The Preservation of Convexity by Geodesics in the Space of Kähler Potentials on Complex Affine Manifolds

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Abstract

On a compact complex affine manifold with a constant coefficient Kähler metric \(\omega_0\), we introduce a concept: \((S, \omega_0)\)-convexity and show that \((S, \omega_0)\)-convexity is preserved by geodesics in the space of Kähler potentials. This implies that if two potentials are both strictly \((S, \omega_0)\)-convex, then the metrics along the geodesic connecting them are non-degenerate.

1 Introduction

Results of this paper provide partial answers to the following questions: First, in the space of Kähler potentials, with the metric introduced by Semmes-Mabuchi-Donaldson, any two points can be joined by a weak geodesic, but metrics along the geodesic may be degenerate. The question is can we pose some conditions on two points, so that metrics along the geodesic connecting them do not degenerate? Second, the maximum rank problem has been extensively studied for a general class of fully nonlinear elliptic equations. But the situation for degenerate elliptic equations has remained unexplored, for example, degenerate complex Monge-Ampère equation. One question is whether, under some conditions, maximum rank property holds for solutions of degenerate complex Monge-Ampère equations?

In this paper, on a complex affine manifold with a constant coefficient metric \(\omega_0\), we introduce a concept \((S, \omega_0)\)-convexity, and show that if two potentials are both strictly \((S, \omega_0)\)-convex, then they can be connected by a geodesic with non-degenerate metric. Under similar condition, we can show the Hessian of the solution to the homogenous complex Monge-Ampère equation on an \(n+1\) dimensional product space has rank \(n\).

In section 1.1 we introduce some basic concepts and present some former results; in section 1.2 we introduce the concept of \((S, \omega_0)\)-convexity, an elliptic perturbation of the homogenous complex Monge-Ampère equation and also present our main results; in section 1.3 we provide a bird’s-eye view of results of each section and show the structure of the paper; in section 1.4 more notation and convention are introduced.

1.1 Background

Given an \(n\) dimensional Kähler manifold \((V, \omega_0)\), we define the space of Kähler potentials:

\[ \mathcal{H} = \{ \phi \in C^\infty(V) | \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}. \]  

(1.1)
A Riemannian metric can be introduced to this space. For $\psi_1, \psi_2 \in T_{\phi}H$, let
\[
<\psi_1, \psi_2>_{\phi} = \int_V \psi_1 \psi_2 (\omega_0 + \sqrt{-1} \partial \overline{\partial} \phi)^n.
\] (1.2)

With the Riemannian metric above, for a curve $\{\varphi(t)| t \in [0,1]\} \subset H$, its energy is
\[
E(\varphi) = \int_0^1 \int_V \varphi_1^2 (\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi)^n dt.
\] (1.3)

In this paper we denote $\omega_0 = \sqrt{-1} b_{\alpha \beta} dz^\alpha \wedge \overline{dz}^\beta$, $\alpha, \beta \in \{1, \ldots, n\}$, and $g_{\alpha \overline{\beta}} = b_{\alpha \beta} + \varphi_{\alpha \overline{\beta}}$. Then the Euler-Lagrange equation for the energy above is
\[
\varphi_{tt} = \varphi_{t\alpha} g^{\overline{\alpha} \overline{\beta}} \varphi_{t\overline{\beta}}.
\] (1.4)

When $\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi > 0$, equation (1.4) is equivalent to
\[
\det \begin{pmatrix}
\varphi_{tt} & \varphi_{t\alpha} \\
\varphi_{t\alpha} & \varphi_{\alpha \overline{\beta}} + b_{\alpha \overline{\beta}}
\end{pmatrix} = 0.
\] (1.5)

Here the curve $\varphi$ is considered as a function defined on $[0,1] \times V$. Let $S = \{\tau = t + \sqrt{-1} \theta|0 \leq t \leq 1\} \subset \mathbb{C}$. (1.6)

We can consider $\varphi$ as a function on $S \times V$ by letting $\varphi(\tau) = \varphi(t)$. Then equation (1.5) becomes a homogenous complex Monge-Ampère equation:
\[
\det \begin{pmatrix}
\varphi_{\tau \tau} & \varphi_{\tau \alpha} \\
\varphi_{\alpha \tau} & \varphi_{\alpha \overline{\beta}} + b_{\alpha \overline{\beta}}
\end{pmatrix} = 0.
\] (1.7)

Denote the projection from $S \times V$ to $V$ by $\pi_V$ and denote $\pi_V(\omega_0)$ by $\Omega_0$. Then equation (1.7) becomes
\[
(\Omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi)^{n+1} = 0.
\] (1.8)

This leads to the study of the following Dirichlet problem on $S \times V$.

**Problem 1.1 (Geodesic Problem).** Given $\varphi_0, \varphi_1 \in H$, find $\Phi \in C^{1,1}(S \times V)$, satisfying
\[
\begin{align*}
(\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi)^{n+1} &= 0, & \text{in } S \times V; \\
\Omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi &\geq 0, & \text{in } S \times V; \\
\Phi_\theta &= 0, & \text{in } S \times V; \\
\Phi &= \varphi_0, & \text{on } \{t = 0\} \times V; \\
\Phi &= \varphi_1, & \text{on } \{t = 1\} \times V.
\end{align*}
\] (1.9-1.13)

For a solution $\Phi$ to Problem 1.1, $\Phi(t,*)$ may not be in $H$, so we consider it as a weak or generalized geodesic connection $\varphi_0$ and $\varphi_1$.

More generally, we can replace $S$ by a Riemann surface $\mathcal{R}$ and consider the following Dirichlet problem. In this paper we only consider the case where $\mathcal{R}$ is a bounded domain in $\mathbb{C}$ with smooth boundary.
Problem 1.2 (A Homogenous Monge-Ampère Equation on General Product Spaces). Given $\mathcal{R}$, a bounded domain in $\mathbb{C}$ with smooth boundary, and $F \in C^\infty(\partial \mathcal{R} \times V)$ satisfying

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} (F(\tau,*)) > 0, \quad \text{for any } \tau \in \partial \mathcal{R},$$

find $\Phi \in C^{1,1}(\mathcal{R} \times V)$ which satisfies

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} (\Phi(\tau,*))^{n+1} = 0, \quad \text{in } \mathcal{R} \times V;$$

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi \geq 0, \quad \text{in } \mathcal{R} \times V;$$

$$\Phi = F, \quad \text{on } \partial \mathcal{R} \times V.$$

Remark 1.1. Problem 1.1 can be reduced to Problem 1.2. Let $f$ be a holomorphic covering map from $\mathcal{S}$ to an annulus $\{ \tau \mid 1 < |\tau| < 2 \}$. If $\Phi$ is a solution to Problem 1.2 with $\mathcal{R}$ being the annulus $\{ \tau \mid 1 < |\tau| < 2 \}$ and $F|_{\{|\tau|=1\}} = \phi_0$, $F|_{\{|\tau|=2\}} = \phi_1$, then $\Phi(f(\tau),z)$ is a solution to Problem 1.1.

Problem 1.2 and 1.1 were introduced by [M87] [S92] [D99]. The existence of $C^{1,1}$ solution was established by [C00] [B12] [CTW19] [CTW17] and it was also shown by [LV13] and [DL12] that the optimal regularity of general solutions is $C^{1,1}$.

Besides regularity, we may ask if

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau,* > 0, \quad \text{for all } \tau \in \mathcal{R},$$

for a solution $\Phi$ to Problem 1.2. This is similar to the maximum rank problem, which asks if

$$\text{rank}(\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau,*)) = n.$$

(1.19)

It’s easy to see that (1.18) implies (1.19). But the inverse implication may not be true. It turns out that, for a general solution, (1.18) or (1.19) may not be valid. An example was constructed in [RN], when $\mathcal{R}$ is a disc and $V$ is $\mathbb{C}P^1$. In this example, a solution $\Phi$ was constructed, which satisfies

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi = 0,$$

in an open set in $\mathcal{R} \times V$. However, we may ask is it possible to find some conditions for the boundary value $F$, so that if they are satisfied then (1.18) is valid.

Theorem 1 of [D02] says that, when $\mathcal{R}$ is a disc, the set of smooth functions $F$ for which a smooth solution to Problem 1.2 exists is open in $C^\infty(\partial \mathcal{R} \times V)$. Actually the proof implies that if the boundary value of $\Phi$ is in this set, then (1.18) is satisfied. The proof also suggests that this set is open in $C^2$ topology. The proof made use of the foliation structure associated to a solution to homogenous complex Monge-Ampère equations. This technique was also used in [L81], to construct pluri-complex Green’s function. In [CFH20], by partially generalizing this technique to the case where $\mathcal{R}$ is an annulus, we proved that if $|\varphi_0|_5 + |\varphi_1|_5$ is small enough, then the geodesic connecting $\varphi_0$ and $\varphi_1$ are $C^4$ and

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(t,* > 0, \quad \text{for all } t \in [0,1].$$

(1.21)

In a recent paper [H22], the author improved the result above, reducing the $C^5$ smallness condition to a $C^2$ smallness condition.

However, [GPT2M] [LV13] [H21] and the Appendix A of [H22] all suggest that a proper condition on $F$, which implies (1.18), may be a convexity condition. In [GPT2M], it was shown that, when
$V$ is a 1-dimensional complex flat torus, if $\varphi_1$ and $\varphi_0$ both satisfy a convexity condition, then the geodesic connecting them has non-degenerate metric. In the Appendix A of [H22], a similar result was proved, for solutions to Problem [1.2] with a very different method. Furthermore, computations of [LV13] and [H21] suggest that the convexity condition is also necessary.

In this paper, when $V$ is a compact complex affine manifold with a constant coefficient metric, we introduce a concept: $(S, \omega_0)$-convexity and show that boundary values satisfying $(S, \omega_0)$-convexity implies (1.18).

1.2 Notation, Constructions and Main Results

In this paper, we will discuss the situation, where $V$ is a compact complex affine manifold with a constant coefficient Kähler metric $\omega_0$. By the definition of complex affine manifold, $V$ is equipped with an atlas, so that all transition maps are affine and holomorphic. Furthermore, we require that $\omega_0$ is a constant metric, which we mean, in any coordinate neighborhood, with coordinates $\{z^\alpha\}_{\alpha=1}^n$, if

$$\omega_0 = \sqrt{-1}b_{\alpha\overline{\beta}}dz^\alpha \wedge d\overline{z}^\beta,$$

then $b_{\alpha\overline{\beta}}$ is constant for any $\alpha, \beta \in \{1, \ldots, n\}$.

With these preparations, we can introduce the concept of $\omega_0$-convexity.

**Definition 1.1 ($\omega_0$-Convexity and Strict $\omega_0$-Convexity).** A function $\varphi \in C^0(V)$ is (strictly) $\omega_0$-convex if, in any coordinate chart, with coordinates $\{z^\alpha\}$,

$$\varphi + b_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta$$

is a (strictly) convex function.

The convexity above has been widely used in many works related to Hessian manifolds, for example, it was called local convexity in [CV01] and called $g$-convexity in [GT21]. However, to estimate the convexity of solutions to Problem (1.2), it’s necessary to extend this concept and introduce the following concept of $(S, \omega_0)$-convexity.

**Definition 1.2 ($(S, \omega_0)$-Convexity and Strict $(S, \omega_0)$-Convexity for $C^0$ Function).** Suppose $S$ is a constant section of $T^*_{2,0}(V)$. Then a function $\varphi \in C^0(V)$ is (strictly) $(S, \omega_0)$-convex if, in any coordinate chart, with coordinates $\{z^\alpha\}$,

$$\varphi + b_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta + \text{Re}(S_{\alpha\beta} z^\alpha \overline{z}^\beta)$$

is a (strictly) convex function.

For a constant section, we mean, in any coordinate chart, the tensor components of $S$ are constant. Obviously, when $S = 0$, $(S, \omega_0)$-convexity is exactly $\omega_0$-convexity. Furthermore, to gauge the convexity, we introduce the concept of modulus of convexity.

**Definition 1.3 (Modulus of $(S, \omega_0)$-Convexity).** Suppose $S$ is a constant section of $T^*_{2,0}(V)$. Then a function $\varphi \in C^0(V)$ is $(S, \omega_0)$-convex of modulus $\geq \mu$ if, in any coordinate chart, with coordinates $\{z^\alpha\}$,

$$\varphi + (1 - \mu)b_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta + \text{Re}(S_{\alpha\beta} z^\alpha \overline{z}^\beta)$$

is a convex function. It’s $(S, \omega_0)$-convex of modulus $> \mu$ if (1.25) is strictly convex.
Remark 1.2. It's easy to see that if \( \varphi \) is \((S, \omega_0)\)-convex of modulus > \( \mu \), for \( \mu \geq 0 \), then it's strictly \((S, \omega_0)\)-convex. Another fact is that \( \varphi \) is \((S, \omega_0)\)-convex of modulus > \( \mu \) implies that it's \((S, \omega_0)\)-convex of modulus \( \geq \mu \). In addition, since \( V \) is compact, \( \varphi \) is \((S, \omega_0)\)-convex of modulus \( \geq \mu \) implies that, for any \( \mu' < \mu \), \( \varphi \) is \((S, \omega_0)\)-convex of modulus > \( \mu' \).

When the function \( \varphi \) is \( C^2 \) continuous, the strict \((S, \omega_0)\)-convexity can be defined using complex second order derivatives. Using Lemma A.1, we will see Definition 1.2 is equivalent to the following.

Definition 1.4 (Strict \((S, \omega_0)\)-Convexity for \( C^2 \) Functions). Suppose \( S \) is a constant section of \( T^*_{2,0}(V) \). Then a function \( \varphi \in C^2(V) \) is strictly \((S, \omega_0)\)-convex if

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0
\]

and the maximal eigenvalue of the tensor

\[
K = (\varphi_{\theta \nu} - S_{\theta \nu}) g^{\gamma \epsilon} (\varphi_{\epsilon \gamma} - S_{\epsilon \gamma}) g^{\beta \kappa} dz^\beta \otimes \frac{\partial}{\partial z^\kappa}
\]

is smaller than 1.

Remark 1.3. Here \( \varphi_{\alpha \beta} dz^\alpha \otimes dz^\beta \) is a well defined section of \( T^*_{2,0}(V) \), because the coordinate transition functions between charts are affine. In addition, using basic linear algebra, we know eigenvalues of \( K \) are real and non-negative.

