ON A NON-ABELIAN POINCARÉ LEMMA

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ABSTRACT. We show that a well-known result on solutions of the Maurer–Cartan equation extends to arbitrary (inhomogeneous) odd forms: any such form with values in a Lie superalgebra satisfying \( d\omega + \omega^2 = 0 \) is gauge-equivalent to a constant,

\[
\omega = gCg^{-1} - dg g^{-1}.
\]

This follows from a non-Abelian version of a chain homotopy formula making use of multiplicative integrals. An application to Lie algebroids and their non-linear analogs is given.

1. Introduction

It is well known that a 1-form with values in a Lie algebra \( g \) (for example, matrix) satisfying the Maurer–Cartan equation

\[
d\omega + \frac{1}{2}[\omega, \omega] = 0
\]

possesses a ‘logarithmic primitive’, locally or for a simply-connected domain:

\[
\omega = -dg g^{-1}
\]

for some \( G \)-valued function, where \( G \) is a Lie group with the Lie algebra \( g \). Here and in the sequel we write all formulas as if our algebras and groups were matrix; it makes the equations more transparent and certainly nothing prevents us from rephrasing them in an abstract form. Therefore the main equation (1) will be also written as

\[
d\omega + \omega^2 = 0.
\]

In the physical parlance, (2) means that the \( g \)-valued 1-form \( \omega \) is a ‘pure gauge’, i.e., the gauge potential \( \omega \) is gauge-equivalent to zero. Mathematically (1) means that the operation (4)

\[
d + \omega
\]

is a flat connection and \( g = g(x) \) in (2) is a choice of a parallel frame by which \( d + \omega \) can be reduced to the trivial connection \( d \).

On the other hand, in the Abelian case (for example, for scalar-valued forms), the Maurer–Cartan equation becomes simply \( d\omega = 0 \),
the equation (2) becomes $\omega = -d \ln g = df$ for a non-zero function $g(x) = e^{-f(x)}$, and the whole statement is just a particular case of the Poincaré lemma for 1-forms.

We show that there is a non-Abelian analog of the Poincaré lemma in full generality. Namely, instead of a one-form we consider an arbitrary odd form $\omega$ with values in a Lie superalgebra $\mathfrak{g}$. ‘Odd’ here means, in the sense of total $\mathbb{Z}_2$-grading (parity) taking account of parity of elements of the Lie superalgebra. The form may very well be inhomogeneous w.r.t. degree, i.e., be the sum of a 0-form, a 1-form, a 2-form, etc., or even be a pseudodifferential form on a supermanifold.

The statement then is as follows. If the odd form $\omega$ satisfies (1) or (3), then it is gauge-equivalent to a constant: there is a $G$-valued form $g$ (necessarily even) such that

$$\omega = gCg^{-1} - dg g^{-1},$$

where $C$ is a constant odd element of the superalgebra $\mathfrak{g}$ satisfying $C^2 = 0$. Here $G$ is a Lie supergroup corresponding to the Lie superalgebra $\mathfrak{g}$.

This is a precise analog of the statement that every closed form is an exact form plus a constant (in a contractible domain). The appearance of the constant $C$ in (5) is a crucial non-trivial feature, distinguishing our statement from the well-known case when $\omega$ is a $\mathfrak{g}$-valued 1-form.

From a ‘connections viewpoint’, the meaning of our statement is that the operation (4), which now can be interpreted as, say, Quillen’s superconnection [9] (it is not an ordinary connection if $\omega$ is not a 1-form), is equivalent, by taking the conjugation with a group element $g$ (an invertible even form taking values in $G$), to $d + C$.

We discuss an application to Lie algebroids and their non-linear analogs. No doubt, the statement has other applications.

We deduce the ‘non-Abelian Poincaré lemma’ from a more general theorem, which may be seen as non-Abelian analog of a homotopy formula for differential forms.

2. ‘Multiplicative direct image’ and a homotopy formula

Consider a (super)manifold $M$ and the direct product $M \times I$ where $I = [0, 1]$. Let $\mathfrak{g}$ be a Lie superalgebra with a Lie supergroup $G$.

Remark 2.1. The Lie superalgebra $\mathfrak{g}$ may be finite-dimensional, but not necessarily. Of course, for infinite-dimensional algebras, interesting for applications, the existence of a corresponding supergroup $G$ and of multiplicative integrals, see below, need to be established in each concrete case. As mentioned above, in the sequel we use the notation mimicking the case of matrix algebras and matrix groups, but everything makes sense in the abstract setting.
We shall define a map
\[ \Omega^{\text{odd}}(M \times I, g) \to \Omega^{\text{even}}(M, G) \]
from odd \( g \)-valued forms on \( M \times I \) to even \( G \)-valued forms on \( M \), corresponding to the projection \( M \times I \to M \). Here a rigorous understanding of an ‘even form with values in some (super)manifold’, say, \( N \) (in our case it is a supergroup \( G \)), is a map of supermanifolds \( \Pi TM \to N \) (in the case under consideration, a map \( \Pi TM \to G \)).

