Research Article

On the Limit Cycles for a Class of Perturbed Fifth-Order Autonomous Differential Equations

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Received 12 September 2021; Accepted 23 November 2021; Published 7 December 2021

Academic Editor: Sundarapandian Vaidyanathan

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We study the limit cycles of the fifth-order differential equation

\[ \dot{x} - \mu x + \dot{x} - \mu x = \varepsilon F(x, \dot{x}, x, t), \]  

with \( \mu \neq 0 \); \( \varepsilon \) is a small real parameter; \( F \in C^2 \) is \( 2\pi \)- periodic in \( t \).

In [5], the authors studied equation (1) with \( F = F(x, \dot{x}, \ddot{x}) \) which is autonomous. They studied the two cases \( \mu \neq 0 \) and \( \mu = 0 \).

In [6], the authors studied the following fourth-order differential equation:

\[ x - (\lambda + \mu)\dot{x} + (1 + \lambda \mu)x - (\lambda + \mu)\ddot{x} + \lambda \mu x = \varepsilon F(x, \dot{x}, x, \ddot{x}, t), \]  

where \( \lambda \) and \( \mu \) are real, \( \varepsilon \) is a small real parameter, and \( F \in C^2 \) is \( 2\pi \)- periodic in \( t \).

In [4], the authors studied equation (2) with \( F = F(x, \dot{x}, \ddot{x}, \dddot{x}) \) which is autonomous.

In this paper, we shall use a result of the averaging theory to study the limit cycles of the following class of fifth-order autonomous ordinary differential equations:

\[ \ddot{x} - ex - d\dot{x} - cx - b\dot{x} - ax = \varepsilon F(x, \dot{x}, x, \ddot{x}, \dddot{x}), \]  

1. Introduction and Statement of the Main Results

The study of the limit cycles is one of the main topics of the qualitative theory of differential equations and dynamical systems. A limit cycle of a differential equation is an isolated periodic orbit of this equation; it means that there is no periodic orbits in the vicinity of this limit cycle. There are several theories and methods for the study of the existence, uniqueness, or number and stability of limit cycles of differential equations which have been developed in trying to answer Hilbert’s sixteenth problem posed in 1900 [1] about the maximum number of limit cycles that a planar polynomial differential system can have.

The averaging theory is one of the most important tools used actually to the study of limit cycles for second and higher order differential equations, you can see in [2–8]. More details on the averaging theory can be found in the books of Sanders and Verhulst [9] and of Verhulst [9].

In [7], the authors studied the limit cycles of the following third-order differential equation

\[ x - (\lambda + \mu)\dot{x} + (1 + \lambda \mu)x - \mu \ddot{x} + \lambda \mu x = \varepsilon F(x, \dot{x}, x, \ddot{x}), \]  

which is autonomous.
where
\[ a = \lambda \mu \delta, \quad b = -(\lambda \mu + \lambda \delta + \mu \delta), \quad c = \lambda + \mu + \delta + \lambda \mu \delta, \quad d = -(1 + \lambda + \lambda \delta + \mu \delta), \quad e = \lambda + \mu + \delta, \]

where the dot means derivative with respect to an independent variable \( t \), \( \epsilon \) is a small enough parameter, and \( F \in C^2 \) is a nonlinear function. Here, the variable \( x \) and the parameters \( \lambda, \mu, \delta \) and \( \epsilon \) are real.

In [8], the authors studied equation (3) with \( F = F(x, \dot{x}, \ddot{x}, \dot{x}, x, t) \) which depends explicitly on the independent variable \( t \). Here, we study the autonomous case using a different approach. Note that our results are distinct and new.

Now, we state our main results for the limit cycles of equation (3).

For the different values of the parameters \( \lambda, \mu, \) and \( \delta \), we distinguish the five following cases.

**Case 1.** \( \lambda \mu \delta \neq 0 \) and \( \lambda \neq \mu \neq \delta \).

**Case 2.** \( \lambda = 0, \mu \delta \neq 0 \), and \( \mu \neq \delta \).

**Case 3.** \( \lambda = 0 \) and \( \mu = \delta \neq 0 \).

**Case 4.** \( \lambda \neq 0 \) and \( \mu = \delta \neq 0 \).

**Case 5.** \( \lambda = \mu = \delta \neq 0 \).

For each one of these cases, we will give a theorem which provides sufficient conditions for the existence of limit cycles of equation (3) and we provide also an application.

There are two other cases (\( \lambda = \mu = 0 \) and \( \delta \neq 0 \)) and (\( \lambda = \mu = \delta = 0 \)) that we cannot study because they are too much degenerated for Theorem 6.

1.1. Case 1: \( \lambda \mu \delta \neq 0 \) and \( \lambda \neq \mu \neq \delta \). In order to state our results for this case, we define the function
\[
\mathcal{F}(r_0) = \frac{1}{2n} \int \sin \theta F(A_1, A_2, A_3, A_4, A_5) \, d\theta,
\]
where
\[
A_1 = \frac{((\lambda \mu \delta - \lambda - \mu - \delta) \cos \theta + (1 - \lambda \mu - \lambda \delta - \mu \delta) \sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)},
\]
\[
A_2 = \frac{((1 - \lambda \mu - \lambda \delta - \mu \delta) \cos \theta + (\lambda + \mu + \delta - \lambda \mu \delta) \sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)},
\]
\[
A_3 = \frac{((\lambda + \lambda \delta \mu \delta - \lambda \delta - \mu \delta) \cos \theta + (\lambda \mu \delta - \lambda - \mu \delta) \sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)},
\]
\[
A_4 = \frac{((\lambda \mu \delta - \lambda - \mu - \delta) \cos \theta + (1 - \lambda \mu - \lambda \delta - \mu \delta) \sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)},
\]
\[
A_5 = \frac{((\lambda \mu \delta - \lambda - \mu - \delta) \cos \theta + (1 - \lambda \mu - \lambda \delta - \mu \delta) \sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)}.
\]

Our main result for this case is the following theorem.

**Theorem 1.** Assume that \( \lambda \mu \delta \neq 0 \) and \( \lambda \neq \mu \neq \delta \). For every positive simple zero \( r_0^* \) of the function \( \mathcal{F}(r_0) \) given by (5) there is a limit cycle \( x(t, \epsilon) \) of equation (3) tending to the periodic solution
\[
x^*(t) = \frac{((\lambda \mu \delta - \lambda - \mu - \delta) \cos t + (1 - \lambda \mu - \lambda \delta - \mu \delta) \sin t) r_0^*}{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)},
\]
\[
(7)
\]
of
\[
\dot{x} - (\lambda + \mu + \delta)x - (1 + \lambda + \lambda \delta + \mu \delta)\dot{x} - (\lambda + \mu + \delta + \lambda \mu \delta)x - (\lambda \mu + \lambda \delta + \mu \delta)\dot{x} - \lambda \mu \delta x = 0,
\]
\[
(8)
\]
when \( \epsilon \to 0 \).

Theorem 1 will be proved in Section 3.1.1.

An application of Theorem 1 is the following.

**Corollary 1.** Assume that \( \lambda \mu \delta \neq 0 \), \( \lambda \neq \mu \neq \delta \), \( \lambda \mu + \lambda \delta + \mu \delta \neq 1 \), and
\[
F(x, \dot{x}, \ddot{x}, \dot{x}, x) = x^5 - x^4 - \dot{x}^2 - x^2 - x - 1.
\]
\[
(9)
\]

Then, there is a limit cycle \( x_1(t, \epsilon) \) of equation (3) tending to the periodic solution
\[
x_1(t) = \frac{2((\lambda \mu \delta - \lambda - \mu - \delta) \cos t + (1 - \lambda \mu - \lambda \delta - \mu \delta) \sin t)}{\sqrt{(1 + \lambda^2)(1 + \mu^2)(1 + \delta^2)}},
\]
\[
(10)
\]
of equation (8) when \( \epsilon \to 0 \).

