Lang’s Height Conjecture and Szpiro’s Conjecture

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Abstract. It is known that Szpiro’s conjecture, or equivalently the $ABC$-conjecture, implies Lang’s conjecture giving a uniform lower bound for the canonical height of nontorsion points on elliptic curves. In this note we show that a significantly weaker version of Szpiro’s conjecture, which we call “prime-depleted,” suffices to prove Lang’s conjecture.

Introduction

Let $E/K$ be an elliptic curve defined over a number field, let $P \in E(K)$ be a nontorsion point on $E$, and write $\mathfrak{D}(E/K)$ and $\mathfrak{F}(E/K)$ for the discriminant and the conductor of $E/K$. In this paper we discuss the relationship between the following conjectures of Serge Lang [7, page 92] and Lucien Szpiro (1983).

**Conjecture 1** (Lang Height Conjecture). There are constants $C_1 = C_1(K) > 0$ and $C_2 = C_2(K)$ such that the canonical height of $P$ is bounded below by

$$\hat{h}(P) \geq C_1 \log N_{K/Q} \mathfrak{D}(E/K) - C_2.$$  

**Conjecture 2** (Szpiro Conjecture). There are constants $C_3$ and $C_4 = C_4(K)$ such that

$$\log N_{K/Q} \mathfrak{D}(E/K) \leq C_3 \log N_{K/Q} \mathfrak{F}(E/K) + C_4.$$  

In [5] Marc Hindry and the author proved that Szpiro’s conjecture implies Lang’s height conjecture. (See [2] [10] for improved constants.) It is thus tempting to try to prove the opposite implication, i.e., prove that Lang’s height conjecture implies Szpiro’s conjecture. Since Szpiro’s conjecture is easily seen to be imply the $ABC$-conjecture of...
Masser and Oesterlé \[9\] (with some exponent), such a proof would be of interest.

It is the purpose of this note to explain how the pigeonhole argument in \[11\] may be combined with the Fourier averaging methods in \[5\] to prove Lang’s height conjecture using a weaker from of Szpiro’s conjecture. Roughly speaking, the “prime-depleted” version of Szpiro’s conjecture that we require allows one to discard a bounded number of primes from $\mathfrak{D}(E/K)$ and $\mathfrak{G}(E/K)$ before comparing them.

We briefly summarize the contents of this paper. In Section \[1\] we describe the prime-depleted Szpiro conjecture and prove that it implies Lang’s height conjecture. Section \[2\] contains various elementary properties of the prime-depleted Szpiro ratio. Finally, in Section \[3\] we state a prime-depleted $ABC$-conjecture and show that it is a consequence of the prime-depleted Szpiro conjecture.

1. The Prime-Depleted Szpiro Conjecture

We begin with some definitions.

**Definition.** Let $\mathfrak{D}$ be an integral ideal and factor $\mathfrak{D} = \prod p^{e_i}$ as a product of prime powers. We write $\nu(\mathfrak{D})$ for the number of factors in the product, i.e., $\nu(\mathfrak{D})$ is the number of distinct prime ideals dividing $\mathfrak{D}$. The Szpiro ratio of $\mathfrak{D}$ is the quantity

$$\sigma(\mathfrak{D}) = \frac{\log N_{K/Q} \mathfrak{D}}{\log N_{K/Q} \prod p_i} = \frac{\sum e_i \log N_{K/Q} p_i}{\sum \log N_{K/Q} p_i}.$$  

(If $\mathfrak{D} = (1)$, we set $\sigma(\mathfrak{D}) = 1$.) More generally, for any integer $J \geq 0$, the $J$-depleted Szpiro ratio of $\mathfrak{D}$ is defined as follows:

$$\sigma_J(\mathfrak{D}) = \min_{I \subseteq \{1, 2, \ldots, \nu(\mathfrak{D})\}} \left( \sigma(\prod_{i \in I} p_i^{e_i}) \right).$$

Thus $\sigma_J(\mathfrak{D})$ is the smallest value that can be obtained by removing up to $J$ of the prime powers divided $\mathfrak{D}$ before computing the Szpiro ratio. We note that if $\nu(\mathfrak{D}) \leq J$, then $\sigma_J(\mathfrak{D}) = 1$ by definition.

