The Banks-Zaks expansion
in perturbative QCD: an update

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Abstract:

The recent QCD calculations of the five-loop $\beta$-function and of $R_{e^+e^-}$ to $O(\alpha_s^4)$ provide one more term in the Banks-Zaks expansion in $(16\frac{1}{2} - n_f)$. There is no longer any hope that the expansion could extend, even crudely, to low $n_f$. Above $n_f \sim 9$, however, the results appear to be reasonably consistent from order to order.
This Letter is to update earlier work [1, 2], taking into account the new results of Baikov, Chetyrkin, and Kühn for the fifth-order $\beta$ function [3] and for $R_{e^+ e^-}$ at fourth order [4]. Contrary to our original hopes, the Banks-Zaks (BZ) expansion [5]-[7] appears to break down around $n_f \sim 9$ or sooner, and does not extrapolate, even crudely, to low $n_f$.

We write the $\beta$ function in the form:

$$\beta(a) \equiv \mu \frac{da}{d\mu} = -2a^2 \left( 1 + ca + c_2a^2 + c_3a^3 + c_4a^4 + \ldots \right),$$

where $a \equiv \alpha_s/\pi$. The coefficients, in the \textit{MS} scheme, are [8]-[11],[3]:

$$2b = 11 - \frac{2}{3}n_f,$$

$$8bc = 102 - \frac{38}{3}n_f,$$

$$32bc_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2,$$

$$128bc_3 = \left( \frac{149753}{6} + 3564\zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27}\zeta_3 \right) n_f$$

$$+ \left( \frac{50065}{162} + \frac{6472}{81}\zeta_3 \right) n_f^2 + \frac{1093}{729}n_f^3,$$

$$512bc_4 = \left( \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \right)$$

$$+ \left( -\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{9}\zeta_4 + \frac{1358995}{27}\zeta_5 \right) n_f$$

$$+ \left( \frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right) n_f^2$$

$$+ \left( -\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right) n_f^3 + \left( \frac{1205}{2916} - \frac{152}{81}\zeta_3 \right) n_f^4.$$

Here $\zeta_s$ is the Riemann zeta-function and $n_f$ is the number of massless quark flavours.

For $n_f$ just below $16\frac{1}{2}$, the $\beta$ function has a zero at $a^* \sim -\frac{1}{5}$, and $a^*$ is asymptotically proportional to $(16\frac{1}{2} - n_f)$. Its limiting form,

$$a_0 = \frac{8}{321} \left( 16\frac{1}{2} - n_f \right),$$

is the natural expansion parameter for the BZ expansion [1]. To proceed, one re-writes all perturbative coefficients, eliminating $n_f$ in favour of $a_0$. The first two $\beta$-function coefficients, which are renormalization-scheme (RS) invariant, become:

$$b = \frac{107}{8}a_0, \quad c = -\frac{1}{a_0} + \frac{19}{4}. \quad (4)$$
Within the class of so-called ‘regular’ schemes [7, 1], which includes $\overline{\text{MS}}$, perturbative coefficients have a polynomial dependence on $n_f$, and we may write

$$c_i = \frac{1}{a_0} \left( c_{i,-1} + c_{i,0}a_0 + c_{i,1}a_0^2 + \ldots \right).$$

(5)

The coefficients, in $\overline{\text{MS}}$, are collected in the table below.

