In this paper, we studied the progression of shallow water waves relevant to the variable coefficient Korteweg–de Vries (vcKdV) equation. We investigated two kinds of cases: when the dispersion and nonlinearity coefficients are proportional, and when they are not linearly dependent. In the first case, it was shown that the progressive waves have some geometric structures as in the case of KdV equation with constant coefficients but the waves travel with time dependent speed. In the second case, the wave structure is maintained when the nonlinearity balances the dispersion. Otherwise, water waves collapse. The objectives of the study are to find a wide class of exact solutions by using the extended unified method and to present a new algorithm for treating the coupled nonlinear PDE's.

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where the function $F$ is a polynomial in its arguments, $z_0$ is a constant.

The traveling wave solutions of (1) satisfy

$$G(u, u', u'', \ldots, u^{(m)}) = 0, \quad u' = \frac{du}{dz} = x - ct.$$  \hspace{1cm} (2)

Some exact solutions of (1) were found, \cite{15, 16}, by extrapolating the auto-Bäcklund transformation. The homogeneous balance method was used to find some exact solutions for evolution equations with variable coefficients \cite{17, 18}.

The extended unified method

In this section, we give a brief description of the extended unified method \cite{9, 14}.

The extended unified method is characterized by two aspects:

- Constructing the necessary conditions for the existence of solutions of an evolution equation.
- Suggesting a new classification to the different structures of solutions, namely:
  
  (i) The polynomial function solutions.
  (ii) The rational function solutions.

By the polynomial function solutions, we mean (for example) a polynomial in a function $\phi(x, t)$ that satisfies an auxiliary equation which may be solved to elementary or to special functions. Similar outlines hold in the rational function solutions.

The polynomial function solutions

In this section, we introduce the steps of computations to find the polynomial function solutions for NLEEs by using the extended unified method as they follow:

**Step 1:** The method asserts that the solution of (1) can be written in the form

$$u(x, t) = \sum_{i=0}^{n} a_i(x, t)\phi_i(x, t),$$  \hspace{1cm} (3)

and $\phi(x, t)$ satisfies the auxiliary equations

$$\phi_p = \sum_{j=0}^{k} b_j(x, t)\phi_j, \quad \phi_p = \sum_{j=0}^{k} c_j(x, t)\phi_j, \quad p = 1, 2,$$  \hspace{1cm} (4)

together with the compatibility equation

$$\phi_{xu} = \phi_{ux},$$  \hspace{1cm} (5)

where $a_i(x, t)$, $b_j(x, t)$ and $c_j(x, t)$ are arbitrary functions in $x$ and $t$.

We mention that, the cases when $p = 1$ and $p = 2$ correspond to explicit or implicit elementary solutions and periodic (trigonometric) or elliptic solutions respectively. To determine the relation between $n$ and $k$, we use the balance condition which is obtained by balancing the highest derivative and the nonlinear term in Eq. (1). The consistency condition determines the values of $k$ such that the polynomial solutions exist.

**Step 2:** By inserting (3) and (4) into (1), we get a set of equations, namely “the principle equations”, which is solved in some of arbitrary functions $a(x, t)$, $b(x, t)$ and $c(x, t)$. The compatibility equation in (5) gives rise to $2k - 1$ equations where $k \geq 2$.

**Step 3:** Solving the auxiliary equations in (4).

**Step 4:** Evaluating the formal exact solution by using (3).

The variable coefficients KdV equation (vcKdV)

Consider the KdV equation with variable coefficients (vcKdV) [19]

$$u_t + f(t)u_{xxx} + g(t)v_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$  \hspace{1cm} (6)

where $f(t) \neq 0$ and $g(t) \neq 0$ are arbitrary functions. We mention that (6) is well known as a model equation describing the progression of weakly nonlinear and weakly dispersive waves in homogeneous media. Eq. (6) arises in various areas of Mathematical Physics and Nonlinear Dynamics. These include Fluid Dynamics with shallow water waves and Plasma Physics. A particular form of (6) when $f(t) = 1$, $g(t) = \frac{1}{\sqrt{t}}$ and by using the following transformation $v = \sqrt{\eta}u$, Eq. (6) becomes the cylindrical KdV equation or the concentric KdV equation [20]

$$\eta_t + \eta v_x + \eta_{xxx} + \frac{1}{2\tau} \eta = 0.$$  \hspace{1cm} (7)

Eq. (7) arises in the study of Plasma Physics. Thus, as a special case the solution of the cylindrical KdV equation will fall out from the solution of (6) that will be obtained, in this paper. Soliton, periodic and Jacobi elliptic function solutions of Eq. (6) have been obtained \cite{10, 21}, when $f(t) = c\eta(t)$, where $c$ is a constant.

