ON THE AMOUNT OF DEPENDENCE IN THE PRIME FACTORIZATION OF A UNIFORM RANDOM INTEGER

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Abstract. How much dependence is there in the prime factorization of a random integer distributed uniformly from 1 to $n$? How much dependence is there in the decomposition into cycles of a random permutation of $n$ points? What is the relation between the Poisson-Dirichlet process and the scale invariant Poisson process? These three questions have essentially the same answers, with respect to total variation distance, considering only small components, and with respect to a Wasserstein distance, considering all components. The Wasserstein distance is the expected number of changes – insertions and deletions – needed to change the dependent system into an independent system.

In particular we show that for primes, roughly speaking, $2 + o(1)$ changes are necessary and sufficient to convert a uniformly distributed random integer from 1 to $n$ into a random integer $\prod_{p \leq n} p^{Z_p}$ in which the multiplicity $Z_p$ of the factor $p$ is geometrically distributed, with all $Z_p$ independent. The changes are, with probability tending to 1, one deletion, together with a random number of insertions, having expectation $1 + o(1)$.

The crucial tool for showing that $2 + \epsilon$ suffices is a coupling of the infinite independent model of prime multiplicities, with the scale invariant Poisson process on $(0, \infty)$. A corollary of this construction is the first metric bound on the distance to the Poisson-Dirichlet in Billingsley’s 1972 weak convergence result. Our bound takes the form: there are couplings in which

$$\mathbb{E} \sum \left| \log P_i(n) - (\log n)V_i \right| = O(\log \log n),$$

where $P_i$ denotes the $i^{th}$ largest prime factor and $V_i$ denotes the $i^{th}$ component of the Poisson-Dirichlet process. It is reasonable to conjecture that $O(1)$ is achievable.

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For the reader impatient to get to business, the main results are given by the bound (72) in Theorem 3 and the bound (81) in Theorem 5. A $500 conjecture, related to Theorem 3, is given by relation (27), and a $100 conjecture, related to Theorem 5, is given by the bound (80).

1. Lecture 1. Growing a random integer

I would like to thank the János Bolyai Mathematical Society and the organizers of this conference for the honor of speaking here, where the spirit of Erdős seems so close. At the conference reception last night Imre Csiszar, student of Rényi, suggested that Erdős, now in heaven, has read the book where all best proofs are given. But I prefer to believe that in heaven, Erdős by choice does not ask to see the book; he only asks of a proof, “Is there
an even better one in the book?” And he watches us fellow mathematicians here on earth, as we lecture and write and discover; by not diving into the book, he can continue to compete against us. For the joy of discovery through one’s own effort is far more rewarding than even reading the book. I would also like to thank my collaborators, Andrew D. Barbour and Simon Tavaré, as their work and ideas pervade these lectures. I hope that our book on logarithmic combinatorial structures [5], in preparation since 1992, will soon see publication!

The guiding question for today’s lecture is to ask, by analogy with the Erdős-Rényi notion of growing a random graph, “can we grow a random integer?” For guidance, we compare with the simpler task of “growing” a random permutation.

NOTATION: \( n \) is always the parameter, rather than the random object. We consider

- a permutation chosen uniformly from \( S_n \)
- an integer \( N \) chosen uniformly from 1 to \( n \).

We may emphasize the role of the parameter \( n \) by writing it explicitly:

\[
P_n(N = i) \equiv P(N(n) = i) = \frac{1}{n} \quad \text{for } i = 1, 2, \ldots, n.
\]

For the prime factorization we write

\[
N(n) = \prod p^{C_p(n)}, \quad \text{so } (C_p(n))_p = (C_2(n), C_3(n), C_5(n), \ldots)
\]

is a dependent process, with \( C_p(n) \) identically zero if \( p > n \).

The baby fact: as \( n \to \infty \)

\[
(C_p(n))_p \Rightarrow (Z_p)_p = (Z_2, Z_3, Z_5, \ldots)
\]

with independent, geometrically distributed coordinates, with \( P(Z_p \geq k) = p^{-k} \).

**Proof.** Given \( j \) distinct primes \( p_1, \ldots, p_j \), and integers \( i_1, \ldots, i_j \geq 0 \), write \( d = p_1^{i_1} \cdots p_j^{i_j} \) so that as events,

\[
\{C_{p_1} \geq i_1, \ldots, C_{p_j} \geq i_j\} = \{d|N\}.
\]

Thus

\[
P_n(d|N) = \frac{1}{n} \left\lfloor \frac{n}{d} \right\rfloor \to \frac{1}{d} = P(Z_{p_1} \geq i_1, \ldots, Z_{p_j} \geq i_j),
\]

and \( j \)-dimensional differencing yields

\[
P_n(C_{p_1} = i_1, \ldots, C_{p_j} = i_j) \to P(Z_{p_1} = i_1, \ldots, Z_{p_j} = i_j).
\]

In fact, the approximation error in (2) is at most \( 1/n \), so the error in (3) is at most \( 2^j/n \) — a crude upper bound which comes from taking the
absolute value inside. The sum over all integers \( d = \prod_{p \leq b} p^{a_p} \) whose largest prime factor is at most \( b \), i.e.

\[
\sum_{d: P^*(d) \leq b} |\mathbb{P}(C_p(n) = a_p \forall p \leq b) - \mathbb{P}(Z_p = a_p \forall p \leq b)|,
\]

gives the total variation distance \( d_{TV}(b, n) \), restricting to primes not exceeding \( b \), and surprisingly, the crude upper bound \( u(b, n) \), formed by taking absolute values inside, gives pretty good information. But the information is not nearly as good as the result of Kubilius; we will describe these bounds later, in sections 1.5.2 and 1.5.3.

Since \( \mathbb{E} Z_p = \frac{1}{p-1} \) with \( \sum \mathbb{E} Z_p = \infty \), it follows from the independence of the \( Z_p \) that

\[
1 = \mathbb{P} \left( \sum Z_p = \infty \right).
\]

Thus we call the multiset, having \( Z_p \) copies of \( p \) for each prime, the natural random infinite multiset of primes. The guiding question for today’s lecture, can we grow random integers, is now stated more precisely as: can we construct \( N(1), N(2), \ldots, N(n), \ldots \) all on a single probability space, together with the natural random infinite multiset of primes, so that the \( C_p(n) \) evolve smoothly, with \( C_p(n) \to Z_p \) as \( n \to \infty \). The answer is yes in a sense; the construction culminating with (75) at the end of Lecture 3 has \( C_p(n) \to Z_p \) in probability, but with probability one, \( \liminf C_p(n) = Z_p \) and \( \limsup C_p(n) = Z_p + 1 \).

### 1.1. Overview: the limits for primes, small and large.

This section presents the material from one of the two transparencies which were shown repeatedly throughout the lectures; the material from the other transparency appears in this writeup as Conjecture 1 in section 2.2 and as Conjecture 2 in section 4.

\( N(n) \) is uniform 1 to \( n \), with

- \( N(n) = \prod p^{C_p(n)} \) — small factors,
- \( N(n) = P_1(n)P_2(n) \cdots, P_1 \geq P_2 \geq \cdots \), prime or 1 — large factors

**LIMITS** in distribution as \( n \to \infty \)

\[
(C_p(n))_p \Rightarrow (Z_p)_p, \quad \text{independent Geometric}
\]

\[
\left( \frac{\log P_1(n)}{\log n}, \frac{\log P_2(n)}{\log n}, \ldots \right) \Rightarrow (V_1, V_2, \ldots), \quad \text{Poisson-Dirichlet}
\]

proved by Billingsley in 1972, where the limit is now known as the Poisson-Dirichlet process, with parameter 1.
HOW CLOSE? One coupling has

\[ E \sum_{p \leq n} |C_p(n) - Z_p| \leq 2 + O \left( \frac{(\log \log n)^2}{\log n} \right), \]  

(5)  

\[ E \sum_{p \leq n} |\log P_i(n) - (V_i)(\log n)| = O(\log \log n). \]  

(6)  

We note that in (5), 2 - \( \epsilon \) is not possible, while in (6), O(1) should be possible. The reason that loglog \( n \) appears in our bound for (6) is made clear by equation (63) in conjunction with the proof of Theorem 5.

1.2. Sketch of the coupling to grow \( N(n) \). Take a size biased permutation, say \( Q_1, Q_2, \ldots \), of the prime factors in \( 2^{Z_2}3^{Z_3}\cdots \).

Let \( J(n) = \) the largest partial product \( Q_1Q_2\cdots Q_j \leq n \). Write \( L \equiv L(n) \) for the number of factors, so that \( J(n) = Q_1Q_2\cdots Q_L(n) \).

Show: \( J(n) \) has approximately the same distribution as \( H(n) \), where by definition \( H(n) \) has the harmonic distribution on \( [n] \),

\[ P(H(n) = i) = \frac{1/i}{1 + 1/2 + \cdots + 1/n} = \frac{1}{ih_n}, \quad i = 1, 2, \ldots, n. \]  

(7)  

Fill in one extra factor, \( P_0(n) \), to be prime or one. Use a random uniform \( U \in (0, 1] \) to choose uniformly from the \( 1 + \pi(n/J(n)) \) possibilities with \( J(n)P_0(n) \leq n \).

Show that the resulting random integer, \( JP_0 \equiv J(n)P_0(n) \) is close to uniform. In fact, from Lemma 3

\[ d_{TV}(J(n), H(n)) = O \left( \frac{1}{\log n} \right), \]  

(8)  

and from Lemma 1

\[ d_{TV}(J(n)P_0(n), N(n)) = O \left( \frac{\log \log n}{\log n} \right). \]  

(9)  

Finally, modify the coupling on the event whose probability is the left side of (9), so that the modified versions of \( JP_0 \) are exactly uniform, for all \( n \).

Noga Alon asked, “Does \( P(J(n)P_0(n) = 1) \sim 1/n? \) The total variation distance bound (9) does not determine the answer. Since \( P(P_0(n) = 1 \mid J(n) = 1) = 1/(1 + \pi(n)) \sim \log n/n \), Alon’s question is equivalent to, “Does \( P(J(n) = 1) \sim 1/\log n? \) Now if \( Z_p = 0 \) for all \( p \leq n \) then we must have \( J(n) = 1 \), and \( P(Z_p = 0 \ \forall p \leq n) = \prod_{p \leq n}(1 - 1/p) \sim e^{-\gamma}/\log n \) by Mertens’ Theorem. Thus Alon’s question is equivalent to asking: does \((1 - e^{-\gamma})/\log n \) give the asymptotic probability that there are one or more primes less than or equal to \( n \) in the infinite multiset and, in the size biased permutation, some prime greater than \( n \) comes before all of them. The answer, which I did not give at the workshop, but was more or less evident
from (48) and (50), is yes; see Lemma 3 for further details. While this affirmative answer might give us hope that $J(n)P_0(n)$ is close to uniform in the sense that for all $i \leq n$, $\mathbb{P}(J(n)P_0(n) = i) \sim 1/n$, it is clearly not so. In fact, (for the coupling in Lecture 3, which is slightly different from the coupling described in this lecture,) for fixed $i$, having $\omega(i)$ distinct prime factors, as $n \to \infty$, $\mathbb{P}(J(n)P_0(n) = i) \sim (1 + \omega(i))/n$; see (59) and (60) for details. Thus $i = 1$ is the only fixed integer having the correct asymptotic probability!

How does $J(n)P_0(n)$ evolve as $n$ grows? Write $p^#$ for the smallest prime larger than $p$, with $1^# = 2$. The following properties hold for all $n > 1$ and for all outcomes.

$$J(n)P_0(n) \in [1, n].$$

Each factor has its own smoothness:

- $J(n)/J(n-1)$ is one or a prime,
- $P_0(n) = P_0(n-1), P_0(n-1)^#$, or 1,

and the jumps in the two factors are linked:

- $J(n) \neq J(n-1)$ implies ($J(n) = n$ and $P_0(n) = 1$),
- $P_0(n) < P_0(n-1)$ implies ($J(n) = n$ and $P_0(n) = 1$).

1.3. Size biased permutations. Several in the audience requested clarification: what is a size biased permutation? Given $k$ objects, with “sizes” or “weights” $r_1, r_2, \ldots, r_k > 0$, we can carry out a size biased permutation using $k$ independent standard exponentially distributed “alarm clocks” $S_1, S_2, \ldots, S_k$ with $\mathbb{P}(S_i > t) = e^{-t}$ for all $t > 0$. The labels $W_i := S_i/r_i$ are exponentially distributed with $\mathbb{P}(W_i > t) = e^{-r_it}$, (we say “rate $r_i$” or “mean $1/r_i$,”) and the labels are independent, with $\mathbb{P}(W_i = W_j) = 0$ for every $i \neq j$. The ranking of the labels induces a size biased permutation of the objects. Observe that $\mathbb{P}(W_i$ is the smallest among $W_1, W_2, \ldots, W_k) = r_i/(r_1 + r_2 + \cdots + r_k)$, which is a familiar fact from the study of finite-state Markov chains in continuous time, where the $r_i$ are jump rates. This calculation shows the distribution of the first item; iterating and using the memoryless property of exponentials gives a product formula for the distribution of the full size biased permutation.

For primes, the size of $p$ is $\log p$, and such a size biased permutation was used in the 1984 Ph.D. thesis of Eric Bach [13] to generate uniformly distributed random integers, factored into primes, and independently by Donnelly and Grimmett in 1993 [22] to give a simple proof of Billingsley’s Poisson-Dirichlet limit for prime factors – discussed at (77) below. In our context, the infinite random multiset of primes, there are infinitely many labels, but with probability one, the only limit point is zero, and there is a
largest label. Our size biased permutation starts with the prime having this largest label; listing the labels from largest downward tends to put smaller primes toward the front of the list.

1.4. An example of the growth of an integer. Take for example an outcome of the experiment with

$$Z_2 = 3, Z_3 = 1, Z_5 = 0, Z_7 = 1, Z_{11} = 1, \ldots$$

and size biased permutation

$$Q_1, Q_2, \ldots, Q_6, \ldots = 3, 2, 2, 11, 2, 7, \ldots$$

The first seven partial products, one through $3 \cdot 2 \cdot 11 \cdot 2 \cdot 7$, shown on a logarithmic scale, are

The arrow pointing to $24024 = 1848 \times 13$ is to show that the next partial product will lie on or to the right of this location. We know this because the information $6 = \sum_{p<13} Z_p$, together with the good luck that the first six primes in the size biased permutation are less than 13, implies that the seventh prime in the size biased permutation will be 13 or greater. In particular, for $1848 \leq n < 13 \times 1848$, we know that $J(n) = 1848$, even though we don’t know the seventh prime in the size biased permutation, accounting for the last line of the table below.

The jumps in $J(n)$ are shown by double horizontal lines in the two tables below.
Recall that $d_{TV}(J(n)P_0(n), N(n)) \to 0$, but the random integer we grow is not exactly uniform. For comparison, random permutations have similar behavior and can be grown exactly. Eric Bach’s procedure gets an exactly uniform random integer, but not with $n$ evolving.
1.5. Notions of distance.

1.5.1. The expected number of insertions and deletions needed. The material below on $d_W$ was delivered at the workshop at the start of Lecture 2; the material on $d_{TV}$ was drawn out in workshop conversations with Joel Spencer, and I prepared a transparency for the lecture but did not deliver it, for lack of time.

We consider the metric $d$ on positive integers which counts the number of changes needed to convert the prime factorization of one integer into that of the other. For example, $d(40, 500) = d(2^35^1, 2^25^3) = 3$, $d(8, 3) = 4$, and $d(i, ip) = 1$ for any integer $i$ and prime $p$. Writing $(i, j)$ for the greatest common divisor, and $\Omega(i)$ for the number of prime factors, including multiplicities, we have in general

\[
d(i, j) = \Omega \left( \frac{i}{(i, j)} \right) + \Omega \left( \frac{j}{(i, j)} \right).
\]

If we think of converting $j$ to $i$, the first term above is the number of insertions needed, and the second term is the number of deletions, so we think of $d$ as the insertion/deletion distance, analogous to the string edit distance of Levenstein [36] or Ulam [16]; see the book of Kruskal and Sankoff [33] for more history.

