Universal algebraic relaxation of velocity and phase in pulled fronts generating periodic or chaotic states

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We investigate the asymptotic relaxation of so-called pulled fronts propagating into an unstable state, and generalize the universal algebraic velocity relaxation of uniformly translating fronts to fronts that generate periodic or even chaotic states. A surprising feature is that such fronts also exhibit a universal algebraic phase relaxation. For fronts that generate a periodic state, like those in the Swift-Hohenberg equation or in a Rayleigh-Bénard experiment, this implies an algebraically slow relaxation of the pattern wavelength just behind the front, which should be experimentally testable.

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Many systems, when driven sufficiently far from equilibrium, spontaneously organize themselves in coherent or incoherent patterns [1]. While the ‘‘selection’’ of a final state pattern can be determined by a variety of dynamical mechanisms, or even the competition thereof, the final state selection by a propagating ‘‘pulled’’ front turns out to be remarkably simple and robust. So-called pulled fronts propagate into a linearly unstable state and are almost literally being ‘‘pulled along’’ by the leading edge of the profile whose dynamics is governed by the linearization about the unstable state [2–5]: Their asymptotic speed is equal to the linear spreading speed \( v^* \) of linear perturbations about the unstable state.

Recently, it was discovered that non-pattern-generating pulled fronts, which asymptotically are uniformly translating, relax to their asymptotic velocity and shape very slowly ‘‘pulled along’’ by the leading edge of the profile whose wavelength just behind a coherent pattern relaxes as \( 1/t \) to its asymptotic value: it is given by

\[
\Lambda(t) = 2\pi \left| \frac{v^* + \dot{X}(t)}{\Omega^* + \dot{\Gamma}(t)} \right| + \mathcal{O}\left(\frac{1}{t^2}\right),
\]

with the frequency \( \Omega^* \) also given below. As \( \dot{X}(t) \) and \( \dot{\Gamma}(t) \) are explicitly given by Eqs. (2) and (3), this immediately yields \( \Lambda(t) \) up to order \( t^{-3/2} \) in time.

Before summarizing our derivation, we explain what we mean by velocity and phase for the various types of fronts.

Uniformly translating pulled fronts. The simplest types of fronts are those for which the dynamical field \( \phi(x,t) \) asymptotically approaches a uniformly translating profile \( \Phi(x-v^*t) \), as happens, e.g., in the celebrated nonlinear diffusion equation \( \partial_t \phi = \partial_x^2 \phi + \phi - \phi^3 \) for fronts propagating into the unstable \( \phi = 0 \) state. If we define level curves as the lines in an \( x,t \) diagram where \( \phi(x,t) \) has a particular value, we can define the velocity \( v(t) \) as the slope of a level curve. For uniformly translating fronts, \( q^* = 0 \) = \text{Im} \( D \); Eq. (2) then reduces to the expression derived for uniformly translating fronts in [5]. The remarkable point is that the expression for \( v(t) \) is in this case completely independent of which level curve one traces. Moreover, it was shown in [5] that the nonlinear front region is slaved to the leading edge of the front whose velocity relaxes according to Eq. (2). This results in

\[
\phi(x,t) = \Phi_{v^*}(\xi x) + \mathcal{O}(t^{-2}), \quad \xi x \ll \sqrt{t},
\]
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Coherent pattern generating fronts. As an example of coherent pattern generating fronts, we consider the so-called Swift-Hohenberg (SH) equation

$$\xi_x = x - v^* t - X(t),$$

where $\Phi_x(\xi)$, $\xi = x - vt$ solves the ordinary differential equation (ODE) for a front propagating uniformly with velocity $v$. $v(t)$ in Eq. (5) is the instantaneous velocity of the front, and the frame $\xi_x$ is shifted by the time dependent quantity $X(t)$. Since the collective coordinate $X(t)$ diverges as $\ln t$ for large $t$ according to Eq. (2), the difference between $\xi_x$ and a uniformly translating frame is crucial; only in the former can we follow the relaxation. Uniformly translating fronts have no phase, hence all terms in Eq. (3) vanish identically.