Corollary 1 will be proved in Section 3.1.2.
1.2. Case 2: $\lambda = 0, \mu \delta \neq 0$, and $\mu \neq \delta$. We define the functions

$$F_1(r_0, Z_0) = \frac{1}{2\pi} \sin \theta F(B_1, B_2, B_3, B_4, B_5) d\theta,$$

$$F_2(r_0, Z_0) = \frac{1}{2\pi} F(B_1, B_2, B_3, B_4, B_5) d\theta,$$

and

$$B_1 = \frac{((-\mu - \delta) \cos \theta + (1 - \mu \delta) \sin \theta) r_0}{(1 + \mu^2)(1 + \delta^2)} + Z_0 \delta,$$

$$B_2 = \frac{((\mu - \delta) \cos \theta + (\mu + \delta) \sin \theta) r_0}{(1 + \mu^2)(1 + \delta^2)},$$

$$B_3 = \frac{((\mu + \delta) \cos \theta + (\mu \delta - 1) \sin \theta) r_0}{(1 + \mu^2)(1 + \delta^2)},$$

$$B_4 = \frac{((-\mu + \delta) \cos \theta + (\mu - \delta) \sin \theta) r_0}{(1 + \mu^2)(1 + \delta^2)},$$

$$B_5 = \frac{((-\mu + \delta) \cos \theta + (1 - \mu \delta) \sin \theta) r_0}{(1 + \mu^2)(1 + \delta^2)}.$$

Our main result for this case is the following theorem.

**Theorem 2.** Assume that $\lambda = 0, \mu \delta \neq 0$, and $\mu \neq \delta$. For every zero $(r_0^*, Z_0^*)$ of the system $F_1(r_0, Z_0) = F_2(r_0, Z_0) = 0$ where $F_1$ and $F_2$ are given by (11) such that

$$\det \left( \frac{\partial^2 (F_1, F_2)}{\partial (r_0^*, Z_0^*)} \right) \mid (r_0^*, Z_0^*) \neq 0,$$

there is a limit cycle $x(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{((-\mu - \delta) \cos t + (1 - \mu \delta) \sin t) r_0^*}{(1 + \mu^2)(1 + \delta^2)} + Z_0^* \delta,$$

of

$$\dot{x} = - (\mu + \delta) x + (1 + \mu \delta) \dot{x} - (\mu + \delta) \dot{x} + \mu \delta \dot{x} = 0,$$

when $\epsilon \to 0$.

**Corollary 2.** Assume that $\lambda = 0, \mu \delta \neq 0, \mu \neq \delta, \mu \delta \neq 1$, and $F(x, \dot{x}, \ddot{x}, \dddot{x}, x) = x^2 - 2x + x^2$, then there is a limit cycle $x_3(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x_3^*(t) = \frac{((-\mu - \delta) \cos t + (1 - \mu \delta) \sin t)}{\sqrt{(1 + \mu^2)(1 + \delta^2)}} + 1,$$

of equation (15) when $\epsilon \to 0$.

**Corollary 2** will be proved in Section 3.

$$\mathcal{F}_1(r_0, Z_0) = \frac{1}{2\pi} \sin \theta F(C_1, C_2, C_3, C_4, C_5) d\theta,$$

$$\mathcal{F}_2(r_0, Z_0) = \frac{1}{2\pi} \sin \theta F(C_1, C_2, C_3, C_4, C_5) d\theta,$$

where

$$C_1 = \frac{(2 \mu \cos \theta + (\mu^2 - 1) \sin \theta) r_0}{(1 + \mu^2)^2} + Z_0 \delta,$$

$$C_2 = \frac{(\mu^2 - 1) \cos \theta - (\mu^2 - 1) \sin \theta)}{(1 + \mu^2)^2} r_0,$$

$$C_3 = \frac{(- 2 \mu \cos \theta - (\mu^2 - 1) \sin \theta) r_0}{(1 + \mu^2)^2},$$

$$C_4 = \frac{(- \mu^2 - 1) \cos \theta + 2 \mu \sin \theta)}{(1 + \mu^2)^2} r_0,$$

$$C_5 = \frac{2 \mu \cos \theta + (\mu^2 - 1) \sin \theta) r_0}{(1 + \mu^2)^2},$$

Our main result for this case is the following theorem.

**Theorem 3.** Assume that $\lambda = 0$ and $\mu = \delta \neq 0$. For every zero $(r_0^*, Z_0^*)$ of the system $F_1(r_0, Z_0) = F_2(r_0, Z_0) = 0$ where $F_1$ and $F_2$ are given by (17) such that

$$\det \left( \frac{\partial (F_1, F_2)}{\partial (r_0^*, Z_0^*)} \right) \mid (r_0^*, Z_0^*) \neq 0,$$

there is a limit cycle $x(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{(2 \mu \cos t + (\mu^2 - 1) \sin t) r_0^*}{(1 + \mu^2)^2} + Z_0^* \delta,$$

of

$$\dot{x} = - 2 \mu \dot{x} + (1 + \mu^2) \ddot{x} - 2 \mu \dddot{x} + \mu^2 \dot{x} = 0,$$

when $\epsilon \to 0$.

**Corollary 3.** Assume that $\lambda = 0, \delta = \mu \neq 0, \mu \neq -1 \pm \sqrt{5}/2$, and $F(x, \dot{x}, \ddot{x}, \dddot{x}, x) = x^2 - x \dot{x} - 1$, then there is a limit cycle $x_3(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x_3^*(t) = \frac{(2 \mu \cos t + (\mu^2 - 1) \sin t) \sqrt{2}}{1 + \mu^2},$$

of equation (21) when $\epsilon \to 0$.

**Corollary 3** will be proved in Section 3.3.2.
1.4. Case 4: $\lambda \neq 0$ and $\mu = \delta \neq 0$. We define the function

$$F(r_0) = \frac{1}{2\pi r_0} \sin \theta F(D_1, D_2, D_3, D_4, D_5)d\theta,$$

where

$$D_1 = \frac{((\lambda + 2\mu - \lambda \mu^2)\cos \theta + (\mu^2 + 2\lambda \mu - 1)\sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)^2},$$

$$D_2 = \frac{((\mu^2 + 2\lambda \mu - 1)\cos \theta - (\lambda + 2\mu - \lambda \mu^2)\sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)^2},$$

$$D_3 = \frac{(-\lambda + 2\mu - \lambda \mu^2)\cos \theta - (\mu^2 + 2\lambda \mu - 1)\sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)^2},$$

$$D_4 = \frac{(-\mu^2 + 2\lambda \mu - 1)\cos \theta + (\lambda + 2\mu - \lambda \mu^2)\sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)^2},$$

$$D_5 = \frac{(\lambda + 2\mu - \lambda \mu^2)\cos \theta + (\mu^2 + 2\lambda \mu - 1)\sin \theta)r_0}{(1 + \lambda^2)(1 + \mu^2)^2}.$$

(24)

Our main result for this case is the following theorem.

**Theorem 4.** Assume that $\lambda \neq 0$ and $\delta = \mu \neq 0$. For every positive simple zero $r_0^*$ of the function $F(r_0)$ given by (23), there is a limit cycle $x(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{((\lambda + 2\mu - \lambda \mu^2)\cos t + (\mu^2 + 2\lambda \mu - 1)\sin t)r_0^*}{(1 + \lambda^2)(1 + \mu^2)^2}.$$

(25)

of

$$\ddot{x} - (\lambda + 2\mu)x + (1 + 2\lambda \mu + \mu^2)\dot{x} - (\lambda + 2\mu + \lambda \mu^2)x = 0,$$

(26)

when $\epsilon \to 0$.

Theorem 4 will be proved in Section 3.4.1.

An application of Theorem 4 is the following.

**Corollary 4.** Assume that $\lambda \neq 0$, $\epsilon = \mu \neq 0$, $\mu^2 + 2\lambda \mu - 1 \neq 0$, and $F(x, \dot{x}, \ddot{x}, x) = x^3 - x - 1$, then there is a limit cycle $x_4(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x_4^*(t) = \frac{2\sqrt{3}(\lambda + 2\mu - \lambda \mu^2)\cos t + (\mu^2 + 2\lambda \mu - 1)\sin t}{3\sqrt{1 + \lambda^2}(1 + \mu^2)}.$$

(27)

of equation (26) when $\epsilon \to 0$.

Corollary 4 will be proved in Section 3.4.2.

1.5. Case 5: $\lambda = \mu = \delta \neq 0$. In order to state our result for this case, we define the function

$$F(r_0) = \frac{1}{2\pi r_0} \sin \theta F(E_1, E_2, E_3, E_4, E_5)d\theta,$$

where

$$E_1 = \frac{((\lambda^3 - 3\lambda)\cos \theta + (1 - 3\lambda^2)\sin \theta)r_0}{(1 + \lambda^2)^3},$$

$$E_2 = \frac{((1 - 3\lambda^2)\cos \theta - (1 - 3\lambda^2)\sin \theta)r_0}{(1 + \lambda^2)^3},$$

$$E_3 = \frac{((-\lambda^3 - 3\lambda)\cos \theta - (1 - 3\lambda^2)\sin \theta)r_0}{(1 + \lambda^2)^3},$$

$$E_4 = \frac{((-1 - 3\lambda^2)\cos \theta + (1 - 3\lambda^2)\sin \theta)r_0}{(1 + \lambda^2)^3},$$

$$E_5 = \frac{((\lambda^3 - 3\lambda)\cos \theta + (1 - 3\lambda^2)\sin \theta)r_0}{(1 + \lambda^2)^3}.$$

(29)

Our main result for this case is the following theorem.

