Diffeomorphic moment-angle manifolds with different Betti numbers

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Abstract

We find here two simple polytopes whose moment-angle manifolds are diffeomorphic and that have different Betti numbers.

Introduction

Moment-angle complexes and manifolds are nowadays become a usual object of toric topology. They have been introduced by Davis and Januszkiewicz [D-J] as "universal spaces" for quasitoric manifolds, and appeared since then in several contexts such as toric topology, polyhedral products, intersections of quadrics... For an overview of the current theory on moment-angle manifolds and related topics, we can refer to [B-P2].

The topology of a moment-angle manifold is completely encoded by the combinatorial structure of the underlying polytope and we know how to compute some topological invariants of a moment-angle manifold, such as cohomology, from combinatoric datas on the polytope.

However, the precise differential structure of a "generic" moment-angle manifold is still poorly understood; only several cases have been settled ([LdM-V], [B-P1], [B-M], [LdM-G], [C-F-W]).

Another challenging problem is to understand when two polytopes give rise to "the same" moment-angle manifolds.

A natural notion of similarity, given differential manifolds, is the notion of diffeomorphism. Two polytopes are said diff-equivalent if their moment-angle manifolds are diffeomorphic. Despite the apparent strenght of this request, a classification of all diff-equivalent polytopes is far out of reach, so we shall focus on less ambitious problems. We rather ask which combinatorial invariants are kept between diff-equivalent polytopes.

The simplest numerical invariant of a polytope is its dimension, and the problem of equality of dimensions of diff-equivalent polytopes seems still open. Let’s just remark that preservation of dimension is equivalent to preservation of the number of facets, as the dimension of the moment-angle manifold defined by a polytope is the sum of the dimension and the number of facets of this one.

Other important invariants of a polytope are its Betti numbers (see [B-P1]).

Each usual Betti number of a moment-angle manifold, which is well know to be a topological invariant, is the sum of some precise Betti numbers of the underlying polytope. Hence, if two polytopes have the same Betti numbers, their associated moment-angle manifolds have the same usual Betti numbers, and we can wonder whether the converse is also true, i.e. if the Betti numbers of the moment-angle manifold determine the Betti numbers of the polytope.
In this paper, we answer negatively to this question, by exhibiting a counterexample, i.e. two diffequivalent polytopes with different Betti numbers.

To achieve this, we consider multiwedges over neighbourly dual polytopes. For such polytopes, the associated moment-angle has a known differential structure, which is completely determined by its usual Betti numbers. Indeed, by the work of López de Medrano and Gitler [LdM-G], any such polytope induces a connected sums of sphere products as moment-angle manifold, so its differential structure is given by the dimensions of the appearing spheres.

We more precisely consider multiwedges over neighbourly dual polytopes with four more facets than their dimension. Indeed, considering multiwedges over polytopes with only three more facets than their dimension cannot produce counterexample, as in this case, Betti numbers of the moment-angle manifold correspond to Betti numbers of the polytope. In the other sense, considering multiwedges over polytopes with many more facets than their dimension implies a "high" total Betti rank and consequently more numbers to manage. In addition, the Gale diagrams of duals of such polytopes take place in high dimensional spaces, which increases difficulty. This explains the choice on the kind of polytopes we deal with.

The smallest example of such a polytope is the hexagon. I originally hoped to decide whether two diffequivalent multiwedges over the hexagon must have the same Betti numbers. I investigated the cases that, in my opinion, would most plausibly provide a counterexample and find none, despite some cases "very close to counterexamples". I cannot exclude to have overlooked something. Perhaps an exhaustive investigation of all cases is yet possible, which would definitely classify the diffequivalent multiwedges over the hexagon, but this is not clear to me due to their abundance.

In higher dimension, the combinatorics of neighbourly dual polytopes with four more facets than their dimension is not unique. There is a broad choice of polytopes and we can ask if performing multiwedges with the same multimindex over different polytopes can produce counterexamples. This is indeed the case. We will consider polytopes for which the difference between Betti numbers can be controlled. With suitable choices of a pair of neighbourly polytopes and of a multimindex, we will get what we are looking for.

The counterexample we produce involves 47-dimensional polytopes with 51 facets, so their common moment-angle manifold has dimension 98. We can suspect this is far from optimal.

Naturally, one counterexample generates many others, for example by simply taking products, we can get as many diffequivalent polytopes as we want, no two of them having the same Betti numbers.

Another noticeable fact is that our two diffequivalent polytopes don’t have equal number of vertices, which proves that the number of vertices is not an invariant of diffequivalence (in contrast, this number, and even the full \( f \)-vector of a polytope, can be recovered from the Betti numbers of this polytope [B-P1] as brought to the author’s mind by A. Bahri and T. Panov).

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1 Recalls

1.1 Betti numbers of polytopes and cohomology of moment-angle manifolds

Given a simple polytope \( P \), with facets \( F_1, ..., F_n \), its moment-angle manifold can be defined by the quotient operation \( P \times T^n/ \sim \) where \((p, (z_1, ..., z_n))\) is identified with \((p, (z'_1, ..., z'_n))\) if \( z_i = z'_i \) whenever \( p \) is not on \( F_i \) (in other terms, each coordinate of \( T^n \) generates a rotation that fixes the preimage of the corresponding facet of \( P \)).
Recall we have the following usual decomposition of the homology of a moment-angle manifold, in terms of subsets of facets [3]:

\[ H_k(Z_P, \mathbb{Z}) \simeq \bigoplus_{X \subset F} \tilde{H}_{k-|X|-1}(P_X, \mathbb{Z}) \]

A homology class of \(Z_P\) is called induced by a subset of facets when it is in the image of the reduced homology of this subset.

We also remark a bigraduation in the homology of the moment-angle manifold, by \(k\) and \(|X|\). Fix two integers \(p\) and \(q\). We note here the Betti numbers of \(P\) in the following way:

\[ b_{p,q} = \bigoplus_{X \subset F, |X|=q} \dim \tilde{H}_p(P_X, \mathbb{Z}) \]

This number was standardly noted \(b_{p-q+1,2q}\) but the used notation is more clear in our context.

