ON THE INTEGRAL REPRESENTATIONS OF RELATIVE $(p, q)$-TH TYPE AND RELATIVE $(p, q)$-TH WEAK TYPE OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we wish to establish the integral representations of relative $(p, q)$-th type and relative $(p, q)$-th weak type of entire and meromorphic functions. We also investigate their equivalence relation under some certain condition.

1. Introduction

For any entire function $f$, $M_f (r)$, a function of $r$ is defined as follows:

$$M_f (r) = \max_{|z|=r} |f(z)|.$$

If an entire function $f$ is non-constant then $M_f (r)$ is strictly increasing and continuous and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_f^{-1} (s) = \infty$.

Whenever $f$ is meromorphic, one can define another function $T_f (r)$ known as Nevanlinna’s Characteristic function of $f$ in the following manner which perform the same role as the maximum modulus function:

$$T_f (r) = N_f (r) + m_f (r),$$

wherever the function $N_f (r, a) \left( \tilde{N}_f (r, a) \right)$ known as counting function of $a$-points (distinct $a$-points) of meromorphic $f$ is defined as

$$N_f (r, a) = \int_{0}^{r} \frac{n_f (t, a) - n_f (0, a)}{t} dt + \tilde{n}_f (0, a) \log r$$

$$\left( \tilde{N}_f (r, a) = \int_{0}^{r} \frac{\tilde{n}_f (t, a) - \tilde{n}_f (0, a)}{t} dt + \tilde{n}_f (0, a) \log r \right),$$

in addition we symbolize by $n_f (r, a) \left( \tilde{n}_f (r, a) \right)$ the number of $a$-points (distinct $a$-points) of $f$ in $|z| \leq r$ and an $\infty$-point is a pole of $f$. In many situations, $N_f (r, \infty)$ and $\tilde{N}_f (r, \infty)$ are symbolized by $N_f (r)$ and $\tilde{N}_f (r)$ respectively. Also the function $m_f (r, \infty)$

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alternatively symbolized by \( m_f(r) \) known as the proximity function of \( f \) is defined in the following way:

\[
m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,
\]

where \( \log^+ x = \max(\log x, 0) \) for all \( x \geq 0 \), and some times one may denote \( m \left( r, \frac{1}{f-a} \right) \) by \( m_f(r, a) \).

The term \( m(r, a) \) which is defined to be the mean value of \( \log^+ \left| \frac{1}{f-a} \right| \) (or \( \log^+ |f| \) if \( a = \infty \)) on the circle \( |z| = r \), receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value ‘a’. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle \( |z| = r \) of the functional value \( f \) from the value ‘a’.

If \( f \) is an entire function, then the Nevanlinna’s Characteristic function \( T_f(r) \) of \( f \) is defined as follows:

\[
T_f(r) = m_f(r).
\]

Moreover, if \( f \) is non-constant entire then \( T_f(r) \) is strictly increasing and continuous functions of \( r \). Also its inverse \( T_f^{-1} : (T_f(0), \infty) \rightarrow (0, \infty) \) exist and is such that \( \lim_{s \to \infty} T_f^{-1}(s) = \infty \). Also the ratio \( \frac{T_f(r)}{r} \) as \( r \to \infty \) is called the growth of \( f \) with respect to \( g \) in terms of the Nevanlinna’s Characteristic functions of the meromorphic functions \( f \) and \( g \).

The order and lower order of an entire function \( f \) which is generally used in computational purpose are classical in complex analysis. L. Bernal \{[1], [2]\} introduced the relative order (respectively relative lower order) between two entire functions to avoid comparing growth just with \( \exp z \). Extending the notion of relative order (respectively relative lower order) Ruiz et al \[5\] introduced the relative \((p, q)\)-th order (respectively relative lower \((p, q)\)-th order) where \( p \) and \( q \) are any two positive integers. Now to compare the growth of entire functions having the same relative \((p, q)\)-th order or relative lower \((p, q)\)-th order, we wish to introduce the definition of relative \((p, q)\)-th type and relative \((p, q)\)-th weak type of an entire function with respect to another entire function and establish their integral representations. We also investigate their equivalence relations under certain conditions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in \[4\] and \[6\].

2. Preliminary remarks and definitions

First of all we state the following two notations which are frequently used in our subsequent study:

\[
\log^{[k]} r = \log \left( \log^{[k-1]} r \right) \text{ for } k = 1, 2, 3, \ldots ; \quad \log^{[0]} r = r
\]

and

\[
\exp^{[k]} r = \exp \left( \exp^{[k-1]} r \right) \text{ for } k = 1, 2, 3, \ldots ; \quad \exp^{[0]} r = r.
\]
Taking this into account, let us denote that
\[ \rho^{(p,q)}_{\alpha}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p]} \alpha^{-1} \beta(r)}{\log^{[q]} r} \quad \text{and} \quad \lambda^{(p,q)}_{\alpha}(\beta) = \liminf_{r \to \infty} \frac{\log^{[p]} \alpha^{-1} \beta(r)}{\log^{[q]} r} \]
where \( p, q \) are any two positive integers and \( \alpha(x), \beta(x) \) be any two positive continuous increasing to \( +\infty \) on \( [x_0, +\infty) \) functions.

If we consider \( \alpha(x) = M_g(x) \) and \( \beta(x) = M_f(x) \) where \( f \) and \( g \) are any two entire functions with index-pairs \( (m, q) \) and \( (m, p) \) respectively where \( p, q, m \) are positive integers such that \( m \geq \max(p, q) \), then the above definition reduces to the definition of relative \((p, q)\)-th order and relative \((p, q)\)-th lower order of an entire function \( f \) with respect to another entire function \( g \) respectively as introduced by Ruiz et. al. \[5\]. Similarly if we take \( \alpha(x) = T_g(x) \) and \( \beta(x) = T_f(x) \) where \( f \) be a meromorphic function and \( g \) be any entire function with index-pairs \( (m, q) \) and \( (m, p) \) respectively where \( p, q, m \) are positive integers such that \( m \geq \max(p, q) \), then the above definition reduces to the definition of relative \((p, q)\)-th order and relative \((p, q)\)-th lower order of a meromorphic function \( f \) with respect to an entire function \( g \) respectively as introduced by Debnath et. al. \[3\]. For detains about index pair one may see \[3\] and \[5\].

