A chain of strongly correlated $SU(2)_4$ anyons: Hamiltonian and Hilbert space of states

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Abstract

One-dimensional lattice model of $SU(2)_4$ anyons containing a transition into the topological ordered phase state is considered. An effective low-energy Hamiltonian is
found for half-integer and integer indices of the type of strongly correlated non-Abelian anyons. The Hilbert state space properties in the considered modular tensor category are studied.

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1 Introduction

In basic models of quantum magnets, the states of particles with half-integer and integer values of spin, which are distributed over lattice sites, correspond to the fundamental or adjoint irreducible representation of the $SU(2)$ group and are classified according to odd or even representations of the permutation group. In spatially one-dimensional or two-dimensional systems, instead of the permutation group, we deal with the braid group and its Abelian and non-Abelian irreducible representations. Therefore, in the generalized Heisenberg or Ising models, neither fermions, nor bosons are distributed over lattice sites, but the quantum states of particles realizing irreducible representations of the braid group and corresponding to fractional statistics.

The set of quantum numbers indicating the type of these so-called anyon states comprises integers specifying the structure of quantum groups, e.g., $SU(2)_k$. Here, $k$ is the $SU(2)_k$ level of the Wess-Zumino-Witten-Novikov theory. This number which coincides with the coefficient in the Chern-Simons action, determines the braiding degree of excitation worldlines in the $(2+1)D$ systems. When $k = 1$, fermion worldlines are parallel. If $k = 2$, worldlines are linked twice and we deal with semi-fermion quasiparticles, the so-called semions. Such anyons are primary fields of the $SU(2)_2$ conformal field theory as well as that of the transverse field Ising model at the critical point [1] [2] [3].

Anyon states with $k = 3$ form the Fibonacci family. Detailed analysis of these non-Abelian states [1] [4] and generalization of the Heisenberg Hamiltonian to the case of interacting Fibonacci anyons are presented in Refs. [5] [6]. Paper [7] is devoted to collective states

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for a higher level of $k$ in the model of interacting $SU(2)_5$ anyons belonging to the sector with integer spins. A characteristic feature of anyon systems for $k > 2$ is the presence of a term describing the three-body interaction in the Hamiltonian. Another important property is the fact that the squared total quantum dimension of states with $k = 1, 2, 4$ is an integer. In this case, the density of braid group representations is such that it does not provide universal quantum calculations \[8, 9\]. For $k \rightarrow \infty$, the Heisenberg Hamiltonian is restored for particles with the spin 1/2, while for particles with an arbitrary spin, we have its polynomial variant \[10\]. To elucidate the Hamiltonian structure of the theory for intermediate values of $k$, in this paper we focus on the theory of anyons in the case of $k = 4$.

Studying of anyon systems for small $k$ is important for several reasons. First of all, due to their possible use for decoherence-free quantum computations based on the rigidity of the wave-function phase of non-Abelian excitations \[11\]. For this reason, anyons of the transverse field Ising model and the $SU(2)_k$ theory with $k = 2$ differing by the sign of the Frobenius-Schur indicator and Fibonacci anyons for $k = 3$ were analyzed \[12\] in detail as probable candidates for the role of non-Abelian states in the fractional quantum Hall effect with the filling factor $\nu = 2 + \frac{k}{k+2}$. Phase effects in interferometry \[13, 14\] and tunneling \[15\] of non-Abelian anyon states for finding experimental consequences were studied in a number of recent papers (see also review \[1\]). Discussion of the criteria of anyon deconfinement \[16\], search of correspondences between the lattice theory at the critical point and the conformal field theory determined on coset-spaces as the source of the nucleation effects in parent two-dimensional spin liquid and edge modes at their boundaries \[7, 17\], as well as detailed description of topology driven quantum phase transitions \[18\] in time-reversal invariant systems (in quantum doubles) are subjects of recent papers in this challenging field.

