Counting families of mutually intersecting sets

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To Cecile, on the occasion of her 65th birthday

Abstract

We determine the number of maximal intersecting families on a 9-set and find 423295099074735261880. We determine the number of independent sets of the Kneser graph \( K(9, 4) \) and find 366996244568643864340. Finally, we determine the number of intersecting families on an 8-set and find 14704022144627161780744368338695925314597520.

1 Introduction

Let \( X \) be a finite set, and \( \mathcal{F} \) a collection of subsets of \( X \). We call \( \mathcal{F} \) linked (or intersecting) when any two (not necessarily distinct) members of \( \mathcal{F} \) have nonempty intersection. We call \( \mathcal{F} \) a maximal linked system (mls) when \( \mathcal{F} \) is linked, but no strictly larger collection of subsets of \( X \) is linked. Let \(|X| = n\). If \( n > 0 \), then a maximal linked system has size \( 2^n - 1 \) (since it contains precisely one of \( A, X \setminus A \) for each subset \( A \) of \( X \)).

For a family \( \mathcal{F} \) of subsets of \( X \), let \( \mathcal{F}^{\text{min}} \) be the collection of inclusion-wise minimal elements of \( \mathcal{F} \). Then \( \mathcal{F}^{\text{min}} \) is an antichain. Let \( \mathcal{F}^{\uparrow} \) be the collection of subsets of \( X \) containing some element of \( \mathcal{F} \). If \( \mathcal{F} \) is an mls, then \( \mathcal{F} = \mathcal{F}^{\text{min}}^{\uparrow} \).

The Kneser graph \( K(n, r) \) is the graph with as vertices the \( r \)-subsets of a fixed \( n \)-set, two vertices being adjacent when they are disjoint. It follows that a coclique (independent set) in \( K(n, r) \) is a collection of mutually intersecting \( r \)-subsets of the given \( n \)-set.

Let \( \lambda(n) \) and \( \Lambda(n) \) be the number of maximal linked systems and linked systems on \( n \) points, respectively. In this note we determine \( \lambda(9) \) and \( \Lambda(8) \). The numbers \( \lambda(n) \) (and, to a lesser degree, \( \Lambda(n) \)) play a rôle in various areas of mathematics. The description in terms of maximal linked systems is from topology (giving the size of the superextension of a finite space). In this setting, \( \lambda(n) \) with \( n \leq 6 \) was determined by G. A. Jensen in 1966, \( \lambda(7) \) was found in [1], and \( \lambda(8) \) in [10]. An equivalent formulation comes from the area of Boolean functions (see below) where \( \lambda(n) \) is the number of self-dual monotone Boolean functions of \( n \) variables. Knuth [6] computes \( \lambda(n) \) for \( n \leq 8 \). Hosten & Morris [4] found that the order dimension of the complete graph \( K_n \) is the smallest \( t \) for which \( \lambda(t - 1) \geq n \), and computed \( \lambda(n) \) for \( n \leq 6 \). Conway and Loeb studied \( \lambda(n) \) in the context of multi-player coalitions and determined \( \lambda(n) \) for \( n \leq 8 \), cf. [8, 9]. The value of \( \Lambda(n) \) for \( n \leq 7 \) was found in [11]. The sequences \( \lambda(n) \) and \( \Lambda(n) \) are given in Sloane’s Encyclopedia of Integer Sequences under A001206 and A051185, respectively.
1.1 Description in terms of Boolean functions

A *Boolean function* in \( n \) variables is a function \( f: \{0, 1\}^n \to \{0, 1\} \). There is a 1-1 correspondence between Boolean functions \( f \) and set systems \( F \) obtained by letting \( f \) be the characteristic function of \( F \). The Boolean function \( f \) is called *monotone* when it cannot decrease (become false) when some variables are increased (made true). The equivalent property for \( F \) is that \( F^\uparrow = F \). The Boolean function \( f \) is called *self-dual* when \( f(1-x_1, ..., 1-x_n) = 1-f(x_1, ..., x_n) \). The equivalent property for \( F \) is that \( F \) contains precisely one element from every complementary pair \( \{A, X \setminus A\} \). Counting maximal linked systems is therefore equivalent to counting self-dual monotone Boolean functions.

