NONEQUILIBRIUM QUANTUM FIELDS
WITH LARGE FLUCTUATIONS

JÜRGEN BERGES AND MARKUS M. MÜLLER
Institute for Theoretical Physics, Heidelberg University,
Philosophenweg 16, 69120 Heidelberg, Germany

We consider the nonequilibrium evolution of an $O(N)$–symmetric scalar quantum field theory using a systematic two–particle irreducible $1/N$–expansion to next-to-leading order, which includes scattering and memory effects. The corresponding “full Kadanoff-Baym equations” are solved numerically without further approximations. This allows one to obtain a controlled nonperturbative description of far-from-equilibrium dynamics and the late-time approach to quantum thermal equilibrium. Employing in addition a first-order gradient expansion for the Wigner transformed correlators we derive kinetic equations. In contrast to standard descriptions based on loop expansions, our equations remain valid for nonperturbatively large fluctuations. As an application, we discuss the fluctuation dominated regime following parametric resonance in quantum field theory.

In recent years we have witnessed an enormous increase of interest in the dynamics of quantum fields out of equilibrium. Strong motivation in elementary particle physics comes, in particular, from current and upcoming relativistic heavy-ion collision experiments, phase transitions in the early universe or astrophysical applications. Here the involved nonequilibrium dynamics is often characterized by large corrections from quantum-statistical fluctuations that are not accessible in a weak coupling or loop expansion. A paradigm for such a situation is provided by the phenomenon of parametric resonance, which represents an important building block for our understanding of the (pre)heating of the early universe after a period of inflation. In this context the resonant amplification of fluctuations leads to explosive particle production with a transition from a classical to a fluctuation dominated regime, characterized by nonperturbatively large occupation numbers inversely proportional to the coupling.

Until recently, classical field theory studies on the lattice have been the only quantitative approach available. These are expected to be valid for not too late times, before the approach to quantum thermal equilibrium sets in. Calculations in quantum field theory had been limited to mean-field
type approximations (leading-order in large-$N$, or Hartree) which neglect scatterings. These approximations are known to fail to describe late-time thermalization and, even at early times, do not give a valid description of the entire amplification regime as pointed out in Refs.\textsuperscript{2,5}.

In Ref.\textsuperscript{2} the first study of parametric resonance in quantum field theory from a next-to-leading order (NLO) calculation in a systematic two-particle irreducible (2PI) $1/N$–expansion\textsuperscript{6,7} was presented, which includes scattering and memory effects. The classical resonant amplification at early times is found to be followed by a collective amplification regime with explosive particle production in a broad momentum range. As a consequence, one observes rapid prethermalization with a particle number distribution monotonous in momentum. In particular, in this regime there are leading contributions from all 2PI loop orders and standard weak coupling or loop expansions break down. For its description it is crucial to employ a non-perturbative approximation as provided by the $1/N$–expansion at NLO\textsuperscript{2}.

In this note, after reviewing the nonperturbative physics involved in the phenomenon of parametric resonance, we present suitable kinetic equations derived\textsuperscript{8} from the 2PI $1/N$–expansion at NLO\textsuperscript{6}. For similar approximation schemes see also Ref.\textsuperscript{9}. Standard descriptions are typically based on a loop expansion of the 2PI effective action. In particular the classical Boltzmann equation can be obtained starting from a three–loop approximation.\textsuperscript{10,11,12} The advantage of our kinetic equations is that their applicability is not limited by the question of the validity of a loop expansion. For sufficiently large $N$,\textsuperscript{13} it is only restricted by the applicability of a gradient expansion employed in the derivation of the kinetic equations. Apart from the example of the fluctuation dominated regime following parametric resonance, as a further important application these equations are expected to be valid near second-order phase transitions. Here the correlation length diverges at the transition in the static limit. The possibility of enhanced fluctuations near such a critical point for the nonequilibrium dynamics has received much attention recently in the context of relativistic heavy-ion collisions.\textsuperscript{14} A detailed discussion will be presented elsewhere.\textsuperscript{8}

