Environmental induced renormalization effects in quantum Hall edge states due to $1/f$ noise and dissipation

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Abstract. We propose a general mechanism for the renormalization of the tunnelling exponents in edge states of the fractional quantum Hall effect. Mutual effects of the coupling with out-of-equilibrium $1/f$ noise and dissipation are considered for both the Laughlin sequence and the composite co- and counter-propagating edge states with Abelian or non-Abelian statistics. For states with counter-propagating modes, we demonstrate the robustness of the proposed mechanism in the so-called disorder-dominated phase. Prototypes of these states, such as $\nu = 2/3$ and $\nu = 5/2$, are discussed in detail, and the rich phenomenology induced by the presence of a noisy environment is presented. The proposed mechanism could help justify the strong renormalizations reported in many experimental observations carried out at low temperatures. We show how environmental effects could affect the relevance of the tunnelling excitations, leading to important implications, in particular for the $\nu = 5/2$ case.

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1. Introduction

States living at the boundary of fractional quantum Hall (FQH) systems represent one of the more intriguing examples of one-dimensional (1D) interacting electron gases [1]. The general theory describing these edge states involves the idea of a chiral Luttinger liquid (χLL) [2, 3]. In particular, for the simple Laughlin sequence [4], at filling factor $\nu = 1/(2n + 1)$ with $n \in \mathbb{N}$, all the properties of the system, including the fractional charge and statistics of edge excitations, are described in terms of a single chiral bosonic field. More involved is the description of states belonging to the Jain sequence [5] at filling factor $\nu = p/(2np + 1)$, in which $n \in \mathbb{N}$ and $p \in \mathbb{Z}$, where the introduction of a charge bosonic field and additional neutral bosonic modes is required by the proposed hierarchical theories leading to a hidden SU($|p|$) symmetry [6, 7]. In recent years much interest was devoted to more exotic states, such as $\nu = 5/2$, where different models were proposed, with excitations supporting both Abelian [9–11] or non-Abelian [12–17] statistics. To date, different empirical observations [18–21] have suggested that the non-Abelian anti-Pfaffian model is a proper candidate for $\nu = 5/2$ even though the debate is still very open [22, 23]. In the effective field theories the peculiar non-Abelian properties are encoded in an additional conformal field, which belongs to the Ising sector [24]. The non-Abelian nature of the excitations of this state kindled interest in the perspective of their possible application to topologically protected quantum computation [24]. A simpler experimental test for all these models is the study of transport properties in a quantum point contact (QPC) geometry [25]. In the absence of interactions between edges and external degrees of freedom, the power-law behaviour of the transport properties in the QPC geometry, as a function of bias or temperature, directly reflects the universal exponents of the χLL theory.

Unfortunately, sometimes strong discrepancies between the predictions of such theories and experimental observations are reported. For example, even for the simple Laughlin sequence [26, 27] the behaviour of the differential conductance, as a function of the voltage, is in qualitative agreement with predictions only at high temperature showing a peak at zero
bias. However, with decreasing the temperature, the observed peak turns into a completely unexpected dip.

Anomalous current/voltage characteristics have been measured also for other filling factors such as $\nu = 2/5$ in the Jain sequence [28]. Furthermore, renormalizations of the $\chi$ LL exponents are sometimes crucial to fully explain the measured crossover of the tunnelling charges at low temperatures [28–35].

Possible explanations for these disagreements have been traced back to the inhomogeneity of the filling factor below the QPC due to the action of the electrostatic gates [27, 36] or to an energy-dependent tunnelling amplitude caused by the extended nature of the contact [37, 38]. Alternatively, various mechanisms leading to the renormalization of the Luttinger parameters through coupling with external environments have been proposed. They range from the coupling with 1D phonons [39, 40], edge reconstruction induced by the smoothness of the confinement potential [41], possible Coulomb interaction between the different edges [42, 43], to interactions with a compressible component of a composite fermion liquid with very small longitudinal conductivity [44, 45] or to the coupling with electromagnetic environments [46–49].

Many of these approaches have focused on the Laughlin case and cannot be easily extended to composite edge states where anomalous behaviours are usually observed. In particular, many of the above mechanisms are not robust against the disorder-induced intra-edge electron tunnelling, an unavoidable effect in real samples responsible for the equilibration of the different channels. This is a crucial ingredient in explaining the universal quantization of the conductance in the presence of counter-propagating modes [7, 15–17, 23, 50].

Recently, Dalla Torre et al [51, 52] observed that the interplay between the $1/f$ noise generated by the external environment and the dissipation induced by the cooling setup could lead to a renormalization of the Luttinger parameter for 1D systems of cold atoms.

In this paper, we will apply this idea to the case of the edge states in the FQH effect, taking into account the peculiar chiral nature of the $\chi$ LL theories and investigating the effects of an external noisy environment. The $1/f$ noise is a quite universal and unavoidable perturbation in any electronic circuitry. The possible sources of the $1/f$ noise in quantum Hall samples depend on the detailed physical properties of the samples such as mobility, doping concentration, temperatures or growing conditions [53]. There are clear signs that, at least for a very high mobility sample, the $1/f$ noise is generated by switching noise determined by the remote ionized dopand [54–59]. These sources of noise, with $1/f$ spectrum, stochastically drive the system into an out-of-equilibrium condition, and the stationary condition is recovered by the dissipation mechanism. For 1D electron systems and $\chi$ LLs, many different dissipative mechanisms have previously been considered: from the coupling with metallic gates used to confine the electron gas [60] to the coupling with the electromagnetic environment [46–48] or with other systems [44, 45].

We will discuss the consequences of the joint presence of $1/f$ noise and dissipative effects that are very common in real systems. The presence of $1/f$ noise drives the problem out-of-equilibrium requiring an appropriate formalism to deal with it. The noise and dissipation could both spoil the scale invariance of the theory, an essential property of the $\chi$ LLs. For example, the energy scale of this breaking associated with the dissipative term will be $E_\gamma \sim h\gamma$, where $\gamma$ is the friction coefficient. Inspired by [51, 52], we will assume that the noise strength $F$ and the friction coefficient $\gamma$ are almost negligible but keeping their ratio $F/\gamma$ finite. The noisy environment (noise and dissipation) modifies the scaling of the $\chi$ LL exponents without breaking the scale invariance for energies bigger than $E_\gamma$. Indeed it is the presence of other energy scales,
such as finite bias $eV$, temperature $k_B T$ or even finite size effect $E_L \sim \hbar v_f / L$, bigger than $E_\gamma$ that allows us to consider only a scale invariant theory. In other words, we will assume the existence of an energy window where the effects of breaking of the scale invariance due to the noisy environment are not observable but their non-equilibrium effects are manifest on the dynamics of the system. We think that this model is appropriate for discussing real systems in which the coupling of the edge states with the noise and dissipation is typically very weak.

A relevant advantage of the proposed mechanism is its validity in the disorder-dominated phase, a key feature in order to apply the model to edge states with counter-propagating channels, such as $\nu = 2/3$ and $\nu = 5/2$. In particular, we will show that the disorder-dominated phase is still possible even in the presence of moderate noise strength by the investigation of the first-order renormalization group (RG) for the disorder term. Furthermore, once the system is assumed to be in the disordered-dominated phase, one can show that the neutral and charge modes are also renormalized in comparison to the standard prediction of the $\chi$LL theories. The aim of this paper is to present a detailed analysis of these facts and their possible consequences for real experiments. This paper discusses, at a general level, the origin of the renormalization mechanism anticipated in [35] for the case $\nu = 5/2$.

This paper is organized as follows. In section 2, we consider the Laughlin sequence. Using this paradigmatic example we introduce the notations and the general methods that we will use later for the composite edge case, which is the main issue of this paper. The effects of the renormalization induced by noisy environments and the possible consequences for the QPC transport are discussed. In section 3, we analyse the effect of the noisy environment on the Jain sequence, limiting ourselves for simplicity to the two-mode cases $\nu = 2/5, \ 2/3$. In particular for the co-propagating case $\nu = 2/5$, we investigate how the scaling becomes non-universal and also dependent on the strength of the intra-edge Coulomb interaction when external noise is present. This is very different from the standard hierarchical result where the scaling dimension is predicted to be independent of interaction between the modes. Discussing the case of counter-propagating modes (i.e. $\nu = 2/3$), we exploit this condition to get the disorder-dominated phase in the presence of a noisy environment. In section 4, the properties of the anti-Pfaffian model for $\nu = 5/2$ as a function of the strength of the $1/f$ noise are analysed, showing the regions of the parameters space where the elementary excitation with non-Abelian nature could dominate. Finally, we discuss the counterintuitive result that the external noise could help in the manipulation of non-Abelian excitation in the QPC geometry. Our conclusions are summarized in section 5.

2. The Laughlin sequence

Let us consider the edge states of a quantum Hall fluid, described in terms of $\chi$LL theories, and investigate the effect induced by the joint presence of an external environment, $1/f$ noise and dissipation. We stress that owing to the presence of $1/f$ noise, we have to face an out-of-equilibrium problem; therefore, in the following, we will employ proper techniques, i.e. the Keldysh contour formalism [61–65].

2.1. The model

We start our analysis considering the Laughlin sequence [4] with filling factor $\nu = 1/(2n + 1)$, in which $n \in \mathbb{N}$. The Lagrangian density of the $\chi$LL for an infinite edge is described in terms of
a single bosonic mode
\[ L_0 = \frac{1}{4\pi v} \partial_i \varphi (-\partial_t - v \partial_x) \varphi, \]  
where \( \varphi \) is a right-moving field along the edge with propagation velocity \( v \).

In view of dealing with an out-of-equilibrium system, we treat the problem in the Keldysh contour formalism. According to the standard path integral formulation, the non-equilibrium forward/backward time branch \( \varphi^{f/b} \). See [63] and [64] for a general treatment of these issues and [65] for the application of these methods to the edge states of the quantum Hall effect.

