OPERATOR DIAGONALIZATIONS OF MULTIPLIER SEQUENCES

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ABSTRACT. We consider a new representation of linear operators on \( \mathbb{R}[x] \). New properties of the Hermite and Laguerre multiplier sequences are discovered with respect to this new representation; it is demonstrated that every Hermite multiplier sequence and Laguerre multiplier sequence can be diagonalized into a sum of hyperbolicity preserving operators, each of which diagonalizes on the standard basis. Interestingly, this does not work for other orthogonal bases, for example, this property fails for the Legendre basis. Many examples and questions are presented with respect to these formulations.

1. INTRODUCTION

The Laguerre-Pólya class, the class of entire functions that are uniform limits of hyperbolic polynomials, \( \mathcal{L}−\mathcal{P} \) (Definition 1), has been of great importance in modern research. It is certainly of substantial interest in pure mathematics \([9, 4, 10, 5]\). A shining example of the importance of the Laguerre-Pólya class is the restatement of the Riemann Hypothesis in terms of \( \mathcal{L}−\mathcal{P} \) \([12, 13, 11, 14]\).

In operator theory, different operator representations raise questions regarding the preservation of \( \mathcal{L}−\mathcal{P} \). For example, it is known that every linear operator on \( \mathbb{R}[x] \) can be written as a differential equation with real polynomial coefficients, Theorem 3. One naturally asks what properties the polynomial coefficients might possess in order to preserve \( \mathcal{L}−\mathcal{P} \) \([1, 4]\)? Another example, suppose we have a linear operator, \( T : \mathbb{R}[x] \to \mathbb{R}[x] \), where \( \{a_n\}_{n=0}^{\infty} \) is a sequence of real eigenvalues corresponding to a sequence of simple polynomial eigenvectors, \( \{B_n(x)\}_{n=0}^{\infty} \), where \( \deg(B_n(x)) = n \) for \( n = 0, 1, 2, \ldots \), i.e. \( T[B_n(x)] = a_nB_n(x) \). What conditions on \( \{B_n(x)\}_{n=0}^{\infty} \) and \( \{a_n\}_{n=0}^{\infty} \) ensure that \( T \) preserves \( \mathcal{L}−\mathcal{P} \) \([2, 3, 16, 15, 7]\)?

The essence of our research is to present a new representation of linear operators and answer questions concerning the preservation of the Laguerre-Pólya class. Our outline is as follows. We will demonstrate that every linear operator can be diagonalized into a unique “sum” of operators that each diagonalize on the standard basis [Theorem 17, Theorem 19]. We then demonstrate the remarkable fact that when we diagonalize a Hermite or Laguerre multiplier sequence, then these summand terms give rise to more Hermite and Laguerre multiplier sequences, respectively [Theorem 31, Theorem 37]. Interestingly, the Legendre basis does not enjoy this property [Example 23]. In addition, new formulas concerning the polynomial coefficients for the differential representation of the Hermite and Laguerre bases operators are discovered [Theorem 32, Theorem 38]. Many examples and questions are given.

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Definition 1. The Laguerre-Pólya class, denoted as $\mathcal{L}-\mathcal{P}$, is the set of entire functions that are uniform limits of hyperbolic polynomials (real valued polynomials with only real zeros). We define $\mathcal{L}-\mathcal{P}^+$ to be the entire functions in $\mathcal{L}-\mathcal{P}$ with Taylor coefficients of the same sign. Likewise, we define $\mathcal{L}-\mathcal{P}^-$ to be the entire functions in $\mathcal{L}-\mathcal{P}$ with alternating Taylor coefficients. The notation, $\mathcal{L}-\mathcal{P}^\pm$, is defined as $\mathcal{L}-\mathcal{P}^\pm = \mathcal{L}-\mathcal{P}^+ \cup \mathcal{L}-\mathcal{P}^-$. Given an interval, $I \subseteq \mathbb{R}$, $\mathcal{L}-\mathcal{P}^\pm I$ will denote functions in $\mathcal{L}-\mathcal{P}^\pm$ that have zeros in $I$, where $\mathcal{L}-\mathcal{P}^\pm$ is either $\mathcal{L}-\mathcal{P}^+$, $\mathcal{L}-\mathcal{P}^-$, or $\mathcal{L}-\mathcal{P}^\pm$.

Definition 2. Let $T$ be a linear operator defined on $\mathbb{R}[x]$. If $T(A) \subseteq A$ for some set of functions $A$, then $T$ is said to preserve $A$. In particular, if $T$ preserves $\mathcal{L}-\mathcal{P}$, $T$ is said to be hyperbolicity preserving.

Theorem 3 ([18], [19, p. 32]). Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then there is a unique sequence of real polynomials, $\{Q_k(x)\}_{k=0}^\infty$, such that,

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k, \quad D := \frac{d}{dx}. \quad (1.1)$$

Furthermore, the sequence, $\{Q_k(x)\}_{k=0}^\infty$, can be calculated by the recursive formula,

$$Q_n(x) = \frac{1}{n!} \left( T[x^n] - \sum_{k=0}^{n-1} Q_k(x) D^k x^n \right). \quad (1.2)$$

Definition 4. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator such that $T[B_n(x)] = a_n B_n(x)$ for every $n \in \mathbb{N}_0$, where $\{a_n\}_{n=0}^\infty$ is a sequence of real numbers and $\{B_n(x)\}_{n=0}^\infty$ is a simple basis of real polynomials, i.e. $\deg(B_n(x)) = n$ for $n \in \mathbb{N}_0$. Then $T$ will be referred to as a diagonal differential operator with respect to $\{B_n(x)\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$. If $\{B_n(x)\}_{n=0}^\infty = \{x^n\}_{n=0}^\infty$ then $T$ is said to be a classic diagonal differential operator. Similarly, if $\{B_n(x)\}_{n=0}^\infty$ is the Hermite, Laguerre, or Legendre polynomials, Definition 7, then $T$ is said to be a Hermite diagonal differential operator, Laguerre diagonal differential operator, or a Legendre diagonal differential operator, respectively.

If $T$ is hyperbolicity preserving, then the sequence, $\{a_n\}_{n=0}^\infty$, will be referred to as a $\{B_n(x)\}_{n=0}^\infty$ multiplier sequence or, for simplicity, a $B_n$-MS. If $T$ is hyperbolicity preserving and $\{B_n(x)\}_{n=0}^\infty = \{x^n\}_{n=0}^\infty$ is the classical basis, then the sequence, $\{a_n\}_{n=0}^\infty$, is referred to as a classic multiplier sequence or C-MS. Similarly, if $T$ is hyperbolicity preserving and $\{B_n(x)\}_{n=0}^\infty$ is the Hermite, Laguerre, or Legendre polynomials, then the sequence, $\{a_n\}_{n=0}^\infty$, is said to be a H-MS, L-MS, or J-MS, respectively.