**Example 3.**

$$\sigma_0(1728) = \frac{\log 1728}{\log 6} \approx 4.16, \quad \sigma_1(1728) = \frac{\log 27}{\log 3} = 3, \quad \sigma_2(1728) = 1.$$

**Conjecture 4** (Prime-Depleted Szpiro Conjecture). Let $K/Q$ be a number field. There exist an integer $J \geq 0$ and a constant $C_3$, depending only on $K$, such that for all elliptic curves $E/K$,

$$\sigma_J(\mathfrak{D}(E/K)) \leq C_3.$$
It is clear from the definition that \( \sigma_0(\mathcal{D}) = \sigma(\mathcal{D}) \). An elementary argument (Proposition 9) shows that the value of \( \sigma_J \) decreases as \( J \) increases,

\[
\sigma_0(\mathcal{D}) \geq \sigma_1(\mathcal{D}) \geq \sigma_2(\mathcal{D}) \geq \cdots.
\]

Hence the prime-depleted Szpiro conjecture is weaker than the classical version, which says that \( \sigma_0(\mathcal{D}(E/K)) \) is bounded independent of \( E \). Before stating our main result, we need one further definition.

**Definition.** Let \( E/K \) be an elliptic curve defined over a number field. The **height of** \( E/K \) **is the quantity**

\[
h(E/K) = \max\{h(j(E)), \log N_{K/Q} \mathcal{D}(E/K)\}.
\]

For a given field \( K \), there are only finitely many elliptic curves \( E/K \) of bounded height, although there may be infinitely many elliptic curves of bounded height defined over fields of bounded degree [13].

We now state our main result, which implies that the \( J \)-depleted Szpiro conjecture implies Lang’s height conjecture.

**Theorem 5.** Let \( K/Q \) be a number field, let \( J \geq 1 \) be an integer, let \( E/K \) be an elliptic curve, and let \( P \in E(K) \) be a nontorsion point. There are constants \( C_1 > 0 \) and \( C_2 \), depending only on \([K : Q]\), \( J \), and the \( J \)-depleted Szpiro ratio \( \sigma_J(\mathcal{D}(E/K)) \), such that

\[
\hat{h}(P) \geq C_1 h(E/K) - C_2.
\]

In particular, the depleted Szpiro conjecture implies Lang’s height conjecture.

**Remark 6.** As in [10], it is not hard to give explicit expressions for \( C_1 \) and \( C_2 \) in terms of \([K : Q]\), \( J \), and \( \sigma_J(\mathcal{D}(E/K)) \), but we will not do so here. In terms of the dependence on the Szpiro ratio, probably the best that comes out of a careful working of the proof is something like

\[
C_1 \gg \sigma_J(\mathcal{D}(E/K))^{cJ}
\]

for some absolute constant \( c \). But until the (depleted) Szpiro conjecture is proven or a specific application arises, such explicit expressions seem of limited utility.

**Proof.** We refer the reader to [14, Chapter 6] for basic material on canonical local heights on elliptic curves. Replacing \( P \) with \( 12P \), we may assume without loss of generality that the local height satisfies

\[
\hat{\lambda}(P; v) \geq \frac{1}{12} \log N_{K/Q} \mathcal{D}(E/K)
\]
for all nonarchimedean places \( v \) at which \( E \) does not have split multiplicative reduction. We factor the discriminant \( \mathcal{D}(E/K) \) into a product

\[
\mathcal{D}(E/K) = \mathcal{D}_1 \mathcal{D}_2
\]

with \( \nu(\mathcal{D}_2) \leq J \) and \( \sigma_J(\mathcal{D}(E/K)) = \sigma(\mathcal{D}_1) \).

We also choose an integer \( M \geq 1 \) whose value will be specified later, and for convenience we let \( d = [K : \mathbb{Q}] \).

