Checks of asymptotia in $pp$ elastic scattering at LHC

Agnes Grau$^a$, Simone Pacetti$^b$, Giulia Pancheri$^c$, Yogendra N. Srivastava$^b$

$^a$Departamento de Física Teórica y del Cosmos, Universidad de Granada, 18071 Granada, Spain
$^b$INFN & Physics Department, University of Perugia, Italy
$^c$INFN Frascati National Laboratories, Via E. Fermi 40, Italy

Abstract

We parametrize TOTEM data for the elastic differential $pp$ cross section at $\sqrt{s} = 7$ TeV in terms of two exponentials with a relative phase. We employ two previously derived sum rules for $pp$ elastic scattering amplitude in impact parameter space to check whether asymptotia has been reached at the LHC. A detailed study of the TOTEM data for the elastic differential cross section at $\sqrt{s} = 7$ TeV is made and it is shown that, within errors, the asymptotic sum rules are satisfied at LHC. We propose to use this parametrization to study forthcoming higher energy data.

Keywords: proton-proton, elastic scattering, total cross section

1. Introduction

The TOTEM experiment has measured the total $pp$ cross section at $\sqrt{s} = 7$ TeV [1] and the value may be consistent [2] with saturation of the Froissart-Martin bound [3, 4], i.e. $\sigma_{\text{total}} \simeq \ln^2 s$. It is also consistent with a range of total cross section predictions, based on a mini-jet QCD model with infrared gluons [5, 6], as well other deductions based on a hard Pomeron component [7]. The saturation of the Froissart bound, if confirmed, could exclude, at least at present energies [8], a different behaviour obtained from hidden extra dimensions [9, 10]. Indeed, according to Ref. [11], the TOTEM result indicates that we have now reached asymptotia as far as the total cross section is considered and thus existence of hidden extra dimensions can be excluded. On the other hand, contradictions in the analysis of Ultra High Energy Cosmic Rays composition [12] might hint to the existence of new physics that modifies baryonic interactions at center-of-mass...
(CM) energies in the 50 ÷ 100 TeV range. Thus the problem of the energy
dependence of the total cross section is not yet beyond all dispute.

Another interesting feature has arisen in connection with TOTEM pub-
lished results for the differential and the total elastic cross section [13]. These
data show that the dip, which had characterized data at ISR in $pp$ scatter-
ing, and which has been only a faint presence in $\bar{p}p$ at the Tevatron, has
reappeared. We propose to use these data to check whether asymptotia has
been reached, through two asymptotic sum rules (ASR) for the elastic am-
plitude in impact parameter space [14]. To check the satisfaction of these
ASR, we shall use the model independent parametrization of the scattering
amplitude [15] proposed in 1973 by Barger and Phillips.

We shall see that this model, with two exponentials and a phase, describes
well both the diffraction peak, as well as the $t$-dependence after the dip at
LHC with CM energy of 7 TeV (LHC7). We also analyze ISR data for $pp$
scattering at $\sqrt{s} = 53$ GeV, and, by comparison, find that the ASR are not
satisfied at ISR, while being almost exactly satisfied at LHC7. We discuss
the approach to asymptotia of the ratio $\sigma_{\text{elastic}}/\sigma_{\text{total}}$ and of the forward slope
and suggest to use this simple model to fit future forward scattering LHC
data.

2. Asymptotic sum rules for the elastic scattering amplitude

Let the elastic amplitude $F(s, t)$ be normalized so that

$$\sigma_{\text{total}}(s) = 4\pi \Im m F(s, 0).$$

At high energy, all particle masses can be ignored and the elastic differential
cross section reads

$$\frac{d\sigma}{dt} = \pi |F(s, t)|^2.$$  \hspace{1cm} (2)

Writing the elastic amplitude as

$$F(s, t) = i \int_0^\infty (b dB) J_0(b \sqrt{-t}) \left[1 - e^{2i\delta_R(b, s)} e^{-2\delta_I(b, s)} \right],$$

inversion of Eq. (3) reads

$$[1 - e^{2i\delta_R(b, s)} e^{-2\delta_I(b, s)}] = -i \frac{1}{2} \int_{-\infty}^0 (dt) J_0(b \sqrt{-t}) F(s, t),$$