By using the transformations $x = x \tau$ and $\tau = \int_{0}^{t} f(t)dt$, $t > 0$, Eq. (6) can be written as

$$v_t + v_{xxx} + h(\tau)v_{xx} = 0,$$  \hspace{1cm} (8)

where $h(\tau) = \frac{f(t)}{f(t)} > 0$.

In this work, we use the unified method and the extended unified method to find exact solutions for Eq. (6) when $g(t) = z\eta(t)$ and $g(t) \neq z\eta(t)$ respectively, where $z$ is a constant.

**When $g(t) = z\eta(t)$**

In this case, Eq. (8) has the traveling wave solution

$$v(x, t) = u(\zeta), \quad \zeta = ax + bt,$$  \hspace{1cm} (9)

where $a$ and $b$ are constants. Thus (8) reduces to

$$a^2u'''' + axu'' + bu' = 0, \quad u' = \frac{du}{d\zeta}.$$  \hspace{1cm} (10)

**I – The polynomial function solutions**

In this case, we write

$$u(\zeta) = \sum_{i=0}^{n} a_i(\zeta)^i, \quad (\phi(\zeta))^p = \sum_{j=0}^{k} c_j(\zeta)^j, \quad p = 1, 2.$$  \hspace{1cm} (11)

**First:** when $p = 1$
When $p = 1$, the balance condition yields $n = 2(k - 1)$, $k > 1$ and the consistency condition gives rise to $k \leq 3$. Thus, in this case, the polynomial function solutions exist when $k = 2, 3$.

(I) When $k = 2$, $n = 2$.

By using any package in symbolic computations, we get the solutions of (10) as

$$u(\xi) = -\frac{b + a^2 R^2(2 + 3\tanh^2(\frac{1}{2} R\xi))}{a\xi},$$

or

$$u(\xi) = -\frac{b + a^2 R^2(2 - 3\tanh^2(\frac{1}{2} R\xi))}{a\xi}, \quad \xi = ax + bt,$$

where $R^2 = 4c_2c_0 - c_1^2 = -R_1^2$ are arbitrary constants. The solution given by (13) is a soliton solution in a moving frame.

Fig. 1a and b represents the solution (13) when $f(t) = 1 + t^2$ in the moving non-inertial frame and in the rest inertial frame respectively.

Fig. 1b shows soliton waves which are moving along the characteristic curve in the $xt$-plane (namely $ax + b\int_0^1 f(t)dt = \text{constant}$). The solution in Fig. 1 represents a bright solitary wave solution which is a usual compact solution with a single peak.

(II) When $k = 3$, $n = 4$.

By using (11), we have

$$u(\xi) = \sum_{i=0}^4 a_i\phi(\xi)^i, \quad \phi(\xi) = \sum_{i=0}^3 c_i\phi(\xi)^i.$$

By a similar way as we did in the previous case, we get the solution of (10) as

$$u(\xi) = -\frac{b + a^2 R^2(2 - 3\tanh^2(\frac{1}{2} R\xi))}{a\xi},$$

$$k(\xi) = \exp\left(-\frac{2R^2(27A^2 + \xi)}{3\xi}\right), \quad \xi = ax + bt,$$

where $R^2 = c_2^2 - 3c_0c_3$ and $A$ are arbitrary constants.

Second: when $p = 2$

In this case, we find the exact polynomial function solutions for (10) in trigonometric or elliptic functions forms. To this end we put $n = 2, k = 1$ or $n = 2, k = 2$ in (11) respectively.

(a) When $k = 2, n = 2$.

