J-invariant of linear algebraic groups

V. Petrov, N. Semenov, K. Zainoulline*

Abstract

Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $X$ be a projective homogeneous $G$-variety such that $G$ splits over the function field of $X$. In the present paper we introduce an invariant of $G$ called $J$-invariant which characterizes the motivic behavior of $X$. This generalizes the respective notion invented by A. Vishik in the context of quadratic forms. As a main application we obtain a uniform proof of all known motivic decompositions of generically split projective homogeneous varieties (Severi-Brauer varieties, Pfister quadrics, maximal orthogonal Grassmannians, $G_2$- and $F_4$-varieties) as well as provide new examples (exceptional varieties of types $E_6$, $E_7$ and $E_8$). We also discuss relations with torsion indices, canonical dimensions and cohomological invariants of the group $G$.

Introduction

Let $G$ be a semisimple linear algebraic group over a field $F$ and $X$ be a projective homogeneous $G$-variety. In the present paper we address the problem of computing the Grothendieck-Chow motive $\mathcal{M}(X)$ of $X$ or, in other words, providing a direct sum decomposition of $\mathcal{M}(X)$.

This problem turns out to be strongly related with several classical conjectures concerning algebraic cycles. For instance, the motivic decomposition of a Pfister quadric plays a major role in the proof of Milnor’s conjecture by V. Voevodsky. The proof of the generalization of this conjecture known as

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the Bloch-Kato conjecture was recently announced by M. Rost and V. Voevodsky. It essentially uses motivic decompositions of the norm varieties which are closely related to projective homogeneous varieties.

Another deep application deals with the famous Kaplansky problem on the values of the $u$-invariant of a field. It has a long history starting from the works of A. Merkurjev and O. Izhboldin. Recently an essential breakthrough in this problem was achieved by A. Vishik [Vi06], where he used the $J$-invariant of an orthogonal group. The present paper was mostly motivated by this result. The invariant that we introduce and study is a generalization of the $J$-invariant of A. Vishik to an arbitrary semisimple algebraic group.

It was first observed by B. Köck [Kö91] that if the group $G$ is split, i.e., contains a split maximal torus, then the motive of $X$ has the simplest possible decomposition – it is isomorphic to a direct sum of twisted Tate motives. The next step was done by V. Chernousov, S. Gille and A. Merkurjev [CGM] and P. Brosnan [Br05]. They proved that if $G$ is isotropic, i.e., contains a split 1-dimensional torus, then the motive of $X$ can always be decomposed as a direct sum of the motives of projective homogeneous varieties of smaller dimensions corresponding to anisotropic groups, thus, reducing the problem to the anisotropic case.

For anisotropic groups only very few partial results are known. In this case the components of a motivic decomposition of $X$ are expected to have a non-geometric nature, i.e., can not be identified with (twisted) motives of some other varieties. The first examples of such decompositions were provided by M. Rost [Ro98]. He proved that the motive of a Pfister quadric decomposes as a direct sum of twisted copies of a certain a priori non-geometric motive $R$ called Rost motive. The motives of Severi-Brauer varieties were computed by N. Karpenko [Ka96]. For exceptional varieties examples of motivic decompositions were provided by J.-P. Bonnet [Bo03] (varieties of type $G_2$) and by S. Nikolenko, N. Semenov, K. Zainoulline [NSZ] (varieties of type $F_4$). Observe that in all these examples the respective group $G$ splits over the generic point of $X$. Such varieties will be called generically split.

In the present paper we provide a uniform proof of all these results. Namely, we prove that (see Theorem 5.17)

**Theorem.** Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $p$ be a prime integer. Let $X$ be a generically split projective homogeneous $G$-variety. Then the Chow motive of $X$ with $\mathbb{Z}/p$-coefficients
is isomorphic to a direct sum

$$\mathcal{M}(X; \mathbb{Z}/p) \cong \bigoplus_{i \in I} \mathcal{R}_p(G)(i)$$

of twisted copies of an indecomposable motive $\mathcal{R}_p(G)$ for some finite multiset $I$ of non-negative integers.

Observe that the motive $\mathcal{R}_p(G)$ depends only on $G$ and $p$ but not on the type of a parabolic subgroup defining $X$. Moreover, considered with $\mathbb{Q}$-coefficients it always splits as a direct sum of twisted Tate motives.

Our proof is based on two different observations. The first is the Rost Nilpotence Theorem. It was originally proven for projective quadrics by M. Rost and then generalized to arbitrary projective homogeneous varieties by P. Brosnan [Br05], V. Chernousov, S. Gille and A. Merkurjev [CGM]. Roughly speaking, this result plays a role of the Galois descent for motivic decompositions over a separable closure $\overline{F}$ of $F$. Namely, it reduces the problem to the description of idempotent cycles in the endomorphism group $\text{End}(\mathcal{M}(X_{\overline{F}}; \mathbb{Z}/p))$ which are defined over $F$.

To provide such cycles we use the second observation which comes from the topology of compact Lie groups. In paper [Kc85] V. Kac invented the notion of $p$-exceptional degrees – the numbers which relate the degrees of mod $p$ basic polynomial invariants and the $p$-torsion part of the Chow ring of a compact Lie group. These numbers have combinatorial nature. By the result of K. Zainoulline [Za06] there is a strong interrelation between $p$-exceptional degrees and the subgroup of cycles in $\text{End}(\mathcal{M}(X_F; \mathbb{Z}/p))$ defined over $F$. To describe this subgroup we introduce the notion of the $J$-invariant of a group $G$ mod $p$ denoted by $J_p(G)$ (see Definition 4.6). In the most cases the values of $J_p(G)$ were implicitly computed by V. Kac in [Kc85] and can easily be extracted from Table 4.13.

It follows from the proof that the $J$-invariant measures the ‘size’ of the motive $\mathcal{R}_p(G)$ and, hence, characterizes the motivic decomposition of $X$. Observe that if the $J$-invariant takes its minimal possible non-trivial value $J_p(G) = (1)$, then the motive $\mathcal{R}_p(G) \otimes \mathbb{Q}$ has the following recognizable decomposition (cf. [Vo03 §5] and [Ro06 §5])

$$\mathcal{R}_p(G) \otimes \mathbb{Q} \cong \bigoplus_{i=0}^{p-1} \mathbb{Q}(i \cdot p^{n-1-1}),$$

where $n = 2$ or 3.
The assignment $G \mapsto \mathcal{R}_p(G)$ can be viewed as a motivic analog of the cohomological invariant of $G$ given by the Tits class of $G$ if $n = 2$ and by the Rost invariant of $G$ if $n = 3$. In these cases the motive $\mathcal{R}_p(G)$ coincides with a generalized Rost motive.

We also generalize some of the results of paper [CPSZ]. Namely, using the motivic version of the result of D. Edidin and W. Graham [EG97] on cellular fibrations we provide a general formula which expresses the motive of the total space of a cellular fibration in terms of the motives of its base (see Theorem 3.7). We also provide several criteria for the existence of liftings of motivic decompositions via the reduction map $\mathbb{Z} \to \mathbb{Z}/m$. We prove that such liftings always exist (see Theorem 2.16).

The paper is organized as follows. In the first section we provide several auxiliary facts concerning motives and rational cycles. Rather technical section 2 is devoted to lifting of idempotents. In section 3 we discuss the motives of cellular fibrations. The next section is devoted to the notion of a $J$-invariant. The proof of the main result is given in section 5. The last two sections are devoted to various applications of the $J$-invariant and examples of motivic decompositions. In particular, we discuss the relations with canonical $p$-dimensions, degrees of zero-cycles and the Rost invariant.

1 Chow motives and rational cycles

In the present section we provide several auxiliary facts concerning algebraic cycles, correspondences and motives which will be extensively used in the sequel. We follow the notation and definitions from [EKM, Ch. XII] (see also [Ma68]).

Let $X$ be a smooth projective irreducible variety over a field $F$. Let $\mathrm{CH}_i(X; \Lambda)$ be the Chow group of cycles of dimension $i$ on $X$ with coefficients in a commutative ring $\Lambda$. For simplicity we denote $\mathrm{CH}(X; \mathbb{Z})$ by $\mathrm{CH}(X)$.

1.1. Following [EKM, §63] an element $\phi \in \mathrm{CH}_{\dim X + d}(X \times Y; \Lambda)$ is called a correspondence between $X$ and $Y$ of degree $d$ with coefficients in $\Lambda$. Let $\phi \in \mathrm{CH}(X \times Y; \Lambda)$ and $\psi \in \mathrm{CH}(Y \times Z; \Lambda)$ be correspondences of degrees $d$ and $e$ respectively. Then their product $\psi \circ \phi$ is defined by the formula $(\mathrm{pr}_{XZ})_*(\mathrm{pr}_{XY}^*(\phi) \cdot \mathrm{pr}_{YZ}^*(\psi))$ and has degree $d + e$. The correspondence product endows the group $\mathrm{CH}(X \times X; \Lambda)$ with a ring structure. The identity element of this ring is the class of the diagonal $\Delta_X$. Given $\phi \in \mathrm{CH}(X \times X; \Lambda)$ of degree $d$ we define a $\Lambda$-linear map $\mathrm{CH}_i(X; \Lambda) \to \mathrm{CH}_{i+d}(Y; \Lambda)$ by $\alpha \mapsto$
(pr\_1\_r\_Y \_r\_X(\alpha) \cdot \phi). This map is called realization of \( \phi \) and is denoted by \( \phi_\_r\_r\_ \). By definition \( (\psi \circ \phi)_\_r\_r\_ = \psi_\_r\_ \circ \phi_\_r\_ \). Given a correspondence \( \phi \) we denote by \( \phi^\_r\_ \) its transpose.

**1.2.** Following [EKM, § 64] let \( \mathcal{M}(X; \Lambda) \) denote the Chow motive of \( X \) with \( \Lambda \)-coefficients and \( \mathcal{M}(X; \Lambda)(n) = \mathcal{M}(X; \Lambda) \otimes \Lambda(n) \) denote the respective twist by the Tate motive. For simplicity by \( \mathcal{M}(X) \) we will denote \( \mathcal{M}(X; \mathbb{Z}) \). Recall that morphisms between \( \mathcal{M}(X; \Lambda)(n) \) and \( \mathcal{M}(Y; \Lambda)(m) \) are given by correspondences of degree \( n - m \) between \( X \) and \( Y \). Hence, the group of endomorphisms \( \text{End}(\mathcal{M}(X; \Lambda)) \) coincides with the Chow group \( \text{CH}_{\text{dim}X}(X \times X; \Lambda) \). Observe that to provide a direct sum decomposition of \( \mathcal{M}(X; \Lambda) \) is the same as to provide a family of pair-wise orthogonal idempotents \( \phi_i \in \text{End}(\mathcal{M}(X; \Lambda)) \) such that \( \sum_i \phi_i = \Delta_X \).

**1.3.** Assume that a motive \( M \) is a direct sum of twisted Tate motives. In this case its Chow group \( \text{CH}(M) \) is a free abelian group. We define its Poincaré polynomial as

\[
P(M, t) = \sum_{i \geq 0} a_i t^i,
\]

where \( a_i \) is the rank of \( \text{CH}_i(M) \).

**1.4 Definition.** Let \( L/F \) be a field extension. We say \( L \) is a splitting field of a smooth projective variety \( X \) if, equivalently, a variety \( X \) splits over \( L \) if the motive \( \mathcal{M}(X; \mathbb{Z}) \) splits over \( L \) as a finite direct sum of twisted Tate motives.

**1.5 Example.** A variety \( X \) over a field \( F \) is called cellular if \( X \) has a proper descending filtration by closed subvarieties \( X_i \) such that each complement \( X_i \setminus X_{i+1} \) is a disjoint union of affine spaces defined over \( F \). According to [EKM, Corollary 66.4] if \( X \) is cellular, then \( X \) splits over \( F \).

In particular, let \( G \) be a semisimple linear algebraic group over a field \( F \) and \( X \) be a projective homogeneous \( G \)-variety. Assume that the group \( G \) splits over the generic point of \( X \), i.e., \( G_{F(X)} = G \times_F F(X) \) contains a split maximal torus defined over \( F(X) \). Then \( X_{F(X)} \) is a cellular variety and, therefore, \( F(X) \) is a splitting field of \( X \). Some concrete examples of such varieties are provided in [3.6].

**1.6.** Assume \( X \) has a splitting field \( L \). We will write \( \text{CH}(\overline{X}; \Lambda) \) for \( \text{CH}(X_L; \Lambda) \) and \( \overline{\text{CH}}(X; \Lambda) \) for the image of the restriction map \( \text{CH}(X; \Lambda) \to \text{CH}(\overline{X}; \Lambda) \).
Similarly, by \( \mathcal{M}(X;\Lambda) \) we denote the motive of \( X \) considered over \( L \). If \( M \) is a direct summand of \( \mathcal{M}(X;\Lambda) \), by \( \overline{M} \) we denote the motive \( M_L \). The elements of \( \overline{CH}(X;\Lambda) \) will be called rational cycles on \( X_L \) with respect to the field extension \( L/F \) and the coefficient ring \( \Lambda \). If \( L' \) is another splitting field of \( X \), then there is a chain of canonical isomorphisms \( \overline{CH}(X_L) \cong \overline{CH}(X_{LL'}) \cong \overline{CH}(X_{L'}) \), where \( LL' \) is the composite of \( L \) and \( L' \). Hence, the groups \( \overline{CH}(X) \) and \( \overline{CH}(X) \) do not depend on the choice of \( L \).

