Random walks on infinite percolation clusters in models with long-range correlations

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Abstract

For a general class of percolation models with long-range correlations on $\mathbb{Z}^d$, $d \geq 2$, introduced in [19], we establish regularity conditions of Barlow [4] that mesoscopic subballs of all large enough balls in the unique infinite percolation cluster have regular volume growth and satisfy a weak Poincaré inequality. As immediate corollaries, we deduce quenched heat kernel bounds, parabolic Harnack inequality, and finiteness of the dimension of harmonic functions with at most polynomial growth. Heat kernel bounds and the quenched invariance principle of [31] allow to extend various other known results about Bernoulli percolation by mimicking their proofs, for instance, the local central limit theorem of [6] or the result of [8] that the dimension of at most linear harmonic functions on the infinite cluster is $d + 1$.

In terms of specific models, all these results are new for random interlacements at every level in any dimension $d \geq 3$, as well as for the vacant set of random interlacements [39, 38] and the level sets of the Gaussian free field [34] in the regime of the so-called local uniqueness (which is believed to coincide with the whole supercritical regime for these models).

1 Introduction

Delmotte [16] proved that the transition density of the simple random walk on a graph satisfies Gaussian bounds and the parabolic Harnack inequality holds if all the balls have regular volume growth and satisfy a Poincaré inequality. Barlow [4] relaxed these conditions by imposing them only on all large enough balls, and showed that they imply large time Gaussian bounds and the elliptic Harnack inequality for large enough balls. Later, Barlow and Hambly [6] proved that the parabolic Harnack inequality also follows from Barlow’s conditions. Barlow [4] verified these conditions for the supercritical cluster

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of Bernoulli percolation on $\mathbb{Z}^d$, which lead to the almost sure Gaussian heat kernel bounds and parabolic Harnack inequality. By using stationarity and heat kernel bounds, the quenched invariance principle was proved in [37, 9, 25], which lead to many further results about supercritical Bernoulli percolation, including the local central limit theorem [6] and the fact that the dimension of harmonic functions of at most linear growth is $d + 1$ [8].

The independence property of Bernoulli percolation was essential in verifying Barlow’s conditions, and up to now it has been the only example of percolation model for which the conditions were verified. On the other hand, once the conditions are verified, the derivation of all the further results uses rather robust methods and allows for extension to other stationary percolation models.

The aim of this paper is to develop an approach to verifying Barlow’s conditions for infinite clusters of percolation models, which on the one hand, applies to supercritical Bernoulli percolation, but on the other, does not rely on independence and extends beyond models which are in any stochastic relation with Bernoulli percolation. Motivating examples for us are random interlacements, vacant set of random interlacements, and the level sets of the Gaussian free field [39, 38, 34]. In all these models, the spatial correlations decay only polynomially with distance, and classical Peierls-type arguments do not apply. A unified framework to study percolation models with strong correlations was proposed in [19], within which the shape theorem for balls [19] and the quenched invariance principle [31] were proved. In this paper we prove that Barlow’s conditions are satisfied by assumptions from [19], which include supercritical Bernoulli percolation, random interlacements at every level in any dimension $d \geq 3$, the vacant set of random interlacements and the level sets of the Gaussian free field in the regime of local uniqueness.

### 1.1 General graphs

Let $G$ be an infinite connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For $x, y \in V(G)$, define the weights

$$
\nu_{xy} = \begin{cases} 
1, & \{x, y\} \in E(G), \\
0, & \text{otherwise},
\end{cases}
\quad \mu_x = \sum_y \nu_{xy},
$$

and extend $\nu$ to the measure on $E(G)$ and $\mu$ to the measure on $V(G)$.

For functions $f : V(G) \to \mathbb{R}$ and $g : E(G) \to \mathbb{R}$, let $\int f \, d\mu = \sum_{x \in V(G)} f(x) \mu_x$ and $\int g \, d\nu = \sum_{e \in E(G)} g(e) \nu_e$, and define $|\nabla f| : E(G) \to \mathbb{R}$ by $|\nabla f|([x, y]) = |f(x) - f(y)|$ for $\{x, y\} \in E(G)$.

Let $d_G$ be the graph distance on $G$, and define $B_G(x, r) = \{ y \in V(G) : d_G(x, y) \leq r \}$. We assume that $\mu(B_G(x, r)) \leq C_0 r^d$ for all $x \in V(G)$ and $r \geq 1$. In particular, this implies that the maximal degree in $G$ is bounded by $C_0$.

We say that a graph $G$ satisfies the volume regularity and the Poincaré inequality if for all $x \in V(G)$ and $r > 0$, $\mu(B_G(x, 2r)) \leq C_1 \cdot \mu(B_G(x, r))$ and, respectively, $\min_{B_G(x, r)} (f - a)^2 d\mu \leq C_2 \cdot r^2 \cdot \int_{E(B_G(x, r))} |\nabla f|^2 d\nu$, with some constants $C_1$ and $C_2$. Graphs satisfying these conditions are very well understood. Delmotte proved in [16] the
equivalence of such conditions to Gaussian bounds on the transition density of the simple random walk and to the parabolic Harnack inequality for solution to the corresponding heat equation, extending results of Grigoryan \[20\] and Saloff-Coste \[35\] for manifolds. Under the same assumptions, he also obtained in \[15\] explicit bounds on the dimension of harmonic functions on \( G \) of at most polynomial growth. Results of this flavor are classical in geometric analysis, with seminal ideas going back to the work of De Giorgi \[14\], Nash \[29\], and Moser \[27, 28\] on the regularity of solutions of uniformly elliptic second order equations in divergence form.

The main focus of this paper is on random graphs, and more specifically on random subgraphs of \( \mathbb{Z}^d, d \geq 2 \). Because of local defects in such graphs caused by randomness, it is too restrictive to expect that various properties (e.g., Poincaré inequality, Gaussian bounds, or Harnack inequality) should hold globally. An illustrative example is the infinite cluster \( \mathcal{C}_\infty \) of supercritical Bernoulli percolation \[21\] defined as follows. For \( p \in [0, 1] \), remove vertices of \( \mathbb{Z}^d \) independently with probability \((1-p)\). The graph induced by the retained vertices almost surely contains an infinite connected component (which is unique) if \( p > p_c(d) \in (0, 1) \), and contains only finite components if \( p < p_c(d) \). It is easy to see that for any \( p > p_c(d) \) with probability 1, \( \mathcal{C}_\infty \) contains copies of any finite connected subgraph of \( \mathbb{Z}^d \), and thus, none of the above global properties can hold.

Barlow \[4\] proposed the following relaxed assumption which takes into account possible exceptional behavior on microscopic scales.

**Definition 1.1.** (\[4\] Definition 1.7) Let \( C_V, C_P, \) and \( C_W \geq 1 \) be fixed constants. For \( r \geq 1 \) integer and \( x \in V(G) \), we say that \( B_c(x, r) \) is \((C_V, C_P, C_W)\)-good if \( \mu(B_c(x, r)) \geq C_V r^d \) and the weak Poincaré inequality

\[
\min_a \int_{B_c(x,r)} (f - a)^2 d\mu \leq C_P \cdot r^2 \int_{E(B_c(x,C_Wr))} |\nabla f|^2 d\nu.
\]

holds for all \( f : B_c(x, C_Wr) \rightarrow \mathbb{R} \).

We say \( B_c(x, R) \) is \((C_V, C_P, C_W)\)-very good if there exists \( N_{B_c(x,R)} \leq R^{d+2} \) such that \( B_c(x, R) \) is \((C_V, C_P, C_W)\)-good whenever \( B_c(y, r) \subseteq B_c(x, R) \), and \( N_{B_c(x,R)} \leq r \leq R \).

**Remark 1.2.** For any finite \( H \subseteq V(G) \) and \( f : H \rightarrow \mathbb{R} \), the minimum \( \min_a \int_H (f-a)^2 d\mu \) is attained by the value \( a = \overline{f} = \frac{1}{|H|} \int_H f d\mu \).

For a very good ball, the conditions of volume growth and Poincaré inequality are allowed to fail on microscopic scales. Thus, if all large enough balls are very good, the graph can still have rather irregular local behavior. Despite that, on large enough scales it looks as if it was regular on all scales, as the following results from \[4, 6, 8\] illustrate.

Let \( X = (X_n)_{n \geq 0} \) and \( Y = (Y_t)_{t \geq 0} \) be the discrete and continuous time simple random walks on \( G \). \( X \) is a Markov chain with transition probabilities \( \frac{v_{xy}}{\mu_x} \) and \( Y \) is the Markov process with generator \( \mathcal{L}_G f(x) = \frac{1}{\mu_x} \sum_y v_{xy} (f(y) - f(x)) \). In words, the walker \( X \) (resp., \( Y \)) waits a unit time (resp., an exponential time with mean 1) at each vertex \( x \), and then jumps to a uniformly chosen neighbor of \( x \) in \( G \). For \( x \in V(G) \), we denote by \( P_x = P_{G,x} \) (resp., \( Q_x = Q_{G,x} \) the law of \( X \) (resp., \( Y \)) started from \( x \). The transition density of
Theorem 1.5. (resp., \( Y \)) with respect to \( \mu \) is denoted by \( p_n(x, y) = p_{G,n}(x, y) = \frac{p_{G,x}[X_n=y]}{\mu_y} \) (resp., \( q_t(x, y) = q_{G,t}(x, y) = \frac{Q_{G,x}[Y_t=y]}{\mu_y} \)).

The first implications of Definition 1.1 are large time Gaussian bounds for \( q_t \) and \( p_n \).

Theorem 1.3. ([4, Theorem 5.7(a)] and [6, Theorem 2.2]) Let \( x \in V(G) \). If there exists \( R_0 = R_0(x, G) \) such that \( B_G(x, R) \) is \( (C_V, C_P, C_W) \)-very good with \( N_{B_G(x,R)}^{3(d+2)} \leq R \) for each \( R \geq R_0 \), then there exist constants \( C_i = C_i(d, C_0, C_V, C_P, C_W) \) such that for all \( t \geq R_0^{3/2} \) and \( y \in V(G) \),

\[
F_t(x, y) \leq C_1 \cdot t^{-\frac{d}{2}} \cdot e^{-C_2 \frac{d \mu}{t}}, \quad \text{if } t \geq d_G(x, y), \tag{1.1}
\]

\[
F_t(x, y) \geq C_3 \cdot t^{-\frac{d}{2}} \cdot e^{-C_4 \frac{d \mu}{t}}, \quad \text{if } t \geq d_G(x, y) \frac{3}{2}, \tag{1.2}
\]

where \( F_t \) stands for either \( q_t \) or \( p_{t|t} + p_{t|t+1} \).

The next result gives an elliptic Harnack inequality.

Theorem 1.4. ([4, Theorem 5.11]) There exists \( C_{ei} = C_{ei}(d, C_0, C_V, C_P, C_W) \) such that for any \( x \in V(G) \) and \( R \geq 1 \), if \( B_C(x, R \log R) \) is \( (C_V, C_P, C_W) \)-very good with \( N_{B_C(x,R \log R)}^{2(d+2)} \leq R \), then for any \( y \in B_C(x, \frac{1}{3}R \log R) \), and \( h : B_C(y, R+1) \to \mathbb{R} \) nonnegative and harmonic in \( B_C(y, R) \),

\[
\sup_{B_C(y, \frac{1}{4}R)} h \leq C_{ei} \cdot \inf_{B_C(y, \frac{1}{2}R)} h. \tag{1.3}
\]

In fact, more general parabolic Harnack inequality also takes place. (For the definition of parabolic Harnack inequality see, e.g., [6, Section 3].)

Theorem 1.5. ([6, Theorem 3.1]) There exists \( C_{phi} = C_{phi}(d, C_0, C_V, C_P, C_W) \) such that for any \( x \in V(G) \), \( R \geq 1 \), and \( R_1 = R \log R \geq 16 \), if \( B_C(x, R_1) \) is \( (C_V, C_P, C_W) \)-very good with \( N_{B_C(x,R_1)}^{2(d+2)} \leq \frac{R_1}{2 \log R_1} \), then for any \( y \in B_C(x, \frac{1}{3}R_1) \), the parabolic Harnack inequality (in both discrete and continuous time settings) holds with constant \( C_{phi} \) for \((0, R^2] \times B_C(y, R) \). In particular, the elliptic Harnack inequality \( [3, 4] \) also holds.

Next result is about the dimension of the space of harmonic functions on \( G \) with at most polynomial growth.

Theorem 1.6. ([6, Theorem 4]) Let \( x \in V(G) \). If there exists \( R_0 = R_0(x, G) \) such that \( B_C(x, R) \) is \( (C_V, C_P, C_W) \)-very good for each \( R \geq R_0 \), then for any positive \( k \), the space of harmonic functions \( h \) with \( \limsup_{d_C(x,y) \to \infty} \frac{h(y)}{d_C(x,y)^k} < \infty \) is finite dimensional, and the bound on the dimension only depends on \( k, d, C_0, C_V, C_P, \) and \( C_W \).

The notion of very good balls is most useful in studying random subgraphs of \( \mathbb{Z}^d \). Up to now, it was only applied to the unique infinite connected component of supercritical Bernoulli percolation, see [4, 6]. Barlow [4, Section 2] showed that on an event of probability 1, for every vertex of the infinite cluster, all large enough balls centered at it are
very good. Thus, all the above results are immediately transfered into the almost sure statements for all vertices of the infinite cluster.

Despite the conditions of Definition 1.1 are rather general, their validity up to now has only been shown for the independent percolation. The reason is that most of the analysis developed for percolation is tied very sensitively with the independence property of Bernoulli percolation. One usually first reduces combinatorial complexity of patterns by a coarse graining, and then balances the complexity out by exponential bounds coming from the independence, see, e.g., [4, Section 2].

The main purpose of this paper is to develop an approach to verifying properties of Definition 1.1 for random graphs which does not rely on independence or any comparison with Bernoulli percolation, and, as a result, extending the known results about Bernoulli percolation to models with strong correlations. Our primal motivation comes from percolation models with strong correlations, such as random interlacements, vacant set of random interlacements, or the level sets of the Gaussian free field, see, e.g., [39, 38, 34].

Remark 1.7. (1) The lower bound of Theorem 1.3 can be slightly generalized by following the proof of [4, Theorem 5.7(a)]. Let \( \epsilon \in (0, \frac{1}{2}] \) and \( K > \frac{1}{\epsilon} \). If there exists \( R_0 = R_0(x, G) \) such that \( B_G(x, R) \) is \((C_V, C_P, C_W)\)-very good with \( N_{B_G(x, R)}^{K(d+2)} \leq R \) for each \( R \geq R_0 \), then for all \( t \geq R_0^{1+\epsilon} \),

\[
F_t(x, y) \geq C_3 \cdot t^{-\frac{d}{2}} \cdot e^{-C_4 \frac{d_G(x, y)^2}{t}}, \quad \text{if } t \geq d_G(x, y)^{1+\epsilon}.
\]

The constants \( C_3 \) and \( C_4 \) are the same as in (1.2), in particular, they do not depend on \( K \) and \( \epsilon \). For \( \epsilon = \frac{1}{2} \) and \( K = 3 \), we recover (1.2). (There is a small typo in the statements of [4, Theorem 5.7(a)] and [6, Theorem 2.2]: \( R_0^{2/3} \) should be replaced by \( R_0^{3/2} \).)

Indeed, the proof of [4, Theorem 5.7(a)] is reduced to verifying assumptions of [4, Theorem 5.3] for some choice of \( R \). The original choice of Barlow is \( R = t^{\frac{2}{3}} \), and it implies (1.2). By restricting the choice of \( N_{B_G(x, R)} \) as above, one notices that all the conditions of [4, Theorem 5.3] are satisfied by \( R = t^{1+\epsilon} \), implying (1.4).

(2) In order to prove the lower bound of (1.2) for the same range of \( t \)'s as in the upper bound (1.1), one needs to impose a stronger assumption on the regularity of the balls \( B_G(x, R) \) (see, for instance, [4, Definition 5.4] of the exceedingly good ball and [4, Theorem 5.7(b)]). In fact, the recent result of [7, Theorem 1.10] states that the volume doubling property and the Poincaré inequality satisfied by large enough balls are equivalent to certain partial Gaussian bounds (and also to the parabolic Harnack inequality in large balls).

(3) Under the assumptions of Theorem 1.5, various estimates of the heat kernels for the processes \( X \) and \( Y \) killed on exiting from a box are given in [6, Theorem 2.1].

(4) Theorem 1.6 holds under much weaker assumptions, although reminiscent of the ones of Definition 1.1 (see [8, Theorem 4]). Roughly speaking, one assumes that the conditions from Definition 1.1 hold with \( N_{B_G(x, R)} \) only sublinear in \( R \), i.e., a volume growth condition and the weak Poincaré inequality should hold only for macroscopic subballs of \( B_G(x, R) \).
1.2 The model

We consider the measurable space \( \Omega = \{0,1\}^{\mathbb{Z}^d}, d \geq 2 \), equipped with the sigma-algebra \( \mathcal{F} \) generated by the coordinate maps \( \{ \omega \mapsto \omega(x) \}_{x \in \mathbb{Z}^d} \). For any \( \omega \in \{0,1\}^{\mathbb{Z}^d} \), we denote the induced subset of \( \mathbb{Z}^d \) by

\[
S = S(\omega) = \{ x \in \mathbb{Z}^d : \omega(x) = 1 \} \subseteq \mathbb{Z}^d.
\]

We view \( S \) as a subgraph of \( \mathbb{Z}^d \) in which the edges are drawn between any two vertices of \( S \) within \( \ell^1 \)-distance 1 from each other, where the \( \ell^1 \) and \( \ell^\infty \) norms of \( x = (x(1), \ldots, x(d)) \in \mathbb{R}^d \) are defined in the usual way by \( |x|_1 = \sum_{i=1}^{d} |x(i)| \) and \( |x|_\infty = \max \{|x(1)|, \ldots, |x(d)|\} \), respectively. For \( x \in \mathbb{Z}^d \) and \( r \in \mathbb{R}_+ \), we denote by \( B(x, r) = \{ y \in \mathbb{Z}^d : |x - y|_\infty \leq r \} \) the closed \( \ell^\infty \)-ball in \( \mathbb{Z}^d \) with radius \( r \) and center at \( x \).

**Definition 1.8.** For \( r \in [0, \infty) \), we denote by \( S_r \), the set of vertices of \( S \) which are in connected components of \( S \) of \( \ell^1 \)-diameter \( \geq r \). In particular, \( S_{\infty} \) is the subset of vertices of \( S \) which are in infinite connected components of \( S \).

1.2.1 Assumptions

On \((\Omega, \mathcal{F})\) we consider a family of probability measures \((\mathbb{P}^u)_{a \leq u \leq b}\) with \( 0 < a < b < \infty \), satisfying the following assumptions P1 – P3 and S1 – S2 from [19]. Parameters \( d, a, \) and \( b \) are considered fixed throughout the paper, and dependence of various constants on them is omitted.

An event \( G \in \mathcal{F} \) is called increasing (respectively, decreasing), if for all \( \omega \in G \) and \( \omega' \in \{0,1\}^{\mathbb{Z}^d} \) with \( \omega(y) \leq \omega(y') \) (respectively, \( \omega(y) \geq \omega(y') \)) for all \( y \in \mathbb{Z}^d \), one has \( \omega' \in G \).

**P1** (Ergodicity) For each \( u \in (a, b) \), every lattice shift is measure preserving and ergodic on \((\Omega, \mathcal{F}, \mathbb{P}^u)\).

**P2** (Monotonicity) For any \( u, u' \in (a, b) \) with \( u < u' \), and any increasing event \( G \in \mathcal{F} \), \( \mathbb{P}^u[G] \leq \mathbb{P}^{u'}[G] \).

