Qualitative investigation of Hamiltonian systems
by application of skew-symmetric differential forms
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A great number of works is devoted to qualitative investigation of Hamiltonian systems. One of tools of such investigation is the method of skew-symmetric differential forms [1-3].

In present work, under investigation Hamiltonian systems in addition to skew-symmetric exterior differential forms, skew-symmetric differential forms, which differ in their properties from exterior forms, are used. These are skew-symmetric differential forms defined on manifolds that are nondifferentiable ones [4]. Such manifolds result, for example, under describing physical processes by differential equations.

This approach to investigation of Hamiltonian systems enables one to understand a connection between Hamiltonian systems and partial differential equations, which describe physical processes, and to see peculiarities of Hamiltonian systems and relevant phase spaces connected with this fact.

1 Connection between Hamiltonian systems and partial differential equations

The connection of Hamiltonian systems with partial differential equations can be understood if one performs the analysis of partial differential equations by means of skew-symmetric differential forms. Such method of investigation has been developed by Cartan [2] in his analysis of integrability of differential equations. In present work we will call attention to some new aspects of such investigation.

Let

\[ F(x^i, u, p_i) = 0, \quad p_i = \partial u / \partial x^i \]  

be a partial differential equation of the first order.

Let us consider the functional relation

\[ du = \theta \]  

where \( \theta = p_i \, dx^i \) (the summation over repeated indices is implied). Here \( \theta = p_i \, dx^i \) is a differential form of the first degree.

The specific feature of functional relation (2) is that in the general case, for example, when differential equation (1) describes any physical processes, this relation turns out to be nonidentical.

The left-hand side of this relation involves a differential, and the right-hand side includes the differential form \( \theta = p_i \, dx^i \). For this relation be identical, the differential form \( \theta = p_i \, dx^i \) must also be a differential (like the left-hand side of relation (2)), that is, it has to be a closed exterior differential form. To do this, it requires the commutator \( K_{ij} = \partial p_j / \partial x^i - \partial p_i / \partial x^j \) of the differential form \( \theta \) has to vanish.
In the general case, from equation (1) it does not follow (explicitly) that the derivatives $p_i = \partial u/\partial x^i$, which obey to the equation (and given boundary or initial conditions of the problem), make up a differential. Without any supplementary conditions the commutator $K_{ij}$ of the differential form $\theta$ is not equal to zero. The form $\theta = p_i \, dx^i$ turns out to be unclosed and is not a differential like the left-hand side of relation (2). Functional relation (2) appears to be nonidentical: the left-hand side of this relation is a differential, whereas the right-hand side is not a differential.

[Nonidentity of such relation has been pointed out in the work [5]. In that case a possibility of using a symbol of differential in the left-hand side of this relation has been allowed.]

[Functional relation (2) can be written in the form
\[
du - p_i \, dx^i = 0 \quad (2')
\]
This is a well-known Pfaff equation for partial differential equation. However, the relation cannot be treated as an equation. To solve the equation means to find the derivatives of equation (2') which make up a differential (and equation (2') is turned to identity). In this case the derivatives of equation (1) that do not obey these conditions are ignored, although they satisfy original equation (1) and boundary and initial conditions. But in the relation all derivatives that satisfy the original equation and boundary or initial conditions are accounted for, and their role in the physical process under consideration is analyzed.]

The nonidentity of functional relation (2) points to a fact that without additional conditions the derivatives of original equation do not make up a differential. This means that the corresponding solution $u$ of the differential equation will not be a function of only variables $x^i$. The solution will depend on the commutator of the form $\theta$, that is, it will be a functional.

To obtain a solution that is a function (i.e., the derivatives of this solution compose a differential), it is necessary to add a closure condition for the form $\theta = p_i \, dx^i$ and for corresponding dual form (in the present case the functional $F$ plays the role of a form dual to $\theta$) [2]:
\[
\begin{align*}
\left\{ \begin{array}{l}
dF(x^i, u, p_i) = 0 \\
d(p_i \, dx^i) = 0 
\end{array} \right.
\end{align*}
\]
[The dual form that corresponds to exterior differential form defines a manifold or structure, on which the exterior form is defined.]