In this paper, which employs the approach developed in Refs. \[6, 7\], we have found the Hamiltonian and the Hilbert space of states in the $SU(2)_4$ theory for chains of half-integer and integer spins indicating sectors of interacting anyons. Since we are interested in the low-energy states, the effective low-energy Hamiltonian has the form of projectors on the ground state with a fixed spin. Some aspects of the problem associated with representation of the Hamiltonian by means of projectors of the Temperley-Lieb algebra have been discussed in Refs. \[5, 18\]. Maps of Hamiltonians \[19, 20\] of the exact solvable quantum models \[11, 21\], study of correlation functions \[22\], and classification of the tolopological order \[23\] and quantum phase transitions are also considered.

We will pay attention to some universal features of externally different models and discuss a correspondence to the conformal field theory in description of the braiding effects. The main employed tools belong to the theory of modular tensor categories, in terms of which the anyon theory is formulated now. Therefore, in the next section we present some technique from the theory of modular tensor categories for describing the Hamiltonian and the eigenstate space considered in the third and the fourth sections. In conclusion, we discuss the unsolved problems related to the universal representation of Hamiltonian dynamics of the exact integrable spin models describing the topological ordered phase states, which are characterized by non-Abelian anyon excitations.

2 Chain of $SU(2)_4$-anyons

Let us consider the Klein-type Hamiltonian family \[24\]
\[ H = - \sum_{i=1}^{N} g_j P_i^{(j)}. \]  

Here, index \( i \) enumerates links of the chain, \( \{g_j\} \) are the coupling constants, and \( P_i^{(j)} \) are projectors on states with the spin \( j \).

In systems with non-Abelian quasiparticles, strongly correlated states form some part of the low-energy Hilbert space. Since we are primarily interested in the low-energy limit, this property of non-Abelian anyons is an argument for representation of effective Hamiltonian \( \textbf{(1)} \) by means of operators with two eigenvalues corresponding to the ground and excited states. Besides, when the considered particles are spatially separated, the low-energy space proves to be degenerate; constrains are imposed on the degree of degeneracy, which are characterized by topological quantum numbers.

One of the methods for describing the topology effects related to braiding of quasiparticle worldlines is based on the use of Temperley-Lieb algebra operators. The Temperley-Lieb algebra operators \( e_i \) satisfy the commutation relations

\[ e_i^2 = d e_i, \]
\[ e_i e_{i+1} e_i = e_i, e_i e_j = e_j e_i, \quad (j \neq i \pm 1). \]

This implies that the operators \( P_i = e_i/d \) are projectors: \( P_i^2 = P_i \). Here, the isotopic parameter is \( d = 2 \cos\left(\frac{\pi}{k+2}\right) \).

The Hamiltonian for a small anisotropy parameter \( x \) can be obtained using the transfer matrix \( T = t_1 t_2 \cdots t_2 N - 1 t_0 t_2 t_4 \cdots t_2 N - 2 t_2 N \) with \( t_i = I_i + x e_i \) and the identity matrix \( I_i \). The Temperley-Lieb algebra operators \( e_i \) are defined here \( \textbf{(2)} \) as

\[ e[i][j_i-1;j_i,j_i+1] = \sum_{j_i'} \left( e[i][j_i-1;j_i,j_i+1] \right)_{j_i} \left. |j_i-1;j_i',j_i+1\rangle \right\rangle \]
\[ e[i][j_i-1;j_i,j_i+1] = \delta_{j_i-1,j_i+1} \sqrt{\frac{S^{\alpha}_{j_i} S^{\alpha}_{j_i+1}}{S^{\alpha}_{j_i-1} S^{\alpha}_{j_i+1}}}, \]

where \( S^{\alpha}_{j_i} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi (2j_i+1)(2j_i'+1)}{k+2}\right) \) is the modular \( S \)-matrix \( SU(2)_k \) of the conformal field theory.

