Properties of finite Gaussians and the
discrete-continuous transition

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Received 29 May 2012, in final form 13 September 2012
Published 9 October 2012
Online at stacks.iop.org/JPhysA/45/425305

Abstract
Weyl’s formulation of quantum mechanics opened the possibility of studying the dynamics of quantum systems both in infinite-dimensional and finite-dimensional systems. Based on Weyl’s approach, generalized by Schwinger, a self-consistent theoretical framework describing physical systems characterized by a finite-dimensional space of states has been created. The used mathematical formalism is further developed by adding finite-dimensional versions of some notions and results from the continuous case. Discrete versions of the continuous Gaussian functions have been defined by using the Jacobi theta functions. We continue the investigation of the properties of these finite Gaussians by following the analogy with the continuous case. We study the uncertainty relation of finite Gaussian states, the form of the associated Wigner quasi-distribution and the evolution under free-particle and quantum harmonic oscillator Hamiltonians. In all cases, a particular emphasis is put on the recovery of the known continuous-limit results when the dimension $d$ of the system increases.

PACS number: 03.65.Aa
Mathematics Subject Classification: 81Q99
(Some figures may appear in colour only in the online journal)

1. Introduction

The continuous Gaussian functions
$$g_\kappa : \mathbb{R} \longrightarrow \mathbb{R}, \quad g_\kappa (x) = e^{-\frac{\kappa}{2} x^2} \quad \kappa \in (0, \infty)$$

play a fundamental role in physics, particularly in quantum mechanics, due to their remarkable properties, among which we mention the following.

(i) With the Fourier transform defined as
$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f(x) \, dx$$
we have
\[ \mathcal{F}[g_\kappa] = \frac{1}{\sqrt{\kappa}} g_\kappa. \]

(ii) The function \( g_1 \) is an eigenfunction of a second-order differential operator
\[ \left(-\frac{d^2}{dx^2} + x^2\right) e^{-\frac{i}{\kappa} x^2} = e^{-\frac{i}{\kappa} x^2}. \]

(iii) The function \( g_\kappa \) is a minimum uncertainty state for the coordinate-momentum
\[ \Delta \hat{x} \Delta \hat{p} = \frac{1}{2}. \]

(iv) The Wigner quasi-distribution \([25]\) corresponding to \( g_\kappa \) defined as
\[ W(x, p) = \frac{1}{\sqrt{\kappa}} e^{-\frac{x^2}{\kappa}} e^{-\frac{p^2}{2}}. \]

Let \( d = 2s + 1 \in \{3, 5, 7, \ldots\} \) be a fixed odd positive integer, and let \( \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} \) be the ring of integers modulo \( d \) for which we use \( \{-s, -s + 1, \ldots, s - 1, s\} \) as a set of ‘standard’ representatives. The number \( d \) represents the dimension of the Hilbert space describing the states of the investigated quantum systems.

The Jacobi theta function \([11, 21, 24]\)
\[ \theta_3(z, \tau) = \sum_{\alpha=-\infty}^{\infty} e^{i\alpha \tau \alpha^2} e^{2\pi i \alpha z} \]
has several remarkable properties among which we mention:
\[ \theta_3(z + m + n\tau, \tau) = e^{-i\pi \tau n^2} e^{-2\pi i nz} \theta_3(z, \tau) \]
\[ \theta_3(z, i\tau) = \frac{1}{\sqrt{\tau}} e^{-\frac{z^2}{\tau}} \theta_3 \left( \frac{z}{i\tau}, \frac{i}{\tau} \right) \]
and \([19]\)
\[ \theta_3 \left( \frac{k}{d'}, \frac{i}{\kappa d} \right) = \frac{1}{\sqrt{\kappa d'}} \sum_{n=-s}^{s} e^{-\frac{2\pi i n k}{d'}} \theta_3 \left( \frac{n}{d'}, \frac{i}{\kappa d} \right). \]

For any \( \kappa \in (0, \infty) \), the function
\[ g_\kappa : \mathbb{Z}_d \rightarrow \mathbb{R}, \quad g_\kappa(n) = \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\alpha^2}{\kappa d}} \]
can be expressed in terms of the Jacobi function \( \theta_3 \) as
\[ g_\kappa(n) = \frac{1}{\sqrt{\kappa d}} \theta_3 \left( \frac{n}{d'}, \frac{i}{\kappa d} \right) \]
and by using the finite Fourier transform
\[ F[\psi](k) = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{\frac{2\pi i n k}{d}} \psi(n). \]
Figure 1. The functions $g_{1/3}$ (left), $g_1$ (centre) and $g_3$ (right) in the case $d = 31$.

Ruzzi’s relation (5) can be written in a form identical to (1), namely

$$F[g_κ] = \frac{1}{\sqrt{κ}} g_1.$$

This property of the function $g_κ$ and the shape of its graph (see figure 1) show that it can be regarded as a finite version of the Gaussian function $g_κ$. We call $g_κ$ a finite Gaussian, and our main purpose is to prove the existence of a finite version for the relations (1)–(4). We recover well-known results for continuous Gaussians for large $d$ values, i.e. at the discrete–continuous transition, and emphasize conditions under which the evolution of continuous Gaussian states can be retrieved from the evolution of their discrete counterparts.

The finite Gaussians $g_κ$ represent a generalization of Mehta’s function $f_0$, and relation (6) is a generalization of Mehta’s relation $F[f_0] = f_0$. More precisely, Mehta has proved [15] that

$$f_k(n) = \sum_{α=-∞}^{∞} e^{-\frac{2}{d} (αd + n)^2} H_k\left(\sqrt{\frac{2π}{d}} (αd + n)\right)$$

defined by using the Hermite polynomials are eigenvectors of the finite Fourier transform

$$F[f_k] = i^k f_k \quad \text{for any } k \in \{0, 1, 2, \ldots\}.$$

Since $H_0(x) = 1$, we have $f_0 = g_1$ and the relation $F[f_0] = f_0$ coincides with $F[g_1] = g_1$. Mehta’s results are mainly based on the following remarks, which we will use in the following.

(a) A periodic function with period $d$, namely

$$Φ : \mathbb{Z} \rightarrow \mathbb{R}, \quad Φ(n) = \sum_{α=-∞}^{∞} φ\left(\sqrt{\frac{2π}{d}} (αd + n)\right).$$

can be defined by starting from any function $φ : \mathbb{R} \rightarrow \mathbb{R}$ for which the series is absolutely convergent. This is similar to a Zak [26] or Weil [23] transform.

(b) The relation

$$\sum_{α=-∞}^{∞} \int_{0}^{(α+1)\sqrt{2πd}} φ(x) \, dx = \int_{-∞}^{∞} φ(x) \, dx$$

is true for any function $φ : \mathbb{R} \rightarrow \mathbb{R}$ for which the integral is convergent.
(c) The relation
\[ \sum_{\alpha = -\infty}^{\infty} \phi(\alpha) = \sum_{\alpha = -s}^{s} \sum_{\alpha = -\infty}^{\infty} \phi(\alpha d + n) \]
is satisfied for any function \( \phi : \mathbb{Z} \rightarrow \mathbb{R} \) for which the series is absolutely convergent.

