Anyon Black Holes

Maryam Aghaei Abchouyeh,* Behrouz Mirza,† Moein Karimi Takaromi, and Younes Yomnesizadeh
Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

We propose a correspondence between an Anyon Van der Waals black hole and a (2+1) dimensional AdS black hole. Anyons are particles with intermediate statistics that interpolate between a Fermi-Dirac statistics and a Bose-Einstein one. A parameter $\alpha$ ($0 < \alpha < 1$) characterises this intermediate statistics of Anyons. The equation of state for the Anyon Van der Waals fluid shows that it has a quasi Fermi-Dirac statistics for $\alpha > \alpha_c$, but a quasi Bose-Einstein statistics for $\alpha < \alpha_c$. By defining a general form of the metric for the (2+1) dimensional AdS black hole and considering the temperature of the black hole to be equal with that of the Anyon Van der Waals fluid, we construct the exact form of the metric for a (2+1) dimensional AdS black hole. The thermodynamic properties of this black hole are consistent with those of the Anyon Van der Waals fluid. For $\alpha < \alpha_c$, the solution exhibits a quasi Bose-Einstein statistics. For $\alpha > \alpha_c$, there is, however, no acceptable event horizon so there is no black hole solution. Thus, the AdS Anyon Van der Waals black holes have only quasi Bose-Einstein statistics.

I. INTRODUCTION

The physics of the black holes has always been one of the most interesting research areas since its appearance as a field [1–7]. It is known that there is an analogy between an AdS black hole and the Van der Waals fluid [8]. The correspondence between the two is important because the thermodynamic behaviour of black holes can be explained by that of the fluid; the conventional thermodynamic phase space (including temperature, entropy, and volume) can also be defined for an AdS black hole. This analogy will be more complete in an extended phase space. The past few years has witnessed an interest in the study of the cosmological constant ($\Lambda$) as a thermodynamic parameter in the first law of thermodynamics [8–13]. Although this assumption seems awkward, there is good reasons why $\Lambda$ should be considered in the first law of thermodynamics. First, including the cosmological constant $\Lambda$ in the the first law of thermodynamics will make it consistent with smarr relation[12] and the vari- cal constant $\Lambda$ in the first law of thermodynamics will be defined in terms of its energy scale.

Once we introduce the cosmological constant as a ther- modynamic parameter, we can define its conjugate variable. Since $\Lambda$ is proportional to the thermodynamic pressure ($P = - \frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}$ using geometric units $G_N = h = c = k = 1$ ), its conjugate must have volume dimensions. This definition will give rise to an additional term, $P\delta V$, in the first law of thermodynamics and the mass of the black hole will be defined in terms of its enthalpy. In this extended phase space, one can write the equation of state for the AdS black hole and compare it with the equation of state for the Van der Waals fluid that reads as follows:

$$T = (P + \frac{a}{v^2})(v - b),$$

where, $P$ is the thermodynamic pressure, $v$ is the specific volume of the fluid $v = V/N$, and $N$ is the degree of freedom ($V$ is the conjugate volume for $P$). In Ref. [14], the authors derived an exact form for the metric of an AdS black hole which has the same thermodynamics as the Van der Waals fluid. In this work, we construct an Anyon Van der Waals fluid whose thermodynamics is exactly consistent with that of an AdS Anyon Van der Waals black hole. What they have in common is that the particles are considered to be either fermions or bosons which obey the Fermi-Dirac or the Bose-Einstein statistics, respectively. For a fluid of the latter type, there can be a Bose-Einstein condensation (For fermions there is no condensation because of Pauli’s exclusion principle). But in the two spatial dimension (2+1) space time, we may have an intermediate statistics [15–20]. The particles that have this intermediate statistics are called Anyons. It is notable that fermions and bosons are the two limits of the Anyons. Thus, we can use a real number $\alpha$ ($0 < \alpha < 1$) to parameterize the intermediate statistics of the Anyons with $\alpha = 0$ corresponding to bosons (the particles that can have the Bose-Einstein condensation), $\alpha = 1$ corresponding to fermions (the particles which obey the Pauli exclusion principle), and $0 < \alpha < 1$ parameterizing the intermediate statistics of the Anyons. Here, we are going to construct a metric for a (2+1) dimensional black hole with statistics consistent with the intermediate statistics of the Anyon fluid. The results show that the Anyon Van der Waals fluid has a quasi Fermi-Dirac statistics for $\alpha_c < \alpha < 1$ and that the AdS Anyon Van der Waals fluid has a quasi Bose-Einstein statistics for $0 < \alpha < \alpha_c$. In the former case, however, there will be no black hole solution. Thus, for $\alpha_c < \alpha < 1$, it is not possible to describe an AdS Anyon Van der Waals black hole by means of an Anyon Van der Waals fluid. The interesting consequence of our work is that AdS Anyon Van der Waals black holes can be ex-

* m.aghaei@ph.iut.ac.ir
† b.mirza@cc.iut.ac.ir
pressed only for \(0 < \alpha < \alpha_c\) and that they have a quasi Bose-Einstein statistics.