We introduce another measurement of the convexity:

Definition 1.5 (Degree of Convexity). Suppose \( S \) is a constant section of \( T^*_{2,0}(V) \) and \( \delta \) is a positive number. Then a function \( \varphi \in C^2(V) \) is \((S, \omega_0)\)-convex of degree > \( \delta \) if, for any constant section \( \Theta \) of \( T^*_{2,0}(V) \) with

\[
\text{maximum eigenvalue of } \left( \Theta_{\alpha \beta} b^{\gamma \epsilon} \Theta_{\beta \gamma} b^{\epsilon \alpha} dz^\alpha \otimes \frac{\partial}{\partial z^\beta} \right) \leq \delta^2
\]

\( \varphi \) is a strictly \((S + \Theta, \omega_0)\)-convex function.

It turns out that the degree of convexity and the modulus of convexity coincide. In Lemma A.2 we show that, for \( \varphi \in C^2(V) \) and \( \delta \geq 0 \), \( \varphi \) is \((S, \omega_0)\)-convex of degree > \( \delta \) if and only if it is \((S, \omega_0)\)-convex of modulus > \( \delta \).

The main results of the paper are the following.

Theorem 1.1 (Estimates for Geodesics). Given \( \varphi_0, \varphi_1 \in \mathcal{H} \), suppose that there is an \( S \) which is a constant section of \( T^*_{2,0}(V) \), so that \( \varphi_0 \) and \( \varphi_1 \) are both \((S, \omega_0)\)-convex of modulus > \( \mu \), for \( \mu > 0 \). Let \( \{ \varphi_t \mid t \in [0, 1] \} \) be the geodesic connecting \( \varphi_0 \) and \( \varphi_1 \). Then, for any \( t \in (0, 1) \), \( \varphi_t \) is \((S, \omega_0)\)-convex of modulus \( \geq \mu \) and, by definition, this implies

\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_t \geq \mu \omega_0,
\]

in the weak sense.

The theorem above is a particular case of the following theorem, according to Remark 1.1.
Theorem 1.2 (Estimates on Product Space). Suppose $F$ is a $C^\infty$ function on $\partial R \times V$ and, for a constant $\mu > 0$ and a constant section $S$ of $T^*_2(\Sigma)$,

$$F(\tau,*) = (S,\omega_0) \text{-convex of modulus } > \mu, \text{ for any } \tau \in \partial R.$$ (1.30)

Let $\Phi$ be the solution to Problem 1.2 with boundary value $F$. Then $\Phi(\tau,*)$ is $(S,\omega_0)$-convex of modulus $\geq \mu$, for any $\tau \in R$, and, as a consequence

$$\omega_0 + \sqrt{-1}\partial\bar{\partial}[\Phi(\tau,*)] \geq \mu \omega_0,$$ (1.31)

in the weak sense.

As we discussed, solutions to Problem 1.2 may only be $C^{1,1}$, but to implement our method we need to use up to 4-th order derivatives. Therefore, we consider an elliptic perturbation of Problem 1.2, for which there are smooth solutions. In many previous works, the following problem was studied:

Problem 1.3 (Non-Degenerate Monge-Ampère Equation on Product Space). Given $R$, a bounded domain in $\mathbb{C}$ with smooth boundary, and $F \in C^\infty(\partial R \times V)$ satisfying

$$\omega_0 + \sqrt{-1}\partial\bar{\partial}(F(\tau,*)) > 0, \text{ for any } \tau \in \partial R,$$ (1.32)

find $\Phi \in C^{1,1}(R \times V)$ which satisfies

$$\begin{align*}
(\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} &= \varepsilon \sqrt{-1}d\tau \wedge d\bar{\tau} \wedge \Omega_0^n, &\text{in } R \times V; \\
\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi &= 0, &\text{in } R \times V; \\
\Phi &= F, &\text{on } \partial R \times V.
\end{align*}$$ (1.33-1.35)

In above, $\varepsilon$ is a positive constant.

However, in this paper we introduce a different perturbation.

Problem 1.4 (An Elliptic Perturbation of Homogenous Complex Monge-Ampère Equation). Suppose $F \in C^\infty(\partial R \times V)$ satisfies that

$$\omega_0 + \sqrt{-1}\partial\bar{\partial}(F(\tau,*)) > 0, \text{ for any } \tau \in \partial R.$$ (1.36)

Find $\Phi \in C^\infty(R \times V)$ satisfying:

$$\begin{align*}
(\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n+1} &= \epsilon \sqrt{-1}d\tau \wedge d\bar{\tau} \wedge \Omega_0^n \cap (\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi)^{n-1}, &\text{in } R \times V; \\
\Omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi &= 0, &\text{in } R \times V; \\
\Phi &= F, &\text{on } \partial R \times V.
\end{align*}$$ (1.37-1.39)

Here $\epsilon$ is a positive constant.

Remark 1.4. Equation (1.37) is equivalent to

$$\Phi_{\tau\bar{\tau}} - \Phi_{\tau\bar{\tau}}g^{\alpha\beta} \Phi_{\alpha\beta} = \epsilon b_{\alpha\beta}g^{\alpha\beta},$$ (1.40)

providing

$$\omega_0 + \sqrt{-1}\partial\bar{\partial}(\Phi(\tau,*)) > 0, \text{ for any } \tau \in R.$$ (1.41)

We also notice that, when $n = 1$, Problem 1.4 is exactly Problem 1.3, with $\varepsilon = \epsilon$. In a previous paper [122], we proved a particular 1-dimensional case of Theorem 1.2, by working with solutions to Problem 1.3.
In Section 4, we prove that under the condition of Theorem 1.2 a smooth solution to Problem 1.4 exists. However, we don’t know if a smooth solution always exists for general boundary value $F$. The convergence of solutions as $\epsilon$ goes to zero is discussed in Section 5.

1.3 Structure of the Paper

In section 2, by differentiating equation (1.40), we find second order derivatives of $\Phi$,
\[ A_{\alpha\beta} = \Phi_{\alpha\beta} + b_{\alpha\beta} \text{ and } B_{\alpha\beta} = \Phi_{\alpha\beta}, \] (1.42)
satisfy two non-linear equations (2.2) and (2.3).

In section 3.1, using $A$ and $B$, we construct $M_S$, which measures the convexity, and $Q_S^{[p]}$, which is a smooth approximation of $M_S$. Then, using equation (2.2) and (2.3), we show, for an elliptic operator $L^{ij} \partial_{ij}$ and $Q_S^{<p>} = \left( Q_S^{[p]} \right)^p$,
\[ L^{ij} \partial_{ij} (Q_S^{<p>}) \geq 0, \] (1.43)
providing $Q_S^{[p]} \leq 1 - \frac{1}{2^p}$.

In section 3.2, using a continuity argument, we show if $M_S < 1$ on $\partial R \times V$, then $M_S < 1$ in $R \times V$. This means the strict $(S,\omega_0)$-convexity on $\partial R \times V$ implies the strict $(S,\omega_0)$-convexity in $R \times V$.

In section 3.3, by altering $S$, we show the $(S,\omega_0)$-convexity of modulus $\mu$ on $\partial R \times V$ implies the $(S,\omega_0)$-convexity of modulus $\mu$ in $R \times V$.

We point out that estimates in section 3.1, 3.2 and 3.3 all depends on some apriori assumptions. These assumptions can be removed after we prove the existence of smooth solutions to Problem 1.4.

In section 4, we establish $C^0$, $C^1$, $C^2$ and $C^{2,\alpha}$ estimates. The methods are standard in PDE. In the proof, the result of section 3.3 plays an important role. Actually, it basically says equation (1.40) is a uniform elliptic equation, providing $\epsilon > 0$. With these estimates, we prove existence of $C^\infty$ solution by the method of continuity in section 4.6.

Finally, we prove estimates for solutions to Problem 1.1 and 1.2 by letting $\epsilon \to 0$.

1.4 A Convention for Tensor Contraction

In this paper, we need to contract a sequence of rank 2 tensors to form new tensors. The computation can be simplified by converting rank 2 tensors to matrices. In this sections, we explain how to do this.

Suppose $A$ is a section of $T_{1,1}(V)$ and $B$ is a section of $T_{2,0}(V)$. In a coordinate chart, $A$ and $B$ can be considered as matrix valued functions, which we still denote by $A$ and $B$. Let
\[ A = (A_{\alpha\beta}), \quad B = (B_{\alpha\beta}), \] (1.44)
where $\alpha$ is the row index and $\beta$ is the column index. As a matrix, when $A$ is invertible, we denote
\[ (A^{\alpha\beta}) = A^{-1}, \] (1.45)
where \( \alpha \) is the column index and \( \beta \) is the row index. Then we have
\[
A^{\alpha \overline{\beta}} A_{\alpha \overline{\beta}} = \delta_{\delta \beta} \tag{1.46}
\]
and we know
\[
A^{\alpha \overline{\beta}} \frac{\partial}{\partial z^\alpha} \otimes \frac{\partial}{\partial z^\beta} \tag{1.47}
\]
is a section of \( T_{1,1}(V) \).

Let
\[
K^\beta_\alpha = B_{\alpha \theta} A^{\theta \overline{\beta}} B_{\mu \rho} A^{\mu \overline{\theta}}, \tag{1.48}
\]
then \( K^\beta_\alpha dz^\alpha \otimes \frac{\partial}{\partial z^\beta} \) is a section of \( T^*_1(V) \otimes T^*_1(V) \). Locally, we can consider \( K \) as a matrix, with
\[
K = (K^\beta_\alpha), \tag{1.49}
\]
where \( \alpha \) is the row index and \( \beta \) is the column index. Then
\[
K = BA^{-1} \tag{1.50}
\]
for a constant \( C \), if, at any point \( p \in V \), the maximum eigenvalue of \( \Theta(p) \) is smaller than \( C \).

## 2 Equations for Second Order Derivatives

In this paper we will mainly work on the product space \( \mathcal{R} \times V \), where \( \mathcal{R} \) is a compact domain in \( \mathbb{C} \) with smooth boundary and \( V \) is the complex affine manifold. We denote the coordinate on \( \mathcal{R} \) by \( \tau \) and denote the coordinates on \( V \) by \( \{z^\alpha\}_{\alpha=1}^n \). Coordinates on \( V \) are indexed by Greek letters, except \( \tau \). The coordinate \( \tau \) on \( \mathcal{R} \) will be considered as the 0-th coordinate and, in some situation, we denote it by \( z^0 \). Thus, the coordinates on \( \mathcal{R} \times V \) will be indexed by Roman letters, running from 0 to \( n \).

Suppose \( \Phi \) is a solution to Problem 1.4. Let
\[
A_{\alpha \overline{\beta}} = \Phi_{\alpha \overline{\beta}} + b_{\alpha \overline{\beta}}, \quad B_{\alpha \beta} = \Phi_{\alpha \beta}. \tag{2.1}
\]
As described in section 1.4, \( A, B \) can be considered as matrices locally. In this section, by differentiating (1.40), we show, as matrices, \( A, B \) satisfy the following equations:
\[
L^\gamma \partial_\gamma A = L^\gamma (\partial_\gamma A)A^{-1}(\partial_\gamma A) + L^\gamma (\partial_\gamma A)A^{-1}(\partial_\gamma B), \tag{2.2}
\]
\[
L^\gamma \partial_\gamma B = L^\gamma (\partial_\gamma A)A^{-1}(\partial_\gamma B) + L^\gamma (\partial_\gamma B)A^{-1}(\partial_\gamma A). \tag{2.3}
\]
Here $L = L^\gamma \partial_\gamma$ is an elliptic operator on $\mathcal{R} \times V$, with
\[
\begin{pmatrix}
L^{\alpha \gamma} & L^{\beta \gamma} \\
L^{\alpha \delta} & L^{\beta \delta}
\end{pmatrix} = \begin{pmatrix}
1 & -\Phi_{\tau \rho}g^{\rho \gamma} \\
-\Phi_{\tau \rho}g^{\rho \tau} & \Phi_{\tau \rho}g^{\rho \gamma} + c_{\gamma \delta}g^{\gamma \delta}g^{\rho \gamma}
\end{pmatrix}.
\tag{2.4}
\]

The computations in this section are similar to the computations of Section 2.1 and 2.2 of [H22]. However, the computations here are much simpler because we are working on a flat affine manifold so $\Phi_{\alpha \beta}$ are coordinate derivatives while in [H22] we need to compute covariant derivatives.

Apply $\partial_\theta$ to (1.15), we get
\[
\Phi_{\theta \gamma \tau} - \Phi_{\theta \tau \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} = -c_{\alpha \beta}g^{\alpha \gamma} \Phi_{\rho \gamma}g^{\rho \gamma}.
\tag{2.5}
\]
Then apply $\partial_\gamma$ to (2.5), we get
\[
\Phi_{\theta \gamma \tau} - \Phi_{\theta \tau \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\tau \rho \gamma}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} = -c_{\alpha \beta}g^{\alpha \gamma} \Phi_{\rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}.
\tag{2.6}
\]
\[
= eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}.
\tag{2.7}
\]
We will use the following convention. $(*,*)_k$ stands for the $k$-th term in the line $(*)$, including the sign. For example,
\[
(2.9)_2 = -c_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma}, \quad (2.10)_1 = \Phi_{\theta \rho \gamma}, \quad (2.10)_4 = -\Phi_{\rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}.
\tag{2.10}
\]

It’s straightforward to verify the following equality:
\[
\begin{align*}
(2.9)_4 + (2.9)_2 + (2.9)_2 + (2.9)_1 - (2.9)_2 &= \Phi_{\theta \rho \gamma}g^{\rho \gamma},
(2.10)_3 + (2.10)_4 + (2.10)_1 + (2.10)_4 - (2.10)_1 &= -\Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}g^{\rho \gamma},
(2.10)_5 + (2.10)_3 + (2.10)_2 + (2.10)_3 - (2.10)_3 &= -\Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}g^{\rho \gamma}.
\end{align*}
\tag{2.11}
\]
This gives us (2.9).

Similarly, we apply $\partial_\tau$ to (2.5) and get
\[
\begin{align*}
\Phi_{\theta \tau \gamma} - \Phi_{\theta \gamma \tau}g^{\rho \gamma} \Phi_{\alpha \tau} + \Phi_{\gamma \tau \rho}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\gamma \tau \rho}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + \Phi_{\gamma \tau \rho}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - \Phi_{\gamma \tau \rho}g^{\rho \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}.
\tag{2.14}
\end{align*}
\]
\[
\begin{align*}
= eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} - eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau} + eb_{\alpha \beta}g^{\alpha \gamma} \Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}.
\tag{2.15}
\end{align*}
\]
It’s straightforward to verify the following equality
\[
\begin{align*}
(2.14)_1 + (2.14)_2 + (2.14)_2 + (2.14)_4 - (2.14)_2 &= \Phi_{\theta \rho \gamma}g^{\rho \gamma},
(2.14)_3 + (2.14)_4 + (2.14)_1 + (2.14)_4 - (2.14)_1 &= -\Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}g^{\rho \gamma},
(2.14)_5 + (2.14)_3 + (2.14)_2 + (2.14)_3 - (2.14)_3 &= -\Phi_{\theta \rho \gamma}g^{\rho \gamma} \Phi_{\alpha \tau}g^{\rho \gamma}.
\tag{2.18}
\end{align*}
\]
This gives us (2.2).
3 Apriori Estimates

In this section, we will prove some estimates for solutions to Problem 1.4 with some apriori assumptions. These assumptions can be removed after we prove the existence of $C^\infty$ solutions to Problem 1.4 in section 4. One estimate in this section, Prop 3.3, will play an indispensable role in our proof of the existence result.