**P3** (Decoupling) Let \( L \geq 1 \) be an integer and \( x_1, x_2 \in \mathbb{Z}^d \). For \( i \in \{1,2\} \), let \( A_i \in \sigma(\{ \omega \mapsto \omega(y) \}_{y \in B(x_i, 10L)}) \) be decreasing events, and \( B_i \in \sigma(\{ \omega \mapsto \omega(y) \}_{y \in B(x_i, 10L)}) \) increasing events. There exist \( R_p, L_p < \infty \) and \( \varepsilon_p, \chi_p > 0 \) such that for any integer \( R \geq R_p \) and \( a < \hat{u} < u < b \) satisfying

\[
u \geq (1 + R^{-\chi_p}) \cdot \hat{u},
\]

if \( |x_1 - x_2|_\infty \geq R \cdot L \), then

\[
\mathbb{P}^u [A_1 \cap A_2] \leq \mathbb{P}^{\hat{u}} [A_1] \cdot \mathbb{P}^{\hat{u}} [A_2] + e^{-f_p(L)},
\]

and

\[
\mathbb{P}^{\hat{u}} [B_1 \cap B_2] \leq \mathbb{P}^u [B_1] \cdot \mathbb{P}^{\hat{u}} [B_2] + e^{-f_p(L)},
\]

where \( f_p \) is a real valued function satisfying \( f_p(L) \geq e^{(\log L)^{\chi_p}} \) for all \( L \geq L_p \).
S1 (Local uniqueness) There exists a function $f_s : (a, b) \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ such that for each $u \in (a, b)$,

there exist $\Delta_s = \Delta_s(u) > 0$ and $R_s = R_s(u) < \infty$

such that $f_s(u, R) \geq (\log R)^{1+\Delta_s}$ for all $R \geq R_s$,

(1.5)

and for all $u \in (a, b)$ and $R \geq 1$, the following inequalities are satisfied:

$$\mathbb{P}^u [S_R \cap B(0, R) \neq \emptyset] \geq 1 - e^{-f_s(u, R)},$$

and

$$\mathbb{P}^u \left[ \text{for all } x, y \in S_{R/10} \cap B(0, R), \right. \left. x \text{ is connected to } y \right. \left. \text{in } S \cap B(0, 2R) \right] \geq 1 - e^{-f_s(u, R)}.$$

S2 (Continuity) Let $\eta(u) = \mathbb{P}^u [0 \in S_{\infty}]$. The function $\eta(\cdot)$ is positive and continuous on $(a, b)$.

Remark 1.9. (1) The use of assumptions P2, P3, and S2 will not be explicit in this paper. They are only used in Lemma 3.2 to prove likeliness of certain patterns in $S_{\infty}$ produced by a multi-scale renormalization. (Of course, they are also used in already known results of Theorems 1.10 and 1.11). Roughly speaking, we use P3 repeatedly on multiple scales for a convergent sequence of parameters $u_k$ and use P2 and S2 to establish convergence of iterations.

(2) If the family $\mathbb{P}^u$, $u \in (a, b)$, satisfies S1, then a union bound argument gives that for any $u \in (a, b)$, $\mathbb{P}^u$-a.s., the set $S_{\infty}$ is non-empty and connected, and there exist constants $C_i = C_i(u)$ such that for all $R \geq 1$,

$$\mathbb{P}^u [S_{\infty} \cap B(0, R) \neq \emptyset] \geq 1 - C_1 \cdot e^{-C_2 \cdot (\log R)^{1+\Delta_s}}.$$  

(1.6)

1.2.2 Examples

Here we briefly list some motivating examples (already announced earlier in the paper) of families of probability measures satisfying assumptions P1 – P3 and S1 – S2. All these examples were considered in details in [19], and we refer the interested reader to [19, Section 2] for the proofs and further details.

(1) Bernoulli percolation with parameter $u \in [0, 1]$ corresponds to the product measure $\mathbb{P}^u$ with $\mathbb{P}^u[\omega(x) = 1] = 1 - \mathbb{P}^u[\omega(x) = 0] = u$. The family $\mathbb{P}^u$, $u \in (a, b)$, satisfies assumptions P1 – P3 and S1 – S2 for any $d \geq 2$ and $p_c(d) < a < b \leq 1$, see [21].

(2) Random interlacements at level $u > 0$ is the random subgraph of $\mathbb{Z}^d$, $d \geq 3$, corresponding to the measure $\mathbb{P}^u$ defined by the equations

$$\mathbb{P}^u[S \cap K = \emptyset] = e^{-u \cdot \text{cap}(K)}, \quad \text{for all finite } K \subset \mathbb{Z}^d,$$

where $\text{cap}(\cdot)$ is the discrete capacity. It follows from [32, 39, 40] that the family $\mathbb{P}^u$, $u \in (a, b)$, satisfies assumptions P1 – P3 and S1 – S2 for any $0 < a < b < \infty$. Curiously, for any $u > 0$, $S$ is $\mathbb{P}^u$-almost surely connected [39], i.e., $S_{\infty} = S$. 

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Vacant set of random interlacements at level \( u > 0 \) is the complement of the random interlacements at level \( u \) in \( \mathbb{Z}^d \). It corresponds to the measure \( \mathbb{P}^u \) defined by the equations
\[
\mathbb{P}^u[K \subseteq S] = e^{-u \cdot \text{cap}(K)}, \quad \text{for all finite } K \subseteq \mathbb{Z}^d.
\]
Unlike random interlacements, the vacant set undergoes a percolation phase transition in \( u \) \([39, 38]\). If \( u < u_\ast(d) \in (0, \infty) \) then \( \mathbb{P}^u \)-almost surely \( S_\infty \) is non-empty and connected, and if \( u > u_\ast(d) \), \( S_\infty \) is \( \mathbb{P}^u \)-almost surely empty. It is known that the family \( \mathbb{P}^u, u \in (a, b) \), satisfies assumptions \( \mathbf{P1} - \mathbf{P3} \) for any \( 0 < a < b < \infty \) \([39, 40]\), \( \mathbf{S2} \) for any \( \frac{1}{u_\ast(d)} < a < b < \infty \) \([41]\), and \( \mathbf{S1} \) for some \( \frac{1}{u_\ast(d)} < a < b < \infty \) \([18]\).

The Gaussian free field on \( \mathbb{Z}^d, \ d \geq 3 \), is a centered Gaussian field with covariances given by the Green function of the simple random walk on \( \mathbb{Z}^d \). The excursion set above level \( h \in \mathbb{R} \) is the random subset of \( \mathbb{Z}^d \) where the fields exceeds \( h \). Let \( \mathbb{P}^h \) be the measure on \( \Omega \) for which \( S \) has the law of the excursion set above level \( h \). The model exhibits a non-trivial percolation phase transition \([12, 34]\). If \( h < h_\ast(d) \in [0, \infty) \) then \( \mathbb{P}^h \)-almost surely \( S_\infty \) is non-empty and connected, and if \( h > h_\ast(d) \), \( S_\infty \) is \( \mathbb{P}^h \)-almost surely empty. It was proved in \([19, 34]\) that the family \( \mathbb{P}^{h_\ast(u_\ast(d)) - h}, h \in (a, b) \), satisfies assumptions \( \mathbf{P1} - \mathbf{P3} \) and \( \mathbf{S2} \) for any \( 0 < a < b < \infty \), and \( \mathbf{S1} \) for some \( 0 < a < b < \infty \).

The last three examples are particularly interesting, since they have polynomial decay of spatial correlations and cannot be studied by comparison with Bernoulli percolation on any scale. In particular, many of the methods developed for Bernoulli percolation do not apply. As we see from the examples, assumptions \( \mathbf{P1} - \mathbf{P3} \) and \( \mathbf{S2} \) are satisfied by all the 4 models through their whole supercritical phases. However, assumption \( \mathbf{S1} \) is currently verified for the whole range of interesting parameters only in the cases of Bernoulli percolation and random interlacements, and only for a non-empty subset of interesting parameters in the last two examples. We call all the parameters \( u \) for which \( \mathbb{P}^u \) satisfies \( \mathbf{S1} \) the regime of local uniqueness (since under \( \mathbf{S1} \), there is a unique giant cluster in each large box). It is a challenging open problem to verify if the regime of local uniqueness coincides with the supercritical phase for the vacant set of random interlacements and the level sets of the Gaussian free field. A positive answer to this question will imply that all the results of this paper hold unconditionally also for the last two considered examples through their whole supercritical phases.

1.2.3 Known results

Below we recall some results from \([19, 31]\) about the large scale behavior of graph distances in \( S_\infty \) and the quenched invariance principle for the simple random walk on \( S_\infty \). Both results are formulated in the form suitable for our applications.

\textbf{Theorem 1.10.} \([19, \text{Theorem 1.3}]\) Let \( d \geq 2 \) and \( \theta_{\text{chd}} \in (0, 1) \). Assume that the family of measures \( \mathbb{P}^u, u \in (a, b) \), satisfies assumptions \( \mathbf{P1} - \mathbf{P3} \) and \( \mathbf{S1} - \mathbf{S2} \). Let \( u \in (a, b) \). There exist \( \Omega_{\text{chd}} \in \mathcal{F} \) with \( \mathbb{P}^u[\Omega_{\text{chd}}] = 1 \), constants \( C_{\text{chd}}, C_{1.10} \) and \( C_{1.10} \) all dependent on
u and $\theta_{\text{chd}}$, and random variables $R_{\text{chd}}(x), x \in \mathbb{Z}^d$, such that for all $\omega \in \Omega_{\text{chd}} \cap \{0 \in S_{\infty}\}$ and $x \in S_{\infty}(\omega)$,

(a) $R_{\text{chd}}(x, \omega) < \infty$,

(b) for all $R \geq R_{\text{chd}}(x, \omega)$ and $y, z \in B_{\mathbb{Z}^d}(x, R) \cap S_{\infty}(\omega)$,

$$d_{S_{\infty}(\omega)}(y, z) \leq C_{\text{chd}} \cdot \max \{d_{\mathbb{Z}^d}(y, z), R_{\text{chd}}^\theta\},$$

(c) for all $z \in \mathbb{Z}^d$ and $r \geq 1$,

$$P_u[R_{\text{chd}}(z) \geq r] \leq C_{1.10} \cdot e^{-\frac{1}{10} \log r^{1+\Delta_S}},$$

where $\Delta_S$ is defined in (1.5).

For $T > 0$, let $C[0, T]$ be the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$, and $\mathcal{W}_T$ the Borel sigma-algebra on it. Let

$$\tilde{B}_n(t) = \frac{1}{\sqrt{n}} \left( X_{\lfloor tn \rfloor} + (tn - \lfloor tn \rfloor) \cdot (X_{\lfloor tn \rfloor+1} - X_{\lfloor tn \rfloor}) \right). \quad (1.7)$$

**Theorem 1.11.** ([31, Theorem 1.1, Lemma A.1, and Section 5]) Let $d \geq 2$. Assume that the family of measures $\mathbb{P}^u, u \in (a, b)$, satisfies assumptions $P1 – P3$ and $S1 – S2$. Let $u \in (a, b)$ and $T > 0$. There exist $\Omega_{\text{qip}} \in \mathcal{F}$ with $\mathbb{P}^u[\Omega_{\text{qip}}] = 1$ and a non-degenerate matrix $\Sigma = \Sigma(u)$, such that for all $\omega \in \Omega_{\text{qip}} \cap \{0 \in S_{\infty}\}$,

(a) there exists $\chi : S_{\infty}(\omega) \to \mathbb{R}^d$ such that $x \mapsto x + \chi(x)$ is harmonic on $S_{\infty}(\omega)$, and $\lim_{n \to \infty} \frac{1}{n} \max_{x \in S_{\infty} \cap B(0, n)} |\chi(x)| = 0$,

(b) the law of $\tilde{B}_n(t)$ on $(C[0, T], \mathcal{W}_T)$ converges weakly (as $n \to \infty$) to the law of Brownian motion with zero drift and covariance matrix $\Sigma$.

In addition, if reflections and rotations of $\mathbb{Z}^d$ by $\pi$ preserve $\mathbb{P}^u$, then the limiting Brownian motion isotropic, i.e., $\Sigma = \sigma^2 \cdot I_d$ with $\sigma^2 > 0$.

**Remark 1.12.** ([31, Theorem 1.1] is stated for the (“blind”) random walk which jumps to a neighbor with probability $\frac{1}{2d}$ and stays put with probability $1 - \frac{1}{2d}$ (number of neighbors). Since the blind walk and the simple random walk are time changes of each other, the invariance principle for one process implies the one for the other (see, for instance, [9, Lemma 6.4]).

### 1.3 Main results

The main contribution of this paper is Theorem 1.13, where we prove that under the assumptions $P1 – P3$ and $S1 – S2$, all large enough balls in $S_{\infty}$ are very good in the sense of Definition 1.1. (In fact, our result is stronger, see Proposition 4.3.) This result has many immediate applications, including Gaussian heat kernel bounds, Harnack inequalities, and finiteness of the dimension of harmonic functions on $S_{\infty}$ with prescribed
polynomial growth, see Theorems 1.3, 1.5, 1.4, 1.6. In fact, all the results from [6, 8] can
be easily translated from Bernoulli percolation to our setting, since (as also pointed out
by the authors) their proofs only rely on (some combinations of) stationarity, Gaussian
heat kernel bounds, and the invariance principle. Among such results are estimates on the
gradient of the heat kernel (Theorem 1.16) and on the Green function (Theorem 1.17),
which will be deduced from the heat kernel bounds by replicating the proofs of 8 The-
orem 6] and 3 Theorem 1.2(a)], the fact that the dimension of at most linear harmonic
functions on $S_\infty$ is $d + 1$ (Theorem 1.18), the local central limit theorem (Theorem 1.19),
and the asymptotic for the Green function (Theorem 1.20), which we derive from the
heat kernel bounds and the quenched invariance principle by mimicking the proofs of 8 The-
orem 5], 3 Theorem 1.1], and 3 Theorem 1.2(b,c)].

We begin by stating the main result of this paper.

**Theorem 1.13.** Let $d \geq 2$ and $\theta_{v_{gb}} \in (0, \frac{1}{2d})$. Assume that the family of measures $P^u$, $u \in (a,b)$, satisfies assumptions P1 – P3 and S1 – S2. Let $u \in (a,b)$. There exist
$\Omega_{v_{gb}} \in F$ with $P^u[\Omega_{v_{gb}}] = 1$, constants $C_V, C_P, C_W$, and $G$ all dependent on
$u$ and $\theta_{v_{gb}}$, and random variables $R_{v_{gb}}(x) \in Z^d$, such that for all $\omega \in \Omega_{v_{gb}} \cap \{0 \in S_\infty\}$ and
$x \in S_\infty(\omega)$,

(a) $R_{v_{gb}}(x, \omega) < \infty$,
(b) for all $R \geq R_{v_{gb}}(x, \omega)$, $B_{S_\infty(\omega)}(x, R)$ is $(C_V, C_P, C_W)$-very good with $N_{B_{S_\infty(\omega)}(x, R)} \leq
P^u[\Omega_{v_{gb}}] = 1$,

(c) for all $z \in Z^d$ and $r \geq 1$,

$$P^u[\Omega_{v_{gb}}] = 1$$

where $\Delta_\infty$ is defined in (1.5).

**Corollary 1.14.** Theorem 1.13 immediately implies that all the results of Theorems 1.3,
1.5, 1.4, and 1.6 hold almost surely for $G = S_\infty$. Since the constants $C_V, C_P,$ and $C_W
in the statement of Theorem 1.13 are deterministic, all the constants in Theorems 1.3, 1.5,
1.4, and 1.6 are also deterministic.

Combining Corollary 1.13 with Theorem 1.10 and Remark 1.7(1), we notice that the
quenched heat kernel bounds of Theorem 1.3 hold almost surely for $G = S_\infty$ with $d_G$
replaced by $d_G$ in (1.11), (1.12), and (1.14). Since we will use the quenched heat kernel
bounds often in the paper, we give a precise statement here.

**Theorem 1.15.** Let $d \geq 2$. Assume that the family of measures $P^u$, $u \in (a,b)$, satisfies
assumptions P1 – P3 and S1 – S2. Let $u \in (a,b)$ and $\epsilon > 0$. There exist
$\Omega_{v_{gb}} \in F$ with $P^u[\Omega_{v_{gb}}] = 1$, constants $C_i = C_i(u)$, $C_{1.13} = C_{1.13}(u, \epsilon)$, and $\epsilon_{1.15} = 1.15(u, \epsilon)$, and
random variables $T_{v_{gb}}(x, \epsilon)$, $x \in Z^d$, such that for all $\omega \in \Omega_{v_{gb}} \cap \{0 \in S_\infty\}$ and
$x \in S_\infty(\omega)$,

(a) $T_{v_{gb}}(x, \epsilon, \omega) < \infty$,
(b) for all $t \geq T_{v_{gb}}(x, \epsilon, \omega)$ and $y \in S_\infty(\omega)$,

$$F_t(x, y) \leq C_1 \cdot t^{-\frac{d}{2}} \cdot e^{-C_2 \frac{D(x, y)^2}{t}}$$

if $t \geq D(x, y)$, (1.9)
Theorem 1.16. Let \( d \geq 2 \). Assume that the family of measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfies assumptions P1 – P3 and S1 – S2. Let \( u \in (a, b) \). There exist constants \( C_i = C_i(u) \), such that for all \( x, x', y \in \mathbb{Z}^d \) and \( n > \max \{ d_{G}(x, y), d_{Z}(x', y') \} \),

\[
\mathbb{P}^u \left[ (p_n(x, y) - p_{n-1}(x', y))^2 \cdot \mathbb{1}_{\{y \in S_n\}} \cdot \mathbb{1}_{\{x \text{ and } x' \text{ are neighbors in } S_n\}} \right] \leq C_1 \cdot e^{-C_2 \frac{d(x,y)^2}{n}}.
\]

The heat kernel bounds of Theorem 1.15 imply also the following quenched estimates on the Green function \( g_G(x, y) = \int_0^\infty q_{G,t}(x, y) dt = \sum_{n \geq 0} p_{G,n}(x, y) \) for almost all \( G = S_\infty \).

It is proved in [6 Theorem 1.2] for supercritical Bernoulli percolation, but extension to our setting is rather straightforward.

Theorem 1.17. Let \( d \geq 3 \). Assume that the family of measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfies assumptions P1 – P3 and S1 – S2. Let \( u \in (a, b) \). There exist constants \( C_i = C_i(u) \) such that for all \( \omega \in \Omega_\text{hk} \) and distinct \( x, y \in S_\infty(\omega) \), if \( d_{Z}(x, y)^2 \geq \min \{ T_\text{hk}(x), T_\text{hk}(y) \} \cdot (1 + C_3 \cdot \log d_{Z}(x, y)) \), then

\[
C_1 \cdot d_{Z}(x, y)^{2-d} \leq g_{S_\infty(\omega)}(x, y) \leq C_2 \cdot d_{Z}(x, y)^{2-d}.
\]

The remaining results are derived from the Gaussian heat kernel bounds and the quenched invariance principle. In the setting of supercritical Bernoulli percolation, all of them were obtained in [6 8], but all the proofs extend directly to our setting.

We begin with results about harmonic functions on \( S_\infty \). It is well known that Theorems 1.13 and Theorem 1.14 imply the almost sure Liouville property for positive harmonic functions on \( S_\infty \). The absence of non-constant sublinear harmonic functions on \( S_\infty \) is even known assuming just stationary of \( S \) (see [8 Theorem 3 and discussion below]). In particular, it implies the uniqueness of the function \( \chi \) in Theorem 1.11(a). The following result about the dimension of at most linear harmonic functions is classical on \( \mathbb{Z}^d \). It was extended to supercritical Bernoulli percolation on \( \mathbb{Z}^d \) in [8 Theorem 5].
Theorem 1.18. Let \( d \geq 2 \). Assume that the family of measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfies assumptions \( P1 - 3 \) and \( S1 - S2 \). Let \( u \in (a, b) \). There exist \( \Omega_{\text{int}} \in \mathcal{F} \) with \( \mathbb{P}^u[\Omega_{\text{int}}] = 1 \) such that for all \( \omega \in \Omega_{\text{int}} \cap \{ 0 \in S_\infty \} \), the dimension of the vector space of harmonic functions on \( S_\infty(\omega) \) with at most linear growth equals \( d + 1 \).