1.2 Counting

Erdős [3] (p. 79) writes: *It does not seem easy to determine \( \lambda(n) \). We could not even get an asymptotic formula. It is an easy exercise to give asymptotic formulas for \( \log_2 \lambda(n) \) and \( \log_2 \Lambda(n) \).*

**Proposition 1.1** ([1])

(i) Let \( \alpha(n) \) be the number of antichains on \( n \) points. Then

\[
\log_2 \lambda(n) \sim \log_2 \alpha(n-1) \sim \frac{2^n}{\sqrt{2\pi n}}.
\]

(ii) Let \( i(n) \) be the number of families on \( n \) points with nonempty intersection. Then

\[
\log_2 \Lambda(n) \sim \log_2 i(n) \sim 2^{n-1}.
\]

The currently known values of \( \lambda(n) \) and \( \Lambda(n) \) are as follows.

**Proposition 1.2** The values of \( \lambda(n) \) and \( \Lambda(n) \) for \( n \leq 8 \) are as given in the table below. Moreover, \( \lambda(9) = 423295099074735261880 \).

| \( n \) | \( \lambda(n) \) | \( \Lambda(n) \) |
|------|----------------|----------------|
| 0    | 1              | 1              |
| 1    | 1              | 2              |
| 2    | 2              | 6              |
| 3    | 4              | 40             |
| 4    | 12             | 1376           |
| 5    | 81             | 1314816        |
| 6    | 2646           | 912818962432   |
| 7    | 1422564        | 291201248266450683035648 |
| 8    | 229809982112   | 14704022144627161780744368338695925293142507520 |

2 Easy bounds

Before one starts counting, it helps to have some idea about the size of the result, so that one can pick a suitable algorithm. Below we give some rough estimates.

**Lemma 2.1** Let \( n \geq 1 \). Then \( \log_2 \lambda(n) \geq \binom{n-1}{\lceil n/2 \rceil-1} \).
Proof. If $n$ is even, say $n = 2m$, then pick arbitrarily one element from each pair $\{A, X \setminus A\}$ of complementary sets of size $m$. This gives $2^e$ linked systems, where $e = \frac{1}{2}\binom{n}{m} = \binom{n-1}{m-1}$. Extend each of these linked systems to a maximal linked system. The mls’s obtained will be pairwise distinct.

If $n$ is odd, say $n = 2m + 1$, then pick arbitrarily one element from each pair $\{A, X \setminus A\}$ of complementary sets where $A$ has size $m$ and contains a fixed element $x_0 \in X$. This gives $2^e$ linked systems, where $e = \binom{2m}{m} - \binom{m}{m-1}$, and the same conclusion follows.

Lemma 2.2 Let $n \geq 1$. Then $\lambda(n) < \alpha(n-1)$.

Proof. Fix $x_0 \in X$. The map $F \mapsto \{A \in F_{\text{min}} \mid x_0 \notin A\}$ is a bijection from mls’s on $n$ points to linked antichains on $n - 1$ points (and $\emptyset$ is an antichain that is not linked).

Since Kleitman [5] shows that $\log_2 \alpha(n) \sim \binom{n}{\lfloor n/2 \rfloor}$, these two lemmas imply part (i) of Proposition 1.1. More precise results were given by Korshunov [7].

For small $n$ the value of $\alpha(n)$ was determined by various authors. One finds 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 5613043722867557907788 for $0 \leq n \leq 8$ ([12]). This is Sloane’s sequence A000372.

Lemma 2.3 Let $n \geq 1$. Then

\begin{align*}
\lambda(n) & \leq \Lambda(n) \leq \lambda(n) 2^{2^{n-1} - \binom{n}{\lfloor n/2 \rfloor}}, \\
& \leq \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{n-k} = i(n) \leq \Lambda(n).
\end{align*}

Proof. (i) Let $F$ be an mls. Then $F$ has size $2^{n-1}$, and hence contains $2^{2^n-1}$ linked subfamilies. This shows that $\Lambda(n) \leq \lambda(n).2^{2^n-1}$. Next, $F_{\text{min}}$ is an antichain, and, by Sperner’s Lemma, has size at most $\binom{n}{\lfloor n/2 \rfloor}$. Each of the at least $2^{2^n-1} - \binom{n}{\lfloor n/2 \rfloor}$ linked families $G$ with $F_{\text{min}} \subseteq G \subseteq F$ determines $F = G^\uparrow$.