Suppose $S$ is a constant section of $T^*_2(V)$ and $F \in C^\infty(\partial R \times V)$ satisfies

$$F(\tau, \ast) \text{ is strictly } (S, \omega_0)-\text{convex, for any } \tau \in \partial R.$$  \hspace{1cm} (3.1)

For a solution $\Phi$ to Problem 1.4 with boundary value $F$, let

$$B = (\Phi_{\alpha\beta}), \hspace{1cm} (3.2)$$
$$A = (\Phi_{\alpha\beta} + b_{\alpha\beta}), \hspace{1cm} (3.3)$$
$$K_S = (B - S)A^{-1}(B - S)A^{-1}, \hspace{1cm} (3.4)$$

and, for any $(\tau, z) \in \mathcal{R} \times V$,

$$M_S(\tau, z) = \text{Maximum Eigenvalue of } K_S(\tau, z). \hspace{1cm} (3.5)$$

Condition (3.1) implies that

$$M_S < 1, \text{ on } \partial \mathcal{R} \times V. \hspace{1cm} (3.6)$$

We want to show that

$$M_S < 1, \text{ in } \mathcal{R} \times V. \hspace{1cm} (3.7)$$

This implies that $\Phi(\tau, \ast)$ is strictly $(S, \omega_0)$-convex for any $\tau \in \partial \mathcal{R}$. However, it’s difficult to directly work with $M_S$, since it may not be differentiable. We introduce the following approximation of $M_S$.

Let

$$Q^{<p>}_S = \text{tr}(K^{p}_S), \hspace{1cm} (3.8)$$
$$Q^{[p]}_S = (Q^{<p>}_S)^{\frac{1}{p}}. \hspace{1cm} (3.9)$$

According to basic calculus, for $\lambda_1, \ldots, \lambda_n \geq 0$,

$$\lim_{p \to +\infty} (\lambda_1^p + \ldots + \lambda_n^p)^{1/p} = \max\{\lambda_1, \ldots, \lambda_n\}, \hspace{1cm} (3.10)$$

so

$$\lim_{p \to +\infty} Q^{[p]}_S = M_S. \hspace{1cm} (3.11)$$

If we can show for $p$ big enough,

$$Q^{[p]}_S \leq \max_{\partial \mathcal{R} \times V} Q^{[p]}_S, \text{ in } \mathcal{R} \times V, \hspace{1cm} (3.12)$$

then we can let $p$ go to $\infty$ and prove (3.7).
In section 3.1 we prove that for the elliptic operator $L = L^\gamma \partial_\gamma$ introduced in section 2,

$$L^\gamma (Q^{<p>}_S) \geq 0, \quad \text{in } \mathcal{R} \times V,$$  \hspace{1cm} (3.13)

providing $K_S \leq 1 - \frac{1}{2p}$. In section 3.2 using a continuity argument we prove

$$Q^{|p|}_S \leq \max_{\partial \mathcal{R} \times V} Q^{|p|}_S, \quad \text{in } \mathcal{R} \times V.$$  \hspace{1cm} (3.14)

In section 3.3 by altering $S$, we prove a convexity estimate, which implies a metric lower bound estimate

$$\omega_0 + \sqrt{-1}(\partial_\gamma \Phi(\tau, *)) \geq \mu \omega_0,$$  \hspace{1cm} (3.15)

for a constant $\mu > 0$.

In section 3.2 and 3.3 we need an apriori assumption that for any $\lambda \in [0, 1]$, Problem 1.4 has a solution $\Phi^\lambda$ with boundary value $\lambda F$ and $\{\Phi^\lambda | \lambda \in [0, 1]\}$ is a continuous curve in $C^4(\overline{\mathcal{R}} \times V)$ in $C^2$ topology.

### 3.1 Computation

Suppose $\Phi$ is a $C^4$ solution to Problem 1.4. In a local coordinate chart, let

$$K = BA^{-1} BA^{-1}$$  \hspace{1cm} (3.16)

and, for a positive integer $p$,

$$Q^{<p>} \equiv \text{tr}(K^p).$$  \hspace{1cm} (3.17)

As matrices, $B, A$ and $K$ depends on the choice of coordinate, while $Q^{<p>}$ is a well defined function on $\mathcal{R} \times V$. In this section, we show

$$L^\gamma (Q^{<p>}) \geq 0, \quad \text{providing } K \leq 1 - \frac{1}{2p}.$$  \hspace{1cm} (3.18)

After proving this, we know for

$$K_S = (B - S)A^{-1} (B - S)A^{-1}$$  \hspace{1cm} (3.19)

and

$$Q^{<p>_S} \equiv \text{tr}(K^p_S),$$  \hspace{1cm} (3.20)

$$L^\gamma (Q^{<p>_S}) \geq 0, \quad \text{providing } K_S \leq 1 - \frac{1}{2p}.$$  \hspace{1cm} (3.21)

This is because $B - S, A$ satisfy the same set of equations as $B, A$ do. Similar to (2.2) (2.3), we have:

$$L^\gamma \partial_\gamma A = L^\gamma (\partial_\gamma A)A^{-1} \partial_\gamma A + L^\gamma \partial_\gamma (B - S)A^{-1} \partial_\gamma (B - S);$$  \hspace{1cm} (3.22)

$$L^\gamma \partial_\gamma (B - S) = L^\gamma (\partial_\gamma A)A^{-1} \partial_\gamma (B - S) + L^\gamma \partial_\gamma (B - S)A^{-1} (\partial_\gamma A).$$  \hspace{1cm} (3.23)
Equations above are equivalent to $(2.2)$ $(2.3)$ because $S$ is a constant section and all derivatives of $S$ are zero.

Before the computation of $L^7 (Q^{<p>})_7$, we do some preparation. First, we note that the conjugate of equation $(2.3)$ is equivalent to

$$L^7 \left( A^{-1} B_i A^{-1} \right)_7 = 0.$$  \hspace{1cm}  \text{(3.24)}

Then we introduce a quantity

$$B_i = B_i - A_i A^{-1} B - B A^{-1} A_i.$$  \hspace{1cm}  \text{(3.25)}

The reason to introduce $B_i$ is to combine some terms with $B_i$ to simplify the computation. Even $B_i$ can be considered as a tensor, we just need to consider it as a symmetric matrix valued function defined in a local coordinate chart. $(3.25)$ is equivalent to

$$A^{-1} B_i A^{-1} = \partial_i \left( A^{-1} B A^{-1} \right).$$  \hspace{1cm}  \text{(3.26)}

When $B_i$ is differentiated by $L^7 \partial_7$, using $(2.2)$ $(2.3)$ and the conjugate of $(2.2)$, we find

$$L^7 \partial_7 B_i = \left( -B_i A^{-1} B_i A^{-1} B - B A^{-1} B_i A^{-1} B_i B_i \right).$$  \hspace{1cm}  \text{(3.27)}

Now we start to compute $L^7 (Q^{<p>})_7$. In the expression of $Q^{<p>}$, we combine some terms together to simplify the computation:

$$Q^{<p>} = \text{tr} \left( B A^{-1} B A^{-1} \right)^p$$  \hspace{1cm}  \text{(3.28)}

$$= \text{tr} \left[ (A^{-1} B A^{-1}) \cdot B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right].$$  \hspace{1cm}  \text{(3.29)}

In $(3.29)$, $(A^{-1} B A^{-1})$ and $B$ both appear $p$ times. When $\partial_i$ act on any of $(A^{-1} B A^{-1})$ (or $B$), we get the same result. So

$$\partial_i Q^{<p>} = p \cdot \text{tr} \left[ \partial_i (A^{-1} B A^{-1}) \cdot B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right]$$  \hspace{1cm}  \text{(3.30)}

$$+ p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) \cdot \partial_i B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right].$$  \hspace{1cm}  \text{(3.31)}

We plug $(3.26)$ into $(3.30)$ and get

$$\partial_i Q^{<p>} = p \cdot \text{tr} \left[ A^{-1} B_i A^{-1} B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right]$$  \hspace{1cm}  \text{(3.32)}

$$+ p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) \cdot \partial_i B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right].$$  \hspace{1cm}  \text{(3.33)}

We reorganize terms in the product of $(3.32)$ $(3.33)$:

$$\partial_i Q^{<p>} = p \cdot \text{tr} \left[ B_i \cdot (A^{-1} B A^{-1}) \cdot B \cdot (A^{-1} B A^{-1}) \cdots B \cdot (A^{-1} B A^{-1}) \cdot B \right]$$  \hspace{1cm}  \text{(3.34)}

$$+ p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) \cdot \partial_i B \cdot (A^{-1} B A^{-1}) \cdot B \cdots (A^{-1} B A^{-1}) \cdot B \right].$$  \hspace{1cm}  \text{(3.35)}

When apply $L^7 \partial_7$ to $Q^{<p>}$, $L^7 \partial_7$ acts on 6 kinds of terms:
(i) \( L^j \partial_j \) acts on \( B_i \) in (3.34).

(ii) \( L^j \partial_j \) acts on \( (A^{-1} B A^{-1}) \) in (3.34).

(iii) \( L^j \partial_j \) acts on \( B \) in (3.34).

(iv) \( L^j \partial_j \) acts on \( (A^{-1} \partial_i B A^{-1}) \) in (3.34).

(v) \( L^j \partial_j \) acts on \( B \) in (3.34).

(vi) \( L^j \partial_j \) acts on \( (A^{-1} B A^{-1}) \) in (3.34).

In the following we do the computation separately.

(i) When \( L^j \partial_j \) acts on \( B_i \) in (3.34), the result is

\[
p \cdot \text{tr} \left[ L^j \partial_j B_i (A^{-1} B A^{-1}) \cdot \left( B \cdot (A^{-1} B A^{-1}) \right)^{p-1} \right] L^j = p \cdot \text{tr} \left[ -B_j A^{-1} B_i A^{-1} \cdot \left( B \cdot (A^{-1} B A^{-1}) \right)^{p} \right] L^j + p \cdot \text{tr} \left[ -A^{-1} B_i A^{-1} B_j \cdot (A^{-1} B A^{-1}) \cdot B \right] L^j.
\]

To get this, we need to use equation (3.27).

(ii) When \( L^j \partial_j \) acts on \( (A^{-1} B A^{-1}) \) in (3.34), we need to use the conjugate of (3.26):

\[
A^{-1} B_j A^{-1} = \partial_j (A^{-1} B A^{-1}).
\]

The result is the sum of \( p \) terms:

\[
p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1} B)^0 A^{-1} B_j A^{-1} (B A^{-1} B A^{-1})^{p-1} \right] L^j + p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1} B)^1 A^{-1} B_j A^{-1} (B A^{-1} B A^{-1})^{p-2} \right] L^j + \ldots + p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1} B)^p A^{-1} B_j A^{-1} (B A^{-1} B A^{-1})^0 \right] L^j.
\]

In above, fictitious terms \((B A^{-1} B A^{-1})^0\) and \((A^{-1} B A^{-1} B)^0\) are added in to make the pattern clearer. We will do this in other parts of the computation either.

(iii) When \( L^j \partial_j \) acts on \( B \) in (3.34), we don’t need to use other equations. The result is the sum of \( p - 1 \) terms:

\[
p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^0 B_j (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^{p-2} \right] L^j + p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^1 B_j (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^{p-3} \right] L^j + \ldots + p \cdot \text{tr} \left[ B_i (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^{p-2} B_j (A^{-1} B A^{-1})(B A^{-1} B A^{-1})^0 \right] L^j.
\]
(iv) When $L^5 \partial_j$ acts on $(A^{-1} \partial_j B A^{-1})$ in (3.35), the result is zero. This is because of the equation (3.24).

(v) When $L^5 \partial_j$ acts on $B$ in (3.35), we don’t need to use other equations. Simply differentiating $B$, we get the following result which is the sum of $p$ terms:

$$p \cdot \text{tr} \left[ A^{-1} \partial_j B A^{-1} (B A^{-1} B) B^{-1} (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-2} \right] L^5$$

(3.46)

$$+ p \cdot \text{tr} \left[ A^{-1} \partial_j B A^{-1} (B A^{-1} B) B^{-1} (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-3} \right] L^5$$

(3.47)

...  

$$+ p \cdot \text{tr} \left[ A^{-1} \partial_j B A^{-1} (B A^{-1} B) B^{-1} (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-1} \right] L^5.$$  

(3.48)

(vi) When $L^5 \partial_j$ acts on $(A^{-1} B A^{-1})$ in (3.35), we need to use (3.39). The result is the sum of $p-1$ terms

$$p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) B (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-2} \right] L^5$$

(3.49)

$$+ p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) B (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-3} \right] L^5$$

(3.50)

...  

$$+ p \cdot \text{tr} \left[ (A^{-1} B A^{-1}) B (A^{-1} B A^{-1}) (B^{-1} A^{-1} B)^{p-1} \right] L^5.$$  

(3.51)

At a point $(\tau_0, z_0) \in R \times V$, we change coordinate on $V$ to diagonalize $A, B$. As discussed in section 1.4, we need to find $P$, so that

$$P A P^* = I, \quad P B P^T = \Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_n) \geq 0.$$  

(3.52)

This can be done by Lemma (3.4). Denote that

$$B_i = (B_{i,\alpha \beta}), \quad \partial_j B = (B_{j,\alpha \beta}).$$  

(3.53)

With these notation, results of (i)-(vi) can be simplified: Result of (i) is:

$$p \left(-\overline{B_{i,\alpha \beta}} \overline{B_{j,\alpha \beta}} (\Lambda^2_\alpha + \Lambda^2_\beta) \right) L^5;$$

(3.54)

result of (ii) is

$$p \left(B_{i,\alpha \beta} \overline{B_{j,\alpha \beta}} (\Lambda^2_\alpha + \Lambda^2_\beta) \right) L^5;$$

(3.55)

result of (iii) is

$$p \left(B_{i,\alpha \beta} \overline{B_{j,\alpha \beta}} (\Lambda_\alpha^3 + \Lambda_\beta^3) \right) L^5;$$

(3.56)

result of (iv) is still 0; result of (v) is

$$p \left(B_{i,\alpha \beta} \overline{B_{j,\alpha \beta}} (\Lambda^2_\alpha + \Lambda^2_\beta) \right) L^5;$$

