The application of gauge invariance and canonical quantization to the internal structure of gauge field systems

Fan Wang\textsuperscript{1,3*}, X.S.Chen\textsuperscript{2,3}, X.F. Lu\textsuperscript{2}, W. M. Sun\textsuperscript{1,3}, and T.Goldman\textsuperscript{4}

\textsuperscript{1} Department of Physics, Nanjing University, and Joint Center of Particle, Nuclear Physics and Cosmology, Nanjing University and PMO, CAS, Nanjing 210093

\textsuperscript{2} Department of Physics, Sichuan University, Chengdu, 610064

\textsuperscript{3} Kavli Inst. for Theor. Phys. China, CAS, Beijing 100190 and

\textsuperscript{4} Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Abstract

It is unavoidable to deal with the quark and gluon momentum and angular momentum contributions to the nucleon momentum and spin in the study of nucleon internal structure. However, we never have the quark and gluon momentum, orbital angular momentum and gluon spin operators which satisfy both the gauge invariance and the canonical momentum and angular momentum commutation relation. The conflicts between the gauge invariance and canonical quantization requirement of these operators are discussed. A new set of quark and gluon momentum, orbital angular momentum and spin operators, which satisfy both the gauge invariance and canonical momentum and angular momentum commutation relation, are proposed. The key point to achieve such a proper decomposition is to separate the gauge field into the pure gauge and the gauge covariant parts. The same conflicts also exist in QED and quantum mechanics and have been solved in the same manner. The impacts of this new decomposition to the nucleon internal structure are discussed.

PACS numbers: 14.20.Dh, 11.15.-q, 12.38.-t

\textsuperscript{*} corresponding author:fgwang@chenwang.nju.edu.cn
I. INTRODUCTION

In quantum physics, any observable is expressed as a Hermitian operator in Hilbert space. The fundamental operators, such as momentum, orbital angular momentum, spin, satisfy the canonical momentum and angular momentum commutation relation. These commutation relations or Lie algebras define the properties of these operators.

Gauge invariance has been recognized as the first principle through the development of the standard model. In classical gauge field theory, gauge invariance principle requires any observable must be expressed in terms of gauge invariant variable. In quantum gauge field theory, in general one only requires the matrix elements of an operator in between physical states to be gauge invariant. However, one usually requires the operators themselves to be gauge invariant. This is called the strong gauge invariance in \( SU(3) \) color group gauge invariant. We will restrict our discussion in strong gauge invariance in this paper and leave the other possibility to the future study [1, 2].

In the study of nucleon (atom) internal structure, it is unavoidable to study the quark, gluon (electron, photon) momentum, orbital angular momentum and spin contributions to the nucleon (atom) momentum and spin. However, we never have the quark, gluon (electron, photon) momentum, orbital angular momentum and spin operators which satisfy both the gauge invariance and canonical momentum and angular momentum commutation relation except the quark (electron) spin. Even it has been claimed in some textbooks that one can not define separately the photon spin and orbital angular momentum operators [3] and almost a proper gluon spin operator search has been given up in the nucleon spin structure study for the last ten years. This situation has left puzzles in quantum mechanics, quantum electrodynamics (QED) and quantum chromodynamics (QCD). For example, the expectation value of the Hamiltonian of hydrogen atom is gauge dependent under a time dependent gauge transformation [4], the meaning of the multipole radiation analysis from atom to hadron spectroscopy would be obscure if the photon spin and orbital angular momentum operators were not well defined, especially the parity of these microscopic states determined from the multipole radiation analysis would be obscure, there will be no way to compare the measured gluon spin contribution to nucleon spin with the theoretically calculated one if one has not a proper gluon spin operator, etc.
In section II the conflict between gauge invariance and canonical quantization of the usual quark gluon (electron photon) momentum, orbital angular momentum and spin operators are discussed from the simple quantum mechanics of a charged particle moving in an electromagnetic field to those of quark and gluon in QCD. In the third section a new set of momentum, orbital angular momentum and spin operators, which satisfy both the gauge invariance and canonical momentum and angular momentum commutation relation, are given. The key point to achieve this is to separate the gauge field into pure gauge and gauge covariant (invariant) parts. The possible impacts of these modification to the nucleon internal structure will be discussed in section IV. The last section is a summary and a prospect of further studies.

II. GAUGE INVARIANCE AND CANONICAL QUANTIZATION OF THE MOMENTUM AND ORBITAL ANGULAR MOMENTUM OPERATORS OF THE FERMION AND GAUGE FIELD PARTS

The conflict in the application of gauge invariance and canonical quantization to the momentum and orbital angular momentum operators of a charged particle moving in the electromagnetic field, a U(1) Abelian gauge field, has existed in quantum mechanics since the establishment of gauge invariance principle. Starting from the Lagrangian of a non-relativistic charged particle with mass $m$, velocity $\vec{v}$ and charge $e$ moving in an electromagnetic field $A^\mu = (A^0, \vec{A})$,

$$\mathcal{L}(m, \vec{v}, e, A^\mu) = \frac{1}{2m}(m\vec{v})^2 - e(A^0 - \vec{v} \cdot \vec{A}),$$

one obtains the canonical momentum, orbital angular momentum and Hamiltonian

$$\vec{p} = m\vec{v} + e\vec{A}, \quad \vec{L} = \vec{r} \times \vec{p}, \quad H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eA^0.$$  

