Mantaining Dynamic Matrices for Fully Dynamic Transitive Closure *

Camil Demetrescu †
Dipartimento di Informatica e Sistemistica
Universit`a di Roma “La Sapienza”, Roma, Italy

Giuseppe F. Italiano ‡
Dipartimento di Informatica, Sistemi e Produzione
Universit`a di Roma “Tor Vergata”, Roma, Italy

Abstract

In this paper we introduce a general framework for casting fully dynamic transitive closure into the problem of reevaluating polynomials over matrices. With this technique, we improve the best known bounds for fully dynamic transitive closure. In particular, we devise a deterministic algorithm for general directed graphs that achieves $O(n^2)$ amortized time for updates, while preserving unit worst-case cost for queries. In case of deletions only, our algorithm performs updates faster in $O(n)$ amortized time.

Our matrix-based approach yields an algorithm for directed acyclic graphs that breaks through the $O(n^2)$ barrier on the single-operation complexity of fully dynamic transitive closure. We can answer queries in $O(n^\epsilon)$ time and perform updates in $O(n^{\omega(1, \epsilon, 1) - \epsilon} + n^{1+\epsilon})$ time, for any $\epsilon \in [0, 1]$, where $\omega(1, \epsilon, 1)$ is the exponent of the multiplication of an $n \times n^{\epsilon}$ matrix by an $n^{\epsilon} \times n$ matrix. The current best bounds on $\omega(1, \epsilon, 1)$ imply an $O(n^{0.58})$ query time and an $O(n^{1.58})$ update time. Our subquadratic algorithm is randomized, and has one-side error.

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†Email: demetres@dis.uniroma1.it. URL: http://www.dis.uniroma1.it/~demetres
Part of this work has been done while visiting AT&T Shannon Laboratory, Florham Park, NJ.

‡Email: italiano@info.uniroma2.it. URL: http://www.info.uniroma2.it/~italiano
Part of this work has been done while visiting Columbia University, New York, NY.
1 Introduction

In this paper we present fully dynamic algorithms for maintaining the transitive closure of a directed graph. A dynamic graph algorithm maintains a given property on a graph subject to dynamic changes, such as edge insertions and edge deletions. We say that an algorithm is fully dynamic if it can handle both edge insertions and edge deletions. A partially dynamic algorithm can handle either edge insertions or edge deletions, but not both: we say that it is incremental if it supports insertions only, and decremental if it supports deletions only. In the fully dynamic transitive closure problem we wish to maintain a directed graph $G = (V, E)$ under an intermixed sequence of the following operations:

$\text{Insert}(x, y)$: insert an edge from $x$ to $y$ in $G$;

$\text{Delete}(x, y)$: delete the edge from $x$ to $y$ in $G$;

$\text{Query}(x, y)$: report yes if there is a path from $x$ to $y$ in $G$, and no otherwise.

Throughout the paper, we denote by $m$ and by $n$ the number of edges and vertices in $G$, respectively.

Research on dynamic transitive closure spans over two decades. Before describing the results known, we list the bounds obtainable with simple-minded methods. If we do nothing during each update, then we have to explore the whole graph in order to answer reachability queries: this gives $O(n^2)$ time per query and $O(1)$ time per update in the worst case. On the other extreme, we could recompute the transitive closure from scratch after each update; as this task can be accomplished via matrix multiplication [1, 18], this approach yields $O(1)$ time per query and $O(n^\omega)$ time per update in the worst case, where $\omega$ is the best known exponent for matrix multiplication (currently $\omega < 2.38$ [2]).

Previous Work. For the incremental version of the problem, the first algorithm was proposed by Ibaraki and Katoh [11] in 1983: its running time was $O(n^3)$ over any sequence of insertions. This bound was later improved to $O(n)$ amortized time per insertion by Italiano [12] and also by La Poutré and van Leeuwen [17]. Yellin [13] gave an $O(m^*\delta_{max})$ algorithm for $m$ edge insertions, where $m^*$ is the number of edges in the final transitive closure and $\delta_{max}$ is the maximum out-degree of the final graph. All these algorithms maintain explicitly the transitive closure, and so their query time is $O(1)$.

The first decremental algorithm was again given by Ibaraki and Katoh [11], with a running time of $O(n^2)$ per deletion. This was improved to $O(m)$ per deletion by La Poutré and van Leeuwen [17]. Italiano [13] presented an algorithm that achieves $O(n)$ amortized time per deletion on directed acyclic graphs. Yellin [13] gave an $O(m^*\delta_{max})$ algorithm for $m$ edge deletions, where $m^*$ is the
initial number of edges in the transitive closure and $\delta_{\text{max}}$ is the maximum out-degree of the initial graph. Again, the query time of all these algorithms is $O(1)$. More recently, Henzinger and King [9] gave a randomized decremental transitive closure algorithm for general directed graphs with a query time of $O(n/\log n)$ and an amortized update time of $O(n \log^2 n)$.

The first fully dynamic transitive closure algorithm was devised by Henzinger and King [9] in 1995: they gave a randomized Monte Carlo algorithm with one-side error supporting a query time of $O(n/\log n)$ and an amortized update time of $O(n \hat{m}^{0.58} \log^2 n)$, where $\hat{m}$ is the average number of edges in the graph throughout the whole update sequence. Since $\hat{m}$ can be as high as $O(n^2)$, their update time is $O(n^{2.16} \log^2 n)$. Khanna, Motwani and Wilson [14] proved that, when a lookahead of $\Theta(n^{0.18})$ in the updates is permitted, a deterministic update bound of $O(n^{2.18})$ can be achieved. Very recently, King and Sagert [16] showed how to support queries in $O(1)$ time and updates in $O(n^{2.26})$ time for general directed graphs and $O(n^2)$ time for directed acyclic graphs; their algorithm is randomized with one-side error. The bounds of King and Sagert were further improved by King [15], who exhibited a deterministic algorithm on general digraphs with $O(1)$ query time and $O(n^2 \log n)$ amortized time per update operations, where updates are insertions of a set of edges incident to the same vertex and deletions of an arbitrary subset of edges. We remark that all these algorithms (except [15]) use fast matrix multiplication as a subroutine.

We observe that fully dynamic transitive closure algorithms with $O(1)$ query time maintain explicitly the transitive closure of the input graph, in order to answer each query with exactly one lookup (on its adjacency matrix). Since an update may change as many as $\Omega(n^2)$ entries of this matrix, $O(n^2)$ seems to be the best update bound that one could hope for this class of algorithms. It is thus quite natural to ask whether the $O(n^2)$ update bound can be actually realized for fully dynamic transitive closure on general directed graphs while maintaining one lookup per query. Another important question, if one is willing to spend more time for queries, is whether the $O(n^2)$ barrier for the single-operation time complexity of fully dynamic transitive closure can be broken. We remark that this has been an elusive goal for many years.

**Our Results.** In this paper, we affirmatively answer both questions. We first exhibit a deterministic algorithm for fully dynamic transitive closure on general digraphs that does exactly one matrix look-up per query and supports updates in $O(n^2)$ amortized time, thus improving over [15]. Our algorithm can also support within the same time bounds the generalized updates of [15], i.e., insertion of a set of edges incident to the same vertex and deletion of an arbitrary subset of edges. In the special case of deletions only, our algorithm achieves $O(n)$ amortized time for deletions and $O(1)$ time for queries: this generalizes to directed graphs the bounds of [13], and improves over [9].

As our second contribution, we present the first algorithm that breaks through
the $O(n^2)$ barrier on the single-operation time complexity of fully dynamic transitive closure. In particular, we show how to trade off query times for updates on directed acyclic graphs: each query can be answered in time $O(n^\omega(1,\epsilon,1)+\epsilon) + n^{1+\epsilon}$, for any $\epsilon \in [0,1]$, where $\omega(1,\epsilon,1)$ is the exponent of the multiplication of an $n \times n^\epsilon$ matrix by an $n^\epsilon \times n$ matrix. Balancing the two terms in the update bound yields that $\epsilon$ must satisfy the equation $\omega(1,\epsilon,1) = 1 + 2\epsilon$. The current best bounds on $\omega(1,\epsilon,1)$ [2, 10] imply that $\epsilon < 0.58$ [20]. Thus, the smallest update time is $O(n^{1.58})$, which gives a query time of $O(n^{0.58})$. Our subquadratic algorithm is randomized, and has one-side error.

All our algorithms are based on a novel technique: we introduce a general framework for maintaining polynomials defined over matrices, and we cast fully dynamic transitive closure into this framework. In particular, our deterministic algorithm hinges upon the equivalence between transitive closure and matrix multiplication on a closed semiring; this relation has been known for over 30 years (see e.g., the results of Munro [18], Furman [8] and Fischer and Meyer [7]) and yields the fastest known static algorithm for transitive closure. Surprisingly, no one before seems to have exploited this equivalence in the dynamic setting: some recent algorithms [4, 12, 16] make use of fast matrix multiplication, but only as a subroutine for fast updates. Differently from other approaches, the crux of our method is to use dynamic reevaluation of products of Boolean matrices as the kernel for solving dynamic transitive closure.

The remainder of this paper is organized as follows. We first formally define the fully dynamic transitive closure problem and we give preliminary definitions in Section 2. A high-level overview of our approach is given in Section 3. In Section 4 we introduce two problems on dynamic matrices, and show how to solve them efficiently. Next, we show how to exploit these problems on dynamic matrices for the design of three efficient fully dynamic algorithms for transitive closure in Section 5, Section 6 and Section 7, respectively. Finally, in Section 8 we list some concluding remarks.

### 2 Fully Dynamic Transitive Closure

In this section we give a more formal definition of the fully dynamic transitive closure problem considered in this paper. We assume the reader to be familiar with standard graph and algebraic terminology as contained for instance in [1, 3].

**Definition 1** Let $G = (V,E)$ be a directed graph and let $TC(G) = (V,E')$ be its transitive closure. The **Fully Dynamic Transitive Closure Problem** consists of maintaining a data structure $G$ for graph $G$ under an intermixed sequence $\sigma = \langle G.\text{Op}_1, \ldots, G.\text{Op}_k \rangle$ of initialization, update, and query operations. Each operation $G.\text{Op}_j$ on data structure $G$ can be either one of the following:
• $G.\text{Init}(A)$: perform the initialization operation $E \leftarrow A$, where $A \subseteq V \times V$.

• $G.\text{Insert}(v, I)$: perform the update $E \leftarrow E \cup \{(u, v) \mid u \in V \land (u, v) \in I\} \cup \{(v, u) \mid u \in V \land (v, u) \in I\}$, where $I \subseteq E$ and $v \in V$. We call this update a $v$-Centered insertion in $G$.

• $G.\text{Delete}(D)$: perform the update $E \leftarrow E - D$, where $D \subseteq E$.

• $G.\text{Query}(x, y)$: perform a query operation on $TC(G)$ by returning 1 if $(x, y) \in E'$ and 0 otherwise.

Few remarks are in order at this point. First, the generalized $\text{Insert}$ and $\text{Delete}$ updates considered here have been first introduced by King in [15]. With just one operation, they are able to change the graph by adding or removing a whole set of edges, rather than a single edge, as illustrated in Figure 1. Second, we consider explicitly initializations of the graph $G$ and, more generally than in the traditional definitions of dynamic problems, we allow them to appear everywhere in sequence $\sigma$. This gives more generality to the problem, and allows for more powerful data structures, i.e., data structures that can be restarted at run time on a completely different input graph. Differently from others variants of the problem, we do not address the issue of returning actual paths between nodes, and we just consider the problem of answering reachability queries.

It is well known that, if $G = (V, E)$ is a directed graph and $X_G$ is its adjacency matrix, computing the Kleene closure $X_G^*$ of $X_G$ is equivalent to computing the (reflexive) transitive closure $TC(G)$ of $G$. For this reason, in this paper, instead
of considering directly the problem introduced in Definition 1, we study an equivalent problem on matrices. Before defining it formally, we need some preliminary notation.

**Definition 2** If $X$ is a matrix, we denote by $I_{X,i}$ and $J_{X,j}$ the matrices equal to $X$ in the $i$-th row and $j$-th column, respectively, and null in any other entries:

$$I_{X,i}[x,y] = \begin{cases} X[x,y] & \text{if } x = i \\ 0 & \text{otherwise} \end{cases}$$

$$J_{X,j}[x,y] = \begin{cases} X[x,y] & \text{if } y = i \\ 0 & \text{otherwise} \end{cases}$$

**Definition 3** Let $X$ and $Y$ be $n \times n$ Boolean matrices. Then $X \subseteq Y$ if and only if $X[x,y] = 1 \Rightarrow Y[x,y] = 1$ for any $x, y \in \{1, \ldots , n\}$.

We are now ready to define a dynamic version of the problem of computing the Kleene closure of a Boolean matrix. In what follows, we assume that algebraic operations $+$ and $-$ are performed modulo $n + 1$ by looking at Boolean values 0 and 1 as integer numbers. Integer results are binarized by converting back nonzero values into 1 and zero values into 0. We remark that in our dynamic setting operator $-$ is just required to flip matrix entries from 1 to 0.

**Definition 4** Let $X$ be an $n \times n$ Boolean matrix and let $X^*$ be its Kleene closure. We define the **Fully Dynamic Boolean Matrix Closure Problem** as the problem of maintaining a data structure $X$ for matrix $X$ under an intermixed sequence $\sigma = \langle X.\text{Op}_1, \ldots , X.\text{Op}_k \rangle$ of initialization, update, and query operations. Each operation $X.\text{Op}_j$ on data structure $X$ can be either one of the following:

- $X.\text{Init}^*(Y)$: perform the initialization operation $X \leftarrow Y$, where $Y$ is an $n \times n$ Boolean matrix.
- $X.\text{Set}^*(i, \Delta X)$: perform the update $X \leftarrow X + I_{\Delta X,i} + J_{\Delta X,i}$, where $\Delta X$ is an $n \times n$ Boolean matrix and $i \in \{1, \ldots , n\}$. We call this kind of update an $i$-centered set operation on $X$ and we call $\Delta X$ Update Matrix.
- $X.\text{Reset}^*(\Delta X)$: perform the update $X \leftarrow X - \Delta X$, where $\Delta X \subseteq X$ is an $n \times n$ Boolean update matrix.
- $X.\text{Lookup}^*(x,y)$: return the value of $X^*[x,y]$, where $x, y \in \{1, \ldots , n\}$.

Notice that $\text{Set}^*$ is allowed to modify only the $i$-th row and the $i$-th column of $X$, while $\text{Reset}^*$ and $\text{Init}^*$ can modify any entries of $X$. We stress the strong correlation between Definition 1 and Definition 4: if $G$ is a graph and $X$ is its adjacency matrix, operations $X.\text{Init}^*$, $X.\text{Set}^*$, $X.\text{Reset}^*$, and $X.\text{Lookup}^*$ are equivalent to operations $G.\text{Init}$, $G.\text{Insert}$, $G.\text{Delete}$, and $G.\text{Query}$, respectively.
3 Overview of Our Approach

In this section we give an overview of the new ideas presented in this paper, discussing the most significant aspects of our techniques.

Our approach consists of reducing fully dynamic transitive closure to the problem of maintaining efficiently polynomials over matrices subject to updates of their variables. In particular, we focus on the equivalent problem of fully dynamic Kleene closure and we show that efficient data structures for it can be realized using efficient data structures for maintaining polynomials over matrices.

Suppose that we have a polynomial over Boolean matrices, e.g., $P(X, Y, Z, W) = X + YZ^2W$, where matrices $X, Y, Z$ and $W$ are its variables. The value $P(X, Y, Z, W)$ of the polynomial can be computed via sum and multiplication of matrices $X, Y, Z$ and $W$ in $O(n^{2.38})$. Now, what kind of modifications can we perform on a variable, e.g., variable $Z$, so as to have the chance of updating the value of $P(X, Y, Z, W)$ in less than $O(n^{2.38})$ time?

In Section 4.1 we show a data structure that allows us to reevaluate correctly $P(X, Y, Z, W)$ in just $O(n^2)$ amortized time after flipping to 1 any entries of $Z$ that were 0, provided they lie on a row or on a column (SetRow or SetCol operation), of after flipping to 0 any entries of $Z$ that were 1 (Reset operation). This seems a step forward, but are this kind of updates of variables powerful enough to be useful our original problem of fully dynamic transitive closure? Unfortunately, the answer is no. Actually, we also require the more general Set operation of flipping to 1 any entries of $Z$ that were 0. Now, if we want to have our polynomial always up to date after each variable change of this kind, it seems that there is no way of doing any better than recomputing everything from scratch.

So let us lower our expectations on our data structure for maintaining $P$, and tolerate errors. In exchange, our data structure must support efficiently the general Set operation. The term “errors” here means that we maintain a “relaxed” version of the correct value of the polynomial, where some 0’s may be incorrect. The only important property that we require is that any 1’s that appear in the correct value of the polynomial after performing a SetRow or SetCol operation must also appear in the relaxed value that we maintain. This allows us to support any Set operation efficiently in a lazy fashion (so in the following we call it LazySet) and is powerful enough for our original problem of fully dynamic transitive closure.

