Large Field Cutoffs Make Perturbative Series Converge

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For $\lambda \phi^4$ problems, convergent perturbative series can be obtained by cutting off the large field configurations. The modified series converge to values exponentially close to the exact ones. For $\lambda$ larger than some critical value, the method outperforms Padé approximants and Borel summations. We discuss some aspects of the semi-classical methods used to calculate the modified Feynman rules and estimate the error associated with the procedure. We provide a simple numerical example where the procedure works despite the fact that the Borel sum has singularities on the positive real axis.

1. INTRODUCTION

It is intuitively clear that for Euclidean scalar field theory with quartic interactions, the suppression $e^{-\lambda \phi^4}$ makes the large field configurations unimportant. This general idea has been tested explicitly by introducing sharp field cutoffs in models which can be solved numerically. A first example is the spectrum of the anharmonic oscillator which can be calculated very accurately \footnote{This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664.} by using Sturm-Liouville theorem and requiring the wave function to vanish at some large value $\phi_{\text{max}}$ rather than at infinity. A second example is the calculation of the zero-momentum correlation functions of Dyson’s hierarchical model. This calculations requires the evaluation of the Fourier transform of the local measure and a sharp field cutoff can then be introduced at the first iteration, leaving the rest unchanged. In both examples, if the field cutoff is taken large enough the errors due to the cutoff can be made exponentially small.

In addition to being a natural procedure in numerical calculations, a large field cutoff cures the lack of convergence of the perturbative series. It removes the instabilities appearing at negative $\lambda$. From a mathematical point of view, $e^{-\lambda \phi^4}$ has a Taylor expansion which is uniformly convergent over the disk $|\phi| \leq \phi_{\text{max}}$ and one can then interchange the sum and the integral. This statement has been checked numerically in Ref. \footnote{This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664.} for the two cases mentioned above. It was also shown that the modifications to the Feynman rules due to the field cutoff could be calculated accurately by using semi-classical methods. In addition, the errors made at a given order, as a function of $\lambda$ and the field cutoff, can be estimated without knowing the numerical answer. At fixed $\lambda$, this allows an optimal choice of the field cutoff. Beyond a certain value of $\lambda$, this procedure outperforms Padé approximants and the Padé-Borel method. This is illustrated in Fig. \footnote{This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664.} and discussed in Ref. \footnote{This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664.}. In the following, we first explain the general ideas with a simple example and then present some aspects not discussed in Ref. \footnote{This research was supported in part by the Department of Energy under Contract No. FG02-91ER40664.}.

2. A SIMPLE EXAMPLE

In order to understand the origin of the divergence and how to remedy it, we first consider the simple integral

$$Z(\lambda) = \int_{-\infty}^{+\infty} d\phi e^{-(1/2)\phi^2 - \lambda \phi^4}. \quad (1)$$

If we expand $e^{-\lambda \phi^4}$, the integrand for the order $p$ contribution is $e^{-(1/2)\phi^2} \phi^{4p}/p!$ and has its maximum when $\phi^2 = 4p$. On the other hand, the truncation of $e^{-\lambda \phi^4}$ at order $p$ is accurate pro-
vided that $\lambda \phi^4 \ll p$. Requiring that the peak of the integrand for the $p$-th order term is within the range of values of $\phi$ for which the $p$-th order truncation provides an accurate approximation, yields the condition $\lambda \ll (16p)^{-1}$. One sees that the range of validity for $\lambda$ shrinks as one increases the order. We can avoid this problem by restricting the range of integration in Eq. (1) to $|\phi| \leq \phi_{\text{max}}$. As the order increases, the peak of the integrand moves across $\phi_{\text{max}}$ and the contribution is suppressed. Similar conclusions were reached in Ref. [4] using Lebesgue's dominated convergence.