2
and one can study properties of the amplitude in impact parameter space from the known behaviour in the angular momentum discrete variable. The Froissart bound implies that there must exist a finite angular momentum value, below which all partial waves are absorbed. Under the stronger hypothesis of total absorption as \( b \to 0 \) in the ultra high energy limit, namely all “low-\( b \)” waves are completely absorbed, we have the two ASR:

\[
SR_1 = \frac{1}{2} \int_{-\infty}^{0} (dt) \Im F(s, t) \to 1; \quad \text{as} \quad s \to \infty, \quad (5)
\]

\[
SR_0 = \frac{1}{2} \int_{-\infty}^{0} (dt) \Re F(s, t) \to 0; \quad \text{as} \quad s \to \infty. \quad (6)
\]

Satisfaction of these rules is a good measure of whether the asymptotic limit has been reached, and would reinforce the statement [2] based on the TOTEM data for total cross section, that we may have reached asymptotia, namely that \( \sigma_{\text{total}} \sim \ln^2 s \), and saturation of the Froissart bound has been observed. Notice that the Froissart-Martin bound is obtained under a hypothesis weaker than total absorption [4] and it would lead to \( SR_1 \to 2; \ SR_0 \to 0 \). As we shall see, phenomenologically Eq.(5), is favoured.

In order to check these ASR, one requires a model for the scattering amplitude. While the imaginary part of the elastic amplitude is solidly anchored to the optical theorem, the real part is more model dependent. A limiting behaviour for the real part of the forward amplitude has been obtained by Khuri and Kinoshita [16], \( i.e., \) for \( s \to \infty \)

\[
\rho(s, 0) = \frac{\Re F(s, 0)}{\Im m F(s, 0)} \approx \frac{\pi}{\ln(s/s_0)}, \quad (7)
\]

if the Froissart bound is saturated. As discussed later, the model of Ref. [17] gives \( \rho(s, 0) = \pi/[2p \ln(s/s_0)] \) with \( p = 1 \) for a total cross section asymptotically growing as \( \sim \ln s \), and \( p = 1/2 \), when the Froissart bound is saturated.

For values of \( q^2 = -t \neq 0 \), there is a result by André Martin [18, 19], which relates the real part of the amplitude to its imaginary part. Given an \( \Im m F(s, t) \), Martin’s method gives an asymptotic expression for \( \Re F(s, t) \), \( i.e. \)

\[
\Re F(s, t) = \rho(s, 0) \frac{d}{dt} [t \Im m F(s, t)]. \quad (8)
\]

We note here that the above equation is valid also if the Froissart bound is not saturated, but \( \sigma_{\text{total}} \sim \ln^{1/p}(s) \), where \( 1/2 \leq p \leq 1 \). The elastic
amplitude still scales with the variable $\tau = t\sigma_{\text{total}}$, the only difference is $ho = \frac{\pi}{2p\ln(s/s_0)}$, with $s_0 = 1 \text{GeV}^2$.

To complete the expression for the real part, one needs to estimate $\rho(s,0)$ for all cases including the case when the Froissart bound is not saturated. The phase of the leading (first term) contribution at $t = 0$ is readily obtained from the $s$-dependence of the total cross section, or equivalently $\Im F(s,0)$, using the prescription $s \rightarrow se^{-i\pi/2}$, which assures analyticity and crossing symmetry of the scattering amplitude [18, 20]. At high energies and very small $t$ values, $s \leftrightarrow u$ crossing symmetry tells us that the leading $C = +$ amplitude is a function of the complex variable $se^{-i\pi/2}$. The Froissart bound requires $\Im F(s,0) \lesssim \ln s$, and we can generally write, asymptotically

$$F(s,0) \rightarrow i \left[ \ln \left( \frac{s}{s_0} e^{-i\frac{\pi}{2}} \right) \right]^{1/p} = i \left[ \ln \left( \frac{s}{s_0} \right) - i \frac{\pi}{2} \right]^{1/p},$$

with $1/2 \leq p \leq 1$ to include both the case of saturation of the Froissart bound and a slower rise, compatible with $\ln s$ behaviour. For large $s$, this may be approximated to

$$F(s,0) \rightarrow i \left[ \ln \left( \frac{s}{s_0} \right) \right]^{1/p} \left[ 1 - \frac{i\pi}{2p\ln(s/s_0)} \right].$$

