Damping of gravitational waves in 2-2-holes

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A 2-2-hole sourced by a relativistic gas has a calculable area-law entropy. This leads to an estimate of the damping of a gravitational wave as it travels to the center of the 2-2-hole and back out again. We identify two frequency dependent effects that greatly diminish the damping. This damping is important for determining the echo signal and for taming the ergoregion instability.

The description of a gravitational wave around a 2-2-hole [1] reduces to a radial equation that describes a wave in a 1D cavity [2, 3]. Low frequencies waves tend to be trapped due to a boundary condition at one end \( r = 0 \) and a potential barrier at the other end \( r = 3GM \). The cavity structure implies a resonance spectrum and possible echoes in the time domain [4–6]. But how is this picture influenced by the gravitational wave interacting with the matter inside the 2-2-hole? As realized in [7] and further considered in [8], a 2-2-hole is most naturally sourced by a relativistic gas. The properties of the gas are well determined, and this enables our question to be addressed.

We start with the Hawking formula [9] for the rate of energy loss or damping rate experienced by a gravitational wave traveling through a medium with shear viscosity \( \eta \),

\[
\frac{1}{E} \frac{dE}{dt} = -16\pi G \eta. \tag{1}
\]

This only applies when the frequency of the wave is much less than the collisional frequency \( \nu_c \) in the medium. This is typically not the case and we shall return to this point shortly. \( \nu_c \) is proportional to the scattering cross section between particles which in turn is proportional to the square of some effective coupling strength. We are considering a highly relativistic gas and so we can write

\[
\nu_c = \tilde{\alpha}^2 T, \tag{2}
\]

where \( \tilde{\alpha} \) is the dimensionless coupling and \( T \) is the temperature. We shall use this relation to define \( \tilde{\alpha} \). The viscosity-to-entropy-density ratio will depend on this coupling,

\[
\frac{\eta}{s} = \mathcal{V}(\tilde{\alpha}). \tag{3}
\]

The classic analysis of viscosity (or the leading perturbative result) gives \( \mathcal{V}(\tilde{\alpha}) \propto \tilde{\alpha}^{-2} \), but we
might not be in the regime of small coupling and so we keep the general function.

We are assuming that the system can be characterized by a single \( \alpha \). This is a simplification because different particle species in the gas can interact with different couplings. Also there could be some dependence of the coupling on the spatially varying temperature due to the energy dependence of running couplings. It remains to refine the present work to include these effects.

The cavity description is easiest when using the tortoise radial coordinate \( x \) defined by
\[
dx = \left( -g_{rr}/g_{tt} \right)^{1/2} dr,
\]
where \( x = 0 \) corresponds to \( r = 0 \) and \( \Delta x \) is the size of the cavity. The matter in the cavity has an entropy density \( s(x) \), the entropy per unit area per unit \( x \). The time \( t \) may be traded for \( x \) since the speed of light is unity in these coordinates. We further trade \( x \) for a dimensionless coordinate \( \tilde{x} = x / (GM) \) and define \( \tilde{s}(\tilde{x}) = GM s(GM \tilde{x}) \) so that
\[
\frac{1}{\tilde{E}} \frac{d\tilde{E}}{d\tilde{x}} = -16\pi \nu(\tilde{\alpha})G \tilde{s}(\tilde{x}).
\]
From this we can define a damping factor \( R_{\text{damp}} \) that gives the fraction of wave energy remaining after the round trip travel time \( \Delta t = 2\Delta x = 2GML \),
\[
R_{\text{damp}} = \exp \left( -32\pi \nu(\tilde{\alpha})G \int_0^L \tilde{s}(\tilde{x})d\tilde{x} \right).
\]
\( R_{\text{damp}} = 1 \) means no damping. \( L \) is the dimensionless length of the cavity. The truncated Kerr black hole is used as an approximation for a spinning 2-2-hole, and this gives \([6, 10]\)
\[
L = 2 \eta \log \left( \frac{GM}{\ell_{\text{Pl}}} \right) \left( 1 + \frac{1 - \chi^2}{2} \right)^{-\frac{1}{2}} (1 + z)
\]
for spin \( \chi \) and redshift \( z \). The redshift factor ensures that the implied \( \Delta t \) is in the lab frame. The cavity may be said to be stretched since \( L \) is around 400 or more. Empirically it seems that \( \eta \approx 1.7 \) \([11]\).

