The oscillations in the lossy medium

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Abstract

The object of the work: to explore dependence mass point oscillatory motion parameters in the following cases:
- without resistance (free oscillations);
- the resistance force is proportional to the velocity vector;
- the resistance force is proportional to the velocity squared.

Used equipment: the work have doing on a personal computer. The oscillatory motion simulation is carried out by the numerical solution of system of differential equations. This equations describe a motion of a particle under an elastic force action and exterior forces (resistance force) with initial values and parameters being entered during the dialogue with the computer.

1 The theoretical part

Let’s begin from the definition: If some physical quantity \( F \) under specified physical conditions is described periodic or almost-periodic function of time one can say that this physical quantity is in oscillatory process or in oscillations.

As is known, a function \( F(t) \) is called periodic if \( F(t) = F(t + T) \). At the oscillatory process the constant \( T = 2 \pi / \omega \) is called an oscillation period and the constant \( \omega \) is called an oscillation frequency (circular or cyclic). Obviously \( T \) is a time interval by means of that the values of function \( F(t) \) are repeated.

If the physical quantity is in oscillations described by the harmonic function of time (i.e. function \( \sin(\omega t) \) or \( \cos(\omega t) \)) the oscillations is called harmonic.

Among all oscillatory processes the special interest is represented those which the man can observe directly without any devices. The most known oscillatory process having so remarkable property is the oscillatory motion.

According to this, the oscillatory motion of a mass point we will call any its motion that the all physical quantities describing motion are periodic (or almost-periodic) functions of time.

The major physical values describing a motion of a mass point are:
- the radius vector of a particle \( \vec{r}(t) \), i.e. its coordinates (we shall remind the equation of a form \( \vec{r} = \vec{r}(t) \) is called the motion equation (or law):

\[ 1 \] The definition almost-periodic functions will be introduced later.
- and the vector of the particle acceleration \( \vec{a}(t) \).

If we take into account that the vectors of velocity and acceleration are defined uniquely by the radius vector \( \vec{r}(t) \) of a mass point, it is possible to formulate the following definition:

\[
\text{any motion of a mass point at which the radius vector of a particle is a periodic (or almost periodic) function of time is called the oscillatory motion.}
\]

### 1.1 Free simple harmonic motions

The elementary oscillatory motion of a mass point is the harmonic oscillatory motion. Thus, according to the definition of harmonic oscillations, we shall call by free simple harmonic motion such oscillatory motion, at which the radius vector of a particle is harmonic function of time. It means, the equation (the law) motion of a mass point that is in a free harmonic oscillatory motion, has a form

\[
\vec{r}(t) = \vec{r}_o \sin(\omega t + \varphi_o).
\]  

(1)

In eq.(1) the constant \( \Phi = \omega t + \varphi_o \) is called by a phase of the oscillatory motion and its value at \( t = 0 \), i.e. \( \varphi_o \), is called by an epoch angle accordingly. The constant \( \vec{r}_o \) is called by an amplitude of the oscillatory motion. From the equation (1) is obvious that the amplitude is the maximal value of radius vector the achievable at those point in time when \( \sin(\omega t + \varphi_o) = 1 \).

Let’s note one important characteristic of the oscillatory motion described by the equation (1). The vector \( \vec{r}_o \) is a constant vector, i.e. does not change neither in magnitude nor in the direction. Therefore the vector \( \vec{r}(t) \) can change only in magnitude (at the expense of function \( \sin(...) \)), but remains parallel to the same line. It means that the harmonic oscillatory motion always has only one degree of freedom. In other words, one coordinate is enough for describing of a harmonic oscillatory motion. For example, coordinates measured along an axis \( OX \). So the vector equation (1) can always be replaced by one equation in the coordinate form

\[
x(t) = x_o \sin(\omega t + \varphi_o),
\]  

(2)

where \( x_o = |\vec{r}_o| \) is the module of the vector \( \vec{r}_o \).

