Strong Nonlinear Coupling due to Induced Photon Interaction on a Si$_3$N$_4$ Chip

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Second-order optical processes lead to a host of applications in classical and quantum optics. With the enhancement of parametric interactions that arise due to light confinement, on-chip implementations promise very-large-scale photonic integration. But as yet there is no route to a device that acts at the single photon level. Here we exploit the $\chi^{(3)}$ nonlinear response of a Si$_3$N$_4$ microring resonator to induce a large effective $\chi^{(2)}$. Effective second-order upconversion (ESUP) of a seed to an idler can be achieved with 74,000 %/W efficiency, indicating that single photon nonlinearity is within reach of current technology. Moreover, we show a nonlinear coupling rate of seed and idler larger than the energy dissipation rate in the resonator, indicating a strong coupling regime. Consequently we observe a Rabi-like splitting, for which we provide a detailed theoretical description. This yields new insight into the dynamics of ultrastrong effective nonlinear interactions in microresonators, and access to novel phenomena and applications in classical and quantum nonlinear optics.

Current strategies for implementing nonlinearity at the single photon level [1-3] have to make use of very complex experimental arrangements, such as high-Q photonic crystal cavities and quantum dots, which are difficult to implement using conventional photonic technologies. Nonetheless they are being actively pursued, in part because of the common assumption that for purely parametric nonlinear optical processes – that is, those that do not involve the excitation of a resonance in the response of a material medium – the intrinsic material nonlinearities would be far too weak to allow for efficient nonlinear interactions between single photons.

This assumption was challenged by Langford et al. [4], who argued that the use of a strong pump for one of the fields of a four-wave mixing (FWM) process in a $\chi^{(3)}$-nonlinear medium would lead to an effective $\chi^{(2)}$ nonlinearity for the remaining three fields. Not directly limited by the intrinsic material properties, this effective $\chi^{(2)}$ interaction can in principle be made arbitrarily large by increasing the pump power. While theoretically this would allow for an induced photon interaction efficient at the single-photon level [4], to date such an effective $\chi^{(2)}$ has experimentally only been demonstrated at relatively low efficiencies using a photonic crystal fiber [4], a chalcogenide nanofiber [5] and a resonant $\chi^{(3)}$-nonlinearity in warm Rb-vapor [6]. Yet such an approach is also compatible with material platforms and light confinement strategies characterized by strong field enhancements and exceptionally large Q factors, which offer the promise of strong effective second-order photon interactions. We shall see below that this promise can be fulfilled.

In any such scenario, the nonlinear processes that can result are richer than those occurring in materials with a natural $\chi^{(2)}$, but are analogous to them. In particular, it is possible to produce an effective second-order upconversion (ESUP) process (see Fig. 1), in which two photons are converted to one photon at a higher photon energy, but with much less than double the original photon energy that would characterize the analogous process of second harmonic generation (SHG). To measure the strength of ESUP we employ the standard figure-of-merit used for quantifying SHG, a normalized efficiency quoted in %/W. While ultra-high Q, mm-sized resonators made from lithium niobate (LN) can reach 300,000 %/W [7, 8], current state-of-the-art values for integrated devices are on the order of few thousand %/mW, such as the 5400 %/W achieved in LN waveguides [9], and the recently reported 2,500 %/W in AIN micro-ring resonators [10].

Here we utilize a high-Q Si$_3$N$_4$ microresonator to generate a strong effective $\chi^{(2)}$. With a pump field at 1590 nm, we induce ESUP of two seed photons at 1400 nm to an idler photon at 1260 nm. All photon energies are at microring resonances, strongly enhancing the conversion. Changing the seed power at 1400 nm over 5 orders of magnitude, and measuring the resulting idler power, we verify the quadratic scaling expected of a second-order process, and measure a normalized efficiency of 74,000%/W. Intriguingly, at higher seed powers of only a few hundreds of microwatts at 1400 nm, where our total conversion efficiency is saturating at around 5%, the nonlinear coupling rate in the resonator is comparable with resonator damping rate. This fulfills a strong coupling condition between the nonlinearly interacting modes, and consequently there occurs a Rabi-like splitting of the modes, which we observe and study both ex-
The use of a high-Q resonator gives rise to dramatic enhancements of both the circulating pump power available and the nonlinear interaction between the three fields. The Si$_3$N$_4$ material was chosen to avoid two-photon absorption, which typically plagues silicon systems. This enables very large unit power ESUP conversion efficiencies, $\eta_{ESUP}$, to be achieved. This figure of merit can be theoretically calculated as

