New wave equation for ultra-relativistic particles

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Abstract

In this letter we present a novel wave equation for ultra-relativistic matter, particles that satisfy the condition, \( p \gg m \), and whose energy-momentum relation can therefore be approximated by the relation, \( E \approx p + \frac{m^2}{2p} \). We discuss the implications of this wave equation and analyse their possible solutions. In particular, it is found that the plane-wave solution is completely consistent with the theory of neutrino oscillations. On the other hand, we also provide the Lagrangian formulation and discuss the associated conservation laws.

I. INTRODUCTION

The Dirac equation is one of the most beautiful creations of human intellect. It opened a new era in particle physics, yielding to many impressive results among which the prediction of antimatier represents a milestone in the history of science. However, in spite of their impressive agreement with experiments, it is not clear whether the Dirac equation will always prevail as an untouchable scientific truth. We specifically refer to the ultra-high energy regimes, i.e, \( p \gg m \), where possible deviations from Lorentz invariance might be present. These possible violations of Lorentz invariance have received attention in the literature and are being extensively studied in the last years, specially in the context of ultra-high energy cosmic-rays.\(^{[1,2]}\)

We should clarify that it is not the purpose of this letter the study of a competitor for the Dirac wave equation. There is no doubt that fermions follow the Dirac equation, whose unrivalled success in the understanding of matter is not subject to discussion. However, fermions such as the electron also satisfy the Schrödinger equation, but in the low-energy limit. Indeed, as is well known, the Dirac equation replaces the Schrödinger equation in the relativistic regime. Then, in the ultra-high energy regime \( p \gg m \), not yet completely explored, maybe a new wave equation could substitute the Dirac equation in the same fashion that the Dirac equation replaces Schrödinger’s wave equation in the appropriate limit.

Indeed, from a purely theoretical point of view, in the regime \( p \gg m \), it seems to be some room for the existence of another matter wave equation, which can be derived from first principles. This matter wave equation also includes spin in a natural way, and perhaps may be useful to explain some properties of ultra-relativistic particles. The paper is organized as follows. In sec.II, we derive the wave equation starting from first principles. In sec.III, we establish the Lagrangian formulation and the Hamiltonian formalism. In sec.IV, we discuss the issue of the Lorentz invariance of the matter wave equations. Finally, in sec.V, we present the conclusions of this work.

II. DERIVATION AND PHYSICAL INTERPRETATION

In order to derive a wave equation in Physics, we have to focus on the energy-momentum relation assumed. In the same fashion that Schrödinger’s equation is derived assuming a classical relation, \( E = p^2/2m \), the Klein-Gordon equation can be obtained taking the relativistic, \( E^2 = m^2 + p^2 \), and the Dirac equation emerges assuming a linear relation, \( E = \alpha p_1 + \beta m \). Then, it seems natural to wonder what wave equation would correspond to the case, \( E = p + \frac{m^2}{2p} \), which is the subject of this work. In the theoretical discussion that follows, we will maintain the constants \( h, c \), in all the expressions unless otherwise noted. Let us begin the discussion with the energy-momentum relation of the Special Theory of Relativity (STR) for a free particle with positive energy

\[
E = \sqrt{c^2 p^2 + m^2 c^4} \tag{1}
\]

Where \( m \) is the rest mass. As is well known, in the non-relativistic limit, \( p \ll mc \), we can approximate this equation as

\[
E \approx mc^2 \left( 1 + \frac{p^2}{2m c^2} \right) = mc^2 + \frac{p^2}{2m} \tag{2}
\]

However, in the ultra-relativistic limit, \( mc \ll p \), equation (1) provides

\[
E \approx cp \left( 1 + \frac{m^2 c^2}{2p^2} \right) = cp + \frac{m^2 c^3}{2p} \tag{3}
\]

Multiplying by \( p \) the last equation and making explicit the operator representation, \( \hat{E} \rightarrow ih \frac{\partial}{\partial t}, \hat{p} \rightarrow -ih \nabla \), we arrive to the following partial differential wave equation

\[
|\nabla| \frac{\partial \psi}{\partial t} = -c |\nabla|^2 \psi + \frac{m^2 c^3}{2h^2} \psi \tag{4}
\]

