Inverse problems and sharp eigenvalue asymptotics for Euler–Bernoulli operators

Andrey Badanin\textsuperscript{1} and Evgeny Korotyaev

Mathematical Physics Department, Faculty of Physics, Ulianovskaya 2, St. Petersburg State University, St. Petersburg, 198904, Russia
E-mail: an.badanin@gmail.com and korotyaev@gmail.com

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Dedicated to the memory of Professor Vladimir Shelkovich, 1949–2013

Abstract

We consider Euler–Bernoulli operators with real coefficients on the unit interval. We prove the following results: (i) The Ambarzumyan-type theorem about the inverse problems for the Euler–Bernoulli operator. (ii) The sharp asymptotics of eigenvalues for the Euler–Bernoulli operator when its coefficients converge to the constant function. (iii) The sharp eigenvalue asymptotics for both the Euler–Bernoulli operator and fourth-order operators (with complex coefficients) on the unit interval at high energy.

Keywords: Euler-Bernoulli operator, fourth order operator, inverse problem, eigenvalue asymptotics

1. Introduction and main results

1.1. Euler–Bernoulli operators

We consider the Euler–Bernoulli operator $\mathcal{E}$ given by

$$\mathcal{E}u = \frac{1}{b} (au')' + Qu,$$

acting on $L^2((0, 1), b(x)dx)$ with the boundary conditions

$$u(0) = u(1) = 0, \quad u'' + 2(a + \beta)u' = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1,$$

where $a, b$ are positive coefficients given by

$$a(x) = e^b \int_0^x u'(s)ds > 0, \quad b(x) = b(0) e^b \int_0^x \beta(s)ds > 0.$$
Without loss of generality we assume $a(0) = 1$. We assume that the functions $\alpha$, $\beta$, $Q$ are real and satisfy

$$(\alpha, \beta, Q) \in \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_0,$$  \hspace{1cm} (1.4)$$

where $\mathcal{H}_m$ is the Sobolev space defined by

$$\mathcal{H}_m = \left\{ f \in L^1(0, 1) : f^{(m)} \in L^1(0, 1) \right\}, \quad m \geq 0.$$  \hspace{1cm} (1.5)$$

The domain of the operator $\mathcal{E}$ is the set

$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(0, 1) : u^{(m)} \in L^1(0, 1), u \text{ satisfies equation (1.2),} \right\}$$

$$\frac{1}{b}(au')' + Q u \in L^2(0, 1).$$  \hspace{1cm} (1.6)$$

It is well known that the operator $\mathcal{E}$ is self-adjoint; see, e.g., [Na, Th 18.5].

The Euler–Bernoulli operator is a specific form of fourth-order operator. It describes the relationship between a beam’s deflection and the applied load; $a$ is the rigidity and $b$ is the density of the beam (see [Gl1]). The boundary conditions (1.2) mean that the ends of the beam are restrained by some special rotational spring devices. If $(\alpha + \beta)|_0 = (\alpha + \beta)|_1 = 0$ (for example, if the coefficients $a, b$ are constant near the ends), then (1.2) implies the boundary conditions for the pinned–pinned beam:

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$  \hspace{1cm} (1.7)$$

To obtain $(\pi n)^4$ in the first term in the asymptotics (1.11), we add the following normalizations:

$$\int_0^1 \left( \frac{b}{a} \right)^{4/3} dx = 1$$  \hspace{1cm} (1.8)$$

without loss of generality. The operator $\mathcal{E}$ is self-adjoint, and its spectrum is real and purely discrete and consists of the eigenvalues $\lambda_n$, $n \in \mathbb{N}$, of multiplicity $\leq 2$, labeled by

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots,$$

counted with multiplicity. In the case of a uniform beam (i.e., $a, b = 1$) conditions (1.2) take the form (1.7), and the corresponding eigenvalues have the form $\lambda_n = (\pi n)^4$, $n \in \mathbb{N}$.

Recall the following famous result of Ambarzumyan (see [LeS, chapter VI]):

Let $\lambda_0 < \lambda_1 < \ldots$ be eigenvalues of the problem

$$-u'' + Q(x)u = \lambda u, \quad x \in [0, 1], \quad u'(0) = u'(1) = 0,$$

where $Q$ is a real continuous function. Then $\lambda_n = (\pi n)^2$ for all $n = 0, 1, 2, \ldots$, iff $Q = 0$.

Note that, in general, the spectrum of the second-order operator does not determine the potential; i.e., Ambarzumyan’s theorem is not valid for other boundary conditions. The results, similar to Ambarzumyan’s theorem, for the weighted second-order operator $\frac{1}{b}u''$ are well known (see the discussions in section 7).

To formulate Ambarzumyan-type results regarding the inverse problems for the Euler–Bernoulli operator we define the following functions:

$$\kappa = \frac{3\alpha + 5\beta}{2\xi}, \quad \xi = \left( \frac{b}{a} \right)^{4/3}. \hspace{1cm} (1.9)$$
Theorem 1.1. Let $Q = 0$, and let the real $(a, b) \in \mathcal{L} \times \mathcal{L}$ satisfy the conditions (1.8) and $\kappa (0) = \kappa (1)$. Then the eigenvalues of the problem (1.1), (1.2) are $\lambda_n = (\pi n)^2$ for all $n \geq 1$ iff $a = b = 1$.

Remark. In contrast with the second-order operators $-v'' + qv$, we conjecture that this result may be extended to Euler–Bernoulli operators with some other boundary conditions (see proposition 7.1).

The proof of theorem 1.1 is based on eigenvalue asymptotics for the operator $\mathcal{E}$. To determine these asymptotics we introduce a constant $\psi_0$ and a function $\kappa$ by

$$
\psi_0 = \kappa (1) - \kappa (0) + \int_0^1 \frac{x(x)}{\xi(x)} dx, \quad x = \frac{5\alpha^2 + 5\beta^2 + 6\alpha\beta}{4} \geq \frac{\alpha^2 + \beta^2}{2} \geq 0. \quad (1.10)
$$

Theorem 1.2. Let $(\alpha, \beta, Q) \in \mathcal{L} \times \mathcal{L} \times \mathcal{L}$ be real and be normalized by (1.8). Then the eigenvalues $\lambda_n$ of the operator $\mathcal{E}$ satisfy

$$
\lambda_n = (\pi n)^2 + 2(\pi n)^2 \psi_0 + \psi_1 - \kappa_n + \frac{o(1)}{n} \quad \text{as} \quad n \to \infty \quad (1.11)
$$

uniformly for bounded subsets of $\mathcal{L} \times \mathcal{L} \times \mathcal{L}$, where $\psi_1$ is a constant given by equation (6.9) and

$$
\kappa_n = \int_0^1 \left( Q(x) \xi(x) + \frac{\alpha''(x) - \beta''(x)}{4\xi'(x)} \right) \cos \left( 2\pi n \int_0^1 \xi(x) dx \right) dx. \quad (1.12)
$$

Remark. (1) If we introduce a new variable $t = t(x) = \int_0^x \xi(s) ds$ in the integral in equation (1.12), then $\kappa_n$ become Fourier coefficients of the function $Q + \frac{b}{4a}(\alpha'' - \beta'')$.

(2) Jian-jun, Kui, and Da-jun [JKD] announced the first two terms in the eigenvalue asymptotics for the Euler–Bernoulli equation with smooth coefficients of the finite interval for other boundary conditions; see more in section 7.

(3) Let $\alpha, \beta$ satisfy $\kappa (1) = \kappa (0)$ (e.g., $a, b$ are periodic). Then the definition of $\psi_0$ given by equation (1.10) shows that the constant $\psi_0 = \psi_0(\alpha, \beta)$ satisfies

$$
\psi_0(\alpha, \beta) = \psi_0(-\beta, -\alpha) = \int_0^1 \frac{x(x)}{\xi(x)} dx \geq 0.
$$

Moreover, this yields $\psi_0(\alpha, \beta) = 0$ iff $\alpha = \beta = 0$. Thus we obtain the result that any periodic perturbation of the coefficients $a, b$ moves all large eigenvalues strictly to the right. Moreover, the eigenvalue asymptotics for the operators $\mathcal{E}(a, b)$ and $\mathcal{E} \left( \frac{1}{b}, \frac{1}{a} \right)$ coincide up to $O(1)$.

(4) The proof of theorem 1.2 is based on the sharp eigenvalue asymptotics for the operator $H = \partial^4 + 2\partial^2 + q$ on $L^2(0, 1)$ from theorem 1.4. The unitary Barcilon–Gottlieb transformation [B2, Go1] (i.e., the Liouville-type transformation for a fourth-order operator) reduces the Euler–Bernoulli operator to the operator $\partial^4 + 2\partial^2 + q$ with some specific $p, q$. We give the derivation of the Barcilon–Gottlieb transformation in section 5.

1.2. Euler–Bernoulli operators with near-constant coefficients

Theorem 1.2 shows that any periodic perturbation of the coefficients $a, b$ shifts all large eigenvalues to the right. To understand the situation for the eigenvalue $\lambda_n$ with the fixed
number $n$ we consider the operator $\mathcal{E}_\varepsilon$ given by

$$\mathcal{E}_\varepsilon u = \frac{1}{b^2}(c, a^r u^r)^r + \varepsilon Q u,$$

where real $c$, $a$, $b$ have the form (1.3), the constant $c_r > 0$ is defined by

$$c_r = \int_0^1 \frac{b(x) a'(x)}{a(x)} dx,$$  \hfill (1.14)

and $\alpha, \beta, Q \in L^1(0, 1)$ are real. Due to equation (1.14) the coefficients $c_r a^r$, $b^r$ are normalized by the identity $\int_0^1 \frac{b(x)}{a(x)} dx = 1$ for all $\varepsilon \in \mathbb{R}$, similar to equation (1.8).

The domain of the operator $\mathcal{E}_\varepsilon$ is the set

$$\mathcal{D}(\mathcal{E}_\varepsilon) = \left\{ u \in L^2(0, 1) : u^r, (a^r u^r)^r \in L^1(0, 1), u(0) = u(1) = u^r(0) = u^r(1) = 0, \right\}.$$

It is well known that the operator $\mathcal{E}_\varepsilon$ is self-adjoint; see [Na, Th 18.5].

Let $\lambda_n(\varepsilon), n \in \mathbb{N}$, be the eigenvalues of the operator $\mathcal{E}_\varepsilon$ labeled by $\lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \leq ...$, counted with multiplicity.

We define Fourier coefficients $\hat{f}_0, \hat{f}_n, \hat{f}_m, n \in \mathbb{N}$ of a function $f$ by

$$\hat{f}_0 = \int_0^1 f(t) dt, \quad \hat{f}_n = \int_0^1 f(t) \cos(2\pi nt) dt, \quad \hat{f}_m = \int_0^1 f(t) \sin(2\pi mt) dt.$$  \hfill (1.15)

**Theorem 1.3.** Let $\alpha, \beta, Q \in L^1(0, 1)$. Then each eigenvalue $\lambda_n(\varepsilon), n \geq 1$ of the operator $\mathcal{E}_\varepsilon$ is analytic in $\{\varepsilon \in \mathbb{C} : |\varepsilon| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$ and satisfies

$$\lambda_n(\varepsilon) = (\pi n)^4 + 2\varepsilon(\pi n)^3 \left( \hat{a}_m - \hat{b}_m \right) + \varepsilon \left( \hat{Q}_0 - \hat{Q}_n \right) + O(\varepsilon^2)$$

as $\varepsilon \to 0$, uniformly in $\alpha, \beta, Q$ for bounded subsets of $L^1(0, 1)$.

**Remark.** (1) The asymptotics (equation (1.16)) show that each perturbed eigenvalue $\lambda_n(\varepsilon)$ remains close to the unperturbed one $\lambda_n(0) = (\pi n)^4$ under small perturbations. The eigenvalues can move left or right. In particular, if $Q = 0$, then equation (1.16) gives

$$\lambda_n'(0) = 2(\pi n)^3 \left( \hat{a}_m - \hat{b}_m \right).$$  \hfill (1.17)

If $\hat{a}_m > \hat{b}_m$, we obtain $\lambda_n(\varepsilon) > \lambda_n(0)$. If $\hat{a}_m < \hat{b}_m$, we obtain $\lambda_n(\varepsilon) < \lambda_n(0)$.

(2) Barcilon–Korotyaev [BK1, BK2] considered fourth-order operators with small coefficients in the periodic case.

Barcilon [B1] considered a boundary value problem

$$((1 + \varepsilon \alpha) u^r)^r = \lambda(1 + \varepsilon \beta) u, \quad u(0) = u'(0) = u^r(1) = (au^r)'(1) = 0.$$  \hfill (1.18)

He solved a problem of reconstruction of coefficients $\alpha, \beta$ by the first term of the perturbation series for eigenvalues as $\varepsilon \to 0$. Let $\mu_k(\varepsilon)$ be the eigenvalues of the problem (1.18). Barcilon proved that the coefficients $\alpha$ and $\beta$ cannot be uniquely determined by the sequence $\mu_k(0), n \in \mathbb{N}$. Moreover, he showed that it is sufficient to know the spectra of three Euler–Bernoulli operators with different boundary conditions to uniquely determine both $\alpha$ and $\beta$.

The asymptotics (1.16) give a solution the Barcilon inverse problems in our case.
Example. Consider our Euler–Bernoulli operator $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$, which corresponds to the case of $a, b$ close to 1. We show that if we know the coefficient $\alpha$ and we know, in addition, that $\beta$ is odd, then $\beta$ can be uniquely recovered by the sequence $\lambda_n'(0)$ given by equation (1.17).

Assume that $\alpha \in L^1(0, 1), Q = 0$ and that for some unknown
\[ \beta \in L^1_{\text{odd}}(0, 1) = \left\{ f \in L^1(0, 1) : f(x) = f(1 - x), x \in (0, 1) \right\} \]
we have the sequence $\lambda_n'(0), n \in \mathbb{N}$ of derivatives of the eigenvalues of the operator $\mathcal{E}_\varepsilon$ at $\varepsilon = 0$. The identity (1.17) gives
\[ \sum_{n=1}^{\infty} \left( a_m - \beta_m \right) \sin 2\pi nx = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n'(0) \sin 2\pi nx, \quad x \in (0, 1). \]

Then $\beta$ is uniquely determined by
\[ \beta(x) = \frac{1}{2} \left( \alpha(x) - \alpha(-x) - \sum_{n=1}^{\infty} \lambda_n'(0) \sin 2\pi nx \right), \quad x \in (0, 1). \quad (1.19) \]

Remark. (1) A similar argument shows that the function $\alpha \in L^1_{\text{odd}}(0, 1)$ can be determined by $\beta \in L^1(0, 1)$ and $\lambda_n'(0), n \in \mathbb{N}$.

(2) For the solution to an inverse spectral problem for a second-order operator on the unit interval under the Dirichlet boundary condition for even (and generic) potentials (see Pöschel–Trubowitz [PT]), the case of a weighted operator was considered by Coleman–McLaughlin [CM].