Example 5. We consider the following diagonal differential hyperbolicity preserving operator, see Theorem 10,

$$T := x^2 D^2 + 2x D + 1, \quad D := \frac{d}{dx}. \quad (1.3)$$

Operator $T$ can be seen to diagonalize since $T[x^n] = (n^2 + n + 1)x^n$. Thus, sequence, $\{n^2 + n + 1\}_{n=0}^\infty$, is referred to as a C-MS. In a confusing fact, sequence $\{n^2 + n + 1\}_{n=0}^\infty$ is also an H-MS and J-MS (but not a L-MS). However, operator $T$ does not diagonalize on the Hermite polynomials or the Legendre polynomials. Thus, while operator $T$ represents a C-MS, H-MS, and J-MS, operator $T$ only diagonalizes
on the classical basis. In general, it is quite rare to have a differential operator diagonalize on two separate bases, but not impossible [2].

**Remark 6.** Given a basis of simple polynomials, \( \{ B_n(x) \}_{n=0}^{\infty} \), where \( B_n(x) \) and \( B_{n+1}(x) \) have real interlacing zeros, for some \( n \in \mathbb{N}_0 \), then it is well known that if we define,

\[
\gamma_k := \begin{cases} 
0 & k \neq n, n+1 \\
\alpha & k = n \\
\beta & k = n+1 
\end{cases}
\] (1.4)

for some \( \alpha, \beta \in \mathbb{R} \), then \( \{ \gamma_k \}_{k=0}^{\infty} \) is a \( B_n \)-MS. Throughout the literature, these types of multiplier sequences are called trivial multiplier sequences and, in general, they are typically avoided [19, 7, 16, 15] since most known characterizations of \( B_n \)-MS’s [Theorem 29, Theorem 35] are not compatible. Since we wish to make extensive use of the already known characterizations for various multiplier sequence classes, for the remainder of our discussions in this paper, all multiplier sequences will be assumed to be non-trivial.

**Definition 7** ([21, pages 157, 187, 201]). We will denote the Hermite, Laguerre, and Legendre polynomials as, \( \{ H_n(x) \}_{n=0}^{\infty} \), \( \{ L_n(x) \}_{n=0}^{\infty} \), and \( \{ J_n(x) \}_{n=0}^{\infty} \), respectively. For each \( n \in \mathbb{N}_0 \), these polynomials are given by the following formulas,

\[
H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n! 2^{n-2k}}{k!(n-2k)!} x^{n-2k}, \tag{1.5}
\]

\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k!} x^k, \quad \text{and} \tag{1.6}
\]

\[
J_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k \binom{n}{k}}{2^n} \binom{2n-2k}{n} x^{n-2k}. \tag{1.7}
\]

**Theorem 8** ([21, pages 173, 188, 204, 258]). The Hermite, Laguerre, and Legendre polynomials satisfy the following differential equations,

\[
\begin{align*}
\left( (-1/2)D^2 + (x)D \right) H_n(x) &= (n)H_n(x), \\
\left( (-x)D^2 + (x-1)D \right) L_n(x) &= (n)L_n(x), \quad \text{and} \\
\left( (x^2 - 1)D^2 + (2x)D \right) J_n(x) &= (n^2 + n)J_n(x).
\end{align*}
\] (1.8)

**Definition 9.** Let \( \{ \gamma_k \}_{k=0}^{\infty} \) be a sequence of real numbers. We define the reversed Jensen polynomials of \( \{ \gamma_k \}_{k=0}^{\infty} \) to be,

\[
g_n^*(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k}. \tag{1.9}
\]

**Theorem 10** ([20]). Let \( \{ \gamma_k \}_{k=0}^{\infty} \) be a sequence of real numbers. We have the following complete characterization for all multiplier sequences.

1. Sequence \( \{ \gamma_k \}_{k=0}^{\infty} \) is a positive or negative multiplier sequence if and only if

\[
\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}^{-\mathcal{P}^+}.
\]
(2) Sequence $\{\gamma_k\}_{k=0}^{\infty}$ is an alternating multiplier sequence if and only if
$$\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}^{-\mathcal{P}^-}.$$ 

We take great care to discuss $\mathcal{L}^{-\mathcal{P}^+}$ sequences separately from $\mathcal{L}^{-\mathcal{P}^-}$ sequences, since, in general, the $Q_k$’s arising from differential operators that diagonalize with a sequence in $\mathcal{L}^{-\mathcal{P}^+}$ and $Q_k$’s arising from differential operators that diagonalize with a sequence in $\mathcal{L}^{-\mathcal{P}^-}$, are of very different natures. The relationship between these two classes is not very well understood, as the following example helps demonstrate.

Example 11. Consider the following hyperbolicity preserving Hermite diagonal differential operators, see Theorem 29,
$$T[H_n(x)] = nH_n(x) \quad \text{and} \quad W[H_n(x)] = (-1)^n nH_n(x). \quad (1.10)$$

We calculate $T$ and $W$,
$$T = (x)D + \left(-\frac{1}{2}\right)D^2, \quad (1.11)$$
and
$$W = (-x)D + \left(2x^2 - \frac{1}{2}\right)D^2 + (-2x^3 + x)D^3 + \cdots. \quad (1.12)$$

The relationship of the $Q_k$’s in $T$ and the $Q_k$’s in $W$ is not obvious. We also observe that $T$ is a finite order differential operator, while $W$ is an infinite order differential operator. This of course makes sense when we note that $\{(-1)^n n\}_{n=0}^{\infty}$ is not interpolatable by a polynomial [2]. This example could have been performed from any simple basis, $\{B_n(x)\}_{n=0}^{\infty}$.

The sensitivity of the two classes, $\mathcal{L}^{-\mathcal{P}^+}$, $\mathcal{L}^{-\mathcal{P}^-}$, can be also be seen in the following theorem, which holds for sequences arising from $\mathcal{L}^{-\mathcal{P}^+}$, but not for sequences arising from $\mathcal{L}^{-\mathcal{P}^-}$.

Theorem 12. Given a positive or negative classic multiplier sequence, $\{\gamma_k\}_{k=0}^{\infty}$. Then, for each $m \in \mathbb{N}_0$,
$$\left\{ \sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k} \right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{ \sum_{k=0}^{m} \binom{m}{k} \gamma_{n+k} \right\}_{n=0}^{\infty}, \quad (1.13)$$
are also positive or negative classic multiplier sequences, respectively.