For the sake of comparing the uniform random $N(n)$ with the infinite random multiset of primes, clearly primes $p > n$ should not be considered. Thus, we code up the relevant part of the multiset by defining

\[
(12) \quad M(n) := \prod_{p \leq n} p^{Z_p}, \text{ with } Z_2, Z_3, \ldots \text{ independent geometric.}
\]

Recall that

\[
N(n) = \prod_p p^{C_p(n)} = \prod_{p \leq n} p^{C_p(n)} \text{ is uniform 1 to } n.
\]

The insertion-deletion distance between these two random integers is a random, nonnegative integer

\[
d(N, M) = \Omega \left( \frac{NM}{(N, M)^2} \right) = \sum_{p \leq n} |C_p(n) - Z_p|.
\]

The Wasserstein distance between two random objects $M$ and $N$, for a given metric $d$, is by definition the infimum, over all couplings, of the expected value of $d(N, M)$. Recall that a coupling means a construction of $M$ and $N$ simultaneously on a single probability space; it is understood that the marginal distributions for $M$ and for $N$ have been specified in advance, but there is no other constraint on their joint distribution. A compactness argument shows that the inf is achieved; see for example [23]. To emphasize
the role of the parameter \( n \), which determines the marginal distributions of \( N(n) \) and \( M(n) \), we define
\[
\begin{align*}
(13) \quad d_W(n) := \min_{\text{couplings}} \mathbb{E} \left[ d(N(n), M(n)) \right].
\end{align*}
\]

Our result is
\[
(14) \quad \lim_{n \to \infty} d_W(n) = 2.
\]

We will prove the hard part of this, that \( \lim \sup d_W(n) \leq 2 \), in Lecture 3, essentially by analyzing the growth of a random integer. The matching lower bound, from \([2]\), is that \( \lim \inf d_W(n) \geq 2 \); this is relatively easy, and the reader is challenged to discover a proof, before considering the hint given by the paragraph following (72). A related discussion appears in section 22 of \([3]\).

For perspective on the content of (14), we note that even the bound \( d_W(n) = O(1) \) is very strong. For instance, by comparing \((C_p(n))_{p \leq n}\) with the independent process \((Z_p)_{p \leq n}\), the following consequences can be derived easily: (see \([2]\))
\[
\begin{align*}
&d_W(n) = o(\log \log n) \text{ implies the Hardy-Ramanujan Theorem for the normal order of the number of prime divisors.} \\
&d_W(n) = o(\sqrt{\log \log n}) \text{ implies the Erdős-Kac Central Limit Theorem.} \\
&d_W(n) = O(1) \text{ gives another proof of the “conjecture of LeVeque,” that the error in the Central Limit Theorem is } O(1/\sqrt{\log \log n}); \text{ the first proof was given by Rényi and Turán in 1957 \([40]\).} \\
&d_W(n) = o(\log \log \log n) \text{ yields that the optimal rate in the simplest case of the Brownian motion convergence of Billingsley and Philipp, \([15, 19, 38]\), for the expected sup norm, is order of } \log \log \log n/\sqrt{\log \log n} \text{, and no smaller.} \\
&\text{The underlying idea is that for coupling Brownian motion with the rate one (centered) Poisson process, with both processes run until the variance is } t, \text{ and without rescaling by } \sqrt{t}, \text{ the coupling distance grows on the order of } \log t. \text{ For primes this is applied with } t := \sum_{p \leq n} 1/p \sim \log \log n. \text{ See Kurtz (1978) \([35]\) for the upper bound, Rio (1994) \([41]\) for the lower bound, and \([2]\) for the connection with primes.}
\end{align*}
\]

1.5.2. The total variation distance. This section gives a “one transparency” overview of the situation involving total variation distance for primes.

NOTATION: \( \beta \in [0, 1] \), fixed or \( \beta = \beta(n) \to 0; \ u \equiv 1/\beta. \)
\[
(15) \quad d_{TV} := d_{TV}(n^\beta, n) := \min_{\text{couplings}} \mathbb{P} \left( (C_p(n))_{p \leq n^\beta} \neq (Z_p)_{p \leq n^\beta} \right).
\]

Kubilius \([34]\) in the 1950’s showed a) below with an upper bound of the form \( \exp(-cu) \), Barban and Vinogradov \([14]\) improved this to the form \( \exp(-cu \log u) \), and Elliott \([21]\) gave the particular constants in b) below.
a) If $\beta \to 0$ then $d_{TV} \to 0$, and

b) $d_{TV} = O\left(\exp(-\frac{1}{5}u \log u) + n^{-1/15}\right)$.

Elliott [21] gives a partial converse: if $\beta \to 1$ then $d_{TV} \not\to 0$, and it is not hard to see the full converse, that $d_{TV} \to 0$ implies $\beta \to 0$. In Spring 1996 I stated a conjecture: that for fixed $\beta$,

c) $H(\beta) := \lim_{n \to \infty} d_{TV}$ exists, and is the same as the limit for permutations. This limit is given explicitly as an integral in [11], and proved to be the limit for permutations in Stark’s 1994 PhD Thesis; see [13].

Later in 1996 Tenenbaum [45] improved b) to

$$d_{TV} = O\left(\exp(-u(\log u + \log \log u - (1 + \log 2 + \epsilon)) + n^{-1})\right)$$

and Arratia and Stark [10] proved c). Soon after Tenenbaum [45] also proved c), with a rate. The limit was further identified [8] as a distance between the restrictions to $[0, \beta]$ of two processes which will be discussed in lectures 3 and 4, the Poisson-Dirichlet process with parameter 1, and the scale invariant Poisson process with intensity $1/x$ $dx$; for every $\beta \in [0, 1]$,

$$H(\beta) = d_{TV}(\{V_i : V_i < \beta\}, \{X_i : X_i < \beta\})$$

$$= \min_{couplings} \mathbb{P}(\{V_i : V_i < \beta\} \neq \{X_i : X_i < \beta\}).$$

1.5.3. The bound from taking absolute values inside. The crude procedure of “taking the absolute values inside” described following [41] shows that the total variation distance $d_{TV}(n^\beta, n)$ is at most $u(n^\beta, n)$ where

$$u(b, n) = \frac{1}{n} \sum_{d \geq 1: P^+(d) \leq b} \left\{\frac{n}{d}\right\} 2^{\omega(d)}.$$ 

The notation here is $\{x\}$ for the fractional part of $x$, $\omega(d)$ for the number of distinct prime factors of $d$, and $P^+(d)$ for the largest prime factor of $d$, with $P^+(1) = 1$. Analysis of $u(b, n)$ (see [2]) shows that when $b, n \to \infty$ together, the threshold for whether $u(b, n)$ tends to zero or infinity is $\log b = (.5 + \epsilon) \log n \log \log \log n / \log \log n$. In fact if $\log b \leq \log n \log \log \log n / (c \log \log n)$ with $c > 2 + a > 2$, then $u(b, n) = o((\log n)^{-a})$, while if $\log b > \log n \log \log \log n / (c \log \log n)$, for $c < 2$, then $u(b, n) \to \infty$. While much weaker than the upper bound of Kubilius, the upper bound $u(b, n)$ on the total variation distance has the virtue that it also serves as an upper bound on the Wasserstein distance: for all $1 \leq b \leq n$,

$$d_{TV}(b, n) \leq d_W(b, n) \leq u(b, n),$$

where $d_W(b, n)$ is the minimum expected number of insertions and deletions needed to convert $\prod_{p \leq b} p^{C_p(n)}$ to $\prod_{p \leq b} p^{Z_p}$. [The $d_W(n)$ in (13) is the special case $b = n$ of this, i.e. $d_W(n) = d_W(n, n)$.] While we know the limit of (16) for $d_{TV}(n^\beta, n)$ as $n \to \infty$ with $\beta \in (0, 1)$ fixed, the corresponding behavior of $d_W(b, n)$ remains unknown.
**Open question** What is the limit of $d_W(n^\beta, n)$ as $n \to \infty$ for fixed $0 < \beta < 1$?

We know that for $\beta = 1$, the limit is 2 — this is (14). For all $\beta \in [0, 1]$, it can be seen from the coupling in Lecture 4 that $\limsup d_W(n^\beta, n) \leq 2\beta$ — but only for $\beta = 1$ is there a matching lower bound. The natural candidate for the limit of the distance is the Wasserstein distance for the limit systems,

$$H_W(\beta) := d_W(\{V_i : V_i < \beta\}, \{X_i : X_i < \beta\})$$

$$\equiv \min_{\text{couplings}} \mathbb{E} |\{V_i : V_i < \beta\} \Delta \{X_i : X_i < \beta\}|.$$

Thus, the open question involves two tasks. First, prove (or disprove!) that the limit exists and equals $H_W(\beta)$.

**Conjecture 0.** $\forall \beta \in [0, 1], H_W(\beta) = \lim d_W(n^\beta, n)$.

The second task in our open question is to find an explicit formula for $H_W(\beta)$. We know only $H_W(1) = 2$, $H_W(\beta) \leq 2\beta$, and trivially, $H_W(0) = 0$ and $H_W$ is monotone.

### 2. Lecture 2. Growing a random permutation

This is a fresh start — the following can be understood easily from scratch, without worrying about the connection with prime factorizations. However, our example for permutations below has been carefully cooked to match the example for primes in Lecture 1.

Write $C_i(n)$ for the number of cycles of length $i$ in our random permutation $\pi \in S_n$, so that always $C_i(n) = 0$ if $i > n$, and $n = \sum \Delta C_i(n)$. The analog of Theorem 1 provable easily with inclusion-exclusion, is that as $n \to \infty$

$$C_i(n) \Rightarrow (Z_i)_i = (Z_1, Z_2, Z_3, Z_4, \ldots)$$

with independent coordinates $Z_i$, Poisson distributed with $\mathbb{E} Z_i = 1/i$. (Note, we are deliberately re-using the same notation that we used for prime factorizations, although the index must be a prime in the latter case. To contrast the two situations, $Z_p$ is either geometrically distributed, with $\mathbb{P}(Z_p \geq k) = p^{-k}$ and $\mathbb{E} Z_p = 1/(p - 1)$, or else $Z_p$ is Poisson, with $\mathbb{E} Z_p = 1/p$.) That there is an independent process limit, without rescaling, for both prime factorizations of random integers, and the cycle structure of random permutations, is the most basic ingredient in the analogy between these two structures.

Historical notes: the similarity between primes and permutations appears to have been noted first in 1976 by Knuth and Trabb Pardo [32]. The similarity they note is based on the common limit behavior for the $i^{th}$ largest prime factor and the $i^{th}$ longest cycle; further aspects of the similarity, are discussed in [6], which sets forth the role of “conditioning” the independent limit process as the fundamental reason for this similarity. In contrast, in
these lectures we will focus on the closeness of primes and permutations to the scale invariant Poisson process, with its relations involving spacings and size biased permutations, as the underlying reason for the similarity. A reasonable attribution for the construction below, based on canonical cycle notation, is Feller 1945 [26], but the explicit connection with cycle lengths seems to appear first in the unpublished lecture notes [20]; it was independently discovered in [15] and elaborated on in [4], which attached the name “Feller coupling” to this construction.

Start with the canonical cycle notation for a permutation $\pi$. For example, the permutation $\pi$ with $1 \mapsto 5, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 4, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 6$ is written as $\pi = (153)(2)(4)(67)$. In writing the canonical cycle notation for a random $\pi \in S_7$, one always starts with “(1 ”, and then makes a seven-way choice, between “(1)(2 “, “(1 2 “, . . . , and “(1 7 “. One continues with a six-way choice, a five-way choice, . . . , a two-way choice, and finally a one-way choice.

Let $\xi_i$ be defined as the indicator function $\xi_i = 1( \text{close off a cycle when there is an } i\text{-way choice})$. Thus

$$\Pr(\xi_i = 1) = \frac{1}{i}, \Pr(\xi_i = 0) = \frac{i - 1}{i}, \text{and } \xi_1, \xi_2, \ldots, \xi_n \text{ are independent.}$$

An easy way to see the independence of the $\xi_i$ is to take $D_i$ chosen from 1 to $i$ to make the $i$-way choice, so that $\xi_i = 1(D_i = 1)$. Absolutely no computation is needed to verify that the map constructing canonical cycle notation, $(D_1, D_2, \ldots, D_n) \mapsto \pi$, from $[1] \times [2] \times \cdots \times [n]$ to $S_n$, is a bijection. The “decision” variables $D_1, D_2, \ldots, D_n$ determine the random permutation on $n$ points, while the Bernoulli variables $\xi_1, \xi_2, \ldots, \xi_n$ determine the cycle structure, and something more — a size biased permutation $SB(n)$ of the cycle lengths! The total number of cycles is

$$K_n := \# \text{ cycles } = \xi_1 + \xi_2 + \cdots + \xi_n, \text{ with } E K_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \gamma + \log n + o(1).$$

[Incidentally, the relations above give a quick and dirty, but pretty, way to see that the entropy $h((C_i(n))_i)$ for the cycle structure of a random permutation of $n$ objects is asymptotically $(\log n)^2/2$. Namely, the entropy of a size biased permutation of $k$ objects is at most the entropy of a uniform permutation of $k$ objects, which is $\log k! \leq k \log k$. Thus the entropy $h(SB(n))$ of a size biased permutation of the $K_n$ cycle lengths of a random $n$-permutation is at most $E K_n \log K_n \sim \log n \log \log n$. The entropy of the Bernoulli$(1/i)$ random variable $\xi_i$ is $h(\xi_i) = -(\frac{1}{i} \log \frac{1}{i} + \frac{i - 1}{i} \log \frac{i - 1}{i}) = \log i - \log(i - 1) + \frac{1}{i} \log(i - 1)$ for $i \geq 2$, so

$$\sum_{i=1}^{n} h(\xi_i) = \sum_{i=2}^{n} \left( \log \frac{i}{i - 1} + \frac{1}{i} \log(i - 1) \right) = \log(n + 1) + \sum_{i=1}^{n-1} \frac{\log i}{i + 1} \sim \frac{(\log n)^2}{2}.$$\)

Since the cycle structure together with the size biased permutation of the cycle lengths determine $\xi_1, \ldots, \xi_n$, we have $(\log n)^2/2 \sim \sum_{i=1}^{n} h(\xi_i) = h((C_i(n))_i)$+
\[ h(\text{SB}(n)) = h((C_i(n))_i) + O(\log n \log \log n), \text{ hence } h((C_i(n))_i) \sim (\log n)^2/2. \]

As an exercise, the reader can try to verify this directly from Cauchy’s formula, and perhaps (open problem?) give an asymptotic expansion!

The coupling which “grows” a random permutation requires a very simple idea: for all values of \( n \), use the same \( D_1, D_2, \ldots \), and hence the same \( \xi_1, \xi_2, \ldots \).

The Feller coupling, as motivated by the process of writing out canonical cycle notation, “reads” \( \xi_1 \xi_2 \cdots \xi_n \) from right to left: the length of the first cycle is the waiting time to the first one, the length of the next cycle is the waiting time to the next one, and so on. The multiset of cycle lengths can be determined without regard to right or left: every \( i \)-spacing in \( 1 \xi_2 \xi_3 \cdots \xi_n 1 \), that is, every pattern of two ones separated by \( i - 1 \) zeros, corresponds to a cycle of length \( i \). The spacing from the rightmost one in \( 1 \xi_2 \xi_3 \cdots \xi_n \) to the “artificial” one at position \( n + 1 \) corresponds to the first cycle in canonical cycle notation, and also to the factor \( P_0 \) in our construction for growing an almost uniform random integer \( J(n) P_0(n) \).