**Theorem 5.** Assume that $\mu = \delta = \lambda \neq 0$. For every positive simple zero $r_0^*$ of the function $F(r_0)$ given by (28), there is a limit cycle $x(t, \epsilon)$ of equation (3) tending to the periodic solution

$$x^*(t) = \frac{((\lambda^3 - 3\lambda)\cos t + (1 - 3\lambda^2)\sin t)r_0^*}{(1 + \lambda^2)^3}.$$

(30)

of

$$\ddot{x} - 3\lambda x + (1 + 3\lambda^2)\dot{x} - (3\lambda + \lambda^3)x + 3\lambda^2\dot{x} - \lambda^3 = 0,$$

(31)

when $\epsilon \to 0$.

Theorem 5 will be proved in Section 3.5.1.

An application of Theorem 5 is the following.

**Corollary 5.** Assume that $\mu = \delta = \lambda \neq 0$, $\lambda \neq \pm \sqrt{3}$, $\lambda \neq \pm \sqrt{3}/3$, and $F(x, \dot{x}, \ddot{x}, x) = -\dot{x}^2 x + x^3 + x^3 - x \dot{x}^2 + x - 1$, then there is a limit cycle $x_5(t, \epsilon)$ of equation (3) tending to the periodic solution
of equation (31) when $\varepsilon \to 0$.

Corollary 5 will be proved in Section 3.5.2.

2. The Main Tool (First-Order Averaging Theory)

In this section, we present the basic result from the averaging theory that we need for proving the main results of this article.

We consider the problem of the bifurcation of $T$-periodic solutions from the differential system

$$\dot{x}(t) = F_0(x, t) + \varepsilon F_1(x, t) + \varepsilon^2 F_2(x, t, \varepsilon),$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. The functions $F_0, F_1: \Omega \times \mathbb{R} \to \mathbb{R}^n$ and $F_2: \Omega \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $C^2$ functions, $T$-periodic is in the variable $t$, and $\Omega$ is an open subset of $\mathbb{R}^n$. We suppose that the unperturbed system

$$\dot{x}(t) = F_0(x, t),$$

has a $k$-dimensional submanifold $\mathcal{Z}$ of periodic solutions.

Let $x(t, z)$ be the solution of the unperturbed system (34) such that $x(0, z) = z$. The linearisation of system (34) along the periodic solution $x(t, z)$ is written as

$$\dot{y} = D_x F_0(x(t, z), t)y.$$

We denote by $M_z(t)$ some fundamental matrices of the linear differential system (35) and by $\xi: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first $k$ coordinates; i.e.,

$$\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k).$$

Theorem 6. Let $\mathcal{V} \subset \mathbb{R}^k$ be open and bounded and $\beta_0: \text{Cl}(\mathcal{V}) \to \mathbb{R}^{n-k}$ be a $C^2$ function. We assume that

(i) $\mathcal{Z} = \{z_a = (\alpha, \beta_0(\alpha)), \alpha \in \text{Cl}(\mathcal{V})\} \subset \Omega$, and that for each $z_a \in \mathcal{Z}$, the solution $x(t, z_a)$ of (34) is $T$-periodic.

(ii) For each $z_a \in \mathcal{Z}$, there is a fundamental matrix $M_{z_a}(t)$ of (35) such that the matrix $M_{z_a}^{-1}(0) - M_{z_a}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a matrix $\Delta_a((n-k) \times (n-k))$ with $\det \Delta_a \neq 0$. We consider the function $\mathcal{F}: \text{Cl}(\mathcal{V}) \to \mathbb{R}^k$

$$\mathcal{F}(a) = \xi\left( T \int_0^T M_{z_a}^{-1}(t) F_1(x(t, z_a), t) dt \right).$$

If there exists $a \in \mathcal{V}$ with $\mathcal{F}(a) = 0$ and $\det((d \mathcal{F}/da)(a)) \neq 0$, then there is a $T$-periodic solution $\phi(t, \varepsilon)$ of the system (33) such that $\phi(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.

Theorem 1 goes back to [10] and [11]; for a shorter proof, see [12].

Note that the periodic orbits provided by Theorem 6 are limit cycles.

3. Proofs of the Results

3.1. Proofs of the Results in Case 1: $\lambda \mu \delta \neq 0$ and $\lambda \neq \mu \neq \delta$

3.1.1. Proof of Theorem 1. We consider equation (3) and put

$$y = \dot{x}, \quad z = \ddot{x}, \quad u = \frac{1}{\mu} x, \quad \text{and} \quad v = x,$$

then equation (3) can be written as

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= u, \\
\dot{u} &= v, \\
\dot{v} &= ax + by + cz + du + ev + \varepsilon F(x, y, z, u, v).
\end{align*}$$

System (37) with $\varepsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i, \lambda, \mu,$ and $\delta$. Using the change of variables
we transform system (37) into the following system: where

\[
\begin{align*}
\dot{X} &= -Y, \\
\dot{Y} &= X + \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{Z} &= \lambda Z + \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{U} &= \mu U + \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{V} &= \delta V + \varepsilon \tilde{F}(X, Y, Z, U, V),
\end{align*}
\]

(39)

\[
\tilde{F} = \tilde{F}(X, Y, Z, U, V) = F(a_1, a_2, a_3, a_4, a_5),
\]

\[
a_1 = \frac{(\lambda \mu \delta - \lambda - \mu - \delta)X}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \lambda \mu - \lambda \delta - \mu \delta)Y}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{Z}{(\lambda^2 + 1)(\lambda - \mu)(\lambda - \delta)} - \frac{U}{(\mu^2 + 1)(\lambda - \mu)(\mu - \delta)} + \frac{V}{(\delta^2 + 1)(\lambda - \delta)(\mu - \delta)}
\]

\[
a_2 = \frac{(1 - \lambda \mu - \lambda \delta - \mu \delta)X}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} - \frac{(\lambda \mu \delta - \lambda - \mu - \delta)Y}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{\lambda Z}{(\lambda^2 + 1)(\lambda - \mu)(\lambda - \delta)} - \frac{\mu U}{(\mu^2 + 1)(\lambda - \mu)(\mu - \delta)} + \frac{\delta V}{(\delta^2 + 1)(\lambda - \delta)(\mu - \delta)}
\]

\[
a_3 = \frac{(\lambda \mu \delta - \lambda - \mu - \delta)X}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} - \frac{(1 - \lambda \mu - \lambda \delta - \mu \delta)Y}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{\lambda^2 Z}{(\lambda^2 + 1)(\lambda - \mu)(\lambda - \delta)} - \frac{\mu^2 U}{(\mu^2 + 1)(\lambda - \mu)(\mu - \delta)} + \frac{\delta^3 V}{(\delta^2 + 1)(\lambda - \delta)(\mu - \delta)}
\]

\[
a_4 = \frac{(1 - \lambda \mu - \lambda \delta - \mu \delta)X}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{(\lambda \mu \delta - \lambda - \mu - \delta)Y}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{\lambda^2 Z}{(\lambda^2 + 1)(\lambda - \mu)(\lambda - \delta)} - \frac{\mu^2 U}{(\mu^2 + 1)(\lambda - \mu)(\mu - \delta)} + \frac{\delta^3 V}{(\delta^2 + 1)(\lambda - \delta)(\mu - \delta)}
\]

\[
a_5 = \frac{(\lambda \mu \delta - \lambda - \mu - \delta)X}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \lambda \mu - \lambda \delta - \mu \delta)Y}{(\lambda^2 + 1)(\mu^2 + 1)(\delta^2 + 1)} + \frac{\lambda^4 Z}{(\lambda^2 + 1)(\lambda - \mu)(\lambda - \delta)} - \frac{\mu^4 U}{(\mu^2 + 1)(\lambda - \mu)(\mu - \delta)} + \frac{\delta^4 V}{(\delta^2 + 1)(\lambda - \delta)(\mu - \delta)}
\]

(40)

Note that the linear part of system (39) is in the real normal Jordan form of the linear part of system (37). We pass now from the Cartesian coordinates \((X, Y, Z, U, V)\) to the cylindrical ones \((r, \theta, Z, U, V)\) with \(X = r \cos \theta\), \(Y = r \sin \theta\), and we obtain

\[
\begin{align*}
\dot{r} &= \varepsilon \sin \theta G(r, \theta, Z, U, V), \\
\dot{\theta} &= 1 + \frac{\varepsilon}{r} \cos \theta G(r, \theta, Z, U, V), \\
\dot{Z} &= \lambda Z + \varepsilon G(r, \theta, Z, U, V), \\
\dot{U} &= \mu U + \varepsilon G(r, \theta, Z, U, V), \\
\dot{V} &= \delta V + \varepsilon G(r, \theta, Z, U, V),
\end{align*}
\]

