The fractional-order discrete COVID-19 pandemic model: stability and chaos

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Abstract This paper presents and investigates a new fractional discrete COVID-19 model which involves three variables: the new daily cases, additional severe cases and deaths. Here, we analyze the stability of the equilibrium point at different values of the fractional order. Using maximum Lyapunov exponents, phase attractors, bifurcation diagrams, the 0-1 test and approximation entropy (ApEn), it is shown that the dynamic behaviors of the model change from stable to chaotic behavior by varying the fractional orders. Besides showing that the fractional discrete model fits the real data of the pandemic, the simulation findings also show that the numbers of new daily cases, additional severe cases and deaths exhibit chaotic behavior without any effective attempts to curb the epidemic.

Keywords COVID-19 model · Commensurate and Incommensurate orders · Chaos · Complexity · Stability

1 Introduction

Coronavirus pandemic (COVID-19) is a contagious illness caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2 virus). The first case of COVID-19 was detected in Wuhan, China, in December 2019. In early 2020, the World Health Organization announced the spread of the pandemic worldwide, resulting in a public health emergency. More than 300 million individuals have been reported infected with the COVID-19 virus since then, with over 5 million deaths occurring in 222 countries and territories [1,2]. The virus is often spread through the air, where a person can catch the infection by breathing air contaminated with nasal and mouth spray droplets that contain the virus [3]. People who are most affected are the elderly or those with chronic illnesses, such as respiratory problems, diabetes, cardiovascular disease, obesity, among others [4,5].

The fast spread of the COVID-19 epidemic has made countries adopt various measures to reduce it, including isolation of infected people, social distancing, full and partial lockdowns, etc. These measures significantly affected economic development. Therefore, many governments have relaxed these policies in order to stimulate the economy. Consequently, the epidemic has not
been brought completely under control. In an effort to prevent the spread of COVID-19 and to immunize the population, some countries have produced vaccines and some of them have been licensed for use. Vaccination is a necessary method for controlling and mitigating disease outbreaks. To date, more than 9.5 billion doses of vaccine have been administered globally, according to official World Health Organization statistics [6].

Fractional calculus has shown to be a valuable tool for mathematical modeling of various open issues in mathematics and physics [7–9]. Discrete fractional calculus has attracted the attention of a large number of scholars in the past few years [10]. Scientists have been increasingly concerned about its applications in secure communication, neural networks, biology and other fields. Recently various complex dynamics are residing in fractional-order iterated map, such as chaos, hyperchaos and coexisting attractors [11–14]. For instance, in [15] Shukla et al. explore the hyperchaotic dynamic of the fractional generalized Hénon map, whereas, in [16], the chaotic dynamics and combined synchronization of three two-dimensional maps have been investigated. Ouannas et al. [17] introduced a three-dimensional fractional iterated map, in which the suggested fractional map was observed to contain hidden attractors, while the presence of chaos in the discrete fractional memristor system has been shown in [18]. Meanwhile, in [19] the authors discovered the rich chaotic behaviors of a new 2D hyperchaotic fractional map with an infinite line of equilibrium. Thanks to these unique characteristics, fractional-order iterated maps have been deeply studied in academic fields.

In epidemiology, fractional-order operators have been widely employed [20–23]. Recent research has focused on the emerging COVID-19 pandemic, which has lately become the subject of much attention [24–27]. A mathematical model for COVID-19 was formulated by the authors in [28], which took into account both asymptomatic and symptomatic groups with declining immunity. Fractional derivatives were used to present the dynamic model of COVID-19 and citizen reactions in [29]. Abdul Kuddus et al. in [30] studied the analysis of the SLIR model of COVID-19 with nonlinear incidence, while the analysis of the SIRD model of COVID-19 based on real data has been discussed by Kottakkaran et al. in [31]. Meanwhile, the discrete mathematical model of COVID-19 has been analyzed in [32]. Other mathematical models related to the COVID-19 pandemic can be found [33–36]. To date, as far as our knowledge, the dynamical analysis of a discrete fractional COVID-19 model based on a Caputo-like difference operator has not been addressed. This got our attention and prompted us to analyze the phenomenon and explore the behavior of a discrete fractional COVID-19 model where the fractional orders are commensurate and incommensurate.

