THREE-DIMENSIONAL LORENTZIAN HOMOGENEOUS RICCI SOLITONS

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Abstract. We study three-dimensional Lorentzian homogeneous Ricci solitons, proving the existence of shrinking, expanding and steady Ricci solitons. For all the non-trivial examples, the Ricci operator is not diagonalizable and has three equal eigenvalues.

1. Introduction

A Ricci soliton is a pseudo-Riemannian manifold \((M, g)\) which admits a smooth vector field \(X\) on \(M\) such that

\[ \mathcal{L}_X g + \text{Ric} = \lambda g, \]

where \(\mathcal{L}_X\) denotes the Lie derivative in the direction of \(X\), \(\text{Ric}\) denotes the Ricci tensor and \(\lambda\) is a real number. A Ricci soliton is said to be a shrinking, steady or expanding, respectively, if \(\lambda > 0\), \(\lambda = 0\) or \(\lambda < 0\).

The description of Ricci solitons can be regarded as a first step in understanding the Ricci flow, since they are the fixed points of the flow. Moreover, they are important in understanding singularities of the Ricci flow. Under suitable conditions, type I singularity models correspond to shrinking solitons, type II models correspond to steady Ricci solitons while type III models correspond to expanding Ricci solitons. We refer to [6] for a survey and further references on the geometry of Ricci solitons.

Theoretical physicists have also been looking into the equation of Ricci solitons in relation with String Theory. A seminal contribution in this direction is due to Friedan [11] (see also [1, 16] for a discussion of physical aspects of the Ricci flow). Although it was first introduced and studied in a Riemannian context, the Ricci soliton equation (1) is currently being investigated in pseudo-Riemannian settings, with special attention to the Lorentzian case [7, 19]. The Ricci soliton equation may also give some insight into the general study of Einstein field equations, of which (1) is a special case.

As proved in [8] (see also [13, 18]), three-dimensional Lie groups do not admit left-invariant Riemannian Ricci solitons. In this paper, we study the corresponding existence problem in Lorentzian signature. We shall conclude that the Lorentzian case is much richer, allowing the existence of expanding, steady and shrinking left-invariant Ricci solitons.

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Three-dimensional locally homogeneous Lorentzian manifolds are either locally symmetric or locally isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric. Moreover, three-dimensional locally symmetric Lorentzian manifolds which are not of constant sectional curvature, are either locally isometric to a Lorentzian product of a real line and a surface of constant Gauss curvature, or they are Walker manifolds with a two-step nilpotent Ricci operator \[3\].

It is clear that, in addition to Einstein spaces, products \(N^k(c) \times \mathbb{R}\) with \(N^k(c)\) of constant sectional curvature are Ricci solitons, both in the Riemannian and the Lorentzian case. In fact, it suffices to consider the Gaussian soliton on the \(\mathbb{R}\)-factor, scaling it by the constant value of the sectional curvature. An immediate calculation shows that the gradient of \(f(t) = \frac{1}{2}t^2\) defines a Ricci soliton on \(N^k(c) \times \mathbb{R}\) for \(\lambda = (k-1)c\). Ricci solitons on Walker manifolds with nilpotent Ricci operator are considered in Section \(3\) proving the existence of expanding, steady and shrinking locally symmetric Ricci solitons.

The existence of left-invariant Lorentzian Ricci solitons on Lie groups is analyzed in Section \(2\). The following result summarizes the classification of Lorentzian homogeneous Ricci solitons in dimension three:

**Theorem 1.** Let \(M\) be a three-dimensional simply connected, complete homogeneous manifold. \(M\) is a Ricci soliton if and only if \(M\) is one of the spaces listed below:

i) a space of constant curvature \(N^3(c)\);
ii) a Lorentzian product \(\mathbb{R} \times N^2(c)\);
iii) a symmetric Walker manifold;
iv) a unimodular Lie group \(G\) with one of the following Lie algebras:

\[
\begin{align*}
\{e_1, e_2\} &= \frac{1}{2}e_2 - (\beta - \frac{1}{2})e_3, \quad \{e_1, e_3\} = -(\beta + \frac{1}{2})e_2 - \frac{1}{2}e_3, \\
\{e_2, e_3\} &= \alpha e_1,
\end{align*}
\]

with either \(\alpha = 0\) or \(\alpha = \beta \neq 0\). If \(\alpha = 0\) then \(G = E(1, 1)\), while if \(\alpha = \beta \neq 0\) then \(G = (1, 2)\) or \(G = SL(2, \mathbb{R})\).

\[
\begin{align*}
\{e_1, e_2\} &= -\frac{1}{\sqrt{2}}e_1 - \alpha e_3, \quad \{e_1, e_3\} = -\frac{1}{\sqrt{2}}e_1 - \alpha e_2, \\
\{e_2, e_3\} &= \alpha e_1 + \frac{1}{\sqrt{2}}e_2 - \frac{1}{\sqrt{2}}e_3.
\end{align*}
\]

If \(\alpha = 0\) then \(G = E(1, 1)\), while if \(\alpha \neq 0\) then either \(G = (1, 2)\) or \(G = SL(2, \mathbb{R})\).

v) a non-unimodular Lie group \(G\) with Lie algebra given by

\[
\begin{align*}
\{e_1, e_2\} &= -\frac{1}{\sqrt{2}} \left( \alpha e_1 + \frac{1}{\sqrt{2}} \beta (e_2 + e_3) \right), \quad \{e_1, e_3\} = -\frac{1}{\sqrt{2}} \left( \alpha e_1 + \frac{1}{\sqrt{2}} \beta (e_2 + e_3) \right), \\
\{e_2, e_3\} &= \frac{1}{\sqrt{2}} \delta (e_2 + e_3).
\end{align*}
\]

In cases iv) and v), \(\{e_1, e_2, e_3\}\) is an orthonormal basis of the corresponding Lie algebra, of signature \((+ + -)\).

