Error budget in systems with time-dependent forcings

P. Sancho

Instituto Nacional de Meteorología, Centro Zonal de Castilla y León, Orión 1, 47071 Valladolid, Spain

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Abstract. The behaviour of the error growth is analyzed in several simple examples of systems with external time-dependent forcings. In some systems oscillations of the error around the saturation level can be observed. A common feature of these examples is the error growth dependence on initial time. In the examples here considered the improvement in the predictability derived from an adequate choice of the initial time is comparable to those obtained by reducing the initial errors.

1 Introduction

The general question of atmospheric predictability has gained increasing attention during the last years. The evolution of atmospheric flow is governed by nonlinear equations, whose solutions exhibit sensitive dependence on initial conditions. Some initial errors, large or small, will amplify and, after some time, render completely unreliable any forecast. Predictability is analyzed by the error budget that describes how fast forecast errors grow on average. A number of studies on error growth have been carried out using atmospheric models of varying complexity, from the simple analytical red-noise atmosphere (Fraedrich and Ziehmann-Schlumbohm, 1994) to simplified general circulation models or operational weather forecasting models (Schubert and Suarez, 1992; Lorenz 1982; Dalcher and Kalnay, 1987).

Atmospheric dynamics is an example of nonautonomous system with time-dependent external forcings. The incident radiation, driving the large-scale motion of the atmosphere, is a periodic phenomenon. In climate dynamics the situation is even more compelling. The periodic variations of the orbital parameters seem to be one of the fundamental causes of climate change. We want to study the incidence of these time-dependent forcings in the theory of error growth. In particular, in this paper, we shall concentrate on some simple examples that can serve as a guide for more general studies. Despite the simplicity of the models we can obtain several interesting conclusions. The first one is the fact that some time-dependent systems do not obey the usual dynamics of the error growth, that is, an initial stage of exponential growth followed by saturation. We show that in the case of the red-noise atmosphere with time-dependent terms the errors undergo, in the mean, an initial stage of exponential growth, but followed now by oscillations around the saturation level of the autonomous system.

A second conclusion derived from this study is the dependence of the error growth on the initial time \( t_0 \) (the time at which initial conditions are imposed). Error growth dependence on several factors have been clearly established in many studies. These factors are the initial error size (Trevisan, 1993), the weather regime and the location on the weather manifold (Keppenne and Nicolis, 1989). If the choice of the initial time modifies the analytical structure of the error growth, then \( t_0 \) can be viewed as a parameter playing an active role in predictability theory. In particular, the time necessary to reach the predictability limit will in general be different for different initial times (even supposing the size of the initial error similar in both cases). We show that in one of the models considered in this paper the improvement derived from an adequate choice of the initial time is comparable to those obtained by the reduction of the initial error size (the analysis error in operative weather forecasting models).

The plan of the paper is as follows. In Sect. 2 we study the impact of a time-dependent forcing in systems that obey the Lorenz law for error growth. In Sect. 3 the red-noise atmosphere with periodic forcing is analyzed. Finally, in the Discussion the main physical ideas involved in these models are considered.

2 Lorenz’s law for error growth

The first attempt to deduce a law of error growth from real atmospheric data is found in the work of Lorenz (Lorenz, 1969). In the mean, the errors undergo an initial stage of exponential growth followed by saturation. As it turns out, this trend can be represented in a qualitative manner by a quadratic law, the logistic equation for the mean error \( X \)

\[
\frac{dx}{dt} = A(X-X^2),
\]

provided that the parameter \( A \) is suitably adjusted.
values of error growth. The difference can be modelled with $N$ the amplitude and used for the constant $A$. At a given time and initial conditions the two curves reach different values of the initial dimensionless error $X_0$. The mean error growth follows Lorenz's law. The solution of $\text{Eq. (2)}$ can represent, for example, the model used by Trevisan et al. with small initial errors (Trevisan et al., 1992).

Now, we introduce a periodic forcing in $\text{Eq. (2)}$

$$\frac{dx}{dt} = \frac{f(x)}{(1+N\sin \omega t)}, \quad (3)$$

with $N$ the amplitude and $\omega$ the frequency of the forcing.

