Roles of Asymptotic Condition and S-Matrix as Micro-Macro Duality in QFT

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Abstract
Various versions of “independence” are actively investigated in quantum probability. In the context of relativistic QFT, we show here that the physical origin of “independence” can be sought in the asymptotic condition through which asymptotic fields and states exhibiting the independence emerge from the non-independent interacting Heisenberg fields in a kind of “central limit”. From the algebraic viewpoint, this condition is equivalent to the on-shell condition to pick up free one-particle modes, which also reduces to Einstein’s famous formula $E = mc^2$. A scenario to reconstruct interacting Heisenberg fields as Micro-objects from these “independent”=free Macro-objects intertwined by an S-matrix as a measurable quantity is formulated according to the Micro-Macro Duality associated with a new notion of a cocycle of K-T operators.

1 Two questions: What do independence and $E = mc^2$ mean?

In the theory of quantum probability, several versions of independence have been formulated with interesting results as generalizations of the bosonic tensor type. My naïve questions here are on which physical grounds they have appeared and what physical meanings they have. At least for the familiar Gaussian cases (=quasi-free states on bosonic CCR or fermionic CAR), my partial answers in the context of relativistic QFT will be given as follows:

1) Emergence of independence through the asymptotic condition $\varphi_H(x) \xrightarrow{x^0=t\rightarrow\mp\infty} \phi^{in/out}(x)$:

From non-independent interacting Heisenberg fields $\varphi_H$, the asymptotic fields $\phi^{as} = \phi^{in/out}$ and asymptotic states generated by $\phi^{as}$ from the vacuum satisfying the independence arise through the asymptotic condition $\varphi_H(x) \xrightarrow{x^0=t\rightarrow\mp\infty} \phi^{in/out}(x)$, which can be interpreted as a sort of “central limit” theorem. The conceptual meaning of this “central limit” theorem can
be placed in the “Micro-Macro Duality” \[3\] in QFT as follows:

\[
\begin{align*}
\text{Micro} & \quad \text{asymptotic condition} & \quad \text{Macro} \\
\varphi_H : \text{generic} & \quad \iff \quad \phi^{as} : \text{universal} & & \text{GLZ expansion of } \varphi_H \text{ in } \phi^{as} \\
(\Box + m^2)\varphi_H = J_H & \iff \varphi_H = \Delta_{\text{ret}} \ast J_H + \phi^{in} & & p^2 = m^2 \iff (\Box + m^2)\phi^{as} = 0
\end{align*}
\]

(NB: There exists another local-net version of “statistical independence” based on the so-called nuclearity condition \[4\] in Algebraic QFT, which should not be confused with the present version.)

2) “Units” of independence identified with particles specified by Einstein’s famous formula \(E = mc^2\) (\(\Rightarrow\) Sec.2):

It is worth noting that this famous formula \(E = mc^2\) is meaningful only for asymptotic fields/states as the on-shell condition \(p^2 = p_{\mu}p^\mu = m^2\) to extract particles as independent = free = non-interacting entities from the interacting Heisenberg fields; in contrast, the latter do not satisfy this formula because of the interactions. The former, asymptotic fields and states, serve as the vocabulary for describing state changes taking place in the scattering processes in such a form as the state transitions described by the S-matrix \(\langle \beta, \text{out} | \alpha, \text{in} \rangle = \langle \beta | S | \alpha \rangle\) from an asymptotic incoming state (in-state, for short) \(| \alpha, \text{in} \rangle\) to an outgoing state (out-state, for short) \(| \beta, \text{out} \rangle\).

In the absence of interactions, however, the on-shell asymptotic fields \(\phi^{as}\) cannot by themselves ignite scattering processes, which necessitates the use of off-shell interacting Heisenberg fields \(\varphi_H\).

3) The logical basis of the asymptotic condition is to be found in the cluster property \[5\]:

\[
\langle \Omega | A_{\vec{x}}(B) \Omega \rangle \to \langle \Omega | A \Omega \rangle \langle \Omega | B \Omega \rangle,
\]

following from the ergodicity valid for a unique vacuum vector \(\Omega\) invariant under spacetime translations \(U(x)\) and from the assumption of the local commutativity (\(\Rightarrow\) Sec.3). In this context, asymptotic fields \(\phi^{as}\) can be understood as quantities to materialize kinematically this factorization (= independence) of correlations without taking limit and they can be decomposed into creation and annihilation operators \(a(\vec{p})a^\ast(\vec{q})\) which represent infinite number of conserved quantities.

