Appendix: History of Singular
and its relation to Zariski’s multiplicity conjecture
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When you call SINGULAR, local on your computer or online, the following heading appears:

SINGULAR
A Computer Algebra System for Polynomial Computations
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In fact, SINGULAR is nowadays a widely used computer algebra system for polynomial computations with special emphasis on the needs of commutative algebra, algebraic geometry, and singularity theory. However, at the beginning this was never planned, we just wanted to solve mathematical problems. Only later when we had been (partially) successful, we decided to create a system also to be used by others. The development of SINGULAR has been strongly motivated and was for a long period mainly driven by mathematical problems in singularity theory. Even its appreciated computational speed is a consequence of singularity problems which are theoretically as well as computationally very hard. It is perhaps of interest to the singularities community to see how it all came about.

It started at a time, when symbolic computations was just beginning to emerge and algorithms, in particular for local computations, were practically not existent. Moreover, our cooperation within two Germanies was anything but easy because a visit from East Germany to West Germany was not possible. Anyway, we could meet in East Berlin and we started a cooperation around 1984.

Pfister (left) and Greuel at Humboldt University Berlin, 1984

The birth of SINGULAR goes back to our efforts to generalize Kyoji Saito’s well known result for hypersurface singularities (cf. [9]):

Theorem 1 (K. Saito, 1971). Let \((X,0)\) be the germ of an isolated complex hypersurface singularity. The following conditions are equivalent:
1. \((X,0)\) is quasi-homogeneous (that is, has a good \(\mathbb{C}^*\)-action).

2. \(\mu(X,0) = \tau(X,0)\).

3. The Poincaré complex of \((X,0)\) is exact.

If \((X,0)\) is given by \(f \in \mathbb{C}\{x_1, \ldots, x_n\}\) then \(\mu(X,0) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/j(f)\) is the Milnor number and \(\tau(X,0) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/\langle f, j(f) \rangle\) the Tjurina number, with \(j(f) = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle\).

Based on results in [2] and [3] we proved in [5] the following generalization of Saito’s result to isolated complete intersection curve singularities (the result in [5] was more general for reduced Gorenstein curve singularities, with the Tjurina number replaced by the Deligne number).

Theorem 2 (G.-M. Greuel, B. Martin, G. Pfister, 1985). If \((X,0)\) is a reduced complete intersection curve singularity, then

\[(X,0)\] quasi-homogeneous \(\iff\) \(\mu(X,0) = \tau(X,0)\).

So we asked ourselves in [5, Problem 1] whether \((X,0)\) is quasi-homogeneous if the Poincaré complex of \((X,0)\) is exact (the other direction is clear). At the beginning we actually conjectured that the answer should be positive. However, we did not succeed in proving it and so we started to look for possible counter examples. But the computations by hand were very time consuming and with the small examples at hand we were unable to find any counter example. Nevertheless, we started not to believe in the conjecture.

To compute potential counter examples with a help of a computer, two main problems appeared: First, we needed Teo Mora’s tangent cone algorithm (a variation of Buchberger’s algorithm for local rings) to compute standard bases for \(O_{X,0}\)-modules. However, no package for this existed at that time, not even for ideals. The second problem was more of a theoretical nature. We needed to compute the kernel of the exterior derivation in the Poincaré complex, which is only \(\mathbb{C}\)-linear but not \(O_X\)-linear and hence not directly tractable by standard bases computations. Fortunately, using a result of Reiffen (see below) we had been able in [5] to reformulate the exactness of the Poincaré complex as a question of computing submodule membership and dimensions of \(O_{X,0}\)-modules.

The first problem was more serious. There was no computer algebra system available which could compute this kind of examples. In 1984 Neuendorf and Pfister (during vacations at the baltic sea) started an implementation of Buchberger’s Gröbner basis algorithm in Basic on a ZX-Spectrum (an 8 bit home PC from Sinclair UK, 1982). It took Pfister and his student Hans Schönemann two more years of development to obtain a Modula-2 implementation of a package, called Buchmora at that time (Buchberger’s and Mora’s algorithm) for Atari computers. Using this implementation the following counter examples were found (cf. [8]).