It follows from expressions \( \textbf{(2)} \) that in the considered weight representation of the operators \( e_i \), anyon degrees of freedom are distributed over the three neighboring links \( (i-1,i,i+1) \) of the chain. Their filling by some value of the spin \( a \) from the anyon-type set \( j = 0, 1/2, 1, \ldots, k/2 \) will be used below as the collective label of the quantum states \( |j_{i-1},j_i,j_{i+1}\rangle \). The relationship between the admissible values from the set \( \{a,b,c\} \) of the angular momentum values is determined by the fusion rules (tensor product) of the two anyon states in the \( SU(2)_k \) theory:

\[ j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2). \]

Composition of the fields truncated by the level \( k \) after fusion is such that instead of the standard result in the case of the \( SU(2) \) tensor product, we have a smaller number of fields at the output.
The fusion rules $a \times b = \sum_c N_{ab}^c$ of primary fields for possible values of the index $j$ for $k = 4$, the quantum dimensions $d_j = [2j + 1]_q = \frac{q^j - q^{-(j+1)/2}}{q^j - q^{-(j+1)/2}} \approx \sin[\pi(2j+1)/k+2]/\sin[\pi/j+k+2]$ (for $q = e^{2\pi i/(k+2)}$) being the largest eigenvalues of the matrix $(N_a)^b_j$ with the total quantum dimension $D = \sqrt{\sum_a d_a^2} = \sqrt{\frac{\pi}{\sin(\pi/k+2)}} = 2\sqrt{3}$, and the conformal scale dimensions $h_j = \frac{j(2j+1)}{k+2}$ determining the topological spin $\theta_j = e^{2\pi i h_j}$ are summarized in the table

| $j$  | $d_0 = 1$ | $h_0 = 0$ | $d_1/2 = \sqrt{3}$ | $h_{1/2} = \frac{1}{8}$ | $d_1 = 2$ | $h_1 = \frac{1}{3}$ | $d_3/2 = \sqrt{3}$ | $h_{3/2} = \frac{5}{8}$ | $d_2 = 1$ | $h_2 = 1$ |
|------|-----------|-----------|--------------------|------------------------|-----------|------------|--------------------|------------------------|-----------|------------|
| $1/2 \times 0 = 1/2$ | $1/2 \times 1/2 = 0 + 1$ | $1 \times 0 = 1$ | $1/2 \times 1 = 1/2 + 3/2$ | $1 \times 1 = 0 + 1 + 2$ | $3/2 \times 0 = 3/2$ | $1/2 \times 3/2 = 1 + 2$ | $1 \times 3/2 = 1/2 + 3/2$ | $3/2 \times 3/2 = 0 + 1$ | $2 \times 0 = 2$ |
| $1/2 \times 1/2 = 1/2$ | $1 \times 2 = 1$ | $3/2 \times 2 = 1/2$ | $2 \times 2 = 0$ |

Table 1: Anyon types numbered by values of the index $j$, quantum and conformal dimensions, and fusion rules for the $SU(2)_k$ theory.

When two fields fuse, the vector space $V_{a_b}^c$ of the dimension $\dim V_{a_b}^c$ equal to $N_{a_b}^c$ occurs. In our case, $N_{a_b}^c = 0$. For the non-Abelian states indexed in the table by the spin values $j$, $d_j > 1$. Non-Abelian anyons are also characterized by a multiplicity of allowed channels of fusion (splitting) ($\sum_c N_{a_b}^c \geq 2$ for some value of the spin $b$).