$$\eta_{ESUP} = \frac{64R^2\gamma_{NL}^2(\Delta\lambda_{FSR})^4}{\pi^4\lambda^4}P_{in}^4 Q^4 \left(\frac{Q}{Q_{ext}}\right)^4$$

in steady state, where $\Delta\lambda_{FSR}$ is the wavelength free spectral range, $P_{in}$ is the pump input power in the bus waveguide, $Q$ is the loaded quality factor and $Q_{ext}$ the extrinsic quality factor due to the coupling of the resonator to the bus waveguide. Corrections to this simplified formula, including the fact that the quality factors and wavelengths $\lambda$ are different at different frequencies, are described in the Methods. Importantly, the scaling of $\eta_{ESUP}$ with $Q^4$ is one power larger than its $\chi^{(2)}$ analogue - the normalized SHG efficiency, which in a resonator only scales with $Q^2$.

We set out to investigate the ESUP experimentally. The chip-integrated microring resonators utilized in this work were made from a high quality Si$_3$N$_4$ film on an SiO$_2$ substrate, grown via low-pressure chemical vapor deposition and electron beam lithography (see [15] for details). The waveguide dimensions (Fig. 2b) of 730 nm x 910 nm (height x width) are carefully chosen: With a zero group velocity dispersion (GVD) at 1410 nm, the waveguide exhibits phase-matching between a strong pump at 1590 nm, the seed field at 1400 nm, and the idler field at 1260 nm. Moreover, the large normal GVD at $\omega_P$ strongly inhibits any undesired parametric process (e.g., spontaneous FWM) from the strong pump. The microring resonator (radius 45 $\mu$m) with a free spectral range of 500 GHz is coupled to the bus waveguide with a gap of 450 nm, and exhibits loaded Q-factors of 250,000 for the pump (over-coupled), 1.2 million for the seed (close to critical coupling), and about 3.5 million for the idler (under-coupled). Using these parameters we theoretically expect a unit power ESUP efficiency of 87,000 ± 17,000 %/W, where the error interval is estimated from the uncertainty on the experimental parameters (e.g., coupled power, quality factors, and propagation losses). This corresponds to a $\chi^{(2)}_{eff} \approx 1$ pm/V (see Methods section), which is comparable in magnitude to the second-order nonlinear susceptibility used in LN WGM-resonators. And although this is two orders of magnitude smaller than the nonlinear susceptibility of semiconductors such as GaAs or GaN, the upconversion efficiency here is among the highest ever reported.

Our measurement scheme is shown in Fig. 2b). Pump and seedlight are generated by tunable external cavity lasers. The pump is amplified with a high power EDFA to mW and combined with the seed with wavelength division multiplexers (WDMs). They are injected into the chip (and collected) via lensed fibers with a total insertion loss of 8.5 dB, resulting in 85 mW of on-chip pump power at 1590 nm. The output is separated.
into the three wavelength bands by cascaded WDMs to be monitored separately on InGaAs photodiodes. The idler output at 1260 nm is further strongly filtered by a low-pass filter yielding total losses of 6.3 dB, including the WDMs. When measuring very low intensities, we optionally filter the idler with a grating monochromator (0.55 nm filter bandwidth), and then detect with superconducting single-photon detectors. We enclose the chip and stabilize the temperature to reduce thermal and mechanical effects.

For our measurements of the idler output power at 1260 nm as a function of the 1400 nm seed wavelength, first the 1590 pump wavelength is tuned into resonance from the short wavelength side to follow the thermal red-shift and achieve thermal locking [16]. We then measure the power scaling characteristics of the ESUP process. To verify the quadratic scaling of ESUP process with respect to seed power, we vary it over almost five orders of magnitude and we measure the generated idler power levels; they vary over eight orders of magnitude. Starting from 350 µW seed power in the bus waveguide and successively reducing it to nanowatt level, the from the quadratically scaling idler powers inferred effective conversion efficiencies (linear scaling) are plotted in Fig. 3. From this we infer an on-chip normalized \( \eta_{\text{ESUP}} = 74,000 \%/W \), in excellent agreement with the theoretically predicted value.

