A note on some identities involving special functions from the hypergeometric solution of algebraic equations

J. L. González-Santander

Department of Mathematics. Universidad de Oviedo. C/ Federico García Lorca, 18. 33007 Oviedo, Asturias. Spain.

Abstract
From the algebraic solution of \( x^n - x + t = 0 \) for \( n = 2, 3, 4 \) and the corresponding solution in terms of hypergeometric functions, we obtain a set of reduction formulas for hypergeometric functions. By differentiation and integration of these results, and applying other known reduction formulas of hypergeometric functions, we derive new reduction formulas of special functions as well as the calculation of some infinite integrals in terms of elementary functions.

Keywords: Reduction formulas of special functions, hypergeometric functions, integrals of special functions

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1. Introduction and preliminaries
In the literature, we found a large body of literature dealing with the trinomial equation. In fact, there are different versions of this kind of equation. For instance, in 1915, Mellin studied the trinomial equation [1]:

\[
y^n + x y^p + 1 = 0, \quad n > p,
\]

where \( n, p \) are positive integers and \( x \in \mathbb{R} \). By using his integral transform, Mellin derived the following series representation [2, Chap.3 Sect.8]:

\[
y (x) = \frac{1}{n} \sum_{r=0}^{\infty} \frac{\Gamma \left( \frac{1+pr}{n} \right)}{\Gamma \left( \frac{1+pr}{n} + 1 - r \right)} \frac{(-x)^r}{r!},
\]

\[\]
where $|x| < (p/n)^{-p/n} (1 - p/n)^{p/n - 1} \leq 2$. In [3], Miller rewrote the solution of (11) for positive real numbers $n, p$ in terms of the Wright function.

In this paper, we are interested in the trinomial equation:

$$x^n - x + t = 0,$$

assuming that $n \geq 2$ is an integer and $t \in \mathbb{C}$. Performing the change of variables $z = t^{-1/n}x$ and $a = -t^{-1/n-1}$, (2) is transformed into:

$$z^n + a z + 1 = 0. \quad (3)$$

Bring and Jerrard proved that any fifth degree equation can be brought to the form given in (3) for $n = 5$ by means of Tschirnhaus transformations [4]. Nonetheless, Abel proved in 1824 the impossibility of solving the general quintic equation by means of radicals [5]. Consequently, mathematicians turned to the idea of searching for analytic solutions. The first success in this direction was achieved in 1858 by Hermite and Kronecker who were able to express the solution to the quintic equation by means of a modular elliptic function (see [6]).

Equation (2) was first solved by Lambert in 1758 as a series development for $x$ in powers of $t$ [7]. Euler’s version of Lambert series [8] is connected to the tree function and the Lambert $W$ function [9]. More recently, Glasser calculated the roots of (2) as a finite sum of generalized hypergeometric functions [10]. In many cases, one of the roots can be expressed as a single hypergeometric function. However, in 1770, Lagrange [11] applied his inversion formula [12, Appendix E] to derive a root $x_n(t)$ of the equation (2) as an expansion in powers of $t$. Next, we present the derivation given by Lagrange.

**Theorem 1 (Lagrange inversion formula).** Consider the variables $x$, $t$, and $r$ related by

$$x = t + r \phi(x), \quad (4)$$

where $\phi(x)$ is analytic in the neighborhood of $x = t$ with $\phi(t) \neq 0$. Consider as well an analytic function $f(x)$ in the neighborhood of $x = t$. Then Lagrange’s formula is

$$f(x) = f(t) + \sum_{k=1}^{\infty} \frac{r^k}{k!} \frac{d^{k-1}}{dt^{k-1}} \left[ f'(t) \phi^k(t) \right]. \quad (5)$$
If we take \( f(x) = x \), \( \phi(x) = x^n \), and \( r = 1 \), (4) becomes (2), and (5) reads as
\[
x_n(t) = x = t \left[ 1 + \sum_{k=1}^{\infty} \frac{(nk)!}{k! (nk - k)!} t^{(n-1)k} \right].
\]

Fortunately, we can recast (6) in hypergeometric form. For this purpose, denote the Pochhammer symbol as \((a)_n = \Gamma (a + n) / \Gamma (a)\), where \( \Gamma (z) \) is the gamma function. Then, we can prove easily by induction that:
\[
(nk)! = \left( \frac{1}{n} \right)_k \left( \frac{2}{n} \right)_k \cdots \left( \frac{n}{n} \right)_k n^{nk}, \tag{7}
\]
hence
\[
(nk - k + 1)! = \left( \frac{2}{n-1} \right)_k \cdots \left( \frac{n}{n-1} \right)_k (n-1)^{(n-1)k}. \tag{8}
\]

Now, let us define the generalized hypergeometric series\(^2\) as follows:

**Definition 2 (Generalized hypergeometric series).**
\[
pFq(a_1, \ldots, a_p | b_1, \ldots, b_q | z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!}. \tag{9}
\]

If none of the parameters \( b_1, \ldots, b_q \) are nonpositive integers and \( p \leq q \), the series (9) converges for all finite values of \( z \) and defines and entire function.

**Remark 3.** Note that
\[
\lim_{z \to 0} pFq \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} | z \right) = 1. \tag{10}
\]

Therefore, according to (7)-(9), we finally rewrite (6) as
\[
x_n(t) = t_\binom{n}{n-1} F_{n-1} \left( \begin{array}{c} \frac{1}{n-1}, \ldots, \frac{n}{n-1} \\ \frac{n}{n-1}, \ldots, \frac{n}{n-1} \end{array} | \left( \frac{nt}{n-1} \right)^{n-1} \right). \tag{11}
\]

\(^2\)For the different cases of convergence of the generalized hypergeometric series see [13, Sect. 16.2].
On the one hand, for $n = 2, 3, 4$, \((11)\) is reduced to (see as well \([14]\)):

\[
\begin{align*}
  x_2(t) &= t_2 F_1\left(\frac{1}{2}, 1 \mid 4t\right), \\
  x_3(t) &= t_2 F_1\left(\frac{1}{3}, \frac{2}{3}, 3 \mid \left(\frac{3t}{2}\right)^2\right), \\
  x_4(t) &= t_3 F_2\left(\frac{1}{4}, \frac{1}{3}, 4, \frac{5}{4} \mid 4 \left(\frac{4t}{3}\right)^3\right).
\end{align*}
\]

where we have simplified the hypergeometric series with common Pochhammer symbols in numerator and denominator in \((13)\) and \((14)\), according to definition \((9)\). On the other hand, it is well-known that the roots of \((12)-(14)\) are expressible in terms of elementary functions. The scope of this paper is just to compare both approaches, and from this comparison, derive some new reduction formulas and definite integrals involving special functions. For this purpose, we will use the following differentiation formulas, that can be easily proved by induction:

\[
\frac{d^n}{dt^n} \left(\frac{1}{t}\right) = (-1)^n n! \frac{1}{t^{n+1}},
\]

\[
\frac{d^n}{dt^n} \left(\sqrt{1-t}\right) = \left(-\frac{1}{2}\right)_n (1-t)^{1/2-n}.
\]

Notice that for $n = 1$ in \((16)\), we have that

\[
\frac{d}{dt} \left(\sqrt{1-t}\right) = \frac{-1}{2\sqrt{1-t}},
\]

thus, knowing that \([15, \text{Eqn. 18.5.7}]\)

\[
(x)_{n+1} = x (x+1)_n,
\]

we conclude

\[
\frac{d^n}{dt^n} \left(\frac{1}{\sqrt{1-t}}\right) = \left(\frac{1}{2}\right)_n (1-t)^{-1/2-n}.
\]

