CLASSIFYING COMPLEMENTED SUBSPACES OF $L_p$ WITH ALSPACH NORM

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This paper is dedicated to Professor Dale E. Alspach on the occasion of his 60th Birthday. Professor Alspach was born in April 30, 1950. He was one of first three Ph.D. students to finish their theses under the direction of William Johnson. Dr. Alspach’s major contribution to Banach Space Theory are the first example of a non-expansive map on a weakly compact, closed convex subset of a Banach space without a fixed point and classification results for complemented subspaces of classical Banach spaces such as $C[0,1]$ and $L_p$, and translation invariant subspaces of $L_1(G)$ for $G$ Abelian. He is currently the Department Head at Oklahoma State University.

Abstract. Understanding the complemented subspaces of $L_p$ has been an interesting topic of research in Banach space theory since 1960. 1999, Alspach proposed a systematic approach to classifying the subspaces of $L_p$ by introducing a norm given by partitions and weights. This paper shows that with Alspach Norm we are able to classify some complemented subspaces of $L_p$, $2 < p < \infty$.

1. Introduction

Since 1960s understanding the complemented subspaces of $L_p$ has been an interesting topic of research in Banach space theory. Early in the work only obvious combinations of $\ell_p$ and $\ell_2$ were known to give examples. In 1972, Rosenthal’s paper on sums of independent random variables was seminal. He created $X_p$ and $B_p$ spaces in this paper. In 1975, Schechtman proved that, up to isomorphism, there are infinitely many complemented subspaces of $L_p$ by constructing tensor products of $X_p$ spaces and in 1979, Bourgain, Rosenthal, and Schechtman proved that, up to isomorphism, there are uncountably many complemented subspaces of $L_p$. 1999, Alspach proposed a systematic approach to classifying the subspaces of $L_p$ by introducing a norm given by partitions and weights. His proposal was the following:

Let $A$ be a countable set and $P = \{N_i\}$ be a partition of $A$ and $W : A \rightarrow (0,1]$ be a function, which we refer to as the weights. Let $x_j \in \mathbb{R}$ for all $j \in A$. Define

$$\|(x_j)_{j \in A}\|_{(P,W)} = \left( \sum_{N \in P} \left( \sum_{j \in N} x_j^2 w_j^2 \right)^{\frac{p}{q}} \right)^{\frac{q}{p}}$$

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Suppose that \((P_k, W_k)_{k \in K}\) is a family of pairs of partitions and weights as above. Define a (possibly infinite) norm on the real-valued functions on \(A\), \((x_i)_{i \in A}\), by
\[
\| (x_i) \| = \sup_{k \in K} \| (x_i) \|_{(P_k, W_k)}
\]
and let \(X\) be the subspace of elements of finite norm. In this case we say that \(X\) has an \textit{Alspach Norm}.

It is important to note that a space with the Alspach norm has a natural unconditional basis and spaces with finite Alspach norm are Banach spaces [AT1]. Alspach’s approach gives a unified description of many well known complemented subspaces of \(L_p\). It is proved that the class of spaces with such norms is stable under \((p, 2)\) sums [AT1]. 2006, Alspach and Tong proved that subspaces of \(L_p\), \(p > 2\), with unconditional bases have equivalent partition and weight norms[AT2]. In this article we will explain what the conditions on partitions and weights will produce certain known complemented subspaces of \(L_p\). Our work is far from complete due to the scope of this approach. Classifying all complemented subspaces of \(L_p\) with unconditional basis is a big challenge and with Alspach norm we made some progress.

From now on we will always assume that \(p > 2\). In the rest of the paper, we use Rosenthal’s \(X_p\) space and \(B_p\) space many times, [R]. Here are the definitions which can be found in Force’s dissertation [F]: \(X_p\) can be realized as the closed linear span in \(L_p\) of a sequence \(\{f_n\}\) of independent symmetric three-valued random variables such that the ratios \(\|f_n\|_2 / \|f_n\|_p\) approaches zero slowly. Another realization of \(X_p\) is as the set of all sequences \(\{x_n\}\) in \(\ell^p\) for which the weighted \(\ell^2\) norm \((\sum |w_n x_n|^2)^{1/2}\) is finite and \((w_n)\) is a fixed sequence that goes to zero slowly. The Banach space \(B_p\) is of the form \((Y_1 \oplus Y_2 \oplus \cdots) \ell_p\), where each space \(Y_n\) is defined similar to \(X_p\) but is isomorphic to \(\ell_2\), and \(\{Y_n\}_{n=1}^\infty\) is chosen so that \(\sup_{n \in \mathbb{N}} d(Y_n, \ell_2) = \infty\), where \(d(Y_n, \ell_2)\) is the Banach-Mazur distance between \(Y_n\) and \(\ell_2\).

