LIFTING FIXED POINTS
OF COMPLETELY POSITIVE SEMIGROUPS

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Abstract. Let \( \{ \phi_s \}_{s \in S} \) be a commutative semigroup of completely positive, contractive, and weak*-continuous linear maps acting on a von Neumann algebra \( N \). Assume there exists a semigroup \( \{ \alpha_s \}_{s \in S} \) of weak*-continuous \(*\)-endomorphisms of some larger von Neumann algebra \( M \supset N \) and a projection \( p \in M \) with \( N = pMp \) such that \( \alpha_s(1-p) \leq 1-p \) for every \( s \in S \) and \( \phi_s(y) = p\alpha_s(y)p \) for all \( y \in N \). If \( \inf_{s \in S} \alpha_s(1-p) = 0 \) then we show that the map \( E : M \to N \) defined by \( E(x) = pxp \) for \( x \in M \) induces a complete isometry between the fixed point spaces of \( \{ \alpha_s \}_{s \in S} \) and \( \{ \phi_s \}_{s \in S} \).

Let \((S, +, 0)\) be a commutative semigroup with unit 0. Consider the partial pre-order on \( S \) induced by the semigroup structure as follows. If \( s, t \in S \) then \( s \leq t \) if and only if there exists \( r \in S \) such that \( s + r = t \). If \( X \) is a Hausdorff topological space and \( f : S \to X \) is a function, then \( \lim_{s \in S} f(s) \) denotes its limit along the directed set \((S, \leq)\), whenever this limit exists.

Let \( M \) be a von Neumann algebra. Let \( CP(M) \) denote the semigroup of all completely positive, contractive and weak*-continuous linear maps \( \beta : M \to M \). Let also \( End(M) \) be the semigroup of all weak*-continuous \(*\)-endomorphisms of \( M \). A family \( \{ \beta_s \}_{s \in S} \subset CP(M) \) is called a semigroup if the map \( s \mapsto \beta_s \) is a unital homomorphism of semigroups from \( S \) into \( CP(M) \).

Suppose now that \( \{ \alpha_s \}_{s \in S} \subset End(M) \) is a semigroup. Let \( p \) be an orthogonal projection in \( M \) such that
\[
\alpha_s(1-p) \leq 1-p \quad \forall s \in S.
\]
Then one can define, for every \( s \in S \), a completely positive mapping on the von Neumann algebra \( N = pMp \) as follows:
\[
\phi_s(x) = p\alpha_s(x)p \quad \forall x \in N.
\]
It is clear that \( \{ \phi_s \}_{s \in S} \subset CP(N) \). A short calculation shows that
\[
\phi_s(px) = p\alpha_s(x)p \quad \forall x \in M,
\]
and using this, one can show that \( \{ \phi_s \}_{s \in S} \) is a semigroup. According to the terminology used in Chapter 8 of [1], where this construction is given for one-parameter semigroups, \( \{ \alpha_s \}_{s \in S} \) is a dilation of \( \{ \phi_s \}_{s \in S} \) and \( p \) is a co-invariant projection for \( \{ \alpha_s \}_{s \in S} \).

We shall prove the following result, which shows that, under a suitable minimality condition, the fixed point spaces of \( \{ \alpha_s \}_{s \in S} \) and \( \{ \phi_s \}_{s \in S} \) are completely isometric. We point out that the minimality condition is always satisfied by the minimal E-dilation of a CP-semigroup as constructed in [1].

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Then there exists $z$ the sense that it admits invariant means.
Since $\alpha$ is a state on the von Neumann algebra $\mathcal{M}$, it follows that $\mathcal{M}$ is a menable, in
This means that $\mathcal{M}$ is a *-homomorphism from $\mathcal{M}$ onto $\mathcal{M}_\alpha$ such that $(\pi \circ \mathcal{M})(x) = x$ for all $x \in \mathcal{M}_\alpha$.