In this research, by using certain theoretical and numerical techniques, we intend to explore and study the dynamic behaviors of the discrete fractional COVID-19 pandemic model with both commensurate and non-commensurate fractional-order values, and we will analyze the stability of the equilibrium points of the system at different fractional values. Here is the structure of the article: The mathematical model for COVID-19 is presented in Sect. 2. In Sect. 3, we define some basic preliminaries to discrete fractional calculus and provide the fractional discrete version of the system. In Sect. 4, the model’s dynamics and the stability of its equilibrium points are explored in the context of the commensurate and non-commensurate instances. In Sect. 5, the 0–1 test approach and the ApEn algorithm are employed to validate the presence of chaos. Finally, Sect. 6 concludes with a discussion of certain results obtained regarding the time-series plot of the discrete fractional COVID-19 system.

### 2 Mathematical model

In [37], Mangiarotti et al. used data from Johns Hudson University [38] on how the COVID-19 pandemic is spreading worldwide, as well as official statistics from the National Health Commission of the People’s Republic of China [39] to develop a new deterministic mathematical COVID-19 pandemic model. They looked at how this pandemic disease spread in Italy, Japan, South Korea and China from January 21 to April 10, 2020. This model has been defined by a nonlinear 3D system as follows:

\[
\begin{align*}
\frac{dC(t)}{dt} &= \beta_1 D^2 + \beta_2 C^2 + \beta_3 S \left( D + \beta_4 C \right), \\
\frac{dS(t)}{dt} &= \beta_5 C + \beta_6 S + \beta_7 D^2, \\
\frac{dD(t)}{dt} &= \beta_8 C D + \beta_9 C S + \beta_10 D + \beta_11 C^2. \\
\end{align*}
\]  

(1)
where \( C(t) \) = number of new daily cases at time \( t \); \( S(t) \) = number of daily additional severe cases (negative or positive) at time \( t \); \( D(t) \) = number of daily deaths at time \( t \).

\[ \beta_i \quad (i = 1, 11) \] denotes the parameters of the system which are listed in Table 1.

It is demonstrated that this integer-order model exhibits chaos by the authors by comparing the results produced from phase portraits and time series plots with the real observed data (see [37,40]). In [41], the Caputo fractional derivatives have been introduced to the previous integer-order model to obtain the following continuous fractional-order system:

\[
\begin{align*}
C D^q C(t) &= \beta_1 D^2 + \beta_2 C^2 + \beta_3 S (D + \beta_4 C), \\
C D^q S(t) &= \beta_5 C + \beta_6 S + \beta_7 D^2, \\
C D^q D(t) &= \beta_8 C D + \beta_9 C S + \beta_10 D + \beta_11 C^2,
\end{align*}
\]

(2)

where \( q_1, q_2, q_3 \) are the fractional order such that \( 0 < q_i < 1 \). \( C D^q \) is the Caputo fractional derivatives which is determined by the following formula:

\[
C D^q h(v) = \frac{1}{\Gamma(1-q_i)} \int_{\mu_0}^{v} (v-\tau)^{-q_i} h(\tau)d\tau.
\]

(3)

It should be noted that when \( q_1 = q_2 = q_3 \), a commensurate fractional-order system is formed; otherwise, an incommensurate fractional-order system is formed. According to the initial conditions \( (C_0, S_0, D_0) = (184, 30, 8) \) and the parameters listed in Table 1, when \( q_1 = q_2 = q_3 = 0.977 \), the commensurate continuous fractional-order COVID-19 model (2) has chaotic attractors, and when \( (q_1, q_2, q_3) = (1, 0.96, 1) \) and \( (q_1, q_2, q_3) = (1, 1, 0.94) \) the incommensurate continuous fractional-order COVID-19 model (2) has also chaotic attractors. Phase portrait of the COVID-19 model (2) plotted in Fig. 1 confirmed the existence of the complex chaotic attractor.

3 Preliminaries and fractional-order discrete model

Before we define our fractional discrete model, we present some basic definitions and theorems of the discrete fractional calculus.

3.1 Preliminaries and basic concepts

**Definition 1** [42] Let \( N_\theta = \{ \theta, \theta + 1, \theta + 2, \ldots \} \) be a time scale, \( \theta \in \mathbb{R} \). The fractional sum of order \( \gamma \) for a function \( H \) is given as

\[
\Delta^\gamma_\theta H(v) = \frac{1}{\Gamma(\gamma)} \sum_{\tau = \theta}^{v} (v - \tau)^{\gamma-1} H(\tau),
\]

(4)

with \( v \in N_\theta + \gamma, \gamma > 0 \).

