Non-equilibrium current cumulants and moments with a point-like defect

Mihail Mintchev\textsuperscript{1}, Luca Santoni\textsuperscript{2} and Paul Sorba\textsuperscript{3}

\textsuperscript{1}Istituto Nazionale di Fisica Nucleare and Dipartimento di Fisica, Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italy
\textsuperscript{2}Scuola Normale Superiore and Istituto Nazionale di Fisica Nucleare, Piazza dei Cavalieri 7, I-56126 Pisa, Italy
\textsuperscript{3}LAPTh, Laboratoire d’Annecy-le-Vieux de Physique Théorique, CNRS, Université de Savoie, BP 110, F-74941 Annecy-le-Vieux Cedex, France

E-mail: mintchev@df.unipi.it

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Abstract

We derive the exact $n$-point current expectation values in the Landauer–Büttiker non-equilibrium steady state of a multi terminal system with star graph geometry and a point-like defect localised in the vertex. The current cumulants are extracted from the connected correlation functions and the cumulant generating function is established. We determine the moments, show that the associated moment problem has a unique solution and reconstruct explicitly the corresponding probability distribution. The basic building blocks of this distribution are the probabilities of particle emission and absorption from the heat reservoirs, driving the system away from equilibrium. We derive and analyse in detail these probabilities, showing that they fully describe the quantum transport problem in the system.

Keywords: quantum fields away from equilibrium, non-equilibrium steady states, quantum transport

(Some figures may appear in colour only in the online journal)

1. Introduction

Current fluctuations represent a fundamental characteristic feature of non-equilibrium quantum transport. Complete information about these fluctuations is provided by the cumulants $C_n$ of the particle current, which generalise the quadratic noise fluctuations to $n \geq 3$ currents. For this reason the study of the sequence $\{C_n : n = 1, 2, \ldots\}$ attracted a significant amount of attention in last two decades. Following the fundamental work of Khlus
In this paper we consider the particle current fluctuations for general class of point-like defects. More precisely we investigate a system with $N$ terminals with the geometry of a star graph as shown in figure 1. Each of the $N$ semi-infinite leads is attached at infinity to the a heat reservoir $R_i$ with (inverse) temperature $\beta_i$ and chemical potential $\mu_i$. The defect, which drives the system out of equilibrium, is localised in the vertex of the graph and is described by a $N \times N$ unitary scattering matrix $\Sigma$. Although relatively simple, the system in figure 1 represents a remarkable laboratory for studying a large class of intriguing quantum phenomena. The deep relation between the particle density cumulants and the Rényi entanglement entropies in an equilibrium configuration at zero temperature has been investigated in [19]. The study of the first cumulant of the particle and heat currents away from equilibrium shows [20] that the junction transforms heat to chemical potential energy and vice versa, depending on the parameters of the heat baths. The explicit form of the second cumulant $C_2$ in the scale invariant limit reveals [21] the existence of nonlinear effects, which lead to reduction or enhancement of the particle and heat noises in certain ranges of the chemical potentials. In what follows we pursue further the investigation of the junction in figure 1, adopting the following strategy. In a quantum field theory framework we first derive, in explicit and closed form, the particle current cumulants $C_n$ for generic $n$. We use this information for reconstructing both the cumulant and the moment generating functions. From these data we finally recover the associated probability distribution, which captures the microscopic characteristic features of the system. In this respect we explicitly determine the probability $p_{ij}$ of a particle with given energy to be emitted from the heat reservoir $R_i$ and absorbed by $R_j$. It turns out that $p_{ij}$ are nontrivial in both cases $i = j$ and $i \neq j$. Our investigation covers all point-like defects in the vertex, which are compatible with a unitary time evolution in the bulk generated by the Schrödinger Hamiltonian. We are thus considering systems in a non-equilibrium steady state with conserved total energy. For a recent progress in the study of open locally interacting systems we refer to [22, 23].
The paper is organised as follows. In the next section we first recall the form of the particle current in the presence of a point-like defect. After this we derive the exact connected \(n\)-point current correlation functions in the Landauer–Büttiker (LB) non-equilibrium steady state. The zero-frequency limit of these functions defines the cumulants \(C_n\). The cumulant generating function \(C(\lambda)\) in the general case with \(N\)-terminals is reconstructed in section 3. We establish here also the explicit form of \(C(\lambda)\) in the scale invariant limit and describe the main properties of the cumulants \(C_3\) and \(C_4\) in this regime. The last part of section 3 provides a comparison with other results in the subject. In section 4 we derive the moments in a single energy channel, the associated probability distribution and the probabilities \(p_{ij}\) mentioned above. Our conclusions are collected in section 5, and the appendices contain some technical details.

2. Current correlation functions and cumulants

2.1. Particle current in the presence of defect

The system in figure 1 is localised on a star graph with coordinates \(\{ (x, i) : x \leq 0, i = 1, \ldots, N\}\), where \(|x|\) denotes the distance from the vertex and \(i\) labels the leads. The main object of our investigation is the particle current \(j(t, x, i)\) flowing along the leads of the junction. In order to determine \(j(t, x, i)\) one should fix both the dynamics in the bulk and the boundary conditions in the vertex of the graph. We consider the wave function

\[
\left(i \partial_t + \frac{1}{2m} \partial_x^2\right) \psi(t, x, i) = 0, \tag{2.1}
\]

with the boundary condition

\[
\lim_{x \to -0^+} \sum_{j=1}^N [\eta (I - U)_{ij} + i (I + U)_{ij} \partial_x] \psi(t, x, j) = 0, \tag{2.2}
\]

where \(U\) is a \(N \times N\) unitary matrix and \(\eta \in \mathbb{R}\) is a parameter with dimension of mass. Equation (2.2) parametrises all self-adjoint extensions of the bulk Hamiltonian \(-\partial_x^2\) to the whole graph and gives rise to non-trivial one-body interactions, which are described by the scattering matrix \([24–26]\)

\[
S(k) = -\left[ \frac{\eta (I - U) - k (I + U)}{\eta (I - U) + k (I + U)} \right], \tag{2.3}
\]

\(k\) being the particle momentum. We stress that the scattering matrices (2.3) provide a physical description of all point-like contact interactions among the leads, which are compatible with a unitary time evolution in the bulk of the system. This is the fundamental requirement in selecting the class of defects considered in this paper.

The solution of (2.1) and (2.2) is given by

\[
\psi(t, x, i) = \sum_{j=1}^N \int_0^\infty \frac{dk}{2\pi} e^{-i\omega(k)t} \chi_j(k; x) a_j(k), \quad \omega(k) = k^2 \frac{2m}{2m}. \tag{2.4}
\]

where

\[
\chi(k; x) = e^{-ikx} I + e^{ikx} S(-k) \tag{2.5}
\]

4 The natural units \(\hbar = c = k_B = 1\) are adopted throughout the paper.
and \( \{ a_i(k), a^*_i(k) : k \geq 0, i = 1, \ldots, N \} \) generate the standard canonical anticommutation relation algebra. With the above definitions the particle current takes the form\(^5\)

\[
j(t, x, i) = \frac{i}{2m} \int_0^\infty \frac{dk}{2\pi} \int_0^\infty \frac{dp}{2\pi} \ e^{i[\omega(k)-\omega(p)]} \times \sum_{l,m=1}^N a^*_m(k) \chi^*_m(k; x)[\partial_x \chi_{l,m}](p; x) \chi_{l,m}(p; x) a_m(p).
\]

Using the orthogonality and completeness of the system \( \{ \chi(k; x) : k \geq 0, x \geq 0 \} \) one can prove [27] the operator Kirchhoff rule

\[
\sum_{i=1}^N j(t, 0, i) = 0,
\]

which is a simple but fundamental feature of the currents flowing in the junction.

