EXACT CONTROLLABILITY OF THE LINEAR ZAKHAROV-KUZNETSOV EQUATION

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Abstract. We consider the linear Zakharov-Kuznetsov equation on a rectangle with a left Dirichlet boundary control. Using the flatness approach, we prove the null controllability of that equation and provide a space of analytic reachable states.

1. Introduction. The Zakharov-Kuznetsov (ZK) equation

\[ u_t + a u_x + \Delta u_x + uu_x = 0, \tag{1} \]

provides a model for the propagation of nonlinear ionic-sonic waves in a plasma. In (1), \( x, t \in \mathbb{R} \) and \( y \in \mathbb{R}^d \) (with \( d \in \{1, 2\} \)) are the independent variables, \( u = u(x, y, t) \) is the unknown, \( u_t = \partial u / \partial t, u_x = \partial u / \partial x, \Delta u = \partial^2 u / \partial x^2 + \sum_{i=1}^{d} \partial^2 u / \partial y_i^2 \), and the constant \( a > 0 \) stands for the sound velocity. The ZK equation is, from the mathematical point of view, a natural extension to \( \mathbb{R}^{d+1} \) of the famous Korteweg-de Vries equation

\[ z_t + az_x + z_{xxx} + zz_x = 0, \tag{2} \]

which has been extensively studied from the control point of view (see e.g. the surveys [2, 17]). If we focus on the situation where (2) is supplemented with the following boundary conditions

\[ z(0, t) = h(t), \quad z(L, t) = z_x(L, t) = 0, \tag{3} \]

where \( L > 0 \) is a given number and \( h \) is the control input, then it was proved in [8, 16] that (2)-(3) was null controllable on the domain \((0, L)\). Due to the smoothing effect, with such a control at the left endpoint the exact controllability can only hold in a space of analytic functions.

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More recently, a space of analytic reachable states was provided in [13] for the linearized KdV equation

\[ z_t + z_x + z_{xxx} = 0 \]

with the same boundary conditions as in (3). The method of proof was based on the flatness approach, as introduced in [12] to study the reachable states of the heat equation. We refer the reader to [9] for the study of the exact controllability of a semilinear heat equation in a space of analytic functions. The aim of the paper is to extend the results given in [13] to the ZK equation.

The wellposedness of various initial boundary value problems for ZK were studied in [6, 7, 10, 11, 18, 19]. Some unique continuation property for ZK derived with a Carleman estimate was done in [3]. Exact controllability results for ZK in the same spirit as those for KdV in [15] are given in [7, 14]. The stabilization of ZK was addressed in [4, 5].

Here, we limit ourselves to the case \( d = 1 \), so that \( y \in \mathbb{R} \). By a translation, we can assume without loss of generality that \( x \in (-1, 0) \) (this will be more convenient when using series to represent the solutions). We set \( \Omega := (-1, 0) \times (0, 1) \). The paper is concerned with the control properties of the system:

\[
\begin{align*}
    u_1 + u_{xxx} + u_{xxy} + au_x &= 0, & (x, y) &\in \Omega, & t &\in (0, T), \\
    u(0, y, t) &= u_y(0, y, t) = 0, & y &\in (0, 1), & t &\in (0, T), \\
    u(-1, y, t) &= \hat{h}(y, t), & y &\in (0, 1), & t &\in (0, T), \\
    u(x, 0, t) &= u(x, 1, t) = 0, & x &\in (-1, 0), & t &\in (0, T), \\
    u(x, y, 0) &= u_0(x, y), & (x, y) &\in \Omega,
\end{align*}
\]

where \( u_0 = u_0(x, y) \) is the initial data and \( h = h(y, t) \) is the control input.

We shall address the following issues:

1. (Null controllability) Given any \( u_0 \in L^2(\Omega) \), can we find a control \( h \) such that the solution \( u \) of (4)-(8) satisfies \( u(., T) = 0 \)?

2. (Reachable states) Given any \( u_1 \in \mathcal{R} \) (a subspace of \( L^2(\Omega) \) defined thereafter), can we find a control \( h \) such that the solution \( u \) of (4)-(8) with \( u_0 = 0 \) satisfies \( u(., T) = u_1 \)?

We shall investigate both issues by the flatness approach and derive an exact controllability in \( \mathcal{R} \) by combining our results.

To state our result, we need introduce notations. A function \( u \in C^\infty([t_1, t_2]) \) is said to be Gevrey of order \( s \geq 0 \) on \([t_1, t_2]\) if there exist some constant \( C, R \geq 0 \) such that

\[ |\partial_t^n u(t)| \leq C (n!)^s R^n \quad \forall n \in \mathbb{N}, \forall t \in [t_1, t_2]. \]

The set of functions Gevrey of order \( s \) on \([t_1, t_2]\) is denoted by \( G^s([t_1, t_2]) \). A function \( u \in C^\infty([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]) \) is said to be Gevrey of order \( s_1 \) in \( x \), \( s_2 \) in \( y \) and \( s_3 \) in \( t \) on \([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]\) if there exist some constants \( C, R_1, R_2, R_3 > 0 \) such that

\[ |\partial_x^{n_1} \partial_y^{n_2} \partial_t^{n_3} u(x, y, t)| \leq C \frac{(n_1!)^{s_1} (n_2!)^{s_2} (n_3!)^{s_3}}{R_1^{n_1} R_2^{n_2} R_3^{n_3}} \quad \forall n_1, n_2, n_3 \in \mathbb{N}, \forall (x, y, t) \in [x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]. \]

The set of functions Gevrey of order \( s_1 \) in \( x \), \( s_2 \) in \( y \) and \( s_3 \) in \( t \) on \([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]\) is denoted by \( G^{s_1, s_2, s_3}([x_1, x_2] \times [y_1, y_2] \times [t_1, t_2]) \).
The first main result in this paper is a null controllability result with a control input in a Gevrey class.

**Theorem 1.1.** Let \( u_0 \in L^2(\Omega) \) and \( s \in [\frac{3}{2}, 2) \). Then there exists a control input \( h \in G^{\frac{1}{2}+s}([0, 1] \times [0, T]) \) such that the solution \( u \) of (4)-(8) satisfies \( u(\cdot, \cdot, T) = 0 \). Furthermore, it holds that

\[
u \in C([0, T]; L^2(\Omega)) \cap G^{\frac{1}{2}+\frac{s}{2}}([0, 1] \times [\varepsilon, T]), \quad \forall \varepsilon \in (0, T).
\]

Introduce the differential operator

\[Pu := \Delta u_x + au_x\]

and the following space

\[\mathcal{R}_{R_1, R_2} := \{ u \in C^\infty([-1, 0] \times [0, 1]); \exists C > 0, \quad |\partial_x^p \partial_y^q u(x, y)| \leq C \frac{1}{R_1^{p} R_2^{q}} \forall p, q \in \mathbb{N}, \forall (x, y) \in \Omega,\]

and \( P^n u(0, y) = \partial_x P^n u(0, y) = P^n u(x, 0) = P^n u(x, 1) = 0, \forall n \in \mathbb{N}, \forall x \in [-1, 0], \forall y \in [0, 1].\)

Our second main result provides a set of reachable states for system (4)-(8).

**Theorem 1.2.** Let \( R_0 := \sqrt[3]{9(a + 2)e^{(3e)^{-1}}} \), and let \( R_1, R_2 \in (R_0, +\infty) \). Then for any \( u_1 \in \mathcal{R}_{R_1, R_2} \), there exists a control input \( h \in G^{1.2}([0, 1] \times [0, T]) \) such that the solution \( u \) of (4)-(8) with \( u_0 = 0 \) satisfies \( u(\cdot, \cdot, T) = u_1 \). Furthermore, \( u \in G^{1.2}([-1, 0] \times [0, 1] \times [0, T]), \) and the trajectory \( u = u(x, y, t) \) and the control \( h = h(y, t) \) can be expanded as series:

\[
u(x, y, t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y), \tag{9}\]

\[
h(y, t) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(-1) z_j^{(i)}(t) e_j(y). \tag{10}\]

We refer the reader to Section 2 for the definitions of the functions \( g_{i,j} \) (\( i \geq 0, j \geq 1 \)) and of the functions \( e_j \) (\( j \geq 1 \)).

Combining Theorem 1.1 and Theorem 1.2, we obtain the following result which implies the exact controllability of (4)-(8) in \( \mathcal{R}_{R_1, R_2} \) for \( R_1 > R_0 \) and \( R_2 > R_0 \).

**Corollary 1.** Let \( R_0 := \sqrt[3]{9(a + 2)e^{(3e)^{-1}}} \), and let \( R_1, R_2 \in (R_0, +\infty) \). Let \( u_0 \in L^2(\Omega) \) and \( u_1 \in \mathcal{R}_{R_1, R_2} \). Then there exists \( h \in G^{1.2}([0, 1] \times [0, T]) \) such that the solution of (4)-(8) satisfies \( u(\cdot, \cdot, T) = u_1 \).

The paper is outlined as follows. Section 2 introduces the eigenfunctions \( e_j \), the generating functions \( g_{i,j} \), and provide some estimates needed in the sequel. The null controllability of ZK is established in Section 3, while the reachable states of ZK are investigated in Section 4.

2. **Preliminaries.** First we introduce the operator

\[Au := -Pu = -\Delta u_x - au_x\]

with domain

\[\mathcal{D}(A) = \{ u \in L^2(\Omega); Pu \in L^2(\Omega), u(-1, y) = u(0, y) = u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1) \text{ and } u(x, 0) = u(x, 1) = 0 \text{ for a.e. } x \in (-1, 0)\}.\]
It is well-known (see e.g. [19]) that the operator $A$ generates a semigroup of contractions in $L^2(\Omega)$. In what follows, we denote $\|f\|_{D(A)} = \|f\|_{L^2(\Omega)} + \|Af\|_{L^2(\Omega)}$ for all $f \in D(A).

It would be natural to expect, as for KdV, that the domain $D(A)$ coincide with the set
\[
\{u \in H^3(\Omega) \cap H^1_0(\Omega); \quad u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1)\},
\]
but this is not the case. The best description (up to date) of $D(A)$ is given in the following lemma.

**Lemma 2.1.** We have the following inclusions:
\[
\{u \in H^2(\Omega) \cap H^1_0(\Omega); \quad u_x(0, y) = 0 \text{ for a.e. } y \in (0, 1)\} \subset D(A), \quad (11)
\]
\[
D(A) \subset \{u \in H^2(\Omega) \cap H^1_0(\Omega); \quad (x + 1)u_x \in H^2(\Omega) \cap H^1_0(\Omega)\}. \quad (12)
\]

**Proof.** The inclusion (11) is obvious. For (12), it follows from [19, Proposition 2] that $D(A) \subset H^2(\Omega) \cap H^1_0(\Omega)$. If $u \in D(A)$, then $f := \Delta u_x + au_x \in L^2(\Omega)$ and hence
\[
\Delta((x + 1)u_x) = (x + 1)\Delta u_x + 2u_{xx} = (x + 1)(f - au_x) + 2u_{xx} \in L^2(\Omega).
\]
On the other hand, we claim that $(x + 1)u_x \in H^1_0(\Omega)$. Indeed, $u_x \in H^1(\Omega)$ and hence $(x + 1)u_x \in H^1(\Omega)$. Moreover, $u(., 0) = u(., 1) = 0$ in $H^\frac{1}{2}(-1, 0)$ gives $u_x(., 0) = u_x(., 1) = 0$ in $H^\frac{1}{2}(-1, 0)$, and finally $(x + 1)u_x)(-1, .) = u_x(0, .) = 0$ in $H^\frac{1}{2}(0, 1)$. By the classical boundary $H^2$ regularity result for the Dirichlet problem on a Lipschitz domain, we infer that $(x + 1)u_x \in H^2(\Omega) \cap H^1_0(\Omega)$. \hfill \Box

**Remark 1.** It can be shown that the inclusion (11) is strict.

The following lemmas will be used several times thereafter.