It is easy to see that the equation (1) is the solution of the differential equation

\[
\frac{d^2\vec{r}}{dt^2} + \omega^2 \vec{r} = 0.
\]  

(3)

For this reason the differential equation (3) is called by the equation of free simple harmonic motions. So, one can say that

free harmonic oscillatory motion of a mass point is any motion described by the equation of free simple harmonic motions (eq.(3)).

Classical example of a free harmonic oscillatory motion is the particle motion with the mass \( m \) due to action of quasi-elastic force (i.e. simulative elastic force)
\( \vec{F} = -k \vec{r} \), where \( k \) is stiffness coefficient. To be convinced of it we shall describe for such a mass point the dynamical equation (i.e. Newton’s second law)

\[
m \vec{a} = -k \vec{r}.
\] (4)

Taking into account that the acceleration is a second-order derivative of the particle radius vector, we shall obtain

\[
\frac{d^2 \vec{r}}{dt^2} + \frac{k}{m} \vec{r} = 0.
\] (5)

Comparing the obtained equation with the equation of free simple harmonic motions (3), we can see that the motion a mass point due to action of quasi-elastic force is really a free harmonic oscillatory motion. And the oscillation cyclic frequency of a mass point is equal

\[
\omega = \sqrt{\frac{k}{m}}.
\] (6)

### 1.2 Damped oscillations

In the previous section we have considered a free harmonic motion and were convinced that due to action of only elastic force the mass point makes just such motion.

Let’s consider now motion a mass point due to action of quasi-elastic forces \( \vec{F} = -k \vec{r} \) in medium under the action of resistance forces. Let, for example, the resistance force is proportional to a vector of the particle velocity \( \vec{F}_c = -b \vec{v} \), where \( b \) is the resistance coefficient. Then the dynamical law (Newton’s second law) for such the mass point will have a form

\[
m \vec{a} = \vec{F} + \vec{F}_c = -k \vec{r} - b \vec{v}.
\] (7)

Taking into account that the velocity is a first-order derivative and that the acceleration is a second-order derivative of the particle radius vector, we shall obtain

\[
\frac{d^2 \vec{r}}{dt^2} + \frac{b}{m} \frac{d \vec{r}}{dt} + \frac{k}{m} \vec{r} = 0.
\] (8)

It is easy to be convinced that the obtained equation coincides with the equation of free simple harmonic motions only at absence of the resistance forces (i.e. at \( b = 0 \)). The solution of the equation (8) varies from the solution of the equation (3) as well. The eq.(3) is the equation of free simple harmonic motions. So, the common solution of the equation (8) will have a form

\[
\vec{r}(t) = \vec{r}_o e^{-\beta t} \sin(\omega t + \varphi_o),
\] (9)

where the following notation for parameters of an oscillatory motion described by the equation (8) are conventional

- damping factor \( \beta = \frac{b}{2m} \)
- oscillation cyclic frequency of the free harmonic oscillatory motion (i.e. at absence of the resistance forces) \( \omega_o = \sqrt{\frac{k}{m}} \) (10)
- oscillation cyclic frequency of the studied harmonic oscillatory motion \( \omega = \sqrt{\omega_o - \beta^2} \)
Let’s note that in these notation the equation (8) will look like

$$\frac{d^2 \vec{r}}{dt^2} + 2\beta \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = 0.$$  \hfill (11)

As well as in the case of free simple harmonic motions the oscillatory motion described by the equation (8) has **only one degree of freedom**. Hence, if to set the direction of constant vector $\vec{r}_0$ parallelly to axis $OX$ of a cartesian frame, the eq.(8) will have a form

$$x(t) = x_0 e^{-\beta t} \sin(\omega t + \varphi_0),$$  \hfill (12)

where, as well as in the equation (2), $x$ is the length of a vector $\vec{r}_0$.

