HOMOTOPY TYPE AND VOLUME OF LOCALLY SYMMETRIC MANIFOLDS

TSACHIK GELANDER

Abstract. We consider locally symmetric manifolds with a fixed universal covering, and construct for each such manifold $M$ a simplicial complex $\mathcal{R}$ whose size is proportional to the volume of $M$. When $M$ is non-compact, $\mathcal{R}$ is homotopically equivalent to $M$, while when $M$ is compact, $\mathcal{R}$ is homotopically equivalent to $M \setminus N$, where $N$ is a finite union of submanifolds of fairly smaller dimension. This reflects how the volume controls the topological structure of $M$, and yields concrete bounds for various finiteness statements which previously had no quantitative proofs. For example, it gives an explicit upper bound for the possible number of locally symmetric manifolds of volume bounded by $v > 0$, and it yields an estimate for the size of a minimal presentation for the fundamental group of a manifold in terms of its volume. It also yields a number of new finiteness results.

Contents

1. Introduction and statements of the main results 2
2. Notations, definitions and background 7
3. Some deformation retracts 11
4. Constructing a simplicial complex in a thick submanifold with nice boundary 15
5. An arithmetic variant of the Margulis lemma 20
6. The proof of 1.5(1) 22
7. Estimating angles at corners of the boundary 27
8. The proof of 1.5(2) in the rank-1 case 31
9. The proofs of 1.5(2) and 1.5(3) 33
10. Some remarks on Conjecture 1.3, Theorem 1.5 and their relations to algebraic number theory 38
11. Estimating the size of a minimal presentation 41
12. A quantitative version of Wang’s theorem 43
13. Some complements 44
References 51

Research partially supported by the Clore and the Marie Curie fellowships.
1. Introduction and statements of the main results

In this article we study relations between the volume and the topological structure of locally symmetric manifolds. We are interested in asymptotic properties, when the volume tends to infinity.

We shall always fix a symmetric space, $S$, of non-compact type without Euclidean de-Rham factors, and consider the class of $S$-manifolds, by which we mean complete Riemannian manifolds locally isometric to $S$, or equivalently, manifolds of the form $M = \Gamma \setminus S$ where $\Gamma$ is a discrete torsion-free group of isometries of $S$. Sometimes we shall restrict our attention to arithmetic manifolds, i.e. to Riemannian manifolds of the form $M = \Gamma \setminus S$, where $\Gamma \leq \text{Isom}(S)$ is a torsion-free arithmetic lattice in the Lie group $\text{Isom}(S)$ of isometries of $S$. A theorem of Borel and Harish-Chandra \cite{BH} says that if $M$ is arithmetic then $\text{vol}(M) < \infty$. When $\text{rank}(S) \geq 2$, Margulis’ arithmeticity theorem \cite{M} gives the converse, i.e. $\text{vol}(M) < \infty$ iff $M$ is arithmetic\footnote{The equivalence between finite volume and arithmeticity is also known for the rank-1 cases $\text{Sp}(n,1)$ and $\text{F}_4^{-20}$ by \cite{IS}, \cite{J}}.

The “complexity” of the topology of locally symmetric manifolds is controlled by the volume. This is illustrated by a theorem of Gromov (see \cite{G} theorem 2) which asserts that the Betti numbers are bounded by a constant times the volume, i.e.

$$\sum_{i=1}^{n} b_i(M) \leq c(S) \cdot \text{vol}(M)$$

for any $S$-manifold $M$. Gromov’s theorem applies also to non-locally symmetric manifolds, under appropriate conditions.

We conjecture (and prove in many cases) that, for locally symmetric manifolds, the volume forces stronger topological restrictions.

Definition 1.1. A $(d, v)$-simplicial complex is a simplicial complex with at most $v$ vertices, all of them of valence $\leq d$.

Remark 1.2. The number of $k$-simplexes in a $(d, v)$-simplicial complex $\mathcal{R}$ is $\leq \frac{v}{k+1} \binom{n}{k}$. Thus, the size of $\mathcal{R}$ (the number of its simplexes) is at most $v \cdot \sum_{k=0}^{d} \frac{1}{k+1} \binom{n}{k}$, and this depends linearly on $v$.

Conjecture 1.3. For any symmetric space of non-compact type $S$, there are constants $\alpha(S), d(S)$, such that any irreducible $S$-manifold $M = \Gamma \setminus S$ (which is assumed also to be arithmetic in the case $\text{dim}(S) = 3$) is homotopically equivalent to a $(d(S), \alpha(S) \text{vol}(M))$-simplicial complex.

Remark 1.4. The analogous statement is false for non-arithmetic manifolds in dimension 3.
In this paper we shall establish the following partial answers to Conjecture 1.3. Since we failed in proving it in full generality, our results cannot be organized in a compact form, and we have to split our statements and formulate strong results under appropriate conditions, and weaker results for more general cases.

**Theorem 1.5.** Let $S$ be a symmetric space of non-compact type. Then:

1. Conjecture 1.3 holds for non-compact arithmetic $S$-manifolds.
2. If $S$ is neither isometric to $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{H}^2$ nor to $\mathbb{H}^3$, then for some constants $\alpha(S), d(S)$, the fundamental group $\pi_1(M)$ of any $S$-manifold $M$ is isomorphic to the fundamental group of some $(d(S), \alpha(S) \text{vol}(M))$-simplicial complex.
3. If $S$ is isometric to $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{H}^2$ or to $\mathbb{H}^3$, then the fundamental group $\pi_1(M)$ of any $S$-manifold $M$ is a quotient of the fundamental group of some $(d(S), \alpha(S) \text{vol}(M))$-simplicial complex.

**Remark 1.6.** For compact arithmetic manifolds, Conjecture 1.3 would follow, by a straightforward argument, if one could prove that the infimum of the lengths of closed geodesics, taken over all compact arithmetic $S$-manifolds, is strictly positive. This conjectured phenomenon is strongly related to some properties of algebraic integers, such as the Lehmer conjecture, which are still a mystery.

Conjecture 1.3 and Theorem 1.5 yield quantitative versions for some classical finiteness statements. We shall describe our main two applications in paragraphs 1.1 and 1.2.

1.1. **A linear bound on the size of a minimal presentation.** It is well known that the fundamental group of a locally symmetric manifold with finite volume, or more generally a lattice in a connected semisimple Lie group, is finitely presented. The compact case is quite standard, but the non-compact case was proved, step by step, by several authors over several years. Garland and Raghunathan proved it in the rank one case. In the higher rank case, the finite generation was proved by Kazhdan by defining and proving property-$T$ when $\text{rank}(G) > 2$, and then by S.P. Wang when $\text{rank}(G) = 2$. Using the finite generation, Margulis proved the classical arithmeticity theorem for higher rank irreducible lattices. For arithmetic groups the finite presentability follows from the reduction theory of Borel and Harish-Chandra. Later, using Morse theory, Gromov gave a geometric proof of the finite presentability by proving that any locally symmetric manifold is diffeomorphic to the interior of a

---

2 Margulis proved arithmeticity for higher rank non-uniform lattices few years before he proved the general case and without using his superrigidity theorem
compact manifold with boundary. We remark that a slight modification of the argument of section 6 below yields a completely elementary proof of this result of Gromov, and hence of finite presentability (see also Remark 6.4).

