BRST cohomology of Yang-Mills gauge fields in the presence of gravity in Ashtekar variables

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Abstract. The BRST transformations for the Yang-Mills gauge fields in the presence of gravity described by Ashtekar variables are obtained by using the so-called Maurer-Cartan horizontality conditions. The BRST cohomology group expressed by the Wess-Zumino consistency condition is solved with the help of an operator $\delta$ introduced by S.P. Sorella which in our case has a very simple form and generates, together with the differential $d$ and the BRST operator $s$, a simpler algebra than in the pure Yang-Mills theory. In this way we shall find the Yang-Mills Lagrangians, the Chern-Simons terms and the gauge anomalies.

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1 Introduction

Gauge fields play a very important role in all theories which describe fundamental interactions [1]. Electroweak theory and quantum chromodynamics (QCD) are examples of Yang-Mills theories. The most efficient way to study the quantization and the renormalization of such theories is given by the introduction of BRST transformations and by calculation of the so-called BRST cohomology. In despite of the fact that gravity could be introduced as a gauge theory associated with local Lorentz invariance [2], its action has a different structure and it is difficult to connect it to a special form of Yang-Mills theory, known as the topological quantum field theory (TQFT) [3] (see also [4]). It is necessary to describe the propagating degree of freedom using some variables that are related naturally to those employed in TQFT. In particular, we should require that these variables are suitable for implementing both diffeomorphism and gauge invariance. In this respect, the variables introduced by Ashtekar [5] satisfy these requirements. In place of the metric of general relativity, the classical Ashtekar variables [5, 6, 7, 8, 9], corresponding to specific Einstein manifolds, consist of simple $SO(3)$-gauge fields satisfying self-dual conditions. We shall analyze the BRST symmetry [10] of 4-dimensional gravity, described by the Ashtekar variables, coupled to the Yang-Mills fields, i.e. the case of Yang-Mills gauge fields in a curved spacetime, and we shall determine the solution of the descent equations identifying the BRST invariants constructed out of them. These quantities can be used to describe the diffeomorphism and gauge invariance of the observables of the theory, and to deliver characterizations of different structures of the manifold or of the space of Yang-Mills fields.

The BRST invariant Lagrangians, possible anomalies and Schwinger terms are nontrivial solutions of the Wess-Zumino consistency condition [11]

$$s\Delta = 0 \quad , \quad \Delta \neq s\hat{\Delta} ,$$

(1.1)

where $\Delta$ and $\hat{\Delta}$ are integrated local polynomials in the fields and their derivatives and $s$ is the nilpotent BRST operator. In particular, the BRST formalism allows the characterization of classical actions and anomalies as BRST invariant functionals. Especially, an action is a BRST invariant functional with ghost number zero, an anomaly corresponds to a BRST invariant functional with ghost number one and a Schwinger term to such one with ghost number two.

Setting $\Delta = \int B$, condition (1.1) translates into the local equation

$$sB + dQ = 0 ,$$

(1.2)

where $Q$ is some local polynomial in the fields and their derivatives and $d = dx^\mu \partial_\mu$ represents the exterior spacetime differential which, together with the BRST operator $s$, obeys the BRST algebra

$$s^2 = d^2 = sd + ds = 0 .$$

(1.3)

The local term $B$ is called nontrivial if

$$B \neq s\hat{B} + d\hat{Q} ,$$

(1.4)
with $\mathcal{B}$ and $\mathcal{Q}$ local polynomials. In this case the integral of $\mathcal{B}$ on spacetime, $\int \mathcal{B}$, is a representative of a cohomology class of the BRST operator $s$. The integrated cohomology problem (1.1) is equivalent to the local one

$$sa = 0 , \quad a \neq s\bar{a} ,$$

where $a$ is a local polynomial in the fields and their derivatives. The local equation (1.2) could be solved in a simple way by using the operator $\delta$ introduced by S.P. Sorella [12] which obeys the decomposition of the exterior spacetime derivative as a BRST commutator

$$d = -[s, \delta] .$$

Actually, the decomposition (1.6) represents one of the crucial features of the topological field theories [13, 14] and the bosonic string and superstring in the Beltrami and super-Beltrami parametrization [15, 16]. As shown in [17, 18], the eq. (1.6) allows a cohomological interpretation of the cosmological constant, of Lagrangians for pure Einstein gravity and generalizations including also torsion, as well as gravitational Chern-Simons terms and anomalies. Due to the existence of the decomposition the study of the cohomology of $s$ modulo $d$ (1.2) is essentially reduced to the study of the local cohomology of $s$ (1.5) which in turn can be systematically analyzed by using the powerful techniques of spectral sequences [19, 20]. In fact, as proven in [21], the solution obtained by utilizing the decomposition (1.6) is completely equivalent to that based on the Russian formula [22, 23, 24, 25, 26, 27, 28, 29, 30], i.e. they differ only by trivial cocycles.

In a first step we will demonstrate that the BRST transformations for the Yang-Mills fields as well as the Ashtekar variables describing gravity, can be derived from Maurer-Cartan horizontality conditions [22, 23, 24, 25, 31, 32]. After that we will prove that the decomposition (1.6) can be successfully extended to Yang-Mills gauge theory coupled to the Ashtekar variables, i.e. to the gravitational fields. Finally, we will see that the operator $\delta$ gives an elegant and straightforward way to classify the cohomology classes of the full BRST operator. As shown in [33], the eq. (1.6) allows a cohomological interpretation of the cosmological constant, of the Ashtekar Lagrangians [3] for pure gravity, of the Capovilla, Jacobson, Dell Lagrangian [34] as well as the gravitational Chern-Simons terms.

## 2 Maurer-Cartan horizontality conditions

The aim of this section is to derive the set of BRST transformations for the Yang-Mills gauge fields in the presence of gravity from Maurer-Cartan horizontality conditions. In a first step this geometrical formalism is used to discuss the simpler case of non-abelian Yang-Mills theory [25].