Proof. For the first sequence we calculate,
$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k} \right) \frac{x^n}{n!} = e^x D^m \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n \in \mathcal{L}^{-\mathcal{P}^+}. \quad (1.14)$$

For the second sequence we calculate,
$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{m} \binom{m}{k} \gamma_{n+k} \right) \frac{x^n}{n!} = e^{-x} D^m e^x \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} x^n \in \mathcal{L}^{-\mathcal{P}^+}. \quad (1.15)$$
\qed
Example 13. We show that Theorem 12 does not hold for \( \mathcal{L}^- \mathcal{P}^- \). Consider the following function in \( \mathcal{L}^- \mathcal{P}^- \), which is obtain by application of the multiplier sequence \( \{ (-1)^k \}_{k=0}^{\infty} \) to the function \( e^x \),

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}.
\]

The sequence,

\[
\{a_n\}_{n=0}^{\infty} = \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} \right\}_{n=0}^{\infty},
\]

has the form,

\[
\{a_n\}_{n=0}^{\infty} = \left\{ 1, 0, -\frac{1}{2}, -\frac{2}{3}, -\frac{5}{8}, \ldots, \frac{887}{5760}, \ldots \right\}.
\]

Hence, \( \{a_n\}_{n=0}^{\infty} \) is not a multiplier sequence, since there is no function in \( \mathcal{L}^- \mathcal{P}^+ \) or \( \mathcal{L}^- \mathcal{P}^- \) with Taylor coefficients that match the sign of \( \{a_n\}_{n=0}^{\infty} \).

Remark 14. Given a function, \( f(x) \in \mathcal{L}^- \mathcal{P}^+ \), then \( e^{\sigma x} f(x) \in \mathcal{L}^- \mathcal{P}^+ \) for every \( \sigma \geq 0 \). This property is preserved under differentiation as well. However, if \( \sigma < 0 \), then \( e^{\sigma x} f(x) \) may not be in \( \mathcal{L}^- \mathcal{P}^+ \) or \( \mathcal{L}^- \mathcal{P}^- \). For example, consider a polynomial \( (x+1) \in \mathcal{L}^- \mathcal{P}^+ \). Then \( e^{\sigma x}(x+1) \), \( \sigma < 0 \), does not have alternating, all positive or all negative Taylor coefficients and hence, is not in \( \mathcal{L}^- \mathcal{P}^+ \) or \( \mathcal{L}^- \mathcal{P}^- \). It is known that if \( \{f^{(k)}(0)\}_{k=0}^{\infty} \) is an increasing or decreasing sequence then \( e^{\sigma x} f(x) \in \mathcal{L}^- \mathcal{P}^+ \) for \( \sigma \geq -1 \). For a much more detailed analysis into the \( \sigma \)'s of functions in \( \mathcal{L}^- \mathcal{P}^+ \) and \( \mathcal{L}^- \mathcal{P}^- \) we refer the reader to work of Craven and Csordas [10].

Theorem 15. Given a sequence of real numbers, \( \{\alpha_k\}_{k=0}^{\infty} \), for each \( n \in \mathbb{N}_0 \), define,

\[
\beta_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k.
\]

Then, for all \( n \in \mathbb{N}_0 \),

\[
\alpha_n = \sum_{k=0}^{n} \binom{n}{k} \beta_k (-1)^{n-k}.
\]

In particular, we have,

\[
e^x \sum_{n=0}^{\infty} \frac{\alpha_n x^n}{n!} = \sum_{n=0}^{\infty} \frac{\beta_n x^n}{n!} \quad \text{and} \quad e^{-x} \sum_{n=0}^{\infty} \frac{\beta_n x^n}{n!} = \sum_{n=0}^{\infty} \frac{\alpha_n x^n}{n!}.
\]

Similarly, if \( \{g_k(x)\}_{k=0}^{\infty} \) are the reversed Jensen polynomials of \( \{\gamma_k\}_{k=0}^{\infty} \), then, for every \( n \in \mathbb{N}_0 \),

\[
\gamma_n = \sum_{k=0}^{n} \binom{n}{k} g_k(-1) \quad \text{and} \quad g_n^*(-1) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(-1)^{n-k}.
\]

Likewise, if \( \{\gamma_k\}_{k=0}^{\infty} \) diagonalizes the classic diagonal differential operator \( T \), then

\[
T[x^n] = \sum_{k=0}^{n} \binom{n}{k} g_k^*(-1) x^k D^k x^n = \gamma_n x^n.
\]
2. Operator Diagonalizations of Diagonalizable Operators

Our main course is present a new representation of diagonal differential operators, Theorem 17. It is of interest to compare Theorem 17 with the already well known standard representation of diagonal differential operators, Theorem 3. We state Theorem 17 in little more generality, by only requiring that the polynomial coefficients satisfy the property, deg($Q_k(x)$) \leq k for each $k \in \mathbb{N}_0$; a property that all diagonal differential operators possess, as Theorem 16 shows.

**Theorem 16.** Given a diagonal differential operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$, with respect to the simple basis, $\{B_n(x)\}_{n=0}^{\infty}$, and sequence of real numbers, $\{a_n\}_{n=0}^{\infty}$, so that $T[B_n(x)] = a_nB_n(x)$ for every $n \in \mathbb{N}_0$, then there is a unique sequence of polynomials, $\{Q_k(x)\}_{k=0}^{\infty} \subseteq \mathbb{R}[x]$, with deg($Q_k(x)$) \leq k, such that,

$$T = \sum_{k=0}^{\infty} Q_k(x)D^k, \quad D := \frac{d}{dx}. \quad (2.1)$$

In particular, each $Q_n(x)$ can be calculated with the recursive formula,

$$Q_n(x) = \frac{1}{B_n^{(n)}(x)} \left(a_nB_n - \sum_{k=0}^{n-1} Q_k(x)B_n^{(k)}(x)\right). \quad (2.2)$$

**Theorem 17.** Given a linear operator, $T : \mathbb{R}[x] \to \mathbb{R}[x],$

$$T = \sum_{k=0}^{\infty} Q_k(x)D^k, \quad (2.3)$$

where deg($Q_k(x)$) \leq k for every $k \in \mathbb{N}_0$. Define the family of sequences,

$$\{a_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^{k} \binom{k}{j} Q_j^{(j+n)}(0) \right\}_{k=0}^{\infty}, \quad n \in \mathbb{N}_0. \quad (2.4)$$

For each $n \in \mathbb{N}_0$, define the classic diagonal differential operator,

$$T_n[x^k] := a_{n,k}x^k. \quad (2.5)$$

Then, formally,

$$T = \sum_{n=0}^{\infty} T_n D^n. \quad (2.6)$$

Furthermore, the representation in (2.6) is unique.