### 1.2 Balanced configurations

**Definition 1.1** Let \(V\) a \(d\)-dimensional vector space over \(\mathbb{R}\). We will call balanced configuration of points in \(V\) a configuration (i.e. a finite set) of points of \(V\) such that:

i) Any \(d\) points of the configuration form a basis of \(V\).

ii) Given \(d-1\) points of the configuration, there are as many points on each side of the hyperplane they span.

We will note \(n\) the number of points of the configuration.

Remark that with this definition (quite restrictive), the number of points cannot have the same parity as \(d\).

For example, the Gale diagram of an even-dimensional neighbourly polytope is a balanced configuration of points.

Here, we’re only interested in the combinatorics of such a configuration, i.e. the subsets of points whose convex hull contains the origin. Two configurations will be called equivalent if they have the same combinatorics.

**In dimension 2** Let’s recall several facts about such configurations:

**Proposition 1.1** In \(\mathbb{R}^2\), a regular polygon centered at the origin with an odd number \(n\) of sides is a balanced configuration of points. Conversely, any balanced configuration of points in a two-dimensional vector space is equivalent to such a one.

So, a balanced configuration of points induces a natural cyclic order on its points, given by the cyclic order of the polygon. It also induces a distance between the points, the distance \(d(A,B)\) between two points \(A\) and \(B\) being both their distance on the boundary of the forementioned polygon (considered as a graph) and the number of points \(C\) such that 0 is in the convex hull of \(\{A,B,C\}\).

**Remark 1.2** Given three points \(A,B,C\), 0 is in their convex hull if and only if \(d(A,B) + d(B,C) + d(C,A) = n\).

Let’s give some definitions:

**Definition 1.2** Two points \(A\) and \(B\) of a balanced configuration of points are said adjacent if \(d(A,B) = 1\). They are said distant if their distance is maximum, i.e. \(\frac{n-1}{2}\).
Definition 1.3 Given two points $A$ and $B$, we note $\phi(A, B)$ the set of points $C$ such that 0 is in the convex hull of $\{A, B, C\}$.

As recalled thereabove $\phi(A, B)$ has $d(A, B)$ elements.

Definition 1.4 Given a point $A$, the two sets of points on each side of the line spanned by $A$ are called the $A$-classes.

Both have $\frac{n-1}{2}$ points.

Remark that two distant points different from $A$ are one in each $A$-class.

In dimension 3

Remark 1.3 Given a balanced configuration of points in a vector space $V$ ($\dim V \geq 2$) and a point $x$ of the configuration, the images of the other points on the quotient vector space $V/\mathbb{R}x$ yield another balanced configuration of points.

Hence, given a balanced configuration of points in a 3-dimensional vector space and one of its points $x$, the quotient by $\mathbb{R}x$ induces a balanced configuration of points in a 2-dimensional vector space. The induced cyclic order is then called the $x$-cyclic order.

Let's give some properties of these orders:

Proposition 1.4 Let $A$ and $B$ two points. Then the $B$-classes for the $A$-order also are the $A$-classes for the $B$-order.

Both are the sets of points on each side of the plane spanned by $A$ and $B$, hence the result by symmetry.

The following proposition and its corollary can be useful in determining if some convex hulls contain the origin.

Proposition 1.5 Consider a balanced configuration of points in $\mathbb{R}^3$, $x$ one of its points, and $u$, $v$, $w$ three other points. Then:

i) If 0 is not in the convex hull of the projections of $u$, $v$ and $w$ on $\mathbb{R}^3/\mathbb{R}x$, then 0 is not in the convex hull of $\{u, v, w, x\}$ in $\mathbb{R}^3$.

ii) If 0 is in the convex hull of the projections of $u$, $v$ and $w$ on $\mathbb{R}^3/\mathbb{R}x$, then 0 is in the convex hull of exactly one of $\{u, v, w, x\}$ and $\{u, v, w, -x\}$.

Proof The point i) is obvious.

If 0 is in the convex hull of the projections of $u$, $v$ and $w$ on $\mathbb{R}^3/\mathbb{R}x$, then some point $y$ of $\mathbb{R}x$, namely $y = \lambda x$, is a positive combination of $u$, $v$ and $w$, and $y \neq 0$ by affine independance. Hence, if $\lambda > 0$, then $0 = y + \lambda(-x)$ is in the convex hull of $\{u, v, w, -x\}$ and if $\lambda < 0$, then $0 = y + (-\lambda)x$ is in the convex hull of $\{u, v, w, x\}$. And 0 cannot be on both convex hulls because the intersection of the two convex hulls lies on the hyperplane containing $u$, $v$ and $w$. $\square$

Corollary 1.6 Assume 0 is in the convex hull of the projections of $u$, $v$ and $w$ on $\mathbb{R}^3/\mathbb{R}x$ and that, for the $u$-cyclic order, $x$ and $v$ are on different $w$-classes.

Then 0 is in the convex hull of $\{u, v, w, x\}$.

Indeed, $-x$ and $v$ are on the same side of the hyperplane spanned by $u$ and $w$. So 0 cannot be in the convex hull of $\{u, v, w, -x\}$. The assertion then derives from the proposition.
1.3 Lexicographic extensions

We present here a particular case of a more general construction, well explained in [P].

Given a balanced configuration of points in a 3-dimensional vector space, the following construction produces other ones with two more points:

Consider three points $u$, $v$, $w$ of the configuration, three signs (i.e. $\pm 1$) $s_u$, $s_v$, $s_w$, and two positive real numbers $0 < \omega << \epsilon << 1$. Put now $p = s_u \cdot u + \epsilon s_v \cdot v + \epsilon^2 s_w \cdot w$ and $q = -p - \omega u - \omega^2 v - \omega^3 w$.