All of these three classical dynamical variables are gauge dependent and so are not observables in classical gauge theory. In the coordinate representation, the momentum $\vec{p}$ is quantized as

$$\vec{p} = \frac{\hat{\vec{v}}}{i}.$$  

no matter what kind of gauge is fixed on even though the classical canonical momentum is gauge dependent. The orbital angular momentum and Hamiltonian are quantized by
replacing $\vec{p}$ by $\vec{\nabla}_i$. These quantized momentum and angular momentum operators satisfy the canonical commutation relation or the Lie algebra:

$$[p_l, p_m] = 0, \quad [L_l, L_m] = i\epsilon_{lmn} L_n, \quad [p_l, L_m] = i\epsilon_{lmn} p_n, \quad l, m, n = 1, 2, 3,$$

(4)

where $\epsilon_{lmn}$ is the rank-three totally antisymmetric tensor and $\epsilon_{123} = 1$. In general, the $[p_l, H] \neq 0$, which is different from the Poincaré algebra of the total momentum $P_l, (l = 1, 2, 3)$ and total $H$ of the whole system where $[P_l, H] = 0$.

However, after a gauge transformation,

$$\psi' = e^{-i\omega(x)}\psi, \quad A^\mu = A^\mu + \partial^\mu\omega(x),$$

(5)

the matrix elements of the above operators transform as follows,

$$\langle\psi'|\vec{p}|\psi\rangle' = \langle\psi|\vec{p}|\psi\rangle - e\langle\psi|\vec{\nabla}\omega(x)|\psi\rangle,$$

$$\langle\psi'|\vec{L}|\psi\rangle' = \langle\psi|\vec{L}|\psi\rangle - e\langle\psi|\vec{r}\times\vec{\nabla}\omega(x)|\psi\rangle,$$

$$\langle\psi'|H'|\psi\rangle' = \langle\psi|H|\psi\rangle + e\langle\psi|\partial_t\omega(x)|\psi\rangle.$$

(6)

It is obvious that the matrix elements of these three operators are all gauge dependent. Therefore they are not measurable and so these operators do not correspond to observables. This problem has left in quantum mechanics since the gauge principle was proposed.

The relativistic version of quantum mechanics has the same problem. The gauge dependence of the expectation value of the Hamiltonian of a charged particle moving in an electromagnetic field under a time dependent gauge transformation was discussed by T. Goldman[4].

This conflict is carried over to QED. Starting from a QED Lagrangian,

$$\mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

(7)

By means of the Noether theorem one obtains the momentum and angular momentum operators as follows:

$$\vec{P} = \vec{P}_e + \vec{P}_{ph} = \int d^3x \bar{\psi} i\frac{\vec{\nabla}}{i} \psi + \int d^3x E^i\vec{\nabla} A^i,$$

(8)

$$\vec{J} = \vec{S}_e + \vec{S}_{ph} + \vec{L}_{ph}$$

$$= \int d^3x \bar{\psi} i\frac{\vec{\nabla}}{i} \psi + \int d^3x \vec{r}\times\bar{\psi} i\frac{\vec{\nabla}}{i} \psi + \int d^3x \vec{E}\times\vec{A} + \int d^3x \vec{r}\times E^i\vec{\nabla} A^i.$$  

(9)
Here $\Sigma^j = \frac{i}{2} \epsilon_{jkl} \gamma^k \gamma^l$. These electron and photon momentums, orbital angular momentums and spins, after quantization, satisfy momentum and angular momentum Lie algebra. However, they are not gauge invariant except the electron spin.

The multipole radiation analysis is the basis of atomic, molecular, nuclear and hadron spectroscopy. The multipole field is based on the decomposition of the electromagnetic field into field with definite orbital angular momentum and spin quantum numbers. If the photon spin and orbital angular momentum operators were gauge dependent, then the physical meaning of the multipole field would be obscure, especially the parity of these microscopic states determined by the measurement of the orbital angular momentum quantum number of the multipole radiation field would be obscure.

QCD has the same problem as QED. The quark gluon momentum, orbital angular momentum and spin operators derived from QCD Lagrangian by Noether theorem have the same form as those of electron and photon, Eqs.(8,9), if one omits the color indices. They satisfy the momentum and angular momentum Lie algebra but they are not gauge invariant except the quark spin.