By using (11), we have

$$u(\xi) = \sum_{i=0}^2 a_i\phi(\xi)^i, \quad \phi^2(\xi) = c_0 + c_2\phi(\xi)^2 + c_4\phi(\xi)^4.$$

By substituting from (16) into (10) and by using the steps of computations that were given in ‘The extended unified method’ section, we get

$$a_2 = -\frac{12a^2c_4}{\pi}, \quad a_1 = 0, \quad a_0 = -\frac{b + 4ad^2}{a\xi}. \quad (17)$$

We mention that $c_i, i = 0, 2, 4$ are arbitrary constants. So the solutions of the auxiliary equation in (16) are classified according to Table 1.

In Table 1, $0 < \eta < 1$ is called the modulus of the Jacobi elliptic functions. Detailed recursion equations for the Jacobi elliptic functions can be found (the readers may refer to Refs. [22,23]). When $\eta \to 0$, $\sin(\xi)$, $\csc(\xi)$ and $\cot(\xi)$ degenerate to $\sin(\xi)$, $\cos(\xi)$ and $1$, respectively; while, when $\eta \to 1$, $\sin(\xi)$, $\csc(\xi)$ and $\cot(\xi)$ degenerate to $\tan(\xi)$, $\sec(\xi)$ and $\csc(\xi)$ respectively.

According to the relation between $c_0$, $c_2$ and $c_4$ in Table 1, we can find the corresponding Jacobi elliptic function solution $\phi(\xi)$.

Finally, the general solution of (10) in terms of the Jacobi elliptic functions is given by

$$u(\xi) = \delta(\xi) + a_0,$$

where $a_2$ and $a_0$ are given by (17).

| $c_4$ | The relation between $(c_0, c_2, c_4)$ and the corresponding $\phi(\xi)$ |
|-------|--------|
| $\eta^2$ | $c_2 = -(1 + c_4), c_0 = 1$ |
| $1 - \eta^2$ | $c_2 = 1 + c_4, c_0 = 1$ |
| $-\eta^2(1 - \eta^2)$ | $c_2 = 1 + 4c_4, c_0 = 1$ |
| $\eta^2 - 1$ | $c_1 = 1 - c_4, c_0 = -1$ |
| $1 - \eta^2$ | $c_1 = -1 + c_4, c_0 = \eta^2$ |
| $-\eta^2$ | $c_1 = 1 - c_4, c_0 = \eta^2 - 1$ |
| $\eta^2$ | $c_1 = -1 - c_4, c_0 = \eta^2$ |
| $1$ | $c_1 = 1 + c_4, c_0 = c_4 - 1$ |
| $-\eta^2$ | $c_1 = 1 + c_4, c_0 = 1$ |
| $1 - \eta^2$ | $c_1 = 1 + c_4, c_0 = c_4 + 1$ |
| $1$ | $c_1 = 1 + c_4, c_0 = -\eta^2(1 - \eta^2)$ |

Fig. 1 $a = 1, b = -1, R_1 = \sqrt{2}.$
and $f(t) = 1 + r^2$ and $\phi(\xi) = \sin(\xi, \eta)$. $\xi = ax + bt$ in the moving non-inertial frame and in the rest inertial frame respectively.

II – The rational function solutions

In this section, we find a rational function solution of (10). To this end, we write

$$u(\xi) = \sum_{i=0}^{n} p_i \phi(\xi) / \sum_{j=0}^{r} q_j \phi(\xi), \quad n \geq r, \quad (19)$$

where $p_i$ and $q_j$ are constants to be determined later, while $\phi(\xi)$ satisfies the previous auxiliary equations in R.H.S. of (11).

In this case, the balance condition is given by $n - r = 2(k-1), k \geq 1$ where $n > r$. While $k$ being free when $n = r$.

Here, we confine ourselves to find the rational solutions when $n = r$ and $k = 1, 2$ together with the auxiliary equation in (11) when $p = 2$.

$(II_1)$ When $k = 1$.

In this case, the rational function solutions will be in the rational trigonometric function or hyperbolic function solutions.