For a matrix (super)group \( G \), a \( G \)-valued form \( g = g(x, dx) \) on an ordinary manifold is an expansion \( g = g_0(x) + dx^a g_a(x) + \ldots \) where the zero-order term is an invertible matrix-function such that \( g_0(x) \) belongs to \( G \) for each \( x \), and the other terms are ‘higher corrections’. Note however that the whole sum \( g(x, dx) \), not only \( g_0(x) \), has to satisfy the equations specifying the group manifold \( G \), so there are relations for the higher terms as well. The whole sum must of course be even in the sense of total parity.

The desired map will be called the multiplicative direct image or multiplicative fiber integral and denoted
\[ T \exp \int_0^1 : \Omega^{\text{odd}}(M \times I, g) \to \Omega^{\text{even}}(M, G) . \]

As the notation suggests, the construction of the multiplicative direct image goes as follows. For a given \( \omega \in \Omega^{\text{odd}}(M \times I, g) \), we consider the decomposition
\[ \omega = \omega_0 + dt \omega_1 \]
where \( t \) is the coordinate on \( I \). The form \( \omega_1 \) is an even \( g \)-valued form on \( M \) depending on \( t \) as a parameter or, alternatively, it can be regarded as a function of \( t \in I \) with values in the Lie algebra (not superalgebra!) \( \Omega^{\text{even}}(M, g) \). It makes sense to consider the Cauchy problem
\[ \begin{cases} g(0) = 1 , \\ \frac{dg(t)}{dt} = -\omega_1 g(t) \end{cases} \]
(the choice of the minus sign is dictated by the geometric tradition). The solution \( g(t) \), which takes values in the group \( \Omega^{\text{even}}(M, G) \), will be denoted
\[ T \exp \int_0^t (-\omega) := g(t) , \]
for \( t \in [0, 1] \), and in particular
\[ T \exp \int_0^1 (-\omega) := g(1) . \]

More detailed notations such as
\[ T \exp \int_0^1 dt (-\omega_1(x, dx, t)) \text{ or } T \exp \int_0^1 (-D(t, dt) \omega(x, dx, t, dt)) \]
are also possible. (In the latter formula $D(t, dt)$ stands for the Berezin integration element w.r.t. the variables $t, dt$. The role of the Berezin integration over the odd variable $dt$ is in isolating the term $\omega_1$.) Equation (9) is the definition of the map (8). We shall need (8) with arbitrary $t$ as well.

Remark 2.2. The multiplicative integrals above are standard multiplicative integrals (cf. [3], [7]) and if so wished they can be expressed as limits of products of exponentials or as series of integrals of time-ordered products. We use the “T exp” notation from among the variety of existing notations. In our case the integrands are 1-forms on $I$ with values in a Lie algebra. The corresponding Lie group $\Omega^{\text{even}}(M, G)$ is the group of the $\Omega(M)$-points of the Lie supergroup $G$ (that is, maps $\Pi TM \to G$).

In the Abelian case, the multiplicative direct image would be the composition of the ordinary direct image $p_* = \int_0^1$ taking forms on $M \times I$ to forms on $M$ (with the same values) and the exponential map.

We shall now establish an analog of the fiberwise Stokes formula for the ordinary direct image

$$d \circ p_* + p_* \circ d = p'_*.$$  

(10)

Here $p'_*, \sigma = \sigma(x, dx, t, 0)|_{t=1}^{t=0}$ is the ‘integral over the fiberwise boundary’ of a form $\sigma(x, dx, t, dt)$ on $M \times I$.

For such an analog, the first term in equation (10) should be replaced by the multiplicative direct image followed by the Darboux derivative

$$g \mapsto \Delta(g) = -dg g^{-1},$$

while the operator $d$ in the second term should be replaced by the non-linear operation of ‘taking curvature’:

$$\omega \mapsto d\omega + \omega^2.$$  

As we shall see, a certain modification arises in the r.h.s. as well. The formula as a whole is slightly more complicated than a naive analog of (10). (At the same time, in hindsight all extra complications allow a geometric interpretation and are geometrically natural.)

**Theorem 1.** Let $\omega = \omega_0 + dt \omega_1 \in \Omega^{\text{odd}}(M \times I, g)$ be an odd $g$-valued form on the direct product $M \times I$. Denote

$$g := \text{T exp} \int_0^1 (-\omega) \in \Omega^{\text{even}}(M, G)$$

and let $g(t)$ be as above, so that $g = g(1)$. Denote

$$\Omega := d\omega + \omega^2,$$

the ‘curvature’. Then

$$-dg g^{-1} + \int_0^1 gg(t)^{-1} \Omega g(t)g^{-1} = \omega_0|_{t=1} - g(\omega_0|_{t=0})g^{-1}$$

(11)
where \( f_0^1 \) in the second term is the ordinary fiberwise integral applied to a \( g \)-valued form on \( M \times I \).

**Remark 2.3.** The form \( \omega \) is odd; obviously, the \( G \)-valued form \( g \) is even, hence the form \(-dg g^{-1}\) is odd. The form \( d\omega + \omega^2 \) is even and the fiberwise integration makes an even form, odd. The term \( \omega_0 \) is odd. Therefore our main equation (11) is an equality between odd \( g \)-valued forms on \( M \).