Using a pigeon-hole principle argument as described in [11], we can find an integer \( k \) with

\[
1 \leq k \leq (6M)^{J + d}
\]

such that for all \( 1 \leq m \leq M \) we have

\[
\hat{\lambda}(mkP; v) \geq c_1 \log \max \{|j(E)|_v, 1\} - c_2 \quad \text{for all } v \in M^\infty_K,
\]

\[
\hat{\lambda}(mkP; v) \geq c_3 \log |N_{K/\mathbb{Q}}(\mathcal{D}(E/K))|_v^{-1} \quad \text{for all } v \in M^0_K \text{ with } p_v | \mathcal{D}_2.
\]

(Here and in what follows, \( c_1, c_2, \ldots \) are absolute positive constants.) Roughly speaking, we need to force \( J + d \) local heights to be positive for all \( mP \) with \( 1 \leq m \leq M \), which is why we may need to take \( k \) as large as \( O(M)^{J + d} \).

We next use the Fourier averaging technique described in [5]; see also [6, 10]. Let \( p_v | \mathcal{D}_1 \) be a prime at which \( E \) has split multiplicative reduction. The group of components of the special fiber of the Néron model of \( E \) at \( v \) is a cyclic group of order

\[
n_v = \text{ord}_v(\mathcal{D}(E/K)),
\]

and we let \( 0 \leq a_v(P) < n \) be the component that is hit by \( P \). (In practice, there is no prefered orientation to the cyclic group of components, so \( a_v(P) \) is only defined up to \( \pm 1 \). This will not affect our computations.) The formula for the local height at a split multiplicative place (due to Tate, see [14, VI.4.2]) says that

\[
\hat{\lambda}(P; v) \geq \frac{1}{2} \mathbb{B} \left( \frac{a_v(P)}{n_v} \right) \log N_{K/\mathbb{Q}} p_v^{n_v}.
\]

In this formula, \( \mathbb{B}(t) \) is the periodic second Bernoulli polynomial, equal to \( t^2 - t + \frac{1}{6} \) for \( 0 \leq t \leq 1 \) and extended periodically modulo 1. As in [5], we are going to take a weighted sum of this formula over \( mP \) for \( 1 \leq m \leq M \).

