Topology

The hyperelliptic mapping class group of a nonorientable surface of genus $g \geq 4$ has a faithful representation into $GL(g^2 - 1, \mathbb{R})$

Le groupe modulaire hyperelliptique d’une surface non orientable de genre $g \geq 4$ a une représentation fidèle dans $GL(g^2 - 1, \mathbb{R})$

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1. Introduction

Let $N_{g,n}$ be a smooth, nonorientable, compact surface of genus $g$ with $n$ punctures. If $n$ is zero, then we omit it from the notation. Recall that $N_g$ is a connected sum of $g$ projective planes and $N_{g,n}$ is obtained from $N_g$ by specifying the set $\Sigma$ of $n$ distinguished points in the interior of $N_g$.

Let $\text{Diff}(N_{g,n})$ be the group of all diffeomorphisms $h: N_{g,n} \to N_{g,n}$ such that $h(\Sigma) = \Sigma$. By $\mathcal{M}(N_{g,n})$ we denote the quotient group of $\text{Diff}(N_{g,n})$ by the subgroup consisting of maps isotopic to the identity, where we assume that maps and isotopies fix the set $\Sigma$. $\mathcal{M}(N_{g,n})$ is called the mapping class group of $N_{g,n}$.

The mapping class group $\mathcal{M}(S_{g,n})$ of an orientable surface $S_{g,n}$ of genus $g$ with $n$ punctures is defined analogously, but we consider only orientation-preserving maps. If we include orientation reversing maps, we obtain the so-called extended mapping class group $\mathcal{M}^e(S_{g,n})$. Suppose that the closed orientable surface $S_{g-1}$, where $g-1 \geq 2$, is embedded in $\mathbb{R}^3$ as shown in Fig. 1, in such a way that it is invariant under reflections across $xy, yz, xz$ planes. Let $j: S_{g-1} \to S_{g-1}$ be the...
symmetry defined by \( j(x, y, z) = (-x, -y, -z) \). Denote by \( C_{\mathcal{M}^+(S_{g-1})}(j) \) the centraliser of \( j \) in \( \mathcal{M}^+(S_{g-1}) \). The orbit space \( S_{g-1}/(j) \) is a nonorientable surface \( N_g \) of genus \( g \) and it is known (Theorem 1 of [3]) that the orbit space projection induces an epimorphism
\[
\pi_j : C_{\mathcal{M}^+(S_{g-1})}(j) \rightarrow \mathcal{M}(N_g)
\]
with kernel \( \ker \pi_j = \langle j \rangle \). In particular
\[
\mathcal{M}(N_g) \cong C_{\mathcal{M}^+(S_{g-1})}(j)/\langle j \rangle.
\]
As was observed in the proof of Theorem 2.1 of [10], projection \( \pi_j \) has a section
\[
i_j : \mathcal{M}(N_g) \rightarrow C_{\mathcal{M}(S_{g-1})}(j) \subset \mathcal{M}(S_{g-1}).
\]
In fact, for any \( h \in \mathcal{M}(N_g) \), we can define \( i_j(h) \) to be an orientation preserving lift of \( h \).

Let \( \varrho \in C_{\mathcal{M}^+(S_{g-1})}(j) \) be the hyperelliptic involution, i.e. the half turn about the \( y \)-axis. The hyperelliptic mapping class group \( \mathcal{M}^h(S_{g-1}) \) is defined to be the centraliser of \( \varrho \) in \( \mathcal{M}(S_{g-1}) \). The hyperelliptic mapping class group turns out to be a very interesting and important subgroup, in particular its finite subgroups correspond to automorphism groups of hyperelliptic Riemann surfaces – see for example [9] and references therein.

Recently, we extended the notion of the hyperelliptic mapping class group to nonorientable surfaces [10], by defining \( \mathcal{M}^h(N_g) \) to be the centraliser of \( \pi_j(\varrho) \) in the mapping class group \( \mathcal{M}(N_g) \). This definition is motivated by the notion of hyperelliptic Klein surfaces – see for example [4,5]. We say that \( \pi_j(\varrho) \) is the hyperelliptic involution of \( N_g \) and by abuse of notation we write \( \varrho \) for \( \pi_j(\varrho) \).