Since the parabolic Harnack inequality for solutions to the heat equation on \( S_\infty \) implies Hölder continuity of \( p_n \) and \( q_t \), it is possible to replace the weak convergence of Theorem \( 1.11 \) by pointwise convergence. \[ 6 \] Theorems 4.5 and 4.6 give general sufficient conditions for the local central limit theorem on general graphs. They were verified in \[ 6 \] Theorem 1.1 for supercritical Bernoulli percolation. Theorems 1.11 and 1.15 allow to check these conditions in our setting leading to the following (same as for Bernoulli percolation) result. For \( x \in \mathbb{R}^d \), \( t > 0 \), the Gaussian heat kernel with covariance matrix \( \Sigma \) is defined as

\[
    k_{\Sigma, t}(x) = (2\pi \det(\Sigma)t)^{-\frac{d}{2}} \cdot \exp\left( -\frac{x'\Sigma^{-1}x}{2t} \right),
\]

where \( x' \) is the transpose of \( x \).

Theorem 1.19. Let \( d \geq 2 \). Assume that the family of measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfies assumptions \( P1 - 3 \) and \( S1 - S2 \). Let \( u \in (a, b) \), \( m = \mathbb{E}^u[\mu_0 \cdot \mathbb{1}_{0 \in S_\infty}] \), and \( T > 0 \). There exist \( \Omega_{\text{int}} \in \mathcal{F} \) with \( \mathbb{P}^u[\Omega_{\text{int}}] = 1 \), and a non-degenerate covariance matrix \( \Sigma = \Sigma(u) \) such that for all \( \omega \in \Omega_{\text{int}} \cap \{ 0 \in S_\infty \} \),

\[
    \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{t \geq T} \left| n^\frac{d}{2} \cdot F_n(0, g_n(x)) - \frac{C(F)}{m} \cdot k_{\Sigma, t}(x) \right| = 0, \quad (1.12)
\]

where \( F \) stands for \( q \) or \( p_{[x]} + p_{[x]+1} \), \( C(F) \) is 1 if \( F = q \) and 2 otherwise, and \( g_n(x) \) is the closest point in \( S_\infty \) to \( \sqrt{nx} \).

Theorems 1.15 and 1.19 imply the following asymptotic for the Green function, extending results of \[ 6 \] Theorem 1.2(b,c) to our setting. For a covariance matrix \( \Sigma \), let \( G_{\Sigma}(x) = \int_0^\infty k_{\Sigma, t}(x)dt \) be the Green function of a Brownian motion with covariance matrix \( \Sigma \). In particular, if \( \Sigma = \sigma^2 \cdot I_d \), then \( G_{\Sigma}(x) = (2\sigma^2\pi^\frac{d}{2})^{-1}\Gamma(d/2 - 1)|x|^{2-d} \) for all \( x \neq 0 \), where \( | \cdot | \) stands for the Euclidean norm on \( \mathbb{R}^d \).

Theorem 1.20. Let \( d \geq 3 \). Assume that the family of measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfies assumptions \( P1 - 3 \) and \( S1 - S2 \). Let \( u \in (a, b) \), \( m \) and \( \Sigma \) as in Theorem 1.19, and \( \epsilon > 0 \). There exist \( \Omega_{\text{ext}} \in \mathcal{F} \) with \( \mathbb{P}^u[\Omega_{\text{ext}}] = 1 \) and a proper random variable \( M = M(\epsilon) \), such that for all \( \omega \in \Omega_{\text{ext}} \cap \{ 0 \in S_\infty \} \),

(a) for all \( x \in S_\infty(\omega) \) with \( |x| \geq M \),

\[
    \frac{(1 - \epsilon)G_{\Sigma}(x)}{m} \leq g_{S_\infty(\omega)}(0, x) \leq \frac{(1 + \epsilon)G_{\Sigma}(x)}{m},
\]

(b) for all \( y \in \mathbb{R}^d \), \( \lim_{k \to \infty} k^{2-d} \cdot \mathbb{E}^u \left[ g_{S_\infty(\omega)}(0, \lfloor ky \rfloor) \right] \quad 0 \in S_\infty \) \( = \frac{G_{\Sigma}(\omega)}{m} \).
Remark 1.21.  (1) Let us emphasize that our method does not allow to replace $(\log r)^{1+\Delta s}$ in (1.8) by $f_s(u, R)$ from S1. In particular, even if $f_s(u, R)$ growth polynomially with $R$, we are not able to improve the bound in (1.8) to stretched exponential. In the case of independent Bernoulli percolation, it is known from [4, Section 2] that the result of Theorem 1.13 holds with a stretched exponential bound in (1.8).

(2) The fact that the right hand side of (1.11) decays faster than any polynomial will be crucially used in the proofs of Theorems 1.16, 1.18, and 1.20. Quenched bounds on the diagonal $p_n(x, x)$ under the assumptions $P1 – P3$ and $S1 – S2$ were obtained in [31] (see Remarks 1.3 (4) and (5) there) for all $n \geq n_0(\omega)$, although without any control on the tail of $n_0(\omega)$.

(3) In the case of supercritical Bernoulli percolation, Barlow showed in [4, Theorem 1] that the bound (1.10) holds for all $t \geq \max\{T_k(x), D(x, y)\}$. The step “from $\epsilon > 0$ to $\epsilon = 0$” is highly nontrivial and follows from the fact that very good boxes on microscopic scales are dense, see [4, Definition 5.4 and Theorem 5.7(b)]. We do not know if such property can be deduced from the assumptions $P1 – P3$ and $S1 – S2$ or proved for any of the specific models considered in Section 1.2.2 (except for Bernoulli percolation). Our renormalization does not exclude the possibility of dense mesoscopic traps in $S_\infty$, but we do not have a counterexample either. For comparison, let us mention that the heat kernel bounds (1.9) and (1.10) were obtained in [31] for the random conductance model with i.i.d. weights, where it is also stated in [31, Remark 3.4] and [1, Remark 4.12] that the lower bound for times comparable with $D(x, y)$ can likely be obtained by adapting Barlow’s proof, but omitted there because of a considerable amount of extra work and few applications.

(4) The first proofs of the quenched invariance principle for random walk on the infinite cluster of Bernoulli percolation [37, 9, 25] relied significantly on the quenched upper bound on the heat kernel. It was then observed in [11] that it is sufficient to control only the diagonal of the heat kernel (proved for Bernoulli percolation in [24]). This observation was essential in proving the quenched invariance principle for percolation models satisfying $P1 – P3$ and $S1 – S2$ in [31], where the desired upper bound on the diagonal of the heat kernel was obtained by means of an isoperimetric inequality (see [31, Theorem 1.2]). Theorem 1.15 allows now to prove the quenched invariance principle of [31] by following the original path, for instance, by a direct adaptation of the proof of [9, Theorem 1.1].

(5) Our proof of Theorem 1.19 follows the approach of [6] in the setting of supercritical Bernoulli percolation, namely, it is deduced from the quenched invariance principle, parabolic Harnack inequality, and the upper bound on the heat kernel. If we replace in (1.12) $\sup_x$ by $\sup_{|x| < K}$ for any fixed $K > 0$, then it is not necessary to assume the upper bound on the heat kernel, see [13, Theorem 1].

(6) A new approach to limit theorems and Harnack inequalities for the elliptic random conductance model under assumptions on moments of the weights and their reciprocals has been recently developed in [2, 3]. It relies on Moser’s iteration and new weighted Sobolev and Poincaré inequalities, and is applicable on general graphs satisfying globally conditions of regular volume growth and an isoperimetric
inequality (see [3, Assumption 1.1]). We will comment more on these conditions in Remark 1.5. The method of [2] was recently used in [30] to prove the quenched invariance principle for the random conductance model on the infinite cluster of supercritical Bernoulli percolation under the same assumptions on moments of the weights as in [2].

1.4 Some words about the proof of Theorem 1.13

Theorem 1.10 is enough to control the volume growth, thus we only discuss here the weak Poincaré inequality. A finite subset \( H \) of \( V(G) \) satisfies the (strong) Poincaré inequality \( P(C, r) \), if for any function \( f : H \to \mathbb{R} \),
\[
\min_a \int_H (f - a)^2 d\mu \leq C \cdot r^2 \cdot \int_{E(H)} |\nabla f|^2 dv.
\]
The well known sufficient condition for \( P(C, r) \) is the following isoperimetric inequality for subsets of \( H \) (see, e.g., [23, Proposition 3.3.10] or [36, Lemma 3.3.7]): there exists \( c > 0 \) such that for all \( A \subset H \) with \( |A| \leq \frac{1}{2} |H| \), the number of edges between \( A \) and \( H \setminus A \) is at least \( \frac{c}{4} |A| \). Thus, if the ball \( B_c(y, r) \) is contained in a subset \( C(y, r) \) of \( V(G) \) such that \( C(y, r) \subset B_{C'}(y, C' r) \) and the above isoperimetric inequality holds for subsets of \( C(y, r) \), then it is easy to see that the weak Poincaré inequality with constants \( C \) and \( C' \) holds for \( B_{C}(y, r) \) (see Claim 4.2). In the case \( G = S_{\infty} \subset \mathbb{Z}^d \), the natural choice is to take \( C(y, r) \) to be the cluster of \( y \) in \( S_{\infty} \cap B(y, r) \), which turns out to be also the largest cluster in \( S \cap B(y, r) \) (here and below, we implicitly assume that \( r \) is large enough). In the setting of Bernoulli percolation, it is known that subsets of \( C(y, r) \) satisfy the above isoperimetric inequality (see [1, Proposition 2.11]). In our setting, Theorem 1.10 implies that \( C(y, r) \subset B_{C}(y, C' r) \), thus we only need to prove the isoperimetric inequality. The first isoperimetric inequality for subsets of \( C(y, r) \) was proved in [31, Theorem 1.2]. It states that for any \( A \subset C(y, r) \) with \( |A| \geq r^\delta \), the number of edges between \( A \) and \( S_{\infty} \setminus A \) is at least \( c |A| \) (thus, also at least \( \frac{c}{4} |A| \)). Note the key difference, the edges are taken between \( A \) and \( S_{\infty} \setminus A \), not just between \( A \) and \( C(y, r) \setminus A \). The above isoperimetric inequality implies certain Nash-type inequalities sufficient to prove a diffusive upper bound on the heat kernel (see [26, Theorem 2], [11, Proposition 6.1], [10, Lemma 3.2], [31, (A.4)]), but it is too weak to imply the Poincaré inequality (see, e.g., [23, Sections 3.2 and 3.3] for an overview of the two isoperimetric inequalities and their relation to various functional inequalities).

Let us also mention that in the setting of Bernoulli percolation, the “weak” isoperimetric inequality admits a simple proof ([10, Theorem A.1]), but the proof of the “strong” one is significantly more involved ([1, Proposition 2.11]). After all said, we have to admit that we are not able to prove the strong isoperimetric inequality for subsets of \( C(y, r) \), and do not know if it holds in our setting. Nevertheless, we can rescue the situation by proving that a certain enlarged set \( C(y, r) \), obtained from \( C(y, r) \) by adding to it all vertices from \( S_{\infty} \) to which it is locally connected, satisfies the desired strong isoperimetric inequality (see Proposition 4.3, Theorem 3.9, and Corollaries 3.11 and 3.17). The general outline of the proof of our isoperimetric inequality for \( C(y, r) \) is similar to the one of the proof of the weak isoperimetric inequality for \( C(y, r) \) in [31], but we have to modify renormalization and coarse graining of subsets of \( C(y, r) \) and rework some arguments to get good control of the boundary and the volume of subsets of \( C(y, r) \) in terms of the boundary and the volume of the corresponding coarse grainings. For instance, it is crucial for us (but not
for [31] that the coarse graining of a big set (say, of size $\frac{1}{2}|\tilde{C}(y, r)|$) should not be too big (see, e.g., the proof of Claim [31]).

We partition the lattice $\mathbb{Z}^d$ into large boxes of equal size. For each configuration $\omega \in \Omega$, we subdivide all the boxes into good and bad. Restriction of $S$ to a good box contains a unique largest in volume cluster, and the largest clusters in two adjacent good boxes are connected in $S$ in the union of the two boxes. Traditionally in the study of Bernoulli percolation, the good boxes are defined to contain a unique cluster of large diameter. In our case, the existence of several clusters of large diameter in good boxes is not excluded. The reason to work with volumes is that the existence of a unique giant cluster in a box can be expressed as an intersection of two events, an increasing (existence of cluster with big volume) and decreasing (smallness of the total volume of large clusters). Assumption $P3$ gives us control of correlations between monotone events, which is sufficient to set up two multi-scale renormalization schemes with scales $L_n$ (one for increasing and one for decreasing events) and conclude that bad boxes tend to organize in blobs on multiple scales, so that the majority of boxes of size $L_n$ contain at most 2 blobs of diameter bigger than $L_{n-1}$ each, but even their diameters are much smaller than the actual scale $L_n$. By removing two boxes of size $r_{n-1}L_{n-1} \ll L_n$ containing the biggest blobs of an $L_n$-box, then by removing from each of the remaining $L_{n-1}$-boxes two boxes of size $r_{n-2}L_{n-2} \ll L_{n-1}$ containing its biggest blobs, and so on, we end up with a subset of good boxes, which is a dense in $\mathbb{Z}^d$, locally well connected, and well structured coarse graining of $S_\infty$. Similar renormalization has been used in [33, 19, 31]. By reworking some arguments from [31], we prove that large subsets of the restriction of the coarse graining to any large box satisfy a $d$-dimensional isoperimetric inequality, if the scales $L_n$ grow sufficiently fast (Theorem 2.14). We deduce from it the desired isoperimetric inequality for large subsets $A$ of $\tilde{C}(y, r)$ (Theorem 3.9) as follows. If $A$ is spread out in $\tilde{C}(y, r)$, then it has large boundary, otherwise, we associate with it a set of those good boxes from the coarse graining, the unique largest cluster of which is entirely contained in $A$. It turns out that the boundary and the volume of the resulting set are comparable with those of $A$. Moreover, if $|A| \leq \frac{1}{2}|\tilde{C}(y, r)|$, then the volume of its coarse graining is also only a fraction of the total volume of the coarse graining of $\tilde{C}(y, r)$. The isoperimetric inequality then follows from the one for subsets of the coarse graining.

1.5 Structure of the paper

In Section 2 we define perforated sublattices of $\mathbb{Z}^d$ and prove that they satisfy an isoperimetric inequality. The main definition there is (2.6), and the main result is Theorem 2.14. In Section 3 we define a coarse graining of $S_\infty$ and use results from Section 2 to study certain extensions of largest clusters of $S_\infty$ in boxes (Definition 3.6). We prove that they satisfy the desired isoperimetric inequality (Theorem 3.9) and the volume growth (Corollary 3.16). In Section 4 we introduce the notions of regular and very regular balls, so that a (very) regular ball is always (very) good, and use it to prove the main result of the paper. In fact, in Proposition 4.3 we prove that large balls are very likely to be very regular, which is stronger than Theorem 1.13. In Section A we sketch the proofs of Theorems 1.16 – 1.20.

Finally, let us make a convention about constants. As already said, we omit from the
notation dependence of constants on $a$, $b$, and $d$. We usually also omit the dependence on $\varepsilon_r$, $\chi_r$, and $\Delta s$. Dependence on other parameters is reflected in the notation, for example, as $c(u, \theta_{q_b})$. Sometimes we use $C$, $C'$, $c$, etc., to denote “intermediate” constants, their values may change from line to line, and even within a line.

2 Renormalization

In this section we define lattices perforated on multiple scales and study their isoperimetric properties. Such lattices will be used in Section 3 as coarse approximations of largest connected components of $S$ in boxes. The main result of the section is Theorem 2.14.

Let $l_n, r_n, L_n$, $n \geq 0$ be sequences of positive integers such that $l_n > r_n$ and $L_n = l_{n-1} \cdot L_{n-1}$, for $n \geq 1$. To each $L_n$ we associate the rescaled lattice $G_n = L_n \cdot \mathbb{Z}^d = \{ L_n \cdot x : x \in \mathbb{Z}^d \}$, with edges between any pair of $(\ell^1)$-nearest neighbor vertices of $G_n$.

2.1 Cascading events

Let $E = (E_{x,L_0} : L_0 \geq 1, x \in G_0)$ be a family of events from some sigma-algebra. For each $L_0 \geq 1$, $n \geq 0$, $x \in G_n$, define recursively the events $\overline{G}_{x,n,L_0}(E)$ by $\overline{G}_{x,0,L_0}(E) = E_{x,L_0}$ and

$$\overline{G}_{x,n,L_0}(E) = \bigcup_{x_1, x_2 \in G_{n-1} \cap (x+0,L_0)^d \mid |x_1 - x_2|_{\infty} \geq r_n - 1 \cdot L_{n-1}} \overline{G}_{x_1,n-1,L_0}(E) \cap \overline{G}_{x_2,n-1,L_0}(E).$$

The events in (2.1) also depend on the scales $l_n$ and $r_n$, but we omit this dependence from the notation, since these sequences will be properly chosen and fixed later.

Definition 2.1. Given sequences $l_n, r_n, L_n$, $n \geq 0$, as above, and two families of events $\overline{D}$ and $\overline{I}$, we say that for $n \geq 0$, $x \in G_n$ is ($\overline{D}, \overline{I}, n$)-bad (resp., ($\overline{D}, \overline{I}, n$)-good), if the event $\overline{G}_{x,n,L_0}(\overline{D}) \cup \overline{G}_{x,n,L_0}(\overline{I})$ occurs (resp., does not occur).

Remark 2.2. Definition 2.1 can be naturally generalized to $k$ families of events $E_1, \ldots, E_k$, for any fixed $k$, and all the results of Section 2 still hold (with suitable changes of constants). For our applications, it suffices to consider only two families of events (see Section 3.1). Thus, for simplicity of notation, we restrict to this special case.

The choice of the families $\overline{D}$ and $\overline{I}$ throughout the paper is either irrelevant for the result (as in Section 2) or fixed (as in Section 3.1). Thus, from now on we write $n$-bad (resp., $n$-good) instead of ($\overline{D}, \overline{I}, n$)-bad (resp., ($\overline{D}, \overline{I}, n$)-good), hopefully without causing any confusions.

Good vertices give rise to certain geometrical structures on $\mathbb{Z}^d$, which we define and study in the remainder of this section.


2.2 Perforated lattices

Throughout this section, we fix sequences \( l_n, r_n, L_n, n \geq 0 \), such that \( l_n \) is divisible by \( r_n \) for all \( n \), two local families of events \( \overline{D} \) and \( \overline{I} \), and integers \( s \geq 0 \) and \( K \geq 1 \). Recall Definition 2.1 of \( n \)-good vertices in \( \mathbb{G}_n \). For \( x \in \mathbb{Z}^d \), define

\[
Q_{K,s}(x) = x + \mathbb{Z}^d \cap [0, KL_s]^d,
\]

and write \( Q_{K,s} \) for \( Q_{K,s}(0) \). We also fix \( x_s \in \mathbb{G}_s \) and assume that all the vertices in \( \mathbb{G}_s \cap Q_{K,s}(x_s) \) are \( s \)-good.

The aim of this section is to define a ubiquitous well structured subset of \( 0 \)-good vertices in \( Q_{K,s}(x_s) \) by perforating the lattice box \( \mathbb{G}_0 \cap Q_{K,s}(x_s) \) on multiple scales and using Definition 2.1.

We first recursively define certain subsets of \( i \)-good vertices in \( \mathbb{G}_i \cap Q_{K,s}(x_s) \) for \( i \leq s \). Let

\[
\mathcal{G}_{K,s,i}(x_s) = \mathbb{G}_s \cap Q_{K,s}(x_s).
\]

By (2.3), all \( z_s \in \mathcal{G}_{K,s,i}(x_s) \) are \( s \)-good.