Introducing the new variable $dt = (1+N\sin \omega t) \, dt$, $\text{Eq. (3)}$ reads

$$\frac{dx}{dt} = \frac{f(x)}{x}, \quad (4)$$

Now the mean error will obey the equation

$$\frac{dx}{dt} = A \,(X-X^2), \quad (5)$$

equivalent to

$$\frac{dx}{dt} = A \,(1+N\sin \omega t) \,(X-X^2). \quad (6)$$

The solution of $\text{Eq. (6)}$ with initial condition $X(t_0) = x_0$ is

$$X(t) = \left[ 1 + \left( \frac{1}{x_0} \right) \exp \left\{ -A \,(t-t_0) + \frac{NA}{\omega} \,(\cos \omega t - \cos \omega t_0) \right\} \right]^{-1}. \quad (7)$$

Figure 1 shows this solution for two different values of the initial dimensionless time $T = \omega t$, $T_0 = 0$ and $T_0 = \pi$. The numerical values used for the constants are $A/\omega = 0.8$, $N \approx 0.25$ and $X_0 = 0.1$ (the initial error size is supposed to be equal in both cases). At a given intermediate time after imposition of initial conditions the two curves reach different values of error growth. The difference can be large, for instance, for $T = 3$ the values of the error are 0.55 and 0.45, and for $T = 4$ 0.79 and 0.66, respectively.

These differences imply different predictability times. The predictability time $t^*$ is defined as the time it takes for an initial error $X_0$ to reach a preassigned value $X'$. This definition can be expressed in a mathematical form as

$$(X')^{-1} = 1 + \left( \frac{1}{x_0} \right) \exp \left\{ -A \,(t^* - t_0) + \frac{NA}{\omega} \,(\cos \omega t^* - \cos \omega t_0) \right\}. \quad (8)$$

This equation cannot be solved analytically. In Table 1 we include the predictability times for different values of $X'$. In all the cases the difference between both predictability times is important.

Finally, in order to compare the predictability improvements derived from the reduction of the initial error size and from the choice of the initial time, we include in $\text{Fig. 1}$ and $\text{Table 1}$ the error growth curve and predictability times for $T_0 = 0$ and initial error $2X_0/3$. We deduce from the curve that the error growth for $T_0 = \pi$ and $X_0$ is smaller for intermediate times. In particular, the predictability times for $X' = 0.5$ are 2.8 and 3.2 for $T_0 = 0$ and $X_0$, that is, improvements of 22% and 39%, respectively. From the predictability point of view a good choice of the initial time is comparable to a large (1/3) reduction of the initial error size.

3 Red-noise atmosphere

Time series observed in the atmosphere are characterized by some of the properties of red-noise processes. Because of this similarity the red-noise atmosphere has been used in many studies as a substitute of the real atmosphere. Recently Fraedrich and Ziehmann-Schulmbohm (Fraedrich and Ziehmann-Schulmbohm, 1994) have developed a predictability experiment in a red-noise atmosphere. By examining the lead-time-dependent error budgets of individual and ensemble forecasts, these authors derive analytically various measures of predictability. Despite the simplicity of the model, the error budgets share some qualitative features that may be compared to those of numerical weather-prediction and climate models.

In this paper we extend the model of these authors by including a temporal dependence in the red-noise process. The dynamics $Y = Y(t)$ consist of a deterministic part and an additive random part $Z$,

$$Y_n = Y_{n-1} + f_n + Z_{n}, \quad (9)$$

where
and a and b are constants, \( n_0 \) indicates the time at which the process starts. The index \( n \) takes the values 0, 1, ... .

The Gaussian white-noise \( z_n \) has zero mean \( \langle z_n \rangle = 0 \), variance \( S_z^2 = \langle z_n^2 \rangle \) and vanishing crossed correlations \( \langle z_m z_n \rangle = 0 \) if \( n \) is different from \( m \).

We also suppose that \( Y_0 \) and \( z_n \) are statistically independent variables, \( \langle z_n Y_0 \rangle = 0 \).

