Hom-associative Ore extensions

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Abstract. We introduce hom-associative Ore extensions as non-associative, non-unital Ore extensions with a hom-associative multiplication, as well as give some necessary and sufficient conditions when such exist. Within this framework, we also construct a family of hom-associative Weyl algebras as generalizations of the classical analogue, and prove that they are simple.

1. Introduction
Hom-Lie algebras and related hom-algebra structures have recently become a subject of growing interest and extensive investigations, in part due to the prospect of providing a general framework in which one can produce many types of natural deformations of (Lie) algebras, in particular $q$-deformations which are of interest both in mathematics and in physics. One of the main initial motivations for this development came from mathematical physics works on $q$-deformations of infinite-dimensional algebras, primarily the $q$-deformed Heisenberg algebras ($q$-deformed Weyl algebras), oscillator algebras, and the Virasoro algebra [1–6, 8, 10, 11, 15–17].

Quasi-Lie algebras, subclasses of quasi-hom-Lie algebras, and hom-Lie algebras as well as their general colored (graded) counterparts were introduced between 2003 and 2005 in [9, 12–14, 24]. Further on, between 2006 and 2008, Makhlouf and Silvestrov introduced the notions of hom-associative algebras, hom-(co, bi)algebras and hom-Hopf algebras and also studied their properties [18–20]. Hom-associative algebras are a generalization of associative algebras with the associativity axiom extended by a linear twisting map. Moreover are they hom-Lie admissible, meaning that the commutator multiplication in any hom-associative algebra yields a hom-Lie algebra [18]. In hom-associative algebras, the associativity is replaced by hom-associativity; hom-coassociativity for a hom-coalgebra can be considered in a similar way.

One of the main tools in these important developments and in many constructions of examples and classes of hom-algebra structures in physics and in mathematics are based on twisted derivations or $\sigma$-derivations, which are generalized derivations twisting the Leibniz rule by means of a linear map. These types of twisted derivation maps are central for the associative Ore extension algebras or rings, introduced in algebra in the 1930s, generalizing crossed product (semidirect product) algebras or rings incorporating both actions and twisted derivations.

Non-associative Ore extensions were introduced in the unital case in [21] (see also [22] for an extension to monoid Ore extensions). In the present article, we generalize this construction to the non-unital case, and investigate when the Ore extensions are hom-associative. We do not make use of any previous results about non-associative Ore extensions, but our construction of simple hom-associative Weyl algebras in Example 1 has some similarities to the simple non-associative Weyl algebras in [21].
2. Preliminaries

In this section, we present some definitions and review some results from the theory of hom-associative algebras and that of non-associative Ore extensions.

First, by a non-unital ring, we mean a ring which is not necessarily unital. In the same way, a non-associative ring is a ring not necessarily associative. \( \mathbb{N} \) will denote the set of non-negative integers, and \( \mathbb{N}_{>0} \) the set of positive integers.

2.1. Hom-associative algebras

Here we define what we mean for an algebraic structure to be hom-associative, and review a couple of results concerning the construction of them.

**Definition 1** (Hom-associative algebra). A hom-associative algebra over an associative, commutative, and unital ring \( R \), is a triple \((M, \cdot, \alpha)\) consisting of an \( R \)-module \( M \), a binary operation \( \cdot : M \times M \to M \) linear in both arguments, and a linear map \( \alpha : M \to M \) satisfying, for all \( a, b, c \in M \),

\[
\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c).
\]

Since \( \alpha \) twists the associativity, we will refer to it as the twisting map.

**Definition 2** (Hom-associative ring). A hom-associative ring is a hom-associative algebra over the ring of integers.

**Definition 3** (Weakly unital algebra). Let \( \mathcal{A} \) be a hom-associative algebra. If for all \( a \in \mathcal{A} \),

\[
e \cdot a = a \cdot e = \alpha(a)\]

for some \( e \in \mathcal{A} \), we say that \( \mathcal{A} \) is weakly unital with weak unit \( e \).

