Writing representations over minimal fields

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Abstract. The chief aim of this paper is to describe a procedure which, given a $d$-dimensional absolutely irreducible matrix representation of a finite group over a finite field $E$, produces an equivalent representation such that all matrix entries lie in a subfield $F$ of $E$ which is as small as possible. The algorithm relies on a matrix version of Hilbert’s Theorem 90, and is probabilistic with expected running time $O(|E:F|d^3)$ when $|F|$ is bounded. Using similar methods we then describe an algorithm which takes as input a prime number and a power-conjugate presentation for a finite soluble group, and as output produces a full set of absolutely irreducible representations of the group over fields whose characteristic is the specified prime, each representation being written over its minimal field.

1. The main algorithm

Let $\rho: G \to \text{GL}(d, E)$ be an absolutely irreducible representation of the group $G$. It is clear that there exists a subfield $F$ of $E$, minimal with respect to inclusion, such that there exists a representation $G \to \text{GL}(d, F)$ equivalent to $\rho$. If $E$ has nonzero characteristic, then $F$ is determined by $\rho$, and coincides with the subfield generated by the character values of $\rho$ (see [2, VII Theorem 1.17]). Indeed, the arguments presented here yield a proof of this fact. If $E$ has characteristic zero, there may be more than one choice for $F$.

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Suppose that $\mathbb{F}$ is a subfield of $\mathbb{E}$ such that $\mathbb{E}$ is a finite Galois extension of $\mathbb{F}$ whose Galois group is cyclic, of order $t$, and generated by $\alpha$. Assume further that the norm map from $\mathbb{E}$ to $\mathbb{F}$ (given by $\lambda \mapsto \lambda^{\alpha} \lambda^{\alpha^2} \cdots \lambda^{\alpha^{t-1}}$) is surjective. This hypothesis certainly holds if $|\mathbb{E}|$ is finite, and this is the case of principal interest to us. Our first objective is to describe a procedure which determines whether an absolutely irreducible representation $\rho: G \to \text{GL}(d, \mathbb{E})$ of a finitely generated group $G$ is equivalent to a representation $G \to \text{GL}(d, \mathbb{F})$, and if so, finds an $A \in \text{GL}(d, \mathbb{E})$ such that $A^{-1} \rho(g) A \in \text{GL}(d, \mathbb{F})$ for all $g \in G$. Note that if $g_1, g_2, \ldots, g_n$ generate $G$, this condition is equivalent to $A^{-1} \rho(g_i) A \in \text{GL}(d, \mathbb{F})$ for all $i \in \{1, 2, \ldots, n\}$.

A basic step in our algorithm involves testing whether two given matrix representations of $G$ are equivalent, and if they are, finding a nonsingular intertwining matrix. The naive approach to this problem involves solving $nd^2$ homogeneous linear equations in $d^2$ unknowns over the field $\mathbb{E}$. Computationally, this has cost $O(nd^6)$. Alternatively, there is a probabilistic algorithm, described by Holt and Rees in [1], which has expected running time $O(d^3)$. (This complexity result, and those throughout this section, assume that the cost of field arithmetic, including applying a field automorphism, is $O(1)$.)

With the notation as above, suppose that $A \in \text{GL}(d, \mathbb{E})$ has the property that $A^{-1} \rho(g) A \in \text{GL}(d, \mathbb{F})$ for all $g \in G$. The automorphism $\alpha$ of $\mathbb{E}$ gives rise to an automorphism of $\text{Mat}(d, \mathbb{E})$ (the algebra of $d \times d$ matrices over $\mathbb{E}$) which we also denote by $\alpha$. Since the fixed subfield of $\alpha$ is $\mathbb{F}$, it is clear that $B \in \text{Mat}(d, \mathbb{E})$ satisfies $B^\alpha = B$ if and only if $B \in \text{Mat}(d, \mathbb{F})$. So $(A^{-1} \rho(g) A)^\alpha = A^{-1} \rho(g) A$ for all $g \in G$, and thus $C = A(A^\alpha)^{-1}$ satisfies

$$C^{-1} \rho(g) C = \rho(g)^\alpha \quad \text{(for all } g \in G).$$  

Since $\rho$ is absolutely irreducible, equation (1) determines $C$ up to a nonzero scalar multiple. The first step in our procedure is, therefore, to use an algorithm such as in [1] to find (if possible) a $C \in \text{GL}(d, \mathbb{E})$ satisfying (1). If no such $C$ exists, then $\rho$ cannot be written over $\mathbb{F}$; so assume henceforth that such a $C$ has been found.

