Higher regularity and uniqueness for inner variational equations.

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Abstract
We study local minima of the $p$-conformal energy functionals,
\[ E^*_A(h) := \int_D A(\mathcal{K}(w, h)) \ J(w, h) \ dw, \quad h|_S = h_0|_S, \]
defined for self mappings $h : \mathbb{D} \to \mathbb{D}$ with finite distortion of the unit disk with prescribed boundary values $h_0$. Here $\mathcal{K}(w, h) = \frac{\|Dh(w)\|_2^2}{J(w, h)}$ is the pointwise distortion functional, and $A : [1, \infty) \to [1, \infty)$ is convex and increasing with $A(t) \approx t^p$ for some $p \geq 1$, with additional minor technical conditions. Note $A(t) = t$ is the Dirichlet energy functional.

Critical points of $E^*_A$ satisfy the Ahlfors-Hopf inner-variational equation
\[ A'(\mathcal{K}(w, h)) h_w h_w = \Phi \]
where $\Phi$ is a holomorphic function. Iwaniec, Kovalev and Onninen established the Lipschitz regularity of critical points. Here we give a sufficient condition to ensure that a local minimum is a diffeomorphic solution to this equation, and that it is unique. This condition is necessarily satisfied by any locally quasiconformal critical point, and is basically the assumption $\mathcal{K}(w, h) \in L^1(D) \cap L^r_{loc}(\mathbb{D})$ for some $r > 1$.

1 Introduction
A mapping $f : \mathbb{D} \to \mathbb{D}$ has finite distortion if
\begin{enumerate}
  \item $f \in W^{1,1}_{loc}(\mathbb{D})$, the Sobolev space of functions with locally integrable first derivatives,
  \item the Jacobian determinant $J(z, f) \in L^1_{loc}(\mathbb{D})$, and
  \item there is a measurable function $K(z) \geq 1$, finite almost everywhere, such that
\end{enumerate}
\[ |Df(z)|^2 \leq K(z) J(z, f), \quad \text{almost everywhere in } \mathbb{D}. \tag{1.1} \]

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See [2, Chapter 20] or [9] for the basic theory of mappings of finite distortion and the associated governing equations; degenerate elliptic Beltrami systems. In (1.1) the operator norm is used. However this norm loses smoothness at crossings of the singular-values of the differential $Df$ and for this reason when considering minimisers of distortion functionals one considers the distortion functional

$$K(z, f) = \begin{cases} \frac{\|Df(z)\|^2}{J(z, f)}, & \text{if } J(z, f) \neq 0 \\ 1, & \text{if } J(z, f) = 0. \end{cases}$$ (1.2)

This was already realised by Ahlfors in his seminal work proving Teichmüller’s theorem and establishing the basics of the theory of quasiconformal mappings, [1, §3, pg 44]. We reconcile (1.1) and (1.2) by noting

$$K(z, f) = \frac{1}{2} \left( K(z) + 1/K(z) \right)$$ almost everywhere, where $K(z)$ is chosen to be the smallest functions such that (1.1) holds.

Let $A : [1, \infty) \to [1, \infty)$ be convex and increasing with

$$pA(t) \leq tA'(t), \quad \text{for some } p > 1. \quad (1.3)$$

The number $p$ here determines the higher regularity assumptions we make. The $A$-mean distortion of a self-homeomorphism of $D$ is defined as

$$E_A(f) := \int_D A(K(z, f)) \, dz, \quad (1.4)$$

The canonical examples are when $A(t) = t^p$ and there we simply write $E_p(f)$. The dual energy functional is

$$E_A^*(h) := \int_D A(K(w, h)) \, J(w, h) \, dw, \quad (1.5)$$

For self homeomorphisms of $D$, $f$ and $h = f^{-1}$, of finite distortion we have

$$E_A^*(h) = E_A(f). \quad (1.6)$$

See [7] or [2, 8, 9] for more information on the change of variables needed here.

We recall the following conjecture in [11].