Also, we will use Leibniz’s differentiation formula \([13, \text{Eqn. 1.4.2}]\) (for the historical origin of this formula, see \([16, \text{p. 143}]\)),

\[
\frac{d^n}{dt^n} [f(t) g(t)] = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k)}(t),
\]
Gauss summation formula \[13,\text{ Eqn. 15.4.20}\] (for the original work of Gauss, see \[17\]),

\[
2F_1 \left( \frac{a, b}{c} \middle| 1 \right) = \frac{\Gamma (c) \Gamma (c - a - b)}{\Gamma (c - a) \Gamma (c - b)}, \tag{20}
\]

\[\text{Re} \ (c - a - b) > 0,\]

and Whipple’s sum \[13,\text{ Eqn. 16.4.7}\],

\[
3F_2 \left( \frac{a, 1-a, c}{d, 2c - d + 1} \middle| 1 \right) = \frac{\pi 2^{1-2c} \Gamma (d) \Gamma (2c - d + 1)}{\Gamma \left( c + \frac{a-d+1}{2} \right) \Gamma \left( c + 1 - \frac{a+d}{2} \right) \Gamma \left( \frac{a+d}{2} \right) \Gamma \left( \frac{d-a+1}{2} \right)}, \tag{21}
\]

\[\text{Re} \ (c) > 0 \text{ or } a \in \mathbb{Z}.\]

For the calculation of the definite integrals, we will use the following result \[12, \text{ Ch. 2. Ex. 11}\]:

\[
\int_0^{\infty} e^{-st} t^{\alpha-1} \, pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| xt \right) \, dt = \frac{\Gamma (\alpha)}{s^\alpha} \, p+1F_q \left( \begin{array}{c} a_1, \ldots, a_p, \alpha \\ b_1, \ldots, b_q \end{array} \middle| \frac{x}{s} \right), \tag{22}
\]

\[p \leq q, \ \text{Re} \ s > 0, \ \text{Re} \ \alpha > 0.\]

This paper is organized as follows. Section 2 equates the solution of (2) for \( n = 2 \) to \( x_2 (t) \). From this result, and using some differentiation formulas of the hypergeometric \( 2F_1 \) function, we obtain a set of reduction formulas of some hypergeometric functions in terms of elementary functions, which extends the classical Schwarz’s list \[18\]. As corollaries, we obtain identities involving the incomplete beta function and the Legendre function. Also, we calculate two infinite integrals involving the lower incomplete gamma function. Section 3 equates the solution of (2) for \( n = 3 \) to \( x_3 (t) \), and from it, we derive a new reduction formula of an hypergeometric \( 2F_1 \) function in terms of elementary functions. Also, we calculate an infinite integral involving the parabolic cylinder function. Section 4 derives a reduction formula of a \( 3F_2 \) function in terms of elementary functions, equating the solution of (2) for \( n = 4 \) to \( x_4 (t) \). From the latter reduction formula, we obtain an identity involving the product of two Legendre functions. Finally, Section 5 collects our conclusions. In the Appendix, we recall the solution of the cubic and the quartic equations.
2. Case $n = 2$

In this case, the algebraic solution of (2) is

$$x_2(t) = \frac{1 \pm \sqrt{1 - 4t}}{2}, \quad (23)$$

hence, selecting the proper root of (23), we can equate it to (12), to obtain

$$2F_1 \left( \frac{1}{2}, 1 \mid t \right) = \frac{2}{t} (1 - \sqrt{1 - t}), \quad (24)$$

which agrees with the result reported in the literature \[19, \text{Eqn. 7.3.2(84)}\]. Notice that (24) can be derived from the binomial theorem \[15, \text{Eqn. 6.14.1}\]. Indeed,

$$\sqrt{1 - t} = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \frac{t^k}{k!} = 1 + \sum_{k=1}^{\infty} \left( -\frac{1}{2} \right)^k \frac{t^k}{k!},$$

hence, applying (17), we have

$$\frac{\sqrt{1 - t - 1}}{t} = \sum_{k=1}^{\infty} \left( -\frac{1}{2} \right)^k \frac{t^{k-1}}{k!} = -\frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{k-1} \frac{(1)_{k-1} t^{k-1}}{(2k-1)(k-1)!},$$

and the result follows. From (24), we obtain next a set of results using the formulas stated in the Introduction.

2.1. First differentiation formula

**Theorem 4.** For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:

$$2F_1 \left( \frac{1}{2} + n, 1 + n \mid \frac{t}{2 + n} \right) \quad (25)$$

$$= \begin{cases} 
2 \left( \frac{-1}{2} \right)^n (n+1)! 
\left( \frac{1}{2} \right)_n \frac{t^{n+1}}{n!} \left[ 1 - \sqrt{1 - t} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{t}{t-1} \right)^k \right], & t \neq 0, 1, \\
1, & t = 0, \\
2, & t = 1, n = 0, \\
\infty, & t = 1, n \geq 1.
\end{cases}$$
Proof. For $t \neq 0, 1$, apply the following differentiation formula for the Gauss hypergeometric function [13, Eqn. 15.5.2]:

$$
\frac{d^n}{dt^n} \left[ \frac{2F_1 \left( a, b \left| c \right| t \right)}{\left( \frac{a}{c} \right)_n} \right] = \left( \frac{a}{c} \right)_n \frac{b}{c} \left( \frac{c}{a} \right)_n \frac{2F_1 \left( a + n, b + n \left| c + n \right| t \right)}{\left( \frac{a + n}{c + n} \right)_n}.
$$

(26)

Therefore, taking $a = \frac{1}{2}, b = 1$ and $c = 2$ in (26) and using (24), we have

$$
\frac{d^n}{dt^n} \left[ 2F_1 \left( \frac{1}{2}, 1 \left| t \right| \right) \right] = 2 \left[ \frac{d^n}{dt^n} \left( \frac{1}{t} \right) - \frac{d^n}{dt^n} \left( \frac{\sqrt{1-t}}{t} \right) \right].
$$

Applying (15)-(16) and (19), after some algebra, we arrive at (25) for $t \neq 0, 1$.

For $t = 0$, apply (10).

For $t = 1$, apply Gauss summation formula (20). This completes the proof.

Corollary 5. For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:

$$
2F_1 \left( 1 + 2n, \frac{3}{2} + n \left| \frac{3 + 2n}{2} \right| t \right) = \begin{cases} 
\frac{(-1)^n (n+1)!}{(\frac{1}{2})^n} \left( \frac{2}{t} \right)^{2(n+1)} & \text{for } t \neq 0, 1 \\
2 - t - 2\sqrt{1-t} \sum_{k=0}^{n} \frac{(-\frac{1}{2})^k}{k!} \left( \frac{t}{2\sqrt{t} - 1} \right)^{2k} & \text{for } t = 0, \\
1, & \text{for } t = 1, n = 0, \\
\infty, & \text{for } t = 1, n \geq 1.
\end{cases}
$$

(27)

Proof. Apply the quadratic transformation [12, Eqn. 3.1.7]:

$$
2F_1 \left( \frac{\alpha, \beta}{2\beta} \left| x \right| \right) = (1 - \frac{x}{2})^{-\alpha} 2F_1 \left( \frac{\alpha}{2}, \frac{\alpha + 1}{2} \left| \left( \frac{x}{2} \right)^2 \right| \right),
$$

taking $\alpha = 2n + 1, \beta = n + \frac{3}{2}$, and $x = \frac{2\sqrt{z}}{1 + \sqrt{z}}$ to arrive at:

$$
2F_1 \left( 1 + 2n, \frac{3}{2} + n \left| \frac{2\sqrt{z}}{1 + \sqrt{z}} \right| \right) = (1 + \sqrt{z})^{2n+1} 2F_1 \left( \frac{1}{2}, n + 1 + \frac{n}{2} \left| z \right| \right).
$$

(28)
In order to obtain the desired result for \( t \neq 0, 1 \), substitute (25) in (28) and perform the change of variables \( t = \frac{2\sqrt{1 + \sqrt{z}}}{1 + 2n} \). According to this last result, the cases given in (27) for \( t = 1 \) are straightforward. However, for \( t = 0 \), we have an indeterminate expression on the RHS of (27). On the one hand, according to (10), we have

\[
\lim_{t \to 0} 2F_1 \left( \begin{array}{c} 1 + 2n, \frac{3}{2} + n \\ 3 + 2n \end{array} \mid t \right) = 1.
\]