**Definition 2** [43] The \( \gamma \)-Caputo fractional difference operator is defined as

\[
\begin{align*}
C D^\gamma_\theta H(v) &= \Delta^\gamma_\theta \frac{\Delta^m H(v)}{\Gamma(m+\gamma)} \\
&= \frac{1}{\Gamma(m+\gamma)} \sum_{\tau = \theta}^{v} (v - \tau)^{(m+\gamma - 1)} \Delta^m H(\tau),
\end{align*}
\]

(5)

where \( v \in N_{\theta+m-\gamma}, \gamma \notin \mathbb{N} \) and \( m = \lceil \gamma \rceil + 1 \). \( \Delta^m H(\tau) \) and \( (v - \tau)^{(m-\gamma - 1)} \) are the \( m \)-th integer difference operator and the falling factorial function, respectively, which are given as

\[
(v - \tau)^{m-\gamma - 1} = \frac{\Gamma(v - \tau)}{\Gamma(v - \tau - m + \gamma + 1)},
\]

(6)

and

\[
\Delta^m H(v) = \Delta(\Delta^m-1 H(\nu))
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} H(v + k), \quad v \in N_\theta.
\]

(7)

Now, in order to establish the stability conditions for the equilibrium points of a fractional discrete system with commensurate fractional-order values, we need the following theorem [44]:

**Theorem 1** Let \( g(v) = (g_1(v), \ldots, g_m(v))^T, 0 < \gamma < 1 \) be a fractional order and \( B \in \mathbb{R}^{m \times m} \). The zero equilibrium point of the commensurate discrete fractional-order system

\[
C D^\gamma_\theta g(v) = B g(v + 1 + \gamma),
\]

(8)

\( \forall v \in N_\theta + 1 - \gamma \) is asymptotically stable if

\[
\lambda_j \in \left\{ \xi \in \mathbb{C} : |\xi| \leq \left( 2 \cos \frac{|\arg \xi| - \pi}{2 - \gamma} \right)^\gamma \right\}
\]

and
Table 1  Values of parameters of the COVID-19 system (1)

| Parameter | Value           | Parameter | Value           |
|-----------|-----------------|-----------|-----------------|
| $\beta_1$ | $-0.10530723$   | $\beta_7$ | $0.16060376$    |
| $\beta_2$ | $2.343 \times 10^{-5}$ | $\beta_8$ | $-0.00011493$   |
| $\beta_3$ | $0.15204$       | $\beta_9$ | $-1.215 \times 10^{-5}$ |
| $\beta_4$ | $-0.01451520$   | $\beta_{10}$ | $0.2844499$    |
| $\beta_5$ | $-0.20517824$   | $\beta_{11}$ | $2.38 \times 10^{-6}$ |
| $\beta_6$ | $0.44040714$    |           |                 |

Fig. 1  Chaotic attractor of the fractional COVID-19 model (2)

\[ |\arg \xi| \geq \frac{\nu \pi}{2} \quad \] ,

where $\lambda_j$ are the eigenvalues of the matrix $B$.

On the other side, the stability theorem of the nonlinear incommensurate discrete fractional system is as follows:

\[ C^{\Delta_{\theta}^\gamma} x_1(\nu) = g_1(x(\nu - 1 + \gamma_1)), \]
\[ C^{\Delta_{\theta}^\gamma} x_2(\nu) = g_2(x(\nu - 1 + \gamma_2)), \quad \nu = 0, 1, \ldots, \]
\[ \vdots \]
\[ C^{\Delta_{\theta}^\gamma} x_n(\nu) = g_n(x(\nu - 1 + \gamma_n)), \]

(10)
where \( g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n \) and \( x(v) = (x_1(v), \ldots, x_n(v))^T \in \mathbb{R}^n \). Assume that \( 0 < \gamma_i < 1 \), \( i = \frac{1}{M}, n \), and \( M \) is the lowest common multiple (LCM) of the denominators \( u_i \) of \( \gamma_i \)'s with \( \gamma_i = \frac{u_i}{u_i}, (u_i, w_i) = 1, w_i, u_i \in \mathbb{Z}_+, i = 1, 2, \ldots, n \). If each root of the equation
\[
det(diag(\lambda^M \gamma_1, \ldots, \lambda^M \gamma_n) - (1 - \lambda^M)J) = 0, \tag{11}
\]
is included within a set \( \mathbb{C} / K^\alpha \), then the trivial solution of system (8) corresponding to \( x_0 = x(0) \) is locally asymptotically stable, where \( \sigma = \frac{1}{M} \). \( J \) is the Jacobian matrix of system (8) and
\[
K^{\alpha} = \left\{ \xi \in \mathbb{C} : |\xi| \leq \left( \frac{2 \cos |\arg \xi|}{\sigma} \right)^\alpha \text{ and } |\arg \xi| \leq \frac{\sigma \pi}{2} \right\}. \tag{12}
\]