1.7. According to [KM06, Remark 5.6] there is the Künneth decomposition \( \overline{CH}(X \times X) = \overline{CH}(X) \otimes \overline{CH}(X) \) and Poincaré duality holds for \( \overline{CH}(X) \). The latter means that given a basis of \( \overline{CH}(X) \) there is a dual one with respect to the non-degenerate pairing \( (\alpha,\beta) \mapsto \deg(\alpha \cdot \beta) \), where \( \deg \) is the degree map.

In view of the Künneth decomposition the correspondence product of cycles in \( \overline{CH}(X \times X) \) is given by the formula \( (\alpha_1 \times \beta_1) \circ (\alpha_2 \times \beta_2) = \deg(\alpha_1\beta_2)(\alpha_2\times \beta_1) \), the realization by \( (\alpha \times \beta)_*(\gamma) = \deg(\alpha\gamma)\beta \) and the transpose by \( (\alpha \times \beta)^t = \beta \times \alpha \).

Sometimes we will use contravariant notation \( \overline{CH}^* \) for Chow groups meaning \( \overline{CH}^i(X) = \overline{CH}_{\dim X-i}(X) \) for irreducible \( X \). The following important fact will be used in the proof of the main theorem (see Lemma 5.7).

1.8 Lemma. Let \( X \) and \( Y \) be two smooth projective varieties such that \( Y \) is irreducible, \( F(Y) \) is a splitting field of \( X \) and \( Y \) has a splitting field. For any \( r \) consider the projection in the Künneth decomposition

\[
\text{pr}_0^*: \overline{CH}^r(X \times Y) = \bigoplus_{i=0}^r \overline{CH}^{r-i}(X) \otimes \overline{CH}^i(Y) \to \overline{CH}^r(X).
\]

Then for any \( \rho \in \overline{CH}^r(X) \) we have \( \text{pr}_0^{-1}(\rho) \cap \overline{CH}^r(X \times Y) \neq \emptyset \).

Proof. Let \( L \) be a common splitting field of \( X \) and \( Y \). The lemma follows from the commutative diagram

\[
\begin{array}{ccc}
\overline{CH}^r(X \times F Y) & \xrightarrow{\text{res}_{L/F}} & \overline{CH}^r(X_L \times_L Y_L) \\
\downarrow & & \downarrow \\
\overline{CH}^r(X_{F(Y)}) & \xrightarrow{\sim} & \overline{CH}^r((X_L)_{L(Y_L)}) \leftarrow \overline{CH}^r(X_L)
\end{array}
\]

where the left square is obtained by taking the generic fiber of the base change morphism \( X_L \to X \); the vertical arrows are taken from the localization sequence for Chow groups and, hence, are surjective; and the bottom horizontal maps are isomorphisms since \( L \) is a splitting field. \( \square \)
1.9 Definition. We say that a field extension $E/F$ is rank preserving with respect to $X$ if the restriction map $\text{res}_{E/F} : \text{CH}(X) \to \text{CH}(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

1.10 Lemma. Assume $X$ has a splitting field. Then for any rank preserving finite field extension $E/F$ we have $[E:F] \cdot \text{CH}(X_E) \subset \text{CH}(X)$.

Proof. Let $L$ be a splitting field containing $E$. Let $\gamma$ be any element in $\text{CH}(X_E)$. By definition there exists $\alpha \in \text{CH}(X_E)$ such that $\gamma = \text{res}_{L/E}(\alpha)$. Since $\text{res}_{E/F} \otimes \mathbb{Q}$ is an isomorphism, there exists an element $\beta \in \text{CH}(X)$ and a non-zero integer $n$ such that $\text{res}_{E/F}(\beta) = n\alpha$. By the projection formula

$$n \cdot \text{cores}_{E/F}(\alpha) = \text{cores}_{E/F}(\text{res}_{E/F}(\beta)) = [E:F] \cdot \beta.$$

Applying $\text{res}_{L/E}$ to the both sides of the identity we obtain

$$n(\text{res}_{L/E}(\text{cores}_{E/F}(\alpha))) = n[E:F] \cdot \gamma.$$ Therefore, $\text{res}_{L/E}(\text{cores}_{E/F}(\alpha)) = [E:F] \cdot \gamma.$

We provide now examples of varieties for which any field extension is rank preserving and, hence, Lemma 1.10 holds.

1.11. Let $G$ be a semisimple linear algebraic group over a field $F$, $X$ be a projective homogeneous $G$-variety. Denote by $D$ the Dynkin diagram of $G$. According to [Ti66] one can always choose a quasi-split group $G_0$ over $F$ with the same Dynkin diagram, a parabolic subgroup $P$ of $G_0$ and a cocycle $\xi \in H^1(F,G_0)$ such that $G$ is isogenic to the twisted form $\xi G_0$ and $X$ is isomorphic to $\xi(G_0/P)$. If $G_0$ is split (see 1.5), then $G$ is called a group of inner type over $F$.

1.12 Lemma. Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $X$ a projective homogeneous $G$-variety. Then any field extension $E/F$ is rank preserving with respect to $X$ and $X \times X$.

Proof. By [Pa94] Theorem 2.2 and 4.2 the restriction map $K_0(X) \to K_0(X_E)$ becomes an isomorphism after tensoring with $\mathbb{Q}$. Now the Chern character $\text{ch} : K_0(X) \otimes \mathbb{Q} \to \text{CH}^*(X) \otimes \mathbb{Q}$ is an isomorphism and respects pull-backs, hence, $E$ is rank preserving with respect to $X$. It remains to note that $X \times X$ is a homogeneous $G \times G$-variety.

1.13 Remark. For even dimensional quadrics with non-trivial discriminant the restriction map $K_0(X) \to K_0(X)$ is not surjective and Lemma 1.12 doesn’t hold.
2 Lifting of idempotents

This section is devoted to lifting of idempotents and isomorphisms. First, we treat the case of general graded algebras. The main results here are Lemma 2.5 and Proposition 2.6. Then, assuming Rost Nilpotence 2.8 we provide conditions to lift motivic decompositions and isomorphisms (Theorem 2.16).

2.1. Let \( A \) be a \( \mathbb{Z} \)-graded ring. Assume we are given two orthogonal idempotents \( \phi_i \) and \( \phi_j \) in \( A^0 \) that is \( \phi_i \phi_j = \phi_j \phi_i = 0 \). We say an element \( \theta_{ij} \) provides an isomorphism of degree \( d \) between idempotents \( \phi_i \) and \( \phi_j \) if \( \theta_{ij} \in \phi_j A^d \phi_i \) and there exists \( \theta_{ji} \in \phi_i A^d \phi_j \) such that \( \theta_{ij} \theta_{ji} = \phi_j \) and \( \theta_{ji} \theta_{ij} = \phi_i \).

2.2 Example. Let \( X \) be a smooth projective irreducible variety over a field \( F \) and \( \text{CH}^\ast(X \times X; \Lambda) \) the Chow ring with coefficients in a commutative ring \( \Lambda \). Set \( A^\ast = \text{End}^\ast(\mathcal{M}(X; \Lambda)) \), where

\[
\text{End}^{-i}(\mathcal{M}(X; \Lambda)) = \text{CH}^\dim X-i(X \times X; \Lambda) = \text{CH}_{\dim X+i}(X \times X; \Lambda), \quad i \in \mathbb{Z}
\]

and the multiplication is given by the correspondence product. By definition \( \text{End}^0(\mathcal{M}(X; \Lambda)) \) is the ring of endomorphisms of the motive \( \mathcal{M}(X; \Lambda) \) (see 1.2). Note that a direct summand of \( \mathcal{M}(X; \Lambda) \) can be identified with a pair \( (X, \phi_i) \), where \( \phi_i \) is an idempotent (see [EKM, ch. XII]). Then an isomorphism \( \theta_{ij} \) of degree \( d \) between \( \phi_i \) and \( \phi_j \) can be identified with an isomorphism between the motives \( (X, \phi_i) \) and \( (X, \phi_j)(d) \).

2.3 Definition. Let \( f : A^\ast \rightarrow B^\ast \) be a homomorphism of \( \mathbb{Z} \)-graded rings. We say that \( f \) lifts decompositions if given a family \( \phi_i \in B^0 \) of pair-wise orthogonal idempotents such that \( \sum_i \phi_i = 1_B \), there exists a family of pair-wise orthogonal idempotents \( \varphi_i \in A^0 \) such that \( \sum_i \varphi_i = 1_A \) and each \( f(\varphi_i) \) is isomorphic to \( \phi_i \) by means of an isomorphism of degree 0. We say \( f \) lifts decompositions strictly if, moreover, one can choose \( \varphi_i \) such that \( f(\varphi_i) = \phi_i \).

We say \( f \) lifts isomorphisms if for any idempotents \( \varphi_1 \) and \( \varphi_2 \) in \( A^0 \) and any isomorphism \( \theta_{12} \) of degree \( d \) between idempotents \( f(\varphi_1) \) and \( f(\varphi_2) \) in \( B^0 \) there exists an isomorphism \( \vartheta_{12} \) of degree \( d \) between \( \varphi_1 \) and \( \varphi_2 \). We say \( f \) lifts isomorphisms strictly if, moreover, one can choose \( \vartheta_{12} \) such that \( f(\vartheta_{12}) = \theta_{12} \).

2.4. By definition we have the following properties of morphisms which lift decompositions and isomorphisms (strictly):
(i) Let \( f: A^* \to B^* \) and \( g: B^* \to C^* \) be two morphisms. If both \( f \) and \( g \) lift decompositions or isomorphisms (strictly), then so does the composite \( g \circ f \).

(ii) If \( g \circ f \) lifts decompositions (resp. isomorphisms) and \( g \) lifts isomorphisms, then \( f \) lifts decompositions (resp. isomorphisms).

(iii) Assume we are given a commutative diagram with \( \ker f' \subset \text{im } h \)

\[
\begin{array}{c}
A^* \xrightarrow{f} B^* \\
h \downarrow \quad \downarrow h' \\
A'^* \xrightarrow{f'} B'^*.
\end{array}
\]

If \( f' \) lifts decompositions strictly (resp. isomorphisms strictly), then so does \( f \).

2.5 Lemma. Let \( A, B \) be two rings, \( A^0, B^0 \) be their subrings, \( f^0: A^0 \to B^0 \) be a ring homomorphism, \( f: A \to B \) be a map of sets satisfying the following conditions:

- \( f(\alpha)f(\beta) \) equals either \( f(\alpha\beta) \) or \( 0 \) for all \( \alpha, \beta \in A \);
- \( f^0(\alpha) \) equals \( f(\alpha) \) if \( f(\alpha) \in B^0 \) or \( 0 \) otherwise;
- \( \ker f^0 \) consists of nilpotent elements.

Let \( \varphi_1 \) and \( \varphi_2 \) be two idempotents in \( A^0 \), \( \psi_{12} \) and \( \psi_{21} \) be elements in \( A \) such that \( \psi_{12}A^0\psi_{21} \subset A^0 \), \( \psi_{21}A^0\psi_{12} \subset A^0 \), \( f(\psi_{21})f(\psi_{12}) = f(\varphi_1) \), \( f(\psi_{12})f(\psi_{21}) = f(\varphi_2) \).

Then there exist elements \( \vartheta_{12} \in \varphi_2A^0\psi_{12}A^0\varphi_1 \) and \( \vartheta_{21} \in \varphi_1A^0\psi_{21}A^0\varphi_2 \) such that \( \vartheta_{21}\vartheta_{12} = \varphi_1 \), \( \vartheta_{12}\vartheta_{21} = \varphi_2 \), \( f(\vartheta_{12}) = f(\varphi_2)f(\psi_{12}) = f(\psi_{12})f(\varphi_1) \), \( f(\vartheta_{21}) = f(\varphi_1)f(\psi_{21}) = f(\psi_{21})f(\varphi_2) \).

Proof. Since \( \ker f^0 \) consists of nilpotents, \( f^0 \) sends non-zero idempotents in \( A^0 \) to non-zero idempotents in \( B^0 \); in particular, \( f(\varphi_1) = f^0(\varphi_1) \neq 0 \), \( f(\varphi_2) = f^0(\varphi_2) \neq 0 \). Observe that

\[ f(\psi_{12})f(\varphi_1) = f(\psi_{12})f(\psi_{21})f(\psi_{12}) = f(\varphi_2)f(\psi_{12}) \]
and, similarly, \( f(\psi_2)f(\varphi_2) = f(\varphi_1)f(\psi_2) \). Changing \( \psi_2 \psi_2 \varphi_1 \) and \( \psi_2 \psi_2 \varphi_2 \) we may assume that \( \psi_2 \in \varphi_2 A \varphi_1 \) and \( \psi_2 \in \varphi_1 A \varphi_2 \). We have

\[
f^0(\varphi_2) = f(\varphi_2) = f(\psi_2)f(\psi_2) = f(\psi_2 \psi_2) = f^0(\psi_2 \psi_2).
\]

Therefore \( \alpha = \psi_2 \psi_2 \varphi_2 \in A^0 \) is nilpotent, say \( \alpha^n = 0 \). Note that \( \varphi_2 \alpha = \alpha \varphi_2 \). Set \( \alpha^\vee = \varphi_2 - \alpha + \ldots + (-1)^{n-1} \alpha^{n-1} \in A^0 \); then \( \alpha \alpha^\vee = \varphi_2 - \alpha^\vee \).

\[
\varphi_2 \alpha^\vee = \alpha^\vee \alpha \varphi_2 \quad \text{and} \quad f(\varphi_2) = f^0(\varphi_2) = f^0(\alpha^\vee) = f(\alpha^\vee).
\]

Therefore setting \( \vartheta_21 = \psi_2 \varphi_2 \alpha^\vee \) we have \( \vartheta_21 \psi_21 \alpha^\vee = (\varphi_2 + \alpha) \alpha^\vee = \alpha^\vee + \varphi_2 - \alpha^\vee = \varphi_2 \)

and \( f(\psi_2)f(\varphi_2) = f(\psi_2)f(\alpha^\vee) = f(\psi_21 \alpha^\vee) = f(\vartheta_21) \). This also implies that \( \vartheta_21 \psi_21 \) is an idempotent.

