Optimal Hardy inequalities associated with multipolar Schrödinger operators

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Abstract We proved some optimal Hardy inequalities in $\mathbb{R}^N$ which is closely related to multipolar Schrödinger operators with mean-value type potentials, these sharp inequalities imply some multipolar type Heisenberg inequalities. We also obtained some improved multipolar Hardy inequalities on bounded domains, moreover, we got the range of the best Hardy constant for a specific Hardy inequality.

1 Introduction

A Hardy type inequality is said that there is a potential $V$ and a positive constant $\mu$ so that the following inequality

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \mu \int_{\mathbb{R}^N} V |u|^2 \, dx,
\end{equation}

holds. This issue is equivalent to study the positivity of Schrödinger operators $-\Delta - \mu V$. Employing Sobolev embedding inequality $C_N \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2$ (with sharp constant $C_N$), one obtain that $-\Delta - \mu V$ is nonnegative if

\begin{equation}
\|V\|_{L^{2^*}(\mathbb{R}^N)} \leq \frac{C_N}{\mu}.
\end{equation}

See [21] for more discussion of the potential energy operator $V$.

When $N \geq 3$, the well-known Hardy potentials $V = |x|^{-2}$, or so-called inverse square potential, does not satisfy (1.2). In this case, we have the classical Hardy inequality with sharp constant

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx.
\end{equation}

We mention that it is easy to see (1.3) and Plancherel formula imply the Heisenberg inequality,

\begin{equation}
\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \int_{\mathbb{R}^N} |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi \geq \frac{(N-2)^2}{4},
\end{equation}

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where \( \|u\|_{L^2_{\mathbb{R}^N}} = 1 \) and \( \hat{a}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u(x) dx \). It’s well known that (1.4) is the beautiful mathematical description of the famous "Uncertainty Principle" in quantum mechanics.

There exists a great amount of literature on the generalization and improvement of (1.3), see [1] [2] [6] [7] [12] [13] [17] [20] [25] [26] [27] and the references therein.

There are also many works related to multipolar potentials \( V(x) = \sum_{i=1}^{n} \frac{\mu_i}{|x-a_i|^n} \) with \( n \) poles \( a_1, a_2, \ldots, a_n \). These type of multipolar potentials are related to the interaction of a finite number of electric dipoles. This form of systems are characterized by Hartree-Fock type model, which is the most commonly used model in Quantum Molecular [23]. These potentials are also applied in other fields such as combustion models and quantum cosmological models.

Consider the quadratic functional with respect to Schrödinger operator \(-\Delta - \sum_{i=1}^{n} \frac{\mu_i}{|x-a_i|^2}\):

\[
Q[u] := \int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^{n} \mu_i \int_{\mathbb{R}^N} \frac{|u|^2}{|x-a_i|^2} dx.
\]

It is complicated to study the positivity of \( Q[u] \) due to the relative position and interaction among the poles. The author in [16] proved that Schrödinger operator \(-\Delta - \sum_{i=1}^{n} \frac{\mu_i}{|x-a_i|^2}\) is positive if and only if \( \sum_{i=1}^{n} \mu_i^+ \leq \frac{(N-2)^2}{4} \) for any configuration of \( a_1, a_2, \ldots, a_n \); conversely, if \( \sum_{i=1}^{n} \mu_i^+ > \frac{(N-2)^2}{4} \), then there exist a configuration of \( a_1, a_2, \ldots, a_n \) such that \( Q[u] \) is not positive. These results then have been improved by authors in [15] that the existence of a configuration so that the quadratic form \( Q[u] \) is positive is equivalent to \( \mu_i \leq \frac{(N-2)^2}{4} \) for any \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \mu_i \leq \frac{(N-2)^2}{4} \). This shows that the critical mass \( \frac{(N-2)^2}{4} \) for certain singular pole \( a_i \) can be infinitely approximated, though all the other \( \mu_j \) are small enough right now. Bosi, Dolbeault, Esteban [3] obtained a lower bound of the spectrum of the Schrödinger operators \(-\Delta - \mu \sum_{i=1}^{n} \frac{1}{|x-a_i|^2}, \mu \in (0, \frac{(N-2)^2}{4}], n \geq 2 \). In other words, consider \( \mu \in (0, \frac{(N-2)^2}{4}], n \geq 2 \), there exists a nonnegative constant \( K_n \leq \pi^2 \) such that

\[
(1.5) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{4K_n + 4(n+1)\mu}{d^2} \int_{\mathbb{R}^N} |u|^2 dx \geq \mu \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|u|^2}{|x-a_i|^2} dx, u \in H^1(\mathbb{R}^N),
\]

where \( d := \min_{1 \leq i \neq j \leq n} |a_i - a_j| \). Their proof depends on the well-known "IMS" truncation method (see [22] [24]). Moreover, in an attempt to remove the lower order term, the author in [5] obtained the following inequality for any \( u \in H^1(\mathbb{R}^N) \) and \( n \geq 2 \):

\[
(1.6) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|u|^2}{|x-a_i|^2} dx + \frac{(N-2)^2}{4n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{|a_i-a_j|^2}{|x-a_i|^2 |x-a_j|^2} |u|^2 dx.
\]
When $x \to a_i$, the total mass near $a_i$ is $\frac{(N-2)^2}{4} \frac{2n-1}{n^2} \frac{1}{|x-a_i|^2}$, which is strictly smaller than $\frac{(N-2)^2}{4} \frac{1}{|x-a_i|^2}$.

