Lyapunov, metric and flag spectra

Mauro Patrão

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Abstract

We introduce the metric spectrum, which measures the exponential rate of approximation to an isolated invariant set of points starting in its stable set, and relate it to the Lyapunov spectrum. We determine the metric spectrum of each Morse component of the finest Morse decomposition of a linear induced flow on a generalized flag manifold.

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1 Introduction

Let $\phi^t$ be a discrete or continuous-time flow defined on a metric space $(X, d)$. If $M \subset X$ is a compact isolated $\phi^t$-invariant set which is not a repeller, we introduce an spectrum, called metric spectrum, which measures the exponential rate of approximation to $M$ of points starting in the stable set of $M$, as illustrated by Figure 1.

![Figure 1: The approximation of $\phi^t(x)$ to $M$.](image)
We show that this spectrum is an invariant of the conjugation classes by bi-Lipschitz maps. In particular, when $X$ is a manifold, the metric spectrum is independent of Riemannian metrics. When the restriction of $\phi^t$ to an open neighborhood of $M$ is conjugated to a linear flow $\Phi^t$, the metric spectrum is given by the negative Lyapunov exponents of $\Phi^t$. In particular, if $M$ is normally hyperbolic, this implies that its metric spectrum is not empty.

Now let $g^t$ be a linear induced flow on a flag manifold $F_\Theta$ of a connected noncompact real semi-simple Lie group $G$. Assuming that $g^t$ is conformal (its unipotent Jordan component is trivial), we determine explicitly the metric spectrum of each Morse component of the finest Morse decomposition. This is done by combining the linearization presented in [7] with some results on the Jordan decomposition of $g^t$ presented in [3]. This result generalizes the following simple situation. Let $G = \text{Sl}(2, \mathbb{R})$, $F_\Theta = \mathbb{P}\mathbb{R}^2$ and

$$g^t = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}.$$ 

We have that $[e_1]$ and $[e_2]$ are, respectively, the attractor and repeller components of $g^t$ in $\mathbb{P}\mathbb{R}^2$. For each $[v] \neq [e_2]$, we have that $g^t[v] \to [e_1]$, when $t \to \infty$. The $\text{SO}(2, \mathbb{R})$-invariant distance $d$ in $\mathbb{P}\mathbb{R}^2$ is such that $d(g^t[v], [e_1]) = \theta_t$, where $\theta_t$ is the angle illustrated in the Figure 2.

![Figure 2: The approximation of $g^t[v]$ to $[e_1]$ in $\mathbb{P}\mathbb{R}^2$.](image)

Since

$$\lim_{t \to \infty} \frac{tg(\theta_t)}{\theta_t} = 1,$$

it follows that

$$\lim_{t \to \infty} \frac{1}{t} \log d(g^t[v], [e_1]) = -2\lambda,$$

where we use that

$$tg(\theta_t) = \frac{v_y}{v_x} e^{-2\lambda t}.$$
For the attractor component of \( g^t \) in the maximal flag manifold \( \mathbb{F} \), we do not need to assume that \( g^t \) is conformal. When \( g^t \) is arbitrary, the metric spectrum of its attractor component in the maximal flag manifold will be called the flag spectrum of \( g^t \).

## 2 Preliminaries

For the theory of semi-simple Lie groups and their flag manifolds we refer to Duistermat-Kolk-Varadarajan [2], Helgason [4], Knapp [5] and Warner [10]. To set notation let \( G \) be a connected noncompact real semi-simple Lie group with Lie algebra \( \mathfrak{g} \). Let \( \text{Ad} : G \to \text{Gl}(\mathfrak{g}) \) be the adjoint representation of \( G \). An element \( g \in G \) acts in the Lie algebra \( \mathfrak{g} \) by the adjoint representation, so that for \( X \in \mathfrak{g} \) we write

\[
gX = \text{Ad}(g)X.
\]

With this notation we have that

\[
g \exp(X)g^{-1} = \exp(gX) \quad \text{and} \quad \exp(X)Y = e^{\text{ad}(X)}Y,
\]

for \( g \in G, X, Y \in \mathfrak{g} \).