Since we are interested in growing with \( n \), we will always read \( \xi_1 \xi_2 \xi_3 \cdots \) from left to right. Recall that \( \xi_1 = 1 \) identically, so \( \xi_1 \xi_2 \xi_3 \cdots \) is a random infinite word in the alphabet \( \{0,1\} \), starting with a 1. Almost surely, this sequence has infinitely many ones, since the \( \xi_i \) are independent with \( \sum_{i \geq 1} E \xi_i = \sum_{i \geq 1} 1/i = \infty \). We define the inter-one spacings \( B_1, B_2, \ldots \in \mathbb{N} \) by the requirement that

\[ \xi_1 \xi_2 \xi_3 \cdots \text{ has ones at } 1, 1 + B_1, 1 + B_1 + B_2, \ldots, \text{ and nowhere else}. \]

The example which matches the example (11) from Lecture 1, which was 3,2,2,11,2,7, \( p \) with \( p \geq 13 \), and partial products 1,3,6,12,132,264,1848, 1848 \( p \), is the sequence

\[ (21) \quad \xi_1 \xi_2 \cdots \xi_{20} = 10111000011000100000, \]

or equivalently,

\[ B_1, \ldots, B_6, B_7 = 2, 1, 1, 5, 1, 4, B_7, \text{ with } B_7 \geq 6. \]

Do you see the correspondence between 3,2,2,11,2,7, \( p \) with \( p \geq 13 \), and 10111000011000100000 \( \cdots \) ? Writing \( p_i \) for \( i^{\text{th}} \) smallest prime \( p_i \), the prime and permutations examples match in that the \( k^{\text{th}} \) prime on the list of primes is \( p_{B_k} \).
In the Feller coupling, the cycle structure of a random \( \pi \in S_n \) has been realized via \( C_i(n) = \# i \)-spacings in \( 1 \xi_2 \xi_3 \cdots \xi_n 1 \). For comparison, consider

\[
C_i(\infty) := \# i - \text{spacings in } 1 \xi_2 \xi_3 \cdots = \sum_{k \geq 1} 1(B_k = i).
\]

An easy calculation, (22) and (23) below, shows \( P(C_i(n) \neq C_i(\infty)) \leq 2/(n + 1) \).

2.1. A size biased permutation of the multiset having \( i \) with multiplicity \( Z_i \sim \text{Poisson}(1/i) \). The above indirect argument, that the \( C_i(\infty) \) are independent, Poisson(1/i) was presented at Oberwolfach one morning in August 1993; Erdős and Svante Janson were in the audience. Svante asked if there were a direct proof; I said I didn’t know of one. Before lunch time, Svante Janson found and presented the following direct argument. Start with the \( Z_i \), given to be independent, Poisson(1/i). Take a random infinite multiset, having \( Z_i \) copies of \( s_i := 0^{i-1}1 \), the string of length and weight \( i \). Take a size biased permutation of this multiset to get a list \( R_1, R_2, \ldots \), so that by construction, the number of \( i \)-spacings in the string \( 1R_1R_2 \cdots \) is \( Z_i \), for each \( i \). Calculation ([12], section 9.1) shows that the random string \( 1R_1R_2 \cdots \) of zeros and ones has the same distribution as \( \xi_1 \xi_2 \cdots \). And as an historical note: there already existed yet another direct argument, by marking Poisson processes. Jim Pitman describes it this way: “As observed in Diaconis-Pitman [20], the fact that the numbers of \( i \)-spacings in the Bernoulli (1/j) sequence are independent Poisson (1/i) is an immediate consequence of the structure of records of a sequence of i.i.d. uniform (0,1) variables \( U_1, U_2, \ldots \). For if \( N_1 < N_2 < \cdots \) are the successive record indices, then by a well known result of Rényi the indicators \( 1(N_k = j \text{ for some } k) \) are independent Bernoulli (1/j), and as shown by Ignatov [29] the numbers of \( i \)-spacings in the record sequence are independent Poisson(1/i).”

2.2. Keeping score for the Feller coupling. Starting from the independent Bernoulli \( \xi_i \) with \( P(\xi_i = 1) = 1/i \), and defining \( Z_i := C_i(\infty) \), we have a coupling of the cycle structures \( (C_i(n))_i \) for \( n = 1, 2, \ldots \), together with the independent Poisson \( Z_i \) with \( E Z_i = 1/i \). Note that for the event that an \( i \)-spacing occurs with right end at \( k \), the probability is a simple telescoping product:

\[
P(\xi_{k-i} \cdots \xi_k = 1 0^{i-1}1) = \frac{1}{k-i} \frac{k-i}{k-(i-1)} \cdots \frac{k-3}{k-2} \frac{2}{k-1} = \frac{1}{(k-1)k}.
\]
We can have $C_i(n) < Z_i$, due to $i$-spacings whose right end occurs after position $n + 1$; the expected number of times this occurs is

$$\sum_{k>n+1} P(\xi_{k-1} \cdots \xi_k = 10^{i-1}) = \sum_{k>n+1} \frac{1}{(k-1)k} = \frac{1}{n+1}$$

The only way that $C_i(n) > Z_i$ can occur is if the “artificial” one at position $n + 1$ in $1\xi_2 \cdots \xi_n 1$ creates an extra $i$-spacing; for each $n$ this can occur for at most one $i$, and it occurs for each $1 \leq i \leq n$ with the same probability,

$$P(\xi_{n-i+1} \cdots \xi_n\xi_{n+1} = 10^{i-1}) = \frac{1}{n-i+1} \frac{n-i+1}{n-i+2} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$  

The length $A(n)$ of the first cycle in canonical cycle notation is precisely the value $i$ for which this “extra” $i$-cycle may occur,

$$A(n) = n + 1 - \max\{j \leq n: \xi_j = 1\},$$

so that with

$$J(n) := \max\left\{\sum_{1}^{l} B_l: \sum_{1}^{l} B_l \leq n - 1\right\}$$

we have $J(n) + A(n) = n$.

We now summarize the Feller coupling in a way which matches the coupling for primes in subsection 1.2. Start with independent Poisson random variables $Z_i$ with $E Z_i = 1/i$. Take the infinite multiset having $Z_i$ copies of $i$. Take a size biased permutation $B_1, B_2, \ldots$ of this multiset. Let $J(n) \in [0, n-1]$ be the largest partial sum not exceeding $n - 1$. (There is no need to calculate the distribution of $J(n)$, but it happens to be exactly uniform on $0, 1, \ldots, n - 1$, which matches the harmonic distribution in (7), in the sense that $\log H(n)$ is approximately uniform on $[0, \log n]$.) Fill in one extra cycle length, $A(n) := n - J(n)$; this corresponds to $P_0(n)$. The resulting cycle structure, with cycles of lengths $B_1, B_2, \ldots, B_L$ and $A(n)$ (where $J(n) = B_1 + \cdots + B_L$), is the cycle structure of a random permutation chosen uniformly from $S_n$.

Write $\mathbf{e}_i := (0, 0, \ldots, 0, 1, 0, \ldots)$ for the unit vector with all zero coordinates except for a one in position $i$. The Feller coupling shows that the Wasserstein insertion-deletion distance for permutations compared to their independent limit process is at most 2, for all $n$, with a monotonicity relation as a bonus:

$$E \sum_{1}^{n} |C_i(n) - Z_i| \leq \frac{2n}{n+1} < 2, \quad \text{and}$$

always $$(C_1(n), C_2(n), \ldots) \leq (Z_1, Z_2, \ldots) + \mathbf{e}_{A(n)}.$$
The result (26) for random permutations is stronger than the analogous result for prime factorizations, (9), in that there is no exceptionally probability for an event where monotonicity may fail. The result for permutations suggests the following conjecture from [6], for which, in the style of Erdős, I now offer a five hundred dollar prize.

**Conjecture 1. ($500 prize offered)** For all \( n \geq 1 \), it is possible to construct \( N(n) \) uniformly distributed from 1 to \( n \), \( M(n) \) defined by (12), and a prime \( P(n) \) such that

\[
27 \quad N(n) | M(n)P(n).
\]

My reason for believing the conjecture to be true is that permutations “fit together perfectly,” as witnessed by the Feller coupling, and that primes do so also. One sense in which “primes fit together perfectly” is that the weights \( \log 2, \log 3, \log 5, \log 7, \log 11, \ldots \) are such that 1) all multisets of primes have distinct weights, and 2) the weights of these multisets are evenly spaced: \( \log 1, \log 2, \log 3, \log 4, \ldots \).

A restatement of Conjecture 1 in the language of stochastic monotonicity, with respect to the partial order of divisors and multiples, is that for every \( n \), for some randomized choice of \( P_0(n) \) to be 1 or a prime factor of \( N(n) \), \( N(n)/P_0(n) \) lies below \( M(n) \) in distribution. A more specific version of the conjecture, from [2], is that \( P_0(n) \) can be chosen as the first prime factor of \( N(n) \) under a size biased permutation.

In terms of the usual combinatorial language of matchings, Conjecture 1 may be stated as follows. For any set \( D \) of positive integers, define

\[
l(D) := \{ i : \exists m \in D, \ p \text{ prime, } i | mp \},
\]

so that for example \( l(\{1\}) \) is the set of primes, together with 1. Write \([n]\) for \( \{1, 2, \ldots, n\} \). The conjecture is that \( \forall n \geq 1, \forall D \subset [n], \)

\[
\frac{1}{n} | l(D) \cap [n] | \geq \sum_{m \in D} \mathbb{P}(M(n) = m) = \prod_{p \leq n} \left( 1 - \frac{1}{p} \right) \sum_{m \in D} \frac{1}{m}.
\]

2.3. At least \( 2 - \epsilon \) indels are needed. The following extensions to (25) aren’t obvious, but help give a full picture of the qualitative behavior of the Feller coupling. Namely, the positive and negative parts of the quantity inside the absolute value in the left side of (25) have

\[
\sum_{i=1}^{n} (C_i(n) - Z_i)^+ \to 1 \quad \text{in probability and expectation, and}
\]

\[
\text{for } k = 0, 1, 2, \ldots, \mathbb{P} \left( \sum_{i=1}^{n} (Z_i - C_i(n))^+ = k \right) \to p_k > 0, \text{ with } \sum k p_k = 1.
\]

In words, the coupling converts \( (C_1(n), \ldots, C_n(n)) \) to \( (Z_1, \ldots, Z_n) \) with, in the limit, one deletion and a random, mean 1 number of insertions, using \( k \) insertions with probability \( p_k \).
A separate argument (see [2]) shows that, for any coupling, Feller or otherwise, with probability approaching one, at least one deletion is necessary, i.e. 
\[ 1 = \lim \mathbb{P}(\sum_{i=1}^{n}(C_i(n) - Z_i)^+ \geq 1) \]. In particular, \[ 1 \leq \lim \inf \mathbb{E} \sum_{i=1}^{n}(C_i(n) - Z_i)^+ \]. Then, since \( \mathbb{E} C_i(n) = 1/n = \mathbb{E} Z_i \) for \( i = 1 \) to \( n \), the number of insertions must, on average, equal the number of deletions. Hence \[ 1 \leq \lim \inf \mathbb{E} \sum_{i=1}^{n}(Z_i - C_i(n))^+ \), so that \( 2 \leq \lim \inf \mathbb{E} \sum_{i=1}^{n}|C_i(n) - Z_i| \). This shows that the Feller coupling, from the point of view of [25], is asymptotically optimal.

3. Lecture 3. Rescaling space – to get a scale invariant Poisson process

3.1. Review: couplings for primes and permutations. To review, we have two couplings for growing a system with parameter \( n \). The coupling for primes grows a random integer \( J(n)P_0(n) \) which is almost uniformly distributed from 1 to \( n \). The coupling for permutations grows a random permutation which is distributed exactly uniformly in \( S_n \).

The coupling for primes takes a list \( Q_1, Q_2, \ldots \) of primes, forms their partial products, and pays attention to \( J(n) = Q_1Q_2 \cdots Q_L(n) \), the largest partial product not exceeding \( n \). On a logarithmic scale, this means that we plot the points \( 0, \log Q_1, \log Q_1 + \log Q_2, \ldots \), and consider the largest point not exceeding \( \log n \). Think of the values \( \log Q_i \) as spacings. We finish by filling in one extra spacing, \( \log P_0(n) \), to get to the point \( \log(J(n)P_0(n)) \) close to, but not to the right of, \( \log n \). The resulting integer is \( J(n)P_0(n) = Q_1Q_2 \cdots Q_LP_0 \), with either \( L \) or \( L+1 \) prime factors, depending on whether \( P_0 = 1 \) or not.

The coupling for permutations takes a list \( B_1, B_2, \ldots \) of positive integers, forms their partial sums, and pays attention to \( J(n) \), the largest partial sum not exceeding \( n - 1 \). We then fill in one extra spacing, of size \( A(n) \), to get to the point \( n = J(n) + A(n) = B_1 + B_2 + \cdots + B_L + A(n) \). The resulting cycle structure has \( L(n) + 1 \) cycles.

We review once again, looking only at the sequence of points plotted, and their spacings. For primes, the points plotted are \( 0, \log Q_1, \log Q_1 + \log Q_2, \ldots \), with spacings \( \log Q_1, \log Q_2, \log Q_3, \ldots \). Conditional on the multiset of spacings, the order in which they are taken is given by a size biased permutation. For each prime \( p \), the number of \( k \) such that \( \log Q_k = \log p \) is \( Z_p \), and the \( Z_p \)'s are independent, geometrically distributed. For permutations, the points plotted are \( 0, B_1, B_1 + B_2, \ldots \), with spacings \( B_1, B_2, B_3, \ldots \). Conditional on the multiset of spacings, the order in which they are taken is given by a size biased permutation. For each positive integer \( i \), the number of \( k \) such that \( B_k = i \) is \( Z_i \), and the \( Z_i \)'s are independent, Poisson(1/i).

The underlying reason that primes and permutations have similar behavior is that for both systems, the spacings have the same logarithmic property: the expected total number of spacings of size at most \( x \) grows like \( \log x \).
primes, this is the property that
\[ E \sum_{\log p \leq x} Z_p = \sum_{p \leq e^x} \frac{1}{p-1} \sim \log x \]
and for permutations,
\[ E \sum_{i \leq x} Z_i = \sum_{i \leq x} \frac{1}{i} \sim \log x. \]

We pursue this further; even the expression \( \gamma + \log x + o(1) \) is common to the two systems — see (41).

3.2. Limits after scaling space. What does the sequence \( \xi_1 \xi_2 \cdots \) look like when viewed from a distance? Encode \( \xi_1 \xi_2 \cdots \in \{0,1\}^\infty \) as the point process, i.e random measure,
\[ \sum_{i \geq 1} \xi_i \delta(i), \]
where \( \delta(i) \) is the measure placing unit mass at \( i \). Our example was \( \xi_1 \xi_2 \cdots = 1011100011000100000 \cdots \leftrightarrow \delta(1)+\delta(3)+\delta(4)+\delta(5)+\delta(10)+\delta(11)+\delta(15)+\cdots. \)

We rescale space: divide the locations by \( x \), to get \( \sum_{i \geq 1} \xi_i \delta(i/x) \) and take \( x \to \infty \). The limit is the “scale invariant” Poisson process \( \mathcal{X} \) on \((0, \infty)\) with intensity \( \frac{1}{x} \, dx \). The intensity gives the expected number of points in any interval \((a, b)\) with \( 0 < a < b < \infty \), which is
\[ E \mathcal{X}(a, b) = \int_a^b \frac{1}{x} \, dx = \log(b/a). \]
The Poisson property is that for any disjoint \( I_1, I_2, \ldots \subset (0, \infty) \), the counts of points in these sets, \( \mathcal{X}(I_1), \mathcal{X}(I_2), \ldots \) are independent, and Poisson distributed. In particular, for \( 0 < a < b < \infty \)
\[ \mathbb{P}(\mathcal{X}(a, b) = 0) = \exp(-E \mathcal{X}(a, b)) = \frac{a}{b}. \]

Likewise, consider the point process \( \sum Z_i \delta_i \). This process encodes the multiset of spacings used in the Feller coupling, without keeping track of the order in which the spacings are used. In our example, \( \xi_1 \xi_2 \cdots = 10111000011000100000 \cdots \), which implies \( Z_1 \geq 3, Z_2 \geq 1, Z_4 \geq 1, Z_5 \geq 1. \) We now reveal more about the outcome in this example, and declaring that \( Z_1 = 3, Z_2 = 1, Z_3 = 0, Z_4 = 1, Z_5 = 1 \), which corresponds to the example for primes, (10). And to have a nice picture, we also declare that \( Z_i = 0 \) for \( i = 6 \) to 13, \( Z_{14} = 2, Z_{15} = Z_{16} = 0, Z_{17} = 1 \), and \( Z_i = 0 \) for \( i = 18 \) to 25. Our random measure is
This process rescaled, \( \sum Z_i \delta(i/x) \), converges in distribution to \( \mathcal{X} \):

\[
\sum_{i \geq 1} Z_i \delta(i/x) \Rightarrow \mathcal{X} \quad \text{as } x \to \infty.
\]

Recall that \( Z_i \) is the number of \( i \)-spacings of \( \xi_1 \xi_2 \cdots \). We have a process \( \xi \) whose rescaled limit is \( \mathcal{X} \), and the spacings of this process \( \xi \) also have rescaled limit \( \mathcal{X} \). Hence it is plausible to guess that \( \mathcal{X} \) is equal in distribution to its own process of spacings.