The result above was improved by the authors in [10] with an optimal weight. Specifically, when $n \geq 2$, they proved the following inequality

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{|a_i - a_j|^2}{|x-a_i|^2 |x-a_j|^2} |u|^2 \, dx,
\end{equation}

where the constant $\frac{(N-2)^2}{n^2}$ is sharp. This inequality provides a sharp positive singular quadratic potential tends gradually to

\begin{equation}
\frac{(N-2)^2}{4} \frac{2n-1}{n^2} \frac{1}{|x-a_i|^2}
\end{equation}
at any $a_i$, $i = 1, \ldots, n$, which is strictly larger than $\frac{(N-2)^2}{4} \frac{1}{|x-a_i|^2}$. So inequality (1.7) can be seen as an improvement of (1.6). By parallelogram rule in $\mathbb{R}^N$, inequality (1.7) is equivalent to inequality

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{n^2} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{x-a_i}{|x-a_i|^2} \cdot \frac{x-a_j}{|x-a_j|^2} \cdot |u|^2 \, dx,
\end{equation}

Later Devyver, Fraas and Pinchover in [11] obtained another multipolar Hardy inequality for any $u \in H^1(\mathbb{R}^N)$ reads as

\begin{equation}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{n^2} \left[ \sum_{i=1}^n \frac{1}{|x-a_i|^2} + \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \frac{|a_i - a_j|^2}{|x-a_i|^2 |x-a_j|^2} \right] |u|^2.
\end{equation}

The potential arise in (1.9) is smaller than that in (1.7) near every poles as it behaves asymptotically like

\begin{equation}
\frac{(N-2)^2}{4} \frac{4n}{(n+1)^2} \frac{1}{|x-a_i|^2}.
\end{equation}

However, authors in [11] proved that the potential in (1.9) is critical, i.e. inequality (1.9) is impossible to be further improved. Actually, they also proved the criticality of the potential correlated with (1.7).

We also mention some other results of multipolar Hardy inequalities: the authors in [9] consider the inequality (1.7) or (1.8) in a domain $\Omega \subset \mathbb{R}^N$; the inequalities (1.7) and (1.8) are studied on Riemannian manifolds in [14] (It is worth mentioning that the potentials in (1.7) and (1.8) are not equivalent anymore in general Riemannian manifolds); the authors in [3] consider multipolar Poincaré-Hardy inequalities on Cartan-Hadamard manifolds, which generalized the results in [3] for single singularity.

Our goal in this paper is to consider mean-value type multipolar potentials so that the corresponding Hardy inequalities holds. This is motivated by noticing that the potential
\[ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{|x-a_i|^2} \] is just an arithmetic mean of \( n \) numbers \( \frac{1}{|x-a_i|^2}, i = 1, 2, \ldots, n \). In section 2 we list the main results of this paper. In section 3 we give the proof of Theorem 2.1 and 2.2. In the last section we obtain some improved multipolar Hardy inequalities on bounded domains.

2 Main results

For \( n \) different points \( a_1, \ldots, a_n \) in \( \mathbb{R}^N \), we denote that \( d := \min_{1 \leq i \neq j \leq n} |a_i - a_j| > 0 \). In this section we consider the following mean-value type potentials

\[
V_\lambda(d_1, d_2, \ldots, d_n) := \begin{cases} 
\left( \sum_{i=1}^{n} \alpha_i |x - a_i|^{-2\lambda} \right)^{\frac{1}{\lambda}}, & \lambda \in \mathbb{R} \setminus \{0\}, \\
\prod_{i=1}^{n} |x - a_i|^{-2\alpha_i}, & \lambda = 0,
\end{cases}
\]

where \( \alpha_i \geq 0, i = 1, 2, \ldots, n, \sum_{i=1}^{n} \alpha_i = 1 \). We call \( V_{-1} \) the powered harmonic mean, \( V_0 \) powered geometric mean, \( V_1 \) powered arithmetic mean and \( V_2 \) powered quadratic mean respectively. It is well-known that \( V_\lambda \) is an increasing function on \( \lambda \), so we have the following inequalities

\[
\min_{1 \leq i \leq n} |x - a_i|^{-2} \leq V_{-1} \leq V_0 \leq V_1 \leq V_2 \leq \max_{1 \leq i \leq n} |x - a_i|^{-2}.
\]

Then we consider two potentials \( V_{+\infty} \) and \( V_{-\infty} \) in \( \mathbb{R}^N \), where

\[
V_{+\infty} := \max_{1 \leq i \leq n} |x - a_i|^{-2},
\]

\[
V_{-\infty} := \min_{1 \leq i \leq n} |x - a_i|^{-2}.
\]

Our main results are as follow:

**Theorem 2.1.** We assert that the following multipolar Hardy inequality holds for any \( u \in H^1(\mathbb{R}^N) \)

\[
(2.1) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_{+\infty} |u|^2 dx,
\]

and the constant \( \frac{(N-2)^2}{4} \) is sharp.

