Random walks with unbounded jumps among random conductances II: Conditional quenched CLT

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Abstract

We study a one-dimensional random walk among random conductances, with unbounded jumps. Assuming the ergodicity of the collection of conductances and a few other technical conditions (uniform ellipticity and polynomial bounds on the tails of the jumps) we prove a quenched conditional invariance principle for the random walk, under the condition that it remains positive until time $n$. As a corollary of this result, we study the effect of conditioning the random walk to exceed level $n$ before returning to 0 as $n \to \infty$.

Keywords: ergodic environment, unbounded jumps, Brownian meander, 3-dimensional Bessel process, hitting probabilities, crossing time, uniform CLT

AMS 2000 subject classifications: 60J10, 60K37

1 Introduction and results

In this paper, we study one-dimensional random walks among random conductances, with unbounded jumps. This is the continuation of the paper [15], where we proved a uniform quenched invariance principle for this model, where “uniform” refers to the starting position of the walk (i.e., one obtains the same estimates on the speed of convergence as long as this position lies in a certain interval around the origin). Here, our main results concern the (quenched) limiting law of the trajectory of the random walk $(X_n, n = 0, 1, 2, \ldots)$ starting from the origin up to time $n$, under condition that it remains positive at the moments $1, \ldots, n$. In Theorem 1.1 we prove that, after suitable rescaling, for a.e. environment it converges to the Brownian meander process, which is, roughly speaking, a Brownian motion conditioned on staying positive up to some finite time, and the main result of the paper [15] will be an important tool for proving Theorem 1.1.

This kind of problem was extensively studied for the case of space-homogeneous random walk, i.e., when one can write $X_n = \xi_1 + \cdots + \xi_n$, where the $\xi$-s are i.i.d. random variables. These random variables are usually assumed to have expectation 0, and to possess some (nice) tail properties. Among the first papers on the subject we mention [1] and [16], where the convergence of the rescaled trajectory to the Brownian meander was proved. Afterwards, finer results (such as local limit theorems, convergence to other processes if the original walk is in the domain of attraction of some stable Lévy process, etc.) for space-homogeneous random walks were obtained, see e.g. [2, 5, 6, 20] and references therein. Also, it is worth noting that in the paper [4] the approach of [16] was substantially simplified by taking
advantage of the homogeneity of the random walk; however, since in our case the random walk is not space-homogeneous, we rather use methods similar to those of [16].

Also, as mentioned in [15], another motivation for this work came from Knudsen billiards in random tubes, see [2][3][4][5]. We refer to Section 1 of [15] for the discussion on the relationship of the present model to random billiards.

Now, we define the model formally. For \( x, y \in \mathbb{Z} \), we denote by \( \omega_{x,y} = \omega_{y,x} \) the conductance between \( x \) and \( y \). Define \( \theta_z \omega_{x,y} = \omega_{x+z,y+z} \), for all \( z \in \mathbb{Z} \). Note that, by Condition K below, the vectors \( \omega_x \) are elements of the Polish space \( l^2(\mathbb{Z}) \). We assume that \( (\omega_x)_{x \in \mathbb{Z}} \) is a stationary ergodic (with respect to the family of shifts \( \theta \)) sequence of random vectors; \( \mathbb{P} \) stands for the law of this sequence. The collection of all conductances \( \omega = (\omega_{x,y}, x, y \in \mathbb{Z}) \) is called the environment. For all \( x \in \mathbb{Z} \), define \( C_x = \sum_y \omega_{x,y} \). Given that \( C_x < \infty \) for all \( x \in \mathbb{Z} \) (which is always so by Condition K below), the random walk \( X \) in random environment \( \omega \) is defined through its transition probabilities

\[
p_\omega(x, y) = \frac{\omega_{x,y}}{C_x};
\]

that is, if \( \mathbb{P}_\omega^x \) is the quenched law of the random walk starting from \( x \), we have

\[
\mathbb{P}_\omega^x[X_0 = x] = 1, \quad \mathbb{P}_\omega^x[X_{k+1} = z \mid X_k = y] = p_\omega(y, z).
\]

Clearly, this random walk is reversible with the reversible measure \( (C_x, x \in \mathbb{Z}) \). Also, we denote by \( \mathbb{E}_\omega^x \) the quenched expectation for the process starting from \( x \). When the random walk starts from 0, we use shortened notations \( \mathbb{P}_\omega, \mathbb{E}_\omega \).

In order to prove our results, we need to make two technical assumptions on the environment:

**Condition E.** There exists \( \kappa > 0 \) such that, \( \mathbb{P}\text{-a.s.}, \omega_{0,1} \geq \kappa \).

**Condition K.** There exist constants \( K, \beta > 0 \) such that \( \mathbb{P}\text{-a.s.}, \omega_{0,y} \leq \frac{K}{1+y^{1+\beta}} \), for all \( y \geq 0 \).

For future reference, note that combining Conditions E and K we have that there exists \( \hat{\kappa} > 0 \) such that \( \mathbb{P}\text{-a.s.}, \sum_{y \in \mathbb{Z}} \omega_{0,y} \leq \hat{\kappa}^{-1} \).

We decided to formulate Condition E this way because, due to the fact that this work was motivated by random billiards, the main challenge was to deal with the long-range jumps. It is plausible that Condition E could be relaxed to some extent; however, for the sake of cleaner presentation of the argument, we prefer not trying to deal with both long-range jumps and the lack of nearest-neighbor ellipticity.

Next, for all \( n \geq 1 \), we define the continuous map \( Z^n = (Z^n(t), t \in \mathbb{R}_+) \) as the natural polygonal interpolation of the map \( k/n \mapsto \sigma^{-1}n^{-1/2}X_k \) (with \( \sigma \) from Theorem 1.1 in [15]). In other words,

\[
\sigma \sqrt{n} Z^n_t = X_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}
\]

with \( \lfloor \cdot \rfloor \) the integer part. Also, we denote by \( W \) the standard Brownian motion.

Now, let \( \mathring{\tau} = \inf\{k \geq 1 : X_k \in (-\infty, 0]\} \) and \( \Lambda_n = \{\mathring{\tau} > n\} = \{X_k > 0 \text{ for all } k = 1, \ldots, n\} \). Consider the conditional quenched probability measure \( Q^n_\omega[\cdot] = \mathbb{P}_\omega[\cdot \mid \Lambda_n] \), for all \( n \geq 1 \). For each \( n \), the random map \( Z^n \) induces a probability measure \( \mu_n^\omega \) on \( (C[0, 1], B_1) \), where \( B_1 \) is the Borel \( \sigma \)-algebra on \( C[0, 1] \) with the supremum norm: for any \( A \in B_1 \),

\[
\mu_n^\omega(A) = Q^n_\omega[Z^n \in A].
\]
Let us next recall the formal definition of the Brownian meander $W^+$. For this, let $W$ be a standard Brownian motion and define $\tau_1 = \sup\{s \in [0, 1] : W(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$. Then,

$$W^+(s) = \Delta_1^{-1/2}|W(\tau_1 + s\Delta_1)|, \quad 0 \leq s \leq 1.$$ 

Now, we are ready to formulate the quenched invariance principle for the random walk conditioned to stay positive, which is the main result of this paper:

**Theorem 1.1** Under Conditions E and K, we have that, $\mathbb{P}$-a.s., $\mu^n_\omega$ tends weakly to $P_{W^+}$ as $n \to \infty$, where $P_{W^+}$ is the law of the Brownian meander $W^+$ on $C[0, 1]$.

As a corollary of Theorem 1.1 we obtain a limit theorem for the process conditioned on crossing a large interval. Define $\dot{\tau}_n = \inf\{k \geq 0 : X_k \in [n, \infty)\}$ and $A'_n = \{\dot{\tau}_n < \tau\}$. We also define $T_n = \inf\{t > 0 : Z^n t = \sigma^{-1}\}$ and the stopped process $Y^n = Z^n_{\tau_n T_n}$. Denoting by $B_3$ the three-dimensional Bessel process (we recall that $\sqrt{1 - s^2}$ belongs to $A'\omega$, which is the main result of this paper: 

**Corollary 1.1** Assume Conditions E and K. We have that, $\mathbb{P}$-a.s., under the law $\mathbb{P}_\omega[\cdot \mid A'_n]$, the couple $(Y^n, T_n)$ converges in law to $(B_3(\cdot \wedge \dot{\sigma}_1), \dot{\sigma}_1)$ as $n \to \infty$.

In the next section, we prove some auxiliary results which are necessary for the proof of Theorem 1.1. Then, in Section 3 we give the proof of Theorem 1.1. Finally, in Section 4 we give the proof of Corollary 1.1.

We will denote by $K_1, K_2, \ldots$ the “global” constants, that is, those that are used all along the paper and by $\gamma_1, \gamma_2, \ldots$ the “local” constants, that is, those that are used only in the subsection in which they appear for the first time. For the local constants, we restart the numeration in the beginning of each subsection. Besides, to simplify notations, if $x$ is not integer, $\mathbb{P}_\omega^x$ must be understood as $\mathbb{P}_\omega^{[x]}$.

## 2 Auxiliary results

In this section, we will prove some technical results that will be needed later to prove Theorem 1.1. Let us introduce the following notations. If $A \subset \mathbb{Z}$,

$$\tau_A = \inf\{n \geq 0 : X_n \in A\} \quad \text{and} \quad \tau_A^+ = \inf\{n \geq 1 : X_n \in A\}. \quad (2)$$

Whenever $A = \{x\}, x \in \mathbb{Z}$, we write $\tau_x$ (respectively, $\tau^+_x$) instead of $\tau_{\{x\}}$ (respectively, $\tau_{\{x\}}^+$).

### 2.1 Auxiliary environments

From some fixed environment $\omega$, we are going to introduce three derived environments denoted by $\omega^{(1)}, \omega^{(2)}$ and $\omega^{(3)}$ which will be important tools for the proofs of the lemmas in the rest of this section.

Fix two disjoint intervals $B = (-\infty, 0]$ and $E = [N, \infty)$ of $\mathbb{Z}$. For some realization $\omega$ of the environment, consider the new environment $\omega^{(1)}$ obtained from $\omega$ by deleting all the conductances $\omega_{x,y}$ if $x$ and $y$ belong to $(B \setminus \{0\}) \cup E$. The reversible measure (up to a constant factor) on this new environment $\omega^{(1)}$ is given by

$$C_0^{(1)} = C_0,$$
Now, we define \( C_{x}^{(1)} = C_{x} \), if \( x \notin B \cup E \),
\[
C_{x}^{(1)} = \sum_{y \notin (B \cup \{0\}) \cup E} \omega_{x,y},
\]
otherwise.