- Set $n = r = 1$ (for instance) in (19), namely

$$u(\xi) = \frac{p_1 \phi(\xi) + p_0}{q_1 \phi(\xi) + q_0}, \quad (20)$$

- Substituting from Eq. (20) together with the auxiliary equation (11) into Eq. (10), we get

$$q_1 = -\frac{ap_1}{b + a'c_2},$$

$$q_0 = -\frac{a(-3a'p_1c_1 + p_0(b + a'c_2))}{(b - 5a'c_2)(b + a'c_2)},$$

$$p_0 = -\frac{p_1(a'^2c_2(c_1 - 5R_2) + b(c_1 + R_2))}{2c_2(b + a'c_2)},$$

where $R_2 = c_1^2 - 4c_2c_0$ and $c_1^2 \geq 4c_2c_0$.

It remains to solve the auxiliary equation in (11). We distinguish between two cases:

Case 1. If $c_2 > 0$. In this case, the solution of the auxiliary equation (11) is

$$\phi(\xi) = \frac{c_1}{2c_2} + \frac{R_2 \cosh(\sqrt{c_2} \xi + A_1)}{2c_2}, \quad \xi = ax + bt,$$

$$\tau = \int_0^1 f(t_1) dt_1,$$  

where $A_1$ is an arbitrary constant. Substituting (22) into (19) we get the solution of (10), namely

$$u(\xi) = -\frac{b - 5a'c_2 + (b + a'c_2) \cosh(\sqrt{c_2} \xi + A_1)}{a \tau (1 + \cosh(\sqrt{c_2} \xi + A_1))}.$$  

Eq. (23) describes a soliton wave solution in the moving frame

Case 2. If $c_2 < 0$. The solution of the auxiliary equation (11) gives

$$\phi(\xi) = \frac{c_1}{2c_2} + \frac{R_2 \sin(\sqrt{-c_2} \xi + A_2)}{2c_2},$$

where $A_2$ is an arbitrary constant. Substituting (24) into (19) we get the solution of (10), namely

$$u(\xi) = -\frac{b - 5a'c_2 + (b + a'c_2) \sin(\sqrt{-c_2} \xi + A_2)}{a \tau (1 + \sin(\sqrt{-c_2} \xi + A_2))}.$$  

The solutions in (23) and (25) show a soliton wave and a periodic wave solution (as in a rational form) respectively.

$(II_2)$ When $k = 2$.

In this case, the solutions will be in the rational elliptic function form.

To obtain this type of solutions we use the auxiliary equation (11) when $k = 2$. By substituting about $u(\xi)$ from (19) together with $\phi(\xi)$ from (11) into Eq. (10) and using the calculations that were given in ‘The extended unified method’ section, we get;

$$p_1 = -\frac{bq_1^2 + a'(c_2q_1^2 + 6c_2q_0^2)}{aq_1},$$

$$p_0 = -\frac{bq_0^2 + a'(6d_2^2c_0^2 + c_2^2q_0^2)}{aq_1},$$

$$q_0 = \sqrt{\frac{R_3 - c_1^2}{c_1^2 - 4c_2c_0}},$$

where $R_3 = c_1^2 - 4c_2c_0$, $c_4 > 0$ and $c_0 < 0$. It remains to solve the auxiliary equation in (11). The solutions of the auxiliary

Fig. 2a and b represents the Jacobi doubly periodic solution (18) when $f(t) = 1 + r^2$ and $\phi(\xi) = \sin(\xi, \eta)$. $\xi = ax + bt$ in the moving non-inertial frame and in the rest inertial frame respectively.

Fig. 2  $x = 1, a = 1, b = -1, c_4 = 0.25, c_0 = 0.$
equation in (11) are classified according to Table 1 under the conditions \(c_0 < 0\) and \(c_4 > 0\).

Finally, the solution of (10) is given by
\[
u(\xi) = \frac{\sqrt{2}( - b R_3 + 5 a^2 c_2 R_3 + 6 a^3 c_1 R_1^2 - (b + a^2(c_2 + 6c_4 R_1^2)) \phi(\xi))}{ax(1 + \sqrt{2} \phi(\xi))}
\]
(27)

Fig. 3a and b represents the solution (27) when \(f(t) = 1 + t^2\) and \(\phi(\xi) = nc(\xi, \eta)\), \(\xi = ax + bt\) in the moving non-inertial frame and in the rest inertial frame respectively.