Besides the particle current operator, we have to fix also the state for evaluating the current expectation values. The physical setting, presented in figure 1, is nicely described by the LB [28, 29] non-equilibrium steady state \( \Omega_{\beta, \mu} \), defined in terms of \( (\beta, \mu) \) and \( \bar{S}(k) \). A simple and intuitive way [27] to construct this state is to use the scattering matrix \( \bar{S}(k) \) in order to extend the tensor product of Gibbs states, relative to the reservoirs \( R_i \) at the level of asymptotic incoming fields, to the outgoing fields. The state, obtained in this way, has both realistic physical properties [28, 29] and an interesting mathematical structure [27, 30, 31]. The basic expectation values of \( \{ a_i(k), a^*_i(k) \} \) in \( \Omega_{\beta, \mu} \), which are needed in what follows, are reported in appendix A.

### 2.2. Current cumulants in the LB state

Let \( L_i \) be an arbitrary but fixed lead and let us consider the \( n \)-point correlation function

\[
\mathcal{W}^{(n)}_{\mu}(t_1, x_1, \ldots, t_n, x_n) = \langle j(t_1, x_1, i) \cdots j(t_n, x_n, i) \rangle_{\bar{S}, \mu},
\]

of the current (2.6), where \( \langle \cdots \rangle_{\bar{S}, \mu} \) denotes the expectation value in the LB state \( \Omega_{\bar{S}, \mu} \). The \( n \)th cumulant in \( L_i \) is defined by the connected part of (2.8),

\[
C^{(n)}_{\mu}(t_1, x_1, \ldots, t_n, x_n) = \langle j(t_1, x_1, i) \cdots j(t_n, x_n, i) \rangle^{\text{conn}}_{\bar{S}, \mu}.
\]

For \( n = 1 \) the correlators (2.8) and (2.9) coincide and have the following well known [28, 29] time and space independent form

\[
\mathcal{W}^1 = C^1 = \int_0^\infty \frac{d\omega}{2\pi} \sum_{l=1}^N (\delta_{ll} - \langle \bar{S}_l(\sqrt{2m\omega}) \rangle^2) d_l(\omega), \quad d_l(\omega) = \frac{1}{1 + e^{\beta(l-\mu)}},
\]

The situation complicates for \( n \geq 2 \). First of all the correlators (2.8) and (2.9) depend on the time differences \( \{ t_k - t_{k+1} : k = 1, \ldots, n-1 \} \), which reflects the invariance under time translations of the LB state. Moreover, since the defect violates translation invariance in space, (2.8) and (2.9) depend separately on all the coordinates \( \{ x_l : l = 1, \ldots, n \} \). It is clear that dealing with this large number of variables becomes complicated with the growing of \( n \). It also turns out that most of them are marginal for the particle transfer we are interested in. One possibility [15–18] of eliminating some space–time variables is the replacement

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5 The * stands for Hermitian conjugation.
\[
 j(t_l, x_l, i) \longmapsto \int_0^T dt_l j(t_l, x_l, i), \quad \forall \ l = 1, \ldots, n, \tag{2.11}
\]

in (2.8), (2.9). The operation (2.11) obviously simplifies the time dependence. Instead of the \((n - 1)\) time variables \(t_l\), we now have only one, namely \(T\). The final step in this scheme is to study the system for a large enough \(T\). Unfortunately, in the presence of a defect the above procedure solves the problem only partially, because the \(x_l\)-dependence persists.

In this paper we adopt an alternative strategy, which generalises to \(n \geq 3\) the definition (see e.g. [16]) of zero-frequency noise. For \(n \geq 2\) we consider the Fourier transforms

\[
 Z_n^i(t_l, x_l, \ldots, x_n; \omega) = \int_{-\infty}^{\infty} dt_{n-1} e^{i\omega(t_{n-1} - \ldots - t_1)} Z_n^i(t_l, x_l, \ldots, t_n, x_n), \quad Z_n^i = \mathcal{W}_n^i, \ C_n^i, \tag{2.12}
\]

and perform the zero-frequency limit

\[
 Z_n^i = \lim_{\nu \to 0^+} Z_n^i(t_l, \ldots, x_n; \nu). \tag{2.13}
\]

We will show below that in this limit the \(x_l\)-dependence drops out and \(Z_n^i\) depends exclusively on the scattering matrix \(\Sigma\) and the heat bath parameters \(b_l, \mu_l\). In fact, using the explicit form of the current (2.6) and the correlation function (A.82) in appendix A, after some algebra, one finds

\[
 \mathcal{W}_n^i = \int_0^\infty \frac{d\omega}{2\pi} \sum_{l_1, \ldots, l_n=1}^N \begin{vmatrix}
 T_{i_1, l_1}^i(\omega) d_{i_1}(\omega) & T_{i_1, l_2}^i(\omega) d_{i_2}(\omega) & \cdots & T_{i_1, l_n}^i(\omega) d_{i_n}(\omega) \\
 -T_{i_2, l_1}^i(\omega) d_{i_1}(\omega) & T_{i_2, l_2}^i(\omega) d_{i_2}(\omega) & \cdots & T_{i_2, l_n}^i(\omega) d_{i_n}(\omega) \\
 \vdots & \vdots & \ddots & \vdots \\
 -T_{i_n, l_1}^i(\omega) d_{i_1}(\omega) & -T_{i_n, l_2}^i(\omega) d_{i_2}(\omega) & \cdots & T_{i_n, l_n}^i(\omega) d_{i_n}(\omega)
 \end{vmatrix}, \tag{2.14}
\]

where

\[
 T_{i_m}^i(\omega) = \delta_{i_m} \delta_{i_m} - S_{i_m}(\sqrt{2m\omega}) S_{i_m}(\sqrt{2m\omega}), \tag{2.15}
\]

d\(_i(\omega)\) is the Fermi distribution (2.10) of the reservoir \(R_i\) and

\[
 d_{i}(\omega) = 1 + d_{i}(\omega) = \frac{e^{i(\omega - \mu_i)}}{1 + e^{i(\omega - \mu_i)}}. \tag{2.16}
\]

As expected, the connected part of (2.14) simplifies and can be conveniently written in terms of traces involving the matrices

\[
 \mathcal{A} \equiv \mathcal{T}', \quad \mathcal{B} \equiv \mathcal{T}(I - \mathcal{D}), \quad \mathcal{D} \equiv \text{diag}[d_{1}(\omega), d_{2}(\omega), \ldots, d_{n}(\omega)]. \tag{2.17}
\]