**Lemma 2.2.** For any $n \in \mathbb{N}^*$ and any $f \in D(A^n)$ with $A^n f \in H^{2(n-1)}(\Omega)$ for $i = 0, 1, \ldots, n$, we have
\[
\partial_y^{2n} f(x, 0) = \partial_y^{2p} f(x, 1) = 0, \quad \forall x \in [-1, 0], \quad \forall p \in \{0, ..., n-1\}. \quad (13)
\]

**Proof.** We proceed by induction on $n$. For $n = 1$, the property (13) is obvious since $f \in D(A)$. Assume now that (13) is true for $n - 1 \geq 1$. If $f \in D(A^n)$ with $A^n f \in H^{2(n-1)}(\Omega)$ for $i = 0, 1, \ldots, n$, then $P f = -A f \in D(A^{n-1})$ with $A^n P f = -A^{n+1} f \in H^{2(n-1)}(\Omega)$ for $i = 0, 1, \ldots, n-1$, so that by (13) applied to $P f$ and $p = n - 2$
\[
\partial_y^{2n-4} Pf(x, 0) = \partial_y^{2n-4} Pf(x, 1) = 0.
\]
This implies
\[
\partial_y^{3} \partial_y^{2n-4} f(x, 0) + \partial_x \partial_y^{2n-2} f(x, 0) + a \partial_x \partial_y^{2n-4} f(x, 0) = 0, \quad (14)
\]
\[
\partial_y^{3} \partial_y^{2n-4} f(x, 1) + \partial_x \partial_y^{2n-2} f(x, 1) + a \partial_x \partial_y^{2n-4} f(x, 1) = 0. \quad (15)
\]
Since (13) is true for $n - 1$, we obtain that $\partial_y^{2p} f(x, 0) = \partial_y^{2p} f(x, 1) = 0$ for $p = 0, 1, \ldots, n-2$, and hence (taking $p = n - 2$ and using (14)-(15))
\[
\partial_x \partial_y^{2n-2} f(x, 0) = \partial_x \partial_y^{2n-2} f(x, 1) = 0.
\]
This means that we have for some constants $C_1$ and $C_2$
\[
\partial_y^{2n-2} f(x, 0) = C_1, \quad \partial_y^{2n-2} f(x, 1) = C_2 \quad \forall x \in [-1, 0].
\]
Note that $\partial_y^{2n-2} f \in H^2(\Omega) \subset C(\Omega)$. On the other hand, it follows from the assumption $f \in D(A)$ that
\[
\partial_y^{2n-2} f(0, y) = 0 \quad \forall y \in [0, 1].
\]
Taking $y = 0$ and next $y = 1$, we see that $C_1 = C_2 = 0$. The proof of Lemma 2.2 is complete.

**Remark 2.** It will be proved in Proposition 1 (see below) that $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$ for all $n \in \mathbb{N}$, so that the conclusion of Lemma 2.2 will be still valid when assuming solely that $f \in \mathcal{D}(A^n)$.

The following lemma is classical. Its proof is omitted.

**Lemma 2.3.** Let $A' = \partial_y^2$ with domain $\mathcal{D}(A') = H^2(0,1) \cap H^1_0(0,1)$. Then for any $m \in \mathbb{N}^*$, it holds

$$\mathcal{D}(A'|_{\mathbb{N}}) = \{ g \in H^m(0,1); \ g^{(2p)}(0) = g^{(2p)}(1) = 0 \text{ for } 0 \leq p \leq \frac{m-1}{2} \}.$$  

Let $h \in L^2(0,1)$ be decomposed as $h(y) = \sum_{j=1}^{\infty} c_j(y)$, and let $m \in \mathbb{N}^*$. Then

$$h \in \mathcal{D}(A'|_{\mathbb{N}}) \iff \sum_{j=1}^{\infty} |\lambda_j^m c_j|^2 < \infty.$$  

Furthermore, for any $h \in \mathcal{D}(A'|_{\mathbb{N}})$, we have

$$\| h^{(q)} \|_{L^2(0,1)}^2 = \sum_{j=1}^{\infty} \lambda_j^q |c_j|^2 \quad \forall q \in \{0, ..., m\}.$$  

We are in a position to state the main result in this section.

**Proposition 1.** For any $n \in \mathbb{N}$, it holds $\mathcal{D}(A^n) \subset H^{2n}(\Omega)$. Furthermore, there exists a constant $B \geq 1$ such that

$$\| u \|_{H^{2n}(\Omega)} \leq B^n \sum_{i=0}^{n} \| P^i u \|_{L^2(\Omega)}, \quad \forall n \in \mathbb{N}, \quad \forall u \in \mathcal{D}(A^n).$$  

**Proof.** Let $\{e_j\}_{j \geq 1}$ be an orthonormal basis in $L^2(0,1)$ such that $e_j$ is an eigenfunction for the Dirichlet Laplacian on $(0,1)$, $\lambda_j$ being the corresponding eigenvalue; that is

$$-e_j''(y) = \lambda_j e_j(y),$$  

$$e_j(0) = e_j(1) = 0.$$  

A classical choice is $e_j(y) = \sqrt{2}\sin(j\pi y)$ and $\lambda_j = (j\pi)^2$ for $j \geq 1$. Following [19], we decompose any function $u \in L^2(\Omega)$ as

$$u(x,y) = \sum_{j=1}^{\infty} \hat{u}_j(x)e_j(y).$$  

Note that $\| u \|_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty} \| \hat{u}_j \|^2$, where we denote $\| h \| = \| h \|_{L^2(-1,0)}$ for all $h \in L^2(-1,0)$ for the sake of simplicity. If $u \in \mathcal{D}(A)$ and $g := \Delta u_x + au_x$, then for any $j \geq 1$

$$\hat{u}_j'' + (a - \lambda_j)\hat{u}_j = \hat{g}_j \quad \text{in} \quad L^2(-1,0)$$  

(17)

where $' = d/dx$. For $n = 0$, (16) is obvious if we pick $C_0 \geq 1$. Let us assume first that $n = 1$. Note that $\hat{u}_j \in H^3(-1,0)$ by (17). Multiplying (17) by $\lambda_j(x+1)\hat{u}_j$, we obtain

$$\frac{3}{2} \lambda_j \int_{-1}^{0} |\hat{u}_j'|^2 dx - (a - \lambda_j) \frac{\lambda_j}{2} \int_{-1}^{0} |\hat{u}_j|^2 dx = \lambda_j \int_{-1}^{0} (x+1)\hat{u}_j\hat{g}_j dx.$$  

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Let $j_0 := \left[\frac{\sqrt{\pi}}{2}\right]$. Then for $j > j_0$, we have $a \leq \lambda_j / 2$ and hence $|a - \lambda_j|\lambda_j / 2 \geq \lambda_j^2 / 4$. Using

$$\left| \lambda_j \int_{-1}^{0} (x + 1) \hat{u}_j \hat{g}_j dx \right| \leq \frac{\lambda_j^2}{8} \int_{-1}^{0} |\hat{u}_j|^2 dx + 2 \int_{-1}^{0} |\hat{g}_j|^2 dx,$$

we infer that for $j > j_0$

$$\frac{3}{2} \lambda_j \int_{-1}^{0} |\hat{u}'_j|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^{0} |\hat{u}_j|^2 dx \leq 2 \int_{-1}^{0} |\hat{g}_j|^2 dx,$$

and that for $1 \leq j \leq j_0$

$$\frac{3}{2} \lambda_j \int_{-1}^{0} |\hat{u}'_j|^2 dx + \frac{\lambda_j^2}{8} \int_{-1}^{0} |\hat{u}_j|^2 dx \leq \Lambda \int_{-1}^{0} |\hat{u}_j|^2 dx + 2 \int_{-1}^{0} |\hat{g}_j|^2 dx, \quad (18)$$

where $\Lambda := \max_{1 \leq j \leq j_0} \left( \frac{\lambda_j^2}{8} + (a - \lambda_j) \frac{\lambda_j}{2} \right)$. Obviously, (18) is valid for any $j \geq 1$. Summing in $j$, we obtain

$$\frac{3}{2} \int_{\Omega} |u_{xy}|^2 dxdy + \frac{1}{8} \int_{\Omega} |u_{y}|^2 dxdy \leq \Lambda \|u\|^2 + 2\|g\|^2. \quad (19)$$

Dividing in (18) by $\lambda_j \geq \pi^2$ and summing in $j$, we obtain

$$\frac{3}{2} \int_{\Omega} |u_{x}|^2 dxdy + \frac{1}{8} \int_{\Omega} |u_{y}|^2 dxdy \leq \frac{\Lambda}{\pi^2} \|u\|^2 + \frac{2}{\pi^2} \|g\|^2. \quad (20)$$

It remains to estimate $\int_{\Omega} |u_{xx}|^2 dxdy$. Multiplying in (17) by $\hat{u}'_j$, we obtain

$$- \int_{-1}^{0} |\hat{u}_j''|^2 dx + \hat{u}'_j \hat{u}_j|_0^0 + (a - \lambda_j) \int_{-1}^{0} |\hat{u}'_j|^2 dx = \int_{-1}^{0} \hat{g}_j \hat{u}'_j dx,$$

and hence

$$\int_{-1}^{0} |\hat{u}_j''|^2 dx \leq |\lambda_j - a| \int_{-1}^{0} |\hat{u}'_j|^2 dx + \|\hat{g}_j\| \|\hat{u}'_j\| + |\hat{u}'_j(-1)\hat{u}_j''(-1)|.$$ 

We are left to estimate $\hat{u}'_j(-1)$ and $\hat{u}_j''(-1)$. Multiplying in (17) by $\lambda_j \hat{u}_j$ results in

$$\lambda_j \frac{\hat{u}_j''(-1)^2}{2} = \lambda_j \int_{-1}^{0} \hat{u}_j \hat{g}_j dx.$$ 

Combined with (18), this yields

$$\frac{\lambda_j}{2} \hat{u}_j''(-1)^2 \leq \|\lambda_j \hat{u}_j \cdot \hat{g}_j\| \leq \frac{1}{4} \|\lambda_j \hat{u}_j\|^2 + \|\hat{g}_j\|^2 \leq 5\|\hat{g}_j\|^2 + 2\Lambda \|\hat{u}_j\|^2. \quad (21)$$

Multiplying in (17) by $x$, we obtain

$$- \int_{-1}^{0} \hat{u}_j'' dx + x\hat{u}_j|_0^0 = - \int_{-1}^{0} \hat{u}_j dx + x\hat{g}_j|_0^0 = \int_{-1}^{0} x\hat{g}_j dx$$

which yields

$$\hat{u}_j''(-1) = -\hat{u}_j''(-1) + (a - \lambda_j) \int_{-1}^{0} \hat{u}_j dx + \int_{-1}^{0} x\hat{g}_j dx,$$

so that

$$|\hat{u}_j''(-1)|^2 \leq 3 \left( |\hat{u}_j''(-1)|^2 + 2(a^2 + |\lambda_j|^2)\|\hat{u}_j\|^2 + \|\hat{g}_j\|^2 \right).$$
Using (18) and (21), we conclude that \(|\hat{u}_j''(-1)|^2 = O(\|\hat{u}_j\|^2 + \|\hat{g}_j\|^2)\). The same is true for \(\|\hat{u}_j''\|^2\). Gathering together the above estimates, we arrive at

\[
\|u\|^2_{H^2(\Omega)} \leq C_1 \left( \|u\|^2_{L^2(\Omega)} + \|Pu\|^2_{L^2(\Omega)} \right)
\]

for some constant \(C_1 = C_1(a) > 0\).