Any associative algebra can be extended to a non-trivial hom-associative algebra, which the following proposition demonstrates:

**Proposition 1** ([25]). Let \( A \) be an associative algebra, \( \alpha \) an algebra endomorphism on \( A \) and define \( * : A \times A \to A \) by \( a * b := \alpha(a \cdot b) \) for all \( a, b \in A \). Then \((A, *, \alpha)\) is a hom-associative algebra.

**Proof.** Linearity follows immediately, while for all \( a, b, c \in A \), we have

\[
\alpha(a) * (b * c) = \alpha(a) * (\alpha(b \cdot c)) = \alpha(\alpha(a) \cdot \alpha(b \cdot c)) = \alpha(\alpha(a \cdot b \cdot c)),
\]

\[
(a * b) * \alpha(c) = \alpha(a \cdot b) * \alpha(c) = \alpha(\alpha(a \cdot b) \cdot \alpha(c)) = \alpha(\alpha(a \cdot b \cdot c)),
\]

which proves that \((A, *, \alpha)\) is hom-associative. \( \square \)

Note that we are abusing the notation in Definition 1 a bit here; \( A \) in \((A, *, \alpha)\) does really denote the algebra and not only its module structure. From now on, we will always refer to this construction when writing \( * \).

**Corollary 1** ([23]). If \( A \) is an associative, unital algebra, then \((A, *, \alpha)\) is a weakly unital hom-associative algebra with weak unit \( 1 \).

**Proof.** \( 1 * x = \alpha(1 \cdot x) = \alpha(x) = \alpha(x \cdot 1) = x * 1. \) \( \square \)

2.2. Non-associative, non-unital Ore extensions

In what follows, we define non-associative, non-unital Ore extensions, together with some new terminology.

**Definition 4** (Left \( R \)-additivity). If \( R \) is a non-associative, non-unital ring, we say that a map \( \beta : R \to R \) is left \( R \)-additive if for all \( r, s, t \in R \),

\[
r \cdot \beta(s + t) = r \cdot (\beta(s) + \beta(t)).
\]
Given a non-associative, non-unital ring \( R \) with left \( R \)-additive maps \( \delta: R \to R \) and \( \sigma: R \to R \), by a non-associative, non-unital Ore extension of \( R \), \( R[X; \sigma, \delta] \), we mean the set of formal sums \( \sum_{i \in \mathbb{N}} a_i X^i \), \( a_i \in R \), with finitely many \( a_i \) non-zero, endowed with the addition

\[
\sum_{i \in \mathbb{N}} a_i X^i + \sum_{i \in \mathbb{N}} b_i X^i = \sum_{i \in \mathbb{N}} (a_i + b_i) X^i, \quad a_i, b_i \in R,
\]

where two polynomials are equal if and only if their corresponding coefficients are equal, and for all \( a, b \in R \) and \( m, n \in \mathbb{N} \), a multiplication

\[
aX^m \cdot bX^n = \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) X^{i+n}. \tag{1}
\]

Here \( \pi_i^m \) denotes the sum of all \( \binom{m}{i} \) possible compositions of \( i \) copies of \( \sigma \) and \( m-i \) copies of \( \delta \) in arbitrary order. Then, for example \( \pi_0^0 = \text{id}_R \) and \( \pi_1^1 = \sigma \circ \delta \circ \delta + \delta \circ \sigma \circ \sigma + \delta \circ \delta \circ \sigma \). We also extend the definition of \( \pi_i^m \) by setting \( \pi_i^m \equiv 0 \) whenever \( i < 0 \), or \( i > m \). Imposing distributivity of the multiplication over addition makes \( R[X; \sigma, \delta] \) a ring.

Note that when \( m = n = 0 \), \( aX^0 \cdot bX^0 = \sum_{i \in \mathbb{N}} (a \cdot \pi_i^0(b)) X^i = (a \cdot b)X^0 \), so \( R \cong RX^0 \) by the isomorphism \( r \mapsto rX^0 \) for any \( r \in R \). Since \( RX^0 \) is a subring of \( R[X; \sigma, \delta] \), we can view \( R \) as a subring of \( R[X; \sigma, \delta] \), making sense of expressions like \( a \cdot bX^0 \).