The above equation gives an estimate for the leading contribution to the parameter $\rho(s,0)$, namely

$$\rho(s,0) = \frac{\Re F(s,0)}{\Im F(s,0)} \approx \frac{\pi}{2p\ln(s/s_0)},$$

where the approximate sign refers to the fact that the real part of the amplitude can receive also (non-leading) contributions from other terms. The Khuri-Kinoshita result, valid when the Froissart bound is saturated, is obtained for $p = 1/2$. To obtain a numerical value for $p$, we use a model [6] built on QCD mini-jets embedded into the eikonal representation. According to this model, QCD mini-jets drive the rise of the cross section, and $k_t$-resummation of ultra-soft gluons transforms the power-law rise of the mini-jet cross sections into an asymptotic logarithmic behaviour. In our model [17]

$$\sigma_{\text{total}} \sim \ln^{1/p}(s),$$

where the parameter $1/2 < p < 1$ can be related to a dressed one-gluon exchange potential rising as $r^{2p-1}$, and our phenomenology of total cross section data [5] leads to values $p = 0.66 \div 0.77$. 
It should be noted that the asymptotic expression for the real part of the scattering amplitude automatically satisfies the second ASR. However, this does not help in checking whether asymptotia has been reached, since the expression for the real part is based on an asymptotic scaling law, which may not be valid.

Another measure for asymptotia, also examined in Ref. [2], can be obtained from the ratio of the elastic to the total cross section. Based on the black and grey disk model, where $\sigma_{\text{total}} = 2\pi A R^2(s)$ and $\sigma_{\text{elastic}} = \pi A^2 R^2(s)$, the black disk limit, $A = 1$, gives, for the ratio, the limiting value $1/2$. However, we have recently shown [6] that the eikonal formulation at high energy underestimates the total inelastic cross section, including in it only uncorrelated (non-diffractive) collisions, whereas the diffractive collisions appear to be part of the eikonal elastic cross section. In fact, it is known that, at high energies, the “elastic” cross section, as defined in eikonal models, is typically overestimated [21]. Thus we propose the following asymptotic limit

$$\mathcal{R} = \frac{\sigma_{\text{elastic}} + \sigma_{\text{diffractive}}}{\sigma_{\text{total}}} = \mathcal{R}_{\text{el}} + \mathcal{R}_{\text{diff}} \to \frac{1}{2}, \quad s \to \infty,$$

where $\sigma_{\text{diffractive}}$ includes both single and double diffractive components. We show in Fig. 1 a compilation of data [22] for the ratio $\sigma_{\text{elastic}}/\sigma_{\text{total}}$ and compare it with the black disk limit and results of Ref. [2]. In the figure we have added to the accelerator data a value extracted from the recent measurement by the Auger Collaboration for the inelastic cross section [23]. We estimated the ratio $\mathcal{R}_{\text{el}}$ at 57 TeV, by using Block and Halzen (BH) [2] value for the total cross section at $\sigma_{\text{total}}^{\text{BH}}(57 \text{ TeV}) = (134.8 \pm 1.5) \text{ mb}$, which is based on the analytic amplitude method of Ref. [24]. We then obtained $\sigma_{\text{elastic}}(57 \text{ TeV}) = \sigma_{\text{total}}^{\text{BH}} - \sigma_{\text{Auger}}^{\text{inelastic}} = (44.8 \pm 11.6) \text{ mb}$. We also show the asymptotic result (green dot) from Ref. [2], which appears in line with the ratio $\sigma_{\text{elastic}}/\sigma_{\text{total}}$ extrapolated from the Auger point for the inelastic cross section. The conclusion in Ref. [2] that we are still not in asymptotia for what concerns this ratio, may however be too pessimistic. Firstly, from the figure, it is quite conceivable that data may approach a limiting value of $1/3$. This would be better in keeping with Eq. (13). Also, estimates for the inelastic cross section are affected by large uncertainties, and often model dependent on extrapolations. Indeed, in order to reconcile cosmic ray composition data, an extrapolation from $p$-air to $pp$ could use higher total cross section values [12]. Given the uncertainties in different simulations at ultra high energies, the estimate for $\mathcal{R}_{\text{el}}$ at 57 TeV could then be lowered (closer
to the TOTEM value), thereby opening the possibility that an asymptotic plateau for $\sigma_{\text{elastic}}/\sigma_{\text{total}}$ may have already been reached. In fact $R_{\text{el}}$ has been rising from lower energies until the value $\sim 1/4$ at the TeVatron, and at LHC7. Data at 8 TeV could indicate whether we have reached a plateau or not.