Where, \( |\nabla|^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \) is the Laplacian operator and, \( |\nabla| \equiv (\partial_x^2 + \partial_y^2 + \partial_z^2)^{1/2} \) its square root. We can express the square root of the Laplacian as, \( |\nabla| = \vec{\sigma} \cdot \nabla \); In cartesian coordinates

\[
|\nabla| = \vec{\sigma} \cdot \nabla = \sigma_x \partial_x + \sigma_y \partial_y + \sigma_z \partial_z \tag{5}
\]
Where, $\sigma_x$, $\sigma_y$, $\sigma_z$ are certain operators. The condition, $(\vec{\sigma} \cdot \nabla)^2 = \nabla^2$ can be written as

$$\left(\sigma_x \partial_x + \sigma_y \partial_y + \sigma_z \partial_z\right) \left(\sigma_x \partial_x + \sigma_y \partial_y + \sigma_z \partial_z\right) - \partial_x^2 + \partial_y^2 + \partial_z^2 = 0 \quad (6)$$

Then, the operators $\sigma_i$ are subjected to the constraints

$$\sigma_i^2 = I \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I \quad (7)$$

This allows to choose a representation of the sigmas given in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (8)$$

Finally, we can write the explicit form of the wave equation (4) is terms of the Pauli matrices as follows

$$\sigma_x \partial_x^2 \psi + \sigma_y \partial_y^2 \psi + \sigma_z \partial_z^2 \psi = -c^2 \nabla^2 \psi + \frac{m^2 c^3}{2\hbar^2} \psi \quad (9)$$

Or in the compact form

$$(\vec{\sigma} \cdot \nabla) \psi = -c^2 \nabla^2 \psi + \frac{m^2 c^3}{2\hbar^2} \psi \quad (10)$$

This explicit emergence of the Pauli matrices (8) in the wave equation indicates that (10) describes particles of spin $1/2$ in the ultra-high energy regime $p \gg mc$. Then, in such ultra-high energy conditions, this wave equation appears as a natural substitute for the Dirac wave equation.

### A. The plane wave solution. Dispersion relation, phase and group velocities of the ultra-relativistic waves.

Given the wave equation (9), it seems natural to look for a solution with a plane-wave structure

$$\psi(r, t) = \left(\frac{\chi}{\phi}\right) e^{i(k \cdot r - \omega t)} \quad (11)$$

Where, $k \cdot r = k_x x + k_y y + k_z z$. Substituting this ansatz in (9) we obtain after a bit of algebra the following matrix equation

$$\begin{pmatrix} k_x \omega - ck^2 - \frac{m^2 c^3}{2\hbar^2} & (k_x - ik_y)\omega \\ (k_x + ik_y)\omega & -k_z \omega - ck^2 - \frac{m^2 c^3}{2\hbar^2} \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (12)$$

Where, $k^2 = k_x^2 + k_y^2 + k_z^2$. In order to obtain non-trivial solutions we must impose the condition, $\det \tilde{A} = 0$. It gives the result

$$\det \tilde{A} = -k_z^2 \omega^2 + \left(ck^2 + \frac{m^2 c^3}{2\hbar^2}\right)^2 - \omega^2(k_x^2 + k_y^2) = 0 \quad (13)$$

Which implies a dispersion relation, $w(k)$ given by

$$w(k) = \pm \left(ck + \frac{m^2 c^3}{2\hbar^2 k}\right) \quad (14)$$

The minus sign corresponds to the negative energy solutions. Indeed, in order to better understand the last result, let us consider the particular case of the one-dimensional propagation along the z-axis. The simplified version of (9) will be

$$\sigma_z \frac{\partial^2 \psi}{\partial z^2} = -c^2 \frac{\partial^2 \psi}{\partial t^2} + \frac{m^2 c^3}{2\hbar^2} \psi \quad (15)$$

