N-Point Deformation of Algebraic K3 Surfaces

Hoi Kim
Topology and Geometry Research Center, Kyungpook National University, Taegu 702-701, Korea

and

Chang-Yeong Lee
Department of Physics, Sejong University, Seoul 143-747, Korea

ABSTRACT

We consider N-point deformation of algebraic K3 surfaces. First, we construct two-point deformation of algebraic K3 surfaces by considering algebraic deformation of a pair of commutative algebraic K3 surfaces. In this case, the moduli space of the noncommutative deformations is of dimension 19, the same as the moduli dimension of the complex deformations of commutative algebraic K3 surfaces. Then, we extend this method to the N-point case. In the N-point case, the dimension of deformation moduli space becomes $19N(N-1)/2$. 

\[^{1}\text{hikim@gauss.knu.ac.kr} \]
\[^{2}\text{cylee@sejong.ac.kr} \]
I. Introduction

Ever since the work of Connes, Douglas, and Schwarz [1] connecting the noncommutative torus and the T-duality in the M theory context appeared in the string/M theory arena, the field related with noncommutative geometry [2] becomes an industry in the string/M theory circle. Notably, noncommutative torus [3, 4] and its varieties have been studied intensively [5, 6, 7, 8]. However, noncommutative versions of the K3 surfaces and the Calabi-Yau(CY) threefolds have been rarely studied [9, 10, 11] (see also [12, 13, 14]). Only recently, noncommutative tori with complex structures have been studied [15, 16].

In Ref.[17], Berenstein, Jejjala, and Leigh initiated an algebraic geometry approach to noncommutative moduli space. Then in Ref.[1], Berenstein and Leigh discussed noncommutative CY threefold from the viewpoint of algebraic geometry. They considered two examples: a toroidal orbifold $T^6/Z_2 \times Z_2$ and an orbifold of the quintic in $\mathbb{CP}^4$, each with discrete torsion [18, 19, 20, 21, 22]. There, they explained the fractionation of branes at singularities from noncommutative geometric viewpoint under the presence of discrete torsion.

In Ref.[1], Berenstein and Leigh considered the $T^6/Z_2 \times Z_2$ case and recovered a large slice of the moduli space of complex structures of the CY threefold from the deformation of the noncommutative resolution of the orbifolds via central extension of the local algebra of holomorphic functions. In the commutative K3 case, the moduli space for the K3 space itself has been known already (see for instance [23]), and even the moduli space for the bundles on K3 surfaces has been studied [24]. In the noncommutative deformation of CY threefolds in Ref.[1], the three holomorphic coordinates $y_i$ anticommute with each other to be compatible with $Z_2$ discrete torsion.

In our previous work [10], we applied this algebraic approach to K3 surfaces in the cases of the orbifolds $T^4/Z_2$. We constructed a family of noncommutative K3 surfaces by algebraically deforming $T^4/Z_2$ in both complex and noncommutative directions altogether. In that construction the dimensions of moduli spaces for the complex structures and the noncommutative deformations were the same 18, which is the dimension of the moduli space of the complex structures of K3 surfaces constructed with two elliptic curves.

However, in the commutative case the complete family of complex deformations of K3
surfaces is of 20 dimension inside which that of the algebraic K3 surfaces is of 19 dimension \[23\]. Thus, in this paper, we first construct a 19 dimensional family of the noncommutative moduli of general algebraic K3 surfaces by considering algebraic deformation of a pair of K3 surfaces. This construction apparently looks similar to the Connes-Lott’s “two-point space” construction of the standard model \[25\]. Thus, we will call it “two-point deformation”. Next, we extend this method directly to the N-point case by deforming N-tuple of commutative algebraic K3 surfaces embedded in \(\mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N)\).

In section II, we construct a two-point deformation for general algebraic K3 surfaces. In section III, we extend the method to the N-point case. In section IV, we conclude with discussion.