The coefficients of the modified series satisfy the bound $|a_p| < \sqrt{2\pi \phi_{\max}^p}/p!$ and the modified series defines an entire function. However, we are now constructing a perturbative series for a problem which is slightly different than the original one. This procedure is justified from the fact that we have an exponential control on the error $\delta Z$ due to the restricted range of integration:

$$\delta Z < 2e^{-\lambda \phi_{\max}^4} \int_{\phi_{\max}}^{\infty} d\phi e^{-(1/2)\phi^2}.$$  \hfill (2)

### 3. SEMI-CLASSICAL ESTIMATES

Approximate bounds of the form of Eq. (2) for lattice models or their continuum limit can be obtained by semi-classical methods. The general idea is that configurations with large field values are usually configurations of large action and they can be described by classical solutions with small fluctuations. A detailed calculation has been performed in Ref. [3] in the case of the anharmonic oscillator. We used a dilute-gas approximation for configurations with one “lump” of large values to estimate the effect of the field cutoff on the ground state. In the continuum limit, at imaginary time and at $\lambda = 0$, the relevant classical solutions are $\phi_{\max} e^{-|\tau - \tau_0|}$ (in $\hbar = m = \omega = 1$ units). Such solutions can be observed in MC simulations for the harmonic oscillator. We first checked that proper thermalization occurred by comparing the distribution of the values of the action with a simple analytical result for quadratic actions as shown in Fig. [3]. A typical configuration with large field values is shown in Fig. [3] together with the classical solution.

Adapting standard semi-classical arguments, we obtained the zero-th order correction to the ground state:

$$\delta E_0^{(0)} \simeq 4\pi^{-1/2}\phi_{\max}^2 \int_{\phi_{\max}}^{\infty} d\phi e^{-\phi^2}.$$  \hfill (3)

The effects of a non-zero $\lambda$ are obtained by replacing $\phi^4$ by classical solution at $\lambda = 0$. One obtains the approximate error at order $n$:

$$|\delta E_0(\lambda)| \simeq \delta E_0^{(n)} e^{-(1/2)\lambda \phi_{\max}^4} + |a_{n+1}|\lambda^{n+1}.$$  \hfill (4)

Both estimates are in very good agreement [3] with numerical results provided that $\phi_{\text{max}} > 2$.

![Figure 1](image1.png)

**Figure 1.** Action histogram for the harmonic oscillator compared to $(\Gamma[N/2])^{-1/2}S^{N/2-1}e^{-S}$ for $N = 400$ sites.

![Figure 2](image2.png)

**Figure 2.** A large field configuration (thin line) compared with $\phi_{\max} e^{-|\tau - \tau_0|}$ (thick line).

The error due the field cutoff decreases when $\lambda$ increases, while the error due to the truncation of the perturbative series at a finite order decreases.
when $\lambda$ decreases. One thus reaches a compromise at intermediate values of $\lambda$ where the total error is minimized. This is illustrated in Fig. 3. By increasing the field cutoff, the optimal value of $\lambda$ decreases. Using the approximate error formula, one can thus at the same time choose an optimal field cutoff and estimate the error.

4. OTHER MODELS

The generality of the basic principle invoked above (uniform convergence of the exponential on a compact neighborhood of the origin) implies that it can be used in many situations. In particular, for models where the Borel sums have singularities on the positive real axis. A simple example is the integral

$$\int_{-\infty}^{+\infty} d\phi e^{-(1/2)\phi^2+((1+a)/3a)\sqrt{\lambda}\phi^3-(1/4a)\lambda\phi^4}. \quad (5)$$

Numerical results for $a = 3/4$ confirm that all the coefficients of the series are positive and that the Padé approximants of the series or of its Borel sums have all their poles on the positive real axis. The series defined with cutoffs in $\phi$ give results which are significantly better than regular perturbation theory when $\lambda > 0.1$. This is illustrated in Fig. 4. Complementary discussions can be found in Refs. [33,34].

In lattice gauge theory, e.g. $SU(2)$ in 4 dimensions, $U_{x,\mu} = e^{igaA_{\mu,\tau}}$ with $A_{\mu}$ contained in a sphere of radius $\pi/a$ for each $\mu$. In the literature on lattice perturbation theory (with the exception of van Baal), one usually replaces $\int d\eta_{\mu}$ by $\int_{-\infty}^{+\infty} dA_{\mu}^i$ since in the continuum limit the range becomes infinite. From our point of view, the lattice spacing $a$ not only regulates the UV divergences but also the large order behavior of perturbation theory. We plan to treat this problem by following a path similar to scalar field theory but with different approximations to treat the massless quadratic theory with a field cutoff.

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