Fixed-Length Strong Coordination Capacity

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Abstract

This paper investigates the problem of synthesizing joint distributions in the finite-length regime. For a fixed blocklength $n$ and an upper bound on the distribution approximation $\epsilon$, we prove a capacity result for fixed-length strong coordination. It is shown analytically that the rate conditions for the fixed-length regime are lower-bounded by the mutual information that appears in the asymptotical condition plus $Q^{-1}(\epsilon) \sqrt{V/n}$, where $V$ is the channel dispersion, and $Q^{-1}$ is the inverse of the Gaussian cumulative distribution function.

I. INTRODUCTION

The problem of cooperation of autonomous devices in a decentralized network, initially raised in the context of game theory by [1], with applications, for instance, to power control [2], is concerned with communication networks beyond the traditional problem of reliable communications. The goal of coordination is in fact to exceed the classical problem of reliably conveying information, and to characterize the set of target joint probability distributions that are implementable by a choice of strategy of the agents. Coordination is then intended as a way of enforcing a prescribed joint behavior of the devices through communication, by synthesizing joint distributions which approximates a target behavior [3, 4]. This topic, presented as “channel simulation” is related to earlier work on “Shannon’s reverse coding theorem” [5] and the compression of probability distribution sources and mixed quantum states [6–8].

The information-theoretic framework for coordination in networks considered in the present paper has been introduced in [3, 4, 9]. In [3, 4, 9] two metrics to measure the level of coordination have been defined: empirical coordination, which requires the empirical distribution of the distributed random states to approach a target distribution with high probability, and strong coordination, which requires the $L^1$ distance of the distribution of sequences of distributed random states to converge to an i.i.d. target distribution. Strong and empirical coordination in the asymptotical regime have been studied in a number of works, namely [4, 9–23], but until [24] there was no attempt to tackle the problem of coordination in the finite-length regime. However, in many realistic systems of interest, non-asymptotic information-theoretic limits are of high practical interest. Originally raised by [25], and following [26, 27] and more recently [28–30], an increasingly large number of papers have brought up finite blocklength information theory limits (see for instance [31–37]), tackling the question of whether the asymptotical results are well-suited to estimate the finite blocklength problems.

Specifically to the coordination problem, [24] drops the simplifying assumption that allows the blocklength to grow indefinitely, and focuses on the trade-offs achievable in the finite blocklength regime. The notion of $(\epsilon, n)$ fixed-length strong coordination demands that, for a fixed codelength $n$ and a given $\epsilon$, the $L^1$ distance between the distribution of sequences of distributed random states and an i.i.d. target distribution is upper-bounded by $\epsilon$ [24]. In a first attempt to derive a capacity region, [24] presents an inner bound for a two-node network comprised of an information source and a noisy channel, in which both nodes have access to a common source of randomness. Even though the achievability scheme outlined in [24] is general enough to be applied to more sophisticated network topologies, deriving an outer bound for the region is a difficult problem even with the less stringent constraint of the asymptotical regime [23]. Hence, to prove an outer bound as well as an inner bound, in this paper we look at the simplest setting for which the strong coordination problem has been solved in the asymptotical regime [4]. Thus, we consider the point-to-point setting comprised of an information source, a rate-limited error-free link, and a uniform source of common randomness available at the encoder and the decoder as depicted in Figure 1, and

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we present both an inner and an outer bound for the $(\epsilon, n)$ fixed-length strong coordination capacity region. In particular, while the inner bound exploits the achievability approach of [24], we delineate an outer bound proof that profits from Neyman-Pearson theory and hypothesis testing techniques [30, 34, 38, 39], and that does not rely merely on the characteristics of the chosen network, and should therefore be well-suited for generalization to different scenarios.

Interestingly, for fixed blocklength $n$ and bound on the $L^1$ distance $\epsilon$, we find rate constraints

$$\text{rate} \geq \text{mutual information} + Q^{-1} \left( \epsilon + O \left( \frac{1}{\sqrt{n}} \right) \right) \sqrt{\frac{V}{n}} + O \left( \frac{\log n}{n} \right)$$

where $V$, referred to as channel dispersion, is a characteristic of the “test channel” that connects the random variables of the problem ($U$ and $V$ in the setting of Fig. 1) with an auxiliary random variable representing the codebook. Then, $Q^{-1}$ is the inverse of the Gaussian cumulative distribution function, and the approximation term is the same recovered by [30, 33, 34] for channel coding and compression in the finite-length regime. Since the approximation term vanishes as $n$ increases, we also recover the capacity result of asymptotical strong coordination [4], therefore answering the standard question in information theory of how the asymptotic limits relate to their fixed blocklength counterparts. We recall that the best known fixed-length capacity results for channel and source coding [30, 33, 34] are also of the form

$$\text{rate constraint of the asymptotical case} + Q^{-1} (\epsilon) \sqrt{\frac{V}{n}} + O \left( \frac{\log n}{n} \right)$$

where $\epsilon$ represents the maximal probability of error and the upper-bound on distortion respectively, meaning that the coordination problem has the same order of approximation as in [30, 33, 34]. Then, even though the fundamental trade-off of (1) was expected, through (1) we make the trade-off more explicit, and we prove that high coordination rate and common randomness are needed to shorten the blocklength and approximate the distribution with higher precision. Not only the better we can approximate the target distribution, the more expensive this is in terms of rate, but the shape of these rate constraints shows the direct dependence of the rate on the “level of coordination” $\epsilon$ inside the approximation term. A similar relation between the rate constraint and the probability of coding error or the level of distortion has been proved in [30, 33, 34].

A. Contributions

The main contributions of this paper are the following.

Problem formulation: We state the definition of $(\epsilon, n)$ fixed length strong coordination as introduced in [24]: for a given blocklength $n$ and bound on the $L^1$ distance $\epsilon$, an i.i.d. distribution is achievable for strong coordination in the non-asymptotic regime if we can approximate it through the coding process up to a margin $\epsilon$. Similarly to [4], we investigate the fixed length strong coordination region $\mathcal{R}(\epsilon, n)$ for the simplest point-to-point setting comprised of an i.i.d. source and a noiseless link, in which encoder and decoder share a source of common randomness.

The characterization of the coordination region involves two stages: achievability and converse, detailed in the following paragraphs. As in [30, 33], the key step of each stage consists in identifying a sequence of independent random variables to which we apply the Berry-Esseeen Central Limit Theorem.

Inner bound: Theorem 2 presents the sufficient conditions for achievability for non-asymptotic strong coordination. Following the approach designed by [24], we use the random binning approach inspired by [40, 41] to
design a random binning and a random coding scheme which are close in $L^1$ distance. By combining the finite-length techniques of [30, 33] and the Berry-Esseen Central Limit Theorem with the properties of random binning [41], we derive an inner bound on the rate conditions that guarantees coordination with given arbitrary blocklength $n$. Interestingly, the rate constraints, proved in Section IV, are consistent with the inner bound for the fixed length strong coordination region of a two-nodes network with a noisy link [24, Theorem 1].

**Outer bound:** In Theorem 3 we derive the outer bound for the same setting of Theorem 2. The proof involves a meta-converse, an approach which has proved to be optimal in several scenarios [42, 43]. The meta-converse exploits results from the Neyman-Pearson hypothesis testing, similarly to [30, 34, 38, 39]. Starting from sequences which are by assumption generated with a distribution close to i.i.d., we consider a randomized test between this distribution and an i.i.d. one. Then, the probabilities of false positive and miss-detection of the test can be bounded using the Berry-Esseen Central Limit Theorem and the assumption that strong coordination holds, leading to rate conditions that asymptotically match the achievability.

Furthermore, by analyzing the two main results, we observe that we can derive a closed result on the capacity region in Remark 1.

**Comparison with lossy compression:** Since coordination is conceptually related to source coding, we compare our results with the non-asymptotic fundamental limits of lossy data compression [33, 34, 44]. Furthermore, coordination and lossy source coding involve different metrics, thus we need to adapt the formulation of compression to derive analogies between the two problems.

**Discussion of the result:** We discuss the new non-asymptotic results by looking at the trade-off between the rate required to achieve strong coordination and the threshold $\epsilon$ which measures the “level of coordination”.

**B. Organization of the paper**

The remainder of the paper is organized as follows. Section II introduces the notation and the model, and recalls the asymptotical result for strong coordination region derived in [4]. In Section III we present the information-theoretic modelling of fixed-length strong coordination together with the main results of this paper: an inner and an outer bound for fixed-length strong coordination. The inner bound is proved in Section IV, while the proof of the outer bound is given in Section V. Finally, the result is further analyzed in Section VI by comparing it to fixed-length lossy compression [33], and by studying the trade-off between the rate-constraint and the level of coordination, measured by the bound on the $L^1$ distance.

II. SYSTEM MODEL AND BACKGROUND

We begin by reviewing the main concepts used in this paper.

**A. Notation and preliminary results**

We define the integer interval $[a, b]$ as the set of integers from $a$ to $b$. Given a random vector $X^n := (X_1, \ldots, X_n)$, we denote $x \in \mathcal{X}^n$ as a realization of $X^n$, and $x_i \in \mathcal{X}$ is the $i$-th component of $x$. We use the notation $\|\cdot\|_1$ and $\mathbb{D}(\cdot\|\cdot\cdot)$ to denote the $L^1$ distance and Kullback-Leibler (KL) divergence respectively. We write $x \sim y$ if $x$ is proportional to $y$. Finally for a set $X$, we denote with $Q_X$ the uniform distribution over $\mathcal{X}$.

We recall some useful definition and results.

**Definition 1:** Given $A$ generated according to $P_A$ and $(A, B)$ generated according to $P_{AB}$

- Information (or entropy density): $h_{PA} := \log \frac{1}{P_A(a)}$;
- Conditional information: $h_{PA|B}(a|b) := \log \frac{1}{P_{A|B}(a|b)}$;
- Information density: $\tau_{PA} := \log \frac{P_{AB}(a, b)}{P_A(a)P_B(b)}$.

Whenever the underlying distribution is clear from the context, we drop the subscript from $h(\cdot)$ and $\tau(\cdot, \cdot)$.

**Lemma 1 (Properties of $L^1$ distance and $K$-L divergence):**

(i) $\|P_A - \hat{P}_A\|_1 \leq \|P_{AB} - \hat{P}_{AB}\|_1$, see [3, Lemma 16],
(ii) $\|P_A - \hat{P}_A\|_1 = \|P_AP_B|A - \hat{P}_AP_B|A\|_1$, see [3, Lemma 17],
(iii) $\|P_AP_B|A - P_A'P_B'|A\|_1 = \tilde{e}$, then there exists $a \in A$ s.t. $\|P_B|A=a - P'_B|A=a\|_1 \leq 2\tilde{e}$, see [45, Lemma 4].

**Definition 2:** A coupling of two probability mass functions $P_A$ and $P_{A'}$ on $A$ is any probability mass function $\hat{P}_{AA'}$ defined on $A \times A$ whose marginals are $P_A$ and $P_{A'}$. 

Proposition 1 (Coupling property [46, I.2.6]): Given \( A \) generated according to \( P_A \), \( A' \) generated according to \( P_{A'} \), any coupling \( \hat{P}_{AA'} \) of \( P_A, P_{A'} \) satisfies

\[
\| P_A - P_{A'} \|_1 \leq 4 \mathbb{P}_{\hat{P}_{AA'}} \{ A \neq A' \}.
\]

Now, we recall the Berry-Esseen Central Limit Theorem.

Theorem 1 (Berry-Esseen Central Limit Theorem [47, Thm. 2]): Given \( n > 0 \) and \( Z_i, i = 1, \ldots, n \) independent r.v.s. Then, for any real \( t \),

\[
\left| \mathbb{P} \left\{ \sum_{i=1}^{n} Z_i > n \left( \mu_n + t \sqrt{\frac{V_n}{n}} \right) \right\} - Q(t) \right| \leq \frac{B_n}{\sqrt{n}},
\]

where

\[
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i],
\]

\[
V_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i],
\]

\[
T_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|Z_i - \mu_i|^3],
\]

\[
B_n = 6 \frac{T_n}{V_n^{3/2}},
\]

and \( Q(\cdot) \) is the tail distribution function of the standard normal distribution.

B. Point-to-point setting

As in [4], we consider two nodes connected by a one-directional error-free link of rate \( R \) and sharing a common source of uniformly distributed randomness \( C \) defined on \( \mathcal{C} \) of rate \( R_0 = \frac{\log|\mathcal{C}|}{n} \). At time \( i = 1, \ldots, n \), the nodes perform \( U_i \) and \( V_i \) respectively. The sequence \( U^n \) is assigned by nature and behaves according to the fixed distribution \( \hat{P}_{U^n} \). Then, the encoder generates a message \( M \) defined on \( \mathcal{M} \) of rate \( R = \frac{\log|\mathcal{M}|}{n} \) as a function of \( U^n \) and the common randomness \( C \), via the stochastic map \( f_n : U^n \times \mathcal{C} \rightarrow \mathcal{M} \). The message is sent through the error-free link of rate \( R \) and the sequence \( V^n \) is generated through the map \( g_n : \mathcal{M} \times \mathcal{C} \rightarrow V^n \) as a function of the message \( M \) and of the common randomness \( C \).

C. Asymptotic case

We recall the definitions of achievability for strong coordination and strong coordination region in the asymptotic regime [4] for the setting of Figure 1. Let \( P_{U^n V^n} \) be the joint distribution induced by the code \( (f_n, g_n) \), a triplet \( (\hat{P}_{UV}, R, R_0) \), composed of the target distribution, the error-free channel rate and the rate of common randomness, is achievable for strong coordination if

\[
\lim_{n \to \infty} \| P_{U^n V^n} - \hat{P}_{UV} \otimes^n \|_1 = 0.
\]

Then, the strong coordination region is the closure of the set of achievable triplets \( (\hat{P}_{UV}, R, R_0) \) [4]. In more detail, the strong coordination region is characterized in [4, Theorem 10] as follows:

\[
\mathcal{R}_{\text{Cuff}} := \left\{ (\hat{P}_{UV}, R, R_0) \mid \begin{array}{l}
\hat{P}_{UV} = \hat{P}_{U} \hat{P}_{V|U} \\
\exists W \in \mathcal{W}, W \text{ generated according to } \hat{P}_{W|UV} \text{ s.t.} \\
\hat{P}_{UWV} = \hat{P}_{U} \hat{P}_{W|U} \hat{P}_{V|W} \\
R \geq I(U; W) \\
R + R_0 \geq I(UV; W) \\
|\mathcal{W}| \leq |\mathcal{U} \times \mathcal{V}| + 1
\end{array} \right\}. 
\]
To derive a closed result, an auxiliary random variable $W$ is introduced, which represents a “description” of the source on which the encoder and the decoder have to agree on in order to generate a distribution close in $L^1$ distance to i.i.d.

### III. NON-ASYMPTOTIC CASE: DEFINITION AND MAIN RESULTS

We recall the notion of $(\epsilon, n)$ fixed-length strong coordination as introduced in [24].