In section 2, we review some elements of the mathematical formalism used in the case of the quantum systems with finite-dimensional Hilbert space in a form suitable for our purpose. We show that in the case of the free evolution, the time-dependent state is periodic in time. The behaviour of the commutator of the position and momentum operators in the limit of large \( d \) is investigated in section 3. Ruzzi has obtained relation (6) by using the properties of \( \theta \)-functions. In section 4, we present an elementary proof based on (a)–(c) for this finite version of relation (1), and show that \( g_1 \) is almost a minimum uncertainty state. In section 5, we investigate numerically the quantum oscillator Hamiltonian. The tendency to have equidistant energy levels becomes evident only for \( d \) large enough. In section 6, we show, for the first time to our knowledge, that the existence of commensurate or equidistant energy levels is a sufficient condition for the occurrence of revivals.

In section 7, by using (a)–(c) as mathematical tools, we prove that the Wigner function corresponding to a finite Gaussian can be written as a sum of four products of finite Gaussians. The obtained formula is our main result, and can be regarded as a finite version of relation (4). In the particular case \( \kappa = 1 \), an expression of the Wigner function corresponding to a finite Gaussian \( g_1 \) has been previously obtained by Marchiolli and Ruzzi [12]. They proved that, up to a multiplicative constant, the Wigner function corresponding to \( g_1 \) is
\[ W(n, m) = \theta_2 \left( \frac{n \cdot i}{d} \right) \theta_3 \left( \frac{m \cdot 2i}{d} \right) \theta_2 \left( \frac{2m \cdot 2i}{d} \right) \theta_4 \left( \frac{2m \cdot 2i}{d} \right), \]
where \( \theta_2 \) and \( \theta_4 \) are the Jacobi functions
\[ \theta_2(z, \tau) = \sum_{\alpha = -\infty}^{\alpha = \infty} e^{i\pi \tau (\alpha + \frac{1}{2})^2} e^{2\pi i (\alpha + \frac{1}{2})z}, \]
and
\[ \theta_4(z, \tau) = \sum_{\alpha = -\infty}^{\alpha = \infty} (-1)^{\alpha} e^{i\pi \tau \alpha^2} e^{2\pi i \alpha z}. \]

2. Quantum systems with finite-dimensional Hilbert space

In the case of a quantum particle moving along a straight line, the possible positions form the set \( \mathbb{R} = (-\infty, \infty) \), and the space describing the states of the system is the infinite-dimensional Hilbert space \( L^2(\mathbb{R}) \) of all the square integrable functions \( \psi : \mathbb{R} \rightarrow \mathbb{C} \).

We obtain a very simplified version of this continuous one-dimensional system by assuming that we can distinguish only a finite number \( d \) of positions for our particle. In this simplified version, the space describing the states of the system is the \( d \)-dimensional Hilbert space \( \mathbb{C}^d \). Assuming that \( d \) is an odd number, \( d = 2s + 1 \), the space \( \mathbb{C}^d \) can be identified with the space \( \mathcal{H} \) of all the functions
\[ \psi : \{-s, -s + 1, \ldots, s - 1, s\} \rightarrow \mathbb{C} \]
by using the one-to-one mapping
\[ \mathcal{H} \rightarrow \mathbb{C}^d : \psi \mapsto (\psi(-s), \psi(-s+1), \ldots, \psi(s-1), \psi(s)). \]

We choose an orthonormal basis \( \{|n\}\}_{n \in \mathbb{Z}_d} \) in \( \mathcal{H} \) and define the ‘position’ operator
\[ Q : \mathcal{H} \rightarrow \mathcal{H}, \quad Q = \frac{2\pi}{d} \sum_{n=-s}^{s} n |n\rangle\langle n|. \]  

(7)

The finite Fourier transform
\[ F : \mathcal{H} \rightarrow \mathcal{H}, \quad F = \frac{1}{\sqrt{d}} \sum_{n,n'=-s}^{s} e^{\frac{2\pi}{d} kn} |n\rangle\langle n'| \]  

allows us to consider a second orthonormal basis \( \{|\tilde{k}\rangle\}_{k \in \mathbb{Z}_d} \), where
\[ |\tilde{k}\rangle = F|k\rangle = F^{+}|−k\rangle = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{\frac{2\pi}{d} kn} |n\rangle \]

and to define the ‘momentum’ operator
\[ P : \mathcal{H} \rightarrow \mathcal{H}, \quad P = \sqrt{\frac{2\pi}{d}} \sum_{k=-s}^{s} k |\tilde{k}\rangle\langle \tilde{k}|. \]

The operators \( Q \) and \( P \) have the same spectrum, namely
\[ S_d = \left\{ −s\sqrt{\frac{2\pi}{d}}, (−s+1)\sqrt{\frac{2\pi}{d}}, \ldots, (s−1)\sqrt{\frac{2\pi}{d}}, s\sqrt{\frac{2\pi}{d}} \right\}. \]

Since
\[ \lim_{d \rightarrow \infty} \left|\frac{2\pi}{d}\right| = 0 \quad \text{and} \quad \lim_{d \rightarrow \infty} (\pm s)\sqrt{\frac{2\pi}{d}} = \pm \infty \]

in the limit \( d \rightarrow \infty \), the spectra of \( Q \) and \( P \) correspond in a certain sense to \((−\infty, \infty)\).

Each state \( |\psi\rangle \in \mathcal{H} \) can be expanded as
\[ |\psi\rangle = \sum_{n=-s}^{s} \psi(n)|n\rangle = \sum_{k=-s}^{s} \tilde{\psi}(k)|\tilde{k}\rangle, \]

where the functions \( \psi : \mathbb{Z}_d \rightarrow \mathbb{C} : n \mapsto \psi(n) \) and \( \tilde{\psi} : \mathbb{Z}_d \rightarrow \mathbb{C} : k \mapsto \tilde{\psi}(k) \) satisfying
\[ \psi(n) = \langle n|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=-s}^{s} e^{\frac{2\pi}{d} kn} \tilde{\psi}(k) \quad \text{and} \quad \tilde{\psi}(k) = \langle \tilde{k}|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{-\frac{2\pi}{d} kn} \psi(n) \]  

(9)

are the corresponding ‘wavefunctions’ in the position and momentum representations [5]. The operators \( Q \) and \( P \) satisfy the relations
\[ FQF^{+} = P \quad \text{and} \quad FPFP^{+} = −Q. \]

The displacement operators \( A, B : \mathcal{H} \rightarrow \mathcal{H}, \)
\[ A = e^{-\frac{2\pi}{d} i k} P = \sqrt{d} \sum_{k=-s}^{s} e^{-\frac{2\pi}{d} i k} |\tilde{k}\rangle\langle \tilde{k}| \quad \text{and} \quad B = e^{\frac{2\pi}{d} i k} Q = \sqrt{d} \sum_{\ell=-s}^{s} e^{\frac{2\pi}{d} i \ell} |\ell\rangle\langle \ell|, \]  

(10)

are single-valued and satisfy the relations
\[ A|\ell\rangle = |\ell + 1\rangle, \quad A|\tilde{k}\rangle = e^{-\frac{2\pi}{d} i k} |\tilde{k}\rangle, \quad A^d = B^d = 1, \]
\[ B|\ell\rangle = e^{\frac{2\pi}{d} i \ell} |\ell\rangle, \quad B|\tilde{k}\rangle = |\tilde{k} + 1\rangle, \quad A^\alpha B^\beta = e^{−\frac{2\pi}{d} i \alpha \delta} B^\beta A^\alpha. \]
The general displacement operators \([22, 20]\)
\[
D(\alpha, \beta) = e^{\pi i \frac{\alpha \beta}{||\alpha||}} A^\alpha B^\beta
\]
where \((\alpha, \beta) \in \mathbb{Z}_d \times \mathbb{Z}_d\)
(11)
define a projective representation of the finite Weyl group. The vectors \(|\alpha, \beta\rangle\)
\[
|\alpha, \beta\rangle = e^{-\pi i \frac{\alpha \beta}{||\alpha||}} \sum_{j=0}^{s} e^{2\pi i \frac{\alpha \beta}{||\alpha||}} |j - \alpha\rangle |j\rangle,
\]
satisfy the resolution of identity [5, 27]
\[
\frac{1}{d} \sum_{\alpha, \beta \in -s} |\alpha, \beta\rangle \langle \alpha, \beta| = \mathbb{I}.
\]
The tight frame \(\{\alpha, \beta\}\)'s can be regarded as a finite system of coherent states [7] labelled
by using the set \(\mathbb{Z}_d \times \mathbb{Z}_d\), directly related to the
finite phase space \(S_d \times S_d\).

