1-based sparsification of energy interactions in two-dimensional turbulent flows

Riccardo Rubini1, Davide Lasagna1† and Andrea Da Ronch1

1Faculty of Engineering and Physical Sciences, University of Southampton, SO17 1BJ, Southampton, United Kingdom

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In this paper, sparsity-promoting regression techniques are employed to automatically identify from data relevant triadic interactions between modal structures in large Galerkin-based models of two-dimensional turbulent flows. The approach produces sparsely-connected models that reproduce the original dynamical behaviour at a much lower computational cost, as fewer triadic interactions need to be evaluated. The key feature of the approach is that dominant interactions are selected systematically from the solution of a convex optimisation problem, with a unique solution, and no a priori assumptions on the structure of scale interactions are required. We demonstrate this approach on models of two-dimensional lid-driven cavity flow at Reynolds number Re = 2 \times 10^4, where fluid motion is chaotic. To understand the role of the subspace utilised for the Galerkin projection on the sparsity characteristics, we consider two families of models obtained from two different modal decomposition techniques. The first uses energy-optimal Proper Orthogonal Decomposition modes, while the second uses modes oscillating at a single frequency obtained from Discrete Fourier Transform of the velocity snapshots. We show that, in both cases, and despite no a-priori physical knowledge is incorporated into the approach, relevant interactions across the hierarchy of modes are identified in agreement with the expected picture of scale interactions in two-dimensional turbulence. Yet, substantial structural changes in the interaction pattern and a quantitatively different sparsity are observed.

1. Introduction

In the classical description of developed turbulent flows, energy is transferred across the hierarchy of coherent structures via nonlinear triadic interactions. Implicit in this picture is the fact that not all interactions have the same importance, but they occur in preferential patterns. In fact, extensive numerical evidence suggests that the nonlinear interaction pattern among coherent structures is sparse. The evolution of structures at a certain length scale depends predominantly upon a subset of all other structures and the influence of interactions with the complementary set of structures can be generally neglected with minor global effects.

Successful attempts to construct a reduced set of equations that exploit this sparsity have been made in the past, often for canonical geometries where triadic interactions are conveniently examined in Fourier space and using a coarse-grained partitioning of the hierarchy of scales. Laval et al. (1999) considered two-dimensional homogeneous decaying turbulence and developed a reduced set of coupled partial differential equations governing the evolution of the large and small scales. In this model only dominant

† Email address for correspondence: davide.lasagna@soton.ac.uk
terms were retained based on observations from direct numerical simulation. With the goal of identifying fundamental mechanisms underlying wall turbulence, Thomas et al. (2015) developed nonlinear reduced models of plane Couette flow directly from the governing equations by first partitioning the flow into a streamwise-averaged mean and a perturbation field, and then neglecting nonlinear interactions among the streamwise varying perturbations, i.e. the perturbation-perturbation nonlinearity (Thomas et al. 2014). The models captured well-established roll-streak dynamical features of turbulence and its statistics in a computationally efficient framework. The models also sustained turbulent dynamics down to minimal configurations where interactions between the streamwise mean flow and only one single streamwise wavenumber are retained.

When reduced-order dynamical representations are derived using Galerkin projection on a low-dimensional subspace identified by a set of modal structures (Fletcher 1984; Rowley & Dawson 2017), triadic interactions are conveniently studied in modal space by examining a third-order coefficient tensor arising from projection of the basis function on the convective term of the Navier-Stokes equations (Noack et al. 2011). Sparsity characteristics have also been observed in this reduced-order setting. Couplelet et al. (2003) constructed Galerkin models of the separated turbulent flow past a backward-facing step using Proper Orthogonal Decomposition (POD) modes (Lumley 1970; Sirovich 1987) and observed that the energy transfer pattern in modal space shares many properties with its counterpart in isotropic homogeneous three-dimensional turbulent flows (Yeung et al. 1995). For instance, the authors observed that interactions are local in modal space and that a direct energy cascade exists. Analogously, Rempfer & Fasel (1994) examined the power budget of POD modes in a transitional boundary layer and observed that interactions in modal space occur predominantly between triads of modes whose sum of modal indices is equal to zero, similar to energy interactions between Fourier modes in homogeneous turbulence. However, classical model order reduction techniques (Rowley & Dawson 2017) have not traditionally exploited this feature. In fact, when modal decompositions such as POD are employed, densely-connected models are usually obtained, as the third-order coefficient tensor is dense for inhomogeneous flows without particular symmetries. This hinders the interpretation of the underlying physics of scale interactions and increases computational costs, as all triadic interactions have to be evaluated for propagating the model forward in time.

The first contribution of this work is that we apply data-driven techniques (Blum & Langley 1997; Brunton et al. 2016; Loiseau & Brunton 2018; Brunton et al. 2019) to identify the subset of relevant triadic interactions in large Galerkin model. Weak triadic interactions are pruned, producing a one-parameter family of sparsely-connected models trading compactness for prediction capabilities. The main aim is to generate reduced order models resolving a wide range of scales while preserving computational efficiency and interpretability by pruning interactions that are not relevant for the dynamics. The cornerstone of the proposed approach is $l_1$-based regression (Friedman et al. 2008; Tibshirani 2013), widely used in the statistical community to extract parsimonious representation of complex datasets containing a subset of predominant features. The non-differentiable, yet convex, nature of the $l_1$ regularisation allows transforming the interaction selection problem into a convex optimisation problem that can be solved efficiently, with unique solution. Since no a priori knowledge of the dynamics is utilised, the approach is fine-grained and relevant interactions are identified in a mode-by-mode fashion across the hierarchy of modes. Sparsity-promoting regression techniques have been recently proposed by Brunton and coworkers (Brunton et al. 2016; Kaiser et al. 2018) in the SINDy framework (Sparse Identification of Nonlinear Dynamics), as a mean to discover parsimonious dynamical representations of systems whose underlying (but
hidden) evolution equations are somehow sparse in the space of the possible functions (Brunton et al. 2016). The difference of our work is that, for the Navier-Stokes equations, sparsity is not an \emph{a priori} intrinsic property either when partial differential equations are considered, or when a projection onto a low-dimensional subspace is obtained. Rather, sparsity is an \emph{a posteriori} feature of turbulent realisations, since various physical mechanisms produce energy interactions according to a preferential pattern. In addition, these methods have been applied, so far, to relatively small Galerkin models (Loiseau & Brunton 2018), and it is not yet understood if they can identify and extract relevant interactions in very large Galerkin models in agreement with the established picture of energy interactions in turbulent flows. In this sense, our approach is closer to the recent work of Nair & Taira (2015), Taira et al. (2016) and Nair et al. (2018). These authors employed network-theoretic sparsification approaches (Newman 2018) to identify key vortex-to-vortex interactions in two-dimensional homogeneous turbulence, obtaining sparse models that capture the essential physics of unsteady fluid flow with a reduced number of interactions between the same large number of states.

