Synchronization Control of Switched Complex Networks with Additive Delays

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Abstract. The synchronization control problem for a class of switched complex networks with additive time-varying delays is studied. Firstly, the switched complex network drive-response system for system node parameter switching is studied. The appropriate Lyapunov function is constructed under the influence of additive time delay. According to Lyapunov stability theory and inequality technique, the sufficient conditions for mean-square exponential synchronization of the system are obtained. And these conclusions are expressed by linear matrix inequality (LMI). Secondly, the matrix inequality is solved by using LMI toolbox to synchronize the master-slave system. Finally, a numerical example is given to verify the effectiveness of the proposed method.

1. Introduction

During the past few decades, study on the synchronization control of complex networks has attracted more and more considerable attention, due to its wide application in many fields [1]. In practical application, complex networks inevitably encounter multiple delays, such as additive delays. The issue of synchronization problem for complex networks with additive delays has become a research hotspot. For example, new approaches of exponential stability for continuous-time system with multiple additive delay components are put forward in [2]. The stability and synchronization of the complex-valued neural networks including additive time-varying delays have been extensively studied in [3] and [4]. In [5], a class of complex networks with additive stochastic time-varying delays is investigated.

In addition, because of the characteristics of network, the node and topology of complex network is hard to keep fixed. A large number of papers on synchronization control of switched complex networks have emerged. Switching in complex networks can be divided into two kinds, deterministic switch [6-8] (e.g., switch constrained by average dwell time) and stochastic switch [9-11] (e.g., stochastic switch with known dwell probability and Markovian switch). To the best of our knowledge, few results have been reported so far on synchronization control of switched complex networks with additive delays [12-13].

The main contributions of this paper are as follows.

1) The drive-response complex network model is with additive time-varying delay and Markovian jump nodes, which is more practical than those considering single time-varying time-delay or Markovian switch.

2) The mean-square exponential stability criteria of the complex network error system is obtained. The structure of this paper is as follows. Section II describes the problems, which is to be
solved in Section III. Section IV gives the simulation results and its application to prove the effectiveness of proposed control, and Section V summarizes the paper.

2. Problem Statement and Preparation

Consider the following an array of driven-response switched complex networks with additive delay:

\[
\begin{align*}
\dot{x}_i(t) &= A_{e(i)}x_i(t) + B_{e(i)}x_i(t - \tau_1(t) - \tau_2(t)) + \sum_{j=1}^{N} a_{ij} \Gamma x_j(t) \\
\dot{y}_i(t) &= A_{e(i)}y_i(t) + B_{e(i)}y_i(t - \tau_1(t) - \tau_2(t)) + \sum_{j=1}^{N} a_{ij} \Gamma y_j(t) + u_i(t)
\end{align*}
\]

(1)

Where \(x_i(t)=[x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n\) and \(y_i(t)=[y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t)]^T \in \mathbb{R}^n\) are state variables of node \(i\) in drive and response network, respectively. \(A_{e(i)}\) and \(B_{e(i)}\) are constant matrices known to have an appropriate dimension, and \(\{r(t), t \geq 0\}\) denotes a switching signal of the upper-right continuous Markov process on a complete probability space \(\Omega, \mathcal{F}, \mathbb{P}\). The evolution of the Markov process \(\{r(t), t \geq 0\}\) taking values in a finite state space \(\mathcal{M} = \{1, 2, \ldots, N\}\) is governed by the following probability, where \(\Pi = \{\pi_{ab}\}, (a, b) \in \mathcal{M}\)

\[P\{r(t + \Delta t) = b | r(t) = a\} = \begin{cases} \pi_{ab}\Delta t + o(\Delta t), a \neq b \\ 1 + \pi_{aa}\Delta t + o(\Delta t), a = b \end{cases}\]

\(A = (a_{ij})_{N \times N}\) is the external coupling matrix of the system nodes. \(\Gamma = (\Gamma_{ij})_{n \times n}\) is the coupling matrix within the system node. \(u_i(t)\) responds to the control input of the system node \(i\). \(\tau_1(t)\) and \(\tau_2(t)\) are time-varying delays satisfying

\[
\begin{align*}
0 &\leq d_{11} \leq \tau_1(t) \leq d_{12} < \infty, \quad \hat{\tau}_1(t) \leq \mu_1 < \infty \\
0 &\leq d_{21} \leq \tau_2(t) \leq d_{22} < \infty, \quad \hat{\tau}_2(t) \leq \mu_2 < \infty \\
\tau(t) &= \tau_1(t) + \tau_2(t), \quad \hat{\tau}(t) \leq \mu < \infty \\
d_1 &= d_{11} + d_{21}, \quad d_2 = d_{12} + d_{22} \\
d &= d_1 + d_2, \quad \mu = \mu_1 + \mu_2
\end{align*}
\]