Now, we define \( C_{B}^{(1)} = \sum_{x \in B} C_{x}^{(1)} \) and for all \( x \in B \), \( \pi_{B}(x) = C_{x}^{(1)}/C_{B}^{(1)} \). Observe that by Conditions E and K, \( C_{B}^{(1)} \) is positive and finite \( \mathbb{P} \)-a.s. Hence \( \pi_{B} \) is \( \mathbb{P} \)-a.s. a probability measure on \( B \). In the same way we define \( \pi_{E} \) on \( E \). For the sake of simplicity we denote \( P_{\omega}^{B} \) (respectively, \( P_{\omega}^{E} \)) instead of \( P_{\omega_{(1)}}^{B} \) (respectively, \( P_{\omega_{(1)}}^{E} \)) for the random walk on \( \omega^{(1)} \) starting with initial distribution \( \pi_{B} \) (respectively, \( \pi_{E} \)).

The same convention will be adopted for environments \( \omega^{(2)} \) and \( \omega^{(3)} \) defined below.

From the environment \( \omega^{(1)} \), we now construct a new environment \( \omega^{(2)} \) by setting if \( x > 0, y > 0 \),
\[
\omega_{x,0}^{(2)} = \sum_{y \in B} \omega_{x,y}^{(1)}, \quad \omega_{0,0}^{(2)} = \sum_{y \in B} \omega_{y,0}^{(1)}, \quad \omega_{x,y}^{(2)} = \omega_{x,y}^{(1)}
\]
and \( \omega_{x,y}^{(2)} = 0 \) otherwise. Defining the reversible measure associated to \( \omega^{(2)} \) as \( C_{x}^{(2)} = \sum_{y \in \mathbb{Z}} \omega_{x,y}^{(2)} \), for \( x \in \mathbb{Z} \), observe in particular that \( C_{0}^{(2)} = C_{B}^{(1)} \) and \( C_{x}^{(2)} = C_{x}^{(1)} \) for \( x > 0 \).

From the environment \( \omega^{(1)} \), we finally create a last environment \( \omega^{(3)} \) by setting if \( x \in (0, N) \),
\[
\omega_{x,N}^{(3)} = \sum_{y \in E} \omega_{x,y}^{(1)}, \quad \omega_{x,0}^{(3)} = \sum_{y \in B} \omega_{x,y}^{(1)}.
\]
Then, let
\[
\omega_{N,0}^{(3)} = \sum_{y \in E} \omega_{0,y}^{(1)}, \quad \omega_{0,0}^{(3)} = \sum_{y \in B} \omega_{0,y}^{(1)}.
\]
For \( x \in (0, N) \) and \( y \in (0, N) \) we just set \( \omega_{x,y}^{(3)} = \omega_{x,y}^{(1)} \) and \( \omega_{x,y}^{(3)} = 0 \) in all other cases. We define the reversible measure associated to \( \omega^{(3)} \) as \( C_{x}^{(3)} = \sum_{y \in \mathbb{Z}} \omega_{x,y}^{(3)} \), for \( x \in \mathbb{Z} \). Observe in particular that \( C_{0}^{(3)} = C_{B}^{(1)}, C_{N}^{(3)} = C_{E}^{(1)} \) and \( C_{x}^{(3)} = C_{x}^{(1)} \) for \( x \in (0, N) \).

### 2.2 Crossing probabilities and estimates on the conditional exit distribution

Fix \( \varepsilon > 0, n \in \mathbb{N} \) such that \( \varepsilon \sqrt{n} \geq 1 \) and take \( N = \lfloor \varepsilon \sqrt{n} \rfloor \) (\( N \) is from section 2.1). Then define the event \( A_{\varepsilon,n} = \{ \tau_{E} < \tau_{B}^{+} \} \) (\( B \) and \( E \) are from section 2.1). For an arbitrary positive integer \( M \) define \( I_{M} = [N, N + M] \).

**Lemma 2.1** For all \( \eta > 0 \) there exists \( M > 0 \) such that \( \mathbb{P} \)-a.s.,
\[
P_{\omega}[X_{\tau_{E}} \in I_{M} \mid A_{\varepsilon,n}] \geq 1 - \eta, \quad \text{for all } n \text{ such that } N > 1.
\]

**Proof.** The proof of this lemma is very similar to the proof of Proposition 2.3 of [15]. Here, we just give the first steps of the proof and then indicate the exact place where it matches with the proof of Proposition 2.3 of [15]. First, we write
\[
P_{\omega}[X_{\tau_{E}} \in I_{M} \mid A_{\varepsilon,n}] = 1 - P_{\omega}[X_{\tau_{E}} \notin I_{M} \mid A_{\varepsilon,n}] = 1 - \sum_{y > N + M} P_{\omega}[X_{\tau_{E}} = y \mid A_{\varepsilon,n}].
\]
By definition of $\omega^{(1)}$ (cf. section 2.1), we can couple the random walks in environments $\omega$ and $\omega^{(1)}$ to show that $P_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = P_{\omega}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$. Then, by construction of $\omega^{(2)}$, we can couple the random walks in environments $\omega^{(1)}$ and $\omega^{(2)}$ to show that $P_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = P_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$.

Thus, we obtain

$$P_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \sum_{x \in B} \pi_B(x) P_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$$

$$= \sum_{x \in B} \pi_B(x) P_{\omega^{(1)}}[\tau_E = y \mid A_{\varepsilon,n}]$$

$$\geq \pi_B(0) P_{\omega^{(1)}}[\tau_E = y \mid A_{\varepsilon,n}]$$

$$= \frac{C_0}{C_B} P_{\omega}[X_{\tau_E} = y \mid A_{\varepsilon,n}].$$

Thus, by (3) we obtain

$$P_{\omega}[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}] \geq 1 - \frac{C_B^{(1)}}{C_0} \sum_{y > A_{\varepsilon,n} + M} P_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}].$$

Note that, by Condition K and (11), $C_B^{(1)}/C_0 \leq \gamma_1$ for some constant $\gamma_1$. The terms $P_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$ can be treated in the same way as the terms $P_{\omega}[X_{\tau_E} = y \mid A_{\varepsilon}]$ of equation (2.6) in [15]. In particular, following the reasoning antecedent equation (2.9) in [15], we can show that

$$P_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \frac{C_B^{(1)} P_{\omega^{(2)}}[\gamma_0 < \tau_E^{(1)}]}{C_0^{(1)} P_{\omega^{(2)}}[\gamma_0 < \tau_E^{(1)}]}.$$

Then, the numerator and denominator of the above equation can be treated by using the same techniques as those used to treat (2.9) in [15].

Lemma 2.2 There exists a positive constant $K_1$ such that we have, $P$-a.s., $P_{\omega}[A_{\varepsilon,n}] \geq K_1 N^{-1}$ for all $n$ such that $N > 1$.

Proof. Recall that $P_{\omega}[A_{\varepsilon,n}] = P_{\omega}[\tau_E < \tau_B^{(1)}]$. We can couple the random walks in environments $\omega$ and $\omega^{(1)}$ (cf. section 2.1) to show that $P_{\omega}[\tau_E < \tau_B^{(1)}] = P_{\omega^{(1)}}[\tau_E < \tau_B^{(1)}]$.

Let us denote by $\Gamma_{z',z''}$ the set of finite paths $(z', z_1, \ldots, z_k, z'')$ such that $z_i \notin B \cup E \cup \{z', z''\}$ for all $i = 1, \ldots, k$. Let $\gamma = (z', z_1, \ldots, z_k, z'') \in \Gamma_{z',z''}$ and define

$$P_{\omega^{(1)}}[\gamma] := P_{\omega^{(1)}}[X_1 = z_1, \ldots, X_k = z_k, X_{k+1} = z''].$$

By reversibility we obtain

$$P_{\omega^{(1)}}[\tau_E < \tau_B^{(1)}] = \sum_{z \in E} \sum_{\gamma \in \Gamma_{z'}} P_{\omega^{(1)}}[\gamma]$$

$$= \sum_{z \in E} \sum_{\gamma \in \Gamma_{z'}} \frac{C_z^{(1)}}{C_0} P_{\omega^{(1)}}[\gamma]$$

$$= \frac{C_E}{C_0} \sum_{z \in E} \pi_{E}(z) \sum_{\gamma \in \Gamma_{z'}} P_{\omega^{(1)}}[\gamma]$$
Now, define \( B' = (-\infty, 1] \). We have
\[
P^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0] = P^E_{\omega(1)}[\tau_B < \tau^+_E, \tau_{B'} < \tau^+_E, X_{\tau_B} = 0]
\]
\[
= P^E_{\omega(1)}[\tau_{B'} < \tau^+_E]P^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0 | \tau_{B'} < \tau^+_E]
\]
\[
\geq P^E_{\omega(1)}[\tau_B < \tau^+_E]P^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0 | \tau_{B'} < \tau^+_E].
\]

Let us treat the term \( p^E_{\omega(1)}[\tau_B < \tau^+_E] \). By definition of \( \omega^3 \) (cf. section 2.1), we can couple the random walks in environments \( \omega(1) \) and \( \omega^3 \) to show that \( p^E_{\omega(1)}[\tau_B < \tau^+_E] = p^N_{\omega(3)}[\tau_0 < \tau^+_N] \). We obtain
\[
C^E(1)p^E_{\omega(1)}[\tau_B < \tau^+_E] = C^E(1)p^N_{\omega(3)}[\tau_0 < \tau^+_N] = C^E(3)p^N_{\omega(3)}[\tau_0 < \tau^+_N] = C_{\text{eff}}(1, N)
\]

where \( C_{\text{eff}}(1, N) \) is the effective conductance between the points 1 and \( N \) of the electrical network associated to \( \omega(3) \) (cf. [12], section 3.4). Using Condition E, we obtain
\[
C_{\text{eff}}(1, N) \geq \left( \sum_{i=1}^{N-1} \omega_{i,i+1}^{-1} \right)^{-1} \geq \frac{\kappa}{N-1}.
\]

Therefore, there exists a constant \( \gamma_1 \) such that, whenever \( N > 1 \)
\[
C^E(1)p^E_{\omega(1)}[\tau_{B'} < \tau^+_E] \geq \frac{\gamma_1}{N}.
\]

Let us treat the term \( p^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0 | \tau_{B'} < \tau^+_E] \). We have by the Markov property
\[
p^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0 | \tau_{B'} < \tau^+_E]
\]
\[
= \sum_{y \in \{0, 1\}} p^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0, X_{\tau_{B'}} = y | \tau_{B'} < \tau^+_E]
\]
\[
= \sum_{y \in \{0, 1\}} p^E_{\omega(1)}[\tau_B < \tau^+_E, X_{\tau_B} = 0, X_{\tau_{B'}} = y, \tau_{B'} < \tau^+_E]p^E_{\omega(1)}[X_{\tau_{B'}} = y | \tau_{B'} < \tau^+_E]
\]
\[
= \sum_{y \in \{0, 1\}} p^B_{\omega}[\tau_B < \tau_E, X_{\tau_B} = 0]p^E_{\omega(1)}[X_{\tau_{B'}} = y | \tau_{B'} < \tau^+_E]
\]
\[
\geq \min_{y \in \{0, 1\}} p^B_{\omega}[\tau_B < \tau_E, X_{\tau_B} = 0]p^E_{\omega(1)}[X_{\tau_{B'}} = y | \tau_{B'} < \tau^+_E]
\]
\[
\geq p^B_{\omega}[\tau_B < \tau_E, X_{\tau_B} = 0].
\]