On the other hand, we calculate the limit \( t \to 0 \) of the RHS of (27), taking into account the formula [15, Eqn. 18:3:4]:

\[
\frac{1}{(1-t)^{\nu}} = \sum_{k=0}^{\infty} (\nu)_k \frac{t^k}{k!},
\]

thereby

\[
\frac{(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t \to 0} \left( \frac{2}{t} \right)^{2(n+1)} \left\{ 2 - t - 2\sqrt{1-t} \left[ \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k}{k!} \left( \frac{t^2}{4(t-1)} \right)^k \right] - \sum_{k=n+1}^{\infty} \frac{(-\frac{1}{2})_k}{k!} \left( \frac{t^2}{4(t-1)} \right)^k \right\} = \frac{(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t \to 0} \left( \frac{2}{t} \right)^{2(n+1)} \left\{ 2 - t - 2\sqrt{1-t} \left[ \frac{2 - t}{2\sqrt{1-t}} - \sum_{k=n+1}^{\infty} \frac{(-\frac{1}{2})_k}{k!} \left( \frac{t^2}{4(t-1)} \right)^k \right] \right\} = -2 \frac{(-\frac{1}{2})_{n+1}}{\left(\frac{1}{2}\right)_n} = 1,
\]

where we have applied (17) for \( x = -\frac{1}{2} \). ■

**Corollary 6.** For \( n = 0, 1, 2, \ldots \) and \( t \in \mathbb{C} \), the following reduction formula holds true:

\[
B \left( 1 + n, \frac{1}{2} - n, t \right) = \begin{cases} 
\frac{2(-1)^n n!}{\left(\frac{1}{2}\right)_n} \left[ 1 - \sqrt{1-t} \sum_{k=0}^{n} \frac{(-\frac{1}{2})_k}{k!} \left( \frac{t}{t-1} \right)^k \right], & t \neq 0, 1, \\
\frac{2(-1)^n n!}{\left(\frac{1}{2}\right)_n}, & t = 1, \\
0, & t = 0.
\end{cases}
\]
where $B(\nu, \mu, z)$ denotes the incomplete beta function [15, Chap. 58].

**Proof.** For $t \neq 0, 1$, in [15, Eqn. 7.3.1(28)], we found:

$$2F_1 \left( \begin{array}{c} a, b \\ b+1 \end{array} \bigg| t \right) = b \ t^{-b} \ B(b, 1-a, t), \tag{31}$$

Therefore, take $a = \frac{1}{2} + n$ and $b = 1 + n$ in (31) and apply (25) to obtain (30).

For $t = 1$, apply the properties of the incomplete beta function [15, Eqns. 58.3.1\&58.1.1]

$$B(\nu, \mu, 1) = B(\nu, \mu) = \frac{\Gamma(\nu) \Gamma(\mu)}{\Gamma(\nu + \mu)},$$

and the formula of the gamma function [15, Eqn. 43.4.4]

$$\Gamma \left( \frac{1}{2} - n \right) = \frac{(-1)^n}{(\frac{1}{2})_n} \sqrt{\pi},$$

to obtain the desired result.

For $t = 0$, apply the definition of the incomplete beta function [15, Eqn.58.3.1], and calculate the limit $t \to 0$ for $n \geq 0$, to obtain:

$$\lim_{t \to 0} B(1 + n, \mu, t) = \lim_{t \to 0} \int_0^t x^n (1 - x)^{\mu-1} \, dx = 0.$$

It is worth noting that we can derive a different elementary representation of $2F_1 \left( \begin{array}{c} \frac{1}{2} + n, 1 + n; 2 + n; t \end{array} \right)$ by using known formulas given in the literature.

**Theorem 7.** For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:

$$2F_1 \left( \begin{array}{c} \frac{1}{2} + n, 1 + n \\ 2 + n \end{array} \bigg| t \right) \begin{cases} 2 (-1)^n (n + 1)! \\ \left(\frac{1}{2}\right)_n t^{n+1} \end{cases} \left[ 1 - (1 - t)^{1/2-n} \sum_{k=0}^{n} \frac{(\frac{1}{2} - n)_k}{k!} t^k \right], \quad t \neq 0, 1, \tag{32}$$

$$= \begin{cases} 1, & t = 0, \\ 2, & t = 1, n = 0, \\ \infty, & t = 1, n \geq 1. \end{cases}$$
Proof. First we prove (32) for $t \neq 0, 1$. Apply Euler’s transformation formula \cite[Eqn. 15.8.1]{13}:

$$2\binom{\alpha, \beta}{\gamma} z = (1 - z)^{\gamma - \alpha - \beta} 2\binom{\gamma - \alpha, \gamma - \beta}{\gamma} z,$$

to obtain

$$2\binom{\frac{1}{2} + n, 1 + n}{2 + n} t = (1 - t)^{1/2 - n} 2\binom{\frac{3}{2}, 1}{2 + n} t. \quad (33)$$

We found in \cite[Eqn. 7.3.1(123)]{19} for $m = 1, 2, \ldots$, and $m - b \neq 1, 2, \ldots$, the formula:

$$2\binom{1, b}{m} z = \frac{(m - 1)! (-z)^{1-m}}{(1 - b)^{m-1}} \left[ (1 - z)^{m-b-1} - \sum_{k=0}^{m-2} \frac{(b - m + 1)_{k} z^{k}}{k!} \right]. \quad (34)$$

Therefore, apply (34) to (33) with $m = n + 2$ and $b = \frac{3}{2}$, taking into account (17) for $x = -\frac{1}{2}$, to arrive at (32) for $t \neq 0, 1$.

Straightforward from (32) for $t \neq 0, 1$, we have a divergent result for $t = 1$, except for $n = 0$.

For $t = 0$, we have an indeterminate expression on the RHS of (32). On the one hand, according to (10), we have

$$\lim_{t \to 0} 2\binom{\frac{1}{2} + n, 1 + n}{2 + n} t = 1. \quad (35)$$

On the other hand, we calculate the limit $t \to 0$ of the RHS of (32) taking into account (29). Thereby

$$\frac{2 (-1)^{n} (n + 1)!}{\left(\frac{1}{2}\right)_{n}} \lim_{t \to 0} \frac{(1 - t)^{1/2 - n}}{t^{n+1}} \left[ \frac{1}{(1 - t)^{1/2 - n}} - \sum_{k=0}^{n} \frac{(1/2 - n)_{k} t^{k}}{k!} \right]$$

$$= \frac{2 (-1)^{n} (n + 1)!}{\left(\frac{1}{2}\right)_{n}} \lim_{t \to 0} \frac{1}{t^{n+1}} \sum_{k=n+1}^{\infty} \frac{(1/2 - n)_{k} t^{k}}{k!}$$

$$= \frac{2 (-1)^{n} \left(\frac{1}{2} - n\right)_{n+1}}{\left(\frac{1}{2}\right)_{n}} = 1,$$

where we have applied the property $\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin \pi z}$ \cite[Eqn. 1.2.2]{20}.

$\blacksquare$
Theorem 8. For \( n = 0, 1, 2, \ldots \) and \( \text{Re} (s + x) > 0 \), we have

\[
\int_0^\infty e^{-st} \frac{\gamma (n + 1, xt)}{t^{3/2}} \, dt = -2\sqrt{n} \sqrt{s - \sqrt{s + x}} \sum_{k=0}^{n} \left( -\frac{1}{2} \right)_k \left( \frac{x}{x + s} \right)^k,
\]

where \( \gamma (\nu, z) \) denotes the lower incomplete gamma function [15, Chap. 45].