The periodic Bernoulli polynomial has a Fourier expansion

\[
\mathbb{B}(t) = \frac{1}{2^\pi^2} \sum_{n \geq 1 \atop n \neq 0} \frac{e^{2\pi i nt}}{n^2} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2\pi nt)}{n^2}.
\]
We also use the formula (Fejér kernel)
\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \cos(mt) = \frac{1}{2(M+1)} \left| \sum_{m=0}^{M} e^{imt} \right|^2 - \frac{1}{2}
\]
Hence
\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \hat{\lambda}(mP; v) \\
\geq \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \cos\left(\frac{2\pi nm a_v(P)}{n_v}\right) \\
= \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi nm a_v(P)}{n_v}\right)}{n^2} \\
= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \cos\left(\frac{2\pi nm a_v(P)}{n_v}\right) \\
= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{2(M+1)} \left| \sum_{m=0}^{M} e^{2\pi in ma_v(P)}/n_v \right|^2 - \frac{1}{2} \right).
\]
We split the sum over \(n\) into two pieces. If \(n\) is a multiple of \(n_v\), then the quantity between the absolute value signs is equal to \(M+1\), and if \(n\) is not a multiple of \(n_v\), we simply use the fact that the absolute value is non-negative. This yields the local estimate
\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \hat{\lambda}(mP; v) \\
\geq \left( \frac{1}{4\pi^2(M+1)} \sum_{n=1}^{\infty} \frac{(M+1)^2}{(nn_v)^2} - \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \log N_{K/Q}p_v^{n_v} \\
= \left( \frac{(M+1)}{24n_v^2} - \frac{1}{24} \right) \log N_{K/Q}p_v^{n_v}.
\]
We next sum the local heights over all primes dividing \(\mathcal{D}_1\),
\[
\sum_{p_v|\mathcal{D}_1} \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \hat{\lambda}(mP; v) \\
\geq \frac{1}{24} \sum_{p_v|\mathcal{D}_1} \left( \frac{(M+1)}{n_v} - n_v \right) \log N_{K/Q}p_v.
\]
We set
\[ M + 1 = 2 \sum_{p_v \mid D_1} n_v \log N_{K/Q} p_v / \sum_{p_v \mid D_1} n_v^{-1} \log N_{K/Q} p_v, \]
which gives the height estimate
\[ \sum_{p_v \mid D_1} M \sum_{m=1}^M \left( 1 - \frac{m}{M+1} \right) \hat{h}(mp; v) \geq \frac{1}{24} \sum_{p_v \mid D_1} n_v \log N_{K/Q} p_v \]
\[ = \frac{1}{24} \sum_{p_v \mid D_1} \log |N_{K/Q} \mathcal{O}(E/K)|_v^{-1}. \]
We also need to estimate the size of \( M \). This is done using the elementary inequality
\[ \left( \sum_{i=1}^n a_i x_i \right) \left( \sum_{i=1}^n a_i x_i^{-1} \right) \geq \left( \sum_{i=1}^n a_i \right)^2, \quad (1) \]
valid for all \( a_i, x_i > 0 \). (This is a special case of Jensen’s inequality, applied to the function \( 1/x \).) Applying (1) with \( x_i = n_v \) and \( a_i = \log N_{K/Q} p_v \) yields
\[ M + 1 \leq 2 \left( \frac{\sum_{p_v \mid D_1} n_v \log N_{K/Q} p_v}{\sum_{p_v \mid D_1} \log N_{K/Q} p_v} \right)^2 = 2\sigma_1 \sigma_j(E/K)^2 = 2\sigma_j(\mathcal{O}(E/K))^2. \]
In particular, the chosen value of \( M \) is bounded solely in terms of \( \sigma_j(\mathcal{O}(E/K)) \).
We now combine the estimates for the local heights to obtain
\[ \sum_{m=1}^M \left( 1 - \frac{m}{M+1} \right) \hat{h}(mkP) \]
\[ \geq \sum_{m=1}^M \left( 1 - \frac{m}{M+1} \right) \left( \sum_{v \mid M^e} + \sum_{p_v \mid \mathcal{O}(E/K)} \right) \hat{\lambda}(mkP; v) \]
\[ = \left( \sum_{v \mid M^e} + \sum_{p_v \mid D_1} + \sum_{p_v \mid D_2} \right) \sum_{m=1}^M \left( 1 - \frac{m}{M+1} \right) \hat{\lambda}(mkP; v) \]
≥ \sum_{v \in M_K^\infty} \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) (c_1 \log \max\{|j(E)|_v, 1\} - c_2) \\
+ \frac{1}{24} \sum_{p_v | D_1} \log |N_{K/Q} \mathfrak{D}(E/K)|_v^{-1} \\
+ \sum_{p_v | D_1} \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) c_3 \log |N_{K/Q} \mathfrak{D}(E/K)|_v^{-1} \\
\ge c_4 h(j(E)) + c_5 \log N \mathfrak{D}(E/K) - c_6. \\

In the last line we have used the fact that \( \mathfrak{D}(E/K) j(E) \) is integral, so \\
\( \sum_{v \in M_K^\infty} \log \max\{|j(E)|_v, 1\} + \sum_{p_v | D_1 D_2} \log |N_{K/Q} \mathfrak{D}(E/K)|_v^{-1} \ge h(j(E)). \)

On the other hand,
\[
\sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) \hat{h}(mkP) = \sum_{m=1}^{M} \left(1 - \frac{m}{M+1}\right) m^2k^2 \hat{h}(P) = \frac{k^2M(M+1)(M+2)}{12} \hat{h}(P).
\]

Adjusting the constants yet again yields
\[
\hat{h}(P) \ge \frac{c_7 h(j(E)) + c_8 \log N_{K/Q} \mathfrak{D}(E/K) - c_9}{k^2 M^3} \ge \frac{c_{10} h(E/K) - c_9}{k^2 M^3}.
\]

Since \( M \) depends only on \( \sigma_J(\mathfrak{D}(E/K)) \) and since \( k \leq (6M)^{J+\delta} \), this gives the desired lower bound for \( \hat{h}(P) \).

\[\square\]

Remark 7. As in [10], a similar argument can be used to prove that \( \#E(K)_{\text{tors}} \) is bounded by a constant that depends only on \( [K : \mathbb{Q}], J, \) and \( \sigma_J(\mathfrak{D}(E/K)) \).

