LINEAR COMBINATIONS OF DIRICHLET SERIES ASSOCIATED WITH THE THUE-MORSE SEQUENCE

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Abstract

In this paper we study sums of Dirichlet series whose coefficients are terms of the Thue-Morse sequence and variations thereof. We find closed-form expressions for such sums in terms of known constants and functions including the Riemann zeta function and the Dirichlet eta function using elementary methods.

1. Introduction

Let \( t_n \) denote the binary digit sum of the positive integer \( n \) modulo 2, in other words, the \( n^{th} \) element of the Thue-Morse sequence \((t_n)_{n\geq0}\), which begins 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, ....

This sequence was first considered by Prouhet [10] and has applications in many different fields of mathematics (see for instance Allouche and Shallit [8] for a detailed overview). A popular variant is the \( \pm 1 \) Thue-Morse sequence, denoted by \( \varepsilon_n \) and defined as \(((−1)^{t_n})_{n\geq0}\). Dirichlet series associated with \( \varepsilon_n \) have been widely studied during the past years, in particular \( \sum_{n\geq0} \varepsilon_n (n+1)^{-s} \) and \( \sum_{n\geq0} \varepsilon_n (n+1)^{-1} \), which converge for \( \Re(s) > 1 \). Allouche and Cohen [3] continued the series analytically and gave the functional equation

\[
\sum_{n\geq0} \varepsilon_n (n+1)^{-s} = \sum_{k\geq1} 2^{-s-k} \binom{s+k-1}{k} \sum_{n\geq0} \frac{\varepsilon_n}{(n+1)^{s+k}},
\]

and showed that it admits non-trivial zeros at \( s = (2k\pi i)/\log 2 \) for any integer \( k \). The series \( \sum_{n\geq1} \varepsilon_n/n^s \) is continued in a similar manner, yielding a set of non-trivial zeros at \( s = i\pi(2k+1)/\log 2 \) (although as noted by Allouche [2], the question of whether these are all the non-trivial zeros is still an open one). These results were
then further extended by Allouche, Mendès France and Peyrière [5] to \( d \)-automatic sequences, for \( d \geq 2 \). Furthermore, Allouche and Cohen [3] and Alkauskas [1] have noticed that the two series above are related through the identity

\[
\sum_{n \geq 0} \frac{\varepsilon_n}{(n+1)^s} = \frac{1 - 2^s}{1 + 2^s} \sum_{n \geq 1} \frac{\varepsilon_n}{n^s},
\]

Since then, these series have been used in various contexts. For instance, Allouche and Cohen [3] used the derivative of \( \sum_{n \geq 0} \frac{\varepsilon_n}{(n+1)^s} \) at \( s = 0 \) to give an alternative proof of the Woods-Robbins product,

\[
\prod_{n \geq 0} \left( \frac{2n+1}{2n+2} \right)^{\varepsilon_n} = \frac{\sqrt{2}}{2},
\]

which has been extended to many different types of infinite products (see Allouche, Cohen, Mendès France and Shallit [4], Allouche and Sondow [9], Allouche, Riasat and Shallit [6] and Tóth [12] for example). Series involving the \( b \)-ary sum-of-digits function \( s_b(n) \) have also been studied. A classic example is

\[
\sum_{n \geq 1} \frac{s_b(n)}{n(n+1)} = \frac{b}{b-1} \log b,
\]

which was proved by Shallit [11], while other identities such as

\[
\sum_{n \geq 1} \frac{s_2(n)(2n + 1)}{n^2(n+1)^2} = \frac{\pi^2}{9}
\]

are due to Allouche and Shallit [7].

