Existence of incomparable pure bipartite states in infinite dimensional systems

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Based on set theoretic ordering properties, a general formulation for constructing a pair of convertibility monotones, which are generalizations of distillable entanglement and entanglement cost, is presented. The new pair of monotones do not always coincide for pure bipartite infinite dimensional states under SLOCC (stochastic local operations and classical communications), demonstrating the existence of SLOCC incomparable pure bipartite states, a new property of entanglement in infinite dimensional systems, with no counterpart in the corresponding finite dimensional systems.

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Entanglement, or nonlocal quantum correlation, is regarded as the key resource which allows many quantum information processing schemes to out-perform their classical counterparts. Better understanding of entanglement is essential for the field of quantum information. Entanglement can be classified by “plasticity” under local operations, such as LOCC (deterministic local operations and classical communications), SLOCC (stochastic LOCC, nondeterministic LOCC) and PPT (positive-partial-transpose-preserving) operations. Therefore the convertibility properties of two different entangled states (in a single copy or multi-copy situation) under local operations are important for the qualitative and quantitative understanding of entanglement.

For finite dimensional bipartite systems, we now have a better understanding of LOCC and SLOCC convertibility based on intensive work in recent years. For example, the condition for convertibility of two pure entangled states in the single copy situation is given by Nielsen’s majorization theorem for LOCC, and is given by Vidal’s theorem for SLOCC. On the other hand, for infinite dimensional systems (or continuous variable systems), there are many open questions on general LOCC and SLOCC convertibility, although there are important works, which have investigated a limited class of local operations (gaussian operations).

Infinite dimensional systems have been expected to offer high potential for quantum information processing. One of the advantages of infinite dimensional systems is the possibility of implementation using quantum optical systems, as shown by the successful demonstration of teleportation. Another advantage is the possibility of (yet unknown) new types of quantum information processing schemes, which do not exist in finite dimensional systems. If such schemes exists, their existence should be related to essential properties of quantum systems, such as entanglement. Thus the discovery of entanglement properties which are unique to infinite dimensional systems is very desirable.

In a previous paper, we proved that Nielsen’s majorization theorem and Vidal’s theorem can be generalized to infinite dimensional states by introducing the concept of $\epsilon$-convertibility. We showed a new classification of infinite dimensional entangled states based on SLOCC convertibility by introducing the rapidity of convergence, which corresponds to Schmidt rank in finite dimensional systems. The classifications obtained are natural extensions of those for finite dimensional systems, thus they did not suggest the existence of new entanglement properties. In this letter, we will show that a new entanglement property in infinite dimensional systems, the existence of SLOCC incomparable pure bipartite states, by developing a formulation for convertibility based on set theoretic ordering properties.

We consider ordering sets, ordered by the convertibility of two general bipartite quantum states (including infinite dimensional states) under general operations. For such sets, we can define a pair of monotones of the given ordering based on set theory. The definition of these monotones is the core element of our formulation. If these two monotones coincide, it indicates that the ordering is a total ordering, namely, at least one of the states can be converted to another. For the case of finite dimensional pure states under SLOCC operations, the two monotones coincide. In contrast, we will show that the two monotones do not always coincide in the infinite dimensional case. Thus the ordering can be non-total (partial) and the two states can be incomparable.

In this letter, we first present the construction of the general formulation. Then, we apply it to infinite dimensional pure states under SLOCC operations. The advantage of our formulation is that it can be applied for many different situations: single-copy or asymptotic (infinitely many-copy) cases, for finite or infinite dimensional systems, mixed or pure states, and under LOCC, SLOCC or PPT operations. It should be noted that the two monotones become distillable entanglement and concentration for mixed states in the asymptotic situation, therefore, they can be considered as a generalization of distillable entanglement and entanglement cost.

In the language of set theory, convertibility of physical states under some operation can be describe by an order...
denoted by “→”. For a set of physical states $S$, the order indicating the existence of physical transformation satisfies the reflective law $a → a$ and the transitive law $a → b$ and $b → c$ imply $a → c$), where $a, b, c ∈ S$. This ordering property is pseudo partial ordering. We consider that two states are in a same equivalence class, if they transform each other $a → b$ and $b → a$. We denote this situation as $a ↔ b$. The quotient set of $S$ by the equivalent class $↔$ is denoted as $(S/ ↔, →)$ and represents the classification based on the given transformation. For the set $(S/ ↔, →)$, we can redefine the ordering $→$ which satisfies the additional condition $a → b$ and $b → a$ imply $a = b$. This ordering property is partial ordering.