(3.57)
It's obvious that when the result of (vi) is
\[ p \left( \frac{B_{i\alpha\beta}B_{j\alpha\beta}}{B_{j\alpha\beta}}(\Lambda_\alpha^{2p-3}\Lambda_\beta + \Lambda_\alpha^{2p-5}\Lambda_\beta^3 + \ldots + \Lambda_\alpha\Lambda_\beta^{2p-3}) \right) L^5. \] (3.58)

Summing up results of (i)-(vi), we get
\[ L^5(Q^{<p>})_{ij} = p \sum_{i,j} L^5(B_{i\alpha\beta}, B_{j\alpha\beta}) W_{\alpha\beta} \left( \frac{B_{j\alpha\beta}}{B_{i\alpha\beta}} \right), \] (3.59)

where
\[ W_{\alpha\beta} = \left( \begin{array}{c} \sum_{k=0}^{p-1} \lambda_\alpha^{2k} \lambda_\alpha^{2p-2-2k} \\ \sum_{k=0}^{p-2} \lambda_\alpha^{2p-3-2k} \lambda_\alpha^{2k+1} \\ \sum_{k=0}^{p-1} \lambda_\alpha^{2k} \lambda_\alpha^{2p-2-2k} - \Lambda_\alpha^{2p} - \Lambda_\alpha^{2p} \end{array} \right). \] (3.60)

It's obvious that when \( W_{\alpha\beta} \geq 0, \) \[3.59\] \( \geq 0. \) In the following, we show when \( \Lambda_\alpha, \Lambda_\beta \leq \sqrt{1 - \frac{1}{2p}}, \)
\( W_{\alpha\beta} \geq 0. \)

According to linear algebra, \( W_{\alpha\beta} \geq 0 \) if and only if \( \text{tr}(W_{\alpha\beta}) \geq 0 \) and \( \text{det}(W_{\alpha\beta}) \geq 0. \) \( \text{tr}(W_{\alpha\beta}) \)
is easy to compute:
\[ \text{tr}(W_{\alpha\beta}) \geq \Lambda_\alpha^{2p-2} + \Lambda_\beta^{2p-2} - \Lambda_\alpha^{2p} - \Lambda_\beta^{2p}. \] (3.61)

It's greater than 0 providing
\[ \Lambda_\alpha, \Lambda_\beta < 1. \] (3.62)

To investigate the sign of \( \text{det}(W_{\alpha\beta}), \) we need to simplify the right-hand side of \[3.60\]. When \( \Lambda_\alpha = \Lambda_\beta, \) denoting \( \Lambda_\alpha = \Lambda_\beta = \lambda, \)
\[ W_{\alpha\beta} = \left( \begin{array}{c} p \lambda^{2p-2} \\ (p-1) \lambda^{2p-2} \\ p \lambda^{2p-2} - 2 \lambda^{2p} \end{array} \right). \] (3.63)

The determinant follows by direct computation:
\[ \text{det}(W_{\alpha\beta}) = (2p - 1)\lambda^{4p-4} \left[ 1 - \frac{2p}{2p - 1} \lambda^2 \right]. \] (3.64)

It's non-negative proving
\[ \lambda^2 \leq 1 - \frac{1}{2p}. \] (3.65)

When \( \Lambda_\alpha \neq \Lambda_\beta, \) we use the summation formula for geometric series to compute the summation in \[3.60\]. We assume \( \Lambda_\alpha^2 > \Lambda_\beta^2. \) Note that we have \( \Lambda_\alpha, \Lambda_\beta \geq 0, \) so when \( \Lambda_\alpha \neq \Lambda_\beta, \) \( \Lambda_\alpha^2 \neq \Lambda_\beta^2. \) The result is
\[ W_{\alpha\beta} = \left( \begin{array}{c} \frac{\Lambda_\alpha^{2p} - \Lambda_\beta^{2p}}{\Lambda_\alpha^2 - \Lambda_\beta^2} \\ -\frac{\Lambda_\alpha^{2p-1}\Lambda_\alpha - \Lambda_\beta^{2p-1}\Lambda_\beta}{\Lambda_\alpha^2 - \Lambda_\beta^2} \\ -\frac{\Lambda_\alpha^{2p-1}\Lambda_\alpha - \Lambda_\beta^{2p-1}\Lambda_\beta}{\Lambda_\alpha^2 - \Lambda_\beta^2} \end{array} \right). \] (3.66)
The determinant follows by straightforward computation:

$$\det(W_{\alpha\beta}) = \frac{(-\Lambda_\beta^{4p} + \Lambda_\beta^{4p-2}) - (-\Lambda_\alpha^{4p} + \Lambda_\alpha^{4p-2})}{A_\beta^2 - A_\alpha^2}. \quad (3.67)$$

Let \( f(x) = -x^{2p} + x^{2p-1} \). Then (3.67) can be simplified as

$$\det(W_{\alpha\beta}) = f(\Lambda_\beta^2) - f(\Lambda_\alpha^2). \quad (3.68)$$

It’s non-negative, providing \( \Lambda_\alpha^2, \Lambda_\beta^2 \) stay on an interval where \( f \) is non-decreasing. By computing the derivative of \( f \) we know \( f \) is non-decreasing on \([0, 1 - \frac{1}{2p}]\). Combining with (3.62) (3.65), we know

$$L^\gamma (Q^{<p>})_{ij} \geq 0, \quad \text{providing } K \leq 1 - \frac{1}{2p}. \quad (3.69)$$

In the computation above, we can replace \( B \) by \( B - S \) and get

$$L^\gamma (Q^{<p>}_S)_{ij} \geq 0, \quad \text{providing } K_S \leq 1 - \frac{1}{2p}. \quad (3.70)$$

As a result, we have the following proposition:

**Proposition 3.1.** Suppose \( \Phi \) is a \( C^4 \) solution to Problem 1.4 and \( S \) is a constant section of \( T^*_{2,0}(V) \). For \( L, K_S, Q^{<p>}_S \) and \( Q^{[p]}_S \), defined by (2.4) (3.4) (3.8) (3.9) respectively, we have

$$L^\gamma \partial_i (Q^{<p>}_S)_{ij} \geq 0, \quad \text{providing } K \leq 1 - \frac{1}{2p}. \quad (3.71)$$

providing \( K_S \leq 1 - \frac{1}{2p}, \) and, as a consequence,

$$Q^{[p]}_S \leq \max_{\partial \mathcal{R} \times V} Q^{[p]}_S, \quad \text{in } \mathcal{R} \times V, \quad (3.72)$$

providing \( K_S \leq 1 - \frac{1}{2p}, \) in \( \mathcal{R} \times V. \)

**Remark 3.1.** Let

$$Q^{[\rho,p]}_S = \left( tr(K_S^{\rho p}) \right)^{\frac{1}{p}}. \quad (3.73)$$

With more complicated computations we can show

$$L^\gamma \left( Q^{[\rho,p]}_S \right)_{ij} \geq 0, \quad (3.74)$$

providing \( K_S \leq 1 - \frac{1}{2p} \). So, by letting \( p \) go to \( \infty \), we know

$$L^\gamma \left( M^{\rho}_S \right)_{ij} \geq 0, \quad (3.75)$$

providing \( K_S \leq 1 - \frac{1}{2p}, \) in the sense of viscosity solution (see section 6 of [CIL92]). However, the current result (3.69) is enough for our use. We can also consider

$$\text{tr}(e^{\rho K_S}) \quad (3.76)$$

and achieve a similar result. In [H22], (3.76) is used in \( n = 1 \) case.
3.2 Preservation of \((S, \omega_0)\)-Convexity by the Method of Continuity

In addition to (3.1), in this section, we make the following assumption.

**Assumption 3.1.** For any \(\sigma \in [0, 1]\), Problem \(1.4\) with boundary value \(\sigma F\) has a solution \(\Phi^\sigma\) and \(\{\Phi^\sigma | \sigma \in [0, 1]\}\) is a continuous curve in \(C^4(\mathcal{R} \times V)\) in \(C^2\) topology.

**Remark 3.2.** According to the ellipticity of equation \(1.40\), which is proved in section 4.1, the solution to Problem \(1.4\) is unique. As a consequence, \(\Phi^0\) must equal to 0.

Let

\[
A_\sigma = (\Phi^\sigma_{\alpha\beta} + b_{\alpha\beta}),
\]

\[
B_{S,\sigma} = (\Phi^\sigma_{\alpha\beta} + \sigma S_{\alpha\beta}),
\]

\[
K_{S,\sigma} = B_{S,\sigma} A_{\sigma}^{-1} B_{S,\sigma} A_{\sigma}^{-1},
\]

\[
Q_{S,\sigma}^{<p>} = \text{tr}(K_{S,\sigma}^p),
\]

\[
Q_{S,\sigma}^{[p]} = \left(Q_{S,\sigma}^{<p>}\right)^{\frac{1}{p}},
\]

and for any \((\tau, z) \in \mathcal{R} \times V\),

\[
M_{S,\sigma}(\tau, z) = \text{Maximum Eigenvalue of } K_{S,\sigma}(\tau, z).
\]

According to the assumption (3.1),

\[
\max_{\partial \mathcal{R} \times V} M_{S,1} < 1.
\]

So we can choose \(p\) large enough, such that

\[
\max_{\partial \mathcal{R} \times V} Q_{S,1}^{[p]} < 1 - \frac{1}{2p}.
\]

This can be done because when \(p \to +\infty\),

\[
1 - \frac{1}{2p} \to 1
\]

and

\[
\max_{\partial \mathcal{R} \times V} Q_{S,1}^{[p]} \to \max_{\partial \mathcal{R} \times V} M_{S,1} < 1.
\]

According to Lemma \(A.7\), for any \((\tau, z) \in \partial \mathcal{R} \times V\), \(Q_{S,\sigma}^{[p]}(\tau, z)\) is a monotone non-decreasing function of \(\sigma\), so \(\max_{\partial \mathcal{R} \times V} Q_{S,\sigma}^{[p]}\) is a monotone non-decreasing function of \(\sigma\). Thus

\[
\max_{\partial \mathcal{R} \times V} Q_{S,\sigma}^{[p]} < 1 - \frac{1}{2p},
\]

for any \(\sigma \in [0, 1]\). Therefore, for no \(\sigma \in [0, 1]\),

\[
\max_{\mathcal{R} \times V} Q_{S,\sigma}^{[p]} \in \left(\max_{\partial \mathcal{R} \times V} Q_{S,\sigma}^{[p]}, 1 - \frac{1}{2p}\right).
\]
This is because if (3.88) is valid, for \( \sigma = \sigma_0 \), then
\[
\max_{\mathcal{R} \times \mathcal{V}} Q_{S,\sigma_0}^{[p]} \leq 1 - \frac{1}{2p},
\]  
(3.89)
and, as a consequence,
\[
\max_{\mathcal{R} \times \mathcal{V}} M_{S,\sigma_0} \leq 1 - \frac{1}{2p}.
\]  
(3.90)
So, using Proposition 3.1 we know
\[
\max_{\mathcal{R} \times \mathcal{V}} Q_{S,\sigma_0}^{[p]} \leq \max_{\partial \mathcal{R} \times \mathcal{V}} Q_{S,\sigma_0}^{[p]},
\]  
(3.91)
which contradicts with (3.88).

The function \( \max_{\mathcal{R} \times \mathcal{V}} Q_{S,\sigma}^{[p]} \) is a continuous function of \( \sigma \), because of Assumption 3.1. And \( \max_{\mathcal{R} \times \mathcal{V}} Q_{S,0}^{[p]} = 0 \), according to the uniqueness of \( C^2 \) solution. So
\[
\max_{\mathcal{R} \times \mathcal{V}} Q_{S,\sigma}^{[p]} \leq \max_{\partial \mathcal{R} \times \mathcal{V}} Q_{S,\sigma}, \quad \text{for any } \sigma \in [0,1].
\]  
(3.92)

As illustrated by Figure 1, the dashed curve
\[
\left\{ (\sigma, \max_{\partial \mathcal{R} \times \mathcal{V}} Q_{S,\sigma}^{[p]}) \mid \sigma \in [0,1] \right\}
\]  
(3.93)
raises up from left to right and stays below the line \( \{ m = 1 - \frac{1}{2p} \} \). The continuous curve
\[
\left\{ (\sigma, \max_{\mathcal{R} \times \mathcal{V}} Q_{S,\sigma}^{[p]}) \mid \sigma \in [0,1] \right\},
\]  
(3.94)
whose left endpoint is \((0, 0)\), cannot intersect with the shadowed area, so it has to stay below the shadowed area. Actually, it has to coincide with the dashed curve. Therefore, we have, for any \(\sigma \in [0, 1]\),

\[
M_{S, \sigma} \leq \max_{\mathcal{R} \times V} Q_{S, \sigma}^{[p]} = \max_{\partial \mathcal{R} \times V} Q_{S, \sigma}^{[p]} \leq \max_{\partial \mathcal{R} \times V} Q_{S, 1}^{[p]}, \quad \text{in } \mathcal{R} \times V. \tag{3.95}
\]

Let \(p \to 0\), we get

\[
M_{S, \sigma} \leq \max_{\partial \mathcal{R} \times V} M_{S, 1}, \quad \text{in } \mathcal{R} \times V. \tag{3.96}
\]

In sum, we have the following Proposition.