Fix a Cartan involution \( \theta \) of \( \mathfrak{g} \), a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{s} \) and a Weyl chamber \( \mathfrak{a}^+ \subset \mathfrak{a} \). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \) be the Cartan decomposition and \( \langle \cdot, \cdot \rangle \) be the Cartan inner product associated to \( \theta \). We let \( \Pi \) be the set of roots of \( \mathfrak{a} \), \( \Pi^+ \) the positive roots corresponding to \( \mathfrak{a}^+ \), \( \Sigma \) the set of simple roots in \( \Pi^+ \).

For each \( \Theta \subset \Sigma \), we denote by \( \mathbb{F}_\Theta \) the associated flag manifold, which is a homogeneous space of \( G \). Let \( b_\Theta \) be the base point, \( P_\Theta \) be its isotropy group and \( p_\Theta \) be its isotropy algebra. Each element \( X \in \mathfrak{g} \) of the Lie algebra induces a differentiable flow in the flag manifold \( \mathbb{F}_\Theta \) given by

\[
(t, x) \mapsto \exp(tX)x, \quad t \in \mathbb{R}, \ x \in \mathbb{F}_\Theta.
\]

This flow is generated by the induced vector field which is denoted by \( X \cdot \) and its value at \( x \) is denoted by \( X \cdot x \). An element \( g \in G \) acts in a tangent vector \( v \in T_x\mathbb{F}_\Theta \) by its differential at \( x \). We denote this action by \( gv = d_xg(v) \). For induced vector fields we have that

\[
g(X \cdot x) = gX \cdot gx.
\]
For a fixed $x \in F_{\Theta}$, the map $X \mapsto X \cdot x$ is a linear map from $g$ to $T_x F_{\Theta}$ whose kernel is the subalgebra of isotropy at $x$.

We have that the compact group $K = \exp(t)$ is transitive in $F_{\Theta}$, its isotropy subgroup at $b_{\Theta}$ is denoted by $K_{\Theta}$ and its elements acts in $g$ by $\langle , \cdot \rangle$-isometries. There exists a $K_{\Theta}$-invariant subalgebra $n_{\Theta}$ which complements $p_{\Theta}$ in $g$. Thus we can identify the tangent space of $F_{\Theta}$ at $b_{\Theta}$ with $n_{\Theta}$ and $\langle , \cdot \rangle$ determines a $K$-invariant Riemannian metric in $F_{\Theta}$ given by

$$|X \cdot x| = |Y|,$$

where $X \cdot x = k(Y \cdot b_{\Theta})$ with $k \in K$. The Weyl group $W = M^*/M$ acts on $a$ by isometries, where $M^*$ and $M$ are, respectively, the normalizer and the centralizer of $a$ in $K$.

We denote by $g^t$, $t \in T = \mathbb{R}$ or $\mathbb{Z}$, the right invariant continuous-time flow generated by $X \in g$ or the discrete-time flow generated by $g \in G$. More precisely, when $T = \mathbb{R}$, we have that $g^t = \exp(tX)$ and, when $T = \mathbb{Z}$, we have that $g^t$ is the $t$-iterate of $g$. The flow induced by $g^t$ on $F_{\Theta}$ is called a linear induced flow. Let $g^t = e^t h^t u^t$ be the multiplicative Jordan decomposition of $g^t$, a commutative composition of linear induced flows. The elliptic, hyperbolic and unipotent components of $g^t$ are, respectively, the flow $e^t$, the flow $h^t = \exp(tH)$ and the flow $u^t = \exp(tN)$, where $H$ is called the hyperbolic type of the flow $g^t$ and $\text{ad}(N)$ is a nilpotent operator. The hyperbolic component is gradient with respect to a given Riemannian metric on $F_{\Theta}$ (see [2], Section 3). The connected components of fixed point set of this flow are given by

$$\text{fix}_{\Theta}(H, w) = K_H w b_{\Theta},$$

where $K_H$ is the centralizer of $H$ in $K$. The stable and unstable sets of $\text{fix}_{\Theta}(H, w)$ with respect to $h^t$ are given, respectively, by

$$\text{st}_{\Theta}(H, w) = N_H^- \text{fix}_{\Theta}(H, w) \quad \text{and} \quad \text{un}_{\Theta}(H, w) = N_H^+ \text{fix}_{\Theta}(H, w)$$

where $N_H^\pm = \exp(n_H^\pm)$,

$$n_H^- = \sum \{ g_\alpha : \alpha(H) < 0 \} \quad \text{and} \quad n_H^+ = \sum \{ g_\alpha : \alpha(H) > 0 \}.$$