In fig[1] the qualitative view of the solution (12) is presented. This figure demonstrate that the studied oscillatory motion represents oscillations with amplitude **decreasing in time** by exponential law (i.e. described by the function $e^{-\beta t}$). Just for this reason an oscillatory motion described by the equation (11), named as **the damped oscillatory motion**. Accordingly, eq.(11) named as **the equation of damped oscillations**. So

**the damped oscillatory motion**

of a mass point

is any motion described by

the equation of damped oscillations (i.e. eq.(11)).

Let’s consider more in detail properties of the damped oscillatory motion. First of all it is obvious that in contrast to the free harmonic oscillatory motion the radius vector of the mass point in damped oscillations (i.e. expression (11) or (12)) is not periodic function of time $\vec{r}(t) \neq \vec{r}(t + T)$. Thus **damped oscillations are not harmonic oscillations**.

According to the definition by H. Bohr (Danish mathematician) the function $f(t)$ satisfying the requirement

$$|f(t + T) - f(t)| < \epsilon,$$  \hfill (13)

where $\epsilon$ is some positive number is named an **almost-periodic** function. Accordingly, $T$ is named an **almost-period** such function. And the **mean value** of an almost-periodic function is always limitary

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) dt < \infty.$$  \hfill (14)

It is easy to be convinced that for $x(t)$ from expression (12)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) dt = 0.$$  \hfill (15)

Moreover, it always is possible to select such positive number $\epsilon$ that the absolute value of the difference $|x(t + T) - x(t)|$ (where $T = 2\pi/\omega$) will be less than this number. So the requirement (13) will be satisfied.
Hence radius vector of the mass point making the damped oscillations is an almost-periodic function with almost-period $T$.

Let’s remind that according to (10) the damped oscillation cyclic frequency $\omega$ of mass point is equal to

$$\omega = \sqrt{\omega_0^2 - \beta^2}. \quad (16)$$

Obviously the quantity $\omega$ has the meaning of oscillation frequency only in the case $\omega_0^2 < \beta^2$. At $\omega_0^2 > \beta^2$ the $\omega$ becomes imaginary and, accordingly, the trigonometrical function $\sin(\omega t)$ is transformed to the hyperbolic function $\sinh(\omega t)$. In this case the solution of the damped oscillations equation (11) becomes

$$\vec{r}(t) = \vec{r}_0 e^{-\beta t} \sinh(\omega t + \varphi_0), \quad (17)$$

or in the coordinate notation

$$x(t) = x_0 e^{-\beta t} \sinh(\omega t + \varphi_0), \quad (18)$$

Such a solution is neither a periodic function no an almost-periodic function. And, therefore, the motion described by the equation of damped oscillations at $\omega_0^2 > \beta^2$ is not an oscillatory motion. This process is named as aperiodic oscillations. The diagram of a function $x(t)$ for an aperiodic process (i.e. described by eq.(18) at $\varphi_0 = 0$) is presented on fig.2.

2 The practices for simulation of physical processes

Before simulation initiation of physical processes it is necessary to familiarize with blanket rules of operation with the digital computer and simple set of usual activities used at operation with the Borland software menu.

In this laboratory work there is an opportunity to get the help information on its problem (and another) at any moment not quitting the program.

To obtain the help information it is necessary to press the key $F1$.

The set of practices performed by the student at the study of oscillatory motions are determined by the teacher and can vary over a wide range.
Let’s consider practices the realization of which is necessary for understanding of features of the mass point oscillatory motion at presence (and absence) of resistance forces. According to the object of the laboratory work there should be two such practices.

2.1 Free simple harmonic motions

In this practice state problem to study an oscillatory motion of a mass point at absence of resistance force. Namely:

- to make sure that a trajectory of a mass point is the harmonic function;
- to find out how the mass point trajectory varies by change of the following parameters:
  - particle mass \( m \) and stiffness coefficient \( k \) in expression for elastic force (eq. (4));
  - initial kinematic parameters of a motion: the mass point coordinates of the origin \( x(0) = x_o \sin(\varphi_o) \) and its initial velocity \( v(0) = x_o \omega \cos(\varphi_o) \) (they are those parameters, you can change by changing value of the oscillations epoch angle \( \varphi_o \));

The practice consists of the following items:

2.1.1

After you entered in the menu and selected necessary laboratory work (i.e. ”The oscillatory motion”) the title page of this work arises. Press \( \text{Enter} \) then the main menu with titles of all practices will arise. By keys \( \uparrow \) and \( \downarrow \) it is necessary to select practice ”Simple harmonic Motions” and press \( \text{Enter} \).