**Remark 1.** Unless \( R \) contains a unit, there is no element \( X \) in the ring \( R[X; \sigma, \delta] \).

The left-distributivity of the multiplication over addition forces \( \delta \) and \( \sigma \) to be left \( R \)-additive: for any \( r, s, t \in R \), \( rX \cdot (s + t) = rX \cdot s + rX \cdot t \), so by expanding the left- and right-hand side and then comparing coefficients proves the statement;

\[
rX \cdot (s + t) = r \cdot \sigma(s + t) X + r \cdot \delta(s + t),
\]

\[
rX \cdot s + rX \cdot t = r \cdot \sigma(s) X + r \cdot \delta(s) + r \cdot \sigma(t) X + r \cdot \delta(t).
\]

3. **Hom-associative Ore extensions of non-associative rings**

The following section is devoted to the question what non-associative, non-unital Ore extensions \( R[X; \sigma, \delta] \) of non-associative, non-unital rings \( R \) are hom-associative?

**Proposition 2** (Hom-associative Ore extension). Let \( R[X; \sigma, \delta] \) be a non-associative, non-unital Ore extension of a non-associative, non-unital ring \( R \). Furthermore, let \( \alpha_i, j(a) \in R \) be dependent on \( a \in R \) and \( i, j \in \mathbb{N}_{>0} \), and put for an additive map \( \alpha: R[X; \sigma, \delta] \to R[X; \sigma, \delta] \),

\[
\alpha(aX^m) = \sum_{i \in \mathbb{N}} \alpha_{i+1,m+1}(a) X^i, \quad \forall a \in R, \forall m \in \mathbb{N}. \tag{2}
\]

Then \( R[X; \sigma, \delta] \) is hom-associative with the twisting map \( \alpha \) if and only if for all \( a, b, c \in R \) and \( k, m, n, p \in \mathbb{N} \),

\[
\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \alpha_{i+1,m+1}(a) \cdot \pi_i^k \cdot (b \cdot \pi_j^n(c)) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_i^{k+j}(\alpha_{j+1,p+1}(c)). \tag{3}
\]

**Proof.** For any \( a, b, c \in R \) and \( m, n, p \in \mathbb{N} \),

\[
\alpha(aX^m) \cdot (bX^n \cdot cX^p) = \alpha(aX^m) \cdot \left( \sum_{q \in \mathbb{N}} (b \cdot \pi_q^n(c)) X^{q+p} \right)
\]
Let Corollary 2. Let $R[X;\sigma,\delta]$ be a hom-associative, non-unital Ore extension of a non-associative, non-unital ring $R$, with the twisting map defined by (2). Then the following assertions hold for all $a, b, c \in R$ and $k, p \in \mathbb{N}$:

\begin{align}
\sum_{i=k-p}^{l_0,a} \alpha_{i+1,1}(a) \cdot \pi_{k-p}^i (b \cdot c) &= (a \cdot b) \cdot \alpha_{k+1,p+1}(c), \\
\sum_{i=k-p-1}^{l_0,a} \alpha_{i+1,1}(a) \cdot (\pi_{k-p-1}^i (b \cdot \sigma(c))) &+ \sum_{i=k-p}^{l_0,a} \alpha_{i+1,1} (\pi_{k-p}^i (b \cdot \delta(c))) = (a \cdot b) \cdot (\delta(\alpha_{k+1,p+1}(c)) + \sigma(\alpha_{k,p+1}(c))) \\
&= (a \cdot b) \cdot (\alpha_{k+1,p+1}(\delta(c)) + \alpha_{k,p+1}(\sigma(c))), \\
\sum_{i=k-p}^{l_1,a} \alpha_{i+1,2}(a) \cdot \pi_{k-p}^i (b \cdot c) &= (a \cdot \sigma(b)) \cdot (\delta(\alpha_{k+1,p+1}(c)) + \sigma(\alpha_{k,p+1}(c))) \\
&+ (a \cdot \delta(b)) \cdot \alpha_{k+1,p+1}(c),
\end{align}

where $\alpha_{0,p+1}(\cdot) := 0$, and $I_{p,a}$ is the smallest natural number, depending on $p$ and $a$, such that $\alpha_{i+1,p}(a) = 0$ for all $i > I_{p,a}$.

