A measure theoretic paradox from a continuous colouring rule

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Abstract

Given a probability space \((X, \mathcal{B}, m)\), measure preserving transformations \(g_1, \ldots, g_k\) of \(X\), and a colour set \(C\), a colouring rule is a way to colour the space with \(C\) such that the colours allowed for a point \(x\) are determined by that point’s location and the colours of the finitely \(g_1(x), \ldots, g_k(x)\) with \(g_i(x) \neq x\) for all \(i\) and almost all \(x\). We represent a colouring rule as a correspondence \(F\) defined on \(X \times C^k\) with values in \(C\). A function \(f : X \to C\) satisfies the rule at \(x\) if \(f(x) \in F(x, f(g_1(x)), \ldots, f(g_k(x)))\). A colouring rule is paradoxical if it can be satisfied in some way almost everywhere with respect to \(m\), but not in any way that is measurable with respect to a finitely additive measure that extends the probability measure \(m\) defined on \(\mathcal{B}\) and for which the finitely many transformations \(g_1, \ldots, g_k\) remain measure preserving. Can a colouring rule be paradoxical if both \(X\) and the colour set \(C\) are convex and compact sets and the colouring rule says if \(c : X \to C\) is the colouring function then the colour \(c(x)\) must lie (\(m\) a.e.) in \(F(x, c(g_1(x)), \ldots, c(g_k(x)))\) for a non-empty upper-semi-continuous convex-valued correspondence \(F\) defined on \(X \times C^k\)? The answer is yes, and we present such an example. We show that this result is robust, including that any colouring that approximates the correspondence by \(\epsilon\) for small enough positive \(\epsilon\) also cannot be measurable in the same finitely additive way. Because non-empty upper-semi-continuous convex-valued correspondences on Euclidean space can be approximated by continuous functions, there are paradoxical colouring rules that are defined by continuous functions.

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*dedicated to Robert Aumann and Jan Mycielski
1 Introduction

In [7], we introduced colouring rules. We demonstrated several paradoxical colouring rules and proved that if there are finitely many colour classes and the measure preserving transformations are invertible, then any colouring of a paradoxical colouring rule has colour classes that jointly, with the measure preserving transformations and the Borel sets, define a measurably $G$-paradoxical decomposition (for the group $G$ generated by the measure preserving transformations), by which we mean the existence of two measurable sets of different measures that are $G$-equidecomposable (see [7], Thm. 1).

In the conclusion of [7], we asked whether a colouring rule could be paradoxical if the colour classes belonged to a finite dimensional convex set and the colouring rule was defined by an upper-semi-continuous convex-valued non-empty correspondence, as described above. We call such colouring rules probabilistic colouring rules. This means, among other things, that the choosing of colours could be according to a maximisation or minimisation of a continuous and affine evaluation of options, with indifference between two options implying indifference between all of their convex combinations.

Our main inspiration is the question whether measure theoretic paradoxes, such as the Banach Tarski Paradox, have any applications to areas beyond mathematics, such as physics or economics. A colouring rule could represent natural forces, and the lack of any measurable solution could represent a radical inability to predict their behaviour. A particular inspiration is the widely held belief in economic theory that although one cannot always accomplish optimisation goals through behaviour that is measurable with respect to a countably additive measure, one can do so with some finitely additive option. The problem with this belief is that there may be knowledge structures to the optimisation that cannot be altered when extending to a finitely additive measure. If those knowledge structures are defined through the use of ergodic operators, measure invariance of those ergodic operators may be required.

The Brouwer Fixed Point Theorem is relevant to probabilistic colouring rules. If $X$ is compact and the correspondence is independent of the location of $x$, meaning that the correspondence $F$ is defined entirely on $g_1(x), \ldots, g_k(x)$, then the fixed point theorem shows there exists a constant colouring function satisfying the colouring rule, hence it cannot be paradoxical. This is done by mapping $C$ to $k$ copies of $C$ through $k$ copies of the identity, and then following the colouring rule back down to $C$ via the correspondence $F$. Therefore we have to consider colouring rules that are dependent on the location in the space.

In Simon and Tomkowicz [ST] we demonstrated a probabilistic colouring rule with a one dimensional continuum of colours such that after the correspondence is approximated by any $\epsilon$ for small enough positive $\epsilon$ the colouring rule still had no Borel measurable solution. This colouring rule was paradoxical only in the sigma-additive, not in the finitely additive, sense.

In the next section we describe the probabilistic colouring rule and show that it is paradoxical. In the third section we look at approximating the colouring rule and apply it to economics. In conclusion we consider related problems.
2 A Probabilistic Paradoxical Colouring Rule

Let $T_1$ and $T_2$ be two non-invertible generators (each generating a semi-group isomorphic to $\mathbb{N}$). We assume that $T_1^3T_2 = T_2T_1^3$ and there are no other relations. Let $G$ be the semi-group generated by $T_1$ and $T_2$. Let $X$ be the set $\{0,1\}^G$. We extend $X$ to $X' = X \times \{a,b,c\}$ and let the symmetric group $S_3$ act on the three elements $\{a,b,c\}$. We assume that $S_3$ commutes with $G$ and define $G'$ to be the semi group so generated. We need the addition of $S_3$ to define the colouring rule according to measure preserving transformations in $G'$. However it is mostly the colouring of $X \times \{a\}$ that matters.

For any $x \in X$ and $g \in G$, $x^g$ stands for the $g$ coordinate in $x$. With $e$ the identity in $G$, the $e$ coordinate of $x$ is $x^e$. There is a canonical right semi-group action on $X$, namely $g(x)^h = x^{gh}$ for every $g,h \in G$. We use the canonical product topology on $X$. For every cylinder determined by particular choices of $\{0, 1\}$ we assign the probability $\left(\frac{1}{2}\right)^k$ where $k$ is the number of those choices determining the cylinder. With this Borel probability measure the semi-group $G$ act measure preserving on $X$. Any semi-group element acts measure preserving on any cylinder, due to the cancellation law, and this can be extended to any Borel set through approximation via cylinders. Likewise we give $X'$ the Borel probability measure where each element in $\{a,b,c\}$ is given equal probability when paired with a Borel set in $X$. The probability of $A \times \{a,b,c\}$ in $X'$ is given the same probability at that given to $A$ in $X$. In this way $G'$ acts measure preserving on $X'$. With $m$ the canonical Borel measure of $X$, let $m'$ be its extension to $X'$.

The subset of $X$ where $g \neq h$ implies that $gx \neq hx$ is of Borel measure one. This follows from the fact that the semi-group is countable. Without loss of generality, we will be interested only in this subset, and we ignore the set of measure 0 where this doesn’t hold.

The set of colours $C$ is $\Delta(\{1, 2, 3\}) := \{p \mid \forall i, p_i \geq 0, p_1 + p_2 + p_3 = 1\}$ where $\delta_i$ is perceived to be all weight to the colour $c_i$. We represent the extremal colours $c_i$ modulo 3, with $i = 1, 2, 3$ rather than 0, 1, 2.

Now we show how to colour each $x \in X'$ with a point in the simplex $\Delta(\{1, 2, 3\})$, with $p = (p_1, p_2, p_3) \in \Delta(\{1, 2, 3\})$ standing for the weights given to the three colours. We colour according to the optimisation of a continuous function, and the resulting optimal solutions are represented by a correspondence $F$, defined on $X \times C^2$, where each $x \in X$ has two descendants $T_1 x$ and $T_2 x$. The continuous functions are defined by matrix multiplications.

For every $x \in X$ we define a matrix $A_x = \begin{pmatrix} 1 & r_{1,2}^x & r_{1,3}^x \\ r_{2,1}^x & 1 & r_{2,3}^x \\ r_{3,1}^x & r_{3,2}^x & 1 \end{pmatrix}$ such that $|r_{i,j}^x| \leq \frac{1}{100}$ for every choice of $x$ and $i,j$, The $r_{i,j}^x$ are continuous functions of $x$ in $X$, and are chosen such that for every $p \in \Delta(\{1, 2, 3\})$ the set of $x$ such that only one row maximises $A_x p$ is a set of Borel measure one. The following is one way to do that. There are six entries of the $r_{i,j}^x$ to determine. We place the elements of $G$ into six infinite ordered collections. For each $l = 1, 2, \ldots$ and $g_l$ the $l$th group elements corresponding $r_{i,j}^x$ we define $r_{i,j}^x := \frac{1}{100} \sum_{l=1}^{\infty} 2^{-l} x^{g_l}$, where the structure of $X$ requires that $x^g \in \{0, 1\}$ for all $g \in G$. 

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We colour a point \((x, b)\) by any convex combination of the \(c_i\) with the \(i = 1, 2, 3\) whose rows maximise \(A_{T_2T_1T_2(x)p}\), where \(p\) is the colour given to \((T_1x, a)\).