From now on, by a non-trivial Ricci soliton we shall mean a Ricci soliton which is neither Einstein nor a product \(\mathbb{R} \times N^k(c)\). The explicit description of three-dimensional non-trivial homogeneous Lorentzian Ricci solitons is given in Sections \(2\) and \(3\) where the causal character of vector fields defining these Ricci solitons is also discussed.
We briefly recall that the Ricci operator $\hat{\text{Ric}}$, being self-adjoint, is always diagonalizable in the Riemannian case, while at each point of a Lorentzian manifold four different cases can occur, known as Segre types. In dimension three, the possible cases are the following \[14\]:

(i) **Segre type** \{11, 1\}: $\hat{\text{Ric}}$ is symmetric and hence diagonalizable. The comma separates the spacelike and timelike eigenvectors. In the degenerate case, at least two of the Ricci eigenvalues coincide.

(ii) **Segre type** \{1$\bar{z}$\bar{z}\}: $\hat{\text{Ric}}$ has one real and two complex conjugate eigenvalues.

(II) **Segre type** \{21\}: $\hat{\text{Ric}}$ has two real eigenvalues (which coincide in the degenerate case), one of which has multiplicity two and each associated to a one-dimensional eigenspace.

(III) **Segre type** \{3\}: $\hat{\text{Ric}}$ has three equal eigenvalues, associated to a one-dimensional eigenspace.

Segre types of the Ricci operator of a three-dimensional homogeneous Lorentzian manifold were discussed in \[5\] (see also the description included in Sections 2 and 3). Taking into account the classification given in Theorem 1, we have at once the following

**Theorem 2.** A complete and simply connected three-dimensional homogeneous Lorentzian manifold is a non-trivial Ricci soliton if and only if the Ricci operator $\hat{\text{Ric}}$ is not diagonalizable and has exactly three equal eigenvalues, that is, $\hat{\text{Ric}}$ is either of Segre type \{3\} or of degenerate Segre type \{21\}.

### 2. Three-dimensional Homogeneous Lorentzian Manifolds

A connected, complete and simply connected three-dimensional homogeneous Lorentzian manifold is a Lie group \[3\]. For the sake of completeness we include a brief description of three-dimensional unimodular and non-unimodular Lie groups. Theorems 1 and 2 will follow from the subsequent analysis. Theorem 1 iv) and v) and Theorem 2 will follow from the subsequent analysis.

#### 2.1. Unimodular Lie groups

Let $\times$ denote the Lorentzian vector product on $\mathbb{R}^3$ induced by the product of the para-quaternions (i.e., $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature \((+ + -)\)). The Lie bracket $[,]$ defines the corresponding Lie algebra $\mathfrak{g}$, which is unimodular if and only if the endomorphism $L$ defined by $[Z, Y] = L(Z \times Y)$ is self-adjoint \[20\]. Considering the different Segre types of $L$, we have the following four classes of unimodular three-dimensional Lie algebras (we follow notation in \[14\]):

2.1.1. **Segre type** \{11, 1\}. If $L$ is diagonalizable with eigenvalues $\{\alpha, \beta, \gamma\}$ with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature \((+ + -)\), the corresponding Lie algebra is given by

\[(2) \quad (\mathfrak{g}_{11}): \quad [e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1.\]

Up to symmetries, the only nonvanishing components of the curvature tensor are given by

\[
\begin{align*}
R_{1221} &= \frac{1}{4} (\alpha^2 + \beta^2 - 3\gamma^2 - 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma), \\
R_{1313} &= \frac{1}{4} (\alpha^2 - 3\beta^2 + \gamma^2 + 2\alpha\beta - 2\alpha\gamma + 2\beta\gamma), \\
R_{2332} &= \frac{1}{4} (3\alpha^2 - \beta^2 - \gamma^2 - 2\alpha\beta - 2\alpha\gamma + 2\beta\gamma),
\end{align*}
\]
and the Ricci operator is diagonalizable (that is, of Segre type \{11, 1\}) with respect to the basis \{e_1, e_2, e_3\} with eigenvalues

\[
\lambda_1 = \frac{1}{2}((\beta - \gamma)^2 - \alpha^2), \quad \lambda_2 = \frac{1}{2}((\alpha - \gamma)^2 - \beta^2), \quad \lambda_3 = \frac{1}{2}((\alpha - \beta)^2 - \gamma^2).
\]

For an arbitrary vector \(X = \sum X_i e_i\), from equation (2) we get

\[
(\mathcal{L}_X g) = \begin{pmatrix}
0 & X_3(\alpha - \beta) & X_2(\gamma - \alpha) \\
X_3(\alpha - \beta) & 0 & X_1(\beta - \gamma) \\
X_2(\gamma - \alpha) & X_1(\beta - \gamma) & 0
\end{pmatrix}.
\]

Hence, by (1), there exist a Ricci soliton of this type if and only if the following system of equations is satisfied:

\[
\begin{align*}
(\beta - \gamma)^2 - \alpha^2 &= 2\lambda, \\
(\alpha - \gamma)^2 - \beta^2 &= 2\lambda, \\
(\alpha - \beta)^2 - \gamma^2 &= 2\lambda, \\
X_1(\beta - \gamma) &= 0, \\
X_2(\alpha - \gamma) &= 0, \\
X_3(\alpha - \beta) &= 0.
\end{align*}
\]

Now, from (3), it is clear that any solution of (4) gives rise to an Einstein metric. Therefore there are no homogeneous non-trivial Ricci solitons of Segre type \{11, 1\}.