If we expand the differentials, we get a set of homogeneous equations with respect to $dx^i$ and $dp_i$ (in the $2n$-dimensional tangent space):
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} \, p_i \, dx^i + \frac{\partial F}{\partial p_i} \, dp_i = 0 \\
dp_i \, dx^i - dx^i \, dp_i = 0 
\end{array} \right.
\end{align*}
\]
Solvability conditions for this system (vanishing of the determinant composed of coefficients at $dx^i$, $dp_i$) have the form:
\[
\frac{dx^i}{\partial F/\partial p_i} = -\frac{dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u}
\]
The relations obtained establish a connection between the differentials of coordinates \( \{ dx^i \} \) and differentials of derivatives \( \{ dp_i \} \), which satisfy the original equation. It is clear that these differentials specify integral curves on which the derivatives of original equations form a differential. In their properties the integral curves in phase space are pseudostructures. The differential, which is defined only on integral curve, is an interior differential. This differential makes up a closed *inexact* exterior form, namely, an exterior form closed only on some pseudostructure (to the pseudostructure it is assigned the dual form).

Since on integral curves defined by relations (5) the derivatives of equation (1) constitute a differential, the relevant solution to original equation is a function rather then a functional. Such solutions that are functions (i.e. depend only on variables) and are defined only on pseudostructures are so-called generalized solutions [6]. Derivatives of generalized solution constitute an exterior form, which is closed on the pseudostructure.

If conditions (5) are not satisfied, the differential form \( \theta = p_i \, dx^i \) is unclosed and is not a differential. The derivatives do not form a differential, the solution that corresponds to such derivatives will depend on the differential form commutator \( K_{ij} \) composed of derivatives. That means that the solution is a functional rather then a function.

Relations (5), which are integrating relations, can be obtained by other means. If we find the characteristics of equation (1), it turns out that relations (5) are conditions which specify characteristics of the equation under consideration. That is, integrating relations (5) are characteristic relations for partial differential equation. (In what follows such relations will be referred to as characteristic relations).

Here it should call attention to some points that will be necessary in carrying out further investigations.

Firstly, the characteristic relations have been obtained from a requirement of vanishing the determinant (set of equations (4)). This means that a change from original equation to the equation that obeys characteristic relations is a degenerate transformation. One can see that this degenerate transformation is a transition from derivatives of original equation in tangent space to derivatives in cotangent space.

And secondly, to obtain the solution that is a function, it is necessary to impose two (rather then one) additional conditions on the original equation; 1) The closure condition of the differential form composed of derivatives of original equation, and 2) the condition of closure of relevant dual form. The first condition is that the closed form is a differential. In this case, as one can see from the results obtained, the closed form can be only an inexact form, that is, this form is closed only on some pseudostructure. The second condition just allows to obtain such pseudostructure.

Now assume that equation (1) does not depend explicitly on \( u \) and is resolved with respect to some variable, for example \( t \), that is, the equation has the form

\[
\frac{\partial u}{\partial t} + E(t, x^j, p_j) = 0, \quad p_j = \frac{\partial u}{\partial x^j}
\]  

(6)
In this case the differential form $\theta$ in functional relation (2) will have the form $\theta = -E dt + p_j dx^j$

Relation (5) (the closure conditions of the differential form $\theta$ and the corresponding dual form) can be written as (in this case $\partial F/\partial p_1 = 1$)

$$\frac{dx^j}{\partial E/\partial p_j} = \frac{-dp_j}{\partial E/\partial x^j} = dt$$

and can be reduced to the form

$$\frac{dx^j}{dt} = \frac{\partial E}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial E}{\partial x^j} \quad (7)$$

These are integral (characteristic) relations for equation (6), which are conditions of integrability of this equation.

From relations (7) it follows that on integral curves the differential of the form $\theta$ equals zero: $d\theta_\pi = d(-E dt + p_j dx^j) = 0$ (here the index $\pi$ corresponds to integral curve, namely, to pseudostructure). This means that the derivatives of equation (6) $p_1 = \partial u/\partial t$, $p_j = \partial u/\partial x^j$ on integral curves obtained from relations (7) make up a closed inexact exterior form $\theta_\pi = (-E dt + p_j dx^j)_\pi$, namely, an interior, on integral curves, differential

$$(-E dt + p_j dx^j)_\pi = d_\pi u$$

The solution to equation (6) corresponding to such derivatives will be a function (generalized) rather than a functional.

