SPHERICITY OF $\kappa$-CLASSES AND POSITIVE CURVATURE VIA BLOCK BUNDLES

GEORG FRENCK
APPENDIX WITH JENS REINHOLD

ABSTRACT. Given a manifold $M$, we completely determine which rational $\kappa$-classes are non-trivial for (fiber homotopy trivial) $M$-bundles over the $k$-sphere, provided that the dimension of $M$ is large compared to $k$. We furthermore study the vector space of these spherical $\kappa$-classes and give an upper and a lower bound on its dimension. The proof is based on the classical approach to studying diffeomorphism groups via block bundles and surgery theory and we make use of ideas developed by Krannich–Kupers–Randal-Williams.

As an application, we show the existence of elements of infinite order in the homotopy groups of the spaces $\mathcal{R}_{\text{Ric}>0}(M)$ and $\mathcal{R}_{\text{Sec}>0}(M)$ of positive Ricci and positive sectional curvature, provided that $M$ is Spin, has a non-trivial rational Pontryagin class and admits such a metric. This is done by showing that the $\kappa$-class associated to the $\hat{A}$-class is spherical for such a manifold.

In the appendix co-authored by Jens Reinhold it is (partially) determined which classes of the rational oriented cobordism ring contain an element that fibers over a sphere of a given dimension.

1. Introduction

1.1. Main results. For a closed oriented manifold $M$ of dimension $d$ let $\text{Diff}(M)$ denote the group of orientation preserving diffeomorphisms of $M$ with $B\text{Diff}(M)$ the associated classifying space. Since manifold bundles with typical fiber $M$ are classified by maps into this space, its cohomology ring is the ring of characteristic classes of $M$-bundles. For any fiber bundle $\pi: E \to B$ with fiber $M$ there is a map

$$H^{4m}(BO(d); \mathbb{Q}) \to H^{4m-d}(B; \mathbb{Q})$$

sending a characteristic class $c \in H^{4m}(BO(d); \mathbb{Q})$ to $\kappa_c(E) := \pi_!(c(T_{\pi}E))$ where $\pi_!: H^*(E) \to H^{*-d}(B)$ is the Gysin-homomorphism and $T_{\pi}E$ is the vertical tangent bundle of $\pi$. $\kappa_c(E)$ is called the generalized Miller-Morita-Mumford class or simply $\kappa$-class associated to $c$. For the universal $M$-bundle $E_M \to B\text{Diff}(M)$ we hence get universal $\kappa$-classes

$$\kappa_c \in H^{4m-d}(B\text{Diff}(M); \mathbb{Q}) \cong \text{Hom}(H_{4m-d}(B\text{Diff}(M)); \mathbb{Q})$$

and we call a $\kappa$-class spherical for $M$ if it pulls back nontrivially along the Hurewicz-Homomorphism $\pi_{4m-d}(B\text{Diff}(M)) \to H_{4m-d}(B\text{Diff}(M))$. 

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In this article we study the sphericity of $\kappa$-classes on fiber homotopy trivial bundles. Let $\text{hAut}(M)/\text{Diff}(M)$ denote the classifying space for fiber homotopy trivial $M$-bundles. Analogously to the above, we define $\kappa_c$ to be $h$-spherical for $M$ if it pulls back non-trivially along

$$
\pi_{4m-d}(\text{hAut}(M)/\text{Diff}(M)) \to \pi_{4m-d}(B\text{Diff}(M)) \to H_{4m-d}(B\text{Diff}(M)).
$$

Clearly, $h$-spherical $\kappa$-classes are spherical. Our main result is the following.

**Theorem A.** Let $d < 4m \leq \min\left(\frac{4d-1}{3}, \frac{3d-5}{2}\right)$ and let $M^d$ be a simply connected manifold. Let furthermore $p = p_{i_1} \cdots p_{i_n}$ be a monomial in universal rational Pontryagin classes of total degree $4m$.

(i) If $p \neq p_m$, then $\kappa_p$ is $h$-spherical for $M$ if and only if there exists an $l$ such that

$$p_{i_1}(TM) \cdots \widehat{p_{i_l}(TM)} \cdots p_{i_n}(TM) \neq 0.$$

(ii) The class $\kappa_{p_m}$ is $h$-spherical for $M$ if and only if $M$ admits some non-trivial rational Pontryagin class.

**Remark 1.1.** (i) **Theorem A** classifies rational Pontryagin numbers of fiber homotopy trivial bundles over spheres, provided that the dimension of the sphere is small compared to the dimension $M$. Namely, there exists such an $M$-bundle bundle $E \to S^{4m-d}$ with $\langle p(E), [E] \rangle \neq 0$ if and only if $p$ satisfies the condition from the theorem.

(ii) For an $l$ as above we have $i_l \geq m - d/4$, otherwise the above product of Pontryagin classes of $M$ would have degree $* - 4i_l > d$.

(iii) It is known that no $\kappa$-class is spherical if $4m > 3d$ or if all rational Pontryagin classes of $M$ vanish (cf. [Wie19, Lemma 2.3] and [HSS14, Proposition 1.9], see also **Theorem A.3**, part (i)).

(iv) It follows from **Proposition A.9** and **Theorem A**, that the $\kappa$-class associated to $p_{i_1}^{n+k}$ is spherical but not $h$-spherical for $\mathbb{C}P^n \times \mathbb{C}P^{2k}$ for every $k > n \geq 3$.

(v) The bound for the degree $4m$ depends on the pseudoisotopy stable range. Therefore improvements of this range also improve the degree bound above. This bound can also be improved, if the manifold $M$ is $\ell$-connected for sufficiently large $\ell$ (cf. **Lemma 2.5**).

We furthermore study the vector space of $h$-spherical $\kappa$-classes for fiber homotopy trivial bundles. Consider the map

$$
\Psi^h_M : H^{4m}(BO(d); \mathbb{Q}) \to \text{Hom}(\pi_{4m-d}(\text{hAut}(M)/\text{Diff}(M)); \mathbb{Q})
$$

given by sending a characteristic class $c$ to the map $f \mapsto \langle \kappa_c, f_*[S^{4m-d}] \rangle$. The dimension of $\text{im} \Psi^h_M$ of course depends on $M$. In order to get estimates on this dimension, consider the map $\mathcal{P} : \mathbb{Q}[p_1(TM)] \to H^{4m}(BO(d); \mathbb{Q})$ given by sending a product $p_{i_1} \cdots p_{i_k}(TM)$ of Pontryagin classes of $M$ to the product $p_{i_1} \cdots p_{i_k} \cdot p_{m-\sum i_j}$ of universal Pontryagin classes. Also, let $i_{\text{min}}$ be the minimum positive integer $i$ such that $p_i(TM) \neq 0$ and $n_{\text{max}}$ be the maximum integer $n \geq 1$ such that $p_{i_{\text{min}}}(TM)^n \neq 0$.

**Theorem B.** If $d < 4m \leq \min\left(\frac{4d-1}{3}, \frac{3d-5}{2}\right)$ and $p_i(TM) \neq 0$ for some $i$, then

$$\text{im} \Psi^h_M = \text{im} \Psi^h_M \circ \mathcal{P} \text{ and } \text{dim} \text{im} \Psi^h_M \geq n_{\text{max}}.$$

If all Pontryagin classes of \( M \) are contained in the truncated polynomial algebra \( \mathbb{Q}[p_{\min}(TM)] \) generated by \( p_{\min}(TM) \), then equality holds in the above lemma: In this case \( \dim(\mathbb{Q}[p_{\min}(TM)]) = n_{\max} + 1 \) and \( \langle L_m \rangle \subset \ker \Psi_M^h \) because every fiber bundle over a sphere has vanishing signature. Hence \( \dim \ker \Psi_M^h \geq 1 \) and the claim follows from the rank-nullity theorem for finite dimensional vector spaces. This shows that in degrees as in Theorem B we have

\[
\dim \im \Psi_{\mathbb{C}P^n}^h = n, \quad \dim \im \Psi_{\mathbb{C}P^{n}} = \left\lfloor \frac{n}{2} \right\rfloor, \quad \dim \im \Psi_{O_{\mathbb{G}G^2}}^h = 2
\]

and hence the lower bound is sharp.

In order to describe the upper bound let \( p(n) \) be the number of partitions of \( n \in \mathbb{N} \) into sums of positive natural numbers. Note that \( \dim H^{4m}(BO; \mathbb{Q}) = p(m) \). Furthermore, for \( k \in \mathbb{N} \) we define \( p(n, k) \) to be the number of partitions of \( n \) into natural numbers \( \leq k \). Note that \( p(n, n) = p(n), \ p(n, 0) = 0, \ p(n, 1) = 1 \) and \( p(n, 2) = 1 + \lfloor n/2 \rfloor \). Furthermore \( p(n, k) = O(n^{k-1}) \). We first make the following observation: If \( i_1, \ldots, i_r \) is such that \( \sum i_j = 4m \) and \( i_j < (4m - d)/4 \) for all \( j \), then \( \Psi_M^h(p_{i_1} \cdots p_{i_r}) = 0 \). The upper bound is then an immediate consequence of this observation and the main difficulty of the following theorem lies in proving sharpness.

**Theorem C.** If \( d < 4m < \min\left(\frac{d-1}{4}, \frac{3d-5}{2}\right) \), then \( \dim \im \Psi_M^h \leq p(m) - p(m, m - \left\lfloor \frac{d+1}{4} \right\rfloor) - 1 \). There exist a simply connected manifold \( M \) in dimensions \( d \equiv 2, 3 (4) \) for which equality holds.

**Remark 1.2.** If \( d \equiv 0 (4) \) there exists a non-connected manifold \( M \) where every component is simply connected such that equality holds.

### 1.2. \( \hat{A} \)-multiplicativity and application to positive curvature

The \( \hat{A} \)-genus is one of the most interesting characteristic numbers. Recall, that an oriented manifold \( M \) is called an \( \hat{A} \)-multiplicative fiber in degree \( k \), if every oriented \( M \)-bundle over \( S^k \) has vanishing \( \hat{A} \)-genus. The results of Wiemeler and Hanke–Schick–Steimle mentioned above (cf. [Wie19, Lemma 2.3] and [HSS14, Proposition 1.9]) imply that manifolds of small dimension compared to \( k \) and manifolds with vanishing rational Pontryagin classes are \( \hat{A} \)-multiplicative fibers in degree \( k \). From the proof of Theorem B and computations done in [FR21, Lemma 2.5] we deduce that these conditions are actually necessary.

**Proposition 1.3.** Let \( k \geq 1 \). A manifold \( M \) of dimension \( d \geq \max(3k + 1, 2k + 5) \) is an \( \hat{A} \)-multiplicative fiber in degree \( k \) if and only if all its rational Pontryagin classes vanish.

The \( \hat{A} \)-genus is particularly interesting for applications to spaces of Riemannian metrics with lower curvature bounds. Let \( M \) be closed and let \( \mathcal{R}_{\text{scal} > 0}(M) \) denote the space of Riemannian metrics on \( M \) of positive scalar curvature, equipped with the Whitney \( C^\infty \)-topology and let \( \mathcal{R}_C(M) \) be a \( \text{Diff}(M) \)-invariant space which admits a \( \text{Diff}(M) \)-equivariant map to \( \mathcal{R}_{\text{scal} > 0}(M) \). Furthermore, let \( \text{Diff}(M, D) \subset \text{Diff}(M) \) denote the subgroup of those diffeomorphisms that fix an embedded disk \( D \subset M \) point-wise.