**Remark 2.4.** The appearance of the conjugation with \( gg(t)^{-1} \) under the integral in the second term at the l.h.s. of (11) and the conjugation with \( g \) applied to \( \omega_0 \) at \( t = 0 \) at the r.h.s. of (11) becomes geometrically transparent if one thinks of \( g(t) \) as of a 'parallel transport' over the fibers \( M \times I \to M \) from \( t = 0 \) to a current \( t \) (see more in Section 5). We have \( gg(t)^{-1} = T \exp \int_0^t (-\omega) \) and it is the transport from current \( t \) to \( t = 1 \), so the whole formula (11) is written at the time \( t = 1 \).

**Proof of Theorem 1.** We shall deduce a formula equivalent to (11), from which (11) will follow by conjugation:

\[
-g^{-1}dg + \int_0^1 g(t)^{-1} \Omega g(t) = g^{-1}(\omega_0|_{t=1})g - \omega_0|_{t=0}.
\]  

(12)

For brevity we shall use the notation \( g_t = g(t) \). Consider the first term in (12). To obtain an expression for it, notice that \( g^{-1}dg = (g_t^{-1}d_x g_t)|_{t=1} \) and \( d_x g_t|_{t=0} = 0 \). By differentiating and using (7) we arrive after a simplification at

\[
\frac{d}{dt} (g_t^{-1}d_x g_t) = -g_t^{-1}d_x \omega_1 g_t.
\]

Therefore

\[
g^{-1}dg = \int_0^1 dt \frac{d}{dt} (g_t^{-1}d_x g_t) = \int_0^1 dt (-g_t^{-1}d_x \omega_1 g_t).
\]  

(13)

Consider now the curvature \( \Omega = d\omega + \omega^2 \). We have

\[
d\omega + \omega^2 = d_x \omega_0 + \omega_0^2 + dt (-d_x \omega_1 + \dot{\omega}_0 + [\omega_1, \omega_0]),
\]

by a direct calculation (where the dot denotes the time derivative). Therefore

\[
\int_0^1 g_t^{-1} \Omega g_t = \int_0^1 dt (-g_t^{-1}d_x \omega_1 g_t) + \int_0^1 dt \left(g_t^{-1}(\dot{\omega}_0 + [\omega_1, \omega_0])g_t\right).
\]  

(14)

It follows by combining (13) and (14) that

\[
-g^{-1}dg + \int_0^1 g_t^{-1} \Omega g_t = \int_0^1 dt \left(g_t^{-1}(\dot{\omega}_0 + [\omega_1, \omega_0])g_t\right).
\]  

(15)
It remains to identify the r.h.s. of (15). To this end, consider \( g^{-1}_t \omega_0 g_t \). By differentiating and taking into account (7) we obtain, after a simplification, that

\[
\frac{d}{dt} \left( g^{-1}_t \omega_0 g_t \right) = g^{-1}_t (\dot{\omega}_0 + [\omega_1, \omega_0]) g_t .
\]  

(16)

This is exactly what we are looking for. Combining now (15) with (16) we arrive at

\[
-g^{-1} dg + \int_0^1 g^{-1} \Omega g_t = \int_0^1 dt \frac{d}{dt} \left( g^{-1}_t \omega_0 g_t \right) ,
\]  

(17)

which gives (12) as desired, since \( g_1 = g \) and \( g_0 = 1 \). To get (11) we apply the conjugation by the element \( g \), and the theorem is proved. □

Theorem 1 implies an analog of the chain homotopy formula for homotopic maps \( M \to N \). Suppose \( F : M \times I \to N \) is a homotopy between maps \( f_0, f_1 : M \to N \), so that \( F(x, t) = f_t(x) \) for \( t = 0, 1 \). Consider an odd form \( \omega \in \Omega^{\text{odd}}(N, g) \) and take its pull-back \( F^* \omega \in \Omega^{\text{odd}}(M \times I, g) \). By applying equation (11) to \( F^* \omega \) and noting that ‘taking curvature’ commutes with pull-backs, we arrive at the following statement.

**Corollary 2.1 (Non-Abelian algebraic homotopy).** The pull-backs along homotopic maps \( f_0, f_1 : M \to N \) of an odd \( g \)-valued form \( \omega \in \Omega^{\text{odd}}(N, g) \) are related by the formula

\[
f_1^* \omega - g (f_0^* \omega) g^{-1} = -dg g^{-1} + \int_0^1 g g(t)^{-1} F^*(d\omega + \omega^2) g(t) g^{-1}
\]  

(18)

where \( g = g(1) \) and

\[
g(t) = T \exp \int_0^t (-F^* \omega) .
\]  

(19)

Here \( F \) is a given homotopy, \( F(x, t) = f_t(x) \).