Then adding $p$ and $q$ to the initial configuration yields another balanced configuration (see [P]). We will note it $[u^{s_u}, v^{s_v}, w^{s_w}]$.

Lexicographic extensions and dual neighbourly polytopes As we have recalled, if we consider a(n even-dimensional) neighbourly dual polytope $P$, the Gale diagram of its dual is a balanced configuration of points.

We also have the converse, i.e. any balanced configuration of points is the Gale diagram of a neighbourly polytope. Indeed, if this configuration has $n$ points in dimension $d$, it is the Gale diagram of some polytope $P^*$ of dimension $n - d - 1$ with $n$ vertices. On each side of a hyperplane spanned by $d - 1$ points of the configuration, there are $\frac{n-d+1}{2}$ points. Hence on any side of any vector hyperplane, there is at least $\frac{n-d+1}{2}$ points. If we remove any $\frac{n-d-1}{2}$ points to the configuration, there is at least one remaining point on any side of any vector hyperplane. So the origin is in the convex hull of the remaining points, which, by properties of Gale diagrams, means that any set of $\frac{n-d}{2} = \dim P^*$ points of $P^*$ determines a simplex of $P^*$, i.e. $P^*$ is actually neighbourly.

Recall also that points of the Gale diagram of the dual $P^*$ of a polytope $P$ correspond to facets of $P$.

1.4 Biflips

Consider a simple $d$-polytope $P$. Recall that a flip from $P$ is a passage to another simple $d$-polytope $Q$ such that there is a simple $d+1$-polytope $T$ having $P$ and $Q$ as disjoint facets and such that there is exactly one vertex $v$ of $T$ that neither lays on $P$ nor on $Q$ (see [T]). There are $d+1$ edges of $T$ containing $v$, $p$ of them having their other extremity on $P$ and $q$ on $Q$. The flip from $P$ to $Q$ is then called a $(p,q)$-flip, with $p + q = d + 1$.

By symmetry, $P$ is then obtained from $Q$ by the inverse $(q,p)$-flip.

If $p, q \geq 2$, then $P$ and $Q$ have the same number of facets and their facets naturally correspond (a facet of $P$, which is the intersection of $P$ with a facet of $T$, corresponds to the intersection with $Q$ of the same facet).

We will use the following notation for a flip: The vertices of $P$ adjacent to $v$ are the vertices of a simplicial face $F_P$ of $P$, intersection of a set $X$ of facets of $P$, called the containing facets. Call $Y$ the set of facets of $P$ that meet $F$ on its boundary, which are called the extremal facets. (We can notice that containing and extremal facets are inversed in the inverse flip).

The flip will then be noted $(X)|Y]$. The inverse flip from $Q$ to $P$ is then $(Y)|X]$.

Combinatorially, if we unite $X$ with $(Y$ but one element), their intersection is a vertex of $P$, whereas if we unite $Y$ with $(X$ but one element), their intersection is a vertex of $Q$. All other intersections of facets giving vertices of $P$ or $Q$ are the same.

Remark 1.7 If $P$ is a $d = 2d'$ neighbourly dual polytope, then any $(p,q)$-flip from $P$, with $p \geq 2$, is a $(d',d' + 1)$-flip.

Indeed, let $(X)|Y]$ such a flip. Then $[Y]$ corresponds to a set of facets that do not intersect in $P$, so has at least $d' + 1$ elements, and the intersection of all elements of $X$ only meets elements of $Y$, so does not meet every other facet. It has then at least $d'$ elements by dual neighbourliness.
**Definition 1.5** We call biflip a pair of flips of the form $(X)[Y]$ and $(Y')[X]$ from a polytope $P$ to another polytope $R$, i.e. such that the containing facets of the first flip are the extremal facets of the second one.

Such a biflip will be noted $(X)[Y,Y']$.

We can notice that the inverse flips from $R$ to $P$ also form a biflip.

**Definition 1.6** Consider a biflip $(X)[Y,Y']$ from a $(d,d+4)$ neighbourly dual polytope. Then it is called a 1 fixed facet biflip, or $1ff$-biflip if $X \cup Y \cup Y'$ contains $d+3$ facets, i.e. all but one.

Remark that this notion is symmetric, i.e. that the resulting polytope $Q$ is also a $(d,d+4)$ neighbourly dual polytope and that $P$ is obtained from $Q$ by a 1ff-biflip (this biflip is then $(X)[Y',Y]$).

### 1.5 Wedges and multiwedges

Given a simple polytope $P$ and a facet $F$ of $P$, the wedge $W_F P$ of $P$ over $F$ is a simple polytope [K-W].

It is possible to perform iterated wedges over a face and even multiwedges by performing a fixed number of wedges over each facet (the order of performance of wedges are indifferent). Hence, such a multiwedge is given by an initial polytope and a multiindex, a family of natural numbers indexed by the facets of this polytope.

Given a polytope $P$ and a facet $F$, we can compare homology of $Z_P$ and of $Z_{W_F P}$. Indeed, if a subset $\mathcal{X}$ not containing $F$ induces a homology class in $Z_P$, it induces a class of the same bidegree for $Z_{W_F P}$, whereas if it contains $F$, it induces in $Z_{W_F P}$ a class in which the bidegree is increased by $(1,1)$ (hence the total degree by 2).

And this phenomenon passes to multiwedges. In particular, the total Betti rank of the moment-angle manifold is not modified by (multi)wedges.

### 2 Wedges and biflips

We consider here two neighbourly dual polytopes with four facets more than their dimension, and who differ from a 1ff-biflip. We compute the difference between the subsets inducing homology in their moment-angle manifolds. Let’s recall that on the boundary of a $d = 2d'$-dimensional neighbourly dual polytope, a union of facets (except all or none) has the homotopy type of a wedge of spheres of dimension $(d' - 1)$.