Let us check that \(D(A^n) \subset H^{2n}(\Omega)\) for \(n \geq 2\). We proceed by induction on \(n\). Assume that \(D(A^n) \subset H^{2p}(\Omega)\) for \(p = 0, 1, \ldots, n-1\) (with \(n-1 \geq 1\)), and pick any \(u \in D(A^n)\). Then \(g = Au \in D(A^{n-1}) \subset H^{2(n-1)}(\Omega)\). Let \(h := (-1)^{n-1}\partial_y^{2(n-1)}g \in L^2(\Omega)\). Then, using Lemmas 2.2 and 2.3, we have that for all \(j \geq 1\)

\[
\lambda_j^{n-1}(\hat{u}_j'' + (a - \lambda_j)\hat{u}_j') = \hat{h}_j. \tag{22}
\]

Multiplying in (22) by \(\lambda_j^n(u + x)\hat{u}_j\), we obtain

\[
\frac{3}{2} \lambda_j^{2n-1} \int_{-1}^{0} |\hat{u}_j'|^2 dx = (a - \lambda_j)\lambda_j^{2n-1} \int_{-1}^{0} |\hat{u}_j|^2 dx = \lambda_j^n \int_{-1}^{0} (u + x)\hat{u}_j\hat{h}_j dx.
\]

This yields

\[
\lambda_j^{2n-1}||\hat{u}_j'||^2 + \lambda_j^{2n}||\hat{u}_j||^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2), \tag{23}
\]

Multiplying in (22) by \(\lambda_j^n\hat{u}_j\) gives

\[
\lambda_j^{2n-1} \frac{\hat{u}_j''(-1)^2}{2} = \lambda_j^n \int_{-1}^{0} \hat{u}_j\hat{h}_j dx
\]

and

\[
\lambda_j^{2n-1}||\hat{u}_j''(-1)||^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2).
\]

From

\[
\lambda_j^{n-1} \hat{u}_j''(-1) = -\lambda_j^{n-1} \hat{u}_j'(-1) + (a - \lambda_j)\lambda_j^{n-1} \int_{-1}^{0} \hat{u}_j dx + \int_{-1}^{0} x\hat{h}_j dx,
\]

we infer that

\[
\lambda_j^{2n-2}||\hat{u}_j''(-1)||^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2).
\]

It follows from

\[
-\lambda_j^{2n-2} \int_{-1}^{0} |\hat{u}_j''|^2 dx + \lambda_j^{2n-2} |\hat{u}_j'|^2 dx + (a - \lambda_j)\lambda_j^{2n-2} \int_{-1}^{0} |\hat{u}_j'|^2 dx = \lambda_j^{n-1} \int_{-1}^{0} \hat{h}_j \hat{u}_j' dx,
\]

that

\[
\lambda_j^{2n-2}||\hat{u}_j''||^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2). \tag{24}
\]

So far, we have proved that

\[
\sum_{j=1}^{\infty} \left( \lambda_j^{2n}||\hat{u}_j||^2 + \lambda_j^{2n-1}||\hat{u}_j'||^2 + \lambda_j^{2n-2}||\hat{u}_j''||^2 \right) < +\infty.
\]

Using Lemma 2.3, this gives that \(\partial_x^{2n}u, \partial_y^{2n-1}\partial_xu\), and \(\partial_y^{2n-2}\partial_x^2u\) belong to \(L^2(\Omega)\). For the other derivatives of order \(2n\), we apply the operator \(\partial_x^{2k}\) (for \(k \in \mathbb{N}\) with \(2k + 3 \leq 2n\)) to each term in (17) to obtain

\[
\hat{u}_j^{(2k+3)} + (a - \lambda_j)\hat{u}_j^{(2k+1)} = \hat{g}_j^{(2k)}.
\]

This yields

\[
\lambda_j^{2n-3-2k}||\hat{u}_j^{(2k+3)}||^2 = O(||\hat{u}_j||^2 + ||\hat{h}_j||^2 + \lambda_j^{2n-3-2k}||\hat{g}_j^{(2k)}||^2).
\]
On the other hand, (17) gives by differentiation with respect to $x$ that
\[ \hat{u}_j^{(4)} + (a - \lambda_j)\hat{u}_j'' = \hat{g}', \]
and we obtain in a similar way that
\[ \lambda_j^{2n-4-2k}\|\hat{u}_j^{(2k+4)}\|^2 = O\left(\|\hat{u}_j\|^2 + \|\hat{h}_j\|^2 + \lambda_j^{2n-4-2k}\|\hat{g}_j^{(2k+1)}\|^2\right). \]
for $k \in \mathbb{N}$ with $2k + 4 \leq 2n$. Thus we conclude that
\[ \sum_{q=0}^{2n} \sum_{j=1}^\infty \lambda_j^{(2n-q)}\|\hat{u}_j^{(q)}\|^2 < +\infty. \]

Using Lemma 2.3, we infer that for $q \in \{0, \ldots, 2n\}$, $\partial^q_y u \in L^2(-1, 0, H^{2n-q}(0, 1))$, and hence that $\partial^q_y \partial^q_y u \in L^2(\Omega)$. We also have that $\partial_y^{2n-1-q} \partial^q_y u \in L^2(\Omega)$ for $q \in \{0, \ldots, 2n - 1\}$. Taking into account the fact that $u \in D(A^{n-1}) \subset H^{2(n-1)}(\Omega)$, we conclude that $u \in H^{2n}(\Omega)$. The proof of the inclusion $D(A^n) \subset H^{2n}(\Omega)$ is complete.

It remains to prove that the constant in the r.h.s. of (16) is indeed of the form $B^n$. This will require a series of lemmas.

**Lemma 2.4.** For any $\varepsilon_0 > 0$, there exists a constant $K = K(\varepsilon_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $f \in H^2(-1, 0)$,
\[ \int_{-1}^0 |f'(t)|^2 \, dt \leq K\varepsilon \int_{-1}^0 |f''(t)|^2 \, dt + K\varepsilon^{-1} \int_{-1}^0 |f(t)|^2 \, dt. \]  

Lemma 2.4 is a direct consequence of [1, Lemma 4.10] (which is concerned with twice continuously functions) by density of $C^2([-1, 0])$ in $H^2(-1, 0)$.

For any $j \in \mathbb{N}^*$, we define the operator $P_j$ by
\[ P_j f := f''' - (\lambda_j - a)f', \quad \forall f \in H^3(-1, 0). \]

**Lemma 2.5.** There exists a constant $C_1 \geq 1$ such that
\[ \|f\|^2_{H^2(-1, 0)} \leq C_1^2 \sum_{j=0}^n \lambda_j^{2n-2i}\|P_j f\|^2, \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{N}^*, \forall f \in H^{3n}(-1, 0). \]  

**Proof.** For $n = 0$, (26) is obvious. For $n = 1$, it follows from the definition of $P_j$ and Lemma 2.4 that
\[ \|f\|^2_{H^2(-1, 0)} = \|f\|^2 + \|f''\|^2 + \|f'''\|^2 \]
\[ \leq C\|f\|^2 + \|f''\|^2 \]
\[ \leq C\|f\|^2 + \frac{1}{\lambda_j}\|f''\|^2 + \lambda_j\|f'\|^2 \]
\[ \leq C\|f\|^2 + \frac{1}{\lambda_j}\|P_j f\|^2 + \lambda_j\|f'\|^2 \]
\[ \leq C\|f\|^2 + \|P_j f\|^2 + \frac{1}{2}\|f''\|^2 + C\lambda_j^2\|f\|^2 \]
\[ \leq C(\lambda_j^2\|f\|^2 + \|P_j f\|^2) + \frac{1}{2}\|f''\|^2. \]

This shows that we can find a constant $C_2 \geq 1$ such that
\[ \|f\|^2_{H^2(-1, 0)} \leq C_2(\lambda_j^2\|f\|^2 + \|P_j f\|^2). \]
Let us prove (26) for \( n \geq 2 \) by induction on \( n \). Assume (26) to be true for \( n - 1 > 0 \). It follows that
\[
\| f \|_{H^{2n}(-1,0)}^2 = \| f \|_{H^{2n-2}(-1,0)}^2 + \| f^{(2n-1)} \|_2^2 + \| f^{(2n)} \|_2^2 \\
\leq \| f \|_{H^{2n-2}(-1,0)}^2 + \| f^{(2n-2)} \|_2^2 \\
\leq \| f \|_{H^{2n-2}(-1,0)}^2 + C_2(\lambda_j^2 \| f^{(2n-2)} \|_2^2 + \| P_j f^{(2n-2)} \|_2^2) \\
\leq 2C_2\lambda_j^2 \| f \|_{H^{2n-2}(-1,0)}^2 + C_2 \| P_j f \|_{H^{2n-2}(-1,0)}^2 \\
\leq 2C_2\lambda_j^2 C_1^{n-1} \sum_{i=0}^{n-1} \lambda_j^2 \| P_j f \|_2^2 + C_2 C_1^{n-1} \sum_{i=0}^{n-1} \lambda_j^2 \| P_j f \|_2^2 \\
\leq 3C_2 C_1^{n-1} \sum_{i=0}^{n} \lambda_j^2 \| P_j f \|_2^2.
\]

If we pick \( C_1 = 3C_2 \), (26) is true for \( n \).

**Lemma 2.6.** There exists a positive constant \( C_3 \) such that
\[
\| u \|_{H^m(\Omega)}^2 \leq C_3 \sum_{m=0}^{n} \sum_{k=0}^{m} \| \partial_x^k \partial_y^{m-k} u \|_{L^2(\Omega)}^2,
\forall n \in \mathbb{N}, \forall u \in D(A^n).)
\]

**Proof.** For any \( p \in \mathbb{N} \), we set
\[
I_p := \sum_{a+b=p} \| \partial_x^a \partial_y^b u \|_{L^2(\Omega)}^2.
\]
Decompose \( u \) as
\[
u(x, y) = \sum_{j=1}^{\infty} \hat{u}_j(x) e_j(y).
(27)
\]
Let us go back to the proof of Lemma 2.6. Pick any \( u \in D(A^n) \), for some \( n \in \mathbb{N} \). Using Lemma 2.2 and applying Lemma 2.3 to the functions \( \partial_x^{m+1-k} u(x,) \) for \( 0 \leq m \leq n - 1, 0 \leq k \leq 2m + 1 \), and \( x \in (-1,0) \), we obtain that
\[
I_{2m+1} = \sum_{k=0}^{2m+1} \| \partial_x^{2m+1-k} \partial_y^k u \|_{L^2(\Omega)}^2 \\
= \sum_{k=0}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^k \| \hat{u}_j^{(2m+1-k)} \|_2^2 \\
= \sum_{j=1}^{\infty} \| \hat{u}_j^{(2m+1)} \|_2^2 + \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j \| \hat{u}_j^{(2m+1-k)} \|_2^2 \\
\leq \sum_{j=1}^{\infty} \lambda_j \| \hat{u}_j^{(2m+1)} \|_2^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k-1} \| \hat{u}_j^{(2m+1-k)} \|_2^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \sum_{j=1}^{\infty} \lambda_j^{k+1} \| \hat{u}_j^{(2m+1-k)} \|_2^2 \\
= \| \partial_x^{2m+1} \partial_y u \|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \| \partial_x^{2m+1-k} \partial_y^{k-1} u \|_{L^2(\Omega)}^2 + \frac{1}{2} \sum_{k=1}^{2m+1} \| \partial_x^{2m+1-k} \partial_y^{k+1} u \|_{L^2(\Omega)}^2 \\
\leq \frac{3}{2} I_{2m+2} + \frac{1}{2} I_{2m}.
\]
where we used Young’s estimate. Thus, we have
\[
\|u\|_{H^2n(\Omega)}^2 = \sum_{m=0}^{n} I_{2m} + \sum_{m=0}^{n-1} I_{2m+1} \\
\leq \sum_{m=0}^{n} I_{2m} + \sum_{m=0}^{n-1} (3I_{2m+2} + \frac{1}{2}I_{2m}) \tag{28}
\]
\[
\leq 3 \sum_{m=0}^{n} I_{2m}.
\]
Next, we consider \(I_{2m}\). For \(m = 0\), \(I_0 = \|u\|_{L^2(\Omega)}^2\). For \(m \geq 1\), we have
\[
I_{2m} = \sum_{k=0}^{m} \|\partial_x^{2k-1}\partial_y^{2m-2k-1}u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{m-1} \|\partial_x^{2k+1}\partial_y^{2m-2k-1}u\|_{L^2(\Omega)}^2, \tag{29}
\]
and it remains to estimate the second term in the r.h.s. of (29). Applying Lemma 2.4, we obtain
\[
\sum_{k=0}^{m-1} \|\partial_x^{2k+1}\partial_y^{2m-2k-1}u\|_{L^2(\Omega)}^2 \\
= \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-1}\|\hat{u}_j^{(2k+1)}\|^2 \\
\leq C \left( \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k-2}\|\hat{u}_j^{(2k+2)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k}\|\hat{u}_j^{(2k)}\|^2 \right) \tag{30}
\]
\[
= C \left( \sum_{k=1}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k}\|\hat{u}_j^{(2k)}\|^2 + \sum_{k=0}^{m-1} \sum_{j=1}^{\infty} \lambda_j^{2m-2k}\|\hat{u}_j^{(2k)}\|^2 \right) \\
\leq C \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k}\|\hat{u}_j^{(2k)}\|^2 \\
= C \sum_{k=0}^{m} \|\partial_x^{2k}\partial_y^{2m-2k}u\|_{L^2(\Omega)}^2.
\]
Combining (28)-(30), the conclusion of Lemma 2.6 follows. \(\square\)

**Lemma 2.7.** There exists a constant \(C_4 \geq 1\) such that
\[
\lambda_j^{2m}\|P_i^j\hat{u}_j\|^2 \leq C_4 \sum_{l=0}^{m} \binom{m}{l} \|P^{i+l}\hat{u}_j\|^2, \forall m, i \in \mathbb{N}, \forall j \in \mathbb{N}^*, \forall u \in \mathcal{D}(A^{m+i}), \tag{31}
\]
where \(\hat{u}_j\) is the Fourier coefficients of \(u\) as in (27).