We consider a real scalar $N$–component quantum field $\varphi_a$ ($a = 1, \ldots, N$) with $\lambda/(4!N) (\varphi_a \varphi_a)^2$ interaction, where summation over repeated indices is implied. All correlation functions of the quantum theory can be obtained from the 2PI generating functional for Green's functions $\Gamma[\phi,G]$, parametrized by the field expectation value $\phi_a(x) = \langle \varphi_a(x) \rangle$ and the co-
Figure 1. Total energy $E_{\text{tot}}$ (solid line) and classical field energy $E_{\text{cl}}$ (dotted line) as a function of time for $\lambda = 10^{-6}$ and $N = 4$ in $3 + 1$ dimensions, from Ref. 2. The dashed line represents the fluctuation part, showing a transition between a classical field and a fluctuation dominated regime. For $t \gtrsim t_{\text{nonpert}}$ there are leading contributions from all loop orders and the employed nonperturbative approximation becomes crucial.

The connected propagator $G_{ab}(x, y)$:

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \ln G^{-1} + \frac{i}{2} \text{Tr} G^{-1}_{0}(\phi) G + \Gamma_{2}[\phi, G].$$

Here $iG_{0,ab}^{-1}(x, y; \phi) \equiv \delta^{2}S[\phi]/\delta\phi_{a}(x)\delta\phi_{b}(y)$, where $S$ is the classical action with $S_{0} = -\int_{C} \phi_{a}(\Box x + m^{2})\phi_{a}/2$ as the free part. We use the notation $\int_{x} \equiv \int_{C} d^{0}x \int d^{4}x$ with $C$ denoting a closed time path along the real axis. At NLO in the $1/N$–expansion of the 2PI effective action one has

$$\Gamma_{2}[\phi, G] = \frac{\lambda}{4!N} \int_{x} G_{aa}(x, x)G_{bb}(x, x) + \frac{i}{2} \text{Tr} \ln B(G).$$

Here the function $I_{\text{resums an infinite number of “chain graphs”}}$. The equations of motion for $\phi_{a}$ and $G_{ab}$ are given by

$$\frac{\delta\Gamma[\phi, G]}{\delta\phi_{a}(x)} = 0, \quad \frac{\delta\Gamma[\phi, G]}{\delta G_{ab}(x, y)} = 0.$$
They have been solved numerically in Ref. 2 on a lattice along the lines of Ref. 6. We decompose the full two-point function using

$$ G_{ab}(x, y) = G^{>}_{ab}(x, y) \Theta_C (x^0 - y^0) + G^{<}_{ab}(x, y) \Theta_C (y^0 - x^0) $$

such that the spectral function $\rho$ and the symmetric propagator $F$ are given by

$$ \rho_{ab}(x, y) = -2 \text{Im} \left( G^{>}_{ab}(x, y) \right) \quad \text{and} \quad F_{ab}(x, y) = \text{Re} \left( G^{>}_{ab}(x, y) \right) $$

For the discussion of parametric resonance in Ref. 2 a system is considered, which is initially in a pure quantum state. Employing spatially homogeneous fields $\phi_a(t = t_0) \sim (6N/\lambda)^{1/2}$, the initial propagator is taken to be diagonal with $F_{ab} = \text{diag} \{ F_{\parallel}, F_{\perp}, \ldots, F_{\perp} \}$ and equivalently for $\rho_{ab}$. The effective transverse particle number density

$$ n_{\perp}(p) + \frac{1}{2} = \left[ F_{\perp}(t, t'; p) \partial_{t'} \partial_t F_{\perp}(t, t'; p) \right]_{t=t'}^{1/2} $$