**Theorem 2.2.** We assert that the following multipolar Hardy inequality holds for any \( u \in H^1(\mathbb{R}^N) \)

\[
(2.2) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_{-\infty} |u|^2 dx,
\]

and the constant \( \frac{(N-2)^2}{4} \) is sharp.
Recall that $V_{-\infty} \leq V_\lambda \leq V_{+\infty}$, we get the following result from Theorem 2.1 and Theorem 2.2.

**Corollary 2.3.** The following inequality holds
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_\lambda |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N),
\]
morover, the constant $\frac{(N-2)^2}{4}$ is sharp.

When $\lambda = 1$ we have:

**Corollary 2.4.** The following multipolar Hardy inequality holds
\[
(2.3) \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^n \int_{\mathbb{R}^N} \alpha_i \frac{|u|^2}{|x-a_i|^2} \, dx, \quad u \in H^1(\mathbb{R}^N),
\]
the constant $\frac{(N-2)^2}{4n}$ is sharp. Especially, we have the following multipolar Hardy inequality
\[
(2.4) \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4n} \sum_{i=1}^n \int_{\mathbb{R}^N} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N),
\]
the constant $\frac{(N-2)^2}{4n}$ is sharp.

By Corollary 2.3 and Hölder inequality, we have the following multipolar type Heisenberg inequality.

**Corollary 2.5.** For any $\lambda \in \mathbb{R}$, the following inequality holds
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \left( \sum_{i=1}^n \alpha_i |x-a_i|^{2\lambda} \right)^\frac{1}{\lambda} |u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N).
\]
Especially, we have
\[
(2.5) \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \left( \sum_{i=1}^n |x-a_i|^2 \right)^\frac{1}{2} |u|^2 \, dx \geq \frac{(N-2)^2}{4n} \int_{\mathbb{R}^N} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N).
\]

**Proof.** In view of Corollary 2.3 we have
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_{-\lambda} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N),
\]
where $V_{-\lambda}^{-1} = \left( \sum_{i=1}^n \alpha_i |x-a_i|^{2\lambda} \right)^\frac{1}{\lambda}$. Thus
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \left( \sum_{i=1}^n \alpha_i |x-a_i|^{2\lambda} \right)^\frac{1}{\lambda} |u|^2 \, dx
\geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_{-\lambda} |u|^2 \, dx \int_{\mathbb{R}^N} \left( \sum_{i=1}^n \alpha_i |x-a_i|^{2\lambda} \right)^\frac{1}{\lambda} |u|^2 \, dx
\geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} |u|^2 \, dx.
Remark 2.6. When \( n = 1 \), we recover the classical Heisenberg inequality \([1.4]\) by Corollary 2.5.

Let \( f(x) \) be a monotone function of one variable with an inverse \( f^{-1} \). Define multipolar potentials as

\[
V_f(a_1, a_2, \ldots, a_n) := f^{-1}\left(\sum_{i=1}^{n} \alpha_i f(|x - a_i|^{-2})\right),
\]

here \( \alpha_i \geq 0, i = 1, 2, \ldots, n \), \( \sum_{i=1}^{n} \alpha_i = 1 \). Then from Theorem 2.1 and 2.2 we affirm:

Corollary 2.7. We assert that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} V_f |u|^2 dx, u \in H^1(\mathbb{R}^N),
\]

where the constant \( \frac{(N - 2)^2}{4} \) is sharp.

3 Proof of Theorem 2.1 and 2.2

Recall the Hardy type identity for \( u \in C^\infty_0(\Omega) \) and \( \varphi \in C^2(\Omega) \),

\[
(3.1) \quad \int_{\Omega} (|\nabla u|^2 + \frac{\Delta \varphi}{\varphi} |u|^2) dx = \int_{\Omega} |\nabla u - \frac{\nabla \varphi}{\varphi} u|^2 dx = \int_{\Omega} |\nabla (u\varphi^{-1})|^2 \varphi^2 dx.
\]

Equality \((3.1)\) leads to different Hardy type inequality along with different choice of test function \( \varphi \). In fact this equality can be more general, see \([8]\), with the same assumption of \( u \) and \( \varphi \) ahead, \( \alpha \in \mathbb{R} \), it holds,

\[
\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \left(\alpha(1 - \alpha)\frac{|\nabla \varphi|^2}{|\varphi|^2} + \alpha \frac{\Delta \varphi}{\varphi}\right) dx + \int_{\Omega} |\nabla (u\varphi^{-1})|^2 \varphi^{2\alpha} dx.
\]

Due to the nonnegativity of the integral \( \int_{\Omega} |\nabla (u\varphi^{-1})|^2 \varphi^{2\alpha} dx \), we deduce Hardy inequality from \((3.1)\), namely,

\[
(3.2) \quad \int_{\Omega} |\nabla u|^2 dx \geq \int_{\Omega} \frac{-\Delta \varphi}{\varphi} |u|^2 dx.
\]

The difficulty is to find an appropriate function \( \varphi \) to obtain the Hardy inequality we want. We also mention that \((3.1)\) also holds for distribution \( \varphi \).