By Condition E and (11), this last probability is bounded from below by the constant \( \kappa \hat{\kappa} \). Thus, combining this last result with \( \text{(4)}, \text{(5)}, \text{(7)} \) and, since by (11) we have \( C_0 \leq \hat{\kappa}^{-1} \), it follows that \( P \)-a.s.,
\[
P_{\omega}[A_{\epsilon,n}] \geq \frac{\gamma_1 \hat{\kappa}^2}{N}.
\]

This concludes the proof of Lemma 2.2. \( \square \)

**Lemma 2.3** There exists a positive constant \( K_2 \) such that we have, \( P \)-a.s.,
\[
e_{\omega}[\tau^+_B \land \tau_E] \leq K_2 N
\]
for all \( n \) such that \( N > 1 \).
Proof. First notice that by construction of \( \omega(1) \) (cf. section 2.1), we can couple the random walks in environments \( \omega \) and \( \omega(1) \) to show that \( E_{\omega} [\tau^+_B \land \tau_E] \leq E_{\omega(1)} [\tau^+_B \land \tau_E] \). Hence, we obtain

\[
E^B_{\omega(1)} [\tau^+_B \land \tau_E] = \sum_{y \in B} \pi_B(y) E^y_{\omega(1)} [\tau^+_B \land \tau_E] = \pi_B(0) E^{0}_{\omega(1)} [\tau^+_B \land \tau_E] + \sum_{y \in B \setminus \{0\}} \pi_B(y) E^y_{\omega(1)} [\tau^+_B \land \tau_E] \\
\geq \pi_B(0) E_{\omega} [\tau^+_B \land \tau_E].
\]

Therefore, we obtain

\[
E_{\omega} [\tau^+_B \land \tau_E] \leq \frac{E^B_{\omega(1)} [\tau^+_B \land \tau_E]}{\pi_B(0)}.
\]

Then, observe that

\[
E^B_{\omega(1)} [\tau^+_B \land \tau_E] = E_{\omega(0)} [\tau^+_0 \land \tau_N].
\]

We are going to bound the right-hand side term of (9) from above. Before this, we make a brief digression to study the invariant measure of a particular process of interest.

Consider the following particle system in continuous time on the interval \([0, N]\) of \( \mathbb{Z} \). Suppose that we have injection (according to some Poisson process) and absorption of particles at states 0 and \( N \). Once injected, particles move according to transition rates given by \( q_{x,y} = \omega^{(3)}_{x,y}/C^{(3)}_x \), for \((x,y) \in \{0, \ldots, N\}^2\), until they reach 0 or \( N \). Suppose that injections at 0 and \( N \) happen accordingly to independent Poisson processes with rates respectively \( \lambda_0 = C^{(3)}_0 \) and \( \lambda_N = C^{(3)}_N \). We are interested in the continuous time Markov process \((\eta(t) = (\eta_0(t), \ldots, \eta_N(t)), t \geq 0)\) with state space \( \Omega = \mathbb{Z}_{\leq 0}^{\{0, \ldots, N\}} \) where \( \eta_i(t) \) represents the number of particles in \( i \) at time \( t \). Hereafter, for \((i, j) \in \{0, \ldots, N\}^2\), we will use the symbol \( \eta^{i,j} \) to denote the configuration obtained from \( \eta \) by moving a particle from site \( i \) to site \( j \), i.e., if for example \( i < j \), \( \eta^{i,j} = (\eta_0, \ldots, \eta_i-1, \ldots, \eta_j+1, \ldots, \eta_N) \). We also define \( \eta^{i,+} = (\eta_0, \ldots, \eta_i+1, \ldots, \eta_N) \) and \( \eta^{i,-} = (\eta_0, \ldots, \eta_i-1, \ldots, \eta_N) \) for \( i \in \{0, \ldots, N\} \). The generator of this process defined by its action on functions \( f : \Omega \to \mathbb{R} \) is given by

\[
\mathcal{L} f(\eta) = \lambda_0(\eta_0)(f(\eta^{0,+}) - f(\eta)) + \sum_{i=0}^{N} \eta_i q_{i,0} (f(\eta^{+,i}) - f(\eta)) \\
+ \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \eta_j q_{j,i} (f(\eta^{j,i}) - f(\eta)) \\
+ \lambda_N (f(\eta^{N,+}) - f(\eta)) + \sum_{i=0}^{N} \eta_i q_{i,N} (f(\eta^{+,i}) - f(\eta)).
\]

Let \( \mu = \otimes_{i=1}^{N} \mu_i \) be the product measure of laws \( \mu_i \) where for each \( i \in \{0, \ldots, N\} \), \( \mu_i \) is a Poisson law with parameter \( C^{(3)}_i \). We can check that for any configurations \( \eta, \eta' \in \Omega \),

\[
L(\eta, \eta') \mu(\eta) = L(\eta', \eta) \mu(\eta')
\]

where \( L(\eta, \eta') \) is the transition rate from the configuration \( \eta \) to \( \eta' \), i.e., \( L(\eta, \eta') = \mathcal{L} f(\eta) \) with \( f(\eta) = \delta_{\eta, \eta'} \). This implies that the probability measure \( \mu \) is reversible and invariant for the Markov process \( \eta \).

Now, consider the model above with injection at rate \( \lambda_0 \) and absorption at 0 and only absorption (without injection) at \( N \). Such a system can be considered as a \( M/G/\infty \) queue where the customers
arrive according to a Poisson process of rate $\lambda_0$ and the service time law is that of the lifetime of a particle in the interval $[0,N]$. Thus, the expected service time of a customer, denoted by $E[T]$, equals $E_{\omega}(3)[\tau_0^+ \land \tau_N]$. By Little’s formula (see e.g. Section 5.2 of [11]) we have

$$E[T] = \frac{E[R]}{\lambda_0}$$

where $E[R]$ is the mean number of particles in the queue in the stationary regime. By a coupling argument, we can see that the distribution of the number of customers in the system in the stationary regime is stochastically dominated by the distribution of the total number of particles in the interval $[0,N]$ in the stationary regime for the particle system with both injection and absorption of particles at states 0 and $N$. It is not difficult to see that this last distribution is $\mu_0 \ast \cdots \ast \mu_N$ (here $\ast$ is the convolution product of measures). Therefore, combining the foregoing observations, we obtain

$$E_{\omega}(3)[\tau_0^+ \land \tau_N] = E[T] = \frac{E[R]}{\lambda_0} \leq \frac{1}{\lambda_0} \sum_{x \in \mathbb{Z}} x \mu_0 \ast \cdots \ast \mu_N(x) = \frac{1}{C_0^{(3)}} \sum_{x=0}^{N} C_x^{(3)}.$$  \hfill (12)

Finally, by (8), (11) and (12) we obtain

$$E_\omega[\tau_B^+ \land \tau_E] \leq \frac{1}{C_0^{(3)}} \sum_{x=0}^{N} C_x^{(3)}.$$  

By Conditions E and K, it holds that there exists a positive constant $K_2$ such that $P$-a.s.,

$$E_\omega[\tau_B^+ \land \tau_E] \leq K_2 N.$$  

This concludes the proof of Lemma 2.3.

\begin{flushright}
\square
\end{flushright}

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To simplify notations, we consider $\sigma = 1$. Our strategy to prove Theorem 1.1 is to use Theorems 3.6 and 3.10 of [13] (which are restated here as Theorems 3.1 and 3.2). These theorems give equivalent conditions for the tightness and convergence of finite dimensional distributions of the conditioned processes $Z^n$ that are easier to verify in our case. In [13], these theorems are stated in a quite general form that can be simplified here. Also, since in our problem all the processes considered have continuous trajectories, we will transpose these theorems on $C[0,1]$ (instead of $D[0,1]$, the Skorokhod space):

**Theorem 3.1** The sequence of measures $(\mu^n_\omega, n \geq 1)$ is tight if and only if

$$\lim_{x \to \infty} \limsup_{n \to \infty} P_\omega[Z^n_1 > x \mid \Lambda_n] = 0 \quad \text{and} \quad \lim_{t \to 0} \limsup_{n \to \infty} P_\omega[Z^n_t > h \mid \Lambda_n] = 0 \quad \text{for each } h > 0.$$  \hfill (13)

We recall that the measures $\mu^n_\omega$ are defined in the introduction. Now, let us define the following conditions:

(i) if $x_n \to x$, then $(P_\omega^{x_n} \sqrt{n}[Z^n \in \cdot], n \geq 1)$ tends weakly to $P^x[W \in \cdot]$ in $C[0,1]$, 

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(ii) let $x_n \geq 0$, for all $n \geq 1$, then $\lim_{n \to \infty} \mathbb{P}_\omega \supseteq \mathbb{P}_\omega \supseteq P_[0, \infty) [Z_s^n > 0, s \leq t_n] = P^x [W_s > 0, s \leq t]$, whenever $x_n \to x$ and $t_n \to t > 0$.

**Theorem 3.2** Suppose (i)-(ii) hold and $(\mu^0_n, n \geq 1)$ is tight. Then, $(\mu^1_n, n \geq 1)$ tends weakly to $W^+$ if and only if

$$
\lim \liminf_{h \to 0} \mathbb{P}_\omega [Z_t^n > h \mid \Lambda_n] = 1 \quad \text{for all } t > 0.
$$

In our case, condition (i) is an immediate consequence of the quenched Uniform CLT (cf. Theorem 1.2 of [15]) which in the rest of this paper will be referred as UCLT. For condition (ii), let $\varepsilon > 0$, we have for all $n$ large enough

$$
P_{\omega}^{x_n} \supseteq [Z_s^n > 0, s \leq t + \varepsilon] \leq P_{\omega}^{x_n} \supseteq [Z_s^n > 0, s \leq t_n] \leq P_{\omega}^{x_n} \supseteq [Z_s^n > 0, s \leq t - \varepsilon].
$$

Thus, condition (ii) follows from the UCLT and the continuity in $t$ of $P^x [W_s > 0, s \leq t]$. Our next step is to obtain the weak limit of the sequence $(P_{\omega}^{x_n} [Z_t^n \in \cdot \mid \Lambda_n], n \geq 1)$. This is the object of Proposition 3.3. Then, we obtain the weak limit of $(P_{\omega}^{x_n} [Z_t^n \in \cdot \mid \Lambda_n], n \geq 1)$ for all $t \in (0, 1)$. This is done in Proposition 3.4. In the last step, we check that (13), (14), and (15) hold to end the proof of Theorem 1.1.