Assume we have a $d = 2d'$-dimensional neighbourly dual polytope $P$ with $d + 4$ facets. Let $\mathcal{F}$ the set of its facets and $\mathcal{F}'$ any nonempty subset of $\mathcal{F}$ with at most $d'$ elements. As $P$ is neighbourly dual, there is a vertex of $P$ on the intersection of all the facets belonging to $\mathcal{F}'$. The union of all these facets is starshaped on such a vertex, so $\mathcal{F}'$ does not contribute to the homology of $Z_P$. By duality, neither does its complement, so all subsets of $\mathcal{F}$ inducing homology in $Z_P$ (except $\emptyset$ and $\mathcal{F}$ itself) have $d' + 1$, $d' + 2$ or $d' + 3$ elements.

Assume now we have two $(d,d+4)$ neighbourly dual polytopes $P_1$ and $P_2$, with $d = 2d'$, obtained from each other by a 1ff-biflip, namely $(X,Y_1,Y_2)$. Call then $Y = Y_1 \cap Y_2$.

Now, $X$ has $d'$ elements, $Y_1$ and $Y_2$ have $d' + 1$ elements and, as the flip is 1ff, $X \cup Y_1 \cup Y_2$ has $d + 3$ elements, so $Y_1 \cup Y_2$ has $d' + 3$ and $Y$ has $d' - 1$ elements. There are then exactly two elements in each that are not in the other. Note $G_1$, $H_1$ the facets that are in $Y_1$, not in $Y_2$ and $G_2$, $H_2$ the facets that are in $Y_2$, not in $Y_1$ and $G$ the facet which is neither in $X$, $Y_1$ nor $Y_2$.

We are looking for the subsets of $d' + 1$ facets that induce homology in one of the two moment-angle manifolds, not in the other. A subset of $d' + 1$ facets induces homology in $Z_{P_1}$ or $Z_{P_2}$ exactly when the intersection of all its members is empty in the corresponding polytope. So, if such a subset induces homology in $Z_{P_1}$ and not in $Z_{P_2}$, the intersection of its members must be destroyed by the 1ff-biflip.
The first flip $(X)[Y_1]$, destroys the intersection of $X \cup \{F\}$ for any $F$ in $Y_1$, and no other intersection of sets with $d' + 1$ elements. The second flip $(Y_2)[X]$ destroys the intersection of $Y_2$ and no other intersection of sets with $d' + 1$ elements, and it reintroduces intersection of $X \cup F$ if $F$, assumed in $Y_1$, is also in $Y_2$.

Hence, there are only three subsets of $d' + 1$ facets whose members intersect in $P_1$ and not in $P_2$, namely $Y_2$, $X \cup \{G_1\}$ and $X \cup \{H_1\}$. By symmetry, there are three subsets of $d' + 1$ facets whose members intersect in $P_2$ and not in $P_1$, namely $Y_1$, $X \cup \{G_2\}$ and $X \cup \{H_2\}$.

Let’s now determine the subsets of $d' + 2$ facets whose contribution to moment-angle homology changes after the biflip. We can remark that the contribution to moment-angle homology of such a subset only depends on the number of its subsets with $d' + 1$ that do not intersect in the polytopes. So if such a subset contributes more to the homology of $Z_{P_1}$ than to the one of $Z_{P_2}$, it must contain one of the three subsets mentioned above. By duality, so does its complement, and one of them must contain $Y_2$, the other must contain $X \cup \{G_1\}$ or $X \cup \{H_1\}$, at least $X$.

So there are only three pairs of possibilities, namely $Y_2 \cup \{G_1\}$ and $X \cup \{H_1,G\}$, $Y_2 \cup \{H_1\}$ and $X \cup \{G_1,G\}$, $Y_2 \cup \{G\}$ and $X \cup \{G_1,H_1\}$.

Note the coherence: $X \cup \{G_1,H_1\}$ actually contains two new subsets of $d' + 1$ elements that do not intersect, but as they globally intersect in $P_1$, they only contribute for one dimension in homology of $Z_{P_2}$.

To sum up, here are the changes in moment-angle homology:

- $Y_2$, $X \cup \{G_1\}$ and $X \cup \{H_1\}$ contribute for one dimension in $H^{d+1}(Z_2)$, not in $H^{d+1}(Z_1)$.
- $Y_2 \cup \{G_1\}$, $X \cup \{H_1, G\}$, $Y_2 \cup \{H_1\}$, $X \cup \{G_1, G\}$, $Y_2 \cup \{G\}$, $X \cup \{G_1, H_1\}$ contribute for one dimension more in $H^{d+2}(Z_2)$ than in $H^{d+2}(Z_1)$.
- (By duality) $X \cup \{G_1, H_1, G\}$, $Y_2 \cup \{H_1, G\}$ and $Y_2 \cup \{G_1, G\}$ contribute for one dimension in $H^{d+3}(Z_2)$, not in $H^{d+3}(Z_1)$.
- $Y_1$, $X \cup \{G_2\}$ and $X \cup \{H_2\}$ contribute for one dimension in $H^{d+1}(Z_1)$, not in $H^{d+1}(Z_2)$.
- $Y_1 \cup \{G_2\}$, $X \cup \{H_2, G\}$, $Y_1 \cup \{H_2\}$, $X \cup \{G_2, G\}$, $Y_1 \cup \{G\}$, $X \cup \{G_2, H_2\}$ contribute for one dimension more in $H^{d+2}(Z_1)$ than in $H^{d+2}(Z_2)$.
- $X \cup \{G_2, H_2, G\}$, $Y_1 \cup \{H_2, G\}$ and $Y_2 \cup \{G_2, G\}$ contribute for one dimension in $H^{d+3}(Z_1)$, not in $H^{d+3}(Z_2)$.

If we now consider a finite sequence of sufficiently distant 1ff-biflips, so that no two subsets of facets inducing change in homology in different biflips are close to each other, then the global change after the sequence of biflips is the addition of each change.

When considering multiwedges over such polytopes, with the same multiindex, the changes are the same concerning dimensions (as vector spaces), but degrees and bidegrees on which the changes occur depend on the multiindex.