To solve this wave equation we take an ansatz similar to (11)

$$\psi(z, t) = \left(\frac{\chi}{\phi}\right) e^{i(kz - wt)} \quad (16)$$

For simplicity, we have denoted the z component of the wave vector $\vec{k}$ simply as $k$. The substitution of (16) in (15) provides the following system

$$\begin{align*}
kw/c \chi &= \left(k^2 + \frac{m^2 c^2}{2\hbar^2}\right) \chi & (17) \\
kw/c \phi &= -\left(k^2 + \frac{m^2 c^2}{2\hbar^2}\right) \phi & (18)
\end{align*}$$

These relations imply

$$w_\chi(k) = ck + \frac{m^2 c^3}{2\hbar^2 k} \quad (19)$$

$$w_\phi(k) = -\left(ck + \frac{m^2 c^3}{2\hbar^2 k}\right) \quad (20)$$

Of course, they are nothing else than the dispersion relations that we deduced for the 3D case in (14). Note that the field, $\phi$ that appears in (11-16) is the piece of the wave function that corresponds to the negative energy solution, i.e., the field associated to the antiparticle.

On the other hand, it is possible to compute directly from (14) the phase and group velocities associated to the positive energy solutions.

$$V_{ph} = \frac{w}{k} = c + \frac{m^2 c^3}{2\hbar^2 k^2} = c(1 + \left(\frac{mc}{\sqrt{2} \hbar k}\right)^2) \quad (21)$$

$$V_g = \frac{dw}{dk} = c - \frac{m^2 c^3}{2\hbar^2 k^2} = c(1 - \left(\frac{mc}{\sqrt{2} \hbar k}\right)^2) \quad (22)$$

Note that in the previous relations $mc << \hbar k$. Then, although the phase velocity of the ultra-relativistic wave satisfies, $V_{ph} \geq c$, the group velocity (the meaningful concept related with the true energy propagation of the wave), cannot exceed the speed of light. We can therefore conclude that the propagation of these waves is causal and consistent with the STR, the superluminal propagation is not possible in this theory. Furthermore, given the
values of $V_g$ and $V_{ph}$, we can assure after a straightforward computation that their product will also be lower than $c^2$

$$V_{ph} \cdot V_g = c^2 \left(1 - \left(\frac{mc}{\sqrt{2\hbar k}}\right)^4\right)$$  \hspace{1cm} (23)$$

On the other hand, note that by means of the dispersion relation (19), the plane-wave solution with positive energy of (16), can be written in the following manner

$$
\chi(z,t) = \chi(0) \exp \left( i(kz - w_z t) \right) \\
= \chi(0) \exp \left( i(kz - kct - \frac{m^2 c^3}{2\hbar k} t) \right) \\
\simeq \chi(0) \exp \left( -i \frac{m^2 c^2 z}{2\hbar k} \right)
$$

(24)

Where we can approximate $z \simeq ct$ if the ultra-relativistic particle travels close to the speed of light. The wave function (24) is a standard result frequently found within the context of the theory of neutrino oscillations[3, 5]. This theory makes the initial assumption that the mass eigenfunctions that describe the propagation of such particles are plane-waves, $|\psi_i(z,t)\rangle = \exp\left(i(k_i z - w_i t)\right)|\nu_i(0)\rangle$, then it is used the approximation (in natural units), $E = p + m^2/2p$ to simplify the argument of the exponential and finally obtain

$$
|\nu_i(z)\rangle = \exp \left( -i \frac{m^2 c^2 z}{2p} \right) |\nu_i(0)\rangle
$$

(25)

Where $z$ is the distance between the neutrino production and detection points. Note that both wave functions have the same structure. It is worth noting that we have been able to derive this standard result following a non-standard approach. Indeed, we have proved that the family of plane waves (25), are only particular solutions of the ultra-relativistic wave equation (4).

On the other hand, eigenstates with different masses propagate at different speeds, this is evident following equation (22) which establishes the dependence of the group velocity upon $m^2$. This fact is also directly derived from the wave equation.