Potentials \( V_{+\infty} \) and \( V_{-\infty} \) are not in \( C^2(\mathbb{R}^N) \), but the set of non-differentiable points of these two potentials is contained in \( \tilde{T} := T \cup \{a_1, a_2, \ldots, a_n\}, T := \{x : \exists i, j \text{ s.t. } |x - a_i| = |x - a_j|\} \). In fact the \( \varphi \) we would choose are in \( C^2(\mathbb{R}^N \setminus \tilde{T}) \cap C(\mathbb{R}^N) \), i.e.

\[
(3.3) \quad \varphi = \max_{1 \leq i \leq n} |x - a_i|^{\frac{2-N}{2}} \text{ or } \min_{1 \leq i \leq n} |x - a_i|^{\frac{2-N}{2}}.
\]
Denote $E$ the set of non-differentiable points of $\varphi$, then $E \subseteq \tilde{T}$. $T$ can be written as

$$T = \bigcup_{1 \leq i < j \leq n} T_{ij},$$

where $T_{ij} := \{x : |x - a_i| = |x - a_j|\}$. $T_{ij}$ is a hyperplane so that its $N$ dimensional Lebesgue measure is zero. Then $\tilde{T}$ is a zero measure set. $\varphi$ is in $C^2_2(\mathbb{R}^N \setminus \tilde{T}) \cap C(\mathbb{R}^N)$. Thus we have the following identity for $\varphi$ in (3.3),

$$\int_{\mathbb{R}^N \setminus \tilde{T}} |\nabla u|^2 dx = \int_{\mathbb{R}^N \setminus \tilde{T}} \frac{-\Delta \varphi}{\varphi} |u|^2 dx + \int_{\mathbb{R}^N \setminus \tilde{T}} |\nabla (u \varphi^{-1})|^2 \varphi^2 dx.$$

**Proof of Theorem 2.1.**

Let

$$\varphi = \max_{1 \leq i \leq n} |x - a_i|^\frac{2-N}{2}.$$

Then we consider a decomposition of $\mathbb{R}^N$ depending on the configuration of $\{a_i\}_{i=1}^n$. Define

$$E_1 = \{x \in \mathbb{R}^N \setminus \{a_1, a_2, \ldots, a_n\} : \varphi(x) = |x - a_i|^\frac{2-N}{2}\},$$

$$\vdots$$

$$E_i = \{x \in \mathbb{R}^N \setminus \{a_1, a_2, \ldots, a_n\} \setminus \bigcup_{k=1}^{i-1} E_k : \varphi(x) = |x - a_i|^\frac{2-N}{2}\}, i = 2, \ldots, n.$$

It is obvious that $E_i$ verify two properties:

$$E_i \cap E_j = \emptyset, i \neq j;$$

$$\bigcup_{i=1}^n E_i = \mathbb{R}^N \setminus \{a_1, a_2, \ldots, a_n\}.$$

For every $x \in E_i^c$,

$$-\frac{\Delta \varphi}{\varphi} = \frac{(N - 2)^2}{4} \frac{1}{|x - a_i|^2}.$$

Note that $\varphi = V_{+\infty}^\frac{N-2}{4}$, and $\frac{N-2}{4} > 0$ when $N \geq 3$. Thus

$$-\frac{\Delta \varphi}{\varphi} = \frac{(N - 2)^2}{4} V_{+\infty}, \text{ in } \mathbb{R}^N \setminus \tilde{T}.$$ 

Thus we deduce inequality (2.1) holds since $\tilde{T}$ is a zero measure set. Moreover we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} V_{+\infty} |u|^2 dx = \int_{\mathbb{R}^N} |\nabla (u \varphi^{-1})|^2 \varphi^2 dx.$$

The gradient in the r.h.s. of (3.5) is in the sense of weak derivative.
Next we prove the optimality of $\frac{(N-2)^2}{4}$. For $\forall x \in B(a_i, \frac{d}{2}) := \{x : |x - a_i| < \frac{d}{2}\}$, and any $j \neq i$,

$$ |x - a_j| \geq |a_i - a_j| - |x - a_i| > \frac{d}{2} \geq |x - a_i|, $$

so $B(a_i, \frac{d}{2}) \subseteq E_i$ for any $1 \leq i \leq n$. Now for this representation we can also define a series of cut-off functions as follow

$$ \psi_{\varepsilon,i} = \begin{cases} 0, & x \in B(a_i, \varepsilon) \cup \mathbb{R}^N \setminus E_i, \\ \frac{\log \frac{|x - a_i|}{\varepsilon^2}}{\log \varepsilon}, & x \in B(a_i, \varepsilon) \setminus B(a_i, \varepsilon^2), \\ 1, & x \in B(a_i, \varepsilon^2) \cap E_i. \end{cases} $$