**Proof.** The proof is by induction on \(m\). For \(m = 0\), (31) is obvious for any \(C_4 \geq 1\).

For \(m = 1\) and \(u \in \mathcal{D}(A^{1+i})\), we have that \(P^i u \in \mathcal{D}(A)\) and, by [19, Lemma 4.1],
\[
(P^i u)(x, y) = \sum_{j=1}^{\infty} (P^i_j \hat{u}_j)(x)e_j(y),
\]
where the function \( P_j^i \hat{u}_j \) satisfies for each \( j \in \mathbb{N}^* \)
\[
\begin{align*}
\begin{cases}
(P_j^i \hat{u}_j)^{\prime\prime} - (\lambda_j - \alpha)(P_j^i \hat{u}_j)' = P_j^{i+1} \hat{u}_j, & x \in (-1, 0), \\
(P_j^i \hat{u}_j)(-1) = (P_j^i \hat{u}_j)(0) = (P_j^i \hat{u}_j)'(0) = 0.
\end{cases}
\end{align*}
\]
Multiplying the first equation in (32) by \( \lambda_j(x + 1)P_j^i \hat{u}_j \) and integrating over \((-1, 0)\) results in
\[
\frac{3}{2} \lambda_j \int_{-1}^{0} |(P_j^i \hat{u}_j)'|^2 dx + (\lambda_j - \alpha) \frac{\lambda_j}{2} \int_{-1}^{0} |P_j^i \hat{u}_j|^2 dx = \lambda_j \int_{-1}^{0} (x + 1)(P_j^i \hat{u}_j)(P_j^{i+1} \hat{u}_j)dx.
\]
After some elementary calculations, we can find a constant \( C_4 = C_4(a) \geq 1 \) such that
\[
\lambda_j \| (P_j^i \hat{u}_j)' \|^2 + \lambda_j^2 \| P_j^i \hat{u}_j \|^2 \leq C_4 (\| P_j^{i+1} \hat{u}_j \|^2 + \| P_j^{i+1} \hat{u}_j \|^2).
\]
Therefore, (31) holds for \( m = 1 \). Pick now any \( m \geq 2 \), and assume that (31) is true for \( m - 1 \geq 0 \). For any \( u \in D(A^{m+i}) \), we have
\[
\lambda_j^{2m} \| P_j^i \hat{u}_j \|^2 = \lambda_j^2 \lambda_j^{2m-2} \| P_j^i \hat{u}_j \|^2 \leq \lambda_j^2 C_4^{-1} \sum_{l=0}^{m-1} \left( \frac{m - 1}{l} \right) \| P_j^{i+l} \hat{u}_j \|^2.
\]
Since \( u \in D(A^{m+i}) \), for any \( l = 0, 1, ..., m - 1 \), system (32) is satisfied with \( P_j^{i+l} \hat{u}_j \) substituted to \( P_j^i \hat{u}_j \), and it follows as above that
\[
\lambda_j^2 \| P_j^{i+l} \hat{u}_j \|^2 \leq C_4 (\| P_j^{i+l} \hat{u}_j \|^2 + \| P_j^{i+l} \hat{u}_j \|^2).
\]
We infer that
\[
\lambda_j^{2m} \| P_j^i \hat{u}_j \|^2 \leq C_4^m \sum_{l=0}^{m-1} \left( \frac{m - 1}{l} \right) (\| P_j^{i+l} \hat{u}_j \|^2 + \| P_j^{i+l} \hat{u}_j \|^2)
\]
\[
= C_4^m (\| P_j^i \hat{u}_j \|^2 + \sum_{l=1}^{m-1} \left( \frac{m - 1}{l} \right) \| P_j^{i+l} \hat{u}_j \|^2
\]
\[
+ \sum_{l=1}^{m-1} \left( \frac{m - 1}{l - 1} \right) \| P_j^{i+l} \hat{u}_j \|^2 + \| P_j^{i+m} \hat{u}_j \|^2)
\]
\[
= C_4^m (\| P_j^i \hat{u}_j \|^2 + \sum_{l=1}^{m-1} \left( \frac{m}{l} \right) \| P_j^{i+l} \hat{u}_j \|^2 + \| P_j^{i+m} \hat{u}_j \|^2)
\]
\[
= C_4^m \sum_{l=0}^{m} \left( \frac{m}{l} \right) \| P_j^{i+l} \hat{u}_j \|^2
\]
where we used Pascal’s Rule. The proof of Lemma 2.7 is achieved.

We are in a position to complete the proof of Proposition 1. The estimate (16) is obvious for \( n = 0 \). Let \( n \geq 1 \). Using Lemmas 2.5, 2.6, and 2.7, we obtain that
\[
\| u \|_{H^{2n}(\Omega)}^2 \leq C_3 \sum_{m=0}^{n} \sum_{k=0}^{m} \| \partial_x^{2k} \partial_y^{2m-2k} u \|_{L^2(\Omega)}^2
\]
\[
= C_3 \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} \| \hat{u}_j^{(2k)} \|^2
\]
\[
\begin{align*}
&\leq C_3 \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \lambda_j^{2m-2k} C_1^k \sum_{i=0}^{k} \lambda_j^{2k-2i} \| P_j \hat{u}_j \|^2 \\
&\leq C_3 C_1^n \sum_{m=0}^{n} \sum_{k=0}^{m} \sum_{j=1}^{\infty} \sum_{i=0}^{k} \lambda_j^{2m-2i} \| P_j \hat{u}_j \|^2.
\end{align*}
\]

Using the fact that \(i \leq k \leq m \leq n\) in the sum above, we obtain
\[
\| u \|^2_{H^m(\Omega)} \leq C_3 C_1^n \sum_{m=0}^{n} \sum_{j=1}^{\infty} \sum_{i=0}^{n} \lambda_j^{2n-2i} \| P_j \hat{u}_j \|^2 \\
\leq C_3 C_1^n (n+1)^2 \sum_{j=1}^{\infty} \sum_{i=0}^{n} \lambda_j^{2n-2i} \| P_j \hat{u}_j \|^2 \\
\leq C_3 C_1^n (n+1)^2 \sum_{j=1}^{\infty} \sum_{i=0}^{n} \lambda_j^{2n-2i} \sum_{l=0}^{n-i} \left( \begin{array}{c} n-i \\ l \end{array} \right) \| P_j^{i+l} \hat{u}_j \|^2 \\
\leq C_3 C_1^n (n+1)^2 \sum_{j=1}^{\infty} \sum_{i=0}^{n} \sum_{l=0}^{\infty} \| P_j^{i+l} \hat{u}_j \|^2 \\
\leq C_3 C_1^n (n+1)^3 \sum_{j=1}^{\infty} \sum_{i=0}^{n} \| P_j^{i+1} \hat{u}_j \|^2 \\
\leq B^n \sum_{l=0}^{n} \| P^l u \|^2_{L^2(\Omega)}
\]

with \(B := 16C_1 C_3 C_4\). Indeed, it is easy to see that \((n+1)^3 \leq 8^n\) for all \(n \in \mathbb{N}\). The proof of Proposition 1 is achieved. \(\Box\)

Recall that \(\lambda_j = (jn)^2\) for \(j \geq 1\). For any \(j \geq 1\), we consider a sequence of generating functions \(g_{i,j}\) \((i \geq 0)\), where \(g_{0,j}\) is the solution of the Cauchy problem
\[
\begin{align*}
\begin{cases}
g''_{0,j}(x) - (\lambda_j - a)g'_{0,j}(x) = 0, & x \in (-1,0), \\
g_{0,j}(0) = g'_{0,j}(0) = 0, & g''_{0,j}(0) = 1,
\end{cases}
\end{align*}
\quad (33)
\]

while \(g_{i,j}\) for \(i \geq 1\) is defined inductively as the solution of the Cauchy problem
\[
\begin{align*}
\begin{cases}
g''_{i,j}(x) - (\lambda_j - a)g'_{i,j}(x) = -g_{i-1,j}(x), & x \in (-1,0), \\
g_{i,j}(0) = g'_{i,j}(0) = g''_{i,j}(0) = 0.
\end{cases}
\end{align*}
\quad (34)
\]

**Proposition 2.** For any \(i \geq 0, j \geq 1\) and \(x \in [-1,0]\), we have
\[
|g_{i,j}(x)| \leq e^{\sqrt{\lambda_j}} \frac{3^{i+1}}{(3i+2)!},
\quad (35)
\]

**Proof.** It follows from (33) and (34) that
\[
\begin{align*}
g_{i,j}(x) &= -\int_0^x g_{0,j}(x - \xi) g_{i-1,j}(\xi) d\xi \\
&= -\int_0^x g''_{0,j}(x - \xi) \left( \int_0^\xi (\int_0^\zeta g_{i-1,j}(\sigma)d\sigma)d\zeta \right) d\xi, \quad i, j \geq 1.
\end{align*}
\]
(1) if \( \lambda_j \leq a \), it is not difficult to obtain that
\[
g_{0,j}(x) = \begin{cases} 
\frac{1}{a - \lambda_j} (1 - \cos(\sqrt{a - \lambda_j} x)), & \lambda_j < a; \\
\frac{1}{2} x^2, & \lambda_j = a,
\end{cases}
\]
this implies
\[
0 \leq g_{0,j}(x) \leq \frac{x^2}{2}, \forall j \geq 1, x \in [-1,0].
\]
Then it follows from [13, Lemma 2.1] that
\[
|g_{i,j}(x)| = \frac{|x|^{3i+2}}{(3i+2)!} e^{\sqrt{x_j} \frac{3i!}{(3i+2)!}}, \forall i \geq 0, \forall j \geq 1, \forall x \in [-1,0].
\]
(2) if \( \lambda_j > a \), we claim that
\[
g_{i,j}(x) \leq \cosh(\sqrt{\lambda_j - ax}) \frac{(-x)^{3i+2} 3^i i!}{(3i+2)!}, \forall i \geq 0, \forall j \geq 1, \forall x \in [-1,0] \tag{36}
\]
which implies (35).
Let us prove (36) by induction on \( i \). For \( i = 0 \),
\[
0 \leq g_{0,j}(x) = \frac{1}{\lambda_j - a} (\cosh(\sqrt{\lambda_j - ax}) - 1)
\leq \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q}}{(2q)!}
\leq \sum_{q=1}^{\infty} \frac{(\lambda_j - a)^{q-1} x^{2q-2} x^2}{(2q-2)!}
= \cosh(\sqrt{\lambda_j - ax}) \frac{x^2}{2!},
\]
so that (36) is true for \( i = 0 \).
Assume now that (36) is true for \( i - 1 \geq 0 \). We can deduce that for \( x \in [-1,0] \)
\[
|g_{i,j}(x)| \leq \int_0^x g_{i-1,j}(x - \xi) \left( \int_0^\xi \left( \int_0^\zeta |g_{i-1,j}(\sigma)| d\sigma \right) d\zeta \right) d\xi
\leq \int_0^x \sum_{p=0}^{\infty} \frac{(\lambda_j - a)^p (x - \xi)^{2p}}{(2p)!}
\times \left( \int_0^\xi \left( \int_0^\zeta 3^{i-1} (i - 1)! \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^q (-\sigma)^{3i-1+2q}}{(2q)!(3i-1)!} d\sigma \right) d\zeta \right) d\xi
= -3^{i-1} (i - 1)! \int_0^x \sum_{p=0}^{\infty} \frac{(\lambda_j - a)^p (x - \xi)^{2p}}{(2p)!}
\times \sum_{q=0}^{\infty} \frac{(\lambda_j - a)^q (-\xi)^{3i+1+2q}}{(2q)!(3i+1+2q)(3i + 2q + 1)} d\xi.
\]
Then, integrating by parts $2p$ times, we obtain
\[
|g_{i,j}(x)| \leq -3^{i-1}(i-1)! \int_0^x \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\lambda_j - a)^{p+q}(-x)^{3i+1+2q+2p}(3i+2q-1)!}{(2q)!(3i-1)!(3i+1+2q+2p)!} d\xi
\]
\[
= 3^{i-1}(i-1)! \sum_{p=0}^\infty \frac{(\lambda_j - a)^{p+q}(-x)^{3i+1+2q+2p}(3i+2q-1)!}{(2q)!(3i-1)!(3i+1+2q+2p)!}.
\]