Actually, doing things lazily while maintaining the desired properties in our data structure for polynomials is the major technical difficulty in Section 4.1. Sections 5 and 6 then show two methods to solve the fully dynamic Boolean matrix closure problem by using polynomials of Boolean matrices as if they were building blocks. The second method yields the fastest known algorithm for fully dynamic transitive closure with constant query time. If we give up maintaining polynomials of degree > 1, using a surprisingly simple lazy technique we can
even support certain kinds of variable updates in subquadratic worst-case time per operation (see Section 4.2). This turns out to be once again applicable to fully dynamic transitive closure, yielding the first subquadratic algorithms known so far for the problem (see Section 7).

4 Dynamic Matrices

In this section we consider two problems on dynamic matrices and we devise fast algorithms for solving them. As we already stated, these problems will be central to designing efficient algorithms for the fully dynamic Boolean matrix closure problem introduced in Definition 4. In more detail, in Section 4.1 we address the problem of reevaluating polynomials over Boolean matrices under modifications of their variables. We propose a data structure for maintaining efficiently the special class of polynomials of degree 2 consisting of single products of Boolean matrices. We show then how to use this data structure for solving the more general problem on arbitrary polynomials. In Section 4.2 we study the problem of finding an implicit representation for integer matrices that makes it possible to update as many as \(\Omega(n^2)\) entries per operation in \(o(n^2)\) worst-case time at the price of increasing the lookup time required to read a single entry.

4.1 Dynamic Reevaluation of Polynomials over Boolean Matrices

We now study the problem of maintaining the value of polynomials over Boolean matrices under updates of their variables. We define these updates so that they can be useful later on for our original problem of dynamic Boolean matrix closure. We first need some preliminary definitions.

**Definition 5** Let \(X\) be a data structure. We denote by \(X_i\) the value of \(X\) at time \(i\), i.e., the value of \(X\) after the \(i\)-th operation in a sequence of operations that modify \(X\). By convention, we assume that at time 0 any numerical value in \(X\) is zero. In particular, if \(X\) is a Boolean matrix, \(X_0 = 0_n\).

In the following definition we formally introduce our first problem on dynamic matrices.

**Definition 6** Let \(\mathcal{B}_n\) be the set of \(n \times n\) Boolean matrices and let

\[
P = \sum_{a=1}^{h} T_a
\]
be a polynomial with \( h \) terms defined over \( \mathcal{B}_n \), where each

\[
T_a = \prod_{b=1}^{k} X_b^a
\]

has degree exactly \( k \) and variables \( X_b^a \in \mathcal{B}_n \) are distinct. We consider the problem of maintaining a data structure \( P \) for the polynomial \( P \) under an intermixed sequence \( \sigma = \langle P.0p_1, \ldots, P.0p_l \rangle \) of initialization, update, and query operations. Each operation \( P.0p_j \) on the data structure \( P \) can be either one of the following:

- \( P.\text{Init}(Z_1, \ldots, Z_h) \): perform the initialization \( X_b^a \leftarrow Z_b^a \) of the variables of polynomial \( P \), where each \( Z_b^a \) is an \( n \times n \) Boolean matrix.
- \( P.\text{SetRow}(i, \Delta X, X_b^a) \): perform the row update operation \( X_b^a \leftarrow X_b^a + I_{\Delta X,i} \), where \( \Delta X \) is an \( n \times n \) Boolean update matrix. The operation sets to 1 the entries in the \( i \)-th row of variable \( X_b^a \) of polynomial \( P \) as specified by matrix \( \Delta X \).
- \( P.\text{SetCol}(i, \Delta X, X_b^a) \): perform the column update operation \( X_b^a \leftarrow X_b^a + J_{\Delta X,i} \), where \( \Delta X \) is an \( n \times n \) Boolean update matrix. The operation sets to 1 the entries in the \( i \)-th column of variable \( X_b^a \) of polynomial \( P \) as specified by matrix \( \Delta X \).
- \( P.\text{LazySet}(\Delta X, X_b^a) \): perform the update operation \( X_b^a \leftarrow X_b^a + \Delta X \), where \( \Delta X \) is an \( n \times n \) Boolean update matrix. The operation sets to 1 the entries of variable \( X_b^a \) of polynomial \( P \) as specified by matrix \( \Delta X \).
- \( P.\text{Reset}(\Delta X, X_b^a) \): perform the update operation \( X_b^a \leftarrow X_b^a - \Delta X \), where \( \Delta X \) is an \( n \times n \) Boolean update matrix such that \( \Delta X \subseteq X_b^a \). The operation resets to 0 the entries of variable \( X_b^a \) of polynomial \( P \) as specified by matrix \( \Delta X \).
- \( P.\text{Lookup}() \): answer a query about the value of \( P \) by returning an \( n \times n \) Boolean matrix \( Y_j \), such that \( M_j \subseteq Y_j \subseteq P_j \), where \( M \) is an \( n \times n \) Boolean matrix whose value at time \( j \) is defined as follows:

\[
M_j = \sum_{\substack{1 \leq i \leq j : \\ 0p_i \neq \text{LazySet}}} (P_i - P_{i-1})
\]

and \( P_i \) is the value of polynomial \( P \) at time \( i \). According to this definition, we allow the answer about the value of \( P \) to be affected by one-side error.

\footnote{In the following, we omit specifying explicitly the dependence of a polynomial on its variables, and we denote by \( P \) both the function \( P(X_1, \ldots, X_k) \) and the value of this function for fixed values of \( X_1, \ldots, X_k \), assuming that the correct interpretation is clear from the context.}
SetRow and SetCol are allowed to modify only the \(i\)-th row and the \(i\)-th column of variable \(X^a_b\), respectively, while LazySet, Reset and Init can modify any entries of \(X^a_b\). It is crucial to observe that in the operational definition of Lookup we allow one-side errors in answering queries on the value of \(P\). In particular, in the answer there have to be no incorrect 1’s and the error must be bounded: Lookup has to return a matrix \(Y\) that contains at least the 1’s in \(M\), and no more than the 1’s in \(P\). As we will see later on, this operational definition simplifies the task of designing efficient implementations of the operations and is still powerful enough to be useful for our original problem of dynamic Boolean matrix closure.

The following lemma shows that the presence of errors is related to the presence of LazySet operations in sequence \(\sigma\). In particular, it shows that, if no LazySet operation is performed, then Lookup makes no errors and returns the correct value of polynomial \(P\).

**Lemma 1** Let \(P\) be a polynomial and let \(\sigma = \langle P \cdot Op_1, \ldots, P \cdot Op_k \rangle\) be a sequence of operations on \(P\). If \(Op_i \neq \text{LazySet}\) for all \(1 \leq i \leq j \leq k\), then \(M_j = P_j\).

**Proof.** The proof easily follows by telescoping the sum that defines \(M_j\):

\[
M_j = P_j - P_{j-1} + P_{j-1} - P_{j-2} + \cdots + P_2 - P_1 + P_1 - P_0 = P_j - P_0 = P_j.
\]

Errors in the answers given by Lookup may appear as soon as LazySet operations are performed in sequence \(\sigma\). To explain how \(M\) is defined mathematically, notice that \(M_0 = 0_n\) by Definition 5 and \(M\) sums up all the changes that the value of \(P\) has undergone up to the \(j\)-th operation, except for the changes due to LazySet operations, which are ignored. This means that, if any entry \(P[x,y]\) flips from 0 to 1 or vice-versa due to an operation \(Op_j\) different from LazySet, so does \(M[x,y]\) and thus \(Y[x,y]\).

As a side note, we remark that it is straightforward to extend the results of this section to the general class of polynomials with terms of different degrees and multiple occurrences of the same variable.

We now focus on the problem of implementing the operations introduced in Definition 6. A simple-minded implementation of the operations on \(P\) is the following:

- Maintain variables \(X^a_b\), terms \(T_a\), and a matrix \(Y\) that contains the value of the polynomial.
- Recompute from scratch \(T_a\) and the value of \(Y = P = T_1 + \cdots + T_h\) after each Init, SetRow, SetCol and Reset that change \(X^a_b\).
- Do nothing after a LazySet operation, except for updating \(X^a_b\). This means that \(Y\) may be no longer equal to \(P\) after the operation.
• Let \textbf{Lookup} return the maintained value of $Y$.

It is easy to verify that at any time $j$, i.e., after the $j$-th operation, $0p_j \neq \text{LazySet}$ implies $Y = P$ and $0p_j = \text{LazySet}$ implies $Y = M$. In other words, the value $Y$ returned by \textbf{Lookup} oscillates between the exact value $P$ of the polynomial and the value $M$ obtained without considering \textbf{LazySet} operations.

With the simple-minded implementation above, we can support \textbf{Init} in $O(h \cdot k \cdot n^2 + h \cdot n^2)$ time, \textbf{SetRow} and \textbf{SetCol} in $O(k \cdot n^2)$ time, \textbf{Reset} in $O(k \cdot n^2 + h \cdot n^2)$ time, and \textbf{Lookup} and \textbf{LazySet} in $O(h \cdot k \cdot n^2)$ time.

The remainder of this section provides more efficient solutions for the problem. In particular, we present a data structure that supports \textbf{Lookup} and \textbf{LazySet} operations in $O(n^2)$ worst-case time, \textbf{SetRow}, \textbf{SetCol} and \textbf{Reset} operations in $O(k \cdot n^2)$ amortized time, and \textbf{Init} operations in $O(h \cdot k \cdot n^2 + h \cdot n^2)$ worst-case time. The space used is $O(h \cdot k^2 \cdot n^2)$. Before considering the general case where polynomials have arbitrary degree $k$, we focus on the special class of polynomials where $k = 2$.

### 4.1.1 Data Structure for Polynomials of Degree $k = 2$

We define a data structure for $P$ that allows us to maintain explicitly the value $Y_j$ of the matrix $Y$ at any time $j$ during a sequence $\langle P, 0p_1, \ldots, P, 0p_l \rangle$ of operations. This makes it possible to perform \textbf{Lookup} operations in optimal quadratic time. We avoid recomputing from scratch the value of $Y$ after each update as in the simple-minded method, and we propose efficient techniques for propagating to $Y$ the effects of changes of variables $X_a$ due to \textbf{SetRow}, \textbf{SetCol} and \textbf{Reset} operations. In case of \textbf{LazySet}, we only need to update the affected variables, leaving the other elements in the data structure unaffected. This, of course, implies that after a \textbf{LazySet} at time $j$, the maintained value $Y_j$ will be clearly not synchronized with the correct value $P_j$ of the polynomial. Most technical difficulties of this section come just from this lazy maintenance of $Y_j$.

Our data structure for representing a polynomial of degree 2 of the form $P = X_{1}^{1} \cdot X_{2}^{1} + \ldots + X_{1}^{h} \cdot X_{2}^{h}$ is presented below.

**Data Structure 1** We maintain the following elementary data structures with $O(h \cdot n^2)$ space:

1. $2h$ matrices $X_1^a$ and $X_2^a$ for $1 \leq a \leq h$;

2. $h$ integer matrices $\text{Prod}_1, \ldots, \text{Prod}_h$ such that $\text{Prod}_a$ maintains a “lazy” count of the number of witnesses of the product $T_a = X_1^a \cdot X_2^a$.

3. an integer matrix $S$ such that $S[x, y] = |\{a : \text{Prod}_a[x, y] > 0\}|$. We assume that $Y_j[x, y] = 1 \iff S[x, y] > 0$. 

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4. 2h integer matrices LastFlip\(_X\), one for each matrix \(X = X^a\). For any entry \(X[x, y] = 1\), LastFlip\(_X[x, y]\) is the time of the most recent operation that caused \(X[x, y]\) to flip from 0 to 1. More formally:

\[
\text{LastFlip}_X[x, y] = \max\{t \mid X_t[x, y] - X_{t-1}[x, y] = 1\}
\]

if \(X_j[x, y] = 1\), and is undefined otherwise;

5. 2h integer vectors LastRow\(_X\), one for each matrix \(X = X^a\). LastRow\(_X[i]\) is the time of the last Init or SetRow operation on the \(i\)-th row of \(X\), and zero if no such operation was ever performed. More formally:

\[
\text{LastRow}_X[i] = \max\{0, t \mid \text{Op}_t = \text{Init}(\ldots) \lor \text{Op}_t = \text{SetRow}(i, \Delta X, X)\}
\]

We also maintain similar vectors LastCol\(_X\);

6. a counter Time of the number of performed operations;

Before getting into the full details of our implementation of operations, we give an overview of the main ideas. We consider how the various operations should affect the data structure. In particular, we suppose that an operation changes any entries of variable \(X^q\) in a term \(T_a = X^q_1 \cdot X^q_2\), and we define what our implementation should do on matrix \(Prod_a\):

SetRow/SetCol: if some entry \(X^q_1[x, y]\) is flipping to 1, then \(y\) becomes a witness in the product \(X^q_1 \cdot X^q_2\) for any pair \(x, z\) such that \(X^q_2[y, z] = 1\). Then we should put \(y\) in the count \(Prod_a[x, z]\), if it is not already counted. Moreover, if some entry \(X^q_1[x, y]\) was already 1, but for some pair \(x, z\) the index \(y\) is not counted in \(Prod_a[x, z]\), then we should put \(y\) in the count \(Prod_a[x, z]\).

LazySet: if some entry \(X^q_1[x, y]\) is flipping to 1, then \(y\) becomes a witness for any pair \(x, z\) such that \(X^q_2[y, z] = 1\). Then we should put \(y\) in the count \(Prod_a[x, z]\), if it is not already counted, but we do not do this.

Reset: if some entry \(X^q_1[x, y]\) is flipping to 0, then \(y\) is no longer a witness for all pairs \(x, z\) such that \(X^q_2[y, z] = 1\). Then we should remove \(y\) from the count \(Prod_a[x, z]\), if it is currently counted.

Note that after performing LazySet there may be triples \((x, y, z)\) such that both \(X^q_1[x, y] = 1\) and \(X^q_2[y, z] = 1\), but \(y\) is not counted in \(Prod_a[x, z]\). Now the problem is: is there any property that we can exploit to tell if a given \(y\) is counted or not in \(Prod_a[x, z]\) whenever both \(X^q_1[x, y] = 1\) and \(X^q_2[y, z] = 1\)?

We introduce a predicate \(P_a(x, y, z)\), \(1 \leq x, y, z \leq n\), such that \(P_a(x, y, z)\) is true if and only if the last time any of the two entries \(X^q_1[x, y]\) and \(X^q_2[y, z]\) flipped from 0 to 1 is before the time of the last update operation on the \(x\)-th row or the \(y\)-th column of \(X^q_1\) and the time of the last update operation on the \(y\)-th row or the \(z\)-th column of \(X^q_2\). In short:
The property $P_a$ answers our previous question and allows it to define the following invariant that we maintain in our data structure. We remark that we do not need to maintain $P_a$ explicitly in our data structure as it can be computed on demand in constant time by accessing $LastFlip$ and $LastRow$.

**Invariant 1** For any term $T_a = X_{1}^{a} \cdot X_{2}^{a}$ in polynomial $P$, at any time during a sequence of operations $\sigma$, the following invariant holds for any pair of indices $x, z$:

$$Prod_a[x, z] = |\{ y : X_{1}^{a}[x, y] = 1 \land X_{2}^{a}[y, z] = 1 \land P_a(x, y, z)\}|$$

According to Invariant [1], it is clear that the value of each entry $Prod_a[x, z]$ is a “lazy” count of the number of witnesses of the Boolean matrix product $T_a[x, z] = (X_{1}^{a} \cdot X_{2}^{a})[x, z]$. Notice that, since $T_a[x, z] = 1 \iff \exists y : X_{1}^{a}[x, y] = 1 \land X_{2}^{a}[y, z] = 1$, we have that $Prod_a[x, z] > 0 \Rightarrow T_a[x, z] = 1$. Thus, we may think of $P_a$ as a “relaxation” property.

We implement the operations introduced in Definition 6 as described next, assuming that the operation $Time \leftarrow Time + 1$ is performed just before each operation:

**Init**

```plaintext
procedure Init(Z_{1}^{1}, Z_{1}^{2}, \ldots, Z_{h}^{1}, Z_{h}^{2})
1. begin
2. for each $a$ do $X_{1}^{a} \leftarrow Z_{1}^{a}$; $X_{2}^{a} \leftarrow Z_{2}^{a}$
3. { initialize members 2–5 of Data Structure [1] }
4. end
```

**Init** assigns the value of variables $X_{1}^{a}$ and $X_{2}^{a}$ and initializes elements 2–5 of Data Structure [1]. In particular, $LastFlip_{X}[x, y]$ is set to $Time$ for any $X[x, y] = 1$ and the same is done for $LastRow_{X}$ and $LastCol$ for any $i$. $Prod_a$ is initialized by computing the product $X_{1}^{a} \cdot X_{2}^{a}$ in the ring of integers, i.e., looking at $X_{b}^{a}$ as integer matrices.