III. LAGRANGIAN FORMULATION

A remarkable feature of Classical Field Theory is that all the well defined matter wave equations can be derived from a Lagrangian density, from which a continuity equation, $\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$ follows. The aim of this section is to prove that the wave equation (10) also derives from a hermitian Lagrangian function. The Lagrangian formalism is a powerful tool, not only to study the symmetry transformations which allows to apply Noether’s theorem to collect the associated conservation laws, but also to build the associated Hamiltonian, a necessary step to carry out the canonical quantization of the field \(^{1}\). Then, let us define the following Lagrangian density

$$
\mathcal{L} = \frac{m^2 c^2}{2\hbar^2} \psi \psi^\dagger + \nabla \psi \cdot \nabla \psi^\dagger + \frac{1}{2c} \left( \frac{\partial \psi}{\partial t} \sigma \cdot \nabla \psi^\dagger + \frac{\partial \psi^\dagger}{\partial t} \sigma \cdot \nabla \psi \right)
$$

(26)

Where $c$ is the speed of light. As is well known, the Pauli matrices are hermitian, $\sigma^i = \sigma_i$, which guarantees the hermiticity of the Lagrangian, $\mathcal{L} = \mathcal{L}^\dagger$. The Euler-Lagrange (E-L) equations for the fields $\psi, \psi^\dagger$, are given by

$$
\partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right] - \frac{\partial \mathcal{L}}{\partial \psi} = 0, \quad \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\dagger)} \right] - \frac{\partial \mathcal{L}}{\partial \psi^\dagger} = 0
$$

(27)

Making explicit the summation over the index $\mu$, the E-L equation associated to the hermitian field $\psi^\dagger$ will be

$$
\partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\dagger)} \right] + \nabla \cdot \left[ \frac{\partial \mathcal{L}}{\partial (\nabla \psi^\dagger)} \right] - \frac{\partial \mathcal{L}}{\partial \psi^\dagger} = 0
$$

(28)

Applying the derivatives of (28) to the Lagrangian density (26), we find

$$
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\dagger)} = \frac{1}{2c} \sigma \cdot \nabla \psi, \quad \frac{\partial \mathcal{L}}{\partial (\nabla \psi^\dagger)} = \frac{1}{2c} \frac{\partial \psi}{\partial t} \sigma + \nabla \psi
$$

(29)

The substitution of these results in (28) gives

$$
0 = \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\dagger)} \right] + \nabla \cdot \left[ \frac{\partial \mathcal{L}}{\partial (\nabla \psi^\dagger)} \right] - \frac{\partial \mathcal{L}}{\partial \psi^\dagger} = \frac{\partial}{\partial t} \left( \frac{1}{2c} \sigma \cdot \nabla \psi \right) + \nabla \cdot \left( \frac{1}{2c} \frac{\partial \psi}{\partial t} \sigma + \nabla \psi \right) - \frac{m^2 c^2}{2\hbar^2} \psi = \frac{1}{c} (\sigma \cdot \nabla) \psi + \nabla^2 \psi - \frac{m^2 c^2}{2\hbar^2} \psi
$$

(30)

We have therefore been able to derive the wave equation (10) from a Lagrangian density by means of the corresponding E-L equations.

A. Global gauge invariance. Noether’s theorem and conserved current

The Lagrangian density (26) is invariant under the transformation

$$
\psi \rightarrow \psi' = e^{i\theta} \psi
$$

(31)

\(^{1}\) The study of the canonical quantization of the field will be the subject of future work
Then, according to Noether’s theorem it must exist a conserved quantity. Indeed, it can be proved that the associated current \( J^\mu \), satisfies the differential equation
\[
\partial_\mu J^\mu = 0,
\]
where
\[
J^\mu = \left[ \frac{\partial L}{\partial (\partial_\mu \psi)} \right] \delta \psi + \left[ \frac{\partial L}{\partial (\partial_\mu \psi^\dagger)} \right] \delta \psi^\dagger
\]
(32)