1.1. Scope of This Paper

No closed-form expression is currently known for Dirichlet series involving only \( t_n \) and \( t_{n-1} \) and combinations thereof in terms of known constants and functions. In this paper, we provide such closed forms and give a formula for generating similar linear combinations. A few examples are

\[
\sum_{n \geq 1} \frac{5t_{n-1} + 3t_n}{n^2} = \frac{2\pi^2}{3},
\]

\[
\sum_{n \geq 1} \frac{9t_{n-1} + 7t_n}{n^3} = 8\zeta(3).
\]

Then, we extend our results to combinations of sequences involving different alphabets \( \{a, b\} \) along the Thue-Morse sequence, that is, sequences with the substitutions
0 \to a \text{ and } 1 \to b \text{ over } t_n, \text{ with } a, b \in \mathbb{R}. \text{ An example is }
\sum_{n \geq 1} \frac{17q_{n-1} + 15r_n}{n^4} = 16\eta(4),

where \(\eta(s)\) denotes the alternating zeta function \(\sum_{n \geq 1} (-1)^{n-1}/n^s\) and \(q_n\) and \(r_n\) denote the sequences defined respectively by \(\{-\sqrt{2}, 1 - \sqrt{2}\}\) and \(\{17\sqrt{2} - 2, 17\sqrt{2} + 13\}\) along the Thue-Morse sequence.

2. Linear Combinations

Throughout the remainder of this paper, we shall use the following notation:

\[ f(s) = \sum_{n \geq 1} \frac{\varepsilon_{n-1}}{n^s}, \quad g(s) = \sum_{n \geq 1} \frac{\varepsilon_n}{n^s}, \quad \varphi(s) = \sum_{n \geq 1} \frac{t_{n-1}}{n^s}, \quad \gamma(s) = \sum_{n \geq 1} \frac{t_n}{n^s}. \]

We begin this section by recalling a result on Dirichlet series whose sign alternates following the Thue-Morse sequence, originally due to Allouche and Cohen [3] and already mentioned in the introduction.

**Lemma 1.** For all \(s \in \mathbb{C}\) with \(\Re(s) > 1\),
\[ f(s) = \frac{1 - 2^s}{1 + 2^s} g(s). \]

While Allouche and Cohen’s proof involves the analytic continuations of \(f\) and \(g\), a simpler proof was proposed by Alkauskas [1] in 2001 that we cannot resist replicating here. This proof relies only on the fact that every positive integer \(n\) can be uniquely represented as the product \(2^k(2m + 1)\) for \(k \geq 0\) and \(m \geq 0\).

**Proof of Lemma 1 ([1]).** We have
\[ f(s) = \sum_{k \geq 0, m \geq 0} \frac{\varepsilon_{2^k(2m + 1) - 1}}{2^{ks}(2m + 1)^s} = \sum_{k \geq 0, m \geq 0} \frac{(-1)^k \varepsilon_m}{2^{ks}(2m + 1)^s} = \frac{2^s}{2^s + 1} \sum_{m \geq 0} \frac{\varepsilon_m}{(2m + 1)^s}. \]

Now we know that
\[ \sum_{m \geq 0} \frac{\varepsilon_m}{(2m + 1)^s} = \sum_{m \geq 1} \frac{\varepsilon_m}{(2m)^s} - \sum_{m \geq 1} \frac{\varepsilon_m}{m^s}, \]
by splitting \(\sum_{m \geq 1} \varepsilon_m/m^s\) into even and odd indexes and using the fact that \(\varepsilon_{2m} = \varepsilon_m\) and \(\varepsilon_{2m+1} = -\varepsilon_m\). The proof follows naturally.

We note that one could also evaluate \(g(s)\) in the same manner, thus yielding a unified proof of Lemma 1. This is left as an exercise for the reader.
Corollary 1. For any holomorphic functions $u$ and $v$, we have

$$u(s)\varphi(s) + v(s)\gamma(s) = \frac{u(s) + v(s)}{2} \zeta(s) - \frac{f(s)}{2} \left( u(s) + v(s) \frac{1 + 2^s}{1 - 2^s} \right).$$

Proof. Using Lemma 1 together with $\varepsilon_n = 1 - 2t_n$, we have

$$\varphi(s) = \frac{1}{2} \zeta(s) - \frac{1}{2} f(s), \quad \gamma(s) = \frac{1}{2} \zeta(s) - \frac{1 + 2^s}{2(1 - 2^s)} f(s).$$

Now taking two holomorphic functions $u$ and $v$, we quickly obtain the above expression for $u(s)\varphi(s) + v(s)\gamma(s)$.