Next, we will show that
\[
3^{i-1}(i-1)!(3i+2q-1)! \leq \frac{3^i!}{p+q+1} \frac{1}{(2p+2q)!(3i+2)!} \quad \forall \ p, q \geq 0, \ i \geq 1.
\]

It is easy to see that (37) is equivalent to
\[
\frac{(3i+2q-1)!}{(2q)!(3i-1)!(3i+2+2q+2p)!} \leq \frac{3^i!}{p+q+1} \frac{1}{(2p+2q)!(3i+2)!} \quad \forall \ p, q \geq 0, \ i \geq 1.
\]

Since the left hand side of (38) is independent of $p$ and the right hand side of (38) is increasing in $p$, we only need to prove (37) for $p = 0$, namely, we need to show that
\[
\frac{(3i+2q-1)!}{(3i-1)!} \leq \frac{3^i!}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall \ q \geq 0, \ i \geq 1,
\]

this is obvious due to the fact that
\[
\frac{(3i+2)!}{3i(3i-1)!(3i+2+2q)!} \leq \frac{(3i+1)(3i+2)}{(3i+2)(3i+2+2q+1)(3i+2+2q+2)} \leq \frac{1}{3i+2q} \leq \frac{1}{q+1}.
\]

Applying (37), we infer that
\[
|g_{i,j}(x)| \leq \frac{(-x)^{3i+2q}!}{(3i+2)!} \sum_{p=0}^\infty \sum_{q=0}^\infty \frac{(\lambda_j - a)^{p+q}x^{2p+2q}}{(p+q+1)(2p+2q)!}
\]
\[
= \frac{(-x)^{3i+2q}!}{(3i+2)!} \sum_{k=0}^\infty \frac{(\lambda_j - a)^k x^{2k}}{(2k)!}
\]
\[
= \cosh(\sqrt{\lambda_j - ax}) \frac{(-x)^{3i+2q}!}{(3i+2)!},
\]

where we have used the fact that for any function $f : \mathbb{N} \to \mathbb{R}_+$, it holds
\[
\sum_{p=0}^\infty \sum_{q=0}^\infty f(p+q) = \sum_{k=0}^\infty (k+1)f(k).
\]

This ends the proof of Proposition 2.

\[\square\]
Remark 3. Compared with the result in [13, Lemma 2.1], it seems that a more natural estimate of \( g_{i,j} \) is
\[
|g_{i,j}(x)| \leq \cosh(\sqrt{\lambda_j - a} R^i(-x)^{3i+2}\frac{(3i+2)!}{(3i-1)!})
\]
for some constant \( R > 0 \). According to the proof of Proposition 2, to prove this result, we need to obtain that
\[
\frac{(3i+2q-1)!}{(3i-1)!} \leq \frac{R}{q+1} \frac{(3i+2+2q)!}{(3i+2)!} \quad \forall q \geq 0, \ \forall i \geq 1.
\]
This is equivalent to
\[
\frac{(q+1)(3i+1)(3i+2)}{(3i+2q)(3i+2q+1)(3i+2q+2)} \leq R \forall q \geq 0, \ \forall i \geq 1.
\]
However, this is impossible if we pick \( q = 3i \).

Using Proposition 2, we can obtain the following corollary which will be used in the proof of the main results.

**Corollary 2.** For any \( i \geq 0, j \geq 1 \) and \( x \in [-1, 0] \), we have
\[
|g_{i,j}(x)| \leq Ce^{\sqrt{\lambda_j}} \frac{1}{(2i)!},
\]
where the constant \( C \) is independent of \( i \) and \( j \).

**Proof.** By Stirling’s formula \( i! \sim (i/e)^i \sqrt{2\pi i} \), and it follows from (35) that for \( i \geq 1 \) and \( j \geq 1 \) we have
\[
|g_{i,j}(x)| \leq e^{\sqrt{\lambda_j}} \frac{3^i!}{(3i+2)!}
\leq Ce^{\sqrt{\lambda_j}} \frac{3^i!}{(3i+1)(3i+2)!} \frac{\sqrt{2\pi i}^3}{\sqrt{2\pi i} \sqrt{4\pi i}} (2i)!i!
\leq Ce^{\sqrt{\lambda_j}} \frac{1}{(2i)!}.
\]
\[ \Box \]

3. Null controllability.

**Proposition 3.** Let \( s \in [0, 2) \), \( 0 < t_1 < t_2 \leq T \) and \( z_j \in G^s([t_1, t_2]) \) satisfy
\[
|z_j^{(i)}(t)| \leq M_j \frac{(i)!}{R^i},
\]
where \( R \) is a positive constant and the positive constants \( M_j \) are such that
\[
\sum_{j=1}^{\infty} M_j e^{\sqrt{\lambda_j}} < \infty.
\]
Then the function \( u \) defined by (9) solves system (4)-(8) and \( u \in G^{\frac{1}{2}} \cdot \tilde{z}^s([-1, 0] \times [0, 1] \times [t_1, t_2]) \).
Proof. As the proof is similar to that of [13, Proposition 2.1], it is only sketched. Let $m, p, q \in \mathbb{N}$. By applying Proposition 1 and (9), we obtain that
\[
|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \leq C \|\partial_t^m u(\cdot, \cdot, t)\|_{H^{p+q+2}}(\Omega)
\]
\[
\leq CB \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \|P_n \partial_t^m u(\cdot, \cdot, t)\|_{L^2(\Omega)}
\]
\[
\leq CB \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x, y) \in \Omega} |\partial_t^m P_n u(x, y, t)|
\]
\[
\leq CB \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x, y) \in \Omega} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |\partial_t^m P_n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))|.
\]
By the definitions of $g_{i,j}$ and $e_j$, it is clear that
\[
\partial_t^m P_n (g_{i,j}(x) z_j^{(i)}(t) e_j(y)) = \begin{cases} 
  z_j^{(i+m)}(t)(-1)^n g_{i-n,j}(x) e_j(y), & i \geq n; \\
  0, & i < n.
\end{cases}
\]
Setting $k = i - n$ and $N = n + m$, arguing as in [13, Proposition 2.1], we infer from Corollary 2 that
\[
\sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} |\partial_t^m P_n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))| = \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} |z_j^{(i+m)}(t) g_{i-n,j}(x) e_j(y)|
\]
\[
\leq C \sum_{j=1}^{\infty} \sum_{k=0}^{N} M_j \frac{(k + N)!}{R^{k+N}} c \sqrt{N_j} \frac{1}{(2k)!}
\]
\[
\leq C \frac{(N!)^s}{(2)^N}
\]
\[
\leq C \frac{(n!)^s (m!)^s}{R_1 R_2^m}
\]
where $R_1 = R_2 = R/4^s$.
Gathering the above estimates together, we obtain that
\[
|\partial_t^m \partial_x^p \partial_y^q u(x, y, t)| \leq C \sum_{n=0}^{[\frac{p+q+2}{2}]+1} \sup_{(x, y) \in \Omega} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |\partial_t^m P_n (g_{i,j}(x) z_j^{(i)}(t) e_j(y))|
\]
\[
\leq C \frac{(p!)^s (q!)^s (m!)^s}{R_1 R_2^m R_3^r}
\]
for some positive constants $R_1, R_2, R_3$. Finally, it is easily seen that $u$ is indeed a solution of the ZK system. \hfill \Box

Let $\pi$ denote the solution of the free evolution for the ZK system:
\[
\begin{align*}
\pi_t + a \pi_x + \Delta \pi_x &= 0, & (x, y) \in (-1, 0) \times (0, 1), & t \in (0, T), \\
\pi(-1, y, t) &= \pi(0, y, t) = \pi_x(0, y, t) = 0, & y \in (0, 1), & t \in (0, T), \\
\pi(x, 0, t) &= \pi(x, 1, t) = 0, & x \in (-1, 0), & t \in (0, T), \\
\pi(x, y, 0) &= u_0(x, y), & x \in (-1, 0), & y \in (0, 1).
\end{align*}
\]
As for KdV, we have a Kato smoothing effect.

**Proposition 4.** Let $u_0 \in L^2(\Omega)$.

(i) System (41) admits a unique solution $\pi \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ and we have

$$
\sup_{t \in [0, T]} \|\pi(\cdot, \cdot, t)\|_{L^2(\Omega)}^2 + \int_0^T \|\pi(\cdot, \cdot, t)\|_{H^1(\Omega)}^2 \, dt \leq C \|u_0\|_{L^2(\Omega)}^2. \tag{42}
$$

(ii) If, in addition, $u_0 \in D(A) \cap H^3(\Omega)$, then $\pi \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$ and we have

$$
\sup_{t \in [0, T]} \|\pi(\cdot, \cdot, t)\|_{H^3(\Omega)}^2 + \int_0^T \|\pi(\cdot, \cdot, t)\|_{H^4(\Omega)}^2 \, dt \leq C \|u_0\|_{H^3(\Omega)}^2. \tag{43}
$$

**Proof.** (i) comes from [19]. Let us proceed with the proof of (ii). For any $u_0 \in D(A) \cap H^3(\Omega)$, we have that $\pi \in C([0, T]; D(A))$ by the semigroup property, and hence $\pi \in C([0, T]; H^2(\Omega) \cap H^1_0(\Omega))$. Let $w_0 = Au_0$ and $w = A\pi$. It is well known that $w$ is the solution of (41) with initial value $w_0 \in L^2(\Omega)$. According to (i), we have

$$
-\Delta \pi_x - a\pi_x = A\pi = w \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)).
$$

Therefore $\pi_x \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Assume finally that $u_0 \in D(A) \cap H^3(\Omega)$, and let us prove that $u \in C([0, T], H^3(\Omega)) \cap L^2(0, T; H^4(\Omega))$. Decompose $u$ as $u(x, y, t) = \sum_{j=1}^{\infty} \hat{u}_j(x, t)e_j(y)$. Then for $j \geq 1$, $\hat{u}_j$ solves

$$
\frac{d\hat{u}_j}{dt} + \hat{u}_j''' + (a - \lambda_j)\hat{u}_j' = 0, \tag{44}
$$

$$
\hat{u}_j(-1, t) = \hat{u}_j(0, t) = \hat{u}_j'(0, t) = 0, \tag{45}
$$

$$
\hat{u}_j(., 0) = \hat{u}^0_j, \tag{46}
$$

where $u_0(x, y) = \sum_{j=1}^{\infty} \hat{u}^0_j(x)e_j(y)$. Multiplying in (44) by $\hat{u}_j$ (resp. by $(x + 1)\hat{u}_j$) and integrating over $(-1, 0)_x \times (0, T)_t$, we obtain respectively

$$
\int_{-1}^0 |\hat{u}_j(x, T)|^2 \, dx + \int_0^T |\hat{u}_j'(t)|^2 \, dt = \int_{-1}^0 |\hat{u}^0_j(x)|^2 \, dx, \tag{47}
$$

$$
\int_{-1}^0 (x + 1)|\hat{u}_j(x, T)|^2 \, dx + 3 \int_0^T \int_{-1}^0 |\hat{u}_j'(x, t)|^2 \, dx \, dt + (\lambda_j - a) \int_0^T \int_{-1}^0 |\hat{u}_j|^2 \, dx \, dt
$$

$$
= \int_{-1}^0 (x + 1)|\hat{u}^0_j(x)|^2 \, dx. \tag{48}
$$

It follows from (47) that for any $k \in \mathbb{N}$

$$
\sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}_j(., t)\|^2 \leq \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}^0_j\|^2, \quad \forall t \in \mathbb{R}_+, \tag{49}
$$

(that is, $\|\partial_t^k u(., t)\|_{L^2(\Omega)}^2 \leq \|\partial_t^k u_0\|_{L^2(\Omega)}^2$ for all $t \in \mathbb{R}_+$), and from (48) that

$$
\int_0^T \sum_{j=1}^{\infty} (\lambda_j^k \|\hat{u}_j'(., t)\|^2 + \lambda_j^{k+1} \|\hat{u}_j(., t)\|^2) \, dt \leq (1 + aT) \sum_{j=1}^{\infty} \lambda_j^k \|\hat{u}^0_j\|^2, \quad \forall T > 0 \tag{50}
$$

(that is, $\int_0^T \|\nabla \partial_t^k u(., t)\|_{L^2(\Omega)}^2 \, dt \leq (1 + aT)\|\partial_t^k u_0\|_{L^2(\Omega)}^2$ for all $T > 0$). We need the following lemma.
Lemma 3.1. Let $a > 0$ and $\lambda > 0$ be given. Let $H^k (k \in \mathbb{N})$ denote the Sobolev space $H^k(-1,0)$, and let $H^3 := \{ u \in H^3(-1,0); \ u(-1) = u(0) = u'(0) = 0 \}$. Let $\| \cdot \|$ denote the norm $\| \cdot \|_{L^2(-1,0)}$.