1.3. Fourth-order operators

We consider an operator $H$ on $L^2(0, 1)$ given by
\[ Hy = y'''' + 2(py')' + qy, \]
under the boundary conditions
\[ y(0) = y''(0) = y(1) = y'(1) = 0. \quad (1.21) \]

The domain of this operator is the set
\[ \mathcal{D}(H) = \left\{ y \in L^2(0, 1) : y'''' \in L^1(0, 1), y(0) = y''(0) = y(1) = y'(1) = 0, \right. \]
\[ \left. y'''' + 2(py')' + qy \in L^2(0, 1) \right\}. \quad (1.22) \]

It is well known that if $p, q$ are real, the operator $H$ is self-adjoint; see [Na, Th 18.5]. In our paper, in general, the operator $H$ is non-self-adjoint since we assume that the functions $p, q \in L^1(0, 1)$ are complex (including real).

It is well known (see, e.g., [Na, Ch 1.2]) that the spectrum of the operator $H$ is purely discrete and consists of the eigenvalues $\lambda_n, n \in \mathbb{N}$ labeled by
\[ |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \ldots. \]
counted with multiplicities. Furthermore, the following eigenvalue asymptotics hold true:

\[ \lambda_n = (mn)^4 + O(n^2) \quad \text{as} \quad n \to \infty \]

(see, e.g., [Na, Ch. I.4]). Note that in the case of complex coefficients we mean the ‘algebraic’ multiplicity, i.e., a total dimension of the root subspace corresponding to the eigenvalue. This ‘algebraic’ multiplicity may be any integer, whereas the ‘geometric’ multiplicity, i.e., the dimension of the eigen subspace, is \( \leq 2 \). In our paper we determine the eigenvalue asymptotics for the operator \( H \).

Note that in the case of real coefficients any self-adjoint fourth-order operator may be written in the form (1.20). Moreover, if the functions \( p, q \) are real, the eigenvalues \( \lambda_n \) satisfy

\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \]

and the ‘algebraic’ multiplicity is \( \leq 2 \) and coincides with the ‘geometric’ multiplicity.

The sharpest eigenvalue asymptotics for the operator \( H \) at the present time were determined by Caudill–Perry–Schueller [CPS]. They proved that in the case of the real coefficients

\[ \int_{0}^{1} (p^2(t) - \hat{p}_0^2) \, dt + \tilde{V}_0 - \tilde{V}_{\hat{c}n} + \frac{\epsilon n}{n^{\gamma+1}}, \]

as \( n \to \infty \) uniformly for bounded subsets of \( \mathcal{H} \times \mathcal{H}^+ \), where \( \hat{p}_0, \tilde{V}_0, \tilde{V}_{\hat{c}n} \) are defined by equation (1.15).

**Remark.** (1) The asymptotics (1.24) were used to prove trace formulas for fourth-order operators on both the circle [BK5] and the unit interval [BK6].

(2) The proof of the asymptotics (1.23) in [CPS] is based on analysis of both the free and the perturbed resolvents. Our approach is different and is based on analysis of a determinant defined by equation (3.16) expressed in terms of the fundamental solutions, which are entire in \( \lambda \). All eigenvalues \( \lambda_n, n \geq 1 \) are zeros of this determinant. Using the sharp asymptotics of the fundamental solutions from [BK4] we determine asymptotics of the determinant. (The corresponding proof is rather technical; see section 4.) Analysis of these asymptotics provides the sharp asymptotics of \( \lambda_n \).

(3) In some papers there are misprints and mistakes in the eigenvalue asymptotics for the second-order and fourth-order operators. We discuss some of them in section 7.
1.4. Historical review

• Second-order operators. There are a lot of results regarding eigenvalue asymptotics for second-order operators: see the book of Levitan–Sargsyan [LeS] and the review of Fulton–Pruess [FP] and the references therein. Atkinson–Mingarelli [AM] obtained the eigenvalue asymptotics \( \lambda_n = (\pi n^2 + o(1)) \) as \( n \to \infty \) for the weighted second-order operator \( b^{-1}u'' \), where the coefficient \( b \) satisfies \( b \in L^1(0, 1) \) and \( b^{-1} \in L^\infty(0, 1) \). Korotyaev [K] determined sharp eigenvalue asymptotics in terms of the Fourier coefficients of \( b'/b \), where the coefficient \( b \) satisfies \( b, b^{-1} \in L^\infty(0, 1) \) and \( b' \in L^2(0, 1) \). Asymptotics for the case of smooth coefficients were determined by asymptotics for the Schrödinger operator using Liouville transformation (see Fulton–Pruess [FP]).

Sufficiently sharp eigenvalue asymptotics for the Schrödinger operator with a matrix potential for the finite interval were determined by Chelkak–Korotyaev for both the Dirichlet boundary conditions [CK1] and the periodic boundary conditions [CK2]. Now we describe the difference between the asymptotics for the systems of second-order equations and the scalar higher-order differential equations. The unperturbed fundamental matrix for the second-order operators (even with the matrix-valued coefficients) have the entries \( \cos \sqrt{\lambda} t \) and \( \sin \sqrt{\lambda} t \). Then all entries of the perturbed fundamental matrix are bounded as \( \lambda \to +\infty \), and their asymptotics can be determined by the standard iteration procedure. But in the higher-order case we meet some additional difficulties, which are quite absent for second-order operators. For example, the unperturbed fundamental matrix for the fourth-order operators has both the bounded entries \( \cos \lambda t \) and \( \sin \lambda t \) and the unbounded entries \( \cosh \lambda t \) and \( \sinh \lambda t \) as \( \lambda \to +\infty \). Therefore, the standard iterations do not give asymptotics of the perturbed fundamental matrix for the fourth-order (and higher-order) operators. Roughly speaking, determining eigenvalue asymptotics for higher-order operators has the standard difficulties of analysis of the systems plus additional difficulties associated with increasing parts of the fundamental matrix.

• Fourth-order operators. We discuss some results for the fourth-order operators. Such operators arise in many physical models. (See, e.g., the book [PeT] and references therein.) Numerous results concerning the regular and singular boundary value problems for higher-order operators are expounded in the books of Atkinson [At] and Naimark [Na].

Eigenvalue asymptotics for fourth-order and higher-order operators on the unit interval are much less investigated than for second-order operators. The first term is well known; see Naimark [Na]. The second term is known due to Caudill–Perry–Schueller [CPS]. Note that Badanin–Korotyaev [BK3] determined the corresponding term in the eigenvalue asymptotics for the general case of the \( 2n \)-order operators on the circle (for the case \( n = 2 \), see [BK2, BK4]). It is more difficult to obtain the next term. The simple case \( \partial^{2n} + q \) was considered by Akhmerova [Ah], Badanin–Korotyaev [BK1], and Mikhailets–Molyboga [MM].

Many papers are devoted to the inverse spectral problems for fourth-order operators. Barcilon [B1] considered the inverse spectral problem for the fourth-order operators on the interval \([0, 1]\) in terms of three spectra. McLaughlin [McL1] studied the inverse spectral problems in terms of the spectrum and the norming constants. Caudill–Perry–Schueller [CPS] described iso-spectral potentials for our fourth-order operators \( H \). Hoppe–Laptev–Östensson [HLO] considered the inverse scattering problem for the fourth order operator on the real line in the case of rapidly decaying \( p, q \) at infinity. Yurko [Yu, Ch 2] recovered coefficients of a fourth-order operator on the unit interval by its Weyl matrix.

• The Euler–Bernoulli equation. Jian-jun, Kui, and Da-jun [JKD] determined two terms in the formal asymptotic eigenvalue expansion for the Euler–Bernoulli operator on the unit
interval. Many papers are devoted to the inverse spectral problems for the Euler–Bernoulli equation: Barcilon [B2], Gladwell [Gl2], Gottlieb [Go1], and McLaughlin [McL1]; see also the book of Gladwell [Gl1] and references therein. Papanicolaou [P] considered the inverse problem for the Euler–Bernoulli equation for a line in the case of periodic $a, b > 0$. Moreover, there is an enormous quantity of physical and engineering literature regarding the Euler–Bernoulli equation; here we mention only some papers related to our subject: Ghanbari [Gh], Gladwell, England and Wang [GEW], Gladwell and Morassi [GIM], Gottlieb [Go2], Guo [Guo], Chang and Guo [ChG], Kambampati, Ganguli and Mani [KGM], Kawano [Ka], Lesnic [Ls], Soh [Soh], and Sundaram and Ananthasuresh [SuA].

1.5. The plan of our paper

The plan of this paper is as follows. In section 2 we consider the Euler–Bernoulli operators with near-constant coefficients and prove theorem 1.3. In sections 3 and 4 we consider the operator $H$ and prove theorem 1.4. In section 5 we consider the Barcilon–Gottlieb transform, i.e., the unitary transformation between the operator $H$ and the Euler–Bernoulli operator. In section 6 we prove theorems 1.1 and 1.2 regarding the operator $\mathcal{E}$. In section 7 we discuss examples and remarks about eigenvalue asymptotics for both second-order and fourth-order operators.

2. Euler–Bernoulli operators with near-constant coefficients

2.1. Fundamental matrix

We consider the operator $\mathcal{E}_\varepsilon$, given by equation (1.13), for small $\varepsilon \in \mathbb{R}$, where real $\alpha, \beta, Q \in L^1(0, 1)$.

Rewrite the equation
\[
c_\varepsilon (a^\varepsilon u')'' + \varepsilon Qu = \lambda b^\varepsilon u
\]
in the vector form
\[
u' = Cu,
\]
where
\[
u = \begin{pmatrix} u \\ u' \\ c_\varepsilon a^\varepsilon u'' \\ c_\varepsilon (a^\varepsilon u') \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & (c_\varepsilon a^\varepsilon)^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ \lambda b^\varepsilon - \varepsilon Q & 0 & 0 & 0 \end{pmatrix},
\]
The fundamental matrix $M(x, \lambda, \varepsilon) = (M_{jk}(x, \lambda, \varepsilon))_{j,k=1}^{4}, (x, \lambda, \varepsilon) \in [0, 1] \times \mathbb{C} \times \mathbb{R}$ of equation (2.1) satisfies
\[
M' = CM, \quad M(0, \lambda, \varepsilon) = \mathbb{I}_4.
\]
In the unperturbed case \( \varepsilon = 0 \) we have
\[
M_0(x, \lambda) = \left( M_{0,j,k}(x, \lambda) \right)_{j,k=1}^4 = \begin{pmatrix}
\varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\
\lambda \varphi_2 & \varphi_1 & \varphi_2 & \varphi_3 \\
\lambda \varphi_3 & \lambda \varphi_4 & \varphi_1 & \varphi_2 \\
\lambda \varphi_2 & \lambda \varphi_3 & \lambda \varphi_4 & \varphi_1
\end{pmatrix}(x, \lambda),
\]
where
\[
\varphi_1 = \varphi_2', \quad \varphi_2 = \frac{s_+}{\varepsilon}, \quad \varphi_3 = \varphi_4', \quad \varphi_4 = \frac{s_-}{\varepsilon}, \quad s_{\pm} = \frac{\sin z \pm \sin z}{2}.
\]
Here we have used the new spectral variable \( z \in \mathbb{C} \) defined by
\[
z = \frac{1}{\varepsilon}, \quad \arg z \in S = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \text{ as } \arg \lambda \in (-\pi, \pi].
\]

**Lemma 2.1.** Let \( \alpha, \beta, Q \in L^1(0, 1) \). Then each matrix-valued function
\[
M(x, \lambda, \cdot), (x, \lambda) \in [0, 1] \times \mathbb{C}
\]
is analytic in \( \varepsilon \in \{ \varepsilon \in \mathbb{C} : |\varepsilon| < \varepsilon_1 \} \) for some \( \varepsilon_1 > 0 \) and satisfies
\[
M(x, \lambda, \varepsilon) = M_0(x, \lambda) + \varepsilon M_1(x, \lambda) + O(\varepsilon^2)
\]
as \( \varepsilon \to 0 \) uniformly in \( (x, \lambda) \) for bounded subsets of \([0, 1] \times \mathbb{C} \), where the matrix
\[
M_1 = (M_{1,j,k})_{j,k=1}^4
\]
has the form
\[
M_{1,j,k}(x, \lambda) = \int_0^x \left( \alpha_1(t) M_{0,j,k}(x-t, \lambda) M_{0,k,l}(t, \lambda) + \beta_1(t, \lambda) M_{0,j,k}(x-t, \lambda) M_{0,k,l}(t, \lambda) \right) dt,
\]
\[
\alpha_1 = -\ln b(0) - 4 \int_0^x \alpha(s) ds - 4 \int_0^1 \int_0^x (\beta(s) - \alpha(s)) ds dx,
\]
\[
\beta_1 = 4 \lambda \int_0^x \beta(s) ds + \lambda \ln b(0) - Q.
\]

**Proof.** The identities (1.3), (1.14) give
\[
a^e(s) = 1 + 4e \int_0^x \alpha(s) ds + O(e^2),
b^e(s) = 1 + e \ln b(0) + 4e \int_0^x \beta(s) ds + O(e^2),
c_\varepsilon = 1 + e \ln b(0) + 4e \int_0^1 \int_0^x (\beta(s) - \alpha(s)) ds dx + O(e^3)
\]
as \( \varepsilon \to 0 \) uniformly in \([0, 1] \). Substituting these asymptotics into equation (2.3) we obtain
\[
C(x, \lambda, \varepsilon) = C_0(\lambda) + \varepsilon C_1(x, \lambda, \varepsilon), \quad \tilde{C}(x, \lambda, \varepsilon) = \tilde{C}_0(x, \lambda) + O(\varepsilon),
\]
as $\varepsilon \to 0$ uniformly in $(x, \lambda)$ for bounded subsets of $[0, 1] \times \mathbb{C}$, where

$$C_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.12)$$

Using the asymptotics (2.11) and (2.4) we obtain

$$M' - C_0 M = \varepsilon \tilde{C} M,$$

which yields

$$M(x, \lambda, \varepsilon) = M_0(x, \lambda) + \varepsilon \int_0^x M_0(x - t, \lambda) \tilde{C}(t, \lambda, \varepsilon) M(t, \lambda) dt.$$

The standard iterations show that $M$ is analytic in $|\varepsilon| < \varepsilon_1$ and satisfies equation (2.8), where

$$M_{1, \ell}(x, \lambda) = \int_0^x \sum_{\ell, m=1}^4 M_{0, \ell'}(x - t, \lambda) C_{1, \ell'm}(t, \lambda) M_{0, m}(t, \lambda) dt.$$

Substituting equation (2.12) here, we obtain equation (2.9). □

### 2.2. Determinant

Introduce the determinant

$$D(\lambda, \varepsilon) = \det \mathcal{M}(\lambda, \varepsilon), \quad \mathcal{M}(\lambda, \varepsilon) = \begin{pmatrix} M_{12} & M_{14} \\ M_{2} & M_{32} \end{pmatrix}(1, \lambda, \varepsilon). \quad (2.13)$$

The unperturbed determinant (i.e., at $\varepsilon = 0$) has the form

$$D(\lambda, 0) \equiv D_0(\lambda) = \det \begin{pmatrix} \phi_2 & \phi_4 \\ \lambda \phi_4 & \phi_2 \end{pmatrix}(1, \lambda) = \frac{\sinh \frac{z}{2} \sin \frac{z}{2}}{\frac{z}{2}^2}. \quad (2.14)$$

The following result is well known. We give the proof for the sake of completeness.

