Fast Sequential Decoding of Polar Codes

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Abstract

An extension of the stack decoding algorithm for polar codes is presented. The paper introduces a new score function, which enables one to accurately compare paths of different length. This results in significant complexity reduction with respect to the original stack algorithm at the expense of negligible performance loss.

1 Introduction

Polar codes were recently shown to be able to achieve the symmetric capacity of memoryless channels, while having low-complexity construction, encoding and decoding algorithms [1]. However, the performance of polar codes of moderate length is substantially worse compared to LDPC and turbo codes used in today communication systems. This is both due to suboptimality of the classical successive cancellation decoding method, and poor minimum distance of polar codes. List decoding algorithm introduced in [2] enables one to implement near maximum likelihood decoding of polar codes with complexity $O(Ln \log n)$, where $n$ is code length and $L$ is list size. Furthermore, the performance of polar codes concatenated with a CRC outer code [2] and polar subcodes [3] under list decoding appears to be better compared to known LDPC and turbo codes.

However, the complexity of the Tal-Vardy list decoding algorithm turns out to be rather high. It can be reduced by employing sequential decoding techniques [4, 5, 6]. These methods avoid construction of many useless low-probability paths in the code tree. Processing of such paths constitutes the most of the computational burden of the list decoder. In this paper we show that by careful weighting of paths of different length, one can significantly reduce the computational complexity of the decoder. The proposed construction of path score function aims to estimate the expected log-likelihood of the codeword of a polar code, which may be obtained as a continuation of the considered path in the code tree. Simulation results indicate that the proposed approach results in significant reduction of the average number of iterations performed by the decoder.

The paper is organized as follows. Section 2 provides some background on polar codes. Section 3 introduces the proposed sequential decoding method. Its improvements are discussed in Section 4. Simulation results illustrating the performance and complexity of the proposed algorithm are provided in Section 5. Finally, some conclusions are drawn.

2 Polar codes

2.1 Code construction

$(n = 2^m, k)$ polar code over $\mathbb{F}_2$ is a linear block code generated by $k$ rows of matrix $A_m = B_m F^{\otimes m}$, where $B_m$ is the bit reversal permutation matrix, $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\otimes m$ denotes $m$-times Kronecker product of the matrix with itself [1]. Hence, a codeword of a classical polar code is obtained as $c_0^{n-1} = u_0^{n-1} A_m$, where $a_i = (a_4, \ldots, a_t)$, $u_i = 0, i \in \mathcal{F}$, $\mathcal{F} \subset \{0, \ldots, n-1\}$ is the set of $n - k$ frozen symbol indices, and the remaining symbols are set to the data symbols being encoded.
In some cases, when we know the minimum distance \( d \) of the code, we describe code parameters as \((n, k, d)\).

It is possible to show that matrix \( A \) transforms the original binary input memoryless output-symmetric channel \( W^{(0)}_m(y|c) \) into bit subchannels \( W^{(i)}_m(y^{n-1}_0, u^{i-1}_0|u_i) \), the capacities of these subchannels converge with \( m \) to 0 or 1 symbols per channel use, and the fraction of subchannels with capacity close to 1 converges to the capacity of \( W^{(0)}_m(y|c) \). Here \( y_0, \ldots, y_{n-1} \) are the noisy symbols obtained by transmitting codeword symbols \( c_0, \ldots, c_{n-1} \) over a binary input memoryless output-symmetric channel \( W(y|c) \).

The conventional approach to construction of an \((n, k)\) polar code assumes that \( \mathcal{F} \) is the set of \( n - k \) indices \( i \) of bit subchannels \( W^{(i)}_m(y^{n-1}_0, u^{i-1}_0|u_i) \) with the highest error probability. It was suggested in [3] to set frozen symbols \( u_i, i \in \mathcal{F} \) not to zero, but to linear combinations of some other symbols, i.e.

\[
  u_i = \sum_{s=0}^{i-1} V_{j_i, s} u_s,
\]

where \( V \) is a \((n - k) \times n\) binary matrix, such that its rows have last non-zero values in distinct columns, and \( j_i \) is the index of the row having the last non-zero element in column \( i \). The obtained codes are referred to as polar subcodes. Polar codes with CRC [2] can be considered as a special case of polar subcodes. Polar subcodes were shown to provide substantially better performance compared to classical polar codes under list decoding. The above described special shape of matrix \( V \) enables one to implement decoding of polar subcodes using straightforward generalizations of the successive cancellation algorithm and its extensions.

### 2.2 The successive cancellation decoding algorithm

Decoding of polar subcodes can be implemented by the successive cancellation (SC) algorithm. It is convenient to describe it in terms of probabilities

\[
  W^{(i)}_m\{u_0^{i-1}|y_0^{n-1}\} = \frac{W^{(i)}_m(y_0^{n-1}, u_0^{i-1}|u_i)}{2W(y_0^{n-1})}
\]

\[
  = \sum_{u_i+1} W^{(n-1)}_m\{u_0^{n-1}|y_0^{n-1}\} = \sum_{u_i+1} \prod_{j=0}^{n-1} W\{(u_0^{n-1}A_m)_j|y_j\}
\]