Assume that \( \mathcal{G}_{K,s,i-1}(x_s) \subset \mathbb{G}_i \) is defined for some \( i \leq s \) so that all \( z_i \in \mathcal{G}_{K,s,i}(x_s) \) are \( i \)-good. By Definition 2.1 for each \( z_i \in \mathcal{G}_{K,s,i}(x_s) \), there exist \( a_{z_i}, b_{z_i} \in (r_{i-1}L_{i-1}) : \mathbb{Z}^d \cap (z_i + [0, L_i)^d) \)

such that all the vertices in

\[
(\mathbb{G}_{i-1} \cap (z_i + [0, L_i)^d)) \setminus ((a_{z_i} + [0, 2r_{i-1}L_{i-1})^d) \cup (b_{z_i} + [0, 2r_{i-1}L_{i-1})^d))
\]

are \( (i-1) \)-good. If the choice is not unique, we choose the pair arbitrarily. All the results below hold for any allowed choice of \( a_{z_i} \) and \( b_{z_i} \). To save notation, we will not mention it in the statements.

Define \( \mathcal{R}_{z_i} \subseteq \mathbb{G}_{i-1} \) to be

(a) \( \mathbb{G}_{i-1} \cap ((a_{z_i} + [0, 2r_{i-1}L_{i-1})^d) \cup (b_{z_i} + [0, 2r_{i-1}L_{i-1})^d)) \) if \( |a_{z_i} - b_{z_i}|_\infty > 2r_{i-1}L_{i-1}, \)

(b) the smallest rectangle in \( \mathbb{G}_{i-1} \) which contains \( \mathbb{G}_{i-1} \cap ((a_{z_i} + [0, 2r_{i-1}L_{i-1})^d) \cup (b_{z_i} + [0, 2r_{i-1}L_{i-1})^d)) \).

Remark 2.3. In the case \( \mathcal{R}_{z_i} \) is defined by (b), its largest side contains at most \( 4r_{i-1} \) vertices.

To complete the construction, let

\[
\mathcal{G}_{K,s,i-1}(x_s) = \mathbb{G}_{i-1} \cap \bigcup_{z_i \in \mathcal{G}_{K,s,i}(x_s)} ((z_i + [0, L_i)^d) \setminus \mathcal{R}_{z_i}).
\]

Note that all \( z_i \in \mathcal{G}_{K,s,i-1}(x_s) \) are \( (i-1) \)-good.

Now that the sets \( \{\mathcal{G}_{K,s,j}(x_s)\}_{j \leq s} \), are constructed by (2.4) and (2.5), we define the multiscale perforations of \( \mathbb{G}_0 \cap Q_{K,s}(x_s) \) by

\[
Q_{K,s,j}(x_s) = \mathbb{G}_0 \cap \bigcup_{z_j \in \mathcal{G}_{K,s,j}(x_s)} (z_j + [0, L_j)^d), \quad j \leq s.
\]

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Here we collect some general facts that we will frequently use.

2.3 General facts about isoperimetric inequalities

We also view the sets \( Q_{K,s,j}(x_s) \) as subgraphs of \( G_0 \) with edges drawn between any two vertices of the set which are at \( \ell^1 \) distance \( L_0 \) from each other.

In the next lemma we summarize some useful properties of \( Q_{K,s,j}(x_s) \)'s. Its proof is immediate from the construction.

\textbf{Lemma 2.4.} Let \( z_j \in G_{K,s,j}(x_s), \alpha = (\alpha_1, \ldots, \alpha_d) \in \{0,1\}^d \) with \( d_\alpha = \sum_i \alpha_i \geq 1 \). For \( y_0 \in G_0 \), let \( Z_{z_j,j,\alpha}(y_0) \) be the intersection of the \( d_\alpha \)-dimensional sublattice \( y_0 + \alpha_1 \mathbb{Z} \times \cdots \times \alpha_d \mathbb{Z} \) with \( (z_j + [0,L_j])^d \). If \( Z_{z_j,j,\alpha}(y_0) \) is non-empty, then for all \( i \leq j \), the set \( Q_{K,s,i}(x_i) \cap Z_{z_j,j,\alpha}(y_0) \)

- has size \( \geq \left( \frac{L_0}{L_i} \right)^{d_\alpha} \cdot \prod_{k=1}^{d_\alpha} \left( \frac{\lceil \alpha_k \rceil}{L_i} \right)^{d_\alpha} \cdot \prod_{k=i+1}^{d} \left( 1 - \frac{4r_{k-1}}{L_i} \right)^{d_\alpha} \),
- is connected in \( G_0 \) if \( d_\alpha \geq 2 \),
- is a disjoint union of boxes \( (x_t + [0,r_{i-1}L_{i-1}])^d \cap Z_{z_j,j,\alpha}(y_0) \) for some \( x_t \)'s in \( (r_{i-1}L_{i-1}) \cdot \mathbb{Z}^d \).

\textbf{Corollary 2.5.} Let \( y_0 \in G_0, \alpha \in \{0,1\}^d \) such that \( d_\alpha \geq 2 \), and \( z_j, z_j' \in G_{K,s,j}(x_s) \) such that \( |z_j - z_j'|_1 = L_j \). If both \( Z_{z_j,j,\alpha}(y_0) \) and \( Z_{z_j',j,\alpha}(y_0) \) are non-empty and \( 8r_i < l_i \) for all \( i < j \), then the set \( Q_{K,s,0}(x_s) \cap (Z_{z_j,j,\alpha}(y_0) \cup Z_{z_j',j,\alpha}(y_0)) \) is connected in \( G_0 \).

\textbf{Proof.} By Lemma 2.4, the sets \( Q_{K,s,0}(x_s) \cap Z_{z_j,j,\alpha}(y_0) \) and \( Q_{K,s,0}(x_s) \cap Z_{z_j',j,\alpha}(y_0) \) are connected in \( G_0 \). Since \( 8r_i < l_i \) for all \( i < j \) and \( |z_j - z_j'|_1 = L_j \), there exist \( x \in Q_{K,s,0}(x_s) \cap Z_{z_j,j,\alpha}(y_0) \) and \( x' \in Q_{K,s,0}(x_s) \cap Z_{z_j',j,\alpha}(y_0) \) such that \( |x - x'|_1 = L_0 \). The proof is complete. \( \square \)

From now on we always assume that for all \( n, l_n \) is divisible by \( r_n \) and \( l_n > 8r_n \). In the remainder of this section we study isoperimetric properties of subsets of \( Q_{K,s,0}(x_s) \) under condition (2.3).

For a graph \( G \) and a subset \( A \) of \( G \), the boundary of \( A \) in \( G \) is the subset of edges of \( G \), \( E(G) \), defined as

\[ \partial_G A = \{ \{x,y\} \in E(G) : x \in A, y \in G \setminus A \}. \]

Our aim is to prove that under assumption (2.3) and some assumptions on \( l_n \) and \( r_n \) (basically stating that \( \sum_{n \geq 0} \frac{r_n}{l_n} \) is sufficiently small), there exist \( \gamma > 0 \) such that for \( A \subset Q_{K,s,0}(x_s) \) such that \( |A| \leq \frac{1}{2} \cdot |Q_{K,s,0}(x_s)| \), \( |\partial_{Q_{K,s,0}(x_s)} A| \geq \gamma \cdot |A|^{\frac{d}{d-4}} \). This is proved in Theorem 2.8 for \( d = 2 \). In the case \( d \geq 3 \), we only prove the results for sufficiently large sets, see Theorem 2.14.

2.3 General facts about isoperimetric inequalities

Here we collect some general facts that we will frequently use.
Lemma 2.6. Let $d \geq 2$, $n_1, \ldots, n_d \geq 1$ integers with $\max_i n_i \leq N \cdot \min_i n_i$, and $C$ a positive real such that $N \cdot C^{\frac{d}{2}} < 1$. Then, for any subset $A$ of $G = \mathbb{Z}^d \cap [0, n_1) \times \cdots \times [0, n_d)$ with $|A| \leq C \cdot |G|$, $$|\partial_G A| \geq \max \left\{ \left( 1 + 2d \cdot (1 - N \cdot C^{\frac{d}{2}})^{-1} \right)^{-1} \cdot |\partial_{\mathbb{Z}^d} A|, \quad (1 - N \cdot C^{\frac{d}{2}}) \cdot |A^{\frac{d}{2^3}}| \right\}.$$ 

Proof. The proof is similar to that of [17, Proposition 2.2]. Let $\pi_i$ be the projection of $\mathbb{Z}^d$ onto the $(d - 1)$ dimensional sublattice of vertices with $i$th coordinate equal to 0. Let $P_i = \pi_i(A)$, $i'$ be a coordinate corresponding to $P_i$ with the maximal size, and $P'' = P_i'$. Let $P'' = P' \cap \pi_{i'}(G \setminus A)$, i.e., the projection of those $i'$-columns that contain vertices from both $A$ and $G \setminus A$. Note that $|\partial_G A| \geq |P''|$ and $|\partial_{\mathbb{Z}^d} A| \leq |\partial_G A| + 2d \cdot |P'|$. Also note that $|P' \setminus P''| \leq \frac{|A|}{n_i'} \leq N \cdot C^{\frac{d}{2}} \cdot |A^{\frac{d}{2^3}}|$. By the Loomis-Whitney inequality, $|A^{\frac{d}{2^3}}| \leq |P'|$. Thus, $|\partial_G A| \geq |P''| \geq (1 - N \cdot C^{\frac{d}{2}}) \cdot |P'| \geq (1 - N \cdot C^{\frac{d}{2}}) \cdot |A^{\frac{d}{2^3}}|$. We prove the claim. 

Remark 2.7. Let $G$ be a finite graph, and assume that for all $A \subseteq G$ with $c_1 \cdot |G| \leq |A| \leq \frac{1}{2} \cdot |G|$, $|\partial_G A| \geq c_2 \cdot |A|^{\frac{d}{2^3}}$. Then for any $A' \subseteq G$ with $\frac{1}{2} \cdot |G| \leq |A'| \leq (1 - c_1) \cdot |G|$, $|\partial_G A'| = |\partial_G(G \setminus A')| \geq c_2 \cdot |G \setminus A'|^{\frac{d}{2^3}} \geq (c_1 c_2) \cdot |A'|^{\frac{d}{2^3}}$. Thus, any such $A'$ also satisfies an isoperimetric inequality, but possibly with a smaller constant.

2.4 Isoperimetric inequality in two dimensions

We first prove an isoperimetric inequality for subsets of $Q_{K,s,0}(x_s)$ in two dimensions.

Lemma 2.8. Let $d = 2$. Let $l_n$ and $r_n$, $n \geq 0$, be integer sequences such that for all $n$, $l_n > 8r_n$, $l_n$ is divisible by $r_n$, and $$\prod_{j=0}^{\infty} \left( 1 - \left( \frac{4r_j}{l_j} \right)^2 \right) \geq \frac{15}{16} \quad \text{and} \quad 9600 \cdot \sum_{j=0}^{\infty} \frac{r_j}{l_j} \leq \frac{1}{10^6}. \tag{2.7}$$

Then for any integers $s \geq 0$, $L_0 \geq 1$, and $K \geq 1$, $x_s \in G_s$, and two families of events $\overrightarrow{D}$ and $\overrightarrow{E}$, if all the vertices in $G_s \cap Q_{K,s}(x_s)$ are $s$-good, then for any $A \subseteq Q_{K,s,0}(x_s)$ such that $1 \leq |A| \leq \frac{1}{2} \cdot |Q_{K,s,0}(x_s) \cap G_s|$, $$|\partial_{Q_{K,s,0}(x_s)} A| \geq \frac{1}{10^6} \cdot |A|^{\frac{1}{4}}.$$

Remark 2.9. (1) Assumptions (2.7) and the constant $\frac{1}{10^6}$ in the result of Lemma 2.8 are not optimal for our proof, but rather chosen to simplify calculations.

(2) We believe that an analogue of Lemma 2.8 holds for all $d \geq 2$, but cannot prove it. There is only one place in the proof where the assumption $d = 2$ is used, see Remark 2.10.

Proof. Fix $s \geq 0$ and $K \geq 1$ integers, $x_s \in G_s$. Recall the definition of $Q_{K,s,i}(x_s)$ from (2.6), and write $Q_i$ for $Q_{K,s,i}(x_s)$ throughout the proof. Note that $Q_s = Q_{K,s}(x_s) \cap G_0$ and for all $i$, $Q_{i-1} \subseteq Q_i$. 

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Let $A$ be a subset of $Q_0$ such that $1 \leq |A| \leq \frac{1}{2} \cdot |Q_s|$. First of all, without loss of generality we can assume that both $A$ and $Q_0 \setminus A$ are connected in $G_0$. (For the proof of this claim, see page 112 in [24, Section 3.1].)

Let $B, B_1, \ldots, B_m$ be all the connected components (in $G_0$) of $Q_s \setminus A$, of which $B$ is the unique component intersecting $Q_0$. Let $A' = A \cup \bigcup_{i=1}^{m} B_i$ and $Q' = Q_s \cup \bigcup_{i=1}^{m} B_i$. Note that $\partial_{Q'_0} A' = \partial_{Q_0} A$ and $|A'| \geq |A|$. Thus, it suffices to prove that $|\partial_{Q'_0} A'| \geq \frac{1}{100} \cdot |A'|^{\frac{1}{2}}$.

Also note that $A'$ is connected in $G_0$ and $Q'_0 \setminus A' = Q_0 \setminus A$.

By Lemma 2.4 and the first part of (2.7),

$$|A'| \leq |A| + |Q_s \setminus Q_0| \leq \frac{9}{16} \cdot |Q_s|.$$ 

Thus, by Lemma 2.6

$$|\partial_{Q_s} A'| \geq \frac{1}{4} \cdot |A'|^{\frac{1}{2}}. \quad (2.8)$$

In the remainder of the proof we show that for some $c > 0$, $|\partial_{Q_s} A'| \geq c \cdot |\partial_{Q_s} A'|$. Let

$$\delta_i = \partial_{Q'_i} A' \setminus \partial_{Q'_{i-1}} A'$$

be the set of edges with one end vertex in $A'$ and the other in $Q_i \setminus Q_{i-1}$. Note that for any $1 \leq i \leq s$,

$$|\partial_{Q'_i} A'| = |\partial_{Q'_i} A'| - \sum_{j=1}^{i} |\delta_j|. \quad (2.9)$$

Let

$$t = \max \left\{ 0 \leq i \leq s : |\partial_{Q_s} A'| \geq \frac{1}{12} \cdot \frac{L_i}{L_0} \right\}.$$ 

We first show that for all $1 \leq i \leq t$,

$$|\delta_i| \leq 9600 \cdot \frac{r_{i-1}}{t_{i-1}} \cdot |\partial_{Q_s} A'|. \quad (2.10)$$

By the definition of $Q_i$'s, the set $Q_i \setminus Q_{i-1}$ can be expressed as the disjoint union of cubes $S_j = G_0 \cap (y_j + [0, r_{i-1} L_{i-1}]^2)$, for some $y_1, \ldots, y_k \in (r_{i-1} L_{i-1}) \cdot \mathbb{Z}^2$, such that every cube $S_j$ is within $\ell^\infty$ distance $L_i$ from at most $100$ $S_j$'s.

Let $N_i$ be the total number of those $S_j$'s which are adjacent (in $G_0$) to $A'$. Since for each $j$, $|\partial_{Q_s} S_j| \leq 8 \cdot \frac{r_{i-1} L_{i-1}}{L_0}$, it follows that $|\delta_i| \leq N_i \cdot 8 \cdot \frac{r_{i-1} L_{i-1}}{L_0}$.

Consider separately the cases $N_i \leq 100$ and $N_i > 100$. If $N_i \leq 100$, then

$$|\delta_i| \leq N_i \cdot 8 \cdot \frac{r_{i-1} L_{i-1}}{L_0} \leq 800 \cdot \frac{r_{i-1}}{t_{i-1}} \cdot \frac{L_i}{L_0} \leq 9600 \cdot \frac{r_{i-1}}{t_{i-1}} \cdot |\partial_{Q_s} A'|. \quad (2.11)$$

Assume now that $N_i > 100$. In this case, $A'$ is adjacent to at least $\lceil \frac{N_i}{100} \rceil \geq 2$ of $S_j$'s which are pairwise at $\ell^\infty$ distance at least $L_i$ from each other. Let $E = \{ y \in Q_s : \{ x, y \} \in \partial_{Q_s} A' \text{ for some } x \in A' \}$ (the exterior boundary of $A'$ in $Q_s$). Since $A'$ and $Q_s \setminus A'$ are connected, by [17, Lemma 2.1(ii)], for any $x, x' \in E$ there exist $z_0 = x, z_1, \ldots, z_m = x' \in E$
such that $|z_k - z_{k+1}|_{\infty} = L_0$ for all $k$ (i.e., $\mathcal{E}$ is $*$-connected). Since $\mathcal{E}$ intersects each of the $[\frac{N}{100}]$ well separated $S_j$’s, we obtain that $|\mathcal{E}| \geq \frac{1}{3} L_0 \cdot \frac{N}{100}$. Therefore,

$$|\delta_i| \leq N_i \cdot 8 \frac{r_i - 1}{L_0} \leq 2400 \cdot \frac{r_i - 1}{l_i - 1} \cdot |\mathcal{E}| \leq 9600 \cdot \frac{r_i - 1}{l_i - 1} \cdot |\partial_{Q_i} A'|. \quad (2.12)$$

Combining $(2.11)$ and $(2.12)$ we get $(2.10)$.

**Remark 2.10.** The only step in the proof of Lemma 2.8 that uses (crucially!) the assumption $d = 2$ is the derivation of $(2.12)$, more precisely, the bound $|\mathcal{E}| \geq \frac{1}{3} L_0 \cdot \frac{N}{100}$.

If $t = s$, then by $(2.9)$ and $(2.10)$,

$$|\partial_{Q_0} A'| = |\partial_{Q_t} A'| - \sum_{i=1}^{t} |\delta_i| \geq \left(1 - 9600 \cdot \sum_{j=0}^{\infty} \frac{r_j}{l_j}\right) \cdot |\partial_{Q_0} A'|. \quad (2.13)$$

Assume next that $t < s$, i.e.,

$$\frac{1}{12} \cdot \frac{L_t}{L_0} \leq |\partial_{Q_t} A'| < \frac{1}{12} \cdot \frac{L_{t+1}}{L_0} \leq \frac{1}{12} \cdot \frac{L_s}{L_0}. \quad (2.14)$$

Consider first the case $|\partial_{Q_t} A'| > 14 \cdot 800 \cdot \frac{z_j}{L_0}$. As in the proof of $(2.10)$, the set $Q_{t+1} \setminus Q_t$ can be expressed as a disjoint union of cubes $S_j = G_0 \cap (y_j + [0, r_i L_i)^2)$, for some $y_1, \ldots, y_k \in (r_i L_i) \cdot \mathbb{Z}^2$, such that every cube is within $\ell_{\infty}$ distance $L_{t+1}$ from at most 100 of the cubes. Since $|\partial_{Q_t} A'| < \frac{1}{12} \cdot \frac{L_{t+1}}{L_0}$, the set $A'$ can be adjacent (in $G_0$) to at most 100 such cubes, which implies that

$$|\delta_{t+1}| \leq 100 \cdot 8 \frac{r_i L_t}{L_0} \leq \frac{1}{14} \cdot |\partial_{Q_t} A'|. \quad (2.15)$$

Next, we estimate $|\partial_{Q_{t+1}} A'|$ from below. By definition, $Q_{t+1}$ is the disjoint union of boxes $G_0 \cap (z_j + [0, L_{t+1})^2)$, $z_j \in G_{K,s,t+1}(x_s)$. Let $A'_{j}$ be the restriction of $A'$ to the box $(z_j + [0, L_{t+1}^2)$. By $(2.3)$ and $(2.14)$, for every $j$,

$$|A'_j| \leq |A'| \leq 16 \cdot |\partial_{Q_j} A'|^2 \leq \frac{1}{9} \cdot |G_0 \cap [0, L_{t+1})^2|.$$

By applying Lemma 2.6 in each of $G_0 \cap (z_j + [0, L_{t+1})^2)$,

$$|\partial_{Q_{t+1}} A'| \geq \sum_j \left|\partial_{G_0 \cap (z_j + [0, L_{t+1})^2)} A'_j\right| \geq \frac{1}{7} \cdot \sum_j \left|\partial_{G_0} A'_j\right| \geq \frac{1}{7} \cdot |\partial_{Q_t} A'|. \quad (2.16)$$

From $(2.9)$, $(2.10)$, $(2.15)$, and $(2.16)$,

$$|\partial_{Q_t} A'| = |\partial_{Q_{t+1}} A'| - \sum_{j=1}^{t+1} |\delta_j| \geq \left(\frac{1}{14} - 9600 \cdot \sum_{j=0}^{\infty} \frac{r_j}{l_j}\right) \cdot |\partial_{Q_0} A'|. \quad (2.17)$$
It remains to consider the case $|\partial_{Q_t} A'| \leq 14 \cdot 800 \cdot \frac{r_t L_t}{L_0}$. We estimate $|\partial_{Q_t} A'|$ from below. By Lemma 2.4, $Q_t$ can be expressed as a disjoint union of boxes $(z_j + [0, r_t L_t)^2)$, $z_j \in (r_t L_t) \cdot Z^2$. Let $A_j'$ be the restriction of $A'$ to the box $(z_j + [0, r_t L_t)^2)$.