### 3.2 Fractional-order discrete COVID-19 model

The fractional-order COVID-19 epidemic model can be written in discrete form by replacing the fractional derivative \( ^C D^\gamma_i \) in the system (2) with the Caputo-like difference operator \( ^C \Delta^\gamma_i \), as shown below:

\[
\begin{align*}
\Delta^\gamma_i C(n) &= C(0) + \frac{1}{\Gamma(\gamma_i)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_i) \left( \beta_1 (D(j))^2 + \beta_2 (C(j))^2 + \beta_3 S(j) (D(j) + \beta_4 D(j)) \right), \\
\Delta^\gamma_i S(n) &= S(0) + \frac{1}{\Gamma(\gamma_2)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_2) \left( \beta_5 C(j) D(j) + \beta_6 S(j) + \beta_7 (D(j))^2 \right), \\
\Delta^\gamma_i D(n) &= D(0) + \frac{1}{\Gamma(\gamma_3)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_3) \left( \beta_8 C(j) D(j) + \beta_9 C(j) S(j) + \beta_{10} D(j) + \beta_{11} (C(j))^2 \right),
\end{align*}
\]

for \( v \in \mathbb{N}_{\theta+1} \). \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are the fractional-order values such that \( \gamma_i \in (0, 1], \; i = 1, 2, 3 \).

The following theorem will allow us to construct the numerical formula for the fractional discrete model:

**Theorem 3** [46] For the fractional difference equation
\[
\begin{align*}
C \Delta^\gamma_i z(v) &= h(v + \gamma_i - 1, \gamma(v + \gamma_i - 1)), \\
\Delta^\kappa z(v) &= z_\kappa, \; m = [\gamma_i] + 1,
\end{align*}
\]
the unique solution of this initial value problem (14) is given by
\[
z(v) = \frac{1}{\Gamma(\gamma_i)} \sum_{\tau=\theta+m-\gamma_i}^{v-\gamma_i} (v - \tau + 1)^{\gamma_i-1} h(\tau + \gamma_i - 1, \gamma(\tau + \gamma_i - 1)), \; v \in \mathbb{N}_{\theta+m},
\]
where
\[
z_0(v) = \sum_{\kappa=0}^{m-1} \frac{(v - \theta)^\kappa}{\Gamma(\kappa + 1)} \Delta^\kappa z(\theta).
\]

Based on this theorem, the numerical model of the discrete fractional COVID-19 system (13) is constructed as follows:

\[
\begin{align*}
C(n) &= C(0) + \frac{1}{\Gamma(\gamma_1)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_1) \left( \beta_1 (D(j))^2 + \beta_2 (C(j))^2 + \beta_3 S(j) (D(j) + \beta_4 D(j)) \right), \\
S(n) &= S(0) + \frac{1}{\Gamma(\gamma_2)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_2) \left( \beta_5 C(j) D(j) + \beta_6 S(j) + \beta_7 (D(j))^2 \right), \\
D(n) &= D(0) + \frac{1}{\Gamma(\gamma_3)} \sum_{j=0}^{n-1} \Gamma(\nu - j + 1 + \gamma_3) \left( \beta_8 C(j) D(j) + \beta_9 C(j) S(j) + \beta_{10} D(j) + \beta_{11} (C(j))^2 \right),
\end{align*}
\]

where \( C(0), S(0) \) and \( D(0) \) denote initial conditions. This is a new class of COVID-19 model which hold “memory effects.” As one can see from equation (17), the states \( C(n), S(n) \) and \( D(n) \) depend on all past variables \( C(0), C(1), \ldots, C(n - 1), S(0), S(1), \ldots, S(n - 1) \) and \( D(0), D(1), \ldots, D(n - 1) \).

### 4 Dynamics analysis

In this part, we will look at whether the previously proposed discrete fractional COVID-19 model (13)
stable or shows chaotic behavior in both cases: the commensurate fractional orders and non-commensurate fractional orders. This investigation will be carried out using a variety of numerical methods, including Lyapunov exponent calculations, bifurcation diagrams and the display of phase portraits in two-dimensional and three-dimensional projections. Also, we apply the 0–1 test technique to confirm whether chaos exists or not.