However, in order to get a concrete estimate on the minimal possible size of such presentation in terms of the volume, we shall use arithmeticity (in the higher rank non-compact case). We obtain the following quantitative version of finite presentability:

**Theorem 1.7.** Assume that $S$ is neither isometric to $\mathbb{H}^3$, $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ nor to $\mathbb{H}^2 \times \mathbb{H}^2$. Then there is a constant $\eta = \eta(S)$ such that for any irreducible $S$-manifold $M$, the fundamental group $\pi_1(M)$ admits a presentation

$$\pi_1(M) \cong \langle \Sigma : W \rangle$$

with both $|\Sigma|, |W| \leq \eta \cdot \text{vol}(M)$, in which all the relations $w \in W$ are of length $\leq 3$.

**Remark 1.8.** We believe (see also Conjecture 1.3) that the assumptions that $S$ is not isometric to $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ and to $\mathbb{H}^2 \times \mathbb{H}^2$, are not really necessary. However, our proof does not work in these cases. It follows, however, from Theorem 1.5(2) that the analogous statements for non-compact $S$-manifolds hold also in these cases.

**Remark 1.9.** The analogous statement for non-arithmetic hyperbolic 3-manifolds is evidently false. However, if Conjecture 1.3 is true, then the analogue of Theorem 1.7 should hold for arithmetic 3-manifolds. By Theorem 1.5 it holds for non-compact arithmetic 3-manifolds. Moreover, it was shown in 1.3 that for any hyperbolic 3-manifold $M$, the sum of the relations length, in any presentation of $\pi_1(M)$, is at least $\frac{\text{vol}(M)}{\pi}$. This means that our upper bound is tight in this case.

In the general case, Theorem 1.5(3) implies the following (weaker) statement for which even the finiteness is in some sense surprising (in dimension 3). For a group $\Gamma$ let $d(\Gamma)$ denotes the minimal size of a generating set.

**Theorem 1.10.** For any $S$, there is a constant $\eta$ (depending on $S$), such that for any $S$-manifold $M$, $d(\pi_1(M)) \leq \eta \cdot \text{vol}(M)$. In other words, for any $v > 0$,

$$\sup \{d(\pi_1(M)) : \text{vol}(M) \leq v\} \leq \eta v.$$

Moreover, if $S$ is not isomorphic to $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ then there is a presentation

$$\pi_1(M) \cong \langle \Sigma : W \rangle$$

with $|\Sigma|, |W| \leq \eta \cdot \text{vol}(M)$.

Note that Theorem 1.10 does not give a bound on the length of the relations $w \in W$. 
1.2. A quantitative version of Wang’s theorem. We apply our results in order to estimate the number of $S$-manifolds with bounded volume (or more generally the number of conjugacy classes of lattices in $G = \text{Isom}(S)$). This can be considered as a continuous analogue to asymptotic group theory which studies the subgroup growth of discrete groups (where “covolume of lattices” extends the notion “index of subgroups”). Unlike the situation in the discrete (finitely generated) case, even the finiteness statements are not clear, and in general not true. However, a classical theorem of H.C. Wang states that if $S$ is not isometric to one of the hyperbolic spaces $\mathbb{H}^2$, $\mathbb{H}^3$, then for any $v > 0$ there are only finitely many irreducible $S$-manifolds with total volume $\leq v$, up to isometries (see [31] 8.1, and paragraph 13.4 below). We remark that Wang’s result and proof do not give explicit estimates.

Denote by $\rho_S(v)$ the number of non-isometric irreducible $S$-manifolds with volume $\leq v$.

By Mostow’s rigidity theorem, a locally symmetric manifold of dimension $\geq 3$ is determined by its fundamental group. Applying Theorem 1.5(2), we obtain:

**Theorem 1.11.** If $\dim(S) \geq 4$ and $S$ is neither isometric to $\text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$ nor to $\mathbb{H}^2 \times \mathbb{H}^2$ then there is a constant $c$, depending on $S$, with respect to which

$$\log \rho_S(v) \leq c \cdot v \log v$$

for any $v > 0$.

This upper bound was first proved for hyperbolic manifolds by M. Burger, A. Lubotzky, S. Mozes and the author in [9], where also a lower bound of the same type was established, proving that this estimate is the true asymptotic behavior in the hyperbolic case.

**Theorem 1.12 (BGLM).** For $n \geq 4$, there are constants $c_n > b_n > 0$ and $v_n > 0$ such that

$$b_n v \log v \leq \log \rho_{\mathbb{H}^n}(v) \leq c_n v \log v,$$

whenever $v > v_n$.

However, we suspect that in the higher rank case, where all manifolds are arithmetic and conjectured to possess the congruence subgroup property, this upper bound is far from the true asymptotic behavior. It seems, in view of [20], that the problem of determining the real asymptotic behavior is closely related to the congruence subgroup problem. It is also related to the analysis of the Galois cohomology of compact extensions of $G = \text{Isom}(S)$.

For hyperbolic manifolds, the weaker upper bound $\rho_{\mathbb{H}^n}(v) \leq v \exp(\exp(\exp(v+n)))$ was proved previously by Gromov [17]. In this paper we establish a first concrete estimate for $\rho_S(v)$ for general $S$. 
In dimension 2 and 3 the analogue of Wang’s finiteness theorem is false. However, as was shown by Borel [4], it remains true when considering only arithmetic manifolds. The following estimate for the number of non-compact arithmetic 3-manifolds follows from Theorem 1.5(1):

**Proposition 1.13.** For some constant $c > 0$, there are at most $c^v$ non-isometric arithmetic non-compact hyperbolic 3-manifolds with volume $\leq v$.

For compact arithmetic 3-manifolds we conjecture that the analogous statement holds, but prove only the following weaker statement:

**Proposition 1.14.** Let $M$ be a compact (arithmetic) hyperbolic 3-manifold. Then for some constant $c(M)$, the number of non-isometric hyperbolic manifolds commensurable to $M$, with volume $\leq v$, is at most $c^{c(M)v}$.

We remark that the analogue of Propositions 1.13 and 1.14 hold also in the cases $S = \text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R})$ and $S = \mathbb{H}^2 \times \mathbb{H}^2$.

