Study of the Spin-weighted Spheroidal Wave Equation in the Case of $s = 3/2$

Kun Dong, Guihua Tian

School of Sciences, Beijing University of Posts and Telecommunications, Beijing, China, 100876.

Abstract

In this thesis we use the means of super-symmetric quantum mechanics to study of the Spin-weighted Spheroidal Wave in the case of $s=3/2$. We obtain some interesting results: the first-five terms of the super-potential, the general form of the super-potential. The ground eigen-function and eigenvalue of the equation are also given. According these results?we make use of the shape invariance property to compute the exited eigenvalues and eigen-functions. These results help us to understand the Spin-weighted Spheroidal Wave and show that it is integral.

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1 Introduction

The spin-weighted spheroidal functions first appeared in the study of the stable problem of Kerr black hole. Now, the spin-weighted spheroidal functions has been widely used in many fields, such as gravitational wave detection, quantum field theory in curved space-time, black hole stable problem; nuclear modeling; spheroidal cavity problem, spheroidal electromagnetic diffraction, scattering and similar problems in acoustic science, etc. Their equation is

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + s + \beta^2 \cos^2 \theta - 2 \beta \cos \theta - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} \right] \Theta(\theta) = 0, \quad (1)$$

The parameter $s$, the spin-weight of the perturbation fields could be $s = 0, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm 1, \pm 2$, and corresponds to the scalar, neutrino, Rarita-schwinger fields, electromagnetic or gravitational perturbations respectively. When $\beta = 0$ they reduce to the spin-weighted spherical equations, whose solutions are the well-known the spin-weighted spherical harmonics. Furthermore, when $\beta = s = 0$, they become to the spherical ones, whose solutions are the famous associated Legendre’s functions $P_l^m$. However when $\beta \neq 0$ and $s \neq 0$, the solutions are the spin-weighted spheroidal functions, which are treated recently by a new method in supersymmetry quantum mechanics. Some nice results are obtained in the case of $s = 0$ and $s = \frac{1}{2}$. Now we continue this study in the case of $s = \frac{3}{2}$. Though the method is somehow identical, the computation involved becomes more complex than before. The course of our calculations could provide useful information for studying them in the case of $s = 1$ and $s = 2$.

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1 e-mail : woailiuyanbin1@126.com
2 e-mail : hua2007@126.com
2 Calculation of super-potential and ground eigenvalue

The boundary conditions for Eq.(1) requires $\Theta$ is finite at $\theta = 0$, $\pi$, this is kind of the Sturm-Liouville problem. As done before, Eq.(1) is transformed in the Schrödinger form for the use of the method of supersymmetry quantum mechanics. Changing the eigenfunction $\Theta$ by

$$\Theta(\theta) = \frac{\Psi(\theta)}{\sqrt{\sin\theta}}.$$  \hspace{1cm} \hspace{0.5cm} (2)

the differential equations for the new depend functions $\Psi$ become

$$\frac{d^2\Psi}{d\theta^2} + \left[ \frac{1}{4} + s + \beta^2 \cos^2\theta - 2s \beta \cos\theta - \frac{(m + s \cos\theta)^2 - \frac{1}{4}}{\sin^2\theta} + E \right] \Psi = 0. \hspace{1cm} \hspace{0.5cm} (3)$$

The corresponding boundary conditions now turn out as $\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0$. The potential in Eq.(3) is

$$V(\theta, \beta, s) = - \left[ \frac{1}{4} + s + \beta^2 \cos^2\theta - 2s \beta \cos\theta - \frac{(m + s \cos\theta)^2 - \frac{1}{4}}{\sin^2\theta} \right], \hspace{1cm} \hspace{0.5cm} (4)$$

Now we will introduce SUSYQM to our problem. According to the theory of the SUSYQM, the form of ground eigenfunction $\Psi_0$ is completely known through the super-potential $W$ by the formula

$$\Psi_0 = N \exp \left[ - \int W d\theta \right]. \hspace{1cm} \hspace{0.5cm} (5)$$

The problem is transformed into solving the super potential $W$. The super potential $W$ is a most important notion in SUSYQM, and it is determined by the $V(\theta, \beta, s)$ through the Reccita’s equation

$$W^2 - W' = V(\theta, \beta, s) - E_0. \hspace{1cm} \hspace{0.5cm} (6)$$

SO, the key work is to solve Eq.(6). As done before, by expanding the super-potential $W$ and the ground eigenvalue $E_0$ into series form of the parameter $\beta$:

$$W = \sum_{n=0}^{\infty} \beta^n W_n \hspace{1cm} \hspace{0.5cm} (7)$$

$$E_0 = \sum_{n=0}^{\infty} E_{0,n;m} \beta^n, \hspace{1cm} \hspace{0.5cm} (8)$$

we solve Eq.(6). The results are:

$$W_0' - W_0^2 = E_{0,0;m} + \frac{3}{2} + \frac{1}{4} - \frac{(m + \frac{3}{2} \cos\theta)^2 - \frac{1}{4}}{\sin^2\theta} \equiv f_0(\theta) \hspace{1cm} \hspace{0.5cm} (9)$$

$$W_1' - 2W_0 W_1 = E_{0,1;m} - 3 \cos\theta \equiv f_1(\theta) \hspace{1cm} \hspace{0.5cm} (10)$$

$$W_2' - 2W_0 W_2 = E_{0,2;m} + \cos^2\theta + W_1^2 \equiv f_2(\theta) \hspace{1cm} \hspace{0.5cm} (11)$$

$$W_n' - 2W_0 W_n = E_{0,n;m} + \sum_{k=1}^{n-1} W_k W_{n-k} \equiv f_n(\theta)(n \geq 3) \hspace{1cm} \hspace{0.5cm} (12)$$
The solution of Eq. (9) is easy to find

\[
E_{0,0;m} = m^2 + m - 15/4
\]
\[
W_0 = -\frac{3/2 + (m + \frac{1}{2})\cos \theta}{\sin \theta}
\]  

