Electric-magnetic Duality of Abelian Gauge Theory on the Four-torus, from the Fivebrane on $T^2 \times T^4$, via their Partition Functions

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Abstract

We compute the partition function of four-dimensional abelian gauge theory on a general four-torus $T^4$ with flat metric using Dirac quantization. In addition to an $SL(4,\mathbb{Z})$ symmetry, it possesses $SL(2,\mathbb{Z})$ symmetry that is electromagnetic S-duality. We show explicitly how this $SL(2,\mathbb{Z})$ S-duality of the 4d abelian gauge theory has its origin in symmetries of the 6d $(2,0)$ tensor theory, by computing the partition function of a single fivebrane compactified on $T^2$ times $T^4$, which has $SL(2,\mathbb{Z}) \times SL(4,\mathbb{Z})$ symmetry. If we identify the couplings of the abelian gauge theory $\tau = \theta/2\pi + i\pi e^2$ with the complex modulus of the $T^2$ torus $\tau = \beta^2 + i\beta_1\beta_2$, then in the small $T^2$ limit, the partition function of the fivebrane tensor field can be factorized, and contains the partition function of the 4d gauge theory. In this way the $SL(2,\mathbb{Z})$ symmetry of the 6d tensor partition function is identified with the S-duality symmetry of the 4d gauge partition function. Each partition function is the product of zero mode and oscillator contributions, where the $SL(2,\mathbb{Z})$ acts suitably. For the 4d gauge theory, which has a Lagrangian, this product redistributes when using path integral quantization.

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1 Introduction

Four-dimensional \( N = 4 \) Yang-Mills theory is conjectured to possess S-duality, which implies the theory with gauge coupling \( g \), gauge group \( G \), and theta parameter \( \theta \) is equivalent to one with \( \tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \) transformed by modular transformations \( SL(2, \mathbb{Z}) \), and the group to \( G^\vee \) [1]-[3], with the weight lattice of \( G^\vee \) dual to that of \( G \). The conjecture has been tested by the Vafa-Witten partition function on various four-manifolds [4]. More recently, a computation of the \( N = 4 \) Yang-Mills partition function on the four-sphere using the localization method for quantization, enables checking S-duality directly [5].

This duality is believed to have its origin in a certain super conformal field theory in six dimensions, the M5 brane \((2,0)\) theory. When the \( 6d \), \( N = (2,0) \) theory is compactified on \( T^2 \), one obtains the \( 4d \), \( N = 4 \) Yang-Mills theory, and the \( SL(2, \mathbb{Z}) \) group of the torus should imply the S-duality of the four-dimensional gauge theory [6]-[9].

In this paper, we compare the partition function of the \( 6d \) chiral tensor boson of one fivebrane compactified on \( T^2 \times T^4 \), with that of \( U(1) \) gauge theory with a \( \theta \) parameter, compactified on \( T^4 \). We use these to show explicitly how the \( 6d \) theory is the origin of S-duality in the gauge theory. Since the \( 6d \) chiral boson has a self-dual three-form field strength and thus lacks a Lagrangian [10], we will use the Hamiltonian formulation to compute the partition functions for both theories.

As motivated by [11], the four-dimensional \( U(1) \) gauge partition function on \( T^4 \) is

\[
Z^{4d,\text{Maxwell}} = \text{tr} e^{-2\pi H^{4d} + 2\pi \gamma^\alpha F_\alpha} = Z_{\text{zero modes}}^{4d} \cdot Z_{\text{osc}}^{4d},
\]

where the Hamiltonian and momentum are

\[
H^{4d} = \int_0^{2\pi} d^3 \theta \left( \frac{e^2}{32\pi^2} g_{\alpha\beta} \Pi^{\alpha} \Pi^{\beta} \right) + \frac{e^2}{32\pi^2} \sqrt{g} \left( \frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4} \right) g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{\theta e^2}{16\pi^2} \sqrt{g} g_{\alpha\beta} e^{\alpha\gamma\delta} F_{\gamma\delta} \Pi^{\beta},
\]

\[
P_\alpha^{4d} = \int_0^{2\pi} d^3 \theta \, \Pi^{\beta} F_{\alpha\beta},
\]

in terms of the gauge field strength tensor \( F_{ij}(\theta^3, \theta^4, \theta^5, \theta^6) \), the conjugate momentum \( \Pi^\alpha \), and the constant parameters \( g_{\alpha\beta}, R_6 \) and \( \gamma^\alpha \) in the metric \( G_{ij} \) of \( T^4 \). They will be derived from the abelian gauge theory Lagrangian, given here for Euclidean signature

\[
I = \int_0^{2\pi} d^3 \theta \, \Pi^{\beta} F_{\alpha\beta} \left( \frac{4\pi}{e^2} \sqrt{g} R_{ij} F_{ij} - \frac{i\theta}{4\pi} \epsilon^{ijkl} F_{ij} F_{kl} \right),
\]

with \( \epsilon^{3456} = 1 \), \( \epsilon^{ijkl} = g \epsilon^{ijkl} \), and \( g = \det(G_{ij}) \).
In contrast, the partition function of the abelian chiral two-form on $T^2 \times T^4$ is [12]

$$Z_{6d, chiral}^{6d} = \text{tr} e^{-2\pi R_0 H + i2\pi \gamma^a \mathcal{P}_a} = Z_{\text{zero modes}}^{6d} \cdot Z_{\text{osc}}^{6d},$$

$$\mathcal{H} = \frac{1}{12} \int_0^{2\pi} d\theta^1 \cdots d\theta^5 \sqrt{G_5} G_5^{mm'} G_5^{nn'} G_5^{pp'} H_mnp(\bar{\theta}, \theta^6) H_{m'n'p'}(\bar{\theta}, \theta^6),$$

$$\mathcal{P}_\alpha = -\frac{1}{24} \int_0^{2\pi} d\theta^1 \cdots d\theta^5 \epsilon_{mnprs} H_{mnp}(\tilde{\theta}, \theta^6) H_{\alpha rs}(\tilde{\theta}, \theta^6)$$

(1.4)

where $\theta^1$ and $\theta^2$ are the coordinates of the two one-cycles of $T^2$. The time direction $\theta^6$ is common to both theories, the angle between $\theta^1$ and $\theta^2$ is $\beta^2$, and $G_5^{mn}$ is the inverse metric of $G_5^{mn}$, where $1 \leq m, n \leq 5$. The eight angles between the two-torus and the four-torus are set to zero.\(^1\)

Section 2 is a list of our results; their derivations are presented in the succeeding sections. In section 3, the contribution of the zero modes to the partition function for the chiral theory on the manifold $M = T^2 \times T^4$ is computed as a sum over ten integer eigenvalues using the Hamiltonian formulation. The zero mode sum for the gauge theory on the same $T^4 \subset M$ is calculated with six integer eigenvalues. We find that once we identify the modulus of the $T^2$ contained in $M$, $\tau = \beta^2 + i\frac{\alpha_0}{\beta^2}$, with the gauge couplings $\tau = \frac{\theta}{2\pi} + i\frac{\alpha_0}{\beta^2}$, then the two theories are related by $Z_{\text{zero modes}}^{6d} = \epsilon Z_{\text{zero modes}}^{4d}$, where $\epsilon$ is due to the zero modes of the scalar field that arises in addition to $F_{ij}$ from the compactification of the 6d self-dual three-form. In section 4, the abelian gauge theory is quantized on a four-torus using Dirac constraints, and the Hamiltonian and momentum are computed in terms of oscillator modes. For small $T^2$, the Kaluza-Klein modes are removed from the partition function of the chiral two-form, and in this limit it agrees with the gauge theory result, up to the scalar field contribution. In Appendix A, we show the path integral quantization gives the same result for the 4d gauge theory partition function as canonical quantization. However, the zero and oscillator mode contributions differ in the two quantizations. In Appendix B, we show how the zero and oscillator mode contributions transform under $SL(2, \mathbb{Z})$ for the 6d theory, as well as for both quantizations of the 4d theory. We prove the partition functions in 4d and 6d are both $SL(2, \mathbb{Z})$ invariant. In Appendix C, the vacuum energy is regularized. In Appendix D, we introduce a complete set of $SL(4, \mathbb{Z})$ generators, and then prove the 4d and 6d partition functions are invariant under $SL(4, \mathbb{Z})$ transformations.

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\(^1\)A different consideration of the fivebrane on $T^2 \times T^4$ in [13] includes the time direction in $T^2$.  

3
2 Statement of the main result

We compute partition functions for a chiral boson on $T^2 \times T^4$ and for a $U(1)$ gauge boson on
the same $T^4$. The geometry of the manifold $T^2 \times T^4$ will be described by the line element,

$$ds^2 = R_2^2(d\theta^2 - \beta^2 d\phi^1)^2 + R_1^2(d\phi^1)^2 + \sum_{\alpha, \beta} g_{\alpha\beta}(d\theta^\alpha - \gamma^\alpha d\phi^6)(d\phi^\beta - \gamma^\beta d\phi^6) + R_6^2(d\phi^6)^2,$$

(2.1)

with $0 \leq \theta^I \leq 2\pi$, $1 \leq I \leq 6$, and $3 \leq \alpha \leq 5$. $R_1$, $R_2$ are the radii for directions $I = 1, 2$ on $T^2$, and $\beta^2$ is the angle between them. $g_{\alpha\beta}$ fixes the metric for a $T^3$ submanifold of $T^4$, $R_6$ is the remaining radius, and $\gamma^\alpha$ is the angle between those. So, from (2.1) the metric is

$$T^2: \quad G_{11} = R_1^2 + R_2^2 \beta^2, \quad G_{12} = -R_1^2 \beta^2, \quad G_{22} = R_2^2;$$

$$T^4: \quad G_{\alpha\beta} = g_{\alpha\beta}, \quad G_{\alpha 6} = -g_{\alpha\beta} \gamma^\beta, \quad G_{66} = R_6^2 + g_{\alpha\beta} \gamma^\alpha \gamma^\beta;$$

$$G_{\alpha 1} = G_{\alpha 2} = 0, \quad G_{16} = G_{26} = 0;$$

(2.2)

and the inverse metric is

$$T^2: \quad G^{11} = \frac{1}{R_1^2}, \quad G^{12} = \frac{\beta^2}{R_1^2}, \quad G^{22} = \frac{1}{R_2^2} + \frac{\beta^2 \beta^2}{R_1^2} = g^{22} + \frac{\beta^2 \beta^2}{R_1^2};$$

$$T^4: \quad G^{\alpha\beta} = g^{\alpha\beta} + \frac{\gamma^\alpha \gamma^\beta}{R_6^2}, \quad G^{\alpha 6} = \frac{\gamma^\alpha}{R_6^2}, \quad G^{66} = \frac{1}{R_6^2};$$

$$G^{1\alpha} = G^{2\alpha} = 0, \quad G^{16} = G^{26} = 0.$$

(2.3)

$\theta^6$ is chosen to be the time direction for both theories. In the 4d expression (1.3) the indices of the field strength tensor have $3 \leq i,j,k,l \leq 6$, whereas in (1.4), the Hamiltonian and momentum are written in terms of fields with indices $1 \leq m,n,p,r,s \leq 5$. The 5-dimensional inverse in directions 1, 2, 3, 4, 5 is $G_5^{mn}$,

$$G_5^{11} = \frac{1}{R_1^2}, \quad G_5^{12} = \frac{\beta^2}{R_1^2}, \quad G_5^{22} = g_5^{22} + \frac{\beta^2 \beta^2}{R_1^2};$$

$$G_5^{\alpha\beta} = g_5^{\alpha\beta}, \quad G_5^{1\alpha} = 0, \quad G_5^{2\alpha} = 0.$$

(2.4)

$g^{\alpha\beta}$ is the 3d inverse of $g_{\alpha\beta}$. The determinants are related by

$$\sqrt{G} = \sqrt{\det G_{IJ}} = R_1 R_2 \sqrt{g} = R_1 R_2 R_6 \sqrt{\tilde{g}} = R_6 \sqrt{G_5},$$

(2.5)

where $G$ is the determinant for 6d metric $G_{IJ}$. $G_5$, $g$ and $\tilde{g}$ are the determinants for the 5d metric $G_{mn}$, 4d metric $G_{ij}$, and 3d metric $g_{\alpha\beta}$ respectively.

The zero mode partition function of the 6d chiral boson on $T^2 \times T^4$ with the metric (2.3) is
where the zero mode eigenvalues of the field strength tensor are integers, and (2.6) factors into a sum on $H_{\alpha\beta\gamma}$ as $H_{345} = n_7$, $H_{12\alpha}$ as $H_{123} = n_8$, $H_{124} = n_9$, $H_{125} = n_{10}$; and a sum over $H_{1\alpha\beta}$ defined as $H_{134} = n_1$, $H_{145} = n_2$, $H_{135} = n_3$ and $H_{2\alpha\beta}$ as $H_{234} = n_4$, $H_{245} = n_5$, $H_{235} = n_6$, as we will show in section 3.

The zero mode partition function of the 4d gauge boson on $T^4$ with the metric (2.2) is

\[
Z_{\text{zero modes}}^{4d} = \sum_{n_4,n_5,n_6} \exp\left(-\frac{e^2 R_6^2}{4g} g_{\alpha\beta} \bar{\Pi}^\alpha \bar{\Pi}^\beta \right) \cdot \sum_{n_1,n_2,n_3} \exp\left(-\frac{\theta e^2 R_6^2}{8\pi} g_{\alpha\beta} \epsilon^{\alpha\gamma\delta} \bar{F}_{\gamma\delta} \bar{\Pi}^\beta \right) \\
\cdot \exp\left(-\frac{e^2 \sqrt{g}}{8} (\theta^2 + \frac{16\pi^2}{e^4}) g_{\alpha\beta} \gamma^4 \bar{F}_{\alpha\beta} \bar{\Pi}^\beta \bar{F}_{\alpha\beta} \right),
\]

(2.7)

where $\bar{\Pi}^\alpha$ take integer values $\bar{\Pi}^\alpha = n_4$, $\bar{\Pi}^4 = n_5$, $\bar{\Pi}^5 = n_6$, and $\bar{F}_{34} = n_1$, $\bar{F}_{35} = n_2$, $\bar{F}_{45} = n_3$, from section 3. We identify the integers

\[
H_{2\alpha\beta} = \bar{F}_{\alpha\beta} \quad \text{and} \quad H_{1\alpha\beta} = \frac{1}{g} \epsilon_{\alpha\beta\gamma} \bar{\Pi}^\gamma,
\]

(2.8)

where $\bar{g} = g R_6^{-2}$ from (2.5), and the modulus

\[
\tau = \beta^2 + i \frac{R_1}{R_2} = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2},
\]

so that as shown in section 3, we have the factorization

\[
Z_{\text{zero modes}}^{6d} = \epsilon Z_{\text{zero modes}}^{4d},
\]

(2.9)

where $\epsilon$ comes from the remaining four zero modes $H_{\alpha\beta\gamma}$ and $H_{12\alpha}$ due to the additional scalar that occurs in the compactification of the 6d self-dual three-form field strength,
From section 4, there is a similar relation between the oscillator partition functions

\[
\lim_{R_1, R_2 \to 0} Z^{6d}_{\text{osc}} = \epsilon' Z^{4d}_{\text{osc}},
\]

where

\[
Z^{6d}_{\text{osc}} = \left( e^{R_6 \pi^{-3} \sum_{\delta \neq 0} \frac{\sqrt{25}}{G_{m,p} n_{\alpha} n_{\beta} n_{\gamma}}^2} \prod_{\vec{p} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g_{\alpha\beta} p_{\alpha} p_{\beta} + p^2 + 2\pi i \gamma n_{\alpha}}}} \right)^3,
\]

\[
Z^{4d}_{\text{osc}} = \left( e^{\frac{1}{2} R_6 \pi^{-2} \sum_{\delta \neq 0} \frac{\sqrt{2}}{G_{m,p} n_{\alpha} n_{\beta} n_{\gamma}}^2} \prod_{\vec{n} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g_{\alpha\beta} n_{\alpha} n_{\beta} - 2\pi i \gamma n_{\alpha}}}} \right)^2,
\]

where \( \vec{p}^2 \equiv \frac{p^2}{R_1^2} + (\frac{\beta^2}{R_2^2} + \frac{\beta^2}{R_2^2})p_2^2 + \frac{2\beta^2}{R_2^2}p_1p_2 \), and \( \epsilon' \) is the oscillator contribution from the additional scalar,

\[
\epsilon' = e^{\frac{1}{2} R_6 \pi^{-2} \sum_{\delta \neq 0} \frac{\sqrt{2}}{G_{m,p} n_{\alpha} n_{\beta} n_{\gamma}}^2} \prod_{\vec{n} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g_{\alpha\beta} n_{\alpha} n_{\beta} - 2\pi i \gamma n_{\alpha}}}}.
\]

Therefore, in the limit of small \( T^2 \), we have

\[
\lim_{R_1, R_2 \to 0} Z^{6d, \text{chiral}} = \epsilon' \epsilon \ Z^{4d, \text{Maxwell}}.
\]

We use this relation between the 6d and 4d partition functions to extract the S-duality of the latter from a geometric symmetry of the former. For \( \tau = \beta^2 + i\frac{R_1}{R_2} = \theta + i\frac{\pi}{2\tau} \), under the \( SL(2, \mathbb{Z}) \) transformations

\[
\tau \rightarrow -\frac{1}{\tau}; \quad \tau \rightarrow \tau - 1,
\]

\( Z^{6d}_{\text{zero modes}} \) and \( Z^{6d}_{\text{osc}} \) are separately invariant, as are \( Z^{4d}_{\text{zero modes}} \) and \( Z^{4d}_{\text{osc}} \), which we will prove in Appendix B. In particular, \( Z^{4d}_{\text{osc}} \) is independent of \( \epsilon^2 \) and \( \theta \). A path integral computation agrees with our \( U(1) \) partition function, as we review in Appendix A [14]. Nevertheless, in the path integral quantization the zero and non-zero mode contributions are rearranged, and although each is invariant under \( \tau \to \tau - 1 \), they transform differently under \( \tau \to -\frac{1}{\tau} \), with \( Z_{\text{zero modes}}^{PI} \to |\tau|^3 Z_{\text{zero modes}}^{PI} \) and \( Z_{\text{non-zero modes}}^{PI} \to |\tau|^{-3} Z_{\text{non-zero modes}}^{PI} \). For a general spin manifold, the \( U(1) \) partition function transforms as a modular form under S-duality [15], but in the case of \( T^3 \) which we consider in this paper the weight is zero.
3 Zero Modes

In this section, we show details for the computation of the zero mode partition functions. The $N = (2, 0)$, 6d world volume theory of the fivebrane contains a chiral two-form $B_{MN}$, which has a self-dual three-form field strength $H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ with $1 \leq L, M, N \leq 6$,

$$H_{LMN}(\vec{\theta}, \theta^6) = \frac{1}{6\sqrt{-G}}G_{LL'}G_{M'M'}G_{N'N'}\epsilon^{LMN'R'S'T'}H_{RST}(\vec{\theta}, \theta^6). \quad (3.1)$$

Since there is no covariant Lagrangian description for the chiral two-form, we compute its partition function from (1.4). As in [12],[16] the zero mode partition function of the 6d chiral theory is calculated in the Hamiltonian formulation similarly to string theory,

$$Z_{6d}^{\text{zero modes}} = tr(e^{-t\mathcal{H} + iy^lP_l}) \quad (3.2)$$

where $t = 2\pi R_6$ and $y^l = 2\pi \frac{G_{66}}{G_{00}}$, with $l = 1,..,5$. However, $y^1$ and $y^2$ are zero due to the metric (2.3). Neglecting the integrations and using the metric (2.4) in (1.4), we find

$$-t\mathcal{H} = -\frac{\pi}{6}R_6R_1R_2\sqrt{gg}^{\alpha\alpha'}g^{\beta\beta'}\lambda'\lambda''H_{\alpha\beta\lambda}H_{\alpha'\beta'\lambda''} - \frac{\pi}{2}R_6R_1R_2\sqrt{gg}^{\alpha\alpha'}g^{\beta\beta'}H_{2\alpha\beta}H_{2\alpha'\beta'}$$

$$-\frac{\pi}{2}R_6R_1R_2\beta^2g^{\alpha\alpha'}\sqrt{g}H_{2\alpha\beta}H_{2\alpha'\beta'} - \frac{\pi}{2}R_6R_1R_2\beta^2g^{\alpha\alpha'}g^{\beta\beta'}H_{1\alpha\beta}H_{1\alpha'\beta'}$$

$$-\frac{\pi}{2}R_6R_1R_2\sqrt{gg}^{\alpha\alpha'}H_{12\alpha}H_{12\alpha'} - \frac{\pi}{4}R_6R_1R_2\sqrt{g}(g^{\alpha\alpha'}g^{\beta\beta'} - g^{\alpha\beta'}g^{\alpha'\beta})H_{1\alpha\beta}H_{1\alpha'\beta'}, \quad (3.3)$$

and the momentum components $3 \leq \alpha \leq 5$ are

$$P_{\alpha} = -\frac{1}{2}e^{\gamma\beta\delta}H_{12\gamma}H_{\alpha\beta\delta} + \frac{1}{2}e^{\gamma\beta\delta}H_{1\gamma\beta}H_{2\alpha\delta}, \quad (3.4)$$

where the zero modes of the ten fields $H_{imp}$ are labeled by integers $n_1, \ldots, n_{10}$ [12]. Then (3.2) is given by (2.6).