Here $\varepsilon > 0$ is small enough. Then we consider $u_\varepsilon = \sum_{i=1}^n u_{\varepsilon,i}$, where

$$ u_{\varepsilon,i} = \psi_{\varepsilon,i} |x - a_i|^{2-N} - \varepsilon. $$

Take $u_\varepsilon$ into (3.5). Firstly,

$$ \int_{\mathbb{R}^N} |\nabla (u_\varepsilon \varphi^{-1})|^2 \varphi^2 \, dx $$

$$ = \sum_{i=1}^n \int_{E_i} |\nabla (u_\varepsilon \varphi^{-1})|^2 \varphi^2 \, dx $$

$$ = \sum_{i=1}^n \int_{E_i} |\nabla (\psi_{\varepsilon,i} |x - a_i|^{-\varepsilon})|^2 |x - a_i|^{-2N} \, dx $$

$$ \leq 2 \sum_{i=1}^n \left( \int_{B(a_i, \varepsilon^2)} \frac{1}{\log \varepsilon} |x - a_i|^{-2\varepsilon} \, dx + \varepsilon^2 \int_{B(a_i, \varepsilon^2)^c} |x - a_i|^{-2\varepsilon} \, dx \right) $$

$$ = 2n \omega_N \left( \frac{1}{\log \varepsilon} \int_{\varepsilon^2}^{\varepsilon} r^{-2\varepsilon-1} \, dr + \varepsilon^2 \int_{\varepsilon^2}^{+\infty} r^{-2\varepsilon-1} \, dr \right) $$

$$ = 2n \omega_N \left( \frac{\varepsilon^{-2\varepsilon} - \varepsilon^{-2\varepsilon}}{2\varepsilon \log \frac{1}{\varepsilon}} + \frac{1}{2} \varepsilon^{1-4\varepsilon} \right) \to 2n \omega_N, \text{ as } \varepsilon \to 0. $$

Then we know that for any $1 \leq i \leq n$, $E_i$ contains a ball with radius $\frac{d}{2}$, so

$$ \int_{\mathbb{R}^N} V_{+\infty} |u_\varepsilon|^2 \, dx = \sum_{i=1}^n \int_{E_i} |x - a_i|^{-2\varepsilon-N} |\psi_{\varepsilon,i}|^2 \, dx $$

$$ \geq \sum_{i=1}^n \int_{B(a_i, \frac{d}{2}) \setminus B(a_i, \varepsilon)} |x - a_i|^{-2\varepsilon-N} \, dx $$

$$ \geq n \omega_N \int_{\varepsilon}^{\frac{d}{2}} r^{-2\varepsilon-1} \, dr $$

$$ = n \omega_N \varepsilon^{-2\varepsilon} - \left( \frac{d}{2} \right)^{-2\varepsilon} \to +\infty, \text{ as } \varepsilon \to 0. $$
Combining (3.6) and (3.7) we have
\[
\lim_{{\varepsilon \to 0}} \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, dx}{\int_{\mathbb{R}^N} V_{+\infty} |U_\varepsilon|^2 \, dx} = \lim_{{\varepsilon \to 0}} \left( \frac{(N-2)^2}{4} + \frac{\int_{\mathbb{R}^N} |(u_\varepsilon \varphi^{-1})|^2 \varphi^2 \, dx}{\int_{\mathbb{R}^N} V_{+\infty} |U_\varepsilon|^2 \, dx} \right) = \frac{(N-2)^2}{4}.
\]
Thus we complete the proof of Theorem 2.1.

Proof of theorem 2.2. Let
\[
\varphi = \min_{1 \leq i \leq n} |x - a_i|^{\frac{2-N}{2}}.
\]
By similar argument we have the following equality for a.e. \(x \in \mathbb{R}^N\),
\[
-\Delta \varphi = \frac{(N-2)^2}{4} V_{-\infty},
\]
Thus inequality (2.2) holds, and
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} V_{-\infty} |u|^2 \, dx = \int_{\mathbb{R}^N} |(u \varphi^{-1})|^2 \varphi^2 \, dx.
\]
It remains to prove the sharpness of the constant. For any \(\varepsilon > 0\), Let
\[
u_\varepsilon = \min_{1 \leq i \leq n} |x - a_i|^{\frac{2-N}{2} - \varepsilon},
\]
when \(n = 1\), the optimality of \(\frac{(N-2)^2}{4}\) has already known. When \(n \geq 2\), \(u_\varepsilon\) belongs to \(D^{1,2}(\mathbb{R}^N)\) with the norm
\[
\|u\|_{D^{1,2}(\mathbb{R}^N)} = \langle \nabla u, \nabla u \rangle,
\]
define \(\tilde{E}_i\) just as in (3.4) by taking \(\varphi(x) = \min_{1 \leq i \leq n} |x - a_i|^{\frac{2-N}{2}}\), note that \(\tilde{E}_i \subseteq B(a_i, \frac{d}{2})\) for any \(i = 1, 2, \ldots, n\), we obtain by direct computation,
\[
\lim_{{\varepsilon \to 0}} \frac{\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, dx}{\int_{\mathbb{R}^N} V_{-\infty} |U_\varepsilon|^2 \, dx} = \lim_{{\varepsilon \to 0}} \frac{1}{\sum_{i=1}^{n} \int_{E_i} |x - a_i|^{-2\varepsilon - N} \, dx} = \frac{(N-2)^2}{4}.
\]
Our results recover that the result in [16] that Schrödinger operator \(-\Delta - \frac{(N-2)^2}{4} V_1 = -\Delta - \frac{(N-2)^2}{4} \sum_{i=1}^{n} \frac{a_i}{|x - a_i|^2}\) is positive, and the constant \(\frac{(N-2)^2}{4}\) cannot be larger, i.e. (2.3) is a sharp Hardy type inequality. But in fact we can add a positive term in the r.h.s. of (2.3). Actually, in (3.1) let
\[
\varphi_1 = \prod_{i=1}^{n} |x - a_i|^{\beta a_i}, \quad \varphi_2 = \sum_{i=1}^{n} a_i |x - a_i|^{\frac{2-N}{2}}.
\]
Theorem 3.1. The following inequality holds for any $u \in H^1(\mathbb{R}^N)$