The temperature profile of the gas is known after solving the quadratic gravity field equations for a non-spinning 2-2-hole. This then determines the total entropy of the gas within the \( 3GM \) radius \([7, 8]\). We can express this as the total entropy per unit area, which is \( \int_0^{\Delta x} s(x)dx \) or
\[
\frac{S}{\text{Area}} = \int_0^L \tilde{s}(\tilde{x})d\tilde{x} = \frac{\zeta}{4G}.
\]
\( \tilde{s}(\tilde{x}) \) is finite and it peaks at \( \tilde{x} = 0 \). It has dependence on the metric functions that have nontrivial scaling behavior, and this is responsible for the area-law scaling for \( S \). The value of \( S \) differs from the black hole result by the factor \( \zeta \approx 0.75N^{\frac{1}{2}}/\sqrt{m_2/m_{\text{Pl}}} \). \( N \) is \( g_b + 7g_f/8 \) where \( g_b \) and \( g_f \) are the number of bosonic and fermionic degrees of freedom of any mass, and \( m_2 \)
is the mass of the unstable spin-2 mode of quadratic gravity. We shall simply carry this result over to spinning 2-2-holes since we lack explicit spinning 2-2-hole solutions. With that caveat, (7) gives a simple result for the damping factor,

\[ R_{\text{damp}} = \exp(-8\pi\mathcal{V}(\bar{\alpha})\zeta). \]  

(8)

Both \( G \) and \( M \) have dropped out, as made possible by having a self-gravitating system. For a typical value of \( \zeta \) we use 2.5, for example when \( N \approx 125 \) and \( m_2 \approx m_{\text{pl}} \).

Since \( \mathcal{V}(\bar{\alpha}) \) has a theoretical lower bound of \( 1/(4\pi) \) [12], the result in (8) implies strong damping independent of the gravitational wave frequency \( \omega \). But this is not correct. To account for an \( \omega \) that is not much less than the collisional frequency \( \nu_c \), a frequency dependent suppression factor \( (1 + (\omega/\nu_c)^2)^{-1} \) should multiply the Hawking rate (1). This factor has been discussed recently in [13]. It is present because the shear induced by the gravitational wave changes sign over the period of the wave, and so the time over which the particle can sample the shear is limited by this time, rather than the collisional time, when \( \omega \) is larger than \( \nu_c \).

Since the viscosity typically already has a \( 1/\nu_c \) dependence, the net effect is that the damping increases linearly with \( \nu_c \) up to \( \nu_c \approx \omega \), after which it falls like \( 1/\nu_c \). But for a given \( \nu_c \), the damping rate is reduced by a factor \( \nu_c^2/\omega^2 \) when \( \omega \gg \nu_c \).

For the truncated Kerr geometry there is a special frequency proportional to the spin, \( \omega_0 \), and the wave in the cavity has an effective frequency \( \omega - \omega_0 \) and period \( 2\pi/|\omega - \omega_0| \). The relevant suppression factor is then \( (1 + ((\omega - \omega_0)/\nu_c)^2)^{-1} \). We take \( \nu_c = \bar{\alpha}^2 T_\infty \) from (2) where \( T_\infty \) is the analog of the Hawking temperature for a 2-2-hole. In fact \( ST_\infty = S_{\text{BH}} T_{\text{Hawking}} \) and so \( T_\infty = T_{\text{Hawking}}/\zeta = 1/(8\pi GM \zeta) \) where once again we use the spinless result. Note that there are proper, \( \tilde{x} \)-dependent versions of \( \omega - \omega_0 \) and \( \nu_c \) (and the temperature) that are all obtained by multiplying by \( \sqrt{-g_{tt}(r(\tilde{x}))} \). But this metric factor cancels out in the ratio \( (\omega - \omega_0)/\nu_c \). In terms of the dimensionless frequency \( \tilde{\omega} = GM \omega \) we now have a frequency dependent damping factor,

\[ R_{\text{damp}}(\tilde{\omega}) = \exp \left( -8\pi \mathcal{V}(\bar{\alpha})\zeta \left[ 1 + \left( 8\pi(\tilde{\omega} - \tilde{\omega}_0) \frac{\zeta}{\bar{\alpha}^2} \right)^2 \right]^{-1} \right). \]  