In section 13 we shall generalize some of our results concerning $S$-manifolds to the larger family of $S$-orbifolds. We shall also indicate how to construct triangulations for rank-1 manifolds which are not necessarily arithmetic.

Let us now give a short and not precise explanation of the basic lines of the proofs of our main results 1.5. The idea is to construct, inside each $S$-manifold $M$, a submanifold with boundary $M'$, which is similar enough to $M$ and for which we can construct a triangulation of size $\leq c \cdot \text{vol}(M)$. In order to construct such a triangulation we shall require a lower bound on the injectivity radius of $M'$ (independent of $M$) and some bounds on the geometry of the boundary $\partial M'$. We shall construct $M'$ inside the $\epsilon_{\gamma}$-thick part, in a way that its pre-image in the universal cover $\tilde{S}$ will be the complement of some “locally finite” union of convex sets, each has a smooth boundary whose curvature is bounded uniformly from below, and the angles at the corners of $\partial M'$ (where the boundaries of two or more such convex sets meet) are bounded uniformly from below.

The non-compact arithmetic case is easier to deal with. The reason is that any non-uniform arithmetic lattice $\Gamma \leq G = \text{Isom}(S)^0$ comes from a rational structure on $G$ (rather then on a compact extension of $G$ as the case might be when $\Gamma$ is uniform). This implies that there is some constant $\epsilon'_{\gamma}$ such that any non-uniform arithmetic $S$-manifold contains no closed geodesics of length $\leq \epsilon'_{\gamma}$ (see section 5). In other words, any non-trivial closed loop which is short enough corresponds to a unipotent element in the fundamental group.

We define (the pre-image in $\tilde{S}$ of) $M'$ to be the complement of the union of appropriate sub-level sets for the displacement functions $\{d_{\gamma}\}$ where $\gamma$ runs over all unipotents in $\pi_1(M)$. The injectivity radius in $M'$ is then uniformly bounded from below, and using the fact that unipotents acts nicely on $S$ and
on $S(\infty)$ we can both estimate the geometry of $\partial M'$, and also construct a deformation retract from $M$ to $M'$.

The situation is more subtle in the compact case. We do not have sufficient information on the thick-thin decomposition. In this case we construct $M'$ and prove that it is diffeomorphic to $M \setminus N$ where $N$ is a finite union of submanifolds of codimension $\geq 3$. $N$ will contain the union of all closed geodesics of length smaller than some fixed constant. The idea is that at any point outside $N$ there is a preferred direction, such that when we move along it, the injectivity radius is increasing most rapidly. This will help us to define a deformation retract from $M \setminus N$ to $M'$. As $N$ has codimension $\geq 3$, $\pi_1(M) \cong \pi_1(M \setminus N) \cong \pi_1(M')$. A main difficulty (which arises also in the compact rank one case) is how to control the geometry of the boundary of $M'$. We shall handle this difficulty in section 7, where the main idea is Lemma 7.1 which says that if two isometries commute then the exterior angle between their sub-level sets is $\geq \frac{\pi}{2}$.

The present paper generalizes the work [9] which treated the special case of real hyperbolic spaces. Although some of the ideas from [9] appear again in the present paper, the situation in the general case is significantly more complicated than the hyperbolic case. The proof in [9] uses the explicit description of a hyperbolic compact thin component of the thick-thin decomposition as a cone over a coaxial Euclidean ellipsoids, as well as some computations in constant curvature. Hence the argument in [9] does not apply to more general rank-1 symmetric spaces. More crucially, in contrast to the situation in the rank-1 case, in the higher rank case there is currently no good understanding of the structure of the thin components in the thick-thin decomposition, and hence new ingredients are required also in the skeleton of the proof.

Most of this paper is devoted to the treatment of the higher rank case. However, whenever it is not required, we shall not make any assumption on the rank. We remark also that many of the arguments in this paper stay valid for general manifolds of non positive curvature. In particular, some parts of this work could be generalized to non symmetric Hadamard spaces.

2. Notations, definitions and background

In this section we shall fix our notations and summarize some basic facts about semisimple Lie groups, symmetric spaces of non-compact type, and manifolds of non-positive curvature. For a comprehensive treatment of these subjects we refer the reader to [27], [2] and [8].

Let $S$ be a symmetric space of non-compact type. We shall always assume, that $S$ has no Euclidean de Rham factors. Let $G = \text{Isom}(S)$ be the Lie group of isometries of $S$. $G$ is center-free, semi-simple without compact
factors, and with finitely many connected components. We denote by $G^0$ the identity component of $G$ with respect to the real topology. There is an algebraic group $G$ defined over $\mathbb{Q}$, such that $G^0$ coincides with the connected component of the group of real points $G(\mathbb{R})^0$. In particular $G^0$ admits, apart from the real topology, a Zariski topology which is defined as the trace in $G^0$ of the Zariski topology in $G$. $G^0$ acts transitively on $S$, and we can identify $S$ with $G^0/K$ where $K \leq G^0$ is a maximal compact subgroup. We remark that there is a bijection between symmetric spaces of non-compact type and connected center free semisimple Lie groups without compact factors.

$S$ is a Riemannian manifold with non-positive curvature such that for each point $p \in S$ there is an isometry $\sigma_p$ of $S$ which stabilizes $p$ and whose differential $d_p(\sigma_p)$ at $p$ is $-1$. The composition of two such isometries $\sigma_p \cdot \sigma_q$ is called a transvection, and is belong to $G^0$. The non-positivity of the sectional curvature means that the distance function $d : S \times S \to \mathbb{R}^+$ is convex and, in fact, its restriction to a geodesic line $c(t) = (c_1(t), c_2(t))$ in $S \times S$ is strictly convex, unless $c_1$ and $c_2$ are contained in a flat plane in $S$.

For a real valued function $f : X \to \mathbb{R}$ on a set $X$ we denote by $\{f < t\}$ the $t$-sub-level set

$$\{f < t\} = \{x \in X : f(x) < t\}.$$

For $\gamma \in G$ we denote by $d_\gamma$ the displacement function

$$d_\gamma(x) = d(\gamma \cdot x, x).$$

This function is convex and smooth outside $\text{Fix}(\gamma)$. In particular the sub-level sets $\{d_\gamma < t\}$ are convex with smooth boundary. We denote by $\text{min}(\gamma)$ the set

$$\text{min}(\gamma) = \{x \in S : d_\gamma(x) = \inf d_\gamma\}.$$

For a set $A \subset S$ we denote by

$$D_A(x) = d(A, x)$$

the distance function. If $A$ is a convex set then the function $D_A$ is convex and smooth at any point outside the boundary of $A$. We denote its $t$-sub-level set by

$$(A)_t = \{D_A < t\}$$

and call it the $t$-neighborhood of $A$. Note that $(A)_{t+s} = (\{(A)_t\})_s$. Similarly, we let $A_{(t,s)}$ denote the $t$-shrinking of $A$,

$$A_{(t,s)} = S \setminus (S \setminus A)_t.$$
If $A$ is closed and convex then for any $x \in S$ there is a unique closest point $p_A(x)$ in $A$. The map $p_A : S \to A$ is called the projection on $A$, and it is distance decreasing, i.e. $d(p_A(x), p_A(y)) \leq d(x, y)$ for any $x, y \in S$.