If the set has the additional property $a → b$ implies $b → a$ ($a → b$ denotes $a → b$ is not true), the set is totally ordered. Total ordering is a convenient property to analyze the convertibility of a system, since there exists a unique measure of ordering for a totally ordered set. For example, the convertibility of pure state $|ψ⟩$ under LOCC transformation in the asymptotic situation is total ordering. Therefore, there is a unique measure, or a monotone, given by the von Neumann entropy of entanglement $E(|ψ⟩)$ in this case [8]. On the other hand, a pair of monotonies, like distillable entanglement and entanglement cost, are useful tools to distinguish the ordering properties (total ordering or partial ordering) of the system [10].

The key idea of our formulation is that we consider a totally ordered subset $\{ξ_i\}$ parameterized by a real number $r$ for a pseudo partial ordering set $S$. Then we can always define a pair of functions $R^−(ψ)$ and $R^+(ψ)$ for a state $ψ ∈ S$, where $R^−(ψ)$ is the supremum of $r$ at which $ψ$ can be transformed to $ξ_i$, and $R^+(ψ)$ is the infimum of $r$ at which $ψ$ can be transformed to $ξ_i$. Mathematically they are expressed as the following: For a pseudo partial ordering set, if there exists a real parameterized total ordering subset $\{ξ_i\}_{r ∈ A} ⊂ S$ where $A ⊂ R$ such that $r_1 ≤ r_2$ if and only if $ξ_{r_1} → ξ_{r_2}$, we can define a pair of functions on $S$ to $A ⊂ R$ as

$$R^−(ψ) = \sup\{r ∈ A | ψ → ξ_r\} \quad (1)$$

$$R^+(ψ) = \inf\{r ∈ A | ψ → ξ_r\} \quad (2)$$

where we define $R^−(ψ) = \inf\{A\}$ for $r ∈ A$ or $ψ → ξ_r = 0$, and $R^+(ψ) = \sup\{A\}$ for $r ∈ A$ or $ψ → ξ_r = 0$.

Although there are many ways to define a monotone for a partial ordering set from a totally ordered subset $\{ξ_i\}_{r ∈ A}$, our definition of functions $R^−(ψ)$ and $R^+(ψ)$ are preferable for analyzing entanglement convertibility. We prove by contradiction that they are the unique monotonies which give lower and upper bounds of any monotonies defined for a given pseudo partial ordering set $S$. For simplicity we only show the proofs for the case of $A$ to be an interval of $R$ in this letter, but the proofs can easily be extended to an arbitrary subset of $R$.

First, we show that $R^−(ψ)$ and $R^+(ψ)$ are monotonies which satisfy $R^−(ψ) ≤ R^+(ψ)$ for all $ψ ∈ S$. Suppose $R^−(ψ) > R^+(ψ)$, then there exists $r_1$ and $r_2$ such that $R^−(ψ) > r_2 > r_1 > R^+(ψ)$. Since $ξ_{r_1} → ψ$ and $ψ → ξ_{r_2}$, $ξ_{r_1} → ξ_{r_2}$. This means $ξ_{r_1} ↔ ξ_{r_2}$. This contradicts the totality ordering of $\{ξ_i\}_A$, therefore we have $R^−(ψ) ≤ R^+(ψ)$ for all $ψ ∈ e$. Next, suppose $R^−(ψ) > R^+(ψ)$ and $ψ → Φ$, there exists $r ∈ A$ such that $R^−(ψ) > r > R^+(ψ)$. Then, $Φ → ξ_r$. Since $ψ → Φ$, we have $ψ → ξ_r$. This contradicts $R^−(ψ) > r$, thus we have proved $ψ → Φ$ implies $R^−(ψ) > R^−(ψ)$. Similarly, we can prove $ψ → Φ$ implies $R^+(ψ) ≥ R^+(ψ)$. In addition, suppose $R^−(ψ) > R^+(ψ)$, then there exists $r ∈ A$ such that $R^−(ψ) > r > R^+(ψ)$. Since $Φ → ξ_r$ and $ξ_r → ψ$, we have $Φ → ψ$. Thus we have proved $R^−(ψ) < R^+(ψ)$ implies $ψ → Φ$.