For an arbitrary linear induced flow on $F_{\Theta}$, we have the following similar description for its finest Morse decomposition (see Proposition 5.1 of [3]).
Proposition 2.1 Let $g^t$ be a linear induced flow on $F_\Theta$. The set
\[
\{\text{fix}_\Theta(H, w) : w \in W\}
\]
is the finest Morse decomposition for $g^t$. Furthermore, the stable and unstable sets of $\text{fix}_\Theta(H, w)$ with respect to $g^t$ are given, respectively, by
\[
st_{\Theta}(H, w) \quad \text{and} \quad \text{un}_{\Theta}(H, w).
\]

Now we define the subspace
\[
l_{x}^\pm = n_H^\pm \cap wn_{\Theta}^-
\] (1)
and the family of subspaces $l_{x}^\pm \subset n_H^\pm$ given by
\[
l_{x}^\pm = kl_{x}^\pm_{w_{\Theta}},
\]
for $x = k w_{\Theta}$, where $k \in K_H$. By Proposition 3.1 of [7], the families $l_{x}^\pm$ are well defined, each $l_{x}^\pm$ is $h^t$-invariant and, for $k \in K_H$, we have
\[
k l_{x}^\pm = l_{kx}^\pm.
\]
Furthermore we have that the map
\[
X \in l_{x}^\pm \mapsto X \cdot x \in l_{x}^\pm \cdot x
\]
is a linear isomorphism, for each $x \in \text{fix}_\Theta(H, w)$.

By Propositions 3.2 and 3.3 of [7], we have that
\[
V_{\Theta}^\pm(H, w) = \bigcup \{l_{x}^\pm \cdot x : x \in \text{fix}_\Theta(H, w)\}
\]
are differentiable subbundles of the tangent bundle of $F_\Theta$ over $\text{fix}_\Theta(H, w)$ such that its Whitney sum
\[
V_{\Theta}(H, w) = V_{\Theta}^+(H, w) \oplus V_{\Theta}^-(H, w)
\] (2)
is the normal bundle of $\text{fix}_\Theta(H, w)$ and whose fiber at $x \in \text{fix}_\Theta(H, w)$ is given by
\[
V_{\Theta}(H, w)_x = l_{x}^+ \cdot x, \quad \text{where} \quad l_{x} = l_{x}^+ \oplus l_{x}^-.
\]
It follows that the map
\[
X \in l_{x} \mapsto X \cdot x \in V_{\Theta}(H, w)_x
\]
is a linear isomorphism, for each \( x \in \text{fix}_\Theta(H, w) \).

Now we define the linearization map by

\[
\psi : V_\Theta(H, w) \to \mathbb{F}_\Theta, \quad \psi(X \cdot x) = \exp(X)x,
\]

where \( X \in \mathfrak{l}_x \) and \( x \in \text{fix}_\Theta(H, w) \). By Theorem 3.4 of [7], we have the following result.

**Theorem 2.2** The map \( \psi : V_\Theta(H, w) \to \mathbb{F}_\Theta \) takes the null section \( V_\Theta(H, w)_0 \) onto \( \text{fix}_\Theta(H, w) \) and satisfies:

i) Its restriction to some neighborhood \( \mathcal{N} \) of \( V_\Theta(H, w)_0 \) in \( V_\Theta(H, w) \) is a diffeomorphism over a neighborhood \( \mathcal{N} \) of \( \text{fix}_\Theta(H, w) \) in \( \mathbb{F}_\Theta \).

ii) Its restrictions to \( V_\pm \Theta(H, w) \) are diffeomorphisms, respectively, onto \( \text{un}_\Theta(H, w) \) and \( \text{st}_\Theta(H, w) \).