**Conjecture 1.1** Let $f_0 : \overline{D} \to \overline{D}$ be a homeomorphism of finite distortion with $E_A(f_0) < \infty$. In the space of homeomorphic mappings of finite distortion with boundary values $f_0$, there is a minimiser $f$ which is also a smooth diffeomorphism.

There is of course a similar conjecture for $h$ and either one implies the other.
1.1 Inner variational equations.

Note that the a priori regularity for $f$ in (1.4) is $W^{1,\frac{2p}{p+1}}(\mathbb{D})$ and for $h$ in (1.5) is $W^{1,2}(\mathbb{D})$. Let $\varphi \in C_0^\infty(\mathbb{D})$ with $\|\nabla \varphi\|_{L^\infty(\mathbb{D})} < \frac{1}{2}$. Then for $t \in (-1,1)$ the mapping $g^t(z) = z + t\varphi(z)$ is a diffeomorphism of $\mathbb{D}$ to itself which extends to the identify on the boundary $S$. If $f$ is a mapping of finite distortion for which $E_A(f) < \infty$, then so is $E_A(f \circ g^t) < \infty$ and they share boundary values. Similarly for $h$ and $E^*_A(h \circ g^t)$.

The functions $t \mapsto E_A(f \circ g^t)$ and $t \mapsto E^*_A(h \circ g^t)$ are smooth function of $t$. Thus if $f$ or $h$ is a minimiser in any reasonable class we have the stationary equations

$$\left. \frac{d}{dt} \right|_{t=0} E_A(f \circ g^t) = 0, \quad \left. \frac{d}{dt} \right|_{t=0} E^*_A(h \circ g^t) = 0.$$ 

It is a calculation to verify that the first equation is equivalent to

$$2p \int_\mathbb{D} K_f A'(K) \frac{\mu_f}{1 + |\mu_f|^2} \varphi \omega d\omega = \int_\mathbb{D} A(K) \varphi \omega d\omega, \quad \forall \varphi \in C_0^\infty(\mathbb{D}). \quad (1.7)$$

and that the second is equivalent to

$$A'(K(w,h)) h_{w\omega} h_{\overline{w}} = \Phi.$$ 

(1.8)

where $\Phi$ is holomorphic. Using an early modulus of continuity estimate Allfors showed that there is always an $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $h \in C(\overline{\mathbb{D}}) \cap W^{1,2}(\mathbb{D})$ minimising (1.5) for quasisymmetric boundary values and therefore solving (1.8). Strictly speaking he used $A(t) = t^p$, $p \geq 2$, but more recent modulus of continuity estimates give the more general result. For this reason we call $\Phi$ the Allfors-Hopf differential.

The two strongest results currently known to us are the Lipschitz regularity of Iwaniec, Kovalev and Onninen [10]

**Theorem 1.1** Let $h \in W^{1,2}(\mathbb{D})$ be a mapping of finite distortion which solves (1.8) for holomorphic $\Phi$. Then $h$ is locally Lipschitz.

Also our earlier result (strictly speaking only for $A(t) = t^p$ but the ideas are exactly the same) [13].

**Theorem 1.2** Let $f$ be a finite distortion function that satisfies the distributional equation (1.7). Assume that $K(z,f) \in L^r_{loc}(\mathbb{D})$, for some $r > p+1$. Then $f$ is a local diffeomorphism in $\mathbb{D}$.

Actually $r$ does not have to be uniform in $\mathbb{D}$. The following corollary is almost immediate.

**Corollary 1.1** Let $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a continuous locally quasiconformal solution to (1.8). Then $h$ is a smooth self-diffeomorphism of $\mathbb{D}$.
We also gave a counterexample to justify some assumptions on the integrability of the distortion.

**Theorem 1.3** There is a Sobolev mapping with \( f \in W^{1, \frac{2m}{m+1}}(\mathbb{D}) \) with Beltrami coefficient \( \mu_f \) satisfying the distributional equation (1.4), and with \( K_f \in L^p(\mathbb{D}) \) \( \bigcup_{q>p} L^{q, \text{loc}}(\mathbb{D}) \). In particular, this mapping \( f \) has \( E_p(f) < \infty \) and solves the distributional equation, but it cannot be locally quasiconformal.

