On One Uniqueness Theorem for M. Rietz Potentials

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We prove that there exists a nonzero holderian function \( f : \mathbb{R} \to \mathbb{R} \) vanishing together with its M. Rietz potential \( f * \frac{1}{|x|^{1-\alpha}} \) in all points of some set of positive length. This result improves the one of D. Beliaev and V. Havin [2].

0. Introduction. Let \( \alpha \) be a real number, \( 0 < \alpha < 1 \). Let \( f : \mathbb{R} \to \mathbb{C} \) be a locally summable function satisfying the following condition:

\[
\int_{\mathbb{R}} \frac{|f(x)|dx}{1 + |x|^{1-\alpha}} < +\infty. \tag{1}
\]

Let

\[
(U_\alpha f)(t) := \int_{\mathbb{R}} \frac{f(x)dx}{|t-x|^{1-\alpha}}, \quad t \in \mathbb{R}.
\]

In case (1) the function \( U_\alpha f \) is defined a.e. on \( \mathbb{R} \). We call it the M. Rietz Potential, and we call \( f \) the density of this potential. We write \( \text{dom}U_\alpha \) (the domain of \( U_\alpha \)) for the set of all locally summable functions satisfying (1).

Let \( V \in \mathbb{R} \) be a measurable set; we denote its length as \( |V| \). The uniqueness theorem mentioned in the title states that if \( f \) satisfies (1) and Hölder’s condition with an exponent more than \( 1 - \alpha \) in some neighborhood of \( V \), while \( |V| > 0 \) and

\[
f|_V = U_\alpha|_V = 0, \tag{2}
\]

then \( f = 0 \) a.e. on \( \mathbb{R} \).

This theorem follows from a slightly more general “uncertainty principle” proven in [5]. It concerns M. Rietz’s potentials of charges (not necessarily absolutely continuous with respect to Lebesgue measure) and \( \alpha \)'s not necessarily from \((0,1)\); for the history of the problem and its connections with the uniqueness problem for Laplace’s equation see also [6] and [2]. Havin [5] posted the following question: is it possible to omit Hölder’s condition on \( f \) near \( V \) in Theorem 1? Moreover, it was still unknown if there exist a nonzero continuous function \( f \in \text{dom}U_\alpha \) and a set \( V \) of positive length satisfying (2).

In [2], it was shown that the answer to the last question is affirmative. However, the function \( f \) constructed in [2], being continuous, does not satisfy any Hölder’s condition.

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In this paper, we build a nonzero Hölderian function \( f \in \text{dom} U_\alpha \) vanishing with its M. Rietz potential \( U_\alpha f \) on some set of positive length.

As well as in [2], we build the desired function using the techniques of "correction" which were proposed by D. Menshov and applied to the problems of potential theory in [1], [3], [4], [7]. Our progress (as compared to [2]) is based on improving the correction process using elementary probabilistic techniques (see Lemma 5 [2]). We also mention that we deal with complex (not only real-valued!) densities \( f \). This detail seems unessential, but it surprisingly simplifies the construction of a desired (real-valued!) density \( f \), even in case [2], when the goal was just to find continuous (not necessarily Hölderian) \( f \).

The main result of this paper is the following theorem.

**Theorem 1** There exist a nonzero function \( f \in \text{dom} U_\alpha \), a set \( V \subset \mathbb{R} \) of positive length and a positive number \( r \), such that \( f \) and \( U_\alpha f \) are identically zeroes on \( V \) and \( f \) satisfies Hölder’s condition with exponent \( r \).

**Remark.** The function \( f \) we build is complex-valued. In order to get a real-valued density, one should just take its real or imaginary part; at least one of them will not be an identical zero.

I am grateful to V. Havin for introducing me to the problem and for useful discussions.

1. **Operator** \( W_\alpha \). We will need an operator that is in some sense inverse to \( U_\alpha \). Let for \( g \in C_0^\infty (\mathbb{R}) \)

\[
(V_\alpha g)(t) := \frac{1}{\alpha} \int_\mathbb{R} g(x) \frac{s\text{gn}(t - x)}{|t - x|^\alpha} dx,
\]

\[
(W_\alpha g)(t) = ((V_\alpha g)(t))', \quad t \in \mathbb{R}.
\]

Let \( g \) be a function defined on \( \mathbb{R} \), \( \lambda > 0 \), \( \varepsilon > 0 \). Denote \((C_\lambda g)(x) = g(\lambda x), g_\varepsilon = \frac{1}{\varepsilon} C_{1/\varepsilon} g \). The next lemma states some properties of \( W \).

**Lemma 1**

1) \( W_\alpha(C_0^\infty(\mathbb{R})) \subset \text{dom} U_\alpha; \)

2) \( U_\alpha W_\alpha g = cg, g \in (C_0^\infty(\mathbb{R})); \)

3) \( W_\alpha C_\lambda = \lambda^\alpha C_\lambda W_\alpha; \)

I am grateful to S. Smirnov for useful discussions concerning this lemma.
4) $\alpha W_\alpha g = -g' * \frac{4m \pi \alpha}{|x|^\alpha}$ (we use $*$ for the convolution on $\mathbb{R}$);

5) $(W_\alpha g)(t) = (g * |x|^{-\beta})(t), g \in \text{dom} W_\alpha, t \notin \text{supp} g.$

Hereinafter $\beta = \alpha + 1$. All statements of this lemma are well-known (2) or obvious (1,3,4,5). See, for example, [2, page 226].

We use the following notation: $I = (-\frac{1}{2}, \frac{1}{2})$; if $Q$ is a bounded interval, then $c_Q$ is its center.

We write $\phi(t)$ for “the finitizator”: $\phi \in C^\infty$, $\text{supp} \phi \subset I$, $\int_I \phi = 1$, $\phi \geq 0$.

In the proof we shall fix positive numbers $p$ and $\lambda$. For a function $h : \mathbb{R} \rightarrow \mathbb{C}$ we introduce its “embedding to the interval $Q$” $h_Q(t) := h(\frac{t-c_Q}{|Q| \lambda})$, $t \in \mathbb{R}$. Finally, let $M_Q(h) := \left(\int_Q |h|^p \right)^{1/p}$.

2. **Main Lemma.** The main tool for the proof of Theorem 1 will be Lemma 4. First, we prove auxiliary lemmas 2 and 3. We state the existence of functions with certain concrete numerical properties. These functions will serve as “building blocks” for our construction. The following proposition is principal for us: for some constant $B < 0$ and some $p > 0$ there exists a function $h \in C^\infty_0$ with arbitrarily small support satisfying $\int_\mathbb{R} (|1 - W_\alpha h|^p - 1) < B$. The meaning of this fact is that it allows us to control both the length of support of $h$ (future correcting term) and its influence on the “amount” of potential $W_\alpha$ of the function we are correcting. We shall rearrange this statement for it to be convenient for our purposes.

The function $h$ will be made from “the finitizator” $\phi$ by means of an appropriate scaling.

Let for $\varepsilon > 0$, $t \in \mathbb{R}$

$$F^{[\varepsilon]}(t) := (W_\alpha \phi_\varepsilon)(t), \quad F^{[0]}(t) := |t|^{-\beta}.$$  

**Lemma 2** If a positive number $p$ is sufficiently small, then

$$J(p) := \int_\mathbb{R} (|1 - F^{[0]}|^p - 1) dx < 0$$

**Proof.** Let $L := \int_\mathbb{R} \log |1 - F^{[0]}|$. Then, as $(a^p - 1)/p$ is monotone in $p$ for any $a > 0$ and converges to $\log a$ as $p \rightarrow 0$, we have $\lim_{p \rightarrow 0} \frac{J(p)}{p} = L$ (note that $||1 - F^{[0]}(t)|^p - 1| \leq c|t|^\beta$, when $|t|$ is large, and if $p < 1/\beta$, then the integral $J(p)$ converges in zero as well). But $L$ can be computed exactly:
\[ L = 2\pi \cot \frac{\pi}{\beta} < 0. \] One can find the computation, for example, in [2, page 233].

From now on the number \( p \) found in the previous lemma will be fixed.

In the next lemma we pass from \( F^{[0]} \) (the potential \( W_\alpha \) of the delta-function) to the potential of some concrete function \( \phi_\varepsilon \). We also introduce a “small complex rotation”: multiply \( \phi_\varepsilon \) by \( e^{i\theta} \) with small \( \theta \). This leads to some technical simplifications in the future.