At phase \( i \) the SC decoder makes decision

\[
  \hat{u}_i = \begin{cases} \arg \max_{u_i, i \notin \mathcal{F}} W^{(i)}_m\{\hat{u}_0^{i-1}, u_i|y_0^{n-1}\}, & i \notin \mathcal{F} \\ \sum_{s=0}^{i-1} V_{j_i, s} \hat{u}_s, & \text{otherwise,} \end{cases}
\]

where \( a.b \) denotes a vector obtained by appending \( b \) to \( a \), and \( j_i \) is the position of the last non-zero element in the \( i \)-th row of matrix \( V \). The probabilities used in this expression can be recursively computed as

\[
  W^{(2i)}_\lambda\{u_0^{2i}|y_0^{N-1}\} = \\
  \sum_{u_{2i+1}=0} W^{(i)}_{\lambda-1}\{u_0^{2i+1} \oplus u_{2i+1}|y_0^{N-1}\} W^{(i)}_{\lambda-1}\{u_{2i+1}^{N-1}|y_0^{N-1}\}
\]

\[
  W^{(2i+1)}_\lambda\{u_0^{2i+1}|y_0^{N-1}\} = \\
  W^{(i)}_{\lambda-1}\{u_0^{2i+1} \oplus u_{2i+1}|y_0^{N-1}\} W^{(i)}_{\lambda-1}\{u_{2i+1}^{N-1}|y_0^{N-1}\}
\]
where \( N = 2^\lambda, 0 < \lambda \leq m, \) \( a_{0,e}^i \) and \( a_{0,o}^i \) denote subvectors of \( a_0^i \), consisting of elements with even and odd indices, respectively. It is convenient to implement these calculations in LLR domain as

\[
L_{\lambda}^{(2i)}(u_0^{2i-1}|y_0^{N-1}) = 2 \tanh^{-1} \left( \frac{\tanh \left( \frac{L_{\lambda-1}^{(i)}(u_{0,e}^{2i-1} \oplus u_{0,o}^{2i-1}|y_0^{2N-1})}{2} \right)}{\tanh \left( \frac{L_{\lambda-1}^{(i)}(u_{0,o}^{2i-1}|y_0^{N-1})}{2} \right)} \right),
\]

\[
L_{\lambda}^{(2i+1)}(u_0^{2i}|y_0^{N-1}) = (-1)^{u_2i} L_{\lambda-1}^{(i)}(u_{0,e}^{2i-1} \oplus u_{0,o}^{2i-1}|y_0^{N-1}) + L_{\lambda-1}^{(i)}(u_{0,o}^{2i-1}|y_0^{N-1}),
\]

where \( L_{\lambda}^{(i)}(u_0^{i-1}|y_0^{N-1}) = \log \frac{W_m^{(i)}(u_0^{i-1}, 0|y_0^{N-1})}{W_m^{(i)}(u_0^{i-1}, 1|y_0^{N-1})} \), \( 0 \leq i < n, 0 \leq \lambda < m \), so that the decision rule for \( i \notin F \) becomes

\[
\hat{u}_i = \begin{cases} 
0, & L_m^{(i)}(u_0^{i-1}|y_0^{n-1}) > 0 \\
1, & \text{otherwise.}
\end{cases}
\]

### 2.3 Improved decoding algorithms

The original successive cancellation decoding algorithm does not provide maximum likelihood decoding. A list decoding algorithm was suggested in [2], and shown to achieve substantially better performance with complexity \( O(Ln \log n) \). However, large values of \( L \) are needed in order to implement near-ML decoding of polar subcodes and polar codes with CRC. This makes the complexity of list decoding of polar codes too high for practical implementation.

On the other hand, one does not need in practice to obtain a list of codewords, but just a single most probable one. The original Tal-Vardy algorithm for polar codes with CRC examines the elements in the obtained list, and discards those with invalid checksum. This algorithm can be easily tailored to process the dynamic freezing constraints before the decoder reaches the last phase, so that the output list contains only valid codewords of a polar subcode. However, even in this case one has to discard \( L - 1 \) codewords from the obtained list, so most of the computational work performed by the Tal-Vardy decoder is just wasted.

This problem was addressed in [4], where a generalization of the stack algorithm was suggested. It provides lower average decoding complexity compared to the Tal-Vardy algorithm. In this paper we revise the stack decoding algorithm for polar (sub)codes, and show that its complexity can be substantially reduced.

### 3 Sequential decoding of polar codes

#### 3.1 The basic algorithm

The objective of the MAP decoder is to find a codeword (or the corresponding information vector \( u_0^{n-1} \), such that it satisfies freezing constraints) with the highest probability \( W_m^{(n-1)}(u_0^{n-1}|y_0^{n-1}) \), where \( y_0^{n-1} \) is the received noisy vector.

The principle of sequential decoding is based on a representation of the set of codewords as a tree [7] [4]. In the case of polar (sub)codes it is convenient to consider the set of binary vectors of length at most \( n \) arranged into a tree. The nodes of the tree correspond to vectors \( u_0^{\phi-1}, 0 \leq \phi < n, \) satisfying [1]. At depth \( \phi \), each node \( u_0^{\phi-1} \) has two children \( u_0^{\phi-1.0} \) and \( u_0^{\phi-1.1} \). The root of the tree corresponds to an empty vector. By abuse of notation, the path from the root of the tree to a node
Figure 1: Stack decoding of \((8, 4)\) polar code

\(u_0^{\phi-1}\) will be also denoted \(v_0^{\phi-1}\), and the words "path" and "node" will be used interchangeably. A decoding algorithm needs to consider only valid nodes, i.e. the nodes satisfying constraints \(1\).

The stack decoding algorithm \([8, 9, 4, 6]\) employs a priority queue \((PQ)\) to store node identifiers together with their scores. PQ is a data structure, which contains tuples \((M, v)\), where \(v\) is some object, and \(M\) is its score, and provides efficient algorithms for the following operations \([10]\):

- push a tuple into the PQ;
- extract (pop) a tuple \((M, v)\) (or just \(v\)) with the highest \(M\);
- remove a given tuple from the PQ.

We assume here that the PQ may contain at most \(\Theta\) elements.

