ON THE EXISTENCE OF BALANCED AND SKT METRICS ON NILMANIFOLDS

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Abstract. On a complex manifold an Hermitian metric which is simultaneously SKT and balanced has to be necessarily Kähler. It has been conjectured that if a compact complex manifold \((M, J)\) has an SKT metric and a balanced metric both compatible with \(J\), then \((M, J)\) is necessarily Kähler. We show that the conjecture is true for nilmanifolds.

1. Introduction

A Riemannian metric \(g\) on a complex manifold \((M, J)\) is compatible with \(J\) (or \(J\)-Hermitian) if \(g(J\cdot, J\cdot) = g(\cdot, \cdot)\). In the present paper we focus on the existence of special Hermitian metrics on complex manifolds. More precisely, we study the existence of strong Kähler with torsion (shortly SKT) and balanced metrics compatible with the same complex structure. We recall that a \(J\)-Hermitian metric is called SKT (or pluriclosed) if its fundamental form \(\omega\) satisfies

\[
\partial \bar{\partial} \omega = 0,
\]

while \(g\) is called balanced if \(\omega\) is co-closed, i.e.

\[
d^* \omega = 0,
\]

where \(d^*\) denotes the formal adjoint operator of \(d\) with respect to the metric \(g\).

SKT metrics were introduced by Bismut in [6] and further studied in many papers (see e.g. [13, 14, 11, 20, 18, 19, 22] and the references therein), while balanced metrics were introduced and firstly studied by Michelsohn in [17], where their existence is characterized in terms of currents. In a subsequent paper Alessandrini and Bassanelli proved that modifications of compact balanced manifolds are always balanced (see [1, 2]) showing a powerful tool for finding examples of balanced manifolds.

It is well-known that if an Hermitian metric \(g\) is simultaneously SKT and balanced, then it is necessarily Kähler (see e.g. [4]). This result has been generalized in [13] showing that a compact SKT conformally balanced manifold has to be Kähler. The next conjecture was stated in [13] and it is about the existence of

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an SKT metric and a balanced metric both compatible with the same complex structure:

**Conjecture.** Every compact complex manifold admitting both an SKT metric and a balanced metric is necessarily Kähler.

The conjecture has been implicitly already proved in literature in some special cases. For instance, Verbitsky has showed in [22] that the twistor space of a compact, anti-self-dual Riemannian manifold has no SKT metrics unless it is Kählerian and Chiose has obtained in [7] a similar result for non-Kähler complex manifolds belonging to the Fujiki class. Furthermore, Li, Fu and Yau have proved in [20] that some new examples of SKT manifolds do not admit any balanced metric. A natural source of non-Kähler manifolds admitting balanced metrics and SKT metrics are given by nilmanifolds, i.e. by compact manifolds obtained as quotients of a simply connected nilpotent Lie group $G$ by a co-compact lattice $\Gamma$. It is well known that a nilmanifold cannot admit Kähler structures unless it is a torus (see for instance [5, 12]). The aim of the paper is to show that the conjecture is true when $(M, J)$ is a complex nilmanifold. By complex nilmanifold we refer to a nilmanifold equipped with an invariant complex structure $J$, i.e. endowed with a complex structure induced by a left-invariant complex structure on $G$.

Our result is the following

**Theorem 1.1.** Let $M = G/\Gamma$ be a nilmanifold equipped with an invariant complex structure $J$. Assume that $(M, J)$ admits a balanced metric $g$ and an SKT metric $g'$ both compatible with $J$. Then $(M, J)$ is a complex torus.

The theorem is trivial in dimension 6 in view of the main result in [10] and it was already proved in [13] when the nilmanifold has dimension 8 by using a classification result proven in [8].

2. **Proof of Theorem 1.1**

In order to prove Theorem 1.1 we need the following lemmas

**Lemma 2.1.** Let $(M = G/\Gamma, J)$ be a complex nilmanifold.

- If $(M, J)$ has a balanced metric, then it has also an invariant balanced metric [9].
- If $(M, J)$ has an SKT metric, then it has also an invariant SKT metric [21].