The second contribution of this paper is that we examine how sparsity of energy interactions depends on the subspace used to generate the Galerkin model. Finding an appropriate subspace for projection is recognised as a challenging task (Noack et al. 2016), and several modal decompositions have been proposed differing in spirit and approach (see Taira et al. (2017) for a recent review). However, the role of the subspace on the organisation of energy interactions has not been explored in the past. To address this question, we examine and compare in this paper energy interactions and sparsity features of two families of Galerkin models. The first uses energy-optimal POD modes while the second uses modes oscillating temporally at a single frequency, obtained using a procedure based on Spectral Proper Orthogonal Decomposition (Sieber et al. 2016) and equivalent to a Discrete Fourier Transform (DFT) of the velocity snapshots. Here, we aim at understanding if the optimal data-representation property of POD also provides the best description in terms of sparsity, even if POD can couple different flow structures with similar energy content (Noack et al. 2016; Towne et al. 2018).

This manuscript is organised as follows. For completeness, section 2 summarises the methodology utilised to generate reduced order models using Galerkin projection, and then discusses how energy interactions in Galerkin models can be examined. Subsequently, the $l_1$-based sparse regression approach is outlined and conceptual differences between our approach and the SINDy approach proposed in Brunton et al. (2016) are reported. In section 3, we demonstrate this methodology by considering relatively large Galerkin models of two-dimensional lid driven cavity flow at a Reynolds number $Re = 2 \times 10^4$, where dynamics is chaotic (Auteri et al. 2002). We first focus on modal decomposition of the flow and then move to energy analysis and sparsification. Conclusions are offered in section 4.

2. Methodology

2.1. Reduced Order Modelling

We consider a space of square integrable velocity vector fields defined over a spatial domain $\Omega$, endowed by the standard inner product

\[ (\mathbf{u}, \mathbf{v}) := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} d\Omega, \]

(2.1)

where $\mathbf{u}, \mathbf{v}$ are two element of such space. The resulting $L^2(\Omega)$ norm is denoted as $||\mathbf{u}||_2 = \sqrt{(\mathbf{u}, \mathbf{u})}$. Using the time averaged velocity field $\bar{\mathbf{u}}(\mathbf{x})$ as a base flow, and denoting
by \( \mathbf{u}'(t, \mathbf{x}) \) the velocity fluctuation \( \mathbf{u}(t, \mathbf{x}) - \bar{\mathbf{u}}(\mathbf{x}) \), an \( N \)-dimensional expansion expressed by the ansatz

\[
\mathbf{u}(t, \mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(t, \mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x}) + \sum_{i=1}^{N} a_i(t) \phi_i(\mathbf{x}), \tag{2.2}
\]
is introduced to describe the space-time velocity field, where \( a_i(t) \) and \( \phi_i(\mathbf{x}) \), \( i = 1, \ldots, N \) are the temporal and global spatial modes, respectively, with \( \| \phi_i(\mathbf{x}) \| = 1 \). These modes may be computed a posteriori from numerical or experimental data or a priori from a characteristic operator of the system (Taira et al. 2017) or from completeness considerations (Noack & Eckelmann 1994). Reduced order models are then derived by projecting the governing equations onto the subspace defined by the modes (Rowley & Dawson 2017). Restricting our analysis to configurations where the boundaries are either no-slip walls or periodic, this procedure results in an autonomous system of coupled nonlinear ordinary differential equations (ODEs)

\[
\sum_{j=1}^{N} M_{ij} \dot{a}_j(t) = \tilde{C}_i + \sum_{j=1}^{N} \tilde{L}_{ij} a_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} \tilde{Q}_{ijk} a_j(t) a_k(t), \quad i = 1, \ldots, N, \tag{2.3}
\]
defining the temporal evolution of the coefficients \( a_i(t) \). Here, we only report the definitions of the quadratic coefficients

\[
\tilde{Q}_{ijk} = (\phi_i, \phi_j \cdot \nabla \phi_k), \tag{2.4}
\]
while expressions for the tensors \( \tilde{C} \) and \( \tilde{L} \) can be found in Noack et al. (2011). The matrix \( M \), with entries \( M_{ij} = (\phi_i, \phi_j) \), takes into account the fact that the spatial modes may not be orthogonal and is introduced here for generality.

If the \( N \) modes span collectively an \( N \)-dimensional subspace, \( M_{ij} \) is invertible and the system \( \text{(2.3)} \) can be rearranged as

\[
\dot{a}_i(t) = C_i + \sum_{j=1}^{N} L_{ij} a_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} Q_{ijk} a_j(t) a_k(t), \quad i = 1, \ldots, N, \tag{2.5}
\]
with

\[
C_i = \sum_{q=1}^{N} M_{iq}^{-1} \tilde{C}_q, \quad L_{ij} = \sum_{q=1}^{N} M_{iq}^{-1} \tilde{L}_{qj} \quad \text{and} \quad Q_{ijk} = \sum_{q=1}^{N} M_{iq}^{-1} \tilde{Q}_{qjk}. \tag{2.6}
\]
As observed by Rempfer & Fasel (1994a), the infinite dimensional matrix \( M_{ij} \) should be first inverted and then truncated to maintain a good prediction accuracy. For the cases discussed in this paper, we have not followed this procedure as we observed that the matrix \( M_{ij} \) has a strong diagonal structure. Hence, the error performed by truncating it to size \( (N, N) \) and then inverting it can be reasonably assumed to be small.

Since the spatial modes satisfy automatically the boundary conditions, the expansion \( \text{(2.2)} \) provides a suitable foundation to examine interactions between coherent structures in complex geometries. Here, we follow established approaches (Rempfer & Fasel 1994b) and analyse such interactions by introducing the modal energies \( e_i(t) = \frac{1}{2} a_i(t) a_i(t), \quad i = 1, \ldots, N \). The instantaneous rate of change is given by

\[
\dot{e}_i(t) = C_i a_i(t) + \sum_{j=1}^{N} L_{ij} a_j(t) a_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} Q_{ijk} a_j(t) a_j(t) a_k(t), \quad i = 1, \ldots, N, \tag{2.7}
\]
obtained by multiplying \( \text{(2.5)} \) by \( a_i(t) \). Note that, in a general case where the modes
do not form an orthonormal set, the domain integral of the kinetic energy of velocity fluctuations is given by

$$E(t) = \frac{1}{2} \int_\Omega u'(t, x)^2 d\Omega = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} a_i(t) a_j(t)$$

(2.8)

and not by a straightforward sum of the terms $e_i(t)$. The right hand side of equation (2.7) is composed of three terms describing energy transfers between the hierarchy of modes. The first two describe variations of energy due to production/dissipation arising from interactions with the mean flow and from viscous effects. The third term defines variations of energy arising from inviscid nonlinear interactions between triads of modes. Following Rempfer & Fasel (1994), these are defined in a time averaged sense by the quadratic interaction tensor $N$ with entries

$$N_{ijk} = Q_{ijk} a_i a_j a_k,$$

(2.9)

where the overbar denotes temporal averaging. The study of this term is the principal focus of the current analysis.

