Learning Concepts Described by
Weight Aggregation Logic

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We consider weighted structures, which extend ordinary relational structures by assigning weights, i.e. elements from a particular group or ring, to tuples present in the structure. We introduce an extension of first-order logic that allows to aggregate weights of tuples, compare such aggregates, and use them to build more complex formulas. We provide locality properties of fragments of this logic including Feferman-Vaught decompositions and a Gaifman normal form for a fragment called FOW₁, as well as a localisation theorem for a larger fragment called FOWA₁. This fragment can express concepts from various machine learning scenarios. Using the locality properties, we show that concepts definable in FOWA₁ over a weighted background structure of at most polylogarithmic degree are agnostically PAC-learnable in polylogarithmic time after pseudo-linear time preprocessing.

1 Introduction

In this paper, we study Boolean classification problems. The elements that are to be classified come from a set $\mathcal{X}$, the instance space. A classifier on $\mathcal{X}$ is a function $c: \mathcal{X} \to \{0, 1\}$. Given a training sequence $T$ of labelled examples $(x_i, b_i) \in \mathcal{X} \times \{0, 1\}$, we want to find a classifier, called a hypothesis, that can be used to predict the label of elements from $\mathcal{X}$ not given in $T$. We consider the following well-known frameworks for this setting from computational learning theory.

In Angluin’s model of exact learning [1], the examples are assumed to be generated using an unknown classifier, the target concept, from a known concept class. The task is to find a hypothesis that is consistent with the training sequence $T$, i.e. a function $h: \mathcal{X} \to \{0, 1\}$ such that $h(x_i) = b_i$ for all $i$. In Haussler’s model of agnostic probably approximately correct (PAC) learning [12], a generalisation of Valiant’s PAC learning model [23], an (unknown) probability distribution $D$ on $\mathcal{X} \times \{0, 1\}$ is assumed and training examples are drawn independently from this distribution. The goal is to find a hypothesis that generalises well, i.e. one is interested in algorithms that return with high probability a hypothesis with a small expected error on new instances drawn from
the same distribution. For more background on PAC learning, we refer to \cite{13, 21}. We study learning problems in the framework that was introduced by Grohe and Turán \cite{10} and further studied in \cite{4, 7, 8, 24}. There, the instance space \(X\) is a set of tuples from a background structure and classifiers are described using parametric models based on logics.

**Our contribution.** We introduce a new logic for describing such classifiers, namely \textit{first-order logic with weight aggregation} (FOWA). It operates on \textit{weighted structures}, which extend ordinary relational structures by assigning weights, i.e. elements from a particular abelian group or ring, to tuples present in the structure. Such weighted structures were recently considered by Toruńczyk \cite{22}, who studied the complexity of query evaluation problems for the related logic \(\text{FO}[\mathbb{C}]\) and its fragment \(\text{FO}_{\mathbb{C}}[\mathbb{C}]\). Our logic FOWA, however, is closer to the syntax and semantics of the first-order logic with counting quantifiers \(\text{FOC}\) considered in \cite{14}. This connection enables us to achieve locality results for the fragments FOW\(_1\) and FOWA\(_1\) of FOWA similar to those obtained in \cite{15, 9}. Specifically, we achieve Feferman-Vaught decompositions and a Gaifman normal form for FOW\(_1\) as well as a localisation theorem for the more expressive logic FOWA\(_1\). We provide examples illustrating that FOWA\(_1\) can express concepts relevant for various machine learning scenarios. Using the locality properties, we show that concepts definable in FOWA\(_1\) over a weighted background structure of at most polylogarithmic degree are agnostically PAC-learnable in polylogarithmic time after pseudo-linear time preprocessing. This generalises the results that Grohe and Ritzert \cite{8} obtained for first-order logic to the substantially more expressive logic FOWA\(_1\).

The main drawback of the existing logic-based learning results is that they deal with structures and logics that are too weak for describing meaningful classifiers for real-world machine learning problems. In machine learning, input data is often given via numerical values which are contained in or extracted from a more complex structure, such as a relational database (cf., \cite{6, 11, 19, 20}). Hence, to combine these two types of information, we are interested in hybrid structures, which extend relational ones by numerical values. Just as in commonly used relational database systems, to utilise the power of such hybrid structures, the classifiers should be allowed to use different methods to aggregate the numerical values. Our main contribution is the design of a logic that is capable of expressing meaningful machine learning problems and, at the same time, well-behaved enough to have similar locality properties as first-order logic, which enable us to learn the concepts in sublinear time.

**Outline.** This paper is structured as follows. Section 2 fixes basic notation. Section 3 introduces the logic FOWA and its fragments FOW\(_1\) and FOWA\(_1\), provides examples, and discusses enrichments of the logic with syntactic sugar in order to make it more user-friendly (i.e. easier to parse or construct formulas) without increasing its expressive power. Section 4 provides locality results for the fragments FOW\(_1\) and FOWA\(_1\) that are similar in spirit to the known locality results for first-order logic and the counting logic FOC\(_1\). Section 5 is devoted to our results on agnostic PAC learning. Section 6 combines the results from the previous sections to obtain our main learning theorem for FOWA\(_1\), and concludes the paper with an application scenario and directions for future work.
2 Preliminaries

**Standard notation.** We write \( \mathbb{R} \), \( \mathbb{Q} \), \( \mathbb{Z} \), \( \mathbb{N} \), and \( \mathbb{N}_0 \) for the sets of reals, rationals, integers, non-negative integers, and positive integers, respectively. For all \( m, n \in \mathbb{N} \), we write \([m, n]\) for the set \( \{k \in \mathbb{N} : m \leq k \leq n\} \), and we let \([m] := [1, m] \). For a \( k \)-tuple \( \bar{x} = (x_1, \ldots, x_k) \), we write \(|\bar{x}|\) to denote its *arity* \( k \). By (), we denote the empty tuple, i.e. the tuple of arity 0. All graphs are assumed to be undirected. For a graph \( G \), we write \( V(G) \) and \( E(G) \) to denote its vertex set and edge set, respectively. For \( V' \subseteq V(G) \), we write \( G[V'] \) to denote the subgraph of \( G \) induced on \( V' \).

**Monoids, groups, semirings, and rings.** Recall that a *monoid* is a set \( M \) that is equipped with a binary operator \( \circ : M \times M \to M \) that is associative and has a neutral element \( e_M \in M \) (i.e. for all \( a, b, c \in M \), we have \( a \circ (b \circ c) = (a \circ b) \circ c \) and \( e_M \circ a = a \circ e_M = a \)). A monoid is *commutative* if \( a \circ b = b \circ a \) holds for all \( a, b \in M \). An *abelian group* is a commutative monoid \((M, \circ)\) where for each \( a \in M \), there is an \( a' \in M \) such that \( a \circ a' = e_M \); by convention, we write \(-a\) for this \( a' \). When referring to an abelian group, we usually write \((S, +, \cdot)\) instead of \((M, \circ)\), and we denote the neutral element by 0\( _S \). A *semiring* is a set \( S \) that is equipped with two binary operators \(+\) and \( \cdot\) such that \((S, +)\) is a commutative monoid with a neutral element \( 0_S \in S \), \((S, \cdot, 1_S)\) is a monoid with a neutral element \( 1_S \in S \), multiplication distributes over addition, and multiplication by \( 0_S \) annihilates \( S \) (i.e. for all \( a, b, c \in S \), we have \( a \cdot (b + c) = (a \cdot b) + (a \cdot c), (b + c) \cdot a = (b \cdot a) + (c \cdot a) \), \( 0_S \cdot a = a \cdot 0_S = 0_S \)). A *ring* is a semiring \((S, +, \cdot, 1_S)\) where \((S, +)\) is an abelian group. A semiring or ring \((S, +, \cdot)\) is called *commutative* if the monoid \((S, \cdot)\) is commutative. In the following, we briefly write \( S \) instead of \((S, +, \cdot)\), and we write \(+_S, \cdot_S, 0_S, 1_S\) to denote the operators \(+\) and \( \cdot\) and their neutral elements.

**Signatures, structures, and neighbourhoods.** A *signature* \( \sigma \) is a finite set of relation symbols. Associated with every \( R \in \sigma \) is an arity \( \text{ar}(R) \in \mathbb{N} \). A *\( \sigma \)-structure* \( A \) consists of a finite non-empty set \( A \) called the *universe* of \( A \) (sometimes denoted \( U(A) \)), and for each \( R \in \sigma \) a relation \( R^A \subseteq A^{\text{ar}(R)} \). The *size* of \( A \) is \(|A| := |A|\). Note that, according to these definitions, all considered signatures and structures are *finite*, signatures are *relational* (i.e. they do not contain any constants or function symbols), and may contain relation symbols of arity 0 (the only two 0-ary relations over a set \( A \) are \( \emptyset \) and \( \{(\)\}\)).

Let \( \sigma' \) be a signature with \( \sigma' \supseteq \sigma \). A *\( \sigma' \)-expansion* of a \( \sigma \)-structure \( A \) is a \( \sigma' \)-structure \( \mathcal{B} \) with universe \( B \) such that \( B = A \) and \( R^\mathcal{B} = R^A \) for every \( R \in \sigma \). If \( \mathcal{B} \) is a \( \sigma' \)-expansion of \( A \), then \( A \) is called the *\( \sigma \)-reduct* of \( \mathcal{B} \). A *substructure* of a \( \sigma \)-structure \( A \) is a \( \sigma \)-structure \( B \) with a universe \( B \subseteq A \) and \( R^B \subseteq R^A \) for all \( R \in \sigma \). For a \( \sigma \)-structure \( A \) and a non-empty set \( B \subseteq A \), we write \( A[B] \) to denote the *induced substructure* of \( A \) on \( B \), i.e. the \( \sigma \)-structure with universe \( B \) and \( R^A[B] = R^A \cap B^{\text{ar}(R)} \) for every \( R \in \sigma \).

The *Gaifman graph* \( G_A \) of a \( \sigma \)-structure \( A \) is the graph with vertex set \( A \) and an edge between two distinct vertices \( a, b \in A \) iff there exists \( R \in \sigma \) and a tuple \( \langle a_1, \ldots, a_{\text{ar}(R)} \rangle \in R^A \) such that \( a, b \in \{a_1, \ldots, a_{\text{ar}(R)}\} \). The structure \( A \) is *connected* if \( G_A \) is connected; the *connected components* of \( A \) are the connected components of \( G_A \). The *degree* of \( A \) is the degree of \( G_A \), i.e. the maximum number of neighbours of a vertex of \( G_A \). The *distance* \( \text{dist}^A(a, b) \) between two elements \( a, b \in A \) is the minimal number of edges of a path from \( a \) to \( b \) in \( G_A \); if no such path exists, we set \( \text{dist}^A(a, b) := \infty \). For a
tuple $\bar{a} = (a_1, \ldots, a_k) \in A^k$ and an element $b \in A$, we let $\text{dist}^A(\bar{a}, b) := \min_{i \in [k]} \text{dist}(a_i, b)$, and for a tuple $\bar{b} = (b_1, \ldots, b_\ell)$, we let $\text{dist}(\bar{a}, \bar{b}) := \min_{j \in [\ell]} \text{dist}(\bar{a}, \bar{b}_j)$.

For every $r \geq 0$, the $r$-ball of $\bar{a}$ in $A$ is the set $N^A_r(\bar{a}) = \{ b \in A : \text{dist}^A(\bar{a}, b) \leq r \}$. The $r$-neighbourhood of $\bar{a}$ in $A$ is the structure $N^A_r(\bar{a}) := A[N^A_r(\bar{a})]$.

### 3 Weight Aggregation Logic

This section introduces our new logic, which we call first-order logic with weight aggregation. It is inspired by the counting logic $\text{FOC}$ and its fragment $\text{FOC}_1$, as introduced in [14, 9], as well as the logic $\text{FO}[\mathbb{C}]$ and its fragment $\text{FO}[\mathbb{C}]_1$, which were recently introduced by Toruńczyk in [22]. Similarly as in [22], we consider weighted structures, which extend ordinary relational structures by assigning a weight, i.e. an element of a particular group or ring, to tuples present in the structure. The syntax and semantics of our logic, however, are closer in spirit to the syntax and semantics of the logic $\text{FOC}_1$, since this will enable us to achieve locality results similar to those obtained in [15, 9].

**Weighted structures.** Let $\sigma$ be a signature. Let $S$ be a collection of rings and/or abelian groups. Let $W$ be a finite set of weight symbols, such that each $w \in W$ has an associated arity $\text{ar}(w) \in \mathbb{N}_{\geq 1}$ and a type $\text{type}(w) \in S$. A $(\sigma, W)$-structure is a $\sigma$-structure $A$ that is enriched, for every $w \in W$, by an interpretation $w^A : A^{\text{ar}(w)} \to \text{type}(w)$, which satisfies the following locality condition: if $w^A(a_1, \ldots, a_k) \neq 0_S$ for $S := \text{type}(w)$, $k := \text{ar}(w)$ and $(a_1, \ldots, a_k) \in A^k$, then $k = 1$ or $a_1 = \cdots = a_k$ or there exists an $R \in \sigma$ and a tuple $(b_1, \ldots, b_{\text{ar}(R)}) \in R^A$ such that $\{a_1, \ldots, a_k\} \subseteq \{b_1, \ldots, b_{\text{ar}(R)}\}$. All notions that were introduced in Section 2 for $\sigma$-structures carry over to $(\sigma, W)$-structures in the obvious way.

We will use the following as running examples throughout this section.

**Example 3.1.** (a) Consider an online marketplace that allows retailers to sell their products to consumers. The database of the marketplace contains a table with transactions, and each entry consists of an identifier, a customer, a product, a retailer, the price per item, and the number of items sold. We can describe the database of the marketplace as a weighted structure as follows. Let $(\mathbb{Q}, +, \cdot)$ be the field of rationals, let $W$ contain two unary weight symbols $\text{price}$ and $\text{quantity}$ of type $(\mathbb{Q}, +, \cdot)$, let $\sigma = \{T\}$, and let $A$ be a $(\sigma, W)$-structure such that the universe $A$ contains the identifiers for the transactions, customers, products, and retailers. For every transaction, let $T^A$ contain the 4-tuple $(i, c, p, r)$ consisting of the identifier for the transaction, the customer, the product, and the retailer. For every transaction identifier $i$, let $\text{price}^A(i)$ be the price per item in the transaction and $\text{quantity}^A(i)$ be the number of items sold.

(b) In a recent survey [19], Pan and Ding describe different approaches to represent social media users via embeddings into a low-dimensional vector space, where the embeddings are based on the users’ social media posts.\footnote{Among other applications, such embeddings might be used to predict a user’s personality or political leaning.}

We represent the available
data by a weighted structure $A$ as follows. Consider the group $(\mathbb{R}^k, +)$, where $\mathbb{R}^k$ is the set of $k$-dimensional real vectors and $+$ is the usual vector addition, and let $W$ contain a unary weight symbol $\text{embedding}$ of type $(\mathbb{R}^k, +)$. Let $\sigma = \{ F \}$ and let $A$ be a $(\sigma, W)$-structure such that the universe $A$ consists of the users of a social network. Let $F^A$ contain all pairs of users $(a, b)$ such that $a$ is a follower of $b$. For every user $a \in A$, let $\text{embedding}^A(a)$ be a $k$-dimensional vector representing $a$’s social media posts.

(c) Consider vertex-coloured edge-weighted graphs, where $R, B, G$ are unary relations of red, blue, and green vertices, $E$ is a binary relation of edges, and where every edge $(a, b)$ has an associated weight that is a $k$-dimensional vector of reals (for some fixed number $k$). Such graphs can be viewed as $(\sigma, W)$-structures $A$, where $\sigma = \{ E, R, B, G \}$. $W$ contains a binary weight symbol $w$ of type $(\mathbb{R}^k, +)$ and $w^A(a, b) \in \mathbb{R}^k$ for all edges $(a, b) \in E^A$.

Fix a countably infinite set vars of variables. A $(\sigma, W)$-interpretation $I = (A, \beta)$ consists of a $(\sigma, W)$-structure $A$ and an assignment $\beta : \text{vars} \rightarrow A$. For $k \in \mathbb{N}_{>1}$, elements $a_1, \ldots, a_k \in A$, and $k$ distinct variables $y_1, \ldots, y_k$, we write $I^{a_1, \ldots, a_k}_{y_1, \ldots, y_k}$ for the interpretation $(A, \beta^{a_1, \ldots, a_k}_{y_1, \ldots, y_k})$, where $\beta^{a_1, \ldots, a_k}_{y_1, \ldots, y_k}$ is the assignment $\beta'$ with $\beta'(y_i) = a_i$ for every $i \in [k]$ and $\beta'(z) = \beta(z)$ for all $z \in \text{vars} \setminus \{ y_1, \ldots, y_k \}$.

The weight aggregation logic FOWA and its restrictions FOWA$_1$ and FOWA$_i$. Let $\sigma$ be a signature, $\mathcal{S}$ a collection of rings and/or abelian groups, and $W$ a finite set of weight symbols. An $\mathcal{S}$-predicate collection is a 4-tuple $(\mathcal{P}, \text{ar}, \text{type}, \text{[]} )$ where $\mathcal{P}$ is a countable set of predicate names and, to each $P \in \mathcal{P}$, $\text{ar}$ assigns an arity $\text{ar}(P) \in \mathbb{N}_{>1}$, $\text{type}$ assigns a type $\text{type}(P) \in \mathcal{S}^{\text{ar}(P)}$, and $\text{[]}$ assigns a semantics $\text{[]} P \subseteq \text{type}(P)$. For the remainder of this section, fix an $\mathcal{S}$-predicate collection $(\mathcal{P}, \text{ar}, \text{type}, \text{[]} )$.

For every $S \in \mathcal{S}$ that is not a ring but just an abelian group, a $W$-product of type $S$ is either an element $s \in S$ or an expression of the form $w(y_1, \ldots, y_k)$ where $w \in W$ is of type $S$, $k = \text{ar}(w)$, and $y_1, \ldots, y_k$ are $k$ pairwise distinct variables in vars. For every ring $S \in \mathcal{S}$, a $W$-product of type $S$ is an expression of the form $t_1 \cdots t_\ell$ where $\ell \in \mathbb{N}_{>1}$ and for each $i \in [\ell]$ either $t_i \in S$ or there exists a $w \in W$ with type($w$) $= S$ and there exist $k := \text{ar}(w)$ pairwise distinct variables $y_1, \ldots, y_k$ in vars such that $t_i = w(y_1, \ldots, y_k)$. By vars($p$) we denote the set of all variables that occur in a $W$-product $p$.