This result implies that there exists a certain “charge” \( Q \equiv \int_V J^\mu d^3x \), which is a constant of motion, i.e., \( dQ/dt = 0 \). For a transformation of the type given by (31) we have
\[
\delta \psi = \psi' - \psi = (e^{i\theta} - 1) \psi \approx i\theta \psi
\]
\[
\delta \psi^\dagger \approx -i\theta \psi^\dagger
\]
(33)

Substituting the results of equations (29) together with these last identities in (32), we obtain
\[
J^0 = \left[ \frac{\partial L}{\partial (\partial_0 \psi)} \right] \delta \psi + \left[ \frac{\partial L}{\partial (\partial_0 \psi^\dagger)} \right] \delta \psi^\dagger
\]
\[
\frac{\partial}{2c} \left[ \left( \bar{\sigma} \cdot \nabla \psi^\dagger \right) \psi - \left( \bar{\sigma} \cdot \nabla \psi \right) \psi^\dagger \right]
\]
(34)

\[
\bar{J} = \left[ \frac{\partial L}{\partial (\nabla \psi)} \right] \delta \psi + \left[ \frac{\partial L}{\partial (\nabla \psi^\dagger)} \right] \delta \psi^\dagger
\]
\[
i \theta \left[ \psi \left( \frac{1}{2c} \frac{\partial}{\partial t} \bar{\sigma} + \nabla \psi^\dagger \right) - \psi^\dagger \left( \frac{1}{2c} \frac{\partial}{\partial t} \bar{\sigma} + \nabla \psi \right) \right]
\]
(35)

It is straightforward to show that these functions satisfy the hermiticity condition, \( J^0 = (\bar{J})^\dagger \), \( \bar{J} = \bar{J}^\dagger \). On the other hand, since the parameter \( \theta \) is an arbitrary constant, we can take \( \theta = 1 \). Finally, the conserved “charge” will be
\[
Q = \int_V J^0 d^3x = \frac{i}{2c} \int_V \left[ \left( \bar{\sigma} \cdot \nabla \psi^\dagger \right) \psi - \left( \bar{\sigma} \cdot \nabla \psi \right) \psi^\dagger \right] d^3x
\]
(36)

In order to verify the robustness of these results, we can check if the divergence \( \partial_\mu J^\mu \) vanishes or not. After a bit of algebra we find
\[
\partial_\mu J^\mu = \partial_0 J^0 + \nabla \cdot \bar{J}
\]
\[
i \psi \left( \frac{1}{c} \bar{\sigma} \cdot \nabla \psi^\dagger \right) + \psi^\dagger \left( \frac{1}{c} \bar{\sigma} \cdot \nabla \psi \right)
\]
\[
= \frac{m^2 c^2}{2\hbar^2} \psi \psi^\dagger - \frac{m^2 c^2}{2\hbar^2} \psi^\dagger \psi = 0
\]
(37)

Therefore, the divergence \( \partial_\mu J^\mu \) vanishes identically as expected. Then, we can conclude that the Lagrangian formulation of the ultra-relativistic wave equation is a consistent theory, and the Lagrangian density (26) has a global gauge symmetry compatible with a conserved current.

### B. Local gauge invariance

The generalization to the \( U(1) \) case is straightforward. As is well known, for a local phase transformation, i.e., \( \psi \rightarrow \psi' = e^{i\theta(x)} \psi \), the usual derivative transforms in the following way
\[
\partial_\mu \psi \rightarrow \partial_\mu \psi' = \partial_\mu (e^{i\theta(x)} \psi) = \partial_\mu (e^{i\theta(x)} \psi + e^{i\theta(x)} \partial_\mu \psi)
\]
\[
= e^{i\theta(x)} (i\partial_\mu \theta(x)) \psi + e^{i\theta(x)} \partial_\mu \psi
\]
\[
= e^{i\theta(x)} (i\partial_\mu \theta(x) + \partial_\mu \psi)
\]
(38)