(13)

With \(W_0\) known, it is easy to give \(W_n\) on according to the knowledge of differential equations,

\[
W_n(\theta) = e^{2\int W_0 d\theta} A_n(\theta) = \left[ \tan \frac{\theta}{2} \sin^{2m+1} \theta \right]^{-3} A_n(\theta)
\]  

(14)

where,

\[
A_n(\theta) = \int f_n(\theta) e^{-2\int W_0 d\theta} d\theta
\]
\[
= \int f_n(\theta) \left[ \tan \frac{\theta}{2} \sin^{2m+1} \theta \right]^3 d\theta
\]
\[
= \int f_n(\theta)(1 - \cos \theta)^3 \sin^{2m-2} \theta d\theta.
\]  

(15)

Because there appears the cubic power or inverse cubic powers \(\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}\) terms in Eqs. (13), (15), the subsequent calculation becomes more complicated than before (that is, the cases of \(s=0\) and \(s=1/2\)). We put it in appendix A. the obtained results are

\[
E_{0,1;m} = -\frac{9}{2m + 2}
\]
\[
E_{0,2;m} = -\frac{8m^3 + 96m^2 + 168m - 1}{(2m + 2)^3(2m + 3)}
\]
\[
E_{0,3;m} = -\frac{36(2m - 1)^2(2m + 5)^2}{(2m + 2)^3(2m + 3)(2m + 4)}
\]  

(16)

\[
E_{0,4;m} = -\frac{4(2m - 1)^2(2m + 5)(16m^5 + 672m^4 + 3320m^3 + 2416m^2 - 6975m - 7942)}{(2m + 2)^7(2m + 4)^2(2m + 3)^3}
\]

\[
W_1(\theta) = a_{1,1} \sin \theta
\]  

(18)
\[
W_2(\theta) = b_{2,1} \sin \theta + a_{2,1} \sin^3 \theta \cos \theta
\]  

(19)
\[
W_3(\theta) = b_{3,1} \sin \theta + b_{3,2} \sin^3 \theta + a_{3,1} \sin \theta \cos \theta
\]  

(20)
\[
W_4(\theta) = b_{4,1} \sin \theta + b_{4,1} \sin^3 \theta + a_{4,1} \sin \theta \cos \theta + a_{4,2} \sin^3 \theta \cos \theta
\]  

(21)

Where,

\[
a_{1,1} = -\frac{3}{2m + 2}
\]  

(22)
\[ b_{2,1} = -\frac{3(2m - 1)(2m + 5)}{(2m + 2)^3(2m + 3)} \]  
(23)

\[ a_{2,1} = \frac{(2m - 1)(2m + 5)}{(2m + 2)^2(2m + 3)} \]  
(24)

\[ b_{3,1} = \frac{108(2m - 1)(2m + 5)}{(2m + 2)^5(2m + 3)(2m + 4)} \]  
(25)

\[ b_{3,2} = \frac{6(2m - 1)(2m + 5)}{(2m + 2)^3(2m + 3)(2m + 4)} \]  
(26)

\[ a_{3,1} = -\frac{36(2m - 1)(2m + 5)}{(2m + 2)^4(2m + 3)(2m + 4)} \]  
(27)

\[ b_{4,1} = \frac{36(2m - 1)(16m^5 + 672m^4 + 3320m^3 + 2416m^2 - 6975m - 7942)}{(2m + 2)^6(2m + 4)^2(2m + 3)} \]  
(28)

\[ b_{4,2} = -\frac{2(2m - 1)(4m^3 + 16m^2 + 25m + 64)}{(2m + 2)^4(2m + 4)^2(2m + 3)} \]  
(29)

\[ a_{4,1} = \frac{12(2m - 1)(16m^5 + 672m^4 + 3320m^3 + 2416m^2 - 6975m - 7942)}{(2m + 2)^5(2m + 4)^2(2m + 3)} \]  
(30)

\[ a_{4,2} = -\frac{4(2m - 1)(4m^3 + 16m^2 + 25m + 64)}{(2m + 2)^4(2m + 4)^2(2m + 3)^2} \]  
(31)

3 Summarize and prove the general formula of super-potential

From the four terms of \( W_1 - W_4 \), we hypothetically summarize a general formula for \( W_n \) as

\[ W_n(\theta) = \sum_{k=1}^{\left[\frac{n}{2}\right]} a_{n,k} \sin^{2k-1} \theta \cos \theta + \sum_{k=1}^{\left[\frac{n}{2}\right]} b_{n,k} \sin^{2k-1} \theta \]  
(32)

Here we use mathematical induction to prove that the guess is true.