2.1.2. Segre type \{1\bar{z}\bar{z}\}. Assume \(L\) has a complex eigenvalue. Then

\[
L = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \gamma & -\beta \\
0 & \beta & \gamma
\end{pmatrix}, \quad \beta \neq 0,
\]

with respect to an orthonormal basis \{e_1, e_2, e_3\} of signature (+ + -). The corresponding Lie algebra is given by

\[
(\mathfrak{g}_{11}) : \quad [e_1, e_2] = \beta e_2 - \gamma e_3, \quad [e_1, e_3] = -\gamma e_2 - \beta e_3, \quad [e_2, e_3] = \alpha e_1.
\]

The non-zero components of the curvature tensor (up to symmetries) are

\[
R_{1221} = R_{1313} = \frac{1}{4}(\alpha^2 + 4\beta^2), \quad R_{2332} = \frac{3}{4}\alpha^2 + \beta^2 - \alpha\gamma, \quad R_{1231} = \beta(\alpha - 2\gamma).
\]

The Ricci operator \(\hat{\text{Ric}}\), with respect to the basis \{e_1, e_2, e_3\}, is described as follows:

\[
\hat{\text{Ric}} = \begin{pmatrix}
-\frac{1}{2}(\alpha^2 + 4\beta^2) & 0 & 0 \\
0 & \frac{1}{2}\alpha(\alpha - 2\gamma) & -\beta(\alpha - 2\gamma) \\
0 & \beta(\alpha - 2\gamma) & \frac{1}{2}\alpha(\alpha - 2\gamma)
\end{pmatrix}, \quad \beta \neq 0.
\]

Hence, \(\hat{\text{Ric}}\) is of Segre type \{1\bar{z}\bar{z}\} if \(\alpha \neq 2\gamma\) and \{11, 1\} if \(\alpha = 2\gamma\). For \(X = \sum X_i e_i\), one has

\[
(\mathcal{L}_X g) = \begin{pmatrix}
0 & X_2\beta + X_3(\alpha - \gamma) & X_3\beta + X_2(\gamma - \alpha) \\
X_2\beta + X_3(\alpha - \gamma) & -2X_1\beta & 0 \\
X_3\beta + X_2(\gamma - \alpha) & 0 & -2X_1\beta
\end{pmatrix},
\]
and thus, we have a homogeneous Ricci soliton of Segre type \( \{1z\bar{z}\} \) if and only if

\[
\begin{align*}
\alpha^2 + 4\beta^2 &= -2\lambda, \\
\alpha^2 - 2\alpha\gamma - 4X_1\beta &= 2\lambda, \\
\alpha^2 - 2\alpha\gamma + 4X_1\beta &= 2\lambda, \\
X_3(\alpha - \gamma) + X_2\beta &= 0, \\
X_2(\alpha - \gamma) - X_3\beta &= 0, \\
\beta(\alpha - 2\gamma) &= 0.
\end{align*}
\]

(5)

Since \( \beta \neq 0 \), the last equation in (3) gives \( \alpha - 2\gamma = 0 \). Hence, the second and third equations simplify to \( -4X_1\beta = 2\lambda \) and \( 4X_1\beta = 2\lambda \), respectively, which imply \( X_1 = \lambda = 0 \). Finally, from the first equation one gets that there are no solutions of (3) with \( \beta \neq 0 \). Therefore there are no homogeneous Ricci solitons of Segre type \( \{1z\bar{z}\} \).

2.1.3. Segre type \( \{21\} \). Assume \( L \) has a double root of its minimal polynomial. Then, with respect to an orthonormal basis \( \{e_1, e_2, e_3\} \) of signature \((++-)\), one has

\[
L = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & \frac{1}{2} + \beta & -\frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} + \beta
\end{pmatrix}
\]

and the corresponding Lie algebra is given by

\[
(g_{II}) : [e_1, e_2] = \frac{1}{2}e_2 - (\beta - \frac{1}{2})e_3, \quad [e_1, e_3] = -(\beta + \frac{1}{2})e_2 - \frac{1}{2}e_3, \quad [e_2, e_3] = \alpha e_1.
\]

The non-zero components of the curvature tensor are given by

\[
R_{1221} = \frac{1}{4}\left(\alpha^2 - 2\alpha + 4\beta\right), \quad R_{1313} = \frac{1}{4}\left(\alpha^2 + 2\alpha - 4\beta\right),
\]

\[
R_{2332} = \frac{1}{4}\alpha(3\alpha - 4\beta), \quad R_{1231} = \frac{1}{4}\alpha - \beta.
\]

Hence the Ricci operator takes the form

\[
\hat{\text{Ric}} = \begin{pmatrix}
\frac{1}{2}\alpha^2 & 0 & 0 \\
0 & \frac{1}{2}(\alpha + 1)(\alpha - 2\beta) & -\frac{1}{2}\alpha + \beta \\
0 & \frac{1}{2}\alpha - \beta & \frac{1}{2}(\alpha - 1)(\alpha - 2\beta)
\end{pmatrix},
\]

with eigenvalues \( \lambda_1 = -\frac{1}{2}\alpha^2 \) and \( \lambda_2 = \lambda_3 = \frac{1}{2}\alpha(\alpha - 2\beta) \). Thus, \( \hat{\text{Ric}} \) is of Segre type \( \{21\} \), degenerate if and only if \( \alpha = 0 \) or \( \alpha = \beta \).

For a vector field \( X = \sum X_ie_i \), we get

\[
(L_Xg) = \begin{pmatrix}
0 & a_{12} & a_{13} \\
a_{12} & -X_1 & X_1 \\
a_{13} & X_1 & -X_1
\end{pmatrix},
\]

where \( a_{12} = \frac{1}{2}(X_2 + X_3(2\alpha - 2\beta - 1)) \) and \( a_{13} = \frac{1}{2}(X_3 + X_2(2\beta - 2\alpha - 1)) \). Necessary and sufficient conditions for the existence of a homogeneous Ricci soliton of Segre
type \{21\} are then given by

\[
\begin{align*}
\alpha^2 &= -2\lambda, \\
\alpha^2 - 2\alpha\beta + \alpha - 2\beta - 2X_1 &= 2\lambda, \\
\alpha^2 - 2\alpha\beta - \alpha + 2\beta + 2X_1 &= 2\lambda, \\
\alpha - 2\beta - 2X_1 &= 0, \\
(2\alpha - 2\beta)X_3 + X_2 - X_3 &= 0, \\
(2\alpha - 2\beta)X_2 + X_2 - X_3 &= 0.
\end{align*}
\]

From the second and forth equation in (7) one gets \(\alpha^2 - 2\alpha\beta - 2\lambda = 0\). Replacing into the first equation, we then obtain \(\alpha(\alpha - \beta) = 0\). Hence, either \(\alpha = 0 \neq \beta\) or \(\alpha = \beta \neq 0\). (We excluded the case \(\alpha = \beta = 0\), since by (6) this corresponds to a flat manifold.)