Then, the Lagrangian density (26) is no longer invariant under this transformation and we must look for a generalization. This generalization is
\[
\mathcal{L} = \frac{m^2 c^2}{2\hbar^2} \psi \psi^\dagger + \mathcal{D}_\mu \psi (\mathcal{D}_\mu \psi)^\dagger + \frac{1}{2c} \left( \mathcal{D}_\mu \psi (\bar{\sigma} \cdot \mathcal{D}_\mu \psi)^\dagger + (\mathcal{D}_\mu \psi)^\dagger \bar{\sigma} \cdot \mathcal{D}_\mu \psi \right)
\]
(39)

Indeed, the covariant derivative, \( \mathcal{D}_\mu \equiv (\partial_\mu + A_\mu) \) is subjected to the transformation rule
\[
\mathcal{D}_\mu \psi \rightarrow \mathcal{D}_\mu' \psi' = (\partial_\mu + A_\mu') e^{i\theta(x)} \psi = e^{i\theta(x)} [i\partial_\mu \theta(x) + \partial_\mu + A_\mu'] \psi
\]
(40)

Then, in order to compensate the term \( i\partial_\mu \theta(x) \), we take the condition \( A_\mu = A_\mu' = A_\mu - i\partial_\mu \theta(x) \). which implies, \( \mathcal{D}_\mu \psi^\dagger = e^{i\theta(x)} \mathcal{D}_\mu \psi \), assuring the invariance of (39)

### C. The Hamiltonian formalism

Armed with a consistent Lagrangian theory, the next logical step after the analysis of the internal transformations such as (31) is the study of the external symmetries and the Hamiltonian formalism. The canonical energy-momentum tensor that comes from the Lagrangian density (26), under space-time translational invariance is
\[
T^\mu_\nu = \frac{\partial L}{\partial (\partial_\mu \psi)} (\partial_\nu \psi) + \frac{\partial L}{\partial (\partial_\mu \psi^\dagger)} (\partial_\nu \psi^\dagger) - \delta^\mu_\nu L
\]
(41)

On the other hand, we can define the Hamiltonian density \( \mathcal{H} \), as
\[
\mathcal{H} = T^0_0 = \frac{\partial L}{\partial \psi} \frac{\partial L}{\partial \psi^\dagger} - L = \pi(x) \psi + \pi^\dagger(x) \psi^\dagger - L
\]
(42)

The canonical momenta, \( \pi(x) \), \( \pi^\dagger(x) \) are given by the following relations
\[
\pi(x) = \frac{\partial L}{\partial \psi} = \frac{1}{2c} \bar{\sigma} \cdot \nabla \psi^\dagger, \quad \pi^\dagger(x) = \frac{\partial L}{\partial \psi^\dagger} = \frac{1}{2c} \bar{\sigma} \cdot \nabla \psi
\]
(43)

Then, substituting the last results in (42) and using (26)
we obtain
\[ H = \pi \psi + \pi^\dagger \psi^\dagger - L = \frac{1}{2c} \left( \dot{\psi} \vec{\sigma} \cdot \nabla \psi^\dagger + \psi^\dagger \vec{\sigma} \cdot \nabla \psi \right) - L \]
\[ = \frac{1}{2c} \left( \dot{\psi} \vec{\sigma} \cdot \nabla \psi^\dagger + \psi^\dagger \vec{\sigma} \cdot \nabla \psi \right) - \frac{m^2 c^2}{2 \hbar^2} \psi \psi^\dagger - \nabla \psi \cdot \nabla \psi^\dagger \]
\[ = -\frac{m^2 c^2}{2 \hbar^2} \psi \psi^\dagger - \nabla \psi \cdot \nabla \psi^\dagger \] (44)