One finds

\[
 \mathcal{C}_n^i = \int_0^\infty \frac{d\omega}{2\pi} \text{Tr}[\mathcal{A}], \tag{2.18}
\]

\[
 \mathcal{C}_n^i = \int_0^\infty \frac{d\omega}{2\pi} \sum_{\sigma_1, \ldots, \sigma_n=1}^n \text{Tr}[\mathcal{A}^\dagger C_n^i_{\sigma_1, \sigma_2} \cdots C_n^i_{\sigma_{n-2}, \sigma_{n-1}} \mathcal{B}^\dagger]. \quad n \geq 2, \tag{2.19}
\]

\[6\] Here and in what follows the bar indicates complex conjugation.
where the sum runs over all permutations \( \mathcal{P}_{n-1} \) of \( n-1 \) elements and
\[
C_{i, \sigma_i, \sigma_{i+1}} = \begin{cases} \mathbb{A}^i, & \sigma_i < \sigma_{i+1}, \\ \mathbb{B}^i, & \sigma_i > \sigma_{i+1}. \end{cases}
\] (2.20)

The trace representation (2.18), (2.19) of the current cumulants with a point-like defect represents a first basic result of our study. It is worth stressing that the above derivation of \( C_{i, \sigma_i, \sigma_{i+1}} \) is purely field theoretical and makes no use of any kind of cumulant generating function. We will show in section 3 that this function can be uniquely reconstructed from (2.18), (2.19).

2.3. The two-lead junction cumulants

In order to better illustrate the compact expressions (2.18), (2.19), it is instructive to report the explicit form of the first few cumulants in the case \( N = 2 \). Without loss of generality we can concentrate on the cumulants \( C_{i, \sigma_i, \sigma_{i+1}} \) in the lead \( L_i \). For notational simplicity we omit here and in what follows the apex 1 in \( C_{i, \sigma_i, \sigma_{i+1}} \). By means of (2.18)–(2.20) one obtains
\[
C_1 = \int_0^\infty \frac{d\omega}{2\pi} \tau c_1, 
\]
(2.21)
\[
C_2 = \int_0^\infty \frac{d\omega}{2\pi} \tau (c_2 - \tau c_1^2), 
\]
(2.22)
\[
C_3 = \int_0^\infty \frac{d\omega}{2\pi} \tau^2 c_1 (1 - 3c_2 + 2\tau c_1^2), 
\]
(2.23)
\[
C_4 = \int_0^\infty \frac{d\omega}{2\pi} \tau^2 [c_2 - 3c_2^2 + 12\tau c_1^2 c_2 - 2\tau c_1^2(2 + 3\tau c_1^2)], 
\]
(2.24)
\[
C_5 = \int_0^\infty \frac{d\omega}{2\pi} \tau^3 c_1 [1 + 30c_2^2 - 15c_2(1 + 4\tau c_1^2) + 4\tau c_1^2(5 + 6\tau c_1^2)], 
\]
(2.25)
\[
C_6 = \int_0^\infty \frac{d\omega}{2\pi} \tau^3 \{c_2[1 + 15c_2(2c_2 - 1)] - 2\tau c_1^2[8 + 15c_2(9c_2 - 5)] + 120\tau^2 c_1^4 (3c_2 - 1) - 120\tau^3 c_1^6\}, 
\]
(2.26)
where the following combinations
\[
c_1(\omega) \equiv d_1(\omega) - d_2(\omega), \quad c_2(\omega) \equiv d_1(\omega) + d_2(\omega) - 2d_1(\omega)d_2(\omega), 
\]
(2.27)
have been introduced for convenience. Moreover, the transmission probability associated with the \( S \)-matrix (2.3) is given by
\[
\tau(\omega) = |S_{12}(\sqrt{2m}\omega)|^2 = \frac{2m\omega(\eta_1 - \eta_2)^2\sin^2(\theta)}{(2m\omega + \eta_1^2)(2m\omega + \eta_2^2)}, \quad \theta \in [0, \pi), 
\]
(2.28)
with
\[
\eta_j \equiv \eta \tan(\alpha_j), \quad \alpha_j \in [-\pi/2, \pi/2), 
\]
(2.29)
\((e^{-2i\alpha_1} - e^{-2i\alpha_2})\) being the eigenvalues of the matrix \( \mathbb{U} \) entering the boundary condition (2.2).

The main properties of \( C_1 \) and \( C_2 \) have been discussed in [28, 29], whereas the non-linear dependence on the chemical potentials has been examined in detail in [21]. Before analysing some \( C_{n \geq 3} \), we will face the problem of deriving a generating function for the cumulants (2.18), (2.19) and the associated probability distribution.
3. Cumulant generating function

We show in this section that in spite of the complicated explicit form of the cumulants \((2.18)\), \((2.19)\), there exists a relatively simple and compact generating function of \(\mathcal{F}_l\). It is instructive to start by extracting the information encoded in \((2.21)-(2.26)\) about \(\mathcal{F}_l\). Following the pioneering work of Khlus [1], Lesovik and Levitov [2, 3], we look for a generating function in the form

\[
\mathcal{C}(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \ln [1 + F_{12}(\tau, d_1, d_2)(e^{i\lambda\tau} - 1) + F_{21}(\tau, d_1, d_2)(e^{-i\lambda\tau} - 1)],
\]

(3.30)

where \(F_{12}, F_{21}\) and \(f\) are unknown functions. Using the standard definition of generating function

\[
\mathcal{C}_n = (-i\partial_\lambda)^n \mathcal{C}(\lambda)|_{\lambda=0}
\]

(3.31)

and the information from the first three cumulants \(C_1, C_2\) and \(C_3\) only, one can easily determine \(F_{12}, F_{21}\) and \(f\). The simple result

\[
F_{12} = \frac{1}{2}(c_2 + c_1\sqrt{\tau}), \quad F_{21} = \frac{1}{2}(c_2 - c_1\sqrt{\tau}), \quad f = \sqrt{\tau}.
\]

(3.32)

leads to the following generating function

\[
\mathcal{C}(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \ln \left[1 + \frac{1}{2}(c_2 + c_1\sqrt{\tau})(e^{i\lambda\sqrt{\tau}} - 1) + \frac{1}{2}(c_2 - c_1\sqrt{\tau})(e^{-i\lambda\sqrt{\tau}} - 1) \right]
\]

\[
= \int_0^\infty \frac{d\omega}{2\pi} \ln \left[1 + ic_1\sqrt{\tau} \sin(\lambda\sqrt{\tau}) + c_2 \cos(\lambda\sqrt{\tau}) - 1 \right].
\]

(3.33)

One can easily also check that (3.33) reproduces perfectly the cumulants \(C_4, C_5\) and \(C_6\) as well and represents therefore a valid candidate for the final result in the case of two leads. The expression (3.33) has been reported without derivation also by Lesovik and Chkhelkatchev [4]. Our goal below will be to generalise (3.33) to the multi terminal junction in figure 1, thus recovering the \(N=2\) formula as a special case. In the comments at the end of this section we will briefly describe an alternative to (3.33), regarding a slightly different setup.