At this point, let us recall some notations of Section 2.2. Fix $\varepsilon > 0$ and define $N = [\varepsilon \sqrt{n}]$. Let $B = (-\infty, 0]$ and $E = [N, +\infty)$. Then, define the event $A_{\varepsilon,n} = \{\tau_B^+ < \tau_E^+\}$. For an arbitrary positive integer $M$ define $I_M = [N, N + M]$. First, let us prove

**Proposition 3.1** We have $\mathbb{P}$-a.s.,

$$
\lim_{n \to \infty} \mathbb{P}_\omega [Z_t^n > x \mid \Lambda_n] = \exp(-x^2/2), \quad \text{for all } x \geq 0.
$$

**Proof.** For notational convenience, let us only treat the case $x = 1$. The generalization to any $x \geq 0$ is straightforward. Fix $\varepsilon \in (0, 1)$, $\delta \in (0, 1)$ and write

$$
\mathbb{P}_\omega [X_n > \sqrt{n} \mid \Lambda_n] = \frac{1}{\mathbb{P}_\omega [\Lambda_n]} \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n]
$$

$$
= \frac{1}{\mathbb{P}_\omega [\Lambda_n]} \left( \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E^+} \in I_M] + \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E^+} \notin I_M] \right)
$$

$$
= \frac{1}{\mathbb{P}_\omega [\Lambda_n]} \left( \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E^+} \in I_M, \tau_E > \delta n] + \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E^+} \in I_M, \tau_E \leq \delta n] + \mathbb{P}_\omega [X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E^+} \notin I_M] \right)
$$

$$
= \frac{\mathbb{P}_\omega [A_{\varepsilon,n}]}{\mathbb{P}_\omega [\Lambda_n]} \left( \mathbb{P}_\omega [X_{\tau_E^+} \in I_M \mid A_{\varepsilon,n}] \mathbb{P}_\omega [\tau_E > \delta n \mid X_{\tau_E^+} \in I_M, A_{\varepsilon,n}] \times \mathbb{P}_\omega [X_n > \sqrt{n}, \Lambda_n, X_{\tau_E^+} \in I_M, A_{\varepsilon,n}, \tau_E > \delta n] + \mathbb{P}_\omega [X_{\tau_E^+} \in I_M \mid A_{\varepsilon,n}] \mathbb{P}_\omega [\tau_E \leq \delta n \mid X_{\tau_E^+} \in I_M, A_{\varepsilon,n}] \times \mathbb{P}_\omega [X_n > \sqrt{n}, \Lambda_n, X_{\tau_E^+} \in I_M, A_{\varepsilon,n}, \tau_E \leq \delta n] + \mathbb{P}_\omega [X_n > \sqrt{n}, \Lambda_n, X_{\tau_E^+} \notin I_M \mid A_{\varepsilon,n}] \right).
$$

(17)
Informally, the rest of the proof consists in using the decomposition (17) in order to find good lower and upper bounds $L_n$ and $U_n$ for $P_\omega[X_n > \sqrt{n} \mid \Lambda_n]$ such that $U_n/L_n \to 1$ as $n \to \infty$. We start with the upper bound. Let us write

$$P_\omega[X_n > \sqrt{n} \mid \Lambda_n] \leq \frac{P_\omega[A_{\epsilon,n}]}{P_\omega[\Lambda_n]} \left( P_\omega[X_{\tau_E} \notin I_M \mid A_{\epsilon,n}] + P_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}] + P_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}, \tau_E \leq \delta n] \right).$$

(18)

Observe that we can bound the term $P_\omega[X_{\tau_E} \notin I_M \mid A_{\epsilon,n}]$ from above using Lemma 2.1: let $\eta > 0$, then we can choose $M$ large enough in such a way that

$$P_\omega[X_{\tau_E} \notin I_M \mid A_{\epsilon,n}] \leq \eta.$$

(19)

Next, let us bound the other terms of the right-hand side of (18) from above. For $P_\omega[A_{\epsilon,n}]/P_\omega[\Lambda_n]$, we write

$$P_\omega[A_{\epsilon,n} \mid \Lambda_n, A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n] = P_\omega[A_{\epsilon,n}]P_\omega[X_{\tau_E} \in I_M \mid A_{\epsilon,n}]P_\omega[\tau_E \leq \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}]P_\omega[\Lambda_n \mid A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n].$$

(20)

Hence,

$$\frac{P_\omega[\Lambda_n]}{P_\omega[A_{\epsilon,n}]} \geq P_\omega[X_{\tau_E} \in I_M \mid A_{\epsilon,n}]P_\omega[\tau_E \leq \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}]P_\omega[\Lambda_n \mid A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n].$$

Again, we use Lemma 2.1 to bound the term $P_\omega[X_{\tau_E} \in I_M \mid A_{\epsilon,n}]$ from below. For the term $P_\omega[\tau_E \leq \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}]$ we write

$$P_\omega[\tau_E \leq \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}] = 1 - P_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}]$$

(21)

and

$$P_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}] = \frac{P_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\epsilon,n}]}{P_\omega[X_{\tau_E} \in I_M \mid A_{\epsilon,n}]} = \frac{P_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\epsilon,n}]}{P_\omega[X_{\tau_E} \in I_M \mid A_{\epsilon,n}]P_\omega[A_{\epsilon,n}]}.$$ 

(22)

We first treat the numerator of (22). By Chebyshev’s inequality we obtain

$$P_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\epsilon,n}] \leq P_\omega[\tau_E^+ \wedge \tau_E > \delta n] \leq \frac{E_\omega[\tau_E^+ \wedge \tau_E]}{\delta n}.$$ 

Using (22) and Lemmas 2.3 and 2.2 we obtain

$$P_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\epsilon,n}] \leq \frac{K_2N^2}{K_1\delta n(1-\eta)}.$$

(23)

Then, we deal with the term $P_\omega[\Lambda_n \mid A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n]$. By the Markov property we obtain

$$P_\omega[\Lambda_n \mid A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n] = \frac{1}{P_\omega[A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n]} \sum_{x \in I_M} \sum_{u=1}^{\lfloor \delta n \rfloor} P_\omega[\Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\epsilon,n}]P_\omega[X_{\tau_E} = x, \tau_E = u, A_{\epsilon,n}]$$

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\[ \geq \min_{x \in I_M} \min_{u \leq [\delta n]} P^\omega_x[\Lambda_{n-u}] \]
\[ = \min_{x \in I_M} P^\omega_x[\Lambda_n]. \] 
(24)

Thus, by (20), (21), (22), (24) and Lemma 2.1 we have

\[
P^\omega_x[\Lambda_n] \geq (1 - \eta) \left( 1 - \frac{K_2 N^2}{K_1 \delta n (1 - \eta)} \right) \min_{x \in I_M} P^\omega_x[\Lambda_n].
\]
(25)

To bound the term \( P^\omega_x[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \leq \delta n] \) from above we do the following. Let us denote by \( \mathcal{E} \) the event \( \{ X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \leq \delta n \} \). Since \( A_{\varepsilon,n} \in \mathcal{F}_{\tau_E} \) the \( \sigma \)-field generated by \( X \) until the stopping time \( \tau_E \), we have by the Markov property and the fact that \( \delta < 1 \),

\[
P^\omega_x[X_n > \sqrt{n}, \Lambda_n \mid \mathcal{E}] = \frac{1}{P^\omega_x[\mathcal{E}]} \sum_{x' \in I_M} \sum_{u=1}^{[\delta n]} P^\omega_x[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} = x', \tau_E = u, A_{\varepsilon,n}]
\]
\[
\leq \max_{x \in I_M} \max_{u \leq [\delta n]} \min_{x' \in I_M} P^\omega_x[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}]
\]
\[
= \max_{x \in I_M} \max_{u \leq [\delta n]} \min_{x' \in I_M} P^\omega_x[X_{n-u} > \sqrt{n}, X_k > 0, 1 \leq k \leq n-u]
\]
\[
= \max_{x \in I_M} \max_{u \leq [\delta n]} P^\omega_x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}].
\]
(26)

Now, fix \( \delta' \in (0,1) \). Then, we use the following estimate for \( x \in I_M \) and \( u \leq [\delta n] \),

\[
P^\omega_x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}] \leq P^\omega_x[\{(X_{n-[\delta n]} > (1 - \delta') \sqrt{n}) \cup \{|X_{n-[\delta n]} - X_{n-u}| > \delta' \sqrt{n}\}\} \cap \Lambda_{n-u}]
\]
\[
\leq P^\omega_x[\{(X_{n-[\delta n]} > (1 - \delta') \sqrt{n})
\]
\[
\cup \{ \max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| > \delta' \sqrt{n}\}\} \cap \Lambda_{n-[\delta n]}].
\]

Hence, we obtain for all \( x \in I_M \) that

\[
\max_{u \leq [\delta n]} P^\omega_x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}] \leq P^\omega_x[X_{n-[\delta n]} > (1 - \delta') \sqrt{n}, \Lambda_{n-[\delta n]}]
\]
\[
+ P^\omega_x[\max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| > \delta' \sqrt{n}, \Lambda_{n-[\delta n]}].
\]
(27)

To sum up, using (19), (20), (23) and (27) we obtain that \( \mathbb{P} \)-a.s.,

\[
P^\omega_x[X_n > \sqrt{n} \mid \Lambda_n] \leq (1 - \eta)^{-1} \left( 1 - \frac{K_2 N^2}{K_1 \delta n (1 - \eta)} \right)^{-1} \left( \min_{x \in I_M} P^\omega_x[\Lambda_n] \right)^{-1} \left( \frac{K_2 N^2}{K_1 \delta n (1 - \eta)} + \eta \right)
\]
\[
+ \max_{x \in I_M} P^\omega_x[X_{n-[\delta n]} > (1 - \delta') \sqrt{n}, \Lambda_{n-[\delta n]}]
\]
\[
+ \max_{x \in I_M} P^\omega_x[\max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| > \delta' \sqrt{n}, \Lambda_{n-[\delta n]}].
\]
(28)

Our goal is now to calculate the \( \limsup \) as \( n \to \infty \) of both sides of (28). Let us first compute \( \limsup_{n \to \infty} (P^\omega_x[\Lambda_n])^{-1} \) for \( x \in I_M \). We have by definition of \( Z^n \)

\[
P^\omega_x[\Lambda_n] = P^\omega_x[X_m > 0, 0 \leq m \leq n] = P^\omega_x[Z^n_t > 0, t \in [0,1]].
\]

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Thus, by the UCLT, we have
\[
\lim_{n \to \infty} P^x \left[ Z^n_t > 0, t \in [0, 1] \right] = P^x \left[ \min_{0 \leq t \leq 1} W(t) > 0 \right]
\]
with \( W \) a standard Brownian motion. Using the reflexion principle (see Chap. III, Prop. 3.7 in [19]), we obtain
\[
P^x \left[ \min_{0 \leq t \leq 1} W(t) > 0 \right] = P^0 \left[ |W(1)| < \varepsilon \right] = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx.
\]
So, we obtain
\[
\lim_{n \to \infty} \min_{x \in I_M} (P^x_\omega[\Lambda_n])^{-1} = \left( \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \right)^{-1} = \left( \frac{2\varepsilon}{\sqrt{2\pi}} + o(\varepsilon) \right)^{-1} \tag{29}
\]
as \( \varepsilon \to 0 \).