**Lemma 3** There exist numbers \( B < 0, \theta_0 > 0 \) and \( \varepsilon_0 > 0 \), such that if \( 0 \leq \theta \leq \theta_0 \) and \( 0 \leq \varepsilon < \varepsilon_0 \), then

\[
J(\varepsilon, \theta) := \int_\mathbb{R} (|1 - e^{i\theta} F^{[\varepsilon]}|^p - 1) < B.
\]

**Proof.** Using the homogeneity property of \( W_\alpha \) (point 3 of Lemma 1), we get

\[
|F^{[\varepsilon]}(t)| \leq C(\alpha) \min\left( \frac{1}{\varepsilon \beta}, \frac{1}{|t|^\beta} \right), \quad t \in \mathbb{R}. \tag{3}
\]

It is clear that \( F^{[\varepsilon]} \) converges to \( F^{[0]} \) pointwise as \( \varepsilon \to 0 \). It follows from Lebesgue dominated convergence theorem that

\[
\lim_{\varepsilon \to 0} J(\varepsilon, 0) = J(p).
\]

Choose \( B < 0 \) to satisfy \( J(\varepsilon, 0) < 2B \) for \( \varepsilon \in (0, \varepsilon_0) \). We are to prove that \( J(\varepsilon, \theta) \xrightarrow{\theta \to 0} J(\varepsilon, 0) \) uniformly in \( \varepsilon \). Indeed,

\[
|J(\varepsilon, \theta) - J(\varepsilon, 0)| = \left| \int_\mathbb{R} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| \leq \int_{|x| > \theta^{-p/2}} \left| |t|^{-\beta} \right| + \int_{|x| < \theta^{-p/2}} \left| |t|^{-\beta} \right| =: J_1 + J_2.
\]

We have

\[
J_1 = \left| \int_{|x| > \theta^{-p/2}} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| \leq 2C \int_{|x| > \theta^{-p/2}} \left| |t|^{-\beta} \right| \xrightarrow{\theta \to 0} 0,
\]

as the integral converges in infinity (and does not depend on \( \varepsilon \)). Then, using the inequality \( |a^p - b^p| \leq |a - b|^p \) for \( a > 0, b > 0 \) and \( p \in (0, 1) \), we get

\[
J_2 = \left| \int_{|x| < \theta^{-p/2}} |1 - e^{i\theta} F^{[\varepsilon]}|^p - |1 - F^{[\varepsilon]}|^p \right| =
\]
\[
= \left| \int_{|x|<\theta^{-p/2}} |e^{-i\theta} - F^{[\varepsilon]}|p - |1 - F^{[\varepsilon]}|p \right| \leq \left| \int_{|x|<\theta^{-p/2}} |e^{-i\theta} - 1|p \right| \leq \int_{|x|<\theta^{-p/2}} \theta^p \theta \to 0.
\]

So, \( J_1 + J_2 \theta \to 0 \) uniformly in \( \varepsilon \in (0, \varepsilon_0) \), and the lemma is proved.

We are now ready to formulate Lemma 4 – the main tool of further construction. We replace the whole line (from Lemma 3) by a bounded interval, the constant 1 by an arbitrary function with a small oscillation, and finally we bring in a (small) positive parameter \( \lambda \). This one is responsible for smallness of the potential \( W_\alpha \) of the correcting term far away from the interval we correct on. Denote \( \gamma(\lambda) := (1 + B\lambda/2)^{1/p} \), where \( B \) is a constant from Lemma 3. Notice that \( 0 < \gamma(\lambda) < 1 \) for any sufficiently small positive \( \lambda \).

If \( f \) is a function defined on some interval \( Q \), we define \( \text{osc}_Q f := \sup_{x,y \in Q} |f(x) - f(y)| \) (the oscillation of \( f \) on \( Q \)).

**Lemma 4** There exist numbers \( \theta > 0, \lambda_0 > 0, \varepsilon_0 > 0 \), such that for any positive \( \lambda < \lambda_0 \) one can find a number \( \kappa > 0 \) with the following property: if \( 0 < \varepsilon < \varepsilon_0 \), \( Q \) is any bounded interval and a continuous complex-valued function \( h \) satisfies

\[
\text{osc}_Q h \leq \kappa |h(c_Q)|,
\]

then

1) \( M_Q(h - h(c_Q)e^{i\theta}F^{[\varepsilon]}_Q) \leq \gamma(\lambda)|h(c_Q)|; \)

2) \( \frac{2}{\theta} |h(t)| \leq |h(t) - h(c_Q)e^{i\theta}F^{[\varepsilon]}_Q(t)| \leq \frac{C}{\varepsilon^2} |h(t)|, \quad t \in Q. \)

Recall that \( F^{[\varepsilon]}_Q(t) = F^{[\varepsilon]}(\frac{t-c_Q}{|Q|A}). \)

**Proof.** Take \( \varepsilon_0 \) and \( \theta \) from Lemma 3. First we get an estimate in point 2:

\[
|h(t) - h(c_Q)e^{i\theta}F^{[\varepsilon]}_Q| \geq |h(c_Q)||1 - e^{i\theta}F^{[\varepsilon]}_Q(t)||h(t) - h(c_Q) | \geq |h(c_Q)|\frac{3\theta}{4} - \kappa |h(c_Q)| = |h(c_Q)|\left(\frac{3\theta}{4} - \kappa\right) \geq |h(t)|\frac{1}{1 + \kappa \left(\frac{3\theta}{4} - \kappa\right)} \geq |h(t)|\frac{\theta}{2}.
\]
for any sufficiently small \( \kappa \). We have used an elementary inequality \( \text{dist}(1, \{ re^{i\theta} : r \in \mathbb{R} \}) = \sin \theta \geq \frac{3\theta}{4} \), if \( \theta > 0 \) is small.

The right-hand inequality in the point 2 follows clearly from (3).

Now we prove point 1:

\[
(M_Q(h - h(c_Q)e^{i\theta}F^{[\varepsilon]}_Q))^p \leq (M_Q(h - h(c_Q)))^p + |h(c_Q)|^p(M_Q(1 - e^{i\theta}F^{[\varepsilon]}_Q))^p \leq (\text{osc}_{c_Q} h)^p + |h(c_Q)|^p(M_{\lambda-1I}(1 - e^{i\theta}F^{[\varepsilon]}_Q))^p.
\]

Notice that

\[
(M_{\lambda-1I}(1 - e^{i\theta}F^{[\varepsilon]}_Q))^p = 1 + \lambda \int_{\lambda^{-1}I} (|1 - e^{i\theta}F^{[\varepsilon]}_Q|^p - 1) = 1 + \lambda \int_{\mathbb{R}} (|1 - e^{i\theta}F^{[\varepsilon]}_Q|^p - 1) - \lambda \int_{\mathbb{R}\setminus\lambda^{-1}I} (|1 - e^{i\theta}F^{[\varepsilon]}_Q|^p - 1).
\]

Taking (3) into account, we get

\[
\left| \lambda \int_{\mathbb{R}\setminus\lambda^{-1}I} (|1 - e^{i\theta}F^{[\varepsilon]}_Q|^p - 1) \right| \leq C\lambda^{\alpha+1} = o(\lambda), \lambda \to 0,
\]

so, if \( \lambda \) is sufficiently small, we have \( M_{\lambda-1I}(1 - e^{i\theta}F^{[\varepsilon]}_Q) \leq 1 + B\lambda/2 < 1 \). Therefore

\[
M_Q(h - h(c_Q)e^{i\theta}F^{[\varepsilon]}_Q)^p \leq |h(c_Q)|^p(1 + \left(\frac{\text{osc}_{c_Q} h}{h(c_Q)}\right)^p + 2B\lambda/3) \leq |h(c_Q)|^p(1 + \kappa^p + 2B\lambda/3).
\]

If \( \kappa \) is sufficiently small, then \( (1 + \kappa^p + 2B\lambda/3) < 1 + B\lambda/2 \). The lemma is proved.

**Remark 1.** Careful examination of the proof shows that one can take \( \kappa \) equal to \( \min((|B|\lambda)^{1/p}, \frac{\theta}{\pi}) \), if \( \theta \) is not too large.

**Remark 2.** It is the left-hand inequality in the first point for what we pass to complex-valued functions. One cannot obtain such an estimate for real-valued functions.

**Remark 3.** By this moment we have fixed parameters \( p \) and \( \theta \). In what follows, constants that we regard as depending on \( \alpha \) may also depend on these parameters. Later we shall fix an appropriate \( \lambda \) and, thus, \( \kappa \) and \( \gamma \).

### 3. General idea of the construction

Now we describe the plan of construction of \( f \) and \( V \) (see statement of Theorem 1). We shall build a sequence of functions \( g_n, g_n = g_{n-1} - r_{n-1} \), and a decreasing sequence of sets \( V_n \subset I \) with the following properties:
1) A nonzero function $g_1$ belongs to $C_0^\infty$, and $\text{supp } g_1 \subset \mathbb{R} \setminus I$;

2) $r_k \in C_0^\infty$, and $\text{supp } r_k \subset I$ for all $k \in \mathbb{N}$;

3) $\sum_{k=1}^{\infty} |\text{supp } r_k| < \frac{1}{4}$;

4) $|\bigcap_{k=1}^{\infty} V_n| > \frac{3}{4}$;

5) $\int_{V_n} |f_n|^{p \rightarrow \infty} \rightarrow 0$; here $f_n := W_\alpha g_n$ and $p$ is the positive number fixed in the previous section.

6) Sequences $g_n$ and $f_n$ converge uniformly on $\mathbb{R}$ to some continuous functions $g$ and $f$ correspondingly, and $g = U_\alpha f$.