Similarly, we compute the zero mode partition function for the 4d $U(1)$ theory from (1.1). We consider the charge quantization condition

$$n_I = \frac{1}{2\pi} \int_{\Sigma_I^L} F = \frac{1}{2\pi} \int_{\Sigma_I^L} \frac{1}{2} F_{\alpha\beta} d\theta^\alpha \wedge d\theta^\beta, \quad n_I \in \mathbb{Z}, \text{ for each } 1 \leq I \leq 3. \quad (3.5)$$

as well as the commutation relation obtained from (4.17)

$$\left[ \int_{\Sigma_I^L} A_{\alpha}(\vec{\theta}, \theta^6) d\theta^\alpha, \int \frac{d^3\theta'}{2\pi} \Pi^{(\theta', \theta^6)} \right] = \frac{i}{2\pi} \int_{\Sigma_I^L} d\theta^\beta = i \delta^\beta_4, \quad (3.6)$$

and use the standard argument [16],[17] to show that the field strength $F_{\alpha\beta}$ and momentum
\( \Pi^\alpha \) zero modes have eigenvalues

\[
F_{\alpha \beta} = \frac{n_{\alpha \beta}}{2\pi}, \quad n_{\alpha \beta} \in \mathbb{Z} \text{ for } \alpha < \beta, \quad \text{and} \quad \Pi^\alpha(\vec{\theta}, \theta^6) = \frac{n^{(\alpha)}}{(2\pi)^2}, \quad n^{(\alpha)} \in \mathbb{Z}^3. \tag{3.7}
\]

Thus we define integer valued modes \( \tilde{F}_{\alpha \beta} \equiv 2\pi F_{\alpha \beta} \) and \( \tilde{\Pi}^\alpha \equiv (2\pi)^2 \Pi^\alpha \). Taking into account the spatial integrations \( d\theta^\alpha \), (1.2) gives

\[
-2\pi \mathcal{Z}^{4d} + i2\pi \gamma^\alpha P^{4d}_\alpha
= -e^2 R^2 \frac{\sqrt{g}}{4} g_{\alpha \beta} \Pi^\alpha \Pi^\beta - e^2 \frac{\sqrt{g}}{8} \left[ \frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4} \right] g^{\alpha \beta} g^{\gamma \delta} \tilde{F}_{\alpha \gamma} \tilde{F}_{\beta \delta} - \theta e^2 R^2 \frac{\sqrt{g}}{8} g_{\alpha \beta} \epsilon^{\alpha \gamma \delta} \tilde{F}_{\gamma \delta} \tilde{F}_{\alpha \beta} \\
+ 2\pi i \gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha \beta}.
\tag{3.8}
\]

where (1.2) itself is derived in section 4. So from (3.8) and (1.1),

\[
\mathcal{Z}^{4d}_{\text{zero modes}} = \sum_{n_4, n_5, n_6} \exp\left\{-\frac{e^2 R^2}{4} \frac{\sqrt{g}}{4\pi^2} g_{\alpha \beta} \Pi^\alpha \Pi^\beta \right\} \cdot \sum_{n_1, n_2, n_3} \exp\left\{-\frac{\theta e^2 R^2}{8} \frac{\sqrt{g}}{8} g_{\alpha \beta} \epsilon^{\alpha \gamma \delta} \tilde{F}_{\gamma \delta} \tilde{F}_{\alpha \beta} \right\} \\
\cdot \exp\left\{-\frac{e^2 \sqrt{g}}{8} \frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4} \right\} g^{\alpha \beta} g^{\gamma \delta} \tilde{F}_{\alpha \gamma} \tilde{F}_{\beta \delta} + 2\pi i \gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha \beta} \right\}, \tag{3.9}
\]

where \( n_I \) are integers, with \( \tilde{F}_{34} = n_1, \tilde{F}_{35} = n_2, \tilde{F}_{45} = n_3, \) and \( \tilde{\Pi}^3 = n_4, \tilde{\Pi}^4 = n_5, \tilde{\Pi}^6 = n_6. \)

(3.9) is the zero mode contribution to the 4d \( U(1) \) partition function (1.1), and is (2.7).

If we identify the gauge couplings \( \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^4} \) with the modulus of \( T^2 \), \( \tau = \beta^2 + i \frac{R_2}{R_1} \), then

\[
e^2 \frac{1}{4\pi} = \frac{R_2}{R_1}, \quad \frac{\theta}{2\pi} = \beta^2, \tag{3.10}
\]

and (3.9) becomes

\[
\mathcal{Z}^{4d}_{\text{zero modes}} = \sum_{n_4, n_5, n_6} \exp\left\{-\frac{\pi R_2 R_2}{R_1 \sqrt{g}} g_{\alpha \beta} \Pi^\alpha \Pi^\beta \right\} \cdot \sum_{n_1, n_2, n_3} \exp\left\{-\frac{\pi \beta^2 R_2 R_2}{R_1 \sqrt{g}} g_{\alpha \beta} \epsilon^{\alpha \gamma \delta} \tilde{F}_{\gamma \delta} \tilde{F}_{\alpha \beta} \right\} \\
\cdot \exp\left\{-\frac{\pi R_2}{2} \sqrt{g} (\beta^2 + \frac{R_2}{R_1}) g^{\alpha \beta} g^{\gamma \delta} \tilde{F}_{\alpha \gamma} \tilde{F}_{\beta \delta} + 2\pi i \gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha \beta} \right\}. \tag{3.11}
\]
Then the last four terms in the chiral boson zero mode sum (2.6) are equal to (2.7) since
\[
-\frac{\pi}{2} R_2 R_6 \sqrt{g} (\frac{R_2^2}{R_1})^2 g^{\alpha\alpha'} g^{\beta\beta'} H_{2\alpha\beta} H_{2\alpha'\beta'} = -\frac{\pi}{2} R_2 R_6 \sqrt{g} \frac{R_1^2}{R_2} g^{\alpha\beta} g^{\gamma\delta} \tilde{F}_{\alpha\gamma} \tilde{F}_{\beta\delta},
\]

\[
-\frac{\pi}{2} R_6 R_2 \sqrt{g} \frac{R_2^2}{R_1} g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{2\alpha'\beta'} = -\frac{\pi}{2} R_6 R_2 \sqrt{g} \frac{R_1}{R_2} g^{\alpha\beta} g^{\gamma\delta} \tilde{F}_{\alpha\gamma} \tilde{F}_{\beta\delta},
\]

\[
i\pi \gamma^\alpha \epsilon^{\gamma\delta} H_{1\gamma\beta} H_{2\alpha\delta} = 2 \pi i \gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha\beta},
\]

\[
-\frac{\pi}{2} R_6 R_2 \frac{R_1}{R_2} \sqrt{g} g^{\alpha\alpha'} g^{\beta\beta'} H_{1\alpha\beta} H_{1\alpha'\beta'} = -\frac{\pi}{2} R_6 R_2 \frac{R_1}{R_2} g^{\alpha\beta} g^{\gamma\delta} \tilde{F}_{\alpha\gamma} \tilde{F}_{\beta\delta},
\]

(3.12)

when we identify the integers
\[
H_{2\alpha\beta} = \tilde{F}_{\alpha\beta} \quad \text{and} \quad H_{1\alpha\beta} = \frac{1}{g} \epsilon_{\alpha\beta\gamma} \tilde{\Pi}^\gamma,
\]

(3.13)

with \(\tilde{g} = g R_6^{-2}\) from (2.5). Thus the 6d and 4d zero mode sums from (2.6) and (2.7) are related by
\[
Z_{6d \text{ modes}} = \epsilon Z_{4d \text{ modes}},
\]

(3.14)

where
\[
\epsilon = \sum_{n_8,n_9,n_{10}} \epsilon_{n_8,n_9,n_{10}} \exp\left\{-\frac{\pi}{2} R_6 \frac{R_1}{R_2} \sqrt{g} g^{\alpha\alpha'} H_{12\alpha} H_{12\alpha'}\right\}
\]

\[
\times \sum_{n_7} \epsilon_{n_7} \exp\left\{-\frac{\pi}{6} R_6 R_1 R_2 \sqrt{g} g^{\alpha\alpha'} g^{\beta\beta'} g^{\gamma\delta} H_{1\alpha\beta} H_{1\alpha'\beta'} - i \pi \gamma^\alpha \epsilon^{\gamma\delta} H_{1\gamma\beta} H_{1\alpha\delta}\right\}.
\]

(3.15)

4 Oscillator modes

To compute the oscillator contribution to the partition function (1.1), we quantize the \(U(1)\) gauge theory with a theta term on the \(T^4\) manifold using Dirac brackets. From (1.3), the equations of motion are \(\partial^i F_{ij} = 0\), since the theta term is a total divergence and does not contribute to them. So in Lorenz gauge, the gauge potential \(A_i\) with field strength tensor \(F_{ij} = \partial_i A_j - \partial_j A_i\) is obtained by solving the equation
\[
\partial^i \partial_i A_j = 0, \quad \text{with} \quad \partial^i A_i = 0.
\]

(4.1)

The potential has a plane wave solution
\[
A_i(\tilde{\theta}, \theta^6) = \text{zero modes} + \sum_{k \neq 0} (f_i(k) e^{ik \cdot \theta} + (f_i(k) e^{ik \cdot \theta})^*)
\]

(4.2)
with momenta satisfying the on shell condition and gauge condition
\[ G^{ij}_L k_i k_j = 0, \quad k^i f_i = 0. \] (4.3)

As in \[11],\[16] the Hamiltonian \(H^4d\) and momentum \(P^4d\) are quantized with a Lorentzian signature metric that has zero angles with the time direction, \(\gamma^\alpha = 0\). So we modify the metric on the four-torus (2.2), (2.3) to be
\[ \tilde{G}_{L\alpha\beta} = g_{\alpha\beta}, \quad \tilde{G}_{L66} = -\frac{1}{R_6^2}, \quad \tilde{G}_{L06} = 0, \quad \tilde{G}_L = \text{det} \tilde{G}_{Lij} = -g. \] (4.4)

Solving for \(k_6\) from (4.3) we find
\[ k_6 = \frac{\sqrt{-\tilde{G}_{L66}^6}}{G^6_L} |k|, \] (4.5)

where \(3 \leq \alpha, \beta \leq 5\), and \(|k| \equiv \sqrt{g^{\alpha\beta}k_\alpha k_\beta}\). Employ the remaining gauge invariance \(f_i \rightarrow f'_i = f_i + k_i \lambda\) to fix \(f'_6 = 0\), which is the gauge choice
\[ A_6 = 0. \]

This reduces the number of components of \(A_i\) from 4 to 3. To satisfy (4.3), we can use the \(\partial^6 F_{i6} = -\partial_6 \partial^\alpha A_\alpha = 0\) component of the equation of motion to eliminate \(f_5\) in terms of \(f_3, f_4, f_5 = -\frac{1}{p^6}(p^3 f_3 + p^4 f_4)\),

leaving just two independent polarization vectors corresponding to the physical degrees of freedom of a four-dimensional gauge theory.

From the Lorentzian Lagrangian and energy-momentum tensor given by
\[ \mathcal{L} = -\frac{1}{2e^2} \sqrt{-\tilde{G}_{L}^6} \tilde{G}^{ik}_L \tilde{G}^{jl}_L F_{ij} F_{kl} + \frac{\theta}{32\pi^2} \epsilon^{ijkl} F_{ij} F_{kl}, \]
\[ T^i_j = \frac{\delta \mathcal{L}}{\delta \partial_i A_k} \partial_j A_k - \delta^i_j \mathcal{L}, \] (4.6)

we obtain the Hamiltonian and momentum operators
\[ H_c \equiv \int d^3 \theta T^6_6 = \int d^3 \theta \left( -\frac{\sqrt{g}}{e^2} \tilde{G}^{66}_L g^{\alpha\beta} F_{6\alpha} F_{6\beta} + \frac{\sqrt{g}}{2e^2} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} - \partial_\alpha \Pi^\alpha A_6 \right), \] (4.7)
\[ P_\alpha \equiv \int d^3 \theta T^6_\alpha = \int d^3 \theta \left( -\frac{2}{e^2} \sqrt{g} \tilde{G}^{66}_L g^{\beta\gamma} F_{6\gamma} F_{\alpha\beta} - \partial_\beta \Pi^\beta A_\alpha + \Pi^6 \partial_\alpha A_6 \right), \] (4.8)
where we have integrated by parts; and the conjugate momentum is

\[ \Pi^{\alpha} = \frac{\delta \mathcal{L}}{\delta \partial_{6} A_{\alpha}} = -\frac{2}{e^{2}} \sqrt{g} \tilde{G}_{\alpha \beta} F_{\gamma \delta} - \frac{\theta}{8\pi^{2}} \epsilon^{\alpha \gamma \delta} F_{\beta \gamma}, \quad \Pi^{6} = \frac{\delta \mathcal{L}}{\delta \partial_{6} A_{6}} = 0. \quad (4.9) \]

Then we have

\[ H_{c} - i\gamma^{\alpha} P_{\alpha} = \int d\theta^{3} \left( \frac{R_{6}^{2} e^{2}}{4 \sqrt{g}} g_{\alpha \beta} (\Pi^{\alpha} + \frac{\theta}{8\pi^{2}} \epsilon^{\alpha \gamma \delta} F_{\gamma \delta}) (\Pi^{\beta} + \frac{\theta}{8\pi^{2}} \epsilon^{\beta \rho \sigma} F_{\rho \sigma}) + \frac{\sqrt{g}}{2e^{2}} g^{\alpha \tilde{\alpha}} g^{\beta \tilde{\beta}} F_{\alpha \beta} F_{\tilde{\alpha} \tilde{\beta}} - i\gamma^{\alpha} (\Pi^{\beta} + \frac{\theta}{8\pi^{2}} \epsilon^{\beta \gamma \delta} F_{\gamma \delta}) F_{\alpha \beta} \right), \quad (4.10) \]

up to terms proportional to \( A_{6} \) and \( \partial_{6} \Pi^{\alpha} \) which vanish in Lorenz gauge. Note the term proportional to \( e^{\beta \gamma \delta} F_{\gamma \delta} F_{\alpha \beta} \) vanishes identically. (4.10) is equal to \( H^{4d} - i\gamma^{\alpha} P_{\alpha}^{4d} \) given in (1.2), and is used to compute the zero mode partition function in (2.7) via (3.8).

To compute the oscillator modes, the appearance of \( \theta \) solely in the combination \( \Pi^{\alpha} + \frac{\theta}{8\pi^{2}} \epsilon^{\alpha \gamma \delta} F_{\gamma \delta} \) in (4.10) suggests we make a canonical transformation on the oscillator fields \( \Pi^{\alpha}(\vec{\theta}, \theta^{6}), A_{\beta}(\vec{\theta}, \theta^{6}) \) [18]. Consider the equal time quantum bracket, suppressing the \( \theta^{6} \) dependence,

\[ \left[ \int d\theta^{3} \epsilon^{\alpha \beta \gamma} F_{\alpha \beta A_{\delta}}, \quad \Pi^{\gamma} (\vec{\theta}) \right] = 2i\epsilon^{\alpha \beta \gamma} F_{\alpha \beta} (\vec{\theta}), \quad (4.11) \]

and the canonical transformation

\[ U(\theta) = \exp \left\{ i \frac{\theta}{32\pi^{2}} \int d\theta^{3} \epsilon^{\alpha \beta \gamma} F_{\alpha \beta A_{\gamma}} \right\}, \quad (4.12) \]

under which \( \Pi^{\alpha}(\vec{\theta}, \theta^{6}), A_{\beta}(\vec{\theta}, \theta^{6}) \) transform to \( \hat{\Pi}^{\alpha}(\vec{\theta}, \theta^{6}), \hat{A}_{\beta}(\vec{\theta}, \theta^{6}) \),

\[ \hat{\Pi}^{\alpha}(\vec{\theta}) = U^{-1}(\theta) \Pi^{\alpha}(\vec{\theta}) U(\theta) = \Pi^{\alpha}(\vec{\theta}) + \frac{\theta}{8\pi^{2}} \epsilon^{\alpha \gamma \delta} F_{\gamma \delta} (\vec{\theta}), \]

\[ \hat{A}_{\beta}(\vec{\theta}) = U^{-1}(\theta) A_{\beta}(\vec{\theta}) U(\theta) = A_{\beta}(\vec{\theta}). \quad (4.13) \]

Therefore the exponent (4.10) contains no theta dependence when written in terms of \( \hat{\Pi}^{\alpha} \), which now reads

\[ (H_{c} - i\gamma^{\alpha} P_{\alpha}) = \int d\theta^{3} \left( -\frac{R_{6}^{2} e^{2}}{4 \sqrt{g}} g_{\alpha \beta} \hat{\Pi}^{\alpha} \hat{\Pi}^{\beta} + \sqrt{g} \left[ \frac{\sqrt{g}}{2e^{2}} g^{\alpha \tilde{\alpha}} g^{\beta \tilde{\beta}} \hat{F}_{\alpha \beta} \hat{F}_{\tilde{\alpha} \tilde{\beta}} - i\gamma^{\alpha} \hat{\Pi}^{\beta} F_{\alpha \beta} \right] \right). \quad (4.14) \]

Thus, for the computation of the oscillator partition function we will quantize with \( \theta = 0 \). Note that had we done this for the zero modes, it would not be possible to pick the zero mode integer charges consistently. Since the zero and oscillator modes commute, we are free to canonically transform the latter and not the former.