II. Two-point deformation

In this section, we first consider the “two-point space” version of noncommutative deformation for general algebraic K3 surfaces in the direct extension of our previous work on noncommutative \(T^4/\mathbb{Z}_2\) \[10\]. General algebraic K3 surfaces are given by the following form and with a point added at infinity.

\[ y^2 = f(x_1, x_2) \tag{1} \]

Here \(f\) is a function with total degree 6 in \(x_1, x_2\).

Now, we compare this with the Kummer surface, the orbifold of \(T^4/\mathbb{Z}_2\) case \[10\]. There we considered \(T^4\) as the product of two elliptic curves, each given in Weierstrass form

\[ y_i^2 = x_i(x_i - 1)(x_i - a_i) \tag{2} \]

with a point added at infinity for \(i = 1, 2\). By the following change of variables, the point at infinity is brought to a finite point:

\[ y_i \rightarrow y_i' = \frac{y_i}{x_i^2}, \]

\[ x_i \rightarrow x_i' = \frac{1}{x_i}. \tag{3} \]
For algebraic K3 surfaces, we first consider a function with total degree 6 in complex variables $u, v, w$, for instance

$$F(u, v, w) = u^2 v^3 w + u^4 v^2.$$  

In a patch where the point at infinity of $w$ can be brought to a finite point, this can be written as

$$F_{\infty} = \left(\frac{u}{w}\right)^2 \left(\frac{v}{w}\right)^3 + \left(\frac{u}{w}\right)^4 \left(\frac{v}{w}\right)^2$$

and may be denoted as

$$f(x_1, x_2) = x_1^2 x_2^3 + x_1^4 x_2^2$$

where $x_1 = \frac{u}{w}, x_2 = \frac{v}{w}$. Then, an algebraic K3 surface is given by

$$y^2 = f(x_1, x_2) = x_1^2 x_2^3 + x_1^4 x_2^2.$$

(4)

Similarly, in a patch where the point at infinity of $u$ can be brought to a finite point, we consider a function

$$F_u = \left(\frac{v}{u}\right)^3 \frac{w}{u} + \left(\frac{v}{u}\right)^2,$$

and this can be written as

$$y'^2 = f'(x'_1, x'_2) = x'_1^3 x'_2 + x'_1^2$$

(5)

where $x'_1 = \frac{u}{v} = \frac{x_2}{x_1}, x'_2 = \frac{w}{v} = \frac{1}{x_1}$. This can be also obtained directly from (4) by dividing it with $x_1^6$

$$y^2 = \frac{x_1^2 x_2^3}{x_1^6} + \frac{x_1^4 x_2^2}{x_1^6} = \left(\frac{x_2}{x_1}\right)^3 \frac{1}{x_1} + \left(\frac{x_2}{x_1}\right)^2.$$  

Thus, in the case of the general algebraic K3, a point at infinity in one patch can be brought to a finite point in another patch by the following change of variables

$$y \rightarrow y' = \frac{y}{x_1^3},$$

(6)

$$x_1 \rightarrow x'_1 = \frac{x_2}{x_1},$$

(7)

$$x_2 \rightarrow x'_2 = \frac{1}{x_1}.$$
We now consider a deformation of algebraic K3 surfaces in noncommutative direction. Following the same reasoning in our previous work [10], we consider two commuting complex variables \(x_1, x_2\) and two noncommuting variables \(t_1, t_2\) such that

\[
\begin{align*}
t_1^2 &= h_1(x_1, x_2), \\
t_2^2 &= h_2(x_1, x_2),
\end{align*}
\]

where \(h_1, h_2\) are commuting functions of total degree 6 in \(x_1, x_2\). To be consistent with the condition that \(t_1^2, t_2^2\) belong to the center, one can allow the following deformation for \(t_1, t_2\).

\[
t_1 t_2 + t_2 t_1 = P(x_1, x_2)
\]

