Probabilistic Total Store Ordering

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Abstract. We present Probabilistic Total Store Ordering (PTSO) – a probabilistic extension of the classical TSO semantics. For a given (finite-state) program, the operational semantics of PTSO induces an infinite-state Markov chain. We resolve the inherent non-determinism due to process schedulings and memory updates according to given probability distributions. We provide a comprehensive set of results showing the decidability of several properties for PTSO, namely (i) Almost-Sure (Repeated) Reachability: whether a run, starting from a given initial configuration, almost surely visits (resp. almost surely repeatedly visits) a given set of target configurations. (ii) Almost-Never (Repeated) Reachability: whether a run from the initial configuration, almost never visits (resp. almost never repeatedly visits) the target. (iii) Approximate Quantitative (Repeated) Reachability: to approximate, up to an arbitrary degree of precision, the measure of runs that start from the initial configuration and (repeatedly) visit the target. (iv) Expected Average Cost: to approximate, up to an arbitrary degree of precision, the expected average cost of a run from the initial configuration to the target. We derive our results through a nontrivial combination of results from the classical theory of (infinite-state) Markov chains, the theories of decisive and eager Markov chains, specific techniques from combinatorics, as well as, decidability and complexity results for the classical (non-probabilistic) TSO semantics. As far as we know, this is the first work that considers probabilistic verification of programs running on weak memory models.

1 Introduction

The classical Sequential Consistency (SC) semantics [1] has been a fundamental assumption in concurrent programming. SC guarantees that process operations are atomic. A write operation, performed by a given process, is immediately visible to all the other processes. However, designers of modern computer systems, in their quest of increased system efficiency, often sacrifice the SC guarantee. Instead, the processes communicate asynchronously, allowing a delay in the propagation of write operations. Due to the propagation delay, written values can become available to processes at different time points, and in an order that may be different from the order in which they are generated. This asynchronous behavior gives rise to new semantics, collectively referred to as weak memory
models [2]. In the presence of weak memory models, programs exhibit new, and often unexpected, behaviors, bringing about complex challenges in the design and analysis of concurrent systems. Even text-book programs may behave erroneously. The classical Dekker mutual exclusion protocol is a case in point. The ubiquity of weak memory models has led to an extensive research effort for the testing and verification of concurrent programs running under such semantics.

Existing works on the verification of programs running on weak memory models, consider safety properties such as state reachability, assertion violation, and robustness. While safety properties are fundamental, we need also to prove liveness properties, i.e., to show that the program indeed makes progress. This is, of course, true already in the case of SC. A program, such as a mutual exclusion protocol, needs to guarantee that each process will eventually reach its critical section. The satisfiability of liveness properties is often dependent on the type of fairness conditions on process executions that are provided by the underlying platform [3,4]. The reason is the presence of concurrency non-determinism, i.e., the inherent non-determinism in program behavior due to the different possible ways in which the scheduler can interleave the processes. The scheduler may always neglect a given process, which means that the process will never make progress (e.g., never reaches its critical section). Therefore, we need the scheduler to follow a fair selection policy that allows each process to advance in its execution. The situation is even more complicated in the case of weak memory models, since we also need to deal with a second source of non-determinism, besides concurrency non-determinism, namely (data) propagation non-determinism. Since write operations are propagated asynchronously, there is in general no way to predict if, when, and in which order, write operations become visible to the processes.

In this paper we present a framework for the verification of liveness properties for concurrent programs running under the classical Total Store Ordering (TSO) semantics [5]. The TSO model puts an unbounded store (write) buffer between each process and the main memory. The buffer carries pending write operations that have been performed by the process. These operations are propagated from the buffer to the shared memory in a FIFO manner. When a process performs a write operation, it appends the operation as a message to its buffer. When a process reads a variable, it searches its buffer for a pending write operation on that variable. If such operations exist then it reads from the most recent one. If no such operation exists, it fetches the value of the variable from the main memory. The TSO propagation mechanism is a typical example of how propagation non-determinism arises: the write operations are propagated to the shared memory non-deterministically, and a process sees the other processes’ write operations only when the latter are available in the memory. Therefore, having a scheduler that fairly selects the processes is not sufficient. We also need to ensure that the write operations propagate to the processes sufficiently often.

Traditional fairness conditions such as strong or weak fairness [3,4,6] cannot capture propagation policies adequately since they irrationally allow slow propagation, i.e., they allow write operations to propagate at a lower rate than
the rate by which they are issued. For instance, strong fairness guarantees that messages are transferred infinitely often from the buffers to the memory. Still, it does not constrain the relative frequency of write and update operations, and hence it does not prevent the buffer contents from growing unboundedly. In such a scenario, more and more un-propagated messages may be clustered inside the buffers, and a given process may, from some point on, be confined only to read its own writes, since it will not see the memory updates by the other processes. Accordingly, verifying liveness properties subject to strong fairness may wrongly deem the system to be incorrect: even if a process is selected infinitely often by the scheduler and write operations are propagated infinitely often to the memory, a given process may incorrectly be judged not to make progress due to slow propagation.

While slow propagation can arise theoretically under the above mentioned fairness conditions, it is almost never observed in practice. Existing platforms implement different policies, such as invalidation or write-back policies, to flush the buffers at regular intervals \[7, 8\]. This prevents the buffer sizes from growing beyond certain sizes, and implicitly ensure propagation fairness. In fact, this is true to the degree that non-SC behaviors are (relatively) rarely observed on TSO platforms \[9, 10\].

In this paper, we perform verification of liveness properties for concurrent programs under TSO using probabilistic fairness \[11\]. As far as we know, this is the first work that considers probabilistic verification of programs running on weak memory models. In our model, both process scheduling and message propagation are carried out according to given probability distributions. We assign a weight (a natural number) to each process. We resolve concurrency non-determinism probabilistically by letting the scheduler select the next process to execute with a probability that reflects the weight of the process compared to the weights of the other processes that are enabled in the same configuration. After each process step, we allow an update step, in which the buffers transfer parts of their contents to the memory. We make the probability distribution equal among all possible update operations in the given configuration\[4\]. As we will see later in the paper, defining the model in this way implies that we assign low probabilities to program runs that unboundedly increase the number of messages inside the buffers. Accordingly, our model is more faithful to real program behavior compared to models induced by non-probabilistic fairness conditions.

We perform a comprehensive analysis of the decidability of verifying liveness properties for concurrent programs running under the TSO semantics, subject to probabilistic fairness. In fact, verifying programs running on the TSO memory model, even with respect to safety properties, poses a difficult challenge. The unboundedness of the buffers implies that the state space of the system is infinite, even in the case where the input program is finite-state \[12, 13\]. Similarly, the operational semantics of our model gives rise to Markov chains with infinite state spaces. Furthermore, in general, liveness properties give rise to more difficult problems than safety properties, since the former are interpreted over

\[4\] Our framework allows several other types of probability distributions (see Sec. \[9\]).
infinite program executions while the latter are interpreted over finite executions. Our results rely on nontrivial combinations of results from the classical theory of (infinite-state) Markov chains \[14\] \[15\], the theories of decisive and eager Markov chains \[16\] \[17\], specific techniques from combinatorics \[18\], as well as, decidability and complexity results for the classical (non-probabilistic) TSO semantics \[19\] \[13\]. Concretely, we show the decidability of the following problems, each of which is defined by giving an initial configuration $\gamma_{\text{init}}$ and a set $\text{Target}$ of process target states.

**Qualitative Analysis** (Sec. 6). In qualitative reasoning, we are interested in knowing whether the given property is satisfied with probability 1 (almost surely satisfied), or with probability 0 (almost never satisfied). We show that the satisfiability of these properties can be reduced to similar problems on the underlying (non-probabilistic) transition systems for classical TSO. The actual probabilities appearing in the induced Markov chains then are inconsequential and only their non-zeroness matters. This is useful whenever the probabilities have not been measured exactly, or the portion of the system giving rise to probabilistic behavior has not been designed yet. We consider the following different flavors of qualitative analysis: Almost-Sure (Repeated) Reachability, whether a run of the system from $\gamma_{\text{init}}$ will almost surely visit (resp. repeatedly visit) $\text{Target}$; Almost-Never (Repeated) Reachability: whether a run of the system from $\gamma_{\text{init}}$ will almost never visit (resp. repeatedly visit) $\text{Target}$. Furthermore, we show that all these problems have non-primitive-recursive complexities.

**Quantitative Analysis** (Sec. 7). The task is to estimate to an arbitrary degree of precision the probability by which a run from $\gamma_{\text{init}}$ (repeatedly) visits $\text{Target}$, rather than only checking whether the probability is equal to one or zero.

**Expected Average Cost** (Sec. 8). We study the expected cost for runs that start from $\gamma_{\text{init}}$ until they reach $\text{Target}$. To that end, we extend our model by providing a cost function that assigns a fixed cost to each instruction in the language. Calculating expected costs of runs has many potential applications. For instance, one might be interested in the mean-time of reaching a target, i.e., the average number of steps before reaching the target \[20\]. In the context of weak memory models, in general, and TSO in particular, one can perform a more refined analysis by also taking into account the fact that specific instructions, e.g., memory fences, have higher costs \[21\]. Incorporating instruction costs in the model makes average cost analysis reflect more faithfully the efficiency of the program compared to an instruction count based metric. There have been several approaches towards optimizing fence implementations in hardware \[22\] \[23\] \[24\], which exploit the fact that non-SC behaviours are rare even in unfenced code. A quantitative analysis of the prevalence of behaviours and cost of executing instructions can help determine the efficacy of such implementations.

\[5\] While repeated reachability is a liveness property, plain reachability in the non-probabilistic case is a safety property. However, in the presence of probabilities, plain reachability measures the probability of convergence towards a target state, and hence it can be considered a form of liveness property. In any case, this is a matter of definition and has no bearing on the rest of the paper.
The supplementary material contains detailed proofs of all the lemmas and theorems.

2 Preliminaries

In this section, we introduce notation, recall basics of transition systems, Temporal logic and Markov chains.

Basic Notation The size of a set $A$ is denoted by $|A|$. We use $A^*$ and $A^\omega$ to denote the set of finite resp. infinite words over (a possibly infinite set) $A$, and let $\epsilon$ be the empty word. For $w \in A^*$, $|w|$ denotes the length of $w$ ($|w| = \infty$ if $w$ is infinite). For $i : 1 \leq i \leq |w|$, we use $w[i]$ to denote the $i^{th}$ element of $w$. We define $\textbf{head}(w) := w[1]$ and $\textbf{tail}(w) := w[2] \cdots w[|w|]$. We use $a \in w$ to denote that $w[i] = a$ for some $i : 1 \leq i \leq |w|$. For words $w_1 \in A^*$ and $w_2 \in (A^* \cup A^\omega)$, we use $w_1 \cdot w_2$ to denote their concatenation. For $k \in \mathbb{N}$, we define $A^k := \{ w \in A^* \mid |w| = k \}$, i.e., it is the set of words over $A$ of length $k$.

Transition Systems A transition system is a pair $\langle \Gamma, \rightarrow \rangle$ where $\Gamma$ is a (potentially) infinite set of configurations, and $\rightarrow \subseteq \Gamma \times \Gamma$ is the transition relation. We write $\gamma \rightarrow \gamma'$ to denote that $\langle \gamma, \gamma' \rangle \in \rightarrow$, and use $\rightarrow$ to be the reflexive transitive closure of $\rightarrow$. For $k \in \mathbb{N}$, we write $\gamma \xrightarrow{k} \gamma'$ to denote that there is a sequence $\gamma_0 \rightarrow \gamma_1 \rightarrow \cdots \rightarrow \gamma_k$ where $\gamma_0 = \gamma$ and $\gamma_k = \gamma'$, i.e., there is a sequence of $k$ transition steps leading from $\gamma$ to $\gamma'$. For $\sim \in \{ <, \leq, = \}$, we write $\gamma \xrightarrow{m} \gamma'$ to denote that $\gamma \xrightarrow{k} \gamma'$ for some $m : 0 \leq m \sim k$.

Temporal Logic A run $\rho$ of transition system $T = \langle \Gamma, \rightarrow \rangle$ is an infinite word $\gamma_0 \gamma_1 \ldots$ of configurations such that $\gamma_i \rightarrow \gamma_{i+1}$ for $i \geq 0$. We use $\rho[i]$ to denote $\gamma_i$. We say that $\rho$ is a $\gamma$-run if $\rho[0] = \gamma$. We use $\textbf{Runs}(\gamma)$ to denote the set of $\gamma$-runs. A path $\pi$ is a finite prefix of a run, and a $\gamma$-path is a finite prefix of a $\gamma$-run. We use the standard notation $\gamma \models T \phi$ to represent that $\gamma$ satisfies the CTL* state formula $\phi$ and $\rho \models T \phi$ to mean that $\rho$ satisfies the path formula $\phi$.

We refer the reader to [25] for details of CTL.

For $\gamma \in \Gamma$ and $G \subseteq \Gamma$, we say that $G$ is reachable from $\gamma$, denoted $\gamma \models T \exists G$, if there is a $\gamma$-run $\rho$ such that $\rho[i] \in G$ for some $i$. For $k \in \mathbb{N}$, $\gamma \in \Gamma$, and $G \subseteq \Gamma$, $\rho \models T \diamond^k G$ says that $\rho$ reaches $G$ first at the $k^{th}$ step. For $\sim \in \{ <, \leq, =, \geq, > \}$, $\rho \models T \diamond^k \diamond^\sim G$ says that $\rho \models T \diamond^m G$ holds for some $m : 0 \leq m \sim k$. The statement $\rho \models T \diamond^k \diamond^\sim G$ says that $\rho$ visits $G$ at the $k^{th}$ step (but possibly earlier).

Markov Chains A Markov chain $C$ is a pair $\langle \Gamma, M \rangle$ where $\Gamma$ is a (potentially infinite) set of configurations, and $M : \Gamma \times \Gamma \rightarrow [0, 1]$ is a transition probability matrix over $\Gamma$, called the probability matrix of $C$, i.e. $M$ satisfies: $\forall a \in A : \sum_{b \in A} M(a, b) = 1$. A Markov chain $C = \langle \Gamma, M \rangle$ induces an underlying transition matrix $M^k$ for each $k \geq 0$, where $M^k(a, b) = \sum_{c \in \Gamma} M(a, c) M(c, b)$. We term infinite sequences as runs and finite sequences as paths. However, traditionally, $CTL^*$ refers to properties of infinite-sequences (our runs) as path-formulae.
system, denoted $C^\downarrow$. We define $C^\downarrow := \langle \Gamma, \rightarrow \rangle$, where $\rightarrow := \{ \langle \gamma, \gamma' \rangle \mid M(\gamma, \gamma') > 0 \}$. The underlying transition system has the same configuration set, with transitions between configurations that have non-zero transition probability under $C$. This allows us to lift the temporal logic concepts defined above to Markov chains.

**Probability Measures** Consider a Markov chain $C = \langle \Gamma, M \rangle$. The probability of taking path $\pi$ is the product of single step probabilities along $\pi$:

$$Prob_C(\pi) := \prod_{i=0,\ldots,|\pi|-1} M(\pi[i], \pi[i+1])$$

For a configuration $\gamma$, we adopt the usual probability space on $\gamma$-runs with the $\sigma$-algebra over cylindrical sets starting from $\gamma$ (see [26] for details). For path formula $\phi$, we define $Prob_C(\gamma \models \phi) = Prob_C(\{ \rho \in \text{Runs}(\gamma) \mid \rho \models_C \phi \})$ (which is measurable by [27]), e.g. given a set $F \subseteq G$, $Prob_C(\gamma \models \Diamond F)$ is the measure of $\gamma$-runs which reach $F$. If $Prob_C(\gamma \models \phi) = 1$ the we say that almost all $\gamma$-runs of $C$ satisfy $\phi$. Following the literature, we say that $\gamma \models_C \phi$ holds almost surely (almost certainly), or that $\phi$ holds almost surely from $\gamma$.

### 3 Concurrent Programs

A (concurrent) program consists of a set of processes that run in parallel and communicate through a set of shared variables. The operation of the program is controlled by a central scheduler that selects the processes to execute one after the other. We assume a finite set $\text{Procs}$ of processes that share a set $\mathcal{X}$ of variables. Fig. 1 gives the grammar for a small but general assembly-like language that we use for defining the syntax of concurrent programs. A program instance, $P$ is described by a set of shared variables, $\text{var}^*$, followed by the codes of the processes, $(\text{proc reg}^* \text{ instr}^*)^*$. Each process $p \in \text{Procs}$ has a finite set $\text{Regs}_p$ of (local) registers. We assume that the sets of registers of the different processes are disjoint, and define $\text{Regs}_P := \bigcup_{p \in \text{Procs}} \text{Regs}_p$.

Each process declares its set of registers, $\text{reg}^*$, followed by a sequence of instructions. We assume that the data domain of $\mathcal{X}$ and $\text{Regs}_P$ is a finite set $\mathcal{V}$, with a special element $0 \in \mathcal{V}$.

**Instructions** An instruction $i$ is of the form $l : s$ where $l$ is a unique (across processes) label and $s$ is a statement. Labels represent program counters of processes and indicate the instruction that the process executes the next time it is scheduled. A read/write statement either writes the value of a register to a shared variable, reads the value of a shared variable into a register, or updates the value of a register by evaluating an expression. We assume a set $\text{expr}$ of expressions
over constants and registers, but not referring to the shared variables. The \textbf{CAS} statement is the standard \textit{compare-and-swap} operation, and if-statements have their usual interpretations. Iterative constructs such as \textit{while} and \textit{for}, as well as \textit{goto}-statements, can be encoded with branching if-statements as usual.

The \textbf{fence} statement, that flushes the contents of the buffer of the process, can be simulated using the \textbf{CAS} statement. The statement \textbf{term} will cause the process to terminate its execution. Sometimes, we will refer to an instruction by its statement, e.g. the instruction \textbf{r}:=\textbf{x}, (where \textbf{r} is a register and \textbf{x} is a shared variable) a \textit{read} instruction, similarly for a \textit{write} instruction, etc. Semantics of these instructions are explained through a set of inference rules in Sec. 4.

\textbf{Labels} We define \textbf{Lbl}_{p} to be the set of labels that occur in the code of the process \textbf{p}, and define \textbf{Lbl}_{P} := \bigcup_{p\in\textbf{Procs}}\textbf{Lbl}_{p}. We assume that \textbf{term} has the label \textbf{l}_{\text{term}}. We define \textbf{Instr}_{p} to be the set of instructions occurring in \textbf{p}, and define \textbf{Instr}_{P} := \bigcup_{p\in\textbf{Procs}}\textbf{Instr}_{p}. For instruction \textbf{i} of the form \textbf{l} : \textbf{s} we define \(\lambda (i) := l\) and \(\text{stmt}(i) := s\). Abusing notation, we also define \(\text{stmt}(l) := s\). For a process \textbf{p} instruction \textbf{i} \in \textbf{Instr}_{p}, with \(\text{stmt}(i) \neq \text{term}\), we define next(\textbf{i}) to be the (unique) instruction next to \textbf{i} in the code of \textbf{p}. For an instruction \textbf{l}_{1} : (\textbf{if a then l}_{2}), we assume, without loss of generality\footnote{We make the restriction for technical convenience. The case where \(l_{1} = l_{2}\) do not introduce conceptual difficulties. However, it simplifies the presentation by eliminating some corner cases when we define probability measures (Sec. 5) and when we introduce our cost model (Sec. 8).}, that \(l_{1} \neq l_{2}\).

\textbf{Scheduler} The scheduler selects the process from \textbf{Procs} to run next. The operational model for classical TSO \cite{28} uses a non-deterministic scheduler. We adopt a scheduler that selects the next process probabilistically. The scheduler policy is defined by a function \(\text{Sched}: \text{Sched}(p) \in \mathbb{N}\) denotes the scheduling weight assigned to to the process \textbf{p}. If \textbf{p} is enabled (i.e. the process can execute the next instruction, formally defined in Sec. 4) then \textbf{p} is scheduled at the next step with a probability that is proportional to \(\text{Sched}(p)\).

\section{Operational Semantics}

The operational model for classical TSO \cite{28} describes the semantics as a transition system. We also take an operational approach. However, we differ in a fundamental aspect: classical TSO models choice between transitions as non-deterministic choice. We on the other hand, model this as probabilistic choice, to get a system called as Probabilistic TSO (PTSO for short). Adding probabilities induces a Markov chain, which governs the behaviours of PTSO.

A program is described by a pair: the set of processes, \textbf{Procs} and the scheduler policy \textbf{Sched}. In this section, we fix such a program \(\mathcal{P} = (\textbf{Procs},\textbf{Sched})\). We develop the operational semantics of \(\mathcal{P}\) under PTSO as an infinite-state Markov chain \(\mathcal{J}_{\mathcal{P}}^{\text{MC}} := (\Gamma_{\mathcal{P}},\mathcal{M}_{\mathcal{P}})\). We begin by defining the set of configurations \(\Gamma_{\mathcal{P}}\) (Sec. 4.1). Then we describe the behavior of \(\mathcal{P}\) under classical TSO using a
We recall the classical semantics of TSO, using a transition system \([\mathcal{P}]^{TS}\) (Sec. 4.2). Finally, we extend the transition system to a Markov chain \([\mathcal{P}]^{MC}\) by giving probability distributions that define govern process scheduling, and memory updates.

### 4.1 Configurations

The central feature of TSO is the store buffer: a FIFO buffer in which pending write operations are queued as messages. The semantics equips each process \(p \in \text{Procs}\) with an unbounded buffer, here called the \(p\)-buffer, that carries pending write operations issued by \(p\), but that have yet not reached the shared memory.

A configuration, \(\langle \lambda, \mathcal{R}, \mathcal{B}, \mathcal{M} \rangle\), describes four attributes: a labeling state \(\lambda\), a register state \(\mathcal{R}\), a buffer state \(\mathcal{B}\), and a memory state \(\mathcal{M}\). We use \(\Gamma_p\) to denote the set of configurations of \(\mathcal{P}\).

A labeling state is a function \(\lambda : \text{Procs} \to \text{Lbl}_p\) that defines, for \(p \in \text{Procs}\), the label \(\lambda(p) \in \text{Lbl}_p\) of the next instruction to be executed by \(p\).

A register state is a function \(\mathcal{R} : \text{Regs}_p \to \mathcal{V}\) that maps each register \(a \in \text{Regs}_p\), to its current value \(\mathcal{R}(a) \in \mathcal{V}\). For an expression \(e\), we use \(\mathcal{R}(e)\) to denote the evaluation of \(e\) against the register state \(\mathcal{R}\).

A single-buffer state \(w\) is a word in \(\langle \mathcal{X} \times \mathcal{V} \rangle^*\), describing the content of the \(p\)-buffer for some process \(p \in \text{Procs}\). The buffer contains a sequence of pending write messages, i.e., pairs of form \(\langle x, v \rangle\) representing a write to \(x\), with value \(v\). A buffer state is a function \(\mathcal{B} : \text{Procs} \to \langle \mathcal{X} \times \mathcal{V} \rangle^*\) that defines, for each process \(p \in \text{Procs}\), a single-buffer state describing the content of the \(p\)-buffer.

A memory state is a function \(\mathcal{M} : \mathcal{X} \to \mathcal{V}\) that assigns to each variable \(x \in \mathcal{X}\) its current value \(\mathcal{M}(x) \in \mathcal{V}\) in the shared memory.

Consider a configuration \(\gamma = \langle \lambda, \mathcal{R}, \mathcal{B}, \mathcal{M} \rangle\). We say that \(\gamma\) is plain if \(\mathcal{B}(p) = \epsilon\) for all \(p \in \text{Procs}\), i.e., all the buffers in \(\gamma\) are empty. We use \(\Gamma_p^{\text{plain}}\) to denote the set of plain configurations of \(\mathcal{P}\). Notice that \(\Gamma_p^{\text{plain}} \subseteq \Gamma_p\) and that \(\Gamma_p^{\text{plain}}\) is finite. For a label \(l \in \text{Lbl}_p\), we write \(l \in \gamma\) if \(\lambda(p) = l\) for some \(p \in \text{Procs}\). We define \(\Gamma_p^l := \{ \gamma \in \Gamma_p \mid l \in \gamma\}\), i.e., configurations in which \(l\) occurs.

For a configuration \(\gamma = \langle \lambda, \mathcal{R}, \mathcal{B}, \mathcal{M} \rangle\) we define the size of \(\gamma\) by \(|\gamma| := \sum_{p \in \text{Procs}} |\mathcal{B}(p)|\), i.e., it is the total number of messages in the buffers in \(\gamma\). For \(\gamma \in \{<, =, >\}\), we define \(\Gamma_p^{\ell} := \{ \gamma \in \Gamma_p \mid |\gamma| \sim \ell\}\), i.e., configurations where the total number of messages, \(m\), relates to \(\ell\) by \(m \sim \ell\).

### 4.2 The Classical TSO Semantics

We recall the classical semantics of TSO, using a transition system \([\mathcal{P}]^{TS} = \langle \Gamma_p, \to_p \rangle\). We define the transition relation \(\to_p\) through the set of inference rules in Fig. 2. The relation \(\to_p\) is the composition of two relations: the relation \(\to_{\text{proc}}\) describes the processes’ execution steps, and the relation \(\to_{\text{update}}\) describes memory updates, where pending writes are propagated to the memory.