A flat subspace is a totally geodesic sub-manifold which is isometric to a Euclidean space. A flat is a maximal flat subspace. Any flat subspace is contained in a flat. All flats in $S$ have the same dimension $\text{rank}(S) = \text{rank}(G)^3$. A geodesic $c \subset S$ is called regular if it is contained in a unique flat.

An element $\gamma \in G$ is unipotent if $\text{Ad}(\gamma) - 1$ is a nilpotent endomorphism. A subgroup of $G$ is called a unipotent subgroup if all its elements are unipotent. If $H \leq G^0$ is a unipotent subgroup then its Zariski closure $\overline{H}^z \leq \mathbb{G}$ is also unipotent, and $\overline{H}^z_{\mathbb{R}}$ is connected. Moreover, a unipotent subgroup $H \leq G^0$ is connected iff $H = \overline{H}^z_{\mathbb{R}}$.

$S(\infty)$ is the set of equivalence classes of geodesic rays, where two rays are considered equivalent if their traces are of bounded Hausdorff distance from each other. There is a structure of a spherical building on $S(\infty)$ - the Tits building of $S$. The chambers of this building are sometimes called Weyl chambers. A maximal unipotent subgroup of $G^0$ is the unipotent radical of some minimal parabolic subgroup. The minimal parabolic subgroups of $G^0$ are the stabilizers (and the fixator) of the chambers of this building, and there are canonical bijections between the set of minimal parabolic subgroups, the set of maximal unipotent subgroups and the set of chambers. All maximal unipotent subgroups are conjugate in $G^0$, or in other words, $G^0$ acts transitively on the set of chambers $W \subset S(\infty)$. Moreover, each parabolic subgroup acts transitively on $S$, and hence $G$ acts transitively on the set of couples $(W, x)$ of a chamber $W \subset S(\infty)$ and a point $x \in S$. There is a canonical metric on $S(\infty)$, called the Tits metric, with respect to which the apartments of the Tits building are isometric to spheres and the chambers are isometric to spherical simplices. The induced action of $G$ on $S(\infty)$ preserves the Tits metric. A point $p \in S(\infty)$ is called regular if it is an interior point of a chamber. For any $p \in S(\infty)$ and $x \in S$ there is a unique geodesic $c$ with $c(0) = x$ and $c(\infty) = p$, and $p$ is regular iff $c$ is regular.

When $\Gamma$ is a group of isometries of $S$, we say that a subset $A \subset S$ is $\Gamma$-precisely invariant if $\gamma \cdot A = A$ whenever $\gamma \cdot A \cap A \neq \emptyset$. If $\Gamma$ acts freely and discretely (i.e. $\Gamma \subset G$ is discrete and torsion free) then we denote by $\Gamma \backslash A$ the image of $A$ in $\Gamma \backslash S$. If $A$ is a connected simply connected $\Gamma$-precisely invariant set then

$$\pi_1(\Gamma \backslash A) \cong \{ \gamma \in \Gamma : \gamma \cdot A = A \}.$$
For a subset $M'$ of $M = \Gamma \setminus S$ we let $\tilde{M}'$ denote its pre-image in $S$, and we shall usually denote by $\tilde{M}'^0$ an arbitrary connected component of $\tilde{M}'$.

For a subgroup $\Gamma \leq G$, a constant $\epsilon > 0$ and a point $x \in S$, we let $\Gamma_\epsilon(x)$ denote the group generated by the “small elements” at $x$

$$\Gamma_\epsilon(x) = \langle \gamma \in \Gamma : d_\gamma(x) \leq \epsilon \rangle.$$ 

Recall the classical Margulis lemma:

**Theorem 2.1 (The Margulis lemma).** There are constants $\epsilon_s > 0$ and $n_s \in \mathbb{N}$, depending only on $S$, such that for any discrete subgroup $\Gamma \leq G$, and any point $x \in S$, the group $\Gamma_\epsilon_s(x)$ contains a subgroup of index $\leq n_s$ which is contained in a connected nilpotent subgroup of $G$.

The $\epsilon$-thick thin decomposition of an $S$-manifold $M = \Gamma \setminus S$ reads

$$M = M_{\geq \epsilon} \cup M_{\leq \epsilon}.$$ 

The $\epsilon$-thick part $M_{\geq \epsilon}$ is defined as the set of all points $x \in M$ at which the injectivity radius is $\geq \frac{\epsilon}{2}$, and the $\epsilon$-thin part $M_{\leq \epsilon}$ is the complement of the interior of the $\epsilon$-thick part. Note that $M_{\leq \epsilon}$ is the set of points in $M$ through which there is a non-contractible closed loop of length $\leq \epsilon$. The pre-image of $M_{\leq \epsilon}$ in $S$ has a nice description as a locally finite union of convex sets

$$\tilde{M}_{\leq \epsilon} = \bigcup_{\gamma \in \Gamma \setminus \{1\}} \{d_\gamma \leq \epsilon\}.$$ 

The Margulis lemma yields an important piece of information on the structure of the $\epsilon$-thick-thin decomposition when $\epsilon \leq \epsilon_s$.

If $\Gamma \subset G$ is a uniform lattice then any element $\gamma \in \Gamma$ is semisimple, i.e. $\text{Ad}(\gamma)$ is diagonalizable over $\mathbb{C}$, or equivalently $\min(\gamma) \neq \emptyset$. If, in addition, $\Gamma$ is torsion free then all its elements are hyperbolic. A semisimple element is hyperbolic iff it has a complex eigenvalue outside the unit disk. A hyperbolic isometry $\gamma$ has an axis, i.e. a geodesic line on which $\gamma$ acts by translation. Any two axes are parallel, and $x \in \min(\gamma)$ iff the geodesic line $x, \gamma \cdot x$ is an axis of $\gamma$. We define a **regular isometry** to be a hyperbolic isometry whose axes are regular geodesics (if one axis is a regular geodesic, then so are all of them, since axes are parallel). If $g \in G$ acts as a regular isometry then $g$ is a regular element in $G$ in the usual sense, but not necessarily vice versa. If $\gamma \in G$ acts as a regular isometry, and $F$ is the unique flat containing an axis of $\gamma$ then $\gamma$ preserves $F$ and acts by translation on it, and in particular $\min(\gamma) = F$. (This follows for example from the facts that the Weyl group $N_G(\gamma)/A$ for $A = C_G(\gamma)$, acts simply transitively on the set of Weyl chambers in $F(\infty)$, and that a regular point $p \in F(\infty)$ determines its ambient Weyl chamber.)