\section{Some elementary properties of the prime-depleted Szpiro ratio}

We start with an elementary inequality.

\textbf{Lemma 8.} \textit{Let} \( n \geq 2 \), \textit{and let} \( \alpha_1, \ldots, \alpha_n \) \textit{and} \( x_1, \ldots, x_n \) \textit{be positive real numbers, labeled so that} \( \alpha_n = \max \alpha_i \). \textit{Then}
\[
\frac{\alpha_1 x_1 + \cdots + \alpha_n x_n}{x_1 + \cdots + x_n} \geq \frac{\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}}{x_1 + \cdots + x_{n-1}},
\]
\textit{with strict inequality unless} \( \alpha_1 = \cdots = \alpha_n \).
Proof. Let \( A = \sum_{i=1}^{n} \alpha_i x_i \) and \( X = \sum_{i=1}^{n} x_i \). Then
\[
A(X - x_n) - (A - \alpha_n x_n)X = (\alpha_n X - A)x_n = \left( \sum_{i=1}^{n} (\alpha_n - \alpha_i)x_i \right)x_n \geq 0. \tag{2}
\]

Hence
\[
A \frac{X}{X - x_n} = A - \frac{\alpha_n x_n}{X - x_n}, \tag{3}
\]
and since the \( x_i \) are assumed to be positive, inequalities (2) and (3) are strict unless the \( \alpha_i \) are all equal. \( \square \)

We apply the lemma to prove some basic properties of the \( J \)-depleted Szpiro ratio.

**Proposition 9.** Let \( J \geq 1 \).

(a) For all integral ideals \( \mathfrak{D} \),
\[
\sigma_{J-1}(\mathfrak{D}) \geq \sigma_J(\mathfrak{D}).
\]
Further, the inequality is strict unless \( \mathfrak{D} \) has the form \( \mathfrak{D} = \mathfrak{J}^e \) for a squarefree ideal \( \mathfrak{J} \).

(b) Assume that \( \nu(\mathfrak{D}) \geq J \). Then there exists an ideal \( \mathfrak{d} \mid \mathfrak{D} \) satisfying
\[
\nu(\mathfrak{d}) = J \quad \text{and} \quad \sigma_J(\mathfrak{D}) = \sigma(\mathfrak{D}/\mathfrak{d}).
\]

(c) Let \( \mathfrak{p} \) be a prime ideal and \( \mathfrak{D} \) an ideal with \( \mathfrak{p} \mid \mathfrak{D} \). Then
\[
\sigma_J(\mathfrak{D}) \geq \sigma_J(\mathfrak{p}^e \mathfrak{D}) \geq \frac{\sigma_J(\mathfrak{D})}{\log N_{K/Q} \mathfrak{p}}.
\]

(d) Let \( \mathfrak{p} \) be a prime ideal and let \( \mathfrak{D} \) an arbitrary ideal (so \( \mathfrak{p} \) is allowed to divide \( \mathfrak{D} \)). Then
\[
(\log N_{K/Q} \mathfrak{p}) \sigma_J(\mathfrak{D}) \geq \sigma_J(\mathfrak{p}^e \mathfrak{D}) \geq \frac{\sigma_J(\mathfrak{D})}{\log N_{K/Q} \mathfrak{p}}.
\]

**Proof.** (a) Write \( \mathfrak{D} = \prod \mathfrak{p}_i^{e_i} \). To ease notation, we let
\[
q_i = \log N_{K/Q} \mathfrak{p}_i.
\]
If \( \nu(\mathfrak{D}) \leq J - 1 \), then \( \sigma_{J-1}(\mathfrak{D}) = \sigma_J(\mathfrak{D}) = 1 \), so there is nothing to prove. Assume now that \( \nu(\mathfrak{D}) \geq J \). Let \( I \subset \{1, 2, \ldots, \nu(\mathfrak{D})\} \) be a set of indices with \( \# I \geq \nu(\mathfrak{D}) - (J - 1) \) satisfying
\[
\sigma_{J-1}(\mathfrak{D}) = \frac{\sum_{i \in I} e_i q_i}{\sum_{i \in I} q_i}.
\]
Let \( k \in I \) be an index satisfying \( e_k = \max \{ e_i : i \in I \} \). Then Lemma 8 with \( \alpha_i = e_i \) and \( x_i = q_i \) yields