We turn now to seed powers larger than a few tens of µW. Given the lack of two-photon absorption, the saturation of the generated idler observed in the main panel of Figure 3 suggests that at such seed powers the nonlinear coupling between the modes can no longer be treated perturbatively. To further investigate this we scan the seed laser across the resonance wavelengths while the idler power is monitored. Typical scans are shown in Fig. 4 for seed power \( P_{\text{seed}} = 50 \) nW (left) and \( P_{\text{seed}} = 300 \) µW (right). We see a gradual broadening of the seed peak as the seed power is increased, until two dips can be clearly resolved. This is in excellent agreement with the theoretical curves calculated starting from Eqs. (1)-(2) (see Methods section) and using the nominal parameters of the system without the need for any fitting.

The splitting indicates a regime of ESUP in which the nonlinear coupling rate \( \Lambda_3 |\beta_p \beta_S| \) between the modes is comparable to the energy loss rate of the resonator, thus resulting in strong nonlinear coupling. Consequently a Rabi-like splitting of the modes occurs, which is analogous to that theoretically investigated by Carusotto and La Rocca [17], who focused on the second-order nonlinear interaction between two fields oscillating at \( \omega \) and \( 2\omega \) in a doubly resonant system. They predict a Rabi-like resonance splitting at the fundamental frequency when the nonlinear coupling rate exceeds the resonator damping rates.
In the ring resonator, the steady state solutions for the mode amplitudes predict the onset of this saturation as well as the presence of the Rabi-like splitting of the resonance at $\omega_S$ (see Methods section). In particular, one can observe the splitting when the “resolvability” $R$, i.e., the ratio of the splitting to the line width, is about unity. This quantity depends on the ring parameters as well as the pump and seed powers according to

$$R = \frac{\Lambda \beta_S \beta_P}{\Gamma_S} \approx \frac{8 \Lambda}{\omega_S} \sqrt{\frac{P_P P_S}{\hbar \omega_S^2 \hbar \omega_P^2}} \frac{Q_P}{\sqrt{Q_{P,ext}^2}} \frac{Q_S^2}{\sqrt{Q_{S,ext}^2}} \quad (4)$$

where $P_{P,S}$ are the pump and seed input powers and $Q_{P,S}^{ext}$ are the quality factor of the resonators determined solely by the external coupling. The last term of the equation is obtained by assuming a linear solution for the pump and the seed powers and is strictly valid for $R < 1$, and yet it allows one to have a rough estimate of when the strong nonlinear coupling regime can be achieved.

In Figure 5, we plot the seed transmission as a function of detuning for several seed powers. It should be noticed that unlike recent studies in which Rabi oscillations have been predicted [18] and Rabi splittings have been experimentally observed [19] for frequency conversion in the strong linear-coupling regime, here the strength of the coupling between the fields at $\omega_S$ and $\omega_I$ depends on the intensity of the seed field itself, and in particular is linear in $\beta_S$. Finally, we stress that this condition has been achieved because our effective second-order nonlinear interaction is controlled by the pump intensity, and occurs in a structure characterized by an exceptionally large field enhancement and Q-factor. It is the combination of these two elements that has allowed the first observation of a nonlinear strong-coupling.