Splitting of the anyon $b$ into anyon states with the spins $a, b, c$ corresponds to the space $V_{ab}^{bc}$, which can be represented in the form of tensor products of two splitted anyon spaces by matching appropriate pairs of indices. This can be done by two isomorphic ways:

$$V_{ab}^{bc} \simeq \bigoplus_e V_{ae}^c \otimes V_{d_c}^{ec} \simeq \bigoplus_f V_{af}^c \otimes V_{bf}^{cf} , \quad (4)$$

To introduce the notion of associativity at the level of splitting spaces, it is necessary to describe the set of unitary isomorphisms between different decompositions, which can be considered as a change of the basis. These isomorphic transformations can be written as a diagram using the so-called $F$-matrix:

$$\sum_f F_{def}^{abc} = \sum_f F_{def}^{abc} , \quad (5)$$

The $F$-matrix satisfies the pentagonal equation

$$\sum_n F_{kpn}^{mlq} F_{lns}^{ipj} F_{lns}^{jkn} = F_{qkr}^{jip} F_{mls}^{iq} , \quad (6)$$
which has the solution \[ \text{[29]} \] proportional to the \( q \)-deformed analog of the 6j-symbols:

\[
F_{j_1 j_2 j_3}^{j_4 j_5 j_6} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{[2j_{12} + 1]_q [2j_{23} + 1]_q} \left\{ j_1 \atop j_3 \right\}_q \left\{ j_2 \atop j \right\}_q \left\{ j_4 \atop j_6 \right\}_q \quad \text{(7)}
\]

\[
\times \sum_n \left\{ \frac{(-1)^n [n + 1]_q!}{[n - j_1 - j_2 - j_12]_q! [n - j_2 - j_3 - j]_q! [n - j_2 - j_3 - j_23]_q! [n - j_1 - j_23 - j]_q!} \right\}
\]

\[
\Delta \left( j_1, j_2, j_3 \right) = \sqrt{\frac{[j_{12} + j_{23} + 1]_q! [j_{12} - j_{23}]_q! [j_{12} - j_{23}]_q!}{[j_{12} + j_{23} + 1]_q!}} \quad [n]_q! = \prod_{m=1}^{n} [m]_q .
\]

In the following two sections, we will calculate the \( F \)-matrix different from zero or unity by using Eq. (7). We have not presented here the expressions for the braiding \( R_{ab}^{bc} \)-matrix of two anyons and the hexagon equation, because the latter is a consequence \[ \text{[30]} \] of the pentagon equation.

3 The Hamiltonian and the Hilbert space of states

When constructing effective Hamiltonian \[ (1) \], we employ the representation \[ \text{[5]} \]

\[
\langle a', f', d'| P^{(j)} | a, e, d \rangle = F_{d' j}^{a' \delta} F_{de j}^{a b c} \delta_{a', a} \delta_{d', d} \quad \text{(8)}
\]

for projectors in the channel with the total spin \( j \), using the \( F \)-matrix. In Eq. (8), the summation over repetitive indices is absent. In the case of the Heisenberg Hamiltonian \( H = \sum_i (S_i S_{i+1}) \) for the spin equal to unity, we have \( (S_i S_{i+1}) = 3P_i^{(2)} + P_i^{(1)} - 2 \). For the Affleck-Kennedy-Lieb-Tasaki model \( H = \sum_i P_i^{(2)} \). In our case, the superscripts \( (b, c) \) in \[ (8) \] will acquire the value 1/2. In the sector with an integer value of the indices, these labels of the anyon species will be as follows: \( b = c = 1 \).

Let us consider first the half-integer values \( b = c = 1/2 \) of the anyon index in the \( F \)-matrix. In this case, the admissible state space \( \{ |a, f, d \rangle \} \) is fourteen-dimensional and represents the following state set shown in the right-hand side of Eq. \[ \text{[5]} \]:

\[
\{ |0, 0, 0 \rangle, |0, 1, 1 \rangle, |1/2, 0, 1/2 \rangle, |1/2, 1, 1/2 \rangle, |1/2, 1, 3/2 \rangle, |1, 1, 0 \rangle, |1, 0, 1 \rangle, |1, 1, 1 \rangle, |1, 1, 2 \rangle, |3/2, 1, 1/2 \rangle, |3/2, 0, 3/2 \rangle, |3/2, 1, 3/2 \rangle, |2, 1, 1 \rangle, |2, 0, 2 \rangle \} .
\]