**Proof.** We are concerning ourselves with operators defined on $\mathbb{R}[x]$, hence convergence discussions are a non-issue. By Theorem 15, for every $n \in \mathbb{N}_0$, we know the differential representation of $T_n$, namely,

$$T_n = \sum_{k=0}^{\infty} \frac{Q_{k+n}^{(k)}(0)}{k!} x^k D^k. \quad (2.7)$$

Note the calculation,

$$\frac{Q_{k+n}^{(k)}(0)}{k!} x^k, \quad (2.8)$$

is precisely the $k^{th}$ term of the polynomial, $Q_{k+n}(x)$. Hence, each summand, in each $T_n$, is one term from some $Q_k(x)$. Furthermore, no two $T_n$’s use the same
term in a particular $Q_k(x)$. Finally, because $\deg(Q_k(x)) \leq k$, we are assured that every term in every $Q_k(x)$ will be present in some $T_n$. □

Example 18. Theorem 17 can be best understood with an informative example. Define the differential operator,

$$T := (a_2x^2 + b_1x + c_0)D^2 + (a_1x + b_0)D + (a_0),$$

(2.9)

where $a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 17, we re-write $T$, in terms of $T_n$'s,

$$T = \left( \frac{Q^{(2)}(0)}{2!} x^2D^2 + \frac{Q^{(1)}(0)}{1!} x^1D + \frac{Q^{(0)}(0)}{0!} x^0D^0 \right) D^0 +$$

$$\left( \frac{Q^{(1)}(0)}{1!} x^1D + \frac{Q^{(0)}(0)}{0!} x^0D^0 \right) D^1 +$$

$$\left( \frac{Q^{(0)}(0)}{0!} x^0D^0 \right) D^2$$

(2.10)

where $a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}$. Using Theorem 17, we re-write $T$, in terms of $T_n$’s,

$$T = (a_2x^2D^2 + a_1xD + a_0) + (b_1xD + b_0)D + (c_0)D^2.$$

Theorem 17 can be extended to arbitrary linear operators on $\mathbb{R}[x]$; reminiscent of a Lauren series from an elementary complex variables book.

Theorem 19. Given an arbitrary linear operator, $T : \mathbb{R}[x] \to \mathbb{R}[x]$,

$$T = \sum_{k=0}^{\infty} Q_k(x)D^k.$$  

(2.11)

Define the family of sequences,

$$\{a_{n,k}\}_{k=0}^{\infty} := \left\{ \sum_{j=0}^{k} \binom{k}{j} Q^{(j)}_{j+n}(0) \right\}_{k=0}^{\infty} n \in \mathbb{Z}.  $$

(2.12)

For each $n \in \mathbb{Z}$, define the classic diagonal differential operator,

$$T_n[x^k] := a_{n,k}x^k.$$  

(2.13)

Then, formally,

$$T = \sum_{n=1}^{\infty} x^nT_{-n} + \sum_{n=0}^{\infty} T_nD^n.$$  

(2.14)

Furthermore, the representation in (2.14) is unique.

Example 20. We provide another example demonstrating Theorem 19. Define the differential operator,

$$T := (a_2x^2 + b_1x + c_0)D^2 + (z_1x^2 + a_1x + b_0)D + (y_0x^2 + z_0x + a_0),$$

(2.15)
where \( y_0, z_1, z_0, a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R} \). Using Theorem 19, we re-write \( T \), in terms of \( T_n \)'s,

\[
T = x^2 \frac{(y_0)}{T_{-2}} + x^1 \frac{(z_1xD + z_0)}{T_{-1}} + \left( \frac{a_2x^2D^2 + a_1xD + a_0}{T_0} \right) D^0 + \left( \frac{b_1xD + b_0}{T_1} \right) D^1 + \left( \frac{c_0}{T_2} \right) D^2.
\]  

(2.16)

Upon realizing the formulation in Theorem 17 we direct our attention to the property of hyperbolicity preservation. If \( T \), in equation (2.6), is hyperbolicity preserving, then what properties do the \( T_n \)'s possess? One might hope that the \( T_n \)'s also enjoy the property of hyperbolicity preservation. This hope would certainly be warranted since, in fact, \( T_0 \) always possess the property of hyperbolicity preserving for diagonal differential operators \([2, 6, 19]\). In addition, classic multiplier sequences and operators of the form \( f(xD) \) and \( f(D) \), from the Hermite-Poulain [17, p. 4] and Laguerre Theorems [17, Satz 3.2], trivially have \( T_n \)'s that are hyperbolicity preserving. However, in general, our hope is false as the next several examples will demonstrate. The following formula will of great use.

**Theorem 21** ([1]). Suppose we have real valued polynomials \( Q_2(x) = a(x-r_1)(x-r_2), Q_1(x) = b(x-r_3), \) and \( Q_0(x) = c \). Then \( T \) is hyperbolicity preserving where,

\[
T := Q_2(x)D^2 + Q_1(x)D + Q_0(x),
\]

(2.17)

if and only if \( a, b, c \) are of the same sign and

\[
b^2 \frac{(r_1-r_3)(r_2-r_3)}{(r_1-r_2)^2} - ac \geq 0.
\]

(2.18)

We take \( \frac{(r_1-r_3)(r_2-r_3)}{(r_1-r_2)^2} = \frac{1}{4} \) when \( r_1 = r_2 = r_3 \). If \( r_1 = r_2 \) and \( r_1 \neq r_3 \), then \( T \) is not hyperbolicity preserving.

**Example 22.** Consider the following differential operator,

\[
T := (x-2)(x+1)D^2 + 3(x+1/2)D + 1.
\]

(2.19)

Operator \( T \) is certainly hyperbolicity preserving, by Theorem 21,

\[
3^2 \left( \frac{(2-(-1/2))((-1/2)-(1))}{((1)-2)^2} \right) - 1 \cdot 1 = \frac{1}{4} \geq 0.
\]

(2.20)

However, \( T_1 = -xD + \frac{3}{2} \) is not a hyperbolicity preserver, since \( T_1[x^2-1] = -\frac{1}{2}x^2 - \frac{3}{2} \).

