Non–standard Construction of Hamiltonian Structures and of the Hamilton–Jacobi equation.

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Examples of non–standard construction of Hamiltonian structures for dynamical systems and the respective Hamilton–Jacobi (H–J) equations, without using Lagrangians, are presented. Alternative H–J equations for Euler top are explicitly exhibited and solved. We demonstrate that the stability criterion used by Bridges in [16], relating the slope of a Casimir function parametrized by the Lagrange multiplier to critical point type, depends on the used Hamiltonian structure and it is inadequate for this reason.

I. INTRODUCTION

The standard construction of Hamiltonian theories starts from a Lagrangian. The procedure is well known and it is the subject of very many textbooks. Nevertheless it is interesting to construct hamiltonian structures for classical systems of differential equations, without using a Lagrangian, which may even fail to exist, starting from the equation of motion only. These approaches usually are not systematic, conducting to non unique hamiltonian structures and deal with non canonical variables (in some systems these variables are more natural than canonical ones).

One systematic, but complicated procedure to construct hamiltonian structures using symmetries of classical equations was introduced by E. Sudarshan and N. Mukunda in [10]. They also comment the possible relevance of alternative hamiltonian of classical systems under the quantization procedure. Another systematic approach due also to Sudarshan and Mukunda [1], but developed by J. Marsden, A. Weinstein, D. Holm and B. Kupershmidt in [12]–[15] is based on definition of Lie – Poisson structures on duals of Lie algebras. More recently S. Hojman, et. al [2]–[8] presented a newly devised method which constitutes a general technique for construction of hamiltonian structures using symmetries and constants of motion of dynamical systems.

Non standard introduction of hamiltonian structures enrich the possibilities we have in the analysis of both classical and quantum systems and for this reason one must be careful interpreting the results we obtained. In the section III of the present work we use this fact in the context of analysis of stability for dynamical systems in the vicinity of a critical point, demonstrating that the criterion of stability used by Bridges in [16] is inadequate.

On the other hand the Hamilton – Jacobi equation (H–J) is one of the cornerstones of the theoretical physics. Its role in the theoretical setting of optics and both classical and quantum mechanic is well established. Its use as a tool for solving problems is also well known.

The usual way of deriving the H–J equation is based on the lagrangian or hamiltonian formalisms. In the section IV of this paper we present one way of constructing the H–J equation starting from equation of motion only, to do this we use the approaches due to S. Hojman et. al. [2]–[8] for non standard construction of hamiltonian structures. In some sense this complement the analysis carry out in the section III, because we only concern here with the study of regular points of dynamical systems and not concerns singular points of it, as in the above mentioned section.

With the help of alternative hamiltonian structures introduced by Hojman’s recipe, we define the Poisson brackets on the phase space \( M \) for dynamical system. Using a Poisson map \( \varphi \), \( M \) is reduced to a symplectic submanifold \( N \). On \( N \) we can define the action and the respective H–J equation.

The hamiltonian structures on \( M \) are so on \( N \). Furthermore the solutions of dynamical system with initial conditions on \( N \) stay all the time on it. For these reasons the solution of H–J equation is a particular solution of dynamical system constrained to \( N \).

There it is not unique way to define hamiltonian structure on \( M \), neither the Poisson map used for reduction of the phase space to submanifold \( N \) is unique. This means that we can define the action in different ways and obtain different H–J equations. In this paper we show that solutions obtained from different H–J equations, from alternative hamiltonian structures on \( M \) yield correct solutions of dynamical systems. This allow us to speak about some kind of equivalence between different H–J equations obtained this way.

In the section II, for illustration purpose of using the method introduced by S. Hojman, we present one example borrowed from D. Armbruster, J. Guckenheimer, and P. Holmes in [1,2,3]. This example was employed in the analysis
of normal forms with $O(2)$ symmetry. Using the complete set of symmetries of this dynamical system, it is introduced two alternative hamiltonian structures. With the help of $O(2)$ symmetry we can reduce the four equations dynamical system in cartesian coordinates to three equations in polar coordinates and the dynamical system obtained this way we will use as one of the example presented in the section II. The physics motivations for the study of these systems can be found in the above listed references.

The section III is dedicated to demonstrated that the criterion of Bridges in [16], that relates the slope of a Casimir function parametrized by the Lagrange multiplier to critical point type depends of the used hamiltonian structures and it is inadequate for this reason.

The section IV is devoted to illustrated the construction of H–J in the way described above. We employ the examples used in the section III and present a complete analysis of the Euler top.

II. ONE EXAMPLE OF NON–STANDARD CONSTRUCTION OF HAMILTONIAN STRUCTURE.

The contents of this Section illustrates partially the results of [2]–[8].