In particular, for flat forms, the pull-backs along homotopic maps are gauge-equivalent:

\[
f_1^* \omega = g (f_0^* \omega) g^{-1} - dg g^{-1},
\]  

(20)

where ‘flat’ means vanishing curvature: \( d\omega + \omega^2 = 0 \). □

### 3. Non-Abelian Poincaré lemma

The above can be applied for obtaining a non-Abelian version of the Poincaré lemma. We can use Corollary 2.1 or argue directly, as follows. Let \( \omega \) be an odd \( g \)-valued form on a contractible supermanifold, for example on a star-shaped domain \( U \subset \mathbb{R}^{n|m} \). Suppose it satisfies the Maurer–Cartan equation:

\[
d\omega + \omega^2 = 0 .
\]  

(21)
Consider a contracting homotopy $H : M \times I \to M$, for example, the map $H : U \times I \to U$ sending $(x, t)$ to $tx$. Take the pull-back $H^* \omega$ of $\omega$ and apply to it the main formula (11). We arrive at

$$-dg^{-1} + 0 = (H^* \omega)|_{t=1, dt=0} - g((H^* \omega)|_{t=0, dt=0})g^{-1}$$

where

$$g = T \exp \int_0^1 (-H^* \omega).$$

Noticing that $(H^* \omega)|_{t=1, dt=0} = \omega$ and $(H^* \omega)|_{t=0, dt=0} = i^* \omega$ where $i$ is the inclusion of the base point to $M$, we get simply

$$-dg^{-1} = \omega - g(i^* \omega)g^{-1}.$$ 

Here $i^* \omega$ is just an odd element of the Lie superalgebra $\mathfrak{g}$.

We have proved the following statement.

**Theorem 2** (Non-Abelian Poincaré Lemma). On a contractible supermanifold, an odd $\mathfrak{g}$-valued form $\omega \in \Omega^\text{odd}(M, \mathfrak{g})$ satisfying the Maurer–Cartan equation (21) is gauge-equivalent to a constant:

$$\omega = gCg^{-1} - dg^{-1},$$

where $C \in \mathfrak{g}_1$ is an odd element of the Lie superalgebra $\mathfrak{g}$ and a ‘multiplicative primitive’ $g \in \Omega^\text{even}(M, G)$ of the form $\omega$ is given by (22). □

In particular, the formula for a ‘multiplicative primitive’ of an odd form satisfying the Maurer–Cartan equation on a star-shaped domain of $\mathbb{R}^{n|m}$, is

$$g = T \exp \int_0^1 (-D(t, dt) \omega(tx, dt x + t dx))$$

$$= T \exp \int_0^1 \left(-dt x^a \frac{\partial \omega}{\partial dx^a}(tx, t dx)\right),$$

very similar to the classical formula for a primitive of a closed form on $\mathbb{R}^n$ or $\mathbb{R}^{n|m}$.

Since gauge transformations preserve the Maurer–Cartan equation, the constant element $C \in \mathfrak{g}_1$ must also satisfy it. Hence the following holds.

**Proposition 3.1.** The constant $C$ (an odd element of the Lie superalgebra $\mathfrak{g}$) in (23) satisfies

$$C^2 = 0 \quad (\text{or } [C, C] = 0),$$

i.e., $C$ is a homological element of the Lie superalgebra $\mathfrak{g}$. □

If we split $\omega$ and other forms as $\omega = \omega_0 + \omega_+$, where $\omega_0 = \omega|_M$ or more precisely $\omega_0 = \pi^*i^* \omega = (\text{here } \pi: \Pi TM \to M$ is the projection...
and \( i : M \to TM \), the zero section), then equation (23) becomes
\[
\omega_0 = g_0 C g_0^{-1} ,
\]
(26)
\[
\omega_+ = (g C g^{-1})_+ - dg g^{-1} .
\]
(27)
In general, \( \omega_0 \) need not be constant; equation (26) implies \( \omega_0 \) being ‘covariantly constant’ w.r.t. to a flat connection: 
\[
d\omega_0 + [\theta, \omega_0] = 0
\]
where \( \theta = -dg_0 g_0^{-1} \).

The constant \( C \) is not unique. Our proof of Theorem 2 gives \( C \) as the value of \( \omega \) at the base point \( x_0 \) of a contractible manifold \( M \). In fact, it can be replaced by any constant conjugate to the value of \( \omega \) at \( x_0 \) w.r.t. the adjoint action of \( G \). It is easy to deduce the following statement.

**Corollary 3.1.** On a contractible manifold, two odd forms satisfying the Maurer–Cartan equation are gauge-equivalent if and only if their values at the base point are conjugate. 

Hence there is a one-to-one correspondence between the gauge equivalence classes of ‘flat forms’ (or flat \( G \)-superconnections) on a contractible manifold and the \( G \)-adjoint orbits of the homological elements in \( g \). More properly this should be stated not as a bijection of sets, but as an isomorphism of the corresponding functors so as to allow arbitrary families.

It is possible to have non-trivial gauge equivalences between different constants, as well as self-equivalences. So the multiplicative primitive \( g \) in (23) is not defined uniquely even for a fixed constant \( C \). If
\[
\omega = h C' h^{-1} - dh h^{-1} ,
\]
(28)
for some other \( C' \in g_1 \) and \( h \in \Omega^{\text{even}}(M, G) \), then there is a relation
\[
C = k C' k^{-1} - dk k^{-1}
\]
(29)
where \( k = g^{-1} h \). The form \( k \in \Omega^{\text{even}}(M, G) \) satisfies
\[
[C, dk k^{-1}] + (dk k^{-1})^2 = 0 .
\]
(30)
Suppose there is a gauge self-equivalence
\[
C = g C g^{-1} - dg g^{-1} ,
\]
(31)
for a homological element \( C \). Equation (31) can be re-written as
\[
dg = -C g + g C ,
\]
(32)
with the clear intuitive meaning of the ‘frame’ \( g \) being parallel w.r.t. the (constant) ‘flat connection’ \( C \).