The mathematical objects defined above correspond in the large
\(d\) limit to those usually
considered in the case of the quantum harmonic oscillator. An extensive list concerning this
correspondence can be found in table 1.

**Example.** The Hamiltonian
\[
H_{\text{free}} : \mathcal{H} \longrightarrow \mathcal{H}, \quad H_{\text{free}} = \frac{1}{2} \mathbf{p}^2
\]
admits the non-degenerate ground level
\[
\lambda_0 = 0 \quad \text{with} \quad |0\rangle \quad \text{a corresponding eigenvector}
\]
and the doubly-degenerate energy levels
\[
\lambda_1 = \frac{\pi}{d} 1^2 \quad \text{with} \quad |\pm 1\rangle \quad \text{orthogonal eigenvectors}
\]
\[
\lambda_2 = \frac{\pi}{d} 2^2 \quad \text{with} \quad |\pm 2\rangle \quad \text{orthogonal eigenvectors}
\]
\[
\vdots
\]
\[
\lambda_s = \frac{\pi}{d} s^2 \quad \text{with} \quad |\pm s\rangle \quad \text{orthogonal eigenvectors}.
\]
Therefore,
\[
H_{\text{free}} = \frac{\pi}{d} \sum_{n=-s}^{s} n^2 |n\rangle \langle n|
\]
and the evolution operator
\[
e^{-iH_{\text{free}}} = \sum_{n=-s}^{s} e^{-\frac{\pi i}{2} n^2} |n\rangle \langle n|
\]
is periodic with period \(2d\)
\[
e^{-i(t+2d)H_{\text{free}}} = e^{-iH_{\text{free}}}.
\]
For any state \(|\psi\rangle \in \mathcal{H}\), the corresponding time-dependent state
\[
\Psi : \mathbb{Z}_d \times \mathbb{R} \longrightarrow \mathbb{C}, \quad \Psi(n, t) = \langle n|e^{-iH_{\text{free}}} |\psi\rangle
\]
is periodic in time
\[
\Psi(n, t + 2d) = \Psi(n, t).
\]
Note that in the continuum limit, when \(d\) tends to infinity, the periodicity of free evolution practically disappears, and one obtains the result known from continuous-configuration quantum mechanics.

A similar periodicity has been obtained in [3] for a free wavepacket moving in a discrete quantum phase space. However, in [3] the discrete-eigenvalues position and momentum operators were defined differently, with the result that the revivals appear for minimum uncertainty states but are only approximate for long time evolution in other cases.
3. On the commutator \([Q, P]\)

In this section, we present a result similar to Floratos [6], in a version adapted to our operators \(Q\) and \(P\), and a numerical estimation of the spectrum of \([Q, P]\). The matrices of \(Q\) and \(F\) in the basis \((|\ell\rangle)_{\ell \in \mathbb{N}}\) are

\[
(q_{\ell \ell'})_{\ell, \ell' \in \mathbb{N}} = \left(\sqrt{\frac{2\pi}{d}} \hat{J}_{\ell \ell'}\right)_{\ell, \ell' \in \mathbb{N}} \quad \text{and} \quad (F_{\ell \ell'})_{\ell, \ell' \in \mathbb{N}} = \left(\frac{1}{\sqrt{d}} \hat{R}_{\ell \ell'}\right)_{\ell, \ell' \in \mathbb{N}}.
\]
Since \( P = F Q F^+ \), the matrix elements of \( P \) in the same basis are
\[
p_{j\ell} = (F \hat{Q} F^+)^{j\ell} = \frac{1}{d} \sqrt{\frac{2 \pi}{d}} \sum_{k=-s}^{s} k e^{\frac{2 \pi i (j-\ell)k}{d}}.
\]
If we differentiate with respect to \( x \) the identity
\[
\sum_{k=-s}^{s} e^{kx} = e^{-sx} \frac{e^{dx} - 1}{e^x - 1}
\]
true for \( x \neq 0 \), then we obtain the relation
\[
\sum_{k=-s}^{s} k e^{kx} = -s e^{-sx} \frac{e^{dx} - 1}{e^x - 1} + e^{-sx} \frac{d \left( e^x - 1 \right)}{(e^x - 1)^2} - 1
\]
which for \( x = \frac{2 \pi i}{d} (j - \ell) \) becomes
\[
\sum_{k=-s}^{s} k e^{\frac{2 \pi i (j-\ell)k}{d}} = e^{-\frac{2 \pi i (j-\ell)s}{d}} \frac{d}{e^{\frac{2 \pi i (j-\ell)}{d}} - 1}.
\]
Since \( s = \frac{d-1}{2} \), the last relation can be written as
\[
\sum_{k=-s}^{s} k e^{\frac{2 \pi i (j-\ell)k}{d}} = (-1)^{j-\ell} \frac{d}{e^{\frac{2 \pi i (j-\ell)}{d}} - 1}
\]
and we have
\[
p_{j\ell} = \begin{cases} 0 & \text{if } j = \ell \\ -\frac{i}{2} \sqrt{\frac{2 \pi}{d}} \frac{(-1)^{j-\ell}}{\sin \frac{\pi}{d} (j-\ell)} & \text{if } j \neq \ell. \end{cases}
\]
The matrix elements of the commutator \([Q, P]\) are
\[
[Q, P]_{j\ell} = \begin{cases} 0 & \text{if } j = \ell \\ -i \frac{\pi}{d} \frac{(j-\ell) (-1)^{j-\ell}}{\sin \frac{\pi}{d} (j-\ell)} & \text{if } j \neq \ell. \end{cases}
\]
and for large \( d \), they can be approximated as follows [6]:
\[
[Q, P]_{j\ell} \approx i (-1)^{j-\ell} (\delta_{j\ell} - 1).
\]
The matrix
\[
(i (-1)^{j-\ell} (\delta_{j\ell} - 1))^2_{j\ell=-s}
\]
has the eigenvectors
\[
\left( \frac{1}{\sqrt{d}} (-1)^{k} e^{\frac{2 \pi i k}{d}} \right)^{s}_{j=-s}
\]
with eigenvalue
\[
i & \text{for } k \in \{-s, \ldots, -1, 1, \ldots, s\} \\
(1-d)i & \text{for } k = 0.
\]
This means that \( d - 1 \) of the eigenvalues of \([Q, P]\) are equal to \( i \) for large \( d \). We can consider that, in a certain sense,
\[
[Q, P] \approx i.
\]
A numerical estimation of the eigenvalues of the commutator \([Q, P]\) in the case \( d = 15 \) can be seen in table 2. Already for this relatively small \( d \) value, a significant number of eigenvalues tend to \( i \).
Table 2. The eigenvalues $\eta_k$ of the commutator $[\hat{Q}, \hat{P}]$ in the case $d = 15$.