4) Universality of “central limit” due to Haag-GLZ expansion (\(\Rightarrow\) Sec.5):

From the asymptotic condition, the Yang-Feldman equation \(\varphi_H = \Delta_{\text{ret}} \ast J_H + \phi^{in}\) can be derived with the Heisenberg source current \(J_H\) formally defined by \(J_H = (\Box + m^2)\varphi_H \[6\]. The analogue of the “Fock expansion” \[7\] in WNA can be found in the Haag-GLZ expansion \[8, 9\],

\[
SA =: (\omega_0 \otimes id)(T(A \otimes 1) \exp(iJ_H \otimes \phi^{in})) :,
\]
where \( \omega_0 = \langle \Omega| \cdots \Omega \rangle \) is the vacuum state and \( S =: (\omega_0 \otimes \text{id})(T(\exp(iJ_H \otimes \phi^a)) : \) is the \( S \)-matrix. By this formula, the Heisenberg observables \( A \) depending on \( \varphi_H \) can be expressed in terms of the asymptotic fields \( \phi^a \).

2 What does \( E = mc^2 \) mean?

While Einstein’s famous equality \( E = mc^2 \) between energy and mass has been regarded as one of the most fundamental consequences of the relativity theory, however, its actual content is simply the “on-shell condition” to pick up 1-particle modes, meaningful only for the independent = free = non-interacting asymptotic fields/states. In fact, taking \( m \) as “moving mass” \( m = m_0 \sqrt{1 - v^2/c^2} \), we have

\[
E = mc^2 = \frac{m_0}{\sqrt{1 - v^2/c^2}} \rightarrow (m_0 c)^2 = \frac{E}{c} - \left( \frac{m_0}{\sqrt{1 - v^2/c^2}} \right)^2 = \left( \frac{E}{c} - (\vec{p})^2 \right)
\]

where \( \frac{m_0}{\sqrt{1 - v^2/c^2}} \vec{v} =: \vec{p} \) is the relativistic 3-momentum and \( p^\mu = (E/c, \vec{p}) \) is the 4-momentum. The meaning of the above equality \( p^2 = p_\mu p^\mu = (\frac{E}{c}, \vec{p}) = (m_0 c)^2 \) can be understood as follows:

i) It is just the mass-shell (or, on-shell) condition to characterize a mass hyperboloid in the \( p \)-space of 4-momenta \( p_\mu \in \mathbb{R}^4 \) carried by the free 1-particle states with a rest mass \( m_0 \). The spacetime geometry inherent to the special relativity is controlled by the Poincaré group \( P_+^\uparrow = \mathbb{R}^4 \rtimes L_+^\uparrow \) (or, its universal covering \( \widetilde{P}_+^\uparrow = \mathbb{R}^4 \rtimes SL(2, \mathbb{C}) \)) defined by the semi-direct product of spacetime translation group \( \mathbb{R}^4 \) and the orthochronous proper Lorentz group \( L_+^\uparrow := \{ \Lambda = (\Lambda^\mu_\nu); \Lambda x \cdot \Lambda y = x \cdot y, \Lambda^0_0 > 0, \det(\Lambda) = +1 \} \) consisting of homogeneous Lorentz transformations \( \Lambda = (\Lambda^\mu_\nu) \in SO(1,3) \) leaving the Minkowski metric \( \eta(x,y) := x \cdot y = x^0 y^0 - \vec{x} \cdot \vec{y} \) invariant, \( \Lambda^T \eta \Lambda = \eta \), without changing the time direction \( \Lambda^0_0 > 0 \). In Wigner’s construction \( \Box \) of irreducible unitary representations of \( P_+^\uparrow \) or \( \widetilde{P}_+^\uparrow \), four orbit families, \( p^2 \geq 0 \) and \( p_\mu = 0 \), appear: \( p^2 = m_0^2 > 0 \) corresponds to a massive particle with a rest mass \( m_0 \), \( p^2 = 0 \) to massless particles, \( p^2 < 0 \) to (unphysical) “tachyons” (with an imaginary mass) and the last one to the vacuum, each of which is induced from one of the corresponding “little groups” \( SU(2), E(2), SU(1,1) \) and \( L_+^\uparrow \).
ii) Through the “first quantization” $p_\mu \to i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial \phi(x)}$, the Klein-Gordon equation $[\hbar^2 \partial_\mu \partial^\mu + (m_0c)^2]\phi(x) = 0$ describes a free scalar field $\phi(x)$ with rest mass $m_0$.

iii) The existence of positive/negative energy solutions $E = \pm \sqrt{(\vec{p}c)^2 + (m_0c)^2}$ of $(E/c)^2 - (\vec{p})^2 = m_0^2c^2$ is related with creation and annihilation operators, particle-antiparticle pairs, time reversal $T$ and PCT invariance.

Thus, the famous equivalence $E = mc^2$ between energy $E$ and mass $m$ gives only partial information for dynamical descriptions of relativistic quantum fields, with off-shell aspects being neglected in spite of their vital importance for non-trivial scattering processes, particle decays and productions, etc., etc.