In the classical (Abelian) situation, the usual formulation of the Poincaré lemma is for forms of fixed positive degree. Forms of degree zero and constants appear as an exception unless inhomogeneous forms are treated. In the non-Abelian case, inhomogeneous \( g \)-valued forms do appear naturally. As soon as we consider them, we are forced to
consider constants arising in (23). These constants play an important role in examples as we shall see now.

4. Examples

In this section we change our notation and denote by $L$ the Lie superalgebra denoted above by $\mathfrak{g}$. (We shall use $\mathfrak{g}$ for a different object.)

In the following examples, the Lie superalgebra $L$ will be the algebra of vector fields $\mathfrak{x}(N)$ on a (super)manifold $N$. In particular, $N$ can be a vector space; then the Lie superalgebra $L = \mathfrak{x}(N)$ carries an extra $\mathbb{Z}$-grading corresponding to degree in the linear coordinates on $N$. It should not be confused with parity. More generally, $N$ can be an arbitrary graded manifold, i.e., a supermanifold with an extra $\mathbb{Z}$-grading in the structure sheaf, independent of parity in general [16]. We refer to such a $\mathbb{Z}$-grading as weight and denote it $w$.

All the considerations in the previous sections remain valid in the graded case.

Indeed, suppose that the Lie superalgebra $L$ is $\mathbb{Z}$-graded. Total weight of the elements of $\Omega(M, L)$ is the sum of ordinary degree of forms on $M$ and weight on $L$. For the operator $d + \omega$ to be homogeneous, we assume that the weight $w(\omega)$ of the form $\omega$ equals +1 (this does not mean that $\omega$ is a 1-form!). Recalling the construction of the multiplicative direct image on $M \times I$ for $\omega = \omega_0 + dt \omega_1$ in Section 2, we see that $w(\omega_1) = 0$. Hence the multiplicative integral there and all the gauge transformations defined with its help have weight zero, i.e., they all preserve weights.

Now let us turn to examples. We use the notions of Lie algebroid theory, for which the standard and encyclopedic source is Mackenzie’s book [6] (see also [8]).

Example 4.1. Consider the Atiyah algebroid of a principal $G$-bundle $P \rightarrow M$, which is a transitive Lie algebroid over $M$. See [6]. (Here $G$ is a Lie group, nothing to do with what it was in the previous sections.) It is defined as $TP/G$ and it inherits the structure of a vector bundle over $M$. There is an epimorphism of vector bundles $TP/G \rightarrow TM$, which is the anchor in the Lie algebroid structure. Let us reverse parity in the fibers and consider the supermanifold $\Pi T P / G$. We can consider it as a fiber bundle over $\Pi T M$,

$$\Pi T P / G \rightarrow \Pi T M.$$  

The standard fiber of (33) can be identified with $\Pi \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. The transition functions have the form

$$\xi_\alpha = g_{\alpha \beta} \xi_\beta g^{-1}_{\alpha \beta} - dg_{\alpha \beta} g^{-1}_{\alpha \beta},$$  

if $g_{\alpha \beta}$ is the cocycle defining the principal bundle $P \rightarrow M$. They are linear in $\xi, dx$, but affine in $\xi$ alone. Here $\xi$, or $\xi_\alpha$ in a particular local trivialization, belongs to $\Pi \mathfrak{g}$. Hence (33) is an affine bundle over $\Pi T M$. 

The Lie algebroid structure of the Atiyah algebroid is encoded in the following homological vector field on $\Pi TP/G$:

$$Q = dx^a \frac{\partial}{\partial x^a} + \frac{1}{2} \xi^i \xi^j C^k_{ji} \frac{\partial}{\partial \xi^k}. \quad (35)$$

Here $x^a$ are local coordinates on $M$ and $\xi^i$ are linear coordinates on $\Pi g$. The tensor $C^k_{ij}$ gives the structure constants of the Lie algebra $g$. One may consider the second term in (35) as a constant element $C$ of the Lie superalgebra $L = \mathcal{X}(\Pi g)$ of the vector fields on the supermanifold $\Pi g$, so $Q$ has the appearance

$$Q = d + C \quad \text{where} \quad C \in \mathcal{X}(\Pi g). \quad (36)$$

Note that $C$ is odd and has weight 1. It satisfies $C^2 = 0$. In a different language, $C$ is the differential of the standard cochain complex of $g$ usually denoted as $\delta$ (the Chevalley–Eilenberg or Cartan differential).

It is a remarkable fact, directly verifiable, that the decomposition (36) survives transformations of the form (34).