(2) (3) (4) (5) (6)

Let \(e_i(t) = x_i(t) - y_i(t)\), then error system can be obtained as

\[
\dot{e}_i(t) = A_{e(i)}e_i(t) + B_{e(i)}e_i(t - \tau_1(t) - \tau_2(t)) + \sum_{j=1}^{N} a_{ij} \Gamma e_j(t) - u_i(t)
\]

(7)

Assuming that the controller mode is synchronized with the response system mode, the controller with the following structure is designed:

\[u_i = Ke_i(t)\]

(8)

**Definition 1**[14] The network (1) is said to be synchronized, if the error system (7) is globally and exponentially stable in the mean-square, which means, there exist scalars \(c > 0\) and \(\alpha > 0\) such that for any initial conditions \(\phi(t), r(0) \in \mathcal{M}\),

\[
E\|\phi(t)\|^2 \leq ce^{-\alpha t} E\|\phi(0)\|^2
\]

Where \(\|\phi(t)\|_\phi = \sup_{-\infty < s < 0} |\phi(s)| + \int_{-\infty}^{0} \|\phi(s)\|^2 ds|^{1/2}\)

**Lemma 1**[15] For any matrix \(R > 0\), constant \(a, b, (b > a)\) and a vector function \(\omega: [a, b] \rightarrow \mathbb{R}^n\), if the integrals concerned are well defined, then the following inequality holds:

\[
(b - a) \int_{a}^{b} x^T(s)R x(s) ds \geq \int_{a}^{b} x^T(s) ds \int_{a}^{b} R x(s) ds.
\]

(9)
3. Main Result
In this section, the system synchronization standard for the drive-response system with time-varying additive delay (1) is obtained by establishing the Lyapunov-Krasovskii function.

**Theorem 1** Under formula (2), formula (3) and formula (4), the network (1) is globally exponentially synchronized if for given scalars $\alpha > 0$ , $\alpha \in M$, there exist matrices $P(\alpha) > 0$, $Z_1 > 0$, $Z_2 > 0$, $U_1 > 0$, $U_2 > 0$, $U_3 > 0$, $U_4 > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Q_4 > 0$, $F$, $J$ such that the following linear matrix inequalities hold:

$$
\varphi(a) = \begin{bmatrix}
\varphi_{11} & \varphi_{12} & 0 & FB_a & 0 & 0 & 0 & 0 \\
* & \varphi_{22} & 0 & eFB_a & U_3 & 0 & Q_3 & 0 \\
* & * & \varphi_{33} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \varphi_{44} & 0 & 0 & 0 & 0 \\
* & * & * & * & \varphi_{55} & U_4 & 0 & 0 \\
* & * & * & * & * & -U_2 - U_4 & 0 & 0 \\
* & * & * & * & * & * & \varphi_{77} & Q_4 \\
* & * & * & * & * & * & * & \varphi_{88}
\end{bmatrix} < 0 \quad (10)
$$

Where $\varphi_{11} = \phi_{11} = -F - F^T$

$$\phi_{11} = (d_{11} e^{\alpha d_{21}} - 1) U_3 + (d_{12} - d_{11}) \frac{e^{\alpha d_{12}} - e^{\alpha d_{11}}}{\alpha} U_4 + d_1 \frac{e^{\alpha d_{12}} - 1}{\alpha} Q_3 + (d_2 - d_1) \frac{e^{\alpha d_{22}} - e^{\alpha d_{12}}}{\alpha} Q_4$$

$$\phi_{12} = -\alpha F^T + FA_a - J + F(A \otimes \Gamma) + P(\alpha)$$

$$\phi_{22} = \phi_{22} + eFA_a - \alpha J + eF(A \otimes \Gamma) + \alpha A_a^T F^T - \alpha F^T + \alpha (A \otimes \Gamma)^T F^T$$

$$\phi_{33} = \phi_{33} = -(1 - \mu_1) e^{\alpha (d_{12} - d_{11})} Z_1, \varphi_{44} = \phi_{44} = - (1 - \mu) e^{\alpha (d_{22} - d_1)} Z_2$$

$$\phi_{55} = \phi_{55} = e^{\alpha (d_{12} - d_{11})} Z_1 + e^{\alpha (d_{12} - d_{11})} U_2 - U_1 - U_3 - U_4$$

Then, the gain matrix can be obtained $K = F^{-1} * J$.