Since \( \varrho \in C_{\mathcal{M}^+(S_{g-1})}(j) \), we have restrictions of \( \pi_j \) and \( i_j \) to the maps
\[
\pi_j : C_{\mathcal{M}^+(S_{g-1})}((j, \varrho)) \rightarrow \mathcal{M}^h(N_g)
\]
\[
i_j : \mathcal{M}^h(N_g) \rightarrow C_{\mathcal{M}(S_{g-1})}((j, \varrho)) \subset \mathcal{M}^h(S_{g-1}).
\]

2. **Linear representations of the hyperelliptic mapping class group**

Mapping class groups of projective plane \( N_1 \) and of Klein bottle \( N_2 \) are finite, hence the first nontrivial case is the group \( \mathcal{M}(N_3) \). This is an interesting case, because it is well known [3,8] that
\[
\mathcal{M}(N_3) = \mathcal{M}(N_3) \cong \text{GL}(2, \mathbb{Z}).
\]
In particular, \( \mathcal{M}(N_3) \) has a faithful linear representation of real dimension 2.

For \( g \geq 4 \), we can produce a faithful linear representation of the hyperelliptic mapping class group \( \mathcal{M}^h(N_g) \) as a composition of the section
\[
i_j : \mathcal{M}^h(N_g) \rightarrow C_{\mathcal{M}(S_{g-1})}((j, \varrho)) \subset \mathcal{M}^h(S_{g-1})
\]
and a faithful linear representation of \( \mathcal{M}^h(S_{g-1}) \) obtained by Korkmaz [6] or by Bigelow and Budney [2]. Recall that both of these representations of \( \mathcal{M}^h(S_{g-1}) \) are obtained form the Lawrence–Krammer representation of the braid group [1,7].

The above argument is immediate, but the resulting representation of \( \mathcal{M}^h(N_g) \) is far from being optimal. In fact, if we use the Bigelow–Budney representation of \( \mathcal{M}^h(S_{g-1}) \) (which has much smaller dimension than the one obtained by Korkmaz), the dimension of the obtained representation of \( \mathcal{M}^h(N_g) \) is equal to
\[
2g \cdot \binom{2g-1}{2} + 2(g-1) = 2(1)(2g^2 - g + 1).
\]
Theorem 1. If \( g \geq 4 \), then the hyperelliptic mapping class group \( \mathcal{M}^h(N_g) \) has a faithful linear representation of real dimension \( g^2 - 1 \).

Proof. Let \( \mathcal{M}^d(S_{0,g+1}) \) be the extended mapping class group of a sphere with \( g + 1 \) punctures \( \{p_1, \ldots, p_{g+1}\} \), and let \( \mathcal{M}^d(S_{0,g},1) \) be the stabiliser of \( p_{g+1} \) with respect to the action of \( \mathcal{M}^d(S_{0,g+1}) \) on the set of punctures. By Theorem 2.1 of [10], the orbit space projection \( \mathcal{M}^h(N_g) \rightarrow \mathcal{M}^d(S_{0,g},1) \) induces an epimorphism
\[
\pi_0: \mathcal{M}^h(N_g) \rightarrow \mathcal{M}^d(S_{0,g},1)
\]
with \( \ker \pi_0 = \langle \rho \rangle \). Moreover, by rescaling the Lawrence–Kramer representation of the braid group \([1]\), Bigelow and Budney constructed in the proof of Theorem 2.1 of \([2]\) a faithful linear representation
\[
\mathcal{L}' : \mathcal{M}(S_{0,g},1) \rightarrow \text{GL}\left(\left(\frac{g}{2}\right), \mathbb{R}\right).
\]
To be more precise, they obtained a representation over \( \mathbb{C} \); however, their argument works without any changes over \( \mathbb{R} \).
Since \( \mathcal{M}(S_{0,g},1) \) is a subgroup of index 2 in \( \mathcal{M}^d(S_{0,g},1) \), the latter group has an induced faithful linear representation of dimension \( 2 \cdot \left(\frac{g}{2}\right) = g^2 - g \). This gives us a linear representation
\[
\mathcal{L}_1 : \mathcal{M}^h(N_g) \rightarrow \text{GL}\left(\left(\frac{g^2 - g}{2}\right), \mathbb{R}\right)
\]
with kernel \( \ker \mathcal{L}_1 = \langle \rho \rangle \). It is straightforward to check that if
\[
\mathcal{L}_2 : \mathcal{M}^h(N_g) \rightarrow H_1(N_g; \mathbb{R}) \subset \text{GL}(g - 1, \mathbb{R})
\]
is a standard homology representation then \( \mathcal{L}_1 \oplus \mathcal{L}_2 \) is a required faithful linear representation of \( \mathcal{M}^h(N_g) \) of dimension \( g^2 - g + g - 1 = g^2 - 1 \). \( \Box \)

Remark 1. The above theorem gives an upper bound \( g^2 - 1 \) on the minimal dimension of a faithful linear representation of the hyperelliptic mapping class group \( \mathcal{M}^h(N_g) \). As we mentioned in the introduction, the hyperelliptic mapping class group \( \mathcal{M}^h(N_3) \) has a faithful linear representation of real dimension 2, hence it seems very unlikely that the obtained bound is sharp.

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