Example 3.2. Recall Example 3.1(a)–(c), and let $x$ and $y$ be variables. Examples of $W$-products are $\text{price}(x) \cdot \text{quantity}(x)$, $\text{embedding}(x)$, and $w(x, y)$. The logic we will define next is capable of expressing the following statements.

(a) Given a first-order formula $\varphi_{\text{group}}(p)$ that defines products of a certain product group based on the structure of their transactions, we can describe the amount of money a consumer $c$ paid on the specified product group via the $S$-term

$$t_{\text{spending}}(c) := \sum \text{price}(i) \cdot \text{quantity}(i) \cdot \exists p \exists r (\varphi_{\text{group}}(p) \land T(i, c, p, r)).$$

This term associates with every consumer $c$ the sum of the product of $\text{price}(i)$ and $\text{quantity}(i)$ for all transaction identifiers $i$ for which there exists a product $p$ and a
retailer \( r \) such that the tuple \((i, c, p, r)\) belongs to the transaction table and \( \varphi_{\text{group}}(p) \) holds. The \( S \)-term
\[
t_{\text{sales}} := \sum \text{price}(i) \cdot \text{quantity}(i) \cdot \exists c \exists p \exists r \left( \varphi_{\text{group}}(p) \land T(i, c, p, r) \right)
\]
specifies the amount all customers have paid on products from the product group. We might want to select the “heavy hitters”, i.e. all customers \( c \) for whom \( t_{\text{spending}}(c) > 0.01 \cdot t_{\text{sales}} \) holds. In our logic, this is expressed by the formula
\[
P > (t_{\text{spending}}(c), 0.01 \cdot t_{\text{sales}})
\]
where \( P > \) is a predicate name of type \((\mathbb{Q}, +, \cdot) \times (\mathbb{Q}, +, \cdot)\) with \( \left[ P > \right] = \{(r, s) \in \mathbb{Q}^2 : r > s\}\).

(b) For vectors \( u, v \in \mathbb{R}^k \), let \( d(u, v) \) denote the Euclidean distance between \( u \) and \( v \). We might want to use a formula \( \varphi_{\text{similar}}(x, y) \) expressing that the two \( k \)-dimensional vectors associated with persons \( x \) and \( y \) have Euclidean distance at most 1. To express this in our logic, we can add the rational field \((\mathbb{Q}, +, \cdot)\) to the collection \( S \) and use a predicate name \( P_{\text{ED}} \) of arity 3 and type \((\mathbb{R}^k, +) \times (\mathbb{R}^k, +) \times (\mathbb{Q}, +, \cdot)\) with \( \left[ P_{\text{ED}} \right] = \{(u, v, q) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{Q} : d(u, v) \leq q\} \). Then,
\[
\varphi_{\text{similar}}(x, y) := P_{\text{ED}}(\text{embedding}(x), \text{embedding}(y), 1)
\]
is a formula with the desired meaning.

(c) For each vertex \( x \), the sum of the weights of edges between \( x \) and its blue neighbours is specified by the \( S \)-term \( t_B(x) := \sum w(x', y). (x' = x \land E(x', y) \land B(y)) \).

We have designed the definition of the syntax of our logic in a way particularly suitable for formulating and proving the locality results that are crucial for obtaining our learning results. To obtain a more user-friendly syntax, i.e. which allows to read and construct formulas in a more intuitive way, one could of course introduce syntactic sugar that allows to explicitly write statements of the form
\[
\begin{align*}
& t_{\text{spending}}(c) > 0.01 \cdot t_{\text{sales}} \quad \text{instead of} \quad P > (t_{\text{spending}}(c), 0.01 \cdot t_{\text{sales}}) \\
& d(\text{embedding}(x), \text{embedding}(y)) \leq 1 \quad \text{instead of} \quad P_{\text{ED}}(\text{embedding}(x), \text{embedding}(y), 1) \\
& \sum y w(x, y). (E(x, y) \land B(y)) \quad \text{instead of} \quad \sum w(x', y). (x' = x \land E(x', y) \land B(y)).
\end{align*}
\]

We now define the precise syntax and semantics of our weight aggregation logic.

**Definition 3.3** (FOWA(\( \mathbb{P} \))[\( \sigma, S, W \)]). For FOWA(\( \mathbb{P} \))[\( \sigma, S, W \)], the set of *formulas* and *\( S \)-terms* is built according to the following rules:

1. \( x_1 = x_2 \) and \( R(x_1, \ldots, x_{ar(R)}) \) are *formulas*, where \( R \in \sigma \) and \( x_1, x_2, \ldots, x_{ar(R)} \) are variables\(^3\)

\(^3\)In particular, if \( ar(R) = 0 \), then \( R() \) is a formula.
(2) If $w \in W$, $S = \text{type}(w)$, $s \in S$, $k = \text{ar}(w)$, and $\bar{x} = (x_1, \ldots, x_k)$ is a tuple of $k$ pairwise distinct variables, then $(s = w(\bar{x}))$ is a formula.

(3) If $\varphi$ and $\psi$ are formulas, then $\neg \varphi$ and $(\varphi \lor \psi)$ are also formulas.

(4) If $\varphi$ is a formula and $y \in \text{vars}$, then $\exists y \varphi$ is a formula.

(5) If $\varphi$ is a formula, $w \in W$, $S = \text{type}(w)$, $s \in S$, $k = \text{ar}(w)$, and $\bar{y} = (y_1, \ldots, y_k)$ is a tuple of $k$ pairwise distinct variables, then $(s = \sum w(\bar{y}).\varphi)$ is a formula.

(6) If $P \in \mathbb{P}$, $m = \text{ar}(P)$, and $t_1, \ldots, t_m$ are $S$-terms such that $(\text{type}(t_1), \ldots, \text{type}(t_m)) = \text{type}(P)$, then $P(t_1, \ldots, t_m)$ is a formula.

(7) For every $S \in S$ and every $s \in S$, $s$ is an $S$-term of type $S$.

(8) For every $S \in S$, every $w \in W$ of type $S$, and every tuple $(x_1, \ldots, x_k)$ of $k := \text{ar}(w)$ pairwise distinct variables in $\text{vars}$, $w(x_1, \ldots, x_k)$ is an $S$-term of type $S$.

(9) If $t_1$ and $t_2$ are $S$-terms of the same type $S$, then so are $(t_1 + t_2)$ and $(t_1 - t_2)$; furthermore, if $S$ is a ring (and not just an abelian group), then also $(t_1 \cdot t_2)$ is an $S$-term of type $S$.

(10) If $\varphi$ is a formula, $S \in S$, and $p$ is a $W$-product of type $S$, then $\sum p.\varphi$ is an $S$-term of type $S$.

Let $I = (A, \beta)$ be a $(\sigma, W)$-interpretation. For every formula or $S$-term $\xi$ of FOWA($\mathbb{P})[\sigma, S, W]$, the semantics $[\xi]^I$ is defined as follows.

1. $[x_1 = x_2]^I = 1$ if $a_1 = a_2$, and $[x_1 = x_2]^I = 0$ otherwise; $[R(x_1, \ldots, x_{\text{ar}(R)})]^I = 1$ if $(a_1, \ldots, a_{\text{ar}(R)}) \in R^A$, and $[R(x_1, \ldots, x_{\text{ar}(R)})]^I = 0$ otherwise; where $a_j := \beta(x_j)$ for $j \in \{1, \ldots, \max\{2, \text{ar}(R)\}\}$.

2. $[(s = w(\bar{x}))]^I = 1$ if $s = w^A(\beta(x_1), \ldots, \beta(x_k))$, and $[(s = w(\bar{x}))]^I = 0$ otherwise.

3. $[\neg \varphi]^I = 1 - [\varphi]^I$ and $[(\varphi \lor \psi)]^I = \max\{[\varphi]^I, [\psi]^I\}$.

4. $[\exists y \varphi]^I = \max\{[\varphi]^I : a \in A\}$.

5. $[(s = \sum w(\bar{y}).\varphi)]^I = 1$ if $s = \sum S\{w^A(\bar{a}) : \bar{a} = (a_1, \ldots, a_k) \in A^k\}$ with $[\varphi]^I_{w_1 \ldots w_k} = 1$ (as usual, by convention, we let $\sum S X = 0_S$ if $X = \emptyset$).

6. $[P(t_1, \ldots, t_m)]^I = 1$ if $(t_1]^I, \ldots, [t_m]^I) \in [P]$, and $[P(t_1, \ldots, t_m)]^I = 0$ otherwise.

7. $[s]^I = s$.

8. $[w(x_1, \ldots, x_k)]^I = w^A(\beta(x_1), \ldots, \beta(x_k))$.

9. $[(t_1 * t_2)]^I = [t_1]^I *_{S} [t_2]^I$, for $* \in \{+, -, \cdot\}$. 

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contains a unary weight symbol with the following restrictions:

\[ \sum_{i} p_i \varphi_i^T = \sum_{A} \{ [p_i]^T_{y_1 \ldots y_k} : a_1, \ldots, a_k \in A \text{ with } [\varphi_i]^T_{y_1 \ldots y_k} = 1 \}, \] where \( \{y_1, \ldots, y_k\} = \text{vars}(p) \) and \( k = |\text{vars}(p)| \) and \( [p]^T = [t_1]^T \cdots [t_k]^T \) if \( p = t_1 \cdots t_k \) is of type \( S \).

An expression is a formula or an \( S \)-term. As usual, for a formula \( \varphi \) and a \((\sigma, W)\)-interpretation \( I \), we will often write \( I \models \varphi \) to indicate that \( [\varphi]^T = 1 \). Accordingly, \( I \not\models \varphi \) indicates that \( [\varphi]^T = 0 \).

The set \( \text{vars}(\xi) \) of an expression \( \xi \) is defined as the set of all variables in \( \text{vars} \) that occur in \( \xi \). The free variables \( \text{free}(\xi) \) of \( \xi \) are defined as follows: \( \text{free}(\xi) = \text{vars}(\xi) \) if \( \xi \) is built according to one of the rules \([1], [2], [7], [8] \): free(¬\( \varphi \)) = free(\( \varphi \)), free((\( \varphi \lor \psi \))) = free(\( \varphi \)) ∪ free(\( \psi \)), free(∃\( y \) \( \varphi \)) = free(\( \varphi \)) \( \setminus \{y\} \), free((s = \( \sum w(y_1, \ldots, y_k) \varphi \))) = free(\( \varphi \)) \( \setminus \{y_1, \ldots, y_k\} \); free(\( P(t_1, \ldots, t_m) \)) = \( \bigcup_{i=1}^{m} \text{free}(t_i) \); free(\( (t_1 \ast t_2) \)) = free(\( t_1 \)) ∪ free(\( t_2 \)) for \( \ast \in \{+, -, \cdot\} \); free(\( \sum p \cdot \varphi \)) = free(\( \varphi \)) \( \setminus \text{vars}(p) \). As usual, we will write \( \xi(\bar{x}) \) for \( \bar{x} = (x_1, \ldots, x_k) \) to indicate that \( \text{free}(\xi) \subseteq \{x_1, \ldots, x_k\} \). A sentence is a FOWA(\( \mathcal{P} \))\( [\sigma, S, W] \)-formula \( \varphi \) with free(\( \varphi \)) = \{0\}. A ground \( S \)-term is an \( S \)-term \( t \) of FOWA(\( \mathcal{P} \))\( [\sigma, S, W] \) with free(\( t \)) = \{0\}.

For a \((\sigma, W)\)-structure \( \mathcal{A} \) and a tuple \( \bar{a} = (a_1, \ldots, a_k) \in A^k \), we write \( \mathcal{A} \models \varphi[\bar{a}] \) or \( (\mathcal{A}, \bar{a}) \models \varphi \) to indicate that for every assignment \( \beta: \text{vars} \to A \) with \( \beta(x_i) = a_i \) for all \( i \in [k] \), we have \( I \models \varphi \), for \( I = (\mathcal{A}, \beta) \). Similarly, for an \( S \)-term \( t[\bar{x}] \) we write \( t[\bar{a}] \) to denote \( [t]^I \).

**Definition 3.4 (FOWA1 and FOW1).** The set of formulas and \( S \)-terms of the logic FOWA1(\( \mathcal{P} \))\( [\sigma, S, W] \) is built according to the same rules as for the logic FOWA(\( \mathcal{P} \))\( [\sigma, S, W] \), with the following restrictions:

- \([5]_1 \): rule \([5] \) can only be applied if \( S \) is finite,
- \([6]_1 \): rule \([6] \) can only be applied if \( |\text{free}(t_1) \cup \cdots \cup \text{free}(t_m)| \leq 1 \).

FOW1(\( \mathcal{P} \))\( [\sigma, S, W] \) is the restriction of FOWA1(\( \mathcal{P} \))\( [\sigma, S, W] \) where rule \([10] \) cannot be applied.

Note that first-order logic FO(\( \sigma \)) is the restriction of FOW1(\( \mathcal{P} \))\( [\sigma, S, W] \) where only rules \([1], [3] \), and \([4] \) can be applied. As usual, we write \( (\varphi \land \psi) \) and \( \forall y \varphi \) as shorthands for \( \neg(\neg \varphi \lor \neg \psi) \) and \( \neg \exists y \neg \varphi \). The quantifier rank \( \text{qr}(\xi) \) of a FOWA(\( \mathcal{P} \))\( [\sigma, S, W] \)-expression \( \xi \) is defined as the maximum nesting depth of constructs using rules \([1] \) and \([3] \) in order to construct \( \xi \). The aggregation depth \( d_{ag}(\xi) \) of \( \xi \) is defined as the maximum nesting depth of term constructions using rule \([10] \) in order to construct \( \xi \).

**Remark 3.5.** FOW1 can be viewed as an extension of first-order logic with modulo-counting quantifiers: if \( S \) contains the abelian group \( \mathbb{Z}/m\mathbb{Z}, + \) for some \( m \geq 2 \), and \( W \) contains a unary weight symbol \( \text{one}_m \) of type \( \mathbb{Z}/m\mathbb{Z} \) such that \( \text{one}_m(a) = 1 \) for all \( a \in A \), then the modulo \( m \) counting quantifier \( \exists^i \text{mod } m \varphi \) (stating that the number of interpretations for \( y \) that satisfy \( \varphi \) is congruent to \( i \) modulo \( m \)) can be expressed in FOW1(\( \mathcal{P} \))\( [\sigma, S, W] \) via \( (i = \sum \text{one}_m(y) \cdot \varphi) \).

FOW1 can be viewed as an extension of the logic FOC1 of \([9] \): if \( S \) contains the integer ring \( \mathbb{Z}, +, \cdot \) and \( W \) contains a unary weight symbol \( \text{one} \) of type \( \mathbb{Z} \) such that \( \text{one}(a) = 1 \),
for all \( a \in A \) on all considered \( (\sigma, W) \)-structures \( A \), then the counting term \(#(y_1, \ldots, y_k) \cdot \varphi\)

of \( \text{FOC}_1 \) (which counts the number of tuples \((y_1, \ldots, y_k)\) that satisfy \(\varphi\)) can be expressed in \( \text{FOWA}_1(\mathbb{P})[\sigma, S, W] \) via the \( S \)-term \( \sum p \varphi \) for \( p := \text{one}(y_1) \cdots \text{one}(y_k) \).

Let us mention, again, that we have designed the precise definition of the syntax of our logic in a way particularly suitable for formulating and proving the locality results that are crucial for obtaining our learning results. To obtain a more user-friendly syntax, i.e. which allows to read and construct formulas in a more intuitive way, it would of course make sense to introduce syntactic sugar that allows to explicitly write statements of the form

- \( #(y_1, \ldots, y_k) \cdot \varphi \) instead of \( \sum p \varphi \) for \( p := \text{one}(y_1) \cdots \text{one}(y_k) \)

- \( (#(y) \cdot \varphi \equiv i \mod m) \) or \( \exists i \mod m y \varphi \) instead of \( (i = \sum \text{one}_m(y) \cdot \varphi) \).

For this, one would tacitly assume that \( S \) contains \((\mathbb{Z}, +, \cdot)\) (or \((\mathbb{Z}/m\mathbb{Z}, +)\)) and \( W \) contains a unary weight symbol \( \text{one} \) of type \( \mathbb{Z} \) (or \( \text{one}_m \) of type \( \mathbb{Z}/m\mathbb{Z} \)) where \( \text{one}^{\text{A}}(a) = 1 \) (= \( \text{one}_m^{\text{A}}(a) \)) for every \( a \in A \) and every considered \( (\sigma, W) \)-structure \( A \).

To close this section, we return to the running examples from Examples 3.1 and 3.2

**Example 3.6.** We use the syntactic sugar introduced at the end of Remark 3.5

(a) The number of consumers who bought products \( p \) from the product group defined by \( \varphi_{\text{group}}(p) \) is specified by the \( S \)-term

\[
t_{\#\text{cons}} := \sum \text{one}(c) \cdot \exists i \exists p \exists r \left( \varphi_{\text{group}}(p) \land T(i, c, p, r) \right);
\]

and using the syntactic sugar described above, this \( S \)-term can be expressed via \( #(c) \cdot \exists i \exists p \exists r \left( \varphi_{\text{group}}(p) \land T(i, c, p, r) \right) \).

The consumers \( c \) who spent at least as much as the average consumer on the products \( p \) satisfying \( \varphi_{\text{group}}(p) \) can be described by the formula

\[
\varphi_{\text{spending}}(c) := P_{\geq} \left( (t_{\text{spending}}(c) \cdot t_{\#\text{cons}}), t_{\text{sales}} \right),
\]

where \( P_{\geq} \) is a binary predicate in \( P \) of type \( \mathbb{Q} \times \mathbb{Q} \) that is interpreted by the \( \geq \)-relation. To improve readability, one could introduce syntactic sugar that allows to express this as \( t_{\text{spending}}(c) \geq t_{\text{sales}} / t_{\#\text{cons}} \). The formula \( \varphi_{\text{spending}}(c) \) belongs to \( \text{FOWA}_1(\mathbb{P})[\sigma, S, W] \).