3 Free= independent vs. interacting= non-independent

A free quantum field $\phi(x)$ as the quantized solution of Klein-Gordon equation $(\Box + m^2)\phi = 0$ describes “particle pictures” in terms of creation and annihilation operators $a(f), a^*(f)$ defined as follows:

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^32\omega_{\vec{p}}}}(a(\vec{p})\exp(-ip_\mu x^\mu) + h.c.),$$

$$a(f) : = i\int \overleftrightarrow{\partial_0} \phi(x)d^3x = \int \overrightarrow{\partial_0}a(\vec{p})d^3p,$$

$$a^*(f) : = i\int \overleftrightarrow{\partial_0} f(x)d^3x = \int a^*(\vec{p})\overleftarrow{\partial_0}f(\vec{p})d^3p = [a(f)]^*$$

$$[a(f), a^*(g)] = \int \overrightarrow{\partial_0}g(\vec{p})d^3p = \langle \vec{f}, \vec{g} \rangle,$$

$$[\phi(x), \phi(y)] = \int \frac{d^4p}{(2\pi)^4}\varepsilon(p^0)\delta(p^2 - m^2)\exp(-ip(x-y)) = i\Delta(x-y; m^2),$$

with $\omega_{\vec{p}} : = \sqrt{\vec{p}^2 + m^2}$ in the “natural unit system” with $\hbar = c = 1$ (rest mass $m_0$ is denoted by $m$, henceforth).

Although free quantum fields $\phi(x)$ with $a^*(\vec{p}), a(\vec{p})$ are customarily believed to be sufficient entities for describing wave-particle dualism inherent in elementary particles, the perpetual creation and annihilation processes of particles require interactions among elementary particles, which is not consistent with the linearity of free field equations. Concerning this point, the famous theorem of Haag [5, 6] has been taken as a kind of “no-go theorem” for the theoretical description of the interactions: Haag theorem: Poincaré (or even, Galilei)-covariant quantum fields related to free fields by a unitary transformation are only free fields. Owing to this theorem, it is meaningless to formulate the interacting Heisenberg fields by means of a unitary transformation of free fields as is commonly done in perturbative approaches. This is in sharp contrast to the quantum systems with finite degrees of freedom.
On the other hand, to describe relativistic scattering processes of elementary particles in a satisfactory way, we need inevitably the following ingredients: Poincaré-covariant quantum fields/their interactions/free asymptotic fields and states. Here, free fields are indispensable as the vocabulary for the description of scattering processes, where an initial state with incoming free particles is changed into a final one with outgoing particles.

Giving up the idea to connect directly Heisenberg and asymptotic fields (as is forbidden by the Haag theorem), we consider the mutual relations between two free asymptotic fields, $\phi^{\text{in}}(x)$ and $\phi^{\text{out}}(x)$ in terms of the unitary $S$-matrix $S$ to control the basis change between the in-state basis $|\alpha, \text{in}\rangle$ and the out-state basis $|\beta, \text{out}\rangle$:

$$|\alpha, \text{in}\rangle = \sum_{\beta} |\beta, \text{out}\rangle S_{\beta, \alpha}$$

with $S_{\beta, \alpha} := \langle \beta, \text{out}|\alpha, \text{in}\rangle = \langle \beta|S|\alpha\rangle$.

The schematic picture can be summarized as follows:

|          | in                      | out                  |
|----------|-------------------------|----------------------|
| asymptotic fields | $\text{Ad}\Theta^{\text{in}} \cap \phi^{\text{in}}(x)$ | $\text{Ad}S^{-1} \& \text{Ad}\Theta \cap \text{Ad}\Theta^{\text{out}} \phi^{\text{out}}(x)$ |
| GLZ \$\not\to\$ \ $t \to -\infty$ | asymp.cond. $\uparrow$ GLZ formula | $t \to +\infty$ $\not\to$ GLZ |
| Heisenberg fields | $\text{Ad}\Theta \cap \varphi_H(x)$ | $\text{Ad}\Theta^{\text{out}} \cap \varphi_H(x)$ |

To treat Heisenberg fields $\varphi_H(x)$, we recapitulate briefly the essence of Wightman axioms for relativistic quantum fields [5, 6] (in the vacuum representation $(\mathcal{P}, \mathcal{H}, U, \Omega)$) in the form of relativistic covariance, local commutativity, cyclicity or ergodicity of vacuum state and spectral condition:

a) [Heisenberg fields] as operator-valued distributions $\mathcal{D}(\mathbb{R}^4) \ni f \mapsto \varphi_H^i(f)$ with values being (unbounded) closable operators acting on a Hilbert space $\mathcal{H}$ are defined on the 4-dimensional Minkowski spacetime $(\mathbb{R}^4, \eta)$, where $\eta$ is the Minkowski metric: $\eta(x, y) := x \cdot y = x^0 y^0 - \vec{x} \cdot \vec{y}$.