**Theorem D.** Let \( M^d \) be closed, simply connected Spin-manifold that has at least one non-vanishing rational Pontryagin class and let \( g \in \mathcal{R}_C(M) \). Let
$k \geq 1$ be such that $(d + k)$ is divisible by 4 and $k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)$. Then the map

$$
\pi_{k-1}(\text{Diff}(M, D)) \otimes \mathbb{Q} \longrightarrow \pi_{k-1}(\mathcal{R}_C(M)) \otimes \mathbb{Q}
$$

induced by the orbit map $f \mapsto f_*g$ is nontrivial.

Theorem D was the original motivation and the main theorem of the first version of the present article. For readers solely interested in the proof of this theorem we recommend the original version which is considerably more focused and available at 2104.10595v1.

Example 1.4. (i) The class of manifolds to which this theorem is applicable contains $\mathbb{C}P^{2n+1}$, $\mathbb{H}P^n$, $\mathbb{O}P^2$, as well as iterated products and connected sums of these with arbitrary Spin-manifolds.

(ii) The most interesting examples of spaces $\mathcal{R}_C(M)$ are the ones of positive or nonnegative sectional curvature and positive Ricci curvature metrics. Theorem D implies, that for $M$ and $k$ as above we have

$$
\pi_{k-1}(\mathcal{R}_{\text{Ric} > 0}(M)) \otimes \mathbb{Q} \neq 0 \quad \pi_{k-1}(\mathcal{R}_{\text{Sec} > 0}(M)) \otimes \mathbb{Q} \neq 0
$$

and

$$
\pi_{k-1}(\mathcal{R}_{\text{Sec} \geq 0}(M)) \otimes \mathbb{Q} \neq 0.
$$

provided the respective spaces are non-empty. The case of non-negative sectional curvature follows from a Ricci-flow argument, see [FR21, Proposition 3.3].

According to [Zil14] the only known examples of positively curved manifolds in dimensions $4k + 3$ for $k \geq 2$ are spheres. Also, all 7-dimensional examples have finite fourth cohomology (cf. [Esc92; Goe14; GKS04]). Therefore, the answer to the following question appears to be unknown. A positive answer would yield the first example of a closed manifold that admits infinitely many pairwise non-isotopic metrics of positive sectional curvature.

Question 1.5. Is there a positively curved manifold of dimension $4k + 3$, $k \geq 1$ with a non-vanishing rational Pontryagin class?

For positive Ricci and nonnegative sectional curvature, lots of examples for such manifolds are known, see Example 1.4.

1.3. Outline of the argument and obstructions to unblocking of block bundles. In [KKR21], Kranich–Kupers–Randal-Williams have proven that the $\kappa$-class associated to $\check{A}_3 \in H^{12}(BO(8); \mathbb{Q})$ is $h$-spherical for $\mathbb{H}P^2$. It turns out that their construction delivers an excellent blueprint for our generalization. Since [KKR21] is written rather densely, we decided to give a more detailed account of their argument in Section 2 before we go on to proving our main results. Let us give an outline of the construction first.

Instead of constructing an actual fiber one constructs a so-called block bundle (we recall the notion of block bundles and block diffeomorphisms in Section 2). The advantage of working with block bundles is that the $k$-th homotopy group $\pi_k(\text{hAut}(M)/\text{Diff}(M))$ of the classifying space for fiber homotopy trivial block bundles is isomorphic to the structure set $\mathcal{S}_\partial(D^k \times M)$ from surgery theory. The latter is accessible through the surgery exact sequence

$$
L_{k+d+1}(\mathbb{Z}\pi_1 M) \longrightarrow \mathcal{S}_\partial(D^k \times M) \longrightarrow N_\partial(D^k \times M) \overset{\sigma}{\longrightarrow} L_{k+d}(\mathbb{Z}\pi_1 M)
$$
where \( \mathcal{N}_0 \) denotes the set of normal invariants. We are interested in the case, where \( M \) is simply connected and \( (d+k) \) is divisible by 4, so the \( L \)-groups are given by 0 on the left and by \( \mathbb{Z} \) on the right. Hence, in order to construct an \( M \)-bundle it suffices to construct a normal invariant \( \eta \) with \( \sigma(\eta) = 0 \). It turns out, that the set of normal invariants is (rationally) isomorphic to the reduced real \( K \)-theory of \( S_k^1 \wedge M_+ \) which allows one to construct a normal invariant and hence a (fiber homotopy trivial) block bundle with prescribed Pontryagin classes. Fiber homotopy trivial block bundles over \( S^k \) can (rationally) be given the structure of an actual fiber bundle if \( k \) is (roughly) smaller than \( d/3 \) by a classical result of Burghelea–Lashof (cf. [BL82]).

The main work in the present article lies in choosing appropriate normal invariants. This way we can ensure that certain Pontryagin classes and numbers of the total space of the corresponding block bundle are zero or nonzero.

One further observation is, that the above construction of block bundles works regardless of the dimension \( k \) of the base sphere. The same cannot be true for actual fiber bundles over spheres, since the tangent bundle of the total space is stably isomorphic to its vertical tangent bundle which is a vector bundle of rank \( d \). Therefore all Pontryagin classes of degree \( * > 2d \) vanish. This has been previously observed in [ER14], where the authors go on to construct an \( \mathbb{HP}^2 \)-block bundle over \( S^{12} \) with \( p_5 \neq 0 \) which for the above reason cannot be “unblocked”. Utilizing the above construction of block bundles we can study this obstruction to “unblocking” more systematically.

Let \( I = (i_1, \ldots, i_s) \) and let \( M \) be a manifold such that \( p_I(TM) = p_{i_1} \cdot \cdots \cdot p_{i_s}(TM) \neq 0 \). If \( I = (0) \) assume additionally that \( M \) has some non-vanishing rational Pontryagin class. Let

\[
\overline{P}_{M,I} : \pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \to \mathbb{Q}
\]

be the map sending a block bundle \( E \) classified by \( f \) to the Pontryagin number \( p_{i_1} \cdot \cdots \cdot p_{i_s}(TM) \neq 0 \). If \( I = (0) \) assume additionally that \( M \) has some non-vanishing rational Pontryagin class. Let

\[
\pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \to \pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \xrightarrow{\overline{P}_{M,I}} \mathbb{Q} \to 0
\]

(i) The following sequence is exact for \( m > \frac{d+2|I|}{2} \).

\[
\pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \to \pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \xrightarrow{\overline{P}_{M,I}} \mathbb{Q} \to 0
\]

(ii) For \( 1 \leq n \leq n_{\text{max}} \) (cf. Equation 2) and \( m > \frac{d+2n_{\text{max}}}{2} \) there exists an \( n \)-dimensional subspace \( \mathcal{N} \subset \pi_{4m-d}(\text{BDiff}(M)) \otimes \mathbb{Q} \) of block-bundles that do not admit the structure of actual fiber bundles.

Remark 1.6. The same is true for fiber homotopy trivial bundles, i.e. the same corollary holds if \( \text{BDiff}(M) \) and \( \text{BDiff}(M) \) are replaced by \( \text{hAut}(M)/\text{Diff}(M) \) and \( \text{hAut}(M)/\text{Diff}(M) \).

1.4. Rationally fibering a cobordism class over a sphere. Given an integer \( k \geq 1 \), it is a classical problem that goes back to Conner–Floyd [CF65] to decide which cobordism class contains a manifold that fibers over \( S^k \). This has been studied in detail for \( k \leq 4 \) [Bur66; Neu71; Kah84a; Kah84b]. However, the classical approach relied on identifications like \( S^2 \cong \mathbb{CP}^1 \) or \( S^4 \cong \mathbb{HP}^1 \) and does not seem to work for larger \( k \). Theorem C can be rephrased to state that any rational cobordism class in degrees \( 4m \geq 20 \) fibers over \( S^4 \), provided
that the signature vanishes. In the Appendix, written with Jens Reinhold, we consider the question for bigger values of $k$ building on the methods developed in the paper. We show that in a given dimension $d \geq 36$ every (rational) cobordism class in the kernel of the signature homomorphism fibers over $S^k$ for every $k \leq 8$. We also obtain results for $k \geq 9$, see Theorem A.3.

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2. Preliminaries

Let $M$ be a closed oriented manifold of dimension $d$ and let $\text{Diff}(M)$ denote the group of orientation preserving diffeomorphisms of $M$. We denote by $\text{BDiff}(M)$ the classifying space for fiber bundles $E \to B$ with fiber $M$ and oriented vertical tangent bundle $T_\pi E$.

2.1. Block diffeomorphisms. In this subsection we give a short overview of block bundles and diffeomorphisms and we explain how to compare them to honest fiber bundles and diffeomorphisms. For $p \geq 0$ let $\Delta^p$ denote the standard topological $p$-simplex.

Definition 2.1. A block diffeomorphism of $\Delta^p \times M$ is a diffeomorphism of $\Delta^p \times M$ that for each face $\sigma \subset \Delta^p$ restricts to a diffeomorphism of $\sigma \times M$. The set of all block diffeomorphisms forms a semisimplicial group denoted by $\tilde{\text{Diff}}(M)$ whose $p$-simplices are the block diffeomorphisms of $\Delta^p \times M$.

The space $\tilde{\text{Diff}}(M)$ of block bundles is defined as the geometric realization of $\tilde{\text{Diff}}(M)$ and the associated classifying space is denoted by $B\tilde{\text{Diff}}(M)$. This space classifies block bundles. Let us recall the definition of a block bundle over a simplicial complex.

Definition 2.2 ([ER14, Definition 2.4]). Let $K$ be a simplicial complex and let $p: E \to |K|$ be continuous. A block chart for $E$ over a simplex $\sigma \subset K$ is a homeomorphism $h_\sigma: p^{-1}(\sigma) \to \sigma \times M$ which for every face $\tau \subset \sigma$ restricts to a homeomorphism $p^{-1}(\tau) \to \tau \times M$. A block atlas is a set $A$ of block charts, at least one over each simplex of $K$, such that transition functions are block diffeomorphisms. $E$ is called a block bundle if it admits a block atlas.

By [ER14, Proposition 3.2] a block bundle $\pi: E \to B$ has a stable analogue of the vertical tangent bundle, i.e. there exists a stable vector bundle $T_\pi^* E \to E$ which is stably isomorphic to the vertical tangent bundle $T_\pi E$ provided that $E$ is an actual fiber bundle. If furthermore $B$ is a manifold, the total space $E$ is again a manifold and there is a stable isomorphism $T_\pi^* E \oplus \pi^*TB \cong_{st} TEB$ (cf. [ER14, Lemma 3.3]).

Next, let us consider the semisimplicial subgroup $\text{Diff}_*(M)$ of those block diffeomorphisms that commute with the projection $\Delta^p \times M \to \Delta^p$. This gives precisely the $p$-simplices of the singular semisimplicial group $\text{Sing}_* \text{Diff}(M)$. 


We have an inclusion $\text{Sing}_\bullet \text{Diff}(M) \subset \tilde{\text{Diff}}_\bullet(M)$ and since the geometric real-
isation of $\text{Sing}_\bullet(X)$ is homotopy equivalent to $X$ for any space $X$ ([HAT, pp.
8]), we get an induced map

$$B\text{Diff}(M) \longrightarrow B\tilde{\text{Diff}}(M).$$

Next, let $\text{hAut}(M)$ denote the group-like topological monoid of (orientation
preserving) homotopy equivalences of $M$ with classifying space $B\text{hAut}(M)$.
Again, let $\tilde{\text{hAut}}(M)$ be the realization of the semisimplicial group of block ho-
motopy equivalences defined analogously to block diffeomorphisms and let $B\tilde{\text{hAut}}(M)$ be the corresponding classifying space. By [Dol63, Thm 6.1] $\text{hAut}(M)$ and $\tilde{\text{hAut}}(M)$ are homotopy equivalent. Consider the following
maps induced by inclusions:

$$B\tilde{\text{Diff}}(M) \to B\tilde{\text{hAut}}(M) \simeq B\text{hAut}(M) \quad B\text{Diff}(M) \to B\text{hAut}(M)$$

and let $\text{hAut}(M)/\text{Diff}(M)$ and $\text{hAut}(M)/\tilde{\text{Diff}}(M)$ denote the respective ho-
motopy fibers. Note that $\text{hAut}(M)/\text{Diff}(M)$ (resp. $\text{hAut}(M)/\tilde{\text{Diff}}(M)$) clas-
sifies $M$-bundles (resp. $M$-block bundles) that are homotopy equivalent to the
trivial bundle through a homotopy that commutes with the projection of the
bundle, i.e. fiber homotopy trivial $M$-bundles (resp. $M$-block bundles). We
have the following comparison result which easily follows from [BL82, Coro-
llary D].

