Corner contribution to percolation cluster numbers

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We study the number of clusters in two-dimensional (2d) critical percolation, \( N_\Gamma \), which intersect a given subset of bonds, \( \Gamma \). In the simplest case, when \( \Gamma \) is a simple closed curve, \( N_\Gamma \) is related to the entanglement entropy of the critical diluted quantum Ising model, in which \( \Gamma \) represents the boundary between the subsystem and the environment. Due to corners in \( \Gamma \) there are universal logarithmic corrections to \( N_\Gamma \), which are calculated in the continuum limit through conformal invariance, making use of the Cardy-Peschel formula. The exact formulas are confirmed by large scale Monte Carlo simulations. These results are extended to anisotropic percolation where they confirm a result of discrete holomorphicity.

I. INTRODUCTION

In the percolation process sites or bonds of a regular lattice (or a graph) are independently open with a probability, \( p \) and one is interested in the statistics of clusters of connected sites. In the thermodynamic limit for dimensions \( d \geq 2 \) there is a critical point at \( p = p_c \), at which a second order phase transition takes place. Critical percolation is scale invariant and its scaling limit is also believed to be conformally invariant. This has been rigorously proved for some lattices\(^1\). Conformal invariance has been used to predict different relations about the order-parameter profiles, correlation functions, crossing probabilities, etc. of critical percolation. Some of the conformal results have been subsequently derived by rigorous mathematical methods, such as by Schramm-Loewner evolution\(^2\).

In percolation the questions one usually asks concern the distribution of clusters, or the properties of the largest ones, but comparatively less attention is paid to boundary effects, such as surfaces, edges and corners. The total number of clusters and its correction terms are generally non-universal, thus they depend on the type of percolation (bond or site) and the details of the lattice. The only exception could be the corner contribution, which we are going to study in this paper.

To be specific we consider bond percolation in a 2d square lattice and denote by \( \Gamma \) a subset of bonds. Initially we take this to be a simple closed curve consisting of straight edges and corners and will generalize later. We are interested in the number of clusters, \( N_\Gamma \), which intersect \( \Gamma \), in the scaling limit when \( \Gamma \) is large but still much smaller than the total size of the system. This problem is closely related to the entanglement entropy of the bond-diluted quantum Ising model which is defined in the same lattice and \( \Gamma \) represents the interface, which separates a subsystem from the rest of the system.

In the following we calculate \( N_\Gamma \) by different methods, both analytically and numerically. In the scaling limit we consider the random cluster representation of the \( Q \)-state Potts model and in the \( Q \to 1 \) limit we calculate \( N_\Gamma \). In particular we obtain a logarithmically divergent contribution due to corners and its prefactor is calculated through conformal invariance. Using different geometries of \( \Gamma \) we have performed large scale Monte Carlo simulations, the results of which are confronted with the conformal prediction. We have also considered anisotropic percolation, in which the probabilities \((p_x, p_y)\) that horizontal and vertical bonds are open are different.

II. POTTS MODEL REPRESENTATION

Bond percolation is well known to be related to the \( Q \to 1 \) limit of the \( Q \)-state Potts model, defined on a lattice with sites \( i = 1, 2, \ldots, n \) and \( m \) nearest neighbor bonds. The partition function is given by

\[
Z(Q) = \sum_s \prod_{ij} \exp\left(K\delta_{s_i, s_j}\right),
\]

where the sum is over all spin configurations, \( s = (s_1, s_2, \ldots, s_n) \), \( s_i = 1, 2, \ldots, Q \), the product runs over all nearest neighbor bonds, \( \delta_{s_i, s_j} \) is the Kronecker symbol and \( K \) is the reduced coupling, which is the ratio of the pair interaction and the temperature. Using the identity: \( \exp\left(K\delta_{s_i, s_j}\right) = 1 + \frac{p}{1-p}\delta_{s_i, s_j} \) with \( p = 1 - e^{-K} \), the sum of products in \( Z \) is written in terms of the so called Fortuin-Kasteleyn clusters\(^2\), denoted by \( F \). In \( F \) the edge of the lattice \( i, j \) is occupied, if a factor \( \frac{1}{1-p}\delta_{s_i, s_j} \) is present and in any connected cluster the spins are in the same orientation. For a given element of \( F \) there are \( N_{\text{tot}}(F) \leq n \) connected components and \( M(F) \leq m \) occupied bonds so that the partition function reads:

\[
Z(Q) = \sum_F Q^{N_{\text{tot}}(F)} \left(\frac{p}{1-p}\right)^{M(F)}
\]
\[ \sim \sum F Q^{N_{\text{tot}}(F)} p^{M(F)} (1 - p)^{n - M(F)} = \langle Q^{N_{\text{tot}}} \rangle. \] (2)