If for all $j$, $|A_j' - \frac{1}{4} \cdot |G_0 \cap [0, r_t L_t)^2|$, then by applying Lemma 2.6 in each of $G_0 \cap (z_j + [0, r_t L_t)^2)$,

$$|\partial_{Q_t} A'| \geq \sum_j |\partial_{G_0 \cap (z_j + [0, r_t L_t)^2)} A_j' - \frac{1}{4} \cdot \sum_j |\partial_{G_0} A_j'| \geq \frac{1}{9} \cdot |\partial_{Q_t} A'|.$$   

Otherwise, there exists $\tilde{z} \in G_t$ such that

- $G_0 \cap (\tilde{z} + [0, r_t L_t)^2) \subset Q_t$ and
- $\frac{1}{4} \cdot |G_0 \cap [0, r_t L_t)^2| \leq |A' \cap (\tilde{z} + [0, r_t L_t)^2)| \leq \frac{3}{4} \cdot |G_0 \cap [0, r_t L_t)^2|$.

Indeed, if none of $z_j$’s satisfies the two requirements, then there exist $j_1$ and $j_2$ such that $|z_{j_1} - z_{j_2}|_{\infty} = r_t L_t$, $|A_{j_1}'| > \frac{3}{4} \cdot |G_0 \cap [0, r_t L_t)^2|$ and $|A_{j_2}'| \leq \frac{3}{4} \cdot |G_0 \cap [0, r_t L_t)^2|$. Then, $\tilde{z} = \lambda \cdot z_{j_1} + (1 - \lambda) \cdot z_{j_2}$ satisfies the two requirements for some $\lambda \in (0, 1)$. (If $r_t$ is divisible by 2, then one can take $\lambda = \frac{1}{2}$.)

By applying Lemma 2.6 to $G_0 \cap (\tilde{z} + [0, r_t L_t)^2)$,

$$|\partial_{Q_t} A'| \geq |\partial_{G_0 \cap (\tilde{z} + [0, r_t L_t)^2)} (A' \cap (\tilde{z} + [0, r_t L_t)^2))| \geq \left( 1 - \frac{\sqrt{3}}{2} \right) \cdot |A' \cap (\tilde{z} + [0, r_t L_t)^2)| \geq \left( 1 - \frac{\sqrt{3}}{2} \right) \cdot \frac{1}{2} \cdot \frac{r_t L_t}{L_0} \geq \frac{1}{16} \cdot \frac{r_t L_t}{L_0} \geq \frac{1}{2} \cdot 10^5 \cdot |\partial_{Q_t} A'|,$$

where the last inequality follows from the case assumption. Combining the two lower bounds with (2.9) and (2.10) gives

$$|\partial_{Q_t} A'| = |\partial_{Q_t} A'| - \sum_{j=1}^t \{|\delta_j| \geq \left( \frac{1}{2} \cdot 10^5 - 9600 \cdot \sum_{j=0}^\infty \frac{r_j}{l_j} \right) \cdot |\partial_{Q_t} A'|. \tag{2.18}$$

Putting together (2.8), (2.13), (2.17), and (2.18), and using the second part of (2.7), we obtain that

$$|\partial_{Q_0} A'| \geq \frac{1}{4} \cdot \left( \frac{1}{2} \cdot 10^5 - 9600 \cdot \sum_{j=0}^\infty \frac{r_j}{l_j} \right) \cdot |A'|^{\frac{1}{2}} \geq \frac{1}{10^5} \cdot |A'|^{\frac{1}{2}}.$$  

\[\square\]

Our next goal is to prove an isoperimetric inequality for all sufficiently large subsets of $Q_{K,e_0}(x_0)$ in any dimension $d \geq 3$. This will be done in Theorem 2.14. We postpone our goal until we prove a selection lemma, which will be useful in the proof of the theorem.
2.5 Selection lemma

The aim of this section is to prove the following lemma.

**Lemma 2.11.** Let \( \frac{6}{7} \leq C_2 < 1 \), and for \( d \geq 2 \), let

\[
C_d = \frac{C_2^{d-1}}{\prod_{j=1}^{d-2} \left( 1 + \frac{3}{9^j} \right)}, \quad \delta_d = \frac{1}{9^{d-2}}.
\]

Let \( R_1, \ldots, R_d \) be positive integers. Then, for any subset \( A \) of \( Q = [0, R_1) \times \cdots \times [0, R_d) \cap \mathbb{Z}^d \) satisfying

\[
1 \leq |A| \leq C_d \cdot |Q|,
\]

there exist \( S_1, \ldots, S_k \), disjoint two dimensional subrectangles of \( Q \) such that

\[
|A \cap \cup_i S_i| \geq \delta_d \cdot |A|
\]

and for all \( 1 \leq i \leq k \),

\[
1 \leq |A \cap S_i| \leq C_2 \cdot |S_i|.
\]

**Corollary 2.12.** Note that \( \prod_{j=1}^{d-2} \left( 1 + \frac{3}{9^j} \right) \leq e^{\sum_{j=1}^{d-1} \frac{3}{9^j}} = e^{\frac{3}{9^{d-2}}} \). Thus, if we take \( C_2 = e^{-\frac{d-2}{9^{d-2}}} > \frac{6}{7} \), then \( C_d > e^{-\frac{3}{9^{d-2}}} > \frac{7}{9} \), and Lemma 2.11 implies that for any \( A \subset Q \) with \( |A| \leq \frac{1}{2} \cdot |Q| \), there exist disjoint two dimensional rectangles \( S_1, \ldots, S_k \) such that \( |A \cap \cup_i S_i| \geq \frac{1}{9^{d-2}} \cdot |A| \) and \( 1 \leq |A \cap S_i| \leq e^{-\frac{d-2}{9^{d-2}}} \cdot |S_i| \).

**Corollary 2.13.** If \( R_1 = \cdots = R_d = R \), and \( |A| \geq c_d \cdot R^d \) for some \( c_d > 0 \), then at least \( \frac{\delta_d c_d R^d}{2} \) of the \( S_i \)'s contain at least \( \frac{\delta_d c_d R^2}{2} \) vertices from \( A \). Indeed, if such a choice did not exist, then we would have

\[
\delta_d c_d R^d \leq \delta_d \cdot |A| \leq |A \cap \cup_i S_i| < R^d \cdot \frac{\delta_d c_d R^d}{2} + \frac{\delta_d c_d R^2}{2} \cdot \left( k - \frac{\delta_d c_d R^d}{2} \right) \leq \delta_d c_d R^d.
\]

**Proof of Lemma 2.11.** The proof is by induction on \( d \). For \( d = 2 \) the statement is obvious. We assume that \( d \geq 3 \).

Consider all two dimensional slices of the form \( [0, R_1) \times [0, R_2) \times x, x \in [0, R_3) \times \cdots \times [0, R_d) \). If among them there exist slices \( S_1, \ldots, S_k \) such that \( |A \cap \cup_i S_i| \geq \delta_d \cdot |A| \) and for all \( i \), \( 1 \leq |A \cap S_i| \leq C_2 \cdot R_1 R_2 \), then we are done.

Thus, assume the contrary. Let \( S_1 \) be the subset of those slices that contain \( > C_2 \cdot R_1 R_2 \) vertices from \( A \), and \( S_2 \) the rest. By assumption, \( |A \cap \cup_{S \in S_2} S| < \delta_d \cdot |A| \). Since

\[
|S_1| \leq \frac{|A|}{C_2 \cdot R_1 R_2} \leq \frac{C_d}{C_2} \cdot \prod_{j=3}^{d} R_j,
\]

we obtain that \( |S_2| \geq (1 - \frac{C_d}{C_2}) \cdot \prod_{j=3}^{d} R_j \).

Consider \( (d - 1) \) dimensional rectangles

\[
x \times [0, R_2) \times \cdots \times [0, R_d), \quad x \in [0, R_1),
\]

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and consider separately their intersections with \( S_1 \) and \( S_2 \).

First, consider intersections with \( S_1 \). Each of these rectangles intersects \( \cup_{S \in S_1} S \) in at most \( R_2 \cdot \frac{|A|}{C_2 \cdot R_1} = \frac{|A|}{C_2 \cdot R_1} \) vertices. Since \( |A \cap \cup_{S \in S_1} S| \geq (1 - \delta_d) \cdot |A| \), at least \( \frac{2}{3} R_1 \) of these \((d - 1)\) dimensional rectangles contain \( \geq \frac{1}{3} \delta \) vertices from \( A \). Indeed, if not, then at least \( \frac{1}{3} R_1 \) of them contain \( < \frac{|A|}{3 R_1} \) vertices from \( A \), and

\[
|A \cap \cup_{S \in S_1} S| < \frac{1}{3} R_1 \cdot |A| + \frac{2}{3} R_1 \cdot \frac{|A|}{C_2 \cdot R_1} = \left( \frac{1}{9} + \frac{2}{3 \cdot C_2} \right) \cdot |A| \leq \frac{8}{9} \cdot |A| \leq (1 - \delta_d) \cdot |A|,
\]

which is a contradiction.

Next, consider intersections with \( S_2 \). Since \( |A \cap \cup_{S \in S_2} S| \leq \delta_d \cdot |A| \), at least \( \frac{2}{3} R_1 \) of these \((d - 1)\) dimensional slices contain \( \leq 3 \delta_d \cdot \frac{|A|}{R_1} \) vertices from \( A \). Indeed, if not, then at least \( \frac{1}{3} R_1 \) of them contain \( > 3 \delta_d \cdot \frac{|A|}{R_1} \) vertices from \( A \), and

\[
|A \cap \cup_{S \in S_1} S| > \frac{1}{3} R_1 \cdot 3 \delta_d \cdot \frac{|A|}{R_1} = \delta_d \cdot |A|,
\]

which is a contradiction.

Therefore, we can choose \( M_1, \ldots, M_{\frac{2}{3} R_1} \) from the above set of \((d - 1)\) dimensional rectangles such that for each \( 1 \leq i \leq \frac{1}{3} R_1 \),

\[
\frac{|A|}{3 \cdot R_1} \leq |A \cap M_i| \leq \frac{|A|}{C_2 \cdot R_1} + 3 \delta_d \cdot \frac{|A|}{R_1} \leq \frac{C_d}{C_2} \cdot \prod_{j=2}^{d} R_j \cdot \left( 1 + \frac{3}{9^{d-2}} \right) = C_{d-1} \cdot \prod_{j=2}^{d} R_j
\]

and

\[
|A \cap \cup_i M_i| \geq \frac{1}{3} R_1 \cdot \frac{|A|}{3 \cdot R_1} = \frac{|A|}{9}.
\]

If \( d = 3 \), then \( M_i \) are disjoint two dimensional rectangles satisfying all the requirements of the lemma. If \( d > 3 \), consider the sets \( A_i = A \cap M_i, 1 \leq i \leq \frac{1}{3} R_1 \). They satisfy assumption of the lemma with \( d \) replaced by \( d - 1 \). Therefore, there exist disjoint two dimensional rectangles \((S_{ij})_{1 \leq j \leq k_i}\) in \( M_i \) such that for all \( 1 \leq j \leq k_i \),

\[
|A_i \cap S_{ij}| \leq C_2 \cdot |S_{ij}|
\]

and

\[
|A_i \cap \cup_j S_{ij}| \geq \delta_{d-1} \cdot |A_i|.
\]

It is now easy to conclude that the two dimensional rectangles \((S_{ij})_{1 \leq j \leq k_i, 1 \leq i \leq \frac{1}{3} R_1}\) satisfy all the requirements of the lemma. Indeed, they are disjoint,

\[
|A \cap \cup_j S_{ij}| = \sum_i |A_i \cup_j S_{ij}| \geq \frac{1}{3} R_1 \cdot \delta_{d-1} \cdot |A_i| \geq \frac{1}{3} R_1 \cdot \delta_{d-1} \cdot \frac{|A|}{3 \cdot R_1} = \delta_d \cdot |A|,
\]

and for each \( i \) and \( j \),

\[
|A \cap S_{ij}| = |A_i \cap S_{ij}| \leq C_2 \cdot |S_{ij}|.
\]

The proof is complete. \( \square \)
2.6 Isoperimetric inequality in any dimension for large enough subsets

The goal here is to prove the following isoperimetric inequality for large enough subsets of $Q_{K,s,0}(x_s)$.

**Theorem 2.14.** Let $d \geq 2$, $c > 0$. Let $l_n$ and $r_n$, $n \geq 0$, be integer sequences satisfying assumptions of Lemma 2.8 and such that

$$\prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^2\right) \geq e^{-\frac{1}{10(d-1)}} \quad \text{and} \quad \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \geq \frac{1 - \frac{1}{2}}{1 - \frac{1}{2d}}. \quad (2.19)$$

Then for any integers $s \geq 0$, $L_0 \geq 1$, and $K \geq 1$, $x_s \in G_s$, and two families of events $\mathcal{D}$ and $\mathcal{I}$, if all the vertices in $G_s \cap Q_{K,s}(x_s)$ are $s$-good, then any $A \subseteq Q_{K,s,0}(x_s)$ with

$$\min \left\{ c \cdot |Q_{K,s} \cap G_0|, \left(\frac{L_s}{L_0}\right)^d \right\} \leq |A| \leq \frac{1}{2} \cdot |Q_{K,s} \cap G_0|$$

satisfies

$$|\partial_{Q_{K,s,0}(x_s)}A| \geq \frac{c^2}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{d}{2}} \cdot \left(1 - e^{-\frac{1}{10(d-1)}}\right) \cdot |A|^{\frac{d-1}{d}}. \quad (2.20)$$

**Proof of Theorem 2.14.** Fix $s \geq 0$ and $K \geq 1$ integers, $x_s \in G_s$, and assume that all the vertices in $G_s \cap Q_{K,s}(x_s)$ are $s$-good. Take $A \subseteq Q_{K,s,0}(x_s)$ such that $|A| \leq \frac{1}{2} \cdot |Q_{K,s} \cap G_0|$. Assume first that $|A| \geq c \cdot |Q_{K,s} \cap G_0|$. By Corollaries 2.12 and 2.13 there exist at least $\frac{c}{2 \cdot 9^d - 2}$ two dimensional subrectangles $S_i$ in $Q_{K,s} \cap G_0$ such that for all $i$, $|A \cap S_i| \geq \frac{c}{2 \cdot 9^d - 2} \cdot \left(\frac{KL_s}{L_0}\right)^2$ and $|A \cap S_i| \leq e^{-\frac{1}{10(d-1)}} \cdot \left(\frac{KL_s}{L_0}\right)^2$. By Lemma 2.4 and the first part of Lemma 2.13, $|Q_{K,s,0}(x_s) \cap S_i| \geq e^{-\frac{1}{10(d-1)}} \cdot \left(\frac{KL_s}{L_0}\right)^2$, which implies that $|A \cap S_i| \leq e^{-\frac{1}{10(d-1)}} \cdot |Q_{K,s,0}(x_s) \cap S_i|$. Thus, by Lemma 2.8 and Remark 2.7 for all $i$,

$$|\partial_{S_i}(A \cap S_i)| \geq \frac{1}{10^6} \cdot \left(1 - e^{-\frac{1}{10(d-1)}}\right) \cdot |A \cap S_i|^{\frac{d}{2}} \geq \frac{1}{10^6} \cdot \left(1 - e^{-\frac{1}{10(d-1)}}\right) \cdot \frac{c}{2} \cdot \frac{KL_s}{L_0} \cdot \frac{3^d - 2}{2^d - 2}.$$ 

Since all $\partial_{S_i}(A \cap S_i)$ are disjoint,

$$|\partial_{Q_{K,s,0}(x_s)}A| \geq \sum_i |\partial_{S_i}(A \cap S_i)| \geq \frac{c}{2} \cdot \frac{KL_s}{L_0} \cdot \frac{3^d - 2}{2^d - 2} \cdot \left(1 - e^{-\frac{1}{10(d-1)}}\right) \cdot |A|^{\frac{d-1}{d}}. \quad (2.20)$$

Thus, Theorem 2.14 is proved for sets with $|A| \geq c \cdot |Q_{K,s} \cap G_0|$. 

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Assume next that $|A| \geq \left( \frac{L}{L_0} \right)^d$. Let

$$\tilde{A}_s = \{ x \in G_s : A \cap (x + [0, L_s]^d) \neq \emptyset \},$$

$$\tilde{A}_s = \left\{ x \in G_s : |A \cap (x + [0, L_s]^d)| \geq \frac{3}{4} \left( \frac{L_0}{L_s} \right)^d \right\}.$$

Note that $|\tilde{A}_s| \geq |A| \cdot \left( \frac{L}{L_0} \right)^d$. We consider separately the cases when $|\tilde{A}_s| \geq \frac{1}{2} \cdot |\tilde{A}_s|$ and $|\tilde{A}_s| \leq \frac{1}{2} \cdot |\tilde{A}_s|$. We begin with the first case.

