Bounded solutions of the finite and
infinite-dimensional dynamical systems.

Pokutnyi O.A.

Institute of mathematics of NAS of Ukraine, Kiev, Tereshchkivska 3,
E-mail: lenasas@gmail.com

Abstract

Invariant torus are constructed under assumption that the homogeneous system admits an
exponential dichotomy on the semi-axes. The main result is closely related with the well-known
Palmer’s lemma and results of Boichuk A.A., Samoilenko A.M.

Key words: exponential dichotomy, bounded solutions, invariant manifold.

Statement of the problem

Consider the linear inhomogeneous system

\[
\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = P(\phi)x + f(\phi),
\]

which defined on the direct product of \(m\)-dimensional torus \(T_m\) or infinite dimensional torus \(T_{\infty}\) and the space \(R^n\) under assumption that \(a(\phi) \in C^1(T_m); P(\phi), f(\phi) \in C(T_m); \phi = (\phi_1, \ldots, \phi_m) \in T_m; x = \text{col}(x_1, \ldots, x_n) \in R^n\). It is known, that the problem of existing and constructing of invariant torus \(x = u(\phi) \in C(T_m), \phi \in T_m\) of the system (1) for all \(f(\phi) \in C(T_m)\) can be solved with using Samoilenko-Green function [1, 2]. For uniqueness it is necessary and sufficient for all \(f(\phi) \in C(T_m)\) that homogeneous system has no degenerate torus

\[
\frac{d\phi}{dt} = a(\phi), \quad \frac{dx}{dt} = P(\phi)x.
\]

It means, that for all \(\phi \in T_m\) the system

\[
\frac{dx}{dt} = P(\phi_t(\phi))x
\]

is exponentially-dichotomous (e-dichotomous) on the whole axis \(R = (-\infty, +\infty)\), i.e. there is exists projector \(C(\phi) = C^2(\phi)\) and not dependent from \(\phi, \tau\) constants \(K \geq 1, \alpha > 0\) such that

\[
\begin{align*}
\|\Omega^t(\phi)C(\phi)\Omega^\tau_\phi(\phi)\| & \leq Ke^{-\alpha(t-\tau)}, \quad t \geq \tau, \\
\|\Omega^\tau_\phi(\phi)(I - C(\phi))\Omega^t(\phi)\| & \leq Ke^{-\alpha(\tau-t)}, \quad \tau \geq t,
\end{align*}
\]

for all \(t, \tau \in R; \Omega^t_\phi(\phi), \Omega^\tau_\phi(\phi) = I_n\) — is \((n \times n)\)-dimensional fundamental matrix of the system (2); \(\phi_t(\phi)\) is a solution of the Koschi problem \(\dot{\phi} = a(\phi), \phi_0(\phi) = \phi\).
Consider the case when the system (3) doesn’t have e-dichotomous on the semi-axes $R$, but e-dichotomous on the semi-axes $R_+$ and $R_-$ with projectors $C_+(\phi)$ and $C_-(\phi)$ ($C_+^2(\phi) = C_+(\phi)$) respectively. It means that (3) for the system (3) the next inequalities are true:

$$
\begin{align*}
\|\Omega_0^L(\phi)C_+(\phi)\Omega_0^R(\phi)\| &\leq K_1 e^{-\alpha_1(t-\tau)}, \quad t \geq \tau, \quad t, \tau \in \mathbb{R}_+; \\
\|\Omega_0^L(\phi)(I - C_+(\phi))\Omega_0^R(\phi)\| &\leq K_1 e^{-\alpha_1(\tau-t)}, \quad \tau \geq t, \quad t, \tau \in \mathbb{R}_+; \\
\|\Omega_0^L(\phi)C_-(\phi)\Omega_0^R(\phi)\| &\leq K_2 e^{-\alpha_2(t-\tau)}, \quad t \geq \tau, \quad t, \tau \in \mathbb{R}_-; \\
\|\Omega_0^L(\phi)(I - C_-(\phi))\Omega_0^R(\phi)\| &\leq K_2 e^{-\alpha_2(\tau-t)}, \quad \tau \geq t, \quad t, \tau \in \mathbb{R}_-;
\end{align*}
$$

is the so called critical case. In this article it is necessary and sufficient conditions for the existence of invariant torus $x = u(\phi) \in C(T_m), \phi \in T_m,$ of the system (3) are obtained in that case. It is necessary and sufficient conditions for inhomogeneity $f(\phi) \in C(T_m)$, which define invariant manifold are obtained.