**Lemma 2.3.** If $k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)$ then the map

$$\pi_k\left(\frac{\text{hAut}(M)}{\text{Diff}(M)}\right) \left[\frac{1}{2}\right] \to \pi_k\left(\frac{\text{hAut}(M)}{\tilde{\text{Diff}}(M)}\right) \left[\frac{1}{2}\right]$$

is (split-)surjective.

**Proof.** By [BL82, Corollary D] there exists a space $S$ such that for $k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)$ we have

$$\pi_k\left(\frac{\text{hAut}(M)}{\text{Diff}(M)}\right) \left[\frac{1}{2}\right] \simeq \pi_{k-1}(\Omega\left(\frac{\text{hAut}(M)}{\text{Diff}(M)}\right)) \left[\frac{1}{2}\right]

\simeq \pi_{k-1}(\Omega\left(\frac{\text{hAut}(M)}{\text{Diff}(M)}\right) \times \Omega S) \left[\frac{1}{2}\right] \to \pi_k\left(\frac{\text{hAut}(M)}{\text{Diff}(M)}\right) \left[\frac{1}{2}\right] \quad \square$$

Therefore, an element of $\pi_k(\text{hAut}(M)/\tilde{\text{Diff}}(M)) \otimes \mathbb{Q}$ yields an $M$-bundle $E \to S^k$ that is fiber homotopy trivial, provided that the dimension of $M$ is high
enough. The advantage of working with $\text{hAut}(M)/\tilde{\text{Diff}}(M)$ instead of $\text{hAut}(M)/\text{Diff}(M)$
stems from the fact, that the former is accessible through surgery theory as
we will review in the Section 2.3.

**Remark 2.4.** Another approach to compare $B\text{Diff}(M)$ and $B\tilde{\text{Diff}}(M)$ is by
using Morlet’s lemma of disjunction as in [KKR21, Lemma]. Consider the
following diagram of (homotopy) fibrations
If $M$ is $\ell$-connected with $\ell \leq d - 4$, then the induced map on homotopy fibers is $(2\ell - 2)$-connected by Morlet’s lemma of disjunction (cf. [BLR75, Corollary 3.2 on page 29]). Now $\pi_k(B\Diff_\partial(D^d)) \cong \pi_0(\Diff_\partial(D^{d+k-1})$ is isomorphic to the finite group of exotic spheres in dimension $(d+k)$. If $d$ is even, then $B\Diff_\partial(D^d)$ is rationally $(d - 5)$-connected by [Ran17, Theorem 4.1]. On the other hand, if $d$ is odd and $k \leq d - 7$ is not divisible by 4, then $\pi_k(B\Diff_\partial(D^d) \otimes \mathbb{Q}$ is trivial by [Kra21, Corollary B].

Therefore, in both these cases $\pi_k(B\Diff_\partial(D^d)/ \Diff(D^d)) \otimes \mathbb{Q}$ is trivial and the same is true for $\pi_k(B\Diff_\partial(M)/ \Diff(M)) \otimes \mathbb{Q}$, provided $k \leq 2\ell - 2$. This implies that the map

$$\pi_{k+1}(B\Diff(M)) \otimes \mathbb{Q} \rightarrow \pi_{k+1}(B\widetilde{\Diff}(M)) \otimes \mathbb{Q}$$

is surjective which also holds for the induced map

$$\pi_{k+1}\left(\frac{\text{hAut}(M)}{\Diff(M)}\right) \otimes \mathbb{Q} \rightarrow \pi_{k+1}\left(\frac{\text{hAut}(M)}{\Diff(M)}\right) \otimes \mathbb{Q}$$

by the five-lemma.

We summarize this discussion about unblocking in the following lemma.

**Lemma 2.5.** Let $M$ be $\ell$-connected for some $\ell \leq d - 4$ and let $k \in \mathbb{N}$ be such that one of the following holds.

(i) $k \leq \min\left(\frac{d-1}{3}, \frac{d-5}{2}\right)$.
(ii) $d$ is even and $k \leq \min(d-4, 2\ell-1)$.
(iii) $d$ is odd, $(k-1)$ is not divisible by 4 and $k \leq \min(d-6, 2\ell-1)$.

Then the following maps are both surjective

$$\pi_k\left(\frac{\text{hAut}(M)}{\Diff(M)}\right) \otimes \mathbb{Q} \rightarrow \pi_k\left(\frac{\text{hAut}(M)}{\Diff(M)}\right) \otimes \mathbb{Q}$$

$$\pi_k(B\Diff(M)) \otimes \mathbb{Q} \rightarrow \pi_k(B\widetilde{\Diff}(M)) \otimes \mathbb{Q}.$$

### 2.2. $\kappa$-classes of block bundles.

In [ER14] the authors define $\kappa$-classes for block bundles generalising $\kappa$-classes of smooth fiber bundles (cf. [ER14, Theorem 1]). Therefore, we get a block-analogue $\widetilde{\Psi}_M^h$ of the map $\Psi_M^h$ from the introduction:

$$\widetilde{\Psi}_M^h : H^{4m}(BO; \mathbb{Q}) \rightarrow \text{Hom}(\pi_{4m-d}\left(\frac{\text{hAut}(M)}{\Diff(M)}\right), \mathbb{Q})$$

$$c \mapsto (f \mapsto \langle f^*\kappa_c, [S^{4m-d}] \rangle).$$
Since our construction of bundles makes use of the identification
\[ \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \cong S_0(D^k \times M). \]
from Section 2.3, it is more natural to prove our main results for block bundles. The corresponding statements about fiber bundles from the introduction then follow from Lemma 2.5.

Remark 2.6. Even though this Lemma 2.5 only states surjectivity, this still is enough to completely determine \((h-)\)sphericity of \(\kappa\)-classes. Every fiber homotopy trivial bundle also is also a fiber homotopy trivial block bundle and the \(\kappa\)-classes for block bundles generalize those for fiber bundles by [ER14, Theorem 1]. Hence, under the assumptions from Lemma 2.5 we deduce that a (rational) \(\kappa\)-class is \(h\)-spherical for \(M\) if and only if this class pulls back non-trivially to \(\pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \).

If \(E \to S^{4m-d}\) is the block bundle classified by \(f: S^{4m-d} \to B\text{Diff}(M)\) and \(c \in H^*(BO; \mathbb{Q})\) we have
\[
\langle f^* \kappa_c, [S^{4m-d}]^\pi \rangle = \langle \pi_1 c(T^*_\pi E), [S^{4m-d}]^\pi \rangle = \langle \pi_1 c(T^*_\pi E), [S^{4m-d}]^\pi \rangle
\]
\[
= \langle c(T^*_\pi E), [E] \rangle = \langle c(\pi^* T S^{4m-d} \oplus T^*_\pi E), [E] \rangle = \langle c(T E), [E] \rangle
\]
since \(TS^{4m-d}\) is stably parallelizable and hence \(c(T S^{4m-d}) = 1\). In order to show that a \(\kappa\)-class \(\kappa_c\) is spherical, it therefore suffices by Lemma 2.5 to construct an \(M\)-block bundle \(E \to S^k\) such that \(c(E) \neq 0\), provided \(k \leq \min(d_{4m-d}, d_{4m})\). The following well known result follows directly from the multiplicative behavior of the signature for fibrations with simply-connected base (cf. [Sch72]).

**Proposition 2.7.** Let \(E \to S^k\) be an \(M^d\)-block bundle. Then \(\text{sign}(E) = 0\).

Assuming that \(E\) is fiber homotopy equivalent to \(M \times S^k\), we get the following vanishing result for other Pontryagin numbers of \(E\). For a proof, see the more general Proposition A.7 in the Appendix.

**Proposition 2.8.** Let \(p[E] := \langle p_{i_1} \cdots p_{i_s} (TE), [E] \rangle\) be a Pontryagin-number of \(E\) with \(i_j < k/4\) for all \(j\). Then \(p[E] = 0\)

2.3. **Surgery theory.** Let \(X\) be a simply connected manifold with boundary \(\partial X\). The structure set \(S(X, \partial X)\) of \((X, \partial X)\) (sometimes written as \(S_0(X)\)) is defined to be the set of equivalence classes of tuples \((W, \partial W, f)\) where \(W\) is a manifold with boundary \(\partial W\) and \(f\) is a homotopy equivalence\(^1\) that restricts to a diffeomorphism on the boundary. Two such tuples \((W_0, \partial W_0, f_0)\) and \((W_1, \partial W_1, f_1)\) are equivalent, if there exists a diffeomorphism \(\alpha: \partial W_0 \to \partial W_1\) such that \(f_0 = f_1 \circ \alpha\). It is a consequence of the \(h\)-cobordism theorem that we have the following isomorphism ([BM13, Section 3.2, pp.33])
\[
\pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \cong S_0(D^k \times M).
\]

\(^1\)Since we assume \(X\) to be simply connected, every homotopy equivalence is simple and we do not need to require this in the definition.
The main result of surgery theory is that the structure set $\mathcal{S}_\partial(D^k \times M)$ fits into an exact sequence of sets known as the surgery exact sequence (cf. [CLM, Theorem 10.21 and Remark 10.22]):

\[ L_{k+d+1}(\mathbb{Z}) \to \mathcal{S}_\partial(D^k \times M) \to \mathcal{N}_\partial(D^k \times M) \xrightarrow{\sigma} L_{k+d}(\mathbb{Z}) \]

Here, $\mathcal{N}_\partial(D^k \times M)$ is the set of normal invariants which is given by equivalence classes of tuples $(W, f, \hat{f}, \xi)$, where $W$ is a $(d + k)$-dimensional manifold with (stable) normal bundle $\nu_W$, $\xi$ is a stable vector bundle over $D^k \times M$ and $f : W \to D^k \times M$ is a map of degree 1 covered by a bundle map $\hat{f} : \nu_W \to \nu_{D^k \times M} \oplus \xi$ such that $(f, \hat{f})$ restricts to the identity on the boundary and the equivalence relation is given by cobordism.

Since we only consider simply connected manifolds, the relevant $L$-groups are 4-periodic and given by (cf. [CLM, Theorem 7.96])

\[ L_n(\mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & \text{if } n \equiv 0 \ (4) \\
\mathbb{Z}/2 & \text{if } n \equiv 2 \ (4) \\
0 & \text{otherwise}
\end{cases} \]

and the map $\sigma$ in the surgery exact sequence (1) is the so-called surgery obstruction map, which in degrees $d + k \equiv 0 \ (4)$ for simply connected $M$ is given by

\[ \sigma(W, f, \hat{f}, \xi) = \frac{1}{8} \left( \text{sign}(W \cup (D^k \times M)) - \text{sign}(S^k \times M) \right) = \frac{1}{8} \text{sign}(W') \]

where sign denotes the signature (cf. [CLM, Lemma 7.170, Exercise 7.188]).