### 3.3. The scale invariant spacing lemma.

Indeed, the process \( \mathcal{X} \) is equal in distribution to its own spacings. Since \( \mathcal{X} \) has no multiple points (unlike \( \sum Z_i \delta(i) \), which can have multiple points), \( \mathcal{X} \) can be encoded as a random set \( \mathcal{X} = \{X_i : i \in \mathbb{Z}\} \subset (0, \infty) \), with the points indexed so that for all \( i \in \mathbb{Z} \), \( X_i < X_{i+1} \). With such indexing, the spacings are the points

\[ Y_i := X_{i+1} - X_i, \]

and the statement that the spacings of \( \mathcal{X} \) have the same distribution as \( \mathcal{X} \) is

\[ \{Y_i : i \in \mathbb{Z}\} \overset{d}{=} \mathcal{X}, \quad \text{and } 1 = \mathbb{P}(Y_i \neq Y_j \ \forall i \neq j \in \mathbb{Z}). \]

The proof of this, from [2], is inspired by Janson’s proof in section 2.1 that if the multiset with \( Z_i \) copies of \( i \), to be used as spacings, is placed in a size biased permutation, then the set of points is encoded by the Bernoulli \((1/i)\) sequence \( 1\xi_2 \xi_3 \cdots \). Viewing this in terms of the limits under spatial rescaling, it suggests that if the points of \( \mathcal{X} \), to be used as spacings, are placed in a size biased permutation, then the set of partial sums also is distributed as the scale invariant Poisson process.

The size biased permutation of \( \mathcal{X} \) has a very pleasing, symmetric expression. Recall from section [1] that a size biased permutation can be generated by using, for an object of weight \( x \), an exponentially distributed label \( W \), with density \( f_W(w) = xe^{-wx} \). To emphasize that the weights are also random, and we have conditioned on the values of these weights, we write this with the notation for a conditional density given \( x \), that is \( f_W|X(w|x) \text{ is } dx = xe^{-wx} \text{ dw}. \) Starting with the scale invariant Poisson process \( \mathcal{X} = \{X_i\} \), with intensity \( f_X(x) \text{ dx } = (1/x) \text{ dx on } (0, \infty) \), and attaching
label \(W_i\) to \(X_i\), the process \(\{(W_i, X_i) : i \in \mathbb{Z}\}\) of points with labels is a Poisson process with intensity \(f_X(x) \, dx \, f_{W|X}(w|x) \, dw = \)

\[ f_{W,X}(w, x) \, dw \, dx = e^{-wx} \, dw \, dx, \quad (w, x) \in (0, \infty)^2. \]

Note the beautiful and perfect symmetry between the points and labels: the distribution of the process is invariant under \((w, x) \mapsto (x, w)\). No such symmetry is possible for our other size biased permutations, since the points have discrete support, such as \(\mathbb{N}\) or \(\{\log p : p \text{ is prime}\}\), while the labels are continuously distributed in \((0, \infty)^2\). The distribution of the process is also invariant under rescaling of the form \((w, x) \mapsto (cw, x/c)\); we apply this with \(c = 2\) in the picture below, and with \(c = \log n\) in the proof of Lemma 2.

The picture shows the Poisson process with intensity \(e^{-wx}\) restricted to the region \((0, b]^2\) with \(b = 5\). The number of points in this region is Poisson with mean \(\int_0^b (1 - e^{-bx}) \, dx\); this mean is 3.796 for \(b = 5\), and would be 8.401 for \(b = 50\).

We show three realizations; the first experiment, shown with solid circles, had 4 points in this region, the second experiment, shown with open circles had 10 points, and the third experiment, shown with “×” had 3 points. This is honest simulation; I did only three runs, even though they hardly look typical to my naive eye. Try to visualize each of the three runs by itself.

The proof of the scale invariant spacing lemma goes as follows. Start with the point process on \((0, \infty)^2\) with intensity \(e^{-wy} \, dw \, dy\), and index the points as \((W_i, Y_i)\) with \(W_i > W_{i+1}\) for all \(i \in \mathbb{Z}\). (The labels \(W_i\) are all distinct, with probability one, and we remove from the probability space the complementary event.) Notice the reverse direction for the deterministic inequality; small indices \(i \mapsto \) large labels \(W_i\), which tend to go with small weights \(Y_i\). This gives us a set of points \(\{Y_i\}\) having the distribution of the scale invariant Poisson process, indexed in order of a size biased permutation, tending from small to large. Define the points \(X_j\) to be the partial sums of the \(Y_i\) in this order:

\[
\text{for } i \in \mathbb{Z}, \quad X_j := \sum_{-\infty < i < j} Y_i
\]
This gives a set of points \(0 < \cdots < X_i < X_{i+1} < \cdots < \infty\), and further calculation shows that the distribution of \(\{X_i : i \in \mathbb{Z}\}\) is that of a scale invariant Poisson process with intensity \((1/x) \, dx\). By construction, the spacings of the \(X_i\) are the points \(Y_i\), with the desired distribution.

3.4. Primes and the scale invariant Poisson. How does the process of points in the coupling for primes appear, viewed from a distance? Recall, the points are \(0, \log Q_1, \log Q_1 + \log Q_2, \cdots\), and their spacings, \(\log Q_1, \log Q_2, \ldots\) are such that \(\log p\) occurs \(Z_p\) times, where the \(Z_p\) are independent and geometrically distributed. A process which encodes the multiset of spacings, in a form suitable for spatial rescaling, is the random measure

\[
\sum_{i \geq 1} \delta(\log Q_i) = \sum_p Z_p \delta(\log p).
\]

The direct analog of (28) would be the statement that

\[
\sum_p Z_p \delta(\log p/x) \to \mathcal{X} \quad \text{as} \quad x \to \infty.
\]

Standard probability theory reduces this to showing that for \(0 < a < b < \infty\), the expected mass that the rescaled measure gives to \((a, b)\) converges to \(\log(b/a)\) as \(x \to \infty\). The mass is \(\sum_{ax < \log p < bx} \mathbb{E} Z_p = \sum e^{ax} < p < e^{bx} 1/(p-1)\), so that sufficient number-theoretic knowledge needed to prove (30) is that as \(y \to \infty\), \(\sum_{p \leq y} 1/p = B + \log \log y + o(1)\), for some constant \(B\), a statement weaker than the prime number theorem.

But the random measure \(\sum_p Z_p \delta(\log p) = \sum_{i \geq 1} \delta(\log Q_i)\) is much closer to \(\mathcal{X}\) than the rescaling relation (30) shows. Namely, we can match up points so that the total amount of displacement needed to convert one sequence to the other is finite (in expectation, and therefore almost surely.) This coupling comes from [2], where it is combined with the total variation distance approximation of the small prime factors of a uniform integer, to give a metrized version of the Poisson process approximation for the “intermediate” prime divisors, from De Koninck and Galambos [25]. The points \(X_i\) of \(\mathcal{X}\) are indexed by \(i \in \mathbb{Z}\), while the points \(Q_i\) are indexed by \(i \in \mathbb{N}\), so we define \(Q_i = 1\) for \(i \leq 0\). The claim is that we can construct the \(Q_i\) and \(X_i\) on a single probability space, so that

\[
\mathbb{E} \sum_{i \in \mathbb{Z}} |X_i - \log Q_i| < \infty.
\]

3.4.1. Ignoring the difference between geometric and Poisson. In (31) the chief obstacle is that \(\mathcal{X}\) is intrinsically a Poisson process, while the prescribed multiplicity of \(p\) in the sequence \(Q_i\) is not Poisson, but rather a geometric, \(Z_p\). We make our task easier if we change the prescription to the following: for every \(p\), \(A_p = \sum_i 1(Q_i = p)\), where the \(A_p\) are independent, Poisson\((1/p)\). The two versions of the prescribed counts, \(Z_p\) and \(A_p\) are close, and can be coupled with \(\mathbb{E} |Z_p - A_p| = 1/(p(p-1))\). To convert the coupling
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with Poisson multiplicities into a coupling with geometric multiplicities, we can move \(|Z_p - A_p|\) copies of \(\log p\), each at most through distance \(\log p\), because there is an infinite supply of points \(\log Q_j = 0\) to swap with. Since

\[
\sum \log p/(p(p - 1)) < \infty,
\]

this perturbation is absorbed by the right side of (31).

A simple way to handle the scale invariant Poisson process on \((0, \infty)\) is to start with the translation invariant Poisson \(L\) with intensity \(1 \, dx\) on \(\mathbb{R}\) — a process for which the number of points in an interval of length \(x\) is Poisson distributed, with mean \(x\). If the points of this process are \(L_i\) for \(i \in \mathbb{Z}\), then setting

(33) \[X_i := e^{L_i}\]
gives the points of the scale invariant Poisson process \(X\). To get \(A_p\) to be Poisson with mean \(1/p\), all we need to do is assign some interval of length \(1/p\), and let \(A_p\) count how many points of \(L\) land in that interval. Using disjoint intervals for different \(p\) makes the \(A_p\) mutually independent.

The error estimate for the prime number theorem (see e.g. [42], or [44] section 4.1) implies the well known estimate

(34) \[\sum_{p \leq x} \frac{1}{p} = B + \log \log x + O\left(\exp\left(-\sqrt{\log x}\right)\right)\]
as \(x \to \infty\), for some \(c > 0\), with constant

\[B := \gamma - \sum_{k \geq 2} \sum_p \frac{1}{kp^k} \approx .261497,\]

where \(\gamma\) is Euler’s constant, \(\gamma \approx .5772\). We use this in the form

(35) \[f(x) := -B + \sum_{p \leq e^x} \frac{1}{p} = \log x + O\left(\exp\left(-\sqrt{x}\right)\right).\]
The best upper bound for (34) is \(O(\exp(-c(\log x)^{3/5}(\log \log x)^{-1/5}))\), but more easily stated estimate leads to the same order error bounds in our work.

We define a function \(g : \mathbb{R} \to \{0, \log 2, \log 3, \ldots\}\) to be essentially the inverse of \(f\). Specifically, \(g\) has \((-\infty, -B] \mapsto 0, (-B, -B + 1/2] \mapsto \log 2, (-B+1/2, -B+1/2+1/3] \mapsto \log 3, \ldots\). Since \(f\) is close to the log function, \(g\) is close to the exponential function. Now let \(h : (0, \infty) \to \{0, \log 2, \log 3, \ldots\}\) be defined by \(h(x) := g(\log x)\), so that \(h\) is close to the identity function on \((0, \infty)\), and can be applied directly to the points \(X_i\) of the scale invariant Poisson process on \((0, \infty)\). We have, under \(h\), \((0, e^{-B}] \mapsto 0, (e^{-B}, e^{-B+1/2}] \mapsto \log 2, (e^{-B+1/2}, e^{-B+1/2+1/3}] \mapsto \log 3, \ldots\). This map \(h\) is such that the number of \(X_i\) with \(h(X_i) = \log p\) is Poisson(1/p), independently for all primes \(p\), and \(h\) is so close to the identity function that

\[\mathbb{E} \sum_{i \in \mathbb{Z}} |X_i - h(X_i)| < \infty.\]
The illustration below shows the “bins” for the translation invariant Poisson process on $\mathbb{R}$, with all arrivals to the left of $-B$ representing ones, arrivals between $-B$ and $-B+1/2$ representing twos, and so on. The length of the bin corresponding to $p$ is $1/p$. We have faked arrivals to correspond to the example (10) from Lecture 1, with $Z_2 = 3$, $Z_3 = 1$, $Z_5 = 0$, $Z_7 = 1$, $Z_{11} = 1$.

The reader should imagine an arrow labeled “exponential map,” pointing to a second picture, labeled “Bins for Poisson ($1/x\, dx$ on $(0,\infty)$),” with dividing markers at $e^{-B}$, $e^{-B+1/2}$, $e^{-B+1/2+1/3}$, and so on.

**Theorem 1.** Let $\mathcal{X}$ be the scale invariant Poisson process on $(0,\infty)$, with intensity $(1/x)\, dx$, and points $X_i, i \in \mathbb{Z}$. Define $Q_i$ by $\log Q_i = h(X_i)$. Then every $Q_i$ is either one or a prime, and for every prime $p$, $A_p := \sum_i 1(Q_i = p)$ is Poisson distributed, with $\mathbb{E} A_p = 1/p$. The $A_p$ are mutually independent, and with $f$ defined by (35),

$$\mathbb{E} \sum_{i \in \mathbb{Z}} |X_i - \log Q_i| = \int_0^\infty |f(x) - \log x| \, dx < \infty. \quad (36)$$

**Proof.** Constructing $\mathcal{X}$ from $\mathcal{L}$ as in the discussion above, so that $X_i = \exp(L_i)$, we have $\log Q_i = h(X_i) = g(L_i)$. This construction makes it obvious that the $A_p$ are independent, Poisson, with $\mathbb{E} A_p = 1/p$. To prove (36), we argue that

$$\mathbb{E} \sum_{i \in \mathbb{Z}} |\log Q_i - X_i| = \mathbb{E} \sum_{i \in \mathbb{Z}} |g(L_i) - \exp(L_i)| = \int_{-\infty}^\infty |g(l) - e^l| \, dl =: a_0, \quad (37)$$

where we have used Fubini to express an expectation of a sum over points of $\mathcal{L}$ in terms of the intensity of $\mathcal{L}$. Now

$$a_0 = \int_{-\infty}^\infty |g(l) - e^l| \, dl = \int_0^\infty |f(x) - \log x| \, dx < \infty, \quad (38)$$

where the equality holds because $a_0$ may be interpreted as the area between the graphs of $g$ and the exponential function, and equals the area between the graphs of $f$ and the logarithm (with vertical segments added to span the jumps of $f$ and of $g$.) The convergence of this last integral at infinity follows from the bound (35), and the convergence of the integral at 0 follows simply as $\int_0^{\log 2} |f(x) - \log x| \, dx = \int_0^{\log 2} (-B - \log x) \, dx < \infty$.

### 3.4.2. Exploiting the relation between geometric and Poisson.

The argument at (32), for changing the coupling with $A_p \sim \text{Poisson}(1/p)$ copies of $p$ to one with $Z_p \sim \text{Geometric}$, is not pretty, but more importantly, it
is a non-explicit procedure and makes it hard to control the size biased permutation of the log $Q_i$ and their partial products. Carrying out the following explicit coupling turned out to be the key to being able to analyze the coupling for primes described in Lecture 1. A simple observation, that the geometric distribution is compound Poisson, makes everything work!

Every baby should know the following standard lemma, which we will apply with $a = 1/p$ and corresponding random variables $Z_p$ and $A_p^{(k)} \equiv A_{p^k}$. The added entropy involved in the integer partition occurring in (39) is discussed later, at (46).