**Example 23.** Consider the Legendre basis of polynomials, \( \{J_n(x)\}_{n=0}^\infty \), that satisfy the differential equation, Theorem 8,

\[
((x^2 - 1)D^2 + 2xD + 1)J_n(x) = (n^2 + n + 1)J_n(x).
\]

(2.21)

Equation (2.21) was first verified to be hyperbolicity preserving in [3]. We re-verify that \( (x^2 - 1)D^2 + 2xD + 1 \) is a hyperbolicity preserver using Theorem 21,

\[
2^2 \left( \frac{(1-0)(0 - (-1))}{(-1-1)^2} \right) - 1 \cdot 1 = 1 - 1 = 0 \geq 0.
\]

(2.22)
Hence, compositions are hyperbolicity preserving. Thus, $T$ is hyperbolicity preserving where, $T[J_n(x)] = (n^2 + n + 1)^3J_n(x)$ and
\[
T = ((x^2 - 1)D^2 + (2x)D + 1)^3
\]
\[
= (x^6 - 3x^4 + 3x^2 - 1)D^6 + (18x^5 - 36x^3 + 18x)D^5 + (101x^4 - 130x^2 + 29)D^4 + (208x^3 - 160x)D^3 + (145x^2 - 57)D^2 + (26x)D + 1
\]  
(2.23)

Consider the highlighted terms of (2.23) and calculate $T_4 = 3x^2D^2 + 18xD + 29$. Operator $T_4$ fails to be hyperbolicity preserving by the following calculation, Theorem 21,
\[
18^2 \left( \frac{1}{4} \right) - 3 \cdot 29 = 81 - 87 = -6 < 0.
\]  
(2.24)

**Example 24.** Consider the shifted Hermite polynomials, $\{H_n(x \pm 3)\}_{n=0}^{\infty}$, and a multiplier sequence for these shifted Hermite polynomials, $\{n^2 + n + 1\}_{n=0}^{\infty}$. Thus $T$ is hyperbolicity preserving where, $T[H_n(x \pm 3)] = (n^2 + n + 1)H_n(x \pm 3)$ and so we calculate the differential form of $T$,
\[
T = \left( \frac{1}{4} \right)D^4 + \left( -x \mp 3 \right)D^3 + \left( x^2 \pm 6x + \frac{15}{2} \right)D^2 + (2x \pm 6)D + 1 .
\]  
(2.25)

From the highlighted items in (2.25) we formulate $T_2 = -xD + \frac{15}{2}$ and note that $T_2$ is not hyperbolicity preserving since $T[2x^8 - 2x^6] = -8x^8 - 3x^6$.

**Example 25.** Consider the shifted Laguerre polynomials, $\{L_n(x + 2)\}_{n=0}^{\infty}$, and a multiplier sequence for these shifted Laguerre polynomials, $\{n\}_{n=0}^{\infty}$. Thus $T$ is hyperbolicity preserving where, $T[L_n(x + 2)] = nL_n(x + 2)$ and
\[
T = (-x - 2)D^2 + (x + 1)D + 0 .
\]  
(2.26)

Consider the operator formed by the highlighted terms, $T_1 = -xD + 1$. Operator $T_1$ fails to preserve hyperbolicity since $T_1[x^2 - 1] = -x^2 - 1$. 


Example 26. A more technical example is the following. Using the Generalized Malo-Schur-Szegő Composition Theorem [8] it can be shown that, given \( p(x) = (x + 1)^3 \),

\[
T = \frac{(-1)^3 p^{(3)}(x)}{3!} D^3 + \frac{(-1)^2 p^{(2)}(x)}{2!} D^2 + \frac{(-1)^1 p^{(1)}(x)}{1!} D^1 + \frac{(-1)^0 p^{(0)}(x)}{0!} D^0
\]

\[
= (-1)D^3
\]

\[
\left(3x + 3\right) D^2 + \left(-3x^2 - 6x - 3\right) D + (x^3 + 3x^2 + 3x + 1),
\]

is hyperbolicity preserving [22, page 47]. Define \( T_1 := 3xD - 3 \) and note that \( T_1[x^2 - 1] = 3x^2 + 3 \), thus \( T_1 \) is not hyperbolicity preserving.

Example 27. Another example involving \( Q_k \)'s where \( \text{deg}(Q_k(x)) > k \) for some of the \( k \)'s. Using the Hermite-Poulain Theorem [17, p. 4] it can be shown that \( T := (x^2 + 2x + 1)D^2 - (x^2 + 2x + 1) \) preservers hyperbolicity. Operator \( T_0 = x^2 D^2 - 1 \) is not a hyperbolicity preserver, since \( T_0[x^2 - 1] = x^2 + 1 \). This example is even more interesting considering the fact that, in general, \( W_0 \) is always hyperbolicity preserving, whenever \( W \) is any arbitrary diagonal differential hyperbolicity preserver [2].

By now the reader has hopefully been convinced that examples 22, 23, 24, 25, 26, and 27 demonstrate the very high sensitivity of the following results; namely, for Hermite or Laguerre multiplier sequences the \( T_n \)'s in Theorem 17 are hyperbolicity preservers. Evenmoreso, it is incredible, that not only will each \( T_n \) be hyperbolicity preserving, the family of sequences, \( \{a_{n,k}\}_{k=0}^\infty \), turn out to be more Hermite or Laguerre multiplier sequences, respectively. In this sense, every Hermite or Laguerre multiplier sequence, generates an entire family of additional Hermite or Laguerre multiplier sequences.

3. OPERATOR DIAGONALIZATIONS OF HERmite MultiPLier SEQUences

Our goal in the section is to demonstrate for hyperbolicity preserving Hermite diagonal differential operators, each \( T_n \) in Theorem 17 is hyperbolicity preserving. This will done in two phases. First we will find a formula for \( a_{n,k} \). Second, we will show that this formula is indeed a Hermite multiplier sequence for each \( n \). We will need a few preliminary calculations.
Lemma 28. For each derivative, using Definition 7, we evaluate the Hermite polynomials at zero,

\[ H_{k+2j+1}^{(k)}(0) = 0 \quad \text{and} \quad H_{k+2j}^{(k)}(0) = \frac{(k + 2j)! 2^k (-1)^j}{j!}. \]  

(3.1)

We provide the complete characterization for the Hermite multiplier sequences due to A. Piotrowski. Compare this statement with Theorem 10.

Theorem 29 ([19, p. 140]). Let \( \{\gamma_k\}_{k=0}^\infty \) be a sequence of real numbers and let \( \{g_k^*(x)\}_{k=0}^\infty \) be the sequence of reversed Jensen polynomials for \( \{\gamma_k\}_{k=0}^\infty \). We have the following complete characterization for all Hermite multiplier sequences.

1. Sequence \( \{\gamma_k\}_{k=0}^\infty \) is a positive or negative Hermite multiplier sequence if and only if

   \[ e^{-x} \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k = e^{-x} \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k \in \mathcal{L}-\mathcal{D}^+ \]

2. Sequence \( \{\gamma_k\}_{k=0}^\infty \) is an alternating Hermite multiplier sequence if and only if

   \[ e^x \sum_{k=0}^\infty \frac{\gamma_k}{k!} x^k = e^x \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k \in \mathcal{L}-\mathcal{D}^- \]

Furthermore, we note that the absolute value of any Hermite multiplier sequence will be non-decreasing.