Fig. 3 shows the propagation of shallow water waves which are seen as elliptic waves.

Indeed, the solutions that were found in the last two cases may cover all solutions which could be obtained by different methods such as a modified tanh–coth method, the Jacobi-elliptic function expansion method, the extended F-expansion method, Exp-function method and \((G'/G)\)-expansion method [24–28].

When \(g(t) \neq z f(t)\)

In this section, we find exact solutions for Eq. (8) when their coefficients are linearly independent (namely \(g(t) \neq z f(t)\)). We think that, to the best of our knowledge, the results that will be found here are completely new.

We confine ourselves to search for polynomial function solutions for (8) when \(p = 1\) (in (4)) by using (3)–(5). So the balancing condition is \(n = 2(k - 1)\), \(k > 1\) and the consistency condition for obtaining these polynomial function solutions holds when \(k = 2, 3\) [14].

In this case, the calculations are carried out by using the extended unified method together with the symbolic computation for treating coupled nonlinear PDE’s according to the following algorithm:

(i) Solve a nonlinear PDE equation among the set of principle or compatibility equation in the highest order (say \(\frac{dx}{dt}\)).
(ii) Solve another equation in \(\frac{dx}{ax}\).
(iii) Use the compatibility equation between (i) and (ii) to eliminate \(\left(\frac{dx}{ax} \text{ and } \frac{d^n y}{dx^n}\right)\), that is by differentiating the obtained equation in (ii) with respect to \(x\) to get \(\frac{d^n y}{dx^n}\) and balances it with the obtained one in (i).
(iv) Solve the obtained equation from (iii) in \(\frac{dx}{ax}\).
(v) Repeat the steps (i)–(iv) to get an equation in the lowest order.
(vi) Use the same steps for PDE’s with mixed partial derivatives.

By this algorithm, the order of the PDE is reduced successively till a solution to the required function is obtained. When \(k = 2, n = 2\).

The steps of the computations by using the extended unified method (when \(p = 1\)) are as they follow;

Step 1: Solving the principle equations.

By substituting from (3) and (4) into Eq. (8), we get the principle equation which splits into a set of equations in the unknown functions \(a_i(x, \tau), b_i(x, \tau)\) and \(c_i(x, \tau)\). For convenience, we use the transformations on \(c_i(x, \tau)\) that simplify the computation
\[
c c_{2i}(x, \tau) = p_i(x, \tau)c_i(x, \tau)\quad c_1(x, \tau) = -p(x, \tau) + C_i(x, \tau),
\]
\[
c_0(x, \tau) = -2C_{11}(x, \tau) + C^2_i(x, \tau) + 4C_{10}(x, \tau)
\]
\[
\tau = \int_0^t f(t_1)dt_1,
\]
and we solve the obtained equations to get \(b_i(x, \tau), i = 0, 1, 2, a_j(x, \tau), j = 1, 2\) and \(C_{ij}(x, \tau)\) respectively. We are left with unsolved single equation among them.

Step 2: Solving the compatibility equations in (5).

These equations read
\[
b_0(x, \tau)c_1(x, \tau) - b_1(x, \tau)c_0(x, \tau) + c_0(x, \tau) - b_{01}(x, \tau) = 0,
\]
\[
2b_0(x, \tau)c_2(x, \tau) - 2b_2(x, \tau)c_0(x, \tau) + c_1(x, \tau) - b_{11}(x, \tau) = 0,
\]
\[
- b_2(x, \tau)c_1(x, \tau) + b_1(x, \tau)c_2(x, \tau) + c_2(x, \tau) - b_{21}(x, \tau) = 0,
\]
(29)
and (28) will be used in (29). Eqs. (29)\(_3\) and (29)\(_2\) were solved to get \(a_{00}(x, \tau)\) and \(a_{01}(x, \tau)\) respectively. The compatibility equation between the obtained results for \(a_{00}(x, \tau)\) and \(a_{01}(x, \tau)\) gives rise to an equation which solves to
where $h_i$ and $k_i$, $i = 0, 1$ are arbitrary constants.