**Proposition (1.1).** If $C \in \text{GL}(d, \mathbb{E})$ satisfies (1), then $CC^\alpha C^{\alpha^2} \cdots C^{\alpha^{t-1}}$ equals $\mu I$ where $\mu \in \mathbb{F}$ and $I$ is the $d \times d$ identity matrix.
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Proof. Since $CC^\alpha C^{\alpha^2} \cdots C^{\alpha^{t-1}}$ conjugates $\rho(g)$ to $\rho(g)^{\alpha^i} = \rho(g)$ for all $g \in G$, it must equal $\mu I$ for some $\mu \in E$, since $\rho$ is assumed to be absolutely irreducible. However,

$$\mu^\alpha I = C(\mu I)^\alpha C^{-1} = C(C^\alpha C^{\alpha^2} \cdots C^{\alpha^{t-1}})C^{-1} = CC^\alpha C^{\alpha^2} \cdots C^{\alpha^{t-1}} = \mu I,$$

and so $\mu \in \mathbb{F}$, as desired. \qed

The computation of $\mu$ can be effected by $t - 1$ vector by matrix multiplications, since if $v$ is the first row of $C$ then $\mu$ is the first component of the row vector $vC^\alpha C^{\alpha^2} \cdots C^{\alpha^{t-1}}$. This has cost $O(td^2)$. If $t$ is large compared with $d$, then $\mu$ may be computed at cost $O((\log t)d^3)$ by using the fact that $C_{2i} = C_i(C_i)^{\alpha^i}$ for each $i$, where $C_i = CC^\alpha \cdots C^{\alpha^{i-1}}$.

Since the norm map from $E$ to $\mathbb{F}$ is assumed to be surjective, there exists a $\nu \in E$ whose norm is $\mu$. We do not address here the practical problem of finding $\nu$ given $\mu$. The methods used for storing field elements and performing field computations obviously affect this issue. (When $|E|$ is bounded, there is an $O(1)$ probabilistic algorithm for computing $\nu$.) Once $\nu$ has been found we may replace $C$ by $\nu^{-1}C$, and assume thereafter that $CC^\alpha \cdots C^{\alpha^{t-1}} = I$.

Lemma (1.2). If $C \in \text{GL}(d, E)$ satisfies $CC^\alpha \cdots C^{\alpha^{t-1}} = I$, then there exists a nonzero column vector $v \in E^d$ such that $Cv^\alpha = v$.

Proof. Let $u_0 \in E^d$ be nonzero, and for $i > 0$ define $u_i = Cu_{i-1}^\alpha$. Observe that $u_t = u_0$. Now since the field automorphisms $\alpha^0, \alpha^1, \ldots, \alpha^{t-1}$ are distinct they are linearly independent, and since the $u_i$ are nonzero it follows that there exists a $\lambda \in E$ such that $v = \sum_{i=0}^{t-1} \lambda^\alpha u_i \neq 0$. Moreover, $Cv^\alpha = \sum_{i=1}^{t} \lambda^\alpha Cu_{i-1}^\alpha = v$, as desired. \qed

The following proposition may be viewed as a generalization of the multiplicative form of Hilbert’s Theorem 90. The corresponding generalization of the additive form is trivially true.

Proposition (1.3). If $C \in \text{GL}(d, E)$ satisfies $CC^\alpha \cdots C^{\alpha^{t-1}} = I$, then there exists an $A \in \text{GL}(d, E)$ with $C = A(A^\alpha)^{-1}$.