b) [Relativistic covariance]: a local net $\mathcal{P} : \mathcal{K} \ni O \mapsto \mathcal{P}(O)$ of *-algebras generated by local fields $\varphi_H^i(f) = \int \varphi_H^i(x)f(x)d^4x$ with $f \in \mathcal{D}(\mathcal{O})$ and their polynomials defined on the net $\mathcal{K}$ of double cones $O = O_{a,b} = (a + V_\uparrow) \cap (b - V_\uparrow)$ (with the forward lightcone $V_\uparrow$) constitute a non-commutative covariant dynamical system,

$$\alpha_{a,\Lambda}(\varphi_H^i(x)) = U(a, \Lambda)\varphi_H^i(x)U(a, \Lambda)^{-1}$$

$$\alpha_{a,\Lambda}(\mathcal{P}(O)) = \mathcal{P}(\Lambda O + a),$$

under the action $\alpha, \mathcal{P}_+ \ni (a, \Lambda) \mapsto \alpha_{a,\Lambda} \in \text{Aut}(\mathcal{P}(\mathbb{R}^4))$, of Poincaré group $\mathcal{P}_+$ (or its covering $\tilde{\mathcal{P}}_+$) and $\mathcal{P}_+ \ni (a, \Lambda) \mapsto U(a, \Lambda) \in U(\mathcal{H})$ is its uni-
tary representation on $\mathfrak{h}$, and $s(\Lambda)^{\dagger}_{i}$ is a finite-dimensional representation of Lorentz group $L^\dagger_+$ associated with each field multiplet $(\varphi^i_H(x))_i$.

c) [Local commutativity]: the absence of propagation of physical effects exceeding the light velocity due to Einstein causality, implies the local commutativity of Heisenberg fields $\varphi^i_H(f)$:

$$[\varphi^i_H(f_1), \varphi^j_H(f_2)] = 0 \quad \text{if} \quad (\text{supp}f_1) \times (\text{supp}f_2)$$

where $\mathcal{O}_1 \times \mathcal{O}_2$ means that any pair of points $x \in \mathcal{O}_1, y \in \mathcal{O}_2$ are spacelike separated: $(x - y)^2 < 0$.

Remark: By this condition, the Fourier transform of Wightman functions

$$\omega_0(\varphi^i_H(x_1) \cdots \varphi^r_H(x_r))$$

as correlation functions of $\varphi^i_H$ in the vacuum state $\omega_0(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ defined in the next d) admits an analytic continuation into a holomorphic function in the complex energy-momentum space. According to it, dispersion relations are valid.

d) [Vacuum state and spectrum condition]:

d-i) Energy-momentum spectrum $\text{Sp}(U(\mathbb{R}^4))$ of spacetime translations $\mathbb{R}^4$ realized on $\mathfrak{h}$ is within the forward light cone, $\text{Sp}(U(\mathbb{R}^4)) \subset \mathbb{V}^+_{\mathbb{R}}$, and the lowest energy is realized by eigenvalue 0 of the vacuum vector $\Omega$: $U(x) := U(x, 1) = \int_{\mathbb{R}^4} \exp(i px) dE(p); \quad U(x)\Omega = \Omega$

Remark: Similarly to $p$-analyticity due to local commutativity, $x$-space analyticity of Wightman functions $\omega_0(\varphi^i_H(x_1) \cdots \varphi^r_H(x_r))$ follows from spectrum condition, which provides powerful tools for structural analysis.

d-ii) Cyclicity $P(\mathbb{R}^4)\Omega = \mathfrak{h}$ of $\Omega \iff$ irreducibility of $P(\mathbb{R}^4) \iff$ uniqueness of vacuum ($U(x)\Psi = \Psi \implies \Psi \propto \Omega$) $\iff$ cluster property:

$$|\omega_0(A(x)B(y)) - \omega_0(A)\omega_0(B)| \to 0 \quad \text{as} \quad (\vec{x} - \vec{y})^2 \to \infty,$$

where $A(x) := \alpha_x(A) = U(x)AU(x)^*,$ $B(y) := \alpha_y(B)$ are the spacetime translates of local observables $A, B \in \mathcal{P}(\mathcal{O})$ by $x, y \in \mathbb{R}^4$, respectively. This follows from partition of unity due to spectral resolution of spacetime translations $U(x)$:

$$1 = |\Omega\rangle\langle \Omega| + \sum_i (1\text{-particle singularities on mass-shell } p^2 = m_i^2)$$

+(absolutely continuous $p$-spectra)