In this random cluster representation \( Q \) is a real parameter and percolation is recovered in the \( Q \rightarrow 1 \) limit, when the critical point in the square lattice is given by \( p_c = 1/2 \). The mean total number of clusters in percolation is:

\[ \langle N_{\text{tot}} \rangle = \left. \frac{\partial \ln Z(Q)}{\partial Q} \right|_{Q=1}. \] (3)

Now impose a boundary condition that all the Potts spins are fixed (say, in the state 1) on \( \Gamma \). The partition function is now:

\[ Z_\Gamma(Q) \sim \langle Q^{N_{\text{tot}} - N_\Gamma} \rangle, \] (4)

where \( N_\Gamma \) is the number of clusters which intersect \( \Gamma \). Hence

\[ \langle N_{\text{tot}} - N_\Gamma \rangle = \left. \frac{\partial \ln Z_\Gamma(Q)}{\partial Q} \right|_{Q=1}. \] (5)

On the other hand at the critical point \( p = p_c \) we can write:

\[ \ln Z(Q) \sim A f_b(Q) \]
\[ \ln Z_\Gamma(Q) \sim A f_b(Q) + L_\Gamma f_s(Q) + C_\Gamma \ln L_\Gamma, \] (6)

where \( A \propto n \) is the total area, \( L_\Gamma \) is the length of \( \Gamma \), and \( f_b \) and \( f_s \) are the bulk and surface free-energy densities, respectively, which are non-universal. The last term represents the corner contribution. Hence we obtain:

\[ \langle N_\Gamma \rangle = -f'_c(1)L_\Gamma - C_\Gamma(1) \ln L_\Gamma. \] (7)

Cardy and Peschel\(^4\) considered these corner contributions to the free energy of a general conformally invariant system in domains with a boundary. Their results apply equally to an exterior boundary, with corners with interior angle \( \gamma_k \), or an interior boundary, with \( \gamma_k \) replaced by \( 2\pi - \gamma_k \). An important property of percolation is its locality, that is the partition function \( Z_\Gamma(Q) \) is exactly the product of the partition function \( Z_{\Gamma,\text{int}}(Q) \) for the interior of \( \Gamma \), and \( Z_{\Gamma,\text{ext}}(Q) \) for the exterior. Thus we may apply the Cardy-Peschel result to each of these. Note that this would not be correct if we had given the clusters intersecting \( \Gamma \) a weight \( \neq 1 \). We therefore find from Ref.\(^4\) that the prefactor of the logarithm in Eq.(6) is given by:

\[ C_\Gamma(Q) = \frac{c(Q)}{24} \sum \left[ \left( \frac{\pi}{\gamma_k} - \frac{\gamma_k}{\pi} \right) + \left( \frac{\pi}{2\pi - \gamma_k} - \frac{2\pi - \gamma_k}{\pi} \right) \right], \] (8)

where \( \gamma_k \) is the interior angle at each corner, and \( c(Q) \) is the central charge of the \( Q \)-state Potts model, and the two sets of terms come from the interior and exterior contribution.\(^2\)

Using \( \frac{c}{\kappa} = (6 - \kappa)(3\kappa - 8) / 2\kappa \) and \( \sqrt{Q} = -2 \cos(4\pi/\kappa) \) we have:

\[ c'(1) = \frac{5\sqrt{3}}{4\pi}. \] (9)

The corner contribution to \( \langle N_\Gamma \rangle \) which is derived here for bond percolation is expected to be universal, thus to be valid for site percolation, too. We are going to compare the conformal results in Eqs. (7,8) and (9) with those of numerical calculations for different forms of \( \Gamma \) in Sec. IV.