|   |   |   |
|---|---|---|
| $c_{1,0}$ | $\frac{19}{1}$ | $4.75$ |
| $c_{2,-1}$ | $-\left( \frac{8}{107} \right) \left( \frac{37117}{768} \right)$ | $-3.61$ |
| $c_{2,0}$ | $\frac{243}{32}$ | $7.59$ |
| $c_{2,1}$ | $\left( \frac{107}{8} \right) \left( \frac{325}{172} \right)$ | $22.6$ |
| $c_{3,-1}$ | $\left( \frac{8}{107} \right) \left( \frac{53981}{1152} + \frac{5335}{32} \zeta_3 \right)$ | $18.5$ |
| $c_{3,0}$ | $\left( \frac{1544327}{18524} - \frac{16171}{268} \zeta_3 \right)$ | $-179$ |
| $c_{3,1}$ | $\left( \frac{107}{8} \right) \left( \frac{2587}{96} + \frac{809}{171} \zeta_3 \right)$ | $451$ |
| $c_{3,2}$ | $-\left( \frac{107}{8} \right)^2 \left( \frac{1093}{3456} \right)$ | $-56.6$ |
| $c_{4,-1}$ | $\left( \frac{8}{107} \right) \left( \frac{1081380511}{663592} + \frac{17251949}{13824} \zeta_3 - \frac{191675}{192} \zeta_5 \right)$ | $156.7$ |
| $c_{4,0}$ | $\left( \frac{154905729}{1327104} - \frac{48015}{512} \zeta_4 - \frac{4489165}{27648} \zeta_3 + \frac{856625}{2304} \zeta_5 \right)$ | $-1005.3$ |
| $c_{4,1}$ | $\left( \frac{107}{8} \right) \left( \frac{33737869}{221184} + \frac{16171}{512} \zeta_4 - \frac{176837}{2304} \zeta_3 - \frac{88415}{2304} \zeta_5 \right)$ | $731.1$ |
| $c_{4,2}$ | $\left( \frac{107}{8} \right)^2 \left( \frac{471499}{110592} - \frac{809}{256} \zeta_4 + \frac{39409}{2304} \zeta_3 - \frac{345}{128} \zeta_5 \right)$ | $3329.0$ |
| $c_{4,3}$ | $\left( \frac{107}{8} \right)^3 \left( \frac{1205}{18432} - \frac{19}{64} \zeta_3 \right)$ | $-697.4$ |

Table 1. β-function coefficients in the $\overline{\text{MS}}$ scheme.

The BZ expansion can be applied to any perturbatively calculable physical quantity of the form:

$$R = a \left( 1 + r_1 a + r_2 a^2 + r_3 a^3 + \ldots \right).$$

(6)

For ‘primary’ quantities calculated in a ‘regular’ scheme the coefficients $r_i$ are polynomials in $n_f$, and hence in $a_0$:

$$r_i = r_{i,0} + r_{i,1}a_0 + r_{i,2}a_0^2 + \ldots$$

(7)

Note that a term $r_{i,j}a_0^p$ or $c_{i,j}a_0^p$ can be assigned a degree $i + j - p$, and all terms in any formula must have matching degree. [We mention that the same decomposition of coefficients is needed in the “large-b” approximation [12, 13], which employs the opposite limit ($b \to \infty$), rather than $b = \frac{107}{8}a_0 \to 0$ as here.]

The prototypical example is the $e^+e^-$ ratio at a cm energy $Q$:

$$R_{e^+e^-}(Q) \equiv \frac{\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

(8)
where, neglecting quark masses, \( R_{e^+e^-} (Q) = 3\Sigma q_i^2 (1 + R_{e^+e^-}) \), and \( R_{e^+e^-} \) has the form of Eq. (6). [We will drop “singlet” terms proportional to \((\Sigma q_i)^2/(3\Sigma q_i^2)\) whose \( n_f \) dependence is ambiguous and depends on the electric charges assigned to the additional, fictitious quarks.] The coefficients, calculated in \( \overline{\text{MS}} \) with the renormalization scale \( \mu \) equal to \( Q \) \( [14, 15, 4] \), are collected in the table below.

| \( r_{1,0} \) | \( r_{1,1} \) | \( r_{1,2} \) | \( r_{2,0} \) | \( r_{2,1} \) | \( r_{2,2} \) | \( r_{3,0} \) | \( r_{3,1} \) | \( r_{3,2} \) | \( r_{3,3} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( \frac{1}{12} \) | \((107/8) (11/4 - 2\zeta_3)\) | \((107/8) (11/4 - 2\zeta_3)\) | \(-12521/288 + 13\zeta_3\) | \((107/8) (401/24 - 53/6\zeta_3 + 25/3\zeta_5)\) | \((107/8)^2 (151/18 - 19/3\zeta_3 - 1/2\zeta_5)\) | \(-3963761/20736 + 577833\zeta_3 - 275/24\zeta_5\) | \((107/8) (38969/128 + 535/32\zeta_2 + 6907/96\zeta_3 + 165/2\zeta_2^3 + 9595/44\zeta_3 - 665/32\zeta_7)\) | \((107/8)^2 (236089/1728 - 97/16\zeta_2 - 13859/96\zeta_3 + 15/2\zeta_2^2 + 445/12\zeta_5)\) | \((107/8)^3 (6131/216 - 33/8\zeta_2 - 203/12\zeta_3 + 3\zeta_2\zeta_3 - 15/2\zeta_5)\) |

### Table 2. Coefficients in \( \mathcal{R}_{e^+e^-} \) in the \( \overline{\text{MS}}(\mu = Q) \) scheme.

The fixed-point condition \( \beta(a^*) = 0 \) always has a solution as a power series in \( a_0 \):

\[
a^* = a_0 \left( 1 + v_1 a_0 + v_2 a_0^2 + v_3 a_0^3 + \ldots \right). \tag{9}
\]