**Proposition 3.2.** Suppose \(S\) is a constant section of \(T_{2,0}^*(V)\) and \(F \in C^2(\partial \mathcal{R} \times V)\) satisfies that

\[
F(\tau, \ast) \text{ is strictly } (S, \omega_0)\text{-convex, for any } \tau \in \partial \mathcal{R}. \tag{3.97}
\]

In addition, we assume that the Assumption 3.1 is satisfied. Then the solution \(\Phi\) to Problem 1.4 with boundary value \(F\), satisfies

\[
\Phi(\tau, \ast) \text{ is strictly } (S, \omega_0)\text{-convex, for any } \tau \in \mathcal{R}. \tag{3.98}
\]

### 3.3 Convexity and Metric Lower Bound Estimates by Altering \(S\)

In this section, we assume that the conditions of Proposition 3.2 are satisfied. In addition, we assume that, for a constant \(\delta > 0\),

\[
F(\tau, \ast) \text{ is } (S, \omega_0)\text{-convex of modulus } \delta, \text{ for any } \tau \in \partial \mathcal{R}, \tag{3.99}
\]

or equivalently, according to Lemma A.2

\[
F(\tau, \ast) \text{ is } (S, \omega_0)\text{-convex of degree } \delta, \text{ for any } \tau \in \partial \mathcal{R}. \tag{3.100}
\]

Condition \(3.100\) says, for any constant section \(\Theta\) of \(T_{2,0}^*(V)\), with

\[
\Theta \Omega \Omega^{-1} \Theta \Omega^{-1} \leq \delta^2, \tag{3.101}
\]

we have

\[
F(\tau, \ast) \text{ is strictly } (S + \Theta, \omega_0)\text{-convex, for any } \tau \in \partial \mathcal{R}. \tag{3.102}
\]

Here \(W = (b_{\alpha \beta})\). So, by Proposition 3.2 for any constant section \(\Theta\) of \(T_{2,0}^*(V)\), satisfying \(3.101\), we have

\[
\Phi(\tau, \ast) \text{ is strictly } (S + \Theta, \omega_0)\text{-convex, for any } \tau \in \mathcal{R}. \tag{3.103}
\]

Therefore,

\[
\Phi(\tau, \ast) \text{ is } (S, \omega_0)\text{-convex of degree } \delta, \text{ for any } \tau \in \mathcal{R}. \tag{3.104}
\]
By Lemma A.2, we know (3.104) is equivalent to
\[ \Phi(\tau, \nu) \text{(S,}\omega_0)\text{-convex of modulus } \delta, \text{ for any } \tau \in \mathcal{R}. \] (3.105)

Then, by the definition of modulus of convexity, Definition 1.3, we know
\[ \omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \nu) > \delta \omega_0, \quad \text{for any } \tau \in \mathcal{R}. \] (3.106)

As a result, we have the following convexity and metric lower bound estimate:

**Proposition 3.3** (Apriori Convexity and Metric Lower Bound Estimate). Suppose that, for a constant \( \delta > 0 \) and a constant section \( S \) of \( T^*_0(V) \), \( F \in C^\infty(\partial \mathcal{R} \times V) \) satisfies
\[ F(\tau, \nu) \text{(S,}\omega_0)\text{-convex of degree } \delta, \text{ for any } \tau \in \partial \mathcal{R}. \] (3.107)

In addition, Assumption 3.1 is satisfied. Then a solution \( \Phi \) to Problem 1.4 with boundary value \( F \) satisfies
\[ \Phi(\tau, \nu) \text{(S,}\omega_0)\text{-convex of degree } \delta, \text{ for any } \tau \in \mathcal{R} \] (3.108)

and, as a consequence,
\[ \omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi(\tau, \nu) > \delta \omega_0, \quad \text{for any } \tau \in \mathcal{R}. \] (3.109)

## 4 Existence of Solutions to the Perturbed Equation

In this section, we prove the existence of smooth solutions to Problem 1.4.

In section 4.1, we discuss some basic properties of equation (1.40), including ellipticity and concavity. Then in section 4.2, we derive a directional partial \( C^2 \) estimate, in the direction of the affine manifold. The estimate allows us to derive \( C^0 \) and \( C^1 \) estimates, in section 4.3. Then with the metric lower bound estimate, Proposition 3.3, we prove \( C^2 \) and \( C^{2,\alpha} \) estimates in section 4.4 and 4.5. Finally, in section 4.6, we prove the existence of smooth solutions.

### 4.1 Basic Properties of the Elliptic Perturbation Equation

First, equation (1.40) is elliptic. This has been indicated by (2.5). More precisely, let \( \Phi^\lambda \) be a family of solutions of equation (1.40) with \( \Phi^0 = \Phi \) and \( \frac{d}{d\lambda} \Phi^\lambda = \Psi \), at \( \lambda = 0 \). Then differentiating
\[ \Phi^\lambda_{\tau\tau} - \Phi^\lambda_{\tau\beta} g^\alpha_\beta \Phi^\lambda_{\alpha\tau} = \epsilon b_{\alpha\beta} g^\alpha_\beta \] (4.1)

with respect to \( \lambda \), at \( \lambda = 0 \), gives
\[ L^i_{ij} \Psi_{ij} = 0, \] (4.2)

where \( L^i_{ij} \) was introduced in section 2 by (2.4). In (4.1), \( g^\alpha_\beta \) is the inverse of \( b_{\alpha\beta} + \Phi^\lambda_{\alpha\beta} \).

Then we consider the concavity. We will show
\[ F(\Phi^\lambda_{ij}) = \log(\Phi^\lambda_{\tau\tau} - \Phi^\lambda_{\tau\beta} g^\alpha_\beta \Phi^\lambda_{\alpha\tau}) - \log(b_{\alpha\beta} g^\alpha_\beta) \] (4.3)
is a concave function of $\Phi_{ij}$, providing
\[
\begin{pmatrix}
\Phi_{\tau \tau} & \Phi_{\tau \pi} \\
\Phi_{\pi \tau} & \Phi_{\alpha \beta} + b_{\alpha \beta}
\end{pmatrix} > 0.
\] (4.4)

Actually, if we denote
\[
F_1(\Phi_{ij}) = \log(\Phi_{\tau \tau} - \Phi_{\tau \pi}g^{\alpha \beta}\Phi_{\alpha \beta}),
\]
(4.5)
\[
F_2(\Phi_{ij}) = -\log(b_{\alpha \beta}g^{\alpha \beta}),
\]
(4.6)
then we can show $F_1$ and $F_2$ are both concave. These computations are in Appendix A.3.

Suppose $\Phi$ is a $C^4$ solution to Problem 1.4 and $X$ is a constant vector field in $\mathcal{R} \times \mathcal{V}$. We apply $\partial X$ to equation (1.40) and get
\[
\Phi_{X \tau \tau} - \Phi_{X \pi \tau}g^{\alpha \beta}\Phi_{\alpha \pi} + \Phi_{\tau \pi \tau}g^{\alpha \beta}\Phi_{\alpha \beta} - \Phi_{\tau \pi \tau}g^{\alpha \beta}\Phi_{X \alpha \tau} = -\epsilon b_{\alpha \beta}g^{\alpha \beta}\Phi_{X \alpha \tau} = 0.
\] (4.7)
With the linearized operator $L^X\partial_{ij}$, (4.7) is simplified to
\[
L^X\partial_{ij}(\Phi_{X}) = 0.
\] (4.8)

Then apply $\partial X$ to equation (4.7), we get
\[
L^X\partial_{ij}(\Phi_{XX}) \geq 0.
\] (4.9)
This is because of the concavity of (4.3). We can also get (4.9) directly by replacing $\partial_\theta$ and $\partial_\gamma$ in (2.6)-(2.9) by $\partial X$.

### 4.2 Affine-Manifold-Directional $C^2$ Estimates

Suppose $X$ is a constant real vector field in $\mathcal{R} \times \mathcal{V}$, parallel to $\mathcal{V}$. This is to say, if we denote the projection from $\mathcal{R} \times \mathcal{V}$ to $\mathcal{R}$ by $\pi_R$, then $(\pi_R)_*(X) = 0$. By equation (4.9), we know
\[
\Phi_{XX} \leq \max_{\partial R \times \mathcal{V}} \Phi_{XX}, \quad \text{in } \mathcal{R} \times \mathcal{V}.
\] (4.10)
Because $\omega_0$ has a lower bound and $\sqrt{-1} \partial \overline{\partial} F(\tau, \ast)$ has a uniform upper bound, we can find a constant $C > 0$, so that
\[
\max_{\partial \mathcal{R} \times \mathcal{V}} \Phi_{XX} = \max_{\partial \mathcal{R} \times \mathcal{V}} F_{XX} \leq C\omega_0(X, JX).
\] (4.11)
Therefore,
\[
\sqrt{-1} \partial \overline{\partial} \Phi(X, JX) = \frac{\Phi_{XX} + \Phi_{JXJX}}{2} \leq C\omega_0(X, JX).
\] (4.12)
This implies
\[
\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, \ast) \leq (1 + C)\omega_0, \quad \text{for any } \tau \in \partial \mathcal{R}
\] (4.13)
and equivalently
\[
(g_{\alpha \beta}) \leq (1 + C)b_{\alpha \beta}.
\] (4.14)
As a consequence, we have
\[
b_{\alpha \beta}g^{\alpha \beta} \cdot \det(g_{\alpha \beta}) \leq n(1 + C)^{n-1} \cdot \det(b_{\alpha \beta}).
\] (4.15)
4.3 $C^0$ and $C^1$ Estimates

To do the $C^0$ and boundary $C^1$ estimates, we construct $\Psi$ and $\Phi^0$, so that

\[ \Psi \leq \Phi \leq \Phi^0, \quad \text{in } \mathcal{R} \times V, \quad (4.16) \]

and

\[ \Psi = \Phi = \Phi^0, \quad \text{on } \partial \mathcal{R} \times V. \quad (4.17) \]

We let $\Phi^0$ be the solution to Problem 1.2 with the boundary condition

\[ \Phi^0 = \Phi, \quad \text{on } \partial \mathcal{R} \times V. \quad (4.18) \]

We easily know that

\[ \Phi \leq \Phi^0, \quad \text{in } \mathcal{R} \times V, \quad (4.19) \]

because $\Phi^0$ is a maximal $\Omega_0$-PSH function.

For the construction of $\Psi$, we need to use the estimate (4.15). Locally, we have

\[ \det(h_{\alpha \beta}) = \epsilon b_{\alpha \beta} g^{\alpha \beta} \cdot \det(g_{\alpha \beta}) \leq \epsilon n (1 + C)^{n-1} \cdot \det(b_{\alpha \beta}). \quad (4.20) \]

So, for a solution $\Psi$ to Problem 1.3 with $\epsilon = \epsilon n (1 + C)^{n-1}$, we have

\[ (\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi)^{n+1} \leq (\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Psi)^{n+1}. \quad (4.21) \]

Thus $\Psi \leq \Phi$, given the boundary condition $\Psi = \Phi$ on $\partial \mathcal{R} \times V$. The global $C^0$ and $C^1$ estimates for $\Psi$ are known by [CTW17] and [B12]. Therefore, we have the global $C^0$ estimate and boundary $C^1$ estimate for $\Phi$.

The $C^1$ interior estimate can be derived from boundary estimates with equation (4.8).

4.4 $C^2$ Estimates

We first prove the boundary $C^2$ estimate, then we use equation (4.9) to derive the interior estimate.

To do the boundary estimate, we need to flatten the boundary. Around $\tau_0 \in \partial \mathcal{R}$, find a holomorphic map

\[ f : B_{\delta'}(\tau_0) \cap \mathcal{R} \to \mathbb{C} = \{ \zeta = \xi + \sqrt{-1} \eta \}, \quad (4.22) \]

for a small $\delta'$. We want that $f' \neq 0$, $f(\tau_0) = 0$,

\[ f(\partial \mathcal{R}) \subset \{ \zeta | \text{Im}(\zeta) = 0 \} \quad (4.23) \]

and

\[ f(B_{\delta'}(\tau_0) \cap \mathcal{R}) \supset B_{\delta'}(0) = \{ |\text{Re}(\zeta)| < \delta, 0 \leq \text{Im}(\zeta) < \delta \}, \quad (4.24) \]

for a small $\delta$. For a point $p_0 \in V$, let $\{ z^\alpha \}$ be a set of coordinates in $B_r(p_0) \subset V$, for a small $r$. Without loss of generality, we can assume that $p_0 = 0$ in this coordinate chart. We also require that the coordinate $z^\alpha$ is properly chosen so that the natural metric on $B_r(0)$, as a subset of $\mathbb{C}^n$, is
the metric $\omega_0$. In the following, we will work in the coordinate chart $B^+_\varepsilon(0) \times B_r(0)$ and estimate second order derivatives at $(0,0)$. For convenience, we denote $B^+_\varepsilon(0) \times B_r(0)$ by $\mathcal{D}$ and denote $\{\text{Im}(\zeta) = 0\} \times B_r(0)$ by $\Gamma$.

In the coordinate chart $\mathcal{D}$, $\Phi(\zeta, \vec{z})$ satisfies

$$\Phi_{\zeta\zeta} - \Phi_{\zeta\beta} g^{\alpha\beta} \Phi_{\alpha\zeta} = \epsilon \cdot k(\zeta) \cdot b_{\alpha\beta} g^{\alpha\beta}. \tag{4.25}$$

Here $k = \frac{1}{|f'|}$, so it is a positive and smooth function $\zeta$.

In the following, we will work in the coordinate chart $\mathcal{D}$, $\Phi(\zeta, \vec{z})$ satisfies

$$\Phi_{\zeta\zeta} - \Phi_{\zeta\beta} g^{\alpha\beta} \Phi_{\alpha\zeta} = \epsilon \cdot k(\zeta) \cdot b_{\alpha\beta} g^{\alpha\beta}. \tag{4.25}$$

Here $k = \frac{1}{|f'|}$, so it is a positive and smooth function $\zeta$.

In the following, we will work in the coordinate chart $\mathcal{D}$, $\Phi(\zeta, \vec{z})$ satisfies

$$\Phi_{\zeta\zeta} - \Phi_{\zeta\beta} g^{\alpha\beta} \Phi_{\alpha\zeta} = \epsilon \cdot k(\zeta) \cdot b_{\alpha\beta} g^{\alpha\beta}. \tag{4.25}$$

Here $k = \frac{1}{|f'|}$, so it is a positive and smooth function $\zeta$.

In the following, we will work in the coordinate chart $\mathcal{D}$, $\Phi(\zeta, \vec{z})$ satisfies

$$\Phi_{\zeta\zeta} - \Phi_{\zeta\beta} g^{\alpha\beta} \Phi_{\alpha\zeta} = \epsilon \cdot k(\zeta) \cdot b_{\alpha\beta} g^{\alpha\beta}. \tag{4.25}$$

Here $k = \frac{1}{|f'|}$, so it is a positive and smooth function $\zeta$.

In the following, we will show $\Phi_{\zeta\eta}(0,0) \leq C_2$, for a constant $C_2$.