Let $V := \bigcap_{k=1}^{\infty} V_n$, $V' := \{x \in I : g(x) = 0\}$. It follows from 1, 3 and 4 that $|V'| > \frac{3}{4}$ and $|V| > \frac{3}{4}$. Therefore $|V \cap V'| > \frac{1}{2}$. It follows from 5 and 6 that $f|_{V'} = 0$. Finally, using 2, we conclude that $g|_{\mathbb{R} \setminus I} = g_1|_{\mathbb{R} \setminus I}$, and, thus, the function $g$ is not identically 0. Hence the set $V \cap V'$ and the function $f$ satisfy all conditions of Theorem 1, except (may be) Hölder’s condition.

Now we describe more precisely the structure of sets $V_n$ and correcting terms $r_k$. Bring in a sequence of positive numbers $\{\delta_n\}_{n=1}^{\infty}$. It will have the following properties: $\delta_1 = 1$, $\frac{\delta_n}{3n+1} \in \mathbb{N}$. We denote the partition of the interval $I$ to intervals of length $\delta_n$ as $H_n$. The set $V_n$ will be obtained as the union $\bigcup_{Q \in G_n} Q$, where $G_n$ is some subset of $H_n$. Roughly speaking, the set $G_n$ consists of all intervals on which we haven’t finish correction yet, in particular, for all $k > n$ there holds $\text{supp } r_k \subset V_n$.

Let us fix a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$, such that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{4}$, and, in addition, $\varepsilon_n$ decay not very fast: $\varepsilon_n^{-1} = O(n^m)$ for some $m > 0$. It will be responsible for the length of supports of $r_n$: there will hold an estimate $|\text{supp } r_n| \leq \varepsilon_n$ for all $n$.

We also demand $\text{supp } g_1 \subset (\frac{1}{2}, \frac{3}{2})$, and, moreover, $\forall t \in I f_1(t) \neq 0$. One can take, for example, $g_1 := \phi(x - 1)$.

Then, we choose some subset $G_n^q \subset G_n$ and let

$$r_n := \sum_{Q \in G_n^q} (\lambda \delta_{n+1})^\alpha f_n(c_Q)(\phi_{\varepsilon_n}) Q e^{i\theta}. \quad (5)$$
So,
\[
W_\alpha r_n = \sum_{Q \in G_{n+1}^g} f_n(c_Q)F_Q^{[\varepsilon_n]} e^{i\theta}.
\]

Note that in such a definition condition 3 will be satisfied by choosing the sequence \(\varepsilon_n\) as described above.

The idea is that if \(\delta_{n+1}\) is sufficiently small, then on each interval \(Q \in G_{n+1}^g\) the oscillation of \(f_n\) is small (there holds estimate (1)), and one can apply Lemma 4 with \(f_n\) as \(h\). Its result, together with an observation that functions \(F_Q^{[\varepsilon]}\) decay sufficiently fast far away from \(Q\), allow us, using the notation 
\[
V_{g n} := \bigcup_{Q \in G_{n}^g} Q,
\]
to prove an estimate
\[
\int_{V_{g n}^+} \left|f_{n+1}\right|^p \leq \eta \int_{V_{n}^g} \left|f_n\right|^p
\]
with some \(\eta \in (0, 1)\). If one chooses \(G_n^g\) appropriately (if on each step they occupy a large part of \(G_n\)), this leads to an estimate of integral over the whole set \(V_n\):
\[
\int_{V_n} \left|f_n\right|^p = O(\eta^2).
\]
Hence condition 5 will be obtained.

**Remark 1.** The choice of \(G_n\) (decreasing \(V_n\) on each step) allows us to make functions \(f_n\) converge not only in the sense of \(L^p(I)\), but uniformly, in particular, we get an estimate \(\left|f_n(c_Q)\right| = O(\eta'^n)\), \(Q \in G_n\), where \(\eta' \in (0, 1)\).

**Remark 2.** If we did not worry about the control over modulus of continuity, we could take \(G_n^g := G_n\). Then (7) automatically follows from (6), and the whole construction becomes more simple. Unfortunately, in order to get Hölder’s condition, one should pick out the set \(G_n^g\) on each step – this is a set of intervals where the oscillation of \(f_n\) is especially small – and make the correction only there.

**4. Remarks on the estimate of modulus of continuity.** In this section, we explain (not quite rigorously) what does estimates of modulus of continuity of \(f\) depend on.

We use the following simple fact: if a sequence of functions \(h_n\) converges on \(\mathbb{R}\) to the function \(h\), whereas \(|h_n - h| \leq C_1 \eta_1^n\), and \(|h'_n| \leq C_2 R^n\) (here \(\eta_1 \in (0, 1)\), \(R > 1\)), then \(h\) satisfies Hölder’s condition with an exponent \(\log \eta_1 / \log \frac{n}{R}\).
The condition $|f_n - f| \leq C_1 \eta_1^n$, will follow from Remark 1 at the end of
the previous section (and, in fact, from estimate (7)). When one estimates
the derivative $f'_n(t)$ of the function $f_n$, the main role is played by the last
added term $W_\alpha r_{n-1}$, or, more precisely, the building block $f_{n-1}(c_{Q_t})F_{Q_t}^{[\varepsilon_{n-1}]}e^{i\theta}$,
where the interval $Q_t \in H_n$ is defined by the statement $t \in Q_t$. From the
homogeneity properties of $W_\alpha$ one can get (for $Q \in H_n$) an estimate

$$|(F_{Q}^{[\varepsilon_{n-1}]})'(t)| \leq \frac{c}{\varepsilon^{-\beta-1} \lambda \delta_n} t \in \mathbb{R}$$

Thus, one can get (say, on $V_n$) an estimate

$$|f'_n| \leq \frac{Cf_n(c_{Q_t})}{\delta_n \varepsilon_{n-1}^{\beta+1}},$$

and so, everywhere,

$$|f'_n| \leq \frac{C\eta_{n}^{m_n}}{\delta_n \varepsilon_{n-1}^{\beta+1}}.$$  

This means that the obtained function $f$ would be Hölderian if numbers
$\delta_{n-1}$ did not grow faster than some geometric series, in other words, if every
time we divided the interval $Q \in H_n$ into the same number of parts. On the
other hand, the exponent $\log \eta_1/\log \eta_1 R$ tends to zero as $R \to \infty$, hence it
is clear that if $\frac{\delta_n}{\delta_{n+1}} \to \infty$ as $n \to \infty$, then we are unable to prove Hölder’s
condition with any exponent.

It is clear, however, that if in order to define $\delta_{n+1}$ we use a natural estimate
(9) (recall what the necessity to choose small $\delta_{n+1}$ is due to: we need to
estimate the oscillation of $f_n$ in order to use Lemma 4 – condition (4)), then
because of the increasing multiplier $\varepsilon_{n-1}^{-\beta}$ we should take $\delta_{n+1}/\delta_n$ tending
to zero to make (4) hold. Therefore we need finer estimates of modulus of
continuity of $f_n$, holding, however, not on the whole set $V_{n+1}$, but on some
”good“ part $V_{n+1}^g$ of it.

Note that the building block $F_{Q}^{[\varepsilon_{n}]}$ and its derivative $(F_{Q}^{[\varepsilon_{n}]}')$ are large in
modulus (as $\varepsilon_{n}^{-\beta}$ and $\delta_{n+1}^{-1}\varepsilon_{n}^{-\beta-1}$ correspondingly) only near the center of $Q$;
if we consider them outside the interval of the length $\tau |Q|$ and with the same
center with $Q$, where $0 < \tau < 1$, then we get estimates $|F_{Q}^{[\varepsilon_{n}]}| \leq C$ and
$|(F_{Q}^{[\varepsilon_{n}]}')| \leq C\delta_{n+1}^{-1}$.

One has an idea - exclude this “bad” central part of $Q$ and correct on the
remaining part only. Unfortunately, if we drop it forever, this will mean that
on each step one eliminates from $V_n$ a subset of length $\tau |V_n|$, and this makes $\cap_{n \in \mathbb{N}} V_n$ have zero length.

Therefore, for each interval from $H_n$ we bring in a system of its “bad” subsets, and on each of them we shall “make a pause” - not correct during the next few steps, until the partition $H_{n+k}$ becomes so fine that the estimates of $f_n$ and its derivative become satisfactory (requirements to this estimates becomes weaker if $n$ grows). The the pause duration depends on the distance of the corresponding subset from the center of the interval, i.e. on how “bad” $f_n$ is on this subset. According to this, $G_{n+1}$ is divided into two parts: $G_{n+1}^g$, where the correction is made on this step, and $G_{n+1}^d$, where we do not do anything for the time being.

After that, we shall estimate an amount of intervals from $G_n$ such that have more than one half of their “ancestors” from $G_{k-1}$, $k = 0, 1, \ldots, n-1$, belong to $G_{n-1}^d$. It turns out that there are only a few of them (if $G_{n-1}^d$ is a small part of $G_n$ for each $k$), and we drop them out. For the rest, we prove an estimate like $|Q|\epsilon_{n}$, using Lemma 4 and fast decay of $F_{Q}$ away from $Q$.

5. Definition of sets $G_{n}^g$. To complete the construction, we should determine the sequence $\delta_n$ and sets $G_n$ and $G_{n}^g$. We need a number of estimates depending on how one picks out “good” subsets $G_{n}^g$ from $G_n$, but not on the way to choose $G_n$ themselves. We prove these estimates in sections 6, 7 and 8. Later, in section 9, we will define the way to choose $G_n$.