In order to refine the above growth scale, now we intend to introduce the definition of an another growth indicator, called relative \((p, q)\) -th type which is as follows:

**Definition 1.** Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \( +\infty \) on \([x_0, +\infty)\) functions. The relative \((p, q)\) th type of \( \beta(x) \) with respect to \( \alpha(x) \) having finite positive relative \((p, q)\) th order \( \rho^{(p,q)}_{\alpha}(\beta) \left( a < \rho^{(p,q)}_{\alpha}(\beta) < \infty \right) \) where \( p \) and \( q \) are any two positive integers is defined as :

\[ \sigma^{(p,q)}_{\alpha}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta(r)}{\log^{[q-1]} r}^{\rho^{(p,q)}_{\alpha}(\beta)} \]

The above definition can alternatively defined in the following manner:

**Definition 2.** Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \( +\infty \) on \([x_0, +\infty)\) functions having finite positive relative \((p, q)\) -th order \( \rho^{(p,q)}_{\alpha}(\beta) \left( a < \rho^{(p,q)}_{\alpha}(\beta) < \infty \right) \) where \( p \) and \( q \) are any two positive integers. Then the relative \((p, q)\) -th type \( \sigma^{(p,q)}_{\alpha}(\beta) \) of \( \beta(x) \) with respect to \( \alpha(x) \) is define as: The integral \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta(r)}{\log^{[q-1]} r}^{\rho^{(p,q)}_{\alpha}(\beta)} \exp \left( \frac{\log^{[q-1]} r}{{\rho}^{(p,q)}_{\alpha}(\beta)} \right) dr \) \((r_0 > 0)\)
converges for \( k > \sigma^{(p,q)}_{\alpha}(\beta) \) and diverges for \( k < \sigma^{(p,q)}_{\alpha}(\beta) \).

Analogously, to determine the relative growth of two increasing functions having same non zero finite relative \((p, q)\) -th lower order, one can introduced the definition of relative \((p, q)\) -th weak type of finite positive relative \((p, q)\) -th lower order \( \lambda^{(p,q)}_{\alpha}(\beta) \) in the following way:

**Definition 3.** Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \( +\infty \) on \([x_0, +\infty)\) functions having finite positive relative \((p, q)\) th lower order \( \lambda^{(p,q)}_{\alpha}(\beta) \)
\( (a < \lambda_{\alpha}^{(p,q)} (\beta) < \infty) \) where \( p \) and \( q \) are any two positive integers. Then the relative \((p,q)\) th weak type of \( \beta(x) \) with respect to \( \alpha(x) \) is defined as:

\[
\tau_{\alpha}^{(p,q)} (\beta) = \liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{(\log^{[q-1]} r)^{\lambda_{\alpha}^{(p,q)} (\beta)}}.
\]

The above definition can also alternatively defined as:

**Definition 4.** Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions having finite positive relative \((p,q)\)-th lower order \( \lambda_{\alpha}^{(p,q)} (\beta) \) \((a < \lambda_{\alpha}^{(p,q)} (\beta) < \infty)\) where \( p \) and \( q \) are any two positive integers. Then the relative \((p,q)\)-th weak type \( \tau_{\alpha}^{(p,q)} (\beta) \) of \( \beta(x) \) with respect to \( \alpha(x) \) is defined as:

\[
The integral \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right)^{\lambda_{\alpha}^{(p,q)} (\beta)} \right)^{k+1} dr (r_0 > 0)} \)
\]

converges for \( k > \tau_{\alpha}^{(p,q)} (\beta) \) and diverges for \( k < \tau_{\alpha}^{(p,q)} (\beta) \).

Now a question may arise about the equivalence of the definitions of relative \((p,q)\)-th type and relative \((p,q)\)-th weak type with their integral representations. In the next section we would like to establish such equivalence of Definition 1 and Definition 2, and Definition 3 and Definition 4 and also investigate some growth properties related to relative \((p,q)\)-th type and relative \((p,q)\)-th weak type of \( \beta(x) \) with respect to \( \alpha(x) \).

### 3. Lemmas.

In this section we present a lemma which will be needed in the sequel.

**Lemma 1.** Let the integral \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right)^{A} \right)^{k+1} dr (r_0 > 0)} \) converges where \( 0 < A < \infty \). Then

\[
\lim_{r \to \infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right)^{A} \right)^{k}} = 0.
\]

**Proof.** Since the integral \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right)^{A} \right)^{k+1} dr (r_0 > 0)} \) converges, then