**Process Transitions** We define the process transition relation \(\to_{\text{proc}} := \bigcup_{p \in \text{Procs}} \to_{p_{\text{proc}}}^p\) as a union of relations each corresponding to one process (the rule
A write instruction \( (x := a) \) assigns the value of the local register \( a \) to the shared variable \( x \). The process appends a write message consisting of \( x \) together with the value \( R(a) \) of \( a \), to the head of the \( p \)-buffer. A read instruction, \( (a := x) \), assigns the value of the shared variable \( x \) to the local register \( a \). The value of \( x \) is either fetched from the \( p \)-buffer (read-from-memory), or from the shared memory (read-from-memory). We capture both cases in one inference rule, using the function \( \text{FetchVal} \) defined as follows. Let \( w \) be the contents of the \( p \)-buffer. We write \( x \in w \) if \( (x,v) \in w \) for some \( v \in V \), and write \( x \notin w \) otherwise. We define (i) \( \text{FetchVal}(x)(w)(M) := v \) if \( x \in w \) and \( w = w_1 \cdot (x,v) \cdot w_2 \) with \( x \notin w_1 \); and (ii) define \( \text{FetchVal}(x)(w)(M) := M(x) \) if \( x \notin w \). In case (i), the value of \( x \) is taken from the latest \( x \)-message from the \( p \)-buffer. In case (ii), no \( x \)-messages exist in the \( p \)-buffer, and the value is read from the shared memory.

The instruction \( b := \text{CAS}(x,a_1,a_2) \) checks whether the \( p \)-buffer is empty and the value of the shared variable \( x \) is equal to the value of the register \( a_1 \). If yes, we assign atomically the value of the register \( a_2 \) to \( x \), and assign the value true to \( b \) (the rule \text{CAS-true}). If the value of \( x \) is different from the value of \( a_1 \) then we do not change the value of \( x \), but assign the value false to \( b \) (the rule \text{CAS-false}). If the \( p \)-buffer is not empty then \( p \) is disabled in the current

\[\begin{array}{ccc}
\text{write} & \text{read} & \text{expr} \\
\begin{array}{l}
\text{stmt}(\lambda(p)) = (x := a) \\
\mathcal{B}' = \mathcal{B}[p \leftarrow (x, R(a)) \cdot \mathcal{B}[p]] \\
\lambda' = \lambda[p \leftarrow \text{next}(\lambda(p))] \\
\end{array} & \\
\begin{array}{l}
\text{stmt}(\lambda(p)) = (a := x) \\
\text{FetchVal}(x)(\mathcal{B}[p])(\mathcal{M}(x)) = v \\
\mathcal{R}' = \mathcal{R}[a \leftarrow v] \\
\lambda' = \lambda[p \leftarrow \text{next}(\lambda(p))] \\
\end{array} & \\
\begin{array}{l}
\text{stmt}(\lambda(p)) = (a := e) \\
\mathcal{R}' = \mathcal{R}[a \leftarrow e] \\
\lambda' = \lambda[p \leftarrow \text{next}(\lambda(p))] \\
\end{array}
\end{array}\]

\[\begin{array}{ccc}
\text{ CAS-true} & \text{ CAS-false} & \text{ disabled} \\
\begin{array}{l}
\text{stmt}(\lambda(p)) = (b := \text{CAS}(x,a_1,a_2)) \\
\mathcal{M}(x) = R(a_1) \\
\mathcal{R}' = \mathcal{R}[b \leftarrow \text{false}] \\
\lambda' = \lambda[p \leftarrow \text{next}(\lambda(p))] \\
\end{array} & \\
\begin{array}{l}
\text{stmt}(\lambda(p)) = (b := \text{CAS}(x,a_1,a_2)) \\
\mathcal{M}(x) \neq R(a_1) \\
\mathcal{R}' = \mathcal{R}[b \leftarrow \text{true}] \\
\lambda' = \lambda[p \leftarrow \text{next}(\lambda(p))] \\
\end{array} & \\
\begin{array}{l}
\text{proc} \\
\end{array}
\end{array}\]

**Fig. 2.** The classical TSO semantics: process transitions (green), update transitions (orange) and overall transition (Full-TSO).
configuration. We define the set of disabled processes at configuration \( \gamma \):

\[
\text{disab}(\gamma) := \{ p \mid (\text{stmt}(p) = \text{term}) \lor ((\text{stmt}(p) = (b := \text{CAS}(x, a_1, a_2))) \land (B(p) \neq \epsilon)) \}
\]

In other words, it is the set of processes that are disabled in \( \gamma \) either because they have terminated or because they are about to perform a \text{CAS} operation and their buffers are not empty. We say that \( p \) is disabled in \( \gamma \) if \( p \in \text{disab}(\gamma) \), and that \( \gamma \) is disabled if all the processes are disabled in \( \gamma \). If a process (resp. configuration) is not disabled then it is enabled. If \( \gamma \) is disabled, we make a dummy transition that does not change \( \gamma \) (the rule \textit{disabled}). Notice that if \( \gamma \xrightarrow{p} \gamma' \) then there is unique process \( p \in \text{Procs} \) such that \( \gamma \xrightarrow{p} \gamma' \).

**Update Transitions** Between two process transitions, the system may perform a (possibly empty) sequence of update steps. The rule \textit{empty-update} describes an empty update step. Each \textit{single-update} step pops one write message at the end of the \( p \)-buffer for some process \( p \) and uses it to update the memory. The \textit{update} rule captures the effect of a sequence of such \textit{single-update} steps. We define the update transition relation \( \xrightarrow{\text{update}} := \bigcup_{\alpha \in \text{Procs}} \alpha \xrightarrow{\text{update}} \) as a union of relations each corresponding to a given sequence of update steps. The word \( \alpha \) gives the sequence of processes that perform the updates. The net effect is that the system (i) pops a sequence of (possibly empty) suffixes from the buffer of each process, (ii) shuffles these into one sequence, and (iii) uses the resulting sequence to update the memory. Notice that each selection of possible suffixes in step (i) may result in several different sequences due to multiple interleavings in step (ii). Observe that \( \rightarrow P \) is deadlock-free, i.e., for each configuration \( \gamma \in \Gamma \), there is at least one configuration \( \gamma' \in \Gamma \) such that \( \gamma \xrightarrow{P} \gamma' \).

### 4.3 Adding Probabilities: PTSO

We define the Markov Chain \( [P]_{MC} = (\Gamma_P, M_P) \). The set \( \Gamma_P \) of configurations is defined as above. The probability matrix \( M_P \) is defined as the composition of two probability distributions: (i) the process probability distribution \( M_{\text{proc}} \) (ii) the update probability distribution \( M_{\text{update}} \) which add probabilities to the process transition relation \( \xrightarrow{\text{proc}} \), and the update transition relation \( \xrightarrow{\text{update}} \) respectively.

**The Process Probability Distribution: the Scheduler** At each program step (\( \rightarrow P \)), a process is selected for execution according to a probability given by the scheduler. In a configuration \( \gamma \), the scheduler selects an enabled process \( p \in \text{enab}(\gamma) \) with a probability that reflects the relative weight of \( p \) compared to those of the other enabled processes, \( \text{Rweight}(\gamma)(p) \):

\[
\text{Rweight}(\gamma)(p) = \begin{cases} 
0 & \text{if } p \in \text{disab}(\gamma) \\
\frac{\text{Sched}(p)}{\sum_{p' \in \text{enab}(\gamma)} \text{Sched}(p')} & \text{if } p \in \text{enab}(\gamma)
\end{cases}
\] (1)

\footnote{The latter transition is not strictly needed, but it is included for technical convenience.}
This gives the probability that \( p \) to execute in the next step from \( \gamma \). For configurations \( \gamma \) and \( \gamma' \), with \( \gamma \xrightarrow{\mathcal{P}_{\text{proc}}} \gamma' \), we define \( M_{\text{proc}}(\gamma, \gamma') := \text{Rweight}(\gamma)(p) \). In other words, we move from \( \gamma \) to \( \gamma' \) with a probability that is given by the relative weight of \( p \) in \( \gamma \). We define \( M_{\mathcal{P}}(\gamma, \gamma') := 0 \) if \( \gamma \not\xrightarrow{\text{proc}} \gamma' \). To account for the case where all the processes are disabled in \( \gamma \), we define \( M_{\text{proc}}(\gamma, \gamma) := 1 \) if \( \gamma \) is disabled.

**Faithfulness** Our model uses a scheduling policy that assigns a fixed scheduling weight, \( \text{Sched}(p) \), to each process \( p \) in the system. This is a case of memoryless scheduling, i.e., the probability distribution over processes does not depend on the execution history. However, we can relax this constraint to allow for any scheduling policy that satisfies the faithfulness condition:

\[
\forall p \in \text{Procs} \quad \text{Rweight}(\gamma)(p) = 0 \iff p \in \text{disab}(\gamma)
\]

In words, at each step, each enabled process should be scheduled with non-zero probability. A scheduler that assigns scheduling weights such that the above condition holds is said to be a faithful scheduler.

**Schedulers with memory** The above criterion allows for schedulers that are more refined as compared to the memoryless scheduler. As an example, on implementations of TSO, processes are often scheduled for multiple consecutive steps since unnecessary context switching wastes processor resources. To reflect this detail, we can consider a scheduler that assigns a higher probability to the previously scheduled process, \( p_{\text{prv}} \). For some choice of constant weights, \( \text{Sched} \), we can define a new choice of weights \( \text{Sched}' \) where \( \lambda > 1 \) is some parameter.

\[
\text{Sched}'(p) = \begin{cases} 
\text{Sched}(p) & \text{if } p \neq p_{\text{prv}} \\
\lambda \cdot \text{Sched}(p) & \text{otherwise}
\end{cases}
\]

In this case, \( p_{\text{prv}} \) is re-scheduled with a weight which is larger by a factor of \( \lambda \). A larger \( \lambda \) implies a stronger tendency to re-schedule a process. This scheduling policy still satisfies faithfulness. One can extend this by formulating more intricate policies, e.g. ones that account for \( k \) previous steps.

To better illustrate the concerns and challenges of verification, we continue to adopt the simple (memoryless) scheduler proposed earlier. However, we emphasize that our results extend to faithful schedulers.

**The Update Probability Distribution:** the Memory update policy Between the process steps, pending messages from the store buffers are propagated to the shared memory (the update transition). The details of this write propagation are implementation-specific, with policies tuned towards system performance. Classical TSO models this update propagation non-deterministically. We, on the other hand, consider a probabilistic update policy. In a similar manner to the scheduling probabilities, the update probability distribution defines the probability by which a configuration \( \gamma \) reaches another configuration \( \gamma' \) through an update step \( (\rightarrow_{\text{update}}) \). Recall that an update step consists of a sequence of (single) update operations. The number of possible update sequences from \( \gamma \) is
finite since the sizes of each buffer is finite. In our model, we assume that the update distribution is the uniform distribution over all possible update sequences. We note that starting from \( \gamma \), different update sequences can lead to the same configuration \( \gamma' \). The reason is that different shufflings of the selected suffixes (see Sec. 4.2) may lead to the same memory state. To reflect this, for configurations \( \gamma \) and \( \gamma' \), we define \( M_{\text{update}}(\gamma, \gamma') := \left\{ \alpha \mid \gamma \xrightarrow{\alpha} \gamma' \right\} \), i.e. the fraction of update sequences that lead to the configuration \( \gamma' \).

**Left-Biasedness** Though we adopt a specific update distribution, we provide a generic condition on that update policy that is sufficient for our results to hold. We call this the left-biasedness property. Here we provide an intuitive description of left-biasedness and defer the formal definition to Sec. 8.

Intuitively, left-biasedness requires that for sufficiently large configurations, the probability that the configuration size reduces in a single \( \rightarrow \) step is strictly greater than \( p > \frac{1}{2} \). Left-biasedness allows a wide class of more refined scheduling policies, e.g., where no message propagation is performed when the number of messages is smaller than a certain value, or where only the messages inside the buffers of some (probabilistically selected) processes are propagated.

Though our results apply more generally to models characterized by faithfulness (scheduler policy), and left-biasedness (update policy), we continue to adopt the fixed-weight (memoryless) scheduler and uniform update policy for reasons described above.

**The Full Probability Distribution.** We combine the process and update probability distributions, to derive the probability matrix \( M_P \), and thus obtain the Markov chain \( J_P^{\text{MC}} \). Consider configurations \( \gamma \) and \( \gamma' \) where \( \gamma \xrightarrow{\text{proc}} \gamma'' \xrightarrow{\text{update}} \gamma' \). Let \( \gamma'' \) be the unique configuration such that \( \gamma \xrightarrow{\text{proc}} \gamma'' \xrightarrow{\text{update}} \gamma' \). Then, we define \( M_P(\gamma, \gamma') := M_{\text{proc}}(\gamma, \gamma'') \cdot M_{\text{update}}(\gamma'', \gamma') \).

**Lemma 1** \( M_P \) is a prob. distribution on \( \Gamma_P \); hence, \( [P]^{\text{MC}} \) is a Markov chain.

**5 PTSO: Concepts and Properties**

Now, we intuit some concepts underlying Probabilistic TSO and its properties.

**PTSO Refines Classical TSO.** After introducing \( [P]^{\text{TS}} \) and \( [P]^{\text{MC}} \) in Sec. 4, we s.t. they are closely related; \( [P]^{\text{TS}} \) is the underlying transition system of \( [P]^{\text{MC}} \).

**Lemma 2** \( ([P]^{\text{MC}})^{\downarrow} = [P]^{\text{TS}} \) for any program \( P \).

In particular, this means that the PTSO system \( [P]^{\text{MC}} \) is a refinement of \( [P]^{\text{TS}} \): a behaviour is observed in \( [P]^{\text{TS}} \) iff it is seen in \( [P]^{\text{MC}} \) with non-zero probability. Whenever the context is clear, we write \( P \) instead of \( [P]^{\text{TS}}, [P]^{\text{MC}} \).

**Label Reachability.** We formulate our verification problems in terms of reachability to instruction labels. To simplify the notation, we identify a label \( l \in \text{Lbl}_P \)}
with the set $\mathcal{I}_P^l$ of configurations in which $l$ occurs. We say that “$l$ is reachable” rather than “$\mathcal{I}_P^l$ is reachable”, and write $\diamond l$ instead of $\diamond \{\gamma \in \mathcal{I}_P \mid l \in \gamma\}$. In [13,12] the authors show that label reachability from a plain configuration is decidable. The following lemma, generalizes this to the case where the source configuration need not be plain and destination can be a particular plain configuration.

**Lemma 3** For a program $\mathcal{P}$, a configuration $\gamma \in \mathcal{I}_P$, and a plain configuration $\gamma' \in \mathcal{I}_P^{\text{plain}}$, it is decidable whether $\gamma \xrightarrow{\mathcal{P}} \gamma'$.

Extending this, we have Lemma 4, we can query whether $\gamma \xrightarrow{\mathcal{P}} \gamma'$ for each $\gamma' \in \mathcal{I}_P^l \cup \mathcal{I}_P^{\text{plain}}$. Decidability of Lemma 4 follows since $\mathcal{I}_P^{\text{plain}}$ is finite and the subroutine is decidable by Lemma 3.

**Lemma 4** For a program $\mathcal{P}$, a configuration $\gamma \in \mathcal{I}_P$, and a label $l \in \text{Lbl}_P$, it is decidable whether $\gamma \xrightarrow{\mathcal{P}} l$.

### 5.1 Left-Orientedness and Attractors

We show that the set of plain configurations $\mathcal{I}_P^{\text{plain}}$ set has an attractor property in the sense of [16]. In our setting, this means that any run of $[\mathcal{P}]^{MC}$ almost surely visits $\mathcal{I}_P^{\text{plain}}$ infinitely often.

*Small and large configurations* To arrive at this result, we consider a generalization of plain configurations, called *small configurations*, denoted $\mathcal{I}_P^{\text{small}}$. $\mathcal{I}_P^{\text{small}}$ consists of configurations with a small number of messages inside their buffers. Concretely, a configuration $\gamma$ is small if $|\gamma| \leq 4$, i.e., the total number of messages inside the buffers does not exceed 4. We define the set of *large configurations* by $\mathcal{I}_P^{\text{large}} := \mathcal{I}_P - \mathcal{I}_P^{\text{small}} = \mathcal{I}_P^{\geq 5}$. We show that the Markov chain $[\mathcal{P}]^{MC}$ is left-oriented in the sense of [29]. That is, for any large configuration $\gamma \in \mathcal{I}_P^{\text{large}}$, the expected change in configuration size for a single $\rightarrow P$ step is negative.

**An illustrative example** We explain the update probability distribution through the code snippet on the right. To begin with let us only consider the process on the left (procL). It executes an infinite loop, writing 1 to variable $x$. Let us consider the evolution of the buffer-sizes of procL, i.e. the number of $(x,1)$ messages in the procL-buffer. Assume that on reaching label 0, procL has 6 messages in its buffer. The $\rightarrow P$ step consists of a process transition, $\rightarrow_{\text{proc}}$ followed by an update transition, $\rightarrow_{\text{update}}$. In the $\rightarrow_{\text{proc}}$ step, the write increases the size of the buffer by one, thus obtaining a buffer of size 7. Following this the $\rightarrow_{\text{update}}$ step may push any number of messages to the memory. Since the update policy chooses uniformly amongst possible update sequences, the resulting configuration has one amongst $\{0,\ldots,7\}$ messages in the procL-buffer, each occurring with an equal probability of $1/8$. The next $\rightarrow_{\text{proc}}$ step (a goto), does not change the buffer size, but the $\rightarrow_{\text{update}}$ step can still propagate messages. The reasoning for the next steps follows similarly.

---

9 This value is an artifact of the probabilistic policies we have adopted in Sec. 4.
Comparison with other notions of fairness At each \(\rightarrow_{\text{proc}}\) step at most one message is added to the process buffers (when the process performs a write), however in the following \(\rightarrow_{\text{update}}\) can still remove large number of messages. Hence, from sufficient large configuration sizes, the system has a tendency to move towards configurations with smaller buffer sizes. Formally, we prove the following lemma, using the left-orienteredness property mentioned earlier.

**Lemma 5** \(\text{Prob}_P \left( \gamma \models \Box \Diamond \Gamma_P^{\text{plain}} \right) = 1\) for all configurations \(\gamma \in \Gamma_P\).

For the above example, PTSO guarantees that the process on the right (procR) eventually reads value 1 into register \(a\). This follows since in a plain configuration, the buffer of procR is empty and hence it can read the value from the memory - this happens almost surely. We highlight that other notions of fairness such as strong fairness in process scheduling (discussed in [30]) as well memory fairness [31], cannot provide this guarantee. In particular, memory fairness from [31], would consider the execution which exactly alternates writes of both processes but procR reads before its own write is pushed memory to be fair and hence permissible.

\[
x = 1 \quad x = 2 \quad a = x \quad // \quad 2 \quad x = 1 \quad x = 2 \quad a = x \quad // \quad 2 \quad x = 1 \quad \cdots
\]

**B-Plain Configurations** We can refine our analysis of the attraction property enjoyed by the set \(\Gamma_P^{\text{plain}}\) of plain configurations. We consider a subset of \(\Gamma_P^{\text{plain}}\) which we call the set of bottom plain configurations, (or B-plain configurations, for short), denoted \(\Gamma_P^{\text{Bplain}}\). Intuitively, a B-plain configuration is a member of a bottom strongly connected component in the graph of plain configurations. Formally, a configuration \(\gamma \in \Gamma_P\) is said to be B-plain if (i) \(\gamma \in \Gamma_P^{\text{plain}}\), and (ii) for any \(\gamma' \in \Gamma_P^{\text{plain}}\), if \(\gamma \rightarrow_P \gamma'\) then \(\gamma' \rightarrow_P \gamma\). Since any run of the system almost surely visits the set of \(\Gamma_P^{\text{plain}}\) infinitely often, it will also almost surely visit a B-plain configuration infinitely often.

**Lemma 6** \(\text{Prob}_P \left( \gamma \models \Box \Diamond \Gamma_P^{\text{Bplain}} \right) = 1\) for all configurations \(\gamma \in \Gamma_P\).

6 Qualitative (Repeated) Reachability

| Given: a program \(\mathcal{P}\), a configuration \(\gamma_{\text{init}} \in \Gamma_P\), a label \(l \in \text{Lbl}_P\) |
| QUAL\_REACH: Determine whether \(\text{Prob}_P \left( \gamma_{\text{init}} \models \Diamond l \right) = 1\) |
| QUAL\_REP\_REACH: Determine whether \(\text{Prob}_P \left( \gamma_{\text{init}} \models \Box \Diamond l \right) = 1\) |

In this section, we perform qualitative reachability analysis for PTSO. Given a program \(\mathcal{P}\), configuration \(\gamma_{\text{init}}\), and label \(l\), we check whether a \(\gamma_{\text{init}}\)-run almost surely reaches \(l\). We also consider qualitative repeated reachability, where, we ask whether a \(\gamma_{\text{init}}\)-run repeatedly visits \(l\) (visits \(l\) infinitely often) w.p. 1. We also consider almost-never variants of the problems, where we check whether the probabilities are 0 rather than 1. We prove that these problems are decidable, and have non-primitive-recursive complexities.
6.1 Almost-Sure Reachability

The qualitative reachability problem, \textsc{Qual\_Reach}, is defined above. The algorithm in Figure 3 solves \textsc{Qual\_Reach} by analyzing the transition system \([\mathcal{P}]^3\), the underlying transition system of PTSO. If \(l\) occurs in \(\gamma\text{\_init}\) then the property trivially holds, and hence we answer positively. Otherwise, the algorithm considers a new program \(\mathcal{P}'\) obtained by replacing the statement labeled \(l\), by a new statement that makes \(\mathcal{P}'\) terminate immediately if \(l\) is reached. Let \(p \in \text{Procs}\) be the unique process such that \(l \in \text{Lbl}_p\). We define \(\mathcal{P} \ominus l := \langle \text{Procs} - \{p\} \cup \{p'\}, \text{Sched}\rangle\) where \(p'\) is a fresh process derived from \(p\) by replacing \(\text{stmt}(l)\) by \(\text{goto \_new term}\) for a fresh label \(\text{goto \_new term} \not\in \text{Lbl}_P\) and adding a \text{term} at label \(\text{term} \_new\). The remaining instructions of \(p'\) are identical to \(p\).

\begin{figure}[h]
\begin{center}
\begin{algorithm}
\textbf{Algorithm: Qual\_Reach}
\begin{algorithmic}[1]
\State \textbf{Input:} \(\mathcal{P}\): program; \(\gamma\text{\_init} \in I_\mathcal{P}\): configuration; \(l \in \text{Lbl}_P\): label.
\State 1 \textbf{if} \(l \in \gamma\text{\_init}\) \textbf{then} \textbf{return} \text{true};
\State 2 \(\mathcal{P}' := \mathcal{P} \ominus l\);
\State 3 \textbf{for each} \(\gamma \in I_\mathcal{P}^{\text{plain}}\) \textbf{do}
\State 4 \textbf{if} \(\gamma\text{\_init} \rightarrow^* \mathcal{P}' \gamma \land \neg (\gamma \rightarrow^* \mathcal{P} l)\)
\State 5 \textbf{then} \textbf{return} \text{false};
\State 6 \textbf{return} \text{true}
\end{algorithmic}
\end{algorithm}
\end{center}
\caption{Almost-sure reachability algorithm.}
\end{figure}

The loop on line 3 cycles through the (finite) set of plain configurations. For each plain configuration \(\gamma\) from the original program \(\mathcal{P}\), we check: (i) Whether \(\gamma\) is reachable from the initial configuration \(\gamma\text{\_init} \in \mathcal{P}'\). By the construction of \(\mathcal{P}'\), this is equivalent to checking whether \(\gamma\) is reachable from \(\gamma\text{\_init}\) without observing label \(l\). (ii) Whether it can reach the label \(l\). If the answer to (i) is \text{yes}, and the answer to (ii) is \text{no}, then we have found a finite path \(\pi\) in \(\mathcal{P}\) that starting from \(\gamma\text{\_init}\), without visiting \(l\), reaches configuration \(\gamma\) from which \(l\) is not reachable. This implies that \(\text{Prob}_\mathcal{P} (\gamma\text{\_init} | \mathcal{P} \models \Diamond l) < 1\). If none of the plain configurations satisfy the condition, then each plain configuration \(\gamma\) reachable from \(\gamma\text{\_init}\) has a path to \(l\). Now by the attractor lemma, any run will almost surely visit \(I_\mathcal{P}^{\text{plain}}\) infinitely often and by the fairness property of Markov chains, it almost surely visits \(l\).

6.2 Almost-Sure Repeated Reachability

For almost-sure repeated reachability we are interested in determining whether the \(\gamma\text{\_init}\)-runs visit \(l\) infinitely often with probability 1. The algorithm for this is similar to the case for almost-sure reachability: we check whether \(\exists\) a plain configuration \(\gamma\) that satisfies \(\gamma\text{\_init} \rightarrow^* \mathcal{P} \gamma \land \neg (\gamma \rightarrow^* \mathcal{P} l)\), in which case we return false. The difference is that we do not need to transform the program as in the case of almost-sure reachability. Details are in the supplementary material.

6.3 Almost-never (Repeated) Reachability

The almost-never variants of the (repeated) reachability problems, \textsc{Never\_Qual\_Reach} resp. \textsc{Never\_Qual\_Rep\_Reach}, ask whether the probabilities equal to 0 rather than 1. The solution to \textsc{Never\_Qual\_Reach} is straightforward, since \(\text{Prob}_\mathcal{P} (\gamma\text{\_init} | \mathcal{P} \models \Diamond l) = 0\) iff \(\neg (\gamma\text{\_init} \rightarrow^* \mathcal{P} l)\). On the other
Given: a program $P$, a configuration $\gamma_{init} \in \Gamma_P$, a label $l \in Lbl_P$

**NEVER_QUAL_REACH:** Determine whether $\text{Prob}_P (\gamma_{init} \models \Diamond l) = 0$

**NEVER_QUAL_REP_REACH:** Determine whether $\text{Prob}_P (\gamma_{init} \models \Box \Diamond l) = 0$

In the case of the **NEVER_QUAL_REP_REACH** problem, the Never-Quantitative Reachability problem requires a search over B-plain configurations $\gamma$ satisfying $\gamma_{init} \xrightarrow{*} P \gamma \xrightarrow{*} l$. Due to space constraints, we defer the algorithm and proofs to the appendix.