**Example 4.2.** Let $E \rightarrow M$ be now an arbitrary transitive Lie algebroid, see [6]. The anchor map $E \rightarrow TM$ is epimorphic, so we can again consider it as a fiber bundle. One can check that it is an affine bundle. Consider the vector bundles over $M$ with reversed parity in the fibers. We have the affine bundle

$$\Pi E \rightarrow \Pi TM. \quad (37)$$

If we denote coordinates in the fiber of (37) by $\xi^i$, and local coordinates on $M$ by $x^a$ as above, so that on $\Pi TM$ the coordinates are $x^a, dx^a, \xi^i$, the changes of coordinates are

$$\begin{cases} 
 x^a = x^a(x'), \\
 dx^a = dx^a' \frac{\partial x^a}{\partial x'^a}, \\
 \xi^i = dx^a T^i_a(x') + \xi^i T^i_v(x'),
\end{cases} \quad (38)$$

with some matrices $T^i_a(x')$ and $T^i_v(x')$. The homological vector field $Q$ on $\Pi E$ defining the structure of a Lie algebroid over $M$ has the form

$$Q = d + \frac{1}{2} \left( \xi^i \xi^j Q^k_{ji}(x) + 2 \xi^i dx^a Q^k_{ai}(x) + dx^a dx^b Q^k_{ba}(x) \right) \frac{\partial}{\partial \xi^k}. \quad (39)$$

in local coordinates, with $d = dx^a \partial / \partial x^a$. Or:

$$Q = d + \omega, \quad (40)$$

where

$$\omega = \frac{1}{2} \left( \xi^i \xi^j Q^k_{ji}(x) + 2 \xi^i dx^a Q^k_{ai}(x) + dx^a dx^b Q^k_{ba}(x) \right) \frac{\partial}{\partial \xi^k}. \quad (41)$$

can be regarded as a (local) form on $M$ with values in the Lie superalgebra $L = \mathcal{X}(\Pi V)$. Here $V$ is the standard fiber of $E \rightarrow TM$. Note
that the form $\omega$ is odd and inhomogeneous w.r.t. ordinary degree. However, it is homogeneous of weight +1 if weight is counted as the sum of form degree and grading of vector fields on $\Pi V$. Claim: the form $\Omega$ is gauge-equivalent to a constant. This amounts to saying that by an appropriate affine transformation of the coordinates $\xi$,

$$\xi = \eta \cdot A(x) + \beta$$

where the matrix $A(x)$ depends only on $x$ and $\beta$ is a 1-form, one can transform the vector field $Q$ on $\Pi E$ to the form

$$Q = d + C \quad \text{where} \quad C = \frac{1}{2} \eta^i \eta^j C_{ij}^k \frac{\partial}{\partial \eta^k} \in \mathfrak{X}(\Pi V), \quad (42)$$

with constant $C_{ij}^k$. This follows by a direct application of Theorem 2.

That the transformation is affine, follows from the preservation of weights and can be seen from an explicit construction of the multiplicative integrals. In particular, $C^2 = 0$, therefore the vector space $V$ is endowed with a structure of a Lie algebra. This encapsulates a substantial part of the transitive Lie algebroid theory \[6\, \text{Ch. 8}\].

The previous example prompts an immediate analogy. A few words should be said before we pass to it.

Recall that a $Q$-manifold is an arbitrary supermanifold endowed with a homological vector field. $Q$-manifolds should be regarded, together with Poisson manifolds and Schouten (odd Poisson) manifolds, as one of the three equally important non-linear generalizations of Lie algebras. Vaintrob noted \[14\] that a Lie algebroid structure on a vector bundle $E$ is very efficiently described by a homological vector field on the parity-reversed total space $\Pi E$. This approach was further developed in particular in \[10\], \[16\], \[11\], and \[15\]. This is what we used above. (We use the notation of \[16\].) For a Lie algebroid the corresponding field $Q$ has weight +1 w.r.t. the linear coordinates in the fibers. Dropping this restriction leads to objects like strongly-homotopy Lie (or $L_\infty$) algebras and algebroids. On the other hand, constructions such as the ‘cotangent construction of the Drinfeld double’ for Lie bialgebroids \[10\] (compare also \[5\]) take us out of the world of vector bundles. Prompted by that, it is proper to replace a lacking linear structure by weight and consider graded manifolds instead of vector bundles \[16\]. In particular, a non-negatively graded $Q$-manifold with $w(Q) = +1$ is the closest analog of a Lie algebroid and should be regarded as the ‘non-linear version’ of such.

**Example 4.3** ("Non-linear transitive Lie algebroids"). Consider a fiber bundle

$$p: E \to \Pi TM \quad (43)$$

A Lie algebra structure has precisely three geometric manifestations: as a linear Poisson bracket on $\mathfrak{g}^*$; as a linear Schouten bracket on $\Pi \mathfrak{g}^*$; and as a quadratic homological vector field on $\Pi \mathfrak{g}$.
in the category of graded manifolds. Here $E$ is a non-negatively graded manifold (in particular, a supermanifold) $M$ is an ordinary supermanifold, with trivial grading, and $\Pi T M$ is considered with the standard vector bundle grading. (More pedantically it should be denoted $\Pi T[1] M$.) Suppose there is a homological vector field $Q \in \mathfrak{X}(E)$ of weight +1 such that $Q$ on $E$ and $d$ on $\Pi T M$ are $p$-related. Then in suitable local coordinates $Q$ has the form

$$Q = dx^a \frac{\partial}{\partial x^a} + Q^i(x, dx, y) \frac{\partial}{\partial y^i},$$

(44)

where $x^a$ are local coordinates on $M$, $dx^a$ are their differentials, and $y^i$ are coordinates in the fiber over $\Pi T M$. The changes of coordinates on $E$ have the form

$$\begin{cases}
x^a = x^a(x'), \\
dx^a = dx^a' \frac{\partial x^a}{\partial x^a'}, \\
y^i = y^i(x', dx', y'),
\end{cases}$$

(45)

The appearance of $d$ as the first term in (44) follows from the condition that $Q$ and $d$ are $p$-related. We have $w(x^a) = 0$, $w(dx^a) = 1$, and $w(y^i) = w^i > 0$ are some positive integers. The coordinate transformations (45) are homogeneous polynomials in $dx^a'$, $y^i'$. The coefficients $Q^i(x, dx, y)$ in (44) have weights $w^i + 1$ respectively. Compare with (39).