| $k$ | $\eta_k$ | $k$ | $\eta_k$ | $k$ | $\eta_k$ |
|-----|----------|-----|----------|-----|----------|
| 0   | -27.276 466 375 122 i | 5   | 0.999 998 706 977 i | 10  | 1.000 016 906 603 i |
| 1   | -4.322 222 514 423 i  | 6   | 0.999 999 967 717 i | 11  | 1.001 534 631 543 i |
| 2   | 0.649 632 619 978 i   | 7   | 0.999 999 998 i   | 12  | 1.067 898 771 074 i |
| 3   | 0.988 901 431 861 i   | 8   | 1.000 000 000 091 i | 13  | 2.560 890 405 316 i |
| 4   | 0.999 822 475 466 i   | 9   | 1.000 000 076 444 i | 14  | 18.329 992 867 471 i |

4. Minimum uncertainty states

Let $\kappa \in (0, \infty)$, $s \in \{1, 2, 3, \ldots\}$ and let $d = 2s + 1$. The function

$$G_\kappa : \mathbb{R} \rightarrow \mathbb{R}, \quad G_\kappa (x) = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}}$$

obtained by starting from the Gaussian function

$$g_\kappa : \mathbb{R} \rightarrow \mathbb{R}, \quad g_\kappa (x) = e^{-\frac{x^2}{2\kappa}}$$

is a periodic function with period $d$. The finite Gaussian

$$g_\kappa : \mathbb{Z}_d \rightarrow \mathbb{R}, \quad g_\kappa (n) = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + n)^2}}$$

which can be regarded as a finite version of $g_\kappa$, satisfies the relation

$$g_\kappa (-n) = g_\kappa (n), \quad \text{for any} \quad n \in \mathbb{Z}_d.$$

Theorem 1 ([19]). We have

$$F[g_\kappa ] = \frac{1}{\sqrt{\kappa}} g_\kappa^* \quad \text{for any} \quad \kappa \in (0, \infty). \quad (13)$$

Proof. The function $G_\kappa (x)$ admits the Fourier expansion

$$G_\kappa (x) = \sum_{\ell = -\infty}^{\infty} a_\ell e^{\frac{2\pi i \ell x}{d}}$$

with

$$a_\ell = \frac{1}{d} \int_0^d e^{-\frac{2\pi i \ell x}{d}} \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} dx = \frac{1}{d} \sum_{\alpha = -\infty}^{\infty} \int_0^d e^{-\frac{2\pi i \ell x}{d}} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} dx$$

By denoting $t = \sqrt{\frac{2\pi}{d}} (\alpha d + x)$ and using the relation

$$\int_{-\infty}^{\infty} e^{it} e^{-a^2} dt = \sqrt{\frac{\pi}{a}} e^{-\frac{t^2}{4a^2}}$$

we obtain [15]

$$a_\ell = \frac{1}{\sqrt{2\pi} d} \sum_{\alpha = -\infty}^{\infty} \int_{\alpha \sqrt{2\pi} d}^{(\alpha + 1) \sqrt{2\pi} d} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} dx$$

$$= \frac{1}{\sqrt{2\pi} d} \sum_{\alpha = -\infty}^{\infty} \int_{\alpha \sqrt{2\pi} d}^{(\alpha + 1) \sqrt{2\pi} d} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} dx$$

$$= \frac{1}{\sqrt{2\pi} d} \int_{-\infty}^{\infty} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} e^{-\frac{\kappa}{\sqrt{2\pi} (\alpha d + x)^2}} dx = \frac{1}{\sqrt{k} \kappa} e^{-\frac{\kappa}{\sqrt{2\pi} d}}$$
whence

\[ G_\kappa (x) = \frac{1}{\sqrt{kd}} \sum_{\ell=-\infty}^{\infty} e^{\pi i \ell x} e^{-\ell^2}. \]

Particularly, we have

\[ g_\kappa (j) = G_\kappa (j) = \frac{1}{\sqrt{kd}} \sum_{\ell=-\infty}^{\infty} e^{\pi i \ell j} e^{-\ell^2} \]

\[ = \frac{1}{\sqrt{kd}} \sum_{n=-s}^{s} \sum_{\alpha=-\infty}^{\infty} e^{\pi i \ell (\alpha j + n)} e^{-\ell^2 (\alpha j + n)^2} \]

\[ = \frac{1}{\sqrt{kd}} \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{\pi i j n} \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\ell^2}{d} (\alpha j + n)^2} \]

whence

\[ \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{\pi i j n} \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\ell^2}{d} (\alpha j + n)^2} = \sqrt{\kappa} \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\ell^2}{d} (\alpha + j)^2}. \]

\( \square \)

Since \( g_\kappa (n) = g_\kappa (n) \) and \( g_\kappa (n) = g_\kappa (n) \), we have

\[ \sum_{n=-s}^{s} n (g_\kappa (n))^2 = \sum_{n=-s}^{s} n (g_\kappa (n))^2 = 0. \]

Therefore, the square of the dispersion of \( Q \) in the state described by the finite Gaussian

\[ \mathbb{g}_\kappa : \mathbb{Z}_d \rightarrow \mathbb{R}, \quad \mathbb{g}_\kappa (n) = \sum_{\alpha=-\infty}^{\infty} e^{-\frac{\ell^2}{d} (\alpha j + n)^2} \]

is

\[ (\Delta Q)^2 = (Q^2) - (Q)^2 = \frac{2\pi \sum_{n=-s}^{s} n^2 (g_\kappa (n))^2}{d \sum_{n=-s}^{s} (g_\kappa (n))^2} \]

and, in view of the relation \( P = FQF^+ \), the square of the dispersion of \( P \) is

\[ (\Delta P)^2 = (P^2) - (P)^2 = \frac{2\pi \sum_{n=-s}^{s} n^2 (g_\kappa (n))^2}{d \sum_{n=-s}^{s} (g_\kappa (n))^2}. \]

We have

\[ \Delta Q \Delta P = \frac{2\pi}{d} \left( \frac{\sum_{n=-s}^{s} n^2 (g_\kappa (n))^2}{\sum_{n=-s}^{s} (g_\kappa (n))^2} \right)^{\frac{1}{2}} \left( \frac{\sum_{n=-s}^{s} n^2 (g_\kappa (n))^2}{\sum_{n=-s}^{s} (g_\kappa (n))^2} \right)^{\frac{1}{2}}. \]

As concerns the expectation value of \( [Q, P] \), by using relation (12), we obtain

\[ \langle [Q, P] \rangle = \frac{\langle \mathbb{g}_\kappa \mathbb{P} \mathbb{P} \mathbb{g}_\kappa \rangle}{\langle \mathbb{g}_\kappa \mathbb{g}_\kappa \rangle} = \sum_{n=-s}^{s} (g_\kappa (n))^2 \left( \sum_{j=-s}^{s} \sum_{\ell=-s}^{s} (-1)^{j-\ell} e^{\pi i j (j-\ell) / 2} \mathbb{g}_\kappa (j) \mathbb{g}_\kappa (\ell) \right). \]

