A tomographic approach to non-Markovian master equations

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Abstract
We propose a procedure based on symplectic tomography for reconstructing the unknown parameters of a convolutionless non-Markovian Gaussian noisy evolution. Whenever the time-dependent master equation coefficients are given as a function of some unknown time-independent parameters, we show that these parameters can be reconstructed by means of a finite number of tomograms. Two different approaches towards reconstruction, integral and differential, are presented and applied to a benchmark model made up of a harmonic oscillator coupled to a bosonic bath. For this model the number of tomograms needed to retrieve the unknown parameters is explicitly computed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
A central issue in modern physics is the investigation of noise as it drastically affects the evolution of quantum systems. Such a general phenomenon is of importance in quantum information science and beyond, as it addresses a fundamental issue in quantum theory [1, 2]. So far, two main dynamical regimes, Markovian and non-Markovian, can usually be distinguished according to the noise time scale (respectively shorter or longer than that of...
system dynamics). Here we address the non-Markovian case. In fact, despite Markovian evolutions have been extensively investigated, in general real noisy dynamics are far from being Markovian. Although gaining much interest in the last years in both theory and experiment [3, 4], a general theory for non-Markovian dynamics is still missing. In this dynamical regime, exact master equations have been derived for the evolution of a Brownian particle linearly coupled to a harmonic oscillator bath, for instance via path integral methods [5, 6] or phase-space and Wigner function computations [7, 8]. Analogous results are derived by means of quantum trajectories, either exactly or in a weak-coupling approximation [9–12]. In the framework of path integral methods, master equations have been derived both for initially correlated states [13, 14] and for factorized initial states in the case of weak nonlinear interactions [15]. In general, all these master equations cover only few cases and are not simple to solve. Indeed, it would be highly desirable to find an approximation scheme fully capturing non-Markovian features. In general, different approximations (for example on the system–bath interaction strength) may lead to irreconcilable dynamics [1].

For Markovian dissipation it has been shown that, by exploiting symplectic tomography [16] and using Gaussian probes, the unknown parameters governing the dynamics of the system can be reconstructed through a limited number of tomographic measurements [17]. In this paper we extend this tomography-based approach to time-dependent dissipative dynamics. We present an experimentally feasible procedure that exploits symplectic tomography and which, under the assumption of Gaussian noise, allows us to reconstruct the unknown time-independent parameters (TIPs) that characterize the time-dependent coefficients of the master equation. Even though the assumption of Gaussian noise might be seen as an idealization, it is actually well fitted for a significant number of models [1, 2]. Also, small deviations from Gaussianity would introduce small and controllable errors. We also note that Gaussian probes are quite straightforward to produce [18].

The paper is organized as follows. In section 2 we briefly recall the procedure that allows us to reconstruct cumulants of a Gaussian state (our probe) undergoing a dissipative dynamics. In section 3 we introduce the class of non-Markovian master equations we study, and derive the expressions for the first and second time-dependent momenta (cumulants). In section 4 we devise two alternative approaches based on quantum tomography that allow us to reconstruct the unknown time-independent master equation parameters. In section 5 we apply these procedures to a benchmark model and compute the amount of measurements needed. In section 6 we summarize and discuss our results. The appendix contains more details on the reconstruction of the cumulants of a Gaussian state through symplectic tomography.

2. From tomograms to cumulants

In [17] we have introduced a procedure that allows us to reconstruct, via a limited number of measurements, the time-independent master equation coefficients (MECs) governing the dynamical dissipative evolution of a quantum system. In the Markovian case, it has been shown that, using a Gaussian probe, the required number of measurements is at most 10. In fact, the dynamical evolution of a Gaussian state is completely determined by the evolution of its first- and second-order cumulants, which are measurable quantities. The unknown MECs enter the dynamical equations of the cumulants, hence can be retrieved by simple inversion, once the latter are measured. The cumulants can be obtained by using symplectic tomography. Indeed, given the Wigner function of a Gaussian state, it can be measured along lines in phase space (i.e. by performing its tomographic map). This allows us to relate the cumulants to the points on the tomogram. By choosing the lines in phase space corresponding, respectively, to the position and momentum probability distributions, one needs at most four points along
each tomogram (i.e. line) to retrieve the first and second cumulant of the associated variable. The same procedure, applied on a line inclined by \( \pi/4 \) in phase space allows us to retrieve the covariance of the two variables by measuring at most two points. Hence, given a Gaussian shape preserving (GSP) Markovian master equation, by using Gaussian probes one can retrieve at any time its evolved cumulants via a finite amount of measurements. In the following, we refer to this as the tomogram–cumulant (T-C) procedure. More details of the T-C procedure are provided in the appendix. As the key ingredient of the T-C procedure is the preservation of Gaussianity, it is therefore a natural step to investigate how this procedure can be generalized and extended to more involved non-Markovian scenarios, still preserving Gaussianity. Indeed our aim is further supported by recent works [19–21], in which it has been proved that it is in principle possible to make tomographic measurements of the probability densities associated with every quadrature in phase space (for example in quantum optics it could be realized by means of homodyne detection). As a final remark we note that other methods to measure the covariance matrix of Gaussian states have been discussed in [22]. However, in this case the amount of required measurements is much higher, the focus being on the reduction of experimental errors.