### 6.4 Decidability and Complexity

The algorithms can be effectively implemented since (i) $\Gamma_P^\text{plain}$ is finite; and (ii) the conditions of the for-loops and if-statements can be checked effectively, as implied by Lemma 4. This gives Theorem 1. Theorem 2 is proved through reductions from the reachability problem under the classical (non-probabilistic) TSO semantics [19]. The non-primitive-recursive lower bounds follow from the corresponding result for reachability of classical TSO.

**Theorem 1.** QUAL_REACH, QUAL_REP_REACH, NEVER_QUAL_REACH, NEVER_QUAL_REP_REACH are all decidable.

**Theorem 2.** QUAL_REACH, QUAL_REP_REACH, NEVER_QUAL_REACH, NEVER_QUAL_REP_REACH all have non-primitive-recursive complexities.

### 7 Quantitative (Repeated) Reachability

In this section we discuss quantitative reachability problems for PTSO. In contrast to qualitative analysis from Sec. 6, the task here is to compute the actual probability. We are not able to compute the probabilities exactly, but we can approximate the probability with an arbitrary degree of precision.

#### 7.1 Approximate Quantitative Reachability

| Given: program $P$, configuration $\gamma_{init} \in \Gamma_P$, label $l \in Lbl_P$, precision value $\epsilon \in \mathbb{R}^+$ |
| --- |
| **QUANT_REACH:** Determine $\theta$ s.t. $\text{Prob}_P (\gamma_{init} \models \Diamond l) \in [\theta, \theta + \epsilon]$ |
| **QUANT_REP_REACH:** Determine $\theta$ s.t. $\text{Prob}_P (\gamma_{init} \models \Box \Diamond l) \in [\theta, \theta + \epsilon]$ |

In the approximate quantitative reachability problem, **QUANT_REACH**, given a precision parameter $\epsilon$, we are interested in determining an approximation $\theta$ satisfying $\theta \leq \text{Prob}_P (\gamma_{init} \models \Diamond l) \leq \theta + \epsilon$.

The algorithm in Fig. 4 solves the problem by successively improving the approximation at each iteration until we are within $\epsilon$-precision of the exact value.
The algorithm maintains two variables: $\text{PosApprx}$ (positive approximation) is an under-approximation of the probability with which $l$ is reachable from $\gamma_{\text{init}}$, and $\text{NegApprx}$ (negative approximation) is an under-approximation of the probability with which $l$ is not reachable from $\gamma_{\text{init}}$. $\text{PosApprx}$ serves as a lower bound on $\theta$, while $1 - \text{NegApprx}$ serves as an upper bound: $\text{PosApprx} \leq \theta \leq 1 - \text{NegApprx}$.

Algorithm: QUANT_REACH

Input: $\mathcal{P}$: program; $\gamma_{\text{init}} \in \mathcal{L}(\mathcal{P})$: configuration; $l \in \text{Lbl}_{\mathcal{P}}$: label; $\varepsilon \in \mathbb{R}^{\geq 0}$: precision.

1. Var
2. $\text{PosApprx}, \text{NegApprx} \in \mathbb{R}$: approximations, waiting $\in (\mathcal{L}(\mathcal{P}) \times \mathbb{R})^*$: queue
3. $\text{PosApprx} := 0$; $\text{NegApprx} := 0$; waiting := $\langle \gamma_{\text{init}}, 1 \rangle$
4. while $\text{PosApprx} + \text{NegApprx} < 1 - \varepsilon$ do
5. 
6. if $l \in \gamma$ then $\text{PosApprx} := \text{PosApprx} + \phi$
7. else if $\neg(\gamma \rightarrow_{\text{P}} l)$ then $\text{NegApprx} := \text{NegApprx} + \phi$
8. else
9. 
10. return $\text{PosApprx}$

Fig. 4. The quantitative reachability algorithm.

The algorithm iteratively improves these approximations until we reach a point where their sum is within $\varepsilon$ from 1 (line 4). In such a case, the desired value of $\theta = \text{PosApprx}$ is an $\varepsilon$-precise approximation.

To calculate the approximations, the algorithm performs forward reachability analysis starting from the initial configuration $\gamma_{\text{init}}$. It generates the set of $\gamma_{\text{init}}$-paths in a breadth-first manner, using the waiting FIFO queue. For each generated path $\pi$ it also calculates the probability of $\pi$. Instead of the whole path $\pi$, $\text{waiting}$ only stores the last configuration, $\gamma$, of $\pi$ and the probability of $\pi$, $\phi$, as a pair $\langle \gamma, \phi \rangle$.

The approximation variables are initialized (line 3) to zero, and $\text{waiting}$ queue is initialized to contain a single pair, $\langle \gamma_{\text{init}}, 1 \rangle$, representing the initial configuration $\gamma_{\text{init}}$ (which occurs with probability one). The while-loop executes until we achieve the desired precision. At each iteration, we check whether we already have reached the desired precision. If not, the algorithm pops the pair $\langle \gamma, \phi \rangle$ from the $\text{waiting}$-queue. There are three possibilities depending on $\gamma$:

1. If $l \in \gamma$ (if-branch, line 6), the current path reaches $l$ and, consequently, we increment $\text{PosApprx}$ by $\phi$, the weight of the current path.
2. If $l$ is not reachable from $\gamma$ (else-if branch, line 7), the measure of runs that reach $l$ starting from $\gamma$ is zero, and hence we increment $\text{NegApprx}$ by $\phi$.
3. If neither of the above hold (line 10), the current path needs to be explored further, we enqueue all successors $\gamma'$ of $\gamma$ into the queue. The probability of the new path to $\gamma'$ is $\phi \cdot M_\mathcal{P}(\gamma, \gamma')$.

To show correctness of the algorithm, let $\text{PosApprx}^{(i)}$ and $\text{NegApprx}^{(i)}$ represent the value of $\text{PosApprx}$ and $\text{NegApprx}$ prior to performing the $i^{\text{th}}$ iteration. We show that in the limit as $i \to \infty$, the value of $\text{PosApprx}^{(i)} + \text{NegApprx}^{(i)}$ tends to 1. Technically this follows by Lemma 5. By this lemma, any $\gamma_{\text{init}}$-run almost surely either (i) reaches a plain configuration from which $l$ is not reachable, or
(ii) repeatedly reaches a plain configuration from which \( l \) is reachable. In case (ii) it will almost surely reach \( l \). This implies that \( \text{Prob}(\gamma_{\text{init}} \models (\Diamond (l \lor \neg \exists \Diamond l))) = 1 \), i.e., an \( \gamma_{\text{init}} \)-run will almost surely either reach \( l \) or reach a configuration from which \( l \) is not reachable, implying that \( \text{PosApprx}(i) + \text{NegApprx}(i) \) tends to 1.

Finally, by Lemma 4 we can effectively check the condition of the if-statement, and hence the algorithm terminates.

The correctness of the approximation on termination follows by the property that \( \text{PosApprx}(i) \) and \( \text{NegApprx}(i) \) are under-approximations of the reach and non-reach probabilities. This follows from the following invariants:

\[
\text{PosApprx}(i) \leq \text{Prob}(\gamma_{\text{init}} \models \Diamond l) \quad \text{NegApprx}(i) \leq \text{Prob}(\gamma_{\text{init}} \models \Diamond \forall \square \neg l)
\]

\[
\text{Prob}(\gamma_{\text{init}} \models \Diamond l) \leq 1 - \text{Prob}(\gamma_{\text{init}} \models \Diamond \forall \square \neg l)
\]

\[
\text{PosApprx}(i) + \text{NegApprx}(i) > 1 - \epsilon \text{ holds on termination}
\]

These imply that, on termination, \( \text{PosApprx} \) is within \( \epsilon \)-precision of \( \theta \).

**Theorem 3.** \( \text{QuantReach} \) is solvable.

### 7.2 Approximate Quantitative Repeated Reachability

In the case of the approximate quantitative repeated reachability problem, we are interested in approximating the probability of visiting a given label \( l \) infinitely often. We develop an algorithm that uses an iterative approximation scheme similar to the reachability case. We defer full details of this algorithm to the supplementary material and instead give an intuitive explanation on how it differs from Sec. 7.1.

This algorithm too maintains approximations \( \text{PosApprx} \) and \( \text{NegApprx} \) and iteratively narrows the error margin until it is smaller than \( \epsilon \). The main difference is in the condition at line 6 of Figure 4. In the case of reachability the lower estimate \( \text{PosApprx} \) is increased when \( l \in \gamma \). In the repeated reachability case, this is not sufficient; we need to ensure that there is no state \( \gamma' \) that is reachable from the current state \( \gamma \) and such that \( l \) is not reachable from \( \gamma' \). The existence of such a \( \gamma' \) implies existence of a non-zero measure continuation of the current run in which \( l \) is not reached infinitely often. Hence, the conditional of the if-statement is modified to: \( \forall \gamma' \in \text{BPlain}. (\gamma \rightarrow^* \gamma') \Rightarrow (\gamma' \rightarrow^* l) \).

We note that naively we would have to check the above condition for all configurations \( \gamma' \in \Gamma_p \), which is infeasible since \( \Gamma_p \) is an infinite set. We address this by using Lem. 6, which shows that runs from all configurations eventually reach a B-plain configuration. Hence it is sufficient to only check the condition for the (finitely many) B-plain configurations, which are precomputed in \( \text{BPlain} \).

**Theorem 4.** \( \text{QuantRepReach} \) is solvable.

### 8 Expected Average Costs

In this section, we develop a cost model for concurrent programs where we assign a cost to the execution of each instruction, the goal begin to approximate the expected cost of runs that reach a given label.
8.1 Computing costs over runs

A cost function \( \text{Cost} : \text{Lbl}_P \rightarrow \mathbb{N}^{>0} \) for program \( P \) defines for each label \( l \in \text{Lbl}_P \) the cost of executing the instruction at \( l \). A particular way to define the function is to assign a cost to each instruction in the programming language, so that \( \text{Cost}(l) \) depends only on \( \text{stmt}(l) \) and not on \( l \) itself. But we consider the general case. We extend \( \text{Cost} \) to runs as follows. Consider configurations \( \gamma = (\lambda, \mathcal{R}, \mathcal{B}, \mathcal{M}) \) and \( \gamma' \) such that \( \gamma \rightarrow_P \gamma' \). If \( \gamma \not\rightarrow_P \gamma' \), for process \( p \), then we define \( \text{Cost}(\gamma, \gamma') := \text{Cost}(\lambda(p)) \). In other words, it is the cost of the instruction executed by \( p \). Recall from Sec. 4 that \( p \) is unique and therefore the function is well-defined. If \( \text{disab}(\gamma) \) or if \( \neg(\gamma \rightarrow_P \gamma') \) then we define \( \text{Cost}(\gamma, \gamma') := 0 \). Consider a run \( \rho \in \{\text{Runs}(\gamma) \mid \rho \models_P \Diamond^i l\} \), i.e. a \( \gamma \)-run that reaches \( l \) for the first time at step \( i \). We define \( \text{Cost}(\rho)(l) = \sum_{1 \leq j \leq |\rho| - 1} \text{Cost}(\rho[j], \rho[j + 1]) \), i.e., the sum of costs of all executed instructions along \( \rho \) up to the first visit to \( l \).

For a configuration \( \gamma \), a label \( l \), and a cost function \( \text{Cost} \), we define a random variable \( X_{\gamma,l,\text{Cost}} : \Omega \rightarrow \mathbb{R} \) over support \( \Omega = \gamma \cdot P^\omega \) as follows:

\[
X_{\gamma,l,\text{Cost}}(\rho) = \begin{cases} 
0 & \rho \not\in \{\text{Runs}(\gamma) \mid \rho \models_P \Diamond^i l\} \\
X_{\gamma,l,\text{Cost}}(\rho)(l) & \text{otherwise}
\end{cases}
\]

**Given:** program \( P \), configuration \( \gamma_{\text{init}} \in P^\text{plain} \), cost function \( \text{Cost} : \text{Lbl}_P \rightarrow \mathbb{N}^{>0} \), label \( l \in \text{Lbl}_P \) s.t. \( \gamma_{\text{init}} \models \emptyset l \), precision value \( \epsilon \in \mathbb{R}^+ \)

**Exp_Ave_Cost:** Determine \( \theta \) s.t. \( E(X_{\gamma_{\text{init}},l,\text{Cost}} \mid \gamma_{\text{init}} \models \exists \emptyset l) \in [\theta, \theta + \epsilon] \)

The expected average cost problem \( E(X_{\gamma,l,\text{Cost}}) \) is defined as the expected cost of reaching \( l \) from \( \gamma \) and \( E(X_{\gamma,l,\text{Cost}} \mid \gamma \models \exists \emptyset l) \) as the conditional expectation over runs that reach \( l \). If \( \neg(\gamma \models_P \exists \emptyset l) \) then the expected cost is not defined. If however \( \gamma \models_P \exists \emptyset l \) then \( E(X_{\gamma,l,\text{Cost}} \mid \gamma \models \exists \emptyset l) = E(X_{\gamma,l,\text{Cost}})/\text{Prob}_P(\gamma \models_P \Diamond \emptyset l) \), which follows since for the non-reaching runs, the cost is zero. We present the expected average cost problem, in the figure above, where we want to approximate \( E(X_{\gamma,l,\text{Cost}} \mid \gamma \models \exists \emptyset l) \) to \( \epsilon \)-precision.

8.2 Eagerness

Our solution to Exp_Ave_Cost relies on the fact that \([P]^{KC}\) satisfies an eagerness property in the sense of [17]. In our setting, eagerness means that the probability of avoiding the target label \( l \) decreases exponentially with the number of steps. Concretely, we show that there are two constants: the eagerness degree \( \mathcal{E}_P \in \mathbb{R}^{>0} \), and the eagerness threshold \( \eta_P \in \mathbb{R}^{>0} \) satisfying the following:

\[
\forall \gamma \in P^\text{small} \forall l \in \text{Lbl}_P \\forall n \geq \eta_P \quad \gamma \models_P \exists \emptyset l \Rightarrow \text{Prob}_P(\gamma \models_P \Diamond^n \emptyset l) \leq (\mathcal{E}_P)^n
\]

i.e. for \( n \geq \eta_P \), the probability of avoiding \( l \) during the first \( n \) steps decreases exponentially with \( n \). The following lemma forms the crux of this section.

**Lemma 7 (Eagerness Lemma)** \( \mathcal{E}_P \) and \( \eta_P \) exist and are computable.
We devote this sub-section to give an overview of the proof of Lemma 7 (the formal proof is provided in the supplementary material). We consider the behavior of runs with respect to the small and large configurations, exploiting the fact that the runs of the system tend to gravitate towards the small configurations. However here we use a property, called left-biasedness (defined in Sec. 8.2), that is stronger than the left-orientedness property of Sec. 5.1.

To prove Lemma 7, we show that, for a small configuration \( \gamma \in \Gamma^\text{small} \), the runs from \( \gamma \) satisfy the following three properties with a high probability: (i) they make their first return to \( \Gamma^\text{small} \) within a small number of steps, (ii) they return to \( \Gamma^\text{small} \) multiple times, within a small number of steps, and (iii) if they eventually reach 1 then they will do that within a few steps. We collect these results to obtain the proof of Lemma 7.

**Gravity: First Return** We recall that buffer sizes can increase by at most one during process transitions, and that any number of messages can be flushed to the memory during an update transition (Sec. 4 and Sec. 5.1). Based on this, we show left-biasedness, defined as follows:

\[
\text{Left-biasedness } \forall \gamma \in \Gamma^\text{large} \text{ the probability of moving from } \gamma \text{ to a smaller configuration is bounded below by } 2/3 \text{ and that of moving to a larger configuration is bounded above by } 1/3, \text{ regardless of } \mathcal{P}.
\]

Using left-biasedness, we show that the set \( \Gamma^\text{small} \) has a gravity property, namely, a run starting from a small configuration will, with a high probability, return to the set \( \Gamma^\text{small} \) (for the first time) within a few number of steps. Formally, we define the gravity parameter \( G_\mathcal{P} \) as follows: \( q := 2/3, p := 1/3 \), and \( G_\mathcal{P} := 2\sqrt{q \cdot p} = 2\sqrt{2/3} \). We prove the following lemma.

**Lemma 8 (Gravity Lemma)** \( \text{Prob}_\mathcal{P} (\gamma =: \mathcal{P} \triangleleft \otimes^n \Gamma^\text{small}) \leq (G_\mathcal{P})^n \), for all \( \gamma \in \Gamma^\text{small} \) and all \( n \in \mathbb{N} \).

The lemma states that, starting from a small configuration, the probability that a run avoids \( \Gamma^\text{small} \) in the next \( n \) steps decreases exponentially with \( n \).

**Multiple Revisits** Notice that the gravity lemma is concerned with the first return to the set of small configurations. We will now apply this argument repeatedly to conclude that, with high probability, multiple re-visits to small configurations take place “quickly”. That is, the set of runs starting from \( \Gamma^\text{small} \) and frequently re-visiting \( \Gamma^\text{small} \) has a high measure. To formalize these arguments, we make the following definition. For \( m, n : 1 \leq m \leq n \), we define \( \text{Visit}_\mathcal{P} (n, m) \) to be the set of runs that visit the set \( \Gamma^\text{small} \) exactly \( m \) times in their first \( n \) steps. We use the \( \text{Visit} \) predicate to partition the set of \( \gamma \)-runs, depending on how often they return to \( \Gamma^\text{small} \) during their first \( n \) steps. We distinguish these as

\[10\] For technical convenience, we use \( n - 1 \) instead of \( n \) in the definition of \( \text{Visit} \). This allows us to avoid some corner cases in the proofs.
Sporadic-Runs (S-Runs): runs that visit the $\Gamma_p^{\text{small}}$ sporadically during their first $n$ steps, and Frequent-Runs (F-Runs): runs that visit $\Gamma_p^{\text{small}}$ frequently during their first $n$ steps. We will derive a constant $\nu \in \mathbb{N}$ (see below) that delineates the border between these sets. We formally define:

\[
\text{SRuns}(\gamma)(n) := \bigcup_{1 \leq m \leq \left\lfloor \frac{n}{\nu} \right\rfloor} \{ \rho \in \text{Runs}(\gamma) \mid \rho \models \text{Visit}_P(n, m) \}
\]

\[
\text{FRuns}(\gamma)(n) := \bigcup_{\left\lfloor \frac{n}{\nu} \right\rfloor + 1 \leq m \leq n} \{ \rho \in \text{Runs}(\gamma) \mid \rho \models \text{Visit}_P(n, m) \}
\]

**Fig. 5.** Figure depicting configuration sequences of S, F and D runs. Green dots represent small configurations, blue dots represent large configurations. All runs start in a small (plain) configuration. Within the first $n$ configurations: the S-run visits $\Gamma_p^{\text{small}}$ at most $\left\lfloor \frac{n}{\nu} \right\rfloor$ times, the F, D runs visit $\Gamma_p^{\text{small}}$ at least $\left\lfloor \frac{n}{\nu} \right\rfloor + 1$ times. A D-run is a special case of an F-run which does not visit label $l$ (red dot) in the first $n$ steps.

The value of $n/\nu$ distinguishes the S-Runs from the F-Runs. Our goal is to give an upper bound on the measure of the S-Runs. For a prefix path $\pi$ of length $n$, there are $(n-1)^{m-1}$ ways to choose the $m-1$ indices along $\pi$ at which $\Gamma_p^{\text{small}}$ is reached (since the run starts from $\Gamma_p^{\text{small}}$). Each of the $m-1$ path fragments between these indices represents one consecutive revisit of $\Gamma_p^{\text{small}}$. By Lemma 8, the measure of the set of such runs is bounded by $(G^P)^{n-m} = \left( \frac{2\sqrt{2}}{3} \right)^{n-m}$, giving

\[
\text{Prob}_P(\text{SRuns}(\gamma)(n)) \leq \sum_{m=1}^\left\lfloor \frac{n}{\nu} \right\rfloor \cdot (n-1)^{m-1} \cdot G^P_{n-m} \leq \left( \frac{\sqrt{8}}{3} \cdot \left( \frac{\nu}{\nu-1} \right) \cdot (2 + \sqrt{3} \cdot \nu)^{\left\lfloor \frac{n}{\nu} \right\rfloor} \right)^n
\]

under the condition that $4 \leq 2 \cdot \nu \leq n$. The second inequality is obtained through algebraic manipulations using $G^P = \frac{2\sqrt{2}}{3}$. Define $f(x) := \sqrt{\frac{8}{3}} \cdot \left( \frac{x}{x-1} \right) \cdot (2 + \sqrt{3} \cdot x)^{\left\lfloor \frac{x}{\nu} \right\rfloor}$. We have $f(150) = 0.986 < 1$. Hence, for parameter $\nu := 150$, defining $E^S_P := f(\nu)$, we have the following lemma, where the bound decays exponentially with $n$ since $E^S_P < 1$.

**Lemma 9 (S-Run Bound)** $\text{Prob}_P(\gamma \models P \text{SRuns}(\gamma)(n)) \leq (E^S_P)^n$, for all $\gamma \in \Gamma_p^{\text{small}}$ and all $n$ such that $300 = 2 \cdot \nu \leq n$.

**Reaching the label l** We now turn our attention to the set of F-Runs. Our goal is to show that if an F-Run reaches $l$ then, with a high probability, it will reach $l$ “quickly”. To that end, we consider the opposite scenario and introduce a subset of the F-Runs which we call Delayed Runs (D-Runs):

\[
\text{DRuns}(\gamma)(l)(n) := \bigcup_{m=\left\lfloor \frac{n}{\nu} \right\rfloor}^n \{ \rho \in \text{Runs}(\gamma) \mid \rho \models \text{Visit}_P(n, m) \}
\]
A D-Run is an F-Run that delays its first visit to the label $l$ until the $n^{th}$ step for some $n$. We show that the measure of D-Runs decreases $n$ increases. Note that $l$ is reachable from all configurations from a path that ends at $l$. Therefore, we consider the set $\mathcal{A} := \{ \gamma \in \Gamma_P^{\text{small}} \mid \gamma \models_P \exists l\}$, of small configurations from which $l$ is reachable. We analyze how often a run starting from a small configuration, visits $\mathcal{A}$ before finally visiting the label $l$. For sets of configurations $G_1, G_2 \subseteq \Gamma_P$, a run $\rho$, and $m \in \mathbb{N}$, we write $\rho \models G_1 \text{Before}^m G_2$ to denote that $\rho$ visits the set $G_1$ at least $m$ times before visiting $G_2$ for the first time. Notice

$$\text{DRuns}(\gamma)(l)(n) \subseteq \bigcup_{m=[\frac{n}{2}]+1}^{n} \{ \rho \in \text{Runs}(\gamma) \mid \rho \models_P \mathcal{A} \text{Before}^m l \}$$

(2)

To upper bound the measure of D-Runs, we start by upper bounding the measure of the set $\{ \rho \in \text{Runs}(\gamma) \mid \rho \models_P \mathcal{A} \text{Before}^m l \}$, i.e., $\gamma$-runs making $m$ visits to $\mathcal{A}$ before visiting $l$. We consider the probability that a run from a small configuration $\gamma$ does visit $l$ before returning to $\gamma$. We can compute a $\mu$ such that

$$0 < \mu \leq \min_{\gamma \in \mathcal{A}} \text{Prob}_P(\gamma \models \bigcirc(l \text{Before}^1 \gamma))$$

(3)

Hence $\mu$ is a lower bound on the measure of runs that start from some configuration in $\gamma \in \mathcal{A}$ and visit $l$ before returning to $\gamma$. To obtain an upper bound on the measure of D-Runs, we show the following inequality:

$$\text{Prob}_P(\text{DRuns}(\gamma)(l)(n)) \leq \sum_{m=[\frac{n}{2}]+1}^{n} \sum_{\gamma \in \mathcal{A}} (1-\mu)^{\left\lfloor \frac{|\mathcal{A}|}{|\mu|} \right\rfloor -1} \leq \frac{|\mathcal{A}|}{(1-\mu)(1-(1-\mu)^{|\mathcal{A}|})} \cdot \left(1-\mu\right)^{-\left|\frac{|\mathcal{A}|}{|\mu|}\right|}$$

The first inequality follows from formulas 2 and 3 while the second is obtained through algebraic techniques. Define $\mathcal{E}_P^\mu$ such that $(1-\mu)\frac{1}{|\mu|} < \mathcal{E}_P^\mu < 1$. Such an $\mathcal{E}_P^\mu$ is computable since $\nu$, $\mathcal{A}$, $\mu$ are computable. Since $(1-\mu)\frac{1}{|\mu|} < \mathcal{E}_P^\mu$, it follows that there is a natural number, denoted by $\eta_P^0$, such that

$$\left(1-\mu\right)^{-\left|\frac{|\mathcal{A}|}{|\mu|}\right|} \leq \left(\mathcal{E}_P^\mu\right)^n$$

for all $n \geq \eta_P^0$. This gives the following lemma.

**Lemma 10 (D-Run Bound)** $\text{Prob}_P(\text{DRuns}(\gamma)(l)(n)) \leq \left(\mathcal{E}_P^\mu\right)^n$, for all $\gamma \in \Gamma_P^{\text{small}}$ and all $n \geq \eta_P^0$.

