On the scaling properties of $2d$ randomly stirred Navier–Stokes equation

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We inquire the scaling properties of the $2d$ Navier-Stokes equation sustained by a forcing field with Gaussian statistics, white-noise in time and with power-law correlation in momentum space of degree $2 - 2 \varepsilon$. This is at variance with the setting usually assumed to derive Kraichnan’s classical theory. We contrast accurate numerical experiments with the different predictions provided for the small $\varepsilon$ regime by Kraichnan’s double cascade theory and by renormalization group (RG) analysis. We give clear evidence that for all $\varepsilon$ Kraichnan’s theory is consistent with the observed phenomenology. Our results call for a revision in the RG analysis of (2$d$) fully developed turbulence.

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In two dimensions, the joint conservation of energy and enstrophy has relevant consequences for the Navier–Stokes equation. In $2d$ it is possible to prove that in the deterministic case the solution of the Cauchy problem exists and is unique [20] and, very recently, that in the stochastic case [3, 5, 12, 13, 14, 15] the solution is a Markov process exponentially mixing in time and ergodic with a unique invariant (steady state) measure even when the forcing acts only on two Fourier modes [22]. At large Reynolds number, the $2d$ scaling properties also differ from the $3d$-case. A long standing hypothesis [25], very recently corroborated by numerical experiments [6, 7], surmises the existence of a conformal invariance in $2d$. A phenomenological theory due to Kraichnan [18] and Batchelor [2] predicts the presence of a double cascade mechanism governing the transfer of energy and enstrophy in the limit of infinite inertial range. Accordingly, an inverse energy cascade with spectrum characterized by a scaling exponent $d_\varepsilon = -5/3$ appears for values of the wave-number $k$ smaller than the typical scale $k_F$ of the forcing scale. For wave-numbers larger than $k_F$ a direct enstrophy cascade should occur. The corresponding energy spectrum scales as $d_\varepsilon = -3 + \ldots$ where the dots here stand for possible logarithmic corrections. As emphasized in [3, 5, 21] Kraichnan’s theory is encoded in three hypotheses, (i) velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points, (ii) Galilean invariant functions and in particular structure functions reach a steady state and (iii) no dissipative anomalies occur for the energy cascade. Under these hypotheses, if the forcing field is homogeneous and isotropic Gaussian and time $\delta$-correlated it is possible to derive asymptotic expressions of the three point structure functions of the velocity field consistent with Kraichnan’s predictions [3, 5, 21]. Very strong laboratory (see [17, 20] and reference therein) and numerical evidences (see e.g. [11] and references therein) support Kraichnan’s theory. However, a first principle derivation of the statistical properties of two dimensional turbulence is still missing. An attempt in this direction has recently been undertaken [14] (see also [24] and [16]) by applying a renormalization group improved perturbation theory (RG) [27] and [1] for review of applications to fluid turbulence) to the randomly stirred Navier–Stokes equation with power law forcing. However how it will be discussed in details in the sequel, RG analysis leads to a scenario not a priori consistent with Kraichnan’s theory. The goal of the present work is to shed light on this issue.

RG starting point is the randomly stirred Navier–Stokes equation

$$(\partial_t + v \cdot \nabla) v^\alpha - \nu_0 \partial^2 v^\alpha = -\partial^\alpha P + F^\alpha - \xi_0 v^\alpha \quad (1)$$

where $\alpha = 1, 2$, $P$ is the pressure enforcing incompressibility and $\xi_0$ is an Eckman type coupling providing for large scale dissipation. The forcing $f$ is a Gaussian field with zero average and correlation

$$\langle f^\alpha(x, t) f^\beta(y, s) \rangle = \delta(t - s) F^\alpha \delta(x - y) \quad (2)$$

$$F^\alpha \beta(x) := \int \frac{d^d p}{(2 \pi)^d} e^{i p \cdot x} \tilde{F}(p) T^\alpha \beta(p) \quad (3)$$

$$\tilde{F}(p) := \frac{F_0 \chi(\frac{p}{m} + \frac{\Phi}{p})}{p^d - m^d - 4 + 2 \varepsilon} \quad (4)$$