The equations of field theory have the form similar to that of equation (6)

$$\frac{\partial s}{\partial t} + H(t, q_j, p_j) = 0, \quad \frac{\partial s}{\partial q_j} = p_j \quad (8)$$

where $s$ is the field function (the state function) for the action functional $S = \int L dt$. Here $L(t, q_j, \dot{q}_j)$ is the Lagrange function, $H$ is the Hamilton function: $H(t, q_j, p_j) = p_j \dot{q}_j - L, p_j = \partial L/\partial \dot{q}_j$.

Corresponding characteristic relations for equation (8) have the form

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \quad (9)$$

that is, they are Hamiltonian systems.

As it is well known, the canonical relations have just such a form.

The analogy between Hamiltonian systems and characteristic relations, as it will be shown below, allows one to see peculiarities of Hamiltonian systems.

Here it should be emphasized that, in spite of equations (6) and (8) have the same form, they fundamentally differ from one another. As it is known, equation (8) is referred to as the Hamilton-Jacobi equation. Unlike equation (6), where no restrictions are imposed on the function $E$ and this function is defined on tangent manifold, in equation (8) the function $H$ is the Hamilton function defined on cotangent manifold, that is, additional conditions are already imposed on this function. These specific features will be considered below.
2 Properties and peculiarities of characteristic relations

As it has been shown above, the characteristic relations were obtained from the first-order partial differential equation under the conditions that the differential form composed of derivatives of this equation and corresponding dual form have to be closed. Only under such conditions the derivatives of original equation make up a differential, and corresponding solutions prove to be functions rather than functionals, that is, they depend only on variables.

Here it should be remembered that we analyze partial differential equations which arise while describing physical processes. Without additional conditions such equations are nonintegrable. The characteristic relations are just additional conditions under which the derivatives of original equation make up a differential. They define integral curves \( \{ x^i(t), p_i(t) \} \) on which the derivatives of original equation make up a closed inexact form, namely, an interior differential.

What are peculiarities of characteristic relations?

We will analyze this by the example of relations (5).

As it was pointed above, characteristic relations (5) were obtained from the condition that the determinant composed of coefficients at \( dx^i \), \( dp_i \) of the set of equations (4) vanish.

This means that the characteristic relations are conditions of degenerate transformation. (It turns out that only under vanishing some determinant, that is, under the condition of degenerate transformation, the derivatives of original equation can make up a differential.)

Peculiarities of characteristic relations (and, as it will be shown below, of Hamiltonian systems) are just connected with the properties of degenerate transformation.

The degenerate transformation can be mathematically presented as a transition from one coordinate system to another, nonequivalent, one. In the case under consideration this is a transition from tangent space, in which the derivatives of original equation are defined, to cotangent space, in which the derivatives of original equation make up a differential.

In the case under consideration the tangent space is not a differentiable manifold. (If the tangent space be differentiable, the differential of differential form \( \theta = p_i \, dx^i \) would be equal to zero, that is, this form would be closed one).

The frame of reference connected with such manifold cannot be an inertial system. In the case of degeneration it takes place a transition from tangent space to a manifold made up by pseudostructures (integral curves). The frame of reference connected with such manifold is a locally inertial one. That is, in this case the degenerate transformation is a transition from the frame of reference which is not inertial (and even cannot be locally inertial) to the locally inertial frame of reference.

(It should be pointed out that, if the tangent manifold be differentiable, the transition from tangent space to cotangent one would be not a degenerate
transformation. This would be a transition from one inertial frame of reference to another inertial frame of reference.)

It turns out that the transition from derivatives of original equation, which are defined in tangent space and do not make up a differential, to derivatives that are defined in cotangent space and make up a differential, is possible as a degenerate transformation. Since in this case cotangent manifold and tangent manifold are not in one-to-one correspondence, the integral curves it can serve only sections of cotangent bundle, namely, pseudostructures.