Then, bring in a parameter $\delta$, such that $\delta^{-1} \in \mathbb{N}$, and let $\delta_n := \delta^n$. Then, bring in a parameter $\tau \in (0, 1)$. Let $\tau^{-1} \in \mathbb{N}$, and, moreover, $\tau / \delta \in \mathbb{N}$. We use the following notation: if $a > 0$ and $Q$ is a bounded interval, then $Q[a] := Q \setminus Q'$, where $Q'$ is an interval of the length $a |Q|$ and with the same center as $Q$.

We are now ready to define the set $G_{n+1}^g$. An interval $Q \in G_{n+1}$ belongs to $G_{n+1}^g$, iff for any $k = 0, 1, \ldots, n-1$ there holds an implication $Q \subset Q' \in G_{n-k}^g \Rightarrow Q \subset Q'[\tau^{k+1}]$. In other words, if in the $(n-k)$-th step we have made a correction on the interval $Q'$, then on the next step the correction is forbidden on the set $Q' \setminus Q'[\tau]$, on the $(n-k+2)$-th step it is forbidden on the set $Q' \setminus Q'[\tau^2]$, and so on. It follows from the condition $\tau / \delta \in \mathbb{N}$ that the interval $Q \in G_{n+1}$, $Q \subset Q' \in G_{n-k}^d$ either lies in $Q'[\tau^{k+1}]$ or does not intersect it.

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3 Top indices $g$ and $d$ are the first letters of “go” and “delay”

4 Recall that it means that $Q'$ belongs to the set of indices of summation in the definition of the corresponding correcting term $r_{n-k-1}$
Indeed, $Q'[\tau^{k+1}] \setminus Q'[^\tau^k]$ are the very “bad” subsets of $Q'$; on the $k$-th of them the “length of pause” is $k$ steps.

Let us make some simple, but important observations. It is easy to see that if $Q$ is a bounded interval and $\text{dist}(t, c_Q) > 3\lambda \varepsilon_n |Q|$, then for any $n \in \mathbb{N}$

$$|F_Q^{[\varepsilon_n]}(t)| \leq C(\alpha) \frac{(\lambda |Q|)^\beta}{|t - c_Q|^{\beta+1}},$$

(11)

and, besides,

$$|(F_Q^{[\varepsilon_n]})'(t)| \leq C(\alpha) \frac{(\lambda |Q|)^\beta}{|t - c_Q|^{\beta+1}},$$

(12)

This means that for all $k \in \mathbb{N}$ and for $t \in Q[\tau^k]$

$$|F_Q^{[\varepsilon_n]}(t)| \leq C_1(\alpha) \frac{\lambda^\beta}{\tau^{k\beta}},$$

(13)

$$|(F_Q^{[\varepsilon_n]})'(t)| \leq C_1(\alpha) \frac{\lambda^\beta}{|Q|^{\frac{1}{\tau^{k\beta}}}}.$$ 

(14)

Indeed, for $\tau^k/2 > 3\lambda \varepsilon_n$ these estimates coincide with the previous ones, and for $\tau^k/2 > 3\lambda \varepsilon_n$ we can use estimates

$$|F_Q^{[\varepsilon_n]}(t)| \leq \frac{C(\alpha)}{\varepsilon_n^\beta}$$

and

$$|(F_Q^{[\varepsilon_n]})'(t)| \leq \frac{C(\alpha)}{|Q| \lambda \varepsilon_n}, \quad t \in \mathbb{R}.$$

We can improve the right-hand inequality in point 2 of Lemma 4 for $t \in Q[\tau^k]$. Indeed, applying (13) instead of (3), we get

$$|h(t) - h(c_Q)e^{ig}F_Q^{[\varepsilon]}(t)| \leq \frac{C(\alpha)\lambda^\beta}{\tau^{k\beta}}|h(t)|, \quad t \in Q[\tau^k].$$

(15)

Let $t \in I$. Denote as $Q_t^n$ an element of $H_n$ defined by the condition $t \in Q_t^n$. Let $D_t^n(t) := \#\{l = 1, \ldots, n : t \in Q_t^{l-k_l} \setminus Q_t^{l-k_l}[^{\tau^k}]\}$. For $t \in V_n$ let $\tilde{D}_n(t) := \#\{l = 1, \ldots, n : Q_t^{l-k_l} \in G_t^d\}$. We know that if $Q_t^{l-k_l} \in G_t^d$, then for some $k_l \in \mathbb{N}$ there holds an inclusion $Q_t^{l-k_l} \subset Q_t^{l-k_l} \setminus Q_t^{l-k_l}[^{\tau^k}]$. Moreover, for $l_1 \neq l_2$ either $k_{l_1} \neq k_{l_2}$, or $l_1 - k_{l_1} \neq l_2 - k_{l_2}$. So, with each natural $l \leq n$, such that $Q_t^{l-k_l} \in G_t^d$, we can associate a pair $(l - k_l, k_l)$, such that $Q_t^{l-k_l} \subset Q_t^{l-k_l} \setminus Q_t^{l-k_l}[^{\tau^k}]$, and such
a mapping will be injective. Hence $\widetilde{D}_n(t) \leq \sum_{k \in \mathbb{N}} D^k_n(t) =: D_n(t)$. The next section is devoted to estimates of lengths of sets $E_n := \{t \in I : D_n(t) \geq n/2\}$.

6. Estimates of lengths of sets $E_n$.

Lemma 5. For $\tau$ sufficiently small there holds an inequality

$$|E_n| \leq \frac{C(\tau)}{n^2},$$

(16)

where $C(\tau)$ tends to zero as $\tau \to 0$.

Proof. Let

$$\xi_i^{(k)} := \sum_{Q \in H_i} \chi_{Q \setminus Q[k]};$$

$$\xi_i^{(k)} := \xi_i^{(k)} - \tau^k.$$  

Then, considering $\xi_i^{(k)}$ as random variables on the probabilistic space $I$ with the measure $dx$, we have $E\xi_i^{(k)} = 0$. We can describe sets $E_n$ in terms of functions $\xi_i^{(k)}$ as follows:

$$x \in E_n \iff \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \xi_i^{(k)}(x) \geq \frac{n}{2}$$

Hence we are to estimate probabilities of the event that sums of random variables $\sum_{i=1}^{n-k} \xi_i^{(k)}$ are large. It is easy to see that random variables $\xi_i^{(1)}$, $i = 1, 2, \ldots$ are independent: it follows from the fact that $\tau/\delta$ is an integer. Unfortunately, for $k > 1$ one cannot say the same thing about $\xi_i^{(k)}$, $i = 1, 2, \ldots$, because $\tau^k/\delta$ is not necessarily an integer. But, for $k > 1$ variables $\xi_i^{(k)}$, $i = 1, 2, \ldots$ are still in some sense “almost independent”, and we use it.

Note that if $j \geq i + k$, then $\xi_i^{(k)}$ is a constant on each interval from $H_j$ (it follows from inclusion $\tau^k/\delta^k \in \mathbb{N}$). Hence we can made the following observation:

1) If $i_1 + k \leq i_2 \leq i_3 \leq i_4$, then $E(\xi_{i_1}^{(k)} \xi_{i_2}^{(k)} \xi_{i_3}^{(k)} \xi_{i_4}^{(k)}) = 0$ (as the function $\xi_{i_2}^{(k)} \xi_{i_3}^{(k)} \xi_{i_4}^{(k)}$ is periodic with a period equal to $\delta^{i_1+k}$, and $\xi_{i_1}^{(k)}$ is constant on each interval from $H_{i_1+k}$).
2) If \( i_1 \leq i_2 \leq i_3 \leq i_4 - k \), then \( E(\xi_{i_1}^{(k)} \xi_{i_2}^{(k)} \xi_{i_3}^{(k)} \xi_{i_4}^{(k)}) = 0 \) (as the function \( \xi_{i_4}^{(k)} \) is periodic with a period equal to \( \delta_{i_3+k} \), and \( \xi_{i_1}^{(k)} \xi_{i_2}^{(k)} \xi_{i_3}^{(k)} \) is constant on each interval from \( H_{i_3+k} \)).