First case: \(\alpha = 0 \neq \beta\). From the first equation in (7) one gets \(\lambda = 0\), the last two equations give \(X_2 = X_3\) and the forth equation yields \(X_1 = -\beta\). Therefore, the (spacelike) vector field

\[
X = -\beta e_1
\]
defines a homogeneous (steady) Ricci soliton. By (7), the Ricci operator is two-step nilpotent but nonvanishing (since \(\beta \neq 0\)), that is, of degenerate Segre type \{21\} with eigenvalue equal to zero.

Second case: \(\alpha = \beta \neq 0\). In this case, one easily gets from (7) that \(\lambda = -\frac{1}{2}\beta^2\), that \(X_1 = -\frac{1}{2}\beta\) and that \(X_2 = X_3\); thus, there exist a one-parameter family of homogeneous expanding Ricci solitons, given by

\[
X = -\frac{1}{2}\beta e_1 + \delta e_2 + \delta e_3, \quad \delta \in \mathbb{R}.
\]

Note that the causality of \(X\) is again fixed and one can only find examples of solitons for \(X\) spacelike but not null or timelike. Since \(\alpha = \beta \neq 0\), (6) yields that the Ricci operator is of degenerate Segre type \{21\}, with one non-zero eigenvalue equal to \(-\frac{1}{2}\alpha^2\).

**Remark 3.** A Lie group of Segre type \{21\} with \(\alpha = 0\) or \(\alpha = \beta\) is locally symmetric if and only if \(\beta = 0\) (see [4]). This shows that previous examples are not locally symmetric.

2.1.4. Segre type \{3\}. Assume \(L\) has a triple root of its minimal polynomial. Then

\[
L = \begin{pmatrix}
\alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \alpha & 0 \\
\frac{1}{\sqrt{2}} & 0 & \alpha
\end{pmatrix}
\]

with respect to an orthonormal basis \(\{e_1, e_2, e_3\}\) of signature \((+ + -)\), and the corresponding Lie algebra is given by

\[
(\mathfrak{g}_{111}): \begin{cases}
[e_1, e_2] = \frac{1}{\sqrt{2}} e_1 - \alpha e_3, \\
[e_2, e_3] = \alpha e_1 + \frac{1}{\sqrt{2}} e_2 - \frac{1}{\sqrt{2}} e_3.
\end{cases}
\]

Hence the non-zero components of the curvature tensor (up to symmetries) are

\[
R_{1221} = \frac{1}{4}(\alpha^2 + 4), \quad R_{1331} = \frac{1}{4} - \frac{1}{4}\alpha^2, \quad R_{2323} = \frac{1}{4}\alpha^2, \\
R_{1231} = 1, \quad R_{1122} = R_{1323} = \frac{1}{\sqrt{2}} \alpha.
\]
The Ricci operator, expressed in terms of the basis \( \{ e_1, e_2, e_3 \} \), becomes

\[
\hat{\text{Ric}} = \begin{pmatrix}
-\frac{1}{2}\alpha^2 & -\frac{1}{\sqrt{2}}\alpha & -\frac{1}{\sqrt{2}}\alpha \\
-\frac{1}{\sqrt{2}}\alpha & -\frac{1}{2}(\alpha^2 + 2) & -1 \\
-\frac{1}{\sqrt{2}}\alpha & 1 & 1 - \frac{1}{2}\alpha^2 \\
\end{pmatrix},
\]

with a single eigenvalue \(-\frac{1}{2}\alpha^2\). If \( \alpha \neq 0 \), then \( \hat{\text{Ric}} \) is of Segre type \( \{3\} \), while \( \hat{\text{Ric}} \) is two-step nilpotent if \( \alpha = 0 \).

For a vector \( X = \sum X_ie_i \), the Lie derivative has the following expression

\[
(\mathcal{L}_X g) = \frac{1}{\sqrt{2}} \begin{pmatrix}
-2(X_2 + X_3) & X_1 & X_1 \\
X_1 & 2X_3 & X_3 - X_2 \\
X_1 & X_3 - X_2 & -2X_2 \\
\end{pmatrix}.
\]

Thus the Ricci soliton condition (1) on \( \mathfrak{g}_{III} \) gives rise to the following system:

\[
\begin{aligned}
\frac{\alpha^2}{2} + \sqrt{2}X_2 + \sqrt{2}X_3 &= -\lambda, \\
\frac{\alpha^2}{2} - \sqrt{2}X_3 + 1 &= -\lambda, \\
\frac{\alpha^2}{2} - \sqrt{2}X_2 - 1 &= -\lambda, \\
\frac{1}{\sqrt{2}}(X_1 - \alpha) &= 0, \\
X_2 - X_3 + \sqrt{2} &= 0.
\end{aligned}
\]

If we subtract half of the second and third equations to the first one in (10), we see that \( X_2 = -X_3 \) and therefore \( \lambda = -\frac{1}{2}\alpha^2 \). Moreover from the forth equation \( X_1 = \alpha \). Hence any Segre type \( \{3\} \) unimodular Lie group is a homogeneous Ricci soliton for

\[
X = \alpha e_1 - \frac{1}{\sqrt{2}} e_2 + \frac{1}{\sqrt{2}} e_3.
\]

**Remark 4.** A vector field \( X \) defining a homogeneous Ricci soliton on \( \mathfrak{g}_{III} \) satisfies \( \langle X, X \rangle = \alpha^2 \) and thus it is either spacelike or null. Correspondingly, the homogeneous Ricci soliton is either expanding or steady. Note also that Segre type \( \{3\} \) unimodular Lie groups are never symmetric (see also [4]).