In the context of polar codes, the stack decoding algorithm operates as follows:

1. Push into the PQ the root of the tree with score 0.
2. Extract from the PQ a node \(v_0^{\phi-1}\) with the highest score.
3. If \(\phi = n\), return codeword \(v_0^{n-1}A_m\) and terminate the algorithm.
4. If the number of valid children of node \(v_0^{\phi-1}\) exceeds the amount of free space in the PQ, remove from it the element with the smallest score.
5. Compute the scores \(M(v_0^{\phi})\) of valid children \(v_0^{\phi}\) of the extracted node, and push them into the PQ.
6. If nodes of depth \(\phi\) were extracted \(L\) or more times from the PQ, remove from it all nodes \(\hat{v}_j^{\phi-1}, j \leq \phi\).
7. Go to step 2.

In what follows, by iteration we mean one pass of the above algorithm over steps 2–7.

The parameter \(L\) has the same impact on the performance of the decoding algorithm as the list size in the Tal-Vardy list decoding algorithm. Step 6 ensures that the algorithm terminates in at most \(Ln\) iterations. This is also an upper bound on the number of entries stored in the PQ. However, the algorithm can work with PQ of much smaller size \(\Theta\). Step 4 ensures that this size is never exceeded.

**Example 1.** Consider stack decoding of \((8, 4)\) classical polar code with \(F = \{0, 1, 2, 4\}\), where path score is defined as \(M(v_0^{\phi}) = W_3^{(\phi)}\left\{v_0^{\phi} | y_0^{7}\right\}\). Figure [1] illustrates the nodes in the code tree.
considered by the above described algorithm. Solid and dashed lines denote the edges $v_i = 0$ and $v_i = 1$, respectively. The values inside circles denote the scores of the corresponding nodes in the code tree for some noisy sequence. The number near each node denotes the iteration number of the above algorithm, where the corresponding node is extracted from the PQ.

It can be seen that after iteration 4 the decoder switches to another branch of the code tree, but at iteration 6 it returns back. It is reasonable to assume that the path $A = (00000)$ containing four frozen symbols (underlined) is more likely to correspond to the solution of the decoding problem, than the slightly more probable path $B = (0001)$, which contains only three frozen symbols. Indeed, for a classical polar code, path probability typically drops significantly after the path is extended with the value of a frozen symbol. Therefore, in order to reduce the decoding complexity, it is tempting to adjust the path score function in such way, so that the decoder does not consider path $B$. This assumption, however, may turn out to be incorrect, and such approach may result in increased decoding error probability.

The performance and complexity of sequential decoding critically depends on the definition of the score function $M(v_0^\phi)$ of the genie sequential decoder, implementing the block MAP criterion, should select at each iteration the node $v_0^\phi$, such that it corresponds to a part of the path $v_0^{n_1}$, which maximizes $W_m^{(n_1)}\{v_0^{n_1}\mid y_0^{n_1}\}$, while satisfying all freezing constraints. In this case, exactly $n$ iterations would be performed. However, this cannot be implemented in a real decoder. Instead, we propose to select at each iteration for processing the path which maximizes an estimate of the probability of the transmitted codeword.

### 3.2 Path score

#### 3.2.1 The structure of the score function

In order to mimic the behaviour of a genie decoder, we propose to define the score function $M(v_0^\phi)$ in such way, so that it reflects the codeword probability $W_m^{(n_1)}\{v_0^{n_1}\mid y_0^{n_1}\}$. In order to derive such an estimate, we propose the following interpretation of the SC and list SC decoding algorithms. Decoding consists in successive construction of the elements of input vector $\hat{u}_0^n$. This is performed by computing LLRs $L_m(\lambda_0(\hat{u}_0^{(\phi-1)})\mid y_0^{n_1})$. If there are no frozen symbols, then one can immediately construct (possibly incorrect) estimates $\hat{u}_\phi$ depending on the sign of $L_m(\lambda_0(\hat{u}_0^{(\phi-1)})\mid y_0^{n_1}), 0 \leq \phi < n$. If there are frozen symbols, then one may need to correct some of these estimates. The SC algorithm can do this only at phases $\phi \in F$ (i.e. it just enforces the values of frozen symbols), and it does not keep track of the corrections performed at prior phases. List SC algorithm keeps the value $W_m(\lambda_0(\phi)\{v_0^{\phi_0}(n_1)\mid \lambda_0(y_0^{n_1})\}$, which reflects the estimates made for a particular path $v_0^\phi$ on prior phases. If this value becomes too low, then it is likely that some of the prior estimates are incorrect, and the path needs to be discarded.

Let us assume that the input vector of the polarizing transformation is $u_0^{n_1}$, and consider some path $v_0^{\phi_1}, 0 \leq \phi < n$. Assuming that it is the correct one, i.e. $v_j = u_j, 0 \leq j < \phi$, we seek for a method for estimating the log-likelihood $L = \log W_m^{(n_1)}\{u_0^{n_1}\mid y_0^{n_1}\}$ of the input vector $u_0^{n_1}$, without explicitly constructing it. In order to obtain such an estimate, we decompose $L$ into two parts. The first one can be efficiently computed from the received vector $y_0^{n_1}$ and initial $\phi$ symbols $v_0^{\phi_1} = u_0^{\phi_1}$. Computing the second part would require knowledge of $u_0^{n_1}$. Since it is not available until decoding is completed, we replace the second part with its mean value over $y_0^{n_1}$. Essentially, the obtained estimate reflects the mean value of the log-likelihood of a correct continuation of $v_0^{\phi_1}$.