### III. ENTANGLEMENT ENTROPY OF THE DILUTED QUANTUM ISING MODEL

The problem studied in Sec. II for percolation is closely related to the entanglement entropy of a bond-diluted quantum Ising model and here we follow Refs.\(^7\),\(^8\). The model is defined by the Hamiltonian:

\[ \mathcal{H} = - \sum_{(ij)} J_{ij} \sigma^x_i \sigma^x_j + \sum_i h \sigma^z_i, \] (10)

in terms of the \( \sigma^x, \sigma^z \) Pauli matrices at site \( i \). The first sum in Eq.(11) runs over nearest neighbors and the \( J_{ij} \) coupling equals \( J > 0 \) with probability \( p \) and equals \( J = 0 \) with probability \( 1 - p \). At the percolation transition point, \( p_c \), for small transverse field, \( h \), there is a line of phase transition the critical properties of which are controlled by the percolation fixed point. The ground state of \( \mathcal{H} \) is given by a set of ordered clusters, which are in the same form as for percolation. Now consider a subsystem, \( A \) the boundary of which is represented by \( \Gamma \) and calculate the entanglement entropy between the subsystem and the environment, which is given by \( S_T = -\text{Tr}(\rho_A \log_2 \rho_A) \) in terms of the reduced density matrix \( \rho_A \). Here we note, that in each ordered cluster the spins are in a maximally entangled GHZ state, thus for a given realization of disorder in the small \( h \) limit all those clusters give a unit (1) contribution to the entanglement entropy, which intersect \( \Gamma \) and contain also at least one point of the environment. We shall call them crossing clusters, thus \( S_T \) is given by the number of crossing clusters and we are interested in their average value: \( \langle S_T \rangle \). Note that \( S_T \leq N_\Gamma \), and thus \( \langle S_T \rangle \leq \langle N_\Gamma \rangle \). The singular corner contributions to \( \langle N_\Gamma \rangle \) in Eq.(7), however, are due to large clusters, which are present typically among the crossing clusters, too. Consequently we expect the asymptotic form of the average entanglement entropy to be

\[ \langle S_T \rangle = a L_\Gamma + b \ln L_\Gamma, \] (11)

where \( a \leq -f'_c(1) \) and \( b = -C_\Gamma'(1) \). The leading term in Eq.(11) represents the so called area law, to which there is a logarithmic correction, which is expected to be universal and given in Eqs. (7,8) and (9).
IV. NUMERICAL RESULTS

We have performed large scale numerical calculations for site and bond percolations at the critical point on the square lattice\(^{18}\). The finite systems we have used have \(L \times L\) sites with \(L\) up to 8192 and with periodic boundary conditions. The different shapes of the subsystems used in the numerical calculations are illustrated in Fig.1. In all cases the linear size of the subsystem is of \(O(L)\), so that \(L_\Gamma \sim L\) and in the following we shall use \(L\) instead of \(L_\Gamma\). For a given shape of the subsystem and thus for its boundary \(\Gamma\) we have calculated the number of crossing clusters, \(S_\Gamma\), which is then averaged: i) for a given percolation sample over the positions of \(\Gamma\) (typically \(10^3\) positions), and ii) over different samples. Typically we have used \(10^5\) samples for each size, \(L\), except the largest ones, where we had at least \(10^4\) samples. From the numerical data on \(\langle S_\Gamma \rangle\) we have deduced estimates for the prefactor \(b\) by the so called difference approach. Here we use the relation:

\[
\Delta S_\Gamma(L) = 2\langle S_\Gamma \rangle(L) - \langle S_\Gamma \rangle(2L) = b \ln L + \text{const}. \tag{12}
\]

If \(\Gamma\) has a special shape, such as a square or a sheared square, (see the three shapes in the first row of Fig.1) we can calculate the corner contribution to \(\langle S_\Gamma \rangle\) directly, by comparing results obtained at different geometries. This type of geometrical approach, which has been used in Ref. 11, will be explained in more detail in the following subsection.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1}
\caption{(Color online) Shapes of the subsystems used in the numerical calculation: square, sheared square, anisotropic square; equilateral triangle, circle and section of a circle. In a lattice we consider the nearest neighbor bonds which are intersected by the contour of the subsystem and for each bond one of its sites belongs to \(\Gamma\) (depending on the local orientation of the surface).}
\end{figure}