**Bounded solutions on the whole axis**

For fixed $\phi \in T_m$ general solutions of the problem

$$
\frac{dx}{dt} = P(\phi_t(\phi))x + f(\phi_t(\phi)), \quad (7)
$$

bounded on the entire semi-axes $R_+$ и $R_-$, have the next form

$$
x(t, \phi, \xi_1) = \begin{cases} 
\Omega_0^L(\phi)C_+(\phi)\xi_1 + \int_0^t \Omega_0^L(\phi)C_+(\phi_t(\phi))f(\phi_t(\phi))d\tau - \\
\int \Omega_0^L(\phi)(I - C_+(\phi_t(\phi)))f(\phi_t(\phi))d\tau, \quad t \geq 0,
\end{cases} \quad (8)
$$

\begin{align*}
\Omega_0^L(\phi)(I - C_-(\phi))\xi_1 + \int_{-\infty}^t \Omega_0^L(\phi)C_-(\phi_t(\phi))f(\phi_t(\phi))d\tau - \\
- \int \Omega_0^L(\phi)(I - C_-(\phi_t(\phi)))f(\phi_t(\phi))d\tau, \quad t \leq 0,
\end{align*}
for all bounded $f$ and

$$x(t, \phi, \xi_2) = \begin{cases} \Omega_t^0(\phi)C_+(\phi)\xi_2 + \int_0^t \Omega^t(\phi)(I - C_+(\phi t(\phi)))f(\phi_t(\phi))d\tau - \\ - \int_0^\infty \Omega^t(\phi)(C_+(\phi t(\phi)))f(\phi_t(\phi))d\tau, & t \geq 0, \\ \Omega_t^0(\phi)(I - C_-(\phi))\xi_2 + \int_{-\infty}^t \Omega^t(\phi)(I - C_- (\phi t(\phi)))f(\phi_t (\phi))d\tau - \\ - \int_{-\infty}^0 \Omega^t(\phi)(C_- (\phi t(\phi)))f(\phi_t (\phi))d\tau, & t \leq 0, \end{cases}$$

but not for all bounded $f$, where

$$C_+(\phi t(\phi)) = \Omega_0^0(\phi)C_+(\phi)\Omega^0_t(\phi), \quad C_-(\phi t(\phi)) = \Omega_0^0(\phi)C_-(\phi)\Omega^0_t(\phi).$$

We say about conditions on $f$ below. Here are some well-known relations

$$\Omega_t^t(\phi s(\phi)) = \Omega_{t+s}^0(\phi), \quad \Omega^t_t(\phi)\Omega^s_s(\phi) = \Omega^0(\phi),$$

$$(\Omega^0_t(\phi))^{-1} = \Omega_t^0(\phi), \quad \phi_t(\phi) = \phi_{t+s}(\phi),$$

which valid for all $t, \tau, s \in \mathbb{R}$, $\phi \in \mathcal{T}_m$.

Solutions \[9\] and \[10\] will be bounded on the entire axis $\mathbb{R}$, if the constant vectors $\xi_1 = \xi_1(\phi) \in \mathbb{R}^n$ and $\xi_2 = \xi_2(\phi) \in \mathbb{R}^n$ satisfy the next algebraic systems, obtained from the \[9\] and \[10\] for $t = 0$:

$$[C_+(\phi) - (I - C_-(\phi))]\xi_1 = \int_{-\infty}^0 C_-(\phi)\Omega^0_t(\phi)f(\phi_t(\phi))d\tau +$$

$$+ \int_0^\infty (I - C_+(\phi))\Omega^0_t(\phi)f(\phi_t(\phi))d\tau.$$  

$$[C_+(\phi) - (I - C_-(\phi))]\xi_2 = \int_{-\infty}^0 (I - C_-(\phi))\Omega^0_t(\phi)f(\phi_t(\phi))d\tau +$$