\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right)^{A} \right)^{k+1} dr < \varepsilon, \text{ if } r_0 > R(\varepsilon)}.
\]
Therefore,
\[
\exp\left((\log^{[q-1]} r_0)^A\right) + r_0 
\int_{r_0} \frac{\log^{[p-2]} \alpha^{-1} \beta(r)}{\left[ \exp\left(\left(\log^{[q-1]} r\right)^A\right) \right]^{k+1}} \, dr < \varepsilon.
\]
Since \(\log^{[p-2]} \alpha^{-1} \beta(r)\) increases with \(r\), so
\[
\exp\left((\log^{[q-1]} r_0)^A\right) + r_0 
\int_{r_0} \frac{\log^{[p-2]} \alpha^{-1} \beta(r)}{\left[ \exp\left(\left(\log^{[q-1]} r\right)^A\right) \right]^{k+1}} \, dr \geq \frac{\log^{[p-2]} \alpha^{-1} \beta(r_0)}{\left[ \exp\left(\left(\log^{[q-1]} r_0\right)^A\right) \right]^{k+1}} \cdot \left[ \exp\left(\left(\log^{[q-1]} r_0\right)^A\right) \right]^{k+1},
\]
i.e., for all sufficiently large values of \(r\),
\[
\exp\left((\log^{[q-1]} r_0)^A\right) + r_0 
\int_{r_0} \frac{\log^{[p-2]} \alpha^{-1} \beta(r)}{\left[ \exp\left(\left(\log^{[q-1]} r\right)^A\right) \right]^{k+1}} \, dr \geq \frac{\log^{[p-2]} \alpha^{-1} \beta(r_0)}{\left[ \exp\left(\left(\log^{[q-1]} r_0\right)^A\right) \right]^{k+1}},
\]
so that
\[
\frac{\log^{[p-2]} \alpha^{-1} \beta(r_0)}{\left[ \exp\left(\left(\log^{[q-1]} r_0\right)^A\right) \right]^{k}} < \varepsilon \text{ if } r_0 > R(\varepsilon).
\]
i.e., \(\lim_{r \to \infty} \frac{\log^{[p-2]} \alpha^{-1} \beta(r)}{\left[ \exp\left(\left(\log^{[q-1]} r\right)^A\right) \right]^k} = 0\).

This proves the lemma. \(\square\)

4. Main Results.

In this section we state the main results of this chapter.

Theorem 1. Let \(\alpha(x)\) and \(\beta(x)\) be any two positive continuous increasing to \(+\infty\) on \([x_0, \infty)\) functions having finite positive relative \((p, q)\)-th order \(\rho^{(p, q)}_{\alpha}(\beta) \left(0 < \rho^{(p, q)}_{\alpha}(\beta) < \infty\right)\)
and relative \((p, q)\)-th type \(\sigma^{(p, q)}_{\alpha}(\beta)\) where \(p\) and \(q\) are any two positive integers. Then Definition 4 and Definition 2 are equivalent.
Proof. Let us consider \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions such that \( \rho^{(p,q)}_\alpha(\beta) \left( 0 < \rho^{(p,q)}_\alpha(\beta) < \infty \right) \) exists for any two positive integers \( p \) and \( q \).

**Case I.** \( \sigma^{(p,q)}_\alpha(\beta) = \infty \).

**Definition 1 \( \Rightarrow \) Definition 2**

As \( \sigma^{(p,q)}_\alpha(\beta) = \infty \), from Definition 1 we have for arbitrary positive \( G \) and for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p-1]} \alpha^{-1} \beta(r) > G \cdot \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)}
\]

\[i.e., \quad \log^{[p-2]} \alpha^{-1} \beta(r) > \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) \right]^G \quad \text{(4.1)}\]

If possible let the integral \( \int_{r_0}^{\infty} \log^{[p-2]} \alpha^{-1} \beta(r) \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) dr \) converge for arbitrary positive \( \varepsilon \) and for a sequence of values of \( r \) tending to infinity.

Then by Lemma 1

\[
\limsup_{r \to \infty} \log^{[p-2]} \alpha^{-1} \beta(r) \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) G = 0.
\]

So for all sufficiently large values of \( r \),

\[
\log^{[p-2]} \alpha^{-1} \beta(r) < \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) \right]^G \quad \text{(4.2)}
\]

Therefore from (4.1) and (4.2) we arrive at a contradiction.

Hence \( \int_{r_0}^{\infty} \log^{[p-2]} \alpha^{-1} \beta(r) \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) dr \) diverges whenever \( G \) is finite, which is the Definition 2.

**Definition 2 \( \Rightarrow \) Definition 1**

Let \( G \) be any positive number. Since \( \sigma^{(p,q)}_\alpha(\beta) = \infty \), from Definition 2, the divergence of the integral \( \int_{r_0}^{\infty} \log^{[p-2]} \alpha^{-1} \beta(r) \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) dr \) gives for arbitrary positive \( \varepsilon \) and for a sequence of values of \( r \) tending to infinity

\[
\log^{[p-2]} \alpha^{-1} \beta(r) > \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)} \right) \right]^{G-\varepsilon}
\]

\[i.e., \quad \log^{[p-1]} \alpha^{-1} \beta(r) > (G - \varepsilon) \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha(\beta)},\]
which implies that
\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)}} \geq G - \varepsilon .
\]

Since \( G > 0 \) is arbitrary, it follows that
\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)}} = \infty .
\]

Thus Definition [1] follows.

**Case II.** \( 0 \leq \sigma^{(p,q)}_\alpha (\beta) < \infty \).

**Definition [1] \Rightarrow Definition [2]**

**Subcase (A).** \( 0 < \sigma^{(p,q)}_\alpha (\beta) < \infty \).

Let \( \alpha (x) \) and \( \beta (x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions such that \( 0 < \sigma^{(p,q)}_\alpha (\beta) < \infty \) exists for any two positive integers \( p \) and \( q \). Then according to the Definition [1] for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \), we obtain that

\[
\log^{[p-1]} \alpha^{-1} \beta (r) < \left( \sigma^{(p,q)}_\alpha (\beta) + \varepsilon \right) \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \]

i.e.,
\[
\log^{[p-2]} \alpha^{-1} \beta (r) < \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^{\sigma^{(p,q)}_\alpha (\beta) + \varepsilon} \]

\[
\frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^k} < \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^{\sigma^{(p,q)}_\alpha (\beta) + \varepsilon} \]

i.e.,
\[
\frac{1}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^k} \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^{\sigma^{(p,q)}_\alpha (\beta) + \varepsilon} \]