Now, let us bound \( \limsup_{n \to \infty} P^x_\omega[X_{n-[\delta n]} > (1-\delta')\sqrt{n}, \Lambda_{n-[\delta n]}] \) from above. We have
\[
P^x_\omega[X_{n-[\delta n]} > (1-\delta')\sqrt{n}, \Lambda_{n-[\delta n]}] \leq P^x_\omega \left[ X_{n-[\delta n]} > (1-\delta')\sqrt{n-\lfloor \delta n \rfloor}, \Lambda_{n-[\delta n]} \right]
= P^x_\omega \left[ Z^n_{1-[\delta n]} > (1-\delta'), Z^n_{1-[\delta n]} > 0, t \in [0, 1] \right].
\]
As \( \delta < 1 \) and \( x \in I_M \), we have by the UCLT,
\[
\lim_{n \to \infty} P^x_\omega \left[ Z^n_{1-[\delta n]} > (1-\delta'), Z^n_{1-[\delta n]} > 0, t \in [0, 1] \right] = P^x_\omega \left[ W(1) > (1-\delta'), \min_{0 \leq t \leq 1} W(t) > 0 \right].
\]
Abbreviate \( \varepsilon' := (1-\delta)^{-\frac{1}{2}} \) and let us compute \( P^x_\omega \left[ W(1) > (1-\delta'), \min_{0 \leq t \leq 1} W(t) > 0 \right] \) for sufficiently small \( \varepsilon \). By the reflexion principle for Brownian motion, we have
\[
P^x_\omega \left[ W(1) > (1-\delta'), \min_{0 \leq t \leq 1} W(t) > 0 \right] = P^x_\omega \left[ W(1) > (1-\delta') \right] - P^x_\omega \left[ W(1) < -(1-\delta') \right]
= P \left[ W(1) > 1 - (\delta' + \varepsilon') \right] - P \left[ W(1) < -1 + (\delta' - \varepsilon') \right]
= \frac{1}{\sqrt{2\pi}} \int_{1-(\delta' + \varepsilon')}^{1-(\delta' - \varepsilon')} e^{-\frac{x^2}{2}} \, dx.
\]
Therefore, we obtain, as \( \varepsilon \to 0 \)
\[
\limsup_{n \to \infty} \max_{x \in I_M} P^x_\omega[X_{n-[\delta n]} > (1-\delta')\sqrt{n}, \Lambda_{n-[\delta n]}] \leq \frac{1}{\sqrt{2\pi}} \int_{1-(\delta' + \varepsilon')}^{1-(\delta' - \varepsilon')} e^{-\frac{x^2}{2}} \, dx
\]
\[
= \frac{2\varepsilon}{\sqrt{2\pi(1-\delta)}} e^{-\frac{1}{2}} + o(\varepsilon). \tag{30}
\]
Then, let us bound \( \limsup_{n \to \infty} P^x_\omega \left[ \max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| > \delta' \sqrt{n}, \Lambda_{n-[\delta n]} \right] \) from above in \( \mathbb{R}^3 \) for \( x \in I_M \). First, observe that
\[
P^x_\omega \left[ \max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| \geq \delta' \sqrt{n}, \Lambda_{n-[\delta n]} \right] \leq P^x_\omega \left[ \max_{u \leq [\delta n]} |X_{n-[\delta n]} - X_{n-u}| \geq \delta' \sqrt{n} \right]
\]
Observe that the right-hand sides of (31) and (32) are equal since (\(P\))

\[
\lim_{n \to \infty} P^\omega \left[ \max_{1 \leq t \leq n} (Z^n_t - \min_{1 \leq s \leq t} Z^n_s) \geq \delta' \right] = P^\omega \left[ \max_{1 \leq t \leq n} (W(t) - \min_{1 \leq s \leq t} W(s)) \geq \delta' \right]
\]  

and

\[
\lim_{n \to \infty} P^\omega \left[ \min_{1 \leq t \leq n} (Z^n_t - \max_{1 \leq s \leq t} Z^n_s) \leq -\delta' \right] = P^\omega \left[ \min_{1 \leq t \leq n} (W(t) - \max_{1 \leq s \leq t} W(s)) \leq -\delta' \right].
\]  

Using the UCLT, we obtain

\[
\lim_{n \to \infty} P^\omega \left[ \max_{1 \leq t \leq n} (Z^n_t - \min_{1 \leq s \leq t} Z^n_s) \geq \delta' \right] = P^\omega \left[ \max_{1 \leq t \leq n} (W(t) - \min_{1 \leq s \leq t} W(s)) \geq \delta' \right]  
\]  

and

\[
\lim_{n \to \infty} P^\omega \left[ \min_{1 \leq t \leq n} (Z^n_t - \max_{1 \leq s \leq t} Z^n_s) \leq -\delta' \right] = P^\omega \left[ \min_{1 \leq t \leq n} (W(t) - \max_{1 \leq s \leq t} W(s)) \leq -\delta' \right].
\]  

Observe that the right-hand sides of (31) and (32) are equal since \((-W)\) is a Brownian motion. Thus, we can compute for example \(P^\omega[\max_{1 \leq t \leq n} (W(t) - \min_{1 \leq s \leq t} W(s)) \geq \delta']\). First, by the Markov property and since the event is invariant by space shifts, we have

\[
P^\omega \left[ \max_{0 \leq t \leq \delta} (W(t) - \min_{0 \leq s \leq t} W(s)) \geq \delta' \right] = P \left[ \max_{0 \leq t \leq \delta} |W(t)| \geq \delta' \right].
\]

Then, by the reflection principle, we have

\[
P \left[ \max_{0 \leq t \leq \delta} |W(t)| \geq \delta' \right] \leq 2P \left[ \max_{0 \leq t \leq \delta} W(t) \geq \delta' \right] = 4P[W(\delta) \geq \delta'].
\]

Using an estimate on the tail of the Gaussian law (cf. [18], Appendix II, Lemma 3.1) we obtain

\[
P \left[ \max_{0 \leq t \leq \delta} |W(t)| \geq \delta' \right] \leq \frac{4\sqrt{\delta}}{\delta' \sqrt{2\pi}} \exp \left\{ -\frac{(\delta')^2}{2\delta} \right\}.
\]

Thus, we find

\[
\lim_{n \to \infty} \max_{x \in I_M} P^\omega \left[ \max_{1 \leq n \leq \delta} |X_n - \delta' \sqrt{n}, \Lambda_{n-\delta} \right] \leq \frac{8\sqrt{\delta}}{\delta' \sqrt{2\pi}} \exp \left\{ -\frac{(\delta')^2}{2\delta} \right\}.  
\]

Finally, combining (28), (29), (30) and (33), we obtain

\[
\lim_{n \to \infty} P^\omega[X_n > \sqrt{n} | \Lambda_n] \leq (1 - \eta)^{-1} \left( 1 - \frac{K_2 \varepsilon^2}{K_1 \delta (1 - \eta)} \right)^{-1} \left( \frac{2\varepsilon}{\sqrt{2\pi}} + o(\varepsilon) \right)^{-1} \left( \frac{K_2 \varepsilon^2}{K_1 \delta (1 - \eta)} \right) + \frac{2\varepsilon}{\sqrt{2\pi(1 - \delta)}} \left( \frac{7}{2\delta} \right) + \frac{8\sqrt{\delta}}{\delta' \sqrt{2\pi}} \exp \left\{ -\frac{(\delta')^2}{2\delta} \right\}.  
\]
Next, let us bound the quantity $P_\omega[X_n > \sqrt{n} | \Lambda_n]$ from below. Using $(17)$, we write
\[
P_\omega[X_n > \sqrt{n} | \Lambda_n] \geq \frac{P_\omega[A_{\epsilon,n}]}{P_\omega[\Lambda_n]} P_\omega[X_{\tau_E} \in I_M | A_{\epsilon,n}, P_\omega[\tau_E \leq \delta n | X_{\tau_E} \in I_M, A_{\epsilon,n}] \\
\times P_\omega[X_n > \sqrt{n}, \Lambda_n | X_{\tau_E} \in I_M, A_{\epsilon,n}, \tau_E \leq \delta n].
\] (35)

As we have already treated the terms $P_\omega[\tau_E \leq \delta n | X_{\tau_E} \in I_M, A_{\epsilon,n}]$ and $P_\omega[X_{\tau_E} \in I_M | A_{\epsilon,n}]$ in $(21)$ and Lemma 2.1 respectively, we just need to bound the terms $P_\omega[A_{\epsilon,n}] / P_\omega[\Lambda_n]$ and $P_\omega[X_n > \sqrt{n}, \Lambda_n | X_{\tau_E} \in I_M, A_{\epsilon,n}, \tau_E \leq \delta n]$ from below.