The conserved currents and their “charges”, such as \( J^0 \) and \( T^0_0 \), are only determined up to a constant. This means that we are free to redefine \( \mathcal{H} \equiv -T^0_0 \), in order to have a positive defined Hamiltonian density. Then, we can adopt
\[ \mathcal{H} = \frac{m^2 c^2}{2 \hbar^2} \psi \psi^\dagger + \nabla \psi \cdot \nabla \psi^\dagger \]
\[ = \frac{m^2 c^2}{2 \hbar^2} \psi \psi^\dagger + \nabla \cdot \left( \psi^\dagger \nabla \psi \right) - \psi^\dagger \nabla^2 \psi \] (45)

With this result, the relation between the Hamiltonian \( H \) and their density \( \mathcal{H} \), is given by
\[ H = \int_V \mathcal{H} d^3 x = \int_V \psi^\dagger \left( -\nabla^2 + \frac{m^2 c^2}{2 \hbar^2} \right) \psi d^3 x \]
\[ + \nabla \cdot \int_V \psi^\dagger \nabla \psi d^3 x \] (46)

The second term vanishes, so we collect the final expression
\[ H = \int_V \psi^\dagger \left( -\nabla^2 + \frac{m^2 c^2}{2 \hbar^2} \right) \psi d^3 x = \int_V \psi^\dagger \tilde{H} \psi d^3 x = \langle \tilde{H} \rangle \] (47)

Since, \( \tilde{p} = -i\hbar \nabla \), we can write the Hamiltonian operator \( \tilde{H} \) as
\[ \tilde{H} = -\nabla^2 + \frac{m^2 c^2}{2 \hbar^2} = \tilde{p}^2 \left/ \hbar^2 \right. + \frac{m^2 c^2}{2 \hbar^2} \] (48)

It is easy to see that the operator \( \tilde{H} \) is hermitian given their own definition, \( \tilde{H}^\dagger = \tilde{H} \). On the other hand, the vanishing of the divergence, \( \partial_\mu T^\mu_\nu = 0 \) implies, \( d/dt(\intV T^0_0 d^3 x) = 0 = d/dt(\langle \tilde{H} \rangle) \). Therefore, the expectation value (47) is a constant of motion. It is interesting to note that although the energy operator \( \tilde{E} = i\hbar \partial_\xi \) and the Hamiltonian operator \( \tilde{H} \) are closely related, in this theory they are not the same algebraic object. Indeed, using (48) we can write the wave equation (10) as
\[ \frac{1}{c} (\vec{\sigma} \cdot \nabla) \frac{\partial \psi}{\partial t} = -\nabla^2 + \frac{m^2 c^2}{2 \hbar^2} \psi = \tilde{H} \psi \] (49)

IV. COMMENTS ON THE LORENTZ INvariance of the Matter Wave EQUATIONS.

It is important to note that, such as the Schrödinger equation, the wave equation (15) is not covariant (with respect to a Lorentz Transformation). This is because the energy and the momentum do not share the same power in the energy-momentum relation, which implies that the spatial and temporal derivatives that appear in the wave equation will not be of the same order. Since we are neglecting in the power series expansion of (3) all the terms beyond \( m^2 c^2 / 2p \), the breakdown of Lorentz Invariance (LI) given by (3) is therefore of order \( \mathcal{O}(mc^4 / p^2) \), very small indeed.

The following question may arise: could particles with an energy-momentum relation such as \( E = cp + m^2 c^3 / 2p \), be correctly described by a non-covariant wave equation? It is interesting to note that the Schrödinger equation, which is also a non covariant wave equation, works well in their regime of application, \( p << mc \). Our case is exactly the opposite, \( mc << p \), i.e., when the momentum is much greater than \( c \) times the rest mass. In the

| Physical Regime | Wave Equation | Lorentz Inv. |
|-----------------|---------------|--------------|
| \( p << mc \)   | \( i\hbar \partial_\xi \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \) | No |
| \( p \sim mc \)  | \( i(\gamma^0 \partial_\theta + \gamma^1 \partial_\xi) \psi - \frac{mc}{\hbar} \psi = 0 \) | Yes |
| \( p \sim mc \)  | \( \frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{m^2 c^2}{\hbar^2} \psi \) | Yes |
| \( p >> mc \)   | \( \sigma_\xi \frac{\partial^2 \psi}{\partial x \partial t} = -c \frac{\partial^2 \psi}{\partial x^2} + \frac{m^2 c^3}{2\hbar^2} \psi \) | No |