(41)

where \(G(r, \theta, Z, U, V) = \tilde{F}(r \cos \theta, r \sin \theta, Z, U, V)\). After dividing by \(\theta\) and simplifying, we find

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon \sin \theta G(r, \theta, Z, U, V) + o(\varepsilon^3), \\
\frac{dZ}{d\theta} &= \lambda Z + \varepsilon \left(1 - \frac{\lambda Z}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^3), \\
\frac{dU}{d\theta} &= \mu U + \varepsilon \left(1 - \frac{\mu U}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^3), \\
\frac{dV}{d\theta} &= \delta V + \varepsilon \left(1 - \frac{\delta V}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^3).
\end{align*}
\]

(42)

System (42) is now of the same form as system (33) with
\[ x = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta, \]

\[ F_0(x, \theta) = \begin{pmatrix} 0 \\ \lambda Z \\ \mu U \\ \delta V \end{pmatrix}, \]

\[ F_1(x, \theta) = \begin{pmatrix} \sin \theta G(r, \theta, Z, U, V) \\ \left(1 - \frac{\lambda Z}{r} \cos \theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \frac{\mu U}{r} \cos \theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \frac{\delta V}{r} \cos \theta\right) G(r, \theta, Z, U, V) \end{pmatrix}. \]

We shall apply Theorem 6 to system (42). System (42) with \( \varepsilon = 0 \) has the 2\( \pi \)-periodic solutions

\[ \begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0. \]

By the notations of Theorem 6, we have that \( k = 1 \) and \( n = 4 \). Let \( r_1 > 0 \) and \( r_2 > 0 \); we take \( \forall = ]r_1, r_2[ \subset \mathbb{R} \), \( a = r_0 \in [r_1, r_2] \), and

\[ \beta_0: [r_1, r_2] \rightarrow \mathbb{R}^3, \quad r_0 \mapsto \beta_0(r_0) = (0, 0, 0). \]

We also take

\[ \mathcal{Z} = \{ z_0 = (r_0, 0, 0, 0), r_0 \in [r_1, r_2] \}. \]

The fundamental matrix \( M_{z_0}(\theta) \) of the linear system (42) with \( \varepsilon = 0 \) with respect to the periodic solution \( z_0 = (r_0, 0, 0, 0) \) satisfying that \( M_{z_0}(0) \) is the identity matrix is

\[ M_{z_0}^{-1}(0) - M_{z_0}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - e^{-2\pi\lambda} & 0 & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} & 0 \\ 0 & 0 & 0 & 1 - e^{-2\pi\delta} \end{pmatrix}. \]

which satisfy the assumption (ii) of Theorem 6. Taking

\[ \xi: \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad : (r, Z, U, V) \mapsto \xi(r, Z, U, V) = r, \]

we must compute the function \( \mathcal{F}(\alpha) \) given by (36), and we obtain

\[ \mathcal{F}(\alpha) = \mathcal{F}(r_0) = \frac{1}{2\pi} \int_0^{2\pi} \sin \theta G(r_0, \theta, 0, 0, 0) d\theta \]

and \( r_1, r_2, A_1, A_2, A_3, A_4, A_5 \) are given by (6). Then, by Theorem 6, for every simple zero \( r_0^* \) of the function \( \mathcal{F}(r_0) \), there exists a limit cycle \( (r, Z, U, V)(\theta, \varepsilon) \) of system (42) such that

\[ (r, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, 0, 0, 0), \quad \text{when} \ \varepsilon \rightarrow 0. \]

Going back through the change of coordinates, we obtain a limit cycle \( (r, \theta, Z, U, V)(t, \varepsilon) \) of system (41) such that

\[ (r, \theta, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, 0, 0, 0), \quad \text{when} \ \varepsilon \rightarrow 0. \]

Finally, we obtain a limit cycle \( x(t, \varepsilon) \) of equation (3) tending to the periodic solution (7) of equation (8) when \( \varepsilon \rightarrow 0. \)

Theorem 1 is proved.

3.1.2. Proof of Corollary 1. If \( F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddot{x}) = x^2 - x^4 - \dddot{x}^2 - x - 1 \), then we have

\[ \mathcal{F}(r_0) = \frac{r_0(-1 + 3\mu_2 + 3\delta)}{16(1 + \lambda_2)(1 + \mu_2)(1 + \delta)}, \]

which have the real positive simple zero

\[ r_0^* = 2\sqrt{5}/5 \sqrt{(1 + \lambda_2)(1 + \mu_2)(1 + \delta)}, \]

with

\[ \frac{df}{dr_0}(r_0^*) = -\frac{34}{9} \frac{-1 + 3\mu_2 + 3\delta}{(1 + \lambda_2)(1 + \mu_2)(1 + \delta)} 
eq 0. \]
The proof of Corollary 1 follows directly by applying Theorem 1 and (10) is obtained by substituting \( r_0 \) in (7).

3.2. Proofs of the Results in Case 2: \( \lambda = 0, \mu \neq 0, \) and \( \mu \neq \delta \)

3.2.1. Proof of Theorem 2. If \( \lambda = 0 \), equation (3) can be written as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= u, \\
\dot{u} &= v, \\
\dot{v} &= -\mu \delta y + (\mu + \delta) z - (1 + \mu \delta) u + (\mu + \delta) v + \varepsilon F(x, y, z, u, v).
\end{align*}
\]

(56)

we transform system (56) into the following system:

\[
\begin{align*}
\dot{X} &= -Y, \\
\dot{Y} &= X + \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{Z} &= \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{U} &= \mu U + \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{V} &= \delta V + \varepsilon \tilde{F}(X, Y, Z, U, V),
\end{align*}
\]

(58)

where

\[
\begin{align*}
b_1 &= \frac{(-\mu - \delta) X}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \mu \delta) Y}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{Z}{\mu \delta} + \frac{U}{\mu (\mu^2 + 1)(\mu - \delta)} - \frac{V}{\delta (\delta^2 + 1)(\mu - \delta)}, \\
b_2 &= \frac{(1 - \mu \delta) X}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \mu \delta) Y}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{U}{\mu (\mu^2 + 1)(\mu - \delta)} - \frac{V}{\delta (\delta^2 + 1)(\mu - \delta)}, \\
b_3 &= \frac{(-\mu - \delta) X}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \mu \delta) Y}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{\mu U}{(\mu^2 + 1)(\mu - \delta)} + \frac{\delta V}{(\delta^2 + 1)(\mu - \delta)}, \\
b_4 &= \frac{(1 - \mu \delta) X}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \mu \delta) Y}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{\mu^2 U}{(\mu^2 + 1)(\mu - \delta)} + \frac{\delta^2 V}{(\delta^2 + 1)(\mu - \delta)}, \\
b_5 &= \frac{(-\mu - \delta) X}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{(1 - \mu \delta) Y}{(\mu^2 + 1)(\delta^2 + 1)} + \frac{\mu U}{(\mu^2 + 1)(\mu - \delta)} - \frac{\delta V}{(\delta^2 + 1)(\mu - \delta)}.
\end{align*}
\]

System (56) has a unique singular point at the origin and the eigenvalues of the linear part of this system are \( \pm i, 0, \mu, \) and \( \delta \). By the linear transformation

\[
\begin{pmatrix}
X \\
Y \\
Z \\
U \\
V
\end{pmatrix} =
\begin{pmatrix}
0 & -\mu \delta & \mu + \delta & -1 & 0 \\
0 & 0 & \mu \delta & -\mu - \delta & 1 \\
\mu \delta & -\mu - \delta & 1 + \mu \delta & -\mu - \delta & 1 \\
0 & -\delta & 1 & -\delta & 1 \\
0 & -\mu & 1 & -\mu & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix},
\]

(57)
Note that the linear part of system (58) is in the real Jordan normal form of the linear part of system (56). We pass now from the Cartesian coordinates \((X, Y, Z, U, V)\) to the cylindrical ones \((r, \theta, Z, U, V)\) with \(X = r \cos \theta, Y = r \sin \theta,\) and we obtain

\[
\begin{align*}
\dot{r} &= \varepsilon \sin \theta G(r, \theta, Z, U, V), \\
\dot{\theta} &= 1 + \frac{\varepsilon}{r} \cos \theta G(r, \theta, Z, U, V), \\
\dot{Z} &= \varepsilon G(r, \theta, Z, U, V), \\
\dot{U} &= \mu U + \varepsilon G(r, \theta, Z, U, V), \\
\dot{V} &= \delta V + \varepsilon G(r, \theta, Z, U, V),
\end{align*}
\]

(60)

where \(G(r, \theta, Z, U, V) = \bar{F}(r \cos \theta, r \sin \theta, Z, U, V)\).