(b) The term \( t_{\#\text{follows}}(x) := #(y) \cdot F(x, y) \) specifies the number of users \( y \) followed by person \( x \). The term \( t_{\text{sum}}(x) := \sum \text{embedding}(y) \cdot F(x, y) \) specifies the sum of the vectors associated with all users \( y \) followed by \( x \). To describe the users \( x \) whose embedding is \( \delta \)-close (for some fixed \( \delta > 0 \)) to the average of the embeddings of users they follow\footnote{Depending on the target of the embeddings, this could mean that the user mostly follows users with a very similar personality or political leaning.} we might want to use a formula \( \varphi_{\text{close}}(x) \) of the form

\[
d \left( \text{embedding}(x), \frac{1}{t_{\#\text{follows}}(x)} \cdot t_{\text{sum}}(x) \right) < \delta.
\]
We can describe this in FOWA\(_1(\mathcal{P})[\sigma, S, W]\) by the formula

\[
\varphi_{\text{close}}(x) := P_{\text{dist}<\delta}(\text{embedding}(x), t_{\#\text{follows}}(x), t_{\text{sum}}(x)),
\]

where \(P_{\text{dist}<\delta}\) is a ternary predicate in \(\mathcal{P}\) of type \(\mathbb{R}^k \times \mathbb{Z} \times \mathbb{R}^k\) consisting of all triples \((\bar{v}, \ell, \bar{w})\) with \(\ell > 0\) and \(d(\bar{v}, \frac{1}{\ell} \cdot \bar{w}) < \delta\).

(c) Recall the term \(t_B(x)\) introduced in Example 3.2 [6] that specifies the sum of the weights of edges between \(x\) and its blue neighbours, and let \(t_R(x)\) be a similar term summing up the weights of edges between \(x\) and its red neighbours (using the syntactic sugar introduced at the end of Example 3.2, this can be described as \(\sum_y w(x, y) (E(x, y) \land R(y))\)). To specify the vertices \(x\) that have exactly 5 red neighbours, we can use the formula \(\varphi_{5\text{red}}(x) := (5 = \#(y). (E(x, y) \land R(y)))\). Let us now assume we are given a particular set \(H \subseteq \mathbb{R}^{2k}\) and we want to specify the vertices \(x\) that have exactly 5 red neighbours and for which, in addition, the \(2k\)-ary vector obtained by concatenating the \(k\)-ary vectors computed by summing up the weights of edges between \(x\) and its blue neighbours and by summing up the weights of edges between \(x\) and its red neighbours belongs to \(H\). To express this, we can use a binary predicate \(P\) of type \(\mathbb{R}^k \times \mathbb{R}^k\) with \([P] = \{(\bar{u}, \bar{v}) \in \mathbb{R}^k \times \mathbb{R}^k : (u_1, \ldots, u_k, v_1, \ldots, v_k) \in H\}\). Then, the FOWA\(_1(\mathcal{P})[\sigma, S, W]\)-formula \(\psi(x) := \varphi_{5\text{red}}(x) \land P(t_B(x), t_G(x))\) specifies the vertices \(x\) we are interested in.

4 Locality Properties of FOW\(_1\) and FOWA\(_1\)

We now summarise locality properties of FOW\(_1\) and FOWA\(_1\) that are similar to well-known locality properties of first-order logic FO and to locality properties of FOC\(_1\) achieved in [9]. This includes Feferman-Vaught decompositions (Section 4.1) and a Gaifman normal form for FOW\(_1\) (Section 4.2), and a localisation theorem for the more expressive logic FOWA\(_1\) (Section 4.3).

For the remainder of this section, let us fix a signature \(\sigma\), a collection \(S\) of rings and/or abelian groups, a finite set \(W\) of weight symbols, and an \(S\)-predicate collection \((\mathcal{P}, \text{ar}, \text{type}, [\ ]))\).

The notion of local formulas is defined as usual [10]: let \(r \in \mathbb{N}\). A FOWA\((\mathcal{P})[\sigma, S, W]\)-formula \(\varphi(\bar{x})\) with free variables \(\bar{x} = (x_1, \ldots, x_k)\) is \(r\)-local (around \(\bar{x}\)) if for every \((\sigma, W)\)-structure \(\mathcal{A}\) and all \(\bar{a} \in A^k\), we have \(\mathcal{A} \models \varphi[\bar{a}] \iff (\mathcal{N}^\mathcal{A}_r(\bar{a}) \models \varphi[\bar{a}]\). A formula is local if it is \(r\)-local for some \(r \in \mathbb{N}\).

For an \(r \in \mathbb{N}\), it is straightforward to construct an FO[\(\sigma\)]-formula \(\text{dist}_{\sigma}^r(x, y)\) such that for every \((\sigma, W)\)-structure \(\mathcal{A}\) and all \(a, b \in A\), we have \(\mathcal{A} \models \text{dist}_{\sigma}^r(a, b) \iff \text{dist}^\mathcal{A}(a, b) \leq r\). To improve readability, we write \(\text{dist}(x, y) \leq r\) for \(\text{dist}_{\sigma}^R(x, y)\), and \(\text{dist}(x, y) > r\) for \(\text{dist}_{\sigma}^R(x, y)\); and we omit the superscript \(\sigma\) when it is clear from the context. For a tuple \(\bar{x} = (x_1, \ldots, x_k)\) of variables, \(\text{dist}(\bar{x}, y) > r\) is a shorthand for \(\bigwedge_{i=1}^k \text{dist}(x_i, y) > r\), and \(\text{dist}(\bar{x}, y) \leq r\) is a shorthand for \(\bigvee_{i=1}^k \text{dist}(x_i, y) \leq r\). For \(\bar{y} = (y_1, \ldots, y_l)\), we use \(\text{dist}(\bar{x}; \bar{y}) > r\) and \(\text{dist}(\bar{x}; \bar{y}) \leq r\) as shorthands for \(\bigwedge_{j=1}^l \text{dist}(x_j, y_j) > r\) and \(\bigvee_{j=1}^l \text{dist}(x_j, y_j) \leq r\), respectively.
The $r$-localisation $\varphi^{(r)}$ of a FOWA$(\mathbb{P})[\sigma, \mathbb{S}, W]$-formula $\varphi(x)$ is the formula obtained from $\varphi$ by replacing every subformula of the form $\exists y \varphi'$ with the formula $\exists y (\varphi' \land \text{dist}(x, y) \leq r)$, replacing every subformula of the form $(s = \sum \overline{w}(y) \cdot \varphi')$, for $\overline{y} = (y_1, \ldots, y_k)$, with the formula $(s = \sum \overline{w}(y) \cdot (\varphi' \land \bigwedge_{j=1}^k \text{dist}(x, y_j) \leq r))$, and replacing every $S$-term of the form $\sum p. \varphi'$ with the $S$-term $\sum p. (\varphi' \land \bigwedge_{j=1}^k \text{dist}(x, y_j) \leq r)$, where $\{y_1, \ldots, y_k\} = \text{free}(\varphi')$. The resulting formula $\varphi^{(r)}(x)$ is $r$-local.

4.1 Feferman-Vaught Decomposition for FOW$_1$

We pick two new unary relation symbols $X, Y$ that do not belong to $\sigma$, and we let $\sigma' := \sigma \cup \{X, Y\}$.

Definition 4.1. Let $A, B$ be ($\sigma, W$)-structures with $A \cap B = \emptyset$. The disjoint sum $A \oplus B$ is the ($\sigma'$, $W$)-structure $C$ with universe $C = A \cup B$, $X^C = A$, $Y^C = B$, $R^C = R^A \cup R^B$ for all $R \in \sigma$, and such that for all $w \in W$ and $k := \text{ar}(w)$ and all $\overline{c} = (c_1, \ldots, c_k) \in C^k$, we have $w^C(\overline{c}) = w^A(\overline{c})$ if $\overline{c} \in A^k$, $w^C(\overline{c}) = w^B(\overline{c})$ if $\overline{c} \in B^k$, and $w^C(\overline{c}) = 0_S$ otherwise (for $S := \text{type}(w)$). The disjoint union $A \sqcup B$ is the ($\sigma, W$)-structure obtained from $C := A \oplus B$ by omitting the relations $X^C, Y^C$.

Definition 4.2. Let $L$ be a subset of FOWA$(\mathbb{P})[\sigma, \mathbb{S}, W]$. Let $k, \ell \in \mathbb{N}$ and let $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_\ell)$ be tuples of $k + \ell$ pairwise distinct variables. Let $\varphi$ be a FOWA$(\mathbb{P})[\sigma', \mathbb{S}, W]$-formula with free($\varphi$) $\subseteq \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$. A Feferman-Vaught decomposition of $\varphi$ in $L$ w.r.t. $(x; y)$ is a finite, non-empty set $\Delta$ of tuples of the form $(\alpha, \beta)$ where $\alpha, \beta \in L$ and free($\alpha$) $\subseteq \{x_1, \ldots, x_k\}$ and free($\beta$) $\subseteq \{y_1, \ldots, y_\ell\}$, such that the following is true for all ($\sigma, W$)-structures $A, B$ with $A \cap B = \emptyset$ and all $\overline{a} \in A^k$, $\overline{b} \in B^\ell$: $A \sqcup B \models \varphi[\overline{a}, \overline{b}] \iff$ there exists $(\alpha, \beta) \in \Delta$ such that $A \models \alpha[\overline{a}]$ and $B \models \beta[\overline{b}]$.

Our first main result provides Feferman-Vaught decompositions for FOW$_1$.

Theorem 4.3 (Feferman-Vaught decompositions for FOW$_1$(P)[sigma, S, W]).

Let $k, \ell \in \mathbb{N}$ and let $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_\ell)$ be tuples of $k + \ell$ pairwise distinct variables. For every FOW$_1$(P)[sigma', S, W]-formula $\varphi$ with free($\varphi$) $\subseteq \{x_1, \ldots, x_k, y_1, \ldots, y_\ell\}$, there exists a Feferman-Vaught decomposition $\Delta$ in $L$ of $\varphi$ w.r.t. $(x; y)$, where $L := L_\varphi$ is the class of all FOW$_1$(P)[sigma, S, W]-formulas of quantifier rank at most $\text{qr}(\varphi)$ which use only those $P \in P$ and $S \in S$ that occur in $\varphi$ and only those $S$-terms that occur in $\varphi$ or that are of the form $s$ for an $s \in S \in S$ where $S$ is finite and occurs in $\varphi$.

Furthermore, there is an algorithm that computes $\Delta$ upon input of $\varphi, x, y$.

The proof proceeds in a similar way as the proof of the Feferman-Vaught decomposition for first-order logic with modulo-counting quantifiers in [15]. Before presenting the theorem’s proof, let us formulate a straightforward corollary of Theorem 4.3.

Corollary 4.4. Let $k, \ell \in \mathbb{N}$ and let $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_\ell)$ be tuples of $k + \ell$ pairwise distinct variables. Upon input of an $r \in \mathbb{N}$ and an $r$-local FOW$_1$(P)[sigma, S, W]-formula $\varphi(x, y)$, one can compute a finite, non-empty set $\Delta$ of pairs $(\alpha(x), \beta(y))$ of
L-formulas, where L is the class of all r-localisations of formulas in the class L_ϕ of Theorem 4.3, such that the following two formulas are equivalent:

\[
\begin{align*}
\Delta &:= \{ \varphi(x, y) \} \\
\Delta &:= \{ \varphi(x, y) \}
\end{align*}
\]

- We know that each \( t \)
- Since (\( s = w(z_1, \ldots, z_m) \)). If \( \{z_1, \ldots, z_m\} \subseteq \{x_1, \ldots, x_k\} \), we can choose \( \Delta := \{(\varphi, \top)\} \).
- Otherwise, we know that \( \{z_1, \ldots, z_m\} \) contains variables from \( \vec{x} \) and variables from \( \vec{y} \); and if \( s = 0 \), we can choose \( \Delta := \{((\top, \top))\} \), and otherwise, we can choose \( \Delta := \{((\bot, \bot))\} \). It is straightforward to verify that \( \Delta \) is a Feferman-Vaught decomposition in FOW\(_1\)[P][σ, S, W] of \( \varphi \) w.r.t. \( (\vec{x}; \vec{y}) \).

For rule (1), let \( \varphi \) be of the form \( P(t_1, \ldots, t_m) \), where \( P \in P \) and \( t_1, \ldots, t_m \) are \( S \)-terms. We know that each \( t_i \) is built using the rules (7)–(10), and that there is one variable \( z \) such that \( \text{vars}(t_i) \subseteq \{z\} \) for all \( i \in [m] \). Thus, if \( z \in \{x_1, \ldots, x_k\} \), we can choose \( \Delta := \{(\varphi, \top)\} \); and if \( z \in \{y_1, \ldots, y_r\} \), we can choose \( \Delta := \{(\top, \varphi)\} \).

For the induction step, we consider formulas built according to the rules (3), (4), and (5) of Definitions 3.3 and 3.4. Rules (3) and (4) can be handled in exactly the same way as for first-order logic (cf., e.g., [2, 17, 5]). Rule (1) can be handled in exactly the same way as in the traditional Feferman-Vaught construction for first-order logic FO (cf., e.g., [2, 17, 5]).

For rule (5), let \( \varphi \) be of the form \( s = \psi(z) \), for a tuple of variables \( z = (z_1, \ldots, z_m) \) and a weight symbol \( w \in W \) whose type \( S \) := type(\( w \)) is finite. For every \( i \in S \), let

\[
\begin{align*}
\chi_i &:= \left( i = \sum w(z). (\psi \land \bigwedge_{j=1}^m X(z_j)) \right) \quad \text{and} \quad \theta_i := \left( i = \sum w(z). (\psi \land \bigwedge_{j=1}^m Y(z_j)) \right).
\end{align*}
\]

Let \( I := \{(i_1, i_2) \in S \times S : i_1 + S i_2 = s\} \). It is straightforward to see that for all \((σ, W)\)-structures \( A \) and \( B \) with \( A \cap B = \emptyset \) and all \( \bar{a} \in A^k \), \( \bar{b} \in B^r \), we have:

\[
(A \oplus B, \bar{a}, \bar{b}) \models (s) \iff (A \oplus B, \bar{a}, \bar{b}) \models \bigvee_{(i_1, i_2) \in I} (\chi_{i_1} \land \theta_{i_2}). \quad (1)
\]

Since \((\chi_{i_1} \land \theta_{i_2})\) is equivalent to \( \neg(\neg \chi_{i_1} \lor \neg \theta_{i_2}) \) and we already know how to handle formulas built using rule (4), we are done once we have shown the following:
Claim 1. For every \( i \in S \), one can compute Feferman-Vaught decompositions \( \Delta_{\chi_i} \) and \( \Delta_{\theta_i} \) in \( \text{FO} \{ \mathbb{P} \} \) of \( \chi_i \) and \( \theta_i \) w.r.t. \( (\bar{x};\bar{y}) \).

To prove the claim, fix an \( i \in S \). We show how to construct \( \Delta_{\theta_i} \) (the construction of \( \Delta_{\chi_i} \) is analogous). By the induction hypothesis, we can construct a Feferman-Vaught decomposition \( \Delta \) in \( \text{FO} \{ \mathbb{P} \} \) of \( \psi \) w.r.t. \( (\bar{x};\bar{y}) \). It is an easy exercise to see that, w.l.o.g., we can assume that the \( \alpha \)s in \( \Delta \) are mutually exclusive, i.e. for every two distinct \( (\alpha, \beta) \) and \( (\alpha', \beta') \) in \( \Delta \), the formula \( (\alpha \land \alpha') \) is unsatisfiable. Let \( \Delta' := \{ (\alpha, (i = \sum \bar{w}(\bar{z}), \bar{\beta})) : \{\alpha, \beta\} \in \Delta \} \). If \( i \neq 0 \), we let \( \Delta_{\theta_i} := \Delta' \). If \( i = 0 \), we have \( \Delta_{\theta_i} := \Delta \cup \{(\bigwedge_{\alpha \in A} \neg \alpha, \top)\} \), where \( A \) := \{\alpha : \text{there exists } \beta \text{ such that } (\alpha, \beta) \in \Delta \} \).

It remains to verify that \( \Delta_{\theta_i} \) is a Feferman-Vaught decomposition of \( \theta_i \). Consider arbitrary \( (\sigma, \mathbb{W}) \)-structures \( \mathbb{A} \) and \( \mathbb{B} \) with \( A \sqcap B = \emptyset \), and let \( \bar{a} \in A^k \), \( \bar{b} \in B^k \). By definition, we have \( \mathbb{A} \oplus \mathbb{B} \models \theta_i[\bar{a}, \bar{b}] \iff \exists i = \sum \{w^B(\bar{c}) : \bar{c} \in M\} \iff \mathbb{B} \models (i = \sum \{w(\bar{z}), \bar{\beta})\}{\bar{b}} \iff \exists \alpha', \beta' \in \Delta \) such that \( \mathbb{A} \models \alpha'[\bar{a}] \) and \( \mathbb{B} \models \beta'[\bar{b}] \).

We show that the two formulas \( \psi_1(\bar{x}, \bar{y}) \) and \( \psi_2(\bar{x}, \bar{y}) \) are equivalent. Let \( \mathbb{A} \) be an \( \text{FO} \{ \mathbb{P} \} \) structure. Using Theorem 4.3, we can compute a Feferman-Vaught decomposition \( \Delta' \) in \( \text{L}_{\varphi} \) of \( \varphi \) w.r.t. \( (\bar{x};\bar{y}) \). Let \( \Delta := \{(\alpha(\bar{x}), \beta(\bar{x}) : (\alpha, \beta) \in \Delta' \} \). We show that the two formulas

\[
\psi_1(\bar{x}, \bar{y}) := \left( \bigwedge_{i=1}^k \bigwedge_{j=1}^\ell \text{dist}(x_i, y_j) > 2r+1 \right) \land \varphi(\bar{x}, \bar{y})
\]

and

\[
\psi_2(\bar{x}, \bar{y}) := \left( \bigwedge_{i=1}^k \bigwedge_{j=1}^\ell \text{dist}(x_i, y_j) > 2r+1 \right) \land \bigvee_{(\alpha(\bar{x}), \beta(\bar{x})) \in \Delta} (\alpha(\bar{x}) \land \beta(\bar{y}))
\]

given in Corollary 4.4 are equivalent.
Let $A$ be a $(σ, W)$-structure, $a ∈ A^k$, and $b ∈ A^l$. If $\text{dist}(a, b) ≤ 2r+1$, then $A \models ψ_1[a, b]$ and $A \not\models ψ_2[a, b]$. Now let $\text{dist}(a, b) > 2r+1$. Then, since $φ$ is $r$-local, $A \models ψ_1[a, b]$ if and only if $N^A_r(a) ∪ N^A_r(b) \models φ[a, b]$. Thus, we obtain

$$A \models ψ_1[a, b] \iff N^A_r(a) ∪ N^A_r(b) \models φ[a, b]$$

$$\iff N^A_r(a) ⊕ N^A_r(b) \models φ[a, b]$$

$$\iff ∃(α, β) ∈ Δ' : N^A_1(α) ∩ N^A_1(β) \models β[b]$$

$$\iff ∃(α, β) ∈ Δ' : N^A_1(α) ⊕ N^A_1(β) \models β(γ[b])$$

We can switch between the disjoint sum and the disjoint union of structures because the considered formulas only use relations from the disjoint union. All in all, this shows that $ψ_1 \equiv ψ_2$.