Consider a dynamical system for the complex variables $Z_1$ and $Z_2$ defined by the following equations:

$$
\dot{Z}_1 = \bar{Z}_1 Z_2 \\
\dot{Z}_2 = -Z_1^2
$$

(1)

The system (1) can be writing in cartesian coordinates:

$$
\dot{X}_1 = X_1 X_2 + Y_1 Y_2 \\
\dot{Y}_1 = X_1 Y_2 - Y_1 X_2 \\
\dot{X}_2 = -X_1^2 + Y_1^2 \\
\dot{Y}_2 = -2X_1 Y_1
$$

(2)

Where:

$$
Z_1 = X_1 + iY_1 \\
Z_2 = X_2 + iY_2
$$

A hamiltonian structure for (1 or 2) consist of an antisymmetric matrix, $J^{ab}$ and a Hamiltonian $H$ such that $J^{ab}$ is the Poisson bracket for the dynamical variables, which are in general non–canonical. In addition to its antisymmetry, the matrix $J^{ab}$ is required to satisfy Jacobi identity and to reproduce, in conjunction with the Hamiltonian $H$ the dynamical equation (1 or 2) i. e.,

$$
J^{ab,d}J^{dc} + J^{bc,d}J^{da} + J^{ca,d}J^{db} \equiv 0
$$

(3)

and,

$$
J^{ab} \frac{\partial H}{\partial x^b} = f^a .
$$

(4)

Where:

$$
f = \begin{pmatrix}
X_1 X_2 + Y_1 Y_2 \\
X_1 Y_2 - Y_1 X_2 \\
-2X_1 Y_1 - X_1^2 + Y_1^2 \\
\end{pmatrix}
$$

(5)

It has been proved [2,3] that one solution to the problem of finding a Hamiltonian structure for a given dynamical system is provided by one constant of the motion which may be used as the Hamiltonian $H$, and a symmetry vector $\eta^a$ which allows for the construction of a Poisson matrix $J^{ab}$. The constant of the motion and the symmetry vector satisfy,

$$
\mathcal{L}_f H = 0 ,
$$

(6)

$$
(\partial_t + \mathcal{L}_f)\eta^a = 0 ,
$$

(7)
respectively, where $L_f$ is the Lie derivative along $f$ (for a definition, see [9], for instance). In addition, it is required that the deformation $K$ of $H$ along $\eta^a$,

$$K \equiv \frac{\partial H}{\partial x^a} \eta^a = \mathcal{L}_\eta H,$$

be non-vanishing.

The Poisson matrix $J^{ab}$ is constructed as the antisymmetrized product of the flow vector $f^a$ and the “normalized” symmetry vector $\eta^b/K$,

$$J^{ab} = \frac{1}{K} (f^a \eta^b - f^b \eta^a).$$

The Poisson matrix so constructed has rank 2 and it is, therefore, singular. Adding together two Poisson matrices constructed according to (9) will not increase its rank. It will just redefine the symmetry vector used to construct it.

One method to increase the rank of such a Poisson matrix is presented in [2].

The system (1) is invariant under the transformation:

$$Z_1 \rightarrow Z_1 \exp(i\Phi) \quad \text{and} \quad Z_2 \rightarrow Z_2 \exp(2i\Phi)$$

In cartesian coordinates the transformation is writing as:

$$\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} R(\Phi) & 0 \\ 0 & R(2\Phi) \end{pmatrix}$$

Where:

$$R(\Phi) = \begin{pmatrix} \cos \Phi & -\sin \Phi \\ \sin \Phi & \cos \Phi \end{pmatrix}$$

From this, we have the symmetry vector:

$$\eta^{(1)} = \begin{pmatrix} -Y_1 \\ X_1 \\ -2Y_2 \\ 2X_2 \end{pmatrix}$$

Another symmetry transformation, involving space coordinates and time of (2) is:

$$\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} X_1 + t(X_1 X_2 + Y_1 Y_2) \\ Y_1 + t(Y_1 X_2 - Y_1 X_1) \\ X_2 + t(-X_1^2 + Y_1^2) \\ Y_2 - t(2X_1 Y_1) \end{pmatrix}$$

This vector we can equivalently write in the form: (see [3])

$$\eta^{(2)} = \begin{pmatrix} X_1 \\ X_2 \\ -t \end{pmatrix}$$

The symmetry vectors $\eta^{(1)}$, $\eta^{(2)}$ and $f$ satisfy the symmetry equation (7) and the Lie algebra:
\[
\begin{align*}
\eta^{(2)} f &= -f \\
\eta^{(1)} \eta^{(2)} &= 0 \\
\eta^{(1)} f &= 0
\end{align*}
\] (14)

For the system (1) we have the following constants of motion:
\[
\begin{align*}
C_1 &= |Z_1|^2 + |Z_2|^2 \\
C_2 &= \frac{1}{2i}(Z_1 \bar{Z}_2 - \bar{Z}_1 Z_2)
\end{align*}
\] (15)

Or in cartesian coordinates:
\[
\begin{align*}
C_1 &= X_1^2 + Y_1^2 + X_2^2 + Y_2^2 \\
C_2 &= 2X_1Y_1X_2 - Y_2(X_1^2 - Y_1^2)
\end{align*}
\] (16)