$$+ \int_0^\infty (C_+(\phi))\Omega^0_t(\phi)f(\phi_t(\phi))d\tau.$$  

Denote by $D(\phi) = C_+(\phi) - (I - C_-(\phi))$ is $(n \times n)$-dimensional matrix, and by $D^+(\phi)$ its Moore-Penrose pseudoinvertible \[5\]; $P_{N(D)}(\phi)$ and $P_{N(D^+)}(\phi)$ are $(n \times n)$-dimensional orthoprojectors:
\[ P^2_{N(D)}(\phi) = P_{N(D)}(\phi) = P^*_{N(D)}(\phi), \]
\[ P^2_{N(D^*)}(\phi) = P_{N(D^*)}(\phi) = P^*_{N(D^*)}(\phi), \]

which project \( R^n \) onto kernel \( N(D) = \text{ker} D(\phi) \) and cokernel \( N(D^*) = \text{ker} D^*(\phi) \) of matrix \( D(\phi) \):

\[ P_{N(D^*)}(\phi) = I - D(\phi)D^+(\phi), \quad P_{N(D)}(\phi) = I - D^+(\phi)D(\phi). \]

System (12) is solvable if and only if the right part of the system (12) belongs to the orthogonal complement of \( N^\perp(D^*(\phi)) = \text{Im} (D(\phi)) \) of the subspace \( N(D^*(\phi)) \). It means that

\[ P_{N(D^*)}(\phi) \left\{ \int_{-\infty}^{0} C_{-}(\phi)\Omega^0_{\tau}(\phi)f(\phi_{\tau}(\phi))d\tau + \right. \\
+ \left. \int_{0}^{\infty} (I - C_{+}(\phi))\Omega^0_{\tau}(\phi)f(\phi_{\tau}(\phi))d\tau \right\} = 0. \tag{14} \]

In this case the general solutions of the system (12), bounded on the entire axis \( R \), have the form (8) with constant \( \xi_1 = \xi_1(\phi) \in R^n \), which defines from the equation (12) by the rule:

\[ \xi_1 = D^+(\phi) \left\{ \int_{-\infty}^{0} C_{-}(\phi)\Omega^0_{\tau}(\phi)f(\phi_{\tau}(\phi))d\tau + \right. \\
+ \left. \int_{0}^{\infty} (I - C_{+}(\phi))\Omega^0_{\tau}(\phi)f(\phi_{\tau}(\phi))d\tau \right\} + P_{N(D)}(\phi)c, \quad c = c(\phi) \in R^n. \tag{15} \]

Substitute (15) in (8), we obtain that for fixed \( \phi \in T_m \) and inhomogeneity \( f(\phi_t(\phi)) \in C(T_m) \), which satisfies condition (14), solutions, bounded on \( R \) of the system (17) have the form
\[ x(t, \phi, c) = \begin{cases} 
C_+(\phi)P_{N(D)}(\phi)c + \int_0^t C_+(\phi)\Omega^0_{D+}(\phi)f(\phi_\tau(\phi))d\tau - \\
\quad - \int_t^\infty (I - C_+(\phi))\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau + \\
\quad + C_+(\phi)D^+(\phi)\left\{ \int_0^\infty C_-(\phi)\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau + \\
\quad + \int_0^t (I - C_+(\phi))\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau \right\}, \quad t \geq 0, \\
(I - C_-(\phi))P_{N(D)}(\phi)c + \int_0^t C_-(\phi)\Omega^0_{D+}(\phi)f(\phi_\tau(\phi))d\tau - \\
\quad - \int_t^\infty (I - C_-(\phi))\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau + \\
\quad + (I - C_-(\phi))D^+(\phi)\left\{ \int_0^\infty C_-(\phi)\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau + \\
\quad + \int_0^t (I - C_+(\phi))\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau \right\}, \quad t \leq 0. 
\end{cases} \]

Since \( P_{N(D')}(\phi)D(\phi) = P_{N(D)}(\phi)\left[ C_+(\phi) - (I - C_-(\phi)) \right] = 0 \), then

\[ P_{N(D')}(\phi)C_+(\phi) = P_{N(D')}(\phi)\left( I - C_-(\phi) \right), \]

and condition (14) is equivalent one of the conditions

\[ P_{N(D')}(\phi) \int_0^\infty C_-(\phi)\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau = 0, \quad (17) \]

\[ P_{N(D')}(\phi) \int_0^\infty (I - C_+(\phi))\Omega^0_D(\phi)f(\phi_\tau(\phi))d\tau = 0. \]

Since

\[ \left[ C_+(\phi) - (I - C_-(\phi)) \right]D^+(\phi) = I - P_{N(D')}(\phi), \]
we obtain
\[ C_+(\phi)D^+(\phi)\{\ldots\} - I\{\ldots\} = (I - C_- (\phi))D^+(\phi)\{\ldots\}, \]
from the condition (14), \{\ldots\} is the expression in (14).