The above map \( \psi : V_\Theta(H, w) \to \mathbb{F}_\Theta \) is a conjugation of the flow \( g^t \) on a neighborhood of the attractor component \( \text{fix}_\Theta(H, w) \) to the linear flow

\[
g^t(X \cdot x) = g^t X \cdot g^t x
\]
on a neighborhood of the null section of \( V_\Theta(H, w) \) if and only if \( V_\Theta(H, w) \) is invariant and \( \psi \) is equivariant by the flow \( g^t \). A sufficient condition is that the map \( x \mapsto l_x \) be equivariant by \( g^t \), i.e., that \( g^t l_x = l_{g^t x} \). For the attractor component \( \text{fix}_\Theta(H, 1) \), by Corollary 3.6 of [7], this happens whenever either \( \Theta \subset \Sigma(H) \) or \( \Sigma(H) \subset \Theta \), where \( \Sigma(H) \) is the annihilator of \( H \) in the simple roots \( \Sigma \). For the other Morse components, by Proposition 5.5 of [3], it is sufficient that \( g^t \) be conformal, i.e., its unipotent component be trivial.

### 3 Lyapunov and metric spectra

Let \( \phi^t \) be a discrete or continuous-time flow defined on a compact metric space \((X, d)\). If \( M \subset X \) is an isolated compact \( \phi^t \)-invariant set, its stable set is given by

\[
\text{st}(M) = \{ x \in X : \omega(x) \subset M \}.
\]

Let us assume that \( M \) is not a repeller, which is equivalent to that \( \text{st}(M) \) is not empty. We say that \( x \in \text{st}(M) \) is metric regular if there exists the following limit

\[
\lambda(\phi^t, x) = \lim_{t \to \infty} \frac{1}{t} \log d(\phi^t(x), M),
\]
called the metric exponent of $\phi^t$ starting at $x$, measuring the exponential rate of approximation of $\phi^t(x)$ to $M$. The metric spectrum of $\phi^t$ relative to $M$ is defined by

$$\Lambda(\phi^t, M) = \{\lambda(\phi^t, x) : x \text{ is metric regular}\}.$$ 

The following result shows that $\Lambda(\phi^t, M)$ is an invariant of the conjugation classes by bi-Lipschitz maps.

**Proposition 3.1** Let $(X, d)$ be a metric space and $\psi : X \to X$ be a bi-Lipschitz map. If $\overline{\phi} = \psi \phi^t \psi^{-1}$, $\overline{x} = \psi(x)$ and $\overline{M} = \psi(M)$, then

$$\lambda(\phi^t, x) = \lambda(\overline{\phi}, \overline{x})$$

and

$$\Lambda(\phi^t, M) = \Lambda(\overline{\phi}, \overline{M}).$$

In particular, $\Lambda(\phi^t, M)$ is independent of equivalent metrics.

**Proof:** Since $\psi$ is a bi-Lipschitz map, there exist positive constants $b, c \in \mathbb{R}$ such that

$$bd(x, y) \leq d(\psi(x), \psi(y)) \leq cd(x, y),$$

for every $x, y \in X$. We have that

$$\lambda(\overline{\phi}, \psi(x)) = \lim_{t \to \infty} \frac{1}{t} \log d(\overline{\phi}(\psi(x)), \psi(M)) = \lim_{t \to \infty} \frac{1}{t} \log d(\psi(\phi^t(x)), \psi(M))$$

and thus

$$\lim_{t \to \infty} \frac{1}{t} \log bd(\phi^t(x), M) \leq \lambda(\psi(x), \overline{\phi}) \leq \lim_{t \to \infty} \frac{1}{t} \log cd(\phi^t(x), M)$$

showing that $\lambda(\phi^t, x) = \lambda(\overline{\phi}, \psi(x))$, since

$$\lambda(\phi^t, x) = \lim_{t \to \infty} \frac{1}{t} \log bd(\phi^t(x), M) = \lim_{t \to \infty} \frac{1}{t} \log cd(\phi^t(x), M).$$

The last assertion follows, since two metrics $d$ and $\overline{d}$ are equivalent if and only if the identity map from $(X, d)$ to $(X, \overline{d})$ is a bi-Lipschitz map. 