A. Square subsystem

The first geometry we consider for \(\Gamma\) is a square of linear size \(\ell = L/2\). The cluster structure of critical site percolation is illustrated in Fig.2, in which one can identify the clusters which intersect \(\Gamma\) and also the crossing clusters. Here we use the geometrical approach, in which the complete square is divided either to four neighboring squares with \(\ell = L/2\) or to orthogonal stripes of size \(L/2 \times L\): altogether four stripes due to the two different orientations. (This is illustrated in Ref.11 in the right panel of Fig.1.) The two subsystems of different shapes have the same total boundary, however for stripes - due to periodic boundary conditions - there is no corner contribution. This is then obtained from the difference:

\[
S_\text{cr}^\square(L) = S_\square(L) - S_\text{str}(L). \tag{13}
\]

Note that by this geometrical method the corner contribution is calculated for each sample, therefore the average values have considerably less noise, than by the difference method.

Making use of the fact that asymptotically \(\langle S_\text{cr}^\square(L) \rangle = b \ln L + \text{const}\), we have calculated effective, size-dependent estimates for the prefactor by two-point fits, by comparing the average corner contributions in finite systems of size \(L\) and \(2L\). The results are presented in Fig.3 both for site and bond percolations. With increasing \(L\) the effective prefactors approach a common, universal limiting value of \(b = -0.077(1)\). This is to be compared with the conformal result in Eqs. (7,8) and (9): \(-C'(1) = -5\sqrt{3}/(36\pi) = -0.07657\), thus the agreement is satisfactory. We note that a previous numerical estimate in Ref.8 for smaller systems has obtained: \(b = -0.06(1)\). The corner contribution is related to the
large scale topology of the clusters, as illustrated in the inset of Fig.4 and discussed in Sec.IV B.

We have also studied the \( p \)-dependence of the corner contribution to \( \langle S^\square \rangle \) outside the critical point by the geometrical approach. As can be seen in Fig.4 \( \langle S^\square \rangle \) has a peak around \( p = p_c \) and close to \( p_c \) the extrapolated curve can be well described by the scaling result:

\[
S^\square_c(p) \simeq b' \ln(p_c - p) + \text{const} ,
\]

where \( b' = b \nu \), with \( \nu = 4/3 \) being the correlation length critical exponent for percolation. Indeed, assuming the form in Eq.(14) we have calculated effective, \( p \)-dependent prefactors by two-point fits, which are shown in the inset of Fig.4. The extrapolated value for \( p \rightarrow p_c \) (0.106(10)) is consistent with the scaling prediction (0.102).

**B. Sheared square subsystem**

The square subsystem used in the previous subsection is sheared now to a parallelogram, having the same surface: \( \ell^2 = L^2/4 \) and its (smaller) angle is \( \gamma \). The geometrical approach to calculate the corner contribution to \( S^\diamond \) can be extended in this case for specific values of \( \gamma \) given by the condition: \( \tan(\gamma) = 1/n \), with \( n = 0, 1, \ldots, \infty \) being an integer. As shown in Fig.5 with increasing \( L \) up to \( L = 4096 \) the numerical results approach the conformal prediction:

\[
C'(1) = -\frac{c'(1)}{12} \left[ 4 - \pi \left( \frac{1}{\gamma} + \frac{1}{\pi - \gamma} + \frac{1}{\pi + \gamma} + \frac{1}{2\pi - \gamma} \right) \right].
\]

Performing an extrapolation with an \( \ln L/L \) correction term we have obtained an excellent agreement, as shown in the inset of Fig.5. For other values of \( \gamma \), which do not fit to the geometrical approach we have made calculations by the difference method. Also in these cases the numerical results are found to agree with the conformal prediction.
C. Anisotropic percolation

Here we consider anisotropic bond percolation, in which the probabilities are \( p_x \) (horizontally) and \( p_y \) (vertically) and the critical point is given by the condition: \( p_x + p_y = 1 \). The system is a diagonally placed square with \( 2L^2 \) sites, and the subsystem is also a diagonally oriented square, having \( L^2/2 \) sites and its boundary contains \( 4 \times L/2 \) sites, see the third figure in the first row of Fig.1. This anisotropic system with symmetric angles, \( \pi/2 \), is conjectured to be equivalent in the scaling limit to an isotropic system with asymmetric angles, \( \gamma \) and \( \pi - \gamma \), such that

\[
\frac{p_y}{p_x} = \frac{\sin((\pi - \gamma)/3)}{\sin(\gamma/3)}.
\]  

(16)

This follows from the requirement that there exists a discretely holomorphic observable\(^{12}\). More recently Grimmett and Manolescu\(^{13}\) have proved that many properties of the scaling limit are the same if a more general inhomogeneously anisotropic lattice is embedded in the plane according to this prescription.