and corresponding longitudinal particle number density $n_{\parallel}$ are defined as in Refs. 6,17. Initially $n_{\perp} \equiv n_{\parallel} \equiv 0$ such that the initial conditions are characterized by zero particle number and large field amplitudes for small coupling $\lambda$. Correspondingly, the conserved total energy $E_{\text{tot}}$ is initially well approximated by the classical field contribution, i.e. $E_{\text{tot}} \simeq E_{\text{cl}}$, as shown in Fig. 1. When the system evolves in time, more and more energy is converted into fluctuations. A detailed discussion of the various characteristic regimes is given in Ref. 2. For times $t \gtrsim t_{\text{nonpert}}$, this leads to nonperturbatively large particle number densities inversely proportional to the coupling. Neglecting
the difference between transverse and longitudinal modes at sufficiently late
times, with \( F_{ab} \simeq F \delta_{ab} \) and \( \rho_{ab} \simeq \rho \delta_{ab} \) we have \(^2\)
\[
F \sim \mathcal{O}(N^0 \lambda^{-1}), \quad \rho \sim \mathcal{O}(N^0 \lambda^0). \tag{8}
\]
The nonperturbatively large enhancement of the statistical propagator \( F \) has the important consequence that any approximation scheme based on a
loop expansion of the 2PI generating functional breaks down. We emphasize
that the 2PI \( 1/N \)–expansion at NLO remains valid as pointed out in Ref. \(^2\).
To discuss this in more detail, we consider here the simplified evolution
equations for \( F \) and \( \rho \) with \( \phi = 0 \). For the general case with a nonvanishing
field expectation value see Ref. \(^2\). One finds \(^6,17\)
\[
\left[ \Box_x + M^2(x) \right] F(x, y) = - \int_0^{x^0} dz^0 \int dz \Sigma_F(x, z) F(z, y)
+ \int_0^{y^0} dz^0 \int dz \Sigma_F(x, z) \rho(z, y), \tag{9}
\]
\[
\left[ \Box_x + M^2(x) \right] \rho(x, y) = - \int_0^{x^0} dz^0 \int dz \Sigma_F(x, z) \rho(z, y).
\]
At NLO in the \( 1/N \)–expansion the effective mass term \( M^2(x) \) is given by
\[
M^2(x) = m^2 + \lambda \frac{N + 2}{6N} F(x, x) \tag{10}
\]
and the self–energies are \(^6\)
\[
\Sigma_F(x, y) = - \frac{\lambda}{3N} \left[ F(x, y) I_F(x, y) - \frac{1}{4} \rho(x, y) I_\rho(x, y) \right], \tag{11}
\]
\[
\Sigma_\rho(x, y) = - \frac{\lambda}{3N} \left[ \rho(x, y) I_F(x, y) + F(x, y) I_\rho(x, y) \right]. \tag{12}
\]
Here the functions \( I_F \) and \( I_\rho \) contain the resummation:
\[
I_F(x, y) = - \frac{\lambda}{3} \Pi_F(x, y) + \frac{\lambda}{3} \int_0^{x^0} dz^0 \int dz I_\rho(x, z) \Pi_F(z, y)
+ \frac{\lambda}{3} \int_0^{y^0} dz^0 \int dz I_F(x, z) \Pi_\rho(z, y), \tag{13}
\]
\[
I_\rho(x, y) = - \frac{\lambda}{3} \Pi_\rho(x, y) + \frac{\lambda}{3} \int_0^{x^0} dz^0 \int dz I_\rho(x, z) \Pi_\rho(z, y), \tag{14}
\]
with
\[
\Pi_F(x, y) = - \frac{1}{2} \left( F^2(x, y) - \frac{1}{4} \rho^2(x, y) \right), \tag{15}
\]
\[
\Pi_\rho(x, y) = - F(x, y) \rho(x, y). \tag{16}
\]
Using (8) in (13) and (14), one observes that each term of the infinite resummation of loops contained in \( I_F \) and \( I_\rho \) contributes with the same order in \( \lambda \). To describe this regime and the late-time behavior a nonperturbative approximation has to be employed, such as the 2PI \( 1/N \)–expansion at NLO.\(^6,7\)

In the fluctuation dominated regime, for \( t \gtrsim t_{\text{nonpert}} \), the dynamics is characterized by a slow drifting of modes\(^6\), which suggests (cf. Ref. \(^17\)) the applicability of a first order gradient expansion with respect to the center coordinate \( X = (x + y)/2 \).\(^9,10,11,12,18\) Exploiting the effective loss\(^6\) of the dependence on the initial time for sufficiently large \( x^0 \) and \( y^0 \), we send \( t_0 \to -\infty \) in Eqs. (9) and (13). Using the retarded and the advanced propagator

\[
G_R(x, y) = \Theta (x^0 - y^0) \rho(x, y),
\]