**Definition 3 ((\epsilon, n) Fixed-length strong coordination):** For a fixed $\epsilon > 0$ and $n > 0$, a triplet $(\bar{P}_{UV}, R, R_0)$ is $(\epsilon, n)$-achievable for strong coordination if there exists a code $(f_n, g_n)$ with common randomness rate $R_0$, such that

$$
\|P_{U^nV^n} - \bar{P}_{UV}^{\infty n}\|_1 \leq \epsilon,
$$

where $P_{U^nV^n}$ is the joint distribution induced by the code. Then, the $(\epsilon, n)$ fixed-length strong coordination region $\mathcal{R}(\epsilon, n)$ is the closure of the set of achievable $(\bar{P}_{UV}, R, R_0)$.

For the setting of Figure 1, the main result of this paper is the following inner and outer bounds for the $(\epsilon, n)$ fixed-length strong coordination region $\mathcal{R}(\epsilon, n)$.

**Theorem 2 (Inner bound for $\mathcal{R}(\epsilon, n)$ – Sufficient Conditions):** Let $\bar{P}_U$ be the given source distribution, then the triplets $(\bar{P}_{UV}, R, R_0)$ that satisfy the following conditions are achievable for $(\epsilon, n)$ fixed-length strong coordination:

$$
\bar{P}_{UV} = \bar{P}_U \bar{P}_{V|U},
$$

$$
\exists W \in \mathcal{W}, W \text{ generated according to } \bar{P}_{W|UV}, \text{ such that } \bar{P}_{UWV} = \bar{P}_U \bar{P}_{W|U} \bar{P}_{V|W},
$$

$$
R \geq I(W; U) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right) \sqrt{\frac{V_{P_{W|U}}}{n}} + O\left(\frac{\log n}{n}\right),
$$

$$
R_0 + R \geq I(W; UV) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right) \sqrt{\frac{V_{P_{W|UV}}}{n}} + O\left(\frac{\log n}{n}\right),
$$

where $Q(t) = \int_0^t \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the tail distribution function of the standard normal distribution, and $V_{\bar{P}_{W|U}}$ and $V_{\bar{P}_{W|UV}}$ are the dispersions of the test channels $\bar{P}_{W|U}$ and $\bar{P}_{W|UV}$ respectively, as defined in [30, Thm. 49]:

$$
V_{\bar{P}_{W|U}} := \min_{\bar{P}_{W|U}} \text{Var}(\pi(W|U)|W) = \min_{\bar{P}_{W|U}} \text{Var}(\pi(W|U)),
$$

$$
V_{\bar{P}_{W|UV}} := \min_{\bar{P}_{W|UV}} \text{Var}(\pi(W|UV)|W) = \min_{\bar{P}_{W|UV}} \text{Var}(\pi(W|UV)),
$$

where $\bar{P}_{W|U}$ and $\bar{P}_{W|UV}$ are the test channels that connect the random variables $U$ and $V$ of the given setting with the auxiliary random variable $W$ representing the codebook, and the last identification follows from the fact that the channels $\bar{P}_{W|U}$ and $\bar{P}_{W|UV}$ have no cost constraint [48, Section 22.3].

**Theorem 3 (Outer bound for $\mathcal{R}(\epsilon, n)$ – Necessary Conditions):** Let $\bar{P}_U$ be the given source distribution, then the triplets $(\bar{P}_{UV}, R, R_0)$ that are achievable for $(\epsilon, n)$ fixed-length strong coordination have to satisfy the following conditions:

$$
\bar{P}_{UV} = \bar{P}_U \bar{P}_{V|U},
$$

$$
\exists W \in \mathcal{W}, W \text{ generated according to } \bar{P}_{W|UV} \text{ such that } \bar{P}_{UWV} = \bar{P}_U \bar{P}_{W|U} \bar{P}_{V|W},
$$

$$
R \geq I(W; U) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right) \sqrt{\frac{V_{P_{W|U}}}{n}} + O\left(\frac{1}{n}\right),
$$

$$
R_0 + R \geq I(W; UV) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right) \sqrt{\frac{V_{P_{W|UV}}}{n}} + O\left(\frac{1}{n}\right) + O\left(\frac{\log \frac{1}{\epsilon}}{n}\right),
$$

$$
|W| \leq |\mathcal{U} \times \mathcal{V}| + 1.
$$
Remark 1 (Closed result – Sufficient conditions are also necessary): By getting a closer look at the rate conditions, we note that by taking the condition of the inner bound (3), we can retrieve a closed result. In fact, for a given $(\epsilon, n)$ as well as target distribution $\bar{P}_UV$, any $(R, R_0)$ that satisfies the condition in (4), also satisfy the conditions in (3), since

$$f(n) = O\left(\frac{\log n}{n}\right) \text{ if } \exists k_1 > 0, \exists n_0, \forall n \geq n_0 \mid f(n) \mid \leq k_1 \frac{\log n}{n},$$

$$g(n) = O\left(\frac{1}{n}\right) \text{ if } \exists k_2 > 0, \exists n_0, \forall n \geq n_0 \mid g(n) \mid \leq k_2 \frac{1}{n},$$

and $|g(n)| \leq k_2 \frac{1}{n} \leq k_2 \frac{\log n}{n} \forall n \geq 2 \Rightarrow g(n) = O\left(\frac{\log n}{n}\right)$.

Remark 2 (Comparison with the asymptotic case): We observe the following analogies between the asymptotic and the fixed-length case:

- The decomposition of the target joint distribution is the same (see (2) and (3), (4)).
- Even though necessary and sufficient conditions for $(\epsilon, n)$ strong coordination lead to different rate constraints, we observe that the constant terms are the same in (3) and in (4), whereas the difference is only in the growth rate of two functions of $n$ that go to zero as $n$ increases, although with different speeds. More precisely, both $O\left(\frac{\log n}{n}\right)$ and $O\left(\frac{1}{n}\right)$ vanish when $n \to \infty$, hence the following terms in the inner bound of (3) and in the outer bound of (4)

$$O\left(\frac{\log n}{n}\right) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right)\sqrt{V_P}$$

$$O\left(\frac{1}{n}\right) + Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right)\sqrt{V_P}$$

(5)

coincide asymptotically to

$$Q^{-1}(\epsilon)\sqrt{\frac{V_P}{n}}.$$  

(6)

Thus the rate conditions of both the inner bound and the outer bound are reduced to

$$R \geq I(W; U) + Q^{-1}(\epsilon)\sqrt{\frac{V_{P_{W|U}}}{n}},$$

$$R_0 + R \geq I(W; UV) + Q^{-1}(\epsilon)\sqrt{\frac{V_{P_{W|UV}}}{n}}.$$  

(7)

- Perhaps more interestingly, as in [30, 33, 34] for channel coding and compression in the finite-length regime, we observe that by letting $n$ tend to infinity, we end up with the same rate conditions of the asymptotic case (2), therefore reproving the results of [4] with different techniques. In fact, in the asymptotic regime $\epsilon$ vanishes when $n \to \infty$, and

$$Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right)\sqrt{V_P} \sim \log\left(\frac{1}{O(\epsilon)}\right)\sqrt{V_P}.$$  

(8)

Then, if for example, $\epsilon \sim \frac{1}{\sqrt{n}}$, (8) becomes

$$\sqrt{V_P} \frac{Q^{-1}\left(\epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right)}{\sqrt{n}} \sim \sqrt{V_P} \frac{Q^{-1}\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{n}} \sim \sqrt{V_P} \frac{\log \sqrt{n}}{\sqrt{n}} \to 0.$$

Hence, we can recover the asymptotic region of (2) from the fixed-length necessary and sufficient conditions of (3) and (4). Moreover, with this choice the bound $\epsilon_{\text{Tot}}$ on the $L^1$ distance between the two distribution goes to zero as $1/\sqrt{n}$. 

IV. INNER BOUND

Outline of the proof of Theorem 2: The achievability proof is based on non-asymptotic output statics of random binning [41] and is decomposed into the following steps:

A. preliminary definitions and results on random binning are recalled;
B. two schemes are defined for a fixed $n$, a random binning and a random coding scheme; using the properties of random binning, it is possible to derive an upper bound on the $L^1$ distance between the i.i.d. random binning distribution $P_{RB}$ and random coding distribution $P_{RC}$, providing a first bound on $\|P_{RB} - P_{RC}\|_1$. Then, a second bound $\epsilon_{Tot}$ is recovered, by reducing the rate of common randomness to obtain the conditions in (3);
C. the term $\epsilon_{Tot}$ is analyzed;
D. the rate conditions are summarized.

Remark 3: Observe that, as we will see in Section IV.B.4, the final bound $\epsilon_{Tot}$ on the $L^1$ distance between $P_{RB}$ and $P_{RC}$ is worse than the one found in Section IV.B.3. However, by worsening the $L^1$ distance, we can reduce the rate of common randomness.

A. Preliminaries on random binning

Let $A$ taking values in $\mathcal{A}$ be partitioned into $2^R$ bins at random, we denote with $\varphi : \mathcal{A} \to [1, 2^R]$, $a \mapsto k$, the realization of such partition (or binning), and we call $K := \varphi(A)$ a random binning of $\mathcal{A}$. With a slight abuse of notation, throughout this text we may refer to the map $\varphi$ as a uniform random binning, if $\varphi(A)$ is a binning of $A$ and the partition into bins is performed uniformly at random through $\varphi$.

Now, let the pair $(A, B)$ generated according to $P_{AB}$ be a discrete source, and $\varphi$ introduced above be a uniform map, we denote the distribution induced by the binning as

$$P_{RB}(a, b, k) := P_{AB}(a, b)\mathbb{1}\{\varphi(a) = k\}. \quad (9)$$

The first objective consists in ensuring that the binning is almost uniform and almost independent from the source so that the random binning scheme and the random coding scheme generate joint distributions that have the same statistics.

Theorem 4 ([41, Thm. 1]): Given $P_{AB}$, for every distribution $P_B$ on $B$ and any $\gamma \in \mathbb{R}^+$, the marginal of $P_{RB}$ in (9) satisfies

$$\mathbb{E}\|P_{RB}(b, k) - Q_K(k)P_B(b)\|_1 \leq \epsilon_{App},$$

$$\epsilon_{App} := P_{AB}(S_\gamma(P_{AB}\|T_B)) + 2^{-\frac{nR}{2}}, \quad (10)$$

where for a set $X$, we denote with $Q_X$ the uniform distribution over $X$ and

$$S_\gamma(P_{AB}\|T_B) := \{(a, b) : h_{P_{AB}}(a, b) - h_{T_B}(b) - nR > \gamma\}. \quad (11)$$

With the previous result we measure in terms of $L^1$ distance how well we can approximate a distribution for which the binning of $A$ is independently generated from $B$ and uniform, of which we characterize the upper bound $\epsilon_{App}$.

Intuitively, the approximation error $\epsilon_{App}$ is small if the number of bins in which we partition $A$ is high, in particular it has to be higher than the conditional information of $A$ given $B$, with the real number $\gamma$ allowing some degree of freedom.

Before stating the second property, we introduce the decoder that we will use, sometimes called the mismatch stochastic likelihood coder (SLC) [49, 50].

Definition 4: Let $T_{AB}$ be an arbitrary probability mass function, and $\varphi : \mathcal{A} \to [1, 2^R]$, $a \mapsto k$ a uniform random binning of $A$. A mismatch SLC is defined by the following induced conditional distribution

$$T_{A|BK}(\hat{a}|b, k) := \frac{T_{A|B}(\hat{a}|b)\mathbb{1}\{\varphi(\hat{a}) = k\}}{\sum_{\hat{a} \in A} T_{A|B}(\hat{a}|b)\mathbb{1}\{\varphi(\hat{a}) = k\}}. \quad (12)$$

Then, the following result is used to bound the error probability of decoding $A$ when the decoder has access to the side information $B$ as well as to the binning $\varphi(A) = K$. 

where $\gamma$ is an arbitrary positive number and
\[ S_\gamma(T_{AB}) := \{ (a, b) : nR - h_{T_{AB}}(a|b) > \gamma \} . \]

While we will use Theorem 4 at the encoder to ensure that the random binning probability distribution approximates well a random coding process, the latter is used at the decoder’s side to minimize the probability of error of generating the wrong sequence. In this context, the bound on the error probability $\epsilon_{\text{Dec}}$ is small if the number of bins is upper bounded by the conditional information. Since at the encoder’s side we had the opposite request, by demanding that the rate (or equivalently the number of bins) is large enough, we would have to find a compromise between two seemingly competing goals. This issue will be resolved by carefully choosing different side information at the encoder and at the decoder, and by playing with different values of $\gamma$.

### B. Fixed-length coordination scheme

The encoder and the decoder share a source of uniform randomness $C \in [1, 2^{nR_C}]$. Moreover, suppose that the encoder and decoder have access not only to common randomness $C$ but also to extra randomness $F$, where $C$ is generated uniformly at random in $[1, 2^{nR_C}]$ with distribution $Q_C$ and $F$ is generated uniformly at random in $[1, 2^{nR_F}]$ with distribution $Q_F$, independently of $C$. The encoder observes the source $U^n$, and selects a message $M$ of rate $R$, which is then transmitted through an error-free link to the decoder. Then, the decoder exploits the message and the common randomness to select $V^n$.

In the rest of this section, we introduce an auxiliary random variable $W^n$ which is not part of the setting such that the Markov chain $U - W - V$ holds. This random variable represents the “description” of the source on which the encoder and the decoder have to agree to produce the right distributions. In order to do so, we consider the i.i.d. target distribution, and we define three binnings of $W^n$, thus inducing a joint distribution on the random variables $(U^n, W^n, V^n)$ and on the binnings, which we call random binning distribution. Then, we define a random coding distribution, and we use the binning properties of Theorem 4 and Theorem 5 to estimate the $L^1$ distance between this random coding distribution and the random binning distribution. Since the marginal of the random binning distribution coincides with the target distribution, we can estimate the upper bound on the $L^1$ distance $\epsilon_{\text{Fel}}$ between the target distribution and the distribution induced by the code. This will be done in two steps: first, we derive an upper bound on the $L^1$ distance by coordinating the sequences $(U^n, W^n, V^n)$; finally we see how to reuse Theorem 5 to reduce the amount of common randomness and coordinate $U^n$ and $V^n$ only.