The mapping \( f \) of Theorem 1.3 has a pseudo-inverse \( h : \mathbb{D} \to \mathbb{D}, h \in C(\mathbb{D}) \cap W^{1, 2}_\text{loc}(\mathbb{D}) \), a monotone mapping for which \( h(f(z)) = z \) for almost every \( z \in \mathbb{D} \).

Unfortunately we do not know if \( h \) is a homeomorphism, if \( h \) has homeomorphic boundary values or even if \( h \) satisfies the Ahlfors-Hopf equation (though this last would follow if \( f \) were a local minimum for (1.5)). In the case \( p = 1 \) and \( A(t) = t \), [10, Example 3.4] provides a Lipschitz solution to the Hopf-Laplace equation \( h_w h_{\overline{w}} = -1 \) with \( J(w, h) \geq 0 \) almost everywhere and yet is not a homeomorphism. This map can be modified so as to be defined on \( \mathbb{D} \), but its image seems unwilling to be modified so as to be a disk without avoiding the singular set and so becoming a diffeomorphism.

We are unaware of a way to connect the two inner-variational equations, even for homeomorphic solutions, without fairly strong a priori assumptions. As for the boundary values, it remains unclear as to what is the exact criterion for a self-homeomorphism \( f_0 \) or \( h_0 \) to admit an appropriate extension of finite energy, so that the family we might consider is not empty (though see [3] in the case \( p = 1 \)). Thus we stick to the class of quasisymmetric mappings which have a quasiconformal extension. Our main results here are the following. We shall always assume that \( p > 1 \) unless otherwise stated.

**Theorem 1.4** Let \( h : \mathbb{D} \to \mathbb{D} \) with \( E^*_A(h) < \infty \) for quasisymmetric boundary values \( h_0 : \mathbb{S} \to \mathbb{S} \). If \( K(w, h) \in L^1(\mathbb{D}) \), then \( h \) is a homeomorphism. If in addition \( h \) is a local minimum for \( E^*_A \) and if \( K(w, h) \in L^r_{\text{loc}}(\mathbb{D}) \) for some \( r > 1 \), then \( h \) is a diffeomorphism.

We remark that Iwaniec’s calculation of the second inner-variation (personal communication) suggests that there are in fact no local maxima. We note a slightly different result. That \( h \) is a homeomorphic local maximum or minimum shows its inverse satisfies the inner distributional equation. This would also be guaranteed if \( h \) satisfies the outer distributional equation

\[
\frac{d}{dt} \int_{\mathbb{D}} K^p(w, g^t \circ h) J(w, g^t \circ h) \, dw = 0
\]

It is a lengthy calculation to reveal this equation is

\[
\int_{\mathbb{D}} K^p_h ((K_h + 1)p - 1) h_w \phi_w \, dw = \int_{\mathbb{D}} K^p_h ((K_h - 1)p - 1) h_w \phi_{\overline{w}} \, dw \quad (1.9)
\]

A finite distortion mapping which is a solution to (1.9) and has \( E^*_A(h) < \infty \) is called an outer variational stationary point.
Theorem 1.5 If $h$ is an outer-variational stationary point with quasisymmetric boundary values, if $K(w; h) \in L^1(D)$ and if $E_q^*(h)$ is locally finite for some $q > p + 1$, then $h$ is a diffeomorphism.

This is in essence a restatement of Theorem 1.2.

We can say a little about uniqueness here too.

Theorem 1.6 Let $h : \mathbb{D} \to \mathbb{D}$, $h \in W^{1,2}(D)$, be a diffeomorphism of $\mathbb{D}$ with homeomorphic boundary values, and with Ahlfors-Hopf differential $\Phi$ as at (1.8). Let $g : \mathbb{D} \to \mathbb{D}$ be continuous and a homeomorphism on $S$. Suppose $g$ is a solution to the Ahlfors-Hopf equation

$$A'(K(w; g))g_{\mathbb{D}} = \Phi,$$

(1.10)

in $\mathbb{D}$ with $g(0) = h(0)$ and $g(1) = h(1)$. Then $g \equiv h$.