Incoherent or chaotic fronts. The third class we consider consists of fronts which leave behind chaotic states. They occur in some regions of parameter space in the cubic complex Ginzburg-Landau equation [10] or in the quintic extension (QCGL) [11] that we consider here,

$$\partial_t A = eA + (1 + i C_2) \partial_x^2 A + (1 + i C_3)|A|^2 A - (1 - i C_3)|A|^4 A.$$

Figure 1(b) shows an example of a pulled front in this equation. Level curves in a space-time diagram can now also both start and end. If we calculate the velocity from the slope of the uppermost level curve, then its average value is again given by Eq. (2) [12], but the oscillations can be quite large. However, our analysis confirms what is already visible in Fig. 1(b), namely, that even a chaotic pulled front becomes more coherent the further one looks into the leading edge of the profile. Indeed we will see that in the leading edge where $|A| \ll 1$ the profile is given by an expression reminiscent of Eq. (8).

$$A(x,t) \approx e^{-i \Omega^* t - i \Gamma(t)} e^{i k x} \psi(\xi_x), \quad 1 \ll \xi_x \ll \sqrt{t}.$$

The fluctuations about this expression become smaller the larger $\xi_x$.

In Figs. 1(c) and 2(c) we show as an example results of our simulations of the SH equation (7) and the QCGL (9). They fully confirm our predictions (2) and (3) for the asymptotic average velocity and phase relaxation.
We now summarize how these results arise for the case of a single (scalar) equation. The extension to the case of coupled equations can be done along the lines of [5].

Calculation of the asymptotic parameters. We first briefly summarize how the linear spreading velocity \( v^* \) and the associated parameters \( \lambda^* \) etc. arise [13,5]. After linearization about the unstable state, the equations we consider can all be written in the form \( \partial_t \phi = \mathcal{L}(\partial_x, \partial^2_x, \cdots) \phi \). For a Fourier mode \( e^{-j\omega t + i k x} \), this yields the dispersion relation \( \omega(k) \). The linear spreading velocity \( v^* \) of steep initial conditions is then obtained by a saddle point analysis of the Green’s function \( G(t) \) of these equations. In the asymptotic frame \( \xi = x - v^* t \), \( G(\xi,t) \) becomes

\[
G(\xi,t) = \int \frac{dk}{2\pi} e^{-i\Omega(k)t + i k \xi} = e^{i k^* \xi - i \Omega^* t} \frac{e^{-i (\xi^2/4Dt)}}{\sqrt{4\pi Dt}}
\]

for large times. Here \( \Omega(k) = \omega(k) - v^* k \), and

\[
\frac{d\Omega(k)}{dk} \bigg|_{k^*} = 0, \quad \text{Im} \Omega(k^*) = 0, \quad D = \frac{i d^2 \Omega(k)}{2dk^2} \bigg|_{k^*}.
\]

The first equation in Eq. (12) is the saddle point condition, while the second one expresses the self-consistency condition that there is no growth in the comoving frame. These equations straightforwardly determine \( v^*, k^* = q^* + i\lambda^* \), \( D \) and the real frequency \( \Omega^* = \Omega(k^*) \) [14].