In the discussion that follows we assume \( \theta = 0 \) and drop the hats. We directly quantize the Maxwell theory on the four-torus with the metric (4.4) in Lorenz gauge using Dirac
The theory has a primary constraint \( \Pi^6(\vec{\theta}, \theta^6) \approx 0 \). We can express the Hamiltonian (4.7) in terms of the conjugate momentum as

\[
H_c = \int d\theta^3 \frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\bar{\alpha}} g^{\beta\bar{\beta}} F_{\alpha\beta} F_{\bar{\alpha}\bar{\beta}}. 
\] (4.15)

The primary Hamiltonian is defined by

\[
H_c = \int d\theta^3 \left( \frac{R_6^2}{4} \frac{e^2}{\sqrt{g}} g_{\alpha\beta} \Pi^\alpha \Pi^\beta + \frac{\sqrt{g}}{2e^2} g^{\alpha\bar{\alpha}} g^{\beta\bar{\beta}} F_{\alpha\beta} F_{\bar{\alpha}\bar{\beta}} - \partial_\alpha \Pi^\alpha A_6 + \lambda_1 \Pi^6 \right), 
\] (4.16)

with \( \lambda_1 \) as a Lagrange multiplier. As in [16], we use the Dirac method of quantizing with constraints for the radiation gauge conditions \( A_6 \approx 0 \), \( \partial^\alpha A_\alpha \approx 0 \), and find the equal time commutation relations:

\[
[\Pi^\beta(\vec{\theta}, \theta^6), A_\alpha(\vec{\theta}, \theta^6)] = -i \left( \delta^\beta_\alpha - g^{\beta\beta'} (\partial_\alpha \frac{1}{g^{\gamma\gamma'}} \partial_{\gamma'} \partial_{\beta'}) \right) \delta^3(\theta - \theta'),
\]

\[
[A_\alpha(\vec{\theta}, \theta^6), A_\beta(\vec{\theta}', \theta^6)] = 0, \quad [\Pi^\alpha(\vec{\theta}, \theta^6), \Pi^\beta(\vec{\theta}', \theta^6)] = 0. 
\] (4.17)

In \( A_6 = 0 \) gauge, the vector potential on the torus is expanded as

\[
A_\alpha(\vec{\theta}, \theta^6) = \text{zero modes} + \sum_{\vec{k} \neq 0, \vec{k} \in \mathbb{Z}_3} \left( f^\kappa_{\alpha} a^{\kappa}_{k \alpha} e^{ik\cdot\theta} + f^*_{\alpha} a^{\ast \kappa}_{k \alpha} e^{-ik\cdot\theta} \right),
\]

where \( 1 \leq \kappa \leq 2, 3 \leq \alpha \leq 5 \) and \( k_6 \) defined in (4.5). The sum is on the dual lattice \( \vec{k} = k_\alpha \in \mathbb{Z}_3 \neq \vec{0} \). Here we only consider the oscillator modes expansion of the potential and the conjugate momentum in (4.9) with vanishing \( \theta \) angle

\[
A_\alpha(\vec{\theta}, \theta^6) = \sum_{\vec{k} \neq 0, \vec{k} \in \mathbb{Z}_3} (a^{\kappa}_{-k \alpha} e^{ik\cdot\theta} + a^{\dagger \kappa}_{k \alpha} e^{-ik\cdot\theta}),
\]

\[
\Pi^\beta(\vec{\theta}, \theta^6) = -i \frac{2 \sqrt{g}}{e^2} \bar{G}_{\alpha\beta} g^{\alpha\beta'} \sum_{\vec{k}} k_6 (a^{\kappa}_{-k \beta'} e^{ik\cdot\theta} - a^{\dagger \kappa}_{k \beta'} e^{-ik\cdot\theta}). 
\] (4.18)

and the polarizations absorbed in

\[
a_{k \alpha} = f^\kappa_{\alpha} a^{\kappa}_{k \alpha}. 
\] (4.19)

From (4.17), the commutator in terms of the oscillators is

\[
\int \frac{d\theta^3 d\theta'}{(2\pi)^6} e^{-ik_\alpha \theta^\alpha} e^{-ik'_{\alpha} \theta'^\alpha} [A_\alpha(\vec{\theta}, 0), A_{\alpha'}(\vec{\theta}', 0)] = [(a^{\dagger \kappa}_{-k \alpha} + a^{\dagger \kappa}_{k \alpha}), (a_{k' \beta'} + a^{\dagger}_{-k' \beta'})] = 0. 
\] (4.20)

We consider the Fourier transform (4.20) of all the commutators (4.17), so the commutator
of the oscillators is found to be:

\[
[a_{\vec{k}, \alpha}, a_{\vec{k}', \beta}^\dagger] = \frac{\epsilon^2}{2\sqrt{g} \tilde{G}_L^{66} k_6} \frac{1}{2(2\pi)^3} (g_{\alpha\beta} - \frac{k_\alpha k_\beta}{g^{\gamma\gamma} k_\gamma k_\gamma'}) \delta_{\vec{k}, \vec{k}'},
\]

\[
[a_{\vec{k}, \alpha}, a_{\vec{k}', \beta}] = 0, \quad [a_{\vec{k}, \alpha}^\dagger, a_{\vec{k}', \beta}^\dagger] = 0.
\] (4.21)

In $A_6 = 0$ gauge, we use (4.18) and (4.21) to evaluate the Hamiltonian and momentum in (4.7) and (4.8):

\[
H_c = \int d^3 \theta \frac{2 \sqrt{g}}{e^2} \left( -\frac{1}{2} \tilde{G}_L^{66} g^{\alpha\alpha'} \partial_6 A_\alpha \partial_6 A_{\alpha'} + \frac{1}{4} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} \right),
\]

\[
P_\alpha = \frac{2}{R_6 e^2} \int_0^{2\pi} d\theta^3 d\theta^4 \sqrt{g} g^{\beta\beta'} F_{6\beta} F_{6\beta}. \quad (4.22)
\]

With (4.18), (4.22) can be expressed in terms of the oscillator modes where time-dependent terms cancel,

\[
H_c = (2\pi)^3 \frac{2 \sqrt{g}}{e^2} \sum_{\vec{k} \in \mathbb{Z}^3 \neq \vec{0}} |k|^2 \left( a_{\vec{k}, \alpha} a_{\vec{k}, \alpha}^\dagger + a_{\vec{k}, \alpha}^\dagger a_{\vec{k}', \alpha} \right),
\]

\[
P_\alpha = -\frac{2 \sqrt{g}}{e^2} \tilde{G}_L^{66} g^{\beta\beta'} (2\pi)^3 \sum_{\vec{k} \in \mathbb{Z}^3 \neq \vec{0}} k_6 k_\alpha \left( a_{\vec{k}, \beta} a_{\vec{k}', \beta}^\dagger + a_{\vec{k}, \beta}^\dagger a_{\vec{k}', \beta} \right). \quad (4.23)
\]

and we have used the on-shell condition $\tilde{G}_L^{66} k_6 k_0 + |k|^2 = 0$, and the transverse condition $k^\alpha a_{\vec{k}, \alpha} = k^\alpha a_{\vec{k}, \alpha}^\dagger = 0$. Then,

\[
-H_c + i\gamma^\alpha P_\alpha = -i \frac{1}{R_6} \frac{2 \sqrt{g}}{e^2} (2\pi)^3 \sum_{\vec{k} \in \mathbb{Z}^3 \neq \vec{0}} |k|( -i R_6 |k| + \gamma^\alpha k_\alpha) g^{\beta\beta'} \left( a_{\vec{k}, \beta} a_{\vec{k}', \beta}^\dagger + a_{\vec{k}, \beta}^\dagger a_{\vec{k}', \beta} \right).
\] (4.24)

Inserting the polarizations as $a_{\vec{k}, \alpha} = f_{\alpha}^{\kappa} a_{\vec{k}}^\kappa$ and $a_{\vec{k}, \alpha}^\dagger = f_{\alpha}^{\lambda*} a_{\vec{k}}^{\lambda*}$ from (4.19) in the commutator (4.21) gives

\[
[a_{\vec{k}, \alpha}, a_{\vec{k}', \beta}^\dagger] = \frac{\epsilon^2 R_6}{4\sqrt{g} |k| (2\pi)^3} \left( g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} \right) \delta_{\vec{k}, \vec{k}'} = f_{\alpha}^{\kappa} f_{\beta}^{\lambda*} [a_{\vec{k}}^{\kappa}, a_{\vec{k}}^{\lambda*}], \quad (4.25)
\]

where we choose the normalization

\[
[a_{\vec{k}}^{\kappa}, a_{\vec{k}}^{\lambda*}] = \delta^{\kappa\lambda} \delta_{\vec{k}, \vec{k}'}, \quad (4.26)
\]

with $1 \leq \kappa, \lambda \leq 2$. Then the polarization vectors satisfy
\[
\begin{align*}
    f^\kappa_\alpha f^{\lambda_\beta}_\delta \delta^{\kappa_\lambda} &= \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \left( g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} \right), \\
    g^{\beta_\beta'} f^\kappa_\beta f^{\lambda_\beta'} \delta^{\kappa_\lambda'} &= \frac{e^2}{4\sqrt{g}} \frac{R_6}{|k|} \left( \frac{1}{(2\pi)^3} \right).
\end{align*}
\]

So the exponent in (1.1) is given by

\[
-H_c + i\gamma^\alpha P_\alpha = -i R_6 \frac{2\sqrt{g}}{e^2} (2\pi)^3 \sum_{k \in Z^3 \neq 0} |k| (\gamma^\alpha k_\alpha - i R_6 |k|) \left( 2a^+_k a^-_{k'} + [a^+_k a^-_{k'}, a^+_{k'} a^-_k] \right)
\]

\[
= -i \sum_{k \in Z^3 \neq 0} \left( \gamma^\alpha k_\alpha - i R_6 |k| \right) a^+_k a^-_k - \frac{i}{2} \sum_{k \in Z^3 \neq 0} \left( -i R_6 |k| \right) \delta^{\kappa\kappa'}.
\]

The \(U(1)\) partition function is

\[
Z^{4d, Maxwell} = tr \exp \{2\pi (-H_c + i\gamma^1 P_1)\} = Z_{\text{zero modes}}^{4d} Z_{\text{osc}},
\]

so from (4.27),

\[
Z_{\text{osc}}^{4d} = tr e^{-2\pi i \sum_{k \in Z^3 \neq 0} \left( \gamma^\alpha k_\alpha - i R_6 |k| \right) a^+_k a^-_k - \pi R_6 \sum_{\tilde{\kappa} \in Z^3 \neq 0} |\tilde{\kappa}| \delta^{\kappa\kappa'}}.
\]

From the usual Fock space argument

\[
tr \omega^{\sum_p p a^+_p a_p} = \prod_p \sum_{k=0}^{\infty} \langle k | \omega^p a^+_p | k \rangle = \prod_p \frac{1}{1 - \omega^p},
\]

we perform the trace on the oscillators,

\[
Z_{\text{osc}}^{4d} = \left( e^{-\pi R_6 \sum_{\tilde{\kappa} \in Z^3} \sqrt{g} \gamma^\alpha n_\alpha n_\beta} \prod_{\tilde{n} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi i \gamma^\alpha n_\alpha - 2\pi i \gamma^\alpha n_\beta}} \right)^2,
\]

\[
Z^{4d, Maxwell} = Z_{\text{zero modes}}^{4d} \cdot \left( e^{-\pi R_6 \sum_{\tilde{\kappa} \in Z^3} \sqrt{g} \gamma^\alpha n_\alpha n_\beta} \prod_{\tilde{n} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g} \gamma^\alpha n_\alpha - 2\pi i \gamma^\alpha n_\alpha}} \right)^2,
\]

where \(Z_{\text{zero modes}}^{4d}\) is given in (2.7). (4.31) and (4.37) are each manifestly \(SL(3, \mathbb{Z})\) invariant due to the underlying \(SO(3)\) invariance we have labeled as \(\alpha = 3, 4, 5\). We use the \(SL(3, \mathbb{Z})\) invariant regularization of the vacuum energy reviewed in Appendix C to obtain

\[
Z^{4d, Maxwell} = Z_{\text{zero modes}}^{4d} \cdot \left( e^{-\frac{1}{2} R_6 \pi^{-2} \sum_{\tilde{n} \neq 0} \frac{\sqrt{g}}{(\gamma^\alpha n_\alpha n_\beta)^2}} \prod_{\tilde{n} \in Z^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g} \gamma^\alpha n_\alpha - 2\pi i \gamma^\alpha n_\alpha}} \right)^2.
\]
which leads to (2.13).

On the other hand, one can evaluate the oscillator trace for the 6d chiral boson from (1.4) as in [12], [16]. The exponent in the trace is

\[-2\pi R_6 \mathcal{H} + i 2\pi \gamma^j P_i = \frac{i\pi}{12} \int_0^{2\pi} d^5 \theta H_{t r s} \epsilon^{l r s m n} H_{6 m n} = \frac{i\pi}{2} \int_0^{2\pi} d^5 \theta \sqrt{-G} H^{6 m n} H_{6 m n} \]

\[-= -i\pi \int_0^{2\pi} d^5 \theta (\Pi^{m n} H_{6 m n} + H_{6 m n} \Pi^{m n}) \]

\[-= -2i\pi \sum_{\vec{p} \neq 0} p_6 C^\kappa_{\vec{p}}^{\gamma} B^\kappa_{\vec{p}} - i\pi \sum_{\vec{p} \neq 0} p_6 \delta^{\kappa\kappa}, \tag{4.33} \]

where \( \Pi^{m n} = -\frac{\sqrt{G}}{4} \Pi^{6 m n} \), and \( \Pi^{6 m n} \) is the momentum conjugate to \( B_{M N} \). In the gauge \( B_{6 n} = 0 \), the normal mode expansion for the free quantum fields \( B_{m n} \) and \( \Pi^{m n} \) on a torus is given in terms of oscillators \( B^\kappa_{\vec{p}} \) and \( C^\kappa_{\vec{p}} \) defined in [12], with the commutation relations

\[[B^\kappa_{\vec{p}}, C^{\lambda}_{\vec{r}}] = \delta^{\kappa\lambda} \delta_{\vec{p}, \vec{r}} \tag{4.34} \]

where \( 1 \leq \kappa, \lambda \leq 3 \) labels the three physical degrees of freedom of the chiral two-form, and \( \vec{p} = (p_1, p_2, p_3) \) lies on the integer lattice \( \mathbb{Z}^3 \). From the on-shell condition \( G^{L M} p_L p_M = 0 \),

\[p_6 = -\gamma^\alpha p_\alpha - i R_6 \sqrt{g^{\alpha \beta} p_\alpha p_\beta + \frac{p_1^2}{R_1^2} + \left( \frac{1}{R_2^2} + \frac{\beta^2}{R_1^2} \right) p_2^2 + 2 \frac{\beta^2}{R_1^2} p_1 p_2}. \tag{4.35} \]

Thus the oscillator partition function of the chiral two-form on \( T^2 \times T^4 \) is obtained by tracing over the oscillators

\[Z^{6d}_{\text{osc}} = \text{tr} e^{-2i\pi \sum_{\vec{p} \neq 0} p_6 C^\kappa_{\vec{p}} B^\kappa_{\vec{p}} - i\pi \sum_{\vec{p} \neq 0} p_6 \delta^{\kappa\kappa}} \]

\[= \left( e^{-\pi R_6 \sum_{\vec{p} \neq 0} \sqrt{g^{\alpha \beta} p_\alpha p_\beta + \frac{p_1^2}{R_1^2} + \left( \frac{1}{R_2^2} + \frac{\beta^2}{R_1^2} \right) p_2^2 + 2 \frac{\beta^2}{R_1^2} p_1 p_2}} \prod_{\vec{p} \neq 0} \frac{1}{1 - e^{-2\pi i p_6}} \right)^3 \]

\[= \left( e^{-\pi R_6 \sum_{\vec{p} \in \mathbb{Z}^3} \sqrt{g^{\alpha \beta} p_\alpha p_\beta + \vec{p}^2}} \prod_{\vec{p} \in \mathbb{Z}^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} p_\alpha p_\beta + \vec{p}^2 + 2\pi i \gamma^\alpha p_\alpha}}} \right)^3, \tag{4.36} \]

where \( \vec{p}^2 \equiv \frac{p_1^2}{R_1^2} + \left( \frac{1}{R_2^2} + \frac{\beta^2}{R_1^2} \right) p_2^2 + 2 \frac{\beta^2}{R_1^2} p_1 p_2 \). Regularizing the vacuum energy in the oscillator sum [12] yields

\[Z^{6d, \text{chiral}} = Z^{6d}_{\text{zero modes}} \cdot \left( e^{R_6 \gamma^{-1} \sum_{\vec{p} \neq 0} \frac{\sqrt{g^{\alpha \beta}}}{(G_m n^m n^p)^2} \prod_{\vec{p} \in \mathbb{Z}^3 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} p_\alpha p_\beta + \vec{p}^2 + 2\pi i \gamma^\alpha p_\alpha}}} \right)^3, \tag{4.37} \]

where \( \bar{n} \in \mathbb{Z}^5 \) is on the dual lattice, \( G_m n^p \) is defined in (2.2), and \( Z^{6d}_{\text{zero modes}} \) is given in (2.6).