$F_0$ is a constant specifying the amplitude of forcing fluctuations and $T^\alpha \beta(p)$ is the transversal projector. The function $\chi$ in (4) is slowly varying for $m \ll p \ll M$ and set infra-red $m$ and ultra-violet cut-offs $M$ scales for the
forcing. Its detailed shape does not affect the scaling predictions of the RG. Irrespective of the spatial dimension $d$, the cumulative spectrum $F_\alpha^\nu(0)$ of the forcing (Einstein convection for repeated vector indexes) diverges for $0 \leq \varepsilon < 2$ as the ultra-violet cut-off $M$ tends to infinity hence providing for stirring at small spatial scales. For $\varepsilon > 2$, $F_\alpha^\nu(0)$ is dominated by small wave-numbers and thus describes infra-red stirring. At $\varepsilon = 0$, the scaling dimensions of the material derivative, dissipation and forcing in (11) coincide for a scaling dimension of the velocity field $d_\varepsilon = 1$ in momentum units. Thus $\varepsilon = 0$ provides a marginal limit around which scaling dimensions can be perturbatively determined with the help of ultra-violet RG. The main result [14, 15, 16] is the existence of a non-Gaussian infra-red stable fixed point of the RG flow yielding for the energy spectrum the prediction

$$E(k) = \varepsilon^{1/3} F_0^{2/3} k^{1-\varepsilon} R (\varepsilon, \frac{m}{k}, \frac{k_0}{k})$$  

(5)

The adimensional function $R$ depends upon infra-red scales $m$ and $k_0 = (\varepsilon \varepsilon_0^\beta/F_0)^{(\varepsilon-2)/\varepsilon}$ and admits a regular expansion in powers of $\varepsilon$ at the RG fixed point [14]. The resummation leading to $k^{1-\varepsilon}$ in (5) is derived from the solution of the Callan–Symanzik equation [27] which in the present case requires scaling at finite $\varepsilon$ to stem from the balance of the two terms in the material derivative with the forcing i.e. from the requirement of Galilean invariance alone. According to [1, 14, 15] composite operators at small $\varepsilon$ does not induce any self-similarity breaking by the infra-red scales $m$ and $k_0$. The conclusion is that the spectrum should scale for small $\varepsilon$ with exponent $d_\varepsilon = 1 - 4\varepsilon/3$ as in the 3d-case [1]. Such conclusion seems at variance with Kraichnan’s theory which instead suggests the occurrence of an inverse cascade at small $\varepsilon$. Namely under the assumptions (i),(ii),(iii) [2, 3] the energy balance equation

$$v^\alpha(x, t) v_\alpha(0, t) - \frac{1}{2} \partial_t \left< |\delta v|^2 x, t \right> \partial \nu = F(x)$$  

(6)

with $\delta v^\alpha(x, t) = v^\alpha(x, t) - \bar{v}^\alpha(0, t)$ yields for $\xi_0 = 0$, $\varepsilon < 2$ and $r^2 := ||x||^2$

$$S_3(r) \approx c_1 F_0 M^{4-2\varepsilon} + c_2 F_0 \frac{r^{3-2\varepsilon}}{r^{3-2\varepsilon}}, \quad m^{-1} \gg r \gg M^{-1}$$  

(7)

$S_3$ denotes the three point velocity longitudinal structure function and $c_i$, $i = 1, 2$ two adimensional coefficients irrelevant for the present argument. Power counting based on (7) then predicts a $d_\varepsilon = -5/3$ inverse cascade spectrum at small $\varepsilon$ with at most sub-leading corrections consistent with the RG prediction.