Comparing coefficients completes the proof. □

**Corollary 2.** Let $R[X;\sigma,\delta]$ be a hom-associative, non-unital Ore extension of a non-associative, non-unital ring $R$, with the twisting map defined by (2). Then the following assertions hold for all $a, b, c \in R$ and $k, p \in \mathbb{N}$:
Proof. We get (4), the first equality in (5), and (6) immediately from the cases \( m = n = 0 \), \( m = 0, n = 1 \), and \( m = 1, n = 0 \) in (3), respectively. The second equality in (5) follows from comparison with (4).

\[ \square \]

**Remark 2.** In case \( k < p \), or \( k > I_{0,a} \) in (4), \((a \cdot b) \cdot \alpha_{k+1,p+1}(c) = 0 \). The statement is analogous for (5) and (6).

There are two immediate consequences of (5):

**Corollary 3.** Let \( R[X; \sigma, \delta] \) be a hom-associative, non-unital Ore extension of a non-associative, non-unital ring \( R \), with the twisting map defined by (2). Then the following assertions hold for all \( a, b, c \in R \) and \( j, p \in \mathbb{N} \):

\[
(a \cdot b) \cdot \sigma(\alpha_{j+1,p+1}(c)) = (a \cdot b) \cdot \alpha_{j+1,p+1}(\sigma(c)), \quad I = \max(I_{p,c}, I_{p,\delta(c)}), \quad (7)
\]

\[
(a \cdot b) \cdot \delta(\alpha_{1,p+1}(c)) = (a \cdot b) \cdot \alpha_{1,p+1}(\delta(c)) = \begin{cases} (a \cdot b) \cdot \alpha_{j+1,j+1}(\delta(c)) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases} \quad (8)
\]

**Proof.** Put \( k = \max(I_{p,c}, I_{p,\delta(c)}) \) and \( k = 0 \) in (5), respectively.

\[ \square \]

**Definition 5** (Homogeneous map). Let \( R[X; \sigma, \delta] \) be a non-associative, non-unital Ore extension of a non-associative, non-unital ring \( R \). Then we say that a map \( \beta: R[X; \sigma, \delta] \to R[X; \sigma, \delta] \) is **homogeneous** if for all \( a \in R \) and \( m \in \mathbb{N} \), \( \beta(aX^m) = \beta(a)X^m \). If \( \alpha: R \to R \) is any (additive) map, we may extend it homogeneously to \( R[X; \sigma, \delta] \) by \( \alpha(aX^m) := \alpha(a)X^m \) (imposing additivity).

**Lemma 1** (Homogeneously extended ring endomorphism). Let \( R[X; \sigma, \delta] \) be a non-associative, non-unital Ore extension of a non-associative, non-unital ring \( R \). If \( \alpha \) is an endomorphism on \( R \), then the homogeneously extended map on \( R[X; \sigma, \delta] \) is an endomorphism if and only if

\[
\alpha(a) \cdot \pi_i^m(\alpha(b)) = \alpha(a) \cdot \alpha(\pi_i^m(b)), \quad \text{for all } i, m \in \mathbb{N} \text{ and } a, b \in R. \quad (9)
\]

**Proof.** Additivity follows from the definition, while for any monomials \( aX^m \) and \( bX^n \),

\[
\alpha(aX^m \cdot bX^n) = \alpha \left( \sum_{i \in \mathbb{N}} a \cdot \pi_i^m(b)X^{i+n} \right) = \sum_{i \in \mathbb{N}} \alpha(a) \cdot \alpha(\pi_i^m(b))X^{i+n},
\]

\[
\alpha(aX^m) \cdot \alpha(bX^n) = \alpha(a)X^m \cdot \alpha(b)X^n = \sum_{i \in \mathbb{N}} \alpha(a) \cdot \pi_i^m(\alpha(b))X^{i+n}.
\]

Comparing coefficients between the two completes the proof.

\[ \square \]

**Corollary 4** (Homogeneously extended unital ring endomorphism). Let \( R[X; \sigma, \delta] \) be a non-associative, unital Ore extension of the non-associative, unital ring \( R \). If \( \alpha \) is an endomorphism on \( R \), and there exists an \( a \in R \) such that \( \alpha(a) = 1 \), then the homogeneously extended map on \( R[X; \sigma, \delta] \) is an endomorphism if and only if \( \alpha \) commutes with \( \delta \) and \( \sigma \).