We have

\[
f^0(\varphi_1) = f(\varphi_1) = f(\vartheta_21)f(\psi_2) = f(\vartheta_21 \psi_2) = f^0(\vartheta_21 \psi_2),
\]

where the last equality holds, since \( f(\vartheta_21 \psi_2) = f^0(\varphi_1) \) belongs to \( B^0 \) and \( f^0 \) satisfies the second condition. Therefore \( \beta = \vartheta_21 \psi_2 - \varphi_2 \in A^0 \) is nilpotent.

Note that \( \beta \varphi_1 = \beta = \varphi_1 \beta \). Now \( \varphi_1 + \beta = (\varphi_1 + \beta)^2 = \varphi_1 + 2\beta + \beta^2 \) and therefore \( \beta(1 + \beta) = 0 \). But \( 1 + \beta \) is invertible and hence we have \( \beta = 0 \). It means that \( \vartheta_21 \psi_2 = \varphi_1 \) and we can set \( \vartheta_1 = \psi_2 \).

\[\Box\]

2.6 Proposition. Let \( f : A^* \to B^* \) be a surjective homomorphism such that the kernel of the restriction of \( f \) to \( A^0 \) consists of nilpotent elements. Then \( f \) lifts decompositions and isomorphisms strictly.

Proof. The fact that \( f \) lifts decompositions strictly follows from \([AF92, Proposition 27.4]\).

Let \( \varphi_1 \) and \( \varphi_2 \) be two idempotents in \( A^0 \) and \( \theta_12 \) be an isomorphism between \( f(\varphi_1) \) and \( f(\varphi_2) \). Let \( \psi_12 \) in \( A \) (resp. \( \psi_21 \)) be a homogeneous lifting of \( \theta_12 \) (resp. \( \theta_21 \)). The proposition follows now from Lemma 2.5. \[\Box\]

2.7 Corollary. Let \( m \) be an integer and \( m = p_1^{n_1} \cdots p_i^{n_i} \) be its prime factorization. Then the product of reduction maps

\[
\text{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) \to \prod_{i=1}^l \text{End}^*(\mathcal{M}(X; \mathbb{Z}/p_i))
\]

lifts decompositions and isomorphisms strictly.
Proof. We apply Proposition 2.6 to the case $A^* = \text{End}^*(\mathcal{M}(X; \mathbb{Z}/p_i^n))$, $B^* = \text{End}^*(\mathcal{M}(X; \mathbb{Z}/p_i))$ and the reduction map $f_i: A^* \to B^*$. We obtain that $f_i$ lifts decompositions and isomorphisms strictly for each $i$. To finish the proof observe that by the Chinese remainder theorem $\text{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) \simeq \prod_{i=1}^l \text{End}^*(\mathcal{M}(X; \mathbb{Z}/p_i^n))$. \hfill \Box

We are coming to the following important definition.

2.8 Definition. Let $X$ be a smooth projective variety over a field $F$. Assume that $X$ has a splitting field (see 1.4). We say that Rost Nilpotence holds for $X$ if the kernel of the restriction map

$$\text{res}_E: \text{End}^*(\mathcal{M}(X_E; \Lambda)) \to \text{End}^*(\mathcal{M}(X; \Lambda))$$

consists of nilpotent elements for all field extensions $E/F$ and all rings of coefficients $\Lambda$.

2.9 Lemma. Let $X$ be a smooth projective variety which splits over any field over which it has a rational point. Then Rost Nilpotence holds for $X$.

Proof. By [EKM, Theorem 67.1] if $\alpha$ is in the kernel of the restriction map $\text{res}_E$ then $\alpha^{\varphi_{\dim X+1}} = 0$. \hfill \Box

2.10 Lemma. Assume that Rost Nilpotence holds for $X$. Then for any field extension $E/F$ the restriction $\text{res}_E: \text{End}^*(\mathcal{M}(X_E; \Lambda)) \to \text{im}(\text{res}_E)$ onto the image lifts decompositions and isomorphisms strictly.

Proof. Apply Proposition 2.6 to the homomorphism $\text{res}_E: A^* \to B^*$ between the graded rings $A^* = \text{End}^*(\mathcal{M}(X_E; \Lambda))$ and $B^* = \text{im}(\text{res}_E)$. \hfill \Box

2.11 Corollary. Assume that Rost Nilpotence holds for $X$. Let $m$ be an integer and $E/F$ be a field extension of degree coprime to $m$ which is rank preserving with respect to $X \times X$ (see 1.3). Then the restriction map

$$\text{res}_{E/F}: \text{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) \to \text{End}^*(\mathcal{M}(X_E; \mathbb{Z}/m))$$

lifts decompositions and isomorphisms.

Proof. By Lemma 1.10 we have $\text{im}(\text{res}_E) = \text{im}(\text{res}_F)$. We apply now Lemma 2.10 and 2.3(ii) with $A^* = \text{End}^*(\mathcal{M}(X; \mathbb{Z}/m))$, $B^* = \text{End}^*(\mathcal{M}(X_E; \mathbb{Z}/m))$ and $C^* = \text{im}(\text{res}_E)$. \hfill \Box
2.12 Definition. Let $V^*$ be a free graded $\Lambda$-module of finite rank and $A^* = \text{End}^*(V^*)$ be its ring of endomorphisms, where $\text{End}^{-d}(V^*)$, $d \in \mathbb{Z}$, is the group of endomorphisms of $V^*$ decreasing the degree by $d$. Assume we are given a direct sum decomposition of $V^* = \bigoplus_i \text{im} \phi_i$ by means of idempotents $\phi_i$ in $A^*$. We say that this decomposition is $\Lambda$-free if all graded components of $\text{im} \phi_i$ are free $\Lambda$-modules. Observe that if $\Lambda = \mathbb{Z}$ or $\mathbb{Z}/p$, where $p$ is prime, then any decomposition is $\Lambda$-free.

2.13 Example. Assume $X$ has a splitting field. Define $V^* = \text{CH}^*(\mathbb{X})$. Then by Poincaré duality and by the Künneth decomposition (see 1.7) we have $\text{End}^*(V^*) = \text{End}^*(\mathcal{M}(\mathbb{X}))$ (see Example 2.2).

2.14 Lemma. The map $\text{SL}_l(\mathbb{Z}) \rightarrow \text{SL}_l(\mathbb{Z}/m)$ induced by the reduction modulo $m$ is surjective.

Proof. Since $\mathbb{Z}/m$ is a semi-local ring, the group $\text{SL}_l(\mathbb{Z}/m)$ is generated by elementary matrices (see [HOM, Theorem 4.3.9]).

2.15 Proposition. Consider a free graded $\mathbb{Z}$-module $V^*$ of finite rank and the reduction map $f : \text{End}^*(V^*) \rightarrow \text{End}^*(V^* \otimes \mathbb{Z}/m)$. Then $f$ lifts $\mathbb{Z}/m$-free decompositions strictly. Moreover, if $(\mathbb{Z}/m)^* = \{\pm 1\}$, then $f$ lifts isomorphisms of $\mathbb{Z}/m$-free decompositions strictly.

Proof. We are given a decomposition $V^k \otimes \mathbb{Z}/m = \bigoplus_i W^k_i$, where $W^k_i$ is the $k$-graded component of $\text{im} \phi_i$. Present $V^k$ as a direct sum $V^k = \bigoplus_i V^k_i$ of free $\mathbb{Z}$-modules such that $\text{rk}_\mathbb{Z} V^k_i = \text{rk}_\mathbb{Z}/m W^k_i$. Fix a $\mathbb{Z}$-basis $\{v^k_{ij}\}_j$ of $V^k_i$. For each $W^k_i$ choose a basis $\{w^k_{ij}\}_j$ such that the linear transformation $D^k$ of $V^k \otimes \mathbb{Z}/m$ sending each $v^k_{ij} \otimes 1$ to $w^k_{ij}$ has determinant 1. By Lemma 2.14 there is a lifting $\tilde{D}^k$ of $D^k$ to a linear transformation of $V^k$. So we obtain $V^k = \bigoplus_i \tilde{W}^k_i$, where $\tilde{W}^k_i = \tilde{D}^k(V^k_i)$ satisfies $\tilde{W}^k_i \otimes \mathbb{Z}/m = W^k_i$. It remains to define $\varphi_i$ on each $V^k$ to be the projection onto $\tilde{W}^k_i$.

Now let $\varphi_1, \varphi_2$ be two idempotents in $\text{End}^*(V^*)$. Denote by $V^k_1$ the $k$-graded component of $\text{im} \varphi_1$. An isomorphism $\theta_{12}$ between $\varphi_1 \otimes 1$ and $\varphi_2 \otimes 1$ of degree $d$ can be identified with a family of isomorphisms $\theta_{12}^k : V^k_1 \otimes \mathbb{Z}/m \rightarrow V^k_2 \otimes \mathbb{Z}/m$. In the case $(\mathbb{Z}/m)^* = \{\pm 1\}$ all these isomorphisms are given by matrices with determinants $\{\pm 1\}$ and, hence, can be lifted to isomorphisms $\vartheta_{12}^k : V^k_1 \rightarrow V^k_2 \otimes \mathbb{Z}/m$. by Lemma 2.14.

Now we are ready to state and to prove the main result of this section.
2.16 Theorem. Let $X$ be a smooth projective irreducible variety over a field $F$. Assume that $X$ has a splitting field of degree $m$ which is rank preserving with respect to $X \times X$. Assume that Rost Nilpotence holds for $X$. Consider only decompositions of $\mathcal{M}(X; \mathbb{Z}/m)$ which become $\mathbb{Z}/m$-free over the splitting field. Then the reduction map

$$f: \text{End}^*(\mathcal{M}(X; \mathbb{Z})) \to \text{End}^*(\mathcal{M}(X; \mathbb{Z}/m))$$

lifts such decompositions. If additionally $(\mathbb{Z}/m)^\times = \{\pm 1\}$, then this map lifts isomorphisms of such decompositions.

Proof. Consider the diagram

\[
\begin{array}{cccc}
\text{End}^*(\mathcal{M}(X; \mathbb{Z})) & \longrightarrow & \text{im}(\text{res}_F)^h & \longrightarrow & \text{End}^*(\mathcal{M}(\overline{X}; \mathbb{Z})) \\
\downarrow f & & \downarrow f & & \downarrow f' \\
\text{End}^*(\mathcal{M}(X; \mathbb{Z}/m)) & \longrightarrow & \text{im}(\text{res}_F)^h & \longrightarrow & \text{End}^*(\mathcal{M}(X; \mathbb{Z}/m))
\end{array}
\]

Recall that we can identify $\text{End}^{-d}(\mathcal{M}(\overline{X}))$ with the group of endomorphisms of $\text{CH}^*(\overline{X})$ which decrease the grading by $d$ (see 2.13). Applying Proposition 2.15 to the case $V^* = \text{CH}^*(\overline{X})$ we obtain that the map $f'$ lifts decompositions strictly. Moreover, if $(\mathbb{Z}/m)^\times = \{\pm 1\}$ then $f'$ lifts isomorphisms strictly.

By Lemma 1.10 $\ker f' \subset \text{im } h$ and, therefore, applying 2.4(iii) we obtain that $f$ lifts decompositions strictly and, moreover, $f$ lifts isomorphisms strictly if $(\mathbb{Z}/m)^\times = \{\pm 1\}$.

Now by Lemma 2.10 the horizontal arrows of the left square lift decompositions and isomorphisms strictly. It remains to apply 2.4(i) and (ii).

3 Motives of fibered spaces

In the present section we discuss motives of cellular fibration. The main result (Theorem 3.7) generalizes and uniformizes the proofs of paper [CPSZ].

3.1 Definition. Let $X$ be a smooth projective variety over a field $F$. We say a smooth projective morphism $f: Y \to X$ is a cellular fibration if it is a locally trivial fibration whose fiber $\mathcal{F}$ is cellular, i.e., has a decomposition into affine cells (see [EKM, §66]).
3.2 Lemma. Let $f : Y \to X$ be a cellular fibration. Then $\mathcal{M}(Y)$ is isomorphic to $\mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.

Proof. We follow the proof of [EG97, Proposition 1]. Define the morphism

$$\varphi : \bigoplus_{i \in I} \mathcal{M}(X)(\text{codim } B_i) \to \mathcal{M}(Y)$$

to be the direct sum $\varphi = \bigoplus_{i \in I} \varphi_i$, where each $\varphi_i$ is given by the cycle $[\text{pr}_Y^*(B_i) \cdot \Gamma_f] \in \text{CH}(X \times Y)$ produced from the graph cycle $\Gamma_f$ and the chosen (non-canonical) basis $\{B_i\}_{i \in I}$ of $\text{CH}(Y)$ over $\text{CH}(X)$. The realization of $\varphi$ coincides exactly with the isomorphism of abelian groups $\text{CH}(X) \otimes \text{CH}(\mathcal{F}) \to \text{CH}(Y)$ constructed in [EG97, Proposition 1]. Then by Manin’s identity principle (see [Ma68, §3]) $\varphi$ is an isomorphism. \hfill \square

3.3 Lemma. Let $G$ be a linear algebraic group over a field $F$, $X$ be a projective homogeneous $G$-variety and $Y$ be a $G$-variety. Let $f : Y \to X$ be a $G$-equivariant projective morphism. Assume that the fiber of $f$ over $F(X)$ is isomorphic to $F(F)$ for some variety $F$ over $F$. Then $f$ is a locally trivial fibration with fiber $F$.