We colour a point \((x, c)\) by any convex combination of the \(c_i\) with the \(i = 1, 2, 3\) whose rows maximise \(A_{T_1T_2T_1(c)p}\), where \(p\) is the colour given to \((T_2x, a)\).

The complex part of the rule is how to colour a point \((x, a)\).

We define two three dimensional matrices \(B_0\) and \(B_1\). The matrix \(B_0\) is used when \(x^e = 0\) and the matrix \(B_1\) is used when \(x^e = 1\). Both matrices have entries \((b_{i,j,k})\) only in \(\{0, 1\}\) such that for each pair of columns \(j, k\) there is only one non-zero \(b_{i,j,k}\) and it is equal to 1. There are two sets of columns and one row; the columns \(j\) and \(k\) correspond to the weights given to the colours \(c_j\) and \(c_k\) of \((x, b)\) and \((x, c)\) respectively.

A row \(i = 1, 2, 3\) of \(B_1\) or \(B_0\) is evaluated in the following way. If \(p \in \Delta(\{1, 2, 3\})\) is the colour given to \((x, b)\) and \(q \in \Delta(\{1, 2, 3\})\) is the colour given to \((x, c)\) then the \(i\)th row is given the value \(\sum_{j=1}^{3} \sum_{k=1}^{3} p_j q_k b_{i,j,k}\). The colouring rule requires that any convex combination of rows is chosen that maximise this row evaluation.

To define these matrices, we determine when the \(b_{i,j,k}\) entry is 1. The \(j\)th column represents the an extremal colour of \((x, b)\) and the \(k\)th column an extremal colour of \((x, c)\).

The matrix \(B_1\), the case of \(x^e = 1\), is easy to define. The \(b_{i,j,k}\) entry is 1 if and only if \(i = j + 1\).

The matrix \(B_0\), the case of \(x^e = 0\), is more complex. If \(k \neq 1\) and \(j \neq 3\), then the \(b_{i,j,k}\) entry is 1 if and only if \(i = j\).

If \(k \neq 1\), then the \(b_{i,3,k}\) entry is 1 if and only if \(i = 1\).

If \(j \neq 3\), then the \(b_{i,j,1}\) entry is 1 if and only if \(i = j + 1\).

The \(b_{i,3,1}\) entry is 1 if and only if \(i = 3\).

This completes the definition of the correspondence \(F\). We will also refer to \(F\) as the colouring rule. Notice that when \(x^e = 0\) then the matrix \(B_0\) has similarity to the conditions of the Hausdorff paradox, requiring that if the colour of \(x\) is an advancement by one on the colour of \(T_1x\) and \(x^e = 0\) then the two points \(x\) and \(T_2x\) are coloured differently, one of these two points is coloured \(\{c_1\}\) and the other takes a colour in \(\{c_2, c_3\}\).

When \((x, a), (x, b)\) or \((x, c)\) gives all weight to an extremal colour, namely \(\delta_{c_i}\) for some \(i \in \{1, 2, 3\}\), then it is called pure.

A colouring \(c : X' \to C\) satisfies the colouring rule \(F\) if the rule holds almost everywhere with respect to the probability distribution \(m'\).

**Lemma 1:** A colouring that satisfies the colouring rule \(F\) is pure almost everywhere with respect to the Borel measure \(m'\).

**Proof:** There is no relation between \(T_2T_1T_2\) and \(T_1\), likewise between \(T_1T_2T_1\) and \(T_2\). Therefore for every choice of \(x\) the matrices at \((T_1^{-1}x, b)\) at \((T_2^{-1}x, c)\) defining the colouring rule \(F\) are the matrices \(A_y\) for all \(y \in X\). The conditional probability on those matrices is the same as the distribution on \(X\). It follows by the construction of these matrices that regardless of the colour \(p\) at \((x, a)\) the set of colours at \((T_1^{-1}x, b)\) and \((T_2^{-1}x, c)\) that are not pure with respect to \(m'\) is a subset of conditional measure 0. By the definition of conditional probability, the conclusion follows for all \((x, b)\) and \((x, c)\). And therefore it follows for \((x, a)\) also.  

q.e.d.
Lemma 1 allows us to perceive a colouring of $X'$ as being primarily a colouring of $X \times \{a\}$. With purity, the colours of $(x, b)$ and $(x, c)$ are merely conveying to $(x, a)$ the colours of $(T_1x, a)$ and $(T_2x, b)$ in a way that those two colours are determining the colour of $(x, a)$. From now on, by a colouring of $X$ we mean a colouring of $X \times \{a\}$, where by the colour for $x$ we mean the colour for $(x, a)$. If a point is coloured purely with $\delta_i$ we will also write that it is coloured with $c_i$. Our main aim is to prove the following theorem:

**Theorem 1:** For any finitely additive $G$-invariant measure $\mu$ on $X'$ extending $m'$ there exists no colouring $c : X' \to C$ that is $\mu$ measurable and satisfies the colouring rule $F$.

**Definitions:** Semi-group elements $g_1, g_2, \ldots, g_k \in G$ are called independent if there are no relations between them. Two points $x, x'$ are called twins if $T_1x = T_1x'$ and $T_2x = T_2x'$, meaning that the differ only by $x^e \neq x'^e$. A point $x \in X$ is coloured randomly if $x$, the twin of $x$, $T_1x$, and $T_2x$ are all pure and $x$ and its twin are coloured differently (meaning that for the twins $x, x'$ the colour of $T_1x$ is advanced by one to define the colour of $x$ where $x^e = 1$ and the colour of $T_1x$ is not advanced to define the colour of $x'$ where $(x')^e = 0$). A colouring that is measurable with respect to any $G$-invariant finitely additive measure $\mu$ is called measurable.

**Lemma 2:** If there is a positive measure of points that are coloured randomly for a measurable colouring satisfying the colouring rule $F$, and $g_1, g_2, \ldots, g_k$ are independent, then the probability of $\{x \mid g_i x \text{ is coloured } c_n, i = 1, 2, \ldots, k\}$ is the product $\prod_{i=1}^k q_n$, where $q_n$ is the probability of $\{x \mid x \text{ is coloured } c_n\}$ for $n = 1, 2, 3$.

**Proof:** First we show that if there is a positive probability of random colouring, then from the stochastic matrix representing the transition of the distribution of colours of $x$ to the colouring of $T_1^{-1}x$ there is a unique eigenvector in $\Delta(\{1, 2, 3\})$ with the eigenvalue 1 and the other eigenspaces correspond to eigenvalues with norms less than 1. Since by measure invariance we can assume that the probability distribution of the colours is the same at $x$ and $T_1^{-1}x$ for the collection of all $x \in X$, this implies that the distribution is determined by this eigenvector.

Let $p, q, r$ be half the probabilities for the random colouring conditioned on the colours $c_1, c_2, c_3$ respectively, with the half referring to the non-advancement of the colours which happens when $x^e = 0$. We assume that at least one of the $p, q, r$ are positive and none are greater than $\frac{1}{2}$. The stochastic matrix in question is

\[
\begin{pmatrix}
    p & 0 & 1 - r \\
    1 - p & q & 0 \\
    0 & 1 - q & r
\end{pmatrix}.
\]

It has the characteristic polynomial $(p - x)(q - x)(r - x) + (1 - r)(1 - q)(1 - p) = (1 - r)(1 - q)(1 - p) + pq r - (qr + qp + pr) x + (p + q + r) x^2 - x^3$. The degree one polynomial $1 - x$ divides this characteristic polynomial, leaving the second degree polynomial $1 - r - p - q + rp + rq + qp + (1 - r - p - q) x + x^2$ as a factor. Letting $b = 1 - r - p - q$ and $d = rp + rq + qp$ we have the roots $\frac{-b + \sqrt{b^2 - 4(b + d)}}{2}$ and $\frac{-b - \sqrt{b^2 - 4(b + d)}}{2}$. Assuming that $b^2 - 4(b + d)$ is not positive, the norm squared of these roots is equal to $b + d$, which is less than 1 because one of $p, q, r$ must be
positive and if, for example, \( r > 0 \) then \( r > rp \). \( b^2 - 4b \) cannot be positive if \( b \) is not negative. Since \( b \geq -\frac{1}{2} \) and \( \frac{1}{2} + \sqrt{2 + \frac{1}{4}} = 2 \), the only way for the norm of \(-\frac{b - \sqrt{b^2 - 4b}}{2} \) or \(-\frac{b + \sqrt{b^2 - 4b}}{2} \) to reach 1 is if \( b = -\frac{1}{2} \) and therefore \( r = p = q = \frac{1}{2} \). But then the roots are really \( -\frac{\sqrt{2}}{2} \) and \( -\frac{\sqrt{2}}{2} \), with norms of \( \frac{1}{2} \).