Proof. Indeed, take \( a_1 = 1 + n \), \( \alpha = \frac{1}{2} + n \) and \( b_1 = 2 + n \) in (22), consider the result (25), as well as [15, Eqn. 43:4:3]

\[
\Gamma \left( n + \frac{1}{2} \right) = \left( \frac{1}{2} \right)_n \sqrt{\pi},
\]

to obtain

\[
\int_0^\infty e^{-st} t^{n-1/2} _1F_1 \left( \frac{1 + n}{2 + n} \bigg| xt \right) \, dt
= 2\sqrt{\pi} (-1)^n (n + 1)! \sqrt{s - \sqrt{s - x}} \sum_{k=0}^{n} \left( -\frac{1}{2} \right)_k \left( \frac{x}{x - s} \right)^k.
\]

However, according to [19, Eqn. 7.11.1(13)], we have

\[
_1F_1 \left( \begin{array}{c} n \\ 1 + n \end{array} \bigg| z \right) = \frac{(-1)^n n!}{z^n} \left[ 1 - e^{-z} \sum_{k=0}^{n-1} \frac{(-1)^k z^k}{k!} \right],
\]

and [15, Eqns. 45:4:2&26:12:2], we have as well

\[
\Gamma (n, z) = (n - 1)! e^{-z} e_{n-1} (z) = (n - 1)! e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{k!},
\]

where \( \Gamma (\nu, z) \) denotes the upper incomplete gamma function and \( e_n (z) \) is the exponential polynomial. Therefore, from (38) and (39), and taking into account that the lower incomplete gamma function satisfies [15, Eqn. 45:0:1]

\[
\gamma (\nu, z) = \Gamma (\nu) - \Gamma (\nu, z),
\]

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we conclude that

$$\left. _1F_1 \right|_{n + 1} \ = \ n \ (z)^{-n} \gamma \ (n, -z), \quad (40)$$

due to (40) in (37), we arrive at (36), as we wanted to prove.  

It is worth noting that we can obtain also (36) from [21, Eqn. 2.10.3(2)]

and (25).

2.2. Second differentiation formula

Definition 9 (Regularized hypergeometric function).

$$\left. _pF_q \right|_{a_1, \ldots, a_p} \ (b_1, \ldots, b_q) \ = \ \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1 + k) \cdots (b_q + k) \ k!} \ z^k. \quad (41)$$

When \( p \leq q + 1 \) and \( z \) is fixed and not a branch point, (41) is an entire function of each of the parameters \( a_1, \ldots, a_p, b_1, \ldots, b_q \) (see [13, Eqn. 15.2.2]).

Theorem 10. For \( n = 0, 1, 2, \ldots \) and \( t \in \mathbb{C} \setminus \{1\} \),

$$\left. _2F_1 \right|_{\frac{1}{2}, 1} \ (1 - n) \ t = \ \frac{(\frac{1}{2})_n}{1 - t} \ \left( \frac{t}{1 - t} \right)^n. \quad (42)$$

Proof. In [13, Eqn. 15.5.4], we found the differentiation formula:

$$\frac{d^n}{dt^n} \left[ t^{c-1} \left( a, b \right) \ c \ t \right] = (c - n)_n \ t^{c-n-1} \left( a, b \right) \ c - n \ t \right), \quad (43)$$

thus taking \( a = \frac{1}{2}, b = 1 \) and \( c = 2 \) in (43) and considering (24), we have

$$\frac{1}{1 - \sqrt{1 - t}} \ = \ \frac{1}{\Gamma (1 - n)} \ t^n \left( a, b \right) \ c - n \ t \right). \quad (44)$$

Apply (16)-(17), and the definition of the regularized hypergeometric function
given in (41) in order to rewrite (44) as (42), as we wanted to prove.  

Remark 11. According to (20), note that for \( t = 1 \), both sides of (42) are divergent.
2.3. Third differentiation formula

**Theorem 12.** For \( n = 0, 1, 2, \ldots \) and \( t \in \mathbb{C} \), the following reduction formula holds true:

\[
2F_1 \left( \frac{1}{2}, 1 \bigg| \frac{2}{2+n} \right) = \begin{cases} 
\frac{2 (n+1)!}{(\frac{3}{2})_n \sqrt{1-t}} \left( \frac{t-1}{t} \right)^{n+1} \left[ 1 - \frac{1}{\sqrt{1-t}} \sum_{k=0}^{n} \frac{(\frac{1}{2})_k}{k!} \left( \frac{t}{t-1} \right)^k \right], & t \neq 0, 1, \\
\frac{2 (n+1)}{2n+1}, & t = 1, \\
1, & t = 0.
\end{cases}
\]  

**Proof.** For the case \( t \neq 0, 1 \), set \( a = \frac{1}{2}, b = 1 \) and \( c = 2 \) in the differentiation formula [13, Eqn. 15.5.6],

\[
\frac{d^n}{dt^n} \left[ (1-t)^{a+b-c} \right] 2F_1 \left( a, b \bigg| c \right) t \right] = \frac{(c-a)_n (c-b)_n}{(c)_n} (1-t)^{a+b-c-n} 2F_1 \left( a, b \bigg| c+n \right) t, \\
n = 0, 1, 2, \ldots
\]

and use the result [21] to arrive at

\[
2 \frac{d^n}{dt^n} \left[ \frac{1}{t} \frac{1}{\sqrt{1-t}} - \frac{1}{t} \right] = \frac{(\frac{3}{2})_n}{(n+1)(1-t)^{n+1/2}} 2F_1 \left( \frac{1}{2}, 1 \bigg| \frac{2}{2+n} \right).
\]

Apply now Leibniz’s differentiation formula [19] and the differentiation formulas [15] and [18]. After some algebra, we obtain (45), as we wanted to prove.

For \( t = 1 \), apply Gauss summation formula [20], to obtain

\[
2F_1 \left( \frac{1}{2}, 1 \bigg| \frac{2}{2+n} \right) = \frac{2 (n+1)}{2n+1},
\]

where (47) only converges for \( \text{Re} \left( \frac{1}{2} + n \right) > 0 \), i.e. for \( n = 0, 1, \ldots \), as we wanted to prove.

Finally, according to [9], for \( t = 0 \) we have

\[
2F_1 \left( \frac{1}{2}, 1 \bigg| \frac{2}{2+n} \right) = 1,
\]

where (47) only converges for \( \text{Re} \left( \frac{1}{2} + n \right) > 0 \), i.e. for \( n = 0, 1, \ldots \), as we wanted to prove.
as we wanted to prove. ■

It is worth noting that we can provide other elementary representations for $2\, F_1 \left( \frac{1}{2}, 1; 2 + n; t \right)$, by using known formulas given in the literature.

**Theorem 13.** For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:

$$
2\, \text{F}_1 \left( \frac{1}{2}, 1 \left\vert \frac{t}{2 (n + 1)!} \right. \right) = \left\{ \begin{array}{ll}
\frac{2}{\sqrt{1 - t}} \left( \frac{t - 1}{t} \right)^{n+1} + \frac{1}{1 - \sqrt{1 - t}} \sum_{k=0}^{n} \frac{(n + 1)_k}{k! 2^{k+n}} \left( 1 - \frac{1}{\sqrt{1 - t}} \right)^{k-n}, & t \neq 0, 1, \\
\frac{2}{2n + 1}, & t = 1, \\
\frac{2}{2n + 1}, & t = 0.
\end{array} \right.
$$

**Proof.** We need to prove (48) for $t \neq 0, 1$. In [22], we found

$$
2\, \text{F}_1 \left( \frac{\alpha}{2}, \frac{\alpha + 1}{2} \left\vert \frac{t}{2 (n + 1)!} \right. \right) = \frac{(-1)^{\alpha + 1} \Gamma \left( \alpha + n + 1 \right) \Gamma \left( \frac{1}{2} + n \right)}{\Gamma \left( \alpha + 1 \right) \Gamma \left( \frac{1}{2} \right) \Gamma \left( n + 1 \right)} \times 2\, \text{F}_1 \left( \frac{-n + \alpha}{2}, \frac{-n + \alpha + 1}{2} \left\vert \frac{1 - \sqrt{1 - t}}{2} \right. \right),
$$

hence for $\alpha = 1$, we obtain

$$
2\, \text{F}_1 \left( \frac{1}{2}, 1 \left\vert \frac{t}{2 (n + 1)!} \right. \right) = \frac{2}{1 + \sqrt{1 - t}} \times 2\, \text{F}_1 \left( \frac{-n, 1}{2 + n} \left\vert \frac{1 - \sqrt{1 - t}}{2} \right. \right).
$$