3. Non-Markovian master equation

We focus on the class of master equations of the form [1]

\[
\frac{\text{d}\hat{\rho}(t)}{\text{d}t} = L(t)[\hat{\rho}(t)] = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}(t)] + \frac{1}{2\hbar} \sum_j \left( [\hat{V}_j(t)\hat{\rho}(t), \hat{V}_j^\dagger(t)] + [\hat{V}_j(t), \hat{\rho}(t)\hat{V}_j^\dagger(t)] \right),
\]

where the generator \( L(t)[\cdot] \) depends on time. In particular we investigate the following class of time-dependent master equations:

\[
\frac{\text{d}\hat{\rho}(t)}{\text{d}t} = -\frac{i}{\hbar} [\hat{H}_0, \hat{\rho}(t)] - \frac{i(\lambda(t) + \delta)}{2\hbar} [\hat{q}, \hat{p}\hat{\rho}(t) + \hat{\rho}(t)\hat{p}] + \frac{i(\lambda(t) - \delta)}{2\hbar} [\hat{\rho}(t)\hat{q} + \hat{q}\hat{\rho}(t)]
\]

\[
- \frac{D_{pp}(t)}{\hbar^2} [\hat{q}, [\hat{q}, \hat{\rho}]] - \frac{D_{qq}(t)}{\hbar^2} [\hat{p}, [\hat{p}, \hat{\rho}]]
\]

\[
+ \frac{D_{qp}(t)}{\hbar^2} ([\hat{q}, [\hat{p}, \hat{\rho}(t)]] + [\hat{p}, [\hat{q}, \hat{\rho}(t)]]).
\]

The master equation (2) is obtained from the general form (1) by choosing a system Hamiltonian of the form

\[
\hat{H} = \hat{H}_0 + \delta (\hat{q}\hat{p} + \hat{p}\hat{q}), \quad \hat{H}_0 = \frac{1}{2m} \hat{p}^2 + \frac{m\omega^2}{2} \hat{q}^2,
\]

and the linear Lindblad operators \( \hat{V}_j(t) \):

\[
\hat{V}_j(t) = a_j(t)\hat{p} + b_j(t)\hat{q}, \quad j = 1, 2.
\]

Using equations (3) and (4) in (1), the time-dependent MECs in (2) hence read

\[
D_{qq}(t) = \frac{\hbar}{2} \sum_{j=1,2} |a_j(t)|^2, \quad D_{pp}(t) = \frac{\hbar}{2} \sum_{j=1,2} |b_j(t)|^2,
\]

\[
D_{qp}(t) = -\frac{\hbar}{2} \text{Re} \sum_{j=1,2} a_j^*(t)b_j(t), \quad \lambda(t) = -\text{Im} \sum_{j=1,2} a_j^*(t)b_j(t),
\]

where \( D_{ij}(t) \)'s and \( \lambda(t) \) are, respectively, the so-called diffusion and friction coefficients.

The master equation in (2) represents a natural generalization of the time-independent master
equation introduced in [23] which describes a GSP evolution of a quantum state. In the
time-dependent case a wide range of models obeys a GSP master equation of the form (2)
[6–8, 10–14]. As a side remark we note here that the literature about non-Markovian master
equations may lead to some ambiguity. Indeed some authors classify as non-Markovian only
those master equations whose generator contains a convolution integral. It has recently been
proved in [24] that these generators can be mapped into convolutionless ones, following a
so-called local approach. Non-Markovianity becomes then characterized by the dependence
of the convolutionless generator on \( t - t_0 \), where \( t_0 \) is the initial time. According to this
approach a time-dependent convolutionless generator as equation (2) could be considered
Markovian. However, following a consistent part of literature, e.g. [5, 7, 8, 10, 11, 15, 25],
we term non-Markovian convolutionless time-dependent generators as the one in (2).