So we ask whether there exists a multiindex for which the bigraded Betti numbers of the polytopes are different, but the Betti numbers of moment-angle manifolds are the same. We have found such a possibility with four biflips, with a multiindex satisfying the values given in the table, section 4.

### 3 Lexicographic extension and biflips

We show here how to construct pairs of polytopes with the required property.
**Definition 3.1** Consider a balanced configuration of points in $\mathbb{R}^3$, $x_1$ one of these points, and $x_2, x_3$ two points that are adjacent for the $x_1$-cyclic order.

Then, the lexicographic extensions $[x_1^{(+)}, x_2^{(+)}, x_3^{(+)}]$ and $[x_1^{(+)}, x_3^{(+)}, x_2^{(-)}]$ are called cousin extensions. Calling $y$ the common distant point of $x_2$ and $x_3$ for the $x_1$-cyclic order, the points $x_1, x_2, x_3, y$ and the two ones we add, $p$ and $q$, are called the special points of the extensions. The other ones will be called indifferent.

**Warning:** The two extensions are not exactly symmetric in $x_2, x_3$, due to the sign $-$ in the second one. The first extension will be called the left extension, the second the right extension.

We assume here that we have an even-dimensional $(d, d + 4)$ neighbourly dual polytope $P$. The Gale diagram of its dual yields a balanced configuration of $d + 4$ points in $\mathbb{R}^3$.

Consider two cousin extensions related to three points $x_1, x_2$ and $x_3$ (corresponding to facets of $P$), and the associated polytopes $P_l$ (left extension) and $P_r$ (right extension). Then:

**Proposition 3.1** The polytopes $P_l$ and $P_r$ are obtained from each other by a 1ff-biflip.

**Proof** We just have to analyse accurately the differences between these two polytopes.

In the sequel, we identify points of balanced configurations and facets of neighbourly polytopes.

We begin with a definition:

**Definition 3.2** Let’s place under the conditions of the proposition. A set of four points of our extensions will be called a changing quatuor if it contains $0$ in its convex hull for one extension, not for the other.

To compute the changing quatuors, we look at the modifications of the cyclic orders when joining $p$ and $q$ to our facets.

- Let $F$ be an indifferent facet, and let’s determine the $F$-cyclic order of the extensions. We just have to place correctly $p$ anq $q$. We see that $p$ is infinitesimally close to $x_1$, so $q$ infinitesimally close to the opposite of $x_1$, hence between the two distant facets of $x_1$ for the $F$-cyclic order. So the positions of $q$ are the same in both extension. Furthermore, $p$ is in the $x_1$-class containing $x_2$ for the first extension and in the $x_1$-class containing $x_3$ for the second one. Now, as $F$ is not $y, x_2$ and $x_3$ are in the same $F$-class for the $x_1$-cyclic order. So they also are in the same $x_1$-class for the $F$-cyclic order.

Hence, the $F$-cyclic orders are equal for both extensions.

Let’s now examine the special facets.

- Consider first $y$. Still $q$ is between the distant facets to $x_1$ for the $y$-cyclic order, but $x_2$ and $x_3$ are then on opposite $x_1$-classes for the $y$-cyclic order, which implies that, for the two $y$-cyclic orders, $p$ is each time adjacent to $x_1$, but once on each side.
Consider $x_2$. Still $q$ is between the distant facets to $x_1$ for the $x_2$-cyclic order, and $p$ is adjacent to $x_1$. We have, for the left extension, modulo $x_2$, $p \equiv x_1 + \epsilon^2 x_3$ so $p$ is in the $x_1$-class containing $x_3$. For the right extension, we have, modulo $x_2$, $p \equiv x_1 + \epsilon x_3$ so $p$ is also in the $x_1$-class containing $x_3$. Hence, the $x_2$-cyclic orders are the same for both extensions.

Consider $x_3$. Still $q$ is between the distant facets to $x_1$ for the $x_3$-cyclic order, and $p$ is adjacent to $x_1$. We have, for the left extension, modulo $x_3$, $p \equiv x_1 + \epsilon x_2$ so $p$ is on $x_2$'s side from $x_1$. For the right extension, we have, modulo $x_3$, $p \equiv x_1 - \epsilon^2 x_2$ so $p$ is, from $x_1$, on the side opposite to $x_2$. Hence, $x_1$ and $p$ are inversed in the $x_3$-cyclic orders of the extensions.

Consider $x_1$. For the left extension, we have, modulo $x_1$, $p \equiv \epsilon x_2 + \epsilon^2 x_3$, so is adjacent to $x_2$, in the
$x_2$-class containing $x_3$, i.e. between $x_2$ and $x_3$. For the right one, $p \equiv \epsilon x_3 - \epsilon^2 x_2$, so is adjacent to $x_3$, on the $x_3$-class not containing to $x_2$. Whereas $q$ is in both cases infinitely close to $-p$, i.e. close to the opposite of $x_2$ in the left extension and to the opposite of $x_3$ in the right one. For the cyclic orders, both corresponding elements are adjacent to $y$, one on each side depending on the extension.

- Consider $p$. As $p$ is infinitesimally close to $x_1$, the $p$-cyclic order on the initial facets but $x_1$ is the same as the initial $x_1$-cyclic order. We have, modulo $p$, for the left extension, $x_1 \equiv x_1 - p = -\epsilon x_2 - \epsilon^2 x_3$, so is infinitesimally close to $-x_2$. In the right one, $x_1 \equiv x_1 - p = -\epsilon x_3 + \epsilon^2 x_2$, so is infinitesimally close to $-x_3$. For the $p$-cyclic order, $x_1$ is each time adjacent to $y$, once on each side.