First, we need to derive an equation satisfied by $\Phi_{\zeta\eta}$. Applying $\partial_\zeta$ to equation (4.25) gives

$$\Phi_{\zeta\eta\eta} - \Phi_{\zeta\eta\beta} g^{\alpha\beta} \Phi_{\alpha\zeta\eta} + \Phi_{\zeta\eta\alpha\beta} \Phi_{\alpha\zeta\eta} = \epsilon k b_{\alpha\beta} g^{\alpha\beta} \Phi_{\zeta\eta\alpha\beta}. \tag{4.28}$$

We introduce the following operator $\mathcal{L}$, which is a scalar function multiple of $I$, after coordinate transformation. Equivalently, $\mathcal{L}$ can also be considered as the linearization operator of (4.25):

$$\mathcal{L} = L^i_\zeta \partial_\zeta^i, \tag{4.30}$$

with

$$L^i_\zeta = \begin{pmatrix} L^0_\zeta & \cdots & L^n_\zeta \\ \cdots & \cdots & \cdots \\ L^n_\zeta & \cdots & L^0_\zeta \end{pmatrix} = \begin{pmatrix} 1 & -\Phi_{\zeta\eta} g^{\rho\sigma} \\ -\Phi_{\zeta\eta} g^{\rho\sigma} & \epsilon k b_{\alpha\beta} g^{\alpha\beta} \Phi_{\zeta\eta\alpha\beta} \end{pmatrix}. \tag{4.31}$$

Here $i, j$ run from 0 to $n$, and the 0-th coordinate is $\zeta$. With this operator, equation (4.29) becomes

$$\mathcal{L}^i_\zeta \partial_\zeta^i(\Phi_{\zeta\eta}) = \epsilon (\partial_\zeta^i k) b_{\alpha\beta} g^{\alpha\beta}. \tag{4.32}$$

Since we have the metric lower bound estimate, Proposition 3.3, we know the right-hand side of (4.32) is bounded. We can assume, for a constant $C$,\n
$$-C \leq \mathcal{L}^i_\zeta \partial_\zeta^i(\Phi_{\zeta\eta}). \tag{4.33}$$

Then we construct a barrier function $u$, so that $u \geq \Phi_{\zeta\eta}$ and $u(0,0) = \Phi_{\zeta\eta}(0,0)$. Thus we can get an upper bound for $\Phi_{\zeta\eta}$. The barrier function is

$$u = l + C_3 \left(|\zeta|^2 + \xi^2\right) + C_4 \eta - C_5 \eta^2 + C_6 (\Phi - \Psi). \tag{4.34}$$
In above, \( l \) is the \( \Gamma \)-directional linearization of \( \Phi_X \) at \((0, 0)\). That’s to say,
\[
\begin{align*}
l(0, 0) &= \Phi_X(0, 0), \\
\partial_{\eta} l &= 0
\end{align*}
\]
and
\[
\nabla_{\Gamma} l(0, 0) = \nabla_{\Gamma} \Phi_X(0, 0).
\]
\( \Psi \) is a solution to Problem 1.3 with \( \varepsilon = \varepsilon n(1 + C)^{n-1} + 1 \) and
\[
\Psi = \Phi, \quad \text{on } \partial \mathcal{R} \times V.
\]
The \( \Psi \) constructed here is even smaller than the \( \Psi \) constructed in section (4.3), so \( \Psi \leq \Phi \). And according to the \( C_2 \) estimate for \( \Psi \) \( [B12] \), we have
\[
\left( \Psi_{\xi} \xi \Psi_{\xi} - \Psi_{\alpha\beta} \Psi_{\alpha\beta} + b_{\alpha\beta} \right) > \frac{1}{C_7} \left( \frac{1}{2} b_{\alpha\beta} \right).
\]
In the following, we show that we can properly choose parameters \( C_3, C_4, C_5, C_6 \), so that
\[
L^{ij} \partial_{ij} u \leq -\tilde{C}, \quad \text{in } \Omega,
\]
and
\[
u \geq \Phi_X, \quad \text{on } \partial \Omega.
\]
Then the comparison principle implies that \( u \geq \Phi_X \) in \( \Omega \).

We compute and estimate \( L^{ij} \partial_{ij} u \) term by term:
(i) \( L^{ij} \partial_{ij}(l + C_4 \eta) = 0 \), because \( l + C_4 \eta \) is a linear function.
(ii) Using the metric lower bound estimate, Proposition 3.3, we have
\[
L^{ij} \partial_{ij}(z^\alpha \eta^2 + \xi^2) = \epsilon k b_{\sigma\tau} g^{\alpha\sigma} g^{\beta\gamma} \eta g^{\alpha\gamma} \Phi_{\xi} + \frac{1}{2} \leq C_8 \epsilon K + g^{\alpha\eta} \Phi_{\xi} g^{\alpha\eta} \Phi_{\xi}.
\]
Here \( C_8 \) depends on the metric lower bound.
(iii) \( L^{ij} \partial_{ij}(-\eta^2) = -\frac{1}{2} \).
(iv) For \( L^{ij} \partial_{ij}(\Phi - \Psi) \), we split it into two terms: \( L^{ij} \partial_{ij}(\Phi) + L^{ij} b_{\alpha\beta} \) and \(- (L^{ij} \partial_{ij}(\Psi) + L^{ij} b_{\alpha\beta}) \).
Using \( b_{\alpha\beta} + \Phi_{\alpha\beta} = g_{\alpha\beta} \), we get
\[
L^{ij} \partial_{ij}(\Phi) + L^{ij} b_{\alpha\beta} = 2\epsilon k b_{\alpha\beta} g^{\alpha\beta}.
\]
For the right-hand side of (4.43), we use the metric lower bound estimate and get
\[
2\epsilon k b_{\alpha\beta} g^{\alpha\beta} \leq 2\epsilon K C_3.
\]
For \( L^{ij} \partial_{ij}(\Psi) + L^{ij} b_{\alpha\beta} \), we use (4.39) and get
\[
L^{ij} \partial_{ij}(\Psi) + L^{ij} b_{\alpha\beta} \geq \frac{1}{C_7} \left( g^{\alpha\eta} \Phi_{\xi} g^{\beta\gamma} \Phi_{\xi} b_{\alpha\beta} \right).
\]
Note that we have already made the assumption, when choosing the coordinate chart, that, in $D$, $b_{\alpha \beta} = \delta_{\alpha \beta}$. So
\[
\mathcal{L}^5 \partial_\gamma (\Psi) + \mathcal{L}^\alpha \eta b_{\alpha \beta} \geq \frac{1}{C_7} \left( g^{\alpha \gamma} \gamma \gamma g^{\rho \sigma} \Phi_{\rho \sigma} \right) .
\] (4.46)

The right-hand side of (4.46) can be used to control the right-hand side of (4.42).

In sum, we have
\[
\mathcal{L}^5 \partial_\gamma u \leq (C_3 \cdot \epsilon K C_8 + C_6 \cdot 2 \epsilon K C_9 - C_5 \frac{1}{2}) + g^{\alpha \gamma} \gamma \gamma g^{\rho \sigma} \Phi_{\rho \sigma}(C_3 - C_6) .
\] (4.47)

We need to choose $C_3, C_5, C_6$ so that
\[
C_6 \geq C_3 \cdot C_7
\] (4.48)
and
\[
C_5 \geq 2(C_3 \cdot 2 \epsilon K C_8 + C_6 \cdot 2 \epsilon K C_9 + \hat{C}) .
\] (4.49)

We also want $u \geq \Phi_X$ on $\partial \Omega$. We need to choose $C_3$ big enough, so that
\[
C_3 \left( \sum_\alpha |z^\alpha|^2 + \xi^2 \right) + l \geq \Phi_X + \left( \sum_\alpha |z^\alpha|^2 + \xi^2 \right), \quad \text{on } \Gamma .
\] (4.50)

This requires
\[
C_3 \geq \max_\Gamma |D^2(\Phi_X | \Gamma)| + 1 .
\] (4.51)

We note that
\[
C_4 \eta - C_5 \eta^2 + C_6 (\Phi - \Psi) = 0 , \quad \text{on } \Gamma ,
\] (4.52)
so $u \geq \Phi_X$ on $\Gamma$, given (4.50) is valid. To make $u \geq \Phi_X$ on $\partial \Omega - \Gamma$, we choose $C_4$ big enough. Given (4.50), we have
\[
u \geq \Phi_X + \delta_1 , \quad \text{on } \partial \Gamma ,
\] (4.53)
for a small positive constant $\delta_1$. Then for a small $\delta_2 \in (0, r)$,
\[
l + C_3 \left( |z^\alpha|^2 + \xi^2 \right) - C_5 \eta^2 + C_6 (\Phi - \Psi) \geq \Phi_X , \quad \text{on } \{ \eta \leq \delta_2 \} \cap \partial \Omega .
\] (4.54)
$\delta_2$ depends on $\delta_1, C_5, C_6$, second order derivatives of $F$ and the norms of gradients of $\Psi, \Phi$. We also have
\[
l + C_3 \left( |z^\alpha|^2 + \xi^2 \right) - C_5 \eta^2 + C_6 (\Phi - \Psi) > \Phi_X - C_{10} , \quad \text{in } \Omega .
\] (4.55)

for a constant $C_{10}$, depending on $C_5, C_6$, second order derivatives of $F$, the $C^1$ norm of $\Phi$ and the $C^0$ norm of $\Psi$. We can choose
\[
C_4 > \frac{C_{10}}{\delta_2} .
\] (4.56)
Then \( u \geq \Phi \) on \( \partial D \).

In sum, we choose \( C_3 \) large enough with condition (4.51), then choose \( C_6 \) with condition (4.48), then choose \( C_5 \) with condition (4.49) and finally choose \( C_4 \) according to condition (4.56).

Thus, we have an upper bound for \( \Phi_{\eta\eta} \). To get the lower bound, we simply replace \( \partial_X \Phi \) by \( \partial_{-X} \Phi \).

Then we can use equation (4.25) to get the estimate of \( \Phi_{\eta\eta} \). Just note that \( 4\Phi_{\zeta\zeta} = \Phi_{\xi\xi} + \Phi_{\eta\eta} \).

So

\[
\Phi_{\eta\eta} = -\Phi_{\xi\xi} + 4\Phi_{\zeta\zeta} g^{\alpha\beta} \partial_{\alpha\beta} \Phi + 4 \epsilon \cdot k(\zeta) \cdot b_{\alpha\beta} g^{\alpha\beta}.
\]  

(4.57)

The estimate of the right-hand side of (4.57) depends on the boundary value \( F \), the metric lower bound estimate and the estimate of \( \Phi_{\eta\eta} \).

Given the boundary estimate, we can go back to the original \((\tau, \vec{z})\) coordinates and use equation (4.9). Similar to section 4.2, for any constant vector \( X \) in \( \mathbb{R} \times V \), we have

\[
\Phi_{XX} \leq \max_{\partial\mathbb{R} \times V} \Phi_{XX}, \quad \text{in } \mathbb{R} \times V.
\]  

(4.58)

For the lower bound estimate of \( \Phi_{XX} \), we have

\[
-\Phi_{XX} = \Phi_{JXJX} - 2\sqrt{-1} \partial \bar{\partial} \Phi (X, JX) = \Phi_{JXJX} - 2(\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi)(X, JX) + 2\Omega_0 (X, JX).
\]  

(4.59)

(4.60)

Then, using \( \Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi \geq 0 \), we get the lower bound of \( \Phi_{XX} \).

In sum, we get

\[
|\Phi|_{C^2(\Sigma \times V)} \leq C_{11},
\]  

(4.61)

for a constant \( C_{11} \), which depends on \( \epsilon, |F|_{C^3(\Sigma \times V)} \), \( \omega_0 \), the metric lower bound estimate and the boundary of \( \mathcal{R} \). In particular, when \( \epsilon \to 0 \), the constant \( C_{11} \) does not go to \( \infty \). However, we don’t need to use this fact.

### 4.5 \( C^{2,\alpha} \) Estimates

With the \( C^2 \) estimate in the previous section, we know the operator \( \mathcal{L} \) and \( L \) are uniform elliptic. We only need to prove the boundary \( C^{2,\alpha} \) estimate. Then, with the uniform ellipticity, concavity, \( C^2 \) estimate and the boundary \( C^{2,\alpha} \) estimate, we can derive the interior \( C^{2,\alpha} \) estimate with standard method [H16].

For the boundary \( C^{2,\alpha} \) estimate we need to flatten the boundary again. Adopting notation of section 4.4, we know that \( \Phi \) satisfies the equation

\[
\mathcal{L}^\partial \partial_{\partial\zeta}(\Phi_X) = \epsilon (\partial_X k) b_{\alpha\beta} g^{\alpha\beta} \quad \text{in } \mathcal{D}.
\]  

(4.62)

We construct a function

\[
\mathcal{F}(\zeta, \bar{z}) = \partial_X F(\text{Re}(\zeta), \bar{z}).
\]  

(4.63)

Then

\[
\Phi_X - F = 0, \quad \text{on } \Gamma;
\]  

(4.64)

\[
\mathcal{L}^\partial \partial_{\partial\zeta}(\Phi_X - F) = \epsilon (\partial_X k) b_{\alpha\beta} g^{\alpha\beta} - \mathcal{L}^\partial \partial_{\partial\zeta} F, \quad \text{in } \mathcal{D}.
\]  

(4.65)
The right-hand side of (4.65) is bounded, according to the \(C^2\) estimate in the previous section, so we can use Theorem 1.2.16 of [H16] and get the \(C^\alpha\) estimate for \(\partial_\nu(\Phi_X - \mathcal{F}) = \Phi_{X^\nu}\) in a small neighborhood of 0 in \(\Gamma\), for an \(\alpha \in (0, 1)\). Using equation (4.57), we get the \(C^\alpha\) estimate for \(\Phi_{\eta^\nu}\).

### 4.6 Existence of Smooth Solutions by the Method of Continuity

Suppose \(F \in C^\infty(\partial\mathcal{R} \times V)\) satisfies condition (3.1), for a constant section \(F\) of \(T_{2,0}^*(V)\). Then according to Lemma A.7,

\[
\sigma F(\tau, s) \text{ is strictly } (S, \omega_0)\text{-convex for any } \tau \in \mathcal{R} \text{ and any } \sigma \in [0, 1] .
\]

Consider the set

\[
\mathcal{S} = \{ \sigma \in [0, 1] | \text{Problem 1.4 with boundary value } s \cdot F \text{ has a solution } \Phi^\sigma, \text{ for any } s \leq \sigma, \quad (4.67)
\]

and \(\Phi^\sigma\) is a continuous curve in \(C^4(\mathcal{R} \times V)\) with \(C^2\) topology}. (4.68)

Obviously, \(\mathcal{S}\) is non-empty, since it contains 0. So, if \(\mathcal{S}\) is both open and closed, then \(\mathcal{S} = [0, 1]\)

Before we prove the openness and closeness, we point out that if a solution \(\Phi\) is in \(C^2,\alpha(\mathcal{R} \times V)\) then we can use the standard bootstrap technique (Theorem 5.1.9 and 5.1.10 of [H16]) to show that \(\Phi\) is actually in \(C^\infty(\mathcal{R} \times V)\). This is because of the openness of the condition (1.38) and the ellipticity of the equation (1.37).

The openness can be proved with standard implicit function theorem. This is because of the condition (1.38) and the ellipticity. Without such condition, the openness can be quite difficult to prove. For example, in [CFH20], we used Nash-Moser inverse function theorem to prove an openness result for geodesic equations.

For the closeness, given \(\{\sigma_i\}_{i \in \mathbb{Z}^+} \subset \mathcal{S}\) with \(\lim_{i \to \infty} \sigma_i = \sigma_\infty\), we need to show that \(\sigma_\infty \in \mathcal{S}\).