Since $\frac{3}{4} \cdot \left( \frac{L}{L_0} \right)^d \cdot |\tilde{A}_s| \leq |A| \leq \frac{1}{2} \cdot |Q_{K,s} \cap G_0|,

$$|\tilde{A}_s| \leq \frac{2}{3} \cdot |Q_{K,s} \cap G_0| \cdot \left( \frac{L_0}{L_s} \right)^d = \frac{2}{3} \cdot |Q_{K,s}(x) \cap G_s|.$$  

By applying Lemma 2.6 to $\tilde{A}_s \subset Q_{K,s}(x) \cap G_s$, we get

$$|\partial Q_{K,s}(x) \cap G_s \tilde{A}_s| \geq \left( 1 - \left( \frac{2}{3} \right)^\frac{d}{2} \right) \cdot |\tilde{A}_s|^{d-1}$$

Take any pair $x \in \tilde{A}_s$ and $y \in (Q_{K,s}(x) \cap G_s) \setminus \tilde{A}_s$ from $\partial Q_{K,s}(x) \cap G_s \tilde{A}_s$, and consider $z \in (Q_{K,s}(x) \cap G_s)$ such that $z + [0, 2L_s]^d \subset Q_{K,s}(x)$ and $x, y \in z + [0, 2L_s]^d$. Note that all the vertices in $(z + [0, 2L_s]^d) \cap G_s$ are $s$-good, and

$$\frac{3}{2^{d+2}} \cdot |(z + [0, 2L_s]^d) \cap G_0| \leq |A \cap (z + [0, 2L_s]^d)| \leq \left( 1 - \frac{1}{2^{d+2}} \right) \cdot |(z + [0, 2L_s]^d) \cap G_0|.$$  

The upper bound, Lemma 2.3, and the second part of (2.19) imply that $|A \cap (z + [0, 2L_s]^d)| \leq \left( 1 - \frac{1}{2^{d+2}} \right) \cdot |Q_{K,s,0}(x) \cap (z + [0, 2L_s]^d)|$. Thus, by (2.20) (with $c = \frac{3}{2^{d+2}}$) and Remark 2.7

$$|\partial Q_{K,s,0}(x) \cap (z + [0, 2L_s]^d)| (A \cap (z + [0, 2L_s]^d))$$

$$\geq \frac{1}{2^{d+3}} \cdot \left( \frac{9}{4.4^{d+2}.27d^2-2 \cdot 10^6} \left( 1 - e^{-\frac{1}{10(d-1)}} \right) \cdot |A \cap (z + [0, 2L_s]^d)|^{\frac{d-1}{d}} \right)$$

$$\geq \frac{3}{4} \cdot \left( \frac{9}{8^{d+3}.27d^2-2 \cdot 10^6} \left( 1 - e^{-\frac{1}{10(d-1)}} \right) \cdot \left( \frac{L_s}{L_0} \right)^d \right).$$

Since every edge from $\partial Q_{K,s,0}(x)A$ belongs to at most $2^d$ cubes $(z + [0, 2L_s]^d)$, $z \in G_s$, and
every such cube contains at most $d2^{d-1}$ edges from $\partial_{Q_{K,s}(x_0) \cap G_0 A_s}$.

$$|\partial_{Q_{K,s,0}(x_0)} A| \geq \frac{1}{2^d} \cdot \sum_{x \in A_s} |\partial_{Q_{K,s,0}(x_0) \cap (z+[0,2L_s])^d} (A \cap (z+[0,2L_s])^d)| \geq \frac{1}{2d} \cdot \frac{1}{d^{2d-1}} \cdot |\partial_{Q_{K,s}(x_0) \cap G_0 A_s}| \geq \frac{1}{2d} \cdot \frac{1}{d^{2d-1}} \cdot \left(1 - \left(\frac{2}{3}\right)^{\frac{r}{\alpha}} \cdot \left(1 - \frac{1}{\alpha} \cdot L_s \cdot |A| \cdot \left(\frac{L_s}{L_0}\right)^d \right) \right),$$

where the last inequality follows from the case assumption.

Next we consider the case $|\bar{A}_s| \leq \frac{1}{2} \cdot |A_s|$. For $x \in A_s \setminus \bar{A}_s$, $1 \leq |A \cap (x+[0,L_s])^d| < \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d$. By Lemma 2.4 and the second part of (2.19), $|Q_{K,s,0}(x_0) \cap (x+[0,L_s])^d| \geq \frac{3}{4} \cdot \left(\frac{L_s}{L_0}\right)^d$. Thus, $(x+[0,L_s])^d$ contains vertices from both $A$ and $Q_{K,s,0}(x_0) \setminus A$. By Lemma 2.21 $Q_{K,s,0}(x_0) \cap (x+[0,L_s])^d$ is connected in $G_0$, thus it contains an edge from $\partial_{Q_{K,s,0}(x_0)} A$.

We conclude that

$$|\partial_{Q_{K,s,0}(x_0)} A| \geq |A_s \setminus \bar{A}_s| \geq \frac{1}{2} \cdot |A_s| \geq \frac{1}{2} \cdot |A| \cdot \left(\frac{L_0}{L_s}\right)^d \geq \frac{1}{2} \cdot |A| \cdot \left(\frac{L_s}{L_0}\right)^d.$$

The proof of Theorem 2.14 in the case $|A| \geq \left(\frac{L_s}{L_0}\right)^d$ is complete by (2.21) and (2.22). □

Remark 2.15. We believe that Theorem 2.14 holds for all $A$ with $|A| \leq \frac{1}{2} \cdot |Q_{K,s} \cap G_0|$. With a more involved proof, we can relax the assumption $|A| \geq \left(\frac{L_s}{L_0}\right)^d$ of Theorem 2.14 to $|A| \geq \left(\frac{L_s}{L_0}\right)^{2d}$. Since this does not give us the result for all $A$, and the current statement of Theorem 2.14 suffices for the applications in this paper, we do not include this proof here.

3 Properties of the largest clusters

In this section we study properties of the largest subset of $S \cap Q_{K,s}$ (where $Q_{K,s}$ is defined in (2.22)). We first define two families of events such that the corresponding perforated lattices defined in (2.6) are unlikely to have big holes (Lemma 3.2) and serve as a “skeleton” of the largest subset of $S \cap Q_{K,s}$. Then, we provide sufficient conditions for the uniqueness of the largest subset of $S \cap Q_{K,s}$ (Lemma 3.3). To avoid problems, which may be caused by roughness of the boundary of the largest subset of $S \cap Q_{K,s}$, we enlarge it by adding to it all the points of $S$ which are locally connected to it (Definition 3.6). For the enlarged
set we prove under some general conditions (Definition 3.8) that its subsets satisfy an isoperimetric inequality (Theorem 3.9 and Corollary 3.11). Under the same condition we prove that the graph distance is controlled by that on \( \mathbb{Z}^d \) (Lemma 3.15), large enough balls have regular volume growth (Corollary 3.16) and have local extensions satisfying an isoperimetric inequality (Corollary 3.17).

### 3.1 Special sequences of events

Recall Definition 1.8 of \( S_r \). For \( u \in (a, b) \), define two families of events \( \overline{D}^u = (\overline{D}^u_{x,L_0}, L_0 \geq 1, x \in \mathbb{G}_0) \) and \( \overline{T}^u = (\overline{T}^u_{x,L_0}, L_0 \geq 1, x \in \mathbb{G}_0) \) as follows.

- The complement of \( \overline{D}^u_{x,L_0} \) is the event that for each \( y \in \mathbb{G}_0 \) with \( |y - x| \leq L_0 \), the set \( S_{L_0} \cap (y + [0, L_0]^d) \) contains a connected component \( C_y \) with at least \( \frac{3}{4} \eta(u)L_0^d \) vertices such that for all \( y \in \mathbb{G}_0 \) with \( |y - x| \leq L_0 \), \( C_y \) and \( C_x \) are connected in \( S \cap ((x + [0, L_0]^d) \cup (y + [0, L_0]^d)) \).
- The event \( \overline{T}^u_{x,L_0} \) occurs if \( |S_{L_0} \cap (x + [0, L_0]^d)| > \frac{5}{4} \eta(u)L_0^d \).

Note that \( \overline{D}^u_{x,L_0} \) are decreasing and \( \overline{T}^u_{x,L_0} \) increasing events. From now on we fix these two local families, and say that \( x \in \mathbb{G}_n \) is \( n \)-bad / \( n \)-good, if it is \( n \)-bad / \( n \)-good for the two local families \( \overline{D}^u \) and \( \overline{T}^u \) in the sense of Definition 2.1. In particular, \( x \in \mathbb{G}_0 \) is 0-good if both \( \overline{D}^0_{x,L_0} \) and \( \overline{T}^0_{x,L_0} \) do not occur.

The following lemma is immediate from the definition of 0-good vertex (see also [19, Lemma 5.2]).

**Lemma 3.1.** Let \( L_0 \geq 1 \) and \( u \in (a, b) \).

(a) For any 0-good vertex \( x \in \mathbb{G}_0 \), connected component \( C_x \) in \( S_{L_0} \cap (x + [0, L_0]^d) \) with at least \( \frac{3}{4} \eta(u)L_0^d \) vertices is defined uniquely.

(b) For any 0-good \( x, y \in \mathbb{G}_0 \) with \( |x - y| \leq L_0 \), (uniquely chosen) \( C_x \) and \( C_y \) are connected in the graph \( S \cap ((x + [0, L_0]^d) \cup (y + [0, L_0]^d)) \).

The following result is essentially [19, Lemmas 4.2 and 4.4]. (Very minor modifications are needed, since the events \( \overline{D}^u_{x,L_0} \) and \( \overline{T}^u_{x,L_0} \) slightly differ from the corresponding events \( \overline{A}^u_{x} \) and \( \overline{B}^u_{x} \) in [19].)

**Lemma 3.2.** Assume that the measures \( \mathbb{P}^u \), \( u \in (a, b) \), satisfy conditions \( P1 - P3 \) and \( S1 - S2 \). Let \( l_0, r_0, \) and \( L_0 \) be positive integers. Let

\[
\theta_{\varepsilon} = \lfloor 1/\varepsilon_p \rfloor, \quad l_n = l_0 \cdot 4^n \theta_{\varepsilon}, \quad r_n = r_0 \cdot 2^n \theta_{\varepsilon}, \quad L_n = l_{n-1} \cdot L_{n-1}, \quad n \geq 1, \quad (3.1)
\]

where \( \varepsilon_p \) is defined in \( P3 \).

For each \( u \in (a, b) \) there exist \( C = C(u) < \infty \) and \( C' = C'(u, l_0) < \infty \) such that for all \( l_0, r_0 \geq C, L_0 \geq C' \), and \( n \geq 0 \),

\[
\mathbb{P}^u [0 \text{ is } n \text{-bad}] \leq 2 \cdot 2^{-2^n}.
\]
3.2 Uniqueness of the largest cluster

Definition 3.3. Let \((L_n)_{n \geq 0}\) be an increasing sequence of scales. For \(x \in \mathbb{Z}^d\) and \(r \geq 1\), let \(C_{K,s,r}(x)\) be the largest connected component in \(\mathcal{S}_r \cap Q_{K,s}(x)\) (with ties broken arbitrarily), and write \(C_{K,s,r} = C_{K,s,r}(0)\).

Fix \(u \in (a, b)\) and two families of events \(\mathcal{D}^u\) and \(\mathcal{F}^u\) as in Section 3.1.

Lemma 3.4. Let \(l_n\) and \(r_n\) be integer sequences such that for all \(n\), \(l_n\) is divisible by \(r_n\), \(l_n > 8r_n\), and

\[
\prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) > 1 + \frac{5}{2}\eta(u).
\] (3.2)

Let \(L_0 \geq 1\), \(K \geq 1\), and \(s \geq 0\) integers, \(x_s \in \mathbb{G}_s\). If all the vertices in \(\mathbb{G}_s \cap Q_{K,s}(x_s)\) are \(s\)-good, then \(C_{K,s,L_0}(x_s)\) is uniquely defined.

Proof. Without loss of generality we assume that \(x_s = 0\). Since all vertices in \(\mathbb{G}_s \cap Q_{K,s}\) are \(s\)-good, we can define the connected (in \(G\)) union of \(Q_{K,s,0}\) by (3.3). By Lemma 3.1 for any \(x \in Q_{K,s,0}\), there is a uniquely defined connected subset \(C_x\) of \(\mathcal{S}_L\) with at least \(\frac{5}{4}\eta(u)L_0^d\) vertices. Since \(Q_{K,s,0}\) is connected in \(\mathbb{G}_0\), by Lemma 3.1 the set \(\bigcup_{x \in Q_{K,s,0}} C_x\) is connected in \(\mathcal{S}_L\) and

\[
\left| \bigcup_{x \in Q_{K,s,0}} C_x \right| \geq \frac{3}{4}\eta(u) \cdot \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K,s}|.
\] (3.3)

On the other hand, since for any 0-good vertex \(x\), the set \(x + [0, L_0)^d\) contains at most \(\frac{5}{4}\eta(u)L_0^d\) vertices from \(\mathcal{S}_L\),

\[
|\mathcal{S}_L \cap Q_{K,s}| \leq \frac{5}{4}\eta(u)L_0^d \cdot |Q_{K,s,0}| + L_0^d \cdot (|Q_{K,s} \cap \mathbb{G}_0| - |Q_{K,s,0}|) \\
\leq \left(\frac{5}{4}\eta(u) + \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right)\right) \cdot |Q_{K,s}| < \frac{3}{2}\eta(u) \cdot \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K,s}|.
\] (3.4)

Thus, \(\bigcup_{x \in Q_{K,s,0}} C_x\) is a connected subset of \(\mathcal{S}_L \cap Q_{K,s}\), and

\[
\left| \bigcup_{x \in Q_{K,s,0}} C_x \right| > \frac{1}{2} \cdot |\mathcal{S}_L \cap Q_{K,s}|.
\] (3.5)

The result follows. □

Corollary 3.5. It is immediate from (3.5) that under the conditions of Lemma 3.4,

\[
\bigcup_{x \in Q_{K,s,0}} C_x \subseteq C_{K,s,L_0}.
\] (3.6)

In particular, for any \(1 \leq K' \leq K'' \leq K\) and \(x', x'' \in \mathbb{G}_s \cap Q_{K,s}\) such that \(Q_{K',s}(x') \subseteq Q_{K'',s}(x'') \subseteq Q_{K,s}(x) \subseteq C_{K',s,0}(x') \subseteq C_{K'',s,0}(x'') \subseteq C_{K,s,0}\).
3.3 Isoperimetric inequality

In this section we prove an isoperimetric inequality for subsets of a certain extension \( \tilde{C}_{K,s,L_0}(x) \) of \( C_{K,s,L_0}(x) \) obtained by adding to \( C_{K,s,L_0}(x) \) all the vertices to which it is locally connected.

**Definition 3.6.** Let \( E_{K,s,r}(x) \) be the set of vertices \( y' \in S \) such that for some \( y \in C_{K,s,r}(x) \), \( y' \) is connected to \( y \) in \( S \cap B(y, 2L_s) \), and define
\[
\tilde{C}_{K,s,r}(x) = C_{K,s,r}(x) \cup E_{K,s,r}(x).
\]

**Remark 3.7.** Mind that \( \tilde{C}_{K,s,r}(x) \) is contained in \( x + [-2L_s, (K + 2)L_s]^d \), but it is different from the largest cluster of \( \mathcal{S}_s \cap (x + [-2L_s, (K + 2)L_s]^d) \).

We study isoperimetric properties of \( \tilde{C}_{K,s,L_0}(x) \) for configurations from the following event.

**Definition 3.8.** Let \( u \in (a, b) \), \( K \geq 1 \) and \( s \geq 0 \) integers, \( x_s \in \mathbb{G}_s \). The event \( \mathcal{H}_{K,s}^u(x_s) \in F \) occurs if
\begin{itemize}
  \item[(a)] all the vertices in \( \mathbb{G}_s \cap (x_s + [-2L_s, (K + 2)L_s]^d) \) are \( s \)-good,
  \item[(b)] any \( x, y \in S_{L_s} \cap Q_{K,s}(x_s) \) with \( |x - y|_\infty \leq L_s \) are connected in \( S \cap B(x, 2L_s) \).
\end{itemize}

We write \( \mathcal{H}_{K,s}^u \) for \( \mathcal{H}_{K,s}^u(0) \).

Here is the main result of this section.

**Theorem 3.9.** Let \( u \in (a, b) \). Assume that the sequences \( l_n \) and \( r_n \) satisfy the conditions of Theorem 2.14 and
\[
\prod_{i=0}^{\infty} \left( 1 - \left( \frac{4r_n}{l_i} \right)^d \right) \geq \frac{1 + \frac{5}{4} \eta(u)}{1 + \frac{4}{3} \eta(u)}. \tag{3.7}
\]
Let \( L_0 \geq 1 \), \( K \geq 1 \), and \( s \geq 0 \) integers, \( x_s \in \mathbb{G}_s \). If \( \mathcal{H}_{K,s}^u(x_s) \) occurs, then \( \tilde{C}_{K,s,L_0}(x_s) \) is uniquely defined and there exists \( \tilde{3.9} = \tilde{3.9} L_0 \in (0, 1) \) such that for any \( A \subseteq \tilde{C}_{K,s,L_0}(x_s) \) with \( L_s^{d(d+1)} |A| \leq \frac{1}{2} \cdot |\tilde{C}_{K,s,L_0}(x_s)| \),
\[
|\partial_{\tilde{C}_{K,s,L_0}(x_s)} A| \geq \tilde{3.9} \cdot |A|^\frac{d}{d+1}.
\]

**Remark 3.10.** With a more careful analysis and assuming that Theorem 2.14 holds for all subsets of size at least \( \left( \frac{r_n}{l_i} \right)^{2d} \) (see Remark 2.15), condition on \( A \) in Theorem 3.9 can be relaxed to \( |A| \geq L_s^{4d} \). Assuring that Theorem 2.14 holds for all subsets (see Remark 2.15), condition on \( A \) in Theorem 3.9 can be relaxed to \( |A| \geq L_s^d \). Since for our purposes the current statement of Theorem 3.9 suffices, we do not prove the stronger statement here.

**Corollary 3.11.** Let \( u \in (a, b) \). Assume that the sequences \( l_n \) and \( r_n \) satisfy the conditions of Theorem 3.9. Assume that \( K \geq L_s^{d(d+1)-1} \). If \( \mathcal{H}_{K,s}^u(x_s) \) occurs, then for any \( A \subseteq \tilde{C}_{K,s,L_0}(x_s) \) with \( |A| \leq \frac{1}{2} \cdot |\tilde{C}_{K,s,L_0}(x_s)| \),
\[
|\partial_{\tilde{C}_{K,s,L_0}(x_s)} A| \geq \frac{3.9}{(K+4)L_s} \cdot |A|.
\]
Claim 3.12. Any \( x, y \in \tilde{C}_{K,s,L_0} \) are connected in \( \tilde{C}_{K,s,L_0} \).

Proof. Fix \( G \) with \( |x| \leq L_s \), then by Theorem 3.9, \( |\partial_{\tilde{C}_{K,s,L_0}(x)}| \geq \frac{(1 + 4)}{(K + 4)L_s} \cdot |A| \). If \( |A| \geq L_s^{d+1} \), then by Theorem 3.9, \( |\partial_{\tilde{C}_{K,s,L_0}(x)}| \geq \frac{3}{|A|^d} \cdot \frac{1}{(K + 4)L_s} \cdot |A| \). The first statement is proved.

The proof of Theorem 3.9 is subdivided into several claims. In Claim 3.12 we prove that \( \tilde{C}_{K,s,L_0} \) is locally connected and in Claims 3.13 and 3.14 we reduce the isoperimetric problem for subsets of \( \tilde{C}_{K,s,L_0} \) to the one for subsets of a perforated lattice.

Claim 3.12. Any \( x, y \in \tilde{C}_{K,s,L_0} \) with \( |x - y| \leq L_s \) are connected in \( \tilde{C}_{K,s,L_0} \cap B(x, 15L_s) \).

Proof. Fix \( x, y \in \tilde{C}_{K,s,L_0} \) with \( |x - y| \leq L_s \), and take \( x', y' \in \tilde{C}_{K,s,L_0} \) such that \( x \) and \( x' \) are connected in \( \tilde{C}_{K,s,L_0} \cap B(x', 2L_s) \), \( y \) and \( y' \) are connected in \( \tilde{C}_{K,s,L_0} \cap B(y', 2L_s) \). By the triangle inequality, \( |x - y'| \leq 5L_s \).

Since all the vertices in \( G_u \cap Q_{K,s} \) are \( s \)-good, there exist \( x'', y'' \in \bigcup z \in Q_{K,s,0} C_z \) such that \( |x' - x''| \leq L_s \) and \( |y' - y''| \leq L_s \). By the definitions of \( \mathcal{H}_{K,s} \) and \( \tilde{C}_{K,s,L_0} \), \( x'' \) is connected to \( x' \) in \( \tilde{C}_{K,s,L_0} \cap B(x', 2L_s) \) and \( y'' \) is connected to \( y' \) in \( \tilde{C}_{K,s,L_0} \cap B(y', 2L_s) \). By the triangle inequality, \( |x'' - y'| \leq 7L_s \). By Corollary 2.5, the fact that all the vertices in \( G_u \cap Q_{K,s} \) are \( s \)-good, and (3.6), \( x'' \) is connected to \( y'' \) in \( \tilde{C}_{K,s,L_0} \cap B(x'', 8L_s) \). Thus, \( x \) is connected to \( y \) in \( \tilde{C}_{K,s,L_0} \cap B(x, 15L_s) \), and the claim is proved.