3. The two exponential model at LHC energies

In order to perform a test of the ASR in a model independent way, we have analyzed the data for the differential elastic cross section through a parametrization of the scattering amplitude, proposed by Barger and Phillips (BP) in 1973 [15].

In 1973, BP proposed two different parametrizations, consisting in using a phase and two exponentials to describe the elastic scattering amplitude as a function of momentum transfer in the range $-t = 0.15 \div 5.0$ GeV$^2$. The first parametrization was given as

$$A(s, t) = i \left[ \sqrt{A(s)} e^{\frac{i}{2} B(s)t} + \sqrt{C(s)} e^{i\phi(s)} e^{\frac{i}{2} D(s)t} \right], \quad (14)$$
\[ \frac{d\sigma}{dt} = A(s)e^{B(s)t} + C(s)e^{D(s)t} + 2\sqrt{A(s)}\sqrt{C(s)}e^{(B(s)+D(s))t/2}\cos \phi, \quad (15) \]

with five \( s \)-dependent real parameters, \( A, B, C, D, \phi \). The total cross section in this model is given as

\[ \sigma_{\text{total}} = 4\sqrt{\pi} \Im m A(s, t = 0) = 4\sqrt{\pi} \left[ \sqrt{A(s)} + \sqrt{C(s)} \cos \phi \right]. \quad (16) \]

Using the above expression, BP fitted data from \( p_{\text{lab}} = 12 \) GeV to 1496 GeV \((\sqrt{s} \approx 5 \div 53 \) GeV\). The values of the parameters at \( \sqrt{s} = 53 \) GeV are given as \( \sqrt{A} = 6.55, \sqrt{C} = 0.034 \) in \( \sqrt{\text{mb}/\text{GeV}^2} \) and \( B = 10.20, D = 1.7 \) in \( \text{GeV}^{-2} \), \( \phi = 2.53 \) rad. The first exponential was seen to have normal Regge shrinking, namely

\[ B = B_0 + 2\alpha' \ln s, \quad \alpha' \approx 0.3, \quad (17) \]

while the second exponential term appeared to be constant in this energy range. With this parametrization, the phase \( \phi \) was fitted to be always larger than \( \pi/2 \), so that the interference term was always negative. The second parametrization interpreted the energy dependence of the parameters in terms of Regge-Pomeron exchange and fits to data were as good as in the first case, but more model dependent. However, Eq. (17) for the forward slope \( B \) poses now a problem, as recently discussed in Ref. [25], since TOTEM data can be described by Eq. (17) only if \( \alpha' \) is not a constant, but has a logarithmic energy dependence. On the other hand, by relinquishing any model interpretation, one can use the model of Eq. (14) as it appears to be the simplest possible way to describe the \( t \)-dependence of \( pp \) elastic scattering amplitude from ISR onwards.

The simplicity of the model in Eq. (14) suggests to use it to describe present LHC data. Reading the TOTEM data from Ref. [13], we show the result in the left panel of Fig. 2. We find that two exponentials and a phase can give a very good description of TOTEM data, with the tail well described by a term \( e^{Dt} \), with \( B/D \approx 4 \) and a relative weight of the two terms at LHC7 \( \sqrt{A/C} \approx 22 \). With this parametrization, the position of the dip (which the model fit gives at \( -t = 0.52 \text{ GeV}^2 \)) and the behaviour around the dip are both well reproduced. This result differs from the proposal in Ref. [7] and also from the suggestion by TOTEM [13] that the tail, past the dip at \( -t = (0.53 \pm 0.01_{\text{stat}} \pm 0.01_{\text{syst}}) \text{ GeV}^2 \), is described as being compatible with a behaviour of the type \( |t|^{-n} \) with \( n = 7.8 \pm 0.3_{\text{stat}} \pm 0.1_{\text{syst}} \). The TOTEM
suggested behaviour is close to a description of the tail through a QCD type contribution from three gluon exchange advanced in Ref. [26]. It is also close to the observation in Ref. [15], that the “second exponential” can be identified with a term proportional to $p_T^{-14}$, valid, according to the authors, for all available $pp$ data for $s > 15 \text{ GeV}^2$. Following these suggestions, and mindful of the fact that the form factor dependence of the amplitude would contribute to the cross section with a term as $(-t)^{-8}$, we have tried a slight modification of the BP model, which consists in substituting the second term in Eq. (14) with a term proportional to the 2nd power of the proton form factor, namely to write

\[ A(s, t) = i \left[ \sqrt{A(s)}e^{Bt/2} + \frac{\sqrt{C(s)}}{(-t + t_0)^4}e^{i\phi} \right]. \]  