A straightforward calculation yields:

\[
v_1 = c_{1,0} + c_{2,-1},
\]

\[
v_2 = (c_{1,0} + 2c_{2,-1})(c_{1,0} + c_{2,-1}) + c_{2,0} + c_{3,-1}, \tag{10}
\]

\[
v_3 = c_{1,0}^3 + 6c_{1,0}^2 c_{2,-1} + c_{1,0}(3c_{2,0} + 4c_{3,-1} + 10c_{2,-1})

+ c_{2,-1}(4c_{2,0} + 5c_{3,-1}) + 5c_{2,-1}^3 + c_{2,1} + c_{3,0} + c_{4,-1}.
\]

Numerically, \( v_1 = 1.1366, v_2 = 23.27, v_3 = 18.10, \) in the \( \overline{\text{MS}} \) scheme. Since \( a^* \) is RS dependent, the good or bad convergence of this series need not concern us.

A physical quantity \( \mathcal{R} \) also has an infrared limit, \( \mathcal{R}^* \), given by a power series in \( a_0 \). Substituting \( a = a^* \) from Eq. (9) into the perturbative expansion of \( \mathcal{R} \) and re-expanding in powers of \( a_0 \) yields

\[
\mathcal{R}^* = a_0 \left( 1 + w_1 a_0 + w_2 a_0^2 + w_3 a_0^3 + \ldots \right), \tag{11}
\]
where

\[ \begin{align*}
  w_1 &= v_1 + r_{1,0}, \\
  w_2 &= v_2 + 2r_{1,0}v_1 + r_{2,0} + r_{1,1}, \\
  w_3 &= v_3 + (2v_2 + v_1^2)r_{1,0} + v_1(2r_{1,1} + 3r_{2,0}) + r_{2,1} + r_{3,0}.
\end{align*} \]  \tag{12}

These coefficients are RS independent. For the $e^+e^-$ case they are

\[ \begin{align*}
  w_1 &= \frac{4177}{2^5(107)}, \\
  w_2 &= \frac{31250575}{2^93(107)^2} - 275\zeta_3, \\
  w_3 &= \frac{2177185161509}{2^{15}3^2(107)^3} - 4232749\zeta_3 + \frac{65125}{2^6(107)}\zeta_5.
\end{align*} \]  \tag{13}

Numerically we find

\[ \mathcal{R}^*_{e^+e^-} = a_0 \left( 1 + 1.22a_0 + 0.23a_0^2 + 25.38a_0^3 + \ldots \right). \]  \tag{14}

While the first three terms raise hopes for a well-behaved series, those hopes are dashed by the last term. See Fig. 1.

![Fig. 1. $\mathcal{R}^*$ as a function of $n_f$ in the BZ expansion. The curves for 1st to 4th order are shown dotted, dashed, dot-dashed, and solid.](image)

A formulation of the BZ expansion for quantities at a general $Q$ was derived in Ref. [1]. First, we write the integrated $\beta$-function equation in the form

\[ b \ln \left( \frac{\mu}{A} \right) = \frac{1}{a} + c \ln(|c|a) + \int_0^a dx \left( \frac{b}{\beta(x)} + \frac{1}{x^2} - \frac{c}{x} \right). \]  \tag{15}
This form, more convenient for \( c \) negative, is completely equivalent to our previous definition of the \( \tilde{\Lambda} \) parameter \([16, 2]\). We use a tilde to distinguish \( \tilde{\Lambda} \) from the conventional definition of the \( \Lambda \) parameter \([17]\). The relation is \( \ln(\Lambda/\tilde{\Lambda}) = (c/b)\ln(2|c|/b) \). The two definitions are not dissimilar for small \( n_f \), but they become infinitely different as \( n_f \to 16\frac{1}{2} \).

In the BZ-expansion context the use of \( \tilde{\Lambda} \) is much more convenient.

As explained in Ref. \([1]\), it is convenient to put the \( \beta \) function into the form

\[
\frac{b}{\beta(x)} = -\frac{1}{x^2} + \frac{c}{x} - \frac{b}{\gamma^*(a^*-x)} + H(x).
\]

where \( \gamma^* \) is the slope of the \( \beta \) function at the fixed point:

\[
\gamma^* \equiv \frac{d\beta(x)}{dx} \bigg|_{x=a^*} = -ba^* \left( 1 + 2ca^* + 3c_2a^*^2 + 4c_3a^*^3 + \ldots \right).
\]

As discussed below, \( \gamma^* \) can be obtained as a series in \( a_0 \). The remainder function \( H(x) \) can be expanded as a power series, \( H_0 + H_1 x + \ldots \), whose coefficients are of order \( a_0 \).