1) **Random binning scheme**: Let $\hat{P}_U \hat{P}_V|U$ be the target distribution, we introduce an auxiliary random variable $W$ such that $U^n$, $W^n$, and $V^n$ are jointly i.i.d. with distribution
\[ \hat{P} := \hat{P}_{U^n} \hat{P}_{W^n|U^n} \hat{P}_{V^n|W^n}. \]

We consider three uniform random binnings for $W^n$:

i) binning $C = \varphi_C(W)$, where $\varphi_C : W^n \rightarrow [1, 2^{nR_C}]$,

ii) binning $M = \varphi_M(W)$, where $\varphi_M : W^n \rightarrow [1, 2^{nR_M}]$,

iii) binning $F = \varphi_F(W)$, where $\varphi_F : W^n \rightarrow [1, 2^{nR_F}]$

and, inspired by [40, 41, 49], we consider a decoder defined according to (12) that reconstructs $\hat{W}^n$:
\[ \hat{T}_{W^n|FCM}(\hat{w}|f, c, m) := \frac{T_{W^n}(\hat{w}) \mathbb{1}_{\{\varphi(\hat{w}) = (f, c, m)\}}}{\sum_{\hat{w} \in W^n} T_{W^n}(\hat{w}) \mathbb{1}_{\{\varphi(\hat{w}) = (f, c, m)\}}}, \]

where $\varphi = (\varphi_C, \varphi_F, \varphi_M)$. This induces the joint distribution
\[ P^{RB} := \hat{P}_{U^n} \hat{P}_{W^n|U^n} \hat{P}_F W^n \hat{P}_C W^n \hat{P}_M W^n \hat{P}_V W^n \hat{T}_{W^n|FCM}. \]

In particular, $P^{RB}_{W^n|FCU^n}$ is well defined.
2) Random coding scheme: The encoder generates $W^n$ according to $P^{RB}_{W^n|FCU^n}$ defined above. At the decoder, $\hat{W}^n$ is generated via the conditional distribution $\hat{T}_{W^n|FCM}$. The decoder then generates $V^n$ according to the distribution $P^{RC}_{V^n|\hat{W}^n}(\hat{v}|\hat{w}) := \hat{P}_{V^n|\hat{W}^n}(\hat{v}|\hat{w})$, where $\hat{w}$ is the output of a decoder defined as in (12). This induces the joint distribution

$$P^{RC} := Q_F Q_C \hat{P}^{RB}_{U^n|FCU^n} \hat{P}^{RB}_{M|W^n} \hat{T}_{W^n|FCM} P^{RC}_{V^n|\hat{W}^n}. \quad (17)$$

Observe that the marginal of the distribution $P^{RB}$ is by construction trivially close in $L^1$ distance to the target distribution $\hat{P}$. We use the properties of random binning to show that the random binning distribution $P^{RB}$ and the random coding distribution $P^{RC}$ are $\epsilon$-close in $L^1$ distance, and therefore so are the marginals of $P^{RC}$ and $\hat{P}$.

3) Strong coordination of $(U^n, V^n, W^n) — Initial bound: By applying Theorem 4 and Theorem 5 to $P^{RB}$ and $P^{RC}$, we have

$$\|P^{RB}_{U^nW^n|CFM} - P^{RC}_{U^nW^n|CFM}\|_1 \leq \epsilon_{App},$$

$$\mathbb{E}[P[\mathcal{E}]] \leq \epsilon_{Dec},$$

where (a) comes from Lemma 1 (ii), and

$$\epsilon_{App} := \hat{P}^{RC}_{U^n|CF}(S_{γ_1}^c) + 2^{-\frac{2α+1}{2}}, \quad (18a)$$

$$\epsilon_{Dec} := \hat{P}^{RC}_{W^n}(S_{γ_2}^c) + 2^{-γ_2}, \quad (18b)$$

with $γ_1$ and $γ_2$ arbitrary positive numbers, and

$$S_{γ_1} := S_{γ_1}(\hat{P}^{RC}_{U^n}|\hat{P}^{RC}_{U^n}) = \{(u, w) : h_P(u, w) - h_P(u) - n(\hat{R} + R_0) > γ_1\}, \quad (19a)$$

$$S_{γ_2} := S_{γ_2}(\hat{P}^{RC}_{W^n}) = \{w : n(R + R_0 + \hat{R}) - h_P(w) > γ_2\} \equiv \{w : n(R + R_0 + \hat{R}) - \sum_{i=1}^{h_P(w)} > γ_2\}, \quad (19b)$$

where (b) comes from the choice of the decoder (15). Then, we have

$$\|P^{RB}_{U^nW^n|CFM|FCM} - P^{RC}_{U^nW^n|CFM} \hat{T}_{W^n|CFM}\|_1 = \|P^{RB}_{U^nW^n|CFM\hat{W}^n} - P^{RC}_{U^nW^n|CFM\hat{W}^n}\|_1 \leq \epsilon_{App} + \epsilon_{Dec}. \quad (20)$$

To conclude, observe that in the random binning scheme we have $V^n$ generated according to $\hat{P}_{V^n|W^n}$, $W^n$ generated according to $\hat{P}_{W^n|U^n}$, while in the random coding scheme we have $V^n$ generated according to $P^{RC}_{V^n|\hat{W}^n}$, $W^n$ generated according to $\hat{T}_{W^n|CFM}$. Then, by applying the coupling result of Proposition 1, we have

$$\|P^{RB} - P^{RC}\|_1 \leq \epsilon_{App} + 5 \epsilon_{Dec}. \quad (21)$$

4) Reducing the rate of common randomness — Final bound: Although in a first instance we have exploited the extra randomness $F$ to coordinate the whole triplet $(U^n, V^n, W^n)$ with maximal $L^1$ distance $\epsilon_{App} + 5 \epsilon_{Dec}$, we now show that we do not need it in order to coordinate only $(U^n, V^n)$. As in [45] we can reduce the required amount of common randomness by having the two nodes agree on a suitable realization of the extra randomness $F$. By fixing $F = f$, we will reduce the rate requirements at the expense of the upper bound on the $L^1$ distance by introducing a third approximation term, ending up with a new upper bound $\epsilon_{Tot}$. Now, we detail under which circumstance such suitable realization of $F$ exists, and we derive the final upper bound on the $L^1$ distance $\epsilon_{Tot}$, To do so, first we apply Theorem 4 to $A = W^n$, $B = (U^n, V^n)$, $P_B = P^{RB}_{U^nV^n}$, $P_{AB} = P^{RB}_{U^nV^nW^n}$, and $K = F$. Then, we have

$$\|P^{RB}_{U^nV^nF} - P^{RB}_{U^nV^n}\|_1 \leq \epsilon_{App,2}, \quad (20)$$

where

$$\epsilon_{App,2} := P^{RB}_{U^nV^nF}(S_{γ_3}^c) + 2^{-\frac{2α+1}{2}}, \quad (21)$$
\[ S_{\gamma_3} := S_{\gamma_3}(P^{RB}_{U=V^W}=F) = \{(u, v, w) : h_{P_{RA}}(u, v, w) - h_{P_{RA}}(u, v) - n\hat{R} > \gamma_3\}. \]  

Now, we recall that by Lemma 1 (i), we have
\[ \|P^{RB}_{U=V^n F} - P^{RC}_{U=V^n F}\|_1 \leq \|P^{RB} - P^{RC}\|_1 \leq \epsilon_{App} + 5 \epsilon_{Dec}, \]
and combining (20) and (22) with the triangle inequality, we have
\[ \|Q_F P^{RB}_{U=V^n} - Q_F P^{RC}_{U=V^n}\|_1 \leq \|P^{RB}_{U=V^n F} - Q_F P^{RB}_{U=V^n}\|_1 + \|P^{RC}_{U=V^n F} - P^{RC}_{U=V^n}\|_1 \]
\[ \leq \epsilon_{App,2} + \epsilon_{App} + 5 \epsilon_{Dec}. \]

Finally by Lemma 1 (iii), there exists an instance \( F = f \), such that
\[ \|P^{RB}_{U=V^n F=F=f} - P^{RC}_{U=V^n F=F=f}\|_1 \leq \epsilon_{Tot}, \]
\[ \epsilon_{Tot} := 2(\epsilon_{App,2} + \epsilon_{App} + 5 \epsilon_{Dec}). \]

**C. Analysis of the \( L^1 \) distance \( \epsilon_{Tot} \)**

Here, we take a closer look at the overall \( L^1 \) distance between the i.i.d. distribution and the random coding one, denoted with \( \epsilon_{Tot} \). First, substituting the explicit expression for \( (\epsilon_{App}, \epsilon_{Dec}, \epsilon_{App,2}) \) of (18a), (18b), and (21a) into (23b), the bound in (23a) becomes
\[ \epsilon_{Tot} = 2\tilde{P}_{U=CF}(S^c_{\gamma_1}) + 10\tilde{P}_{W^n}(S^c_{\gamma_2}) + 2P^{RB}(S^c_{\gamma_3}) + 2 \left[ 2^{-\frac{21}{2}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{23}{2}} \right]. \]

By the union bound and De Morgan’s law, we have
\[ \epsilon_{Tot} \leq 10 \left[ \tilde{P}_{U=CF}(S^c_{\gamma_1}) + \tilde{P}_{W^n}(S^c_{\gamma_2}) + P^{RB}(S^c_{\gamma_3}) \right] + 2 \left[ 2^{-\frac{21}{2}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{23}{2}} \right]
\[ = 10 \left[ \tilde{P}(S^c_{\gamma_1} \cup S^c_{\gamma_2} \cup S^c_{\gamma_3}) \right] + 2 \left[ 2^{-\frac{21}{2}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{23}{2}} \right]
\[ \leq 10 \left[ \tilde{P}((S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c) \right] + 2 \left[ 2^{-\frac{21}{2}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{23}{2}} \right]. \]

In the next paragraph, we investigate \((S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c\), to understand which rate conditions are dominant to minimize the measure of the set as a function of \( \gamma_i, i = 1, 2, 3 \). In a second instance, we choose the parameters \((\gamma_2, \gamma_2, \gamma_3)\) such that \( \epsilon_{Tot} \) defined above is small.

1) Analysis of \((S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c\): First, we write explicitly the set:
\[ (S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c = \left\{ (u, v, w) : \begin{array}{l}
 h(u, v, w) - h(u, v) - n\hat{R} > \gamma_3 \\
 h(u, w) - h(u) - n(R_0 + \tilde{R}) > \gamma_1 \\
 n(R_0 + \tilde{R} + R) - h(w) > \gamma_2
\end{array} \right\}^c \]
\[ = \left\{ (u, v, w) : \begin{array}{l}
 h(u, v, w) - n\hat{R} > \gamma_3 \\
 h(w|u) - n(R_0 + \tilde{R}) > \gamma_1 \\
 n(R_0 + \tilde{R} + R) - h(w) > \gamma_2
\end{array} \right\}^c \]
\[ = \left\{ (u, v, w) : \begin{array}{l}
 n\hat{R} < h(w|uv) - \gamma_3 \\
 n(R_0 + R) > i(w; uv) + \gamma_2 + \gamma_3 \\
 nR > i(w; u) + \gamma_2 + \gamma_1
\end{array} \right\}^c \]
\[
\begin{align*}
&= \left\{ (u, v, w) : nR < h(w|uv) - \gamma_3 \right\} \cup \left\{ (u, v, w) : n(R_0 + R) > \iota(w; uv) + \gamma_2 + \gamma_3 \right\} \cup \left\{ (u, v, w) : nR > \iota(w; u) + \gamma_2 + \gamma_1 \right\} \\
&= \left\{ (u, v, w) : nR \geq h(w|uv) - \gamma_3 \right\} \cup \left\{ (u, v, w) : n(R_0 + R) \leq \iota(w; uv) + \gamma_2 + \gamma_3 \right\} \cup \left\{ (u, v, w) : nR \leq \iota(w; u) + \gamma_2 + \gamma_1 \right\}, \tag{26}
\end{align*}
\]

Now, recall that in Section IV.B.4 the extra common randomness \( F \) of rate \( \bar{R} \) has been fixed to an instance \( F = f \). Hence, to minimize the measure of the set \( (S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c \), we only need to minimize the second and third terms of (26). We define the sets

\[
S_{i(w;uv)} := \left\{ (u, v, w) : n(R_0 + R) \leq \iota(w; uv) + \gamma_2 + \gamma_3 \right\},
\]

and we treat them separately in the following.

a) Analysis of \( S_{i(w;uv)} \): We observe that, since the distribution \( \bar{P} \) is i.i.d., the terms \( Z_i = \iota_{\bar{P}}(w_i; u_iv_i) \), are mutually independent for \( i = 1, \ldots, n \). Then, we consider the following inequality

\[
n(R + R_0) > \sum_{i=1}^{n} \frac{\mathbb{E}_{\bar{P}_{wuv}}[\iota_{\bar{P}}(w_i; u_iv_i)]}{\mu_n} + Q^{-1}(\epsilon_1) \sum_{i=1}^{n} \text{Var}_{\bar{P}_{wuv}}(\iota_{\bar{P}}(w_i; u_iv_i)) + \gamma_2, \tag{29}
\]

where \( \mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i] \), \( V_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i] \) and \( Q(\cdot) \) is the tail distribution function of the standard normal distribution. We prove that, assuming that (29) holds, we can successfully bound \( S_{i(w;uv)} \). In fact, the chain of inequalities

\[
\sum_{i=1}^{n} \iota_{\bar{P}}(W; Uv) > n(R + R_0) - (\gamma_2 + \gamma_3) > n\left(\mu_n + t\sqrt{\frac{V_n}{n}}\right)
\]

implies that, if (29) holds, \( S_{i(w;uv)} \) is contained in

\[
\left\{ (u, v, w) : \sum_{i=1}^{n} \iota_{\bar{P}}(w_i; u_iv_i) > n\mu_n + n\epsilon_1 \sqrt{\frac{V_n}{n}} \right\}, \tag{30}
\]

Therefore, if we find an upper bound on (30), we have an upper bound on \( S_{i(w;uv)} \) as well. To obtain that, we apply Theorem 1 (Berry-Esseen CLT) to the right-hand side of (30), and we choose

\[
Q(t) = \epsilon_1, \\
\epsilon_1^* = \epsilon_1 + \frac{B_n}{\sqrt{n}}, \tag{31}
\]

where, as in the statement of Theorem 1 (Berry-Esseen CLT), \( B_n = 6 \frac{T_n}{\sqrt{V_n/2}} \), and \( T_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|Z_i - \mu_i|^3] \). Then, we have

\[
\left| \mathbb{P} \left\{ \sum_{i=1}^{n} \iota_{\bar{P}}(w_i, u_i, v_i) > n\mu_n + nQ^{-1}(\epsilon_1) \sqrt{\frac{V_n}{n}} \right\} - \epsilon_1 \right| \leq \frac{B_n}{\sqrt{n}},
\]

\[
\Rightarrow \mathbb{P} \left\{ \sum_{i=1}^{n} \iota_{\bar{P}}(w_i, u_i, v_i) > n\mu_n + nQ^{-1}(\epsilon_1) \sqrt{\frac{V_n}{n}} \right\} \leq \epsilon_1^*.
\tag{32}
\]

Finally, (32) combined with (30) implies \( \bar{P}(S_{i(w;uv)}) \leq \epsilon_1^* \).