Since \( D(\phi)P_{N(D)}(\phi) = [C_+(\phi) - (I - C_- (\phi))]P_{N(D)}(\phi) = 0 \), then
\[ C_+(\phi)P_{N(D)}(\phi) = (I - C_- (\phi))P_{N(D)}(\phi). \]

Similarly the system (13) is solvable if and only if the right part of system (13) belongs to the orthogonal complement of \( N^\perp(D^* (\phi)) = \text{Im } (D(\phi)) \) of the subspace \( N(D^* (\phi)) \). It means that
\[
P_{N(D^*)}(\phi) \left\{ \int_{-\infty}^{0} (I - C_- (\phi))\Omega^0_\tau (\phi) f(\phi_\tau (\phi)) d\tau + \int_{0}^{\infty} (C_+(\phi))\Omega^0_\tau (\phi) f(\phi_\tau (\phi)) d\tau \right\} = 0. \quad (18)\]

In this case the general solutions of the system (13), bounded on the entire axis \( R \), have the form (9) with constant \( \xi_2 = \xi_2(\phi) \in R^n \), which defines from the equation (13) by the rule:
\[
\xi_2 = D^+(\phi) \left\{ \int_{-\infty}^{0} (I - C_- (\phi))\Omega^0_\tau (\phi) f(\phi_\tau (\phi)) d\tau + \int_{0}^{\infty} (C_+(\phi))\Omega^0_\tau (\phi) f(\phi_\tau (\phi)) d\tau \right\} + P_{N(D)}(\phi)c, \quad c = c(\phi) \in R^n. \quad (19)\]

Substitute (19) in (9), we obtain that for fixed \( \phi \in T_m \) and inhomogeneity \( f(\phi_\tau (\phi)) \in C(T_m) \), which satisfies condition (18), solutions, bounded on \( R \) of the system (7) have the form
\[ x(t, \phi, c) = \begin{cases} 
C_+(\phi)P_{N(D)}(\phi)c + \int_0^t (I - C_+(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau - \\
\int_t^\infty (C_+(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau + \\
+C_+(\phi)D^+(\phi) \left\{ \int_0^0 (I - C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau + \\
\int_0^\infty (C_+(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau \right\}, & t \geq 0, \\
(I - C_-(\phi))P_{N(D)}(\phi)c + \int_0^- t (I - C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau - \\
\int_0^\infty (C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau + \\
+(I - C_-(\phi))D^+(\phi) \left\{ \int_0^0 (I - C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau + \\
\int_0^\infty (C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau \right\}, & t \leq 0. 
\end{cases} \]

Since \( P_{N(D^+)}(\phi)D(\phi) = P_{N(D^+)}(\phi)\left[ I - C_+(\phi) + C_-(\phi) \right] = 0 \), then

\[ P_{N(D^+)}(\phi)(I - C_+(\phi)) = P_{N(D^+)}(\phi)(C_-(\phi)), \]

and condition (15) is equivalent one of the conditions

\[ P_{N(D^+)}(\phi) \int_0^\infty (I - C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau = 0, \]
\[ P_{N(D^+)}(\phi) \int_0^- (-\infty) (I - C_-(\phi))\Omega^0_\tau(\phi)f(\phi_\tau(\phi))d\tau = 0. \]

Since

\[ (I - C_+(\phi) + C_-(\phi))D^+(\phi) = I - P_{N(D^+)}(\phi), \]
we obtain

\[(I - C_+(\phi))D^+(\phi)\{\ldots\} - I\{\ldots\} = (C_-(\phi))D^+(\phi)\{\ldots\},\]

from the condition (18), \{\ldots\} is expression in (18).