The Hilbert space \( \{ |a, e, d \rangle \} \) belonging to the left-hand side of Eq. \[ \text{[5]} \] consists of the following states:

\[
\{ |0, 0, 0 \rangle, |0, 1, 1 \rangle, |1/2, 0, 1/2 \rangle, |1/2, 1, 1/2 \rangle, |1/2, 1, 3/2 \rangle, |1, 1, 0 \rangle, |1, 0, 1 \rangle, |1, 1, 1 \rangle, |1, 1, 2 \rangle, |3/2, 1, 1/2 \rangle, |3/2, 0, 3/2 \rangle, |3/2, 1, 3/2 \rangle, |2, 1, 1 \rangle, |2, 0, 2 \rangle \} .
\]
where the elements of the F-matrices. In calculation of the elements of the matrices in these formulas, we used explicit expression \( (17) \) for the \( q \)-deformed values of the \( 6j \)-symbols.

Let us employ the given values of the F-matrix to calculate the projectors by means of the F-matrix. In the case of \( b = c = 1/2 \) and the level \( k = 4 \), the following two projectors prove to be nonzero:

\[
P^{(0)} = \text{diag} (0, 0, 0, 0, 0, 0, 0, 0, 0, 0) ,
\]

\[
P^{(1)} = \text{diag} (0, 0, B, 1, 0, 0, 0, 0, C, 0) ,
\]

where the \( 2 \times 2 \) matrices \( A, B, C \) are present in diagonals and have the forms

\[
A = \left( \begin{array}{cc} \left( F_{3/4}^{+} \right)^2 & F_{1}^{+} F_{3/4}^{+} \\ F_{3/4}^{-} F_{1/4}^{-} & F_{3/4}^{+} F_{3/4}^{-} \end{array} \right) = \frac{1}{3} \left( \begin{array}{cc} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{array} \right) ,
\]

\[
B = \left( \begin{array}{cc} F_{3/4}^{+} F_{3/4}^{-} & F_{1}^{+} F_{1/4}^{-} \\ F_{1/4}^{+} F_{3/4}^{-} & F_{1}^{+} F_{3/4}^{-} \end{array} \right) = \frac{1}{3} \left( \begin{array}{cc} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{array} \right) ,
\]

\[
C = \left( \begin{array}{cc} F_{3/4}^{+} F_{1/4}^{-} & F_{1}^{+} F_{3/4}^{-} \\ F_{1/4}^{+} F_{3/4}^{-} & F_{1}^{+} F_{3/4}^{-} \end{array} \right) = \frac{1}{3} \left( \begin{array}{cc} -\sqrt{2} & -2 \\ 2 & \sqrt{2} \end{array} \right) .
\]
The enumeration in Eqs. (15), (16) corresponds to listing of states in Eq. (10). As a result, the Hamiltonian of the chain of the $SU(2)_4$ anyons is as follows:

$$H = -\sum_{i=1}^N \left( g_0 P_i^{(0)} + g_1 P_i^{(1)} \right)$$

with projectors from Eqs. (15), (16).

4 Integer anyon index

In this section, we shall consider integer values of the index $c = b = 1$ of anyon types. In this case, the admissible state space $\{ |a, f, d \rangle \}$ is nineteen-dimensional and represents the state set shown in the right-hand side of Eq. (3):

$$\{ |0, 0, 0 \rangle, |0, 1, 1 \rangle, |0, 2, 2 \rangle, |1/2, 0, 1/2 \rangle, |1/2, 1, 1/2 \rangle, |1/2, 1, 3/2 \rangle,$$

$$|1/2, 2, 1/2 \rangle, |1, 0, 1 \rangle, |1, 1, 1 \rangle, |1, 1, 2 \rangle, |1, 2, 1 \rangle, |1, 1, 0 \rangle, |1, 1, 2 \rangle,$$