4 Independence of asymptotic fields due to on-shell asymptotic condition as “central limit” theorem

From the cluster property and the local commutativity, follows the asymptotic condition $\varphi^i_H(x) \xrightarrow{\rho = \tau \to \pm \infty} \phi^i_{\text{in/out}}(x)$ (as weak convergence), according to which asymptotic fields $\phi^i_{\text{as}}$ materialize kinematically the factorization (= independence) of correlations without taking limits: any $n$-point
functions of $\phi^{as}$ are factorized into the products of two-point functions,
\[ \omega_0(\phi^{as} \phi^{as} \cdots \phi^{as}) = \sum \omega_0(\phi^{as} \phi^{as}) \cdots \omega_0(\phi^{as} \phi^{as}), \]
which is known as the “quasi-freeness” of the vacuum state $\omega_0$ with respect to $\phi^{as}$, familiar in
the form of “Wick theorem” or the independence of Gaussian type. The creation and annihilation operators
$a_k, a_k^*$ contained in $\phi^{as}$ constitute an infinite number of conserved quantities,
as they are given by the spatial integral of current densities $i\phi^{as}(x) \partial_\mu f(x)$ which are conserved,
\[ \partial_\mu [\phi^{as}(x) \partial_\mu f(x)] = \phi^{as}(x) \partial_\mu f(x) = \phi^{as}(x)(\Box + m^2)f(x) = 0 \]
by the on-shell conditions: $(\Box + m^2)\phi^{as}(x) = 0 = (\Box + m^2)f(x)$.
Thus, the independence embodied in the asymptotic fields $\phi^{as}$ is seen to emerge from interacting Heisenberg fields $\varphi_H$ via the asymptotic condition as a kind of “central limit” theorem. In this context, what corresponds to “Langevin equation” can be found in the Yang-Feldman equation [6] to connect the Heisenberg field $\varphi_H(x)$ and the asymptotic field $\phi^{as}(x)$:
\[ \varphi_H(x) = \int \Delta_{ret}(x-y;m^2)J_H(y)d^4y + \phi^{in}(x) = [\Delta_{ret} * J_H + \phi^{in}](x) \]
\[ = \int \Delta_{adv}(x-y;m^2)J_H(y)d^4y + \phi^{out}(x) = [\Delta_{adv} * J_H + \phi^{out}](x), \]
where $J_H = (\Box + m^2)\varphi_H$ is the Heisenberg source current and $\Delta_{ret/adv}(x-y;m^2)$: retarded/ advanced Green’s functions (i.e., principal solutions) of the Klein-Gordon equation defined by
\[ (\Box_x + m^2)\Delta_{ret/adv}(x-y;m^2) = \delta(x-y), \]
\[ \Delta_{ret/adv}(x-y;m^2) = 0 \quad \text{for} \quad x_0 \leq y_0. \]
In the Yang-Feldman equation, the asymptotic fields $\phi^{in/out}$ and Heisenberg source current $J_H$ appear, respectively, as the residue and the quotient in the division of $\varphi_H$ by $\Delta_{ret/adv}$. What is more important is that $J_H$ gives the residues at the on-shell pole $\frac{1}{p^2 - m^2}$ to determine matrix elements of scattering amplitudes, as will be seen in the next formula.

5 Mutual relations between Heisenberg fields and asymptotic fields controlled by Micro-Macro Duality

From the asymptotic condition and the LSZ reduction formulae [2], one can derive the Haag-GLZ formulæ [8] to express Heisenberg operators $A$ in
terms of the Wick products $\phi^{as} \cdots \phi^{as}$ of asymptotic fields:

$$S A = \exp(\phi^{in}(\Box + m^2) \frac{\delta}{\delta J}) : \omega_0(T(A \exp(i\varphi_H J)) \mid_{J=0}$$

$$= \sum_{k=0}^{\infty} \frac{i^k}{k!} \int dx_1 \cdots \int dx_k (\Box x_1 + m^2) \cdots (\Box x_1 + m^2) \omega_0(T(A \varphi_H(x_1) \cdots \varphi_H(x_k))$$

$$\times : \phi^{in}(x_1) \cdots \phi^{in}(x_k) :,$$

$$S = \exp(\phi^{as}(\Box + m^2) \frac{\delta}{\delta J}) : \omega_0(T(\exp(i\varphi_H J)) \mid_{J=0},$$

which is similar to the Fock expansion formula \([7]\) known in WNA. While the formulae of this type have long been known simply as those to expand the Heisenberg operators in the Wick products of $\phi^{as}$, what should be emphasized here are the following novel points:

1) We can reformulate these equalities into the following form \([9]\):

$$S A = : (\omega_0 \otimes id)(T(A \otimes 1) \exp(iJ_H \otimes \phi^{in})) :,$$

$$A = S^{-1} : (\omega_0 \otimes id)(T[A \otimes 1] \exp(iJ_H \otimes \phi^{in})) :$$

$$= : (\omega_0 \otimes id)(T[A \otimes 1] \exp(iJ_H \otimes \phi^{out}) : S^{-1},$$

$$S = : (\omega_0 \otimes id)(T \exp(iJ_H \otimes \phi^{in})) := (\omega_0 \otimes id)(T \exp(iJ_H \otimes \phi^{out}) : .$$

The roles played by the system consisting of Heisenberg operators $\varphi_H$ and those of asymptotic fields $\phi^{in}$ and $\phi^{out}$ are clearly separated here in the operational context in such a way that the former is an unknown target system to be detected and analyzed by means of the latter ones functioning as probe systems. The relevant coupling terms among them are specified by $\exp(iJ_H \otimes \phi^{as})$, with the first tensor factor to be time-ordered and the second one Wick ordered, which are mutually in duality, as will be seen by the relation between $\frac{\delta}{\delta \phi^{in}(x)}$ and $\phi^{in}(x)$ in 3). In this connection, the comparison to the notion of instruments in quantum measurements will be instructive: while a basic scheme for instruments can be seen in

neutral state of probe system measured values

\(\nearrow\) coupling between system & probe \(\searrow\)

initial state of system \(\rightarrow\) state changes \(\rightarrow\) final state of system

the corresponding one for scattering processes in QFT is given in sharp
It is remarkable that \( \exp(iQ) \) to the second tensor factor is the operator to create a coherent state in terms of a conserved charge. Perhaps, they can be unified by combining the level of the object system in the instrument with that of the asymptotic fields \( \phi^{as} \), which results in the successive measurement processes where the state changes taking place at the level of asymptotic fields in scattering processes of quantum fields are monitored by measuring such observables as the particle momenta or spins through the instruments.

2) The mathematical and conceptual meanings of the coupling term \( \exp(iJ_H \otimes \phi^{in}) \): We first note by simple computation that the on-shell condition \((\Box + m^2)\phi^{as} = 0\) for the asymptotic fields implies the following equalities:

\[
J_H \otimes \phi^{as} = (\Box + m^2)\varphi_H \otimes \phi^{as} = (\Box + m^2)\varphi_H \otimes \phi^{as} - \varphi_H \otimes (\Box + m^2)\phi^{as} = \Box \varphi_H \otimes \phi^{as} - \varphi_H \otimes \Box \phi^{as} - \partial^\mu \partial_\mu \varphi_H \otimes \phi^{as} = -\partial^\mu \varphi_H \otimes \partial_\mu \phi^{as}.
\]

Combing this with the asymptotic condition we can further rewrite this quantity \( iJ_H \otimes \phi^{in} \):

\[
i \int_{\mathbb{R}^4} d^4x J_H(x) \otimes \phi^{in}(x) = -i \int_{\mathbb{R}^4} d^4x \partial_\mu [\varphi_H \otimes \partial_\mu \phi^{in}] = -i \int_{\partial \mathbb{R}^4} dS_\mu [\varphi_H \otimes \partial_\mu \phi^{in}] = -i \int_{x^0 = +\infty} d^3x [\varphi_H \otimes \partial_0 \phi^{in}] + i \int_{x^0 = -\infty} d^3x [\varphi_H \otimes \partial_0 \phi^{in}] = -i(S^{-1} \otimes 1)iQ(S \otimes 1) + iQ = -(S^{-1} \otimes 1)[iQ, S \otimes 1] = -(S^{-1} \otimes 1)ad(iQ)(S \otimes 1),
\]

in terms of a conserved charge \( Q \) defined by

\[
iQ := i \int d^3x [\phi^{in} \otimes \partial_\mu \phi^{in}] = \sum_k [(a_k^{in})^* \otimes a_k^{in} - a_k^{in} \otimes (a_k^{in})^*].
\]

It is remarkable that \( \exp(iQ) \) (with the Wick product \( \cdots \)) to be applied to the second tensor factor is the operator to create a coherent state (or, exponential vector) \( \exp(iQ)\ket{\Omega} \) from the vacuum \( \ket{\Omega} \) with the Wick-ordered...
commutative parameters $a^m_k$. In the context of WNA, these quantities will correspond to the \textit{U-functionals} \cite{10} constituting a commutative algebra with respect to the Wick product.