Now apply [19, Eqn. 7.3.1(179)]

$$
2\, \text{F}_1 \left( \frac{-n, 1}{m} \left\vert \frac{z}{m} \right. \right) = \frac{-n! (z - 1)^{m-2}}{(m)_n z} \left[ (1 - z)^{n+1} - \sum_{k=0}^{m-2} \frac{(n + 1)_k}{k!} \left( \frac{z}{z - 1} \right)^k \right],
$$

taking $m = n + 2$ and $z = \frac{1 - \sqrt{1 - t}}{1 + \sqrt{1 - t}}$. Knowing that $(n + 2)_n = \frac{n^{2n}}{n!} \left( \frac{3}{2} \right)_n$, after some algebra, we arrive at (48) for $t \neq 0, 1$, as we wanted to prove. ■

**Theorem 14.** For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula
holds true:

\[ 2F_1 \left( \frac{1}{2}, 1 \left| \frac{t}{2+n+1} \right. \right) = \begin{cases} \frac{2 (n+1)!}{(\frac{3}{2})_n (-t)^{n+1}} \left(1-t\right)^{n+1/2} + \sum_{k=0}^{n} \frac{(-n-\frac{1}{2})_{k} k^k}{k!}, & t \neq 0, 1, \\ \frac{2 (n+1)}{2 (n+1)}, & t = 1, \\ \frac{2n+1}{1}, & t = 0. \end{cases} \] (49)

**Proof.** We need to prove (49) for \( t \neq 0, 1 \). For this purpose, apply (34) taking \( m = n + 2 \) and \( b = \frac{1}{2} \) and use (17) for \( x = \frac{1}{2} \). \( \blacksquare \)

**Corollary 15.** For \( n = 1, 2, \ldots \) and \( t \in \mathbb{C} \setminus \{1\} \), we have

\[ P_{-n}^{-n} \left( \frac{1}{\sqrt{1-t}} \right) = \begin{cases} \frac{1}{2^n \left(\frac{1}{2}\right)_n} \left( \frac{t-1}{t} \right)^{n/2} \left(1 - \frac{1}{\sqrt{1-t}} \sum_{k=0}^{n-1} \frac{(\frac{1}{2})_k k!}{k!} \left( \frac{t}{t-1} \right)^k \right), & t \neq 0, \\ 0, & t = 0. \end{cases} \] (50)

where \( P_{\mu}^{\nu} (z) \) denotes the Legendre function [23, Chap. III].

**Proof.** For \( t \neq 0, 1 \), we found, in [19, Eqn. 7.3.1(101)],

\[ 2F_1 \left( \alpha, a+\frac{1}{2} \left| \frac{c}{c} \right. \right) = 2^{c-1} \Gamma (c) (-t)^{(1-c)/2} (1-t)^{(c-1)/2-a} P_{2a-c}^{1-c} \left( \frac{1}{\sqrt{1-t}} \right). \] (51)

Therefore, taking \( a = \frac{1}{2} \) and \( c = 2+n \) in (51), and considering (45) and (17), we eventually arrive at (50), as we wanted to prove.

For \( t = 0 \), apply the hypergeometric representation of the Legendre function [13, Eqn. 14.3.15]

\[ P_{-n}^{-n} (x) = 2^{-n} (x^2 - 1)^{n/2} 2F_1 \left( \mu - \nu, \mu + \nu + 1 \left| \frac{1-x}{2} \right. \right), \]

to conclude that \( P_{-n}^{-n} (1) = 0 \), for \( n = 1, 2, \ldots \) \( \blacksquare \)
Corollary 16. For \( n = 1, 2, \ldots \) and \( x, p \in \mathbb{C} \), we have

\[
\int_0^\infty \frac{e^{-pt}}{t^{1/2+n}} \gamma(n, xt) \, dt
= \begin{cases}
\frac{-\sqrt{\pi} \, (n-1)! \, (-p)^{n-1}}{(\frac{1}{2})_n} \left[ \sqrt{p} - \sqrt{p + x} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{k!} \left(\frac{-x}{p}\right)^k \right], & p \neq 0, \\
\frac{2\sqrt{\pi} x^{n-1/2}}{2n-1}, & p = 0.
\end{cases}
\]

(52)

Proof. For \( p \neq 0 \), take \( a_1 = 1, \alpha = \frac{1}{2} \) and \( b_1 = 1 + n \) in (22), consider the reduction formula of the Kummer function [19, Eqn. 7.11.1(14)], i.e.

\[
_1F_1\left( \begin{array}{c}
1 \\
1 + n
\end{array} \left| xt \right. \right) = \frac{n e^{xt}}{(xt)^n} \gamma(n, xt),
\]

and apply the result given in (45) and the property (17), to arrive after some algebra at (52), as we wanted to prove.

For \( p = 0 \), rewrite the result obtained above as

\[
\int_0^\infty \frac{e^{-pt}}{t^{1/2+n}} \gamma(n, xt) \, dt
= \frac{\sqrt{\pi} \, (n-1)! \, (-1)^n}{(\frac{1}{2})_n} \left[ p^{n-1/2} - \sqrt{p + x} \sum_{k=0}^{n-2} \frac{\left(\frac{1}{2}\right)_k}{k!} (-x)^k p^{n-1-k} + \frac{\left(\frac{1}{2}\right)_{n-1}}{(n-1)!} (-x)^k \right],
\]

and take \( p = 0 \), to obtain the desired result. \( \blacksquare \)

It is worth noting that we can obtain (52) from [21, Eqn. 2.10.3(2)] and (45).

2.4. Fourth differentiation formula

Theorem 17. For \( n = 1, 2, \ldots \) and \( t \in \mathbb{C} \), we have

\[
_2\tilde{F}_1\left( \begin{array}{c}
\frac{1}{2} - n, 1 - n \\
2 - n
\end{array} \left| t \right. \right) = \begin{cases}
2 \left(\frac{1}{2}\right)_n t^{n-1}, & n \geq 1, \\
1, & n = 1.
\end{cases}
\]

(53)
**Proof.** Set \( a = \frac{1}{2}, \ b = 1 \) and \( c = 2 \) in the differentiation formula [13, Eqn. 15.5.9],

\[
\frac{d^n}{dt^n} \left[ t^{c-1} (1 - t)^{a+b-c} \right] 2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \left| t \right. \right) = (c - n)_n \ t^{c-n-1} (1-t)^{a+b-c-n} \ 2F_1 \left( \begin{array}{c} a - n, b - n \\ c - n \end{array} \left| t \right. \right),
\]

\( n = 0, 1, 2, \ldots \)

and apply the result given in (24), to obtain

\[
2 \frac{d^n}{dt^n} \left( \frac{1}{\sqrt{1-t}} - 1 \right) = \frac{t^{1-n} (1-t)^{-1/2-n}}{\Gamma (2-n)} \ 2F_1 \left( \begin{array}{c} \frac{1}{2} - n, 1 - n \\ 2 - n \end{array} \left| t \right. \right).
\]

According to (18) for \( n \geq 1 \) and the definition of the regularized generalized hypergeometric function given in (41), we finally get (53).

For \( t = 0 \) and \( n = 1 \), we obtain a indeterminate expression. However, according to (9), we have that

\[
2F_1 \left( \begin{array}{c} a, 0 \\ b \end{array} \left| t \right. \right) = 1,
\]

thus we obtain the desired result for \( n = 1 \).  

**Corollary 18.** The following identity holds true for \( n = 1, 2, \ldots \) and \( t \in \mathbb{C} \),

\[
P^{n-1}_n (t) = - (-2)^n \left( \frac{1}{2} \right)_n t (1-t^2)^{(n-1)/2}.
\]

**Proof.** Set \( a = \frac{1}{2} - n \) and \( c = 2 - n \) in (51), and take into account (53), to obtain

\[
P^{n-1}_{-n-1} \left( \frac{1}{\sqrt{1-t}} \right) = - (-2)^n \left( \frac{1}{2} \right)_n \left( \frac{t}{t - 1} \right)^{(n-1)/2},
\]

which, according to the property [23, Eqn. 3.3.1(1)]:

\[
P^\mu_{-\nu-1} (z) = P^\mu_{\nu} (z),
\]

is equivalent to (55).  