Let us start with the term $P_\omega[A_{\epsilon,n}] / P_\omega[\Lambda_n]$. Observe that
\[
P_\omega[\Lambda_n] = P_\omega[\Lambda_n, \tau_E \leq \delta n] + P_\omega[\Lambda_n, \tau_E > \delta n] \\
= P_\omega[\Lambda_n, A_{\epsilon,n}, \tau_E \leq \delta n] + P_\omega[\Lambda_n, A_{\epsilon,n}, \tau_E > \delta n] + P_\omega[\Lambda_n, A_{\epsilon,n}^c, \tau_E > \delta n] \\
\leq P_\omega[\Lambda_n, A_{\epsilon,n}, \tau_E \leq \delta n] + P_\omega[\Lambda_n, A_{\epsilon,n}, \tau_E > \delta n] + P_\omega[\Lambda_n, A_{\epsilon,n}^c] \\
\leq P_\omega[A_{\epsilon,n}][P_\omega[\Lambda_n | A_{\epsilon,n}, \tau_E \leq \delta n, X_{\tau_E} \in I_M] + P_\omega[\Lambda_n, A_{\epsilon,n}, \tau_E > \delta n, X_{\tau_E} \in I_M] \\
+ 2P_\omega[X_{\tau_E} \notin I_M, A_{\epsilon,n}] + P_\omega[\Lambda_n, A_{\epsilon,n}^c] \\
\leq P_\omega[A_{\epsilon,n}][P_\omega[\Lambda_n | A_{\epsilon,n}, \tau_E \leq \delta n, X_{\tau_E} \in I_M] + 2P_\omega[X_{\tau_E} \notin I_M | A_{\epsilon,n}] \\
+ P_\omega[\tau_E > \delta n | X_{\tau_E} \in I_M, A_{\epsilon,n}] + \frac{P_\omega[\Lambda_n, A_{\epsilon,n}^c]}{P_\omega[A_{\epsilon,n}]}].
\] (36)

From the first equality in $(24)$ we obtain
\[
P_\omega[\Lambda_n | A_{\epsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n] \leq \max_{x \in I_M} \max_{u \leq [\delta n]} P_\omega^x[\Lambda_n - u] \leq \max_{x \in I_M} P_\omega^x[\Lambda_n - [\delta n]].
\] (37)

Now, let us treat the term $P_\omega[A_{\epsilon,n}, A_{\epsilon,n}^c]$. First, observe that by definition of $A_{\epsilon,n}$ we have
\[
P_\omega[A_{\epsilon,n}, A_{\epsilon,n}^c] \leq P_\omega[\tau_B^+ \wedge \tau_E > n].
\]

Then, by Chebyshev’s inequality we obtain
\[
P_\omega[\tau_B^+ \wedge \tau_E > n] \leq \frac{E_\omega[\tau_B^+ \wedge \tau_E]}{n}.
\]

By Lemma 2.3 we obtain
\[
P_\omega[\tau_B^+ \wedge \tau_E > n] \leq \frac{K_2 N}{n}.
\] (38)

Thus, by $(23)$, $(36)$, $(37)$, $(38)$ and Lemmas 2.1 and 2.2 we obtain
\[
\frac{P_\omega[A_{\epsilon,n}]}{P_\omega[\Lambda_n]} \geq \left( \max_{x \in I_M} P_\omega^x[\Lambda_n - [\delta n]] + 2n + \frac{K_2 N^2}{K_1 \delta n(1 - \eta)} + \frac{K_2 N^2}{K_1 n} \right)^{-1}.
\] (39)

Let us find a lower bound for $P_\omega[X_n > \sqrt{n}, \Lambda_n | X_{\tau_E} \in I_M, A_{\epsilon,n}, \tau_E \leq \delta n]$ in $(35)$. Since $A_{\epsilon,n} \in F_{\tau_E}$ we have by the Markov property,
\[
P_\omega[X_n > \sqrt{n}, \Lambda_n | X_{\tau_E} \in I_M, A_{\epsilon,n}, \tau_E \leq \delta n] \\
\geq \min_{x \in I_M} \min_{u \leq [\delta n]} P_\omega[X_n > \sqrt{n}, \Lambda_n | X_{\tau_E} = x, \tau_E = u, A_{\epsilon,n}]
\]
\[
\begin{align*}
    &= \min_{x \in I_M} \min_{u \leq [\delta n]} P^x_{\omega} [X_{n-u} > \sqrt{n}, X_k > 0, 1 \leq k \leq n-u] \\
    &= \min_{x \in I_M} \min_{u \leq [\delta n]} P^x_{\omega} [X_{n-u} > \sqrt{n}, \Lambda_{n-u}].
\end{align*}
\]

For \( x \in I_M \) and \( u \leq [\delta n] \) we write
\[
\begin{align*}
P^x_{\omega} [X_{n-u} > \sqrt{n}, \Lambda_{n-u}] &\geq P^x_{\omega} [X_{n} > (1 + \delta') \sqrt{n}, X_{n-u} \leq \delta' \sqrt{n}, \Lambda_{n-u}] \\
&\geq P^x_{\omega} [X_{n} > (1 + \delta') \sqrt{n}, \max_{u \leq [\delta n]} |X_n - X_{n-u}| \leq \delta' \sqrt{n}, \Lambda_{n-u}] \\
&\geq P^x_{\omega} [X_{n} > (1 + \delta') \sqrt{n}, \max_{u \leq [\delta n]} |X_n - X_{n-u}| \leq \delta' \sqrt{n}, \Lambda_n] \\
&\geq P^x_{\omega} [X_{n} > (1 + \delta') \sqrt{n}, \Lambda_n] - P^x_{\omega} \left[ \max_{u \leq [\delta n]} |X_n - X_{n-u}| > \delta' \sqrt{n} \right].
\end{align*}
\]

To sum up, by (35), (39), (41), (21), (23) and Lemma 2.1 we obtain that \( \mathbb{P} \)-a.s.,
\[
\begin{align*}
P_{\omega} [X_n > \sqrt{n} | \Lambda_n] &\geq (1 - \eta) \left( 1 - \frac{K_2 N^2}{K_1 \delta n (1 - \eta)} \right) \\
&\times \left( \max_{x \in I_M} P^x_{\omega} [\Lambda_{n-[\delta n]}] + 2 \eta + \frac{K_2 N^2}{K_1 \delta n (1 - \eta)} + \frac{K_2 N^2}{K_1 n} \right)^{-1} \\
&\times \left( \min_{x \in I_M} P^x_{\omega} [X_n > (1 + \delta') \sqrt{n}, \Lambda_n] - \max_{x \in I_M} P^x_{\omega} \left[ \max_{u \leq [\delta n]} |X_n - X_{n-u}| > \delta' \sqrt{n} \right] \right).
\end{align*}
\]

Let us now compute \( \lim \inf_{n \to \infty} \) of both sides of (42). First, by (29) we have
\[
\lim_{n \to \infty} \max_{x \in I_M} P^x_{\omega} [\Lambda_{n-[\delta n]}] = \frac{2 \varepsilon}{\sqrt{2 \pi (1 - \delta)}} + o(\varepsilon)
\]
as \( \varepsilon \to 0 \). Then, by the UCLT and after some elementary computations similar to those which led to (30) and (33) we obtain
\[
\lim_{n \to \infty} \min_{x \in I_M} P^x_{\omega} [X_n > (1 + \delta') \sqrt{n}, \Lambda_n] = \frac{1}{\sqrt{2 \pi}} \int_{1+(\delta'-\varepsilon)}^{1+(\delta'+\varepsilon)} e^{-x^2/2} dx = \frac{2 \varepsilon}{\sqrt{2 \pi}} e^{-\frac{\varepsilon}{2}} + o(\varepsilon)
\]
as \( \varepsilon \to 0 \), and
\[
\lim_{n \to \infty} \sup_{x \in I_M} \max_{u \leq [\delta n]} |X_n - X_{n-u}| > \delta' \sqrt{n} \] \[\leq \frac{8 \sqrt{\delta}}{\delta' \sqrt{2 \pi}} \exp \left\{ - \frac{(\delta')^2}{2 \delta} \right\}.
\]
Thus, combining (42) with (13), (44) and (45) leads to
\[
\begin{align*}
\lim_{n \to \infty} \min_{x \in I_M} P^x_{\omega} [X_n > \sqrt{n} | \Lambda_n] &\geq (1 - \eta) \left( 1 - \frac{K_2 \varepsilon^2}{K_1 \delta (1 - \eta)} \right) \\
&\times \left( \frac{2 \varepsilon}{\sqrt{2 \pi (1 - \delta)}} + o(\varepsilon) + 2 \eta + \frac{K_2 \varepsilon^2}{K_1 \delta n (1 - \eta)} + \frac{K_2 \varepsilon^2}{K_1 n} \right)^{-1} \\
&\times \left( \frac{2 \varepsilon}{\sqrt{2 \pi}} e^{-\frac{\varepsilon}{2}} + o(\varepsilon) - \frac{8 \sqrt{\delta}}{\delta' \sqrt{2 \pi}} \exp \left\{ - \frac{(\delta')^2}{2 \delta} \right\} \right).
\end{align*}
\]

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Now take \( \eta = \varepsilon^2 \), \( \delta = \varepsilon^\frac{1}{2} \) and \( \delta' = \varepsilon^\frac{1}{3} \) and let \( \varepsilon \to 0 \) in (34) and (46) to prove (16). \( \square \)

The next step is to show the weak convergence of \( (P_\omega[Z^n_t \in \cdot \mid \Lambda_n], n \geq 1) \) for all \( t \in (0,1) \). We start by recalling the transition density function from (0,0) to \((t,y)\) of the Brownian meander (see [16]):

\[
q(t,y) = t^{-\frac{3}{2}} y \exp\left(-\frac{y^2}{2t}\right) \tilde{N}(y(1-t)^{-\frac{3}{2}})
\]

for \( y > 0, 0 < t \leq 1 \), where

\[
\tilde{N}(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}} du
\]

for \( x \geq 0 \). We will prove the following

**Proposition 3.2** We have \( \mathbb{P} \)-a.s., for all \( x \geq 0 \) and \( 0 < t < 1 \),

\[
\lim_{n \to \infty} P_\omega[Z^n_t \leq x \mid \Lambda_n] = \int_0^x q(t,y)dy.
\]

**Proof.** First notice the following. For all \( \tilde{\varepsilon} > 0 \) we have

\[
P_\omega \left[Z^n_{[nt]} \leq x - \tilde{\varepsilon} \mid \Lambda_n \right] \\
\leq P_\omega \left[Z^n_{[nt]} \leq x - \tilde{\varepsilon}, \left| Z^n_{[nt]+1} - Z^n_{[nt]} \right| \leq \tilde{\varepsilon} \mid \Lambda_n \right] + P_\omega \left[\left| Z^n_{[nt]+1} - Z^n_{[nt]} \right| > \tilde{\varepsilon} \mid \Lambda_n \right] \\
\leq P_\omega[Z^n_t \leq x \mid \Lambda_n] + P_\omega[\Lambda_n]^{-1} P_\omega[|X_{[nt+1]} - X_{[nt]}| > \tilde{\varepsilon}\sqrt{n}].
\]

By (28), (29), Lemma 2.2 and Condition K, the second term of (48) tends to 0 as \( n \to \infty \). Hence, assuming that the following limits exist, we deduce that

\[
\lim_{n \to \infty} P_\omega \left[Z^n_{[nt]} \leq x - \tilde{\varepsilon} \mid \Lambda_n \right] \leq \lim_{n \to \infty} P_\omega[Z^n_t \leq x \mid \Lambda_n] \leq \lim_{n \to \infty} P_\omega \left[Z^n_{[nt]} \leq x + \tilde{\varepsilon} \mid \Lambda_n \right]
\]

for all \( \tilde{\varepsilon} > 0 \). Now, suppose that we have for all \( x \geq 0 \) and \( 0 < t < 1 \),

\[
\lim_{n \to \infty} P_\omega \left[Z^n_{[nt]} \leq x \mid \Lambda_n \right] = \int_0^x q(t,y)dy.
\]