The strength of this second-order nonlinear response raises the question of whether the field associated with just a single pair of photons at $\omega_S$ would be sufficient for a complete pair conversion to a photon at $\omega_I$ in our system. In such a regime, new phenomena are expected given the energy quantization of the optical field [20]. So far, much of the research has been focused on atomic-like systems, either quantum dots or with atomic clouds, while non-resonant photon-photon interactions in bulk nonlinear materials have been far from approaching unit efficiency. For instance, SHG driven by photon pairs has been demonstrated in LN waveguides, but only yielding a very low probability $p_{2 \rightarrow 1}$ of a two-photon conversion [21]. To further increase $p_{2 \rightarrow 1}$ one can take advantage of light confinement in micro resonators, either directly exploiting a natural second-order material nonlinearity or a strong effective second-order response, the latter of which we have done in this work. In Fig. 6 we summarize several nonlinear resonator systems and show $p_{2 \rightarrow 1}$ as a function of the resonator quality factor $Q$, heuristically defined as $p_{2 \rightarrow 1} = \eta \frac{2E_{sp}}{\tau_{cavity}} = 4 \eta \frac{\hbar \omega_P^2}{Q} \approx Q$ (where $2E_{sp}$ is the energy of two photons to be converted, $\tau_{cavity}$ the cavity dwelling time, and $\eta$ is the efficiency of the corresponding parametric process). We stress that one would expect quantization effects to appear when $p_{2 \rightarrow 1}$ approaches a significant fraction of unity. Currently, our system is characterized by $p_{2 \rightarrow 1} = 4.7 \times 10^{-7}$ (star in Fig. 6), which is the largest value ever reported. This is despite the fact that our device has not yet been optimized towards optimal coupling for all wavelength, with strong over- and undercoupling for the pump and idler resonances, respectively. Achieving critical coupling for all wavelength at the same intrinsic $Q$ and moderately increasing the pump power to 500mW would directly yield an enhancement of $p_{2 \rightarrow 1}$ by 2 orders of magnitude. Moreover, because $\eta$ scales with the 4th power of $Q$ (see equation 5) it follows, that $p_{2 \rightarrow 1}$ scales with $Q^3$. Thus a 30 times higher $Q$ would result heuristically already in $p_{2 \rightarrow 1} \approx 1$.

This is challenging, for it requires bus-ring coupling optimization at different wavelengths and the control and compensation of nonlinear phase matching at higher pump powers, but these have individually been shown already on the Si$_3$N$_4$ platform, and even more encouragingly $Q$ factors close to 70 million have been recently obtained [22]. Based on these simplified arguments we thus conclude that approaching the $p_{2 \rightarrow 1} \approx 1$ regime is within reach of current technology.

In conclusion, we demonstrated induced photon interactions based on effective second-order upconversion with a normalized efficiency of 74,000 %/W, and experimentally reached the regime of strong nonlinear coupling for on-chip powers as low as few microwatts, in excellent agreement with our theoretical model. The Si$_3$N$_4$ platform is a mature and highly promising technology for obtaining two-photon interaction at the single-pair level.
Figure 6. Comparison of two-photon conversion probability $p_{2 \rightarrow 1}$ for different resonators and material platforms: (green squares) whispering gallery mode resonators (radius $> 1$ mm) $[7, 8, 23, 26]$, (blue circles) microdisk resonators (radius < 100 μm) $[10, 25, 27–31]$, (star) this work. The probability is defined heuristically as $p_{2 \rightarrow 1} = \eta \tau_{\text{cavity}} / Q$ (where $2E_{\text{sp}}$ is the energy of two photons to be converted and $\tau_{\text{cavity}}$ is the cavity dwelling time, and $\eta$ is the efficiency of the corresponding parametric process)): indeed, for $p_{2 \rightarrow 1}$ approaching unity, quantization effects are expected to appear. Dashed lines show the expected $p_{2 \rightarrow 1}$ as a function of Q and on-chip pump power, (blue) for a $\chi^{(2)}$ (orange) for ESUP. Importantly for $\chi^{(2)}$ devices $p_{2 \rightarrow 1}$ only scales quadratically with Q, whereas using ESUP the scaling with Q is cubic. Moreover, the red dotted line indicates the trend for a 100x improvement possible without increasing Q. Together with a significant improvement in Q up to 10s of Million as motivated by $[22]$, the extrapolation heuristically illustrates that approaching the $p_{2 \rightarrow 1} \simeq 1$ regime is within reach of current Si$_3$N$_4$ technology.

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METHODS

To understand the dynamics of the nonlinear induced photon interaction and ESUP process, we use a quantum mechanical Hamiltonian formalism that fully captures the behavior of the system both in the few-photon limit, and in the classical regime of strong coherent beams. A detailed development of the formalism we employ can be found elsewhere in the literature [22]; here we present a summary.

For the pump, signal, and idler, we identify intra-resonator annihilation operators $b_J$ with $J \in \{P, S, I\}$ for the three resonances that participate in the ESUP process. The Hamiltonian $H_{\text{ring}}$ for the isolated resonant modes in the ring is then given by

$$H_{\text{ring}} = \sum_J \hbar \omega_J b_J^\dagger b_J - \hbar \Lambda_3 \left(b_P^\dagger b_I^\dagger b_S b_S + \text{H.c.}\right),$$

in which we have retained only those nonlinear terms that participate in the ESUP process. In particular, this
Hamiltonian does not include terms that represent self- and cross-phase modulation between the three modes. However, for slowly-varying field envelope functions, the effect of these processes is well-described by an effective shift in the resonant frequencies $\omega_j$.