---

17
3. Case \( n = 3 \)

In this case, (2) becomes

\[ x^3 - x + t = 0. \]  

(57)

In order to solve (57), we apply the solution of the cubic equation given in Appendix A, considering in (A.1) the negative sign ‘−’, \( m = \frac{1}{3} \) and \( n = \frac{1}{2} \), i.e.

\[
x_3(t) = \frac{1}{\sqrt{3}} \begin{cases} 
\cosh \left( \frac{1}{3} \cosh^{-1}(z) \right) - i\sqrt{3} \sinh \left( \frac{1}{3} \cosh^{-1}(\sqrt{z}) \right), & z \geq 1, \\
\cos \left( \frac{1}{3} \cos^{-1}(\sqrt{z}) \right) - \sqrt{3} \sin \left( \frac{1}{3} \cos^{-1}(\sqrt{z}) \right), & z \leq 1.
\end{cases}
\]  

(58)

where \( z = 3 \left( \frac{2t}{3} \right)^2 \). Therefore, from (13) and (58) we have

\[
2F_1 \left( \frac{1}{3}, \frac{2}{3}, \frac{z}{2} \right) = \frac{3}{2\sqrt{z}} \begin{cases} 
\cosh \left( \frac{1}{3} \cosh^{-1}(\sqrt{z}) \right) - i\sqrt{3} \sinh \left( \frac{1}{3} \cosh^{-1}(\sqrt{z}) \right), & z \geq 1, \\
\cos \left( \frac{1}{3} \cos^{-1}(\sqrt{z}) \right) - \sqrt{3} \sin \left( \frac{1}{3} \cos^{-1}(\sqrt{z}) \right), & z \leq 1.
\end{cases}
\]  

(59)

Note that we can simplify (59) considering that

\[
\frac{3}{\sqrt{z}} \sin \left( \frac{1}{3} \sin^{-1}(\sqrt{z}) \right) = \frac{3}{\sqrt{z}} \sin \left( \frac{\pi/2 - \cos^{-1}\sqrt{z}}{3} \right) = \frac{3}{2\sqrt{z}} \left\{ \cos \left( \frac{\cos^{-1}\sqrt{z}}{3} \right) - \sqrt{3} \sin \left( \frac{\cos^{-1}\sqrt{z}}{3} \right) \right\}.
\]

Since

\[
\cos^{-1} x = \begin{cases} 
i \cosh^{-1} x, & x \geq 1, \\
-\cosh^{-1} x, & x \leq 1,
\end{cases}
\]

and \( \cos(ix) = \cosh x \), and \( \sin(ix) = i \sinh x \), we conclude that \( \forall z \in \mathbb{C} \),

\[
2F_1 \left( \frac{1}{3}, \frac{2}{3}, \frac{z}{2} \right) = \frac{3}{\sqrt{z}} \sin \left( \frac{1}{3} \sin^{-1}(\sqrt{z}) \right).
\]  

(60)

The result given (60) can be obtained from [23, Eqn. 2.8(12)]:

\[
2F_1 \left( \frac{1+a}{2}, \frac{1-a}{2}, \frac{1}{3} \sin^2 z \right) = \frac{\sin a z}{a \sin z},
\]

taking \( a = \frac{1}{3} \). Nonetheless, by differentiation, we obtain from (60) the following interesting identity.
Theorem 19. For \( n = 1, 2, \ldots \) and \( z \in \mathbb{C}\setminus\{0, 1\} \), we have:

\[
\begin{align*}
2 \tilde{F}_1 \left( \frac{1}{3}, \frac{2}{3} - n \mid z \right) &= 6 z^{n-1/2} \frac{d^n}{dz^n} \left[ \sqrt{\pi} z^{-1/2} \sin \left( \frac{1}{3} \sin^{-1} \sqrt{z} \right) \right] \\
&= 6 z^{n-1/2} \frac{d^n}{dz^n} \left[ \sqrt{\pi} \sum_{k=1}^{n} \sin \left( \frac{\sin^{-1} \sqrt{z}}{3} + \frac{\pi k}{2} \right) B_{n,k} (h_1(z), \ldots, h_{n-k+1}(z)) \right],
\end{align*}
\]

where \( B_{n,k}(x_1,\ldots,x_{n-k+1}) \) denotes the Bell polynomial \([24, p. 133]\). Also, we have defined

\[
h_s(z) = \frac{(-i)^{s-1}(s-1)!}{6[z(1-z)]^{s/2}} P_{s-1} \left( \frac{1-2z}{2\sqrt{z(z-1)}} \right),
\]

where \( P_n(x) \) is a Legendre polynomial.

Proof. Set \( a = \frac{1}{3}, b = \frac{2}{3}, \) and \( c = \frac{3}{2} \) in (63) to obtain

\[
\begin{align*}
\frac{1}{3} \frac{d^n}{dz^n} \left[ \sqrt{\pi} z^{-1/2} \sin \left( \frac{1}{3} \sin^{-1} \sqrt{z} \right) \right] &= \frac{\sqrt{\pi} z^{1/2-n}}{6} 2 \tilde{F}_1 \left( \frac{1}{3}, \frac{2}{3} - n \mid z \right).
\end{align*}
\]

and substitute (60) in (64), to get

\[
2 \tilde{F}_1 \left( \frac{1}{3}, \frac{2}{3} - n \mid z \right) = 6 z^{n-1/2} \frac{d^n}{dz^n} \left[ \sqrt{\pi} \sum_{k=1}^{n} \sin \left( \frac{\sin^{-1} \sqrt{z}}{3} + \frac{\pi k}{2} \right) B_{n,k} (h_1(z), \ldots, h_{n-k+1}(z)) \right].
\]

In order to calculate the \( n \)-th derivative given in (65), we apply Faà di Bruno’s formula \([24, p. 137]\):

\[
\frac{d^n}{dz^n} f \left[ g(z) \right] = \sum_{k=1}^{n} f^{(k)} \left[ g(z) \right] B_{n,k} \left( g'(z), g''(z), \ldots, g^{(n-k+1)}(z) \right),
\]

Set \( f(z) = \sin z \) and \( g(z) = \frac{1}{3} \sin^{-1} \sqrt{z} \) in (66) and take into account the differentiation formula \([25, Eqn. 1.1.7(7)]\):

\[
\frac{d^n}{dz^n} \sin^{-1} (a\sqrt{z}) = \frac{(-i)^{n-1}}{2} (n-1)!a^n (z-a^2z^2)^{-n/2} P_{n-1} \left( \frac{1-2a^2z}{2a\sqrt{a^2z^2-z}} \right),
\]

\( n \geq 1, \)

and substitute (60) in (64), to get

\[
2 \tilde{F}_1 \left( \frac{1}{3}, \frac{2}{3} - n \mid z \right) = 6 z^{n-1/2} \frac{d^n}{dz^n} \left[ \sqrt{\pi} \sum_{k=1}^{n} \sin \left( \frac{\sin^{-1} \sqrt{z}}{3} + \frac{\pi k}{2} \right) B_{n,k} (h_1(z), \ldots, h_{n-k+1}(z)) \right].
\]

to arrive at (61), as we wanted to prove. ■
Remark 20. On the one hand, according to Gauss summation formula (20), the regularized hypergeometric function given in (61) is divergent for \( z = 1 \) and \( n = 1, 2, \ldots \). On the other hand, for \( z = 0 \), (62) and (63) yield indeterminate expressions. However, according to (10), \( \, _2F_1 \left( \frac{1}{2}, \frac{2}{3}; \frac{3}{2} - n; 0 \right) = 1 \) for \( n = 1, 2, \ldots \).