We can now construct some alternative Poisson structures taking as Hamiltonian \( H = C_1 \) and the Poisson matrix:
\[
J_{ab} = \frac{1}{K}(f^a \eta^{(2)b} - f^b \eta^{(2)a})
\] (17)

Where \( K \) is the deformation of \( H = C_1 \) along \( \eta^{(2)} \),
\[
K \equiv \frac{\partial H}{\partial x^a} \eta^a = \mathcal{L}_\eta H = 2C_1
\] (18)

Or taking \( C_2 \) as a Hamiltonian \( H \), and Poisson matrix:
\[
J_{ab} = \frac{1}{Q}(f^a \eta^{(2)b} - f^b \eta^{(2)a})
\] (19)

Where \( Q \) is the deformation of \( H = C_2 \) along \( \eta^{(2)} \),
\[
Q \equiv \frac{\partial H}{\partial x^a} \eta^a = \mathcal{L}_\eta H = 3C_2
\] (20)

The deformations of both \( C_1 \) and \( C_2 \) along \( \eta^{(1)} \) are vanishing and for this reason we can not use this vector in the construction of Poisson matrix with the above procedure.

Finally we note that introducing polar coordinates and using the \( O(2) \) symmetry, the system (1) can be writing as:
\[
\begin{align*}
\dot{r}_1 &= r_1 r_2 \cos \Theta \\
\dot{r}_2 &= \pm r_2 \cos \Theta \\
\dot{\Theta} &= -\left( 2r_2 \pm \frac{r_1^2}{r_2} \right) \sin \Theta
\end{align*}
\] (21)

Where: \( Z_j = r_j \exp \Phi_j \), \( \Theta = 2\Phi_1 - \Phi_2 \)

### III. Hamiltonian Structures and Linear Stability Criterion.

In this section we use another way to construct Poisson structures based on the knowledge of constants of motion for the dynamical system under consideration (See [3] for detail) to demonstrate that stability criterion used by Bridges in [16] strongly depends of the used Poisson structures. Let us consider as the first example the one we presented in the preceding section.

**Example I:**
\[
\begin{align*}
\dot{X}_1 &= X_1 X_2 \cos \Phi \\
\dot{X}_2 &= \pm X_1^2 \cos \Phi \\
\dot{\Phi} &= -\left( 2X_2 \pm \frac{X_1^2}{X_2} \right) \sin \Theta
\end{align*}
\] (22)
The phase space of (22) is:

\[ M = \{ \Phi, X_1, X_2 : (X_1, X_2) \in \mathbb{R}^{2+}, 0 \leq \Phi \leq 2\pi \} \]

The integrals of motion for this system are:

\[ C_1 = \frac{1}{2}(X_1^2 \mp X_2^2) \tag{23} \]
\[ C_2 = X_1^2 X_2 \sin \Phi \tag{24} \]

Taking as a Hamiltonian \( H = C_1 \) and the function \( \Psi = C_2 \) we have the following Poisson structure:

I Poisson Structure:

\[ J^{(1)ab} = \frac{1}{X_1X_2} \varepsilon^{abc} \frac{\partial \Psi}{\partial X^c} \]

\[ \| J^{(1)ab} \| = \begin{pmatrix}
0 & X_1 \cos \Phi & -\frac{X_1}{X_2} \sin \Phi \\
-X_1 \cos \Phi & 0 & 2 \sin \Phi \\
\frac{X_1}{X_2} \sin \Phi & -2 \sin \Phi & 0
\end{pmatrix} \tag{25} \]

It is not difficult to see that (25) together with \( H = C_1 \) provide a Hamiltonian formulation for (22), and that \( \Psi \) is the Casimir function (i.e., function which has vanishing Poisson bracket relations with any other dynamical quantity. For detail, see [1,11] for instance) for this Poisson structure.

In other words we see that,

\[ \dot{X} = \{ X, H \}_M \]

Where: \( X = (X_1, X_2, \Phi) \)

Let \( \lambda \) be a Lagrange multiplier, then a necessary condition for \( X \) to be a critical point of (22) given a Hamiltonian \( H \) and a Casimir \( \Psi \) is that:

\[ \nabla H(X) - \lambda \nabla \Psi(X) = 0 \]

Or

\[ \begin{pmatrix}
X_1 \\
X_2 \\
0
\end{pmatrix} - \lambda \begin{pmatrix}
2X_1X_2 \sin \Phi \\
X_1^2 \sin \Phi \\
X_2^2 \cos \Phi
\end{pmatrix} = 0 \]

From this equations we have the critical point:

\[ P_c = \left( \frac{1}{\sqrt{2\lambda}}, \frac{1}{2\lambda^2}, \frac{\pi}{2} \right) \tag{26} \]

On the other hand, the derivative of \( \Psi_{P_c} \) and \( H_{P_c} \) with respect to Lagrange multiplier are:

\[ \frac{d\Psi_{P_c}}{d\lambda} = -\frac{3}{4\lambda^4} \tag{27} \]
\[ \frac{dH_{P_c}}{d\lambda} = -\frac{3}{4\lambda^3} \tag{28} \]