Therefore
\[
\int_{r_0}^\infty \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho^{(p,q)}_\alpha (\beta)} \right) \right]^k} dr \quad (r_0 > 0) \]
converges for \( k > \sigma^{(p,q)}_\alpha (\beta) \).
Again by Definition 1, we obtain for a sequence values of \( r \) tending to infinity that
\[
\log^{[p-1]} \alpha^{-1} \beta (r) > \left( \sigma_{(p,q)} (\beta) - \varepsilon \right) \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)}
\]
i.e., \( \log^{[p-2]} \alpha^{-1} \beta (r) > \left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{\sigma_{(p,q)} (\beta) - \varepsilon} \). (4.3)

So for \( k < \sigma_{(p,q)} (\beta) \), we get from (4.3) that
\[
\log^{[p-2]} \alpha^{-1} \beta (r) > \frac{1}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{k - \left( \sigma_{(p,q)} (\beta) - \varepsilon \right)}}.
\]

Therefore
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{k + 1}} dr (r_0 > 0) \text{ diverges for } k < \sigma_{(p,q)} (\beta).
\]

Hence
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{k + 1}} dr (r_0 > 0) \text{ converges for } k > \sigma_{(p,q)} (\beta) \text{ and diverges for } k < \sigma_{(p,q)} (\beta).
\]

Subcase (B). \( \sigma_{(p,q)} (\beta) = 0 \).

When \( \sigma_{(p,q)} (\beta) = 0 \) for any two positive integers \( p \) and \( q \), Definition 1 gives for all sufficiently large values of \( r \) that
\[
\log^{[p-1]} \alpha^{-1} \beta (r) < \varepsilon.
\]

Then as before we obtain that
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{k + 1}} dr (r_0 > 0) \text{ converges for } k > 0 \text{ and diverges for } k < 0.
\]

Thus combining Subcase (A) and Subcase (B), Definition 2 follows.

**Definition 2** \( \Rightarrow \) **Definition 1**

From Definition 2 and for arbitrary positive \( \varepsilon \) the integral
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{\sigma_{(p,q)} (\beta)+\varepsilon+1}} dr (r_0 > 0) \text{ converges. Then by Lemma 1 we get that}
\]
\[
\limsup_{r \to \infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\left[ \exp \left( \left( \log^{[q-1]} r \right)^{\rho_{(p,q)} (\beta)} \right) \right]^{\sigma_{(p,q)} (\beta)+\varepsilon}} = 0.
\]
So we obtain all sufficiently large values of \( r \) that

\[
\frac{\log^{p-2}{\alpha^{-1}}{\beta}(r)}{\left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right]} < \varepsilon
\]

\( i.e., \) \( \log^{p-2}{\alpha^{-1}}{\beta}(r) < \varepsilon \cdot \left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right] \)

\( i.e., \) \( \log^{p-1}{\alpha^{-1}}{\beta}(r) < \log \varepsilon + \left( \sigma^{(p,q)}_{\alpha}(\beta) + \varepsilon \right) \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \)

\( i.e., \) \( \limsup_{r \to \infty} \frac{\log^{p-1}{\alpha^{-1}}{\beta}(r)}{\left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta)} \leq \sigma^{(p,q)}_{\alpha}(\beta) + \varepsilon . \)

Since \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[
\limsup_{r \to \infty} \frac{\log^{p-1}{\alpha^{-1}}{\beta}(r)}{\left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta)} \leq \sigma^{(p,q)}_{\alpha}(\beta) .
\] (4.4)

On the other hand the divergence of the integral \( \int_{r_0}^{\infty} \frac{\log^{p-2}{\alpha^{-1}}{\beta}(r)}{\left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right]} dr \) \( (r_0 > 0) \) implies that there exists a sequence of values of \( r \) tending to infinity such that

\[
\frac{\log^{p-2}{\alpha^{-1}}{\beta}(r)}{\left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right]} > \frac{1}{\left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right]^{1+\varepsilon}}
\]

\( i.e., \) \( \log^{p-2}{\alpha^{-1}}{\beta}(r) > \left[ \exp \left( \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \right) \right] \sigma^{(p,q)}_{\alpha}(\beta) - 2\varepsilon \)

\( i.e., \) \( \log^{p-1}{\alpha^{-1}}{\beta}(r) > \left( \sigma^{(p,q)}_{\alpha}(\beta) - 2\varepsilon \right) \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) \)

\( i.e., \) \( \log^{p-1}{\alpha^{-1}}{\beta}(r) \left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta) > \left( \sigma^{(p,q)}_{\alpha}(\beta) - 2\varepsilon \right) . \)

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[
\limsup_{r \to \infty} \frac{\log^{p-1}{\alpha^{-1}}{\beta}(r)}{\left( \log^{q-1}{r} \right) \rho^{(p,q)}_{\alpha}(\beta)} \geq \sigma^{(p,q)}_{\alpha}(\beta) .
\] (4.5)
So from (4.4) and (4.5), we obtain that
\[
\limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \alpha (\beta)} = \sigma_{(p,q)} (\beta).
\]
This proves the theorem.

**Theorem 2.** Let \( \alpha (x) \) and \( \beta (x) \) be any two positive continuous increasing to \(+\infty \) on \([x_0, +\infty)\) functions having finite positive relative \((p, q)\) -th lower order \( \lambda_{(p,q)} (\beta) \) \((0 < \lambda_{(p,q)} (\beta) < \infty)\) and relative \((p, q)\) -th weak type \( \tau_{(p,q)} (\beta) \) where \( p \) and \( q \) are any two positive integers. Then Definition 3 and Definition 4 are equivalent.