**Proof of Lemma 7** We now give a sketch of the proof of the eagerness property.

Choose a value $\mathcal{E}_P^{SD}$ such that, $\max(\mathcal{E}_P, \mathcal{E}_P^{SD}) < \mathcal{E}_P^{SD} < 1$. From Lemma 9 and Lemma 10 it follows that for some constant $\eta_P^{SD} > \max(\eta_P, 300)$, $\text{Prob}_P(\gamma \models \bigcirc^n l) \leq (\mathcal{E}_P^{SD})^n$, for all $n > \eta_P^{SD}$ (sufficiently large). The final step is to extend the argument to the set of $\gamma$-runs that reach $l$ in $n$ or more steps (as required by Lemma 7).

$$\text{Prob}_P(\gamma \models \bigcirc^n l) = \sum_{k=n}^{\infty} \text{Prob}_P(\gamma \models \bigcirc^n l) \leq \sum_{k=n}^{\infty} (\mathcal{E}_P^{SD})^k = \frac{(\mathcal{E}_P^{SD})^n}{1-\mathcal{E}_P^{SD}}$$

Choose $\mathcal{E}_P$ (exists since $\mathcal{E}_P^{SD} < \mathcal{E}_P < 1$) such that $\mathcal{E}_P^{SD} < \mathcal{E}_P < 1$. There exists an $\eta_P$ such that $\frac{(\mathcal{E}_P^{SD})^n}{1-\mathcal{E}_P^{SD}} \leq (\mathcal{E}_P)^n$ for all $n \geq \eta_P$, and hence $\text{Prob}_P(\gamma \models \bigcirc^n l) \geq (\mathcal{E}_P)^n$ for all $n \geq \eta_P$ (sufficiently large). This gives us the result.
8.3 The Algorithm

Now we proceed to describe the algorithm. The goal is to approximate
\( E (X_{\gamma_{\text{init}}, \text{Cost}} \mid \gamma_{\text{init}} \models \exists \emptyset l) \). The scheme followed by the algorithm is similar to the
quantitative section: it iteratively improves an approximations until it is \( \varepsilon \)-precise. However, the implementation is much more challenging since we need to
maintain error margins on both the cost and the probabilities. It performs forward reachability
analysis, starting from \( \gamma_{\text{init}} \), and generating successively
longer \( \gamma_{\text{init}} \)-paths, in a breadth-first manner.

The variable \textbf{waiting} contains triples of form \( \langle \gamma, \psi, \phi \rangle \) corresponding to \( \gamma_{\text{init}} \)-paths waiting to be analysed. For such a path \( \pi \), \( \gamma \) is the last configuration of \( \pi \), \( \psi \) is the cost of \( \pi \), and \( \phi \) is the probability of taking \( \pi \). We initialize \textbf{waiting} to
contain a triple corresponding to the empty path from \( \gamma_{\text{init}} \): \( \langle \gamma_{\text{init}}, 0, 1 \rangle \). Prior to the \( i \)-th iteration loop (line 10), \textbf{waiting} contains triples corresponding to paths
of length \( i \). At each loop iteration the triples in \textbf{waiting} are analysed and the
triples for paths one step deeper are generated for the next iteration.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} \( P \): program; \( \gamma_{\text{init}} \in T_P \): configuration \( l \in \text{Lbl}_P \): label with \( \gamma_{\text{init}} \models \exists \emptyset l \); \( \text{Cost} : \\text{Instr}_P \rightarrow \mathbb{R} \): cost function; \( \varepsilon \in \mathbb{R}^{>0} \): precision;
\State \textbf{Var:} \( \text{waiting}, \text{waiting}' \) \in \((\mathbb{I}_P \times \mathbb{R} \times \mathbb{R})^* \): queues;
\State \( \text{CostApprx} \in \mathbb{R} \): approximation of \( E (X_{\gamma_{\text{init}}, \text{Cost}}) \);
\State \( \text{ProbApprx} \in \mathbb{R} \): under-approximation of \( \text{Prob}_{\text{Apprx}} (\gamma \models P \emptyset l) \);
\State \( \text{CostError} \in \mathbb{R} \), \( \text{ProbError} \in \mathbb{R} \): over-approximations of errors;
\State \( k, n \in \mathbb{N} \);
\State \( k := \text{MaxCost}(\text{Cost}); n := 0; \)
\State \( \text{CostApprx} := 0; \text{ProbApprx} := 0; \text{waiting} := \langle \gamma_{\text{init}}, 0, 1 \rangle; \)
\State \( \text{CostError} := \frac{1}{(1 - \varepsilon)^2}; \text{ProbError} := \frac{1}{1 - \varepsilon}; \)
\Repeat
\State \( n := n + 1; \text{waiting}' := \emptyset; \)
\For {i = 1 to \text{|waiting|}}
\State \( \langle \gamma, \psi, \phi \rangle := \text{waiting}[i]; \)
\If {i \in \gamma}
\State \( \text{CostApprx} := \text{CostApprx} + \psi \cdot \phi; \text{ProbApprx} := \text{ProbApprx} + \phi; \)
\Else
\State \( \text{for all } \gamma' : \gamma \rightarrow P \gamma' \text{ do} \)
\State \( \text{waiting}' := \text{waiting}' \cdot \langle \gamma', \psi + \text{Cost}(\gamma, \gamma'), \phi \cdot \text{Prob}(\gamma, \gamma') \rangle; \)
\State \( \text{CostError} := \text{CostError} \cdot \mathcal{E}_P; \text{ProbError} := \text{ProbError} \cdot \mathcal{E}_P; \)
\State \( \text{waiting} := \text{waiting}'; \)
\EndFor
\Until {\left( \frac{\text{CostApprx} + \text{CostError}}{\text{ProbApprx} + \text{ProbError}} \right) < \varepsilon \land (\text{CostError} > 0) \land (n \geq n_P);}
\State \textbf{return} \( \frac{\text{CostApprx}}{\text{ProbApprx}} \)
\Endalgorithmic
\end{algorithm}

\textbf{Fig. 6.} The expected average cost algorithm.

The iterations calculate increasingly precise approximations of
\( E (X_{\gamma_{\text{init}}, \text{Cost}}) \), and of \( \text{Prob}_{\text{P}} (\gamma_{\text{init}} \models P \emptyset l) \), maintained in variables \text{CostApprx}
and \text{ProbApprx}, respectively. We maintain two additional variables (\text{CostError}
and \text{ProbError}) that help us to provide an upper bound on the estimation
errors. Defining \text{MaxCost}(\text{Cost}) := \max \{ \text{Cost} (l) \mid l \in \text{Lbl}_P \} , we explain the
correctness of the algorithm with a number of invariants.
Lemma 11. The algorithm maintains the following invariants where invariants 1, 2, 3, 6 hold for all $i > 0$ and invariants 3, 4 hold for all $i \geq \eta_P$.

1. $\text{CostApprx}^{(i)} = \sum_{\{\rho \in \text{Runs}(\gamma) \mid \rho \models \Diamond^i \}} \text{Cost}(\rho) \cdot \text{ProbP}(\rho)$.
2. $\text{ProbApprx}^{(i)} = \text{ProbP}(\gamma_{\text{init}} \models \Diamond^i)$.
3. $\text{CostApprx}^{(i)} \leq E(X_{\gamma,1,\text{Cost}}) \leq \text{CostApprx}^{(i)} + \text{CostError}^{(i)}$.
4. $\text{ProbApprx}^{(i)} \leq \text{ProbP}(\gamma \models \Diamond l) \leq \text{ProbApprx}^{(i)} + \text{ProbError}^{(i)}$.
5. $\text{CostError}^{(i)} = \text{MaxCost}(\text{Cost}) \cdot \frac{\epsilon^i_P}{(1-\epsilon_P)^2}$.
6. $\text{ProbError}^{(i)} = \frac{\epsilon^i_P}{1-\epsilon_P}$.

Invariants 5 and 6 imply that as $i \to \infty$ $\text{CostError}^{(i)}$ and $\text{ProbError}^{(i)}$ tend to 0. Hence, $\lim_{i \to \infty} \left( \frac{\text{CostApprx}^{(i)} + \text{CostError}^{(i)}}{\text{ProbApprx}^{(i)}} - \frac{\text{CostApprx}^{(i)} - \text{CostError}^{(i)}}{\text{ProbApprx}^{(i)} + \text{ProbError}^{(i)}} \right) = 0$ implying termination. Since $n \geq \eta_P$ when the algorithm terminates, by invariants 3 and 4 it follows that $\text{CostApprx}^{(n)} \leq E(X_{\gamma,1,\text{Cost}}) \leq \text{CostApprx}^{(n)} + \text{CostError}^{(n)}$ and $\text{ProbApprx}^{(n)} \leq \text{ProbP}(\gamma \models \Diamond l) \leq \text{ProbApprx}^{(n)} + \text{ProbError}^{(n)}$. Combining these two inequalities and the termination condition of the algorithm, we get the following:

$$\frac{\text{CostApprx}^{(n)}}{\text{ProbApprx}^{(n)} + \text{ProbError}^{(n)}} \leq \frac{E(X_{\gamma,1,\text{Cost}})}{\text{ProbP}(\gamma \models \Diamond l)} < \frac{\text{CostApprx}^{(n)}}{\text{ProbApprx}^{(n)} + \text{ProbError}^{(n)}} + \epsilon$$

Hence on termination, $\theta := \frac{\text{CostApprx}^{(n)}}{\text{ProbApprx}^{(n)} + \text{ProbError}^{(n)}}$ is within $\epsilon$-precision of the true value, implying correctness of the algorithm. We get the following theorem.

Theorem 5. The above algorithm solves $\text{Exp\_Ave\_Cost}$. 

Related Work. Only recently there has been an increased interest in the formulation and verification of liveness properties for weak memory models. In [31], they factor the system into a process and memory subsystems and define notions of fairness for either. This is reminiscent of our approach, where we consider probabilistic policies for process scheduling and memory update. Their model on the other hand is non-probabilistic and they have weaker fairness guarantees, which we describe in more detail in Sec. 5.1. The liveness verification problem for TSO has been considered in [30], where they show undecidability for various liveness properties. However, once again work with non-probabilistic notions of fairness. We show in this paper, that with stronger (probabilistic) fairness, reachability and repeated reachability problems become decidable.

In [12], they show the undecidability of the repeated reachability problem, without fairness conditions, for finite-state programs running under the TSO semantics. In contrast, we show that checking repeated reachability qualitatively is decidable (Sec. 6.2), and that we can even compute the measure of runs satisfying the property with arbitrary precision (Sec. 7.2).

There has been a huge amount of work on the verification of finite-state Markov chains (see, e.g., [20,32}). Since the buffers in TSO are unbounded, we
however, get an infinite-state Markov chain. There is also a substantial literature on the verification of infinite-state Markov chains, where specialized techniques are developed for particular classes of systems. Several works have considered probabilistic push-down automata and probabilistic recursive machines \[33,34,35\]. However, these techniques don’t apply in our case since push-down automata cannot encode the FIFO store-buffer data-structure.

Works such as \[36,16,37,38\] develop algorithmic and complexity results for checking termination and reachability for systems such as probabilistic VASS, probabilistic Petri nets, probabilistic multi-counter systems. Again, these models are different from ours and cannot encode FIFO queues.

The works closest to ours are those on probabilistic lossy channel systems \[16,17\]. These works also rely on the frameworks of decisive and eager Markov chains. However, lossy channel systems and TSO are fundamentally different, and the manner in which we instantiate the frameworks of decisive/eager Markov chains differs. The decidability of verification for probabilistic extensions of lossy channels is sensitive to the definition of the message losses. In the case of lossy channel systems, if messages are only allowed to be lost at one end of the channel (a model that is close to our notion of message updates), then all non-trivial verification problems become undecidable for probabilistic lossy channel systems \[39\]. Therefore, although there is a reduction from TSO to lossy channel systems in the case of non-probabilistic models \[12\], we know of no such reduction between the corresponding probabilistic models.

Finally, the concept of decisiveness has been extended to more general models such as generalized semi-Markov processes, stochastic timed automata \[40\], and lossy channel-based stochastic games \[41\].

9 Conclusions, Discussions, and Perspectives

We presented \(\text{PTSO}\), a probabilistic extension of the classical TSO semantics. We have shown decidability/computability results for a wide a range of properties such as quantitative and qualitative reachability/repeated reachability and expected average costs. As far as we know, this is the first study of probabilistic verification for weak memory models, and opens many avenues for future work.

\textit{Refined Probability Distributions.} For ease of presentation, we developed our results in the context of specific scheduling and update policies. However, we emphasize that our results carry-over to policies satisfying faithfulness and left-orientedness, which are fairly weak conditions. Hence we believe that developing more refined models that better capture behaviours of TSO implementations, using techniques such as parameter estimation, is interesting future work.

\textit{General Cost Models} Similar can be said for cost models: our algorithm works for all cost functions such that the cost of a path is exponentially bounded by its length. In particular, developing cost models that closely mimic usage of processor resources, e.g. cost based on read from local store-buffer vs. read from memory, can be useful to gain a better understanding of the implementation.
Other Memory Models Finally, we are interested in extending our approach to other weak memory models such as RA/SRA, POWER, ARM.

References

1. L. Lamport. How to make a multiprocessor that correctly executes multiprocess programs. *IEEE Trans. on Computers*, C-28:690–691, 1979.
2. Sarita V. Adve and Kourosh Gharachorloo. Shared memory consistency models: A tutorial. *IEEE Computer*, 29(12):66–76, 1996.
3. Nissim Francez. *Fairness*. Texts and Monographs in Computer Science. Springer, 1986.
4. Zohar Manna and Amir Pnueli. *The temporal logic of reactive and concurrent systems - specification*. Springer, 1992.
5. Peter Sewell, Susmit Sarkar, Scott Owens, Francesco Zappa Nardelli, and Magnus O. Myreen. x86-tso: a rigorous and usable programmer’s model for x86 multiprocessors. *Commun. ACM*, 53(7):89–97, 2010.
6. M.Z. Kwiatkowska. Survey of fairness notions. *Information and Software Technology*, 31(7):371–386, 1989.
7. Alberto Ros and Stefanos Kaxiras. Racer: TSO consistency via race detection. In 49th Annual IEEE/ACM International Symposium on Microarchitecture, MICRO 2016, Taipei, Taiwan, October 15-19, 2016, pages 33:1–33:13. IEEE Computer Society, 2016.
8. Marco Elver and Vijay Nagarajan. TSO-CC: consistency directed cache coherence for TSO. In *HPCA 2014*, pages 165–176. IEEE, 2014.
9. Jade Alglave, Luc Maranget, Susmit Sarkar, and Peter Sewell. Litmus: Running tests against hardware. In Parosh Aziz Abdulla and K. Rustan M. Leino, editors, *Tools and Algorithms for the Construction and Analysis of Systems - 17th International Conference, TACAS 2011, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2011, Saarbrücken, Germany, March 26-April 3, 2011. Proceedings*, volume 6605 of *Lecture Notes in Computer Science*, pages 41–44. Springer, 2011.
10. Changhui Lin, Vijay Nagarajan, and Rajiv Gupta. Efficient sequential consistency using conditional fences. In Valentina Salapura, Michael Gschwind, and Jens Knoop, editors, *19th International Conference on Parallel Architectures and Compilation Techniques, PACT 2010, Vienna, Austria, September 11-15, 2010*, pages 295–306. ACM, 2010.
11. Luca de Alfaro. From fairness to chance. *Electron. Notes Theor. Comput. Sci.*, 22:55–87, 1999.
12. Mohamed Faouzi Atig, Ahmed Bouajjani, Sebastian Burckhardt, and Madanlal Musuvathi. On the verification problem for weak memory models. In Manuel V. Hermenegildo and Jens Palsberg, editors, *Proceedings of the 37th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2010, Madrid, Spain, January 17-23, 2010*, pages 7–18. ACM, 2010.
13. Parosh Aziz Abdulla, Mohamed Faouzi Atig, Ahmed Bouajjani, and Tuan Phong Ngo. A load-buffer semantics for total store ordering. *Logical Methods in Computer Science*, 14(1), 2018.
14. W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 1 of *Texts in Statistical Science*. John Wiley, 3rd edition, 1968.
15. V. G. Kulkarni. *Modeling and Analysis of Stochastic Systems*. Texts in Statistical Science. CRC Press, 2nd edition, 2009.

16. Parosh Aziz Abdulla, Noomene Ben Henda, and Richard Mayr. Decisive markov chains. *LMCS*, 3(4), 2007.

17. Parosh Aziz Abdulla, Noomene Ben Henda, Richard Mayr, and Sven Sandberg. Eager markov chains. In Susanne Graf and Wenhui Zhang, editors, *Automated Technology for Verification and Analysis, 4th International Symposium, ATVA 2006, Beijing, China, October 23-26, 2006.*, volume 4218 of Lecture Notes in Computer Science, pages 24–38. Springer, 2006.

18. Pante Stănică. Good lower and upper bounds on binomial coefficients. *Journal of Inequalities in Pure and Applied Mathematics*, 2(3), 2001.

19. Mohamed Faouzi Atig, Ahmed Bouajjani, Sebastian Burckhardt, and Madanlal Musuvathi. What’s decidable about weak memory models? In Helmut Seidl, editor, *Programming Languages and Systems - 21st European Symposium on Programming, ESOP 2012, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2012, Tallinn, Estonia, March 24 - April 1, 2012. Proceedings*, volume 7211 of Lecture Notes in Computer Science, pages 26–46. Springer, 2012.

20. Christel Baier and Joost-Pieter Katoen. *Principles of Model Checking (Representation and Mind Series)*. The MIT Press, 2008.

21. Carl G. Ritchey and Scott Owens. Benchmarking weak memory models. In *Proceedings of the 21st ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming, PPoPP ’16, New York, NY, USA, 2016*. Association for Computing Machinery.

22. Changhui Lin, Vijay Nagarajan, and Rajiv Gupta. Fence scoping. In *SC ’14: Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis*, pages 105–116, 2014.

23. Yuelu Duan, Abdullah Muzahid, and Josep Torrellas. Weefence: Toward making fences free in tso. In *Proceedings of the 40th Annual International Symposium on Computer Architecture, ISCA ’13*, page 213–224, New York, NY, USA, 2013. Association for Computing Machinery.

24. Changhui Lin, Vijay Nagarajan, and Rajiv Gupta. Efficient sequential consistency using conditional fences. In *Proceedings of the 19th International Conference on Parallel Architectures and Compilation Techniques, PACT ’10*, page 295–306, New York, NY, USA, 2010. Association for Computing Machinery.

25. E.M. Clarke, O. Grumberg, and D. Peled. *Model Checking*. MIT Press, Dec. 1999.

26. J.G. Kemeny, J.L. Snell, and A.W. Knapp. *Denumerable Markov Chains*. D Van Nostad Co., 1966.

27. M.Y. Vardi. Automatic verification of probabilistic concurrent finite-state programs. In *FOCS85*, pages 327–338, 1985.

28. Scott Owens, Susmit Sarkar, and Peter Sewell. A better x86 memory model: x86-tso. In Stefan Berghofer, Tobias Nipkow, Christian Urban, and Makarius Wenzel, editors, *Theorem Proving in Higher Order Logics, 22nd International Conference, TPHOLs 2009, Munich, Germany, August 17-20, 2009. Proceedings*, volume 5764 of Lecture Notes in Computer Science, pages 391–407. Springer, 2009.

29. Christel Baier, Nathalie Bertrand, and Philippe Schnoebelen. A note on the attractor-property of infinite-state markov chain. *Inf. Process. Lett.*, 97(2):58–63, January 2006.

30. Chao Wang, Gustavo Petri, Yi Lv, Teng Long, and Zhiming Liu. Decidability of liveness on the TSO memory model. *CoRR*, abs/2107.09930, 2021.
31. Ori Lahav, Egor Namakonov, Jonas Oberhauser, Anton Podkopaev, and Viktor Vafeiadis. Making weak memory models fair. *ArXiv*, abs/2012.01067, 2020.

32. Marta Z. Kwiatkowska, Gethin Norman, and David Parker. PRISM 4.0: Verification of probabilistic real-time systems. In Ganesh Gopalakrishnan and Shaz Qadeer, editors, *Computer Aided Verification - 23rd International Conference, CAV 2011, Snowbird, UT, USA, July 14-20, 2011. Proceedings*, volume 6806 of *Lecture Notes in Computer Science*, pages 585–591. Springer, 2011.

33. Kousha Etessami and Mihalis Yannakakis. Recursive markov decision processes and recursive stochastic games. *J. ACM*, 62(2):11:1–11:69, 2015.

34. Tomás Brázdil, Stefan Kiefer, Antonín Kucera, and Ivana Hutarová Vareková. Runtime analysis of probabilistic programs with unbounded recursion. *J. Comput. Syst. Sci.*, 81(1):288–310, 2015.

35. Javier Esparza, Antonín Kucera, and Richard Mayr. Model checking probabilistic pushdown automata. In *19th IEEE Symposium on Logic in Computer Science (LICS 2004)*, 14-17 July 2004, Turku, Finland, *Proceedings*, pages 12–21. IEEE Computer Society, 2004.

36. Tomás Brázdil, Krishnendu Chatterjee, Antonín Kucera, Petr Novotný, and Dominik Velan. Deciding fast termination for probabilistic VASS with nondeterminism. In Yu-Fang Chen, Chih-Hong Cheng, and Javier Esparza, editors, *Automated Technology for Verification and Analysis - 17th International Symposium, ATVA 2019, Taipei, Taiwan, October 28-31, 2019, Proceedings*, volume 11781 of *Lecture Notes in Computer Science*, pages 462–478. Springer, 2019.

37. Tomás Brázdil, Stefan Kiefer, Antonín Kucera, Petr Novotný, and Joost-Pieter Katoen. Zero-reachability in probabilistic multi-counter automata. In Thomas A. Henzinger and Dale Miller, editors, *Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS ’14, Vienna, Austria, July 14 - 18, 2014*, pages 22:1–22:10. ACM, 2014.

38. Tomás Brázdil, Stefan Kiefer, and Antonín Kucera. Efficient analysis of probabilistic programs with an unbounded counter. *J. ACM*, 61(6):41:1–41:35, 2014.

39. Parosh Aziz Abdulla, Christel Baier, S. Purushothaman Iyer, and Bengt Jonsson. Simulating perfect channels with probabilistic lossy channels. *Inf. Comput.*, 197(1-2):22–40, 2005.

40. Nathalie Bertrand, Patricia Bouyer, Thomas Brihaye, and Pierre Carlier. Analysing decisive stochastic processes. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, volume 55 of *LIPIcs*, pages 101:1–101:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

41. Parosh Aziz Abdulla, Noemene Ben Henda, Luca de Alfaro, Richard Mayr, and Sven Sandberg. Stochastic games with lossy channels. In Roberto M. Amadio, editor, *Foundations of Software Science and Computational Structures, 11th International Conference, FOSSACS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29 - April 6, 2008. Proceedings*, volume 4962 of *Lecture Notes in Computer Science*, pages 35–49. Springer, 2008.

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## A Glossary of Notation

In this section we provide a glossary of notation.

| Notation | Meaning | Reference |
|----------|---------|-----------|
| Transition Systems | γ and I | One and a set of configuration(s) | 2 |
| | →, ⊳, → | Single, multi and k-step reachability | 3 |
| Temporal Logic | ρ ⊬ T G | ρ reaches G at the kth step | 2 |
| | ρ ⊬ T G | ρ reaches G first at the kth step | 2 |
| | ρ ⊬ T G | ρ reaches G at the kth step (possibly before) | 2 |
| | →, → | Simple and k-step reachability | 2 |
| Concurrent Programs | P | A program | 3 |
| | p, Procs | A process, set of processes | 3 |
| | Regs_p, Regs_P | Registers of a process, registers of a program | 3 |
| | Lbl_p, Lbl_P | Labels of a process, registers of a program | 3 |
| Operational Semantics | γ = ⟨λ, R, B, M⟩ | Labelling, Register, Buffer, Memory | 4.1 |
| | | components of γ | 4.1 |
| | | Size of (buffers of) a configuration | 4.1 |
| | Ip | All configurations of P | 4.1 |
| | I_p^plain | Plain (empty buffer) configurations of P | 4.1 |
| | [P]_TS | Transition system of P | 4.2 |
| | [P]_MC | Markov chain for P | 4.3 |
| | Sched (p), Rweight (γ) (p) | Weight and relative weight for scheduling | 4.3 |
| Costs | Cost (l) | Cost of instruction at l | 3 |
| | Cost (γ, γ′) | Single step cost | 3.5 |
| | Cost (p) (l) | Cost of run ρ | 3.5 |
| | X_{γ,l,cost} (ρ) | Random variable for cost over runs | 3.2 |
| | E_P | Eagerness parameter for P | 3.4 |
| | η_P | Eagerness bound for P | 3.4 |
| | G_P | Gravity parameter for P | 3.4 |
We now prove Lemma 1.

**Lemma 1** \( M_P \) is a prob. distribution on \( \Gamma_P \); hence, \([P]^{MC}\) is a Markov chain.

**Proof.** From any configuration \( \gamma \), each transition consists of 2 parts: a process transition \( \gamma \xrightarrow{P_{\text{proc}}} \gamma_p \) followed by an update transition \( \gamma_p \xrightarrow{\text{update}} \gamma' \). Assuming that the configuration \( \gamma \) is enabled, the process transition can be done by any enabled process. This is followed by considering all possible update transitions. Consider an enabled process \( p \). Then \( \gamma \xrightarrow{P_{\text{proc}}} \gamma_p \) happens with probability \( \text{Rweight}(\gamma)(p) \). From \( \gamma_p \), we consider all sequences of processes which can result in an update. Let \( S_{\gamma_p, \gamma'} = \{ w \mid \gamma_p \xrightarrow{w} \gamma' \} \) be the set of sequences resulting in a fixed configuration \( \gamma' \) and let \( S_{\gamma_p} = \{ w \mid \exists \gamma' \gamma_p \xrightarrow{w} \gamma' \} \) be all possible sequences labelling an update from \( \gamma_p \). Then the probability of reaching configuration \( \gamma' \) from \( \gamma_p \) after an update is \( \frac{|S_{\gamma_p, \gamma'}|}{|S_{\gamma_p}|} \). Thus, the probability to reach some configuration from \( \gamma_p \) after an update is \( \sum_{\gamma'} \frac{|S_{\gamma_p, \gamma'}|}{|S_{\gamma_p}|} = 1 \), since \( \cup_{\gamma'} S_{\gamma_p, \gamma'} = S_{\gamma_p} \).