Theorem 30. Let \( \{\gamma_k\}_{k=0}^\infty \) be a Hermite multiplier sequence. Then there is a sequence of polynomials, \( \{Q_k(x)\}_{k=0}^\infty \), and sequence of classic diagonal differential operators, \( \{T_n\}_{n=0}^\infty \), such that,

\[ T[H_n(x)] := \left( \sum_{k=0}^\infty Q_k(x)D^k \right) H_n(x) = \left( \sum_{k=0}^\infty T_kD^k \right) H_n(x) = \gamma_n H_n(x). \]

Then, for each \( n \in \mathbb{N}_0 \),

\[ \{a_{2n+1,m}\}_{m=0}^\infty = \{0\}_{m=0}^\infty \]

and

\[ \{a_{2n,m}\}_{m=0}^\infty = \left\{ \sum_{k=0}^m \binom{m}{k} \frac{(-1)^n}{n!2^m} \left( \sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j} \right) \right\}_{m=0}^\infty, \]

where \( T_n[x^m] = a_{n,m} x^m \) for every \( n, m \in \mathbb{N}_0 \).

Proof. We begin with the remarkable formula of Forgács and Piotrowski that calculates the \( Q_k \)'s in any Hermite diagonal differential operator [15],

\[ Q_k(x) = \sum_{j=0}^{[k/2]} \frac{(-1)^j}{j!(k-2j)!2^{k-j} g_{k-j}^*(-1) H_{k-2j}(x)}. \]  

(3.2)

This formula provides the following calculations for all \( k, n \in \mathbb{N}_0 \),

\[ Q_{k+2n+1}^{(k)}(0) = 0, \]  

(3.3)

and

\[ Q_{k+2n}^{(k)}(0) = \frac{(-1)^n}{n!2^n} \sum_{j=0}^n \binom{n}{j} \frac{g_{k+n+j}^*(-1)}{2^j}. \]  

(3.4)
Equations (3.3) and (3.4) could have been calculated using the recursive formula, equation (2.2), if one knew, a priori, the importance of the \( g_{k+1}^\ast \)'s in formula (3.2). However, this dependence was not made apparent until formula (3.2) was uncovered.

Let us now verify (3.3) and (3.4). Equation (3.3) is obvious from formula (3.2) and the fact that the Hermite polynomials alternate between even and odd polynomials. We now establish (3.4) using formula (3.2) and Lemma 28.

\[
q^{(k)}_{k+2n}(0) = \sum_{j=0}^{[k+2n/2]} \frac{(-1)^j}{j!(k + 2n - 2j)!2^{k+2n-j}g_{k+2n-j}v(-1)H^{(k)}_{k+2n-2j}(0)}
\]

\[
= \sum_{j=0}^{n} \frac{(-1)^j}{j!(k + 2(n-j))!2^{k+n+(n-j)}g_{k+n+(n-j)}(-1)H^{(k)}_{k+2(n-j)}(0)}
\]

\[
= \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!(k + 2j)!2^{k+n+j}g_{k+n+j}(-1)H^{(k)}_{k+2j}(0)}
\]

\[
= \sum_{j=0}^{n} \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \left( \frac{g_{k+n+j}(-1)}{2^j} \right)
\]

We finish the proof by using formula (2.4).

\[\square\]

**Theorem 31.** Suppose \( \{\gamma_k\}_{k=0}^\infty \) is a Hermite multiplier sequence and let \( \{g_k^\ast(x)\}_{k=0}^\infty \) be the reversed Jensen polynomials of \( \{\gamma_k\}_{k=0}^\infty \). Then, for each \( n \in \mathbb{N}_0 \),

\[
\{b_{n,m}\}_{m=0}^\infty := \left\{ \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{(-1)^n}{n!2^n} \left( \sum_{j=0}^{n} \left( \frac{n}{j} \right) \frac{g_{k+n+j}(-1)}{2^j} \right) \right\}_{m=0}^\infty,
\]

(3.5)

is a Hermite multiplier sequence.

**Proof.** It is absolutely astonishing how simple this proof is, despite how complicated formula (3.5) is. By assumption, \( \{\gamma_k\}_{k=0}^\infty \) is a Hermite multiplier sequence. Hence, By Theorem 29, if

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k := \sum_{k=0}^{\infty} \frac{g_k^\ast(-1)}{k!} x^k,
\]

(3.6)

then, either \( f(x) \in \mathcal{L}^\mathcal{P}^+ \) or \( e^{2x}f(x) \in \mathcal{L}^\mathcal{P}^- \). We wish to show that, \( \{b_{n,m}\}_{m=0}^\infty \), is a Hermite multiplier sequence, thus we must show that, by Theorem 29, if

\[
h_n(x) := \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \left( \frac{m}{k} \right) b_{n,k}(-1)^{m-k} \right) \frac{x^m}{m!},
\]

(3.7)
then, either \( h_n(x) \in \mathcal{L} - \mathcal{P}^+ \) or \( e^{2x}h(x) \in \mathcal{L} - \mathcal{P}^- \). We use Theorem 15 and perform the following calculation,

\[
\begin{align*}
  h_n(x) &= \sum_{k=0}^{\infty} \frac{(-1)^n}{n!2^n} \sum_{j=0}^{n} \binom{n}{j} \frac{g_{k+n+j}(-1)}{2^j} \frac{x^k}{k!} \\
  &= \frac{(-1)^n}{n!2^n} \sum_{j=0}^{\infty} \binom{n}{j} \frac{1}{2^j} \sum_{k=0}^{\infty} \left( \frac{g_{k+n+j}(-1)}{k!} \right) x^k \\
  &= \frac{(-1)^n}{n!2^n} \sum_{j=0}^{\infty} \binom{n}{j} \frac{1}{2^j} D^{n+j} f(x) \\
  &= \frac{(-1)^n}{n!4^n} D^n \left( \sum_{j=0}^{\infty} \binom{n}{j} D^j 2^{n-j} \right) f(x) \\
  &= \frac{(-1)^n}{n!4^n} D^n (2 + D)^n f(x) \\
  &= \frac{(-1)^n}{n!4^n} D^n e^{-2x} D^n e^{2x} f(x)
\end{align*}
\]

**(3.8)**

Hence, if \( f(x) \in \mathcal{L} - \mathcal{P}^+ \), then \( h_n(x) \in \mathcal{L} - \mathcal{P}^+ \) and if \( e^{2x}f(x) \in \mathcal{L} - \mathcal{P}^- \), then \( e^{2x}h_n(x) \in \mathcal{L} - \mathcal{P}^- \). □

Equation (3.8) yields much more information than simply Theorem 31, in particular we derive the recursive formula,

\[
  h_n(x) = \frac{-1}{4^n} D e^{-2x} D e^{2x} h_{n-1}(x).
\]  

**(3.9)**

Hence, the hyperbolicity preservation of \( T_n \) with a Hermite multiplier sequence is enough to show that \( T_{n+1} \) is also hyperbolicity preserving with a Hermite multiplier sequence.