One now inserts (16) into (15) and performs the integration. One can then eliminate \( a \) and \( a^* \) in favour of \( \mathcal{R} \) and \( \mathcal{R}^* \). In fact, since the result must be RS invariant, one can — without loss of generality — short-cut this step by utilizing the “effective-charge” RS \([18]\) in which \( a \equiv \mathcal{R} \). In \( n^{th} \) order of the BZ expansion this leads to the formula \([1]\):

\[
\rho_1(Q) = \frac{1}{\mathcal{R}} + c \ln(|c| \mathcal{R}) + \frac{b}{\gamma^*(n)} \ln \left( 1 - \frac{\mathcal{R}}{\mathcal{R}^*(n)} \right) + \sum_{i=0}^{n-4} \frac{H_i^{(EC)} \mathcal{R}^{i+1}}{i+1}.
\]

On the left-hand side, \( \rho_1(Q) \) is the RS invariant

\[
\rho_1(Q) \equiv b \ln \left( \frac{\mu}{\Lambda} \right) - r_1 \equiv b \ln \left( \frac{Q}{\Lambda_{\mathcal{R}}} \right),
\]

where \( \Lambda_{\mathcal{R}} \) is a characteristic scale specific to the particular physical quantity \( \mathcal{R} \). It is related to the \( \tilde{\Lambda} \) parameter of some reference scheme (eg. \( \overline{\text{MS}} \)) by an exactly calculable factor \( \exp(r_1(\mu=Q)/b) \) involving the \( r_1 \) coefficient in that scheme, evaluated at \( \mu = Q \). On the right-hand side the terms involving the \( H_i^{(EC)} \) coefficients of the effective-charge scheme are only relevant in fourth order and beyond. Thus, for the first three orders the equation takes the same form, just with the parameters \( \gamma^* \) and \( \mathcal{R}^* \) approximated to the appropriate order. At \( 4^{th} \) order there is an extra term \( H_0^{(EC)} \mathcal{R} \), with \( H_0^{(EC)} = H_0^{(EC)} a_0 + O(a_0^2) \), where

\[
H_0^{(EC)} = \rho_{4,-1} + 2\rho_{2,-1}\rho_{3,-1} + \rho_{3,-1}^3 - c_4,-1 + c_2,-1(2c_3,-1 + r_{1,0}^2 - r_{2,0}) + c_2^3,-1
\]

\[
- c_2,-1r_{1,0} - c_3,-1r_{1,0} - r_{1,0}^3 + 2r_{1,0}r_{2,0} - r_{3,0},
\]

\[
= \frac{243227350299721}{2^{15}3^4(107)^3} - \frac{5729638277}{2^73^3(107)^2} \zeta_3 - \frac{81125}{2^3(107)} \zeta_5 \approx -164.8
\]
(in the first line, the $\rho_{i,j}$ are the $\beta$-function coefficients of the EC scheme).

The BZ expansion for $\gamma^*$ is obtained straightforwardly by substituting the expansion of $a^*$ (Eqs. (9) and (10)) into (17). This gives:

$$\gamma^* = ba_0 \left( 1 + g_1a_0 + g_2a_0^2 + g_3a_0^3 + \ldots \right),$$

with

$$g_1 = c_{1,0},$$
$$g_2 = c_{1,0}^2 - c_{2,-1}^2 - c_{3,-1},$$
$$g_3 = c_{1,0}^3 - 4c_{2,-1}^3 - 5c_{1,0}c_{2,-1}^2 - 4c_{1,0}c_{3,-1} - 2c_{2,-1}c_{2,0} - 6c_{2,-1}c_{3,-1} - c_{3,0} - 2c_{4,-1}.\quad (22)$$

It is noteworthy that certain terms of degree $n$ are absent in $g_n$: $g_1$ does not contain $c_{2,-1}$; $g_2$ does not contain $c_{2,0}$ or $c_{2,-1}c_{1,0}$; and $g_3$ does not contain $c_{2,1}$ or $c_{2,0}c_{1,0}$ or $c_{2,-1}c_{1,0}^2$.