**Lemma 2.2.** Let $\alpha, \beta, Q \in L^1(0, 1)$. Then the spectrum of $\mathcal{E}_\varepsilon$ satisfies the identity

$$\sigma(\mathcal{E}_\varepsilon) = \{ \lambda \in \mathbb{C} : D(\lambda, \varepsilon) = 0 \}. \quad (2.15)$$

Moreover, the algebraic multiplicity of each eigenvalue $\lambda$ of $\mathcal{E}_\varepsilon$ is equal to the multiplicity of $\lambda$ as a zero of $D$.

**Proof.** Any vector solution $u$ of equation (2.2) satisfies the identity

$$u(1, \lambda, \varepsilon) = M(1, \lambda, \varepsilon) u(0, \lambda, \varepsilon) \quad \forall \lambda \in \mathbb{C}. \quad (2.16)$$

Let $\lambda$ be an eigenvalue of $\mathcal{E}_\varepsilon$, and let $u(x, \lambda, \varepsilon)$ be the corresponding eigenfunction. Then the boundary conditions (1.13) give

$$u(0, \lambda, \varepsilon) = \begin{pmatrix} 0 & g_0, 0, h_0 \end{pmatrix}^T, \quad u(1, \lambda, \varepsilon) = \begin{pmatrix} 0, g_1, 0, h_1 \end{pmatrix}^T,$$
where \( g_j = u'(j, \lambda, \varepsilon) \) and \( h_j = c_j(a' u'(j, \lambda, \varepsilon))' \) for \( j = 0, 1 \). The identity (2.16) implies
\[
M(1, \lambda, \varepsilon)(0, g_0, 0, h_0) = (0, g_1, 0, h_1) .
\] (2.17)
This formula yields \( D(\lambda, \varepsilon) = 0 \). Conversely, if \( D(\lambda, \varepsilon) = 0 \), then the system (2.17) has the non-trivial solution \( (g_0, h_0) \), which gives the eigenfunction \( u(t, \lambda, \varepsilon) \). This proves equation (2.15).

Let \( \lambda \) be an eigenvalue of \( \mathcal{E} \). It is well known (see, e.g., [Na, Ch. I.2.3.VI]) that the total dimension of the root subspace of \( \lambda \), i.e., the algebraic multiplicity of \( \lambda \), coincides with the multiplicity of \( \lambda \) as a zero of \( D \).

**Lemma 2.3.** Let \( \alpha, \beta, Q \in L^1(0, 1) \), and let \( \varepsilon \in \mathbb{R} \) be sufficiently small. Then
(i) The determinant \( D \), given by equation (2.13), satisfies
\[
D(\lambda, \varepsilon) = D_0(\lambda) + \varepsilon D_1(\lambda) + O(\varepsilon^2)
\] (2.18)
uniformly for bounded \( \lambda \in \mathbb{C} \), where \( D_0 \) is given by equation (2.14) and \( D_1 \) is an entire function given by
\[
D_1(\lambda) = \int_0^1 \gamma_1(x, \lambda) g(x, \lambda) dx , \quad \gamma_1(x, \lambda) = \alpha_1(x) + \frac{\beta_1(x)}{\lambda} ,
\] (2.19)
where
\[
g = \frac{1}{4\varepsilon}(\cosh z - \cosh (1 - 2\varepsilon))\sin z + (\cos z - \cos (1 - 2\varepsilon))\sinh z .
\] (2.20)
Moreover,
\[
D_1((\pi n)^3) = \frac{(-1)^{n+1} \sinh \pi n}{2(\pi n)^2} \left( \tilde{a}_n - \tilde{b}_n + \frac{\tilde{Q}_n - \tilde{Q}_n}{2(\pi n)^3} \right) \quad \forall \ n \in \mathbb{N} .
\] (2.21)
(ii) The function \( D(\cdot, \varepsilon) \) has exactly one simple zero in each domain \( |\lambda^2 - \pi n| < 1, n \in \mathbb{N} \).

**Proof.** i) The asymptotics (2.8) imply that the matrix \( \mathcal{M} \), given by equation (2.13), satisfies
\[
\mathcal{M}(\lambda, \varepsilon) = \mathcal{M}_0(\lambda) + \varepsilon \mathcal{M}_1(\lambda) + O(\varepsilon^2)
\] (2.22)
as \( \varepsilon \to 0 \) uniformly for bounded subsets of \( \mathbb{C} \), where, due to equation (2.5),
\[
\mathcal{M}_0 = \left( M_{0,ik} \right)_{j,k=1}^2 = \begin{pmatrix} \varphi_2 & \varphi_4 \\ \lambda \varphi_4 & \varphi_2 \end{pmatrix}(1) , \quad \mathcal{M}_1 = \left( M_{1,ik} \right)_{j,k=1}^2 = \begin{pmatrix} M_{1,12} & M_{1,14} \\ M_{1,32} & M_{1,34} \end{pmatrix}(1) ,
\] (2.23)
and \( \mathcal{M}_1 \) has the form (2.9), here and below \( M(x) = M(x, \lambda, \ldots) \). Substituting the asymptotics (2.22) into the definition (2.13), we obtain equation (2.18), where \( D_1 \) is an entire function, given by
\[
D_1 = \det \begin{pmatrix} \mathcal{M}_{0,11} & \mathcal{M}_{1,12} \\ \mathcal{M}_{0,21} & \mathcal{M}_{1,22} \end{pmatrix} + \det \begin{pmatrix} \mathcal{M}_{1,11} & \mathcal{M}_{0,12} \\ \mathcal{M}_{1,21} & \mathcal{M}_{0,22} \end{pmatrix}
\] (2.24)
We prove (2.21). The identity (2.9) and the definition of $\mathcal{M}_i$ in equation (2.23) yield
\[
\mathcal{M}_1 = \int_0^1 \left( \alpha_1(x)A(x, \lambda) + \beta_1(x, \lambda)B(x, \lambda) \right)dx,
\]
where $\alpha_1, \beta_i$ are given by equation (2.10):
\[
A = \begin{pmatrix}
M_{0,12}(1 - x)M_{0,32}(x) & M_{0,12}(1 - x)M_{0,34}(x) \\
M_{0,32}(1 - x)M_{0,32}(x) & M_{0,32}(1 - x)M_{0,34}(x)
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
M_{0,14}(1 - x)M_{0,12}(x) & M_{0,14}(1 - x)M_{0,14}(x) \\
M_{0,34}(1 - x)M_{0,12}(x) & M_{0,34}(1 - x)M_{0,14}(x)
\end{pmatrix}.
\]
Using equation (2.5) we obtain
\[
A = \begin{pmatrix}
\lambda \varphi_2(1 - x)\varphi_4(x) & \varphi_2(1 - x)\varphi_2(x) \\
\lambda^2 \varphi_4(1 - x)\varphi_4(x) & \lambda \varphi_4(1 - x)\varphi_2(x)
\end{pmatrix},
\]
\[
B = \begin{pmatrix}
\varphi_2(1 - x)\varphi_2(x) & \varphi_2(1 - x)\varphi_4(x) \\
\varphi_2(1 - x)\varphi_2(x) & \varphi_2(1 - x)\varphi_4(x)
\end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} A_{22} & \frac{1}{\lambda} A_{21} \\ \lambda A_{12} & A_{11} \end{pmatrix}.
\]
The identities (2.24), (2.25) give
\[
D_1 = \int_0^1 \left( \alpha_1(x)g(x) + \beta_1(x)\tilde{g}(x) \right)dx,
\]
where
\[
g = \det \begin{pmatrix} \mathcal{M}_{0,11} & \mathcal{M}_{0,12} \\ \mathcal{M}_{0,21} & \mathcal{M}_{0,22} \end{pmatrix} + \det \begin{pmatrix} \mathcal{A}_{11} & \mathcal{M}_{0,12} \\ \mathcal{A}_{21} & \mathcal{M}_{0,22} \end{pmatrix},
\]
\[
\tilde{g} = \det \begin{pmatrix} \mathcal{M}_{0,11} & \mathcal{B}_{12} \\ \mathcal{M}_{0,21} & \mathcal{B}_{22} \end{pmatrix} + \det \begin{pmatrix} \mathcal{B}_{11} & \mathcal{M}_{0,12} \\ \mathcal{B}_{21} & \mathcal{M}_{0,22} \end{pmatrix}.
\]
Using equations (2.23), (2.27) we obtain
\[
g = \det \begin{pmatrix} \varphi_2(1) & \mathcal{A}_{12}(x) \\ \lambda \varphi_4(1) & \mathcal{A}_{22}(x) \end{pmatrix} + \det \begin{pmatrix} \mathcal{A}_{11}(x) & \varphi_4(1) \\ \mathcal{A}_{21}(x) & \varphi_2(1) \end{pmatrix}
\]
\[
= \det \begin{pmatrix} \varphi_2(1) & \lambda \mathcal{A}_{12}(x) + \mathcal{A}_{21}(x) \\ \varphi_4(1) & \mathcal{A}_{11}(x) + \mathcal{A}_{22}(x) \end{pmatrix},
\]
\[
\tilde{g} = \frac{1}{\lambda} \left( \det \begin{pmatrix} \varphi_2(1) & \frac{1}{\lambda} \mathcal{A}_{12}(x) \\ \lambda \varphi_4(1) & \mathcal{A}_{11}(x) \end{pmatrix} + \det \begin{pmatrix} \mathcal{A}_{22}(x) & \varphi_4(1) \\ \lambda \mathcal{A}_{12}(x) & \varphi_2(1) \end{pmatrix} \right) = \frac{g}{\lambda}.
\]
Substituting equation (2.30) into equation (2.28) we obtain equation (2.19).

We prove equation (2.20). The identities (2.26) and (2.6) give
\[
\lambda \mathcal{A}_{12}(x) + \mathcal{A}_{21}(x) = \lambda \varphi_2(1 - x)\varphi_2(x) + \lambda^2 \varphi_4(1 - x)\varphi_4(x) = \varphi_2^2(x) + \mathcal{A}_{11}(x),
\]
\[
\lambda \mathcal{A}_{11}(x) + \mathcal{A}_{22}(x) = \lambda \varphi_4(1 - x)\varphi_2(x) + \lambda \varphi_2(1 - x)\varphi_4(x) = \varphi_4^2(x) + \mathcal{A}_{22}(x).
\]
where

\[ g_1 = \frac{\cosh z - \cosh z(1 - 2x)}{4}, \quad g_2 = \frac{\cos z - \cos z(1 - 2x)}{4}. \]

Thus equations (2.31) and (2.29) give

\[ g = \det \begin{pmatrix} q_2(1) & z^2 \left(g_1(x) - g_2(x)\right) \\ q_4(1) & g_1(x) + g_2(x) \end{pmatrix}. \]

The identities

\[ q_2(1) - z^2q_4(1) = \frac{s_+(1) - s_-(1)}{z} = \frac{\sin z}{z}, \]

\[ q_2(1) + z^2q_4(1) = \frac{s_+(1) + s_-(1)}{z} = \frac{\sinh z}{z}, \]

imply equation (2.20).

We prove equation (2.21). The identity (2.19) gives

\[ D_1(\pi n)^4 = \frac{(-1)^n \sinh \pi n}{4\pi n} \int_0^1 \gamma(x, (\pi n)^4)(1 - \cos 2\pi nx) dx. \]

Using equation (2.10) we deduce that

\[ D_1(\pi n)^4 = \frac{(-1)^n \sinh \pi n}{4\pi n} \left( \frac{\hat{Q}_0 - \hat{Q}_0}{(\pi n)^4} - 4 \int_0^1 \int_0^1 (\beta(s) - \alpha(s)) ds \cos 2\pi nx dx \right). \]

Integration by parts yields equation (2.21).

(ii) The standard arguments based on the asymptotics (2.18) and Rouché’s theorem give the statement. \( \square \)

### 2.3. Proof of theorem 1.3

Due to lemma 2.1, the function \( D(\lambda, \varepsilon) \), given by equation (2.13), is analytic in \((\lambda, \varepsilon) \in \mathbb{C} \times \{ |\varepsilon| < \varepsilon_1 \}\) for some \( \varepsilon_1 > 0 \). Let \( |\varepsilon| < \varepsilon_1 \). Then, due to lemma 2.3ii, each eigenvalue \( \lambda_n(\varepsilon), n \geq 1 \) is a simple zero of \( D(\cdot, \varepsilon) \) and therefore is analytic in \( \varepsilon \).

We determine eigenvalue asymptotics for the operator \( E_{\varepsilon} \) as \( \varepsilon \to 0 \). Let \( z_{n}(\varepsilon) = \lambda_{n}^{V}(\varepsilon), n \in \mathbb{N} \). Lemma 2.3ii yields \( z_{n}(\varepsilon) = \pi n + \delta, \) where \( \delta = \delta(\varepsilon) \) satisfies the estimate \( |\delta| < 1 \) for all sufficiently small \( \varepsilon \). Substituting these asymptotics into equation (2.18) we obtain

\[ D(\lambda_n(\varepsilon)) = \frac{(-1)^n \sinh \pi n}{(\pi n)^2} \delta + \varepsilon D_1(\pi n)^4 + O(\delta + |\varepsilon|)^2). \]

The identity \( D(\lambda_n(\varepsilon)) = 0 \) implies \( \delta = O(\varepsilon) \), which yields

\[ 0 = D(\lambda_n(\varepsilon)) = \frac{(-1)^n \sinh \pi n}{(\pi n)^2} \delta + \varepsilon D_1(\pi n)^4 + O(\varepsilon^2), \]
and then
\[ \delta = e^{-1} \frac{(\pi n)^2 D_1}{\sinh \pi n} + O(e^2). \] (2.32)

Substituting equation (2.21) into equation (2.32) and using \( z_n(\varepsilon) = \pi n + \delta \) we deduce that
\[ z_n(\varepsilon) = \pi n + \frac{\varepsilon}{2} \left( \alpha_{mn} - \beta_{mn} \right) + \frac{\varepsilon \left( Q_0 - \hat{Q}_m \right)}{4(\pi n)^3} + O(e^2), \]
which yields equation (1.16).

3. Monodromy matrix for the operator \( H \)

3.1. Fundamental matrix

Consider the operator \( H \), given by equations (1.20), (1.21), where complex \((p, q) \in \mathcal{H} \times \mathcal{H}\).

We rewrite the equation
\[ y^{(4)} + 2(pq')y' + qy = \lambda y, \quad \lambda \in \mathbb{C} \] (3.1)
in the vector form
\[ y' - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix} y = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2p & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix} y, \quad \text{where} \quad y = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}. \] (3.2)

The corresponding matrix equation has the unique \( 4 \times 4 \)-matrix-valued solution
\[ M(t, \lambda) = \left( M_{jk}(t, \lambda) \right)_{j,k=1}^4, \quad M(0, \lambda) = \mathbb{I}_4, \]
where \( \mathbb{I}_4 \) is the \( 4 \times 4 \) identity matrix. The matrix-valued function \( M(t, \lambda) \) is called the fundamental matrix. Each function \( M(t, \lambda) \), \( t \in \mathbb{R} \), is entire in \( \lambda \) and real at real \( \lambda, p, q \). The function \( M(1, \lambda) \) is called the monodromy matrix. In the unperturbed case \( p = q = 0 \) we have \( M = M_0 \), where \( M_0 \) is given by equation (2.5).