1. For an enabled configuration \( \gamma \), \( \sum_{p \in \text{enab}(\gamma)} \sum_{\gamma' \in \Gamma_P} M_{\text{proc}}(\gamma, \gamma_p) \cdot M_{\text{update}}(\gamma_p, \gamma') \), where \( \gamma_p \) is the configuration such that \( \gamma \xrightarrow{P_{\text{proc}}} \gamma_p \xrightarrow{\text{update}} \gamma' \) can be written as

\[
\sum_{p \in \text{enab}(\gamma)} \text{Rweight}(\gamma)(p) \sum_{\gamma'} \frac{|S_{\gamma_p, \gamma'}|}{|S_{\gamma_p}|} = \sum_{p \in \text{enab}(\gamma)} \frac{|S_{\gamma_p, \gamma'}|}{|S_{\gamma_p}|} \frac{\text{Sched}(p)}{\sum_{p' \in \text{enab}(\gamma)} \text{Sched}(p')} = 1
\]

2. For the case when \( \gamma \) is disabled, by definition, we have \( M_{\text{proc}}(\gamma, \gamma) := 1 \) and \( M_{\text{proc}}(\gamma, \gamma_p) := 0 \) for \( \gamma_p \neq \gamma \). Further, from \( \gamma \), we can consider all possible update transitions resulting in a configuration \( \gamma' \). This gives us \( \sum_{\gamma' \in \Gamma_P} M_{\text{proc}}(\gamma, \gamma) \cdot M_{\text{update}}(\gamma, \gamma') \) which can be seen to be

\[
1 \cdot \sum_{\gamma'} \frac{|S_{\gamma', \gamma'}|}{|S_{\gamma}|} = 1
\]

Thus, in all cases, we have shown that \( M_P \) is a probability distribution, and the induced transition system is a Markov Chain.

Recall that in section 4.2 we introduced the notion of a transition system \([P]^\text{TS} = \langle \Gamma_P, \to_P \rangle\) given a program \( P \). The Markov chain associated to the program \( P \) has been introduced in Section 4.3 as \([P]^{MC} = \langle \Gamma_P, M_P \rangle\). Now, we formally show that \([P]^\text{TS}\) is the same as the transition system \(([P]^{MC})^\downarrow\) induced by \([P]^{MC}\).

We begin with the proof for Lemma 2 which says that the transition system induced by the Markov chain is the same as the transition system induced by the program.

**Lemma 2** \(([P]^{MC})^\downarrow = [P]^\text{TS}\) for any program \( P \).
Proof. First we show that if $\langle \gamma, \gamma' \rangle \in \rightarrow_{\langle \mathcal{P} \rangle^{\text{mc}}}^+$, then $\langle \gamma, \gamma' \rangle \in \rightarrow_{\mathcal{P}^{\text{TS}}}$. Whenever $\langle \gamma, \gamma' \rangle \in \rightarrow_{\langle \mathcal{P} \rangle^{\text{mc}}}^+$, we have $M_\mathcal{P}(\gamma, \gamma') > 0$.

1. Consider the case when $\gamma$ is enabled. Then there exists some $\gamma''$ such that $M_{\text{proc}}(\gamma, \gamma'') > 0$ and $M_{\text{update}}(\gamma'', \gamma') > 0$. Since $M_{\text{proc}}(\gamma, \gamma'') > 0$, we have $\text{Rweight}(\gamma)(p) > 0$ for some $p$ which resulted in obtaining $\gamma''$ from $\gamma$. Hence, for process $p$ we have $\gamma' \xrightarrow{p_{\text{proc}}} \gamma''$. Similarly, there is a sequence $w$ such that $\gamma'' \xrightarrow{w_{\text{update}}} \gamma'$, i.e., $\gamma'' \xrightarrow{w_{\text{update}}} \gamma'$. Composing the two, we obtain $\gamma \xrightarrow{\mathcal{P}^{\text{TS}}} \gamma'$.

2. The second case is when $\gamma$ is not enabled. Then $M_{\text{proc}}(\gamma, \gamma) = 1$, by definition. An update transition can still be done from $\gamma$ (empty update if all buffers are empty). In any case, the resultant configuration $\gamma'$ after an update is such that $M_{\text{update}}(\gamma, \gamma') > 0$. Thus, as above, composing the two, we obtain $\gamma \xrightarrow{\mathcal{P}^{\text{TS}}} \gamma'$ in $\mathcal{P}^{\text{TS}}$.

Next, we show that if $\langle \gamma, \gamma' \rangle \in \rightarrow_{\langle \mathcal{P} \rangle^{\text{mc}}}^+$, then $M_{\mathcal{P}}(\gamma, \gamma') > 0$. As above, there are two cases depending on whether $\gamma$ is enabled or not.

1. Assume $\gamma$ is enabled. Then $\rightarrow_{\mathcal{P}^{\text{TS}}}$ is a composition of $\rightarrow_{\text{proc}}$ and $\rightarrow_{\text{update}}$. There exists some process $p$ and a sequence $w$ such that $\gamma' \xrightarrow{p_{\text{proc}}} \gamma'' \xrightarrow{w_{\text{update}}} \gamma'$. Hence $M_{\text{proc}}(\gamma, \gamma'') = \text{Rweight}(\gamma)(p) > 0$ and $M_{\text{update}}(\gamma'', \gamma') > 0$ since $\{w \mid \gamma'' \xrightarrow{w_{\text{update}}} \gamma'\} > 0$. Hence $M_{\mathcal{P}}(\gamma, \gamma') > 0$.

2. If $\gamma$ is not enabled, then all processes are disabled in $\gamma$. In this case, the only transition in $\mathcal{P}^{\text{TS}}$ is $\gamma \xrightarrow{\text{proc}} \gamma$, followed by an update transition leading to some $\gamma'$. Hence, $M_{\mathcal{P}}(\gamma, \gamma') > 0$.

Thus we have shown that there is a transition between a pair of configurations in $\mathcal{P}^{\text{TS}}$ iff there exists a transition between them in $\rightarrow_{\langle \mathcal{P} \rangle^{\text{mc}}}^+$.

Thanks to Lemma 2 whenever $\langle \mathcal{P} \rangle^{\text{mc}}$ has a transition with non zero probability, $\mathcal{P}^{\text{TS}}$ has the same transition. Thus, it suffices to check reachability in $\mathcal{P}^{\text{TS}}$.

The reachability between plain configurations (those which have all buffers empty) follows from [12]. We prove Lemma 3 by reducing reachability from a given configuration to a plain configuration to the reachability problem between two plain configurations. Likewise, label reachability is known to be decidable when starting from a plain configuration in classical TSO semantics [13]. We can prove Lemma 4 in a similar manner (as in Lemma 3) by reducing the label reachability from a given configuration to label reachability from a plain configuration, and then invoking [13].

Lemma 3 For a program $\mathcal{P}$, a configuration $\gamma \in \Gamma_{\mathcal{P}}$, and a plain configuration $\gamma' \in \Gamma_{\mathcal{P}}^{\text{plain}}$, it is decidable whether $\gamma \xrightarrow{\mathcal{P}} \gamma'$.

Proof. Given a program $\mathcal{P}$, and a configuration $\gamma = \langle \lambda, \mathcal{R}, \mathcal{B}, \mathcal{M} \rangle$, and a plain configuration $\gamma' = \langle \lambda', \mathcal{R}', \mathcal{B}', \mathcal{M}' \rangle$, can we reach $\gamma'$ from $\gamma$? Assume that there are $n$ processes $p_1, \ldots, p_n$, with shared variables $x_1, \ldots, x_m$. The bufferstate $\mathcal{B}$
consists of words \(w_1, w_2, \ldots, w_n\) where \(w_i\) is the buffer content of process \(i\) in \(\mathcal{B}\). Note that each \(w_i\) is a finite length word. Assume that \(|w_i| = k_i\), and \(w_i\) has the form \((x_{i1}, v_{i1}) \ldots (x_{ik_i}, v_{ik_i})\).

We modify \(\mathcal{P}\) to a program \(\mathcal{P}'\) by (i) modifying the code of \(p_1, \ldots, p_n\) and by adding a new process \(p_{n+1}\), (ii) we introduce two new local registers \(r, r'\) in \(p_1\), a new shared variable \(\text{flag}\), initializing all of them to 0. We reduce the reachability of \(\gamma\) to \(\gamma'\) in \(\mathcal{P}\) to the reachability between two plain configurations in \(\mathcal{P}'\).

1. Assume that \(\mathcal{M}\) is given by \(x_i \mapsto u_i\), for all \(1 \leq i \leq m\). We alter \(p_1\) by adding some instructions before all the existing instructions in \(p_1\).
   The new instructions added to \(p_1\) are the following. We begin with a while loop which checks \(\text{flag}\) is 0. Inside the loop, we have the following. For \(1 \leq i \leq m\), we add the instructions \(r' = u_i; CAS(x_i, r, r')\). When we finish executing all \(m\) of these instructions, this results in the memory state as given by \(\mathcal{M}\). This is followed by writing \(v_{11}\) to \(x_{11}\), \ldots, \(v_{1k_1}\) to \(x_{1k_1}\), obtaining the buffer content of \(p_1\) in \(\mathcal{B}\) and also writing the appropriate values to all registers of \(p_1\) as in \(\mathcal{R}\). This is followed by setting \(\text{flag}\) to 1. The while loop is broken at this point. The next instruction checks if \(\text{flag}\) is \(n + 1\), and if so, goes to the label \(\ell_1\) (the control location of \(p_1\) in \(\lambda\)).

2. Processes \(p_i\) for \(2 \leq i \leq n\) are modified as follows. Add a new register \(r_i\)
   3. for each \(p_i\), and initialize to 0. We begin with a while loop which checks \(\text{flag}\) is \(i - 1\). Inside the loop, we have the following. Using \(r_i\), we write \(v_{ik_i}\) to \(x_{ik_i}\), and so on until we write \(v_{ik_i}\) to \(x_{ik_1}\), obtaining the buffer content of \(p_i\) in \(\mathcal{B}\). Then we write the appropriate values to all registers of \(p_i\) as in \(\mathcal{R}\). This is followed by setting \(\text{flag}\) to \(i\). The while loop is broken at this point. The next instruction checks if \(\text{flag}\) is \(n + 1\), and if so, goes to the label \(\ell_i\) (the control location of \(p_i\) in \(\lambda\)).

4. For \(p_{n+1}\), we begin with a while loop which checks if \(\text{flag}\) is \(n\). Inside the loop, it reads the memory and checks that it agrees with \(\mathcal{M}\). This is possible as the buffer of \(p_{n+1}\) is empty. After the check, it changes the \(\text{flag}\) to \(n + 1\), breaking the while loop and reaching the terminal instruction of \(p_{n+1}\).

The modified program \(\mathcal{P}'\) has polynomially many extra instructions at the beginning of each process. Starting from the initial configuration \(\gamma_{\text{init}}\) consisting of all initial labels of instructions in all processes \(p_1, \ldots, p_{n+1}\), with all variables and registers having value 0, and all empty buffers, \(\mathcal{P}'\) first executes these extra instructions in all processes. The new shared variable \(\text{flag}\) moves the processes 1 to \(n + 1\) in order until we obtain (i) the memory state as in \(\mathcal{M}\), (ii) buffer contents of all processes \(p_1, \ldots, p_n\) as in \(\mathcal{B}\), (iii) local registers of all processes \(p_1, \ldots, p_n\) as in \(\mathcal{R}\). When \(\text{flag}\) becomes \(n + 1\), all processes \(p_1, \ldots, p_n\) move to the control locations given by \(\lambda\) in \(\gamma\). Ignoring \(\text{flag}\) as well as the two new local registers added to \(p_1\), and \(p_{n+1}\), the configuration of \(\mathcal{P}'\) at this point is \(\gamma = \langle \lambda, \mathcal{R}, \mathcal{B}, \mathcal{M}\rangle\), the configuration given to us. Define a configuration \(\gamma'' = \langle \lambda'', \mathcal{R}'', \mathcal{B}'', \mathcal{M}''\rangle\) as follows.

1. The labeling \(\lambda''\) agrees with \(\lambda'\) for processes \(p_1, \ldots, p_n\),
2. \(\mathcal{B}''\) agrees with \(\mathcal{B}'\) in terms of buffer states of \(p_1, \ldots, p_n\),
3. $\mathcal{M}''$ agrees with $\mathcal{M}'$, the memory state wrt $x_1, \ldots, x_m$

4. $\mathcal{R}''$ agrees with $\mathcal{R}'$, the register state wrt the registers in $\mathcal{P}$. In addition,

5. the buffer of $p_{n+1}$ is empty in $\mathcal{B}''$,

6. The new shared variable $\text{flag} \mapsto n + 1$ in the memory state $\mathcal{M}''$,

7. The new registers $r, r'$ of $p_1$ are such that $r$ has value 0 and $r'$ has value $u_m$ in $\mathcal{R}''$,

8. The new register $r_i$ of $p_i$ for $2 \leq i \leq m$ has value $v_{ik_i}$ in $\mathcal{R}''$.

9. The label of $p_{n+1}$ in $\lambda''$ is the term instruction in $p_{n+1}$.

We now ask the reachability query from $\gamma_{\text{init}}$ to $\gamma''$ in $\mathcal{P}'$ which is known to be decidable [12], since both $\gamma_{\text{init}}$ and $\gamma''$ are plain. Note that $\mathcal{P}'$ starts simulating $\mathcal{P}$ only when it reaches a configuration whose projection to the processes $p_1, \ldots, p_n$, modulo the new registers and $\text{flag}$, is $\gamma$. Indeed, if $\gamma''$ is reachable in $\mathcal{P}'$ from $\gamma_{\text{init}}$, then it must be that (i) Ignoring values of new registers and $\text{flag}$, $\gamma$ is reachable from $\gamma_{\text{init}}$ in $\mathcal{P}'$ (this follows by construction), and (ii) $\gamma''$ is reachable in $\mathcal{P}'$ from $\gamma$. Indeed since $\gamma''$ when projected to the old registers and shared variables and $p_1, \ldots, p_n$ is $\gamma'$, we obtain the reachability of $\gamma'$ from $\gamma$ in $\mathcal{P}$.

Finally we prove Lemma 5.

**Lemma 5** $\text{Prob}_\mathcal{P}(\gamma \models \square \diamond \Gamma^\text{plain}_\mathcal{P}) = 1$ for all configurations $\gamma \in \Gamma_\mathcal{P}$.

**Proof.** We will show that $\Gamma^\text{plain}_\mathcal{P}$ is a finite attractor in the sense of [16]. An attractor [16], is a set of configurations which is eventually reached with probability 1 from every configuration in the Markov Chain $[\mathcal{P}]^{MC}$.

$\Gamma^\text{plain}_\mathcal{P}$ contains configurations with empty buffers. We intuitively want to show that the system behaviours tend to concentrate towards these. We make this notion precise through the concept of configuration size, $|\gamma|$. It is sufficient to show that the expected value of $|\gamma|$ at each step decreases (when transitioning from configurations with sufficiently large buffers). The expectation is over the possible transitions from $\gamma$.

We show this in two steps. We first show that

$$\text{Prob}_\mathcal{P}(\gamma \models \square \diamond \Gamma_{\mathcal{P}}^{\leq k}) = 1$$

for constant $k = 4$ for all programs $\mathcal{P}$ and configurations $\gamma \in \Gamma_{\mathcal{P}}$. Recall that $\Gamma_{\mathcal{P}}^{\leq k}$ is the set of configurations with size (sum of buffer lengths) at most $k = 4$. Then we use this to prove the statement of the lemma. First to show that $\Gamma_{\mathcal{P}}^{\leq 4}$ is an attractor, we use the following result from [29].

**Left-oriented Markov chains ([29])** Baier et al. consider (infinite) Markov chains where the state space $S$ is partitioned into non-negative integer labelled levels $\cup_{i \in \mathbb{N}} S_i$. For $s \in S$, the level of $s$, $\text{level}(s)$ is $i$ if $s \in S_i$. Then $E(s) = \sum_{j=0}^{\infty} \text{Prob}(s, S_j) \cdot j$ defines the expected next level for state $s$. The Markov chain is called left oriented iff there is a positive constant $\Delta$ such that
\[ \mathbb{E}(s) \leq \text{level}(s) - \Delta \text{ for all states } s \notin S_0, \] that is, for all states \( s \) at level 1 or more. Theorem 2.1 from [29] shows that for any left oriented Markov chain, the leftmost level \( S_0 \) is an attractor.

We leverage Theorem 2.1 from [29] in our proof. To do this, we show that our Markov Chain \( [P]_{\text{MC}} \) is left oriented. The “levels” in \( [P]_{\text{MC}} \) are the configuration sizes, except for the set \( \Gamma_{P}^{\leq 4} \). We formalize this as an abstraction function \( \text{level} \) from the configuration set \( \Gamma_{P} \) to (non-negative) integers.

\[
\text{level}(\gamma) = 0 \quad \text{if } \gamma \in \Gamma_{P}^{\leq 4}
\]
\[
\text{level}(\gamma) = |\gamma| \quad \text{otherwise}
\]

For a configuration \( \gamma \) in \( \{ \gamma \mid \text{level}(\gamma) > 0 \} \), let \( X_\gamma \) be the random variable representing the next configuration obtained after a single transition from \( \gamma \). We then show that for all configurations in \( \{ \gamma \mid \text{level}(\gamma) > 0 \} \), the single step expected change in the \( \text{level} \) is negative. That is, for all \( \gamma \in \Gamma_{P}^{\geq 5} \),

\[
\mathbb{E}(\text{level}(X_\gamma)) \leq |\gamma| - \Delta
\]

where \( \Delta \) is a positive constant where, \( \mathbb{E}(\text{level}(X_\gamma)) = \sum_{\gamma'} \text{level}(\gamma') \cdot M_{\text{P}}(\gamma, \gamma') \)

We show that \( \sum_{\gamma'} |\gamma'| \cdot M_{\text{P}}(\gamma, \gamma') \leq |\gamma| - \Delta, \) (we replace \( \text{level}(\gamma') \) by \( |\gamma'| \) since \( \text{level}(\gamma') \leq |\gamma'| \)). Additionally, \( |\gamma| \) can be written as \( |\gamma| \cdot \sum_{\gamma'} M_{\text{P}}(\gamma, \gamma') \), and hence we want to show:

\[
\sum_{\gamma'} (|\gamma'| - |\gamma|) \cdot M_{\text{P}}(\gamma, \gamma') < -\Delta
\]

Showing this helps us to conclude that \( [P]_{\text{MC}} \) is left oriented. Now, we focus on showing this.

\( [P]_{\text{MC}} \) is left-oriented Consider a transition from the configuration \( \gamma^t \) (with \( |\gamma^t| \geq 5 \)), to the configuration \( \gamma^{t+1} \) as a result of the process \( (\to_{\text{proc}}) \) and update \( (\to_{\text{update}}) \) sub-transitions. For a particular \( \gamma^{t+1} \), let \( \gamma' \) be the intermediate configuration satisfying \( \gamma^t \to_{\text{proc}} \gamma' \to_{\text{update}} \gamma^{t+1} \). We can write the single step change in expectation when going from \( \gamma^t \) to \( \gamma^{t+1} \) as follows.

\[
\sum_{\gamma^{t+1}} (|\gamma^{t+1}| - |\gamma^t|) \cdot M_{\text{P}}(\gamma^t, \gamma^{t+1})
\]

\[
= \sum_{\gamma'} M_{\text{proc}}(\gamma', \gamma') \cdot \left( |\gamma'| - |\gamma^t| + \sum_{\gamma^{t+1}} M_{\text{update}}(\gamma', \gamma^{t+1}) \cdot (|\gamma^{t+1}| - |\gamma'|) \right)
\]

The first term \( |\gamma'| - |\gamma^t| \) is the number of elements added to the buffer in \( \gamma^t \to_{\text{proc}} \gamma' \) and the expectation is over the \( \to_{\text{proc}} \) probabilities while the second term \( |\gamma^{t+1}| - |\gamma'| \) is the number of elements removed from the buffer in \( \gamma' \to_{\text{update}} \gamma^{t+1} \) and the expectation is over the \( \to_{\text{update}} \) probabilities.
Now we see that (by Figure 2) in any \( \gamma' \rightarrow_{\text{proc}} \gamma \) step either one element is added to the buffer (in case of a write transition) or the buffer remains the same (for all other transitions). In particular, even when the disabled rule is taken, the configuration remains the same. We have the following.

\[
\forall \gamma' \quad \gamma' \rightarrow_{\text{proc}} \gamma \quad \Rightarrow \quad 0 \leq |\gamma'| - |\gamma| \leq 1
\]

Since this holds for all \( \gamma, \gamma' \) pairs, it certainly holds for expected values, and we can substitute in the above equation:

\[
\sum_{\gamma'+1}(|\gamma'+1| - |\gamma|) \cdot M_{P}(\gamma,\gamma'+1)
\leq \sum_{\gamma'} M_{\text{proc}}(\gamma',\gamma') \cdot \left(1 + \sum_{\gamma'+1} M_{\text{update}}(\gamma',\gamma'+1) \cdot (|\gamma'+1| - |\gamma'|)\right)
\]

As for the update part of the transition, we consider a uniform distribution over all possible update sequences from the state \( \gamma' \). Due to the combinatorial term involved, the probability of an update that propagates more writes to the memory is strictly greater than that of an update which propagates shorter lengths. Even this conservative reasoning implies that the update rule leads to a configuration with at most half the size with at least probability of \( \frac{1}{2} \), giving us the following bound. Below, \( \text{Prob}_{\rightarrow_{\text{update}}}(x) \) denotes the probability of an update where \( x \) specifies how many elements from the buffer are pushed.

\[
\sum_{\gamma'+1} M_{\text{update}}(\gamma',\gamma'+1) \cdot (|\gamma'+1| - |\gamma'|)
\leq -|\gamma'| \cdot \sum_{i=0}^{\frac{|\gamma'|}{2}} i \cdot \text{Prob}_{\rightarrow_{\text{update}}}(i \text{ elements of the buffer are updated})
\leq -\frac{|\gamma'|}{4} \cdot \text{Prob}_{\rightarrow_{\text{update}}}(\text{at least } \frac{|\gamma'|}{2} \text{ elements are updated}) \leq -\frac{|\gamma'|}{4}
\]

Here the first inequality follows from the fact that we are discounting for cases where less than \( |\gamma'|/2 \) elements are updated. The second inequality follows from the fact that the probability of updating at least \( |\gamma'|/2 \) elements is greater than \( \frac{1}{2} \). We note that this holds for each configuration \( \gamma' \), and hence substituting this
bound in the earlier expression, we get (noting that $|\gamma'| \geq |\gamma^t| \geq 5$),

$$\sum_{\gamma^{t+1}}(|\gamma^{t+1}| - |\gamma^t|) \cdot M_{\gamma^t,\gamma^t+1} \leq \sum_{\gamma'} M_{\gamma^t,\gamma'} \cdot \left(1 - \frac{|\gamma'|}{4}\right)$$

$$\leq \sum_{\gamma'} M_{\gamma^t,\gamma'} \cdot \left(1 - \frac{5}{4}\right)$$

$$\leq \left(1 - \frac{5}{4}\right) \cdot \sum_{\gamma'} M_{\gamma^t,\gamma'} \leq -\frac{1}{4}$$

This proves that $[\mathcal{P}]^{\text{MC}}$ is left-oriented (for $\Delta = \frac{1}{3}$). Hence by invoking Theorem 2.1 from [29] we get that, the level 0 set is an attractor. That is $\text{Prob}_P(\gamma \models T \square \Diamond \Gamma^{\leq 4}_P) = 1$.

Now we want to show that $\text{Prob}_P(\gamma \models \square \Diamond \Gamma^{\text{plain}}_P) = 1$. But this follows directly from the notion of probabilistic fairness (for details see Theorem 10.25 of [20]). We have (even for infinite Markov chains $\mathcal{M}$) and (finite) sets of configurations $T, T'$,

$$\text{Prob}_\mathcal{M} (s \models \square \Diamond T) = \text{Prob}_\mathcal{M} \left( s \models \bigwedge_{T' \subseteq \text{Post}^*(T)} \square \Diamond T' \right)$$

We note that the sets $\Gamma^{\text{plain}}_P$ and $\Gamma^{\leq 4}_P$ are indeed finite (since the buffer sizes are bounded) and $\Gamma^{\text{plain}}_P$ is reachable from $\Gamma^{\leq 4}_P$, since the update rule can simply empty all buffers with non-zero probability. Hence instantiating $s = \gamma$, $T = \Gamma^{\leq 4}_P$, $T' = \Gamma^{\text{plain}}_P$ we have that $\text{Prob}_P(\gamma \models \square \Diamond \Gamma^{\text{plain}}_P) = 1$ as desired.

B Qualitative Reachability: Supplementary material for Sec. 6

In this section, we discuss the results for qualitative reachability and repeated reachability, filling in the details omitted in the main paper. In particular, we provide algorithms for the almost-sure repeated reachability and almost-never repeated reachability.

Then, we go on to prove Theorems 1 and 2.

B.1 Almost-Sure Repeated Reachability

The qualitative repeated reachability problem, differs from reachability in that now we are interested in the property $\square \Diamond l$ i.e., whether $l$ will be reached infinitely often. In a similar manner to the case of reachability, the algorithm of Fig. 7 analyzes the transition system $[\mathcal{P}]^{TS}$. 
The difference between the two algorithms is that we do not need to transform the program by removing the label $l$ here, since reaching $l$ a finite number of times does not affect repeated reachability. Therefore, we perform analysis directly on the input program $P$. As before, the loop on line 1 generates all the plain configurations one by one, and performs the same tests as in the qualitative reachability algorithm of Fig. 3. More precisely, the algorithm terminates and returns a negative answer if it finds a plain configuration that is reachable from $\gamma_{\text{init}}$ but that cannot reach $l$. Otherwise, it returns a positive answer. To see the correctness of the algorithm, we observe that it answers negatively only if it finds a path with a positive probability from $\gamma_{\text{init}}$ to a plain configuration from which $l$ is not reachable. Using a similar reasoning to the case of reachability, this implies that the measure of runs from $\gamma_{\text{init}}$ that reach $l$ is smaller than one. Therefore, the measure of runs from $\gamma_{\text{init}}$ that repeatedly reach $l$ is also smaller than one. In the other direction, if the algorithm answers positively then there is no plain configuration that is reachable from $\gamma_{\text{init}}$ but that cannot reach $l$. Using Lemma 5 as in Sec. 6.1, we conclude that any run $\rho$ from $\gamma_{\text{init}}$ will almost surely repeatedly visit some plain configuration $\gamma$ from which $l$ is reachable. Consequently, $\rho$ almost surely visits $l$ infinitely often.