Proof. The result is true when $d = 1$ by the multiplicative form of Hilbert’s Theorem 90. Proceeding by induction, assume that $d > 1$. By Lemma (1.2)
there exists a nonzero vector \( v \) such that \( Cv^\alpha = v \), and if \( B \) is an invertible matrix with \( v \) as its first column then

\[
B^{-1}CB^\alpha = \begin{pmatrix} 1 & u \\ 0 & C_1 \end{pmatrix}
\]

where \( C_1 \in \text{GL}(d - 1, \mathbb{E}) \) satisfies \( C_1C_1^\alpha \cdots C_1^{\alpha^{d - 1}} = I \). By the inductive hypothesis, there exists an \( A_1 \in \text{GL}(d - 1, \mathbb{E}) \) such that \( C_1 = A_1(A_1^\alpha)^{-1} \), and it follows that

\[
\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}^{-1} B^{-1}CB^\alpha \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}^\alpha = \begin{pmatrix} 1 & u_1 \\ 0 & I \end{pmatrix}
\]

where \( u_1 = u(A_1^{-1})^\alpha \) satisfies \( \sum_{i=0}^{d-1} u_1^\alpha^i = 0 \). It follows from the additive form of Hilbert’s Theorem 90 that there exists a row vector \( u_2 \) with \( u_1 = u_2 - u_2^\alpha \), and then

\[
A = B \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & I \end{pmatrix}
\]

has the required property \( C = A(A^\alpha)^{-1} \).

Note that if \( C = A(A^\alpha)^{-1} \) then the map \( \text{Mat}(d, \mathbb{E}) \rightarrow \text{Mat}(d, \mathbb{E}) \) given by

\[
X \mapsto X + CX^\alpha + CC^\alpha X^\alpha^2 + \cdots + CC^\alpha \cdots C^{\alpha^{d-2}} X^\alpha^{d-1}
= A(A^{-1}X + (A^{-1}X)^\alpha + \cdots + (A^{-1}X)^{\alpha^{d-1}})
\]

has image consisting of all matrices of the form \( AY \) with \( Y \in \text{Mat}(d, \mathbb{E}) \). These are exactly the matrices \( A' \in \text{Mat}(d, \mathbb{E}) \) such that \( (A^{-1}A')^\alpha = A^{-1}A' \), or equivalently, \( C(A')^\alpha = A' \). If \( X \) is chosen arbitrarily and \( X \mapsto AY = A' \), then the probability that \( Y \) is invertible (so that \( C = A'((A')^{-1}) \) is \( |\text{GL}(d, \mathbb{E})|/|\text{Mat}(d, \mathbb{E})| \). It follows that a reasonable procedure for finding an \( A \) satisfying the equation \( C = A(A^\alpha)^{-1} \) is to choose \( X \in \text{Mat}(d, \mathbb{E}) \) randomly and compute \( A = X + CX^\alpha + CC^\alpha X^\alpha^2 + \cdots + CC^\alpha \cdots C^{\alpha^{d-2}} X^\alpha^{d-1} \), repeating if necessary until an invertible \( A \) is found. (One may show that 

\[
1 - |F|^{-1} \geq |\text{GL}(d, \mathbb{F})|/|\text{Mat}(d, \mathbb{F})| > 1 - |F|^{-1} - |F|^{-2} \geq 1/4.
\]

Observe that \( C = A(A^\alpha)^{-1} \) combines with equation (1) to give

\[
A^{-1} \rho(g)A = (A^{-1} \rho(g)A)^\alpha \quad (\text{for all } g \in G).
\]
It follows that $A^{-1}\rho(g)A \in \text{GL}(d, \mathbb{F})$ for each $g$, and we have achieved our goal of constructing a representation equivalent to $\rho$ with image contained in $\text{GL}(d, \mathbb{F})$. Note that if $A_i = X + CX^\alpha + CC^\alpha X^{\alpha^2} + \cdots + CC^\alpha \cdots C^{\alpha^i-2} X^{\alpha^i-1}$ then $A_{i+1} = X + CA_i^\alpha$, and it follows that $A_t$ can be evaluated with $t - 1$ matrix multiplications and $t - 1$ matrix additions. It can be seen, therefore, that our procedure has expected running time $O(|E : \mathbb{F}|d^3)$.