Comparing the 4d and 6d oscillator traces (4.31) and (4.36), the 6d chiral boson sum has a cube rather than a square, corresponding to one additional polarization, and it contains
Kaluzo-Klein modes. In Appendix D, we prove that the product of the zero mode and the oscillator mode partition function for the 4d theory in (4.32) is \( SL(4, \mathbb{Z}) \) invariant. In (D.48) we give an equivalent expression,

\[
Z^{4d, Maxwell} = Z_{\text{zero modes}}^{4d} \cdot \left( \prod_{n \neq 0} e^{-2\pi R_6 < H > p_{\perp}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + 2\pi i \gamma^\alpha n_\alpha}}} \right)^2,
\]

where \( 4 \leq a \leq 5 \), with \( < H >_{p_{\perp}} \) defined in (C.3).

In Appendix D, we also prove the \( SL(4, \mathbb{Z}) \) invariance of the 6d chiral partition function (4.37), using the equivalent form (D.65),

\[
Z^{6d, \text{chiral}} = Z_{\text{zero modes}}^{6d} \cdot \left( \prod_{n \neq 0} e^{-2\pi R_6 < H > p_{\perp}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + 2\pi i \gamma^\alpha n_\alpha}}} \right)^3,
\]

with \( < H >_{p_{\perp}} \) in (D.64), and \( \tilde{n}^2 = (n_1)^2 + \frac{1}{R_3^2} + \frac{(\beta^2)^2}{R_1^2} n_2^2 + 2\frac{\beta^2}{R_1} n_2 n_1 \). In the limit when \( R_1 \) and \( R_2 \) are small with respect to the metric parameters \( g_{\alpha \beta}, R_6 \) of the four-torus, the contribution from each polarization in (4.38) and (4.39) is equivalent. To see this limit, we can separate the product on \( n = (n_1, n_2, n_a) \neq 0 \) in (4.39), into \( (n_1 = 0, n_2 = 0, n_a \neq (0,0)), (n_1 \neq 0, n_2 \neq 0, \text{all } n_a), (n_1 = 0, n_2 \neq 0, \text{all } n_a), (n_1 \neq 0, n_2 = 0, \text{all } n_a) \) to find, at fixed \( n_3 \),

\[
\prod_{n_1 \neq 0, n_2 \neq 0, (n_a \in \mathbb{Z}^2)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + (\frac{(\beta^2)^2}{R_1} n_2^2 + 2\frac{\beta^2}{R_1} n_2 n_1 + 2\pi i \gamma^\alpha n_\alpha}}}
\]

\[
= \prod_{n_a \in \mathbb{Z}^2 \neq (0,0)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + 2\pi i \gamma^\alpha n_\alpha}}}
\]

\[
\cdot \prod_{n_1 \neq 0, n_2 \neq 0, (n_a \in \mathbb{Z}^2)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + (\frac{(\beta^2)^2}{R_1} n_2^2 + 2\frac{\beta^2}{R_1} n_2 n_1 + 2\pi i \gamma^\alpha n_\alpha}}}
\]

\[
\cdot \prod_{n_1 = 0, n_2 \neq 0, (n_a \in \mathbb{Z}^2)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + (\frac{(\beta^2)^2}{R_1} n_2^2 + 2\pi i \gamma^\alpha n_\alpha}}}
\]

\[
\cdot \prod_{n_2 = 0, n_1 \neq 0, (n_a \in \mathbb{Z}^2)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^\alpha \beta n_\alpha n_\beta + (\frac{(\beta^2)^2}{R_1} n_2^2 + 2\pi i \gamma^\alpha n_\alpha}}}
\]

(4.40)
Thus for $T^2$ smaller than $T^4$, the last three products reduce to unity, so

$$\prod_{n_\perp \in \mathbb{Z}^4 \neq 0} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta} + 2\pi i \gamma^\alpha n_\alpha}} \prod_{n_\perp \in \mathbb{Z}^2 \neq (0,0)} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha\beta} n_\alpha n_\beta} + 2\pi i \gamma^\alpha n_\alpha}}.$$

(4.41)

The regularized vacuum energies in (C.3) and (D.64),

$$<H>_{p_\perp \neq 0} = -\pi^{-1} \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^a 2\pi n) \frac{K_1(2\pi n R_3 |p_\perp|)}{n},$$

for $|p_\perp| \equiv \sqrt{g^{ab} n_\alpha n_\beta}$,

$$<H>_{6d}^{p_\perp \neq 0} = -\pi^{-1} \sum_{n=1}^{\infty} \cos(p_\alpha \kappa^a 2\pi n) \frac{K_1(2\pi n R_3 |p_\perp|)}{n},$$

for $|p_\perp| \equiv \sqrt{n^2 + g^{ab} n_\alpha n_\beta}$,

(4.42)

have the same form of spherical Bessel function, but the argument differs by modes $(p_1, p_2)$. Again separating the product on $n_\perp = (n_1, n_2, n_3)$ in (4.39), into $(n_1 = 0, n_2 = 0, n_3 \neq (0,0)), (n_1 \neq 0, n_2 \neq 0, n_3), (n_1 = 0, n_2 \neq 0, n_3), (n_1 \neq 0, n_2 = 0, n_3)$ we have

$$\prod_{n_\perp \in \mathbb{Z}^4 \neq (0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}^{6d}} = \prod_{n_\perp \in \mathbb{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{p_\perp}^{6d}} \cdot \prod_{n_1 \neq 0, n_2 \neq 0, n_3 \in \mathbb{Z}} e^{-2\pi R_6 <H>_{p_\perp}^{6d}} \cdot \prod_{n_1 = 0, n_2 \neq 0, n_3 \in \mathbb{Z}} e^{-2\pi R_6 <H>_{p_\perp}^{6d}} \cdot \prod_{n_1 \neq 0, n_2 = 0, n_3 \in \mathbb{Z}} e^{-2\pi R_6 <H>_{p_\perp}^{6d}}.$$

(4.43)

In the limit $R_1, R_2 \to 0$, the last three products are unity. For example, the second is unity because for $n_1, n_2 \neq 0$,

$$\lim_{R_1, R_2 \to 0} \sqrt{n^2 + g^{ab} n_\alpha n_\beta} \sim \sqrt{n^2},$$

$$\lim_{R_1, R_2 \to 0} (|p_\perp| K_1(2\pi n R_3 |p_\perp|)) = \lim_{R_1, R_2 \to 0} \sqrt{n^2} K_1(2\pi n R_3 (\sqrt{n^2})) = 0,$$

since $\lim_{x \to \infty} x K_1(x) \sim \sqrt{x} e^{-x} \to 0$ [21]. So (4.43) leads to

$$\lim_{R_1, R_2 \to 0} \prod_{n_\perp \in \mathbb{Z}^4 \neq (0,0,0)} e^{-2\pi R_6 <H>_{p_\perp}^{6d}} = \prod_{n_\perp \in \mathbb{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{p_\perp}}.$$

(4.45)
Thus in the limit when $T^2$ is small with respect to $T^4$,

$$
\lim_{R_1, R_2 \to 0} \prod_{n_\perp \in \mathbb{Z}^2 \neq (0,0)} e^{-2\pi R_6 <H>_{\perp}} \prod_{n_\parallel \in \mathbb{Z}} 1 - e^{-2\pi R_6 \sqrt{(g^{\alpha\beta} n_\alpha n_\beta + \frac{n_1^2}{R_1^2} + \frac{(\beta)^2}{R_1^2} n_2^2 + 2 \frac{\beta}{R_1} n_2 n_1 + i2\pi \gamma^\alpha n_\alpha} \frac{1}{R_6}}.
$$

(4.46)

So we have shown the partition functions of the chiral theory on $T^2 \times T^4$ and of gauge theory on $T^4$, agree in the small $T^2$ limit upon neglecting the less interesting contribution $\epsilon'_{\text{osc}}$,

$$
\lim_{R_1, R_2 \to 0} Z_{6d, \text{chiral}}^{6d} = \epsilon'_{\text{osc}} \cdot Z_{4d}^{4d},
$$

(4.47)

which is (2.11). Again, $\epsilon'$ is equivalently the oscillator contribution from one polarization, that is

$$
\epsilon' = \left( e^{\frac{i}{2} R_6 \pi^{-2} \sum_{\beta_0 \neq 0} \sum_{\alpha=0}^{\beta_0} (g^{\alpha\beta} n_\alpha n_\beta + \frac{\beta^2}{R_1^2} n_2^2 + 2 \frac{\beta}{R_1} n_2 n_1 + i2\pi \gamma^\alpha n_\alpha) \frac{1}{R_6}} \right).
$$

(4.48)

The relation between the $4d$ gauge and $6d$ tensor partition function is shown in the small $T^2$ limit,

$$
\lim_{R_1, R_2 \to 0} Z_{6d, \text{chiral}}^{6d} = \epsilon'_{\text{osc}} \cdot Z_{4d, \text{Maxwell}}^{4d},
$$

(4.49)

which is (2.15). $\epsilon'$ is the partition function of a real scalar field in $4d$, and is independent of the gauge coupling $\tau$.

5 S-duality of $Z_{4d, \text{Maxwell}}^{4d}$ from $Z_{6d, \text{chiral}}^{6d}$

In Appendices B and D we show explicitly how the $SL(2, \mathbb{Z}) \times SL(4, \mathbb{Z})$ symmetry of the partition function of the $6d$ tensor field of the M-fivebrane of $N = (2,0)$ theory compactified on $T^2 \times T^4$ implies the $SL(2, \mathbb{Z})$ S-duality of the $4d$ $U(1)$ gauge field partition function. These computations use the Hamiltonian formulation. In Appendix A we review the path integral formalism for the $4d$ zero and non-zero mode partition functions, and give their relations to the quantities computed in the Hamiltonian formulation. The results are summarized here.

$$
Z_{4d, \text{zero modes}} = (\text{Im } \tau)^{\frac{3}{2}} g^{\frac{1}{2}} R_6^{\frac{1}{2}} Z_{4d, \text{zero modes}}^{PI},
$$

(5.1)

$$
Z_{4d, \text{osc}} = (\text{Im } \tau)^{-\frac{3}{2}} g^{-\frac{1}{2}} R_6^{\frac{1}{2}} Z_{4d, \text{osc}}^{PI}.
$$

(5.2)
\[ Z_{\text{zero modes}}^{4d} \rightarrow Z_{\text{zero modes}}^{4d} \]
\[ Z_{\text{zero modes}}^{4d} \rightarrow Z_{\text{zero modes}}^{4d} \]
\[ Z_{\text{non-zero modes}}^{4d} \rightarrow |\tau|^3 Z_{\text{non-zero modes}}^{4d} \]
\[ Z_{\text{zero modes}}^{4d} \rightarrow |\tau|^3 Z_{\text{zero modes}}^{4d} \]
\[ Z_{\text{zero modes}}^{4d} \rightarrow Z_{\text{zero modes}}^{4d} \]
\[ Z_{\text{non-zero modes}}^{4d} \rightarrow Z_{\text{non-zero modes}}^{4d} \]
\[ \text{under } S \]
\[ \text{under } T \quad (5.3) \]

and

\[ Z_{\text{osc}}^{4d} \rightarrow Z_{\text{osc}}^{4d} \]
\[ Z_{\text{non-zero modes}}^{4d} \rightarrow |\tau|^3 Z_{\text{non-zero modes}}^{4d} \]
\[ Z_{\text{osc}}^{4d} \rightarrow Z_{\text{osc}}^{4d} \]
\[ Z_{\text{non-zero modes}}^{4d} \rightarrow Z_{\text{non-zero modes}}^{4d} \]
\[ \text{under } S \]
\[ \text{under } T. \quad (5.4) \]

\( S \) and \( T \) are the generators of the duality symmetry \( SL(2, \mathbb{Z}) \), \( S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau - 1 \), where \( \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \) is also given by the modulus of the two-torus, \( \tau = \beta^2 + i \frac{R_1}{R_2} \).

### 6 Conclusions and Discussion

We computed the partition function of the abelian gauge theory on a general four-dimensional torus \( T^4 \) and the partition function of a chiral boson compactified on \( T^2 \times T^4 \). The coupling for the 4d gauge theory, \( \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \), is identified with the complex modulus \( \tau = \beta^2 + i \frac{R_1}{R_2} \) of the two-torus \( T^2 \) in directions 1 and 2. Assuming the metric of \( T^2 \) is much smaller than \( T^4 \), the 6d partition function factorizes to a partition function for gauge theory on \( T^4 \) and a contribution from the extra scalar arising from compactification.\(^2\) The 6d partition function has a manifest \( SL(2, \mathbb{Z}) \times SL(4, \mathbb{Z}) \) symmetry. Therefore the \( SL(2, \mathbb{Z}) \) symmetry with the group action on the coupling, \( \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \), known as S-duality becomes manifest in the 4d Maxwell theory. Presumably this happens for an arbitrary four manifold, but we chose \( T^4 \) in order to generate explicit formulas, i.e. explicit functions of \( \tau \) and the 4d metric.

The 6d chiral two-form has no Lagrangian, so we use the Hamiltonian approach to compute both the 4d and 6d partition functions. For 4d gauge theory, the integration of the electric and magnetic fields as observables around one- and two-cycles respectively take integer values due to charge quantization. We sum over all possible integers to get the zero mode partition function. For the oscillator modes, we quantize the gauge theory using the Dirac method with constraints. In 6d, the partition function follows from \([12],[16]\).

We have also given the path integral result for the 4d partition function. It agrees with the partition function obtained in the Hamiltonian formulation. However, the path integral factors into zero modes and oscillator modes differently, which leads to different \( SL(2, \mathbb{Z}) \) transformation properties for the components. The 6d and 4d partition functions share the same \( SL(2, \mathbb{Z}) \times SL(4, \mathbb{Z}) \) symmetry.

If we consider supersymmetry, compactification of the 6d theory on \( T^2 \) leads to \( N = 4 \) gauge theory in the limit of small \( T^2 \). On the other hand, an \( N = 2 \) theory of class S \([22],[23]\) arises when the 6d, \( (2, 0) \) theory is compactified on a punctured Riemann surface with genus \( g \). Here the mapping class group of the Riemann surfaces acts as a generalized S-duality on 4d super-Yang-Mills theory \([24]-[26]\).

---

\(^2\)The Lagrangian for this single 4d scalar with a Lorentzian signature metric is
\[
\mathcal{L} = \frac{R_0 e^2}{R_1 R_2} \left( \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{1}{2} g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi \right).
\]
In another direction, we can study the 2d field theory present when 6d theory is compactified on a four-dimensional manifold. A 2d-4d correspondence, relating a generalized gauge theory partition function and a 2d correlation function, acting between $N = 4$ gauge theory on $S^4$ and a 2d Toda-Liouville conformal theory on $T^2$, holds for the radius of $S^4$ fixed to 1 [24],[5],[27]-[29]. It is difficult to see how the 2d-4d correspondence works for the gauge field on $T^4$ because the 4d oscillator partition function is most naturally viewed as that of a 2d theory on a $T^2$ in directions 3 and 6, whereas the 4d zero mode sum is equivalent to a 2d zero mode partition function on a $T^2$ in directions 1 and 2. For an arbitrary 4d metric, the theory may be too rich for a 2d-4d pairing. A 2d-4d relation can also be analyzed from a topological point of view [4],[30],[31]. Finding explicit results, such as we have derived for $T^2 \times T^4$, for these more general investigations would be advantageous.
A Comparison of the 4d \textit{U}(1) partition function in the Hamiltonian and path integral formulations

For convenience in comparing the 4d gauge theory with the 6d chiral theory in sections 2 and 3, we quantized both using canonical quantization. Since a Lagrangian exists for the 4d gauge theory, it is useful to verify that its path integral quantization agrees with canonical quantization. We find the two quantizations distribute zero and oscillator mode contributions differently, and thus these factors transform differently under the action of \( SL(2, \mathbb{Z}) \). We summarize the path integral quantization results from [9], [14], [15], [32]. Following [9], [15], the two-form zero mode part, \( F_{2\pi} \equiv m = \sum_I m_I \alpha_I \), (A.1)

where \( m_I \) are integers. Define \((m, n)\) to be the intersection form such that \((m, n) = \int m \wedge n\), and thus

\[
(m, m) = \frac{1}{16\pi^2} \int d^4 \theta e^{ijkl} F_{ij} F_{kl}
\]

\[
(m, *m) = \frac{1}{8\pi^2} \int d^4 \theta \sqrt{g} F^{ij} F_{ij}.
\]

(A.2)

So the action (1.3) is given as

\[
I = \frac{4\pi^2}{e^2} (m, *m) - \frac{i\theta}{2} (m, m) = \frac{1}{2e^2} \int d^4 \theta \sqrt{g} F^{ij} F_{ij} - \frac{i\theta}{32\pi^2} \int d^4 \theta e^{ijkl} F_{ij} F_{kl}.
\]

(A.3)

The zero mode partition function from the path integral formalism can be expressed as a lattice sum over the integral basis of \( m_I \) [9], [15],

\[
Z_{\text{zero modes}}^{PI} = \sum_{m_I \in \mathbb{Z}^6} \exp \left[ - \frac{4\pi^2}{e^2} (m, *m) + \frac{i\theta}{2} (m, m) \right]
\]

\[
= \sum_{m_I \in \mathbb{Z}^6} \exp \left[ \frac{i\pi}{2} \tau \left( (m, m) + (m, *m) \right) - \frac{i\pi}{2} \tau \left( - (m, m) + (m, *m) \right) \right],
\]

(A.4)

where \( \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \), and we have chosen the \( \theta \) dependence of the action as in [9]. Alternatively the zero mode sum be can written in terms of the metric using (A.3)

\[
Z_{\text{zero modes}}^{PI} = \sum_{F_{ij} \in \mathbb{Z}^6} \exp \left\{ \left[ - \frac{\pi}{2} R_6 \sqrt{g} \alpha^\beta \gamma^\delta \widetilde{F}_{\alpha\gamma} \widetilde{F}_{\beta\delta} - \frac{\pi \sqrt{g}}{R_6} \gamma^\delta \widetilde{F}_{\delta\beta} \gamma^\gamma \widetilde{F}_{\gamma\beta} - \frac{\pi \sqrt{g}}{R_6} \gamma^\alpha \gamma^\beta \widetilde{F}_{\alpha\beta} \right] \right\}
\]

\[
+ 2\pi \frac{\sqrt{g}}{R_6} \alpha^\beta \gamma^\gamma \widetilde{F}_{\gamma\beta} - \frac{\theta e^2}{8\pi} \epsilon^{\alpha\beta\gamma} \widetilde{F}_{\alpha\gamma} \widetilde{F}_{\beta\gamma} \frac{4\pi}{e^2} \}
\]