First it is easy to see the assumption (32) is the same as that of \( W_1 \) when \( N = 1 \). Under the condition that all \( W_N \) meet the requirement of (32) whenever \( N \leq n - 1 \), we will try to solve the differential equation for \( W_n \) to verify that it also can be written as that of (32) and be determined by the terms \( W_k, k < n \). The results are

\[ b_{n,l} = \sum_{p=l+1}^{\left[\frac{n}{2}\right]} \frac{3(2m + 2l - 2)j_{n,p}l(2m + 2p - 4,p - l - 1)}{(2m + 2l + 1)(2m + 2l + 3)(2m + 2p - 3)} + \frac{g_{n,l}}{2m + 2l} \delta_{l,1}, \quad l \geq 1 \]  
(33)

\[ a_{n,l} = -\sum_{p=l+1}^{\left[\frac{n}{2}\right]} \frac{(2m + 2l - 2)(2m + 2l)j_{n,p}l(2m + 2p - 4,p - l - 1)}{(2m + 2l + 1)(2m + 2l + 3)(2m + 2p - 3)} \]  
(34)
where

\[ j_{n,p} = \frac{[h_{n,p}(2m + 2p) - 3g_{n,p}](2m + 2p + 1)}{(2m + 2p - 2)(2m + 2p)}. \]  

(35)

The terms \( h_{n,p}, g_{n,p} \) are determined by the coefficients \( a_{k,j}, b_{k,j}, k < n \) of \( W_k, k < n \):

\[ h_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[ b_{k,p-j}b_{n-k,j} + a_{k,p-j}a_{n-k,j} - a_{k,p-1-j}a_{n-k,j} \right] \]  

(36)

\[ g_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[ b_{k,p-j}a_{n-k,j} + a_{k,p-j}b_{n-k,j} \right] \]  

(37)

4 The ground eigenfunction

Finally, according to the shape invariant potential, we obtain the wave functions of the exited state \( \Theta_0 \).

\[ W = W_0 + \sum_{n=1}^{\infty} \left[ \cos \theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \sin^{2k-1} \theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} \sin^{2k-1} \theta \right] \beta^n \]  

(38)

The ground eigenfunction becomes

\[ \Theta_0 = N \exp \left[ -\int W d\theta \right] \]

\[ = N \exp \left[ -\int W_0 d\theta - \sum_{n=1}^{\infty} \int W_n d\theta \right] \]

\[ = N \exp \left[ \int \frac{3/2 + (m + \frac{1}{2}) \cos \theta \sin \theta}{\sin \theta} d\theta - \beta^n \sum_{n=1}^{\infty} \int (\cos \theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \sin^{2k-1} \theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} \sin^{2k-1} \theta) d\theta \right] \]

\[ = N (1 - \cos \theta)^{\frac{1}{2}} \sin^{m-1} \theta \exp \left[ -\beta^n \sum_{n=1}^{\infty} \left( \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k} \theta}{2k} + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k - 1, \theta) \right) \right] \]  

(39)

The ground eigenvalue is

\[ E_{0,n;m} = E_{0,0;m} + \sum_{n=1}^{\infty} \beta^n E_{0,n;m} = m^2 + m - 15/2 + \sum_{n=1}^{\infty} \beta^n E_{0,n;m} \]  

(40)

5 The excited eigenfunctions

In the following, we will compute the excited eigenfunctions. As done in Ref. [11], we hope to extend the study of the recurrence relations by the means of super-symmetric quantum mechanics to Eq. (3).
The super-potential $W$ connects the two partner potential $V_{\pm}$ by

$$V^+(\theta) = W^2(\theta) \mp W'(\theta).$$  \hfill (41)

The shape-invariance properties mean that the pair of partner potentials $V^\pm(x)$ are similar in shape and differ only in the parameters, that is

$$V^+(\theta; a_1) = V^-(\theta; a_2) + R(a_1),$$  \hfill (42)

where $a_1$ is a set of parameters, $a_2$ is a function of $a_1$ (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of $\theta$.

We must introduce the parameters $A_{i,j}$, $B_{i,j}$ into the super-potential $W$ in order to study the shape-invariance properties of the spin-weighted spheroidal equations as:

$$W(A_{n,j}, B_{n,j}, \theta) = -A_{0,0} (m + \frac{1}{2}) \cot \theta - \frac{3}{2} B_{0,0} \csc \theta + \sum_{n=1}^{\infty} \beta^n W_n(A_{n,j}, B_{n,j}, \theta),$$  \hfill (43)

where

$$W_n(A_{n,j}, B_{n,j}, \theta)
\begin{align*}
&= \sum_{j=1}^{[\frac{n+1}{2}]} \bar{b}_{n,j} \sin^{2j-1} \theta + \cos \theta \sum_{j=1}^{[\frac{n}{2}]} \bar{a}_{n,j} \sin^{2j-1} \theta
\end{align*}$$  \hfill (44)

with

$$\bar{a}_{n,j} = A_{n,j} a_{n,j}, \quad \bar{b}_{n,j} = B_{n,j} b_{n,j}$$  \hfill (45)

Then, $V^\pm(A_{n,j}, B_{n,j}, \theta)$ are defined as

$$V^\pm(A_{n,j}, B_{n,j}, \theta) = W^2(A_{n,j}, B_{n,j}, \theta) \pm W'$$

$$= \sum_{n=0}^{\infty} \beta^n V^\pm_n(A_{i,j}, B_{n,j}, \theta).$$  \hfill (46)