[Examples of pseudostructures and surfaces generated by them are characteristics, cohomology, eikonal surfaces, surfaces made up by shock wave fronts, potential surfaces, pseudo-Euclidean and pseudo-Riemannian spaces and so on.]

Since the differential form composed of derivatives of original equation can be closed only on pseudostructure, this form is an inexact exterior differential form, that is, only an interior (on pseudostructure or on integral curve) differential. And this means that corresponding solutions to original equation, which are functions, are defined discretely, namely, only on pseudostructures.

As it was already pointed out, the solutions, which are defined on pseudostructures and are functions, are so-called generalized solutions. The derivatives of the generalized solution make up the exterior form, which is closed on the pseudostructure. (Under description of physical processes in material systems such solutions are state functions because they have a differential.)

Since the functions, that are the generalized solutions, are defined only on the pseudostructures (on integral curves), they have discontinuities in derivatives in the directions normal to pseudostructures.

To understand with what such discontinuities are connected and how much is their value, one has to focus his attention to the following fact. The derivatives of original equation simultaneously make up two skew-symmetric differential forms, namely, one is an unclosed differential form composed of derivatives of original equation and defined on tangent manifold, and the second is a closed inexact exterior form defined on sections of cotangent bundles (on pseudostructures). The form closed on pseudostructure is an interior differential, and this enables one to obtain the solution (generalized) to original equation on pseudostructure. And the discontinuities that have the derivatives of this solutions in the direction normal to pseudostructure are specified by the commutator of the first-order unclosed differential form. (Nonclosure of the first differential form is connected with the fact that the tangent manifold corresponding to original equation is not differentiable. In the case when tangent manifold is differentiable, both first and second differential forms are closed and the derivatives of solutions to original equation have no discontinuities. In should be noted that, for differential equations describing physical processes, the tangent manifold cannot be differentiable. For this reason the solutions of all differential equations describing physical processes have the above described functional properties[4]).

[The above considered functional properties of the set of differential equations follow from kinematic and dynamical conditions of consistency [7]. In the
appendix to the work [8] the results of calculating values of discontinuities of derivatives for entropy and sound speed in the gas dynamic problem are presented.

Thus, it turns out that in general case the derivatives of partial differential equations compose a differential only under degenerate transformation. The characteristic relations are the conditions of such degenerate transformation. The derivatives of differential equation obeying characteristic relations make up an interior (on pseudostructure defined by characteristic relation) differential and relevant solutions to original equation are functions on pseudostructure. In this case the derivatives normal to pseudostructure undergo a discontinuity. Such specific features of characteristic relations and the solutions corresponding to such relations enable one to see some peculiarities of Hamiltonian systems and their relations to the equations of mathematical physics.

3 Analysis of Hamiltonian systems

Hamiltonian systems arise in the problems of functional extremum which have wide application in quantum field theory and in the problems of classical mechanics at the basis of which it lie such dynamic principles as the principle of minimal action, the principle of virtual motions and so on.

Hamiltonian system (9) appears under the Legendre transformation: $H(t, q_j, p_j) = p_j \dot{q}_j - L$, $p_j = \partial L/\partial \dot{q}_j$, which converts the Lagrange function $L(t, q_j, \dot{q}_j)$ defined on tangent manifold $\{q_j, \dot{q}_j\}$ into the Hamilton function $H(t, q_j, p_j)$ defined on cotangent manifold $\{q_j, p_j\}$.

The Hamiltonian system is connected with the Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

(10)

which specifies a curve that is an extremal of the functional.

The connection of Hamiltonian systems with the Lagrange equation can be traced by comparing the differential of Hamilton function $H(p, q, t)$ with the differential of the function $(p \dot{q} - L)$. (Such comparison is presented in the work [1]. However, in present case we shall focus our attention on some points of such comparison.)