It is useful to write \( \varphi^{f/b} = (\varphi^{cl} \pm \varphi^q)/\sqrt{2} \), where \( \varphi^{cl} (\varphi^q) \) represents the so-called classical (quantum) component of the field [64]. In terms of the classical-quantum basis the bosonic Green’s functions (GFs) are enclosed in the matrix
\[ G_{ab}(x, t) = -i\langle \varphi^a(x, t) \varphi^b(0, 0) \rangle = \begin{pmatrix} G^R(x, t) & G^K(x, t) \\ G^A(x, t) & 0 \end{pmatrix}, \]  
where \( a, b = cl, q \). Here \( G^R \), \( G^A \) and \( G^K \) are the retarded, advanced and Keldysh GFs, respectively [64]. In terms of these fields and in Fourier transform, defined as
\[ \varphi^{cl/q}(q, \omega) = \int dx dt \ e^{i(qx - \omega t)} \varphi^{cl/q}(x, t), \]  
the free Keldysh action, deduced from (1), reads
\[ S_0 = \frac{1}{2} \sum_{q \neq 0, a} (\Phi^*(q, \omega)) \cdot G^{-1}_0(q, \omega) \cdot \Phi(q, \omega), \]  
with the vector \( \Phi = (\varphi^{cl}, \varphi^q)^T \). The matrix kernel of the action is [64]
\[ G^{-1}_0(q, \omega) = \frac{q}{2\pi v} \begin{pmatrix} 0 & \omega - i\epsilon - vq \\ (\omega + i\epsilon) - vq & 2i\epsilon sgn(\omega) \end{pmatrix}, \]  
where \( \epsilon \to 0 \) is the standard regularization factor and the top-left 0 component corresponds to the standard continuum limit of the Keldysh action [64]. Inverting the Kernel matrix and taking the cl–q component \( (G^0)_{cl,q} \), we obtain, according to (2), the retarded GF for the free bosonic fields
\[ G^R_0(q, \omega) = \frac{2\pi v}{q} \left( \frac{1}{(\omega + i\epsilon) - vq} \right). \]  
and analogously for the advanced GF, \( G^K_0 = (G^0)_{q,cl} \). From the linear response theory, one can show that the current along the Hall bar is given by \( I = v g_0 V_H \) with \( V_H \) the Hall potential and \( g_0 = e^2/h \) the quantum of conductance [7].

We now discuss the influence of \( 1/f \) noise term at low energies [56, 58]. This contribution can be described in terms of a classical stochastic external potential \( f(x, t) \) that describes the effective interaction of the edge with the localized trapped charges. The correlation function of the external force is \( K(q, \omega) = \langle f^*(q, \omega) f(q, \omega) \rangle = F/|\omega| \), where \( F \) is the strength of the noise. For simplicity the noise is assumed \( \delta \)-correlated in space, a natural assumption for short-range impurities in the low energy/long wavelength limit. Note that the presence of this time-dependent external force brings the system out of equilibrium.

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6 The regularization factor, responsible for the causality structure, will be disregarded once the dissipation term is introduced.
The external Gaussian random force $f(x, t)$ couples directly with the electron density $\rho = \partial_t \rho/(2\pi)$ of the $\chi$LL. In the Keldysh formalism this interaction is\footnote{Note that the coupling strength can be always reduced to this form by an appropriate redefinition of the parameter $F$.}

$$S_{f,\psi} = \int dt \left( \mathcal{L}_{f,\psi} - \mathcal{L}_{f,\psi0} \right) = \sqrt{2} \sum_{q, \omega} (iq) f^*(q, \omega) \varphi^q(q, \omega),$$

with $\mathcal{L}_{f,\psi} \propto f(x, t) \partial_t \varphi(x, t)$ and where, in the first identity, we write the Keldysh action in terms of the fields $\varphi^{f/b}$ and, in the second one, in terms of the quantum component $\varphi^q$ only. This result is standard in the Keldysh formalism and comes directly from the fact that a purely classical external force couples only with the quantum component $\varphi^q$ of the field [64]. The total $1/f$ effective Keldysh action [63] $S_{1/f}$ for the bosonic field $\varphi$ derives from the functional integration

$$e^{iS_{1/f}} = \int Df \ e^{-\frac{1}{2} \sum_{q, \omega} K^{-1}(q, \omega)|f(q, \omega)|^2} e^{iS_{f,\psi}},$$

which averages on the disorder realizations of the noise potential $f(x, t)$. The averaged effective Keldysh action $S_{1/f}$ can be written in a form similar to (3) with kernel [51]

$$G_{1/f}^{-1}(q, \omega) = \begin{bmatrix} 0 & 0 \\ 0 & +2iq^2 F/|\omega| \end{bmatrix}.$$  

Here, only the Keldysh $q$–$q$ component is different from zero. This is a direct consequence of the fact that $1/f$ noise brings the system out of equilibrium. In this case the usual relations between retarded, advanced and Keldysh GFs, dictated by the fluctuation-dissipation theorem, are no longer valid.

The system, under the external driving force ($1/f$ noise), will reach a stationary condition only in the presence of a dissipative mechanism that drains the energy accumulated in the system. Various mechanisms may introduce dissipation into the edge states [44–48, 60]. Here we limit ourselves to considering the most general assumption with a dissipative term, induced by the external bath, generalizing the Caldeira–Leggett approach to the $\chi$LL [46, 66]. The 1D edge mode can be coupled with oscillators through the current density $j \propto \partial_t \varphi$ or the charge density $\rho \propto \partial_t \rho$. Hereafter, we will discuss the Keldysh action for a generic spectral function of the bath. Later on, we will focus only on the ohmic behaviour. The general Lagrangian density, which couples edge and harmonic oscillator modes, is $\mathcal{L}_{\xi, \varphi} \propto \xi(x, x_\perp = 0, t) \partial_{x_\perp} \varphi(x, t)$, where $\mu = t$ ($\mu = x$) describes the coupling with the current (charge) density. The field $\xi(x, x_\perp, t)$ represents a bath of oscillators with extra spatial degrees of freedom $x_\perp$ orthogonal to the 1D system [46].

Integrating out the harmonic bath degrees of freedom, it is easy to obtain the usual Matsubara Euclidean effective action [64] ($\beta = (k_B T)^{-1}$)

$$S_{\text{diss}}^E = \frac{1}{2\beta} \sum_{q, \omega_n} D^{-1}(q, i\omega_n) |\varphi(q, i\omega_n)|^2$$

with $\omega_n = 2\pi n / \beta$. The spectral function $D^{-1}(q, i\omega_n)$ encodes all the dynamical information about the external bath and the coupling mechanism [46]. In the Keldysh contour formalism
this dissipative contribution \( S_{\text{diss}} \) can be written as in (3) with the kernel [64]

\[
\mathcal{G}_{\text{diss}}^{-1}(q, \omega) = \begin{bmatrix}
0 & -i\gamma \omega \\
+i\gamma \omega & +2i\gamma |\omega| \end{bmatrix},
\]

(10)

\([D^{R/A}(\omega, q)]^{-1}\) being the retarded/advanced analytic continuation of the spectral function \( D^{-1}(q, i\omega_n) \). In this case the Keldysh component of the dissipation is computed by using the fluctuation-dissipation theorem [64]

\[
[D^{-1}]^K = ([D^{R}]^{-1} - [D^{A}]^{-1}) \coth(\beta \omega/2)
\]

(11)

that must be satisfied by a bath in thermal equilibrium [51].

As stated before, in this paper we will only consider a specific type of dissipation, the ohmic one. The form of the bath spectral function for such a case is \( D^{-1}(q, i\omega_n) = \gamma |\omega_n| \), with \( \gamma \) the friction coefficient. At zero temperature the Keldysh kernel becomes

\[
\mathcal{G}_{\text{diss}}^{-1}(q, \omega) = \begin{bmatrix}
0 & -i\gamma \omega \\
+i\gamma \omega & +2i\gamma |\omega| \end{bmatrix}.
\]

(12)

Finally, the total Keldysh action is \( S_{\text{tot}} = S_0 + S_{1/f} + S_{\text{diss}} \) with total kernel for \( \varphi \)

\[
\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \mathcal{G}_{1/f}^{-1} + \mathcal{G}_{\text{diss}}^{-1}
\]

(cf (4), (8) and (12)). Inverting the kernel, the non-equilibrium Keldysh GFs at zero temperatures read

\[
\mathcal{G} = \frac{-2i\gamma |\omega| (1 + (\tilde{F}/\tilde{\gamma}) q^2/|\omega|^2)}{(q(\omega - v q) - i\tilde{\gamma} \omega)^{-1}} \begin{bmatrix}
(q(\omega - v q) + i\tilde{\gamma} \omega)^{-1} & 0 \\
0 & 0
\end{bmatrix},
\]

(14)

where \( \tilde{F} = 2\pi v F \) is the rescaled strength of the noise and \( \tilde{\gamma} = 2\pi v \gamma \) the rescaled friction coefficient. The regularization factor \( \epsilon \) in (4) is suppressed because the causal structure is already guaranteed by the dissipative contribution\(^8\).

It is worthwhile to underline that both the 1/f noise and the dissipation terms are relevant perturbations in the RG sense with massive coupling constants, namely \( \text{dim} [\tilde{F}] = \text{dim} [\tilde{\gamma}] = 1 \) (\( \text{dim} [\cdots] \) indicates the canonical mass dimension). The relevance of these terms will completely spoil the scale invariance property that characterizes the standard \( \chi LL \) theory for the edge states. For example, the dissipative term introduces a characteristic relaxation time \( \tau \sim \tilde{\gamma}^{-1} \) which determines the energy \( E_\gamma \sim \hbar \tau^{-1} \) below which the scale invariance is broken. One can demonstrate that for a noisy environment only \emph{weakly} coupled with the edge, i.e. \( \tilde{\gamma} \to 0 \), but with the ratio \( \tilde{F} / \tilde{\gamma} \) constant [51] the scale invariance is preserved. In this case, \( E_\gamma \to 0 \) and we can safely neglect the breaking of the scale invariance because other energy scales will cut off the RG flow. Indeed, in this case, the combined action of the two environmental effects leads only to a \emph{marginal} perturbation of the theory. Consequently, the conductance in the Hall bar will be quantized. Indeed, in the limit discussed before, with \( \tilde{\gamma} \to 0 \), it is easy to verify that the retarded (advanced) GF \( \mathcal{G}^R(t) \) (\( \mathcal{G}^A(t) \)), namely the anti-transform of the off-diagonal entry \( \mathcal{G}_{\text{cl},q} \) (\( \mathcal{G}_{q,\text{cl}} \)) of the matrix in (14), coincides with the results obtained from the retarded (advanced) GFs of the free theory [2, 3] given in (5). Therefore the linear response shows that a weakly

\(^8\) Note that in the chiral theory this step is less obvious than in the non-chiral one [51] due to the difference in analytical structure between the dissipative term \( \approx i\gamma \omega \) and the standard regularizing term \( \approx i\epsilon q \).
coupled noisy environment does not modify the conductance of the system with respect to the free $\chi$LL theory.