**B.2 Almost-Never (Repeated) Reachability**

The almost-never variants of the (repeated) reachability problems, NEVER\_QUAL\_REACH resp. NEVER\_QUAL\_REP\_REACH, ask whether the probabilities equal to 0 rather than 1. The solution to NEVER\_QUAL\_REACH is straightforward, since $\text{Prob}_P (\gamma_{\text{init}} = \Diamond l) = 0$ iff $\neg (\gamma_{\text{init}} \overset{*}{\Rightarrow} P l)$ the latter is decidable by Lemma 4.

We give the algorithm for solving the almost-never repeated reachability problem in Fig. 8. The algorithm searches for B-plain configurations that are reachable from $\gamma_{\text{init}}$ and from which $l$ is reachable. If it detects such a configuration, it returns a negative answer. Otherwise, it returns a positive answer. The outer loop cycles through all plain configurations. For each such configuration $\gamma$ that is reachable from $\gamma_{\text{init}}$, the inner loop checks whether $\gamma$ is a B-plain configuration (by trying to search for a plain configuration which is reachable but without a path back to $\gamma$).

---

### Algorithm: NEVER\_QUAL\_REP\_REACH

**Input:** $P$: program; $\gamma_{\text{init}} \in I_P$: configuration; $l \in \text{Lbl}_P$: label.

1. for each $\gamma \in I_P^{\text{plain}}$ do
   2. if $\gamma_{\text{init}} \overset{*}{\Rightarrow} P \gamma$ and $\neg (\gamma \overset{*}{\Rightarrow} P l)$ then return false
   3. return true

---

**Fig. 7.** Almost-sure repeated reachability.

---

### Algorithm: NEVER\_QUAL\_REP\_REACH

**Input:** $P$: program; $\gamma_{\text{init}} \in I_P$: configuration; $l \in \text{Lbl}_P$: label.

1. for each $\gamma \in I_P^{\text{plain}}$ do
   2. if $\gamma_{\text{init}} \overset{*}{\Rightarrow} P \gamma$ then
      3. flag := true
      4. for each $\gamma' \in I_P^{\text{plain}}$ do
         5. if $\gamma' \overset{*}{\Rightarrow} P \gamma'$ and $\neg (\gamma' \overset{*}{\Rightarrow} P \gamma)$ then flag := false
      6. if flag = true and $\gamma \overset{*}{\Rightarrow} P l$ then return false

---

**Fig. 8.** Almost-never repeated reachability algorithm.
We intuitively explain the algorithm as follows. Suppose there is a \( \gamma \in I^{B\text{-plain}}_P \) such that \( \gamma_{\text{init}} \xrightarrow{\pi} \gamma \xrightarrow{\pi} l \). By Lemma 5 and the fact that \( \gamma \) is a B-plain configuration, any run from \( \gamma \) will almost surely visit \( \gamma \) infinitely often. Hence by the fairness property for Markov chains it follows that the run almost surely visits \( l \) infinitely often and we return false. Conversely, assume there is no B-plain configuration satisfying \( \gamma_{\text{init}} \xrightarrow{\pi} \gamma \xrightarrow{\pi} l \). By Lemma 6, we know that any run \( \rho \) from \( \gamma_{\text{init}} \) will visits some B-plain configuration \( \gamma \) infinitely often. Since \( l \) is not reachable from \( \gamma \) it follows that \( \rho \) will almost never visit \( l \).

### B.3 Proofs for Sec. 6

**Theorem 1.** \textsc{Qual Reach}, \textsc{Qual Rep Reach}, \textsc{Never Qual Reach}, \textsc{Never Qual Rep Reach} are all decidable.

**Reachability**

**Proof.** Decidability follows by proving that the algorithm given in Figure 3 gives the correct answer and terminates, which we now do.

**Correctness (Algorithm returns false)** When the algorithm returns false, we know that there exists a plain configuration \( \gamma \) and a finite length path \( \gamma_{\text{init}} \xrightarrow{\pi} \gamma \), such that \( \neg(\gamma \xrightarrow{\pi} l) \). Since it is a finite path, it is taken with a non-zero probability, say \( p \). Then we know that 
\[
\Pr_P(\gamma_{\text{init}} \models \square \Diamond l) = 1 - p < 1
\]
and we are done.

**Correctness (Algorithm returns true)** Let the set of plain configurations (in the original program \( P \)) reachable from \( \gamma_{\text{init}} \) be \( I^{r\text{-plain}}_P \). Given that there are finitely many plain configurations, \( |I^{r\text{-plain}}_P| \) is also finite. Since the algorithm returned true we know that \( l \) is reachable from each configuration in \( I^{r\text{-plain}}_P \), with a finite length path and hence some non-zero probability. Let \( p \) denote the minimum of these reachability probabilities over the (finite) set \( I^{r\text{-plain}}_P \). We must have \( p > 0 \), since \( I^{r\text{-plain}}_P \neq \emptyset \). Hence starting from any configuration in \( I^{r\text{-plain}}_P \), the probability of reaching \( l \) is at least \( p > 0 \). Now, Lemma 5 implies that the set \( I^{r\text{-plain}}_P \) is reached infinitely often, \( \Pr_P(\gamma_{\text{init}} \models \square \Diamond I^{r\text{-plain}}_P) = 1 \).

Then the result follows from the fairness theorem for Markov chains which says,
\[
\Pr_P(\gamma \models \square \Diamond \Gamma) = \Pr_P\left(\gamma \models \bigwedge_{\gamma' \in \text{Post}^*(\Gamma)} \square \Diamond \gamma'\right).
\]

As a corollary, we get,
\[
\Pr_P(\gamma \models \square \Diamond \Gamma) \leq \Pr_P(\gamma \models \square \Diamond \Gamma') \quad \text{for } \Gamma' \subseteq \text{Post}^*(\Gamma).
\]

In particular, instantiating \( \gamma = \gamma_{\text{init}}, \Gamma = I^{r\text{-plain}}_P, \Gamma' = \{\gamma' \mid l \in \gamma'\} \), we get
\[
\Pr_P(\gamma_{\text{init}} \models \square \Diamond I^{r\text{-plain}}_P) = 1 \leq \Pr_P(\gamma_{\text{init}} \models \square \Diamond l) \leq \Pr_P(\gamma_{\text{init}} \models \Diamond l).
\]
Both the inequalities must be equalities which completes the proof for correctness.

**Termination** The set of plain configurations is finite, and hence the loop performs finite iterations. At each iteration, both of the queries are decidable as discussed in Lemma 4 and Lemma 3. This shows termination and proves the theorem.

**Repeated Reachability**

**Proof.** This proof is similar to its reachability counterpart. The algorithm returns true only when all plain configurations $γ \in \Gamma_\text{r-plain}^P$ reachable from $γ_{\text{init}}$, are such that 1 is also reachable from $γ$. By Lemma 5, we know that any run from $γ_{\text{init}}$ visits almost surely, some plain configuration infinitely often. In our case, any run from $γ_{\text{init}}$ visits some configuration from $γ \in \Gamma_\text{r-plain}^P$ infinitely often; since 1 is reachable from $γ$, the run from $γ_{\text{init}}$ which reaches 1 will visit almost surely, $γ$ again and again, and hence 1 also, infinitely often, almost surely. In particular, we show correctness and termination for the algorithm in Figure 7.

**Correctness** Proof of correctness is identical to the earlier. In particular, at the last step we also showed

$$\text{Prob}_P(γ_{\text{init}} \models □\Diamond 1) = 1 \leq \text{Prob}_P(γ_{\text{init}} \models □\Diamond l)$$

where the inequality must be an equality, owing to the probability values.

**Termination** The set of plain configurations is finite, and hence the loop performs finite iterations. At each iteration, both of the queries are decidable as discussed in Lemma 4 and Lemma 3. This shows termination and proves the theorem.

**Almost-Never Repeated Reachability**

**Proof. Correctness.** (Algorithm returns false) When the algorithm returns false, we know that there exists a B-plain configuration $γ$ reachable from $γ_{\text{init}}$ and a finite length path $γ \rightarrow_P l$. Since it is a finite path, it is taken with a non-zero probability, say $p$. By Lemma 5 and the fact that $γ$ can reach back to itself, we know that any run from $γ_{\text{init}}$ visits $γ$ infinitely often. Since $l$ is reachable from $γ$ with probability $p > 0$, we can reach $l$ infinitely often with probability > 0. Thus, $\text{Prob}_P(γ_{\text{init}} \models □\Diamond l) > 0$ and we are done.

(Algorithm returns true) Assume that all B-plain configurations $γ$ reachable from $γ_{\text{init}}$ are such that $¬((γ \rightarrow_P l))$. Then we know by Lemma 5, $γ$ is visited infinitely often on any run from $γ_{\text{init}}$; since $l$ is not reachable from $γ$, it is not repeatedly reachable along any run from $γ_{\text{init}}$. If $l$ were repeatedly reachable on some run from $γ_{\text{init}}$, then since some B-plain configuration $γ$ is also visited infinitely often along that run, it would result in reaching $l$ from $γ$, contradicting the non reachability of $l$ from $γ$). Thus, the probability of repeatedly reaching $l$ from $γ_{\text{init}}$ is 0: $\text{Prob}_P(γ_{\text{init}} \models □\Diamond l) = 0.$
Termination The set of plain configurations is finite, and hence the loop performs finite iterations. At each iteration, both of the queries (checking if a plain configuration \( \gamma \) is B-plain : that is, it is reachable from itself, as well as whether \( l \) is reachable from \( \gamma \)) are decidable as discussed in Lemma 4 and Lemma 3. This shows termination and proves the theorem.

Theorem 2. \texttt{QUAL\_REACH}, \texttt{QUAL\_REP\_REACH}, \texttt{NEVER\_QUAL\_REACH}, \texttt{NEVER\_QUAL\_REP\_REACH} all have non-primitive-recursive complexities.

Reachability

Proof \((\text{Proof})\). The proof follows from a reduction from the (non-probabilistic) control-state reachability problem for TSO, which is known to be non-primitive recursive \([12]\).

Concretely, consider an instance program \( P \) of the (non-probabilistic) reachability problem for TSO. We ask whether \( \gamma_{\text{init}} \xrightarrow{}^* P \top \) for some label \( \top \in \text{Lbl}_{pr} \) (we ask for reachability of process \( pr \), and other processes can be in arbitrary labels).

Reduction construction For the reduction, we construct a program \( P' \) obtained from \( P \) by modifying process \( pr \), and adding a new process \( p_{\text{new}} \). All other processes of \( P \) remain unchanged. \( P' \) has all the shared variables and local registers as \( P \), and a new shared variable \( x \), as well as a new local register \( a \). As always, all shared variables and local registers are initialized to 0. Assume the finite data domain of \( P \) (and \( P' \)) consists of values \( \{v_0, v_1, \ldots, v_n\} \), where \( v_0 = 0 \).

1. \( p_{\text{new}} \) consists of a sequence of instructions which starts by checking if \( a \) is \( v_0 \), and if so, goes to the next two instructions which are \( a := v_1; x := a \). This is followed by an instruction which checks if \( a \) is \( v_1 \), and if so, goes to the next two instructions which are \( a := v_2; x := a \). This is continued till we reach the instruction which checks if \( a \) is \( v_n \) and if so, goes to the first instruction. Thus, \( p_{\text{new}} \) is a loop which repeatedly writes values \( v_0, \ldots, v_n \) to \( x \).

2. Now we discuss the modifications in \( pr \).
   - We add two fresh instruction labels \( l_{\text{win}} \) and \( l_{\text{lose}} \) to \( pr \) such that
     \[
     l_{\text{win}} : \text{if } (a \geq 0) \text{ then } l_{\text{win}}, \quad l_{\text{lose}} : \text{if } (a \geq 0) \text{ then } l_{\text{lose}}
     \]
   - Immediately before each instruction label \( l \neq \top \) in \( pr \), we add two fresh instructions labeled \( l_{\text{pre}1} \) and \( l_{\text{pre}2} \) as follows.
     \[
     l_{\text{pre}1} : a := x, \quad l_{\text{pre}2} : \text{if}(a = v_i) \text{ then } l_{\text{win}}
     \]
     Thus, if the value of \( a \) is not \( v_i \), control moves to \( l \); otherwise to \( l_{\text{win}} \).
   - Immediately before \( \top \), we add two fresh instructions labeled \( l_{\text{pre}1} \) and \( l_{\text{pre}2} \) as follows.
     \[
     l_{\text{pre}1} : a := x, \quad l_{\text{pre}2} : \text{if}(a = v_i) \text{ then } l_{\text{lose}}
     \]
     Thus, if the value of \( a \) is not \( v_i \), control moves to \( \top \); otherwise to \( l_{\text{lose}} \).

We see that \( \neg(l_{\text{win}} \xrightarrow{}^* P' \top) \) and \( \neg(l_{\text{lose}} \xrightarrow{}^* P' l_{\text{win}}) \).
Equivalence For this program $\mathcal{P}$, we ask the probabilistic qualitative reachability problem: $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Diamond l_{\text{win}}) = 1$? We claim that an answer to this question, allows us to decide reachability to $l^*$ in $\mathcal{P}$.

$(\Rightarrow)$ If the answer to this is yes, we know that $\neg(\gamma_{\text{init}} \not\rightarrow_P l^*)$, since a (finite-length) path to $l^*$, implies a finite length path to $l_{\text{pre1}}, l_{\text{pre2}}$. This implies a finite and hence non-zero probability path to reach $l_{\text{lose}}$, from which there is no path to $l_{\text{win}}$.

$(\Leftarrow)$ On the other hand, if $l^*$ is not reachable in $\mathcal{P}$, we have $\gamma_{\text{init}} \models \Box(lbl_{\mathcal{P}' \setminus \{l^*\}})$ and hence $\gamma_{\text{init}} \models \Box(l_{\text{lose}}, l^*, l_{\text{pre1}}, l_{\text{pre2}})$ in $\mathcal{P}'$. In the extended program $\mathcal{P}'$, define $\Gamma_{l^*}^{\text{win}}$ as the set of configurations with $\lambda(p_{\mathcal{P}'}) = l_{\text{win}}$. Additionally, we have that all reachable configurations $\text{Lbl}_{\mathcal{P}' \setminus \{l_{\text{lose}}, l^*, l_{\text{pre1}}, l_{\text{pre2}}\}}$ can reach $\Gamma_{l^*}^{\text{win}}$ in a single step (with a non-zero probability). Thus, $l_{\text{win}}$ is reachable in a single step with non zero probability from all configurations which are reachable from $\gamma_{\text{init}}$. This implies that the probability to reach $\Gamma_{l^*}^{\text{win}}$ from $\gamma_{\text{init}}$ is 1, which proves the lemma.

Repeated Reachability

Proof (Proof). This proof is identical to its reachability counterpart. In particular note that $l_{\text{win}}, l_{\text{lose}}$ are associated with self loop transitions, so, reachability and repeated reachability for $l_{\text{win}}, l_{\text{lose}}$ are equivalent problems for program $\mathcal{P}'$.

Never Reachability

Proof (Proof). We can use the construction above, once again. Note that by construction, reaching $l_{\text{win}}$ is same as never reaching label $l_{\text{lose}}$. We have already established the reduction from reachability to some $l$ to the problem $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Diamond l_{\text{win}}) = 1$?. Since $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Diamond l_{\text{win}}) = 1$? iff $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Box \Diamond l_{\text{lose}}) = 0$?, we now have the reduction from reachability in classical TSO to the never reachability in probabilistic TSO.

Never Repeated Reachability

Proof (Proof). Once again, by construction, reaching $l_{\text{win}}$ is same as never repeatedly reaching label $l_{\text{lose}}$. Thus, $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Diamond l_{\text{win}}) = 1$? iff $\text{Prob}_{\mathcal{P}'}(\gamma_{\text{init}} \models \Box \Diamond l_{\text{lose}}) = 0$?, we now have the reduction from reachability in classical TSO to the never repeated reachability in probabilistic TSO.

C Quantitative Reachability: Supplementary material for Sec. 7

In this section, we provide the algorithm for the approximate repeated reachability problem and then provide proofs of correctness and termination.
C.1 Algorithmic details for Approximate Quantitative Repeated Reachability

In the case of the approximate quantitative repeated reachability problem, QUANT_REP_REACH, our task is to approximate the probability of visiting a given label infinitely often. We provide an algorithm for approximate quantitative reachability in Figure 9.

The algorithm for repeated reachability is very similar to the one for reachability. The main difference compared to algorithm of Fig. 4 is the condition of the if-statement (line 19). Instead of checking whether we have reached label $l$, we now increase the value of PosApprx if there is no $\gamma$-path to a configuration from which $l$ is not reachable. To check this condition, we first compute set of B-plain configurations and store them in BPlain. We go through the B-plain configurations that are reachable from the current configuration $\gamma$ one by one. We increase the value of PosApprx if $l$ is reachable from all such configurations.

Again, we let $\text{PosApprx}^{(i)}$ and $\text{NegApprx}^{(i)}$ represent the value of $\text{PosApprx}$ resp. $\text{NegApprx}$ prior to performing the $i^{th}$ iteration. The partial correctness of the algorithm, follows from the following properties of the algorithm: (i) The value of $\text{PosApprx}$ increases only by weights of $\gamma_{\text{init}}$-paths that visit configuration $\gamma$ from which all reachable B-plain configurations can in turn reach $l$. We argue that any $\gamma$-run will almost surely repeatedly reach $l$. To see that, we know by Lemma 6 that $\rho$ will almost surely visit the set of B-plain configurations. By finiteness of the set, $\rho$ will almost surely visit a particular B-plain configuration $\gamma' \in I_{B\text{plain}}$ infinitely often. Since $\gamma' \models \emptyset l$ it follows that $\rho$ will almost surely visit $l$ infinitely often. It follows that $\text{PosApprx}^{(i)} \leq \text{Prob}_P (\gamma_{\text{init}} \models \Box \emptyset l)$. (ii) We increase the value of $\text{NegApprx}$ only by weights of $\gamma_{\text{init}}$-paths that end up at a configuration $\gamma$ from which $l$ is not reachable. Since $(\gamma \models \emptyset l) = \emptyset$, and hence

\begin{algorithm}
\caption{QUANT_REP_REACH}
\textbf{Input:} $P$: program; $\gamma_{\text{init}} \in I_P$: configuration; $l \in \text{Llb}_P$: label; $\varepsilon \in \mathbb{R}^{>0}$: precision.
\begin{algorithmic}
\State $\text{Var}$ $\text{flag, posflag} \in \mathbb{B}$: Boolean flags;
\State $\text{BPlain} \subseteq I_P^{\text{plain}}$: subset of the plain configurations;
\State $\text{PosApprx}, \text{NegApprx} \in \mathbb{R}$: under-approximations;
\State $\text{waiting} \in (I_P \times \mathbb{R})^*: \text{queue};$
\State $\text{PosApprx} := 0$; $\text{NegApprx} := 0$;
\State $\text{BPlain} := \emptyset$; $\text{waiting} := (\gamma_{\text{init}}, 1)$;
\For {each $\gamma \in I_P^{\text{plain}}$}
\State flag := true;
\For {each $\gamma' \in I_P^{\text{plain}}$}
\If {$(\gamma \rightarrow_p \gamma')$ and $\neg(\gamma' \rightarrow_p \gamma)$}
\State flag := false;
\EndIf
\EndFor
\If {flag = true then}
\State $\text{BPlain} := \text{BPlain} \cup \{\gamma\}$
\EndIf
\EndFor
\While {$\text{PosApprx} + \text{NegApprx} < 1 - \varepsilon$}
\State $(\gamma, \phi)$ := head($\text{waiting}$);
\State waiting := tail($\text{waiting}$);
\State posflag := true;
\For {each $\gamma' \in \text{BPlain}$}
\If {$(\gamma \rightarrow_p \gamma') \land \neg(\gamma' \rightarrow_p l)$}
\State posflag := false;
\EndIf
\EndFor
\If {posflag = true then}
\State $\text{PosApprx} := \text{PosApprx} + \phi$;
\Else
\If {$(\neg(\gamma \rightarrow_p l))$}
\State $\text{NegApprx} := \text{NegApprx} + \phi$;
\EndIf
\EndIf
\EndWhile
\State for each $\gamma'$ with $\gamma \rightarrow_p \gamma'$ do
\State waiting := waiting $\cdot (\gamma', \phi \cdot \text{M}_P (\gamma, \gamma'))$;
\EndFor
\State return $\text{PosApprx}$
\end{algorithmic}
\end{algorithm}
also \((\gamma \models \Box l) = \emptyset\). Therefore, \(\text{NegApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \neg \Box l)\). (iii) If the algorithm terminates after the \(i^{th}\) iteration, the condition of the while-loop implies that \(\text{PosApprx}^{(i)} + \text{NegApprx}^{(i)} \geq 1 - \varepsilon\). From (i), (ii), and (iii), it follow that if the termination point is \(i\) then:

\[
\text{PosApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \Box l) \leq 1 - \text{NegApprx}^{(i)} \leq \text{PosApprx}^{(i)} + \varepsilon
\]

Therefore, on termination, \(\text{PosApprx}\) is within \(\varepsilon\)-precision of \(\theta\).

### C.2 Proofs of correctness for Sec. 7

**Theorem 3.** \textsc{Quant\_Reach} is solvable.

*Proof.* Decidability follows by proving that the algorithm given in Figure \[\] gives the correct answer and terminates, which we now do.

**Correctness** We have that \(\text{PosApprx}^{(i)}\) is monotone in \(i\) and that

\[
\text{PosApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \Diamond l)
\]

since we only accumulate probabilities of distinct paths reaching \(l\) in \(\text{PosApprx}^{(i)}\). On the other hand, for \(\text{NegApprx}^{(i)}\) we have the following inequality,

\[
\text{NegApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \neg \Diamond l) = 1 - \text{Prob}_P(\gamma_{\text{init}} \models \Diamond l).
\]

This in turn follows from the fact that \(\text{NegApprx}^{(i)}\) accumulates probabilities of distinct (infinite) paths which never will reach \(l\). Hence, \(\text{NegApprx}^{(i)} + \text{PosApprx}^{(i)} \geq 1 - \varepsilon\) implies that

\[
\text{PosApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \Diamond l) \leq 1 - \text{NegApprx}^{(i)} \leq \text{PosApprx}^{(i)} + \varepsilon
\]

showing that \(\text{PosApprx}^{(i)}\) approximates \(\text{Prob}_P(\gamma_{\text{init}} \models \Diamond l)\) to \(\varepsilon\) precision and proving correctness of the algorithm.

**Termination** This proof crucially uses the existence of a finite set \(I^\text{plain}_P\) which is reached repeatedly with probability one and the fairness theorem on Markov chains. The fairness theorem says that for a Markov chain with (set of) states \(S_1\) and \(S_2\), if there is a non-zero probability path from \(S_1\) to \(S_2\), then the probability of taking a path which reaches \(S_1\) infinitely often but which never reaches \(S_2\) is zero.

As introduced earlier in the main text, we denote by \(\text{PosApprx}^{(i)}, \text{NegApprx}^{(i)}\) the iterates at the \(i\) iteration of the while-loop and additionally we define

\[
\text{UndetApprx}^{(i)} = 1 - \text{PosApprx}^{(i)} - \text{NegApprx}^{(i)}
\]

If we can show that \(\lim_{i \to \infty} \text{UndetApprx}^{(i)} = 0\) we are done since we have that

\[
\text{PosApprx}^{(i)} \leq \text{Prob}_P(\gamma_{\text{init}} \models \Diamond l) \leq \text{PosApprx}^{(i)} + \text{UndetApprx}^{(i)}.
\]
We now work towards this goal. First we observe that the algorithm performs a breadth-first traversal of the space of configurations. Hence at each loop iteration, the configuration which is dequeued from the waiting-queue is associated with a certain depth of search. We denote this depth by $\text{depth}(i)$ for iteration $i$. Conversely, for each depth $d$, there exists a maximal loop-iteration $i$ that considers a configuration at that depth, denoted by $\text{maxiter}(d) = \max_i \{ \text{depth}(i) = d \}$. At each depth, there are only finitely many configurations which are considered, and hence, max is over a finite set, and is well defined. The finiteness of the number of configurations at each depth follows from the observation that the Post set of each configuration $\gamma$ is finite (concretely it can be represented as a polynomial in the size of the configuration, $|\gamma|$). This implies that $\lim_{i \to \infty} \text{depth}(i) = \infty$. Hence, the limits, whether taken over the loop iteration count $i$ or over the depth of search $\text{depth}(i)$ directly are equal. Hence we switch to the limits over the depth of search that is more convenient to reason about.

$$\lim_{i \to \infty} \text{UndetApprx}(i) = \lim_{d \to \infty} \text{UndetApprx}(\text{maxiter}(d))$$

Now for a depth $d$, consider the set of all paths of length $d$: $\gamma_{\text{init}} \cdot \Gamma^d$. A path $\rho \in \gamma_{\text{init}} \cdot \Gamma^d$ can be one of three types: (1) which have reached 1, i.e $\exists i, \pi[i] = 1$ (2) which cannot reach 1: $\neg(\pi[d] \rightarrow P \ 1)$ and (3) undetermined (where none of (1,2) hold). The paths from (1) and (2) have probabilities corresponding to $\text{PosApprx}(\text{maxiter}(d))$ and $\text{NegApprx}(\text{maxiter}(d))$. The probability that a path belongs to (3) on the other hand is given by $\text{UndetApprx}(\text{maxiter}(d))$.