**Proof.** This follows from Lemma 1 by choosing \( a \) so that \( \alpha(a) = 1 \): if \( \alpha \) commutes with \( \delta \) and \( \sigma \), then \( \pi_i^m(\alpha(b)) = \alpha(\pi_i^m(b)) \). On the other hand, if \( \pi_i^m(\alpha(b)) = \alpha(\pi_i^m(b)) \), then by choosing \( m = 1 \) and \( i = 0 \), \( \pi_0^1(\alpha(b)) = \delta(\alpha(b)) \), and \( \alpha(\pi_0^1(b)) = \alpha(\delta(b)) \). If choosing \( m = 1 \) and \( i = 1 \), \( \pi_1^1(\alpha(b)) = \sigma(\alpha(b)) \), \( \alpha(\pi_1^1(b)) = \alpha(\sigma(b)) \).
4. Hom-associative Ore extensions of hom-associative rings

In this section, we will continue our previous investigation, but narrowed down to hom-associative Ore extensions of hom-associative rings.

**Corollary 5.** Let \( R[X; \sigma, \delta] \) be a non-associative, non-unital Ore extension of a hom-associative, non-unital ring \( R \), and extend the twisting map \( \alpha: R \to R \) homogeneously to \( R[X; \sigma, \delta] \). Then \( R[X; \sigma, \delta] \) is hom-associative if and only if for all \( a, b, c \in R \) and \( l, m, n \in \mathbb{N} \),

\[
\sum_{i \in \mathbb{N}} \alpha(a) \cdot \pi_i^m (b \cdot \pi_{l-i}^n(c)) = \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_{l}^{i+n} (\alpha(c)). \tag{10}
\]

**Proof.** A homogeneous \( \alpha \) corresponds to \( \alpha_{i+1,m+1}(a) = \alpha(a) \cdot \delta_{i,m} \) and \( \alpha_{j+1, p+1}(c) = \alpha(c) \cdot \delta_{j,p} \) in Proposition 2, where \( \delta_{i,m} \) is the Kronecker delta. Then the left-hand side reads

\[
\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \alpha_{i+1,m+1}(a) \cdot \pi_{i-k}^j (b \cdot \pi_{j-p}^n(c)) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \alpha(a) \cdot \delta_{i,m} \cdot \pi_{i-k}^j (b \cdot \pi_{j-p}^n(c))
\]

\[
= \sum_{j \in \mathbb{N}} \alpha(a) \cdot \pi_{i-k}^j (b \cdot \pi_{j-p}^n(c))
\]

\[
= \sum_{i \in \mathbb{N}} \alpha(a) \cdot \pi_{i}^m (b \cdot \pi_{n-i}^p(c))
\]

and the right-hand side

\[
\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_{i-k}^{l+n} (\alpha_{j+1,p+1}(c)) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_{i-k}^{l+n} (\alpha(c) \cdot \delta_{j,p})
\]

\[
= \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_{i-k}^{l+n} (\alpha(c))
\]

\[
= \sum_{i \in \mathbb{N}} (a \cdot \pi_i^m(b)) \cdot \pi_{i}^{l+n} (\alpha(c))
\]

which completes the proof. \( \square \)

**Remark 3.** If \( \alpha = \text{id}_R \), i.e. if \( R \) is associative, then (10) becomes

\[
a \cdot \sum_{i \in \mathbb{N}} \pi_i^m (b \cdot \pi_{i-l}^n(c)) = a \cdot \sum_{i \in \mathbb{N}} \pi_i^m(b) \cdot \pi_{l+i}^{n} (c), \tag{11}
\]

which is the necessary and sufficient condition for an Ore extension of an associative ring to be associative. The unital case, corresponding to \( a = 1 \), has been proved in e.g. [26] using a counting argument and in [27] in a more direct way.