Since \(D(\phi)P_{N(D)}(\phi) = [C_+(\phi) - (I - C_-(\phi))]P_{N(D)}(\phi) = 0,\) then

\[(I - C_+(\phi))P_{N(D)}(\phi) = (C_-)P_{N(D)}(\phi).\]

Consider the case, when homogeneous system (3) does not have bounded and unbounded solutions

\[C_+(\phi)P_{N(D)}(\phi) = (I - C_-)P_{N(D)}(\phi) = 0.\]

Then (16) and (20) we can rewrite in the form

\[x(t, \phi) = (G_t(f))(\phi), \tag{22}\]

\[
\begin{align*}
(G_t(f))(\phi) &= \Omega^t_0(\phi) \\
&= \begin{cases} \\
\int_0^t C_+(\phi)\Omega^\tau_0(\phi)f(\phi_\tau(\phi))d\tau - & t \geq 0, \\
- \int_0^\infty (I - C_+(\phi))\Omega^\tau_0(\phi)f(\phi_\tau(\phi))d\tau + \\
+ C_+(\phi)D^+(\phi) \left\{ \int_0^\tau C_-(\phi)\Omega^\rho_\tau(\phi)f(\phi_\rho(\phi))d\rho \right\} \end{cases} \\
&\quad + \int_0^\tau (I - C_+(\phi))\Omega^\rho_\tau(\phi)f(\phi_\rho(\phi))d\rho, \quad t \geq 0, \\
&\quad \int_{-\infty}^0 C_-(\phi)\Omega^\tau_0(\phi)f(\phi_\tau(\phi))d\tau - \\
&\quad - \int_0^\tau (I - C_-(\phi))\Omega^\rho_\tau(\phi)f(\phi_\rho(\phi))d\rho + \\
&\quad + [C_+(\phi)D^+(\phi) - I] \left\{ \int_{-\infty}^0 C_-(\phi)\Omega^\rho_\tau(\phi)f(\phi_\rho(\phi))d\rho \right\} \\
&\quad + \int_0^\tau (I - C_+(\phi))\Omega^\rho_\tau(\phi)f(\phi_\rho(\phi))d\rho, \quad t \leq 0,
\end{align*}\]

and in the second case
which obtained from (22) for $t$

Criterion of existence of invariant torus of nonhomogeneous system

As shown below, under conditions

We show, that the expression

Under conditions (15), (19) solutions, bounded on $R$, of the system (7) for fixed $\phi \in T_m$ have the form (22).

We show, that the expression

$$x(0, \phi) = u(\phi) = (G_t(f))(\phi),$$

which obtained from (22) for $t = 0$, define for all $\phi \in T_m$ invariant torus of the system (1).

Criterion of existence of invariant torus of nonhomogeneous system

As shown below, under conditions

$$P_{N(D^+)}(\phi) \int_{-\infty}^{+\infty} C_-(\phi)\Omega^0_t(\phi)f(\phi_\tau(\phi))d\tau = 0,$$

$$P_{N(D^+)}(\phi) \int_{-\infty}^{+\infty} (I - C_-(\phi))\Omega^0_t(\phi)f(\phi_\tau(\phi))d\tau = 0,$$
the nonhomogeneous system \([7]\) have bounded solutions on \(R\) in the form \([22]\) for fixed \(\phi \in T_m\). Conditions \([24]\) and \([26]\) on solutions \(\phi_t(\phi)\) define invariant set. Substitute \(\phi_t(\phi)\) instead of \(\phi\) and show that conditions \([24]\) and \([26]\) hold for all \(t \in R\) and \(\phi \in T_m\). From the relations for \(D(\phi) = C_+(\phi) - (I - C_-(\phi))\) we obtain the next equality

\[
D(\phi_t(\phi)) = \Omega^0_0(\phi)D(\phi)\Omega^0_t(\phi) \quad \forall t \in R, \quad \forall \phi \in T_m.
\]  

(26)

Direct check shows, that for all \(t \in R\) and \(\phi \in T_m\) the matrix

\[
D^{-}(\phi_t(\phi)) = [\Omega^0_0(\phi)D(\phi)\Omega^0_t(\phi)]^{-} = \Omega^0_0(\phi)D^{-}(\phi)\Omega^0_t(\phi)
\]  

(27)

is generalized-invertible to the matrix \(D(\phi_t(\phi))\) and satisfies the next relations \([5]\)

\[
D^{-}(\phi_t(\phi))D(\phi_t(\phi))D^{-}(\phi_t(\phi)) = D^{-}(\phi_t(\phi)),
\]

\[
D(\phi_t(\phi))D^{-}(\phi_t(\phi))D(\phi_t(\phi)) = D(\phi_t(\phi)).
\]

