Validity of Non-Hermitian fluctuation theorem in open Quantum system with unbroken PT-symmetry

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Abstract
In a recent paper, Deffner and Saxena (2015 Phys. Rev. Lett. 114 150601) showed that quantum Jarzynski equality generalizes to \(\mathcal{PT}\)– symmetric quantum mechanics with unbroken PT symmetry. Later Zeng and Yong (2017 Journal of Phys. Commun. 1 031001) extended this work to Crook’s fluctuation theorem. In another recent paper, Andrew Smith et al. 2018 discusses non-equilibrium work relation in open quantum system. In this paper, we will discuss the validity of Non-Hermitian fluctuation theorem in open quantum system, in a region of unbroken PT-symmetry.

1. Introduction

In recent time, the far from equilibrium phenomena have attracted a lot of attention in the scientific community. In 1993 Evans et al. and in 1994 Gallavotti et al. have developed a series of fluctuation theorems, making the earliest breakthroughs in the field. In his seminal paper [1], Jarzynski derived a very fundamental relationship between non-equilibrium work and free energy difference. The relation reads as

\[
\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}
\]

here \(W\) stands for work done when a driving force acts on the system and drives the system out of the equilibrium, and \(\Delta F\) stands for the change in free energy during the process. In 1999, Gravin E Crooks, in his papers [3,4], extended this relation and derived a more powerful relationship which is known as Crook’s fluctuation theorem. The theorem states

\[
\frac{P_F(+\beta W)}{P_R(-\beta W)} = e^{\beta(W - \Delta F)}
\]

In recent years, a lot of works have been done in generalizing these result for the Quantum system [5,9-11]. But, although a remarkable progress has been made in discussing fluctuation theorems for closed quantum systems, there are many questions yet to discussed for Quantum Open system. In 2013, Rastegin [17] has discussed these non-equilibrium equalities with unital quantum channels. In 2017, Smith et al. [19] discussed the experimental verification of it. In their paper, Smith et al. considered the correlation between system and bath very weak, so that
while decoherence plays an important role in the process, dissipation can be neglected.

In 1998, C. M. Bender [12,13] has discussed about a special non-hermitian Hamiltonian. The interesting thing about these set of Hamiltonians is that despite being non-hermitian, in a specific condition, they produce real spectra. The underlying condition they need to follow, is to have a $\mathcal{PT}$- symmetry. In 2015, Deffner and Saxena (2015 Phys. Rev. Lett. 114 150601) showed that quantum Jarzynski Equality generalizes to $\mathcal{PT}$- symmetric quantum mechanics in a region of unbroken $\mathcal{PT}$- symmetry. Later Zeng and Yong (2017 Journal of Phys. Commun. 1 031001) extended this work to Crook’s fluctuation theorem. In this paper, we consider a Quantum open system which is described by a $\mathcal{PT}$ – symmetric non-Hermitian Hamiltonian. The correlation between the system and bath is weak so that we can ignore dissipation [19]. We will show that Jarzynski Equality and Crook’s fluctuation theorem still hold true.

2. Non-Hermitian Hamiltonians with PT symmetry

According to the earlier conviction, in quantum mechanics, a Hamiltonian to give real spectra must follow the condition of Hermiticity. The Hermiticity of a Hamiltonian can be expressed mathematically as

$$H = H^\dagger$$  \hspace{1cm} (1)

Here $\dagger$ signifies matrix transposition followed by complex conjugation. Once we have this relationship, it can be easily shown that the Eigen values of $H$ are real signifying real spectra. Up until the last decade, it was thought that Hermiticity is a necessary condition for a Hamiltonian in quantum mechanics to have real spectra. But recently C. M. Bender [12,13] showed that even if the Hamiltonian is non-Hermitian, it can have real energy spectrum provided $H$ has an unbroken space-time ($\mathcal{PT}$ – symmetry) symmetry. In this case (1) can be replaced by the following expression

$$H = H^{\mathcal{PT}}$$

Here $\mathcal{P}$ stands for the space-reflection operator or parity operator and $T$ stands for time-reversal operator. The conditions followed by $P$ and $T$ operators are as follows [13]

$$P \hat{x} P = -\hat{x} \hspace{1cm} P \hat{p} P = -\hat{p}$$
$$T \hat{x} T = \hat{x} \hspace{1cm} T \hat{p} T = -\hat{p}$$