The values of these invariants are

$$g_1 = \frac{19}{4},$$
$$g_2 = \frac{633325687}{2^{1033}(107)^2} - \frac{5335}{2^2(107)} \zeta_3,\quad (23)$$
$$g_3 = -\frac{66670528901419}{2^{13}3^4(107)^3} - \frac{1920043907}{2^63^4(107)^2} \zeta_3 + \frac{191675}{2^23(107)} \zeta_5.$$

Numerically the $\gamma^*$ series is:

$$\gamma^* = ba_0 \left( 1 + 4.75a_0 - 8.98a_0^2 - 43.89a_0^3 + \ldots \right).\quad (24)$$

The results, at different orders, are shown in Fig. 2.

Note that $\gamma^*$ is the ‘critical exponent’ in the relation $R^* - R \propto Q^{\gamma^*}$ that describes how $R$ approaches $R^*$ as $Q \to 0$. ($\gamma^*$ is the infrared limit of an RS-invariant ‘effective exponent’ $\gamma(Q) \equiv 1 + Q \frac{d^2R}{dQ^2}/\frac{dR}{dQ} = \frac{d\beta}{da} + \beta(a) \frac{d^2R}{dQ^2}/\frac{dR}{dQ}$ [19].) As pointed out by Grunberg [7], the $g_n$ coefficients are RS invariants and are universal, in the sense that they are not specific to a particular physical quantity $R$.

Numerically inverting Eq. (18) provides $R$ as a function of $Q$. In the BZ region, $n_f \gtrsim 9$, the resulting $R(Q)$ has the general form sketched in Fig. 3. At large $Q$ the result naturally agrees with ordinary perturbation theory to the corresponding order. For $Q \sim \tilde{\Lambda}_R$ there is a large “sloping plateau” region, and at ultra-low energies there is a “spike” reaching up to $R^{*n}$. 

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Fig. 2. $\gamma^*$ as a function of $n_f$ in the BZ expansion. The curves for 1st to 4th order are shown dotted, dashed, dot-dashed, and solid.

Fig. 3. Typical appearance of $R$ as a function of $Q$ in the BZ region ($n_f \gtrsim 9$) showing the “spike” at very low energies, the “sloping plateau” region, and the slow approach to asymptotic freedom at very high energies (this region is shown on a log scale). The plateau value $R_p$ is generally about 0.8 times $R^*$ but depends on $n_f$ and the BZ-expansion order.

We conclude by showing, in Fig. 4, a comparison of the 4th order BZ expansion with the $R_{e^+e^-}$ results of Ref. [20] in optimized perturbation theory (OPT) [16] and in the EC scheme [18] to order $\alpha_s^4$. Contrary to the conjecture of Refs. [1, 2], it now appears that the “freezing” behaviour of $R_{e^+e^-}^*$ at low $n_f$ [21, 22] is not an extrapolation from the BZ region, but a distinct phenomenon. At low $n_f$ one finds that $\gamma^*$ is around 2 or 3, so that $R$ “freezes,” becoming nearly constant in the infrared region, while it falls rapidly around

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1 At low $n_f$ it appears that different physical quantities may have rather different infrared behaviours [23], unlike the BZ region where there is a high degree of universality.
Fig. 4. \( R^* \) as a function of \( n_f \) in the 4\(^{th} \) order BZ expansion (dashed curve) compared with OPT (large points) and EC (small points) results from Ref. [20]. The OPT points are shown as blue circles when they arise from a fixed point and as red squares when they arise from an “unfixed point.” Error bars indicate the estimated uncertainty in the OPT results. (They are not shown for \( n_f = 7,\ldots,11 \), where they would extend well beyond the bounds of the plot.) The dotted blue curve represents \( R^* = 0.9/b \), a purely speculative guess at the large-\( b \) form.

Q \simeq \tilde{\Lambda}_R. In the BZ region, by contrast, \( \gamma^* \) is small (\( \lesssim 1 \)), resulting in the infrared “spike” of Fig. 3 and the sloping plateau around \( Q \simeq \tilde{\Lambda}_R \).

The OPT and EC results in Fig. 4 agree remarkably well at both low and high \( n_f \). In the intermediate region \( 7 \lesssim n_f \lesssim 13 \) they actually differ only at the very lowest energies, because OPT indicates a much more dramatic “spike” in the infrared, of very uncertain size — it could well be even bigger than predicted. This is because the infrared limit in OPT here does not result from a fixed point but from an “unfixed point” and a “pinch mechanism” that leads to \( (R^* - R) \propto 1/|\ln Q|^2 \), corresponding to \( \gamma^* = 0 \). For details, see Ref. [20].

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