Let

$$\hat{\pi}_0^{n_1} = \arg \max \left\{ \pi_0^{n_1} \mid W_m^{(n_1)}\{\pi_0^{n_1}\mid y_0^{n_1}\} \right\}$$

(8)
be the most likely continuation of path \( v_0^{(1)} \) (which does not necessarily correspond to a valid codeword), and let

\[
R_m^{(1)}(v_0^{(1)}|y_0^{n-1}) = \log W_m^{(n-1)} \{v_0^{(1)}|y_0^{n-1}\}
\]

be its log-likelihood. Let us further define the modified log-likelihood ratios

\[
S_m^{(0)}(v_0^{(1)}, y_0^{n-1}) = R_m^{(0)}(v_0^{(1)}|y_0^{n-1}) - R_m^{(0)}(v_0^{(0)}, 1|y_0^{n-1}),
\]

and the penalty function

\[
\tau(S, v) = \begin{cases} 0, & \text{sgn}(S) = (-1)^v \\ -|S|, & \text{otherwise}. \end{cases}
\]

It can be seen that

\[
R_m^{(0)}(v_0^{(0)}|y_0^{n-1}) = R_m^{(0)}(v_0^{(1)}|y_0^{n-1}) + \tau(S_m^{(0)}(v_0^{(1)}, y_0^{n-1}), v_0).
\]

Indeed, if \( v_0 = \tilde{v}_0 \), then one has \( \text{sgn} S_m^{(0)}(v_0^{(1)}|y_0^{n-1}) = (-1)^{v_0} \), and the most likely continuations of \( v_0^{(1)} \) and \( v_0^{(0)} \) are identical. Otherwise, \( -|S_m^{(0)}(v_0^{(1)}, y_0^{n-1})| \) is exactly the difference between the log-likelihoods of the most probable continuations of \( v_0^{(1)} \) and \( v_0^{(0)} \).

Let us now assume that \( v_0^{(1)} = u_0^{(1)} \), i.e. consider a path in the code tree, which agrees with the transmitted vector in \( \phi \) initial symbols. By recursive application of (10), one obtains

\[
\mathbb{L}(v_0^{(1)}|y_0^{n-1}) = R_m^{(0)}(v_0^{(1)}|y_0^{n-1}) + \sum_{i=0}^{n-1} \tau(S_m^{(i)}(u_0^{i-1}, y_0^{n-1}), u_i).
\]

The first term in this expression is the log-likelihood of path \( v_0^{(1)} \), and the second term \( \mathbb{L}(\phi) \) reflects the penalty for estimates \( \tilde{v}_i \neq u_i, i \geq \phi \), i.e. the values of \( u_i \) which do not agree with the sign of the corresponding log-likelihood ratios. It will be shown below that the first term can be computed very efficiently. However, the second term can be obtained only when the decoder finds the actual vector \( y_0^{n-1} \). In order to be able to compare incomplete paths \( v_0^{(1)} \), we propose to approximate \( \mathbb{L}(\phi) \) with its expectation over the received noisy values \( y_0^{n-1} \), i.e.

\[
\mathbb{L}(v_0^{(1)}|y_0^{n-1}) \approx \mathbb{L}(\phi) = \sum_{i=0}^{n-1} \mathbb{E}_{y_0^{n-1}} \left[ \tau(S_m^{(i)}(u_0^{i-1}, y_0^{n-1}), u_i) \right],
\]

so that

\[
\mathbb{L}(v_0^{(1)}|y_0^{n-1}) \approx \mathbb{L}(\phi) = R_m^{(0)}(v_0^{(1)}|y_0^{n-1}) + \mathbb{L}(\phi).
\]

Observe that

\[
R_m^{(0)}(v_0^{(1)}|y_0^{n-1}) = \rho(y_0^{n-1}) + \sum_{i=0}^{n-1} \tau(S_m^{(i)}(v_0^{i-1}, y_0^{n-1}), v_i),
\]

where

\[
\rho(y_0^{n-1}) = \max_{v_0^{i-1} \in \mathcal{F}_2} \log W_m^{(n-1)} \{v_0^{i-1}|y_0^{n-1}\}
\]

is a quantity independent of \( v_0^{(1)} \).
In order to obtain a numerically stable score function for the sequential decoder, we propose to further subtract from both sides of (11) the value of

$$A = \rho(y_0^{n-1}) + \sum_{i=0}^{n-1} E_{y_0^{i-1}} \left[ \tau(S_m^{(i)}(u_0^{i-1}, y_0^{n-1}), u_i) \right],$$
which is independent from $v_0^{\phi-1}$. This results in

$$M_b(v_0^{\phi-1}, y_0^{n-1}) = L(v_0^{\phi-1}, y_0^{n-1}) - A = \hat{R}(v_0^{\phi-1}|y_0^{n-1}) - \Psi_b(\phi - 1),$$
(12)

where $\hat{R}(v_0^{\phi-1}|y_0^{n-1}) = R_m^{(\phi)}(v_0^{\phi-1}, y_0^{n-1}) - \rho(y_0^{n-1})$, and

$$\Psi_b(\phi) = \sum_{i=0}^{\phi} E_{y_0^{i-1}} \left[ \tau(S_m^{(i)}(u_0^{i-1}, y_0^{n-1}), u_i) \right].$$

From (10) one obtains that

$$\hat{R}(v_0^{\phi}|y_0^{n-1}) = \hat{R}(v_0^{\phi-1}|y_0^{n-1}) + \tau(S_m^{(\phi)}(v_0^{\phi-1}, y_0^{n-1}), v_\phi),$$
(13)

where $\hat{R}(\emptyset|y_0^{n-1}) = 0$.

