**Exact Inner Metric and Microscopic State of AdS$_3$-Schwarzschild BHs**

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Through full solvability of 2+1 dimensional general relativity we derive out exact dynamic inner metric of collapsing stars with inhomogeneous initial mass distribution but joining with outside Anti-de-Sitter-Schwarzschild black holes smoothly. We prove analytically by standard quantum mechanics that the log-number of such solutions, or microscopic states of the system is proportional to the perimeter of the outside black holes. Key formulas for generalizing to 3+1D Schwarzschild black holes are also presented. Our result provides a bulk space viewpoint to questions on what the microscopic degrees of freedom are and who their carriers are in various holographic and/or asymptotic symmetry methods to black hole entropies. It may also shed light for singularity theorem and cosmic censorship related researches.

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**Introduction:** In most works talking about the origin of Bekenstein-Hawking entropy, it is believed that matters consisting of the black hole (BH) do not matter. What matters here is the gravitation field itself. A mostly cited reason for this belief is that, the matter’s contribution can hardly be area law type, while holographic features of the gravitation theory make such contributions easy. For this reason, people invented many notion and methods such as asymptotic symmetry and soft hair et al to this question. Among these works, A. Stroming’s calculation of the entropy of 2+1 dimensional Banados-Teitelboim-Zanelli (BTZ) black holes is the most simply minded one: asymptotic symmetries of gravitations under consideration are nothing but those of two dimensional conformal field theory with appropriate central charges, micro-states of black holes in the former correspond to excitations of the latter with the same total energy in the Hilbert space and will exhibit perimeter — area in 2+1 dimensions — law as expected. By this working line, microscopic degrees of freedom of black holes are defined at infinites. Simple as it is, it tells us very few on what they are describing and what their carriers are and contradicts even with some other approaches emphasizing excitations of the horizon.

However, logic possibilities that matters consisting the black hole being the main contributor of its Bekenstein-Hawking entropy are never excluded. Instead, string theory fuzzy ball pictures strongly hint that black hole metrics with essential singularities are just overdone extrapolation of their outside geometry. Inside their horizon, essential singularities are resolved and matters/energy consisting them are regularly living there as strings, whose moving modes just form the basis of Bekenstein-Hawking entropy. In a series of works, we throw away the hyper-physic notion such as extra-dimension and supersymmetry et al and considered the simple idea that matters consisting the black hole are not accreted on the central point statically but are oscillating around there under gravitations. This idea resolves the Schwarzschild singularity into a periodically density-divergent point thus giving matters consisting the black hole a chance to carry information without the singularity theorem’s violation. In 4 and 5 dimensions, by looking matters inside the horizon of a dust ball as composites of many concentric spherical shells and quantify them canonically, we find that the number of distinguishable quantum states of them is finite and scales rather well as exponent of the horizon area with the equal mass Schwarzschild black holes. In the current work, we will use exact solvability of 2+1 dimensional general relativity to give these microscopic states’ definition and enumeration an exact and more persuasive formulation.

In 2+1 dimensional asymptotically flat space-time, Newton’s gravity has linear-inverse force $F \propto r^{-1}$, and linear-log potential $V \propto \log r$. As results, stars of any mass in this theory have infinite escaping speed, making the whole space-time behave as those in black holes do. In general relativities, Ricci tensor in 2+1 dimensional space-time has equal number of independent components as Riemann tensor does, so Ricci-flat means Riemann-flat too. As results, all particles with mass less than $(2G)^{-1}$ in the asymptotically flat spacetime will lead to locally flat geometry with conical singularities on the central point, while those with mass larger than $(2G)^{-1}$ always lead to spacetime with wrong signatued metric. However, in asymptotically AdS case, the background cosmological constant provides a parabolic potential with minimal values zero. Adding effects of matters which are always trying to lower the potential everywhere, black holes with finite size can be formed,

$$ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\phi^2$$

$$h = 1 + \frac{r^2}{\ell^2} - 2GM$$

with the horizon radius given by $r_h = \sqrt{2GM - \ell^2}$ and the corresponding Bekenstein-Hawking entropy given by the perimeter law formula

$$S_{BH} = \frac{k_B A_{h}^{(1)}}{4G} = \frac{k_B 2\pi r_h}{4G} \frac{\ell^{2} < GM}{\ell} \left( \frac{k_B \pi^2 M \ell^2}{2G} \right)^{\frac{1}{2}}$$