Equation (9) can be expressed in terms of the initial condition \( Y_0 \) as

\[
Y_n = Y_0 H_{n-1} + z_n + \sum_{i=1}^{n-1} z_{H_{i-1}} + f_i f_{i+1} \ldots f_j
\]

where

\[
H_{i,j} = f_i f_{i+1} \ldots f_j
\]

is the product of the \( f \)'s between \( i \) and \( j \).

3.1 Persistence forecasts

A first approach to the problem of predictability is provided by persistence forecasts. Persistence predicts the future states \( Y_n \) using the initially given state \( Y_0 \).

The error budget of persistence forecasts is described by the evolution of the error variance \( E_n = \langle (Y_n - Y_0)^2 \rangle \), where the sample average is taken over all the verification pairs. After simple statistical manipulations, \( E_n \) becomes

\[
E_n = E_0 S_z^2 + B_z S_z^2,
\]

where

\[
E_0 = (1 - H_{n_m})^2, \quad B_z = 1 + \sum_{i=1}^{n_m - 1} H_{i-1}^2, \quad S_z^2 = \langle (Y_0 - Y_n)^2 \rangle
\]

is the initial variance of the variable \( Y \) (we suppose a zero mean \( \langle Y_0 \rangle = 0 \)).

In Fig. 1 we present Eq. (13) for two different values of the initial time \( n_0 \). We have taken for \( a \) and \( b \) the values 0.6 and 0.5 and the frequency is \( \omega = 0.85 \). Moreover, the third curve in the figure shows the same process with \( b = 0 \), that is, with no temporal dependence.

The three curves show an initial stage of exponential growth. The two time-dependent systems have important quantitative differences, for instance, we have for \( n = 2 \)

\[
E_2(n_0 = 0) = 1.3 \quad \text{and} \quad E_2(n_0 = n) = 0.98.
\]

These differences are also reflected into the predictability times. In systems with a discrete time variable the predictability time \( n^* \) is defined as the larger value of \( n \) for which the error is smaller than a preassigned predictability limit. For instance, taking the predictability limit as 1 we have \( n^*(n_0 = 0) = 1 \) and \( n^*(n_0 = n) = 2 \). In many studies the variance \( S_z^2 \) serves as a predictability threshold and, consequently, is taken as the predictability limit. With this choice of the predictability limit we would have the same predictability times in both cases. The behaviour of autonomous and nonautonomous systems differs when the error of the autonomous process reaches the saturation level. At this stage, the systems with time dependent forcings show an oscillatory behaviour around a level close to the typical saturation level of the autonomous system. The two oscillations are similar, showing only a phase delay between the values of the two curves. This is a large value of the amplitude if we compare with the value of the saturation level 1.14. Moreover, the maximum and minimum values of the error at this stage are equal for both choices of the initial time, 1.41 and 0.97.
We use the notation $g$ gives nonautonomous system. As remarked earlier, the oscillatory behaviour of the error in the exponential growth. Taking the variance curves. We observe again an initial stage of exponential growth on initial time. This result justify the view of consider the initial time as an active parameter in error growth theory. This dependence of the error growth on initial times can be easily understood by taking into account the fact that at different initial times the external forcings are different. We are placed at different regions in the mathematical space of external forcings, and the respective error dynamics are modified. Moreover, the numerical estimations of Sect. 2 show that in some cases the improvement of the predictability time obtained by an adequate choice of the initial time is comparable to those obtained by a large reduction of the initial error. As we shall discuss in the next point, the error dynamics at the second stage will also depend, in general, on the initial time.

4 Discussion

We have presented an analysis of the dynamics of error growth in nonautonomous systems. The analysis is restricted to some very simple examples. The study of systems with time-dependent forcings can be justified from, at least, two points of view. Firstly, because atmospheric and climate dynamics are examples of nonautonomous dynamics driven by external, periodic forcings. Secondly, from a purely error growth theory point of view, the time-dependent terms are also necessary. For instance, Nicolis has suggested (Nicolis, 1992), that the logistic-like models of error growth must be augmented by time-dependent forcings in order to reflect the coupling of the error dynamics with the structure of the phase space.