We propose to use $M_b(u_0^{\phi}, y_0^{n-1})$ as a path score in the sequential decoding algorithm. Essentially, this quantity shows how far the modified log-likelihood $\hat{R}(v_0^{\phi}|y_0^{n-1})$ of path $v_0^{\phi}$ diverges from its expected value $\Psi_b(\phi)$, obtained under the assumption that $v_0^{\phi} = u_0^{\phi}$, where $u_0^{n-1}$ is the actual input vector of the polarizing transformation used by the transmitter. Positive values of $M_b(v_0^{\phi}, y_0^{n-1})$ correspond to paths, which are closer to a particular received sequence than the average over all possible channel realizations.

### 3.2.2 Efficient calculation of the score function

In order to obtain a simple expression for the proposed path score, observe that $R_m^{(\phi)}(u_0^{\phi}|y_0^{n-1})$ can be computed as

$$R_{\alpha}^{(2i)}(u_0^{2i}|y_0^{N-1}) = \max_{u_{2i+1} \in \mathcal{F}_2} \left( R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i+1} \oplus u_{0_{o,i}}^{2i+1}|y_0^{N-1}) + R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i+1}|y_0^{N-1}) \right),$$

$$R_{\lambda}^{(2i)}(u_0^{2i+1}|y_0^{N-1}) = R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i+1} \oplus u_{0_{o,i}}^{2i+1}|y_0^{N-1}) + R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i+1}|y_0^{N-1}) + R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i+1}|y_0^{N-1}),$$

where $0 < \lambda \leq m, N = 2^\lambda$, and initial values for these recursive expressions are given by $R_0^{(0)}(b|y_j) = \log W_0^{(0)} \{ b|y_j \}, b \in \{0,1\}$. Hence, one obtains

$$S_{\lambda}^{(2i)}(u_0^{2i-1}, y_0^{N-1}) = \max(J(0) + K(0), J(1) + K(1))$$
$$- \max(J(1) + K(0), J(0) + K(1))$$
$$= \max(J(0) - J(1) + K(0) - K(1), 0)$$
$$= \max(K(0) - K(1), J(0) - J(1))$$

$$S_{\lambda}^{(2i+1)}(u_0^{2i}, y_0^{N-1}) = J(u_{2i}) + K(0) - J(u_{2i} + 1) - K(1)$$

where $J(c) = R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i-1} \oplus u_{0_{o,i}}^{2i-1}, c|y_0^{N-1}), K(c) = R_{\lambda-1}^{(i)}(u_{0_{e,i}}^{2i-1}, c|y_0^{N-1}).$
It can be obtained from these expressions that the modified log-likelihood ratios are given by

\[
S^{(2i)}(u^{2i-1}_0|y^{N-1}_0) = Q(a, b) = \text{sgn}(a) \text{sgn}(b) \min(|a|, |b|),
\]

where \(a = S^{(i)}_{\lambda - 1}(u^{2i-1}_{0,e} \oplus u^{2i-1}_{0,o}, y^{N-1}_0)\), \(b = S^{(i)}_{\lambda - 1}(u^{2i-1}_{0,o}, y^{N-1}_0)\). These expressions can be readily recognized as the min-sum approximation for (6)–(7) [11]. However, these are exact values, which reflect the probability of the most likely continuation of a given path in the code tree. Hence, computing path score given by (12) requires only summation and comparison operations.

3.2.3 The heuristic function

The function \(\Psi_b(\phi)\) is equal to the expected value of the log-likelihood of a length-\(\phi\) part of the correct path. Employing this function enables one to estimate how far a particular path has diverted from the expected behaviour of a correct path. Hence, it is similar to the heuristic function used in the \(A^*\) shortest path search algorithm [12, 13]. The heuristic function can be computed offline under the assumption of zero codeword transmission. Indeed, the cumulative density functions \(F^{(i)}_{\lambda}(x)\) of \(S^{(i)}_{\lambda}\) are given by [14]

\[
F^{(2i)}_{\lambda}(x) = \begin{cases} 
2F^{(i)}_{\lambda - 1}(1 - F^{(i)}_{\lambda - 1}(-x)), & x < 0 \\
2F^{(i)}_{\lambda - 1}(x) - (F^{(i)}_{\lambda - 1}(-x))^2 - (F^{(i)}_{\lambda - 1}(x))^2, & x > 0
\end{cases}
\]

where \(F^{(0)}_{\lambda}(x)\) is the CDF of the channel output LLRs. Then one can compute

\[
\Psi_b(\phi) = -\sum_{i=0}^{\phi} \int_{-\infty}^{0} F^{(i)}_{\lambda}(x) dx.
\]

The heuristic function \(\Psi(\phi)\) depends only on \(m\) and channel properties, so it can be used for decoding of any polar (sub)code of a given length. Figure 2 illustrates the heuristic function for the case of BPSK modulation and AWGN channel with different noise standard deviations \(\sigma\).

3.2.4 Discussion

Finally, we illustrate the meaning of \(\tilde{R}(v^\phi_0|y^{n-1}_0)\). Let

\[
E(c^{n-1}_0, s^{n-1}_0) = -\sum_{i=0}^{n-1} \tau(S_i, c_i)
\]
be the ellipsoidal weight (also known as correlation discrepancy) of vector \( c_{n-1}^n \in \mathbb{F}_2^n \) with respect to LLR vector \( S_{n-1}^n \) \[15\] \[16\]. It is possible to show that the ML decoding problem for the case of transmission of codewords of a code \( C \) over a memoryless channel can be formulated as

\[ \hat{u}_{n-1}^n = \arg \min_{c_{n-1}^n \in C} E(c_{n-1}^n | S_{n-1}^n). \]

**Lemma 1.** For any \( c_{0}^{2n-1} \in \mathbb{F}_2^{2n-1} \)

\[ E(c_{0}^{2n-1}, S_{0}^{2n-1}) = E(c_{0,e}^{2n-1} + c_{0,o}^{2n-1}, \tilde{S}_{0}^{n-1}) + E(c_{0,o}^{n-1}, \overline{S}_{0}^{n-1}), \]

where \( \tilde{S}_i = Q(S_{2i}, S_{2i+1}), \overline{S}_i = P(c_{2i}, S_{2i}, S_{2i+1}). \)

**Proof.** It is sufficient to prove the statement for \( n = 1 \). The result follows by examining all possible combinations of \( c_0, c_1 \in \mathbb{F}_2 \).