![FIG. 6: (Color online) Estimates for the prefactor \(-b\) for anisotropic percolation for finite systems with \( L = 32, 64, \ldots, 1024 \) from below. The opening angle of the equivalent sheared square, \( \gamma \), is given by Eq. (16) and the full line represents the conformal result. In the inset the ratio of the finite-size results and the conformal results are given.](image)

In the anisotropic system we have chosen the probabilities in such a way, that the corresponding angle, \( \gamma \), has satisfied the condition: \( \tan(\gamma) = 1/n \), with \( n = 0, 1, \ldots, \infty \). In this way we could directly compare the results for anisotropic systems with those obtained for sheared squares in Sec. IV.B. The numerical results for the prefactor obtained on finite systems up to \( L = 1024 \) are shown in Fig.6. For the three largest systems there is no systematic size-dependence of our data, the error is purely statistical and the numerical results agree well with the conformal prediction.

D. Corner probability

For the (sheared) square subsystem we used the geometrical method, in which the \( L \times L \) system is divided for four \( L/2 \times L/2 \) squares and for four \( L/2 \times L \) stripes and the difference in the number of crossing clusters in Eq. (13) is just the corner contribution. Having a general (connected) cluster it could occupy different parts of the four quadrants and its topology could be of four different types as illustrated in the inset of Fig.3. Among these the I, II and IV type of topology gives identical contribution both for stripes and squares. Clusters with topology III, however, are crossing clusters for all the four possible stripes, but these are crossing clusters only for three out of four squares. (In the left-bottom square there is no crossing.) Let us denote the occurrence probability of type III cluster as \( P_3(L) \), which is given by the ratio of such clusters in a \( L \times L \) square, which have points in three quadrants, but have no point in the fourth one. As argued above \( P_3(L) \) is proportional to the corner contribution of crossing squares, more precisely:

\[
P_3(L) = \frac{1}{4} S_{\text{cr}}^3(L) = -\frac{1}{4} C'(1) \ln L + \text{const.}
\]  

(17)

This results is valid for sheared squares with angle \( \gamma \), as well as for anisotropic percolation with a square subsystem. In these cases the appropriate results of \( S_{\text{cr}}^3(L) \) have to be used. Interestingly this corner probability has a logarithmic \( L \)-dependence and its prefactor is known exactly.

E. Other subsystem geometries

We have also studied subsystems with different geometries: equilateral triangle, circle and section of a circle, these are illustrated in the second row of Fig.1. In these cases the largest linear scale of the subsystem is fixed to \( L/2 \), while an angle was varied. In the calculations the difference approach in Eq. (12) has been used.

1. Equilateral triangle

For an equilateral triangle with a base angle \( \gamma \) the extrapolated results for \(-b\) are shown in Fig.7, which is compared with the conformal prediction:

\[
C'(1) = -\frac{c'(1)}{24} \left[ 6 - \pi \left( \frac{2}{\gamma} + \frac{1}{\pi - 2\gamma} + \frac{1}{\pi + 2\gamma} + \frac{2}{2\pi - \gamma} \right) \right].
\]  

(18)

As seen in Fig. 4 there is a satisfactory agreement, although the statistical error of the numerical results is comparatively large, in particular for small and large angles.
2. Circle and section of a circle

According to the conformal results, there is no logarithmic correction to $S_\bigcirc$ for a circle shaped subsystem. Indeed the numerical results in the inset of Fig. 8 are in agreement with this statement: $\Delta S_\bigcirc$ approaches a finite limiting value of 0.65(5).

In contrast, for a section of a circle with an angle $0 < \gamma < \pi$ there is a logarithmic correction, the prefactor of which is given by conformal invariance:

$$C'(1) = -\frac{c'(1)}{12} \left[ 2 - \pi \left( \frac{1}{\gamma} + \frac{1}{2\pi - \gamma} \right) \right]$$  \hspace{1cm} (19)

The numerical results in Fig. 8 are in agreement with the conformal prediction, although the statistical error of the numerical results is comparatively large.