Linearizing the Poisson system (22) about a critical point \( P_c \): \( X^i = P_c + \xi^i \), satisfying \( \nabla H = \lambda \nabla \Psi \) we have:

\[ \dot{\xi}^i = \frac{\partial}{\partial X^m} \left[ \frac{1}{X_1X_2} (\nabla \times \nabla H)^i \right]_{P_c} \xi^m = \left[ \frac{1}{X_1X_2} \nabla \Psi \times (D^2 H - \lambda D^2 \Psi)_{P_c} \right] \xi^m \tag{29} \]

Where:

\[ \frac{1}{X_1X_2} \nabla \Psi \times (D^2 H - \lambda D^2 \Psi)_{P_c} = \begin{pmatrix}
0 & 0 & -\frac{1}{2\sqrt{2}\lambda^7} \\
0 & 0 & \frac{1}{2\lambda^7} \\
2\sqrt{2} & -4 & 0
\end{pmatrix} \tag{30} \]
One eigenvalue of (30) is zero and the other two are given by:

$$\mu^2 = - \frac{3}{\lambda^2} = 4\lambda^2 \frac{d^2 \Psi}{d\lambda^2} \quad (31)$$

Where we have used (27).

This prove that for the I Poisson structure of dynamical system (22) the critical point is elliptic (purely imaginary eigenvalues) when $d\Psi/d\lambda < 0$ and hyperbolic when $d\Psi/d\lambda > 0$. This statement is agree with the stability criterion used by Bridges in [16]. But this criterion depends of the used Poisson structure as we will see, taking another Poisson structure the criterion change.

Another Poisson structure for (22) is obtained taking $C_2$ as a Hamiltonian $H$ and $C_1$ as a Casimir $\Psi$.

II Poisson Structure:

$$J^{(2)ab} = \frac{1}{X_1 X_2} \varepsilon^{abc} \frac{\partial \Psi}{\partial X^c}$$

$$\|J^{(2)ab}\| = \begin{pmatrix}
0 & 0 & \pm \frac{1}{X_2} \\
0 & 0 & \mp \frac{1}{X_2} \\
-\frac{1}{X_1} & \mp \frac{1}{X_1} & 0
\end{pmatrix} \quad (32)$$

The critical point of (22) using the second Poisson structure is obtained from the equation:

$$\nabla H(\lambda) - \beta \nabla \Psi(\lambda) = 0$$

From this, we have:

$$P_c = \left( \frac{\beta}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}, \frac{\pi}{2} \right) \quad (33)$$

Note that directly from the equation of motion (22) the critical point must satisfy the relation: $X_1^2 = 2X_2^2, \Phi = \frac{\pi}{2}$. The derivative of $\Psi_{P_c}$ and $H_{P_c}$ with respect to Lagrange multiplier are:

$$\frac{d\Psi_{P_c}}{d\beta} = \frac{3}{4} \beta \quad (34)$$

$$\frac{dH_{P_c}}{d\beta} = \frac{3}{4} \beta^2 \quad (35)$$

Liniarizing the Poisson system (22) about a critical point $P_c$ we have the matrix:

$$-\frac{1}{X_1 X_2} \nabla \Psi \times (D^2 H - \beta D^2 \Psi)_{P_c} = \begin{pmatrix}
0 & 0 & -\frac{\beta^2}{2
\sqrt{2}} \\
0 & 0 & \frac{\beta^2}{2
\sqrt{2}} \\
2\sqrt{2} & -4 & 0
\end{pmatrix} \quad (36)$$

The eigenvalues of (36) are zero and:

$$\mu^2 = -3\beta^2 = -4\lambda^2 \frac{d^2 \Psi_{P_c}}{d\beta} \quad (37)$$

Where we have used (34). This prove that for the II Poisson structure of dynamical system (22) the critical point is elliptic (purely imaginary eigenvalues) when $d\Psi/d\beta > 0$ and hyperbolic when $d\Psi/d\beta < 0$. This statement is completely opposite to the mentioned above stability criterion used by Bridges in [16]. This fact confirms that this criterion depends of the used Poisson structure.

As the second example illustrating this fact we use the system presented by T. Bridges in [16].