The signature of $W'$ can be computed via Hirzebruch’s signature theorem, which constructs a power series

\[ \mathcal{L}(x_1, x_2, \ldots) = 1 + s_1 x_1 + \cdots + s_i x_i + \cdots + s_i, x_i - x_j + \cdots \]

such that $\text{sign}(W') = \langle \mathcal{L}(p_1(TW'), p_2(TW'), \ldots), [W'] \rangle$. Here $p_i(TW')$ are the Pontryagin classes of $W'$. Note that $f, \hat{f}$, and $\xi$ can be extended trivially to $W'$. Since the map $f$ is of degree one, evaluating the Pontryagin classes of $W'$ against $[W']$ yields the same result as evaluating the Pontryagin classes of $-\xi \oplus T(S^k \times W)$, where $-\xi$ denotes the (stable) orthogonal complement to $\xi$.

In order to further analyze $\mathcal{N}_\partial(D^k \times M)$, let us define $G(n) = \{ f : S^{n-1} \to S^n \}$ as the $n$-spherical homotopy equivalence and $BG := \text{colim}_{n \to \infty} BG(n)$. Here, $BG$ is the classifying space for stable spherical fibrations whereas $BO$ is the classifying space for stable vector bundles. The inclusion $O(n) \hookrightarrow G(n)$ induces a map $BO \to BG$ and we denote the homotopy fiber by $G/O$. By [CLM, Remark 10.28] there is an identification

\[ \mathcal{N}_\partial(D^k \times M) \cong [S^k \wedge M_+, G/O]_* \]

Here $M_+$ is $M$ with a disjoint base point and $\wedge$ denotes the smash product of pointed spaces given by $(X, x) \wedge (Y, y) := (X \times Y)/(X \times \{y\} \cup \{x\} \times Y)$. The functor $S^k \wedge (\_)$ is adjoint to the $k$-fold loop space functor $\Omega^k(\_)$ and so we get $[S^k \wedge M_+, G/O]_* \cong [M, \Omega^k G/O]$. Now $\Omega^{k+1}BG$ is the homotopy fiber of
the map $\Omega^k G/O \to \Omega^k BO$. By obstruction theory (cf. [Hat02, p. 418]) the obstructions to the lifting problem

$$
\begin{array}{c}
\Omega^k G/O \\
\downarrow \\
\Omega^k BO
\end{array}
\xrightarrow{\text{M}}
$$

live in the groups $H^{i+1}(M;\pi_i(\Omega^{k+1}BG)) \cong H^{i+1}(M;\pi_\ast(\Omega^{k+1}BG))$. The homotopy groups of $\pi_k(BG)$ are isomorphic to the shifted stable homotopy groups of spheres $\pi_{k-1}^s$ by [CLM, p. 135]. By Serre’s finiteness theorem, these groups are finite for $k \geq 2$ and hence all obstruction groups vanish rationally, since we assumed that $k \geq 1$. Since maps $(M,\Omega^k BO)$ is an $H$-space, we see that for every (pointed) map $f: M \to \Omega^k BO$, some multiple of $f$ can be lifted to $\Omega^k G/O$. Therefore it suffices for us to specify an element in

$$
[S^k \land M_+, BO]_* = \text{KO}^0(S^k \land M_+)
$$

in order to get a normal invariant. Next, consider the isomorphism given by the Pontryagin character:

$$
\text{ph}(\_):= \text{ch}(\_ \otimes \mathbb{C}): \text{KO}^0(S^k \land M_+) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{i \geq 0} H^{4i}(S^k \land M_+; \mathbb{Q})
$$

$$
\cong u_k \cdot \bigoplus_{i \geq 0} H^{4i-k}(M; \mathbb{Q})
$$

for $u_k$ the cohomological fundamental class in $H^k(S^k)$. The $i$-th component of the Pontryagin character is given by

$$
\text{ph}_i(\xi) = \text{ch}_2(\xi \otimes \mathbb{C}) = \frac{1}{(2i)!} \left( (-2i)c_2(\xi) + f(c_1(\xi), \ldots, c_{2i-1}(\xi)) \right)
$$

$$
= \frac{(-1)^{i+1}}{(2i-1)!} p_i(\xi)
$$

where $f(c_1(\xi), \ldots, c_{2i-1}(\xi))$ is a polynomial in Chern classes of $\xi$ homogenous of degree $2i$ which vanishes since all products in $\bar{H}^\ast(S^k \land M_+; \mathbb{Q})$ are trivial. Hence, for any collection $(x_i) \in H^{4i-k}(M; \mathbb{Q})$ and $(A_i) \in \mathbb{Q}$ there exists a $\lambda \in \mathbb{Z} \setminus \{0\}$ and a normal invariant $(W,f,\hat{f},\hat{\xi}) \in \mathcal{N}_{\partial}(D^k \times M)$ such that

$$
p_i(\xi') = (-1)^{i+1}(2i-1)!\lambda A_i \cdot u_k \cdot x_i,
$$

where $u_k$ denotes the cohomological fundamental class of $S^k$. Note that any product $p_i(\xi')p_i(\xi')$ vanishes and that $(-1)^{i+1}(2i-1)! \neq 0$ for all choices of $i$. Hence, after replacing $A_i$ by $\frac{(-1)^{i+1}A_i}{(2i-1)!}$ we may assume that the Pontryagin classes of $\xi'$ have the form

$$
p_i(\xi') = \lambda A_i \cdot u_k \cdot x_i
$$

Note that $\lambda$ depends on the collection $(A_i)$ so we cannot absorb it into the $A_i$’s. This allows us to construct a normal invariant and hence an element of $\pi_k(\text{hAut}(M)/\text{Diff}(M))$ such that the underlying stable vector bundle has prescribed Pontryagin classes, which we will do in the following section.
As a final observation in this section, we note that the surgery obstruction map \( \sigma: \mathcal{N}_D(D^k \times M) \rightarrow \mathbb{Z} \) is always surjective by Wall’s realization theorem (cf. [CLM, Theorem 7.192]). Therefore we have

\[
\dim \ker(\sigma: \mathcal{N}_D(D^k \times M) \rightarrow \mathbb{Z}) \otimes \mathbb{Q} = \dim \bigoplus_{i \geq 0} H^{4i-k}(M; \mathbb{Q}) - 1
\]

and hence \( \dim \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \otimes \mathbb{Q} = \sum_{i \geq 0} b_{4i-k}(M; \mathbb{Q}) - 1 \). Since two isomorphic \( M \)-bundles have isomorphic vertical tangent bundles this observation together with Lemma 2.5 implies the following corollary.

**Corollary 2.9.** For \( k \leq \min(\frac{d-1}{2}, \frac{d-5}{2}) \) we have

\[
\dim \pi_k(B\text{Diff}(M)) \otimes \mathbb{Q} \geq \sum_{i \geq 0} b_{4i-k}(M; \mathbb{Q}) - 1.
\]

### 3. Prescribing Pontryagin classes

**3.1. Proof of Theorem A.** In this section we prove Theorem A. It follows from the following “block-version” in combination with Lemma 2.5 (see also Remark 2.6).

**Theorem 3.1.** Let \( M^d \) be a simply connected manifold with \( p_i(TM) \neq 0 \) for some \( i \). Let furthermore \( p = p_{i_1} \cdots p_{i_n} \) be a monomial in universal Pontryagin classes of total degree \( d + k \). Then there exists an fiber homotopy trivial \( M \)-block-bundle \( E \rightarrow S^k \) with \( p(E) \neq 0 \) if and only if there exists an \( l \) such that

\[
p_{i_1}(TM) \cdots p_{i_l}(TM) \cdots p_{i_n}(TM) \neq 0.
\]

**Proof.** We will prove this by distinguishing two cases: \( n = 1 \) and \( n \geq 2 \). In order to prove case 1 we need to show that there always is an element in \( \pi_k \left( \frac{\text{hAut}(M)}{\text{Diff}(M)} \right) \) with \( p_m \neq 0 \) which is the a special case of Lemma 3.3. The second case follows directly from Lemma 3.2. \( \square \)

**Lemma 3.2.** Let \( p_m \neq p = p_{i_1} \cdots p_{i_n} \in H^{4m}(BSO; \mathbb{Q}) \) be a monomial in universal Pontryagin classes. Then

\[
\tilde{\Psi}_M^h(p) \neq 0 \iff \exists \ell: i_\ell \geq \frac{k}{4}: p_{i_1}(TM) \cdots p_{i_\ell}(TM) \cdots p_{i_n}(TM) \neq 0
\]

**Proof.** We first show the “\( \Rightarrow \)” implication. Let \( \ell \) be such that \( i_\ell \) is maximal with the property above and let \( x \in H^*(M; \mathbb{Q}) \) be such that \( p_{i_1}(TM) \cdots p_{i_\ell}(TM) \cdots p_{i_n}(TM) \cdot x = u_M \in H^d(M; \mathbb{Q}) \). By Section 2.3 there exists a normal invariant \( \eta \) such that the underlying stable (extended) vector bundle \( \xi \rightarrow S^k \times M \) has the following Pontryagin classes:

\[
p_0(-\xi) = 1 \quad p_{i_\ell}(-\xi) = \lambda \cdot x \cdot u_k \quad p_m(-\xi) = \lambda A \cdot u_M \cdot u_k
\]

for \( u_k \) the cohomological fundamental class of \( S^k \), \( A \in \mathbb{Q} \) to be chosen later and \( \lambda \in \mathbb{Z} \setminus \{0\} \) determined by \( A \). All other Pontryagin classes of \( \xi \) vanish.
Note that by assumption $i_\ell < m$. Then
\[
p_{i_1} \cdots p_{i_s}(TM \oplus -\xi) = \prod_{j=1}^{s} \sum_{n=0}^{i_j} p_n(-\xi) \cdot p_{i_j-n}(TM)
= p_{i_\ell}(-\xi) \cdot \sum_{r : j_1=\ell} \prod_{q\neq \ell} p_q(TM) \cdot \sum_{r : j_1=\ell} 1
= \lambda a_\ell \cdot u_M \cdot u_k \neq 0.
\]
Finally, we need to choose $A$, such that the surgery obstruction vanishes. Consider Hirzebruch’s signature formula:
\[
\sigma(\eta) = \text{sign}(W') = \langle \mathcal{L}(W'), [W'] \rangle = \langle \mathcal{L}(W'), f_*[S^k \times M] \rangle
= \langle \mathcal{L}(S^k), \mathcal{L}(TM \oplus \xi), [S^k \times M] \rangle = \langle \mathcal{L}(TM \oplus \xi), [S^k \times M] \rangle
= s_m \cdot \lambda \cdot A + \lambda \cdot z,
\]
where $z$ is some number independent of $A$ and $s_m$ is the leading coefficient of $\mathcal{L}$. Since all coefficients in the $\mathcal{L}$-polynomial are nonzero by [BB18], we can choose $A := \frac{1}{s_m}$ so that $\sigma(\eta)$ vanishes independently of $\lambda$. Hence there is a corresponding element $f \in \pi_k(\text{hAut}(M)/\text{Diff}(M))$ such that
\[
\langle f^*\kappa_p, [S^k] \rangle = \langle \lambda a_\ell \cdot u_M \cdot u_k, [S^k \times M] \rangle = \lambda a_\ell \neq 0.
\]
For the other implication \(\Rightarrow\) let us assume that $p_{i_1}(TM) \cdots p_{i_s}(TM) \cdot \cdots p_{i_\ell}(TM) = 0$ for all $i_\ell \geq \frac{k}{4}$. Since products of higher Pontryagin classes of $\xi$ are trivial, we can compute
\[
p_{i_1} \cdots p_{i_s}(TM \oplus -\xi) = \prod_{j=1}^{s} \sum_{n=0}^{i_j} p_n(-\xi) \cdot p_{i_j-n}(TM)
= \sum_{j=1}^{s} \sum_{n=1}^{i_j} p_n(-\xi) p_{i_j-n}(TM) \prod_{r \neq j} p_r(TM).
\]
The product $\prod_{r \neq j} p_r(TM)$ vanishes if $i_j \geq \frac{k}{4}$ by assumption. If $i_j < \frac{k}{4}$, then $p_n(-\xi) = 0$ for all $1 \leq n \leq i_j$ since every higher Pontryagin class of $\xi$ is of the form $u_k \cdot *$ and hence is of index at least $k/4$. It follows that $\Psi^\ell_k(p) = 0$. \(\square\)

For the second case and the proof of the estimates in Theorem B, we recall the following definitions
\[
i_{\min} := \min\{i \geq 1 : p_i(TM) \neq 0\}
n_{\max} := \max\{n \in \mathbb{N} : p_{i_{\min}}(TM)^n \neq 0\}.
\]
By our assumption dim $\mathbb{Q}[p_i(TM)] \geq 2$ the set $\{i \geq 1 : p_i(TM) \neq 0\}$ is actually non-empty and $n_{\max} \geq 1$.