Let 
\[
x' = (-2L_s, \ldots, -2L_s) \in G_u \quad \text{and} \quad K' = K + 4.
\]
Since all the vertices in \( G_u \cap Q_{K',s,0} \) are \( s \)-good, we can define the set \( Q_{K',s,0} (x'') \) of \( s \)-good vertices in \( G_u \cap Q_{K',s,0} \) as in (2.6). By the fact that (3.7) implies (3.2), Lemma 3.1, Corollary 2.5 (3.6), and the definition of \( \tilde{C}_{K,s,L_0} \),

\[
\bigcup_{x \in Q_{K',s,0}(x')} C_x \subseteq \tilde{C}_{K,s,L_0}.
\]

The next two claims allow to reduce the isoperimetric problem for subsets of \( \tilde{C}_{K,s,L_0} \) to the isoperimetric problem for subsets of \( Q_{K',s,0}(x'') \). For \( A \subseteq \tilde{C}_{K,s,L_0} \), let \( A \) be the set of all \( x \in Q_{K',s,0}(x'') \) such that \( C_x \subseteq A \), and \( A' \) the set of \( x \in A \) such that there exists \( y \in \tilde{C}_{K,s,L_0} \setminus A \) with \( |x - y| \leq L_s \).

Claim 3.13.

\[
|\partial_{\tilde{C}_{K,s,L_0}} A| \geq \max \left\{ \frac{1}{2^d} \cdot |\partial_{\tilde{C}_{K',s,0,s}(x'')} A|, \frac{|A'|}{31 \cdot L_s^d} \right\}
\]

and

\[
|A| \leq 2 \cdot 3^d \cdot L_0^d \cdot |A| + |A'|.
\]

Proof. We begin with the proof of (3.3). For any \( x \in A \) and \( y \in Q_{K',s,0}(x'') \setminus A \) such that \( |x - y| \leq L_0 \), \( C_x \subseteq A \) and \( C_y \not\subseteq A \). By Lemma 3.1 and (3.8), \( C_x \) and \( C_y \) are connected in

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\( \tilde{C}_{K,s,L_0} \cap ((x + [0,L_0]^d) \cup (y + [0,L_0]^d)) \). Each path in \( \tilde{C}_{K,s,L_0} \) connecting \( C_x \) and \( C_y \setminus A \) contains an edge from \( \partial \tilde{C}_{K,s,L_0} A \). This implies that

\[
|\partial \tilde{C}_{K,s,L_0} A| \geq \frac{1}{2d} \cdot |\partial_{Q_{K',s,0}(x'_s)} A|.
\]  

(3.11)

Next, by the definition of \( A' \), for any \( x \in A' \), there exists \( y \in \tilde{C}_{K,s,L_0} \setminus A \) such that \( |x-y|_\infty \leq L_s \). By Claim 3.12, \( x \) and \( y \) are connected in \( \tilde{C}_{K,s,L_0} \cap B(x,15L_s) \). In particular, the ball \( B(x,15L_s) \) contains an edge from \( \partial \tilde{C}_{K,s,L_0} A \). Since every edge from \( \partial \tilde{C}_{K,s,L_0} A \) is \( \ell_\infty \) distance \( 15L_s \) from at most \((32L_s)^d \) vertices of \( A' \),

\[
|\partial \tilde{C}_{K,s,L_0} A| \geq \frac{|A'|}{(31 \cdot L_s)^d}.
\]  

(3.12)

Inequalities (3.11) and (3.12) imply (3.9).

We proceed with the proof of (3.10). We need to show that

\[
|A \setminus A'| \leq 2 \cdot 3^d \cdot L_0^d \cdot |A|.
\]  

(3.13)

Let \( z \in A \setminus A' \). By the definition of \( \tilde{C}_{K,s,L_0} \), there exists \( z_s \in 2_G \cap Q_{K',s,0}(x'_s) \) such that \( z_s + [0,L_s]^d \subset B(z,L_s) \). By the definition of \( A' \) and (3.8), for any \( x \in Q_{K',s,0}(x'_s) \), \( C_x \subset A \). Thus, \( Q_{K',s,0}(x'_s) \cap (z_s + [0,L_s]^d) \subset A \). By Lemma 2.4 and (3.7),

\[
|Q_{K',s,0}(x'_s) \cap (z_s + [0,L_s]^d)| \geq \frac{1}{2} \cdot \left( \frac{L_s}{L_0} \right)^d.
\]

Thus,

\[
|A \cap B(z,L_s)| \geq \frac{1}{2} \cdot \left( \frac{L_s}{L_0} \right)^d,
\]

and we conclude that

\[
\frac{1}{2} \cdot \left( \frac{L_s}{L_0} \right)^d \cdot |A \setminus A'| \leq \{|z \in A \setminus A', x \in A : x \in B(z,L_s)| \} \leq |B(0,L_s)| \cdot |A|,
\]

which implies (3.13).

Claim 3.14.

\[
\max \left\{ |\partial_{Q_{K',s,0}(x'_s)} A|, \frac{|A'|}{L_s^d} \right\} \geq \frac{1}{9} \cdot \left( \frac{2}{3} \right)^{\frac{d-1}{2}} \cdot \left( 1 - e^{-\frac{1}{16(d-1)}} \right).
\]  

(3.14)

Let \( \gamma_{2.14} \) be the isoperimetric constant from Theorem 2.14 (\( c = 1 \)):

\[
\gamma_{2.14} \geq \frac{1}{2d \cdot 32^d \cdot 27^d \cdot 10^6} \cdot \left( 1 - \left( \frac{2}{3} \right)^{\frac{d-1}{2}} \right) \cdot \left( 1 - e^{-\frac{1}{16(d-1)}} \right).
\]
Proof. If $|A|^{d-1} < \frac{|A|}{L_s^d}$, then (3.14) trivially holds. Thus, we assume that $|A|^{d-1} \geq \frac{|A|}{L_s^d}$. We will deduce (3.14) from Theorem 2.14. By (3.12),

$$|A| \leq 2 \cdot 3^d \cdot L_0^d \cdot |A| + L_s^d \cdot |A|^{\frac{d-1}{d}} \leq 3^d \cdot L_s^d \cdot |A|.$$ 

Since $|A| \geq L_s^d \cdot 2^{d+1}$, we obtain that $|A| \geq \left(\frac{|A|}{L_s^d}\right)^{d-1}$.

Since $A \subseteq Q_{K',s,0}(x'_s)$, for all $x \in A$, $|C_x| \geq \frac{3}{4} \eta(u) L_0^d$. Thus, $|A| \geq \frac{3}{4} \eta(u) L_0^d \cdot |A|$. Since also all the vertices in $Q_s \cap Q_{K',s}(x'_s)$ are s-good, we obtain as in (3.4) that

$$|A| \leq \frac{1}{2} |C_{K,s,L_0}| \leq \frac{1}{2} \left(\frac{5}{4} \eta(u) + 1 \cdot \inf \left|1 - \left(\frac{4r_i}{l_i}\right)^d\right|\right) \cdot |Q_{K',s}(x'_s)|$$

$$\leq \frac{2}{3} \eta(u) \cdot \prod_{i=0}^{\infty} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \cdot |Q_{K',s}(x'_s)| \leq \frac{2}{3} \eta(u) L_0^d \cdot |Q_{K',s,0}(x'_s)|,$$

where the last inequality follows from Lemma 2.3. Thus, $|A| \leq \frac{3}{8} \cdot |Q_{K',s,0}(x'_s)|$. By Theorem 2.14 and Remark 2.7,

$$|\partial Q_{K',s,0}(x'_s)| \geq \frac{1}{9} \cdot |A|^{d-1},$$

completing the proof of (3.14). \qed

We are now ready to prove Theorem 3.9. It easily follows from Claims 3.13 and 3.14.

Proof of Theorem 3.9. By (3.9), (3.10), and (3.14),

$$|\partial Q_{K,s,L_0} A| \leq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \max \left\{ |A|^{d-1}, \frac{|A'|}{L_s^d} \right\} \leq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \max \left\{ |A|^{d-1}, \frac{|A'|}{L_s^d} \right\} \leq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \frac{|A|^{d-1}}{L_s^d}.$$ 

On the one hand, if $L_0^d \cdot |A| \geq |A'|$, then

$$|\partial Q_{K,s,L_0} A| \geq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \frac{|A|^{d-1}}{L_0^d} \cdot |A|^{\frac{d-1}{d}} \geq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \frac{|A|^{d-1}}{L_0^d} \cdot |A|^{\frac{d-1}{d}}.$$ 

On the other hand, if $L_0^d \cdot |A| \leq |A'|$, then by (3.10), $|A'| \geq \frac{1}{3^{d+1}} \cdot |A| \geq \frac{1}{3^{d+1}} \cdot L_s^{d(d+1)} \geq L_s^d$, and

$$|\partial Q_{K,s,L_0} A| \leq \frac{1}{9 \cdot 3^{d-1}} \cdot \frac{2.14}{2.7} \cdot \frac{|A'|^{\frac{d}{d}}}{3^d \cdot L_s^d} \geq \frac{1}{9 \cdot 93^d} \cdot \frac{2.14}{2.7}.$$ 

The proof of Theorem 3.9 is complete with $3.9 = \frac{1}{9 \cdot 93^d \cdot L_0^d} \cdot 2.14$. \qed

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3.4 Graph distance

In this section we study the graph distances $d_S$ in $S$ between vertices of $C_{K,s,L_0}(x_s)$ for configurations in $\mathcal{H}_{K,s}^n(x_s)$. As consequences, we prove that large enough balls centered at vertices of $C_{K,s,L_0}(x_s)$ have regular volume growth (Corollary 3.16) and allow for local extensions which satisfy an isoperimetric inequality (Corollary 3.17). These results will be used in Section 4 to prove our main result.

**Lemma 3.15.** Let $d \geq 2$ and $u \in (a,b)$. Let $l_n$ and $r_n$, $n \geq 0$, be integer sequences such that for all $n$, $l_n > 16r_n$ and $\prod_{n \geq 0} \left( 1 + \frac{32r_n}{l_n} \right) \leq 2$. Let $L_0 \geq 1$, $K \geq 1$, and $s \geq 0$ integers, $x_s \in \mathcal{G}_s$. There exists $C = C_{3.15} = C(3.15)\leq 3.15(1)\leq 3.15(2)$ such that if $\mathcal{H}_{K,s}^n(x_s)$ occurs, then for all $y, y' \in C_{K,s,L_0}(x_s)$,

$$d_S(y, y') \leq C \cdot \max \{ |y - y'|_{C_1}, L_s \}.$$ 

**Proof.** Let $y_s, y'_s \in Q_{K,s}(x_s) \cap \mathcal{G}_s$ be such that $(y_s + [0, L_s]^d) \subset B(y, L_s)$ and $(y'_s + [0, L_s]^d) \subset B(y', L_s)$. By [19, Lemma 5.3] (applied to sequences $l_n$ and $4r_n$), there exist $y_0 \in Q_{K,s,0}(x_s) \cap (y_s + [0, L_s]^d)$ and $y_0 \in Q_{K,s,0}(x_s) \cap (y'_s + [0, L_s]^d)$ which are connected by a nearest neighbor path in $Q_{K,s,0}(x_s)$ of at most $\prod_{n \geq 0} \left( 1 + \frac{32r_n}{l_n} \right) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$ vertices.

Using the definition of 0-good vertices and Lemma 3.1, any vertices $\tilde{y} \in C_{y_0}$ and $\tilde{y}' \in C_{y'_0}$ are connected by a nearest neighbor path in $S$ of at most $L_0^d \cdot \prod_{n \geq 0} \left( 1 + \frac{32r_n}{l_n} \right) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$ vertices. Finally, by the definition of $\mathcal{H}_{K,s}^n(x_s)$, $y$ is connected to $y_0$ in $S \cap B(y, L_s)$ and $y'$ is connected to $y'_0$ in $S \cap B(y', L_s)$. Thus, $y$ is connected to $y'$ by a nearest neighbor path in $S$ of at most $2 \cdot |B(0, L_s)| + L_0^d \cdot \prod_{n \geq 0} \left( 1 + \frac{32r_n}{l_n} \right) \cdot \frac{|y_s - y'_s|_1 + L_s}{L_0}$ vertices. Since $|y_s - y'_s|_1 \leq d \cdot |y - y'|_{C_1} + 2dL_s$, the result follows. \hfill \square

**Corollary 3.16.** In the setup of Lemmas 3.4 and 3.15 there exists $C = C(3.15)(u, L_0) > 0$ such that for any $C = C_{3.15} \leq C(3.15)$, $r \leq KL_s$, and $y \in C_{K,s,L_0}(x_s)$,

$$\mu(B_S(y, r)) \geq C \cdot r^d.$$ 

**Proof.** Let $K' = \max \{ k : kL_s \leq \frac{r}{3.15} \}$. There exists $y_s \in Q_{K,s}(x_s) \cap \mathcal{G}_s$ such that $Q_{K',s}(y_s) \subset B(y, \frac{r}{3.15}) \cap Q_{K,s}(x_s)$. Since $\mathcal{H}_{K,s}^n(x_s)$ occurs, we can define $Q_{K,s,0}(x_s)$ as in (2.6), and by (3.6),

$$\bigcup_{x \in Q_{K,s,0}(x_s) \cap Q_{K',s}(y_s)} C_x \subset C_{K,s,L_0}(x_s).$$

Since also $\bigcup_{x \in Q_{K,s,0}(x_s) \cap Q_{K',s}(y_s)} C_x \subset B(y, \frac{r}{3.15})$, Lemma 3.15 implies that

$$\bigcup_{x \in Q_{K,s,0}(x_s) \cap Q_{K',s}(y_s)} C_x \subset B_S(y, r).$$


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By Lemma 2.4 and using the fact that $|C_x| \geq \frac{3}{4} \eta(u) L_s^d$, we conclude that

$$|B_s(y,r)| \geq \frac{3}{4} \eta(u)(K'L_s)^d \cdot \prod_{i \geq 0} \left(1 - \left(\frac{4r_i}{l_i}\right)^d\right) \geq \frac{3}{8} \eta(u) \cdot \frac{1}{(2C_{\frac{3}{4}})^d} \cdot r^d.$$ 

Since $\mu(B_s(y,r)) \geq |B_s(y,r)|$, the result follows with $c_{\frac{3}{4}} = \frac{3}{8} \eta(u) \cdot \frac{1}{(2C_{\frac{3}{4}})^d}$. \(\square\)

**Corollary 3.17.** In the setup of Theorem 1.13 and Lemma 2.14, if $\mathcal{H}_{5K_s}^u(x'_s)$ occurs with $x'_s = x_s + (-2KL, \ldots, -2KL)$, then for all $L_s^{(d+1)} \leq r \leq K'L_s$ and $y \in C_{K,s,L_0}(x_s)$, there exists $\mathcal{F}_{B_s}(y,r)$ such that $B_s(y,r) \subseteq \mathcal{F}_{B_s}(y,r) \subseteq B_s(y,8C_{\frac{3}{4}})$ and for all $A \subseteq B_s(y,r)$ with $|A| \leq \frac{1}{2} \cdot |C_{B_s}(y,r)|$, $|\partial \mathcal{F}_{B_s}(y,r) A| \geq \frac{359}{8r} |A|$.

**Proof.** Let $K' = \min\{k : kL_s \geq 2r + 1\} + 1$. (Note that $K'L_s \leq 4r$.) For $y \in C_{K,s,L_0}(x_s)$, let $y_s \in G_s \cap Q_{5K_s}(x'_s)$ be such that $B(y,r) \subseteq Q_{K'}(y_s) \subseteq Q_{5K_s}(x'_s)$. Since $\mathcal{H}_{K'}(y_s)$ occurs, by Corollary 3.11 $B_s(y,r) \subseteq C_{K',s,L_0}(y_s) \subseteq C_{K',s,L_0}(y_s)$.

By Lemma 3.15 for $r \geq L_s$, $\mathcal{C}_{K',s,L_0}(y_s) \subseteq B_s(y,359(K'+4)L_s) \subseteq B_s(y,8C_{\frac{3}{4}})$. By Corollary 3.11 since $K' \geq L_s^{(d+1)}$, for any $A \subseteq \mathcal{C}_{K',s,L_0}(y_s)$ with $|A| \leq \frac{1}{2} |\mathcal{C}_{K',s,L_0}(y_s)|$, $|\partial \mathcal{C}_{K',s,L_0}(y_s) A| \geq \frac{359}{(K'+4)L_s} \cdot |A| \geq \frac{359}{8r} \cdot |A|$.

The proof is complete by taking $c_{B_s}(y,r) = \mathcal{C}_{K',s,L_0}(y_s)$.

\(\square\)

### 4 Proof of Theorem 1.13

In this section we collect together, on the one hand, the deterministic results that large enough balls have regular volume growth (Corollary 3.16) and allow for local extensions satisfying an isoperimetric inequality (Corollary 3.17), and on the other, the bound on the probability of (un)successful renormalization (Lemma 3.2), to deduce Theorem 1.13. In fact, the result that we prove here is stronger. In Definition 4.1 we introduce the notions of regular and very regular balls, so that (very) regular ball is always (very) good (see Claim 4.2), and then prove in Proposition 4.3 that large balls are likely to be very regular. The main result is an immediate consequence of Proposition 4.3.

**Definition 4.1.** Let $C_V$, $C_P$, and $C_W \geq 1$ be fixed constants. For $r \geq 1$ integer and $x \in V(G)$, we say that $B_G(x,r)$ is $(C_V,C_P,C_W)$-regular if $\mu(B_G(x,r)) \geq C_V r^d$ and there exists a set $C_{B_G}(x,r)$ such that $B_G(x,r) \subseteq C_{B_G}(x,r) \subseteq B_G(x,C_Wr)$ and for any $A \subseteq C_{B_G}(x,r)$ with $|A| \leq \frac{1}{2} \cdot |C_{B_G}(x,r)|$, $|\partial C_{B_G}(x,r) A| \geq \frac{1}{r \sqrt{C_P}} |A|$.

We say $B_G(x,r)$ is $(C_V,C_P,C_W)$-very regular if there exists $N_{B_G}(x,r) \leq r^{\frac{1}{2}}$ such that $B_G(y,r)$ is $(C_V,C_P,C_W)$-regular whenever $B_G(y,r) \subseteq B_G(x,r)$, and $N_{B_G}(x,r) \leq r \leq r$.

**Claim 4.2.** If $B_G(x,r)$ is $(C_V,C_P,C_W)$-regular, then it is $(C_V,C_P,C_W)$-good.
Proof. By [23 Proposition 3.3.10] and Remark 1.2
\[
\min a \int_{\mathbb{B}_G(x,r)} (f - a)^2 d\mu = \int_{\mathbb{B}_G(x,r)} (f - \mathcal{F}_{\mathbb{B}_G(x,r)})^2 d\mu \leq C_P \cdot r^2 \cdot \int_{E(\mathbb{B}_G(x,r))} |\nabla f|^2 dv.
\]
Thus, again by Remark 1.2
\[
\min a \int_{B_G(x,r)} (f - a)^2 d\mu \leq \int_{B_G(x,r)} (f - \mathcal{F}_{\mathbb{B}_G(x,r)})^2 d\mu \leq C_P \cdot r^2 \cdot \int_{E(\mathbb{B}_G(x,r))} |\nabla f|^2 dv.
\]

Theorem 1.13 is immediate from Claim 1.2 and the following proposition.