Note that we do not require that $g = h$ on $S$ in the hypotheses. In fact diffeomorphic minimisers are locally absolute minimisers for their boundary values.

Theorem 1.7 Let $\Omega_1, \Omega_2$ be Jordan domains and let $h : \Omega_1 \to \Omega_2$ be a diffeomorphic minimiser of $E_A^*$ for its boundary values. If $D = D(z_0, r) \subset \Omega_2$ is any disk and $\varphi : h^{-1}(D) \to \mathbb{D}$ is a Riemann map, then $h_*(z) = \frac{1}{r}(h \circ \varphi^{-1})(z) - z_0 : D \to \mathbb{D}$ is the unique minimiser for its boundary values.

Proof. Suppose there is $g$ with the same or smaller energy in $D$ for the boundary values of $h_*$. Set

$$\tilde{h} = \begin{cases} h(z), & z \in \mathbb{D} \setminus \{h^{-1}(D)\}, \\ rg(\varphi(z)) + z_0, & z \in \{h^{-1}(D)\} \end{cases} \quad (1.11)$$

Then $\tilde{h}$ is a mapping of finite distortion and has energy no more than $h$, a minimiser (the only issue is the $W^{1,1}_{loc}$ regularity across the set $\partial h^{-1}(D)$, but this is a smooth Jordan curve). So $h$ is also a minimiser, and therefore has a holomorphic Ahlfors-Hopf differential. This differential must be $\Phi$ since $\tilde{h}$ agrees with $h$ on $D \setminus h^{-1}(D)$. The result now follows from Theorem 1.6. \hfill \Box

2 Diffeomorphisms; proof of Theorem 1.4.

Since $A(t) \geq t$ we see $h$ lies in the Sobolev space $W^{1,2}(\mathbb{D})$ as $E_A^*(h) < \infty$. We observe that since $p > 1$, the estimate

$$\frac{\|Dh\|^{2p-2}}{J(w, h)^{p-1}} \frac{|\mu|}{1 + |\mu|^2} \approx |\Phi|$$

implies that $h$ is a mapping of finite distortion.
Let \( h_* : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D} \) be a quasiconformal extension of the quasisymmetric boundary values \( h_0 \). We can ensure that \( h_*(z) = z \) for all sufficiently large \( z \) by the quasiconformal version of the Schönhflies theorem, [4] or [5, §7]. Then

\[
H(w) = \begin{cases} 
  h(w), & w \in \mathbb{D}, \\
  h_*(w), & w \in \mathbb{C} \setminus \mathbb{D},
\end{cases}
\]

is a mapping of finite distortion, \( H(w) - w \in W^{1,2}(\mathbb{C}) \) and \( \mu_H = H_{\overline{w}}/H_w \) is compactly supported. Further \( \mathbb{K}(w, H) - 1 \in L^1(\mathbb{C}) \). Then [2, Theorem 20.2.1] provides a homeomorphic entire principal solution (we have to make the minor adjustment of replacing \( \mathbb{D} \) by a larger disk in which \( \mu \) is compactly supported) to the Beltrami equation \( g \circ g^t = \mu_H g \) and the Stoilow factorisation shows this solution must be \( H \) up to a similarity. Thus \( H \), and hence \( h \), is a homeomorphism.

Next we suppose that \( h \) is a homeomorphic local maximum or local minimum and set \( f = h^{-1} : \mathbb{D} \to \mathbb{D} \). We have \( E^*_A(h) = E_A(f) \) and also, for \( g^t(z) = z + t\varphi(z) \) with compactly supported test function \( \varphi \), \( |\nabla \varphi| < 1 \),

\[
E_A(f \circ g^t) = E^*_A((g^t)^{-1}h).
\]

Then \( h \) a local minimum for \( E^*_A \) implies that \( \frac{d}{dt}|_{t=0} E_A(f \circ g^t) = 0 \) and \( f \) satisfies the distributional equation (1.7). Next, the Alhfors-Hopf differential is

\[
\Phi = A'(\mathbb{K}(w, h))h_w h_{\overline{w}} = A'(\mathbb{K}(w, h))|h_w|^2 \overline{h_w} \\
|\Phi| \geq c_0 \mathbb{K}(w, h)^p J(w, h)
\]