According to the \(C^2,\alpha\) estimate, the sequence of solutions \(\Phi^{\sigma_i}\) satisfy

\[
|\Phi^{\sigma_i}|_{C^{2,\alpha}} \leq C
\]

and

\[
\Omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi^{\sigma_i} \geq \frac{1}{C}(\Omega_0 + \sqrt{-1}d\tau \wedge d\bar{\tau}),
\]

for a constant \(C\), which depends on \(\epsilon\). It’s easy to know that \(\Phi^{\sigma_i}\) is a Cauchy sequence in \(C^0(\mathcal{R} \times V)\). So, using interpolation, we can find \(\Phi^{\sigma_\infty} \in C^{2,\frac{\alpha}{2}}(\mathcal{R} \times V)\) and

\[
\Phi^{\sigma_i} \to \Phi^{\sigma_\infty}, \quad \text{in } C^{2,\frac{\alpha}{2}} \text{ norm.}
\]

The \(C^{2,\frac{\alpha}{2}}\) convergence implies \(\Phi^{\sigma_\infty}\) satisfies equation (1.37) and condition (1.38). Therefore, \(\Phi^{\sigma_\infty}\) is a solution to Problem 1.4 with boundary value \(\sigma_\infty F\).

It remains to show that, for any \(\lambda \in [0, \sigma_\infty]\), there is a solution \(\Phi^\lambda\) and \(\{\Phi^\lambda|\lambda \in [0, \sigma_\infty]\}\) is a continuous curve with \(C^2\) topology.

In the following, when we say solution \(\Phi^\delta\), we always mean a solution to Problem 1.4 with boundary value \(\theta F\). For any \(\nu < \sigma_\infty\), let \(\delta_{\nu} = \frac{\sigma_\infty - \nu}{2}\). There is a \(\sigma_k > \nu + \delta_{\nu}\) because \(\sigma_k \to \sigma_\infty\). \(\sigma_k\) being in \(\mathcal{S}\) implies solution \(\Phi^{\nu}\) with boundary value \(\nu F\) exists and \(\{\Phi^\delta|s \in [0, \nu + \delta_{\nu}]\}\) is a continuous curve with \(C^2\) topology. So

\[
\{\Phi^\lambda|\lambda \in [0, \sigma_\infty]\} = \bigcup_{\nu < \sigma_\infty} \{\Phi^\lambda|\lambda \in [0, \nu + \delta_{\nu}]\}
\]

(4.72)
is $C^2$ continuous everywhere. Therefore, for any sequence $\nu_k \to \sigma_\infty$, solution $\Phi^{\nu_k}$ has uniform $C^{2,\alpha}$ estimate. It’s easy to know $\Phi^{\nu_k}$ converges to $\Phi^{\sigma_\infty}$ in $C^0$ norm, so, by interpolation, we know $\Phi^{\nu_k}$ converges to $\Phi^{\sigma_\infty}$ in $C^2$ norm and the curve $\{ \Phi^\lambda | \lambda \in [0, \sigma_\infty] \}$ is $C^2$ continuous everywhere.

In sum, we have the following.

**Theorem 4.1 (Existence of Smooth Solutions to Problem 1.4 and Convexity Estimates).** Suppose that, for a constant $\delta > 0$ and a constant section $S$ of $T^*_2(V)$, $F \in C^\infty(\partial \mathcal{R} \times V)$ satisfies

$$F(\tau, *)$$ is $(S, \omega_0)$-convex of modulus $> \delta$, for any $\tau \in \partial \mathcal{R}$. \hfill (4.73)

Then Problem 1.4 with boundary value $F$ has a unique and smooth solution $\Phi$. In addition, $\Phi(\tau, *)$ is $(S, \omega_0)$-convex of modulus $> \delta$, for any $\tau \in \mathcal{R}$, \hfill (4.74)

and, consequently,

$$\omega_0 + \sqrt{-1} \partial \overline{\partial} \Phi(\tau, *) > \delta \omega_0, \quad \text{for any } \tau \in \mathcal{R}. \hfill (4.75)$$

5 Estimates For Homogenous Monge-Ampère Equations

In this section, we prove estimates for solutions to Problem 1.2 and Problem 1.1. Solutions we talk about in this section all have the same boundary value $F$ which satisfies condition (1.30).

Suppose $\Phi^\epsilon$ is the solution to Problem 1.4 and $\Phi^0$ is the solution to Problem 1.2. We will show that $\Phi^\epsilon$ converges to $\Phi^0$ in $C^0$ norm and $\Phi^0$ satisfies estimates, which are satisfied by $\Phi^\epsilon$.

Let $\Psi^\epsilon$ be the solution to Problem 1.3 with $\epsilon = \epsilon n (1 + C)^{n-1}$, where the constant $C$ is from (4.20). Then we know $\Psi^\epsilon \leq \Phi^\epsilon \leq \Phi^0$, in $\mathcal{R} \times V$. \hfill (5.1)

According to estimates in [C00] or [B12], for solutions to Problem 1.3 $\Psi^\epsilon \to \Phi^0$ in $C^0$ norm, so $\Phi^\epsilon \to \Phi^0$ in $C^0$ norm. In particular, for any $\tau \in \mathcal{R}$,

$$\Phi^\epsilon(\tau, *) \to \Phi^0(\tau, *), \quad \text{in } C^0 \text{ norm}. \hfill (5.2)$$

According to Theorem 4.1, in every local coordinate chart,

$$\Phi^\epsilon(\tau, *) + (1 - \mu) b_{\alpha \beta} z^\alpha z^\beta + \text{Re} \left( S_{\alpha \beta} z^\alpha z^\beta \right) \hfill (5.3)$$

is a convex function, for any $\tau \in \mathcal{R}$. Then, because of the $C^0$ convergence of $\Phi^\epsilon \to \Phi^0$, we can replace $\Phi^\epsilon$ by $\Phi^0$ in (5.3) and get

$$\Phi^0(\tau, *) + (1 - \mu) b_{\alpha \beta} z^\alpha z^\beta + \text{Re} \left( S_{\alpha \beta} z^\alpha z^\beta \right) \hfill (5.4)$$

is a convex function, for any $\tau \in \mathcal{R}$. Thus $\Phi^0(\tau, *)$ is $(S, \omega_0)$-convex of modulus $\geq \mu$, for any $\tau \in \mathcal{R}$.

For the metric lower bound estimate, the proof is standard, we only need to do an integration by parts. Theorem 4.1 implies that, for any positive function $\eta$,

$$\int_V (\omega_0(1 - \mu) + \sqrt{-1} \partial \overline{\partial} \Phi^\epsilon(\tau, *)) \wedge \omega_0^{n-1} \eta \geq 0, \hfill (5.5)$$

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for any \( \tau \in \mathcal{R} \). Then for any \( \eta \) with sufficiently small support, we can find \( \rho_0 \), so that
\[
\omega_0 = \sqrt{-1} \partial \bar{\partial} \rho_0, \quad \text{in the support of } \eta. \tag{5.6}
\]

Thus
\[
\int_V (\rho_0 (1 - \mu) + \Phi'(\tau, *)) \land \omega_0^{n-1} \land \sqrt{-1} \partial \bar{\partial} \eta \geq 0, \tag{5.7}
\]
for any \( \tau \in \mathcal{R} \). Let \( \epsilon \to 0 \), we get
\[
\int_V (\rho_0 (1 - \mu) + \Phi^0(\tau, *)) \land \omega_0^{n-1} \land \sqrt{-1} \partial \bar{\partial} \eta \geq 0, \tag{5.8}
\]
for any \( \tau \in \mathcal{R} \). So
\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi^0(\tau, *) \geq \mu \omega_0, \tag{5.9}
\]
in the weak sense, for any \( \tau \in \mathcal{R} \).

Thus, Theorem 1.2 is proved.

Remark 5.1. Here, we cannot use Lemma A.2 to directly derive metric lower bound estimate from convexity estimate, because \( \Phi^0(\tau) \) may not be \( C^2 \) continuous and the degree of \( (S, \omega_0) \)-convexity may not be well defined.

A Algebra Lemmas

A.1 Lemmas for Convexity Estimate

In this appendix, we show that if \( \varphi \) is \( C^2 \) then Definition 1.2 and Definition 1.4 are equivalent. Furthermore, the modulus of \( (S, \omega_0) \)-convexity and the degree of \( (S, \omega_0) \)-convexity also coincide. The main results are Lemma A.1 and Lemma A.2.

Lemma A.1 (Equivalent Definitions of Strict \((S, \omega_0)\)-Convexity). Suppose that \( \varphi \) is a \( C^2 \) continuous function on \( V \). Then it satisfies the condition of Definition 1.2 if and only if it satisfies the condition of Definition 1.4.

Lemma A.2 (Equivalence between Modulus of Convexity and Degree of Convexity). Suppose that \( \varphi \) is a \( C^2 \) continuous function on \( V \) and \( S \) is a constant section of \( T_{2,0}^*(V) \). Then \( \varphi \) is \( (S, \omega_0) \)-convex of degree \( > \delta \) if and only if it is \( (S, \omega_0) \)-convex of modulus \( > \delta \).

In the proof of these Lemmas and also in other parts of the paper, we need to use the following Autonne-Takagi factorization and its corollary Lemma A.4. The following Autonne-Takagi factorization is the Corollary 4.4.4(c) of [HJ13].

Lemma A.3 (Autonne-Takagi Factorization). Given a complex valued symmetric matrix \( S \), there is a unitary matrix \( U \) such that
\[
S = U^T \Sigma U, \tag{A.1}
\]
in which \( \Sigma \) is a non-negative diagonal matrix. And obviously the Hermitian matrix \( S \bar{S} \) has a decomposition
\[
S \bar{S} = U^T \Sigma^2 \bar{U}. \tag{A.2}
\]
With Autonne-Takagi Factorization, we can prove the following lemma.

**Lemma A.4.** Suppose \( A \) is an \( n \times n \) positive definite Hermitian matrix and \( B \) is a complex \( n \times n \) symmetric matrix. Then we can find an \( n \times n \) invertible matrix \( P \), so that

\[
PAP^* = I, \quad PBP^T = \Lambda, \tag{A.3}
\]
in which \( \Lambda \) is a non-negative diagonal matrix.

**Proof of Lemma A.4.** Since \( A \) is a positive definite Hermitian matrix, we can find an invertible matrix \( Q \) so that

\[
QAQ^* = I. \tag{A.4}
\]

Then we apply Autonne-Takagi factorization to \( QBQ^T \) and find \( R \in U(n) \) so that

\[
R(QBQ^T)R^T = \Lambda. \tag{A.5}
\]

\( P = RQ \) satisfies condition \( \Lambda \).

The proof of Lemma A.4 essentially depends on the following lemma.

**Lemma A.5.** Suppose \( U, V, W \) are real valued \( n \times n \) matrices and \( U, V \) are symmetric. Let

\[
A = \frac{1}{4}(U + W) + \frac{\sqrt{-1}}{4}(V - V^T), \tag{A.6}
\]

\[
B = \frac{1}{4}(U - W) - \frac{\sqrt{-1}}{4}(V + V^T). \tag{A.7}
\]

Then

\[
\begin{pmatrix} U & V \\ V^T & W \end{pmatrix} > 0 \tag{A.8}
\]

if and only if

\[
A > 0 \quad \text{and} \quad B A^{-1} B A^{-1} < 1. \tag{A.9}
\]

We first prove Lemma A.5, then Lemma A.4 follows easily.

**Proof of Lemma A.5.** \( \Rightarrow \) Given \( U, V, W \) satisfying \( A.8 \) we can construct a strictly convex quadratic polynomial on \( \mathbb{C}^n \), with coordinate \( z^\alpha = x^\alpha + \sqrt{-1}y^\alpha \),

\[
H(z) = U_{\alpha\beta}x^\alpha x^\beta + 2V_{\alpha\beta}x^\alpha y^\beta + W_{\alpha\beta}y^\alpha y^\beta. \tag{A.10}
\]

\( H \) is a strictly PSH function, since it’s strictly convex. And it’s straightforward to check that

\[
\partial_{\alpha\beta} H = A_{\alpha\beta} \quad \text{and} \quad \partial_{\alpha\beta} H = B_{\alpha\beta}. \tag{A.11}
\]

Therefore, we know \( A > 0 \). Then using Lemma A.4 we find matrix \( P \), so that

\[
PAP^* = I, \quad PBP^T = \Lambda. \tag{A.12}
\]
in which \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is non-negative. We consider a new set of coordinates \( \{\zeta^\alpha\} \), with

\[
z^\alpha = P^\beta_\alpha \zeta^\beta. \tag{A.13}
\]

With this new coordinate

\[
(\partial_{\zeta^\alpha \zeta^\beta} H) = (P^\mu_\alpha P^\rho_\beta \partial_{z^\mu z^\rho} H) = P B P^T = \Lambda; \tag{A.14}
\]

\[
(\partial_{\zeta^\alpha \zeta^\beta} H) = (P^\mu_\alpha P^\rho_\beta \partial_{z^\mu z^\rho} H) = P A P^* = I. \tag{A.15}
\]

In above, \( P = (P^\beta_\alpha) \), where \( \alpha \) is the row index and \( \beta \) is the column index. It’s obvious that the linear change of coordinates \( (A.13) \) does not affect the fact that \( H \) is a convex function. So the restriction of \( H \) to a complex line

\[
L_\alpha = \{\zeta^\alpha = \tau, \zeta^\mu = 0, \text{ for } \mu \neq \alpha | \tau \in \mathbb{C}\} \tag{A.16}
\]

is still a convex function. Therefore,

\[
H|_{L_\alpha} = \tau \tau + \lambda_\alpha \frac{\tau^2 + \bar{\tau}^2}{2} \tag{A.17}
\]

is a convex function of \( \tau \). This implies \( |\lambda_\alpha| < 1 \). So \( \Lambda^2 < 1 \) and

\[
B A^{-1} B A^{-1} = P^{-1} \Lambda^2 P < 1. \tag{A.18}
\]

(\( \Leftarrow \)) For another direction, we use Lemma (A.4) to diagonalize \( A, B \) simultaneously. Since \( A > 0 \), we can find \( R \) so that

\[
B = R A R^T, \quad \text{and} \quad A = R R^*. \tag{A.19}
\]

Let \( R = R_1 + \sqrt{-1} R_2 \). Then

\[
B = R_1 \Lambda R_1^T - R_2 \Lambda R_2^T + \sqrt{-1} (R_2 \Lambda R_1^T + R_1 \Lambda R_2^T); \tag{A.20}
\]

\[
A = R_1 R_1^T + R_2 R_2^T + \sqrt{-1} (R_2 R_1^T - R_1 R_2^T). \tag{A.21}
\]

We can get \( U, V, W \) by adding \( (A.6) \) and \( (A.7) \) or subtracting one from another:

\[
U = 2 \text{Re}(A + B), \quad W = 2 \text{Re}(A - B), \quad V = 2 \text{Im}(A - B). \tag{A.22}
\]

Then plug \( (A.20) \) and \( (A.21) \) into \( (A.22) \), we get

\[
\begin{pmatrix}
U \\
V^T \\
W
\end{pmatrix} = 2
\begin{pmatrix}
R_1 \\
-R_2 \\
R_1
\end{pmatrix}
\begin{pmatrix}
I + \Lambda \\
I - \Lambda \\
R_1^T \\
R_2^T \\
R_1^T
\end{pmatrix}. \tag{A.23}
\]

Therefore, \( \Lambda^2 < 1 \) implies \( (A.8) \). Similar to \( (A.18) \), we have

\[
B A^{-1} B A^{-1} = R \Lambda^2 R. \tag{A.24}
\]

So \( B A^{-1} B A^{-1} < 1 \) implies \( \Lambda^2 < 1 \). \( \square \)
Lemma A.1 follows immediately, by letting $A = (\varphi_{\alpha\beta})$ and $B = (\varphi_{\alpha\beta}) + S$.