4.1 Commensurate fractional order

4.1.1 Stability analysis of equilibrium points

Here we explore the stability of the equilibrium points of the discrete fractional COVID-19 model (13) with commensurate fractional order. To calculate the equilibrium points of the system, we solve the following system of equations:

\[
\begin{align*}
\beta_1 D^2 + \beta_2 C^2 + \beta_3 S (D + \beta_4 C) &= 0, \\
\beta_5 C + \beta_6 S + \beta_7 D^2 &= 0, \\
\beta_8 C D + \beta_9 C S + \beta_{10} D + \beta_{11} C^2 &= 0.
\end{align*}
\]

(18)

Take the parameters \( \beta_i \) of the system listed in Table 1, we find that there are two equilibrium points, \( E_0 = (0, 0, 0) \) and \( E_1 = (4313, 594, 62) \). The stability condition of the equilibrium point \( E_1 \) is given by the following theorem:

**Theorem 4** The equilibrium point \( E_1 \) of the model (13) is locally asymptotically stable if \( \gamma \leq 0.6815 \).

**Proof** The Jacobian matrix of the fractional system (13) is:

\[
J = \begin{pmatrix}
2\beta_2 C + \beta_3 \beta_4 S & \beta_3 D + \beta_3 \beta_4 C & 2\beta_3 D + \beta_3 S \\
\beta_5 & \beta_6 & 2\beta_7 D \\
\beta_8 D + \beta_9 S + 2\beta_{11} C & \beta_9 C & \beta_8 C + \beta_{10}
\end{pmatrix}.
\]

(19)

The eigenvalues of the matrix \( J \) at the equilibrium point \( E_1 \) are \( \lambda_1 = -1.07544 \), \( \lambda_2 = 0.0979083 + 0.632971i \) and \( \lambda_3 = 0.0979083 - 0.632971i \). Therefore, by substituting the eigenvalues \( \lambda_2 \) and \( \lambda_3 \) into the condition stated in Theorem 1, we obtain: \( |\lambda| \leq \left( 2 \cos \frac{|\arg \lambda_i| - \pi}{2 - \gamma} \right)^\gamma \). Hence, we get \( \gamma \leq 0.6815 \). Thus, the equilibrium point \( E_1 \) is asymptotically stable if \( \gamma \leq 0.6815 \).

To validate this important finding, let us consider \( \gamma = 0.63 \). From this, we can deduce that \( |\lambda_j| \leq \left( 2 \cos \frac{|\arg \lambda_j| - \pi}{2 - \gamma} \right)^\gamma \) and \( |\arg \lambda_j| \geq \frac{\pi}{2} \) \( \forall j = 1, 2, 3 \).

Based on Theorem 1, the equilibrium point \( E_1 \) of the discrete COVID-19 model with commensurate order (13) is asymptotically stable. To see this, Fig. 2 depicts the stability case of the system subject to the initial conditions \( (C(0), S(0), D(0)) = (184, 30, 8) \).

Now, let \( \gamma = 0.73 \). So, we can deduce that \( |\lambda_j| > \left( 2 \cos \frac{|\arg \lambda_j| - \pi}{2 - \gamma} \right)^\gamma \) for \( j = 2, 3 \). According to Theorem 1, the equilibrium point \( E_1 \) of the fractional discrete COVID-19 model with commensurate order (13) is unstable; then this model may have chaotic attractors.

4.1.2 Bifurcation diagrams and Lyapunov exponents

To explore the dynamics of the commensurate fractional discrete COVID-19 model given in (13) regarding the fractional-order \( \gamma \), take the initial conditions \( (C(0), S(0), D(0)) = (184, 30, 8) \) and system’s parameter as in Table 1. The bifurcation diagram of the discrete fractional system (13) where \( \beta_1 \) varies in the interval \([−0.5, 0]\) and with various fractional-order values \( \gamma \) is depicted in Fig. 4. We notice that when \( \gamma = 0.73 \) the system is stable at \( -0.4933 < \beta_1 < -0.1615 \) and then with increasing \( \beta_1 \), the state \( C(n) \) of the model (13) exhibits irregular movements in conjunction with periodic windows in the interval \( \beta_1 \in [−0.1615, 0.076] \), and when \( \beta_1 \in [−0.076, −0.0435] \) the system (13) is chaotic. When \( \gamma \) decreases, the system has more stability, and the chaotic region shift to the right. For example, for \( \gamma = 0.69 \), the system is stable at the interval \( \beta_1 \in [−0.5, −0.1087] \) and has a chaotic region at \( \beta_1 \in [−0.028, −0.0034] \). To provide more clarity about the dynamic behavior, Fig. 5a depicts the bifurcation diagram of the commensurate fractional discrete COVID-19 model (13) for initial conditions \( (C(0), S(0), D(0)) = (184, 30, 8) \) and with the same parameter values of the system listed in Table 1 where \( \beta_1 = -0.043 \). As can be observed, the system exhibits stable behavior at first, but as the value of the fractional order increases, the system loses its stability with the appearance of periodic motion before becoming
chaotic in the interval $\gamma \in [0.708, 0.7303]$. If we keep increasing the fractional order, the system converges toward infinity.