Second, we show that $R^−(ψ)$ and $R^+(ψ)$ are lower and upper bounds of monotonies, respectively. Suppose there was another monotone $R_0(ψ)$ defined for a given ordering such that $ψ → Φ$ implies $R_0(ψ) ≥ R_0(ψ)$ satisfying $R_0(ξ_r) = r$ for all $r ∈ A$. If $R_0(ψ) < R^−(ψ)$, there exists a real number $r ∈ A$ such that $R_0(ψ) < r < R^−(ψ)$, then we obtain $ψ → ξ_r$. On the other hand, from $R_0(ψ) < r$ we have $ψ → ξ_r$ by the monotonicity of $R_0(ψ)$. This is also a contradiction. Similarly we can prove $R_0(ψ) ≤ R^+(ψ)$. Thus $R^−(ψ) ≤ R_0(ψ) ≤ R^+(ψ)$ for all $ψ ∈ S$.

From the properties of $R^−(ψ)$ and $R^+(ψ)$, we can immediately derive the following important results: If the quotient set $(S/ ↔, →)$ is totally ordered, namely, $ψ → Φ$ implies $ψ → ψ$ is satisfied, then for all $ψ ∈ S$, we have $R^−(ψ) = R^+(ψ)$. On the other hand, if there exists $ψ ∈ S$ such that $ψ ≤ R^−(ψ)$, then $(S/ ↔, →)$ is not totally ordered, and $ψ$ is incomparable to all $ξ_r$ with $R^−(ψ) < r < R^+(ψ)$.

Many important known results of entanglement theory can be re-derived only from simple ordering properties and the existence of the real parameterized total ordering subset. To demonstrate the power of our formulation, we apply the formulation to the following four situations of finite (d) dimensional systems; A. LOCC operation for pure states in the single copy situation: $\{|ξ_i⟩\}_{k=1}^{d}$ is the maximally entangled state in d dimensional systems [12]; B. SLOCC operation for pure states: $\{|ξ_i⟩\}_{k=1}^{d}$ is also the maximally entangled state in d dimensional systems; C. LOCC operations for pure states in the asymptotic situation: $\{|ξ_i⟩\}_{k=1}^{d}$ is a subset with $E(|ξ_i⟩) = s$ [11]; D. LOCC operations for mixed states in the asymptotic situation: $\{|ξ_i⟩\}_{k=1}^{d}$ is a subset of pure states with $E(|ξ_i⟩) = s$ [8]. We summarize the representation of $R^−$ and $R^+$ for these four situations in Table 1. In the situations A and D, we see that the sets are not totally ordered and $R^−$ (distillable entanglement) and $R^+$ (entanglement cost) are limits of other monotonies [11]. Furthermore for D, we see that there is a set of states such that $R^− = 0$ but $R^+ > 0$, the bound entangled states [13].

Now we concentrate on the investigation of SLOCC convertibility (with non-zero probability) of infinite dimensional pure states in the single-copy situation. In general, entanglement of a pure state $|a⟩$ is characterized by the sequence of Schmidt coefficients $\{λ_i^a\}$ ($0 ≤ i ≤ d$ and $0 ≤ i ≤ ∞$ for finite d and infinite dimensional systems, respectively). Recall that the corresponding result
for finite dimensional systems is given by B in Table I. The two monotones coincide with the Schmidt rank, the number of non-zero Schmidt coefficients (Vidal’s theorem 3). With a simple extension of the result obtained for finite dimensional systems, it is not possible to determine convertibility between the states with infinite Schmidt ranks.

Since analysis of convertibility between the “genuine” infinite dimensional states (with infinite Schmidt ranks) are our main aim, we adopt Vidal’s theorem to infinite dimensional systems as following: Theorem 1 (Vidal): $|\psi\rangle \in \mathcal{H}$ can be converted to $|\phi\rangle \in \mathcal{H}$ by SLOCC with non-zero probability in the single-copy situation if and only if there exists $0 < \epsilon < 1$, $g_0(n)/g_0(n) \geq \epsilon$ for all $n \in N$, where $g_0(n) = \sum_{i=0}^{\infty} \lambda_i^n$ is a function defined in terms of Schmidt coefficients $\{\lambda_i\}_{i=0}^\infty$ of a genuine infinite dimensional state $|\phi\rangle$.