The BRST transformations of the 1-form gauge connection $A^A = A^A_\mu dx^\mu$ and the 0-form ghost field $c^A$ are given by

$$sA^A = dc^A + f^{ABC} c^B A^C ,$$

$$sc^A = \frac{1}{2} f^{ABC} c^B c^C ,$$

(2.1)\footnote{Here, capital Latin indices are denoting gauge group indices.}
with
\[ s^2 = 0 , \quad (2.2) \]
where \( f^{ABC} \) are the structure constants of the corresponding gauge group \( G \). As usual, the adopted grading is given by the sum of the form degree and of the ghost number. In this sense, the fields \( \mathcal{A}^A \) and \( c^A \) are both of degree one, their ghost number being respectively zero and one. A \( p \)-form with ghost number \( q \) will be denoted by \( \Omega^p_q \), its total grading being \( (p + q) \). The 2-form field strength \( \mathcal{F}^A \) is given by
\[ \mathcal{F}^A = \frac{1}{2} \mathcal{F}^A_{\mu
u} dx^\mu dx^\nu = d\mathcal{A}^A + \frac{1}{2} f^{ABC} \mathcal{A}^B \mathcal{A}^C , \quad (2.3) \]
and
\[ d\mathcal{F}^A = f^{ABC} \mathcal{F}^B \mathcal{A}^C , \quad (2.4) \]
is its Bianchi identity. The BRST transformations (2.1) can be interpreted as a Maurer-Cartan horizontality condition if one introduces the combined gauge-ghost field
\[ \tilde{\mathcal{A}}^A = \mathcal{A}^A + c^A , \quad (2.5) \]
and the generalized nilpotent differential operator
\[ \tilde{d} = d - s , \quad \tilde{d}^2 = 0 . \quad (2.6) \]
Notice that both \( \tilde{\mathcal{A}}^A \) and \( \tilde{d} \) have degree one. The nilpotency of \( \tilde{d} \) in (2.6) just implies the nilpotency of \( s \) and \( d \), and furthermore fulfills the anticommutator relation
\[ \{ s, d \} = 0 . \quad (2.7) \]

With \( \tilde{d} \) and \( \tilde{\mathcal{A}} \) we can build up the generalized degree-two field strength \( \tilde{\mathcal{F}}^A \):
\[ \tilde{\mathcal{F}}^A = \tilde{d}\tilde{\mathcal{A}}^A + \frac{1}{2} f^{ABC} \tilde{\mathcal{A}}^B \tilde{\mathcal{A}}^C , \quad (2.8) \]
which, from eq.(2.6), obeys the generalized Bianchi identity
\[ \tilde{d}\tilde{\mathcal{F}}^A = f^{ABC} \tilde{\mathcal{F}}^B \tilde{\mathcal{A}}^C . \quad (2.9) \]
The Maurer-Cartan horizontality condition reads then
\[ \tilde{\mathcal{F}}^A = \mathcal{F}^A . \quad (2.10) \]
Now it is very easy to check that the BRST transformations (2.1) can be obtained from the horizontality condition (2.10) by simply expanding \( \tilde{\mathcal{F}}^A \) in terms of the elementary fields \( \mathcal{A}^A \) and \( c^A \) and collecting the terms with the same form degree and ghost number.

### 2.1 Yang-Mills gauge fields coupled to gravity in Ashtekar variables

If we want to study the BRST cohomology of the Yang-Mills fields in the presence of gravity we should generalize the horizontality condition (2.10) and specify the functional
space the BRST operator $s$ acts upon. The latter is chosen to be the space of local polynomials which depend on the 1-forms $(e^a, A^a_b, A^A)$, where $e^a$, $A^a_b$ and $A^A$ being respectively the vielbein, the Ashtekar connection and the Yang-Mills gauge field

$$
e^a = e^a_\mu dx^\mu ,
A^a_b = A^a_{b\mu} dx^\mu ,
A^A = A^A_\mu dx^\mu ,$$

(2.11)

and on the 2-forms $(T^a, F^a_b, F^A)$, whereby $T^a$, $F^a_b$ and $F^A$ denoting the Ashtekar torsion, the Ashtekar field strength and the Yang-Mills field strength

$$T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu dx^\nu = d\epsilon^a + A^a_b \epsilon^b = D\epsilon^a ,$$

$$F^a_b = \frac{1}{2} F^a_{\mu\nu} dx^\mu dx^\nu = dA^a_b + A^a_c A^c_b ,$$

$$F^A = \frac{1}{2} F^A_{\mu\nu} dx^\mu dx^\nu = dA^A + \frac{1}{2} f^{ABC} A^B A^C ,$$

(2.12)

with the covariant exterior derivative

$$D = d + A + A .$$

(2.13)

The tangent space indices $(a, b, c, ...)$ are referred to the group $SO(1, 3)$.

Applying the exterior derivative $d$ to both sides of eqs. (2.12) one gets the Bianchi identities

$$DT^a = dT^a + A^a_b T^b = F^b_e ,$$

$$DF^a_b = dF^a_b + A^a_c F^c_b - A^c_b F^a_c = 0 ,$$

$$D F^A = dF^A + f^{ABC} A^B A^C = 0 .$$

(2.14)

To write down the gravitational Maurer-Cartan horizontality conditions for this case one introduces a further ghost, as done in [22, 23, 31], the local translation ghost $\eta^a$ having ghost number one and a tangent space index. As explained in [22, 31] (see also [14, 35]), the field $\eta^a$ represents the ghost of local translations in the tangent space.

The local translation ghost $\eta^a$ can be related to the ghost of local diffeomorphisms $\xi^a$ by

$$\xi^a = E^a_\mu \eta^\mu , \quad \eta^a = \xi^a \epsilon^a ,$$

(2.15)

where $E^a_\mu$ denotes the inverse of the vielbein $e^a_\mu$, i.e.

$$e^a_\mu E^a_b = \delta^a_b ,$$

$$e^a_\mu E^a_\nu = \delta^\nu_\mu .$$

(2.16)

Proceeding now as for the pure Yang-Mills case, one defines the nilpotent differential operator $\tilde{d}$ of degree one

$$\tilde{d} = d - s ,$$

(2.17)
and the generalized vielbein-ghost field \( \hat{e}^a \), the extended Ashtekar connection \( \hat{A}^a_b \), and the generalized non-abelian Yang-Mills gauge field \( \hat{A}^A \)

\[
\begin{align*}
\hat{e}^a &= e^a + \eta^a , \\
\hat{A}^a_b &= \hat{A}^a_b + e^a_b , \\
\hat{A}^A &= \hat{A}^A + e^A ,
\end{align*}
\]

(2.18)

with the ashtekar ghost \( e^a_b \) and where \( \hat{A}^a_b \) and \( \hat{A}^A \) are given by

\[
\begin{align*}
\hat{A}^a_b &= A^a_{bm} e^m = A^a_b + A^a_{bm} \eta^m , \\
\hat{A}^A &= A^A_m e^m = A^A + A^A_m \eta^m ,
\end{align*}
\]