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The purpose of this work is to show, by general relativity and quantum mechanics, that this entropy arises from the way that \( M \) can be distributed and moving inside the horizon, i.e. the way it can be looked as time dependent functions satisfying \( M(t,r) \) with the former understood as the mass contained in the horizon of the black hole. Looking at the 'hole' as many concentric rings, each of them has co-moving observers following orbit \( \{ t(\tau), r(\tau), \phi \text{ fixed}\} \) determined by

\[
\begin{align*}
\dot{r}_i t^2 - h_i^{-1} \dot{r}^2 &= 1, \\
\dot{r}_i t &= \text{const} \Leftrightarrow \dot{r} + \Gamma^{(i)}_{tr} \dot{t} \dot{r} = 0
\end{align*}
\]

where \( M_i \) is the mass inside, including the \( i \)-th ring itself and \( \Gamma^{(i)}_{tr} \) is the Christoffel symbol associated with metrics \( ds^2 = -h_i dt^2 + h_i^{-1} dr^2 - r^2 d\phi^2 \). Eq. (13) has oscillatory solutions \( r = r_{\text{in}} \cos(\omega_i \lambda + \varphi_i) \) with \( r_{\text{in}}^2 = 2GM_i + \gamma_i^2 - 1 \) and \( t = \int h_i^2 [r(\lambda)] \gamma_i d\lambda \). When \( r_{\text{in}}^2 < (2GM_i - 1)\ell^2 \), \( h_i \) will be negative and the corresponding \( \lambda \) can be chosen to be purely imaginary. This does not affect the oscillatory behavior of \( r(t) \). As results, the mass function \( M(t,r) \) can be written down explicitly

\[
M(t,r) = \sum_i m_i \Theta \{ r - r_{\text{in}} \cos(\omega_i \lambda(t) + \varphi_i) \} \\
\sum_i m_i = M_{\text{tot}}, r_{\text{in}}^2 = 2GM_i + \gamma_i^2 - 1
\]

where \( \Theta[\cdots] \) is the usual Heaviside step function and we have labeled all the rings with their masses \( m_i \), oscillation amplitudes \( r_{\text{in}} \) and initial phases \( \varphi_i \). Obviously, physics unveiled in this way are completely the same as those in the previous paragraph. Translating the function (10) into the \( M(\rho) \) in (11) is also a routine work except for some apparently singular coordinate transformation from \( \{ t, r \} \) to \( \{ \tau, \rho \} \). By classic general relativity, the function form of both \( M(\rho) \) and \( M(t,r) \) are uncountably infinite. However, at quantum levels, things are different and they form just the basis of microscopic state counted by the Bekenstein-Hawking entropy.

Quantum viewpoint: Taking an arbitrary mass ring from (11) as an example, we can get its quantum description as follows: introducing a wave function \( \psi(r) \) to denote the probability amplitude it be measured at radius \( r \) and operatorising the radial momentum \( \hbar \partial_r \equiv p_r \) as \( p_i = ih \partial_r \), then the quantum version of equation (9) will tell us

\[
[\{i\hbar m_i^2 \partial_r \}^2 + 1 + \frac{r^2}{\ell^2} - 2GM_i - \gamma_i^2] \psi(r) = 0
\]

1 Non-trivial \( \phi(\lambda) \) means orbits with nonzero angular momentum, the corresponding objects will not fall across the central point in any finite proper time. This means that classic black hole consisting of concentric shells each with nonzero angular momentum but adding up to zero are prohibited by the singularity theorem.
However, when such effects are considered, the relation neglects gravitation and relativity effects, assuming that (15)-(16) is a flat space quantum oscillation, with the interpretation of mass in classic. This is our quantum version of classic mass function (10). According to the standard quantum mechanics textbooks, we know that due to the wave function's square integrability, the energies of the mass shell are quantized, $E_i = (GMi + \frac{\gamma^2 - 1}{2})mi\ell h = n_i + \frac{1}{2}$, with $n_i = 0, 1, 2 \cdots$ and the corresponding wave function given by, here $N_i$ is the normalization constant

$$\psi_i[E_i,E_e] = N_i e^{-r^2/\pi} \text{HermiteH} \left[ \frac{E_i \ell}{\hbar} - 1 \right] \left( \frac{\hbar^2}{m_i^2} \right)$$