(18)

With $t_0$ a free parameter, this allows a comparison with the parametrization of the tail in Ref. [13], but the fit worsens, and the tail is everywhere better described by an exponential. We show this fit in the right hand panel of Fig. 2. From here on, we shall use the two exponential model and proceed to check the ASR. Since the phase $\phi \neq \pi$, for the satisfaction of both ASR

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Five-parameter fit of $pp$ data for the elastic differential cross section at LHC7. At left the fit uses the two exponential model from Ref. [15], at right a form factor term is used to describe the tail, according to Eq. (18). Parameter values in the two cases are indicated. TOTEM data were read from Ref. [13].}
\end{figure}
the model of Eq. (14) needs to be modified so as to include a real part in the leading amplitude $i\sqrt{A(s)}$, i.e. the leading term (at $t \approx 0$) should acquire an extra phase. Rewriting the amplitude of Eq. (14) as

$$A(s,t) = [iA_I + A_R] e^{\frac{i}{2}B(s)t} + \left[i\sqrt{C(s)} \cos \phi - \sqrt{C} \sin \phi \right] e^{\frac{i}{2}D(s)t},$$

one has

$$\rho(s,0) = \frac{A_R - \sqrt{C} \sin \phi}{A_I + \sqrt{C} \cos \phi}. \quad (20)$$

We use the result for the leading contribution to $\rho(s, t = 0)$ to give a real part to the first term, i.e.

$$\hat{\rho}(s, t = 0) = \frac{A_R}{A_I} = \frac{\pi}{2p \ln(s/s_0)}. \quad (21)$$

With this, we now find

$$SR_0 = \sqrt{\frac{A}{1 + \rho^2}} \frac{1}{\sqrt{\pi B}} \hat{\rho} - \frac{\sqrt{C} \sin \phi}{\sqrt{\pi D}} = 0.045 \div 0.066, \quad (22)$$

for $p = 0.77$ and 0.66 respectively, and

$$SR_1 = \sqrt{\frac{A}{1 + \rho^2}} \frac{1}{\sqrt{\pi B}} + \frac{\sqrt{C}}{\sqrt{\pi D}} \cos \phi = 0.94, \quad (23)$$

with $A = A_R^2 + A_I^2$, and inserting the parameter values from the figure, with $\pi/2 < \phi < \pi$. One observes a good satisfaction of both ASR. Notice however, that, in general, the conjecture for $\rho(s, 0)$ proposed in Eq. (11) cannot be valid at lower energies, because the approximation for the amplitude in Eq. (9) is only valid at asymptotic energies. Thus, for uniformity with the ISR case, curves in Fig. 2 have been done with $A_R = 0$. Applying the above model to the ISR data for $pp$ scattering at $\sqrt{s} = 53$ GeV [27], we obtain the fit shown in the left panel of Fig. 3. We notice that the application of the ASR to $pp$ scattering at these lower energies, gives $SR_1 \approx 0.7$ for the parameter values obtained by the fit. As expected, the ASR for the imaginary part of the amplitude in $b$-space is not yet satisfied at ISR. As for the second ASR at ISR, the expression for the real part of the amplitude as, discussed in Sec. 2,
cannot be applied here, because at ISR we are far from an asymptotic regime. Before leaving this discussion, we point out that the value obtained for the total cross section in this model is below the experimental value. This is particularly true for ISR energies, where the fit gives $\sigma_{\text{total}} = (35.6 \pm 1.7) \text{ mb}$ to be compared with values with very small errors, such as $(42.71 \pm 0.35) \text{ mb}$ [28]. At LHC7 the total cross section is better reproduced by the fit, with a value of $(91 \pm 5) \text{ mb}$, although still a bit low. From Figs. 2 and 3 we conclude that this model is approximately valid at lower energies, better apt to describe data at LHC7 and onwards.

4. The slope parameter in the two exponential model

TOTEM released values for $\sigma_{\text{total}}$, $\sigma_{\text{elastic}}$, the slope $B_{\text{exp}}$ and the position of the dip. The total cross section is reproduced by the five-parameter fit ($\sigma_{\text{total}}$ is let unconstrained) of the two exponential model within $10\%$, and we have a good determination of the position of the dip, which, in our fit, occurs at $-t_{\text{dip}} = 0.52 \text{ GeV}^2$.