3.1. Evolution of the cumulants

We begin by writing the dynamical equations, obtained from (2), for the cumulants of a
Gaussian state. They can be expressed in a compact matrix form as

\[
\frac{d}{dt} S(t) = (M - \lambda(t)I_2) S(t),
\]

\[
\frac{d}{dt} X(t) = (R - 2\lambda(t)I_3) X(t) + D(t),
\]

where \( I_2(3) \) is the two(three)-dimensional identity matrix. The vectors \( S(t) \) and \( X(t) \)
correspond, respectively, to the first- and second-order cumulants:

\[
S(t) = \frac{1}{\sqrt{\hbar}} \left( \sqrt{m\omega} \langle \hat{q}\rangle, \frac{\langle \hat{p}\rangle}{\sqrt{m\omega}} \right),
\]

\[
X(t) = \frac{1}{\hbar} \left( \frac{m\omega \Delta q^2}{\langle \sigma_{q,p} \rangle}, \frac{\Delta p^2 / (m\omega)}{\langle \sigma_{q,p} \rangle} \right).
\]

The matrices \( M \) and \( R \) contain the Hamiltonian parameters:

\[
M = \begin{pmatrix} \delta & \omega \\ -\omega & -\delta \end{pmatrix}, \quad R = \begin{pmatrix} 2\delta & 0 & 2\omega \\ 0 & -2\delta & -2\omega \\ -\omega & \omega & 0 \end{pmatrix}
\]

and, finally, \( D(t) \) is the diffusion vector

\[
D(t) = \frac{2}{\hbar} \begin{pmatrix} m\omega D_{qq} \\ D_{pq} \\ m\omega D_{pp} \end{pmatrix}.
\]

As in the time-independent case, the first cumulant dynamical evolution (6) depends only
on \( \lambda(t) \) also entering, together with the diffusion coefficients, the second-order cumulants
equation (7). Hence, the friction coefficient can be retrieved by inverting (6). By carrying out
a formal integration one obtains the following expression:

\[
\int_0^t dt' \lambda(t') = \ln \left( \frac{S_j(0)}{S_j(t)} \right),
\]

where the suffix \( j = 1, 2 \) labels the two components of the vector

\[
\tilde{S}(t) = e^{-iM} S(t).
\]

Analogously, (7) can be rewritten as

\[
\frac{d}{dt} \tilde{X}(t) = \tilde{D}(t),
\]
where
\[ \tilde{X}(t) = e^{2 \int_0^t dt' \lambda(t') D(t')} e^{-itR X(t)}, \]
\[ \tilde{D}(t) = e^{2 \int_0^t dt' \lambda(t') D(t')} e^{-itR D(t)}. \]

(14)

It can be shown [23] that the transformations in (12) and (14) are always invertible, provided one sets the quantity \( \eta \equiv \sqrt{\delta^2 - \omega^2} \) to \( i\Omega \), whenever \( \eta^2 < 0 \). The formal solution of (13) is given by
\[ \tilde{X}(t) = \tilde{X}(0) + \int_0^t dt' \tilde{D}(t'). \]

(15)

Inverting the transformation in (14), one can write
\[ \int_0^t dt' e^{-2 \int_0^t dt' \lambda(t')} e^{(t-t')R} D(t') = X(t) - e^{itR} e^{-2 \int_0^t dt' \lambda(t')} X(0). \]

(16)

Both in (11) and (16), measurable quantities appear on the right-hand side, whereas the (unknown) coefficients are on the left-hand side.

4. Reconstruction of time-independent parameters

Hereafter, we will assume that the MECS \( \lambda(t), D_{qq}(t), D_{pp}(t) \) and \( D_{qp}(t) \) have a known functional form. This implies that the non-Markovian master equation with certain expressions for the MECS has been previously derived within some approximation scheme (e.g. by means of a microscopic derivation and subsequent dynamical assumptions). The time-dependent MECS are thus the function of a set of TIPs whose value is \textit{a priori} unknown. In this section we propose two alternative procedures aiming at reconstructing the TIPs by means of symplectic tomography.

4.1. Integral approach

Here we introduce an approach based on formal integration of the dynamical equations (11) and (16). The right-hand side of both equations involves experimental inputs (\( S(t) \) and \( X(t) \)) and the known Hamiltonian parameters (\( R \)). The left-hand sides, once the MECS are known, can be regarded as functions of the TIPs. Hence the set of TIPs can be in principle obtained by inverting these relations. Unfortunately, in general, an analytical inversion of (11) and (16) may represent a highly involved task. In fact, even if we assume a known MEC time dependence, we could be unable to either compute analytically the integrals on the right-hand side of (11) and (16), or to invert the equations, or even both. All these problems anyway can be overcome by resorting to numerical computation. To provide an example of how to apply this procedure, in the next section we apply it to a specific model.