**Proposition 4.3.** Let \( d \geq 2 \), \( u \in (a,b) \), and \( \theta_{vgb} \in (0, \frac{1}{d+1}) \). Assume that the family of measures \( \mathbb{P}_u \), \( u \in (a,b) \), satisfies assumptions \( \mathbf{P1} \) - \( \mathbf{P3} \) and \( \mathbf{S1} \) - \( \mathbf{S2} \). There exist \( C_V, C_P, C_W \), and \( C_5 \) all depending on \( u \) and \( \theta_{vgb} \), such that for all \( R \geq 1 \),
\[
\mathbb{P}_u \left[ B_\mathbb{S}(0, R) \text{ is } (C_V, C_P, C_W)\text{-very regular} \mid 0 \in \mathcal{S}_\infty \right] \geq 1 - C_5 e^{- C_5 (\log R)^{1+\Delta_S}}.
\]

**Proof.** Let \( R \geq 1 \), take the scales as in (3.1) so that the conditions of Lemma 3.4. Theorem 3.9, and Lemma 3.15 are satisfied. Without loss of generality, we can assume that \( R^\theta_{vgb} \geq \max(C_{5K,s} L^d_0, L_0^{d+1}) \). Let \( s = \max\{s' : \max\{C_{5K,s'}, L_0^{d+1}\} \leq R^\theta_{vgb}\} \), \( K = \min\{k : kL_s \geq 2R + 1\} + 1 \), \( x_s \in \mathcal{G}_s \) such that \( B(0, R) \subseteq Q_{K,s}(x_s) \), and \( x_s' = x_s + (-2KL_s, \ldots, -2KL_s) \).

If the event \( \mathcal{H}_{5K,s}(x_s') \cap \{0 \in \mathcal{S}_\infty\} \) occurs, then \( B_\mathbb{S}(0, R) \subseteq \mathcal{C}_{5K,s,L_0}(x_s) \). Therefore, for all \( y \in B_\mathbb{S}(0, R) \) and \( R^\theta_{vgb} \leq r \leq R \), by Corollaries 3.10 and 3.11, the ball \( B_\mathbb{S}(y, r) \) is \((a_{3.16}, \overline{8C}_{3.15})\)-regular. Thus, the ball \( B_\mathbb{S}(0, R) \) is \((a_{3.16}, \overline{8C}_{3.15})\)-very regular with \( N_{B_\mathbb{S}(0,R)} \leq R^\theta_{vgb} \).

**Remark 4.4.** Consider a subgraph \( G \) of \( \mathbb{Z}^d \) encoded by \( \omega \in \{0, 1\}^{2^d} \), i.e., \( x \in V(G) \) if and only if \( \omega(x) = 1 \). The above argument implies that the condition \( \omega \in \mathcal{H}_{5K,s}(x_s') \cap \{0 \in \mathcal{S}_\infty\} \) is sufficient for the ball \( B_G(0, R) \) being \((a_{3.16}, \overline{8C}_{3.15})\)-very good with \( N_{B_G(0,R)} \leq R^\theta_{vgb} \).

Let \( C_V = a_{3.16} C_P = \overline{8C}_{3.15} \), and \( C_W = 8C_{3.15} \). The result will follow once we show that there exist \( c = c(u, \theta_{vgb}, \varepsilon) > 0 \) and \( C = C(u, \theta_{vgb}, \varepsilon) < \infty \) such that for all \( R \geq 1 \),
\[
\mathbb{P}_u \left[ \mathcal{H}_{5K,s}(x_s') \mid 0 \in \mathcal{S}_\infty \right] \geq 1 - C e^{- C (\log R)^{1+\Delta_S}}. \tag{4.1}
\]

By Definition 3.8, Lemma 3.2, and \( \mathbf{S1} \), there exists \( C = C(u) < \infty \) such that
\[
\mathbb{P}_u \left[ \mathcal{H}_{5K,s}(x_s') \right] \leq (5K + 4)^d \cdot 2 \cdot 2^{-2s} + (5KL_s)^d \cdot C \cdot e^{-f_3(u, 2L_s)}.
\]

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By (3.1) and the choice of $s$, for all $R \geq C^{(3.15)}_{3.15} \cdot L_0^{d(d+1)/\theta_{vgb}}$,

$$\left(\frac{R}{C^{(3.15)}_{3.15}}\right)^{\theta_{vgb} \frac{d(d+1)}{d+1}} \leq L_{s+1} = l_s \cdot L_s \leq l_0 \cdot 4 \cdot (L_s)^{1+2\theta_{sc}},$$

which implies that

$$L_s \geq \frac{1}{4l_0} \left(\frac{R}{C^{(3.15)}_{3.15}}\right)^{\theta_{vgb} \frac{d(d+1)}{d+1}}. \tag{4.2}$$

By (3.1) and (4.2), there exists $c = c(\theta_{vgb}, \theta_{sc}, l_0, L_0) > 0$ such that for all $R \geq C^{(3.15)}_{3.15} \cdot L_0^{d(d+1)/\theta_{vgb}}$,

$$s \geq c \cdot (\log R)^{1+\Delta_S} \tag{4.3}$$

Using (1.5), (4.2), and (4.3), we deduce that there exist $c' = c'(u, \theta_{vgb}, \theta_{sc}) > 0$ and $C' = C'(u, \theta_{vgb}, \theta_{sc}, l_0, L_0) < \infty$ such that for all $R \geq C'$,

$$2^s \geq (\log R)^{1+\Delta_S} \quad \text{and} \quad f_s(u, 2L_s) \geq c'(\log R)^{1+\Delta_S}. \tag{4.4}$$

By the choice of $K$, $KL_s \leq 4R$. Therefore, there exist $c'' = c''(u, \theta_{vgb}, \theta_{sc}) > 0$ and $C'' = C''(u, \theta_{vgb}, \theta_{sc}, l_0, L_0) < \infty$ such that for all $R \geq C''$,

$$\mathbb{P}^u[H_{K,s}(x_s')^c] \leq C'' e^{-c''(\log R)^{1+\Delta_S}}. \tag{4.4}$$

Since $\mathbb{P}^u[0 \in S_\infty] = \eta(u) > 0$, (4.4) implies (4.1). The proof is complete. \qed

**Remark 4.5.** As we already mentioned in Remark 1.21(6), a new approach to the random conductance model on general graphs satisfying some regularity assumptions has been recently developed in [2, 3]. The main assumption on graphs there is [3, Assumption 1.1], which is reminiscent of Definition 4.1, but stronger. The main difference is that we do not require that an isoperimetric inequality is satisfied by subsets of a ball, but by those of a local extension of the ball. In fact, we do not know how to show (and if it is true) that subsets of balls satisfy the desired isoperimetric inequality of [3, Assumption 1.1] in our setting. It would be very interesting to see if the machinery developed in [2, 3] can be applied to graphs with all large balls being very regular in the sense of Definition 4.1.

**A Proofs of Theorems 1.16–1.20**

In this section we give proof sketches of Theorems 1.16, 1.17, 1.18, 1.19, and 1.20. Their proofs are straightforward adaptations of main results in [6, 8] from Bernoulli percolation to our setup.

**Proof of Theorem 1.16.** The proof is essentially the same as that of [8, Theorem 6]. The only minor care that is required comes from the fact that the bound (1.11) is not stretched exponential. Since this fact is used several times, we provide a general outline of the proof.
As in the proof of [8, Theorem 6], by stationarity P1 and the ergodicity of \( \mathcal{S}_\infty \) with respect to the shift by \( X_1 \) (see, e.g., [9, Theorem 3.1]), it suffices to prove that

\[
\mathbb{E}^u \left[ (p_{2n}(0, x) - p_{2n-1}(X_1, x))^2 \cdot 1_{x \in \mathcal{S}_\infty} \right] \leq \frac{C}{n^{d+1}} \cdot e^{-c d \frac{u^2}{n}},
\]

where \( C \) and \( c \) only depend on \( d \) and \( u \). If \( d_{2d}(0, x) \geq n^\frac{1}{4} (\log n)^{\frac{1+\Delta_M}{2}} \), where \( \Delta_M \) is defined in (1.5), then by the general upper bound on the heat kernel (see, e.g., [4, (1.5)]),

\[
\mathbb{E}^u \left[ (p_{2n}(0, x) - p_{2n-1}(X_1, x))^2 \cdot 1_{x \in \mathcal{S}_\infty} \right] \leq C' \cdot e^{-c' d \frac{u^2}{n}} \leq \frac{C''}{n^{d+1}} \cdot e^{-c'' d \frac{u^2}{n}}.
\]

Thus, we can assume that \( d_{2d}(0, x) \leq n^\frac{1}{4} (\log n)^{\frac{1+\Delta_M}{2}} \).

Let \( N = N(\omega) = \max \{ T_{\text{sh}}(y) : y \in B_{Z \eta}(0, n) \} \). By (1.11),

\[
\mathbb{E}^u \left[ (p_{2n}(0, x) - p_{2n-1}(X_1, x))^2 \cdot 1_{x \in \mathcal{S}_\infty} \cdot 1_{N(\omega) \geq n} \right] \leq \mathbb{E}^u \left[ N(\omega) \geq n \right] \leq C n^d \cdot e^{-c (\log n)^{1+\Delta_M}} \leq \frac{C''}{n^{d+1}} \cdot e^{-c'' d \frac{u^2}{n}}.
\]

It remains to bound \( \mathbb{E}^u \left[ (p_{2n}(0, x) - p_{2n-1}(X_1, x))^2 \cdot 1_{x \in \mathcal{S}_\infty} \cdot 1_{N(\omega) \leq n} \right] \). As in [8, Section 2], define the quenched entropy of the simple random walk on \( \mathcal{S}_\infty \) by \( H_n = \sum_x \phi(p_{S_x, n}(0, x)) \), where \( \phi(0) = 0 \) and \( \phi(t) = -t \log t \) for \( t > 0 \), and the mean entropy by \( H_n = \mathbb{E}^u[H_n] \).

By a general argument in the proof of [8, Theorem 6], the heat kernel upper bound (1.9) implies that

\[
\mathbb{E}^u \left[ (p_{2n}(0, x) - p_{2n-1}(X_1, x))^2 \cdot 1_{x \in \mathcal{S}_\infty} \cdot 1_{N(\omega) \leq n} \right] \leq (H_n - H_{n-1}) \cdot \frac{C}{n^d} \cdot e^{-c d \frac{u^2}{n}}.
\]

The proof of [8, Theorem 6] is completed by showing in [8, Lemma 20] that \( H_n - H_{n-1} \leq \frac{C}{n} \). Thus, in order to finish the proof of Theorem 1.15, it suffices to prove that \( H_n - H_{n-1} \leq \frac{C}{n} \) in our setting too. This is a simple consequence of Theorem 1.15. Indeed, by writing \( H_n \) as the sums over \( x \) with \( d_{2d}(0, x) \leq n \) and \( d_{2d}(0, x) \geq n \), applying (1.9) and (1.10) to the summands in the first sum, and showing smallness of the second sum by using, for instance, the general upper bound on the heat kernel (see, e.g., [11, (1.5)]), we prove that for all \( n \geq T_{\text{sh}}(0) \),

\[
H_n \leq d \log(2n) + O(1).
\]

For \( n \leq T_{\text{sh}}(0) \), we use the crude bound \( H_n \) \( \leq d \log(2n) \) (see the proof below [8, (25)])]. By integrating \( H_n \) and using (1.11), we get that \( H_n = \frac{d}{2} \log n + O(1) \), which implies that \( H_n - H_{n-1} \leq \frac{C}{n} \) for some \( C \). Since \( H_n - H_{n-1} \) is decreasing by [8, Corollary 10], we conclude that \( H_n - H_{n-1} \leq \frac{2C}{n} \), finishing the proof of Theorem 1.16.

**Proof of Theorem 1.17.** The proof of Theorem 1.17 is literally the same as the proof of [6, Theorem 1.2(a)]. For the upper bound, one splits the Green function into the integrals over \([0, \min \{T_{\text{sh}}(x), T_{\text{sh}}(y)\}] \) and \([\min \{T_{\text{sh}}(x), T_{\text{sh}}(y)\}, \infty) \). Using general bounds on the heat kernel (see [6] (6.4) and (6.5)), one shows that the first integral is \( o(d_{2d}(x, y)^{2-d}) \), and by (1.9), the second integral is bounded by \( C d_{2d}(x, y)^{2-d} \). For the lower bound, one estimates the Green function from below by the integral of heat kernel over \([d_{2d}(x, y)^2, \infty) \), applies (1.10), and arrives at the desired bound. \( \square \)
Proof of Theorem 1.18. The proof of Theorem 1.18 is identical to the one of [8, Theorem 5]. The constant functions and the projections of $x + \chi(x)$ (see Theorem 1.11(a)) on coordinates of $\mathbb{Z}^d$ are independent harmonic functions with at most linear growth. Thus, the dimension of such functions is at least $(d + 1)$. It remains to show that the above functions form a basis. Let $h$ be a harmonic function $h$ on $S_\infty$ with at most linear growth and $h(0) = 0$, and assume that it is extended on $\mathbb{R}^d$ (see above [8, Proposition 19]). By Theorem 1.13 and the upper bound on the heat kernel (1.9), the proof of [8, Proposition 19] goes through without any changes in our setting, implying that the sequence $h_n(\cdot) = 1_n h(n\cdot)$ is uniformly bounded and equicontinuous on compacts. Thus, there exists a sequence $n_k$ such that $h_{n_k}$ converges uniformly on compact sets to a continuous function $\tilde{h}$. By using the quenched invariance principle of Theorem 1.11 one obtains by repeating the proof of [8, Theorem 5] that $\tilde{h}$ is harmonic in $\mathbb{R}^d$. Since $\tilde{h}$ has at most linear growth and $\tilde{h}(0) = 0$, it is linear. Therefore, the function $f(x) = h(x) - \tilde{h}(x + \chi(x))$ is harmonic on $S_\infty$ and for every $\varepsilon > 0$ and all large enough $k$, $|f(x)| \leq \varepsilon n_k$ for all $x \in B_{S_\infty}(0, n_k/\varepsilon)$. By (1.9), $B_{S_\infty}(0, f(X_{nk}^2) \leq \varepsilon n_k^2$ for all large $k$. The proof of [8, Theorem 5] is finished by applying [8, Corollary 21] which states that $f$ must be constant. The proof of [8, Corollary 21] is rather general and only uses the fact that the mean entropy $H_n$ (see the proof of Theorem 1.16) satisfies $H_n - H_{n-1} \leq C_n$. We already proved this bound in the proof of Theorem 1.16. Thus, [8, Corollary 21] holds in our setting, and we conclude that $f$ must be constant. The proof is complete. \hspace{1cm} \Box

Proof of Theorem 1.19. Theorem 1.19 was proved in the case of supercritical Bernoulli percolation in [6, Theorem 1.1] by first providing general assumptions [6, Assumption 4.4] for the local limit theorem on infinite subgraphs of $\mathbb{Z}^d$ (see [6, Theorems 4.5 and 4.6]), and then verifying these assumptions for the infinite cluster of Bernoulli percolation. [6, Assumption 4.4] is tailored for random subgraphs of $\mathbb{Z}^d$ with laws invariant under reflections with respect to coordinate axes and rotations by $\pi/2$. These assumptions only simplify the expression for the heat kernel of the limiting Brownian motion, and can be naturally extended to the case without such symmetries.

We only consider the case of discrete time random walk (the continuous time case is the same). As in [6, Theorem 4.5], to prove Theorem 1.19 it suffices to show that there exist an event $\Omega' \in \mathcal{F}$ with $\mathbb{P}[\Omega'] = 1$, positive constants $\delta$, $C_i$, and $C_H$, and a covariance matrix $\Sigma$, such that for all $\omega \in \Omega' \cap \{0 \in S_\infty\},$

(a) for any $y \in \mathbb{R}^d$ and $r > 0$, as $n \to \infty$, $P_{S_\infty, 0} \left[ \tilde{B}_n(t) \in (y + [-r, r]^d) \right]$ converges to $\int_{y+[-r,r]^d} k_{\Sigma, t}(y') dy'$ uniformly over compact subsets of $(0, \infty)$ ($\tilde{B}_n(t)$ is as in (1.7)),

(b) there exists $T_1 = T_1(\omega) < \infty$ such that for all $n \geq T_1$ and $x \in S_\infty$, $p_{n}(0, x) \leq C_1 \cdot n^{-\frac{d}{2}} \cdot e^{-c_2 \frac{d_{S_\infty, 0}(0, x)^2}{n}},$

(c) for each $y \in S_\infty$, there exists $R_H(y) = R_H(y, \omega) < \infty$ such that the parabolic Harnack inequality holds with constant $C_H$ in $(0, R^2 \times B_{S_\infty}(y, R)$ for all $R \geq R_H(y)$,

(d) for $h(r) = \max \{r' : \exists y \in [-r, r]^d$ such that $S_\infty \cap (y + [-r', r']^d) = \emptyset \}$, the ratio $\frac{h(r)}{r}$ tends to 0 as $r \to \infty$.\hspace{1cm} 39
(e) for any $x \in \mathbb{Z}^d$ and $r > 0$, \( \lim_{n \to \infty} \frac{u(S_\infty \cap (\sqrt{n}x+[-\sqrt{n}r, \sqrt{n}r]^d))}{(2\sqrt{n})^d} = \mathbb{E}^u[\mu_0 \cdot 1_{0 \in S_\infty}] \),

(f) for each $x \in \mathbb{Z}^d$ and $r > 0$, there exists $T_2(x) = T_2(x, \omega) < \infty$ such that for all $n \geq T_2$, and $x', y' \in S_\infty \cap (\sqrt{n}x + [-\sqrt{n}r, \sqrt{n}r]^d)$, \( d_{S_\infty}(x', y') \leq C \cdot \max\{d_{\mathbb{Z}^d}(x', y'), n^{\frac{1}{2} - \delta}\} \),

(g) for $x \in \mathbb{Z}^d$ and $R_H$ as in (c), \( \lim_{n \to \infty} n^{-\frac{1}{2}}R_H(g_n(x)) = 0 \).

It is easy to see that the above assumptions are satisfied in our setting:

(a) follows from Theorem 1.11,

(b) follows from (1.9),

(c) follows from Theorems 1.5 and 1.13,

(d) follows from stationarity, (1.6), and the Borel-Cantelli lemma,

(e) follows from a spatial ergodic theorem [22, Theorem 2.8 in Chapter 6], since the sequence of boxes \((\sqrt{n}x + [-\sqrt{n}r, \sqrt{n}r]^d)_{n \geq 1}\) is regular in the sense of [22, Definition 2.4 in Chapter 6] (see [3, Lemma 5.1]),

(f) follows from Theorem 1.10,

(g) follows from (1.8), Theorem 1.5, and the Borel-Cantelli lemma.

The proof of Theorem 1.19 is complete.

Proof of Theorem 1.20. Statement (a) follows from Theorem 1.19 and (1.9) by repeating the proof of [6, Theorem 1.2(b)] without any changes. For the statement (c) we use bounds [6, (6.30) and (6.31)] and (1.11), to get

\[
\frac{(1 - \varepsilon)G(0)}{m} \mathbb{E}\left[\left. M \leq |x| \right| x \in S_\infty \right] \leq \mathbb{E}\left[\left. g_{S_\infty}(0, x) \right| 0 \in S_\infty \right]
\leq \frac{(1 + \varepsilon)G(x)}{m} + \frac{C'}{m} \mathbb{E}\left[\left. M > |x| \right| 0 \in S_\infty \right]
\leq \mathbb{E}\left[\left. g_{S_\infty}(0, x)^2 \right| 0 \in S_\infty \right]^{\frac{1}{2}} \cdot e^{-c'(\log |x|)^{1+\Delta_S}},
\]

where $M$ is defined in the statement of Theorem 1.20. As in [6, (6.17)], by (1.9),

\[
g_{S_\infty}(0, x) \leq g_{S_\infty}(0, 0) \leq T_0(0) + \int_{T_0(0)}^{\infty} C't^{-\frac{d}{2}} dt \leq (1 + 2C')T_0(0).
\]

Combining this bound with (1.11), we obtain that \( \mathbb{E}\left[\left. g_{S_\infty}(0, x)^2 \right| 0 \in S_\infty \right] < C'' \). Let $x = ky$. Since $G(x) = k^{2-d}G(y)$, by taking limits $k \to \infty$ and then $\varepsilon \to 0$, we complete the proof of statement (c).

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