The following linear algebra lemma is essential for the equivalence between modulus of convexity and degree of convexity.

**Lemma A.6.** Suppose $A, G$ are positive definite $n \times n$ Hermitian matrices, $B$ is a complex valued symmetric matrix and $\mu$ is a positive constant. Then

$$A > \mu G, \text{ and } B(A - \mu G)^{-1} B(A - \mu G)^{-1} < 1$$

(A.25)

if and only if

$$A > 0, \text{ and } (B - \Theta)A^{-1} B - \Theta A^{-1} < 1, \text{ for any symmetric } \Theta, \text{ with } \Theta G^{-1} \Theta G^{-1} \leq \mu^2.$$  

(A.26)

**Proof of Lemma A.6.** We first provide a proof in the $n = 1$ case. The lemma in this case is very intuitive and it best demonstrates the idea. In this case, $B, \Theta$ are complex numbers, $A,G$ are positive numbers, and, without loss of generality, we can assume $G = 1$. As illustrated by Figure 2, the set

$$\{(B, A) | A > 0, |B - \Theta| < A\}$$

(A.27)

is a cone with the corner at $(\Theta, 0)$. The condition that, for all $\Theta$ with $|\Theta| \leq \mu$,

$$A > 0, \text{ and } (B - \Theta)A^{-1} B - \Theta A^{-1} < 1, \text{ for any symmetric } \Theta, \text{ with } \Theta G^{-1} \Theta G^{-1} \leq \mu^2.$$  

(A.26)

Figure 2: Metric Lower Bound Estimate

$$(B, A) \subset \{(B, A) | A > 0, |B - \Theta| < A\}$$

(A.28)

is equivalent to

$$(B, A) \subset \bigcap_{|\Theta| \leq \mu} \{(B, A) | A > 0, |B - \Theta| < A\}. $$

(A.29)

The elementary geometry tells us that

$$\bigcap_{|\Theta| \leq \mu} \{(B, A) | A > 0, |B - \Theta| < A\} = \{A > \mu + |B|\}.$$  

(A.30)
Therefore, (A.29) is equivalent to
\[ A > \mu \quad \text{and} \quad \left| \frac{B}{A - \mu} \right| < 1, \tag{A.31} \]
which is condition (A.25).

In general dimension, we need to construct quadratic polynomials from \( A, B, G \), similar to the proof of Lemma A.1. Given an Hermitian matrix \( H \) and a symmetric matrix \( S \), let
\[ K^H_B(z) = H_{\alpha\beta} z^\alpha z^\beta + \text{Re}(S_{\alpha\beta} z^\alpha z^\beta). \tag{A.32} \]

According to Lemma A.1, \( A, B, G \) and \( \mu \) satisfying (A.25) is equivalent to that
\[ K^{A-\mu G}_B \] is a strictly convex function on \( \mathbb{C}^n \); (A.33)
\( A, B, G \) and \( \mu \) satisfying (A.26) is equivalent to that
\[ K^{A-\mu G}_B \] is a strictly convex function on \( \mathbb{C}^n \), for any symmetric \( \Theta \), with \( \Theta G^{-1} \Theta G^{-1} \leq \mu^2 \). (A.34)

Because \( K^{A-\mu G}_B \) and \( K^{A-\mu G}_B \) are both quadratic polynomials, they are strictly convex if and only if they are positive on \( \mathbb{C}^n - \{0\} \). So we need to show
\[ K^{A-\mu G}_B > 0, \text{ on } \mathbb{C}^n \setminus \{0\} \tag{A.35} \]
if and only if
\[ K^{A-\mu G}_B > 0, \text{ on } \mathbb{C}^n \setminus \{0\}, \text{ for any symmetric } \Theta, \text{ with } \Theta G^{-1} \Theta G^{-1} \leq \mu^2. \tag{A.36} \]

Equivalently, we need to show
\[ K^A_B(z) > \mu G_{\alpha\beta} z^\alpha z^\beta, \text{ on } \mathbb{C}^n \setminus \{0\} \tag{A.37} \]
if and only if
\[ K^A_B(z) > \text{Re}(\Theta_{\alpha\beta} z^\alpha z^\beta), \text{ on } \mathbb{C}^n \setminus \{0\}, \text{ for any symmetric } \Theta, \text{ with } \Theta G^{-1} \Theta G^{-1} \leq \mu^2. \tag{A.38} \]

This is valid because
\[ \sup_{\Theta \text{ symmetric}, \Theta G^{-1} \Theta G^{-1} \leq \mu^2} \text{Re}(\Theta_{\alpha\beta} z^\alpha z^\beta) = \mu G_{\alpha\beta} z^\alpha z^\beta. \tag{A.39} \]

To prove (A.39), we find \( P \), so that \( P G P^* = I \), and we let
\[ \frac{P \Theta P^T}{\mu} = S. \tag{A.40} \]

Then we find (A.39) is equivalent to
\[ \sup_{\Theta \text{ symmetric}, \Theta G^{-1} \Theta G^{-1} \leq \mu^2} \text{Re}(S_{\alpha\beta} z^\alpha z^\beta) = \delta_{\alpha\beta} z^\alpha z^\beta. \tag{A.41} \]

(A.41) can be easily proved with basic linear algebra.

Lemma A.2 follows immediately, by letting \( A = (\varphi_{\alpha\beta}) \) and \( B = (\varphi_{\alpha\beta}) + S \).
A.2 Monotonicity

In this section, we prove the following algebra lemma.

Lemma A.7 (A Monotonicity Lemma). Suppose $A_0$ and $A$ are Hermitian matrices satisfying
\[
A_0 > 0 \quad \text{and} \quad A_0 + A > 0,
\]
and $B$ is a symmetric matrix. Let
\[
K_t = B(A_0 + tA)^{-1} B(A_0 + tA)^{-1}. \tag{A.43}
\]
Then $t^{2p} \text{tr}(K_t^p)$ and the maximum eigenvalue of $t^2 K_t$ are both non-decreasing functions of $t$. Here $p$ is any positive integer.

Proof. First of all, we note that condition (A.42) implies
\[
A_0 + tA > 0, \quad \text{for any } t \in (0, 1). \tag{A.44}
\]
So, in (A.43), $(A_0 + tA)^{-1}$ exists. In the following we compute
\[
\frac{d}{dt} \left[ t^{2p} \text{tr}(K_t^p) \right] \tag{A.45}
\]
and show it’s non-negative. We need to simultaneously diagonalize $A$ and $A_0 + tA$ to simplify the computation.

For $t_0 \in [0, 1]$, find $P$ so that
\[
PAP^* = \Lambda, \quad P(A_0 + t_0A)P^* = I. \tag{A.46}
\]
Here $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_\alpha \in \mathbb{R}$. Let
\[
H = PBP^*, \tag{A.47}
\]
them $H$ is a symmetric matrix. Plug (A.46) and (A.47) into (A.43) to simplify the expression of $K_{t_0}$. We get, at $t = t_0$,
\[
K_{t_0} = P^{-1}HH^*P, \tag{A.48}
\]
and so
\[
P K_{t_0}^{p-1} P^{-1} = (HH^*)^{p-1}. \tag{A.49}
\]
Then we compute the derivative,
\[
\frac{d}{dt} \left[ \text{tr}(K_t^p) t^{2p} \right] \tag{A.50}
\]
\[
= \frac{d}{dt} \left[ t^{2p} \text{tr} \left( B(A_0 + tA)^{-1} B(A_0 + tA)^{-1} \right) \right] \tag{A.51}
\]
\[
= 2p \cdot t^{2p-1} \text{tr}(K_t^p) - p \cdot t^{2p} \text{tr} \left[ B(A_0 + tA)^{-1} A (A_0 + tA)^{-1} B(A_0 + tA)^{-1} K_t^{p-1} \right] \tag{A.52}
\]
\[
- p \cdot t^{2p} \text{tr} \left[ B(A_0 + tA)^{-1} B (A_0 + tA)^{-1} A(A_0 + tA)^{-1} K_t^{p-1} \right]. \tag{A.53}
\]

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Plug (A.46), (A.47) and (A.49) into the expression above, we get, at $t = t_0$,

$$
\frac{d}{dt} \left[ t^{2p} \text{tr}(K_t^p) \right] = p \cdot t_0^{2p-1} \left( 2 \cdot \text{tr} \left[ (HH^*)^p \right] - t_0 \cdot \text{tr} \left[ (H^*H)^p \Lambda \right] - t_0 \cdot \text{tr} \left[ (HH^*)^p \Lambda \right] \right). \tag{A.54}
$$

Using the fact that $H$ is symmetric, we know

$$
\text{tr} \left[ (H^*H)^p \Lambda \right] = \text{tr} \left[ \Lambda (HH^*)^p \right] = \text{tr} \left[ (HH^*)^p \Lambda \right]. \tag{A.55}
$$

So

$$
\frac{d}{dt} \left[ t^{2p} \text{tr}(K_t^p) \right] = 2p \cdot t_0^{2p-1} \left( \text{tr} \left[ (HH^*)^p \right] - t_0 \cdot \text{tr} \left[ (HH^*)^p \Lambda \right] \right) \tag{A.56}
$$

$$
= 2p \cdot t_0^{2p-1} \left( \text{tr} \left[ (HH^*)^p (I - t_0 \Lambda) \right] \right). \tag{A.57}
$$

It’s obvious that $HH^*$ is semi-positive definite, and, according to (A.46),

$$
I - t_0 \Lambda = PA_0P^* > 0. \tag{A.58}
$$

Therefore, we get

$$
\frac{d}{dt} \left[ t^{2p} \text{tr}(K_t^p) \right] \geq 0. \tag{A.59}
$$

\[\square\]

### A.3 Concavity

In this appendix, we show the operator (4.3) is concave. Suppose $A$ is an $(n + 1) \times (n + 1)$ positive definite Hermitian matrix and $G$ is an $n \times n$ positive definite Hermitian matrix. Denote the lower right $n \times n$ block of $A$ by $A$. Let

$$
F_1(A) = \log \left( A_{0\bar{0}} - A_{0\bar{0}} A^{\alpha \bar{\beta}} A_{\alpha \bar{0}} \right) \tag{A.60}
$$

and

$$
F_2(A) = - \log \left( G_{\alpha \bar{\beta}} A^{\alpha \bar{\beta}} \right). \tag{A.61}
$$

We will prove

**Lemma A.8.** $F_1$ is a concave function of $A$ in the space of positive definite $(n + 1) \times (n + 1)$ Hermitian matrices; $F_2$ is a concave function of $A$ in the space of positive definite $n \times n$ Hermitian matrices.

**Proof.** For the concavity of $F_1$, actually, we can show

$$
f_1(A) = A_{0\bar{0}} - A_{0\bar{0}} A^{\alpha \bar{\beta}} A_{\alpha \bar{0}} \tag{A.62}
$$

is a concave function of $A$. Let $H$ be an $(n + 1) \times (n + 1)$ Hermitian matrix. Similar to $A$, denote the lower-right block of $H$ by $H$. Let

$$
q(t) = f_1(A + tH), \tag{A.63}
$$

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for \( t \) close to 0. We will show that \( q''(0) \leq 0 \). To simplify the computation, we diagonalize \( \mathcal{A} \) and \( \mathcal{H} \) simultaneously. Find an \( n \times n \) matrix \( P \), so that

\[
P \mathcal{A} P^* = I; \quad P \mathcal{H} P^* = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n).
\]  

(A.64)

Let

\[
P = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}.
\]  

(A.65)

Note that another expression for \( f_1 \) is

\[
f_1(A) = \frac{\det A}{\det \mathcal{A}}.
\]  

(A.66)

So

\[
q(t) = \frac{\det(A + t\mathcal{H})}{\det(A + t\mathcal{H})} = \frac{\det[P(A + t\mathcal{H})P^*]}{\det[P(A + t\mathcal{H})P^*]}.
\]  

(A.67)

Denote that

\[
P \mathcal{A} P^* = \begin{pmatrix} a_{0\overline{0}} & \cdots & u_{0\overline{0}} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ u_{\alpha\overline{0}} & I \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}, \quad P \mathcal{H} P^* = \begin{pmatrix} h_{0\overline{0}} & \cdots & v_{0\overline{0}} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ v_{\alpha\overline{0}} & \Lambda \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}.
\]  

(A.68)

With these simplifications,

\[
q(t) = a_{0\overline{0}} + th_{0\overline{0}} - \sum_{\alpha} \frac{(u_{0\overline{0}} + tv_{0\overline{0}})(u_{\alpha\overline{0}} + tv_{\alpha\overline{0}})}{1 + t\lambda_{\alpha}}.
\]  

(A.69)

Straightforward computation gives

\[
q''(0) = -\sum_{\alpha} (u_{0\overline{0}}\lambda_{\alpha} - v_{0\overline{0}})(u_{\alpha\overline{0}}\lambda_{\alpha} - v_{\alpha\overline{0}}) \leq 0.
\]  

(A.70)

Therefore, \( f_1 \) is concave, and, consequently, \( F_1 = \log(f_1) \) is concave.

For the concavity of \( F_2 \), actually, we can show

\[
f_2 = \frac{1}{\text{tr}(\mathcal{G}A^{-1})}
\]  

(A.71)

is a concave function of \( \mathcal{A} \). This is a simple consequence of the well-known fact that

\[
\frac{1}{\text{tr}(A^{-1})} = \frac{\det(A)}{\sigma_{n-1}(A)}
\]  

(A.72)

is a concave function of \( \mathcal{A} \). Therefore, \( F_2 = \log(f_2) \) is concave.

\( \square \)
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