If $A \subset S$ is a closed convex set, and $\alpha \in G$ preserves $A$, i.e. $\alpha \cdot A = A$, then the projection $P_A : S \to A$ commutes with $\alpha$, and since $P_A$ does not increase
distances, we have \( d_\alpha(P_A(x)) \leq d_\alpha(x) \). In particular, \( \alpha \) is semisimple iff \( \min(\alpha) \cap A \neq \emptyset \). Since for any isometry \( i \) of a flat subspace \( F \), \( \min(i) \neq \emptyset \), it follows that if \( \alpha \in G \) preserves a flat subspace then \( \alpha \) is semisimple. If \( g \in G^0 \) is semisimple, then its centralizer group \( C_{G^0}(g) \) is a closed reductive group. If \( g \in G \) is a transvection with axis \( c \), then
\[
C_{G^0}(g) = \{ h \in G^0 : h \cdot c \text{ is parallel to } c \}.
\]

If \( \Gamma \subset G \) is a non-uniform lattice, then it is almost generated by unipotents. \( \gamma \) is unipotent iff \( \inf(d_\gamma) = 0 \) while \( d_\gamma \) never vanishes on \( S \). In particular a unipotent element stabilizes a point at infinity but not in \( S \).

If \( f : G^0 \to H \) is a surjective Lie homomorphism, and \( g \in G \) is semisimple (resp. unipotent) then \( f(g) \) is semisimple (resp. unipotent).

3. Some deformation retracts

In several places in this paper we will need to produce a deformation retract from a submanifold (maybe with boundary) to a subset of it which is defined by the condition that all the functions from some given family are larger than a given constant. A typical example of such a situation is given by the manifold itself and the \( \epsilon \)-thick part. The family of functions in this example is indexed by the fundamental group. To each homotopy class of closed loops corresponds the function for which the value at any point \( x \) is the minimal length of a loop from the class, which passes through \( x \). The \( \epsilon \)-thick part is exactly the subset where all these functions are \( \geq \epsilon \). The Margulis lemma yields information about the set of functions which are \( < \epsilon \) at \( x \), which can sometimes be exploited in order to define a deformation retract to the \( \epsilon \)-thick part. Another example is given by a subset \( M' \) of the manifold \( M \) and its \( \epsilon \)-shrinking \( )M'(\epsilon \). If the complement of \( M' \) is given by the union of subsets satisfying certain properties, it is sometimes convenient to use the distance functions from these subsets in order to define a deformation retract to \( )M'(\epsilon \).

The idea is to construct a continuous vector field which makes an acute angle at any point \( x \) with the gradients of all the functions from the given family which are small at \( x \), and then to let points flow along its integral curves. This idea has been used in an elegant way in other places (c.f. [2]). The situation concerned here, however, is more subtle since we consider manifolds with boundary, and have to make sure that on the boundary, the vector field is pointing towards the interior.

**Definition 3.1.** For a family of real valued continuous functions \( \mathcal{F} = \{ \phi_i \}_{i \in I} \) on a manifold \( Y \) (with or without boundary) and \( \epsilon > 0 \), we define the \( (\mathcal{F}, \epsilon) \)-thick part (or simply the \( \mathcal{F} \)-thick part) \( \mathcal{F}_{\geq \epsilon} \) as follows
\[
\mathcal{F}_{\geq \epsilon} = \bigcap_{\phi \in \mathcal{F}} \{ \phi > \epsilon \}.
\]
When $\mathcal{F}$ is given, we denote by $\Psi_{x,\tau} (x \in Y, \tau \in \mathbb{R})$ the subset

$$\Psi_{x,\tau} = \{ \phi \in \mathcal{F} : \phi(x) \leq \tau \}.$$  

Abusing notations, we shall sometimes write $\Psi_{x,\tau}$ for the corresponding set of indices $\Psi_{x,\tau} = \{ i \in I : \phi_i(x) \leq \tau \}$. We say that the family $\mathcal{F}$ is **locally finite** if $\Psi_{x,\tau}$ is finite for any $x \in Y$ and $\tau \in \mathbb{R}$. We say that $\tau \in \mathbb{R}$ is a **critical value** of a continuous function $\phi$ if $\{ \phi > \tau \} \neq \{ \phi \geq \tau \}$, i.e. if $\tau$ is a value of a local maximum of $\phi$.

If $Z \subset Y$ is a submanifold with smooth boundary, and $\phi$ a real valued function on $Z$. The gradient $\nabla \phi$ is well defined on $Z$ wherever it is unique (on the boundary it is the tangent vector with length and direction equal to the value of the maximal directional derivative and the direction at which it occurs). The function $\phi$ is said to be smooth (or $C^1$) if its gradient $\nabla \phi$ is a continuous function on the whole of $Z$. When $\phi$ is smooth, its directional derivative with respect to $v \in T_x(Y)$ is given by the usual formula $v \cdot \nabla \phi(x)$.

In later sections, we will consider families $\mathcal{F}$ which consist of functions of the following 2 types:

- The displacement function $d_\gamma(x) = d(x, \gamma \cdot x)$ associated with a non-trivial isometry $\gamma$ of $S$.
- The distance function $D_A(x) = d(x, A)$ from a given closed convex set with smooth boundary.

In both cases the function is convex, non-negative, and without positive critical values. The function $d_\gamma$ is smooth on $\{d_\gamma > 0\}$, and the function $D_A$ is smooth on $\{D_A > 0\}$.

**Lemma 3.2 (A deformation retract which increases functions).** Consider a submanifold without boundary $Y \subset S$.

- Let $\mathcal{F} = \{ \phi_i \}_{i \in I}$ be a locally finite family of non-negative continuous functions on $Y$.
- Assume that for each $\phi \in \mathcal{F}$ with $\{ \phi = 0 \} \neq \emptyset$, the set $\{ \phi > 0 \}$ is a submanifold with smooth boundary.
- Assume that each $\phi \in \mathcal{F}$ is $C^1$ on $\{ \phi > 0 \}$.
- Let $L = \mathcal{F}_{\geq 0} = \bigcap_{\phi \in \mathcal{F}} \{ \phi > 0 \}$.
- Let $\beta : [0, 3\epsilon] \to \mathbb{R}^{>0}$ be a given continuous positive function (in case all $\phi \in \mathcal{F}$ are strictly positive we allow $\beta$ to be defined only on $(0, 3\epsilon]$).
- Assume that $\epsilon$ is not a critical value for any $\phi \in \mathcal{F}$.
- Assume that for any $x \in Y$ there is a unit tangent vector $\hat{n}(x) \in T_x(S)$ such that

$$\hat{n}(x) \cdot \nabla \phi(x) \geq \beta(\phi(x))$$

for any $\phi \in \Psi_{x,3\epsilon}$.  