The total differential of the Hamilton function $H(p, q, t)$ is written in the form

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt$$

And the total differential of the Hamilton function expressed in terms of the Lagrange function $H = p \dot{q} - L$ has the form

$$dH = q dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt$$
These expressions will be identical under the condition
\[\dot{q} = \frac{\partial H}{\partial p}, \quad \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}, \quad \frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}\] (11)

From Lagrange equation (10) it follows that \(\partial L/\partial q = \dot{p}\). Replacing in the second relation (11) \(\partial L/\partial q\) by \(\dot{p}\), we obtain
\[\frac{dp}{dt} = -\frac{\partial H}{\partial q}\]
which corresponds to the second relation of Hamiltonian system. That is, the second relation of Hamiltonian system is just the Lagrange equation.

But from relations (11) one can see that under changing from the Lagrange function to the Hamilton function in addition to the relation corresponding to the Lagrange equation it arises one more relation, namely, the first relation (11), which corresponds to the first relation for Hamiltonian system (9). The physical meaning of such difference between Hamiltonian system and the Lagrange equation will be analyzed below.

Thus, the connection of the Lagrange equation with Hamiltonian system is seen. The transition from the Lagrange equation to Hamiltonian system is a transition from tangent manifold to cotangent one. [Tangent and cotangent manifolds for Lagrangian system are tangent and cotangent bundles of configuration space]. When tangent manifold is a differentiable one, such transition is a degenerate transformation. The transition from tangent manifold to cotangent one is one-to-one mapping, and Hamiltonian system and the Lagrange equation are identical.

While deriving the Lagrange equation for mechanical system it was assumed that constraints are ideal holonomic ones. In this case configuration space and tangent manifold of Lagrangian system are differentiable manifolds [1], and the transition from tangent space to cotangent one is a nondegenerate transformation.

In the case of nonholonomic constraints the tangent manifold of Lagrangian system will be not a differentiable manifold. In this case the transition from tangent manifold to cotangent one, that is, the transition from the Lagrange function to the Hamilton function and, correspondingly, from the Lagrange equation to Hamiltonian system, is possible only as a degenerate transformation. This means that the transition to subset of cotangent manifold composed of pseudostructures (sections of cotangent bundles) is only possible. That is, Hamiltonian system can be realized only discretely, namely, on pseudostructures.

In essence, Hamiltonian system will turn out to be a characteristic relation for the Lagrange equation and will have the same peculiarities as the characteristic relations.

In general case (when constrains are nonholonomic) the Lagrange equation is nonintegrable equation. The solutions to the Lagrange equations define a curve which is an extremal of the functional. But for these curves be integral
curves, the conditions of integrability have to be satisfied. The availability of closed exterior form serves as the integrability condition.

The condition of maximum of the action functional $S$, from which the Lagrange equation has been obtained, is one of conditions being necessary in definition of closed form. But for the differential form be closed, it is necessary that the relevant dual form (determining manifold or structure on which the skew-symmetric differential form is defined) be closed. The first relation of Hamiltonian system is just such a condition. In the case when tangent manifold of Lagrangian system is differentiable (this is possible only for holonomic constrains), this condition is satisfied automatically. Thus, Hamiltonian system is equivalent to the Lagrange equation. In general case the tangent manifold of Lagrangian system is not a differentiable manifold, and hence the Lagrange equation can become integrable one only under additional conditions. In this case the first relation for Hamiltonian system proves to be such additional condition of integrability of Lagrange equation. (In the calculus of variations to such additional condition there corresponds the condition of transversality.)

Thus, one can see that the Lagrange equation is equivalent to Hamiltonian system only if the conditions of integrability are satisfied. In the case of nonholonomic constrains when tangent manifold of Lagrangian system is not differentiable one, such correspondence is satisfied only under degenerate transformation. The correspondence between Hamiltonian system and the Lagrange equation is not identical.

What peculiarities in Hamiltonian system appear in the case when the tangent manifold is nondifferentiable manifold and the transition from Lagrange function to Hamilton function turns out to be degenerate transformation?

As it was already pointed out, under degenerate transformation the transition from tangent space is possible only to pseudostructures. This means that as phase space formatted it can be only the subset of cotangent manifold composed of pseudostructures (sections of cotangent bundles). That is, in this case as the phase space it can serve only cotangent bundle sections of manifold of Lagrangian system.

What properties has such phase space?

To answer this question, let us study a relation between Hamiltonian system and the Hamilton-Jacobi equation.