Example II:

$$\dot{X}_1 = 2X_3$$

$$\dot{X}_2 = 2Q(X_1)X_3$$

$$\dot{X}_3 = X_2 + Q(X_1)X_1 \quad (38)$$
Where $Q(X_1)$ is a real polynomial in $X_1$. The phase space of (38) is: $M = \mathbb{R}^3^+$

The integrals of motion for this system are:

\begin{align*}
C_1 &= X_3^2 - X_1X_2 \\
C_2 &= X_2 - \int_0^{X_1} Q(s) ds
\end{align*}

I Poisson Structure $H = C_1$, and $\Psi = C_2$.

\[ J^{(1)ab} = \epsilon^{abc} \frac{\partial \Psi}{\partial X^c} \]

\[ \|J^{(1)ab}\| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & Q(X_1) \\ -1 & -Q(X_1) & 0 \end{pmatrix} \]

(41)

With this Poisson structure the equation of motion (38) take the simple form: $\dot{X} = \nabla \Psi \times \nabla H$. When $\nabla \Psi$ and $\nabla H$ are parallel the system (38) has a critical point.

\[ \nabla H(X) + \lambda \nabla \Psi(X) = 0 \]

(42)

From this, we have:

\[ P_c = (\lambda, -\lambda Q(\lambda), 0) \]

The derivative of $\Psi_{P_c}$ and $H_{P_c}$ with respect to Lagrange multiplier are:

\[ \frac{dH_{P_c}}{d\lambda} = 2\lambda Q(\lambda) + \lambda^2 \frac{dQ}{d\lambda} \]

\[ \frac{d\Psi_{P_c}}{d\lambda} = -\left[ Q(\lambda) + \lambda \frac{dQ}{d\lambda} \right] \]

(43)

Linizarizing the Poisson system (38) about a critical point $P_c$: $X^i = P_c + \xi^i$, we have:

\[ \dot{\xi} = [\nabla \Psi \times (D^2H + \lambda D^2\Psi)_{P_c}]\xi \]

(44)

Where:

\[ \nabla \Psi \times (D^2H - \beta D^2\Psi)_{P_c} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2Q(\lambda) \\ \lambda \frac{dQ}{d\lambda} + Q(\lambda) & 1 & 0 \end{pmatrix} \]

(45)

The eigenvalues of (46) are zero and:

\[ \mu^2 = 2 \left( 2Q + \lambda \frac{dQ}{d\lambda} \right) = -2 \frac{d\Psi_{P_c}}{d\lambda} \]

(46)

Where we have used (44). We have proved that for the I Poisson Structure of dynamical system (38) the critical point is elliptic when $d\Psi/d\lambda > 0$ and hyperbolic when $d\Psi/d\beta < 0$.

II Poisson Structure $H = C_2$, and $\Psi = C_1$.

\[ J^{(2)ab} = \epsilon^{abc} \frac{\partial \Psi}{\partial X^c} \]

\[ \|J^{(2)ab}\| = \begin{pmatrix} 0 & 2X_3 & X_1 \\ -2X_3 & 0 & -X_2 \\ -X_1 & X_2 & 0 \end{pmatrix} \]

(47)

With the help of the Poisson structure (48) T. Bridges in [16] obtained the opposite criterion to the one proved above. This means that we can not use the slope of a Casimir function parametrized by the Lagrange multiplier to establish the critical point type, independently of the used hamiltonian structure.
IV. THE HAMILTON–JACOBI EQUATIONS

Consider the following equation of motion
\[
\frac{dx^a}{dt} = f^a(x^b) \quad a, b = 1, \ldots, N
\]  \hspace{1cm} (49)

If we have a hamiltonian structure on the phase space \( M \) of (49) with a Casimir function \( \Psi \), then the Casimir determine the geometry of the phase space. \( \Psi \) is, of course, a constant of the motion of dynamical system (49). The equation \( \Psi(x^b) = C \) for different values of \( C \) define a foliation of the phase space \( M \). Each leaf of the foliation is an invariant submanifold, furthermore it is a symplectic submanifold \( N \) of the phase space \( M \). Using a Poisson map \( \phi \), i.e. a transformation \( \phi: M \to N \) that conserve the Poisson bracket: \( \{ F(x), H(x) \}_M = \{ F, H \}_N(\phi) \), for any functions \( F, H: N \to \mathbb{R} \), we can define canonical variables, and an action on \( N \). The Hamilton–Jacobi (H–J) equation for this action yields a particular solution of dynamical system constrained to \( N \). Let’s illustrate the above procedure using the first example (22) of the previous section with the II Poisson structure (32). For this structure we have the following Poisson brackets:

\[
\{ X_1, X_2 \}_M = 0 \quad \{ X_1, \Phi \}_M = \frac{1}{X_1} \quad \{ X_2, \Phi \}_M = \pm \frac{1}{X_2} \hspace{1cm} (50, 51, 52)
\]

From (52) we have:
\[
X_1 \{ X_1, \Phi \}_M = \{ X_1, X_1 \Phi \} = 1 \hspace{1cm} (53)
\]

This suggest the Poisson map \( \phi : M \to N \), given by the inverse of:
\[
X_1 = q, \quad \Phi = \frac{p}{q}, \quad X_2 = \sqrt{2C_1 \pm q^2} \hspace{1cm} (54)
\]