(9)

This additional factor in the exponent quite dramatically reduces its magnitude as \( \tilde{\omega} \) moves away from \( \tilde{\omega}_0 \), and thus reduces damping, i.e. keeps \( R_{\text{damp}} \) closer to unity. It also means that a weaker coupling \( \bar{\alpha} \) implies less damping, at least for \( \tilde{\omega} \) not very close to \( \tilde{\omega}_0 \).

There is another effect that reduces damping for \( \tilde{\omega} \) very close to \( \tilde{\omega}_0 \). A (nearly) trapped wave in the cavity is a superposition of the quasi-normal modes, and these QNMs are very close to being standing waves. Their overall amplitude only slowly changes due to an outgoing wave of small amplitude on the outside. Standing waves have zero net flux across any point
\( \bar{x} \) and the energy density can vary with \( \bar{x} \) and vanish at the nodes of the wave. If \( \psi(\bar{\omega}, \bar{x}) \) is the wave amplitude for standing wave, it is typical that the energy density profile \( e(\bar{\omega}, \bar{x}) \) is proportional to \( |\psi|^2 \). When the wave satisfies a Dirichlet boundary condition at \( \bar{x} = 0 \), as it does for a spinless 2-2-hole, then the energy density will be small near the \( \bar{x} = 0 \) end of the cavity over a distance that grows with the wavelength.

The amount of wave energy being lost locally will vary from point to point, and this will be proportional to the product of the local densities of the wave energy and the matter entropy. We may define an entropy profile \( \hat{s}(\bar{x}) \) such that \( \int_0^L \hat{s}(\bar{x}) d\bar{x} = 1 \). The total damping rate thus has another \( \bar{\omega} \) dependent factor given by the overlap between these two profiles \( \int_0^L \hat{s}(\bar{x}) e(\bar{\omega}, \bar{x}) d\bar{x} \). As we have said, the entropy profile peaks at \( \bar{x} = 0 \). A numerical example of the entropy density as a function of \( r, s(r) \), was shown in Fig. 18 of [7]. When \( r \) is used instead of \( x \), most of the cavity size is compressed into a small region where \( (g_{rr}/g_{tt})^{1/2} \) spikes to a large finite value just slightly outside \( 2GM \). This spike is the source of the dominant \( \log(M) \) enhanced contribution to the cavity size \( L \). Fig. 18 shows that most of the entropy is located at smaller \( r \). The interior contribution to \( L \) that is independent of \( \log(M) \) is \( 2\sqrt{a_2/b_2} \approx 17.2 \), and so Fig. 18 shows that \( \hat{s}(17.2) \approx 0.1\hat{s}(0) \). From this we construct an approximate estimate of the entropy profile \( \hat{s}(\bar{x}) \) as shown in Fig. 1. Below we consider cavity sizes of \( L = 400 \) and \( 600 \), and thus Fig. 1 shows that the entropy density is concentrated near the \( \bar{x} = 0 \) end of the cavity.