Applying Lemma 1 recursively, one obtains that

\[ \hat{R}(u_{n-1}^n | y_{n-1}^n) = -E(u_{n-1}^n A_m | S), \]

where \( S = (S_0^{(0)}(y_0), \ldots, S_0^{(0)}(y_{n-1})) \). This means that the proposed score function reflects the ellipsoidal weight of the transmitted codeword.

### 3.2.5 Summary of the proposed approach

We propose to implement decoding of polar (sub)codes by employing the stack algorithm described in Section 3.1, making use of the score function \( M_0(v_0^{n-1}, y_0^{n-1}) \) given by (12). The first component of the score function is given by (13), i.e. it is the accumulated penalty for phases 0, \( \ldots, \phi - 1 \). Phase penalty is equal (up to the sign) to the absolute value of LLR \( S_i(v_i^{j-1}, y_i^{j-1}) \), if its sign does not agree with \( v_i \), and zero otherwise. The second component of the score function is the expected value of the penalty for these phases. The LLRs are recursively computed using the min-sum method, as described in (14)–(15).

Observe that the proposed approach can be used both with classical polar codes, polar subcodes \[3\] and polar codes with CRC \[2\].

### 4 Improvements

Experiments show that the performance of the Tal-Vardy algorithm and its derivatives (including the proposed algorithm) for polar codes with CRC and polar subcodes is dominated not by the maximum likelihood decoding error events, but by the events corresponding to the correct path being prematurely killed. In such case the decoding error is unavoidable, and all subsequent calculations performed by the decoder are useless. This is particularly harmful in the case of sequential decoding, since this causes the decoder to perform substantially more iterations than in the case of successful decoding. Therefore we propose some techniques in order to detect such events, and either adjust the decoder parameters, or terminate decoding.

#### 4.1 Adaptive list size

In the case of infinite size \( \Theta \) of the priority queue the performance of the proposed algorithm depends mostly on parameter \( L \). Setting \( L = \infty \) results in near maximum likelihood decoding at the cost of excessive complexity and memory consumption. It was observed in experiments that in most cases the decoding can be completed successfully with small \( L \), and only rarely a high value of \( L \) is required.

\[ ^2 \text{Similarly to the case of sequential decoding of convolutional codes, employing a non-zero heuristic function may prevent the stack algorithm from providing ML decoding performance.} \]
REMOVESHORTPATHSADAPTIVE($M_0, \Psi$)

1. Remove from PQ all pairs ($M, l$) : $\psi_l \leq \Psi$
2. if $L < L_{\text{max}}$
   3. then Save all $s$ removed pairs ($M, l$)
   4. if $s > 0$
      5. then $\kappa \leftarrow \kappa + 1$
   6. if $\kappa > \kappa_1$
      7. then $\kappa_1 \leftarrow \kappa + \kappa_0; L \leftarrow 2L$
   8. Push into PQ saved pairs ($M, l$) : $M > M_0$
9. else For each removed pair ($M, l$) kill path $l$

Figure 3: Adaptive list size implementation

It was also observed that if the decoder needs to kill short paths too many times while processing a single noisy vector, this most likely means that the correct path is already killed, and decoding error is unavoidable.

Therefore, in order to reduce the decoding complexity, we propose to change list size $L$ adaptively. We propose to start decoding of a noisy vector with small $L$, and increase it, if the decoder is likely to have killed the correct path. In order to do this, we propose to keep track of the number of times $\kappa$ the algorithm actually removes some paths at step 6. If this value exceeds some threshold $\kappa_0$, then the decoding may need to be restarted with larger $L$, similarly to [17].

However, more efficient approach is possible. In order to avoid repeating the same calculations, we propose not to kill paths permanently at step 6, but remove them from the PQ, and save the corresponding pairs ($M_b, l$) in a temporary array, where $l$ is an identifier of some path $v_0^{\phi-1}$, and $M_b = M_0(v_0^{\phi-1}, y_0^{n-1})$. Such paths will be referred to as suspended paths. If $\kappa$ exceeds some threshold $\kappa_0$, instead of restarting the decoder, we propose to double list size $L$ and re-introduce into the PQ suspended paths with the score better than the score $M_0$ of the current path. This is performed until $L$ reaches some upper bound $L_{\text{max}}$. Figure 3 illustrates this method. Variables $\kappa$ and $\kappa_1$ should be set to 0 and $\kappa_0$, respectively, at step 1 of the sequential decoding algorithm presented in Section 3.1. Here $\kappa_0$ is a parameter of the decoding algorithm.

The amount of storage for storing suspended paths can be reduced, if one saves on line 3 of the REMOVESHORTPATHSADAPTIVE algorithm only the paths with sufficiently high score, and kills the remaining paths. Furthermore, $\kappa_0$ threshold can be also selected adaptively depending on $\kappa$.

4.2 Early termination

The sequential decoder may kill the correct path at step 6 of the algorithm presented in Section 3.1. In this case, the decoder cannot return the correct codeword, but it continues to perform useless calculations, inspecting many wrong paths in the code tree. It is desired to detect quickly such case and abort decoding.