Due to the lack of any relation between the \( g_i \), with respect to the Borel probability distribution whether the \((g_ix)^c\) are equal to 0 or 1 are independent choices over all the \( x \in X \), hence also with any finitely additive measure extending the Borel measure. Assuming invariance of joint distributions of colour combinations before and after applying \( T_1^{-1} \), using induction on the number of the \( g_i \), and that the independence of the \( g_i \) implies the independence of the \( T_1^ng_i \) for all positive \( n \), we complete the proof with the following claim:

**Claim:** Assume that there are two one-stage stochastic processes on two finite sets \( S \) and \( T \) respectively and a stochastic process defined on \( S \times T \) such that the transitions are defined independently by transitions on \( S \) and \( T \). Assume for each of the \( S \) and \( T \) processes that there is only one unique invariant distribution/eigenvector corresponding to an eigenvalue of norm 1. Furthermore assume that for each \( s \in S \) and \( t \in T \) that there is a positive probability that there is a transition to something other than \( s \) and \( t \) respectively. Then there is one unique invariant joint distribution on \( S \times T \) defined by the independent distributions on \( S \) and \( T \) respectively.

**Proof of Claim:** Let \((s,t)\) be any pair of states in \( S \times T \); we want to prove that any invariant probability for \((s,t)\) is the same as \( ab \) where \( a \) is the invariant probability for \( s \) and \( b \) is the invariant probability for \( t \). Let \( q \) be an invariant probability for \((s,t)\); we want to show that \( q = ab \). Let \( l_a \) be the probability, conditioned on the state being at \( a \), of leaving \( a \) on the next stage. Let \( l_b \) be the same probability for the state \( b \). Let \( r_a \) be the probability of not being at \( a \) and returning to \( a \) on the next stage. Let \( r_b \) be the same probability for the state \( b \). We can calculate \( q \) by \( q = q\cdot l_aq - l_bq + l_alq + (1 - a)r_a(1 - lb)b + (1 - b)rb(1 - la)a + (1 - b)rb(1 - a)r_a \). If these distributions on \( S \) and \( T \) are invariant independently, then \( q = ab \) is another solution for an invariant probability for \((s,t)\). Therefore we can also write \( ab = ab - l_aab - lb ab + l_alb + (1 - a)r_a(1 - lb)b + (1 - b)rb(1 - la)a + (1 - b)rb(1 - a)r_a \), the same formula but with \( q \) replaced by \( ab \). But then we can write \( 0 = q(l_a + lb - la lb) = ab(l_a + la - la lb) \). With \( l_a > 0 \) and \( lb > 0 \) we have \( l_a + la - la lb > 0 \) and \( q = ab \). q.e.d.

Notice that if \( h = jg \) for semi-group elements \( g, h, j \) then the choice of \( gx \) for any \( x \in X \) will determine the \( hx \), including the value of \((hx)^c\), and therefore the \((gx)^c\) and \((hx)^c\) are dependent for the various choices of \( x \in X \) in the maximal way that two variables can be dependent. If \( kh = jg \) for some semi-group elements \( g, h, j, k \), then starting at \( kh(x) = jg(x) \) the dependence from \( h \) and \( jg \) can have residual influences on the joint colour distribution at the set \( \{(gx, hx) \mid x \in X \} \) before they reach their limit distributions. This residual dependence implies that the joint colour distribution at \( \{(T_1x, T_2x) \mid x \in X \} \) should have some dependence coming from \( T_1^2T_2 = T_2T_1^2 \). However from a different perspective they should be independent! This contradiction drives our proof of Theorem 1.

**Lemma 3:** There is no measurable colouring satisfying the colouring rule \( F \)
such that there is a positive measure of points that are coloured randomly.

**Proof:** Let $C_z$ be the stochastic matrix $C_z = \begin{pmatrix} \frac{1-z}{2} & 0 & \frac{2-z}{2} \\ \frac{z+1}{2} & \frac{1-z}{2} & 0 \\ 0 & \frac{z+1}{2} & \frac{z}{2} \end{pmatrix}$. Given an independent distribution of colours for the pair $T_1x$ and $T_2x$, the matrix determines the distribution of colours at $x$ where $z$ is the probability for the colour $c_1$ at $T_2x$.

First consider the pair of independent elements $T_2T_1$ and $T_2^2$ and let $(y_1, y_2, y_3)$ be the global probabilities for the colours $c_1, c_2, c_3$, respectively. From Lemma 2 the joint distribution of the colours of $T_2T_1x$ and $T_2^2x$ are that determined by the products of the $y_i$, meaning that the probability of $T_2T_1x$ and $T_2^2x$ coloured $c_i$ and $c_j$ respectively are $y_i \cdot y_j$. But as the colouring is measurable, using that $T_2$ is measure preserving, we have to assume that the probability of the $T_1x$ and $T_2x$ coloured $c_i$ and $c_j$ respectively is also $y_i \cdot y_j$. Applying the matrix $C_z$ with $z = y_1$ for $C_{y_1}(y_1, y_2, y_3)^t = (y_1, y_2, y_3)^t$ we get the following three equations with three variables:

\[
\begin{align*}
y_1 &= \frac{1}{2}(y_3) + \frac{1}{2}((1 - y_1)y_1 + (1 - y_1)y_3) \\
y_2 &= \frac{1}{2}(y_1) + \frac{1}{2}((1 - y_1)y_2 + y_1^2) \\
y_3 &= \frac{1}{2}(y_2) + \frac{1}{2}(y_1y_3 + y_1y_2).
\end{align*}
\]

This solves to $y_1 = 3 - \sqrt{7}$, $y_2 = 3 - \sqrt{7}$, and $y_3 = 2\sqrt{7} - 5$. These are the global probabilities for the colours $c_1, c_2, c_3$ respectively. Approximately this is the triple (.35425, .35425, .2915). The reason for a smaller probability for the colour $c_3$ results from the tendency to move away from this colour with greater probability than toward it.

Second consider the seven semi-group elements.

\[
\begin{align*}
g_1 &:= T_2^4T_1T_2^3, \\
g_2 &:= T_2^4T_1T_2^2T_1T_2, \\
g_3 &:= T_2^4T_1T_2T_1^2T_2, \\
g_4 &:= T_2^4T_1T_2^2T_1^3, \\
g_5 &:= T_2^4T_1T_2T_1^4, \\
g_6 &:= T_2^4T_1T_2^2T_1^2, \\
g_7 &:= T_2^4T_1T_2^2T_1.
\end{align*}
\]

We show that they are independent, meaning that there is no relations between them. Make the revision that $T_1$ is invertible and $T_1^3$ is the identity. If the resulting elements still have no relations between them, there was no relation before this revision. We can now proceed with the assumption that there are no relations other than $T_1^3 = e$.

We have the reductions

\[
h_1 := T_2^4T_1T_2^3,
\]

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\[ h_2 := T_2^4 T_1 T_2^2 T_1 T_2, \]
\[ h_3 := T_2^4 T_1 T_2^2 T_1 T_2, \]
\[ h_4 := T_2^4 T_1^2 T_2, \]
\[ h_5 := T_2^2 T_1 T_2^3 T_1, \]
\[ h_6 := T_2^2 T_1 T_2^3 T_1, \]
\[ h_7 := T_2^2 T_1^2 T_2 T_1. \]

Notice that any combinations of the \( h_i \) are separated by \( T_2^4 \) or a higher power of \( T_2 \), so that we can identify when one element ends and the next begins. Removing these bookends of \( T_2^4, T_2^3, T_2^2, \) or \( T_2^2 \), we are left with the interiors \( T_1, T_1 T_2^2 T_1, T_1 T_2^2 T_1^2, T_1 T_2 T_1 \). All but the \( T_1 T_2 T_1 \) (from \( h_5 \)) are repeated twice. The two uses of \( T_1 \) are distinguished by the powers of \( T_2 \) succeeding them (\( T_2^2 \) vs \( T_2^3 \)). Likewise the two uses of \( T_1 T_2^2 T_1 \) can be distinguished by the power of \( T_2 \) succeeding them (\( T_2^2 \) vs \( T_2^3 \)) and the same is true of \( T_1 T_2^2 T_1^2 \) (\( T_2^3 \) vs \( T_2^4 \)). It follows that all seven elements \( g_1, \ldots, g_7 \) also must be independent.