(2.19)

with the 0-forms \( A^a_{bm} \) and \( A^A_m \) defined by the expansion of the 0-form Ashtekar connection \( A^a_{b\mu} \) and the 0-form Yang-Mills gauge field \( A^A_{\mu} \) in terms of the vielbein \( e^a_{\mu} \), i.e.:

\[
\begin{align*}
A^a_{b\mu} &= A^a_{bm} e^m_{\mu} , \\
A^A_{\mu} &= A^A_m e^m_{\mu} .
\end{align*}
\]

(2.20)

As it is well-known, the last formulas stem from the fact that the vielbein formalism allows to transform locally the spacetime indices of an arbitrary tensor \( N_{\mu\nu\rho\sigma...} \) into flat tangent space indices \( N_{abcd...} \) by means of the expansion

\[
N_{\mu\nu\rho\sigma...} = N_{abcd...} e^a_{\mu} e^b_{\nu} e^c_{\rho} e^d_{\sigma} ... .
\]

(2.21)

Vice versa one has

\[
N_{abcd...} = N_{\mu\nu\rho\sigma...} E^\nu_a E^\rho_b E^\sigma_c E^\tau_d ... .
\]

(2.22)

According to eqs.(2.12), the generalized Ashtekar torsion field, the generalized Ashtekar field strength and the generalized Yang-Mills field strength are given by

\[
\begin{align*}
\tilde{T}^a &= \tilde{d} \hat{e}^a + \hat{A}^a_b \hat{e}^b = \tilde{D} \hat{e}^a , \\
\tilde{F}^a_b &= \tilde{d} \hat{A}^a_b + \hat{A}^a_c \hat{A}^c_b , \\
\tilde{F}^A &= \tilde{d} \hat{A}^A + \frac{1}{2} f^{ABC} \hat{A}^B \hat{A}^C ,
\end{align*}
\]

(2.23)

and are easily seen to obey the generalized Bianchi identities

\[
\begin{align*}
\tilde{D} \tilde{T}^a &= \tilde{d} \tilde{T}^a + \hat{A}^a_b \tilde{T}^b = \tilde{F}^a_b \hat{e}^b , \\
\tilde{D} \tilde{F}^a_b &= \tilde{d} \tilde{F}^a_b + \hat{A}^a_c \tilde{F}^c_b - \hat{A}^c_b \tilde{F}^a_c = 0 , \\
\tilde{D} \tilde{F}^A &= \tilde{d} \tilde{F}^A + f^{ABC} \hat{A}^B \hat{F}^C = 0 ,
\end{align*}
\]

(2.24)

with

\[
\tilde{D} = \tilde{d} + \hat{A} + \hat{A}
\]

(2.25)

the generalized covariant derivative.

With these definitions the Maurer-Cartan horizontality conditions for the Yang-Mills gauge fields in the presence of gravity in terms of Ashtekar fields may be expressed in the

\footnote{Remark that the 0-form \( A^a_{bm} \) does not possess any symmetric or antisymmetric property with respect to the lower indices (bm).}
following way: $\tilde{e}$ and all its generalized covariant exterior differentials can be expanded over $\tilde{e}$ with classical coefficients,

$$e^a = \delta^a_b \tilde{e}^b \equiv \text{horizontal} ,$$

(2.26)

$$\tilde{T}^a(\tilde{e}, \tilde{A}) = \frac{1}{2} T^a_{mn}(e, A)e^m \tilde{e}^n \equiv \text{horizontal} ,$$

(2.27)

$$\tilde{F}^a_b(\tilde{A}) = \frac{1}{2} F^a_{bmn}(A)e^m \tilde{e}^n \equiv \text{horizontal} ,$$

(2.28)

$$\tilde{F}^A(\tilde{A}) = \frac{1}{2} F^A_{mn}(A)e^m \tilde{e}^n \equiv \text{horizontal} .$$

(2.29)

Through eq. (2.21), the 0-forms $T^a_{mn}, F^a_{bmn},$ and $F^A_{mn}$ are defined by the vielbein expansion of the 2-forms of the Ashtekar torsion, the Ashtekar field strength and the Yang-Mills field strength of eqs. (2.12),

$$T^a = \frac{1}{2} T^a_{mn} e^m e^n ,$$

$$F^a_b = \frac{1}{2} F^a_{bmn} e^m e^n ,$$

$$F^A = \frac{1}{2} F^A_{mn} e^m e^n ,$$

(2.30)

and the 0-form $D_m$ of the covariant exterior derivative $D$ is given by

$$D = e^m D_m .$$

(2.31)

Notice also that eqs. (2.19) are nothing but the horizontality conditions for the Ashtekar connection and the Yang-Mills gauge field expressing the fact that $\tilde{A}$ and $\tilde{\tilde{A}}$ themselves can be expanded over $\tilde{e}$.

The horizontality conditions (2.26)-(2.29) are equivalent with the statements

$$\tilde{e}^a = \exp(i\xi)e^a = e^a + i\xi e^a ,$$

(2.32)

$$\tilde{T}^a = \exp(i\xi)T^a = T^a + i\xi T^a + \frac{1}{2} i\xi i\xi T^a ,$$

(2.33)

$$\tilde{F}^{ab} = \exp(i\xi)F^{ab} = F^{ab} + i\xi F^{ab} + \frac{1}{2} i\xi i\xi F^{ab} ,$$

(2.34)

$$\tilde{F}^A = \exp(i\xi)F^A = F^A + i\xi F^A + \frac{1}{2} i\xi i\xi F^A ,$$

(2.35)

since $e^a$ is a 1-form, while $T^a, F^{ab}$ and $F^A$ are 2-forms.

Eqs. (2.26)-(2.29) define the Maurer-Cartan horizontality conditions for the Yang-Mills gauge fields in the presence of gravity in terms of Ashtekar variables and when expanded in terms of the elementary fields ($e^a, A^a, A^A, \eta^a, c^a_b, c^A$), give the nilpotent BRST transformations corresponding to the diffeomorphism transformations, the local Lorentz rotations and the gauge transformations.