Overall, what we have accomplished is a new property of Hermite multiplier sequences. At first glance, it might seem as though we have a new characterization for Hermite multiplier sequences, however this is not the case. For example, suppose that each \( T_n \) is hyperbolicity preserving with Hermite multiplier sequences, then in fact \( T \) itself will be hyperbolicity preserving. But this is obvious, since if

\[
  T[H_n(x)] = a_n H_n(x),
\]

then \( T_0 \) diagonalizes on the same sequence,

\[
  T_0[x^n] = a_n x^n.
\]

**(3.10)**

In other words, to say that \( T_0 \) is hyperbolicity preserving with a Hermite multiplier sequence, is simply another way of saying that \( \{a_n\}_{n=0}^{\infty} \) is a H-MS, which is another way of saying that \( T \) is hyperbolicity preserving. So, there is a converse statement of Theorem 31, but only trivially.

As an interesting anecdote, we can use calculation (3.3) and (3.4) to re-write the \( Q_k \)'s in formula (3.2) in terms of the standard basis.
Theorem 32. Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of numbers and \( \{Q_k(x)\}_{k=0}^{\infty} \) be a sequence of polynomials, such that,

\[
T[H_n(x)] := \left( \sum_{k=0}^{\infty} Q_k(x)D^k \right) H_n(x) = a_n H_n(x) \quad n \in \mathbb{N}_0.
\]  

(3.12)

Then for each \( m \in \mathbb{N}_0 \),

\[
Q_m(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(-1)^k}{k!2^k} \left( \sum_{j=0}^{k} \binom{k}{j} \frac{g^*_m(-1)}{2^j} \right) \frac{x^{m-2k}}{(m-2k)!},
\]  

(3.13)

where \( \{g^*_k(x)\}_{k=0}^{\infty} \) are the associated reversed Jensen polynomials of \( \{a_n\}_{n=0}^{\infty} \).

4. Operator Diagonalizations of Laguerre Multiplier Sequences

The objective of this section is exactly the same as that of the previous. We provide a few preliminary remarks for Laguerre multiplier sequences, we then find a formula for the \( a_{n,k}'s \), and finally we show that the \( a_{n,k}'s \) that arise from a Laguerre multiplier sequence yield more Laguerre multiplier sequences.

Lemma 33. For each derivative, we evaluate the Laguerre polynomials at zero,

\[
L_n^{(k)}(0) = \binom{n}{k}(-1)^k.
\]  

(4.1)

Lemma 34. Let \( n, m, \) and \( p \) be integers. We then have the following combinatorial identity,

\[
\sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \binom{n+1}{k+m-j} = \binom{n+1}{p} \binom{n+1}{m} - \binom{n+1-m}{p-m} \binom{p}{n+1-m}.
\]

Theorem 35 (\cite{7}). Let \( \{\gamma_k\}_{k=0}^{\infty} \) be a sequence of real numbers and let \( \{g^*_k(x)\}_{k=0}^{\infty} \) be the reversed Jensen polynomials of \( \{\gamma_k\}_{k=0}^{\infty} \). We have the following complete characterization for all Laguerre multiplier sequences.

(1) Sequence \( \{\gamma_k\}_{k=0}^{\infty} \) is a positive or negative Laguerre multiplier sequence if and only if

\[
\sum_{k=0}^{\infty} g^*_k(-1)x^k \in \mathbb{R}[x] \cap \mathcal{L}_{-P}^+[\mathcal{O}].
\]

(2) There are no alternating Laguerre multiplier sequences.

Furthermore, it is known that all Laguerre multiplier sequences are increasing or decreasing polynomial interpolated multiplier sequences.

Theorem 36. Let \( \{\gamma_k\}_{k=0}^{\infty} \) be a Laguerre multiplier sequence. Then there is a sequence of polynomials, \( \{Q_k(x)\}_{k=0}^{\infty} \), and a sequence of classic diagonal differential operators, \( \{T_k\}_{k=0}^{\infty} \), such that,

\[
T[L_n(x)] := \left( \sum_{k=0}^{\infty} Q_k(x)D^k \right) L_n(x) = \left( \sum_{k=0}^{\infty} T_kD^k \right) L_n(x) = \gamma_n L_n(x).
\]  

\[
T[H_n(x)] := \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \binom{k}{j} \frac{g^*_m(-1)}{2^j} \right) \frac{x^{m-2k}}{(m-2k)!}.
\]
Then, 

\[ \{a_{n,m}\}_{m=0}^{\infty} := \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{n}}{n!} \left( \sum_{j=0}^{n} \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^* (-1) \right) \right\}_{m=0}^{\infty}, \]

where \( T_n[x^m] = a_{n,m} x^m \) for every \( n, m \in \mathbb{N}_0 \).

**Proof.** Consider the following formula,

\[ Q^{(k)}_{m+n}(0) = \frac{(-1)^n}{n!} \left( \sum_{j=0}^{n} \binom{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^* (-1) \right). \] (4.2)

To ease the verification process, we first re-write the above formula as follows,

\[ Q^{(m)}_{n}(0) = \sum_{p=0}^{n} (-1)^{n-m} \binom{n-m}{p-m} \binom{p}{n-m} g_{p}^* (-1). \] (4.3)

We will now verify formula (4.3), *tour de force*, by induction. Suppose for every \( m \in \mathbb{N}_0 \) and \( k \in \{0, 1, \ldots, n\} \), formula (4.3) holds for \( Q^{(m)}_{k+n}(0) \). We now calculate \( Q^{(m)}_{n+1}(0) \) using the recursive formula (2.2) and (1.22), and Lemma 33 and 34,

\[ Q^{(m)}_{n+1}(0) = \frac{1}{L_{n+1}^{(m)}} \left[ a_{n+1} L_{n+1}^{(m)} - \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{d^m}{dx^m} \left( Q^{(k)}_{k+n+1} \right) \right] \]

\[ = (-1)^{n+1} \binom{n+1}{p} g_{p}^* (-1) \binom{n+1}{m} (-1)^{m} \]

\[ - \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} Q^{(j)}_{k} \left[ L_{n+1}^{(k+1)-j} \right] \]

\[ = (-1)^{n+1} \binom{n+1}{p} g_{p}^* (-1) \binom{n+1}{m} (-1)^{m} \]

\[ - \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \left( \sum_{p=0}^{n+1} \binom{k-j}{p-j} \binom{p}{k-j} (-1)^{k-j} g_{p}^* (-1) \right) \]

\[ \left( \binom{n+1}{k+m-j} (-1)^{k+m-j} \right) \]

\[ = \sum_{p=0}^{n+1} \left( (-1)^{n+1-m} \binom{n+1}{p} \binom{n+1}{m} \right) \]

\[ - \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \binom{k-j}{p-j} \binom{p}{k-j} \left( \binom{n+1}{k+m-j} \right) g_{p}^* (-1). \]
and perform the following calculations,

\[
\begin{align*}
\frac{(-1)^n}{n!} \sum_{j=0}^{n} \left( \sum_{k=0}^{\infty} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^* (-1) \right) x^k \\
\end{align*}
\]