![FIG. 1. The entropy profile.](image)

We return to the wave energy profile when the truncated Kerr black hole is used to approximate a spinning 2-2-hole. We use a boundary condition at \( \bar{x} = 0 \) analogous to the Dirichlet

\[ \hat{s}(\bar{x}) = \frac{1}{9} \frac{1}{1 + 0.03\bar{x}^2} \]

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\(^1\) This is from the leading behavior of the metric at small \( r \), which is \(-g_{00} = b_2r^2 + \ldots \) and \( g_{rr} = a_2r^2 + \ldots \).
boundary condition of the spinless 2-2-hole. The form of the wave amplitude is then

$$\psi(\tilde{\omega}, \tilde{x}) \propto \exp(-i(\tilde{\omega} - \tilde{\omega}_0)\tilde{x}) - R(\tilde{\omega}) \exp(i(\tilde{\omega} - \tilde{\omega}_0)\tilde{x}).$$

(10)

$R(\tilde{\omega})$ is a known real function determined by requiring that the net flux, the difference between the ingoing and outgoing fluxes, vanishes [2]. From this we determine the energy density profile (normalized so that its $\tilde{x}$-average value is unity),

$$e(\tilde{\omega}, \tilde{x}) = \frac{4R(\tilde{\omega}) \sin((\tilde{\omega} - \tilde{\omega}_0)\tilde{x})^2 + (R(\tilde{\omega}) - 1)^2}{1 + R(\tilde{\omega})^2}.$$

(11)

This does not necessarily have points where the energy density vanishes. But the key is that $R(\tilde{\omega}_0) = 1$, and as well $R(\tilde{\omega})$ is not rapidly varying. Thus for frequencies close to $\tilde{\omega}_0$ (where the wavelength is large) the energy profile is still small where the entropy profile is large. And as $\tilde{\omega} \to \tilde{\omega}_0$ the overlap integral still tends to zero and $R_{\text{damp}} \to 1$. In fact the overlap integral has little impact unless $\tilde{\omega}$ is close to $\tilde{\omega}_0$, and so there is very little dependence on the behavior of $R(\tilde{\omega})$ away from $\tilde{\omega}_0$.

Including the overlap integral gives our final result for the damping factor,

$$R_{\text{damp}}(\tilde{\omega}) = \exp\left(-8\pi \mathcal{V}(\tilde{a}) \zeta \left[1 + \left(8\pi(\tilde{\omega} - \tilde{\omega}_0)\frac{\zeta}{\tilde{a}^2}\right)^2\right]^{-1} \int_0^L \hat{s}(\tilde{x}) e(\tilde{\omega}, \tilde{x}) d\tilde{x}\right).$$

(12)

In Fig. 2 we compare $R_{\text{damp}}$ to $R_{\text{BH}}$, the reflection amplitude for a Kerr black hole. $|R_{\text{BH}}| > 1$ signals superradiant amplification and a possible ergoregion instability for the horizonless 2-2-hole. The product $|R_{\text{BH}}| R_{\text{damp}}$ is also shown in the figure and when $|R_{\text{BH}}| R_{\text{damp}} < 1$, the damping is sufficient to remove the instability. For the modeling of the spectra in [11] we simply used a constant $R_{\text{damp}}$ (0.995 and 0.992 for $\chi = 2/3$ and 0.81 respectively). Our new more realistic $R_{\text{damp}}(\tilde{\omega})$ quite quickly approaches unity for frequencies away from $\tilde{\omega}_0$. The $R_{\text{damp}}(\tilde{\omega})$ dependent curves that are visible in Fig. 2 are essentially only dependent on the combination $\mathcal{V}(\tilde{a})\tilde{a}^4$. When this has a value of 0.005 the left plot with $\chi = 2/3$ shows that the instability is easily removed. In the right plot with higher spin $\chi = 0.81$ we see that the same value is just on the stability edge. Given that event GW170729 has such a spin, this sets a bound $\mathcal{V}(\tilde{a})\tilde{a}^4 \gtrsim 0.005$.