Because of the lack of gauge invariant quark, gluon momentum operators, the present operator product expansion (OPE) used the following two operators as quark and gluon momentum operators (the color indices are suppressed),

$$\vec{P} = \vec{P}_q + \vec{P}_g = \int d^3x \bar{\psi} \vec{D} \psi + \int d^3x \vec{E} \times \vec{B},$$

$$\vec{D} = \vec{\nabla} - ig\vec{A}.$$ (10)

Both the quark and gluon "momentum" operators $\vec{P}_q$ and $\vec{P}_g$ defined in Eq.(10) are gauge invariant but neither the quark "momentum" $\vec{P}_q$ nor the gluon "momentum" $\vec{P}_g$ satisfies the momentum algebra, for example,

$$[D_l, D_m] = -ig(\partial_l A_m - \partial_m A_l) - ig^2 C_{abc} A_l^a A_m^b T^c,$$ (11)

where $C_{abc}$ is the $SU(3)$ group structure constant. The $\vec{P}_g$ does not satisfy the momentum algebra either in the interacting quark-gluon field, i.e., QCD case. Therefore neither the $\vec{P}_q$ nor the $\vec{P}_g$ used in the OPE is the real momentum operator.

A gauge invariant decomposition of quark, gluon angular momentum has been proposed $^5$ and used in the study of nucleon spin structure in the last ten years,

$$\vec{J} = \vec{S}_q + \vec{L}_q + \vec{J}_g.$$
\[
\int d^3x \psi^\dagger \frac{\Sigma}{2} \psi + \int d^3x \times \psi^\dagger \frac{\vec{D}}{i} \psi + \int d^3x \vec{x} \times (\vec{E} \times \vec{B}),
\]

(12)

Here the color indices have been suppressed. Each term in this decomposition is gauge invariant, however, they do not satisfy the angular momentum algebra except the quark spin. In addition, there is no gluon spin in this decomposition. Experimentally the gluon spin contribution to nucleon spin is under intensive study, PHENIX, STAR, COMPASS, HERMES, and others are measuring the gluon spin contribution to nucleon spin. Neither the decomposition, Eq.(9), nor the decomposition, Eq.(12), gives a gluon spin operator which satisfies both gauge invariance and angular momentum algebra. There is also no quark, gluon orbital angular momentum operator which satisfies both gauge invariance and orbital angular momentum algebra. These situations hindered the study of the nucleon spin structure.

III. A NEW SET OF MOMENTUM, ORBITAL ANGULAR MOMENTUM AND SPIN OPERATORS FOR THE FERMION AND GAUGE FIELD PARTS

A. Decomposing the gauge field \( A_\mu \) into pure gauge part \( A_{\text{pure}} \) and gauge invariant (covariant) part \( A_{\text{phys}} \)

Let us start from the simpler QED case. It is well known that to use gauge potential \( A_\mu \) to describe the electromagnetic field the \( A_\mu \) is not unique, i.e., there is gauge freedom. Under a gauge transformation,

\[
A'^\mu = A^\mu + \partial^\mu \omega(x),
\]

(13)

one obtains a new gauge potential \( A'^\mu \) from \( A^\mu \). \( A^\mu \) and \( A'^\mu \) describe the same electromagnetic field,

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A'_\nu - \partial_\nu A'_\mu.
\]

(14)

Such a gauge freedom is necessary because the gauge potential \( A^\mu \) plays two role in gauge field theory: the first is to provide a pure gauge field \( A_{\text{pure}} \) to compensate the induced field due to the phase change in a local gauge transformation of the Fermion field \( \psi'(x) = e^{-i\omega(x)} \psi(x) \) which must be varied with the arbitrarily changed phase parameter \( \omega(x) \); the second is to provide a physical field \( A_{\text{phys}} \) for the physical interaction between Fermion field and gauge field which should be gauge invariant (covariant) under gauge transformation. The pure
gauge potential $A_{pure}$ should not contribute to electromagnetic field,

$$ F_{\mu\nu}^{pure} = \partial^{\mu} A^{\nu}_{pure} - \partial^{\nu} A^{\mu}_{pure} = 0. \quad (15) $$

This equation can not fix the $A_{pure}$ uniquely. One has to find additional condition to fix it. The spatial part of Eq.(15) is

$$ \nabla \times \vec{A}_{pure} = 0, \quad (16) $$

which means $\vec{A}_{pure}$ does not contribute to magnetic field. This equation can be expressed in another form,

$$ \nabla \times \vec{A}_{phys} = \nabla \times \vec{A}. \quad (17) $$

A natural choice of the additional condition in QED case is

$$ \nabla \cdot \vec{A}_{phys} = 0, \quad (18) $$

which is the transverse wave condition and we know that the $\vec{A}_{phys}$ part is the physical one from the Coulomb gauge quantization. Combining these two conditions, Eqs.(17) and (18), under the natural boundary condition,

$$ \vec{A}_{phys}(|x| \to \infty) = 0, \quad (19) $$

for any given set of gauge field $\vec{A}$, one can decompose it uniquely as follows,

$$ \vec{A} = \vec{A}_{pure} + \vec{A}_{phys}, \quad (20) $$

where

$$ \vec{A}_{phys}(x) = \vec{\nabla} \times \frac{1}{4\pi} \int d^3 x' \frac{\vec{\nabla}' \times \vec{A}(x')}{|\vec{x} - \vec{x}'|}, $$