We also have, in both cases, modulo $p$, $q \equiv q + p$, which, in the first extension, equals $-\omega x_1 - \omega^2 x_2 - \omega^3 x_3 \equiv (\epsilon - \omega)x_2 + (\epsilon^2 - \omega^3)x_3$, so $q$ is infinitesimally close to $x_2$, on $x_3$’s side, i.e. between both, and, in the second extension, equals $-\omega x_1 - \omega^2 x_2 - \omega^3 x_3 \equiv (\epsilon - \omega)x_3 - (\epsilon^2 + \omega^3)x_2$, so $q$ is infinitesimally close to $x_3$, on the opposite of $x_2$’s side. So for the $p$-cyclic order, $x_1$ is each time adjacent to $y$, once on each side.

- Let’s finally consider $q$. As $q$ is infinitesimally close to $-x_1$, the cyclic order on the initial facets but $x_1$ is the same as the initial $x_1$-cyclic order.

We have, in the left extension, $q = -(1 + \omega)x_1 - (\epsilon + \omega^2)x_2 - (\epsilon^2 + \omega^3)x_3$. So, modulo $q$, we have $(1 + \omega)x_1 \equiv (1 + \omega)x_1 + q = -(\epsilon + \omega^2)x_2 - (\epsilon^2 + \omega^3)x_3$. So, for the $q$-cyclic order, 1 is infinitesimally close to $-x_2$.

In the right extension, we have $q = -(1 + \omega)x_1 - (\epsilon + \omega^2)x_3 + (\epsilon^2 - \omega^3)x_2$. So, modulo $q$, we have $(1 + \omega)x_1 \equiv (1 + \omega)x_1 + q = -(\epsilon + \omega^2)x_3 + (\epsilon^2 - \omega^3)x_3$. So, for the $q$-cyclic order, 1 is infinitesimally
close to $-x_3$. 

So, for the $q$-cyclic order, $x_1$ is each time adjacent to $y$, once on each side.

We have, modulo $q$, in the left extension, $p \equiv p + \frac{1}{1 + \omega} q = (1 - \frac{1 + \omega}{1 + \omega}) x_1 + (\epsilon - \frac{\epsilon + \omega^2}{1 + \omega}) x_2 + (\epsilon^2 - \frac{\epsilon^2 + \omega^3}{1 + \omega}) x_3 = \frac{\omega}{1 + \omega} [(\epsilon - \omega)x_2 + (\epsilon^2 - \omega^3)x_3] = \frac{\omega(x - \omega)}{1 + \omega}(x_2 + (\epsilon + \omega)x_3)$.

For the $q$-cyclic order, $p$ is between $x_2$ and $x_3$.

In the right one, modulo $q$, we have $p \equiv p + \frac{1}{1 + \omega} q = (1 - \frac{1 + \omega}{1 + \omega}) x_1 + (\epsilon - \frac{\epsilon + \omega^2}{1 + \omega}) x_3 - (\epsilon^2 - \frac{\epsilon^2 + \omega^3}{1 + \omega}) x_2 = \frac{\omega}{1 + \omega} [(\epsilon - \omega)x_3 - (\epsilon^2 + \omega^3)x_2]$. For the $q$-cyclic order, $p$ is adjacent to $x_3$, on the opposite side to $x_2$.

So, for the $q$-cyclic order, $p$ is each time adjacent to $x_3$, once on each side.

To sum up, the only inversions in the cyclic orders occurring when switching the extensions are:

- For the $x_1$-cyclic order, $x_3$ with $p$ and $y$ with $q$.
- For the $x_3$-cyclic order, $x_1$ with $p$.
- For the $y$-cyclic order, $x_1$ with $p$.
- For the $p$-cyclic order, $x_3$ with $q$ and $x_1$ with $y$.
- For the $q$-cyclic order, $x_3$ with $p$ and $x_1$ with $y$.

We now can deduce that the polytopes are obtained from each other by a 1ff-biflip.

Consider the following sets of facets:

In the initial $x_1$ cyclic order, $x_3$ determines two classes, namely $\mathcal{X}$, containing $x_2$ and $\mathcal{X}'$, containing $y$. Put $\mathcal{Y} = \mathcal{X}' \setminus \{y\}$, $\mathcal{Y}_1 = \mathcal{Y} \cup \{p, y\}$ and $\mathcal{Y}_2 = (\mathcal{Y} \cup \{x_3, x_1\})$. Let’s check that $P_2$ is obtained from $P_1$ by the biflip $(\mathcal{X})[\mathcal{Y}_1, \mathcal{Y}_2]$, which is a 1ff-biflip because $q$ is the only facet neither belonging to $\mathcal{X}$, $\mathcal{Y}_1$ nor $\mathcal{Y}_2$.

The complement of $\mathcal{X} \cup \mathcal{Y}_1$ contains $x_1$, $x_3$ and $q$. The complement of $\mathcal{X} \cup \mathcal{Y}_2$ contains $y$, $p$ and $q$.

Passing to Gale diagrams, we then have to show that sets of the form $\{x_1, x_3, q, F\}$ are changing quatuors, 0 being in their convex hull when $F$ is in $\mathcal{X}$ for the left extension, and in $\mathcal{Y}_1$ for the right one, that sets of the form $\{y, p, q, F\}$ are changing quatuors, 0 being in their convex hull when $F$ is in $\mathcal{Y}_2$ for the left extension, and in $\mathcal{X}$ for the right one, and that there are no other changing quatour.

In the left extension, we have, $\phi_{x_1}(x_3, q) = \mathcal{X}$. We deduce that, if $F$ is in $\mathcal{Y}_1$, then 0 is not in the convex hull of $\{x_1, x_3, q, F\}$.

In the right extension, $\phi_{x_1}(x_3, q) = \mathcal{Y}_1$. So, if $F$ is in $\mathcal{X}$, then 0 is not in the convex hull of $\{x_1, x_3, q, F\}$.  

\[\]
In the left extension, we have, \( \phi_p(y, q) = \mathcal{Y}_2 \). We deduce that, if \( F \) is in \( \mathcal{X} \), then 0 is not in the convex hull of \( \{y, p, q, F\} \).