3) Applying the above formula $A = S^{-1} : (\omega_0 \otimes id)(T[A \otimes 1] \exp(i J_H \otimes \phi^{in})) :$ to the Heisenberg source current $J_H = A$, we reproduce the expression

$$(\Box + m^2)\varphi_H = J_H = S^{-1} : (\omega_0 \otimes id)(T[J_H \otimes 1] \exp(i J_H \otimes \phi^{in})) : = S^{-1} \frac{\delta}{i \delta \phi^{in}(x)} S,$$

once obtained by Bogoliubov, Medvedev and Polivanov \cite{6}. Through the above relations, the asymptotic condition $\varphi_H \xrightarrow{t \to \pm \infty} \phi^{in/out}$ to be identified with the \textit{on-shell condition} $\Box + m^2 = 0 = m^2 - p^2$ is seen to play the essential role from the algebraic viewpoint: it extracts the asymptotic fields $\phi^{in/out}$ from the algebra of interacting Heisenberg fields $\varphi_H$ as the \textbf{fixed points under the Lie subgroup} $\Gamma$ generated by $\phi^{in}$ or $Q$ in the infinite-dimensional Heisenberg Lie group generated by $\phi^{in}$ and $\frac{\delta}{i \delta \phi^{in}(x)}$ with $\phi^{in}$ taken as an element belonging to the above commutative algebra with the Wick product. Under the action of $\Gamma$, $a_k^{in}$ and $(a_k^{in})^*$ are infinitely many conserved charges (to characterize the “integrability” of the probe system consisting of $\phi^{as}$). Thus the group $\Gamma$ generated by (the Wick-ordered) $a_k^{as}$ and $(a_k^{as})^*$ characterizes the aspects of the symmetry associated with the macroscopic on-shell situation described by $\phi^{as}$. The essential features here show the strong similarity to that of WNA if the functional derivatives $\frac{\delta}{i \delta \phi^{in}(x)}$ are put in parallel with the Hida derivatives \cite{10}.

4) \textbf{Breakdown of invariance} under $\Gamma$ due to the interactions: The symmetry $\Gamma$ preserved in the probe systems consisting of $\phi^{as}$ is broken in the total system containing the interacting Heisenberg fields $\varphi_H$, essentially due to the presence of $\frac{\delta}{i \delta \phi^{in}(x)}$ or the Heisenberg source current $J_H(x) = S^{-1} \frac{\delta}{i \delta \phi^{in}(x)} S = (\Box + m^2)\varphi_H(x) \neq 0$, which introduces the effects coming from the \textbf{off-shell} aspects of the theory. The non-trivial existence of the coupling term $i J_H \otimes \phi^{in} = -(S^{-1} \otimes 1)[iQ, S \otimes 1]$ is equivalent to the non-triviality $S \neq 1$ of the S-matrix which implies the difference between $\phi^{in}$ and $\phi^{out}$ due to $[iQ, S \otimes 1] \neq 0$. It is interesting to note the parallelism of this situation with the mutual relation between the first and second laws in Newtonian mechanics: since the interaction term due to a(n external) force is switched off in the stage of the first law, the system enjoys a symmetry to conserve the momentum (or velocity), similarly to the above constancy of $\phi^{in}$ or $Q$. At the stage of the second law, the symmetry inherent to the first law is \textit{broken} by introducing the (external) force $F$, which causes the \textbf{state changes} described by the \textit{changes} of the momentum $p$ according to the second law, $dp/dt = F$, in the essential use of the vocabulary provided.
by the first law. The state changes in QFT due to \( J_H = (\Box + m^2)\varphi_H \neq 0 \) is similarly described by the very breaking term \( iJ_H \otimes \phi^m = -(S^{-1} \otimes 1)[iQ, S \otimes 1] \) in terms of the non-trivial S-matrix \( S \neq 1 \). At first sight, what is conserved, momentum \( p \) and \( \phi \) as (contrasted with the “momentum variables” \( \frac{\delta}{i\delta \phi^{as}} \)), seems to be opposite between the Newtonian and the QFT cases, but actually it is not the case, because a particle picture with conserved \( \vec{p} \) in the Newtonian first law is embedded also in \( \phi^{as} \) through the creation and annihilation operators, \( a(\vec{p})^* \) and \( a(\vec{p}) \).

5) **Reconstruction** of \( \varphi_H \) from \( \phi^{in/out} \) intertwined by \( S \): The essence

of the GLZ-Fock expansion of the Heisenberg fields \( \varphi_H \) or \( A \) in terms of the asymptotic fields \( \phi^{as} \) can be seen in the “inverse problem” to reconstruct the former from its fixed-point subalgebra(s) \( \{ \varphi_H \}^F = \{ \phi^{in} \} \) or \( \{ \phi^{out} \} \) through the **co-action** of \( \Gamma \) (which is in parallel to the so-called “inverse scattering method” in quantum mechanics to determine a potential term responsible for scattering processes from the scattering data). This kind of **duality** relation between \( \varphi_H \) and \( \phi^{as} \) ensures the **universality** of the asymptotic fields \( \phi^{as} \) in spite of its speciality as statistically independent free objects. What is conceptually more important is such a possibility to re-construct an interacting theory of relativistic quantum fields \( \varphi_H \) from the knowledge of an S-matrix \( S \) intertwining the asymptotic in- and out-fields \( \phi^{in/out} \).

