Abstract. This note concerns the category \( \square \) of cartesian cubes with connections, equivalently the full subcategory of posets on objects \([1]^n\) with \( n \geq 0 \). We show that the idempotent completion of \( \square \) consists of finite complete posets. It follows that cubical sets, i.e. presheaves over \( \square \), are equivalent to presheaves over finite complete posets. This yields an alternative exposition of a result by Kapulkin and Voevodsky that simplicial sets form a subtopos of cubical sets.

1. Preliminaries

1.1. Idempotents. We briefly recall some of the basic theory of idempotents and their splittings. A classical reference is [BD86].

The following notions are relative to a category \( E \). An idempotent \((A, f)\) consists of an endomorphism \( f: A \to A \) satisfying \( f^2 = f \):

\[
\begin{array}{c}
A \\
\downarrow f \\
\downarrow f \\
A
\end{array}
\]

A retract \((A, B, r, s)\) consists of a pair of morphisms \( r: A \to B \) (the retraction) and \( s: B \to A \) (the section) such that \( rs = \text{id}_B \):

\[
\begin{array}{c}
A \\
\downarrow s \\
\downarrow \text{id} \\
B
\end{array}
\]

Any retract \((A, B, s, r)\) induces an idempotent \( sr: A \to A \). An idempotent is said to split if it is induced by a retract diagram in this fashion. The splitting of \( f: A \to A \) is also characterized as an equalizer or a coequalizer of \( f \) and \( \text{id}_A \). Retract diagrams are preserved by any functor, so these are absolute (co)limits. The category \( E \) is called idempotent complete if all its idempotents split.

Definition 1.1. A functor \( F: C \to D \) exhibits \( D \) as an idempotent completion of \( C \) if \( F \) is fully faithful, \( D \) is idempotent complete, and every object \( B \in D \) arises as a retract of an object \( FA \) with \( A \in C \).

The idempotent completion is also known as the Cauchy completion or Karoubi envelope. It can be characterized using a universal property, but we will not need this here.

We write \( \hat{\mathcal{C}} \) for the category of presheaves over a category \( \mathcal{C} \). The main property of the idempotent completion relevant to our intended application is the following.

Proposition 1.2 (essentially [BD86, Theorem 1]). Let \( F: C \to D \) be an idempotent completion. Then the induced pullback functor \( F^*: \hat{D} \to \hat{C} \) is an equivalence.
1.2. Cube category. We write $\text{Poset}$ for the category of small posets.

Following [KV], we define $\square$ as the full subcategory of $\text{Poset}$ on powers $[1]^n$ of the walking arrow $[1] = \{0 \to 1\}$ with $n \geq 0$. Equivalently, in the style of [CCHM16], we may view $\square$ as the opposite of the category of bounded distributive lattices free on a finite set. One may also explicitly describe $\square$ as a category of cubes with symmetries, diagonals, and (upper and lower) connections. Cubical sets using this notion of cube category are sufficiently rich to interpret all main features of cubical type theory as introduced in [CCHM16], including recent extensions to higher inductive types [CHM18].

2. Idempotent completion of $\square$

Let $\text{Poset}_{\text{compl.fin}}$ denote the full subcategory of complete finite posets; note that it has a small (countable) skeleton. Recall that every complete finite poset is also cocomplete. Thus, we may also describe $\text{Poset}_{\text{compl.fin}}$ as the full subcategory of posets on finite bounded lattices.

The walking arrow $[1]$ is a finite bounded lattice, hence are its finite powers $[1]^n$ with $n \geq 0$. We thus obtain a fully faithful inclusion $\square \to \text{Poset}_{\text{compl.fin}}$.

**Theorem 2.1.** The inclusion $\square \to \text{Poset}_{\text{compl.fin}}$ is an idempotent completion.

**Proof.** The category of finite posets is finitely complete, hence in particular idempotent complete. To transfer this property to $\text{Poset}_{\text{compl.fin}}$, it suffices to check given a retraction $A \to B$ of (finite) posets that $B$ is complete as soon as $A$ is. This is Lemma 2.2 below.

It remains to check that every complete finite poset arises as a retract of $[1]^n$ with $n \geq 0$. This is Lemma 2.4 below.

**Lemma 2.2.** Retracts in $\text{Poset}$ preserve completeness.

**Proof.** Let

$$
\begin{array}{c}
\begin{array}{c}
A \\
B \\
\end{array} \\
\begin{array}{c}
S \\
R \\
\end{array}
\end{array}
\xymatrix{
A \ar[r]^R & B \\
B \ar[r]_{\text{Id}} & B
}
$$

be a retract of posets. We assume $A$ complete and will show $B$ complete.

Given a diagram $F : C \to B$, we need to show that $B \downarrow F$ has a terminal object. By functoriality of comma category formation, the above retract diagram lifts to a retract diagram

$$
\begin{array}{c}
\begin{array}{c}
A \downarrow SF \\
B \downarrow F \\
\end{array} \\
\begin{array}{c}
S \downarrow C \\
R \downarrow C
\end{array}
\end{array}
\xymatrix{
A \downarrow SF \\
B \downarrow F \\
B \downarrow F \ar[r]_{\text{Id}} & B \downarrow F
}
$$

of comma categories, which are again posets. By completeness of $A$, we have a terminal object in $A \downarrow SF$. The conclusion then follows from Lemma 2.3 below.