Since in this work we focus on CW input beams, this approximation is well justified, and we assume the frequencies $\omega_j$ in already contain these corrections. The nonlinear coupling constant $\Lambda_3$ is related to the third-order nonlinearity of the waveguide material, and is estimated by

$$\Lambda_3 \approx \frac{\hbar (\omega_p \omega_S^2 \omega_j^{1/2} v_g^2)\gamma_{NL}}{2 \pi R},$$

(6)

where $v_g$ is the group velocity in the ring waveguide, $R$ the ring radius, and $\gamma_{NL}$ the nonlinear parameter; for the ring used in these experiments we find $\Lambda_3 \approx 10 \text{ s}^{-1}$.

For each ring mode operator $b_j$, we also introduce a corresponding input and output channel field operator $\psi_j(t)$ and $\psi_{j>}(t)$ at the ring-channel coupling point, normalized such that $\psi_j(\psi^\dagger_j(t) \psi_j(t))$ is the mean incoming (outgoing) photon flux in the side channel at frequencies near $\omega_j$, with $v_j$ the corresponding group velocity. The time dependencies of operators are understood in the usual Heisenberg picture context. The coupling of these fields to the ring can be described using standard cavity input-output theory. Defining ring-channel coupling coefficients $\gamma_j$ to each channel field/ring mode pair, the input-output relation for these field operators is given by

$$\psi_{j>}(t) = \psi_j(t) - \frac{i\gamma_j}{v_j} b_{j>}(t),$$

(7)

with $b_{j>}(t)$ the time-dependent ring mode operator in the Heisenberg picture.

The equations of motion in the classical limit for the ring mode amplitudes can be found by calculating the full Heisenberg equations of motion and replacing each operator by its expectation value. In what follows we remove the fast optical time dependence and consider slowly-varying quantities only. This gives

$$\frac{d\beta_S}{dt} = -\Gamma_S \beta_S - i\gamma_s^\ast a_S e^{-i\Delta_r},$$

$$\frac{d\beta_J}{dt} = -\Gamma_J \beta_J + i\Lambda_2(t) \beta_S^\ast e^{i\Delta_r},$$

(8)

for the seed and idler modes, where $\beta_J$ is the classical ring mode amplitude for mode $J$, and $\Gamma_J$ is the full damping rate for mode $J$ (including both coupling to the channel and to scattering from the resonator), related to the full loaded quality factor $Q_J$ for that mode via $\Gamma_J = \omega_J/(2Q_J)$. Here we have assumed that the seed is driven by a coherent beam with amplitude $a_S e^{-i\Delta_r}$ in the channel, corresponding to a seed input power $P_S = \hbar \omega_S^2 |a_S|^2$ and detuning $\Delta_S$ from resonance. The detuning $\Delta_{res} = 2\omega_S - \omega_P - \omega_P$ accounts for any deviation of the resonances from perfectly equal spacing in frequency. The effective second-order coupling coefficient is defined as $\Lambda_2(t) = \Lambda_3 \beta_P(t)$, with $\beta_P(t)$ the pump amplitude in the resonator.

In this work we assume a CW pump, so that the magnitude of $\Lambda_2$ is constant; The squared pump magnitude $|\beta_P|^2$ (corresponding to the mean pump photon number in the resonator) in this regime is given by

$$|\beta_P|^2 = \frac{4 P_P Q_{ext}^2}{\hbar^2 \omega_P^2 Q_{res}^2},$$

(9)

where $P_P$ is the input pump power in the channel, and $Q_{ext}^2$ is the extrinsic quality factor of the pump associated with the channel-ring coupling only, given by $Q_{ext}^2 = \omega_P v_P/|\gamma_P|^2$. This expression assumes that the pump is sufficiently strong to render any back-action or depletion effects on the pump mode negligible. The degradation factor $u_P = (1 + 4P_P^2)^{-1}$ ranges from 0 to 1, with $l_P$ the pump detuning from resonance in units of the pump resonance full-width-half-max linewidth, $l_P = \Delta_P/(2\Gamma_P)$ with $\Delta_P$ the pump detuning from resonance. We define the steady amplitude $\Lambda_2$ via $\Lambda_2(t) = \Lambda_2 e^{-i\Delta_P} t$.