$$ \vec{A}_{pure}(x) = \vec{A} - \vec{A}_{phys}(x). \quad (21) $$

We have to emphasize that for fixed $\vec{A}(x)$, the integration can be done and the obtained $\vec{A}_{phys}(x)$ is a local function of space-time $x$ and a functional of $A_{\mu}(x)$. It is easy to prove that these two parts transform as follows in a gauge transformation Eq.(13),

$$ \vec{A}'_{phys} = \vec{A}_{phys}, $$

$$ \vec{A}'_{pure} = \vec{A}_{pure} - \vec{\nabla} \omega(x). \quad (22) $$
The time component $A^0$ can be decomposed in the same manner. From the condition $F_{\text{pure}}^{i0} = 0$, one obtains

$$
\partial_i A^0_{\text{phys}} = \partial_i A^0 + \partial_i (A^i - A^i_{\text{phys}}),
$$

$$
A^0_{\text{phys}} = \int_{-\infty}^x dx^i (\partial_i A^0 + \partial_i A^i - \partial_i A^i_{\text{phys}}),
$$

(23)

The $A^\mu_{\text{pure}} = A^\mu - A^\mu_{\text{phys}}$ can also be obtained from Eq.(16), (18) and (23) directly,

$$
\vec{A}_{\text{pure}} = \vec{\nabla} \phi(x),
$$

$$
\phi(x) = -\frac{1}{4\pi} \int d^3 x' \frac{\vec{\nabla}' \cdot \vec{A}'(x')}{|\vec{x} - \vec{x}'|} + \phi_0(x),
$$

$$
A^0_{\text{pure}} = \partial_t \phi(x),
$$

(24)

where $\phi_0(x)$ satisfies the condition,

$$
\nabla^2 \phi_0(x) = 0,
$$

(25)

and is determined by the boundary condition.

To decompose the gauge potential $A_\mu = A_\mu^a T^a$ for the gluon field is more complicated than QED case. We first define the pure gauge potential $A^\mu_{\text{pure}}$ (hereafter we omit the color indices if not necessary) by the same condition, i.e., it does not contribute to color electromagnetic field,

$$
F_{\text{pure}}^{\mu\nu} = \partial^\mu A^\nu_{\text{pure}} - \partial^\nu A^\mu_{\text{pure}} + ig[A^\mu_{\text{pure}}, A^\nu_{\text{pure}}] = 0.
$$

(26)

In order to make this defining condition looks similar to Eq.(16), we introduce a notation,

$$
\vec{D}_{\text{pure}} = \vec{\nabla} - ig\vec{A}_{\text{pure}},
$$

$$
\vec{D}_{\text{pure}} \times \vec{A}_{\text{pure}} = \vec{\nabla} \times \vec{A}_{\text{pure}} - ig\vec{A}_{\text{pure}} \times \vec{A}_{\text{pure}} = 0.
$$

(27)

The additional condition is even more complicated, i.e., one does not have a natural choice as Eq.(18) in QED. We make the following choice[6],

$$
\vec{D}_{\text{pure}} = \vec{\nabla} - ig[\vec{A}_{\text{pure}}, ]
$$

$$
\vec{D}_{\text{pure}} \cdot \vec{A}_{\text{phys}} = \vec{\nabla} \cdot \vec{A}_{\text{phys}} - ig[A^i_{\text{pure}}, A^i_{\text{phys}}] = 0.
$$

(28)

The summation over the vector component $i$ has been assumed in the above equation and the following ones. Please note that in the above adjoint representation of the new covariant
derivative operator $\vec{D}$, the bracket $[A^i_{\text{pure}}, A^i_{\text{phys}}]$ is not the quantum bracket but a color $SU_c(3)$ group one,

$$[A^i_{\text{pure}}, A^i_{\text{phys}}] = iC_{abc}A^{ib}_{\text{pure}}A^{ic}_{\text{phys}}T^a.$$  \hspace{1cm} (29)

These equations can be rewritten as follows,

$$\vec{\nabla} \cdot A^i_{\text{phys}} = ig[A^i - A^i_{\text{phys}}, A^i_{\text{phys}}] = ig[A^i, A^i_{\text{phys}}],$$

$$\vec{\nabla} \times A^i_{\text{phys}} = \vec{\nabla} \times A - ig(\vec{A} - \vec{A}^\text{phys}) \times (\vec{A} - \vec{A}^\text{phys}),$$ \hspace{1cm} (30)

$$\partial_t A^0_{\text{phys}} = \partial_t A^0 + \partial_t (A^i - A^i_{\text{phys}}) - ig[A^i - A^i_{\text{phys}}, A^0 - A^0_{\text{phys}}].$$

These equations can be solved perturbatively: in the zeroth order, i.e., assuming $g = 0$, these equations are the same as those of QED, one can obtain the zeroth order solution; then taking into account the nonlinear coupling through iteration one obtains a perturbative solution as a power expansion in $g$.