The paper is organized as follows:

In Sec. II, we present a review of the AdS Van der Waals black hole and the equations for both its energy density and pressure. In Sec. III, the equation of state for the Anyons is introduced. In Sec. IV, the exact form of the metric that is consistent with the AdS Anyon Van der Waals fluid is obtained, the equations of the energy density and the pressure of the black hole are derived, and the behaviour of the energy density and the pressure are analysed. It is interesting that there are black hole solutions that only correspond to the semi Bose Einstein statistics. We present the results and conclusions in Sec. V.

II. VAN DER WAALS BLACK HOLE

In [8], the authors constructed a metric for an AdS black hole that has a similar thermodynamic behavior to that of the Van der Waals fluid and checked the validity of energy conditions for this black hole. This metric construction is based on the AdS Black hole similarity to Van der Waals fluid together with the assumption that the cosmological constant is a thermodynamic variable. In this extended phase space, the relation \(P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}\) hold between the thermodynamic pressure and the cosmological constant \(\Lambda\). By assuming this equation to be true, we should identify the conjugate variable for the pressure proportional to \(\Lambda\); obviously, the natural choice is volume. So, the equation for the mass of the black hole should be modified from \(\delta M = T\delta S\) to:

\[
\delta M = T\delta S + V\delta P + ..., \quad (2)
\]

This is why, in an extended phase space for the AdS black hole, the mass of the black hole is related to its enthalpy [22]. Using Eq. (2), one can see that the thermodynamic volume \(V\) can be obtained from:

\[
V = \left(\frac{\partial M}{\partial P}\right)_{s,...}. \quad (3)
\]

Now that we have the metric of the black hole, we can write the equation of state for the black hole as \(P = P(V,T)\) and compare it with that for the fluid. For simplicity, one can assume the metric to be:

\[
ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \quad (4)
\]

\[
f = \frac{r^2}{L^2} - \frac{2M}{r} - h(r,P),
\]

where, \(M\) is the mass of the black hole. We assume this metric to be a solution for the Einstein field equation \(G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}\). The energy momentum tensor is defined in an orthonormal basis by \(T^{\mu\nu} = \rho e^\mu_0 e^\nu_0 + \sum_i p_i e^\mu_i e^\nu_i\), where \(\rho\) is the black hole energy density and \(p\) is its pressure. So, the thermodynamic pressure and the energy density of the black hole can be calculated by using the metric in Eq. (4):

\[
\rho = -p_1 = \frac{1 - f - rf'}{8\pi r^2} + P \quad (5)
\]

\[
p_2 = p_3 = \frac{rf'' + 2f'}{16\pi r} - P, \quad (6)
\]

with the prime denoting the derivative with respect to \(r\). One should define the function \(f\) such that the equation of state for the black hole is consistent with that of the Van der Waals fluid. The specific volume and temperature of the black hole are defined as functions of the black hole horizon and the thermodynamic pressure,

\[
v = \frac{k}{4\pi r^2} \left[\frac{4\pi r^3 - r_+^2 \frac{\partial h(r_+,P)}{\partial P}}{2}\right] \quad (7)
\]

\[
T = \frac{f'}{4\pi} = 2r_+P - \frac{h(r_+,P)}{4\pi} - \frac{1}{4\pi} \frac{\partial h(r_+,P)}{\partial r_+}, \quad (8)
\]

where, for a \(d\) space time dimension, \(k = \frac{4(4-1)}{\pi}\), \(v = k\frac{V}{N}\) and \(N = \frac{1}{\pi r^2}\). Since we expect the equation of state for the AdS black hole to be consistent with that of the Van der Waals fluid, we should compare the equation of state obtained from Eqs. (7) and (8) with that of the Van der Waals fluid. The direct relation between the specific volume and pressure of Van der Waals fluid and the temperature of the black hole will be obtained by combining Eqs. (7),(8), and (1):

\[
2Pr_+ - \frac{h}{4\pi r_+} - \frac{h'}{4\pi} \left( P + \frac{a}{r^2} \right) (v - b) = 0. \quad (9)
\]

By setting \(h(r,P) = A(r) - PB(r)\), one can find an solution for Eq. (9). This leads to an equation in the form of \(F_1(r)P + F_2(r) = 0\) in which \(F_1\) and \(F_2\) are the functions of \(A\) and \(B\) and their derivatives. By setting \(F_1(r) = 0\) and \(F_2(r) = 0\) separately, one can derive the solution for \(h(r,P)\); hence, we will have the solutions for the energy density and pressure of the black hole.