The well-known uncertainty relation originating from the Schwarz inequality

\[ \Delta Q \Delta P \geq \frac{1}{2} |\langle [Q, P] \rangle| \tag{14} \]

is satisfied, and for \( \kappa = 1 \) the difference

\[ \Delta Q \Delta P - \frac{1}{2} |\langle [Q, P] \rangle| \approx 0 \tag{15} \]
The state $g_1$ is a quasi-minimum uncertainty state.

$$d / \Delta Q / \Delta P \left| \langle [Q, P] \rangle \right| \Delta Q \Delta P - d / \left| \langle [Q, P] \rangle \right|$$

| $d$  | $\Delta Q \Delta P$ | $\frac{1}{2} \left| \langle [Q, P] \rangle \right|$ | $\Delta Q \Delta P - \frac{1}{2} \left| \langle [Q, P] \rangle \right|$ |
|------|-----------------|------------------|------------------|
| 3    | 0.442 597 763 118 52 | 0.442 597 763 118 52 | 0.0 \times 10^{-75} |
| 5    | 0.497 099 938 415 60 | 0.496 206 497 579 54 | 0.000 893 440 |
| 7    | 0.499 859 143 647 43 | 0.499 851 404 927 77 | 7.738 719 663 \times 10^{-4} |
| 9    | 0.499 993 279 725 81 | 0.499 990 989 929 68 | 2.289 796 128 \times 10^{-4} |
| 11   | 0.499 999 684 160 91 | 0.499 999 654 409 67 | 2.975 123 667 \times 10^{-4} |
| 13   | 0.499 999 985 327 38 | 0.499 999 980 263 67 | 5.063 715 121 \times 10^{-4} |
| 15   | 0.499 999 999 324 43 | 0.499 999 999 243 81 | 8.061 781 262 \times 10^{-11} |

Table 4. The eigenvalues of $H = \frac{1}{2}(P^2 + Q^2)$.

| $d$  | $d = 3$ | $d = 5$ | $d = 7$ | $d = 9$ | $d = 11$ | $d = 13$ |
|------|---------|---------|---------|---------|---------|---------|
| –    | –       | –       | –       | –       | –       | 15.685 806 |
| –    | –       | –       | –       | –       | –       | 12.088 829 |
| –    | –       | –       | –       | –       | –       | 12.908 813 |
| –    | –       | –       | –       | –       | –       | 9.802 541 |
| –    | –       | –       | –       | –       | –       | 9.713 488 |
| –    | –       | –       | –       | –       | –       | 10.156 706 |
| –    | –       | –       | 7.601 849 | 7.799 516 | 7.588 461 |
| –    | –       | –       | 7.433 857 | 5.929 737 | 6.324 626 | 6.469 345 |
| –    | –       | –       | 5.501 405 | 5.772 956 | 5.541 025 | 5.505 452 |
| –    | –       | 4.745 031 | 4.092 770 | 4.414 645 | 4.489 404 | 4.498 956 |
| –    | –       | 3.512 928 | 3.629 951 | 3.514 121 | 3.501 381 | 3.500 114 |
| –    | –       | 2.094 395 | 2.273 277 | 2.472 337 | 2.497 725 | 2.499 837 |
| –    | –       | 1.651 797 | 1.538 153 | 1.502 561 | 1.500 166 | 1.500 009 |
| 0.442 597 | 0.496 978 | 0.499 856 | 0.499 993 | 0.499 999 | 0.499 999 |

except for a few small values of $d$. Numerical results concerning the case $\kappa = 1$ are presented in table 3.

Note that for different definitions of the position and momentum operators in a discrete quantum phase space [3], the minimum uncertainty of these operators is dependent on the discretization step (the exact formula is known as the generalized uncertainty principle) and approaches the result for the continuum case only for an infinitely fine discretization; the generalized uncertainty principle can be obtained from a quantum mechanical model with discrete eigenvalues for the coordinate operator [2]. In contrast, in our case the uncertainty is approximately minimum for quite small $d$ values. Moreover, in [14] the uncertainty relation was shown to reach its minimum value only in particular cases, but approximate expansions of the unitary position and momentum operators on particular states were used.

5. Finite-dimensional quantum system of oscillator type

The Hamiltonian $H = \frac{1}{2}(P^2 + Q^2)$ is of harmonic oscillator type, but a certain similitude between the behaviour of our quantum system with finite-dimensional Hilbert space and the standard harmonic oscillator exists only for a large enough dimension $d$. Particularly, the tendency to have equidistant energy levels becomes evident only for $d$ large enough (see table 4 and figure 2).

The finite Gaussian $g_1$ is a quasi-eigenstate of the oscillator-type Hamiltonian $H = \frac{1}{2}(P^2 + Q^2)$. 
Figure 2. The energy levels of $H = \frac{1}{2}(P^2 + Q^2)$.

Table 5. The state $\psi_1$ is a quasi-eigenstate of $H = \frac{1}{2}(P^2 + Q^2)$.

| $d$ | $\lambda$ | $(H\psi_1 - \lambda\psi_1)(\pm 1)$ | $(H\psi_1 - \lambda\psi_1)(\pm 2)$ | $(H\psi_1 - \lambda\psi_1)(\pm 3)$ | $(H\psi_1 - \lambda\psi_1)(\pm 4)$ | $(H\psi_1 - \lambda\psi_1)(\pm 5)$ |
|-----|-----------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| 3   | 0.442 598 | $2.2 \times 10^{-16}$             | $1.2 \times 10^{-3}$              | $2.8 \times 10^{-4}$              | $5.8 \times 10^{-4}$              | $1.1 \times 10^{-4}$              |
| 5   | 0.489 794 | $1.2 \times 10^{-2}$              | $8.5 \times 10^{-4}$              | $1.5 \times 10^{-4}$              | $3.1 \times 10^{-4}$              | $4.3 \times 10^{-5}$              |
| 7   | 0.498 096 | $2.8 \times 10^{-3}$              | $1.3 \times 10^{-4}$              | $1.6 \times 10^{-4}$              | $4.8 \times 10^{-5}$              | $5.1 \times 10^{-5}$              |
| 9   | 0.499 638 | $5.8 \times 10^{-4}$              | $1.3 \times 10^{-4}$              | $2.0 \times 10^{-4}$              | $5.5 \times 10^{-4}$              | $1.0 \times 10^{-4}$              |
| 11  | 0.499 93  | $1.1 \times 10^{-4}$              | $3.1 \times 10^{-4}$              | $2.0 \times 10^{-4}$              | $5.5 \times 10^{-4}$              | $1.0 \times 10^{-4}$              |

We have (see table 5)

$$H\psi_1 \approx \lambda\psi_1 \quad \text{for} \quad \lambda = \frac{H\psi_1(0)}{\psi_1(0)}.$$  

This relation can be regarded as an approximate finite version of the relation

$$\frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right) e^{-\frac{1}{2}x^2} = \frac{1}{2} e^{-\frac{1}{2}x^2}.$$

6. On the occurrence of revivals

**Theorem 2.** If $H$ has $k \geq 2$ commensurate energy levels, then there exist revivals.

**Proof.** If the energy levels $\varepsilon_1$, $\varepsilon_2$, ..., $\varepsilon_k$ are commensurate, then $\varepsilon_2/\varepsilon_1$, $\varepsilon_3/\varepsilon_1$, ..., $\varepsilon_k/\varepsilon_1$ are rational numbers and can be represented as fractions. If $m$ is the least common multiple of the denominators of these fractions, then there exist the integers $\ell_2$, $\ell_3$, ..., $\ell_k$ such that

$$\frac{\varepsilon_j}{\varepsilon_1} = \frac{\ell_j}{m} \in \mathbb{Q} \quad \text{for any} \quad j \in \{2, 3, ..., k\}.$$