**Lemma 1.** Let $Z$ be geometric with parameter $a \in (0, 1)$, i.e. $P(Z \geq k) = a^k$. Let $A \equiv A^{(1)}$ be Poisson with $E(A) = a$, and more generally, for $k = 1, 2, 3, \ldots$, let $A^{(k)}$ be Poisson with $E(A^{(k)}) = a^k/k$, with the $A^{(k)}$ independent. Then

$$Z \overset{d}{=} \sum_{k \geq 1} kA^{(k)} = A + \sum_{k \geq 2} kA^{(k)}.$$  

**Proof** Recall that the probability generating functions of geometric and Poisson are given by

$$E(s^Z) = \sum_{j \geq 0} s^j (1-a)a^j = \frac{1-a}{1-as}, \quad E(s^A) = \sum_{j \geq 0} s^j e^{-a} a^j / j! = e^{a(s-1)},$$

with $|s| < 1$, so that $E(s^A) = e^{a(s^k-1)}$ and $E(s^{kA^{(k)}}) = e^{((as)^k-a^k)/k}$. Writing

$$\log \frac{1-a}{1-as} = \sum_{k \geq 1} \frac{(as)^k - a^k}{k}$$

proves (39).

Notice also that the last expression in (39) demonstrates stochastic domination $Z_p \geq_d A_p$, so that in the expressions $|Z_p - A_p|$ in the argument before (32), the absolute value signs weren’t needed!

For us,

$$Z_p = \sum_{k \geq 1} kA_{p^k}$$

with $p$ prime, $k \geq 1$, and

$$q = p^k, \quad A_q \sim \text{Poisson}, \quad E(A_q) = \frac{1}{kq}.$$  

The coupling for primes described in Lecture 1 had a multiset with $Z_p$ copies of $p$, taken in order of a size biased permutation. A far more tractable coupling has a multiset of primes and prime powers, with $A_q$ copies of the prime power $q = p^k$, and the objects $Q_i$ (re-using the same notation $Q_i$ as before) to be taken in size biased permutation are these prime powers. In the remainder of this, Section 3.4.2, we will study the coupling of this multiset of prime powers to the scale invariant Poisson. Then in Section 3.5 we will
take a size biased permutation, to give our second coupling for growing a random integer. We will analyze this second coupling in detail; the first coupling for growing a random integer, from Lecture 1, can be analyzed as a perturbation of the second coupling — but we won’t present the details of the comparison, which are similar in spirit to proofs of lemmas 5 and 6.

The modified version of (34) is

$$\sum_{q=p^k \leq x} \frac{1}{kq} = \gamma + \log \log x + O\left(\exp(-c\sqrt{\log x})\right)$$

and our modified version of (35) is

$$f(x) := -\gamma + \sum_{q=p^k \leq e^x} \frac{1}{q^k} = \log x + O\left(\exp(-c\sqrt{x})\right).$$

Likewise, we modify $g$ to be essentially the inverse of this $f$, with $g(t) = 0$ if $t \in (-\infty, -\gamma]$, and if $q > 1$ is a prime power, $l = \log q$ and $\log x \in (f(l-), f(l)]$ then $g(x) = l$. We define $h = g \circ \log$, so that our modified version of the map $h$, which can be applied to the points of the scale invariant Poisson process, has

$$\begin{align*}
(0, e^{-\gamma}) &\mapsto 0, & (e^{-\gamma}, e^{-\gamma+1/2}) &\mapsto \log 2, & (e^{-\gamma+1/2}, e^{-\gamma+1/2+1/3}) &\mapsto \log 3, \\
(\exp(-\gamma+1/2+1/3), \exp(-\gamma+1/2+1/3+1/8]) &\mapsto \log 4,
\end{align*}$$

and so on. Notice that in the last endpoint given above we have $1/8 = 1/(kq)$ for $q = p^k = 4$. To summarize formally, $h$ is given by the recipe: $h(x) = 0$ if $0 < x \leq e^{-\gamma}$, and if $q > 1$ is a prime power, $l = \log q$ and $\log x \in (f(l-), f(l)]$, then $h(x) = l$, with $f$ as in (41).

**Theorem 2.** Let $\mathcal{X}$ be the scale invariant Poisson process on $(0, \infty)$, with intensity $(1/x) \, dx$, and points $X_i, i \in \mathbb{Z}$. Define points $Q_i$ by $\log Q_i = h(X_i)$, for the function $h$ described at (42). Then every $Q_i$ is either one or a prime power, and for every $q = p^k$, with $p$ prime and $k \geq 1$, $A_q := \sum_i 1(Q_i = q)$ is Poisson distributed, with $\mathbb{E} A_q = 1/(kq)$. The $A_q$ are mutually independent, and with $f$ defined by (41),

$$\mathbb{E} \sum_{i \in \mathbb{Z}} |X_i - \log Q_i| = \int_0^\infty |f(x) - \log x| \, dx < \infty.$$ 

**Proof** Constructing $\mathcal{X}$ from $\mathcal{L}$ as in the discussion above, so that $X_i = \exp(L_i)$, we have $\log Q_i = h(X_i) = g(L_i)$. This construction makes it obvious that the $A_q$ are independent, Poisson, with $\mathbb{E} A_{p^k} = 1/(kq)$. To prove (43), we argue that

$$\mathbb{E} \sum_{i \in \mathbb{Z}} |\log Q_i - X_i| = \mathbb{E} \sum_{i \in \mathbb{Z}} |g(L_i) - \exp(L_i)| = \int_{-\infty}^\infty |g(l) - e^l| \, dl =: b_0,$$
where we have used Fubini to express an expectation of a sum over points of \( \mathcal{L} \) in terms of the intensity of \( \mathcal{L} \). Now
\[
(45) \quad b_0 = \int_{-\infty}^{\infty} |g(l) - e^l| \ dl = \int_{0}^{\infty} |f(x) - \log x| \ dx < \infty,
\]
where the equality holds because \( b_0 \) may be interpreted as the area between the graphs of \( g \) and the exponential function, and equals the area between the graphs of \( f \) and the logarithm. The convergence of this last integral at infinity follows from the bound (41), and the convergence of the integral at 0 follows simply as \( \int_{0}^{\log 2} |f(x) - \log x| \ dx = \int_{0}^{\log 2} (-\gamma - \log x) \ dx < \infty \).

3.5. The size biased permutation of the multiset having \( \log p^k \) with multiplicity \( A_{p^k} \sim \text{Poisson}(1/(kp^k)) \). Just as the multiset with \( Z_p \) copies of each prime \( p \), for independent \( Z_p \sim \text{geometric}(1/p) \), is the “natural random infinite multiset of primes,” the multiset with \( A_q \) copies of each prime power \( q = p^k > 1 \), for independent \( A_q \sim \text{Poisson}(1/(kq)) \), is the natural random infinite multiset of prime powers. The latter multiset may be viewed as the former, with auxiliary randomization, picking a partition of the integer \( Z_p \), independently for each \( p \).

[Incidentally, the amount of additional information in our partitioning of the \( Z_p \) is small; it is approximately .612433379 bits, computed as follows. Recall that for a discrete random variable \( X \) with \( \mathbb{P}(X = X_i) = p_i > 0, \sum p_i = 1 \), the entropy is \( h(X) := -\sum p_i \log p_i \). For the geometrically distributed \( Z \) in (39), \( h(Z) = -\log(1 - a) - a/(1 - a) \log a = \sum_{k \geq 1} (a^k/k)(1 + \log(1/a^k)) \), and for \( A \sim \text{Poisson}(x) \) we have \( h(A) = x + x \log(1/x) + \sum_{j \geq 2} \mathbb{P}(A \geq j) \log j \), which we apply with \( x = a^k/k \) for \( k = 1, 2, 3, \ldots \). Recall that the entropy of an independent process, such as \( (A^{(1)}, A^{(2)}, \ldots) \), is the sum of the entropies of the coordinates. Thus in (39), the additional information needed to partition \( Z \) into \( \sum_{k \geq 1} kA^{(k)} \) is \( d(a) := \sum_{k \geq 1} h(A^{(k)}) - h(Z) = \)
\[
(46) \quad \sum_{k \geq 1} \left( \frac{a^k}{k} + \frac{a^k}{k} \log \frac{k}{a^k} + \sum_{j \geq 2} \mathbb{P}(A^{(k)} \geq j) \log j \right) - \sum_{k \geq 1} \frac{a^k}{k} (1 + \log \frac{1}{a^k})
\]

\( = a^2 \log 2 + O(a^2 \log a) \) as \( a \to 0+ \). Numerical evaluation, with logs taken base 2, gives \( d(1/2) \approx .375076 \) as the additional information needed to partition \( Z_2 \), approximately .13879 for \( Z_3 \), and approximately .612433379 for the sum over all primes.]

In Lecture 1 we described a procedure for “growing” a random integer, based on a size biased permutation of the multiset having each prime \( p \), taken as an object of weight \( \log p \), with multiplicity \( Z_p \sim \text{Geometric}(1/p) \). While relatively simple to describe, this coupling is rather hard to analyze directly. The reason that it is hard to analyze directly is that the size biased permutation involves using exponentially distributed labels \( W_i \), and the resulting two-dimensional process, with points of the form \( (W_i, \log Q_i) \),
does not have a simple structure. By changing the objects to be prime powers \( q = p^k \), with weight \( \log q \) and multiplicity \( A_q \sim \text{Poisson } 1/(kq) \) the total number of factors of \( p \) is still distributed as \( Z_p \sim \text{Geometric } (1/p) \), because \( \sum_{k \geq 1} kA_k \sim Z_p \), but now the size biased permutation is tractable.

The size biased permutation is tractable because attaching conditionally independent labels \( W_i \) to the Poisson process with points \( \log Q_i \) yields a two-dimensional Poisson process with points \( (W_i, \log Q_i) \) – this is an instance of the “labeling” theorem; see for example [31].

This process \( \{(W_i, \log Q_i) : i \in \mathbb{N} \} \) is very close to the Poisson process of Section 3.3 on \((0, \infty)^2 \) with intensity \( e^{-wq} dw dy \), which was used to give a size biased permutation of the scale invariant Poisson process. However, our two-dimensional process now is neither discrete nor continuous. It is supported on a one dimensional subset of the positive quadrant, formed by the half lines \( w > 0, y = \log q \) for some prime power \( q = p^k > 1 \). Its intensity on the line \( y = \log q \) is the product of the discrete intensity, \( f_Y(y) = 1/(kq) \) for the weight \( y \), times the continuous density for exponentially distributed label \( w \), \( f_{W|Y}(w|y) dw = ye^{-wy} dw \). That is, for \( y = \log q \), \( q = p^k \),

\[
(47) \quad f_{W,Y}(w, y) dw = f_Y(y) f_{W|Y}(w|y) dw = \frac{\log q}{kq} e^{-wy} dw.
\]

The next lemma gives an exact expression for the density of \( J(n) \), and will let us see in Lemma 3 that the distribution of \( J(n) \) is close to the harmonic distribution \((7)\) on \( 1, 2, \ldots, n \); in \((50)\), \( d(\log n^\beta) \) is close to \( d\beta/\beta \), and the zeta function has a simple pole at one, so the expression in \((50)\) is close to

\[
(48) \quad \frac{1}{i} \int_{\beta > 1-\alpha} \frac{d\beta}{\beta} \int_{c>0} \beta e^{-\beta c} e^{-ac} \log n \ dc = \frac{1}{i} \log n.
\]

**Lemma 2.** For the \( Q_i \) taken in order of decreasing labels \( W_i \), let \( J(n) \) be the largest partial product with \( J(n) \leq n \). Then for \( i = n^\alpha = 1 \) to \( n \), with \( q = p^k = n^\beta \) ranging over prime powers,

\[
(49) \quad \mathbb{P}(J(n) = i) = \sum_{q: \beta > 1-\alpha} \frac{1}{kq} \int_{c>0} \beta e^{-\beta c} \frac{i^{-1-c/\log n}}{\zeta(1+c/\log n)} \ dc
\]

\[
(50) \quad = \frac{1}{i} \sum_{q: \beta > 1-\alpha} \frac{1}{kq} \int_{c>0} \beta e^{-\beta c} \frac{e^{-ac}}{\zeta(1+c/\log n)} \ dc.
\]

**Proof** Write the labels \( W_i \) as \( W_i = S_i/\log Q_i \) as in Section 1.3 with \( S_1, S_2, \ldots \) being iid standard exponentials, independent of \( Q_1, Q_2, \ldots \). For any prime power \( q > 1 \) and constant \( t \geq 0 \), the number \( A_q(t) := \sum_i 1(Q_i = q, S_i/\log q > t) \) of occurrences of \( q \) with label greater than \( t \) is Poisson distributed with \( \mathbb{E} A_q(t) = A_q \mathbb{P} (S_i/\log q > t) = 1/(kq) \). The \( A_q(t) \)
jointly for all $q$ are independent. For each prime $p$ let

$$Z_p(t) := \sum_{k \geq 1} k A_p^k(t).$$

Note that by (39) the distribution of $Z_p(t)$ is geometric with $P(Z_p(t) \geq j) = (1/p^{1+t})^j$ and

$$P(Z_p(t) = j) = \left(1 - \frac{1}{p^{1+t}}\right)^j.$$ 

For any $t > 0$, consider the product $I_t$ of all primes and prime powers having label strictly greater than $t$, i.e.

$$I_t := \prod_{q=p^k} q^{A_q(t)} = \prod p^{Z_p(t)}.$$

Using (52) and the independence of the $Z_p(t)$ over all primes $p$, for any $i \geq 1$ we have

$$P(I_t = i) = \prod (1 - p^{-1-t}) = \frac{i^{-1-t}}{\zeta(1+t)}.$$ 

With probability one, all labels $W_i$ are distinct. For any $t > 0$ there are, with probability one, only finitely many labels greater than $t$ — this follows from (53), which summed over $i = 1, 2, \ldots$ yields 1. There are infinitely many labels, with probability one, because the total intensity of our Poisson process is infinite — it is \(\sum_{q=p^k} E A_q = \sum_{q=p^k} 1/(kq) > \sum p(1/p) = \infty\). For these three reason combined, with probability one, as $t$ decreases from infinity to zero, the partial products $I_t$ increase from 1 to infinity, and each increase corresponds to factoring in one new factor $Q_i$, taken in decreasing order of their labels $L_i = S_i/\log Q_i$.

Let $Q^*(n)$ be the new factor that first takes the partial product beyond $n$, and let $T(n)$ be its label. Our product $J(n)$ will equal $I_t$ for $t = T(n)$. We consider the joint distribution of $(Q^*(n), T(n), J(n))$. Write $i = n^\alpha$ for the test value for $J(n)$, $q = p^k = n^\beta$ for the test value for $Q^*(n)$, and $c$ for $T(n) \log n$, which, as $\log n$ times the label for $q$, is exponentially distributed with rate $\log q/\log n = \beta$. Summing over $q$ and integrating over $c$ yields (49) as the marginal distribution of $J(n)$, and simplifying yields (50).

**Lemma 3.** Let $H(n)$ have the harmonic distribution (7) on 1 to $n$, so that for $i \leq n$, $P(H(n) = i) = 1/(ih_n)$. For the $J(n)$ in Lemma 2, based on a size biased permutation of the natural infinite Poisson multiset of prime powers,

$$d_{TV}(J(n), H(n)) := \sum_{i \leq n} |P(J(n) = i) - P(H(n) = i)| = O\left(\frac{1}{\log n}\right).$$
Furthermore, the relative error in approximating the density of \( J(n) \) by the harmonic density is \( O(1/\log n) \), uniformly:

\[
\max_{1 \leq n \leq \infty} \left| \frac{\mathbb{P}(J(n) = i)}{\mathbb{P}(H(n) = i)} - 1 \right| = O \left( \frac{1}{\log n} \right).
\]

\textbf{Proof} Define \( d_n(i) := (i \log n) \mathbb{P}(J(n) = i) \) so that (50) can be rewritten as

\[
d_n(i) = \sum_{\beta > 1 - \alpha} \frac{1}{kq} \int_{c>0} \beta \ e^{-\beta c} \frac{e^{-\alpha c} \log n}{\zeta(1 + c/\log n)} \, dc.
\]

Since \( \mathbb{P}(H(n) = i) = 1/(N_h) = 1/(i \log n) (1 + O(1/\log n)) \), showing (55) is equivalent to showing \( \max_{1 \leq i \leq n} |d_n(i) - 1| = O(1/\log n) \).