Previous analysis proves iv) of Theorem 1. Lie groups having unimodular Lie algebras compatible with the Ricci soliton equation (1), and listed in Theorem 1, can be deduced from [20] (see also [3]). The results we proved are summarized in the following

**Theorem 5.** The following are all non-trivial homogeneous Lorentzian Ricci solitons realized as unimodular Lorentzian Lie groups \( G \):

a) \( G = E(1,1) \), with Lie algebra as in Theorem 1–(iv.1), \( \alpha = 0 \neq \beta \). The homogeneous Ricci soliton is steady and defined by a spacelike vector field [5].

b) \( G = O(1,2) \) or \( SL(2, \mathbb{R}) \), with Lie algebra as in Theorem 1–(iv.1), \( \alpha = \beta \neq 0 \). The homogeneous Ricci soliton is expanding and defined by a spacelike vector field [4].

c) \( G = O(1,2) \) or \( SL(2, \mathbb{R}) \), with Lie algebra as in Theorem 1–(iv.2), \( \alpha \neq 0 \). The homogeneous Ricci soliton is expanding and defined by a spacelike vector field [11].

d) \( G = E(1,1) \), with Lie algebra as in Theorem 1–(iv.2), \( \alpha = 0 \). The homogeneous Ricci soliton is steady and defined by a null vector field [11].
Remark 6. Ricci solitons listed in Theorem [5] are locally conformally flat if and only if they correspond to \( G = E(1, 1) \), which is not locally symmetric.

2.2. Non-unimodular Lie groups. Following [10], non-unimodular Lorentzian Lie algebras of non-constant sectional curvature are given, with respect to a suitable basis \( \{e_1, e_2, e_3\} \), by

\[
(12) \quad (\mathfrak{g}_{IV}): \quad [e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2,
\]

where \( \alpha + \delta \neq 0 \) and one of the following holds:

IV.1 \( \{e_1, e_2, e_3\} \) is orthonormal with \( \langle e_1, e_1 \rangle = -\langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = -1 \) and the structure constants satisfy \( \alpha \gamma - \beta \delta = 0 \).

IV.2 \( \{e_1, e_2, e_3\} \) is orthonormal with \( \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = 1 \) and the structure constants satisfy \( \alpha \gamma + \beta \delta = 0 \).

IV.3 \( \{e_1, e_2, e_3\} \) is a pseudo-orthonormal basis with

\[
\langle \cdot, \cdot \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}
\]

and the structure constants satisfy \( \alpha \gamma = 0 \).

We analyze the three cases separately.

2.2.1. Type IV.1. The non-zero components of the curvature tensor are given by

\[
\begin{align*}
R_{1212} &= \frac{1}{4} (\beta^2 + \gamma^2 + 4\alpha \delta - 2\beta \gamma), \\
R_{1313} &= \frac{1}{4} (4\alpha^2 - 3\beta^2 + \gamma^2 + 2\beta \gamma), \\
R_{2332} &= \frac{1}{4} (\beta^2 - 3\gamma^2 + 4\delta^2 + 2\beta \gamma).
\end{align*}
\]

Hence the Ricci operator is diagonalizable with eigenvalues

\[
\begin{align*}
\lambda_1 &= \frac{1}{2}(\beta^2 - \gamma^2 - 2\alpha(\alpha + \delta)), \\
\lambda_2 &= \frac{1}{2}(\gamma^2 - \beta^2 - 2\delta(\alpha + \delta)), \\
\lambda_3 &= \frac{1}{2}((\beta - \gamma)^2 - 2(\alpha^2 + \delta^2)).
\end{align*}
\]

The Lie derivative of the metric for an arbitrary vector \( X = \sum X_i e_i \) is given by

\[
(L_X g) = \begin{pmatrix} -2\alpha X_3 & X_3(\beta - \gamma) & X_1 \alpha + X_2 \gamma \\
X_3(\beta - \gamma) & 2X_3 \delta & -X_1 \beta - X_2 \delta \\
X_1 \alpha + X_2 \gamma & -X_1 \beta - X_2 \delta & 0
\end{pmatrix},
\]

and thus necessary and sufficient conditions for the existence of a homogeneous Ricci soliton [11] on \( \mathfrak{g}_{IV.1} \) are given by

\[
\begin{align*}
\beta^2 - \gamma^2 - 2\alpha(\alpha + \delta) + 4X_3 \alpha &= 2\lambda, \\
\gamma^2 - \beta^2 - 2\delta(\alpha + \delta) + 4X_3 \delta &= 2\lambda, \\
(\beta - \gamma)^2 - 2(\alpha^2 + \delta^2) &= 2\lambda, \\
X_1 \alpha + X_2 \gamma &= 0, \\
X_1 \beta + X_2 \delta &= 0, X_3(\beta - \gamma) = 0.
\end{align*}
\]