4.2. Differential approach

The tomographic T-C procedure recalled in section 2 allows us to measure not only the cumulants of a given Gaussian state but also their first time derivatives. Indeed, we can estimate the derivative through the incremental ratio by measuring each cumulant at two different times \( t \) and \( t + \delta t \). For instance
\[ \frac{d}{dt} q_i^2 \sim \frac{\Delta q_i^2}{\delta t}, \]

where the amount of time \( \delta t \) is defined as the smallest time interval which allows us to experimentally distinguish two different values of the given cumulant. Once we substitute derivatives with their approximations, the two sets of equations (6) and (7) are not differential
anymore. Being the cumulants and their approximate derivatives at given times experimental inputs, we insert in (6) and (7) the time-dependent coefficients $\lambda(t)$ and $D(t)$ (and in case $m(t)$, $\omega(t)$, $\delta(t)$), which involve the unknown TIPs and in this way the two sets of equations (6) and (7) reduce to algebraic equations which can always be solved, at least numerically. Due to this simplification, this approach is also suitable when dealing with more complicated generators than the one considered here in equation (2), e.g. when the Lamb shift contribution is explicitly taken into account such that the Hamiltonian parameters become time dependent: $m(t)$, $\omega(t)$, $\delta(t)$. We note that the two sets (6) and (7) consist of five equations which must be fulfilled at any chosen time. Therefore, considering them at different times, we can derive a system made up of an arbitrary number of equations. The number of equations must then be chosen as the minimum amount of equations needed to uniquely determine the TIPs, which is clearly model dependent. In the following section, we show how to apply this procedure to a benchmark model.

5. A benchmark model

In the following we refer to a specific model of a quantum Brownian particle discussed in [25]. We show how to apply in this specific case the two general procedures presented in the previous section. The model consists of an Ohmic reservoir made of harmonic oscillators, linearly coupled to a single harmonic oscillator of frequency $\omega$ (our system particle) through the coupling constant $\alpha$, with a Lorentz–Drude cutoff [1] $\omega_c$ and at temperature $T$. Starting from a superoperatorial version of the Hu–Paz–Zhang master equation [5], a secular master equation of the form (2) is obtained in the weak-coupling limit (up to the second order in $\alpha$), with the following coefficients:

$$\lambda(t) = \frac{\alpha^2 \omega \omega_c}{\omega_c^2 + \omega^2} \left\{ 1 - e^{-\omega_c t} \left[ \cos(\omega t) + \frac{\omega_c}{\omega} \sin(\omega t) \right] \right\},$$

$$\Delta(t) = \frac{2 \alpha^2 \omega^2}{\omega_c^2 + \omega^2} \frac{kT}{\hbar} \left\{ 1 - e^{-\omega_c t} \left[ \cos(\omega t) - \frac{\omega_c}{\omega} \sin(\omega t) \right] \right\},$$

the last having this form for high temperatures $T$. This master equation is of Lindblad-type when the coefficients $\Delta(t) \pm \lambda(t)$ are positive at all times. The Lindblad–non-Lindblad border as a function of the temperature $T$ and the frequency cutoff $\omega_c$ has been analysed in [25]. As Gaussianity is preserved, by choosing a Brownian particle initially in a Gaussian state, the T-C procedure can be employed at any time. The coefficients $\lambda(t)$ and $\Delta(t)$ reach stationary values for $t \gg \frac{1}{\omega_c}$:

$$\lambda(t) \rightarrow \frac{\alpha^2 \omega \omega_c}{\omega_c^2 + \omega^2}, \quad \Delta(t) \rightarrow \frac{2 \alpha^2 \omega^2}{\omega_c^2 + \omega^2} \frac{kT}{\hbar}.$$  

In this specific model, the unknown TIPs are the coupling constant $\alpha$, the temperature $T$ and the frequency cutoff $\omega_c$. Usually, when studying quantum Brownian motion, one assumes $\omega_c/\omega \gg 1$, corresponding to a Markovian reservoir, with $\omega_c \rightarrow \infty$. In this limit, the
5.1. Example: integral approach

Here we apply the integral procedure (section 4.1) to the benchmark model. In this case the left-hand side of (16) is not simply analytically computable. Thus, we must use (11) to reconstruct all the TIPs it involves, and then numerically integrate the left-hand side of (16).

The left-hand side of (11) is given by

\[
\int_0^t dt' \lambda(t') = \frac{\alpha^2 \omega_c^2 \omega^2}{(\omega_c^2 + \omega^2)^2} \left[ \omega t \frac{\omega_c^2 + \omega^2}{\omega^2} - 2 \frac{\omega_c}{\omega} + e^{-\omega_c t} \left\{ 2 \frac{\omega_c}{\omega} \cos(\omega t) + \frac{\omega_c^2 - \omega^2}{\omega^2} \sin(\omega t) \right\} \right].
\]

(20)