By Lemma 2 the probability of \{ \( x \mid g_i x \) coloured \( c_{n_i} \) \} for any choice of \( n_1, \ldots, n_7 \) is equal to the product \( \prod_{i=1}^{7} y_{n_i} \). Due to the measure preserving property of \( T_2^4 T_1 T_2 \), we can assume that the probability of \{ \( x \mid g'_i x \) coloured \( c_{n_i} \) \} for any choice of \( n_1, \ldots, n_7 \) is equal to the product \( \prod_{i=1}^{7} y_{n_i} \) for the elements

\[ g'_1 := T_2^2, \]
\[ g'_2 := T_2 T_1 T_2, \]
\[ g'_3 := T_2 T_1^2 T_2, \]
\[ g'_4 := T_2 T_1^3, \]
\[ g'_5 := T_1^4, \]
\[ g'_6 := T_2 T_1^2, \]
\[ g'_7 := T_2 T_1. \]

Now consider the defining condition that \( g'_i x = T_1^3 T_2 x = T_1 T_2^3 x \) for all \( x \). For each of the three colour choices at \( T_1^3 T_2 x \) we determine the distribution on the colours of \( x \) by building up to a joint distribution for \( T_1 x, T_2 x, \) and \( x \) through the two pathways, the left pathway going through three applications of \( T_1^{-1} \) and the right pathway going first through \( T_2^{-1} \) followed by two applications of \( T_1^{-1} \). Due to the above mutual independence of colour distributions of the \( g'_i x \) and their locations in relation to the two pathways we can use the matrices \( C_z \) to determine these distributions, including the last step of determining the colour distribution of \( x \), conditioned on the colour of \( T_1^3 T_2 x \). There are three matrices most relevant to our calculations, \( C_1, C_0, \) and \( C_{g_1} \).

The distribution of colours for \( T_2 x \), conditioned on a choice of colour for \( T_1^3 T_2 \) is determined by

\[
C^{y_1}_{g_1} = \begin{pmatrix}
\frac{1}{8} (50\sqrt{7} - 129) & \frac{1}{8} (23 - 8\sqrt{7}) & \frac{1}{8} (19\sqrt{7} - 49) \\
\frac{1}{8} (216 - 81\sqrt{7}) & \frac{1}{8} (50\sqrt{7} - 129) & \frac{1}{8} (23 - 8\sqrt{7}) \\
\frac{1}{8} (31\sqrt{7} - 79) & \frac{1}{8} (91 - 34\sqrt{7}) & \frac{1}{8} (11 - 3\sqrt{7})
\end{pmatrix}.
\]
To determine the distribution of colours for $T_1x$ we need to calculate $C_{y_1}^2C_0(y_1, y_2, y_3)$ and $C_{y_1}^2C_1(y_1, y_2, y_3)$, the former for what happens when $T_1^3T_2x$ is coloured $c_2$ or $c_3$ and the latter for $T_1^3T_2x$ coloured $c_1$. For the former we get

\[
\begin{pmatrix}
\frac{1}{4}(11 - 4\sqrt{7}) & \frac{1}{4}(5\sqrt{7} - 11) & \frac{1}{4}(\sqrt{7} - 1) \\
\frac{1}{4}(6\sqrt{7} - 15) & \frac{1}{4}(11 - 4\sqrt{7}) & \frac{1}{4}(5\sqrt{7} - 11) \\
\frac{1}{4}(23 - 8\sqrt{7}) & \frac{1}{4}(4 - \sqrt{7}) & \frac{1}{2}(8 - 3\sqrt{7})
\end{pmatrix}
\begin{pmatrix}
\frac{3}{2}\sqrt{7} - \frac{1}{2} \\
\frac{3}{2}\sqrt{7} - \frac{1}{2} \\
\frac{3}{2}\sqrt{7} - \frac{1}{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{8}(117\sqrt{7} - 307), \frac{1}{8}(258 - 97\sqrt{7}), \frac{1}{8}(77\sqrt{7} - 201).
\end{pmatrix}
\]

For the latter we get

\[
\begin{pmatrix}
\frac{1}{4}(11 - 4\sqrt{7}) & \frac{1}{4}(5\sqrt{7} - 11) & \frac{1}{4}(\sqrt{7} - 1) \\
\frac{1}{4}(6\sqrt{7} - 15) & \frac{1}{4}(11 - 4\sqrt{7}) & \frac{1}{4}(5\sqrt{7} - 11) \\
\frac{1}{4}(23 - 8\sqrt{7}) & \frac{1}{4}(4 - \sqrt{7}) & \frac{1}{2}(8 - 3\sqrt{7})
\end{pmatrix}
\begin{pmatrix}
\frac{1}{4}(5\sqrt{7} - 2) \\
\frac{3}{2}\sqrt{7} - \frac{1}{2} \\
\frac{1}{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{8}(95\sqrt{7} - 248), \frac{1}{8}(429 - 161\sqrt{7}), \frac{1}{8}(66\sqrt{7} - 173).
\end{pmatrix}
\]

For each of the three possibilities, $T_1^3T_2x = T_2T_1^3x$ coloured $c_1$, $c_2$, or $c_3$, we do the calculations for the three colours at $T_1^3T_2x$ separately and recombine the results according to $(y_1, y_2, y_3)$.

For $T_1^3T_2x$ coloured $c_1$, we have to calculate $C_{\frac{1}{8}(50\sqrt{7} - 129)}(\frac{1}{8}(95\sqrt{7} - 248), \frac{1}{8}(429 - 161\sqrt{7}), \frac{1}{8}(66\sqrt{7} - 173)) = \frac{1}{128}(43635\sqrt{7} - 115411), \frac{1}{128}(178381 - 67402\sqrt{7}), \frac{1}{128}(23767\sqrt{7} - 62842))$

For $T_1^3T_2x$ coloured $c_2$, we have to calculate $C_{\frac{1}{8}(23 - 8\sqrt{7})}(\frac{1}{8}(117\sqrt{7} - 307), \frac{1}{8}(258 - 97\sqrt{7}), \frac{1}{8}(77\sqrt{7} - 201)) = \frac{1}{32}(9856 - 3721\sqrt{7}), \frac{1}{64}(13429\sqrt{7} - 35509), \frac{1}{64}(15861 - 5987\sqrt{7}))$.

For $T_1^3T_2x$ coloured $c_3$, we have to calculate $C_{\frac{1}{8}(19\sqrt{7} - 49)}(\frac{1}{8}(117\sqrt{7} - 307), \frac{1}{4}(258 - 97\sqrt{7}), \frac{1}{8}(77\sqrt{7} - 201)) = \frac{1}{16}(57 - 19\sqrt{7}), \frac{1}{16}(57 - 19\sqrt{7}), \frac{1}{16}(57 - 19\sqrt{7})$.
\[
\left( \frac{1}{64}(10663\sqrt{7} - 28183), \frac{1}{64}(41681 - 15746\sqrt{7}), \frac{1}{64}(5083\sqrt{7} - 13434) \right).
\]

Combining these three vectors according to the distribution \((3-\sqrt{7}, 3-\sqrt{7}, 2\sqrt{7}-5)\) of the three colours at \(T_1^3T_2x\) we get a different distribution for the colours at \(x\), namely \(\left( \frac{1}{4}(75588 - 28561\sqrt{7}), \frac{1}{4}(95189\sqrt{7} - 251801), \frac{1}{4}(100753 - 38067\sqrt{7}) \right)\), a contradiction. In decimals this corresponds approximately to \((.35464, .35468, .29050)\), (different from the original \((.35425, .35425, .2915) = (3 - \sqrt{7}, 3 - \sqrt{7}, 2\sqrt{7} - 5))\).

q.e.d.

The source of the discrepancy in the final distribution at the top comes from different starting distributions at \(T_2T_1^3x\) and \(T_1^3T_2x\). If there was no commuting of \(T_1^3\) and \(T_2\), there would be two different points and the probability of a start of \(c_i\) at \(T_1^3T_2x\) and \(c_j\) at \(T_2T_1^3x\) would be \(y_iy_j\) with \((y_1, y_2, y_3) = (3 - \sqrt{7}, 3 - \sqrt{7}, 2\sqrt{7} - 5)\). But with the commuting there are no mixed starts.

Finally with the above lemmas, we can prove Theorem 1.

**Proof of Theorem 1:** By the three lemmata, a measurable colouring satisfying the colouring rule must have no positive subset of points coloured randomly. That means, with any probability approaching 1, that the colours cycle through with \(x\) coloured \(c_{i+1}\) whenever \(T_1\) is coloured \(c_i\). So with the measure preserving application of \(T_1\) we see that \(\frac{1}{3}\) of the space is coloured \(c_1\). On the other hand, by \(T_2\) being measure preserving, as with the Hausdorff paradox, this implies that the probability of the space coloured \(c_1\) must be arbitrarily close to \(\frac{2}{3}\), a contradiction. q.e.d.

There is of course a non-measurable colouring of the space. Start at some \(x\). Let the colours cycle through \(c_i\) by the repeated application of \(T_1^{-1}\), increasing the colour by one, and \(T_1\), decreasing the colour by one. Choose some \(y\) already coloured this way and colour \(T_2y\) so that that \(T_2y\) is coloured \(c_1\) if and only if \(y\) is not coloured \(c_1\). Extend this choice for \(T_2y\) with the repeated application of \(T_1^{-1}\) and \(T_1\) to \(T_2y\). Notice that the commuting of \(T_1^3\) and \(T_2\) does not get in the way of this pattern. This can be continued, but there is a general solution to the existence of a non-measurable colouring which includes this type of colouring. This general solution is presented in the next theorem. It demonstrates the other half of the argument that \(F\) is a paradoxical colouring rule, that the colouring rule \(F\) can be satisfied.