A partition is called \textit{discrete} if every subset has only one element. A partition is called \textit{indiscrete} if the partition is the whole set. A partition which is not discrete and not indiscrete is called a \textit{regular} partition. Partition \(P_1\) is called a \textit{refinement} of \(P_2\) if every element in \(P_2\) is a union of elements in \(P_1\). From now on, we treat the discrete partition with constant weight of 1 as trivial and it will be included in the discussion but will not be counted towards the number of partitions. For instance we say that Rosenthal’s \(X_p\) space [R] is an example of Banach space with Alspach norm given by one partition and weights. In the context of subspaces of \(L_p\) with unconditional basis there is always a lower \(\ell_p\) estimate and the discrete partition ensures that the spaces we consider also have a lower \(\ell_p\) estimate.

### 2. One Regular Partition

\textbf{Definition 2.1.} Let \(A\) and \(P = (P, W)\) be defined as above. A Banach space \(X\) is said to have Alspach norm with one partition and weights if
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$\| (x_j) \|_X = \max \left\{ \left( \sum x_j^p \right)^{\frac{1}{p}}, \left( \sum_{N \in P} \left( \sum_{j \in N} x_j^2 w_j^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\}$

There is a complete classification of the spaces with Alspach norm and one regular partition.

**Proposition 2.2.** Let $P = \{ N_i : i \in B \}$ where $B$ is an index set. Let $|N_i|$ be the cardinality of $N_i$. Let $I = \{ i : |N_i| = \infty \}$. Then

1. If $|I| < \infty$, then $X$ is isomorphic to one of $\ell_p$, $X_p$, $\ell_2$, or $\ell_2 \oplus \ell_p$.
2. If $|I| = \infty$, then $X$ is isomorphic to one of $\ell_p$, $X_p$, $\ell_2 \oplus \ell_p$, $B_p$, $(\sum \ell_2)_{\ell_p}$, $(\sum \ell_2)_{\ell_p} \oplus X_p$, $B_p \oplus X_p$, or $(\sum X_p)_{\ell_p}$.

To prove the proposition we need following three lemmas. Below $X_i = X_i^{[N_i]} = [e_j : j \in N_i]$ where $(e_j)$ is the natural basis of $X$.

**Lemma 2.3.** Let $I$ be defined as above, then

$$X \sim \left( \sum_{i \in I} X_i \right) \oplus \left( \sum_{i \not\in I} X_i^{[N_i]} \right)_{\ell_p}$$

**Lemma 2.4.** If $B \setminus I$ is finite, then

$$\ell_p \oplus \left( \sum_{i \not\in I} X_i^{[N_i]} \right)_{\ell_p} \sim \ell_p$$

**Lemma 2.5.** If $B \setminus I$ is infinite, then

$$\left( \sum_{i \not\in I} X_i^{[N_i]} \right)_{\ell_p} \sim \ell_p$$

Proof: For each $i \notin I$, $X_i$ is a finite dimensional version of one of the spaces considered by Rosenthal and thus is isomorphic to a complemented subspace of $\ell_p$ and the norm of the projection is independent of $i$. This implies

$$\left( \sum X_i^{[N_i]} \right)_{\ell_p} \overset{c}{\cong} \left( \sum \ell_p \right)_{\ell_p}.$$  

Since $(\sum \ell_p)_{\ell_p} \sim \ell_p$, then

$$\left( \sum X_i^{[N_i]} \right)_{\ell_p} \overset{c}{\cong} \ell_p.$$  

Since every infinite dimensional complemented subspace of $\ell_p$ is isomorphic to $\ell_p$, then

$$\left( \sum X_i^{[N_i]} \right)_{\ell_p} \sim \ell_p.$$  

After a messy computation based on splitting the argument into several cases depending on the isomorphic type of the $\ell_p$ sum of $X_i$ for $i \in I$ the results in the proposition follow.
3. Admissible Partitions

Definition 3.1. A family of partitions and weights $\mathcal{P}$ is said to be admissible if there are partitions and weights $(P_0, W_0)$, $(P_1, W_1)$, and $(P_2, W_2)$ in $\mathcal{P}$ such that $(P_0, W_0)$ is the discrete partition with weight constantly 1, $(P_1, W_1)$ is a regular partition and weight and $P_2$ is the indiscrete partition with weight $W_2 = (w_{2,j})$.