Spatial modes obtained from classical decompositions have generally global support over the domain (see e.g. Taira et al. (2017)). The result is that the evolution equations (2.5) are not strictly sparse in the sense employed by Brunton et al. (2016). In fact, unless particular symmetries apply, the tensor $Q$ is generally dense, i.e. most of its entries are different from zero and the right hand side of (2.5) contains all monomial terms in the modal amplitudes $a_i(t)$ up to order two. However, as anticipated in the introduction, in turbulent realisations of the Navier-Stokes equations only a subset of all triadic interactions contributes to a significant degree to the overall energy budget (Couplet et al. 2003; Rempfer & Fasel 1994b). In this sense, sparsity is a primarily an 
_a posteriori_ feature of solutions, i.e. a feature of the quadratic interaction tensor $N$.

The approach developed in this work starts from this fundamental observation and aims to generate a sparse Galerkin model, defined by a sparse coefficient tensor $Q^s$, that is a good approximation of the original dynamical system in the sense that the mismatch between the transfer tensors $N^s$ and the original $N$ obtained from the definitions (2.4, 2.6) is as small as possible across the hierarchy of modes.

### 2.2. Sparse regression

To construct a sparse Galerkin system, we use a procedure akin to that utilised in previous work for calibrating Galerkin models from data (Perret et al. 2006; Cordier et al. 2010; Xie et al. 2018) and more recently for the identification of sparse dynamical systems (Brunton et al. 2016). In the first step, we assume that $N_t$ snapshots of the velocity field are available from simulation and arrange samples of the temporal coefficients $a_i(t_j), i = 1, \ldots, N$ and $j = 1, \ldots, N_t$, into the data matrix $A \in \mathbb{R}^{N_t \times N}$, with entries $A_{ij} = a_i(t_j)$. Similarly, we construct the modal acceleration matrix $\dot{A} \in \mathbb{R}^{N_t \times N}$ containing the time derivative of the temporal coefficients obtained by projecting the modes $\phi_i(x)$ on snapshots of the Eulerian acceleration field $\partial_t u(t_j, x)$ and correcting such projections with $M$ when modes are not orthogonal (see also Rempfer & Fasel 1994b). We then exploit the polynomial structure of the Galerkin system (2.5) to construct the database matrix $\Theta(A) \in \mathbb{R}^{N_t \times q}$

$$\Theta(A) = \begin{pmatrix}
1 & a_1^1 & \ldots & a_N^1 & a_1^1 a_1^1 & \ldots & a_N^1 a_N^1 \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & a_1^{N_t} & \ldots & a_N^{N_t} & a_1^{N_t} a_1^{N_t} & \ldots & a_N^{N_t} a_N^{N_t}
\end{pmatrix},$$

(2.10)
called nonlinear feature library in Brunton et al. (2016), where $q = (N+1) + N(N+1)/2$ is the total number of features, the sum of constant, linear and quadratic interactions. The number of quadratic coefficients is only $N(N+1)/2$ because the interaction between mode $i$ and $j$ is considered only once in (2.10). As discussed later on in the paper, this avoids columns of $\Theta(A)$ becoming linearly dependent, which would in turn result in numerical stability issues in the solution regression problem (see e.g. Perret et al. (2006) and Cordier et al. (2010)).

Arranging the projection coefficients tensors $C$, $L$ and $Q$ associated to the $i$-th mode into a coefficient vector $\beta_i \in \mathbb{R}^q$, the Galerkin system (2.3) can be equivalently expressed as

$$\dot{A}_i = \Theta(A)\beta_i, \quad i = 1, \ldots, N,$$

(2.11)

where $\dot{A}_i$ is the $i$-th column of the modal acceleration matrix. The key idea is that if some nonlinear interactions are more important than others, then the corresponding entries of the coefficient vector $\beta_i$ can be shrunk to zero with minor effects on the predictive ability of the resulting model. The challenge is to find a systematic method to identify the dominant interactions and prune unnecessary coefficients whilst calibrating the remaining model coefficients such as to preserve the overall energy budget. Here, we adopt an established sparsity-promoting regression technique known as LASSO regression (Least Absolute Shrinkage Selection Operator, see Tibshirani (1996)). In short, it leads to a set of $N$ optimisation problems of the form

$$\min_{\beta_i} ||\Theta(A)\beta_i - \dot{A}_i||_2^2 + \gamma_i ||\beta_i||_1, \quad i = 1, \ldots, N,$$

(2.12)

one for each mode, where $|| \cdot ||_p$ denotes the $l_p$ norm of a vector. The first term in the objective function in (2.12) produces calibrated models that have minimum prediction error on the modal acceleration (see discussion in Cordier et al. (2010) and Couplet et al. (2005)). The second term penalises large model coefficients, regularises the regression and encourages sparsity in the solution by shrinking exactly to zero coefficients in $\beta_i$ corresponding to interactions in $\Theta(A)$ with little dynamical influence. Ideally, to prune unnecessary coefficients, a penalisation term proportional to the cardinality of $\beta_i$, $\text{card}(\beta_i)$, would formally be more correct (Jovanović et al. 2012, 2014). However, the resulting optimisation problem would be computationally intractable even for Galerkin models of modest dimensions. In fact, this penalisation is usually relaxed to the computationally tractable $l_1$ term (Ramirez et al. 2013). Regardless, the optimisation problems (2.12) are convex and thus have an unique solution. In addition, the approach lends naturally to parallelisation, since the optimisation problems can be solved independently for each mode. In initial stages of the research, we have found approaches based on sequential thresholded least-squares (Brunton et al. 2016; Zhang & Schaeffer 2019; Loiseau & Brunton 2018) to be not sufficiently robust. Hence, solutions of (2.12) have been computed using the sklearn (Pedregosa & Varoquaux 2011) library, which implements a sub-gradient descent algorithm to manage the non differentiability of the $l_1$ norm.

The weight $\gamma_i$ in equation (2.12) is an arbitrary coefficient that can be manipulated to trade prediction ability (when it is small) for sparsity (when it is large). To formalise these concepts we introduce the global reconstruction error $\epsilon$

$$\epsilon = \sum_{i=1}^{N} \frac{||\Theta(A)\beta_i - \dot{A}_i||_2^2}{||\dot{A}_i||_2^2}$$

(2.13)
and the density of the system $\rho$

$$\rho = \frac{1}{Nq} \sum_{i=1}^{N} \text{card}(\beta_i).$$

(2.14)

In equation (2.13), the absolute reconstruction error $|\Theta(A)\beta_i - \dot{A}_i|^2$ is normalised with the mean squared acceleration $|\dot{A}_i|^2$ to balance the global reconstruction error across the hierarchy, which would be otherwise dominated by the most energetic modes. On the other hand, the density $\rho$ ranges from 0, when all interactions have been pruned, to 1, for a fully connected model. Note that for large models, the density is dominated by the quadratic tensor $Q$. A one-parameter family of models can be generated by varying the regularisation weight $\gamma_i$, producing a Pareto front (Schmidt & Lipson 2009) on the $\rho$-$\epsilon$ plane. Since only a subset of triadic interactions is relevant, the expectation is that a sweet spot appears on this curve, defining an ‘optimal’ penalisation $\gamma_i$.