From the Keldysh GF $G^K = (\mathcal{G})_{cl,cl}$ (anti-transform in time of the top-left entry of (14)), we can define the bosonic correlation function $\tilde{G}^K(t) = G^K(t) - G^K(0)$ with

$$\tilde{G}^K(t) = iv \ln \left[1 + \omega^2 c t^2\right],$$

where $\omega_c = v/a$, with $a$ a finite length cut-off, and

$$g = \left(1 + \frac{\tilde{F}}{v^2 \gamma}\right).$$

Comparing (15) with the same quantity calculated from the free $\chi$LL theory described in (4), i.e. $\tilde{G}^0(t) = iv \ln \left[1 + \omega^2 c t^2\right]$, we see that the functional dependence remains exactly the same, but with an additional renormalization factor $g$. In the limit of vanishing noise, one obtains $g = 1$ recovering the standard result.

### 2.2. Scaling dimension renormalizations

The above result leads to extremely important physical consequences. As a remarkable example, we can consider a generic $m$-agglomerate quasiparticle (qp) annihilation operator in the bosonized form $\Psi_1^m(x) = e^{i m \varphi(x)} \sqrt{2/\pi a}$ (17) and the two-point greater/lesser GFs

$$C_m^>(g, t) = \langle \Psi_1^m(t) \Psi_1^m(0) \rangle = -C_m^<(-t).$$

These quantities determine the tunnelling densities of states of the edges and consequently the transport properties in a QPC geometry. They can be expressed in terms of the bosonic correlation function $\tilde{G}^>(t) = G^>(t) - G^>(0)$ with $G^>(t) = (G^K(t) + G^K(t) - G^K(t))/2$ and retarded, advanced and Keldysh GFs obtained from (14). At zero temperature, one then has [67, 68]

$$C_m^>(g, t) = e^{im^2 \tilde{G}^>(t)} = \left[1 + \omega^2 c t^2\right]^{\frac{\omega_c}{2 \gamma}} e^{-im^2 v \phi(t)},$$

where

$$\phi(t) = \tan^{-1} \left[ \frac{\omega_c t}{\sqrt{1 + \omega^2 c t^2}} \right] \longrightarrow \frac{\pi}{2} \text{sgn}(t).$$

From the comparison of the previous expressions with the results obtained for the free $\chi$LL, it is possible to see that the renormalization factor $g$ only influences the absolute value of the GF. We explicitly indicate the peculiar functional dependence on $g$ in the left-hand term of (19). The phase instead, as expected, is not affected, being related to the universal statistical properties of the excitations.

We define now the scaling dimension $\Delta(m)$ of the $m$-agglomerate operator $\Psi_1^m(x)$ as the long-time behaviour of the two-point GF $|C_m^>(t)|_{t \to \infty} \approx |t|^{-2\Delta(m)}$. This quantity is

$$\Delta(m) = g \Delta_0(m) = g^2 \frac{m^2}{2};$$

New Journal of Physics 14 (2012) 093032
note that the scaling of the raw theory $\Delta_0(m)$ is renormalized by the factor $g$. This result induces a modification of the power-law behaviour of the transport properties, with respect to the free (unrenormalized) case.

For simplicity we calculate only the single quasiparticle (single-qp) contribution to the back scattering current, the most dominant one in the Laughlin sequence, for the weak-backscattering regime. Note that, for the Laughlin model, the renormalization mechanism cannot affect the relevance of the excitations. We will see that for the models with composite edges this will not be in general the case.

We model the QPC in terms of a local tunnelling term at $x = 0$ between the right- (R) and left- (L) moving edges such as $H_I = t \Psi_R^{(1)} \Psi_L^{(1)} + \text{h.c.}$ [47, 69–71]. We also assume that the edges are affected by different environments and consequently they may have different renormalization parameters $g_{R/L}$ for the right-moving edge (R) and the left-moving one (L).

The average current at zero temperature, at the lowest order in the tunnelling, reads ($\hbar = 1$)

$$
\langle I_B \rangle = e^* \left( \frac{|t|^2}{2\pi a} \right)^2 \int_{-\infty}^{\infty} dt \ e^{iEt} C^>(g_R, t) C^< (g_L, -t),
$$

where $E = e^* V$ is the energy involved in the tunnelling, with $V$ being the bias and $e^* = \mu e$ the single-qp charge. From (19), one has

$$
C^>(g_R, t) C^< (g_L, -t) = \left( \frac{1}{1 - i\nu t} \right)^{\nu(\tilde{g} - 1)} \left( \frac{1}{1 + i\nu t} \right)^{\nu(\tilde{g} + 1)},
$$

with $\tilde{g} = (g_R + g_L)/2$. From this result, one can calculate the expression of the current at zero temperature in (22), obtaining

$$
\langle I_B \rangle = e^\theta (E) \left( \frac{|t|^2}{a^2 \omega_c} \right) \frac{(E/\omega_c)^{2\nu\tilde{g} - 1}}{\Gamma[\nu(\tilde{g} - 1)]\Gamma[\nu(\tilde{g} + 1)]} N,
$$

where $\Gamma[x]$ is the Gamma function and $N = 2F_1[1, 1 - \nu(\tilde{g} - 1), 1 - \nu(\tilde{g} + 1), 1 + \nu(\tilde{g} - 1), -1]$ is a constant with $2F_1[a, b, c, z]$ the hypergeometric function [25].

The power-law behaviour of the back scattering current at zero temperature is therefore $\langle I_B \rangle \propto V^{2\nu\tilde{g} - 1}$ with a renormalized exponent $\nu\tilde{g}$. In the following, we will always assume that the renormalization phenomenon affects identically the right and left edges. A generalization to the case of different couplings can be done straightforwardly.

Note that in (16) the strength of the renormalization can take any value $g \geq 1$. The same formula suggests that, for a fixed environmental contribution ($\tilde{F}/\tilde{\gamma}$ constant), the renormalization would be typically stronger for slow propagating modes, due to the explicit dependence on the inverse of the squared mode velocity in the expression. In principle this renormalization could also reach high values with important modifications of the power-law behaviour of transport properties [26–28].

As mentioned before, other mechanisms could explain the same renormalization of the exponents [36–43, 46–48]. However, some of these mechanisms (such as coupling with phonons) contain intrinsic limitations on the strength renormalization, differently from our model where the only real limitation is the requirement that $g \geq 1$.

How can one directly test the validity of the proposed scenario? A direct microscopical evaluation of the coupling constant might be almost impossible, as usually happens in the effective theories. Furthermore, in this case, $\tilde{F}/\tilde{\gamma}$ is the ratio of two very small quantities, making its evaluation even more difficult. One possibility, instead, could be to modulate the...
renormalization strength value of (16) using an external parameter and investigating how this affects the measured Luttinger exponents. From the formula for \( g \), we see which are the quantities we should modulate: the noise strength \( \tilde{F} \), the friction coefficient \( \tilde{\gamma} \) or the mode velocity \( v \). Even through we can imagine methods to modulate the edge velocity and the friction coefficient of the dissipative environment, we do not see how to make sure that those modulations do not also affect other relevant quantities in the system making the analysis less direct. The \( 1/f \) noise instead may be less problematic. It could be modulated by acting on the electron density \[72\] or alternatively by introducing a modulated additional \( 1/f \) noise signal in the source contact. In the second case, one has to look at the weak-backscattering transport properties at very low frequencies \( \omega \ll v/L \) where \( L \) is the total length of the edge between contacts.  

In the next sections, we will observe that the discussed model can be simply generalized to more complex FQH states, such as composite edge states, and more importantly, it also reveals as robust to the presence of disorder.

3. Composite edges: the Jain sequence

3.1. The model

Here, we focus on the effects of the out-of-equilibrium noise source in the case of multichannel edge states. The prototype of these Hall states is represented by the Jain sequence \([5]\) with filling factor \( \nu = p/(2np + 1) \), in which \( n \in \mathbb{N} \) and \( p \in \mathbb{Z} \). Following the hierarchical construction \([3]\), one has one charged bosonic mode, analogous to the one described for the Laughlin sequence, and \( |p| - 1 \) additional neutral modes which propagate either in the same direction \( (p > 0) \) or in the opposite one \( (p < 0) \). For simplicity we restrict the discussion to the case of two edge modes \( (|p| = 2) \), underlying the differences between co-propagating modes \( (p > 0, \nu = 2/5) \) and counter-propagating ones \( (p < 0, \nu = 2/3) \) \([3]\). The edge states in the former case are described in terms of two co-propagating bosonic charged fields with different filling factors \( \nu_1 = 1/3 \) and \( \nu_2 = 1/15 \), such as \( \nu = \nu_1 + \nu_2 = 2/5 \), while in the latter case the bosonic fields, with \( \nu_1 = 1 \) and \( \nu_2 = 1/3 \), respectively, propagate in opposite directions leading to \( \nu = \nu_1 - \nu_2 = 2/3 \).