Choosing the proper frame and transformation. Equation (11) confirms that a localized initial condition will grow out and spread asymptotically with the velocity \( v^* \) given by Eq. (12). Our aim now is to understand the convergence of a pulled front due to the interplay of the linear spreading and the nonlinearities. The Green’s function expression (11) gives three important hints in this regard: First of all, \( G(\xi,t) \) is asymptotically of the form \( e^{-i k^* \xi - i \Omega^* t} \) times a crossover function whose diffusive behavior is betrayed by the Gaussian form in Eq. (11). Hence, if we write our dynamical fields as \( A = e^{i k^* \xi - i \Omega^* t} \psi(\xi,t) \) for the QCGL (9) or \( u = e^{i k^* \xi - i \Omega^* t} \psi(\xi,t) + \text{c.c.} \) for the real field \( u \) in Eq. (7), we expect that the dynamical equation for \( \psi(\xi,t) \) obeys a diffusion-type equation. Second, as we have argued in [5], for the relaxation analysis one wants to work in a frame where the crossover function \( \psi \) becomes asymptotically time independent. This is obviously not true in the \( \xi \) frame, due to the factor \( 1/\sqrt{t} \) in Eq. (11). However, this term can be absorbed in the exponential prefactor, by writing \( t \to e^{i k^* \xi - i \Omega^* t} = e^{i k^* \xi - i \Omega^* t} e^{i \ln t} \). Hence, we introduce the logarithmically shifted frame \( \xi_X = \xi - X(t) \) [5], as already used in Eq. (6). Third, we find a feature specific for pattern forming fronts: the complex parameters, and \( D \) in particular, lead us to introduce the global phase \( \Gamma(t) \). We expand \( \Gamma(t) \) like \( X(t) \) [5],

\[
X(t) = \frac{c_1}{t} + \frac{c_{3/2}}{t^{3/2}} + \cdots, \quad \Gamma(t) = \frac{d_1}{t} + \frac{d_{3/2}}{t^{3/2}} + \cdots
\]

and analyze the long time dynamics by performing a ‘leading edge transformation’ to the field \( \psi \).

QCGL: \( A = e^{i k^* \xi_X - i \Omega^* t - i \Gamma(t)} \psi(\xi_X,t) \),

SH: \( u = e^{i k^* \xi_X - i \Omega^* t - i \Gamma(t)} \psi(\xi_X,t) + \text{c.c.} \)

Steep initial conditions imply that \( \psi(\xi_X,t) \to 0 \) as \( \xi_X \to \infty \). The determination of the coefficients in the expansions (13) of \( X \) and \( \Gamma \) is the main goal of the subsequent analysis, as this then directly yields Eqs. (2) and (3).

Understanding the intermediate asymptotics. Substituting the leading edge transformation (14) into the nonlinear dynamical equations, we get

\[
\partial_x \psi = D \partial_x^2 \psi + \sum_{n=3} D_n \partial_x^n \psi
\]

\[
+ [X(t)(\partial_x^2 + i k^*) + i \Gamma(t)] \psi - N(\psi),
\]

FIG. 2. (a) and (b) Simulation of the QCGL equation as in Fig. 1b for times \( t = 35 \) to 144. (a) shows \( |X| \) (16) as a function of \( \xi_X \). (b) shows \( |\psi| \), which in region I builds up a linear slope \( \psi \approx \alpha \xi_X \), and in region III decays like a Gaussian widening in time. The lines in region II show the maxima of \( \psi(\xi_X,t) \) for fixed \( t \) and their projection \( \xi_X - \sqrt{t} \) into the \( (\xi_X,t) \) plane. (c) shows the scaling plot for the phase relaxation. From left to right: SH (dashed lines) for \( n = \sqrt{e}, 0.01 \sqrt{e}, \) and 0.0001 \( \sqrt{e} \) (\( e = 5 \)), and QCGL (dotted lines) for \( |A| = 0.002, 0.0002, \) and 0.00002. Plotted is \( \Gamma(t) T_F/c_1 \) vs \( 1/\tau \). Here \( \tau = t/T_F \), and \( T_F = T_q [1 + \lambda^* \text{Im} D^{-1/2}(q^* \text{Re} D^{-1/2})] \). The solid line again is the universal asymptote \( -1/\tau + 1/\tau^2 \).
with \(D_n=\frac{-i\pi!}{\pi^2}\frac{d^n\Omega/(dk)^n}{n!}\) the generalization of \(D\) in Eq. (12) [of course, for the QCGL, \(\Omega(k)\) is quadratic in \(k\), so \(D_n=0\)]. In this equation, \(N\) accounts for the nonlinear terms; e.g., for the QCGL, we simply have
\[
N=e^{-2\lambda^*\xi}x^2\psi\left[1-(1+iC_\lambda)+(1-iC_\lambda)e^{-2\lambda^*\xi}\psi\right].
\] (16)

The expression for the SH equation is similar.