It has been shown above the analogy between Hamiltonian system and the characteristic relation for the first-order partial differential equation. The Hamilton-Jacobi equation is an equation of similar type. However, in the Hamilton-Jacobi equation it is apriori assumed a fulfilment of additional conditions, namely, the conditions of integrability. In this equation the function $H$ is Hamilton function, that is, a function defined on cotangent and not on tangent manifold. This fact points to a correspondence between the Hamilton-Jacobi equation for state function and Hamiltonian system.

Since the Hamiltonian system is fulfilled only on pseudostructures, the solutions to the Hamilton-Jacobi equation, which define the state function, can be only generalized functions. That is, the state function is defined only on pseudostructures, and the derivatives of state function have discontinuities in the
direction normal to pseudostructure, namely, to the phase trajectory. Just with this fact the peculiarities of phase trajectories and phase space of Hamiltonian system are connected.

It is known that in the case when the tangent manifold is differentiable and hence when the transition from tangent space to cotangent space is one-to-one mapping, in the extended phase space \( \{t, q_j, p_j\} \) there exists the Poincare invariant \( ds = -H dt + p_j dx^j \) (the differential \( ds \) directly follows from the Hamilton-Jacobi equation). [Differential is invariant under gauge transformations (conserving the differential). The canonical transformations are examples of such transformations.]

In the case when tangent manifold is not differentiable manifold (and hence when the transition from tangent space to cotangent space is degenerate) both Hamiltonian system and the Hamilton-Jacobi equation will be fulfilled only on pseudostructures, the Poincare invariant will be also fulfilled only on pseudostructures, namely, on integral curves. In the directions normal to integral curves the differential \( ds \), which corresponds to the Poincare invariant, will be discontinuous.

Invariants on pseudostructures set up invariant structures, which are connected with conservation laws.

Closed exterior forms are invariants. The closed exterior form is conservative quantity because the differential of closed form equals zero. This means that the closed form reflects conservation laws. And the closed inexact exterior form (a form closed on pseudostructure) describes a conservative object, namely, pseudostructure with conservative quantity. Such object is a physical structure and corresponds to conservation law. Phase trajectories with invariants (with closed forms) make up invariant structures, which are physical structures corresponding to conservation laws. Discontinuities (jumps) of invariants explain a discreteness of physical structures. (It can be noted that such invariant structures are an example of differential-geometric G-structure).

As it was already pointed out above, Hamiltonian system is nothing more than canonical relations.

It is known that canonical relations execute nondegenerate transformations, namely, transformations which conserve a differential. The connection of Hamiltonian system with characteristic and canonical relations discloses a duality of Hamiltonian system. From one hand, in the case when tangent manifold is not differentiable manifold, Hamilton system represents a characteristic relation, which is obtained as a condition of degenerate transformation. And from another hand, Hamiltonian system is canonical relations, which execute nondegenerate transformation. The degenerate transformation is a transition from tangent space \( (q_j, \dot{q}_j) \) to cotangent manifold \( (q_j, p_j) \). And the nondegenerate transformation is a transition in cotangent space from some pseudostructure (phase trajectory) \( (q_j, p_j) \) to another pseudostructure \( (Q_j, P_j) \). [The formula of canonical transformation can be written as \( p_j dq_j = P_j dQ_j + dW \), where \( W \) is the generating function].

Thus, it turns out that Hamiltonian systems, from one hand (in the case when the tangent manifold \( \{q_j, \dot{q}_j\} \) is not differentiable one) are characteris-
tic relations, which execute degenerate transformations describing a transition from tangent manifold on which there is no invariant structure, to cotangent space, on which there is invariant structure. And from other hand, Hamiltonian systems are canonical relations, which execute nondegenerate transformations of invariant structures.

The transition from tangent space to cotangent one under degenerate transformation when the closed exterior form is realized describes an origination of invariant structure. And the nondegenerate transformation (with the help of canonical relations) is a transition from one invariant structures to another invariant structure. (This demonstrates the connection of degenerate and non-degenerate transformation.)

Nondegenerate transformations can be described by pseudogroups, in particular, by Lie pseudogroups. But the group theory is not sufficient for describing a behavior of Lagrangian systems in the case of real physical processes.

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