Let \(X^*\) be the \(x\) in \(X\) where \(g \neq h\) implies that \(gx \neq hx\).

**Theorem 2:** There are pure colourings of \(X^*\) that satisfy the colouring rule \(F\).

**Proof:** Let a subset \(A\) of \(X\) be called **closed** if whenever \(T(x) \in A\) and \(T_2(x) \in A\) then also \(x\) is in \(A\). Let \(\overline{A}\) stand for the closure of \(A\). A subset \(A\) of \(X\) is called **pyramidic** if whenever \(x \in A\) then the \(G\) orbit \(Gx\) is in \(A\). Notice that the closure of a pyramidic set is pyramidic.

For any pyramidic set \(B\) and any pure colouring of \(B\) consistent with the colouring rule there is a deterministic way to colour the closure \(\overline{B}\) according to the colouring rule. We define a partially ordered set on the pairs \((B, c)\) where \(B\) is a pyramidic and closed set and \(c\) is a colouring of \(B\) according to the rule. We say that \((B, c) \geq (B', c')\) if \(B\) contains \(B'\) and \(c\) restricted to \(B'\) is \(c'\).
With Zorn’s lemma there is a maximal element to any tower of the partial order. We show that it is not possible for \((B, c)\) to be maximal however there is some \(x \in X^*\) such that \(x \notin B\).

Consider the orbit \(Gx\) and the sequence \(A_l\) where \(A_l\) is the subset of \(Gx\) such that \(y \in A_l\) if and only if \(y = gx\) with \(g\) a word of length \(l\) and \(y \notin B\). In any way we colour \(A_l\), the colouring rule extends to a colouring of the closure of \(Gx \cup B\) that follows the rule and does not change the colours of \(B\). Let \(f_l\) be a sequence of such colourings of \(Gx \cup B\). As \(Gx\) is a countable set, we can find a point-wise convergent subsequence of the \(f_l\) defining a colouring function \(f\) on \(Gx \cup B\). We then extend this function \(f\) to a pure colouring on the closure of \(Gx \cup B\). q.e.d.

3 Optimality, Stability, and a Bayesian game

The colouring rule \(F\) is already formulated as a problem of local optimisation according to an objective function. We can relax the rules, so that for some given \(\delta > 0\) it is required that the colours chosen at all points are within \(\delta\) of optimality. We call this point wise \(\delta\)-optimality. But that is only one concept of approximate optimality, that at each individual point there is no gain by more than \(\delta\) through a different choice of colour. We seek a slightly broader concept. For each \(x \in X\) let \(t(x)\) be the possible improvement in the objective function at \(x\), keeping the colouring for all other \(y \neq x\) fixed. Let \(\mu\) be a \(G\)-invariant finitely additive extension. A colouring is \(\gamma\)-stable if the \(\mu\)-expectation of \(t(x)\) is no more than \(\gamma \geq 0\), meaning that there is no finite disjoint collection \(A_1, \ldots, A_n\) of \(\mu\) measurable sets such that the objection function can be improved by at least \(t_i\) at all points in \(A_i\) and \(\sum_{i=1}^{*} \mu(A_i) t_i\) is greater than \(\gamma\).

Another way of understanding \(\gamma\)-stability is that \(X\) is an uncountable space of human society or molecules, and the solution is \(\gamma\)-stable if the gains from the individual deviations do not add up to an expectation of \(\gamma\).

There are two ways that a measurable colouring must obey \(\gamma\)-stability. First, the set where there is significant divergence from optimality must be small. Second, where divergence from optimality exists in a subset of large measure, that divergence must be small (the first concept we presented). That can be formalised in the following way: if a colouring is \(\epsilon \cdot \delta\)-stable, then the subset where it diverges from optimality by more than \(\delta\) cannot be of measure more than \(\epsilon\).

By the continuity of Borel measure, with \(p \in \Delta(\{1, 2, 3\})\) fixed and as \(\delta\) goes down to 0 the set of \(x\) where two rows of \(A_x\) have expectations within \(\delta\) of the optimal choice falls in Borel measure to 0. As the finitely additive measure must be an extension of the Borel measure, the same is true for the finitely additive measure. This allows the following corollary.

**Corollary:** For small enough \(\gamma\) there is no finitely additive \(G\)-invariant measure extending the Borel measure with a \(\gamma\)-stable measurable colouring for the colouring rule \(F\), and hence there is a paradoxical colouring rule defined by a continuous function.

**Proof:** Let \(\delta > 0\) be fixed, and consider the subset \(X_\delta\) of \(X\) where only one row is optimal and the other two are not within \(\delta\) of being optimal. As \(\delta\) goes to zero the
probability (according to $m'$) of $X' \setminus X_\delta$ goes to zero. Let $\rho(\delta)$ be a function of $\delta$ such that $\rho(\delta)$ goes to 0 as $\delta$ goes to 0 and $\rho(\delta)$ is greater than the probability of $X' \setminus X_\delta$ for all positive $\delta$. We break the measurable colourings of $X'$ into two cases, (1) those involving at least $\frac{1}{100}$ of the space coloured randomly and (2) those with less than $\frac{1}{5}$ of the space coloured randomly. For every $\epsilon > 0$ require that the probability for not having a pure colour or that the one extremal colour chosen is not optimal is less than $\epsilon$. We have only to determine a positive $\epsilon$ and positive $\delta$ small enough so that $\rho(\delta) < \epsilon$ and $\epsilon$ is small enough to assure, for both cases, a discrepancy in the measure of the set where purity holds and $c_1$ is chosen. For the latter case (2) this is easy, $\epsilon < \frac{1}{1000}$ suffices for a contradiction (as the subset coloured $c_1$ would have to be simultaneously below $\frac{2}{3}$ and above $\frac{2}{7}$). For the former case (1), due to the continuity of the determinant and therefore also the characteristic polynomial, following the argument in the proof of Lemma 2 the $\frac{1}{100}$ of the space using random generation implies the existence of a $d > 0$ and an $\overline{r} > 0$ such that the other eigenvalues (other than 1) in the transition matrix of the colours have norms less than $1 - d$ for all $0 < \epsilon < \overline{r}$. Due to $G$-invariance of any proposed finitely additive measure, and assuming $\epsilon < \overline{r}$, starting at any distribution on colours and applying $T_1^{-1}$ there is uniform minimal convergence rate to a unique invariant distribution, meaning that at each stage the probability for the distribution and the limit distribution is no more than $1 - d$ times what is was on the previous stage. As the transitions from the random process determined by $y^e = 0$ or $y^e = 1$ will remain independent at the various $y = g_i(x)$ for independent elements $g_1, \ldots, g_k$ and they will dominate any distributional dependency from a small set of measure less than $\epsilon$ of not following the colouring rule $F$, we can repeat the arguments of Lemma 2, showing that the joint distributions of the $g_i(x)$ for the seven $g'_i$ are independent in the limit as $\epsilon$ goes to 0. The same can be done for the colour distribution on the whole space, that there is convergence to independent distributions of colours with $T_1$ and $T_2$, using the independence of $T_2T_1$ and $T_2^2$ and measure invariance. Due to the constant $d$ and the existence of unique invariant distributions, as $\epsilon$ goes to 0 the convergence to equalities of the three equations with three variables $y_1, y_2, y_3$ from Lemma 3 implies that the unique invariant distributions on colours converges to the same fixed point distribution $(y_1, y_2, y_3) = (3 - \sqrt{7}, 3 - \sqrt{7}, 2\sqrt{7} - 5)$. Although the calculations from which one determines the distribution of colours at $x$ from the point $T_1^3T_2x = T_2T_1^3x$ don’t hold perfectly due to a subset of size up to $\epsilon$ where the colour rule does not hold, nevertheless with $\epsilon$ small enough they show a persistant $\frac{1}{5000}$ discrepancy in the probability given to the colour $c_1$. Having determined positive $\epsilon < \overline{r}$ and $\rho(\delta) < \epsilon$ small enough for both cases, we choose $\delta$ small enough to guarantee $\rho(\delta) < \epsilon$. The lack of $2\delta\epsilon$-stability follows.