(A.5)
where $F_{ij} = 2\pi F_{ij} = m_I$ are integers due to the charge quantization (A.1), and where we have taken into account the integrations $\int d^4\theta = (2\pi)^4$ in (A.5). To compare the zero mode partition functions from the Hamiltonian and path integral formalisms, we rewrite the Hamiltonian formulation result (2.7) as

$$Z_{\text{zero modes}}^{4d} = \sum_{\tilde{\Pi}^\alpha, \tilde{F}_{\alpha\beta}} \exp \left[ -\frac{e^2 R_6}{4\sqrt{g}} g_{\alpha\beta} \left( \bar{\Pi}^\alpha + i \frac{4\pi \sqrt{g}}{e^2 R_6} g^{\alpha\delta} \bar{F}_{\delta\lambda} \gamma^\lambda + \frac{\theta \epsilon^{\alpha\gamma\delta}}{4\pi} \bar{F}_{\gamma^\delta} \right) \cdot \left( \bar{\Pi}^\beta + i \frac{4\pi \sqrt{g}}{e^2 R_6} g^{\beta\delta'} \bar{F}_{\delta'\lambda} \gamma^\lambda + \frac{\theta \epsilon^{\beta\gamma\delta'}}{4\pi} \bar{F}_{\gamma^\delta'} \right) ight. \\
\left. - \frac{4\pi^2 \sqrt{g}}{e^2 R_6} g^{\delta\delta'} \bar{F}_{\delta\beta} \gamma^\beta \bar{F}_{\delta'\beta'} \gamma^\beta' - \frac{2\pi^2}{e^2} \sqrt{g} g^{\alpha\beta} \gamma^\delta \bar{F}_{\alpha\gamma} \bar{F}_{\beta\delta} \right]. \quad (A.6)$$

After Poisson resummation,

$$\sum_{n \in \mathbb{Z}^6} \exp \left[ -\pi (n + x) \cdot A \cdot (n + x) \right] = (\det A)^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}^6} e^{-\frac{\pi n \cdot A^{-1} n}{4}} e^{i \pi n \cdot x}, \quad (A.7)$$

where $A_{\alpha\beta} \equiv \frac{e^2 R_6}{4\pi \sqrt{g}} g_{\alpha\beta}$ and $x^\alpha \equiv \frac{4\pi \sqrt{g}}{e^2 R_6} g^{\alpha\delta} \bar{F}_{\delta\lambda} \gamma^\lambda + \frac{\theta \epsilon^{\alpha\gamma\delta}}{4\pi} \bar{F}_{\gamma^\delta}$, we get the Hamiltonian expression as

$$Z_{\text{zero modes}}^{4d} = \left( \frac{e^2 R_6}{4\pi \sqrt{g}} \right)^{-\frac{1}{2}} \frac{g_f^4}{R_6^2} \sum_{\tilde{\Pi}^\alpha, \tilde{F}_{\alpha\beta}} \exp \left\{ -\frac{4\pi^2 \sqrt{g}}{e^2 R_6} g^{\alpha\beta} \bar{\Pi}^\alpha \Pi_{\beta} - \frac{i \theta}{2} \bar{\Pi}^\alpha \epsilon^{\alpha\gamma\delta} \bar{F}_{\gamma^\delta} + \frac{8\pi^2 \sqrt{g}}{e^2 R_6} g^{\alpha\beta} \bar{\Pi}^\alpha \bar{F}_{\beta\gamma^\delta} \\
- \frac{4\pi^2 \sqrt{g}}{e^2 R_6} g^{\delta\delta'} \bar{F}_{\delta\beta} \gamma^\beta \bar{F}_{\delta'\beta'} \gamma^\beta' - \frac{2\pi^2 R_6}{e^2} \sqrt{g} g^{\alpha\beta} \gamma^\delta \bar{F}_{\alpha\gamma} \bar{F}_{\beta\delta} \right\} \right. \\
= (\text{Im} \tau)^{\frac{3}{2}} \frac{g_f^4}{R_6^2} \sum_{\Pi^\alpha, F_{\alpha\beta}} Z^{\Pi}_{\text{zero modes}}, \quad (A.8)$$

where $\Pi^\alpha$ is the integer value of $\bar{\Pi}^\alpha$, and we identify $\Pi^\alpha$ with $\bar{F}_{\alpha\alpha}$ in (A.5). Then

$$Z^{\Pi}_{\text{zero modes}} = (\text{Im} \tau)^{\frac{3}{2}} \frac{R_6^2}{g_f^4} Z_{\text{zero modes}}^{4d}, \quad (A.9)$$

which is (5.1).

We review from [14] how the non-zero mode partition function is defined by a path integral,

$$Z^{\Pi}_{\text{non-zero modes}} = \int_{\mathcal{A}} D\mathcal{A} e^{-I}. \quad (A.10)$$

Performing the functional integration with the Fadeev-Popov approach, [14] regularizes the
path integral by
\[
Z_{\text{non-zero modes}}^{PI} = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \left( \frac{g}{\text{vol} T^4} \right)^\frac{1}{2} \left[ \det(\Delta_0) \det(2\pi \text{Im} \tau \Delta_1) \right] \frac{1}{2} = \left( \frac{g}{(2\pi)^2 \sqrt{g}} \right)^\frac{1}{2} (2\pi \text{Im} \tau)^\frac{b_1 - 1}{2} \frac{\det \Delta_0}{\det \Delta_1^{\frac{1}{2}}},
\]
where \( b_1 = 4 \) is the dimension of the group \( H^1(T^4) \). \( \Delta_p = (d^\dagger d + dd^\dagger)_p \) is the Laplacian operator acting on the \( p \)-form. \( g = \det G_{ij} \). So \( \Delta_0 = -G^{ij} \partial_i \partial_j \), and \( \det(\Delta_1) = \det(\Delta_0)^4 \).

Thus
\[
Z_{\text{non-zero modes}}^{PI} = \frac{g^\frac{1}{4}}{\sqrt{2\pi}} (\text{Im} \tau)^\frac{1}{2} \det \Delta_0^{-1}.
\]

The determinant can be computed
\[
\det \Delta_0^{-\frac{1}{2}} = \exp\{-\frac{1}{2} \text{tr} \ln A\},
\]
\[
\exp\{-\frac{1}{2} \text{tr} \ln \Delta_0\} = \exp\left(-\frac{1}{2} \text{tr} \ln \left( -G^{66} \partial_6^2 - 2G_{6\alpha} \partial_6 \partial_\alpha - G^{\alpha\beta} \partial_\alpha \partial_\beta \right) \right)
= \exp\left(-\frac{1}{2} \sum_{n_\alpha \neq \hat{0}} \sum_{n_6} \ln \left( \frac{1}{R_6^2} (n_6^2 + \gamma^{\alpha} n_\alpha)^2 + g^{\alpha\beta} n_\alpha n_\beta \right) \right)
= \exp\left(-\frac{1}{2} \sum_{n_\alpha \neq \hat{0}} \sum_{n_6} \ln \left( \frac{1}{R_6^2} (n_6 + \gamma^{\alpha} n_\alpha)^2 + g^{\alpha\beta} n_\alpha n_\beta \right) \right).
\]

Let \( \mu(E) \equiv \sum_{n_6} \ln \left( \frac{1}{R_6^2} (n_6 + \gamma^{\alpha} n_\alpha)^2 + E^2 \right) \), where \( E^2 \equiv g^{\alpha\beta} n_\alpha n_\beta \), \( \rho = 2\pi R_6 \),
\[
\frac{\partial \mu(E)}{\partial E} = \sum_{n_6} \frac{2E}{R_6^2} \frac{1}{(n_6 + \gamma^{\alpha} n_\alpha)^2 + E^2} = \frac{\rho \sinh(\rho E)}{\cosh(\rho E) - \cos(2\pi \gamma^{\alpha} n_\alpha)}
= \partial_E \ln \left[ \cosh(\rho E) - \cos(2\pi \gamma^{\alpha} n_\alpha) \right].
\]

After integration, we have
\[
\mu(E) = \ln \left[ \cosh(\rho E) - \cos(2\pi \gamma^{\alpha} n_\alpha) \right] + \ln \left( R_6^2 \sqrt{\frac{2}{\pi}} \right).
\]
where the constant \( \ln \left( R_6 \sqrt{\frac{2}{\pi}} \right) \) maintains \( SL(4, \mathbb{Z}) \) invariance of the partition function. So,

\[
det \Delta_0^{-\frac{1}{2}} = \exp \left( -\frac{1}{2} \text{tr} \ln \Delta_0 \right) = e^{-\frac{1}{2} \sum_{\alpha \neq 0} \mu(\mu)} = \frac{(2\pi)^{\frac{1}{4}}}{R_6} \prod_{n_\alpha \in \mathbb{Z}^3 \neq 0} \frac{1}{\sqrt{2} \sqrt{\cosh(\rho E) - \cos(2\pi \gamma \alpha n_\alpha)}} = \frac{(2\pi)^{\frac{1}{4}}}{R_6} \prod_{n_\alpha \in \mathbb{Z}^3 \neq 0} e^{-\frac{\rho E}{2} - e^{-\rho E + 2\pi i \gamma \alpha n_\alpha}}. \tag{A.17}
\]

Therefore, using (A.12), we have

\[
Z_{\text{non-zero modes}}^{PI} = (\text{Im } \tau)^{\frac{3}{2}} \frac{4^{\frac{1}{4}} g_1^4}{R_6^2} Z_{\text{osc}}^{4d}, \tag{A.18}
\]

which is (5.2).

Together with (A.9), the partition functions from the two quantizations agree but they factor differently into zero and oscillator modes,

\[
Z_{\text{Maxwell}}^{4d} = Z_{\text{zero modes}}^{4d} Z_{\text{osc}}^{4d} = Z_{\text{zero modes}}^{PI} \cdot Z_{\text{non-zero modes}}^{PI}. \tag{A.19}
\]

\[B\] \( SL(2, \mathbb{Z}) \) invariance of the \( Z^{4d, \text{chiral}} \) and \( Z^{4d, \text{Maxwell}} \) partition functions

The S-duality group \( SL(2, \mathbb{Z}) \) group has two generators \( S \) and \( T \) which act on the parameter \( \tau \) to give

\[
S : \tau \rightarrow -\frac{1}{\tau}, \quad T : \tau \rightarrow \tau - 1. \tag{B.1}
\]

Since \( \tau = \beta^2 + i \frac{R_1}{R_2} = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2} \), the transformation \( S \) corresponds to

\[
R_1 \rightarrow R_1 |\tau|^{-1}, \quad R_2 \rightarrow R_2 |\tau|, \quad \beta^2 \rightarrow -|\tau|^{-2} \beta^2, \tag{B.2}
\]

and \( T \) corresponds to

\[
\beta^2 \rightarrow \beta^2 - 1. \tag{B.3}
\]

Or equivalently

\[
S : \quad \frac{4\pi}{e^2} \rightarrow \frac{4\pi}{e^2} |\tau|^{-2}, \quad \theta \rightarrow -\theta |\tau|^{-2} \\
T : \quad \theta \rightarrow \theta - 2\pi, \tag{B.4}
\]

which for \( \theta = 0 \) is the familiar electromagnetic duality transformation \( \frac{4\pi}{e^2} \rightarrow \frac{4\pi}{e^2} \).
6d partition function

The 6d chiral boson zero mode partition function (2.6),

\[
Z_{\text{6d modes}}^{6d} = \sum_{n_8,n_9,n_{10}} \exp\left\{-\frac{\pi R_0}{R_1 R_2} \sqrt{g} \, g^{\alpha' \alpha'} H_{12 \alpha} H_{12 \alpha'} \right\}
\cdot \sum_{n_7} \exp\left\{-\frac{\pi}{6} R_6 R_1 R_2 \sqrt{g} \, g^{\alpha' \alpha'} g^{\beta' \beta'} g^{\delta' \delta'} H_{\alpha \beta \delta} H_{\alpha' \beta' \delta'} - i \pi \gamma \alpha \epsilon^{\gamma \delta} H_{12 \gamma} H_{\alpha \beta \delta} \right\}
\cdot \sum_{n_4,n_5,n_6} \exp\left\{-\frac{\pi}{2} R_6 R_1 R_2 \sqrt{g} \left( \frac{1}{R_2^2} + \frac{\beta^{22}}{R_1^2} \right) g^{\alpha' \alpha'} g^{\beta' \beta'} H_{2 \alpha \beta} H_{2 \alpha' \beta'} \right\}
\cdot \sum_{n_1,n_2,n_3} \exp\left\{-\frac{\pi R_6 R_2}{4 R_1} \sqrt{g} (g^{\alpha' \alpha'} g^{\beta' \beta'} - g^{\alpha \beta} g^{\alpha' \beta'}) H_{1 \alpha \beta} H_{1 \alpha' \beta'} \right\}
\]

(B.5)

where \( H_{134} = n_1, H_{145} = n_2, H_{135} = n_3, H_{234} = n_4, H_{245} = n_5, H_{235} = n_6, H_{345} = n_7, H_{123} = n_8, H_{124} = n_9, H_{125} = n_{10} \), is invariant under both \( S \) and \( T \). To show the invariance using (B.2,B.3) we group the exponents in (B.5) into two sets,

\[
-\frac{\pi R_6}{R_1 R_2} \sqrt{g} g^{\alpha' \alpha'} H_{12 \alpha} H_{12 \alpha'} - \frac{\pi}{6} R_6 R_1 R_2 \sqrt{g} g^{\alpha' \alpha'} g^{\beta' \beta'} H_{1 \beta \delta} H_{1 \beta' \delta'} - i \pi \gamma \alpha \epsilon^{\gamma \delta} H_{12 \gamma} H_{1 \beta \delta},
\]

(B.6)

and

\[
-\frac{\pi}{2} R_6 R_1 R_2 \sqrt{g} \left( \frac{1}{R_2^2} + \frac{\beta^{22}}{R_1^2} \right) g^{\alpha' \alpha'} g^{\beta' \beta'} H_{2 \alpha \beta} H_{2 \alpha' \beta'} - \frac{\pi R_6}{R_1} R_2 \sqrt{g} g^{\alpha' \alpha'} g^{\beta' \beta'} H_{1 \alpha \beta} H_{2 \alpha' \beta'}
\]

(B.7)

(B.6) has no \( \beta^2 \) dependence and therefore is invariant under \( T \). (B.7) transforms under \( T \) to become

\[
-\frac{\pi}{2} R_6 R_1 R_2 \sqrt{g} \left( \frac{1}{R_2^2} + \frac{\beta^{22}}{R_1^2} \right) g^{\alpha' \alpha'} g^{\beta' \beta'} H_{2 \alpha \beta} H_{2 \alpha' \beta'} - \frac{\pi R_6}{R_1} R_2 \sqrt{g} g^{\alpha' \alpha'} g^{\beta' \beta'} H_{1 \alpha \beta} H_{2 \alpha' \beta'}
\]

(B.8)

which is equivalent to (B.7) in the sum where we shift the integer zero mode field strength \( H_{1 \alpha \beta} \) to \( H_{1 \alpha \beta} - H_{2 \alpha \beta} \).
Under \( S \), we see (B.6) as a function of \( R_1 R_2 \) is invariant, and find (B.7) transforms to
\[
-\frac{\pi R_6 R_2}{R_1} \sqrt{g} g^{\alpha \alpha'} g^{\beta \beta'} H_{2 \alpha \beta} H_{2 \alpha' \beta'} + \frac{R_6}{R_1} R_2 \sqrt{g} g^{\beta \beta'} g^{\alpha \alpha'} H_{1 \alpha \beta} H_{2 \alpha' \beta'} \\
+ i\pi \gamma^{\alpha \alpha'} H_{1 \gamma \beta} H_{2 \alpha \delta} - \frac{\pi}{2} R_1 R_6 R_2 \sqrt{g} (g^{22} + \frac{\beta^{22}}{R_1^2}) g^{\alpha \alpha'} g^{\beta \beta'} H_{1 \alpha \beta} H_{1 \alpha' \beta'}. \tag{B.9}
\]
So by shifting the integer field strength tensors \( H_{1 \alpha \beta} \rightarrow H_{2 \alpha \beta} \) and \( H_{2 \alpha \beta} \rightarrow -H_{1 \alpha \beta} \), the sum on (B.7) is left invariant by \( S \). Thus we have proved \( SL(2, \mathbb{Z}) \) invariance of the 6d zero mode partition function (2.6), and that its factors \( \epsilon \) and \( Z^4_{\text{zeromodes}} \) in (2.9) are separately \( SL(2, \mathbb{Z}) \) invariant.

For the oscillator modes (4.36), the only term that transforms in the sum and product is
\[
\tilde{p}^2 \equiv \frac{p_1^2}{R_1} + (g^{22} + \frac{\beta^{22}}{R_1^2}) p_2^2 + \frac{2 \beta^2}{R_1^2} p_1 p_2, \tag{B.10}
\]
which is invariant under \( T \) by shifting the momentum \( p_1 \rightarrow p_1 + p_2 \). With the \( S \) transformation, \( \tilde{p}^2 \) becomes
\[
p_1^2 (g^{22} + \frac{\beta^{22}}{R_1^2}) + \frac{1}{R_1^2} p_2^2 - \frac{2 \beta^2}{R_1^2} p_1 p_2, \tag{B.11}
\]
and by also exchanging the momentum \( p_1 \rightarrow p_2 \) and \( p_2 \rightarrow -p_1 \), the term remains the same. So the 6d oscillator partition function (4.36) is \( SL(2 \mathbb{Z}) \) invariant, which holds also for regularized vacuum energy as given in (4.37).

4d \( U(1) \) partition function

In the Hamiltonian formulation, \( SL(2, \mathbb{Z}) \) leaves invariant the \( U(1) \) oscillator partition function (4.30), since it is independent of \( \epsilon^2 \) and \( \theta \). We have also checked above, starting from 6d, that the zero mode 4d partition function (2.7) is invariant. Thus the \( U(1) \) partition function from the Hamiltonian formalism is S-duality invariant.

The S-duality transformations on the corresponding quantities in the path integral quantization can be derived from (A.9) and (A.18). Since \( \text{Im} \tau \rightarrow \frac{1}{|\tau|^3} \text{Im} \tau \) under \( S \), and is invariant under \( T \), we have
\[
\begin{align*}
Z^4_{\text{zeromodes}} & \rightarrow Z^4_{\text{zeromodes}}, & Z^4_{\text{zero modes}} & \rightarrow |\tau|^3 Z^4_{\text{zero modes}} & \text{under } S \\
Z^4_{\text{zero modes}} & \rightarrow Z^4_{\text{zeromodes}}, & Z^4_{\text{zero modes}} & \rightarrow Z^4_{\text{zero modes}} & \text{under } T \tag{B.12}
\end{align*}
\]
and
\[
\begin{align*}
Z^4_{\text{osc}} & \rightarrow Z^4_{\text{osc}}, & Z^4_{\text{osc}} & \rightarrow |\tau|^{-3} Z^4_{\text{osc}} & \text{under } S \\
Z^4_{\text{osc}} & \rightarrow Z^4_{\text{osc}}, & Z^4_{\text{osc}} & \rightarrow Z^4_{\text{osc}} & \text{under } T, \tag{B.13}
\end{align*}
\]
which is (5.3) and (5.4).