Here $\hat{x}$ and $\hat{p}$ are position and momentum operators respectively. Also

$$T i T = -i$$

Here $i$ is the complex number.
Since we don’t have the property of Hermiticity any more, many properties of quantum mechanics need to be modified [21,12,13,18]. The new bra-ket relationship for non-Hermitian quantum mechanics can be expressed as [21,12,13,18]

\[ |a⟩ \leftrightarrow ⟨Ba| = ⟨a|B^\dagger = ⟨a|B \]

Here B is a Hermitian operator such that B = B^\dagger

And hence the normalization condition can be expressed as \( ⟨a|B|a⟩ = 1 \). The completeness can be expressed as \( \sum_i |a_i⟩⟨a_i|B = 1 \).

The quantum dynamical equation, in the context of non-Hermitian formalism, can be written as [13],

\[
i\hbar \frac{\partial |ψ⟩}{\partial t} = (H(t) + A(t))|ψ⟩
\]

In the above equation, H(t) is the time dependent non-Hermitian Hamiltonian and A(t) is the time dependent gauge field term. The above modified equation preserves the unitarity of the non-Hermitian system.

3. Quantum channels

In this section, we will define Quantum Channel. The material of this section is borrowed from [17].

Let’s \( \mathcal{H} \) be the Hilbert space and \( \mathcal{L}(\mathcal{H}) \) is the space of linear operators on the Hilbert space \( \mathcal{H} \).

Now let us consider two Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). \( \Phi \) is a linear map between \( \mathcal{L}(\mathcal{H}_A) \) and \( \mathcal{L}(\mathcal{H}_B) \) such that \( \Phi: \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B) \).

For all \( X \in \mathcal{L}(\mathcal{H}_A) \), the linear map can be written as

\[
\Phi(X) = \sum_u K_u X K_u^\dagger
\]

Here \( K_u \) is the kraus operator. The adjoint of this linear map is written as

\[
\Phi^\dagger(Y) = \sum_u K_u^\dagger Y K_u
\]

In case of open quantum system, the dynamics is represented by density matrix rather than wave function. The density operator can be written as [20]

\[
\rho = \sum_i w_i |i⟩⟨i|
\]

where the system is in state \( i \) with probability \( w_i \). For a pure state, \( \rho \) can be written as [20]

\[
\rho = |i_{\text{pure}}⟩⟨i_{\text{pure}}|
\]

We can use the formalism of linear map mentioned above to establish the relationship between the density matrices in \( \mathcal{L}(\mathcal{H}_A) \) and \( \mathcal{L}(\mathcal{H}_B) \). If \( \rho_A \) is the input then \( \Phi(\rho_A) \) is the output. The output density matrix can be written as [17]

\[
\rho_B = \text{Tr}(\Phi(\rho_A))^{-1} \Phi(\rho_A)
\]

from (2), it can easily be shown that

\[
\Phi(1_A) = \sum_u K_u K_u^\dagger
\]
4. Jarzynski Equality in non-Hermitian (unbroken $\mathcal{PT}$-symmetric region) quantum open system

For classical system, Jarzynski equality can be written as [1,2]

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$

If we consider the correlation between the system and the bath is weak, we can assume that only decoherence is present and dissipation can be neglected [19]. Since there is no dissipation, two-point energy measurement scheme can be considered [9-11]. Let’s the initial Eigen state be $\varphi_m(t_0)$ and final $\varphi_n(t_f)$. Now considering only decoherence and neglecting dissipation [19], the work done by the external force in the system can be written as,

$$W = E_n(t_f) - E_m(t_0)$$

Now, applying the conditions of $\mathcal{PT}$-symmetric quantum system,

$$\langle e^{-\beta W} \rangle = \sum_{m,n} \frac{e^{-\beta E_m(t_0)}}{Z(t_0)} \langle \varphi_n(t_f)| B_{t_f} \Phi(\rho_A) | \varphi_n(t_f) \rangle \ e^{-\beta (E_n(t_f) - E_m(t_0))}$$

Here $\langle \varphi_n(t_f)| B_{t_f} \Phi(\rho_A) | \varphi_n(t_f) \rangle$ is the transition probability.