**Lemma 2.3.** Retracts in $\text{Poset}$ preserve the existence of terminal objects.

**Proof.** Let

$$
\begin{array}{c}
\begin{array}{c}
A \\
B \\
\end{array} \\
\begin{array}{c}
S \\
R \\
\end{array}
\end{array}
\xymatrix{
A \ar[r]^R & B \\
B \ar[r]_{\text{Id}} & B
}
$$
be a retract of posets. Given a terminal object $1_A$ in $A$, we will verify that $1_B \overset{\text{def}}{=} R1_A$ is terminal in $B$. Given $X \in B$, a map $X \to 1_B$ is obtained by taking the map $SX \to 1_A$ (using terminality of $1_A$) and applying $R$. \hfill \Box

Note the Lemma 2.3 really only works for posets, not categories in general. For a counterexample, note that the terminal category is a retract of the walking pair of arrows.

**Lemma 2.4.** Let $C \in \text{Poset}$ have set of objects $|C|$ and decidable hom-sets. Then $C$ is a retract of $[1]^{|C|}$.

**Proof.** We use the posetal Yoneda embedding $y: C \to [1]^{C^{\text{op}}}$. As for categories, it is the universal map from $C$ to a cocomplete poset. Since $C$ is already cocomplete, there exists a unique cocontinuous functor $R$ making the following diagram commute (using that $C$ is skeletal):

![Diagram](https://example.com/diagram.png)

This exhibits $C$ as a retract of $[1]^{C^{\text{op}}}$.

In a second step, we show that $[1]^{C^{\text{op}}}$ is a retract of $[1]^{|C|}$. Seeing $|C|$ as a discrete poset, we have an inclusion $I: |C| \to C$ that is bijective on objects. Thus, the restriction functor $(-) \circ I: [1]^{C^{\text{op}}} \to [1]^{|C|}$ is fully faithful. Together with its left adjoint $\text{Lan}_I$, it thus forms a reflection. Since $C$ is skeletal, the counit isomorphism of this reflection is valued in identities, i.e. we have a retract

![Diagram](https://example.com/diagram.png)

The goal follows by composition of retracts. \hfill \Box

3. **Application to comparing cubical and simplicial sets**

Let $\Delta$ denote the simplex category. We may use Theorem 2.1 to give a different exposition of the proof by Kapulkin and Voevodsky of the following theorem:

**Theorem 3.1** ([KV, Section 1]). There is an essential geometric embedding $\hat{\Delta} \to \hat{\square}$.

Note that essentiality of the geometric embedding, meaning a further left adjoint to the inverse image functor, was not observed in the cited reference.

**Proof.** Observe first that every object $[n]$ of $\Delta$ is a complete finite poset. We have fully faithful embeddings of index categories as follows:

![Diagram](https://example.com/diagram.png)

using Theorem 2.1 for the annotation of the right arrow.
Upon taking presheaves, we therefore obtain essential geometric embeddings as follows:

\[
\begin{array}{c}
\Delta \\
\downarrow \\
\approx \\
\downarrow
\end{array}
\]

Here, we used Proposition 1.2 to derive the equivalence on the right. Thus, we obtain an essential geometric embedding \( \Delta \to \square \) as desired.

Evaluating the resulting inverse image functor \( \hat{\Delta} \to \hat{\square} \), a representable \( y([1]^n) \) is first mapped to the representable \( y([1]^n) \) in \( \text{Poset}_{\text{compl.fin}} \) and then restricted to \( \Delta \). The resulting presheaf sends \([m] \in \Delta \) to \( \text{Poset}([m],[1]^n) \cong \hat{\Delta}(y[m],(\Delta^1)^n) \), i.e. coincides with the standard triangulation of the \( n \)-cube. By naturality and cocontinuity, the inverse image functor is thus the standard triangulation functor of cubical sets, guaranteeing that our construction coincides with the one of [KV].

The left adjoint \( \hat{\Delta} \to \hat{\square} \) to the inverse image functor sends a representable \( y[m] \) first to the representable \( y[m] \) in \( \text{Poset}_{\text{compl.fin}} \) and then restricts it to \( \square \). The resulting presheaf sends \([1]^n \) to \( \text{Poset}([1]^n,[m]) \). Following the below remark, this may also be seen as the quotient of \( y([1]^m) \) by the endomorphism of \([1]^m\) reordering each \( x \) in an ascending fashion.

**Remark 3.2.** Write the poset \([n]\) as \( \{0 \to \ldots \to n\} \). We can exhibit an object \([n] \in \Delta \) explicitly as a retract in \( \text{Poset} \) of \([1]^n \) by sending \( k \in [n] \) to \( (0, \ldots, 0, 1, \ldots, 1) \in [1]^n \) and \( x \in [1]^n \) to \( \sum_{n-k} x_i \in [n] \). This observation suffices to recognize \( \Delta \) as a full subcategory of the idempotent completion of \( \square \), using only the fact that \( \text{Poset} \) is idempotent complete and hence contains the idempotent completion of \( \square \) as a full subcategory. In turn, this is enough to carry out the proof of Theorem 2.1, not requiring full use Theorem 2.1 or its ingredients Lemmata 2.2 to 2.4. We believe this mirrors best the original proof in [KV].

**References**

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