26
A 2d-4d correspondence for $Z_{\text{zero modes}}^{PI}$

We remark here that the zero mode contribution to the partition function for the 4d Maxwell field is equivalent to the zero mode contribution to the partition function for a 2d worldsheet action [9],

$$S = \int d^2\sigma [\sqrt{h} h^{\mu\nu} G_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta + e^{\mu\nu} B_{\alpha\beta} \partial_\mu X^\alpha \partial_\nu X^\beta],$$  \hspace{1cm} (B.14)

where $1 \leq \mu, \nu \leq 2$ and $3 \leq \alpha, \beta \leq 5$.

$$Z_{\text{zero modes}}^{2d} = \sum_{(p_L, p_R) \in \Gamma_{3,3}} e^{i\pi \tau (p_L)^2 - i \pi \tau (p_R)^2} = \sum_{(p_L, p_R) \in \Gamma_{3,3}} e^{i\pi \tau (p_L)^2 - i \pi \tau (p_R)^2 - 4\pi^2 (p_L)^2 + (p_R)^2)}$$

$$= \sum_{n_\alpha \in \mathbb{Z}, m_\beta \in \mathbb{Z}_2} e^{i\theta n_\alpha m^\alpha - 4\pi^2 (m^\alpha m^\beta G_{\alpha\beta} + (n_\alpha - B_{\alpha\beta}) G_{\alpha\beta}(n_\beta - B_{\beta\gamma} m^\gamma))}$$

$$= Z_{\text{zero modes}}^{PI}$$

where $Z_{\text{zero modes}}^{PI}$ is given in (A.5). The 2d metric is $h_{11} = R_1^2 + R_2^2\beta^2\beta^2$, $h_{12} = -R_2^2\beta^2$, $h_{22} = R_2^2$, and $\tau = \beta^2 + i\frac{R_1}{R_2} = \theta + i\frac{y}{\pi}$. The nine parameters of the moduli space $\frac{O(3,3)}{O(3) \times O(3)}$ of the Lorentzian lattice $\Gamma_{3,3}$ are given by the 4d gauge theory metric as

$$G_{\alpha\beta} = \frac{R_6}{\sqrt{\tilde{g}}} g_{\alpha\beta}, \hspace{0.5cm} B_{\alpha\beta} = \frac{e_{\alpha\beta\gamma\lambda} \gamma^\lambda}{\tilde{g}}.$$

The integers are identified with the Maxwell field components as

$$\bar{F}_{\alpha} = n_\alpha, \hspace{0.5cm} \bar{F}_{\alpha} = \frac{e_{\alpha\beta\gamma\delta} m^\delta}{\tilde{g}}.$$

The points $(p_{L\gamma}, p_{R\delta})$ on the Lorentzian lattice $\Gamma_{d,d}$ are $[33]$

$$p_{L\gamma} = n_\alpha e^{\gamma}, \hspace{0.5cm} p_{R\gamma} = n_\alpha S^{\gamma}, \hspace{0.5cm} p_{L\gamma}^2 = d \sum_{\gamma=1}^{d} p_{L\gamma} p_{L\gamma}, \hspace{0.5cm} p_{R\gamma}^2 = d \sum_{\gamma=1}^{d} p_{R\gamma} p_{R\gamma}, \hspace{0.5cm} \sum_{\gamma=1}^{d} e^{*\alpha} e^{*\gamma} = \frac{1}{2} G_{\alpha\beta}, \hspace{0.5cm} p_L^2 - p_R^2 = 2n_\alpha m^\alpha,$$

$$p_L^2 + p_R^2 = m^\alpha m^\beta G_{\alpha\beta} + (n_\alpha - B_{\alpha\beta} m^\beta) G_{\alpha\beta}(n_\beta - B_{\beta\gamma} m^\gamma),$$

for $1 \leq \alpha, \beta, \gamma, \delta \leq d$.

However, the non-zero mode partition function of the 2d theory (B.14)

$$Z_{\text{non-zero modes}}^{2d} = (\eta(\tau) \tilde{\eta}(\tilde{\tau}))^{-3}$$  \hspace{1cm} (B.15)

is not the 4d non-zero mode partition function (A.18), although they both transform in the same way under the $SL(2,\mathbb{Z})$ duality transformation. Indeed (A.18) is more naturally described by a 2d scalar theory with massless and massive modes on a two-torus in the directions 3 and 6, as we show in Appendix D.
The sum in (4.30) is divergent. We regularize the vacuum energy following [12,16]. For
\[< H > = \frac{1}{2} \sum_{p, \alpha \in \mathbb{Z}^3} \sqrt{g^{\alpha \beta} p_\alpha p_\beta}, \]
the \(SL(3, \mathbb{Z})\) invariant regularized vacuum energy becomes
\[< H > = -\frac{1}{4\pi^3} \sqrt{g} \sum_{n^\alpha \in \mathbb{Z}^3 \neq 0} \frac{1}{(g^{\alpha \beta} n^\alpha n^\beta)^2} = -4\pi \sqrt{g} \sum_{n \in \mathbb{Z}^3 \neq 0} \frac{1}{|2\pi n|^4}. \] (C.1)

For the proof of \(SL(4, \mathbb{Z})\) invariance in Appendix D, it is also useful to write the regularized sum (C.1), as
\[< H > = \sum_{p^\perp \in \mathbb{Z}^2} < H >_{p^\perp = 0} + \sum_{p^\perp \in \mathbb{Z}^2 \neq 0} < H >_{p^\perp}, \] (C.2)
where \(p^\perp = p_\alpha \in \mathbb{Z}^2, \ a = 4, 5,\) and
\[< H >_{p^\perp = 0} = \frac{1}{2} \sum_{p \in \mathbb{Z}} \sqrt{g^{33} p^3} = \frac{1}{R_3} \sum_{n = 1}^{\infty} n = \frac{1}{R_3} \zeta(-1) = -\frac{1}{12R_3};\]
\[< H >_{p^\perp \neq 0} = |p^\perp|^2 R_3 \sum_{n = 1}^{\infty} \cos(p_\alpha \kappa^\alpha 2\pi n) \left[K_2(2\pi n R_3 |p^\perp|) - K_0(2\pi n R_3 |p^\perp|)\right]. \] (C.3)

\(|p^\perp| = \sqrt{p_\alpha p_\kappa g^{ab}},\) using the 2d inverse metric as defined in Appendix D.

D \(SL(4, \mathbb{Z})\) invariance of \(Z^{4d,\text{Maxwell}}\) and \(Z^{6d,\text{chiral}}\)

**Rewriting the 4d metric (3,4,5,6)**

From (2.2) the metric on the four-torus, for \(\alpha, \beta = 3, 4, 5,\) is
\[G_{\alpha \beta} = g_{\alpha \beta}, \quad G_{\alpha 6} = -g_{\alpha \beta} \gamma^\beta, \quad G_{66} = R_6^2 + g_{\alpha \beta} \gamma^\alpha \gamma^\beta. \] (D.1)

We can rewrite this metric using \(a, b = 4, 5,\)
\[g_{33} \equiv R_3^2 + g_{ab} \kappa^a \kappa^b, \quad g_{a3} \equiv -g_{ab} \kappa^b, \quad g_{ab} \equiv g_{ab}, \quad (\gamma^3) \kappa^a - \gamma^a \equiv -\gamma^a, \] (D.2)
\[G_{33} = R_3^2 + g_{ab} \kappa^a \kappa^b, \quad G_{36} = -(\gamma^3) R_3^2 + g_{ab} \kappa^b \gamma^a, \quad G_{3a} = -g_{ab} \kappa^b, \]
\[G_{ab} = g_{ab}, \quad G_{66} = R_6^2 + (\gamma^3) R_3^2 + g_{ab} \gamma^a \gamma^b, \] (D.3)

The 3d inverse of \(g_{\alpha \beta}\) is
\[g^{ab} = \bar{g}^{ab} + \frac{\kappa^b R_3}{R_3^2}, \quad g^{a3} = \frac{\kappa^a R_3}{R_3^2}, \quad g^{33} = \frac{1}{R_3^2}, \] (D.4)
where $\tilde{g}^{ab}$ is the $2d$ inverse of $g_{ab}$. 

$$g \equiv \det G_{ij} = R_6^2 \det g_{\alpha\beta} \equiv R_6^2 \tilde{g} = R_6^2 R_3^2 \det g_{ab} \equiv R_6^2 R_3^2 \tilde{g}.$$ 

The line element can be written as

$$ds^2 = R_6^2 (d\theta^6)^2 + \sum_{\alpha, \beta = 3, 4, 5} g_{\alpha\beta} (d\theta^\alpha - \gamma^\alpha d\theta^6)(d\theta^\beta - \gamma^\beta d\theta^6)$$

$$= R_3^2 (d\theta^3 - (\gamma^3) d\theta^6)^2 + R_6^2 (d\theta^6)^2 + \sum_{a, b = 4, 5} g_{ab} (d\theta^a - \gamma^a d\theta^6 - \kappa^a d\theta^3) (d\theta^b - \gamma^b d\theta^6 - \kappa^b d\theta^3).$$ 

We define

$$\tilde{\tau} \equiv \gamma^3 + i \frac{R_6}{R_3}.$$ 

The 4d inverse is

$$\tilde{G}^{33} = \frac{1}{R_6^2} \tilde{G}_{66} |\tilde{\tau}|^2, \quad \tilde{G}^{66} = \frac{1}{R_6^2}, \quad \tilde{G}^{36} = \frac{\gamma^3}{R_6^2}, \quad \tilde{G}^{3a} = \frac{\kappa^a |\tilde{\tau}|^2}{R_6^2} + \frac{\gamma^3 \tilde{\gamma}^a}{R_6^2},$$

$$\tilde{G}^{ab} = \tilde{g}^{ab} + \frac{\kappa^a \kappa^b}{R_6^2} |\tilde{\tau}|^2 + \frac{\gamma^3 (\tilde{\gamma}^a \kappa^b + \kappa^a \tilde{\gamma}^b)}{R_6^2} + \frac{\gamma^3 \tilde{\gamma}^a \kappa^b}{R_6^2}. \quad \tilde{G}^{6a} = \frac{\gamma^a}{R_6^2} = \frac{\gamma^3 \kappa^a + \tilde{\gamma}^a}{R_6^2}. \quad \tilde{G}^{6b} = \frac{\gamma^b}{R_6^2} = \frac{\gamma^3 \kappa^b + \tilde{\gamma}^b}{R_6^2}.$$ 

**Generators of $\text{GL}(n, \mathbb{Z})$**

The $\text{GL}(n, \mathbb{Z})$ unimodular group can be generated by three matrices [34]. For $\text{GL}(4, \mathbb{Z})$ these can be taken to be $U_1, U_2$ and $U_3$,

$$U_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad U_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that every matrix $M$ in $\text{GL}(4, \mathbb{Z})$ can be written as a product $U_1^{n_1} U_2^{n_2} U_3^{n_3} U_1^{n_4} U_2^{n_5} U_3^{n_6} \ldots$, for integers $n_i$. Matrices $U_1$, $U_2$ and $U_3$ act on the basis vectors of the four-torus $\alpha_i$ where $\tilde{\alpha}_i \cdot \tilde{\alpha}_j \equiv \alpha^k_i \alpha^l_j G_{kl} = G_{ij},$

$$\tilde{\alpha}_3 = (1, 0, 0, 0)$$

$$\tilde{\alpha}_6 = (0, 1, 0, 0)$$

$$\tilde{\alpha}_4 = (0, 0, 1, 0)$$

$$\tilde{\alpha}_5 = (0, 0, 0, 1).$$
For our metric (D.3), the $U_2$ transformation

$$
\begin{pmatrix}
\tilde{\alpha}_3' \\
\tilde{\alpha}_6' \\
\tilde{\alpha}_4' \\
\tilde{\alpha}_5'
\end{pmatrix} = U_2
\begin{pmatrix}
\tilde{\alpha}_3 \\
\tilde{\alpha}_6 \\
\tilde{\alpha}_4 \\
\tilde{\alpha}_5
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

(D.10)

results in $\tilde{\alpha}_3' \cdot \tilde{\alpha}_3' \equiv \alpha_3' \alpha_3' \gamma_{ij} = G_{33} = G_3'$, $\tilde{\alpha}_3' \cdot \tilde{\alpha}_6' \equiv \alpha_3' \alpha_6' \gamma_{ij} = G_{36} = G_{36}'$, etc. So $U_2$ corresponds to

$$
R_3 \rightarrow R_3, \ R_6 \rightarrow R_6, \ \gamma^3 \rightarrow \gamma^3 - 1, \ \kappa^a \rightarrow \kappa^a, \ \tilde{\gamma}^a \rightarrow \tilde{\gamma}^a + \kappa^a, \ g_{ab} \rightarrow g_{ab},
$$

(D.11)

or equivalently

$$
R_6 \rightarrow R_6, \ \gamma^3 \rightarrow \gamma^3 - 1, \ g_{\alpha\beta} \rightarrow g_{\alpha\beta}, \ \gamma^a \rightarrow \gamma^a,
$$

(D.12)

which leaves invariant the line element (D.5) if $d\theta^3 \rightarrow d\theta^3 - d\theta^6$, $d\theta^6 \rightarrow d\theta^6$, $d\theta^a \rightarrow d\theta^a$. $U_2$ is the generalization of the usual $\tilde{\tau} \rightarrow \tilde{\tau} - 1$ modular transformation. The 3d inverse metric $g^{\alpha\beta} \equiv \{g^{ab}, g^{a3}, g^{33}\}$ does not change under $U_2$. It is easily checked that $U_2$ is an invariance of the 4d Maxwell partition function (4.32) as well as the 6d chiral boson partition function (4.37). It leaves the zero mode and oscillator contributions invariant separately.

The other generator, $U_1$ is related to the $SL(2, \mathbb{Z})$ transformation $\tilde{\tau} \rightarrow -(\tilde{\tau})^{-1}$ that we discuss as follows:

$$
U_1 = U'M_3
$$

(D.13)

where $M_3$ is a $GL(3, \mathbb{Z})$ transformation given by

$$
M_3 =
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

(D.14)

and $U'$ is the matrix corresponding to the transformation on the metric parameters (D.16),

$$
U' =
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(D.15)
Under $U'$, the metric parameters transform as

$$R_3 \rightarrow R_3|\tilde{\tau}|, \quad R_6 \rightarrow R_6|\tilde{\tau}|^{-1}, \quad \gamma^3 \rightarrow -\gamma^3|\tilde{\tau}|^{-2}, \quad \kappa^a \rightarrow \tilde{\gamma}^a, \quad \tilde{\gamma}^a \rightarrow -\kappa^a, \quad g_{ab} \rightarrow g_{ab}. \quad (D.16)$$

Or equivalently,

$$G_{ab} \rightarrow G_{ab}, \quad G_{a3} \rightarrow G_{a6}, \quad G_{a6} \rightarrow -G_{a3}, \quad G_{33} \rightarrow G_{66}, \quad G_{66} \rightarrow G_{33}, \quad G_{36} \rightarrow -G_{36}, \quad \tilde{G}_{4}^{ab} \rightarrow \tilde{G}_{4}^{ab}, \quad \tilde{G}_{4}^{a3} \rightarrow \tilde{G}_{4}^{a6}, \quad \tilde{G}_{4}^{63} \rightarrow -\tilde{G}_{4}^{a6}, \quad \tilde{G}_{4}^{33} \rightarrow \tilde{G}_{4}^{33}, \quad \tilde{G}_{4}^{66} \rightarrow |\tilde{\tau}|^2 \tilde{G}_{4}^{66}, \quad \tilde{G}_{4}^{36} \rightarrow -\tilde{G}_{4}^{36}, \quad \tilde{\tau} \rightarrow -\frac{1}{\tilde{\tau}}. \quad (D.17)$$

where $4 \leq a,b \leq 5$, and

$$\tilde{\tau} \equiv \gamma^3 + \frac{iR_6}{R_3}, \quad |\tilde{\tau}|^2 = (\gamma^3)^2 + \frac{R_6^2}{R_3^2}. \quad (D.18)$$

The transformation (D.16) leaves invariant the line element (D.5) when $d\theta^3 \rightarrow d\theta^6,$ $d\theta^6 \rightarrow -d\theta^3$, $d\theta^a \rightarrow d\theta^a$. The generators have the property $\det U_1 = -1, \det U_2 = 1,$ $\det U_3 = -1, \det U' = 1, \det M_3 = -1.$