Then there is a deformation retract from $L$ to the $F$-thick part $F_{\geq \epsilon}$. (In particular it follows that $F_{\geq \epsilon} \neq \emptyset$.)

If $\Gamma \leq \text{Isom}(S)$ is a discrete subgroup, and if $L$ and the family $F$ are $\Gamma$-invariant, in the sense that $\gamma \cdot L = L$, and the function $\gamma \cdot \phi(x) := \phi(\gamma^{-1} \cdot x)$ belongs to $F$ for any $\phi \in F$, $\gamma \in \Gamma$, then there exists such a deformation retract which is $\Gamma$-invariant (in the obvious sense) and hence induces a deformation retract between the corresponding subsets $\Gamma \backslash L$ and $\Gamma \backslash F_{\geq \epsilon}$ of $M = \Gamma \backslash S$.

**Proof.** We will define an appropriate continuous vector field on $L$. The desired deformation retract will be the flow along this vector field.

Let $\delta(x)$ denote

$$\delta(x) = \min_{\phi \in F} \phi(x).$$

For any non-empty subset $\Psi \subset F$ for which $\cap_{\phi \in \Psi} \{ \phi \leq 3\epsilon \} \neq \emptyset$ and any point $x$ of this intersection, let $\hat{f}(x, \Psi) \in T_x(S)$ be a unit tangent vector which maximizes the expression

$$\min\{ \hat{f} \cdot \nabla \phi(x) : \phi \in \Psi \},$$

and let

$$\beta_{\Psi}(x) = \min_{\phi \in \Psi} \beta(\phi(x)).$$

Then for any $\phi \in \Psi$

$$\hat{f}(x, \Psi) \cdot \nabla \phi(x) \geq \min_{\phi \in \Psi} \hat{f}(x, \Psi) \cdot \nabla \phi(x) \geq \min_{\phi \in \Psi} \hat{n}(x) \cdot \nabla \phi(x) \geq \min_{\phi \in \Psi} \beta(\phi(x)) = \beta_{\Psi}(x).$$

Moreover, it follows from the strict convexity of the Euclidean unit disk that $\hat{f}(x, \Psi)$ is uniquely determined, and consequently, that for a fixed $\Psi$, the vector field $\hat{f}(x, \Psi)$ is continuous on $\cap_{\phi \in \Psi} \{ \phi \leq 3\epsilon \}$.

The desired vector field is defined as follows:

$$\hat{V}(x) = \sqrt{2(\epsilon - \delta(x))} \cdot \hat{f}(x, \Psi),$$

where the sum is taken over all non-empty finite subsets $\Psi \subset F$.

Notice that $\beta_{\Psi}(x)$ is strictly positive, and all the terms in each summand are continuous, and $\hat{V} \equiv 0$ on the $F$-thick part

$$F_{\geq \epsilon} = \cap_{\phi \in F} \{ \phi > \epsilon \} = \cap_{\phi \in F} \{ \phi \geq \epsilon \} = \{ \delta \geq \epsilon \}.$$
The term $\sqrt{2(\epsilon - \delta(x))} \vee 0$ takes care of the continuity on the boundary $\{\delta = \epsilon\} = \partial\{\delta \leq \epsilon\}$ of the $F$-thin part. The terms $\left(\frac{3\epsilon - \max_{\phi \in \Psi} \phi(x)}{\epsilon}\right) \vee 0$ guarantee that all the non-zero summands correspond to sets which are contained in the finite set $\Psi_{x,3\epsilon}$. In particular the summation is finite for any $x \in L$, and $\hat{f}(x,\Psi)$ is well defined for any non-zero summand. The terms $\left(\min_{\phi \in \Psi} \phi(x) - \epsilon\right) \vee 0$ guarantee that all the non-zero summands correspond to $\Psi$'s which contains $\Psi_{x,\epsilon,3\epsilon}$.

If $\delta(x) < \epsilon$ then $\Psi_{x,\epsilon} \neq \emptyset$, and, when looking only on the summand corresponding to $\Psi_{x,2\epsilon}$, we see that

$$\nabla \phi(x) \cdot \nabla V(x) \geq \sqrt{2(\epsilon - \delta(x))}$$

for any $\phi \in \Psi_{x,\epsilon} \subset \Psi_{x,2\epsilon}$, and in particular for any $\phi$ with $\phi(x) = \delta(x)$.

It follows that if $x(t)$ is an integral curve of $\nabla V$ with $\delta(x(0)) < \epsilon$, then

$$\frac{d}{dt}(\delta(x(t))) \geq \sqrt{2(\epsilon - \delta(x(t)))}.$$ 

(To be more precise, since $\delta(x(t))$ is not necessarily differentiable, we should write $\liminf_{\tau \to 0} \frac{\delta(x(t+\tau)) - \delta(x(t))}{\tau}$ instead of $\frac{d}{dt}(\delta(x(t)))$ in the last inequality.)

Thus, for $t = \sqrt{2(\epsilon - \delta(x(0)))}$ we have $\delta(x(t)) = \epsilon$.

Since $x \in L$ belongs to $\partial L$ iff $\delta(x) = 0$, and hence, the vector field $\nabla V$ points everywhere towards the interior $\text{int}(L)$, it follows from the Peano existence theorem of solution for ordinary differential equations, that for any $x \in L$ there is an integral curve $c_x(t)$ of $\nabla V$, defined for all $t \geq 0$ with $c_x(0) = x$ and with $c_x(t) \in \text{int}L$ for $t > 0$. Stability of the solutions implies that $c_x(t)$ depends continuously on $x$ and $t \geq 0$.

As a conclusion, we get that the flow along $\nabla V$ for $\sqrt{2\epsilon}$ time units defines a deformation retract from $L$ to $F_{\geq \epsilon} = \{\delta \geq \epsilon\}$.

If $L$ and $F$ are $\Gamma$-invariant, then $\nabla V(x)$, as it is defined above, is also $\Gamma$-invariant. Hence it induces a vector field on $\Gamma \setminus L$, and a deformation retract from $\Gamma \setminus L$ to $\Gamma \setminus F_{\geq \epsilon}$. □

A second kind of deformation retracts that we shall often use is the following. Let $A \subset S$ be a closed convex set, and $B \subset S$ a set containing $A$. We say that $B$ is star-shaped with respect to $A$ if for any $b \in B$ the geodesic segment connecting $b$ to its closest point (the projection) in $A$ is contained in $B$. In that case we can define a deformation retract from $B$ to $A$ by moving $b$ at a constant rate (equal to the initial distance) towards its projection in
A. We call this the star-contraction from $B$ to $A$. From the convexity of the distance function we conclude that the star contraction from $B$ to $A$ is distance decreasing, and hence continuous, and if the Hausdorff distance $\text{Hd}(A, B)$ is finite, then the star-contraction gives a homotopy equivalence between $(B, \partial B)$ and $(A, \partial A)$.