After dividing by \(\dot{\theta}\) and simplifying, we find

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon \sin \theta G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dZ}{d\theta} &= \varepsilon G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dU}{d\theta} &= \mu U + \varepsilon \left(1 - \frac{\mu U}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dV}{d\theta} &= \delta V + \varepsilon \left(1 - \frac{\delta V}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2).
\end{align*}
\]

(61)

System (61) is now of the same form as system (33) with

\[
\begin{align*}
x &= \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta,
\end{align*}
\]

\[
F_0(x, \theta) = \begin{pmatrix} 0 \\ 0 \\ \mu U \\ \delta V \end{pmatrix},
\]

\[
F_1(x, \theta) = \begin{pmatrix} \sin \theta G(r, \theta, Z, U, V) \\ G(r, \theta, Z, U, V) \\ \left(1 - \frac{\mu U}{r} \cos \theta\right) G(r, \theta, Z, U, V) \\ \left(1 - \frac{\delta V}{r} \cos \theta\right) G(r, \theta, Z, U, V) \end{pmatrix}.
\]

(62)

By the notations of Theorem 6, we have that \(k = 2\) and \(n = 4\). Let \(R > 0\); we take

\[
\mathcal{V} = \{(r_0, Z_0) : 0 < r_0^2 + Z_0^2 < R\} \subset \mathbb{R}^2,
\]

(64)

\[
\alpha = (r_0, Z_0) \in \mathcal{V}, \text{ and }
\]

\[
\beta_0 : C^1(\mathcal{V}) \rightarrow \mathbb{R}^2, \quad (r_0, Z_0) \mapsto \beta_0(r_0, Z_0) = (0, 0).
\]

(65)
We also take
\[ \mathcal{L} = \{ \mathbf{z}_n = (r_0, Z_0, 0, 0), (r_0, Z_0) \in \mathcal{V} \}. \]  

(66)

The fundamental matrix \( M_{x_n}(\theta) \) of the linear system (61) with \( \varepsilon = 0 \) with respect to the periodic solution \( \mathbf{z}_n = (r_0, Z_0, 0, 0) \) satisfying that \( M_{x_n}(0) \) is the identity matrix is
\[
M_{x_n}(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & e^\mu \theta & 0 \\
0 & 0 & 0 & e^{\delta \theta}
\end{pmatrix}.
\]  

(67)

We have
\[
M_{x_n}^{-1}(0) - M_{x_n}^{-1}(2\pi) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 - e^{-2\mu} & 0 \\
0 & 0 & 0 & 1 - e^{-2\delta}
\end{pmatrix},
\]  

(68)

which satisfy the assumption (ii) of Theorem 6. Taking
\[ \xi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \]

(69)

we must compute the function \( \mathcal{F}(a) \) given by (36), and we obtain
\[
\mathcal{F}(a) = \mathcal{F}(r_0, Z_0) = \frac{1}{2\pi} \left( \sin \theta G(r_0, \theta, Z_0, 0, 0) \right),
\]

(70)

\[
d\theta = \left( \begin{array}{c}
\mathcal{F}_1(r_0, Z_0) \\
\mathcal{F}_2(r_0, Z_0)
\end{array} \right),
\]

(71)

where
\[
\mathcal{F}_1(r_0, Z_0) = \frac{1}{2\pi} \left( \int B_1, B_2, B_3, B_4, B_5 \right) d\theta,
\]

and \( B_1, B_2, B_3, B_4, \) and \( B_5 \) are given by (12). Then, by Theorem 6, for every simple zero \( (r_0^*, Z_0^*) \) of the function \( \mathcal{F}(r_0, Z_0) \) there exists a limit cycle \( (r, Z, U, V)(\theta, \varepsilon) \) of system (61) such that
\[
(r, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0), \text{ when } \varepsilon \rightarrow 0. \]  

(72)

Going back through the change of coordinates, we obtain a limit cycle \( (r, \theta, Z, U, V)(t, \varepsilon) \) of system (60) such that
\[
(r, \theta, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{ when } \varepsilon \rightarrow 0. \]  

(73)

We have a limit cycle \( (X, Y, Z, U, V)(t, \varepsilon) \) of system (58) such that
\[
(X, Y, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0, 0), \text{ when } \varepsilon \rightarrow 0 \]  

(74)

Finally, we obtain a limit cycle \( x(t, \varepsilon) \) of equation (3) tending to the periodic solution (14) of equation (15) when \( \varepsilon \rightarrow 0 \).

Theorem 2 is proved.

3.2.2. Proof of Corollary 2. If \( F(x, \dot{x}, \ddot{x}, \cdot) = x^2 - 2x + x^2 \), then we have
\[
\mathcal{F}_1(r_0, Z_0) = \frac{r_0(\mu \delta - 1)(\mu \delta - Z_0)}{\mu \delta (1 + \mu)(1 + \delta)}.
\]

(75)

The system \( \mathcal{F}_1(r_0, Z_0) = \mathcal{F}_2(r_0, Z_0) = 0 \) has the zero \( (r_0^*, Z_0^*) = (\sqrt{1 + \mu^2}(1 + \delta^2), \mu \delta) \) such that
\[
\det \left( \frac{\partial (\mathcal{F}_1, \mathcal{F}_2)}{\partial (r_0, Z_0)} \right) \big|_{(r_0, Z_0) = (r_0^*, Z_0^*)} = \frac{2(\mu \delta - 1)}{\mu \delta (1 + \mu)(1 + \delta)} \neq 0.
\]

(76)

The proof of Corollary 2 follows directly by applying Theorem 2 and (16) is obtained by substituting \( (r_0^*, Z_0^*) \) in (14).

3.3. Proofs in Case 3: \( \lambda = 0 \)

3.3.1. Proof of Theorem 3. If \( \lambda = 0, \mu = \delta \neq 0 \), then equation (3) can be written as
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= u, \\
\dot{u} &= v, \\
\dot{v} &= -\mu^2 y + 2\mu z - (1 + \mu^2) u + 2\mu v + \varepsilon F(x, y, z, u, v).
\end{align*}
\]  

(77)

System (77) has a unique singular point at the origin and the eigenvalues of the linear part of this system are \( \pm i, 0 \), and \( \mu \). By the linear transformation
\[
\begin{pmatrix}
X \\
Y \\
Z \\
U \\
V
\end{pmatrix} = \begin{pmatrix}
0 & \mu^2 & -2\mu & 1 & 0 \\
0 & 0 & -\mu^2 & 2\mu & -1 \\
0 & -\mu^2 & -2\mu & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & -\mu & 1 & -\mu & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix},
\]  

(78)

we transform system (77) into the following system:
\[
\begin{align*}
\dot{X} &= -Y, \\
\dot{Y} &= X - \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{Z} &= \varepsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{U} &= \mu U + V, \\
\dot{V} &= \mu V + \varepsilon \tilde{F}(X, Y, Z, U, V),
\end{align*}
\]  
where \( \tilde{F} = \tilde{F}(X, Y, Z, U, V) = F(c_1, c_2, c_3, c_4, c_5) \),

\[
\begin{align*}
c_1 &= \frac{2 \mu X}{(\mu^2 + 1)^2} + \frac{(\mu^2 - 1)Y}{(\mu^2 + 1)^2} - \frac{U}{(\mu^2 + 1)^2} - \frac{Z}{\mu^2 + 1} - \frac{(3 \mu^2 + 1)V}{\mu^2 (\mu^2 + 1)^2}, \\
c_2 &= \frac{(\mu^2 - 1)X}{(\mu^2 + 1)^2} + \frac{2 \mu Y}{(\mu^2 + 1)^2} + \frac{U}{\mu^2 + 1} - \frac{2 \mu V}{(\mu^2 + 1)^2}, \\
c_3 &= -\frac{2 \mu X}{(\mu^2 + 1)^2} - \frac{(\mu^2 - 1)Y}{(\mu^2 + 1)^2} - \frac{\mu U}{(\mu^2 + 1)} + \frac{(\mu^2 - 1)V}{(\mu^2 + 1)^2}, \\
c_4 &= -\frac{(\mu^2 - 1)X}{(\mu^2 + 1)^2} + \frac{2 \mu Y}{(\mu^2 + 1)^2} + \frac{\mu^2 U}{(\mu^2 + 1)^2} + \frac{2 \mu V}{(\mu^2 + 1)^2}, \\
c_5 &= \frac{2 \mu X}{(\mu^2 + 1)^2} + \frac{(\mu^2 - 1)Y}{(\mu^2 + 1)^2} + \frac{\mu U}{(\mu^2 + 1)^2} + \frac{\mu^2 (\mu^2 + 3)V}{(\mu^2 + 1)^2}.
\end{align*}
\]