Now we write

\[
P(|\sum_{i=1}^{n} \xi_i^{(k)}| > \varepsilon) \leq \frac{E(\xi_1^{(k)} + \ldots + \xi_n^{(k)})^4}{\varepsilon^4} = \\
\sum_{(i_1,i_2,i_3,i_4)\in\{1,\ldots,n\}^4} E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)})
\]

First note that \( E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)}) \leq E|\xi_{i_1}^{(k)}| = 2\tau^k(1 - \tau^k) \leq 2\tau^k \) (the first inequality follows from the inequality \( |\xi_i^{(k)}| < 1 \) for all \( i \) and \( k \)). Second, if \((j_1,j_2,j_3,j_4)\) is a non-decreasing permutation of the numbers \((i_1,i_2,i_3,i_4)\), then a term \( E(\xi_{i_1}^{(k)}\xi_{i_2}^{(k)}\xi_{i_3}^{(k)}\xi_{i_4}^{(k)}) \) may differ from zero only if \( j_2 - j_1 < k \) and \( j_4 - j_3 < k \) (by the above observation). But the number of such fours \((j_1,j_2,j_3,j_4)\) does not exceed \( \frac{n(n-1)k^2}{2} \). Hence, the number of nonzero terms in the numerator in (17) does not exceed \( 41^2\frac{n(n-1)}{2}k^2 \). Therefore

\[
P(|\sum_{i=1}^{n} \xi_i^{(k)}| > \varepsilon) \leq \frac{Cn^2k^2\tau^k}{\varepsilon^4}.
\]

Let \( E_n^k := \{t \in I : D_n^k(t) \geq n(\sqrt{4\tau^k} + \tau^k)\} \). If \( \tau \) is so small that \( \sum_{1}^{\infty} \sqrt{4\tau^k} + \tau^k < \frac{3}{2} \), then \( |E_n| \leq \sum_{k=0}^{\infty} |E_n^k| \). Finally, note that \( D_n^k(t) = \sum_{i=1}^{\infty} \xi_i^{(k)}(t) = \sum_{i=1}^{n} \xi_i^{(k)}(t) + n\tau^k \), so, using (18), we get

\[
|E_n| \leq \sum_{k=1}^{\infty} \sum_{i=1}^{n} \xi_i^{(k)}(t) \leq \sum_{k=1}^{\infty} Ck^2\tau^k \frac{\varepsilon^4}{n^22^k}.
\]

The lemma follows from this estimate.

7. Estimates of \( f_n \). In this section, we prove some estimates for \( f_n \)'s, main of them are estimate (24), which allows us to apply Lemma 4, and estimate (20), which shows that for \( \lambda \) sufficiently small the terms \( f_n(c_{Q'})E_{Q'}^{[c_{n}]}e^{i\theta} \), corresponding to \( Q' \neq Q \), do not change the situation on \( Q \) essentially. All the estimates we prove do not depend on the choice of \( G_n \), but depend on how
we pick out subsets $G^g_n$ (namely, we use estimates (13) and (14)). The sets $G_n$, as mentioned above, will be defined later. Let

$$T_{n+1}(t) := \sum_{Q \in G^g_n, Q \neq Q_{t+1}^g} f_n(c_Q) F_Q^{[e_n]} e^{i\theta}.$$  \tag{19}$$

Recall that an interval $Q_{t+1}^g \in H_{n+1}$ is defined by condition $t \in Q_{t+1}^g$.

**Lemma 6** There exists a positive number $\rho = \rho(\alpha)$, such that for $\lambda > 0$ sufficiently small and for $\delta = \delta(\lambda) > 0$ sufficiently small there holds the following:

1) for all $t \in I$

$$|T_{n+1}(t)| \leq \frac{c(\alpha)\lambda^\beta |f_n(t)|}{\rho};$$  \tag{20}$$

2) if $k \leq n$, $x \in V^g_{n+1}$, $y \in I$ and $|x - y| \leq \delta^k$, then

$$|f_n(x)| \leq \frac{|f_n(y)|}{\rho^{n-k+1}};$$  \tag{21}$$

3) for all $t \in I$, there holds an estimate

$$|T'_{n+1}(t)| \leq \frac{c(\alpha)\lambda^\beta |f_n(t)|}{\delta^{n+1}\rho};$$  \tag{22}$$

4) for all $t \in V^g_{n+1}$, there holds an estimate

$$|f'_n(t)| \leq \frac{c_1(\alpha)\lambda^\beta |f_n(t)|}{\delta^n};$$  \tag{23}$$

5) for all $Q \in G^g_{n+1}$, there holds an estimate

$$\text{osc}_Q f_n \leq \kappa |f_n(c_Q)|.$$  \tag{24}$$

**Proof.** Inequality (24) clearly follows from (23), if we take $\delta$ sufficiently small (an interval $Q$ in (24) is of the length $\delta^{n+1}$). We derive (20) and (22) from (21), which, in turn, follows from (20) and (24) for preceding $n$. Finally, (23) follows from (24) and (22) for preceding $n$. 

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The base of induction – (24) and (21) for \( n = 1 \) – is provided by the condition \( f_1(t) \neq 0, \ t \in I \) (see Section 3) and the choice of \( \rho \) and \( \delta \) sufficiently small.

Derive (20) and (22) from (21). Fix \( t \in I \). Denote as \( G_{\epsilon} \) the set of all intervals \( Q' \in H_{n+1} \) satisfying the property \( \text{dist}(c_{Q'}, Q_{n+1}^*) \geq \epsilon \). Denote
\[
\sigma_{\epsilon}(t) := \sum_{Q' \in G_{\epsilon}} |F_{Q'}^{[\epsilon]}(t)|,
\]
\[
\sigma^*_{\epsilon}(t) := \sum_{Q' \in G_{\epsilon}} |(F_{Q'}^{[\epsilon]})'(t)|.
\]

We need estimates
\[
\sigma_{\epsilon}(t) \leq c(\alpha) \lambda^\beta \left( \frac{\delta^{n+1}}{\epsilon} \right)^\alpha (25)
\]
and
\[
\sigma^*_{\epsilon}(t) \leq c(\alpha) \lambda^\beta \frac{\delta^{(n+1)\beta}}{\epsilon^{\beta+1}}. (26)
\]

One can obtain them by estimating each term by (11) and (12) correspondingly, and then estimating the sum by an integral. The detailed proof of the first one can be found in [2, page 234], the second one can be proved in the same way.

In order to get the above estimate of \( |T| \) and \( |T'| \), we shall divide terms in the right-hand side of (19) into several groups according to their distance from the point \( t \). For each group, we estimate \( |f(c_Q)| \) by means of (21) (the closer to \( t \) the interval \( Q \) is, the better is this estimate) and then apply (25) (correspondingly, (20) for \( |T'| \)), which, by contrast, becomes better when \( \epsilon \) grows.

So, let \( G_{n+1}^q := \bigsqcup_{k \leq n+1} G^{[k]} \), where \( G^{[n+1]} := G_{n+1}^q \setminus G_{\delta^n}, \ G^{[k]} := (G_{n+1}^q \cup G_{\delta^k}) \setminus G_{\delta^{k-1}} \). For \( y \in G^{[k]} \) we have \( |f_n(y)| \leq |f_n(t)|/\rho^{n-k+2} \) and \( \text{dist}(G^{[k]}, t) \geq \delta^k/2 \), therefore
\[
|T_{n+1}(t)| \leq \sum_k \sum_{Q' \in G^{[k]}} |F_{Q'}^{[\epsilon]}(t)||f_n(t)|/\rho^{n-k+2} \leq
\]
\[
\leq c(\alpha) \lambda^\beta |f_n(t)| \sum_k \frac{\delta^{(n+1)\beta}}{\delta^{k+2}\rho^{n-k+2}} \leq c(\alpha) \lambda^\beta |f_n(t)| \rho^{-1} \sum_k \left( \frac{\delta^\beta}{\rho} \right)^{n-k+1}.
\]
First assume $k$ because of (24)), the last one – from (20). Now, the inequality follows from point 2 of Lemma 4 (which is applicable

\[ |T'_{n+1}(t)| \leq \sum_{k} \sum_{Q' \in G^{k}} |(F^{[x]}_{Q'})'(t)||f_{n}(t)|/\rho^{n-k+2} \leq \]

\[ \leq c(\alpha) \frac{\lambda^{\beta} |f_{n}(t)|}{\delta^{n+1}} \sum_{k} \frac{\delta^{(n+1)(\beta+1)}}{\delta k^{\beta+1} \rho^{n-k+2}} \leq c(\alpha) \frac{\lambda^{\beta} |f_{n}(t)|}{\delta^{n+1} \rho} \sum_{k} \left( \frac{\delta^{\beta+1}}{\rho} \right)^{n-k+1}. \]

Taking $\delta$ according to the condition $\frac{\delta^{\beta}}{\rho} < \frac{1}{2}$, we get (20) and (22).

Now let us prove that (21) and (23) follow from (20), (22) and (24) for the previous $n$. We need an estimate

\[ |f_{n}(x)| \leq \frac{4}{\theta} |f_{n+1}(x)| \leq \ldots \leq \left( \frac{4}{\theta} \right)^{k} |f_{n+k}(x)|, \ x \in I, \ n, k \in \mathbb{N}, \ (27) \]

which holds for $\lambda$ sufficiently small. Let us prove it. Let $Q_{x}^{n+1} \in G_{n+1}^{\varrho}$. Then we have

\[ |f_{n}(x)| \leq \frac{2}{\varrho} |f_{n}(x) - f_{n}(c_{Q_{x}})e^{i\varphi} F^{[x]}_{Q_{x}}(x)| \leq \]

\[ \leq \frac{2}{\varrho} (|f_{n+1}(x)| + |T_{n+1}(x)|) \leq \frac{2}{\varrho} (|f_{n+1}(x)| + c(\alpha) \lambda^{\beta} \rho^{-1} |f_{n}(x)|). \]

The first inequality follows from point 2 of Lemma 4 (which is applicable because of (24)), the last one – from (20). Now, the inequality $|f_{n}(x)| \leq \frac{4}{\theta} |f_{n+1}(x)|$ follows from the last estimate, if $2c(\alpha) \lambda^{\beta} \varrho^{-1} < \frac{1}{2}$. If $Q_{x}^{n+1} \notin G_{n+1}^{\varrho}$, then it follows from (20) even easier. So, (27) is proved.