4.2 Gaifman Normal Form for $FOW_1$

We now turn to a notion of Gaifman normal form for $FOW_1$.

**Definition 4.5.** A basic-local sentence in $FOW_1(ℙ)[σ, S, W]$ is a sentence of the form $∃x_1 ⋯ ∃x_ℓ \big( A_{1≤i<j≤ℓ} \text{dist}(x_i, x_j) > 2r ∧ A_{i=1}^{r} λ(x_i) \big)$, where $ℓ ∈ 2N$, $r ∈ N$, $λ(x)$ is an $r$-local $FOW_1(ℙ)[σ, S, W]$-formula, and $x_1, …, x_ℓ$ are $ℓ$ pairwise distinct variables.

A local aggregation sentence in $FOW_1(ℙ)[σ, S, W]$ is a sentence of the form $( s = ∑ w(ϕ).λ(ϕ) )$, where $w ∈ W$, $s ∈ S := \text{type}(w)$, $ℓ = ar(w)$, $ϕ = (y_1, …, y_ℓ)$ is a tuple of $ℓ$ pairwise distinct variables, and $λ(ϕ)$ is an $r$-local $FOW_1(ℙ)[σ, S, W]$-formula.

A $FOW_1(ℙ)[σ, S, W]$-formula in Gaifman normal form is a Boolean combination of local $FOW_1(ℙ)[σ, S, W]$-formulas, basic-local sentences in $FOW_1(ℙ)[σ, S, W]$, and local aggregation sentences in $FOW_1(ℙ)[σ, S, W]$.

Our next main theorem provides a Gaifman normal form for $FOW_1$.

**Theorem 4.6** (Gaifman normal form for $FOW_1(ℙ)[σ, S, W]$). Every $FOW_1(ℙ)[σ, S, W]$-formula $φ$ is equivalent to an $FOW_1(ℙ)[σ, S, W]$-formula $γ$ in Gaifman normal form with $\text{free}(γ) = \text{free}(φ)$. Furthermore, there is an algorithm that computes $γ$ upon input of $φ$.

The proof proceeds similarly as Gaifman’s original proof for first-order logic $FO$ ([3], see also [5] Sect. 4.1), but since subformulas are from $FOW_1(ℙ)[σ, S, W]$, we use Corollary 4.4 instead of Feferman-Vaught decompositions for $FO$ (cf. [5] Lemma 2.3). Furthermore, for formulas built according to rule [5]1, we proceed in a similar way as for the modulo-counting quantifiers in the Gaifman normal construction of [15].

The remainder of Section 4.2 is devoted to the proof of Theorem 4.6.
Proof of Theorem 4.6
The proof proceeds by induction on the construction of $\varphi$. The cases where formulas are built according to the rules (1), (2), (3) of Definition 3.3 are trivial. A formula $\varphi$ that is built according to rule (6) is of the form $P(t_1, \ldots, t_m)$, where $P \in \mathcal{P}$ and $t_1, \ldots, t_m$ are $S$-terms built using the rules (7)–(9) — thus, $\varphi$ is 0-local.

If $\varphi$ is of the form $\exists y \varphi'$, we can argue in the same way as in Gaifman’s original proof for first-order logic ([3], see also [5, Sect. 4.1]), but since $\varphi'$ is from $\text{FOW}_1(\mathcal{P}|\sigma, S, \mathcal{W})$, we use Corollary 4.4 instead of Feferman-Vaught decompositions for first-order logic (cf. [5 Lemma 2.3]).

For formulas built according to rule (5), of Definition 3.4 we proceed in a similar way as for the modulo-counting quantifiers in the Gaifman normal construction of [15]. Let $\varphi$ be of the form $(s = \sum w(\bar{y}) \cdot \varphi'(\bar{x}, \bar{y}))$, for a tuple of variables $\bar{y} = (y_1, \ldots, y_\ell)$ and a weight symbol $w \in \mathcal{W}$ whose type $S := \text{type}(w)$ is finite, and let $\bar{x} = (x_1, \ldots, x_k)$ be the free variables of $\varphi$ (note that $k$ might be 0). By the induction hypothesis, we can transform $\varphi'$ into an equivalent formula in Gaifman normal form, and we can assume w.l.o.g. that this formula is of the form $\bigwedge^n_{j=1} (\chi_j \land \lambda_j(\bar{x}, \bar{y}))$, where each $\chi_j$ is an $\text{FOW}_1(\mathcal{P}|\sigma, S, \mathcal{W})$-sentence in Gaifman normal form and each $\lambda_j(\bar{x}, \bar{y})$ is $r$-local, for some $r \in \mathbb{N}$. For every $J \subseteq [n]$, let

$$
\chi_J := \bigwedge_{j \in J} \chi_j \land \bigwedge_{j \in [n] \setminus J} \neg \chi_j \quad \text{and} \quad \lambda_J(\bar{x}, \bar{y}) := \bigvee_{j \in J} \lambda_j(\bar{x}, \bar{y}).
$$

Clearly, $\bigvee^n_{j=1} (\chi_j \land \lambda_j(\bar{x}, \bar{y}))$ is equivalent to $\bigvee_{\emptyset \neq J \subseteq [n]} (\chi_J \land \lambda_J(\bar{x}, \bar{y}))$, the $(\chi_J)_{J \subseteq [n]}$ are mutually exclusive sentences in Gaifman normal form, and $\lambda_J(\bar{x}, \bar{y})$ is $r$-local. Let

$$
\tilde{\varphi} := \bigvee_{\emptyset \neq J \subseteq [n]} \left( \chi_J \land (s = \sum w(\bar{y}) \cdot \lambda_J(\bar{x}, \bar{y})) \right).
$$

The following is straightforward to prove.

Claim 1. If $s \neq 0_S$, then $\varphi$ is equivalent to $\tilde{\varphi}$. If $s = 0_S$, then $\varphi$ is equivalent to $(\tilde{\varphi} \lor \chi_0)$.

To complete the proof of Theorem 4.6 it suffices to consider an arbitrary non-empty $J \subseteq [n]$ and the $r$-local formula $\lambda(\bar{x}, \bar{y}) := \lambda_J(\bar{x}, \bar{y})$ and show how to transform the formula $\psi(\bar{x}) := (s = \sum w(\bar{y}) \cdot \lambda(\bar{x}, \bar{y}))$ into an equivalent formula in Gaifman normal form. If $k = 0$, we are done since $\psi$ is a local aggregation sentence in $\text{FOW}_1(\mathcal{P}|\sigma, S, \mathcal{W})$. If $k > 0$, we proceed as follows. Let $r' := 2r+1$ and $I := \{(i_1, i_2) \in S \times S : i_1 + i_2 = s\}$. Then, $\psi(\bar{x})$ is equivalent to $\bigvee_{(i_1, i_2) \in I} (\psi'_{i_1} \land \psi''_{i_2})$, where

$$
\psi'_{i_1}(\bar{x}) := (i_1 = \sum w(\bar{y}) \cdot \left( \lambda(\bar{x}, \bar{y}) \land \left( \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r' \right) \right)),
$$

$$
\psi''_{i_2}(\bar{x}) := (i_2 = \sum w(\bar{y}) \cdot \left( \lambda(\bar{x}, \bar{y}) \land \left( \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r' \right) \right)).
$$
Note that the formula $\psi'_2(\bar{x})$ is local (namely, $(r'+1+r)$-local; this is because tuples $\bar{a}$ in a $(\sigma, W)$-structure $A$ with $w^A(\bar{a}) \neq 0_S$ must form a clique in the Gaifman graph of $A$).

It remains to transform $\psi''_2$ into an equivalent formula in Gaifman normal form. To achieve this, we use Corollary 4.4 to obtain a finite, non-empty set $\Delta$ of pairs $(\alpha(\bar{x}), \beta(\bar{y}))$ of $r$-local $\FOW_1(W)[\sigma, S, W]$-formulas such that 

$$
\bigl(\lambda(\bar{x}, \bar{y}) \land \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r'\bigr)
$$

is equivalent to 

$$
\bigl(\bigvee_{(\alpha, \beta) \in \Delta}(\alpha(\bar{x}) \land \beta(\bar{y})) \land \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r'\bigr).
$$

W.l.o.g., we can assume that the as in $\Delta$ are mutually exclusive, i.e., for any two distinct $(\alpha, \beta)$ and $(\alpha', \beta')$ in $\Delta$, the formula $\alpha \land \beta'$ is unsatisfiable. Thus, $\psi''_2(\bar{x})$ is equivalent to the formula 

$$
\tilde{\psi}_{i_2}(\bar{x}) := \bigvee_{(\alpha, \beta) \in \Delta} \left( \alpha(\bar{x}) \land \left( i_2 = \sum \mathcal{W}(\bar{y}).(\beta(\bar{y}) \land \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r'\right) \right).
$$

Let $A := \{ \alpha : \text{there exists } \beta \text{ such that } (\alpha, \beta) \in \Delta \}$. The following is straightforward to prove:

**Claim 2.** If $i_2 \neq 0_S$, then $\psi''_2(\bar{x})$ is equivalent to $\tilde{\psi}_{i_2}(\bar{x})$. If $i_2 = 0_S$, then $\psi''_2(\bar{x})$ is equivalent to 

$$
\left( \tilde{\psi}_{i_2}(\bar{x}) \lor \bigwedge_{\alpha \in A} \neg \alpha(\bar{x}) \right).
$$

To complete the proof of Theorem 4.6 it suffices to consider an arbitrary $r$-local formula $\beta(\bar{y})$ and transform the formula 

$$
\mu(\bar{x}) := \left( i_2 = \sum \mathcal{W}(\bar{y}).(\beta(\bar{y}) \land \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r'\right)
$$

into an equivalent $\FOW_1(W)[\sigma, S, W]$-formula in Gaifman normal form. This is not difficult: let $J := \{(j_1, j_2) \in S \times S : j_1 \neq j_2 = i_2\}$. Then, $\mu(\bar{x})$ is equivalent to 

$$
\bigvee_{(j_1, j_2) \in J} \left( \mu'_{j_1} \land \mu''_{j_2}(\bar{x}) \right),
$$

where

$$
\mu'_{j_1} := \left( j_1 = \sum \mathcal{W}(\bar{y}).\beta(\bar{y}) \right)
$$

and 

$$
\mu''_{j_2}(\bar{x}) := \left( j_2 = \sum \mathcal{W}(\bar{y}).(\beta(\bar{y}) \land \neg \left( \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{\ell} \text{dist}(x_i, y_j) > r'\right) \right).
$$

Now, $\mu'_{j_1}$ is a local aggregation sentence in $\FOW_1(W)[\sigma, S, W]$, and $\mu''_{j_2}$ is local (namely, $(r'+1+r)$-local; this is because tuples $\bar{a}$ in a $(\sigma, W)$-structure $A$ with $w^A(\bar{a}) \neq 0_S$ must form a clique in the Gaifman graph of $A$). This completes the proof of Theorem 4.6. \[\square\]

### 4.3 Localisation Theorem for $\FOWA_1$

Our next main theorem provides a locality result for the logic $\FOWA_1$, which is a logic substantially more expressive than $\FOW_1$.

**Theorem 4.7** (Localisation Theorem for $\FOWA_1$). For every $\FOWA_1(W)[\sigma, S, W]$-formula $\varphi(x_1, \ldots, x_k)$ (with $k \geq 0$), there is an extension $\sigma'$ of $\sigma$ with relation symbols of arity $\leq 1$, and a $\FOW_1(W)[\sigma', S, W]$-formula $\varphi'(x_1, \ldots, x_k)$ that is a Boolean combination of local formulas and statements of the form $R()$ where $R \in \sigma'$ has arity 0, for which
the following is true: there is an algorithm\footnote{with $P$- and $S$-oracles, so that operations $+_S$, $\cdot_S$ for $S \in S$ and checking if a tuple belongs to $[P]$ for $P \in P$ can be done in constant time} that, upon input of a $(\sigma, W)$-structure $\mathcal{A}$, computes in time $|\mathcal{A}| \cdot d^{O(1)}$, where $d$ is the degree of $\mathcal{A}$, a $\sigma_\cdot$-expansion $\mathcal{A}^\varphi$ of $\mathcal{A}$ such that for all $\bar{a} \in A^k$ it holds that $\mathcal{A}^\varphi \models \varphi[\bar{a}] \iff \mathcal{A} \models \varphi[\bar{a}]$.

The remainder of Section 4.3 is devoted to the proof of Theorem 4.7. Our approach is to decompose FOWA$_1$-expressions into simpler expressions that can be evaluated in a structure $\mathcal{A}$ by exploring for each element $a$ in the universe of $\mathcal{A}$ only a local neighbourhood around $a$. This is achieved by a decomposition theorem (Theorem 4.15), which is a generalisation of the decomposition for FOC$_1(P)$ provided in [9 Theorem 6.6].

4.3.1 Connected local terms

The following well-known lemma summarises easy facts concerning neighbourhoods.

**Lemma 4.8.** Let $\mathcal{A}$ be a $(\sigma, W)$-structure, $r \geq 0$, $k \geq 1$, and $\bar{a} = (a_1, \ldots, a_k) \in A^k$.

$N_r^A(a_1, a_2)$ is connected $\iff$ dist$^A(a_1, a_2) \leq 2r + 1$.

If $N_r^A(\bar{a})$ is connected, then $N_r^A(a) \subseteq N_r^{A_{r+(k-1)(2r+1)}}(a_i)$, for each $i \in [k]$.

For every $k \in \mathbb{N}_{\geq 1}$, we let $\mathcal{G}_k$ be the set of all undirected graphs $G$ with vertex set $[k]$. For a graph $G \in \mathcal{G}_k$, a number $r \in \mathbb{N}$, and a tuple $\bar{y} = (y_1, \ldots, y_k)$ of $k$ pairwise distinct variables, we consider the formula

$$\delta^G_{r} \mathcal{G}_k \bar{y} := \bigwedge_{\{i,j\} \in E(G)} \text{dist}^G(y_i, y_j) \leq r \land \bigwedge_{\{i,j\} \notin E(G)} \text{dist}^G(y_i, y_j) > r.$$  

Note that $\mathcal{A} \models \delta^G_{2r+1} \mathcal{G}_k \bar{a}$ means that the connected components of the $r$-neighbourhood $N_r^A(\bar{a})$ correspond to the connected components of $G$. Clearly, the formula $\delta^G_{2r+1} \mathcal{G}_k \bar{y}$ is $r$-local around its free variables $\bar{y}$.

The main ingredient of our decomposition of FOWA$_1(P)[\sigma, S, W]$-expressions are the connected local terms (cl-terms, for short), defined as follows.

**Definition 4.9 (cl-Terms).** Let $r \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 1}$.

A **basic cl-term (of radius $r$ and width $k$)** is an $S$-term of the form

$$\sum p \cdot (\psi(y_1, \ldots, y_k) \land \delta^G_{2r+1} \mathcal{G}_k \bar{y})$$

where $\text{vars}(p) \subseteq \{y_1, \ldots, y_k\}$, $\bar{y} = (y_1, \ldots, y_k)$ is a tuple of $k$ pairwise distinct variables, $\psi(y_1, \ldots, y_k)$ is an FOWA$_1(P)[\sigma, S, W]$-formula that is $r$-local around $\bar{y}$, and $G \in \mathcal{G}_k$ is connected. A **cl-term (of radius $\leq r$ and width $\leq k$)** is built from basic cl-terms (of radius $\leq r$ and width $\leq k$) by using rules (7)–(9) of Definition 3.3.

Note that cl-terms are “easy” with respect to query evaluation in the following sense.
Lemma 4.10. For every fixed cl-term $t(z_1, \ldots, z_\ell)$ (with $\ell \geq 0$), there is an algorithm which, upon input of a $(\sigma, \mathbf{W})$-structure $\mathcal{A}$, can compute, within precomputation time $|\mathcal{A}| \cdot d^{O(1)}$ where $d$ is the degree of $\mathcal{A}$, a data structure that, whenever given a tuple $(a_1, \ldots, a_\ell) \in A^\ell$, returns the value $t^A[a_1, \ldots, a_\ell]$ in constant time.

Proof. It suffices to prove the lemma for basic cl-terms. The statement for general cl-terms then follows by induction. Consider a basic cl-term $u(z_1, \ldots, z_\ell)$ of the form $\sum p.(\psi(y_1, \ldots, y_k) \land \delta_G^{\sigma, \ell+1}(y_1, \ldots, y_k))$. Recall from Definition 4.9 that $G$ is a connected graph and $\{z_1, \ldots, z_\ell\} \subseteq \{y_1, \ldots, y_k\}$. Let $S \in \mathbb{S}$ be the type of the $\mathbf{W}$-product $p$. We can assume w.l.o.g. that $(z_1, \ldots, z_\ell) = (y_1, \ldots, y_k)$. Consequently, $\text{vars}(p) = \{y_{\ell+1}, \ldots, y_k\}$.

Given a $(\sigma, \mathbf{W})$-structure $\mathcal{A}$ and an element $c_1 \in A$, we can explore the $R$-neighbourhood of $c_1$ for $R := r + (k-1)(2r+1)$ (cf. Lemma 4.8) and thereby compute the set $M_{c_1}$ of all $\bar{a} = (a_1, \ldots, a_k) \in A^k$ with $a_1 = c_1$ such that $(\mathcal{A}, \bar{a}) \models (\psi \land \delta_G^{\sigma, \ell+1})$. For each such tuple $\bar{a}$, we compute and store the value $v_{\bar{a}} := p^A[a_{\ell+1}, \ldots, a_k] \in S$. Then, we group the tuples in $M_{c_1}$ by their prefix $(a_1, \ldots, a_\ell)$ of length $\ell$, and for each group, we compute the $+\text{-sum}$ $s_{c_1(a_1, \ldots, a_\ell)}$ of the values $v_{\bar{a}}$ of all tuples $\bar{a} \in M_{c_1}$ that have the same prefix $(a_1, \ldots, a_\ell)$.