Proof. By the assumptions, we have $Y \times_X \text{Spec } F(X) \simeq (F \times X) \times_X \text{Spec } F(X)$ as schemes over $F(X)$. Since $F(X)$ is a direct limit of $\mathcal{O}(U)$ taken over all non-empty affine open subsets $U$ of $X$, by [EGA IV, Corollaire 8.8.2.5] there exists $U$ such that $f^{-1}(U) = Y \times_X U$ is isomorphic to $(F \times X) \times_X U \simeq F \times U$ as a scheme over $U$. Since $G$ acts transitively on $X$ and $f$ is $G$-equivariant, the map $f$ is a locally trivial fibration. \hfill \square

3.4 Corollary. Let $X$ be a projective homogeneous $G$-variety, $Y$ be a projective variety such that $Y_F(X) \simeq \mathcal{F}_F(X)$ for some variety $\mathcal{F}$ over $F$. Then the projection map $X \times Y \to X$ is a locally trivial fibration with fiber $\mathcal{F}$. Moreover, if $\mathcal{F}$ is cellular, then $\mathcal{M}(X \times Y) \simeq \mathcal{M}(X) \otimes \mathcal{M}(\mathcal{F})$.

Proof. Apply Lemma 3.3 to the projection map $X \times Y \to X$ and use Lemma 3.2. \hfill \square

3.5 Lemma. Let $G$ be a semisimple linear algebraic group over $F$, $X$ and $Y$ be projective homogeneous $G$-varieties corresponding to parabolic subgroups $P$ and $Q$ of the split form $G_0$, $Q \subseteq P$. Denote by $f : Y \to X$ the map induced by the quotient map. If $G$ splits over $F(X)$ then $f$ is a cellular fibration with fiber $\mathcal{F} = P/Q$. 

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Proof. Since $G$ splits over $F(X)$, the fiber of $f$ over $F(X)$ is isomorphic to $(P/Q)_{F(X)} = F_{F(X)}$. Now apply Lemma 3.3 and note that $F$ is cellular. 

3.6 Example. Let $P = P_\Theta$ be the standard parabolic subgroup of a split simple group $G_0$, corresponding to a subset $\Theta$ of the respective Dynkin diagram $\mathcal{D}$ (our enumeration of roots follows Bourbaki). In this notation the Borel subgroup corresponds to the empty set. Let $\xi$ be a cocycle in $H^1(F, G_0)$. Set $G = \xi G_0$ and $X = \xi(G_0/P)$. In particular, $G$ is a group of inner type and $X$ is the respective projective homogeneous $G$-variety. Denote by $q$ the degree of a splitting field of $G$ and by $d$ the index of the associated Tits algebra (see Table II). For groups of type $D_n$, we set $d$ to be the index of the Tits algebra associated with the vector representation.

Analyzing Tits indices of $G$ we see that $G$ becomes split over $F(X)$ if the subset $\mathcal{D} \setminus \Theta$ contains one of the following vertices $k$ (cf. [KR94, §7]):

| $G_0$ | $^1A_n$ | $B_n$ | $C_n$ | $^1D_n$ | $G_2$ |
|-------|--------|-------|-------|--------|-------|
| $k$   | $\gcd(k, d) = 1$ | $k = n$; | $k$ is odd; | $k = n - 1, n$ if $d = 1$; | any $k$ in the Pfister case |
|       | any $k$ in the Pfister case |     |     | any $k$ in the Pfister case |       |
| $G_0$ | $F_4$  | $^1E_6$ | $E_7$ | $E_8$ |
| $k$   | $k = 1, 2, 3$; | $k = 3, 5$; | $k = 2, 5$; | $k = 2, 3, 4, 5$; |
|       | any $k$ if $q = 3$ | $k = 2, 4$ if $d = 1$; | $k = 3, 4$ if $d = 1$; | any $k$ if $q = 3$ |
|       | $k = 1, 6$ if $q$ is odd | $k \neq 7$ if $q = 3$ | $q = 5$ |       |

(here by the Pfister case we mean the case when the cocycle $\xi$ corresponds to a Pfister form or its maximal neighbor)

Case-by-case arguments of paper [CPSZ] show that under certain conditions the Chow motive of a twisted flag variety $X$ can be expressed in terms of the motive of a minimal flag. These conditions cover almost all twisted flag varieties corresponding to groups of types $A_n$ and $B_n$ together with some examples of types $C_n$, $G_2$ and $F_4$. The following theorem together with Table 3.6 provides a uniform proof of these results and extends them to some other types.

3.7 Theorem. Let $Y$ and $X$ be taken as in Lemma 3.5. Then the Chow motive $\mathcal{M}(Y)$ of $Y$ is isomorphic to a direct sum of twisted copies of the motive $\mathcal{M}(X)$, i.e.,

$$\mathcal{M}(Y) \simeq \bigoplus_{i \geq 0} \mathcal{M}(X)(i)^{\oplus c_i}.$$
where $\sum c_i t^i = P(\mathrm{CH}_*(\overline{Y}), t)/P(\mathrm{CH}_*(\overline{X}), t)$ is the quotient of the respective Poincaré polynomials.

**Proof.** Apply Lemmas 3.5 and 3.2. \hfill \Box

**3.8 Remark.** An explicit formula for $P(\mathrm{CH}_*(\overline{X}), t)$ involves degrees of the basic polynomial invariants of $G_0$ and is provided in [Hi82] Ch. IV, Cor. 4.5.

### 4 J-invariant and its properties

Fix a prime integer $p$. To simplify the notation we denote by $\mathrm{Ch}(X)$ the Chow ring of a variety $X$ with $\mathbb{Z}/p$-coefficients and by $\overline{\mathrm{Ch}}(X)$ the image of the restriction map $\mathrm{CH}(X; \mathbb{Z}/p) \to \mathrm{CH}(X; \mathbb{Z}/p)$.

**4.1.** Let $G_0$ be a split semisimple linear algebraic group over a field $F$ with a split maximal torus $T$ and a Borel subgroup $B$ containing $T$. Let $G = \xi G_0$ be a twisted form of $G_0$ given by a cocycle $\xi \in H^1(F, G_0)$. Let $\mathfrak{X} = \xi(G_0/B)$ be the corresponding variety of complete flags. Observe that the group $G$ splits over any field $K$ over which $\mathfrak{X}$ has a rational point, in particular, over the function field $F(\mathfrak{X})$. According to [De74] the Chow ring $\overline{\mathrm{Ch}}(\mathfrak{X})$ can be expressed in purely combinatorial terms and, therefore, depends only on the type of $G$ but not on the base field $F$.

**4.2.** Let $\hat{T}$ denote the group of characters of $T$ and $S(\hat{T})$ be the symmetric algebra. By $R$ we denote the image of the characteristic map $c: S(\hat{T}) \to \overline{\mathrm{Ch}}(\mathfrak{X})$ (see [Gr58] (4.1)). According to [KM06] Thm.6.4 there is an embedding

$$R \subseteq \overline{\mathrm{Ch}}(\mathfrak{X}),$$

where the equality holds if the cocycle $\xi$ corresponds to a generic torsor.

**4.3.** Let $\overline{\mathrm{Ch}(G)}$ denote the Chow ring with $\mathbb{Z}/p$-coefficients of the group $G_0$ over a splitting field of $\mathfrak{X}$. Consider the pull-back induced by the quotient map

$$\pi : \overline{\mathrm{Ch}(\mathfrak{X})} \to \overline{\mathrm{Ch}(G)}$$

According to [Gr58] p. 21, Rem. 20 $\pi$ is surjective with its kernel generated by $R^+$, where $R^+$ stands for the subgroup of the non-constant elements of $R$.  

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4.4. An explicit presentation of $\text{Ch}(\overline{G})$ is known for all types of $G$ and all torsion primes $p$ of $G$ (see [Gr58, Definition 3]). Namely, by [Ke85, Theorem 3] it is a quotient of the polynomial ring in $r$ variables $x_1, \ldots, x_r$ of codimensions $d_1 \leq d_2 \leq \ldots \leq d_r$ coprime to $p$, modulo an ideal generated by certain $p$-powers $x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}$ ($k_i \geq 0$, $i = 1, \ldots, r$)

$$\text{Ch}^*(\overline{G}) = (\mathbb{Z}/p)[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}).$$

(2)

In the case where $p$ is not a torsion prime of $G$ we have $\text{Ch}^*(\overline{G}) = \mathbb{Z}/p$, i.e., $r = 0$.

Note that a complete list of numbers $\{d_i p^{k_i}\}_{i=1}^{r}$ called $p$-exceptional degrees of $G_0$ is provided in [Ke85, Table II]. Taking the $p$-primary and $p$-coprimary parts of each $p$-exceptional degree from this table one restores the respective $k_i$ and $d_i$.

4.5. We introduce two orders on the set of additive generators of $\text{Ch}(\overline{G})$, i.e., on the monomials $x_1^{m_1} \cdots x_r^{m_r}$. To simplify the notation, we will denote the monomial $x_1^{m_1} \cdots x_r^{m_r}$ by $x^M$, where $M$ is an $r$-tuple of integers $(m_1, \ldots, m_r)$. The codimension of $x^M$ will be denoted by $|M|$. Observe that $|M| = \sum_{i=1}^{r} d_i m_i$.

- Given two $r$-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$ we say $x^M \preceq x^N$ (or equivalently $M \preceq N$) if $m_i \leq n_i$ for all $i$. This gives a partial ordering on the set of all monomials ($r$-tuples).
- Given two $r$-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$ we say $x^M \leq x^N$ (or equivalently $M \leq N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \leq n_i$ for the greatest $i$ such that $m_i \neq n_i$. This gives a well-ordering on the set of all monomials ($r$-tuples) known as the $\text{DegLex}$ order.

Now we are ready to give the main definition of the present paper.

4.6 Definition. Let $G = \xi G_0$ be the twisted form of a split semisimple algebraic group $G_0$ over a field $F$ by means of a cocycle $\xi \in H^1(F, G_0)$ and $\mathfrak{X} = \xi(G_0/B)$ the respective variety of complete flags. Let $\mathcal{C}(G)$ denote the image of the composite

$$\text{Ch}(\mathcal{X}) \xrightarrow{\text{res}} \text{Ch}(\overline{\mathcal{X}}) \xrightarrow{\pi} \text{Ch}(\overline{G})$$

Since both maps are ring homomorphisms, $\mathcal{C}(G)$ is a subring of $\text{Ch}(\overline{G})$. 17
For each $1 \leq i \leq r$ set $j_i$ to be the smallest non-negative integer such that the subring $\mathcal{C}h(G)$ contains an element $a$ with the greatest monomial $x_i^{p^j_i}$ with respect to the $\text{DegLex}$ order on $\mathcal{C}h(G)$, i.e., of the form

$$a = x_i^{p^j_i} + \sum_{x^M \preceq x_i^{p^j_i}} c_M x^M, \quad c_M \in \mathbb{Z}/p.$$ 

The $r$-tuple of integers $(j_1, \ldots, j_r)$ will be called the $J$-invariant of $G$ modulo $p$ and will be denoted by $J_p(G)$.

Observe that if the Chow ring $\mathcal{C}h(G)$ has only one generator, i.e., $r = 1$, then the $J$-invariant is equal to the smallest non-negative integer $j_1$ such that $x_i^{p^j_i} \in \mathcal{C}h(G)$.

**4.7 Example.** From the definition it follows that $J_p(G_E) \leq J_p(G)$ for any field extension $E/F$. Moreover, $J_p(G) \leq (k_1, \ldots, k_r)$ by (2).

According to (1) the $J$-invariant takes its maximal possible value $J_p(G) = (k_1, \ldots, k_r)$ if the cocycle $\xi$ corresponds to a generic torsor. Later on (see Corollary 6.7) it will be shown that the $J$-invariant takes its minimal value $J_p(G) = (0, \ldots, 0)$ if and only if the group $G$ splits by a finite field extension of degree coprime to $p$.

The next example explains the terminology ‘$J$-invariant’.

**4.8 Example.** Let $\phi$ be a quadratic form with trivial discriminant. In [Vi05, Definition 5.11] A. Vishik introduced the notion of the $J$-invariant of $\phi$, a tuple of integers which describes the subgroup of rational cycles on the respective maximal orthogonal Grassmannian. This invariant provides an important tool for study of algebraic cycles on quadrics. In particular, it was one of the main ingredients used by A. Vishik in his significant progress on the solution of Kaplansky’s Problem. More precisely, in the notation of paper [Vi06] the $J$-invariant of a quadric corresponds to the upper row of its elementary discrete invariant (see [Vi06, Definition 2.2]).

An equivalent but ‘dual’ (in terms of non-rationality of cycles) definition of $J(\phi)$ was provided in [EKM, § 88]. Using Theorem 3.7 one can show that $J(\phi)$ introduced in [EKM] can be expressed in terms of $J_2(O^+(\phi)) = (j_1, \ldots, j_r)$ as follows:

$$J(\phi) = \{2^l d_i \mid i = 1, \ldots, r, \ 0 \leq l \leq j_i - 1\}.$$ 

Since all $d_i$ are odd, $J_2(O^+(\phi))$ is uniquely determined by $J(\phi)$. 