Here the right hand side should be a polynomial and free of poles in each patch. Thus, under the change of variables \((7)\)

\[
\begin{align*}
x_1 &\longrightarrow x'_1 = \frac{x_2}{x_1}, \\
x_2 &\longrightarrow x'_2 = \frac{1}{x_1},
\end{align*}
\]

\(t_i\) should be changed into

\[
t_i \longrightarrow t'_i = \frac{t_i}{x'_1}, \quad \text{for } i = 1, 2.
\]

This is due to the fact that \(t\)'s transform just like \(y\) in \((8)\). Therefore, \(P\) transforms as

\[
P(x_1, x_2) \longrightarrow x'_1^6 P'(\frac{x_2}{x_1}, \frac{1}{x_1}).
\]

This implies that \(P'\) should be of total degree 6 in \(x'_1, x'_2\), at most. Interchanging the role of \(P\) and \(P'\) one can see that \(P\) should be also of total degree 6 in \(x_1, x_2\).

The above structure can be understood in the following manner. If we do not impose the condition \((9)\), and if we have only one of \(t_i\)'s satisfying the condition \((8)\), then we have only one copy of an algebraic K3 surface. If we have both \(t_i\)'s without the condition \((9)\), then we have two copies of K3 surfaces. If we have both \(t_i\)'s and impose the condition \((9)\), then we have a noncommutatively deformed K3 surfaces in which the above mentioned two K3 surfaces intertwined each other everywhere on their surfaces, becoming fuzzy. This seems to be similar to the “two-point space” version of the Connes-Lott model [24]. In the
Connes-Lott model, every point of the space becomes fuzzy due to the 1-to-2 correspondence at each point in the space, where the two corresponding points at each classical location are pre-fixed. On the other hand, ours are more or less like position $x$ and momentum $p$ in quantum mechanics at every point in the space. However, since we started with two copies of the classical space just like the Connes-Lott model, and combined them to become a noncommutative space, we will call our construction “two-point deformation” though our construction is not exactly the same as the Connes-Lott’s in its nature.

Now, we count the dimension of the moduli space of our deformation. In our previous work for noncommutative $T^4/\mathbb{Z}_2$ \cite{10}, $t_1$ for $y_1 y_2$ and $t_2$ for $y_2 y_1$ were all invariants of the K3 surface. The dimensions of the moduli spaces of these deformations were 18 for both the noncommutative and complex deformation cases, matching the moduli space dimension of the complex deformation for $T^4/\mathbb{Z}_2$. In the present case, from eq.\ref{11} we can see that the dimension of the moduli spaces of these deformations are 19 for both the noncommutative and complex deformation cases. In fact, to show this we need to count the dimension of the polynomials of degree 6 in three variables up to constant modulo projective linear transformations of three variables. We get $19 = 28 - 1 - 8$, where 28 is the dimension of polynomials of degree 6 in three variables and 1 and 8 correspond to a constant and $PGL(3, \mathbb{C})$, respectively.

III. N-point deformation

In this section, we follow the method in the previous section and consider the “N-point space” of the noncommutative deformation of the general algebraic K3 surfaces.

First, we consider $N$-tuple of commutative algebraic K3 surfaces

$$
t_i^2 = h_i(x_1, x_2), \quad \vdots \quad t_N^2 = h_N(x_1, x_2),
$$

where $h_1, \cdots, h_N$ are commuting functions of total degree 6 in $x_1, x_2$. This can be regarded as embedding the i-th copy of algebraic K3 surface $X_i$ in $\mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_i)$ as $t_i^2 = h_i(x_1, x_2)$.
of a degree 6 polynomial. Locally the algebra representing the functions on $X_i$ can be expressed as $\mathbb{C}[x_1, x_2, t_i]/I_i$, where $I_i$ is a principal ideal generated by $t_i^2 - h_i(x_1, x_2)$ and $\mathbb{C}[x_1, x_2, t_1, \cdots, t_N]$ is a local polynomial algebra of $\mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N)$. Thus embedding $X_i$ in $\mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N)$ induces a natural quotient map from $\mathbb{C}[x_1, x_2, t_1, \cdots, t_N]$ to $\mathbb{C}[x_1, x_2, t_i]/I_i$ by putting $t_j$ as 0 for $j \neq i$.