By using (11) and (20), we obtain the following transcendental equation for the coupling strength \( \alpha^2 \):

\[
\alpha^2 = \ln \left( \frac{\tilde{S}_j(0)}{\tilde{S}_j(t)} \right) \left\{ \frac{(\omega_c^2 + \omega^2)^2}{\omega_c^2 \omega^2} \left[ \omega t \frac{\omega_c^2 + \omega^2}{\omega^2} - 2 \frac{\omega_c}{\omega} + e^{-\omega_c t} \left\{ 2 \frac{\omega_c}{\omega} \cos(\omega t) + \frac{\omega_c^2 - \omega^2}{\omega^2} \sin(\omega t) \right\} \right] \right\}^{-1},
\]

(21)

where the ratio \( \tilde{S}_j(0)/\tilde{S}_j(t) \) is the experimentally measurable quantity. Hence, by performing two distinct measurements of this ratio we can evaluate (21) at two different times. We thus obtain a system of two numerically solvable equations, which allows us to retrieve the TIPs \( \alpha \) and \( \omega_c \). To provide a concrete evidence of the validity of this procedure, we show two numerical examples in figure 1. Indeed we retrieve the TIP \( \alpha^2 = 0.01 \) in two different dynamical regimes, respectively, close to the Markovian (figure 1(a)) and non-Markovian (figure 1(b)) limits. The last missing parameter is the temperature \( T \) entering the coefficient thermalization time [25] is inversely proportional to the coupling strength, while for an out-of-resonance engineered reservoir with \( \omega_c/\omega \ll 1 \) (i.e. highly non-Markovian), the thermalization process is slowed down.

Figure 1. We show how to indirectly measure the TIPs \( \omega_c \) and \( \alpha^2 \), in two different regimes respectively close to the Markovian (a) and non-Markovian dynamics (b). Each line refers to an experimental measure of \( \ln(\tilde{S}_j(0)/\tilde{S}_j(t)) \) at a specific time \( \omega t \). In both regimes, the TIP values are found at the intersection of the two lines. In the example close to the Markovian regime (a) if we measure \( 3.03 \times 10^{-3} \) at \( \omega t_1 = 0.5 \) (solid line) and \( 9.70 \times 10^{-2} \) at \( \omega t_2 = 10 \) (dashed line), we retrieve \( \alpha^2 = 0.01 \) and \( \omega_c/\omega = 10 \). Analogously, for the example close to the non-Markovian regime (b) if we measure \( 4.55 \times 10^{-5} \) at \( \omega t_1 = 0.5 \) (solid line) and \( 1.84 \times 10^{-2} \) at \( \omega t_2 = 10 \) (dashed line), we retrieve \( \alpha^2 = 0.01 \) and \( \omega_c/\omega = 0.5 \).
Δ(t). By using (16), we obtain the following equation:
\[
\frac{kT}{\hbar \omega} = \left[ X_j(t) - e^{-2 \int_0^t dt' \lambda(t')} (e^{i R} X(0))_j \right] \frac{\omega_c^2 + \omega^2}{2 \omega^2 \omega_c} \left\{ \int_0^t dt' e^{-2 \int_0^{t'} dt'' \lambda(t'')} \sum_{l=1}^2 (e^{i (t-t') R})_{j,l} \right. \\
\times \left. \left[ 1 - e^{-\omega_c t} \left( \cos(\omega t') - \frac{\omega}{\omega_c} \sin(\omega t') \right) \right] \right\}^{-1}, \quad (22)
\]
where \( j = 1, 2, 3 \) denote the vector components, and \((e^{i (t-t') R})_{j,l}\) are the matrix elements of the matrix \( e^{i (t-t') R} \). The explicit expression of the integral appearing in the first line of (22) is provided in (20). In general, the remaining integrals are not analytically computable. However, since all the parameters involved have been previously reconstructed, these integrals can be computed numerically.

**Number of tomographic measurements.** Let us now explicitly compute the number of tomographic measurements needed to apply the integral approach to the benchmark model. To reconstruct \( \alpha \) and \( \omega_c \) each of the quantities \( \tilde{S}_{1,2}(t) = (e^{-i M} S)_{1,2}(t) \) must be measured once but not at the same time, as shown by (21) and figure 1. Each \( \tilde{S}_{1,2}(t) \) is a function of the first cumulants of both position and momentum. According to the T-C procedure (see the appendix), the reconstruction of a first cumulant involves at most four tomographic points. Thus, \( \alpha \) and \( \omega_c \) can be obtained via, in the worst case, 16 measures. Furthermore, being \( e^{-i M} \) an orthogonal transformation, \( e^{-i M} \tilde{S}(t) \) is by itself a first cumulant along a time-dependent direction in phase space. Hence, if time-dependent tomographic measurements (i.e. measurements in a frame rotating as \( e^{-i M} \)) are allowed, the number of required tomograms decreases to 8, as we would only need a single first cumulant (\( \tilde{S}_1(t) \) or \( \tilde{S}_2(t) \)).