**Lookup**

```plaintext
function Lookup()
1. begin
2. return $Y$ s.t. $Y[x, y] = 1 \iff S[x, y] > 0$
3. end
```

**Lookup** simply returns a binarized version $Y$ of matrix $S$ defined in Data Structure [1].
SetRow

\[
\text{procedure SetRow}(i, \Delta X, X^a_b)
\]
1. \textbf{begin}
2. \(X^a_b \leftarrow X^a_b + I_{\Delta X,i}\)
3. \{update LastFlip\_X^a_b}\
4. \textbf{if} \(b = 1\) \textbf{then}\
5. \quad \textbf{for each} \(x: X^a_1[i,x] = 1\) \textbf{do}\
6. \quad \quad \textbf{for each} \(y: X^a_2[x,y] = 1\) \textbf{do}\
7. \quad \quad \quad \textbf{if not} \(P_a(i,x,y)\) \textbf{then}\
8. \quad \quad \quad \quad \text{Prod}_a[i,y] \leftarrow \text{Prod}_a[i,y] + 1\
9. \quad \quad \quad \textbf{if} \text{Prod}_a[i,y] = 1 \textbf{then} \(S[i,y] \leftarrow S[i,y] + 1\)\
10. \quad \textbf{else} \{b = 2: similar to P\_SetCol\(i, \Delta X, X^a_1)\}\n11. \quad \text{LastRow}_{X^a_1}[i] \leftarrow \text{Time}\
12. \textbf{end}\

After performing an \(i\)-centered insertion in \(X^a_b\) on line 2 and after updating LastFlip\_\(X^a_2\) on line 3, SetRow checks on lines 5–7 for any triple \((i, x, y)\) such that the property \(P_a(i,x,y)\) is still not satisfied, but will be satisfied thanks to line 11, and increases \(\text{Prod}_a\) and \(S\) accordingly (lines 8–9).

SetCol

\[
\text{procedure SetCol}(i, \Delta X, X^a_b)
\]
1. \textbf{begin}
2. \(X^a_b \leftarrow X^a_b + J_{\Delta X,i}\)
3. \{update LastFlip\_X^a_b}\
4. \textbf{if} \(b = 1\) \textbf{then}\
5. \quad \textbf{for each} \(x: X^a_1[i,x] = 1\) \textbf{do}\
6. \quad \quad \textbf{for each} \(y: X^a_2[x,y] = 1\) \textbf{do}\
7. \quad \quad \quad \textbf{if not} \(P_a(i,x,y)\) \textbf{then}\
8. \quad \quad \quad \quad \text{Prod}_a[x,y] \leftarrow \text{Prod}_a[x,y] + 1\
9. \quad \quad \quad \textbf{if} \text{Prod}_a[x,y] = 1 \textbf{then} \(S[x,y] \leftarrow S[x,y] + 1\)\
10. \quad \textbf{else} \{b = 2: similar to P\_SetRow\(i, \Delta X, X^a_1)\}\n11. \quad \text{LastCol}_{X^a_1}[i] \leftarrow \text{Time}\
12. \textbf{end}\

Similar to SetRow.

LazySet

\[
\text{procedure LazySet}(\Delta X, X^a_b)
\]
1. \textbf{begin}
2. \(X^a_b \leftarrow X^a_b + \Delta X\)
3. \{update LastFlip\_X^a_b}\
4. \textbf{end}\

LazySet simply sets to 1 any entries in \(X^a_b\) and updates LastFlip\_\(X^a_2\). We remark that no other object in the data structure is changed.
Reset

procedure Reset(\(\Delta X, X^a_b\))
begin
if \(b = 1\) then
for each \(x, y : \Delta X[x, y] = 1\) do
if \(\max\{LastRow_{X^a_1}[x], LastCol_{X^a_2}[y]\} \geq LastFlip_{X^a_1}[x, y]\) then
for each \(z : X^a_1[y, z] = 1\) do
if \(P_a(x, y, z)\) then
\(Prod_a[x, z] \leftarrow Prod_a[x, z] - 1\)
else if \(Prod_a[x, z] = 0\) then \(S[x, z] \leftarrow S[x, z] - 1\)
for each \(z : X^a_1[y, z] = 1\) ∧ \(LastCol_{X^a_2}[z] > LastFlip_{X^a_1}[x, y]\) do
if \(P_a(x, y, z)\) then
\(Prod_a[x, z] \leftarrow Prod_a[x, z] - 1\)
else if \(Prod_a[x, z] = 0\) then \(S[x, z] \leftarrow S[x, z] - 1\)
end
\(X^a_b \leftarrow X^a_b - \Delta X\)
end
In lines 2-14, using \(LastRow_{X^a_1}, LastCol_{X^a_2}, \) and \(LastFlip_{X^a_1},\) \(\text{Reset}\) updates \(Prod_a\) and \(S\) so as to maintain Invariant \([1]\). Namely, for each reset entry \((x, y)\) specified by \(\Delta X\) (line 3), it looks for triples \((x, y, z)\) such that \(P(x, y, z)\) is going to be no more satisfied due to the reset of \(X^a_b[x, y]\) to be performed (lines 5-6 and lines 10–11); \(Prod_a\) and \(S\) are adjusted accordingly (lines 7–8 and lines 12-13).

The distinction between the two cases \(\max\{LastRow_{X^a_1}[x], LastCol_{X^a_2}[y]\} \geq LastFlip_{X^a_1}[x, y]\) and \(\max\{LastRow_{X^a_1}[x], LastCol_{X^a_2}[y]\} < LastFlip_{X^a_1}[x, y]\) in line 4 and in line 9, respectively, is important to achieve fast running times as it will be discussed in the proof of Theorem \([2]\). Here we only point out that if the test in line 4 succeeds, then we can scan any \(z\) s.t. \(X^a_1[y, z] = 1\) without affecting the running time. If this is not the case, then we need to process only indices \(z\) such that the test \(LastCol_{X^a_2}[y, z] > LastFlip_{X^a_1}[x, y]\) is satisfied, and avoid scanning other indices.

For this reason line 10 must be implemented very carefully by maintaining indices \(z\) in a list and by using a move-to-front strategy that brings index \(z\) to the front of the list as any operation \(\text{Init}(\ldots), \text{SetRow}(\ldots, \ldots)\) or \(\text{SetCol}(\ldots, \ldots)\) is performed on \(z\). In this way indices are sorted according to the dates of operations on them. As last step, \(\text{Reset}\) resets the entries of \(X^a_b\) as specified by \(\Delta X\) (line 15).

\[\triangleleft \triangleleft \triangleleft\]

The correctness of our implementation of operations \(\text{Init}, \text{SetRow}, \text{SetCol}, \text{LazySet}, \text{Reset}\) and \(\text{Lookup}\) is discussed in the following theorem.

**Theorem 1** At any time \(j\), \(\text{Lookup}\) returns a matrix \(Y_j\) that satisfies the relation \(M_j \subseteq Y_j \subseteq P_j\) as in Definition \([2]\).

**Proof.** We first remind that \(Y\) is the binarized version of \(S\) as follows from the implementation of \(\text{Lookup}\).
To prove that $Y \subseteq P$, observe that $\text{SetRow}$ increases $\text{Prod}_a[i,y]$ (line 8), and possibly $S$ (line 9), only if both $X_1^a[i,x] = 1$ and $X_2^a[x,y] = 1$: this implies that $T_a[i,y] = 1$ and $P[i,y] = 1$.

To prove that $M \subseteq Y$, notice that at time $j$ after performing an operation $\text{Op}_j = \text{SetRow}(i, \Delta X, X_a^b)$ on the $i$-th row of $X_1^a$, $P(i,x,y)$ is satisfied for any triple $(i,x,y)$ such that $X_1^a[i,x] = 1$ and $X_2^a[x,y] = 1$. This implies that $T_a[i,y] = 1$ and $P[i,y] = 1$.

From the definition of $M$ in Definition 6 we have that:

$$M_j[i,y] - M_{j-1}[i,y] = 1 \iff P_j[i,y] - P_{j-1}[i,y] = 1.$$ 

This proves the relation $M \subseteq Y$. A similar argument is valid also for $\text{SetCol}$, while $\text{LazySet}$ does not affect $S$ at all.

To complete the proof we remark that $Y = P$ just after any $\text{Init}$ operation and that $\text{Reset}$ leaves the data structure as if reset entries were never set to 1. Indeed, $\text{Reset}$ can be viewed as a sort of “undo” procedure that cancels the effects of previous $\text{SetRow}$, $\text{SetCol}$ or $\text{Init}$ operations.

We now analyze the complexity of our implementation of the operations on polynomials.

**Theorem 2** Any $\text{Lookup}$, $\text{SetRow}$, $\text{SetCol}$ and $\text{LazySet}$ operation requires $O(n^2)$ time in the worst case. Any $\text{Init}$ requires $O(h \cdot n^\omega + h \cdot n^2)$ worst-case time, where $\omega$ is the exponent of matrix multiplication. The cost of any $\text{Reset}$ operation can be charged to previous $\text{SetRow}$, $\text{SetCol}$ and $\text{Init}$ operations. The maximum cost charged to each $\text{Init}$ is $O(h \cdot n^3)$. The space required is $O(h \cdot n^2)$.

**Proof.** It is straightforward to see from the pseudocode of the operations that any $\text{SetRow}$, $\text{SetCol}$ and $\text{LazySet}$ operation requires $O(n^2)$ time in the worst case.

$\text{Init}$ takes $O(h \cdot n^\omega + h \cdot n^2)$ in the worst case: in more detail, each $\text{Prod}_a$ can be directly computed via matrix multiplication and any other initialization step requires no more than $O(n^2)$ worst-case time.

To prove that the cost of any $\text{Reset}$ operation can be charged to previous $\text{SetRow}$, $\text{SetCol}$ and $\text{Init}$ operations, we use a potential function

$$\Phi_a = \sum_{x,y} \text{Prod}_a[x,y]$$

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associated to each term \( T_a \) of the polynomial. From the relation:

\[
Prod_a[x, z] = \{ y : X_1^a[x, y] = 1 \land X_2^a[y, z] = 1 \land \mathcal{P}_a(x, y, z) \}
\]
given in Invariant 1, it follows that \( 0 \leq Prod_a[x, z] \leq n \) for all \( x, z \). Thus, \( 0 \leq \Phi_a \leq n^3 \).

Now, observe that \text{SetRow} increases \( \Phi_a \) by at most \( n^2 \) units per operation, while \text{Init} increases \( \Phi_a \) by at most \( n^3 \) units per operation. Note that \text{LazySet} does not affect \( \Phi_a \). We can finally address the case of \text{Reset} operations. Consider the distinction between the two cases \( \max \{ \text{LastRow}_{X_1^a}[x], \text{LastCol}_{X_1^a}[y] \} \geq \text{LastFlip}_{X_1^a[x,y]} \) in line 4 and \( \max \{ \text{LastRow}_{X_1^a}[x], \text{LastCol}_{X_1^a}[y] \} < \text{LastFlip}_{X_1^a[x,y]} \) in line 9. In the first case, we can charge the cost of processing any triple \((x, y, z)\) to some previous operation on the \( x \)-th row of \( X_1^a \) or to some previous operation on the \( y \)-th column of \( X_1^a \); in the second case, we consider only those \((x, y, z)\) for which some operation on the \( z \)-th column of \( X_2^a[y, z] \) was performed after both \( X_1^a[x, y] \) and \( X_2^a[y, z] \) were set to 1. In both cases, any \text{Reset} operation decreases \( \Phi_a \) by at most \( n \) units for each reset entry of \( X_b^a \), and this can be charged to previous operations which increased \( \Phi_a \).

The complex statement of the charging mechanism encompasses the dynamics of our data structure. In particular, we allow \text{Reset} operations to charge up to an \( O(n^3) \) cost to a single \text{Init} operation. Thus, in an arbitrary mixed sequence with any number of \text{Init}, \text{Reset} takes \( O(n^3) \) amortized time per update. If, however, we allow \text{Init} operations to appear in \( \sigma \) only every \( \Omega(n) \) \text{Reset} operations, the bound for \text{Reset} drops down to \( O(n^2) \) amortized time per operation.

As a consequence of Theorem 2, we have the following corollaries that refine the analysis of the running time of \text{Reset} operations.

**Corollary 1** If we perform just one \text{Init} operation in a sequence \( \sigma \) of length \( \Omega(n) \), or more generally one \text{Init} operation every \( \Omega(n) \) \text{Reset} operations, then the amortized cost of \text{Reset} is \( O(n^2) \) per operation.

**Corollary 2** If we perform just one \text{Init} operation in a sequence \( \sigma \) of length \( \Omega(n^2) \), or more generally one \text{Init} operation every \( \Omega(n^2) \) \text{Reset} operations, and no operations \text{SetRow} and \text{SetCol}, then the amortized cost of \text{Reset} is \( O(n) \) per operation.

In the following, we show how to extend the previous techniques in order to deal with the general case of polynomials of degree \( k > 2 \).
4.1.2 Data Structure for Polynomials of Degree \(k > 2\)

To support terms of degree \(k > 2\) in \(P\), we consider an equivalent representation \(\hat{P}\) of \(P\) such that the degree of each term is 2. This allows us to maintain a data structure for \(\hat{P}\) with the operations defined in the previous paragraph.

**Lemma 2** Consider a polynomial

\[
P = \sum_{a=1}^{h} T_a = \sum_{a=1}^{h} X_1^a \cdots X_k^a
\]

with \(h\) terms where each term \(T_a\) has degree exactly \(k\) and variables \(X_b^a\) are Boolean matrices. Let \(\hat{P}\) be the polynomial over Boolean matrices of degree 2 defined as

\[
\hat{P} = \sum_{a=1}^{h} \sum_{b=0}^{k} L_{b,b-1}^a \cdot R_{b,k-b-1}^a
\]

where \(L_{b,j}^a\) and \(R_{b,j}^a\) are polynomials over Boolean matrices of degree \(\leq 2\) defined as

\[
L_{b,j}^a = \begin{cases} 
X_{b-j}^a \cdot L_{b,j-1}^a & \text{if } j \in [0, b-1] \\
I_n & \text{if } j = -1
\end{cases}
\]

\[
R_{b,j}^a = \begin{cases} 
R_{b,j-1}^a \cdot X_{b+1+j}^a & \text{if } j \in [0, k-b-1] \\
I_n & \text{if } j = -1
\end{cases}
\]

Then \(P = \hat{P}\).

**Proof.** To prove the claim, it suffices to check that

\[
T_a = \sum_{b=0}^{k} L_{b,b-1}^a \cdot R_{b,k-b-1}^a
\]

Unrolling the recursion for \(L_{b,b-1}^a\), we obtain:

\[
L_{b,b-1}^a = X_1^a \cdot L_{b-1,b}^a = X_1^a \cdot X_2^a \cdot L_{b-2,b}^a = \cdots = X_1^a \cdot X_2^a \cdots X_b^a \cdot I_n
\]

Likewise, \(R_{b,k-b-1}^a = I_n \cdot X_{b+1}^a \cdots X_k^a\) holds. Thus, by idempotence of the closed semiring of Boolean matrices, we finally have:

\[
\sum_{b=0}^{k} L_{b,b-1}^a \cdot R_{b,k-b-1}^a = \sum_{b=0}^{k} X_1^a \cdots X_b^a \cdot X_{b+1}^a \cdots X_k^a = X_1^a \cdots X_k^a = T_a.
\]

Since \(\hat{P}\), \(L_{b,j}^a\) and \(R_{b,j}^a\) are all polynomials of degree \(\leq 2\), they can be represented and maintained efficiently by means of instances of Data Structure 1. Our data structure for maintaining polynomials of degree \(> 2\) is presented below:
\[ T = X_1 \ldots X_{b-1} X_b \downarrow \downarrow L_{b,1} \ldots \downarrow \downarrow L_{b,b-1} \]

\[ X_{b+1} \downarrow \downarrow R_{b,1} \]

\[ X_{b+2} \ldots X_k \]

\[ \cdots \]

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always limited to a row or a column and thus can be efficiently handled by means of operations like \texttt{SetRow}_{k=2} and \texttt{SetCol}_{k=2}.