If one assumes a trivial boundary condition for the pure gauge field $A_{\text{pure}}$, then one can use the following equations to obtain a perturbative solution too,

$$\vec{\nabla} \times A^i_{\text{pure}} = igA^i_{\text{pure}} \times A^i_{\text{pure}},$$

$$\vec{\nabla} \cdot A^i_{\text{pure}} = \vec{\nabla} \cdot A - ig[A^i_{\text{pure}}, A^i],$$ \hspace{1cm} (31)

$$\partial_t A^0_{\text{pure}} = -\partial_t A^0_{\text{pure}} + ig[A^i_{\text{pure}}, A^0_{\text{pure}}].$$

Under a gauge transformation,

$$\psi' = U\psi,$$

$$A'_\mu = UA_\mu U^\dagger - \frac{i}{g}U\partial_\mu U^\dagger,$$  \hspace{1cm} (32)

where $U = e^{-ig\omega}$. The $\vec{A}_{\text{pure}}$ and $\vec{A}_{\text{phys}}$ will be transformed as

$$\vec{A}'_{\text{phys}} = U\vec{A}_{\text{phys}} U^\dagger,$$

$$\vec{A}'_{\text{pure}} = U\vec{A}_{\text{pure}} U^\dagger - \frac{i}{g}U\partial_\mu U^\dagger.$$  \hspace{1cm} (33)

**B. Quantum mechanics**

We have mentioned in the introduction part that even in quantum mechanics, there are already puzzles related to the fundamental operators, the matrix elements of canonical
momentum, orbital angular momentum and Hamiltonian of a charged particle moving in an electromagnetic field are all not gauge invariant. In order to get rid of these puzzles, gauge invariant operators have been introduced,

\[ \vec{P} = \vec{p} - e\vec{A}, \quad \vec{L} = \vec{x} \times \vec{P}. \]  

(34)

It is easy to check that the matrix elements of these operators are gauge invariant. However, as we have pointed out in Eq.(11), that the gauge invariant "momentum" \( \vec{P} \) does not satisfy the canonical momentum Lie algebra, so they are not the proper momentum. The gauge invariant "orbital angular momentum" \( \vec{L} \) does not satisfy the angular momentum Lie algebra either.

Based on our proposed gauge field decomposition in the above section, we introduce another set of momentum and orbital angular momentum operators which satisfy both gauge invariance and the corresponding commutation relation,

\[ \vec{p}_{\text{pure}} = \vec{p} - e\vec{A}_{\text{pure}}, \]

\[ \vec{L}_{\text{pure}} = \vec{x} \times \vec{p}_{\text{pure}}. \]  

(35)

The long-standing puzzle, the gauge non-invariance of the expectation value of the Hamiltonian\[4\] can be solved in the same manner. For the non-relativistic quantum mechanics, the new Hamiltonian is

\[ H = \frac{(\vec{p} - e\vec{A}_{\text{pure}} - e\vec{A}_{\text{phys}})^2}{2m} + e(A^0 - \partial_t \phi(x)). \]  

(36)

The last term is a pure gauge term, it cancels the unphysical energy appearing in \( eA^0 \) induced by the pure gauge term and then guarantees the expectation value of this Hamiltonian gauge invariant. It is a direct extension of Eq.(35) to the zeroth momentum component.

The Dirac Hamiltonian has the same unphysical energy part and has to be canceled in the same manner as that for the Schrödinger Hamiltonian. Here we have done a check: starting from a QED Lagrangian with both electron and proton, under the infinite proton mass approximation, we derived the Dirac equation of electron and the gauge invariant Hamiltonian of the electron part and verified the difference between the Dirac Hamiltonian obtained from the Dirac equation and the gauge invariant one from the energy-momentum tensor.
Our study shows that the canonical momentum, orbital angular momentum and the Hamiltonian used in quantum mechanics are not observables, one must subtract the pure gauge part, the unphysical one, from these operators as we did in Eq.(35) and (36) to obtain the observable ones. In Coulomb gauge, where the $A_{\mu pure}^\mu = 0$, the momentum, orbital angular momentum and Hamiltonian operators in Eqs.(35,36) are simplified to the familiar form used in quantum mechanics. This justifies the quantum mechanics calculation with Coulomb gauge.

C. QED

We have explained that the momentum and angular momentum operators of the Fermion and gauge field part, Eqs.(8) and (9), derived from the QED Lagrangian by means of Noether theorem are not gauge invariant except the Fermion spin. One can obtain a gauge invariant decomposition by adding a surface term to Eqs.(8) and (9) or from the Belinfante symmetric energy-momentum tensor,

$$\vec{P} = \vec{P}_e + \vec{P}_{ph} = \int d^3 x \psi^\dagger \frac{\vec{D}}{i} \psi + \int d^3 x \vec{E} \times \vec{B}$$

$$\vec{J} = \vec{S}_e + \vec{L}_e + \vec{J}_{ph} = \int d^3 x \psi^\dagger \frac{\vec{S}}{2} \psi + \int d^3 x \vec{x} \times \psi^\dagger \frac{\vec{D}}{i} \psi + \int d^3 x \vec{x} \times (\vec{E} \times \vec{B}).$$

There are two problems with this decomposition: (1), $\vec{P}_e, \vec{P}_{ph}, \vec{L}_e$ and $\vec{J}_{ph}$ do not satisfy the momentum and angular momentum commutation relations even though in free electromagnetic field theory the photon momentum $\vec{P}_{ph}$ and angular momentum $\vec{J}_{ph}$ do; (2), there is no separate photon spin and orbital angular momentum operators and this feature will ruin the multipole radiation analysis as we discussed in the second section.