Under $M_3$, the metric parameters transform as

$$R_6 \rightarrow R_6, \quad \gamma^3 \rightarrow -\gamma^4, \quad \gamma^a \rightarrow \gamma^a+1, \quad g_{ab} \rightarrow g_{a+1,b+1}, \quad g_{a3} \rightarrow -g_{a+1,4}, \quad g_{33} \rightarrow g_{44}, \quad g_{a6} \rightarrow g_{a+1,6}, \quad g_{a6} \rightarrow g_{a+1,6}, \quad g_{33} \rightarrow g_{44}, \quad g_{66} \rightarrow g_{66}, \quad g_{36} \rightarrow -g_{46}, \quad \tilde{G}_{4}^{ab} \rightarrow \tilde{G}_{4}^{a+1,b+1}, \quad \tilde{G}_{4}^{a3} \rightarrow -\tilde{G}_{5}^{a+1,4}, \quad \tilde{G}_{4}^{63} \rightarrow -\tilde{G}_{5}^{a+1,4}, \quad \tilde{G}_{4}^{33} \rightarrow \tilde{G}_{4}^{33}, \quad \tilde{G}_{4}^{36} \rightarrow -\tilde{G}_{4}^{36}, \quad \tilde{G}_{4}^{66} \rightarrow \tilde{G}_{4}^{66}, \quad \det \tilde{G}_{4} = R_6 \tilde{g}, \quad \det \tilde{G}_{4} \rightarrow \det \tilde{G}_{4}. \quad (D.19)$$

where $4 \leq a,b \leq 5$, and $a+1 \equiv 3$ for $a = 5$. We see that $M_3$ takes $Z_{\text{zero modes}}^{ad}$ to its complex conjugate as follows. Letting the $M_3$ transformation (D.19) act on (2.7), we find that the three subterms in the exponent

$$-\frac{e^2}{8}R_6 \sqrt{\tilde{g}} \left( \frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^4} \right) \left( g^{aa'}g^{bb'}\tilde{F}_{ab}\tilde{F}_{a'\alpha'} + 4g^{aa'}g^{b3}\tilde{F}_{ab}\tilde{F}_{a'3} + 2g^{aa'}g^{33}\tilde{F}_{a3}\tilde{F}_{a'3} - 2g^{a3}g^{a3}\tilde{F}_{a3}\tilde{F}_{a'3} \right),$$

$$-\frac{e^2R_6}{4\sqrt{g}} \tilde{\Pi}^\alpha g_{\alpha\beta} \tilde{\Pi}^\beta,$$

$$-\frac{\theta e^2R_6}{8\pi^2\sqrt{g}} g_{\alpha\beta} e^{\alpha\gamma\delta} \tilde{F}_{\gamma\delta} \tilde{\Pi}^\beta, \quad (D.19)$$

are separately invariant under (D.18) if we replace the integers $\tilde{F}_{\alpha\beta} \in Z^3, \tilde{\Pi}^\alpha \in \mathbb{Z}^3$ by

$$\tilde{F}_{ab} \rightarrow \tilde{F}_{a+1,b+1}, \quad \tilde{F}_{a3} \rightarrow -\tilde{F}_{a+1,4}, \quad \tilde{\Pi}^3 \rightarrow \tilde{\Pi}^4, \quad \tilde{\Pi}^a \rightarrow -\tilde{\Pi}^{a+1}. \quad (D.20)$$
However, acted on by $M_3$ with the field shift (D.20), the term
\[
2\pi i\gamma^\alpha \bar{\Pi}^\beta \bar{F}_{\alpha\beta} \rightarrow -2\pi i\gamma^\alpha \bar{\Pi}^\beta \bar{F}_{\alpha\beta}
\]
changes sign. Thus we have
\[
M_3 : \quad Z_{\text{zero modes}}^{4d} \rightarrow Z_{\text{zero modes}}^{4d^*}
\]

**The action of $U'$ on $Z_{\text{zero modes}}^{4d}$**

Next we show that under $U'$, $Z_{\text{zero modes}}^{4d}$ transforms to $|\bar{\tau}|^2 Z_{\text{zero modes}}^{4d}$. From (A.5) and (A.9), we have
\[
Z_{\text{zero modes}}^{4d} = \left(\frac{4\pi}{e^2}\right)^{\frac{1}{2}} \frac{\gamma^4}{R_6} \sum_{F_{ij} \in \mathbb{Z}^4} \exp\left\{-\frac{2\pi^2}{e^2} R_6 \sqrt{\bar{g}g} g^{ij} \bar{g} \bar{F}_{ij} \bar{F}_{ij} - \frac{i}{2} \theta \bar{e} \bar{\alpha} \bar{\beta} \bar{F}_{6\alpha} \bar{F}_{6\beta}\right\},
\]
from which it will be easy to see how it transforms under the $U'$ transformation. Under $U'$ from (D.16), the coefficient transforms as
\[
U' : \quad \left(\frac{4\pi}{e^2}\right)^{\frac{1}{2}} \frac{\gamma^4}{R_6} \rightarrow \left(\frac{4\pi}{e^2}\right)^{\frac{1}{2}} \frac{\gamma^4}{R_6} |\bar{\tau}|^2.
\]

The Euclidean action for the zero mode computation is invariant under $U'$, as we show next by first summing $i = \{3, a, 6\}$, with $4 \leq a \leq 5$.
\[
-\frac{2\pi^2 R_6 \sqrt{\bar{g}}}{e^2} \frac{R_1}{R_2} g^{ij} \bar{g} \bar{F}_{ij} \bar{F}_{ij}
= -\frac{2\pi^2 R_6 \sqrt{g}}{e^2} \left(\bar{G}_{ab}^{ac} \bar{G}_{bc}^{bd} \bar{F}_{ab} \bar{F}_{ac} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} + 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd}
- 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} - 4\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd}
+ 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} - 2\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} - 4\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd}
+ 4\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} - 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 4\bar{G}_{ab}^{ac} \bar{G}_{bd}^{cd} \bar{F}_{ab} \bar{F}_{cd} + 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd} + 2\bar{G}_{ab}^{ac} \bar{G}_{cd}^{bd} \bar{F}_{ab} \bar{F}_{cd}\right)\]
\]
\[
(D.25)
\]
Letting the $U'$ transformation (D.16) act on (D.25), we see the first term in the exponent of (D.23) changes to
The second term in the exponential of (D.23) is a topological term, and is left invariant under the action of $U'$ by inspection. If we replace the integers $\tilde{F}_{3a} \to \tilde{F}_{6a}$ and $\tilde{F}_{ab} \to -\tilde{F}_{3ab}$, the two terms are left invariant, so the sum
\[
\sum_{\tilde{F}_{ij} \in \mathbb{Z}^6} e^{-\frac{2\pi^2 R_6 \sqrt{g}}{e^2} \left( \tilde{G}^{4a3} \tilde{G}^{4b3} \tilde{F}_{ab} \tilde{F}_{a'b'} + 4 \tilde{G}^{4a3} \tilde{G}^{4b3} \tilde{F}_{ab} \tilde{F}_{a3} - 4 \tilde{G}^{4a3} \tilde{G}^{4b3} \tilde{F}_{ab} \tilde{F}_{a6} + \frac{2}{|\tau|^2} \tilde{G}^{4a3} G_4 G_{a3} \tilde{F}_{a3} \tilde{F}_{a3} \right) \right.}
\]
\[
-2 G_4 \tilde{G}^{4a3} \tilde{F}_{a3} \tilde{F}_{a3} - 4 \tilde{G}^{4a3} G_4 \tilde{F}_{a3} \tilde{F}_{a6} + 4 \tilde{G}^{4a3} G_4 \tilde{F}_{a3} \tilde{F}_{a6} - 4 \tilde{G}^{4a3} G_4 \tilde{F}_{a3} \tilde{F}_{a6} + 4 \frac{4}{|\tau|^2} \tilde{G}^{4a3} G_4 \tilde{F}_{a3} \tilde{F}_{a6} + 2 |\tau|^2 G_4 \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} - 2 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} - 4 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} + 4 \frac{4}{|\tau|^2} \tilde{G}^{6a} \tilde{F}_{a3} \tilde{F}_{a3} \tilde{F}_{a6} + 4 |\tau|^2 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} - 2 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} - 2 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} + 2 \tilde{G}^{6a} \tilde{G}^{6a} \tilde{F}_{a6} \tilde{F}_{a6} \right).}
\]
(D.26)

is invariant. Thus we have shown that under the $U'$ transformation (D.16),
\[
Z_{\text{zero modes}}^{4d}(R_3|\tau, R_6|\tau|^{-1}, g_{ab}, -g^{a3}|\tau|^{-2}, g^{33}, -\kappa^a) = |\tau|^2 Z_{\text{zero modes}}^{4d}(R_3, R_6, g_{ab}, \gamma^3, \kappa^a, \gamma^a).
\]
(D.28)

Also from (D.23), we can write (D.22) as
\[
M_3 : Z_{\text{zero modes}}^{4d} (e^2, \theta, G_{ij}) \to Z_{\text{zero modes}}^{4d} (e^2, -\theta, G_{ij}),
\]
and thus under the $GL(4, \mathbb{Z})$ generator $U_1$,
\[
Z_{\text{zero modes}}^{4d} \to |\tau|^2 \left( Z_{\text{zero modes}}^{4d} \right)^*.
\]
(D.30)

The residual factor $|\tau|^2$ is sometimes referred to as an $SL(2, \mathbb{Z})$ anomaly of the zero mode partition function, because $U'$ includes the $\tau \to -\frac{1}{\tau}$ transformation. Finally we will show how this anomaly is canceled by the oscillator contribution.

Under $U_3$, the metric parameters transform as
\[
R_6 \to R_6, \quad \gamma^3 \to -\gamma^3, \quad \gamma^a \to \gamma^a, \quad g_{ab} \to g_{ab}, \quad g_{a3} \to g_{a3}, \quad g_{33} \to g_{33},
\]
\[
g^{ab} \to g^{ab}, \quad g^{a3} \to -g^{a3}, \quad g^{33} \to g^{33}, \quad \det g_{\alpha \beta} = \tilde{g}, \quad \tilde{g} \to \tilde{g}. \quad \text{Or equivalently,}
\]
\[
G_{ab} \to G_{ab}, \quad G_{a3} \to -G_{a3}, \quad G_{a6} \to G_{a6}, \quad G_{33} \to G_{33}, \quad G_{66} \to G_{66}, \quad G_{36} \to -G_{36},
\]
\[
G_{4} \to G_{4}, \quad G_{a4} \to -G_{a4}, \quad G_{a6} \to G_{a6}, \quad G_{36} \to G_{36}, \quad G_{4} \to -G_{4}, \quad G_{66} \to \tilde{G}_{66},
\]
\[
\det \tilde{G}_4 = R_6 \tilde{g}, \quad \det \tilde{G}_4 \to \det \tilde{G}_4.
\]
(D.31)
where \(4 \leq a, b \leq 5\) and \(\tilde{G}^{\alpha\beta}\) is the 3d inverse. We can check that \(Z_{\text{zero modes}}^{4}\) becomes its complex conjugate under \(U_3\) given in (D.31) as follows. Letting the \(U_3\) transformation (D.31) act on (2.7), we find that three of the terms in the exponent

\[
-\frac{e^2 R_6}{8} \sqrt{g} \left( \frac{\theta^2}{4\pi^2} + \frac{16\pi^2}{e^2} \right) \left( g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'} + 4g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha3} F_{\alpha'3} + 2g^{\alpha\alpha'} g^{3\beta3} F_{\alpha3} F_{\alpha'3} - 2g^{\alpha3} g^{\beta3} F_{\alpha3} F_{\alpha'3} \right),
\]

\[
-\frac{e^2 R_6}{4\sqrt{g}} \tilde{\Pi}^\alpha g_{\alpha\beta} \tilde{\Pi}^\beta,
\]

\[
-\frac{\theta e^2 R_6}{8\pi \sqrt{g}} g_{\alpha\beta} \epsilon^{\alpha\gamma\delta} \tilde{F}_{\gamma\delta} \tilde{\Pi}^\beta,
\]

(D.32)

are separately invariant under (D.18), if we replace the the integers \(\tilde{F}_{\alpha\beta} \in \mathbb{Z}^3, \tilde{\Pi}^\alpha \in \mathbb{Z}^3\) by

\[
\tilde{F}_{ab} \rightarrow \tilde{F}_{ab}, \quad \tilde{F}_{a3} \rightarrow -\tilde{F}_{a3}, \quad \tilde{\Pi}^3 \rightarrow \tilde{\Pi}^3, \quad \tilde{\Pi}^a \rightarrow -\tilde{\Pi}^a,
\]

(D.33)

However the subterm

\[
2\pi i \gamma^\alpha \tilde{\Pi}^3 \tilde{F}_{\alpha\beta} \rightarrow -2\pi i \gamma^\alpha \tilde{\Pi}^\beta \tilde{F}_{\alpha\beta}
\]

acted by \(U_3\) with the field shift in (D.33). Therefore the zero mode partition function goes to its complex conjugate under \(U_3\).

**Appropriate generators for \(SL(4, \mathbb{Z})\)**

We claim that \(U_1^2, U_2\) and \(U_1 U_3\) generate the group \(SL(4, \mathbb{Z})\) since \(GL(n, \mathbb{Z})\) is generated by \(U_1, U_2\) and \(U_3\) or alternatively \(R_1 = U_1, R_2 = U_2^{-1}U_2\) and \(R_3 = U_3\), i.e., any element in \(GL(n, \mathbb{Z})\) \(U\) can be written as

\[
U = R_1^{n_1} R_2^{n_2} R_3^{n_3} R_1^{n_4} R_2^{n_5} R_3^{n_6}....
\]

(D.35)

It is understood that \(SL(n, \mathbb{Z})\) is generated by even numbers of \(R_1, R_2\) and \(R_3\). Thus, the possible set of group generators for \(SL(n, \mathbb{Z})\) is

\[
R_1^2, R_2^2, R_3^2, \quad R_1 R_2, R_2 R_3, R_3 R_1, \quad R_2 R_1, R_3 R_2, R_1 R_3
\]

(D.36)

with the properties that \(R_2^2 = 1\) and \(R_3^2 = 1\). A smaller set of the \(SL(4, \mathbb{Z})\) generators is

\[
R_1^2, R_1 R_3, R_2 R_3,
\]

(D.37)

since other generators in (D.36) can be expressed with the generators in (D.39) through the following relations

\[
R_1 R_2 = R_1 R_3 (R_2 R_3)^{-1}, \quad R_2 R_1 = (R_1 R_2)^{-1} R_1^2
\]

\[
R_3 R_2 = (R_2 R_3)^{-1}, \quad R_3 R_1 = (R_1 R_3)^{-1} R_1^2.
\]

(D.38)
Notice that
\[
\{R_1^2, R_1R_3, R_2R_3\} = \{U_1^2, U_1U_3, U_2^{-1}\}. \tag{D.39}
\]

These three matrices generate \(SL(4, \mathbb{Z})\). They can be shown to generate Trott’s twelve generators \(B_{ij}[35]\).

Since we have tested the invariance of the zero mode partition function under \(U_2\), we only need to check invariance under \(U_1U_3\) and \(U_1^2\). For \(U_1U_3\), as previously we separate \(U_1\) into \(U'\) and \(M_3\),
\[
U_1U_3 = U'M_3U_3 = U'(M_3U_3). \tag{D.40}
\]
Since both \(M_3\) and \(U_3\) take \(Z_{4d}^{\text{zero modes}}\) to its complex conjugate, \(M_3U_3\) is an invariance of the zero mode partition function. Thus from (D.28),
\[
U_1U_3 : Z_{4d}^{\text{zero modes}} \rightarrow |\bar{\tau}|^2 Z_{4d}^{\text{zero modes}}. \tag{D.41}
\]

\(U_1^2\) acts on \(Z_{4d}^{\text{zero modes}}\)

Since we have shown before
\[
U_1 : Z_{4d}^{\text{zero modes}} \rightarrow |\bar{\tau}|^2 Z_{4d}^{\text{zero modes}}. \tag{D.42}
\]
then
\[
U_1^2 : Z_{4d}^{\text{zero modes}} \rightarrow Z_{4d}^{\text{zero modes}}. \tag{D.43}
\]

To summarize, we have
\[
U_2 : Z_{4d}^{\text{zero modes}} \rightarrow Z_{4d}^{\text{zero modes}}; \\
U_1U_3 : Z_{4d}^{\text{zero modes}} \rightarrow |\bar{\tau}|^2 Z_{4d}^{\text{zero modes}}; \\
U_1^2 : Z_{4d}^{\text{zero modes}} \rightarrow Z_{4d}^{\text{zero modes}}. \tag{D.44}
\]

One can derive a similar transformation property for \(Z_{6d}^{\text{zero modes}}\) using (2.9),
\[
U_2 : Z_{6d}^{\text{zero modes}} \rightarrow Z_{6d}^{\text{zero modes}}; \\
U_1U_3 : Z_{6d}^{\text{zero modes}} \rightarrow |\bar{\tau}|^3 Z_{6d}^{\text{zero modes}}; \\
U_1^2 : Z_{6d}^{\text{zero modes}} \rightarrow Z_{6d}^{\text{zero modes}}. \tag{D.45}
\]
which follows from transformations on the factor \(\epsilon\), given in (2.10). By inspection \(\epsilon\) is invariant under \(U_2\) and \(M_3\), and transforms as
\[
U' : \epsilon \rightarrow |\bar{\tau}|\epsilon. \tag{D.46}
\]
This can be seen by Poisson resummation since \( \epsilon \) can be written as
\[
\epsilon = \sum_{n_a} \exp\left\{ -\frac{\pi R_6 \sqrt{g}}{R_1 R_2} g^{ab} n_a n_b - \frac{\pi R_6 \sqrt{g}}{R_3 R_1 R_2 |\tau|} \eta^a \eta^b n_a n_b \right\} \exp\left\{ - \pi (N + x) \cdot A \cdot (N + x) \right\},
\]
\[
= |\tau|^{-1} U' \epsilon,
\]
where
\[
H_{12\alpha} = n_\alpha, \quad H_{\alpha\beta\delta} = \frac{\epsilon_{\alpha\beta\delta}}{g} m, \quad m, n_\alpha \in \mathbb{Z}^4,
\]
\[
A = \left( \frac{R_6 \sqrt{g}}{R_3 R_1 R_2} \frac{i \gamma^3}{i \gamma^3} \frac{R_6 R_1 R_2}{R_3 \sqrt{g}} \right), \quad \text{det} A = |\tau|^2, \quad N = \left( n_3 \right)_m, \quad x = \left( \kappa a n_a + \gamma^3 \bar{n}_a \bar{n}_a \right) \left( \frac{R_6 \sqrt{g} \bar{\gamma}^3 \bar{n}_a}{R_3 R_1 R_2 |\tau|} \right).
\]

**U' acts on \( Z_{osc}^4 \)**

To derive how \( U' \) acts on \( Z_{osc}^4 \), we first separate the product on \( \bar{\eta} = (n, n_\alpha) \neq \bar{0} \) into a product on (all \( n \), but \( n_\alpha \neq (0,0) \)) and on \( (n \neq 0, n_\alpha = (0,0)) \). Then using the regularized vacuum energy (C.1) expressed as sum over zero and non-zero transverse momenta \( p_\perp = n_\alpha \) in (C.2), we find that (4.32) becomes
\[
Z_{4d,\text{Maxwell}} = Z_{\text{zero modes}} \cdot \left( \frac{\pi R_6}{e^{\frac{\pi R_6}{R_3}}} \prod_{n \neq 0} \frac{1}{1 - e^{-2\pi R_6 |n| - 2\pi i \gamma^3 n}} \right)^2 \cdot \left( \prod_{n_\alpha \in \mathbb{Z}^2 \neq (0,0)} e^{-2\pi R_6 \langle H \rangle_{p_\perp}} \prod_{n \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{a\beta} n_a n_\beta - 2\pi i \gamma^3 n_\alpha}}} \right)^2.
\]

As in [12] we observe the middle expression above can be written in terms of the Dedekind eta function \( \eta(\tau) \equiv e^{\frac{\pi R_6}{R_3}} \prod_{n \neq 0} \left( 1 - e^{2\pi i \tau} \right) \), with \( \tau = \gamma^3 + i \frac{R_6}{R_3} \),
\[
\left( \frac{\pi R_6}{e^{\frac{\pi R_6}{R_3}}} \prod_{n \neq 0} \frac{1}{1 - e^{-2\pi R_6 |n| - 2\pi i \gamma^3 n}} \right)^2 = (\eta(\tau) \bar{\eta}(\bar{\tau}))^{-2}.
\]