The crucial ingredient, \( iJ_H \otimes \phi^{in} = -(S^{-1} \otimes 1)[iQ, S \otimes 1] \), as a coupling term in \( (\omega_0 \otimes id)(T((A \otimes 1) \exp(iJ_H \otimes \phi^{in}))) = SA \) can be determined, at least in its integrated form, by the knowledge of \( \phi^{as} \) and \( S \), the former of which is easily constructible as free objects. The latter one is the highly non-trivial object to be determined phenomenologically from the experimental data of particle scattering processes to within certain limits of exactitude. Once the functional dependence of \( S \) on \( \phi^{in} \) is specified, the formula \( J_H(x) = S^{-1} \frac{\delta}{i\delta \phi^{in}(x)} S \) allows a local quantity \( J_H(x) \) to be determined.

Therefore, the whole scheme to control the mutual relations between \( \varphi_H \) (Micro) and \( \phi^{as} \) (Macro) can be understood in the context of “**Micro-Macro duality**”\(^3\) via the Fourier-Galois duality between the fixed-point subalgebra and the recovery of the total algebra as the Galois extension. A **new feature** found here is the problem related with the appearance of **two** fixed point subalgebras, \( \{ \phi^{in} \} \) and \( \{ \phi^{out} \} \), which are mutually equivalent by the intertwining actions of the S-matrix, \( Ad(S) \) and \( Ad(S^{-1}) \):

\[
S \phi^{out} S^{-1} = \phi^{in}, \quad S^{-1} \phi^{in} S = \phi^{out}.
\]

In this connection, the coupling terms : \( T \exp(iJ_H \otimes \phi^{as}) \) : cannot be directly regarded as a Kac-Takesaki operator but should be interpreted as a kind of a **cocycle** between two such. This aspects will further be elaborated.

6) **PCT invariance & Borchers classes**: While the (weak) local com-
mutativity is inevitably violated between $\varphi_H$ and $\phi^{in/out}$ and between $\phi^{in}$ and $\phi^{out}$, the fields $\varphi_H$, $\phi^{in}$ and $\phi^{out}$ enjoy the local commutativity within each system. Then the vacuum $\omega_0$ is invariant, $\omega_0 \circ \theta = \omega_0 = \omega_0 \circ \theta_{as}$, under the PCT transformations $\theta, \theta_{as}$ given by $\theta(\varphi_H(x)) = \gamma \varphi_H(-x)^*$, $\theta_{as}(\phi^{as}(x)) = \gamma \phi^{as}(-x)^*$ (with $\gamma \in T$), and hence, $\theta, \theta_{as}$ are implemented, respectively, by anti-unitary PCT operators $\Theta$ and $\Theta_{as}$ s.t. $\theta(\varphi_H(x)) = \Theta \varphi_H(x) \Theta$, $\theta_{as}(\phi^{as}(x)) = \Theta_{as} \phi^{as}(x) \Theta_{as}$ and $\Theta \Omega = \Theta_{as} \Omega = \Omega$ [5, 6]. Then $S\phi^{out}(x) S^{-1} = \phi^{in}(x) = \Theta \gamma^{-1} \phi^{out}(-x)^* \Theta = \Theta \Theta^{out} \phi^{out}(x) \Theta$ implies

$$S = \Theta^{in} \Theta = \Theta \Theta^{out}, \quad S \Theta^{out} = \Theta = \Theta^{in} S,$$

under the assumption of asymptotic completeness. These relations exhibit more detailed structures of $S$ in terms of PCT operators. Thus, quantum fields with the same PCT operator $\Theta$ have the same S-matrix $S = \Theta^{in} \Theta = \Theta \Theta^{out}$: this explains the “ambiguities” in the choice of Heisenberg fields interpolating asymptotic fields $\phi^{in/out}$ connected by a given S-matrix $S$ in such a form as the Borchers classes [6] characterized by the relative local commutativity to share the same PCT operator $\Theta$. The considerations on this aspect will be crucial for discussing the above points of 2) – 5).

7) For physical applications in the scheme of the above 1), it would be interesting to utilize the above formula $S A =: (\omega_0 \otimes id)(T(A \otimes 1) \exp(iJ_H \otimes \phi^{in}))$ : with a specific choice of $A$ such as the electromagnetic current $A = J_\mu(x)$ to analyze the processes to measure the form factor $\langle \beta, \text{out} | J_\mu(x) | \alpha, \text{in} \rangle = \langle \beta | S J_\mu(x) | \alpha \rangle$, or, in more general contexts of “weak values” [11] $\langle \beta, \text{out} | A | \alpha, \text{in} \rangle = \langle \beta | S A | \alpha \rangle$, as suggested by Hosoya and Shikano.

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