(80)

Note that the linear part of system (81) is in the real Jordan normal form of the linear part of system (77). We pass now from the Cartesian coordinates \((X, Y, Z, U, V)\) to the cylindrical ones \((r, \theta, Z, U, V)\) with \(X = r \cos \theta, Y = r \sin \theta\), and we obtain

\[
\begin{align*}
\dot{r} &= -\varepsilon \sin \theta G(r, \theta, Z, U, V), \\
\dot{\theta} &= 1 + \frac{\varepsilon}{r} \cos \theta G(r, \theta, Z, U, V), \\
\dot{Z} &= \varepsilon G(r, \theta, Z, U, V), \\
\dot{U} &= \mu U + V, \\
\dot{V} &= \mu V + \varepsilon G(r, \theta, Z, U, V),
\end{align*}
\]  
where \(G(r, \theta, Z, U, V) = \tilde{F}(r \cos \theta, r \sin \theta, Z, U, V)\).

After dividing by \(\theta\) and simplifying, we find

\[
\begin{align*}
\frac{dr}{d\theta} &= -\varepsilon \sin \theta G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dZ}{d\theta} &= \varepsilon G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dU}{d\theta} &= \mu U + V + \frac{\mu U + V}{r} \cos \theta G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dV}{d\theta} &= \mu V + \varepsilon \left(1 + \frac{\mu V}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2).
\end{align*}
\]  

(82)

System (82) is now of the same form as system (33) with

\[
F_0(x, \theta) = \begin{pmatrix}
  0 \\
  0 \\
  \mu U + V \\
  \mu V \\
\end{pmatrix},
\]

(83)

and

\[
F_1(x, \theta) = \begin{pmatrix}
-\sin \theta G(r, \theta, Z, U, V) \\
G(r, \theta, Z, U, V) \\
\frac{\mu U + V}{r} \cos \theta G(r, \theta, Z, U, V) \\
\left(1 + \frac{\mu V}{r} \cos \theta\right) G(r, \theta, Z, U, V)
\end{pmatrix}.
\]

We shall apply Theorem 6 to system (82). System (82) with \(\varepsilon = 0\) has the 2\(\pi\)-periodic solutions

\[
\begin{pmatrix}
  r(\theta) \\
  Z(\theta) \\
  U(\theta) \\
  V(\theta)
\end{pmatrix} = \begin{pmatrix}
  r_0 \\
  Z_0 \\
  0 \\
  0
\end{pmatrix},
\]

\(\forall r_0 > 0, \forall Z_0 \in \mathbb{R}\).

(84)
By the notations of Theorem 6, we have that $k = 2$ and $n = 4$. Let $R > 0$; we take
\[ V = \{ (r_0, Z_0) : 0 < r_0^2 + Z_0^2 < R \} \subset \mathbb{R}^2. \]  \hfill (85)
\[ \alpha = (r_0, Z_0) \in V \] and
\[ \beta : C_l(V) \rightarrow \mathbb{R}^2, \]\[ : (r_0, Z_0) \rightarrow \beta_0(r_0, Z_0) = (0, 0). \]  \hfill (86)
We also take
\[ \mathcal{L} = \{ z = (r_0, Z_0, 0, 0), (r_0, Z_0) \in V \}. \]  \hfill (87)
The fundamental matrix $M_{\alpha}(\theta)$ of the linear system (82) with $\varepsilon = 0$ with respect to the periodic solution $\mathbf{z}_0 = (r_0, Z_0, 0, 0)$ satisfying that $M_{\alpha}(0)$ is the identity matrix is
\[ M_{\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{pmatrix}. \]  \hfill (88)
We have
\[ M_{\alpha}^{-1}(0) - M_{\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -e^{-2\pi} & 2\pi e^{-2\pi} \\ 0 & 0 & 0 & 1 - e^{-2\pi} \end{pmatrix}, \]  \hfill (89)
which satisfy the assumption (ii) of Theorem 6. Taking
\[ \xi : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \]\[ : (r, Z, U, V) \rightarrow \xi(r, Z, U, V) = (r, Z), \]  \hfill (90)
we must compute the function $\mathcal{F}(\alpha)$ given by (36), and we obtain
\[ \mathcal{F}(\alpha) = \mathcal{F}(r_0, Z_0) = \frac{1}{2\pi} \left\{ -\sin \theta G(r_0, \theta, Z_0, 0, 0) \right\} d\theta \]\[ = \left( \mathcal{F}_1(r_0, Z_0) \right) - \left( \mathcal{F}_2(r_0, Z_0) \right), \]  \hfill (91)
where
\[ \mathcal{F}_1(r_0, Z_0) = \frac{1}{2\pi} - \sin \theta F(C_1, C_2, C_3, C_4, C_5) d\theta, \]  \hfill (92)
\[ \mathcal{F}_2(r_0, Z_0) = \frac{1}{2\pi} F(C_1, C_2, C_3, C_4, C_5) d\theta, \]  \hfill (92)
and $C_1, C_2, C_3, C_4$, and $C_5$ are given by (18). Then, by Theorem 6, for every simple zero $(r_0^*, Z_0^*)$ of the function $\mathcal{F}(r_0, Z_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \varepsilon)$ of system (82) such that
\[ (r, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0), \] \hfill (93)
when $\varepsilon \rightarrow 0$.

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (81) such that
\[ (r, \theta, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0, 0), \] \hfill (94)
when $\varepsilon \rightarrow 0$.

We have a limit cycle $(X, Y, Z, U, V)(t, \varepsilon)$ of system (79) such that
\[ (X, Y, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, Z_0^*, 0, 0, 0), \] \hfill (95)
when $\varepsilon \rightarrow 0$.

Finally, we obtain a limit cycle $x(t, \varepsilon)$ of equation (3) tending to the periodic solution (20) of equation (21) when $\varepsilon \rightarrow 0$.

Theorem 3 is proved.

3.3.2. Proof of Corollary 3. If $F(x, \dot{x}, \ddot{x}, \dddot{x}, x) = x^2 - xx - 1$, then we have
\[ \mathcal{F}_1(r_0, Z_0) = \frac{1}{(1 - \mu + \mu^2)r_0Z_0}, \] \hfill (96)
\[ \mathcal{F}_2(r_0, Z_0) = \frac{r_0^2}{2(1 + \mu^2)^2} - \frac{Z_0^2}{\mu^2} + 1. \]

The system $\mathcal{F}_1(r_0, Z_0) = \mathcal{F}_2(r_0, Z_0) = 0$ has the zero $(r_0^*, Z_0^*) = (\sqrt{2}(1 + \mu^2), 0)$ such that
\[ \det \begin{pmatrix} \partial \mathcal{F}_1 \partial \mathcal{F}_2 \end{pmatrix} = \begin{pmatrix} (r_0, Z_0) \end{pmatrix} = \begin{pmatrix} -2(\mu^2 + \mu - 1) \mu^3(1 + \mu^2)^2 \neq 0. \] \hfill (97)

The proof of Corollary 3 follows directly by applying Theorem 3 and (22) is obtained by substituting $(r_0^*, Z_0^*)$ in (20).