The last inequality holding by virtue of (1.3) and only on \( E = \{ w : |\mu_h(w)| \geq \frac{1}{2} \} \) for \( c_0 \) a positive constant. In particular we see that \( \mathbb{K}(w, h)^p J(w, h) \) is locally bounded on the set \( \mathbb{K}(w, h) \geq \frac{1}{2} \). If \( \mathbb{K}(w, h) \in L^r(V) \) for some relatively compact set \( V \), then \( \mathbb{K}(w, h)^p J(w, h) \in L^1(V) \) as the Jacobian is locally integrable. Then

\[
\int_{h^{-1}(V)} \mathbb{K}(z, f)^{p+r}(z, f) = \int_{\mathbb{D}} \mathbb{K}(w, h)^{p+r} J(w, h) < \infty
\]

and since \( h : \mathbb{D} \to \mathbb{D} \) is a homeomorphism we see our hypotheses imply \( \mathbb{K}(z, f) \in L^q_{loc}(\mathbb{D}) \) for some \( q > 1+p \) and hence \( f : \mathbb{D} \to \mathbb{D} \) is a diffeomorphism by Theorem [2]. The result now follows. □

\section{3 Uniqueness: Proof of Theorem 1.6}

The proof of uniqueness will be separated into two parts. First we will find a degenerate elliptic Beltrami equation so that \( h \) has the Hopf differential \( \Phi \) if and only if

\[
h_{\overline{w}} = B(w, h_w).
\]
Here $B(w, \xi) : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ is defined implicitly and is smooth away from the set $\xi = 0$. We then give the ellipticity bounds on the nonlinear equation (3.1). We discuss the Schauder bounds and smoothness elsewhere.

Second, we use the ellipticity bounds, together with the existence of a diffeomorphic solution to establish the following lemma.

**Lemma 3.1** Let $B(w, \xi)$ as above and $h : \overline{\mathbb{D}} \to \mathbb{D}$ a continuous $W^{1,2}(\mathbb{D})$ solution to (3.1) and homeomorphic on $\mathbb{S}$. Let $g$ be a homeomorphism from $\overline{\mathbb{D}}$ to $\mathbb{D}$, a diffeomorphism from $\mathbb{D}$ to $\overline{\mathbb{D}}$, lies in $W^{1,2}(\mathbb{D})$ and that also satisfies equation (3.1). Then $\eta := g - h$ is a locally quasiregular mapping.

Given this lemma, uniqueness quickly follows in exactly the same way as in [2, §9.2.2, pp 267] using the total variation of the boundary values [2, Lemma 9.2.2] and the the Stoilow factorisation theorem, as in [2, Lemma 9.2.3].

### 3.1 The equation; $B(w, \xi)$ and its ellipticity properties

We may assume that $\Phi$ is not identically 0. We make the simplifying assumption that $A(t) = t^p$. The ideas in the general case are the same, we consider the level curves of the function $(x, y) \mapsto A'((x^2 + y^2)/(x^2 - y^2))xy$, but some of the formulas become a little unwieldy and estimates not as clean.

We begin by considering the level curves of the function

$$(x, y) \mapsto \left(\frac{x^2 + y^2}{x^2 - y^2}\right)^{p-1} xy.$$

This function is defined on the region $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < x\}$. For fixed $x > 0$ the function $y \mapsto (\frac{x^2 + y^2}{x^2 - y^2})^{p-1} xy$ is strictly increasing for $0 < y < x$:

$$\frac{\partial}{\partial y} \left[ \left(\frac{x^2 + y^2}{x^2 - y^2}\right)^{p-1} xy \right] = \left(\frac{x^2 + y^2}{x^2 - y^2}\right)^{p-1} x + (p-1) \left(\frac{x^2 + y^2}{x^2 - y^2}\right)^{p-2} \frac{4x^3y^2}{(x^2 - y^2)^2} > 0.$$

Hence for any $k > 0$ there is a unique solution $y$ so that

$$\left(\frac{x^2 + y^2}{x^2 - y^2}\right)^{p-1} xy = k.$$

The implicit function theorem guarantees the level curves are simple arcs in the $x, y$ plane and on the level curve we have that $y$ can be expressed as a function of $x$, $y = A_k(x)$. 