The function $g_0(n)$ plays the central role in the construction of the monotones $R^-$ and $R^+$ for infinite dimensional states. By definition, a sequence of the functions $\{g_0(n)\}_{n \in N}$ satisfy four conditions, (strict) positivity $g_0(n) > 0$, (strict) monotonicity $g_0(n) > g_0(n+1)$, convexity $g_0(n+1) \leq \{g_0(n) + g_0(n+2)/2\}$, and normalization $g_0(0) = 1$. Conversely, for a given sequence of functions $\{g_0(n)\}_{n=0}^\infty$, there exist a genuine infinite dimensional state $|\phi\rangle$, where the Schmidt coefficients are given by $\lambda_i = g_0(n) - g_0(n+1)$ if and only if $\{g_0(n)\}_{n \in N}$ satisfies the strict positivity, strict monotonicity, convexity and normalization conditions.

According to Theorem 1, if a real parameterized subset $\{\xi_r\}_{r \in A} \subset \mathcal{H}$ is totally ordered, $g_{\xi_r}(n)$ must satisfies $\lim_{n \to \infty} (g_{\xi_r}(n)/g_{\xi_{r+1}}(n)) > 0$ if and only if $r_1 \leq r_2$ for all $r_1$ and $r_2$. From the property of $g_{\xi_r}(n)$, we can construct the monotones $R^-$ and $R^+$

$$R^-(|\psi\rangle) = \inf\{r \in A| \lim_{n \to \infty} g_{\psi}(n)/g_{\xi_r}(n) = 0\} \quad (3)$$

$$R^+(|\psi\rangle) = \inf\{r \in A| \lim_{n \to \infty} g_{\psi}(n)/g_{\xi_r}(n) = \infty\} \quad (4)$$

for all $\{g_{\xi_r}(n)\}_{r \in A, n \in N}$ satisfying the three conditions: I. Strict monotonicity for all $r \in A$, II. Convexity for all $r \in A$ and $n \in N$, and III. $\lim_{n \to \infty} (g_{\xi_{r+1}}(n)/g_{\xi_{r+2}}(n)) > 0$ is equivalent to $r_1 \geq r_2$. The proof is given by the following: If $\{g_{\xi_r}(n)\}_{r \in A, n \in N}$ satisfies the conditions I, II, and III, the corresponding set of states $\{\xi_r\}_{r \in A}$ for $\{g_{\xi_r}(n)\}_{r \in A, n \in N}$ exists and is totally ordered. Since $A \in \mathbb{R}$ is assumed to be an interval, the two functions $R^-(|\psi\rangle$ and $R^+(|\psi\rangle$ defined by Eqs. (3) and (4), can be represented by Eqs. (3) and (4) by using Theorem 1.

Now we show that there exists a pairs of genuine infinite dimensional states which are incomparable to each other. We prove that the two monotones $R^-(|\psi\rangle$ and $R^+(|\psi\rangle$ given by Eqs. (3) and (4) do not necessarily coincide with each other for infinite dimensional systems, by constructing an example.

We consider a twice continuously differentiable function $g(x)$, which is the continuous counterpart of $g(n)$, since a continuous function is more convenient for analytical investigation. The conditions for $g(x)$ to relate to a genuine infinite dimensional state is now given by $g(x) > 0$ (strict positivity), $g'(x) < 0$ (strict monotonicity), $g''(x) \geq 0$ (convexity), and $g(0) = 1$ (normalization), for all $x$. If $g(x)$ satisfies the above conditions except the normalization condition, we can easily normalize $g(x)$. Thus we omit the normalization condition for simplicity. Since convertibility is determined only by the ratio of functions, we introduce another function $d(x)$, which is given by $d(x) = p(x)g(x)$. Let $d(x)$ satisfy the same conditions as $g(x)$. Then $p(x) > 0$, $f'(x)p(x) + f(x)p'(x) < 0$, $f''(x)p(x) + 2f'(x)p'(x) + f(x)p''(x) \geq 0$ and $p(1) = 1$ are to be satisfied.

Now we set our function to be $g(x) = e^{-x}$. Our choice of $g(x)$ represents one of the most tractable genuine infinite dimensional entangled states, the two mode squeezed state $\psi_q = 1/\sqrt{q} \sum_{n=0}^\infty q^n |n\rangle \otimes |n\rangle$, where $q$ is a squeezing parameter. We give a construction of a function $d(x)$ which indicates the existence of incomparable genuine infinite dimensional states. In this case, the conditions for $p(x)$ become simple, $p(x) > 0$, $p(x) - p'(x) > 0$, and $p(x) - 2p'(x) + p''(x) \geq 0$.