**Lemma 2.2** ([8]). Let $(M = G/\Gamma, J)$ be a complex nilmanifold of real dimension $2n$. If $(M, J)$ has an SKT metric, then $G$ is (at most) 2-step nilpotent, and there exists a complex $(1, 0)$-coframe $\{\alpha^1, \ldots, \alpha^n\}$ on $\mathfrak{g}$ satisfying the following structure equations

$$
\begin{align*}
\{ & d\alpha^j = 0, \quad j = 1, \ldots, k, \\
& d\alpha^j = \sum_{r,s=1}^{k} \left( \frac{1}{2} c_{rs}^j \alpha^r \wedge \alpha^s + c_{\bar{r}s}^j \alpha^r \wedge \bar{\alpha}^s \right), \quad j = k + 1, \ldots, n,
\end{align*}
$$

for some $k \in \{1, \ldots, n - 1\}$ and with $c_{rs}^j, c_{\bar{r}s}^j \in \mathbb{C}$. 


Now we can prove Theorem 1.1

**Proof of Theorem 1.1.** Suppose that \((M = G/\Gamma, J)\) is not a complex torus, i.e. that \(G\) is not abelian and denote by \(g\) the Lie algebra of \(G\). Assume that \((M, J)\) admits a balanced metric \(g\) and also an SKT metric \(g'\) both compatible with \(J\). Then in view of Lemma 2.1, we may assume both \(g\) and \(g'\) invariant and regarding them as scalar products on \(g\). This allows us to work at the level of the Lie algebra \(g\). As a consequence of Lemma 2.2, the existence of the SKT metric implies that Lie algebra \(g\) is 2-step nilpotent and that \((g, J)\) has a \((1, 0)\)-coframe \(\{\alpha^1, \ldots, \alpha^n\}\) satisfying the following structure equations

\[
\begin{cases}
  d\alpha^j = 0, & j = 1, \ldots, k, \\
  d\alpha^j = \sum_{r,s=1}^{k} \left( \frac{1}{2} c^j_{rs} \alpha^r \wedge \alpha^s + c^j_{r,s} \alpha^r \alpha^s \right).
\end{cases}
\]

for some \(k \in \{1, \ldots, n-1\}\) and with \(c^j_{rs}, c^j_{r,s} \in \mathbb{C}\). Here we use the notation \(\bar{\alpha}^i = \alpha^i\) and \(\alpha^{r_1 \cdots r_p s_1 \cdots s_q} = \alpha^{s_1} \wedge \ldots \wedge \alpha^{s_p} \wedge \alpha^{s_1} \wedge \ldots \alpha^{s_q}\). We may assume without restrictions that the coframe \(\{\alpha^i\}\) is unitary with respect to the balanced metric \(g\). Indeed, since in the Gram-Schmidt process the spaces spanned by the first \(r\) elements of the original basis are preserved, we can modify the coframe \(\{\alpha^r\}\) making it unitary with respect to \(g\) and satisfying the same structure equations as in (1) with different structure constants. In this way \(g\) writes as

\[
g = n \sum_{r=1}^{n} \alpha^r \otimes \bar{\alpha}^r
\]

and the balanced condition can be written in terms of \(c^j_{r,s}\)'s as

\[
\sum_{r=1}^{k} c^j_{r,s} = 0,
\]

for every \(l > k\) (see also [3, Lemma 2.1]).

Next we focus on the SKT metric \(g'\). Since \(g\) is 2-step nilpotent, we have \(\partial \bar{\partial} \alpha^r = 0\), \(r = 1, \ldots, n\). If

\[
\omega' = n \sum_{i,j=1}^{n} a_{ij} \alpha^i \wedge \bar{\alpha}^j
\]

is the fundamental form of \(g'\), then the SKT condition \(\partial \bar{\partial} \omega' = 0\) writes as

\[
\sum_{i,j=k+1}^{n} a_{ij} \left( \partial \bar{\partial} \alpha^i \wedge \partial \bar{\partial} \alpha^j - \partial \alpha^i \wedge \bar{\partial} \alpha^j - \bar{\partial} \alpha^i \wedge \partial \alpha^j \right) = 0.
\]