Proof: Define the following form of Lyapunov-Krasovskii function:

$$V(e_j, r(t)) = \sum_{l=1}^{4} V_l(e_j, r(t)) \quad (11)$$

Where $e_j(s) = e(t + s), s \in [-\delta, 0]$

$$V_1(e_j, r(t)) = e^{\alpha t} e^T(t) P(r(t)) e(t)$$

$$V_2(e_j, r(t)) = \int_{-\delta}^{-d_1} e^{\alpha(s+d_1)} e^T(s) Z_1 e(s) ds + \int_{-\delta}^{-d_1} e^{\alpha(s+d_1)} e^T(s) Z_2 e(s) ds$$

$$V_3(e_j, r(t)) = \int_{-\delta}^{-d_1} e^{\alpha(s+d_1)} e^T(s) U_1 e(s) ds + \int_{-\delta}^{-d_1} e^{\alpha(s+d_1)} e^T(s) U_2 e(s) ds$$

$$+ \int_{-\delta}^{-d_1} \int_{-\delta}^{-d_1} e^{\alpha \theta} \hat{e} \hat{e}^T(s) U_3 \hat{e}(s) + e^{\alpha(s+d_2)} e^T(s) U_4 \hat{e}(s) ds$$

$$V_4(e_j, r(t)) = \int_{-\delta}^{-d_1} e^{\alpha(s+d_2)} e^T(s) Q_1 e(s) ds + \int_{-\delta}^{-d_1} e^{\alpha(s+d_2)} e^T(s) Q_2 e(s) ds$$

$$+ \int_{-\delta}^{-d_1} \int_{-\delta}^{-d_1} e^{\alpha \theta} \hat{e} \hat{e}^T(s) Q_3 \hat{e}(s) + e^{\alpha(s+d_3)} e^T(s) Q_4 \hat{e}(s)$$

The weak infinite operator $L$ of the stochastic process $\{(e_j, r(t)), t \geq 0\}$ along the system (7) is
defined as

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} [E[V(e_{r+\Delta t}, r(t + \Delta t)) | e_r, r(t)] - V(e_r, r(t))]$$  \hspace{1cm} (12)$$

Then for \( r(t) = a, a \in M \), \( LV(e_r, a) = \sum_{t=1}^{4} LV_t(e_r, a) \)

Where

$$LV_1(e_r, a) = e^{\alpha t} \left( 2P(a)\dot{e}(t) + (\alpha P(a) + \sum_{b \in M} \pi_{ab} P(b))e(t) \right)$$

$$LV_2(e_r, a) = e^{\alpha t} \left( (t - d_{11})Z_1 \dot{e}(t - d_{11}) - (1 - \beta_1(t))e^{\alpha t} \left( t - \tau_1(t) \right)Z_1 \dot{e}(t - \tau_1(t)) \right)$$

$$+ e^{\alpha t} \left( (t - d_{12})Z_2 \dot{e}(t - d_{12}) - (1 - \beta_2(t))e^{\alpha t} \left( t - \tau_2(t) \right)Z_2 \dot{e}(t - \tau_2(t)) \right)$$

$$LV_3(e_r, a) = e^{\alpha t} \left[ e^{\alpha t} \left( t - d_{11} \right)U_1 \dot{e}(t - d_{11}) + e^{\alpha t} \left( t - d_{12} \right)U_2 \dot{e}(t - d_{12}) \right]$$

$$+ e^{\alpha t} \left( t - d_{12} \right)U_2 \dot{e}(t - d_{12}) + e^{\alpha t} \left( t - d_{11} \right)U_1 \dot{e}(t - d_{11})$$

$$- t \dot{e}(t - d_{12})U_2 \dot{e}(t - d_{12}) + e^{\alpha t} \left( t - d_{11} \right)U_1 \dot{e}(t - d_{11})$$

$$- d_{11} \sum_{t=1}^{4} \dot{e}^T(s)U_3 \dot{e}(t - d_{11}) \sum_{t=1}^{4} \dot{e}^T(s)U_4 \dot{e}(t)$$