**Proof.** Let us consider \( \alpha (x) \) and \( \beta (x) \) be any two positive continuous increasing to \(+\infty \) on \([x_0, +\infty)\) functions such that \( \lambda_{(p,q)} (\beta) \) \((0 < \lambda_{(p,q)} (\beta) < \infty)\) exists for any two positive integers \( p \) and \( q \).

**Case I.** \( \tau_{(p,q)} (\beta) = \infty. \)

**Definition 3 \( \Rightarrow \) Definition 4.**

As \( \tau_{(p,q)} (\beta) = \infty \), from Definition 3, we obtain for arbitrary positive \( G \) and for all sufficiently large values of \( r \) that
\[
\log^{[p-1]} \alpha^{-1} \beta (r) > G \cdot \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)}
\]
i.e.,
\[
\log^{[p-2]} \alpha^{-1} \beta (r) > \left[ \exp \left( \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)} \right) \right]^{G}.
\]

Now if possible let the integral
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)} \right)^{G}} \, dr \ (r_0 > 0)
\]
be converge.

Then by Lemma 1,
\[
\liminf_{r \to \infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)} \right)^{G}} = 0.
\]

So for a sequence of values of \( r \) tending to infinity we get that
\[
\log^{[p-2]} \alpha^{-1} \beta (r) < \left[ \exp \left( \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)} \right) \right]^{G}.
\]

Therefore from (4.6) and (4.7), we arrive at a contradiction.

Hence
\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} \alpha (\beta) \right)^{\lambda_{(p,q)} (\beta)} \right)^{G}} \, dr \ (r_0 > 0)
\]
diverges whenever \( G \) is finite, which is the Definition 4.

**Definition 4 \( \Rightarrow \) Definition 3.**

Let \( G \) be any positive number. Since \( \tau_{(p,q)} (\beta) = \infty \), from Definition 4, the divergence
Since $G > \text{Subcase (C)}$, 0 $\Rightarrow$ Definition 3, Case II.

Thus Definition 3 follows.

of the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]}{\alpha^{-1}\beta(r)}}{\exp\left(\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)} dr$ ($r_0 > 0$) gives for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[p-2]}{\alpha^{-1}\beta(r)} > \left\lceil \exp\left(\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right) \right\rceil^{G-\varepsilon}$$

i.e., $\log^{[p-1]}{\alpha^{-1}\beta(r)} > (G - \varepsilon)\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$, which implies that

$$\liminf_{r \to \infty} \frac{\log^{[p-1]}{\alpha^{-1}\beta(r)}}{\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} \geq G - \varepsilon.$$ 

Since $G > 0$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log^{[p-1]}{\alpha^{-1}\beta(r)}}{\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} = \infty.$$ 

Thus Definition 4 follows.

Case II. $0 \leq \tau_{\alpha}^{(p,q)}(\beta) < \infty$.

Definition 3 $\Rightarrow$ Definition 4.

Subcase (C). $0 < \tau_{\alpha}^{(p,q)}(\beta) < \infty$.

Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $0 < \tau_{\alpha}^{(p,q)}(\beta) < \infty$ exists for any two positive integers $p$ and $q$. Then according to the Definition ??, for a sequence of values of $r$ tending to infinity we get that

$$\log^{[p-1]}{\alpha^{-1}\beta(r)} < \left(\tau_{\alpha}^{(p,q)}(\beta) + \varepsilon\right)\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$$

i.e., $\log^{[p-2]}{\alpha^{-1}\beta(r)} < \left\lceil \exp\left(\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right) \right\rceil^{\tau_{\alpha}^{(p,q)}(\beta) + \varepsilon}$

$$\frac{\log^{[p-2]}{\alpha^{-1}\beta(r)}}{\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} < \left(\frac{\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}{\left(\log^{[q-1]}{r}\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}\right)^{\frac{1}{\tau_{\alpha}^{(p,q)}(\beta) + \varepsilon}}.$$
Therefore \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} a^{-1} \beta(r)}{\exp\left(\frac{1}{\log^{[q-1]} r} \lambda_{[\alpha]}^{(p,q)}(\beta)\right)} \) converges for \( k > \tau_{[\alpha]}^{(p,q)}(\beta) \).

Again by Definition 3 we obtain for all sufficiently large values of \( r \) that

\[
\log^{[p-1]} a^{-1} \beta(r) > \left( \tau_{[\alpha]}^{(p,q)}(\beta) - \epsilon \right) \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)}
\]

i.e., \( \log^{[p-2]} a^{-1} \beta(r) > \left[ \exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right) \right]^{\tau_{[\alpha]}^{(p,q)}(\beta) - \epsilon} \).

(4.8)

Therefore \( \tau_{[\alpha]}^{(p,q)}(\beta) \), we get from (4.8) that

\[
\frac{\log^{[p-2]} a^{-1} \beta(r)}{\left[ \exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right) \right]^k} > \frac{1}{\left[ \exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right) \right]^{k \left( \tau_{[\alpha]}^{(p,q)}(\beta) - \epsilon \right)}}.
\]

Therefore \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} a^{-1} \beta(r)}{\exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right)} \) diverges for \( k < \tau_{[\alpha]}^{(p,q)}(\beta) \).

Hence \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} a^{-1} \beta(r)}{\exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right)} \) diverges for \( k > \tau_{[\alpha]}^{(p,q)}(\beta) \) and converges for \( k < \tau_{[\alpha]}^{(p,q)}(\beta) \) for \( k < \tau_{[\alpha]}^{(p,q)}(\beta) \).

**Subcase (D).** \( \tau_{[\alpha]}^{(p,q)}(\beta) = 0 \).

When \( \tau_{[\alpha]}^{(p,q)}(\beta) = 0 \) for any two positive integers \( p \) and \( q \), Definition 3 gives for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log^{[p-1]} a^{-1} \beta(r)}{\left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)}} < \epsilon.
\]

Then as before we obtain that \( \int_{r_0}^{\infty} \frac{\log^{[p-2]} a^{-1} \beta(r)}{\exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right)} \) diverges for \( k < 0 \) and converges for \( k > 0 \).