3.1. General result for \(N\) terminals

The argument in what follows is based on the fact that the particle transport in our system can be separated in statistically independent processes with fixed energy. In fact, excitations with different energies propagate in the graph in figure 1 in a fully independent way, because the only interaction, localised in the vertex, leaves the energy invariant (see \(A.84\)). For this reason one can focus first on a single energy channel \(\omega\), thus dealing with a system with finite degrees of freedom. This fact significantly simplifies the problem and allows one to derive explicitly the single energy channel cumulant generating function \(\mathcal{C}_\omega(\lambda)\) in the lead \(L_\omega\). The final step is to integrate over all energies,

\[
\mathcal{C}(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \mathcal{C}_\omega(\lambda),
\]

(3.34)

using at this stage the well known property that the total cumulant of a process, which can be decomposed in statistically independent subprocesses, is the sum of the cumulants of each of the latter.
In order to obtain the particle current of a single energy channel \( \omega \geq 0 \), we modify the integration measure in the general expression of the particle current (2.6) according to

\[
dk \, dp \longrightarrow dk \, dp \, (2\pi)^2 \, \delta \left( \frac{k^2}{2m} - \omega \right) \delta (k - p),
\]

which selects the contribution with energy \( \omega \). This operation leads to the simple time and position independent expression

\[
j^*_\omega = \sum_{l,m=1}^N a^*_l \, \mathbb{T}^i_{lm}(\sqrt{2m}\omega) a_m,
\]

where \( \{a_i, a^*_i\} \) are standard fermionic oscillators:

\[
[a_i, a^*_j] = \delta_{ij}, \quad [a_i, a_j] = [a_i^*, a_j^*] = 0, \quad \langle a_i^* a_j \rangle_{\beta,\mu} = \delta_{ij} d_j(\omega).
\]

Now, the generating function of cumulants in the channel \( \omega \) and lead \( L_i \) is given by

\[
C^i_\omega(\lambda) = \ln \langle e^{i\lambda \xi} \rangle_{\beta,\mu}.
\]

The expectation value \( \langle e^{i\lambda \xi} \rangle_{\beta,\mu} \) can be evaluated explicitly. The key points of the computation, which leads to the final result

\[
\langle e^{i\lambda \xi} \rangle_{\beta,\mu} = \det \left[ \mathbb{1} + (e^{i\lambda \xi} + \sqrt{2m\omega}) \mathbb{D}(\omega) \right],
\]

\[
\mathbb{D}(\omega) \equiv \text{diag} [d_1(\omega), d_2(\omega), \ldots, d_N(\omega)],
\]

are given in appendix C. In the lead \( L_1 \) one finds

\[
C(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \ln \left\{ 1 + i c_1^{\text{eff}} \sqrt{\tau} \sin (\lambda \sqrt{\tau}) + c_2^{\text{eff}} \left[ \cos (\lambda \sqrt{\tau}) - 1 \right] \right\}.
\]

Here

\[
\tau(\omega) = \sum_{i=2}^N \tau_i(\omega), \quad \tau_i(\omega) \equiv |S_{1i}(\sqrt{2m}\omega)|^2,
\]

is the total transmission probability between the lead \( L_1 \) and the remaining \( N - 1 \) leads \( L_i \) and \( c_1^{\text{eff}}, c_2^{\text{eff}} \) are obtained from (2.27) by the substitution

\[
d_2(\omega) \longrightarrow d_2^{\text{eff}}(\omega) \equiv \sum_{i=2}^N \frac{\tau_i(\omega)}{\tau(\omega)} \, d_i(\omega),
\]

which represents an effective distribution where \( d_i(\omega) \) with \( i \geq 2 \) are weighted by the ratios \( \tau_i(\omega)/\tau(\omega) \in [0, 1] \). As expected, the expression (3.40) reproduces (3.33) for \( N = 2 \).

Summarising, we derived in explicit form the generating function of the cumulants (2.18), (2.19) for the Schrödinger junction (2.1), (2.2) with \( N \geq 2 \) leads. The novelty with respect to the two terminal case (3.33) is the effective Fermi distribution (3.42), which captures the presence of all \( N - 1 > 1 \) reservoirs.

### 3.2. Scale invariant limit

The \( \omega \)-integration in (3.33), (3.40) with general \( \mathbb{S} \)-matrix of the form (2.3) cannot be performed in a closed analytic form. For this reason it is instructive to select among (2.3) the *scale-invariant* matrices, which incorporate the universal transport properties of the system [32] while being simple enough to be analysed explicitly. The scale invariant (critical) elements in the family (2.3) are fully classified [33] and belong to the orbits...
of the adjoint action of the unitary group $U(N)$ on the diagonal matrices $\mathbb{S}_d$. As expected, the critical $S$-matrices are $\omega$-independent, which in the case $\beta_1 = \beta_2 \equiv \beta$ allows the computation of the integrals in (3.33), (3.40) explicitly. In fact, introducing the variables

$$\gamma_j \equiv e^{-\beta j}, \quad \Lambda(\lambda) \equiv i \sqrt{\tau} (\gamma_2 - \gamma_1) \sin(\lambda \sqrt{\tau}) + (\gamma_2 + \gamma_1) \cos(\lambda \sqrt{\tau}),$$

where $\tau = |S_{12}|^2$ now is constant, one gets in the case $N = 2$

$$C(\lambda) = \frac{1}{2\pi \beta} \left( \text{Li}_2(-\gamma_1) + \text{Li}_2(-\gamma_2) \right) - \text{Li}_2 \left[ \frac{-2}{\Lambda(\lambda) - \sqrt{\Lambda^2(\lambda) - 4\gamma_1 \gamma_2}} \right] - \text{Li}_2 \left[ \frac{-2}{\Lambda(\lambda) + \sqrt{\Lambda^2(\lambda) - 4\gamma_1 \gamma_2}} \right],$$

(3.45)

$L_i$ being the dilogarithm function.

The results of [21] suggest the investigation of (3.45) as a function of

$$\mu_\pm = (\mu_1 + \mu_2)/2,$$

(4.6)

$\mu_+ \in \mathbb{R}$ playing the role of control parameter. For $\mu_+ = 0$ the expression (3.45) greatly simplifies in the low temperature limit $\beta \to \infty$. In fact, one finds

$$\lim_{\beta \to \infty} C(\lambda) \mid_{\mu_+ = 0} = \frac{1}{2\pi} \ln \left[ \cos(\lambda \sqrt{\tau}) + i \varepsilon (\mu_-) \sqrt{\tau} \sin(\lambda \sqrt{\tau}) \right],$$

(4.7)

$\varepsilon$ being the sign function. The result (4.47) was derived for $\mu_- > 0$ independently by Levitov and Lesovik [3, 18] and provides therefore a valuable check on (3.45). For the first few cumulants one obtains from (4.47)

$$C_1 = \frac{\tau \mu_-}{2\pi}, \quad C_2 = \frac{\tau (1 - \tau) |\mu_-|}{2\pi},$$

$$C_3 = \frac{\tau^2 (\tau - 1) \mu_-}{\pi}, \quad C_4 = \frac{\tau^2 (\tau - 1)(1 - 3\tau) |\mu_-|}{\pi}.$$ 

(4.8)

From (4.47) one infers that for any $n \geq 0$

$$C_{2n+1} \propto C_1, \quad \mu_+ = 0, \quad \beta \to \infty,$$

(4.9)

which provides an interesting signature for experiments. In fact, experimental evidence for the linear dependence of $C_1$ on the current $C_i$ in this regime has been reported in [34].