Introduce a diagonal \( 4 \times 4 \) matrix \( \omega \) given by
\[ \omega = \text{diag}(\omega_1, \omega_2, \omega_3, \omega_4), \quad \omega_1 = -\omega_3 = i, \quad \omega_2 = -\omega_4 = 1; \] (3.3)
here and following \( \text{diag}(a_1, a_2, a_3, a_4) = \text{diag}(a_j)_{j=1}^4 = (a_j \delta_{jk})_{j,k=1}^4 \). Define the set \( S^+ \) by
\[ S^+ = \left\{ z \in \mathbb{C} : \arg z \in \left[ 0, \frac{\pi}{4} \right] \right\} = \left\{ \lambda \in \mathbb{C}_+ \right\}, \]
where the variable \( z \) is given by equation (2.7). We have the following estimates:
\[ \Re(i\omega_1 z) \leq \Re(i\omega_2 z) \leq \Re(i\omega_3 z) \leq \Re(i\omega_4 z) \quad \forall \quad z \in S^+. \] (3.4)
Introduce a unitary $4 \times 4$-matrix $U$ by

$$U = \frac{1}{2} \left( \omega \mathbb{I} \right)^{j=1}_{j=k=1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & 1 & -1 & -i \\ -1 & 1 & 1 & -1 \\ -i & 1 & -1 & i \end{pmatrix}$$

(3.5)

and a diagonal $4 \times 4$-matrix-valued function

$$v(z) = \text{diag} \left( v_j(z) \right)_{j=1}^4, \quad v_j = v_j(z) = \omega_j + \frac{\bar{p}_0}{2z^2} \omega_j \quad \forall \quad j \in \mathbb{N}_4 = \{1, 2, 3, 4\}$$

for $z \in S^+ \setminus \{0\}$. Note that

$$e^{iv_jz} = e^{iv_jz} \left( 1 + O \left( |z|^{-1} \right) \right) \quad \forall \quad j \in \mathbb{N}_4$$

(3.6)

as $|z| \to \infty$, $z \in S^+$.

Define functions $x_{jk}(s, z), j, k \in \mathbb{N}_4$, $s \in [0, 1]$ by

$$x_{jk}(s, z) =
\begin{cases}
  e^{i(\omega_j - \omega_k)z(1 - s)} \left( 1 + \frac{i(\sigma_j - \sigma_k)}{2z} \int_s^1 p(t) \, dt \right), & j < k \\
  -1, & j = k, \\
  -e^{-i(\omega_j - \omega_k)z} \left( 1 - \frac{i(\sigma_j - \sigma_k)}{2z} \int_0^s p(t) \, dt \right), & j > k
\end{cases}

(3.7)

and a $4 \times 4$-matrix-valued function

$$W(t, z) = \mathbb{I}_4 - \frac{p(t)}{z^2} W_1 - \frac{p'(t)}{z^3} W_2,$n

(3.8)

where

$$W_1 = \frac{1}{4} \begin{pmatrix} 0 & 1 + i & 1 & -1 \\ 1 - i & 0 & -1 & 1 \\ 1 - i & 0 & -1 & 1 \\ -1 + i & 0 & -1 & 1 \end{pmatrix}, \quad W_2 = \frac{1}{8} \begin{pmatrix} 0 & -2 & -2 & -1 \\ 2i & 0 & i & 2i \\ -2i & -i & 0 & -2i \\ 1 & 2 & 2 & 0 \end{pmatrix}

(3.9)

and

$$X(t, z) = \left( X_{jk}(t, z) \right)_{j,k=1}^4 = p^*(t) W_2 + \frac{iq(t)}{4} \begin{pmatrix} -i & -i & -i & -i \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & i & i & i \end{pmatrix}$$

$$+ \frac{i p^2(t)}{8} \begin{pmatrix} i & i & i & 0 \\ 1 & 1 & 0 & 1 \\ -i & 0 & -1 & -1 \\ 0 & -i & -i & -i \end{pmatrix} + \frac{3p(t)p'(t)}{16z} \begin{pmatrix} -1 & 1 & -1 & -i \frac{2}{3} \\ 1 + i & -1 & \frac{2}{3} & 1 + i \\ i & i & i & i \frac{2}{3} & 1 + i & -1 & -i \end{pmatrix}.

(3.10)

Recall that the spaces $\mathcal{H}_m$ are given by equation (1.5). In the following we assume, without loss of generality, that $q$ belongs to the Sobolev space $\mathcal{H}_0^0$, where $\mathcal{H}_m^0, m \geq 0$ are given by
\( \mathcal{H}_m^0 = \left\{ f \in \mathcal{H}_m : \int_0^1 f(t) dt = 0 \right\} \).

Sharp asymptotics of the monodromy matrix \( M(1, \lambda) \) were determined in [BK4] for the case of the real 1-periodic coefficients \( p, p^* \in L^1(\mathbb{R}/\mathbb{Z}) \); see lemma 3.1 and the relations (3.15), (3.13), (3.8), (3.21), (2.14) in [BK4]. The proof from [BK4] may be extended to the case of complex coefficients \( (p, q) \in \mathcal{H}_2 \times \mathcal{H}_0^0 \) to obtain the following results.

**Lemma 3.1.** Let \((p, q) \in \mathcal{H}_2 \times \mathcal{H}_0^0\) and let \( r > 0 \) be sufficiently large. Then
\( M(1, \lambda) = \mathcal{Z}(z)A(z)e^{iz\xi(z)}B(z)\mathcal{Z}^{-1}(z) \quad \forall \ z \in S_r^+ = \left\{ z \in S^+ : |z| > r \right\}, \) (3.11)
where
\[ \mathcal{Z}(z) = \text{diag}(1, iz, (iz)^2, (iz)^3) \]
and the \( 4 \times 4 \)-matrix-valued functions \( A, B \) are analytic and uniformly bounded in \( S_r^+ \). Moreover,
\[ A(z) = \left( A_{jk}(z) \right)_{j,k=1}^4 = U\phi(z), \quad B(z) = \left( B_{jk}(z) \right)_{j,k=1}^4 = \Phi(z)U^*, \] (3.12)
\[ \phi(z) = \left( \phi_{jk}(z) \right)_{j,k=1}^4 = W(1, z) \left[ \mathbb{I}_4 + \frac{G(1, z)}{z^3} + \frac{O(1)}{|z|^5} \right], \]
\[ \Phi(z) = \left( \Phi_{jk}(z) \right)_{j,k=1}^4 = \left[ \mathbb{I}_4 - \frac{G(0, z)}{z^3} + \frac{O(1)}{|z|^5} \right]W^{-1}(0, z) \] (3.13)
as \( |z| \to \infty, z \in S^+ \) uniformly for bounded subsets of \( \mathcal{H}_2 \times \mathcal{H}_0^0 \), where each \( 4 \times 4 \)-matrix-valued function \( G(t, z) = \left( G_{jk}(t, z) \right)_{j,k=1}^4, t \in \{0, 1\} \) is analytic and uniformly bounded in \( S_r^+ \) and satisfies
\[ G_{jk}(1, z) = 0 \quad \text{as} \quad j \geq k, \quad G_{jk}(0, z) = 0 \quad \text{as} \quad j < k, \] (3.14)
\[ g_{jk}(z) \equiv \begin{cases} G_{jk}(1, z) & j < k \\ G_{jk}(0, z) & j \geq k \end{cases} = \int_0^1 \chi_{jk}(s, z)X_{jk}(s, z) ds. \] (3.15)

**3.2. Determinant**

We define the determinant \( D \) by
\[ D = \det \mathcal{M}, \quad \mathcal{M}(\lambda) = \begin{pmatrix} M_{12} & M_{14} \\ M_{22} & M_{24} \end{pmatrix}(1, \lambda). \] (3.16)

The function \( D \) is entire in \( \lambda \) since \( M \) is entire. The unperturbed determinant (i.e., at \( p = q = 0 \)) has the form (2.14).

The arguments from the proof of lemma 2.2 show that the spectrum of \( H \) satisfies the identity
\[ \sigma(H) = \{ \lambda \in \mathbb{C} : D(\lambda) = 0 \}. \] (3.17)
Moreover, the algebraic multiplicity of each eigenvalue \( \lambda \) of \( H \) is equal to the multiplicity of \( \lambda \) as a zero of \( D \).
Now we will determine asymptotics of the determinant $D(\lambda)$. Our proof is based on the sharp asymptotics of the monodromy matrix from our paper [BK4] by using the matrix form of the standard Birkhoff approach.

The formula (3.11) represents the monodromy matrix as a product of the simple diagonal matrices $Z, Z^{-1}$, the bounded matrices $A, B$, and the diagonal exponential factor $e^{izv}$. The following result is an immediate consequence of the identity (3.11).

**Lemma 3.2.** Let $(p, q) \in \mathcal{H} \times \mathcal{H}$, then the function $D(\lambda)$, given by equation (3.16), satisfies

$$D(\lambda) = -\frac{1}{z^2} \sum_{j,k \in \mathbb{N}_4, j < k} e^{(i(j+k)v)} \det \left( \alpha_{jk}(z) \beta_{jk}(z) \right), \quad \forall \ z \in S_r^+, \quad (3.18)$$

for some sufficiently large $r > 0$. Here the $2 \times 2$-matrix-valued functions $\alpha_{jk}, \beta_{jk}$ have the form

$$\alpha_{jk} = \begin{pmatrix} A_{jk} & A_{jk} \\ A_{jk} & A_{jk} \end{pmatrix}, \quad \beta_{jk} = \begin{pmatrix} B_{jk} & B_{jk} \\ B_{jk} & B_{jk} \end{pmatrix}, \quad j, k \in \mathbb{N}_4, \quad (3.19)$$

and $A_{jk}, B_{jk}$ are given by equation (3.12).

**Proof.** The identity (3.11) gives

$$D(\lambda) = -\frac{1}{z^2} \sum_{j,k \in \mathbb{N}_4, j < k} e^{(i(j+k)v)} \det \left( \alpha_{jk}(z) \beta_{jk}(z) \right),$$

where $\gamma_{jk} = \begin{pmatrix} A_{jk} & A_{jk} \\ A_{jk} & A_{jk} \end{pmatrix}$. Then

$$D(\lambda) = \det \left( \begin{pmatrix} M_{12} & M_{14} \\ M_{32} & M_{34} \end{pmatrix} \right)(1, \lambda) = \left( \begin{array}{cc} 1 & 0 \\ 0 & (iz)^{-2} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & (iz)^{-3} \end{array} \right).$$

Using the identities

$$\det \sum_{j,k \in \mathbb{N}_4} e^{(i(j+k)v)} \gamma_{jk} = \sum_{j,k \in \mathbb{N}_4} e^{(i(j+k)v)} \gamma_{jk} = \sum_{j,k \in \mathbb{N}_4} e^{(i(j+k)v)} \left( \det \gamma_{jk} + \det \gamma_{kj} \right)$$

and $\det \gamma_{jk} + \det \gamma_{kj} = \det (\alpha_{jk} \beta_{jk})$, we obtain equation (3.18). \(\square\)

Now we determine rough asymptotics of the determinant.

**Lemma 3.3.** Let $(p, q) \in \mathcal{H} \times \mathcal{H}$, then the function $D$, defined by equation (3.16), satisfies

$$D(\lambda) = D_0(\lambda) + e^{\text{Re} z + \text{Im} z} \frac{O(1)}{z^3}, \quad (3.20)$$

as $|\lambda| \to \infty, z \in S$ uniformly for bounded subsets of $\mathcal{H} \times \mathcal{H}$, where $D_0$ is given by equation (2.14).
Proof. Let $|z| \to \infty$, $z \in S^+$. Substituting equation (3.8) into equation (3.13) we obtain
\[ \Theta(z) = \mathbb{I}_4 + O \left( |z|^{-1} \right), \quad \Phi(z) = \mathbb{I}_4 + O \left( |z|^{-1} \right). \]
Then the identities (3.12) imply
\[ A(z) = U + O \left( |z|^{-1} \right), \quad B(z) = U^* + O \left( |z|^{-1} \right), \]
which, jointly with equation (3.18), yield
\[ \lambda \sum_{\lambda \in \mathbb{N}} e^{i(\nu_j + \nu_k)} \det \left( \begin{array}{cc} U_{ij} & U_{ik} \\ U_{kj} & U_{kk} \end{array} \right) \left( \begin{array}{cc} U_{ij} & U_{ij} \\ \overline{U}_{kj} & \overline{U}_{kj} \end{array} \right) + O \left( 1 \right) \right) = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right). \]
Definition (3.5) gives
\[ \det \left( \begin{array}{cc} U_{ij} & U_{ik} \\ U_{kj} & U_{kk} \end{array} \right) = \frac{1}{16} \det \left( \begin{array}{cc} 1 & 1 \\ \frac{1}{\omega_j} & \frac{1}{\omega_k} \end{array} \right) \left( \frac{\sigma_j}{\sigma_k} - \frac{1}{\sigma_j} \right). \]
and using equation (3.21) we have
\[ D(\lambda) = -\frac{1}{8z^2} \sum_{j<k} e^{i(\nu_j + \nu_k)} \left( \frac{\sigma_j}{\sigma_k} - \frac{1}{\sigma_j} \right) + O \left( |z|^{-1} \right). \]
Then, due to equation (3.3), (3.6) we obtain
\[ D(\lambda) = -\frac{i}{2z^2} \left( \cosh((1 + i)z) \left( 1 + \frac{O(1)}{|z|} \right) - \cosh((1 - i)z) \left( 1 + \frac{O(1)}{|z|} \right) \right), \]
which yields equation (3.20) in $S^+$. By the identity $D(\lambda, \mathbb{I}_p, \mathbb{I}_q) = \overline{D}(\lambda, p, q)$, we extend equation (3.20) from $S^+$ to $S$.

We prove the counting lemma for the determinant $D(\lambda)$.

**Lemma 3.4.** Let $(p, q) \in \mathcal{H}_2 \times \mathcal{H}_p^0$. For any sufficiently large integer $N \geq 1$ the entire function $D(\lambda)$ has exactly $N$ zeros, counting with multiplicities, in the disc $\{ |\lambda| < \pi^4 (N + \frac{1}{4}) \}$; and for each integer $n > N$ it has exactly one simple zero in the domain $\{ |z - \pi n| < \frac{\pi}{4} \}$. There are no other zeros.

Proof. Let $N \in \mathbb{N}$ be sufficiently large, and let $N' > N$ be an integer. Let $\lambda \in \mathbb{C}$ belong to the contours
\[ |z| = \pi \left( N + \frac{1}{2} \right), \quad |z| = \pi \left( N' + \frac{1}{2} \right), \quad |z - \pi n| = \frac{\pi}{4}, \quad n > N. \]
Then the estimates \(|\sinh z| > \frac{e^{Re z}}{4}\) and \(|\sin z| > \frac{e^{|Im z|}}{4}\) and the asymptotics (3.20) give
\[
|D(\lambda) - D_0(\lambda)| < |D_0(\lambda)|
\]
for all contours. Hence, by Rouche’s theorem, the function \(D\) has the same number of zeros as the function \(D_0\) in each bounded domain and in the remaining unbounded domain. Since \(D_0\) has exactly one simple zero at each point \(\lambda = (\pi n)^4, n \in \mathbb{N},\) and since \(N' > N\) can be chosen arbitrarily large, the lemma follows.