In this work, we are going to construct new types of black holes whose statistics completely matches that of the Anyon Van der Waals fluid. Anyons are \((2+1)\) dimensional particles with a statistic that interpolates between Bose-Einstein and Fermi-Dirac statistics. As mentioned, the parameter \(\alpha\) is used to identify these particles so that we expect the Anyons to obey the Pauli exclusion principle for \(\alpha_c < \alpha < 1\) and to have a Bose-Einstein condensation for \(0 < \alpha < \alpha_c\).
III. EQUATION OF STATE FOR ANYONS

In the four dimensional space time, we have only Fermi-Dirac and Bose-Einstein statistics. But in the (2+1) dimensional one, there can be particles with intermediate statistics. For these particles, the statistics interpolate between bosons and fermions. These particles with intermediate statistics are called Anyons. In the following, we will introduce the equation of state for Anyons. Generalizing the equation of state for bosons and fermions into the intermediate form will yield the equation of state for Anyons.

The number of quantum states for N bosons or fermions which occupy G state is:

\[ W_b = \frac{(G + N - 1)!}{N!(G - 1)!}, \quad W_f = \frac{G!}{N!(G - N)!}, \]  

where, \( G \) is the thermodynamic pressure, \( P \) and \( E \) are the chemical potential and the temperature, respectively. By some straightforward calculation, we will have:

\[ N = \sum N_i \quad (12) \]
\[ E = \sum \varepsilon_i N_i \quad (13) \]

With \( \varepsilon_i \) being the energy of each one of the \( N_i \) particles, the partition function for the grand canonical ensemble will be given by:

\[ Z = \sum_{N_i} W(N_i) \exp \left( \sum_i N_i (\mu_i - \varepsilon_i)/kT \right) \]  

where, \( \mu \) and \( T \) are the chemical potential and the temperature, respectively. By some straightforward calculations and assuming that \( E = PV \), we can obtain the equation of state for Anyons as follows [17, 23]:

\[ PV = NkT \left[ 1 + (2\alpha - 1)N\lambda^2/4V \right], \]  

(15)

where, \( a \) is a positive constant representing the attraction between the particles of the Van der Waals fluid and \( b \) is the volume of a single molecule \( (a = 0, b = 0 \) corresponds to the ideal fluid). In this case, we can find the value of \( \alpha = \alpha_c = 1/2 \) using the calculation similar to that in [15]. \( \alpha_c \) is the boundary between the quasi Bose-Einstein and quasi Fermi-Dirac statistics of the fluid so that, for \( 0 < \alpha < \alpha_c \), the fluid has a quasi Bose-Einstein statistics, whereas for \( \alpha_c < \alpha < 1 \), Pauli’s exclusion principle has the leading role. So the fluid has a quasi Fermi-Dirac statistics.

In the following Section, we are going to study an AdS black hole whose thermodynamics corresponds to Eq. (16).

IV. ANYON VAN DER WAALS BLACK HOLE

The thermodynamic properties of the AdS black holes have been studied extensively [24–27]. We will construct a metric for a (2+1) dimensional AdS black hole whose thermodynamic properties are similar to those of an Anyon Van der Waals fluid [17]. The general metric for a (2+1) dimensional black hole will be [28]:

\[ ds^2 = -f dr^2 + \frac{dr^2}{f} + r^2 d\phi^2 \]  

(17)

where, the unknown function \( h(r, \phi) \) should be obtained in such a way that the thermodynamic properties of this black hole will be consistent with those of the Anyon Van der Waals fluid. Let us again assume this metric to be a solution for the Einstein field equation \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \) in (2+1) space-time. Using the energy momentum tensor already defined in Sec. II, the energy density and the pressure of the black hole are derived as follows:

\[ \rho = -p_1 = -\frac{\rho'}{16\pi} + P \]  

(18)

\[ p_2 = -\frac{\rho''}{16\pi} - P, \]  