**Corollary 6.** Let \( R[X; \sigma, \delta] \) be a hom-associative, non-unital Ore extension of a hom-associative, non-unital ring \( R \), with the twisting map \( \alpha: R \to R \) extended homogeneously to \( R[X; \sigma, \delta] \). Then, for all \( a, b, c \in R \),

\[
(a \cdot b) \cdot \delta(\alpha(c)) = (a \cdot b) \cdot \alpha(\delta(c)), \tag{12}
\]

\[
(a \cdot b) \cdot \sigma(\alpha(c)) = (a \cdot b) \cdot \alpha(\sigma(c)), \tag{13}
\]

\[
\alpha(a) \cdot \delta(b \cdot c) = \alpha(a) \cdot (\delta(b) \cdot c + \sigma(b) \cdot \delta(c)), \tag{14}
\]

\[
\alpha(a) \cdot \sigma(b \cdot c) = \alpha(a) \cdot (\sigma(b) \cdot \sigma(c)). \tag{15}
\]
Proof. Using the same technique as in the proof of Corollary 5, this follows from Corollary 2 with a homogeneous $\alpha$. 

For the two last equations, it is worth noting the resemblance to the associative case; if $R$ is an associative, unital ring, then it is necessary that $\sigma$ be an endomorphism and $\delta$ a $\sigma$-derivation, i.e. an additive map satisfying $\delta(b \cdot c) = \delta(b) \cdot c + \sigma(b) \cdot \delta(c)$ for all $b, c \in R$, for $R[X; \sigma, \delta]$ to be an associative, unital Ore extension.

Corollary 7. Assume $\alpha : R \to R$ is the twisting map of a hom-associative ring $R$, and extend the map homogeneously to $R[X; \sigma, \delta]$. Assume further that $\alpha$ commutes with $\delta$ and $\sigma$. Then $R[X; \sigma, \delta]$ is hom-associative if and only if for all $a, b, c \in R$ and $l, m, n \in \mathbb{N}$,

$$\alpha(a) \cdot \sum_{i \in \mathbb{N}} \pi_i^m \left( b \cdot \pi_{i-l}^n (c) \right) = \alpha(a) \cdot \sum_{i \in \mathbb{N}} \left( \pi_i^m(b) \cdot \pi_{i-l}^{i+n} (c) \right).$$

Proof. Using Corollary 5, we know that $R[X; \sigma, \delta]$ is hom-associative if and only if for all $a, b, c \in R$ and $l, m, n \in \mathbb{N}$,

$$\sum_{i \in \mathbb{N}} \alpha(a) \cdot \pi_i^m \left( b \cdot \pi_{i-l}^n (c) \right) = \sum_{i \in \mathbb{N}} \left( a \cdot \pi_i^m(b) \right) \cdot \pi_{i-l}^{i+n} (\alpha(c)).$$

However, since $\alpha$ commutes with both $\delta$ and $\sigma$, and $R$ is hom-associative, the right-hand side can be rewritten as

$$\sum_{i \in \mathbb{N}} \left( a \cdot \pi_i^m(b) \right) \cdot \pi_{i-l}^{i+n} (\alpha(c)) = \sum_{i \in \mathbb{N}} \left( a \cdot \pi_i^m(b) \right) \cdot \pi_{i-l}^{i+n} (\alpha(c)).$$

As a last step, we use left-distributivity to pull out $\alpha(a)$ from the sums.

We refer to Remark 3 for the resemblance with the associative case. With this in mind, one may further ask oneself whether it is possible to construct non-trivial hom-associative Ore extensions, starting from associative rings? The answer is affirmative, and the remaining part of the article will be devoted to show that.

Proposition 3. Let $R[X; \sigma, \delta]$ be an associative, non-unital Ore extension of an associative, non-unital ring $R$, and $\alpha : R \to R$ a ring endomorphism that commutes with $\delta$ and $\sigma$. Then $(R[X; \sigma, \delta], \ast, \alpha)$ is a hom-associative, non-unital Ore extension with $\alpha$ extended homogeneously to a ring endomorphism on $R[X; \sigma, \delta]$.