The key point is to try to find some quantities $C_{i,j}, D_{i,j}$ to make the relations

$$V^+_n(A_{i,j}, B_{n,j}, \theta) = V^-_n(C_{i,j}, D_{n,j}, \theta) + R_{n,m}(A_{i,j}, B_{n,j})$$  \hfill (47)

retain with $R_{n,m}(A_{i,j}, B_{n,j}) = R_{n,m}$ pure quantities. In the previous, we know the general formula with $W_n$ in the case of $s = 3/2$ is same as $s = 1/2$. So that, we can refer to the results of previous article:

$$D_{n,p} = \frac{D_{0,0} a_{n,p}}{\alpha_p b_{n,p}} C_{n,p} - \frac{U_{n,p}}{\alpha_p b_{n,p}}$$  \hfill (48)

$$C_{n,p-1} = \left( \alpha_p + \frac{D_{0,0}^2}{\alpha_p} \right) a_{n,p} C_{n,p} + \frac{\bar{U}_{n,p}}{\alpha_p} - \frac{D_{0,0} a_{n,p}}{\alpha_p} U_{n,p}$$

$$+ \frac{1}{(\alpha_p - 1) a_{n,p-1}} C_{n,p} + \frac{\bar{U}_{n,p} - D_{0,0} a_{n,p}}{(\alpha_p - 1) a_{n,p-1}},$$  \hfill (49)

$$p = 2, 3, \ldots, \left[ \frac{n + 2}{2} \right]$$  \hfill (50)
where,

\[ U_{n,p} = -\alpha_p D_{n,p} b_{n,p} - D_{0,0} C_{n,p} a_{n,p} \]  
\[ \tilde{U}_{n,p} = -\alpha_p C_{n,p} a_{n,p} - D_{0,0} D_{n,p} b_{n,p} + (\alpha_p - 1) C_{n,p-1} a_{n,p-1} \]

\[ \alpha_p = \left[ (2m + 1)C_{0,0} + (2p - 1) \right] \]  

The only difference is the initial value. Now, we will give the initial value for the cases \( n = 0, 1, 2 \):

\[ C_{0,0} = A_{0,0} + \frac{2}{2m + 1} \]  
\[ D_{0,0} = B_{0,0} \]  
\[ D_{1,1} = \frac{(2m + 1)A_{0,0} - 1}{(2m + 1)A_{0,0} + 3} B_{1,1} \]

and

\[ D_{2,1} = \frac{(2m + 1)A_{0,0} - 1}{(2m + 1)A_{0,0} + 3} B_{2,1} \]

\[ + \frac{18B_{0,0}B_{2,1}}{(2m + 1)A_{0,0} + 3} \]

\[ + \frac{24[(2m + 1)A_{0,0} + 1]B_{0,0}B_{2,1}}{[(2m + 1)A_{0,0} + 3][2m + 1]A_{0,0} + 4} \]

\[ - \frac{[(2m + 1)A_{0,0} + 3]^3[(2m + 1)A_{0,0} + 4]}{[(2m + 1)A_{0,0} + 3][2m + 1]A_{0,0} + 4} \]

\[ C_{2,1} = \frac{8[(2m + 1)A_{0,0} + 1]B_{2,1}^2}{[(2m + 1)A_{0,0} + 3][2m + 1]A_{0,0} + 4} \]

\[ + \frac{(2m + 1)A_{0,0} - 2}{(2m + 1)A_{0,0} + 4} A_{2,1} \]

with

\[ R_{0,m}(A_{0,0}) = (2m + 1)A_{0,0} + 1, \]  
\[ R_{1,m}(A_{0,0}, B_{0,0}, B_{1,1}) = -\frac{12B_{0,0}B_{1,1}}{(2m + 1)A_{0,0} + 3} \]  
\[ R_{2,m}(A_{0,0}, B_{0,0}, B_{1,1}, B_{2,1}, A_{2,1}) \]

\[ = \left[ -\frac{4B_{0,0}B_{2,1}}{(2m + 1)A_{0,0} + 3} + AB_{1,1}^2 + BA_{2,1} \right] \]

where

\[ A = \frac{72B_{0,0}^2 - 8[(2m + 1)A_{0,0} - 1][2m + 1]A_{0,0} + 3}{[(2m + 1)A_{0,0} + 3]^3[2m + 1]A_{0,0} + 4} \]

\[ B = \frac{54B_{0,0}^2 - 2[(2m + 1)A_{0,0} - 1][2m + 1]A_{0,0} + 3}{[(2m + 1)A_{0,0} + 3][2m + 1]A_{0,0} + 4} \]
Then, the excited eigen-values \( E_{l,m} \) and eigenfunctions \( \Psi_l \) is obtained by the recurrence relation:

\[
E_{l,m}^- = E_{0,m} + \sum_{k=1}^{l} R(a_k, b_k),
\]

\[
E_{0,m} = m(m + 1) - \frac{15}{2} + \sum_{n=1}^{\infty} E_{0,n,m} \beta^n
\]

\[
R(a_k, b_k) = R_{0,m} + \sum_{n=1}^{\infty} \beta^n R_{n,m}(a_k, b_k),
\]

\[
a_1 = (A_{i,j}, B_{i,j}), \ a_2 = (C_{i,j}, D_{i,j}), \ldots,
\]

\[
\Psi_0 \propto \exp \left[ - \int^\theta_{\theta_0} W(A_{n,j}, B_{n,j}, \theta) d\theta \right],
\]

\[
A^\dagger = -\frac{d}{d\theta} + W(A_{n,j}, B_{n,j}, \theta)
\]

\[
\Psi_n^- = A^\dagger (A_{n,j}, B_{n,j}, \theta) \Psi_{n-1}^- (C_{n,j}, D_{n,j}, \theta), \quad n = 1, 2, 3, \ldots
\]

In conclusion, we have proved the shape-invariance properties for the spin-weighted equations in the case of \( s = \frac{3}{2} \) and obtain the recurrence relations for them. By these results we can get the exited eigenvalue and eigenfunction.