The weights can be chosen independently for each index $i$. This can be useful to modulate sparsity across the hierarchy of structures. We consider two different modulation strategies. In strategy S1, the weight is constant for all modes, $\gamma_i = \gamma$. This strategy sparsifies more aggressively the equations of motion of low-energy modes, because the $l_1$ penalisation term has a higher importance than the $l_2$ component. Hence, we also introduce strategy S2, where the weight is normalised with respect to the mean squared modal acceleration as $\gamma_i = |\dot{A}_i|^2 \gamma$. This is equivalent to solving problem (2.12) using the relative error in (2.13) as least-squares component of the objective function. This strategy results in a more balanced sparsification across the hierarchy of modes and avoids earlier truncation, i.e. when all coefficients are set to zero by the LASSO.

One additional modification of this approach is that discussed in Loiseau & Brunton (2018), namely to enforce that the nonlinear term in the sparsified Galerkin model conserves energy (see e.g. Balajewicz et al. (2013) for a formal definition). This can be achieved by introducing a set of constraints on the coefficients vectors $\beta_i$, leading to a single optimisation problems of larger dimension. In a preliminary study performed on small systems, where this approach is feasible, we observed that the difference in energy conservation of models obtained from the constrained and the unconstrained problems is small in relative terms. This occurs because the temporal coefficients in the data matrix $A$ are originally obtained from an energy conserving nonlinearity, and the regression “discovers” this property from data. Hence, in further investigations we always solved the unconstrained form of problems (2.12).
3. Results

As a demonstration of the ideas discussed in the introduction, we apply this methodology to the two-dimensional flow in a lid-driven square cavity. The Reynolds number is defined as \( Re = \frac{LU}{\nu} \) where \( L, U \) are the cavity dimension and the lid velocity, respectively, while \( \nu \) is the kinematic viscosity. We investigated a chaotic regime (Auteri et al. 2002) at \( Re = 2 \times 10^4 \). The domain is defined in nondimensional Cartesian coordinates \( x = (x, y) \) and the velocity field is defined by the components \( u(x, t) = (u(x, t), v(x, t)) \). For visualisation purposes, we introduce the out-of-plane vorticity \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \).

Numerical simulations were performed in OpenFOAM with the incompressible flow solver icofoam. A grid independence study has been performed, comparing the solutions obtained for increasingly finer meshes with the numerical results of Erturk et al. (2005). As validation we observed a good agreement in the mean values and the Our final mesh at this Reynolds number is composed of 300 \( \times \) 300 cells, with refinement near the cavity boundaries. Three snapshots of the vorticity field obtained in the present simulation are shown in figure 1. The dominant feature in this regime is the shear layer separating the main vortex core from the recirculation areas located in the cavity corners. Most of the dynamically interesting features in this regime originate from the bottom right corner of the cavity, as the secondary vortex in the recirculation zone sheds erratically, and produces wave-like disturbances advected along the shear layer by the mean flow. The characteristic non-dimensional frequency of this motion is \( fL/U = 0.7 \). From our simulations, we extract \( N_t = 1500 \) velocity snapshots using a nondimensional sampling period \( \Delta t = 0.1 \). These settings are sufficient to adequately time-resolve the fast scales as well as to include many shedding events at the bottom right corner, making the regression problems (2.12) statistically reliable.

3.1. Modal Decomposition

First, we consider models generated using POD modes. POD produces economic reduced order models, but has the well-known shortcoming of mixing together fluid motions at different temporal/spatial scales (Mendez et al. 2019). Second, we consider models generated from modes oscillating at a single frequency obtained from a procedure that is equivalent to a Discrete Fourier Transform (DFT) of the velocity snapshots. For practical convenience, we obtain the two distinct sets of modes using the same technique, based on the approach proposed by Sieber et al. (2016). Briefly, the method considers...
the temporal correlation matrix $R \in \mathbb{R}^{N_t \times N_t}$, with entries

$$R_{ij} = \frac{1}{N_t} \langle u'(t_i, x), u'(t_i, x) \rangle,$$  \hspace{1cm} (3.1)

and then defines a filtered correlation matrix $S$, with elements

$$S_{ij} = \sum_{k=-N_f}^{k=N_f} g_k R_{i+k,j+k}$$  \hspace{1cm} (3.2)

given by the application of the filter coefficient vector $g$ along the diagonals of the correlation matrix. An ordered set of temporal coefficients $a_i = [a_i(t_1), \ldots, a_i(t_{N_t})]$ and associated mode energies $\lambda_i$ is then obtained from the eigendecomposition of $S$,

$$S a_i = \lambda_i a_i,$$  \hspace{1cm} (3.3)

so that $\lambda_i \delta_{ij} = a_i^\top a_j$. As discussed in [Sieber et al. (2016)], when the filter is extended over the entire dataset and in the limit of number of samples tending to infinity, the filtered correlation matrix converges to a Toeplitz, circulant matrix. Then, its eigenvalues trace the power spectral density of the underlying data set. On the other hand, the eigenvectors $a_i$ corresponds to the Fourier basis. This procedure generates conjugate pairs of modal structures with same energy oscillating at a single frequency. These can be viewed as a set of modal oscillators exhibiting periodic fluctuations [Taira et al. (2017)] and tracing fluid motion at on a two-dimensional subspace. In practice, for a finite-length dataset, we filter the temporal correlation matrix assuming periodicity using a box-car filter, as suggested in [Sieber et al. (2016)]. Hereafter, we will refer to the modal structures identified by this procedure as DFT modes.

One important consideration is that, unlike Dynamic Mode Decomposition (see [Rowley et al. (2009); Schmid (2010); Chen et al. (2012)]), DFT lacks the ability to discern and identify dominant frequency components. Instead, a number of modes equal to the number of snapshots utilised is produced, oscillating in conjugate pairs at specific frequencies determined by the sampling period $\Delta t$ and observation time $T$ (Mendez et al. 2019). This property, picket fencing, results in frequencies that are integer multiples of the fundamental frequency $f_1 = T^{-1}$, up to the Nyquist component $f_{Nyq} = (2\Delta t)^{-1}$. In addition, unlike for POD, as the length of the dataset is increased, the number of energy-relevant modes increases and low-frequency modes with little dynamical importance appear. The approach we use here is to divide the dataset into five partition of thirty time units, covering an average of 20 cycles of the dominant oscillatory component, and providing sufficient frequency resolution to distinguish small scale spectral features. In addition, two possible ways of sorting pairs of modal structures are possible, i.e. by energy content (using the eigenvalues $\lambda_i$) or by frequency. Models obtained with the two sorting schemes will be referred to as DFT$e$ and DFT$f$, respectively.