Moreover, we can simplify (29) with the following identifications.
Remark 4 (Mutual Information and Channel Dispersion): Similarly to [51, Section IV.A], observe that, since \((u_i, w_i, v_i)\) are generated i.i.d. according to the same distribution \(P_{WUV}\), we have

\[
\mu_n := \frac{1}{n} \sum_{i=1}^{n} E_{P_{WUV}}[t_P(w_i; u_i, v_i)] \\
= E_{P_{WUV}}[t_P(w; u, v)] \\
= I_P(W; UV),
\]

\[
V_n := \frac{1}{n} \sum_{i=1}^{n} \text{Var}_{P_{WUV}}(t_P(w_i; u_i, v_i)) \\
= \text{Var}_{P_{WUV}}(t_P(W; UV))
\]

and \(V_{P_{WUV}} = \min_{P_{WUV}} \left[\text{Var}_{P_{WUV}}(t_P(W; UV))\right] = \min_{P_{WUV}} \left[\text{Var}_{P_{WUV}}(t_P(W; UV)|W)\right]\) is the dispersion of the channel \(P_{W|UV}\) as defined in [30, Thm. 49].

Then, (29) can be rewritten as

\[
n(R + R_0) > nI(P(W; UV)) + n Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|UV}}}{n}} + (\gamma_2 + \gamma_3).
\]

b) Analysis of \(S_{i(w:u)}\): Similarly, we use Theorem 1 (Berry-Esseen CLT) to estimate \(P(S_{i(w:u)})\). If we apply the same reasoning to \(Z_i' = t_P(w_i, U_i)\) for \(i = 1, \ldots, n\), and \(\mu'_n = \frac{1}{n} \sum_{i=1}^{n} E[Z_i'] = I(W; U), V'_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i'] = V_{P_{W|U}}, T'_n = \frac{1}{n} \sum_{i=1}^{n} E[Z_i' - \mu'_n]^2, B'_n = 6 \frac{T'_n}{V'_n},\) we find that \(P(S_{i(w:u)}) \leq \epsilon'_2 = \epsilon_2 + \frac{B'_n}{n}\) if

\[
nR > nI_P(W; U) + n Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + (\gamma_2 + \gamma_1).
\]

A more detailed proof can be found in Appendix A.

2) Choice of \((\gamma_1, \gamma_2, \gamma_3)\) and rate conditions: If we choose \((\gamma_1, \gamma_2, \gamma_3) = (\log n, \frac{1}{2} \log n, \log n)\), then the bound (24) on the \(L^1\) distance becomes

\[
\|P^{RB}_{U^R V_n} - P^{RC}_{U^C V_n}\|_1 \leq \epsilon_{\text{Tot}}, \\
\epsilon_{\text{Tot}} = 10 \bar{P}((S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c) + 2 \left(2^{-\frac{\gamma_1+1}{2}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{\gamma_3+1}{2}}\right) \\
= 10 \bar{P}((S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c) + \frac{10 + 2\sqrt{2}}{\sqrt{n}} \\
\leq 10 \bar{P}(S_{i(w:uv)}) + 10 P(S_{i(w:u)}) + \frac{10 + 2\sqrt{2}}{\sqrt{n}} \\
\leq 10 (\epsilon'_1 + \epsilon'_2) + \frac{10 + 2\sqrt{2}}{\sqrt{n}} \\
= 10 (\epsilon_1 + \epsilon_2) + \frac{10(1 + B_n + B'_n) + 2\sqrt{2}}{\sqrt{n}} \\
= 10 (\epsilon_1 + \epsilon_2) + O\left(\frac{1}{\sqrt{n}}\right).
\]

With this choice for \(\gamma_i\), the rate conditions become

\[
R + R_0 > I_P(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|UV}}}{n}} + \frac{3 \log n}{2n},
\]
\[ R > I_p(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{p_{W|U}}}{n}} + \frac{3\log n}{2n}. \]  

From now on, we drop the subscript \(\bar{P}\) from \(I(\cdot, \cdot)\) to simplify the notation.

V. OUTER BOUND

For a fixed \(\epsilon > 0\) and \(n > 0\), we consider a triplet \((\bar{P}_{UV}, R, R_0)\) which is \((\epsilon, n)\)-achievable for strong coordination, and we want to prove that it is contained in (4). Since \((\bar{P}_{UV}, R, R_0)\) is achievable, there exist a code \((f_n, g_n)\) that induces a distribution \(P_{\bar{U}^n \bar{V}^n}\) such that

\[ \|P_{\bar{U}^n \bar{V}^n} - \bar{P}_{\bar{U}}^{\otimes n} \bar{P}_{\bar{V}|\bar{U}}^{\otimes n}\|_1 \leq \epsilon. \]  

Now, we consider \(n\) i.i.d. pairs \((C, M)\) of the message and the common randomness, generated by the stochastic code via \(P_{(C,M)|U^n \bar{V}^n}\), and a time index \(T\) uniformly distributed from 1 to \(n\). Observe that since \(U_T\) and \(V_T\) are conditionally independent through \((C, M, T)\), we have:

\[ P_{(C,M,T)}|U_T \bar{V}_T \bar{P}_{\bar{U}^n \bar{V}^n} = \bar{P}_{\bar{V}_T|(C,M,T)} \bar{P}_{\bar{U}^n \bar{V}^n}|U_T \bar{P}_{\bar{U}^n} \bar{P}_{\bar{V}^n}|U_T. \]  

Then, using (39) and the properties of the \(L_1\) distance [3, Lemma 17], (38) becomes

\[ \epsilon \geq \|P_{U_T \bar{V}_T} - \bar{P}_{\bar{U}^n \bar{V}^n}|U_T\|_1 \]
\[ = \|P_{U_T|(M,C,T)} \bar{V}_T - P_{(C,M,T)}|U_T \bar{V}_T \bar{P}_{\bar{U}^n \bar{V}^n}|U_T\|_1 \]
\[ = \|P_{U_T|(M,C,T)} \bar{V}_T - \bar{P}_{\bar{V}_T|(C,M,T)} \bar{P}_{\bar{U}^n \bar{V}^n}|U_T \bar{P}_{\bar{U}^n} \bar{P}_{\bar{V}^n}|U_T\|_1 \]  

with \(P_{U_T|(M,C,T)} \bar{V}_T := P_{U^V^n}|U^W^n(V=U^T^n).\)

We denote \((C, M, T)\) as \((W_T, T)\), \(W_t\) as \((C, M)\) for each \(t \in [1, n]\) and \(W\) as \((W_T, T) = (C, M, T)\), and we will see from the following that this identification satisfies the conditions in (4). Before proceeding, we observe that with this notation we find the following equivalent to (40)

\[ \|P_{U^n \bar{V}^n} - \bar{P}_{\bar{U}^n \bar{V}^n}|U^n\|_1 = \|P_{U^n \bar{W}^n |U^n \bar{V}^n} - \bar{P}_{\bar{U}^n \bar{W}^n |U^n \bar{V}^n} \bar{P}_{\bar{V}^n}|U^n\|_1 \]
\[ = \|P_{U^n \bar{W}^n} - \bar{P}_{\bar{U}^n \bar{W}^n |U^n \bar{V}^n} \bar{P}_{\bar{V}^n}|U^n\|_1 \leq \epsilon \]  

and we later refer to (41) as the \((\epsilon, n)\)-strong coordination assumption. Note that we have yet to prove that this is the appropriate choice for \(W\), that is that \(W\) verifies all the constraints of (4). First, observe that by definition of encoder and decoder the following Markov chains hold:

\[ U_t - (C, M) - V_t \Leftrightarrow U_t - W_t - V_t, \]
\[ U_T - (C, M, T) - V_T \Leftrightarrow U - W - V. \]

In the next section, we present a proof of the rate constraints based on a meta-converse, following the approach of [30]. More precisely, we show the bound on the rate \(R\), while the proof for the bound on \(R + R_0\) and the cardinality bound are developed in Appendix D and Appendix E respectively.

A. First bound – \(R\)

Similarly to [30], for an observation \(w = (m, c)\) composed of a message and an instance of the common randomness, we define the hypothesis:

\[ \mathcal{H}_0 \text{ w generated according to } \sum_u P_{U^n W^n}(u, w), \]
\[ \mathcal{H}_1 \text{ w generated according to } \sum_u \bar{P}_{U}^{\otimes n} \bar{P}_{W}^{\otimes n}(u, w), \]
where $P_{U^nW^n}$ is the coding distribution $\epsilon$-close to i.i.d. by assumption, and $P_{U^nW^n}^{\otimes n}$ is the i.i.d. target distribution. We now carry out a thought experiment: we consider a test with a (possibly) stochastic decision rule between the distributions $P_{U^nW^n}$ and $P_{U^nW^n}^{\otimes n}$: a test is defined by a random transformation 

$$P_{Z|U^nW^n} : U^n \times W^n \to \{\mathcal{H}_0, \mathcal{H}_1\},$$

where $\mathcal{H}_0$ indicates that the test chooses $P_{U^nW^n}$, and $\mathcal{H}_1$ indicates that the test chooses $P_{U^nW^n}^{\otimes n}$. The purpose of this randomized test is to bound the error probability of such test declaring that a pair message-common randomness, here identified with the sequence $w$, is generated according to the product distribution $P_{U^nW^n}$, while $w$ is by assumption generated according to $P_{U^nW^n}$. Later on, the bound on this error probability will be reduced to the rate conditions of the outer bound (4) by applying the Theorem 1 (Berry-Esseen CLT) and the $(\epsilon, n)$-strong coordination assumption (41). Hence, we start by defining the probability of type-I error (probability of choosing $\mathcal{H}_1$ when the true hypothesis is $\mathcal{H}_0$) and type-II error (probability of choosing $\mathcal{H}_0$ when the true hypothesis is $\mathcal{H}_1$) as

$$P_e^I(P_{Z|U^nW^n}) := \mathbb{P}\{\mathcal{H}_1|\mathcal{H}_0\} = \sum_u P_{\mathcal{H}_1}(u)P_{W^n}(w)P_{Z|U^nW^n}(\mathcal{H}_1|u,w),$$

$$P_e^II(P_{Z|U^nW^n}) := \mathbb{P}\{\mathcal{H}_0|\mathcal{H}_1\} = \sum_u P_{U^nW^n}(u,w)P_{Z|U^nW^n}(\mathcal{H}_1|u,w).$$

Similar to [30], we denote the minimum type-I error for a maximum type-II error $1 - \alpha$:

$$\beta_{\alpha} := \min_{P_{Z|U^nW^n}}: P_e^I(P_{Z|U^nW^n}) \leq 1 - \alpha$$

$$\min_{P_{Z|U^nW^n}}: \sum_u P_{U^nW^n}(u,w)P_{Z|U^nW^n}(\mathcal{H}_1|u,w) \leq 1 - \alpha$$

where the error probability $\alpha$ will be defined later. For the error probabilities $\alpha$ and $\beta_{\alpha}$, [48, Section 12.4] proves the following relations:

- upper bound on $\min P_e^I(P_{Z|U^nW^n})$ $\beta_{\alpha} \leq \frac{1}{\gamma_0}$, with $\gamma_0$ s.t. $\mathbb{P}_{P_{U^nW^n}} \left\{ \log \frac{P_{U^nW^n}}{P_{U^nW^n}^{\otimes n}} > \log \gamma_0 \right\} \geq \alpha$, (44a)

- lower bound on $\min P_e^I(P_{Z|U^nW^n})$ $\alpha \leq \mathbb{P}_{P_{U^nW^n}} \left\{ \log \frac{P_{U^nW^n}}{P_{U^nW^n}^{\otimes n}} > \log \gamma \right\} + \gamma \beta_{\alpha} \forall \gamma > 0$, (44b)

Then, our goal is to use the inequalities (44a) and (44b) to prove the rate constraint

$$nR \geq nI(W;U) + Q^{-1}(\epsilon)\sqrt{nV_{P_{W|U}}} - x$$

(45)

where $\epsilon$ is the approximation term, and $x \in \mathbb{R}$ is a parameter which will be defined later. First, we split (45) into an upper and a lower bound on the logarithm of the error probability $\beta_{\alpha}$:

upper bound on $\log \beta_{\alpha}$ $\log \frac{1}{\beta_{\alpha}} \geq nI(W;U) + Q^{-1}(\epsilon)\sqrt{nV_{P_{W|U}}}$

lower bound on $\log \beta_{\alpha}$ $nR + x \geq \log \frac{1}{\beta_{\alpha}}$. (46b)

Then, the proof of (45) is divided in the following steps, detailed in the next sections:

(i) Proof of the upper bound on $\log \beta_{\alpha}$: we use the upper bound on $\min P_e^I(P_{Z|U^nW^n})$ of (44a) combined with the $(\epsilon, n)$-strong coordination assumption (41) and Theorem 1 (Berry-Esseen CLT) to derive the following upper bound on the logarithm $\beta_{\alpha}$ of (46a) by choosing the parameter $\gamma_0$;

(ii) Proof of the lower bound on $\log \beta_{\alpha}$: we use the lower bound on $\min P_e^I(P_{Z|U^nW^n})$ of (44b) combined with the $(\epsilon, n)$-strong coordination assumption (41) and classical information theory properties to derive the lower bound on $\beta_{\alpha}$ of (46b) by choosing the parameter $\gamma$;

(iii) Proof of the rate constraint: we combine (46a) and (46b) proved in the previous steps and we derive (45). Before proceeding, observe that by the $(\epsilon, n)$-strong coordination assumption (41) and the properties of $L^1$
distance [3, Lemma 16], we have
\[ \| P_{U^nW^n} - P_{U^nW^n} \|_1 \leq \| P_{U^nW^n} - \bar{P}_{U^nW^n} \|_1 = \varepsilon, \]
\[ \Rightarrow \forall (u, w) \ | P_{U^nW^n}(u, w) - \bar{P}_{U^nW^n}(u, w) | \leq \varepsilon \]
\[ \Rightarrow \forall (u, w) \exists \ v_{uw} \leq \varepsilon \text{ such that } | P_{U^nW^n}(u, w) - \bar{P}_{U^nW^n}(u, w) | = v_{uw} \]
which we can distinguish into two cases:

- **Case 1** \( P_{U^nW^n}(u, w) = \bar{P}_{U^nW^n}(u, w) + v_{uw}; \)
- **Case 2** \( P_{U^nW^n}(u, w) = \bar{P}_{U^nW^n}(u, w) - v_{uw}. \)

Thus, we prove steps (i)–(iii) separately for both cases. With a slight abuse of notation from now on we will drop the index from \( v_{uw} \) and use \( \varepsilon \) instead, since \( v_{uw} \) just has to be smaller than \( \varepsilon \). Similarly, we omit the pairs \( (u, w) \) in order to simplify the notation.