The Lagrangian densities are

\[ L_\zeta = \sum_{j=1,2} -\frac{1}{4\pi v_j} \partial_x \varphi_j \left( (\zeta)^{j+1} \partial_t \varphi_j + v_j \partial_x \varphi_j \right), \tag{25} \]

where \( \zeta = \pm \) indicates the co-propagating \( (\zeta = +) \) or counter-propagating \( (\zeta = -) \) case, and \( v_1, v_2 \) are the velocities of the modes and effectively contain the information on intra-edge interactions. The field commutation relations are \( [\varphi_j(x), \varphi_k(x')] = i \delta_{jk} \eta_k v_k \text{sgn}(x - x') \) where \( \eta_k = (\zeta)^{k+1} \) is related to the direction of propagation of the fields \( (j, k = 1, 2) \).

The two modes are close to each other and interact via the density–density coupling (inter-edge interaction)

\[ L_{12} = \frac{v_{12}}{2\pi v_1 v_2} \partial_x \varphi_1 \partial_x \varphi_2, \tag{26} \]

with strength \( v_{12} \), where \( v_{12} = \sqrt{v_1 v_2} \). Note that \( v_1, v_2 \) and \( v_{12} \) are non-universal parameters related to the intra- and inter-channel interaction strengths.

Anyway we always have to satisfy the condition \( E_\gamma \ll \omega \) to not see the breaking of the scale invariance induced by noisy terms.
Observations showed that, in the fractional regime, the extension of edge channels can also be very wide [73–76]. For example, the total edge width of \( v = 2/3 \) (two channels), in some experiments, appears wider than the \( v = 1/3 \) (single channel), suggesting a more structured nature of the former [75]. We make the reasonable assumption that, for a sufficiently smooth confining potential, the edge is composed of two channels localized in slightly different positions. Therefore one reasonably assumes that they effectively ‘feel’ different noisy environments. Indeed, in general, the trapped charges in the substrate beneath the Hall bar affect the \( \varphi_1 \) and \( \varphi_2 \) modes in a different way. One can introduce two distinct 1/f noise fields \( f_{1/2}(x, t) \) operating on the edge, with noise strength \( F_{1/2} \) and spectrum \( K_{1/2}(\omega, q) = F_{1/2}/|\omega| \), respectively. We will also consider two different ohmic dissipations with friction coefficients \( \gamma_1/2 \). Also in this case the total Keldysh action can be written in a form analogous to (3), but in terms of the four-component vector \( \Phi = (\varphi_1^c, \varphi_2^c, \varphi_1^q, \varphi_2^q)^T \), due to the presence of the two fields \( \varphi_1/2 \). The Keldysh kernel now reads

\[
\mathcal{G}_{\zeta}^{-1} = \frac{1}{2\pi} \begin{pmatrix}
0 & 0 & \frac{q(\omega - v_1 q) - i\tilde{\gamma}_1 \omega}{v_1} & \frac{v_1 \gamma_2^2}{v_2} \\
0 & 0 & \frac{v_1 \gamma_2^2}{v_2} & \frac{q(\omega - v_2 q) - i\tilde{\gamma}_2 \omega}{v_2} \\
\frac{v_1 \gamma_2^2}{v_2} & \frac{v_1 \gamma_2^2}{v_2} & \frac{2i|\omega| v_1}{v_2} & \frac{2i|\omega| v_2}{v_2} \\
\frac{q(\omega - v_1 q) + i\tilde{\gamma}_1 \omega}{v_1} & \frac{q(\omega - v_2 q) + i\tilde{\gamma}_2 \omega}{v_2} & \frac{2i|\omega| v_1}{v_2} & \frac{2i|\omega| v_2}{v_2}
\end{pmatrix},
\]  

(27)

where \( \tilde{\gamma}_1 = 2\pi v_1 \gamma_1 \) and \( \tilde{F}_i = 2\pi v_i F_i \) with \( i = 1, 2 \). The \( \tilde{\gamma}_1 \) and \( \tilde{F}_i \) can be interpreted, respectively, as effective strengths of the friction coefficient and the noise intensity for the two channels. They give a real measure of the environment strength taking into account the difference in \( v_i \) between the two channels. In conclusion, even when the two modes ‘see’ exactly the same noisy environment \( \gamma_1 = \gamma_2 \) and \( F_1 = F_2 \), for example the channels are located in the exactly same position or the long-range nature of the interaction made the position of the two channels practically undistinguishable, the two channels still feel different effective strengths because typically \( v_1 \neq v_2 \).

3.2. Interaction effects

We will now discuss how the presence of a noisy environment could affect the renormalization of the \( \chi \) edge exponents in composite edge systems. We will start this analysis by considering the presence of the interaction between the channels described in (26). Let us start discussing the case of co-propagating channels such as \( v = 2/5 \). For such cases, indeed, the clean system, i.e. with no static disorder along the edge, properly describes the physics of the edge states. In the next subsection we will investigate the renormalization effects for \( v = 2/3 \) (counter-propagating modes) assuming instead the presence of the static disorder, which is crucial in the equilibration process between counter-propagating modes.

The Keldysh action kernel for \( v = 2/5 \) is given by \( \mathcal{G}_{\zeta}^{-1} \) in (27) with \( v_1 = 1/3 \) and \( v_2 = 1/15 \). To better analyse the problem it is useful to make a rescaling \( \Sigma \) and a rotation \( \mathcal{R}(\theta) \) of the fields \( \varphi_{1/2} \). In the new basis \( \varphi'_{1/2} \) we have

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} = \begin{pmatrix}
\sqrt{v_1} & 0 \\
0 & \sqrt{v_2}
\end{pmatrix} \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{pmatrix} \begin{pmatrix}
\varphi'_1 \\
\varphi'_2
\end{pmatrix},
\]  

(28)
where the angle $\theta$ satisfies $\tan(2\theta) = 2v_{12}/(v_1 - v_2)$. The new field eigenmodes $\varphi'_{1/2}$ are decoupled with respect to the density–density interaction and have different velocities

$$v'_{1,2} = \left(\frac{v_1 + v_2 \pm \sqrt{(v_1 - v_2)^2 + 4v_{12}^2}}{2}\right).$$  

(29)

From the above relations we obtain a criterion, called stability, which requires these velocities to be always positive [7]. This is reflected in a constraint $v_{12}^2 \leq v_1 v_2$ between the intra- and inter-mode couplings.

At the same time the dissipative and $1/f$ terms acquire off-diagonal contributions due to the transformation in (28). In particular, we can see what happens to those terms by focusing on the $2 \times 2$ bottom-left block matrix, which coincides with the q–cl component of the kernel in (27). In the new $\varphi'$ basis it becomes

$$(G'^{-1})_{q,\text{cl}} = \frac{q}{2\pi} \begin{bmatrix} \omega - v'_1 q & 0 \\ 0 & \omega - v'_2 q \end{bmatrix} + (\tilde{G}'_{\text{diss}}^{-1})_{q,\text{cl}},$$

(30)

where the dissipative contribution is no longer diagonal and reads

$$(G'^{-1})_{\text{diss}}_{q,\text{cl}} = R^T(\theta) \cdot \begin{bmatrix} i\tilde{\gamma}_1 \omega & 0 \\ 0 & i\tilde{\gamma}_2 \omega \end{bmatrix} \cdot R(\theta)$$

(31)

with the rotation $R(\theta)$ defined in (28).

Note that the advanced component ($2 \times 2$ top-right block matrix) $(G'^{-1})_{\text{cl},q}$ can be easily derived from the previous result by complex conjugation, $(G'^{-1}(\omega, q))_{\text{cl},q} = ((G'^{-1}(\omega, q))_{q,\text{cl}})^\ast$. The Keldysh component $(G'^{-1})_{q,q}$ of (27) ($2 \times 2$ bottom-right block matrix) transforms in the new basis according to (31)

$$\frac{(G'^{-1})_{q,q}}{2i|\omega|} = R^T(\theta) \cdot \begin{bmatrix} \tilde{\gamma}_1 + \tilde{F}_1 \frac{q^2}{|\omega|^2} & 0 \\ 0 & \tilde{\gamma}_2 + \tilde{F}_2 \frac{q^2}{|\omega|^2} \end{bmatrix} \cdot R(\theta),$$

(32)

where the linear dependence on the coefficients $\tilde{F}_1, \tilde{F}_2$ is now explicit.

In the limit of weak contribution of noise and dissipation (see section 2), i.e. $\tilde{\gamma}_i, \tilde{F}_i \to 0$ but keeping the ratios $\tilde{F}_i/\tilde{\gamma}_i$ constant, it is possible to calculate all the Keldysh GFs following the same approach used for the Laughlin case.

To simplify the discussion, we consider only the case when the effective friction coefficients of the dissipative contributions are the same, $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}$, allowing only different strengths for the $1/f$ noise. This assumption greatly simplifies our discussion without affecting the key results.

For $\tilde{\gamma} \to 0$ one recovers again the standard form of the retarded/advanced GFs for composite edges. A direct consequence of this fact is that the argument discussed in [7], where the conductance of a multichannel edge of interacting co-propagating modes is calculated using the retarded GFs, is still valid here. Therefore, also for the multichannel edge, the Hall bar current is correctly quantized $I = v_{g_0} V_{H}$, independently of the noise and the value of the inter-edge interaction $v_{12}$.

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10 A more general analysis is also possible, but does not add any further insight to the issue under discussion.

11 Note that this is no more true for counter-propagating modes (see later).
The presence of an external noisy environment modifies the scaling dimension of the excitations. Now we analyse this point, taking care of the presence of inter-edge coupling \( v_{12} \). In particular, we will show that the scaling properties are no longer universal and depend, in general, on inter-edge coupling. This result is different from the standard theory [7] that predicts only universal scaling properties in the co-propagating case. It is important to stress that this is a direct consequence of the presence of a noisy external environment. To better demonstrate this fact, we will consider now all the possible excitations and their scaling dimensions.