Based on the decomposition of the gauge potential into pure gauge and the physical parts, Eq.(20), we obtain the following decomposition,

$$\vec{P} = \vec{P}^p = \vec{P}^p_e + \vec{P}^p_{ph} = \int d^3 x \psi^\dagger \frac{\vec{D}_{pure}^i}{i} \psi^l + \int d^3 x E^i \vec{D}_{pure}^i \vec{A}_{phys}^i.$$ 

$$\vec{J} = \vec{S}_e + \vec{L}_e + \vec{J}_{ph} = \int d^3 x \psi^\dagger \frac{\vec{S}}{2} \psi + \int d^3 x \vec{x} \times \psi^\dagger \frac{\vec{D}_{pure}^i}{i} \psi^l + \int d^3 x \vec{E} \times \vec{A}_{phys}^i + \int d^3 x \vec{x} \times (\vec{E} \times \vec{B}_{pure}^i \vec{A}_{phys}^i).$$
Here the operator $\vec{D}_{\text{pure}}$ and $\vec{D}_{\text{pure}}$ are the same as given in Eqs.(27) and (28) but with $g$ replaced by $e$. Because of the Abelian property of the $U(1)$ gauge field, the adjoint representation of the operator $\vec{D}$ is simplified to be a simple $\vec{\nabla}$. It is not hard to check that each operator in the above decomposition, Eq.(39) and (40) is gauge invariant and satisfies the momentum, angular momentum commutation relation. In Coulomb gauge, the operators in Eqs.(39,40) are simplified to operator forms in Eqs.(8,9). This justifies that the gauge non-invariant operators of Eqs.(8,9) can be used in Coulomb gauge to get the gauge invariant results.

The photon spin and orbital angular momentum operators are well defined as shown in Eq.(40). The multipole radiation analysis is theoretically sound now as it should be.

D. QCD

One can copy results for QED, the Eqs.(39,40), to QCD to obtain the quark, gluon momentum, orbital angular momentum and spin operators which satisfy both the gauge invariance and the canonical momentum and angular momentum commutation relations.

A decomposition of the form of Eq.(10,12) has been used in the nucleon spin structure study for the last ten years\[5\]. Each operator in those decomposition is gauge invariant and so corresponds to observable. However, because they do not satisfy the momentum, angular momentum Lie algebra so the measured ones are not the quark, gluon momentum and orbital angular momentum and can not be connected to those used in hadron spectroscopy.

The gluon spin operator had been searched for more than ten years in the nucleon spin structure study and no satisfactory one was obtained. Now one has the gluon spin operator $\vec{S}_g = \int d^3x \vec{E} \times \vec{A}_{\text{phys}}$ which can be used to calculate the matrix element for a polarized nucleon state $|N(p,s)\rangle$ to obtain the gluon spin contribution to nucleon spin and compare it with the measured ones.

IV. REEXAMINATION OF THE NUCLEON INTERNAL STRUCTURE

For the past years the nucleon internal structure has been studied based on momentum, angular momentum operators given in Eq.(10,12). These operators are gauge invariant but do not satisfy the momentum and angular momentum Lie algebra. This led to a distorted
picture of the nucleon internal structure. For example, that the quark and gluon carry half of the nucleon momentum in the asymptotic limit has been a deeply rooted picture of nucleon internal momentum structure. Using the new quark, gluon momentum operator we recalculated their scale evolution and obtained the new mixing matrix

$$\gamma_P = -\frac{\alpha_s}{4\pi} \left( -\frac{2}{9} n_g \frac{2}{3} n_f \right),$$

which gives the new asymptotic limit for the renormalized gluon momentum,

$$\vec{P}_g^R = \frac{1}{2} \frac{n_g}{n_g + 3n_f} \vec{P}_{total}.$$  (42)

For typical gluon number $n_g = 8$ and quark flavor number $n_f = 5$, the above equation gives $\vec{P}_g^R \simeq \frac{1}{8} \vec{P}_{total}$. This is distinctly different from the renowned results $\vec{P}_g^R \simeq \frac{1}{2} \vec{P}_{total}$. This latter result is obtained from the mixing matrix

$$\gamma_P = -\frac{\alpha_s}{4\pi} \left( -\frac{8}{9} n_g \frac{4}{3} n_f \right),$$

which is obtained by means of the quark and gluon momentum operators given in Eq.(10). The mixing matrix element of Eq.(43) leads to the well known asymptotic limit,

$$\vec{P}_g^R = \frac{2n_g}{2n_g + 3n_f} \vec{P}_{total}.$$  (44)

However, the $\vec{P}_g$ and $\vec{P}_q$ used in this quark gluon momentum scale evolution calculation are not the proper momentum operators, part of the quark momentum had been shifted to the gluon and gave the superficial large gluon momentum contribution to nucleon momentum.