TABLE I. Matter Wave Equations in 1+1 Dimensions

opposite limits, \( mc << p \) and \( mc >> p \) we can expand the energy-momentum relation (1). As a result, the energy and the momentum will not have the same power in the final approximation. This means that from a theoretical point of view, there is some room for the existence of non covariant wave equations as table I illustrates. So, it is not impossible that due to this reason, for \( mc << p \) a non-covariant wave equation could be appropriate to describe successfully some properties of such particles, exactly in the same fashion that occurs with the Schrödinger equation in the contrary limit, \( mc >> p \). Indeed, as it was shown in (3), we emphasize that the equation \( E = cp + m^2 c^3 / 2p \) is obtained from the relativistic energy-momentum relation \( E^2 = m^2 c^4 + p^2 c^2 \) in the limit \( mc << p \). Therefore, the wave equation that
derives from this energy-momentum relation cannot be in contradiction with the STR, even not being strictly LI. At any rate, experiments have the last word. If a wave equation explains the experimental results but fails to be Lorentz Invariant, then the problem is not necessarily in the wave equation.

It is well known that there is no clue so far of Lorentz Invariance Violations (LIV) in particle physics experiments. However, a huge amount of theoretical and phenomenological effort has been carried out in the last years to study this possibility, specially in the context of the cosmic rays, which are conformed by ultra-relativistic particles.[1, 2]

V. FINAL REMARKS

In this work we have presented a wave equation that works for particles whose energy can be approximated by, (ignoring constants), the relation \( E \approx p + \frac{m^2}{2p} \).

If such energy-momentum relation encloses a “hidden” wave equation, then this wave equation can only be the one that we have introduced in this paper, which is a hyperbolic second order linear PDE with well-behaved physical solutions. As we have demonstrated, it can be useful to explain some properties of ultra-relativistic particles. For instance, the family of plane-wave functions usually employed in the theory of neutrino oscillations are only particular solutions of the wave equation discussed here. On the other hand, such as the Dirac equation, this wave equation describes particles of spin 1/2. Indeed, the spin operators are incorporated in a natural way by means of the Pauli matrices, which emerge explicitly in the square root of the Laplacian that appears in the derivation of the wave equation. Then, the study of the behaviour of this wave equation under different interactions \( V(x) \), will allow to enlarge the possible family of solutions, which may be useful to improve the understanding of ultra-relativistic processes, including perhaps the ultra-high energy cosmic-rays. In addition, a detailed Lagrangian formulation of the wave equation was also provided. In particular, we have proved that this is a consistent theory where, through the symmetries of the Lagrangian density, some standard and well defined conservation laws are derived in a natural way. The Hamiltonian formalism was also discussed, and we have found that the Hamiltonian operator that comes from the canonical formalism differs from the energy operator. Indeed, although both operators are closely related, they are not exactly the same object in this theory. Using the Hamiltonian formalism, it can be demonstrated that in the free case the conserved quantity corresponds to the expectation value of the operator \( -\nabla^2 + m^2 c^2 / 2\hbar^2 \).

In conclusion, we point out that all the consistent matter wave equations in Physics derive of non-trivial energy-momentum relations. Indeed, if we think of the space of possible non-trivial energy-momentum configurations, we will realize that it is quite constrained: It seems that there are only four consistent possibilities: i). The non-relativistic \( E = p^2 / 2m \). ii). The linear \( E = \alpha p_i + \beta m \). iii). The quadratic \( E^2 = \frac{p^2}{2} + m^2 \). iv). The ultra-relativistic \( E = p + m^2 / 2p \). The first three options are all associated with consistent wave equations that describe particles with different properties in their appropriate physical regime. The study of option iv) deserves an analysis, and has been the subject of this work. We suggest that the existence of another matter wave equation is a very interesting possibility that deserves to receive further attention.

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