Comparing (10) and (11), we easily see that in classic theories the mass/energy of a composite ring is determined by its oscillation amplitude, while in quantum theories it has no direct relation with the amplitude but are integer multiples of the oscillation frequency. Nevertheless, (10) tells us that $E_i$ has two origins, the first is the gravitational part $GM_i m_i$, while the second is the kinetic contribution $\frac{\gamma^2 - 1}{2} m_i$. For all shells released inside the horizon $\gamma_i < -1$, these two term contribute contrarily to $E_i$. The eigen energy and wave function quotient above tell us that the ring under consideration can only be at a series of quantum state marked by $n_i$, while the disk or black hole as a whole can only be at states featured by direct products of its composite rings' wave function. The probability that we find mass $M$ inside the radius $r$ circle at time $t$ is

$$P[M,t,r] \propto \int_0^{E_i} \left[ \frac{\hbar^2}{m_i} \int_0^{E_i} \psi_i[E_i,E_e] \right]^2 2\pi t d\hat{r}$$

This is our quantum version of classic mass function (10). However, we must be cautioned with subtleties involved in the physical interpretation of $t$ and $E_i$. If we take the viewpoint that (15) and (16) is a flat space quantum oscillator, then they have the meaning of time and energy definitely. However, such a viewpoint is valid only locally according to the equivalent principle. Away from any pre-specified reference point, we must consider the redshift effect on them.

Consider the relation between $m_i$ and $M_{tot}$, when we neglect gravitation and relativity effects, $\sum m_i = M_{tot}$. However, when such effects are considered, the relation becomes $\sum E_i \frac{G}{\ell} = M_{tot}$. The factor $\frac{G}{\ell}$ on the left hand side is due to the redshift because $E_i$ from (10) can be understood energy only locally on the origin, while $M_{tot}$ is an invariant in AdS with time warps $\frac{\ell}{G}$. Considering this fact, energies of the composite ring and the mass of the whole disk or black hole should be written as

$$\sum (n_i + \frac{1}{2}) \hbar \omega = M_{tot} \frac{\ell}{G}$$

From this relation and the famous partition number formula of Ramanujan (31), we know that due to the wave function's square integrability, the energies of the mass shell are quantized, $E_i = (GMi + \frac{\gamma^2 - 1}{2})mi\ell h = n_i + \frac{1}{2}$, with $n_i = 0, 1, 2 \cdots$ and the corresponding wave function given by, here $N_i$ is the normalization constant

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$$\sum (n_i + 1/2) \hbar \omega = M_{tot} \ell / G$$

This is nothing but the origin of Bekenstein-Hawking entropy given in the perimeter law formula (3) except for a pure numeric factor of $\frac{1}{4}$, i.e. $k_B \log W = (\frac{1}{4}) \frac{1}{2} S_{BH}$, which arises from our imprecise estimation of the redshift factor $\frac{\ell}{G}$ in (10). In approaches basing on asymptotic symmetries, this factor is interpreted as central charges of the corresponding conformal field theory (12). Here, we see that it may have a different mechanism of origin. No matter where its origin is from, and even no matter it originates or not, we emphasize here that the key feature of Bekenstein-Hawking entropy $S_{BH} \propto M_{tot}^2$ follows simply from the radial distribution modes counting of matters consisting and moving inside the horizon of an AdS-Schwarzschild black hole.

On this microscopic origin of Bekenstein Hawking entropy, the most direct question is, why do we neglect the black hole consisting particles' random motion degrees of freedom. The answer is related with the singularity theorem, according to which all matters falling into the horizon or consisting of a Schwarzschild black hole are to pass through the central point at finite proper time. So they have zero angular momentum and no non-radial motion is allowed. Except singularity theorem, another reason denying the random motion's contribution to $S_{BH}$ is that, in AdS-Schwarzschild black holes $S_{BH} = 2\pi r_h \propto M_{tot}^2$, while in 3+1-Schwarzschild case $S_{BH} = 4\pi r_h^2 \propto M^2 > N$, with $N$ being the number of particles in the system and $S_{mand} \propto N$ by general statistic mechanics. Obviously the scaling law produced by this contribution to $S_{BH}$ is contradicted to each other in 2+1-AdS-Schw and 3+1-Schw black holes.