To measure \( T \) we should evaluate one of the second cumulants at a given time. Following the T-C procedure this amounts to two tomographic points. However, the required second cumulant has been already obtained when reconstructing the corresponding first cumulant, hence the temperature can be retrieved without further effort. This argument also holds for time-dependent measurements. In fact, the reconstruction formula (22) has been derived from (16), which can be recast in terms of the variances in the rotating frame. The temperature can then be obtained using the variance along the same time-dependent direction of the measured first cumulant.

In conclusion, in order to implement the integral approach in the benchmark model, according to whether we can perform time-dependent measurements or not, we need 8 or at most 16 tomographic points.

**5.2. Example: differential approach**

We now skip to the differential procedure (section 4.2). In (18) the dependence on \( \alpha^2 \) and \( \omega_c \) is factorized; hence using (6) one gets
\[
\alpha^2 \sim \frac{1}{\langle \hat{q} \rangle} \left( \frac{\langle \hat{p} \rangle - \langle \hat{q} \rangle_{t+\delta t}}{\delta t} \right) \omega_c^2 + \frac{\omega^2}{\omega_c^2} \left[ 1 - e^{-\omega_c t} \left( \cos(\omega t) + \frac{\omega}{\omega_c} \sin(\omega t) \right) \right]^{-1}. \quad (23)
\]
Since \( \alpha^2 \) and \( \omega_c \) are time independent, they can be determined by solving (23) for two different times \( t_1 \) and \( t_2 \) and looking at the intersection of the two different solutions. This procedure requires to measure the cumulants \( \langle \hat{q} \rangle_{t_1}, \langle \hat{q} \rangle_{t_1+\delta t}, \langle \hat{p} \rangle_{t_1} \) and \( \langle \hat{q} \rangle_{t_2} \) at \( t = t_1, t_2 \) and to solve (23) numerically, as shown in figure 2. In other words, the first two TIPs, \( \alpha^2 \) and \( \omega_c \), can be determined by measuring six quantities. As for the integral procedure, we retrieve the TIP \( \alpha^2 = 0.01 \) in the two cases, corresponding to the extreme dynamical regimes, Markovian,
The reconstruction of temperature $T$ is based on equation (24). We need the variance of the position at time $t$ and at time $t + \delta t$, and the covariance of $\hat{q}$ and $\hat{p}$ at time $t$. However, according to the T-C procedure (appendix), the variance is required to obtain the position average. This implies that the second cumulant has been already measured during the previous reconstruction and there is no need to measure it again. Retrieving the covariance requires two more tomographic points.

Number of tomographic measurements. We now count the number of tomographic measurements to apply the differential approach to this example. The reconstruction of both $\alpha$ and $\omega_c$ is based on (23), which must then be evaluated at two different times $t_i$ and $t_2$, see figure 2. Each evaluation of (23) requires two measurements of the average position, at times $t_i$ and $t_i + \delta t$, and one of the average momentum at time $t_i$, where $i = 1, 2$. This implies reconstructing six first cumulants. As each first cumulant requires four tomographic points (see the appendix), the total number of needed tomographic points amounts to 24.
In conclusion, in order to implement the differential approach in the benchmark model, we need 26 tomograms.

Comparison between the two approaches. Let us now briefly compare the two procedures described in this section. On one hand the differential approach requires more experimental measurements compared to the integral one while, on the other, the latter procedure is more involved from a computational point of view. Indeed, it may happen that to compute the first members of equations (11)–(16) some numerical or analytical approximations are needed, thus reducing the accuracy of the reconstruction. In this case, the differential approach should be preferred as it is very simple from the point of view of analytical computation. Clearly, if the computation of the integral functions in equations (11)–(16) does not present remarkable difficulties, the integral procedure proves better as, requiring less measurements, it involves a lower number of interactions with the physical system. For example, in our benchmark model, the differential approach requires 26 measurements, while the integral approach requires 16 time-independent measurements or only eight time-dependent measurements.

One could summarize by saying that the integral procedure is more advantageous in terms of number of measurements, but requires the ability of solving potentially involved analytical expressions. The differential approach, instead, is more advantageous from the point of view of versatility, as it allows us to deal in a straightforward way with complex generators, at the expenses of a higher number of measurements. In conclusion, the choice between the two strategies introduced in this paper strictly depends on the specific model under investigation.