2.1.2

You will pass in the first dialog box ”Parameters of the system”. In this box you must set the particle mass and stiffness coefficient in SI units and write down those values to table 1. Let’s note that the ending of input in all dialog boxes is possible by two paths:

- by pressing the key \( \text{Enter} \);
- by activation of the dialog box button \( \text{Ok} \) (with the help of the device \( \text{Mouse} \)).

2.1.3

In the following dialog box (according to its title ”Epoch angle”) you should choose an epoch angle \( \varphi_o \) of a mass point oscillatory motion. Then write down this value to table 1 and press \( \text{Enter} \).

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2In the line of the context-sensitive help (bottom line of the display) range of values is indicated within the bounds of all parameters numerical values, you can change.
2.1.4
You will see the diagram representing a trajectory of a free harmonic oscillatory motion of a mass point with parameters chosen by you.

2.1.5
You will pass at the next dialog box "Change of mass". Enter another value of the particle mass (in comparison with the value entered in item 2.1.3 i.e. in the dialog box "Parameters of the system" ). You will see two diagrams corresponding to different values of the particle mass with unchangeable others parameters. By pressing any key (according to the message in the bottom line) you will return to the same dialog box again. Iterate the described activities for one more value of mass. As a result you will see three diagrams corresponding to three values of the particle mass is in a simple harmonic motions.

2.1.6
You will pass at the next dialog box "Change of K". Enter another value of the stiffness coefficient \( k \) (in comparison with value entered in item 2.1.3 i.e. in the dialog box "Parameters of the system" ). You will see two diagrams corresponding to different values of the stiffness coefficient with unchangeable others parameters. By pressing any key (according to the message in the bottom line) you will return to the same dialog box again. Iterate the described activities for one more stiffness coefficient. In result you will see three diagrams corresponding to three values of the stiffness coefficient of the quasi-elastic force (by due to action of this force the mass point is in a simple harmonic motions).

2.1.7
You will pass at the next dialog box "Change of epoch angle". Enter another value of the epoch angle (in comparison with value entered in item 2.1.3 i.e. in the dialog box "Epoch angle" ). You will see two diagrams corresponding to different values of the epoch angle with unchangeable others parameters. By pressing any key (according to the message in the bottom line) you will return to the same dialog box again. Iterate the described activities for one more value of epoch angle. In result you will see three diagrams corresponding to three values of the epoch angle of the simple harmonic motions.

So the first practice is ended and you will be returned to the main menu. Let’s note that after you exit out of the first practice you can not enter there once again. Therefore, if you do not accept results of this practice and you want to iterate it you should start the program once again.

2.2 Damped oscillations
In this practice state problem to study an oscillatory motion of a mass point with the resistance force is proportional to a vector of velocity. Namely:

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\(^3\)If the obtained diagrams satisfy the object of the practice (in your opinion) then sketch these diagrams in yours writing-book and press any key (according to the message in the bottom line).
- to make sure that the trajectory of a mass point is the non harmonic function, but almost-periodic function;

- to find out how the mass point trajectory varies by change of the following parameters:
  - particle mass $m$, stiffness coefficient $k$ in expression for the elastic force (eq. (1)) and the resistance coefficient $b$ for the resistance force (eq. (3))
  - initial kinematic parameters of a motion: the mass point coordinates of the origin $x(0) = x_o \sin(\varphi_o)$ and its initial velocity $v(0) = x_o \omega \cos(\varphi_o)$ (they are those parameters you can change by changing value of the oscillations epoch angle $\varphi_o$);

The practice consists of the following items:

2.2.1
After you exited out of the first practice you will see the main menu with titles of all practices again. By keys ↑ and ↓ it is necessary to select the practice "Damped oscillations" and press Enter.