**Lemma 3.3.** For every $\ell = 1, \ldots, n_{\max}$ there exists a normal invariant $\eta_\ell$ with underlying stable vector bundle $\xi_\ell \to D^k \times M$ with the following property: For $\xi'_\ell$ the extension of $\xi_\ell$ by the trivial bundle to $S^k \times M$ we have that
\[
\langle p_{i_{\min}}(TM \oplus -\xi'_\ell) p_{r-i_{\min}}(TM \oplus -\xi'_\ell), [S^k \times M] \rangle \neq 0 \iff r = \ell \text{ or } r = 0
\]
and $\sigma(\eta) = 0$. For $\ell = 1$ we furthermore have that 

$$
\langle p_{\min}(TM \oplus -\xi_1), p_{m-\min}(TM \oplus -\xi_1)', [S^k \times M]\rangle \quad \text{and} \quad \langle p_m(TM \oplus -\xi_1), [S^k \times M]\rangle
$$

are the only non-vanishing elementary Pontryagin numbers of $TM \oplus -\xi_1$.

**Proof.** Let $u_M \in H^{4m-k}(M; \mathbb{Q})$ denote the cohomological fundamental class of $M$. Since the cup product induces a perfect pairing

$$H^j(M; \mathbb{Q}) \times H^{(m-j)-k}(M; \mathbb{Q}) \rightarrow \mathbb{Q},$$

there exists a class $x := x_{\max} \in H^{(m-\min n_{\max})-k}(M; \mathbb{Q})$ such that $x \cdot p_{\min}(TM)^{n_{\max}} = u_M$. For $r = 0, \ldots, n_{\max}$ we define $x_r := x \cdot p_{\min}(TM)^{n_{\max}-r}$. Then

$$x_r \cdot p_{\min}(TM)^r = x \cdot p_{\min}(TM)^{n_{\max}-r} \cdot p_{\min}(TM)^r = u_M.$$  

By the discussion in Section 2 we know that for every collection $A_0, A_1, \ldots, A_{n_{\max}} \in \mathbb{Q}$ there exists a $\lambda \in \mathbb{Z} \setminus \{0\}$ and a normal invariant $\eta = (W, f, \tilde{f}, \xi)$ such that the (extended) stable vector bundle $\xi'$ has only the following (rational) Pontryagin classes:

$$p_0(-\xi') = 1$$

$$p_m(-\xi') = \lambda A_r \cdot u_k \cdot x_r \quad \text{for} \quad r = 0, \ldots, n_{\max}$$

Since $\min i < m$ and $p_r(TM) = 0$ for all $0 < r < \min i$, we have\footnote{Since we are only interested in rational Pontryagin classes we have $p(V \oplus W) = p(V) \cdot p(W)$.}

$$p_{\min}(TM) = \sum_{a=0}^{\min i} p_a(TM) \cdot p_{m-a}(TM) = p_{\min}(TM) = p_{\min}(TM)$$

We will now distinguish two cases: $m = (n_{\max}+1)\min i$ and $m > (n_{\max}+1)\min i$. In the former case we have $p_{\min}(-\xi') = \lambda A_{n_{\max}} \cdot u_k \cdot x_{n_{\max}}$ and compute:

$$p_{\min}(TM \oplus -\xi')^{n_{\max}} \cdot p_{m-n_{\max}} \cdot \min i (TM \oplus -\xi') = p_{\min}(TM \oplus -\xi')^{n_{\max}}$$

$$= (p_{\min}(-\xi') + p_{\min}(TM))^{n_{\max}} = m \cdot p_{\min}(TM)^{n_{\max}} \cdot p_{\min}(-\xi')$$

For $0 \leq r < n_{\max}$ we have:

$$p_{\min}(TM \oplus -\xi')^{r} \cdot p_{m-\min r}(TM \oplus -\xi')$$

$$= (p_{\min}(-\xi') + p_{\min}(TM))^{r} \sum_{a=0}^{m-\min r} p_a(TM) \cdot p_{m-\min r-a}(-\xi')$$

(3)

$$= (p_{\min}(-\xi') + p_{\min}(TM))^{r} \cdot \left( p_{m-\min r}(-\xi') + \sum_{a=0}^{m-\min r} p_a(TM) \cdot p_{m-\min r-a}(-\xi') \right)$$

$$= p_{\min}(TM)^{r} \cdot \lambda A_r \cdot u_k \cdot x_r + \ast u_k \cdot u_M,$$
where \( * \) is a linear expression in the variables \( \lambda A_{r+1}, \ldots, \lambda A_{n_{\text{max}}} \). Now for \( b, a_1, \ldots, a_{n_{\text{max}}} \in \mathbb{Q} \), consider the following system of equations

\[
\begin{align*}
b &= \langle p_m(TM \oplus -\xi'), [S^k \times M] \rangle \\
a_1 &= \langle p_{i_{\text{min}}}(TM \oplus -\xi') \cdot p_{m-i_{\text{min}}}(TM \oplus -\xi'), [S^k \times M] \rangle \\
&\vdots \\
a_r &= \langle p_{i_{\text{min}}}(TM \oplus -\xi')^r \cdot p_{m-r \cdot i_{\text{min}}}(TM \oplus -\xi'), [S^k \times M] \rangle \\
&\vdots \\
a_{n_{\text{max}}} &= \langle p_{i_{\text{min}}}(TM \oplus -\xi')^{n_{\text{max}}} \cdot p_{i_{\text{min}}}(TM \oplus -\xi'), [S^k \times M] \rangle
\end{align*}
\]

By the above computation this is a linear system of equations in the variables \( A_0, A_1, \ldots A_{n_{\text{max}}} \) and it has the following form

\[
\begin{pmatrix}
1 & * & * & * \\
0 & \ddots & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & m
\end{pmatrix}
\begin{pmatrix}
\lambda A_0 \\
\vdots \\
\lambda A_{n_{\text{max}}}
\end{pmatrix}
= 
\begin{pmatrix}
b \\
\vdots \\
a_{n_{\text{max}}}
\end{pmatrix}
\]

The matrix \( B \) is invertible and hence, there exist \( A_1, \ldots A_{n_{\text{max}}} \) such that \( a_i = 0 \) if and only if \( i \neq \ell \). Note, that \( \lambda \) is not yet determined as it also depends on \( A_0 \), but the condition of \( a_i \) being zero or nonzero is independent of \( \lambda \). Furthermore note, that since \( B \) is triangular, the values of \( a_i \) are independent of \( A_0 \). Finally, we need to choose \( A_0 \), such that the surgery obstruction vanishes. Consider Hirzebruch’s signature formula:

\[
\sigma(\eta) = \text{sign}(W') = \langle \mathcal{L}(W'), [W'] \rangle = \langle \mathcal{L}(W'), f_s[S^k \times M] \rangle
= \langle \mathcal{L}(S^k) \cdot \mathcal{L}(TM \oplus \xi), [S^k \times M] \rangle = \langle \mathcal{L}(TM \oplus \xi), [S^k \times M] \rangle
= s_m \cdot \lambda \cdot A_0 + \lambda \cdot z
\]

where \( z \) is some number independent of \( A_0 \) and \( s_m \) is the leading coefficient of \( \mathcal{L} \). Since all coefficients in the \( \mathcal{L} \)-polynomial are nonzero by [BB18], we can choose \( A_0 = \frac{z}{s_m} \) so that \( \sigma(\eta) \) vanishes independently of \( \lambda \).

The case \( m > (n_{\text{max}} + 1) \cdot i_{\text{min}} \) is very similar. By a computation similar to (3) we have

\[
p_{i_{\text{min}}}(TM \oplus -\xi')^{n_{\text{max}}} \cdot p_{m-n_{\text{max}} \cdot i_{\text{min}}}(TM \oplus -\xi')
= p_{i_{\text{min}}}(TM)^{n_{\text{max}}} \cdot \lambda A_{n_{\text{max}}} \cdot u_k \cdot x_{n_{\text{max}}}
= \lambda A_{n_{\text{max}}} \cdot u_k \cdot u_M
\]

This implies that the respective matrix \( B \) has the same form as above with \( m \) in the lower right corner replaced by 1. The rest of the argument is verbatim to the first case.
If \( \ell = 1 \) the bundle \( \xi' \) can be chosen to only have three non-vanishing Pontryagin classes, namely
\[
\begin{align*}
p_0(\xi'_1) &= 1 \\
p_{m-\min}(\xi'_1) &= \lambda A_1 \cdot u_k \cdot x \\
p_m(\xi'_1) &= \lambda A_0 \cdot u_k \cdot u_M
\end{align*}
\]

As noted above, every Pontryagin number of \( TM \oplus -\xi'_1 \) contains precisely one Pontryagin class of \( \xi'_1 \). Since \( p_{\min}(TM) \) is the smallest Pontryagin class of \( M \), we deduce that the only possibly non-vanishing Pontryagin numbers of \( TM \oplus -\xi'_1 \) are \( (p_{\min}(TM) \oplus -\xi'_1) \cdot p_{m-1}(TM \oplus -\xi'_1), \ [S^k \times M] \) and \( (p_m(TM) \oplus -\xi'_1), \ [S^k \times M] \). \( \square \)

### 3.2. The image of \( \tilde{\Psi}_M^h \)

Consider the algebra \( \mathbb{Q}[p_i(TM)] \) generated by the Pontryagin classes of \( M \). **Theorem B** again follows from a “block-analogue” thereof together with **Lemma 2.5** and **Remark 2.6**. Recall the map
\[
P : \mathbb{Q}[p_i(TM)] \to H^*(BO(d); \mathbb{Q})
\]
from the introduction given by sending \( p_i \cdots p_m(TM) \) to \( p_i \cdots p_m \cdot p_s/4 - \sum i_j \).