6. Conclusions

In this paper we have proposed an experimentally feasible procedure to reconstruct the unknown time-independent master equation parameters in a non-Markovian scenario. To this end we have adopted an approach based on symplectic tomography. While previously the case of Markovian dynamics has been investigated [17], here the procedure is generalized and extended to the more involved convolutionless non-Markovian case. In particular, our analysis is focused on the class of GSP master equations. We have addressed the situation in which the time-dependent MECs are the analytic functions of some unknown TIPs. The key point of our approach lies in using Gaussian states as probes, as information on the dissipative dynamics can be inferred via a limited number of tomograms. We have proposed two alternative procedures, integral and differential, to reconstruct the unknown quantities. In order to provide an explicit example of how these different approaches work, we have applied them to a benchmark model made up of a harmonic oscillator coupled to a bosonic bath, whose unknown parameters are the coupling constant, the temperature and the bath frequency cutoff. In this case the number of needed tomograms ranges between 16 (at most) in the integral approach and 26 in the differential approach.

Besides measuring the unknown parameters, our procedure proves to be useful also in case those are already known. Indeed it could be employed as a preliminary consistency test for the adopted master equation. In fact, the reconstruction procedure assumes that the time-dependent MECs are the previously known functions of a set of time-independent quantities. This is for example the case of a microscopical derivation (and related approximations) of the master equation. In this perspective, the agreement between the measured and theoretically expected TIPs provides a necessary validity condition for the adopted approximation scheme. Along the same line of thought one could also envisage an extension of this approach to the reconstruction of the whole set of time-dependent MECs. This could provide a sound, reliable and complete experimental check of the goodness of the approximation scheme underlying...
a master equation. A detailed investigation of this wider scenario will be the subject of a distinct study [26]. Our proposal opens up several interesting questions which are going to be the subject of further future investigation. In fact how our approach can be recast within an estimation theory perspective represents a relevant open scenario. A similar analysis has indeed been performed in [27] for a single-parameter Markovian master equation. Another relevant point to investigate is whether the proposed protocol can be enhanced by employing entangled Gaussian states as a probe. Finally, whether or not the proposed procedure can be generalized and employed in presence of memory kernels is a challenging question. Indeed, reconstructing the unknown parameters of Gaussian noisy evolutions with memory represents a highly involved and interesting task.

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Appendix A. The T-C procedure

Tomographic maps [16, 28] allow us to reconstruct the state or some other properties of a physical system, both in a classical and in a quantum regime. In general, tomography-based techniques stem from a probabilistic perspective. Indeed, given a quantum state $\hat{\rho}(t)$, its Wigner function [29, 30] provides a generalization on phase space of a classical probability distribution and is defined as

$$W(q, p, t) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dy \exp \left( \frac{i2py}{\hbar} \right) \hat{\rho}(q - y, q + y, t).$$

(A.1)

The previous equation can be read as a map between real phase-space functions and density matrices. In particular, whenever the dynamics of a quantum system initially in a Gaussian state obeys the master equation (2), the associated Wigner function is a Gaussian function itself and reads

$$W(q, p, t) = \frac{1}{2\pi \sqrt{\Delta q_t^2 \Delta p_t^2 - \sigma^2(q, p)}} \exp \left[ -\frac{\Delta q_t^2 (p - \langle \hat{p} \rangle_t)^2 + \Delta p_t^2 (q - \langle \hat{q} \rangle_t)^2}{2[\Delta q_t^2 \Delta p_t^2 - \sigma^2(q, p)]} \right]$$

$$- \frac{2\sigma(q, p)(q - \langle \hat{q} \rangle_t)(p - \langle \hat{p} \rangle_t)}{2[\Delta q_t^2 \Delta p_t^2 - \sigma^2(q, p)]}. \quad (A.2)$$

Given the Wigner distribution of a quantum system, the Radon transform [31] represents the key ingredient to perform a tomographic analysis. This invertible integral transformation allows us to retrieve the marginal probability densities of the system, i.e. the probability density along straight lines in phase space:

$$X - \mu q - \nu p = 0. \quad (A.3)$$

The formal expression of the Radon transform, for a generic quantum state, is then given by

$$\sigma(X, \mu, \nu) = \langle \delta(X - \mu q - \nu p) \rangle = \int_{\mathbb{R}^2} W(q, p, t) \delta(X - \mu q - \nu p) \ dq \ dp. \quad (A.4)$$
In the case of the Gaussian function (A.2), it becomes
\[
\varpi(X, \mu, \nu) = \frac{1}{\sqrt{2\pi} \sqrt{\Delta q^2 \mu^2 + \Delta p^2 \nu^2 + 2\sigma(q, p)_t \mu \nu}} \times \exp \left[ -\frac{(X - \mu \langle\hat{q}\rangle_t - \nu \langle\hat{p}\rangle_t)^2}{2[\Delta q^2 \mu^2 + \Delta p^2 \nu^2 + 2\sigma(q, p)_t \mu \nu]} \right].
\] (A.5)

The second cumulants always obey the constrain \(\Delta q^2 \mu^2 + \Delta p^2 \nu^2 + 2\sigma(q, p)_t \mu \nu > 0\) as a consequence of the Robertson–Schrödinger relation [32]. This matrix inequality is a generalization of the Heisenberg principle. The advantage of using Gaussian probes to investigate a dissipative dynamics arises also within a statistical perspective. One could in fact wonder whether, due to experimental errors, a violation of the uncertainty principle might be observed. This may happen if measurements are performed on states almost saturating the Robertson–Schrödinger inequality, i.e. on the minimum uncertainty states which are pure. However, our measurements are performed on states undergoing a dissipative evolution, i.e. on states typically far from being pure hence from saturating the inequality. Furthermore any additional noise of statistical origin will have the effect of moving the reconstructed state further away from the boundary, as noted in [22].