The Lyapunov exponents are an essential tool that is used in combination with bifurcation diagram to demonstrate chaos in fractional discrete systems. The Jacobian matrix technique is employed to estimate the maximum Lyapunov exponents (see [47]), which are computed in a manner similar to that used to determine the states in the discrete system (13). We define the tangent map $J_i$ as:

$$J_i = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

(20)

where

$$a_i(n) = a_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1+\gamma)}{\Gamma(n-j)} \left( 2a_1 c_i(j) D(j) + 2a_2 a_i(j) C(j) + 2a_3 b_i(j) (D(j) + a_4 C(j)) + a_5 a_i(j) + a_6 b_i(j) + 2a_7 c_i(j) D(j) \right),$$

$$b_i(n) = b_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1+\gamma)}{\Gamma(n-j)} \left( a_5 a_i(j) + a_6 b_i(j) + 2a_7 c_i(j) D(j) \right),$$

$$c_i(n) = c_i(0) + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n-1} \frac{\Gamma(n-1+\gamma)}{\Gamma(n-j)} \left( a_8 (a_i(j) D(j) + c_i(j) C(j))) + a_9 (a_i(j) + b_i(j) C(j)) + a_{10} c_i(j) + 2a_{11} a_i(j) C(j) \right).$$

(21)
Then, the Lyapunov exponents can be given by:

\[ \lambda_\kappa(x_0) = \lim_{i \to \infty} \frac{1}{i} \ln |\lambda_\kappa^{(i)}|, \quad \text{for } \kappa = 1, 2, 3. \]  

(22)

\( \lambda_\kappa \) are the eigenvalues of the Jacobian matrix \( J_i \).

In order to compute the maximum Lyapunov exponents (MLE) of the discrete fractional COVID-19 model (13), we used a MATLAB script and the obtained results are shown in Fig. 5a in accordance with the initial conditions \((C(0), S(0), D(0)) = (184, 30, 8)\) and with the same parameters of the system mentioned in Table 1. One can note that the system has negative LEs at minimum values of \( \gamma \), and when \( \gamma \) increases, the LEs have positive values, meaning that the COVID-19 model (13) transitions from a stable state to a chaotic state, which confirms the results obtained from the bifurcation diagram shown in Fig. 5b. To further provide clarity about this findings, Fig. 6 illustrates the phase diagrams of the discrete fractional COVID-19 model (13) in the 2D and 3D plane for various fractional values.
4.2 Incommensurate fractional order

In this subsection, we examine the dynamics of the incommensurate fractional discrete COVID-19 model in a manner similar to the case of commensurate orders. In particular, we explore the effects of the incommensurate fractional values on the dynamic behavior of the discrete COVID-19 model. To see this, Fig. 7 depicts the bifurcation diagram and the MLE of the incommensurate discrete fractional COVID-19 system (13) where \( \gamma_2 \) is the bifurcation parameter and \( \gamma_1 = 0.6 \) and \( \gamma_3 = 0.7 \). This diagram is drawn with the initial conditions \((C(0), S(0), D(0)) = (184, 30, 8)\) and the parameters listed in Table 1 where \( \beta_1 = -0.043 \). When \( \gamma_2 < 0.569 \), we can see that the discrete model is stable. When \( \gamma_2 \) increases, the MLE changes its values between positive and negative, meaning that chaos occurs with the appearance of certain periodic orbits. Furthermore, when \( \gamma_2 \in [0.6943, 0.7194] \), the maximum LE has positive numbers and the system becomes chaotic. Additionally, to evaluate the dynamic behaviors of the discrete fractional COVID-19 model when...
Fig. 6 Phase portrait of the commensurate discrete fractional COVID-19 model (13)

Fig. 7 a Bifurcation diagram of the incommensurate discrete fractional COVID-19 model (13) versus $\gamma_2$ for $\gamma_1 = 0.6$ and $\gamma_3 = 0.7$. b Maximum Lyapunov exponents

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the fractional order $\gamma_3$ varies, Fig. 8 shows the bifurcation diagram and its corresponding MLE for $\gamma_1 = 0.6$ and $\gamma_2 = 0.7$. Clearly, with increasing $\gamma_3$, we see a shift from stable to chaotic dynamics in the fractional discrete COVID-19 pandemic model (13). In particular, the system is stable when $\gamma_3 < 0.54$ where the MLE is negative, whereas the system is chaotic when $\gamma_3 \in [0.689, 0.731]$ where the MLE is positive. These findings illustrate that the incommensurate orders influence the dynamic behavior of the COVID-19 model (13). For completeness, the phase attractors of the solutions of the discrete fractional COVID-19 model with incommensurate orders (13) are shown in Fig. 9.