Let $E : \mathcal{M} \to \mathcal{M}$ be defined by

Then the following hold true:

1. For each $y \in C^*(\mathcal{N})$ there exists the limit (in the strong operator topology)
   $\pi(y) = \lim_{s \in S} \alpha_s(y)$
   and the map $y \mapsto \pi(y)$ is a *-homomorphism from $C^*(\mathcal{N})$ onto $\mathcal{M}_\alpha$ such
   that $(\pi \circ E)(x) = x$ for all $x \in \mathcal{M}_\alpha$.
2. $E$ induces a complete isometry between $\mathcal{M}_\alpha$ and $\mathcal{N}$.
3. For each $y \in C^*(\mathcal{N})$ there exists the limit
   $\Phi(y) = \lim_{s \in S} \phi_s(y)$
   and the map $y \mapsto \Phi(y)$ is completely positive, idempotent, $\text{Ran}(\Phi) = \mathcal{N}$,
   and $E \circ \pi = \Phi$ on $C^*(\mathcal{N})$.

Proof. First, we show that $E(\mathcal{M}_\alpha) = \mathcal{N}$. It is clear that $E(\mathcal{M}_\alpha) \subset \mathcal{N}$. Let $\mu$ be
an invariant mean on $S$. This means that $\mu$ is a state on the von Neumann algebra $\ell^\infty(S)$ of all complex-valued bounded functions on $S$ that remains invariant under translations. It is well known [5] that any commutative semigroup is amenable, in
the sense that it admits invariant means.

Let $y \in \mathcal{N}$. For each $\gamma$ in the predual $\mathcal{M}_s$ of $\mathcal{M}$, let $f_\gamma \in \ell^\infty(S)$ be defined by

Then there exists $z \in \mathcal{M}$ such that

$$(z, \gamma) = \mu(f_\gamma) \quad \forall \gamma \in \mathcal{M}_s.$$ 

Since $\alpha_s$ are weak * continuous and $\{\alpha_s\}_{s \in S}$ is a semigroup, it follows that $z \in \mathcal{M}_\alpha$.
Moreover $pwp = y$ and this shows that $E(\mathcal{M}_\alpha) = \mathcal{N}$.

In order to go further, we need to use the minimality assumption on $\{\alpha_s\}_{s \in S}$.
Suppose now that $w \in \mathcal{M}_\alpha$. Since $\lim_{t \in S} \alpha_t(1 - p) = 0$ we see that

$$\lim_{s \in S} \alpha_s(pwp) = w.$$ 

Let $\{\phi_s\}_{s \in S} \subset \mathcal{M}_\alpha$ be the compression of $\{\alpha_s\}_{s \in S}$ to $N = p\mathcal{M}p$ defined by

$\phi_s(x) = p\alpha_s(x)p \quad \forall x \in N$.

Let $M^\alpha = \{x \in M : \alpha_t(x) = x, \forall t \in S\}$
and

$N^\phi = \{x \in N : \phi_t(x) = x, \forall t \in S\}$
and let $C^*(\mathcal{N})$ be the $C^*$-subalgebra of $N$ generated by $N^\phi$. Let $E : \mathcal{M} \to \mathcal{M}$ be defined by

$E(x) = pwp \quad \forall x \in \mathcal{M}$. 

Since \( E(M^\alpha) = N^\phi \) it follows that the limit
\[
\pi(y) = \lim_{s \in S} \alpha_s(y)
\]
extists for every \( y \in N^\phi \) and that \( \pi \circ E = id \) on \( M^\alpha \). In particular \( E \) is completely isometric on \( M^\alpha \). All the other assertions are straightforward consequences of what we have already proved.

\[
\square
\]

This result and its proof provide, in particular, an alternate and simplified approach to the lifting theorem for fixed points of completely positive maps from [6].

In the case when \( S \) is either a commutative, countable and cancellative semigroup or \( S = \mathbb{R}^d_+ \) for some \( d \geq 1 \), and \( \{\alpha_s\}_{s \in S} \) are unit preserving, part 2 of Theorem 1 follows directly from Proposition 4.4 together with Theorem 4.5 from [4]. In the case when \( \{\phi_s\}_{s \in S} \) is the semigroup induced by the unilateral shift on the Hardy space \( H^2 \), the existence of the limit in part 3 of Theorem 1 is proved in [2].