It is instructive to derive the charge transferred through the junction in the time interval $[0, T]$. Since the LB state is stationary, one has in general

$$Q_T = \int_0^T dt \left\langle j(t, x, i) \right\rangle_{\beta, \mu} = TC_i.$$ 

(5.0)

Restoring the electric charge according to $j \mapsto ej$ and $\mu_- = eV$, where $V$ is the applied voltage, one gets from (4.48)

$$Q_T = \frac{e^2 \tau}{2\pi} VT, \quad e_{\text{eff}} = e\sqrt{\tau},$$

(5.1)

where the effective charge $e_{\text{eff}}$ has been introduced. We see that switching on the defect ($\tau < 1$) causes a finite renormalisation of the charge $e \mapsto e_{\text{eff}}$ with respect to the case in which the defect is absent ($\tau = 1$). This purely quantum phenomenon is induced by the non-trivial reflection probability $(1 - \tau)$ from the defect. In this respect the appearance of the
effective charge $e_{\text{eff}}$ in the probability distribution (4.67), reconstructed in section 4 below, is not surprising. It is worth mentioning that the same charge renormalisation effect has been observed in [3, 4] as well.

Relevant non-linear effects show up [21] for $\mu_+ = 0$. Consider for instance the current, which can be written in the form

$$C_1 = \frac{\tau \mu_-}{2\pi} + \frac{\tau}{2\pi \beta} \ln \left[ \frac{e^{\beta \mu_-} + e^{-\beta \mu_-}}{1 + e^{\beta (\mu_+ - \mu_-)}} \right].$$

The second term vanishes for $\mu_+ = 0$, but is nontrivial otherwise and captures the nonlinear behavior close to the origin $\mu_- = 0$, shown in the left panel of figure 2. Remarkably enough, such nonlinearity has been experimentally observed in the junctions studied in [35]. The reduction and enhancement [21] of the noise power $C_2$ for $\mu_+ < 0$ and $\mu_+ > 0$ are also consequences of the nonlinearity in $\mu_-.$

It follows from (3.45) that for $\mu_+ \neq 0$ the cumulants $C_3$ and $C_4$ are respectively odd and even nonlinear functions of $\mu_-.$ For the slopes of the asymptotes at $\mu_- = \pm \infty$ one finds

$$\lim_{\mu_- \to \pm \infty} \frac{C_3}{\mu_-} = \pm \frac{\tau^2 (\tau - 1)}{\pi}, \quad \lim_{\mu_- \to \pm \infty} \frac{C_4}{\mu_-} = \pm \frac{\tau^2 (\tau - 1)(1 - 3\tau)}{\pi},$$

which depend on $\tau$ but not on $\beta$ and $\mu_+.$ The central and the right panel of figure 2 illustrate the behavior of $C_3$ and $C_4$ for three different values of $\mu_+.$

3.3. Comments

The form of the current correlations (2.8), (2.9) clearly depends on the non-equilibrium system under consideration. In the present investigation the cumulants (2.18), (2.19) refer to the family of point-like defects defined by the boundary condition (2.2) or equivalently, by the scattering matrix (2.3). In the paper [5] Levitov, Lee and Lesovik (LLL) investigated a different setup. First of all, instead of the boundary condition (2.2), which fixes the self-adjoint extension of the operator $-\partial_x^2$ in our framework, these authors introduced in (2.1) a minimal coupling

$$i \partial_x \rightarrow i \partial_x + A(x).$$

The physical idea, inspiring this modification, is to implement a kind of quantum galvanometer, based on the interaction of the current (2.6) with the classical potential $A(x).$ In addition, instead of the zero frequency limit (2.13) of the current expectation values (2.12), LLL focus on
\[ \int_0^\tau dt_1 \cdots \int_0^\tau dt_n \langle j(t_1, x_1, i) \cdots j(t_n, x_n, i) \rangle \lambda, \mu \] (3.55)

at \( x_1 = \cdots = x_n = 0 \). In the special case

\[ A(x) \sim \lambda \delta(x) \] (3.56)

with a coupling proportional to the counting parameter \( \lambda \), LLL obtained, for sufficiently large \( T [5] \)

\[ C_L(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \ln \left[ 1 + \frac{1}{\tau} (c_2 + \tau c_1)(e^{i\lambda} - 1) + \frac{1}{2} (c_2 - \tau c_1)(e^{-i\lambda} - 1) \right] \]

\[ = \int_0^\infty \frac{d\omega}{2\pi} \ln \{ 1 + i\tau \sin(\lambda) + \tau c_2[\cos(\lambda) - 1] \}. \] (3.57)

(See also [8, 18] and the contributions to [15]). As observed already in [4], the difference between (3.33) and (3.57) is not surprising, because these results concern two inequivalent settings. Notice in this respect the twofold role of \( \lambda \) in [5] as a coupling constant (3.56) and a counting parameter, whereas the system (2.1), (2.2) and the cumulants (2.18), (2.19) are \( \lambda \)-independent and \( \lambda \) enters in (3.33), (3.40) only as an auxiliary parameter in the spirit of conventional generating functions. Nevertheless, comparing (3.33) and (3.57), we observe that these results coincide when the defect in the junction disconnects the leads (\( \tau = 0 \)), or when the defect is absent (\( \tau = 1 \)). Moreover, quite remarkably the first two cumulants generated by (3.33) and (3.57) coincide for any \( \tau \). Therefore the transferred charge (3.50), (3.51) and the zero frequency quantum noise power \( C_2 \) coincide in both schemes. For distinguishing the two settings, one should concentrate on the higher cumulants \( C_{n \geq 3} \). It turns out that they have the same monomials in \( c_1 \) and \( c_2 \), but with different powers of \( \tau \). One has for instance

\[ C_{L,3} = \int_0^\infty \frac{d\omega}{2\pi} \tau c_1 (1 - 3\tau c_2 + 2\tau c_1^2), \] (3.58)

\[ C_{L,4} = \int_0^\infty \frac{d\omega}{2\pi} \tau [c_2 - 3\tau c_1^2 + 12\tau^2 c_1^3 c_2 - 2\tau c_2^2 (2 + 3\tau^2 c_1^3)], \] (3.59)

to be compared with (2.23), (2.24). The difference in the \( \tau \)-behaviour between \( C_3 \) and \( C_{L,3} \) has been studied in detail in [4]. It was argued there that both \( C_3 \) and \( C_{L,3} \) are in principle observable, but in different experimental setups.

It is natural to expect that the form (3.57) of \( C_L \) depends on the specific choice (3.56) of the external potential, which implements the effective description of the charge detector. It has been suggested in [6] that localising \( A(x) \) at \( x = 0 \) minimises the disturbance of the system due to the measuring device. The localisation of \( A(x) \) in one point can be achieved however in many different ways, using general linear combinations of the delta function and its derivatives. The study of the freedom associated with \( A(x) \) is beyond the scope of the present paper, which is focussed on the properties of the cumulants (2.18), (2.19), uniquely defined by the connected current correlation functions (2.9) in the quantum boundary value problem (2.1), (2.2). In the next section we recover the probability distribution, associated with these cumulants, and provide a microscopic physical interpretation for it.