For a better understanding of this point let us discuss in detail the horizontality condition (2.27) for the Ashtekar torsion. Making use of eqs. (2.17), (2.18), (2.19) and of
the definition (2.23), one verifies that eq. (2.27) gives
\[ de^a - se^a + d\eta^a - s\eta^a + A_b^a e^b + c^a_b e^b + A^a_{bm} \eta^m e^b + A^a_{bm} \eta^m \eta^b = \]
\[ = \frac{1}{2} T^a_{mn} e^m \eta^n + T^a_{mn} e^m \eta^n + \frac{1}{2} T^a_{mn} \eta^m \eta^n , \] (2.36)
from which, collecting the terms with the same form degree and ghost number, one easily obtains the BRST transformations for the tetrad 1-form \( e^a \) and for the local translation ghost \( \eta^a \):
\[ se^a = d\eta^a + A^a_{bm} \eta^m e^b - T^a_{mn} e^m \eta^n , \]
\[ s\eta^a = c^a_b \eta^b + A^a_{bm} \eta^m \eta^b - \frac{1}{2} T^a_{mn} \eta^m \eta^n . \] (2.37)
These equations, when rewritten in terms of the variable \( \xi^\mu \) of eq. (2.15), take the more familiar form
\[ se^a_{\mu} = c^a_{\nu} e^b_{\mu} - L_{\xi} e^a_{\mu} , \]
\[ s\xi^\mu = -\xi^\lambda \partial_\lambda \xi^\mu = -\frac{1}{2} L_{\xi} \xi^\mu , \] (2.38)
where \( L_{\xi} \) denotes the ordinary Lie derivative along the direction \( \xi^\mu \), i.e.
\[ L_{\xi} e^a_{\mu} = \xi^\lambda \partial_\lambda e^a_{\mu} + (\partial_\mu \xi^\lambda) e^a_{\lambda} . \] (2.39)
It is apparent now that eq. (2.37) represents the tangent space formulation of the usual BRST transformations corresponding to local Lorentz rotations and diffeomorphisms.

One sees then that the Maurer-Cartan horizontality conditions (2.26)-(2.29) together with eq. (2.23) carry in a very simple and compact way all the information relative to the gravitational Yang-Mills gauge algebra. Indeed, it is easy to expand eqs. (2.26)-(2.29) in terms of \( e^a \) and \( \eta^a \) and work out the BRST transformations of the remaining fields \( (A^a_b, A^A, T^a, F^a_b, F^A, c^a_b, c^A) \).

However, in view of the fact that we will use as fundamental variables the 0-forms \( (A^a_{bm}, A^A, T^a_{mn}, F^a_{bmn}, F^A_{mn}) \) rather than the 1-forms \( A^a_b \) and \( A^A \) and the 2-forms \( T^a, F^a_b \) and \( F^A \) let us proceed by introducing the partial derivative \( \partial_a \) with a flat tangent space index. According to the formulas (2.21) and (2.22), the latter is defined by
\[ \partial_a \equiv E^a_u \partial_\mu , \] (2.40)
and
\[ \partial_\mu = e^a_\mu \partial_a , \] (2.41)
so that the intrinsic exterior differential \( d \) becomes
\[ d = dx^\mu \partial_\mu = e^a \partial_a . \] (2.42)
Let us emphasize that the introduction of the operator \( \partial_a \) and the use of the 0-forms \( (A^a_{bm}, A^A, T^a_{mn}, F^a_{bmn}, F^A_{mn}) \) allows for a complete tangent space formulation of the gravitational Yang-Mills gauge algebra. This step, as we shall see later, turns out to be very useful in the analysis of the corresponding BRST cohomology. Moreover, as one can easily understand, the knowledge of the BRST transformations of the 0-form sector \( (A^a_{bm}, A^A, T^a_{mn}, F^a_{bmn}, F^A_{mn}) \) together with the expansions (2.20), (2.30) and the eqs. (2.37) completely characterize the transformation law of the forms \( (A^a_b, A^A, T^a, F^a_b, F^A) \).
2.2 BRST transformations and Bianchi identities

Let us finish this section by giving, for the convenience of the reader, the BRST transformations and the Bianchi identities which one can find by using the Maurer-Cartan horizontality conditions \((2.26)-(2.29)\) and from eqs.\((2.23)\) and \((2.24)\) for each form sector and ghost number.

- **Form sector two, ghost number zero** \((T^a, F_b^a, F^A)\)

\[
ST^a = c_b^a T^b + A_{bk}^a \eta^k T^b - F_b^a \eta^b \\
+ A_{bm}^a \eta^m e^b + F_{bm}^a \eta^m e^b + (dT_{mn}^a) e^m \eta^n \\
- T_{mn}^a e^m d\eta^n + T_{mn}^a T^m \eta^n - T_{kn}^a A_k^m e^m \eta^n ,
\]

\[
sF_b^a = c_c^b F_c^a - c_b^c F_c^a + A_{ck}^a \eta^k F_c^a \\
+ A_{cm}^a F_{mn}^c e^m \eta^n - A_{c}^b F_{cmn}^a e^m \eta^n + (dF_{mn}^a) e^m \eta^n \\
+ F_{bmn}^a T^m \eta^n - F_{ckn}^a A_k^m e^m \eta^n - F_{bmn}^a e^m d\eta^n ,
\]

\[
sF^A = (dF^A_{mn}) e^m \eta^n + F^A_{mn} T^m \eta^n - F^A_{mn} A_{mn}^m \eta^n - (dF^A_{mn}) e^m \eta^n \\
+ f_{ABC} C^B F^C + f_{ABC} A^B_m \eta^m F^C + f_{ABC} A^B F^C_{mn} e^m \eta^n . \tag{2.43}
\]

For the Bianchi identities one has

\[
dT^a + A_b^a T^b = F_b^a e^b ,
\]

\[
dF^a + A_b^a F^b - A_b^c F_c^a = 0 ,
\]

\[
dF^A + f_{ABC} A^B_m \eta^m F^C + 0 . \tag{2.44}
\]

- **Form sector one, ghost number zero** \((e^a, A_b^a, A^A)\)

\[
sc^a = d\eta^a + A_{b}^a \eta^b + c_b^a \eta^b + A_{bm}^a e^b + T_{mn}^a \eta^m e^n \\
- A_{c}^a F_c^a + c_c^a A^c_b + e^a c_b^c + (dA_{bm}^a) \eta^m + A_{mn}^a d\eta^n \\
+ A_{cm}^a A_b^m \eta^m + A_{mn}^a A_b^m \eta^m - F_{bmn}^a e^m \eta^n ,
\]

\[
sA^A = dA^A + (dA^A_m) \eta^m + A^A_m d\eta^m + f_{ABC} A^B e^C \\
+ f_{ABC} A^B A^m M^m - F_{mn} e^m \eta^n . \tag{2.45}
\]