If \( X_3 = 0 \), then the first three equations in (13) imply \( \lambda_1 = \lambda_2 = \lambda_3 \). On the other hand, if \( X_3 \neq 0 \), then the last equation in (13) gives \( \beta = \gamma \). Since \( \alpha \gamma - \beta \delta = 0 \), from \( \alpha + \delta \neq 0 \) and (13) we obtain \( \alpha = \delta \) and hence \( \lambda_1 = \lambda_2 = \lambda_3 \). Thus, all solutions of (13) are Einstein, and hence of constant sectional curvature.
2.2.2. Type IV.2. Assume the non-unimodular Lie algebra \( \mathfrak{g}_{IV} \) has a basis as given in IV.2. Then, a straightforward calculation shows that the non-zero components of the curvature tensor are given by
\[
\begin{align*}
R_{1212} &= \alpha\delta - \frac{1}{3}(\beta + \gamma)^2, \\
R_{1331} &= \frac{1}{4}(4\alpha^2 + 3\beta^2 - \gamma^2 + 2\beta\gamma), \\
R_{2323} &= \frac{1}{4}(\beta^2 - 3\gamma^2 - 4\delta^2 - 2\beta\gamma).
\end{align*}
\]
Therefore, the Ricci operator is diagonalizable with eigenvalues
\[
\lambda_1 = \frac{1}{2}(\beta^2 - \gamma^2 + 2\alpha(\alpha + \delta)), \\
\lambda_2 = \frac{1}{2}(\gamma^2 - \beta^2 + 2\delta(\alpha + \delta)), \\
\lambda_3 = \frac{1}{2}((\beta + \gamma)^2 + 2(\alpha^2 + \delta^2)).
\]
A straightforward calculation from (12), using the fact that the structure constants satisfy \( \alpha\gamma + \beta\delta = 0 \) and \( \alpha + \delta \neq 0 \), shows that the Lie derivative of the metric with respect to a vector \( X = \sum X_i e_i \) is given by
\[
(\mathcal{L}_X g) = \begin{pmatrix}
2X_3\alpha & X_3(\beta + \gamma) & -X_1\alpha - X_2\gamma \\
X_3(\beta + \gamma) & 2X_3\delta & -X_1\beta - X_2\delta \\
-X_1\alpha - X_2\gamma & -X_1\beta - X_2\delta & 0
\end{pmatrix}.
\]
Then a Ricci soliton must satisfy
\[
\begin{align*}
\beta^2 - \gamma^2 + 2\alpha(\alpha + \delta) + 4X_3\alpha &= 2\lambda, \\
\gamma^2 - \beta^2 + 2\delta(\alpha + \delta) + 4X_3\delta &= 2\lambda, \\
(\beta + \gamma)^2 + 2(\alpha^2 + \delta^2) &= 2\lambda, \\
X_1\alpha + X_2\gamma &= 0, \\
X_1\beta + X_2\delta &= 0, \\
X_3(\beta + \gamma) &= 0.
\end{align*}
\]
A similar analysis to that developed for type IV.1 shows that homogeneous Ricci solitons of type IV.2 necessarily are of constant sectional curvature.

2.2.3. Type IV.3. Let now \( \mathfrak{g}_{IV} \) admit a pseudo-orthonormal basis as in IV.3. We then consider the orthonormal basis
\[
\tilde{e}_1 := e_1, \quad \tilde{e}_2 := \frac{1}{\sqrt{2}}(e_2 - e_3), \quad \tilde{e}_3 := \frac{1}{\sqrt{2}}(e_2 + e_3),
\]
with signature \((+ + -)\). Then the non-zero components of the curvature tensor are given by
\[
\begin{align*}
R_{1212} &= \frac{1}{4}(2\alpha\delta - 2\alpha^2 - \gamma(2\beta + \gamma)), \\
R_{1313} &= \frac{1}{4}(2\alpha\delta - 2\alpha^2 + \gamma(\gamma - 2\beta)), \\
R_{2323} &= -\frac{3}{4}\gamma^2.
\end{align*}
\]
The Ricci operator in the new basis \( \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \) becomes
\[
\hat{\text{Ric}} = \begin{pmatrix}
-\frac{1}{2}\gamma^2 & 0 & 0 \\
0 & \frac{1}{2}(\alpha(\alpha - \delta) + \gamma(\gamma - \beta)) & \frac{1}{2}(\alpha(\alpha - \delta) + \beta(\beta + \gamma)) \\
0 & -\frac{1}{2}(\alpha(\alpha - \delta) + \beta\gamma) & \frac{1}{2}(\alpha(\alpha - \delta) + \gamma(\beta + \gamma))
\end{pmatrix},
\]
which has eigenvalues \( \lambda_1 = -\frac{1}{2}\gamma^2 \) and \( \lambda_2 = \lambda_3 = \frac{1}{2}\gamma^2 \). Thus, \( \hat{\text{Ric}} \) is of Segre type \( \{21\} \).
For an arbitrary vector $X = \sum X_i \hat{e}_i$, the Lie derivative of the metric becomes

$$
(L_X g) = \begin{pmatrix}
\sqrt{2}\alpha (X_3 - X_2) \\
a_{12} X_1 \beta + \sqrt{2} X_3 \delta \\
 \alpha \delta - \beta \gamma + 2 X_1 \beta + 2 \sqrt{2} X_3 \delta = 2 \lambda, \\
2 \gamma X_2 - X_2 \beta + X_3 \beta = 0, \\
\beta \gamma - X_2 \beta - \sqrt{2} X_2 \delta - \sqrt{2} X_3 \delta = 0,
\end{pmatrix}
$$

where

$$
a_{12} = \frac{1}{2} \left( X_3 (\beta + 2 \gamma) + \sqrt{2} X_1 \alpha - X_2 \beta \right),
\alpha \delta - \beta \gamma + 2 X_1 \beta + 2 \sqrt{2} X_3 \delta = 2 \lambda,
\alpha = 0, \beta = 0, \gamma = 0 \text{ separately. Note that if } \alpha = \gamma = 0, \text{ then the metric } g_{IV} \text{ is flat.}
$$

Assume first that $\alpha = 0 \neq \gamma$. Then, (1) holds if and only if

$$
\begin{cases}
\gamma^2 - 2 \lambda, \\
\gamma^2 - \beta \gamma + 2 X_1 \beta + 2 \sqrt{2} X_3 \delta = 2 \lambda, \\
\gamma^2 - \beta \gamma - 2 X_1 \beta - 2 \sqrt{2} X_2 \delta = 2 \lambda, \\
-2 \gamma X_3 + X_2 \beta - X_3 \beta = 0, \\
2 \gamma X_2 - X_2 \beta + X_3 \beta = 0, \\
\beta \gamma - X_2 \beta - \sqrt{2} X_2 \delta - \sqrt{2} X_3 \delta = 0,
\end{cases}
$$

for a vector $X = \sum X_i \hat{e}_i$. From the forth and fifth equation in (16) we get $X_2 = X_3$, which implies, using the second and third equations, that (16) admits no solutions.