4. Moments and probability distribution

Since the cumulants (2.18), (2.19) concern a quantum field theory system with unitary time evolution, one can expect that they correspond to a well defined probability distribution. In
order to show that this is indeed the case, we reconstruct below this distribution from its moments, following a standard procedure in probability theory \[36\]. In our case the moment generating function at fixed energy \(\omega\) for the two-lead junction can be extracted from equation (C.92) in appendix C. One has

\[
\chi_{\omega}(\lambda) = 1 + ic_1\sqrt{\tau} \sin(\lambda\sqrt{\tau}) + c_2[\cos(\lambda\sqrt{\tau}) - 1],
\]

where \(c_1(\omega)\) and \(\tau(\omega)\) are given by (2.27)–(2.29). The moments \(m_n : n = 0, 1, \ldots\) are inferred from the expansion

\[
\chi_{\omega}(\lambda) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} m_n
\]

and have the simple general form

\[
\begin{cases} 
1, & n = 0, \\
 c_1^k \tau^k, & n = 2k - 1, \quad k = 1, 2, \ldots, \\
 c_2^k \tau^k, & n = 2k, \quad k = 1, 2, \ldots.
\end{cases}
\]

One can verify that (4.62) and the cumulants at energy \(\omega\), given by the integrands of (2.18), (2.19) (see also (2.21)–(2.26)), satisfy the conventional relations between moments and cumulants. The nice surprise is that for generic \(n\) the moment \(m_n\) is much simpler then the corresponding cumulant. This fact represents a great technical advantage for solving the moment problem, namely for determining a probability distribution \(\varphi(\xi)\) such that

\[
m_n = \int_{\mathcal{D}} d\xi \, \xi^n \varphi(\xi).
\]

There exist \[36\] three possible choices for the domain \(\mathcal{D}\): the whole line \(\mathcal{D} = \mathbb{R}\), the half line \(\mathcal{D} = \mathbb{R}_+\) and a compact interval \(\mathcal{D} = [a, b]\). A necessary and sufficient condition for the existence of \(\varphi\) on \(\mathbb{R}\) is \[36\] the non-negativity of the Hankel determinants

\[
\mathbb{H}_n \equiv \begin{vmatrix} 
m_0 & m_1 & \cdots & m_n \\
m_1 & m_2 & \cdots & m_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n+1} & \cdots & m_{2n} 
\end{vmatrix} \geq 0.
\]

From (4.63) one gets

\[
\mathbb{H}_0 = 1, \quad \mathbb{H}_1 = \tau(c_2 - c_1^2 \tau), \quad \mathbb{H}_2 = \tau^3(1 - c_2)(c_2^2 - c_1^2 \tau), \quad \mathbb{H}_{n \geq 3} = 0.
\]

Using the explicit form of \(c_1\) and that \(\tau \in [0, 1]\), one can show that \(\mathbb{H}_2\) and \(\mathbb{H}_3\) are non-negative. Therefore the probability distribution \(\varphi\) on \(\mathbb{R}\) exists. Combining the results of \[36\] with

\[
\mathbb{H}_2' \equiv \begin{vmatrix} 
m_1 & m_2 \\
m_2 & m_3 
\end{vmatrix} = \tau^2(c_1^2 \tau - c_2) \leq 0,
\]

one concludes that the domains \(\mathbb{R}_+\) and \([a, b]\) are excluded and one is left therefore with the so called Hamburger moment problem \(\mathcal{D} = \mathbb{R}\). Moreover, since \(\mathbb{H}_{n \geq 3} = 0\) the general theory \[36\] implies that \(\varphi\) is localised in three different points on the \(\xi\)-line. In fact, employing (4.60)–(4.63), one finds
\[ \varphi(\xi) = \int_{-\infty}^{\infty} d\lambda \frac{d\lambda}{2\pi} e^{-i\lambda \xi} \chi_\omega(\lambda) \]
\[ = \frac{1}{2}(c_2 - c_1)\delta(\xi + e\sqrt{\tau}) + (1 - c_2)\delta(\xi) + \frac{1}{2}(c_2 + c_1)\delta(\xi - e\sqrt{\tau}), \]  
(4.67)

where the charge \( e \) has been restored for clarifying the physical interpretation of \( \varphi \). The normalisation condition

\[ \int_{-\infty}^{\infty} d\xi \varphi(\xi) = 1 \]  
(4.68)

is easily verified. One can show in addition that the coefficients of the delta distributions in (4.67) take value in the interval \([0, 1]\) and have therefore a direct probabilistic interpretation.

In analysing this fundamental point it is useful to distinguish the two cases \( t = 0 \) and \( t = 0 \).

The basic microscopic process, which takes place in the system for \( t = 0 \), is the emission of a particle with energy \( \omega \) from a reservoir \( R_i \) and its absorption from \( R_j \). Let us denote by \( p_{ij} \) the relative probability. Recalling that \( \varphi \) concerns the lead \( L_1 \), the three terms in (4.67) correspond to the three elementary processes of emission and absorption relative to the reservoir \( R \), namely:

i. \( p_{12} = \frac{1}{2}(c_2 - c_1)\sqrt{\tau} \) is the probability for a particle to be emitted from \( R_1 \) and absorbed by \( R_2 \);

ii. \( p_{11} = (1 - c_2) \) is the probability for a particle to be emitted from and absorbed by \( R_1 \);

iii. \( p_{21} = \frac{1}{2}(c_2 + c_1)\sqrt{\tau} \) is the probability for a particle to be emitted from \( R_2 \) and absorbed by \( R_1 \).

The variation \( \xi \) of the charge in \( L_1 \), involved in these processes, is fixed by the support of the delta functions in (4.67) and is \( \xi = e_{\text{eff}}, \xi = 0 \) and \( \xi = -e_{\text{eff}} \) respectively, \( e_{\text{eff}} \) being defined in (3.51). These values of \( \xi \) are consistent with the physical interpretation of (i)–(iii).

It is worth stressing that \( p_{11} \) is \( \tau \)-independent as intuitively expected.

For \( \tau = 0 \) the two leads are disconnected and one expects therefore that \( p_{12} = p_{21} = 0 \) and \( p_{11} = 1 \). This simple physical observation is confirmed by the fact that the three terms in (4.67) collapse in one, namely:

\[ \varphi(\xi)|_{\tau=0} = \delta(\xi). \]  
(4.69)

The property (4.69) implies also that as functions of \( \tau \) the probabilities \( p_{ij} \) are discontinuous at \( \tau = 0 \), when the system divides in two parts.

The probability distribution \( \varphi \) in \( L_2 \) is obtained by implementing \( d_1 \leftrightarrow d_2 \), or equivalently (see (2.27)) by the substitutions \( c_1 \rightarrow -c_1 \) and \( c_2 \rightarrow c_2 \) in (4.67). As expected, under this operation one has \( p_{22} \leftrightarrow p_{21} \). We deduce moreover that \( p_{22} = p_{11} \), which completes the picture of the two lead junction at a microscopic level.