Next, we provide the elementary representations of (61) for \( n = 1, 2 \):

\[
\, _2F_1 \left( \frac{1}{3}, \frac{2}{3}; \z \right) = \cos \left( \frac{1}{2} \sin^{-1} \sqrt{\frac{1}{3}} \right) \frac{1}{\sqrt{1 - z}},
\]

and

\[
\, _2F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{2} \right) z \right) = \frac{(3 - 6z) \cos \left( \frac{1}{3} \sin^{-1} \sqrt{\frac{1}{3}} \right) + \sqrt{-z} (z - 1) \sin \left( \frac{1}{3} \sin^{-1} \sqrt{\frac{1}{3}} \right)}{3 (1 - z)^{3/2}}.
\]

Theorem 21. For Re \((2p - x) > 0\), the following definite integral holds true:

\[
\int_0^\infty e^{-pt} t^{5/6} D_{1/3} \left( -\sqrt{2x} t \right) dt = 2 \frac{\Gamma \left( \frac{1}{3} \right) }{(2p + x)^{1/6}} \left[ \cos \left( \frac{1}{3} \cos^{-1} \sqrt{\frac{2x}{2p + x}} \right) - \sin \left( \frac{1}{3} \sin^{-1} \sqrt{\frac{2x}{2p + x}} \right) \right],
\]

where \( D_\nu (z) \) denotes the parabolic cylinder function [26, Chap. VIII].

Proof. Set \( a_1 = \frac{1}{3} \), \( b_1 = \frac{3}{2} \), and \( \alpha = \frac{2}{3} \) in (22)\(^3\), taking into account (60), to obtain

\[
\int_0^\infty e^{-st} t^{1/3} \, _1F_1 \left( \frac{1}{2}; \frac{1}{2}; \frac{x t}{s^{1/6}} \right) dt = \frac{3 \Gamma \left( \frac{2}{3} \right) }{s^{1/6} \sqrt{x}} \sin \left( \frac{1}{3} \sin^{-1} \sqrt{\frac{x}{s}} \right).
\]

Apply now the following formula with \( a = \frac{1}{3} \) [19, Eqn. 7.11.1(10)]:

\[
\, _1F_1 \left( \frac{a}{3}; \frac{2}{3} x \right) = \frac{2a^{-5/2} \sqrt{\pi z}}{\sqrt{\pi z}} \Gamma \left( a - \frac{1}{2} \right) e^{x/2} \left[ D_{1-2a} \left( -\sqrt{2z} \right) - D_{1-2a} \left( \sqrt{2z} \right) \right],
\]

\(^3\)It is worth noting that the other choice, i.e. \( a_1 = \frac{2}{3} \) and \( \alpha = \frac{1}{3} \), leads to non-convergent integrals.
hence the RHS of (68) becomes:

\[
\int_0^\infty e^{-st} \frac{1}{t^{1/3}} \, \text{I}_1 \left( \frac{1}{3}, \frac{x}{t} \right) \, dt = \frac{2^{-13/6}}{\sqrt{\pi} x} \Gamma \left( \frac{-1}{6} \right) \tag{69}
\]

\[
\left[ \int_0^\infty e^{-s-x/2t} \frac{1}{t^{5/6}} D_1/3 \left( -\sqrt{2xt} \right) \, dt - \int_0^\infty e^{-s-x/2t} \frac{1}{t^{5/6}} D_1/3 \left( \sqrt{2xt} \right) \, dt \right].
\]

Consider now the definite integral [26, Eqn. 8.3(11)]:

\[
\int_0^\infty e^{-zt} \frac{1}{t^{\beta/2}} D_{\nu} \left( 2\sqrt{kt} \right) \, dt = \frac{2^{1-\beta-\nu/2} \sqrt{\pi} \Gamma (\beta)}{\Gamma \left( \nu+\beta+1 \right)} \frac{1}{s^{1/6}} \text{I}_1 \left( \frac{1}{3}, 1, \frac{x}{s} \right),
\]

Re \( \beta > 0 \), Re \( z/k > 0 \),

and the reduction formula [19, Eqn. 7.3.1(83)]:

\[
_2F_1 \left( \frac{a}{2}, -\frac{a}{2}, z \right) = \cos \left( 2a \sin^{-1} \sqrt{z} \right),
\]

to arrive at

\[
\int_0^\infty e^{-s-x/2t} \frac{1}{t^{5/6}} D_1/3 \left( \sqrt{2xt} \right) \, dt = \frac{2^{5/6} \Gamma \left( \frac{1}{3} \right)}{s^{1/6}} \cos \left( \frac{1}{3} \cos^{-1} \sqrt{\frac{x}{s}} \right). \tag{70}
\]

Therefore, taking into account (68)-(70), as well as [20, Eqns. 1.2.1&3]:

\[
\frac{\Gamma \left( \frac{1}{3} \right)}{\Gamma \left( -\frac{1}{6} \right) \Gamma \left( \frac{1}{3} \right)} = \frac{-1}{6 \times 2^{1/3} \sqrt{\pi}},
\]

after some algebra, we conclude (67), as we wanted to prove. ■

4. Case \( n = 4 \)

In this case, (2) becomes

\[
x^4 - x + t = 0. \tag{71}
\]

To solve (71), we consider \( p = 0, q = -1 \) and \( r = t \) in the solution of the quartic equation given in Appendix B, i.e. (B.1). Thereby, (B.5) and (B.6) become

\[
\gamma = \frac{1}{2} \left( \alpha^2 + \frac{1}{\alpha} \right), \tag{72}
\]

\[
\beta = \frac{t}{\gamma}. \tag{73}
\]
Therefore, setting $\xi = \alpha^2$, the resolvent cubic (B.4) is

$$\xi^3 - 4t\xi - 1 = 0,$$

which can be solved taking in (A.1) the ‘−’ sign, $m = \frac{4t}{3}$ and $n = -\frac{1}{2}$. Thereby, according to (A.2) and (A.3), and defining $z = 4 \left(\frac{4t}{3}\right)^3$, we arrive at

$$\xi (z) = \begin{cases} 
-2^{2/3}z^{1/6}\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{1}{\sqrt{z}}\right)\right), & z \leq 1, \\
-2^{2/3}z^{1/6}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{1}{\sqrt{z}}\right)\right), & z \geq 1.
\end{cases} \quad (74)$$

Note that both branches in (74) are equivalent, if we consider $z \in \mathbb{C}$, thus let us define the following function:

**Definition 22.**

$$g (z) = -z^{1/6}\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{1}{\sqrt{z}}\right)\right). \quad (75)$$

By inspection, the solution of (2) for $n = 4$ corresponding to (14) is just the solution $x_1$ in (B.2), i.e.

$$x_1 = \frac{1}{2} \left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right). \quad (76)$$

Therefore, from (14) on the one hand, and from (72)-(76) on the other hand, we finally obtain:

**Theorem 23.** For $z \in \mathbb{C}$, we have

$$\begin{aligned} 
\mathbf{3}_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{3}{3} \\ \frac{3}{3}, \frac{3}{3}, \frac{3}{3} \end{array} | z \right) & = \frac{4}{3}z^{-1/3} \\
& \quad \left[ \sqrt{g (z)} + \frac{3z^{1/3}\sqrt{g (z)}}{1 - 2 [g (z)]^{1/2}} - \sqrt{g (z)} \right]. \quad (77)
\end{aligned}$$

**Remark 24.** It is worth noting that the numerical evaluation of the LHS of (77) seems to fail for $z = 1$, since this point is a branch point. However, taking $a = \frac{1}{4}$, $c = \frac{1}{2}$ and $d = \frac{1}{3}$ in Whipple’s sum (21), we obtain:

$$\begin{aligned} 
\mathbf{3}_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{3}{3} \\ \frac{3}{3}, \frac{3}{3}, \frac{3}{3} \end{array} | 1 \right) & = \frac{\pi \Gamma \left(\frac{5}{4}\right) \Gamma \left(\frac{1}{4}\right) \Gamma \left(\frac{1}{4}\right) \Gamma \left(\frac{5}{4}\right) \Gamma \left(\frac{5}{4}\right) \Gamma \left(\frac{5}{4}\right) \Gamma \left(\frac{5}{4}\right)}{\Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right) \Gamma \left(\frac{19}{24}\right)} & = \frac{4}{3},
\end{aligned}$$

which is the result that we obtain on the RHS of (77).
Corollary 25. For $|z| < 1$, we have
\[ 3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{5}{6} ; \frac{1}{6} \\ \frac{5}{3}, \frac{1}{3} \end{array} \right) | z \right) = \frac{1}{\sqrt{1-z}} H \left( \frac{-4z}{(1-z)^2} \right), \quad (78) \]

where
\[ H(t) = \frac{4}{3} t^{-1/3} \left[ \sqrt{g(t)} + \frac{3^{1/3} \sqrt{g(t)}}{1 - 2 [g(t)]^{3/2}} - \sqrt{g(t)} \right]. \]