\[
\sigma_{J-1}(\mathfrak{D}) = \frac{\sum_{i \in I} e_i q_i}{\sum_{i \in I} q_i} \geq \frac{\sum_{i \in I, i \neq k} e_i q_i}{\sum_{i \in I, i \neq k} q_i} \geq \sigma_J(\mathfrak{D}).
\]

Further, Lemma 8 says that the inequality is strict unless all of the \( e_i \) are equal, in which case \( \mathfrak{D} \) is a power of a squarefree ideal.

(b) If \( \mathfrak{D} = \mathfrak{I}^e \) is a power of a squarefree ideal, then \( \sigma_J(\mathfrak{D}) = \sigma(\mathfrak{D}/\mathfrak{I}^e) \) for every ideal \( \mathfrak{I} \mid \mathfrak{J} \) satisfying \( \nu(\mathfrak{I}) = J \), so the assertion to be proved is clear. We may thus assume that \( \mathfrak{D} \) is not a power of a squarefree ideal.

Suppose in this case that \( \sigma_J(\mathfrak{D}) = \sigma(\mathfrak{D}/\mathfrak{d}) \) for some \( \mathfrak{d} \mid \mathfrak{D} \) with \( \nu(\mathfrak{d}) \leq J - 1 \). Then

\[
\sigma_{J-1}(\mathfrak{D}) \leq \sigma(\mathfrak{D}/\mathfrak{d}) = \sigma_J(\mathfrak{D}),
\]

contradicting the fact proven in (a) that \( \sigma_{J-1}(\mathfrak{D}) > \sigma_J(\mathfrak{D}) \) (strict inequality).

(c) We always have

\[
\sigma_J(p^e \mathfrak{D}) \leq \sigma_{J-1}(\mathfrak{D}),
\]

since in computing \( \sigma_J(p^e \mathfrak{D}) \), we can always remove \( p \) and \( J - 1 \) other primes from \( \mathfrak{D} \). If this inequality is an equality, we’re done. Otherwise the value of \( \sigma_J(p^e \mathfrak{D}) \) is obtained by removing \( J \) primes from \( \mathfrak{D} \). Continuing with the notation from (a) and letting \( q = \log N_{K/Q}p \), this means that there is an index set \( I \) with \( \#I \geq \nu(\mathfrak{D}) - J \) such that

\[
\sigma_J(\mathfrak{D}) = \frac{e q + \sum_{i \in I} e_i q_i}{q + \sum_{i \in I} q_i} \geq \frac{q + \sum_{i \in I} e_i q_i}{q + \sum_{i \in I} q_i} = \frac{q + X}{q + Y},
\]

where to ease notation, we write \( X \) and \( Y \) for the indicated sums.

If \( Y = 0 \), then also \( X = 0 \) and \( \nu(\mathfrak{D}) \leq J \), so \( \sigma_J(p^e \mathfrak{D}) \) equals either \( e \) or 1. In either case, it is greater than \( \sigma_J(\mathfrak{D}) = 1 \). So we may assume that \( Y > 0 \), which implies that \( Y \geq \log 2 \).

We observe that

\[
\frac{X}{Y} = \frac{\sum_{i \in I} e_i q_i}{\sum_{i \in I} q_i} \geq \sigma_J(\mathfrak{D}).
\]
Hence
\[ \sigma_J(\mathfrak{D}) = \frac{X}{Y} \cdot \frac{1 + q/X}{1 + q/Y} \geq \frac{\sigma_J(\mathfrak{D})}{1 + q/Y} \geq \frac{\sigma_J(\mathfrak{D})}{3q} . \]
(The final inequality is true since \( q \geq \log 2 \) and \( Y \geq \log 2 \).) This proves that \( \sigma_J(\mathfrak{D}) \) is greater than the smaller of \( \sigma_{J-1}(\mathfrak{D}) \) and \( \sigma_J(\mathfrak{D})/3q \). But from (a) we have \( \sigma_{J-1}(\mathfrak{D}) \geq \sigma_J(\mathfrak{D}) \), so the latter is the minimum.