The currently available numerical resources permit to contrast the two apparently discordant predictions [15] and [7] with the actual Navier–Stokes phenomenology. To attain this goal we integrated the Navier–Stokes equation (11) for the vorticity field ($\omega = \alpha \beta \partial^2 v^\alpha$) with a standard, fully-dealiased pseudospectral method in a doubly periodic square domain of resolution $1024^2$. Time evolution was computed by means of a standard second-order Runge–Kutta scheme. We repeated our numerical experiments for different values of the hyperviscosity $(-1)^{p+i} \nu_0 \partial^p v$ including $p = 1$, standard viscosity as in (11), and $p = 4, 6$ obtaining qualitatively identical results. The outcomes evince that for $0 \leq \varepsilon < 2$ and sufficiently small viscosity, an inverse energy cascade with exponent consistent with $d_\varepsilon = -5/3$ is observed (see Fig. 1). If the viscosity increases, finite resolution effects prevent the observation of an inertial range as the Kolmogorov scale becomes of the order of the infra-red dissipation scale. In such a case a dissipative spectrum appears with scaling exponent consistent with the prediction $d_\varepsilon = 1 - 2\varepsilon$ dictated by the balance between forcing and dissipation (see again Fig. 1). In agreement with (7), local balance scaling becomes dominant in the range $2 < \varepsilon < 3$ (see Fig. 2). There, energy spectra scale with an exponent compatible with $d_\varepsilon = 1 - 4\varepsilon/3$. Finally, for $\varepsilon > 3$ a direct enstrophy cascade invades the whole inertial range. The phenomenology just described is confirmed by the inspection of the energy and enstrophy fluxes in the inertial range (see Fig. 3). They are respectively defined by $\Pi_E(k) \propto k^{2-\varepsilon} \langle (v^\alpha(-q) (v \cdot \partial k) (v \cdot \partial k) \rangle (q)$ and $\Pi_Z(k) \propto k^{2-\varepsilon} \langle (v^\alpha(-q) (v \cdot \partial k) \rangle (q)$ where $\langle \rangle$ denotes the Fourier transform and $k_0$ is chosen such that $\Pi_E(k)$ and $\Pi_Z(k)$ are positive quantities in the inertial range. As spectra and structure functions, fluxes highlight the existence versus $\varepsilon$ of three distinct scaling regimes the origin of which can be understood by contrasting (9) with the energy and enstrophy inputs.
turbative expansion. Instead scaling seems to be related see discussion in section 9.6.4 of [13]) entailed by per-
currence of a vorticity dissipative anomaly [4]. These results provide an indirect positive test for the oc-
compatible with a weakly anomalous scaling (Fig. 5). Higher order vorticity structure function are
support normal scaling of velocity structure functions agreement with our numerical observations.

Since stirring occurs mainly at small spatial scales an in-
verse energy cascade sets in the whole inertial range. In
the range 2 < ε < 3 the energy input becomes infra-red divergent whilst I_E is still ultra-violet divergent in the
absence of cut-offs. In this situation, the steady state is
attained when the energy and enstrophy fluxes balance scale by scale the corresponding inputs (see Fig. 2). In
this ε-range the energy spectrum scales in agreement with
the exponent which can be extrapolated but no longer fully justified using the RG. Finally, for ε > 3 both the
correlation admit a regular Taylor expansion for
The hypotheses (i), (ii), (iii) thus recover S_3(x) ∝ r^3 i.e. Kraichnan’s scaling for the direct enstrophy cascade in agreement with our numerical observations.

As far as intermittency is concerned, our numerics support normal scaling of velocity structure functions
(Fig. 4). Higher order vorticity structure function are compatible with a weakly anomalous scaling (Fig. 5).
These results provide an indirect positive test for the occurrence of a vorticity dissipative anomaly [4].

In conclusion, our numerics fully support the validity of Kraichnan’s theory for all values of the Hölder ex-
ponent of power law forcing in 2d. RG scaling prediction is not observed in the range where it was supposed to appear. It seems to us that this in not trivially a conse-
quicence of low Reynolds number (O(ε^{1/2}) for F_0 = O(ε) see discussion in section 9.6.4 of [13]) entailed by per-
turbative expansion. Instead scaling seems to be related to the convolution of the response field with the fore-
ing kernel in the Schwinger-Dyson equation [1, 27] giving rise to a non-local scaling field of dimension d_0 = −1/3 independently of ε. This point however deserves more theoretical inquiry.

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FIG. 4: Structure function of longitudinal velocity increments $S_4(r)$ (squares) and $S_6(r)$ (circles) vs. $S_2(r)$. The lines represent the non-intermittent scalings $S_4(r) \sim S_2(r)^2$ (solid line) and $S_6(r) \sim S_2(r)^3$ (dashed line). (a) $\epsilon = 1$; (b) $\epsilon = 2.5$; (c) $\epsilon = 4$.

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FIG. 5: Structure function of vorticity increments $S_\omega(n)(r)$ for $n = 1, 2, 3, 4$ (square, circles, triangles, polygons). In the inset we show the scaling exponents $\zeta_n$. Here $\epsilon = 4$.

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