It is instructive to study the behavior of the probabilities \( p_{ij} \) because they provide fundamental information about the elementary processes in the system and uniquely fix the probability distribution, thus determining all moments and cumulants. Since

\[ p_{11} + p_{12} + p_{21} = 1, \]  
(4.70)

it is enough to focus on the pair \( \{p_{12}, p_{21}\} \).

For describing the low temperature limit \( \beta_1 = \beta_2 = \beta \rightarrow \infty \) it is convenient to adopt the variables \( \mu_{\pm} \) defined by (3.46), assuming without loss of generality that \( \mu_- \geq 0 \). Then one finds...
\[ \lim_{\beta \to \infty} p_{12} = \begin{cases} 0, & \mu_+ < \omega - \mu_-, \\ (1 - \sqrt{T})/2, & \omega - \mu_- < \mu_+ < \omega + \mu_-, \\ 0, & \mu_+ > \omega + \mu_. \end{cases} \quad (4.71) \]

and

\[ \lim_{\beta \to \infty} p_{21} = \begin{cases} 0, & \mu_+ < \omega - \mu_-, \\ (1 + \sqrt{T})/2, & \omega - \mu_- < \mu_+ < \omega + \mu_-, \\ 0, & \mu_+ > \omega + \mu_. \end{cases} \quad (4.72) \]

At the boundary points \( \mu_+ = \omega \pm \mu_- \) one has instead

\[ \lim_{\beta \to \infty} p_{12} = (1 - \sqrt{T})/4, \quad \lim_{\beta \to \infty} p_{21} = (1 + \sqrt{T})/4, \quad \mu_+ \neq 0, \]

\[ \lim_{\beta \to \infty} p_{12} = \lim_{\beta \to \infty} p_{21} = 1/4, \quad \mu_- = 0. \quad (4.73) \]

We conclude that at low temperatures the process of emission and absorption from the same reservoir is favored for \( \mu_+ \) outside the interval \([\omega - \mu_-, \omega + \mu_-]\), which is illustrated by the left panel of figure 3. The probabilities \( p_{21} \) and \( p_{12} \) are instead dominating if \( \mu_+ \in (\omega - \mu_-, \omega + \mu_-) \), as shown in the right panel of the same figure. Finally, equation (4.73) implies that in the low temperature limit \( p_{11} = 1/2 \) for \( \mu_+ = \omega \pm \mu_- \).

At high temperatures one has

\[ \lim_{\beta \to 0} p_{12} = \lim_{\beta \to 0} p_{21} = 1/4, \quad (4.74) \]

which is manifest in figure 3.
At high energies one finds
\[
\lim_{\omega \to \infty} p_{12} = \lim_{\omega \to \infty} p_{21} = 0, \tag{4.75}
\]
showing that in this regime the probability of emission of a particle from one reservoir and its absorption from the other one is negligible. This feature is illustrated by the blue and red curves in the plots of figure 4 for constant \( \tau = 0.5 \).

Summarising, the solution of the moment problem at hand is a superposition of three delta functions. The coefficients are positive and have direct physical interpretation, representing the probabilities of emission and absorption of particles from the heat reservoirs of the system.

We conclude with the following two observations. First, in terms of the basic probabilities \( \{\beta_{ij}\} \), the cumulant generating function (3.33) takes the form
\[
\mathcal{C}(\lambda) = \int_0^\infty \frac{d\omega}{2\pi} \ln \{1 + i(p_{21} - p_{12})\sin(\lambda\sqrt{\tau}) + (p_{21} + p_{12})[\cos(\lambda\sqrt{\tau}) - 1]\}
\]
\[
= \int_0^\infty \frac{d\omega}{2\pi} \ln [p_{11} + p_{21} e^{i\lambda\sqrt{\tau}} + p_{12} e^{-i\lambda\sqrt{\tau}}]. \tag{4.76}
\]
Second, the distribution (4.67) has a straightforward extension to the multi terminal system \( N \geq 3 \). In this case the probability distribution \( \varphi(\xi) \) in the lead \( L_1 \) is obtained from (4.67) by replacing \( c_i \) by the effective quantities \( c_{i e f} \) according to (3.42) and fixing \( \tau \) by (3.41). Now \( p_{12} \) is the probability of the emission of a particle from \( R_1 \) and absorption by any of the reservoirs \( \{R_2, R_3, \ldots, R_N\} \), whereas \( p_{21} \) gives the probability of the inverse process.

5. Outlook and conclusions

We developed a quantum field theory approach for the derivation of the \( n \)-point particle current correlation functions for the system in figure 1, modelling a quantum wire junction away from equilibrium. The system has the geometry of a star graph with free particle propagation along the leads and a defect interaction localised in the vertex. The new achievements in this framework are:

a. the closed and exact form (2.18), (2.19) of the cumulants \( C_n \) for generic \( n \);
b. the cumulant generating function (3.40)–(3.42) in the \( N \)-terminal case;
c. the cumulant generating function (3.45) in the scale invariant limit;
d. the moments (4.62) and the associated probability distribution (4.67);
e. the exact emission-absorption probabilities \( p_{ij} \).

These results clearly indicate that the probabilities \( p_{ij} \), which describe the fundamental microscopic processes in the system and are the final goal of our investigation, represent the core of the quantum transport problem in consideration. In fact, our analysis shows that \( p_{ij} \), supplemented by the defect transmission \( \tau \), are the building blocks of the probability distribution and uniquely determine all moments and cumulants. We established the explicit form of \( p_{ij} \) and analysed in detail their dependence on the heat bath parameters \( \{\beta_i, \mu_i\} \).

The field theoretical framework, developed in this paper, is universal and can be applied to other systems as well. It will be interesting for instance to extend the above results to Majorana fermions, which attract much attention [37, 38] with their remarkable physical properties and potential applications in topological quantum information [39]. In this context
the quantum transport of Majorana fermions along the edge of a topological superconductor represents a fascinating problem, which can be faced by the above methods.

Appendix A. Correlation functions in the LB state

In the computation of the current correlators (2.8) we need the 2\(n\)-point functions
\[
\langle a_i^+(k_i)a_{m_1}(p_1) \cdots a_i^+(k_n)a_{m_n}(p_n) \rangle_{\beta,\mu}
\]
for positive momenta. In this case
\[
\langle a_i^+(k_i)a_{m_1}(p_1) \cdots a_i^+(k_n)a_{m_n}(p_n) \rangle_{\beta,\mu} = \frac{1}{Z} \text{Tr}[\text{e}^{-K}a_i^+(k_i)a_{m_1}(p_1) \cdots a_i^+(k_n)a_{m_n}(p_n)],
\]
where
\[
K = \int_0^\infty \frac{dk}{2\pi} \sum_{i=1}^N \beta_i [\omega(k) - \mu_i] a_i^+(k_i)a_i(k), \quad Z = \text{Tr}(\text{e}^{-K}).
\]

For the two-point functions one gets (\(k > 0, p > 0\))
\[
\langle a_i^+(k)a_{m}(p) \rangle_{\beta,\mu} = 2\pi \delta(k - p)\tilde{\delta}_{im}d_i[\omega(k)] \equiv \Delta_{im}(k, p),
\]
\[
\langle a_{m}(p)a_i^+(k) \rangle_{\beta,\mu} = 2\pi \delta(k - p)\tilde{\delta}_{im}[1 - \hat{d}_i[\omega(k)]] \equiv \tilde{\Delta}_{im}(k, p),
\]
where \(d_i(\omega)\) is the Fermi distribution (2.10) of the reservoir \(R_i\).