We now have all the tools to establish our first result in this paper.

Theorem 1. Let $\zeta(s) = \sum_{n \geq 1} 1/n^s$ denote the Riemann zeta function defined for complex $s$ with $\Re(s) > 1$. Then

$$(2^s + 1)\varphi(s) + (2^s - 1)\gamma(s) = 2^s \zeta(s).$$

Note that both Dirichlet series on the left-hand side converge for $\Re(s) > 1$ since the sequence $(t_n)_{n \geq 0}$ takes only finitely many values.

We shall now give two proofs of this identity. The first uses Corollary 1, while the second relies on index-splitting (used also within the context of the Woods-Robbins product by Allouche, Mendès France and Peyrière [5], for instance).

Proof 1 of Theorem 1. We take Corollary 1 with $u(s) = 2^s + 1$ and $v(s) = 2^s - 1$. The proof immediately follows.

Proof 2 of Theorem 1. We begin by splitting $\gamma(s)$ and $\varphi(s)$ into odd and even indexes. On the one hand, we have

$$\gamma(s) = 2^{-s} \gamma(s) - \sum_{n \geq 0} \frac{t_n}{(2n + 1)^s} + (1 - 2^{-s}) \zeta(s).$$

On the other hand,

$$\varphi(s) = 2^{-s} \zeta(s) - 2^{-s} \varphi(s) + \sum_{n \geq 0} \frac{t_n}{(2n + 1)^s}.$$

Now taking the sum of these two equations yields

$$\gamma(s) + \varphi(s) = 2^{-s} (\gamma(s) - \varphi(s)) + \zeta(s),$$

and a simple rearrangement of the terms concludes this proof.
Here we note that whenever $t_{n-1} = t_n = 0$, the corresponding $n^{\text{th}}$ term disappears from both Dirichlet series on the left-hand side. The first few missing terms are thus $n = 6, 10, 18, 24, \ldots$ (sequence A248056 in the OEIS). Our result above implies several interesting examples, which we have already mentioned in the introduction.

**Example 1.** We have the following equalities:

(a) $$\sum_{n \geq 1} \frac{5t_{n-1} + 3t_n}{n^2} = \frac{2\pi^2}{3},$$

(b) $$\sum_{n \geq 1} \frac{9t_{n-1} + 7t_n}{n^3} = 8\zeta(3).$$

There are several ways to extend these results, which we will do in the following sections.

### 2.1. Generalization to Different Alphabets Along the Thue-Morse Sequence

In the following paragraphs, we extend Theorem 1 to linear combinations of series involving various alphabets $\{a, b\}$ along the Thue-Morse sequence, i.e., sequences with the substitutions $0 \to a$ and $1 \to b$, with $a, b \in \mathbb{R}$, over the $t_n$ sequence. In particular, we show that there exist alphabets which give rise to identities involving the alternating zeta function $\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}$, others that give rise to identities involving the Riemann zeta function and, finally, alphabets that produce simple linear combinations with other Dirichlet series.

**Theorem 2.** Let $k, \ell > 0$ be real numbers and define the sequences $q_n = t_n - k$ and $r_n = t_n + \ell$ for all $n > 0$. Furthermore, consider the function $\lambda(s; k, \ell) = 2^s - (2^s(k - \ell) + (k + \ell))$ for some complex $s$ with $\Re(s) > 1$. We then have

(1) $$\sum_{n \geq 1} \frac{q_{n-1}}{n^s} = \frac{1 - 2^s}{1 + 2^s} \sum_{n \geq 1} \frac{r_n}{n^s}$$ if $\lambda(s; k, \ell) = 0,$

(2) $$(2^s + 1) \sum_{n \geq 1} \frac{q_{n-1}}{n^s} + (2^s - 1) \sum_{n \geq 1} \frac{r_n}{n^s} = 2^s \zeta(s)$$ if $\lambda(s; k, \ell) = 2^s,$