As for the slope parameter, one cannot immediately compare the value
of this model parameter $B$ with TOTEM data, where it is defined as

$$\frac{d\sigma_{el}}{dt} = \frac{d\sigma_{el}}{dt} \bigg|_{t=0} e^{B_{\text{exp}} t}$$  \hspace{1cm} (24)$$

and is measured in two $t$-intervals, with $B_{\text{exp}} = (20.1 \pm 0.2^{\text{stat}} \pm 0.3^{\text{syst}}) \text{ GeV}^{-2}$ for smaller $|t|$ values, i.e. $0.02 < -t < 0.33 \text{ GeV}^2$, and $B_{\text{exp}} = (23.6 \pm 0.5^{\text{stat}} \pm 0.4^{\text{syst}}) \text{ GeV}^{-2}$ from the interval $0.36 < -t < 0.47 \text{ GeV}^2$. Such difference is well understood in the two exponential model. With the definition for the effective slope [20]

$$B_{\text{eff}}(s, t) \equiv \frac{d}{dt} \ln \left( \frac{d\sigma_{el}}{dt} \right),$$

one obtains a function changing with $t$, i.e.

$$B_{\text{eff}}(s, t) = \frac{A B e^{B t} + C D e^{D t} + \sqrt{A} \sqrt{C} (B + D) e^{(B + D) t/2} \cos \phi}{d\sigma_{el}/dt}.$$  \hspace{1cm} (26)$$

Using the parameter values from the left hand panel of Fig. 2, we show in the right-hand panel of Fig. 3 the $t$-dependence of the slope $B_{\text{eff}}$ at LHC7, compared with the values given by the TOTEM experiment in the two different $t$ intervals, whose extension is indicated by the horizontal bars. At $t = 0$, one finds

$$B_{\text{eff}}(7 \text{ TeV}, 0) = \frac{d}{dt} \ln \left( \frac{d\sigma_{el}}{dt} \right) \bigg|_{t=0} = 17.7 \text{ GeV}^{-2}, \quad \text{(five-parameter fit)}.$$  \hspace{1cm} (27)$$

Notice that the figure is obtained by neglecting the contribution from $\hat{A}_R$. Clearly this approximation is not valid where the real or the imaginary part of the amplitude approaches zero. It is possible to identify the two $t$-values where this will happen before the dip, namely where the real part of the amplitude is zero, and a subsequent one, close or very close to the dip, where the imaginary part is zero. At LHC7, these two points occur at

$$-t_R = \frac{2}{B - D} \ln \left( \frac{\hat{A}_R}{\sin \phi \sqrt{C}} \right) = 0.28 \div 0.30 \text{ GeV}^2,$$

$$-t_I = \frac{2}{B - D} \ln \left( \frac{\sqrt{A/C}}{| \cos \phi |} \right) = 0.5 \text{ GeV}^2,$$  \hspace{1cm} (28)$$
where the uncertainty in $t_R$ depends on the value for $\hat{\rho}$ and we have neglected $\hat{\rho}^2$ in Eq. [29] and, again, we have used the parameter values indicated in the left hand panel of Fig. [2].

An asymptotic value for the slope $B_{\text{eff}}(s,t)$, can be obtained through the ASR, already well satisfied at LHC7. Thus, in the exact limits $SR_0 = 0$, $SR_1 = 1$, for asymptotic energies at LHC and beyond, one obtains for the effective slopes

$$B_{\text{eff}}(s,t_R) = \frac{\sigma_{\text{total}}}{4\pi},$$  \hspace{1cm} (30)

$$B_{\text{eff}}(s,t_I) = B + D - \frac{\sigma_{\text{total}}}{4\pi}.$$  \hspace{1cm} (31)

We find, in agreement with TOTEM, that past $t_R$, where the real part of the amplitude is zero, the slope is $\sim 20 \text{ GeV}^{-2}$, and that it increases to higher values, to then start decreasing as it approaches $t_I$ where the imaginary part goes to zero in correspondence with the occurrence of the dip.