Where \( N \) is the submanifold defined by: \( \Psi(X_1, X_2, \Phi) = C_1 \)
We can show that \( \phi \) is a Poisson map, moreover it define canonical brackets on \( N \):
\[
\{ X_1, X_1 \Phi \}_M = \{ q, \frac{pq}{q} \}_N = \{ q, p \}_N = 1 \\
\{ X_1, X_2 \}_M = \{ q, \sqrt{2C_1 \pm q^2} \}_N = 0 \\
\{ X_2, \Phi \}_M = \{ \sqrt{2C_1 \pm q^2}, \frac{p}{q} \}_N = \pm \frac{1}{\sqrt{2C_1 \pm q^2}} = \pm \frac{1}{X_2} \hspace{1cm} (55, 56, 57)
\]

On \( N \) in \((p, q)\) coordinates the Hamiltonian \( H \) take the form:
\[
H = q^2 \sqrt{2C_1 \pm q^2} \sin \frac{p}{q} \hspace{1cm} (55)
\]

Using this Hamiltonian and introducing the action by the transformation: \( q \to q \) and \( p \to \frac{\partial S}{\partial q} \). We have the H–J:
\[
\frac{\partial S}{\partial t} + q^2 \sqrt{2C_1 \pm q^2} \sin \frac{p}{q} \frac{\partial S}{\partial q} = 0 \\
\hspace{1cm} (56)
\]

Taking the solution of (56) in the form:
\[
S = Q(q) - Et \\
\hspace{1cm} (57)
\]

Replacing in the H–J equation we find:
\[
\frac{d}{dt} \frac{\partial Q}{\partial E} = \int_0^q \frac{\dot{q} dq}{\sqrt{q^2(2C_1 \pm q^2) - E^2}} \\
\hspace{1cm} (57)
\]
The integral (57) is an elliptic integral and has solution in terms of special functions (the cnoidal and snoidal functions). The solutions \( q(t) \) and from the Poisson map (64), the solutions of dynamical system (22) lie on concentric closed curves constrained to the submanifold \( N \). We can deduce immediately that all solutions of (22) are periodic except for the stationary point (33).

Let’s take now the second example (38) of the previous section, with the Poisson structure (41). We have the following Poisson bracket on the phase space \( M \):

\[
\{X_1, X_3\}_M = 1 \tag{58}
\]
\[
\{X_2, X_3\}_M = Q(X_1) \tag{59}
\]
\[
\{X_1, X_2\}_M = 0 \tag{60}
\]

Using the Poisson map \( \varphi : M \to N \), given by the inverse of:

\[
X_1 = q, \quad X_3 = p, \quad X_2 = C_1 + \int_0^q Q(s)ds \tag{61}
\]

where \( N \) is the submanifold defined by: \( \Psi(X_1, X_2, X_3) = C_1 \), we obtain, the canonical brackets on \( N \) with the Hamiltonian:

\[
H = p^2 - q \left( C_1 + \int_0^q Q(s)ds \right) \tag{62}
\]

Introducing the action by the transformation: \( q \to \hat{q} \) and \( p \to \frac{\partial S}{\partial q} \). We have the H–J equation:

\[
\frac{\partial S}{\partial t} + \left( \frac{\partial S}{\partial q} \right)^2 - q \left( C_1 + \int_0^q Q(s)ds \right) = 0 \tag{63}
\]

Taking the solution of (63) in the form:

\[
S = K(q) - Et
\]

Replacing in the H–J equation we find:

\[
2(t - t_0) = \frac{\partial K}{\partial E} = \int_0^q \frac{dq}{\sqrt{\hat{q} \left( C_1 + \int_0^q Q(s)ds \right)}} + E \tag{64}
\]

Taking as an example \( Q(s) = 1 \) we find the following solution for (38):

\[
X_1 = \left[ Ae^{2(t-t_0)} - \frac{C_1}{2} \right] - E \tag{65}
\]
\[
X_2 = \left[ Ae^{2(t-t_0)} + \frac{C_1}{2} \right] - E
\]
\[
X_3 = \left[ Ae^{2(t-t_0)} - \frac{C_1}{2} \right] \left[ Ae^{2(t-t_0)} + \frac{C_1}{2} \right]
\]

Where the solution is obtained for: \( E - \frac{C_1^2}{4} \geq 0 \) and \( A = \sqrt{E + \frac{C_1^2}{2}} \). For other expressions of the polynomial \( Q(s) \), the solutions of (38) are expressed in terms of elliptic functions.

Alternatively we can use, with this example the Poisson structure (48). For this structure we have the following Poisson brackets on \( M \):

\[
\{X_1, X_2\}_M = 2X_3 \tag{66}
\]
\[
\{X_1, X_3\}_M = X_3 \tag{67}
\]
\[
\{X_3, X_2\}_M = X_3 \tag{68}
\]

Using the Poisson map \( \varphi : M \to N \), given by the inverse of:
\[ X_1 = e^{-q}, \ X_3 = p, \ X_2 = e^q p^2 \] (69)

where \( N \) is the submanifold defined by: \( \Psi(X_1, X_2, X_3) = C_2 \), we obtain the canonical brackets on \( N \) with the Hamiltonian:

\[ H = qp^2 - \int_0^e^{-q} Q(s) ds \] (70)

The obtained H–J equation is:

\[ \frac{\partial S}{\partial t} + e^q \left( \frac{\partial S}{\partial q} \right)^2 - \int_0^e^{-q} Q(s) ds = 0 \] (71)

From (71) we find an implicit equation for \( q(t) \):

\[ 2(t - \tilde{t}_0) = \int_0^q \frac{dq}{\sqrt{e^{-q} \left( \int_0^e^{-q} Q(s) ds + E \right)}} \] (72)

and from \( q(t) \) we obtain a solution of (38).