We now focus on the characteristics of the modal structures obtained by these two methods. We denote the normalised cumulative sum of the eigenvalues $\lambda_i$ of the (filtered) correlation matrix as

$$e(n) = \sum_{i=1}^{n} \lambda_i / \sum_{i=1}^{N_t} \lambda_i,$$  \hspace{1cm} (3.4)

describing the fraction of the fluctuation kinetic energy captured by the first $n$ elements of the expansion (2.2). This quantity is shown in figure 2-(a) for the POD and for the two possible DFT sorting schemes. As expected, a larger energy is captured by the POD basis. For the DFT decomposition, the energy-based sorting is more efficient at data
Table 1. Normalised cumulative energy distribution $e(n)$ for POD and DFT modes, where the latter are sorted by energy content ($\text{DFT}_e$) or by frequency ($\text{DFT}_f$).

| $n$  | 1  | 5  | 10 | 15 | 20 | 26 | 50 | 75 | 80 | 95 | 100 | 300 |
|------|----|----|----|----|----|----|----|----|----|----|------|-----|
| POD  | 0.26 | 0.74 | 0.85 | 0.89 | 0.9 | 0.95 | 0.98 | 0.99 | 0.995 | 0.998 | 0.999 |
| $\text{DFT}_e$ | 0.17 | 0.49 | 0.62 | 0.74 | 0.8 | 0.88 | 0.92 | 0.95 | 0.97 | 0.98 | 1 |
| $\text{DFT}_f$ | 0.02 | 0.07 | 0.15 | 0.39 | 0.43 | 0.47 | 0.91 | 0.96 | 0.97 | 0.98 | 1 |

Figure 2. Panel (a): cumulative sum of the first 100 eigenvalues of $S_{ij}$ for the three decompositions considered. Panel (b): distribution of the modal energies of DFT modes sorted by frequency.

compression, although the difference vanishes for large $n$, since for low energy modes the two sorting schemes are equivalent. The modal energies associated to the $\text{DFT}_f$ modes are shown in figure 2(b) as a function of the modal index $i$. The distribution is characterised by a continuous component, with modal energy decaying with frequency, and a discrete component, with a fundamental peak for the pair of modes $(31, 32)$ and its first few harmonics. The peak, at a non-dimensional frequency $f = 0.7$, is physically originated from the high-energy structures transported along the shear layer by the rotation of the main vortical structure. This can be observed in panels (a) and (b) of figure 3 showing the vorticity field $\omega$ of the DFT mode pair $(31, 32)$. This pair of modes describes a vorticity perturbation having the form of a wave travelling along the edge of the main vortex. Hence, the spatial structure of the two modes is shifted in the direction of the shear layer by half wave. Travelling-wave structures in cavity flows have already been observed by Poliashenko & Aidun [1995], Auteri et al. [2002] and recently characterised with Koopman analysis by Arbabi & Mezić [2017]. The two leading POD modes, reported in panels (c) and (d) of figure 3, have the same energy and capture a similar travelling-wave pattern described by the leading DFT mode pair.

3.2. Energy Analysis

To provide a more robust foundation to understand the sparsification results reported in sections 3.3 and 3.5, we first focus on the analysis of the average energy interactions. The structure of the interaction tensor $N$ for a large POD-based model with $N = 75$, reconstructing more than 99% of the fluctuation kinetic energy, is reported in figure 4 showing the magnitude of the interactions for three slices for $i = 1, 10$ and 75, in panels (a), (b) and (c), respectively. All entries of the tensor $N$ are generally nonzero,
Figure 3. Vorticity field of the most energetic pair of DFT modes, panels (a) and (b), and of the first two POD modes, panels (c) and (d).

Figure 4. Magnitude of the average interaction tensor coefficients $N_{ijk}$ for three POD modes across the spectrum, $i = 1, 10$ and 75 in panel (a), (b) and (c) respectively, for a model resolving 99% of the fluctuation kinetic energy. Panel (d) shows the coefficient $\chi_i(n)$ as a function of the normalised cutoff $n$ for the same three modes.

although the strength of the interactions varies across several orders of magnitude. This is a combined result of the projection coefficients tensor $Q$, whose entries are typically non zero, and of the complex spectral structure of the temporal coefficients $a_i(t)$. The most important feature of figure 4 is that interactions are highly organised and there exists a subset of interactions that are more active. Specifically, for any mode $i$, triadic interactions can be classified as illustrated in panel (a) in four different categories by introducing a cutoff modal index $n$. The subset of interactions denoted as LL corresponds to nonlinear energy transfer involving pairs of low index modes, HL and LH denote interactions involving high-low/low-high modes, while HH denotes the subset of interactions involving pairs of high modes. We observe that the areas corresponding to LL and HL/LH are the most active. If we map low/high modal indices to large/small scales, this result is in agreement with the picture of energy transfer between scales in homogeneous isotropic two-dimensional turbulence (Ohkitani 1990; Laval et al. 1999),
Figure 5. Panel (a): Magnitude of the average interaction tensor $N_{ijk}$ for $i = 100$, with the three characteristics branches, showing that active interactions come in $2 \times 2$ blocks corresponding to matching triads of modes. The small inset focuses on the interactions of branches C and U. Panel (b): magnitude of the average interaction tensor where the three branches of panel (a) have been unfolded on a larger plane spanned by the coordinates $l$ and $\eta$. The inset shows details of the interactions of the branch U in the plane $\eta - l$.

where the large scales interact with the small ones in a non-local fashion. In addition, interactions are not symmetric with respect to a swap of indices $j,k$. This can be quantified by computing the coefficient

$$\chi_i(n) = \sum_{j=1}^{n} \sum_{k=1}^{N} N_{ijk}/\sum_{j=1}^{N} \sum_{k=1}^{N} N_{ijk},$$

representing the relative dynamical importance of the subset of interactions LL + HL and LL + LH. Panel (d) of figure 4 shows $\chi_i$ for $i = 1, 10$ and 75 as a function of the normalised cutoff $n$. The interaction subset HL is up to four times more important than the subset LH. This is a consequence of the asymmetry of the projection coefficients $Q_{ijk}$, which arise from the fact that the convective transport of structure $\phi_k(x)$ operated by the structure $\phi_j(x)$ is more intense when the modal structure $\phi_j(x)$ describes large-scale flow features.

We now consider energy analysis of a large, full-resolution DFT$_f$ model constructed from five partitions of thirty time units as discussed in section 3.1. The model is composed of all $N = 300$ modes, corresponding to 150 distinct frequencies. We perform modal decomposition and energy analysis on each partition separately, and then average the mean energy transfer rate tensor $N$ over the five partitions. Figure 5-(a) shows the mean transfer rate distribution for mode $i = 100$. Energy interactions in the DFT model are very sparsely distributed on a thin horseshoe-shaped structure composed of three branches (denoted in the figure as L, C and U) of $2 \times 2$ blocks, and all other mean energy transfer rates interactions are identically zero. This pattern results from the joint effect of the oscillatory nature of the temporal coefficients and the quadratic nonlinearity of system (2.3), which can only be satisfied by triads of modes having matching temporal wave numbers. A less pronounced horseshoe-shaped distribution of the energy interactions has been previously observed in energy analysis of POD-based models of three-dimensional transitional boundary layers Rempfer & Fasel (1994a, b). These authors noticed that low-energy modal structures resembles Fourier modes in the spanwise direction and thus coefficients $Q_{ijk}$ and energy interactions are nonzero only for specific triads of modes. In the present case, this pattern is determined exclusively by the temporal coefficients as
Figure 6. Panels (a) and (b): absolute and relative strength of the energy interactions between pairs of DFT modes for a model with \( N = 300 \) visualised on the plane \( m, n \), with the additional coordinates \( l \) and \( \eta \). Panels (c): relative energy interactions for the first and last mode pairs.