This transforms under \( U' \) in (D.16) as
\[
(\eta(-\bar{\tau}^{-1}) \bar{\eta}(-\bar{\tau}^{-1}))^{-2} = |\tau|^{-2} (\eta(\tau) \bar{\eta}(\bar{\tau}))^{-2},
\]
where \( \eta(-\bar{\tau}^{-1}) = (i \tau)^{\frac{1}{2}} \eta(\tau) \). In this way the anomaly of the zero modes in (D.28) is canceled.
by (D.50). Lastly we demonstrate the third expression in (D.48) is invariant under $U'$,

$$
\left( \prod_{n_a \in \mathbb{Z}^2, n \neq 0} \frac{e^{-2\pi R_6 < H >_\perp}}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} n_\alpha n_\beta} - 2\pi i \gamma^a n_a}} \right)^2 = PI, \tag{D.51}
$$

where $PI$ is the modular invariant 2d path integral of two massive scalar bosons of mass $\sqrt{\tilde{g}^{ab} n_a n_b}$, coupled to a worldsheet gauge field, on a two-torus in directions 3,6. Following [12], with more detail in (D.68), we extract from (4.30)

$$
Z_{osc}^{4d} = \left( e^{-\pi R_6 \sum_{n_3 \in \mathbb{Z}^3} \sqrt{g^{\alpha \beta} n_\alpha n_\beta}} \prod_{n_3 \in \mathbb{Z}^3} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} n_\alpha n_\beta} - 2\pi i \gamma^a n_a}} \right)^2 \tag{D.52}
$$

the 2d path integral of free massive bosons coupling to the gauge field, where $n_a$ is fixed and non-zero,

$$
\langle PI \rangle^\frac{1}{2} \equiv e^{-\pi R_6 \sum_{n_3 \in \mathbb{Z}^3} \sqrt{g^{\alpha \beta} n_\alpha n_\beta}} \prod_{n_3 \in \mathbb{Z}^3} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} n_\alpha n_\beta} - 2\pi i \gamma^a n_a}} = \prod_{s \in \mathbb{Z}} \frac{e^{\frac{\beta' E}{2}}}{1 - e^{-\beta' E + 2\pi i (\gamma^3 s + \gamma a n_a)}} \quad \text{where } s \equiv n_3, \quad E \equiv \sqrt{g^{\alpha \beta} n_\alpha n_\beta}, \quad \beta' \equiv 2\pi R_6
$$

$$
= \prod_{s \in \mathbb{Z}} \sqrt{2} \frac{1}{\cosh(\beta' E - \cos 2\pi (\gamma^3 s + \gamma^a n_a))} \quad \text{for } n_a \rightarrow -n_a
$$

$$
= e^{-\frac{1}{2} \sum_{s \in \mathbb{Z}} \left( \ln [\cosh(\beta' E - \cos 2\pi (\gamma^3 s + \gamma^a n_a))] + \ln 2 \right)} = e^{-\frac{1}{2} \sum_{s \in \mathbb{Z}} \nu(E)}, \tag{D.53}
$$

where

$$
\sum_{s \in \mathbb{Z}} \nu(E) = \sum_{s \in \mathbb{Z}} \left( \ln [\cosh(\beta' E - \cos 2\pi (\gamma^3 s + \gamma^a n_a))] + \ln 2 \right)
$$

$$
= \sum_{s \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} \ln \left[ \frac{4\pi^2}{\beta'^2} (r + \gamma^3 s + \gamma^a n_a)^2 + E^2 \right]. \tag{D.54}
$$

We can show directly that (D.54) is invariant under $U'$, since

$$
E^2 = g^{\alpha \beta} n_\alpha n_\beta = g^{33} s^2 + 2 g^{3a} s a + g^{ab} n_a n_b = \frac{1}{R_3^2} (s + \kappa^a n_a)^2 + \tilde{g}^{ab} n_a n_b,
$$

$$
\frac{4\pi^2}{\beta'^2} (r + \gamma^3 s + \gamma^a n_a)^2 = \frac{1}{R_6^2} (r + \tilde{\gamma}^a n_a + \gamma^3 (s + \kappa^a n_a))^2, \tag{D.55}
$$

then

$$
\frac{4\pi^2}{\beta'^2} (r + \gamma^3 s + \gamma^a n_a)^2 + E^2 = \frac{1}{R_6^2} (s + \kappa^a n_a)^2 |\tilde{\gamma}|^2 + \frac{1}{R_6^2} (r + \tilde{\gamma}^a n_a)^2 + \frac{2\gamma^3}{R_6^2} (r + \tilde{\gamma}^a n_a)(s + \kappa^a n_a) + \tilde{g}^{ab} n_a n_b. \tag{D.56}
$$
So we see the transformation $U'$ given in (D.16) leaves (D.56) invariant if $s \to r$ and $r \to -s$. Therefore (D.54) is invariant under $U'$, so that $(PI)^{\frac{3}{2}}$ given in (D.53) is invariant under $U'$.

$M_3$ acts on $Z_{osc}^{4d}$

$M_3$ leaves the $Z_{osc}^{4d}$ invariant as can be seen from (D.48) by shifting the integer $n_\alpha$ as

$$n_3 \to -n_4, \quad n_\alpha \to n_{\alpha+1}.$$  \hspace{1cm} (D.57)

So, under $U_1 = U'M_3$,

$$Z_{osc}^{4d} \to |\tau|^{-2} Z_{osc}^{4d}.$$  \hspace{1cm} (D.58)

$U_2$ is an invariance of the oscillator partition function by inspection.

$U_3$ acts on $Z_{osc}^{4d}$

$U_3$ leaves the $Z_{osc}^{4d}$ invariant as can be seen from (D.48) by shifting the integers $n_\alpha$ as

$$n_3 \to -n_3, \quad n_\alpha \to n_\alpha.$$  \hspace{1cm} (D.59)

Thus, the oscillator partition function transforms under the $SL(4,Z)$ generators \{U$_1^2$, U$_1$U$_3$, U$_2$\} as

$$U_2 : Z_{osc}^{4d} \to Z_{osc}^{4d},$$

$$U_1 U_3 : Z_{osc}^{4d} \to |\tau|^{-2} Z_{osc}^{4d},$$

$$U_1^2 : Z_{osc}^{4d} \to Z_{osc}^{4d}.$$  \hspace{1cm} (D.60)

So together with (D.44) we have established invariance under (D.39), and thus proved the partition function for the 4d Maxwell theory on $T^4$, given alternatively by (4.32) or (D.48), is invariant under $SL(4,Z)$, the mapping class group of $T^4$.

$U'$ acts on $Z_{osc}^{6d}$

For the 6d chiral theory on $T^2 \times T^4$, where $< H >^{6d} \equiv \frac{1}{2} \sum_{\vec{p} \in Z^5} \sqrt{G_{lm} n_l n_m}$ appears in (4.36), the $SL(3,Z)$ invariant regularized vacuum energy [12] becomes,

$$< H >^{6d} = -\frac{1}{2\pi^4} \sqrt{G_5} \sum_{\vec{n} \neq 0} \frac{1}{(G_{lm} n_l n_m)^3}$$

$$= -32\pi^2 \sqrt{G_5} \sum_{\vec{n} \neq 0} \frac{1}{(2\pi)^6} \left(g_{\alpha\beta} n^\alpha n^\beta + (R_1^2 + R_2^2)\beta^2\right)(n^1)^2 - \beta^2 R_1^2 n^1 n^2 + R_2^2 (n^2)^2.$$  \hspace{1cm} (D.61)

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and can be decomposed similarly to (C.2),
\[
<H>^d = \sum_{p_\perp \in \mathbb{Z}^4} <H>^d_{p_\perp} = <H>^d_{p_\perp=0} + \sum_{p_\perp \in \mathbb{Z}^4 \neq 0} <H>^d_{p_\perp},
\]
(D.62)

where
\[
<H>^d_{p_\perp=0} = -\frac{1}{12R_3},
\]
\[
<H>^d_{p_\perp \neq 0} = |p_\perp|^2 R_3 \sum_{n=1}^{\infty} \cos(p_\perp R_6) [K_2(2\pi n R_3 |p_\perp|) - K_0(2\pi n R_3 |p_\perp|)]
\]
\[
= -\pi^{-1}|p_\perp|R_3 \sum_{n=1}^{\infty} \cos(p_\perp R_6) \frac{K_1(2\pi n R_3 |p_\perp|)}{n},
\]
(D.64)

with \( \sqrt{(n_1)^2 + 2n_2^2 + (4/n_1^2 + 2n_2^2)n_2^2 + \bar{g}^{\alpha \beta} n_\alpha n_\beta} = 0 \) and on \( (n_1, n_2, n_4, n_5) \) $\in \mathbb{Z}^4$.

The $U'$ invariance of (4.37) follows when we separate the product on $\bar{n} \in \mathbb{Z}^5 \neq \bar{0}$ into a product on $(n_3 \neq 0, n_\perp \equiv (n_1, n_2, n_4, n_5) = (0, 0, 0, 0))$, and on (all $n_3$, but $n_\perp = (n_1, n_2, n_4, n_5) \neq (0, 0, 0, 0)$). Then
\[
Z^d_{osc} = \left(e^{\frac{\pi R_6}{6R_3}} \prod_{n_3 \in \mathbb{Z} \neq 0} \frac{1}{1 - e^{2\pi i (n_3^2 + \frac{R_6}{R_3} |n_3|)}} \right)^3
\]
\[
\cdot \left( \prod_{n_\perp \in \mathbb{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 \sum_{p_\perp} <H>^d_{p_\perp}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{n_3^2 + \bar{g}^{\alpha \beta} n_\alpha n_\beta + 2\pi \gamma^2 n_\alpha}}} \right)^3
\]
\[
= (\eta(\bar{\tau}) \bar{\eta}(\bar{\tau}))^{-3}
\]
\[
\cdot \left( \prod_{(n_1, n_2, n_4, n_5) \in \mathbb{Z}^4 \neq (0,0,0,0)} e^{-2\pi R_6 \sum_{p_\perp} <H>^d_{p_\perp}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{g^{\alpha \beta} n_\alpha n_\beta + 2\pi \gamma^2 n_\alpha}}} \right)^3,
\]
(D.65)

where $\bar{\tau} = n_3^2 + i\frac{R_6}{R_3}$, and $\bar{n}^2 = \frac{n_1^2}{R_1^2} + 2\frac{n_2^2}{R_1^2} n_1 n_2 + (\frac{1}{R_2^2} + \frac{n_2^2}{R_1^2}) n_2^2$. Under $U'$,
\[
\eta(\bar{\tau}) \bar{\eta}(\bar{\tau}) \rightarrow |\bar{\tau}| \eta(\bar{\tau}) \bar{\eta}(\bar{\tau}).
\]
(D.66)

$U'$ leaves invariant the part of the 6d oscillator partition function (D.65) at fixed $n_\perp \neq 0,$

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since

$$e^{-2\pi R_6 < H^4 >_{n_1 \searrow \emptyset}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{\delta^a n_a \delta^b n_b + \frac{1}{R_1} + 2 \frac{\alpha^2}{R_1} n_1 n_2 + (\frac{\alpha^2}{R_1} + \frac{\beta^2}{R_1}) n_2^2 + i 2 \pi \gamma^a n_a}}}$$

is the square root of the partition function on $T^2$ (now in the directions 3,6) of a massive complex scalar with $m^2 \equiv G^{11} n_1^2 + G^{22} n_2^2 + 2G^{12} n_1 n_2 + G^{ab} n_a n_b$, $4 \leq a, b \leq 5$, that couples to a constant gauge field $A^\mu \equiv i G^{a \mu} n_i$ with $\mu, \nu = 3, 6$; $i, j = 1, 2, 4, 5$. The metric on this $T^2$ is $h_{33} = R_3^2$, $h_{66} = R_6^2$ + $(\gamma^3)^2 R_3^2$, $h_{36} = -\gamma^3 R_3^2$. Its inverse is $h^{33} = \frac{1}{R_3^2} + (\gamma^3)^2$, $h^{66} = \frac{1}{R_6^2}$ and $h^{36} = \frac{\gamma^3}{R_6}$. The manifestly $SL(2, \mathbb{Z})$ invariant path integral is

$$P.I. = \int d\phi d\bar{\phi} e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6 h^{\mu \nu} (\partial_\mu + A_\mu) \bar{\phi} (\partial_\nu - A_\nu) \phi + m^2 \phi}$$

$$= \int d\phi d\bar{\phi} e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6 \phi(-\frac{(\gamma^3)^2}{R_3^2} \partial_3^2 \delta_6 - 2 \frac{\alpha^2}{R_6^2} \delta_6 + 2 \frac{\beta^2}{R_6^2} \delta_6 + 2G^{11} 1 n_1 n_2 + 2G^{22} 2 n_2 n_2 + 2G^{12} 1 n_2 n_2 + G^{ab} n_a n_b )}$$

$$= \frac{\sqrt{\det\left[\begin{matrix} \frac{\alpha^2}{R_6^2} & 2G^{11} 1 n_1 n_2 + 2G^{22} 2 n_2 n_2 + 2G^{12} 1 n_2 n_2 + G^{ab} n_a n_b \end{matrix}\right]}}{1 - e^{-\int_0^{2\pi} d\theta^3 \int_0^{2\pi} d\theta^6 \phi(-\frac{(\gamma^3)^2}{R_3^2} \partial_3^2 \delta_6 - 2 \frac{\alpha^2}{R_6^2} \delta_6 + 2 \frac{\beta^2}{R_6^2} \delta_6 + 2G^{11} 1 n_1 n_2 + 2G^{22} 2 n_2 n_2 + 2G^{12} 1 n_2 n_2 + G^{ab} n_a n_b )}}$$

$$= e \sum_{r \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \ln[\frac{4\pi^2}{\beta'} \int d\phi d\bar{\phi} \phi^2 (r + \gamma^3 s + \gamma^a n_a)^2 + E^2]$$

where from (2.3), $G^{11} = \frac{R_3^2}{R_3^2}$, $G^{22} = \frac{R_6^2}{R_6^2}$, $G^{12} = \frac{\beta^2}{R_6^2}$, $G^{ab} = g^{ab} + \frac{\gamma^a \gamma^b}{R_6^2}$, $G^{3a} = g^{3a} + \frac{\gamma^a}{R_6^2}$, $G^{6a} = \frac{\gamma^a}{R_6^2}$, $G^{63} = \frac{\gamma^3}{R_6^2}$, and $\partial_3 \phi = -i \gamma^a \partial_3 \phi = -i r \phi$, $s = n_3$, and $\beta' = 2\pi R_6$. The sum on $r$ is

$$\nu(E) = \sum_r \ln\left[\frac{4\pi^2}{\beta'} (r + \gamma^3 s + \gamma^a n_a)^2 + E^2\right],$$

with $E^2 \equiv G^{11} n_1 n_m = G^{11} n_1 n_1 + G^{12} n_1 n_2 + G^{22} n_2 n_2 + G^{6} n_3 n_3$, and $G^{11} = \frac{R_3^2}{R_3^2}$, $G^{12} = \frac{\beta^2}{R_6^2}$, $G^{22} = \frac{R_6^2}{R_6^2}$, $G^{6} = 0$, $G^{3a} = g^{3a} = \frac{\gamma^3}{R_6^2}$. We evaluate the divergent sum $\nu(E)$ on $r$ by

$$\frac{\partial \nu(E)}{\partial E} = \sum_r \frac{2E}{\beta'} \ln\left[\cosh \beta' E - \frac{\gamma^3 s + \gamma^a n_a}{2\pi r}\right]$$

$$= \frac{\partial \nu(E)}{\partial E} \left[\cosh \beta' E - \frac{\gamma^3 s + \gamma^a n_a}{2\pi r}\right],$$

(D.69)

(D.70)
using the sum \( \sum_{n \in \mathbb{Z}} \frac{2y}{(2\pi n + z)^2 + y^2} = \frac{\sinh y}{\cosh y - \cos z} \). Then integrating (D.70), we choose the integration constant to maintain modular invariance of (D.68),

\[
\nu(E) = \ln [\cosh \beta' E - \cos 2\pi (\gamma^3 s + \gamma^a n_a)] + \ln 2. \tag{D.71}
\]

It follows for \( s = n_3 \) that (D.68) gives

\[
(P.I.)^{1/2} = \prod_{s \in \mathbb{Z}} \frac{1}{\sqrt{2} \sqrt{\cosh \beta' E - \cos 2\pi (\gamma^3 s + \gamma^a n_a)}}
= \prod_{s \in \mathbb{Z}} \frac{e^{-\beta' E}}{1 - e^{-\beta' E + 2\pi i (\gamma^3 s + \gamma^a n_a)}}
= e^{-\pi R_6 \sum_{s \in \mathbb{Z}} \sqrt{G_{3}^{nm} n_l n_m}} \prod_{s \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{G_{3}^{nm} n_l n_m}} + 2\pi i \gamma^3 s + 2\pi i \gamma^a n_a}
= e^{-2\pi R_6 \langle H \rangle_{n_3}} \prod_{n_3 \in \mathbb{Z}} \frac{1}{1 - e^{-2\pi R_6 \sqrt{G_{3}^{nm} n_l n_m} + 2\pi i \gamma^3 n_3 + 2\pi i \gamma^a n_a}}, \tag{D.72}
\]

which is (D.67). Its invariance under \( U' \) follows from the \( U' \) invariance of (D.54), which differs from (D.69) only by an additional contribution of \( \tilde{n}^2 \) to the mass \( m^2 \).

Hence (D.72) and thus (D.67) are invariant under \( U' \).

Furthermore \( Z_{\text{osc}}^{6d} \) is invariant under \( M_3, U_2, U_3 \) by inspection.

Using the same approach for proving \( SL(4, \mathbb{Z}) \) symmetry of the 4d partition function, we have shown the 6d oscillator partition function for the chiral boson given by (4.36), or equivalently (D.65), transforms as

\[
U_2 : Z_{\text{osc}}^{6d} \rightarrow Z_{\text{osc}}^{6d},
U_1 U_3 : Z_{\text{osc}}^{6d} \rightarrow |\tau|^{-3} Z_{\text{osc}}^{6d},
U_1^2 : Z_{\text{osc}}^{6d} \rightarrow Z_{\text{osc}}^{6d}. \tag{D.73}
\]

Together with (D.45), the 6d partition function \( Z_{\text{osc}}^{6d,chiral} \equiv Z_{\text{zero modes}}^{6d} Z_{\text{osc}}^{6d} \) is \( SL(4, \mathbb{Z}) \) invariant.
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