1. There exists a constant $C > 0$ such that
$$
\sum_{k=0}^{3} \lambda^k \| \partial_x^{3-k} y \|^2 \leq C (\|y'''\| + (a-\lambda)\|y''\|^2 + \lambda^3 \|y\|^2) \quad \forall y \in H^3, \ \forall \lambda \geq \lambda_0. \quad (51)
$$

2. There exists a constant $C' > 0$ such that
$$
\sum_{k=0}^{4} \lambda^k \| \partial_x^{4-k} y \|^2 \leq C' \left( \|y^{(4)}\| + (a-\lambda)\|y''\|^2 + \|y'''\| + (a-\lambda)\|y''\|^2 + \lambda^4 \|y\|^2 \right) \quad \forall y \in H^3 \cap H^4, \ \forall \lambda \geq \lambda_0. \quad (52)
$$

**Proof of Lemma 3.1.** 1. Pick any $y \in H^3$ and any $\lambda \geq 0$. By the Interpolation Theorem and Young inequality, we have that
$$
\lambda^2 \|y\|^2 \leq C \lambda^2 \|y\| \frac{2}{3} \|y''\| \frac{2}{3} \leq \varepsilon \|y'''\|^2 + C \varepsilon \lambda^2 \|y\|^2,
$$
$$
\lambda \|y''\|^2 \leq C \lambda \|y\| \frac{2}{3} \|y''\| \frac{2}{3} \leq \varepsilon \|y'''\|^2 + C \varepsilon \lambda^2 \|y\|^2.
$$
We infer that if $\lambda \geq \lambda_0 > 0$
$$
\|y'''\|^2 \leq 2 \|y'''\| + (a-\lambda)\|y''\|^2 + 2(a-\lambda)^2 \|y'\|^2 \leq 2 \|y'''\| + (a-\lambda)\|y''\|^2 + 2\varepsilon \|y'''\|^2 + 2C \varepsilon \lambda^2 \|y\|^2 \leq 2 \|y'''\| + (a-\lambda)\|y''\|^2 + 2\varepsilon \|y'''\|^2 + C \varepsilon \lambda^2 \|y\|^2
$$
and (51) follows by picking $\varepsilon < 1/4$.

2. Pick now any $y \in H^3 \cap H^4$ and any $\lambda \geq 0$. Then we have
$$
\lambda^3 \|y\|^2 \leq C \lambda^3 \|y\| \frac{2}{3} \|y''\| \frac{2}{3} \leq \varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C \varepsilon \lambda^4 \|y\|^2,
$$
$$
\lambda \|y''\|^2 \leq C \lambda \|y\| \frac{2}{3} \|y''\| \frac{2}{3} \leq \varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C \varepsilon \lambda^4 \|y\|^2.
$$
On the other hand, we have that for $\lambda \geq \lambda_0 > 0$
$$
\|y^{(4)}\|^2 \leq 2 \|y^{(4)}\| + (a-\lambda)\|y''\|^2 + 2(a-\lambda)^2 \|y''\|^2 \leq 2 \|y^{(4)}\| + (a-\lambda)\|y''\|^2 + 2\varepsilon (\|y^{(4)}\|^2 + \|y'''\|^2) + C \varepsilon \lambda^4 \|y\|^2
$$
and (52) follows by picking $\varepsilon < 1/4$ and by using (51).

Assuming that $u_0 \in D(A) \cap H^3(\Omega)$ and using (49) and (51), we obtain that for any $t \in [0,T]$ (with a constant $C$ that may vary from line to line)
$$
\| u(\cdot, \cdot, t) \|^2_{H^3(\Omega)} = \| u(\cdot, \cdot, t) \|^2_{H^3(\Omega)} + \sum_{k=0}^{3} \lambda^k \| \partial_x^{3-k} u(\cdot, \cdot, t) \|^2_{L^2(\Omega)} \leq C \| u_0 \|^2_{D(A)} + \sum_{k=0}^{3} \sum_{j=1}^{\infty} \lambda^k \| \partial_x^{3-k} \hat{u}_j(\cdot, \cdot, t) \|^2 \leq C \| u_0 \|^2_{D(A)} + C \sum_{k=0}^{3} \sum_{j=1}^{\infty} \left( \| \hat{u}_j''(\cdot, \cdot, t) + (a-\lambda)\| \hat{u}_j'(\cdot, \cdot, t) \|^2 + \lambda^3 \| \hat{u}_j'(\cdot, \cdot, t) \|^2 \right)
Without loss of generality, we assume that \( T \) from Proposition 1 that
\[
\frac{\|u(\cdot, t)\|_{H^3(\Omega)}}{\|u_0\|_{H^3(\Omega)}} = \frac{\|u(\cdot, t)\|_{H^3(\Omega)}}{\|u_0\|_{H^3(\Omega)}} + \sum_{k=0}^{4} \left| \partial_y^k \partial_x^{4-k} u(\cdot, t) \right|_{L^2(\Omega)}^2
\]
and it is clear that \( \int_0^T \|u(\cdot, t)\|_{H^3(\Omega)} dt \leq C\|u_0\|_{H^3(\Omega)}^2 \). Using (52), we obtain
\[
\int_0^T \sum_{k=0}^{4} \left| \partial_y^k \partial_x^{4-k} u(\cdot, t) \right|_{L^2(\Omega)}^2 dt = \int_0^T \sum_{k=0}^{4} \sum_{j=1}^{\infty} \lambda_j^k \left| \partial_y^k \partial_x^{4-k} \hat{u}_j(\cdot, t) \right|^2 dt \leq C \int_0^T \left( \|\hat{u}(\cdot, t)\|_{H^2(\Omega)} + \left| \partial_y^k \hat{u}(\cdot, t) \right|_{L^2(\Omega)}^2 \right) dt \leq C \int_0^T \left( \|Au(\cdot, t)\|_{H^2(\Omega)} + \left| \partial_y^k u(\cdot, t) \right|_{L^2(\Omega)}^2 \right) dt \leq C\|u_0\|_{H^3(\Omega)}^2
\]
where we used (50) with \( k = 3 \). This completes the proof of the proposition.

Interpolating between (42) and (43), we obtain
\[
\sup_{t \in [0, T]} \|\hat{u}(\cdot, t)\|_{H^2(\Omega)} + \int_0^T \|\hat{u}(\cdot, t)\|_{H^2(\Omega)}^2 dt \leq C\|u_0\|_{H^2(\Omega)}^2
\]
This gives
\[
\|\hat{u}(\cdot, t)\|_{H^{n+1}(\Omega)} \leq C\|u_0\|_{H^n(\Omega)}, \quad \text{for } n \in \{0, 1, 2, 3\}.
\]

Proceeding as in [13, Proposition 2.2], we can show that if \( u_0 \in L^2(\Omega) \), then \( \hat{u}(t) \in D(A^n) \) for any \( t \in (0, T) \) and \( n \in \mathbb{N} \), and it holds
\[
\|A^n \hat{u}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C^n}{t^n} \|u_0\|_{L^2(\Omega)}.
\]
Without loss of generality, we assume that \( T = 1 \). Then for any \( p, q \in \mathbb{N} \), we infer from Proposition 1 that
\[
|\partial_y^p \partial_x^q \hat{u}(x, y, t)| \leq \|\hat{u}(\cdot, t)\|_{H^{n+1}(\Omega)} \leq \|\hat{u}(\cdot, t)\|_{H^{n+1}(\Omega)} \leq \frac{C^n}{t^n} \|u_0\|_{L^2(\Omega)}
\]
for some \( R_1, R_2 > 0 \). This means that \( \hat{u}(\cdot, t) \in G^{1/2}(\mathbb{R}^4([-1, 0] \times [0, 1])) \) for any \( t \in (0, T) \).
Let
\[ f_j(t) := \int_0^1 e_j(y) \partial^2_x \pi(0, y, t) dy. \]

**Lemma 3.2.** For any \( j \geq 1 \) and \( n \geq 0 \), there exist positive constants \( R_1, R_2 \) and \( C \) such that
\[ |f_j^{(n)}(t)| \leq \frac{C}{(\pi j)^{n+3}} t^{-\frac{3}{2}(n+\frac{j}{2})} \frac{(n!)^{\frac{j}{2}}}{R_1 R_2^j}. \]

**Proof.** Without loss of generality, we can assume that \( T = 1 \). Since \( \pi(x, t) \in \mathcal{D}(A^n) \) for any \( t \in (0, T) \) and \( n \in \mathbb{N} \), it follows from Lemma 2.2 that
\[ \partial_x^2 \partial_{y^n} \pi(x, 0, t) = \partial_x^2 \partial_{y^n} \pi(x, 1, t) = 0, \quad \forall x \in [-1, 0], \forall t \in (0, T), \forall n \in \mathbb{N}. \]

Then, integrating by parts \( j \)-times, we deduce that
\[ f_j(t) = \sqrt{2} \int_0^1 \sin(j \pi y) \partial^2_x \pi(0, y, t) dy 
= \frac{\sqrt{2}}{j \pi} \int_0^1 \cos(j \pi y) \partial^2_x \partial_y \pi(0, y, t) dy 
= \frac{\sqrt{2}}{(j \pi)^2} \int_0^1 \sin(j \pi y) \partial^2_x \partial_y^2 \pi(0, y, t) dy 
= \begin{cases} \frac{\sqrt{2}}{(j \pi)^2} \int_0^1 \sin(j \pi y) \partial^2_x \partial_y^2 \pi(0, y, t) dy, & \text{if } j \text{ is even;} \\
- \frac{\sqrt{2}}{(j \pi)^2} \int_0^1 \cos(j \pi y) \partial^2_x \partial_y^2 \pi(0, y, t) dy, & \text{if } j \text{ is odd.} \end{cases} \]

To estimate \( |f_j^{(n)}(t)| \), it remains to estimate \( |\partial^n_x \partial^2_y \pi(0, y, t)| \). Let
\[ l = \left\lfloor \frac{j + 4}{2} \right\rfloor + 1. \]

Taking (16) (with \( u = P^n \pi \)) and (53) into account, we obtain that
\[ |\partial^n_x \partial^2_y \pi(x, y, t)| \leq \frac{C}{(j \pi)^{n+4}} \|P^n \pi(\cdot, \cdot, t)\|_{H^{n+4}(\Omega)} 
\leq CB^l \sum_{k=0}^{n-l} \|P^{n+k} \pi(\cdot, \cdot, t)\|_{L^2(\Omega)} 
\leq CB^l \sum_{k=0}^{n-l} \|P^k \pi(\cdot, \cdot, t)\|_{L^2(\Omega)} 
\leq CB^l \sum_{k=0}^{n-l} \frac{C^k k^{\frac{j}{2}}}{t^{\frac{j}{2}k}} \|u_0\|_{L^2(\Omega)} 
\leq CB^l \sum_{k=0}^{n-l} \frac{C^{n+l} (n + 1) (n + l) \frac{3}{4} (n+4)}{t^{\frac{3}{4}(n+l)}} \|u_0\|_{L^2(\Omega)} 
\leq Ct^{-\frac{3}{4}(n+\frac{3}{2})} \frac{(n!)^{\frac{j}{2}} (j!)^{\frac{j}{2}}}{R_1 R_2^j} \|u_0\|_{L^2(\Omega)} \]
for some \( R_1, R_2 > 0 \).
Combining (54) and (55), we obtain
\[
|f_j^{(n)}(t)| \leq \frac{C}{(j\pi)^{j}} \sup_{y \in [0, 1]} |\partial^m_x \partial_y^n \pi(0, y, t)| \\
\leq \frac{C}{(j\pi)^{j}} t^{-\frac92(n+\frac{j}{2}+3)} (nl)^{\frac{j}{2}} (j!)^{\frac{j}{2}} R_1^n R_2^2.
\]

Now, we can prove the first main result in this paper.