The tensor \( Q \) constructed from projection modes does not possess any structure and its coefficients have a similar statistical distribution to that obtained using the POD modes.

To facilitate the interpretation of the energy interaction pattern, we follow Rempfer & Fasel (1994\textsuperscript{b}) and Arbabi & Mezić (2017) and define oscillatory modal structures

\[
\mathbf{u}_l(t, \mathbf{x}) = a_{2l-1}(t)\phi_{2l-1}(\mathbf{x}) + a_{2l}(t)\phi_{2l}(\mathbf{x}),
\]

numbered by the index \( l \) and tracing fluid motion at a single frequency on a two-dimensional subspace. Their modal energy is

\[
e_l(t) = \frac{1}{2} \left( a_{2l-1}^2(t) + a_{2l}^2(t) \right) + a_{2l-1}(t)a_{2l}(t)(\phi_{2l-1}(\mathbf{x}), \phi_{2l}(\mathbf{x})).
\]

Numerical experiments show that, for large number of snapshots, pairs of modes \( \phi_{2l-1}(\mathbf{x}) \) and \( \phi_{2l}(\mathbf{x}) \) tend to be orthogonal. Hence, considering the evolution equation for the modal energy \( e_l(t) \sim \frac{1}{2} (a_{2l-1}^2(t) + a_{2l}^2(t)) \) leads to the condensed triadic interaction tensor \( \hat{\mathbf{N}} \) of size \((N/2, N/2, N/2)\) with entries

\[
\hat{\mathbf{N}}_{lmn} = \sum_{i=l}^{l+1} \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} N_{ijk},
\]

lumping together the \( 2 \times 2 \) blocks of interactions at matching triads of figure 5-(a). In addition, the three branches L, C and U can be unfolded and conveniently visualised on a two-dimensional plane spanned by the coordinate \( l \), the modal structure index, and \( \eta = m - n \), representing the distance in modal space between pairs of temporal wavenumbers. This unfolding process is shown in panel (b) of figure 5 and when repeated for all modal structures leads to the distribution shown in figure 6-(a). In figure 6-(b), we report the average transfer rate \( \hat{\mathbf{N}}_{lmn} \) normalised with the total average transfer rate for each structure, the quantity

\[
\hat{T}_l = \sum_{m=1}^{N/2} \sum_{n=1}^{N/2} \hat{\mathbf{N}}_{lmn}, \quad l = 1, \ldots, N/2,
\]

to illustrate more clearly the relative strength of the interactions. In figure 6-(c), the normalised mean transfer rate for \( l = 1 \) and 150 is reported. Interactions between triads of pairs of DFT modes are organised in agreement with the physics of scale interactions...
previously discussed for POD models. In absolute terms, the most relevant interactions are clearly those located near the origin of the coordinates. These correspond to low-index modes where nonlinear interactions with other low-index modes dominate, while interactions with the high-index modes, for larger $\eta$, are less important. This suggests that a sparsification approach based on pruning the interactions involving the high-index modes, i.e. the small scales, would be effective. By contrast, for high-index modes, relevant energy interactions are organised in bands along the axes $m$ and $n$ and involve energy exchange between low-index modes and high-energy, high-index modes. This suggests that the dynamics of the small scales is driven primarily by non-local interactions with the largest structures of the flow and not by small-scale/small-scale quadratic interactions. The slight asymmetry visible in panel (c) arises from the structure of the coefficients tensor $Q_{ijk}$ and has the same physical origin as that observed in figure 4 for the POD model.

### 3.3. Sparsification of POD-based models

We now apply the methodology presented in section 2 to three POD-based models resolving 90%, 95% and 99% of the kinetic energy, respectively (see Table 1 for details). Because the size of the database matrix $\Theta(A)$ grows quadratically with the number of modes, the number of possible interactions $q$ can easily become larger than the number of available snapshots $N_t$, resulting in an underdetermined regression problem and overfitting. This is a well understood issue in data analysis and requires cross validation techniques to ensure the statistical reliability of the result \cite{Friedman2008}. In this work, we employed $K$-fold cross validation, using typically $K = 10$. Briefly, the database is first divided into $K$ folds. The model is trained using $K - 1$ blocks and the reconstruction error $\epsilon$ of equation (2.13) is obtained from the fold that was left out. This procedure is iterated over all folds, obtaining the mean and the standard deviation of $\epsilon$.

Figures 7-(a,b,c) show the sparsification curves on the $\rho - \epsilon$ plane for the three POD models considered. The mean of $\epsilon$ across the folds is displayed as a thick black line, while the grey dashed line indicates plus or minus one standard deviation. These curves have been obtained by solving problem (2.12) using strategy S1 and progressively increasing the regularisation weight $\gamma$. When low weights are used, dense systems with
good prediction accuracy are obtained. The opposite is true for large weights, identifying points in the left part of the graph characterised by low density and poor prediction accuracy. As postulated in section 2, the curves show a sweet spot at around $\rho \approx 0.2$, displaying a plateau for $\rho \gtrsim 0.2$, while the error $\epsilon$ grows quickly when $\rho \lesssim 0.2$. These results indicate that it is possible to prune about 80% of the quadratic interactions in model (2.3) without influencing the average prediction accuracy. It is worth pointing out that the Galerkin model obtained directly from projection (indicated as a red square) has worse prediction accuracy then the calibrated, sparsified model with $\rho = 1$.

The mean reconstruction error decreases as the resolved energy increases, moving from panel (a) to panel (c), as more modes participate in capturing the dynamics of velocity fluctuations. In addition, larger models can be more effectively sparsified, as the sparsification curve drops more rapidly. This results from the non-local structure of energy interactions shown in figure 4. When one additional low-energy mode is included, the number of relevant interactions to be retained in the model is only $O(N)$ and not $O(N^2)$, i.e. all non-local interactions with the rest of the hierarchy denoted as LL, LH and HL in figure 4-(a). Since the total number of possible interactions grows as $O(N^3)$, larger models can be more effectively sparsified. This is conceptually in agreement with the observations of Taira et al. (2016) on the sparsification properties of discrete vortex models. We also observe that the mean prediction error does not necessarily decrease monotonically when the density increases. This phenomenon is particularly visible for the model in panel (b) but all models reproduce the same behaviour. This is a symptom that the number of available snapshots (1500) is potentially not large enough for the number of coefficients ($q = N \times (N + 1)/2 + N + 1 = 2926$ for the model in panel (c)) and overfitting would have occured if no cross-validation had been performed.

3.4. Energetic interactions identified by the regression

To visualise the sparsity pattern identified by the regression as the regularisation weight in equation (2.12) is increased, we introduce the tensor $\gamma$ with entries $\gamma_{ijk}$ defined as the value of the regularisation weight $\gamma$ at which the corresponding coefficient $Q_{ijk}$ is shrunk to zero by the LASSO. Figures 8-(a,b,c) show three slices of $\gamma$ for modes $i = 1, 10$ and 75, respectively, for the largest POD model considered. Here, strategy S1 is used to set the regularisation weight. The first interactions to disappear are the small-scale/small-scale interactions.
interactions. Increasing the penalisation, interactions that are local in modal space are progressively pruned, leaving only non-local interactions involving triadic exchanges with the low-index modes for large penalisations. Interestingly, this pattern does not change qualitatively nor quantitatively as the modal index $i$ increases. In fact, a comparable number of interactions is retained across the hierarchy and the governing equations of all modes are sparsified by an equal amount. Hence, sparsification has not produced mode truncation, which would have occurred if all coefficients of some low-energy modes had been shrunk to zero by the LASSO. This behaviour can be justified by noting that the mean squared acceleration $||\dot{A}_i||_2^2$ of the POD modes varies only slightly with $i$. In fact, the sparsification pattern does not change significantly when strategy S2 is used to vary the regularisation weight across the hierarchy.