For a given $d$, denote the set of these undetermined paths from (3) as $\text{Undet}(d)$. Let $U = \{ \pi \mid \forall d, \pi[0] \cdots \pi[d] \in \text{Undet}(d) \} \subseteq \gamma_{\text{init}} \cdot \Gamma^\infty$. These are the set of infinite paths from $\gamma_{\text{init}}$, for which all finite prefixes are undetermined w.r.t reachability to 1. Now we know that $\text{Prob}_P(\gamma_{\text{init}} \models \Box \Diamond \Gamma^\text{plain}_P) = 1$. Partition $\Gamma^\text{plain}_P = \Gamma^\text{pos-plain}_P \cup \Gamma^\text{neg-plain}_P$ into two: (1) $\Gamma^\text{pos-plain}_P = \{ \gamma \mid \gamma \rightarrow P \ 1 \}$ and (2) $\Gamma^\text{neg-plain}_P = \{ \gamma \mid \neg(\gamma \rightarrow P \ 1) \}$. We have the following since $\Gamma^\text{plain}_P$ is reached repeatedly with probability one.

$$\text{Prob}_P(\gamma_{\text{init}} \models U) = \text{Prob}_P(\gamma_{\text{init}} \models (U \land \Box \Diamond \Gamma^\text{plain}_P))$$

However, any path satisfying $\pi \models \Box \Gamma^\text{neg-plain}_P$ cannot belong to $U$ (if it belongs to $U$, each finite prefix is undetermined wrt 1, contradicting $\Box \Gamma^\text{neg-plain}_P$ which is determined to not reach 1), we must have the following.

$$\text{Prob}_P(\gamma_{\text{init}} \models U) = \text{Prob}_P(\gamma_{\text{init}} \models (U \land \Box \Diamond \Gamma^\text{pos-plain}_P))$$

Since $\Gamma^\text{plain}_P$ is finite so it $\Gamma^\text{pos-plain}_P$ and we have

$$\text{Prob}_P(\gamma_{\text{init}} \models U) \leq \sum_{\gamma \in \Gamma^\text{pos-plain}_P} \text{Prob}_P(\gamma_{\text{init}} \models (U \land \Box \Diamond \gamma))$$

where the sum is over a finite set. Now $U$ consists of paths which never reach 1. However these paths reach $\Gamma^\text{pos-plain}_P$ infinitely often. Consequently, since each
\( \gamma \in \Gamma_{p}^{\text{pos-plain}} \) has a finite path (with non-zero probability) to \( l \), and by the fairness theorem on Markov chains we must have, for all \( \gamma \in \Gamma_{p}^{\text{pos-plain}} \),

\[
\Pr_{p}(\gamma_{\text{init}} \models (U \land \Box \Diamond \gamma)) = 0,
\]

and hence,

\[
\Pr_{p}(\gamma_{\text{init}} \models U) \leq 0
\]

To finish the proof note that the approximation margin term \( \text{UndetApprx}^{(d)} \) approaches \( \Pr_{p}(\gamma_{\text{init}} \models U) \) as \( d \) approaches infinity (since at a given value of \( d \) it expresses the probability of taking paths which are undetermined for \( d \) steps). Hence we have

\[
\lim_{d \to \infty} \text{UndetApprx}^{(\text{maxiter}(d))} = \Pr_{p}(\gamma_{\text{init}} \models U) = 0.
\]

Proving the requisite claim and hence the theorem.

**Theorem 4.** \textsc{Quant Rep Reach} is solvable.

**Proof.** Decidability follows by proving that the algorithm given in Figure 9 gives the correct answer and terminates, which we now do.

**Correctness** The proof of correctness is similar to that for Theorem 3. In particular, the only difference is that \( \text{PosApprx}^{(i)} \) and \( \text{NegApprx}^{(i)} \) now estimate (from below) the probabilities \( \Pr_{p}(\gamma_{\text{init}} \models \Diamond l) \) and \( \Pr_{p}(\gamma_{\text{init}} \models \neg \Box \Diamond l) \). The remaining analysis follows replacing reachability by repeated reachability.

**Termination** The proof is similar to the termination argument for Theorem 3. There are two main differences. First the definition of the undetermined set of infinite paths \( (U \text{ in Theorem 3}) \) changes: we call this set \( V \) here. Secondly we must use a stronger variant of the fairness theorem on Markov chains, which says that for two (sets of) states \( S_{1}, S_{2} \) if we have a non-zero probability path from \( S_{1} \) to \( S_{2} \) then the probability of taking infinite paths which reach \( S_{1} \) infinitely often, but reach \( S_{2} \) only finitely often is zero.

To begin, we once again define \( \text{UndetApprx}^{(i)} = 1 - \text{PosApprx}^{(i)} - \text{NegApprx}^{(i)} \) and invoke the finite branching of the transition system to go from limit over the iteration count to limit over the depth of search. The following two relations hold.

\[
\text{PosApprx}^{(i)} \leq \Pr_{p}(\gamma_{\text{init}} \models \Diamond l) \leq \text{PosApprx}^{(i)} + \text{UndetApprx}^{(i)}
\]

\[
\lim_{i \to \infty} \text{UndetApprx}^{(i)} = \lim_{d \to \infty} \text{UndetApprx}^{(\text{maxiter}(d))}
\]

Then once again it remains to prove: \( \lim_{d \to \infty} \text{UndetApprx}^{(\text{maxiter}(d))} = 0 \).

Each path \( \rho \in \gamma_{\text{init}} \cdot \Gamma_{d} \) (from the ones considered up to depth \( d \)) fall into three (disjoint) sets: (1) which henceforth will reach \( l \) infinitely often, i.e.\( \exists i \in [0 \ldots d], \pi[i] \models \forall \Box \exists l \) (2) which cannot reach \( l \): \( \neg(\pi[d] \rightarrow p l) \) and (3) undetermined (in none of the sets (1,2)). The probability that a path belongs to (3) is given by \( \text{UndetApprx}^{(\text{maxiter}(d))} \).
For a given $d$, denote the set of these undetermined paths from (3) as $\text{RepUndet}(d)$. Let $V = \{ \pi | \forall d, \pi[0] \cdots \pi[d] \in \text{RepUndet}(d) \} \subseteq \gamma_{\text{init}} \cdot I^\omega$. These are infinite paths from $\gamma_{\text{init}}$, for which all finite prefixes are undetermined w.r.t repeated reachability to $l$. Invoking $\text{Prob}_P(\gamma_{\text{init}} \models \Box \Diamond \gamma_{\text{plain}}) = 1$ we partition $I_{\text{plain}}^P = I_{\text{pos-plain}}^P \uplus I_{\text{neg-plain}}^P$: (1) $I_{\text{pos-plain}}^P = \{ \gamma | \gamma \xrightarrow{-} P \}$ and (2) $I_{\text{neg-plain}}^P = \{ \gamma | \neg (\gamma \rightarrow_P l) \}$. We have the following since $I_{\text{plain}}^P$ is reached repeatedly with probability one.

$$\text{Prob}_P(\gamma_{\text{init}} \models V) = \text{Prob}_P(\gamma_{\text{init}} \models (V \land \Box \Diamond \gamma_{\text{plain}}))$$

However, any path satisfying $\pi \models \Diamond I_{\text{neg-plain}}^P$ cannot belong to $V$ (as it is determined to not reach $l$, it cannot be in $V$), we must have the following.

$$\text{Prob}_P(\gamma_{\text{init}} \models V) = \text{Prob}_P(\gamma_{\text{init}} \models (V \land \Box \Diamond I_{\text{pos-plain}}^P))$$

Since $I_{\text{plain}}^P$ is finite so it $I_{\text{pos-plain}}^P$ and we have

$$\text{Prob}_P(\gamma_{\text{init}} \models V) \leq \sum_{\gamma \in I_{\text{pos-plain}}^P} \text{Prob}_P(\gamma_{\text{init}} \models (V \land \Box \Diamond \gamma))$$

where the sum is over a finite set. Now $V$ consists of paths which never reach $l$. However these paths reach $I_{\text{pos-plain}}^P$ infinitely often. Consequently, since each $\gamma \in I_{\text{pos-plain}}^P$ has a finite path (with non-zero probability) to $l$, and by the (extended) fairness theorem on Markov chains we must have the probability of repeatedly reaching $\gamma \in I_{\text{pos-plain}}^P$ yet reaching $l$ only finitely many times is zero:

$$\text{Prob}_P(\gamma_{\text{init}} \models (V \land \Box \Diamond \gamma)) = 0,$$

and hence,

$$\text{Prob}_P(\gamma_{\text{init}} \models V) \leq 0$$

Again note that the approximation margin term $\text{UndetApprx}^{(d)}$ approaches $\text{Prob}_P(\gamma_{\text{init}} \models V)$ as $d$ approaches infinity (since at a given value of $d$ it expresses the probability of taking paths which are undetermined for $d$ steps). Hence we have

$$\lim_{d \to \infty} \text{UndetApprx}^{(\text{maxiter}(d))} = \text{Prob}_P(\gamma_{\text{init}} \models V) = 0.$$
D.1 The Gambler’s Ruin Problem

We consider the family of Gambler’s Ruin Markov chains $C_{p,q}^G = \langle \mathbb{N}, M_{p,q}^G \rangle$. The family is parameterized by two positive real numbers $p, q \in \mathbb{R}^+ \text{ such that } p + q = 1$. For each instantiation of the parameters, we get a concrete Markov chain. The set of configurations is the set of natural numbers, and the probability matrix is parameterized by $p$ and $q$. More precisely, we have $M_{p,q}^G(i, i + 1) = p$ for $i > 0$, $M_{p,q}^G(i, i - 1) = q$ for $i > 0$, and $M_{p,q}^G(0, 0) = 1$. In other words, the left-most configuration 0 is a sink (a configuration which we cannot leave). In configurations different from 0, we move right with probability $p$ and move left with probability $q$. If $q \geq p$ we say that the Markov chain is “left-oriented”; otherwise we say it “right-oriented”. The following lemma is classical.

**Lemma D1** If $p \leq q$ then $\text{Prob}_{C_{p,q}^G}(k \models \diamondsuit 0) = 1$ for all $k \in \mathbb{N}$.

Lemma [D1] tells us that if the Markov is left-oriented then, from any configuration, we will almost surely reach the sink state.

The following lemma states that if the Markov chain is left-oriented then the probability of reaching a left segment of the chain, within a given number of steps, is higher if we are closer to the left.

**Lemma D2** If $p \leq q$ then $\text{Prob}_{C_{p,q}^G}(k + 1 \models \diamondsuit n 0) \leq \text{Prob}_{C_{p,q}^G}(k \models \diamondsuit n 0)$ for all $k, n \in \mathbb{N}$.

**Proof.** We use induction on $n$. For the base case, with $n = 0$, we know that $\text{Prob}_{C_{p,q}^G}(k + 1 \models \diamondsuit 0) = \text{Prob}_{C_{p,q}^G}(k \models \diamondsuit 0)$. Hence, Lemma [D1] immediately implies the result. For the induction step, we consider two sub-cases. If $k = 0$ then $\text{Prob}_{C_{p,q}^G}(k \models \diamondsuit n 0) = 0$, and the result holds trivially. In the second case, we assume that $k > 0$.

\[
\begin{align*}
\text{Prob}_{C_{p,q}^G}(k \models \diamondsuit n + 10) &= q \cdot \text{Prob}_{C_{p,q}^G}(k - 1 \models \diamondsuit n 0) + p \cdot \text{Prob}_{C_{p,q}^G}(k + 1 \models \diamondsuit n 0) \quad \{\text{Definition of } C^G\} \\
&\leq q \cdot \text{Prob}_{C_{p,q}^G}(k \models \diamondsuit n 0) + p \cdot \text{Prob}_{C_{p,q}^G}(k + 2 \models \diamondsuit n 0) \quad \{\text{Induction Hypothesis}\} \\
&= \text{Prob}_{C_{p,q}^G}(k + 1 \models \diamondsuit n + 10) \quad \{\text{Definition of } C^G\}
\end{align*}
\]

**Corollary 1.** $p \leq q$ and $k_1 \leq k_2$ imply $\text{Prob}_{C_{p,q}^G}(k_1 \models \diamondsuit n 0) \leq \text{Prob}_{C_{p,q}^G}(k_2 \models \diamondsuit n 0)$.

The following lemma is an instantiation of equation (4.14), page 352, in [14]. It gives an upper bound on the probability of avoiding the sink in the $n^{th}$ step, starting from position 1.

**Lemma D3** $\text{Prob}_{C_{p,q}^G}(1 \models \diamondsuit n 0) = \frac{1}{n} \cdot \left(\frac{n}{n+1}\right) \cdot p^{\frac{n-1}{2}} \cdot q^{\frac{n+1}{2}}$, if $n$ is odd, and $\text{Prob}_{C_{p,q}^G}(1 \models \diamondsuit n) = 0$, if $n$ is even.

We use Lemma [D3] to give an upper bound on the probability of avoiding the sink in the next $n$ steps, starting from position 1.
Lemma D4 \( \text{Prob}_{C_{p,q}} \left( 1 \models \hat{\varepsilon}^n 0 \right) \leq \frac{3}{\sqrt{n}} \cdot (4 \cdot p \cdot q)^{\lfloor \frac{n}{2} \rfloor}, \) for all \( n \geq 2, p, \) and \( q. \)

Proof.

\[
\begin{align*}
\text{Prob}_{C_{p,q}} \left( 1 \models \hat{\varepsilon}^n 0 \right) &= \frac{1}{n} \cdot \left( \frac{n}{n+1} \right) \cdot p^{\frac{n+1}{2}} \cdot q^{\frac{n+1}{2}} \quad \{\text{Lemma D3}\} \\
&= \sum_{m=\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} \frac{1}{2m+1} \cdot \left( \frac{2 \cdot m + 1}{m+1} \right) \cdot p^m \cdot q^{m+1} \quad \{n \geq 2\} \\
&= \sum_{m=\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} \frac{1}{2m+1} \cdot \left( \frac{2 \cdot m}{m} \right) \cdot p^m \cdot q^{m+1} \quad \{\text{Algebra}\} \\
&\leq \sum_{m=\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} \frac{1}{2m+1} \cdot \frac{1}{\sqrt{2m}} \cdot 2^{2m} \cdot p^m \cdot q^{m+1} \quad \{\text{LNS}\} \\
&\leq \frac{q}{\sqrt{\pi}} \cdot \sum_{m=\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} \frac{1}{2m+1} \cdot (4 \cdot p \cdot q)^m \quad \{\text{Algebra}\} \\
&\leq \frac{q}{\sqrt{\pi}} \cdot \left( \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor} \cdot \sqrt{\left\lfloor \frac{n}{2} \right\rfloor} \cdot (4pq)^{\frac{n}{2}} \right) + \int_{\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} \frac{1}{m \cdot \sqrt{m}} \cdot (4pq)^m \, dm \quad \{\text{Approximating } \sum \text{ by } \int\} \\
&\leq \frac{q}{\sqrt{\pi}} \cdot \left( \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor} \cdot \sqrt{\left\lfloor \frac{n}{2} \right\rfloor} \cdot (4pq)^{\frac{n}{2}} \right) + 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor^{-\frac{1}{2}} \cdot (4pq)^{\frac{n}{2}} \quad \{\text{Overapprox, since } \int \frac{e^{-x}}{\sqrt{x}} \, dx = -2 \frac{e^{-x}}{\sqrt{x}} - 2 \int \frac{e^{-x}}{\sqrt{x}} \, dx\} \\
&\leq \frac{3}{\sqrt{n}} \cdot (4 \cdot p \cdot q)^{\lfloor \frac{n}{2} \rfloor} \quad \{n \geq 2\}
\end{align*}
\]

D.2 Gravity

In this sub-section, we give the details of Lemma 8.

We define

\[
q_P := \min_{\gamma \in \Gamma_P^5} \sum_{\gamma' \in \Gamma_P^{5,4}} M_P (\gamma, \gamma')
\]

In other words, it is the smallest probability by which a configuration of size 5 will decrease its buffer size in the next transition step, and thus moves to a small configurations.

In Lemma D5, we first show that \( q_P \) is always \( \text{bounded from below } \hat{q} = 2/3. \) We will then use this bound \( \hat{q} \) in our further development.

Lemma D5 For all programs \( \mathcal{P}, \) \( q_P \geq \hat{q}. \)

Proof. The transition from \( \gamma \in \Gamma_P^5 \) is composed of two parts, the \( \rightarrow_{\text{proc}} \) and \( \rightarrow_{\text{update}} \) transitions. The \( \rightarrow_{\text{proc}} \) transition can lead to a configuration \( \gamma' \) of size either 5 (when the process does not take a write transition) or 6 (when the process does take a write transition). Then \( \rightarrow_{\text{update}} \) transition essentially pushes writes from the buffer such that all possible update words are given equal weight. We treat the cases with configuration size 5, 6 separately.
We consider all possible distributions of buffer sizes across the processes. Depending upon the number of processes, \(|\text{Procs}|\), we have different cases. We only need to consider atmost 5 processes since beyond this, the remaining processes must have empty buffers. The possible distributions are as follows (since order is immaterial, we represent distribution as a set):

\[
\{1,1,1,1,1\}, \{2,1,1,1,0\}, \{2,2,1,0,0\}, \{3,1,1,0,0\}, \{3,2,0,0,0\}, \{4,1,0,0,0\}, \{5,0,0,0,0\}.
\]

For each of these distributions, the number of non-empty update words is clearly greater than the number of empty update words (a singleton set, \(\epsilon\)), i.e. 1. Since we choose uniformly across all update words, we choose the non-empty word with probability greater than 0.5 and hence reach a configuration in \(\Gamma_p^{\leq 4}\) w.p. greater than 0.5.

\[|\gamma'| = 6\] Now it suffices to consider 6 processes. The possible distributions of the buffer sizes across these are as follows (since order is immaterial, we represent distribution as a set):

\[
\{1,1,1,1,1,1\}, \{2,1,1,1,1,0\}, \{2,2,1,1,0,0\}, \{2,2,2,0,0,0\}, \{3,1,1,1,0,0\}, \{3,2,1,0,0,0\}, \\
\{3,3,0,0,0,0\}, \{4,1,1,0,0,0\}, \{4,2,0,0,0,0\}, \{5,1,0,0,0,0\}, \{6,0,0,0,0,0\}.
\]

Once again the number of update words of length less or equal to 1 for each of these distributions are 7, 6, 5, 4, 5, 4, 3, 4, 3, 3, 2 respectively. This is clearly less than half the total number of update words of each configuration. On choosing an update word of length greater than one, we reach a configuration in \(\Gamma_p^{\leq 4}\). Since we choose uniformly amongst these words, and since words longer than 1 outnumber those less or equal to 1, we reach a configuration in \(\Gamma_p^{\leq 4}\) w.p. greater than 0.5.

Since we show this without making any assumption on the intermediate configuration \(\gamma'\) (except for the size), we can conclude that \(q_p > 0.5\). On enumerating the exact update word counts for each of the above cases, we verify that \(\hat{q} = \frac{2}{3}\) satisfies the needed constraints.

Henceforth, we will continue to use the symbol \(\hat{q}\), instead of the concrete value, to make the terms in the presentation clearer to understand. However, we highlight that we the concrete value of \(\frac{2}{3}\) that can be substituted in place of \(\hat{q}\). The next lemma states that probability of decreasing the size of the buffer is at least \(\hat{q}\) for all configuration of size at least 4.

**Lemma D6** For any \(\gamma \in \Gamma_p^{\geq 5}\), we have \(\sum_{\gamma' \in \Gamma_p^{< |\gamma|}} m_p(\gamma, \gamma') \geq \hat{q}\).

**Proof.** By a similar reasoning as the earlier, we see that starting from the configuration \(\gamma\), following the \(\rightarrow_{\text{proc}}\) transition, the intermediate configuration \(\gamma'\) has size \(\gamma' \in \{|\gamma|, |\gamma| + 1\}\). We can consider the update words for both possibilities.

\[|\gamma'| = |\gamma|\] For this the only empty update word leads to a configuration in \(\Gamma_p^{= |\gamma|}\). On the other hand, the total number of update words is atleast greater than the configuration size (atleast one possible update word for each number of single updates). Hence we have that the probability to reach \(\Gamma_p^{\leq |\gamma|}\) from \(\gamma'\) is
\[
\frac{|\gamma|}{|\gamma|+1} \geq \frac{5}{6} \quad \text{(since } |\gamma| \geq 5) \text{. This gives us the following for all } \gamma \in \Gamma_{P}^{\geq 5}.
\]
\[
\sum_{\gamma' \in \Gamma_{P}^{<|\gamma|}} \text{M}_{\text{update}}(\gamma, \gamma') > \frac{2}{3} \hat{q}.
\]

Let the distribution of buffer contents across processes be \(\{b_0, b_1, \cdots\}\). Then the number of update words of length \(|\gamma'| = \sum b_i\) is given by the multinomial coefficient \(\binom{|\gamma'|}{b_0, b_1, \cdots}\). For this case update words of length 0 or 1 lead to a configuration not in \(\Gamma_{P}^{<|\gamma|}\). The number of these words is \(1 + \sum_i 1\) where \(1\) is the indicator function. It is clear that the multinomial coefficient is greater than the this expression by atleast a factor of 2 for \(|\gamma'| \geq 5\). This follows from the fact that under the constraints \(\sum_i b_i = |\gamma'|\) and \(\sum_i 1 = c\) for some fixed \(c\), the largest value of \(\prod_i b_i!\) (and hence the smallest value of the multinomial coefficient) is given by the distribution \(\{b_i\} = \{|\gamma'|-c+1, 1, 1, \cdots, 1, 0, 0, \cdots\}\) \(c-1\) times.

Hence the probability that the length of the update word is atmost 1 is less than \(\frac{1}{3}\), and we have the following.
\[
\sum_{\gamma' \in \Gamma_{P}^{<|\gamma|-1}} \text{M}_{\text{update}}(\gamma, \gamma') > \frac{2}{3} \hat{q}.
\]

Since we showed the above two inequalities for all configurations, this holds for any possible \(\rightarrow_{\text{proc}}\) transition and hence we have the result as desired.

We define specific Gambler’s Ruin’s Markov chain, induced by the program \(P\), namely \(C_{P}^{G} := C_{\hat{p}, \hat{q}}^{G}\). From Lemma D1 and Lemma D5 we get the following lemma.

**Lemma D7** \(\text{Prob}_{C_{P}^{G}}(k |\models \emptyset 0) = 1\), for all \(k \in \mathbb{N}\) and \(\ell \geq 5\).

We consider the probability of reaching the set \(\Gamma_{P}^{\text{small}}\) of small configurations. To that end, we define the function
\[
y: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R} \text{ where } y(k, n) := \max_{\gamma \in \Gamma_{P}^{\leq k}} \text{Prob}_{P} (\gamma |\models \emptyset^n \Gamma_{P}^{\text{small}})
\]

In other words, it is the maximum of the probability measures by which runs from configurations of size \(k\) can avoid small configurations in the next \(n\) steps. The following lemma relates this probability with the corresponding probability in the Gambler Ruin’s problem. Essentially, the lemma abstracts the set of configurations of \([P]^{MC}\) to the configurations in \(C_{P}^{G}\) as given by the following level function, which was first introduced in \(A\) level(\(\gamma\)) = 0 if \(\gamma \in \Gamma_{P}^{\leq 4}\), and level(\(\gamma\)) = \(|\gamma|\) otherwise.

The following lemma follows directly from the fact that the size of the configuration will never increase by more than one in PTSO (which is the case when a write \(\rightarrow_{\text{proc}}\) transition is taken, and no element of any buffer is pushed to the memory).

**Lemma D8** \(\sum_{\gamma' \in \Gamma_{P}^{\leq n+1}} \text{M}_{P}(\gamma, \gamma') = 1\), for all \(\gamma \in \Gamma_{P}^{n}\) for all \(n\).
Lemma D9  \( y(k + 4, n) \leq \text{Prob}_{C_p} (k \models \hat{\Theta}^n 0) \), for all \( k, n \in \mathbb{N} \).

\[ \text{Proof.} \text{ We use induction on } n. \text{ In the base case, we have } n = 0. \text{ By Lemma D1 and Lemma D5, it follows that } \text{Prob}_{C_p} (k \models \hat{\Theta}^0 0) = 1, \text{ and the result follows immediately.} \]

\[ \text{For the induction step we consider two cases, namely when } k = 1 \text{ and when } k \geq 1. \text{ If } k = 1 \text{ then } y(k + 4, n + 1) = 0 \text{ and the results follows immediately. If } k \geq 1, \text{ we fix } \gamma \in \mathbb{G}^{k+4}, \text{ where } k \geq 1, \text{ such that } y(k + 3, n + 1) = \text{Prob}_{C_p} (\gamma \models \hat{\Theta}^{n+1} \mathbb{G}^{k+4}). \text{ Such a configuration exists by the definition of } y. \]

\[ y(k + 4, n + 1) = \{ \text{Definition of } \gamma \} \]

\[ = \{ \text{Lemma D8} \} \]

\[ = \{ \text{Lemma D8} \} \]

\[ \sum_{j=0}^{k+3} \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \cdot \text{Prob}_\mathbb{P} (\gamma' \models \hat{\Theta}^n \mathbb{G}^{j+4}) \]

\[ + \sum_{j=0}^{k+3} \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \cdot \text{Prob}_\mathbb{P} (\gamma' \models \hat{\Theta}^n \mathbb{G}^{j+4}) \]

\[ + \sum_{j=0}^{k+3} \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \cdot \text{Prob}_\mathbb{P} (\gamma' \models \hat{\Theta}^n \mathbb{G}^{j+4}) \]

\[ \leq \{ \text{Definition of } y \} \]

\[ \sum_{j=0}^{k+3} y(j, n) \cdot \left( \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + y(k + 4, n) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + y(k + 5, n) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+5}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ \leq \{ \text{Induction Hypothesis} \} \]

\[ \sum_{j=0}^{k+3} \text{Prob}_{C_p} (j - 4 \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + \text{Prob}_{C_p} (k \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + \text{Prob}_{C_p} (k + 1 \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+5}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ \leq \{ \text{Corollary D} \} \]

\[ \text{Prob}_{C_p} (k - 1 \models \hat{\Theta}^n 0) \cdot \left( \sum_{j=0}^{k+3} \sum_{\gamma \in \mathbb{G}^{j+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + \text{Prob}_{C_p} (k + 1 \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+4}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + \text{Prob}_{C_p} (k + 1 \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+5}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ = \{ \text{Lemma D8} \} \]

\[ \text{Prob}_{C_p} (k - 1 \models \hat{\Theta}^n 0) \cdot \left( \sum_{\gamma \in \mathbb{G}^{k+3}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ + \text{Prob}_{C_p} (k + 1 \models \hat{\Theta}^n 0) \cdot \left( 1 - \sum_{\gamma \in \mathbb{G}^{k+3}} \text{M}_\mathbb{P} (\gamma, \gamma') \right) \]

\[ \leq \{ \text{Lemma D5 and Lemma D6} \} \]

\[ \hat{q} \cdot \text{Prob}_{C_p} (k - 1 \models \hat{\Theta}^n 0) + \hat{p} \cdot \text{Prob}_{C_p} (k + 1 \models \hat{\Theta}^n 0) \]

\[ = \{ \text{Definition of } C_p \} \]

\[ \text{Prob}_{C_p} (k \models \hat{\Theta}^{n+1} 0) \]
From Lemma \[D9\] and Lemma \[D4\] we get the following lemma.