3.4. Proofs in Case 4: $\lambda \neq 0$ and $\mu = \delta \neq 0$

3.4.1. Proof of Theorem 4. If $\lambda \neq 0$ and $\mu = \delta \neq 0$, then equation (3) can be written as
\[ \begin{cases} \dot{x} = y, \\
\dot{y} = z, \\
\dot{z} = u, \\
\dot{u} = v, \\
\dot{v} = \lambda \mu \dot{x} - \left( \mu^2 + 2\lambda \mu \right)y + \left( \lambda + 2\mu + \lambda \mu^2 \right)z \\
- \left( \mu^2 + 2\lambda \mu + 1 \right)u + (\lambda + 2\mu)v + eF(x, y, z, u, v). \end{cases} \] \hfill (98)

System (98) with $\varepsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i, \lambda$, and $\mu$. Using the change of variables
we transform the system (98) into the following system:

\[
\begin{align*}
F &= \bar{F}(X, Y, Z, U, V) = F(d_1, d_2, d_3, d_4, d_5), \\
d_1 &= \frac{(-\lambda^2 + 2\mu + \lambda)X}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{(\mu^2 + 2\lambda - 1)Y}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{Z}{(\lambda^2 + 1)(\lambda - \mu)^2} - \frac{U}{(\mu^2 + 1)(\lambda - \mu)^2} - \frac{(3\mu^2 - 2\lambda + \mu + 1)V}{(\mu^2 + 1)^2}, \\
d_2 &= \frac{(-\lambda^2 + 2\mu + \lambda)X}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{(\mu^2 + 2\lambda - 1)Y}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{\lambda Z}{(\lambda^2 + 1)(\lambda - \mu)^2} - \frac{\mu U}{(\mu^2 + 1)(\lambda - \mu)^2} - \frac{(2\mu^2 - \lambda^2 + \lambda)V}{(\mu^2 + 1)^2}, \\
d_3 &= \frac{(-\lambda^2 + 2\mu + \lambda)X}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{(\mu^2 + 2\lambda - 1)Y}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{\lambda^2 Z}{(\lambda^2 + 1)^2} - \frac{\mu U}{(\mu^2 + 1)(\lambda - \mu)^2} - \frac{(\mu^3 - \mu + 2\lambda)V}{(\mu^2 + 1)^2}, \\
d_4 &= \frac{(-\lambda^2 + 2\mu + \lambda)X}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{(\mu^2 + 2\lambda - 1)Y}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{\lambda^2 Z}{(\lambda^2 + 1)(\lambda - \mu)^2} - \frac{\mu U}{(\mu^2 + 1)^2} - \frac{(\mu^2 \lambda^2 - 2\mu + 3\lambda)V}{(\mu^2 + 1)^2}, \\
d_5 &= \frac{(-\lambda^2 + 2\mu + \lambda)X}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{(\mu^2 + 2\lambda - 1)Y}{(\lambda^2 + 1)(\mu^2 + 1)} + \frac{\lambda^4 Z}{(\lambda^2 + 1)^2} - \frac{\mu^4 U}{(\mu^2 + 1)^2} - \frac{\mu^2 (-\mu^3 + 2\mu^2 - 3\mu + 4\lambda)V}{(\mu^2 + 1)^2}.
\end{align*}
\]

Note that the linear part of system (100) is in the real normal Jordan form of the linear part of system (98). We pass now from the Cartesian coordinates \((X, Y, Z, U, V)\) to the cylindrical ones \((r, \theta, Z, U, V)\) with \(X = r \cos \theta\), \(Y = r \sin \theta\), and we obtain

\[
\begin{align*}
\dot{r} &= -\varepsilon \sin \theta G(r, \theta, Z, U, V), \\
\dot{\theta} &= 1 + \frac{\varepsilon \cos \theta G(r, \theta, Z, U, V)}{r}, \\
\dot{Z} &= \lambda Z + \varepsilon G(r, \theta, Z, U, V), \\
\dot{U} &= \mu U + V, \\
\dot{V} &= \mu V + \varepsilon G(r, \theta, Z, U, V),
\end{align*}
\]

where \(G(r, \theta, Z, U, V) = \bar{F}(r \cos \theta, r \sin \theta, Z, U, V)\).

After dividing by \(\theta\) and simplifying, we find

\[
\begin{align*}
\frac{dr}{d\theta} &= -\varepsilon \sin \theta G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dZ}{d\theta} &= \lambda Z + \varepsilon \left(1 + \frac{\lambda Z}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dU}{d\theta} &= \mu U + V + \varepsilon \left(1 + \frac{\mu U + V}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dV}{d\theta} &= \mu V + \varepsilon \left(1 + \frac{\mu V}{r} \cos \theta\right) G(r, \theta, Z, U, V) + o(\varepsilon^2).
\end{align*}
\]

System (103) is now of the same form as system (33) with
We shall apply Theorem 6 to system (103). System (103) with \( \varepsilon = 0 \) has the \( 2\pi \)-periodic solutions

\[
\begin{pmatrix}
  r(
  \theta
), \\
  Z(\theta), \\
  U(\theta), \\
  V(\theta)
\end{pmatrix}
= \begin{pmatrix}
  r_0, \\
  0, \\
  0, \\
  0
\end{pmatrix}, \quad \forall r_0 > 0.
\]

By the notations of Theorem 6, we have that \( k = 1 \) and \( n = 4 \). Let \( r_1 > 0 \) and \( r_2 > 0 \); we take \( \forall = \{ r_1, r_2 \} \subset \mathbb{R} \), \( a = r_0 \in [r_1, r_2] \), and

\[
\beta_0: [r_1, r_2] \rightarrow \mathbb{R}^3, \\
: r_0 \mapsto \beta_0(r_0) = (0, 0, 0).
\]

We also take

\[
\mathcal{K} = \{ z_a = (r_0, 0, 0, 0), \quad r_0 \in [r_1, r_2] \}.
\]

The fundamental matrix \( M_{z_a}(\theta) \) of the linear system (103) with \( \varepsilon = 0 \) with respect to the periodic solution \( z_a = (r_0, 0, 0, 0) \) satisfying that \( M_{z_a}(0) \) is the identity matrix is

\[
M_{z_a}^{-1}(0) - M_{z_a}^{-1}(2\pi) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - e^{-2\pi \lambda} & 0 & 0 \\
0 & 0 & 1 - e^{-2\pi \mu} & 2\pi e^{-2\pi \mu} \\
0 & 0 & 0 & 1 - e^{-2\pi \mu}
\end{pmatrix}
\]

which satisfy the assumption (ii) of Theorem 6. Taking

\[
\begin{align*}
\xi: \mathbb{R} \times \mathbb{R}^3 & \rightarrow \mathbb{R}, \\
(r, Z, U, V) & \mapsto \xi(r, Z, U, V) = r,
\end{align*}
\]

we must compute the function \( \mathcal{F}(r_0) \) given by (34), and we obtain

\[
\mathcal{F}(r_0) = \frac{1}{2\pi} \sin \theta G(r_0, \theta, 0, 0, 0) \, d\theta,
\]

and \( D_1, D_2, D_3, D_4, \) and \( D_5 \) are given by (24). Then, by Theorem 6, for every simple zero \( r_0^* \) of the function \( \mathcal{F}(r_0) \), there exists a limit cycle \((r, Z, U, V)(\theta, \varepsilon)\) of system (103) such that

\[
(r, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, 0, 0, 0), \text{ when } \varepsilon \rightarrow 0. \quad (112)
\]

Going back through the change of coordinates, we obtain a limit cycle \((r, \theta, Z, U, V)(t, \varepsilon)\) of system (102) such that

\[
(r, \theta, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, 0, 0, 0, 0), \text{ when } \varepsilon \rightarrow 0. \quad (113)
\]

We have a limit cycle \((X, Y, Z, U, V)(t, \varepsilon)\) of system such that

\[
(X, Y, Z, U, V)(0, \varepsilon) \rightarrow (r_0^*, 0, 0, 0, 0), \text{ when } \varepsilon \rightarrow 0. \quad (114)
\]

Finally, we obtain a limit cycle \( x(t, \varepsilon) \) of equation (3) tending to the periodic solution (25) of equation (26) when \( \varepsilon \rightarrow 0. \)

Theorem 4 is proved.

3.4.2. Proof of Corollary 4. If \( F(x, \dot{x}, \ddot{x}, x) = x^3 - x - 1 \), then we have

\[
\mathcal{F}(r_0) = \frac{r_0 \left( -1 + 2\mu + \mu^2 \right) \left( 4(1 + \lambda^2)(1 + \mu^2)^2 - 3r_0^2 \right)}{8(1 + \lambda^2)^2(1 + \mu^2)^4},
\]

which have the real positive simple zero \( r_0^* = 2\sqrt{\frac{3}{5}}/\sqrt{(1 + \lambda^2)(1 + \mu^2)} \) with

\[
\frac{dF}{dr_0}(r_0^*) = \frac{1 - 2\lambda \mu - \mu^2}{(1 + \lambda^2)(1 + \mu^2)} \neq 0.
\]
The proof of Corollary 4 follows directly by applying Theorem 4 and (27) is obtained by substituting $r^*_0$ in (25).