- **Form sector zero, ghost number zero** \((A_b^a, A_m^a, T^a_m, F_{mn}^a, F^A_{mn})\)

\[
SA_{bm}^a = - \partial_m c^a_b + c^a_c A_b^c - c^a_b A_c^m - e^a c^c_b A_{cm} - c^k A_{bk}^a - \eta^k \partial_k A_{bm}^a \\
- \partial_m A^a_c - f_{ABC} A^B M^C - e^k A^A_k - \eta^k \partial_k A^m \\
+ T_{mn}^a = c^a_c T^k_m - c^a_k T^c_m - e^c_k T^a m - \eta^k \partial_k T_{mn}^a ,
\]

\[
sF_{bmn}^a = c^a_c F_{bmn}^c - c^a_b F_{cmn}^c - c^a_k F_{bkn}^c - e^a_k F_{bmn}^c - \eta^k \partial_k F_{bmn}^a \\
+ f_{ABC} A^B M^C - e^a_m F^A_m + c^a_k F^A_k - \eta^k \partial_k F^A_{mn} . \tag{2.46}
\]
The Bianchi identities (2.44) are projected on the 0-form Ashtekar torsion \( T^a_{mn} \), on the 0-form Ashtekar field strength \( F^a_{bmn} \) and on the 0-form Yang-Mills field strength \( F^A_{mn} \) to give

\[
dT^a_{mn} = (\partial_l T^a_{mn}) e^l \\
= (F^a_{lmn} + F^a_{mnl} + F^a_{nml} \\
- A^a_{bl} T^b_{mn} - A^a_{bm} T^b_{nl} - A^a_{bn} T^b_{lm} \\
+ T^a_{kn} T^k_{ml} + T^a_{km} T^k_{ln} + T^a_{kl} T^k_{nm} \\
- T^a_{kn} A^k_{lm} - T^a_{km} A^k_{nl} - T^a_{kl} A^k_{mn} \\
+ T^a_{kn} A^k_{ml} + T^a_{kl} A^k_{nm} + T^a_{km} A^k_{ln} \\
- \partial_m T^a_{nl} - \partial_n T^a_{lm}) e^l ,
\]

\[
dF^a_{bmn} = (\partial_l F^a_{bmn}) e^l \\
= (-A^a_{cl} F^c_{bmn} - A^a_{cm} F^c_{bml} - A^a_{cn} F^c_{blm} \\
+ A^c_{bl} F^a_{cmm} + A^c_{bm} F^a_{cln} + A^c_{bn} F^a_{clm} \\
+ F^a_{bkn} T^k_{ml} + F^a_{bkm} T^k_{ln} + F^a_{bkl} T^k_{nm} \\
- F^a_{bkm} A^k_{ln} - F^a_{bkl} A^k_{mn} - F^a_{bkl} A^k_{mn} \\
+ F^a_{bkm} A^k_{ml} + F^a_{bkl} A^k_{nm} + F^a_{bkm} A^k_{ln} \\
- \partial_m F^a_{bnl} - \partial_n F^a_{blm}) e^l ,
\]

\[
dF^A_{mn} = (\partial_l F^A_{mn}) e^l \\
= (f^{ABC} F^B_{mn} A^C_n + f^{ABC} F^B_{nl} A^C_m + f^{ABC} F^B_{lm} A^C_n \\
- F^A_{kn} T^k_{lm} - F^A_{kl} T^k_{mn} - F^A_{km} T^k_{ln} \\
+ F^A_{kn} A^k_{lm} + F^A_{km} A^k_{ln} + F^A_{kl} A^k_{mn} \\
- F^A_{kn} A^k_{ml} - F^A_{kl} A^k_{nm} - F^A_{km} A^k_{ln} \\
- \partial_m F^A_{nl} - \partial_n F^A_{lm}) e^l .
\]  

(2.47)

One has also the equations

\[
 dA^a_{bm} = (\partial_n A^a_{bn}) e^n \\
= (-F^a_{bmn} + A^a_{cn} A^c_{bn} - A^a_{cm} A^c_{bn} \\
+ A^a_{bk} T^k_{mn} - A^a_{bk} A^k_{mn} + A^a_{bk} A^k_{mn} + \partial_m A^a_{bn}) e^n ,
\]

\[
 dA^A_m = (\partial_n A^A_m) e^n \\
= (-F^A_{mn} + f^{ABC} A^B_m A^C_n + A^A_{km} T^k_{mn} \\
- A^A_{km} A^k_{mn} + A^A_{km} A^k_{mn} + \partial_m A^A_n) e^n .
\]  

(2.48)

- **Form sector zero, ghost number one** \((\eta^a, c^a_b, c^A)\)

\[
 s \eta^a = A^a_{bm} \eta^m \eta^b + c^a_b \eta^b - \frac{1}{2} T^a_{mn} \eta^m \eta^n ,
\]

\[
 s c^a_b = c^a_c c^c_b - \eta^k \partial_k c^a_b ,
\]

\[
 s c^A = \frac{1}{2} f^{ABC} c^B C - \eta^k \partial_k c^A .
\]  

(2.49)
• Algebra between $s$ and $d$

From the above transformations it follows:

$$s^2 = 0, \quad d^2 = 0,$$  \hfill (2.50)

and

$$\{s, d\} = 0.$$  \hfill (2.51)

3 Solution of the descent equations

The question of finding the invariant Lagrangians, the anomalies and the Schwinger terms for the Yang-Mills gauge field theory coupled to four-dimensional gravity in Ashtekar variables can be solved in a purely algebraic way by solving the BRST consistency condition in the space of the integrated local field polynomials. In order to solve this problem we have to find out the nontrivial solution of the Wess-Zumino consistency condition \[11\]

$$s\Delta = 0,$$  \hfill (3.1)

where $\Delta$ is an integrated local field polynomial, i.e. $\Delta = \int B$. The condition (3.1) translates into the local equation

$$sB + dQ = 0,$$  \hfill (3.2)

where $Q$ is some local polynomial and $d = dx^\mu \partial_\mu$ is the nilpotent exterior spacetime derivative which anticommutes with the nilpotent BRST operator $s$

$$s^2 = d^2 = sd + ds = 0$$  \hfill (3.3)

and it is acyclic (i.e. its cohomology group vanishes).

The local equation (3.2), due to (3.3) and the acyclicity of $d$, generates a tower of descent equations

$$sB + dQ^1 = 0$$

$$sQ^1 + dQ^2 = 0$$

$$\ldots$$

$$sQ^{k-1} + dQ^k = 0$$

$$sQ^k = 0$$  \hfill (3.4)

with $Q^i$ local polynomials in the fields.

