $M$ has the claimed cohomology ring. \hfill $\square$

Remark 2.4. There is an exact sequence (cf. [2])

$$H_3(\mathbb{Z}/2) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}_- \to H_3(M) \to H_3(\mathbb{Z}/2),$$

where $\mathbb{Z}_-$ is the nontrivial $\mathbb{Z}[\mathbb{Z}/2]$-module. By this exact sequence, $H_2(M)$ is either $\mathbb{Z}_2$ or trivial. $H^3(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ implies that $H_2(M) = 0$. This was shown in [4, Lemma 2.1] by geometric arguments.
Lemma 2.7. The induced map \( \pi_2(M) \xrightarrow{\sim} M \). Proof. Apply the Leray-Serre spectral sequence for integral cohomology to the fibration \( M \) that the total Stiefel-Whitney class of \( v_2(M) \) is at the second Wu class is
\[
\tilde{\text{ Sq}}_2. \text{ The Steenrod square }
\]
The involution \( \tau \) is orientation reversing, therefore \( M \) is nonorientable and \( w_1(M) = t \). The Steenrod square \( S^2 : H^4(M; \mathbb{Z}_2) \rightarrow H^6(M; \mathbb{Z}_2) \) is trivial, this can be seen by looking at \( M_0 \), whose 4-dimensional cohomology classes are pulled back from \( S^4 \). Therefore the second Wu class is \( v_2(M) = 0 \). Thus by the Wu formula \( w(M) = \text{Sq}_i(M) \) it is seen that the total Stiefel-Whitney class of \( M \) is \( w(M) = 1 + t + t^2 \). □

Let \( \pi : H\mathbb{C}P^3 \rightarrow M \) be the projection map, the \( \pi^*p_1(M) = p_1(H\mathbb{C}P^3) \).

Lemma 2.6. The total Stiefel-Whitney class of \( M \) is \( w(M) = 1 + t + t^2 \), where \( t \in H^1(M; \mathbb{Z}_2) \) is the generator.

Proof. The involution \( \tau \) is orientation reversing, therefore \( M \) is nonorientable and \( w_1(M) = t \). The Steenrod square \( S^2 : H^4(M; \mathbb{Z}_2) \rightarrow H^6(M; \mathbb{Z}_2) \) is trivial, this can be seen by looking at \( M_0 \), whose 4-dimensional cohomology classes are pulled back from \( S^4 \). Therefore the second Wu class is \( v_2(M) = 0 \). Thus by the Wu formula \( w(M) = \text{Sq}_i(M) \) it is seen that the total Stiefel-Whitney class of \( M \) is \( w(M) = 1 + t + t^2 \). □

Lemma 2.7. The induced map \( \pi^* : H^4(M) \rightarrow H^4(H\mathbb{C}P^3) \) is an isomorphism.

Proof. Apply the Leray-Serre spectral sequence for integral cohomology to the fibration \( M \rightarrow \mathbb{R}P^\infty \), the \( E_2 \)-terms are \( E_2^{p,q} = H^p(\mathbb{R}P^\infty;\mathbb{H}^q(M)) \). It is known that \( H^3(M) = H^5(M) = 0 \) and \( H^5(M) = H^5(Q) = 0 \) (for \( H^*(Q) \), see [6, pp.265]), therefore \( E_2^{0,4} = H^4(M) \) is the only nonzero term in the line \( p + q = 4 \). This shows that the edge homomorphism is an isomorphism, which is just \( \pi^* \). □

Therefore the first Pontrjagin class of \( M \) is \( p_1(M) = (24j + 4)u \) \( (j \in \mathbb{Z}) \), where \( u = \pi^*(x^2) \) is the canonical generator of \( H^4(M) \).

3. Classification of the orbit spaces

By Proposition 2.5, every orbit space \( M \) is homotopy equivalent to \( M_0 \). Thus the set of conjugation classes of free involutions on homotopy \( \mathbb{C}P^3 \) is in 1 : 1 correspondence to the set of diffeomorphism classes of smooth manifolds homotopy equivalent to \( M_0 \). Denote the latter by \( \mathcal{M}(M_0) \). Let \( \mathcal{O}(M_0) \) be the smooth structure set of \( M_0 \), \( \text{Aut}(M_0) \) be the set of homotopy classes of self-equivalences of \( M_0 \). There is an action of \( \text{Aut}(M_0) \) on
\[ \mathcal{S}(M_0) \] with orbit set \( M(M_0) \). (Since the Whitehead group of \( \mathbb{Z}_2 \) is trivial, we omit the decoration \( s \) all over.)