In case that $\ell = 0$, $u$ is a ground term and we have $u^A = \sum_S \{s_{c_1(\cdot)} : c_1 \in A\}$. In case that $\ell \geq 1$, whenever given an arbitrary tuple $(a_1, \ldots, a_\ell) \in A^\ell$, we can determine $v^A[a_1, \ldots, a_\ell]$ as follows: let $c_1 := a_1$, if $M_{c_1}$ contains a tuple with prefix $(a_1, \ldots, a_\ell)$ then $v^A[a_1, \ldots, a_\ell] = s_{c_1(a_1, \ldots, a_\ell)}$, and otherwise $v^A[a_1, \ldots, a_\ell] = 0_S$.

Thus, upon input of a $(\sigma, \mathbf{W})$-structure $\mathcal{A}$, we can, within precomputation time $|\mathcal{A}| \cdot d^{O(1)}$ where $d$ is the degree of $\mathcal{A}$, compute a data structure which, whenever given a tuple $(a_1, \ldots, a_\ell) \in A^\ell$, returns the value $v^A[a_1, \ldots, a_\ell]$ in constant time. 

Our decomposition of FOWA$_1(\mathcal{P})[\sigma, \mathbb{S}, \mathbf{W}]$-expressions proceeds by induction on the construction of the input expression. The main technical tool for the construction is the following lemma.

Lemma 4.11. Let $r \geq 0$, $k \geq 1$, and let $\vec{y} = (y_1, \ldots, y_k)$ be a tuple of $k$ pairwise distinct variables. Let $\psi(\vec{y})$ be an FOWA$_1(\mathcal{P})[\sigma, \mathbb{S}, \mathbf{W}]$-formula that is $r$-local, and consider an $\mathbb{S}$-term $u(z_1, \ldots, z_m)$ of the form $\sum p.\psi(y_1, \ldots, y_k)$, where $p$ is a $\mathbf{W}$-product, $m \geq 0$, and $\{z_1, \ldots, z_m\} \subseteq \{y_1, \ldots, y_k\}$. There exists a cl-term $\hat{u}(z_1, \ldots, z_m)$ of radius $\leq r$ and width $\leq k$, such that $\hat{u}^A[\vec{a}] = u^A[\vec{a}]$ holds for every $(\sigma, \mathbf{W})$-structure $\mathcal{A}$ and every $\vec{a} \in A^m$. Furthermore, there is an algorithm which, upon input of $r$ and $u$, constructs $\hat{u}$.

Proof. For a $(\sigma, \mathbf{W})$-structure $\mathcal{A}$ and a formula $\theta(\vec{y})$, we consider the set
\[
S^A_\theta := \{ \vec{a} = (a_1, \ldots, a_k) \in A^k : \mathcal{A} \models \theta[\vec{a}] \}.
\]
Note that for every graph $G \in \mathcal{G}_k$, the formula
\[
\psi_G(\vec{y}) := \psi(\vec{y}) \land \delta_G^{\sigma, 2r+1}(\vec{y})
\]
is $r$-local around $\vec{y}$. Furthermore, for every $(\sigma, \mathbf{W})$-structure $\mathcal{A}$, the set $S^A_\psi$ is the disjoint union of the sets $S^A_{\psi_G}$ for all $G \in \mathcal{G}_k$. Therefore, $u$ is equivalent to the $+\text{-sum}$, over all
To complete the proof, it therefore suffices to show that, for every $G \in \mathcal{G}_k$, the $S$-term $u^\psi_G := \sum p_\psi(y_1, \ldots, y_k)$. To complete the proof of Lemma 4.11, it therefore suffices to show that, for every $G \in \mathcal{G}_k$, the $S$-term $u^\psi_G$ is equivalent to a cl-term of radius $r$. We prove this by an induction on the number of connected components of $G$. Precisely, we show that the following statement $(\ast)_c$ is true for every $c \in \mathbb{N}_{\geq 1}$.

$(\ast)_c$: For every $k \geq c$, for every tuple $\vec{y} = (y_1, \ldots, y_k)$ of $k$ pairwise distinct variables, for every $r \geq 0$, for every FOW$_1(\mathcal{P}[\sigma, S, W]$-formula $\psi(\vec{y})$ that is $r$-local around $\vec{y}$, for every $W$-product $p$ with $\text{vars}(p) \subseteq \{y_1, \ldots, y_k\}$, and for every graph $G \in \mathcal{G}_k$ that has at most $c$ connected components, the $S$-term $u^\psi_G := \sum p_\psi(y_1, \ldots, y_k)$ is equivalent to a cl-term of radius $r$.

The induction base for $c = 1$ is trivial: it involves only connected graphs $G$, for which by Definition 4.9, $u^\psi_G$ is a basic cl-term.

For the induction step from $c$ to $c+1$, consider a $k \geq c+1$ and a graph $G = (V, E) \in \mathcal{G}_k$ that has $c+1$ connected components. Let $V'$ be the set of all vertices of $V$ that are connected to the vertex 1, and let $V'' := V \setminus V'$.

Let $G' := G[V']$ and $G'' := G[V'']$ be the induced subgraphs of $G$ on $V'$ and $V''$, respectively. Clearly, $G$ is the disjoint union of $G'$ and $G''$, $G'$ is connected, and $G''$ has $c$ connected components.

To keep notation simple, we assume (without loss of generality) that $V' = \{1, \ldots, \ell\}$ and $V'' = \{\ell+1, \ldots, k\}$ for an $\ell$ with $1 \leq \ell < k$. For a tuple $\vec{v} = (v_1, \ldots, v_k)$, we let $\vec{v}' := (v_1, \ldots, v_\ell)$ and $\vec{v}'' := (v_{\ell+1}, \ldots, v_k)$.

Now consider a number $r \geq 0$ and the formula $\delta_{G,2r+1}^\sigma(\vec{y})$ for $\vec{y} = (y_1, \ldots, y_k)$. For every $\sigma$-structure $A$ and every tuple $\vec{a} = (a_1, \ldots, a_k) \in A^k$ with $A \models \delta_{G,2r+1}^\sigma(\vec{a})$, the $r$-neighbourhood $N^A_\sigma(\vec{a})$ is the disjoint union of the $r$-neighbourhoods $N^A_\sigma(\vec{a}')$ and $N^A_\sigma(\vec{a}'')$.

Let $\psi(\vec{y})$ be an FOW$_1(\mathcal{P}[\sigma, S, W]$-formula that is $r$-local. By using Corollary 4.4, we can compute a decomposition of $\psi(\vec{y})$ into a formula $\hat{\psi}(\vec{y})$ of the form

$$\bigvee_{i \in I} \left( \psi_i'(\vec{y}') \land \psi_i''(\vec{y}'') \right),$$

where $I$ is a finite non-empty set, each $\psi_i'(\vec{y}')$ is an FOW$_1(\mathcal{P}[\sigma, S, W]$-formula that is $r$-local around $\vec{y}'$, each $\psi_i''(\vec{y}'')$ is an FOW$_1(\mathcal{P}[\sigma, S, W]$-formula that is $r$-local around $\vec{y}''$, and for every $(\sigma, W)$-structure $A$ and every $\vec{a} \in A^k$ with $A \models \delta_{G,2r+1}^\sigma(\vec{a})$, the following is true: there exists at most one $i \in I$ such that $(A, \vec{a}) \models \left( \psi_i'(\vec{y}') \land \psi_i''(\vec{y}'') \right)$, and $A \models \psi[\vec{a}] \iff A \models \hat{\psi}[\vec{a}]$. This implies that the set $S^A_{\hat{\psi}^G}$ is the disjoint union of the sets $S^A_{\psi_i' \land \psi_i'' \land \delta_{G,2r+1}^\sigma}$ for all $i \in I$.

Now let $p$ be an arbitrary $W$-product with $\text{vars}(p) \subseteq \{y_1, \ldots, y_k\}$, and consider the $S$-term $u^\psi_G := \sum p_\psi(y_1, \ldots, y_k)$. From the above reasoning, it follows that $u^\psi_G$ is equivalent to the $+$-sum, over all $i \in I$, of the $S$-terms

$$u^\psi_{G,i} := \sum p_\psi'(\vec{y}') \land \psi_i''(\vec{y}'') \land \delta_{G,2r+1}^\sigma(\vec{y})).$$

To complete the proof, it suffices to show that $u^\psi_{G,i}$ is equivalent to a cl-term of radius $r$. 

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By the definition of the formula $\delta^\sigma_{G,2r+1}(\bar{y})$, we obtain that the formula $\psi_i'(\bar{y}') \land \psi_i''(\bar{y}'') \land \delta^\sigma_{G,2r+1}(\bar{y})$ is equivalent to the formula
\[
\psi_i'(\bar{y}') \land \delta^\sigma_{G',2r+1}(\bar{y}') \land \left( \psi_i''(\bar{y}'') \land \delta^\sigma_{G'',2r+1}(\bar{y}'') \land \bigwedge_{j'' \in V''} \text{dist}^\sigma(y_{j'}, y_{j''}) > 2r+1 \right).
\]
(2)

Therefore, for every $(\sigma, W)$-structure $A$, we have
\[
S^A_{\psi_i' \land \psi_i'' \land \delta^\sigma_{G,r}} = \left( S^A_\theta \times S^A_\theta' \right) \setminus T^A, \quad \text{for} \quad T^A := \left\{ \bar{a} \in A^k : A \models \theta'[^{\bar{a}}_i], \ A \models \theta''[^{\bar{a}''}_i], (A, \bar{a}) \not\models \bigwedge_{j'' \in V''} \text{dist}^\sigma(y_{j'}, y_{j''}) > 2r+1 \right\}.
\]

Let $H$ be the set of all graphs $H \in G_k$ with $H \neq G$, but $H[V'] = G'$ and $H[V''] = G''$. Clearly, every $H \in H$ has at most $c$ connected components. Furthermore, for every $(\sigma, W)$-structure $A$, the set $T^A$ is the disjoint union over all $H \in H$ of the sets
\[
T^A_H := \left\{ \bar{a} \in A^k : A \models \theta'[^{\bar{a}}_i], \ A \models \theta''[^{\bar{a}''}_i], A \models \delta^\sigma_H,2r+1[\bar{a}] \right\}.
\]

Now let us have a closer look at the $W$-product $p$ used in the $S$-term $u_G^{\psi,i}$. We let $Y' := \text{vars}(p) \cap \{y_1, \ldots, y_k\}$ and $Y'' := \text{vars}(p) \cap \{y_{k+1}, \ldots, y_l\}$.

Case 1: $p$ contains a factor $\omega(\bar{z})$ for some $\omega \in W$ and a tuple $\bar{z}$ that contains variables from both $Y'$ and $Y''$. Then, for every $(\sigma, W)$-structure $A$ and every $\bar{a} \in S^A_{\psi_i' \land \psi_i'' \land \delta^\sigma_{G,r}}$, we know that $[p]^{(A,\bar{a})} = 0_S$, where $S \in S$ is the type of $p$. Hence, $u_G^{\psi,i}$ is equivalent to the $S$-term $0_S$, and we are done.

Case 2: If case 1 does not apply, then $p$ is of the form $p'_1 \cdot p'_2 \cdots \cdot p'_j \cdot p''_j$, where $j \geq 1$ and $\text{vars}(p'_i) \subseteq Y'$ and $\text{vars}(p''_j) \subseteq Y''$ for each $i \in [j]$.

Now let us consider an arbitrary $(\sigma, W)$-structure $A$ and fix an assignment $\beta$ to the free variables of $u_G^{\psi,i}$. Evaluating $u_G^{\psi,i}$ in $(A, \beta)$ means computing the value
\[
s^{(A,\beta)} := \sum_S \left\{ [p]^{(A,\bar{a})} : \bar{a} \in S^A_{\psi_i' \land \psi_i'' \land \delta^\sigma_{G,r}} \text{ such that } \bar{a} \text{ agrees with } \beta \text{ on free}(u_G^{\psi,i}) \right\}.
\]

We already know that $S^A_{\psi_i' \land \psi_i'' \land \delta^\sigma_{G,r}} = (S^A_\theta \times S^A_\theta') \setminus (\bigcup_{H \in H} T^A_H)$, where the sets $T^A_H$ for $H \in H$ are pairwise disjoint and contained in $S^A_\theta \times S^A_{\theta'}$. Therefore, $s^{(A,\beta)} = \sum_S \left\{ [p]^{(A,\bar{a})} : \bar{a} \in S^A_\theta \times S^A_{\theta'} \text{ such that } \bar{a} \text{ agrees with } \beta \text{ on free}(u_G^{\psi,i}) \right\}$.

Furthermore, since $p = p'_1 \cdot p''_1 \cdots \cdot p'_j \cdot p''_j$, we obtain that
\[
\sum_S \left\{ [p]^{(A,\bar{a})} : \bar{a} \in S^A_\theta \times S^A_{\theta'} \text{ such that } \bar{a} \text{ agrees with } \beta \text{ on free}(u_G^{\psi,i}) \right\} = 
\]
\[
\prod_{i=1}^j \left( \sum_S \left\{ [p'_i]^{(A,\bar{a}')} : \bar{a}' \in S^A_\theta \text{ such that } \bar{a}' \text{ agrees with } \beta \text{ on free}(\theta') \setminus \text{vars}(p') \right\} \right) \cdot \sum_S \left\{ [p'']^{(A,\bar{a}'') : \bar{a}'' \in S^A_{\theta'} \text{ such that } \bar{a}'' \text{ agrees with } \beta \text{ on free}(\theta'') \setminus \text{vars}(p'') \right\} \right) .
\]
Therefore, $u_{G}^{\psi.J}$ is equivalent to
\[
\prod_{i=1}^{\hat{J}} \left( \frac{\sum_{i} \delta_{i}'(\bar{y}') \cdot \left( \sum_{i} \delta_{i}''(\bar{y}'') \right) - \sum_{H \in H} \sum_{H} p.(\delta_{i}'(\bar{y}') \land \delta_{i}''(\bar{y}'') \land \delta_{H,2i+1}(\bar{y}))}{=} t'_{i}, \quad t''_{i}, \quad t_{H}} \right)
\]
By the induction hypothesis $(\ast)_{c}$, each of the terms $t'_{i}$, $t''_{i}$, and $t_{H}$ is equivalent to a cl-term of radius $r$. Hence, also $u_{G}^{\psi.J}$ is equivalent to a cl-term of radius $r$. This completes the proof of Lemma 4.11.

As an easy consequence of Lemma 4.11 we obtain

**Lemma 4.12.** Let $s \geq 0$ and let $\chi_{1}, \ldots, \chi_{s}$ be arbitrary sentences that can be evaluated in $(\sigma, W)$-structures. Let $r \geq 0$, $k \geq 1$, and let $\bar{y} = (y_{1}, \ldots, y_{k})$ be a tuple of $k$ pairwise distinct variables. Let $\varphi(\bar{y})$ be a Boolean combination of the sentences $\chi_{1}, \ldots, \chi_{s}$ and of $\text{FOW}_{1}(\mathcal{P})[\sigma, S, W]$-formulas that are $r$-local around their free variables $\bar{y}$. Consider an $S$-term $u(z_{1}, \ldots, z_{m})$ of the form $\sum p.\varphi(y_{1}, \ldots, y_{k})$, where $p$ is a $W$-product, $m \geq 0$, and $\{z_{1}, \ldots, z_{m}\} \subseteq \{y_{1}, \ldots, y_{k}\}$. For every $J \subseteq [s]$, there is a cl-term $\hat{u}_{J}$ (of radius $\leq r$ and width $\leq k$) such that for every $(\sigma, W)$-structure $A$, there is exactly one set $J \subseteq [s]$ such that

\[
\mathcal{A} \models p.\chi_{J} := \bigcap_{J \subseteq [s]} \chi_{J} \land \bigcap_{J \subseteq [s]} \neg \chi_{J}.
\]

and for this set $J$, we have $\hat{u}_{J}^{A}[\bar{a}] = u^{A}[\bar{a}]$ for every $\bar{a} \in A^{m}$. Furthermore, there is an algorithm which upon input of $r$, $u$, and $J$ constructs $\hat{u}_{J}$.

**Proof.** We can assume w.l.o.g. that $\varphi(\bar{y})$ is of the form

\[
\bigvee_{J \subseteq [s]} \left( \chi_{J} \land \psi_{J}(\bar{y}) \right)
\]

where, for each $J \subseteq [s]$, $\psi_{J}(\bar{y})$ is an $\text{FOW}_{1}(\mathcal{P})[\sigma, S, W]$-formula that is $r$-local around its free variables $\bar{y}$.

For every $J \subseteq [s]$ let $\hat{u}_{J}$ be the cl-term obtained by Lemma 4.11 for the term $u_{J} := \sum p.\psi_{J}(\bar{y})$. Recall that $u = \sum p.\varphi(\bar{y})$.

Now consider an arbitrary $J \subseteq [s]$ and a $\sigma$-structure $\mathcal{A}$ with $\mathcal{A} \models \chi_{J}$. Clearly, for every $\bar{a} \in A^{m}$ we have

\[
u^{A}[\bar{a}] = (\sum p.\psi_{J}(\bar{y}))^{A}[\bar{a}] = \hat{u}_{J}^{A}[\bar{a}].
\]

Hence, the proof of Lemma 4.12 is complete.

---

5We do not restrict attention to $\text{FOW}_{1}(\mathcal{P})[\sigma, S, W]$-sentences here—the $\chi_{j}$s may be sentences of any logic, e.g., $\text{FOWA}(\mathcal{P})[\sigma, S, W]$. 

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4.3.2 A connected local normalform for FOW

By combining Lemma 4.11 with the Gaifman locality Theorem 4.13, we obtain the following normal form for FOW, which may be of independent interest. From now on, we assume that whenever S contains the integer ring (Z, +, ·), there is a weight symbol one ∈ W of type Z such that, in every (σ, W)-structure A that we consider, we have one(a) = 1 ∈ Z for all a ∈ A.

Theorem 4.13 (cl-Normalform). Let S contain the integer ring (Z, +, ·). Every formula ϕ̄(x) of FOW1(σ, S, W) is equivalent to a Boolean combination of FOW1(σ, S, W)-formulas ϕ(x) that are local around their free variables x, of local aggregation sentences in FOW1(σ, S, W), and of statements of the form “y ≥ 1”, for a ground cl-term g of type Z.

Furthermore, there is an algorithm which transforms an input FOW1(σ, S, W)-formula ϕ(x) into an equivalent such formula ϕ′(x) and outputs the radius of each ground cl-term in ϕ′ as well as a number r such that every local formula in ϕ′ is r-local.