4. Eigenvalue asymptotics for the operator \(H\)

4.1. Preliminaries

To determine high-energy eigenvalue asymptotics for the operator \(H\) we need to improve the asymptotics (3.20). Substituting equation (3.6) into the identity (3.18) and using equation (3.3) we obtain
\[
D(\lambda) = -\frac{e^{\imath z}}{z^2} \left( e^{\imath z} d_2(z) + \frac{e^{\imath z}}{2} d_3(z) + O(e^{-Re z}) \right)
\] (4.1)
as \(|z| \to \infty, z \in S^*\) uniformly for bounded subsets of \(\mathcal{H}_2 \times \mathcal{H}_0^0\), where
\[
d_j(z) = \det(\alpha_{j4}(z)\beta_{j4}(z)), \quad j = 2, 3.
\] (4.2)
Hereafter we will need the following auxiliary identities.

**Lemma 4.1.** Let \((p, q) \in \mathcal{H}_2 \times \mathcal{H}_0^0\). The functions \(d_j, j = 2, 3\) satisfy the following identities:
\[
d_j = -\frac{i}{4}\det(\xi_j\eta_j),
\] (4.3)
where
\[
\xi_j = \begin{pmatrix} \Theta_{14} + \Theta_{44} & \Theta_{1j} + \Theta_{4j} \\ \Theta_{21} + \Theta_{34} & \Theta_{2j} + \Theta_{3j} \end{pmatrix}, \quad \eta_j = \begin{pmatrix} \Phi_{j2} - \Phi_{j3} & \Phi_{j1} - \Phi_{j4} \\ \Phi_{42} - \Phi_{43} & \Phi_{41} - \Phi_{44} \end{pmatrix}
\] (4.4)
and \(\Theta_{jk}, \Phi_{jk}\) are given by equation (3.13).

**Proof.** Let \(j = 2, 3\). The identities (3.12), (3.19) provide
\[
\alpha_{j4} = u_1\theta_{1j} + u_2\theta_{2j}, \quad \beta_{j4} = \phi_{j4}u_3 + \phi_{j2}u_4,
\] (4.5)
where
\[
\theta_{1j} = \begin{pmatrix} \Theta_{1j} & \Theta_{14} \\ \Theta_{4j} & \Theta_{44} \end{pmatrix}, \quad \theta_{2j} = \begin{pmatrix} \Theta_{2j} & \Theta_{24} \\ \Theta_{3j} & \Theta_{34} \end{pmatrix}, \quad \phi_{j1} = \begin{pmatrix} \Phi_{j1} & \Phi_{j4} \\ \Phi_{41} & \Phi_{44} \end{pmatrix}, \quad \phi_{j2} = \begin{pmatrix} \Phi_{j2} & \Phi_{j3} \\ \Phi_{42} & \Phi_{43} \end{pmatrix}
\] (4.6)
\[
u_1 = \begin{pmatrix} U_{11} & U_{14} \\ U_{31} & U_{34} \end{pmatrix}, \quad u_1 = \begin{pmatrix} U_{12} & U_{13} \\ U_{22} & U_{23} \end{pmatrix}, \quad u_2 = \begin{pmatrix} U_{21} & U_{24} \\ U_{41} & U_{44} \end{pmatrix}, \quad u_3 = \begin{pmatrix} U_{22} & U_{23} \\ U_{42} & U_{43} \end{pmatrix}
\]
Definition (3.5) gives
\[ u_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad u_3 = \frac{i}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad u_1 = u_4 = \frac{f}{2}, \quad \text{where} \quad J = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \]
and we have
\[ u_3 u_2 = u_4 u_1 = 0, \quad i u_3 u_1 = u_4 u_2 = \frac{f}{2}. \]
Then the identities (4.5) imply
\[ \beta_{j4} a_{j4} = \frac{1}{2} \left( \phi_{j2} J \theta_{j2} - i \phi_{j1} J \theta_{j1} \right). \]
Substituting equation (4.6) into this identity we obtain
\[ \beta_{j4} a_{j4} = \frac{1}{2} \left( \Phi_{j1} - \Phi_{j4} \Phi_{j2} - \Phi_{j3} \right) \left( \begin{array}{cc} -i & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \Theta_{j1} + \Theta_{j4} & \Theta_{j3} + \Theta_{j4} \\ \Theta_{j2} + \Theta_{j3} & \Theta_{j2} + \Theta_{j3} \end{array} \right). \]
The identity (4.2) gives equation (4.3).

We will determine asymptotics of the functions \( d_j(z) \).

**Lemma 4.2.** Let \((p, q) \in \mathcal{H} \times \mathcal{H}_0^0\).

(i) The functions \( d_j(z) \) satisfy the asymptotics
\[
\begin{align*}
d_2(z) &= \frac{i}{4} \left( 1 + \frac{1 + i}{8z^3} (p'(1) - p'(0)) - \frac{g_{22}(z) - f_2(z)}{z^3} \right) + O(1) |z|^5, \\
d_3(z) &= \frac{i}{4} \left( 1 + \frac{1 - i}{8z^3} (p'(1) - p'(0)) + \frac{f_1(z) + f_2(z)}{z^3} \right) + O(1) |z|^5.
\end{align*}
\]
as \(|z| \to \infty\), \(z \in S^+\) uniformly for bounded subsets of \( \mathcal{H} \times \mathcal{H}_0^0\), where
\[
f_1 = g_{23} + g_{32} - g_{33}, \quad f_2 = g_{41} + g_{14} - g_{44} + \frac{3}{16z} \left( p^2(1) - p^2(0) \right)
\]
and \(g_{jk}\) are given by equation (3.15).

(ii) Let \(z \in S^+\) for some sufficiently large \(r > 0\). Then
\[
g_{j\ell}(z) = -\frac{io \alpha_{j\ell} p_1}{8} + \frac{3}{32z} \left( p^2(1) - p^2(0) \right) \quad \forall \quad j \in \mathbb{N}_4, \quad \ell \in \mathbb{N}_4, \quad p_1 = \int_0^1 p^2(t) dt. \]

**Proof.** (i) Let \(|z| \to \infty\) in \(S^+\). The identities (3.8), (3.13), and \(W_i^2 = -\frac{1}{16} \Pi_4\) imply
\[
\Theta(z) = \Pi_4 - \frac{p(1)}{z^2} W_1 + \frac{1}{z^3} \left( -p'(1) W_2 + G(1, z) \right) + \frac{O(1)}{|z|^5}.
\]
\[ \Phi(z) = \mathbb{I} + \frac{p(0)}{z^2} W_1 + \frac{1}{z^3} \left( p'(0) W_2 - G(0, z) - \frac{3p^2(0)}{16z} \mathbb{I}_n \right) + O(1) \frac{1}{|z|^5}. \] (4.11)

Substituting the identities (3.9), (3.14) into the asymptotics (4.10), (4.11) and using equation (4.4) we obtain

\[ \xi_2(z) = X_1 + \frac{p'(1)}{8z^3} X_2 + \frac{1}{z^3} \left( g_{14} - g_{12} \right) (z) + O(1) \frac{1}{|z|^5}, \]

\[ \xi_3(z) = X_1 + \frac{p'(1)}{8z^3} X_2 + \frac{1}{z^3} \left( g_{14} - g_{12} \right) (z) + O(1) \frac{1}{|z|^5}, \]

\[ \eta_2(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left( Y_1 + \frac{p'(0)}{8z^3} Y_2 - \frac{1}{z^3} \left( g_{22} - g_{21} \right) \right) (z) + O(1) \frac{1}{|z|^5}, \]

\[ \eta_3(z) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left( Y_1 + \frac{p'(0)}{8z^3} Y_2 - \frac{1}{z^3} \left( g_{33} - g_{32} \right) \right) (z) + O(1) \frac{1}{|z|^5}, \]

where

\[ X_1 = \mathbb{I}_2 - \frac{p(1)}{4z^2} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \]

\[ Y_1 = \mathbb{I}_2 + \frac{p(0)}{4z^2} \begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} - \frac{3p^2(0)}{16z^4} \mathbb{I}_2, \quad Y_2 = \begin{pmatrix} -i & 0 \\ 0 & -1 \end{pmatrix}. \]

The standard formula for any \( n \times n \) matrix \( A \),

\[ \det(\mathbb{I}_n + \varepsilon A) = 1 + \varepsilon \text{Tr} A + \frac{\varepsilon^2}{2} \left( \text{Tr} A^2 - \text{Tr} A^2 \right) + O(\varepsilon^3) \]

as \( |\varepsilon| \to 0, \quad \varepsilon \in \mathbb{C} \)
yields

\[ \det \xi_2(z) = 1 + x_1(1) + \frac{g_{14}(z)}{z^3} + O(1) \frac{1}{|z|^5}, \]

\[ \det \xi_3(z) = 1 + x_2(1) + \frac{g_{33}(z)}{z^3} + O(1) \frac{1}{|z|^5}, \]

\[ \det \eta_2(z) = -1 + x_1(0) - \frac{g_{44}(z) - g_{42}(z)}{z^3} + O(1) \frac{1}{|z|^5}, \]

\[ \det \eta_3(z) = 1 - x_2(0) + \frac{g_{44}(z) - g_{42}(z)}{z^3} + O(1) \frac{1}{|z|^5}, \]

where

\[ x_1(t) = \frac{(1 + i)p'(t)}{8z^3} + \frac{3p^2(t)}{16z^4}, \quad x_2(t) = \frac{(1 - i)p'(t)}{8z^3} + \frac{3p^2(t)}{16z^4}. \]

Substituting these asymptotics into equation (4.3) we obtain equation (4.7).
(ii) The identities (3.7) yield $x_j = -1$, which together with equation (3.15) gives $g_0 = -\int_0^1 X_j ds$. Substituting equation (3.10) into the last identity and using $\int_0^1 q(t)dt = 0$ we obtain equation (4.9).

\[\square\]

4.2. Sharp asymptotics of the determinant

We will determine the sharp asymptotics of the determinant.

**Lemma 4.3.** Let $(p, q) \in \mathcal{H}_2 \times \mathcal{H}_0^0$.

(i) Let $|\lambda| \to \infty$, $z \in S$. Then the function $D(\lambda)$ satisfies

\[D(\lambda) = E(z)\left(\sin w(z) + \frac{p'(1) - p'(0)}{8z^3} \cos w(z) + \frac{F(z)}{z^3}\right),\]  

(4.12)

where $E, w$ are given by

\[E(z) = \frac{e^{z - z_0}}{2z^2}, \quad w(z) = z + \frac{\hat{p}_0}{2z}\]  

(4.13)

and the function $F(z)$ is analytic in $S$ and satisfies

\[|F(z)| \leq C|e^{\text{Im} z}| \quad \forall \quad z \in \{S: |z| > r\}\]  

(4.14)

for some $C > 0$ and sufficiently large $r > 0$.

(ii) Let $z_n = \lambda_n \bar{\tau} \in S$, $n \in \mathbb{N}$, where $\lambda_n$ are the eigenvalues of $H$. Then

\[w(z_n) = \pi n + O\left(n^{-2}\right) \quad \text{as} \quad n \to \infty\]  

(4.15)

uniformly for bounded subsets of $\mathcal{H}_2 \times \mathcal{H}_0^0$.

(iii) Let $(p, q) \in \mathcal{H}_{2+}\times \mathcal{H}_j^0$, $j \in \{0, 1\}$, and let $n \to \infty$. Then

\[F(z_n) = \frac{(-1)^j}{8} \left(\int_0^1 p^2(t)dt + 2\hat{\nu}_n\right) + \hat{F}_n,\]  

(4.16)

where

\[\hat{F}_n = \begin{cases} \frac{o\left(n^{-1}\right)}{} & \text{if } j = 0 \\ O\left(n^{-2}\right) & \text{if } j = 1 \end{cases}\]  

(4.17)

uniformly for bounded subsets of $\mathcal{H}_{2+j} \times \mathcal{H}_j^0$.

**Proof.** (i) Let $|z| \to \infty$, $z \in S^+$. The asymptotics (4.7) and (4.1) imply equation (4.12), where

\[F(z) = \frac{(p'(1) - p'(0))\sin v_2 z}{8} - \varphi_1(z) + f_2(z)\sin v_2 z + \frac{O\left(e^{\text{Im} z}\right)}{|z|^2},\]  

(4.18)
\[ \varphi_1(z) = \frac{g_{22}(z)e^{\nu_{2}z} + f_{1}(z)e^{-\nu_{2}z}}{2i}. \]  

(4.19)

The asymptotics (4.18) yield equation (4.14). Using the identity \( D(\lambda, p, q) = D(\lambda, p, q) \) we extend (4.12), (4.14) from \( S^+ \) to \( S \).

(ii) The asymptotics (4.12) imply

\[ D(\lambda) = E(z) \left( \sin w(z) + \frac{O(e^{\text{Im}z})}{|z|^3} \right) \quad \text{as} \quad |z| \to \infty, \quad z \in S. \]  

(4.20)

Lemma 3.4 gives \( |z_n - \pi n| < \frac{\pi}{4} \) for all sufficiently large \( n \in \mathbb{N} \). If \( n \to \infty \), then the second identity in equation (4.13) implies

\[ w(z_n) = \left( 1 + \frac{\rho_n}{2z_n^2} \right) z_n = \pi n + \epsilon_n, \quad |\epsilon_n| < 1. \]  

(4.21)

The substitution of these asymptotics into equation (4.20) and the identity (3.17) \( D(\lambda_n) = 0 \) give

\[ D(\lambda_n) = (-1)^n E(z_n) \left( \sin \epsilon_n + O(n^{-3}) \right), \]  

(4.22)

which yields \( \epsilon_n = O(n^{-3}) \) since \( |\epsilon_n| < 1 \). Thus the asymptotics (4.21) imply equation (4.15).