In lines 3–4 \texttt{SetCol} propagates via \texttt{SetCol}_{k=2} the changes of the \(i\)-th column of \(X^a_b\) to \(L^a_{b,1}\), then the changes of the \(i\)-th column of \(L^a_{b,1}\) to \(L^a_{b,2}\), and so on through the recursive decomposition:

\[
\begin{align*}
L^a_{b,0} &= X^a_b \cdot I_n = X^a_b \\
L^a_{b,1} &= X^a_{b-1} \cdot L^a_{b,0} = X^a_{b-1} \cdot X^a_b \\
L^a_{b,2} &= X^a_{b-2} \cdot L^a_{b,1} = X^a_{b-2} \cdot X^a_{b-1} \cdot X^a_b \\
&
\vdots \\
L^a_{b,b-1} &= X^a_1 \cdot L^a_{b,b-2} = X^a_1 \cdots X^a_{b-2} \cdot X^a_{b-1} \cdot X^a_b
\end{align*}
\]

Likewise, in lines 5–6 it propagates via \texttt{SetRow}_{k=2} a null matrix of changes of the \(i\)-th row of \(X^a_{b+1}\) to \(R^a_{b,1}\), then the changes (possibly none) of the \(i\)-th row of \(R^a_{b,1}\) (due to the late effects of some previous \texttt{LazySet}) to \(R^a_{b,2}\), and so on through the recursive decomposition:

\[
\begin{align*}
R^a_{b,0} &= I_n \cdot X^a_b = X^a_{b+1} \\
R^a_{b,1} &= R^a_{b,0} \cdot X^a_{b+1} = X^a_{b+1} \cdot X^a_{b+2} \\
R^a_{b,2} &= R^a_{b,1} \cdot X^a_{b+2} = X^a_{b+1} \cdot X^a_{b+2} \cdot X^a_{b+3} \\
&
\vdots \\
R^a_{b,k-b-1} &= R^a_{b,k-b-2} \cdot X^a_k = X^a_{b+1} \cdot X^a_{b+2} \cdot X^a_{b+3} \cdots X^a_k
\end{align*}
\]

We remark that both loops in lines 3–4 and in lines 5–6 reveal, gather and propagate any 1’s that appear in the intermediate polynomials due to the late effects of some previous \texttt{LazySet}. In particular, even if the presence of lines 5–6 may seem strange because \(\Delta X^a_{b+1} = 0_n\), these lines are executed just for this reason.

Finally, in lines 7–8 changes of \(L^a_{b,b-1}\) and \(R^a_{b,k-b-1}\) are propagated to \(Y\), which represents the maintained value of \(\hat{P}\), and in lines 9–10 new 1’s are lazily inserted in any other polynomials that feature \(X^a_b\) as a variable.

We omit the pseudocode for \texttt{SetRow} because it is similar to \texttt{SetCol}.

\textbf{Reset, LazySet, Init, Lookup}

\texttt{Reset}(\(\Delta X, X^a_b\)) can be supported by propagating via \texttt{Reset}_{k=2} any changes of \(X^a_b\) to any intermediate polynomial \(L^a_{u,v}\) and \(R^a_{u,v}\) that contains it, then changes of such polynomials to any polynomials which depend on them and so on up to \(Y\).

\texttt{LazySet}(\(\Delta X, X^a_b\)) can be supported by performing \texttt{LazySet}_{k=2} operations on each polynomial \(L^a_{u,v}\) and \(R^a_{u,v}\) that contains \(X^a_b\).

\texttt{Init}(\(Z^1_1, \ldots, Z^k_1\)) can be supported by invoking \texttt{Init}_{k=2} on each polynomial \(L^w_{u,v} \), \(R^w_{u,v}\) and by propagating the intermediate results up to \(Y\).

\texttt{Lookup}() can be realized by returning the maintained value \(Y\) of \(\hat{P}\).
To conclude this section, we discuss the correctness and the complexity of our operations in the case of polynomials of arbitrary degree.

**Theorem 3** At any time $j$, Lookup returns a matrix $Y_j$ that satisfies the relation $M_j \subseteq Y_j \subseteq P_j$ as in Definition 8.

**Proof.** Since $\hat{P} = P$ by Lemma 3, we prove that:

$$\hat{P}_j \supseteq Y_j \supseteq M_j = \sum_{1 \leq i \leq j : \text{Op}_i \neq \text{LazySet}} (\hat{P}_i - \hat{P}_{i-1}).$$

To this aim, it is sufficient to prove that any 1 that appears (or disappears) in the correct value of $\hat{P}$ due to an operation different from LazySet appears (or disappears) in $Y$ as well, and that any entry of $Y$ equal to 1 is also equal to 1 in $\hat{P}$.

- **SetCol/SetRow:** assume a SetCol operation is performed on the $i$-th column of variable $X_{ab}^a$ (see Figure 2). By induction, we assume that all new 1’s are correctly revealed in the $i$-th column of our data structure for $L_{b,j}^a$ after the $j$-th iteration of SetCol$_{k=2}$ in line 4. Notice that $\Delta L_{b,j}^a = J \Delta L_{b,j}^a$, that is changes of $L_{b,j}^a$ are limited to the $i$-th column: this implies that these changes can be correctly propagated by means of a SetCol operation to any polynomial that features $L_{b,j}^a$ as a variable. As a consequence, by Theorem 1, the $j + 1$-th iteration of SetCol$_{k=2}$ in line 4 correctly reveals all new 1’s in our data structure for $L_{b,j}^a$, and again these new 1’s all lie on its i-th column. Thus, at the end of the loop in lines 3–4, all new 1’s appear correctly in the $i$-th column of $L_{b,b-1}^a$. Similar considerations apply also for $R_{b,k-1}^a$. To prove that lines 7–8 insert correctly in $Y$ all new 1’s that appear in $\hat{P}$ and that $Y \subseteq \hat{P}$ we use again Theorem 1 and the fact that any 1 that appears in $\hat{P}$ also appears in $L_{b,b-1}^a \cdot R_{b,k-1}^a$. Indeed, for any entry $\hat{P}[x,y]$ that flips from 0 to 1 due to a change of the $i$-th column of $X_{b}^a$ or the $i$-th row of $X_{b}^a$ there is a sequence of indices $x = u_0, u_1, \ldots, u_{b-1}, u_b = i, u_{b+1}, \ldots, u_{k-1}, u_k = y$ such that $X_{j}^a[u_{j-1}, u_j] = 1$, $1 \leq j \leq k$, and either one of $X_{b}^a[u_{b-1}, i]$ or $X_{b}^a[i, u_{b+1}]$ just flipped from 0 to 1 due to the SetRow/SetCol operation. The proof for SetRow is completely analogous.

- **Reset:** assume a Reset operation is performed on variable $X_{b}^a$. As Reset$_{k=2}$ can reset any subset of entries of variables, and not only those lying on a row or a column as in the case of SetRow$_{k=2}$ and SetCol$_{k=2}$, the correctness of propagating any changes of $X_{b}^a$ to the polynomials that depend on it easily follows from Theorem 1.

- **Init:** each Init operation recomputes from scratch all polynomials in Data Structure 3. Thus $Y = \hat{P}$ after each Init operation.
Theorem 4  Any Lookup and LazySet operation requires $O(n^2)$ time in the worst case. Any SetRow and SetCol operation requires $O(k \cdot n^2)$ amortized time, and any Init operation takes $O(h \cdot k \cdot n^2 + h \cdot n^2)$ worst-case time. The cost of any Reset operation can be charged to previous SetRow, SetCol and Init operations. The maximum cost charged to each Init is $O(h \cdot k \cdot n^3)$. The space required is $O(h \cdot k^2 \cdot n^2)$.

Proof. The proof easily follows from Theorem 2. □

As in the previous paragraph, we have the following corollaries.

Corollary 3  If we perform just one Init operation in a sequence $\sigma$ of length $\Omega(n)$, or more generally one Init operation every $\Omega(n)$ Reset operations, then the amortized cost of Reset is $O(k \cdot n^2)$ per operation.

Corollary 4  If we perform just one Init operation in a sequence $\sigma$ of length $\Omega(n^2)$, or more generally one Init operation every $\Omega(n^2)$ Reset operations, and we perform no operations SetRow and SetCol, then the amortized cost of Reset is $O(k \cdot n)$ per operation.

4.2  Maintaining Dynamic Matrices over Integers

In this section we study the problem of finding an implicit representation for a matrix of integers that makes it possible to support simultaneous updates of multiple entries of the matrix very efficiently at the price of increasing the lookup time required to read a single entry. This problem on dynamic matrices will be central to designing the first subquadratic algorithm for fully dynamic transitive closure that will be described in Section 7. We formally define the problem as follows:

Definition 7  Let $M$ be an $n \times n$ integer matrix. We consider the problem of performing an intermixed sequence $\sigma = \langle M.\text{Op}_1, \ldots, M.\text{Op}_l \rangle$ of operations on $M$, where each operation $M.\text{Op}_j$ can be either one of the following:

- $M.\text{Init}(X)$: perform the initialization $M \leftarrow X$, where $X$ is an $n \times n$ integer matrix.
- $M.\text{Update}(J, I)$: perform the update operation $M \leftarrow M + J \cdot I$, where $J$ is an $n \times 1$ column integer vector, and $I$ is a $1 \times n$ row integer vector. The product $J \cdot I$ is an $n \times n$ matrix defined for any $1 \leq x, y \leq n$ as:

$$ (J \cdot I)[x, y] = J[x] \cdot I[y] $$
- \textbf{M\.Lookup}(x, y): return the integer value \( M[x, y] \).

It is straightforward to observe that \textbf{Lookup} can be supported in unit time and operations \textbf{Init} and \textbf{Update} in \( O(n^2) \) worst-case time by explicitly performing the algebraic operations specified in the previous definition.

In the following we show that, if one is willing to give up unit time for \textbf{Lookup} operations, it is possible to support \textbf{Update} in \( O(n^{\omega(1,\epsilon,1) - \epsilon}) \) worst-case time for each update operation, for any \( \epsilon \), \( 0 \leq \epsilon \leq 1 \), where \( \omega(1,\epsilon,1) \) is the exponent of the multiplication of an \( n \times n^\epsilon \) matrix by an \( n^\epsilon \times n \) matrix. Queries on individual entries of \( M \) are answered in \( O(n^\epsilon) \) worst-case time via \textbf{Lookup} operations and \textbf{Init} still takes \( O(n^2) \) worst-case time.

We now sketch the main ideas behind the algorithm. We follow a simple lazy approach: we log at most \( n^\epsilon \) update operations without explicitly computing them and we perform a global reconstruction of the matrix every \( n^\epsilon \) updates. The reconstruction is done through fast rectangular matrix multiplication. This yields an implicit representation for \( M \) which requires us to run through logged updates in order to answer queries about entries of \( M \).

\textbf{Data Structure}

We maintain the following elementary data structures with \( O(n^2) \) space:

- an \( n \times n \) integer matrix \texttt{Lazy} which maintains a lazy representation of \( M \);
- an \( n \times n^\epsilon \) integer matrix \( \texttt{Buf}_J \) in which we buffer update column vectors \( J \);
- an \( n^\epsilon \times n \) integer matrix \( \texttt{Buf}_I \) in which we buffer update row vectors \( I \);
- a counter \( t \) of the number of performed \textbf{Update} operations since the last \textbf{Init}, modulo \( n^\epsilon \).

Before proposing our implementation of the operations introduced in Definition 4, we discuss a simple invariant property that we maintain in our data structure and that guarantees the correctness of the implementation of the operations that we are going to present. We use the following notation:

\textbf{Definition 8} We denote by \( \texttt{Buf}_J(j) \) the \( n \times j \) matrix obtained by considering only the first \( j \) columns of \( \texttt{Buf}_J \). Similarly, we denote by \( \texttt{Buf}_I(i) \) the \( i \times n \) matrix obtained by considering only the first \( i \) rows of \( \texttt{Buf}_I \).

\textbf{Invariant 2} At any time \( t \) in the sequence of operations \( \sigma \), the following invariant is maintained:

\[ M = \texttt{Lazy} + \texttt{Buf}_J(t) \cdot \texttt{Buf}_I(t). \]
Update

procedure Update(J, I)
1. begin
2. \( t \leftarrow t + 1 \)
3. if \( t \leq n^\epsilon \) then
4. \( Buf_J[t, \cdot] \leftarrow J \)
5. \( Buf_I[\cdot, t] \leftarrow I \)
6. else
7. \( t \leftarrow 0 \)
8. \( Lazy \leftarrow Lazy + Buf_J \cdot Buf_I \)
9. end

Update first increases \( t \) and, if \( t \leq n^\epsilon \), it copies column vector \( J \) onto the \( t \)-th column of \( Buf_J \) (line 4) and row vector \( I \) onto the \( t \)-th row of \( Buf_I \) (line 5). If \( t > n^\epsilon \), there is no more room in \( Buf_J \) and \( Buf_I \) for buffering updates. Then the counter \( t \) is reset in line 7 and the reconstruction operation in line 8 synchronizes \( Lazy \) with \( M \) via rectangular matrix multiplication of the \( n \times n^\epsilon \) matrix \( Buf_J \) by the \( n^\epsilon \times n \) matrix \( Buf_I \).

Lookup

procedure Lookup(x, y)
1. begin
2. return \( Lazy[x, y] + \sum_{j=1}^{t} Buf_J[x, j] \cdot Buf_I[j, y] \)
3. end

Lookup runs through the first \( t \) columns and rows of buffers \( Buf_J \) and \( Buf_I \), respectively, and returns the value of \( Lazy \) corrected with the inner product of the \( x \)-th row of \( Buf_J(t) \) by the \( y \)-th column of \( Buf_I(t) \).

Init

procedure Init(X)
1. begin
2. \( Lazy \leftarrow X \)
3. \( t \leftarrow 0 \)
4. end

Init simply sets the value of \( Lazy \) and empties the buffers by resetting \( t \).

The following theorem discusses the time and space requirements of operations Update, Lookup, and Init. As already stated, the correctness easily follows from the fact that Invariant \( \Box \) is maintained throughout any sequence of operations.

**Theorem 5** Each Update operation can be supported in \( O(n^\omega(1, \epsilon, 1) - \epsilon) \) worst-case time and each Lookup in \( O(n^\epsilon) \) worst-case time, where \( 0 \leq \epsilon \leq 1 \) and \( \omega(1, \epsilon, 1) \)
is the exponent for rectangular matrix multiplication. \textbf{Init} requires $O(n^2)$ time in the worst case. The space required is $O(n^2)$.

\textbf{Proof}. An amortized update bound follows trivially from amortizing the cost of the rectangular matrix multiplication $Buf_J \cdot Buf_I$ against $n^\epsilon$ update operations. This bound can be made worst-case by standard techniques, i.e., by keeping two copies of the data structures: one is used for queries and the other is updated by performing matrix multiplication in the background.

As far as \textbf{Lookup} is concerned, it answers queries on the value of $M[x, y]$ in $\Theta(t)$ worst-case time, where $t \leq n^\epsilon$. \hfill \square

\textbf{Corollary 5} If $O(n^\omega)$ is the time required for multiplying two $n \times n$ matrices, then we can support \textbf{Update} in $O(n^{2-(3-\omega)\epsilon})$ worst-case time and \textbf{Lookup} in $O(n^\epsilon)$ worst-case time. Choosing $\epsilon = 1$, the best known bound for matrix multiplication ($\omega < 2.38$) implies an $O(n^{1.38})$ \textbf{Update} time and an $O(n)$ \textbf{Lookup} time.

\textbf{Proof}. A rectangular matrix multiplication between a $n \times n^\epsilon$ matrix by a $n^\epsilon \times n$ matrix can be performed by computing $O((n^{1-\epsilon})^2)$ multiplications between $n^\epsilon \times n^\epsilon$ matrices. This is done in $O((n^{1-\epsilon})^2 \cdot (n^\epsilon)\omega)$. The amortized time of the reconstruction operation $\text{Lazy} \leftarrow \text{Lazy} + Buf_J \cdot Buf_I$ is thus $O \left( \frac{(n^{1-\epsilon})^2 \cdot (n^\epsilon)^\omega + n^2}{n^\epsilon} \right) = O(n^{2-(3-\omega)\epsilon})$. The rest of the claim follows from Theorem 5. \hfill \square

5 Transitive Closure Updates in $O(n^2 \log n)$ Time

In this section we show a first method for casting fully dynamic transitive closure into the problem of reevaluating polynomials over Boolean matrices presented in Section 4.1.

Based on the technique developed in Section 4.1, we revisit the dynamic graph algorithm given in [15] in terms of dynamic matrices and we present a matrix-based variant of it which features better initialization time while maintaining the same bounds on the running time of update and query operations, i.e., $O(n^2 \cdot \log n)$ time per update and $O(1)$ time per query. The space requirement of our algorithm is $M(n) \cdot \log n$, where $M(n)$ is the space used for representing a polynomial over Boolean matrices. As stated in Theorem 4, $M(n)$ is $O(n^2)$ if $h$ and $k$ are constant.

In the remainder of this section we first describe our data structure and then we show how to support efficiently operations introduced in Definition 4 for the equivalent problem of fully dynamic Boolean matrix closure.

5.1 Data Structure

As it is well known, the Kleene closure of a Boolean matrix $X$ can be computed from scratch via matrix multiplication by computing $\log_2 n$ polynomials $P_k =$
\[ P_{k-1} + P_{k-1}^2, \ 1 \leq k \leq \log_2 n. \] In the static case where \( X^* \) has to be computed only once, intermediate results can be thrown away as only the final value \( X^* = P_{\log_2 n} \) is required. In the dynamic case, instead, intermediate results provide useful information for updating efficiently \( X^* \) whenever \( X \) gets modified.