1) **Proof of the upper bound on \( \log \beta_{\alpha} \) (46a)** - Case 1 \( P_{U^nW^n} = \bar{P}_{U^nW^n} + \varepsilon \): By the \((\varepsilon, n)\)-strong coordination assumption (41), we have
\[ \log (P_{U^nW^n}(u, w)) = \log \left( \bar{P}_{U^nW^n}(u, w) + \varepsilon \right) = \log \left( \bar{P}_{U^nW^n}(u, w) \right) + \log \left( 1 + \frac{\varepsilon}{\bar{P}_{U^nW^n}(u, w)} \right). \]

Thus, we rewrite the left-hand side of the upper bound on \( \min P_{\alpha}(P Z_{U^nW^n}) \) in (44a) as
\[ \mathbb{P} \{ \log P_{U^nW^n} \geq \log \bar{P}_{U^nW^n} + \log \gamma_0 \} =: \mathbb{P} \left\{ \log \bar{P}_{U^nW^n} + \log \left( 1 + \frac{\varepsilon}{\bar{P}_{U^nW^n}} \right) \geq \log \bar{P}_{U^nW^n} + \log \gamma_0 \right\} = \mathbb{P} \left\{ \log \frac{\bar{P}_{U^nW^n}}{\bar{P}_{U^nW^n}} \geq \log \gamma_0 - \log \left( 1 + \frac{\varepsilon}{\bar{P}_{U^nW^n}} \right) \right\}. \]

Given the parameter \( \varepsilon \), we choose the parameter \( \gamma_0 \) as:
\[ \log \gamma_0 := n \mu_n + Q^{-1} (\varepsilon) \sqrt{n V_n} + \log \left( 1 + \frac{\varepsilon}{\bar{P}_{U^nW^n}} \right) \]
where, as in the statement of Theorem 1 (Berry-Esseen CLT), \( \sum_{i=1}^{n} \log(\bar{P}_{U^nW^n} \bar{P}_{U^nW^n}) =: \sum_{i=1}^{n} Z_i \) is the sum of \( n \) i.i.d. random variables, and \( \mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i], V_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i], T_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|Z_i - \mu_i|^3], B_n = 6 \frac{V_n^{3/2}}{V_n^{1/2}}, \) and \( Q(\cdot) \) is the tail distribution function of the standard normal distribution. Moreover, we can rewrite these terms by using the identification of the following remark:

**Remark 5 (Mutual Information and Channel Dispersion):** Similarly to [30], we observe that for the discrete i.i.d. distributions \( \bar{P}_{U^nW^n} \bar{P}_{U^nW^n} \) and \( \bar{P}_{U^nW^n} \bar{P}_{U^nW^n} \), we have:
\[ \mu_n = \mathbb{D}(\bar{P}_{U^nW^n} \bar{P}_{U^nW^n}) = I(W; U), \]
\[ V_n = \sum_{u,w} \bar{P}_{U^n}(u) \bar{P}_{W^n}(w|u) \left[ \log \frac{\bar{P}_{U^n}(u) \bar{P}_{W^n}(w|u)}{\bar{P}_{U^n}(u) \bar{P}_{W^n}(w)} \right]^2 - \mathbb{D}(\bar{P}_{U^n} \bar{P}_{W^n})^2 = V_{\bar{P}_{W^n}}, \]
\[ T_n = \sum_{u,w} \bar{P}_{U^n}(u) \bar{P}_{W^n}(w|u) \left[ \log \frac{\bar{P}_{U^n}(u) \bar{P}_{W^n}(w|u)}{\bar{P}_{U^n}(u) \bar{P}_{W^n}(w)} \right] - \mathbb{D}(\bar{P}_{U^n} \bar{P}_{W^n}) \left[ \bar{P}_{U^n} \bar{P}_{W^n} \right], \]
\[ B_n = 6 \frac{T_n}{V_n^{3/2}}. \]
Now, observe that we have chosen parameter $\gamma_0$ in (49) appropriately such, that with the identifications of Remark 5, the probability of error (48) becomes

$$\mathbb{P}\left\{ \sum_{i=1}^{n} \log \frac{\tilde{P}_{U|W}P_{W}}{\tilde{P}_{U}P_{W}} \geq n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} \right\}$$

$$= \mathbb{P}\left\{ \sum_{i=1}^{n} \log \frac{\tilde{P}_{U|W}P_{W}}{\tilde{P}_{U}P_{W}} \geq nI(U;W) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|U}}} \right\}. \tag{51}$$

Since $\sum_{i=1}^{n} \log(\tilde{P}_{U|W}/\tilde{P}_{U}) = \sum_{i=1}^{n} Z_i$ is the sum of $n$ i.i.d. random variables, the next step is to bound the probability in (51) using Theorem 1 (Berry-Esseen CLT):

$$\mathbb{P}\left\{ \log \prod_{i=1}^{n} \frac{\tilde{P}_{U|W}P_{W}}{\tilde{P}_{U}P_{W}} \geq nI(U;W) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|U}}} \right\} - \epsilon \geq - \frac{B_n}{\sqrt{n}}$$

$$\iff \mathbb{P}\left\{ \log \prod_{i=1}^{n} \frac{\tilde{P}_{U|W}P_{W}}{\tilde{P}_{U}P_{W}} \geq nI(U;W) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|U}}} \right\} \geq \left( \epsilon - \frac{B_n}{\sqrt{n}} \right). \tag{52}$$

Then, we identify

$$\alpha := \epsilon - \frac{B_n}{\sqrt{n}}, \tag{53}$$

and by combining the upper bound on $\min P_{\epsilon}^{l}(P_{Z|U=W^n})$ of (44a) with the parameter $\gamma_0$ as chosen in (49), we obtain:

$$\log \frac{1}{\beta_{\alpha}} \geq \log \gamma_0 = n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} + \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}/\tilde{P}_{U|W}P_{W}^{\otimes n}} \right). \tag{54}$$

2) Proof of the lower bound on $\log \beta_{\alpha}$ (46b) – Case 1 ($P_{U=W^n} = \tilde{P}_{U|W}P_{W}^{\otimes n} + \epsilon$): First, we observe that

$$nR = H(M) \overset{(a)}{=} nI(U;W) \overset{(b)}{=} nI(U;W) + Q^{-1}(y)\sqrt{nV_n}, \quad \frac{1}{2} < y < 1, \tag{55}$$

where (a) is proved in Appendix B and (b) comes from the fact that $Q^{-1}(y)\sqrt{nV_n} \leq 0$ for every $\frac{1}{2} < y < 1$.

Now, we recall that by the lower bound on $\min P_{\epsilon}^{l}(P_{Z|U=W^n})$ of (44b), for every $\gamma > 0$ we have

$$\beta_{\alpha} \geq \frac{1}{\gamma} \left[ \alpha - \mathbb{P}_{P_{U=W^n}} \left\{ \log \frac{\tilde{P}_{U|W}P_{W}^{\otimes n}}{\tilde{P}_{U}P_{W}^{\otimes n}} > \log \gamma \right\} \right]$$

$$\overset{(c)}{=} \frac{1}{\gamma} \left[ \alpha - \mathbb{P}_{P_{U=W^n}} \left\{ \log \frac{\tilde{P}_{U|W}P_{W}^{\otimes n} + \epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} > \log \gamma \right\} \right]$$

$$= \frac{1}{\gamma} \left[ \alpha - \mathbb{P}_{P_{U=W^n}} \left\{ \log \frac{\tilde{P}_{U|W}P_{W}^{\otimes n} + \epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} > \log \gamma - \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} \right) \right\} \right],$$

$$\iff \log \beta_{\alpha} \geq \log \frac{1}{\gamma} \left[ \alpha - \mathbb{P}_{P_{U=W^n}} \left\{ \log \frac{\tilde{P}_{U|W}P_{W}^{\otimes n} + \epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} > \log \gamma - \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} \right) \right\} \right], \tag{56}$$

where in (c) we have used the ($\epsilon, n$)-strong coordination assumption (41). Then, similarly to Section V.A.1, we choose appropriately the parameter $\gamma$:

$$\log \gamma = H(M) + \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}P_{W}^{\otimes n}} \right). \tag{57}$$
With this choice of $\gamma$, we plug (55) into (56), and we have
\[
\log \beta_\alpha \geq \log \frac{1}{\gamma} + \log \left[ \alpha - \mathbb{P}_{U^n W^n} \left\{ \log \frac{\tilde{P}_U^{\otimes n} \tilde{P}_W^{\otimes n}}{P_U^{\otimes n} P_W^{\otimes n}} > \log \gamma - \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) \right\} \right] \\
= - \left[ H(M) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) \right] + \log \left[ \alpha - \mathbb{P}_{U^n W^n} \left\{ \log \frac{\tilde{P}_U^{\otimes n} \tilde{P}_W^{\otimes n}}{P_U^{\otimes n} P_W^{\otimes n}} > H(M) \right\} \right] \\
\geq - \left[ H(M) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) \right] + \log \left[ \alpha - \mathbb{P}_{U^n W^n} \left\{ \log \frac{\tilde{P}_U^{\otimes n} \tilde{P}_W^{\otimes n}}{P_U^{\otimes n} P_W^{\otimes n}} > nI(U;W) + Q^{-1}(y) \sqrt{nV_n} \right\} \right] \\
= -H(M) - \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) + \log \left( \alpha - y - B_n \sqrt{n} \right) \\
\tag{58}
\]
which is equivalent to the lower bound on $\log \beta_\alpha$ of (46b) if we identify
\[
x = \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) - \log \left( \alpha - y - B_n \sqrt{n} \right), \\
\tag{59}
\]
since
\[
nR + x = nR + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) - \log \left( \alpha - y - B_n \sqrt{n} \right) \\
= H(M) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) - \log \left( \alpha - y - B_n \sqrt{n} \right) \geq \log \frac{1}{\beta_\alpha}. \tag{60}
\]

3) Proof of the rate constraint – Case 1 ($P_{U^n W^n} = \tilde{P}_U^{\otimes n} \tilde{P}_W^{\otimes n} + \epsilon$): Now, we can conclude the proof of this part of the outer bound by combining (54) and (60). In fact, for $1/2 < y < 1$ we have
\[
\text{H}\left(M\right) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right) - \log \left( \alpha - y - B_n \sqrt{n} \right) \geq n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_W^{\otimes n}} \right),
\]
which is equivalent to
\[
R \geq \mu_n + Q^{-1}(\epsilon) \sqrt{\frac{V_n}{n} + \frac{\log \left( \alpha - y - B_n \sqrt{n} \right)}{n}}. \tag{61}
\]

4) Proof of the rate constraint – Case 2 ($P_{U^n W^n} = \tilde{P}_U^{\otimes n} \tilde{P}_W^{\otimes n} - \epsilon$): The proofs of the upper bound and of the lower bound on $\log \beta_\alpha$ of (46a) and (46b) are similar to the one of Section V.A.1 and Section V.A.2, and are therefore deferred to Appendix C.

Remark 6 (Speed of convergence): Note that for both case 1 and case 2 we retrieve the same rate condition as in (61), and the term
\[
\frac{\log \left( \alpha - y - B_n \sqrt{n} \right)}{n} \leq \frac{\alpha - y - B_n \sqrt{n}}{n} - \frac{B_n}{n \sqrt{n}} = O \left( \frac{1}{n} \right).
\]
which goes to zero faster than the term $\log n/n$ in the achievability.
B. Second bound – \( R + R_0 \)

The proof is similar to the one of Section V.A, and it is deferred to Appendix D.

VI. DISCUSSION ON THE RESULT

A. Comparison with fixed-length lossy compression

In [4, 20] the authors show that empirical coordination in the asymptotic regime yields the rate-distortion result of Shannon [52]. Empirical coordination is the weaker form of coordination which requires the joint histogram of the devices’ distributed random states to approach a target distribution in \( L^1 \) distance with high probability [4], thus capturing an “average behavior” of the agents. This metric of choice can be specialized to the probability of distortion, therefore connecting empirical coordination with source coding [4, 20]. In this paper however we have considered the strong coordination metric, which requires the joint distribution of sequences of distributed random states to converge to an i.i.d. target distribution in \( L^1 \) distance instead [4], hence dealing with a different and more stringent constraint which demands a positive rate of common randomness. Nonetheless, by looking at the known results for fixed-length rate-distortion [33], we can derive similarities with our case of study.

First, we recall the setting and notation of fixed-length lossy compression [33]. The output of a source \( S \) generated according to \( P_S \) with alphabet \( \mathcal{M} \) is mapped to one of the \( M \) codewords from \( \mathcal{M} \), and a lossy code consists of a pair of mappings \( f : \mathcal{M} \mapsto \{1, \ldots, M\} \) and \( c : \{1, \ldots, M\} \mapsto \mathcal{M} \). Then, a distortion measure \( d : \mathcal{M} \times \mathcal{M} \mapsto [0, \infty) \) is used to quantify the performance of a lossy code. Given the decoder \( c \), the best encoder maps the source output to the codeword which minimises the distortion. Then, \( \epsilon \) is the excess-distortion probability if

\[
\mathbb{P}\{d(S, c(f(S))) > d\} \leq \epsilon.
\]

The minimum achievable code size at excess-distortion probability \( \epsilon \) and distortion \( d \) is defined by

\[
M^*(d, \epsilon) = \min\{M : \exists (M, d, \epsilon) \text{ code}\},
\]

\[
R(n, d, \epsilon) = \frac{1}{n} \log M^*(M, d, \epsilon).
\]

Then, the following achievability results are presented in [33, 44].

**Theorem 6 (Achievability for fixed-length lossy compression [44, Thm. 2.21]):** There exists an \((M, d, \epsilon)\) code with

\[
\epsilon \leq \inf_{P_{Z|S}} \left\{ \mathbb{P}\{d(S, Z) > d\} + \inf_{\gamma > 0} \left\{ \sup_{z \in \mathcal{M}} \mathbb{P}\{\epsilon_{S|Z}(S; z) \geq \log M - \gamma\} \right\} + \exp(-\gamma) \right\}.
\]

**Theorem 7 (Gaussian approximation for fixed-length lossy compression [33, Thm. 12]):** When the source is memoryless,

\[
R(n, d, \epsilon) = R(d) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + \Theta\left(\frac{\log n}{n}\right)
\]

\[
\geq \min I(S; Z) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + \Theta\left(\frac{\log n}{n}\right)
\]

where \( V \) is the dispersion term, and \( f(n) = \Theta\left(\frac{\log n}{n}\right) \) indicates that \( f \) is bounded both above and below by \( \frac{\log n}{n} \) asymptotically: \( \exists k_1 > 0, \exists k_2 > 0, \exists n_0 \) such that \( \forall n > n_0, k_1 \frac{\log n}{n} \leq f(n) \leq k_2 \frac{\log n}{n} \).

Now, if we choose as a distance \( d(\cdot) \) the \( L^1 \) distance, the strong coordination condition

\[
\|P_{U^n V^n} - \hat{P}_{U^n V^n}\|_1 \leq d
\]

implies the rate-distortion condition (62). Then, we can interpret the strong coordination problem outlined in Section II as two connected “stronger” rate-distortion problems depicted in Figure 2 and Figure 3 respectively:

- **Rate-distortion Problem 1:** first, at the encoder we have to generate a pair \((U, V)\) which is close in \( L^1 \) distance to the one generated according to the fixed i.i.d. distribution \( \hat{P}_{UV} \);
• Rate-distortion Problem 2: in a second instance, the decoder has to reconstruct the source \( U \) to produce \( V \) via the conditional distribution \( \hat{P}_{V|U} \).

\[
\begin{array}{c}
P_U \hat{P}_{V|U} \\
\text{rate } R + R_0 \\
\begin{array}{c}
\text{Enc.} \\
(U^n, V^n) \\
\text{Dec.} \\
W^n
\end{array}
\end{array}
\]

Figure 2. Rate-distortion Problem 1: Compression of the source \((U^n, V^n)\) with a link of rate \( R + R_0 \).

\[
\begin{array}{c}
P_U \hat{P}_U \hat{P}_{V|U} \\
\text{rate } R \\
\begin{array}{c}
\text{Enc.} \\
U^n \\
\text{Dec.} \\
W^n \\
\hat{P}_{V|W} \\
V^n
\end{array}
\end{array}
\]

Figure 3. Rate-distortion Problem 2: Compression of the source \( U^n \) with a link of rate \( R \) and reconstruction of \( V^n \).