Since diffeomorphisms between Riemannian manifolds are locally bi-Lipschitz maps, we have the following result.
Corollary 3.2 Let \( \psi : X \to \overline{X} \) be a diffeomorphism between the Riemannian manifolds \((X,d)\) and \((\overline{X},d)\). If \( \overline{\phi^t} = \psi \phi^t \psi^{-1} \), \( \overline{x} = \psi(x) \) and \( \overline{M} = \psi(M) \), then
\[
\lambda(\phi^t, x) = \lambda(\overline{\phi^t}, \overline{x})
\]
and
\[
\Lambda(\phi^t, M) = \Lambda(\overline{\phi^t}, \overline{M}).
\]
In particular, \( \Lambda(M, \phi^t) \) is independent of Riemannian metrics.

**Proof:** Since \( X \) is locally compact and \( M \) is a compact subset, there exists an compact neighborhood \( B \) of \( M \). Since \( \psi \) is a diffeomorphism, it is a locally bi-Lipschitz map and its restriction to \( B \) is a bi-Lipschitz map onto its image. \( \square \)

Now we establish the connection between the Lyapunov and the metric spectra. We recall that for a linear flow \( \Phi^t \) of a normed vector bundle \((V,|\cdot|)\), we say that \( v \in V \) is **Lyapunov regular** if there exists the following limit
\[
\lambda_L(\Phi^t, v) = \lim_{t \to \infty} \frac{1}{t} \log |\Phi^t v|,
\]
called the **Lyapunov exponent of** \( \Phi^t \) **starting at** \( v \), measuring the exponential rate of growth of \(|\Phi^t v|\). The **Lyapunov spectrum of** \( \Phi^t \) is defined by
\[
\Lambda_L(\Phi^t, V) = \{ \lambda_L(\Phi^t, v) : v \text{ is Lyapunov regular} \}.
\]
The **stable Lyapunov spectrum of** \( \Phi^t \) is defined by
\[
\Lambda_L(\Phi^t, S) = \{ \lambda_L(\Phi^t, v) : v \in S \text{ and } v \text{ is Lyapunov regular} \}.
\]
where \( S \) is the stable set of the null section of \( V \).

**Proposition 3.3** Let \( (X,d) \) be a Riemannian manifold and \( \psi : N \to A \) be a diffeomorphism between some open neighborhood \( N \) of the null section \( Z \) of the normal bundle \( V \) of \( M \) and some open neighborhood \( A \) of \( M \). If \( M = \psi(Z) \) and \( \psi^{-1} \phi^t \psi \) is the restriction to \( N \) of a linear flow \( \Phi^t \) of \( V \), then
\[
\lambda(\phi^t, \psi(v)) = \lambda(\Phi^t, v),
\]
for all \( v \in S \), Lyapunov regular, and
\[
\Lambda(\phi^t, M) = \Lambda_L(\Phi^t, S).
\]
Furthermore, we have that \( \Lambda(\phi^t, M) \) is not empty.
Proof: There exist $s > 0$ such that $\Phi^s v \in N$ and $\phi^s(\psi(v)) \in A$. Since $\lambda(\Phi^t, v) = \lambda(\Phi^t, \Phi^s v)$ and $\lambda(\phi^t, \psi(v)) = \lambda(\phi^t, \phi^s(\psi(v)))$, we can assume that $v \in N$ and $\psi(v) \in A$. By Corollary 3.2, we have that

$$\lambda(\Phi^t, v) = \lambda(\Phi^t|_N, v) = \lambda(\phi^t|_A, \psi(v)) = \lambda(\phi^t, \psi(v)).$$

Thus it remains to prove that $\lambda(\Phi^t, v) = \lambda_L(\Phi^t, v)$, for all $v \in S$, metric regular. Let $\tilde{d}$ be the Sasaki metric on the normal bundle $V$ (see Section 4.2 of [1]). We can choose $N$ such that $\tilde{d}(v, Z) = |v|$, for all $v \in N$. Thus, for all $v \in S$, we have that $v$ is metric regular if and only if $v$ is Lyapunov regular and, in this case, $\lambda(\Phi^t, v) = \lambda_L(\Phi^t, v)$.