The 2\(n\)-point function can be expressed in terms of (A.80), (A.81) as a determinant. For \(k_i > 0, p_j > 0\) one has:
\[
\langle a_i^+(k_i)a_{m_1}(p_1) \cdots a_i^+(k_n)a_{m_n}(p_n) \rangle_{\beta,\mu} = \begin{vmatrix}
\Delta_{i,m_1}(k_1, p_1) & \Delta_{i,m_1}(k_1, p_2) & \cdots & \Delta_{i,m_1}(k_1, p_n) \\
\Delta_{i,m_2}(k_2, p_1) & \Delta_{i,m_2}(k_2, p_2) & \cdots & \Delta_{i,m_2}(k_2, p_n) \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{i,m_n}(k_n, p_1) & \Delta_{i,m_n}(k_n, p_2) & \cdots & \Delta_{i,m_n}(k_n, p_n) \\
\end{vmatrix}.
\]

The validity of (A.82) can be proven by induction. For \(n = 1\) the determinant correctly reproduces (A.80). Using the definition (A.78), (A.79) one can show that for \(n \geq 2\)
\[
\langle a_i^+(k_i)a_{m_1}(p_1) \cdots a_i^+(k_n)a_{m_n}(p_n) \rangle_{\beta,\mu} = \sum_{k=1}^n \langle a_i^+(k_i)a_{m_1}(p_1) \rangle_{\beta,\mu} \langle \cdots \rangle_{\beta,\mu},
\]
where the primed expectation value \(\langle \cdots \rangle_{\beta,\mu}\) indicates that the elements \(a_i^+(k_i)\) and \(a_{m_1}(p_1)\) are removed from the string. Iterating this step in \(\langle \cdots \rangle_{\beta,\mu}\) one obtains the Laplace expansion of the determinant in (A.82) along the first row, which concludes the argument.

Finally, one extends [27] the correlation functions (A.80)–(A.82) to negative momenta by using the reflection-transmission constraints
Appendix B. Some basic properties of the T-matrix

The Hermitian matrices $\mathbb{T}_1$, defined by (2.15), play a fundamental role. For illustrating their properties it is enough to focus on $\mathbb{T}_1$ associated with lead $L_1$. First of all, there exists a $N \times N$ unitary matrix $U$ diagonalising $\mathbb{T}_1$. After some algebra one finds

$$U \mathbb{T}_1 U^\dagger = \text{diag}[0, 0, \ldots, -\sqrt{\tau(\omega)}, \sqrt{\tau(\omega)}], \quad (B.85)$$

where

$$\tau(\omega) = \sum_{i=2}^{N} \tau_i(\omega), \quad \tau_i(\omega) \equiv |\mathbb{S}_{i1}(\sqrt{2m\omega})|^2, \quad (B.86)$$

is the total transmission probability between the lead $L_1$ and the remaining leads $L_i$ with $2 \leq i \leq N$. We conclude that $N - 2$ of the eigenvalues of $\mathbb{T}_1$ vanish, the nontrivial two being $\pm \sqrt{\tau(\omega)}$. A complete system of eigenvectors is given by

$$v_1 = (0, -\mathbb{S}_{N1}, 0, \ldots, 0, \mathbb{S}_{21}),$$

$$v_2 = (0, 0, -\mathbb{S}_{N1}, \ldots, 0, \mathbb{S}_{31}),$$

$$\ldots \ldots \ldots$$

$$v_{(N-2)} = (0, 0, \ldots, -\mathbb{S}_{N1}, \mathbb{S}_{(N-1)1}), \quad (B.87)$$

with eigenvalue 0 and

$$v_+ = (\pm \sqrt{\tau} - \tau, \mathbb{S}_{11}\mathbb{S}_{21}, \mathbb{S}_{11}\mathbb{S}_{31}, \ldots, \mathbb{S}_{11}\mathbb{S}_{N1}), \quad (B.88)$$

with eigenvalues $\pm \sqrt{\tau}$. Orthonormalising the system (B.87), (B.88), one determines the diagonalising matrix $U$ in explicit form.

Appendix C. Generating function in a single energy channel

We summarise here the key points of the computation of the expectation value (3.39) which leads to the basic result (3.40). Without loss of generality we concentrate on the lead $L_1$, the extension to other leads being straightforward. The problem consists in computing

$$\langle \psi^{(1)}_{l,\beta} \rangle_{l,\mu} = \langle \psi^{(1)} \sum_{j=1}^{\mu} a_j^\dagger a_j \rangle_{\beta,\mu}. \quad (C.89)$$

It is convenient for this purpose to change the basis in the algebra (3.37) according to

$$b_j = \sum_{j=1}^{N} U_j a_j, \quad b_j^\dagger = \sum_{j=1}^{N} a_j^\dagger U^\dagger_{j\mu}, \quad (C.90)$$

where $U$ is the matrix diagonalising $\mathbb{T}_1$ according to (B.85). In the new basis the expectation value (C.89) reads

$$\langle \psi^{(1)}_{l,\beta} \rangle_{\beta,\mu} = \left\{ \prod_{j=1}^{N} e^{\lambda_l T_{ji} b_j^\dagger} \right\}_{\beta,\mu} = \left\{ \prod_{j=1}^{N} [1 + (\psi^{(1)}_{ji} - 1) b_j^\dagger b_j] \right\}_{\beta,\mu}. \quad (C.91)$$
In the case $N = 2$ one has $\mathbb{1}_d = \text{diag}(\sqrt{\tau}, \sqrt{\tau})$ and one finds
\[
\langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} = \det \left[ 1 + \langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} - I \right] \left( b_1^\dagger b_1 + b_2^\dagger b_2 \right)_{\beta, \mu} + c_2 \cos(\lambda_1 \sqrt{\tau}) - 1.
\]
Combining (C.92) with (3.38), (3.34) one gets the final result (3.33).

The extension to a generic $N > 2$ is attained by expanding the product in (C.91) and expressing the multiple correlators of the operators $\{b_i, b_i^\dagger\}$ in terms of two-point functions through the analogue of formula (A.82). One recognises at this point that the series can be re-summed as a determinant
\[
\langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} = \det[I + \langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} - I] \mathbb{D},
\]
where $\mathbb{D}$ is the diagonal matrix (2.17) of the Fermi distributions of the reservoirs. In the basis in which $\mathbb{D}$ is diagonal the result (C.93) takes the form
\[
\langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} = \det[I + \langle \exp(\lambda_1^\dagger \mathbb{1}) \rangle_{\beta, \mu} - I] U \mathbb{D} U^*.
\]
Using the explicit form of $U$, which can be deduced from (B.87), (B.88), one obtains from (C.94) the cumulant generating function (3.40)–(3.42) in the multi terminal case.

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