2. Absolutely irreducible representations of soluble groups

Suppose that we are given a consistent power-conjugate presentation for a finite group $G$. That is, $G$ is generated by $g_1, g_2, \ldots, g_n$, where $n$ is the composition length of $G$, with defining relations

\begin{align*}
g_i^{p_i} &= v_i & (1 \leq i \leq n) \\
g_i^{-1}g_jg_i &= w_{ij} & (1 \leq i < j \leq n)
\end{align*}

where each $p_i$ is a prime and each $v_i$ is a word in the generators $g_j$ for $i < j \leq n$, and each $w_{ij}$ is a word in the $g_k$ for $i < k \leq n$. It is clear that a group has such a presentation if and only if it is finite and soluble. Specifically, if $G_i$ is the subgroup of $G$ generated by $g_i, g_{i+1}, \ldots, g_n$, then

\[ (\ast) \quad G = G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} = \{1\} \]

is a subnormal series, and for each $i$ the quotient $G_i/G_{i+1}$ has order dividing $p_i$. Given that $n$ is the composition length of $G$, it follows that $(\ast)$ is a composition series and the order of $G_i/G_{i+1}$ is exactly $p_i$. We will show how the natural algorithm for constructing the absolutely irreducible representations of $G$ (in a fixed nonzero characteristic), by working up the composition series $(\ast)$, can be readily adapted to ensure that each representation is written over its minimal field. We consider that we have constructed a representation of the group $G_i$ once we have computed matrices representing the generators $g_i, g_{i+1}, \ldots, g_n$.

For ease of exposition we let $\mathbb{K}$ be a fixed algebraic closure of a field of prime order, and deal henceforth only with subfields of $\mathbb{K}$. Assume, inductively, that we have constructed representations $\sigma_1, \sigma_2, \ldots, \sigma_s$ of the group
such that

(i) each $\sigma_i$ is absolutely irreducible and written over its (unique) minimal subfield of $K$, and

(ii) every absolutely irreducible representation of $G_2$ over $K$ is equivalent to exactly one of the $\sigma_i$.

Henceforth, to simplify the notation, we write $H = G_2$, $a = g_1$ and $p = p_1$.

The absolutely irreducible $K$-representations of $H$ are permuted by $G$ via

$$\sigma^g(h) = \sigma(ghg^{-1})$$

for all $h \in H$ and $g \in G$. The first step is to find, for each $i$, which of the representations $\sigma_1, \sigma_2, \ldots, \sigma_s$ is equivalent to the representation $\sigma_a^i$. If $\sigma_a^i$ is equivalent to $\sigma_i$, then there exists a representation of $G$ extending $\sigma_i$; the minimal field for any such extension will be an extension of the field of $\sigma_i$. If $\sigma_a^i$ is not equivalent to $\sigma_i$, then $\sigma_i$ will be $G$-conjugate to $p = |G : H|$ of the representations $\sigma_k$. In this case the representation of $G$ induced from $\sigma_i$ is absolutely irreducible; however, its minimal field may be smaller than that of $\sigma_i$. Since $G$-conjugate representations of $H$ yield equivalent induced representations of $G$, one representative only should be chosen from each $G$-conjugacy class.

Case 1. Assume that $E$ is a finite field, and $\sigma: H \to GL(d, E)$ is an absolutely irreducible representation, with minimal field $E$, such that $\sigma^a$ is equivalent to $\sigma$.