For the Yang-Mills case, these equations can be solved by means of a transgression procedure generated by the Russian formula \[22, 23, 24, 25, 26, 27, 28, 29, 30\].

More recently a new and efficient way of finding nontrivial solutions of the tower (3.4) has been proposed by S.P. Sorella \[12\] and successfully applied to the study of the Yang-Mills cohomology \[21\], the gravitational anomalies \[30\] and the algebraic structure.
of gravity with torsion [17]. The basic ingredient of the method is an operator $\delta$ which allows us to express the exterior derivative $d$ as a BRST commutator, i.e.:

$$d = -[s, \delta].$$

(3.5)

Now it is easy to see that, once the decomposition (3.5) has been found, repeated application of the operator $\delta$ on the polynomial $Q$ which is a nontrivial solution of the last equation of (3.4) gives an explicit and nontrivial solution for the other cocycles $Q_i$ and for $B$. If $B$ has ghost number one then it is called an anomaly and if it has ghost number zero then it represents an invariant Lagrangian. In other word using the operator $\delta$ we can calculate the solution of the cohomology $H(s \mod d)$ if we know the solution of the cohomology $H(s)$. Actually, as has been shown in [21], the cocycles obtained by the descent equations (3.4) turn out to be completely equivalent to those which are based on the Russian formula.

For the Yang-Mills fields coupled with gravity in the Ashtekar variables the operator $\delta$ introduced in (3.5) can be defined by

$$\delta \eta^a = -e^a,$$

$$\delta \Phi = 0 \quad \text{for} \quad \Phi = (e^a, A^{ab}, A^A, T^a, F^{ab}, F^A, c^{ab}, c^A).$$

(3.6)

Now it is easy to verify that $\delta$ is of degree 0 and obeys the following algebraic relations

$$d = -[s, \delta], \quad [d, \delta] = 0.$$  

(3.7)

In order to solve the tower (3.4) we shall make use of the following identity

$$e^\delta s = (s + d)e^\delta,$$

(3.8)

which is a direct consequence of (3.7) (see [21]).

Let us consider now the solution of eqs. (3.4) with a given ghost number $G$ and form degree 4, i.e. a solution of the tower

$$s\Omega_4^G + d\Omega_3^{G+1} = 0,$$

$$s\Omega_3^{G+1} + d\Omega_2^{G+2} = 0,$$

$$s\Omega_2^{G+2} + d\Omega_1^{G+3} = 0,$$

$$s\Omega_1^{G+3} + d\Omega_0^{G+4} = 0,$$

$$s\Omega_0^{G+4} = 0$$

(3.9)

with $(\Omega_4^G, \Omega_3^{G+1}, \Omega_2^{G+2}, \Omega_1^{G+3}, \Omega_0^{G+4})$ local polynomials in the variables $(e^a, A^{ab}, A^A, \eta^a, c^{ab}, c^A)$ which, without loss of generality, will be always considered as irreducible elements, i.e. they cannot be expressed as the product of several factored terms. In particular $\Omega_4^0, \Omega_3^1$ and $\Omega_2^2$ correspond, respectively to an invariant Lagrangian, an anomaly and a Schwinger term.

Due to the identity (3.8) we can obtain the higher cocycles $\Omega_q^{G+4-q}(q = 1, 2, 3, 4)$ once a nontrivial solution for $\Omega_0^{G+4}$ is known. Indeed, if one applies the identity (3.8) on $\Omega_0^{G+4}$ one gets

$$(s + d)\left(e^\delta \Omega_0^{G+4}(\eta^a, c^{ab}, c^A, A^{ab}, A^A, T^a, F^{ab}, F^A)\right) = 0.$$ 

(3.10)
But as one can see from eq.(3.3), the operator $\delta$ acts as a shift on the ghost $\eta^a$ with an amount $(-e^a)$ and eq.(3.10) can be rewritten as

$$(s + d)\Omega_0^{G+4}(\eta - e, c, A, T, F) = 0.$$  \hfill (3.11)

Thus the expansion of the 0-form cocycle $\Omega_0^{G+4}(\eta^a - e^a, c^a, A^a, A^A, T^a, F^{ab}, F^A)$ in power of the 1-form tetrads $e^a$ yields all the cocycles $\Omega_q^{G+4-q}$.

4 Examples

This section is devoted to apply the previous algebraic setup and to discuss some explicit examples. We want to emphasize the cohomological origin of the Lagrangian which describes Yang-Mills fields in the presence of gravity, the topological Yang-Mills Lagrangian as well as of the Chern-Simons terms for this theory.

4.1 The Yang-Mills Lagrangian in the presence of gravity

The simplest local BRST polynomial which one can construct from the Yang-Mills fields and the local translation ghost is

$$\Omega_0^4 = \frac{1}{4} Tr(\mathcal{F}^{mn}\mathcal{F}_{mn}) \frac{1}{4!} \varepsilon_{abcd} e^a e^b e^c e^d,$$  \hfill (4.1)

with $\varepsilon_{abcd}$ the totally antisymmetric invariant tensor of $SO(1,3)$. Taking into account that in 4-dimensional spacetime the product of 5 ghost fields $\eta^a$ automatically vanishes, it is easy to check that $\Omega_0^4$ identifies a cohomology class of the BRST differential, i.e.

$$s \Omega_0^4 = 0, \quad \Omega_0^4 \neq s \tilde{\Omega}_0^3.$$  \hfill (4.2)

The 0-form cocycle corresponds to the invariant Yang-Mills Lagrangian in the presence of gravitational fields

$$\Omega_0^4 = \delta^4 \Omega_0^4 = \frac{1}{4} Tr(\mathcal{F}^{mn}\mathcal{F}_{mn}) \frac{1}{4!} \varepsilon_{abcd} e^a e^b e^c e^d$$
$$= \frac{1}{4} Tr(\mathcal{F}^{mn}\mathcal{F}_{mn}) e^d x = \frac{1}{4} Tr(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) e^m e^n e_d x$$
$$= \frac{1}{4} Tr(\mathcal{F}_{\mu\nu}\mathcal{F}_{\tau\sigma}) g^{\mu\tau} g^{\nu\sigma} \sqrt{-g} d^4 x,$$  \hfill (4.3)

where $e = det(e^a_\mu) = \sqrt{-g}$ denotes the determinant.