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## 6 The details of the calculation of the first several terms of the superpotential

In order to calculate Eqs. (14)-(15), we needs the following integral formulae [13]:

\[
P(2m, \theta) = \int \sin^{2m} \theta d\theta = -\frac{\cot \theta}{2m + 1} \left[ \sum_{k=0}^{m-1} \bar{I}(2m, k) \sin^{2m-2k} \theta \right] + \frac{(2m-1)!!}{(2m)!!} \]

where

\[
\bar{I}(2m, k) = \frac{(2m + 1)(2m - 1) \cdots (2m - 2k + 1)}{2m(2m - 2) \cdots (2m - 2k)} \quad (k \geq 0),
\]

the repeating results of the above equation

\[
P(2m, \theta) = \frac{2m - 1}{2m} P(2m - 2, \theta) - \frac{\cos \theta \sin^{2m-1} \theta}{2m},
\]

8
\[ P(2m + 2, \theta) = \frac{(2m - 1)(2m + 1)}{2m(2m + 2)} P(2m - 2, \theta) - \frac{\cos \theta \sin^{2m+1}}{2m + 2} - \frac{(2m + 1) \cos \theta \sin^{2m+1}}{2m(2m + 2)} \]

\[ P(2m + 4, \theta) = \frac{(2m - 1)(2m + 1)(2m + 3)}{2m(2m + 2)(2m - 4)} P(2m - 2, \theta) - \frac{\cos \theta \sin^{2m+3}}{2m + 4} - \frac{(2m + 3) \cos \theta \sin^{2m+1}}{(2m + 2)(2m + 4)} - \frac{(2m + 1)(2m + 3) \cos \theta \sin^{2m+1}}{2m(2m + 2)(2m - 4)} \]

By the above equation, we can summarize the general formula

\[ P(2m + 2n - 2) = -\cos \theta \sum_{l=1}^{n} \frac{\bar{I}(2m + 2n - 2, n - l)}{2m + 2n - 1} \sin^{2m+2l-3} \theta + \frac{2m - 1}{2m + 2n - 1} \bar{I}(2m + 2n - 2, n - 1) P(2m - 2, \theta) \]

This formula can be proved by mathematical induction. By the help of Eqs. (13), (75), \( A_1(\theta) \) is now simplified as

\[ A_1(\theta) \]

\[ = \int (E_{0,1;m} - 3\cos \theta)(1 - \cos \theta)^3 \sin^{2m-2} \theta \, d\theta \]

\[ = \frac{(4E_{0,1;m} + 12) P(2m - 2, \theta) - (3E_{0,1;m} + 15) P(2m, \theta) + 3P(2m + 2, \theta)}{4E_{0,1;m} + 12 \sin^{2m-1} \theta} - \frac{(9 + E_{0,1;m}) \sin^{2m+1} \theta}{2m + 1} \]

\[ = \frac{4(E_{0,1;m} + 12) \sin^{2m-1} \theta}{2m + 1} - \frac{(9 + E_{0,1;m}) \sin^{2m+1} \theta}{2m + 1} \]

\[ - \frac{3 \cos \theta \sin^{2m+1}}{2m + 2} + \frac{[(2m + 2)(3E_{0,1;m} + 15) - (2m + 1)] \cos \theta \sin^{2m-1} \theta}{2m(2m + 2)} \]

Please note that the coefficient of \( P(2m - 2, \theta) \), according to Eq. (5) and Eq. (71), we can see that \( \Psi(\theta) \) is infinite at the boundaries \( \theta = 0, \pi \). This result does not meet the boundary conditions that \( \Psi(\theta) \) should finite at \( \theta = 0, \pi \). So that the coefficient of the term \( P(2m - 2, \theta) \) should be zero.

\[ E_{0,1;m} = -\frac{9}{2m + 2} \]

and it further provides the concise form for \( A_1 \)

\[ A_1(\theta) = -\frac{12}{2m + 2} \sin^{2m-1} \theta (\cos \theta - 1) + \frac{3}{2m + 2} \sin^{2m+1} \theta (1 - \cos \theta) + \frac{3}{2m + 2} \sin^{2m+1} \]

(78)
With the help of Eq. (14), it is easy to obtain the first order \( W_1(\theta) \)

\[
W_1(\theta) = A_1(\theta) (1 + \cos \theta)^3 (\sin \theta)^{-2m-4} = -\frac{3}{2m+2} \sin \theta 
\] (79)

Now, we can see \( W_1(\theta) \) is convergence at the boundaries \( \theta = 0, \pi \). So that, the result of \( E_{0,1;m} \) is appropriate. By the same steps with more complex calculation than \( W_1(\theta) \), we can also get \( E_{0,2;m}, E_{0,3;m}, W_3 \) and \( E_{0,4;m}, W_4 \).