In the bosonized form a generic excitation is written in terms of a linear combination of the bosonic field \( \phi_{1/2} \) as

\[
\Psi^{(a_1, a_2)}(x) \propto e^{i[a_1 \phi_1(x) + a_2 \phi_2(x)]}
\]  

with \( a_{1/2} \) the coefficients that determine the considered excitation [31, 32, 34]. Here, we just recall that the charges of all the excitations are integer multiples of the fundamental single qp, e.g. for \( \nu = 2/5 \), the fundamental charge is \( e^* = e/5 \) (e the electron charge). The statistical properties of the excitations are directly connected with the values of \( a_1 \) and \( a_2 \) and the commutation relations of the \( \phi_{1/2} \) fields [32, 34], while the scaling dimensions depend additionally, as we will see, on the presence of a noisy environment. The two-point correlation function of the operator is [67]

\[
C_{a_1, a_2}(t) = \langle \Psi^{(a_1, a_2)}(t) \Psi^{(a_1, a_2)\dagger}(0) \rangle = e^{\sum_{j=1,2} a_j \tilde{G}_{\nu}^{\gamma}(t) a_j},
\]  

where, in the second equality, we introduced the greater GFs \( \tilde{G}_{\nu}^{\gamma}(t) = G_{\nu}^{\gamma}(t) - G_{\nu}^{\gamma}(0) \) such that \( \tilde{G}_{\nu}^{\gamma}(t) = -i \langle \phi_j(t) \phi_k(0) \rangle \) for the \( \phi_j \) fields with \( j, k = 1, 2 \).

In the new basis \( \phi'_{1/2} \), taking the limit of weak coupling with the environment \( \tilde{\gamma}, \tilde{F}_1, \tilde{F}_2 \to 0 \), the Keldysh GFs read \( (j, k = 1, 2) \)

\[
\tilde{G}_{jk}'(t) = i \delta_{jk} g'_{\gamma} \ln[1 + \omega_{\gamma,j}^2 t^2],
\]  

where the cut-offs are \( \omega_{\gamma,j} = v_j'/a \) with \( j = 1, 2 \). The ‘mixed’ terms \( \tilde{G}_{jk}'(t) \) with \( j \neq k \) vanish due to the assumption \( \tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma} \) on the dissipative friction coefficients. The renormalization coefficients for the two normal modes are now

\[
g'_j = \left( 1 + \frac{\tilde{F}_+ - (-)^j \tilde{F}_- \cos(2\theta)}{2v_j^2 \tilde{\gamma}} \right),
\]  

with \( \tilde{F}_\pm = \tilde{F}_1 \pm \tilde{F}_2 \) and the mode velocities \( v_j^2 \) given in (29). Note that the renormalization parameters depend on the coupling strength \( v_{12} \) and the mode velocities, through the angle \( \theta \). This appears to be quite a natural generalization of the result given in (16). It is important to note that these renormalizations are present, even assuming a completely identical effective environment in the channels \( \tilde{F}_1 = \tilde{F}_2 \).

From this result one can calculate \( \tilde{G}_{jk}'(t) \) with the same procedure used for the Laughlin sequence, in the form of (19), after the proper replacement of the renormalization parameter \( v_g \to g'_1, g'_2 \) and of the cut-offs \( \omega_k \to v_{1/2}'/a, v_{1/2}'/a \). The greater GF in (34) can be expressed as

\[
\tilde{G}_{\nu}^{\gamma}(t) = \Sigma \cdot \mathcal{R}(\theta) \cdot \tilde{G}_{\nu}^{\gamma}(t) \cdot \mathcal{R}^\dagger(\theta) \cdot \Sigma,
\]  

where we used the compact matrix notation with \( \tilde{G}_{\nu}^{\gamma}(t) = (\tilde{G}_{\nu}^{\gamma}(t))_{jk} \) with the rescaling matrix \( \Sigma \) and the rotation matrix \( \mathcal{R}(\theta) \) defined in (28).
From the long-time behaviour of the two-point correlation function of (34) one can calculate the scaling dimension $\Delta^{(\alpha_1, \alpha_2)}$ of the $\Psi^{(\alpha_1, \alpha_2)}(x)$ operator 
\[
\Delta^{(\alpha_1, \alpha_2)} = \frac{1}{2} \left[ v_1 \alpha_1^2 \left( g_1' \cos^2(\theta) + g_2' \sin^2(\theta) \right) + v_2 \alpha_2^2 \left( g_2' \cos^2(\theta) + g_1' \sin^2(\theta) \right) \right] + \sqrt{v_1 v_2} \alpha_1 \alpha_2 \sin(2\theta) \left[ g_1' - g_2' \right],
\] 
where we see an explicit dependence on the coupling $v_{12}$ and the mode velocities $v_{1,2}$, via the angle $\theta$ and the renormalization factors $g_i'$. Note that, in the absence of $1/f$ noise ($g_1' = g_2' = 1$), the scaling dimensions reduce to the standard $\Delta_0^{(\alpha_1, \alpha_2)} = (v_1 \alpha_1^2 + v_2 \alpha_2^2)/2$ obtained for hierarchical theories. Furthermore, if the renormalizations of the normal modes are exactly the same $g_1' = g_2' = g$ the scaling dimension becomes independent of the angle $\theta$ (i.e. the coupling $v_{12}$) with $\Delta^{(\alpha_1, \alpha_2)} = g \Delta_0^{(\alpha_1, \alpha_2)}$.

In conclusion, we showed that the scaling of a generic excitation in the presence of a noisy environment is influenced by the strength of the coupling $v_{12}$ even for co-propagating modes. This strongly differs from the standard result [7] where the scaling is independent of the coupling. This fact shows that scaling dimensions in the presence of weakly coupled $1/f$ noise are, in general, no more universal, i.e. no more determined only by the coefficients $(\alpha_1, \alpha_2)$, the filling factor $\nu$ and the model of composite edges we consider. As a remarkable consequence, in the presence of environmental effects, the relevance between the excitations could differ from the raw theory. We have already discussed the potential effects of assuming not universal values for the $\chi_{LL}$ exponent for composite edge theories [31, 32]. This rich phenomenology, triggered by the assumption of renormalizations, may be explained eventually in terms of the model presented.

We will now see that the most important advantage of the presented model is its robustness with respect to static impurity disorder along the edge. Indeed, all the results up to now are essentially based on the assumption of a clean edge, without any contribution coming from static disorder. We will also see that including such a contribution the role of inter-mode interactions will be less important but the scaling dimensions will be still affected by renormalization effects due to the noisy environment.

### 3.3. Disorder effects

A more realistic discussion of edge states in real samples would require the inclusion of static disorder along the edge\footnote{The $1/f$ contribution considered before can be realistically interpreted as a small time-dependent correction to the disorder profile considered here.}. The disorder plays a fundamental role in recovering the proper quantization of the Hall conductance for counter-propagating modes. See the seminal paper by Kane and Fisher [7] for a detailed discussion of the disorder-dominated phase in the hierarchical theories.

In the following, we will analyse the case of $\nu = 2/3$ where the two channels are counter-propagating. The discussions of the disorder effects in the composite edge, presented here, can be generalized also to the whole Jain sequence. In the next section we adapt the argument even to the $\nu = 5/2$ state.

The Keldysh action for the multichannel edge state at $\nu = 2/3$ under the influence of $1/f$ noise and dissipation is given by the kernel $G_{\nu}^{-1}$ of (27) with $v_1 = 1$ and $v_2 = 1/3$. The effect of static disorder on the $\chi_{LL}$ channels can be naturally described by adding two more terms to the action.
The first one describes the coupling of two static disorder potential profiles $V_i(x)$ with the charge densities $\rho_i(x) = \partial_x \varphi_i(x)/(2\pi)$ of the two channels $\varphi_i$ composing the edge $v = 2/3$ with $i = 1, 2$. The Lagrangian of this term, which affects locally the two channels, is $\mathcal{L}_{V,\varphi} \propto \sum_i V_i(x) \partial_x \varphi_i(x)$. These forward scattering terms can be easily eliminated from the action by a simple redefinition of the $\varphi_i(x)$ fields and will be neglected in the following [7].

The second term describes the effect of the disorder in terms of impurity scattering, i.e. how the disorder potential mediates the electron transfer between the two counter-propagating modes. These tunnelling terms have the main effect to equilibrate the two channels when they are at different potentials restoring the proper value of the quantized conductance [7]. This random tunnelling term is

$$\mathcal{L}_{\text{rdm}} = \xi(x)e^{i(\varphi_2(x)+3\varphi_1(x))} + \text{h.c.},$$

with $\xi(x)$ being a complex random tunnelling amplitude. This process leads to the destruction of an electron into the $v = 1$ channel ($e^{i\varphi_1(x)}$) and its creation ($e^{i3\varphi_2(x)}$) into the $v = 1/3$ one and vice versa [7, 50]. For simplicity it is assumed that $\xi(x)$ is a Gaussian random variable $\delta$-correlated in space satisfying

$$\langle \xi^*(x)\xi(y) \rangle_{\text{ens}} = W \delta(x - y),$$

where $\langle \cdots \rangle_{\text{ens}}$ indicates the ensemble average over the realizations of disorder.

To further analyse the disorder terms, it is convenient to express the system in the basis $\varphi'_{1/2}(x)$ that diagonalizes the problem with respect to inter-edge coupling. We follow essentially the same procedure used for the $v = 2/5$ case. The transformation now is given by the composition of a rescaling $\Sigma$ and a Lorentz boost $B(\chi)$ with rapidity $\chi$, instead of the standard rotation used for co-propagating modes. The relation between the old fields $\varphi_{1,2}$ with the new ones $\varphi'_{1,2}$ is

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{bmatrix} \sqrt{v_1} & 0 \\ 0 & \sqrt{v_2} \end{bmatrix} \cdot \begin{bmatrix} \cosh(\chi) & \sinh(\chi) \\ \sinh(\chi) & \cosh(\chi) \end{bmatrix} \cdot \begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix},$$

where $\tanh(2\chi) = -2v_{12}/(v_1 + v_2)$.