$$\langle e^{-\beta W} \rangle = \sum_{m,n} \frac{e^{-\beta E_m(t_0)}}{Z(t_0)} \ e^{-\beta (E_n(t_f) - E_m(t_0))} \langle \varphi_n(t_f)| B_{t_f} \Phi(\varphi_m(t_0)) | \varphi_n(t_f) \rangle$$

$$= \frac{1}{Z(t_0)} \sum_{m} e^{-\beta E_n(t_f)} \langle \varphi_n(t_f)| B_{t_f} \Phi \langle \sum_m |\varphi_m(t_0)\rangle \langle \varphi_m(t_0)|B_{t_0}\rangle | \varphi_n(t_f) \rangle$$

$$= \frac{1}{Z(t_0)} \sum_{m} e^{-\beta E_n(t_f)} \langle \varphi_n(t_f)| B_{t_f} \Phi (1) | \varphi_n(t_f) \rangle$$

$$= \frac{1}{Z(t_0)} \sum_{m} e^{-\beta E_n(t_f)} \langle \varphi_n(t_f)| B_{t_f} | \varphi_n(t_f) \rangle$$

$$= \frac{1}{Z(t_0)} \sum_{m} e^{-\beta E_n(t_f)}$$

$$= \frac{Z(t_f)}{Z(t_0)}$$

$$= e^{-\beta \Delta F}$$

Since, free energy can be written as $F = \frac{1}{\beta} \ln Z$ and this is the mathematical representation of
5. Crook’s fluctuation theorem in non-Hermitian (unbroken $\mathcal{PT}$-symmetric region) quantum open system

The probability distribution of work can be written [10,11]

$$P_{t_0t_f}(W) = \sum_{m,n} \delta(W - (E_n(t_f) - E_m(t_0))) P(|\varphi_m(t_0)\rangle) P(|\varphi_n(t_f)\rangle)$$

Here, $P(|\varphi_m(t_0)\rangle)$ denotes the probability of the system found in the Eigen state $|\varphi_m(t_0)\rangle$ and $P(|\varphi_n(t_f)\rangle)$ is the transition probability. $P(|\varphi_m(t_0)\rangle)$ can be written as [20]

$$P(|\varphi_m(t_0)\rangle) = \frac{e^{-\beta E_m(t_0)}}{Z(t_0)}$$

The transition probability for the Hermitian quantum mechanics can be written as

$$P(|\varphi_m(t_0)\rangle \rightarrow |\varphi_n(t_f)\rangle) = \langle \varphi_n(t_f)|\Phi|\varphi_n(t_f)\rangle$$

Considering the first pure energy state of the system to be $|\varphi_m(t_0)\rangle$.

Hence the transition probability for non-Hermitian system in unbroken $\mathcal{PT}$-symmetric region can be written as

$$P(|\varphi_m(t_0)\rangle \rightarrow |\varphi_n(t_f)\rangle) = \langle \varphi_n(t_f)|B_{t_f}\Phi(|\varphi_m(t_0)\rangle\langle\varphi_m(t_0)|B_{t_0})|\varphi_n(t_f)\rangle$$

Next, we will follow the same method adopted by Talkner and Hanggi [10,11]. The Fourier transformation of the work distribution is

$$\tilde{P}_{t_0t_f}(u) = \int dW e^{iuW} P_{t_0t_f}(W)$$

$$= \sum_{m,n} e^{iu(E_n(t_f) - E_m(t_0))} e^{-\beta E_m(t_0)}/Z(t_0) \langle \varphi_n(t_f)|B_{t_f}\Phi(|\varphi_m(t_0)\rangle\langle\varphi_m(t_0)|B_{t_0})|\varphi_n(t_f)\rangle$$

(3)

The time reversed distribution can be written [10,11], setting $\vartheta = u + i\beta$

$$\tilde{P}_{t_f t_0}(\vartheta) = \int dW e^{i\vartheta W} P_{t_f t_0}(W)$$

$$= \sum_{m,n} e^{i\vartheta(E_m(t_0) - E_n(t_f))} e^{-\beta E_n(t_f)}/Z(t_f)$$
\[ \langle \varphi_m(t_0) | B_{t_0} \Phi(|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} \rangle | \varphi_m(t_0) \rangle \]  

(4)