If $|\psi_1\rangle$, $|\psi_2\rangle$, ..., $|\psi_k\rangle$ are the eigenstates corresponding to $\varepsilon_1$, $\varepsilon_2$, ..., $\varepsilon_k$, that is,

$$H|\psi_j\rangle = \varepsilon_j|\psi_j\rangle \quad \text{for any} \quad j \in \{1, 2, ..., k\},$$

then the time-dependent state

$$\Psi: \mathbb{Z}_d \times \mathbb{R} \rightarrow \mathbb{C}, \quad \Psi(n, t) = \langle n | e^{-iHt} | \psi_j \rangle,$$

corresponding to an arbitrary state of the form (certain coefficients may be 0)

$$|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle + \cdots + \alpha_k|\psi_k\rangle$$
is periodic with the period $\frac{2\pi}{\varepsilon_1}$. Indeed,

$$e^{-iH}|\psi\rangle = \alpha_1 e^{-i\varepsilon_1}|\psi_1\rangle + \alpha_2 e^{-i\varepsilon_2}|\psi_2\rangle + \cdots + \alpha_k e^{-i\varepsilon_k}|\psi_k\rangle$$

and we have

$$e^{-i\varepsilon + \frac{2\pi}{\varepsilon_1}H}|\psi\rangle = e^{-iH}|\psi\rangle$$

whence

$$\Psi\left(n, t + \frac{2m\pi}{\varepsilon_1}\right) = \Psi(n, t).$$

**Theorem 3.** *If $H$ has $k \geq 3$ equidistant energy levels $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$, then there exist revivals.*

**Proof.** If $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle$ are corresponding eigenstates

$$H|\psi_j\rangle = \varepsilon_j|\psi_j\rangle$$

for any $j \in \{1, 2, \ldots, k\}$, then the time-dependent state

$$\Psi : \mathbb{Z}_d \times \mathbb{R} \rightarrow \mathbb{C}, \quad \Psi(n, t) = |n\rangle e^{-iH}|\psi\rangle$$

corresponding to an arbitrary state of the form (certain coefficients may be 0)

$$|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle + \cdots + \alpha_k|\psi_k\rangle$$

is periodic with the period $\frac{2\pi}{\varepsilon_2 - \varepsilon_1}$. Indeed,

$$e^{-iH}|\psi\rangle = \alpha_1 e^{-i\varepsilon_1}|\psi_1\rangle + \alpha_2 e^{-i\varepsilon_2}|\psi_2\rangle + \cdots + \alpha_k e^{-i\varepsilon_k}|\psi_k\rangle$$

and, up to a phase factor, we have

$$e^{-i\varepsilon + \frac{2\pi}{\varepsilon_2 - \varepsilon_1}H}|\psi\rangle = e^{-iH}|\psi\rangle$$

whence

$$\Psi\left(n, t + \frac{2\pi}{\varepsilon_2 - \varepsilon_1}\right) = \Psi(n, t).$$

The number $\frac{2\pi}{\varepsilon_2 - \varepsilon_1}$ is a period for any coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$, but generally, it is not a ‘fundamental’ period. For example, in the particular case

$$|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_3|\psi_3\rangle,$$

there exists a smaller period, namely $\frac{\pi}{\varepsilon_3 - \varepsilon_1}$. □

In the case of our finite-dimensional oscillator, we have equidistant levels and hence revivals only for $d$ large enough. Once again, the discrete–continuum transition recovers the standard results for the quantum harmonic oscillator. But, more importantly, the proposition above relates the revivals to the condition of equidistant energy levels. From a physical point of view, this relation is extremely important: unlike for free evolution, where the revival period was determined by $d$ but the energy spectrum had no equidistant levels and thus the revivals disappeared in the continuous limit, for the harmonic oscillator case $d$ does not explicitly enter the expression of the revival period. This period does not disappear in the large $d$ limit; in contrast, a large $d$ guarantees equidistant energy levels, which determine a sort of physical feedback necessary for revivals.
7. Discrete Wigner function

Let $\kappa \in (0, \infty)$, $s \in \{1, 2, 3, \ldots\}$ and let $d = 2s + 1$. The periodic function, $G_+^\kappa: \mathbb{R} \longrightarrow \mathbb{R}$,

$$G_+^\kappa(x) = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa}{2} \left( \sqrt{\frac{d}{\pi}} \left( (\alpha + \frac{1}{2})d + x \right) \right)^2},$$

with period $d$ allows us to define the function $g_+^\kappa: \mathbb{Z}_d \longrightarrow \mathbb{R}$,

$$g_+^\kappa(n) = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa\pi}{d} \left( (\alpha + \frac{1}{2})d + n \right)^2} = \sum_{\alpha = -\infty}^{\infty} e^{-\frac{\kappa\pi}{d} \left( (\alpha - \frac{1}{2})d + n \right)^2}.$$

Since

$$g_+^\kappa(n) = G_+^\kappa \left( n + s + \frac{1}{2} \right) \quad \text{and} \quad g_+^\kappa(n) = G_+^\kappa(n),$$

the function $g_+^\kappa$ is a kind of translated finite Gaussian (see figure 3). By direct computation, one can prove the relations

$$g_+^\kappa(-n) = g_+^\kappa(n), \quad \text{and} \quad g_+^\kappa(2n) = g_+^\kappa(n) + g_2^\kappa(n)$$

and

$$\sum_{\alpha = -\infty}^{\infty} (-1)^{\alpha} e^{-\frac{\kappa\pi}{d} \left( \alpha d + 2n \right)^2} = g_+^\kappa(n) - g_4^\kappa(n).$$
Lemma 1. The finite Fourier transform of $\mathcal{g}^\pm_{2\kappa}$ satisfies the relation
\[
\mathcal{F}[\mathcal{g}^\pm_{2\kappa}](2m) = \frac{1}{\sqrt{2\kappa}}(\mathcal{g}^\pm_2(m) - \mathcal{g}^\pm_2(m)).
\] (16)