The whole structure consisting of the fiber bundle $E \to \Pi T M$ and the field $Q$ with these properties should be regarded as a transitive non-linear Lie algebroid. The bundle projection $p: E \to \Pi T M$ plays the role of the anchor. A construction of this type with an extra restriction — see below — was put forward in [13, 4]. They introduced $Q$-bundles $p: E \to B$ where both $E$ and $B$ are $Q$-manifolds and the projection is a $Q$-morphism (the respective vector fields are $p$-related) with a condition of ‘local triviality’ in the following strong sense. The standard fiber $F$ is a $Q$-manifold and there is a bundle atlas where in local trivializations the homological vector field $Q_E$ on $E$ takes the form of the sum

$$Q_E = Q_B + Q_F$$

(46)

where $Q_B$ and $Q_F$ are the homological vector fields on $B$ and $F$; the transition functions are such that they respect the decomposition $[46]$.

The main case is when the $Q$-manifold $B$ equals $\Pi T M$ for some $M$. So it is a transitive non-linear Lie algebroid in our sense subject to the condition of the existence of a bundle atlas for $E \to \Pi T M$ with the described properties. We shall refer to that, in particular to equation (46),

2Comparing with the previous example, $E$ now stands for what was $\Pi E$.

3There are more conditions to be mentioned: the total space and the base are graded manifolds and the homological fields have weights +1 — in fact, in [13, 4] it is assumed that grading induces parity, which is absolutely unnecessary; and some Lie group of transformations preserving $[10]$ is fixed as a part of data.
as to a “Strobl (or Kotov–Strobl) gauge”. Our Theorem 2 therefore implies that every transitive non-linear Lie algebroid admits a Strobl gauge. That means that it is possible to transform coordinates \( y^i \) to \( z^j \) by a homogeneous transformation depending on \( x^a, dx^a \) as parameters in each bundle chart so that the vector field \( (44) \) becomes

\[
Q = dx^a \frac{\partial}{\partial x^a} + Q^i(z) \frac{\partial}{\partial z^i}
\]

(no dependence on \( x^a, dx^a \) in the second term). The graded Lie super-algebra \( L \) here is the algebra of all graded vector fields \( \mathfrak{X}(F) \) on the standard fiber \( F \). In particular, we conclude that constructions of [4] are valid for arbitrary transitive non-linear Lie algebroids, without extra restrictions.

An appearance of fiber bundles of the form \( E \to \Pi T M \) may look artificial. It helps to put it into a broader perspective as follows. Consider an arbitrary non-negatively graded \( Q \)-manifold \( E \) where \( w(Q) = +1 \). As noted before, it should be regarded as a general non-linear Lie algebroid. At the first glance, there is no fiber bundle there. Recall however that any non-negatively graded manifold \( E \) gives rise to a finite tower of fibrations [16]

\[
E = E_N \to E_{N-1} \to \ldots \to E_2 \to E_1 \to E_0,
\]

where functions on \( E_0 \) have weight 0, \( E_1 \to E_0 \) is a vector bundle, and \( E_{k+1} \to E_k \) for higher \( k \) are affine bundles. This can be assembled into a single fiber bundle

\[
E \to M,
\]

where \( M = E_0 \), the coordinates on the standard fiber have positive weights and the transition functions are homogeneous polynomials. The supermanifold \( M \) is embedded into \( E \) as a ‘zero section’. The homological vector field \( Q \in \mathfrak{X}(E) \) in bundle coordinates has the form

\[
Q = Q^a(x, y) \frac{\partial}{\partial x^a} + Q^i(x, y) \frac{\partial}{\partial y^i},
\]

where \( x^a \) are local coordinates on the base \( M \), and \( y^i \) are coordinates on the fiber. We define a map

\[
a^*: \ E \to \Pi T M
\]

by the formula \( a^*(x^a) = x^a, a^*(dx^a) = Q^a(x, y) \) and call it the anchor for \( E \). (Well-defined because we have a bundle, so the transformation of the coordinates is of the form \( x^a = x^a(x'), y^i = y^i(x', y') \).) It is a bundle map over \( M \), and a \( Q \)-morphism (by a direct check using \( Q^2 = 0 \)). In the case of ordinary Lie algebroids it coincides with the usual anchor after the parity reversion. If the anchor \( E \to \Pi T M \) is a surjective submersion, we call the non-linear Lie algebroid \( E \) transitive, as in the ordinary case [6], and we arrive at the setup of Example 4.3.
Remark 4.1. A tautological ‘anchor’ makes sense for any $Q$-manifold $M$ without assumption of grading: it is the vector field $Q$ itself considered as a map $M \to \Pi TM$, which is a $Q$-morphism. Of course such an ‘anchor’ carries information about all the algebraic structures contained in the vector field $Q$, so the name becomes a bit arbitrary.