Proof. By Theorem 4.16, it suffices to translate a basic local sentence into a statement of the form “y ≥ 1” for a ground cl-term g of type Z.

For a basic local sentence χ := ∃y1 · · · ∃yk ϑ(y1, . . . , yk) with ϑ(y1, . . . , yk) :=

\[ \bigwedge_{1 \leq i < j \leq k} \text{dist}^\sigma(y_i, y_j) > 2r \land \bigwedge_{1 \leq i \leq k} \psi(y_i), \]

let gχ be the ground term gχ := \( \sum p.\vartheta(y_1, \ldots, y_k) \) for p := one(y1) · · · one(yk).

Note that \( \vartheta(y_1, \ldots, y_k) \) is r-local around its free variables. Hence, by Lemma 4.11, we obtain a ground cl-term ̂gχ such that ̂gχA = gχA for every (σ, W)-structure A. Furthermore, A |= χ ⇐⇒ ̂gχA ≥ 1 ⇐⇒ ̂gχA ≥ 1. This completes the proof of Theorem 4.13.

We use the notion cl-normalform to denote the formulas ϕ′(x) provided by Theorem 4.13. Note that cl-normalforms do not necessarily belong to FOW1(σ, S, W), but can be viewed as formulas in FOWA(σ, S, W), where there contains a unary predicate P≥1 of type Z with \([P_{≥1}] := N_{≥1}\). Then, statements of the form “y ≥ 1” can be expressed via P≥1(g).

4.3.3 A decomposition of FOWA1-expressions

Our decomposition of FOWA1(σ, S, W) utilises Theorem 4.13 and is based on an induction on the maximal nesting depth of term constructions of the form \( \sum p.\psi \) (i.e., constructions by rule (10) of Definition 3.3). We call this nesting depth the aggregation depth (for short: ag-depth) dag(ξ) of a given formula or term ξ. Formally, dag(ξ) is defined as follows:

1. \( d_{\text{ag}}(\varphi) := 0 \), if \( \varphi \) is a formula of the form \( x_1 = x_2 \) or \( R(x_1, \ldots, x_{\text{at}(R)}) \)
2. \( d_{\text{ag}}((s = w(x))) := 0 \)

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Also, there is an algorithm which transforms an input formula $\phi$ into such a Boolean combination of

\begin{align*}
(3) \quad d_{\text{ag}}(\neg \varphi) & := d_{\text{ag}}(\varphi) \quad \text{and} \quad d_{\text{ag}}((\varphi \lor \psi)) := \max\{d_{\text{ag}}(\varphi), d_{\text{ag}}(\psi)\} \\
(4) \quad d_{\text{ag}}(\exists y \varphi) & := d_{\text{ag}}(\varphi) \\
(5) \quad d_{\text{ag}}((s = \sum \bar{w}(\bar{y}).\varphi)) & := d_{\text{ag}}(\varphi) \\
(6) \quad d_{\text{ag}}(P(t_1, \ldots, t_m)) & := \max\{d_{\text{ag}}(t_1), \ldots, d_{\text{ag}}(t_m)\}, \\
(7) \quad d_{\text{ag}}(s) & := 0, \quad \text{for all } s \in S \\
(8) \quad d_{\text{ag}}(\bar{w}(\bar{x})) & := 0 \\
(9) \quad d_{\text{ag}}((t_1 * t_2)) & := \max\{d_{\text{ag}}(t_1), d_{\text{ag}}(t_2)\}, \quad \text{for } * \in \{+, -, \} \\
(10) \quad d_{\text{ag}}(\sum p.\varphi) & := d_{\text{ag}}(\varphi) + 1.
\end{align*}

The base case of our decomposition of $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$ is provided by the following lemma. The proof utilises Theorem 4.13.

**Lemma 4.14.** Let $S$ contain the integer ring $(\mathbb{Z}, +, \cdot)$. Let $\varphi$ be a $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$-formula of the form $P(t_1, \ldots, t_m)$ with $P \in \mathbb{P}$, $m = \text{ar}(P)$, and where $t_1, \ldots, t_m$ are $S$-terms of ag-depth at most 1. Then, $\varphi$ is equivalent to a Boolean combination of

(i) formulas of the form $P(t'_1, \ldots, t'_m)$, for cl-terms $t'_1, \ldots, t'_m$ with $\text{free}(t'_i) = \text{free}(t_i)$ for all $i \in [m]$, 

(ii) local aggregation sentences in $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$, and 

(iii) statements of the form “$g \geq 1$” for ground cl-terms $g$ of type $\mathbb{Z}$.

Also, there is an algorithm which transforms an input formula $\varphi$ into such a Boolean combination $\varphi'$ and which outputs the radius of each cl-term and each local formula in $\varphi'$.

**Proof.** From Definition 3.4 we know that either $\text{free}(\varphi) = \emptyset$ or $\text{free}(\varphi) = \{x\}$ holds for a variable $x$. Furthermore, we know that for every $i \in [m]$, the $S$-term $t_i$ is built by using rules (7)–(9) and $S$-terms $\theta'$ of the form $\sum p.\theta$, for a $W$-product $p$ such that $\text{free}(\theta) \setminus \text{vars}(p) \subseteq \{x\}$. Let $\Theta'$ be the set of all these $S$-terms $\theta'$ and let $\Theta$ be the set of all the according formulas $\theta$.

By assumption, we have $d_{\text{ag}}(\varphi) \leq 1$. Therefore, every $\theta \in \Theta$ has ag-depth 0. Thus, each such $\theta$ is an $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$-formula. By Theorem 4.13, for each $\theta \in \Theta$, we obtain an equivalent formula $\varphi(\theta)$ in cl-normalform. Let $\Phi$ be the set of all these $\varphi(\theta)$.

For each $\theta$ in $\Theta$, the formula $\varphi(\theta)$ is a Boolean combination of (a) $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$-formulas that are local around the free variables of $\theta$, and (b) local aggregation sentences in $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$, and (c) statements of the form “$g \geq 1$” for a ground cl-term $g$ of type $\mathbb{Z}$.

Let $\chi_1, \ldots, \chi_s$ be a list of all statements of the forms (b) or (c), such that each formula in $\Phi$ is a Boolean combination of statements in $\{\chi_1, \ldots, \chi_s\}$ and of $\text{FOWA}_1(\mathbb{P})[\sigma, S, W]$-formulas that are local around their free variables. For every $J \subseteq [s]$ let $\chi_J := \bigwedge_{j \in J} \chi_j \land \bigwedge_{j \notin [s] \setminus J} \neg \chi_j$. 

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Let \( r \in \mathbb{N} \) be such that each of the local FOWA \((P)[\sigma, S, W]\)-formulas that occur in a formula in \( \Phi \) is \( r \)-local around its free variables. For each \( \theta' \) in \( \Theta' \) of the form \( \sum p.\theta \), we apply Lemma 4.12 to the term

\[
t'(\theta') := \sum p.\varphi(\theta)
\]

and obtain for every \( J \subseteq [s] \) a cl-term \( i'_J(\theta') \) for which the following is true:

- If free(\( \theta' \)) = \( \emptyset \), then \( (\theta')^A = (i'_J(\theta'))^A \) for every \((\sigma, W)\)-structure \( A \) with \( A \models \chi_J \).
- If free(\( \theta' \)) = \( \{x\} \), then \( (\theta')^A[a] = (i'_J(\theta'))^A[a] \) for every \((\sigma, W)\)-structure \( A \) with \( A \models \chi_J \) and every \( a \in A \).

Thus, for each \( J \subseteq [s] \), we have

\[
(\chi_J \land P(t_1,\ldots,t_m)) \equiv (\chi_J \land P(t_{1,J},\ldots,t_{m,J}))
\]

where, for every \( i \in [m] \), we let \( t_{i,J} \) be the cl-term obtained from \( t_i \) by replacing each occurrence of a term \( \theta' \in \Theta' \) with the term \( i'_J(\theta') \). In summary, we obtain the following:

\[
\varphi = P(t_1,\ldots,t_m) \equiv \bigvee_{J \subseteq [s]} (\chi_J \land P(t_1,\ldots,t_m)) \\
\equiv \bigvee_{J \subseteq [s]} (\chi_J \land P(t_{1,J},\ldots,t_{m,J})) =: \varphi'.
\]

The formula \( \chi_J \) is a Boolean combination of local aggregation sentences in FOWA \((P)[\sigma, S, W]\) and of statements of the form “\( g \geq 1 \)” for ground cl-terms \( g \) of type \( Z \). Furthermore, all terms \( t_{i,J} \) are cl-terms with free(\( t_{i,J} \)) \( \subseteq \) free(\( t_i \)), and we can easily modify them to achieve that free(\( t_{i,J} \)) = free(\( t_i \)). Thus, the proof of Lemma 4.14 is complete.

We are now ready for the decomposition theorem for FOWA \(_1\), which can be viewed as a generalisation of the decomposition theorem for FOC \(_1\) provided in [9].

**Theorem 4.15** (Decomposition of FOWA \(_1\)). Let \( S \) contain the integer ring \((\mathbb{Z},+,-,\cdot)\). Let \( z \) be a fixed variable in \( \text{vars} \). For every \( d \in \mathbb{N} \) and every FOWA \((P)[\sigma, S, W]\)-formula \( \varphi(\bar{x}) \) of ag-depth \( d_{ag}(\varphi) = d \), there exists a sequence \( (L_1,\ldots,L_{d+1},\varphi') \) with the following properties.

1. \( L_i = (\tau_i, \iota_i) \), for every \( i \in \{1,\ldots,d+1\} \), where
   - \( \tau_i \) is a finite set of relation symbols of arity \( \leq 1 \) that do not belong to \( \sigma_{i-1} := \sigma \cup \bigcup_{j < i} \tau_j \), and
   - \( \iota_i \) is a mapping that associates with every symbol \( R \in \tau_i \) a formula \( \iota_i(R) \)
     - (i) of the form \( P(t_1,\ldots,t_m) \), where \( P \in \mathbb{P} \), \( m = \text{ar}(P) \), and \( t_1,\ldots,t_m \) are cl-terms of signature \( \sigma_{i-1} \), such that free(\( t_j \)) \( \subseteq \{z\} \) for each \( j \in [m] \), or
Moreover, there is an algorithm which constructs such a sequence $D$ of statements of the form "$g \geq 1$" for a ground cl-term $g$ of signature $\sigma_{i-1}$ and of type $Z$.

If $R$ has arity 0, then $\iota(R)$ has no free variable. If $R$ has arity 1, then $z$ is the unique free variable of $\iota_z(R)$ (thus, $\iota_z(R)$ is of the form (i)).

**(II)** $\varphi'(\bar{x})$ is a Boolean combination of (A) FOW$_1(\mathbb{P})[\sigma_{d+1}, \mathcal{S}, \mathcal{W}]$-formulas $\psi(\bar{x})$ that are local around their free variables $\bar{x}$, where $\sigma_{d+1} := \sigma \cup \bigcup_{1 \leq i \leq d+1} \tau_i$, and (B) statements of the form $R()$ where $R$ is a 0-ary relation symbol in $\sigma_{d+1}$. In case that free($\varphi$) = $\emptyset$, $\varphi'$ only contains statements of the latter form.

**(III)** For every $(\sigma, \mathcal{W})$-interpretation $I = (A, \beta)$, we have $I \models \varphi$ iff $I_{d+1} \models \varphi'$, where $I_{d+1} = (A_{d+1}, \beta)$, and $A_{d+1}$ is the $\sigma_{d+1}$-expansion of $A$ defined as follows: $A_0 := A$, and for every $i \in [d+1]$, $A_i$ is the $\sigma_i$-expansion of $A_{i-1}$, where for every unary $R \in \tau_i$, we have $R^{A_i} := \{ a \in A : (A_{i-1}, a) \models \iota_i(R) \}$ and for every 0-ary $R \in \tau_i$ we have $R^{A_i} := \emptyset$ if $A_{i-1} \models \iota_i(R)$, and $R^{A_i} := \emptyset$ if $A_{i-1} \models \iota_i(R)$.

Moreover, there is an algorithm which constructs such a sequence $D = (L_1, \ldots, L_{d+1}, \varphi')$ for an input formula $\varphi$ and outputs the radius of each cl-term in $D$ as well as a number $r$ such that every local formula in $\varphi'$ is $r$-local around its free variables.

**Proof.** We proceed by induction on $i$ to construct for all $i \in [0, d]$ a tuple $L_i = (\tau_i, \iota_i)$ and a FOWA$_1(\mathbb{P})[\sigma_i, \mathcal{S}, \mathcal{W}]$-formula $\varphi_i(\bar{x})$ of ag-depth $(d-i)$, such that for every $(\sigma, \mathcal{W})$-interpretation $I = (A, \beta)$ and the interpretation $I_i := (A_i, \beta)$, we have $I \models \varphi$ $\iff$ $I_i \models \varphi_i$.

For $i = 0$, we are done by letting $\tau_0 := \emptyset$, $\sigma_0 := \sigma$, $\varphi_0 := \varphi$, and $\iota_0$ be the mapping with empty domain. Now assume that for some $i < d$, we have already constructed $L_i = (\tau_i, \iota_i)$ and $\varphi_i$. To construct $L_{i+1} = (\tau_{i+1}, \iota_{i+1})$ and $\varphi_{i+1}$, we proceed as follows.

Let $\Pi$ be the set of all FOWA$_1(\mathbb{P})[\sigma_i, \mathcal{S}, \mathcal{W}]$-formulas of ag-depth $\leq 1$ of the form $P(t_1, \ldots, t_m)$, for $P \in \mathbb{P}$, that occur in $\varphi_i$.

Now consider an arbitrary formula $\pi$ in $\Pi$ of the form $P(t_1, \ldots, t_m)$. From Definition 3.4 we know that there is a variable $y$ such that free($t_j$) $\subseteq \{y\}$ for every $j \in [m]$. By Lemma 4.14, $\pi$ is equivalent to a Boolean combination $\pi'$ of

(a) formulas of the form $P(t'_1, \ldots, t'_m)$, for cl-terms $t'_1, \ldots, t'_m$ of signature $\sigma_i$, where free($t'_j$) = free($t_j$) $\subseteq \{y\}$ for each $j \in [m]$,

(b) statements of the form "$g \geq 1$" for ground cl-terms $g$ of signature $\sigma_i$, and

(c) local aggregation sentences in FOW$_1(\mathbb{P})[\sigma_i, \mathcal{S}, \mathcal{W}]$.

For each statement $\chi$ of the form (b) or (c), we include into $\tau_{i+1}$ a 0-ary relation symbol $R_\chi$, we replace each occurrence of $\chi$ in $\pi'$ with the new atomic formula $R_\chi()$, and we let $\iota_{i+1}(R_\chi) := \chi$. For each statement $\chi$ in $\pi$ of the form (a), we proceed as follows. If free($\chi$) = $\emptyset$, then we include into $\tau_{i+1}$ a 0-ary relation symbol $R_\chi$, we replace each occurrence of $\chi$ in $\pi'$ with the new atomic formula $R_\chi()$, and we let $\iota_{i+1}(R_\chi) := \chi$. If free($\chi$) = $\{y\}$, then we include into $\tau_{i+1}$ a unary relation symbol $R_\chi$, we replace each
occurrence of \( \chi \) in \( \pi' \) with the new atomic formula \( R_\chi(y) \), and we let \( \iota_{i+1}(R_\chi) \) be the formula obtained from \( \chi \) by consistently replacing every free occurrence of the variable \( y \) with the variable \( z \). We write \( \pi'' \) for the resulting formula \( \pi' \).

Clearly, \( \pi'' \) is of signature \( \sigma_{i+1} := \sigma_i \cup \tau_i \), it has ag-depth 0, and for every \( \sigma \)-interpretation \( \mathcal{I} = (A, \beta) \) and \( \mathcal{I}_i := (A_i, \beta) \) and \( \mathcal{I}_{i+1} := (A_{i+1}, \beta) \), we have: \( \mathcal{I} \models \pi \iff \mathcal{I}_{i+1} \models \pi'' \).

The induction step is completed by letting \( \varphi_{i+1} \) be the formula obtained from \( \varphi_i \) by replacing every occurrence of a formula \( \pi \in \Pi \) with the formula \( \pi'' \). It can easily be verified that \( \varphi_{i+1} \) is an \( \text{FOWA}_1(\mathbb{P})[\sigma_{i+1}, S, W] \)-formula of ag-depth \( d_{ag}(\varphi_i) - 1 = ((d-i)-1) = (d-(i+1)) \) and that \( \mathcal{I}_i \models \varphi_i \iff \mathcal{I}_{i+1} \models \varphi_{i+1} \).

By the above induction, we have constructed \( L_1, \ldots, L_d \) and an \( \text{FOWA}_1(\mathbb{P})[\sigma_d, S, W] \)-formula \( \varphi_d \) of ag-depth 0. Hence, \( \varphi_d \) is an \( \text{FOWA}_1(\mathbb{P})[\sigma_d, S, W] \)-formula. Theorem 4.13 yields an equivalent formula \( \tilde{\varphi} \) of signature \( \sigma_d \) in cl-normalform. That is, \( \tilde{\varphi} \) is a Boolean combination of

(A) \( \text{FOW}_1(\mathbb{P})[\sigma_d, S, W] \)-formulas that are local around their free variables \( \bar{x} \),

(B) local aggregation sentences in \( \text{FOW}_1(\mathbb{P})[\sigma_d, S, W] \), and

(C) statements of the form “\( g \geq 1 \)”, for a ground cl-term \( g \) of type \( Z \) and of signature \( \sigma_d \).

For each statement \( \chi \) of the form (B) or (C), we include into \( \tau_{d+1} \) a new relation symbol \( R_\chi \) of arity 0, we replace each occurrence of \( \chi \) in \( \tilde{\varphi} \) with the new atomic formula \( R_\chi() \), and we let \( \iota_{d+1}(R_\chi) := \chi \). Letting \( \varphi' \) be the resulting formula \( \tilde{\varphi} \) completes the proof.

We call the sequence \( (L_1, \ldots, L_{d_{ag}(\xi)+1}, \varphi') \) that Theorem 4.15 provides for a formula \( \varphi \) in \( \text{FOWA}_1(\mathbb{P})[\sigma, S, W] \) a cl-decomposition of \( \varphi \).

4.3.4 Proof of Theorem 4.7

By combining Theorem 4.15 with Lemmas 4.10 and 4.11, we can now prove Theorem 4.7.