(19)

and for the 2+1 dimensional space-time, \( P = -\frac{\Lambda}{8\pi} \) [29]. Furthermore, using the Einstein field equation and the
energy momentum tensor definition will yield the energy conditions as follows:

\[
\begin{align*}
\text{Weak} : & \quad \rho \geq 0 , \ \rho + p_r \geq 0 \quad (20) \\
\text{Strong} : & \quad \rho + \sum_i p_i \geq 0 \quad (21) \\
\text{Dominant} : & \quad \rho \geq |p_i| \quad (22)
\end{align*}
\]

The validity of these energy conditions can be ensured by calculating the exact function for \( h(r, P) \). Now, to calculate the function \( h(r, P) \) the temperature of the black hole and the temperature of the Anyon Van der Waals fluid are assumed to be equal. We can, thus, find a \( 2+1 \) dimensional black hole whose thermodynamics is exactly similar to that of an Anyon Van der Waals fluid. By assuming that \( h(r, P) = A(r) - PB(r) \) and using the equality between Eq. (8) and the temperature obtained form Eq. (15), we will have:

\[
- \frac{2akm(v - b) + v(2kmPv(v - b) + (\pi - 2\pi\alpha)h)}{2kmv^2} - \frac{A'(r)}{4\pi} + \frac{PB'(r)}{4\pi} + 4Pr = 0,
\]

where, \( v = \frac{4(B(r) + 8\pi r^2)}{\sqrt{4\pi}} \). The above equation is a differential one in the form \( F_1(r) + PF_2(r) = 0 \) in which the functions \( F_1 \) and \( F_2 \) depend on the functions \( A \) and \( B \) and their first derivatives. The functions \( F_1(r) \) and \( F_2(r) \) should vanish independently so that two differential equations will remain to be solved for \( A(r) \) and \( B(r) \) (By setting \( k = 8 \)):

\[
\begin{align*}
A(r) &= \frac{15r(-\pi akm + 16\pi^2\alpha h - 8\pi^2 h)}{64bk} + C_1 \quad (24) \\
B(r) &= \frac{4}{15}\pi r(b - 30r) + C_2 r^{16}. \quad (25)
\end{align*}
\]

\( C_1 \) and \( C_2 \) are integration constants and can be taken to be equal to zero. Thus, the function \( f(r, P) \) will take the following form:

\[
f(r, P) = -\left( \frac{15r(-\pi akm + 16\pi^2\alpha h - 8\pi^2 h)}{64bk} - \frac{1}{15}P(4\pi r(b - 30r)) \right) - M + 8\pi Pr^2. \quad (26)
\]

Eq. (18), the behaviour of \( \rho \) (the energy density of the black hole) with respect to \( r \) for different values of \( \alpha \) is depicted in Fig. 1. As shown, for \( \alpha < \alpha_c = \frac{1}{2} \), energy density decreases with decreasing \( \alpha \). From Eq. (16), the quasi-Bose-Einstein statistics is expected for \( \alpha < \alpha_c \). In our case, we have used \( E = PV \), which is the equation of state for a non-relativistic fluid and the Bose-Einstein condensation is not expected for the fluid because the non-relativistic \( 2+1 \) dimensional fluid cannot have this condensation [30].

The black hole horizon is where the function \( f(r) \) vanishes. Fig. 2 depicts the evolution of the function \( f(r) \) with respect to \( r \). It is interesting that for \( \alpha > \alpha_c \) where the fluid has a Fermi-like behaviour, there is no horizon, and, therefore, no black hole solution (Fig. 2). So, our method implies that black holes only exhibit a Bose-Like behaviour in \( 2+1 \) dimensions. The important point is that, for the energy conditions to be satisfied in \( 2+1 \) dimension, the value of \( \alpha \) in which the energy density is positive outside the horizon can be determined based on the values of other parameters. For the values of \( C_1, C_2, P, a \) and \( b \) here, the interval for the \( \alpha \) values would be between 0.3 and 0.5.
We may also study this statistical property of black holes in higher dimensions. In the 2+1 dimension, the particles have intrinsic properties of Anyons, but in a 3+1 dimension space-time, the particles may have an effective Haldane statistics as an effective intermediate one. So, the statistics of the collective particles may be checked. The energy density of the Black hole with an intermediate statistics in 3+1 dimensions is depicted in Fig. (3). It may be noted that the fluid can be in the Bose-Einstein condensation state in 3+1 dimensions. As seen in Fig. (3), in the 3+1 dimension and for $\alpha < 0.5$, the behaviour of the fluid is Bose-like while it is Fermi-like for $\alpha > 0.5$. The important difference in this case is that there are black hole solutions (horizons) in the 3+1 dimensions for all values of $\alpha$. For the energy conditions to be satisfied, the energy density and the pressure of the black hole need to be checked as in the previous case. With $C_1 = C_2 = 0$, $a = \frac{1}{2\pi}$, $b = 1$, $M = 10$, $P = 0.1$, $\alpha$ must be between 0.3 and 0.5.