(d) Let \( \mathfrak{D} = p^j \mathfrak{D}' \) with \( p \nmid \mathfrak{D}' \). Then writing \( q = \log \mathcal{N}_K/\mathcal{Q}_p \) as usual and applying (c) several times, we have
\[ \sigma_J(p^{e+1}\mathfrak{D}) = \sigma_J(p^{e+1} \mathfrak{D}') \leq \sigma_J(\mathfrak{D}') \leq q \sigma_J(p^{i} \mathfrak{D}') = q \sigma_J(\mathfrak{D}). \]
Similarly
\[ \sigma_J(p^{e}\mathfrak{D}) = \sigma_J(p^{e+1} \mathfrak{D}') \geq \frac{\sigma_J(\mathfrak{D}')}{q} \geq \frac{\sigma_J(p^{i} \mathfrak{D}')}{q} = \frac{\sigma_J(\mathfrak{D})}{q}. \]

3. The Prime-Depleted Szpiro and ABC Conjectures

In this section we describe a prime-depleted variant of the ABC-conjecture and show that it is a consequence of the prime-depleted Szpiro conjecture. For ease of notation, we restrict attention to \( K = \mathbb{Q} \) and leave the generalization to arbitrary fields to the reader.

**Conjecture 10** (Prime-Depleted ABC-conjecture). There exist an integer \( J \geq 0 \) and a constant \( C_5 \) such that if \( A, B, C \in \mathbb{Z} \) are integers satisfying
\[ A + B + C = 0 \quad \text{and} \quad \gcd(A, B, C) = 1, \]
then
\[ \sigma_J(ABC) \leq C_5. \]

The classical ABC-conjecture (with non-optimal exponent) says that \( \sigma(ABC) \) is bounded, which is stronger than the prime-depleted version, since \( \sigma_J(ABC) \) is less than or equal to \( \sigma(ABC) \).

**Proposition 11.** If the prime-depleted Szpiro conjecture is true, then the prime-depleted ABC-conjecture is true.

*Proof.* We suppose that the prime-depleted Szpiro conjecture is true, say with \( J \) primes deleted. Let \( A, B, C \in \mathbb{Z} \) be as in the statement of the depleted ABC-conjecture. We consider the Frey curve
\[ E : y^2 = x(x + A)(x - B). \]
An easy calculation \([15 \text{ VIII.11.3}]\) shows that the minimal discriminant of \( E \) is either \( 2^4(ABC)^2 \) or \( 2^{-8}(ABC)^2 \), so in any case we can
write $\mathcal{D}(E/\mathbb{Q}) = 2^e(ABC)^2$ for some exponent $e \in \mathbb{Z}$. Then using Proposition $9$ we find that

$$\sigma_J(\mathcal{D}(E/\mathbb{Q})) = \sigma_J(2^e(ABC)^2) \geq \frac{\sigma_J((ABC)^2)}{\log 2} = \frac{2\sigma_J(ABC)}{\log 2}.$$ 

Hence the boundedness of $\sigma_J(\mathcal{D}(E/\mathbb{Q}))$ implies the boundedness of $\sigma_J(ABC)$. □

**Remark** 12. The Szpiro and ABC-conjectures have many important consequences, including asymptotic Fermat (trivial), a strengthened version of Roth’s theorem [11, 16], the infinitude of non-Wieferich primes [12], non-existence of Siegel zeros [4], Faltings’ theorem (Mordell conjecture) [3, 16], . . . . (For a longer list, see [8].) It is thus of interest to ask which, if any, of these results follows from the prime-depleted Szpiro conjecture. As far as the author has been able to determine, the answer is none of them, which would seem to indicate that the prime-depleted Szpiro conjecture is qualitatively weaker than the original Szpiro conjecture.

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