Combining (50) and (51), we obtain (18) since the limit distribution \( q(t,x) \) is absolutely continuous. Our goal is now to show (51). For this, observe that

\[
P_\omega \left[Z^n_{[nt]} \leq x \mid \Lambda_n \right] = \frac{1}{P_\omega[\Lambda_n]} \int_0^{\frac{\sqrt{nt}}{\sqrt{n}}} P_\omega[Z^{[nt]}_1 \in dy, \Lambda_{[nt]}, X_k > 0, [nt] < k \leq n] \\
= \frac{P_\omega[\Lambda_{[nt]}]}{P_\omega[\Lambda_n]} \int_0^{\frac{\sqrt{nt}}{\sqrt{n}}} P_\omega^{\sqrt{nt}} \left[Z^n_s > 0, 0 \leq s \leq 1 - \frac{[nt]}{n} \right] P_\omega[Z^{[nt]}_1 \in dy \mid \Lambda_{[nt]}].
\]

By (25), (39), (29), and (43) we have

\[
\lim_{n \to \infty} \frac{P_\omega[\Lambda_{[nt]}]}{P_\omega[\Lambda_n]} = t^{-\frac{3}{2}}.
\]
Using part (v) of the UCLT and Dini’s theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain
\[
\lim_{n \to \infty} \mathbb{P}^\omega \sqrt{nt} \left[ Z^n_s > 0, 0 \leq s \leq 1 - \frac{|nt|}{n} \right] = \mathbb{P}^z \left[ \min_{s \in [0,1-t]} W_s > 0 \right] = \mathbb{P}[|W_{1-t}| < z] = \tilde{N} \left( \left( \frac{t}{1-t} \right)^{\frac{1}{2}} \right)
\]
uniformly in \( z \) on every compact set of \( \mathbb{R}_+ \). By Proposition 3.1, we have
\[
\lim_{n \to \infty} \mathbb{P}_w[Z^n] \leq x | \Lambda_{|nt|}] = \int_0^x y e^{-\frac{y^2}{2}} dy.
\]
Now, applying Lemma 2.18 of [16] to (52), we obtain
\[
\text{Finally, make the change of variables } u = t^{\frac{1}{2}}y \text{ to obtain the desired result.} \quad \square
\]
We can now use Propositions 3.1 and 3.2 to easily check that (13), (14) and (15) of Theorems 3.1 and 3.2 are satisfied. This ends the proof of Theorem 1.1.

4 Proof of Corollary 1.1

In this last part, for the sake of brevity, we will use the same notation for a real number \( x \) and its integer part \( \lfloor x \rfloor \). The interpretation of the notation should be clear by the context where it is used. We also suppose without loss of generality that \( \sigma = 1 \). Let us first introduce some spaces needed in the proof of Corollary 1.1.

For any \( l > 0 \), let \( C_0([0,l]) \) the space of continuous functions \( f \) from \([0,l]\) into \( \mathbb{R} \) such that \( f(0) = 0 \). We endow this space with the metric
\[
d(f, g) = \sup_{x \in [0,l]} |f(x) - g(x)|
\]
and the Borel sigma-field on \( C_0([0,l]) \) corresponding to the metric \( d \).

Then, let \( C_0(\mathbb{R}_+) \) the space of continuous functions \( f : \mathbb{R}_+ \to \mathbb{R} \) such that \( f(0) = 0 \). We endow this space with the metric
\[
d(f, g) = \sum_{n=1}^{\infty} 2^{-n+1} \min\{1, \sup_{x \in [0,n]} |f(x) - g(x)|\}
\]
and the Borel sigma-field on \( C_0(\mathbb{R}_+) \) corresponding to the metric \( d \). Next, let \( G \) be the set of functions of \( C_0(\mathbb{R}_+) \) for which there exists \( x_0 \) (depending on \( f \)) such that \( f(x_0) = 1 \). Let us also define the set \( H \) as the set of functions of \( C_0(\mathbb{R}_+) \) such that there exists \( x_1 = x_1(f) = \min\{s > 0 : f(s) = 1\} \) and \( f(x) = 1 \) for all \( x \geq x_1 \); observe that \( G \) and \( H \) are closed subsets of \( C_0(\mathbb{R}_+) \). We define the continuous map \( \Psi : G \to H \) by
\[
\Psi(f)(x) = \begin{cases} f(x) & \text{for } x \leq x_1, \\ 1 & \text{for } x > x_1. \end{cases}
\]
Now, Corollary 1.1 can be restated as follows: under the conditions of Theorem 1.1 we have \( \mathbb{P} \)-a.s., for all measurable \( A \subset H \) such that \( \mathbb{P}[B_3(\cdot \wedge \varrho_1) \in \partial A] = 0 \) and all \( a \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{P}_w[Y^n \in A, T_n \leq a | \Lambda'_n] = \mathbb{P}[B_3(\cdot \wedge \varrho_1) \in A, \varrho_1 \leq a]. \quad (54)
\]
Before proving this last statement, let us start by denoting \( R = \{Y^n \in A\} \). We will bound the term \( P_\omega[R, T_n \leq a \mid \Lambda_n'] \) from above and below, for sufficiently large \( n \).

We start with the upper bound. Let \( M > 0 \) be an integer and \( I_M = [n, n + M] \). We obtain

\[
P_\omega[R, T_n \leq a \mid \Lambda_n'] = \frac{1}{P_\omega[\Lambda_n']} P_\omega[R, T_n \leq a, \Lambda_n']
= \frac{1}{P_\omega[\Lambda_n']} \left( P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} \in I_M] + P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} \notin I_M] \right)
\leq \frac{1}{P_\omega[\Lambda_n']} P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} \in I_M] + P_\omega[X_{\hat{r}_n} \notin I_M \mid \Lambda_n']
\]

for all sufficiently large \( n \). The second term of the right-hand side of (55) can be treated easily. Indeed, by the same method we used to prove Lemma 2.1, we can show that, \( \mathbb{P}\text{-a.s.}, \) for all \( \eta > 0 \), there exists \( M > 0 \) such that

\[
P_\omega[X_{\hat{r}_n} \notin I_M \mid \Lambda_n'] \leq \eta
\]

for all \( n \geq 1 \). Let \( c > 2a \) and observe that \( R \cap \{T_n \leq a\} \in \mathcal{F}_{\hat{r}_n} \), where \( \mathcal{F}_{\hat{r}_n} \) is the sigma-field generated by \( X \) until time \( \hat{r}_n \). For the first term of the right-hand side of (55), we have by the Markov property

\[
\frac{1}{P_\omega[\Lambda_n']} P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} \in I_M] = \sum_{u=0}^{M} \frac{1}{P_\omega[\Lambda_n']} P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} = n + u]
= \sum_{u=0}^{M} \frac{P^{n+u}[\Lambda_{(c-a)n^2}]}{P_\omega[\Lambda_n'] P^{n+u}[\Lambda_{(c-a)n^2}]} P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} = n + u]
\leq \sum_{u=0}^{M} \frac{1}{P_\omega[\Lambda_n'] P^{n+u}[\Lambda_{(c-a)n^2}]} P_\omega[R, T_n \leq a, \Lambda_n', X_{\hat{r}_n} = n + u].
\]

Next, let us define the event \( E = \{X_1 > 0, \ldots, X_{\hat{r}_n} > 0, \ldots, X_{\hat{r}_n+(c-a)n^2} > 0\} \). Using the Markov property, we can write

\[
P_\omega[E] \leq \sum_{v=0}^{M} P_\omega[X_1 > 0, \ldots, X_{\hat{r}_n} = n + v, \ldots, X_{\hat{r}_n+(c-a)n^2} > 0] + P_\omega[X_{\hat{r}_n} \notin I_M \mid \Lambda_n']
= \sum_{v=0}^{M} P_\omega[\Lambda_n', X_{\hat{r}_n} = n + v] P^{n+v}[\Lambda_{(c-a)n^2}] + P_\omega[X_{\hat{r}_n} \notin I_M \mid \Lambda_n'] P_\omega[\Lambda_n'].
\]

But, by the UCLT, we have for all \( \varepsilon > 0 \) that uniformly in \( v \in [0, M] \) and \( u \in [0, M] \),

\[
\left| P^{n+u}[\Lambda_{(c-a)n^2}] - P^{n+v}[\Lambda_{(c-a)n^2}] \right| \leq \varepsilon
\]

for all \( n \) sufficiently large. Therefore, we obtain for all \( u \in [0, M] \),

\[
P^{n+u}[\Lambda_{(c-a)n^2}] P_\omega[\Lambda_n'] \geq P_\omega[E] - (\varepsilon + \eta) P_\omega[\Lambda_n']
\]

for all \( n \) sufficiently large. Now, let us bound the first term of the right-hand side of (58) from below. Fix some \( \delta > 0 \). We write

\[
P_\omega[E] \geq P_\omega[E, \hat{r}_n \leq (a + \delta)n^2]
\]
Finally, by (55), (56), (58) and (59) we obtain $\mathbb{P}$-a.s.,
\[
P_\omega[R, T_n \leq a \mid \Lambda'_n] \leq (P_\omega[\Lambda((c+\delta)n^2+3)] (P_\omega[\Lambda((c+\delta)n^2+3)])^{-1} P_\omega[\Lambda((c+\delta)n^2) P_\omega(T_n \leq a \mid \Lambda_{cn^2})]^{-1} + \eta \tag{60}
\]
for all sufficiently large $n$.