In spite of the simplicity of the models here considered, the results obtained can be viewed as a preliminary step in the study of error growth in nonautonomous systems. Two principal conclusions have been derived from these models:

1) In the initial stage of exponential growth the error dynamics is sensitive to the choice of the initial time. As the predictability limit is reached at this stage, the different quantitative behaviours for different initial times imply a dependence of the predictability time on the initial time. This result justify the view of consider the initial time as an active parameter in error growth theory. This dependence of the error growth on initial time can be easily understood by taking into account the fact that at different initial times the external forcings are different. We are placed at different regions in the mathematical space of external forcings, and the respective error dynamics are modified. Moreover, the numerical estimations of Sect. 2 show that in some cases the improvement of the predictability time obtained by an adequate choice of the initial time is comparable to those obtained by a large reduction of the initial error. As we shall discuss in the next point, the error dynamics at the second stage will also depend, in general, on the initial time.

2) Autonomous and nonautonomous systems undergo an initial stage of exponential growth. Therefore, in spite of some quantitative differences, the underlying dynamics must be equalized in both cases and, consequently, the oscillations are modified by the autonomous terms. After this initial stage the behaviour of both types of systems is, also qualitatively, different. Instead of the saturation stage typical of autonomous systems we observe in some nonautonomous models an oscillatory behaviour. Note that the model considered in this case is not the same as that considered in the oscillations. This behaviour can be easily understood taking into account the type of temporal forcing introduced. The temporal term multiplies the right hand side of the differential equation and, consequently, the equation can be factored. We can see the temporal term as a modification of the mathematical measure of the variable time. We cannot expect that this simple modification of the system can modify qualitatively the error
The oscillations exhibited by nonautonomous systems can be explained by the contribution of several terms to the error dynamics. The autonomous terms contribute to the dynamics stabilizing the error and constraining the error variation to a finite interval (instead of a saturation level). On the other hand, the nonautonomous terms introduce an oscillatory temporal dependence on the error growth. A parameterization of the error growth that reproduces its main properties at this stage is

\[ E_n = ESL + A \cos(\Omega n + \phi), \] (24)

where ESL is the effective saturation level, defined as the level around which the error oscillates. \( \Omega, \phi \) and \( A \) are the frequency, phase delay and amplitude of the oscillation.

For persistence forecasts and \( n_0 = 0 \) and \( n_0 = \pi \), Eq. (24) reads \( 1.19 + 0.22 \cos(n/7 + 2.43) \) and \( 1.19 + 0.22 \cos(n/7 + 2.86) \). The two following features are noted:

(a) The effective saturation level differs from the saturation level of the autonomous system. This difference reflects the coupling between the autonomous and nonautonomous terms in the error dynamics. This coupling is a consequence of the nonlinearity of the system.

(b) The phase delay depends on the initial time. This dependence can be viewed as a manifestation of the fact that the system reaches the second stage at different times as a function of the initial time.

The parameterizations of the error growth in ensemble-mean forecasts at this second stage for \( n_0 = 0 \) and \( n_0 = \pi \) are \( 1.86 + 0.34 \cos(n/7 + 2.14) \) and \( 1.6 + 0.34 \cos(n/7 + 1.57) \). Now, the effective saturation level differs for different initial times. This difference reflects that the coupling between autonomous and nonautonomous terms depends on the choice of the initial time. In the case of persistence forecasts, the forecast is always the same, \( \gamma_n \), for any initial time; the coupling is independent of the initial time. On the other hand, in the case of ensemble-mean forecasts the members of the ensemble are, in general, different and the coupling can depend on the initial time.

The conclusions obtained with the simple models here presented must be tested with more complex and realistic systems. In particular, the amplitudes of the error oscillations obtained in this paper are large because of the large ratio \( b/a \) (deterministic autonomous/nonautonomous terms) used. In realistic models we must expect smaller amplitudes. Also, we must study systems with several simultaneous periodic forcings, as it is the case in atmospheric and climate dynamics.

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