Thus combining Subcase (C) and Subcase (D), Definition 4 follows.

**Definition 4 ⇒ Definition 3**

From Definition 4 and for arbitrary positive \( \epsilon \) the integral

\[
\int_{r_0}^{\infty} \frac{\log^{[p-2]} a^{-1} \beta(r)}{\exp\left( \left( \log^{[q-1]} r \right)^{\lambda_{[\alpha]}^{(p,q)}(\beta)} \right)} dr (r_0 > 0) \]

converges. Then by Lemma 1 we get
that
\[
\liminf_{r \to \infty} \frac{\log^{p-2} \alpha^{-1} \beta (r)}{\exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) {\tau_{\alpha}^{(p, q)} (\beta) + \varepsilon}} = 0 .
\]
So we get for a sequence of values of \( r \) tending to infinity that
\[
\log^{p-2} \alpha^{-1} \beta (r)
\]
\[
\exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) \left( {\tau_{\alpha}^{(p, q)} (\beta) + \varepsilon} \right)
\]
\[
i.e., \quad \log^{p-2} \alpha^{-1} \beta (r) < \varepsilon \cdot \left[ \exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) \right]^{\tau_{\alpha}^{(p, q)} (\beta) + \varepsilon}
\]
\[
i.e., \quad \log^{p-1} \alpha^{-1} \beta (r) < \log \varepsilon + \left( {\tau_{\alpha}^{(p, q)} (\beta) + \varepsilon} \right) \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right)
\]
\[
i.e., \quad \liminf_{r \to \infty} \frac{\log^{p-1} \alpha^{-1} \beta (r)}{\log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta)} \leq {\tau_{\alpha}^{(p, q)} (\beta) + \varepsilon} .
\]
Since \( \varepsilon ( > 0 ) \) is arbitrary, it follows from above that
\[
\liminf_{r \to \infty} \frac{\log^{p-1} \alpha^{-1} \beta (r)}{\log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta)} \leq {\tau_{\alpha}^{(p, q)} (\beta)} . \tag{4.9}
\]
On the other hand the divergence of the integral
\[
\int_{r_0}^{\infty} \frac{\log^{p-2} \alpha^{-1} \beta (r)}{\exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) {\tau_{\alpha}^{(p, q)} (\beta) - \varepsilon + 1}} dr ( r_0 > 0 )
\]
implies for all sufficiently large values of \( r \) that
\[
\log^{p-2} \alpha^{-1} \beta (r)
\]
\[
\exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) \left( {\tau_{\alpha}^{(p, q)} (\beta) - 2\varepsilon} \right)
\]
\[
i.e., \quad \log^{p-2} \alpha^{-1} \beta (r) > \left[ \exp \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) \right]^{\tau_{\alpha}^{(p, q)} (\beta) - 2\varepsilon} \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right)
\]
\[
i.e., \quad \log^{p-1} \alpha^{-1} \beta (r) \left( \log^{[q-1]} r \lambda_{\alpha, \alpha}^{(p, q)} (\beta) \right) > \left( {\tau_{\alpha}^{(p, q)} (\beta) - 2\varepsilon} \right)
\]
As $\varepsilon (> 0)$ is arbitrary, it follows from above that
\[
\liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \rho^{(p,q)}(\beta)} \geq \tau^{(p,q)}_{\alpha}(\beta). \tag{4.10}
\]
So from (4.9) and (4.10) we obtain that
\[
\liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \rho^{(p,q)}(\beta)} = \tau^{(p,q)}_{\alpha}(\beta).
\]
This proves the theorem. \hfill \square

Next we introduce the following two relative growth indicators which will also enable help our subsequent study.

**Definition 5.** Let $\alpha (x)$ and $\beta (x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive relative $(p, q)$-th order $\rho^{(p,q)}_\alpha (\beta)$ \((a < \rho^{(p,q)}_\alpha (\beta) < \infty)\) where $p$ and $q$ are any two positive integers. Then the relative $(p, q)$-th lower type of $\beta (x)$ with respect to $\alpha (x)$ is defined as :
\[
\varphi^{(p,q)}_{\alpha}(\beta) = \liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \rho^{(p,q)}(\beta)}. \tag{4.11}
\]

The above definition can alternatively be defined in the following manner:

**Definition 6.** Let $\alpha (x)$ and $\beta (x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive relative $(p, q)$-th order $\rho^{(p,q)}_\alpha (\beta)$ \((a < \rho^{(p,q)}_\alpha (\beta) < \infty)\) where $p$ and $q$ are any two positive integers. Then the relative $(p, q)$-th lower type $\varphi^{(p,q)}_{\alpha}(\beta)$ of $\beta (x)$ with respect to $\alpha (x)$ is defined as: The integral
\[
\int_{r_0}^{\infty} \exp \left( \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \rho^{(p,q)}(\beta)} \right) r^{k+1} dr (r_0 > 0) \text{ converges for } k > \varphi^{(p,q)}_{\alpha}(\beta) \text{ and diverges for } k < \varphi^{(p,q)}_{\alpha}(\beta).
\]

**Definition 7.** Let $\alpha (x)$ and $\beta (x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive relative $(p, q)$-th lower order $\lambda^{(p,q)}_\alpha (\beta)$ \((a < \lambda^{(p,q)}_\alpha (\beta) < \infty)\). Then the growth indicator $\pi^{(p,q)}_{\alpha}(\beta)$ of $\beta (x)$ with respect to $\alpha (x)$ is defined as :
\[
\pi^{(p,q)}_{\alpha}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\log^{[q-1]} \lambda^{(p,q)}(\beta)}. \tag{4.12}
\]