Now, in the region labeled I in Fig. 2, finite width in the presence of a sink localized at some finite value of \(\xi\). The ensuing dynamics of \(\psi\) to the right of the sink can be understood with the aid of Figs. 2(a) and 2(b), which are obtained directly from the time-dependent numerical simulations of the QCGL (9). To extract the intermediate asymptotic behavior illustrated by these plots, we integrate Eq. (15) once to get
\[
\frac{d}{dt}\int_{-\infty}^{\xi} d\xi' \psi = D\frac{d}{d\xi} \psi + \sum_{n=3}^{\infty} \frac{D_n}{n-1} \frac{d^{n-1}}{d\xi} \psi - \int_{-\infty}^{\xi} d\xi' N(\psi).
\] (17)

Now, in the region labeled I in Fig. 2(b), we have for fixed \(\xi\) and \(t\to\infty\) that the terms proportional to \(X\) and \(\Gamma\) can be neglected upon averaging over the fast fluctuations; the same holds for the term on the left. Since the integral converges quickly to the right due to the exponential factors in \(N\), we then get immediately, irrespective of the presence of higher order spatial derivatives

\[
\lim_{t\to\infty} D \frac{\partial \overline{\psi}}{\partial \xi} = \int_{-\infty}^{\infty} d\xi N(\overline{\psi}) = \alpha D.
\] (18)

Here, the overbar denotes a time average (necessary for the case of a chaotic front). Thus, for large times in region I, \(\overline{\psi} = \alpha \xi_D + \beta\) in dominant order. Moreover, from the diffusive nature of the equation, our assertion that the fluctuations of \(\psi\) rapidly decrease to the right of the region where \(N\) is nonzero comes out naturally. In other words, provided that the time-averaged sink strength \(\alpha\) is nonzero, \(\alpha \neq 0\), one will find a buildup of a linear gradient in \(\sqrt{\psi}\) in region I, independent of the precise form of the nonlinearities or of whether or not the front dynamics is coherent. This behavior is clearly visible in Fig. 2(b). We can understand the dynamics in regions II and III along similar lines. In region III the dominant terms in Eq. (15) are the one on the left and the first one on the second line, and the crossover region II which separates regions I and III moves to the right according to the diffusive law \(\xi_0 \sim D\sqrt{t}\).

Systematic expansion. These considerations are fully corroborated by our extension of the analysis of [5]. Anticipating that \(\psi\) falls off for \(\xi_0 \gg 1\), we split off a Gaussian factor by writing \(\psi(\xi_0,t) = G(z,t)e^{-z}\) in terms of the similarity variable \(z = \xi_0^2/(4Dt)\), and expand
\[
G(z,t) = t^{1/2}g_{-1/2}(z) + g_0(z) + t^{-1/2}g_{1/2}(z) + \cdots.
\] (19)

This, together with the expansion (13) for \(X(t)\) and \(\Gamma(t)\), the left ‘boundary condition’ that \(\psi(\xi_0,t\to\infty) = \alpha \xi_0 + \beta\) and the condition that the functions \(g(z)\) do not diverge exponentially, then results in the expressions (2) for \(X(t)\) and (3) for \(\Gamma\) [9]. For the QCGL, the analysis immediately implies the result (10) for the front profile in the leading edge. In addition for the SH equation, one arrives at Eq. (8) for the shape relaxation in the front interior along the lines of [5]: Starting from the ODEs for the \(U^a\), one finds upon transforming to the frame \(x^b\) that to \(O(t^{-2})\), the time dependence only enters parametrically through \(v(t)\). This then yields Eq. (8).

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