In this section we consider a slightly different definition of polynomials \( P_1, \ldots, P_{\log_2 n} \) with the property that each of them has degree \( \leq 3 \):

**Definition 9** Let \( X \) be an \( n \times n \) Boolean matrix. We define the sequence of \( \log_2 n + 1 \) polynomials over Boolean matrices \( P_0, \ldots, P_{\log_2 n} \) as:

\[
P_k = \begin{cases} 
X & \text{if } k = 0 \\
P_{k-1} + P_{k-1}^2 + P_{k-1}^3 & \text{if } k > 0
\end{cases}
\]

Before describing our data structure for maintaining the Kleene closure of \( X \), we discuss some useful properties.

**Lemma 3** Let \( X \) be an \( n \times n \) Boolean matrix and let \( P_k \) be formed as in Definition 9. Then for any \( 1 \leq u, v \leq n \), \( P_k[u, v] = 1 \) if and only if there is a path \( u \leadsto v \) of length at most \( 3^k \) in \( X \).

**Proof.** The proof is by induction on \( k \). The base (\( k = 0 \)) is trivial. We assume by induction that the claim is satisfied for \( P_{k-1} \) and we prove that it is satisfied for \( P_k \) as well.

**Sufficient condition:** Any path of length up to \( 3^k \) between \( u \) and \( v \) in \( X \) is either of length up to \( 3^{k-1} \) or it can be obtained as concatenation of three paths of length up to \( 3^{k-1} \) in \( X \). Since all these paths are correctly reported in \( P_{k-1} \) by the inductive hypothesis, it follows that \( P_{k-1}[u, v] = 1 \) or \( P_{k-1}^2[u, v] = 1 \) or \( P_{k-1}^3[u, v] = 1 \). Thus \( P_k[u, v] = P_{k-1}[u, v] + P_{k-1}^2[u, v] + P_{k-1}^3[u, v] = 1 \).

**Necessary condition:** If \( P_k[u, v] = 1 \) then at least one among \( P_{k-1}[u, v] \), \( P_{k-1}^2[u, v] \) and \( P_{k-1}^3[u, v] \) is 1. If \( P_{k-1}[u, v] = 1 \), then by the inductive hypothesis there is a path of length up to \( 3^{k-1} < 3^k \). If \( P_{k-1}^2[u, v] = 1 \) then there are two paths of length up to \( 3^{k-1} \) whose concatenation yields a path no longer than \( 3^k \). Finally, if \( P_{k-1}^3[u, v] = 1 \), then there are three paths of length up to \( 3^{k-1} \) whose concatenation yields a path no longer than \( 3^k \).

**Lemma 4** Let \( X \) be an \( n \times n \) Boolean matrix and let \( P_k \) be formed as in Definition 9. Then \( X^* = I_n + P_{\log_2 n} \).

**Proof.** The proof easily follows from Lemma 3 and from the observation that the length of the longest simple path in \( X \) is no longer than \( n-1 < 3^{\log_3 n} \leq 3^{\log_2 n} \). \( I_n \) is required to guarantee the reflexivity of \( X^* \).
Data Structure 3  We maintain an $n \times n$ Boolean matrix $X$ and we maintain the $\log_2 n$ polynomials $P_1 \ldots P_{\log_2 n}$ of degree 3 given in Definition 3 with instances of Data Structure 3 presented in Section 4.1.

As we will see in Section 5.2, the reason for considering the extra term $P_{k-1}$ in our data structure is that polynomials need to be maintained using not only SetRow/SetCol, but also LazySet. As stated in Definition 3, using LazySet yields a weaker representation of polynomials, and this forces us to increase the degree if complete information about $X^*$ has to be maintained. This aspect will be discussed in more depth in the proof of Theorem 4.

5.2 Implementation of Operations

In this section we show that operations $\text{Init}^*$, $\text{Set}^*$, $\text{Reset}^*$ and $\text{Lookup}^*$ introduced in Definition 4 can all be implemented in terms of operations $\text{Init}$, LazySet, SetRow, and SetCol (described in Section 4.1) on polynomials $P_1 \ldots P_{\log_2 n}$.

Init*  

procedure $\text{Init}^*(X)$
begin
  1. $Y \leftarrow X$
  2. for $k = 1$ to $\log_2 n$ do
  3.     $P_k.\text{Init}(Y)$
  4.     $Y \leftarrow P_k.\text{Lookup}()$
  5. end

$\text{Init}^*$ performs $P_k.\text{Init}$ operations on each $P_k$ by propagating intermediate results from $X$ to $P_1$, then from $P_1$ to $P_2$, and so on up to $P_{\log_2 n}$.

Lookup*  

procedure $\text{Lookup}^*(x, y)$
begin
  1. $Y \leftarrow P_{\log_2 n}.\text{Lookup}()$
  2. return $I_n + Y[x, y]$
end

$\text{Lookup}^*$ returns the value of $P_{\log_2 n}[x, y]$. 
Set* procedure $\text{Set}^*(i, \Delta X)$
1. begin
2. $\Delta Y \leftarrow \Delta X$
3. for $k = 1$ to $\log_2 n$ do
4. $P_k$.LazySet($\Delta Y, P_{k-1}$)
5. $P_k$.SetRow($i, \Delta Y, P_{k-1}$)
6. $P_k$.SetCol($i, \Delta Y, P_{k-1}$)
7. $\Delta Y \leftarrow P_k$.Lookup()
8. end

$\text{Set}^*$ propagates changes of $P_{k-1}$ to $P_k$ for any $k = 1$ to $\log_2 n$. Notice that any new 1’s that appear in $P_{k-1}$ are inserted in the object $P_k$ via LazySet, but only the $i$-th row and the $i$-th row column of $P_{k-1}$ are taken into account by SetRow and SetCol in order to determine changes of $P_k$. As re-inserting 1’s already present in a variable is allowed by our operations on polynomials, for the sake of simplicity in line 7 we assign the update matrix $\Delta Y$ with $P_k$ and not with the variation of $P_k$.

Reset* procedure $\text{Reset}^*(\Delta X)$
1. begin
2. $\Delta Y \leftarrow \Delta X$
3. for $k = 1$ to $\log_2 n$ do
4. $Y \leftarrow P_k$.Lookup()
5. $P_k$.Reset($\Delta Y, P_{k-1}$)
6. $\Delta Y \leftarrow Y - P_k$.Lookup()
7. end

$\text{Reset}^*$ performs $P_k$.Reset operations on each $P_k$ by propagating changes specified by $\Delta X$ to $P_1$, then changes of $P_1$ to $P_2$, and so on up to $P_{\log_2 n}$. Notice that we use an auxiliary matrix $Y$ to compute the difference between the value of $P_k$ before and after the update and that the computation of $\Delta Y$ in line 6 always yields a Boolean matrix.

### 5.3 Analysis

In what follows we discuss the correctness and the complexity of our implementation of operations $\text{Init}^*$, $\text{Set}^*$, $\text{Reset}^*$, and $\text{Lookup}^*$ presented in Section 5.2. We recall that $X$ is an $n \times n$ Boolean matrix and $P_k$, $0 \leq k \leq \log_2 n$, are the polynomials introduced in Definition 9.

**Theorem 6** If at any time during a sequence $\sigma$ of operations there is a path of length up to $2^k$ between $x$ and $y$ in $X$, then $P_k[x, y] = 1$.  

Proof. By induction. The base is trivial. We assume that the claim holds inductively for $P_{k-1}$, and we show that, after any operation, the claim holds also for $P_k$.

- **Init***: since any **Init** operation rebuilds from scratch $P_k$, the claim holds from Lemma 3.

- **Set***: let us assume that a **Set** operation is performed on the $i$-th row and column of $X$ and a new path $\pi$ of length up to $2^k$, say $\pi = \langle x, \ldots, i, \ldots, y \rangle$, appears in $X$ due to this operation. We prove that $P_k[x, y] = 1$ after the operation.

  Observe that $P_k\text{LazySet}(\Delta P_{k-1}, P_{k-1})$ puts in place any new 1’s in any occurrence of the variable $P_{k-1}$ in data structure $P_k$. We remark that, although the maintained value of $P_k$ in data structure $P_k$ is not updated by **LazySet** and therefore the correctness of the current operation is not affected, this step is very important: indeed, new 1’s corresponding to new paths of length up to $2^{k-1}$ that appear in $X$ will be useful in future **Set** operations for detecting the appearance of new paths of length up to $2^k$.

  If both the portions $x \sim i$ and $i \sim y$ of $\pi$ have length up to $2^{k-1}$, then $\pi$ gets recorded in $P_{k-1}^2$, and therefore in $P_k$, thanks to one of $P_k\text{.SetRow}(i, \Delta P_{k-1}, P_{k-1})$ or $P_k\text{.SetCol}(i, \Delta P_{k-1}, P_{k-1})$. On the other hand, if $i$ is close to (but does not coincide with) one endpoint of $\pi$, the appearance of $\pi$ may be recorded in $P_{k-1}^d$, but not in $P_{k-1}^2$. This is the reason why degree 2 does not suffice for $P_k$ in this dynamic setting.

- **Reset***: by inductive hypothesis, we assume that $P_{k-1}[x, y]$ flips to zero after a **Reset** operation only if no path of length up to $2^{k-1}$ remains in $X$ between $x$ and $y$. Since any $P_k\text{.Reset}$ operation on $P_k$ leaves it as if cleared 1’s in $P_{k-1}$ were never set to 1, $P_k[x, y]$ flips to zero only if no path of length up to $2^k$ remains in $X$.

    \[\square\]

We remark that the condition stated in Theorem 6 is only sufficient because $P_k$ may keep track of paths having length strictly more than $2^k$, though no longer than $3^k$. However, for $k = \log_2 n$ the condition is also necessary as no shortest path can be longer than $n = 2^k$. Thus, it is straightforward to see that a path of any length between $x$ and $y$ exists at any time in $X$ if and only if $P_{\log_2 n}[x, y] = 1$.

The following theorem establishes the running time and space requirements of operations **Init***, **Set*** and **Reset***.

**Theorem 7** Any **Init*** operation can be performed in $O(n^{\omega} \cdot \log n)$ worst-case time, where $\omega$ is the exponent of matrix multiplication; any **Set*** takes $O(n^2 \cdot \log n)$
amortized time. The cost of \texttt{Reset*} operations can be charged to previous \texttt{Init*} and \texttt{Set*} operations. The maximum cost charged to each \texttt{Init} is $O(n^3 \cdot \log n)$. The space required is $O(n^2 \cdot \log n)$.

\textbf{Proof.} The proof follows from Theorem \[\text{[1]}\] by considering the time bounds of operations on polynomials described in Section \[\text{[14]}\]. As each maintained polynomial has constant degree $k = 3$, it follows that the space used is $O(n^2 \cdot \log n)$.

\textbf{Corollary 6} If we perform just one \texttt{Init*} operation in a sequence $\sigma$ of length $\Omega(n)$, or more generally one \texttt{Init} operation every $\Omega(n)$ \texttt{Reset} operations, then the amortized cost of \texttt{Reset} is $O(n^2 \cdot \log n)$ per operation.

\textbf{Corollary 7} If we perform just one \texttt{Init*} operation in a sequence $\sigma$ of length $\Omega(n^2)$, or more generally one \texttt{Init} operation every $\Omega(n^2)$ \texttt{Reset} operations, and we perform no operations \texttt{SetRow} and \texttt{SetCol}, then the amortized cost of \texttt{Reset} is $O(n \cdot \log n)$ per operation.

In the traditional case where $\text{Op}_1 = \text{Init*}$ and $\text{Op}_i \neq \text{Init*}$ for any $i > 1$, i.e., \texttt{Init*} is just performed once at the beginning of the sequence of operations, previous corollaries state that both \texttt{Set*} and \texttt{Reset*} are supported in $O(n^2 \cdot \log n)$ amortized time. In the decremental case where only \texttt{Reset*} operations are performed, the amortized time is $O(n \cdot \log n)$ per update.

$$\triangleright$$

The algorithm that we presented in this section can be viewed as a variant which features very different data structures of the fully dynamic transitive closure algorithm presented by King in \[\text{[12]}\].

King’s algorithm is based on a data structure for a graph $G = (V, E)$ that maintains a logarithmic number of edge subsets $E_0, \ldots, E_{\log_2 n}$ with the property that $E_0 = E$ and $(x, y) \in E_i$ if there is a path $x \leadsto y$ of length up to $2^i$ in $G$. Moreover, if $y$ is not reachable from $x$ in $G$, then $(x, y) \notin E_i$ for all $0 \leq i \leq \log_2 n$.

The maintained values of our polynomials $P_0, \ldots, P_{\log_2 n}$ here correspond to the sets $E_0, \ldots, E_{\log_2 n}$.

The algorithm by King also maintains $\log_2 n$ forests $F_0, \ldots, F_{\log_2 n-1}$ such that $F_i$ uses edges in $E_i$ and includes $2n$ trees Out$_i(v)$ and In$_i(v)$, two for each node $v \in V$, such that Out$_i(v)$ contains all nodes reachable from $v$ using at most 2 edges in $E_i$, and In$_i(v)$ contains all nodes that reach $v$ using at most 2 edges in $E_i$. For each pair of nodes, also a table Count$_i$ is maintained, where Count$_i[x, y]$ is the number of nodes $v$ such that $x \in$ In$_i(v)$ and $y \in$ Out$_i(v)$. Now, $E_i$ is maintained so as to contain edges $(x, y)$ such that Count$_{i-1}[x, y] > 0$. Trees In$_i(v)$ and Out$_i(v)$ are maintained for any node $v$ by means of deletions-only data structures \[\text{[1]}\] which are rebuilt from scratch after each $v$-centered insertion of edges.
Our data structures for polynomials over Boolean matrices $P_i$ play the same role as King’s forests $F_i$ of $In_i$ and $Out_i$ trees and of counters $Count_i$.

While King’s data structures require $O(n^3 \cdot \log n)$ worst-case initialization time on dense graphs, the strong algebraic properties of Boolean matrices allow us to exploit fast matrix multiplication subroutines for initializing more efficiently our data structures in $O(n^{\omega} \cdot \log n)$ time in the worst case, where $\omega = 2.38$.

6 Transitive Closure Updates in $O(n^2)$ Time

In this section we show our second and more powerful method for casting fully dynamic transitive closure into the problem of reevaluating polynomials over Boolean matrices presented in Section 4.1.

This method hinges upon the well-known equivalence between transitive closure and matrix multiplication on a closed semiring and yields a new deterministic algorithm that improves the best known bounds for fully dynamic transitive closure. Our algorithm supports each update operation in $O(n^2)$ amortized time and answers each reachability query with just one matrix lookup. The space used is $O(n^2)$.

6.1 Data Structure

Let $X$ be a Boolean matrix and let $X^*$ be its Kleene closure. Before discussing the dynamic case, we recall the main ideas behind the algorithm for computing statically $X^*$.

**Definition 10** Let $B_n$ be the set of $n \times n$ Boolean matrices and let $X \in B_n$. Without loss of generality, we assume that $n$ is a power of 2. Define a mapping $\mathcal{F} : B_n \rightarrow B_n$ by means of the following equations:

$$
\begin{align*}
E &= (A + BD^*C)^* \\
F &= EBD^* \\
G &= D^*CE \\
H &= D^* + D^*CEBD^*
\end{align*}
$$

(1)

where $A, B, C, D$ and $E, F, G, H$ are obtained by partitioning $X$ and $Y = \mathcal{F}(X)$ into sub-matrices of dimension $\frac{n}{2} \times \frac{n}{2}$ as follows:

$$
X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}
$$

The following fact is well known [18]: if $X$ is an $n \times n$ Boolean matrix, then $\mathcal{F}(X) = X^*$.

Another equivalent approach is given below:
Definition 11  Let $\mathcal{B}_n$ be the set of $n \times n$ Boolean matrices, let $X \in \mathcal{B}_n$ and let $G : \mathcal{B}_n \rightarrow \mathcal{B}_n$ be the mapping defined by means of the following equations:

\[
\begin{align*}
E &= A^* + A^* B H C A^* \\
F &= A^* B H \\
G &= H C A^* \\
H &= (D + C A^* B)^*
\end{align*}
\]

where $X$ and $Y = G(X)$ are defined as:

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}
\]

It is easy to show that, for any $X \in \mathcal{B}_n$, $G(X) = F(X)$. Both $F(X)$ and $G(X)$ can be computed in $O(n^\omega)$ worst-case time \[18\], where $\omega$ is the exponent of Boolean matrix multiplication.

We now define another function $H$ such that $H(X) = X^*$, based on a new set of equations obtained by combining Equation (1) and Equation (2). Our goal is to define $H$ in such a way that it is well-suited for efficient reevaluation in a fully dynamic setting.