To provide further clarification, in order to better understand the different complex dynamic behaviors of the discrete fractional COVID-19 model (13) with incommensurate orders, we analyze the stability of equilibrium point $E_1$ utilizing the stability condition stated in Theorem 2. We suppose $(\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.6, 0.7)$ and take the parameters $\beta_i, \; (i = 1, 2, \ldots, 11)$ listed in Table 1. Notice that the lowest common multiple $M = 10$, and so

$$
\det \left( diag(\lambda^6, \lambda^6, \lambda^7) - (1 - \lambda^{10})J(E_1) \right) = 0, \quad (23)
$$

$$
\Rightarrow
-0.4412 \lambda^{30} - 0.5072 \lambda^{27} + 0.7068 \lambda^{26}
-0.6684 \lambda^{23} - 0.2112 \lambda^{22} + 1.3236 \lambda^{20}
+ \lambda^{19} + 1.0143 \lambda^{17} - 1.4136 \lambda^{16}
+0.6684 \lambda^{13} + 0.2112 \lambda^{12} - 1.3236 \lambda^{10}
-0.5072 \lambda^7 + 0.7068 \lambda^6 + 0.4412 = 0. \quad (24)
$$

The solutions of equation (24) are

$$
\lambda = \begin{pmatrix}
-1.37063 \\
-1.0949 \\
-0.922024 \\
-0.800994 \pm 0.6976i \\
-0.780686 \pm 0.516317i \\
-0.73874 \pm 0.414755i \\
-0.41184 \pm 0.920063i \\
-0.224906 \pm 0.992982i \\
-0.144078 \pm 0.862137i \\
0.226464 \pm 1.00189i \\
0.31096 \pm 0.857914i \\
0.351222 \pm 1.24091i \\
0.670865 \pm 0.568715i \\
0.774981 \pm 0.513305i \\
0.935074 \pm 0.12645i \\
0.973855 \pm 0.609252i \\
1.10319
\end{pmatrix} \quad (25)
$$

Thus, one can see that for $j = 26, 27, 30$ we get $|\lambda_j| > \left(2 \cos \left(10|\arg \lambda_j|\right)\right)^{1/10}$, while the other solutions $\lambda_j$ of the system (24) satisfy $|\arg \lambda_j| > \frac{\pi}{20}$. This implies that $\lambda_j \in \mathbb{C}/K^n$, $(j = 1, 30)$. Hence, by Theorem 2, the equilibrium point $E_1$ of the discrete COVID-19 model with incommensurate frac-

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**Fig. 8** a Bifurcation diagram of the incommensurate fractional discrete COVID-19 model (13) versus $\gamma_3$ for $\gamma_1 = 0.6$ and $\gamma_2 = 0.7$. b maximum Lyapunov exponents
(a) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.6, 0.7)\)

(b) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.66, 0.7)\)

(c) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.7)\)

(d) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.67)\)

(e) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.69)\)

(f) \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.72)\)

Fig. 9 Phase portraits of the incommensurate discrete fractional COVID-19 model (13)
Fig. 10 Time evolution of states of the system (13) for \((\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.6, 0.7)\)
discrete COVID-19 model (13) may show a chaotic attractor, and the time evolution plotted in Fig. 11 confirms this results.

5 0 – 1 test and complexity

5.1 0 – 1 test

To discriminate between chaotic and regular behavior in dynamical systems, one can apply the 0–1 test method [48]. The test uses a series of data as input, and the output approaches 0 or 1 based on whether the dynamics are chaotic or not. To describe the method, we define the translation variables using the time series $C(k)$ for $k = 1, 2, \ldots, N$, as follows:

$$p_\sigma(m) = \sum_{k=1}^{m} C(k) \cos(k\sigma),$$

$$q_\sigma(m) = \sum_{k=1}^{m} C(k) \sin(k\sigma),$$

(28)

where $\sigma \in (0, \pi)$ is a random number. The plotting of $p_\sigma$ and $q_\sigma$ in the $(p_\sigma - q_\sigma)$ plane is used to detect if the chaos occurs or not. If the behavior of $p_\sigma$ and $q_\sigma$ is bounded, the dynamics of the system are regular, but if the behavior indicates Brownian-like behavior, the dynamics of the system are chaotic. Define the mean square displacement as follows:
Fig. 12 The 0-1 test of the discrete fractional COVID-19 model (13)
\[ M_\sigma(m) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \left[ (p_\sigma(k+m) - p_\sigma(k))^2 + (q_\sigma(k+m) - q_\sigma(k))^2 \right], \quad m \leq \frac{N}{10}. \]  

The asymptotic growth rate \( K_\sigma \) is represented by

\[ K_\sigma = \lim_{m \to \infty} \frac{\log M_\sigma(m)}{\log m} \]  

The asymptotic growth rate allows for a distinction when the dynamics of the fractional COVID-19 model (13) are non-chaotic or chaotic. \( K \) close to 0 suggests that the system is non-chaotic and \( K \) close to 1 suggests that the system is chaotic.