The surgery exact sequence for \( M_0 \) is

\[ L_7(\mathbb{Z}_2^{-}) \rightarrow \mathcal{S}(M_0) \rightarrow [M_0, G/O] \rightarrow L_6(\mathbb{Z}_2^{-}). \]

By [11] Theorem 13A.1, \( L_7(\mathbb{Z}_2^{-}) = 0 \), \( L_6(\mathbb{Z}_2^{-}) \cong \mathbb{Z}_2 \), where \( c \) is the Kervaire-Arf invariant. Since \( \dim M_0 = 6 \) and \( PL/O \) is 6-connected, we have an isomorphism \( [M_0, G/O] \cong [M, G/PL] \). For a given surgery classifying map \( g : M_0 \rightarrow G/PL \), the Kervaire-Arf invariant is given by the Sullivan formula ([9], [11, Theorem 13B.5])

\[
c(M_0, g) = \langle w(M_0) \cdot g^* \kappa, [M_0] \rangle \\
= \langle (1 + t + t^2) \cdot g^*(1 + Sq^2 + Sq^2)(k_2 + k_6), [M_0] \rangle \\
= \langle g^*k_6, [M_0] \rangle.
\]

Now since \( M_0 \) has only 2-torsion, and modulo the groups of odd order we have

\[ G/PL \simeq Y \times \prod_{i \geq 2} (K(\mathbb{Z}_2, 4i - 2) \times K(\mathbb{Z}, 4i)), \]

where \( Y = K(\mathbb{Z}_2, 2) \times Sq^2 K(\mathbb{Z}, 4) \), we have \( [M_0, G/PL] = [M_0, Y] \times [M_0, K(\mathbb{Z}_2, 6)] \). \( k_6 \) is the fundamental class of \( K(\mathbb{Z}_2, 6) \). Therefore the surgery exact sequence implies

**Lemma 3.1.** \( \mathcal{S}(M_0) \cong [M_0, Y] \).

The projection \( \pi : \mathbb{CP}^3 \rightarrow M_0 \) induces a homomorphism \( \pi^* : [M_0, Y] \rightarrow [\mathbb{CP}^3, Y] \), and \( [\mathbb{CP}^3, Y] \) is isomorphic to \( \mathbb{Z} \) through the splitting invariant \( s_4 \) ([11, Lemma 14C.1]). Let \( \Phi = s_4 \circ \pi^* \) be the composition.

**Lemma 3.2.** There is a short exact sequence \( \mathbb{Z}_2 \rightarrow [M_0, Y] \xrightarrow{\Phi} 2\mathbb{Z} \).

**Proof.** We have \([\mathbb{CP}^3, Y] = [\mathbb{CP}^2, Y]\), and according to Sullivan [9], the exact sequence

\[ L_4(1) \xrightarrow{2} [\mathbb{CP}^2, Y] \rightarrow [\mathbb{CP}^1, Y] \]

is non-splitting. Let \( p : Y \rightarrow K(\mathbb{Z}_2, 2) \) be the projection map, then for any \( f \in [\mathbb{CP}^3, Y] \), \( s_4(f) \in 2\mathbb{Z} \) if and only if \( p \circ f : \mathbb{CP}^3 \rightarrow K(\mathbb{Z}_2, 2) \) is null-homotopic. Now by Lemma 2.3 the homomorphism \( H^2(M_0; \mathbb{Z}_2) \rightarrow H^2(\mathbb{CP}^3; \mathbb{Z}_2) \) is trivial. Therefore for any \( g \in [M_0, Y] \), the composition \( p \circ g \circ \pi \) is null-homotopic, thus \( \text{Im} \Phi \subset 2\mathbb{Z} \). On the other hand, since \( \pi^* : H^4(M_0; \mathbb{Z}) \rightarrow H^4(\mathbb{CP}^3) \) is an isomorphism, any map \( f : \mathbb{CP}^3 \rightarrow K(\mathbb{Z}, 4) \) factors through some \( g' : M_0 \rightarrow K(\mathbb{Z}, 4) \). Let \( i : K(\mathbb{Z}, 4) \rightarrow Y \) be the fiber inclusion, since \( s_4(i \circ f) \) takes any value in \( 2\mathbb{Z} \), so does \( \Phi(i \circ g') \).