Because the matrices $A_k$ change continuously on the Cantor set $X$ and the two matrices $B_0$ and $B_1$ are defined on disjoint clopen sets (hence together change continuously), and because the $T_1$ and $T_2$ are continuous functions, two points close to each other are close in terms of the consequences of colour choice. With the colouring rule $F$ defined through the optimisation of a continuous function, optimised at the correspondence $F$, and with uniform continuity of the optimising functions (from $X'$ and $C$ compact), any sequence of colouring rule correspondences $F_1, F_2, \ldots$ that approximate the colouring rule $F$ are also approximating its its optimisation.
With the correspondence $F$ non-empty, upper-semi-continuous and convex valued, for every sequence of positive $\gamma_1, \gamma_2, \ldots$ converging to 0 there is a sequence of continuous functions $f_i : X' \times C^2 \to C$ that approximate the correspondence $F$ and the satisfaction of $f_i$ implies point wise $\gamma_i$ optimality with respect to the correspondence $F$. By the above, there is a positive $\overline{\gamma}$ where satisfaction of $f_i$ through measurable colouring is no longer possible when $\gamma_i < \overline{\gamma}$. Therefore we get eventually paradoxical colouring rules defined by continuous functions. q.e.d.

We don’t use the full force of $\gamma$-stability in showing that there are paradoxical colouring rules from continuous functions. However there is one application of $\gamma$-stability that does use that the probability of significant deviation from optimality is limited in probability.

An important part of economic theory is the study of incomplete information. The idea is that some economic agent has some information that the others do not have, and this private information has to be used carefully to that player’s advantage. It could be only one player with private information or it could be all the players. Often this situation can be modelled as a Bayesian game.

The connection to the above colouring rule $F$ is that there is a Bayesian game played on the same probability space $X'$ for which local optimising behaviour by a player is equivalent to satisfaction of the colouring rule $F$ at an appropriate point and equilibrium behaviour is equivalent to satisfaction of the colouring rule $F$ almost everywhere. Furthermore, a $\gamma$-equilibrium of the game is equivalent to a colouring with the $\gamma$-stability property.

Our interest in paradoxical colouring rules came originally from game theory, from the desire to show that all, not just some, equilibria of a game are not measurable. R. Simon [5] showed that there is a Bayesian game which had no Borel measurable equilibria, though it had non-measurable equilibria. The infinite dihedral group, an amenable group, acted on the equilibria in a way that prevented any equilibrium from being measurable. R. Simon and G. Tomkowicz [6] showed that there is a Bayesian game with non-measurable equilibria but no Borel measurable $\epsilon$-equilibrium for small enough positive $\epsilon$. That construction involved the action of a non-amenable semi-group.

A few words are necessary concerning the way a Bayesian game is played. There is a probability space $(\Omega, \mathcal{F}, m)$; nature chooses a point $x$ in the space $\Omega$ according to the probability distribution $m$ defined on a sigma algebra $\mathcal{F}$. There are two approaches to defining the information, strategies and payoffs of a player $j$.

In one approach, for each player $j$ there is a sigma algebra $\mathcal{F}_j$ smaller than $\mathcal{F}$ such that the strategy of player $j$ is a function measurable with respect to $\mathcal{F}_j$.

With the other approach we assume that a player $j$ has a partition $\mathcal{P}_j$ of the space $\Omega$ – if nature chooses some $x \in \Omega$ the player $j$ learns that nature’s choice lies in the $B \in \mathcal{P}_j$ such that $x \in B$. If two points $x, y$ belong to the same partition member $B$ then player $j$ cannot distinguish between $x$ and $y$ and must act identically at $x$ and $y$.

The different approaches result in different ways to understand what is the strategy and payoff of a player.

With the measurable function approach the evaluation of a player’s strategy is determined by the strategies of the other players doing the same, and because all
their sigma algebras are contained in $\mathcal{F}$, the evaluation of the result goes through the probability distribution $m$. We call this the Harsanyi approach.

The partition approach we call the Bayesian approach. With the Bayesian approach each player $j$ has a probability distribution on each set in $\mathcal{P}_j$ and evaluates its actions according to the actions of other players inside of the appropriate member of $\mathcal{P}_j$. Notice that the Bayesian approach doesn’t really need a probability measure $m$ for the whole space, though we include it because we want to link up the two approaches.

Though the two approaches are different, they can be related. If every member of $\mathcal{P}_j$ is in $\mathcal{F}$, we may move from the Bayesian approach to the Harsanyi approach. A sigma algebra $\mathcal{F}_j$ for player $j$ is defined in the canonical way; a set $A \in \mathcal{F}$ is in $\mathcal{F}_j$ if and only for every set $B \in \mathcal{P}_j$ $A \cap B$ is either $B$ or the empty set. If additionally the player $j$’s probability distributions on each member of $\mathcal{P}_j$ form a regular conditional probability with respect to $m$ and the $\mathcal{F}_j$ so generated above, we complete the move to the Harsanyi approach.

Throughout we assume that the payoff of each player is affine with respect to changes in any one player’s strategy, both with the Harsanyi measurable perspective and with the local Bayesian perspective. One can consider more complex payoff structures, but even in the most trivial information structures the existence of an equilibrium is not guaranteed when optimality doesn’t occur in a convex set.

Both approaches to what defines a strategy and a payoff have their strengths and weaknesses. The Bayesian approach is more inclusive because it does not require that strategies are measurable. But in general, the Bayesian approach is more problematic. With the Bayesian approach, an evaluation of an action by player $j$ in some $A \in \mathcal{P}_j$ may be impossible because within $A$ the strategies of the other players may not be measurable with respect to the probability distribution player $j$ has in the set $A$. But if each member in $\mathcal{P}_j$ is finite there is not a problem. Also if there is sufficient structure to the collection $\mathcal{P}_j$, and we have not yet determined the local probability distributions for each player, we can determine a probability distribution on each member $B$ in $\mathcal{P}_j$ as a regular conditional probability with support on $B$ (see [1]), and therefore add a link between the Harsanyi approach to the Bayesian approach.

When the Harsanyi and Bayesian approaches are linked by a regular conditional probability, the importance of measure preserving invariance to finitely additive extensions can be observed. A regular conditional probability on some $A \in \mathcal{P}_j$ must respect (almost everywhere) any measure preserving transformations taking place within the set $A$. By this we mean that if $B$ is a measurable subset in $A \in \mathcal{P}_j$ such that $T^{-1}(B)$ is also contained in $A$, then (almost everywhere) the regular conditional probability for that player at that set must give $B$ and $T^{-1}(B)$ the same measure. If a finitely additive measure on $\Omega$ that extend the original measure $m$ doesn’t respect the local probability distributions of the players, the game is distorted and the players’ interests would be no longer represented. If these local probability distributions are defined by measure preserving transformations then it makes sense that the finitely additive measure must keep those transformations measure preserving.

An ergodic game, (full definition in [5]), is one where the most important proper-
ties is that for each player $j$ each member of $\mathcal{P}_j$ is finite and the player’s local belief at each such member of $\mathcal{P}_j$ form a regular conditional probability. With ergodic games, neither is the Bayesian approach nor the Harsanyi approaches problematic. Strictly speaking, the Bayesian game we present below is not ergodic because the partition members of $\mathcal{P}_j$ for one of the players are not finite. We define a quasi-ergodic game to have all the same properties of an ergodic game with the relaxation that some members $B$ of $\mathcal{P}_j$ may be infinite, however for every such infinite $B$ there a finite subset $B' \subseteq B$ such that inside the set $B \setminus B'$ the player $j$ has no influence over its payoff. Our Bayesian game described below is quasi-ergodic.

The difference between the two approaches, their different types of strategies and evaluations, gives an added depth to optimisation and stability. In [5] and [6] we defined a Harsanyi $\epsilon$-equilibrium for a positive $\epsilon$: all players in a Harsanyi $\epsilon$-equilibrium choose measurable strategies with respect to their sigma algebras as defined above and there is no measurable deviation by some player to another measurable strategy resulting in an expected gain of more than $\epsilon$ in global evaluation. But there is another type of equilibria, the Bayesian. A Bayesian $\epsilon$-equilibrium is a way for each player to play that is $\epsilon$-optimal for each set in its partition with respect to its local probability distribution on that set. Because the Bayesian equilibrium concept does not require measurable strategies (only that strategies are constant for a player on each set in its partition), there can be Bayesian equilibria where there are no Harsanyi equilibria. This is true for the example in [5], which is also an ergodic game.

There is an added complication to the relation between Harsanyi and Bayesian equilibria when moving to approximate equilibria. Hellman [3] showed that there is a two person ergodic game without a Borel-measurable Bayesian $\epsilon$-equilibrium for sufficient small positive $\epsilon$. For a positive $\epsilon$, a Harsanyi $\epsilon$-equilibrium can be much easier to find than a Harsanyi equilibrium. If the Bayesian game is defined with an amenable structure (for example through the actions of an amenable group or semi-group) there will be a Harsanyi $\epsilon$-equilibrium for every $\epsilon > 0$ even though there may be no Harsanyi equilibrium [4] (as happens with Hellman’s example). This is because a Harsanyi $\epsilon$-equilibrium could employ a very small set where the deviation from local $\epsilon$-equilibrium is significant, for example of measure less than $\frac{\epsilon}{B}$ where the deviation in payoff optimality can be no more than $B$. By performing this deviation the measurable behaviour elsewhere of a Harsanyi $\epsilon$-equilibrium could be supported. The existence of such a small set and its role in supporting a Harsanyi $\epsilon$-equilibrium is related to Folner’s condition for amenability.