We have following result

Proposition 3.2. Assume $X$ be a sequence space of finite Alspach norm with an admissible family of partitions and weights and only one regular partition and weight. Then

1. If $\inf_j w_{2,j} \geq \delta > 0$, then $X \sim \ell_2$.
2. Suppose $\sum_j (w_{2,j})^{\frac{p}{2p-2}} < \infty$. Let $P_1 = \{N_i : i \in \mathbb{N}\}$. Let $|N_i|$ be the cardinality of $N_i$. Let $I = \{i : |N_i| = \infty\}$. Then $X$ is isomorphic to one of the spaces listed in Proposition 2.2 (1).
3. If we combine first two cases, i.e. there is some $\delta > 0$, such that $\{j : w_{2,j} \geq \delta\}$ and $\{j : w_{2,j} < \delta\}$ are infinite and $\sum_{w_{2,j} = \delta} (w_{2,j})^{\frac{p}{2p-2}} < \infty$, then $X$ is a direct sum of $\ell_2$ from (1) and one of the spaces from (2).

Proof:

1. Since $\inf w_{2,j} \geq \delta > 0$, then
   $$\delta \left( \sum_j x_j^2 \right)^{\frac{1}{2}} \leq \|x_j\|_X \leq \left( \sum_j x_j^2 \right)^{\frac{1}{2}}$$

2. Since $\sum_j (w_{2,j})^{\frac{p}{2p-2}} < \infty$, then we can apply Hölder’s inequality
   $$\left( \sum x_j^2 w_{2,j}^2 \right)^{\frac{1}{2}} \leq \left( \sum x_j^p \right)^{\frac{1}{2}} \left( \sum w_{2,j}^{\frac{p}{2p-2}} \right)^{\frac{2p-2}{2p}}$$

   Thus
   $$\|x_j\|_X \sim \|x_j\|_{P_1}$$

Apply the results from proposition 2.2 to the rest of the proof of (2).

4. Two Partitions Related by Refinement

Let $A$ be any index set. Let $P_1$ and $P_2$ be two partitions of $A$ and let $W_1 = (w_{1,k})$ and $W_2 = (w_{2,k})$ be two sequences of weights. Assume that $P_1$ is a refinement of $P_2$ and that $w_{1,j} \geq w_{2,j}$ for all $j \in A$. For a fixed $N \in P_1$, notice that

$$W_N = \sup \left\{ \left( \sum x_j^2 w_{2,j}^2 \right)^{\frac{1}{2}} / \left( \sum x_j^2 w_{1,j}^2 \right)^{\frac{1}{2}} \right\}$$

$$\leq \sup_{k \in N} \sup_{w_{1,k}} \left( \sum x_j^2 w_{1,1,j}^2 \right)^{\frac{1}{2}} / \left( \sum x_j^2 w_{1,k}^2 \right)^{\frac{1}{2}}$$

$$= \sup_{k \in N} \sup_{w_{1,k}} \left( \sum x_j^2 w_{1,1,j}^2 \right)^{\frac{1}{2}} / \left( \sum x_j^2 w_{1,k}^2 \right)^{\frac{1}{2}}$$

the supremum in the first two lines are taken over all sequences $(x_j)$ for $x_j = 0, j \not\in N$ and $x_j \neq 0$ for finitely many $j$. Taking $x_j = 1$ and $x_k = 0$ for $k \neq j$, shows
that $W_N = \sup_{k \in N} w_{2,k}$. Now notice that for fixed $M \in P_2$,

$$\sum_{N \in M} \sum_{j \in N} x_j^2 w_{2,j} \leq \sum_{N \in M} W_2^2 \sum_{j \in N} x_j^2 w_{1,j}$$

$$\leq \left( \sum_{N \in M} W_2^{2p} \right)^{\frac{p}{2}} \left( \sum_{N \in M} \left( \sum_{j \in N} x_j^2 w_{1,j}^p \right)^{\frac{p}{2}} \right)^{2/p}$$

Then by Using Hölder’s inequality

$$\|x_j\| = \max \left\{ \left( \sum x_j^p \right)^{\frac{1}{p}} , \left( \sum_{N \in M} \left( \sum_{i \in N} x_i^2 w_{1,i}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}, \left( \sum_{M \in P_2} \left( \sum_{j \in M} x_j^2 w_{2,j}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \right\}$$

$$= \max \left\{ \left( \sum x_j^p \right)^{\frac{1}{p}} , \left( \sum_{N \in M} \left( \sum_{i \in N} x_i^2 w_{1,i}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}, \left( \sum_{M \in P_2} \left( \sum_{N \in M} \sum_{j \in N} x_j^2 w_{2,j}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \right\}$$

$$\leq \max \left\{ \left( \sum x_j^p \right)^{\frac{1}{p}} , \left( \sum_{N \in M} \left( \sum_{i \in N} x_i^2 w_{1,i}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}, \left( \sum_{M \in P_2} \left( \sum_{N \in M} \sum_{j \in N} x_j^2 w_{1,j}^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \right\}$$