By using the obtained result for $a_{0i}(x, t)$, we found that it satisfies the unsolved equation in the principle ones also. Thus we are only left with Eq. (29), which is a nonlinear PDE in $C_0(x, t), C_1(x, t)$ and $C_2(x, t)$. Consequently, we have two arbitrary functions, namely $C_2(x, t)$ and $C_1(x, t)$, so that no loss of generality if we take $C_2(x, t) = 1$ and $C_1(x, t) = 0$. Thus (29) is closed in $C_0(x, t)$. This equation is satisfied by taking $C_0(x, t) = A_j h_j^2(t) - \frac{1}{4}$ or when $C_0(x, t) = A_j h_j^2(t)$, where $A_3$ and $A_4$ are constants.

**Step 3: Solving the auxiliary equations in (4).**

In this step Eq. (4) is solved in the new variables according to the following two cases;

(i) When $C_0(x, t) = A_j h_j^2(t) - \frac{1}{4}$

\[
\phi_1(x, t) = \frac{(h_0 + 2\tau - h_j x^2)\cos(\mu_j(x, t)) - \sqrt{h_j}(h_0 + 2\tau)\sin(\mu_j(x, t))}{\sqrt{h_0 + 2\tau}(h_0 + 2\tau)\cos(\mu_j(x, t)) - \sqrt{h_j}(h_0 + 2\tau)\sin(\mu_j(x, t))},
\]

where $\mu_j(x, t) = \sqrt{h_j(4h_0 + x^2)}$, $h_j > 0$ is a constant or

\[
\phi_2(x, t) = \frac{(h_0 + 2\tau - h_j x^2)\cosh(\mu_j(x, t)) + \sqrt{h_j}(h_0 + 2\tau)\sinh(\mu_j(x, t))}{\sqrt{h_0 + 2\tau}(h_0 + 2\tau)\cosh(\mu_j(x, t)) + \sqrt{h_j}(h_0 + 2\tau)\sinh(\mu_j(x, t))},
\]

where $\mu_j(x, t) = \sqrt{h_j(4h_0 + x^2)}$, $h_j > 0$ is a constant.

(ii) When $C_0(x, t) = A_j h_j^2(t)$

\[
\phi_3(x, t) = \frac{A_j h_j^2(\exp(2H(\tau)(2A_4 h_j^2 + x)) - 2\sqrt{h_0 + 2\tau}H(\tau))}{(h_0 + 2\tau)H(\tau)\exp(2H(\tau)(2A_4 h_j^2 + x)) + 2H(\tau)\sqrt{h_0 + 2\tau}}.
\]

where $H(\tau) = \sqrt{\frac{4A_4}{h_0 + 2\tau}}$ and $A_4 < 0$, $h_0$ are arbitrary constants.

**Step 4: Finding the formal solution.**

Finally, the solutions of (8) according to the cases (i) and (ii) respectively are given by

\[
v_1(x, t) = \frac{1}{2\sqrt{h_0 + 2\tau}(\sqrt{h_0 + 2\tau} \cos(\mu_j(x, t)) - \sqrt{h_j}(h_0 + 2\tau) \sin(\mu_j(x, t)))}
\times (Q_1(x, t) + ((h_0 + 2\tau)x + h_j^2(12h_0 + 24\tau - x^2)) \cos(2\mu_j(x, t))
- 2\sqrt{h_0 + 2\tau}x^2 \sin(2\mu_j(x, t)))
\]

\[
v_2(x, t) = \frac{1}{2\sqrt{h_0 + 2\tau}(\sqrt{h_0 + 2\tau} \cos(\mu_j(x, t)) + \sqrt{h_j}(h_0 + 2\tau) \sin(\mu_j(x, t)))}
\times (Q_2(x, t) + ((h_0 + 2\tau)x + h_j^2(12h_0 + 24\tau - x^2)) \cos(2\mu_j(x, t))
+ 2\sqrt{h_0 + 2\tau}x^2 \sin(2\mu_j(x, t))),
\]