Note that we can calculate the scaling dimension for a generic qp operator $\Psi^{(\alpha_1,\alpha_2)}(x)$ defined in (33) following the same steps as in the previous section. It is also useful, without losing generality, to assume that $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}$ but keeping as free parameters the strengths of the $1/f$ noise $\tilde{F}_{1/2}$. We find the scaling dimension

$$\Delta^{(\alpha_1,\alpha_2)} = \frac{1}{2} \left\{ v_1 \alpha_1^2 \left[ g'_1 \cosh^2(\chi) + g'_2 \sinh^2(\chi) \right] + v_2 \alpha_2^2 \left[ g'_1 \cosh^2(\chi) + g'_2 \sinh^2(\chi) \right] + \sqrt{v_1v_2} \alpha_1 \alpha_2 \sinh(2\chi) \left[ g'_1 + g'_2 \right] \right\},$$

where the contribution of the Lorentz boost is explicit in the terms $\cosh^2(\chi)$ and $\sinh^2(\chi)$. The renormalization factors of the new modes $g'_j$ are now ($j = 1, 2$)

$$g'_j = \left( 1 + \frac{\tilde{F}_+ - (-)^j \tilde{F}_- \operatorname{sech}(2\chi)}{2v_j^2 \tilde{\gamma}} \right),$$

where the $v'_j$ are eigenmode velocities and $\tilde{F}_\pm$ are defined after (36). Note that a direct comparison with the co-propagating expression shows that, in the counter-propagating case,
the sech(2χ) takes the role of the cos(2θ) of (36), but the form of the renormalization factors remains essentially the same.

We now address the role of the disorder terms, starting with the inspection of the scaling \( \Delta[\mathcal{O}] = \Delta^{(1,3)} \) of the tunnelling disorder operator \( \mathcal{O} \propto e^{i[\psi_1(x) + 3\psi_2(x)]} \) introduced in (39). One can demonstrate that the first-order RG flow equation for the inter-edge disorder strength \( W \) in (40) is [7, 50, 78]

\[
\frac{dW}{dl} = (3 - 2\Delta[\mathcal{O}])W. \tag{44}
\]

In principle, the non-equilibrium RG flow analysis can be extended also to higher orders but, in that case, a much richer physics is expected [52]. Here we limit the discussion only to first order to keep the discussion as simple as possible.

In particular, looking at equation (44), one can see that with \( \Delta[\mathcal{O}] < 3/2 \) the disorder is a relevant contribution and the system is driven in the so-called disorder-dominated phase. For such a phase the Hall bar conductance \( g \) is universal (i.e. independent of the environment and the intra- and inter-mode couplings) and is properly quantized at the value \( g = v F_0 \) [50]. All the discussions of quantum Hall states with counter-propagating modes are typically done assuming that the system is exactly in this phase.

In contrast, if the scaling of the electron tunnelling is \( \Delta[\mathcal{O}] > 3/2 \), the disorder is irrelevant and, at the fixed point, the conductance is no more universal depending on the intra- and inter-edge interactions [50].

In general the environmental effects, through the parameters \( F_i / \gamma_i \) and the intra- and inter-mode couplings, could affect the scaling \( \Delta[\mathcal{O}] \), making the discussion quite cumbersome. As a simple check, we need to first recover the standard result obtained in the absence of \( 1/f \) noise, namely the ratios \( F_i / \gamma_i, F_2 / \gamma_2 \to 0 \). From (42) and the definition of the operator \( \mathcal{O} \), one obtains that

\[
\Delta[\mathcal{O}]^0 = \lim_{F_i \to 0} \Delta^{(1,3)} = 2 \frac{v_1 + v_2 - \sqrt{3}v_{12}}{\sqrt{(v_1 + v_2)^2 - 4v_{12}^2}}, \tag{45}
\]

which coincides with the result of Kane et al [50]. It is convenient now to measure the velocities in units of \( \nu_1 \). For the values of couplings where the stability criterion \( v_{12}^2 \leq v_1v_2 \) is satisfied, we can determine when \( \Delta[\mathcal{O}]^0 < 3/2 \). In figure 1, we report, in the plane \( (v_2 / \nu_1, v_{12} / \nu_1) \), the regions where this condition is fulfilled. The area is delimited by the stability (solid black) curve and two (dashed black) lines representing respectively the maximum/minimum value of \( v_{12} / \nu_1 \) compatible with the disorder-dominated phase. These two lines are given by \( v_{12} / \nu_1 = (4\sqrt{3}/21 \pm \sqrt{5}/14)(1 + v_2 / \nu_1) \).

Now we can evaluate how this area changes under the presence of a noisy environment. One could expect, in analogy with the renormalizations induced by coupling with phonons [39], that the effects of an external environment lead always to an enhancement of the scaling dimension \( \Delta[\mathcal{O}] \geq \Delta[\mathcal{O}]^0 \). A direct consequence of this fact would be the progressive reduction of the region of existence of the disorder-dominated phase. This is explicit from the figure where we calculated the regions where \( \Delta[\mathcal{O}] < 3/2 \), using (42) varying \( \tilde{F}_i / (\nu_1^2 \tilde{\gamma}) = 0, 0.1, 0.2, 0.3 \) for a fixed ratio \( \tilde{F}_2 / (\nu_1^2 \tilde{\gamma}) = 0 \). Note that for very strong noise the disordered-dominated phase could be completely washed out.

We conclude this discussion by observing that, for moderate noise strength, the disorder-dominated phase is still present even if the conditions on the inter- and intra-mode coupling
Figure 1. Coloured areas represent the disorder-dominated phase for $\nu = 2/3$ in the parameters space $(v_2/v_1, v_{12}/v_1)$ for different strengths of the noise. In light blue is shown the case without the noisy environment [50], i.e. $\Delta^0_0 < 3/2$ (see in the text). This area is limited by the stability criterion (solid line) and the two dashed lines representing respectively the maximum/minimum value of the ratio $v_{12}/v_1$ to get the disordered phase $\Delta^0_0 < 3/2$. Other coloured areas represent $\Delta^0_0 < 3/2$ with the successive reduction of the disordered phase due to the increasing of noise strength $\tilde{F}_1/(v_1^2 \tilde{y}) = 0$ (lighter blue), 0.1, 0.5, 1 (darker blue) with $\tilde{F}_2 = 0$.

are modified. Our analysis generalizes some of the results of [50] in the presence of a noisy environment.

The robustness of the proposed model for the renormalization of the exponents in the presence of disorder along the edge is one of the most important results of this paper.

We will now discuss quantitatively how renormalizations are affected by noise intensity. In the disorder-dominated phase the system naturally decouples in charged and neutral contributions [7, 50]. Therefore, it is convenient to change the basis from the original $\varphi_{1/2}$ to the charged $\varphi_\rho$ and neutral fields $\varphi_\sigma$

$$
\begin{pmatrix}
\varphi_\rho \\
\varphi_\sigma
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{3}(\varphi_1 + \varphi_2) \\
3\varphi_2 + \varphi_1
\end{pmatrix}
$$

(46)

as obtained from the transformation in (41) with $\tanh(\chi^*) = -\sqrt{1/3}$. The action is expressed in the form of (27), but with the propagation velocities $v_i$ with index $i = \rho, \sigma, \rho\sigma$ given by

$$
\begin{bmatrix}
v_\rho \\
v_\rho\sigma \\
v_\sigma
\end{bmatrix} = B(\chi^*)^T \begin{bmatrix}
v_1 \\
v_{12} \\
v_2
\end{bmatrix} B(\chi^*)
$$

(47)
with \( B(\chi^*) \) being the Lorentz boost in (41).

The same transformation defines the noise strengths \( \tilde{F}_i \), in the new basis as

\[
\begin{bmatrix}
\tilde{F}_\rho & \tilde{F}_{\rho\sigma} \\
\tilde{F}_{\rho\sigma} & \tilde{F}_\sigma
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
3\tilde{F}_1 + \tilde{F}_2 & \sqrt{3}(\tilde{F}_1 + \tilde{F}_2) \\
\sqrt{3}(\tilde{F}_1 + \tilde{F}_2) & \tilde{F}_1 + 3\tilde{F}_2
\end{bmatrix}
\]

(48)

in terms of the coefficients \( \tilde{F}_{1/2} \) of (27). An equivalent transformation can be written for the friction coefficients of the dissipative bath, with the introduction of the quantities \( \tilde{\gamma}_\rho, \tilde{\gamma}_\sigma, \tilde{\gamma}_{\rho\sigma} \), as linear combinations of \( \tilde{\gamma}_{1/2} \).

The off-diagonal terms of the action containing \( v_{\rho\sigma} \) are irrelevant in the RG sense and can be neglected at the fixed point. This was clearly shown in [7]. In the limit of weak coupling such that \( \tilde{\gamma}_1, \tilde{F}_i \to 0 \), but keeping constant the ratios \( \tilde{F}_i/\tilde{\gamma}_i \), with \( i = 1, 2 \), the environmental contributions are marginal in the RG sense [52]. This shows that, at the fixed point of the disorder-dominated phase, we could safely neglect the residual coupling between charged and neutral modes but we have to include the noisy environmental contributions. In the following, we will take \( v_{\rho\sigma} = 0 \) taking explicitly the dissipative and \( 1/f \) noise terms into account.

We observe that, in the case of the coupling with 1D phonon \( \varphi_{ph} \) modes [39], one has also a term analogous to the one proportional to \( v_{\rho\sigma} \) discussed above. The canonical mass dimension of the phonons in 1 + 1 dimensions is the same as that of a chiral bosonic field \( \text{dim}[\varphi_{ph}] = \text{dim}[\varphi_{1,2}] \). As a natural consequence, this coupling term becomes RG irrelevant in the disordered dominated fixed point as already discussed. This shows that, even if the coupling with phonons could in principle generate renormalizations of the scaling exponent, in the disorder-dominated phase the phonons are effectively decoupled from the system and their renormalization effects do not survive against disorder. This indicates that our model is qualitatively different and presents concrete advantages in comparison with other mechanisms, especially for all those cases when counter-propagating modes are present and, consequently, the disorder-dominated phase has to be considered.

We can now evaluate the GFs along the same lines followed in the previous section for \( v = 2/5 \). Also in this case we assume that \( \tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma} \) and consider the strengths of the \( 1/f \) noises \( \tilde{F}_{1/2} \) as free parameters.