Theorem 3 (G. Pfister, H. Schönemann, 1989). Let \((X_{k,l}, 0)\) be the germ of the unimodal space curve singularity \(FT_{k,l}\) of the classification of C.T.C. Wall (cf.
defined by the equations
\[xy + z^{l-1} = xz + y^2 + y^{k-1} = 0, \quad (4 \leq l \leq k, \ 5 \leq k).\]

Then the Poincaré complex
\[0 \to \mathbb{C} \to \mathcal{O}_{X_{lk},0} \to \Omega^1_{X_{lk},0} \to \Omega^2_{X_{lk},0} \to \Omega^3_{X_{lk},0} \to 0\]
is exact, but \((X_{lk},0)\) is not quasi-homogeneous.

**Proof.** To show that \((X_{lk},0)\) is not quasi-homogeneous, it suffices to show
\[\mu(X_{lk},0) = \tau(X_{lk},0) + 1 = k + l + 2\]
by the following formulas. Let \((X,0) \subset (\mathbb{C}^3,0)\) be the space curve singularity defined by \(f = g = 0\), with \(f, g \in \mathbb{C}\{x, y, z\}\). Then

- \(\mu(X,0) = \dim_{\mathbb{C}}(\Omega^1_{X,0}/d\mathcal{O}_{X,0}) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle f, M_1, M_2, M_3 \rangle - \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle\),
- \(\tau(X,0) = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\}/\langle f, g, M_1, M_2, M_3 \rangle\),

with \(M_1, M_2, M_3\) the 2-minors of the Jacobian matrix \(\begin{pmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \end{pmatrix}\).

On the other hand, a result of Reiffen says:

- The Poincaré complex is exact iff
  1. \(\langle f, g \rangle \cdot \Omega^3_{\mathbb{C}^3,0} \subset d(\langle f, g \rangle \cdot \Omega^2_{\mathbb{C}^3,0})\), and
  2. \(\mu(X,0) = \dim_{\mathbb{C}}(\Omega^2_{X,0}) - \dim_{\mathbb{C}}(\Omega^3_{X,0})\).

All these statements could be checked with the Buchmora algorithm. \(\square\)

Encouraged by this success and having a computer algebra system that was able to compute in local rings, we tried to find a counterexample to Zariski’s multiplicity conjecture (Zariski had posed this as a question, which he supposed to have quick answer by topologists, cf. [11]).

**Conjecture 4** (O. Zariski, 1971). Two hypersurface singularities (given by convergent power series) with the same topological type have the same multiplicity.

A weaker version of this conjecture says:

*In a \(\mu\)-constant deformation of an isolated hypersurface singularity the multiplicity is constant.*

The conjecture was already known for reduced plane curve singularities and the weaker conjecture for isolated quasi-homogeneous hypersurface singularities (cf. [4]). The methods of [4] are in principal applicable to any isolated hypersurface singularity, but we failed to prove the weak Zariski’s conjecture in general.
Due to the many unsuccessful efforts by us and others we were (and are still) convinced that Zariski’s conjecture might not be true.

Hence, we tried to find a counterexample. The main problem is the difficulty to construct examples of $\mu$-constant deformations. Since Zariski’s conjecture is true in the semi quasi-homogeneous case and for plane curve singularities, the examples to test should be somewhat complicated. We used the Newton diagram to construct families of surface singularities where the multiplicity drops and with Newton diagram becoming degenerate but with rather small degeneracy area, hoping that the Milnor number would stay constant. Among others we tried a series of examples of the following form:

$$F_t = x^a + y^b + z^{3c} + x^{c+2}y^{c-1} + x^{c-1}y^{c-1}z^3 + x^{c-2}y^{c}(y^2 + tx)^2$$

The multiplicity can be read of from the equation, but for the Milnor number we had to use a computer and the package Buchmora. However, this and other examples took hours to compute. Whenever we met, in East Germany (often in Pfister’s dacha close to Berlin) or at conferences outside West Germany, we tried to improve the algorithm by checking different local orderings and trying to optimize the selection strategies during the standard basis computation (producing huge tables with timings). The selection strategies for different orderings, which we finally preferred, are still in use in the present version of Singular.