We seek a self-consistent stationary solution of Eqs. (12). Thus we search for a steady state solution of the form $\beta_S(t) = \beta_S e^{-i\Delta_S t}$, $\beta_J(t) = \beta_J e^{-i\Delta_J t}$, with the over-bar denoting constant quantities. By substitution it directly follows that this is only possible when the idler field oscillates with detuning $\Delta_J = \Delta_{res} - \Delta_P + 2\Delta_S$; for what remains we estimate $\Delta_{res} = \Delta_P = 0$ so $\Delta_J = 2\Delta_S$. The constant idler amplitude can then be written in terms of the seed amplitude as

$$\beta_J = \frac{i\Lambda_2^\ast \beta_S}{-i\Delta_J + \Gamma_J}. $$

(10)

Inserting this back into the equation for the the seed, we obtain

$$\left( -i\Delta_S + \Gamma_S + \frac{2|\beta_S|^2}{-i\Delta_J + \Gamma_J} \right) \beta_S = -i\gamma_S a_S. $$

(11)

To solve for $\beta_S$ we first need a value for $|\beta_S|^2$. This can be obtained by taking the modulus squared of (11), which yields an implicit cubic equation for $|\beta_S|^2 = N_S$:

$$a_3 N_S^3 + a_2 N_S^2 + a_1 N_S + a_0 = 0, $$

(12)

where $a_3 = 4|\beta_S|^4$, $a_2 = 4|\beta_S|^2 (\Gamma_S \gamma_J - \Delta_S \Delta_J)$, $a_1 = (\Gamma_S \gamma_J - \Delta_S \Delta_J)^2 + (\Delta_J \Gamma_S + \Delta_S \Gamma_J)^2$, and $a_0 = (\Gamma_J + \Delta_J^2) P_S/(\hbar Q_{ext}^2)$. Note the explicit dependence of $a_0$ on the extrinsic quality factor for the seed resonance, making the coupling condition (under, over, or critically coupled) of the seed a relevant parameter that influences the conversion dynamics; indeed, the predictions of our theory were found to depend quite sensitively on this coupling condition.

For a given set of ring parameters a positive real root $N_S$ of Eq. (12) can be found for each seed power and detuning, representing a steady intra-resonator solution for the seed and idler amplitudes. The seed amplitude can be calculated from (11), and can be substituted in (10) to
find the idler amplitude. The input-output relations can then be used to derive the output powers of the seed and the idler into the channel. The transmission ratio of the seed is given by

\[ T = \left| \frac{1 - 2\Gamma_S}{(-i\Delta_S + \Gamma_S)(-i\Delta_I + \Gamma_I) + 2|\Lambda_2|^2 N_S} \right|^2, \]  

and the corresponding generated idler power output to the channel can be calculated from \( P_{\text{gen}} = 2\hbar \omega_I \Gamma_I |\beta_I|^2 \). In this expression and \[ 13 \] the quantities \( \Gamma_S \) and \( \Gamma_I \) are the channel-ring coupling rates for the seed and idler resonances, related to the corresponding extrinsic quality factors via \( \Gamma_j = \omega_j/(2Q_{j\text{ext}}) \).

Viewed as a function of \( \Delta_S \), the transmission ratio \( T \) represents a seed transmission spectrum; this was used to produce the theoretical curves in Fig. 4 in the main body. The corresponding generated power in the idler at each point in these curves can be calculated, forming the theoretical spectral curves for the generated idler in Fig. 4 as well as the power scaling in Fig. 5.