If the two goals are fulfilled the strong coordination requirements are met, and each one implies lossy compression of a source. Then, at the encoder, we can interpret the strong coordination problem as “distorting” a source of i.i.d. distribution \( \hat{P}_{UV} \), by exploiting a link of rate \( R + R_0 \) as in Figure 2. Then, the constraint on rate of the coordination problem

\[
R + R_0 \geq \min I(UV; W) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + O\left(\frac{\log n}{n}\right) \tag{66}
\]

is similar and implies the rate-distortion condition of Theorem 7:

\[
R + R_0 \geq \min I(UV; W) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + \Theta\left(\frac{\log n}{n}\right). \tag{67}
\]

Moreover, as in Figure 3, if the decoder is able to reconstruct \( U^n \) reliably, then it can generate \( V^n \) by using the i.i.d. distribution, and strong coordination would be achieved. By allowing \( W^n \) to be the reliable reconstruction of \( U^n \) at the decoder, we can also reformulate this in terms of “stronger” rate-distortion, and we find

\[
R \geq \min I(U; W) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + O\left(\frac{\log n}{n}\right), \tag{68}
\]

which implies the rate-distortion condition of Theorem 7:

\[
R \geq \min I(U; W) + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + \Theta\left(\frac{\log n}{n}\right). \tag{69}
\]

Then, the constraints (67) and (69) which ensure lossy compression are similar to the rate conditions (66) and (68) in Theorem 2 for achievability in fixed-length strong coordination, with the only difference being the order of approximation. However, as we will see in Section IV, we prove the achievability with rate constraints

\[
\text{rate} \geq \text{mutual information} + \sqrt{\frac{V}{n} Q^{-1}(\epsilon)} + \text{constant} \cdot \frac{\log n}{n} + O\left(\frac{n}{\log n}\right)
\]

which is consistent with (67) and (69). We keep the term \( O\left(\frac{\log n}{n}\right) \) in Theorem 2 because, being less restrictive, it allows us to derive the closed result for the fixed-length coordination region of Corollary 1.

Notice that there is no rate of common randomness in this second constraint. This is because the two problems have to be solved together, and once that all the possible distributions \( \hat{P}_{U|V} \) are generated at the “encoder side” in Rate-distortion Problem 1, the decoder’s role merely relies on generating the correct random variable. This concept is better explained in the following remarks.

Remark 7 (Stochasticity at the decoder does not help): Note that we can always represent discrete stochastic decoders as discrete deterministic decoders with auxiliary randomness \( S \) that takes value in \([1, 2^{nR}]\). Then, instead
of the stochastic decoder function $\text{dec}$, we can consider the deterministic decoder $\text{dec}'$, that exploits external randomness $S$. Then, when focusing on the probability of error $p_e$, we have

$$p_e = \mathbb{P}\{d(U^n, W^n) > d\} = \mathbb{E}_S[\mathbb{P}\{d(U^n, W^n) > d \, | \, S\}] = \mathbb{E}_S[p_e(S)].$$

(70)

Since each realization $s$ of $S$ gives a deterministic decoder, and the average over all $s$ is equal to $p_e$ by (70), there exists at least one choice $s^*$ for which $p_e(s^*) \leq p_e$. Because of this, and because the choice of the deterministic decoder only concerns reliable reconstruction and not approximating the target distribution, we can assume that the decoder is deterministic without loss of generality.

Remark 8 (The encoder has to be stochastic): While we can suppose that the decoder is deterministic, the encoding function should be stochastic to achieve the whole coordination region. This is because we not only want to characterize the rates such that the rate condition holds, but also the target distributions $\hat{P}_{UV}$. When restricting the case to deterministic functions, we would restrict the choice of distributions $\hat{P}_{UV}$ that can be coordinated. More in details, $W^n$ generated according to $\hat{P}_{W|UV}^{\otimes n}$ comes from:

$$[1, 2^{nR}] \times [1, 2^{nR_0}] \times U^n \times V^n \xrightarrow{\text{enc.}} W^n,$$

and if the encoder is a deterministic function, $\text{enc}(c, m, u, v) = w$ with probability 1, whereas if the encoder is stochastic, $\text{enc}(c, m, u, v) = w$ with probability $P_{V|U}^{\otimes n}(v|u)$. Thus, the “deterministic encoder choice” restricts the possibilities for $\hat{P}_{UV^n|V^n}$ and therefore for the target distributions $\hat{P}_{UV^n|V^n}$, since the realization $(u, w, v)$ would be generated with probability

$$\frac{1}{2^{nR_C}} \frac{1}{2^{nR_M}} \hat{P}_{U}^{\otimes n}(u) \hat{P}_{V|U}^{\otimes n}(v|u)$$

instead of

$$\frac{1}{2^{nR_C}} \frac{1}{2^{nR_M}} \hat{P}_{U}^{\otimes n}(u) \hat{P}_{V|U}^{\otimes n}(v|u) \hat{P}_{W|UV}^{\otimes n}(v|u, v).$$

B. Trade-off between $\epsilon_{\text{Tot}}$ and rate

Observe that in order to minimize $\epsilon_{\text{Tot}}$, in the achievability proof we can choose $\epsilon_1^*$ and $\epsilon_2^*$ equal to zero. On the other hand, this would require more common randomness since $Q^{-1}(\cdot)$ increases as its argument approaches zero. Note that one can minimize $\epsilon_{\text{Tot}}$ (for example, we can have $\epsilon_{\text{Tot}} = \text{constant} \cdot 2^{-n}$) simply by choosing different $(\gamma_1, \gamma_2, \gamma_3)$ in Section IV.C.2 in the achievability, but this increases the rate conditions (37). If, for example we choose $(\gamma_1, \gamma_2, \gamma_3) = (2cn, cn, 2cn)$ for every constant $c$, the rate conditions become

$$R + R_0 > I(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|UV}}}{n}} + 3c,$$

$$R > I(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + 3c,$$

(71)

and therefore match the rate constraints of the outer bound (4). With this choice, the bound (24) on the $L^1$ distance decreases exponentially:

$$\|P_{U^nV^n}^{\text{RB}} - P_{U^nV^n}^{\text{RC}}\|_1 \leq \epsilon_{\text{Tot}},$$

$$\epsilon_{\text{Tot}} = 10\hat{P}(S_{\gamma_1} \cap S_{\gamma_2} \cap S_{\gamma_3})^c + 2 \left(2^{-\frac{2\gamma_1+1}{4}} + 5 \cdot 2^{-\gamma_2} + 2^{-\frac{5\gamma_1+1}{4}}\right) \leq 10(\epsilon_1 + \epsilon_2) + 2^{1-c} \left(2^\frac{1}{2} + 5\right) 2^{-n}.$$  

(72)
Suppose instead that we want to recover the same conditions of the outer bound (4). Then we can choose \((\gamma_1, \gamma_2, \gamma_3) = (2, 1, 2)\). With this choice for \(\gamma_i\), the rate conditions become

\[
R + R_0 > I(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|U}V}}{n}} + \frac{3}{n},
\]

\[
R > I(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + \frac{3}{n},
\]

and therefore match the rate constraints of the outer bound (4). With this choice, the bound (24) on the \(L^1\) distance becomes

\[
\epsilon_{\text{Tot}} \leq 10(\epsilon_1 + \epsilon_2) + \left(2^{\frac{1}{2}} + 5\right).
\]

More to this point, in the achievability we can choose \((\gamma_1, \gamma_2, \gamma_3) = \left(\frac{2c}{n^k}, \frac{c}{n^k}, \frac{2c}{n^k}\right)\) for any constant \(c\) and any \(k \geq 0\). With this choice for \(\gamma_i\), the rate conditions become

\[
R + R_0 > I(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|U}V}}{n}} + \frac{3c}{n^{k+1}},
\]

\[
= I(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|U}V}}{n}} + O\left(\frac{1}{n}\right),
\]

\[
R > I(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + \frac{3}{n \sqrt{n}},
\]

\[
= I(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + O\left(\frac{1}{n}\right)
\]

because \(\frac{3c}{n^{k+\pi}} = O\left(\frac{1}{n}\right)\), since \(\forall k \geq 0\) we have \(\frac{3c}{n^{k+\pi}} \leq \frac{3c}{n}\), and the bound on the \(L^1\) distance becomes

\[
\epsilon_{\text{Tot}} \leq 10(\epsilon_1 + \epsilon_2) + 2\left(2^{\frac{1}{2}} + 5\right)2^{-\frac{\pi}{2}}.
\]

Figure 4. Trade-off between \((\gamma_1, \gamma_2, \gamma_3)\) and \(\epsilon_{\text{Tot}}\) for \(\epsilon_1 = \epsilon_2 = 0\)

We can generalize this by fixing \((\gamma_1, \gamma_2, \gamma_3) = (2x, x, 2x)\), the rate conditions and the bound on the \(L^1\) distance become

\[
R + R_0 > I(W; UV) + Q^{-1}(\epsilon_1) \sqrt{\frac{V_{P_{W|U}V}}{n}} + \frac{3x}{n},
\]
\[ R > I(W; U) + Q^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n} + \frac{3x}{n}}, \]

\[ \epsilon_{\text{Tot}} \leq 10 (\epsilon_1 + \epsilon_2) + 2 \left( 2^{\frac{3}{2}} + 5 \right) 2^{-x}. \]  

(77)

and the trade-off between \((\gamma_1,\gamma_2,\gamma_3)\), and therefore the rate conditions, and \(\epsilon_{\text{Tot}}\) is depicted in Figure 4.

APPENDIX A

DETAILED ANALYSIS OF \(S_{i(w,u)}\)

We observe that, since the distribution \(P\) is i.i.d., the terms \(Z'_i = i_P(w_i; u_i)\), are mutually independent for \(i = 1, \ldots, n\). Then, we consider the following inequality

\[ nR > \sum_{i=1}^{n} \mathbb{E}_{P_{W|U}}[i_P(w_i; u_i)] + Q^{-1}(\epsilon_2) \sum_{i=1}^{n} \text{Var}_{P_{W|U}}(i_P(w_i; u_i)) + \gamma_2, \]  

(78)

where \(\mu'_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i]\), \(V'_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i]\) and \(Q(\cdot)\) is the tail distribution function of the standard normal distribution. We prove that, assuming that (78) holds, we can successfully bound \(S_{i(w,u)}\). In fact, the chain of inequalities

\[ \sum_{i=1}^{n} i_P(W; U) > nR - (\gamma_1 + \gamma_2) > n \left( \mu'_n + t \sqrt{\frac{V'_n}{n}} \right) \]

implies that, if (78) holds, \(S_{i(w,u)}\) is contained in

\[ \left\{ (u, w) : \sum_{i=1}^{n} i_P(w_i; u_i) > n\mu'_n + n Q^{-1}(\epsilon_2) \sqrt{\frac{V'_n}{n}} \right\}. \]  

(79)

Therefore, if we find an upper bound on (79), we have an upper bound on \(S_{i(w,u)}\) as well. To obtain that, we apply Theorem 1 (Berry-Esseen CLT) to the right-hand side of (79), and we choose

\[ Q(t) = \epsilon_2, \]

\[ \epsilon_2^* = \epsilon_2 + \frac{B'_n}{\sqrt{n}}. \]

(80)

where, as in the statement of Theorem 1 (Berry-Esseen CLT), \(B'_n = 6 \frac{T_n}{V'_{3/2}}\), and \(T'_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|Z'_i - \mu'_i|^3]\). Then, we have

\[ \mathbb{P} \left\{ \sum_{i=1}^{n} i_P(w_i; u_i) > n\mu'_n + n Q^{-1}(\epsilon_2) \sqrt{\frac{V'_n}{n}} \right\} - \epsilon_2 \leq \frac{B'_n}{\sqrt{n}}, \]

\[ \Rightarrow \mathbb{P} \left\{ \sum_{i=1}^{n} i_P(w_i; u_i) > n\mu'_n + n Q^{-1}(\epsilon_2) \sqrt{\frac{V'_n}{n}} \right\} \leq \epsilon_2^*. \]  

(81)

Finally, (81) combined with (79) implies \(P(S_{i(w,u)}) \leq \epsilon_2^*.\) Moreover, we can simplify (78) with the following identifications: similarly to [41], observe that

\[ \mu'_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{P_{W|U}}[i_P(w_i; u_i)] = \mathbb{E}_{P_{W|U}}[i_P(w; u)] = I(W; U), \]

\[ V'_n := \frac{1}{n} \sum_{i=1}^{n} \text{Var}_{P_{W|U}}(i_P(W; U)) \]

(82a)
\[
\text{Var}_{P_W}(t_P(W;U)),
\]
and \(V_{P_W} = \min_{P_W} \left[ \text{Var}_{P_W}(t_P(W;U)) \right] = \min_{P_W} \left[ \text{Var}_{P_W}(t_P(W;U)|W) \right] \) is the dispersion of the channel \( P_{W|U} \) as defined in [30, Thm. 49]. Hence, (78) can be rewritten as
\[
nR > nI(W;U) + nQ^{-1}(\epsilon_2) \sqrt{\frac{V_{P_{W|U}}}{n}} + (\gamma_1 + \gamma_2). \tag{83}
\]

### APPENDIX B

**Proof of \( H(M) \geq nI(U;W) \)**

We have
\[
H(M) \geq H(M|C) \geq I(U^n;M|C) = \sum_{i=1}^{n} I(U_i;M|U^{i-1}C) 
\]
\[
= \sum_{i=1}^{n} I(U_i;MC|U^{i-1}) - \sum_{i=1}^{n} I(U_i;C|U^{i-1})
\]
\[
\geq \sum_{i=1}^{n} I(U_i;MC|U^{i-1}) - \sum_{i=1}^{n} I(U_i;C) \overset{(a)}{=} \sum_{i=1}^{n} I(U_i;MC|U^{i-1})
\]
\[
= \sum_{i=1}^{n} I(U_i;MCU^{i-1}) - \sum_{i=1}^{n} I(U_i;U^{i-1})
\]
\[
\overset{(d)}{=} \sum_{i=1}^{n} I(U_i;MCU^{i-1}) \geq \sum_{i=1}^{n} I(U_i;MC)
\]
\[
\overset{\text{c}}{=} \sum_{i=1}^{n} I(U_i;W_i) \overset{\text{f}}{=} nI(U;W) \tag{84}
\]
where (a) and (b) follow from the i.i.d. nature of the channel, and (c) and (d) from the identifications \( W_i = (C, M) \) for each \( t \in [1,n] \) and \( W = (W_T, T) = (C, M, T) \).