When moving to finitely additive measures, there are structures to a Bayesian game that enable paradoxical decompositions. To understand this, we take the Bayesian approach and an ergodic game where the information sets of the players are defined as the orbits of finite groups that generate a non-amenable group $G$ and whose elements are measure preserving. There are two intermediate levels between the whole probability space and individual points in that space. One level is the beliefs of the players as defined by the partitions $\mathcal{P}_j$ and the probability distributions on each partition member. As all sets in all $\mathcal{P}_j$ are finite, this level is very close to the individual points. The other and higher level (involving larger sets) is the collection of subsets that the players know in common, the meet partition
of the \( \mathcal{P}_j \), the largest partition smaller than each of the \( \mathcal{P}_j \). The subset that the players know in common, the meet partition, is the orbit of \( G \). There may be no probability distribution supported on this set, not even a finitely additive one, that is \( G \)-invariant. With the Bayesian approach, the game is played out on these orbits of \( G \). The partitions forming each player’s knowledge may be countably generated (the result of countably many refinements of finite partitions of the space) while the meet partition may fail to be countably generated. This could frustrate any attempt to create Hansanyi equilibria from Bayesian equilibria, including the broader context of finitely additive measures. Herein lies the special contribution of Bayesian games to measure theoretic paradoxes. Nothing pathological about the information and payoff structures of the individual players (from the finite group actions) is necessary for the game to be paradoxical with respect to equilibria and finitely additive measures.

The following two player Bayesian game has equilibria, yet fails to have measurable \( \epsilon \)-equilibria for sufficiently small \( \epsilon > 0 \), by which we mean measurable with respect to any finitely additive extension of the Borel measure that is invariant with respect to the semi-group used to define the information structure of the game.

The most important connection between the information structure of a game and colouring rules is stated above, that if \( x, y \) belong to the same information set of a player, then that player must behave identically at \( x \) and \( y \). It is this transmission of behaviour over an overlapping system of partitions defined by ergodic operators that connects a colouring rule to equilibrium behaviour.

We use the same space \( X' = X \times \{a, b, c\} \) as above. We define two overlapping partitions of \( \Omega = X' \) corresponding to two players \( I \) and \( II \).

The information sets of Player \( I \) are the sets of the form \( \{(x, a)\} \cup T_1^{-1}(x) \times \{b\} \cup T_2^{-1}(x) \times \{c\} \). The information sets of Player \( II \) are the sets of size three of the form \( \{x\} \times \{a, b, c\} \). Player \( I \) considers each of these three sets equally likely. Likewise Player \( II \) consider each of the three points equally likely.

The payoffs to Player \( II \) take place at the points \((x, b)\) and \((x, c)\), but with a separate analysis for these two points. Player \( II \) has nine actions to choose from, the set of \((j, k)\) corresponding to \( j = 1, 2, 3 \) and \( k = 1, 2, 3 \). The payoff for Player \( II \) is the sum of the payoffs resulting from \((x, b)\) and \((x, c)\); one applies the matrix \( A_y \) for \( y = T_1T_2T_1x \) to the behaviour of Player \( I \) at \((x, b)\) and the matrix \( A_z \) for \( z = T_2T_1T_2x \) to the behaviour of Player \( I \) at \((x, c)\). The maximising of the sum for Player \( II \) is accomplished independently. If \( p \in \Delta(\{1, 2, 3\}) \) is chosen by Player \( I \) at \((x, b)\) and \( q \in \Delta(\{1, 2, 3\}) \) is chosen by Player \( I \) at \((x, c)\) (usually different because they come from different information sets of Player \( I \)), and Player \( II \) chooses the distribution \( q_{j,k} \in \Delta(\{1, 2, 3\} \times \{1, 2, 3\}) \) then the payoff for Player \( II \) is \( rA_y + sA_z \), where \( y, z \) are defined as above, \( r = \sum_k q_{j,k} \) is Player \( II \)'s marginal distribution on the first coordinate and \( s = \sum_j q_{j,k} \) is Player \( II \)'s marginal distribution on the second coordinate.

The payoff to Player \( I \) takes place entirely at the point labeled \( a \). We would like to say that it follows the three dimensional matrix of the colouring rule \( F \), however Player \( II \) may not choose an independent distribution. Nevertheless we can still define a payoff for Player \( I \) according to these matrices. We replace the above evaluation \( \sum_{j=1}^3 \sum_{k=1}^3 p_jq_kb_{i,j,k} \) by the evaluation \( \sum_{j=1}^3 \sum_{k=1}^3 q_{j,k}b_{i,j,k} \) for the
ith row, where \( q_{j,k} \) is the probability that Player II chooses the combination of \( j \) with \( k \).

The same argument for purity applies, that if the players are optimising then they are choosing almost everywhere pure strategies, meaning that Player I puts all weight on only one row and Player II puts all weight on only one combination of a \( j \) with a \( k \).

Notice how the causation of players’ actions follows through the space. However Player I behaves at a point \((x,a)\) this behaviour gets translated identically to 
\[ T_{1}^{-1} x \times \{b\} \]  
and to 
\[ T_{2}^{-1} x \times \{c\} \]. Player II follows suit at 
\[ T_{1}^{-1} x \times \{b\} \]  
and at 
\[ T_{2}^{-1} x \times \{c\} \] with an attempt to copy Player I’s action. However these points belong in uncountably many different information sets of Player II, and the structure of the matrices \( A_x \) imply that almost everywhere the response is pure. When these various actions of Player II happen at a common point \((y,a)\) (translated from both \((y,c)\) and \((y,b)\)), Player I responds accordingly and transmits this response further to 
\[ T_{1}^{-1}(y) \times \{b\} \]  
and 
\[ T_{2}^{-1}(y) \times \{c\} \]. The structure of the informations sets mirrors precisely the structure of descendants used to define the colouring rule \( F \). When the behaviour of Player II is pure, the difference in the payoffs of Player I between the game and the optimisation of the colouring rule \( F \) disappears. With Player II’s independent evaluation of the two matrices and the use of marginals to evaluate Player II’s payoff, there is no difference to the optimising of payoffs by Player II and the choosing of optimal rows that define the colouring rule \( F \). Once we have marginalised in probability the set where the players are not choosing pure strategies, the argument that this game does not have measurable \( \gamma \)-equilibria for sufficiently small positive \( \gamma \) is the same as that for the lack of \( \gamma \)-stability of any measurable colouring,

4 Conclusion

We conjecture that paradoxical probabilistic colouring rules exist when defined with group action. The difficulty seems to lie with the analysis of a stochastic process that isn’t reducible to a combinatorial argument via purity.

Does every colouring satisfying our colouring rule \( F \), or every colouring satisfying a paradoxical colouring rule, imply the existence of a measurably \( G \)-paradoxical decomposition (\( \Pi \)) using sets of the sigma algebra generated by the colour classes, the Borel sets, and the action of the semi-group generated by the measure preserving transformations defining the descendants?

There is a problem with applying our above colouring rule \( F \) to the group action context. We could revise the colouring rule to one on \( X = \{0,1\}^G \) with the group \( G = C_{2} \ast C_{3} \), the context of the Hausdorff paradox. The problem is that the colouring of each point could be so cleverly balanced as to allow for the non-purity of colours throughout the countable orbits of \( G \). Putting these various orbits together may still result in a failure of finitely additive measurability, but the argument for this, if true, is opaque. Surrounding a three-cycle with points coloured purely does not stop the colouring of the three-cycle with non-pure colours, as by Brouwer’s Fixed Point Theorem there would always be a colouring, not necessarily pure, satisfying
the rule inside the three cycle. One would have to argue that such a colouring inside the cycle must influence the colouring of the three points surrounding that cycle, and as a consequence other cycles, such that completion of the colouring throughout the orbit, in combination with other orbits, would not be possible in a measurable way. Such an argument is plausible, however seems very difficult.