$$LV_4(e_r, a) = e^{\alpha t} \left( e^{\alpha t} \left( t - d_{21} \right)Q_1 \dot{e}(t - d_{21}) + e^{\alpha t} \left( t - d_{21} \right)Q_2 \dot{e}(t - d_{21}) \right)$$

$$+ e^{\alpha t} \left( t - d_{21} \right)Q_2 \dot{e}(t - d_{21}) + e^{\alpha t} \left( t - d_{21} \right)Q_2 \dot{e}(t - d_{21})$$

$$- d_{21} \sum_{t=1}^{4} \dot{e}^T(s)Q_3 \dot{e}(t - d_{21}) \sum_{t=1}^{4} \dot{e}^T(s)Q_4 \dot{e}(t)$$

By formula (2), formula (3), formula (4) and Lemma 1, one has

$$LV(e_r, a) \leq e^{\alpha t} \left[ \dot{e}^T(t)(d_1) \frac{e^{\alpha t} - 1}{\alpha} U_3 + (d_{12} - d_{11}) \frac{e^{\alpha t} - 1}{\alpha} U_4 + d_1 \frac{e^{\alpha t} - 1}{\alpha} Q_3 \right]$$

$$+ \sum_{b \in M} \pi_{ab} P(b) + e^{\alpha t} \left( t - d_{11} \right)U_1 \dot{e}(t - d_{11}) + e^{\alpha t} \left( t - d_{12} \right)U_2 \dot{e}(t - d_{12})$$

$$+ e^{\alpha t} \left( t - d_{21} \right)Q_1 \dot{e}(t - d_{21}) + e^{\alpha t} \left( t - d_{21} \right)Q_2 \dot{e}(t - d_{21})$$

$$+ \sum_{t=1}^{4} \dot{e}^T(s)U_3 \dot{e}(t - d_{11}) \sum_{t=1}^{4} \dot{e}^T(s)U_4 \dot{e}(t)$$

Where

$$\xi(t) = \left[ \dot{e}^T(t), e^T(t - \tau_1(t)), e^T(t - \tau_2(t)), e^T(t - d_{11}), e^T(t - d_{12}), e^T(t - d_{21}) \right]^T$$
Definition \( X(a) = [-I, A_y - K + A \otimes \Gamma, 0, B_y, 0, 0, 0, 0] \), then the error system (7) can be written as \( X(a) \cdot \xi(t) = 0 \), let \( \gamma = [F^T, aF^T, 0, 0, 0, 0, 0]^T \), \( F \cdot K = J \), then the formula (10) can be written as \( \varphi(a) = \phi(a) + \gamma \cdot X(a) + X^T(a) \cdot \gamma^T \).

This completes the proof.

4. Numerical Example

Considering the driver-response model shown in formula (1), three nodes are taken.

It is assumed that \( \pi_{11} = -0.3 \), \( \pi_{12} = 0.3 \), \( \pi_{21} = 0.7 \), \( \pi_{22} = -0.7 \). The other parameters are as follows:

\[
A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & 1 \\ 1 & 0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.3 & 0.5 \\ 0.4 & -0.5 \end{bmatrix},
\]

\[
A = \begin{bmatrix} 0.2 & -0.1 & -0.1 \\ -0.1 & 0.2 & -0.1 \\ -0.1 & -0.1 & 0.2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

By using LMI to solve the matrix inequality (10) in Theorem 1, the gain matrix of the controller is obtained:

\[
K = \begin{bmatrix} 44.8657 & 6.1340 \\ -13.4316 & 34.9503 \end{bmatrix}.
\]

In the synchronization error waveform of figure 1, it can be clearly seen that the drive-response system can realize synchronization through the feedback controller.

5. Conclusions

The synchronous control problem of a class of switched systems with additive time-varying delays is
studied. Considering the coupling of three nodes, a complex switched network-driven-response system with node parameter switching is studied. Under the influence of additive delay, the synchronization problem of switched network is transformed into the problem of stability of error system to realize the synchronization of drive-response system by constructing the appropriate Lyapunov function, according to the theory of Lyapunov stability and using the theory and method of switched system for reference. Finally, a numerical example is given, and the LMI toolbox is used to solve the matrix inequality, which proves the effectiveness of the proposed method and the correctness of the given result.

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