The asymptotic nucleon spin structure is obtained based on the decomposition Eq.(9), a QED analog of QCD angular momentum decomposition. The authors had pointed out that the quark and gluon orbital angular momentum and the gluon spin operators are not gauge invariant. As we have mentioned in the beginning, in the present gauge field theory an observable must be expressed in terms of a gauge invariant operator. The gauge dependent operators used in this analysis are not measurable ones. Therefore this asymptotic limit of nucleon spin content should be reexamined.

Another nucleon internal structure parameters are the parton distribution function (PDF). For example, the quark PDF in a target $A$ is defined as,

$$P_{q/A}(\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx^-}{2\pi} e^{-ix^-} \langle \bar{\psi}(0, x^-, 0_\perp) \gamma^+ P \exp\{ig \int_0^{x^-} dy^- A^+(0, y^-, 0_\perp)\} \psi(0) \rangle_A,$$  (45)
where a gauge link (Wilson line) is inserted to achieve the gauge invariance. Based on our
gauge field decomposition discussed in the third section, the above gauge link not only in-
cludes the necessary pure gauge $A_{\text{pure}}$ part to achieve the gauge invariance, but also includes
the physical part $A_{\text{phys}}$ which induced a physical coupling and makes the PDF defined in
Eq.(45) an interaction-involving one. The interaction term is more clear in the momentum
relation,

$$\int_{-\infty}^{\infty} d\xi \xi \mathcal{P}_{q/A}(\xi) = \frac{1}{2(P^+)^2} \langle \bar{\psi} \gamma^+ iD^+ \psi \rangle_A.$$  \hspace{1cm} (46)

This is just the + component of $\vec{P}_q$ in Eq.(10). Here the gauge field in $D^+$ originates exactly
from the gauge link in Eq.(45).

To obtain a gauge invariant quark PDF, a gauge link with the pure gauge part is enough,

$$\mathcal{P}_{q/A}(\xi) = \frac{1}{2(P^+)^2} \langle \bar{\psi} \gamma^+ iD^+ \psi \rangle_A.$$  \hspace{1cm} (47)

this PDF will not include the redundant physical gauge interaction and the integration gives
the real quark momentum defined in Eq.(39) (the QCD quark and gluon momentums have
exactly the same expression as those of QED, only the subscript e and ph are replaced by q
and g).

$$\int_{-\infty}^{\infty} d\xi \xi \mathcal{P}_{q/A}(\xi) = \frac{1}{2(P^+)^2} \langle \bar{\psi} \gamma^+ iD^+_{\text{pure}} \psi \rangle_A.$$  \hspace{1cm} (48)

Analogously, the conventional gluon PDF

$$\mathcal{P}_{g/A}(\xi) = \frac{1}{\xi P^+} \int_{-\infty}^{\infty} \frac{dx^-}{2\pi} e^{-i\xi P^+ x^-} \langle F^+(0, x^-, 0_) \rangle \mathcal{P} \exp\{ig \int_0^{x^-} dy^- A^+_{\text{pure}}(0, y^-, 0_)\} \langle P \rangle.$$  \hspace{1cm} (49)

can be replaced accordingly as

$$\mathcal{P}_{g/A}(\xi) = \int_{-\infty}^{\infty} \frac{dx^-}{2\pi} e^{-i\xi P^+ x^-} \langle F^+(0, x^-, 0_) \rangle \mathcal{P} \exp\{ig \int_0^{x^-} dy^- A^+_{\text{pure}}(0, y^-, 0_)\} \langle A^i_{\text{phys}}(0) \rangle.$$  \hspace{1cm} (50)

where besides the pure gauge link, the physical component $A^i_{\text{phys}}$ is used instead of $F^+_{\nu}$ as
the gauge invariant variable. The second moments of $\mathcal{P}_{g/A}$ and $\mathcal{P}_{g/A}$ relate to Poynting and
the proper gluon momentum in Eq.(12) and (39).

Our approach is also convenient in constructing the gauge invariant polarized and
transverse-momentum dependent PDFs with clear particle number interpretation, and off-
forward PDFs which can be measured to infer the proper orbital angular momentums in
Eq.(40). For example the polarized gluon PDF can be defined gauge invariantly as

$$\mathcal{P}_{\Delta g/A}(\xi) = \int_{-\infty}^{\infty} \frac{dx^-}{2\pi} e^{-i\xi P^+ x^-} \langle F^{+i}(0, x^-, 0_) \rangle \mathcal{P} \exp\{ig \int_0^{x^-} dy^- A^+_{\text{pure}}(0, y^-, 0_)\} \langle \epsilon_{ij} A^j_{\text{phys}}(0) \rangle.$$  \hspace{1cm} (51)

with a first moment relating to the gauge invariant gluon spin in Eq.(40).
V. SUMMARY AND PROSPECT

Since the establishment of gauge invariance principle, we enjoy that the total momentum, angular momentum and the Lorentz boosting operators of a gauge system satisfy both the gauge invariance and Poincaré algebra. However, we never have the separate momentum, orbital angular momentum operators of the Fermion (electron in QED, quark in QCD) and boson (photon in QED, gluon in QCD) part which satisfy both the gauge invariance and the canonical momentum, angular momentum Lie algebra. We have the electron, quark spin operator but we never have the photon and gluon spin operators which satisfy both the gauge invariance and spin Lie algebra. Even it had been claimed in some textbooks that it is impossible to have a well defined photon spin[3]. The nucleon spin structure study needs the gluon spin operator, but after about ten years effort in searching for a gluon spin operator since the so-called proton spin crisis, such an effort has almost been given up for the last ten years. In this report we proposed a new set of quark (electron), gluon (photon) momentum, orbital angular momentum and spin operators which satisfy both the gauge invariance and the canonical momentum, angular momentum Lie algebra.