2.2.2
You will pass in the first dialog box "Parameters of the system". In this box you must set the particle mass, stiffness coefficient and resistance coefficient in SI units$^2$ and write down those values to table 2.

2.2.3
In the following dialog box (according to its title "Epoch angle") you should choose an epoch angle $\varphi_o$ of mass point oscillatory motion$^2$. Then write down this value to table 2 and press Enter.

2.2.4
You will see the diagram representing a trajectory of a damped oscillatory motion of a mass point with parameters chosen by you$^3$.

2.2.5
You will pass at the next dialog box "Change of the parameters". Here you can change values of two parameters: the particle mass $m$ and resistance coefficient $b$. In contrast to the first practice, you can have this box as much as long. Because after each new diagram (for the next pair of parameters $m$ and $b$) you will be returned here. However, as well as in the first practice, at the display draw no more three diagrams. Therefore we recommend to act as follows:

- at first draw three diagrams with different values of the particle mass $m$ and the constant resistance coefficient $b^3$;
- then draw three diagrams with different values of the resistance coefficient \( b \) and the constant particle mass \( m \); 

- then select the such underload resistance coefficient \( b_{\text{min}} \) (with the constant particle mass \( m \)) for which the particle motion will become aperiodic; write down values \( m \) and \( b_{\text{min}} \) to table 2; 

- iterate operations of the previous item for another two values of the particle mass \( m \); 

- then enter pairwise obtained values of mass \( m \) and coefficient \( b_{\text{min}} \) (from table 2) so to obtain all three aperiodic motions on one picture (i.e. in one frame) and sketch these diagrams in your writing-book;

So the second practice is ended. In order to return to the main menu (if you have the dialog box ”Change of parameters”) is necessary to press the key \( \text{Esc} \) (or make active the dialog box button \( \text{Exit} \) by the device \( \text{Mouse} \)).

3 Return

3.1 Contents of the return

The return should include the following items:

1. Object of work.

2. Summary theoretical part.

3. Practices:

- **Free simple harmonic motions.**

  - **TABLE 1**
    - Diagrams \( x = x(t) \) with different \( m \) for all three trajectories in one frame.
  
  - Diagrams \( x = x(t) \) with different \( k \) for all three trajectories in one frame.
  
  - Diagrams \( x = x(t) \) with different \( \phi_0 \) for all three trajectories in one frame.

- **Damped oscillations.**

  - **TABLE 2**
    - Diagrams \( x = x(t) \) with different \( m \) for all three trajectories in one frame.
  
    - Diagrams \( x = x(t) \) with different \( b \) for all three trajectories in one frame.
  
    - Diagrams \( x = x(t) \) with different \( b_{\text{min}} \) for all three trajectories in one frame.

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\(^4\) **Title page** of the return at the laboratory work on physical processes simulation one should draw up on the same rules that the title page of the return at the laboratory work is done in chair T&EPh experimental laboratories.
4. Conclusion

3.2 Design of the tables.

The tables used in the return should be designed by following ways.

| Change | Change $k$ | Change $\phi_o$ |
|--------|------------|-----------------|
| $m,$   |            |                 |
| $k,$   |            |                 |
| $\phi_o,$ |        |                 |

| Change | Change $b$ | Change $b_{min}$ |
|--------|------------|------------------|
| $m,$   |            |                  |
| $b,$   |            |                  |
| $b_{min}$ |        |                  |
| $\phi_o,$ |        |                  |

4 Conclusion

Let’s mark, that this paper is written on the basis of the previous works carried out on chair T&EPPh under the author leadership (or direct participation).

References

[1] Korotchenko K.B., Sivov U.A.

[2]