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Figure 1. The graph of the level curve \( \frac{x^2 + y^2}{x^2 - y^2} xy = 10 \).

On this curve
\[
(p - 1) \log \left( \frac{x^2 + A_k^2(x)}{x^2 - A_k^2(x)} \right) + \log x + \log A_k(x) = \log k.
\]
(3.2)

With \( x = |h_w| \) and \( y = |h_{\overline{w}}| \) we have
\[
|h_{\overline{w}}| = A_k(|h_{\overline{w}}|), \quad h_{\overline{w}}h_w \frac{\Phi}{|\Phi|} = A_k(|h_{\overline{w}}|)h_w,
\]
and hence we find a nonlinear Beltrami equation for \( h \) as
\[
h_w = \frac{\Phi}{|\Phi|} A_k(|h_{\overline{w}}|) h_{\overline{w}} |h_w| = B(w, h_{\overline{w}}).
\]
(3.3)

Next, the ellipticity properties of equation (3.3). We drop the subscript on \( A \),
\[
|B(w, \zeta) - B(w, \xi)| = \left| A(|\zeta|) \frac{\zeta}{|\zeta|} - A(|\xi|) \frac{\xi}{|\xi|} \right|.
\]
(3.4)

We set \( V(x) = A(x)/x \). After division, equation (3.4) reads as
\[
\frac{|B(w, \zeta) - B(w, \xi)|}{|\zeta - \xi|} = \frac{|V(|\zeta|)\zeta - V(|\xi|)\xi|}{|\zeta - \xi|}.
\]
(3.5)

We put \( |\zeta| = t, |\xi| = s, a = V(t) \) and \( b = V(s) \). Then there is a \( \theta \in [0, 2\pi] \) such that \( \zeta \cdot \xi = st \cos(\theta) \), and
\[
\frac{|V(|\zeta|)\zeta - V(|\xi|)\xi|^2}{|\zeta - \xi|^2} = \frac{a^2t^2 + b^2s^2 - 2abst \cos(\theta)}{t^2 + s^2 - 2st \cos(\theta)} := F(\theta).
\]
(3.6)

We differentiate (3.6) with respect to \( \theta \) to see
\[
\frac{d}{d\theta} F(\theta) = \frac{2st(abt^2 + abs^2 - a^2t^2 - b^2s^2)}{(t^2 + s^2 - 2st \cos(\theta))^2} \sin(\theta) = \frac{2st(a - b)(s^2b - t^2a)}{(t^2 + s^2 - 2st \cos(\theta))^2} \sin(\theta).
\]

Here we claim that
\[
(a - b)(s^2b - t^2a) \geq 0.
\]
(3.7)
Recall $V(x) = A(x)/x$. We also define $W(x) = xA(x)$. Then (3.2) gives us the relation
\[
\log k = (p - 1) \log \left[ \frac{1 + V^2(x)}{1 - V^2(x)} \right] + 2 \log x + \log V(x),
\]
\[
= (p - 1) \log \left[ \frac{x^4 + W^2(x)}{x^4 - W^2(x)} \right] + \log W(x),
\]
which we differentiate to see that
\[
V'(x) \left[ \frac{4(p - 1)V(x)}{1 - V^4(x)} + \frac{1}{V(x)} \right] = -\frac{2}{x},
\]
\[
W'(x) \left[ \frac{4(p - 1)x^4W(x)}{x^8 - W^4(x)} + \frac{1}{W(x)} \right] = \frac{8(p - 1)x^3W^2(x)}{x^8 - W^4(x)}.
\]
So $V$ is decreasing and $W$ is increasing. Now assume $t \leq s$, then
\[
a = V(t) \geq V(s) = b, \quad s^2b = W(s) \geq W(t) = t^2a,
\]
and vice versa. So (3.7) follows. Assume $\zeta \neq \xi$, we then have
\[
\frac{d}{d\theta} \left[ \frac{|V(\zeta)|\zeta - V(\xi)\xi|^2}{|\zeta - \xi|^2} \right] = G(|\zeta|,|\xi|,\cos(\theta)) \sin \theta,
\]
where $G(|\zeta|,|\xi|,\cos(\theta))$ is always non-negative. Then, in a period $\theta \in [0,2\pi]$, $F(\theta)$ is increasing when $\theta$ is moving from 0 to $\pi$ and decreasing when $\theta$ is moving from $\pi$ to $2\pi$, so we get the maximum of $F(\theta)$ at $\theta = \pi$. In particular we can now write (3.4) as
\[
\frac{|B(w,\zeta) - B(w,\xi)|}{|\zeta - \xi|} \leq \frac{|V(t)t + V(s)s|}{|t + s|} = \frac{|A(t) + A(s)|}{|t + s|} \leq \max\{\frac{A(|\zeta|)}{|\zeta|},\frac{A(\xi)}{|\xi|}\},
\]
whenever $\zeta \neq \xi$.