(iii) Let \( |z| \to \infty, \quad z \in S^+ \). Substituting the identities (3.7), (3.10) into equation (3.15), using equation (3.3) and integrating by parts, we obtain

\[ g_{23}(z) = -\frac{i}{4} \int_0^1 e^{2z(1-s)} \left( 1 + \frac{i}{z} \int_s^1 p(t)dt \right) V(s)ds + O(1), \]  

\[ g_{32}(z) = -\frac{i}{4} \int_0^1 e^{2z|z|} \left( 1 + \frac{i}{z} \int_0^s p(t)dt \right) V(s)ds + O(1). \]  

(4.23)

The asymptotics (4.23), (4.9), (4.8), and (4.19) yield

\[ \varphi_1(z) = \frac{P_2}{8} \cos v_2z + \frac{3(p^2(1) - p^2(0))}{32z} \sin v_2z \]  

\[ -\frac{1}{4} \int_0^1 \cos z(1 - 2s)V(s)ds + \tilde{\varphi}_1(z), \]  

(4.24)

where

\[ \tilde{\varphi}_1(z) = \begin{cases} o(e^{\text{Im}z}|z|^{-1}), & j = 0 \\ O(e^{\text{Im}z}|z|^{-2}), & j = 1. \end{cases} \]  

(4.25)

Similar arguments give

\[ g_{14}(z) = -\frac{1}{4} \int_0^1 e^{-2z(1-s)} \left( 1 + \frac{1}{z} \int_s^1 p(t)dt \right) V(s)ds + O(1), \]  

\[ g_{41}(z) = -\frac{1}{4} \int_0^1 e^{-2z|z|} \left( 1 + \frac{1}{z} \int_0^s p(t)dt \right) V(s)ds + O(1). \]
which yields
\[ f_3(z) = \frac{P_1}{8} - \frac{e^{-z}}{2} \int_0^1 \cosh(z(1 - 2s)V(s))ds + \tilde{f}_2(z), \quad (4.26) \]
where
\[ \tilde{f}_2(z) = \begin{cases} o\left(|z|^{-1}\right), & j = 0 \\ O\left(|z|^{-2}\right), & j = 1. \end{cases} \quad (4.27) \]
Substituting equations (4.24), (4.26) into equation (4.18) we obtain
\[ F(z) = \frac{P_1}{8} \cos v_2z + \frac{1}{4} \int_0^1 \cos z(1 - 2s)V(s)ds + \varphi_2(z)\sin v_2z + \tilde{F}(z), \quad (4.28) \]
where
\[ \varphi_2(z) = \left(\frac{p'(1) - p'(0)}{8} - \frac{3(p^2(1) - p^2(0))}{32z}\right) + \frac{P_2}{8} - \frac{e^{-z}}{2} \int_0^1 \cosh z(1 - 2s)V(s)ds, \]
\[ \tilde{F}(z) = \tilde{\varphi}_1(z) + \tilde{\varphi}_2(z)\sin v_2z + \frac{O(e^{imz})}{|z|^2}. \]
The asymptotics (4.25), (4.27) give
\[ \tilde{F}(z) = \begin{cases} o\left(e^{imz}|z|^{-1}\right), & j = 0 \\ O\left(e^{imz}|z|^{-2}\right), & j = 1 \end{cases} \quad (4.29) \]
as \[ |z| \to \infty, \quad z \in S^+ \]. Using the identity \( D(\xi, \eta) = D(\lambda, p, q) \) we extend the identity (4.28) and the asymptotics (4.29) from \( S^+ \) to \( S \). The relations (4.28), (4.29) give equation (4.16), (4.17).

4.3. Proof of theorem 1.4

Let \( n \to \infty \). The asymptotics \( w(z_n) = \pi n + \epsilon_n \) and \( \epsilon_n = O(n^{-3}) \) from equations (4.15) and (4.12) imply
\[ D(\lambda_n) = (-1)^j E(z_n) \left( \epsilon_n + \frac{p'(1) - p'(0)}{8(\pi n)^3} + \frac{(-1)^j F(z_n)}{(\pi n)^3} + O(1) \right). \quad (4.30) \]
The asymptotics (4.16), (4.17) and the identities (4.30) and \( D(\lambda_n) = 0 \) yield
\[ \epsilon_n = -\frac{P}{8(\pi n)^3} - \frac{\tilde{v}_n}{4(\pi n)^3} + \tilde{\epsilon}_n, \quad \tilde{\epsilon}_n = \begin{cases} o\left(n^{-4}\right), & j = 0 \\ O\left(n^{-5}\right), & j = 1 \end{cases}, \quad (4.31) \]
where \( P = \int_0^1 (p^2 + p^*)dt \). The identities (4.13) and the asymptotics (4.15) give
\[ z_n + \frac{\bar{P}_0}{2z_n} = \pi n - \frac{P}{8(\pi n)^3} - \frac{\tilde{v}_n}{4(\pi n)^3} + \tilde{\epsilon}_n. \]

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and then
\[ z_n = \frac{mn}{\pi} - \frac{\hat{p}_0}{2} - \frac{P + 2\hat{p}_0^2}{8(m\pi)^3} - \frac{\hat{v}_n}{4(m\pi)^3} + \hat{c}_n + \frac{O(1)}{n^3}, \tag{4.32} \]
which yields equation (1.24).

5. The Barcilon–Gottlieb transformation

In this section we consider the transformation of the operator \( \mathcal{E} \) into the operator \( H \). To define this transformation we introduce the new variable \( \xi \in [0, 1] \) by
\[ t(x) = \int_0^x \xi(s)ds, \quad \forall \ x \in [0, 1], \quad \text{where} \quad \xi = \left( \frac{b}{a} \right)^\frac{1}{2} > 0. \tag{5.1} \]
Let \( x(t) \) be the inverse function for \( t(x), x \in [0, 1] \). Introduce the unitary Barcilon–Gottlieb transformation \( U : L^2((0, 1), b(x)dx) \to L^2((0, 1), dt) \) by
\[ u(x) \mapsto y(t) = (Uu)(t) = \rho(x(t))u(x(t)) \quad \forall \ t \in [0, 1], \tag{5.2} \]
where
\[ \rho = \frac{1}{\sqrt{a b}} > 0. \tag{5.3} \]

In lemma 5.1 we will show that the operator \( \mathcal{E} \) is unitarily equivalent to the operator \( H \), defined by equations (1.20), (1.21), where the functions \( p(t), q(t), t \in [0, 1] \) have the forms
\[ p = \varphi - \sigma^2 - 2\sigma', \tag{5.4} \]
\[ q = -\sigma'' + (\sigma')^2 + \frac{4}{3}(\sigma^3)' + \sigma^4 - 2(\varphi \sigma)' - 2\varphi \sigma^2 + v, \tag{5.5} \]
and
\[ \varphi(t) = \frac{1}{2\xi(x)} \left( \frac{\eta_+(x)}{\xi(x)} \right)' + \frac{\eta_-(x)}{\xi^2(x)}, \quad \sigma(t) = \frac{s(x)}{\xi(x)}, \quad v(t) = \frac{s(x)}{\xi(x)} \quad \text{at} \quad x = x(t), \tag{5.6} \]
\[ s = \frac{\alpha + 3\beta}{2}, \quad \eta = \eta_+\eta_-, \quad \eta_\pm = \beta \pm \alpha. \tag{5.7} \]
The values \( \alpha, \beta \) are given by equation (1.3). Due to \( (\alpha, \beta, Q) \in \mathcal{H}_3 \times \mathcal{H}_3 \times \mathcal{H}_0 \), the coefficients \( p, q \) are well defined.

** Lemma 5.1.** Let the operator \( \mathcal{E} \) be defined by equations (1.1), (1.2), where the real \( (\alpha, \beta, Q) \in \mathcal{H}_3 \times \mathcal{H}_3 \times \mathcal{H}_0 \). Let the operator \( H \) be defined by equations (1.20), (1.21), where the coefficients \( p, q \) have the forms (5.4), (5.5). Then the operators \( \mathcal{E} \) and \( H \) are unitarily equivalent and satisfy
\[ \mathcal{E} = U^{-1}HU, \tag{5.8} \]
where the operator \( U \) is defined by equation (5.2).
Proof. The identity (5.1) gives \( dt = \xi dx \), which yields \( f' = \xi f' \); here and hereafter for shortness we write \( f' = \frac{df}{dx}, f'' = \frac{df'}{dt} \).

Due to \( \alpha, \beta \in \mathcal{H} \), we have \( \xi, \rho \in \mathcal{H} \). Definition (5.2) shows that \( U(\mathcal{H}) = \mathcal{H} \).

Let \( u \in \mathcal{H} \) and let \( y = Uu \). Let, in addition, \( y(0) = y^*(0) = 0 \). Then

\[
\begin{align*}
    u(0) = 0, \quad u'(0) &= \left. \left( \frac{y}{\rho} \right)' \right|_0 = \frac{y'}{\rho} \bigg|_0,
\end{align*}
\]

and

\[
\begin{align*}
    u^{xx}(0) &= \left. \left( \frac{y'}{\rho} \right)^{xx} \right|_0 = \left( \frac{y''}{\rho} - \frac{2\rho' y'}{\rho^2} \right) \bigg|_0 = \left( \frac{\xi y'}{\rho} - 2u' \right) \bigg|_0 = \frac{\xi y'}{\rho} - 2u' \bigg|_0 = (\eta - 2x)u' \bigg|_0 = -2\eta u'.
\end{align*}
\]

Conversely, let \( u(0) = (u^{xx} + 2\eta_u u') |_0 = 0 \). Then \( y(0) = 0 \) and

\[
\begin{align*}
    y^{xx}(0) &= \left( 2\rho'u' + \rho u' \right) \bigg|_0 = \left( \frac{2\rho y'}{\xi^2} + \frac{(\xi^2 u')}{\xi^2} \right) \bigg|_0 = \frac{\rho}{\xi^2} (u^{xx} + (2s - \eta)u') \bigg|_0 = 0.
\end{align*}
\]

Similar relations at the point \( x = 1 \) hold true, which proves that the function \( y \) satisfies the boundary the conditions (1.21) iff the function \( u \) satisfies the boundary conditions (1.2).

Assume that

\[
\rho \left( \frac{1}{b} \left( au^{xx} \right)^{xx} + Qu \right) \bigg|_{x = 1(t)} = y^{(4)} + 2(py')' + qy \quad \forall \quad t \in [0, 1]. \tag{5.9}
\]

Then \( \frac{1}{b} (au^{xx})^{xx} + Qu \in L^2(0, 1) \) iff \( y^{(4)} + 2(py')' + qy \in L^2(0, 1) \). We obtain the identity

\[
U(\mathcal{D}(\mathcal{E})) = \mathcal{D}(H), \tag{5.10}
\]

where \( \mathcal{D}(\mathcal{E}), \mathcal{D}(H) \) are the domains of the operators \( \mathcal{E}, H \) given by equations (1.6), (1.22).

The identities (5.9), (5.10) show that \( UEu = H = HUu \), which yields equation (5.8).

We need to prove equation (5.9). We obtain

\[
\frac{1}{b} (au^{xx})^{xx} + Qu = \frac{\tilde{c}}{b} \left( \xi (c(\xi u'))' \right)' + Qu, \quad c = a\xi. \tag{5.11}
\]

Moreover, due to \( \rho = c^2\xi \), we obtain

\[
\left( c(\xi u')' \right)' = \left( c(\xi u' + \xi' u') \right)' = \left( \xi c\xi u' + (c\xi')u' + (c\xi')u' \right) = \left( \rho^2 u'' + (\rho^2 u'' + (c\xi')u') = \left( \rho^2 u'' \right)' + \left( c(\xi')u' \right)' \right) \tag{5.12}
\]

and

\[
\frac{(c\xi')\xi}{\rho^2} = \frac{c^2\xi'}{c^2\xi} = \frac{c^2\xi'}{c^2\xi} \tag{5.13}
\]
Using the identities 
\[ \xi' = \eta, \quad c' = \frac{\alpha'}{\alpha} + \frac{\xi'}{\xi} = \frac{4\alpha + \eta}{\xi}, \]
where \( \eta, \alpha \) are given by equation (5.7), we obtain 
\[ \frac{(c\xi')^2}{\rho^2} = \left( \frac{\eta'}{\xi} \right)^2 + \left( \frac{4\alpha + \eta}{\xi^2} \right)^2 = \left( \frac{\eta}{\xi} \right)^2 + \frac{2\eta}{\xi^2} = 2\phi. \]
Then equation (5.12) gives 
\[ \xi' \left( c(\xi')' \right)' = \left( \rho^2 u' \right)' + \left( 2\rho^2 \varphi u' \right)'. \]
The substitution of this expression into equation (5.11) and the identity \( \rho \xi = b \) imply 
\[ \frac{1}{b} \left( au_{\xi \xi} \right)' + Qu = \frac{1}{\rho^2} \left( \left( \rho^2 u' \right)' + \left( 2\rho^2 \varphi u' \right) \right) + Qu. \quad (5.13) \]
The identities \( y = \rho u \) and (5.13) give 
\[ \rho \left( \frac{1}{b} \left( au_{\xi \xi} \right)' + Qu \right) = \frac{1}{\rho} \left( \left( \rho^2 u' \right)' + \left( 2\rho^2 \varphi u' \right) \right) + Qy. \quad (5.14) \]
The identities (1.8), (5.3), (5.7) yield \( \rho(x(t)) = \rho(0)e^{\int_0^t \sigma(x)ds} \); then 
\[ \frac{\rho'}{\rho} = \sigma, \quad \frac{\rho''}{\rho} = \sigma^2 + \sigma'. \quad (5.15) \]
Using equation (5.15) we obtain 
\[ \frac{1}{\rho} \left( \rho^2 \varphi ' \left( \frac{y}{\rho} \right) \right)' = \frac{1}{\rho} \left( \rho \varphi (y' - \sigma y)' = (\varphi y')' - \left( \varphi \sigma^2 + (\varphi \sigma)' \right)y. \quad (5.16) \]
Moreover, 
\[ \frac{1}{\rho} \left( \rho^2 \varphi ' \left( \frac{y}{\rho} \right) \right)' \\
= \frac{1}{\rho} \left( \rho \left( y' - 2\sigma y' + (\sigma^2 - \sigma')y \right) \right)' \\
= \varphi' + 2 \left( \sigma^2 - \frac{2(\sigma \rho')}{\rho} \right) y' + 2 \rho \left( \left( (\sigma^2 - \sigma') \rho' \right)' - (\sigma \rho') y' + \left( (\sigma^2 - \sigma')^2 \rho \right)' \right)' \\
= \varphi' + 2 \left( \sigma^2 - \frac{2(\sigma \rho')}{\rho} \right) y' + 2(\sigma')^2 + 2\sigma^2 y' - \sigma^2 \\
+ 2(2\sigma' - \sigma) \frac{\sigma'}{\rho} + \left( \sigma^2 - \sigma' \right) \frac{2\sigma'}{\rho^2} y \\
= \varphi' - 2 \left( 2\sigma' + \sigma^2 \right) y' + \left( (\sigma')^2 - \sigma^2 + 4\sigma^2 \sigma' + \sigma^4 \right)y. \quad (5.17) \]
Substituting equations (5.16), (5.17) into equation (5.14) and using the definitions (5.4), (5.5) we obtain equation (5.9).
6. Eigenvalue asymptotics for the Euler–Bernoulli operator

We need the following identity.

Lemma 6.1. Let $\alpha, \beta \in \mathcal{H}_3$. Then the function $p$ has the form

$$p = -\frac{\kappa}{\xi^2} - \kappa',$$

(6.1)

where $\kappa$ and $\kappa'$ are given by equations (1.10) and (1.9) respectively.

Proof. The identities (5.6) and $s^2 - \eta = \kappa$ imply

$$\sigma^2 - \varphi = \frac{s^2 - \eta}{\xi^2} - \frac{1}{2} \left( \frac{\eta_0}{\xi} \right) = \frac{\kappa}{\xi^2} - \frac{1}{2} \left( \frac{\eta_0}{\xi} \right).$$

(6.2)

Substituting equation (6.2) into equation (5.4) and using the identity $\frac{1}{\xi^2} (\eta_0 - 4s) = -\kappa$, we obtain

$$p = -\frac{\kappa}{\xi^2} + \left( \frac{\eta_0}{\xi^2} - 2\sigma \right) = -\frac{\kappa}{\xi^2} + \left( \frac{\eta_0 - 4s}{\xi^2} \right) = -\frac{\kappa}{\xi^2} - \kappa'.$$

We begin to prove the main theorems.

Proof of theorem 1.2. Our proof is based on the eigenvalue asymptotics (1.24) for the operator $H$ (see hereafter in this section) and the Barcilon–Gottlieb transformation (see section 5). Let $\alpha, \beta \in \mathcal{H}_3$, and let $p, q$ have the form (5.4), (5.5). Then $(p, q) \in \mathcal{H}_2 \times \mathcal{H}_0$.