For the last assertion, we first observe that $Z$ is a $\Phi^t$-invariant and isolated, since $M$ is a $\phi^t$-invariant and isolated. Thus the restriction of $S$ over some chain transitive component of the flow induced by $\Phi^t$ on the base $M$ is a subbundle (see Theorem 2.13 of [9]). Thus we can apply Oseledec theorem (see [6]) to the restriction of $\Phi^t$ to this subbundle, showing that $\Lambda_L(\Phi^t, S)$ is not empty.

The previous proposition and Theorem 1 of [8] imply the following result.

**Corollary 3.4** If $M$ is normally hyperbolic, then $\Lambda(\phi^t, M)$ is not empty.

## 4 Flag spectrum

In this section, we determine explicitly the metric spectrum of a linear induced flow $g^t$ relative to a Morse component $\text{fix}_\Theta(H, w)$. First we need to introduce some suitable constructions, which are related to the constructions presented in the Section 2. We denote

$$\Lambda_\Theta(H, w) = \{\alpha(H) : g_\alpha \subset l_{wb_\Theta}\},$$

where

$$l_{wb_\Theta} = (n^+_H \oplus n^-_H) \cap w n^-_\Theta.$$

Now writing

$$\Lambda_\Theta(H, w) = \{\lambda_1 > \cdots > \lambda_{n^w_\Theta}\},$$

for each $i \in \{1, \ldots, n^w_\Theta\}$, we denote

$$b_i = \{X : \text{ad}(H)X = \lambda_i X\}$$

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and define the family of subspaces given by

$$\tilde{l}_x^i = l_x \cap \sum \{b_j : j \geq i\}. \quad (5)$$

**Proposition 4.1** For each $i \in \{1, \ldots, n^w_\Theta\}$, we have that

$$\tilde{l}_x^i = k^i l_x \tilde{w}_\Theta$$

for $x = k \tilde{w}_\Theta$, where $k \in K_H$. Furthermore, if $H$ is the hyperbolic type of $g^t$ and

(i) $w$ and $\Theta$ are arbitrary, with $g^t$ conformal or

(ii) $w = 1$ and $\Theta \subset \Sigma(H)$ or $\Sigma(H) \subset \Theta$, with $g^t$ arbitrary,

then

$$g^t \tilde{l}_x^i = \tilde{l}_{g^t x}^i.$$

**Proof:** First we note that the eigenspaces of $\text{ad}(H)$ are invariant by the centralizer $G_H$ of $H$ in $G$, since the elements of $G_H$ commute with $\text{ad}(H)$. The first claim follows, since $l_x = k l_x \tilde{w}_\Theta$, for $x = k \tilde{w}_\Theta$, where $k \in K_H \subset G_H$. By Corollary 3.6 and 3.9 of [7] and Proposition 5.5 of [3], if (i) or (ii) are verified, then we that $g^t l_x = l_{g^t x}$. Thus the second claim follows, since $g^t \in G_H$. \(\square\)

Following the proof of Proposition 3.2 of [7], for each $i \in \{1, \ldots, n^w_\Theta\}$, we have that

$$V^i_\Theta(H, w) = \bigcup \{l_x^i \cdot x : x \in \text{fix}_\Theta(H, w)\}$$

is a differentiable subbundle of the tangent bundle of $V_\Theta(H, w)$. The norm in $V_\Theta(H, w)$ is the restriction of the norm in the tangent bundle of $\mathbb{F}_\Theta$ induced by the Riemannian metric introduced in Section 2. We need to prove an elementary fact about the norm $| \cdot |$ in $V_\Theta(H, w)$.

**Lemma 4.2** If $v = X \cdot x \in V_\Theta(H, w)$, where $X \in l_x$ and $x \in \text{fix}_\Theta(H, w)$, then $|v| = |X|$.