Now, let $k \leq n + 1$, $x \in V_{n+2}^{\varrho}$, $y \in I$ and $|x - y| \leq \delta^{k}$. Let $k' := \max\{l \leq n : x \in V_{l+1}^{\varrho}\}$. From the fact that $x$ is again in a “good” set $V_{n+2}^{\varrho}$, it follows that

\[ x \in Q_{x}^{k'+1}[\tau^{n-k'+1}]. \ (28) \]

First assume $k \leq k'$. Then

\[ |f_{n+1}(x)| = |f_{n}x| + |T_{n+1}(x)| \leq |f_{n}(x)|(1 + c(\alpha) \frac{\lambda^{\beta}}{\rho}) \leq \ldots \leq \]

\[ \leq |f_{k'+1}(x)|(1 + c(\alpha) \frac{\lambda^{\beta}}{\rho})^{n-k'} \leq \]

\[ \leq (|f_{k'}(x)| + |T_{k'+1}(x)| + |f_{k'}(c_{Q_{x}^{k'}})| |F^{[x]}_{Q_{x}^{k'+1}}|)(1 + c(\alpha) \frac{\lambda^{\beta}}{\rho})^{n-k'} \leq \]

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Applying (13) and taking into account (28), we write on:

\[ |f_{n+1}(x)| \leq |f_k'(x)| \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} + (1 + \kappa) |F_{Q_{k'+1}}^{[\varepsilon_n]}| \right) \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'} \]

Applying (13) and taking into account (28), we write on:

\[ |f_{n+1}(x)| \leq |f_k'(x)| \left( 1 + C(\alpha) \lambda^\beta \left( \frac{1}{\rho} + \frac{(1 + \kappa)}{\tau(n-k'+1)\beta} \right) \right) \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'} \leq \]

\[ \leq \frac{|f_k'(y)|}{\rho^{k-k+1}} \left( 1 + C(\alpha) \lambda^\beta \left( \frac{1}{\rho} + \frac{(1 + \kappa)}{\tau(n-k'+1)\beta} \right) \right) \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'} \leq \]

\[ \leq |f_{n+1}(y)| \frac{1}{\rho^{k-k+1}} \left( 1 + C(\alpha) \lambda^\beta \left( \frac{1}{\rho} + \frac{(1 + \kappa)}{\tau(n-k'+1)\beta} \right) \right) \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'}. \]

We have used the induction assumption (21) for \( n = k' \) and (27). Now, if we choose \( \rho \) so that \( \sqrt{\rho} < \theta/50 \) and \( \sqrt{\rho} < \tau^2/50(C(\alpha)) \), we get

\[ |f_{n+1}(x)| \leq |f_{n+1}(y)| \frac{1}{\rho^{k-k+1}} \left( 1 + C(\alpha) \lambda^\beta \left( \frac{1}{\rho} + \frac{(1 + \kappa)}{\tau(n-k'+1)\beta} \right) \right) \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'}. \]

Then we choose \( \lambda = \lambda(\rho) \) so that \( (1 + c(\alpha) \frac{\lambda^\beta}{\rho}) < 2 \), and get

\[ |f_{n+1}(x)| \leq \frac{|f_{n+1}(y)|}{\rho^{n-k'1+k'-k+1}}, \]

and we are done.

Now we prove (23). Let \( t \in V_{n+2}^n \). Let, as in the proof of (21), \( k' := \max\{l \leq n : x \in V_{l+1}^n \} \). Again there holds (28). We have:

\[ |(f_{n+1})'(t)| \leq |f_n'(t)| + |T_{n+1}(t)| \leq \cdots \leq \]

\[ \leq |f_k'(t)| + |f_k'(c Q_{k'+1})||F_{Q_{k'+1}}^{[\varepsilon_n]}| \left( 1 + c(\alpha) \frac{\lambda^\beta}{\rho} \right)^{n-k'} + \sum_{l=k'+1}^{n+1} |T_l'(t)|. \]

Applying the induction assumption (23) for \( n = k' \) and (27), we have

\[ |f_k'(t)| \leq c_1(\alpha) \lambda^\beta \left( \frac{|f_k'(t)|}{\delta^{k'}} \right) \leq c_1(\alpha) \lambda^\beta (4/\theta)^{n-k'1} \left( \frac{|f_{n+1}(t)|}{\delta^{k'}} \right). \]

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We estimate the second term in (29) using (24), then (28) and (14) and finally (27), in the following way:

\[
|f_{k'}(c_{Q_{k'}+1})'((F[ε_{k'}]_{Q_{k'}+1})'(t)| \leq (1 + \kappa)|f_{k'}(t)| \frac{c(α)λ^β}{δ^{k'+1}τ^β(n-k'+1)} \leq (1 + \kappa)(4/θ)^{n-k'+1}|f_{n+1}(t)| \frac{c(α)λ^β}{δ^{k'+1}τ^β(n-k'+1)}.
\]

Finally, using (22) and then again (27), we get

\[
|T_{l+1}'(t) \leq \frac{c(α)λ^β|fi(t)|}{ρδ^{l+1}} \leq (4/θ)^{n+1-l}c(α)λ^β|f_{n+1}(t)|
\]

Taking \(δ < \theta τ^β/(4A)\) and substituting all the estimates into (29), we get

\[
|f_{n+1}'(t)| \leq \frac{λ^β|f_{n+1}(t)|}{δ^{n+1}}(c_{1}(α)/A^{n-k'+1} + c(α)\frac{4(1 + \kappa)}{θ^β} + \frac{4}{ρθ} \sum_{k=0}^{n-k'+1} A^{-k}).
\]

Taking \(A > 2\), we get (23), if \(c_{1}(α)\) is sufficiently large.

8. The end of the proof. In order to complete the construction, we should define the sets \(G_n\). Fix a parameter \(τ\) in order to satisfy inequality \(C(τ) \sum_{n=1}^{∞} \frac{1}{n^2} < 1/8\), where \(C(τ)\) is a constant from Lemma 5.

We define the sets \(G_{n+1}\) as follows: \(G_1 := H_1 = \{I\}\); an interval \(Q \in H_{n+1}\) belongs to \(G_{n+1}\), if \(Q \subset V_n\),

\[
M_Q(f_n) \leq K_n η^n,
\]

where \(K_n\) and \(η\) will be defined later and, besides,

\[
\widetilde{D}_n(c_Q) \leq n/2.
\]

(in fact, of course, \(\widetilde{D}_n\) is a constant on \(Q\)). Note that the choice of \(τ\) and Lemma 5 guarantee that the total (for all \(n\)’s) length of \(Q \in H_{n+1}\), \(Q \subset V_n\), not included in \(G_{n+1}\) because of violation of the condition (31), does not exceed \(1/8\).

Lemma 7 There exists a constant \(C'(α)\) such that for all sufficiently small \(λ\) the following inequalities hold:
1) If $Q \in G_{n+1}^g$, then
\[ \int_Q |f_{n+1}|^p \leq X \int_Q |f_n|^p. \]

2) If $Q \in G_{n+1}^d$, then
\[ \int_Q |f_{n+1}|^p \leq Y \int_Q |f_n|^p, \]

where $X := \gamma(\lambda)^p(1 + C'(\alpha)\lambda^\beta)^p = (1 + B\lambda)(1 + C'(\alpha)\lambda^\beta)^p$, $Y := (1 + C'(\alpha)\lambda^\beta)^p$.

Proof. Letting $P_{n+1}(t) := f_n(t) - f_n(c_{Q_t})e^{i\theta}F_{Q_t}(t)$, we get for $Q \in G_{n+1}^g$:
\[ \int_Q |f_{n+1}|^p = \int_Q |P_{n+1} + T_{n+1}|^p \leq \int_Q |P_{n+1}|^p (1 + \frac{|T_{n+1}|}{|P_{n+1}|})^p. \]

Applying point 2 of Lemma 4 and the estimate (20), we get
\[ \int_Q |f_{n+1}|^p \leq \int_Q |P_{n+1}|^p (1 + 2c(\alpha)\lambda^\beta)^p, \]

and then, estimating the integral using point 1 of Lemma 4, we have
\[ \int_Q |f_{n+1}|^p \leq (1 + B\lambda)(1 + C'(\alpha)\lambda^\beta)^p \int_Q |f_n|^p. \]

Recall that applicability of Lemma 4 is provided by (24).

The second case is even easier.

Lemma 8 If $\lambda$ is sufficiently small, then for all $n$ there holds an inequality
\[ \int_{V_n} |f_n|^p \leq \eta^n \int_{V_1} |f_1|^p \quad (32) \]

with some $\eta \in (0, 1)$.