Proof. The periodic function $G^\pm_{2\kappa}(x)$ admits the Fourier expansion
\[
G^\pm_{2\kappa}(x) = \sum_{\ell=-\infty}^{\infty} c_\ell e^{\frac{2\pi i}{2\kappa} \ell x}
\]
with
\[
c_\ell = \frac{1}{d} \int_0^d e^{-\frac{2\pi i}{2\kappa} \ell x} \sum_{n=0}^{\infty} e^{-\sqrt{\kappa}((\alpha + \frac{1}{2})d + x)} \, dx = \frac{1}{d} \sum_{n=0}^{\infty} \int_0^d e^{-\frac{2\pi i}{2\kappa} \ell x} e^{-\sqrt{\kappa}((\alpha + \frac{1}{2})d + x)} \, dx.
\]
By denoting $t = \sqrt{\frac{2\pi}{d}} ((\alpha + \frac{1}{2})d + x)$, we obtain
\[
c_\ell = \frac{1}{2\pi d} \sum_{n=0}^{\infty} \int_{(\alpha - \frac{1}{2})d}^{(\alpha + \frac{1}{2})d} e^{-\frac{2\pi i}{2\kappa} \ell x} e^{-\sqrt{\kappa}((\alpha + \frac{1}{2})d + x)} \, dt = \frac{(-1)^\ell}{\sqrt{2\pi d}} \sum_{n=0}^{\infty} \int_{(\alpha - \frac{1}{2})d}^{(\alpha + \frac{1}{2})d} e^{-it\sqrt{\kappa}} e^{-\sqrt{\kappa}((\alpha + \frac{1}{2})d + x)} \, dt = \frac{(-1)^\ell}{\sqrt{2\pi d}} \int_{-\infty}^{\infty} e^{-it\sqrt{\kappa}} e^{-\sqrt{\kappa}((\alpha + \frac{1}{2})d + x)} \, dt = \frac{(-1)^\ell}{\sqrt{\kappa}} e^{-\frac{\pi i}{\kappa} \ell^2}.
\]
whence
\[
G^\pm_{2\kappa}(x) = \frac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} e^{\frac{2\pi i}{2\kappa} \ell x} (-1)^\ell e^{-\frac{\pi i}{\kappa} \ell^2}.
\]
Particularly, we have
\[
\mathcal{g}^\pm_{2\kappa}(j) = G^\pm_{2\kappa}(j) = \frac{1}{\sqrt{2\kappa d}} \sum_{\ell=-\infty}^{\infty} e^{\frac{2\pi i}{2\kappa} \ell j} (-1)^\ell e^{-\frac{\pi i}{\kappa} \ell^2} = \frac{1}{\sqrt{2\kappa d}} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} e^{\frac{2\pi i}{2\kappa} \ell j (\alpha + \frac{1}{2})d + n} (-1)^{\ell(\alpha d + n)} e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + n)^2} = \frac{1}{\sqrt{2\kappa d}} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} e^{\frac{2\pi i}{2\kappa} \ell j (\alpha + \frac{1}{2})d + n} (-1)^{\ell n} e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + n)^2} = \frac{1}{\sqrt{2\kappa d}} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^n e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + n)^2} = \frac{(-1)^n}{\sqrt{2\kappa}} \sum_{\ell=-\infty}^{\infty} e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + n)^2}.
\]
whence
\[
\mathcal{F}[\mathcal{g}^\pm_{2\kappa}](n) = \frac{(-1)^n}{\sqrt{2\kappa}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + n)^2}
\]
and we obtain
\[
\mathcal{F}[\mathcal{g}^\pm_{2\kappa}](2m) = \frac{1}{\sqrt{2\kappa}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell e^{-\frac{\pi i}{\kappa} \ell^2 (\alpha d + 2m)^2} = \frac{1}{\sqrt{2\kappa}} (\mathcal{g}^\pm_2(m) - \mathcal{g}^\pm_2(m)).
\]
□
Lemma 2. If the numbers \(N_{\alpha, \beta}\) are such that the series are absolutely convergent, then

\[
\sum_{\alpha, \beta = -\infty}^{\infty} N_{\alpha, \beta} = \sum_{\mu, \eta = -\infty}^{\infty} N_{\mu + \eta, \mu - \eta} + \sum_{\mu, \eta = -\infty}^{\infty} N_{\mu + \eta + 1, \mu - \eta}.
\]

Proof. We separate the sum as [13]

\[
\sum_{\alpha, \beta = -\infty}^{\infty} N_{\alpha, \beta} = \sum_{\alpha, \beta \text{ both even or odd}} N_{\alpha, \beta} + \sum_{\alpha, \beta \text{ one even and other odd}} N_{\alpha, \beta}
\]

and use the substitutions \((\alpha, \beta) = (\mu + \eta, \mu - \eta)\) and \((\alpha, \beta) = (\mu + \eta + 1, \mu - \eta)\), respectively. □

The function \(W : \mathbb{Z}_d \times \mathbb{Z}_d \rightarrow \mathbb{C}\)

\[
W(n, m) = \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} g_k(n - k) g_k(n + k)
\]

\[
= \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\mu, \eta = -\infty}^{\infty} e^{-\frac{\pi i}{d} (a d + n - k)^2} e^{-\frac{\pi i}{d} (b d + n + k)^2}
\]

is called the discrete Wigner function corresponding to \(g_k\). It is well determined by its restriction to the unit cell \([-s, -s + 1, \ldots, s - 1, s] \times \{-s, -s + 1, \ldots, s - 1, s\}\) directly related to the finite phase space \(S_d \times S_d\).

Theorem 4. The discrete Wigner function \(W\) is a sum of products of finite Gaussians

\[
W(n, m) = \frac{1}{\sqrt{2\pi d}} g_{2s}(n) \left(g_{\frac{1}{2}}(m) + g_{\frac{1}{2}}^\dagger(m)\right) + \frac{1}{\sqrt{2\pi d}} g_{2s}^\dagger(n) \left(g_{\frac{1}{2}}(m) - g_{\frac{1}{2}}^\dagger(m)\right).
\]

Proof. By using theorem 1, lemma 1 and lemma 2, we obtain

\[
W(n, m) = \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\mu, \eta = -\infty}^{\infty} e^{-\frac{\pi i}{d} ((\mu + \eta) d + n - k)^2} e^{-\frac{\pi i}{d} ((\mu - \eta) d + n + k)^2}
\]

\[
+ \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\mu, \eta = -\infty}^{\infty} e^{-\frac{\pi i}{d} ((\mu + \eta + 1) d + n - k)^2} e^{-\frac{\pi i}{d} ((\mu - \eta) d + n + k)^2}
\]

\[
= \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\mu, \eta = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n - k)^2} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n + k)^2}
\]

\[
+ \frac{1}{d} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\mu, \eta = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n - k)^2} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n + k)^2}
\]

\[
= \frac{1}{\sqrt{d}} \sum_{\mu = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n - k)^2} \frac{1}{\sqrt{d}} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\eta = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n - k)^2}
\]

\[
+ \frac{1}{\sqrt{d}} \sum_{\mu = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n - k)^2} \frac{1}{\sqrt{d}} \sum_{k = -s}^{s} e^{\frac{2\pi i mk}{d}} \sum_{\eta = -\infty}^{\infty} e^{-\frac{2\pi i}{d} ((\mu + \frac{1}{2}) d + n + k)^2}
\]
The Wigner function $W(n, m)$ in the case $\kappa = \frac{1}{3}$, $d = 31$.

\[
= \frac{1}{\sqrt{d}} g_{2\kappa}(n) F[g_{2\kappa}](2m) + \frac{1}{\sqrt{d}} \theta_{2\kappa}^+(n) \frac{1}{\sqrt{d}} \sum_{k=-s}^{s} e^{i \pi mk} \sum_{\eta=-\infty}^{\infty} e^{-2\sqrt{d} g_{2\kappa} \left(\eta - \frac{1}{d}d+s\right)^2} \\
= \frac{1}{\sqrt{2\kappa d}} g_{2\kappa}(n) g_{\frac{1}{2}}^+(2m) + \frac{1}{\sqrt{d}} \theta_{2\kappa}^+(n) F[g_{2\kappa}^+](2m) \\
= \frac{1}{\sqrt{2\kappa d}} g_{2\kappa}(n) \left( g_{\frac{1}{2}}(m) + g_{\frac{1}{2}}^+(m) \right) + \frac{1}{\sqrt{2\kappa d}} \theta_{2\kappa}^+(n) \left( g_{\frac{1}{2}}(m) - g_{\frac{1}{2}}^+(m) \right).
\]

The shape of the Wigner function $W$, directly related to the shape of the functions involved in expression (18), depends on the value of the parameter $\kappa$. For example, for $d$ large enough and $\frac{1}{2} < \kappa < 2$, the Wigner function $W$ has three peaks placed around $(0, 0)$, $(0, s)$, $(s, 0)$ and an anti-peak around $(s, s)$ (see figures 3 and 4). For other values of $\kappa$, the shape of the Wigner function $W$ may be very different (see figures 5 and 6).