We have following result:

**Proposition 4.1.** Let partition $P_1$ be a refinement of partition $P_2$. Let $W_1 = (w_{1,k})$ and $W_2 = (w_{2,k})$ be two corresponding sequences of weights. Assume that $w_{1,j} \geq w_{2,j}$ for all $j \in A$ if $w_{1,j} \geq w_{2,j}$ for all $j \in A$.

$$\sup_{M} \left\{ \sum_{N \in M} W_2^{2p} \right\} < \infty$$

then $X$ can be classified by the behavior of $W_1$.

**Proposition 4.2.** Let $X$ be the Banach space with finite norms given by two partitions $P_1$ and $P_2$ and we also assume for every $N \in P_1$ there is an $M \in P_2$ such that $N \subset M$ and $w_{1,k} \geq w_{2,k}$ for all $k \in N$. We have following results:

1. Assume $\inf(w_{2,k}) = \delta > 0$.
   1. If $|P_2| < \infty$ we have

$$\delta |P_2|^\frac{1}{p} \|x_j\|_2 \leq \delta \left( \sum_{M \in P_2} \left( \sum_{j \in M} x_j^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq \|x_j\|_X \leq \|x_j\|_2.$$
Therefore, 

\[ X \sim \ell_2. \]

(b) If \(|P_2| = \infty\) and \(|M| < \infty\) for all \(M \in P_2\),

\[ \|x_j\|_X \sim \left( \sum_{M \in P_2} \left( \sum_{j \in M} x_j^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}. \]

Let \(Y_n\) represent the space generated by the inner sum which implies that \(X \sim (\sum Y_n)_p\). Since \(X\) is infinite dimensional we conclude \(X \sim \ell_p\).

(c) If \(|P_2| = \infty\), \(|M| = \infty\) for at least one and at most finitely many \(M \in P_2\), then \(X\) will be a direct sum of the spaces, \(X \sim \ell_2 \oplus \ell_p\).

(d) If \(|P_2| = \infty\) and \(|M| = \infty\) for infinitely many \(M \in P_2\), then

\[ \|x_j\|_X \sim \left( \sum_{|M| = \infty} \left( \sum_{j \in M} x_j^2 \right)^{\frac{1}{2}} + \sum_{|M| < \infty} \left( \sum_{j \in M} x_j^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}}. \]

and \(X \sim (\sum \ell_2)_{\ell_p}\).

(2) Assume \(\inf(w_{1k}) = \gamma > 0\) and \(\inf(w_{2k}) = 0\)

(a) If \(|P_1| < \infty\), we have

\[ \gamma |P_1|^{\frac{1}{p} - \frac{1}{2}} \|x_j\|_2 \leq \gamma \left( \sum_{N \in P_1} \left( \sum_{j \in N} x_j^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \leq \|(x_j)\|_X \leq \|(x_j)\|_2 \]

so \(X \sim \ell_2\).

(b) If \(|P_1| = \infty\) and \(|N| < \infty\) for all \(N \in P_1\), and for every \(M \in P_2\), let \(C_M = \sum_{n \in M} w_{2n}^{2n} < \infty\) and assume \(\sup_{M \in P_2} C_M < \infty\), then \(X \sim \ell_p\).

REFERENCES

[1] Dale Alspach. Tensor Products and Independent Sums of \(L_p\) Spaces, \(1 < p < \infty\). Number 660. Memoirs of the American Mathematical Society, 1999.
[2] Dale Alspach and Simei Tong. Subspaces of \(L_p\) with \(p > 2\), determined by partitions and weights. Studia Mathematica, 159(2):207–227, 2003.
[3] Dale Alspach and Simei Tong. Subspaces of \(L_p\), for \(p > 2\), with unconditional bases have equivalent partition and weight norms. Archiv der Mathematik, 86:73–78, 2006.
[4] Gregory Michael Force. Constructions of \(L_p\), \(1 < p \neq 2 < \infty\). PhD thesis, Oklahoma State University, OK, 1995.
[5] Haskell Rosenthal. On the subspaces of \(L_p\) for \(p > 2\) spanned by sequences of independent random variables. Israel Journal of Mathematics, 8, 1970.
[6] [S] G Schechtman. Examples of $\mathcal{L}_p$ spaces ($1 < p \neq 2 < \infty$). *Israel Journal of Mathematics*, 22(2), 1975.

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