Lemma 5  Let $\mathcal{B}_n$ be the set of $n \times n$ Boolean matrices, let $X \in \mathcal{B}_n$ and let $H : \mathcal{B}_n \rightarrow \mathcal{B}_n$ be the mapping defined by means of the following equations:

\[
\begin{align*}
P &= D^* \\
E_1 &= (A + B P^2 C)^* \\
E_2 &= E_1 B H_2^2 C E_1 \\
F_1 &= E_2^2 B P \\
F_2 &= E_1 B H_2^2 \\
G_1 &= P C E_1^2 \\
G_2 &= H_2^2 C E_1 \\
H_1 &= P C E_1^2 B P \\
H_2 &= (D + C E_1^2 B)^* \\
E &= E_1 + E_2 \\
F &= F_1 + F_2 \\
G &= G_1 + G_2 \\
H &= H_1 + H_2
\end{align*}
\]

where $X$ and $Y = H(X)$ are defined as:

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}
\]

Then, for any $X \in \mathcal{B}_n$, $H(X) = X^*$.

Proof. We prove that $E_1 + E_2$, $F_1 + F_2$, $G_1 + G_2$ and $H_1 + H_2$ are sub-matrices of $X^*$:

\[
X^* = \begin{pmatrix} E_1 + E_2 & F_1 + F_2 \\ G_1 + G_2 & H_1 + H_2 \end{pmatrix}
\]

We first observe that, by definition of Kleene closure, $X = X^* \Rightarrow X = X^2$. Thus, since $E_1 = (A + B P^2 C)^*$, $H_2 = (D + C E_1^2 B)^*$ and $P = D^*$ are all closures,
then we can replace $E_1^2$ with $E_1$, $H_2^2$ with $H_2$ and $P^2$ with $P$. This implies that $E_1 = (A + BPC)^* = (A + BD^* C)^*$ and then $E_1 = E$ by Equation 1. Now, $E$ is a sub-matrix of $X^*$ and encodes explicitly all paths in $X$ with both end-points in $V_1 = \{1, \ldots, \frac{n}{2}\}$, and since $E_2 = EB(D + CEB)^* CE$, then $E_2 \subseteq E$. It follows that $E_1 + E_2 = E + E_2 = E$. With a similar argument, we can prove that $F_1 + F_2$, $G_1 + G_2$ and $H_1 + H_2$ are sub-matrices of $X^*$. In particular, for $H = H_1 + H_2$ we also need to observe that $D^* \subseteq H_2$. \hfill \Box

Note that $H$ provides a method for computing the Kleene closure of an $n \times n$ Boolean matrix, provided that we are able to compute Kleene closures of Boolean matrices of size $\frac{n}{2} \times \frac{n}{2}$. The reason of using $E_1^2$, $H_2^2$ and $P^2$ instead of $E_1$, $H_2$ and $P$ in Equation (3), which is apparently useless, will be clear in Lemma 7 after presenting a fully dynamic version of the algorithm that defines $H$.

In the next lemma we show that a Divide and Conquer algorithm that recursively uses $H$ to solve sub-problems of smaller size requires asymptotically the same time as computing the product of two Boolean matrices.

**Theorem 8** Let $X$ be an $n \times n$ Boolean matrix and let $T(n)$ be the time required to compute recursively $H(X)$. Then $T(n) = O(n^\omega)$, where $O(n^\omega)$ is the time required to multiply two Boolean matrices.

**Proof.** It is possible to compute $E$, $F$, $G$ and $H$ with three recursive calls of $H$, a constant number $c_m$ of multiplications, and a constant number $c_s$ of additions of $\frac{n}{2} \times \frac{n}{2}$ matrices. Thus:

$$T(n) \leq 3T(\frac{n}{2}) + c_m M(\frac{n}{2}) + c_s (\frac{n}{2})^2$$

where $M(n) = O(n^\omega)$ is the time required to multiply two $n \times n$ Boolean matrices. Solving the recurrence relation, since $\log_2 3 < \max\{\omega, 2\} = \omega$, we obtain that $T(n) = O(n^\omega)$ (see e.g., the Master Theorem in [3]). \hfill \Box

The previous theorem showed that, even if $H$ needs to compute one more closure than $F$ and $G$, asymptotically the running time does not get worse.

\hfill \textit{\textcircled{}}

In the following, we study how to reevaluate efficiently $H(X) = X^*$ under changes of $X$. Our data structure for maintaining the Kleene closure $X^*$ is the following:

**Data Structure 4** We maintain two $n \times n$ Boolean matrices $X$ and $Y$ decomposed in sub-matrices $A$, $B$, $C$, $D$, and $E$, $F$, $G$, $H$:

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$
We also maintain the following 12 polynomials over \( n \times n \) Boolean matrices with the data structure presented in Section 4.1:

\[
\begin{align*}
Q &= A + BP^2C & E_2 &= E_1BH_2^2CE_1 & E &= E_1 + E_2 \\
F_1 &= E_2^2BP & F_2 &= E_1BH_2^2 & F &= F_1 + F_2 \\
G_1 &= PCE_1^2 & G_2 &= H_2^2CE_1 & G &= G_1 + G_2 \\
H_1 &= PCE_1^2BP & R &= D + CE_1^2B & H &= H_1 + H_2
\end{align*}
\]

and we recursively maintain 3 Kleene closures \( P, E_1 \) and \( H_2 \):

\[
P = D^* \quad E_1 = Q^* \quad H_2 = R^*
\]

with instances of size \( \frac{n}{2} \times \frac{n}{2} \) of Data Structure 2 presented in Section 4.1.

It is worth to note that Data Structure 2 is recursively defined: \( P, E_1 \) and \( H_2 \) are Kleene closures of \( \frac{n}{2} \times \frac{n}{2} \) matrices. Also observe that the polynomials \( Q, F_1, G_1, H_1, E_2, F_2, G_2, R, E, F, G \) and \( H \) that we maintain have all constant degree \( \leq 6 \). In Figure 3 we show the acyclic graph of dependencies between objects in our data structure: there is an arc from node \( u \) to node \( v \) if the polynomial associated to \( u \) is a variable of the polynomial associated to \( v \). For readability, we do not report nodes for the final polynomials \( E, F, G, H \). A topological sort of this graph, e.g., \( \tau = (P, Q, E_1, R, H_2, F_1, G_1, H_1, E_2, F_2, G_2, E, F, G, H) \), yields a correct evaluation order for the objects in the data structure and thus gives a method for computing \( \mathcal{H}(X) \).

We remark that our data structure has memory of all the intermediate values produced when computing \( \mathcal{H}(X) \) from scratch and maintains such values upon updates of \( X \). As it was already observed in Section 5, maintaining intermediate results of some static algorithm for computing \( X^\ast \) is a fundamental idea for updating efficiently \( X^\ast \) whenever \( X \) gets modified.

Since our data structure reflects the way \( \mathcal{H}(X) \) is computed, it basically represents \( X^\ast \) as the sum of two Boolean matrices: the first, say \( X_1^\ast \), is defined by submatrices \( E_1, F_1, G_1, H_1 \), and the second, say \( X_2^\ast \), by submatrices \( E_2, F_2, G_2, H_2 \):

\[
X_1^\ast = \begin{bmatrix}
E_1 \\
G_1 \\
F_1 \\
H_1
\end{bmatrix} \quad X_2^\ast = \begin{bmatrix}
E_2 \\
G_2 \\
F_2 \\
H_2
\end{bmatrix}
\]

In the next section we show how to implement operations \( \text{Init}^\ast, \text{Set}^\ast, \text{Reset}^\ast \) and \( \text{Lookup}^\ast \) introduced in Definition 4 in terms of operations \( \text{Init}, \text{LazySet}, \text{SetRow} \) and \( \text{SetCol} \) (see Section 4.1) on the polynomials of Data Structure 4.

### 6.2 Implementation of Operations

From a high-level point of view, our approach is the following. We maintain \( X_1^\ast \) and \( X_2^\ast \) in tandem (see Figure 4): whenever a \( \text{Set}^\ast \) operation is performed on \( X \),
we update $X^*$ by computing how either $X_1^*$ or $X_2^*$ are affected by this change. Such updates are lazily performed so that neither $X_1^*$ nor $X_2^*$ encode complete information about $X^*$, but their sum does. On the other side, Reset* operations update both $X_1^*$ and $X_2^*$ and leave the data structures as if any reset entry was never set to 1.

We now describe in detail our implementation. To keep pseudocodes shorter and more readable, we assume that implicit Lookup and Lookup* operations are performed in order to retrieve the current value of objects so as to use them in subsequent steps. Furthermore, we do not deal explicitly with base recursion steps.
Init* procedure Init*(Z)
1. begin
2. \( X \leftarrow Z \)
3. \( P.\text{Init}^*(D) \)
4. \( Q.\text{Init}(A, B, P, C) \)
5. \( E_1.\text{Init}^*(Q) \)
6. \( R.\text{Init}(D, C, E_1, B) \)
7. \( H_2.\text{Init}^*(R) \)
8. \( F_1.\text{Init}(E_1, B, P) \)
9. \{ similarly for \( G_1, H_1, E_2, F_2, G_2 \), and then for \( E, F, G, H \) \}
10. end

Init* sets the initial value of \( X \) (line 2) and initializes the objects in Data Structure 4 according to the topological order \( \tau \) of the graph of dependencies as explained in the previous subsection (lines 3–9).

Set*

Before describing our implementation of Set*, we first define a useful shortcut for performing simultaneous SetRow and SetCol operations with the same \( i \) on more than one variable in a polynomial \( P \)
procedure P.Set(i, ΔX_1, . . . , ΔX_q)
1. begin
2. P.SetRow(i, ΔX_1, X_1)
3. P.SetCol(i, ΔX_1, X_1)
4. :
5. P.SetRow(i, ΔX_q, X_q)
6. P.SetCol(i, ΔX_q, X_q)
7. end

Similarly, we give a shortcut for performing simultaneous LazySet operations on more than one variable in a polynomial P:

procedure P.LazySet(ΔX_1, . . . , ΔX_q)
1. begin
2. P.LazySet(ΔX_1, X_1)
3. :
4. P.LazySet(ΔX_q, X_q)
5. end

We also define an auxiliary operation LazySet* on closures that performs LazySet operations for variables A, B, C and D on the polynomials Q, R, F_1, G_1, H_1, E_2, F_2, and G_2 and recurses on the closure P which depend directly on them. We assume that, if M is a variable of a polynomial maintained in our data structure, \( \Delta M = M_{curr} - M_{old} \) is the difference between the current value \( M_{curr} \) of M and the old value \( M_{old} \) of M.

procedure LazySet*(ΔX)
1. begin
2. \( X \leftarrow X + \Delta X \)
3. Q.LazySet(ΔA, ΔB, ΔC)
4. R.LazySet(ΔB, ΔC, ΔD)
5. \{ similarly for F_1, G_1, H_1, E_2, F_2, and G_2 \}
6. P.LazySet*(ΔD)
7. end

Using the shortcuts Set and LazySet and the new operation LazySet*, we are now ready to define Set*.

\(^2\)For the sake of simplicity, we use the same identifier LazySet for both the shortcut and the native operation on polynomials, assuming to use the shortcut in defining Set*.,
procedure Set\(^*\)(i, ∆X)
1. begin
2. \(X ← X + IΔX,i + JΔX,i\)
3. if \(1 ≤ i ≤ \frac{n}{2}\) then
4. Q.Set\((i, \Delta A, \Delta B, \Delta C)\)
5. E\(_{1}\).Set\(*\)(i, ∆Q)
6. F\(_{1}\).Set\((i, \Delta E_{1}, \Delta B)\)
7. G\(_{1}\).Set\((i, \Delta C, \Delta E_{1})\)
8. H\(_{1}\).Set\((i, \Delta C, \Delta E_{1}, \Delta B)\)
9. R.Set\((i, \Delta C, \Delta E_{1}, \Delta B)\)
10. H\(_{2}\).LazySet\(*\)(\(\Delta R\))
11. G\(_{2}\).LazySet\((\Delta C, \Delta E_{1})\)
12. F\(_{2}\).LazySet\((\Delta E_{1}, \Delta B)\)
13. G\(_{2}\).LazySet\((\Delta E_{1}, \Delta B, \Delta C)\)
14. else \{ \(\frac{n}{2} + 1 ≤ i ≤ n\) \}
15. \(i ← i - \frac{n}{2}\)
16. P.Set\(*\)(i, ∆D)
17. R.Set\((i, \Delta B, \Delta C, \Delta D)\)
18. H\(_{2}\).Set\(*\)(i, ∆R)
19. G\(_{2}\).Set\((i, \Delta H_{2}, \Delta C)\)
20. F\(_{2}\).Set\((i, \Delta B, \Delta H_{2})\)
21. E\(_{2}\).Set\((i, \Delta B, \Delta H_{2}, \Delta C)\)
22. Q.Set\((i, \Delta B, \Delta H_{2}, \Delta C)\)
23. E\(_{1}\).LazySet\(*\)(\(\Delta Q\))
24. F\(_{1}\).LazySet\((\Delta B, \Delta P)\)
25. G\(_{1}\).LazySet\((\Delta P, \Delta C)\)
26. H\(_{1}\).LazySet\((\Delta B, \Delta P, \Delta C)\)
27. E.Init\((E_{1}, E_{2})\)
28. F.Init\((F_{1}, F_{2})\)
29. G.Init\((G_{1}, G_{2})\)
30. H.Init\((H_{1}, H_{2})\)
31. end

Set\(^*\) performs an \(i\)-centered update in \(X\) and runs through the closures and the polynomials of Data Structure 4 to propagate any changes of \(A, B, C, D\) to \(E, F, G, H\). The propagation order is \((Q, E_{1}, F_{1}, G_{1}, H_{1}, R, H_{2}, G_{2}, F_{2}, E_{2}, E, F, G, H)\) if \(1 ≤ i ≤ \frac{n}{2}\) and \((P, R, H_{2}, G_{2}, F_{2}, E_{2}, Q, E_{1}, F_{1}, G_{1}, H_{1})\) if \(\frac{n}{2} + 1 ≤ i ≤ n\) and is defined according to a topological sort of the graph of dependencies between objects in Data Structure 4 shown in Figure 3.

Roughly speaking, Set\(^*\) updates the objects in the data structure according to the value of \(i\) as follows:

1. If \(1 ≤ i ≤ \frac{n}{2}\), fully updates \(Q, R, E_{1}, F_{1}, G_{1}, H_{1}\) (lines 4–9) and lazily updates \(E_{2}, F_{2}, G_{2}, H_{2}\) (lines 10–13). See Figure 3(a).
Figure 5: Portions of Data Structure affected during a Set* operation when:
(a) $1 \leq i \leq \frac{n}{2}$; (b) $\frac{n}{2} + 1 \leq i \leq n$. 
2. If $\frac{n}{2} + 1 \leq i \leq n$, fully updates $P, Q, R, E_2, F_2, G_2, H_2$ (lines 16–22) and lazily updates $E_1, F_1, G_1, H_1$ (lines 23–26). See Figure 5 (b).

We highlight that it is not always possible to perform efficiently full updates of all the objects of Data Structure 4. Actually, some objects may change everywhere, and not only in a row and column. Such unstructured changes imply that we can only perform lazy updates on such objects, as they cannot be efficiently manipulated by means of $i$-centered SetRow and SetCol operations.

We now explain in detail the operations performed by $\text{Set}^*$ according to the two cases $1 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + 1 \leq i \leq n$.

**Case 1:** $1 \leq i \leq \frac{n}{2}$.

In this case an $i$-centered update of $X$ may affect the $i$-th row and the $i$-th column of $A$, the $i$-th row of $B$ and the $i$-th column of $C$, while $D$ is not affected at all by this kind of update (see Figure 4). The operations performed by $\text{Set}^*$ when $1 \leq i \leq \frac{n}{2}$ are therefore the following:

**Line 2:** an $i$-centered set operation is performed on $X$.

**Line 4:** $Q = A + BP^2C$ is updated by performing SetRow and SetCol operations for any variables $A, B$ and $C$ being changed. $P = D^*$ does not change since, as already observed, $D$ is not affected by the change. Notice that new 1’s may appear in $Q$ only in the $i$-th row and column due to this operation.

**Line 5:** $\text{Set}^*$ is recursively called to propagate the changes of $Q$ to $E_1$. We remark that $E_1$ may change also outside the $i$-th row and column due to this operation. Nevertheless, as we will see in Lemma 4 the fact that $E_1$ is a closure implies that new 1’s appear in a very structured way. This will make it possible to propagate changes efficiently to any polynomial that, in turn, depends on $E_1$.

**Lines 6–9:** polynomials $F_1, G_1, H_1$ and $R$ are updated by performing SetRow and SetCol operations for any variables $E_1, B$ and $C$ being changed. We recall that such operations take into account only the entries of $\Delta E_1$ lying in the $i$-th row and in the $i$-th column, albeit other entries may be non-zero. Again, Lemma 4 and Lemma 5 will show that this is sufficient.