Now assume now $\alpha \neq 0 = \gamma$. Then, (11) reduces to the following system of equations

$$
\begin{cases}
2 \sqrt{2} \alpha (X_2 - X_3) = -2 \lambda, \\
\alpha \delta - \alpha^2 + 2 X_1 \beta + 2 \sqrt{2} X_3 \delta = 2 \lambda, \\
\alpha^2 - \alpha \delta - 2 X_1 \beta - 2 \sqrt{2} X_3 \delta = 2 \lambda, \\
\sqrt{2} X_1 \alpha - X_2 \beta + X_3 \beta = 0, \\
\alpha^2 - \alpha \delta - 2 X_1 \beta = \sqrt{2} \delta (X_2 + X_3),
\end{cases}
$$

for a vector $X = \sum X_i \hat{e}_i$. We subtract the third equation to the second one and conclude, using the first equation, that either $2 \alpha = \delta$ or $X_2 = X_3$. We analyze both cases separately.

Set first $\alpha = \frac{1}{2} \delta \neq 0$. Then there exists homogeneous Ricci solitons for

$$
X = -\frac{2 \beta \lambda}{\delta^2} \hat{e}_1 - \frac{\delta^4 + 8(\delta^2 - 2 \beta^2) \lambda}{8 \sqrt{2} \delta^3} \hat{e}_2 - \frac{\delta^4 - 8(\delta^2 + 2 \beta^2) \lambda}{8 \sqrt{2} \delta^3} \hat{e}_3.
$$

Note that the corresponding solitons may be expanding, steady or shrinking depending on the value of $\lambda$, which can be chosen with absolute freedom.

Set now $X_2 = X_3$. Then necessarily $\lambda = 0$ and $X_1 = 0$. The remaining equation $\alpha^2 - \alpha \delta - 2 \sqrt{2} X_2 \delta = 0$ in (16) gives rise to homogeneous steady Ricci solitons for

$$
X = \frac{\alpha^2 - \alpha \delta}{2 \sqrt{2} \delta} (\hat{e}_2 + \hat{e}_3).
$$

In all cases above the Ricci operator is two-step nilpotent and the metric is non-symmetric whenever $\alpha \delta (\alpha - \delta) \neq 0$. Furthermore, for the particular choice $\delta = 0 \neq \alpha$ the resulting metric is symmetric but not of constant curvature [4].

The results of this subsection prove case v) of Theorem 4 and are summarized in the following
Theorem 7. A non-unimodular Lie group $G$ equipped with a left-invariant Lorentzian metric is a non-trivial homogeneous Ricci soliton if and only if its non-unimodular Lie algebra $\mathfrak{g}_{IV}$ satisfies $\alpha \neq 0 = \gamma$.

Steady Ricci solitons, defined by null vector fields (18), exist for any choice of $\alpha \neq 0$, $\beta$, $\delta$.

In the special case $\delta = 2\alpha$, there exist expanding, steady and shrinking Ricci solitons, defined by vector fields (17), whose causal character depends on $\lambda$.

Remark 8. Nontrivial Ricci solitons in Theorem 7 are locally conformally flat if and only if $\gamma = \beta = 0$, in which case they are not locally symmetric. Therefore, Theorem 7 provides examples of complete locally conformally flat expanding, steady and shrinking Ricci solitons.

3. Three-dimensional Walker manifolds

We now consider three-dimensional Lorentzian manifolds $(M, g)$ admitting a parallel null vector field $\mathcal{U}$. We refer to [2, 9] and references therein for more information on Walker manifolds. It has been shown by Walker [21] that there exist adapted coordinates $(t, x, y)$ where the Lorentzian metric tensor expresses as

\begin{equation}
\begin{pmatrix}
0 & 0 & 1 \\
0 & \varepsilon & 0 \\
1 & 0 & f(x, y)
\end{pmatrix},
\end{equation}

for some function $f(x, y)$, where $\varepsilon = \pm 1$ and the parallel null vector field is $\mathcal{U} = \frac{\partial}{\partial t}$. Then, the associated Levi-Civita connection is described by

\begin{equation}
\nabla_{\partial_t} \partial_y = \frac{1}{2} f_x \partial_t, \quad \nabla_{\partial_t} \partial_y = \frac{1}{2} f_y \partial_t - \frac{1}{2\varepsilon} f_x \partial_x.
\end{equation}

As shown in [9], the Ricci tensor $\text{Ric}$ and the Ricci operator $\hat{\text{Ric}}$ of a metric (19), expressed in the coordinate basis, take the form

\begin{equation}
\text{Ric} = -\frac{1}{2\varepsilon} f_{xx} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \hat{\text{Ric}} = -\frac{1}{2\varepsilon} f_{xx} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{equation}

Hence, if $f_{xx} = 0$, then the metric (19) is flat, while for $f_{xx} \neq 0$ the Ricci operator $\hat{\text{Ric}}$ is two-step nilpotent.