Finally we present a similar analysis for the Euler top. The free body equations of motion are:

\[ \dot{\mathbf{L}} = \mathbf{\Omega} \times \mathbf{L}, \quad \mathbf{\Omega} = I \mathbf{L} \] (73)

Where \( I = \text{diag}(I_1, I_2, I_3) \) is the diagonalized moment of inertia tensor, \( I_1, I_2, I_3 > 0 \). \( \mathbf{L} \) is the angular momentum, and \( \mathbf{\Omega} \) is the angular velocity.

The dynamical system (73) has the following conserved quantities:

\[ C_1 = \frac{1}{2} \left( \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} \right) \] (74)
\[ C_2 = L_1^2 + L_2^2 + L_3^2 \] (75)

This system is Hamiltonian with \( H = C_1, \ \Psi = C_2 \) and the following Poisson matrix:

\[ J^{(1)ab} = \frac{1}{2} \varepsilon^{abc} \frac{\partial \Psi}{\partial X^c} \]

\[ \|J^{(1)ab}\| = \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{pmatrix} \] (76)

The phase space of (73) is \( M = \mathbb{R}^3 \). \( M \) is a Poisson manifold with the brackets:

\[ \{L_1, L_2\}_M = L_3 \]
\[ \{L_2, L_3\}_M = L_1 \]
\[ \{L_3, L_1\}_M = L_2 \] (77)

Mapping \( \varphi : M \to N \), with the Poisson map defined by:

\[ L_1 = \sqrt{\lambda - p^2 \cos q} \]
\[ L_2 = \sqrt{\lambda - p^2 \sin q} \]
\[ L_3 = p \] (78)

and \( N \) being a submanifold defined by the equation: \( \Psi(L_1, L_2, L_3) = C_2 \). We obtain the canonical brackets from (77), for the variables \( p \) and \( q \) on the symplectic submanifold \( N \).

Introducing the action by the transformation: \( q \to \tilde{q} \) and \( p \to \frac{2\tilde{q}}{\tilde{q}} \). We have the H–J equation:
\[
\frac{\partial S}{\partial t} + \frac{1}{2I_1} \left[ \lambda - \left( \frac{\partial S}{\partial q} \right)^2 \right] \cos^2 q + \frac{1}{2I_2} \left[ \lambda - \left( \frac{\partial S}{\partial q} \right)^2 \right] \sin^2 q + \frac{1}{2I_3} \left( \frac{\partial S}{\partial q} \right)^2 = 0 \tag{79}
\]

Taking the solution of (79) in the form:

\[ S = Q(q) - Et \]

Replacing in the H–J equation and considering the conditions:

\[ I_3 > I_2 > I_1, \quad 2EI_1 < \lambda < 2I_3E \]

We find the solution of (73):

\[ L_1 = I_1^{1/2} \left( \frac{2EI_3 - \lambda}{I_3 - I_2} \right) \text{cn} \tau \]
\[ L_2 = I_2^{1/2} \left( \frac{2EI_3 - \lambda}{I_3 - I_2} \right) \text{sn} \tau \]
\[ L_3 = I_3^{1/2} \left( \frac{\lambda - 2EI_1}{I_3 - I_1} \right) \text{dn} \tau \]
\[ \tau = \sqrt{\frac{(\lambda - 2EI_1)(I_3 - I_2)}{I_1I_2I_3}}(t - t_0) \tag{80} \]

Where \( \text{sn} \tau, \text{cn} \tau, \) and \( \text{dn} \tau \) are the Jacobi functions. Note that (80) is agree with the solution obtained by L. D. Landau in [19].

Another Poisson structure can be obtained taking a Hamiltonian \( H = \frac{1}{2}C_2 \), a Casimir \( \Psi = C_1 \) and a Poisson matrix:

\[ J^{(2)ab} = \varepsilon^{abc} \frac{\partial \Psi}{\partial X^c} \]
\[ \|J^{(2)ab}\| = \begin{pmatrix} 0 & \frac{L_3}{I_3} & -\frac{L_2}{I_2} \\ -\frac{L_3}{I_3} & 0 & -\frac{L_1}{I_1} \\ -\frac{L_2}{I_2} & -\frac{L_1}{I_1} & 0 \end{pmatrix} \tag{81} \]

The phase space of (73) with this structure, is a Poisson manifold with the brackets:

\[ \{L_1, L_2\}_M = \frac{L_3}{I_3} \]
\[ \{L_2, L_3\}_M = \frac{L_1}{I_1} \]
\[ \{L_3, L_1\}_M = \frac{L_2}{I_2} \tag{82} \]