Equation (3) can be written in terms of the structure constants as

\[
\sum_{i,j=k+1}^{n} \sum_{r,s,u,v=1}^{k} a_{ij} \left( c^j_{r,s} \bar{c}^i_{u,v} + \frac{1}{4} d^j_{r,s} d^j_{u,v} \right) \alpha^{r,s,u,v} = 0.
\]
By considering the component along $\alpha^{rs\bar{s}\bar{r}}$ in the above expression we get

$$
\sum_{i,j=k+1}^{n} a_{ij} \left( c_{rr}\bar{c}_{s\bar{s}} + \frac{1}{4} c_{rs}\bar{c}_{s\bar{r}} - c_{sf}\bar{c}_{s\bar{f}} - \frac{1}{4} c_{sr}\bar{c}_{s\bar{r}} + c_{ss}\bar{c}_{s\bar{s}} + \frac{1}{4} c_{sr}\bar{c}_{s\bar{r}} - c_{\bar{r}s}\bar{c}_{s\bar{r}} - \frac{1}{4} c_{rs}\bar{c}_{s\bar{r}} \right) = 0,
$$

i.e. the condition

$$
\sum_{i,j=k+1}^{n} a_{ij} \left( c_{rr}\bar{c}_{s\bar{s}} - c_{sf}\bar{c}_{s\bar{f}} + c_{ss}\bar{c}_{s\bar{s}} - c_{\bar{r}s}\bar{c}_{s\bar{r}} + c_{rs}\bar{c}_{s\bar{r}} \right) = 0.
$$

Now by taking the sum of (4) for $r, s = 1, \ldots, k$ and keeping in mind the balanced assumption (2) we get

$$
\sum_{i,j=k+1}^{n} a_{ij} \left( 2c_{sf}\bar{c}_{s\bar{f}} + c_{rs}\bar{c}_{s\bar{r}} \right) = 0.
$$

Let us consider now the $(1, 0)$-vectors $X_{sr}$ and $X_{r\bar{s}}$ on $\mathfrak{g}$ defined by

$$
X_{rs} = \sum_{i=k+1}^{n} c_{rs} X_i, \quad X_{r\bar{s}} = \sqrt{2} c_{r\bar{s}} X_i
$$

where $\{X_1, \ldots, X_n\}$ is the dual frame to $\{\alpha^1, \ldots, \alpha^n\}$. We have

$$
\sum_{r,s=1}^{k} \left( \omega'(X_{rs}, X_{r\bar{s}}) + \omega'(X_{r\bar{s}}, X_{s\bar{r}}) \right) = \sum_{i,j=k+1}^{n} \sum_{r,s=1}^{k} a_{ij} \left( 2c_{sf}\bar{c}_{s\bar{f}} + c_{rs}\bar{c}_{s\bar{r}} \right)
$$

and so equation (5) implies $X_{rs} = X_{r\bar{s}} = 0$ for every $r, s = 1, \ldots, k$. So every $\alpha^r$ is closed and $\mathfrak{g}$ is abelian, contradicting the first assumption in the proof. \hfill \Box

**Remark 2.3.** Theorem 1.1 is trivial in dimension 6. Indeed, in view of [10] and Lemma 2.1, if a 6-dimensional complex nilmanifold $(M, J)$ has an SKT metric, then every invariant $J$-Hermitian metric on $(M, J)$ is SKT and so every invariant balanced metric compatible with $J$ is automatically Kähler. More generally, the same argument works when $M$ has arbitrary real dimension $2n$, but $k = n - 1$.

**Remark 2.4.** Another case where Theorem 1.1 is trivial is when $k = 1$. In this case the Lie algebra $\mathfrak{g}$ is necessary isomorphic to $\mathfrak{h}_3^R \oplus \mathbb{R}^{2n-3}$, where $\mathfrak{h}_3^R$ is the real 3-dimensional Heisenberg algebra, and the result follows.

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