Let us now consider the tomograms corresponding to the position and momentum probability distribution functions \((X = q\) and \(X = p)\):
\[
\varpi(X, 1, 0) = \frac{1}{\Delta q_t \sqrt{2\pi}} \exp \left[ -\frac{(X - \langle\hat{q}\rangle_t)^2}{2\Delta q_t^2} \right].
\] (A.6)
\[
\varpi(X, 0, 1) = \frac{1}{\Delta p_t \sqrt{2\pi}} \exp \left[ -\frac{(X - \langle\hat{p}\rangle_t)^2}{2\Delta p_t^2} \right].
\] (A.7)

The lines individuated by the choices \((\mu, \nu) = (1, 0)\) and \((\mu, \nu) = (0, 1)\) correspond to the tomograms depending on the time average and variance respectively of the position and momentum. In order to determine the latter quantities we must invert (A.6) and (A.7) for different values of \(X\), i.e. for a given number of points to measure along a tomogram. By considering first the direction \(\mu = 1, \nu = 0\) and by inverting (A.6), we obtain
\[
(X - \langle\hat{q}\rangle_t)^2 = 2\Delta q_t^2 \ln \frac{1}{\varpi(X, 1, 0) \Delta q_t \sqrt{2\pi}}.
\] (A.8)

If we know the sign of \(\langle\hat{q}\rangle_t\), then we need only the value of the tomogram \(\varpi(0, 1, 0)\) to get \(\langle\hat{q}\rangle_t\), otherwise we need another point. In this way we get \(\langle\hat{q}\rangle_t\), as a function of \(\Delta q_t\):
\[
\langle\hat{q}\rangle_t = \pm \Delta q_t \sqrt{\frac{2\ln \frac{1}{\varpi(0, 1, 0) \Delta q_t \sqrt{2\pi}}}{\varpi(X, 1, 0) \Delta q_t \sqrt{2\pi}}}. 
\] (A.9)

Using (A.9), (A.6) becomes an equation for \(\Delta q_t\) only and it can be rewritten as
\[
2\Delta q_t^2 \ln \frac{1}{\varpi(X, 1, 0) \Delta q_t \sqrt{2\pi}} = \left( X \mp \Delta q_t \sqrt{\frac{2\ln \frac{1}{\varpi(0, 1, 0) \Delta q_t \sqrt{2\pi}}}{\varpi(X, 1, 0) \Delta q_t \sqrt{2\pi}}} \right)^2.
\] (A.10)

This equation is transcendental, therefore we can only solve it numerically. For each \(X\) and corresponding \(\varpi(X, 1, 0)\) there may be two values of \(\Delta q_t\) satisfying the previous equation. In order to identify one of the two solutions, it is enough to consider two points, \(\{(X_1, \varpi(X_1, 1, 0))\}\) and \(\{(X_2, \varpi(X_2, 1, 0))\}\), and to choose the common solution for the variance. Hence, whether we know or not the sign of the average \(\langle\hat{q}\rangle_t\), we need three or four
points to determine $\langle \hat{q} \rangle_t$ and $\Delta q_t$ in (A.6). Analogously, we need other three or four points for $\langle \hat{p} \rangle_t$ and $\Delta p_t$ in (A.7). We now compute the covariance $\sigma(q, p)_t$. To this end we consider the tomogram

$$\pi(X, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\Delta q_t^2 + \Delta p_t^2 + 2\sigma(q, p)_t}} \exp \left[ -\frac{(X - \langle \hat{q} \rangle_t + \langle \hat{p} \rangle_t)^2}{\Delta q_t^2 + \Delta p_t^2 + 2\sigma(q, p)_t} \right].$$

This is a Gaussian whose average is already determined. Indeed, according to the previous steps, we need two more points of this tomograms to determine the spread $(\Delta q_t^2 + \Delta p_t^2)/2 + \sigma(q, p)_t$ from which we can retrieve $\sigma(q, p)_t$.

Hence, the first- and second-order momenta of a Gaussian state can be measured at an arbitrary time $t$ by means of eight or at most ten points belonging to three tomograms.

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