We note that the ASR imply that the slope $B_{\text{eff}}(s,t_R)$ from the two exponential model should grow as $\sigma_{\text{total}}$, thus as $\ln^2 s$, if the Froissart bound is saturated, or as $\ln^{1/p} s$ for a slower rise. We see that the same behaviour is also true asymptotically for the (constant in $t$) leading parameter $B$ of the two exponential model, for which we also have the well known result

$$R_{\text{el}}(s) = \frac{\sigma_{\text{elastic}}(s)}{\sigma_{\text{total}}(s)} \approx \frac{\sqrt{A}}{4\sqrt{\pi} B} \approx \frac{\sigma_{\text{total}}}{16\pi B}$$  \hspace{1cm} (32)

recently also discussed in Ref. [29] on general grounds.

We conclude that, if the two exponential model is a good representation of the amplitude at LHC in the range $0 \leq -t \leq 3 \text{ GeV}^2$, and if the total cross section rises faster than $\ln s$, the slope of the leading term in this model should grow faster than $\ln(s/s_0)$. Notice that in Ref. [25], an analysis of the effective slope from NA8 to TOTEM seems to indicate a rise according to $\ln^2 s$. Thus, just as in the case of the total cross section, the question of the energy dependence of the effective slope is still very much open.

In conclusion, we stress the following points concerning the slope parameter:

1. One can not speak about a unique slope if there is more than one term with different $t$ dependence in the elastic amplitude, as required by the presence of a dip, in the differential cross section.
2. Even in the region before the dip, the necessity of at least two terms, whose interference gives rise to the dip, is felt and TOTEM itself provides two different values for the slope in two different regions of $t$. Hence it is not meaningful to consider only a single slope and its asymptotic behavior. In a two component model such as the one considered above, all one can discuss is the relative growth of the two slopes with energy.

3. Even in the rather simple model analyzed here, we find that the slope near $t = 0$ is less than that around $t = -0.2$ GeV$^2$, which is less than that around $t = -0.4$ GeV$^2$ and hence the question of the growth with energy becomes a strong function of the $t$ range under consideration.

The curve in the right hand panel of Fig. 3 shows that at LHC this model predicts an increase of the slope parameter in the range $0 < -t \lesssim 0.4$ GeV$^2$. This is in contrast with the behaviour observed at lower energies, notably at ISR for very small $t$–values [30, 31]. Notice however that our findings at LHC are in agreement with those expected in Ref. [32] at asymptotic energies.

Conclusion

We have utilized two asymptotic sum rules derived earlier for the elastic scattering amplitude in $pp$ scattering [14] and have tested them using a well known parametrization of the elastic differential cross section with two exponentials with a relative phase. Our study of TOTEM data at LHC7 finds that this model can describe well the data before and past the observed dip. Similar conclusions about the behaviour past the dip, have also been advanced in Ref. [33].

A possible interpretation of this model is that the exponential behaviour exhibited both before and after the dip corresponds to a resummation of soft terms accompanying the leading $C = +1$, two gluon exchange, and $C = -1$, three gluon exchange term, with non leading terms giving rise to $\phi \neq \pi$.

Given the general nature of the parametrization discussed here, we recommend its usage for further studies of the behaviour of the scattering amplitude at higher energies.

Acknowledgement

We thank E. Lomon and R. Godbole for useful suggestions and enlightening discussions about elastic scattering data. One of us is grateful to the
Center for Theoretical Physics of MIT for hospitality. Work partially supported by Spanish MEC (FPA2006-05294, FPA2010-16696) and by Junta de Andalucia (FQM 101). This work has been supported in part by the Spanish Consolider Ingenio 2010 Programme CPAN (CSD2007-00042). YS would like to thank his colleagues on the Auger Collaboration for discussions.

References

[1] G. Latino et al., First Results from the TOTEM Experiment, arXiv:1110.1008.

[2] M. Block and F. Halzen, Phys.Rev.Lett.107 (2011) 212002, arXiv:1109.2041 [hep-ph].

[3] M. Froissart, Phys. Rev. 123 (1961) 1053.

[4] A. Martin, Phys. Rev. 129 (1963) 1432-1436; A. Martin and F. Cheung, “Analyticity Properties and Bounds of the Scattering Amplitudes”, Eq.(6.7), page 34, Gordon and Breach Science Publishers, N. Y., (1970).

[5] A. Achilli, R. Hegde, R. M. Godbole, A. Grau, G. Pancheri, Y.N. Srivastava, Phys.Lett. B659 (2008) 137, arXiv:0708.3626 [hep-ph].

[6] A. Achilli, R. M. Godbole, A. Grau, G. Pancheri, O. Shekhovtsova, Y. N. Srivastava, Phys.Rev. D84 (2011) 094009, arXiv:1102.1949 [hep-ph].