Now we assume that theorem 1.4 is valid. Then the asymptotics (1.24) and lemma 5.1 yields

$$\lambda_n = (\pi n)^4 - 2(\pi n)^2 \hat{p}_0 + \psi_i - \hat{V}_a + o\left( n^{-1} \right) \quad \text{as} \quad n \to \infty,$$

(6.3)

where

$$\psi_i = \hat{V}_0 - \frac{1}{2} \int_0^1 \left( p^2(t) - \hat{p}_0^2 \right) dt,$$

(6.4)

$\hat{p}_0, \hat{q}_0, \hat{V}_a$ are given by equation (1.15), $V = q - \frac{p^2}{2}$, and $y_i' = \frac{dy}{dx}$, $y' = \frac{dy}{dt}$. The identity (6.1) gives

$$\hat{p}_0 = \int_0^1 p(t) dt = \kappa(0) - \kappa(1) - \int_0^1 \frac{\kappa}{\xi^2} dt = -\psi_0,$$

(6.5)

where $\psi_0$ is defined by equation (1.10). The relations (5.4), (5.5) give

$$V = q - \frac{p^2}{2} = 2(\sigma')^2 + 4\sigma' \sigma^2 + \sigma^4 - 2(\varphi \sigma') - 2\varphi \sigma^2 - \frac{\varphi'}{2} + \sigma \varphi + \nu.$$

Let $n \to \infty$. Using $\sigma^*, \varphi^* \in L^1(0, 1)$, we have

$$\hat{V}_a = \int_0^1 \left( \nu(t) - \frac{\varphi^*(t)}{2} \right) \cos 2\pi n t dt + o\left( n^{-1} \right).$$

(6.6)
Due to equation (6.14) and $\eta_{\pm}^\prime \xi \in L^1(0, 1)$ we obtain

$$
\hat{V}_n = \int_0^1 \left( u(t) - \frac{\eta_{\pm}^\prime(t)}{4\xi(x(t))} \right) \cos 2\pi nt \, dt + o\left(n^{-1}\right)
$$

$$
= \int_0^1 \left( Q(x) - \frac{(\eta_{\pm})^{\prime\prime\prime}}{4\xi^4(x)} \right) \xi(x) \cos 2\pi nt \, dx + o\left(n^{-1}\right) = \gamma_n + o\left(n^{-1}\right), \tag{6.7}
$$

where $\gamma_n$ is defined by (1.12). Substitution of equations (6.5), (6.7) into the asymptotics (6.3) yields equation (1.11).

We prove the Ambarzumyan-type theorem regarding the inverse problem for the operator $\mathcal{E}$.

**Proof of theorem 1.1.** The identities $a = b = 1$ give $\alpha = \beta = 0$, which yields $\lambda_n = (\pi n)^4$. Conversely, let $k(x) = k(1)$ and let $\lambda_n = (\pi n)^4$ for all $n \in \mathbb{N}$. The asymptotics (1.11) imply $\psi_0 = 0$; then equation (1.10) yields $\varpi = 0$. The identity (1.10) gives $\alpha = \beta = 0$, i.e., $a = b = 1$, which proves the statement.

The identity (6.4) shows that the constant $\psi_1$ in the asymptotics (1.11) has the form

$$
\psi_1 = \int_0^1 \psi(t) \, dt, \quad \text{where} \quad \psi = V - \frac{p^2 - \hat{p}_0^2}{2}, \quad V = \frac{q - p^2}{2}, \tag{6.8}
$$

where $p, q$ are given by equations (5.4), (5.5). In proposition 6.2 we express the constant $\psi_1$ in terms of $\alpha, \beta$.

**Proposition 6.2.** Let $\alpha, \beta \in \mathcal{H}_3$. Then

$$
\psi_1 = \mathcal{A}(1) - \mathcal{A}(0) + \int_0^1 \mathcal{B}(x) \xi(x) \, dx + \frac{\psi_0^2}{2} + Q_0, \tag{6.9}
$$

where $\psi_0$ is given by equation (1.10), $Q_0 = \int_0^1 Q(x) \xi(x) \, dx$ and the functions $\mathcal{A}(x), \mathcal{B}(x)$ have the form

$$
\mathcal{A} = \frac{1}{6} \left( \frac{2\alpha^3}{3} - \frac{\eta_{\pm}^3}{2} - 2\eta_{\pm} - (s - \eta_{\pm})_{\xi} \eta_{\xi} + (s - \eta_{\pm})_{\xi} \eta_{\xi} - \frac{1}{4} \right), \tag{6.10}
$$

$$
\mathcal{B} = \frac{1}{8} \left( \left( \eta_{\pm} \right)_{\xi}^2 - \eta^2 - 2\varpi \right)^2 - 8 \left( \left( \eta_{\pm} \right)_{\xi}^2 - 2\varpi \right), \tag{6.11}
$$

$\varpi$ and $s, \eta_{\pm}, \eta$ are given by equations (1.10) and (5.7), respectively.

**Proof.** The identity (5.5) yields

$$
q(t) = \left( -\sigma^2 + \frac{4}{3} \sigma^3 - 2\varpi \sigma \right)^\prime + (\sigma^\prime)^2 + \sigma^4 - 2\varpi \sigma^2 + \nu.
$$
Substituting this equality into equation (6.8), we obtain
\[
\psi = \left( -\sigma + \frac{4}{3} \sigma^3 - 2 \varphi \sigma - \frac{p^2}{2} \right) + (\sigma')^2 + \sigma^4 - 2 \varphi \sigma^2 - \frac{p^2}{2} + \frac{\hat{p}_0^2}{2} + u.
\]

Using equation (5.4) and the identities
\[
(\sigma')^3 + \sigma^4 - 2 \varphi \sigma^2 - \frac{p^2}{2} = (\sigma')^2 + \sigma^4 - 2 \varphi \sigma^2 - \frac{(\varphi - \sigma^2 - 2 \sigma')^2}{2} = \frac{(\sigma^2 - \varphi)^2}{2} - (\sigma' - \varphi)^2 - \frac{2}{3} (\sigma')^4,
\]
we obtain
\[
\psi = \mathcal{A}' + \mathcal{B} + \frac{\psi_0^2}{2} + u,
\]
where
\[
\mathcal{A} = \frac{2}{3} \sigma^3 - 2 \varphi \sigma + \frac{(\sigma^2 - \varphi)^2}{2}, \quad \mathcal{B} = \frac{(\sigma^2 - \varphi)^2}{2} - (\sigma' - \varphi)^2.
\] (6.13)

Thus equations (6.12) and (6.8) yield equation (6.9).

We show equations (6.10), (6.11). The identities (5.6) and \( \xi' = \eta = \eta \) give
\[
\varphi = 1 - \frac{1}{2} \left( \frac{\eta_s'}{\xi^2} \right) + \frac{\eta}{\xi^2} = 1 - \frac{1}{2} \left( \frac{\eta_s'}{\xi^2} - \frac{\eta^2}{\xi^2} \right) + \frac{\eta}{\xi^2},
\]
\[
\varphi' = \frac{\eta_{ss}'}{2 \xi^2} - \frac{3 \eta_s \eta_{s'}}{2 \xi^3} + \frac{\eta^3}{\xi^2} + \frac{\eta'}{\xi^2} - \frac{2 \eta \eta_{ss}}{\xi^2}.
\] (6.14)

Substituting these identities into equation (6.13) and using \( \sigma = \frac{\xi}{\xi} \) we obtain
\[
\mathcal{A} = \frac{2}{3} \sigma^3 - \frac{\eta^3}{2 \xi^3} + \frac{\eta \eta_s}{\xi^3} - \frac{2 \eta \xi}{\xi^3} + \frac{s - \eta_s}{\xi} - \frac{\eta_s'}{\xi} + \frac{3 \eta_s \eta_{ss}}{4 \xi^2} - \frac{\eta_s}{2 \xi^2}.
\]

Using
\[
\eta_{ss} = \frac{\eta_s'}{\xi^2}, \quad \eta_{ss'} = \frac{\eta_{ss'}}{\xi^2}, \quad \eta_{ss} = \frac{\eta_{ss'}}{\xi^2}, \quad \eta_{ss'} = \frac{\eta_{ss'}}{\xi^2},
\]
and the formulas \( \eta_s - 2 \eta = -2 \eta, \eta_s^2 - \eta = -2 \eta \eta_s \) we obtain equation (6.10).

The identities (5.6) yield
\[
\sigma' = \varphi = \left( \frac{s}{\xi} - \frac{\eta_s}{\xi^2} \right) = \frac{\eta_s}{\xi^2}, \quad \sigma'' = \frac{(\eta_{ss})'}{\xi^2} - \frac{\eta_{ss'}}{\xi^2},
\]
\[
\sigma^2 - \varphi = \frac{x}{\xi^2} - \frac{1}{2} \left( \frac{\eta_s}{\xi^2} \right)' = \frac{2x + \eta_s' - \eta_{s'}}{2 \xi^2}.
\]

Substituting these equalities into equation (6.13) we obtain equation (6.11). \( \square \)
7. Examples and remarks

7.1. Fourth-order operator

We consider an example of the operator $H$.

**Example 1: the square of a second-order operator.** Consider the operator

$$hy = -y'' - py, \quad y(0) = y(1) = 0, \quad p \in \mathcal{H}$$

in the unit interval $[0, 1]$. Let $\alpha_n$, $n \in \mathbb{N}$, be the eigenvalues of this operator labeled by $|\alpha_1| \leq |\alpha_2| \leq |\alpha_3| \leq ...$, counted with multiplicity. It is well known (see [FP, equation (4.21)]) that

$$\alpha_n = (\pi n)^2 - \tilde{p}_0 + \frac{1}{(2\pi n)^2} \left( \int_0^1 \left( p^2 + p'' \right) dt - \tilde{p}_0^2 \right) + \frac{O(1)}{n^4} \quad \text{as} \quad n \to \infty. \quad (7.1)$$

This gives the following asymptotics for the eigenvalues $\lambda_n = \alpha_n^2$ of the operator $\chi = \partial^2 + p\partial + p''$,

$$\lambda_n = \alpha_n^2 = (\pi n)^4 - 2(\pi n)^2\overline{p}_0 + \frac{1}{2} \left( \int_0^1 \left( p^2 + p'' \right) dt + \tilde{p}_0^2 \right) + \frac{O(1)}{n^2} \quad (7.2)$$

as $n \to \infty$. The asymptotics (7.2) are in agreement with equation (1.24) since $V = q - \frac{p''}{2} = p^2 + \frac{p''}{2}$ in our case, and then

$$\tilde{V}_0 - \frac{1}{2} \int_0^1 (p^2 - \tilde{p}_0^2) dt = \frac{1}{2} \left( \int_0^1 (p^2 + p'') dt + \tilde{p}_0^2 \right).$$

**Remarks 1.** Fulton and Pruess [FP, equation (4.21)] determined the correct asymptotics

$$\alpha_n = (\pi n)^2 - \tilde{p}_0 + \frac{P - \tilde{p}_0^2}{(2\pi n)^2} + \frac{O_2}{(2\pi n)^4} + \frac{O(1)}{n^6}, \quad \text{(7.3)}$$

where $P = \int_0^1 (p^2 + p'') dt$,

$$O_2 = 2\tilde{p}_0 \left( 12P - 11\tilde{p}_0^2 \right) - 2\int_0^1 p^3(s) ds + \int_0^1 p^2(s) ds - \left( 6pp' + p'' \right) \bigg|_0^1.$$

**Remarks 2.** Pöschel and Trubowitz (see [PT, problem 2.3]) determined the asymptotics

$$\alpha_n = (\pi n)^2 + \frac{1}{(2\pi n)^2} \int_0^1 (p^2 + p'') dt + \frac{o(1)}{n^2}$$

in the class of real $p$, $p'' \in L^2(0, 1)$, $\tilde{p}_0 = 0$. (Note that there is a misprint in the sign in the second term of the asymptotics in [PT].)

**Remarks 3.** Dikii [D1, p 189] considered an operator $h$ with the real $p \in C^\infty[0, 1]$ and determined the following asymptotics:

$$\alpha_n = (\pi n)^2 - \tilde{p}_0 + \frac{1}{(2\pi n)^2} \left( \|p\|^2 - 4\tilde{p}_0^2 + \frac{1}{3}(p'(1) - p'(0)) \right) + \ldots \quad (7.4)$$
This also is in a disagreement with equation (7.1). The coefficients 4 and $\frac{1}{3}$ in equation (7.4) are mistaken; see [FP, remark 4.5].

**Remarks 4.** Sadovnichii [S, pp 308–9] considered the operator $H = \partial^4 + 2\partial p \partial + q$, where $p, q \in C^\infty_0[0, 1]$, $p^{(j)}(0) = p^{(j)}(1) = q^{(2j-1)}(0) = q^{(2j-1)}(1) = 0 \quad \forall \quad j \in \mathbb{N}$. (7.5)

Let $\lambda_n, n \in \mathbb{N}$ be the eigenvalues of this operator labeled by $\lambda_1 \leq \lambda_2 \leq ...$, counted with multiplicities. Sadovnichii wrote (without the proof) the following asymptotics:

$$
\lambda_n = (\pi n)^4 = 2(\pi n)^2 p_0 + \frac{c_1}{n^2} + \frac{c_2}{n^3} + ...
$$

(7.6) as $n \to \infty$, where $c_1, c_2$ are some undetermined constants. Consider the operator $h^2 = (-\partial^2 - p)^2 = \partial^4 + 2\partial p \partial + p^* + p_0$, where $p$ satisfies equation (7.5). In this case the asymptotics (7.6) give

$$
\lambda_n = (\pi n)^4 = 2(\pi n)^2 p_0 + \|p\|^2 + \frac{c_1}{n^2} + \frac{c_2}{n^3} + ...
$$

(7.7)

On the other hand, the asymptotics (7.2) of $\alpha_n^2$ yield in this case

$$
\lambda_n = \alpha_n^2 = (\pi n)^4 - 2(\pi n)^2 p_0 + \frac{\|p\|^2 + p_0^2}{2} + O(1).
$$

(7.8)

The third term in the asymptotics (7.7) is in disagreement with the corresponding term in equation (7.8). Therefore, the term $p_0$ in equation (7.6) is incorrect.

**7.2. The Euler–Bernoulli operator**

We give some remarks and examples for the Euler–Bernoulli operator.