3.5. Proofs in Case 5: $\lambda = \mu = \delta \neq 0$

3.5.1. Proof of Theorem 5.

If $\lambda = \mu = \delta \neq 0$, then equation (3) can be written as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= u, \\
\dot{u} &= v, \\
\dot{v} &= \lambda^2 x - 3\lambda^2 y + (3\lambda + \lambda^3) z - (3\lambda^2 + 1) u + 3\lambda v + \epsilon F(x, y, z, u, v).
\end{align*}
\]

(System (117))

System (117) with $\epsilon = 0$ has a unique singular point at the origin and the linear part of this system has the eigenvalues $\pm i$ and $\lambda$. Using the change of variables

\[
\begin{pmatrix}
X \\
Y \\
Z \\
U \\
V
\end{pmatrix} =
\begin{pmatrix}
\lambda^3 - 3\lambda^2 & 3\lambda & -1 & 0 \\
0 & -\lambda^3 & 3\lambda^2 & -3\lambda & 1 \\
1 & 0 & 1 & 0 & 0 \\
-\lambda & 1 & -\lambda & 1 & 0 \\
\lambda^2 - 2\lambda & \lambda^2 + 1 & -2\lambda & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
u \\
v
\end{pmatrix},
\]

we transform system (117) into the following system:

\[
\begin{align*}
\dot{X} &= -Y, \\
\dot{Y} &= X + \epsilon \tilde{F}(X, Y, Z, U, V), \\
\dot{Z} &= \lambda Z + U, \\
\dot{U} &= \lambda U + V, \\
\dot{V} &= \lambda V + \epsilon \tilde{F}(X, Y, Z, U, V),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{F} &= F(X, Y, Z, U, V) = F(e_1, e_2, e_3, e_4, e_5), \\
e_1 &= -\lambda(\lambda^2 - 3)X + (1 - 3\lambda^2)Y + (\lambda^2 + 1)^2 Z - 2\lambda(\lambda^2 + 1)U + (3\lambda^2 - 1)V \\
e_2 &= (1 - 3\lambda^2)X - \lambda(\lambda^2 - 3)Y + \lambda(\lambda^2 + 1)^2 Z + (1 - \lambda^4)U + \lambda(\lambda^2 - 3)V \\
e_3 &= -\lambda(\lambda^2 - 3)X - (1 - 3\lambda^2)Y + \lambda^2(\lambda^2 + 1)^2 Z + 2\lambda(\lambda^2 + 1)U + (1 - 3\lambda^2)V \\
e_4 &= -(1 - 3\lambda^2)X + \lambda(\lambda^2 - 3)Y + \lambda^2(\lambda^2 + 1)^2 Z + \lambda^2(\lambda^2 + 1)(\lambda^2 + 1)U - \lambda(\lambda^2 - 3)V \\
e_5 &= \lambda(\lambda^2 - 3)X + (1 - 3\lambda^2)Y + \lambda^4(\lambda^2 + 1)^2 Z + 2\lambda^3(\lambda^2 + 2)(\lambda^2 + 1)U + \lambda^3(3\lambda^2 + \lambda^4 + 6)V.
\end{align*}
\]
Note that the linear part of system (119) is in the real normal Jordan form of the linear part of system (51). We pass now from the Cartesian coordinates \( (X, Y, Z, U, V) \) to the cylindrical ones \( (r, \theta, Z, U, V) \) with \( X = r \cos \theta \), \( Y = r \sin \theta \), and we obtain
\[
\begin{aligned}
\dot{r} &= \varepsilon \sin \theta G(r, \theta, Z, U, V), \\
\dot{\theta} &= 1 + \varepsilon \frac{r}{r} \cos \theta G(r, \theta, Z, U, V), \\
\dot{Z} &= \lambda Z + U, \\
\dot{U} &= \lambda U + V, \\
\dot{V} &= \lambda V + \varepsilon G(r, \theta, Z, U, V),
\end{aligned}
\] (121)
where \( G(r, \theta, Z, U, V) = \bar{F}(r \cos \theta, r \sin \theta, Z, U, V) \).

After dividing by \( \dot{\theta} \) and simplifying, we find
\[
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon \sin \theta G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dZ}{d\theta} &= \lambda Z + U - \varepsilon \left( \frac{\lambda Z + U}{r} \cos \theta \right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dU}{d\theta} &= \lambda U + V - \varepsilon \left( \frac{\lambda U + V}{r} \cos \theta \right) G(r, \theta, Z, U, V) + o(\varepsilon^2), \\
\frac{dV}{d\theta} &= \lambda V + \varepsilon \left( 1 - \frac{\lambda V}{r} \cos \theta \right) G(r, \theta, Z, U, V) + o(\varepsilon^2). \\
\end{aligned}
\] (122)

System (122) is now of the same form as system (33) with
\[
\begin{aligned}
x &= \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \\
F_0(x, \theta) &= \begin{pmatrix} 0 \\ \lambda Z + U \\ \lambda U + V \\ \lambda V \end{pmatrix}, \\
F_1(x, \theta) &= \begin{pmatrix} -\sin \theta G(r, \theta, Z, U, V) \\ -\frac{\lambda Z + U}{r} \cos \theta G(r, \theta, Z, U, V) \\ -\frac{\lambda U + V}{r} \cos \theta G(r, \theta, Z, U, V) \\ 1 - \frac{\lambda V}{r} \cos \theta G(r, \theta, Z, U, V) \end{pmatrix}, \\
F_2(x, \theta) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
\] (123)

We shall apply Theorem 6 to system (122). System (122) with \( \varepsilon = 0 \) has the \( 2\pi \)-periodic solutions
\[
\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall r_0 > 0.
\] (124)

By the notations of Theorem 6, we have that \( k = 1 \) and \( n = 4 \). Let \( r_1 > 0 \) and \( r_2 > 0 \); we take \( \psi = ]r_1, r_2[ \subset \mathbb{R} \), \( \alpha = r_0 \in [r_1, r_2] \), and
\[
\beta_0: [r_1, r_2] \to \mathbb{R}^3, \\
\beta_0: r_0 \mapsto \beta_0(r_0) = (0, 0, 0).
\] (125)

We also take
\[
\beta_0: [r_1, r_2] \to \mathbb{R}^3, \\
\beta_0: r_0 \mapsto \beta_0(r_0) = (0, 0, 0).
\] (125)
The fundamental matrix $M_{z_k}(\theta)$ of the linear system (122) with $\epsilon = 0$ with respect to the periodic solution $z_k = (r_0, 0, 0, 0)$ satisfying that $M_{z_k}(0)$ is the identity matrix is

$$M_{z_k}(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i\theta} & 0 & 0 \\
0 & 0 & e^{i\theta} & 0 \\
0 & 0 & 0 & e^{i\theta}
\end{pmatrix}. \quad (127)
$$

We have

$$M_{z_k}^{-1}(0) - M_{z_k}^{-1}(2\pi) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 - e^{-2\pi\lambda} & 2\pi e^{-2\pi\lambda} & -2\pi^2 e^{-2\pi\lambda} \\
0 & 0 & 1 - e^{-2\pi\lambda} & 2\pi e^{-2\pi\lambda} \\
0 & 0 & 0 & 1 - e^{-2\pi\lambda}
\end{pmatrix}, \quad (128)
$$

which satisfy the assumption (ii) of Theorem 6. Taking

$$\xi: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (r, Z, U, V) \mapsto \xi(r, Z, U, V) = r, \quad (129)
$$

we must compute the function $\mathcal{F}(a)$ given by (36), and we obtain

$$\mathcal{F}(r_0) = \int_0^{2\pi} \sin \theta G(r_0, \theta, 0, 0, 0) d\theta, \quad (130)
$$

$$\mathcal{F}(a) = \int_0^{2\pi} \sin \theta G(E_1, E_2, E_3, E_4) a d\theta,
$$

and $E_1, E_2, E_3, E_4$, and $E_5$ are given by (29). Then, by Theorem 6, for every simple zero $r_0^*$ of the function $\mathcal{F}(r_0)$, there exists a limit cycle $(r, Z, U, V)(\theta, \epsilon)$ of system (122) such that

$$(r, Z, U, V)(0, \epsilon) \longrightarrow (r_0^*, 0, 0, 0), \text{ when } \epsilon \longrightarrow 0. \quad (131)
$$

Going back through the change of coordinates, we obtain a limit cycle $(r, \theta, Z, U, V)(t, \epsilon)$ of system (121) such that

$$(r, \theta, Z, U, V)(0, \epsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{ when } \epsilon \longrightarrow 0. \quad (132)
$$

We have a limit cycle $(X, Y, Z, U, V)(t, \epsilon)$ of system (119) such that

$$(X, Y, Z, U, V)(0, \epsilon) \longrightarrow (r_0^*, 0, 0, 0, 0), \text{ when } \epsilon \longrightarrow 0. \quad (133)
$$

Finally, we obtain a limit cycle $x(t, \epsilon)$ of equation (3) tending to the periodic solution (30) of equation (31) when $\epsilon \longrightarrow 0.$
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