In the right extension, \( \phi_p(y, q) = \mathcal{X} \). So, if \( F \) is in \( \mathcal{Y}_2 \), then 0 is not in the convex hull of \( \{y, p, q, F\} \).

Consider the \( x_3 \)-classes in the \( q \)-cyclic order for the right extension. We have \( \mathcal{X} \cup \{x_1\} \) and \( \mathcal{Y} \cup \{p, y\} = \mathcal{Y}_1 \). Hence, by corollary 1.6 if \( F \) is in \( \mathcal{Y}_1 \) then 0 is actually in the convex hull of \( \{x_1, x_3, q, F\} \) for the right extension.

Consider the \( x_3 \)-classes in the \( q \)-cyclic order for the left extension. We have \( \mathcal{X} \cup \{p\} \) and \( \mathcal{Y} \cup \{x_1, y\} = \mathcal{Y}_1 \). Hence, by corollary 1.6 if \( F \) is in \( \mathcal{X} \) then 0 is actually in the convex hull of \( \{x_1, x_3, q, F\} \) for the left extension.

Consider the \( y \)-classes in the \( q \)-cyclic order for the right extension. We have \( \mathcal{X} \cup \{x_1\} \) and \( \mathcal{Y} \cup \{p, x_3\} \). Hence, by corollary 1.6 if \( F \) is in \( \mathcal{X} \) then 0 is actually in the convex hull of \( \{y, p, q, F\} \) for the right extension.

Consider the \( y \)-classes in the \( q \)-cyclic order for the left extension. We have \( \mathcal{X} \cup \{p\} \) and \( \mathcal{Y} \cup \{x_1, x_3\} = \mathcal{Y}_2 \). Hence, by corollary 1.6 if \( F \) is in \( \mathcal{Y}_2 \) then 0 is actually in the convex hull of \( \{y, p, q, F\} \) for the left extension.

We then have seen that the required changes between the two extensions occur. Let’s now verify there are no other ones.

Consider four points. If neither \( p \) nor \( q \) is not one of these points, there is nothing to show. If \( p \) is one of these points, not \( q \), then 0 is in their convex hull if and only if this is also true when replacing \( p \) by \( x_1 \), and this in both extensions. Hence, a set of four points not containing \( q \) cannot be a changing quatuor.

So we can consider \( q \) and three other points. If neither of them is \( x_1 \) nor \( p \), then 0 is in their convex hull if and only if this is also true when replacing \( q \) by \(-x_1\), and this in both extensions. Such a set cannot be a changing quatuor.

If (at least) two of these points are nonspecial or \( x_2 \), then they induce the same cyclic orders in both extensions and, by corollary 1.6 we can conclude that this set is not a changing quatuor.

Hence a changing quatuor must contain \( q \) and at least two points among \( x_1, x_3, p, y \). If it contains \( x_1, x_3 \) and \( q \) or \( p, y \) and \( q \), we have proved it is a changing quatuor.

Only remains the case of \( q, (x_1 \text{ or } x_3), (y \text{ or } p) \) and a last point.

Now, \( \phi_q(x_1, y) \) equals \( \{p\} \) for the left extension and \( \{x_3\} \) for the right one. This induces no new changing quatuor.

Also, \( \phi_q(x_3, p) \) equals \( \{y\} \) for the left extension and \( \{x_1\} \) for the right one. This induces no new changing quatuor.

Consider \( \{x_3, y, q, F\} \), \( F \) not being \( x_1 \) nor \( p \). As the only inversion in the \( x_3 \)-cyclic orders are 1 with \( p \), \( F \) is in \( \phi_{x_3}(y, q) \) for both extensions or for none. Also, the \( F \)-cyclic orders are equal in both extensions, so, by corollary 1.6 \( \{x_3, y, q, F\} \) is not a changing quatuor.

Consider \( \{x_1, p, q, F\} \), \( F \) not being \( x_3 \) nor \( y \). For the first extension, we have \( \phi_q(x_1, p) = \mathcal{X} \cup \{y\} \) and for the second, \( \phi_q(x_1, p) = \mathcal{X} \cup \{x_3\} \). So \( F \) is in \( \phi_q(x_1, p) \) for both extensions or for none. As the \( F \)-cyclic orders are equal in both extensions, we again have, by corollary 1.6 \( \{x_1, p, q, F\} \) is not a changing quatuor.

All the cases have been examined. So we are done. \( \square \)

We now consider multiple pairs of extensions, i.e. we start with a neighbourly dual polytope or a balanced configuration of points and we perform several pairs of cousin extensions. We can ask if they can be performed together.

**Proposition 3.2** Assume we have a balanced configuration of points, and a family \( ce_1, \ldots, ce_k \) of cousin extensions of this configuration. Then, if their special facets (points) are all different, we can get two extensions by performing them in any order and any sense (for each pair or cousins, we can choose the left or right one for the first extension and the other for the second). The resulting extensions do not depend on the order in which the extensions have been performed (i.e. in some sense, these extensions commute), it just depends on the choices done for the senses.
And the resulting polytopes are obtained by each other by a sequence of \( k \) consecutive 1ff-biflips.

Proof To verify that the final cyclic orders do not depend on the order of performance of the extensions, we just have to see that there is no ambiguity on the combinatorics of the polytope, provided the different \( \epsilon \)'s and \( \omega \)'s are small enough, independently of their relative sizes.

Indeed, consider four points for whose we try to determine if 0 is in their convex hull.

1. If the added points among these four ones come from a unique extension, there is no ambiguity.

2. If among the four points there is some \( p \) (resp. \( q \)) without the corresponding \( q \) (resp. \( p \)) nor \( x_1 \), then the considered point can be replaced by the corresponding \( x_1 \) (resp. \( -x_1 \)) so that "its" extension can be peformed at any time.

3. If among the four points there is some \( p \) or \( q \) accompanied with the other without the corresponding \( x_1 \), then \( p \) can be replaced by "its" \( x_1 \) and \( q \) by "its" \( x_2 \) or \( x_3 \) depending only on the sense of this extension.