The place where everything started: Pfister’s dacha in the GDR
Among the above series of examples there was unfortunately no counter example. We found e.g. for \((a,b,c) = (40,30,8)\): 

\[ m(F_0) = 17, \; m(F_t) = 16, \; \mu(F_0) = 10661, \; \mu(F_t) = 10655. \]

The computations for \(\mu\) took many hours (today within a few seconds), but smaller Milnor numbers could be excluded by heuristical arguments. A significant speed up of the computation of standard bases for local orderings was needed and we decided to make a further step towards a more professional development of a computer algebra package.

In 1989 Buchmora was renamed to SINGULAR. It was jointly developed by groups from Berlin (Pfister) and Kaiserslautern (Greuel) within a DFG priority program 1990–1996. Within this program we could hire Hans Schönemann, who moved to Kaiserslautern in 1990, right after the unification of Germany. SINGULAR was ported to Unix (still in Modula-2) and a first user manual was released. In 1993 Pfister moved to Kaiserslautern and we decided to rewrite the code in C/C++, carried out mainly by Schönemann. Within the DFG priority program the SINGULAR programming language was developed and many libraries had been established. Around 1996 Olaf Bachmann joined the team in Kaiserslautern and with his help it was possible to improve the code of SINGULAR significantly, mainly by adapting the data structures and the memory management, which increased the speed drastically.

In spite of these improvements, no counter example was found! But by analyzing the above examples a partial solution to Zariski’s conjecture was published in [7], including the first publication of a standard basis algorithm for arbitrary mixed monomial orderings (implemented in Singular since 1993):

**Proposition 5** (G.-M. Greuel, G. Pfister, 1996). Let 

\[ F_t(x_1,\ldots,x_n) = G_t(x_1,\ldots,x_{n-1}) + x_n^2 H_t(x_1,\ldots,x_n) \]

be a family of isolated hypersurface singularities. Let \(G_0\) be semiquasihomogeneous or let \(n = 3\). If the family has constant Milnor number and the multiplicity of \(G_t\) is smaller or equal to the multiplicity of \(H_t + 2\) then the multiplicity of \(F_t\) is constant.

To conclude, let us remark, that the failure to find a counter example to Zariski’s conjecture was the most important reason for the development of SINGULAR as it is now. First of all, for many years it was the main motivation to improve its speed, since the possible counter examples were complicated to compute. Secondly, it was a very good theoretical problem that convinced the referees to support the development of SINGULAR for many years.

**SINGULAR – Some History**

- 1984 Neuendorf/Pfister: Implementation of the Gröbner basis algorithm in Basic on a ZX-Spectrum.
• 1990 Schönemann moved to KL, porting to Unix

• 1993 Pfister moved to KL, C/C++ version.

• 1996–2000 Greuel/Pfister: symbolic/numerical algorithms in Singular, joint with electrical engineers and a Mathematic package ”Analog Insydes”.

• 1997/1998 Singular release 1.0 -1.2, with multivariate polynomial factorization, gcd, syzygies, free resolutions, communication links, primary decomposition and normalization.

• 2002 Book: A SINGULAR Introduction to Commutative Algebra. By G.-M. Greuel and G. Pfister, with contributions by O. Bachmann, C. Lossen and H. Schönemann.

• 2004 First Richard D. Jenks Memorial Prize for Excellence in Software Engineering awarded to Singular at ISSAC in Santander.

• 2004 Greuel/Levandovsky: The subsystem PLURAL for non-commutative polynomial algebras is included in Singular.

• 2008 interface to the computer algebra system “Sage”.

• 2009 Decker moves to KL, with Greuel/Pfister/Schönemann one of the leaders of the Singular development.

• 2016 The ”Oscar” system includes a Julia package for the Singular library.

• Singular has been supported by Deutsche Forschungsgemeinschaft (DFG), Stiftung Rheinland-Pfalz für Innovation, and Volkswagen Stiftung.

• Singular is free software, available at https://www.singular.uni-kl.de/

SINGULAR-team with Pfister, Schönemann, Lossen, Decker, Greuel, ...
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