Mass and angular momenta of Kerr anti-de Sitter spacetimes in Einstein–Gauss–Bonnet theory

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Abstract
We compute the mass and angular momenta of rotating anti-de Sitter spacetimes in Einstein–Gauss–Bonnet theory of gravity using a superpotential derived from standard Noether identities. The calculation relies on the fact that the Einstein and Einstein–Gauss–Bonnet vacuum equations are the same when linearized on maximally symmetric backgrounds and uses the recently discovered D-dimensional Kerr–anti-de Sitter solutions to Einstein’s equations.

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In [1], Gibbons, Lu, Page and Pope found D-dimensional Kerr–anti-de Sitter solutions of Einstein’s equations in vacuo. In [2], Gibbons, Perry and Pope calculated the angular momenta $J_{(D)i}$ of these rotating spacetimes using Komar’s integrals. They also obtained the mass $E_{(D)}$ of rotating black holes using the first law of thermodynamics. In [3], the mass of these rotating spacetimes was computed using the covariant Katz–Bičák–Lynden Bell (KBL) superpotential [4], that is by means of classical Noether identities. The results confirm those of [2] and hold whether the source is a black hole or a ‘star’ because they do not depend on the actual source of curvature\textsuperscript{4}.

In Einstein–Gauss–Bonnet (EGB) theory in five dimensions, the static spherically symmetric Schwarzschild-like solution was obtained in [5] and its mass was computed by various methods, including the use of the KBL superpotential extended to the Einstein–Gauss–Bonnet Lagrangian; see [6].

The rotating Kerr-like solution of the EGB equations is not known (apart from the topologically special solution given in [7]). However, on one hand, the Einstein and Einstein–Gauss–Bonnet vacuum equations are the same when linearized on a maximally symmetric background (see [3] and below): hence the D-dimensional Kerr–anti-de Sitter solutions to

\textsuperscript{4} In the second version of their paper, Gibbons Perry and Pope showed that the mass can also be obtained using the classical Ashtekar–Magnon–Das conformal boundary formula.
the Einstein field equations found in [1] are also asymptotic solutions to the Einstein–Gauss–Bonnet equations. On the other hand, only the asymptotic form of the metric is required to define and compute the mass and angular momenta of a given spacetime when using superpotentials derived from Noether identities.

The object of this paper is to use these two remarks to compute the mass and angular momenta of Kerr–AdS spacetimes in EGB theory by means of the KBL superpotential proposed in [6].

The Einstein–Gauss–Bonnet Lagrangian can be written as (see, e.g., [6])

\[ L = -\frac{2}{\Lambda} + R + \alpha R^\mu\nu\rho\sigma P_{\mu\nu\rho\sigma} \]

where \( \Lambda \) is a (negative) cosmological constant, \( \alpha \) a coupling constant which is zero in Einstein’s theory and \( R_{\mu\nu\rho\sigma}, R_{\mu\nu}, \) and \( R \) are the Riemann tensor, Ricci tensor and curvature scalar of the metric \( g_{\mu\nu} \). Brackets stand for anti-symmetrization: \( f_{\mu\nu} = \frac{1}{2}(f_{\mu\nu} - f_{\nu\mu}) \).

The equations of motion derived from that Lagrangian are

\[ R_{\mu\nu} + 2\alpha R^\rho\lambda\sigma P_{\nu\rho\lambda\sigma} - \frac{1}{2} \delta_{\mu}^\nu R + \delta_{\mu}^\nu \Lambda \frac{l^2}{\Lambda^2} L = 0. \]

We linearize them, that is, we set

\[ g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}. \]

We choose the background \( \overline{g}_{\mu\nu} \) to be the metric of a maximally symmetric spacetime \( \overline{M} \) solving the field equations (2). Since they are quadratic in the Riemann tensor there are two solutions, defined by

\[ \overline{R}_{\mu\nu\rho\sigma} = -\frac{1}{\overline{L}^2}(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}) \quad \text{where} \quad \frac{1}{\overline{L}^2} = \frac{1}{2\tilde{\alpha}} \left( 1 \pm \sqrt{1 - \frac{4\tilde{\alpha}}{l^2}} \right) \]

with \( \tilde{\alpha} \equiv (D - 3)(D - 4)\alpha \) and \( l^2 \equiv -\frac{(D-1)(D-2)}{M} \), \( D \) being the dimension of spacetime. We shall choose the lower sign for \( \overline{L}^2 \) so that the limit when \( \alpha = 0 \) solves the Einstein equations. (When \( \Lambda \) and \( \alpha \) are such that \( \sqrt{1 - 4\tilde{\alpha}/l^2} = 0 \) the two roots coalesce into a single double root.) It is an exercise to show that at linear order the field equations (2) read (a result already obtained in [5] for \( \Lambda = 0 \))