In the limit of \( \tilde{\gamma} \to 0 \), the retarded/advanced GFs are exactly the same as the ones in [7] and, consequently, the edge conductance returns the appropriate quantized value of \( g = v_0q_0 \). For the Keldysh GF contributions of the charged and neutral fields we obtain, in the disordered phase, a result identical to (35), where the only non-zero GFs are the \( G_{\rho\rho}^0 \) and \( G_{\sigma\sigma}^0 \). These are characterized by the cut-off energies \( \omega_i = v_i/a \) with \( i = \rho, \sigma \) and by renormalization parameters

\[
g_{\rho} = \left( 1 + \frac{\tilde{F}_\rho}{2v_\rho^2\tilde{\gamma}} \right), \quad g_{\sigma} = \left( 1 + \frac{\tilde{F}_\sigma}{2v_\sigma^2\tilde{\gamma}} \right).
\]

(49)

They coincide with (43) choosing \( \chi = \chi^* \) and using the definition of (48). Their values depend on the noise strength \( \tilde{F}_i \), the dissipation \( \tilde{\gamma} \) and the neutral and charged mode velocities \( v_\rho \) and \( v_\sigma \). Figure 2 shows how the charged renormalization parameter (black curves) and the neutral one (grey curves) depend on the ratio \( v_\sigma/v_\rho \) for different values of the \( 1/f \) noise strengths. At a fixed noise strength and with increasing the ratio \( v_\sigma/v_\rho \), the charged mode renormalization rises while the neutral one decreases. This behaviour is directly connected with the dependence on the inverse of the squared mode velocities of (49). When the mode velocities become small, the renormalization parameters increase rapidly.

New Journal of Physics 14 (2012) 093032 (http://www.njp.org/)
figure 2. renormalization parameters \( g_\rho \) (black curves) and \( g_\sigma \) (grey curves) as a function of the ratio \( v_\sigma / v_\rho \). different line styles correspond to different noise strengths \( \tilde{f}_{1}/(v_1^2 \gamma) \) = 1 (dashed), 2 (dot-dashed) and 3 (dotted) having kept fixed the value of \( \tilde{f}_{2}/(v_2^2 \gamma) = 0.1 \).

Interestingly, it is also possible to obtain the counterintuitive condition \( g_\sigma > g_\rho \). this is an interesting result because, for all the experimental fitting we have done up to now \([31, 33, 35]\), we have seen values of the renormalizations that are \( g_\sigma \gtrsim g_\rho \). following physical intuition, it appears natural to assume the opposite (i.e. \( g_\sigma \lesssim g_\rho \)) because neutral bosonic modes seem less easily coupled with the environment with respect to the charged ones. nevertheless, this intuition fails because the neutral bosonic modes, in the composite edges, derive from a particle–hole combination between the two modes. they are affected by the differences in the effective strength of the noisy environments (differential mode) where the charge modes, instead, are influenced by the common mode only. in any case the dependence of the renormalizations (49) on the neutral and charged mode velocities shows that, even in the case of a completely symmetric effective noise environment for the two channels, we could still have different renormalizations. in this last case indeed, the assumption \( g_\sigma \gtrsim g_\rho \) follows directly from the physical natural assumption \( v_\sigma \lesssim v_\rho \) and from equations (49).

in conclusion, the dependence of the renormalization parameters on the noise strengths \( \tilde{f}_{1/2} \) (see (48)) guarantees the possibility to obtain even very high renormalization value for almost any values of the velocity ratio \( v_\rho / v_\sigma \). note that these high values seem sometimes necessary to fully explain the experimental observations \([33]\).

we conclude this section by showing that, rescaling the two fields \( \varphi_c = \sqrt{2/3}\varphi_\rho \) and \( \varphi_n = \sqrt{2}\varphi_\sigma \), it is possible to write all qp operators as \([33, 50]\)

\[
\Psi^{(m,l)} \propto e^{i(m/2)\varphi_c + i(l/2)\varphi_n},
\]

(50)

with the coefficients \( m, l \in \mathbb{Z} \) and with the same parity. these operators destroy an \( m \)-agglomerate, namely an excitation with charge \( me^* \) in which \( e^* = e/3 \) is the minimal charge allowed by the model. their scaling dimensions become

\[
\Delta(m, l) = \frac{1}{2} \left[ \left( \frac{2}{3} \right) g_\rho \left( \frac{m}{2} \right)^2 + 2 g_\sigma \left( \frac{l}{2} \right)^2 \right],
\]

(51)
where the $g_\rho$ and $g_\sigma$ renormalize the charge and neutral sectors of the excitation separately. Obviously, we recover the scaling dimension reported in the literature [2, 33] in the absence of noise for $g_\rho = g_\sigma = 1$. The last formula shows that in the disorder-dominated phase, the presence of a noisy environment naturally leads to different renormalizations for the neutral and charged modes. A consequence of this fact is a possibility to change the relevance of the excitations and, indeed when $g_\rho, g_\sigma \neq 1$, this could happen due to the environmental effects we are discussing. This phenomenology could have a deep impact on transport properties of the QPC, especially in the weak-backscattering regime where the dominant excitations are different from the electrons. The possibilities opened by our model for composite edges are fully compatible with the extremely rich phenomenology observed in QPC transport at low temperatures for these systems. In [32, 33], we have discussed in detail the experiments on noise and transport in QPC for $\nu = 2/3$. To fully match the theory with the data, the presence of the renormalization parameters $g_\rho, g_\sigma \geq 1$ in the disorder-dominated phase was indeed assumed.

In conclusion, here we have shown that a noisy environment can be considered a valid renormalization mechanism, robust to unavoidable disorder effects.

4. Composite edges: the $\nu = 5/2$ case

4.1. The anti-Pfaffian model

Another relevant example of composite edge states is represented by $\nu = 5/2$. Possible descriptions have been proposed for this state predicting both Abelian [10] and non-Abelian [13–16, 77] statistical properties for the elementary excitations. Particularly interesting is the so-called anti-Pfaffian model [15, 16], supporting non-Abelian statistics, that seems to be as indicated by experimental evidence a proper description for this state [18, 19]. According to this model, the edge states are described as a narrow region at $\nu = 3$ with nearby a Pfaffian edge of holes with $\nu = 1/2$ [16]. Assuming the second Landau level to be the ‘vacuum’, the edge is modelled in terms of a single $\nu = 1$ bosonic branch $\varphi_1$ and a counter-propagating $\nu = 1/2$ Pfaffian branch [14], composed of a bosonic mode $\varphi_2$ and a Majorana fermion $\psi$.

The Lagrangian for the free system is $\mathcal{L}_0 = \mathcal{L}_- + \mathcal{L}_{12} + \mathcal{L}_\psi$ where the bosonic contributions $\mathcal{L}_-$ and $\mathcal{L}_{12}$ are given in (25) and (26), respectively, with $\nu_1 = 1$ and $\nu_2 = 1/2$. The Lagrangian describing the free evolution of the Majorana fermion $\psi$ in the Ising sector is

$$\mathcal{L}_\psi = i\psi \left(-\partial_t + v_\psi \partial_x\right) \psi,$$

(52)

with propagation velocity $v_\psi$. In addition to the free theory we have the coupling of the bosonic modes $\varphi_{1/2}$ with the different noisy environments (1/f noise and the dissipative ohmic bath) surrounding them. Also, in this case we consider coupling with the 1/f noise strengths $\tilde{F}_i$ and friction coefficients of the dissipative baths $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}$. Note that the noise and the dissipation couple electrostatically with $\varphi_{1/2}$ but not with the neutral Ising sector of the theory that is decoupled from the electromagnetic environment. The total Keldysh bosonic action coupled with the noisy environment has the kernel $\tilde{G}^{-1}$ of (27) with $\nu_1 = 1$ and $\nu_2 = 1/2$. The Lagrangian density is completed by the addition of the disorder term

$$\mathcal{L}_{\text{rdm}} = \xi(x) \psi(x) e^{i[\varphi_1(x) + 2\varphi_2(x)]} + \text{h.c.},$$

(53)

which describes the random electron tunnelling processes which equilibrate the two branches, in full analogy with $\nu = 2/3$. The complex random coefficients $\xi(x)$, Gaussian distributed, satisfy also (40). This unavoidable contribution guarantees that the appropriate value of the Hall
resistance is recovered in the disorder-dominated phase. The RG flow equation for the disorder term $W$ is the same as that of (44), with $\Delta_\varphi$ the scaling dimension of the tunnelling operator $\mathcal{O} \propto \psi e^{i[\varphi_1+2\varphi_2]}$. Consequently, investigating when $\Delta_\varphi < 3/2$, it identifies the conditions for a disorder-dominated phase of $\nu = 5/2$ [15, 16].

We first identify the conditions for the existence of the disorder-dominated phase as a function of the couplings $v_i$ and the noisy environment. The scaling is

$$\Delta_\varphi = 1/2 + \Delta^{(1,2)},$$

where the first term in the sum represents the contribution of the Ising sector (Majorana fermion) and the second one is the bosonic contribution of (42) with $v_1 = 1$ and $v_2 = 1/2$. The bosonic contribution can indeed be derived by following exactly the same steps considered for $\nu = 2/3$. The scaling dimension, in general, depends on the renormalization parameters $g'_i$, defined in (43). Without the noisy environment $g'_1 = g'_2 = 1$, we then recover

$$\Delta^0 = \lim_{k \to 0} \Delta_\varphi = \frac{1}{2} \left( 1 + \frac{3(v_1 + v_2) - 4\sqrt{2}v_{12}}{\sqrt{(v_1 + v_2)^2 - 4v_{12}^2}} \right),$$

which is the scaling dimension of the intra-edge electron tunnelling reported in the literature [15].

The region of existence of the disorder-dominated phase ($\Delta_\varphi < 3/2$) is represented in figure 3 for different velocities of the modes ($v_2/v_1$, $v_{12}/v_1$). The lines delimiting the area are the same as those discussed in the previous section: the stability condition (black solid curve) and the two lines (black dashed) that limit the values of $v_{12}/v_1$, i.e. $v_{12}/v_1 = (3\sqrt{2} \pm \sqrt{3})(1 + v_2/v_1)/12$.

The discussion hereafter goes in parallel with what we have done for $\nu = 2/3$. The noisy environment will further restrict the set of values of intra- and inter-mode couplings where the system is dominated by the disordered phase. In the figure this is represented by the progressive reduction of the coloured area: from lighter blue to darker blue when the environmental noise increases. If the noise becomes strong enough the disorder-dominated phase could even disappear.