We now estimate the term $P_\omega[R, T_n \leq a \mid \Lambda'_n]$ from below. Let us write
\[
P_\omega[R, T_n \leq a \mid \Lambda'_n] = \frac{1}{P_\omega[\Lambda'_n]} P_\omega[R, T_n \leq a, \Lambda'_n] \geq \frac{1}{P_\omega[\Lambda'_n]} P_\omega[R, T_n \leq a, \Lambda'_n, X_{\tilde{\tau}_n} \in I_M] \tag{61}
\]
Then, we have by the Markov property
\[
\frac{1}{P_\omega[\Lambda'_n]} P_\omega[R, T_n \leq a, \Lambda'_n, X_{\tilde{\tau}_n} \in I_M] = \sum_{u=0}^{M} \frac{1}{P_\omega[\Lambda'_n]} P_\omega[R, T_n \leq a, \Lambda'_n, X_{\tilde{\tau}_n} = n + u] = \sum_{u=0}^{M} \frac{P^{n+u}[\Lambda((c-\delta)n^2)]}{P_\omega[\Lambda'_n] P^{n+u}[\Lambda((c-\delta)n^2)]} P_\omega[R, T_n \leq a, \Lambda'_n, X_{\tilde{\tau}_n} = n + u] \geq \sum_{u=0}^{M} \frac{1}{P_\omega[\Lambda'_n] P^{n+u}[\Lambda((c-\delta)n^2)]} P_\omega[R, T_n \leq a, \Lambda_{cn^2}, X_{\tilde{\tau}_n} = n + u]. \tag{62}
\]
Again using the Markov property, we can write
\[
P_\omega[E] \geq \sum_{v=0}^{M} P_\omega[X_1 > 0, \ldots, X_{\tilde{\tau}_n} = n + v, \ldots, X_{\tilde{\tau}_n+(c-\delta)n^2} > 0] \geq \sum_{v=0}^{M} P_\omega[\Lambda'_n, X_{\tilde{\tau}_n} = n + v] P^{n+u}[\Lambda((c-\delta)n^2)].
\]
Using (62), we obtain for all $u \in [0, M]$,
\[
P^{n+u}[\Lambda((c-\delta)n^2)] P_\omega[\Lambda'_n] \leq P_\omega[E] + \varepsilon P_\omega[\Lambda'_n] \tag{63}
\]
for all sufficiently large $n$. Then, as $\tilde{\tau}_n \geq 1$, we have
\[
P_\omega[E] \leq P_\omega[\Lambda((c-\delta)n^2)]. \tag{64}
\]
Finally, by (61), (62), (63) and (64), we obtain $\mathbb{P}$-a.s.,
\[
P_\omega[R, T_n \leq a \mid \Lambda'_n] \geq \frac{(P_\omega[\Lambda((c-\delta)n^2)])^{-1} P_\omega[\Lambda_{cn^2} P_\omega[R, T_n \leq a \mid \Lambda_{cn^2}]]^{-1}}{1 + \varepsilon P_\omega[\Lambda'_n] (P_\omega[\Lambda((c-\delta)n^2)])^{-1}} \tag{65}
\]
for all sufficiently large $n$. 

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Our intention is now to take the lim sup as \( n \to \infty \) in (60). Before this, observe that by (25), (39), (29) and (43) we have for some constant \( \varepsilon \leq 1 \),

\[
\lim_{n \to \infty} \frac{P_\omega[A_{(c-a)n^2}]}{P_\omega[A_{((c+\delta)n^2+3}]} = \sqrt{\frac{c + \delta}{c - a}}, \tag{66}
\]

\[
\limsup_{n \to \infty} \frac{P_\omega[A'_n]}{P_\omega[A_{((c+\delta)n^2+3}]} \leq \limsup_{n \to \infty} \frac{P_\omega[A_{z,n^2}]}{P_\omega[A_{((c+\delta)n^2+3}]} \leq \gamma_1 \sqrt{c + \delta} \tag{67}
\]

for some constant \( \gamma_1 \). By the usual scaling, from the Brownian meander \( W^+ \) on \([0, 1]\) it is possible to define the Brownian meander \( W^+_t \) on any finite interval \([0, t]\): \( W^+_t(\cdot) := \sqrt{t}W^+(\cdot/t) \). Thus, Theorem 11.1 implies that

\[
\lim_{n \to \infty} P_\omega[\hat{r}_n \leq (a + \delta)n^2 | \Lambda_{((c+\delta)n^2+3} = P\left[ \sup_{0 \leq s \leq (a + \delta)} W^+_{c+a}(s) \geq 1 \right]. \tag{68}
\]

Denoting by \( U_a \) the measurable set of functions \( f \) in \( H \) such that \( f(a) = 1 \) and by \( \pi_l \) the projection map from \( C_0([R_+, 0]) \) onto \( C_0([0, l]) \), we have

\[
P_\omega[R_n \leq a | \Lambda_{(c-a)n^2} = P_\omega[Z^2_{\Lambda_{T_n}} \in A \cap U_a | \Lambda_{(c-a)n^2}]
= P_\omega[Z^2_{\Lambda_{T_n}} \in \Psi^{-1}(A \cap U_a) | \Lambda_{(c-a)n^2}]
= P_\omega[Z^2_{\Lambda_{(c-a)}} \in \pi_{c-a}(\Psi^{-1}(A \cap U_a)) | \Lambda_{(c-a)n^2}].
\]

The next step is to show that

\[
\lim_{n \to \infty} P_\omega[Z^2_{\Lambda_{(c-a)}} \in \pi_{c-a}(\Psi^{-1}(A \cap U_a)) | \Lambda_{(c-a)n^2} = P[W^+_{c-a} \in \pi_{c-a}(\Psi^{-1}(A \cap U_a))], \tag{69}
\]

where \( W^+_{c-a} \) is the Brownian meander on \([0, c - a]\). As the law of the Brownian meander on \([0, c - a]\) is absolutely continuous with respect to the law of the three dimensional Bessel process \( B_3 \) on \([0, c - a]\) (see [17] section 4), to prove (69) we will show that

\[
P[B_3(\cdot \land c - a) \in \partial(\pi_{c-a}(\Psi^{-1}(A \cap U_a))) = 0. \tag{70}
\]

Observe that, as \( \pi_{c-a} \) is a projection, we have

\[
P[B_3(\cdot \land c - a) \in \partial(\pi_{c-a}(\Psi^{-1}(A \cap U_a)))] \leq P[B_3(\cdot \land c - a) \in \pi_{c-a}\partial(\Psi^{-1}(A \cap U_a))]
= P[B_3 \in \partial(\Psi^{-1}(A \cap U_a))].
\]

Now, as \( \Psi \) is a continuous map, we have

\[
P[B_3 \in \partial(\Psi^{-1}(A \cap U_a))] \leq P[B_3 \in \Psi^{-1}(\partial(\Psi^{-1}(A \cap U_a)))]
\leq P[B_3 \in \Psi^{-1}(\partial(A \cup U_a))]
\leq P[B_3(\cdot \land q_1) \in \partial A] + P[q_1 = a]. \tag{71}
\]

By hypothesis, \( P[B_3(\cdot \land q_1) \in \partial A] = 0 \). As the law of \( q_1 \) is absolutely continuous with respect to the Lebesgue measure (see [17] Theorem 4), we also have \( P[q_1 = a] = 0 \). This proves (69).

Then, we want to take the lim inf as \( n \to \infty \) in (65). Before this, notice that

\[
\lim_{n \to \infty} \frac{P_\omega[A_{mn^2}]}{P_\omega[A_{(c-a)n^2}]} = \sqrt{\frac{c - a}{c}}, \tag{72}
\]
Let us make the change of variable $y$ for $\gamma$ for some constant $c_0$, where $W \in \{\gamma \in (0, 1) : \gamma = \sqrt{c - a}\}$. Then, define $V_l = \{W_l^+ \in \pi_l(\Psi^{-1}(A \cap U_a))\}$ for $l \in \{c - a, c\}$. Combining (66), (67), (68), (69), (72), (73) and (74) we see that

\[
\frac{P[V_c] \sqrt{c - a}}{1 + \gamma_2 \sqrt{c - a}} \leq \lim inf P[V_c | R, T_n \leq a | \Lambda_n] \leq \lim sup P[V_c | R, T_n \leq a | \Lambda_n] \leq P[V_{c-a}] \sqrt{\frac{c + \delta}{c - a}} + \eta.
\]

Now, take $\varepsilon = \eta = c^{-1}$ and $\delta = \sqrt{c}$ and let $c$ tend to infinity. Since

\[
P[W_{l}^+ \in \pi_l(\Psi^{-1}(A \cap U_a)) = P[W_{l}^+ (\cdot \wedge a) \in \pi_a(\Psi^{-1}(A \cap U_a)),
\]

we have by Lemma 11-1 of [3] the limit $P[V_{l}] = P[B_{l} (\cdot \wedge \theta_1) \in A, \theta_1 \leq a]$ for $l \in \{c - a, c\}$.  

The last thing we have to check to obtain (75) is that

\[
\lim_{c \to \infty} P[\sup_{0 \leq s \leq (a + \delta)} W_{c+d}(s) < 1] = 0.
\]

First, we start by noting that by scaling property

\[
P\left[\sup_{0 \leq s \leq (a + \delta)} W_{c+d}(s) < 1\right] = P\left[(c + \delta)^{\frac{1}{2}} \sup_{0 \leq s \leq (a + \delta)} W^{+}\left(\frac{s}{c + \delta}\right) < 1\right]
\]

\[= P\left[\sup_{0 \leq s \leq \frac{c + \delta}{c + \delta}} W^{+}(s) < (c + \delta)^{-\frac{1}{2}}\right]
\]

\[\leq P\left[W^{+}\left(\frac{a + \delta}{c + \delta}\right) \leq (c + \delta)^{-\frac{1}{2}}\right]
\]

where $W^{+}$ is a Brownian meander on $[0, 1]$. This last term is easily computable using the transition density function from $(0, 0)$ of $W^{+}$ given in (77). Let $u = \frac{a + \delta}{c + \delta}$,

\[
P\left[W^{+}\left(\frac{a + \delta}{c + \delta}\right) \leq (c + \delta)^{-\frac{1}{2}}\right] = \int_{0}^{(c+\delta)^{-\frac{1}{2}}} u^{-\frac{1}{2}} x \exp\left(-\frac{x^2}{2u}\right) \tilde{N}(x(1-u)^{-\frac{1}{2}}) dx
\]

Let us make the change of variable $y = (c + \delta)^{\frac{1}{2}} x$ in the right-hand side integral. Then, we obtain

\[
P\left[W^{+}\left(\frac{a + \delta}{c + \delta}\right) \leq (c + \delta)^{-\frac{1}{2}}\right] = \int_{0}^{1} \frac{(c + \delta)^{\frac{1}{2}}}{(a + \delta)^{\frac{1}{2}}} y \exp\left(-\frac{y^2}{2(a + \delta)}\right) \tilde{N}(y(c - a)^{-\frac{1}{2}}) dy.
\]
Now, making the change of variable $z = (c - a)^{1/2}u$ in the following integral

$$
\tilde{N}(y(c-a)^{-1/2}) = \sqrt{2/\pi} \int_0^y (c-a)^{-1/2} \exp \left( -\frac{u^2}{2} \right) du
$$

we obtain

$$
P \left[ W^+ \left( \frac{a + \delta}{c + \delta} \right) \leq \left( c + \delta \right)^{-1/2} \right] = \left( \frac{c + \delta}{c - a} \right)^{1/2} \frac{1}{(a + \delta)^{1/2}} \int_0^1 \int_0^y y \exp \left( -\frac{y^2}{2(a + \delta)} \right) \exp \left( -\frac{z^2}{2(c - a)} \right) dz dy
$$

$$
\leq \left( \frac{c + \delta}{c - a} \right)^{1/2} \frac{1}{(a + \delta)^{1/2}} \int_0^1 \int_0^y y dz dy
$$

$$
= \frac{1}{3} \left( \frac{c + \delta}{c - a} \right)^{1/2} \frac{1}{(a + \delta)^{1/2}}. \tag{77}
$$

Taking $\delta = \sqrt{c}$ and letting $c \to \infty$ in (77), we obtain (76). This concludes the proof of (54). \hfill \Box

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