(ii) Since a family with nonempty intersection is linked, $i(n) \leq \Lambda(n)$. The formula for $i(n)$ follows by inclusion-exclusion. Since $\binom{n}{k}2^{n-k}$ decreases with $k$, the first two terms give a lower bound for the sum.

This lemma implies part (ii) of Proposition 1.1.

Finally, let us note a 1-1 correspondence. Above we saw a 1-1 correspondence between mls’s on $n$ points and linked antichains on $n - 1$ points. There is also a 1-1 correspondence between linked antichains on $n - 1$ points and linked antichains on $n$ points ‘in the bottom half $H$ of the Boolean lattice’, where $H$ consists of the subsets of $X$ of size less than $n/2$, together with an arbitrary choice of precisely one element from each complementary pair $\{A, X \setminus A\}$ of sets of size $n/2$. Indeed, we can let $F$ correspond with $(F \cap H) \cup \{X \setminus A \mid A \in F \setminus H\}$.

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3 Computation of $\lambda(8)$

In Section 5, $\lambda(8)$ is found as a side result of the computation of $\Lambda(8)$. The algorithm used in [10] enumerated upwardly closed linked systems $L$ of sets of size at most 3, and for each $L$ counted the number $a$ of complementary pairs $\{A, X \setminus A\}$ of 4-sets, such that both $A$ and $X \setminus A$ meet all elements of $L$. Now $L$ is contained in precisely $2^a$ mls’s. One finds $\lambda(8) = \sum_L 2^a(L) = 229809982112$.

4 Computation of $\lambda(9)$

We count linked antichains $A$ on 9 points, of which all elements have size at most 4. Let $A'$ be the subcollection of $A$ consisting of the sets of size at most 3.

Classifying all linked antichains $B$ on 9 points, of which all elements have size at most 3 under the action of the symmetric group $\text{Sym}(9)$ shows that there are 15952 orbits. For each orbit, pick a representative $B$ and count extensions to linked antichains $A$ for which $A' = B$. For example, if $B$ contains a singleton $\{x\}$, then it contains nothing else, and $A = B$. (There are 9 such $A$.) A similar conclusion holds when $B$ contains three pairs $\{x,y\}$, $\{x,z\}$ and $\{y,z\}$. For largish $B$ it is very easy to find all extensions $A$. The worst case is that where $B$ is empty, so that $A$ is a coclique in the Kneser graph $K(9,4)$.

4.1 Counting independent sets in a sparse graph

Counting independent sets in a sparse graph is a popular topic. People mostly prove complexity results: for regular graphs of degree at most 4, approximate counting is easy, for degree at least 6 it is difficult (cf. [2]). Here we have degree 5, and we want the exact count.

A recursive algorithm that keeps a current set of vertices $S$, and when $S$ contains an edge $xy$ calls itself with $S \setminus \{x\}$ and with $S \setminus N(x)$, where $N(x)$ is the set of neighbours of $x$, can count $2^{|S|}$ independent sets when $S$ is independent. This is good, but too slow.

A better version of the algorithm will go down the recursion when some vertex $x$ has degree at least 2 in $S$, and count $2^a3^b$ when $S$ contains $a$ isolated points and $b$ isolated edges. This is still too slow.

The algorithm used in practice uses recursion when some vertex $x$ has degree at least 3 in $S$, and counts the proper number when all vertices in $S$ have degree at most 2, so that $S$ induces a union of paths and cycles. (Let $p(n)$ be the number of cocliques in the path $P_n$ on $n$ vertices, and $c(n)$ the number of cocliques in the cycle $C_n$ on $n$ vertices. Then $p(0) = 1$, $p(1) = 2$, $p(m) = p(m - 1) + p(m - 2)$ for $m \geq 2$, and $c(m) = p(m - 1) + p(m - 3)$ for $m \geq 3$. The proper number is the product of numbers $p(m)$ and $c(m)$, one for each connected component $P_m$ or $C_m$.)

The result of doing this on $K(9,4)$ was $366996244568643864340$, and together with the $56298854506091397540$ extensions of nonempty $B$ we find that $\lambda(9) = 423295099074735261880$. 
5 Computation of $\Lambda(8)$

We follow the setup of Pogosyan, Miyakawa & Nozaki [11]. Let $2^X$ be the power set of $X$, of size $2^n$. If $n$ is even, fix an element $x_0 \in X$. Let $\mathcal{H}$, the bottom half of $2^X$, consist of the subsets of $X$ of size less than $n/2$, or of size precisely $n/2$ and not containing $x_0$. Then for each complementary pair $\{A, X \setminus A\}$ precisely one element is in $\mathcal{H}$. Let there be $k(r, n)$ linked antichains contained in $\mathcal{H}$.