Using the Poisson map \( \varphi : M \rightarrow N \), defined by:

\[ L_1 = \sqrt{\frac{\lambda - p^2}{I_2I_3}} \cos q \]
\[ L_2 = \sqrt{\frac{\lambda - p^2}{I_1I_3}} \sin q \]
\[ L_3 = \frac{p}{\sqrt{I_1I_2}} \tag{83} \]

the phase space \( M \) is reduced to the submanifold \( N \), defined by the equation: \( \Psi(L_1, L_2, L_3) = C_1 \). \( N \) is symplectic in the variables \( p \) and \( q \). With the Poisson map (83) we obtain the canonical brackets for \( p \) and \( q \) from (82).

Introducing the action by the transformation: \( q \rightarrow q \) and \( p \rightarrow \frac{2\pi}{\partial \Psi} \). We have the H–J equation:
\[
\frac{\partial S}{\partial t} + \frac{1}{2I_2I_3} \left[ \lambda - \left( \frac{\partial S}{\partial q} \right)^2 \right] \cos^2 q + \frac{1}{2I_1I_3} \left[ \lambda - \left( \frac{\partial S}{\partial q} \right)^2 \right] \sin^2 q + \frac{1}{2I_1I_2} \left( \frac{\partial S}{\partial q} \right)^2 = 0
\] (84)

Taking the solution of (84) in the form:

\[ S = Q(q) - Et \]

Note that in this additive separation of the variables, the constant parameter \( E \) is not the usual energy.

Replacing in the H–J equation (84) and considering the conditions:

\[ I_3 > I_2 > I_1, \quad 2EI_1I_2 < \lambda < 2EI_3I_2, \quad \lambda = 2I_1I_2I_3C_1 \]

We find the solution of (73):

\[
L_1 = \frac{1}{I_2^{1/2}} \left( \frac{\lambda - 2EI_1I_2}{I_3 - I_1} \right) cn \tilde{\tau} \\
L_2 = \frac{1}{I_1^{1/2}} \left( \frac{\lambda - 2EI_1I_2}{I_3 - I_2} \right) sn \tilde{\tau} \\
L_3 = \frac{1}{I_2^{1/2}} \left( \frac{2EI_3I_2 - \lambda}{I_3 - I_1} \right) dn \tilde{\tau} \\
\tilde{\tau} = \frac{1}{I_1I_2I_3} \sqrt{I_1(I_3 - I_2)(2EI_2I_3 - \lambda)(t - \tilde{t}_0)}
\] (85)

It is a straightforward matter to realize that (85) is a solution of (73) and become to (80) taking:

\[ \lambda = 2I_1I_2I_3C_1, \quad 2E = C_2 \]

V. CONCLUSIONS

In section II we have been able to construct two different structures based on one symmetry vector \( \eta \) and two conserved quantities \( C_1 \) and \( C_2 \) for one dynamical system. The Poisson structures \( J^{(1)} \) and \( J^{(2)} \) are built as the antisymmetric product of the evolution vector \( f \) and the symmetry vector \( \eta \) normalized using the deformation \( K \) and \( Q \) of \( C_1 \) and \( C_2 \) respectively.

In section III we showed that the stability criterion involving the derivative of a Casimir function parametrized by a Lagrange multiplier strongly depends on the used Poisson structure. The situation is even worse, because we can construct infinitely many Poisson structures, using the Poisson matrix defined by:

\[ J^{ab} = \mu \varepsilon^{abc} \frac{\partial D}{\partial X^c} \]

Where \( D = D(C_1, C_2) \) is an arbitrary function of \( C_1 \) and \( C_2 \), and a Hamiltonian \( H = H(C_1, C_2) \), which is a function independent of \( D \). We have many Poisson structures depending on the different choices of arbitrary functions \( D \) and \( H \). (See for details [1,3]).

In the section IV, the Hamilton–Jacobi equations for a number of simple classical systems were obtained using non–standard Hamiltonian, constructed with the Hojman’s techniques, developed in [2]–[8].

It is important to note that in the reduction of the phase space \( M \) of a dynamical system, the different values of the Casimir functions define a foliation of \( M \). Each sheet, specified by the values of the casimir contains entire trajectories, i. e. solution of the equation of motion, and cannot be connected to any other sheet, only states within a sheet are transformable into one another. In other words each selection of the Poisson structure and then each sheet yield a different physical system, though they all share the same equation of motion. Further, since classical H–J theory forms a link with quantum mechanics it is of particular importance to examine alternative appoaches of constructing H–J equation for one classical system, to see if they lead to a razonable quantum mechanical systems.

Alternative H–J equations for Euler top are explicitly exhibited and solved. We demonstrated that both obtained H–J equation yield the correct solutions for the equation of motion, thought the actions must be different.

More examples will be discussed in forthcoming articles.
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