**Lemma 3.4.** \( \text{im} \tilde{\Psi}_M^h \circ \mathcal{P} = \text{im} \tilde{\Psi}_M^h \text{ and dim im } \tilde{\Psi}_M^h \geq n_{\max}. \)

**Proof.** Obviously \( \text{im} \tilde{\Psi}_M^h \circ \mathcal{P} \subset \text{im} \tilde{\Psi}_M^h \) and it suffices to show \( \text{im} \tilde{\Psi}_M^h \subset \text{im} \tilde{\Psi}_M^h \circ \mathcal{P} \). Let \( p = p_{i_1} \cdots p_{i_s} \in H^{4m}(BSO; \mathbb{Q}) \) be a product of universal Pontryagin classes with \( \tilde{\Psi}_M^h(p) \neq 0 \). It suffices to show that there exists an \( \ell \in \{1, \ldots, s\} \) such that \( p_{i_1}(TM) \cdots p_{i_\ell} \overline{(TM)} \cdots p_{i_s}(TM) \neq 0 \). This again follows from **Lemma 3.3** for \( s = 1 \) and from **Lemma 3.2** for \( s \geq 2 \). The lower bound also follows immediately from **Lemma 3.3.** \( \square \)

**Achieving the upper bound.** The upper bounds on \( \text{dim im } \tilde{\Psi}_M^h \) following immediate from **Lemma 3.4** are
\[
\begin{align*}
\text{dim im } \tilde{\Psi}_M^h &\leq \text{dim } \mathbb{Q}[p_i(TM)] - 1 \\
\text{dim im } \tilde{\Psi}_M^h &\leq \text{dim } H^{4m}(BSO; \mathbb{Q}) - 1,
\end{align*}
\]
where \(-1\) originates from \( (L_m) \subset \ker \tilde{\Psi}_M^h \). If \( k = 4m - d \geq 5 \), we also observe the following obstruction.

**Proposition 3.5.** Let \( p = p_{i_1} \cdots p_{i_s} \in H^{4m}(BSO; \mathbb{Q}) \). If \( i_j < m - \frac{d}{4} \) for all \( j \in \{1, \ldots, s\} \), then \( \tilde{\Psi}_M^h(p) = 0 \).

**Proof.** If \( \tilde{\Psi}_M^h(p) \) were nonzero by **Theorem 3.1** there exists an \( i_j \geq k/4 \) such that \( p_{i_1}(TM) \cdots p_{i_j}(TM) \cdots p_{i_s}(TM) \neq 0 \). However the degree of this product is \( 4(m - i_j) > d \) and hence the product has to vanish for degree reasons leading to a contradiction. \( \square \)

Recall that \( p(n) \) is defined to be the number of partitions and the number of those partitions into natural numbers \( \leq n' \) is \( p(n, n') \).

**Lemma 3.6.** \( \text{dim im } \tilde{\Psi}_M^h \leq p(m) - p(m, m - \lfloor \frac{d+1}{4} \rfloor) - 1. \)
We will now show that this upper bound is sharp, i.e. there exists a manifold for which equality holds. In order to do so, let $M$ be a manifold with simply connected components such that all possible products of rational Pontryagin classes of $TM$ are nonzero and all of them are linearly independent (see the example below for a construction of such a manifold in dimensions $d \equiv 2, 3 \ (4)$).

For $I = (i_1, \ldots, i_s)$ let $|I| := \sum i_j$ and let us introduce the short notation

$$PI = p_{i_1} \cdots p_{i_s}.$$ 

Since the cup product induces a perfect pairing there exist elements $x_J \in H^{d-4|J|}(M)$ for every collection $J$ with $|J| < \frac{d}{4}$ such that $x_J \cdot p_J(TM) \neq 0$ if and only if $J = J'$.

**Example 3.7.** For $n \geq 1$ let $M_1^n, \ldots, M_{s_n}^n$ be a basis for $\Omega_{4n} \otimes \mathbb{Q}$ with the property that each of those only has one non-trivial Pontryagin number. Since $\Omega_{s} \otimes \mathbb{Q}$ is generated by $\mathbb{CP}^{2n}$ we may choose $M_1^n$ to be simply connected.

Consider the following $d$-dimensional manifold:

$$M := \prod_{n=1}^{d/4} \bigoplus_{j=1}^{s_n} M_j^n \times S^{d-4n}$$

This manifold has all possible products of Pontryagin classes and they are all linearly independent, since $H^*(M; \mathbb{Q}) = \bigoplus_{n=1}^{d/4} \bigoplus_{j=1}^{s_n} H^*(M_j^n \times S^{d-4n})$ and for every $I = (i_1, \ldots, i_s)$ there is a unique $j \in \{1, \ldots, s_{|I|}\}$ such that $p_I(TM_j^{|I|}) \neq 0$. Note that for $d \neq 1 \ (4)$ every component of $M$ is simply connected. If $d \equiv 2, 3 \ (4)$, we can even assume $M$ to be simply connected by performing connected sums. If $d \equiv 0, 1 \ (4)$ this is not possible since Pontryagin products of top degree would then live in the 1-dimensional space $H^d(M; \mathbb{Q})$ (resp. in the 0-dimensional space $H^{d-1}(M; \mathbb{Q})$).

**Lemma 3.8.** Let $M$ be as above and let $I = (i_1, \ldots, i_s)$ with $|I| = m$ and $i_j \geq m - \frac{d}{4}$ for some $j$. Then there exists a normal invariant $\eta \in N_0(D^{4m-d} \times M)$ with (extended) underlying stable vector bundle $\xi'$ such that

$$\langle p_I(TM \oplus -\xi), [S^{4m-d} \times M] \rangle \text{ and } \langle p_m(TM \oplus -\xi), [S^{4m-d} \times M] \rangle$$

are the only non-vanishing elementary Pontryagin numbers of $TM \oplus -\xi$ and $\sigma(\eta) = 0$.

**Proof.** Without loss of generality let $i_1$ be the biggest element of $I$ and let $a_1$ be the number of elements in $I$ equal to $i_1$. By assumption $i_1 \geq m - \frac{d}{4}$ and $p_{I \setminus \{i_1\}}(TM) \neq 0$. By Section 2.3 there exists a normal invariant $\eta$ such that the underlying stable (extended) vector bundle $\xi' \rightarrow S^k \times M$ has the following Pontryagin classes:

$$p_0(-\xi') = 1,$$

$$p_{i_1}(-\xi') = \lambda \cdot x_{I \setminus \{i_1\}} \cdot u_{4m-d},$$

$$p_i(-\xi') = 0 \quad \text{for } i < i_1,$$

$$p_i(-\xi') = \lambda \cdot u_{4m-d} \sum_{J: |J|=m-i} A_{J}x_{J} \quad \text{for } i > i_1.$$
for $x_J$ as defined above Example 3.7 and $A_J$ to be determined later. Note that the degree of $x_J$ is $d - 4|J| = 4i - (4m - d)$. We have seen in the proof of Lemma 3.2 that

$$p_I(TM \oplus -\xi') = \lambda a_1 \cdot u_M \cdot u_k \neq 0.$$  

It remains to show that all other monomial Pontryagin numbers are trivial. Now let $I' = (i'_1, \ldots, i'_t)$ be different collection with again $|I'| = m$, $i'_1$ the maximum and $a'_1$ the number of elements in $I'$ equal to $i'_1$. Then

$$p_{I'}(TM \oplus -\xi') = \prod_{j=1}^t p_{i'_j}(TM \oplus -\xi') = \prod_{j=1}^t \sum_{a=0}^{i'_j} p_a(-\xi')p_{i'_j-a}(TM)$$

$$= \prod_{j=1}^t \left( p_{i'_j}(TM) + \sum_{a=i'_1}^{i'_j} p_a(-\xi')p_{i'_j-a}(TM) \right).$$

If $i'_j < i_1$ for all $j$, then the sum on the right vanishes and for degree reasons so does the entire expression. If $i'_1 = i_1$, then we get

$$p_{I'}(TM \oplus -\xi') = \prod_{j: i'_j < i'_1} p_{i'_j}(TM) \prod_{j: i'_j = i'_1} \left( p_{i'_j}(TM) + p_{i'_j}(-\xi') \right)$$

$$= \lambda a'_1 \cdot u_{4m-d} \cdot x_{I' \setminus \{i'_1\}} \cdot p_{I' \setminus \{i'_1\}}(TM)$$

By the choice of $x_J$, we have that $x_{I' \setminus \{i'_1\}} \cdot p_{I' \setminus \{i'_1\}}(TM) = 0$ unless $I = I'$ and therefore $p_{I'}(TM \oplus -\xi') = 0$. Therefore we may assume without loss of generality that $i'_1 > i_1$. The strategy is to choose the coefficients $A_J$ by downwards induction with respect to $|J|$. Let $J = (j_1, \ldots, j_r)$ with $|J| \geq 1$ and let us assume that $A_J$ is already chosen for all $J'$ with $|J'| > |J|$. By the choice of the Pontryagin classes of $-\xi'$, this implies that $p_{ij}(-\xi')$ is already determined for all $i < 4(m - |J|) =: i_J$. If there exists a $j \in J$ such that $j > i_J$, we set $A_J = 0$. If not, let $I' := (i_J, J) := (i'_1, \ldots, i'_t)$ and note that by assumption $|I'| = 4m$ and $i'_1$ is the largest entry of $I'$. We again denote the number of indices agreeing with $i'_1$ by $a'_1$. We compute

$$p_{I'}(TM \oplus -\xi') = \prod_{t=1}^t \left( p_{i'_t}(TM) + \sum_{a=i_1}^{i'_t} p_a(-\xi')p_{i'_t-a}(TM) \right)$$

$$= \prod_{t: i'_t = i'_1} \left( p_{i'_t}(TM) + p_{i'_t}(-\xi') + \sum_{a=i_1}^{i'_t-1} p_a(-\xi')p_{i'_t-a}(TM) \right) \cdot \prod_{t: i'_t < i'_1} \left( p_{i'_t}(TM) + \sum_{a=i_1}^{i'_t} p_a(-\xi')p_{i'_t-a}(TM) \right).$$
It remains to specify Lemma 3.3. Proposition 1.3. we can choose Wie19. there exists an Example 3.7. we have Lemma 3.3. Lemma 2.5. HSS14.

Let $M$ be an oriented, simply connected manifold of dimension $d \geq \max(3k + 1, 2k + 5)$ that has at least one non-vanishing rational Pontryagin class. If $d + k \equiv 0 \pmod{4}$, then there exists a smooth, oriented $M$-bundle $E \to S^k$ that is fiber homotopy equivalent to the trivial bundle and satisfies $\tilde{A}(E) \neq 0$.

Proof. By Lemma 2.5 and Lemma 3.3 there exists an $M$-bundle $E \to S^k$ that has only two non-vanishing elementary Pontryagin numbers, namely $p_m$ and $p_{i\text{min}} \cdot p_{m-i\text{min}}$. By [FR21, Lemma 2.5], this implies that $\tilde{A}(E) \neq 0$. 

Remark 4.2. (i) This generalizes to HSS14, Theorem 1.4 and provides an upgrade: the result in loc.cit. is “based on abstract existence results [and] does not yield an explicit description of the diffeomorphism type of the fiber manifold” HSS14, p. 337). In contrast, our result states, that it is correct for generic manifolds.

(ii) By [HSS14, Proposition 1.9] and [Wie19, Lemma 2.3] a bundle $M \to E \to S^k$ is rationally nullcobordant, if all rational Pontryagin classes vanish or if $\dim(M) < \frac{k}{2}$. This shows that both assumptions on $M$
from Proposition 4.1 (and hence from Theorem D) are actually necessary, even though the dimension bound is not optimal (cf. Remark 2.4).

Next, let us investigate, if the bundles we constructed have cross-sections with trivial normal bundle. This is sometimes desirable for applications to positive curvature as it allows fiber-wise connected sums. We have the following result.

**Lemma 4.3.** If $i_{\text{min}} < d/4$, then the bundle from Proposition 4.1 has a cross-section with trivial normal bundle.