$$|3/2, 1, 1/2 \rangle, |3/2, 2, 1/2 \rangle, |3/2, 0, 3/2 \rangle, |3/2, 1, 3/2 \rangle,$$

$$|2, 2, 0 \rangle, |2, 1, 1 \rangle, |2, 0, 2 \rangle \}.$$ 

The Hilbert state space $\{ |a, c, d \rangle \}$ shown in the left-hand side of Eq. (3) consists of the following states:

$$\{ |0, 1, 0 \rangle, |0, 1, 1 \rangle, |0, 1, 2 \rangle, |1/2, 1/2, 1/2 \rangle, |1/2, 3/2, 1/2 \rangle, |1/2, 1/2, 3/2 \rangle,$$

$$|1/2, 3/2, 3/2 \rangle, |1, 0, 1 \rangle, |1, 1, 1 \rangle, |1, 1, 2 \rangle, |1, 2, 1 \rangle, |1, 1, 0 \rangle, |1, 1, 2 \rangle,$$

$$|3/2, 3/2, 1/2 \rangle, |3/2, 3/2, 2/2 \rangle, |3/2, 1/2, 3/2 \rangle, |3/2, 3/2, 3/2 \rangle,$$

$$|2, 1, 0 \rangle, |2, 1, 1 \rangle, |2, 1, 2 \rangle \}.$$

The one-dimensional $F$-matrices equal to unity correspond to the following sequence of indices:

$$F_{010}^{011} = F_{121}^{012} = F_{141}^{232} = F_{021}^{122} = 1.$$

The four two-dimensional $F$-matrices $\hat{M}_j$ are equal to

$$\hat{M}_\frac{1}{2} = F_{\frac{1}{2} c f}^{\frac{1}{2} 1} = \begin{pmatrix} F_{\frac{1}{2} 0}^{\frac{1}{2} 0} & F_{\frac{1}{2} 1}^{\frac{1}{2} 0} \\ F_{\frac{1}{2} 0}^{\frac{1}{2} 1} & F_{\frac{1}{2} 1}^{\frac{1}{2} 1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\hat{M}_\frac{3}{2} = F_{\frac{3}{2} c f}^{\frac{3}{2} 1} = \begin{pmatrix} F_{\frac{3}{2} 0}^{\frac{3}{2} 0} & F_{\frac{3}{2} 1}^{\frac{3}{2} 0} \\ F_{\frac{3}{2} 0}^{\frac{3}{2} 1} & F_{\frac{3}{2} 1}^{\frac{3}{2} 1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\hat{M}_\frac{5}{2} = F_{\frac{5}{2} c f}^{\frac{5}{2} 1} = \begin{pmatrix} F_{\frac{5}{2} 0}^{\frac{5}{2} 0} & F_{\frac{5}{2} 1}^{\frac{5}{2} 0} \\ F_{\frac{5}{2} 0}^{\frac{5}{2} 1} & F_{\frac{5}{2} 1}^{\frac{5}{2} 1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$
The $F$-symbol also contains a three-dimensional matrix having the form

\[
\hat{M}_1 = F^{(11)}_{1ef} = \begin{pmatrix}
F_{100}^{1} & F_{101}^{1} & F_{102}^{1} \\
F_{110}^{1} & F_{111}^{1} & F_{112}^{1} \\
F_{120}^{1} & F_{121}^{1} & F_{122}^{1}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & -\sqrt{2} & 1 \\
-\sqrt{2} & 0 & \sqrt{2} \\
1 & \sqrt{2} & 1
\end{pmatrix}.
\]  

(28)