\[
v_3(x, t) = \frac{1}{\sqrt{h_0 + 2\tau}}
\times \left( x - 6A_4 h_j^2 + \frac{(\exp(2H(\tau)(2A_4 h_j^2 + x)) - 2H(\tau)\sqrt{h_0 + 2\tau})}{(\exp(2H(\tau)(2A_4 h_j^2 + x)) + 2H(\tau)\sqrt{h_0 + 2\tau})} \right),
\]

where $Q_1(x, t) = 12h_0 h_j^2 + 24h_0 x + 2\tau x - 24h_j^2 x^2 + h_j^4 x^4$, $Q_2(x, t) = 12h_0 h_j^2 + 24h_j^2 x^2 + h_0 x + 2\tau x - 24h_j^2 x^2 + h_j^4 x^4$, $\tau = \int_0^t g(t) \, dt$, $\tau > 0$ and $x_j, i = 1, 2$ are constants.

We mention that the solutions which are given in (34)–(36) satisfy Eq. (8).

Fig. 4a and b represents the solutions in (34) and (35) when $g(t) = 1 + t^2$ respectively.

The solution in Fig. 4a shows the interaction between soliton, solitary and periodic waves (a highly dispersed periodic-soliton waves). While the solution in Fig. 4b shows a soliton wave coupled to two solitary waves the intersection between soliton, kink and anti-kink waves.

**II. When $k = 3, n = 4$.**

By using the same steps in the previous case (when $k = n = 2$), we get the solution of (8) as

\[
v(x, t) = \frac{1}{h_0 Q(\tau)(2Q(\tau) + A_0 h_0(s_1 + x + s_0 Q(\tau)))}
\times ((4(s_1 + x)A_0 h_0 + A_1 h_0^2(12h_0 + 24\tau - s_0^2(s_1 + x)))Q(\tau))
+ A_0 h_0(s_1 + x)^2(4Q(\tau) + A_0 h_0(s_1 + x + 2s_0 Q(\tau))),
\]

where $Q(\tau) = \sqrt{h_0 + 2\tau}$ and $s_0, h_0, A_0, i = 0, 1$ are arbitrary constants. Again, we verified that the solution in (37) satisfies Eq. (8).
Conclusions

The Korteweg–de Vries equation with variable coefficients which describes the shallow water wave propagation through a medium with varying dispersion and nonlinearity coefficients was studied. The extended unified method for finding exact solutions to this equation has been outlined. We have shown that water waves propagate as traveling solitary (or elliptic) waves with anomalous dispersion. This holds when the coefficients of the nonlinear and dispersion terms are linearly dependent (or comparable). For linearly independent coefficients, the water waves behave in similarity waves with a breakdown of wave propagation. This holds when the dispersion coefficients prevail the nonlinearity. Some of these solutions show “winged” soliton (anti-soliton) or wave train solutions. The obtained solutions here are completely new. The extended unified method can be used to find exact solutions of coupled evolution equations, but we think that parallel computations should be used because they require a very lengthy computation. Indeed, they cannot be transformed to traveling wave equations.