Application of the Eq. (23) to the ideal fluid for which $a = b = 0$ yields the same results and there would only exist quasi Bosonic AdS Anyon Van der Waals Black holes.

V. CONCLUSION

In the three spatial dimensions, the metric of the space-time can be assumed to be in the form of Eq. (4). One can then obtain the exact form of the function $f(r)$ in such a way that the metric can describe the Van der Waals black hole [14]. This allows the functions of the energy density and pressure of these black holes to be calculated and their behaviour to be studied. For (2+1) dimensional AdS black holes, the general form of the metric should be considered in 2 spatial dimensions as in Eq. (17), in which there can be particles with intermediate statistics. The two boundaries of the intermediate statistics are bosons ($\alpha = 0$) and fermions ($\alpha = 1$). Thus, $0 < \alpha < 1$ for the statistics interpolating between these two cases. The particles that have this intermediate statistics are called Anyons and they can exist in two spatial dimensions. Therefore, the (2+1) dimensional AdS black holes consist of an Anyon fluid.

We developed an exact form of the metric for the (2+1) dimensional AdS Anyon Black holes in which the Anyons obey the equation of state for the Van der Waals fluid. The equation of state was introduced in Eq. (16) for an Anyon Van der Waals fluid. The metric in Eq. (17) as well as the descriptions of the temperature and the volume of the black hole were then used to obtain the exact form of the metric. Based on Eqs. (5) and (19), the exact form of $f(r)$ yields enough information not only to study the behaviour of the energy density and pressure of the black hole but also to investigate the existence of a horizon for the Black hole.

Results show that for $M = 10$, $a = \frac{1}{2\pi}$, $b = 0.1$ and $0 < \alpha < \alpha_c = 0.5$, energy density decreases with decreasing value of $\alpha$ and we expect the fluid to have a quasi Bose-Einstein statistics of $\alpha$. For $\alpha_c = 0.5 < \alpha < 1$, we expect a Fermi-Dirac statistics for the fluid and the energy density of the black holes to increase with increasing value of $\alpha$ (Fig. 1). As shown in Fig. 2, for $\alpha_c = 0.5 < \alpha < 1$, there is no horizon for the Black hole assumed. This implies that there will be no real black hole in this situation. Thus, it may be concluded that in the case studied here and for $M = 10$, $a = \frac{1}{2\pi}$, $b = 0.1$ there will be a (2+1) dimensional AdS Anyon Van der Waals black hole with a behavior similar to the Bose-Einstein statistics only for $\alpha > \alpha_c = 0.5$.

FIG. 2. The function $f(r)$ is plotted for $\alpha = 0.2$ (Red), $\alpha = 0.4$ (Blue), $\alpha = 0.6$ (Green dashed) and $\alpha = 0.8$ (purple dashed) for $M = 10$, $a = \frac{1}{2\pi}$, $b = 0.1$ and $P = 10$. The horizon of the black hole is where the function $f(r)$ vanishes ($f(r) = 0$). It is seen that there exists no acceptable horizon and, thereby, no black hole for $\alpha > \alpha_c = 0.5$. For $\alpha < 0.5$, however, we can determine the position of the event horizon.

$\alpha < 0.5$ corresponds to the era when the fluid has a quasi Bose-Einstein statistics. In this condition, the energy density of the black hole increases with $\alpha$.

FIG. 3. Pressure of the anyon Van der Waals black hole for 3+1 dimensions in the cases of $\alpha = 0.2$ (Red), $\alpha = 0.3$ (Blue), $\alpha = 0.5$ and $\alpha = 0.7$ (Green) for $M = 10$, $b = 1$, $a = \frac{1}{\pi}$ and $P = 0.1$. 

$\alpha = 0.7$ and $\alpha = 0.7$ (Green) for $M = 10$, $b = 1$, $a = \frac{1}{\pi}$ and $P = 0.1$. 

$\alpha = 0.3$ (Red), $\alpha = 0.5$ and $\alpha = 0.7$ (Green) for $M = 10$, $b = 1$, $a = \frac{1}{\pi}$ and $P = 0.1$. 

$\alpha = 0.7$ and $\alpha = 0.7$ (Green) for $M = 10$, $b = 1$, $a = \frac{1}{\pi}$ and $P = 0.1$.
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