(28)

From the conditions

\[
D(\phi_t(\phi))D^{-}(\phi_t(\phi)) = I - P_{N(D)}(\phi_t(\phi)),
\]

\[
D^{-}(\phi_t(\phi))D(\phi_t(\phi)) = I - P_{N(D^*)}(\phi_t(\phi))
\]

we get expressions for projectors \(P_{N(D)}(\phi_t(\phi))\) and \(P_{N(D^*)}(\phi_t(\phi))\) onto kernel and cokernel of matrix \(D(\phi)\) on solutions \(\phi_t(\phi)\) of the respectively Koschi problem for all \(t \in R\) and \(\phi \in T_m\):

\[
P_{N(D)}(\phi_t(\phi)) = \Omega^0_0(\phi)P_{N(D)}(\phi)\Omega^0_t(\phi) = \Omega^0_0(\phi)[I - D^{-}(\phi)D(\phi)]\Omega^0_t(\phi),
\]

\[
P_{N(D^*)}(\phi_t(\phi)) = \Omega^0_0(\phi)P_{N(D^*)}(\phi)\Omega^0_t(\phi) = \Omega^0_0(\phi)[I - D(\phi)D^{-}(\phi)]\Omega^0_t(\phi).
\]

(29)

We can choose that \(D^{-}(\phi) = D^+(\phi)\). In that case projectors \(P_{N(D)}(\phi)\) and \(P_{N(D^*)}(\phi)\) will be orthoprojectors.

For all \(t \in R\) and \(\phi \in T_m\) we have

\[
P_{N(D^*)}(\phi_t(\phi)) \int_{-\infty}^{+\infty} C_-(\phi_t(\phi))\Omega^0_t(\phi_t(\phi))f(\phi_t(\phi))d\tau =
\]

\[
= \Omega^0_0(\phi)P_{N(D^*)}(\phi) \int_{-\infty}^{+\infty} C_-(\phi)\Omega^0_t(\phi)\Omega^0_t(\phi)\phi_t(\phi)d\tau = 0
\]

and

\[
\Omega^0_0(\phi)P_{N(D^*)}(\phi) \int_{-\infty}^{+\infty} (I - C_-(\phi))\Omega^0_t(\phi)\phi_t(\phi)d\tau = 0
\]
From the conditions (10), (12), (29) we have
\[ u(\phi_t(\phi)) = (G_0(f))(\phi_t(\phi)) = (G_t(f))(\phi) \]
for all \( t \in R \) and \( \phi \in T_m \). It shows that \( u(\phi_t(\phi)) \in C^1(T_m) \), and the set \( u(\phi) \) defines invariant torus of the system (1).

In such a way we have the following theorem.

Theorem. Let the system (3) is e-dichotomous on both semi-axes \( R_+ \) u \( R_- \) with projectors \( C_{\pm}(\phi) \), which satisfy the next equalities for \( \phi_t(\phi) \)
\[ C_{\pm}(\phi_t(\phi)) = \Omega_0^t(\phi)C_{\pm}(\phi)\Omega_0^t(\phi), \quad C_{\pm}^2(\phi) = C_{\pm}(\phi). \]
The system (1) has invariant torus if and only if nonhomogeneit \( f(\phi) \in C(T_m) \) satisfies conditions (24), (25). If the homogeneous system (3) does not have bounded and unbounded solutions, i.e. the next condition is true
\[ C_{+}(\phi)P_{N(D)}(\phi) = (I - C_{-}(\phi))P_{N(D)}(\phi) = 0, \]
then expression
\[ u(\phi) = (G_0(f))(\phi) \]
which obtain from (22), for \( t = 0 \), defines for all \( \phi \in T_m \) invariant torus of system (1).