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We have the following reduction formula (cf. [EKM] Cor. 88.7) in the case of quadrics).

**4.9 Proposition.** Let \( G \) be a semisimple group of inner type over a field \( F \) and \( \mathcal{X} \) the variety of complete \( G \)-flags. Let \( Y \) be a projective variety such that the map \( \text{CH}^l(Y) \to \text{CH}^l(Y_F(x)) \) is surjective for all \( x \in X \) and \( l \leq n \). Then \( j_i(G) = j_i(G_F(Y)) \) for all \( i \) such that \( d_{p^l}(G_F(Y)) \leq n \).

**Proof.** By [EKM] Lemma 88.5 the map \( \text{CH}^l(X) \to \text{CH}^l(X_F(Y)) \) is surjective for all \( l \leq n \) and, therefore \( j_i(G) \leq j_i(G_F(Y)) \). The converse inequality is obvious. \( \square \)

**4.10 Corollary.** \( J_p(G) = J_p(G_F(Y)) \).

**Proof.** Take \( Y = \mathbb{P}^1 \) and apply Proposition 4.9. \( \square \)

**4.11.** To find restrictions on the possible values of \( J_p(G) \) we use Steenrod \( p \)-th power operations introduced by P. Brosnan. Recall (see [Br03]) that if the characteristic of the base field \( F \) is different from \( p \) then one can construct Steenrod \( p \)-th power operations

\[
S^l : \text{Ch}^*(X) \to \text{Ch}^{*+l(p-1)}(X), \quad l \geq 0
\]

such that \( S^0 = \text{id} \), the restriction \( S^l|_{\text{Ch}^i(X)} \) coincides with taking to the \( p \)-th power, \( S^l|_{\text{Ch}^i(X)} = 0 \) for \( l > i \), and the total operation \( S^* = \sum_{l \geq 0} S^l \) is a homomorphism of \( \mathbb{Z}/p \)-algebras compatible with pull-backs. In particular, Steenrod operations preserve rationality of cycles.

In the case of projective homogeneous varieties over the field of complex numbers \( S^l \) is compatible with its topological counterparts: the reduced power operation \( \mathcal{P}^l \) if \( p \neq 2 \) and the Steenrod square \( \mathcal{S}q^{2l} \) if \( p = 2 \) (over complex numbers \( \text{Ch}^*(\mathcal{X}) \) can be viewed as a subring of the singular cohomology \( H^{2*}_{\text{sing}}(\mathcal{X}, \mathbb{Z}/p) \)). Moreover, \( \text{Ch}^*(\mathcal{G}) \) may be identified with the image of the pull-back map \( H^{2*}_{\text{sing}}(\mathcal{X}, \mathbb{Z}/p) \to H^{2*}_{\text{sing}}(\mathcal{G}, \mathbb{Z}/p) \). An explicit description of this image and formulae describing the action of \( \mathcal{P}^l \) and \( \mathcal{S}q^{2l} \) on \( H^{*}_{\text{sing}}(\mathcal{G}, \mathbb{Z}/p) \) can be found in [MT91] for exceptional groups and in [EKM] for classical groups.

The action of the Steenrod operations on \( \text{Ch}(\mathcal{X}) \) and on \( \text{Ch}(\mathcal{G}) \) can be described in purely combinatorial terms (see [DZ07]) and, hence, doesn’t depend on the choice of a base field \( F \).

The following lemma provides an important technical tool for computing possible values of the \( J \)-invariant of \( G \).
4.12 Lemma. Assume that in $\text{Ch}^*(G)$ we have $S^i(x_i) = x_m^{p_i}$, and for any $i' < i$ $S^{i'}(x_{i'}) < x_m^{p_i}$ with respect to the DegLex order. Then $j_m \leq j_i + s$.

Proof. By definition there exists a cycle $\alpha \in \text{Ch}(\mathfrak{X})$ such that the leading term of $\pi(\alpha)$ is $x_i^{p_i}$. For the total operation we have

$$S(x_i^{p_i}) = S(x_i)^{p_i} = S^0(x_i)^{p_i} + S^1(x_i)^{p_i} + \ldots + S^d(x_i)^{p_i}.$$ 

In particular, $S^{p_i}(x_i^{p_i}) = S^i(x_i)^{p_i}$. Applying $S^{p_i}$ to $\alpha$ we obtain a rational cycle whose image under $\pi$ has the leading term $x_m^{p_{i+s}}$. \qed

4.13. We summarize information about restrictions on the $J$-invariant into the following table (numbers $r$, $d_i$ and $k_i$ are taken from [Kc85, Table II]). Recall that $r$ is the number of generators of $\text{Ch}^*(G)$, $d_i$ are their codimensions and $k_i$ define the $p$-power relations.

| $G_n$ | $\mu_n$, $m$ | $p$ | $r$ | $d_i$ | $k_i$ | $j_i$ |
|-------|---------------|-----|-----|-------|-------|------|
| $SL_n$, $\mu_n$, $m$ | $n$ | $|p|/m$ | 1 | 1 | $p^k1$ | $n$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $PGSp_n$, $2 \mid n$ | 2 | $[n+1]/4$ | 2i - 1 | $\log_2 n + 1$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $SO_n$ | 2 | $[\frac{n-3}{2}]$ | 2i + 1 | $\log_2 n + 1$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $Spin_n$ | 2 | $[\frac{n-3}{2}]$ | 2i + 1 | $\log_2 n + 1$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $POGO_{2n}$, $n > 1$ | 2 | $[n+2]/2$ | 1, $i = 1$ | $2k1, k1$ | $n$ | $\log_2 n + 1$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $Spin^n_{2n}$, $2 \mid n$ | 2 | $[\frac{n-3}{2}]$ | 2i + 1, $i \geq 2$ | $\log_2 n + 1$ | $j_i \geq j_{i+1}$ if $2 \mid (i+1)$, $j_i \leq j_{i+1} + 1$ |
| $G_2$, $F_4$, $E_6$ | 2 | 3 | 1 | 1 | 1 | 1 |
| $F_4$, $E_6^{sc}$, $E_7$ | 3 | 1 | 4 | 1 | 1 | 1 |
| $E_6^{ad}$ | 3 | 2 | 1, 4 | 2, 1 | 1 | 1 |
| $E_7^{ad}$ | 2 | 4 | 1, 3, 5, 9 | 1, 1, 1 | 1 | 1 |
| $E_8$ | 2 | 4 | 3, 5, 9, 15 | 3, 2, 1, 1 | 1 | 1 |
| $E_8$ | 3 | 2 | 4, 10 | 1, 1 | 1 | 1 |
| $E_8$ | 5 | 1 | 6 | 1 | 1 | 1 |

The last column of the table follows from Lemma 4.12 and, hence, requires char $(F) \neq p$ restriction. All other columns are taken directly from [Kc85, Table II] and are independent on the characteristic of the base field.
5 Motivic decompositions

In the present section we prove the main result of this paper (Theorem 5.17). In the beginning of this section we describe a basis of the subring of rational cycles of \( \text{Ch}(\mathfrak{X} \times \mathfrak{X}) \), where \( \mathfrak{X} \) is the variety of complete flags. The key results here are Propositions 5.3 and 5.10. As a consequence, we obtain a motivic decomposition of \( \mathfrak{X} \) (Theorem 5.13) in terms of certain motive \( \mathcal{R}_p(G) \). Then using motivic decompositions of cellular fibrations obtained above (Theorem 3.7) we generalize Theorem 5.13 to arbitrary generically split projective homogeneous varieties. At the end we discuss some properties of the motives \( \mathcal{R}_p(G) \).

5.1. We use the notation of the previous section. Let \( G \) be a semisimple group of inner type over \( F \) and \( \mathfrak{X} \) the respective variety of complete flags. Let \( R \subseteq \text{Ch}(\mathfrak{X}) \) be the image of the characteristic map. Consider the quotient map \( \pi: \text{Ch}(\mathfrak{X}) \rightarrow \text{Ch}(G) \). Fix preimages \( e_i \) of \( x_i \) in \( \text{Ch}(\mathfrak{X}) \). For an \( r \)-tuple \( M = (m_1, \ldots, m_r) \) set \( e^M = \prod_{i=1}^r e_i^{m_i} \). Set \( N = (p^{k_1} - 1, \ldots, p^{k_r} - 1) \) and \( d = \dim \mathfrak{X} - |N| \).

5.2 Lemma. The Chow ring \( \text{Ch}(\mathfrak{X}) \) is a free \( R \)-module with a basis \( \{e^M\} \), \( M \preceq N \).

Proof. Note that the subgroup \( R^+ \) of the non-constant elements of \( R \) is a nilpotent ideal in \( R \). Applying the Nakayama Lemma we obtain that \( \{e^M\} \) generates \( \text{Ch}(\mathfrak{X}) \). By [Kc85, (2)] \( \text{Ch}(\mathfrak{X}) \) is a free \( R \)-module, hence, for the Poincaré polynomials we have

\[
P(\text{Ch}^*(\mathfrak{X}), t) = P(\text{Ch}^*(G), t) \cdot P(R^*, t).
\]

Substituting \( t = 1 \) we obtain that

\[
\text{rk Ch}(\mathfrak{X}) = \text{rk Ch}(G) \cdot \text{rk } R.
\]

To finish the proof observe that \( \text{rk Ch}(G) \) coincides with the number of generators \( \{e^M\} \).

5.3 Proposition. The pairing \( R \times R \rightarrow \mathbb{Z}/p \) given by \( (\alpha, \beta) \mapsto \deg(e^N \alpha \beta) \) is non-degenerated, i.e., for any non-zero element \( \alpha \in R \) there exists \( \beta \) such that \( \deg(e^N \alpha \beta) \neq 0 \).
Proof. Choose a homogeneous basis of $\text{Ch}(\overline{X})$. Let $\alpha^\vee$ be the Poincaré dual of $\alpha$ with respect to this basis. By Lemma 5.2 $\text{Ch}(\overline{X})$ is a free $R$-module with the basis $\{e^M\}$, hence, expanding $\alpha^\vee$ we obtain

$$\alpha^\vee = \sum_{M \preceq N} e^M \beta_M,$$

where $\beta_M \in R$.

Note that if $M \neq N$ then $\text{codim} \alpha \beta_M > d$, therefore, $\alpha \beta_M = 0$. So we can set $\beta = \beta_N$. \hfill $\square$

From now on we fix a homogeneous $\mathbb{Z}/p$-basis $\{\alpha_i\}$ of $R$ and the dual basis $\{\alpha_i^\#\}$ with respect to the pairing introduced in Proposition 5.3.

5.4 Corollary. For $|M| \leq |N|$ we have

$$\text{deg}(e^M \alpha_i \alpha_j^\#) = \begin{cases} 1, & M = N \text{ and } i = j; \\ 0, & \text{otherwise}. \end{cases}$$

Proof. If $M = N$, then it follows from the definition of the dual basis. Assume $|M| < |N|$. If $\text{deg}(e^M \alpha_i \alpha_j^\#) \neq 0$, then $\text{codim}(\alpha_i \alpha_j^\#) > d$, in contradiction with the fact that $\alpha_i \alpha_j^\# \in R$. Hence, we are reduced to the case $M \neq N$ and $|M| = |N|$. Since $|M| = |N|$, $\text{codim}(\alpha_i \alpha_j^\#) = d$ and, hence, $R^+ \alpha_i \alpha_j^\# = 0$. On the other hand there exists $i$ such that $m_i \geq p^{k_i}$ and $e^{p^{k_i}} \in \text{Ch}(\overline{X}) \cdot R^+$. Hence, $e^M \alpha_i \alpha_j^\# = 0$. \hfill $\square$

5.5 Definition. Given two pairs $(L,l)$ and $(M,m)$, where $L,M$ are $r$-tuples and $l,m$ are integers, we say $(L,l) \leq (M,m)$ if either $L \preceq M$, or in the case $L = M$ we have $l \leq m$. We introduce a filtration on the ring $\text{Ch}(\overline{X})$ as follows:

The $(M,m)$-th term $\text{Ch}(\overline{X})_{M,m}$ of the filtration is the $\mathbb{Z}/p$-subspace spanned by the elements $e^I \alpha$ with $I \leq M$, $\alpha \in R$ homogeneous, $\text{codim} \alpha \leq m$.

Define the associated graded ring as follows:

$$A^{**} = \bigoplus_{(M,m)} A^{M,m},$$

where $A^{M,m} = \text{Ch}(\overline{X})_{M,m}/ \bigcup_{(L,l) \preceq (M,m)} \text{Ch}(\overline{X})_{L,l}$.

By Lemma 5.2 if $M \preceq N$ the graded component $A^{M,m}$ consists of the classes of elements $e^M \alpha$ with $\alpha \in R$ and $\text{codim} \alpha = m$. In particular, $\text{rk} A^{M,m} = \text{rk} R^m$. Comparing the ranks we see that $A^{M,m}$ is trivial when $M \not\preceq N$. \hfill 22
Consider the subring \( \overline{\text{Ch}(\mathfrak{X})} \) of rational cycles with the induced filtration. The associated graded subring will be denoted by \( A_{rat}^{*,*} \). From the definition of the \( J \)-invariant it follows that the elements \( e_i^{p^i}, i = 1, \ldots, r \), belong to \( A_{rat}^{*,*} \).