In order to simplify (56), we apply the following with \( \delta = c/\log n \). From the well known \( \zeta(1 + \delta) = (1/\delta) + \gamma + O(\delta) \) as \( \delta \to 0+ \), we get \( 1/\zeta(1 + \delta) = \delta - O(\delta^2) \) as \( \delta \to 0+ \). It follows that \( \exists C_1, 1/\zeta(1 + \delta - \delta) \leq C_1 \delta^2 \) for all \( \delta > 0 \). This motivates us to consider a first order approximation to the right side of (53), defined by

\[
e_n(i) := \sum_{\beta > 1 - \alpha} \frac{1}{kq} \int_{c>0} \beta \ e^{-\beta c} \ c \ e^{-\alpha c} \, dc.
\]

Our goal is to show that uniformly in \( 1 \leq i \leq n \), \( e_n(i) = 1 + O(1/\log n) \); having this, the error estimate for \( d_n(i) \) versus \( e_n(i) \) will be virtually the same computation.

Note that since \( \int_{c>0} c e^{-(\alpha + \beta) c} \, dc = (\alpha + \beta)^{-2} \), and \( \beta = \log q/\log n \), we have

\[
e_n(i) = \frac{1}{\log n} \sum_{q>n^{1-\alpha}} \frac{\log q}{kq} (\alpha + \beta)^{-2}.
\]

Instead of (41) we only need a crude bound, due to Chebyshev, that

\[
R(x) := \sum_{q=p^{k} \leq x} \frac{\log q}{kq} - \log x = O(1).
\]

Fix \( n \) and \( i = n^{\alpha}, 1 \leq i \leq n \), and define

\[
S_t := \sum_{n^{1-\alpha} < q \leq n^{t}} \frac{\log q}{kq} = (t - (1 - \alpha)) \log n - R(n/i) + R(n^{t}),
\]

so that by Abel summation

\[
e_n(t) = \frac{1}{\log n} \int_{t \in (1-\alpha, \infty)} \frac{1}{\log n} \int_{t \in (1-\alpha, \infty)} S_t \ (\alpha + t)^{-2} = \frac{1}{\log n} \int_{t \in (1-\alpha, \infty)} S_t \ 2(\alpha + t)^{-3} \ dt.
\]

The contribution at infinity to the Abel summation is zero since \( S_t \sim t \log n \) as \( t \to \infty \). From \( \sup_{x>0} R(x) < \infty \), and \( \int_{1-\alpha, \infty} 2(t + \alpha - 1) \ (\alpha + t)^{-3} \ dt = 1 \) it follows that \( \max_{1 \leq i \leq n} |e_n(i) - 1| = O(1/\log n) \).
Finally, using \(|1/\zeta(1+\delta) - \delta| \leq C_1 \delta^2\), we have

\[
|d_n(i) - e_n(i)| \leq \frac{C^2}{\log n} \sum_{q: \beta > 1-\alpha} \frac{1}{kq} \int_{e^c}^{\infty} \beta \ e^{-\beta c} \ e^{-\alpha c} \ dc,
\]

\[
= \frac{C_1^2}{(\log n)^2} \int_{(1-\alpha, \infty)} dS_t \ 2(\alpha + t)^{-3} \ dt = O\left(\frac{1}{\log n}\right).
\]

3.6. **Filling in the extra prime factor.** As in Lecture 1, we take \(P_0(n)\) to be one or prime (and not a prime power!), such that \(J(n)P_0(n) \leq n\), picking uniformly over the \(1 + \pi(n/J(n))\) possibilities.

With the notation \(f_n(i) := \mathbb{P}(J(n)P_0(n) = i)\), the total variation distance in the next lemma is \(d_{TV}(J P_0, N) =\)

\[
\sum_{1 \leq i \leq n} \left( f_n(i) - \frac{1}{n} \right)^+ = \sum_{1 \leq i \leq n} \left( f_n(i) - \frac{1}{n} \right)^- = \frac{1}{2} \sum_{1 \leq i \leq n} \left| f_n(i) - \frac{1}{n} \right|.
\]

**Lemma 4.** The total variation distance between the distribution of \(J(n)P_0(n)\) and the uniform distribution satisfies

\[
d_{TV}(N(n), J(n) P_0(n)) = O\left(\frac{\log \log n}{\log n}\right).
\]

**Proof**

Since \(P_0\) is one or prime, for \(1 \leq m \leq n\),

\[
f_n(m) := \mathbb{P}(J(n)P_0(n) = m) = \sum_{p|m}^* \mathbb{P}(J(n) = \frac{m}{p}) \frac{1}{1 + \pi(np/m)},
\]

where \(\sum^*\) indicates that the index \(p\) ranges over prime divisors of \(m\) also allowing \(p = 1\). Using \(1/(i \log n)\) as an approximation for \(\mathbb{P}(J(n) = i)\), we consider the simpler expression

\[
g_n(m) := \frac{1}{n} \sum_{p|m}^* \frac{1}{\log n} \frac{np}{m} \frac{1}{1 + \pi(np/m)}.
\]

We have

\[
\sup_{1 \leq m \leq n} \left| 1 - f_n(m)/g_n(m) \right| = O(1/ \log n)
\]

thanks to (55), and hence also \(\sum_{m \leq n} |f_n(m) - g_n(m)| = O(1/ \log n)\). Thus (58) is equivalent to \(\sum_{m \leq n} |g_n(m) - \frac{1}{n}| = O(\log \log n/ \log n)\).

From the well known error bound for the prime number theorem, \(\pi(x) = \text{li}(x) + O(xe^{-c\sqrt{\log x}})\), together with the approximation \(\text{li}(x) = (x/ \log x)[1 + 1/ \log x + O((\log x)^{-2})]\) we have

\[
r(x) := \left| \frac{x}{1 + \pi(x)} - (\log x - 1) \right| = O(1/ \log x).
\]
Thus a good approximation to $g_n(m)$ will be given by

$$h_n(m) := \frac{1}{n \log n} \sum_{p|m}^\ast \left( \log \left( \frac{np}{m} \right) - 1 \right).$$

Writing $s(i)$ for the largest squarefree divisor of $i$, and $\omega(i)$ for the number of distinct prime divisors of $i$, we have

$$h_n(m) = \frac{\log(s(m)) + (1 + \omega(m))(\log(n/m) - 1)}{n \log n}. \tag{61}$$

With $N \equiv N(n)$ to represent the uniform distribution on 1 to $n$, the total variation distance in $[55]$ is approximately

$$\frac{1}{2} \sum_{m \leq n} \left| h_n(m) - \frac{1}{n} \right| = \frac{1}{2} \mathbb{E} \left| \frac{\log(s(N)) - \log n}{\log n} + \frac{1 + \omega(N)}{\log n} (\log(n/N) - 1) \right|. \tag{62}$$

Before completing the proof of (58), we outline the analysis to focus on the source of the loglog $n$ factor in (58), and the reason that it cannot be decreased. The net contribution from the first term inside the expectation in (62) is $O(1/\log n)$, and for the second term, the two factors are approximately uncorrelated, with $\mathbb{E} (1 + \omega(N))/\log n \sim \log\log n/\log n$ for the first factor. The second factor has $\mathbb{E} |\log (n/N) - 1| \to \mathbb{E} |S - 1| = 2/e$, where $S$ has the standard exponential distribution, with $\mathbb{P}(S > t) = e^{-t}$. Thus it should be possible to show that

$$d_{TV}(N(n), J(n) P_0(n)) \sim \frac{1}{e} \left( \frac{\log\log n}{\log n} \right). \tag{63}$$

The estimates for the simpler task (58) in place of (63) are as follows. We need only $K := \sup_{x \geq 1} r(x) < \infty$ to conclude that $|g_n(m) - h_n(m)| \leq 1/(n \log n) \sum_{p|m} r(np/m) \leq K (1 + \omega(m))/(n \log n)$, and hence $\sum_{1 \leq m \leq n} |g_n(m) - h_n(m)| \leq K \mathbb{E} (1 + \omega(N(n)))/\log n = O(\log\log n/\log n)$. In (62), for the first term with $\mathbb{E} \log(s(N)) - \log n$ we have $\mathbb{E} \log N(n) - \log n \to 1$ by Stirling’s formula, and $\mathbb{E} \log(N(n)/s(N(n))) \leq \sum_{p \leq n} \log p \mathbb{E} (C_p(n) - 1)^+ \leq \sum_{p} \log p \sum_{k \geq 2} p^{-k} < \infty$. For the second term, we apply Cauchy-Schwarz, with $\mathbb{E} (1 + \omega(N(n)))^2 \sim (\log\log n)^2$ and $\mathbb{E} (\log(n/N(n))^2 \to 1$. These bounds combine to show that $\sum_{m \leq n} |h_n(m) - \frac{1}{n}| = O(\log\log n/\log n)$. Together with our previous bounds comparing $f, g$, and $h$, we have proved (58).

3.7. **Keeping score: 1 insertion and on average $1 + O((\log\log n)^2/\log n)$ deletions suffice for primes.** We need to control the expected number of deletions used to convert $M(n)$ to $N(n)$, which correspond to the number of primes not exceeding $n$ but occurring in the size biased permutation after the partial product $J(n)$. A priori it seems reasonable to believe that one would have to calculate something along the lines of (22) for permutations,
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crossed with (49) for primes. Happily, the idea of “matching intensity” can be used to finesse the calculation.

From (58) in Lemma 4 it follows (see section 3.8 iff you want to know the details) that on a single probability space we can construct independent $Z_p$, together with $J(n)$ and $P_0(n)$, and an exactly uniform $N(n)$, so that always $J(n)|M(n)$, and the good event

(64) $E_n = \{ J(n)P_0(n) = N(n) \}$

has

(65) $\mathbb{P}(E_n^c) = d_{TV}((J(n) P_0(n), \ N(n))) = O \left( \frac{\log \log n}{\log n} \right)$.

On the uncoupled event, $E_n^c$, how many prime factors, that might contribute to $d_W(n)$, can we expect to see? Recall our notation from section 1.5.1. Lemma 6 in [2] states that for events $E$ of small, but not too small probability, the expected number of prime factors is $O((\log \log n) \mathbb{P}(E))$. The precise statement is: uniformly in $\delta \in [0, 1]$,

(66) $\sup_{E: \mathbb{P}(E) \leq \delta} \mathbb{E} \left( 1(E) \Omega(N(n)) \right) = O \left( \max \left( \delta \log \log n, \frac{1}{\log n} \right) \right)$.

With $E = E_n^c$ and $\delta = \mathbb{P}(E_n^c)$, the combination of (65) with (66) shows that

(67) $\mathbb{E} \left( 1(E_n^c) \Omega \left( \frac{N}{(N, M)} \right) \right) \leq \mathbb{E} \left( 1(E_n^c) \Omega(N) \right) = O \left( \frac{(\log \log n)^2}{\log n} \right)$.

Now $J(n)|M(n)$ always, and on $E_n$ we have $N(n) = J(n)P_0(n)$ so that $N/(N, M) | P_0$, so that

(68) $\mathbb{E} \left( 1(E_n) \Omega \left( \frac{N}{(N, M)} \right) \right) \leq 1$.

Adding gives

(69) $\mathbb{E} \Omega \left( \frac{N(n)}{(N(n), M(n))} \right) \leq 1 + O \left( \frac{(\log \log n)^2}{\log n} \right)$.

Now any coupling has

$\mathbb{E} \Omega \left( \frac{N(n)}{(N(n), M(n))} \right) - \mathbb{E} \Omega \left( \frac{M(n)}{(N(n), M(n))} \right) = O(1/\log n)$,

because (see e.g. [44] p. 41)

(70) $\mathbb{E} \Omega(N(n)) - \mathbb{E} \Omega(M(n)) = O(1/\log n)$.

Combining this with (69) yields

(71) $\mathbb{E} \Omega \left( \frac{M(n)}{(N(n), M(n))} \right) \leq 1 + O \left( \frac{(\log \log n)^2}{\log n} \right)$.

Adding (69) and (71) proves
**Theorem 3.** The coupling of section 3.3 based on a size biased permutation of the natural Poisson multiset of prime powers, and extended to include a uniform random integer $N(n)$, has

$$
E \sum_{p \leq n} |C_p(n) - Z_p| \leq 2 + O \left( \frac{(\log \log n)^2}{\log n} \right),
$$

and hence $d_W(n) \leq 2 + O \left( (\log \log n)^2 / \log n \right)$.

A separate argument (see [2]) shows that, for any coupling, with probability approaching one, at least one insertion is necessary to convert $M(n)$ to $N(n)$, i.e. $1 = \lim \mathbb{P}(\sum_{p \leq n}(C_p(n) - Z_p)^+ \geq 1$. In particular, $1 \leq \lim \inf \mathbb{E} \sum_{p \leq n}(C_p(n) - Z_p)^+$. Using (70), the average number of deletions is within $O(1/\log n)$ of the number of insertions. Hence $1 \leq \lim \inf \mathbb{E} \sum_{p \leq n}(Z_p - C_p(n))^+$, so that $2 \leq \lim \inf \mathbb{E} \sum_{p \leq n}|C_p(n) - Z_p|$. This shows that that \( \lim d_W(n) = 2 \), and that our coupling, from the point of view of the insertion-deletion metric in section 1.5.1 is asymptotically optimal.

### 3.8. Extending the coupling to $N(n)$, constructively.

(The reader is invited to skip past this section, which defends the claim at (64).) Our coupling as described so far is fairly natural and explicit. It starts with independent Poisson $A_q$ for $q = p^k$. These determine the prime powers $Q_i$ and the $Z_p$ such that $M(n) := \prod_{p \leq n} p^{Z_p} = \prod_{i:Q_i = p^k,p \leq n} Q_i$, where the $Z_p$ for primes $p$ are independent geometric. Use independent exponentially distributed $S_1, S_2, \ldots$ to give a size biased permutation of these prime powers $q$; this determines $J(n)$, a divisor of $M(n)$. A single uniformly distributed random variable $U$, independent of the $Q_i$ and $S_i$ can then be used to determine $P_0$, via the recipe: with $K(n) := 1 + \pi(n/J(n))$, let $P_0 = 1$ if $K(n)U \leq 1$, and let

$$
P_0 = p_i \text{ if } K(n)U \in (i, i + 1],
$$

where $p_i$ denotes the $i^{th}$ smallest prime. Equation (58) gives an upper bound on the total variation distance between the distributions of $J(n)P_0(n)$ and of $N(n)$, and (57) emphasizes that this is just about the distribution of $J(n)P_0(n)$.

We have constructed $J(n)P_0(n)$ so far, and we have not yet constructed $N(n)$. It is a standard coupling argument that there exist couplings in which the good event $E_n = \{ J(n)P_0(n) = N(n) \}$ has the maximum possible probability, with $\mathbb{P}(E_n^c) = d_{TV}(JP_0, \ N)$. It is also true, but less obvious, that there exist such couplings which extend our already given construction of $((Q_i, S_i)_{i \geq 1}, P_0)$. We note further that joint distribution of $((Q_i, S_i)_{i \geq 1}, P_0, N)$ for such an extension is **not uniquely determined**. The next paragraph offers a constructive choice of joint distribution.