F. Line segment

According to the derivation in Sec. II, $\Gamma$ does not have to be a closed curve. For example it could be a straight line of length $\ell$. In that case we have only a corner contribution from two exterior angles, each $\gamma = 2\pi$, so that $C_\Gamma(Q) = -c(Q)/8$.

To study this problem we can use the geometrical approach, when the line is oriented parallel with one of the axes of the square lattice. Then the corner contribution is related to the difference between the cluster numbers obtained for a periodic line of length $L$ and that of two segments of lengths $\ell = L/2$. In this case the corner contribution is simply half of the number of common clusters between the two line segments.

In the numerical calculation we have used the difference approach for a line segment of length $\ell = L/2$, which is placed in random positions and random orientations with respect to the axes of the periodic square lattice of size $L \times L$. The average of $\Delta S_\bigcirc(L)$ is shown in Fig. 9 as a function of $\ln L$. The logarithmic dependence is clearly visible with a prefactor estimated as $b = 0.086(1)$, which agrees fairly well with the conformal prediction: $-C_\Gamma'(1) = \frac{5\sqrt{3}}{32\pi} \approx 0.08615$. We have also checked the $p$-dependence of $S_\bigcirc^\tau(p)$, which is shown in the inset of

FIG. 7: (Color online) Prefactor of the logarithm for an equilateral triangle as a function of the base angle $\gamma$ compared with the conformal result in Eq. (18). The numerical results are extrapolated up to $L = 4096$ for site percolation. The statistical error is larger than the difference between the conformal and the numerical results.

FIG. 8: (Color online) Prefactor of the logarithm $-b$ for a section of a circle as a function of the angle $\gamma$. The full line represents the conformal result in Eq. (19). The differences between the conformal and numerical results are of the order of the statistical error. In the inset the corner contribution is shown for a circle: there is no systematic radius dependence. The numerical results are extrapolated up to $L = 4096$ (in the main figure) and up to $L = 8192$ (in the inset) for site percolation.

FIG. 9: (Color online) $\Delta S$ for the line segment obtained by the difference approach for different lengths, $L$. In the inset the $p$-dependence of $S_\bigcirc^\tau$ is shown close to the critical point for finite systems with $L = 64, 128, 256$ and 512 from below.
Fig. 9. A singularity is developed at the critical point as \( L \rightarrow \infty \).

We note that if the line segment is put to the open boundary of the system, then the prefactor is different, being \( b = \sqrt{\frac{3}{4\pi}} \approx 0.1378 \) as obtained by conformal invariance\(^8,14\) and numerically \( b = 0.15(2)\). 

V. DISCUSSION

In this paper we considered the logarithmic terms in the mean number \( N_\Gamma \) of clusters in critical percolation which intersect a curve \( \Gamma \) in the cases when it has sharp corners or end points. We have shown that the Cardy-Peschel result\(^4\) can be simply applied to compute the universal coefficients of these logarithmic terms, and that accurate numerical estimates agree very well with this, for a variety of shapes for \( \Gamma \). We also considered anisotropic bond percolation on the square lattice and showed that if Eq. (16) is used to deform the lattice then the predictions again agree with numerics if the correct effective corner angle is used. We have also pointed out a relation between the corner contribution to \( N_\Gamma \) and the statistics of cluster shapes.

Our study is related to recent investigations of shape dependent terms of different thermodynamic quantities, mainly the free energy, of 2\(d\) critical systems. For critical percolation, however, there is no logarithmic corner contribution to the free energy, since the central charge is \( c(Q = 1) = 0 \). In the latter problem the corner contribution to the cluster numbers and its higher moments are of interest, which are related to derivatives of \( c(Q) \) at \( Q = 1 \).

The excellent agreement of the numerical data with the theory serves as confirmation that critical percolation is indeed conformally invariant, of the Coulomb gas predictions for \( c(Q = 1) \), and of the formula\((10)\). Extensions are obviously possible: for example by taking further derivatives with respect to \( Q \) we can find results for the corner contributions to the higher moments of the distribution of the cluster numbers \( N_\Gamma \). We note that recently\(^15\) an explicit example has been found of a correlation function in percolation which contains a multiplicative logarithm. As with our result, this may be understood\(^16\) from the necessity of having to take a suitable derivative with respect to \( Q \) at \( Q = 1 \) in the Potts model.

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