The above definition can also alternatively defined as:
Definition 8. Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions having finite positive relative \((p,q)\) -th order lower type \(\rho_{\alpha}^{(p,q)}(\beta)\) \((0 < \rho_{\alpha}^{(p,q)}(\beta) < \infty)\) and relative \((p,q)\) -th order upper type \(\sigma_{\alpha}^{(p,q)}(\beta)\) where \(p\) and \(q\) are any two positive integers. Then the growth indicator \(\tau_{\alpha}^{(p,q)}(\beta)\) of \(\beta(x)\) with respect to \(\alpha(x)\) is defined as: The integral
\[
\int_{r_0}^{\infty} \left[ \frac{\log^{[p-2]} \alpha^{-1}(r)}{\exp \left( \frac{\log^{[q-1]} \beta^{-1}(r)}{r^{\rho_{\alpha}^{(p,q)}(\beta)}} \right) } \right] \, dr \quad (r_0 > 0)
\]
converges for \(k > \tau_{\alpha}^{(p,q)}(f)\) and diverges for \(k < \tau_{\alpha}^{(p,q)}(f)\).

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma \(\ref{lem:lemma1}\) and in the line of Theorem \(\ref{thm:theorem1}\) and Theorem \(\ref{thm:theorem2}\) respectively.

Theorem 3. Let \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions having finite positive relative \((p,q)\) -th order lower type \(\rho_{\alpha}^{(p,q)}(\beta)\) \((0 < \rho_{\alpha}^{(p,q)}(\beta) < \infty)\) and relative \((p,q)\) -th lower type \(\sigma_{\alpha}^{(p,q)}(\beta)\) where \(p\) and \(q\) are any two positive integers. Then Definition \(\ref{def:definition7}\) and Definition \(\ref{def:definition8}\) are equivalent.

Theorem 4. \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions having finite positive relative \((p,q)\) -th order lower type \(\lambda_{\alpha}^{(p,q)}(\beta)\) \((0 < \lambda_{\alpha}^{(p,q)}(\beta) < \infty)\) and the growth indicator \(\tau_{\alpha}^{(p,q)}(\beta)\) where \(p\) and \(q\) are any two positive integers. Then Definition \(\ref{def:definition7}\) and Definition \(\ref{def:definition8}\) are equivalent.

Theorem 5. \( \alpha(x) \) and \( \beta(x) \) be any two positive continuous increasing to \(+\infty\) on \([x_0, +\infty)\) functions with \(0 < \lambda_{\alpha}^{(p,q)}(\beta) \leq \rho_{\alpha}^{(p,q)}(\beta) < \infty\) where \(p\) and \(q\) are any two positive integers. Then

\[
(i) \quad \sigma_{\alpha}^{(p,q)}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[ \log^{[q-1]} \beta^{-1}(r) \right]^{\rho_{\alpha}^{(p,q)}(\beta)}}.
\]

\[
(ii) \quad \bar{\sigma}_{\alpha}^{(p,q)}(\beta) = \liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[ \log^{[q-1]} \beta^{-1}(r) \right]^{\rho_{\alpha}^{(p,q)}(\beta)}}.
\]

\[
(iii) \quad \tau_{\alpha}^{(p,q)}(\beta) = \liminf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\log^{[q-1]} \beta^{-1}(r)^{\lambda_{\alpha}^{(p,q)}(\beta)}}
\]

and

\[
(iv) \quad \bar{\tau}_{\alpha}^{(p,q)}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[ \log^{[q-1]} \beta^{-1}(r) \right]^{\lambda_{\alpha}^{(p,q)}(\beta)}}.
\]

Proof. Taking \( \beta(r) = R \), theorem follows from the definitions of \( \sigma_{\alpha}^{(p,q)}(\beta), \bar{\sigma}_{\alpha}^{(p,q)}(\beta), \tau_{\alpha}^{(p,q)}(\beta) \) and \( \bar{\tau}_{\alpha}^{(p,q)}(\beta) \) respectively. \( \square \)
In the following theorem we obtain a relationship between $\sigma^{(p,q)}_{\alpha}(\beta)$, $\bar{\sigma}^{(p,q)}_{\alpha}(\beta)$, $\tau^{(p,q)}_{\alpha}(\beta)$ and $\bar{\tau}^{(p,q)}_{\alpha}(\beta)$.

**Theorem 6.** Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such $\rho^{(p,q)}_{\alpha}(\beta) = \lambda^{(p,q)}_{\alpha}(\beta)$ $(0 < \lambda^{(p,q)}_{\alpha}(\beta) = \rho^{(p,q)}_{\alpha}(\beta) < \infty)$ where $p$ and $q$ are any two positive integers, then the following quantities

$(i)$ $\sigma^{(p,q)}_{\alpha}(\beta)$, $(ii)$ $\tau^{(p,q)}_{\alpha}(\beta)$, $(iii)$ $\bar{\sigma}^{(p,q)}_{\alpha}(\beta)$ and $(iv)$ $\bar{\tau}^{(p,q)}_{\alpha}(\beta)$

are all equivalent.

**Proof.** From Definition 1 it follows that the integral \[ \int_{r_0}^{\infty} \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} dr \quad (r_0 > 0) \]

converges for $k > \tau^{(p,q)}_{\alpha}(\beta)$ and diverges for $k < \tau^{(p,q)}_{\alpha}(\beta)$. On the other hand, Definition 2 implies that the integral \[ \int_{r_0}^{\infty} \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} dr \quad (r_0 > 0) \]

converges for $k > \sigma^{(p,q)}_{\alpha}(\beta)$ and diverges for $k < \sigma^{(p,q)}_{\alpha}(\beta)$.

$(i) \Rightarrow (ii)$.