The projectors $\hat{P}^{(j)}$ with $j = 0, 1, 2$ and $e = 0, 1, 2, f = 0, 1, 2$ determined by Eq. (28) in the three-dimensional subspace are equal to

\[
\hat{P}^{(0)} = F^{(11)}_{100} F^{(11)}_{10f} = \begin{pmatrix}
F_{00} F_{00} & F_{00} F_{01} & F_{00} F_{02} \\
F_{10} F_{00} & F_{10} F_{01} & F_{10} F_{02} \\
F_{20} F_{00} & F_{20} F_{01} & F_{20} F_{02}
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
1 & -\sqrt{2} & 1 \\
-\sqrt{2} & 2 & -\sqrt{2} \\
1 & -\sqrt{2} & 1
\end{pmatrix},
\]

(29)

\[
\hat{P}^{(1)} = F^{(11)}_{110} F^{(11)}_{11f} = \begin{pmatrix}
F_{01} F_{10} & F_{01} F_{11} & F_{01} F_{12} \\
F_{11} F_{10} & F_{11} F_{11} & F_{11} F_{12} \\
F_{21} F_{10} & F_{21} F_{11} & F_{21} F_{12}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 0 & -1 \\
0 & 0 & 0 \\
-1 & 0 & 1
\end{pmatrix},
\]

(30)

\[
\hat{P}^{(2)} = F^{(11)}_{120} F^{(11)}_{12f} = \begin{pmatrix}
F_{02} F_{20} & F_{02} F_{21} & F_{02} F_{22} \\
F_{12} F_{20} & F_{12} F_{21} & F_{12} F_{22} \\
F_{22} F_{20} & F_{22} F_{21} & F_{22} F_{22}
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
1 & \sqrt{2} & 1 \\
\sqrt{2} & 2 & \sqrt{2} \\
1 & \sqrt{2} & 1
\end{pmatrix}.
\]

(31)

The projectors $\hat{P}^{(j)}$ dependent on the matrix $\hat{M}_j$ in two-dimensional subspaces with half-integer indices $a$ and $d$ are zero: $\hat{P}^{(1/2)} = \hat{P}^{(3/2)} = \hat{P}^{(5/2)} = 0$.

As a result, for $k = 4$ and $a = b = 1$ we have $\hat{P}^{(j)} = 0$ for half-integer values of the anyon type and the following nonzero projectors in the integer-valued sector

\[
\hat{P}^{(0)} = \text{diag} \left( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right),
\]

(32)

\[
\hat{P}^{(1)} = \text{diag} \left( 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 \right),
\]

(33)

\[
\hat{P}^{(2)} = \text{diag} \left( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right).
\]

(34)

where the three-dimensional matrices $\hat{P}^{(j)}$ are located on diagonals.

All aforesaid means that the Hamiltonian of the chain of the $SU(2)_4$-anyons for the ladder indices $b = c = 1$ has the form

\[
H = - \sum_{i=1}^{N} \left( g_0 \hat{P}^{(0)}_i + g_1 \hat{P}^{(1)}_i + g_2 \hat{P}^{(2)}_i \right)
\]

(35)

with projectors from Eqs. (32), (33), and (34).
5 Conclusion

The features of the Hamiltonian dynamics in the case of the $SU(2)_4$ theory can be revealed by comparing it with the $SU(2)_3$ theory of Fibonacci anyons and the $SU(2)_5$ theory \[6\]. In the latter case for $b = c = 1$, of interest is the absence into total projectors (32)-(34) of the contribution made by projectors in the two-dimensional subspace. In contrast with the Hamiltonian of Fibonacci anyons, the one-dimensional projector is absent in the Hamiltonian of the $SU(2)_4$ theory, when the zero energy is assigned to the state with $j = 1$.