$$
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N - 2)^2}{4} \sum_{i=1}^{n} \int_{\mathbb{R}^N} \alpha_i \frac{|u|^2}{|x-a_i|^2} dx 
+ \frac{(N - 2)^2}{4} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \alpha_i \alpha_j \frac{|a_i - a_j|^2}{|x-a_i|^2|x-a_j|^2} |u|^2 dx.
$$

(3.9)

Theorem 3.2. There holds

$$
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq (N - 2)^2 \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \alpha_i \alpha_j \frac{|a_i - a_j|^2}{|x-a_i|^2|x-a_j|^2} |u|^2 dx, \ u \in H^1(\mathbb{R}^N).
$$

Theorem 3.3. For any $u \in H^1(\mathbb{R}^N)$ there holds

$$
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N - 2)^2}{4} \sum_{i=1}^{n} \int_{\mathbb{R}^N} \alpha_i \frac{|u|^2}{|x-a_i|^2} dx 
+ \frac{(N - 2)^2}{4} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \alpha_i \alpha_j \frac{|a_i - a_j|^2}{|x-a_i|^2|x-a_j|^2} |u|^2 dx.
$$

Compute directly we have

$$
\int_{\mathbb{R}^N} |\nabla (w\varphi_2^{-1})|^2 \varphi_1^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx + [\beta^2 + \beta(N - 2)] \sum_{i=1}^{n} \int_{\mathbb{R}^N} \alpha_i \frac{|u|^2}{|x-a_i|^2} dx 
- \alpha^2 \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \alpha_i \alpha_j \frac{|a_i - a_j|^2}{|x-a_i|^2|x-a_j|^2} |u|^2 dx,
$$

(3.8)

and

$$
\int_{\mathbb{R}^N} |\nabla (w\varphi_2^{-1})|^2 \varphi_2^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{(N - 2)^2}{4} \sum_{i=1}^{n} \int_{\mathbb{R}^N} \alpha_i \frac{|u|^2}{|x-a_i|^2} dx 
- \frac{(N - 2)^2}{4} \sum_{1 \leq i < j \leq n} \int_{\mathbb{R}^N} \alpha_i \alpha_j \frac{|a_i - a_j|^2}{|x-a_i|^2|x-a_j|^2} |u|^2 dx.
$$

The term \((|x-a_i|^2 - |x-a_j|^2) (|x-a_i|^{2-N} - |x-a_j|^{2-N})\) \(\geq 0\) when \(N \geq 3\). Thus we obtain the generalization of inequalities (3.4) and (3.7) by letting \(\beta = \frac{2-N}{2}\) and \(\beta = 2 - N\) respectively in (3.8), also an improvement of inequality (2.3).
Inequality \([3.9\)] does not break the optimality of constant \([N-2\]^2\) in \([2.3\)] because the potential \([\sum_{1\leq i<j\leq n}\frac{|a_i-a_j|^2}{|x-a_i||x-a_j|}]\) cannot be compared with \(V_1\) near infinity. Actually it behaves asymptotically like

\[
\sum_{1\leq i<j\leq n}\frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2} \sim O\left(\frac{1}{|x|^4}\right), |x| \to \infty.
\]

4 Some improvements on bounded domains

The classical Hardy inequality which corresponds to \(V = |x|^{-2}\) and \(\mu = \frac{(N-2)^2}{4}\) in \([1.1]\) is

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} dx, \quad \forall u \in H_0^1(\Omega)
\]

where \(N \geq 3\) and \(\Omega\) is an open subset of \(\mathbb{R}^N\) containing the origin. The constant is optimal and never achieved. When \(\Omega = \mathbb{R}^N\), it is impossible to add a strictly positive term in the r.h.s. of \([4.1]\). But if \(\Omega\) is bounded, Brezis and Vázquez firstly in \([6]\) obtained an improvement of \([4.1]\), the so-called Hardy-Poincaré inequality

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} dx + \frac{h_2}{R_{\Omega}^2} \int_{\Omega} |u|^2 dx, \quad \forall u \in H_0^1(\Omega)
\]

where \(R_{\Omega} = \left(\frac{\Omega}{\omega_N}\right)^{\frac{1}{N}}\), \(\omega_N\) is the volume of \(N\)-dimensional unit ball; \(h_2\) is the first eigenvalue of Laplace operator in the unit disk of \(\mathbb{R}^2\). In addition, they proved that when \(\Omega\) is a ball, the constant \(\frac{h_2}{R_{\Omega}^2}\) is sharp and never attained. They also obtained another improvement. When \(N \geq 3\) and \(1 < q < 2^* = \frac{2N}{N-2}\), then for any \(u \in H_0^1(\Omega)\),

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} dx + C(\Omega) \left(\int_{\Omega} |u|^q dx\right)^{\frac{2}{q}}.
\]

Motivated by \([4.2]\) and \([4.3]\), we have the following two similar improvements in the case of multiple singularities.