Let \( h : M_0 \rightarrow K(\mathbb{Z}_2, 2) \) be a map corresponding to the nontrivial cohomology class in \( H^2(M_0; \mathbb{Z}_2) \). By obstruction theory, there is a lifting \( g : M_0 \rightarrow Y \). By the previous argument, there is also a map \( g' : M_0 \rightarrow Y \) such that \( \Phi(g) = \Phi(g') \), but \( p \circ g' : M_0 \rightarrow K(\mathbb{Z}_2, 2) \) is null-homotopic. Therefore the kernel of \( \Phi \) consists of two elements.

**Remark 3.3.** In [7] Petrie showed that every homotopy \( \mathbb{CP}^3 \) admits free involution. It was pointed out by Dovermann, Masuda and Schultz [3, pp. 4] that since the class \( G \)
is in fact twice the generator of $H^4(S^3)$. Petrie’s computation actually shows that every $HCP^2_{2k}$ admits free involution, which is consistent with Lemma 3.2.

The set of diffeomorphism classes of manifolds homotopy equivalent to $M_0$, $\mathcal{M}(M_0)$, is the orbit set $\mathcal{M}(M_0)/Aut(M_0)$. In general, the determination of the action of $Aut(M_0)$ on the structure set is very difficult. But in our case, the situation is quite simple, since

**Lemma 3.4.** The group of self-equivalences $Aut(M_0)$ is the trivial group.

**Proof.** A special CW-complex structure of $M_0$ was given in [1] pp. 885: $M_0$ is a $\mathbb{R}P^2$-bundle over $S^4$, therefore it is the union of two copies of $\mathbb{R}P^2 \times D^4$, glued along boundaries. Choose a CW-complex structure of $\mathbb{R}P^2$, we have a product CW-structure on one copy of $\mathbb{R}P^2 \times D^4$, and by shrinking the other copy of $\mathbb{R}P^2 \times D^4$ to the core $\mathbb{R}P^2$, we get a CW-complex structure on $M_0$, whose 2-skeleton is $\mathbb{R}P^2$.

Let $\varphi \in Aut(M_0)$ be a self homotopy equivalence of $M_0$. By cellular approximation, we may assume that $\varphi$ maps $\mathbb{R}P^2$ to $\mathbb{R}P^2$. It is easy to see that $[\varphi]_{\mathbb{R}P^2}$ is homotopic to $\text{id}_{\mathbb{R}P^2}$. Therefore, by homotopy extension, we may further assume that $\varphi|_{\mathbb{R}P^2} = \text{id}_{\mathbb{R}P^2}$. The obstruction to construct a homotopy between $\varphi$ and $\text{id}_{M_0}$, which is the identity on $\mathbb{R}P^2$, is in $H^i(M, \mathbb{R}P^2; \pi_1(M_0))$. Since $\pi_i(M_0) = 0$ for $3 \leq i \leq 6$ and $H^1(M_0, \mathbb{R}P^2; \mathbb{Z}) = H^2(M_0, \mathbb{R}P^2; \mathbb{Z}) = 0$, all the obstruction groups are zero. Therefore $\varphi \simeq \text{id}_{M_0}$. □

Combine Lemma 3.1, Lemma 3.2 and Lemma 3.4, we have a classification of manifolds homotopy equivalent to $M_0$.

**Theorem 3.5.** Let $M^6$ be a smooth manifold homotopy equivalent to $M_0$, then $p_1(M) = (48j + 4)u$, where $u \in H^1(M; \mathbb{Z})$ is the canonical generator: for each $j \in \mathbb{Z}$, up to diffeomorphism, there are two such manifolds with the same $p_1 = 48j + 4$.

Theorem 3.1 and Corollary 1.2 are direct consequences of this theorem.

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