**Proof of Theorem 1.1.** Pick any \( \tau \in (0, T), s \in [3/2, 2) \) and let
\[
z_j(t) = \phi_s \left( \frac{T - t}{T - \tau} \right) f_j(t), \quad 0 \leq t \leq T,
\]
where
\[
\phi_s(\rho) = \begin{cases} 
1 & \text{if } \rho \leq 0, \\
0 & \text{if } \rho \geq 1, \\
\frac{e^{-\frac{1}{(1-\rho)^{s}}}}{e^{\frac{1}{(1+\rho)^{s}}} + e^{-\frac{1}{(1-\rho)^{s}}}} & \text{if } \rho \in (0, 1)
\end{cases}
\]
with \( M > 0 \) and \( \sigma = (s - 1)^{-1} \). As \( \phi_s \) is Gevrey of order \( s \), there exist \( R_\phi > 0 \) such that
\[
|\phi_s(\rho)| \leq \frac{C(p)^s}{R_\phi^s} \quad \forall p \in \mathbb{N}, \rho \in \mathbb{R}.
\]

Then, applying Lemma 3.2, for any \( \varepsilon \in (0, T) \) and \( t \in [\varepsilon, T] \), we have
\[
|z_j^{(i)}(t)| \leq \sum_{n=0}^{i} \binom{i}{n} |\partial^i_x - n \cdot \phi_s \left( \frac{T - t}{T - \tau} \right) | \left| f_j^{(n)}(t) \right|
\]
\[
\leq C \sum_{n=0}^{i} \binom{i}{n} \left( \frac{(i-n)!}{R_\phi^{i-n}} \left( \frac{1}{T - \tau} \right)^{i-n} \left( \frac{1}{(j\pi)^{j}} \right)^{j} t^{-\frac92(n+\frac{j}{2}+3)} (nl)^{\frac{j}{2}} (j!)^{\frac{j}{2}} \right)^{n} R_1^n R_2^2
\]
\[
\leq C \frac{1}{(j\pi)^{j}} \varepsilon^{-\frac92(n+\frac{j}{2}+3)} (j!)^{\frac{j}{2}} R_2^j \sum_{n=0}^{i} \binom{i}{n} \left( \frac{(i-n)!}{R_\phi^{i-n}} \left( \frac{1}{T - \tau} \right)^{i-n} \varepsilon^{-\frac92(n+\frac{j}{2}+3)} (nl)^{\frac{j}{2}} (j!)^{\frac{j}{2}} \right)^{n} R_1^n R_2^2
\]
\[
\leq C \frac{1}{(j\pi)^{j}} \varepsilon^{-\frac92(n+\frac{j}{2}+3)} (j!)^{\frac{j}{2}} R_2^j \sum_{n=0}^{i} \binom{i}{n} \left( \frac{1}{T - \tau} \right)^{i-n} \varepsilon^{\frac92(n+\frac{j}{2}+3)} (nl)^{\frac{j}{2}} (j!)^{\frac{j}{2}} \right)^{n} R_1^n R_2^2
\]
\[
\leq M_j \left( \frac{(j!)^{\frac{j}{2}}}{R_2^j} \right)^{i-n} \varepsilon^{\frac92(n+\frac{j}{2}+3)} (nl)^{\frac{j}{2}} (j!)^{\frac{j}{2}} \right)^{n} R_1^n R_2^2
\]
where \( M_j \) satisfies (40). Let
\[
u(x, y, t) = \begin{cases} 
u_0(x, y) & \text{if } x \in [-1, 0], y \in [0, 1], t = 0, \\
\sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x) z_j^{(i)}(t) e_j(y) & \text{if } x \in [-1, 0], y \in [0, 1], t \in (0, T].
\end{cases}
\]
Then, it is easy to see that \( u(\cdot, T) = 0 \). By Proposition 3, \( u \in G^{\frac{1}{2}, \frac{1}{2}}([-1, 0] \times [0, 1] \times \{\varepsilon, T\}) \) for any \( \varepsilon \in (0, T) \). Furthermore, we have
\[
\begin{align*}
&u_t + au_x + \Delta u_x = 0 = \nabla u + a\nabla x + \Delta \nabla x \quad \text{in } \Omega \times (0, T), \\
u(0, y, t) = 0 = \nabla(0, y, t), \quad \forall y \in [0, 1], \quad \forall t \in (0, \tau), \\
\partial_x u(0, y, t) = 0 = \partial_x \nabla(0, y, t), \quad \forall y \in [0, 1], \quad \forall t \in (0, \tau), \\
\partial^2_x u(0, y, t) = \sum_{j=1}^{\infty} z_j(t)e_j(y) = \partial^2_x \nabla(0, y, t), \quad \forall y \in [0, 1], \quad \forall t \in (0, \tau).
\end{align*}
\]

It follows from Holmgren theorem that \( u(x, y, t) = \nabla(x, y, t) \) for any \( (x, y, t) \in [-1, 0] \times [0, 1] \times (0, \tau) \). In particular, \( u \in C([0, T]; L^2(\Omega)) \) and \( h = 0 \) for \( t \in [0, \tau) \), so that \( u \in G^{\frac{1}{2}, \frac{1}{2}}([-1, 0] \times [0, 1] \times [0, T]) \). The proof of Theorem 1.1 is complete. \( \square \)

4. Reachable states.

Proposition 5. For any \( j \geq 1 \), assume that \( z_j \in G^2([0, T]) \) is such that
\[
|z_j^{(i)}(t)| \leq M_j \left( \frac{2i}{R^2} \right)^{n-j} \forall i \geq 0, \quad t \in [0, T],
\]
where \( R > 1 \) and \( M_j \) satisfies (40). Then the function \( u \) defined by (9) solves system (4)-(8) and \( u \in G^{1, 1, 2}([-1, 0] \times [0, 1] \times [0, T]) \).

Proof. According to the proof of Proposition 3, for any \( m, p, q \in \mathbb{N} \), we have
\[
|\partial^m_t \partial^p_x \partial^q_y u(x, y, t)| \leq CB^{|p+q+2|+1} \sum_{n=0}^{\infty} \sup_{(x,y) \in \Omega} \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t)g_{i-n,j}(x)e_j(y)|.
\]
Let \( k = 2i - 2n \) and \( N = 2n + 2m \). We can obtain by the same arguments as in [13, Proposition 3.1] that
\[
\begin{align*}
\sum_{j=1}^{\infty} \sum_{i=n}^{\infty} |z_j^{(i+m)}(t)g_{i-n,j}(x)e_j(y)| &\leq \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} M_j \left( \frac{2i + 2m}{R^{2i + 2m}} \right)! \frac{Ce^{\sqrt{\lambda_j}}}{(2i - 2n)!} \\
&= \sum_{j=1}^{\infty} CM_j e^{\sqrt{\lambda_j}} \sum_{k=0}^{\infty} \frac{(k + N)!}{R^{k + N} k!} \\
&\leq C \sum_{k=0}^{\infty} \frac{(k + N)!}{R^{k + N} k!} \\
&= C \sum_{k=0}^{\infty} \frac{(k + 1) \cdots (k + N)}{R^{k + N}} \\
&\leq C \left( \frac{\alpha c}{R^2} \right)^N N! \sqrt{N} \\
&\leq C \frac{(2n)! (2m)!}{R_1^2 R_2^m} ,
\end{align*}
\]
where \( R_1, R_2 \) are two positive constants, \( \sigma \in (0, 1) \) and
\[
\alpha = \sup_{k \geq 0} \frac{k + 2}{(R^{\sigma - 1})^{k+1}}.
\]
It follows from the above estimates that
\[
|\partial_t^m \partial_x^p \partial_y^n u(x, y, t)| \leq CB^{(\frac{p+1}{2})+1} \left( \sum_{n=0}^{\frac{p+1}{2}} \frac{(2n)!}{R_1^n R_2^m} \right)
\]
\[
\leq C p! q!(m!)^2 \frac{H}{R_1^n R_2^m R_3^n}
\]
for some positive constants \( \tilde{R}_1, \tilde{R}_2 \) and \( \tilde{R}_3 \). This ends the proof of Proposition 5.

As a particular case of [12, Proposition 3.6] (with \( a_0 = 1, a_p = [2p(2p - 1)]^{-1} \) for \( p \geq 1 \), we have the following result.

**Proposition 6.** Let \( \{d_q\}_{q \geq 0} \) be a sequence of real numbers such that
\[
|d_q| \leq CH^q(2q)! \quad \forall \quad q \geq 0
\]
for some \( H > 0 \) and \( C > 0 \). Then for all \( \tilde{H} > e^{e^{-1}}H \), there exists a function \( f \in C^\infty(\mathbb{R}) \) such that
\[
f^{(q)}(0) = d_q \quad \forall \quad q \geq 0,
\]
\[
|f^{(q)}(x)| \leq C \tilde{H}^q(2q)! \quad \forall \quad q \geq 0, \quad x \in \mathbb{R}.
\]

Let
\[
\mathcal{X} := \{ u \in C^\infty([-1, 0] \times [0, 1]); \quad P^n u(0, y) = \partial_x P^n u(0, y) = P^n u(x, 0) = P^n u(x, 1) = 0, \quad \forall n \in \mathbb{N}, \ \forall x \in [-1, 0], \ \forall y \in [0, 1]\}.
\]
A result similar to Lemma 2.2 can be derived.

**Lemma 4.1.** For any \( n \in \mathbb{N} \), we have
\[
\partial_y^{2n} f(x, 0) = \partial_y^{2n} f(x, 1) = 0, \quad \forall f \in \mathcal{X}, \quad \forall x \in [-1, 0]. \quad (56)
\]

**Proof.** We proceed by induction on \( n \). For \( n = 0 \), (56) is obvious since \( f \in \mathcal{X} \). Assume now that (56) is true for \( n - 1 \geq 0 \). If \( f \in \mathcal{X} \), then \( Pf \in \mathcal{X} \), so that by the induction hypothesis
\[
\partial_y^{2n-2} Pf(x, 0) = \partial_y^{2n-2} Pf(x, 1) = 0.
\]
This implies
\[
\partial_x^2 \partial_y^{2n-2} f(x, 0) + \partial_x \partial_y^{2n} f(x, 0) + a \partial_x \partial_y^{2n-2} f(x, 0) = 0,
\]
\[
\partial_x^2 \partial_y^{2n-2} f(x, 1) + \partial_x \partial_y^{2n} f(x, 1) + a \partial_x \partial_y^{2n-2} f(x, 1) = 0.
\]
Since (56) is true for \( n - 1 \), we obtain that
\[
\partial_x \partial_y^{2n} f(x, 0) = \partial_x \partial_y^{2n} f(x, 1) = 0.
\]
This means that for some constants \( C_1 \) and \( C_2 \),
\[
\partial_y^{2n} f(x, 0) = C_1, \quad \partial_y^{2n} f(x, 1) = C_2 \quad \forall x \in [-1, 0].
\]
On the other hand, we infer from the assumption \( f \in \mathcal{X} \) that
\[
\partial_y^{2n} f(0, y) = 0 \quad \forall y \in [0, 1].
\]
Taking \( y = 0 \) and next \( y = 1 \), we see that \( C_1 = C_2 = 0 \). The proof of Lemma 4.1 is complete. □
Lemma 4.2. If $f \in \mathcal{X}$ is such that
\[
\int_0^1 e_l(y)P^nf(0,y)dy = \int_0^1 e_l(y)\partial_xP^nf(0,y)dy = \int_0^1 e_l(y)\partial^2_xP^nf(0,y)dy = 0
\]
for any $l \geq 1$ and any $n \geq 0$, then
\[
\int_0^1 e_l(y)\partial^m_xf(0,y)dy = 0
\]
holds for any $l \geq 1$ and any $m \geq 0$.