Similarly, we introduce a filtration on the ring \( \text{Ch}(\mathfrak{X} \times \mathfrak{X}) \) as follows:

The \( (M, m) \)-th term of the filtration is the \( \mathbb{Z}/p \)-subspace spanned by the elements \( e^I \alpha \times e^L \beta \) with \( I + L \leq M \), \( \alpha, \beta \in R \) homogeneous and \( \text{codim} \alpha + \text{codim} \beta \leq m \).

The associated graded ring will be denoted by \( B_{rat}^{*,*} \). By definition \( B_{rat}^{*,*} \) is isomorphic to the tensor product of graded rings \( A_{rat}^{*,*} \otimes_{\mathbb{Z}/p} A_{rat}^{*,*} \). The graded subring associated to \( \text{Ch}(\mathfrak{X} \times \mathfrak{X}) \) will be denoted by \( B_{rat}^{*,*} \).

5.6. The key observation is that due to Corollary 5.4 we have

\[
\text{Ch}(\mathfrak{X} \times \mathfrak{X})_{M,m} \circ \text{Ch}(\mathfrak{X} \times \mathfrak{X})_{L,l} \subset \text{Ch}(\mathfrak{X} \times \mathfrak{X})_{M+L-N,m+l-d}
\]

and, therefore, we have a correctly defined composition law

\[
\circ: B^{M,m} \times B^{L,l} \to B^{M+L-N,m+l-d}
\]

and the realization map (see 1.1)

\[
*: B^{M,m} \times A^{L,l} \to A^{M+L-N,m+l-d}
\]

In particular, \( B^{N+,+d,*} \) can be viewed as a graded ring with respect to the composition and \( (\alpha \circ \beta)* = \alpha* \circ \beta* \). Note also that both operations preserve rationality of cycles.

The proof of the following result is based on the fact that the group \( G \) splits over \( F(\mathfrak{X}) \).

5.7 Lemma. The classes of the elements \( e_i \times 1 - 1 \times e_i \) in \( B_{rat}^{*,*}, i = 1, \ldots, r \), belong to \( B_{rat}^{*,*} \).

Proof. Fix an \( i \). Since \( G \) splits over \( F(\mathfrak{X}) \), \( F(\mathfrak{X}) \) is a splitting field of \( \mathfrak{X} \) and by Lemma 1.8 there exists a cycle in \( \overline{\text{Ch}^d(\mathfrak{X} \times \mathfrak{X})} \) of the form

\[
\xi = e_i \times 1 + \sum_s \mu_s \times \nu_s + 1 \times \mu,
\]
where \( \text{codim } \mu_s, \text{codim } \nu_s < d_i \). Then the cycle

\[
\text{pr}^*_{13}(\xi) - \text{pr}^*_{23}(\xi) = (e_i \times 1 - 1 \times e_i) \times 1 + \sum_s (\mu_s \times 1 - 1 \times \mu_s) \times \nu_s
\]

belongs to \( \overline{\text{Ch}}(X \times X \times X) \), where \( \text{pr}_{ij} \) denotes the projection on the product of the \( i \)-th and \( j \)-th factors. Applying Corollary 5.4 to the projection \( \text{pr}_{12} : X \times X \times X \rightarrow X \times X \) we conclude that there exists a (non-canonical) \( \text{Ch}(X \times X) \)-linear isomorphism \( \text{Ch}(X \times X \times X) \simeq \text{Ch}(X \times X) \otimes \text{Ch}(X) \), where \( \text{Ch}(X \times X) \) acts on the left-hand side via \( \text{pr}^*_{12} \). This gives rise to a \( \text{Ch}(X \times X) \)-linear retraction \( \delta \) to the pull-back map \( \text{pr}^*_{12} \). Since the construction of the retraction preserves base change, it preserves rationality of cycles. Hence, passing to a splitting field we obtain a rational cycle

\[
\bar{\delta}(\text{pr}^*_{13}(\xi) - \text{pr}^*_{23}(\xi)) = e_i \times 1 - 1 \times e_i + \sum_s (\mu_s \times 1 - 1 \times \mu_s) \bar{\delta}(1 \times 1 \times \nu_s)
\]

whose image in \( B^{\ast \ast} \) is \( e_i \times 1 - 1 \times e_i \).

We will write \( (e \times 1 - 1 \times e)^M \) for the product \( \prod_{i=1}^r (e_i \times 1 - 1 \times e_i)^{m_i} \) and \( \binom{M}{L} \) for the product of binomial coefficients \( \prod_{i=1}^r \binom{m_i}{l_i} \). We assume that \( \binom{m_i}{l_i} = 0 \) if \( l_i > m_i \). In the computations we will extensively use the following two formulae (the first follows directly from Corollary 5.4 and the second one is a well-known binomial identity).

5.8. Let \( \alpha \) be an element of \( R^\ast \) and \( \alpha^\# \) be its dual with respect to the non-degenerate pairing from 5.3, i.e. \( \text{deg}(e^N \alpha \alpha^\#) = 1 \). Then we have

\[
((e \times 1 - 1 \times e)^M (\alpha^\# \times 1)) \ast (e^L \alpha) = \binom{M}{M + L - N} (-1)^{M + L - N} e^{M + L - N}.
\]

Indeed, expanding the brackets in the left-hand side, we obtain

\[
\left( \sum_{I \leq M} (-1)^I \binom{M}{I} e^{M - I} \alpha^\# \times e^I \right) \ast (e^L \alpha),
\]

and it remains to apply Corollary 5.4

5.9 (Lucas’ Theorem). The following identity holds

\[
\binom{n}{m} \equiv \prod_{i \geq 0} \binom{n_i}{m_i} \mod p,
\]

where \( m = \sum_{i \geq 0} m_i p^i \) and \( n = \sum_{i \geq 0} n_i p^i \) are the base \( p \) presentations of \( m \) and \( n \).
Let \( J = J_p(G) = (j_1, \ldots, j_r) \) be the \( J \)-invariant of \( G \) (see Definition 4.6). Set \( K = (k_1, \ldots, k_r) \).

**5.10 Proposition.** Let \( \{\alpha_i\} \) be a homogeneous \( \mathbb{Z}/p \)-basis of \( R \). Then the set of elements \( \mathcal{B} = \{e^{p^jL}\alpha_i \mid L \leq p^{K-J} - 1\} \) forms a \( \mathbb{Z}/p \)-basis of \( A^*_\text{rat} \).

**Proof.** According to Lemma 5.2 the elements from \( \mathcal{B} \) are linearly independent. Assume \( \mathcal{B} \) does not generate \( A^*_\text{rat} \). Choose an element \( \omega \in A^m_{\text{rat}} \) of the smallest index \((M, m)\) which is not in the linear span of \( \mathcal{B} \). By definition of \( A^m_{\text{rat}} \) (see Definition 5.5) \( \omega \) can be written as \( \omega = e^M\alpha \), where \( M \preceq N \), \( \alpha \in R^m \) and \( M \) can not be presented as \( M = p^jL' \) for an \( r \)-tuple \( L' \). The latter means that in the decomposition of \( M \) into \( p \)-primary and \( p \)-coprimary components \( M = p^S L \), where \( M = (m_1, \ldots, m_r) \), \( S = (s_1, \ldots, s_r) \), \( L = (l_1, \ldots, l_r) \) and \( p \mid l_k \) for \( k = 1, \ldots, r \), we have \( J \not\preceq S \). Choose an \( i \) such that \( s_i < j_i \). Denote \( M_i = (0, \ldots, 0, m_i, 0, \ldots, 0) \) and \( S_i = (0, \ldots, 0, s_i, 0, \ldots, 0) \), where \( m_i \) and \( s_i \) stand at the \( i \)-th place.

Set \( T = N - M + M_i \). By Lemma 5.7 and 5.8 together with observation 5.6 the element

\[
((e \times 1 - 1 \times e)^T(\alpha^\# \times 1))(e^M\alpha) = \left(\frac{p^{k_i} - 1}{m_i}\right)(-1)^{m_i}e^{m_i}
\]

belongs to \( A^m_{\text{rat}} \). By 5.9 we have \( p \nmid \left(\frac{p^{k_i} - 1}{m_i}\right) \) and, therefore, this element is non-trivial. Moreover, since \( s_i < j_i \), this element is not in the span of \( \mathcal{B} \). Since \( (M, m) \) was chosen to be the smallest index and \((M_i, 0) \preceq (M, m) \) we obtain that \((M, m) = (M_i, 0) \). Repeating the same arguments for \( T = N - M_i + p^S_i \) we obtain that \( M_i = p^S_i \), i.e., \( l_i = 1 \).

Now let \( \gamma \) be a representative of \( \omega = e_i^{p^j} \) in \( \overline{\mathfrak{h}}(X) \). Then its image \( \pi(\gamma) \) in \( \mathfrak{h}(G) \) has the leading term \( x_i^{p^j} \) with \( s_i < j_i \). This contradicts the definition of the \( J \)-invariant. \( \square \)

**5.11 Corollary.** The elements

\[
\{e \times 1 - 1 \times e)^S(e^pJ\alpha_i \times e^{p^j(p^{K-J} - 1 - M)\alpha_j^\#}) \mid L, M \preceq p^{K-J} - 1, S \preceq p^j - 1\}
\]

form a \( \mathbb{Z}/p \)-basis of \( B^*_\text{rat} \). In particular, the ones such that \( S = p^j - 1 \) and \( L = M \) form a basis of \( B^N_{\text{rat}} \).

**Proof.** According to Lemma 5.2 these elements are linearly independent and their number is \( p^{2(K-J)(\text{rk } R)^2} \). They are rational by Definition 5.3 and
Lemma 5.7. Applying Corollary 3.4 to the projection $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ we obtain that

$$\text{rk } B_{\text{rat}}^* = \text{rk } \text{Ch}(\mathfrak{X} \times \mathfrak{X}) = \text{rk } \text{Ch}(\mathfrak{X}) \cdot \text{rk } \text{Ch}(\mathfrak{X}),$$

where the latter coincides with $\text{rk } A_{\text{rat}}^* p^{|K|} \text{rk } R = p^{2|K|-|J|}(\text{rk } R)^2$ by Lemma 5.2 and Proposition 5.10.

5.12 Lemma. The elements

$$\theta_{L,M,i,j} = (e \times 1 - 1 \times e)^{p^j-1}(e^{p^j L} \alpha_i \times e^{p^j(p^{|K|-|J|-M}| \alpha_j^\#}), L, M \ll p^{|K|-|J|-1},$$

belong to $B_{\text{rat}}^*$ and satisfy the relations $\theta_{L,M,i,j} \circ \theta_{L',M',i',j'} = \delta_{LM} \delta_{ij} \theta_{L',M',i',j}$ and $\sum_{L,i} \theta_{L,i,i} = \Delta_{\mathfrak{X}}$.

Proof. Expanding the brackets and using the identity $(p^j-1) \equiv (-1)^i \mod p$, we see that

$$\theta_{L,M,i,j} = \sum_{I \ll p^j-1} e^{p^j L+I} \alpha_i \times e^{N-M-1} M-1 \alpha_j^\#,$$

and the composition relation follows from Corollary 5.4. By definition we have

$$\sum_{L,i} \theta_{L,i,i} = \sum_{I \ll N,i} \alpha_i \times e^{N-I} \alpha_i^\#.$$

By Corollary 5.4 the latter sum acts trivially on all basis elements of $\text{Ch}(\mathfrak{X})$ and, hence, coincides with the diagonal.

We are now ready to provide a motivic decomposition of the variety of complete flags.

5.13 Theorem. Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $\mathfrak{X}$ be the variety of complete $G$-flags. Let $p$ be a prime. Assume that $J_p(G) = (j_1, \ldots, j_r)$. Then the motive of $\mathfrak{X}$ is isomorphic to the direct sum

$$\mathcal{M}(\mathfrak{X}; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus c_i},$$

where the motive $\mathcal{R}_p(G)$ is indecomposable, its Poincaré polynomial over a splitting field is given by

$$P(\mathcal{R}_p(G), t) = \prod_{i=1}^r \frac{1 - t^{d_i p^i}}{1 - t^{d_i}}, \quad \text{(3)}$$

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and the integers $c_i$ are the coefficients of the quotient
\[
\sum_{i \geq 0} c_i t^i = P(\text{Ch}^*(\overline{x}), t)/P(\overline{R}_p(G), t).
\]

Proof. Consider the projection map
\[
f^0 : \overline{\text{Ch}}(\overline{x} \times \overline{x})_{N,d} \to B_{\text{rat}}^{N,d}.
\]
Observe that the kernel of $f^0$ is nilpotent. Indeed, any element $\xi$ from $\ker f^0$ belongs to $\text{Ch}(\overline{x} \times \overline{x})_{M,m}$ for some $(M, m) \preceq (N, d)$ which depends on $\xi$. Then by [5.6] its $i$-th composition power $\xi^i$ belongs to the graded component $\text{Ch}(\overline{x} \times \overline{x})_{iM-(i-1)N,im-(i-1)d}$, and, therefore, becomes trivial for $i$ big enough.

By Lemma 5.12 the elements $\theta_{L,L,i,j}$ form a family of pairwise-orthogonal idempotents whose sum is the identity. Therefore, by Proposition 2.6 there exist pair-wise orthogonal idempotents $\varphi_{L,i}$ in $\text{Ch}(\overline{x} \times \overline{x})$ which are mapped to $\theta_{L,L,i,i}$ and whose sum is the identity.

Recall (see [1.1]) that given two correspondences $\phi$ and $\psi$ in $\overline{\text{Ch}}(\overline{x} \times \overline{x})$ of degrees $c$ and $c'$ respectively its composite $\phi \circ \psi$ has degree $c + c'$. Using this fact we conclude that the homogeneous components of $\varphi_{L,i}$ of codimension $\dim \overline{x}$ are pair-wise orthogonal idempotents whose sum is the identity. Hence, we may assume that $\varphi_{L,i}$ belong to $\text{Ch}^{\dim \overline{x}}(\overline{x} \times \overline{x})$.