\[
G_A(x, y) = -\Theta (y^0 - x^0) \rho(x, y)
\]
as well as corresponding definitions for the retarded and advanced self–energies \( \Sigma_{R,A} \) and the resummation functions \( I_{R,A} \), one can then also send the upper limits to infinity and Fourier transform with respect to the relative coordinate \( s = x - y \). For the two–point functions we write

\[
\hat{F}(X, k) = \int d^4s \, e^{iks} F \left( X + \frac{s}{2}, X - \frac{s}{2} \right),
\]

\[
\hat{\rho}(X, k) = -i \int d^4s \, e^{iks} \rho \left( X + \frac{s}{2}, X - \frac{s}{2} \right).
\]

Here we introduced a factor \(-i\) in the definition of the Wigner transformed spectral function to obtain a real \( \hat{\rho}(X, k) \). The advanced and the retarded propagators satisfy \( \hat{G}_R^*(X, k) = \hat{G}_A(X, k) \). Again, this property also holds for the corresponding self–energies. With the notation

\[
\left( \hat{f} \ast \hat{g} \right)(X, k) = \int \frac{d^4q}{(2\pi)^4} \hat{f}(X, k - q) \hat{g}(X, q)
\]

and the definition of the Poisson brackets

\[
\left\{ \hat{f}(X, k); \hat{g}(X, k) \right\}_{PB} = \left[ \partial_{k_\mu} \hat{f}(X, k) \right] \left[ \partial_{X^\nu} \hat{g}(X, k) \right] - \left[ \partial_{X^\nu} \hat{f}(X, k) \right] \left[ \partial_{k_\mu} \hat{g}(X, k) \right],
\]

one finds from a first–order gradient expansion of Eqs. (9) – (16) the kinetic equations:
\[
\left[ 2k^\mu \partial_{X^\mu} + \left( \partial_{X^\mu} M^2 (X) \right) \partial_{k^\mu} \right] \tilde{F} (X, k) \\
= \tilde{F} (X, k) \tilde{\Sigma}_\phi (X, k) - \tilde{\Sigma}_F (X, k) \tilde{\varrho} (X, k) \\
+ \left\{ \tilde{\Sigma}_F (X, k) ; \text{Re} \tilde{G}_R (X, k) \right\}_{PB} + \left\{ \text{Re} \tilde{\Sigma}_R (X, k) ; \tilde{F} (X, k) \right\}_{PB},
\]
(17)

\[
\left[ 2k^\mu \partial_{X^\mu} + \left( \partial_{X^\mu} M^2 (X) \right) \partial_{k^\mu} \right] \tilde{\varrho} (X, k) \\
= \left\{ \tilde{\Sigma}_\phi (X, k) ; \text{Re} \tilde{G}_R (X, k) \right\}_{PB} + \left\{ \text{Re} \tilde{\Sigma}_R (X, k) ; \tilde{\varrho} (X, k) \right\}_{PB}.
\]
(18)

Here the Wigner transformed self-energies are given by
\[
\tilde{\Sigma}_F (X, k) = -\frac{\lambda}{3N} \left( \tilde{F} * \tilde{I}_F \right) (X, k) + \frac{1}{4} \left( \tilde{\varrho} * \tilde{I}_\varrho \right) (X, k),
\]
(19)

\[
\tilde{\Sigma}_\phi (X, k) = -\frac{\lambda}{3N} \left( \tilde{F} * \tilde{I}_\varrho \right) (X, k) + \left( \tilde{\varrho} * \tilde{I}_F \right) (X, k)
\]
(20)

and the equations for the resummation functions read
\[
\tilde{I}_\varrho (X, k) = -\lambda \left\{ \tilde{I}_\varrho (X, k) \left[ 1 - \tilde{I}_R (X, k) \right] + \tilde{I}_\varrho (X, k) \left( \tilde{G}_A * \tilde{F} \right) (X, k) \\
- \frac{i}{2} \left\{ \tilde{I}_R (X, k) ; \tilde{I}_\varrho (X, k) \right\}_{PB} + \frac{i}{2} \left\{ \tilde{I}_\varrho (X, k) ; \left( \tilde{G}_A * \tilde{F} \right) (X, k) \right\}_{PB} \right),
\]
(21)

\[
\tilde{I}_F (X, k) = -\lambda \left\{ \tilde{I}_F (X, k) \left[ 1 - \tilde{I}_R (X, k) \right] + \tilde{I}_F (X, k) \left( \tilde{G}_A * \tilde{F} \right) (X, k) \\
+ i \left\{ \tilde{I}_R (X, k) ; \tilde{I}_F (X, k) \right\}_{PB} + \frac{i}{2} \left\{ \tilde{I}_F (X, k) ; \left( \tilde{G}_A * \tilde{F} \right) (X, k) \right\}_{PB} \right).
\]
(22)