**Theorem 4.1.** Let \(\Omega \subset \mathbb{R}^N (N \geq 3)\) be a bounded domain, \(a_i (i=1,2,...,n)\) be \(n\) different points in \(\Omega\). There holds

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} V_{+\infty} |u|^2 dx + \frac{1}{n} \frac{h_2}{R_{\Omega}^2} \int_{\Omega} |u|^2 dx, \quad \forall u \in H_0^1(\Omega).
\]

**Theorem 4.2.** Let \(\Omega \subset \mathbb{R}^N (N \geq 3)\) be a bounded domain, \(a_i (i=1,2,...,n)\) be \(n\) different points in \(\Omega\), \(1 < q < 2^*\). There exists a positive constant \(C(q,\Omega)\) such that the following inequality holds for any \(u \in H_0^1(\Omega)\)

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} V_{+\infty} |u|^2 dx + \frac{1}{n} C(q,\Omega) \left(\int_{\Omega} |u|^q dx\right)^{\frac{2}{q}}.
\]
The proof of Theorem 4.1 is similar to that of Theorem 4.2. We only prove Theorem 4.2 here.

**Proof of Theorem 4.2.**

Let \( u = v \max_{1 \leq i \leq n} |x - a_i|^\frac{N-2}{2}, \) \( u_i = v|x - a_i|^\frac{N-2}{2}, \) supp\( v \subseteq \Omega. \) We observe that \( u_i = u \) in \( E_i \cap \Omega. \) Thus using (3.1),

\[
\int_\Omega |\nabla u|^2 \, dx - \frac{(N-2)^2}{4} \sum_{i=1}^{n} \int_{\Omega} V_{\infty} |u|^2 \, dx
\]

\[
= \int_\Omega |\nabla v|^2 \max_{1 \leq i \leq n} |x - a_i|^{2-N} \, dx
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} \int_\Omega |\nabla v|^2 |x - a_i|^{2-N} \, dx
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \int_\Omega |\nabla u_i|^2 \, dx - \frac{(N-2)^2}{4} \int_\Omega \frac{|u_i|^2}{|x - a_i|^2} \, dx \right)
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} C(\Omega) \left( \int_\Omega |u_i|^q \, dx \right)^\frac{2}{q}
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} C(\Omega) \left( \int_{E_i \cap \Omega} |u|^q \, dx \right)^\frac{2}{q}
\]

\[
\geq \frac{1}{n} C(q, \Omega) \left( \int_{\Omega} |u|^q \, dx \right)^\frac{2}{q}.
\]

We complete the proof.

The constant in (4.1) is optimal in bounded domains containing the origin. However, if \( a_i \in \Omega, i = 1, \ldots, n, n \geq 2, \) and \( \Omega \) is a bounded open subset of \( \mathbb{R}^N. \) Then from (3.9) we obtain

\[
\int_\Omega |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \sum_{i=1}^{n} \int_{\Omega} \alpha_i |u|^2 |x - a_i|^2 \, dx,
\]

where \( \alpha_i = \alpha_i + \frac{1}{2} \sum_{j=1, j \neq i}^{n} \alpha_i \alpha_j |a_i - a_j|^2/diam\Omega^2, \) \( \alpha_i \geq 0, i = 1, 2, \ldots, n, \) \( \sum_{i=1}^{n} \alpha_i = 1, \) diam\( \Omega \) denotes the diameter of \( \Omega. \) When \( n \geq 2, \) \( \frac{(N-2)^2}{4} \sum_{i=1}^{n} \alpha_i \) is strictly larger than \( \frac{(N-2)^2}{4}. \) Thus the constant \( \frac{(N-2)^2}{4} \) in (2.3) is not optimal. We aim to find a better potential in a bounded domain or deduce the range the optimal constant. Motivated by [9], we obtain the following result.