(3) $$(2^s + 1) \sum_{n \geq 1} \frac{q_{n-1}}{n^s} + (2^s - 1) \sum_{n \geq 1} \frac{r_n}{n^s} = 2^s \eta(s)$$ if $\lambda(s; k, \ell) = 2^s - 2.$

**Proof.** Define the sequences $q_n = t_n - k$ and $r_n = t_n + \ell$ for all $n > 0$ and real $k, \ell > 0$. We immediately have

$$\sum_{n \geq 1} \frac{q_{n-1}}{n^s} = \varphi(s) - k\zeta(s), \quad \sum_{n \geq 1} \frac{r_n}{n^s} = \gamma(s) + \ell\zeta(s).$$
Thus,
\[(2^s + 1) \sum_{n \geq 1} \frac{q_n - 1}{n^s} + (2^s - 1) \sum_{n \geq 1} \frac{r_n}{n^s} = \zeta(s) \left(2^s - (2^s(k - l) + (k + l))\right).\]

Inspired by the coefficient of \(\zeta(s)\) on the right-hand side, we define the function \(\lambda(s; k, \ell) = 2^s - (2^s(k - \ell) + (k + \ell))\) for real \(k, \ell > 0\) and complex \(s\) with \(\Re(s) > 1\). It is obvious that if \(\lambda(s; k, \ell) = 0\), the \(\zeta(s)\) term on the right-hand side vanishes, thus proving (1). If \(\lambda(s; k, \ell) = 2^s\), we have identities of the same type as in Theorem 1, proving (2), and if \(\lambda(s; k, \ell) = 2^s - 2\) then the identity \(\eta(s) = (1 - 2^{1-s}) \zeta(s)\) quickly establishes (3).

Solutions to each of these cases lead to interesting illustrative examples. A first set of “simple” solutions – in terms of \(k\) and \(\ell\) only – are easy to find. For the case (1) above, we have \(k = \frac{1}{2}, \ell = -\frac{1}{2}\), which leads to the classic Woods-Robbins product by differentiation of the resulting series identity at \(s = 0\), as already noted by Allouche and Cohen [3]. The solution of the second case, \(k = 0, \ell = 0\), results in our identity in Theorem 1, and finally that of the third case \((k = 1, \ell = 1)\) results in
\[(2^s + 1) \sum_{n \geq 1} \frac{q_n - 1}{n^s} + (2^s - 1) \sum_{n \geq 1} \frac{r_n}{n^s} = 2^s \eta(s),\]
where \(q_n \to \{-1, 0\}\) and \(r_n \to \{1, 2\}\) along the Thue-Morse sequence.

Another, perhaps more interesting set of solutions can be found by taking \(s\) into account as well.

**Proposition 1.** Define the sequences \(q_n = t_n - k\) and \(r_n = t_n + \ell\) for all \(n > 0\) and real \(k, \ell > 0\). We have the following equalities:

(a) \[5 \sum_{n \geq 1} \frac{q_n - 1}{n^2} = -3 \sum_{n \geq 1} \frac{r_n}{n^2},\]
\(q_n \to \{-1, 0\}, r_n \to \{\frac{1}{3}, \frac{4}{3}\}\)

(b) \[9 \sum_{n \geq 1} \frac{q_n - 1 + 7r_n}{n^3} = 8 \zeta(3),\]
\(q_n \to \{-1, 0\}, r_n \to \{\frac{9}{7}, \frac{16}{7}\}\)

(c) \[17 \sum_{n \geq 1} \frac{q_n - 1 + 15r_n}{n^4} = 16 \eta(4),\]
\(q_n \to \{-\sqrt{2}, 1 - \sqrt{2}\}, r_n \to \{\frac{17\sqrt{2} - 2}{15}, \frac{17\sqrt{2} + 13}{15}\}\)

Proof. The general solution of the equation \(\lambda(s; k, \ell) = 0\) (i.e., case (1) in Theorem 2) in the reals is
\[s = \frac{\log \left(\frac{k + \ell}{2k + k + 1}\right)}{\log 2},\]
with \(k \neq \ell + 1\) and \(k + \ell \neq 0\). So for instance we can take \(k = 1\) and \(\ell = \frac{1}{3}\), yielding the sequences \(q_n = t_n - 1\) and \(r_n = t_n + \frac{1}{3}\), for all \(n \geq 0\), i.e., the sequences...
defined respectively by $\{-1, 0\}$ and $\{\frac{1}{3}, \frac{2}{3}\}$ along the Thue-Morse sequence. A simple substitution above gives $s = 2$, thereby proving our statement (a).