We now focus on the sparse model with $\rho = 0.3$, nearby the sweet spot of the curves in figure 7-(c), and capturing 99% of kinetic energy. Figure 9 shows the time averaged energy transfer rate associated to the $i$-th mode

$$T_i = \sum_{j=1}^{N} \sum_{k=1}^{N} N_{ijk}. \quad (3.10)$$

The empty circles correspond to the projection model (indicated as $\rho = 1$), while the dashed line corresponds to the sparse model. Data is reported for one every two modes. Results from the projection model show that the net energy transfer is negative for the first few modes, and that it changes sign at $i \sim 10$. This indicates that the sparse model correctly reproduces global trends of the mean energy transfer across modal structures, predicting that the first few modes extract energy from the mean flow and feed high-index modes via triadic interactions. The regression has identified a sparse coefficient tensor $Q^s$ producing a pattern of interactions that resembles that of the mean energy transfer rate tensor $\mathbf{N}$ shown in figures 4-(a,b,c). This is illustrated in figure 10 showing the base ten logarithm of $N^s_{ijk}$ computed as in (2.9) with the coefficient tensor $Q^s$. It can be observed that most weak small-scale/small-scale interactions have been pruned. One shortcoming of the approach is that the asymmetry of the interaction pattern observed in figure 4 and the physical mechanism that originates it are invisible to the regression and the pattern of figure 10 is now symmetric with respect to a swap of the indices $j, k$. We have verified a-posteriori that the sparse tensor $Q^s_{ijk}$ also satisfies to a very good accuracy the original energy conservation property of the convective term of the Navier-Stokes equations, even
3.5. Sparsification set up for DFT-based models

Before moving to the sparsification of DFT-based ROMs, we briefly discuss three technicalities arising from the oscillatory nature of the temporal coefficients. As an illustrative example, we consider a small-sized model constructed with $N = 26$ modes and perform sparsification as discussed in section 2 with a relatively small regularisation weight ($\gamma = 10^{-14}$). The first key result is that all the entries of the constant and quadratic coefficient tensors $C$ and $Q$ are set to zero, while the linear tensor $L$ has a
characteristic bidiagonal structure, shown in figure 11-(a). The system identified by the regression is equivalent to a set of $N/2$ decoupled linear oscillators in the form

$$
\begin{bmatrix}
\dot{a}_{2l-1} \\
\dot{a}_{2l}
\end{bmatrix} = \omega_l \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_{2l-1} \\
a_{2l}
\end{bmatrix}, \quad l = 1, \ldots, N/2,
$$

(3.11)
coupling pairs of temporal coefficient oscillating at the same angular frequency $\omega_l = 2\pi l/T$, with $T$ being the observation time. The eigendecomposition of the tensor $L$, shown in figure 11-(b), is trivial. Eigenvalues are all imaginary and come in pairs that are integer multiples of the fundamental frequency $\omega_1 = 2\pi/T$. While this result is consistent with recent ideas on Koopman operator theory (Mezić 2013), where nonlinear dynamics are modelled with a linear system of larger dimension, all information on nonlinear energetic interactions has been lost in the process since the nonlinear part of the system $Q_{ijk}$ has been completely eliminated by the regression. This result is due to the fact that, when temporal coefficients are sine/cosine pairs, there is a column of $\Theta(A)$ that is exactly parallel to the target $A_i$, since time differentiation is equivalent to a permutation of sine/cosine pairs. As pointed out in Brunton et al. (2019) incorporating and enforcing known flow physics is a challenge and opportunity for machine learning algorithms. In order to address this first aspect, we introduce a physically-motivated approach based on considerations of the time averaged energy budget of system (2.7). Since the temporal coefficients have zero mean and are uncorrelated in time, we obtain

$$
\sum_{j=1}^{N} \sum_{k=1}^{N} Q_{ijk}a_ia_ja_k + a_ia_iL_{ii} = 0,
$$

(3.12)
i.e. only the diagonal element of the linear term participates in the mean power budget. Hence, for the sparsification of DFT-based models we use a modified database matrix that only contains the column associated to the diagonal part of the linear term.

The second aspect is that for DFT models the database matrix $\Theta(A)$ is not full rank and some of the columns of this matrix are linearly dependent. In this case, the LASSO is known to select one column at random (according to the particular ordering of the columns) and sets to zero regression coefficients of the other linearly dependent columns (Tibshirani 2013; Hastie et al. 2015). Machine learning techniques often come without guarantees for robustness (Brunton et al. 2019), implying that physical insight obtained with these tools might be questionable. To avoid this problem, we constructed a reduced database matrix, using only the columns associated to interactions on the three branches of figure 3. The reduced database matrix is full rank, as can be seen in panel (b) of figure 11 showing the singular values of the full database matrix defined by equation (2.10) and of the reduced matrix. The important consequence is that the solution of the LASSO problem (2.12) is unique (Tibshirani 1996), and can be thus compared with the available physical knowledge of scale interactions in turbulent flows. In addition, the computational complexity of sparsifying the entire Galerkin model only grows as $O(N^2)$ instead of $O(N^3)$, as for POD models, because the reduced database matrix contains a number of interactions equal to $q = 2(N + 1)$ at most. As a result, cross-validation techniques to avoid over-fitting was not necessary.

The third aspect of DFT-based models is that, as anticipated, the number of modes is not uniquely defined by the energy resolution but depends on the overall observation time. Long observation times would be beneficial to reach statistical significance but would result in low-energy/low-frequency modes that do not contribute significantly to the overall dynamics. In practice, we have divided the original dataset into $M$ partitions and performed DFT for each of them separately. Then, we stacked vertically the modal
Figure 12. Panel (a): sparsification curves for models obtained by three different observation times and resolving 100% of the kinetic energy ($e(n) = 1$). Panel (b): sparsification performed with $T = 30$ with three different energy resolutions $e(n)$.

acceleration matrices and the reduced database matrices from the partitions and solved (2.12) for a common coefficients vector.

3.6. Sparsification of DFT-based models

We now move to the sparsification of DFT-based models. Here, we introduce the modified density $\rho_{DFT}$ spanning the range [0,1] and representing the number of active coefficients with respect to the total number of active interactions on the three branches of figure 5. For large models, the approximation $\rho_{DFT} \approx \frac{2}{3} \rho_N$ can be used.