Now, we consider the deformation of this embedded space in the noncommutative direction as in the two-point case. In order to be consistent with the condition that $t_2^2, \cdots, t_N^2$ belong to the center along with $x_1, x_2$, we can allow the following deformation for $t_1, \cdots, t_N$.

$$t_i t_j + t_j t_i = P_{ij}(x_1, x_2), \quad \text{for } i, j = 1, \cdots, N, \ i \neq j. \quad (13)$$

Here the right hand side should be a polynomial and free of poles in each patch. Thus, under the change of variables (7)

$$x_1 \rightarrow x_1' = \frac{x_2}{x_1},$$
$$x_2 \rightarrow x_2' = \frac{1}{x_1},$$

$t_i$’s should be changed into

$$t_i \rightarrow t_i' = \frac{t_i}{x_1}, \quad \text{for } i = 1, \cdots, N. \quad (14)$$

This is due to the fact that $t_i$’s transform just like $y$ in (3). Therefore, $P_{ij}$ transforms as

$$P_{ij}(x_1, x_2) \rightarrow x_1^6 P_{ij}'(\frac{x_2}{x_1}, \frac{1}{x_1}). \quad (15)$$

This implies that $P_{ij}'$ should be of total degree 6 in $x_1', x_2'$, at most. Interchanging the role of $P_{ij}$ and $P_{ij}'$ one can see that $P_{ij}$ should be also of total degree 6 in $x_1, x_2$.

If we forget the embedded $N$ K3 surfaces given by the constraints (12) for the time being, the above defined $\{P_{ij}(x_1, x_2)\}$ given by (13) define a deformation of the ambient space $\mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N)$. So, we can understand that imposing the condition of the change of chart (4),(3) compatible to the complex structures coming from (12) induces a restriction on $P_{ij}$ being of total degree 6 in $x_1, x_2$. We might call this deformation a deformation of $N$ K3 surfaces. The choice of $P_{ij}$ is independent of the choice of $h_i$, which means that the deformations of the classical complex structure and of the noncommutative
structure are independent of each other as expected. Now, we count the dimension of
the moduli space of our deformation. In the two-point case of the previous section, the
dimension of the moduli space of the deformation was 19. In that case, we counted the
dimension of the polynomials of degree 6 in three variables up to constant modulo projective
linear transformations of three variables. Thus, we got $19 = 28 - 1 - 8$, where 28 is the
dimension of polynomials of degree 6 in three variables and 1 and 8 correspond to a constant
and $PGL(3, \mathbb{C})$, respectively. Thus, in the N-point case the dimension of the moduli space
of the deformation is the number of independent $P_{ij}$ times the deformation dimension of the
two-point case. Namely, we have $19N(N-1)/2$ as the dimension of the deformation moduli
for the N-point deformation case.

IV. Discussion

In this paper, we deformed $N$ K3 surfaces in the noncommutative sense and computed the
dimension for the moduli space.

In the first part of the paper, we constructed the two-point deformation of algebraic K3
surfaces by considering algebraic deformation of a pair of commutative algebraic K3 surfaces.
Doing this, we used the same method as in the case of the Kummer K3 surface[10] which
is the $\mathbb{Z}_2$ quotient of two elliptic curves $E_1, E_2$ where $E_i$ satisfies $y_i^2 = f_i(x_i)$. In Ref.[10],
we defined $t_1 = y_1y_2$, $t_2 = y_2y_1$ and introduced the deformation $t_1t_2 + t_2t_1 = P_{12}(x_1, x_2)$. In
that case $t_1$, $t_2$ were functions on the Kummer K3 surface, so that the deformation was a
noncommutative deformation of one Kummer K3 surface. However, the moduli dimension
of that deformation was of 18 [10], not the same as the moduli dimension of algebraic K3
surfaces. Here, we recovered the same moduli dimension of deformation, 19 by algebraically
deforming a pair of algebraic K3 surfaces in a manner similar to the Connes-Lott construction
[25].