7  the calculation for the general form of \( W_n \)

Back to Eqs. (12), (14), (15), one needs to simplify the term \( \sum_{k=1}^{n-1} W_k W_{n-k} \) in order to calculate \( W_n \). Whenever \( 1 \leq k \leq n-1 \), one has \( 1 \leq n-k \leq n-1 \) and \( W_k(\theta) \), \( W_{n-k}(\theta) \) could be written in the form of (32). That is,

\[
W_k(\theta) = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} a_{k,i} \sin^{2i-1} \theta \cos \theta + \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,i} \sin^{2i-1} \theta \\
W_{n-k}(\theta) = \sum_{j=1}^{\lfloor \frac{n-k}{2} \rfloor} a_{n-k,j} \sin^{2j-1} \theta \cos \theta + \sum_{j=1}^{\lfloor \frac{n-k-1}{2} \rfloor} b_{n-k,j} \sin^{2j-1} \theta 
\] (80)

For the sake of later use, the facts are true

\[
a_{i,j} = 0, j < 1 or j > \lfloor \frac{i}{2} \rfloor; \quad b_{i,j} = 0, j < 1 or j > \lfloor \frac{i+1}{2} \rfloor
\] (81)

for \( i < n \). Thus

\[
\sum_{k=1}^{n-1} W_k W_{n-k} \\
= \sum_{k=1}^{n-1} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{n-k}{2} \rfloor} b_{k,i} b_{n-k,j} \sin^{2i+2j-2} \theta + \sum_{k=1}^{n-1} \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{n-k-1}{2} \rfloor} a_{k,i} b_{n-k,j} \sin^{2i+2j-2} \theta \cos^2 \theta \\
+ \cos \theta \sum_{k=1}^{n-1} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{n-k}{2} \rfloor} a_{k,i} b_{n-k,j} \sin^{2i+2j-2} \theta + \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{n-k-1}{2} \rfloor} b_{k,i} a_{n-k,j} \sin^{2i+2j-2} \theta 
\] (82)

\[
= \sum_{p=2}^{n-1} \sum_{k=1}^{n-1-p} \sum_{j=1}^{n-1-k} b_{k,p-j} b_{n-k,j} \sin^{2p-2} \theta + \sum_{p=2}^{n-1} \sum_{k=1}^{n-1-p} \sum_{j=1}^{n-1-k} a_{k,p-j} a_{n-k,j} \sin^{2p-2} \theta \cos^2 \theta \\
+ \cos \theta \left[ \sum_{p=2}^{n-1} \sum_{k=1}^{n-1-p} \sum_{j=1}^{n-1-k} b_{k,p-j} a_{n-k,j} \sin^{2p+1} \theta + \sum_{p=2}^{n-1-p} \sum_{k=1}^{n-1-p} \sum_{j=1}^{n-1-k} a_{k,p-j} b_{n-k,j} \sin^{2p+1} \theta \right] \\
= \sum_{p=2}^{\lfloor \frac{n}{2} \rfloor} \left[ h_{n,p} + g_{n,p} \cos \theta \right] \sin^{2p-2} \theta
\] (83)
where \( g_{n,p} \) and \( h_{n,p} \) are constant coefficients:

\[
h_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[ b_{k,p-j} b_{n-k,j} + a_{k,p-j} a_{n-k,j} - a_{k,p-1-j} a_{n-k,j} \right] \tag{84}
\]

\[
g_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[ b_{k,p-j} a_{n-k,j} + a_{k,p-j} b_{n-k,j} \right] \tag{85}
\]

\[
\bar{c}_1 = \left[ \frac{k+1}{2} \right] + \left[ \frac{n-k+1}{2} \right], \quad \bar{c}_2 = \left[ \frac{k}{2} \right] + \left[ \frac{n-k}{2} \right] \tag{86}
\]

\[
\bar{c}_3 = \left[ \frac{k+1}{2} \right] + \left[ \frac{n-k}{2} \right], \quad \bar{c}_4 = \left[ \frac{k}{2} \right] + \left[ \frac{n-k+1}{2} \right] \tag{87}
\]

It is easy to see

\[
\bar{c}_1 = \frac{n}{2} + 1, \quad \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \frac{n}{2}, \text{ when } n \text{ is even} \tag{88}
\]

\[
\bar{c}_2 = \frac{n-1}{2}, \quad \bar{c}_1 = \bar{c}_3 = \bar{c}_4 = \frac{n+1}{2}, \text{ when } n \text{ is odd} \tag{89}
\]

Hence, one has

\[
g_{n,p} = 0, \quad p < 1 \text{ or } p > \frac{n}{2}, \quad h_{n,p} = 0, \quad p < 1 \text{ or } p > \frac{n+1}{2}, \text{ when } n \text{ is even} \tag{90}
\]

\[
g_{n,p} = h_{n,p} = 0, \quad p < 1 \text{ or } p > \frac{n+1}{2}, \text{ when } n \text{ is odd.}
\]