We finish the proof with formula (2.4).

\[ \Box \]

**Theorem 37.** Suppose \( \{ \gamma_k \}_{k=0}^{\infty} \) is a Laguerre multiplier sequence and let \( \{ g_k^* (x) \}_{k=0}^{\infty} \) be the reversed Jensen polynomials of \( \{ \gamma_k \}_{k=0}^{\infty} \). Then, for each \( n \in \mathbb{N}_0 \),

\[
\{ a_{n,m} \}_{m=0}^{\infty} := \left\{ \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n} \frac{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^* (-1) \right) x^m \right\}_{m=0}^{\infty}
\]

is a Laguerre multiplier sequence.

**Proof.** By assumption, \( \{ \gamma_k \}_{k=0}^{\infty} \) is a Laguerre multiplier sequence. Hence, By Theorem 35,

\[
f(x) = \sum_{k=0}^{\infty} \frac{f(k)(0)}{k!} x^k := \sum_{k=0}^{\infty} g_k^*(-1)x^k \in \mathbb{R}[x] \cap \mathcal{L}^{\mathcal{P}^+}[-1,0].
\] (4.4)

To show that, \( \{ a_{n,m} \}_{m=0}^{\infty} \) is a Laguerre multiplier sequence we must show that, by Theorem 35,

\[
h_n(x) := \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \left( \sum_{j=0}^{n} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) x^m \right) \in \mathbb{R}[x] \cap \mathcal{L}^{\mathcal{P}^+}[-1,0].
\] (4.5)

We use Theorem 15 and perform the following calculations,

\[
h_n(x) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{n} \frac{n}{j} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) \right) x^k
\]

\[
= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \frac{n}{j} \sum_{k=0}^{\infty} \frac{(k+j)!}{((k+j)-n)!} g_{k+j}^*(-1) x^k
\]

\[
= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \frac{n}{j} \sum_{k=0}^{\infty} \frac{f(k+j)(0)}{(k+j)-n!} x^k
\]

\[
= \frac{(-1)^n}{n!} \sum_{j=0}^{n} \frac{n}{j} x^{n-j} D^n f(x)
\]

\[
= \frac{(-1)^n}{n!} (1+x)^n D^n f(x).
\] (4.6)

Hence, if \( f(x) \in \mathbb{R}[x] \cap \mathcal{L}^{\mathcal{P}^+}[-1,0] \), then \( h_n(x) \in \mathbb{R}[x] \cap \mathcal{L}^{\mathcal{P}^+}[-1,0] \). \[ \Box \]

Similar to the Hermite case, equation (4.6) also provides a recursive formula,

\[
h_n(x) := \frac{-1}{n} (x+1)^n D(x+1)^{1-n} h_{n-1}(x).
\] (4.7)
Thus, again, the hyperbolicity preservation of $T_n$ with a Laguerre multiplier sequence, is enough to establish that $T_{n+1}$ is hyperbolicity preserving with a Laguerre multiplier sequence.

From (4.3) calculations we can also provide a formula for the Laguerre basis that is analogous to Theorem 32.

**Theorem 38.** Let $T$ be a diagonal differential operator with respect to the Laguerre basis, $\{L_n(x)\}_{n=0}^{\infty}$, and a sequence of real numbers, $\{a_n\}_{n=0}^{\infty}$, as in $T[L_n(x)] = a_n L_n(x)$ for $n \in \mathbb{N}_0$. By Theorem 16 there is a sequence of real polynomials, $\{Q_k(x)\}_{k=0}^{\infty}$, such that $T = \sum_{k=0}^{\infty} Q_k(x) D^k$. Then for each $n \in \mathbb{N}_0$,

$$Q_n(x) = \sum_{k=0}^{n} \left( \sum_{p=0}^{n} (-1)^{n-k} \binom{n-k}{p-k} \binom{p}{n-k} g^*_p(-1) \right) x^k,$$

(4.8)

where $\{g^*_k(x)\}_{k=0}^{\infty}$ are the associated reversed Jensen polynomials of $\{a_n\}_{n=0}^{\infty}$.

### 5. Open Problems

**Problem 1.** A common question amongst the literature is to find the properties of the $Q_k$’s such that

$$T := \sum_{k=0}^{\infty} Q_k D^k$$

(5.1)

will have the property of hyperbolicity preservation. We instead ask a parallel question; what are the properties needed for classic diagonal differential operators, $T_n$’s, to form hyperbolicity preservers, as in

$$T = \sum_{k=0}^{\infty} T_n D^n \ ?$$

(5.2)

**Problem 2.** The separate consideration of $\mathcal{L} - \mathcal{P}^+$ and $\mathcal{L} - \mathcal{P}^-$ is of interest, with respect to establishing properties concerning the $Q_k(x)$’s from a differential operator. We ask a follow up question to Forgacs and Piotrowski [15]: given a hyperbolicity preserving Hermite diagonal differential operator,

$$T[H_n(x)] := \left( \sum_{k=0}^{\infty} Q_k(x) D^k \right) H_n(x) = a_n H_n(x).$$

(5.3)

Must each $Q_k(x)$ have only real zeros? If we additionally assume that $\{a_n\}_{n=0}^{\infty}$ arises from $\mathcal{L} - \mathcal{P}^+$ then the answer is already known in the affirmative. Thus, our question is pertaining to those sequences that arise from $\mathcal{L} - \mathcal{P}^-$. It seems a careful consideration of Theorem 32 might prove useful.

**Problem 3.** Extend the formulas found in Theorem 32 and 38 to more bases. Is there a general formula for finding the $Q_k$’s in a differential operator?

**Problem 4.** Do the shifted Laguerre polynomials, $\{L_n(x - \alpha)\}_{n=0}^{\infty}$, possess the same property found in Theorem 36 and Theorem 37.

**Problem 5.** Find all hyperbolicity preservers that can be diagonalized into classic hyperbolicity preservers, as in Theorem 17 or 19.
Problem 6. What property do the Hermite polynomials and Laguerre polynomials possess, which allows their multiplier sequences to be diagonalized into more Hermite and Laguerre multiplier sequences? In particular, why do other orthogonal bases seem to fail, Example 23?

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