4.2 The topological Yang-Mills Lagrangian

We start with the local ghost polynomial

$$\tilde{\Omega}_0^3 = Tr \left( \tilde{\mathcal{F}} \tilde{c} - \frac{1}{2} \tilde{c} \tilde{c} \tilde{c} \right) = Tr \left( s\tilde{c} \tilde{c} - \frac{2}{3} \tilde{c} \tilde{c} \tilde{c} \right).$$  \hfill (4.4)
where we have used the following redefined ghost variables
\[ \hat{c}^A = c^A + A_m \eta^m, \quad \hat{F}^A = \frac{1}{2} \mathcal{F}^A_m \eta^m \eta^n , \] (4.5)
which have the BRST transformations
\[ s\hat{c}^A = \frac{1}{2} f^{ABC} \hat{c}^B \hat{c}^C - \hat{F}^A, \quad s\hat{F}^A = f^{ABC} \hat{c}^B \hat{F}^C . \] (4.6)
Under the action of the operator \( \delta \) they transform as
\[ \delta \hat{c}^A = -A^A, \quad \delta \hat{F}^A = -\mathcal{F}^A_m \eta^m \eta^n . \] (4.7)
Starting with the trivial cocycle
\[ \Omega_0^4 = s\hat{\Omega}_0^3 = Tr(\hat{\mathcal{F}} \hat{\mathcal{F}}) = \frac{1}{4} Tr(F_{kl}F_{mn})\eta^k \eta^l \eta^m \eta^n , \] (4.8)
we obtain the topological Yang-Mills invariant Lagrangian:
\[ \Omega_4^0 = \frac{1}{4!} \delta^4 \Omega_0^4 = Tr(\mathcal{F} \mathcal{F}) \]
\[ = \frac{1}{4} Tr(F_{kl}F_{mn})e^k e^l e^m e^n = \frac{1}{4} Tr(F_{kl}F_{mn})\epsilon^{klmn} e^4 x \]
\[ = \frac{1}{4} Tr(F_{\mu\nu}F_{\tau\sigma})dx^\mu dx^\nu dx^\tau dx^\sigma = \frac{1}{4} Tr(F_{\mu\nu}F_{\tau\sigma})\epsilon^{\mu\nu\tau\sigma} d^4 x . \] (4.9)

4.3 Chern-Simons terms, gauge anomalies and Ashtekar variables

In our theory we can define two types of Chern-Simons Lagrangians: one for the Yang-Mills gauge field \( A^A \) and the other for the Ashtekar connection \( A^a_b \).

For the sake of clarity and to make contact with the results obtained in [12, 37], let us discuss in detail the construction of the 3-dimensional Chern-Simons term. In this case the descent equations take the form
\[ s\Omega_3^0 + d\Omega_2^1 = 0 , \]
\[ s\Omega_2^1 + d\Omega_1^2 = 0 , \]
\[ s\Omega_1^2 + d\Omega_0^3 = 0 , \]
\[ s\Omega_0^3 = 0 , \] (4.10)
and thus their solutions are:
\[ \Omega_1^2 = \delta \Omega_0^3 , \]
\[ \Omega_2^1 = \frac{\delta^2}{2!} \Omega_0^3 , \]
\[ \Omega_3^0 = \frac{\delta^3}{3!} \Omega_0^3 . \] (4.11)
In order to find a solution for \( \Omega^3_0 \) we use again the redefined ghost variables:

\[
\hat{c}^A = A^A_m \eta^m + c^A, \quad \hat{F}^A = \frac{1}{2} F^A_{mn} \eta^m \eta^n.
\] (4.12)

For the cocycle \( \Omega^3_0 \) one then has

\[
\Omega^3_0 = \frac{1}{3!} f^{ABC} \hat{c}^A \hat{c}^B \hat{c}^C - \hat{F}^A \hat{c}^A,
\] (4.13)

from which \( \Omega^3_1, \Omega^3_2, \) and \( \Omega^3_3 \) are computed to be

\[
\Omega^3_1 = -\frac{1}{2} f^{ABC} A^A \hat{c}^B \hat{c}^C + \hat{F}^A A^A + F^A_{mn} \eta^m \eta^n \hat{c}^A,
\] (4.14)

\[
\Omega^3_2 = \frac{1}{2} f^{ABC} A^A A^B \hat{c}^C - F^A_{mn} \eta^m A^A - F^A \hat{c}^A,
\] (4.15)

\[
\Omega^3_3 = -\frac{1}{6} f^{ABC} A^A A^B A^C + \hat{F}^A A^A.
\] (4.16)

In particular, expression (4.16) gives the familiar 3-dimensional Chern-Simons term. Finally, let us remark that the cocycle \( \Omega^3_1 \) of eq. (4.15), when referred to dimension \( N = 2 \), reduces to the expression

\[
\Omega^1_2 = -(dA^A) c^A, \quad \text{for } N = 2,
\] (4.17)

which directly gives the 2-dimensional gauge anomaly.

Let us proceed to give the construction of the 5-dimensional Ashtekar Chern-Simons term. The descent equations are given by

\[
s \Omega^0_5 + d \Omega^1_4 = 0,
\]

\[
s \Omega^1_4 + d \Omega^2_3 = 0,
\]

\[
s \Omega^2_3 + d \Omega^3_2 = 0,
\]

\[
s \Omega^3_2 + d \Omega^4_1 = 0,
\]

\[
s \Omega^4_1 + d \Omega^5_0 = 0,
\]

\[
s \Omega^5_0 = 0,
\] (4.18)

and the cocycles are obtained by using Sorella’s method [12]:

\[
\Omega^1_4 = \delta \Omega^5_0,
\]

\[
\Omega^3_2 = \frac{\delta^2}{2!} \Omega^5_0,
\]

\[
\Omega^3_2 = \frac{\delta^3}{3!} \Omega^5_0,
\]

\[
\Omega^1_4 = \frac{\delta^4}{4!} \Omega^5_0,
\]

\[
\Omega^0_5 = \frac{\delta^5}{5!} \Omega^5_0.
\] (4.19)
In order to find a solution for the last equation of the tower given in eq.(4.18) we use the redefined Ashtekar ghost
\[ \hat{c}^a_b = A^{a}_{bm} \eta^m + \hat{c}^a_b , \]  
which, from eq.(3.6), transforms as
\[ \delta \hat{c}^a_b = -A^a_b . \] (4.21)