We now show that $\varphi_{L,i}$ are indecomposable. By Corollary 5.11 and Lemma 5.12 the ring $(B_{\text{rat}}^{*,*}, \circ)$ can be identified with a product of matrix rings over $\mathbb{Z}/p$
\[
B_{\text{rat}}^{N,d} \simeq \prod_{s=0}^d \text{End}((\mathbb{Z}/p)^\beta^{K-s_j} \cdot R^s).
\]
By means of this identification $\theta_{L,L,i,i} : e^{p^i M} \alpha_j \mapsto \delta_{L,M} \delta_{i,j} e^{p^i L} \alpha_i$ is an idempotent of rank 1 and, therefore, is indecomposable. Since the kernel of $f^0$ is nilpotent, the $\varphi_{L,i}$ are indecomposable as well.

Next we show that $\varphi_{L,i}$ is isomorphic to $\varphi_{M,j}$. In the ring $B_{\text{rat}}^{*,*}$ mutually inverse isomorphisms between them are given by $\theta_{L,M,i,j}$ and $\theta_{M,L,j,i}$. Let
\[
f : \overline{\text{Ch}}(\overline{x} \times \overline{x}) \to B_{\text{rat}}^{*,*}
\]
be the leading term map; it means that for any $\gamma \in \overline{\text{Ch}}(\overline{x} \times \overline{x})$ we find the smallest degree $(I, s)$ such that $\gamma$ belongs to $\overline{\text{Ch}}(\overline{x} \times \overline{x})_{I,s}$ and set $f(\gamma)$ to be the image of $\gamma$ in $B_{\text{rat}}^{I,s}$. Note that $f$ is not a homomorphism but satisfies the
condition that \( f(\xi) \circ f(\eta) \) equals either \( f(\xi \circ \eta) \) or 0. Choose preimages \( \psi_{L,M,i,j} \) and \( \psi_{M,L,j,i} \) of \( \theta_{L,M,i,j} \) and \( \theta_{M,L,j,i} \) by means of \( f \). Applying Lemma 2.5 we obtain mutually inverse isomorphisms \( \vartheta_{L,M,i,j} \) and \( \vartheta_{M,L,j,i} \) between \( \varphi_{L,i} \) and \( \varphi_{M,j} \). By the definition of \( f \) it remains to take their homogeneous components of the appropriate degrees.

Applying now Lemma 2.9 and Corollary 2.10 to the restriction map

\[
\text{res}_F: \text{End}(\mathcal{M}(X; \mathbb{Z}/p)) \to \overline{\text{End}}(\mathcal{M}(X; \mathbb{Z}/p))
\]

and the family of idempotents \( \varphi_{L,i} \) we obtain a family of pair-wise orthogonal idempotents \( \phi_{L,i} \in \text{End}(\mathcal{M}(X; \mathbb{Z}/p)) \) such that

\[
\Delta_X = \sum_{L,i} \phi_{L,i}.
\]

Since \( \text{res}_{F, \mathbb{F}_s}/\mathbb{F} \) lifts isomorphisms, for the respective motives we have \((X, \varphi_{L,i}) \simeq (X, \varphi_{0,0})(|L| + \text{codim} \alpha_i)\) for all \( L \) and \( i \) (see 2.2). The twists \(|L| + \text{codim} \alpha_i\) can be easily recovered from the explicit formula for \( \vartheta_{L,L,i,i} \) (see Lemma 5.12). Denoting \( \mathcal{R}_p(G) = (X, \varphi_{0,0}) \) we obtain the desired motivic decomposition.

Finally, consider the motive \( \mathcal{R}_p(G) \) over a splitting field. The idempotent \( \theta_{0,0,0,0} \) splits into the sum of pair-wise orthogonal (non-rational) idempotents \( e^I \times e^{N-I} \# \), \( I \preceq p^l - 1 \). The motive corresponding to each summand is isomorphic to \((\mathbb{Z}/p)(|I|)\). Therefore, we obtain the decomposition into Tate motives

\[
\mathcal{R}_p(G) \simeq \bigoplus_{I \preceq p^l - 1} (\mathbb{Z}/p)(|I|),
\]

which gives formula (3) for the Poincaré polynomial.

As a direct consequence of the proof we obtain

**5.14 Corollary.** Any direct summand of \( \mathcal{M}(X; \mathbb{Z}/p) \) is isomorphic to a direct sum of twisted copies of \( \mathcal{R}_p(G) \).

**Proof.** Indeed, in the ring \( B_{i;i_i}^{N,d} \) any idempotent is isomorphic to a sum of idempotents \( \theta_{L,L,i,i} \), and the map \( f^0 \) lifts isomorphisms.

**5.15 Remark.** Corollary 5.14 can be viewed as a particular case of the Krull-Schmidt Theorem proven by V. Chernousov and A. Merkurjev (see [CM06, Corollary 9.7]).
5.16 Definition. Let $G$ be a linear algebraic group over a field $F$ and $X$ a projective homogeneous $G$-variety. We say $X$ is *generically split* if the group $G$ splits over the generic point of $X$.

The main result of the present paper is the following

5.17 Theorem. Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $p$ be a prime integer. Let $X$ be a generically split projective homogeneous $G$-variety. Then the motive of $X$ with $\mathbb{Z}/p$-coefficients is isomorphic to the direct sum

$$
\mathcal{M}(X; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus a_i},
$$

where $\mathcal{R}_p(G)$ is an indecomposable motive; Poincaré polynomial $P(\overline{\mathcal{R}_p(G)}, t)$ is given by (3) and, hence, only depends on the $J$-invariant of $G$; the $a_i$’s are the coefficients of the quotient polynomial

$$
\sum_{i \geq 0} a_i t^i = P(CH^*(X), t)/P(\overline{\mathcal{R}_p(G)}, t).
$$

Proof. Let $\mathfrak{X}$ be the variety of complete $G$-flags. According to Theorem 3.7 the motive of $Y = \mathfrak{X}$ is isomorphic to a direct sum of twisted copies of the motive of $X$. To finish the proof we apply Theorem 5.13 and Corollary 5.14.

We now provide several properties of $\mathcal{R}_p(G)$ which will be extensively used in the applications.

5.18 Proposition. Let $G$ and $G'$ be two semisimple algebraic groups of inner type over $F$, $\mathfrak{X}$ and $\mathfrak{X}'$ be the corresponding varieties of complete flags.

(i) (base change) For any field extension $E/F$ we have

$$
\mathcal{R}_p(G)_E \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G_E)(i)^{\oplus a_i},
$$

where $\sum a_i t^i = P(\overline{\mathcal{R}_p(G)}, t)/P(\overline{\mathcal{R}_p(G_E)}, t)$.

(ii) (transfer argument) If $E/F$ is a field extension of degree coprime to $p$ then $J_p(G_E) = J_p(G)$ and $\mathcal{R}_p(G_E) = \mathcal{R}_p(G_E)$. Moreover, if $\mathcal{R}_p(G_E) \simeq \mathcal{R}_p(G'_E)$ then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$. 

(iii) (comparison lemma) If $G$ splits over $F(\mathfrak{X}')$ and $G'$ splits over $F(\mathfrak{X})$ then $R_p(G) \simeq R_p(G')$.

Proof. The first claim follows from Theorem 5.13 and Corollary 5.14. To prove the second claim note that $E$ is rank preserving with respect to $\mathfrak{X}$ and $\mathfrak{X} \times \mathfrak{X}$ by Lemma 1.12. Now $J_p(G_E) = J_p(G)$ by Lemma 1.10 and hence $R_p(G_E) = R_p(G)_E$ by the first claim. The remaining part of the claim follows from Corollary 2.11 applied to the variety $\mathfrak{X} \coprod \mathfrak{X}'$.

We now prove the last claim. The variety $\mathfrak{X} \times \mathfrak{X}'$ is the variety of complete $G \times G'$-flags. By Corollary 3.14 applied to the projections $\mathfrak{X} \times \mathfrak{X}' \to \mathfrak{X}$ and $\mathfrak{X} \times \mathfrak{X}' \to \mathfrak{X}'$ we can express $\mathcal{M}(\mathfrak{X} \times \mathfrak{X}'; \mathbb{Z}/p)$ in terms of $\mathcal{M}(\mathfrak{X}; \mathbb{Z}/p)$ and $\mathcal{M}(\mathfrak{X}'; \mathbb{Z}/p)$. The latter motives can be expressed in terms of $R_p(G)$ and $R_p(G')$. Now the claim follows from the Krull-Schmidt theorem (see Corollary 5.14). □

5.19 Corollary. We have $R_p(G) \simeq R_p(G_{an})$, where $G_{an}$ is the semisimple anisotropic kernel of $G$.

Finally, we provide conditions which allow to lift a motivic decomposition of a generically split homogeneous variety with $\mathbb{Z}/m$-coefficients to a decomposition with $\mathbb{Z}$-coefficients.

5.20. Let $m$ be a positive integer. We say a polynomial $g(t)$ is $m$-positive, if $g \neq 0$, $P(R_p(G), t) \mid g(t)$ and the quotient polynomial $g(t)/P(R_p(G), t)$ has non-negative coefficients for all primes $p$ dividing $m$.

5.21 Proposition. Let $G$ be a semisimple linear algebraic group of inner type over a field $F$ and $X$ be a generically split projective homogeneous $G$-variety. Assume that $X$ splits by a field extension of degree $m$. Let $f(t)$ be an $m$-positive polynomial dividing $P(M(X), t)$ which can not be presented as a sum of two $m$-positive polynomials. Then the motive of $X$ with integer coefficients splits as a direct sum

$$\mathcal{M}(X; \mathbb{Z}) \simeq \bigoplus_i R_i(c_i), \quad c_i \in \mathbb{Z},$$

where $R_i$ are indecomposable and $P(R_i, t) = f(t)$ for all $i$. Moreover, if $m = 2, 3, 4$ or 6, then all motives $R_i$ are isomorphic up to twists.

Proof. First, we apply Corollary 2.17 to obtain a decomposition with $\mathbb{Z}/m$-coefficients. By Lemma 1.12 our field extension is rank preserving so we can apply Theorem 2.16 to lift the decomposition to the category of motives with $\mathbb{Z}$-coefficients. □
6 Applications of the $J_p$ and of the motive $R_p$

Let $G$ be a semisimple group of inner type over $F$ and $\mathfrak{X}$ the variety of complete $G$-flags.

First, we obtain the following expression for the canonical $p$-dimension of $\mathfrak{X}$ (see [EKM §90]).

6.1 Proposition. In the notation of Theorem 5.13 we have

$$\text{cd}_p(\mathfrak{X}) = \sum_{i=1}^r d_i (p^{i_i} - 1).$$

Proof. Follows from Proposition 5.10 and [KM06, Theorem 5.8].

Let $X$ be a smooth projective variety which has a splitting field.

6.2 Lemma. For any $\phi, \psi \in \text{CH}^*(X \times X)$ one has

$$\deg((\text{pr}_2)_*(\phi \cdot \psi^t)) = \text{tr}((\phi \circ \psi)_*).$$

Proof. Choose a homogeneous basis $\{e_i\}$ of $\text{CH}^*(X)$. Let $\{e^\vee_i\}$ be its Poincaré dual. Since both sides of the relation under proof are bilinear, it suffices to check the assertion for $\phi = e_i \times e^\vee_j$ and $\psi = e_k \times e^\vee_l$. In this case both sides of the relation are equal to $\delta_{il} \delta_{jk}$.

Denote by $d(X)$ the greatest common divisor of the degrees of all zero cycles on $X$ and by $d_p(X)$ its $p$-primary component.

6.3 Corollary. Let $m$ be an integer. For any $\phi \in \overline{\text{CH}}(X \times X; \mathbb{Z}/m)$ we have

$$\gcd(d(X), m) \mid \text{tr}(\phi_*).$$

Proof. Set $\psi = \Delta_X$ and apply Lemma 6.2.

6.4 Corollary. Assume that $\mathcal{M}(X; \mathbb{Z}/p)$ has a direct summand $M$. Then

1. $d_p(X) \mid P(M, 1);$

2. if $d_p(X) = P(M, 1)$ and the kernel of the restriction $\text{End}(\mathcal{M}(X)) \to \text{End}(\mathcal{M}(X))$ consists of nilpotents, then $M$ is indecomposable.
Proof. Set $q = d_p(X)$ for brevity. Let $M = (X, \phi)$. By Corollary 2.7, there exists an idempotent $\varphi \in \text{End}(M(X); \mathbb{Z}/q)$ such that $\varphi \mod p = \phi$. Then $\text{res}(\varphi) \in \text{End}(\mathcal{M}(X); \mathbb{Z}/q)$ is a rational idempotent. Since every projective module over $\mathbb{Z}/q$ is free, we have

$$\text{tr}(\text{res}(\varphi)_*) = \text{rk}_{\mathbb{Z}/q}(\text{res}(\varphi)_*) = \text{rk}_{\mathbb{Z}/p}(\text{res}(\phi)_*) = P(\overline{M}, 1) \mod q,$$

and the first claim follows from Corollary 6.3. The second claim follows from the first one, since the second assumption implies that for any non-trivial direct summand $M'$ of $M$ we have $P(\overline{M'}, 1) < P(\overline{M}, 1)$.