Then we considered the extension of this method to the N-point case. Notice that
in the N-point deformation case, $t_j$ in the $t_it_j + t_jt_i = P_{ij}(x_1, x_2)$ is not a function on
the $i$-th copy of commutative K3 surface $X_i$ for $i \neq j$. Rather this can be thought of as
noncommutative deformation of $N$ K3 surfaces or a noncommutative deformation of the
ambient space \( \mathbb{P}^2(x_1, x_2) \times \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N) \) compatible to the complex structure of each K3 surface. In the N-point case, we obtained \( 19N(N-1)/2 \) as the dimension of deformation moduli.

When \( N=3 \), it is interesting whether we can find an analogue of the classical hyperk"ahler structure of K3 surface. First, we recall the property of the moduli space of Ricci flat metrics on a K3 surface \( S \). If a given metric \( g \) satisfies \( g(Jv, Jw) = g(v, w) \) for any tangent vector \( v, w \), then we say that the metric \( g \) is compatible with the complex structure \( J \). If the two form \( \Omega(\cdot, \cdot) = g(J\cdot, \cdot) \) is closed, then it is called a K"ahler metric and \( \Omega \) is called a K"ahler form. Any given Ricci-flat metric \( g \) induces a Hodge * operator on \( H^2(S, \mathbb{R}) \cong \mathbb{R}^{3,19} \) by which \( H^2(S, \mathbb{R}) \) can be decomposed as a direct sum of two eigenspaces, self dual part (eigenvalue 1) of dimension 3 and anti-self dual part (eigenvalue \( -1 \)) of dimension 19.

In this setting, for the given Ricci-flat metric \( g \), the self dual part \( \Lambda^+ \) is a 3-dimensional real vector space consisting of vectors whose self intersection is positive. Different compatible structures \( J \) to \( g \) correspond to different unit vectors in \( \Lambda^+ \), and they form \( S^2 \) isomorphic to \( \mathbb{P}^1 \). Here we choose 3 orthogonal unit vectors \( \Omega_1, \Omega_2, \Omega_3 \) in \( \Lambda^+ \) such that corresponding complex structures \( J_1, J_2, J_3 \) satisfy the relation \( J_iJ_j = \epsilon_{ijk}J_k \) for \( i, j, k = 1, 2, 3 \). This is called a hyperk"ahler structure on \( S \). We wonder whether we can see the 3-point deformation case as the deformation of this hyperk"ahler structure on \( S \) by relating \( t_i \)'s with \( J_i \)'s.

Fr"ohlich et al \[26, 27\] defined a spectral triple for this hyperk"ahler case introducing the operators \( \partial, \bar{\partial}, T^i, \bar{T}^i, \ i = 1, 2, 3 \) acting on the differential forms. Here, \( \partial = \frac{1}{2}(D - i\bar{D}) \), where \( D \) is the Dirac operator and \( T^i, \ i = 1, 2, 3 \) are operators coming from the hyperk"ahler structure. Then they extended this definition to the noncommutative case. We also wonder whether we can relate our \( t_i \) with their \( T_i \).