We choose $p(x)$ to be parameterized by $r$ as $p_r(x) = (\log x)^r[\sin(\log x + 1) + (\log x)^{-1}]$ where $0 < r < +\infty$. We define two functions $m_r(x) \equiv p_r(x) - p_{r-1}(x)$ and $c_r(x) \equiv p_r(x) - 2p_{r-1}(x) + p_{r-2}(x)$ for evaluating monotonicity and convexity, respectively. Then we have

$$m_r(x) = \{[(\log x)^{1+r} - (1+r)(\log x)^{r-1}]\sin(\log x) + 1 + (\log x)^{-1}O(\log x)\}$$

$$c_r(x) = \{\sin(\log x) + 1\}[(\log x)^{1+r} - 2(1+r)(\log x)^{r-1}]$$

$$+ (\log x)^{-1}O(\log x)^{1+r}x^{-1}$$

$$+ (\log x)^{-1}O(\log x)^{1+r}x^{-1}$$

For all $0 < r_1 < r_2 < \infty$, there exists $x_{r_1, r_2} > 0$ such that $m_r(x) > 0$ and $c_r(x) \geq 0$ for all $x \geq x_{r_1, r_2}$, and $r \in [r_1, r_2]$. That is, the function $p_r(x + x_{r_1, r_2})$ satisfies the positivity, monotonicity and convexity conditions. Therefore we can consider a state $|\xi_r\rangle$ represented by the function $d_r(x) = p_r(x + x_{r_1, r_2})g(x)$. The ratio of the functions $g(x)/d_r(x) = 1/p_r(x + x_{r_1, r_2})$ determines convertibility between the two states $|\psi_q\rangle$ and $|\xi_r\rangle$. To evaluate the ratio, we rewrite $p_r(x + x_{r_1, r_2})$ in the dis-
crette form: $p_r(n') = p_r(\Delta n + x_{r_1,r_2})$ where $\Delta = - \log q$. Then we can easily show that $\lim_{n \to \infty} 1/p_r(n) = 0$ and $\lim_{n \to \infty} 1/p_r(n) = \infty$. Defining $R^-$ and $R^+$ from $(\{\xi_r\})_{r \in (r_1, r_2)}$, we obtain $R^-(\psi) = r_1$ and $R^+(\psi) = r_2$. The two states $|\psi\rangle$ and $(\xi_r)$ for all $r \in [r_1, r_2]$ are now shown to be incomparable under SLOCC.

The state which corresponds to the function $d(x)$ in our example may not be feasible to create with present technology. However we can show the existence of incomparable states by choosing other forms of the function for $p(x)$, if $g(x)$ is a function converging as fast as or faster than exponential functions. Since the conditions for incomparable states are not related to the Schmidt basis, we can choose a Schmidt basis which is easy to control in experiments. We still have the possibility to find more feasible incomparable states.

Another point related to feasibility in the laboratories is that our functions $R^-(|\psi\rangle)$ and $R^+(|\psi\rangle)$ are discontinuous for the usual topology of Hilbert space. This discontinuity is caused by the discontinuity of the SLOCC convertibility itself. Since we cannot completely determine Schmidt coefficients of the states in realistic situations, we cannot apply our discussion immediately to such situations. However, we can say that the maximum probability to convert $|\psi\rangle$ where $\| |\psi\rangle - |\psi'\rangle \| < \epsilon$ for small $\epsilon$ to $|\phi\rangle$ and the probability of the inverse process are both very small, if $|\psi\rangle$ and $|\phi\rangle$ are incomparable under SLOCC, because the maximum probability of convertibility under SLOCC itself is continuous. In other words, this incomparability-like property appears in the limit of large dimensional space.

In this paper, we have developed a general formulation for constructing a pair of convertibility monotones using order properties. The monotones are considered as generalizations of distillable entanglement and entanglement cost. This formulation can be applied to many different situations to analyze entanglement convertibility. We have applied the formulation to SLOCC convertibility for genuine infinite dimensional pure states in the single-copy situation. By constructing an example, we have proved the existence of SLOCC incomparable pure bipartite states, a new property of entanglement in infinite dimensional systems. In contrast, incomparable pure states only exists for multipartite systems (such as GHZ and W states for three qubit states) in finite dimensional systems.

One of the important remarks in this letter is that the ordering property under SLOCC convertibility is changed fundamentally, from total ordering to non-total (partial) ordering, with the shift in dimensionality from finite to infinite. It had been widely believed that the fundamental entanglement properties of finite and infinite dimensional systems are similar. However, we have shown that there exists a significant difference in convertibility. Our result encourages the search for other fundamental differences between finite or infinite dimensional systems.

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