**Proof:** We have that $X \in l_x = k l_x \tilde{w}_\Theta$ and that $x = k \tilde{w}_\Theta$, for some $k \in K_H$. Thus $Y \in k^{-1} X \in l_x \tilde{w}_\Theta$ and

$$v = k(Y \cdot \tilde{w}_\Theta) = kw^{-1} \cdot b_\Theta.$$
Thus, by the definition of $|\cdot|$, we have that

$$|v| = |w^{-1}Y| = |Y| = |k^{-1}X| = |X|,$$

where we use that $K$ acts in $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$-isometries.

Now we prove a strong version of Oseledec theorem (see [6]), determining explicitly the Oseledec decomposition of $V_{\Theta}(H, w)$ relative to the flow $g^t$.

**Theorem 4.3** Let $H$ be the hyperbolic type of $g^t$. For each

(i) arbitrary $w$ and $\Theta$, when $g^t$ is conformal or

(ii) $w = 1$ and $\Theta \subset \Sigma(H)$ or $\Sigma(H) \subset \Theta$, when $g^t$ is arbitrary,

each $i \in \{1, \ldots, n_w^w\}$ and each $v \in V_i^i(H, w) - V_{i-1}^i(H, w)$, we have that

$$\lambda_i(g^t, v) = \lambda_i.$$

In particular, every point of $V_{\Theta}(H, w)$ is Lyapunov regular and

$$\Lambda_L(g^t, V_{\Theta}(H, w)) = \Lambda_{\Theta}(H, w).$$

**Proof:** Using Proposition 4.1 and the definition of $V_i^i(H, w)$, we have that $v = k(X \cdot w_{\Theta})$, for some $k \in K_H$ and some $X \in f_{w_{\Theta}}^i - f_{w_{\Theta}}^{i-1}$. Writing $X = \sum_{\alpha} X_{\alpha}$, with $X_{\alpha} \in \mathfrak{g}_{\alpha}$, by the equations (4) and (5), we have that

$$\lambda_i = \max\{\alpha(H) : X_{\alpha} \neq 0\}.$$

On the other hand, by Lemma 4.2, we have that

$$|g^t v| = |g^t kX \cdot g^t k w_{\Theta}| = |g^t kX|.$$

Let $g^t = e^t u^t h^t$ be the multiplicative Jordan decomposition of the flow $g^t$. We have that $e^t$ acts on $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$-isometries, that $u^t = \exp(tN)$ with $\text{ad}(N)$ a nilpotent operator and that $h^t$ commutes with $K_H$. Thus we have that

$$|g^t k X| = |e^t u^t h^t k X| = |u^t k h^t X| = |e^{\text{ad}(N)} k h^t X|,$$

since $\exp(tN)Z = e^{\text{ad}(N)}Z$. Hence

$$|g^t k X| = |e^{\text{ad}(N)} k h^t X| = |k^{-1} e^{\text{ad}(N)} k h^t X| = |e^{tN} h^t X|,$$

(7)
where $\mathcal{N} = k^{-1}\text{ad}(N)k$ is also a nilpotent operator.

By equations (6) and (7), denoting $\lambda = \lambda_i$, it is sufficient to show that

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log |e^{t\mathcal{N}} h^t X|.$$  

(8)

Since $h^t X_\alpha = e^{(H) t} X_\alpha$, we have that

$$e^{t\mathcal{N}} h^t X = \sum_{\alpha} e^{(H) t} e^{t\mathcal{N}} X_\alpha = \sum_{\alpha} e^{(H) t} \sum_{n} \frac{t^n}{n!} \mathcal{N}^n X_\alpha.$$  

(9)

Denote

$$Y = \sum_{\alpha(H) = \lambda} X_\alpha$$

and

$$l = \max \{ n : \mathcal{N}^n Y \neq 0 \} \leq m,$$

where $m \in \mathbb{N}$ is such that $\mathcal{N}^{m+1} = 0$. In order to get equation (8), we just need to prove that

$$\lim_{t \to \infty} \frac{|e^{t\mathcal{N}} h^t X|}{\frac{1}{l!} |\mathcal{N}^l Y|} = 1,$$