Proof. To prove that (58) holds for any $l \geq 1$ and any $m \geq 0$, it is sufficient to show that for any $M \in \mathbb{N}$, (58) holds for any $l \geq 1$ and any $m \leq 3M + 2$. We proceed by induction on $M$.

For $M = 0$, we can take $n = 0$ in (57) to see that (58) holds for any $l \geq 1$ and $m \leq 2$.

Assume that (58) is true for any $l \geq 1$ and any $m \leq 3M - 1$. We claim that (58) holds for any $l \geq 1$ and $m = 3M, 3M + 1, 3M + 2$. Indeed, taking $n = M$ in (57), we have
\[
0 = (-1)^M \int_0^1 e_l(y)P^Mf(0,y)dy
\]
\[
= \int_0^1 e_l(y)(\partial^2_x + \partial^2_y + a)^M\partial^M_xf(0,y)dy
\]
\[
= \int_0^1 e_l(y)\partial^M_xf(0,y)dy
\]
\[
+ \int_0^1 e_l(y) \sum_{k=0}^{M-1} \binom{M}{k} \sum_{i=0}^{M-k} \binom{M-k}{i} a^{M-k-i}\partial_x^{2k+M}\partial_y^{2i}f(0,y)dy.
\]
(59)

Since $f \in \mathcal{X}$, it follows from Lemma 4.1 that
\[
\partial_y^{2n}f(x,0) = \partial_y^{2n}f(x,1) = 0, \quad \forall x \in [-1,0], \quad \forall n \in \mathbb{N}.
\]

Then, we obtain by integrations by parts that for $k \in \{0,\ldots,M-1\}$ and $i \in \{0,\ldots,M-k\}$
\[
\int_0^1 e_l(y)\partial_x^{2k+M}\partial_y^{2i}f(0,y)dy = (-1)^i(l\pi)^2i\int_0^1 e_l(y)\partial_x^{2k+M}f(0,y)dy = 0.
\]

In the last step, we used the fact that $2k + M \leq 3M - 1$. Thus, we infer from (59) that
\[
\int_0^1 e_l(y)\partial^M_xf(0,y)dy = 0, \quad \forall l \geq 1.
\]

We can show in the same way that (58) is true for $m = 3M + 1, 3M + 2$ by using the fact that
\[
\int_0^1 e_l(y)\partial_xP^Mf(0,y)dy = \int_0^1 e_l(y)\partial^2_xP^Mf(0,y)dy = 0, \quad \forall l \geq 1.
\]

The proof of Lemma 4.2 is complete. \qed

Now, we are in a position to prove the second main result in this paper.
Proof of Theorem 1.2. Assume that $R := \min\{R_1, R_2\} > R_0 = \sqrt[3]{9(a + 2)e^{(3e)^{-1}}}$ and pick any $u_1 \in \mathcal{R}_{R_1, R_2}$. We intend to expand $u_1$ in the following form:

$$u_1(x, y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j} g_{i,j}(x)e_j(y),$$

where

$$b_{i,j} = (-1)^i \int_0^1 e_j(y) \partial_x^2 P^i u_1(0, y) dy.$$ 

Since $u_1 \in \mathcal{R}_{R_1, R_2} \subset \mathcal{X}$, we have that $P^i u_1 \in \mathcal{X}$ for any $i \in \mathbb{N}$. By Lemma 4.1, we infer that

$$\partial_x^{2n} P^i u_1(x, 0) = \partial_y^{2n} P^i u_1(x, 1) = 0, \forall x \in [-1, 0].$$

Then, by integration by parts, we have

$$|b_{i,j}| = \left| \int_0^1 e_j(y) \partial_x^2 P^i u_1(0, y) dy \right| \leq \frac{C}{(j\pi)^j} \sup_{(x,y) \in \Omega} |\partial_x^2 \partial_y^i P^i u_1(x, y)|.$$ 

Next, we estimate $|\partial_x^2 \partial_y^i P^i u_1(x, y)|$.

$$|\partial_x^2 \partial_y^i P^i u_1(x, y)| = |\partial_x^2 \partial_y^i \sum_{n=0}^\infty \left( \begin{array}{c} i \\ n \end{array} \right) (\partial_x^2 + \partial_y^2)^n (a \partial_x)^{i-n} u_1(x, y)|$$

$$= |\partial_x^{i+2} \partial_y^i \sum_{n=0}^\infty \left( \begin{array}{c} i \\ n \end{array} \right) a^{i-n} (\partial_x^2 + \partial_y^2)^n u_1(x, y)|$$

$$= |\partial_x^{i+2} \partial_y^i \sum_{n=0}^\infty \left( \begin{array}{c} i \\ n \end{array} \right) a^{i-n} \sum_{m=0}^n \left( \begin{array}{c} n \\ m \end{array} \right) \partial_x^{2m} \partial_y^{2n-2m} u_1(x, y)|$$

$$\leq \sum_{n=0}^i \sum_{m=0}^n \left( \begin{array}{c} i \\ n \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) a^{i-n} |\partial_x^{2m+i+2} \partial_y^{2n-2m} u_1(x, y)|$$

$$\leq C \sum_{n=0}^i \sum_{m=0}^n \left( \begin{array}{c} i \\ n \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) a^{i-n} \frac{(2m + i + 2)! \left( \frac{\sqrt{3}}{2} \right)^{(2n - 2m + j)!}}{R_1^{2m+i+2} R_2^{2n-2m+j}}$$

$$\leq C \sum_{n=0}^i \sum_{m=0}^n \left( \begin{array}{c} i \\ n \end{array} \right) \left( \begin{array}{c} n \\ m \end{array} \right) a^{i-n} \frac{(2m + i + 2)! \left( \frac{\sqrt{3}}{2} \right)^{(2n - 2m + j)!}}{R_2^{2n+i+j+2}}.$$ 

We notice that

$$(2m + i + 2)! (2n - 2m + j)!$$

$$= \left( \begin{array}{c} 2m + i + 2 \\ 2 \end{array} \right) \left( \begin{array}{c} 2n - 2m + j \\ j \end{array} \right) 2!j! (2m + i)! (2n - 2m)!$$

$$\leq \left( \begin{array}{c} 2m + i + 2 \\ 2 \end{array} \right) \left( \begin{array}{c} 2n - 2m + j \\ j \end{array} \right) 2!j! (2n + i)!,$$

where we used the fact that

$$\left( \begin{array}{c} 2n + i \\ 2m + i \end{array} \right) = \frac{(2n + i)!}{(2m + i)! (2n - 2m)!} \geq 1.$$ 

According to [9, Lemma A.1], we have

$$\left( \begin{array}{c} 2m + i + 2 \\ 2 \end{array} \right) \left( \begin{array}{c} 2n - 2m + j \\ j \end{array} \right) \leq \left( \begin{array}{c} 2n + i + j + 2 \\ j + 2 \end{array} \right).$$
This implies
\[
(2m+i+2)! (2n-2m+j)! \leq \left( \frac{2n + i + j + 2}{j + 2} \right) 2! j! (2n + i)!
= \frac{(2n + i + j + 2)! j! (2n + i)!}{(j + 2)! (2n + i)!}
\leq (2n + i + j + 2)!
\]

Combining the above estimates, we infer that
\[
|b_{i,j}| \leq C \frac{(j+2)!}{(j+2)! R^{3j+2}} 2^{pi} (2a)! (2j)! (2a)! i! n^n a_i^{i-n} \frac{(2n+i+j+2)!}{R^{2n+i+j+2}}
\leq C \frac{(j+2)!}{(j+2)! R^{3j+2}} 2^{pi} (2a)! (2j)! (2a)! i! n^n a_i^{i-n} \frac{(2n+i+j+2)!}{R^{2n+i+j+2}}
\]

where \( M_j \) satisfies (40).

By Proposition 6, for any \( j \geq 1 \), there exists a function \( h_j \in G^2([0,T]) \) and a number \( \bar{R} > 1 \) such that
\[
h_j^{(i)}(T) = b_{i,j} \quad \forall \ i \geq 0,
\]
\[
|h_j^{(i)}(t)| \leq M_j \frac{(2j)!}{R^{2j}} \quad \forall \ i \geq 0, \ t \in [0,T].
\] (60)

Pick any \( \tau \in (0,T), s \in (1,2) \) and let
\[
g(t) = 1 - \phi_s \left( \frac{t - \tau}{T - \tau} \right) \text{ for } t \in [0,T].
\]

Setting
\[
z_j(t) = h_j(t)g(t) \quad \forall \ t \in [0,T],
\]
following the method developed in [12, Theorem 3.2], and taking into account the fact that \( s < 2 \), we see that \( z_j \) satisfies
\[
z_j^{(i)}(T) = b_{i,j} \quad \forall \ j \geq 1, \ i \geq 0,
\]
\[
z_j^{(i)}(0) = 0 \quad \forall \ j \geq 1, \ i \geq 0,
\] (61)
\[
|z_j^{(i)}(t)| \leq CM_j \frac{(2j)!}{R^{2j}} \quad \forall \ j \geq 1, \ i \geq 0, \ t \in [0,T].
\]
where \( \hat{R} \) is the same as in (60) and \( C \) is a positive constant independent of \( i \) and \( j \).

Let \( u \) be as in (9). According to (61), we have \( u_0 = 0 \) and

\[
    u(x, y, T) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} g_{i,j}(x)z_j^{(i)}(T)e_j(y) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j}g_{i,j}(x)e_j(y).
\]

By Proposition 5, \( u \) solves system (4)-(8) and \( u \in G^{1,1,2}([-1,0] \times [0,1] \times [0,T]) \).

Let

\[
    h(y, t) = u(-1, y, t) \forall y \in [0,1], \forall t \in [0,T].
\]

Then \( h \in G^{1,2}([0,1] \times [0,T]) \).

Finally, for any \( l \geq 1 \) and \( n \geq 0 \), we have

\[
    \int_0^1 e_l(y)P^n u(0, y, T)dy = \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j}(-1)^n g_{i-n,j}(0)e_j(y)dy = 0
\]

\[
    = \int_0^1 e_l(y)P^n u_1(0, y)dy,
\]

\[
    \int_0^1 e_l(y)\partial_x^n u(0, y, T)dy = \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j}(-1)^n g_{i-n,j}(0)e_j(y)dy = 0
\]

\[
    = \int_0^1 e_l(y)\partial_x^n u_1(0, y)dy,
\]

\[
    \int_0^1 e_l(y)\partial_x^n u_1(0, y)dy = \int_0^1 e_l(y) \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} b_{i,j}(-1)^n g_{i-n,j}(0)e_j(y)dy = 0
\]

Since \( u(\cdot, \cdot, T), u_1 \in X \), it follows from Lemma 4.2 that

\[
    \int_0^1 e_l(y)[\partial_x^m u(0, y, T) - \partial_x^m u_1(0, y)]dy = 0 \quad \forall l \geq 1, \forall m \geq 0,
\]

and hence

\[
    \partial_x^m u(0, y, T) - \partial_x^m u_1(0, y) = 0 \quad \forall m \geq 0, \forall y \in [0,1].
\]

Since the map \( x \to u(x, y, T) - u_1(x, y) \) is in \( G^1([-1,0]) \) (i.e. is analytic) for any \( y \in [0,1] \), we infer that

\[
    u(x, y, T) = u_1(x, y) \forall (x, y) \in [-1,0] \times [0,1].
\]

The proof of Theorem 1.2 is complete. \( \square \)

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