3+1D generalizations: In 3+1 Schwarzschild black holes, completely parallel with (10) (53) (50) and (10) (17), we have

$$ds^2 = -dt^2 + \frac{1}{a[t,\theta]} \left[ 1 - \frac{2GM}{\ell^2} \right] \frac{d\hat{r}^2}{\ell^2} + a[\tau,\vartheta]^2 \vartheta^2 d\Omega_2$$

$$a[\tau,\vartheta] = [1 - \frac{3}{2} \frac{2GM[\vartheta]}{\vartheta^3}]^{\frac{1}{2}} \propto M[\vartheta \leq \vartheta] = M_{tot}$$
\[
\rho[\tau, \varphi] = \frac{M'[\varphi]/8\pi g^2}{a^2 + \frac{3GMr^2}{4g^2} - \left(\frac{GM}{c^2}\right)^{\frac{3}{2}}}, \quad p = 0
\]  
(23)

\[
-\frac{h^2}{2m_i} \phi_x^2 - \frac{GMm_i}{x} - E_i \psi(x) = 0,
\]
(24)

\[
E_i = \frac{\gamma^2 - 1}{2} m_i = \frac{(GMm_i)^2 m_i}{n_i^2 h^2}, \quad n_i = 1, 2, \ldots
\]
(25)

\[
\psi_i^{E_i}[r] = N_i e^{-\frac{r}{\bar{r}}} L_{\text{Laguerre}}[n_i-1, 1, 2\bar{r}], \quad \bar{r} \equiv \frac{m_i r}{1 - \gamma_i^2}
\]
(26)

The only difference is that, the mass function now is oscillating in a \((1 - \frac{r}{\bar{r}})^2\) pattern of the harmonic \(\cos(\ell r)\) one. However, this difference is completely irrelevant for the Schwarzschild singularity’s resolving at both classic and quantum levels. Also the same as AdS3-Schwarzschild holes, \(\psi_i^{E_i}[r]_{\text{shell}} \neq 0\) provides us a rather intuitive fuzzyball picture for black holes. That is, we have always nonzero — although very small — probabilities to find the composite shells outside the horizon. Requiring all compositing shells’ energy add up to \(-M_{\text{tot}}\) yields similar equality like \(\sum_i \frac{G^2 M_i^2 m_i^3}{n_i h^2} = M_{\text{tot}}\). But in this case we have no adoptable Ramanujan formula to derive the area law entropy indeed follows from this radial mass’ moving modes counting. Basing on this picture, we provide substantial evidence that the area law entropy indeed follows from this radial mass’ moving modes counting.

Conclusion and Discussion: Come back to our beginning talk about the origin of Bekenstein-Hawking entropy. Both our exact solution and analytical proof in the main text support that the radial collective motion modes of matters consisting of an AdS3-Schwarzschild black hole is the main contributor to the area (perimeter in 2+1D spacetime) law feature of Bekenstein-Hawking formulas. The popular belief that such object’s contribution should be proportional to the volume (area in 2+1D) of the system is a doctrine completely. Classic particles inside the horizon carrying non-zero angular momentum do not go across the central point in any finite proper time, so is prohibited by singularity theorems. They makes no contribution to \(S_{\text{BH}}\).

Comparing two set of equal mass binary black hole system fixed on z-axis and separated by \(2r_h = 4Gm\), the former consists of two point like singular black holes while the latter consists of two oscillatory uniform density regular ones, their quadrupoles can be calculated easily

\[
D_{zz}^p = 2mr_h^2, \quad D_{ij(\neq zz)}^b = 0
\]

\[
D_{zz}^b = \frac{32}{9} mr_h^2, \quad D_{xx}^b = D_{yy}^b = \frac{8}{9} mr_h^2, \quad D_{ij(\neq j)}^b = 0
\]
(27)

When this two set of binary system rotate around their own central vertical line and radiate gravitational waves \([34][36]\), the wave will have unequal amplitude and can be measured by us. So our proposition in this and previous works \([16][18]\) that matters consisting black holes are regularly living and oscillating inside the horizon instead of singularly accumulating on the central point is verifiable through gravitational wave observations from the binary black hole merging events. This disprovability or allowing inside-horizon structure be measurable to outside observers does not violate causality, because when quantum effects are considered, the black hole horizon is highly blurred. This is easy to understand from the wave function \([17]\) and \([26]\)’s nonzero tail outside the horizon.

From pure theoretic aspects, our work provides a bulk space answer to questions such as what the microscopic degrees of freedom are and who their carriers are in various holographic and/or asymptotic symmetry methods to black hole entropies. It may also shed light for singularity theorem and cosmic censorship related researches.

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2 Formulas \([21][22][23]\) are generalizations of the Oppenheimer-Snyder collapsing star \([32][33]\). Our formulas describe dynamic collapsing stars with inhomogeneous but spherical symmetric initial mass distribution whose outside geometry joins to the Schwarzschild metric smoothly. Without pressure, we cannot set the initial speed of the system to be zero mathematically. In real collapsing stars, pressures originating from the Pauli exclusion principle can support a zero initial speed collapsing. \((1 - \frac{r}{\bar{r}})^2\) is the behavior of the oscillation across the central point.
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