(10)

since

$$\lim_{t \to \infty} \frac{1}{t} \log \left( \frac{e^{\lambda t} l!}{l!} |\mathcal{N}^l Y| \right) = \lambda.$$  

Using equation (11), we have that

$$\frac{|e^{t\mathcal{N}} h^t X|}{\frac{1}{l!} |\mathcal{N}^l Y|} = \frac{\left| \sum_{\alpha} e^{(H) t} \sum_{n} \frac{t^n}{n!} \mathcal{N}^n X_\alpha \right|}{\frac{1}{l!} |\mathcal{N}^l Y|} = \frac{|U_t + V_t|}{\frac{1}{l!} |\mathcal{N}^l Y|}$$  

(11)

where

$$U_t = \sum_{\alpha(H) = \lambda} e^{(H) t} \sum_{n=0}^{m} \frac{t^{n-l}}{n!} \mathcal{N}^n X_\alpha = \sum_{n=0}^{l} \frac{t^{n-l}}{n!} \mathcal{N}^n Y$$

and

$$V_t = \sum_{\alpha(H) < \lambda} e^{(H) t} \sum_{n=0}^{m} \frac{t^{n-l}}{n!} \mathcal{N}^n X_\alpha.$$  

It is immediate that

$$\lim_{t \to \infty} U_t = \frac{1}{l!} \mathcal{N}^l Y \quad \text{and} \quad \lim_{t \to \infty} V_t = 0.$$  

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and thus equation (11) implies the equation (10).

The last assertion follows, since we have that $V_{\Theta}(H, w)$ is given by the disjoint union of $V_{\Theta}^i(H, w) - V_{\Theta}^{i-1}(H, w)$, for $i \in \{1, \ldots, n_{\Theta}\}$.

We denote
$$\Lambda_{\Theta}^-(H, w) = \Lambda_{\Theta}(H, w) \cap \mathbb{R}^-$$
and, for each $\lambda_i \in \Lambda_{\Theta}^-(H, w)$, we define
$$\text{st}_{\lambda_i}^i(H, w) = \psi(V_{\Theta}^i(H, w) - V_{\Theta}^{i-1}(H, w)),$$
where $\psi : V_{\Theta}(H, w) \rightarrow \mathbb{F}_{\Theta}$ is the map presented in Theorem 2.2. It is immediate that the stable set of $\text{fix}_{\Theta}(H, w)$ relative to the flow $g^i$ is given by the following disjoint union
$$\text{st}_{\Theta}(H, w) = \bigcup\{\text{st}_{\lambda_i}^i(H, w) : \lambda_i \in \Lambda_{\Theta}^-(H, w)\}.$$  

The following result determines explicitly the metric spectrum of $g^i$ relative to the Morse components.

**Corollary 4.4** Let $H$ be the hyperbolic type of $g^i$. For each

(i) arbitrary $w$ and $\Theta$, when $g^i$ is conformal or

(ii) $w = 1$ and $\Theta \subset \Sigma(H)$ or $\Sigma(H) \subset \Theta$, when $g^i$ is arbitrary

and each $x \in \text{st}_{\lambda_i}^i(H, w)$, we have that
$$\lambda(g^i, x) = \lambda_i.$$  

In particular, every point of $\text{st}_{\Theta}(H, w)$ is metric regular and
$$\Lambda(g^i, \text{fix}_{\Theta}(H, w)) = \Lambda_{\Theta}^-(H, w).$$

**Proof:** Under the above hypothesis, we have that $V_{\Theta}(H, w)$ is invariant and $\psi : V_{\Theta}(H, w) \rightarrow \mathbb{F}_{\Theta}$ is equivariant by the flow $g^i$. Hence the corollary is an immediate consequence of Propositions 2.1 and 3.3, and Theorems 2.2 and 4.3.

The previous result allows us the following definition. The *flag spectrum* of an arbitrary linear induced flow $g^i$ is given by $\Lambda^-(H, 1)$, its metric spectrum relative to the unique attractor component $\text{fix}(H, 1)$ in the maximal flag manifold $\mathbb{F}$, where $\Theta = \emptyset$. 

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