The above use of the relation $T_3^2T_2 = T_2T_3^2$ was introduced due to the trouble caused by the Brouwer Fixed Point Theorem. Assume there is a probabilistic colouring rule where the probability space $X$ is a Cantor set and the colour is determined by elements $g_1, \ldots, g_k$ of a semi-group of measure preserving $G$ consisting of only non-invertible elements generated by $T_1, \ldots, T_m$. Assume further that the rule forces the purity of colours almost everywhere with the purity of the $g_ix$ forcing the purity of $x$, and the rule breaks down into finitely many parts defined by finitely many clopen sets $A_1, \ldots, A_l$ that partition $X$. Let $H$ be the finite subset of $G$ such that $H = \{ h \mid g_i = hj \text{ for some } j \in G \}$, with $|H| = n$. We can consider all the possible joint distributions on the colours of $hy$ for all the $h \in H$ and $y = T_ix$ for $i = 1, \ldots, m$ conditioned on the membership of the $y$ in the different $A_j$. If there are relations between the $T_i$, some joint distribution starting at some $y = T_jx$, and therefore this approach could lead nowhere. However, given no relation between the $T_i$, the colouring rule generates in a continuous way (from purity and the fixed probabilities for each of the $A_j$) a joint distribution on the colours of $hx$ for all the $h \in H$ conditioned on $x \in A_j$ for the various $j = 1, \ldots, l$. The Euclidean space on which we apply Brouwer's fixed point theorem would have dimension $l \cdot (r^n - 1)$ where $r$ is the number of extremal colours. Though that dimension could be very large, it is still finite and we could apply Brouwer's fixed point theorem to get a fixed point of joint distributions, each conditioned on membership in the $A_1, \ldots, A_l$. We would be robbed of our best tool to demonstrate the lack of an invariant measure. Is there a way around the Brouwer fixed point theorem that allows for continuous paradoxical colouring rules while maintaining free generation by non-invertible generations?

Another approach is to single out one non-invertible semi-group element $T$ and interpret $T^{-1}$ as the passage of one unit of time. The idea is that with every passage of a unit of time there are uncountable variations for continuation, with some conditional probability distribution governing these variations. In this context one could study how colouring develops over time. One could use a paradoxical colouring rule for which static satisfaction implies finitely additive non-measurability (assuming semi-group invariance), and a large area satisfying the rule enjoys some form of relative stability with respect to the passage of time. Is it possible to define a paradoxical colouring rule this way as something that with high probability, with respect to some concept of non-measurable random starts, will move toward paradoxical structures of some form? On the one hand, local obedience to the rule could be self perpetuating. On the other hand, non-amenability of the structure means that there is a large boundary to any rule obedient area with the potential to destroy that configuration quickly. With our above example of a paradoxical colouring rule, we show that if there is global satisfaction of the rule, it cannot be invariant measurable. We don't provide an understanding of how that paradoxical
structure could come into existence.

We conjecture that there are two-player ergodic Bayesian games without finitely additive approximate equilibria. The discovery of such a Bayesian game would answer all the open problems in the conclusion of [6].

In [7] we ask the following: if for a finite sequence of sets $A_1, A_2, \ldots, A_n$ there is no $G$-invariant measure for which all the sets are measurable (where $G$ is the semi-group acting on the probability space), does this imply that one of these sets is absolutely non-measurable, meaning that there is no finitely additive $G$-invariant measure such that this set is measurable? This is a very relevant question to game theory. We could easily redefine the above Bayesian game to be a three player game, with Player $II$ divided into two different players, one evaluating only at points of the form $(x, b)$ and other one evaluating only at points of the form $(x, c)$. For Player $I$, who evaluates the colour at $(x, a)$, we could replace the payoff of 1 with the payoff of 2 wherever that results from the choice of the colour $c_1$. We could do that same for the player evaluating at $(x, b)$ for the colour $c_2$ and for the other player when the colour is $c_3$ at an $(x, c)$. In this way, given a positive answer to the above question, the game would become one for which non-measurable equilibria do exist, but none that gives an expected payoff of any kind for at least one of the players (which we consider to be based on some private global evaluation not shared with the other players).

One could raise an objection with our above Bayesian game and our conclusion of no $G$-invariant equilibria. Although with respect to any finitely additive measure that is $G$-invariant there are no measurable strategies in equilibrium, nevertheless a finitely additive measure that makes all subsets measurable could be the basis for any equilibrium of the game (for which we have proved existence with Theorem 2). Indeed Player $I$ receives a non-zero payoff at only one point in each of its information sets and Player $II$ evaluates actions independently at two different points in each of its corresponding information sets. One could argue that the local optimisation process remains intact which switching to such a finitely additive measure. The problem with this perspective is that the resulting global expected payoff for a player from such a measure would bear no relation to an integration over the space of the expected payoffs as perceived locally by the players, since the conditional probability distributions on their information sets re-confirm the measure invariance of the generators of the semi-group; therefore the probability distributions on their information sets would have to change when passing to a measure that allowed the global expected payoffs to be measurable. In the context of game theory, the paradox of our Bayesian game is that one can have equilibria or one can have payoffs as player expectations of measurable functions, but not both simultaneously.

There is another answer to the above objection, observed from a simple change to the payoffs of the game, a change that keeps the game quasi-ergodic. The lack of a measurable equilibrium from a $G$-invariant measure centres on a discrepancy of at least $\frac{1}{5,000}$ for what global probability should be given to actions corresponding to the colour $c_1$. For Player $I$ its information sets give one point $(x, a)$ one-third
probability and one-third each to uncountable sets of the form $T_1^{-1}x \times \{b\}$ and $T_2^{-1}x \times \{c\}$, with the payoff of 0 for all actions at those two sets. Let $M$ be a very large positive number. For the $(x, a)$ point the payoff for Player $I$ for the action corresponding to the colour $c_2$ could be increased for all combinations of columns by $M$ and likewise the action corresponding to $c_2$ decreased by $M$ uniformly at the set $T_1^{-1}x \times \{b\}$. At the same point, the payoff for Player $I$ for the action corresponding to $c_3$ could be increased for all combinations of columns by $M$ and likewise the action corresponding to $c_3$ decreased by $M$ uniformly at the set $T_2^{-1} \times \{c\}$. Likewise we do something similar for Player $II$; at $(x, b)$ we increase the payoff for action corresponding to $c_2$ for all columns by $2M$ and decrease it by $M$ uniformly at the other two points $(x, a)$ and $(x, c)$ and do the same for Player $II$ for the action corresponding to $c_3$ at $(x, c)$ and the other two points. Notice that nothing is changed strategically as long as the players do not lose their local probability distributions on their information sets. Now the definition of an equilibrium of the game must respect the players’ local probability evaluations of the information sets, and these in turn support the measure preserving properties of $T_1$, $T_2$, and the $S_3$ (that permutes the $(x, a), (x, b)$ and $(x, c)$). With $M$ large enough, any finitely additive measure for which the strategies of the players are measurable would distort the local evaluations of the players sufficiently to violate the equilibrium property.

We could consider the following three example, which here are based on group action. Let $G$ be the group freely generated by $\tau$ and $\sigma$ with $\tau^2 = e$ and $\tau^n = e$ for $n \geq 3$. There are $n$ colours $c_1, \ldots, c_n$, represented modulo $n$.

Rule 1:
If $x^e = 0$:
(1) if $\tau^{-1}x$ is coloured with $c_i$, $i \neq n$, and $\sigma x$ is not coloured $c_1$ then $x$ is coloured $c_i$,
(2) if $\tau^{-1}x$ is coloured with $c_i$ and either $i = n$ or $\sigma x$ is coloured $c_1$ then $x$ is coloured $c_{i+1}$.

$x^e = 1$: if $\tau^{-1}x$ is coloured with $c_1$ then $x$ is coloured $c_{i+1}$.

Rule 2:
If $x^e = 1$ and $\tau^{-1}x$ is coloured with $c_i$ then colour $x$ with $c_{i+1}$.
Given $x^e = 0$ and $\tau^{-1}x$ coloured $c_i$:
(1) if $\sigma x$ is not coloured $c_1$ then colour $x$ with $c_1$,
(2) if $\sigma x$ is coloured $c_1$ then colour $x$ with $c_{i+1}$.

Rule 3:
The number $n$ is 3.
Given $\tau^{-1}x$ is coloured $c_i$ and $\tau x$ is coloured $c_j$:
(1) if $i = j$ then colour $x$ with $c_{i+1}$,
(2) if $i = j + 1$ $i + 1 \neq 1$ and $\sigma x$ is coloured $c_1$ then colour $x$ with $c_{i+1}$,
(3) if $i = j + 1$ $i + 1 = 1$ and $\sigma x$ is not coloured $c_1$ then colour $x$ with $c_{i+1}$,
(4) in all other cases, colour $x$ with $c_j$.

All three rules are very interesting, but we are not sure where they could lead in terms of establishing probabilistic paradoxical colouring rules or other interesting colouring structures. It is relatively easy to show that Rule 1 is paradoxical for
\( n \geq 5 \) (from a discrepancy for the probability of the colour \( c_1 \)), but it is not clear if it could be the basis of a probabilistic paradoxical rule. The third rule, defined in [2] and proven there to be paradoxical, would need to be revised to a rule dependent on location in order for it to be the basis of a probabilistic colouring rule.

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