Proof. Denote as $\Theta_n$ the set $\{0, 1\}^n$ (the set of all ordered sets $v = (v_1, \ldots, v_n)$ of the length $n$ of zeroes and unities). Define for $v \in \Theta_n$ the set $G^{(v)} \subset G_n$ as the set of all $Q$’s such that for all $k$ $v_k = 1$ if and only if
$Q \subset V^g$. Let for $v \in \Theta_n$ $Z(v) := \prod_{k=1}^{n} X^{v_k} Y^{1-v_k}$. Let by definition $G_1 := I$ and prove by induction in $n$ the estimate

$$\int_I |f_1|^p \geq X \sum_{v \in \Theta_n} Z(v)^{-1} \int_{G(v)} |f_n|^p.$$ (33)

The base is obvious. Suppose (33) holds for some $n$. Then

$$Z(v)^{-1} \int_{G(v)} |f_n|^p = Z(v)^{-1} \left( \int_{G(v,1)} |f_n|^p + \int_{G(v,0)} |f_n|^p \right) \geq$$

$$\geq Z(v)^{-1} \left( X^{-1} \int_{G(v,1)} |f_{n+1}|^p + Y^{-1} \int_{G(v,0)} |f_{n+1}|^p \right) =$$

$$= Z(v,1)^{-1} \int_{G(v,1)} |f_{n+1}|^p + Z(v,0)^{-1} \int_{G(v,0)} |f_{n+1}|^p.$$ (we used Lemma 7). Thus we have proved the inductive step. Then, for $\lambda$ sufficiently small we have $X < 1$, $Y > 1$. If $n > N_0$, $v \in \Theta_n$, then $\sum_k v_k < n/2$ implies $G(v) = \emptyset$ (it follows from (31)), therefore the right-hand side in (33) is not less than $X^{-\frac{n}{2} + 1} Y^{-\frac{n}{2}} \int V_n |f_n|^p$, and (32) follows provided $\lambda$ is sufficiently small.

Now we are ready to finish the setup of the construction. What we should do is to make precise the condition (30). We take $\eta$ from Lemma 8. As, by Lemma 8, $\int V_n |f_n|^p \leq C \eta^n$, the total length of all intervals $Q \in H_{n+1}$, such that $Q \subset V_n$ and on $Q$ the condition (30) fails, does not exceed $\frac{C}{K_n}$. For $K_n$ we take a sequence growing as some power of $n$ and satisfying the condition $\sum_{n=1}^{\infty} \frac{C}{K_n} < \frac{1}{8}$. We get that total length of all intervals dropped out of $V_n$ on all steps because of violation of (30) is less than $\frac{1}{8}$. But one can tell the same thing about the length of all intervals dropped out because of violation of (31). Hence $|\bigcap_{i=1}^{\infty} V_n| > \frac{3}{4}$. It follows from (30) and an estimate (24) of the oscillation of $f_n$ that if $Q \in G_{n+1}$, then $|f_n(c_Q)| \leq C' \eta^n$. Hence, taking into account (3) and (24) we get

$$|f_n(t) - f_{n+1}(t)| \leq \frac{C_1 \eta^n}{\varepsilon_n^{\beta}} + C_2 \lambda^\beta \eta^n, t \in \mathbb{R}$$ (34)

The first term in the right-hand side corresponds to the building block $F^{[\varepsilon_n]}_{G_{n+1}}$ (if there is any), the second – to all the others. It follows from this estimate
that, $f_n$ converges uniformly on $\mathbb{R}$ to some function $f$. We also need an estimate
\[
|f_n(t) - f_{n+1}(t)| \leq \frac{\lambda^\beta \delta_{n+1}^{\alpha} C^n \eta^{n+1}}{|t|^\beta}, \quad t \notin 3I,
\] (35)
which follows from (11). Now, (35) implies that
\[
|f_n(t)| \leq c|t|^{-\beta}, \quad |t| \geq 3/2,
\] (36)
so we can, fixing $t$, write
\[
\int_{\mathbb{R}} \frac{f_n(s)ds}{|s-t|^{1-\alpha}} = \int_{|s| \leq \max(2|t|,3/2)} \frac{f_n(s)ds}{|s-t|^{1-\alpha}} + \int_{|s| \geq \max(2|t|,3/2)} \frac{f_n(s)ds}{|s-t|^{1-\alpha}}.
\]
In the first term, the passage to the limit in the integral is provided by uniform convergence of $f_n$ as $n \to \infty$, in the second one, the function under integral is majorized by $c|s|^{-2}$. Hence
\[
\lim_{n \to \infty} g_n = \lim_{n \to \infty} \int_{\mathbb{R}} \frac{f_n(s)ds}{|s-t|^{1-\alpha}} = \int_{\mathbb{R}} \frac{f(s)ds}{|s-t|^{1-\alpha}}
\]
(The first inequality follows from Lemma 1). Finally note that the functions $g_n$ converge uniformly to $g$:
\[
|r_n| \leq \frac{C^n \eta^{n+1}}{\varepsilon_n}.
\]
So, the program announced in the beginning of Section 3 is completed. Let us now show that the function $f$ satisfies Hölder’s condition.

In this argument, constants may depend on all parameters except $n$. Of course, we use the proposition from the beginning of Section 4. The condition $|f_n - f| \leq C_1 \eta^n$ follows easily from (34).

Let us estimate the derivative of the correcting term on the $n$-th step:
\[
|(W\alpha r_n)'(t)| = \left| \frac{d}{dt} \left( \sum_{Q \in G^o_{n+1}} f_n(c_Q) F_{Q}^{[\varepsilon_n]}(t) e^{i\theta} \right) \right| \leq \max_{Q \in G^o_{n+1}} |f_n(c_Q)| \sum_{Q \in G^o_{n+1}} |(F_{Q}^{[\varepsilon_n]}(t))'| \leq \max_{Q \in G^o_{n+1}} |f_n(c_Q)| \left( |(F_{Q}^{[\varepsilon_n]}(t))'| + \sum_{Q \in H_{n+1}, Q \neq Q_{i+1}^n} |(F_{Q}^{[\varepsilon_n]}(t))'| \right).
\]
We know that \( \max_{Q \in G_{n+1}} |f_n(c_Q)| \leq C' \eta'^n \). Then, we use (8) for the first term in the brackets, and for the sum \(- (26) \) with \( \epsilon := \frac{\delta}{2} \). Thus we have

\[
|\left(W_n r_n\right)'(t)| \leq C' \eta'^n \left( \frac{C_1}{\delta n+1 \epsilon_n} + \frac{C_2}{\delta n+1} \right) \leq \frac{C_3}{\delta n+1}
\]

Hence \( |f_n'(t)| \leq \frac{C_3}{1-\delta'} \delta'^{-n-1} \leq \frac{C_3}{\delta n+1} \), and Hölder’s property for \( f \) is proved.

9. Remark on the order of choice of the parameters. Recall the order we chose our parameters in. Given \( \alpha \in (0,1) \), we fix \( p \) and \( \theta \) (Lemmas 3 and 4). Here Lemma 4 holds for all sufficiently small \( \lambda \) and \( \epsilon \). Then we fix a sequence \( \epsilon_n \). Independently of other parameters we fix \( \tau \). Lemma 6 holds for all sufficiently small \( \lambda \) and \( \delta \), independently of the choice of \( G_n \) (here how small \( \lambda \) and \( \delta \) should depend on \( \rho \) from this lemma, and \( \rho \) itself only depends on \( \alpha, \theta, p, \tau \), and the setup function \( g_1 \)). Now we choose \( \lambda \) such that the multiplier \( \eta \) in front of the integral in the right-hand side of (32) is less than 1 (decreasing \( \lambda \) killing the influence of “tails” \( T_n \) in comparison with the correction effect provided by point 1 of Lemma 2. Note that the last effect decays when \( \lambda \) decreases as well \(( \gamma(\lambda) \xrightarrow{\lambda \to 0} 1! \) ), but the “tails” dies faster). Finally, we fix \( \delta \) in order to provide (24).

So far our considerations did not depend on \( G_n \), therefore we did not need to define them. Now we fix a sequence \( K_n \) and this finishes the definition of our construction.

10. The case of negative \( \alpha \). One can ask whether there is an analogue of Theorem 1 for other M.Rietz’s kernels. In the case of the kernel \( |x|^{-\beta} \), \( 1 < \beta < 2 \), the answer is affirmative; moreover, as the convolution with such a kernel is, at least formally, the inverse operator to \( U_{\beta-1} \) (see Lemma 1, points 2 and 5), the example in essential coincides with the one built above.

**Theorem 2** There exist a nonzero continuous function \( g : \mathbb{R} \to \mathbb{C}, \) supp \( g \subseteq 3I \) and a set \( E \) of positive measure, such that \( t \in E \implies g(t) = 0, \int g(x) |t - x|^{-\beta} dx = 0 \). The last integral converges absolutely for every \( t \in E \). The function \( g \) satisfies Hölder’s condition with an exponent \( \beta - 1 \).