We have used the fact (81) and substituted the quantities \( \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4 \) by the maximum \( \left[ \frac{n}{2} \right] + 1 \) of them in last line in the above equation. Taking \( f_n(z) = E_{0,n;m} + \sum_{k=1}^{n-1} W_k W_{n-k} \) into Eqs. (15) we can have

\[
A_n(\theta) = \int \left[ E_{0,n;m} + \sum_{p=2}^{\left\lceil \frac{n}{2} \right\rceil + 1} \left[ h_{n,p} + g_{n,p} \cos \theta \right] \sin^{2p-2} \theta (1 - \cos \theta)^3 \sin^{2m-2} \theta d\theta \right]
\]

\[
= \int 4E_{0,n;m} \sin^{2m-2} \theta d\theta - \int 3E_{0,n;m} \sin^{2m} \theta d\theta
\]

\[
- \frac{4E_{0,n;m}}{2m-1} \sin^{2m-1} \theta + \frac{E_{0,n;m}}{2m+1} \sin^{2m+1} \theta
\]

\[
+ \sum_{p=2}^{\left\lceil \frac{n}{2} \right\rceil + 1} \frac{4g_{n,p} - 4h_{n,p}}{2m+2p-3} \sin^{2m+2p-3} \theta + \sum_{p=2}^{\left\lceil \frac{n}{2} \right\rceil + 1} \frac{h_{n,p} - 3g_{n,p} \sin^{2m+2p-1} \theta}{2m+2p-1}
\]

\[
+ \sum_{p=2}^{\left\lceil \frac{n}{2} \right\rceil + 1} (4h_{n,p} - 4g_{n,p}) \sin^{2m+2p-4} \theta d\theta
\]
\[
\begin{align*}
  &\sum_{p=2}^{[\frac{n}{2}]+1} \int (5g_{n,p} - 3h_{n,p}) \sin^{2m+2p-2} \theta d\theta - \sum_{p=2}^{[\frac{n}{2}]+1} \int g_{n,p} \sin^{2m+2p} \theta d\theta \\
  &= 4E_{0,n;m} P(2m - 2, \theta) - 3E_{0,n;m} P(2m, \theta) - \frac{4E_{0,n;m}}{2m - 1} \sin^{2m-1} \theta + \frac{E_{0,n;m}}{2m + 1} \sin^{2m+1} \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} 4g_{n,p} - 4h_{n,p} \sin^{2m+2p-3} \theta + \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p} - 3g_{n,p} \sin^{2m+2p-1} \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} (4h_{n,p} - 4g_{n,p}) P(2m + 2p - 4, \theta) \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} (5g_{n,p} - 3h_{n,p}) P(2m + 2p - 2, \theta) - \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p} P(2m + 2p, \theta) \\
  &= 4E_{0,n;m} P(2m - 2, \theta) - 3E_{0,n;m} P(2m, \theta) - \frac{4E_{0,n;m}}{2m - 1} \sin^{2m-1} \theta + \frac{E_{0,n;m}}{2m + 1} \sin^{2m+1} \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} 4g_{n,p} - 4h_{n,p} \sin^{2m+2p-3} \theta + \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p} - 3g_{n,p} \sin^{2m+2p-1} \theta \\
  &\quad - \sum_{p=2}^{[\frac{n}{2}]+1} \left[ \frac{5g_{n,p} - 3h_{n,p}}{2m + 2p - 2} + \frac{(2m + 2p - 1)g_{n,p}}{(2m + 2p - 2)(2m + 2p)} \right] \sin^{2m+2p-3} \theta \cos \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p}}{2m + 2p} \sin^{2m+2p-1} \theta \cos \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} j_{n,p} P(2m + 2p - 4, \theta) \\
  &= 4E_{0,n;m} P(2m - 2, \theta) - 3E_{0,n;m} P(2m, \theta) - \frac{4E_{0,n;m}}{2m - 1} \sin^{2m-1} \theta + \frac{E_{0,n;m}}{2m + 1} \sin^{2m+1} \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} 4g_{n,p} - 4h_{n,p} \sin^{2m+2p-3} \theta + \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p} - 3g_{n,p} \sin^{2m+2p-1} \theta \\
  &\quad - \sum_{p=2}^{[\frac{n}{2}]+1} i_{n,p} \sin^{2m+2p-3} \theta \cos \theta + \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p}}{2m + 2p} \sin^{2m+2p-1} \theta \cos \theta \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} j_{n,p} P(2m + 2p - 4, \theta) \\
  &\quad + \sum_{p=2}^{[\frac{n}{2}]+1} j_{n,p} P(2m + 2p - 4, \theta)
\end{align*}
\]
For the coefficient of divergent term must be zero, that is

\[
i_{n,p} = \frac{5g_{n,p} - 3h_{n,p}}{2m + 2p - 2} + \frac{(2m + 2p - 1)g_{n,p}}{(2m + 2p - 2)(2m + 2p)}
\]

(92)

\[
j_{n,p} = 4(h_{n,p} - g_{n,p}) + \frac{2m + 2p - 3}{2m + 2p - 2}[5g_{n,p} - 3h_{n,p}]
\]

\[\quad - \frac{(2m + 2p - 3)(2m + 2p - 1)}{(2m + 2p)(2m + 2p - 2)}g_{n,p}
\]

\[\quad = 4(h_{n,p} - g_{n,p}) + (2m + 2p - 3)i_{n,p}
\]

(93)

\[
j_{n,p} = \frac{[h_{n,p}(2m + 2p) - 3g_{n,p}](2m + 2p + 1)}{(2m + 2p - 2)(2m + 2p)}
\]

(94)

According to Eq.(75), one has

\[
P(2m + 2p - 4) = -\cos \theta \sum_{l=1}^{p-1} \frac{I(2m + 2p - 4, p - l - 1)}{2m + 2p - 3} \sin^{2m+2l-3} \theta
\]

\[\quad + \frac{2m - 1}{2m + 2p - 3} I(2m + 2p - 4, p - 2) p(2m - 2, \theta)
\]

(95)