\[ \overline{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu}^\nu R + \frac{l^2}{\Lambda^2} \overline{L}^2 = O(h^2) \]

where (\( \overline{\nabla}_{\mu} \) being the covariant derivative associated with \( \overline{g}_{\mu\nu} \))

\[ R_{\mu\nu} = -(D - 1)\frac{1}{\overline{L}^2} (\delta_{\mu}^\nu - h_{\mu}^\nu) + \frac{1}{2} \left[ \left( \nabla_{\mu} (\nabla_{\nu} h_{\rho}^{\mu\nu} + \nabla_{\nu} h_{\rho}^{\mu\nu} - \nabla_{\rho} h_{\mu}^{\nu\nu}) - \nabla_{\rho} \nabla_{\nu} h_{\mu}^{\nu\nu} \right) + O(h^2) \]

is the linearized Ricci tensor on \( \overline{M} \). Hence the linearized Einstein–Gauss–Bonnet equations are the same as the linearized pure Einstein equations with effective cosmological constant \( \Lambda_e \equiv \Lambda \frac{l^2}{\overline{L}^2} \), and up to the overall factor \( \sqrt{1 - 4\tilde{\alpha}/l^2} \) that we shall assume not to be zero.

The general D-dimensional Kerr–AdS metrics solutions to Einstein’s equations found in [1] therefore also solve the EGB equations at linear order. As an example, the five-dimensional Einstein–Kerr-AdS metric reads, in Kerr–Schild ellipsoidal coordinates \( x^\mu = (t, r, \theta, \phi, \psi) \)

\[ ds^2 = dx^2 + \frac{2m}{U} (h_{\mu} dx^\mu)^2 \]
with $d\mathfrak{s}^2$ the AdS line element of $\mathcal{M}$:

$$d\mathfrak{s}^2 = -\frac{1 + r^2/L^2}{\Sigma_a \Sigma_b} \, dt^2 + \frac{r^2 \rho^2}{(1 + r^2/L^2)(r^2 + a^2)(r^2 + b^2)} \, dr^2 + \frac{\Delta_\rho}{\Delta_\theta} \, d\theta^2$$

$$+ \frac{r^2 + a^2}{\Sigma_a} \sin^2 \theta \, d\phi^2 + \frac{r^2 + b^2}{\Sigma_b} \cos^2 \theta \, d\psi^2$$

(8)

where $\Delta_\rho \equiv \Sigma_a \cos^2 \theta + \Sigma_b \sin^2 \theta$, where $\rho^2 \equiv r^2 + L^2(1 - \Delta_\rho)$ and where $\Sigma_a$ and $\Sigma_b$ are related to the rotation parameters $a$ and $b$ by $\Sigma_a \equiv 1 - a^2/L^2$, $\Sigma_b \equiv 1 - b^2/L^2$. As for the function $U$ and the null vector $h_\mu$ they are given by $U = \rho^2$ and

$$h_\mu \, dx^\mu = \frac{\Delta_\rho}{\Sigma_a \Sigma_b} \, dt + \frac{r^2 \rho^2}{(1 + r^2/L^2)(r^2 + a^2)(r^2 + b^2)} \, dr + \frac{a \sin^2 \theta}{\Sigma_a} \, d\phi + \frac{b \cos^2 \theta}{\Sigma_b} \, d\psi.$$  

(9)

Note that the full Ricci tensor of the Kerr–Schild metric (7)–(9) is linear in $\hat{d}$, defined as

$$\hat{d} \equiv \Sigma_a \rho \sin^2 \theta + \Sigma_b \rho \sin^2 \theta,$$

where $\hat{d}$ is the killig vector associated with time translations with respect to a non-rotating sphere at infinity. If the vector $\xi^\mu$ is evaluated on the AdS spacetime $\mathcal{M}$, then the associated charges are the angular momenta.