Here \( \tilde{\Pi}_{F,\varrho} (X, k) \) are the Wigner transforms of the functions defined in Eqs. (15) and (16). At this order of the gradient expansion the retarded propagator fulfills the algebraic equation
\[
\tilde{G}_R (X, k) = \left[ -k^2 + M^2 (X) + \tilde{\Sigma}_R (X, k) \right]^{-1}.
\]
(23)

Similarly the retarded self-energy and the retarded resummation function satisfy
\[ \Sigma_R (X, k) = -\frac{\lambda}{3N} \left[ \left( \tilde{F} * \tilde{I}_R \right) (X, k) + \left( \tilde{G} \tilde{R} \tilde{F} \right) (X, k) \right] \]  

(24)

and

\[ \tilde{I}_R (X, k) = \frac{\lambda}{3} \left[ 1 - \tilde{I}_R (X, k) \right] \left( \tilde{G} \tilde{R} \tilde{F} \right) (X, k) \]

\[ -\frac{i\lambda}{6} \left\{ \tilde{I}_R (X, k), \left( \tilde{G} \tilde{R} \tilde{F} \right) (X, k) \right\}_{PB}. \]  

(25)

We emphasize that Eqs. (17) – (25) represent a closed set of equations. In their range of applicability these equations have the advantage that they do not involve an integration over time history. This is important for an efficient description of the late–time behaviour of quantum fields. A detailed discussion will be presented in Ref. 8.

We thank Julien Serreau for collaboration on related work 2,7 and many discussions.

References
1. J.H. Traschen and R.H. Brandenberger, Phys. Rev. D42 (1990) 2491; L. Kofman, A.D. Linde and A.A. Starobinsky, Phys. Rev. Lett. 73 (1994) 3195.
2. J. Berges, J. Serreau, hep–ph/0208070.
3. S.Yu. Khlebnikov and I.I. Tkachev, Phys. Rev. Lett. 77 (1996) 219; T. Prokopec and T.G. Roos, Phys. Rev. D55 (1997) 3768.
4. D. Boyanovsky, H.J. de Vega, R. Holman and J.F.J. Salgado, Phys. Rev. D54 (1996) 7570.
5. L. Kofman, A.D. Linde and A.A. Starobinsky, Phys. Rev. D56 (1997) 3258.
6. J. Berges, Nucl. Phys. A699 (2002) 847.
7. G. Aarts, D. Ahrensmeier, R. Baier, J. Berges, J. Serreau, to appear in Phys. Rev. D, hep–ph/0201308.
8. J. Berges, M. M. Müller, in preparation.
9. Yu. B. Ivanov, J. Knoll, D. N. Voskresensky, Ann. Phys. 293 (2001) 126.
10. G. Baym, L. P. Kadanoff, Quantum Statistical Mechanics, Benjamin, New York (1962).
11. P. Danielewicz, Ann. Phys. 152 (1984) 239.
12. E. Calzetta, B. L. Hu, Phys. Rev. D37 (1988) 2878.
13. G. Aarts, J. Berges, Phys. Rev. Lett. 88 (2002) 041603.
14. B. Berdnikov and K. Rajagopal, Phys. Rev. D61 105017 (2000).
15. J. M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. D10 (1974) 2428.
16. L.V. Keldysh, Sov. Phys. JETP 20 (1965) 1018.
17. G. Aarts, J. Berges, Phys. Rev. D64 (2001) 105010.
18. Jean–Paul Blaizot, Edmond Iancu, Phys. Rept. 359 (2002) 355
19. J. Berges, J. Cox, Phys. Lett. B517 (2001) 369.