In figure 12-(a), sparsification curves for three models obtained with observation times $T = 10, 30$ and 50 (with $M = 15, 5$ and 3 partitions of the full dataset, respectively), at full energy resolution, are reported. Strategy S1, where the regularisation weight is maintained constant for all modes is used. We observe that the error decreases monotonically with the observation time. This is a consequence of the larger number of frequencies that interact quadratically to reconstruct the original DNS acceleration data. For the larger model obtained at $T = 50$, 70% of the triadic interactions can be pruned with no major effects on the overall prediction error. If the full coefficient tensor $Q$ is considered, this correspond to a remarkably low density of 0.0015. Figure 12-(b) shows the sparsification curves for models obtained with observation time $T = 30$, for three different energy resolutions, $e(n) = 0.9, 0.95$ and 0.99. Interestingly, we notice that the curves do not present a plateau for high densities as opposed to the full resolution mode shown in panel (a) and the POD sparsification curves of figure 7. This is the combined effect of the dramatic decrease in the number of modes at lower energy resolutions (see table 1) and the inherent efficient description of energy interactions in DFT-based models compared to POD.

We now compare strategies S1 and S2 on the full resolution model obtained with observation time $T = 30$. Results of this analysis are reported in figure 13. The top/bottom panels are obtained with the strategy S1/S2. Panel (a) shows the tensor $\hat{\gamma}$, obtained by processing and visualising the full tensor $\gamma$ using the same technique utilised for the interaction tensor $\hat{N}$ in figure 5. Panel (b) shows the density of individual ordinary differential equations for a selected number of modal structures as a function of the overall model density $\rho_{DFT}$, while the sparsified interaction tensor $\hat{N}^s$ for $\rho_{DFT} = 0.7$ is shown in panel (c). When the regularisation weight is maintained constant, the sparsification pattern emerging from the tensor $\hat{\gamma}$ follows the distribution of the mean energy transfer rate of figure 6-(a). In particular, despite the signature of nonlocality is still visible in the
pattern, the sparsification is highly skewed across the spectrum because the equations for high-frequency modes are excessively sparsified for moderate penalisations as opposed to those of low-frequency, high-energy modes. This behaviour is better seen in the individual density curves in panel (b). Specifically, the density $\rho_l$ of the last mode pair ($l = 150$) drops quite pronouncedly to much lower density than average at $\rho_{DFT} \approx 0.5$. Panel (d) shows the sparsification pattern obtained with the second strategy. We observe that, in this case, the interactions are retained according to their relative strength producing a sparsification pattern that follows the relative energy transfer rate reported in figure 6-(b). This results in a more balanced sparsification across the spectrum, we the modal density $\rho_l$ decreases more uniformly for all modes as the global density is decreased, as shown in panel (e). The mean energy transfer rate of the models sparsified using the two strategies, with $\rho_{DFT} = 0.7$, is reported in panels (c) and (f). Globally, the structure and intensity of energy interactions is preserved by the LASSO, although strategy S1 has more aggressively sparsified the high-index modes and truncated the equations of the last five pairs of modes.

As a final remark, we have observed in sparsification of larger DFT models that, although the LASSO is able to successfully identify the dominant subset of energy interactions, the complexity of the optimisation problem makes an accurate reconstruction of the numerical values of the system coefficients challenging. This is due to the spectral properties of the database matrix $\Theta$ which deteriorate as the number of modes considered grows [Cordier et al. 2010]. A potential solution to this issue would be to use elastic-net regression [Friedman et al. 2008] which combines an $l_1$ term with an $l_2$ (Tikhonov) penalisation. This would provide a better trade-off between sparsification and stability of the reconstructed coefficients.
4. Concluding remarks

In this paper, we have applied recent methods for the identification of sparse dynamical systems from data to sparsify nonlinear triadic interactions in projection-based reduced order models of turbulent flows. Our work is motivated by established knowledge of scale interactions in turbulence, whereby dynamics at a certain length scale depend most prominently on a subset of other length scales. Computationally, our methodology is based on $l_1$-based regression methods and is scalable to large models defined by hundreds of modal structures. These methods are used to recast the problem of identifying relevant triadic interactions into a convex optimisation problem for which scalable, efficient solvers can be used. The overarching aim is to develop large reduced order models covering a wide range of length scales, but where computational efficiency and physical interpretability have been preserved by pruning weak triadic interactions.

In this analysis we considered two-dimensional lid-driven cavity flow at Reynolds number $Re = 2 \times 10^4$. We generated two families of reduced order models by Galerkin projection of the Navier-Stokes equations onto the subspace spanned by Proper Orthogonal Decomposition and Discrete Fourier Transform modes. The goal was to understand the role of the subspace utilised for projection on the structure and sparsity of energy interactions between modes. As discussed in Brunton et al. (2019) an open problem in applying machine learning algorithms to fluids problems is to successfully incorporate known flow physics. In our case, we have observed that for DFT-based models, it has become necessary to manually modify the database matrix in order to ensure the uniqueness of the solution and preserve the full nonlinear character of mode interactions. The analysis of the average energy transfer rates between modal structures has shown that, for both models, a small subset of most relevant interactions exists, in agreement with the established picture of scale interactions in two-dimensional flows. Such a sparsity is exclusively an a-posteriori feature of the solutions and not an a-priori property of the evolution equations. In fact, the model coefficients identified by the Galerkin projection are typically non zero. Our results show that, in both cases, there exists a sweet-spot on the $\rho - \epsilon$ curve where the sparsification approach recovers correctly this subset at little costs on the prediction accuracy. In addition, the non-local nature of triadic interactions and the overall conservation properties of the convective term of the Navier-Stokes equations are well captured by the approach. We have also observed that the effectiveness of the sparsification grows with the number of modes (energy resolution). This is a result of the non-local nature of scales interactions, where the dynamics of small-scale features is dominated by the advection of the large modes, rather than by small-scale/small-scale nonlinearity. Hence, while the total number of new interactions in the quadratic interactions tensor grows quadratically with the number of modes, the number of relevant interactions only grows linearly. Our expectation is that sparsification become more effective as the Reynolds number increases, as a results of the increased range of scales. The characterisation of the sparsificability as a function of the Reynolds number in flow in two and three dimensional domain will be the object of a separate work.

Nevertheless, a major difference between the two decompositions is that energy interactions between triads of DFT modes are highly localised in modal space because of the oscillatory nature of the temporal coefficients. On the other hand, for POD modes, temporal coefficients contain a wider range of frequencies and energy transfers are inevitably more distributed in modal space. As a result, the number of active interactions grows only as $O(N^2)$ for the DFT model rather then as $O(N^3)$ for the POD. Therefore, we conclude that the sparsity is not necessarily invariant under different representations.
of the same phenomenon. This observation would suggest that it might be possible to develop a modal decomposition that identifies a set of maximally independent modal structures, i.e. where the resulting quadratic coefficient tensor is sparse. However, this appears to be a nonlinear optimisation problem, with the associated convergence and uniqueness issues.

The next step is to apply this methodology to a three-dimensional turbulent flow test case. Here, the approach discussed in this paper should be able to recover the local nature of triadic interactions in three dimensional configurations. One question to address is whether the joint effect of larger separation of scales at higher Reynolds numbers and the locality of the interactions will result in higher sparsity at higher Reynolds numbers.

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