We present a recipe for constructing $N(n)$ as a function of the random variables used above, together with some auxiliary randomization. Take two
additional uniform random variables $U_1, U_2$ with $U, U_1, U_2, Q_1, Q_2, \ldots, S_1, S_2, \ldots$ independent. We define a deterministic function $r_n(u, u_1, u_2, q_1, q_2, \ldots, s_1, s_2, \ldots)$ such that for all $\omega \in \Omega$, $N(n) := r_n(U, U_1, U_2, Q_1, Q_2, \ldots, S_1, S_2, \ldots)$ and $\mathbb{P}(N(n) \neq J(n)P_0(n)) = d_{TV}(JP_0, N)$. The recipe $r_n$ is determined by two requirements. First, let $b_n(i) := \min(f_n(i), \frac{1}{n})/f_n(i)$, and let $E_n$ be the event
\[(74) \quad E_n := \{U_1 \leq b_n(J(n)P_0(n))\}.
\]
Note that $\mathbb{P}(E_n) = \sum_{1 \leq i \leq n} f_n(i)b_n(i) = \sum \min(f_n(i), \frac{1}{n}) = 1 - d_{TV}(JP_0, N)$.

On the event $E_n$, we define $N(n)$ by $N(n) = J(n)P_0(n)$. Second, let $G_n(j) := \sum_{i \leq j} (f_n(i) - \frac{1}{n})^{-} / d_{TV}(JP_0, N)$, so that by (57), $G_n(n) = 1$.

On the event $E_n^c$, we define $N(n)$, to have one of the values $i$ for which $f_n(i) < 1/n$, by setting:
\[(75) \quad \text{on } E_n^c, \quad N(n) = j \text{ if and only if } G_n(j - 1) < U_2 \leq G_n(j).
\]
It follows that for $i = 1$ to $n$, $\mathbb{P}(N(n) = i) = 1/n$, and that $E_n$ satisfies (64) and (65).

The above construction yields $N(n)$, uniformly distributed from 1 to $n$, together with random integers $J(n)$ and $P_0(n)$ that evolve smoothly with $n$ growing, such that the event $E_n = \{N(n) = J(n)P_0(n)\}$ has the maximal possible probability, namely $1 - d_{TV}((J(n) P_0(n), N(n)))$. Is it the case that with probability one, for all sufficiently large $n$, we have $N(n) = J(n)P_0(n)$? This is not a trivial question, as the events $E_n$ are not nested, and the sum of their probabilities is infinite.

**Theorem 4.** For the above construction,
\[1 = \mathbb{P}(E_n \text{ eventually}).\]

**Proof** First note that as events, $\{J(n) \to \infty\} = \{\sum Z_p = \infty\}$, and hence $1 = \mathbb{P}(J(n) \to \infty)$. Recall our use of two fixed uniform random variables, $U$ in (73) and $U_1$ in (74). For $\delta > 0$ we will show that
\[(76) \quad \{U_1 < 1 - \delta, U > \delta, \text{ and } J(n) \to \infty\} \subset \{E_n \text{ eventually}\}
\] and hence $\mathbb{P}(E_n \text{ eventually}) \geq (1 - \delta)^2$.

Assume we are given an outcome in the event on the left side of (76). Since the $J(n)$ are product processes, $J(n) \to \infty$ ensures that $\omega(J(n)) \to \infty$ as $n \to \infty$. To have $E_n$ fail, we must have $b_n(J(n)P_0(n)) < 1 - \delta$, and hence $n f_n(J(n)P_0(n)) > 1/(1 - \delta) > 1 + \delta$. For $n$ and $\omega(m)$ both large, arguing as in the proof of Theorem (3) $f_n(m)/h_n(m) \to 1$ so that $f_n(m) > 1 + \delta$ implies that $n h_n(m) > 1 + \delta/2$. [The hypothesis that $\omega(m)$ is large is needed to ensure that terms of $g_n(m)$ having $x = np/m$ small, where we cannot guarantee that $x/(1 + \pi(x))$ is close to $(\log x - 1)$, make a relatively negligible contribution.] Using only the bound $\log s(m) \leq \log n$ in (61), this implies that for sufficiently large $n$ and $\omega(m)$, $(1 + \omega(m)) \log(n/m) > (\delta/2) \log n$.

Pick $x_0 > 1/(2\delta)$ and large enough that $x > x_0$ implies $\pi(2\delta x) > (\delta)(1 + \pi(x))$. Since $\sup_{1 \leq i \leq n} \omega(i) = o(\log n)$, $(1 + \omega(m)) \log(n/m) > (\delta/2) \log n$. 

implies that for sufficiently large \( n, n/m > x_0 \). Thus for sufficiently large \( n \),
if \( E_n \) fails then \( x = n/J(n) > x_0 \), But \( U > \delta \) now implies \( P_0(n) \geq 2\delta n/J(n) \),
which would contradict \( n/m > x_0 \) with \( m = J(n)P_0(n) \). This shows that
for the given outcome, there is an \( N_0, \) (depending on the outcome through
the values of \( J(1), J(2), \ldots \) and \( U,U_1, \)) such that for all \( n > N_0, E_n \) occurs.

4. Lecture 4: The Distance to the Poisson-Dirichlet

For an integer \( N(n) \) distributed uniformly from 1 to \( n \), write
\[
N(n) = P_1(n)P_2(n) \cdots P_K(n) = P_1(n)P_2(n) \cdots, \quad P_1 \geq P_2 \geq \cdots,
\]
where \( K = \Omega(N(n)) \) is the number of prime factors of \( N \), and every \( P_i(n) \)
is either one or prime. Billingsley \[17\] in 1972 proved that the Poisson-
Dirichlet process gives the limit in distribution for the sizes of the large
prime factors,
\[
\left( \frac{\log P_1(n)}{\log n}, \frac{\log P_2(n)}{\log n}, \ldots \right) \Rightarrow (V_1, V_2, \ldots),
\]
where \((V_1, V_2, \ldots)\) has the Poisson-Dirichlet distribution with parameter 1.
The marginal distribution of the largest component is given by Dickman’s
function \( \rho \), in the form \( \mathbb{P}(V_1 \leq 1/u) = \rho(u) \); see \[44\] Chapter III.5. The
characterization of the limit which is useful for us is
\[
(V_1, V_2, \ldots) \overset{d}{=} \text{RANK}(1 - X_1, X_1 - X_2, X_2 - X_3, \ldots)
\]
where RANK is the function which sorts the coordinates in nonincreasing
order, and \( X_1, X_2, \ldots \) are those points of the scale invariant Poisson process
which fall in \((0,1)\), indexed with \( 1 > X_1 > X_2 > \cdots > 0 \). For other
characterizations of the Poisson-Dirichlet, see for example \[8, 39\]. Donnelly
and Grimmett \[22\] gave a very nice proof of (77) by showing that a size
biased permutation of the left side of (77) converges in distribution to
\((1 - X_1, X_1 - X_2, X_2 - X_3, \ldots)\), and then using the continuity of RANK on the
simplex.

We asked: how close are the right and left sides of (77)? One notion of
approximation, from Knuth and Trabb Pardo \[32\], is that for fixed \( i \) and
t \( \in (0,1) \), as \( n \to \infty \),
\[
\mathbb{P}\left( \frac{\log P_i(n)}{\log n} \leq t \right) = \mathbb{P}(V_i \leq t) + O\left( \frac{1}{\log n} \right).
\]
They also give a version with a \( O(1/\log n) \) correction term, of the form:
for fixed \( i \geq i \), and fixed \( t \in (0,1) \), \( \mathbb{P}(\log P_i(n)/(\log n) \leq t) = \mathbb{P}(V_i \leq t) \\
+ r_i(t)/\log n + O(1/(\log n)^2) \). That a similar result holds for the joint
finite dimensional distributions, together with an expansion in negative powers of
the logarithm, has recently been shown by Tenenbaum \[46\]. To state this,
for \( k \geq 1 \) write \( F_n(\alpha_1, \ldots, \alpha_k) = \mathbb{P}(\log P_i(n)/(\log n) \geq \alpha_i \) for \( i = 1 \) to \( k \)\),
and \( \phi_0(\alpha_1, \ldots, \alpha_k) = \mathbb{P}(V_i \geq \alpha_1, \ldots, V_k \geq \alpha_k) \), so that Billingsley’s result
(77) is equivalent to: for all \( k \geq 1 \) and \( \alpha_1, \ldots, \alpha_k \in (0,1) \), \( F_n(\alpha_1, \ldots, \alpha_k) = 

φ₀(α₁, . . . , αₖ) + o(1). Tenenbaum’s result is the following: for every k ≥ 1 there exist functions φ₁, φ₂, . . . , continuous except on finitely many hyperplanes, such that,

\[ F_n(α₁, . . . , αₖ) = \sum_{0 ≤ h ≤ H} φ_h(α₁, . . . , αₖ) (\log n)^h + O \left( \frac{1}{(αₖ \log n)^{H+1}} \right) \]

holds for all fixed (α₁, . . . , αₖ), and uniformly outside an exceptional set of (α₁, . . . , αₖ) with measure \( O(\log \log n / \log n) \), specified in [46].

A very natural and useful choice of metric is the \( l₁ \) distance, since this controls the approximation of the set of logarithms of all divisors, see [27, 37], by its limit, see [3], section 22. Approximations in this metric are not comparable to results such as (79); an analogous situation is that, from knowledge that the difference between the two sides of (3) is at most \( 1/n \), Kubilius’ fundamental lemma does not follow as a consequence.

A very natural and useful choice of metric is the \( l₁ \) distance, since this controls the approximation of the set of logarithms of all divisors, see [27, 37], by its limit, see [3], section 22. For the metrized approximation question, it is not necessary to divide by \( \log n \). For a proof or disproof of the following conjecture about the \( l₁ \) distance, from [6], I now offer a one hundred dollar prize.

**Conjecture 2. (§100 prize offered)** For all \( n ≥ 1 \), it is possible to construct \( N(n) \) uniformly distributed from 1 to \( n \), and the Poisson-Dirichlet process \((V₁, V₂, . . .)\), on one probability space, so that

\[ \mathbb{E} \sum |\log P_i(n) - (\log n)V_i| = O(1). \]

Note that the liminf of the left side of (80) is at least 1, because \( \mathbb{E} \sum \log P_i(n) = \mathbb{E} \log N(n) = \log n - 1 + o(1) \), from Stirling’s \( n! \sim (n/e)^n \sqrt{2πn} \), while \( \mathbb{E} \sum (\log n)V_i = \log n \). We finished our workshop lectures by describing a coupling that achieves \( O(\log \log n) \) in place of \( O(1) \) in (80); here in the writeup we also present the proof that this coupling works as claimed.

**Theorem 5.** The coupling of the Poisson-Dirichlet with a random integer \( N(n) \) uniformly distributed from 1 to \( n \), described in section 4.2, achieves

\[ \mathbb{E} \sum |\log P_i(n) - (\log n)V_i| = O(\log \log n). \]

Historical notes: all logarithmic combinatorial structures have a Poisson-Dirichlet limit for the fractions of system size in the largest, second largest, third largest, . . . , components. This was shown in 1977 for permutations, by Kingman [30], and independently by Vershik and Schmidt [47, 48]. It was shown for random mappings — where the Poisson-Dirichlet limit has parameter 1/2 — by Aldous [11] in 1983. It was shown for a wide class of combinatorial structures by Hansen [28] in 1994, and with local limit bounds, for a very general scheme, in [7].
The analog of Conjecture 1 is false for permutations, for the simple reason that $nV_i$ has a distribution with a continuous density, while the size $L_i(n)$ of the $i^{th}$ largest cycle has integer support, so that under any conceivable coupling, for every $i$, $\liminf_n \mathbb{E}|L_i(n) - nV_i| \geq 1/4$. For $i > (1 + \epsilon) \log n$, one can match $nV_i$ with $L_i(n) = 0$, and the net result is that $\liminf_n (1/\log n) \mathbb{E} \sum_i |L_i(n) - nV_i| \geq 1/4$. It is shown in [9] that the coupling for permutations which is analogous to our coupling in section 4.1 achieves this lower bound, with

$$\mathbb{E} \sum_i |\log L_i(n) - nV_i| \sim \frac{1}{4} \log n.$$  

4.1. Growing a random integer from the Poisson-Dirichlet. This section gives a third coupling for growing a random integer $J(n)P_0(n)$ which is close in distribution to the uniform random integer $N(n)$. Our first coupling, in Lecture 1, determined $J(n)$ from a size biased permutation of the multiset with $Z_p$ spacings of size $\log p$, with $Z_p$ geometrically distributed. Our second coupling, in Lecture 3, used instead a size biased permutation of the Poisson multiset with $A_p$ spacings of size $\log p^k$, with $A_p \sim \text{Poisson}(1/(kp^k))$. This Poisson multiset can be constructed by applying the deterministic function $h$ at (42) to the points of the scale invariant Poisson process, so the second coupling may be viewed as constructing $J(n)$ from a deterministic function of the scale invariant Poisson, together with the auxiliary randomization of a size biased permutation.

For our third coupling, this Poisson multiset, and its permutation, will be given by a deterministic function applied to the spacings of the Poisson-Dirichlet process. These spacings are the points of the scale invariant Poisson process, in order of a size biased permutation. The sizes $\log p^k$ are slightly different from the sizes of the Poisson-Dirichlet spacings, so the ordering of spacings in our third coupling is a perturbation of that in our second coupling.

Start with the Poisson process on $(0, \infty)^2$, with intensity $e^{-wy} dw dy$, from section 3 with points $\{(W_i, Y_i), i \in \mathbb{Z}\}$, but reverse the direction of the indexing, so that $W_i < W_{i+1}$ for all $i \in \mathbb{Z}$. Defining $X_i := \sum_{j \geq i} Y_j$ gives the scale invariant Poisson process $\mathcal{X} = \{X_i : i \in \mathbb{Z}\}$, indexed in decreasing order, with $X_{i+1} < X_i$. Now shift the indexing of $\{(W_i, Y_i), i \in \mathbb{Z}\}$, so that $X_1$ appears as the first point to the left of $\log n$. To summarize, we have the points of the scale invariant Poisson process $\mathcal{X}$, indexed so that

$$0 < \cdots < X_3 < X_2 < X_1 < \log n \leq X_0 < X_{-1} < \cdots .$$

The Poisson-Dirichlet from (74), scaled up by $\log n$, is realized as

$$((\log n)V_1, (\log n)V_2, \ldots) = \text{RANK}(\log n - X_1, X_1 - X_2, \ldots).$$
The points $Y_i = X_i - X_{i+1}$ for $i \in \mathbb{Z}$ are the points of the scale invariant Poisson process, and using $h$ from (42), we construct random $Q_i$ by

$$\log Q_i = h(Y_i), \quad \text{with } Y_i = X_i - X_{i+1}, \ i \in \mathbb{Z}. \quad (85)$$

An outcome where $e^{-\gamma} \geq Y_i$ for $i = 2, 6, 7, \ldots$, so that $Q_1, Q_3, Q_4, Q_5$ are prime powers; but $1 = Q_2 = Q_6 = Q_7 = \ldots$.

$\log Q_i$ is close to $Y_i$, and $\log P_0(n)$ is close to $\log n - X_1$.

| $0$ | $e^{-\gamma}$ | $Y_5$ | $Y_4$ | $Y_3$ | $Y_2$ | $Y_1$ | $\log n$ |
|-----|---------------|-------|-------|-------|-------|-------|---------|
| $X_6$ | $X_5$ | $X_4$ | $X_3 X_2$ | $X_1$ | $\ldots$ |

[The following information is motivational, and not part of our proof. Like the $Q_1, Q_2, \ldots$ in section 3.5, for every $q = p^k$ we have that $A_q := \sum_{i \in \mathbb{Z}} 1(Q_i = k)$ is Poisson$(1/(kq))$, with the $A_q$ mutually independent, but there are several differences in the indexing scheme: here the $Q_i$ are indexed from right to left, there may be instances of $Q_i = 1$ in between occurrence of proper prime powers, and most significantly, the sizes used for the size biased permutation are the $X_i - X_{i+1}$ and not the $\log Q_i$. We will need to show that since $h$ is close to the identity function, the second and third couplings are close, in that with high probability, they produce the same $J(n)$.]