Conflict of interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

References

[1] Kumar H, Chand F. Optical solitary wave solutions for the higher order nonlinear Schrödinger equation with self-steepening and self-frequency shift effects. Opt Laser Technol 2013;54:265–73.
[2] Kumar H, Chand F. Applications of extended F-expansion and projective Ricatti equation methods to (2 + 1)-dimensional soliton equations. AIP Adv 2013;3(3), 032128-032128-20.
[3] Jianbin L, Kongqing Y. The extended F-expansion method and exact solutions of nonlinear PDEs. Chaos Soliton Fract 2004;22:111–21.
[4] Sheng Z. New exact solutions of the KdV–Burgers–Kuramoto equation. Phys Lett A 2006;358:414-20.
[5] Zayed EME, Abdelaziz MAM. Applications of a generalized extended \((G'/G)\)-expansion method to find exact solutions of two nonlinear Schrödinger equations with variable coefficients. Acta Phys Pol A 2012;121:573–80.
[6] Wang ML, Li XZ, Zhang JL. The \((G'/G)\)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys Lett A 2008;372:417-23.
[7] Wazwaz AM. New kinds of solitons and periodic solutions to the generalized KdV equation. Numer Meth Part D E 2006;23:247-55.
[8] Wazwaz AM. New solitary wave solutions to the Karamoto–Sivashinsky and the Kawahara equations. Appl Math Comput. 2006;182:1642–50.
[9] Abdel-Gawad HI. Towards a unified method for exact solutions of evolution equations. An application to reaction diffusion equations with finite memory transport. J Stat Phys 2012;147:506–18.
[10] Fu ZT, Liu SD, Liu SK. New exact solutions to KdV equations with variable coefficients or forcing. Appl Math Mech 2004;25:73–9.
[11] Xu GQ, Liu ZB. Explicit solutions to the coupled KdV equations with variable coefficients. Appl Math Mech 2005;26:101–7.
[12] Kumar H, Malik A, Chand F. Soliton solutions of some nonlinear evolution equations with time-dependent coefficients. Pramana – J Phys 2012;80(2):361–7.
[13] Kumar H, Malik A, Chand F. Analytical spatiotemporal soliton solutions to \((3 + 1)\)-dimensional cubic–quintic nonlinear Schrödinger equation with distributed coefficients. J Math Phys 2012;53, 103704-103704-13.
[14] Abdel-Gawad HI, Elazab NS, Osman M. Exact solutions of space dependent Korteweg–de Vries equation by the extended unified method. J Phys Soc Jpn 2013;82:044004.
[15] Mingliang W, Yueming W, Yubin Z. An auto-Bäcklund transformation and exact solutions to a generalized KdV equation with variable coefficients and their applications. Phys Lett A 2002;303(1):45–51.
[16] Gai XL, Gao YT, Yu X, Wang L. Painleve’ property, auto-Bäcklund transformation and analytic solutions of a variable-coefficient modified Korteweg–de Vries model in a hot magnetized dusty plasma with charge fluctuations. Appl Math Comput 2011;218(2):271–9.
[17] Wang ML, Zhou YB. The periodic wave solutions for the Klein–Gordon–Schrödinger equations. Phys Lett A 2003;318:84–92.
[18] Wang ML, Li XZ. Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations. Phys Lett A 2005;343:48.
[19] El GA, Grimshaw RHJ. Generation of undular bores in the shelves of slowly-varying solitary waves. CHAOS 2002;12:1015.
[20] Zhang S. Exact solution of a KdV equation with variable coefficients via exp-function method. Nonlinear Dynam. 2007;52(1–2):11–7.
[21] Zhang S. Application of Exp-function method to a KdV equation with variable coefficients. Phys Lett A 2007;365:448–53.
[22] Bowman F. Introduction to elliptic functions with applications. London: Universities; 1959.
[23] Prasolov V, Solovyev Y. Elliptic functions and elliptic integrals. Providence: American Mathematical Society; 1997.
[24] Wazzan Luwai. A modified tanh–coth method for solving the KdV and the KdV–Burgers’ equations. Commun Nonlinear Sci 2009;14(2):443–50.
[25] Liu S, Fu Z, Liu S, Zhao Q. Jacobi elliptic function expansion integrals. Providence: American Mathematical Society; 1997.
[26] Wang ML, Zhou YB. The periodic wave solutions for the generalized KdV and the KdV–Burgers’ equations. Commun Nonlinear Sci 2007;52(1–2):11–7.
[27] Zhang S. Application of Exp-function method to a KdV equation with variable coefficients. Phys Lett A 2007;365:448–53.
[28] Bowman F. Introduction to elliptic functions with applications. London: Universities; 1959.
[29] Prasolov V, Solovyev Y. Elliptic functions and elliptic integrals. Providence: American Mathematical Society; 1997.
[30] Wazzan Luwai. A modified tanh–coth method for solving the KdV and the KdV–Burgers’ equations. Commun Nonlinear Sci 2009;14(2):443–50.
[31] Liu S, Fu Z, Liu S, Zhao Q. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 2001;289(1):69–74.
[32] Liu J, Yang K. The extended F-expansion method and exact solutions of nonlinear PDEs. Chaos Soliton Fract 2004;22(1):111–21.
[33] Ebaid A. Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method. Phys Lett A 2007;365(3):213–9.
[34] Wang M, Li X, Zhang J. The \((G'/G)\)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Phys Lett A 2008;372(4):417–23.