### APPENDIX C

**Proof of the bound on \( \hat{R} \): Case 2**

#### A. Proof of the upper bound on \( \log \beta_\alpha (46a) \) - Case 2 \( (P_{U^nW^n} = P_{U^{\otimes n}}^{\otimes n} P_{W|U}^{\otimes n} - \epsilon) \)

We have
\[
\log (P_{U^nW^n}(u, w)) = \log \left( \tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w) - \epsilon \right)
\]
\[
= \log \left( \tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w) \right) + \log \left( 1 - \frac{\epsilon}{\tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w)} \right)
\]
\[
= \log \left( \tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w) \right) - \log \left( \frac{\tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w)}{\tilde{P}_{U^{\otimes n}} P_{W|U}^{\otimes n}(u, w) - \epsilon} \right). \tag{85}
\]
Thus, the following holds
\[
\mathbb{P} \left\{ \log P_{U^nW^n} \geq \log P_{U^{\otimes n}} P_{W}^{\otimes n} + \log \gamma_0 \right\}
\]
\[
= \mathbb{P} \left\{ \log P_{U^{\otimes n}} P_{W|U}^{\otimes n} - \log \left( \frac{P_{U^{\otimes n}} P_{W|U}^{\otimes n}}{P_{U^{\otimes n}} P_{W|U}^{\otimes n} - \epsilon} \right) \geq \log P_{U^{\otimes n}} P_{W}^{\otimes n} + \log \gamma_0 \right\}
\]
\[
= \mathbb{P} \left\{ \log P_{U^{\otimes n}} P_{W|U}^{\otimes n} \geq \log P_{U^{\otimes n}} P_{W}^{\otimes n} + \log \gamma_0 + \log \left( \frac{P_{U^{\otimes n}} P_{W|U}^{\otimes n}}{P_{U^{\otimes n}} P_{W|U}^{\otimes n} - \epsilon} \right) \right\}
\]
Then, we set
\[ B = \Pr\left\{ \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U} \geq \log \gamma + \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} - \epsilon \right\} \]
\[ = \Pr\left\{ \sum_{i=1}^{n} \log \frac{\bar{P}_n \bar{P}_n}{P_U P_W} \geq \log \gamma + \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} - \epsilon \right\}. \tag{86} \]

Since \( \sum_{i=1}^{n} \log (\bar{P}_n \bar{P}_n / \bar{P}_n \bar{P}_n) =: \sum_{i=1}^{n} Z_i \) is the sum of \( n \) i.i.d. random variables, the next step is to evaluate the probability in (48) using Theorem 1 (Berry-Esseen CLT). To accomplish that, we choose appropriately the parameter \( \gamma \):

\[ \log \gamma_0 := n \mu_n + Q^{-1} (\epsilon) \sqrt{nV_n} - \log \left( \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} - \epsilon \right) \tag{87} \]

where, as in Theorem 1, \( \mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i] \), \( V_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i] \), \( T_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|Z_i - \mu_i|^3] \), \( B_n = 6 \frac{T_n}{\sqrt{V_n}} \), and \( Q(\cdot) \) is the tail distribution function of the standard normal distribution. Observe that by (87) and Remark 5, (86) becomes

\[ \Pr\left\{ \sum_{i=1}^{n} \log \frac{\bar{P}_n \bar{P}_n}{P_U P_W} \geq n \mu_n + Q^{-1} (\epsilon) \sqrt{nV_n} \right\} \geq 1 - \frac{B_n}{\sqrt{n}} =: \alpha. \tag{88} \]

Then, we have

\[ \log \frac{1}{\beta_n} \geq \log \bar{\gamma} = n \mu_n + Q^{-1} (\epsilon) \sqrt{nV_n} - \log \left( \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} - \epsilon \right). \tag{89} \]

**B. Proof of the lower bound on \( \log \beta_n \) (46b) - Case 2 \( \bar{P}_n \bar{P}_n = \bar{P}_n \otimes n \bar{P}_n[U] - \epsilon \)**

By (44b) for every \( \gamma > 0 \) we have

\[ \frac{1}{\gamma} \left[ \alpha - \Pr_{\bar{P}_n \bar{P}_n} \left\{ \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W} > \log \gamma \right\} \right] \geq \frac{1}{\gamma} \left[ \alpha - \Pr_{\bar{P}_n \bar{P}_n} \left\{ \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} - \epsilon > \log \gamma \right\} \right] \]
\[ \geq \frac{1}{\gamma} \left[ \alpha - \Pr_{\bar{P}_n \bar{P}_n} \left\{ \log \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U]} > \log \gamma + \log \left( \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U] - \epsilon} \right) \right\} \right]. \]

Then, we set

\[ \log \gamma = H(M) - \log \left( \frac{\bar{P}_n \otimes n \bar{P}_n}{P_U \otimes n P_W[U] - \epsilon} \right) \]

and, as proved in (55),

\[ nR = H(M) \geq nI(U; W) \geq nI(U; W) + Q^{-1}(y) \sqrt{nV_n} \quad \frac{1}{2} < y < 1 \]
which implies
\[
\log \beta_\alpha \geq \log \frac{1}{\gamma} + \log \left[ \alpha - \mathbb{P}_{P_{U^nW^n}} \left\{ \log \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n}} > \log \gamma + \log \left( \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n} - \epsilon} \right) \right\} \right]
\]
\[
\geq -H(M) + \log \left( \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n} - \epsilon} \right) + \log \left[ \alpha - \mathbb{P}_{P_{U^nW^n}} \left\{ \log \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n}} > nI(U; W) + Q^{-1}(y) \sqrt{nV_n} \right\} \right]
\]
\[
= -H(M) + \log \left( \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n} - \epsilon} \right) + \log \left( \alpha - y - B_n \right)
\]

(90)

and similarly to (60), (46b) holds for this case as well.

C. Proof of the rate constraint – Case 2 \((P_{U^nW^n} = \tilde{P}_{U^n} \tilde{P}_{W^n} - \epsilon)\)

Now, we conclude the proof by combining (89) and (90). For \(1/2 < y < 1\), we have
\[
\frac{H(M)}{nR} = \log \left( \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n} - \epsilon} \right) + \log \left( \alpha - y - B_n \right) \geq n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} - \log \left( \frac{\tilde{P}_{U^n} \tilde{P}_{W^n}}{P_{U^n} P_{W^n} - \epsilon} \right)
\]

which is equivalent to
\[
R \geq \mu_n + Q^{-1}(\epsilon) \sqrt{\frac{V_n}{n}} + \frac{\log \left( \alpha - y - B_n \right)}{n}.
\]

(91)

APPENDIX D

PROOF OF THE BOUND ON \(R + R_0\)

As in Section V.A, for an observation \(w = (m, c)\), we define the hypothesis:

\[H_0: \text{w generated according to } P_{U^nV^nW^n}(w) = \sum_{u,v} P_{U^n}(w) P_{U^nV^nW^n}(u,w,v) = \sum_{u,v} P_{U^nW^n}(u,w,v),\]

\[H_1: \text{w generated according to } \tilde{P}_{U^nW^n}(w) = \sum_{u,v} \tilde{P}_{U^nV^nW^n}(u,w,v),\]

where \(\tilde{P}\) is the i.i.d. target distribution. Then, we consider a randomized test between the distributions \(P_{U^nW^nV^n}\) and \(\tilde{P}_{U^nV^nW^n}\):
\[P_{Z[U^nW^nV^n]} : U^n \times V^n \times V^n \rightarrow \{H_0, H_1\},\]

where \(H_0\) indicates that the test chooses \(P_{U^nW^nV^n}\), and \(H_1\) indicates that the test chooses \(\tilde{P}_{U^nV^nW^n}\). The probability of type-I error (probability of choosing \(H_1\) when the true hypothesis is \(H_0\)) and type-II error (probability of choosing \(H_0\) when the true hypothesis is \(H_1\)) are

\[
P_e^I(P_{Z[U^nW^nV^n]}) := \mathbb{P}\{H_1|H_0\} = \sum_{u,v} \tilde{P}_{U^n}(u,v) \tilde{P}_{W^n}(w) P_{Z[U^nW^nV^n]}(H_0|u,w,v),
\]

(92a)

\[
P_e^{II}(P_{Z[U^nW^nV^n]}) := \mathbb{P}\{H_0|H_1\} = \sum_{u,v} P_{U^nW^nV^n}(u,w,v) P_{Z[U^nW^nV^n]}(H_1|u,w,v).
\]

(92b)

Similar to (43), we denote with \(\beta_{\alpha'}^{I}\) the minimum type-I error for a maximum type-II error \(1 - \alpha':\)

\[
\beta_{\alpha'}^{I} := \min_{P_{Z[U^nW^nV^n]}: P_{e}^{I}(P_{Z[U^nW^nV^n]} \leq 1 - \alpha')}
\]

(93)
where the error probability \( \alpha' \) will be defined later. The following relations between \( \alpha' \) and \( \beta_{\alpha'}' \), proved in [48, Section 12.4], hold:

\[
\beta_{\alpha'}' \leq \frac{1}{\gamma_0}, \quad \text{if} \quad \gamma_0 \text{ is such that } \Pr_{U^nW^nV^n} \left\{ \log \frac{P_{U^nW^nV^n}}{P_{U^nV^n}P_{W^n}} > \log \gamma_0 \right\} \geq \alpha',
\]

(94a)

\[
\alpha' \leq \Pr_{U^nW^nV^n} \left\{ \log \frac{P_{U^nW^nV^n}}{P_{U^nV^n}P_{W^n}} > \log \gamma \right\} + \gamma \beta_{\alpha'}' \quad \forall \gamma > 0.
\]

(94b)

Similarly to Section V.A, we prove the rate constraint by separately deriving an upper and a lower bound on \( \log \beta_{\alpha'}' \):

\[
\text{upper bound on } \log \beta_{\alpha'}' \quad \log \frac{1}{\beta_{\alpha'}'} \geq nI(W;UV) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|UV}}},
\]

(95a)

\[
\text{lower bound on } \log \beta_{\alpha'}' \quad n(R + R_0) + x \geq \log \frac{1}{\beta_{\alpha'}'},
\]

(95b)

for a certain \( x \in \mathbb{R} \) which will be defined later. Then, the proof of the rate constraint is divided in the following steps, detailed in the next sections:

(i) **Proof of the upper bound on \( \log \beta_{\alpha'}' \):** we use the upper bound on \( \min \) \( P_{Z|U^nW^nV^n} \) of (94a) combined with the \((\epsilon, n)\)-strong coordination assumption (41) and Theorem 1 (Berry–Essen CLT) to derive the following upper bound on the logarithm \( \beta_{\alpha'}' \) of (95a) by choosing the parameter \( \gamma_0 \);

(ii) **Proof of the lower bound on \( \log \beta_{\alpha'}' \):** we use the lower bound on \( \min \) \( P_{Z|U^nW^nV^n} \) of (94b) combined with the \((\epsilon, n)\)-strong coordination assumption (41) and classical information theory properties to derive the lower bound on \( \beta_{\alpha'}' \) of (95b) by choosing the parameter \( \gamma \);

(iii) we combine (95a) and (95b) proved in the previous steps and we derive the rate constraint.

Moreover, as before by the \((\epsilon, n)\)-strong coordination assumption (41) we have

\[
\|P_{U^nW^nV^n} - \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}\|_1 \leq \epsilon,
\]

\[
\Rightarrow \forall (u, w, v) \quad |P_{U^nW^nV^n}(u, w, v) - \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v)| \leq \epsilon
\]

\[
\Rightarrow \forall (u, w, v) \exists \epsilon_{uvw} \leq \epsilon \text{ such that } |P_{U^nW^nV^n}(u, w, v) - \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v)| = \epsilon_{uvw}
\]

which we can distinguish into two cases:

- **Case 1** \( P_{U^nW^nV^n}(u, w, v) = \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v) + \epsilon_{uvw} \);

- **Case 2** \( P_{U^nW^nV^n}(u, w, v) = \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v) - \epsilon_{uvw} \).

Thus, we prove steps (i)–(iii) separately for both cases. With a slight abuse of notation from now on we will drop the index from \( \epsilon_{uvw} \) and use \( \epsilon \) instead, since \( \epsilon_{uvw} \) just has to be smaller that \( \epsilon \). Similarly, we omit the pairs \((u, w, v)\) in order to simplify the notation.

**A. Proof of the upper bound on \( \log \beta_{\alpha'}' \) (95a) – Case 1** \( (P_{U^nW^nV^n} = \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n} + \epsilon) \)

By the \((\epsilon, n)\)-strong coordination assumption (41), we have

\[
\log (P_{U^nW^nV^n}(u, w, v)) = \log \left( \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v) \right) + \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v)} \right).
\]

(96)

Then, similarly to (51), the following holds

\[
\Pr \left\{ \log P_{U^nW^nV^n} \geq \log \tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n} + \log \gamma_0 \right\}
\]

\[
= \Pr \left\{ \log \left( \frac{\tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}}{P_{U^nV^n}P_{W^n}} \right) \geq \log \gamma_0 - \log \left( 1 + \frac{\epsilon}{\tilde{P}_{U}^{\otimes n} \tilde{P}_{W|U}^{\otimes n} \tilde{P}_{V|W}^{\otimes n}(u, w, v)} \right) \right\}
\]

\[
= \Pr \left\{ \sum_{i=1}^{n} \log \frac{P_{U}^{i}P_{W|U}^{i}P_{V|W}^{i}}{P_{U}^{i}P_{W}^{i}} \geq nI(UV;W) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|UV}}} \right\}
\]

(97)
where the last equality follows from choosing the parameter $\gamma_0$ as
\[
\log \gamma_0 := n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} + \log \left(1 + \frac{\epsilon}{P_{U \cap W|U} P_{V|W}}\right) \tag{98}
\]
and, as in the statement of Theorem 1 and Remark 5, from the identifications:
\[
\sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} \log(\tilde{P}_U \tilde{P}_W|U \tilde{P}_V|W) / \tilde{P}_{UV} \tilde{P}_W,
\]
\[
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z_i] = \mathbb{D}(\tilde{P}_U \tilde{P}_W|U \tilde{P}_V|W) \| \tilde{P}_{UV} \tilde{P}_W) = I(W; UV),
\]
\[
V_n = \frac{1}{n} \sum_{i=1}^{n} \text{Var}[Z_i] = V_{\tilde{P}_W|UV}
\]
\[
= \sum_{u,w,v} \tilde{P}_U(u) \tilde{P}_W|U(w|u) \tilde{P}_V|W(v|w) \left[ \log \frac{\tilde{P}_W|U(w|u) \tilde{P}_V|W(v|w)}{\tilde{P}_{UV}(u,v) \tilde{P}_W(w)} \right] \tag{99}
\]
\[
T_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|Z_i - \mu_i\|^3]
\]
\[
= \sum_{u,w,v} \tilde{P}_U(u) \tilde{P}_W|U(w|u) \tilde{P}_V|W(v|w) \left[ \log \frac{\tilde{P}_W|U(w|u) \tilde{P}_V|W(v|w)}{\tilde{P}_{UV}(u,v) \tilde{P}_W(w)} \right] - \mathbb{D}(\tilde{P}_U \tilde{P}_W|U \tilde{P}_V|W) \| \tilde{P}_{UV} \tilde{P}_W)^3 \tag{100}
\]
\[
B_n = \frac{6 T_n}{V_n^{3/2}},
\]
and $Q(\cdot)$ is the tail distribution function of the standard normal distribution.