**Proof.** We take for \( g \) the function built in Theorem 1 (with \( \beta - 1 \) for \( \alpha \)). Let

\[
E := V \cap (I \setminus S), \quad \text{where } S := \bigcup_{n \in \mathbb{N}} \bigcup_{Q \in G_{n+1}} 3\epsilon_n (Q - c_Q) + c_Q
\]
The idea is that $|S| < \frac{3}{4}$, but now $\text{supp } g_n$ is contained “rather deep” inside $S$:

$$\text{dist}(\text{supp } g_n, E) \geq \varepsilon_n \delta^{n+1}. \quad (37)$$

We know (point 5 of Lemma 1), that for $t \in E$

$$f_n(t) = \int_{\mathbb{R}} g_n(x)|t - x|^{-\beta} dx \xrightarrow{n \to \infty} 0.$$

To prove Theorem 2, it is sufficient to justify the passage to the limit in the integral. For the summable majorant we take a function

$$\tilde{g}(x) := |t - x|^{-\beta} \sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|.$$

Let us estimate the $n$-th term:

$$|g_{n+1} - g_n| \leq \sum_{Q \in G^n_{n+1}} |f_n(c_Q)|(\delta^{n+1} \lambda)^\alpha (\phi_{\varepsilon_n Q}) \leq C' \eta^m (\delta^{n+1} \lambda)^\alpha \sum_{Q \in G^n_{n+1}} (\phi_{\varepsilon_n Q}).$$

Check summability of the majorant:

$$\int_{\mathbb{R}} |t - x|^{-\beta} |g_{n+1}(x) - g_n(x)| \leq C' \eta^m (\delta^{n+1} \lambda)^\alpha \sum_{Q \in C^n_{n+1}} \int_{\mathbb{R}} |t - x|^{-\beta} \phi_{\varepsilon_n Q}(x) dx \leq C' \eta^m (\delta^{n+1} \lambda)^\alpha \sum_{Q \in G^n_{n+1}} |t - x^*_Q|^{-\beta} \int_{\mathbb{R}} \phi_{\varepsilon_n Q}(x) dx \leq C' \eta^m (\delta^{n+1} \lambda)^\beta \sum_{Q \in G^n_{n+1}} |t - x^*_Q|^{-\beta}. \quad (38)$$

Here $x^*_Q$ denotes a point of the support of $\phi_{\varepsilon_n Q}(x)$, closest to $t$. Using (37) and the fact that the distance between two different points $x^*_Q$ and $x^*_Q'$ is no less than $\delta^{n+1}$, we get

$$\sum_{Q \in G^n_{n+1}} |t - x^*_Q|^{-\beta} \leq 2 \sum_{k=0}^{\infty} (\varepsilon_n \delta^{n+1} + k\delta^{n+1})^{-\beta} \leq 2\delta^{(n+1)(-\beta)} \sum_{k=0}^{\infty} |\varepsilon_n + k|^{-\beta} \leq 2\delta^{(n+1)(-\beta)} (\varepsilon_n^{-\beta} + \sum_{k=1}^{\infty} k^{-\beta}) \leq C \delta^{(n+1)(-\beta)} \varepsilon_n^{-\beta} \quad (39)$$
Substituting (39) to (38) and summing over all $n$, we get

$$\int_{\mathbb{R}} \tilde{g}(t)dt \leq C'' \lambda^{-\beta} \sum_{n=1}^{\infty} \eta^m \varepsilon_n^{-\beta} < +\infty$$

We should now only prove that $g$ satisfies Hölder’s condition with an exponent $\alpha = \beta - 1$. In fact this is a property of the potential $U_\alpha$ of any bounded function for which it is defined. To prove it, take $t > 0$ and write the following estimate:

$$\int_{\mathbb{R}} ||x|^{\alpha-1} - |x-t|^{\alpha-1}|dx = \int_{(-t;2t)} + \int_{\mathbb{R}\setminus(-t;2t)} =: J_1 + J_2.$$ 

Estimate each term:

$$J_1 \leq \int_{(-t;2t)} |x|^{\alpha-1} + \int_{(-t;2t)} |x-t|^{\alpha-1} = \frac{2}{\alpha} (1 + 2^\alpha) t^\alpha;$$

$$J_2 \leq 2 \int_{(t;+\infty)} x^{\alpha-1} - (x+t)^{\alpha-1} \leq 2(\alpha - 1) \int_{(t;+\infty)} t x^{\alpha-2} \leq 2 t^\alpha.$$ 

When estimating $J_2$ we have used the inequality $|h(x+t)-h(x)| \leq t \sup_{s \in (x,x+t)} |h'(s)|$ for a smooth function $h$. From this estimates we get

$$|(U_\alpha f)(t+\delta)-(U_\alpha f)(t)| \leq \sup_{\mathbb{R}} |f| \int_{\mathbb{R}} (|t+\delta-x|^{\alpha-1}-|t-x|^{\alpha-1})dx \leq C(\alpha)(\sup_{\mathbb{R}} |f|)\delta^\alpha.$$ 

The theorem is proven.

**Remark 1.** Proving the smoothness of $g$ we didn’t use all the information we had about $f$. In fact $f$, besides it is bounded and belongs to the domain of $U_\alpha$, satisfies Hölder’s condition with some exponent $r > 0$. Using this and well-known techniques of estimating operators similar to M. Rietz potential (see, for example, [8]), one can prove that $g$ satisfies Hölder’s condition with an exponent $\beta - 1 + r$.

**Remark 2.** The theorem proven in [3] states that for the potentials $U_\alpha$, $-1 < \alpha < 0$, uniqueness holds if the density $g$ belongs to $C^{1+\varepsilon}$ with some $\varepsilon > 0$. Theorem 2 shows that the last condition cannot be replaced by Hölder’s condition with an exponent $-\alpha$. So, there is a gap between the two results, which decreases when $\alpha \to -1$. If $\alpha \leq -1$, one does not need any supplementary smoothness condition: the mere existence of the potential is sufficient (see, for example, [6]).
For the cases $\alpha = 0$ and $\alpha > 1$ (except odd integers for which the uniqueness does not hold in any sense), the question whether one can omit the smoothness conditions imposed in [5] remains open.

11. Extension of the results to the multidimensional case. The result of Theorem 1 can be extended to the case of M. Rietz potentials in spaces $\mathbb{R}^d$ for $d > 1$. In this case for $\alpha \in (0, d)$ we consider a set of all measurable functions $f$, satisfying the condition

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{1 + |x|^{d-\alpha}} \, dx < +\infty$$

(40)

(as above, we denote this set $\text{dom } U_\alpha$). We let

$$U_\alpha f := f \ast |x|^{d-\alpha}, \ f \in \text{dom } U_\alpha,$$

where $\ast$ denotes the convolution in $\mathbb{R}^d$. The case of major interest is $d = 2; \ \alpha = 1$ (Newton’s potential of the charge concentrated in the plane). Note, however, that in case of $d > 1$ there is not any analogue of the uniqueness theorem mentioned in the introduction.

There holds the following generalization of Theorem 1:

**Theorem 3** For all $d \in \mathbb{N}$ and for all $\alpha \in (0, d)$ there exist a nonzero function $f : \mathbb{R}^d \to \mathbb{R}, \ f \in \text{dom } U_\alpha$ and a set $E \subset \mathbb{R}^n$ of positive Lebesgue measure satisfying the condition $f|_E = 0, U_\alpha f|_E = 0$, and Hölder’s condition with some positive exponent.

The proof of this theorem is quite similar to the one of Theorem 1. We highlight some details differing in the multidimensional case.

We need an operator $W_\alpha$, “the inverse operator” to $U_\alpha$. The precise expression of this operator (see, for example, [2, page 241]) does not matter for us, the only thing we need is that if we now denote as $\beta$ the number $d + \alpha$, then points 1,2,3,5 of Lemma 1 still hold.

The role of $I$ will be played by the cube $I^d$, and we shall consequently divide it to congruent cubes with the side equal to $\delta^n$. Instead of “the finitizator” $\phi(x)$ we take the function $\phi(|x|)$

The computations made in [2, page 241] show that lemma 2 still holds in the multidimensional case. Lemmas 3 and 4 can be derived from it quite similarly to the above.
Taking into account that now $\beta = d + \alpha$, the most of computations in the multidimensional case will repeat one-dimensional literally, if we also replace derivative by gradient everywhere. So, because of point 3 for lemma 1 there still hold (3), (11), (12), (13), (14) (now for the cube $Q$ with sides parallel to the coordinate axes the symbol $Q[a]$ denotes $Q \setminus Q'$, where $Q'$ is a cube obtained from $Q$ by homothety with the center $c_Q$ and the dilation factor $a$).

The remaining part of the construction is the same. Note that estimates (25) and (26), playing the key role in the proof of lemma 6, and seeming to depend on the dimension, in fact hold in the literally same form.

Remark. Theorem 2 can be extended to the multidimensional case as well: for $d < \beta < 2d$ there exist a nonzero continuous function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ and a set $E$ of positive measure, such that if $t \in E$, then $g(t) = 0$. Indeed, $\int_{\mathbb{R}^d} g(x)|t-x|^{-\beta}dx = 0$. The last integral converges absolutely for all $t \in E$. The function $g$ satisfies Hölder’s condition with an exponent $\min\{\beta - d; 1\}$. The only difference in the proof is the estimate (39), where sums becomes multiple (of order $d$). They still will converge, because now $\beta > d$.

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This smoothness estimate can be obtained by a simple method similar to the one used in the proof of Theorem 2. Using the techniques mentioned in the remark after Theorem 2, one can prove the inclusion $g \in C^{\beta-d+r}$ (in case when $\beta - d + r$ is an integer, we understand it as the corresponding Zygmund class)
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