Here, the 0-1 test of the COVID-19 model (13) has been performed directly on the series data \( C(k) \). Figure 12 depicts the \( p-q \) plots for regular and chaotic dynamics of the discrete fractional COVID-19 model (13) for various fractional-order values \( \gamma_1, \gamma_2 \) and \( \gamma_3 \). In particular, one can see that when \( \gamma = 0.69, (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.6, 0.7) \) and \( (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.67) \), Fig. 12a, c and d displays bounded like trajectories, which indicates that the system is periodic, whereas when \( \gamma = 0.73, (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.72) \) and \( (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.69) \), Figures 12b, e and f show Brownian-like trajectories, which confirms the chaotic behavior of the discrete fractional COVID-19 model. It can be seen that the findings of the 0-1 test are quite consistent with the previous results of the MLE and bifurcation diagrams.

5.2 Approximate entropy (ApEn)

In this subsection, we use the approximate entropy (ApEn) algorithm to quantify the level of complexity of the discrete fractional COVID-19 model (13). Generally, the more complex time series are the ones with higher \( ApEn \) values. The \( ApEn \) is calculated using the following formula [49]:

\[ ApEn = \Psi^m(r) - \Psi^{m+1}(r), \]  

where \( \Psi^m(r) \) is given by

\[ \Psi^m(r) = \frac{1}{n-m+1} \sum_{j=1}^{n-m+1} \log C_j^m(r), \]  

\begin{table}[h]
\centering
\caption{Approximate entropy test of the fractional discrete COVID-19 model (13)}
\begin{tabular}{cccc}
\hline
\( \gamma_1 \) & \( \gamma_2 \) & \( \gamma_3 \) & ApEn \\
\hline
0.69 & 0.69 & 0.69 & 0.2775 \\
0.73 & 0.73 & 0.73 & 0.5332 \\
0.6 & 0.6 & 0.7 & 0.2239 \\
0.6 & 0.7 & 0.67 & 0.2473 \\
0.6 & 0.7 & 0.69 & 0.3998 \\
0.6 & 0.7 & 0.72 & 0.4932 \\
\hline
\end{tabular}
\end{table}

and \( r \) is the tolerance defined as \( r = 0.2 \text{std}(C) \), where \( \text{std}(C) \) is the standard deviation.

The approximate entropy results of the fractional COVID-19 system (13) in the two cases where the fractional-order values are commensurate and incommensurate are given in Table 2. As can be observed, the results of \( ApEn \) have higher values when \( (\gamma_1, \gamma_2, \gamma_3) = (0.73, 0.73, 0.73), (0.6, 0.7, 0.72) \) and \( (0.6, 0.7, 0.69) \), which means that the series of the system are more complex in these values. Therefore, this finding is in agreement with the MLE results shown previously and thus confirms the occurrence of chaos in the COVID-19 system (13).

6 Discussion

Here, we conclude our analysis using the time-series plots in order to get a better view and more comprehension of the previous chaotic results of the COVID-19 epidemic model. Figure 13a, b, and c depicts the time evolution of new daily cases \( C(n) \) (blue line), additional severe cases \( S(n) \) (green line) and deaths \( D(n) \) (red line) for \( \gamma_1 = \gamma_2 = \gamma_3 = 0.73, (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.74, 0.7) \), and \( (\gamma_1, \gamma_2, \gamma_3) = (0.6, 0.7, 0.74) \), respectively. It is clear that the number of new cases, additional severe cases and death cases continue to exhibit chaotic dynamical behavior. The real data of COVID-19 as well as the drawn findings from the continuous fractional model (2) are listed in Table 3. The real data on the pandemic disease outbreak in China were obtained between January 21 and April 10, 2020. By comparing with the results obtained from the real data, it can be seen that the results drawn from the discrete fractional COVID-19 model for the min and max of the new daily cases and deaths and the results drawn from the continuous fractional model for the min and
max of the new daily cases and deaths are identical to the results of the real data, whereas the expected maximum of the additional severe cases derived from the discrete fractional model is closer to the real data than the expected number obtained from the continuous fractional COVID-19 model.

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