Now, since $\sum_{i=1}^{n} \log(\tilde{P}_U \tilde{P}_W|U \tilde{P}_V|W) / \tilde{P}_{UV} \tilde{P}_W) = \sum_{i=1}^{n} Z_i$ is the sum of $n$ i.i.d. random variables, we bound (97) using Theorem 1 (Berry-Esseen CLT):
\[
\mathbb{P} \left\{ \log \prod_{i=1}^{n} \frac{\tilde{P}_U \tilde{P}_W|U \tilde{P}_V|W}{\tilde{P}_{UV} \tilde{P}_W} \geq n I(UV; W) + Q^{-1}(\epsilon) \sqrt{nV_{\tilde{P}_W|UV}} \right\} \geq \left( \epsilon - \frac{B_n}{\sqrt{n}} \right) \tag{99}
\]
and we identify
\[
\alpha' := \epsilon - \frac{B_n}{\sqrt{n}} \tag{100}
\]
and by combining (94a) with (98), we obtain:
\[
\log \frac{1}{\beta'_{\alpha'}} \geq \log \gamma_0 = n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} + \log \left(1 + \frac{\epsilon}{P_{U \cap W|U} P_{V|W}}\right) \tag{101}
\]

B. Proof of the lower bound on $\log \beta'_{\alpha'}$ (95b) - Case 1 ($P_{U=W \cap V=}$ $P_{U \cap W|U} P_{V|W} + \epsilon$)

First, we prove that
\[
n(R + R_0) = H(M, C) \geq n I(UV; W) - 4n \left( \log |U \times V| + \log \frac{1}{\epsilon} \right) \tag{102}
\]
\[
\geq n I(UV; W) - 4n \left( \log |U \times V| + \log \frac{1}{\epsilon} \right) + Q^{-1}(y) \sqrt{nV_n}, \quad \frac{1}{2} < y < 1.
\]
To prove (a), observe that
\[
H(M, C) \geq I(U^n V^n; MC) = \sum_{i=1}^{n} I(U_i V_i; MC|U^{i-1}V^{i-1})
\]
\[
= \sum_{i=1}^{n} I(U_i V_i; MC U^{i-1}V^{i-1}) - \sum_{i=1}^{n} I(U_i V_i; U^{i-1}V^{i-1})
\]
\[
\geq \sum_{i=1}^{n} I(U_i V_i; MC U^{i-1}V^{i-1}) - ng(\epsilon)
\]
\[
\geq \sum_{i=1}^{n} I(U_i V_i; MC) - ng(\epsilon)
\]
\[
= \sum_{i=1}^{n} I(U_i V_i; W_i) - ng(\epsilon) \geq nI(UV; W) - 2ng(\epsilon)
\]
\[
= nI(UV; W) - 4n\epsilon \left( \log |\mathcal{U} \times \mathcal{V}| + \log \frac{1}{\epsilon} \right)
\]
(103)

where, as in [9], the term \( g(\epsilon) \) in (c) and (e) is defined as
\[
g(\epsilon) := 2\epsilon \left( \log |\mathcal{U} \times \mathcal{V}| + \log \frac{1}{\epsilon} \right),
\]
(104)

and the inequalities (c) and (e) are proved in [9, Lemma VI.3]. Moreover (d) and (e) use the identifications \( W_t = (C, M) \) for each \( t \in [1, n] \) and \( W = (W_T, T) = (C, M, T). \) Finally, (102) is proved in [9, Lemma VI.3] since (b) comes from the fact that \( Q^{-1}(y)\sqrt{nV_n} \leq 0 \) for every \( \frac{1}{2} < y < 1. \)

Now, we recall that by (94b) for every \( \gamma > 0, \) we have
\[
\beta_{\alpha'} \geq \frac{1}{\gamma} \left[ \alpha' - P_{P_{U^n W^n V^n} \cap P_{P_{U^n W^n V^n}} P^n_{W^n V^n} > \log \gamma} \right]
\]
\[
= \frac{1}{\gamma} \left[ \alpha' - P_{P_{U^n W^n V^n}} \left\{ \log \frac{P^n_{U^n W^n V^n}}{P^n_{U^n V^n} P^n_{W^n}} > \log \gamma - \log \left( 1 + \frac{\epsilon}{P^n_{U^n W^n V^n} P^n_{W^n U^n V^n} P^n_{V^n W^n}} \right) \right\} \right].
\]
(105)

Then, we set
\[
\log \beta_{\alpha'} \geq \log \frac{1}{\gamma} + \log \left[ \alpha' - P_{P_{U^n W^n V^n}} \left\{ \log \frac{P^n_{U^n W^n V^n}}{P^n_{U^n V^n} P^n_{W^n}} > \log \gamma - \log \left( 1 + \frac{\epsilon}{P^n_{U^n W^n V^n} P^n_{W^n U^n V^n} P^n_{V^n W^n}} \right) \right\} \right]
\]
\[
= - \left[ H(M, C) + 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P^n_{U^n W^n V^n} P^n_{W^n U^n V^n} P^n_{V^n W^n}} \right) \right]
\]
\[
+ \log \left[ \alpha' - P_{P_{U^n W^n V^n}} \left\{ \log \frac{P^n_{U^n W^n V^n}}{P^n_{U^n V^n} P^n_{W^n}} > H(M, C) + 2ng(\epsilon) \right\} \right]
\]
\[
\geq - \left[ H(M, C) + 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P^n_{U^n W^n V^n} P^n_{W^n U^n V^n} P^n_{V^n W^n}} \right) \right]
\]
\[
+ \log \left[ \alpha' - P_{P_{U^n W^n V^n}} \left\{ \log \frac{P^n_{U^n W^n V^n}}{P^n_{U^n V^n} P^n_{W^n}} > nI(UV; W) + Q^{-1}(y)\sqrt{nV_n} \right\} \right]
\]
\[ -H(M, C) - 2ng(\epsilon) - \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) + \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \]. \tag{107}

Similarly to (60), we identify
\[ x = 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \]
and we observe that (107) is equivalent to
\[ n(R + R_0) + x = n(R + R_0) + 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \]
\[ = H(M, C) + 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \geq \log \frac{1}{\beta_{\alpha'}}. \tag{108} \]

C. Proof of the rate constraint – Case 1 \((P_{U^n W^n V^n} = P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n} + \epsilon)\)

Now, we conclude the proof of this part by combining (101) and (108). For \(1/2 < y < 1\) we have
\[ \frac{H(M, C)}{n(R + R_0)} + 2ng(\epsilon) + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \]
\[ \geq n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} + \log \left( 1 + \frac{\epsilon}{P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}} \right) \]
which is equivalent to
\[ R + R_0 \geq \mu_n + Q^{-1}(\epsilon) \sqrt{\frac{V_n}{n}} + \frac{\log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right)}{n} - 2g(\epsilon) \]
\[ = \mu_n + Q^{-1}(\epsilon) \sqrt{\frac{V_n}{n}} + \frac{\log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right)}{n} - 4\epsilon \left( \log |U \times V| + \log \frac{1}{\epsilon} \right). \tag{109} \]

D. Proof of the upper bound on \(\log \beta_{\alpha'}\) (95a) – Case 2 \((P_{U^n W^n V^n} = P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n} - \epsilon)\)

We have
\[ \log (P_{U^n W^n V^n}(u, w, v)) = \log \left( P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}(u, w, v) \right) - \log \left( P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n}(u, w, v) - \epsilon \right). \tag{110} \]

Thus, the following holds
\[ \mathbb{P} \left\{ \log P_{U^n W^n V^n} \geq \log P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n} + \log \gamma_0 \right\} \]
\[ = \mathbb{P} \left\{ \sum_{i=1}^n \log \frac{P_U P_{W|U} P_{V|W}}{P_U P_W} \geq \log \gamma_0 + \log \left( P_U^{\otimes n} P_{W|U}^{\otimes n} P_{V|W}^{\otimes n} - \epsilon \right) \right\} \]
\[ = \mathbb{P} \left\{ \sum_{i=1}^n \log \frac{P_U P_{W|U} P_{V|W}}{P_{U W|U} P_W} \geq nI(UV; W) + Q^{-1}(\epsilon) \sqrt{nV_{P_{W|U} W}} \right\} \]
\[ \geq \epsilon - \frac{B_n}{\sqrt{n}} =: \alpha'. \tag{111} \]
by Theorem 1 (Berry-Esseen CLT) and
\[
\log \gamma_0 := n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right). \tag{112}
\]

Then, we have
\[
\log \frac{1}{\beta_{\alpha'}} \geq \log \gamma_0 = n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right). \tag{113}
\]

**E. Proof of the lower bound on** $\log \beta_{\alpha'}$ **(95b) – Case 2** ($P^{\otimes n}_{U^n V^n} = \tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon$)

By (94b) $\forall \gamma > 0$ we have
\[
\beta_{\alpha'} \geq \frac{1}{\gamma} \left[ \alpha' - \mathbb{P}_{P^{\otimes n}_{U^n V^n}} \left\{ \log \frac{P^{\otimes n}_{U^n V^n}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{V|W}} > \log \gamma \right\} \right]
= \frac{1}{\gamma} \left[ \alpha' - \mathbb{P}_{P^{\otimes n}_{U^n V^n}} \left\{ \log \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}} > \log \gamma + \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right) \right\} \right].
\]

Then, we set
\[
\log \gamma = H(M, C) - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right)
\]
and we recall the following chain of inequalities, proved in (102)
\[
n(R + R_0) = H(M, C) \geq nI(UV; W) - 4n\epsilon \left( \log |U \times V| + \log \frac{1}{\epsilon} \right) + Q^{-1}(y)\sqrt{nV_n}, \quad \frac{1}{2} < y < 1 \tag{114}
\]
which implies
\[
\log \beta_{\alpha'} \geq \log \frac{1}{\gamma} + \log \left[ \alpha' - \mathbb{P}_{P^{\otimes n}_{U^n V^n}} \left\{ \log \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}} > \log \gamma + \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right) \right\} \right]
\geq -H(M, C) - \left\{ 4n\epsilon \left( \log |U \times V| + \log \frac{1}{\epsilon} \right) - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right) \right\}. \tag{115}
\]

**F. Proof of the rate constraint – Case 2** ($P^{\otimes n}_{U^n V^n} = \tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon$)

Now, by combining (115) with (113), for $1/2 < y < 1$ we have
\[
\left( H(M, C) \right)_{\frac{n(R + R_0)}{n}} + 4n\epsilon \left( \log |U \times V| + \log \frac{1}{\epsilon} \right) - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right) - \log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right)
\geq n\mu_n + Q^{-1}(\epsilon) \sqrt{nV_n} - \log \left( \frac{\tilde{P}^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W}}{P^{\otimes n}_{U} \tilde{P}^{\otimes n}_{W|U} \tilde{P}^{\otimes n}_{V|W} - \epsilon} \right)
\]
which is equivalent to
\[
R + R_0 \geq \mu_n + Q^{-1}(\epsilon) \sqrt{\frac{V_n}{n}} + \frac{\log \left( \alpha' - y - \frac{B_n}{\sqrt{n}} \right)}{n} - 4\epsilon \left( \log |U \times V| + \log \frac{1}{\epsilon} \right). \tag{116}
\]
Appendix E

Proof of the Cardinality Bound

Here we prove the cardinality bound for the outer bound in Theorem 3.

First, we state the Support Lemma [53, Appendix C].

Lemma 2: Let $\mathcal{A}$ a finite set and $\mathcal{W}$ be an arbitrary set. Let $\mathcal{P}$ be a connected compact subset of probability mass functions on $\mathcal{A}$ and $P_{\mathcal{A}|\mathcal{W}}$ be a collection of conditional probability mass functions on $\mathcal{A}$. Suppose that $h_i(\pi)$, $i = 1, \ldots, d$, are real-valued continuous functions of $\pi \in \mathcal{P}$. Then for every $W$ defined on $\mathcal{W}$ there exists a random variable $W'$ with $|\mathcal{W}'| \leq d$ and a collection of conditional probability mass functions $P_{\mathcal{A}|\mathcal{W}'} \in \mathcal{P}$ such that

$$
\sum_{w \in \mathcal{W}} P_W(w) h_i(P_{\mathcal{A}|\mathcal{W}}(a|w)) = \sum_{w \in \mathcal{W}'} P_{W'}(w) h_i(P_{\mathcal{A}|\mathcal{W}'}(a|w)) \quad i = 1, \ldots, d.
$$

Now, we consider the probability distribution $\bar{P}_U \bar{P}_{W|U} \bar{P}_{V|W}$ that is $\epsilon$-close in $L^1$ distance to the i.i.d. distribution. We identify $\mathcal{A}$ with $\{1, \ldots, |\mathcal{A}|\}$ and we consider $\mathcal{P}$ a connected compact subset of probability mass functions on $\mathcal{A} = \mathcal{U} \times \mathcal{V}$. Similarly to [20], suppose that $h_i(\pi)$, $i = 1, \ldots, |\mathcal{A}| + 1$, are real-valued continuous functions of $\pi \in \mathcal{P}$ such that:

$$
h_i(\pi) = \begin{cases} 
\pi(i) & \text{for } i = 1, \ldots, |\mathcal{A}| - 1, \\
H(U) & \text{for } i = |\mathcal{A}|, \\
H(V|U) & \text{for } i = |\mathcal{A}| + 1.
\end{cases}
$$

Then by Lemma 2 there exists an auxiliary random variable $W'$ taking at most $|\mathcal{U} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| + 1$ values such that:

$$
H(U|W) = \sum_{w \in \mathcal{W}} P_W(w) H(U|W = w) = \sum_{w \in \mathcal{W}'} P_{W'}(w) H(U|W' = w) = H(U|W'),
$$

$$
H(V|UW) = \sum_{w \in \mathcal{W}} P_W(w) H(V|UW = w) = \sum_{w \in \mathcal{W}'} P_{W'}(w) H(V|UW' = w) = H(V|UW').
$$

The constraints on the conditional distributions, the rate constraints and the Markov chain $U - W - V$ are therefore still verified since we can write

$$
I(U; W) = H(U) - H(U|W),
$$

$$
I(UV; W) = H(UV) - H(UV|W) = H(U) + H(V|U) - H(U|W) + H(V|UW),
$$

$$
I(U; V|W) = H(U|W) - H(U|VW) = 0.
$$

Note that we are not forgetting any constraints: once the distribution $\bar{P}_{UV}$ and the Markov chain $U - W - V$ are preserved, the dispersion term of the channels $\bar{P}_{W|U}$ and $\bar{P}_{W|UV}$ are fixed as well.

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