**Theorem 4.3.** Let \( N \geq 3 \) and \( \Omega \subset \mathbb{R}^N \) be a bounded domain with \( n \) different poles \( a_1, \ldots, a_n \in \Omega, n \geq 2. \) Given \( \gamma_i > 0, i = 1, \ldots, n. \) If the following sharp Hardy inequality

\[
\int_\Omega |\nabla u|^2 \, dx \geq C^* (\Omega) \sum_{i=1}^{n} \int_{\Omega} \gamma_i |u|^2 |x - a_i|^2 \, dx,
\]
holds for any $u \in H^1_0(\Omega)$, then we have

\[ C^*(\Omega) \sum_{i=1}^{n} \gamma_i > \frac{(N-2)^2}{4}, \quad \text{and} \quad \max_{1 \leq i \leq n} C^*(\Omega) \gamma_i \leq \frac{(N-2)^2}{4}. \]

**Proof.** The first inequality of (4.4) is obtained by the above discussion. The second inequality could be deduced by Hardy inequality (4.1). Assume there is a $\gamma_k$ such that

\[ C^*(\Omega) \gamma_k > \frac{(N-2)^2}{4}. \]

Choose a ball $B(a_k, \epsilon)$, $\epsilon$ small enough such that $B(a_k, \epsilon) \subset \Omega$, and

\[ \sum_{i=1}^{n} \gamma_i \frac{1}{|x-a_i|^2} = C^*(\Omega) \gamma_k \frac{1}{|x-a_k|^2} (1 + o(1)). \]

Thus for any $u \in C^\infty_0(B(a_k, \epsilon))$,

\[ \int_{\Omega} |\nabla u|^2 dx \geq C^*(\Omega) \gamma_k (1 + o(1)) \int_{\Omega} \frac{|u|^2}{|x-a_k|^2} dx > \frac{(N-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x-a_k|^2} dx. \]

This is contradicted with (4.1). We complete the proof. \qed

The following proposition reveals that there exists subset $U$ of $\Omega$ which contain all the poles such that the Hardy constant of the multipolar Hardy potentials $\sum_{i=1}^{n} \frac{1}{|x-a_i|^2}$ in $U$ can be close to $\frac{(N-2)^2}{4}$ infinitely.

**Proposition 4.4.** Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with $n$ poles $a_1, \ldots, a_n$, $n \geq 2$, and $V_\ast = \sum_{i=1}^{n} \frac{1}{|x-a_i|^2}$. Then for any $\epsilon > 0$, there exists a domain $U_\epsilon \subset \Omega$ such that the following inequality holds for any $u \in C^\infty_0(U_\epsilon)$,

\[ \int_{U_\epsilon} |\nabla u|^2 dx \geq \left( \frac{(N-2)^2}{4} - \epsilon \right) \int_{U_\epsilon} V_\ast |u|^2 dx. \]

**Proof.** We take $U_\epsilon = \bigcup_{i=1}^{n} B(a_i, r_\epsilon) \subset \Omega$, where $r_\epsilon$ small enough so that $B(a_i, r_\epsilon) \cap B(a_j, r_\epsilon) = \emptyset$ for any $i \neq j$. From (2.1) we have

\[ \int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} W_1 |u|^2 dx, \]

where $W_1 = V_\ast V_\ast$. In view of the behavior of $V_\ast$ and $V_\ast$ near each pole we have

\[ \lim_{x \to a_i} W_1(x) = 1, i = 1, \ldots, n. \]

Then for any $\epsilon > 0$, we choose $r_\epsilon$ small enough, so that

\[ |W_1(x) - 1| < \delta(\epsilon), \quad \forall x \in B(a_i, r_\epsilon), \quad i = 1, \ldots, n, \]

\[ |W_1(x) - 1| < \delta(\epsilon), \quad \forall x \in B(a_i, r_\epsilon), \quad i = 1, \ldots, n. \]
where \( \delta(\epsilon) = \frac{4\epsilon}{(N-2)^2} \). Since \( U_\epsilon \) is composed of \( n \) connected branch, for any \( u \in C_0^\infty(U_\epsilon) \), we can denote \( u = \sum_{i=1}^n u_i \), here \( u_i \in C_0^\infty(B(a_i, r_\epsilon)) \). Then, combining (4.5) and (4.6) we have

\[
\int_\Omega |\nabla u|^2 \, dx = \sum_{i=1}^n \int_{B(a_i, r_\epsilon)} |\nabla u_i|^2 \, dx \\
\geq \sum_{i=1}^n \frac{(N-2)^2}{4} \int_{B(a_i, r_\epsilon)} W_1 V_* |u_i|^2 \, dx \\
\geq \sum_{i=1}^n \left( \frac{(N-2)^2}{4} (1 - \delta(\epsilon)) \right) \int_{B(a_i, r_\epsilon)} V_* |u_i|^2 \, dx \\
= \sum_{i=1}^n \left( \frac{(N-2)^2}{4} \right) \int_{B(a_i, r_\epsilon)} V_* |u_i|^2 \, dx \\
= \left( \frac{(N-2)^2}{4} \right) \int_\Omega V_* |u|^2 \, dx.
\]

The proof of Proposition 4.4 is completed. \( \square \)

We end this paper by concluding a problem presented in [9].

**Corollary 4.5.** Let \( N \geq 3 \) and \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( a_1, \ldots, a_n \in \Omega \), \( n \geq 2 \). Then for the following optimization problem

\[
\mu_\Omega := \inf_{u \in D_{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega V_* |u|^2 \, dx},
\]

we have

\[
(4.7) \quad \frac{(N-2)^2}{4n} < \mu_\Omega \leq \frac{(N-2)^2}{4}.
\]

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