6.5. Denote by $n(G)$ the greatest common divisor of degrees of all finite splitting fields of $G$ and by $n_p(G)$ its $p$-primary component. Note that $n(G) = d(X)$ and $n_p(G) = d_p(X)$.

We obtain the following estimate on $n_p(G)$ in terms of the $J$-invariant (cf. [EKM, Prop. 88.11] in the case of quadrics).

6.6 Proposition. Let $G$ be a semisimple linear algebraic group of inner type with $J_p(G) = (j_1, \ldots, j_r)$. Then

$$n_p(G) \leq p^{\sum_j j_i}.$$

Proof. Follows from Theorem 5.13 and Corollary 6.4.

6.7 Corollary. The following statements are equivalent:

- $J_p(G) = (0, \ldots, 0)$;
- $n_p(G) = 1$;
- $\mathcal{R}_p(G) = \mathbb{Z}/p$.

Proof. If $J_p(G) = (0, \ldots, 0)$ then $n_p(G) = 1$ by Proposition 6.6. If $n_p(G) = 1$ then there exists a splitting field $L$ of degree $m$ prime to $p$ and, therefore, $\mathcal{R}_p(G) = \mathbb{Z}/p$ by transfer argument 5.18(ii). The remaining implication is obvious.

7 Examples

In the present section we provide examples of motivic decompositions of projective homogeneous varieties obtained using Theorem 5.17.
The case \( r = d_1 = 1 \). According to Table 4.13 this corresponds to the case when \( G \) is of type \( A_n \) or \( C_n \). Let \( A \) be a central simple algebra corresponding to \( G \). We have \( A = M_m(D) \), where \( D \) is a division algebra of index \( d \geq 1 \) over a field \( F \). Let \( p \) be a prime divisor of \( d \). Observe that according to Table 4.13 \( J_p(G) = (j_1) \) for some \( j_1 \geq 0 \). Let \( X_\Theta \) be the projective homogeneous \( G \)-variety given by a subset \( \Theta \) of vertices of the respective Dynkin diagram such that \( p \nmid j \) for some \( j \notin \Theta \) (cf. Example 3.6). Then by Theorem 5.17 we obtain that
\[
\mathcal{M}(X_\Theta; \mathbb{Z}/p) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\Theta_{a_i}},
\]
where \( \mathcal{R}_p(G) \) is indecomposable and
\[
\mathcal{R}_p(G) \simeq \bigoplus_{i=0}^{p^{j_1}-1} (\mathbb{Z}/p)(i).
\]

We now identify \( \mathcal{R}_p(G) \). Using the comparison lemma (see Proposition 5.18) we conclude that \( \mathcal{R}_p(G) \) only depends on \( D \), so we may assume \( m = 1 \). By Table 4.13 we have \( p^{j_1} | d \), but on the other hand by Proposition 6.6 we have \( n_p(G) \leq p^{j_1} \). Therefore, \( p^{j_1} \) is a \( p \)-primary part of \( d \).

We have \( D \simeq D_p \otimes_F D' \), where \( p^{j_1} = \text{ind}(D_p) \) and \( p \nmid \text{ind}(D') \). Passing to a splitting field of \( D' \) of degree prime to \( p \) and using Proposition 5.18 we conclude that the motives of \( X_\Theta \) and \( \text{SB}(D_p) \) are direct sums of twisted \( \mathcal{R}_p(G) \). Comparing the Poincaré polynomials we conclude that

7.1 Lemma. \( \mathcal{M}(\text{SB}(D_p); \mathbb{Z}/p) \simeq \mathcal{R}_p(G) \).

Applying Proposition 5.21 to \( X = \text{SB}(D) \) and comparing the Poincaré polynomials of \( \mathcal{M}(X) \) and \( \mathcal{R}_i \) we obtain that

7.2 Corollary. The motive of \( \text{SB}(D) \) with integer coefficients is indecomposable.

7.3 Remark. Indeed, we provided a uniform proof of the results of paper [Ka96]. Namely, the decomposition of \( \mathcal{M}(\text{SB}(A); \mathbb{Z}/p) \) (see [Ka96, Cor. 1.3.2]) and indecomposability of \( \mathcal{M}(\text{SB}(D); \mathbb{Z}) \) (see [Ka96, Thm. 2.2.1]).

The case \( r = 1 \) and \( d_1 > 1 \). According to Table 4.13 this holds if \( p = 2 \): \( G_2, F_4, E_6 \) or \( G \) is a strongly inner form of type \( B_3, B_4, D_4, D_5; \)
$p = 3$: $G$ is a group of type $F_4$, $E_7$ or strongly inner form of type $E_6$;

$p = 5$: $G$ is a group of type $E_8$.

We say a group $G$ is strongly inner over a field $F$ if it is the twisted form by means of a cocycle from $H^1(F; G_0)$, where $G_0$ is the simply-connected split group over $F$ of the same type as $G$ (see [4.1]).

Observe that in all these cases $J_p(G) = (0)$ or $(1)$. Let $X$ be a generically split projective homogeneous $G$-variety (cf. Example 3.6). By Theorem 5.17 we obtain the decomposition

$$\mathcal{M}(X; \mathbb{Z}/2) \simeq \bigoplus_{i \geq 0} \mathcal{R}_p(G)(i)^{\oplus \alpha_i},$$

where the motive $\mathcal{R}_p(G)$ is indecomposable and (cf. [Vo03, (5.4-5.5)])

$$\mathcal{R}_p(G) \simeq \bigoplus_{i=0}^{p-1} (\mathbb{Z}/p)(i \cdot (p + 1)).$$

We now identify $\mathcal{R}_p(G)$. Let $r$ be the Rost invariant as defined in [Me03] and $r_p$ denote its restriction to the $p$-primary closure of $F$.

**7.4 Lemma.** Let $G$ be a simple linear algebraic group over $F$ satisfying $r = 1$ and $d_1 > 1$ and $p$ be a prime. Then $r_p(G)$ is trivial iff $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$.

**Proof.** According to [Ga01, Theorem 0.5], [Ch94] and [Gi00, Theoreme 10] the invariant $r_p(G)$ is trivial iff the group $G$ splits over the $p$-primary closure of $F$. By Corollary 6.7 the latter is equivalent to the fact that $\mathcal{R}_p(G) \simeq \mathbb{Z}/p$. \qed

**7.5 Lemma.** Let $G$ and $G'$ be simple linear algebraic groups over $F$ satisfying $r = 1$ and $d_1 > 1$. If $r_p(G) = r_p(G')c$ for some $c \in (\mathbb{Z}/p)^\times$, then $\mathcal{R}_p(G) \simeq \mathcal{R}_p(G')$.

**Proof.** By transfer arguments (see Proposition 5.18) it is enough to prove this over a $p$-primary closure of $F$. Let $X$ and $X'$ be the respective varieties of complete flags. Observe that the invariant $r_p(G)$ becomes trivial over the function field $F(X)$. Since $r_p(G) = r_p(G')c$, it becomes trivial over $F(X')$ as well. By Lemma 7.4 $X$ splits over $F(X')$. Similarly $X'$ splits over $F(X)$.

Therefore by Lemma 1.8 there exists a cycle $\phi$ in $\overline{\text{Ch}_{\dim}X}(X \times X')$ of the form $\phi = 1 \times pt + \sum_{\text{codim}_{\alpha_i} > 0} \alpha_i \times \beta_i$. Observe that by definition $\phi_*: pt_X \mapsto$
Similarly, interchanging $X$ and $X'$ we obtain a cycle $\phi' \in \text{Ch}_{\dim X}'(X' \times X)$ such that $\phi'_*: pt_{X'} \mapsto pt_X$. Restricting of $\phi$ and $\phi'$ to the direct summands $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$ of $\mathcal{M}(X)$ and $\mathcal{M}(X')$ respectively we obtain the rational maps $\phi_R: \mathcal{R}_p(G) \to \mathcal{R}_p(G')$ and $\phi'_R: \mathcal{R}_p(G') \to \mathcal{R}_p(G)$.

Since the motive $\mathcal{R}_p(G)$ is indecomposable and $\text{rk Ch}^i(\mathcal{R}_p(G)) \leq 1$ for all $i$, the ring of rational endomorphisms of $\mathcal{R}_p(G)$ is generated by the identity endomorphism $\Delta$. The same holds for the ring of rational endomorphisms of $\mathcal{R}_p(G')$. Since $(\phi'_R)_* \circ (\phi_R)_*: pt_X \mapsto pt_X$, the composite $\phi'_R \circ \phi_R = \Delta$. Similarly we obtain $\phi_R \circ \phi'_R = \Delta'$. By Rost Nilpotence, since $\phi_R$ and $\phi'_R$ are rational, the motives $\mathcal{R}_p(G)$ and $\mathcal{R}_p(G')$ are isomorphic.

$\mathbb{Z}$-coefficients. Let $G$ be a group of type $F_4$ or a strongly inner form of type $E_6$ which doesn’t split by field extensions of degrees 2 and 3. Observe that such a group splits by an extension of degree 6. Let $X$ be a generically split projective homogeneous $G$-variety. Then according to Proposition 5.21 the Chow motive of $X$ with integer coefficients splits as a direct sum of twisted copies of an indecomposable motive $\mathcal{R}(G)$ such that

$$\mathcal{R}(G) \otimes \mathbb{Z}/2 = \bigoplus_{i=0,1,2,6,7,8} \mathcal{R}_2(G)(i), \quad P(\mathcal{R}_2(G), t) = 1 + t^3,$$

$$\mathcal{R}(G) \otimes \mathbb{Z}/3 = \bigoplus_{i=0,1,2,3} \mathcal{R}_3(G)(i), \quad P(\mathcal{R}_3(G), t) = 1 + t^4 + t^8,$$

$$P(\mathcal{R}(G), t) = 1 + t + t^2 + \ldots + t^{11}.$$ 

7.6 Remark. In particular, we provided a uniform proof of the main results of papers [Bo03] and [NSZ], where the cases of $G_2$- and $F_4$-varieties were considered.

7.7 Remark. Using Proposition 5.21 one can construct other liftings of the motivic decompositions of $X$. Thus, the Krull-Schmidt theorem fails in the category of Chow motives with $\mathbb{Z}/6$-coefficients.

The case $r > 1$. According to Table 4.13 this holds for groups $G$ of types $B_n$ and $D_n$ and exceptional types $E_7, E_8$ for $p = 2$ and $E_6^{ad}, E_8$ for $p = 3$.

Pfister case. Let $G = O^+(\phi)$, where $\phi$ is a $k$-fold Pfister form or its maximal neighbor. Assume $J_2(G) \not\equiv (0, \ldots, 0)$. In view of Corollary 6.7 this holds iff
\( n_2(G) \neq 1 \). By Springer’s Theorem the latter holds iff \( \phi \) is not split. By Theorem 5.17 we obtain the decomposition

\[
\mathcal{M}(X; \mathbb{Z}/2) \simeq \bigoplus_{i \geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i},
\]

where \( \mathcal{R}_2(G) \) is indecomposable. Moreover, by Theorem 2.16 the same decomposition holds with \( \mathbb{Z} \)-coefficients.

Now we compute \( J_2(G) \). Let \( Y \) be a projective quadric corresponding to \( \phi \). Then \( G \) splits over \( F(Y) \) and \( Y \) splits over \( F(x) \) for any \( x \in X \). It is known that \( \text{CH}^l(Y) \) for \( l < 2^{k-1} - 1 \) is generated by \( \text{CH}^1(Y) \) and, therefore, is rational. By Proposition 4.9 and Table 4.13 we see that \( j_i(G) = 0 \) for \( 0 \leq i < r \), where \( r = 2^{k-2} \). Therefore, \( J_2(G) = (0, \ldots, 0, 1) \) and \( P(\mathcal{R}_2(G), t) = 1 + t^{2k-1-1} \). Finally, by Corollary 5.14 the motive \( \mathcal{R}_2(G) \) coincides with the motive introduced in [Ro98] which is called the Rost motive.

In this way we obtain the Rost decomposition of the motive of a Pfister quadric and of its maximal neighbor.

**Maximal orthogonal Grassmannian.** Let \( G = O^+(q) \), where \( q: V \to F \) is an arbitrary anisotropic regular quadratic form and \( X \) is a connected component of the respective maximal orthogonal Grassmannian. The variety \( X \) is generically split, hence, by Theorem 5.17 we have the decomposition

\[
\mathcal{M}(X; \mathbb{Z}/2) \simeq \bigoplus_{i \geq 0} \mathcal{R}_2(G)(i)^{\oplus a_i},
\]

where the motive \( \mathcal{R}_2(G) \) is indecomposable. Comparing the Poincaré polynomials of \( \mathcal{M}(X; \mathbb{Z}/2) \) and \( \mathcal{R}_2(G) \) we obtain the following particular cases:

- If the group \( G \) corresponds to a generic cocycle (see 4.2), the motive \( \mathcal{M}(X; \mathbb{Z}/2) \) is isomorphic to \( \mathcal{R}_2(G) \) and, hence, is indecomposable. This corresponds to the maximal value of the \( J \)-invariant.

- If \( q \) is a Pfister form or its maximal neighbor, by the previous example \( \mathcal{R}_2(G) \) coincides with the Rost motive. This corresponds to the minimal non-trivial value of the \( J \)-invariant.

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Viktor Petrov
PIMS, Department of Mathematical and Statistical Sciences,
University of Alberta, Edmonton, AB T6G 2G1, Canada

Nikita Semenov
Mathematisches Institut der LMU München, Germany

Kirill Zainoulline
Mathematisches Institut der LMU München,
Theresienstr. 39, D-80333 München, Germany