Compute a matrix $A \in GL(d, E)$ such that $A\sigma(h)A^{-1} = \sigma(aha^{-1})$ for all $h \in H$. As $\sigma$ is absolutely irreducible and $a^p \in H$, so $A^p = \mu \sigma(a^p)$ for some $\mu$ in $E^\times$ (the multiplicative group of $E$). If the characteristic of $E$ equals $p$, then $\mu$ has a unique $p$th root $\nu \in E^\times$. Indeed, $\nu$ is a power of $\mu$ since $p$ is coprime to $|E^\times|$. In this case there is a unique representation $\rho$ of $G$ extending $\sigma$, given by $\rho(a) = \nu^{-1}A$ and $\rho(h) = \sigma(h)$ for all $h \in H$. Suppose alternatively that the characteristic of $E$ is not $p$. In this case $\nu^p = \mu$ has exactly $p$ solutions $\nu_1, \ldots, \nu_p$ in $K$, and correspondingly there are $p$ pairwise inequivalent extensions $\rho_1, \ldots, \rho_p$ of $\sigma$ given by defining $\rho_i(a) = \nu_i^{-1}A$. For each $i$, the extension field $E(\nu_i)$ is the minimal field for $\rho_i$. If $|E^\times|$ is coprime to $p$, then one of the solutions of $\nu^p = \mu$ lies in the field $E$, while the remaining $p - 1$ solutions generate the same field, which is the smallest extension of
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\( \mathbb{E} \) whose order is congruent to 1 modulo \( p \). If \( |\mathbb{E}^\times| \) is a multiple of \( p \), then all solutions of \( \nu^p = \mu \) generate the same extension \( \mathbb{E}' \) of \( \mathbb{E} \). Note that \( |\mathbb{E}' : \mathbb{E}| \) is 1 or \( p \), and \( \mathbb{E}' \) is the smallest extension of \( \mathbb{E} \) whose order is congruent to 1 modulo \( p|\nu| \).

**Case 2.** Assume that \( \mathbb{E} \) is a finite field, and \( \sigma : H \to GL(d, \mathbb{E}) \) is an absolutely irreducible representation, with minimal field \( \mathbb{E} \), such that \( \sigma^\alpha \) is not equivalent to \( \sigma \).

Let \( k \) be the degree of \( \mathbb{E} \) over its prime subfield. If \( k \) is not a multiple of \( p \), then \( \mathbb{E} \) is the minimal field for the induced representation \( \sigma^G \). If \( k \) is a multiple of \( p \), then \( \mathbb{E} \) has an automorphism \( \alpha \) of order \( p \) whose fixed subfield, \( \mathbb{F} \), is uniquely defined by \( |\mathbb{E} : \mathbb{F}| = p \). In this case, if the representation \( \sigma^\alpha : h \mapsto \sigma(h)^\alpha \) is not equivalent to one of the \( G \)-conjugates of \( \sigma \), then \( \mathbb{E} \) is still the minimal field for \( \sigma^G \); however, if \( \sigma^\alpha \) is equivalent to a \( G \)-conjugate of \( \sigma \) then one can readily show that \( \sigma^G \) is equivalent to \((\sigma^G)^\alpha \), and so the minimal field of \( \sigma^G \) is \( \mathbb{F} \).

We present an explicit construction for an \( \mathbb{F} \)-representation equivalent to \( \sigma^G \) in the case that \( \sigma^\alpha \) is equivalent to a \( G \)-conjugate of \( \sigma \). Replacing \( \alpha \) by a power of itself, we may assume that \( \sigma^\alpha \) is equivalent to \( \sigma^a \). Find an \( A \in GL(d, \mathbb{E}) \) such that

\[
(2) \quad A \sigma(h)^\alpha A^{-1} = \sigma(aha^{-1}) \quad \text{(for all } h \in H),
\]

and note that, by absolute irreducibility, \( AA^\alpha \cdots A^{\alpha^p - 1} = \mu \sigma(a^p) \) for some \( \mu \in \mathbb{E} \). As in Proposition (1.1) we see that \( \mu \in \mathbb{F} \), since

\[
\mu^\alpha \sigma(a^p)^\alpha = A^\alpha A^{\alpha^2} \cdots A^{\alpha^p} = A^{-1}(AA^\alpha A^{\alpha^2} \cdots A^{\alpha^p - 1})A = \mu(A^{-1} \sigma(a^p)A) = \mu(A^{-1} \sigma(aa^p a^{-1})A) = \mu \sigma(a^p)^\alpha,
\]

where the last step follows from (2). Hence replacing \( A \) by \( \nu^{-1} A \), where \( \nu \in \mathbb{E}^\times \) satisfies \( \nu \nu^\alpha \cdots \nu^{\alpha^p - 1} = \mu \), we may assume that

\[
(3) \quad AA^\alpha \cdots A^{\alpha^p - 1} = \sigma(a^p).
\]
The regular representation of $E$ considered as an $F$-algebra yields an $F$-algebra monomorphism $\phi: E \to \text{Mat}(p, F)$, and since $\alpha$ is an $F$-automorphism of $E$ there is an $M \in \text{GL}(p, F)$ satisfying $M^p = I$ and

$$M^{-1}\phi(\lambda)M = \phi(\lambda^\alpha) \quad (\text{for all } \lambda \in E).$$