More generally, if $A \subset B_1 \subset B_2 \subset S$, and if the $B_i$’s are closed and star-shaped with respect to $A$, we can define a deformation retract from $B_2$ to $B_1$ by letting any $b \in B_2 \setminus B_1$ flow in the direction of its projection $P_A(b)$ in $A$ at constant speed $s$, where $s$ equals to the length of the segment $[b, P_A(b)] \cap (B_2 \setminus B_1)$. By a similar procedure one can show that there is a deformation retract from $S \setminus B_1$ to $S \setminus B_2$. In this way we obtain:

**Lemma 3.3 (A generalized star-contraction).** Assume that

- $A$ is convex and closed,
- $A \subset B_1 \subset B_2$ where $B_i$ are closed and star-shaped with respect to $A$.
- $\text{Hd}(A, B_2) < \infty$.

Then there is a deformation retract from $(B_2, \partial B_2)$ to $(B_1, \partial B_1)$. Similarly, there is a deformation retract from $S \setminus B_1$ to $S \setminus B_2$.

4. **Constructing a simplicial complex in a thick submanifold with nice boundary**

Throughout this section, $M = \Gamma \setminus S$ is a fixed $S$-manifold with finite volume, $M' \subset M$ a connected submanifold with boundary, and $\epsilon > 0$ is fixed. The main result of this section is Lemma 4.1.

We wish to formulate some convenient conditions on $M'$, under which $M'$ is homotopically equivalent to a simplicial complex $\mathcal{R}$ whose combinatorics is bounded in terms of $\text{vol}(M)$. More precisely, we would like $\mathcal{R}$ to be a $(d, \alpha \cdot \text{vol}(M'))$-simplicial complex, where $d$ and $\alpha$ are some constants depending only on $S$ and $\epsilon$ (see Definition 4.1).

To construct $\mathcal{R}$ we will use a “good covering” argument. Recall that a cover of a topological space $T$ is called a **good cover** if any non empty intersection of sets of the cover is contractible. In this case the simplicial complex $\mathcal{R}$ which corresponds to the nerve of the cover is homotopically equivalent to $T$. By definition, the vertices of $\mathcal{R}$ correspond to the sets of the cover, and a collection of vertices form a simplex when the intersection of the corresponding sets is non-empty (see [7], theorem 13.4).

In a manifold with injectivity radius radius bounded uniformly from below by $\epsilon$, such a good cover is achieved by taking $\epsilon$-balls for which the set of centers form an $\epsilon/2$ (say) discrete net. In our case, in order to be able to use $\epsilon$-balls (or more precisely $\epsilon/c$-balls for some constant $c$) we shall require that $M'$ lies inside $M_{\geq \epsilon}$. However, $M'$ is not a manifold but a manifold with boundary,
and balls may not be “nice” subsets - they may not be convex or even contractible, and an intersection of balls may not be connected. Therefore we should be more careful. Our problems arise only near the boundary (far away from the boundary balls are nice). So we need some control on the geometry of the boundary.

**Lemma 4.1.** Let $M = \Gamma \backslash S$ be a fixed $S$-manifold with finite volume, let $M' \subset M$ be a connected submanifold with boundary, and let $\epsilon > 0$ be fixed.

- Assume that $M'$ is contained in the $\epsilon$-thick part $M_{\geq \epsilon}$.
- Write $X = M \setminus M'$, and assume that its pre-image $\tilde{X} \subset S$ under the universal covering map is a locally finite union of convex open sets with smooth boundary $\tilde{X} = \bigcup_{i \in I} \tilde{X}_i$ (by locally finite we mean that every compact set in $S$ intersects only finitely many $X_i$’s).
- Assume that $M'$ is homotopically equivalent to its $\frac{\epsilon}{2}$-shrinking $M'(\frac{\epsilon}{2})$.
- Let $b > 1$ be a constant which depends only on $S$ and on $\epsilon$.
- Assume that for any point $x \in S \setminus \tilde{X}$ with $d(x, \tilde{X}) \leq \epsilon$, there is a unit tangent vector $\tilde{n}(x) \in T_x(S)$, such that for each $i \in I$ with $d(x, X_i) = d(x, \tilde{X})$, the inner product of $\tilde{n}(x)$ with the gradient $\nabla D_{X_i}(x)$ satisfies $\tilde{n}(x) \cdot \nabla D_{X_i}(x) > \frac{1}{b}$.

Then there are constants $\alpha$ and $d$, depending only on $S$ and on $\epsilon$, such that $M'$ is homotopically equivalent to some $(d, \alpha \cdot \text{vol}(M'))$-simplicial complex.

Throughout this section we use the notation of the statement of Lemma 4.1.

The proof of Lemma 4.1 relies on a uniform estimate on the distance between shrinkings of $M'$.

**Proposition 4.2.** For any $\tau$ and $\delta$ with $\epsilon \geq \tau + \delta \geq \tau > 0$ we have

$$(M'_{(\tau+\delta)} \cup) \supset M'_{(\tau)}.$$ 

The proposition can also be stated as follows: For any such $\tau$ and $\delta$, the Hausdorff distance between the corresponding sets satisfies

$$\text{Hd}(M'_{(\tau+\delta)}, M_{(\tau)}) \leq b\delta.$$ 

In other words, in order to prove the proposition, we need to show that for any $x \in M_{(\tau)} \setminus M_{(\tau+\delta)} = (X)_{\tau+\delta} \setminus (X)_{\tau}$ there is $y \notin (X)_{\tau+\delta}$ with $d(x, y) \leq b\delta$.

**Proof.** Let $x \in (X)_{\tau+\delta} \setminus (X)_{\tau}$. Choose a lifting $\tilde{x} \in (\tilde{X})_{\tau+\delta} \setminus (\tilde{X})_{\tau}$ of $x$. As $\tilde{x} \in (\tilde{X})_{\tau+\delta} \setminus (\tilde{X})_{\tau}$, there is $\tau_1$ ($\tau + \delta \geq \tau_1 \geq \tau$) such that $\tilde{x} \in \partial(\tilde{X})_{\tau_1}$.