To achieve this a key point is to separate the gauge field into pure gauge and physical parts: the former one is unphysical and can be gauged away as in Coulomb gauge, it is used to compensate the induced unphysical gauge field due to the local gauge transformation of the Fermion field to keep the gauge invariance; the physical part is responsible for the physical coupling between Fermion and boson field. It is physical and should be gauge invariant (covariant). We provide a method to do this separation both for the Abelian $U(1)$ and the non-Abelian $SU(3)$ gauge field.

Our proposed momentum operators are different from the canonical ones. The latter ones of the Fermion (electron in QED and quark in QCD) are not gauge invariant and so do not represent observables because they include the unphysical pure gauge field contribution. The new ones subtract the unphysical pure gauge field contribution and so they are gauge invariant and correspond the observables. The Poynting vector used for the boson (photon in QED and gluon in QCD) is not the proper momentum operator either because they do not satisfy the momentum Lie algebra in the interacting field case.

We achieved to obtain a gauge invariant orbital angular momentum and spin operators of the photon and gluon by means of the physical part of the gauge field, Eq.(40), which
provides the theoretical basis of the widely used multipole radiation analysis, the photon spin and orbital angular momentum used in quantum computation and communication study, the gluon spin contribution in the nucleon spin structure study.

In Coulomb gauge, the new momentum and orbital angular momentum operators proposed in this paper coincide with the usually used ones. This explains why the quantum mechanics calculations obtain the right results. It is because usually these calculations are performed in the Coulomb gauge. The multipole radiation calculation is also performed in Coulomb gauge and so the results are correct too.

The Poincaré algebra can not be fully maintained for the momentum, angular momentum and Lorentz boosting operators of the individual Fermion and boson part of an interacting gauge field system. What is the meaning of these observables if they are not Lorentz covariant? We have shown that the momentum and angular momentum algebra can be maintained simultaneously with the gauge invariance. How much part of the Poincaré algebra can be maintained for the operators of the interacting Fermion and boson separately, especially the Lorentz covariance can be maintained to what extent are left for further study.

For a quantum gauge field system, in general one only requires the matrix elements of the operator corresponding to a physical observable between physical states to be gauge invariant. The strong gauge invariance requirement might be a too strong one. This should be studied further.

The new asymptotic limit of quark and gluon parton momentums of a nucleon have been obtained, the immediate problem is the asymptotic limit of the quark and gluon orbital angular momentums and spins.

The gluon spin contribution to the nucleon spin is under measurement in different labs. A lattice QCD calculation with the gauge invariant gluon spin operator is called for.

How to relate the new PDFs to the measured cross sections in deep inelastic scattering and virtual Compton scattering should be examined.

In summary, the nucleon internal structure is better to be reexamined based on the new quark, gluon momentum, orbital angular momentum, spin operators and parton distribution functions and our picture of the nucleon internal structure might be modified.
Acknowledgments

We thank Prof. X. D. Ji, K. F. Liu, J.W. Qiu, S. J. Wang and Dr. J.P. Chen for stimulating discussions. This research is supported by NSFC under Grant No. 90503011, 10875082, U.S. DOE under Contract No. W-7405-ENG-36 and in part by the PKIP of CAS under Grant No. KJCX2.YW.W10

[1] F. Strocchi and A. S. Wightman, J. Math. Phys. 15, 2198 (1974).
[2] X. S. Chen and F. Wang, hep-ph/9802346
[3] J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Springer-Verlag, Berlin, 1976).
V.B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, Quantum Electrodynamics (Pergamon, Oxford, 1982).
C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, Photons and Atoms (Wiley, New York, 1987).
[4] T. Goldman, Phys. Rev. D15, 1063 (1977).
[5] X. Ji, Phys. Rev. Lett. 78, 610 (1997);
X. S. Chen and F. Wang, Commun. Theor. Phys. 27, 121 (1997).
[6] X. S. Chen, X. F. Lü, W. M. Sun, F. Wang and T. Goldman, Phys. Rev. Lett. 100, 232002 (2008). arXiv:0806.3166[hep-ph]
[7] X. S. Chen, X. F. Lü, W. M. Sun, F. Wang and T. Goldman, Phys. Rev. Lett. 103, 062001 (2009). arXiv:0904.0321[hep-ph].
[8] X. Ji, J. Tang and P. Hoodbhoy, Phys. Rev. Lett. 76, 740 (1996).