The vacuum state $|0, 0\rangle$ used in [12] coincides in our notation with the function $\frac{e^{i\pi |\langle \vec{g} |n\rangle|}}{|\langle \vec{g} |n\rangle|}$. The three-dimensional plot of the Wigner function corresponding to $|0, 0\rangle$ presented in [12] is similar to our figure 4. In [16] and [17], the authors investigate the Wigner function corresponding to the ground state in the case of the Hamiltonians

\[
\hat{H}_0 = -\cos \hat{U} - \cos \hat{V}, \quad \hat{H}_1 = (1/2)\hat{U}^2 + (1/2)\hat{V}^2
\]
Figure 5. The Wigner function $W(n, m)$ in the case $\kappa = 10$, $d = 31$.

where

$$\hat{U} = \sum_{n=0}^{d-1} n|n\rangle\langle n|, \quad \hat{V} = \sum_{n=0}^{d-1} n|\tilde{n}\rangle\langle \tilde{n}|.$$  

The obtained three-dimensional plots of the Wigner function are also similar to our figure 4. A possible explanation for this coincidence is the following. From the relations

$$F\hat{U}F^+ = \hat{V}, \quad F\hat{V}F^+ = -\hat{U},$$

we obtain the relations

$$F\hat{H}_0 = \hat{H}_0 F \quad \text{and} \quad F\hat{H}_1 = \hat{H}_1 F$$

which show that a non-degenerate ground state must be an eigenstate of $F$. We know that $F[g_1] = g_1$, and we think that in the case of the two Hamiltonians the ground state is almost identical to $g_1$ (see our table 5).

By using Schwarz inequality, we obtain

$$W(0,0) = \frac{1}{d} \sum_{k=-s}^{s} g_k(-k\bar{g}_k(k)) \leq \frac{1}{d} \sqrt{\sum_{k=-s}^{s} (g_k(-k))^2 \sum_{k=-s}^{s} (g_k(k))^2}$$

$$= \frac{1}{d} \sum_{k=-s}^{s} (g_k(k))^2 \approx \frac{1}{d} \sum_{k=-s}^{s} (g_k(k))^2 \leq \frac{2\delta_d + 1}{d}(g_0(0))^2$$
Figure 6. The Wigner function $W(n, m)$ in the case $\kappa = 30, d = 31$.

Table 6. The number $2s_d + 1$ of $k$ with $(g_\kappa (k))^2 > 0.0001$.

| $d$  | $2s_d + 1$ | $\frac{2s_d + 1}{d}$ | $d$  | $2s_d + 1$ | $\frac{2s_d + 1}{d}$ |
|------|-----------|----------------------|------|-----------|----------------------|
| 31   | 13        | 0.41...              | 401  | 49        | 0.12...              |
| 101  | 25        | 0.24...              | 801  | 69        | 0.08...              |
| 201  | 35        | 0.17...              | 1001 | 77        | 0.07...              |

where, for example, $s_d = \max \{ k \mid (g_\kappa (k))^2 > 0.0001 \}$. Since

$$g_\kappa (0) = \sum_{\alpha = -\infty}^{\infty} e^{-\kappa \pi d \alpha^2} = \sum_{\alpha = -\infty}^{\infty} e^{-(\sqrt{\kappa \pi d} \alpha)^2}$$

is a decreasing function of $d$, and the numerical results suggest that (see table 6)

$$\lim_{d \to \infty} \frac{2s_d + 1}{d} = 0,$$

we obtain

$$\lim_{d \to \infty} W(0, 0) = 0.$$

The discrete Wigner function $W(n, m)$ defined by (17) does not approximate the continuous Wigner function (4) for large $d$. In contrast to this, we have

$$\lim_{d \to \infty} g_\kappa (0) = \lim_{d \to \infty} \sum_{\alpha = -\infty}^{\infty} e^{-(\sqrt{\kappa \pi d} \alpha)^2} = 1$$

in agreement with the value at the origin of the continuous Gaussian $e^{-\frac{\pi}{2} \chi^2}$.
8. Discussions and conclusions

The most straightforward way to define a Gaussian-type function on the set \{-s, -s + 1, \ldots, s - 1, s\} or \(S_d\) is to consider the restriction of the Gaussian \(g_\kappa(x) = e^{-\frac{\kappa}{2}x^2}\) to these sets

\[
\text{f: } \{-s, -s + 1, \ldots, s - 1, s\} \rightarrow \mathbb{R}, \quad \text{f}(n) = e^{-\frac{n^2}{\kappa}}
\]

respectively,

\[
\text{f: } S_d \rightarrow \mathbb{R}, \quad \text{f}\left(n\sqrt{\frac{2\pi}{d}}\right) = e^{-\frac{n^2}{\kappa}}.
\]

The functions obtained in this way behave similar to the continuous Gaussians only for large values of \(d = 2s + 1\).

Our approach is based on an alternative method. By using a Weil–Zak-type transform, we firstly generate a periodic function \(G_\kappa\) with period \(d\), and then we define our finite Gaussian as a restriction of \(G_\kappa\) to \(\mathbb{Z}\), namely \(g(n) = G_\kappa(n)\). Our finite Gaussians start to behave similar to a continuous Gaussian from relatively small values of \(d\). In addition, they have several remarkable mathematical properties.

The discreteness of the configuration space has a particular appeal in both classical and quantum physics, particularly in phase space [18] or in attempts to unify general relativity and quantum mechanics [12]. On the other hand, the results in a discrete configuration space are required to match standard physical results in the continuum limit. Because discrete physical systems are of interest in quantum mechanics or in classical physics in connection with the problem of sampling, the characterization of such systems and of their evolution has already received some attention. For example, a discrete quantum phase space has been studied in [8, 9], the continuous limit being obtained by decreasing the lattice spacing in both momentum and position space, while in [2] the dynamics on a discrete quantum phase space has been investigated by defining a Hamiltonian with the appropriate classical limit. The results in this paper are different from previous ones and demonstrate the importance of the choice of some particular states in the discrete–continuum transition. As an additional example in this respect, we mention that a discrete model of the quantum harmonic oscillator, for instance, can be introduced such that the energy spectrum is equally spaced and the spectra of both momentum and position operators are denumerable non-degenerate [1]. This last model, too, recovers the results for the ordinary harmonic oscillator in an appropriate limit, as do the discrete models of the quantum harmonic oscillator in terms of Kravchuk polynomials [10] or Harper functions [4].

Summarizing, the correct continuous limit of discrete models can be obtained in many situations. The choice of Gaussian states render the discrete-continuous transition smoother in the sense that many results known from the continuous case are obtained for reasonably large \(d\) values or can easily be extrapolated from the results for finite \(d\) values. Moreover, Gaussian states have advantages over other states in phase space representations of physical systems since, as shown in section 7, the Wigner distribution function has a particularly simple expression in this case. In conclusion, finite Gaussian states represent a useful mathematical tool for the study of quantum or classical physical systems.

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