[7] A. Donnachie, P.V. Landshoff, Elastic Scattering at the LHC, arXiv:1112.2485 [hep-ph]; Z.Phys. C2 (1979) 55, Erratum-ibid. C2 (1979) 372.

[8] J. Swain, A. Widom and Y. Srivastava, Asymptotic High Energy Total Cross Sections and Theories with Extra Dimensions, arXiv:1104.2553 [hep-ph].

[9] D. Amati, M. Ciafaloni, and G. Veneziano, preprint S.I.S.S.A. 121 EP, Sep. 1988.

[10] K. Agashe and A. Pomarol, New J. Phys. 12 (2010) 075010; doi: 10.1088/1367-2630/12/7/075010 (2010).
[11] M. Block and F. Halzen, ‘Soft’ Hadronic Cross Sections Challenge Hidden Dimensions, e-Print: arXiv:1201.0960 [hep-ph] [submitted for publication].

[12] N. Shaham and T. Piran, The UHECRs Composition Problem: Evidence for a New Physics at 100 TeV?, arXiv:1204.1488 [astro-ph.HE].

[13] G. Antchev et al., Europhys.Lett. 96 (2011) 21002; e-Print: arXiv:1110.1395 [hep-ex].

[14] G. Pancheri, Y.N. Srivastava and N. Staffolani, Eur. Phys. J. C42 (2005) 303-308; Acta Phys. Polon. B36 (2005) 749-754, arXiv:0411007 [hep-ph].

[15] R. J.N. Phillips and V. D. Barger, Phys.Lett. B46 (1973) 412-414.

[16] N.N. Khuri, T. Kinoshita, Phys.Rev. 137 (1965) B720-B729.

[17] A. Achilli, R. Hegde, R. M. Godbole, A. Grau, G. Pancheri, Y.N. Srivastava, Phys.Lett. B659 (2008) 137-143, arXiv:0708.3626 [hep-ph].

[18] A. Martin, Lett.Nuovo Cim. 7S2 (1973) 811-812, Lett.Nuovo Cim. 7 (1973) 811-81.

[19] G. Auberson, T. Kinoshita, Andre Martin, Phys.Rev. D3 (1971) 3185-3194.

[20] M. Block and R. Cahn, Rev. Mod. Phys. 57 (1985) 563.

[21] P. Lipari and M. Lusignoli, Phys.Rev. D80 (2009) 074014, e-Print: arXiv:0908.0495 [hep-ph].

[22] http://pdg.lbl.gov/2011/hadronic-xsections/hadron.html

[23] Auger Collaboration, arXiv:1107.4804v1 [astro-ph.HE]; M. Mostafa, XXXI Physics in Collisions, Vancouver 2011, arXiv:1111.2661v1 [astro-ph.HE]; R.Ulrich, 32nd International Cosmic Ray Conference (ICRC 2011), ICRC2011 ICRC11. 11-18 Aug 2011. Beijing, China.

[24] M.M. Block and F. Halzen, Phys.Rev. D73 (2006) 054022, e-Print: hep-ph/0510238.

[25] V.A. Schegelsky, M.G. Ryskin, The diffraction cone shrinkage speed up with the collision energy, arXiv:1112.3243 [hep-ph].
[26] A. Donnachie, P.V. Landshoff, Phys.Lett. B387 (1996) 637-641, arXiv: 9607377 [hep-ph].

[27] E. Nagy, R.S.Orr, W. Schmidt-Parzefall, K. Winter, A. Brandt et al., Nucl. Phys. B150 (1979) 221; A. Breakstone, H.B. Crawley, G.M. Dallavalle, K. Doroba, D. Drijard et al., Phys.Rev.Lett. 54 (1985) 2180; A. Breakstone et al., Nucl.Phys. B248 (1984) 253.

[28] U. Amaldi et al., Nucl. Phys. B145 (1978) 367.

[29] D.A. Fagundes and M.J. Menon, Nucl.Phys. A880 (2012) 1-11, arXiv:1112.5115 [hep-ph].

[30] M. Ambrosio et al., Phys. Lett. B115 (1982) 495.

[31] N. Amos et al., Nucl. Phys. B 262 (1985) 689.

[32] P. Desgrolard, J. Kontros, A.I. Lengyel and E.S. Martynov, Nuovo Cim. A110 (1997) 615-630.

[33] S.M. Troshin and N.E.Tyurin, arXiv:1203.5137.