Proof. Since $\alpha$ is an endomorphism on $R$ that commutes with $\delta$ and $\sigma$, (9) holds, so by Lemma 1, the homogeneously extended map $\alpha$ on $R[X; \sigma, \delta]$ is an endomorphism. Referring to Proposition 1, $(R[X; \sigma, \delta], \ast, \alpha)$ is thus a hom-associative ring. Furthermore, we see that $\ast$ is the multiplication (1) of a non-associative, non-unital Ore extension, since for all $a, b \in R$ and $m, n \in \mathbb{N}$,

$$aX^m \ast bX^n = \alpha \left( \sum_{i \in \mathbb{N}} \left( a \cdot \pi_i^m(b) \right) X_i^{i+n} \right) = \sum_{i \in \mathbb{N}} \alpha \left( a \cdot \pi_i^m(b) \right) X_i^{i+n} = \sum_{i \in \mathbb{N}} \left( a \ast \pi_i^m(b) \right) X_i^{i+n}. $$

\[\square\]
Remark 4. Note in particular that if $R[X; \sigma, \delta]$ is unital, then $(R[X; \sigma, \delta], *, \alpha)$ is weakly unital with weak unit 1 due to Corollary 1.

Proposition 4 (Hom-associative $\sigma$-derivation). Let $A$ be an associative algebra, $\alpha$ and $\sigma$ algebra endomorphisms, and $\delta$ a $\sigma$-derivation on $A$. Assume $\alpha$ commutes with $\delta$ and $\sigma$. Then $\sigma$ is an algebra endomorphism and $\delta$ a $\sigma$-derivation on $(A, *, \alpha)$.

Proof. Linearity follows immediately, while for any $a, b \in A$,
\[
\sigma(a * b) = \sigma(\alpha(a \cdot b)) = \alpha(\sigma(a) \cdot \sigma(b)) = \alpha(\sigma(\alpha \cdot \delta(a)) + \alpha(\delta(a) \cdot b) + \delta(a) \cdot b) = \alpha(\sigma(a) \cdot \delta(b)) + \alpha(\delta(a) \cdot b) = \alpha(\sigma) \cdot \delta(b) + \delta(a) \cdot b,
\]
which completes the proof. \qed

Example 1 (Hom-associative Weyl algebras). Consider the first Weyl algebra exhibited as an associative, unital Ore extension, $K[Y][X; \text{id}_{K[Y]}, \delta] =: A$, where $K$ is a field and $\delta = \frac{d}{dY}$, i.e. the standard derivation on $K[Y]$. Clearly any algebra endomorphism $\alpha$ on $K[Y]$ commutes with $\text{id}_{K[Y]}$, but what about $\delta$? Since $\alpha(\delta(Y)) = \alpha(1) = 1$, we need to have $\delta(\alpha(Y)) = 1$ which implies $\alpha(Y) = Y + k$, for some $k \in K$. On the other hand, if $\alpha$ is an algebra endomorphism such that $\alpha(Y) = Y + k$ for any $k \in K$, then for any monomial $aY^m$,
\[
\alpha(\delta(aY^m)) = am\alpha(Y^{m-1}) = am^{m-1}Y = am(Y + k)^{m-1},
\]
\[
\delta(\alpha(aY^m)) = a\delta(\alpha(Y)) = a\delta((Y + k)^m) = am(Y + k)^{m-1}.
\]

Hence any algebra endomorphism $\alpha$ on $K[Y]$ that satisfies $\alpha(Y) = Y + k$ for any $k \in K$ will commute with $\delta$ (and any algebra endomorphism that commutes with $\delta$ will be on this form). Since $\alpha$ commutes with $\delta$ and $\sigma$, we know from Corollary 4 that $\alpha$ extends to a ring endomorphism on $A$ as well by $\alpha(aX^m) = \alpha(a)X^m$. Linearity over $K$ follows from the definition, so in fact $\alpha$ extends to an algebra endomorphism on $A$. Appealing to Proposition 3 and Remark 4, we thus have a family of hom-associative, weakly-unital Ore extensions $(A, *, \alpha_k)$ with weak unit 1, where $k \in K$ and $\alpha_k$ is the $K$-algebra endomorphism defined by $\alpha_k(p(Y)^m) = p(Y + k)^m$ for all polynomials $p(Y) \in K[Y]$ and $m \in \mathbb{N}$. Since Proposition 4 assures $\delta$ to be a $K$-linear $\sigma$-derivation on any member $(A, *, \alpha_k)$ as well, we call these hom-associative Weyl algebras, including the associative Weyl algebra in the member corresponding to $k = 0$.