We now assume that $h$, $g$ are finite distortion homeomorphisms solutions to (3.3) and consider the function $\eta = g - h$.

At all but a discrete set of points $w \in \Omega$, we have $\Phi(w) \neq 0$. If $h_w = g_w$, then (3.3) gives $h_{\bar{w}} = g_{\bar{w}}$; if $h_w \neq g_w$, then by (3.8),
\[
|\mu_g| = \frac{|\eta_{\bar{w}}|}{\eta_w} = \frac{|g_{\bar{w}} - h_{\bar{w}}|}{h_w - g_w} \leq \max\{|\mu_g|,|\mu_h|\}.
\]

Also note
\[
|\eta_w|^2 - |\eta_{\bar{w}}|^2 = |g_w - h_w|^2 - |g_{\bar{w}} - h_{\bar{w}}|^2
= J(z,g) + J(z,h) - 2Re[\bar{g}_w h_w - g_{\bar{w}} h_{\bar{w}}] \in L^1(\Omega).
\]
These facts imply that \( \eta \) is a finite distortion function.

If we assume further that \( g \) is diffeomorphic from \( D \) to \( D \), then in any compact subset \( \Omega \subset D \), we have

\[
|\Phi| \leq M < \infty, \quad |g_{\bar{w}}| \geq \varepsilon > 0, \quad |\mu_g| \leq k < 1.
\]

Now at any point \( w \in \Omega \), by (3.8),

\[
\frac{|h_{\bar{w}} - g_{\bar{w}}|}{|h_{w} - g_{w}|} \leq \frac{|h_{\bar{w}}| + |g_{\bar{w}}|}{|h_{w}| + |g_{w}|} \leq \frac{|h_{\bar{w}}| + |g_{\bar{w}}|}{|g_{w}|}.
\]

So we can choose a \( \delta > 0 \) such that if \( |h_{\bar{w}}| < \delta \), then

\[
\frac{|h_{\bar{w}} - g_{\bar{w}}|}{|h_{w} - g_{w}|} \leq \frac{|g_{\bar{w}}|}{|g_{w}|} + \frac{1 - k}{2} < 1.
\]

Note this \( \delta \) depends only on \( \varepsilon, M \) and \( k \) but not a specific point \( w \in \Omega \). On the other hand, if \( |h_{\bar{w}}| > \delta \), then

\[
\mathbb{K}_h^{-1}|h_{\bar{w}}h_{\bar{w}}| = |\Phi| \leq M,
\]

which gives \( \mathbb{K}_h < (\frac{M}{\delta^p})^{-\frac{1}{p-1}} \), so at this point we have

\[
|\mu_\eta| \leq \max \left\{ \frac{1 + |\mu_g|}{2}, \sqrt{\left( \frac{M}{\delta^p} \right)^{\frac{1}{p-1}} - 1} \right\} < 1.
\]

This estimate holds locally uniformly and thus proves that \( \eta \) is locally quasiregular and completes the proof of Lemma 3.1. \( \square \)

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