Also for $\nu = 5/2$, at the fixed point of the disorder-dominated phase, the system naturally decouples into a charged bosonic mode $\varphi_c = \varphi_1 + \varphi_2$ with velocity $v_\rho$ and a neutral counter-propagating sector (another bosonic mode) $\varphi_n = \varphi_1 - 2\varphi_2$ and one Majorana fermion $\psi$ with the same velocity $v_\sigma$ [15, 16]. It is again natural to introduce the charged $g_\rho$ and neutral $g_\sigma$ renormalization parameters, according to (49). The renormalizations can be very strong, for realistic values of the ratio $v_\sigma/v_\rho$, and satisfying the condition $g_\sigma > g_\rho$ as we anticipated for $\nu = 2/3$. In conclusion, our model fits well also with the values of the renormalizations proposed in [35] for $\nu = 5/2$ [30].

4.2. Agglomerate dominance

Here, we will discuss the effects of a noisy environment on the relevance of excitations in the anti-Pfaffian model. Using the charged and neutral modes basis, one can express the more general qp operator as [15, 35]

$$\Psi(x,l,m) \propto \chi(x) e^{i[m(\varphi_{c}+l/2\varphi_{n})]},$$

where the integer coefficients $m, l$ and the Ising field operator $\chi(x)$ define the admissible excitations. In the Ising sector, $\chi(x)$ can be $I$ (identity operator), $\psi(x)$ (Majorana fermion)
Figure 3. Coloured areas represent the disorder-dominated phase for \( \nu = 5/2 \) in the parameter space \((v_2/v_1, v_{12}/v_1)\) for different strengths of the noise. In light blue is shown the case without the noisy environment \([50], \Delta_0^0 < 3/2\) (see the text). This area is limited by the stability condition (solid line) and the two dashed lines representing respectively the maximum/minimum value of the ratio \(v_{12}/v_1\) to get the disordered phase. Other coloured areas represent the successive reduction of the disordered phase due to the increasing of noise strength \(\tilde{F}_1/v_1^2\tilde{\gamma} = 0\) (lighter blue), 0.1, 0.5, 1 (darker blue) keeping fixed \(\tilde{F}_2 = 0\).

or \(\sigma (x)\) (spin operator). The monodromy condition forces \(m, l\) to be even integers for \(\chi = I, \psi\) and odd integers for \(\chi = \sigma\). The charge associated with the above operator is \((m/4)e\) depending on the charged mode contribution only, while its scaling dimension is

\[
\Delta(\chi, m, l) = \frac{1}{2} \delta_\chi + \frac{g_\rho}{16} m^2 + \frac{g_\sigma}{8} l^2,
\]

(57)

with \(\delta_I = 0, \delta_\psi = 1\) and \(\delta_\sigma = 1/8\). Note that, as stated before, the contribution of the Ising sector to the scaling dimension is not affected by any renormalization.

We adopted the previous formula to predict the scaling dimension and the transport properties in the experiment done by the Heiblum group at Weizmann [35]. We found good agreement with the experiment where, at the lowest temperatures, the dominant excitation is the 2-agglomerate \(2e^* = e/2\), which is described by the operator \(\Psi^{(1,2,0)}\). Our explanation [35] clarifies why the anomalous increasing of the effective charge is observed at extremely low temperatures.

Let us see now when the noise environmental parameters determine the dominance of the 2-agglomerate. In general, the excitation with the lowest scaling dimension dominates the properties in the low-energy sector. Without any renormalization \((g_\rho = g_\sigma = 1)\) the scaling
Figure 4. Three-dimensional picture of the region (coloured) where the inequality $g_\rho < (1 + 2 g_\sigma)/3$ is fulfilled, namely where the 2-agglomerate dominates with respect to the single qp. On the vertical axis is reported the ratio $v_\sigma/v_\rho$, while in the plane the renormalization factors $\tilde{F}_1/(\tilde{\gamma} v^2_1)$ and $\tilde{F}_2/(\tilde{\gamma} v^2_1)$ are reported. The plane $v_\sigma/v_\rho = 1$ is highlighted by a thick line.

dimensions are exactly the same: $\Delta(I, 2, 0) = \Delta(\sigma, 1, \pm 1) = 1/4$. So, only the presence of environmental renormalization will determine the dominance of one excitation over the other. The effect of renormalizations is indeed crucial to make the single-qp excitation—described by the operator $\Psi^{(\sigma, 1, \pm 1)}$ with charge $e^* = e/4$—less relevant than the agglomerate\textsuperscript{13}.

The agglomerate with charge $e/2$ will be dominant over the single-qp if $\Delta(I, 2, 0) < \Delta(\sigma, 1, \pm 1)$ so we obtain the inequalities\textsuperscript{[35]}

$$g_\rho < \frac{1 + 2 g_\sigma}{3}. \quad (58)$$

In figure 4, we show the domain where the agglomerate $\Psi^{(I, 2, 0)}$ is dominant over the single-qp $\Psi^{(\sigma, 1, \pm 1)}$. We see that agglomerates are more readily dominant for $v_\sigma/v_\rho < 1$—the regime probably valid in the real samples. Conversely, when $v_\sigma/v_\rho \gtrsim 2$, the dominance of agglomerates is possible only at very small values of noise strength as shown by the peak in the figure.

Note that, for small values of $\tilde{F}_2$ and strong enough $\tilde{F}_1$, it is also possible to have the dominance of the single-qp for $v_\sigma/v_\rho < 1$. In the last figure this corresponds to the volume underneath the plane identified by the thick line that coincides with $v_\sigma/v_\rho = 1$ and above the light-blue surface.

In conclusion, the dominance of the agglomerate is quite common and only in the case of neutral modes velocity similar to the charged modes and in the presence of a noisy environment\textsuperscript{13} The single-qps can be, in general, seen as a superposition of two different excitations differing in the sign of the bosonic neutral mode contribution, but with exactly the same scaling dimension.

\textsuperscript{13} The single-qps can be, in general, seen as a superposition of two different excitations differing in the sign of the bosonic neutral mode contribution, but with exactly the same scaling dimension.
could the single-qp be more dominant. Anyway, we want to mention that the excitation that dominates at very low energy, potentially could not be dominant also at higher energies, i.e. by increasing the bias or temperature. This explains why the single-qp seems to be the dominant charge carrier in measurements carried out at higher temperatures [18, 20], as we have discussed in more detail in [35].

We conclude this section commenting on the need for a correct identification of the dominant excitations at low energy. We recall that one of the most important properties of anti-Pfaffian (Pfaffian) states for \( \nu = 5/2 \) is the possibility to support excitations which satisfy non-Abelian statistics. Indeed, the single-qp is represented by the operator \( \Psi^{(\sigma, 1, \pm 1)} \) that, due to the peculiar fusion rule in the Ising sector \( \sigma \times \sigma = I + \psi \), is intrinsically non-Abelian. On the other hand, the agglomerate is Abelian, being represented in terms of the operator \( \Psi^{(I, 2, 0)} \), i.e. with an identity operator \( I \) on the Ising sector.

Therefore, the dominance of the agglomerates with respect to the single-qp could have important consequences for the real possibility to manipulate non-Abelian excitations with the help of QPC setups. The hope to encode topological protected quantum computation protocols in this system may be potentially affected by this issue. Counterintuitively, given the previous analysis, a noisy environment could become a helpful resource leading, in some regions of the parameter space, to the dominance of the non-Abelian single-qp.

In perspective, we wish to mention that our approach and also many of the discussed results could be recovered also for other models, such as the Pfaffian or the Abelian ‘331’ model [10, 22]. This shows that, for a large class of models of edges states, the renormalization phenomenon induced by the noisy environment could play an important role influencing the physics in the low-energy regime.

5. Conclusions

We have presented a renormalization mechanism of the tunnelling exponent in the \( \chi \) LL theories for edge states, based on the joint effects of the weak coupling with out-of-equilibrium \( 1/f \) noise and dissipation. The model is very general and could be applied to many different states, such as in the Jain sequence or even the anti-Pfaffian model for \( \nu = 5/2 \).

Considering the paradigmatic case of the Laughlin sequence, we showed how a noisy environment could modify the Luttinger exponents. The direct consequences of this renormalization are derived for the QPC current in the weak-backscattering regime, mainly focusing on the effects on the power-law behaviour as a function of bias.

In the Jain sequence and in particular for \( \nu = 2/5 \), we have investigated how the scaling dimensions of the excitations are affected by the interplay between the inter-channel couplings and the noisy environment. Here, the possibility of a change in the dominance of the excitations is reported. We have already considered [31, 32, 34] the rich phenomenology induced by these facts and we found a good match with the experimental observations.

The case of counter-propagating modes has been analysed in detail. We investigated how the noisy environment modifies the conditions of stability of the disorder-dominated phase by using first order RG analysis. We demonstrated that the disorder-dominated phase is still possible for moderate values of the noise strength. We also verified that, for a system at the fixed point of the disorder-dominated phase, the renormalization mechanism could operate changing the \( \chi \) LL exponents. This is a crucial result, because the quantum Hall edge theories with counter-propagating modes require the presence of static disorder to guarantee
the equilibration along the edge and the proper universal value of the quantum resistance experimentally observed. This robustness makes our model a good candidate for a realistic renormalization mechanism of the Luttinger exponent, while other models, such as the coupling with 1D phonons or other bosonic baths, might not survive in the presence of disorder.

In the last part of the paper, we discussed the $\nu = 5/2$ case considering the non-Abelian anti-Pfaffian model for the edge states. In analogy with the previous analysis, we studied the effect of external environments and their role in the disorder-dominated phase.

Our proposal for the renormalization mechanism seems to be applicable to a plethora of cases giving a convincing and rather simple unified perspective. Our results suggest that the values of the Luttinger exponents, which typically in the literature are related to universal features of the adopted theoretical models, have to be considered with care owing to the presence of unavoidable noisy environments that can modify, even consistently, some of the predictions.

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