Proof of Theorem 4.7

Use Theorem 4.15 to compute a cl-decomposition \( D = (L_1, \ldots, L_{d+1}, \varphi') \) of \( \varphi \), for \( d := d_{ag}(\varphi) \). This formula \( \varphi' \) is the desired formula. We let \( \sigma_{\varphi} := \sigma_{d+1} := \sigma \cup \bigcup_{1 \leq i \leq d+1} \tau_i \).

We also let \( A_{\varphi} := A_{d+1} \). To compute \( A_{\varphi} \), we proceed as follows.

Let \( A_0 := A \). For each \( i \in [d+1] \), compute the \( \sigma_i \)-expansion of \( A_{i-1} \). To achieve this, consider for each \( R \in \tau_i \) the formula \( \iota_i(R) \). This formula is of signature \( \sigma_{i-1} \) and (I) of the form \( P(t_1, \ldots, t_m) \) for a \( P \in \mathbb{P} \) and cl-terms \( t_1, \ldots, t_m \), or (II) of the form \( g \geq 1 \) where \( g \) is a ground cl-term of type \( Z \), or (III) a local aggregation sentence, i.e. of the form \( \sum w(y) \lambda(y) \) for a local \( \text{FOW}_1(\mathbb{P})[\sigma_{i-1}, S, W] \)-formula \( \lambda \)—and by Lemma 4.11, \( \sum w(y) \lambda(y) \) is equivalent to a ground cl-term. Thus, in all three cases, \( \iota_i(R) \) is a very simple statement that concerns one or several cl-terms and that involves at most one free variable. By using Lemma 4.10, we can compute in time \( |A| \cdot d^{O(1)} \) for each such cl-term \( t \) the values \( t^A[a] \) for all \( a \in A \) (resp., the value \( t^A, \) if \( t \) is ground). Then, we combine the values and use a \( \mathbb{P} \)-oracle to check for each \( a \in A \) whether \( \iota_i(R) \) is satisfied by \( (A_{i-1}, a) \), and we store the new relation \( R_{A_{\varphi}} \) accordingly.
5 Learning Concepts on Weighted Structures

Throughout this section, fix a collection $S$ of rings and/or abelian groups, an $S$-predicate collection $(P, ar, type, [\,])$, and a finite set $W$ of weight symbols.

Furthermore, fix numbers $k, \ell \in \mathbb{N}$. Let $L$ be a logic (e.g. FO, FOW$_1(P)$, FOWA$_1(P)$, FOWA(P)), let $\sigma$ be a signature, and let $\Phi \subseteq L[\sigma, S, W]$ be a set of formulas $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = k$ and $|\bar{y}| = \ell$. For a $(\sigma, W)$-structure $A$, we follow the same approach as [4, 7, 10, 24] and consider the instance space $X = A^k$ and concepts from the concept class

$$C(\Phi, A, k, \ell) := \{ [\varphi(\bar{x}, \bar{y})]_A(\bar{x}, \bar{v}) : \varphi \in \Phi, \bar{v} \in A^\ell \},$$

where $[\varphi(\bar{x}, \bar{y})]_A^A(\bar{x}, \bar{v})$ is defined as the mapping from $A^k$ to $\{0, 1\}$ that maps $\bar{a} \in A^k$ to $[\varphi(\bar{a}, \bar{v})]_A^A$, which is 1 iff $A \models \varphi[\bar{a}, \bar{v}]$. Given a training sequence $T = ((\bar{a}_1, b_1), \ldots, (\bar{a}_t, b_t))$ from $(A^k \times \{0, 1\})^t$, we want to compute a hypothesis that consists of a formula $\varphi$ and a tuple of parameters $\bar{v}$ and is, depending on the approach, consistent with the training sequence or probably approximately correct.

Instead of allowing random access to the background structure, we limit our algorithms to have only local access. That is, an algorithm may only interact with the structure via queries of the form “Is $\bar{a} \in R^A$?”, “Return $w^A(\bar{a})$” and “Return a list of all neighbours of $a$ in the Gaifman graph of $A$”. Hence, in this model, algorithms are required to access new vertices only via neighbourhood queries of vertices they have already seen. This enables us to learn a concept from examples even if the background structure is too large to fit into the main memory. To obtain a reasonable running time, we intend to find algorithms that compute a hypothesis in sublinear time, measured in the size of the background structure. This local access model has already been studied for relational structures in [10, 24] for concepts definable in FO or in FOCN(P). Modifications of the local access model for strings and trees have been studied in [4, 7].

In many applications, the same background structure is used multiple times to learn different concepts. Hence, similar to the approaches in [4, 7], we allow a precomputation step to enrich the background structure with additional information. That is, instead of learning on a $(\sigma, W)$-structure $A$, we use an enriched $(\sigma^*, W)$-structure $A^*$, which has the same universe as $A$, but $\sigma^* \supseteq \sigma$ contains additional relation symbols. The hypotheses we compute may make use of this additional information and thus, instead of representing them via formulas from the fixed set $\Phi$, we consider a set $\Phi^*$ of formulas of signature $\sigma^*$. These formulas may even belong to a logic $L^*$ different from $L$. We study the following learning problem.

**Problem 5.1** (Exact Learning with Precomputation). Let $\Phi \subseteq L[\sigma, S, W]$ and $\Phi^* \subseteq L^*[\sigma^*, S, W]$ such that, for every $(\sigma, W)$-structure $A$, there is a $(\sigma^*, W)$-structure $A^*$ with $U(A^*) = U(A)$ that satisfies $C(\Phi, A, k, \ell) \subseteq C(\Phi^*, A^*, k, \ell)$, i.e. every concept that can be defined on $A$ using $\Phi$ can also be defined on $A^*$ using $\Phi^*$. The task is as follows.

**Given** a training sequence $T = ((\bar{a}_1, b_1), \ldots, (\bar{a}_t, b_t)) \in (A^k \times \{0, 1\})^t$ and, for a $(\sigma, W)$-structure $A$, local access to the associated $(\sigma^*, W)$-structure $A^*$,
return a formula $\varphi^* \in \Phi^*$ and a tuple $\bar{v} \in A^\ell$ of parameters such that the hypothesis $[[\varphi^*(\bar{x}, \bar{y})]]_{A^\ell}^{A^*}(\bar{x}, \bar{v})$ is consistent with $T$, i.e., it maps $\bar{a}_i$ to $b_i$ for every $i \in [t]$.

The algorithm may reject if there is no consistent classifier using a formula from $\Phi$ on $A$.

Next, we examine requirements for $\Phi$ and $\Phi^*$ that help us solve Problem 5.1 efficiently. Following the approach presented in [8], to obtain algorithms that run in sublinear time, we study concepts that can be represented via a set of *local* formulas $\Phi$ with a finite set $\Phi^*$ of normal forms. Using Feferman-Vaught decompositions and the locality of the formulas, we can then limit the search space for the parameters to those that are in a certain neighbourhood of the training sequence. Recall that $\Phi$ is a set of formulas $\varphi(\bar{x}, \bar{y}) \in L[\sigma, S, W]$ with $|\bar{x}| = k$ and $|\bar{y}| = \ell$. In the following, we require $\Phi$ to have the following property.

**Property 5.2.** There are a signature $\sigma^*$, a logic $L^*$, an $r \in \mathbb{N}$, and a finite set of $r$-local formulas $\Phi^* \subseteq L^*[\sigma^*, S, W]$ such that the following hold.

1. For every $(\sigma, W)$-structure $A$, there is a $(\sigma^*, W)$-structure $A^*$ with $U(A^*) = U(A)$ such that, for every $\varphi(\bar{x}, \bar{y}) \in \Phi$, there is a $\varphi^*(\bar{x}, \bar{y}) \in \Phi^*$ with $A \models \varphi[\bar{a}, \bar{b}] \iff A^* \models \varphi^*\bar{a}, \bar{b}$ for all $\bar{a} \in A^k$, $\bar{b} \in A^\ell$.

2. Every $\varphi^* \in \Phi^*$ has, for every partition $(\bar{z}, \bar{z}')$ of the free variables of $\varphi^*$, a Feferman-Vaught decomposition in $\Phi^*$ w.r.t. $(\bar{z}, \bar{z}')$.

3. For all $\varphi^*_1, \varphi^*_2 \in \Phi^*$, the set $\Phi^*$ contains formulas equivalent to $\lnot \varphi^*_1$ and to $(\varphi^*_1 \lor \varphi^*_2)$.

This property suffices to solve Problem 5.1.

**Theorem 5.3** (Exact Learning with Precomputation). There is an algorithm that solves Problem 5.1 with local access to a structure $A^*$ associated with a structure $A$ in time $f_{\Phi^*}(A^*) \cdot (\log n + d + t)^{O(1)}$, where $A$, $A^*$, $\Phi$, and $\Phi^*$ are as described in Property 5.2, $t$ is the number of training examples, $n$ and $d$ are the size and the degree of $A^*$, and $f_{\Phi^*}(A^*)$ is an upper bound on the time complexity of model checking for formulas in $\Phi^*$ on $A^*$.

We prove the theorem in Section 5.1.

Apart from exact learning with precomputation, we also study hypotheses that generalise well in the following sense. The generalisation error of a hypothesis $h: A^k \to \{0, 1\}$ for a probability distribution $D$ on $A^k \times \{0, 1\}$ is

$$\text{err}_D(h) := \Pr_{(\bar{a}, b) \sim D}(h(\bar{a}) \neq b).$$

We write $\text{rat}(0, 1)$ for the set of all rationals $q$ with $0 < q < 1$. A hypothesis class $\mathcal{H} \subseteq \{0, 1\}^{A^k}$ is *agnostically PAC-learnable* if there is a function $t_H: \text{rat}(0, 1)^2 \to \mathbb{N}$ and a learning algorithm $\mathcal{L}$ such that for all $\varepsilon, \delta \in \text{rat}(0, 1)$ and for every distribution $D$ over
on the time complexity of model checking for formulas in \( \Phi \) class \( A \) to an algorithm that, given \( t \in \text{Property 5.2} \). There is an \( s \) in \( \text{Property 5.2} \). With the same argument, rules (7) and (8) only produce a finite number of \( (\text{Agnostic PAC Learning with Precomputation}) \) well-known properties of first-order logic, \( \Phi \) has \( \text{Property 5.2} \) (e.g. via \( L \) in \( \Phi \), which implies the claim.

We show by induction on the nesting depth of constructs using rules (4), (5), and (10) that there are, up to logical equivalence, only finitely many (sub-)formulas and \( S \)-terms used in \( \Phi \), which implies the claim.

Claim 1. Up to logical equivalence, \( \Phi \) only contains a finite number of formulas.

Proof. Since the maximum nesting depth of constructs using rules (1) and (5) as well as the maximum nesting depth of constructs using rule (10) from Definition 3.3 is bounded by \( q \) and every construct using rule (4) adds one new variable, rule (5) adds at most \( q \) new variables, and rule (10) adds at most \( q^2 \) new variables, every subformula of a formula in \( \Phi \) has at most \( k + \ell + q^2 + q^3 \) free variables. With finitely many free variables and \( \sigma \), \( S \), and \( W \) being finite as well, rules (1), (2), and (5) only produce a finite number of formulas. With the same argument, rules (7) and (8) only produce a finite number of \( S \)-terms. Therefore, in \( \Phi \), which implies the claim.

The following theorem, which we prove in Section 5.2, provides an agnostic PAC learning algorithm.

**Theorem 5.4 (Agnostic PAC Learning with Precomputation).** Let \( A, A^*, \) and \( \Phi^* \) be as in Property 5.2. There is an \( s \in \mathbb{N} \) such that, given local access to \( A^* \), the hypothesis class \( \mathcal{H} := C(\Phi^*, A^*, k, \ell) \) is agnostically PAC-learnable with \( t_{\mathcal{H}}(\varepsilon, \delta) = s \cdot \left\lceil \frac{\log(n/\delta)}{\varepsilon^2} \right\rceil \) via an algorithm that, given \( t_{\mathcal{H}}(\varepsilon, \delta) \) examples, returns a hypothesis of the form \( (\varphi^*, \bar{v}^*) \) with \( \varphi^* \in \Phi^* \) and \( \bar{v}^* \in A^k \) in time \( f_{\Phi^*}(A^*) \cdot (\log n + d + \frac{1}{\varepsilon} + \log \frac{1}{\delta})^{O(1)} \) with only local access to \( A^* \), where \( n \) and \( d \) are the size and the degree of \( A^* \), and \( f_{\Phi^*}(A^*) \) is an upper bound on the time complexity of model checking for formulas in \( \Phi^* \) on \( A^* \).
If there are only finitely many $S$-terms, then, with a bounded nesting depth, rule (4) only yields a finite number of new $S$-terms. Thus, since $P$ is also finite, rule (6) only produces a finite number of formulas of the form $P(t_1, \ldots, t_m)$. Hence, rule (3) only creates a finite number of formulas up to logical equivalence. (Consider them being in a normal form analogous to CNF.)

Applying rule (4) or rule (5) to a set of finitely many formulas only creates finitely many new formulas. Then, rule (10) only yields finitely many $S$-terms. This completes the proof of Claim 1.

For each of these finitely many formulas $\varphi$, we apply Theorem 4.7 to obtain an extension $\sigma_\varphi$ of $\sigma$, a $\sigma_\varphi$-expansion $A^\varphi$ of $A$, and a local FOW$_1(P)[\sigma, S, W]$-formula $\varphi'$. Then we let $\sigma^*$ be the union of all the $\sigma_\varphi$, we let $A^*$ be the $\sigma^*$-expansion of $A$ whose $\sigma_\varphi$-reduct coincides with $A^\varphi$ for every $\varphi$, and we let $\Phi'$ be the set of all the formulas $\varphi'$. Choose a number $r \in \mathbb{N}$ such that each of the $\varphi' \in \Phi'$ is $r$-local.

We can repeatedly apply Theorem 4.3, take the $r$-localisations $\alpha^{(r)}$, $\beta^{(r)}$ of the resulting formulas $\alpha, \beta$, and take Boolean combinations to obtain an extension $\Phi^*$ of $\Phi'$ such that $\Phi^*$ satisfies statements (2) and (3) of Property 5.2 and contains only $r$-local formulas.

Claim 2. One can stop the process after finitely many steps and thus obtain a finite extension $\Phi^*$.

Proof. When applying Theorem 4.3 to a formula $\varphi$ w.r.t. $(\bar{x}; \bar{y})$, the Feferman-Vaught decomposition only contains new formulas if $\{x_1, \ldots, x_k\} \subset \text{free}(\varphi)$ and $\{y_1, \ldots, y_l\} \subset \text{free}(\varphi)$. Hence, if one only applies Theorem 4.3 and takes the $r$-localisations of the resulting formulas, then one can stop the process after finitely many steps. Let $\Phi^{(0)} = \Phi, \Phi^{(1)}, \ldots, \Phi^{(m)}$ be the resulting sets of formulas from this finite process.

Let $\varphi$ be a Boolean combination of formulas from $\Phi^{(i)}$. If $\varphi = \varphi_1 \lor \varphi_2$ with $\varphi_1, \varphi_2 \in \Phi^{(i)}$, then $\Delta_\varphi = \Delta_{\varphi_1} \cup \Delta_{\varphi_2} \subseteq (\Phi^{(i+1)})^2$. If $\varphi = \neg \psi$ with $\psi \in \Phi^{(i)}$ and $\Delta_\psi = \{(\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)\}$, then $\Delta_{\varphi} = \{(\alpha_A, \beta_{[s]} \setminus A) : A \subseteq [s] \subseteq (\Phi^{(i+1)})^2$ with $\alpha_A = \bigwedge_{i \in A} \neg \alpha_i$ and $\beta_A = \bigwedge_{i \in A} \neg \beta_i$. Inductively, it follows that all formulas used in the Feferman-Vaught decomposition of $\varphi$ w.r.t. $(\bar{x}; \bar{y})$ are Boolean combinations of formulas in $\Phi^{(i+1)}$.

Hence, the result of the overall process is the set of Boolean combinations of $r$-localisations of formulas in $\Phi^{(m)}$. Since the set of Boolean combinations of finitely many formulas is, up to logical equivalence, again finite, the process stops after finitely many steps with a finite extension $\Phi^*$. This completes the proof of Claim 2.

This $\Phi^*$ witnesses that $\Phi$ has Property 5.2.

5.1 Exact Learning with Precomputation

Section 5.1 is devoted to the proof of Theorem 5.3.

Let $A$ be a $(\sigma, W)$-structure and let $A^*$ and $\Phi^*$ be as in Property 5.2. To prove Theorem 5.3 we present an algorithm that follows similar ideas as the algorithm presented
in [8]. Note, however, that [8] focuses on first-order logic, whereas our setting allows to achieve results for considerably stronger logics.

While the set of possible formulas \( \Phi^* \) already has constant size, we have to reduce the parameter space to obtain an algorithm that runs in sublinear time. Since the formulas in \( \Phi^* \) are \( r \)-local, we show that it suffices to consider parameters in a neighbourhood of the training sequence with a fixed radius. The main ingredient is the following result, which uses a Feferman-Vaught decomposition and allows us to analyse the parameters we choose by splitting them into two parts with disjoint neighbourhoods. For \( \bar{a} \in A^{|\bar{s}|} \), let \( \tp_{\Phi^*}(\bar{A}^*, \bar{a}) \coloneqq \{ \varphi^*(\bar{z}) \in \Phi^* : \bar{A}^* \models \varphi^*[\bar{a}] \} \).

**Lemma 5.6** (Local Composition Lemma). For numbers \( k, \ell \), let \( \bar{a}, \bar{a}' \in A^k, \bar{b}, \bar{b}' \in A^\ell \), \( \text{dist}^{A^*}(\bar{a}, \bar{a}') > 2^r + 1 \), \( \text{dist}^{A^*}(\bar{b}, \bar{b}') > 2^r + 1 \), \( \tp_{\Phi^*}(\bar{A}^*, \bar{a}) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}') \), and \( \tp_{\Phi^*}(\bar{A}^*, \bar{b}) = \tp_{\Phi^*}(\bar{A}^*, \bar{b}') \). Then \( \tp_{\Phi^*}(\bar{A}^*, \bar{a}, \bar{b}) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}', \bar{b}') \).