Examples. Consider the problem of the existence of invariant manifold of the system
\[ \dot{\varphi} = 1, \quad (30) \]
\[ \dot{x}_1(t) = \text{th}(\varphi)x(t) + f_1(\varphi), \quad (31) \]
\[ \dot{x}_2(t) = -\text{th}(\varphi)x_2(t) + f_2(\varphi), \quad (32) \]
\[ \dot{x}_3(t) = \text{th}(\varphi)x_3(t) + f_3(\varphi). \quad (33) \]
This system has the following characteristics:
\[ \Omega_0^t(\varphi) = \left( \begin{array}{cc} \frac{\text{ch}(\varphi_t(\varphi))}{\text{ch}(\varphi)} & 0 \\ 0 & \frac{\text{ch}(\varphi_t(\varphi))}{\text{ch}(\varphi)} \end{array} \right), \]
\[ C_{+}(\varphi) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), C_{-}(\varphi) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \]
\[ D(\varphi) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) = D^+(\varphi), \]
\[ P_{N(D)}(\varphi) = P_{N(D^*)}(\varphi) = I. \]
In this case the first condition of solvability have the form
\[ \int_{-\infty}^{+\infty} \frac{\text{ch}_{\varphi}}{\text{ch}(\varphi_t(\varphi))}f_1(\varphi_t(\varphi))d\tau = 0, \]
and under this condition the system has the invariant torus in the form

\[ u_1(\varphi) = \left( - \int_0^{+\infty} \frac{\text{ch}(\varphi) f_1(\varphi, \varphi)}{\text{ch}(\varphi)} d\tau \right) \]

and under the second condition of solvability

\[ \int_{-\infty}^{+\infty} \text{ch}(\varphi) f_2(\varphi) d\tau = 0 \]

the system has the invariant torus

\[ u_2(\varphi) = \left( - \int_0^{+\infty} \frac{\text{ch}(\varphi) f_2(\varphi)}{\text{ch}(\varphi)} d\tau \right) \]

If, for example \( f_1(\varphi) = \frac{\text{sh}(\varphi)}{\text{ch}^3(\varphi)} \) and \( f_2(\varphi) = \frac{\text{sh}(\varphi)}{\text{ch}^4(\varphi)} \), then the given system has invariant torus in the glued form

\[ u(\varphi) = \left( - \frac{1}{5\text{ch}^3(\varphi)} \frac{1}{2\text{ch}^4(\varphi)} \right) \]

We mention that given theory works in the case of infinite dimensional space. Here is an example. Consider the countable system of differential equations in the space \( BC(\mathbb{R}, l_2) \) or \( BC(\mathbb{R}, l_{2\text{loc}}) \) in the next form

\[ \dot{\varphi}(t) = 1, \]

\[ \frac{dx(t)}{dt} = P(\varphi_1(t)) x(t) + f(\varphi(t)), \]

where

\[ x(t) = (x_1(t), x_2(t), ...) \in l_2 \text{ for all } t, \]

and

\[ f(\varphi) = (f_1(\varphi), f_2(\varphi), ...), \]

where

\[ P(\varphi) = \text{diag}\{\text{th}(\varphi), \text{th}(\varphi), -\text{th}(\varphi), -\text{th}(\varphi), ...\}. \]

\[ x(t) = (x_1(t), x_2(t), ...), f(t) = (f_1(t), f_2(t), ...) \in BC(\mathbb{R}, l_2). \]

Here \( BC(\mathbb{R}, l_2) \) and \( BC(\mathbb{R}, l_{2\text{loc}}) \) are the spaces of bounded and continuous on the whole axis functions with values in \( l_2 \) or \( l_{2\text{loc}} \).

Matriciant of the system has the form

\[ \Omega_0(\varphi) = \text{diag}\{ \frac{\text{ch}(\varphi)}{\text{ch}^2(\varphi)}, \frac{\text{ch}(\varphi)}{\text{ch}^3(\varphi)}, \frac{\text{ch}(\varphi)}{\text{ch}(\varphi)^2}, \frac{\text{ch}(\varphi)}{\text{ch}(\varphi)^3}, \frac{\text{ch}(\varphi)}{\text{ch}(\varphi)^4}, ... \}. \]