5. Discussion

We have studied inhomogeneous odd forms on a supermanifold $M$ taking values in a Lie superalgebra $\mathfrak{g}$ and showed that if they satisfy the Maurer–Cartan equation $d\omega + \omega^2 = 0$, then, on a contractible supermanifold, they are gauge-equivalent to constants:

$$\omega = g C g^{-1} - dg g^{-1},$$

(52)

where $g$ is an even $G$-valued form and $C \in \mathfrak{g}_1$ such that $C^2 = 0$ (Theorem 2). Everything works also in the case of extra $\mathbb{Z}$-gradings on the supermanifold $M$ and the algebra $\mathfrak{g}$. The statement follows from a more general claim concerning homotopy properties of $\mathfrak{g}$-valued pseudodifferential forms (Theorem 1 and Corollary 2.1). Our statement is particularly useful in examples related with Lie algebroids and their generalizations, but there is little doubt about its broader significance.

The main novelty compared with the well-known case of 1-forms is the possibility to treat inhomogeneous (or pseudodifferential) odd forms and the appearance of a constant in (52), both features crucial for applications.

We may note that there is a paper [1] concerned with a certain ‘non-Abelian de Rham theory’ in a sense suggested there; it may be interesting to explore whether there are any relations between it and our work.

The results of the present paper can be formulated in a more abstract algebraic framework. (This was not our goal, since we were mainly interested in geometric applications such as to Lie algebroids and their non-linear analogs.) It is possible to replace the odd forms taking values in a Lie superalgebra by odd elements of some abstract differential Lie superalgebra (possibly of an $L_\infty$-algebra). The notion of gauge equivalence carries over to this case verbatim, and forms on a cylinder $M \times I$ are modelled by the tensor product with the algebra generated by $t$ and $dt$.

After this paper was finished, we learned about the “secret” work by Schlessinger and Stasheff [12], still unpublished, and the recent preprint of Chuang and Lazarev [2], in which such an abstract setting is employed. Stasheff pointed out to us that equation (20) in Corollary 2.1 reminded him of a more formal formula in [12]. The Main Homotopy Theorem of [12] states that ‘homotopic Maurer–Cartan elements are (gauge-)equivalent’. A proof of that for pro-nilpotent dg Lie algebras

\footnote{Then a geometric approach based on homological fields should be used.}
is given in [2]. This statement is an abstract algebraic analog of the zero curvature case of our Corollary 2.1. The method used in [2] — formal path-ordered integrals with values in a completion of the universal enveloping algebra — is remarkably close to ours.

In conclusion, we wish to comment on the geometric meaning of the integrals arising above.

For an ordinary 1-form, one can integrate it over paths, for a closed form obtaining a function of the upper limit, which is a primitive of the form. Similarly, for a 1-form $\omega \in \Omega^1(M, \mathfrak{g})$ with values in a Lie (super)algebra, one can take multiplicative integrals over paths, which represent the parallel transport w.r.t. the corresponding connection. In the flat case this gives a $G$-valued function $g$ on $M$ such that $\omega = \Delta(g)$. The picture seems to break down when a 1-form is replaced by a form of higher degree, or even more, by an inhomogeneous or pseudodifferential form. The integrals over $t$ appearing in the proof of the Poincaré lemma seem to lose an immediate geometric meaning — how can one integrate something that is not a 1-form over a path? (Here there is no difference between the Abelian case with ordinary integrals and the non-Abelian case of multiplicative integrals.) The key of course is that we deal with a family of paths parametrized by endpoints, not individual paths. Even if what we integrate is not a 1-form, the integrals over paths in a family do make sense, giving a form, rather than a function, on the space of parameters (which is our supermanifold $M$). In the non-Abelian case the result of multiplicative integration is an element of $\Omega^{\text{even}}(M, G)$. Therefore (24) and similar formulas have a geometric interpretation as the parallel transports over paths in a family with base $M$ with respect to a flat Quillen superconnection $d + \omega$. Such a parallel transport is described by a $G$-valued form on $M$.

It comes about naturally to drop the condition of flatness and consider the ‘universal’ family, i.e., the space of all paths on $M$ (of some appropriate class, such as, e.g., piecewise smooth). It is a path groupoid $\text{Paths}(M)$. More precisely, consider a principal $G$-bundle $P \to M$ over $M$, where $G$ is a supergroup and endowed with Quillen’s superconnection $\nabla$. Then the parallel transport corresponding to $\nabla$ is an even form on the path groupoid $\text{Paths}(M)$ on $M$ (arrows are paths up to reparametrization) with values in the transitive Lie groupoid $\Phi(P)$ associated with the principal bundle $P \to M$, i.e., it is a map

$$\tau : \Pi T \text{Paths}(M) \to \Phi(P),$$

which is a morphism of Lie groupoids.

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5It may be natural to consider bundles over $\Pi TM$ from the start.
comments on style as well as mathematics. I am much grateful to A. Lazarev for attracting my attention to his paper [2].

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