**Remarks 5.** Jian-jun, Kui, and Da-jun [JKD] considered the Euler–Bernoulli equation $(au^r)' = \lambda bu$, where $a, b$ are smooth positive coefficients, under the following four boundary conditions:

$$
u(0) = u'(0) = u''(1) = (au^r)'|_{x=1} = 0 \quad \text{(clamped–free beam)},
$$

(7.9)

$$
u(0) = u''(0) = u''(1) = (au^r)'|_{x=1} = 0 \quad \text{(clamped–sliding beam)},
$$

(7.10)

$$
u(0) = u'(0) = u'(1) = u''(1) = 0 \quad \text{(clamped–pinned beam)},
$$

(7.11)

$$
u(0) = u'(0) = u(1) = u'(1) = 0 \quad \text{(clamped–clamped beam)}. \quad (7.12)
$$

In fact, they considered the four operators. For these four operators they announced the following asymptotics:

$$
\lambda_n = \alpha_n^4 + 4\alpha_n^2 \int_0^1 \left( \frac{5\xi^4}{4\xi^2 - 15(\xi^2)^2 - 3(a')^2/8\xi a^2 + a^2/2a^2} \right) dx + O(1)
$$

(7.13)
as \( n \to \infty \), where

\[
a_n = \begin{cases} 
\pi \left( n - \frac{1}{4} \right) & \text{for conditions (7.9)} \\
\pi \left( n - \frac{1}{2} \right) & \text{for conditions (7.10)} \\
\pi \left( n + \frac{1}{4} \right) & \text{for conditions (7.11)} \\
\pi \left( n + \frac{1}{2} \right) & \text{for conditions (7.12)} 
\end{cases}
\]  
(7.14)

see [JKD, equation (3.8)]. Direct calculations show that

\[
2 \int_0^1 \left( \frac{5\xi^4}{4\xi^2} - \frac{15(\xi^2)^2}{8\xi^3} - \frac{3(a^2)^2}{8\xi^2} + \frac{a^4}{2a_2^2} \right) dx = \psi_0,
\]

where \( \psi_0 \) is given by equation (1.10). Then, assuming that the asymptotics (7.13) are correct, we obtain

\[
\lambda_n = a_n^4 + 2a_n^2 \psi_0 + O(1)
\]  
(7.15)

as \( n \to \infty \). These asymptotics make it possible to extend the results of theorem 1.1 to the Euler–Bernoulli operator with the boundary conditions (7.9)–(7.12).

**Proposition 7.1.** Let \( Q = 0 \), and let the real \((\alpha, \beta) \in \mathcal{H}_2 \times \mathcal{H}_2 \) satisfy the conditions (1.8) and \( \kappa(0) = \kappa(1) \). Assume that the eigenvalues \( \lambda_n, n \in \mathbb{N} \) of the Euler–Bernoulli operator (1.1) under one of the boundary conditions (7.9)–(7.12) satisfy the asymptotics (7.13), with the coefficients \( a_n \) given by equation (7.14). Then the eigenvalues \( \lambda_n = (\pi n)^2 \) for all \( n \geq 1 \) iff \( a = b = 1 \).

**Proof.** Using equation (7.15) and repeating the arguments from the proof of theorem 1.1 we obtain the statement. \( \square \)

**Remarks 6.** Consider the weighted second-order operator \( h_w \) given by

\[
h_wu = -\frac{1}{b} u''_w, \quad x \in [0, 1], \quad u(0) = u(1) = 0,
\]  
(7.16)

with real periodic coefficients \( b \in \mathcal{H}_2, b > 0 \) normalized by \( \int_0^1 b^2 ds = 1 \). Let \( \lambda_1 < \lambda_2 < \ldots \) be the eigenvalues of this operator. The standard Liouville transformation

\[
t(x) = \int_0^x b(s)^{1/2} ds, \quad y(t) = b^{1/2}(x(t))u(x(t))
\]
yields the result that the operator \( h_w \) is unitarily equivalent to the operator \( \tilde{h} \) given by

\[
\tilde{h}y = -y''_w + \left( \beta^2 + \beta' \right)y, \quad t \in [0, 1], \quad y(0) = y(1) = 0,
\]

where \( \beta = \frac{b'}{4b} \). The standard eigenvalue asymptotics for the second-order operator give

\[
\lambda_n = (\pi n)^2 + \int_0^1 \beta^2(t) dt + o(1) \quad \text{as} \quad n \to \infty;
\]
see [PT]. Assume that \( \lambda_n = (\pi n)^2 \) for all \( n \in \mathbb{N} \). Then the last asymptotics imply \( \beta = 0 \), which yields \( b = 1 \). We obtain a result similar to the result of theorem 1.1 for the operator (7.16): the eigenvalues \( \lambda_n = (\pi n)^2 \) for all \( n \in \mathbb{N} \) iff \( b = 1 \).

**Example 2: the Euler–Bernoulli operators with periodic coefficients.** Consider the case of 1-periodic \( a, b \). Let \( a, \beta, a^n, b^n \in L^1(\mathbb{R}/\mathbb{Z}) \) satisfy equation (1.8), and let \( Q = 0 \). The identities (1.10), (6.9) give

\[
\psi_0 = \int_0^1 \frac{x(x)}{\xi(x)} dx > 0, \quad \psi_1 = \int_0^1 \mathcal{B}(x) \xi(x) dx + \frac{\psi_0^2}{2}. \tag{7.17}
\]

Consider some particular cases.

**Example 2.1.** Let, in addition, \( ab = 1 \). In this case we have \( \alpha = -\beta \), and the identities (1.8), (1.10), (5.7), (6.11) give

\[\xi = a^{-1} > 0, \quad \eta_+ = -2a, \quad \eta_+ = 0, \quad s = -\alpha, \quad \varepsilon = a^2, \quad \mathcal{B} = \frac{1}{2}a^2\left(\alpha' + 6\alpha^2\right)^2.\]

The Euler–Bernoulli operator

\[\mathcal{E}u = a\left(u_{xx}ight)' + h_0^2 u,\]

where the operator \( h_0 u = -au_{xx} \) with the boundary condition \( u(0) = u(1) = 0 \). The identities (7.17) yield

\[
\psi_0 = \int_0^1 a^3 a^2 dx, \quad \psi_1 = \frac{1}{2} \int_0^1 a^3 \left(\alpha' + 6\alpha^2\right)^2 dx + \frac{1}{2} \left( \int_0^1 a^4 a^2 dx \right)^2 \geq 0.
\]

The identity (1.12) gives

\[
\gamma_n = \frac{1}{2} \int_0^1 a^3 a^2 \cos \left(2\pi n \int_0^1 \frac{ds}{a^2(s)}\right) dx.
\]

Substituting these identities into equation (1.11) we obtain the eigenvalue asymptotics for this case. Note that the eigenvalues \( \sqrt{\lambda_n} \) of the operator \(-u_{xx}, u(0) = u(1) = 0\) satisfy

\[
\sqrt{\lambda_n} = (\pi n)^2 + \psi_0 + \frac{\psi_1 - \psi_0^2 - \gamma_n}{2(\pi n)^2} + \frac{o(1)}{n^3} \quad \text{as} \quad n \to \infty.
\]

**Example 2.2.** Consider the case \( a = 1 \). Then \( \alpha = 0 \) and the identities (1.8), (1.10), (5.7), (6.11) give

\[\xi = b^2 > 0, \quad \eta_+ = \eta_+ = \beta, \quad \varepsilon = \frac{2}{3} \beta^2, \quad \mathcal{B} = -\frac{1}{8b} \left(7(b')^2 - 25b'^2 + \frac{79}{4} \beta^2\right).\]

The identities (7.17) yield

\[
\psi_0 = \frac{5}{4} \int_0^1 b^2 dx, \quad \psi_1 = \frac{1}{8} \int_0^1 b^2 \left(7(b')^2 - 25b'^2 + \frac{79}{4} \beta^2\right) dx + \frac{25}{32} \left( \int_0^1 \frac{\xi'}{b^2} dx \right)^2.
\]
The identity (1.12) gives

\[ \gamma_n = -\frac{1}{4} \int_0^1 \beta^\alpha(x) \cos \left( 2\pi n \int_0^1 \frac{1}{b}\frac{1}{b^\alpha(x)} \right) \frac{dx}{b^\alpha(x)}. \]

Substituting these identities into equation (1.11) we obtain the eigenvalue asymptotics for this case.

Example 2.3. Consider the operator

\[ \mathcal{E}u = \frac{1}{a} (au''_{xx})_{xx}. \]

Then \( a = b \) and \( \alpha = \beta \). The identities (1.8), (1.10), (5.7), (6.11) give

\[ \xi = 1, \quad \eta_+ = 2\alpha, \quad \eta_- = 0, \quad \chi = 4\alpha^2, \quad \mathcal{B} = -4 \left( (\alpha')^2 - 2\alpha^4 \right). \]

The identities (7.17) yield \( \psi_0 = 4 \int_0^1 \alpha^2 dx = 4 \| \alpha \|^2 \),

\[ \psi_1 = -4 \int_0^1 \left( (\alpha')^2 - 2\alpha^2 \right) dx + 8 \left( \int_0^1 \alpha^2 dx \right)^2 = 8 \left( \| \alpha^2 \|^2 + \| \alpha \|^4 \right) - 4 \| \alpha \|^2, \]

where \( \| f \|^2 = \int_0^1 f^2 dx \). Depending on \( \alpha \), the constant \( \psi_1 \) may be positive or negative. The definition (1.12) gives

\[ \gamma_n = 0. \]

Substituting these identities into equation (1.11) we obtain

\[ \lambda_n = (\pi n)^2 + 8 \| \alpha \|^2 (\pi n)^2 + 8 \left( \| \alpha^2 \|^2 + \| \alpha \|^4 \right) - 4 \| \alpha \|^2 + \frac{\alpha(1)}{n} \quad \text{as} \quad n \to \infty. \]

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References

[Am] Amara J B 2009 Sturm theory for the equation of vibrating beam J. Math. Anal. Appl. 349 1–9
[Ah] Akhmerova E F 2011 Asymptotics of the spectrum of nonsmooth perturbations of differential operators of order 2m Math. Notes 90 813–23
[At] Atkinson F V 1964 Discrete and Continuous Boundary Problems vol 214 (New York: Academic Press)
[AM] Atkinson F V and Mingarelli A 1987 Asymptotics of the number of zeros and the eigenvalues of general weighted Sturm-Liouville problems J. Reine Angew Math. 375 380–93
[BK1] Badanin A and Korotyaev E 2005 Spectral asymptotics for periodic fourth-order operators Int. Math. Res. Not. 45 2775–814
[BK2] Badanin A and Korotyaev E 2011 Spectral estimates for periodic fourth-order operators St. Petersburg Math. J 22 703–36
[BK3] Badanin A and Korotyaev E 2012 Even order periodic operators on the real line Int. Math. Res. Not. 2012 1143–94
[BK4] Badanin A and Korotyaev E 2014 Sharp eigenvalue asymptotics for fourth-order operators on the circle J. Math. Anal. Appl. 417 804–18
[BK5] Badanin A and Korotyaev E 2013 Trace formula for fourth-order operators on the circle Dynamics of PDE 10 343–52
[BK6] Badanin A and Korotyaev E 2014 Trace formulas for fourth-order operators on unit interval, II arXiv:1412.4951
[B1] Barcilon V 1974 On the uniqueness of inverse eigenvalue problems Geophys. J. Int. 38 287–98
[B2] Barcilon V 1987 Sufficient conditions for the solution of the inverse problem for a vibrating beam Inverse Problems 3 181–93
[CPS] Caudill L F Jr, Perry P A and Schueller A W 1998 Isospectral sets for fourth-order ordinary differential operators SIAM J. Math. Anal. 29 935–66
[ChG] Chang J D and Guo B Z 2007 Identification of variable spacial coefficients for a beam equation from boundary measurements Automatica 43 732–7
[CM] Coleman C F and McLaughlin J R 1993 Solution of the inverse spectral problem for an impedance with integrable derivativi: I. Comm. Pure Appl. Math. 46 145–84
[D1] Dikii L A 1955 The zeta function of an ordinary differential equation on a finite interval Izvestiya Rossiiskoi Akademi Nauk. Seriya Matematicheskaia 19 187–200
[FP] Fulton C T and Pruess S A 1994 Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems J. Math. Anal. Appl. 188 297–340
[Gh] Ghanbari K 2005 On the isospectral beams Electronic J. Diff. Eqs. Conf. 12 57–64
[Gl1] Gladwell G M 2004 Inverse Problems in Vibration vol 119 (Berlin: Springer)
[Gl2] Gladwell G M L 2002 Isospectral vibrating beams Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 458 2691–703
[GEW] Gladwell G M L, England A H and Wang D 1987 Examples of reconstruction of an Euler-Bernoulli beam from spectral data J. Sound Vib. 119 81–94
[GiM] Gladwell G M and Morassi A 2010 A family of isospectral Euler-Bernoulli beams Inverse Problems 26 035006
[Go1] Gottlieb H P W 1987 Isospectral Euler-Bernoulli beam with continuous density and rigidity functions Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 413 235–50
[Go2] Gottlieb H P W 1993 Inhomogeneous annular plates with exactly beam-like radial spectra IMA J. Appl. Math. 50 107–12
[Guo] Guo B Z 2002 Riesz basis property and exponential stability of controlled euler-bernuilli beam equations with variable coefficients SIAM J. Control Optim. 40 1905–23
[HLO] Hoppe J, Laptev A and Östensson J 2006 Solitons and the removal of eigenvalues for fourth-order differential operators Int. Math. Res. Not. 85050 1–14
[JKD] Jian-jun J, Kui H and Da-jun W 1989 The asymptotic properties of high frequencies for bars, beams and membranes Appl. Math. Mech. 10 1187–93
[KGM] Kambampati S, Ganguli R and Mani V 2012 Determination of isospectral nonuniform rotating beams J. Appl. Mech. 79 061016
[Ka] Kawano A 2014 Uniqueness in the identification of asynchronous sources and damage in vibrating beams Inverse Problems 30 065008
[K] Korotyaev E 2003 Periodic weighted operators J. Differ. Equ. 189 461–86
[LeS] Lesnic D 2006 Determination of the flexural rigidity of a beam from limited boundary measurements J. Appl. Math. Comp. 20 17–34
[LeS] Levitan B M and Sargsyan I S 1991 Sturm-Liouville and Dirac Operators (Dordrecht: Kluwer)
[McL1] McLaughlin J R 1978 An inverse eigenvalue problem of order four-an infinite case SIAM J. Math. Anal. 9 395–413
[McL2] McLaughlin J R 1984 On constructing solutions to an inverse Euler-Bernoulli problem Inverse Problems of Acoustic and Elastic Waves 341–7
[MM] Mikhailets V A and Molyboga V 2012 On the spectrum of singular perturbations of operators on the circle Math. Notes 91 588–91
[Na] Naimark M 1968 Linear Differential Operators: I, II. (New York: Ungar)
[P] Papanicolaou V 2004 An inverse spectral result for the periodic Euler-Bernoulli equation Indiana Univ. Math. J. 53 223–42
[PeT] Peletier L A and Troy W C 2001 Spatial patterns: higher-order models in physics and mechanics Progr. Nonlinear Differential Equations Appl. vol 45 (Boston, MA: Birkhauser Boston Inc.)
[PT] Pöschel J and Trubowitz E 1987 Inverse Spectral Theory (Boston: Academic Press)
[RS] Reed M and Simon B 1975 Methods of Modern Mathematical Physics: vol 2. Fourier Analysis, Self-Adjointness (New York: Academic Press)

[S] Sadovnichii V A 1967 The trace of ordinary differential operators of high order Matematicheskii Sbornik 114 293–317

[Soh] Soh C W 2009 Isospectral Euler–Bernoulli beams via factorization and the Lie method Int. J. Non-Linear Mech. 44 396–403

[SuA] Sundaram M M and Ananthasuresh G K 2013 A note on the inverse mode shape problem for bars, beams, and plates Inverse Problems in Science and Engineering 21 1–16

[Yu] Yurko V A 2002 Method of Spectral Mappings in the Inverse Problem Theory vol 31 (Berlin: Walter de Cruyter)