We obtain for the 0-form cocycle \( \Omega^5_0 \) in five dimensions
\[ \Omega^5_0 = -\frac{1}{10} \hat{c}^a_b \hat{c}^c_d e^e_a + \frac{1}{4} F^{a}_{bmn} \eta^m \eta^n \hat{c}^b_c \hat{c}^c_d e^e_a \]
\[ - \frac{1}{4} F^{a}_{bmn} \eta^m \eta^n F^{b}_{ckl} \eta^k \hat{c}^c_a , \] (4.22)

which leads to the 5-dimensional Chern-Simons term in Ashtekar variables
\[ \Omega^5_0 = \frac{1}{5!} \delta^5 \Omega^5_0 = \frac{1}{10} A^a_b A^b_c A^c_d A^d_e A^e_a - \frac{1}{2} F^{a}_{b} A^b_c A^c_d A^d_a + F^{a}_{b} F^{b}_{c} A^c_a . \] (4.23)

5 Appendices:

Appendix A is devoted to demonstrate the computation of some commutators involving the tangent space derivative \( \partial_a \). In appendix B one finds some relations concerning the determinant of the tetrad and the \( \varepsilon \)-tensor.

A Commutator relations

In order to find the commutator of two tangent space derivatives \( \partial_a \), we make use of the fact that the usual spacetime derivatives \( \partial_\mu \) have a vanishing commutator:
\[ [\partial_\mu, \partial_\nu] = 0 . \] (A.1)

From
\[ \partial_\mu = e^n_\mu \partial_n \] (A.2)

one gets
\[ [\partial_\mu, \partial_\nu] = 0 = [e^n_\mu \partial_m, e^n_\nu \partial_n] \]
\[ = e^m_\mu e^n_\nu [\partial_m, \partial_n] + e^m_\mu (\partial_m e^n_\nu) \partial_n - e^n_\nu (\partial_n e^m_\mu) \partial_m \]
\[ = e^m_\mu e^n_\nu [\partial_m, \partial_n] + (\partial_\mu e^n_\nu - \partial_\nu e^m_\mu) \partial_k \]
\[ = e^m_\mu e^n_\nu [\partial_m, \partial_n] + (T^{k}_{\mu\nu} - A^k_{n\mu} e^n_\nu + A^k_{m\nu} e^m_\mu) \partial_k \]
\[ = e^m_\mu e^n_\nu (T^{k}_{mn} + A^k_{mn} - A^k_{nm}) \partial_k \]
\[ + e^m_\mu e^n_\nu [\partial_m, \partial_n] , \] (A.3)

so that
\[ [\partial_m, \partial_n] = -(T^{k}_{mn} + A^k_{mn} - A^k_{nm}) \partial_k . \] (A.4)
For the commutator of $d$ and $\partial_m$ we get
\[
[d, \partial_m] = [e^n \partial_n, \partial_m] = -(\partial_m e^k) \partial_k - e^n [\partial_m, \partial_n] = -(\partial_m e^k) \partial_k + e^n (T_{mn} + A^k_{mn} - A^k_{nm}) \partial_k ,
\]
and one has therefore
\[
[d, \partial_m] = (T^k_{mn} e^n + A^k_{mn} e^n - A^k_{nm} e^n - (\partial_m e^k)) \partial_k .
\]

Analogously, from
\[
[s, \partial_{\mu}] = 0
\]
one easily finds
\[
[s, \partial_m] = (\partial_m \eta^k - e^k_m) \partial_k + \eta^n [\partial_m, \partial_n] = (\partial_m \eta^k - e^k_m - T^k_{mn} \eta^n - A^k_{mn} \eta^n + A^k_{nm} \eta^n) \partial_k .
\]

### B Determinant of the vielbein and the $\varepsilon$-tensor

The definition of the determinant of the vielbein $e^a_{\mu}$ is given by
\[
e = det(e^a_{\mu}) = \frac{1}{4!} \varepsilon_{a_1 a_2 a_3 a_4} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} e^{a_1}_{\mu_1} e^{a_2}_{\mu_2} e^{a_3}_{\mu_3} e^{a_4}_{\mu_4} .
\]

One can easily verify that the BRST transformation of $e$ reads
\[
se = -\partial_\lambda (\xi^\lambda e) .
\]

For the case of $SO(1,3)$ one has
\[
e^0 e^1 e^2 e^3 = \frac{1}{4!} \varepsilon_{a_1 a_2 a_3 a_4} e^{a_1} e^{a_2} e^{a_3} e^{a_4} = \frac{1}{4!} \varepsilon_{a_1 a_2 a_3 a_4} e^{a_1} e^{a_2} e^{a_3} e^{a_4} dx^\mu dx^\nu dx^\rho dx^\tau
\]
\[
= \frac{1}{4!} \varepsilon_{a_1 a_2 a_3 a_4} e^{a_1} e^{a_2} e^{a_3} e^{a_4} dx^0 dx^1 dx^2 dx^3 = ed^4 x = \sqrt{-g} d^4 x ,
\]
where $g$ denotes the determinant of the metric tensor $g_{\mu \nu}$
\[
g = det(g_{\mu \nu}) .
\]

The $\varepsilon$-tensor has the usual norm
\[
\varepsilon_{a_1 a_2 a_3 a_4} e^{a_1} e^{a_2} e^{a_3} e^{a_4} = -4! ,
\]
and obeys the following relation under partial contraction of two indices
\[
\varepsilon_{abcd} \varepsilon^{mmcd} = -2 (\delta^m_a \delta^m_b - \delta^m_a \delta^m_b) ,
\]
and in general the contraction of two $\epsilon$-tensors is given by the determinant of $\delta$-tensors in the following way:

$$\varepsilon_{a_1a_2a_3a_4}\epsilon^{b_1b_2b_3b_4} = -\begin{vmatrix}
\delta^{b_1}_{a_1} & \delta^{b_2}_{a_1} & \delta^{b_3}_{a_1} & \delta^{b_4}_{a_1} \\
\delta^{b_1}_{a_2} & \delta^{b_2}_{a_2} & \delta^{b_3}_{a_2} & \delta^{b_4}_{a_2} \\
\delta^{b_1}_{a_3} & \delta^{b_2}_{a_3} & \delta^{b_3}_{a_3} & \delta^{b_4}_{a_3} \\
\delta^{b_1}_{a_4} & \delta^{b_2}_{a_4} & \delta^{b_3}_{a_4} & \delta^{b_4}_{a_4}
\end{vmatrix}.$$  \hspace{1cm} (B.7)

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