Next, let $X = (A(t, x, y), B(t, x, y), C(t, x, y))$ be an arbitrary vector field on $M$. A straightforward calculation from (20) shows that the Lie derivative of the metric $(\mathcal{L}_X g)$ expresses in the coordinate basis as follows:

\begin{equation}
\mathcal{L}_X g = \begin{pmatrix}
2C_t & \varepsilon B_t + C_x & A_t + C_y + fC_t \\
\varepsilon B_t + C_x & 2\varepsilon B_x & A_x + \varepsilon B_y + fC_x \\
A_t + C_y + fC_t & A_x + \varepsilon B_y + fC_x & Bf + 2(A_y + fC_y)
\end{pmatrix}.
\end{equation}
Now, from (21) and (22) we obtain the following necessary and sufficient conditions for a strict Walker metric (19) to be a Ricci soliton:

\[
\begin{align*}
2C_t &= 0, \\
C_x + \varepsilon B_t &= 0, \\
C_y + A_t + fC_t &= \lambda, \\
2B_x &= \lambda, \\
A_x + \varepsilon B_y + fC_x &= 0, \\
2A_y + f_x B + f_y C - \frac{1}{2\varepsilon}f_{xx} &= f(\lambda - 2C_y).
\end{align*}
\]

(23)

The first equation in (23) gives \(C = C(x, y)\) and simplifies the third one. Since \(C\) does not depend on \(t\), we can easily integrate the second and third equations in (23) and get

\[
A = (\lambda - C_y)t + G(x, y), \quad B = -\frac{C_x t}{\varepsilon} + H(x, y).
\]

Therefore, the fourth equation in (23) now gives

\[
-2tC_{xx} + 2\varepsilon H_x = \varepsilon \lambda.
\]

(24)

Since (24) must hold for any value of \(t\), it implies at once \(C_{xx} = 0\) and \(2H_x = \lambda\). By integration we then have \(C = u(y)x + v(y)\) and \(H = \frac{1}{2}\lambda x + w(y)\). Then the fifth equation in (23) becomes

\[
fu(y) - 2tu'(y) + \varepsilon w'(y) + G_x = 0
\]

from where it follows that \(u(y)\) is constant: \(u(y) = \alpha\). Then, system (23) now reduces to

\[
\begin{align*}
\alpha f + \varepsilon w'(y) + G_x &= 0, \\
2f v'(y) - \lambda f - 2tv''(y) + f_y (\alpha x + v(y)) + 2G_y + f_x (w(y) + \frac{1}{2}x - \varepsilon \alpha t) &= \frac{1}{2\varepsilon} f_{xx}.
\end{align*}
\]

(25)

The second equation in (25) holds for any value of \(t\). Hence \(\frac{\alpha f}{2} = -2v''(y)\) and thus \(\alpha f_{xx} = 0\). Since \(f_{xx} = 0\) if and only if the Walker metric is flat, we assume \(\alpha = 0\) and (25) reduces to

\[
\begin{align*}
\varepsilon w'(y) + G_x &= 0, \\
2f v'(y) - \lambda f - 2tv''(y) + f_y v(y) + 2G_y + f_x (w(y) + \frac{1}{2}x) &= \frac{1}{2\varepsilon} f_{xx}.
\end{align*}
\]

(26)

The first equation in (26) gives \(G(x, y) = -\varepsilon x w'(y) + \mu(y)\). Then, since the second equation in (26) must be independent of \(t\), one gets \(v''(y) = 0\) and hence \(v(t) = \beta y + \gamma\).

Finally we conclude that there exist nontrivial Ricci solitons given by Walker metrics (19) if and only if the vector field \(X\) takes the form

\[
X(t, x, y) = \left( t(\lambda - \beta) - \varepsilon x w'(y) + \mu(y), \frac{1}{2} \lambda x + w(y), \beta y + \gamma \right),
\]

for some real constants \(\beta, \gamma\) and smooth functions \(w(y)\) and \(\mu(y)\), satisfying the partial differential equation

\[
2\beta f - \lambda f + 2u'(y) - 2\varepsilon x w''(y) + f_y (\beta y + \gamma) + f_x (\frac{1}{2} x + w(y)) = \frac{1}{2\varepsilon} f_{xx}.
\]

(27)

One can not expect the partial differential equation (27) to admit solutions in general. We now turn our attention to the special case when the Walker metric is
locally symmetric. Locally symmetric Walker metrics \[19\] are characterized by the fact that their defining function \(f(x, y)\) is given by (see \[2, 9\])
\[
(28) \quad f(x, y) = x^2\kappa + xP(y) + Q(y),
\]
for arbitrary functions \(P\) and \(Q\), and constant \(\kappa\) which vanishes if and only if the metric is flat. When \(f\) satisfies \[25\], a straightforward calculation shows that equation \[27\] becomes
\[
(29) \quad 0 = 2x^2\beta + x(2\beta P(y) - \frac{1}{2}\lambda P(y) + 2\kappa w(y) + (\beta y + \gamma)P'(y) - 2\varepsilon w''(y) - \frac{2}{2} + 2\beta Q(y) - \lambda Q(y) + P(y)w(y) + (\beta y + \gamma)Q'(y) + 2\mu'(y).
\]
But \(29\) must hold for all values of \(x\). Therefore, it gives \(\beta = 0\) (excluding the flat case \(\kappa = 0\)) and reduces to the system
\[
(30) \quad \begin{cases} 2\varepsilon w''(y) - 2\kappa w(y) = \gamma P'(y) - \frac{1}{2}\lambda P(y), \\ 2\mu'(y) = \frac{\varepsilon}{2} - P(y)w(y) + \lambda Q(y) - \gamma Q'(y). \end{cases}
\]
The second equation in \(30\), by direct integration, permits to express \(\mu(y)\) in terms of \(w(y)\) and \(P(y), Q(y)\). The first equation in \(30\) is a second order linear ordinary differential equation for \(w(y)\), with constant coefficients, determined by the smooth function \(\gamma P'(y) - \frac{1}{2}P(y)\). A standard theorem ensures the existence of solutions for such an equation. Therefore, we proved the following

**Theorem 9.** Any three-dimensional symmetric Walker metric \[19\] is a Ricci solit- ton, which can be expanding, steady or shrinking and is defined by vector fields
\[
(31) \quad X(t, x, y) = \left(\lambda t - \varepsilon x w'(y) + \mu(y), \frac{1}{2}\lambda x + w(y), \gamma\right),
\]
where \(\lambda\) and \(\gamma\) are real constants and the functions \(w(y)\) and \(\mu(y)\) are arbitrary solutions of \(30\). In general, the causal character of \(X\) may vary with the point.

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