**Proof.** Let $\text{triv}: S^k \to S^k \times M$ be the trivial section. Since the bundle $E$ from Proposition 4.1 is fiber homotopy equivalent to the trivial bundle via $f: S^k \times M \simeq E$ we get a section $s := f \circ \text{triv}: S^k \to E$. We have

$$s^* p_n(TE) = \text{triv}^* \left( \sum_{i=0}^{n} p_i(TM) \cdot p_{n-i}(-\xi') \right)$$

$$= \sum_{i=0}^{n} \text{triv}^* p_i(TM) \cdot \text{triv}^* p_{n-i}(-\xi') = \text{triv}^* p_n(-\xi')$$

with $\xi'$ as in the proof of Lemma 3.3. Recall, that the only non-vanishing Pontryagin classes of $\xi'$ are $p_{m-i_{\text{min}}}$ and $p_m$ and let $\nu_s$ denote the normal bundle of $s$. Since the rank of this bundle is bigger than $k$, the bundle $\nu_s$ is stable in the sense that it is classified by an element in

$$\pi_k(BO) = \text{KO}^{-k}(pt) \cong \begin{cases} \mathbb{Z} & \text{for } k \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{for } k \equiv 1, 2 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Since we are only interested in the problem rationally, it suffices to consider the case $k \equiv 0 \mod 4$. It follows, that $\nu_s$ is trivial if $p_{k/4}(\nu_s) = 0$ and as $p(S^k) = 1$, the Pontryagin class $p_{k/4}$ of $\nu_s$ satisfies

$$p_{k/4}(\nu_s) = p_{k/4}(s^* TE) = s^* p_{k/4}(TE) = \text{triv}^* p_{k/4}(\xi) = 0$$

since by our assumption $k/4 < d/4 - i_{\text{min}} = m - i_{\text{min}}$ and $p_m - i_{\text{min}}$ and $p_m$ are the only Pontryagin classes of $\xi$. \hfill \square

**Remark 4.4.** If $d \neq 0 \mod 4$, the requirement from the lemma is automatically full-filled. If $d \equiv 0 \mod 4$ and $i_{\text{min}} = d/4$, then $M$ has only one non-vanishing Pontryagin number, namely $p_{d/4}(TM), [M]$). Since all coefficients in the $A$-polynomial are nonzero by [BB18], we have $\hat{A}(M) = a \cdot (p_{d/4}(TM), [M]) \neq 0$ for some $a \in \mathbb{Z} \setminus \{0\}$. If additionally $M$ admits a Spin-structure, then by the Lichnerowicz-formula and the Atiyah–Singer index theorem [AS63; Lic63], $M$ does not support a metric of positive scalar curvature. Hence, for a Spin-manifold of positive scalar curvature, we have $i_{\text{min}} < d/4$ and Lemma 4.3 applies.

4.2. Spin-structures and positive (scalar) curvature. Let $M$ be Spin and let $\text{BDiff}^{\text{Spin}}(M)$ be the classifying space for $M$-bundles with a Spin-structure on the vertical tangent bundle\footnote{A model for $\text{BDiff}^{\text{Spin}}(M)$ is given by $\text{BDiff}^{\text{Spin}}(M) := \{(N, \ell_N), M \cong N \subset \mathbb{R}^\infty, \ell_N \in \text{Bun}(TN, \theta^* U_d)\}$}. By [Ebe06, Lemma 3.3.6] the homotopy fiber
of the forgetful map $B\text{Diff}^{\text{Spin}}(M) \to B\text{Diff}(M)$ is a $K(\mathbb{Z}/2, 1)$ if $M$ is simply connected. Therefore the induced map

$$\pi_n(B\text{Diff}^{\text{Spin}}(M)) \otimes \mathbb{Q} \to \pi_n(B\text{Diff}(M)) \otimes \mathbb{Q}$$

is an isomorphism and we may assume without loss of generality that the bundles from Section 3 carry a Spin-structure on the vertical tangent bundle and hence on the total space, provided that $M$ admits one. Theorem D then follows from Proposition 4.1 by a standard argument that goes back to Hitchin [Hit74] (see [HSS14, Remark 1.5] or [FR21, Proposition 3.7]).

Applying Theorem D, we get the following classification for the push-forward action on metrics of positive scalar curvature which uses rigidity results from [ER19] and [Fre21].

**Corollary 4.5.** Let $M$ be a simply connected, $d$-dimensional Spin-manifold of positive scalar curvature and let $k \leq \min(2(d-3), d-\frac{5}{2})$.

(i) Then the orbit map $\pi_{k-1}(\text{Diff}(M, D)) \to \pi_{k-1}(\mathcal{R}_{\text{scal}>0}(M))$ factors through a finite group if and only if $(d+k)$ is not divisible by four or all rational Pontryagin classes of $M$ vanish.

(ii) If $k = 1$, then the same holds for $\text{Diff}(M)$ instead of $\text{Diff}(M, D)$.

**Proof.** The orbit maps

$$\pi_{k-1}(\text{Diff}(M, D)) \to \pi_{k-1}(\mathcal{R}_{\text{scal}>0}(M))$$

both are rationally trivial if all Pontryagin classes of $W$ vanish (cf. [ER19, Theorem F] and [Fre21, Theorem A]). The rest follows from Theorem D. □

Theorem D also allows to recover the main result from [HSS14] (loc.cit. Theorem 1.1 a)) and is actually slightly more precise on the dimension restriction.

**Corollary 4.6.** Let $k \geq 1$ and let $N$ be a Spin-manifold of positive scalar curvature with $\dim(N) = d \geq \max(3k + 1, 2k + 5)$ and $d + k \equiv 0 \mod 4$. Then the group $\pi_{k-1}(\mathcal{R}_{\text{scal}>0}(N))$ contains an element of infinite order.

**Proof.** Let $K$ be a $K3$-surface. Then for $n := d-4 \geq 2$, the manifold $K \times S^n$ satisfies the hypothesis of Theorem D and there is a $K \times S^n$-bundle $E \to S^k$ that has non-vanishing $\hat{A}$-genus and admits a cross section with trivial normal bundle. Gluing in the trivial $N \setminus D^d$-bundle along this cross section yields a $N\#(K \times S^{d-4})$-bundle over $S^k$ with non-vanishing $\hat{A}$-genus. Hence the group $\pi_{k-1}(\mathcal{R}_{\text{scal}>0}(N\#(K \times S^{d-4})))$ contains an element of infinite order. Since $N$ is cobordant to $N\#(K \times S^{d-4})$ in $\Omega^d_{\text{Spin}}(B\pi_1(N))$, the corresponding spaces of positive scalar curvature metrics are homotopy equivalent by [EF21, Theorem 1.5]. □

**Remark 4.7.** A more general result without any dimension restriction has been proven by Botvinnik–Ebert–Randal-Williams [BER17]. The methods from loc.cit. are however not constructive and do not give a way to decide if the obtained elements arise from the orbit of the action $\text{Diff}(M) \curvearrowright \mathcal{R}_{\text{scal}>0}(M)$. Furthermore it is unclear if those elements originate from the spaces $\mathcal{R}_{\text{Ric}>0}(M)$ or $\mathcal{R}_{\text{Sec}>0}(M)$.

for $\theta: B\text{Spin}(d) \to B\text{SO}(d)$ the 2-connected cover, $U_d \to B\text{SO}(d)$ the universal oriented vector bundle and $\text{Bun}(\underline{\underline{\cdot}})$ the space of bundle maps.
Appendix A. Rationally fibering a cobordism class over a sphere with Jens Reinhold

This appendix promotes the problem of studying the ideal of oriented cobordism classes that have a representative fibering over a sphere of fixed dimension. Such a class also fibers over any manifold of smaller dimension, see Proposition A.5. An answer thus has consequences for other bases, too. We are only interested in the rational version. It turns out that the results from the present paper can be used to say something new about this problem, which has been solved (even integrally) for dimensions at most 4 some time ago: in this case the rational answer is that a cobordism class fibers over $S^k$ for $k \leq 4$ if and only if its signature vanishes [Bur66; Neu71; Kah84a; Kah84b]. A variant of the analogous problem without orientations was originally introduced by Connor and Floyd [CF65]. We describe a construction that goes beyond the way bundles arise in the preceding paper. Unfortunately it seems as if even both ideas combined are not sufficient to solve the problem completely unless $k \leq 8$. We first outline a more concrete version of the problem. Let $\Omega_*^\ast$ denote the (graded) oriented cobordism ring.

**Definition A.1.**
(i) An oriented cobordism class $\alpha \in \Omega_*$ is said to fiber over a manifold $B$ if there is an oriented smooth fiber bundle $M \rightarrow E^d \rightarrow B$ such that $[E] = \alpha$.
(ii) For $k \geq 1$, let $A^k_* \subset \Omega_*$ denote the graded subgroup spanned by cobordism classes that fiber over $S^k$.
(iii) For given $k, m \geq 1$, define $c^k(m) \in \mathbb{Z}$ by

$$c^k(m) := \dim_Q(\Omega_{4m} \otimes Q) - \dim_Q(A^k_{4m} \otimes Q) - 1$$

Forming disjoint unions and products, we see that $A^k_*$ is an ideal in $\Omega^\ast$ and we may ask what these ideals are depending on $k$. As the signature of any manifold that fibers over a sphere vanishes, the two maps

$$A^k_* \otimes Q \hookrightarrow \Omega_* \otimes Q \xrightarrow{\varepsilon} Q$$

compose to 0. As there exist manifolds of non-zero signature in any dimension divisible by 4, this implies $c^k(m) \geq 0$. We may ask if the above sequence is exact in sufficiently high degrees, or equivalently (see part (ii)):

**Problem A.2.**
(i) Describe the ideals $A^k_* \subset \Omega_*$ for all values of $k$.
(ii) Is $c^k(m) = 0$ for fixed $k$ and sufficiently large $m$?

We will see below that we (at least) need to restrict to degrees $m \geq k/2$ for (ii) to be true: there are more constraints than the vanishing of the signature in lower degrees, see Proposition A.6. The following is our contribution towards an answer to Problem A.2.

**Theorem A.3.** Let $k \geq 1$ be fixed.

(i) We have $c^k(m) = \dim \Omega_{4m} - 1$ for $m < \frac{3k}{4}$, and $c^k(m) \geq 1$ for $m < \frac{k}{2}$.
(ii) For $5 \leq k \leq 8$ we have $c^k(m) = 0$ for $m > k$.
(iii) For $9 \leq k \leq 12$ we have $c^k(m) \leq 6$ in degrees $m > k$.

Regarding the last item we note that from computer-aided calculations we know that $c^k(m) = 0$ for $k \leq 12$ and $m \leq 500$, see Remark A.10.
We prove Theorem A.3 below. Before doing so, let us elaborate on the consequences of the preceding paper regarding a partial answer to Question A.2 for bigger values of $k$: sharpness of the upper bound from Theorem C (see also Remark 1.2) can be reformulated as $c^k(m) \leq p(m, [(k-1)/4])$. Note that $p(n, \ell) = p(n-\ell, \ell) + p(n, \ell-1)$. Using $p(n, 1) = 1$ a simple induction shows that $p(n, \ell) = \mathcal{O}(n^{\ell-1})$, which yields the following consequence of Theorem C.

**Corollary A.4.** For $k \geq 1$ and $m > k$, we have

$$c^k(m) \leq p(m, [(k-1)/4]) = \mathcal{O}(m^{(k-1)/4 - 1}).$$

The rest of the appendix is devoted to proving Theorem A.3.