**Lemma D10** \( y(k + 4, n) \leq \frac{3 \hat{q}}{\sqrt{\pi}} \cdot (4 \cdot \hat{p} \cdot \hat{q})^{\frac{n}{2}}, \) for all \( n \geq 2 \).

Now we have the ingredients to formally prove Lemma \[8\]

**Lemma 8 (Gravity Lemma)** \( \text{Prob}_P(\gamma \models_P \bigcirc^n \Gamma_P^{\text{small}}) \leq (G_P)^n, \) for all \( \gamma \in \Gamma_P^{\text{small}} \) and all \( n \in \mathbb{N} \).

**Proof.** If \( n = 0 \) then \( \text{Prob}_P(\gamma \models_P \bigcirc^0 \Gamma_P^{\text{small}}) \leq 1 \leq (G_P)^0 \)

If \( n = 1 \) then, by Lemma \[D10\] we have

\[
\begin{align*}
\text{Prob}_P(\gamma \models_P \bigcirc^1 \Gamma_P^{\text{small}}) & \leq \hat{p} \quad \{\text{Definition of } \hat{p}\} \\
& \leq \sqrt{4 \cdot \hat{p} \cdot \hat{q}} \quad \{\text{Since } \hat{q} > \hat{p}\} \\
& \leq G_P
\end{align*}
\]

If \( n \geq 2 \).

\[
\begin{align*}
\text{Prob}_P(\gamma \models_P \bigcirc^n \Gamma_P^{\text{small}}) & \leq \hat{p} \cdot y(4, n + 1) \quad \{\text{Definition of } y \text{ and } \hat{p}\} \\
& \leq \hat{p} \cdot \text{Prob}_{C_{G_P}}(1 \models \bigcirc^{n+1} 0) \quad \{\text{Algebra and } 4 \cdot \hat{p} \cdot \hat{q} < 1\} \\
& \leq \frac{3 \hat{q} \hat{p}}{\sqrt{\pi}} \cdot (4 \cdot \hat{p} \cdot \hat{q})^{\frac{n}{2}} \quad \{\text{Lemma } [D9]\} \\
& \leq (4 \cdot \hat{p} \cdot \hat{q})^{\frac{n}{2}} \quad \{3 < \pi \text{ and } \hat{p} \cdot \hat{q} < 1\} \\
& \leq (G_P)^n \quad \{\text{Definition of } G_P\}
\end{align*}
\]
D.3 S-Runs and F-Runs

We first define the predicate Visit formally. For a natural number $1 \leq m \leq n$, let $n \oplus m \subseteq (\mathbb{N}^{>0})^m$ be the set of words $w = i_1, \ldots, i_m$ of length $m$, over the set of positive natural numbers, such that $i_1 + \cdots + i_m = n$. Notice that

$$|n \oplus m| = \binom{n - 1}{m - 1}$$

We define $n^{\oplus} := \cup_{1 \leq m \leq n} n \oplus m$. For $w \in n^{\oplus}$, we define $\text{Visit}_P (n, w)$ to be the set of runs of the form $\gamma_0 \cdot \pi_0 \cdot \gamma_1 \cdot \cdots \cdot \gamma_m \cdot \pi_m \cdot \rho'$, such that the following conditions are satisfied

\begin{itemize}
  \item $m = |w|.$
  \item $\forall i: 0 \leq i \leq m : w[i] = |\pi_i| + 1.$
  \item $\forall i: 0 \leq i \leq m : \gamma_i \in \Gamma_P^{\text{small}}.$
  \item $\forall i: 0 \leq i \leq m : \forall j: 1 \leq j \leq |\pi_i| : \pi_i[j] \not\in \Gamma_P^{\text{small}}.$
\end{itemize}

We define $\text{Visit}_P (n, m) := \cup_{w \in (n^{\oplus})} \text{Visit}_P (n, w)$. Intuitively, $\text{Visit}_P (n, m)$ is the set of runs whose prefixes of length $n$ visit the set of small configurations exactly $m$ times.

**Lemma D11** For every $m, n : 1 \leq m \leq n$, and $\gamma \in \Gamma_P^{\text{small}}$, we have $\text{Prob}_P (\gamma \models_P \text{Visit}_P (n, w)) \leq (G_P)^{n - m}$

**Proof.** We use induction on $m$.

The base case corresponds to $m = 1$, i.e. $w \in (n^{\oplus})$.

$$\text{Prob}_P (\gamma \models_P \text{Visit}_P (n, w)) = \text{Prob}_P (\gamma \models_P \bigcap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus}) \cap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus})) \leq (G_P)^{n - 1} \quad \{\text{Lemma 8}\}$$

Let $w = \gamma_0 \cdot \pi_0 \cdot \gamma_1 \cdot \cdots \cdot \gamma_m \cdot \pi_m \cdot \rho'$. Define $v := \gamma_1 \cdot \cdots \cdot \gamma_m \cdot \pi_m \cdot \rho'$. We know that

$$\text{Prob}_P (\gamma \models_P \text{Visit}_P (n, w)) = \sum_{\gamma' \in \Gamma_P^{\text{small}}} \text{Prob}_P (\gamma \models_P \bigcap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus}) \cap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus})) \cdot \text{Prob}_P (\gamma' \models_P \text{Visit}_P (n - |\pi_0| - 1, v)) \quad \{\text{Definition of Visit}\}$$

$$\leq \sum_{\gamma' \in \Gamma_P^{\text{small}}} \text{Prob}_P (\gamma \models_P \bigcap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus}) \cap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus})) \cdot \text{Prob}_P (\gamma' \models_P \text{Visit}_P (n - |\pi_0| - 1^{\oplus} - (m - 1))) \quad \{\text{Induction Hypothesis}\}$$

$$\leq (G_P)^{n - |\pi_0| - m} \cdot \sum_{\gamma' \in \Gamma_P^{\text{small}}} \text{Prob}_P (\gamma \models_P \bigcap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus}) \cap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus})) \quad \{\text{Algebra}\}$$

$$\leq (G_P)^{n - |\pi_0| - m} \cdot \text{Prob}_P (\gamma \models_P \bigcap \text{small} \Gamma_P (n^{\oplus} - n^{\oplus})) \quad \{\text{Definition of Prob}_P\}$$

$$\leq (G_P)^{n - |\pi_0| - m} \cdot (G_P)^{|\pi_0|} \quad \{\text{Lemma 8}\}$$

$$\leq (G_P)^{n - m} \quad \{\text{Algebra}\}$$
We now recall and give the proof of Lemma 9.

**Lemma 9 (S-Run Bound)** \(\text{Prob}_P (\gamma \models_P \text{SRuns}(\gamma)(n)) \leq (e^8)^n\), for all \(\gamma \in \Gamma_P^{\text{small}}\) and all \(n\) such that \(300 = 2 \cdot \nu \leq n\).

**Proof.**

\[
\text{Prob}_P (\text{SRuns}(\gamma)(n)) = \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \text{Prob}_P (\gamma \models_P \Diamond^n \wedge \text{Visit}_P (n, m)) \quad \{\text{Definition of s-runs}\}
\]
\[
\leq \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \text{Prob}_P (\gamma \models_P \text{Visit}_P (n, m)) \quad \{\text{Definition of } \text{Prob}_P\}
\]
\[
= \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \sum_{w \models n \oplus m} \text{Prob}_P (\gamma \models_P \text{Visit}_P (n, m)) \quad \{\text{Definition of } \oplus\}
\]
\[
= \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \sum_{w \models n \oplus m} g_P^{n-m} \quad \{\text{Lemma D11}\}
\]
\[
\leq \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n-1}{m-1} \cdot g_P^{n-m}\right) \quad \{\text{Algebra}\}
\]
\[
= g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n-1}{m-1} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n}{m} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n}{m} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n}{m} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n}{m} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \sum_{m=1}^{\lfloor \frac{n}{\nu} \rfloor} \left( \frac{n}{m} \cdot g_P^{-m}\right) \right) \quad \{\text{Algebra}\}
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \cdot n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra: } \sum_{b=0}^{\infty} \left( \frac{G}{b} \cdot x^b = (1 + x)^G \right)\}
\]
\[
\leq g_P^n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra}\}
\]
\[
\leq g_P^n \cdot \left( \frac{n}{\nu} \right)^{\frac{n}{2}} \cdot \nu^{\frac{n}{2}} \cdot \left( 1 + \left( \frac{\nu}{\nu^n} \right)^{\frac{n}{2}} \right) \quad \{\text{Algebra}\}
\]
\[
= \left( \frac{\nu}{\nu-1} \right)^n \cdot \left( 2 \cdot \nu \right)^{\frac{1}{\nu}} \cdot \left( \frac{1}{\nu} + \frac{1}{\nu^n} \right)^{\frac{1}{\nu}} \quad \{4 \leq 2 \cdot \nu \leq n\}
\]
\[
\leq g_P^n \cdot \left( \left( \frac{\nu}{\nu-1} \right)^n \cdot \left( 2 \cdot \nu \right)^{\frac{1}{\nu}} \cdot \left( \frac{1}{\nu} + \frac{1}{\nu^n} \right)^{\frac{1}{\nu}} \right) \quad \{\text{Algebra}\}
\]
\[
\leq \left( \left( \frac{\nu}{\nu-1} \right)^n \cdot \left( 2 \cdot \nu \right)^{\frac{1}{\nu}} \cdot \left( \frac{1}{\nu} + \frac{1}{\nu^n} \right)^{\frac{1}{\nu}} \right) \cdot g_P^n \quad \{\text{Algebra}\}
\]
\[
\leq (e^8)^n \quad \{\text{Definition}\}
\]
Next, we define a bound on \( D_{\text{Runs}}(\gamma)(l)(n) \). We characterize the set of runs that visit the set of small configurations “many times” before visiting \( l \). For sets of configurations \( G_1, G_2 \subseteq \Gamma_P \), a run \( \rho \), and \( m \in \mathbb{N} \), we write \( \rho \models G_1 \text{Before}^m G_2 \) to denote that \( \rho = \pi \cdot \rho' \) for some path \( \pi \) and run \( \rho' \), \( \pi \) is of the form \( \pi_1 \cdot \gamma_1 \cdots \cdot \pi_m \cdot \gamma_m \), and the following conditions are satisfied

\[ \triangleright \gamma_i \in G_1 \text{ for all } i : 1 \leq i \leq m. \]
\[ \triangleright \pi[i] \notin G_2 \text{ for all } i : 0 \leq i \leq |\pi|. \]

In other words \( \rho \) visits the set \( G_1 \) at least \( m \) times before visiting \( G_2 \) for the first time. We usually write \( \text{Before} \) instead of \( \text{Before}^1 \), and write \( \gamma \text{Before}^k G \) instead of \( \{\gamma\} \text{Before}^k G \). Define the set \( \mathcal{A} \) of small configurations from which \( l \) is reachable.

\[ \mathcal{A} := \Gamma_P^{\text{small}} \cap (\gamma \models_P \exists (l)) \]

Consider a \( \mu \) satisfying (well defined since \( \mathcal{A} \) is finite),

\[ 0 < \mu \leq \min_{\gamma \in \mathcal{A}} \text{Prob}_P(\gamma \models \bigcirc (l \text{Before } \gamma)). \]

This means that \( \mu \) is a lower bound on the measure of runs that start from some configuration in \( \gamma \in \mathcal{A} \) and visit \( l \) before visiting \( \gamma \).

**Lemma D12** \( \text{Prob}_P(\gamma \models \gamma \text{Before}^m l) \leq (1 - \mu)^{m-1} \) for each \( \gamma \in \mathcal{A} \).

**Proof.** By induction on \( m \). The base case, when \( m = 1 \) is trivial.

For the induction step, we observe that, by definition, we have

\[ \text{Prob}_P(\gamma \models \gamma \text{Before}^2 l) = \text{Prob}_P(\gamma \models \bigcirc (\gamma \text{Before } l)) \leq (1 - \mu) \]

By the induction hypothesis we know that

\[ \text{Prob}_P(\gamma \models \gamma \text{Before}^{m-1} l) \leq (1 - \mu)^{m-2} \]

We obtain

\[ \text{Prob}_P(\gamma \models \gamma \text{Before}^m l) = \text{Prob}_P(\gamma \models \gamma \text{Before}^2 l) \cdot \text{Prob}_P(\gamma \models \gamma \text{Before}^{m-2} l) \leq (1 - \mu) \cdot (1 - \mu)^{m-2} = (1 - \mu)^m \]
We now recall and give the proof of Lemma 10.

**Lemma 10 (D-Run Bound)** \( \text{Prob} (D\text{Runs} (\gamma) (l) (n)) \leq (\mathcal{E}^D_P)^n \), for all \( \gamma \in \Gamma^\text{small}_P \) and all \( n \geq n^P_P \).

**Proof.** There are two possible cases: (i) \( \gamma \in \Gamma^\text{small}_P - A \). From the definitions, it follows that \( \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \text{Prob} (\gamma \models_p \checkmark^n \land \text{Visit}_P (n, m)) = 0 \). (ii) \( \gamma \in A \). We analyze this case below.

For any \( m \) we have that

\[
\begin{align*}
\text{Prob} (D\text{Runs} (\gamma) (l) (n)) & = \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \text{Prob} (\gamma \models_p \checkmark^n \land \text{Visit}_P (n, m)) \quad \text{(Definition of F-Runs)} \\
& \leq \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \text{Prob} (\gamma \models_A \text{Before}^m) \quad \text{(Definitions of \( \checkmark^n \), Visit, and Before)} \\
& \leq \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \sum_{\gamma' \in A} \text{Prob} (\gamma' \models \gamma' \text{Before} [\frac{m}{|A|}]) \\
& \quad \text{(Finiteness of \( A \) and pigeonhole principle)} \\
& \leq \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \sum_{\gamma' \in A} \text{Prob} (\gamma' \models \gamma' \text{Before} [\frac{m}{|A|}]) \cdot \text{Prob} (\gamma' \models \gamma' \text{Before} [\frac{m}{|A|}]) \quad \text{(Definition of Before)} \\
& \leq \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \sum_{\gamma' \in A} \text{Prob} (\gamma' \models \gamma' \text{Before} [\frac{m}{|A|}]) \quad \text{(Prob (\( \cdot \)) \leq 1)} \\
& \leq \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n \sum_{\gamma' \in A} (1 - \mu)^{\frac{m}{|A|} - 1} \quad \text{(Lemma D12)} \\
& \leq \frac{|A|}{1 - \mu} \cdot \sum_{m=\lceil \frac{n}{L} \rceil + 1}^n (1 - \mu)^{\frac{m}{|A|}} \\
& = \frac{|A|}{1 - \mu} \cdot \frac{(1 - \mu)^{\frac{n}{|A|} - 1} - (1 - \mu)^{\frac{n+1}{|A|}}}{1 - (1 - \mu)^{\frac{n}{|A|}}} \quad \text{(Algebra)} \\
& \leq \frac{|A|}{1 - \mu} \cdot \frac{(1 - \mu)^{\frac{n}{|A|} - 1} - (1 - \mu)^{\frac{n+1}{|A|}}}{1 - (1 - \mu)^{\frac{n}{|A|}}} \quad \text{(Algebra)} \\
& = \frac{|A|}{1 - \mu} \cdot \frac{(1 - \mu)^{\frac{n}{|A|} - 1} - (1 - \mu)^{\frac{n+1}{|A|}}}{1 - (1 - \mu)^{\frac{n}{|A|}}} \quad \text{(Algebra)} \\
& = \frac{|A|}{|A|} \cdot \frac{|A|}{1 - (1 - \mu)^{\frac{n}{|A|}}} \cdot \left( (1 - \mu)^{\frac{1}{|A|}} \right)^n \quad \text{(Algebra)} \\
& = (\mathcal{E}^D_P)^n \quad \text{(Definition)}
\end{align*}
\]
D.4 Eagerness: Existence and Computability

The results from D.3 give us all the ingredients that were necessary to prove Lemma 7. We briefly discussed the proof idea for this lemma in the main paper. We give here the proof with full details.

Lemma 7 (Eagerness Lemma) \( E_P \) and \( \eta_P \) exist and are computable.

Existence We start off by showing the existence of \( E_P \) and \( \eta_P \).

Proof. For values \( E_P^S, E_P^D < 1 \), we have,

For \( n \geq 300 \):
\[
\sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \text{Prob} (\gamma \models p \diamond^n \land \text{Visit}_P (n, m)) \leq (E_P^S)^n \quad \{\text{Lemma 7}\}
\]

For \( n \geq \eta_P^D \):
\[
\sum_{m=\lfloor \frac{n}{2} \rfloor}^{n} \text{Prob} (\gamma \models p \diamond^n \land \text{Visit}_P (n, m)) \leq (E_P^D)^n \quad \{\text{Lemma 10}\}
\]

Choose (exists since \( E_P^S, E_P^D < 1 \)) a value \( E_P^S \) such that, \( \max(E_P^S, E_P^D) < E_P^S < 1 \).

It follows that for some value \( n^1 \), \( \text{Prob} (\gamma \models p \diamond^n \land \text{Visit}_P (n, m)) \leq (E_P^S)^n \) for all \( n \geq n^1 \).

Define \( \eta_P^D = \max(\eta_P^D, 300, n^1) \). From the earlier two results, we get the following,

For \( n \geq \eta_P^D \):
\[
\text{Prob} (\gamma \models p \diamond^n) \leq (E_P^S)^n
\]

The final step is to extend the argument to the set of \( \gamma \)-runs that reach \( l \) in \( n \) or more steps (as required by Lemma 7).

\[
\text{Prob} (\gamma \models p \diamond^n) = \sum_{k=n}^{\infty} \text{Prob} (\gamma \models p \diamond^n) \leq \sum_{k=n}^{\infty} (E_P^S)^k = \frac{(E_P^S)^n}{1-E_P^S}
\]

Choose (exists since \( E_P^S < 1 \)) \( E_P^S < E_P^S < 1 \). There exists an \( \eta_P \geq \eta_P^D \) such that \( \frac{(E_P^S)^n}{1-E_P^S} \leq (E_P^S)^n \) for all \( n \geq \eta_P \), and hence we have,

For \( n \geq \eta_P \):
\[
\text{Prob} (\gamma \models p \diamond^n) \geq (E_P^S)^n
\]

This gives us the result.

Computability Now we show that computability of these terms. We proceed systematically along the dependencies and illustrate how each term can be computed, not just for our model but for arbitrary models.

Proof. \( \triangleright \hat{q} := \min_{\gamma \in \Gamma_P^{=3}} \sum_{\gamma' \in \Gamma_P^{\text{small}}} M_P (\gamma, \gamma') \). Computable since the sets \( \Gamma_P^{=4} \) and \( \Gamma_P^{\leq 3} \) are finite, and for any two configurations \( \gamma, \gamma' \in \Gamma \), we can compute \( M_P (\gamma, \gamma') \). In fact, for us, this has the constant value of 2/3.

\( \triangleright \hat{p} := 1 - \hat{q} \). Computable since \( \hat{q} \) is computable.

\( \triangleright \mathcal{E}_P := 2 \sqrt{\hat{p} \cdot \hat{q}} \). Computable since \( \hat{q} \) and \( \hat{p} \) are computable.

\( \triangleright \mathcal{A} \) is computable since \( \mathcal{A} = \Gamma_P^{\text{small}} \cap \{ \rho \in \text{Runs} (\gamma) \mid \rho \models p \exists \Box (l) \} \), the set \( \Gamma_P^{\text{small}} \) is finite (and explicitly given), and the property \( \gamma \models p \exists \Box (l) \) is decidable by Lemma 4.
Compute $\nu$ such that

$$\left(\frac{\nu}{\nu - 1}\right) \left(2 \cdot \nu\right)^{\frac{1}{\nu}} \cdot \left(\frac{1}{\nu} + \frac{1}{G_P}\right)^{\left\lfloor \frac{1}{\nu} \right\rfloor} \cdot G_P \leq 1$$

This is possible since the function is monotone and its limit approaches $G_P < 1$, i.e.,

$$\lim_{\nu \to \infty} \left(\frac{\nu}{\nu - 1}\right) \left(2 \cdot \nu\right)^{\frac{1}{\nu}} \cdot \left(\frac{1}{\nu} + \frac{1}{G_P}\right)^{\left\lfloor \frac{1}{\nu} \right\rfloor} \cdot G_P = G_P < 1$$

Define $E_P^1 := \left(\frac{\nu}{\nu - 1}\right) \left(2 \cdot \nu\right)^{\frac{1}{\nu}} \cdot \left(\frac{1}{\nu} + \frac{1}{G_P}\right)^{\left\lfloor \frac{1}{\nu} \right\rfloor} \cdot G_P$

This is possible since both $G_P$ and $\nu$ are computable. In fact for our model, the constant value of $\nu = 150$ suffices.

Define $E_P^2$ such that $(1 - \mu)^{\frac{|\mathcal{A}|}{|\mathcal{A}|}} < E_P^2 < 1$. We can compute $E_P^2$ since $\nu, \mathcal{A}, \mu$ are computable. Since $(1 - \mu)^{\frac{|\mathcal{A}|}{|\mathcal{A}|}} < E_P^2$ it follows that there is a natural number, which we denote by $\eta_P^2$ such that $(E_P^2)^n \leq \left(1 - (1 - \mu)^{\frac{|\mathcal{A}|}{|\mathcal{A}|}}\right) \cdot \left(1 - \mu\right)^{\frac{|\mathcal{A}|}{|\mathcal{A}|}}$.

Define $E_P$ such that $\max(E_P^1, E_P^2) < E_P < 1$. It follows that there is a natural number, which we denote by $\eta_P$ such that $(E_P)^n < \left(\max(E_P^1, E_P^2)\right)^n$, for all $n \geq \eta_P$.

This concludes the proof of computability and hence gives us Lemma 7.
D.5 Proving the invariants

We now prove Lemma 11 that states the validity of the invariants.

**Lemma 11** The algorithm maintains the following invariants where invariants (1, 2, 5, 6) hold for all \( i > 0 \) and invariants (3, 4) hold for all \( i \geq \eta_P \).

The invariants (1), (2), (5), and (6) follow directly from the definitions. Below, we show Invariant (3) and (4).

**Invariant (3):** Let \( k = \text{MaxCost}(\text{Cost}) \).

\[
E(X, l, \text{Cost}) - \text{CostApprx}^{(n)} = \sum_{i=0}^\infty \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Cost}(\rho) \cdot \text{Prob}_{P}^{\rho} - \text{CostApprx}^{(n)} \quad \{\text{Definition}\}
\]

\[
= \sum_{i=0}^\infty \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Cost}(\rho) \cdot \text{Prob}_{P}^{\rho} - \sum_{i=0}^n \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Cost}(\rho) \cdot \text{Prob}_{P}^{\rho} \quad \{\text{Invariant 1}\}
\]

\[
\leq k \cdot \sum_{i=0}^\infty i \cdot \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Prob}_{P}^{\rho} - k \cdot \sum_{i=0}^n \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Prob}_{P}^{\rho} \quad \{\rho \models \varphi^i \implies \text{Cost}(\rho) \leq k \cdot i\}
\]

\[
\leq k \cdot \sum_{i=n}^\infty i \cdot \text{Prob}_{P}^{\gamma_{\text{init}} | \varphi^i} \quad \{\text{Lemma 7 and } n \geq \eta_P\}
\]

\[
\leq k \cdot \sum_{i=n}^\infty \frac{E^i_P}{(1 - E_P)^2} \quad \{E_P < 1\}
\]

\[
= \text{CostError}^{(n)} \quad \{\text{Invariant 5}\}
\]

**Invariant (4):**

\[
\text{Prob}_{P}^{\pi} - \text{ProbApprx}^{(n)} = \sum_{i=0}^\infty \sum_{\rho \in \{\rho \in \text{Runs}(\gamma_{\text{init}}) | |\rho| = \varphi^i\}} \text{Prob}_{P}^{\rho} - \text{ProbApprx}^{(n)} \quad \{\text{By definition}\}
\]

\[
= \sum_{i=0}^\infty \text{Prob}_{P}^{\gamma_{\text{init}} | \varphi^i} - \sum_{i=0}^n \text{Prob}_{P}^{\gamma_{\text{init}} | \varphi^i} \quad \{\text{Invariant 2}\}
\]

\[
= \sum_{i=n}^\infty \text{Prob}_{P}^{\gamma_{\text{init}} | \varphi^i} \quad \{\text{Lemma 7}\}
\]

\[
\leq \sum_{i=n}^\infty E^i_P \quad \{\text{Lemma 7}\}
\]

\[
\leq \frac{E^n_P}{1 - E_P} \quad \{E_P < 1\}
\]

\[
= \text{ProbError}^{(n)} \quad \{\text{Invariant 6}\}
\]