For $n = 8$ we found the following values:

| $r$ | 0 | 1 | 2  | 3  | 4  | 5  |
|-----|---|---|----|----|----|----|
| $k(r, 8)$ | 1 | 127 | 5103 | 110901 | 1442910 | 12564636 |
| $r$ | 6 | 7 | 8  | 9  | 10 | 11 |
| $k(r, 8)$ | 78501094 | 365924948 | 1302838180 | 3609216800 | 14155324680 |
| $r$ | 12 | 13 | 14 | 15 | 16 | 17 |
| $k(r, 8)$ | 21054328876 | 26807793040 | 29932703320 | 29875293476 | 27014411074 | 22319717630 |
| $r$ | 18 | 19 | 20 | 21 | 22 | 23 |
| $k(r, 8)$ | 16932275290 | 11821639550 | 7598222786 | 4489816356 | 2432135090 | 1202614280 |
| $r$ | 24 | 25 | 26 | 27 | 28 | 29 |
| $k(r, 8)$ | 539687680 | 218192464 | 78745884 | 25082260 | 6952300 | 1647520 |
| $r$ | 30 | 31 | 32 | 33 | 34 | 35 |
| $k(r, 8)$ | 326312 | 52416 | 6545 | 595 | 35 | 1 |

Since each mls $\mathcal{F}$ is uniquely determined by the linked antichain $\mathcal{F}^\text{min} \cap \mathcal{H}$, we see that $\lambda(n) = \sum_r k(r, n)$. In this particular case we see that the numbers given add up to $\lambda(8) = 229809982112$, a good check. The values $k(0, n) = 1$ and $k(1, n) = 2^n - 1$ are obvious.

The largest intersecting antichain in $\mathcal{H}$ for even $n$ is the collection of elements of $\mathcal{H}$ of size $n/2$. (This collection is linked since the sets do not contain $x_0$.) For odd $n$ the largest intersecting antichains are the collections of sets of size $(n - 1)/2$ containing some fixed element $x \in X$. In both cases the size of a largest intersecting antichain equals $m := (n - 1)/2 - 1$.

For a linked system $\mathcal{F}$, consider the linked antichain $\mathcal{G} = \mathcal{F}^\text{min} \cap \mathcal{H}$ and put $r = |\mathcal{G}|$. The number of $\mathcal{F}$ giving rise to the same $\mathcal{G}$ equals $2^{2n-r}$. Indeed, there are $2^{n-1-r}$ pairs $\{A, X \setminus A\}$ with $A \in \mathcal{H} \setminus \mathcal{G}$. For such a pair, it possible that $A \in \mathcal{F}$ only if there is a $B \in \mathcal{G}$ with $B \subset A$. In this case $(X \setminus A) \cap B = \emptyset$, so $X \setminus A \not\in \mathcal{F}$, while we can freely choose whether $A \in \mathcal{F}$. On the other hand, if there is no such $B$, then $A \not\in \mathcal{F}$ while we can freely choose whether $X \setminus A \in \mathcal{F}$. (Note that $2^X \setminus \mathcal{H}$ is linked, and adding sets that properly contain a set that is present already cannot invalidate the property of being linked.) This shows that $\Lambda(n) = \sum_{r=0}^{m} 2^{2n-r} k(r, n)$. For $n = 8$ this equals $2^{28} \cdot 148476812083249435 = 1470402214461780744368338695925293142507520$.

Since $k(m, n) = 1$ if $n$ is even, and $k(m, n) = n$ if $n$ is odd, so that $k(m, n)$ is odd in all cases, it follows that the precise power of 2 that divides $\Lambda(n)$ is $2^{2n-1-m}$.

6 History

Parts of the above are from [1] and [10]. These four authors had agreed to submit a joint paper, but nothing came of it. Today my three coauthors† are

†W. H. Mills (1921-2007), C. F. Mills (1951-2000), A. Verbeek (1946-1990)
no longer alive, and the results of [1, 10] have been published by others. The present note improves all previous results known to me.

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