For the coinciding coupling constants in the Hamiltonian (35), we have the model describing the critical state; in the continuous limit, it coincides with the rational conformal field theory with the central charge \(c = 4/5\). For the opposite sign of the coupling constants, the considered case \(k = 4\) in the continuous limit corresponds to the $Z_4$ theory of parafermions with the central charge \(c = 1\). If the coupling constants do not coincide, we can analyze the type of gapped phase states by assigning the zero energy to the phase state with the spin \(j = 1\) in (20), i.e., by setting \(g_1 = 0\) in (20) and \(g_0 = 0\) with the spin \(j = 0\) in (35). This can be done by changing the angle \(\theta\) for parametrization of the coupling constants \(g_1 = \sin\theta, g_2 = \cos\theta\) in (35) as it was done in Refs. \[7, 6\]. We plan to consider this problem in a separate paper. In this paper, we focus on the study of the Hamiltonian representation of exact integrable anyon systems in the form of projectors in a more general formulation.

When discussing weight representation (2) of the Temperley-Lieb algebra, we have already mentioned that the Hamiltonian in the extremely anisotropic case $x \to 0$ at the boundary of some regimes of the exact integrable $RSOS$-model \[31, 32\] can be obtained using the $R$-matrix. This matrix satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (36)$$

In this equation, we employed standard notations $R_{12} = R \otimes R \otimes 1$, etc. with the $R$ matrix realizing self-mapping of the space $V \otimes V \mapsto$ in itself.

In its turn, the $R$-matrix of the models describing topological ordered phase states can be constructed using the solution \[33\]

$$R_{12,34} = (F_{11}^{t_1})^{-1} F_{13} F_{24} (F_{23}^{-1})^{t_2} \quad (37)$$

of the pentagon equation

$$F_{12} F_{13} F_{23} = F_{23} F_{12} \quad (38)$$

Here, $R_{12,34} \in W \otimes W$, where $W = V \otimes V^*$, and the indices $t$ and $t_i$ mean, respectively, complete and partial transpositions in the $i$-th space. The indices of the $F$-matrices in Eqs. (37) and (38) belong to the four faces of tetrahedron. In the general case, the $F$-matrix also depends on the spectral parameter $x$ \[33, 34\], the small value of which yields the Hamiltonian. For example, the case \[34\]

$$R(x) = [x - 1]_q + [x]_q e_1 \quad (39)$$

yields the known answers \[18, 35\].

Let us consider the $(2 + 1)D$ models and the tensor $T_{ijkl}$ which depends on the indices $(i, j, k, l)$ belonging to the tetrahedron faces. Each of these indices is the collective label comprising variables from the sets $(a, b, c, d, e, f)$, $(i_s, j_s, k_s, l_s)$ with $s = 1, 2, 3, 4$ of the...
indices belonging respectively to edges and vertices of each tetrahedron face. Employing the $T_{ijkl}$-tensors, we can find the partition sum \[ Z = \text{Tr} \ e^{-\beta H} = \sum_{ijkl} T_{jfei} T_{hjkT_{qklr}} T_{lits} \cdots = t \text{Tr} \otimes_i T \] (40) by computing the tensor trace $t \text{Tr} \otimes_i T$. To carry out the calculations, anzats for the $T_{ijkl}$-tensor can be written as

$$T_{[i][j][k][l]} = F_{def}^{abc} \cdot \lambda(i_1, j_1, k_1) \cdot \lambda(i_2, j_2, l_2) \cdot \lambda(i_3, k_3, l_3) \cdot \lambda(j_4, k_4, l_4).$$ (41)

Since we are usually interested in the system behavior at the distances much larger than the lattice constant, now we consider also the final stage of the renormalization group flow, which does not contain data about the distribution of the degrees of freedom at small distances. The fixed surface at this final stage of the renormalization group flow in the terms of the functions from Eq. \[41\] corresponds to the condition that all functions of local variables tend to be constant. In other words, all the functions are $\lambda \to 1$ up to normalization. This conjecture means that topological universality of the behavior at large distances occurs in the infrared limit due to special contribution to the partition sum, which depends only on the $F$-matrix. Practical implementation of the computation program of the result of such normalization group flow is related to the solution of the problem of finding a convenient relation between the variables belonging to tetrahedron faces and the variables belonging to tetrahedron edges and vertices. We plan to consider this problem in future.

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