Hence,

\[
A_n(\theta)
\]

\[
= 4E_{0,n,m}P(2m - 2, \theta) + 3E_{0,n,m}\left[\frac{\cos \theta \sin^{2m-1} \theta}{2m} - \frac{2m - 1}{2m} P(2m - 2, \theta)\right]
\]

\[\quad - \frac{4E_{0,n,m}}{2m - 1} \sin^{2m-1} \theta + \frac{E_{0,n,m}}{2m + 1} \sin^{2m+1} \theta
\]

\[\quad + \sum_{p=2}^{\left[\frac{3}{2}\right]+1} \frac{4g_{n,p} - 4h_{n,p}}{2m + 2p - 3} \sin^{2m+2p-3} \theta + \sum_{p=2}^{\left[\frac{3}{2}\right]+1} \frac{h_{n,p} - 3g_{n,p}}{2m + 2p - 1} \sin^{2m+2p-1} \theta
\]

\[\quad - \sum_{p=2}^{\left[\frac{3}{2}\right]+1} i_{n,p} \sin^{2m+2p-3} \theta \cos \theta + \sum_{p=2}^{\left[\frac{3}{2}\right]+1} \frac{g_{n,p}}{2m + 2p} \sin^{2m+2p-1} \theta \cos \theta
\]

\[\quad + \sum_{p=2}^{\left[\frac{3}{2}\right]+1} j_{n,p} \left[\cos \theta \sum_{l=1}^{p-1} \frac{I(2m + 2p - 4, p - l - 1)}{2m + 2p - 3} \sin^{2m+2l-3} \theta
\]

\[\quad + \frac{2m - 1}{2m + 2p - 3} I(2m + 2p - 4, p - 2) P(2m - 2, \theta)\right]
\]

(96)

For the coefficient of divergent term must be zero, that is

\[
b_1 = \frac{(2m + 3)}{2m} E_{0,n,m} + \sum_{p=2}^{\left[\frac{3}{2}\right]+1} j_{n,p} \frac{2m - 1}{2m + 2p - 3} I(2m + 2p - 4, p - 2) = 0
\]

(97)
then

\[
E_{0,n,m} = -\frac{2m}{2m+3} \sum_{p=2}^{\left[\frac{n}{2}\right]+1} j_{n,p} \frac{2m-1}{2m+2p-3} \bar{I}(2m+2p-4, p-2)
\]

\[
= -\sum_{p=2}^{\left[\frac{n}{2}\right]+1} \frac{2m(2m-1)[h_{n,p}(2m+2p)-3g_{n,p}](2m+2p+1)}{(2m+3)(2m+2p-3)(2m+2p-2)(2m+2p)} \bar{I}(2m+2p-4, p-2)
\]

Taking Eq.(98) into Eq.(??), we can get

\[
A_n(\theta) = -\frac{4E_{0,n,m}}{2m-1} \sin^{2m-1} \theta + \frac{E_{0,n,m}}{2m} \sin^{2m+1} \theta + \frac{3E_{0,n,m}}{2m} \sin^{2m+1} \theta \cos \theta
\]

\[
+ \sum_{p=2}^{\left[\frac{n}{2}\right]+1} \frac{4g_{n,p} - 4h_{n,p} \sin^{2m+2p-3} \theta + \left[\frac{n}{2}\right]+1 h_{n,p} - 3g_{n,p}}{2m+2p-1} \sin^{2m+2p-1} \theta \cos \theta
\]

\[
- \sum_{p=2}^{\left[\frac{n}{2}\right]+1} i_{n,p} \sin^{2m+2p-3} \theta \cos \theta + \sum_{p=2}^{\left[\frac{n}{2}\right]+1} j_{n,p} \cos \theta \sum_{l=1}^{p-1} \frac{\bar{I}(2m+2p-4, p-l-1)}{2m+2p-3} \sin^{2m+2l-3} \theta
\]

(99)

It is easy to see the factor \((1 - \cos \theta)^3\) in the integrand in above calculation make the form of \(A_n(\theta)\) very complicated. More calculation are need to treat the problem now. The results are

\[
W_n(\theta) = \cos \theta \sum_{l=1}^{\left[\frac{n}{2}\right]} a_{n,l} \sin^{2l-1} \theta + \sum_{l=1}^{\left[\frac{n-1}{2}\right]} b_{n,l} \sin^{2l-1} \theta
\]

(100)

\[
b_{n,l} = \sum_{p=l+1}^{\left[\frac{n}{2}\right]+1} \frac{3(2m + 2l - 2)j_{n,p} \bar{I}(2m+2p-4, p-l-1)}{(2m+2l-1)(2m+2l+3)(2m+2p-3)} + \frac{g_{n,l}}{2m+2l} \zeta_{l,1}, l \geq 1
\]

(101)

\[
a_{n,l} = -\sum_{p=l+1}^{\left[\frac{n}{2}\right]+1} \frac{(2m + 2l - 2)(2m + 2l)j_{n,p} \bar{I}(2m+2p-4, p-l-1)}{(2m+2l-1)(2m+2l+3)(2m+2p-3)}, l \geq 1
\]

(102)
where \( \xi_{l,1} = 0, l = 1 \) and otherwise \( \xi_{l,1} = 1, l \neq 1 \). We delay the process of proved in the appendix-1. Therefore, we have proved the correctness of our induction about the general formula with \( W_n \). That is to say, any \( n \geq 1, W_n \) satisfy the form of general formula. Interestingly, the general formula with \( W_n \) in the case of \( s = 3/2 \) is same as \( s = 1/2 \).

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