**Lines 10–13:** $H_2 = R^*$ is not updated, but new 1’s that appear in $R$ are lazily inserted in the data structure of $H_2$ by calling LazySet*. Then LazySet operations are carried out on polynomials $G_2, F_2, E_2$ to insert in the data structures that maintain them any new 1’s that appear in $C, E_1$ and $B$.

**Lines 27–30.** Recompute polynomials $E, F, G$ and $H$ from scratch. This is required as $F_1, G_1$ and $H_2$ may change everywhere and not only in a row.
and a column. Differently from the case of $E_1$, whose change is structured as it is a closure, we cannot exploit any particular structure of $\Delta F_1$, $\Delta G_1$ and $\Delta H_2$ for reducing ourselves to use $\text{SetRow}$ and $\text{SetCol}$ and we are forced to use $\text{Init}$. Note that, since $E$, $F$, $G$ and $H$ have all degree 1, this is not a bottleneck in terms of running time.

**Case 2:** $\frac{n}{2} + 1 \leq i \leq n$.

In this case an $i$-centered update of $X$ may affect only the $i$-th row and the $i$-th column of $D$, the $i$-th row of $C$ and the $i$-th column of $B$, while $A$ is not affected at all by this kind of update (see Figure 4). Operations performed by $\text{Set}^*$ are completely analogous to the case $1 \leq i \leq \frac{n}{2}$, except for the fact that we need to rescale the index $i$ in line 15 and we have also to perform a recursive call to update $P$ in line 16.

**Reset**

Before describing our implementation of $\text{Reset}^*$, we define a useful shortcut\textsuperscript{3} for performing simultaneous $\text{Reset}$ operations on more than one variable in a polynomial $P$.

```plaintext
procedure P.Reset($\Delta X_1, \ldots, \Delta X_q$)
1. begin
2. P.Reset($\Delta X_1, X_1$)
3. 
4. P.Reset($\Delta X_q, X_q$)
5. end
```

Using this shortcut, we are now ready to define $\text{Reset}^*$. We assume that, if $M$ is a variable of a polynomial maintained in our data structure, $\Delta M = M_{\text{old}} - M_{\text{curr}}$ is the difference between the value $M_{\text{old}}$ of $M$ just before calling $\text{Reset}^*$ and the current value $M_{\text{curr}}$ of $M$.

```plaintext
procedure Reset^*(\Delta X)
1. begin
2. $X \leftarrow X - \Delta X$
3. P.Reset($\Delta D$)
4. Q.Reset($\Delta A, \Delta B, \Delta P, \Delta C$)
5. $E_1$.Reset^*($\Delta Q$)
6. R.Reset($\Delta D, \Delta C, \Delta E_1, \Delta B$)
7. $H_2$.Reset^*($\Delta R$)
8. $F_1$.Reset($\Delta E_1, \Delta B, \Delta P$)
9. { similarly for $G_1, H_1, E_2, F_2, G_2$, and then for $E, F, G, H$ }
10. end
```

\textsuperscript{3}For the sake of simplicity, we use the same identifier $\text{Reset}$ for both the shortcut and the native operation on polynomials, assuming to use the shortcut in defining $\text{Reset}^*$.
Reset* resets any entries of $X$ as specified by $\Delta X$ and runs through the closures and the polynomials in the data structure to propagate any changes of $A, B, C, D$ to $E, F, G, H$. The propagation is done according to a topological order $\tau$ of the graph of dependencies shown in Figure 3 and is the same order followed by Init*, which has a similar structure. Actually, we could think of Reset* as a function that “undoes” any previous work performed by Init* and Set* on the data structure, leaving it as if the reset entries of $X$ were never set to 1.

**Lookup* procedure**

```plaintext
procedure Lookup*(x, y)
1. begin
2. return $Y[x, y]$
3. end
```

Lookup* simply returns the maintained value of $Y[x, y]$.

### 6.3 Analysis

Now we discuss the correctness and the complexity of our implementation. Before providing the main claims, we give some preliminary definitions and lemmas that are useful for capturing algebraic properties of the changes that polynomials in our data structure undergo during a Set* operation.

The next definition recalls a property of Boolean update matrices that is related to the operational concept of $i$-centered update.

**Definition 12** We say that a Boolean update matrix $\Delta X$ is $i$—centered if $\Delta X = I_{\Delta X,i} + J_{\Delta X,i}$, i.e., all entries lying outside the $i$-th row and the $i$-th column are zero.

If the variation $\Delta X$ of some matrix $X$ during an update operation is $i$-centered and $X$ is a variable of a polynomial $P$ that has to be efficiently reevaluated, then we can use P.SetRow and P.SetCol operations which are especially designed for doing so. But what happens if $X$ changes by a $\Delta X$ that is not $i$-centered? Can we still update efficiently the polynomial $P$ without recomputing it from scratch via Init? This is the case of $E_1$ and $\Delta E_1$ while performing a Set* update with $1 \leq i \leq \frac{n}{2}$. In the following we show that, under certain hypotheses on $X$ and $\Delta X$ (which are satisfied by $E_1$ and $\Delta E_1$), we can still solve the problem efficiently.

While the property of being $i$-centered is related to an update matrix by itself, the following two definitions are concerned with properties of an update matrix $\Delta X$ with respect to the matrix $X$ to which it is applied:

**Definition 13** If $X$ is a Boolean matrix and $\Delta X$ is a Boolean update matrix, we say that $\Delta X$ is $i$-transitive with respect to $X$ if $I_{\Delta X,i} = I_{\Delta X,i} \cdot X$ and $J_{\Delta X,i} = X \cdot J_{\Delta X,i}$. 
Definition 14 If \( X \) is a Boolean matrix and \( \Delta X \) is a Boolean update matrix, we say that \( \Delta X \) is \( i \)-complete with respect to \( X \) if
\[
\Delta X = J_{\Delta X,i} \cdot I_{\Delta X,i} + X \cdot I_{\Delta X,i} + J_{\Delta X,i} \cdot X.
\]

Using the previous definitions we can show that the variation of \( X^* \) due to an \( i \)-centered update of \( X \) is \( i \)-transitive and \( i \)-complete.

Lemma 6 Let \( X \) be a Boolean matrix and let \( \Delta X \) be an \( i \)-centered update matrix. If we denote by \( \Delta X^* \) the matrix \( (X + \Delta X)^* - X^* \), then \( \Delta X^* \) is \( i \)-transitive and \( i \)-complete with respect to \( X^* \).

Proof. The following equalities prove the first condition of \( i \)-transitivity:
\[
I_{\Delta X^*,i} \cdot X^* = I_{(X + \Delta X)^*,x^*,i} \cdot X^* = I_{(X + \Delta X)^* - X^*,i} = I_{\Delta X^*,i}.
\]
The other conditions can be proved analogously. The hypothesis that \( \Delta X \) is \( i \)-centered is necessary for the \( i \)-completeness.

The following lemma shows under what conditions for \( \Delta X \) and \( X \) it is possible to perform operations of the kind \( X \leftarrow X + \Delta X \) on a variable \( X \) of a polynomial by reducing such operations to \( i \)-centered updates even if \( \Delta X \) is not \( i \)-centered.

Lemma 7 If \( X \) is a Boolean matrix such that \( X = X^* \) and \( \Delta X \) is an \( i \)-transitive and \( i \)-complete update matrix with respect to \( X \), then \( X + \Delta X = (X + I_{\Delta X,i} + J_{\Delta X,i})^2 \).

Proof. Since \( X = X^* \) it holds that \( X = X^2 \) and \( X = X + I_{\Delta X,i} \cdot J_{\Delta X,i} \). The proof follows from Definition 13 and Definition 14 and from the facts that:
\[
I_{\Delta X,i}^2 \subseteq I_{\Delta X,i}, \quad J_{\Delta X,i}^2 \subseteq J_{\Delta X,i}, \quad \text{and} \quad \Delta X = \Delta X + I_{\Delta X,i} + J_{\Delta X,i}.
\]

It follows that, under the hypotheses of Lemma 7, if we replace any occurrence of \( X \) in \( P \) with \( X^2 \) and we perform both \( P.SetRow(i, I_{\Delta X,i}, X) \) and \( P.SetCol(i, J_{\Delta X,i}, X) \), then new 1’s in \( P \) correctly appear. This is the reason why in Data Structure 4 we used \( E_1^2, H_2^2 \), and \( P^2 \) instead of \( E_1, H_2 \), and \( P \), respectively.

Before stating the main theorem of this section which establishes the correctness of operations on our data structure, we discuss a general property of polynomials and closures over Boolean matrices that will be useful in proving the theorem.

Lemma 8 Let \( P \) and \( Q \) be polynomials or closures over Boolean matrices and let \( \hat{P} \) and \( \hat{Q} \) be relaxed functions such that \( \hat{P}(X) \subseteq P(X) \) and \( \hat{Q}(Y) \subseteq Q(Y) \) for any values of variables \( X \) and \( Y \). Then, for any \( X \):
\[
\hat{Q}(\hat{P}(X)) \subseteq Q(P(X))
\]
Proof. Let \( \hat{Y} = \hat{P}(X) \) and \( Y = P(X) \). By definition, we have: \( \hat{Y} \subseteq Y \) and \( \hat{Q}(\hat{Y}) \subseteq Q(\hat{Y}) \). By exploiting a monotonic behavior of polynomials and closures over Boolean matrices, we have: \( \hat{Y} \subseteq Y \Rightarrow \hat{Q}(\hat{Y}) \subseteq Q(\hat{Y}) \Rightarrow \hat{Q}(\hat{P}(X)) \subseteq Q(P(X)) \). \( \square \)

Theorem 9 Let \( H \) be the function defined in Lemma 5, let \( X \) and \( Y \) be the matrices maintained in Data Structure 4, and let \( M \) be a Boolean matrix whose value at any time \( j \) is defined as:

\[
M_j = \sum_{1 \leq i \leq j : \text{Op}_i \neq \text{LazySet}^*} H(X_i) - H(X_{i-1}).
\]

If we denote by \( X_j \) and \( Y_j \) the values of \( X \) and \( Y \) after the \( j \)-th operation, respectively, then the relation \( M_j \subseteq Y_j \subseteq H(X_j) \) is satisfied.

Proof. The proof is by induction on the size \( n \) of matrices in Data Structure 4. The base is trivial. We assume that the claim holds for instances of size \( n^2 \) and we prove that it holds also for instances of size \( n \).

- \( \text{Op}_j = \text{Init}^* \): since \( \text{Init}^* \) performs \( \text{Init} \) operations on each object, then \( Y_j = H(X_j) \).

- \( \text{Op}_j = \text{Set}^* \): we first prove that \( Y_j \subseteq H(X_j) \). Observe that \( Y \) is obtained as a result of a composition of functions that relax the correct intermediate values of polynomials and closures of Boolean matrices in our data structure allowing them to contain less 1's. Indeed, by the properties of \( \text{Lookup} \) described in Section 4.1, we know that, if \( P \) is the correct value of a polynomial at any time, then \( P.\text{Lookup}(x, y) = 1 \Rightarrow P[x, y] = 1 \). Similarly, by inductive hypothesis, if \( K \) is a Kleene closure of an \( n^2 \times n^2 \) Boolean matrix, then at any time \( K.\text{Lookup}^*(x, y) = 1 \Rightarrow K[x, y] = 1 \). The claim then follows by Lemma 8, which states that the composition of relaxed functions computes values containing at most the 1's contained in the values computed by the correct functions.

To prove that \( M_j \subseteq Y_j \), based on the definition of \( M \), it suffices to verify that \( \Delta H(X) \subseteq \Delta Y \), where \( \Delta H(X) = H(X_j) - H(X_{j-1}) \) and \( \Delta Y = Y_j - Y_{j-1} \). In particular, we prove that if \( H[x, y] \) flips from 0 to 1 due to operation \( \text{Set}^* \), then either \( X_1^*[x, y] \) flips from 0 to 1 (due to lines 4–8 when \( 1 \leq i \leq \frac{n}{2} \)), or \( X_2^*[x, y] \) flips from 0 to 1 (due to lines 17–21 when \( \frac{n}{2} + 1 \leq i \leq n \)).

Without loss of generality, assume that the \( \text{Set}^* \) operation is performed with \( 1 \leq i \leq \frac{n}{2} \) (the proof is completely analogous if \( \frac{n}{2} + 1 \leq i \leq n \)).

As shown in Figure 4, sub-matrices \( A \), \( B \) and \( C \) may undergo \( i \)-centered updates due to this operation and so their variation can be correctly propagated through \( \text{SetRow} \) and \( \text{SetCol} \) operations to polynomial \( Q \) (line 4) and
to polynomials $F_1$, $G_1$ and $H_1$ (lines 6–8). As $\Delta Q$ is also $i$-centered due to line 4, any variation of $Q$, that is assumed to be elsewhere correct from previous operations, can be propagated to closure $E_1$ through a recursive call of $\text{Set}^*$ in line 5. By the inductive hypothesis, this propagation correctly reveals any new 1’s in $E_1$. We remark that $E_1$ may contain less 1’s than $E$ due to any previous $\text{LazySet}$ operations done in line 23.

Observe now that $E_1$ occurs in polynomials $F_1$, $G_1$ and $H_1$ and that $\Delta E_1$ is not necessarily $i$-centered. This would imply that we cannot propagate directly changes of $E_1$ to these polynomials, as no efficient operation for doing so was defined in Section 4.1. However, by Lemma 6, $\Delta E_1$ is $i$-transitive and $i$-complete with respect to $E_1$. Since $E_1 = E_1^*$, by Lemma 7, performing both $\text{SetRow}(i, I_{\Delta E_1}, i, \Delta E_1)$ and $\text{SetCol}(i, J_{\Delta E_1}, i, \Delta E_1)$ operations on data structures $F_1$, $G_1$ and $H_1$ in lines 6–8 is sufficient to correctly reveal new 1’s in $F_1$, $G_1$ and $H_1$.

Again, note that $F_1$, $G_1$ and $H_1$ may contain less 1’s than $F$, $G$ and $H$, respectively, due to any previous $\text{LazySet}$ operations done in lines 23–26. We have then proved that lines 4–8 correctly propagate any $i$-centered update of $X$ to $X^*$.

To conclude the proof, we observe that $E_1$ also occurs in polynomials $E_2$, $F_2$, $G_2$, $R$ and indirectly affects $H_2$. Unfortunately, we cannot update $H_2$ efficiently as $\Delta R$ is neither $i$-centered, nor $i$-transitive/$i$-complete with respect to $R$. So in lines 9–13 we limit ourselves to update explicitly $R$ and to log any changes of $E_1$ by performing $\text{LazySet}$ operations on polynomials $G_2$, $F_2$, and $E_2$ and a $\text{LazySet}^*$ operation on $H_2$. This is sufficient to guarantee the correctness of subsequent $\text{Set}^*$ operations for $\frac{n}{2} + 1 \leq i \leq n$.

- $\text{Op}_j = \text{Reset}^*$: this operation runs in judicious order through the objects in the data structure and undoes the effects of previous $\text{Set}^*$ and $\text{Init}^*$ operations. Thus, any property satisfied by $Y$ still holds after performing a $\text{Reset}^*$ operation.

\[ \square \]

**Corollary 8** Let $X$ be an instance of Data Structure 4 and let $\sigma = \langle X.\text{Op}_1, \ldots, X.\text{Op}_k \rangle$ be a sequence of operations on $X$. If $\text{Op}_i \neq \text{LazySet}^*$ for all $1 \leq i \leq j \leq k$, then $M_j = \mathcal{H}(X_j)$.

**Proof.** Since $\mathcal{H}(0_n) = 0_n^* = 0_n$, the proof easily follows by telescoping the sum that defines $M_j$: $M_j = \mathcal{H}(X_j) - \mathcal{H}(X_{j-1}) + \mathcal{H}(X_{j-1}) - \mathcal{H}(X_{j-2}) + \cdots + \mathcal{H}(X_2) - \mathcal{H}(X_1) + \mathcal{H}(X_1) - \mathcal{H}(X_0) = \mathcal{H}(X_j) - \mathcal{H}(X_0) = \mathcal{H}(X_j). \square$

To conclude this section, we address the running time of operations and the space required to maintain an instance of our data structure.
Theorem 10 Any $\text{Init}^*$ operation can be performed in $O(n^\omega)$ worst-case time, where $\omega$ is the exponent of matrix multiplication; any $\text{Set}^*$ takes $O(n^2)$ amortized time. The cost of $\text{Reset}^*$ operations can be charged to previous $\text{Init}^*$ and $\text{Set}^*$ operations. The maximum cost charged to each $\text{Init}^*$ is $O(n^3)$. The space required is $O(n^2)$.