Finally, we wonder whether we can find a sort of Clifford structures on \( \mathbb{P}^2(x_1, x_2) \) with the fiber \( \mathbb{P}^1(t_1) \times \cdots \times \mathbb{P}^1(t_N) \). This may be considered by regarding \( t_it_j + t_jt_i = P_{ij}(x_1, x_2), \ i, j = 1, \cdots, N, \ i \neq j \) and \( t_i^2 = h_i(x_1, x_2), \ i = 1, \cdots, N, \) not as constraints giving noncommutative deformation and complex structures for K3 surfaces but as the components of a symmetric matrix giving the metric on the fiber.
Acknowledgments

This work was supported by KOSEF Interdisciplinary Research Grant No. R01-2000-00022.

References

[1] A. Connes, M.R. Douglas, and A. Schwarz, JHEP 9802, 003 (1998), hep-th/9711162.
[2] A. Connes, Noncommutative geometry (Academic Press, New York, 1994).
[3] A. Connes and M. Rieffel, Contemp. Math. 62, 237 (1987).
[4] M. Rieffel, Can. J. Math. Vol. XL, 257 (1988).
[5] D. Brace, B. Morariu, and B. Zumino, Nucl. Phys. B 545, 192 (1999), hep-th/9810099; P.-M. Ho, Y.-Y. Wu, and Y.-S. Wu, Phys. Rev. D 58, 026006 (1998), hep-th/9712201.
[6] C. Hofman and E. Verlinde, Nucl. Phys. B 547, 157 (1999), hep-th/9810219.
[7] E. Kim, H. Kim, N. Kim, B.-H. Lee, C.-Y. Lee, and H. S. Yang, Phys. Rev. D 62, 046001 (2000), hep-th/9912272.
[8] See, for instance, N. Seiberg and E. Witten, JHEP 9909, 032 (1999), hep-th/9908142 and references therein for the development in this direction.
[9] D. Berenstein and R. G. Leigh, Phys. Lett. B 499, 207 (2001), hep-th/0009209.
[10] H. Kim and C.-Y. Lee, “Noncommutative K3 surfaces”, hep-th/0105263.
[11] A. Belhaj and E. H. Saidi, “On noncommutative Calabi-Yau hypersurfaces”, hep-th/0108143.
[12] A. Konechny and A. Schwarz, Nucl. Phys. B 591, 667 (2000), hep-th/9912183.
[13] A. Konechny and A. Schwarz, JHEP 0009, 005 (2000), hep-th/0005174.
[14] E. Kim, H. Kim, and C.-Y. Lee, J. Math. Phys. 42, 2677 (2001), hep-th/0005205.
[15] A. Schwarz, Lett. Math. Phys. 58, 81 (2001).

[16] Y. Manin, “Theta functions, quantum tori and Heisenberg groups”, math.AG/0011197.

[17] D. Berenstein, V. Jejjala, and R. Leigh, Nucl. Phys. B 589, 196 (2000), hep-th/0005087;
    Phys. Lett. B 493, 162 (2000), hep-th/0006168.

[18] C. Vafa, Nucl. Phys. B 273, 592 (1986).

[19] C. Vafa and E. Witten, J. Geom. Phys. 15, 189 (1995), hep-th/9409188.

[20] M. R. Douglas, “D-branes and discrete torsion”, hep-th/9807234.

[21] M. R. Douglas and B. Fiol, “D-branes and discrete torsion II”, hep-th/9903031.

[22] J. Gomis, JHEP 0005, 006 (2000), hep-th/0001200.

[23] P. S. Aspinwall, “K3 surfaces and string duality”, TASI-96 lecture notes, hep-th/9611137.

[24] S. Mukai, “On the moduli space of bundles on K3 surfaces, I” in Vector Bundles on
    Algebraic Varieties, edited by M. Atiah et al. (Oxford Univ. Press, Oxford, 1985).

[25] A. Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) 18B, 29 (1990).

[26] J. Fröhlich, O. Grandjean, and A. Recknagel, Commun. Math. Phys. 193, 527 (1998).

[27] J. Fröhlich, O. Grandjean, and A. Recknagel, Commun. Math. Phys. 203, 119 (1999).