In the limit of a weak seed input we can ignore the back-action on the seed amplitude from its coupling to the idler, which enables a simple analytic formula to be derived for the generated idler power. Dropping the nonlinear term in the first of \[ (1) \] and solving for \( |\beta_I|^2 \), we find (after using the input-output relation \[ 7 \]) for the generated idler power output in the channel

\[ P_{\text{gen}} = \left( \frac{256\Lambda_2^2}{\hbar^2 \omega_S^4 \omega_P^2} \left( P_S^2 P_P(Q_P Q_I Q_S^2) \right) \frac{Q_P}{Q_{\text{ext}}^P} \frac{Q_I}{Q_{\text{ext}}^I} \left( \frac{Q_S}{Q_{\text{ext}}^S} \right)^2 \right) u_P u_{\text{net}}, \]

where \( u_{\text{net}} = (1 + 4|\beta_{\text{net}}|^4)^{-1} \) is the degradation factor between 0 and 1 associated with the net detuning \( l_{\text{net}} \) (in units of idler resonance FWHM linewidth) associated with both the pump and \( \Delta_{\text{res}} \). For our experiment both \( u_P \) and \( u_{\text{net}} \) were estimated to be approximately unity, corresponding to a perfectly resonant system. This expression for the generated power demonstrates the quadratic scaling of the generated idler power with the seed input, and can be used to predict the unit power ESUP conversion efficiency \( \eta_{\text{ESUP}} \) as defined in the main body (eq. 3).

\[ \eta_{\text{ESUP}} = \frac{P_{\text{gen}}}{P_S^2}. \]

Our formula takes into account different frequencies, coupling ratios and quality factors for each resonance; the rough estimate of \( \eta_{\text{ESUP}} \) in the main body is found by taking these all to be roughly equal across the different resonances.

A few subtleties should be noted: first, the steady state solutions described in this section for our experimental parameters are stable, as verified by a first-order fluctuation analysis. However, unstable regimes can be reached with realistic physical parameters, the presence of which were found to sensitively depend on the seed coupling ratio. Second, in order to obtain good agreement with the experimentally measured values, it was necessary to include the effects of back-scattering of the seed resonance in the resonator; this effect was strong enough to be directly measured in our experiments. In particular, the presence of back-scattering changes the procedure by which the coupling ratios are extracted from measured linear transmission spectra of the microring resonator. We modeled this phenomenologically with the inclusion of a term \( H_{\text{BS}} = \hbar q(b_S b_S^\dagger + \text{H.c.}) \) in the ring Hamiltonian, with \( b_S^\dagger \) the creation operator for the counter-propagating mode at the seed frequency in the ring that couples to the channel in the usual way, and where \( q \) is a backscattering strength parameter.

Both of these effects highlight the care which must be taken when carrying out a mathematical analysis of nonlinear optics in high-Q resonators, especially when considering strongly driven phenomena like ESUP.

Finally, to calculate the effective equivalent second-order nonlinearity \( \chi_{\text{eff}}^{(2)} \) for our system, we study the nonlinear part of the ring Hamiltonian that would describe a true degenerate SHG/SPDC interaction,

\[ H_{\text{SHG}} = -\hbar \Lambda_{\text{SHG}} \left( b_P b_P^\dagger b_S^\dagger + \text{H.c.} \right), \]

where \( F \) describes the fundamental and \( \text{SH} \) the second harmonic mode. The constant \( \Lambda_{\text{SHG}} \) describes the strength of the second-order nonlinearity for this interaction, and is given by

\[ \Lambda_{\text{SHG}} = \frac{\hbar^{1/2} \omega_F^{3/2} \chi^{(2)}}{2n^3 \epsilon_0^{1/2} (L \Lambda)^{1/2}}, \]

where \( \omega_F \) is the frequency of the fundamental, \( \chi^{(2)} \) the relevant component of the second-order nonlinear ten-
sor, $n$ the material refractive index (assumed not to vary between the fundamental and second harmonic frequencies), $L$ the ring length, and $A$ the effective area.

The Hamiltonian (14) is formally identical to the Hamiltonian (15) that describes the ESUP process in the case of a CW pump. Thus we can identify

$$|\Lambda_3\beta| = \Lambda_{\text{SHG}}.$$ 

By calculating the pump amplitude $|\beta|$, and the coupling strength $|\Lambda_3|$ in our experiment, using the the expression (15) for $\Lambda_{\text{SHG}}$, we can extract the magnitude of $\chi^{(2)}$ that would be required in this system to produce a comparable unit power conversion efficiency; this is precisely the effective equivalent second-order nonlinearity $\chi^{(2)}_{\text{eff}}$. We find for a pump power of 85 mW, using $\Lambda_3 = 10 \text{ s}^{-1}$, that

$$\chi^{(2)}_{\text{eff}} \approx 0.6 \text{ pm/V}.$$