Observe that for a correct path $u_0^{n-1}$ one has $E_{y_0^{n-1}} \left[ M_b(u_0^{n-1}, y_0^{n-1}) \right] = 0$, and, if the decoder does not make an error, for any other path $\overline{u}_0^{n-1}$ one has $M_b(\overline{u}_0^{n-1}, y_0^{n-1}) < M_b(u_0^{n-1}, y_0^{n-1})$. Figure 4 illustrates the distributions of final path scores for the case of correct and incorrect decoding.

At any phase $\phi$ the probability that $v_0^{\phi-1}$ is a part of the correct path is an increasing function of the score $M_b(v_0^{\phi-1}, y_0^{n-1})$. Therefore, we propose to abort decoding if the score $M_b(v_0^{\phi-1}, y_0^{n-1})$ of a path $v_0^{\phi-1}$ extracted from the PQ at some iteration is below some threshold $T$. This may increase the decoding error probability. The threshold should be selected so that the probability of correct path $u_0^{n-1}$ satisfying the described early termination criterion, i.e. $M_b(u_0^{n-1}, y_0^{n-1}) < T$, should be sufficiently small. This requires one to study the distribution of $M_b(v_0^{\phi-1}, y_0^{n-1})$ at each phase $\phi$. Since this is a difficult problem, we propose to select $T$ based on the distribution of
µ = M_b(u_0^{n-1}, y_0^{n-1}). Namely, we propose to set T to the value of p_{MAP}-quantile of the distribution of µ, where p_{MAP} is the codeword error probability of the MAP decoder.

It follows from (18) that

\[ M_b(u_0^{n-1}, y_0^{n-1}) = \sum_{i=0}^{n-1} \left( \tau(S_i, c_i) - E_{y_i} \left[ \tau(S_i, c_i) \right] \right), \]

where \( S_i = S_0^{(0)}(y_i) \) and \( c_0^{n-1} = u_0^{n-1}A_m \). Assuming that zero codeword was transmitted, one can derive the probability density function of log-likelihood ratios \( S_0^{(0)}(y_i) \), and compute the PDF of \( M_b(u_0^{n-1}, y_0^{n-1}) \) as n-times convolution of the PDFs of \( \tau(S_0^{(0)}(y_i), 0) - E_{y_i} \left[ \tau(S_0^{(0)}(y_i), 0) \right] \).

The decoding error probability \( p_{MAP} \) of the MAP decoder can be estimated by running simulations using the proposed sequential decoding algorithm with very large \( L \). This enables one to derive the termination threshold \( T \), which depends only on channel and code properties.

Figure 5 illustrates the termination threshold for the case of (1024, 736) polar code and AWGN channel with BPSK modulation. It can be seen that the threshold function can be well approximated by

\[ T \approx \begin{cases} \frac{a_C}{\sigma^2} + b_C, & \sigma > t_C \\ \frac{c_C}{\sigma}, & \sigma \leq t_C \end{cases} \]

for some parameters \( a_C, b_C, t_C \), which depend on the code \( C \), and can be obtained by curve fitting techniques.
Table 1: Average decoding complexity of (1024, 512) code, ×10^3 operations

| $E_b/N_0$, dB | Summations | Comparisons |
|--------------|-------------|-------------|
|              | New approach | Path score | New approach | Path score |
| 0.5          | 63.8        | 133         | 90.2         | 218         |
| 1            | 29.9        | 73          | 41.6         | 122         |
| 1.5          | 14          | 32          | 19.3         | 54          |
| 2            | 12.6        | 18          | 11.8         | 31          |

Let $p_{seq}$ and $p_T$ be the codeword error probabilities of the sequential decoding algorithm presented in Section 3.1 and the sequential algorithm, which discards all paths $v_{n-1}^n$ with $M_b(v_{0}^{n-1}, y_{0}^{n-1}) < T$, respectively. Obviously, $p_{MAP} \leq p_{seq} \leq p_T$.

Then one obtains

$$p_T = P\{\mu < T\} + P\{\mathcal{E}|\mu \geq T\} \leq p_{MAP} + p_{seq} \leq 2p_{seq},$$

where $\mathcal{E}$ is the event of sequential decoder error.

Note that the proposed early termination method may abort the decoding even if $M_b(v_{0}^{n-1}, y_{0}^{n-1}) \geq T$, but at some $\phi$ it happens that $M_b(u_{0}^{\phi-1}, y_{0}^{n-1}) < T$, i.e. the probability of sequential decoding error with early termination is not less than $p_T$. However, the below presented numeric results suggest that the performance loss is negligible.

5 Numeric results

![Figure 6: The impact of the score function on the decoder performance and complexity](image-url)

The performance and complexity of the proposed decoding algorithm were investigated in the case of BPSK modulation and AWGN channel. The results are reported for polar codes with 16-bit CRC (polar-CRC) and polar subcodes\(^3\) (ps). The size of the priority queue was set in all cases to $\Theta = L_n$.

Figure 6 illustrates the decoding error probability and average number of iterations performed by the sequential decoder for the case of the proposed path score $M_b$, and the probability-domain implementation of the Niu-Chen stack decoding algorithm\(^4\), which employs path score

$$M_1(v_{0}^{\phi}, y_{0}^{n-1}) = W_{m}^{\phi} \left\{ v_{0}^{\phi} | y_{0}^{n-1} \right\},$$

\(^3\)In order to ensure reproducibility of the results, we have set up a web site containing the specifications of the considered polar subcodes.
Figure 7: Performance and decoding complexity of (2048, 1024) codes

Figure 8: Performance and complexity of sequential decoding with list size adaptation
as well as the min-sum approximation for this score given by

\[ M_2(v_0^\phi, y_0^{n-1}) = \hat{R}(v_0^\phi | y_0^{n-1}). \]

Recall, that the proposed path score is equal to \( M_6(v_0^\phi, y_0^{n-1}) = M_2(v_0^\phi, y_0^{n-1}) + \Psi(\phi) \). Observe that the Niu-Chen algorithm was shown to achieve exactly the same performance as the Tal-Vardy list decoding algorithm, provided that the size of the priority queue \( \Theta \) is sufficiently high.