We now turn our attention to the solution of $\lambda(s; k, \ell) = 2^s$ (i.e., case (2) in Theorem 2), which in the reals is

$$s = \frac{\log(-k+\ell)}{\log 2},$$

with $k - \ell \neq 0$ and $k + \ell \neq 0$. Let now $q_n = t_n - 1$ and $r_n = t_n + \frac{9}{7}$, for all $n \geq 0$, in other words the sequences defined respectively by $\{-1, 0\}$ and $\{\frac{9}{7}, \frac{16}{7}\}$ along the Thue-Morse sequence. This means that we have $k = 1$ and $\ell = \frac{9}{7}$, which yields $s = 3$ in the equation above and thus proves statement (b).

Finally, the solution of the equation $\lambda(s; k, \ell) = 2^s - 2$ in the reals, corresponding to case (3) in Theorem 2 is

$$s = \frac{\log(-k+\ell-2)}{\log 2},$$

with $k - \ell \neq 0$ and $k + \ell \neq 2$, which allows us to choose $k = \sqrt{2}$ and $\ell = \frac{17\sqrt{2}-2}{15}$. This gives the sequences $q_n = t_n - \sqrt{2}$ and $r_n = t_n + \frac{17\sqrt{2}-2}{15}$, for all $n \geq 0$, i.e., the sequences defined respectively by $\{-\sqrt{2}, 1-\sqrt{2}\}$ and $\{\frac{17\sqrt{2}-2}{15}, \frac{17\sqrt{2}+13}{15}\}$ along the Thue-Morse sequence. Thus we have $s = 4$ and identity (c) as claimed.

\[\Box\]

3. Conclusion and Further Work

In this paper we have found closed forms for certain linear combinations of Dirichlet series associated with the Thue-Morse sequence in terms of known constants and functions. However, closed forms for the individual series remain elusive.

Question 1. Do the series $\sum_{n \geq 1} \frac{t_n}{n^s}$ and $\sum_{n \geq 1} \frac{t_{n-1}}{n^s}$ for $s \in \mathbb{C}$ with $\Re(s) > 1$ admit closed forms in terms of known constants and functions?

Despite our efforts, we have not been able to find a set of linear combinations allowing us to eliminate either $f$ or $g$ from Theorem 1.

3.1. Extension to Other Sequences

In some of our proofs we used an index-splitting method to find expressions for series involving the Thue-Morse sequence and variations thereof. The same method can possibly be applied to other series whose coefficients are generated by finite automata. A few examples are listed below, and the proofs are left to the reader.
Example 2. Let $\delta_n = t_n - t_{n-1}$ for all $n \geq 1$ and $\varepsilon_n = (-1)^{t_n}$ the ±1 Thue-Morse sequence. Then for all $s$ with $\Re(s) > 1$,

$$\sum_{n\geq 1} \frac{\delta_n}{n^s} = \frac{4^s}{4^s - 1} \sum_{n\geq 0} \frac{\varepsilon_n}{(2n + 1)^s}.$$ 

Example 3. Let $\sigma_n$ denote the “period-doubling sequence” (A096268 in the OEIS), defined by the recurrence $\sigma_{2n} = 0, \sigma_{4n+1} = 1, \sigma_{4n+3} = \sigma_n$. Then for all $s$ with $\Re(s) > 1$,

$$\sum_{n\geq 1} \frac{\sigma_n ((4n + 3)^s - n^s)}{(4n^2 + 3n)^s} = 4^{-s} \zeta\left(s, \frac{1}{4}\right),$$

where $\zeta(s, a)$ denotes the Hurwitz zeta function.

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