A well-known fact about the associative Weyl algebras are that they are simple, meaning that any non-zero ideal of such an algebra is necessarily the algebra itself. This fact is also true in the case of the non-associative Weyl algebras introduced in [21], and it turns out that the hom-associative Weyl algebras as described above have this property as well.

Proposition 5. The hom-associative Weyl algebras are simple.

Proof. The main part of the proof follows the line of reasoning for the associative case; let $(A, *, \alpha_k)$ be any hom-associative Weyl algebra, and $I$ any non-zero ideal of it. Let $p = \sum_{i \in \mathbb{N}} p_i(Y)X^i \in I$ be an arbitrary non-zero polynomial with $p_i(Y) \in K[Y]$, and put $m := \max_i(\deg(p_i(Y)))$. Then
\[
X * p = \sum_{i \in \mathbb{N}} X * p_i(Y)X^i = \sum_{i \in \mathbb{N}} 1 * (p'_i(Y)X^i + p_i(Y)X^{i+1})
\]
\[
= \sum_{i \in \mathbb{N}} \alpha_k(p'_i(Y)X^i + p_i(Y)X^{i+1}) = \sum_{i \in \mathbb{N}} (\alpha_k(p'_i(Y)X^i) + \alpha_k(p_i(Y)X^{i+1}))
\]
\[
p * X = \sum_{i \in \mathbb{N}} p_i(Y)X^i * X = \sum_{i \in \mathbb{N}} p_i(Y) * X^{i+1} = \sum_{i \in \mathbb{N}} \alpha_k(p_i(Y)X^{i+1}),
\]
where \( \prime \) denotes differentiation with respect to \( Y \). By using the commutator \([\cdot, \cdot]\),

\[
[X, p] = \sum_{i \in \mathbb{N}} \alpha_k \left( p_i'(Y)X^i \right) = \sum_{i \in \mathbb{N}} p_i'(Y + k)X^i.
\]

Since \( \max_i(\deg(p_i'(Y + k))) = m - 1 \), by applying the commutator to the resulting polynomial with \( X \) \( m \) times, we get a polynomial \( \sum_{j \in \mathbb{N}} a_j X^j \) of degree \( n \), where \( a_n \in K \) is non-zero. Then

\[
\sum_{j \in \mathbb{N}} a_j X^j = \sum_{j \in \mathbb{N}} a_j \pi_i'(Y)X^i = \sum_{j \in \mathbb{N}} (a_j \ast X^{j-1} + a_j \ast YX^j),
\]

\[
Y \ast \sum_{j \in \mathbb{N}} a_j X^j = \alpha_k \left( \sum_{j \in \mathbb{N}} \pi_j YX^j \right) = \sum_{j \in \mathbb{N}} \alpha_k \left( a_j YX^j \right) = \sum_{j \in \mathbb{N}} a_j \ast YX^j.
\]

Therefore \( \deg \left( \sum_{j \in \mathbb{N}} a_j X^j, Y \right) = n - 1 \), where \( \deg(\cdot) \) now denotes the degree of a polynomial in \( X \). By applying the commutator to the resulting polynomial with \( Y \) \( n \) times, we get \( a_n \ast 1 \in I \):

\[
a_n \ast 1 = \alpha_k(a_n) = a_n \in I \quad \Rightarrow \quad a_n^{-1} \ast (a_n \ast 1) = a_n^{-1} \ast a_n = \alpha_k(1) = 1 \in I.
\]

Take any polynomial \( q = \sum_{i \in \mathbb{N}} q_i(Y)X^i \) in \( (A, \ast, \alpha_k) \). Then

\[
1 \ast \sum_{i \in \mathbb{N}} q_i(Y - k)X^i = \sum_{i \in \mathbb{N}} q_i(Y)X^i = q \in I
\]

and therefore \( I = (A, \ast, \alpha_k) \). \( \square \)

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