It can be seen that employing score \( M_2 \) results in a marginal performance loss, but significant reduction of the average number of iterations performed by the decoder. This is due to existence of multiple paths \( v_0^{n-1} \) with low probability \( W_m^{(n)} \{v_0^{n-1} | y_0^{n-1}\} \), which add up (see (2)) to non-negligible probabilities \( W_m^{(\phi)} \{v_0^{\phi-1} | y_0^{n-1}\} \). Hence, employing path score \( M_1 \) causes the decoder to inspect many incorrect paths \( v_0^{\phi-1} \). At sufficiently high SNR the most probable vector \( \tilde{v}_0^{n-1} \), given by (5), with high probability satisfies almost all freezing constraints, so that the value given by \( M_2 \) score function turns out to be close to the final path score. This enables the decoder to avoid visiting many incorrect nodes in the code tree.

Even more significant complexity reduction is obtained if one employs the proposed path score \( M_b \). The proposed path score enables one to correctly compare the probabilities of paths \( v_0^{\phi-1} \) of different length \( \phi \). This results in an order of magnitude reduction of the average number of iterations. Observe that the performance of the decoder employing the proposed score \( M_b \) is essentially the same as in the case of score \( M_2 \). Table 1 provides comparison of the average number of arithmetic operations performed by the decoder implementing the proposed approach, as well as the one presented in our prior work (6). It can be seen that the proposed decoding algorithm has substantially lower complexity. This is due to simpler structure of the proposed score function.

Figure 7 presents the performance and average decoding complexity of (2048, 1024) codes. For further comparison, we report also the performance with adaptive list size (ALS) reproduced from (17), performance of a CCSDS LDPC code, and the complexity of the min-sum implementation of the Tal-Vardy algorithm with fixed list size. It can be seen that for the case of a polar code with CRC the performance loss of the sequential decoding algorithm with respect to the Niu-Chen algorithm and list decoding with adaptive list size is more significant than in the case of (1024, 512) code. This loss reflects the above discussed suboptimality of the min-sum decoding algorithm, which is the basis of the proposed approach. However, this performance loss can be easily compensated by employing appropriately designed polar subcodes. It is not clear if the algorithm presented in (17) can be adjusted to decode generic polar subcodes, since it relies on checking CRC of the obtained data vector in order to detect if another decoding attempt with larger list size is needed. It can be also seen that the average number of summation and comparison operations in the case of the proposed decoding algorithm quickly converges to the complexity of the min-sum SC algorithm. Observe also, that polar subcodes under the proposed sequential decoding algorithm with \( L = 32 \) provide the performance comparable to the state-of-the-art LDPC code, and with larger \( L \) far outperform it.

Figure 8 illustrates the performance and complexity of the sequential decoding algorithm with score \( M_b \) with list size adaptation (LSA) method described in Section 4.1. Here list size \( L \) was allowed to grow from 32 to \( L_{max} = 128 \). It was doubled after \( \kappa_0 = 20 \) iterations, such that at least one node was removed at step 6 of the algorithm described in Section 3.1. It can be seen that the proposed list size adaptation method enables one to achieve essentially the same performance as in the case of non-adaptive algorithm with \( L = L_{max} \), but with lower average complexity. Observe that the proposed implementation of list size adaptation does not require restarting the decoder from scratch, as in the case of the techniques considered in (14, 18).

Figure 9 illustrates the probability distribution of the number of iterations performed by the decoder in the case of correct and incorrect decoding. It can be seen that the distribution of the number of iterations in the event of correct decoding has rather heavy tail, and employing list size adaptation results in higher tail probabilities. In the event of decoding error the number of iterations may become quite high. However, there is very small probability that the decoding would be correct if the decoder has performed more than 20000 iterations. This can be also used for early termination of decoding.
Figure 9: Probability distribution of the number of decoder iterations

Figure 10: Performance and complexity of sequential decoding with early termination
Table 2: Early termination parameters

| Code    | $a_C$  | $b_C$  | $t_C$ |
|---------|--------|--------|-------|
| (1024,736) | -108.27 | 50.84  | 12    |
| (1024, 512) | -116.37 | 121.41 | 43    |

Figure [10] illustrates the performance and complexity of the sequential decoding algorithm with and without the proposed early termination method. The parameters of the early termination threshold function (19) for the considered codes are given in Table 2. It can be seen that for a code with sufficiently high minimum distance $d$ employing the early termination method does not result in any noticeable performance loss. For a code with lower minimum distance there is some performance loss at high SNR. It can be also seen that the early termination condition enables one to significantly reduce the decoding complexity in the low-SNR region, where decoding error probability is high. This can be used to implement HARQ and adaptive coding protocols. Observe also, that the performance and complexity of the decoding algorithms employing the exact and approximate termination threshold functions are very close.

6 Conclusions

In this paper a novel decoding algorithm for polar (sub)codes was proposed. The proposed approach relies on the ideas of sequential decoding. The key contribution is a new path score definition, which enables one to accurately compare paths of different lengths. This enables one to significantly reduce the average number of iterations performed by the decoder with negligible performance loss. The proposed path score function requires only summation and comparison operations, so it is easier to evaluate compared to the original score function. It was also shown that the performance of the proposed decoding algorithm can be substantially improved by recovering previously removed nodes, and resuming their processing with increased list size. The improvement comes at the cost of small increase of the average decoding complexity. The average decoding complexity in the low-SNR region can be reduced by employing the proposed early termination method.

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