Proof. Since all the polynomials in Data Structure 4 are of constant degree and involve a constant number of terms, the amortized cost of any $\text{SetRow}$, $\text{SetCol}$, $\text{LazySet}$, and $\text{Reset}$ operation on them is quadratic in $n^2$ (see Theorem 4). Let $T(n)$ be the time complexity of any $\text{Set}^*$, $\text{LazySet}^*$ and $\text{Reset}^*$ operation. Then:

$$T(n) \leq 3T\left(\frac{n}{2}\right) + \frac{cn^2}{4}$$

for some suitably chosen constant $c > 0$. As $\log_2 3 < 2$, this implies that $T(n) = O(n^2)$.

$\text{Init}^*$ recomputes recursively $\mathcal{H}$ from scratch using $\text{Init}$ operations on polynomials, which require $O(n^\omega)$ worst-case time each. We can then prove that the running time of $\text{Init}^*$ is $O(n^\omega)$ exactly as in Theorem 8.

To conclude the proof, observe that if $K(n)$ is the space used to maintain all the objects in Data Structure 4, and $M(n)$ is the space required to maintain a polynomial with the data structure of Section 4.1, then:

$$K(n) \leq 3K\left(\frac{n}{2}\right) + 12M(n).$$

Since $M(n) = O(n^2)$ by Theorem 4, then $K(n) = O(n^2)$. \qed

Corollary 9 If we perform just one $\text{Init}^*$ operation in a sequence $\sigma$ of length $\Omega(n)$, or more generally one $\text{Init}^*$ operation every $\Omega(n)$ $\text{Reset}^*$ operations, then the amortized cost of $\text{Reset}^*$ is $O(n^2)$ per operation.

Corollary 10 If we perform just one $\text{Init}^*$ operation in a sequence $\sigma$ of length $\Omega(n^2)$, or more generally one $\text{Init}^*$ operation every $\Omega(n^2)$ $\text{Reset}^*$ operations, and we perform no $\text{Set}^*$ operations, then the amortized cost of $\text{Reset}^*$ is $O(n)$ per operation.

In the traditional case where $0p_1 = \text{Init}^*$ and $0p_i \neq \text{Init}^*$ for any $i > 1$, i.e., $\text{Init}^*$ is performed just once at the beginning of the sequence of operations, previous corollaries state that both $\text{Set}^*$ and $\text{Reset}^*$ are supported in $O(n^2)$ amortized time. In the decremental case where only $\text{Reset}^*$ operations are performed, the amortized time is $O(n)$ per update.
7 Breaking Through the $O(n^2)$ Barrier

In this section we present the first algorithm that supports both updates and queries in subquadratic time per operation, showing that it is actually possible to break through the $O(n^2)$ barrier on the single-operation complexity of fully dynamic transitive closure. This result is obtained by means of a new technique that consists of casting fully dynamic transitive closure into the problem of dynamically maintaining matrices over integers presented in Section 4.2. As already shown in Section 5 and in Section 6, dynamic matrices, thanks to their strong algebraic properties, play a crucial role in designing efficient algorithms for the fully dynamic transitive closure problem.

The remainder of this section is organized as follows. In Section 7.1 we present a subquadratic algorithm for directed acyclic graphs based on dynamic matrices that answers queries in $O(n^{\epsilon})$ time and performs updates in $O(n^{\omega(1, \epsilon, 1) - \epsilon} + n^{1+\epsilon})$ time, for any $0 \leq \epsilon \leq 1$, where $\omega(1, \epsilon, 1)$ is the exponent of the multiplication of an $n \times n^\epsilon$ matrix by an $n^\epsilon \times n$ matrix. According to the current best bounds on $\omega(1, \epsilon, 1)$, we obtain an $O(n^{0.58})$ query time and an $O(n^{1.58})$ update time. The algorithm we propose is randomized, and has one-side error.

7.1 Counting Paths in Acyclic Directed Graphs

In this section we study a variant of the fully dynamic transitive closure problem presented in Definition 1 and we devise the first algorithm that supports both update and query in subquadratic time per operation. In the variant that we consider, the graph that we maintain is constrained to be acyclic; furthermore, Insert and Delete operations work on single edges rather than on set of edges. We shall discuss later how to extend our algorithm to deal with more than one edge at a time.

Definition 15 Let $G = (V, E)$ be a directed acyclic graph and let $TC(G) = (V, E')$ be its transitive closure. We consider the problem of maintaining a data structure $G$ for the graph $G$ under an intermixed sequence $\sigma = \langle G.\text{Op}_1, \ldots, G.\text{Op}_k \rangle$ of update and query operations. Each operation $G.\text{Op}_j$ on the data structure $G$ can be either one of the following:

- $G.\text{Insert}(x, y)$: perform the update $E \leftarrow E \cup \{(x, y)\}$, such that the graph obtained after the update is still acyclic.
- $G.\text{Delete}(x, y)$: perform the update $E \leftarrow E - \{(x, y)\}$, where $(x, y) \in E$.
- $G.\text{Query}(x, y)$: perform a query operation on $TC(G)$ by returning 1 if $(x, y) \in E'$ and 0 otherwise.

In this version of the problem, we do not deal explicitly with initialization operations.
Data Structure

In [16] King and Sagert showed that keeping a count of the number of distinct paths between any pair of vertices in a directed acyclic graph $G$ allows it to maintain the transitive closure of $G$ upon both insertions and deletions of edges. Unfortunately, these counters may be as large as $2^n$: to perform $O(1)$ time arithmetic operations on counters, an $O(n)$ wordsize is required. As shown in [16], the wordsize can be reduced to $2c \lg n$ for any $c \geq 5$ based on the use of arithmetic operations performed modulo a random prime number. This yields a fully dynamic randomized Monte Carlo algorithm for transitive closure with the property that “yes” answers on reachability queries are always correct, while “no” answers are wrong with probability $O(\frac{1}{n^c})$. We recall that this algorithm performs reachability queries in $O(1)$ and updates in $O(n^2)$ worst-case time on directed acyclic graphs.

We now present an algorithm that combines the path counting approach of King and Sagert with our technique of implicit matrix representation. Both techniques are very simple, but surprisingly their combination solves a problem that has been open for many years.

**Data Structure 5** We keep a count of the number of distinct paths between any pair of vertices in graph $G$ by means of an instance $M$ of the dynamic matrix data structure described in Section 4.3. We assume that $M[x, y]$ is the number of distinct paths between node $x$ and node $y$ in graph $G$. Since $G$ is acyclic, this number is well-defined.

Implementation of Operations

We now show how to implement operations Insert, Delete and Query in terms of operations Update and Lookup on our data structure as described in Section 4.2. We assume all arithmetic operations are performed in constant time.

Insert

```
procedure Insert(x, y)
1. begin
2. 
3. for $z = 1$ to $n$ do
4. 
5. 
6. 
7. end
```

Insert first puts edge $(x, y)$ in the graph and then, after querying matrix $M$, computes two vectors $J$ and $I$ such that $J[z]$ is the number of distinct paths $z \rightsquigarrow x$ in $G$ and $I[z]$ is the number of distinct paths $y \rightsquigarrow z$ in $G$ (lines 3–5).
Finally, it updates $M$ in line 6. The operation performed on $M$ is $M \leftarrow M + J \cdot I$: this means that the number $M[u, v]$ of distinct paths between any two nodes $(u, v)$ is increased by the number $J[u]$ of distinct paths $u \sim x$ times the number $I[v]$ of distinct paths $y \sim v$, i.e., $M[u, v] \leftarrow M[u, v] + J[u] \cdot I[v]$.

**Delete**

```latex
\begin{verbatim}
procedure Delete(x, y)
1. begin
2. \hspace{0.5cm} E ← E − \{(x, y)\}
3. \hspace{0.5cm} for z = 1 to n do
4. \hspace{1.0cm} J[z] ← M.Lookup(z, x)
5. \hspace{1.0cm} I[z] ← M.Lookup(y, z)
6. \hspace{0.5cm} M.Update(−J, I)
7. end
\end{verbatim}
```

Delete is identical to Insert, except for the fact that it removes the edge $(x, y)$ from the graph and performs the update of $M$ in line 6 with $−J$ instead of $J$. The operation performed on $M$ is $M \leftarrow M − J \cdot I$: this means that the number $M[u, v]$ of distinct paths between any two nodes $(u, v)$ is decreased by the number $J[u]$ of distinct paths $u \sim x$ times the number $I[v]$ of distinct paths $y \sim v$, i.e., $M[u, v] \leftarrow M[u, v] − J[u] \cdot I[v]$.

**Query**

```latex
\begin{verbatim}
procedure Query(x, y)
1. begin
2. \hspace{0.5cm} if M.Lookup(x, y) > 0 then return 1
3. \hspace{0.5cm} else return 0
4. end
\end{verbatim}
```

Query simply looks up the value of $M[x, y]$ and returns 1 if the current number of distinct paths between $x$ and $y$ is positive, and zero otherwise.

\[\text{We are now ready to discuss the running time of our implementation of operations Insert, Delete, and Query.} \]

**Theorem 11** Any Insert and any Delete operation can be performed in $O(n^{\omega(1,\epsilon,1)−\epsilon} + n^{1+\epsilon})$ worst-case time, for any $0 \leq \epsilon \leq 1$, where $\omega(1,\epsilon,1)$ is the exponent of the multiplication of an $n \times n^\epsilon$ matrix by an $n^\epsilon \times n$ matrix. Any Query takes $O(n^\epsilon)$ in the worst case. The space required is $O(n^2)$.

**Proof.** We recall that, by Theorem [2], each entry of $M$ can be queried in $O(n^\epsilon)$ worst-case time, and each Update operation can be performed in $O(n^{\omega(1,\epsilon,1)−\epsilon})$ worst-case time. Since $I$ and $J$ can be computed in $O(n^{1+\epsilon})$ worst-case time by means of $n$ queries on $M$, we can support both insertions and deletions in
\( O(n^{\omega(1, \epsilon, 1) - \epsilon} + n^{1+\epsilon}) \) worst-case time, while a reachability query for any pair of vertices \((x, y)\) can be answered in \( O(n^\epsilon) \) worst-case time by simply querying the value of \( M[x, y] \).

Corollary 11 Any Insert and any Delete operation requires \( O(n^{1.58}) \) worst-case time, and any Query requires \( O(n^{0.58}) \) worst-case time.

Proof. Balancing the two terms in the update bound \( O(n^{\omega(1, \epsilon, 1) - \epsilon} + n^{1+\epsilon}) \) yields that \( \epsilon \) must satisfy the equation \( \omega(1, \epsilon, 1) = 1 + 2\epsilon \). The current best bounds on \( \omega(1, \epsilon, 1) \) [4, 10] imply that \( \epsilon < 0.58 \) [20]. Thus, the smallest update time is \( O(n^{1.58}) \), which gives a query time of \( O(n^{0.58}) \).

The algorithm we presented is deterministic. However, as the numbers involved may be as large as \( 2^n \), performing arithmetic operations in constant time requires wordsize \( O(n) \). To reduce wordsize to \( O(\log n) \) while maintaining the same subquadratic bounds (\( O(n^{1.58}) \) per update and \( O(n^{0.58}) \) per query) we perform all arithmetic operations modulo some random prime number as explained in [10]. Again, this produces a randomized Monte Carlo algorithm, where “yes” answers on reachability queries are always correct, while “no” answers are wrong with probability \( O(1/n^c) \) for any constant \( c \geq 5 \).

It is also not difficult to extend our subquadratic algorithm to deal with insertions/deletions of more than one edge at a time. In particular, we can support any insertion/deletion of up to \( O(n^{1-\eta}) \) edges incident to a common vertex in \( O(n^{\omega(1, \epsilon, 1) - \epsilon} + n^{2-(\eta-\epsilon)}) \) worst-case time. We emphasize that this is still \( o(n^2) \) for any \( 1 > \eta > \epsilon > 0 \). Indeed, rectangular matrix multiplication can be trivially implemented via matrix multiplication: this implies that \( \omega(1, \epsilon, 1) < 2 - (2 - \omega)\epsilon \), where \( \omega = \omega(1, 1, 1) < 2.38 \) is the current best exponent for matrix multiplication [3].

8 Conclusions

In this paper we have presented new time and space efficient algorithms for maintaining the transitive closure of a directed graph under edge insertions and edge deletions. As a main contribution, we have introduced a general framework for casting fully dynamic transitive closure into the problem of dynamically reevaluating polynomials over matrices when updates of variables are performed. Such technique has turned out to be very flexible and powerful, leading both to revisit the best known algorithm for fully dynamic transitive closure [13] from a completely different perspective, and to design new and faster algorithms for the problem.

In particular, efficient data structures for maintaining polynomials over Boolean matrices allowed us to devise the fairly complex deterministic algorithm described in Section 6, which supports updates in quadratic amortized time and queries with
just one matrix lookup. Our algorithm improves the best bounds for fully dy-
namic transitive closure achieved in [16] and is the fastest algorithm with constant
query time known in literature for this problem.

In addition, a surprisingly simple technique for efficiently maintaining dy-
namic matrices of integers under simultaneous updates of multiple entries, com-
bined with a previous idea of counting paths in acyclic digraphs [16], yielded the
randomized algorithm presented in Section 7.1: this algorithm, for the first time
in the study of fully dynamic transitive closure, breaks through the $O(n^2)$ barrier
on the single-operation complexity of the problem.

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References

[1] A.V. Aho, J.E. Hopcroft, and J.D. Ullman. The Design and Analysis of
Computer Algorithms. Addison Wesley, 1974.

[2] D. Coppersmith and S. Winograd. Matrix multiplication via arithmetic
progressions. Journal of Symbolic Computation, 9:251–280, 1990.

[3] T.H. Cormen, C.E. Leiserson, and R.L. Rivest. Introduction to Algorithms.
The MIT Press, 1990.

[4] C. Demetrescu. Fully Dynamic Algorithms for Path Problems on Directed
Graphs. PhD thesis, Department of Computer and Systems Science, Uni-
versity of Rome “La Sapienza”, February 2001.

[5] C. Demetrescu and G.F. Italiano. Fully dynamic transitive closure: Breaking
through the $O(n^2)$ barrier. In Proc. of the 41st IEEE Annual Symposium on
Foundations of Computer Science (FOCS’00), pages 381–389, 2000.

[6] S. Even and Y. Shiloach. An on-line edge-deletion problem. Journal of the
ACM, 28:1–4, 1981.

[7] M. J. Fischer and A. R. Meyer. Boolean matrix multiplication and transitive
closure. In Conference Record 1971 Twelfth Annual Symposium on Switch-
ing and Automata Theory, pages 129–131, East Lansing, Michigan, 13–15
October 1971. IEEE.
[8] M.E. Furman. Application of a method of fast multiplication of matrices in the problem of finding the transitive closure of a graph. *Soviet Math. Dokl.*, 11(5), 1970. English translation.

[9] M. Henzinger and V. King. Fully dynamic biconnectivity and transitive closure. In *Proc. 36th IEEE Symposium on Foundations of Computer Science (FOCS’95)*, pages 664–672, 1995.

[10] X. Huang and V.Y. Pan. Fast rectangular matrix multiplication and applications. *Journal of Complexity*, 14(2):257–299, June 1998.

[11] T. Ibaraki and N. Katoh. On-line computation of transitive closure for graphs. *Information Processing Letters*, 16:95–97, 1983.

[12] G. F. Italiano. Amortized efficiency of a path retrieval data structure. *Theoretical Computer Science*, 48(2–3):273–281, 1986.

[13] G. F. Italiano. Finding paths and deleting edges in directed acyclic graphs. *Information Processing Letters*, 28:5–11, 1988.

[14] S. Khanna, R. Motwani, and R. H. Wilson. On certificates and lookahead on dynamic graph problems. In *Proc. 7th ACM-SIAM Symp. Discrete Algorithms*, pages 222–231, 1996.

[15] V. King. Fully dynamic algorithms for maintaining all-pairs shortest paths and transitive closure in digraphs. In *Proc. 40th IEEE Symposium on Foundations of Computer Science (FOCS’99)*, 1999.

[16] V. King and G. Sagert. A fully dynamic algorithm for maintaining the transitive closure. In *Proc. 31st ACM Symposium on Theory of Computing (STOC’99)*, pages 492–498, 1999.

[17] J. A. La Poutré and J. van Leeuwen. Maintenance of transitive closure and transitive reduction of graphs. In *Proc. Workshop on Graph-Theoretic Concepts in Computer Science*, pages 106–120. Lecture Notes in Computer Science 314, Springer-Verlag, Berlin, 1988.

[18] I. Munro. Efficient determination of the transitive closure of a directed graph. *Information Processing Letters*, 1(2):56–58, 1971.

[19] D. M. Yellin. Speeding up dynamic transitive closure for bounded degree graphs. *Acta Informatica*, 30:369–384, 1993.

[20] U. Zwick. All pairs shortest paths in weighted directed graphs - exact and almost exact algorithms. In *Proc. of the 39th IEEE Annual Symposium on Foundations of Computer Science (FOCS’98)*, pages 310–319, Los Alamitos, CA, November 8–11 1998.