4. If among the four points there is some \( p \) (resp. \( q \)) accompanied with the corresponding \( x_1 \) but not with "its" \( q \) (resp. \( p \)), then we can replace \( p \) (resp. \( q \)) by \( x_2 \) or \( x_3 \) (resp. \( -x_2 \) or \( -x_3 \)) depending only on the sense of this extension.

5. If finally, we have \( p, q \) and \( x_1 \) in the four points, then there is no ambiguity because there is only one facet remaining (if it is some \( p \) or \( q \), we are in fact in the case 2).

Hence, the order of performance of the extensions is indifferent. We now just have to check that the two final polytopes are obtained from each other by the correct sequence of biflips.

We can proceed by induction on \( k \). Assume this is true for \( k \) cousin extensions, and consider \( k + 1 \) ones, giving two polytopes \( T_{k+1} \) and \( T'_{k+1} \). Consider then the polytope \( T''_{k+1} \) obtained by changing only the last choice on the family of extension giving \( T'_{k+1} \) (making this choice the same as the one giving \( T_{k+1} \)). Then, by what precedes, \( T'_{k+1} \) and \( T''_{k+1} \) are obtained from each other by a 1ff-biflip. Now, this last extension could also have been performed at first, so that \( T_{k+1} \) and \( T''_{k+1} \) are both obtained from the same polytope by a sequence of \( k \) consecutive 1ff-biflips with different special facets. By induction assumption, they are obtained from each other by a sequence of \( k \) consecutive 1ff-biflips. So \( T_{k+1} \) and \( T'_{k+1} \) are obtained from each other by a sequence of \( k + 1 \) consecutive 1ff-biflips, which concludes the induction. \( \square \)

4 Explicit counterexample

We give here an explicite example of two polytopes with diffeomorphic moment-angle manifolds and different Betti numbers.

Consider the dual cyclic polytope \( P = C_{16,20}^* \), with the natural cyclic order on its facets.

The Gale diagram of its dual is a balanced configuration of points in dimension 3. If we choose the (point corresponding to) facet 1, it induces a cyclic order on the other ones. For this cyclic order, facets 2 and 4 are adjacent, their common distant facet being 3.

We now produce four pairs of cousin extensions, the first one with \( x_1 = 1, \ x_2 = 2 \) and \( x_3 = 4 \). The special facets are then, in addition with the two new ones, 1, 2, 3 (corresponding to \( y \)) and 4.

The three other pairs of extensions are obtained by shifting one, two and three times the facets by 5, hence no two pairs among them have any common special facet.

So we can consider extensions, and associated polytopes, obtained by performing these cousin extensions together. We will denote these two polytopes \( P'_1 \) and \( P'_2 \).
For reasons of aesthetics, we alternate the senses of the extensions, i.e., for $P'_1$ we consider the left extensions in the first and third pair and the right extension for the second and forth ones, and we inverse for $P'_2$. So, we easily see that shifting by 5 the facets induces an isomorphism between $P'_1$ and $P'_2$.

Now our counterexample $P_1$ and $P_2$ will be obtained by multiwedges over respectively $P'_1$ and $P'_2$, the indices of the multiwedges being the same and given by the following table:

| Facet index | 1 2 3 4 5 6 7 8 9 10 |
|------------|----------------------|
| Facet      | 11 12 13 14 15 16 17 18 19 20 |
| index      | 1 0 0 1 0 0 3 1 2 0 |
| Facet      | $p_1$ $q_1$ $p_2$ $q_2$ $p_3$ $q_3$ $p_4$ $q_4$ |
| index      | 1 0 0 1 0 0 1 1 |

The total number of wedges done equals 23, hence we get two polytopes of dimension 47 with 51 facets. Their moment-angle manifolds have dimension 98.

We can compute their Betti numbers, what characterizes them, up to diffeomorphism, and we assert they are the same in both cases. We indeed find that both moment-angle manifolds are diffeomorphic to

\[ (33686) \]

We moreover can count the number of vertices of each polytope. The polytope $P_1$ has two vertices more (33686) than $P_2$ (33684). This proves, as announced, that these two polytopes have different $f$-vectors.

### References

[B] I. Baskakov Cohomology of K-powers of spaces and the combinatorics of simplicial divisions
Russian math. surveys 57 (2002), no. 5, p.989-990

[B-M] F. Bosio and L. Meersseman Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes
Acta math., vol 197(2006), no.1, p. 53-127

[B-P1] V. Buchstaber and T. Panov Torus actions and their applications in topology and combinatorics
University lectures series, 24. Amer. math. soc., providence, RI, 2002.
[B-P2] V. Buchstaber and T. Panov Toric topology
arXiv:1210.2368

[C-F-W] L. Chen, F. Fan and X. Wang The topology of the moment-angle manifolds. On a conjecture of S. Gitler and S. Lopez
arXiv:1406.6756

[D-J] M. Davis and T. Januszkiewicz Convex polytopes, Coxeter orbifolds and torus actions
Duke Math. Journal, vol 62 (1991), no. 2, p. 417-451

[K-W] Klee and Walkup The d-step conjecture for polyhedra of dimension d < 6
Acta math., vol 117, (1967), p. 53-78

[LdM-V] S. López de Medrano and A. Verjovsky A new family of complex, compact, non-symplectic manifolds
Bol. Soc. Brasil. Mat. (N.S.) 28 (1997), no. 2, 253-269.

[LdM-G] S. López de Medrano and S. Gitler Intersections of quadrics, moment-angle manifolds and connected sums
Geom. Topol. 17 (2013), no. 3, 1497-1534.

[P] A. Padrol Constructing neighborly polytopes through balanced point configurations
www.renyi.hu/conferences/ec11/.../ArnauPadrol.pdf

[T] V. Timorin An analogue of the Hodge-Riemann relations for simple convex polyhedra
Russian math. surveys 54 (1999), no. 2, p. 381-426

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