Now, using the property of linear map explained above
\[ \langle \varphi_n(t_f) | B_{t_f} \Phi(|\varphi_m(t_0)\rangle \langle \varphi_m(t_0)|B_{t_0} \rangle | \varphi_n(t_f) \rangle = \]

\[ \langle \varphi_n(t_f) | B_{t_f} \sum_u K_u (|\varphi_m(t_0)\rangle \langle \varphi_m(t_0)|B_{t_0} \rangle K_u^+ | \varphi_n(t_f) \rangle = \]

\[ = \sum_u \langle \varphi_n(t_f) | B_{t_f} K_u | \varphi_m(t_0) \rangle \langle \varphi_m(t_0)|B_{t_0} \rangle K_u^+ | \varphi_n(t_f) \rangle = \]

\[ = \sum_u \langle \varphi_m(t_0)|B_{t_0} \rangle K_u^+ | \varphi_n(t_f) \rangle \langle \varphi_n(t_f)|B_{t_f} K_u | \varphi_m(t_0) \rangle = \]

\[ = \langle \varphi_m(t_0)|B_{t_0} \sum_u K_u^+ (|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} K_u | \varphi_m(t_0) \rangle = \]

\[ = \langle \varphi_m(t_0)|B_{t_0} \Phi(|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} \rangle | \varphi_m(t_0) \rangle \]  

(5)

Now from (3)
\[ \tilde{p}_{t_0 t_f}(u) = \sum_{m,n} e^{iu(E_n(t_f) - E_m(t_0))} e^{-\beta E_m(t_0)/Z(t_0)} \langle \varphi_n(t_f) | B_{t_f} \Phi(|\varphi_m(t_0)\rangle \langle \varphi_m(t_0)|B_{t_0} \rangle | \varphi_n(t_f) \rangle \]

\[ \Rightarrow \ Z(t_0) \tilde{p}_{t_0 t_f}(u) = \sum_{m,n} e^{iu(E_n(t_f) - E_m(t_0))} e^{-\beta E_m(t_0)/Z(t_0)} \]

\[ = \langle \varphi_m(t_0)|B_{t_0} \Phi(|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} \rangle | \varphi_m(t_0) \rangle \]

(6)

Again
\[ \tilde{p}_{t_f t_0}(\vartheta) = \sum_{m,n} e^{i\vartheta(E_m(t_0) - E_n(t_f))} e^{-\beta E_n(t_f)/Z(t_f)} \langle \varphi_m(t_0)|B_{t_0} \Phi(|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} \rangle | \varphi_m(t_0) \rangle \]

\[ \Rightarrow \ Z(t_f) \tilde{p}_{t_f t_0}(\vartheta) = \sum_{m,n} e^{i\vartheta(E_m(t_0) - E_n(t_f))} e^{-\beta E_n(t_f)/Z(t_f)} \]

\[ = \langle \varphi_m(t_0)|B_{t_0} \Phi(|\varphi_n(t_f)\rangle \langle \varphi_n(t_f)|B_{t_f} \rangle | \varphi_m(t_0) \rangle \]
Now considering the condition (5), from (6) and (7), we can say that

\[ Z(t_0) \tilde{P}_{t_0t_f}(u) = Z(t_f) \tilde{P}_{t_f t_0}(\theta) \]

\[ \Rightarrow \frac{\tilde{P}_{t_0t_f}(u)}{\tilde{P}_{t_f t_0}(\theta)} = \frac{Z(t_f)}{Z(t_0)} \]

\[ \Rightarrow \frac{\tilde{P}_{t_0t_f}(u)}{\tilde{P}_{t_f t_0}(u+i\beta)} = \frac{Z(t_f)}{Z(t_0)} \]

After taking the inverse Fourier transformation

\[ \frac{\tilde{P}_{t_0t_f}(W)}{\tilde{P}_{t_f t_0}(-W)} = \frac{Z(t_f)}{Z(t_0)} e^{\beta W} = e^{\beta(W-\Delta F)} \]

And this is nothing but the mathematical representation of Crook’s fluctuation theorem.

The above derivation is valid when we have unbroken PT-symmetry. In case of broken PT-symmetry the dynamics is no longer unitary [14] and it can be concluded that the crook’s theorem and Jarzynski Equality are not valid for the Quantum open system with broken PT-symmetry.

6. Conclusion

In the above discussion, it has been shown that both Jarzynski Equality and Crook’s Fluctuation Theorem valid in non-Hermitian open quantum system with unbroken PT-symmetry. In this work, the coupling between the system and the Bath is considered to be weak so that we can assume dissipation to be negligible compared to decoherence. A future goal would be to examine the validity of these theorems when the coupling is strong and both dissipation and decoherence are significant.
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