Projectors have the form

\[ C_+(\varphi) = \text{diag}\{0, 0, 1, 1, ...\}, C_-(\varphi) = \text{diag}\{1, 1, 0, 0, ...\}. \]
Matrixes $D(\varphi) = D^+(\varphi) = 0$, and $P_{N(D)} = P_{N(D^*)} = I$, where $I$ is the identity matrix. Condition of the solvability for the first type of torus has the form
\[
\int_{-\infty}^{+\infty} \frac{f_i(\varphi_+(\varphi))}{ch(\varphi_+(\varphi))} d\tau = 0, \quad i = 1, 2,
\]
and invariant torus has the form
\[
x = u_1(\varphi) = (-\int_{0}^{+\infty} \frac{ch(\varphi) f_1(\varphi_+(\varphi))}{ch(\varphi_+(\varphi))} d\tau, -\int_{0}^{+\infty} \frac{ch(\varphi) f_2(\varphi_+(\varphi))}{ch(\varphi_+(\varphi))} d\tau, 0, ...).
\]
Condition of the solvability for the second type has the form
\[
\int_{-\infty}^{+\infty} f_i(\varphi_+(\varphi)) ch(\varphi_+(\varphi)) d\tau = 0, \quad i \geq 3,
\]
and invariant torus has the form
\[
x = u_2(\varphi) = (0, 0, -\int_{0}^{+\infty} \frac{f_3(\varphi_+(\varphi)) ch(\varphi_+(\varphi))}{ch(\varphi)} d\tau, ..., -\int_{0}^{+\infty} \frac{f_i(\varphi_+(\varphi)) ch(\varphi_+(\varphi))}{ch(\varphi)} d\tau, ...),
\]
or in the glued form
\[
x = u(\varphi) = (-\int_{0}^{+\infty} \frac{ch(\varphi) f_1(\varphi_+(\varphi))}{ch(\varphi_+(\varphi))} d\tau, -\int_{0}^{+\infty} \frac{ch(\varphi) f_2(\varphi_+(\varphi))}{ch(\varphi_+(\varphi))} d\tau, \\
-\int_{0}^{+\infty} \frac{f_3(\varphi_+(\varphi)) ch(\varphi_+(\varphi))}{ch(\varphi)} d\tau, ..., -\int_{0}^{+\infty} \frac{f_i(\varphi_+(\varphi)) ch(\varphi_+(\varphi))}{ch(\varphi)} d\tau, ...).
\]
If, for example, $f_i(\varphi) = \frac{sh\varphi}{ch^{i-1}(\varphi)}$, $i \geq 1$, then we have
\[
x = u_1(\varphi) = \left(-\frac{1}{2ch^2(\varphi)}, -\frac{1}{3ch^3(\varphi)}, 0, ... \right),
\]
and
\[
x = u_2(\varphi) = (0, 0, -\frac{1}{3ch^4(\varphi)}, ..., -\frac{1}{ich^{i+1}(\varphi)}, ...),
\]
or in the glued form
\[
x = u(\varphi) = \left(-\frac{1}{2ch^2(\varphi)}, -\frac{1}{3ch^3(\varphi)}, -\frac{1}{3ch^4(\varphi)}, ..., -\frac{1}{ich^{i+1}(\varphi)}, ... \right).
\]
Here is denotions as in [8].
Bibliography

[1] Samoilenko A.M. Elements of mathematical theory of multifrequency oscillations. – M.: Science, 1987. – 304 p. (in russian).

[2] Mitropolsky Yu.O., Samoilenko A.M., Kulik V.L. Investigation of dichotomy of linear system of differential equations with Lyapunov functions. – Kiev, 1990. – 270 p. (in russian)

[3] Palmer K. J. Exponential dichotomies and transversal homoclinic points // J. Different. Equat. – 1984. – 55. – P. 225 – 256.

[4] Boichuk A. A. Solutions of weakly nonlinear differential equations bounded on the whole line // Nonlinear Oscillations. – 1999. – 2, No 1. – P. 3 – 10.

[5] Boichuk A. A., Samoilenko A. M. Generalized inverse operators and fredholm boundary value problems. – Utrecht; Boston: VSP, 2004. – 317 p.

[6] Boichuk A.A. Condition of existence of unique Green-Samoilenko function of the invariant torus problem // Ukrainian Math. Journ. – 2001. – 53, No4. – p. 556–559.

[7] Boichuk A. Bounded solutions of differential equations in Banach space // Colloq. Different. and Difference Equat. dedicat. Prof. Jaroslav Kurzweil 80-th Birthday: Abstrs (Brno, Czech Republic, Sept. 5 – 8, 2006). – P. 35.

[8] Boichuk A.A. Criterion of existence of unique invariant torus of linear extension dynamical systems. – Ukrainian Math. Journal, 2007, V.59, №1. – p.3 – 13.

Institute of mathematics of NAS of Ukraine, Kiev, 01601, Tereshenkiuska str. 3, lenasas@gmail.com