Proof. Take $\alpha = \frac{1}{4}, \lambda = \frac{1}{3},$ and $\mu = -\frac{1}{3}$ in the quadratic transformation [27];
\[ 3F_2 \left( \begin{array}{c} 2\alpha, 2\alpha + \lambda, 2\alpha + \mu \\ 1 - \lambda, 1 - \mu \end{array} \right) | x \right) = (1 - x)^{-2\alpha} 3F_2 \left( \begin{array}{c} \alpha, \alpha + \frac{1}{2}, 1 - 2\alpha - \lambda - \mu \\ 1 - \lambda, 1 - \mu \end{array} \right) \left( \frac{-4x}{(1-x)^2} \right),\]

to obtain,
\[ 3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{5}{6} ; \frac{1}{6} \\ \frac{5}{3}, \frac{1}{3} \end{array} \right) | 1 \right) = \frac{1}{\sqrt{1-x}} 3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{3}{4} ; \frac{1}{3} \\ \frac{5}{6}, \frac{4}{3} \end{array} \right) \left( \frac{-4x}{(1-x)^2} \right). \quad (79) \]

From (77) and (79), we arrive at (78), as we wanted to prove. \[ \blacksquare \]

Remark 26. We can calculate the LHS of (78) for the branch point $z = 1$
taking $a = \frac{1}{6}, c = \frac{1}{2},$ and $d = \frac{2}{3},$ resulting in
\[ 3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{5}{6} ; \frac{1}{6} \\ \frac{5}{3}, \frac{1}{3} \end{array} \right) | 1 \right) = \frac{\pi \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{4}{3} \right)}{\Gamma \left( \frac{5}{12} \right) \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{13}{12} \right)} \approx 1.24081. \]

Corollary 27. From the result (77), we obtain the following identity involving
the product of two Legendre functions:
\[ P_{-1/6}^{-1/3} \left( \sqrt{\frac{2}{1 + \sqrt{1-z}}} \right) P_{-1/6}^{-1/3} \left( \sqrt{\frac{2}{1 + \sqrt{1-z}}} \right) \quad (80) \]
\[ = \frac{\sqrt{6} (1 + \sqrt{1-z})}{\pi z^{1/3}} \left[ \sqrt{g(z)} + \frac{3z^{1/3} \sqrt{g(z)}}{1 - 2 [g(z)]^{3/2}} - \sqrt{g(z)} \right]. \]
**Proof.** We found in the literature [19, Eqn. 7.4.1(10)]:

\[
3F_2 \left( \begin{array}{c} a, 1 - a, \frac{1}{2} \\ b, 2 - b \end{array} \bigg| z \right) = 2F_1 \left( \begin{array}{c} a - a \\ 2 - b \end{array} \bigg| \frac{1 - \sqrt{1 - z}}{2} \right) \cdot 2F_1 \left( \begin{array}{c} a, 1 - a \\ b \end{array} \bigg| \frac{1 - \sqrt{1 - z}}{2} \right),
\]

thus, taking \( a = \frac{1}{4} \) and \( b = \frac{3}{3} \), we have

\[
3F_2 \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4}, \frac{3}{3} \\ \frac{3}{3}, \frac{3}{3} \end{array} \bigg| z \right) = 2F_1 \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{3}{3} \end{array} \bigg| \frac{1 - \sqrt{1 - z}}{2} \right) \cdot 2F_1 \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{3}{3} \end{array} \bigg| \frac{1 - \sqrt{1 - z}}{2} \right).
\] (81)

Also, setting \( a = \frac{1}{4} \) and \( c = \frac{2}{3}, \frac{4}{3} \) in (51), we have

\[
2F_1 \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{3}{3} \end{array} \bigg| z \right) = 2^{1/3} \Gamma \left( \frac{2}{3} \right) z^{1/6} (1 - z)^{-5/12} P_{-1/6}^{1/3} \left( \frac{1}{\sqrt{1 - z}} \right), \quad (82)
\]

\[
2F_1 \left( \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{4}{3} \end{array} \bigg| z \right) = 2^{1/3} \Gamma \left( \frac{4}{3} \right) z^{-1/6} (1 - z)^{-1/12} P_{-5/6}^{-1/3} \left( \frac{1}{\sqrt{1 - z}} \right). \quad (83)
\]

Therefore, inserting (82) and (83) in (81), taking into account the property (56), and knowing, according to [15, Eqn. 43:4:5], that \( \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{4}{3} \right) = \frac{2\pi}{\sqrt{3}} \), we obtain (80), as we wanted to prove.

\[\blacksquare\]

**5. Conclusions**

We have considered the solution of \( x^n - x + t = 0 \) for \( n = 2, 3, 4 \), both in terms of hypergeometric functions as well as in terms of elementary functions. Thereby, we have obtained some reduction formulas of hypergeometric functions. In order to extend the latter results, we have applied the differentiation formulas (26), (43), (46) and (54), as well as the integration formula stated in (22). Consequently, we have derived new identities and infinite integrals involving special functions, i.e. the incomplete beta function, the lower incomplete gamma function, the parabolic cylinder function and the Legendre function. All the results presented in this paper have been tested with MATHEMATICA and are available at [https://bit.ly/2PyPz6Y](https://bit.ly/2PyPz6Y).
Appendix A. The solution of the cubic equation

According to [28], in the solution of the depressed cubic equation:

\[ x^3 \pm 3mx + 2n = 0, \quad m > 0, \quad (A.1) \]

we may distinguish the following cases:

**Case I** Sign ‘+’ in (A.1). One real root and two complex roots:

\[
\begin{align*}
x_1 &= -2\sqrt{m} \sinh \left( \frac{\sinh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right), \\
x_{2,3} &= \sqrt{m} \left[ \sinh \left( \frac{\sinh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \pm i \sqrt{3} \cosh \left( \frac{\sinh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \right].
\end{align*}
\]

**Case II** Sign ‘−’ in (A.1) and \( n^2 - m^3 > 0 \). One real root and two complex roots.

\[
\begin{align*}
x_1 &= -2\sqrt{m} \cosh \left( \frac{\cosh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right), \quad (A.2) \\
x_{2,3} &= \sqrt{m} \left[ \cosh \left( \frac{\cosh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \pm i \sqrt{3} \sinh \left( \frac{\cosh^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \right].
\end{align*}
\]

**Case III** Sign ‘−’ in (A.1) and \( n^2 - m^3 < 0 \). Three real roots.

\[
\begin{align*}
x_1 &= -2\sqrt{m} \cos \left( \frac{\cos^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right), \quad (A.3) \\
x_{2,3} &= \sqrt{m} \left[ \cos \left( \frac{\cos^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \pm \sqrt{3} \sin \left( \frac{\cos^{-1} \left( \frac{n}{m^{3/2}} \right)}{3} \right) \right].
\end{align*}
\]

Appendix B. The solution of the quartic equation

According to Descartes solution of the quartic equation [29], the four solutions of the depressed quartic equation:

\[ x^4 + px^2 + qx + r = 0, \quad (B.1) \]
are given by:

\[ x_{1,2} = \frac{1}{2} \left( -\alpha \pm \sqrt{\alpha^2 - 4\beta} \right), \quad (B.2) \]
\[ x_{3,4} = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 - 4\gamma} \right), \quad (B.3) \]

where \( \alpha \) is a solution of the \textit{resolvent bicubic equation}:

\[ \alpha^6 + 2p\alpha^4 + (p - 4r)\alpha^2 - q^2 = 0, \quad (B.4) \]

and

\[ \gamma = \frac{1}{2} \left( p + \alpha^2 + \frac{q}{\alpha} \right), \quad (B.5) \]
\[ \beta = \frac{r}{\gamma}. \quad (B.6) \]

Note that the resolvent equation can be solved in \( \alpha^2 \) with the solution described in Appendix A.

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