Now it is obvious that all the quantities in the expression

\[
\left[ \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} - \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} \right] \]

are of non negative type. So

\[
\int_{r_0}^{\infty} \left[ \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} - \frac{\log^{[p-2]}_{\alpha-1}\beta(r)}{\exp\left(\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}_{\alpha}(\beta)}\right)^{k+1}} \right] dr \quad (r_0 > 0) \geq 0
\]
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\[ i.e., \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right) \lambda_{\alpha}^{(p,q)} (\beta) \right)} dr \geq \]

\[ \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1} \beta (r)}{\exp \left( \left( \log^{[q-1]} r \right) \rho_{\alpha}^{(p,q)} (\beta) \right)} dr \text{ for } r_0 > 0. \]

\[ i.e., \tau_{\alpha}^{(p,q)} (\beta) \geq \sigma_{\alpha}^{(p,q)} (\beta). \quad (4.11) \]

Further as \( \rho_{\alpha}^{(p,q)} (\beta) = \lambda_{\alpha}^{(p,q)} (\beta) \), therefore we get that

\[ \sigma_{\alpha}^{(p,q)} (\beta) = \lim \sup_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \rho_{\alpha}^{(p,q)} (\beta)} \]

\[ \geq \lim \inf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \rho_{\alpha}^{(p,q)} (\beta)} = \lim \inf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \lambda_{\alpha}^{(p,q)} (\beta)} = \tau_{\alpha}^{(p,q)} (\beta). \]

\[ (4.12) \]

Hence from (4.11) and (4.12) we obtain that

\[ \sigma_{\alpha}^{(p,q)} (\beta) = \tau_{\alpha}^{(p,q)} (\beta). \quad (4.13) \]

(ii) \(\Rightarrow\) (iii).

Since \( \rho_{\alpha}^{(p,q)} (\beta) = \lambda_{\alpha}^{(p,q)} (\beta) \), we get that

\[ \tau_{\alpha}^{(p,q)} (\beta) = \lim \inf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \lambda_{\alpha}^{(p,q)} (\beta)} = \lim \inf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \rho_{\alpha}^{(p,q)} (\beta)} = \tau_{\alpha}^{(p,q)} (\beta). \]

(iii) \(\Rightarrow\) (iv).

In view of (4.13) and the condition \( \rho_{\alpha}^{(p,q)} (\beta) = \lambda_{\alpha}^{(p,q)} (\beta) \), it follows that

\[ \overline{\sigma_{\alpha}^{(p,q)} (\beta)} = \lim \inf_{r \to \infty} \frac{\log^{[p-1]} \alpha^{-1} \beta (r)}{\left( \log^{[q-1]} r \right) \rho_{\alpha}^{(p,q)} (\beta)} \]

\[ i.e., \overline{\sigma_{\alpha}^{(p,q)} (\beta)} = \tau_{\alpha}^{(p,q)} (\beta) \]

\[ i.e., \overline{\sigma_{\alpha}^{(p,q)} (\beta)} = \sigma_{\alpha}^{(p,q)} (\beta) \]
i.e., $\sigma^{(p,q)}_{\alpha}(\beta) = \limsup_{r \to \infty} \frac{\log^{[\alpha-1]} \log^{[q-1]} r^{\rho^{(p,q)}_{\alpha}\beta}(r)}}{\log^{[q-1]} r^{\lambda^{(p,q)}_{\alpha\beta}(r)}}$

i.e., $\overline{\sigma}^{(p,q)}_{\alpha}(\beta) = \limsup_{r \to \infty} \frac{\log^{[p-1]} \log^{[q-1]} r^{\rho^{(p,q)}_{\alpha\beta}(r)}}{\log^{[q-1]} r^{\lambda^{(p,q)}_{\alpha\beta}(r)}}$

i.e., $\sigma^{(p,q)}_{\alpha}(\beta) = \overline{\sigma}^{(p,q)}_{\alpha}(\beta)$.

\((iv) \Rightarrow (i)\).

As $\rho^{(p,q)}_{\alpha}(\beta) = \lambda^{(p,q)}_{\alpha\beta}(\beta)$, we obtain that

$$
\limsup_{r \to \infty} \frac{\log^{[p-1]} \log^{[q-1]} r^{\lambda^{(p,q)}_{\alpha\beta}(r)}}{\log^{[q-1]} r^{\lambda^{(p,q)}_{\alpha\beta}(r)}} = \limsup_{r \to \infty} \frac{\log^{[p-1]} \log^{[q-1]} r^{\rho^{(p,q)}_{\alpha\beta}(r)}}{\log^{[q-1]} r^{\rho^{(p,q)}_{\alpha\beta}(r)}} = \sigma^{(p,q)}_{\alpha}(\beta).
$$

Thus the theorem follows. \(\square\)

**Remark 1.** If we consider $\alpha(x) = M_f(x)$ and $\beta(x) = M_g(x)$ where $f$ and $g$ are any two entire functions with index-pairs $(m, q)$ and $(m, p)$ respectively where $p, q, m$ are positive integers such that $m \geq \max(p, q)$, then the above results reduces for the relative $(p, q)$-th growth indicators such as relative $(p, q)$-th type, relative $(p, q)$-th weak type etc. of an entire function $f$ with respect to another entire function $g$.

**Remark 2.** If we take $\alpha(x) = T_f(x)$ and $\beta(x) = T_g(x)$ where $f$ be a meromorphic function and $g$ be any entire function with index-pairs $(m, q)$ and $(m, p)$ respectively where $p, q, m$ are positive integers such that $m \geq \max(p, q)$, then the above theorems reduces for relative $(p, q)$-th growth indicators such as relative $(p, q)$-th type, relative $(p, q)$-th weak type etc. of a meromorphic function $f$ with respect to an entire function $g$.

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