We close with the following result which shows that part 3 of the previous theorem holds true even without assuming the existence of a dilation.

**Theorem 2.** Let \( N \subset B(H) \) be a von Neumann algebra on some Hilbert space \( H \). Let \( (S,+,0) \) be a commutative semigroup with unit. Let \( \{\phi_s\}_{s \in S} \) be a semigroup of completely positive, contractive and weak*-continuous linear maps on \( N \). Let
\[
N^\phi = \{ x \in N : \phi_t(x) = x, \forall t \in S \}
\]
and let \( C^*(N^\phi) \) be the \( C^* \)-subalgebra of \( N \) generated by \( N^\phi \). Then for each \( y \in C^*(N^\phi) \) there exists the strong-operator limit
\[
\Phi(y) = \lim_{s \in S} \phi_s(y)
\]
and the map \( y \mapsto \Phi(y) \) is completely positive, contractive, idempotent, and moreover \( \text{Ran}(\Phi) = N^\phi \).

**Proof.** Let \( \mu \) be an invariant mean on \( S \). Let \( y \in N \). For each \( \gamma \) in the predual \( N_\gamma \) of \( N \), let \( f_\gamma \in \ell^\infty(S) \) be defined by
\[
f_\gamma(s) = (\phi_s(y), \gamma) \quad s \in S.
\]
Then there exists \( z \in N \) such that
\[
(z, \gamma) = \mu(f_\gamma) \quad \forall \gamma \in N_\gamma.
\]
Since \( \{\phi_s\}_{s \in S} \) are weak* continuous and \( \{\phi_s\}_{s \in S} \) is a semigroup, it follows that \( z \in N^\phi \). Let us denote \( z = \rho(y) \). The mapping \( \rho : N \rightarrow N \) is completely positive, contractive, idempotent, and \( \text{Ran}(\rho) = N^\phi \). Moreover
\[
\phi_s \circ \rho = \rho \circ \phi_s = \rho.
\]
Let \( \Phi : C^*(N^\phi) \rightarrow N^\phi \) be the restriction of \( \rho \) to \( C^*(N^\phi) \). A well known result from [3] shows that
\[
\Phi(\Phi(x)y) = \Phi(\Phi(x)\Phi(y)) \quad \forall x, y \in C^*(N^\phi).
\]
This easily implies that \( \ker(\Phi) \) is the closed left ideal in \( C^*(N^\phi) \) generated by all the operators of the form \( xy - \Phi(xy) \) with \( x, y \in N^\phi \). Moreover by polarization we see that \( \ker(\Phi) \) is the closed ideal of \( C^*(N^\phi) \) generated by all the operators of the form \( x^* x - \Phi(x^* x) \) with \( x \in N^\phi \).
Let \( x \in N^\phi \). Since \( \{\phi_s\}_{s \in S} \) are completely positive, they satisfy the Kadison-Schwarz inequality, therefore
\[
\phi_s(x^*x) - x^*x \geq 0
\]
for all \( s \in S \) hence the net \( \{\phi_s(x^*x)\}_{s \in S} \) is monotone increasing. It follows from the way \( \Phi \) is constructed that
\[
\Phi(x^*x) = so - \lim_{s \in S} \phi_s(x^*x).
\]
Let \( y = \Phi(x^*x) - x^*x \). It follows that
\[
so - \lim_{s \in S} \phi_s(y) = 0.
\]
Let \( a \in C^*(N^\phi) \) and let \( h \in H \). Then
\[
\|\phi_s(ay^{1/2}h)\|^2 \leq (\phi_s(y^{1/2}a^*ay^{1/2})h, h) \leq \|a\|^2(\phi_s(y)h, h).
\]
This shows that \( \lim_{s \in S} \|\phi_s(z)h\| = 0 \) for every \( z \in \ker \Phi \) therefore
\[
\Phi(w) = so - \lim_{s \in S} \phi_s(w)
\]
for every \( w \in C^*(N^\phi) \). This completes the proof.

\[\Box\]

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