**Elementary observations.** We first collect some elementary facts about the ideals $A_k^i \subset \Omega_s$.

**Proposition A.5.** A cobordism class that fibers over $S^k$ fibers over any $(k-1)$-manifold $B$.

**Proof.** Cutting out a $(k+1)$-disk from a nullbordism of $B \times S^1$, we see that $S^k$ and $B \times S^1$ can be joined by a connected oriented cobordism $W$. Applying obstruction theory to a relative CW-decomposition of $(W, S^k)$ and using that the obstructions lie in $H^j(W, S^k; \pi_{j-1}(S^k)) = 0$ for all $j$, we see that $W$ retracts onto $S^k$. For any smooth bundle $M \to E \to S^k$, we can thus extend the classifying map $S^k \to B\text{Diff}(M)$ to $W$. Restricting this extension to the other end of the cobordism gives a bundle $M \to E' \to B \times S^1$ with $[E'] = [E] \in \Omega_{4m}$. Since $E'$ clearly also fibers over $B$, this finishes the proof.  

Proposition A.5 implies that $(A_k^i)_{k \geq 1}$ forms a decreasing chain of ideals of $\Omega_s$. We next prove part (i) of Theorem A.3.

**Proof.** (c.f. [Wie19, Lemma. 2.3]) We need to show that for any smooth bundle $\pi: E^{4m} \to S^k$ with $d := (4m-k)$-dimensional fiber $M$ such that $4m < \frac{3}{2}k$, we have $[E] = 0 \in \Omega_{4m} \otimes \mathbb{Q}$.

Since the tangent bundle $TE$ is stably isomorphic to the vertical tangent bundle $T_\pi E$ whose dimension is $d$, we deduce that only Pontryagin classes $p_i$ with $i \leq 2d$ can be non-zero.

Analyzing the Serre spectral sequence of the fibration $M \to E \to S^k$ yields that $E$ has no cohomology in degrees $d < s < k$. The assumption implies that $k > 2d$, hence we deduce that all monomials in Pontryagin classes of $E$ in degrees at least $k$ vanish. In particular, all composite Pontryagin numbers of $E$ are zero. Together with the result on the vanishing signature that we recalled above, we deduce that $E$ is rationally nullbordant.

The second part of the assertion immediately follows from Proposition A.6 that we state and prove next.

**Proposition A.6.** Let $\pi: E^{4m} \to S^k$ be a fiber bundle with $4m < 2k$. Then $p_i(TE) = 0$ for all $i > 2m - \frac{k}{2}$. (The inequality ensures this number is smaller than $m$.)

**Proof.** Let $T_\pi E$ be the vertical tangent bundle of $\pi$. We have

$$p_i(TE) = p_i(T_\pi E \oplus TS^k) = p_i(T_\pi E).$$

Now $\text{rank}(T_\pi E) = 4m-k$ and so any $p_i(T_\pi E)$ with $i \geq \frac{1}{2}(4m-k)$ vanishes.  

Bundles that are trivial as fibrations. Note that constructions arising from block bundles yield bundles that are trivial as fibrations. For such bundles the following vanishing result holds, which implies that the analogue of Problem A.2 (ii) for fiber-homotopically trivial bundles has a negative answer.

**Proposition A.7.** For a fiber-homotopically trivial bundle $M \to E^{4m} \to B$ whose base space $B$ is $4\ell$-connected and $p \in H(BSO(4m); \mathbb{Q})$ a monomial in Pontryagin classes $p_i$ with $i \leq \ell$, the Pontryagin number $p(E)$ vanishes.

**Proof.** From the assumption that the bundle is trivial as a fibration, we deduce that $E \sim M \times B$. In particular, we get a retraction $E \to M$ for the inclusion of a fiber. But $B$ is $4\ell$-connected, hence all Pontryagin classes $p_i$ with $i \leq \ell$ pull back along this map. Since $H^{2m}(M) = 0$, this implies the assertion. □

Constructing a bundle that is non-trivial as a fibration. In this section, we construct for any $m \geq 1$ a bundle $\mathbb{C}P^m \to E \to S^{2m}$ so that $p_1^m(E) \neq 0$ if $m \geq 3$. We have seen in Proposition A.7 that the latter is not possible for bundles that are trivial as fibrations.

**Construction A.8.** Let $m \geq 1$. We construct a smooth $\mathbb{C}P^m$-bundle over $S^{2m}$ as follows. The topological group $GL_m(\mathbb{C})$ acts on

$$\mathbb{C}P^m = \{[z_0 : z_1 : \ldots : z_m] \mid z_i \in \mathbb{C} \text{ not all 0}\}$$

by acting linearly on the last $m$ projective coordinates. This action fixes the point $\ast := [1 : 0 : \ldots : 0]$ and induces a map

$$BGL_m(\mathbb{C}) \to BDiff(\mathbb{C}P^m, \ast).$$

The action of a differential on the tangent space of this fixed point produces a map

$$BDiff(\mathbb{C}P^m, \ast) \to BGL_{2m}(\mathbb{R}),$$

and it is evident that the composition of these two maps is the canonical map $BGL_m(\mathbb{C}) \to BGL_{2m}(\mathbb{R})$ induced from seeing $\mathbb{C}$ as a 2-dimensional real vector space. We now choose a complex $m$-dimensional vector bundle over $S^{2m}$, classified by a map $S^{2m} \to BGL_m(\mathbb{C})$, whose underlying $2m$-dimensional real vector bundle $\xi$ has a non-zero Euler number. When composed with the previous map, we obtain a map classifying a smooth bundle $\mathbb{C}P^m \to E \to S^{2m}$.

**Proposition A.9.** If $m \geq 3$, then the bundle from Construction A.8 satisfies $p_1(E)^m \neq 0$.

**Proof.** Choose a generator $\alpha \in H^2(\mathbb{C}P^m)$. Then there exists a unique class $\beta \in H^2(E)$ that pulls back to $\alpha$ along the inclusion $j: \mathbb{C}P^m \to E$ of the fiber. From the Serre spectral sequence of the bundle $\mathbb{C}P^m \to E \to S^{2m}$ which collapses since the $E^2$ page is supported in even degrees, we see that $\beta^m$ is Poincaré dual to a non-zero multiple of $\pi^*[S^{2m}]$, where $[S^{2m}] \in H_{2m}(S^{2m})$ denotes the fundamental class. We thus get that $\beta^m \in H^{4m}(E)$ is Poincaré dual to the Euler number of $\xi$, and hence non-zero. Since $S^{2m}$ has a trivial tangent bundle, we deduce $j^*p_1(E) = p_1(\mathbb{C}P^m) = (m + 1)\alpha^2$, hence $p_1(E) = (m + 1)\beta$. Hence indeed $p_1^m(E) = (m + 1)^2m \beta^m \neq 0$. □

We next prove part (ii) of Theorem A.3.
Proof. Assume that $5 \leq k \leq 8$ and $m > k$. Then Corollary A.4 says that $c^k(m) \leq 1$. We want to improve this to $c^k(m) = 0$. To do so, observe that the proof of Corollary A.4, which was simply a reformulation of (the sharpness of the upper bound of) Theorem C, only involved bundles that are trivial as fibrations. For any such bundle, we know from Proposition A.7 that $p^m_1(E) = 0$. However, the bundle arising from Construction A.8 satisfies $p^m_2(E) \neq 0$, we have thus found another element in $A^k_{4m}$ (not arising from block bundles) and so we have finished the proof.

Finally, we prove part (iii) of Theorem A.3.

Proof. For $i = 1, \ldots, m$, let $E_i \to S^{2i}$ denote the $\mathbb{CP}^i$-bundle from Construction A.8. First note that for $i \geq 5$, we have

$$p_2(E_i) = (j^*)^{-1}p_2(\mathbb{CP}^{2i}) = \frac{i}{2}(j^*)^{-1}p_1(\mathbb{CP}^{2i})^2 = \frac{i}{2}p_1(E_i)^2$$

Next, let $Q_i$ be a manifold of dimension $4(m - i)$ such that $p^m_{1-i}(Q_i) \neq 0$ is the only non-vanishing Pontryagin number and let $X_i := E_i \times Q_i$. We consider the following matrix:

$$B^m := \left( p^j_2(X_i) \cdot p^{m-2j}_1(X_i) \right)_{i=6, \ldots, m}$$

If $\text{rank}(B^m) \geq \left\lceil \frac{m}{2} \right\rceil + 1 - a$, then $c^k(m) \leq a$ for $k \leq 12$.

$$p^j_2(X_i) \cdot p^{m-2j}_1(X_i) = p^j_2(E_i \times Q_i) \cdot p^{m-2j}_1(E_i \times Q_i)$$

$$= \left( p_2(E_i) + p_1(E_i)p_1(Q_i) + p_2(Q_i) \right)^j \cdot \left( p_1(E_i) + p_1(Q_i) \right)^{m-2j}$$

By our choice of $Q_i$, any product containing $p_2(Q_i)$ will vanish and therefore we can go on with our computation.

$$= \left( p_2(E_i) + p_1(E_i)p_1(Q_i) \right)^j \cdot \left( p_1(E_i) + p_1(Q_i) \right)^{m-2j}$$

$$= p_1(Q_i)^{m-i} \cdot \sum_{n=0}^{\left\lfloor \frac{m}{2} \right\rfloor} p_2(E_i)^n \cdot p_1(E_i)^{j-n} \cdot p_1(E_i)^{i-j-n} \cdot \binom{j}{n} \cdot \binom{m-2j}{i-j+n}$$

$$= p_1(Q_i)^{m-i} \cdot \sum_{n=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \frac{i}{2} \right)^n \cdot \binom{m-2j}{i-j+n} \cdot p_1(E_i)^{2n} \cdot p_1(E_i)^{j-n} \cdot p_1(E_i)^{i-j+n}$$

$$= p_1(Q_i)^{m-i} \cdot \sum_{n=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \frac{i}{2} \right)^n \cdot \binom{m-2j}{i-j+n}$$

and hence it suffices to compute or estimate the rank of the following matrix:

$$A^m = (A^m_{ij})_{i=6, \ldots, m; j=0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor} := \left( \sum_{n=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{i}{2} n \binom{m-2j}{i-j+n} \right)_{i=6, \ldots, m}$$
Note that for \( j > i \) we have \((m-2j)/(i+j+n)\) = 0 for all \( n \geq 0 \) and hence \( A_{ij} = 0 \). Therefore the matrix \( A \) has the following form, where the asterisks represent non-zero entries.

\[
A = \begin{pmatrix}
  \ast & \ldots & \ast & 0 & \ldots & 0 \\
  \vdots & \ddots & \ast & 0 & \ldots & \ast \\
  \ast & \ldots & \ast & \ast & \ldots & \ast \\
  \ast & \ldots & \ast & \ast & \ldots & \ast \\
  \ast & \ldots & \ast & \ast & \ldots & \ast \\
  \ast & \ldots & \ast & \ast & \ldots & \ast \\
\end{pmatrix}
\]

In the first row, there are 7 non-zero entries, so the rank of \( A \) is at least \( \lfloor \frac{m}{2} \rfloor - 5 \).

**Remark A.10.** Computer calculations, for which we thank Marek Kaluba, have shown that \( c^k(m) = 0 \) for \( k \leq 12 \) and \( m \leq 500 \). This can be rephrased in the following way: For every \( k \leq 12 \) and any oriented manifold \( M \) of dimension at most 2000 there exists a \( \lambda \in \mathbb{N} \) such that the \( \lambda \)-fold connected sum of \( M \) with itself is cobordant to a fiber bundle \( E \to S^k \).

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Mathematisches Institut, Einsteinstr. 62, 48149 Münster, Germany

jens.reinhold@uni-muenster.de
jens.reinhold@posteo.de

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Email address: georg.frenck@kit.edu
Email address: math@frenck.net

INSTITUT FÜR ALGEBRA UND GEOMETRIE, ENGLERSTR. 2, 76131 KARLSRUHE, GERMANY