Define $J^*(n)$ by

$$J^*(n) = \prod_{i \geq 1} Q_i. \quad (86)$$

It is conceivable that $J^*(n) > n$ if $X_1$ is close enough to $\log n$ and $h$ gets applied in places with $h(y) > y$; in this case we will prescribe $P_0^*(n) = 1$. When $J^*(n) \leq n$, take $P_0^*(n)$ to be one or prime, such that $J^*(n)P_0^*(n) \leq n$, picking uniformly over the $1 + \pi(n/J^*(n))$ possibilities.

We have two tasks. The first, carried out in Lemma 5, is to show that the prime factors $P_1^*(n), P_2^*(n), \ldots$ of this random integer $J^*(n)P_0^*(n)$, listed in nonincreasing order, give a vector of logarithms close to the Poisson-Dirichlet. The second task, carried out in Lemma 6, is to show, like Lemma 4, that the random integer we have constructed is close to uniform. From these two lemma, Theorem 5 follows easily. We will state the lemmas, then give the proof of Theorem 5 to finish this section. The next section provides the proofs of the two lemmas.

**Lemma 5.**

$$\mathbb{E} \sum_{i \geq 1} |\log P_i^*(n) - (\log n) V_i| = O(1). \quad (87)$$
Lemma 6.

\[ d_{TV}(\{J^*(n)P_0^*(n), N(n)\} = O\left(\frac{\log \log n}{\log n}\right). \]

**Proof of Theorem 5** As in section 3.8 the coupling of \( J^*(n)P_0(n) \) with the Poisson-Dirichlet process can be extended to include \( N(n) \), distributed uniformly on 1 to \( n \), in such a way that \( N(n) = J^*(n)P_0^*(n) \) except on a “bad” event of probability equal to the total variation distance \( d_{TV}(\{J^*(n)P_0^*(n), N(n)\}) \).

On the bad event we have no control over the \( l_1 \) distance, apart from the trivial bound: it is at most \( 2\log n \). Multiplying by an upper bound on the probability of the bad event, from Lemma 6 gives us the main contribution to the error, of size \( O(\log \log n) \). On the complement of the bad event, the contribution is \( O(1) \) by Lemma 5. Adding these errors gives \( 88 \).

4.2. *Proofs of the lemmas for Theorem 5.*

**Proof of Lemma 5** There are three contributions to the \( l_1 \) distance: first \( Y_i \) versus \( \log Q_i \), second a contribution from \( Q_i \) which are prime powers but not prime, and third \( \log P_0^*(n) \) versus \( (\log n - X_1) \). Observe that for any vectors \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) in \([0, \infty)^N \cap l_1\), the \( l_1 \) distance is not increased if the coordinates of both vectors are sorted: \( ||\text{RANK}(x) - \text{RANK}(y)||_1 \leq ||x - y||_1 \).

Consider the random variable

\[ D = \sum_{i \in \mathbb{Z}} |h(Y_i) - Y_i|. \]

It has

\[ \mathbb{E}D = \mathbb{E}\sum_{i \in \mathbb{Z}} |h(Y_i) - Y_i| = b_0 < \infty, \]

using \( 89 \) and the scale invariant spacing lemma.

The first contribution to our expected \( l_1 \) distance is handled by: for all \( n \),

\[ \mathbb{E}\sum_{i \geq 1} |\log Q_i - Y_i| = \mathbb{E}\sum_{i \geq 1} |h(Y_i) - Y_i| \leq \mathbb{E}D = b_0 < \infty. \]

To handle the second contribution, we “split up” any prime powers \( p^k \) with \( k > 1 \) which may occur among the \( Q_i \), defining \( Q_1^*, Q_2^*, \ldots \) to be one or prime, so that \( J(n) = \prod_{i \geq 1} Q_i = \prod_{i \geq 1} Q_i^* \). To do this, start with \( Q_j^* = 1 \) whenever \( Q_j = 1 \); some of these will be changed. Always take \( Q_j^* = p \) when \( Q_i = p^k \). For any \( Q_i = p^k \) with \( k > 1 \), take \( k-1 \) indices \( j \) for which \( Q_j^* = 1 \) and change these to \( Q_j^* = p \). With \( x = (\log Q_1, \log Q_2, \ldots) \) and \( y = (\log Q_1^*, \log Q_2^*, \ldots) \) we have \( ||x - y|| = \sum_{p^k, i} 2(k - 1)(\log p)1(Q_i = p^k) \leq \sum_{q=p^k, k>1} 2(\log q)A_q \).

Note that \( \mathbb{E}\sum_{q=p^k, k>1} (\log q)A_q = \sum_{q=p^k, k>1} (\log q)/(kq) < \infty \).

To handle the third contribution: using \( c = (\log n - \sum h(Y_i))^+ \), our recipe for \( P_0^*(n) \) is to choose uniformly over the \( 1 + \pi(e^c) \) numbers which are one,
or a prime at most $e^{x}$. Writing $P_{c}$ and $E_{c}$ for such a choice of $P_{0}$, we have $\sup_{c \geq 0} E_{c}(c - \log P_{0}) < \infty$, as a simple consequence of the prime number theorem, that $\pi(e^{x}) \sim e^{x}/x$ as $x \to \infty$. Using $X_{1} = \sum_{i \geq 1} Y_{i}$, we have $|\log n - X_{1} - (\log n - \sum_{i \geq 1} h(Y_{i})| = |\sum_{i \geq 1} (Y_{i} - h(Y_{i})|$, with expectation bounded by $b_{0}$, using (90). Combining yields $\sup_{n} E|\log P_{0}^{*}(n) - (\log n - X_{1})| < \infty$.

Combining these three contributions, and using the $l_{1}$ contraction property of the function RANK, proves (87).

\section*{Proof of Lemma 6}

In contrast to (83) and (85), we now re-index the $\{Y_{i} : i \in \mathbb{Z}\}$ so that

\[ \cdots < Y_{-2} < Y_{-1} < Y_{0} \leq e^{-\gamma} < Y_{1} < Y_{2} < \cdots \]

The choice of location of $e^{-\gamma}$ has the effect that $Q_{1}, Q_{2}, \ldots$ are the primes and prime powers, while $1 = Q_{i}$ for $i \leq 0$. The indexing of the $Y_{i}$ in the order of their own values has the effect that, with

\[ S_{i} := W_{i}/Y_{i}, \]

the $S_{i}$ for $i \in \mathbb{Z}$ are, conditional on the values of $Y_{i}, i \in \mathbb{Z}$, mutually independent, standard exponentials — and this would not have been true under the previous indexing, where $W_{i} < W_{i+1}$.

We construct the second coupling, of section 3.5, from this multiset $\{Q_{1}, Q_{2}, \ldots\}$. For the size biased permutation of the log $Q_{i}$, we use the exponentially distributed labels

\[ \hat{W}_{i} := S_{i}/\log Q_{i} = W_{i} \frac{Y_{i}}{h(Y_{i})}, \]

so that $J(n)$ is the largest partial product of the $Q_{i}$ not exceeding $n$, with the $Q_{i}$ taken in order of decreasing labels $\hat{W}_{i}$. In contrast, $J^{*}(n)$ is a partial product of $Q_{1}, Q_{2}, \ldots$ taken in order of decreasing $W_{i}$. Because $h$ is close to the identity, eventually the permutation induced by the $W_{i}$ agrees with the permutation induced by the $\hat{W}_{i}$. We will show that

\[ \mathbb{P}(J^{*}(n) \neq J(n)) = O\left(\frac{\log \log n \log n}{\log n}\right), \]

and thus $d_{TV}(J^{*}(n)P_{0}^{*}(n), J(n)P_{0}(n)) = O(\log \log n/\log n)$. Combined with Lemma 4 this yields $d_{TV}(J^{*}(n)P_{0}^{*}(n), N(n)) = O(\log \log n/\log n)$. As a remark, we believe that the quantity in (93) is actually $O(1/\log n)$, but since this improved bound would not improve the overall result, we settle for the looser bound.

First, we show that the effect in (86) of stopping at the largest $X_{i} < \log n$, rather than stopping with the largest partial product not exceeding $n$, is negligible. Consider the “good” event

\[ G = \{D \geq \log \log n \text{ or } X \cap (\log n - \log \log n, \log n + \log \log n) \neq \emptyset \}, \]
with $D$ given by (39). We will show that

$$\mathbb{P}(G^c) = O\left(\frac{\log \log n}{\log n}\right),$$

To show this, first observe that $\mathbb{P}(\mathcal{X} \cap (\log n - \log \log n, \log n + \log \log n) \neq \emptyset) = O(\log \log n/\log n)$, since the intensity of $\mathcal{X}$ is $1/x \, dx$. Second, observe that $\mathbb{P}(D \geq \log \log n) = O(1/\log n)$, which follows from showing $\mathbb{E} e^{\beta D} < \infty$ with some $\beta > 1$. In fact, $\mathbb{E} e^{\beta D} < \infty$ for all $\beta$; with $g$ as defined following (41), and using (33), we have $D = \sum_{i \in \mathbb{Z}} |g(L_i) - \exp(L_i)|$ so that

$$\mathbb{E} e^{\beta D} = \exp\left(\int_{-\infty}^{\infty} (e^{\beta (g(l) - e^l)} - 1) \, dl\right).$$

The contribution to the integral from the neighborhood of $-\infty$ is finite, using $g(l) = 0$ there, and contribution to the integral from the neighborhood of $\infty$ is finite, using $g(l) - e^l = O(e^l \exp(-c e^l/2))$ as $l \to \infty$, which follows from (41).

Next, we consider a bad event on which the two permutations, one induced by the sizes $W_i$, and the other induced by the sizes $W_i := W_i Y_i/h(Y_i)$ as in (92), might disagree in a nontrivial way, i.e. giving the opposite ordering to $Q_i, Q_j > 1$, out beyond the place where the partial sum of the $Y_i$ exceeds $(\log n)/2$. Let

$$B = \{\exists i \neq j, Y_i, Y_j > e^{-\gamma}, W_i < W_j, \hat{W}_i \geq \hat{W}_j, \sum Y_k 1(W_k \geq W_i) > (\log n)/2\}.$$

For $n$ so large that $\log \log n < (\log n)/2$, we have

$$\{J^*(n) \neq J(n)\} \subset G^c \cup B,$$

so that it only remains to show $\mathbb{P}(B) = O(\log \log n/\log n)$.

Let

$$T_w = \sum_{i \in \mathbb{Z}} Y_i 1(W_i > w).$$

The distribution of $T_w$ is exponential, with $\mathbb{E} T_w = 1/w$ — see for example [31], under the “Moran process”, or [2], where this is used as an ingredient in the proof of the scale invariant spacing lemma. We say that $((W, Y), (W', Y'))$ is a “potential witness” to the bad event $B$ if $y, y' > e^{-\gamma}, w' < w$, the Poisson process $\{(W_i, Y_i)\}$ has points at $(w, y)$ and $(w', y')$, and no points $(W_k, Y_k)$ with $w' < W_k < w$, and $T_w + y + y' > (\log n)/2$, and

$$\log\left(\frac{w}{w'}\right) \leq |\log(y/h(y))| + |\log(y'/h(y'))|.$$

Let $N_B$ denote the number of potential witnesses, and observe that $B \subset \{N_B > 0\}$. Thus, it only remains to calculate that $\mathbb{E} N_B = O(\log \log n/\log n)$; and in fact we will show that it is $O(1/\log n)$.

Now $\mathbb{E} N_B$ is merely a four-fold integral, so the reader is invited to take our claim at face value; but for those declining our invitation, here are
the details. Conditional on having points \((W_i, Y_i)\) and \((W_j, Y_j)\) with \(W_i = w, W_j = w'\), the joint distribution of \(Y_i, Y_j, T_w\) is that of three independent exponentials, with means \(1/w, 1/w'\), and \(1/w\) respectively. Let 
\[c_0 := \exp(2 \sup \{ |\log(y/h(y))| : y > e^{-\gamma} \})\]
so that for a potential witness, \(w/w' \leq c_0\), and \(Y_j\) lies below an exponential with mean \(c_0/w\). This gives us, for the conditional probability, that 
\[\text{Pr}_{w,w'}(Y_i + Y_j + T_w > \log n/2) \leq 3 \exp((\log n/6)/6c_0).\]

We need some monotonicity for the next simplification. From (97) we have that 
\[h(x) = x + O(xe^{-c\sqrt{x}}) \quad \text{as } x \to \infty,\]
so that for some constants \(c_1, c_2 > 0\), for all \(y > e^{-\gamma}\), \(|\log(h(y)/y)| < c_1e^{-c_2\sqrt{y}}\). Thus we can relax the notion of “potential witness,” replacing \(\text{Pr}_{w,w'}(Y_i + Y_j + T_w > \log n/2)\) with the condition 
\[\log\left(\frac{w}{w'}\right) \leq r(y) + r(y'), \quad \text{where } r(y) = c_1e^{-c_2\sqrt{y}}.\]

Write \(N_R\) for the number of potential witnesses in this relaxed sense, so that \(N_B \leq N_R\). Now the indicator of the inequality \((98)\) is a decreasing function of \((y, y')\), while the indicator \(1(y+y'+t > \log n/2)\) is an increasing function, so that we have negative correlations (with respect to \(Y_i, Y_j\), and \(T_w\), which are conditionally independent given \(w, w'\)): 
\[\text{Pr}_{w,w'}\left(Y_i + Y_j + T_w > \log n/2, \log\left(\frac{w}{w'}\right) \leq r(Y_i) + r(Y_j)\right)\]
\[\leq \text{Pr}_{w,w'}\left(Y_i + Y_j + T_w > \log n/2\right) \text{Pr}_{w,w'}\left(\log\left(\frac{w}{w'}\right) \leq r(Y_i) + r(Y_j)\right).\]

We use the monotonicity of \(r(\cdot)\) again, to justify 
\[\text{Pr}_{w,w'}\left(\log\left(\frac{w}{w'}\right) \leq r(Y_i) + r(Y_j)\right) \leq \text{Pr}\left(\log\left(\frac{w}{w'}\right) \leq r(S_i/w) + r(S_j/w)\right),\]
where \(S_i, S_j\) represent independent standard exponentials.

Recalling that \(\{W_k : k \in \mathbb{Z}\}\) form a copy of the scale invariant Poisson process \(\mathcal{X}\), and \(\text{Pr}(\mathcal{X} \cap (w', w) = \emptyset) = w'/w\), 
\[\mathbb{E} N_R \leq \int \int_{w' < w} \frac{dw'}{w'} \frac{dw}{w} \frac{dw'}{w'} 3 \exp(-w \log n/(6c_0)) \text{Pr}\left(\log\left(\frac{w}{w'}\right) \leq r(S_i/w) + r(S_j/w)\right).\]
Recall further that for two consecutive points \(W_j < W_i\) of the scale invariant Poisson process, conditional on \(W_i = w\) the distribution of \(W_j\) is that of \(Uw\), where \(U\) uniformly distributed in \((0, 1)\). Thus the right hand side above is equal to 
\[\int_{w > 0} \frac{dw}{w} 3 \exp(-w \log n/(6c_0)) \text{Pr}\left(\log\left(\frac{1}{U}\right) \leq r(S_i/w) + r(S_j/w)\right).\]
Since $-\log U$ is exponentially distributed, with density bounded above by one, we have
\[
\mathbb{E} N_R \leq \int_{w>0} \frac{dw}{w} 3 \exp(-w \log n/(6c_0)) \mathbb{E} 2r\left(\frac{S_i}{w}\right)
\]
\[= 6c_1 \int_{w>0} \exp(-w \log n/(6c_0)) dw \int_{y>0} e^{-w y} e^{-c_2 \sqrt{y}} dy\]
\[< 6c_1 \int_{y>0} e^{-c_2 \sqrt{y}} dy \int_{w>0} \exp(-w \log n/(6c_0)) dw = O\left(\frac{1}{\log n}\right).
\]
This completes the proof of (88).

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