In Fig. 3 we look at the full range of the product $R_{\text{damp}} |R_{\text{BH}}|$ in a region closer to $\tilde{\omega}_0$, to see the impact of the overlap integral. We see that it significantly reduces damping close to $\tilde{\omega}_0$. The two plots show that in this region we can also see the $\tilde{a}$ dependence of the damping for fixed $\mathcal{V}(\tilde{a})\tilde{a}^4$. The plot on the right has $\mathcal{V}(\tilde{a}) = 1/(4\pi)$, the minimum value, and a corresponding

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2 The QNM frequencies form a discrete set $\tilde{\omega} = \tilde{\omega}_n$. For the long-lived modes the effective boundary condition from the potential barrier is close to being Neumann and so for these modes $\tilde{\omega}_n - \tilde{\omega}_0 \approx \frac{\pi}{4}(1 + n)$ for any integer $n$. 
FIG. 2. The damping factor $R_{\text{damp}}$ is compared to the Kerr black hole reflection amplitude $|R_{\text{BH}}|$ and their product. The difference in the two plots is due mainly the different spin $\chi$ and not the different cavity size $L$. The frequency where $R_{\text{damp}}$ spikes back up is $\tilde{\omega}_0$. The values $L = 600$ and $\chi = .81$ are inspired by event GW170729 [11].

larger $\tilde{\alpha}$, and we see that this produces the minimum amount of damping for fixed $\mathcal{V}(\tilde{\alpha})\tilde{\alpha}^4$.

FIG. 3. The vertical and horizontal plot ranges are changed to show the effect of the overlap integral on the product $R_{\text{damp}}|R_{\text{BH}}|$. The vertical lines show QNM frequencies and their spacing $\pi/L$. The two plots show different choices of $\mathcal{V}$ and $\tilde{\alpha}$ with $\mathcal{V}\tilde{\alpha}^4 = 0.005$.

In Fig. 4 we show the effect that the new $R_{\text{damp}}(\tilde{\omega})$ has on the observable spectrum. This is the spectrum reconstructed from a real projection of the strain $h(t)$ [3, 11], given that the strain data involves such a projection. Relative to the previously used constant $R_{\text{damp}}$, we see the increased suppression around $\tilde{\omega}_0$ and less suppression elsewhere. The spectrum has a two component structure and the lower “negative frequency” component of this spectrum is now relatively enhanced due to essentially no damping at negative frequencies. The $\tilde{\alpha}$ dependence for fixed $\mathcal{V}(\tilde{\alpha})\tilde{\alpha}^4$ illustrated in the Fig. 3 has very little effect on these spectra since it is occurring in the region around $\tilde{\omega}_0$ ($\sim 250$ or $340$ Hz respectively) where the spectrum is already suppressed. Thus the observed spectra essentially just constrain $\mathcal{V}(\tilde{\alpha})\tilde{\alpha}^4$.

We have seen that there may be some critical value of the spin above which the ergoregion instability is not completely damped. It could be that the instability extracts energy from the rotational energy and that an 2-2-hole with a high spin will spin down to near the critical
spin. The interesting question is how violent this process is. Another question is the effect of accretion on the instability. Accretion means that matter at a temperature much higher than $T_{\infty}$ is entering the cavity. Until this matter equilibrates with the matter already in the cavity, it can cause gravitational wave damping unsuppressed by either of the two suppression mechanisms we have identified. It remains to determine the extent to which this can increase stability.

Our analysis suggests that $\mathcal{V}(\tilde{a})\tilde{a}^4$ cannot be much larger or much smaller than 0.005. If it was much larger then there would be more pronounced damping in a larger region around $\omega_0$, and this may be less consistent with the findings in [11]. If it was much smaller then the smaller damping may lead to an instability for a spin around 0.81, as for GW170729. This restriction on $\mathcal{V}(\tilde{a})\tilde{a}^4$ is interesting since it characterizes the fundamental interactions among particles at proper temperatures that range up to Planck energies and beyond. More theoretical knowledge of the coupling dependence of the viscosity-to-entropy ratio $\mathcal{V}(\tilde{a})$ could then separately constrain both this ratio and the value of the effective coupling.

FIG. 4. The spectra are shown as amplitude spectral densities. The left plots use the old constant $R_{\text{damp}}$ and the right plots use the new frequency dependent $R_{\text{damp}}$ when $\nu = 1$ and $\tilde{a}^4 = 0.005$. We have converted frequencies to Hz by using a mass of $50M_\odot$. All spectra are normalized so that the peak strain in the time domain is $5 \times 10^{-22}$. $\kappa$ is another parameter used to model the spectra [3, 11].
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