(We remark that computing $\phi$ and $M$ is best done when the elements of $E$ are represented as polynomials over $F$ modulo an irreducible polynomial. In this case, the assumption in Section 1, that field arithmetic in $E$ can be performed in constant time, does not hold.) Let $\Phi: \text{Mat}(d, E) \to \text{Mat}(pd, F)$ be defined by $\Phi((\lambda_{i,j})) = (\phi(\lambda_{i,j}))$, and define $S \in \text{GL}(d, F)$ to be the diagonal sum of $d$ copies of $M$. Then $\Phi$ is an $F$-algebra monomorphism, and

$$(4) \quad S^{-1}\Phi(X)S = \Phi(X^\alpha) \quad (\text{for all } X \in \text{Mat}(d, E)).$$

It now follows that there is a representation $\rho: G \to \text{GL}(pd, F)$ such that $\rho(a) = \Phi(A)S^{-1}$ and $\rho(h) = \Phi(\sigma(h))$ for all $h \in H$, since

$$\rho(a)^p = (\Phi(A)S^{-1})^p$$

$$= \Phi(A)(S^{-1}\Phi(A)S)\cdots(S^{-1(p-1)}\Phi(A)S^{p-1})S^{-p}$$

$$= \Phi(A)\Phi(A^\alpha)\cdots\Phi(A^{\alpha^{p-1}}) \quad (\text{using (4) and } S^p = I)$$

$$= \Phi(\sigma(a^p)) \quad (\text{by (3)})$$

$$= \rho(a^p)$$

and

$$\rho(a)\rho(h)\rho(a)^{-1} = \Phi(A)S^{-1}\Phi(\sigma(h))S\Phi(A)^{-1}$$

$$= \Phi(A)\Phi(\sigma(h)^\alpha)\Phi(A^{-1}) \quad (\text{by (4)})$$

$$= \Phi(\sigma(aha^{-1})) \quad (\text{by (2)})$$

$$= \rho(aha^{-1}).$$

It remains to check that $\rho$ is equivalent to $\sigma^G$. It is clear that there exists a $T \in \text{GL}(p, E)$ such that

$$T\phi(\lambda)T^{-1} = \text{diag}(\lambda, \lambda^\alpha, \ldots, \lambda^{\alpha^{p-1}})$$
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for all $\lambda \in \mathbb{E}$. Furthermore, if $v_i$ denotes the $(i+1)$th row of $T$ and $V_i$ denotes the subspace of $\mathbb{E}^{pd}$ comprising the elements of the form $(\lambda_1 v_i, \lambda_2 v_i, \ldots, \lambda_d v_i)$ where $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{E}$, then

(i) $\mathbb{E}^{pd} = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1}$,
(ii) each $V_i$ is $\rho(H)$-invariant, inducing an action equivalent to $\sigma^{a^i}$, and
(iii) $V_i \rho(a) = V_{i+1}$, where the subscripts are read modulo $p$.

Note that (ii) follows from $v_i \phi(\lambda) = \lambda^{a^i} v_i$, and (iii) follows from the equation $\rho(a)\rho(h)\rho(a)^{-1} = \rho(aha^{-1})$. These conditions guarantee that $\rho$ is equivalent to $\sigma^G$, as required. We have thus achieved our goal of constructing the absolutely irreducible representations of $G$ over their minimal fields.

References

[1] Derek F. Holt and Sarah Rees, Testing modules for irreducibility, J. Aust. Math. Soc. (A) 57 (1994), 1–16.

[2] B. Huppert and N. Blackburn, Finite Groups II, Springer-Verlag, Berlin, 1982.