**Proof.** Let \( \varphi^*(\bar{x}, \bar{y}) \in \tp_{\Phi^*}(\bar{A}^*, \bar{a}, \bar{b}) \). Then, with Property \([5.2][2] \), \( \varphi^* \) has a Feferman-Vaught decomposition \( \Delta \) in \( \Phi^* \) w.r.t. \( (\bar{x}, \bar{y}) \), and thus, \( \text{dist}^{A^*}(\bar{a}) + \text{dist}^{A^*}(\bar{b}) = \varphi^*[\bar{a}, \bar{b}] \) if and only if there exists \( (\alpha, \beta) \in \Delta \) such that \( \text{dist}^{A^*}(\bar{a}) = \alpha[\bar{a}] \) and \( \text{dist}^{A^*}(\bar{b}) = \beta[\bar{b}] \). Since \( \bar{A}^* \models \varphi^*[\bar{a}, \bar{b}] \) and \( \varphi^*, \alpha \) and \( \beta \) are \( r \)-local, it follows that \( \bar{A}^* \models \varphi^*[\bar{a}'] \) and \( \bar{A}^* \models \beta[\bar{b}'] \) if and only if there exists \( (\alpha, \beta) \in \Delta \) such that \( \text{dist}^{A^*}(\bar{a}) = \alpha[\bar{a}'] \) and \( \text{dist}^{A^*}(\bar{b}) = \beta[\bar{b}] \). Hence, \( \alpha \in \tp_{\Phi^*}(\bar{A}^*, \bar{a}) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}') \) and \( \beta \in \tp_{\Phi^*}(\bar{A}^*, \bar{b}) = \tp_{\Phi^*}(\bar{A}^*, \bar{b}') \). We obtain \( \bar{A}^* \models \bigwedge_{(\alpha, \beta) \in \Delta} \alpha[\bar{a}'] \land \beta[\bar{b}'] \) and thus \( \bar{A}^* \models \varphi^*[\bar{a}', \bar{b}'] \). \( \square \)

The next lemma shows that it suffices to search in a reduced parameter space to find a consistent hypothesis. For \( S \subseteq A \) and an element \( b \in A \), let \( \text{dist}^{A^*}(b, S) := \min_{a \in S} \text{dist}^{A^*}(b, a) \). For \( R \geq 0 \), set \( N^A_R(S) := \bigcup_{n \in S} N^A_R(n) \). Also, for a training sequence \( T = ((\bar{a}_1, b_1), \ldots, (\bar{a}_t, b_t)) \in (A^k \times \{0, 1\})^t \), let \( N^A_R(T) := N^A_R(S) \), where \( S \) is the set of all \( a \in A \) that occur in one of the \( \bar{a}_i \).

**Lemma 5.7.** Let \( T = ((\bar{a}_1, b_1), \ldots, (\bar{a}_t, b_t)) \in (A^k \times \{0, 1\})^t \) be consistent with some classifier in \( C(\Phi^*, \bar{A}^*, k, \ell) \). Then there are a formula \( \varphi^*(\bar{x}, \bar{y}) \in \Phi^* \) and a tuple \( \bar{v}^* \in N^A_{(2^r+1)}(T)^t \) such that \( \|\varphi^*(\bar{x}, \bar{y})\|^*_{\bar{A}^*}(\bar{x}, \bar{v}^*) \) is consistent with \( T \).

**Proof.** The proof is similar to the proof of the analogous statement in [8] for the special case of FO, but relies on Property \([5.2][2] \) and Lemma \([5.6] \).

Let \( \varphi(\bar{x}, \bar{y}) \in \Phi^* \) and \( \bar{v} = (v_1, \ldots, v_t) \in \bar{A}^* \) such that \( \|\varphi(\bar{x}, \bar{y})\|^*_{\bar{A}^*}(\bar{x}, \bar{v}) \in C(\Phi^*, \bar{A}^*, k, \ell) \) is consistent with \( T \). Let \( N^{(0)} := N^{A^*}(T) \). Now we inductively define \( v^{(i)} \) and \( N^{(i)} \) for \( i \geq 1 \) as follows. Given \( N^{(i-1)} \), if there is a \( v \in \{v_1, \ldots, v_t\} \setminus \{v^{(1)}, \ldots, v^{(i-1)}\} \) such that \( \text{dist}^{A^*}(v, N^{(i)}) \leq r+1 \), then we set \( v^{(i)} := v \) and \( N^{(i)} := N^{(i-1)} \cup N^{A^*}(v^{(i)}) \). If there is no such \( v \), then we set \( m := i-1 \) and stop.

W.l.o.g. let \( v^{(i)} = v_i \) for \( i \in [m] \). Let \( \bar{v}^< := (v_1, \ldots, v_m) \) and \( \bar{v}^> := (v_{m+1}, \ldots, v_t) \). Then \( \bar{v}^< \in (N^{A^*}_{(2^r+1)}(T))^{\bar{m}} \).

**Claim 1.** Let \( i, j \in [t] \) such that \( \tp_{\Phi^*}(\bar{A}^*, \bar{a}_i \bar{v}^<) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}_j \bar{v}^<) \). Then \( b_i = b_j \).

**Proof.** From the construction, it follows that \( \text{dist}^{A^*}(\bar{v}^>, N^{(m)}) > r+1 \) and \( N^{A^*}(\bar{v}^>) \subseteq N^{(m)} \). Hence, with \( N^{A^*}(\bar{a}_i) \subseteq N^{(m)} \), we obtain \( \text{dist}^{A^*}(\bar{v}^>, \bar{a}_i \bar{v}^<) > 2^r+1 \) for every \( s \in [t] \). With Lemma \([5.6] \) it follows that \( \tp_{\Phi^*}(\bar{A}^*, \bar{a}_i \bar{v}) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}_j \bar{v}^>) = \tp_{\Phi^*}(\bar{A}^*, \bar{a}_j \bar{v}^< \bar{v}^>) = \)
The formula \[ t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{y}) \] Thus, in particular, \( \varphi \in t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{y}) \iff \varphi \in t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{v}) \). Since \( \varphi(\bar{x}, \bar{y}) \)^{A^*} (\bar{x}, \bar{v}) \) is consistent with \( T \), this implies that \( b_i = b_j \).

We let \( \bar{g}^< := (y_1, \ldots, y_m) \) and choose

\[
\varphi^< := \bigvee_{i \in \{t\}, b_i = 1} \bigwedge_{\gamma(\bar{x}, \bar{g}^<) \in t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{v})} \gamma(\bar{x}, \bar{y}^<)
\]

The formula \( \varphi^< \) is a Boolean combination of formulas in \( \Phi^* \) and thus, according to Property [5.2], there is a formula \( \varphi^* \in \Phi^* \) that is equivalent to \( \varphi^< \). The free variables of \( \varphi^* \) are among \( \bar{x} \) and \( \bar{y}^< \), and since \( \bar{y}^< \) is a prefix of \( \bar{y} \), we can safely write \( \varphi^* (\bar{x}, \bar{y}) \). We turn \( \bar{v}^< = (v_1, \ldots, v_m) \) into a tuple \( \bar{v}^* \in N^{A^*_t}(T)^t \) by choosing an arbitrary \( v \in N^{A^*_t}(T) \) and filling the missing \((t-m)\) positions with the value \( v \).

By the choice of \( \varphi^< \), the following is true for all \( j \in \{t\} \): if \( \mathcal{A}^* \models \varphi^* [\bar{a}_j, \bar{v}] \), then there is a positive example \( \bar{a}_j \) with \( \mathcal{A}^* \models \bigwedge_{\gamma(\bar{x}, \bar{g}^<) \in t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{v})} \gamma(\bar{a}_j, \bar{y}^<) \). Thus \( t \varphi \cdot (\mathcal{A}^*, \bar{a}_j, \bar{v}) \leq \gamma(\bar{a}_j, \bar{v}^<) \) for some positive example \( \bar{a}_j \); and with Claim [1] we can conclude that \( b_j = 1 \). Conversely, if \( b_j = 1 \), then \( \mathcal{A}^* \models \bigwedge_{\gamma(\bar{x}, \bar{g}^<) \in t \varphi \cdot (\mathcal{A}^*, \bar{a}, \bar{v})} \gamma(\bar{a}_j, \bar{v}^<) \) and hence \( \mathcal{A}^* \models \varphi^* [\bar{a}_j, \bar{v}] \). Thus, \( \varphi^*(\bar{x}, \bar{y})^{A^*}(\bar{x}, \bar{v}) \) is consistent with \( T \).

We can now prove Theorem [5.3].

**Proof of Theorem [5.3].** We show that the algorithm depicted on the left-hand side of Figure [1] fulfills the requirements given in Theorem [5.3]. The algorithm goes through all tuples \( \bar{v}^* \in (N^{A^*_t}(T))^t \) and all formulas \( \varphi^*(\bar{x}, \bar{y}) \in \Phi^* \). A hypothesis \( \varphi^*(\bar{x}, \bar{y})^{A^*}(\bar{x}, \bar{v}) \) is consistent with the training sequence \( T \) if and only if \( \varphi^*(\bar{a}_i, \bar{v})^{A^*} = b_i \) for all \( i \in \{t\} \).
Since $\Phi^*$ only contains $r$-local formulas, this holds if and only if $[\varphi^*(\bar{a}_i, \bar{v}^*)]^{N\mathcal{A}^*}(\bar{a}_i, \bar{v}^*) = b_i$ for every $i \in [t]$. Hence, the algorithm only returns a hypothesis if it is consistent. Furthermore, if there is a consistent hypothesis in $\mathcal{C}(\Phi, \mathcal{A}, k, \ell)$, then by Property 5.2, there is also a consistent hypothesis in $\mathcal{C}(\Phi^*, \mathcal{A}^*, k, \ell)$, and Lemma 5.7 ensures that the algorithm then returns a hypothesis.

It remains to show that the algorithm satisfies the running time requirements while only using local access to the structure $\mathcal{A}^*$. For all $\bar{a} \in \mathcal{A}^k$ and $\bar{v}^* \in \mathcal{A}^\ell$, we can bound the size of their neighbourhood by $|N^\mathcal{A}^*(\bar{a}\bar{v}^*)| \leq (k + \ell) \cdot \sum_{i=0}^{\ell} d^i \leq (k + \ell) \cdot (1 + d^{\ell+1})$. Therefore, the representation size of the substructure $N^\mathcal{A}^*(\bar{a}\bar{v}^*)$ is in $O((k + \ell) \cdot d^{\ell+1} \cdot \log n)$. Thus, the consistency check in lines 4–8 runs in time $f_{\mathcal{A}*}(\mathcal{A}^*) \cdot t \cdot O((k + \ell) \cdot d^{\ell+1} \cdot \log n)$. The algorithm checks up to $|N|^{\ell} \cdot |\Phi^*| \in O((\ell k d(2r+1)^{\ell+1})^{\ell} \cdot |\Phi^*|)$ hypotheses with $N = N(2r+1)(T)$. All in all, since $k$, $\ell$, $r$ are considered constant, the running time of the algorithm is in $f_{\mathcal{A}*}(\mathcal{A}^*) \cdot (\log n + d + \ell) O(1)$ and it only uses local access to the structure $\mathcal{A}^*$.

5.2 Agnostic PAC Learning with Precomputation

Section 5.2 is devoted to the proof of Theorem 5.4.

To obtain a hypothesis that generalises well, we follow the Empirical Risk Minimization rule (ERM) [21, 25], i.e. our algorithm should return a hypothesis $h$ that minimises the training error

$$\text{err}_T(h) := \frac{1}{|T|} \cdot |\{(\bar{a}, b) \in T : h(\bar{a}) \neq b\}|$$

on the training sequence $T$. To prove Theorem 5.4, we use the following result from [21].

Lemma 5.8 (Uniform Convergence [21]). Let $\mathcal{H}$ be a finite class of hypotheses $h: \mathcal{A}^k \rightarrow \{0, 1\}$. Then $\mathcal{H}$ is agnostically PAC-learnable using an ERM algorithm and

$$t_{\mathcal{H}}(\varepsilon, \delta) := \left[\frac{2 \log(2 |\mathcal{H}|/\delta)}{\varepsilon^2}\right].$$

Proof of Theorem 5.4. We show that the algorithm depicted on the right-hand side of Figure 1 fulfils the requirements from Theorem 5.4. The algorithm goes through all tuples $\bar{v}^* \in (N^{\mathcal{A}^*}(\mathcal{A}^*)^{\ell}$ and all formulas $\varphi^*(\bar{x}, \bar{y}) \in \Phi^*$ and counts the number of errors that $[\varphi^*(\bar{x}, \bar{y})]^{\mathcal{A}^*}(\bar{x}, \bar{v}^*)$ makes on $T$. Then it returns the hypothesis with the minimal training error.

Since $\Phi^*$ and $\mathcal{A}^\ell$ are finite, $\mathcal{H} = \mathcal{C}(\Phi^*, \mathcal{A}^\ell, k, \ell)$ is finite. Thus, using Lemma 5.8, $\mathcal{H}$ is agnostically PAC-learnable with $t_{\mathcal{H}}(\varepsilon, \delta) = \left[\frac{2 \log(2 |\mathcal{H}|/\delta)}{\varepsilon^2}\right] \leq \left[\frac{4\ell d \log(|\Phi^*|) \log(n/\delta)}{\varepsilon^2}\right].$

The running time analysis works as in the proof of Theorem 5.3. The algorithm returns a hypothesis in time $f_{\mathcal{A}*}(\mathcal{A}^*) \cdot (\log n + d + t) O(1)$. For a training sequence of length $t = t_{\mathcal{H}}(\varepsilon, \delta)$, we obtain a running time in $f_{\mathcal{A}*}(\mathcal{A}^*) \cdot (\log n + d + \log(1/\delta) + 1/\varepsilon) O(1)$.}

6 Putting Things Together

Let the collections $\mathcal{P}$ and $\mathcal{S}$ be finite (but $\mathcal{S}$ may contain infinite rings or abelian groups), fix a finite set $S$ of elements $s \in S \in \mathcal{S}$, fix a $q \in \mathbb{N}$, and let $\Phi := \Phi_{q,k+\ell,\mathcal{S}}$ be the
set of FOWA$_1$($P$, $\sigma$, $S$, $W$)-formulas defined in Remark 5.5. Let $\Phi^*$, $\sigma^*$, and $A^*$ (for all ($\sigma$, $W$)-structures $A$) be as described in Remark 5.5. By Theorem 4.7, $A^*$ can be computed from $A$ in time $|A|d^{O(1)}$, where $d$ is the degree of $A$. By Remark 5.5, the formulas in $\Phi^*$ are $r$-local for a fixed number $r$, and this implies that model checking for a formula in $\Phi^*$ on $A^*$ can be done in time polynomial in $d$. Combining this with Theorems 5.3 and 5.4 yields the following.

**Theorem 6.1.** Let $n$ and $d$ denote the size and the degree of $A$.

1. There is an algorithm that solves Exact Learning with Precomputation for $\Phi$ and $\Phi^*$ with local access to a structure $A^*$ associated with a structure $A$ in time $(\log n + d + t)^{O(1)}$, where $t$ is the number of training examples.

2. There is an $s \in \mathbb{N}$ such that, given local access to a structure $A^*$ associated with a structure $A$, the hypothesis class $H := C(\Phi^*, A^*, k, \ell)$ is agnostically PAC-learnable with $t_H(\varepsilon, \delta) = s \cdot \left[\frac{\log(n/\delta)}{\varepsilon^2}\right]$ via an algorithm that, given $t_H(\varepsilon, \delta)$ examples, returns a hypothesis of the form $(\varphi^*, \bar{\upsilon}^*)$ with $\varphi^* \in \Phi^*$ and $\bar{\upsilon}^* \in A^\ell$ in time $(\log n + d + \frac{1}{\varepsilon} + \log \frac{1}{\delta})^{O(1)}$ with only local access to $A^*$.

Additionally, the algorithms can be chosen such that the returned hypotheses can be evaluated in time $(\log n + d)^{O(1)}$.

We conclude with an example that illustrates an application scenario for Theorem 6.1.

**Example 6.2.** Recall the ($\sigma$, $W$)-structure $A$ for the online marketplace from part (a) of Examples 3.1, 3.2, and 3.6. Retailers can pay the marketplace to advertise their products to consumers. Since the marketplace demands a fee for every single view of the advertisement, retailers want the marketplace to only show the advertisement to those consumers that are likely to buy the product. One possible way to choose suitable consumers is to consider only those who buy a variety of products from the same or a similar product group as the advertised product and who are thus more likely to try new products that are similar to the advertised one. At the same time, the money spent by the chosen consumers on the product group should be above average.

In the previous examples, we have already seen a formula $\varphi_{\text{spending}}(c)$ that defines consumers who have spent at least as much as the average consumer on the product group. The formula depends on a formula $\varphi_{\text{group}}(p)$ that defines a certain group of products based on the structure of their transactions. Due to the connection between graph neural networks and the Weisfeiler-Leman algorithm described in [18], we may assume that there is a formula in FO[$\sigma$] that at least roughly approximates such a product group. Likewise, we might assume that there is a formula $\varphi_{\text{variety}}(c)$ in FO[$\sigma$] that defines consumers with a wide variety of products bought from a specific product group. However, it is a non-trivial task to design such formulas by hand. It is even not clear whether there exist better rules for finding suitable consumers. Meanwhile, we can easily show the advertisement

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*All mentioned algorithms are assumed to have $P$- and $S$-oracles, so that operations $+_S \cdot_S$ for $S \in S$ and checking if a tuple is in $[P]$ for $P \in P$ takes time $O(1)$.  

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to consumers and then check whether they buy the product. Thus, we can generate a list with positive and negative examples of consumers. Since the proposed rule can be defined in FOWA\(_1\) as \(\varphi_{\text{advertise}}(c) := (\varphi_{\text{variety}}(c) \land \varphi_{\text{spending}}(c))\), we can use one of the learning algorithms from Theorem 6.1 to find good definitions for \(\varphi_{\text{variety}}(c)\) and \(\varphi_{\text{group}}(p)\) or to learn an even better definition for \(\varphi_{\text{advertise}}(c)\) in FOWA\(_1\)[\(\sigma, S, W\)] from examples.

We believe that our results can be generalised to an extension of FOWA\(_1\) where constructions of the form \(P(t_1, \ldots, t_m)\) are not restricted to the case that \(|V| = 1\) for \(V := \text{free}(t_1) \cup \cdots \cup \text{free}(t_m)\), but may also be used in a guarded setting of the form (\(P(t_1, \ldots, t_m) \land \bigwedge_{v, w \in V} \text{dist}(v, w) \leq r\)). It would also be interesting to study non-Boolean classification problems, where classifiers are described by \(\Sigma\)-terms defined in a suitable fragment of FOWA. We plan to do this in future work.

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