To obtain the mass and angular momenta of the 5D-Kerr-like solution of EGB theory we inserted metric (7)–(9) in the definitions (10)–(13) and took the large $r$ limit (using Maple and $GR$ tensor). The result is

$$M(\delta) = \sqrt{1 - \frac{4a}{L^2} \left[ \frac{m \pi}{4(\Sigma_a \Sigma_b)^2} (2 \Sigma_a + 2 \Sigma_b - \Sigma_a \Sigma_b) \right]}, \quad \text{and} \quad J_{5a} = \sqrt{1 - \frac{4a}{L^2} \left[ \frac{\pi ma}{2 \Sigma_a \Sigma_b} \right]}.$$  

(14)
We calculated similarly the mass and angular momentum in \( D = 6, 7, 8 \) dimensions in the case when there is only one non-zero rotation parameter. The results can be cast under the form \( (D = 5, 6, 7, 8) \)

\[
M_D = \sqrt{1 - \frac{4\bar{a}}{\ell^2} \frac{m \mathcal{V}_{D-2}}{4\pi \Sigma_a^2} \left[ 1 + \frac{(D - 4)}{2} \Sigma_a \right]}, \quad J_D = \sqrt{1 - \frac{4\bar{a}}{\ell^2} \frac{ma \mathcal{V}_{D-2}}{4\pi \Sigma_a^2}}
\]

(15)

where \( \mathcal{V}_{D-2} \) is the volume of the \((D - 2)\)-unit sphere. Finally, if one extrapolates to all dimensions the asymptotic expressions for the Kerr–AdS Riemann tensor obtained with Maple and GR tensor in \( D = 5, 6, 7, 8 \) dimensions\(^5\), one finds (by hand) the mass and angular momenta of the general \( D \)-dimensional Kerr-like solutions of Einstein–Gauss–Bonnet theory as

\[
M_{2n} = \sqrt{1 - \frac{4\bar{a}}{\ell^2} \left[ \frac{m \mathcal{V}_{2n-2}}{4\pi \Sigma} \sum_{i=1}^{i=n-1} \frac{1}{\Sigma_i} \right]}, \quad M_{2n+1} = \sqrt{1 - \frac{4\bar{a}}{\ell^2} \left[ \frac{m \mathcal{V}_{2n-2}}{4\pi \Sigma} \left( \sum_{i=1}^{i=n} \frac{1}{\Sigma_i} - \frac{1}{2} \right) \right]}
\]

\[
J_{(D)i} = \sqrt{1 - \frac{4\bar{a}}{\ell^2} \left[ \frac{m \mathcal{V}_{D-2}}{4\pi \Sigma} a_i \right]},
\]

(16)

where \( \Sigma = \prod_{i=1}^{i=n-1} \Sigma_i \) for \( D = 2n \) and \( \Sigma = \prod_{i=1}^{i=n} \Sigma_i \) for \( D = 2n + 1 \). Note that neither the background nor the vectors \( k^\mu_E \) and \( k^\mu_{GB} \) enter the calculation of the angular momenta. However, they are not given simply by Komar’s integrals because the first term in (12) is not zero. Expressions (14)–(16) have the right limits. When \( a = 0 \) (Einstein’s theory) they reduce to the results obtained in [2, 3]. When there is no rotation \( (a_t = 0, \Sigma_i = 1) \) masses (16) reduce to the result obtained in [6, 8] (noting that the coefficients of metric (7) tend to \( g_{t\ell} \simeq 1/g_{rr} \simeq r^2/L^2 - 2m/r^{D-3} \)).

The decisive check of formulae (14)–(16), that is of the pertinence of proposal (12) made in [6] for the vector \( k^\mu_{GB} \), will be possible when the full geometry of the Kerr-like rotating black holes is known, by seeing if, with such definitions, the first law of thermodynamics is still satisfied.

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\(^5\) The structure of the relevant components of the asymptotic Riemann tensor is easily guessed and depends on two constants only. For example: \( \lim_{\ell \to \infty} (R^{00}_{00} - \bar{R}^{00}_{00}) = \frac{m L^2}{\ell^2} (D - 1) \left[ c_1 (D - 3) W - c_2 \right] \), where \( W = \sum_{i=0}^{D/2} \mu_i^2 \) with \( \sum_{i=0}^{D/2} \mu_i^2 = 1 \) and \( \Sigma_{D/2} = 1 \) if \( D \) is even.

\(^6\) Since the overall factor in (14)–(16) is the same as that which appears in (5) one may conjecture that in the general Lovelock theory in \( D \) dimensions (whose Lagrangian is the sum of the first \( D/2 \) dimensionally continued Euler forms) the expressions for the masses and angular momenta will also be proportional to their values in Einstein’s theory with an overall coefficient which is zero when the equation for the maximally symmetric space curvature has one single root of maximal multiplicity \([D/2]\).
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