Let $c(t)$ be the piecewise geodesic curve of constant speed $b$, passing through $\tilde{x}$, which is defined inductively (for $t \geq \tau_1$) as follows: At time
\[ t_1 = \tau_1 \text{ set } c(t_1) = \tilde{x} \text{ and define its one sided derivative by} \]
\[ \frac{d}{dt} c(t_1) = b\tilde{n}(\tilde{x}), \]
and define
\[ I_c(t_1) = \{ i \in I : x \in \partial (X_i)_{t_1} \}. \]
Identify \( c \) with the constant speed geodesics determined by this condition, as long as \( t > t_1 \) and \( c(t) \notin \overline{(X_i)_t} \) for any \( i \notin I_c(t_1) \). Let \( t_2 \) be the first time (if such exist) \( t_1 \leq t < \tau + \delta \) at which \( c(t) \) hits some \( \overline{(X_i)_t} \) for \( i \notin I_c(t_1) \). As the collection \( \{ X_i \}_{i \in I} \) is locally finite, \( t_2 \) is well defined and strictly bigger than \( t_1 \). We claim that \( c(t_2) \in \partial (\tilde{X})_{t_2} \), and that \( c(t) \notin \overline{(X_i)_t} \) for \( t_1 < t < t_2 \). To prove this, we need to show that \( c(t) \notin \overline{(X_i)_t} \) for any \( t_1 \leq t \leq t_2 \) and \( i \in I_c(t_1) \). Fix such \( i \) and observe that
\[ \frac{d}{dt} |_{t=t_1} D_{X_i}(c(t)) = b\tilde{n}(x) \cdot \nabla D_{X_i}(x) > b\frac{1}{\delta} = 1. \]
Since the convex function \( D_{X_i}(c(t)) \) has non-decreasing derivative we get that \( \frac{d}{dt} \left( D_{X_i}(c(t)) \right) > 1 \) for any \( t_2 > t > t_1 \), and hence, \( D_{X_i}(c(t)) > D_{X_i}(c(t_1)) + (t - t_1) = t \) for any such \( t \). Thus, the point \( c(t) \) is outside \( \partial ((X_i)_{t_1}) = (X_i)_t \).

Then we define the second piece of \( c \) by the condition that its one sided derivative \( \frac{d}{dt_+} c(t_2) \) satisfies \( \frac{d}{dt_+} c(t_2) = b\tilde{n}(c(t_2)) \), and we define
\[ I_c(t_2) = \{ i \in I : c(t_2) \in \partial (X_i)_{t_2} \}. \]

Note that since \( t_2 \) is smaller than \( \tau + \delta < \epsilon \) and \( c(t_2) \notin \partial (X_2) \), the direction \( \tilde{n}(c(t_2)) \) is well defined. In this way, we continue to define \( c \) inductively.

If \( t_1 \) converge to some \( t_\omega_0 < \tau + \delta \), then we define \( c(t_\omega_0) \) to be the limit of \( c(t_i) \) and \( \frac{d}{dt_+} c(t_\omega_0) = b\tilde{n}(c(t_\omega_0)) \) and \( I_c(t_\omega_0) = \{ i \in I : c(t_\omega_0) \in \partial (X_i)_{t_\omega_0} \} \), and denote the next index by \( \omega_0 + 1 \), and so on. By this way we obtain a piecewise geodesic curve with at most countably many pieces, connecting \( \tilde{x} = c(\tau_1) \) to \( \tilde{y} = c(\tau + \delta) \), with \( \tilde{y} = c(\tau + \delta) \notin \overline{(\tilde{X})_{\tau + \delta}} \). Since the length of \( c([\tau_1, \tau + \delta]) \) is \( b(\tau + \delta - \tau_1) \leq b\delta \) this proves the proposition. \( \square \)

**Corollary 4.3.** Let \( \tau \) and \( \delta \) be as in Proposition 4.2. Let \( \mathcal{C} \) be a collection of balls of radius \((b + 1)\delta\) for which the set of centers \( \mathcal{C}' \) form a maximal \( \delta \)-discrete subset of \( M(\tau + \delta) \). Then \( \mathcal{C} \) is a cover of \( M(\tau) \).

**Proof.** As \( \mathcal{C}' \) is maximal \( (\mathcal{C}')_\delta \supseteq M(\tau + \delta) \). By Proposition 4.2 we have

\[ (M(\tau + \delta))_{b\delta} \supseteq (M'(\tau + \delta))_{b\delta} \supseteq M'(\tau). \]

Thus

\[ \cup_{C \in \mathcal{C}} C = (\mathcal{C}'(b+1)\delta) = ((\mathcal{C}')_{b\delta}) \supseteq (M'(\tau + \delta))_{b\delta} \supseteq M'(\tau). \]

\( \square \)
In the sequel we will take \( \tau = \frac{\epsilon}{2} \). The next proposition provides a uniform bound on the curvature of the smooth pieces of \( \partial M(\frac{\epsilon}{2}) \).

**Proposition 4.4.** For any \( X_i \) and any point \( x \in \partial(X_i)_{\frac{\epsilon}{2}} \), the \( \frac{\epsilon}{2} \)-ball, whose boundary sphere is tangent at \( x \) to \( \partial(X_i)_{\frac{\epsilon}{2}} \), which lies on the same side of \( \partial(X_i)_{\frac{\epsilon}{2}} \) as \( (X_i)_{\frac{\epsilon}{2}} \), is contained in \( (X_i)_{\frac{\epsilon}{2}} \).

**Proof.** The distance between \( x \) and its closest point \( p_{X_i}(x) \) in the closed convex set \( X_i \) is easily seen to be \( \frac{\epsilon}{2} \), and the \( \frac{\epsilon}{2} \)-ball centered at \( p_{X_i}(x) \) is the required one. \( \Box \)

The following proposition follows directly from the definition of a deformation retract.

**Proposition 4.5.** Let \( B' \subset B \) be topological spaces, and let \( F_t \ (t \in [0, 1]) \) be a deformation retract of \( B \) such that \( F_t(b) \in B' \) for any \( b \in B' \), \( t \in [0, 1] \). Then \( F_t|_{B'} \) is a deformation retract of \( B' \).

Let \( B_r \) denote an arbitrary ball in \( S \) of radius \( r \), and \( B_r(x) \) the ball of radius \( r \) centered at \( x \). The following proposition follows from the fact that the volume of a ball of radius \( r \) is a convex function of \( r \) (because the surface area of the \( r \)-sphere is an increasing function of \( r \)).

**Proposition 4.6.** There is a constant \( m \) such that for any \( \delta < 1 \),

\[
m \cdot \text{vol}(B_{\delta/2}) \geq \text{vol}(B_{(b+1)\delta}).
\]

Thus any \( \delta \)-discrete subset of \( B_{(b+1)\delta} \) consists of at most \( m \) elements.

For a finite set \( \{y_1, ..., y_t\} \subset S \) we denote by \( \sigma(y_1, ..., y_t) \) its Chebyshev center, i.e. the unique point \( x \) which minimizes the function \( \max_{1 \leq i \leq t} d(x, y_i) \).

We will soon take \( B \) to be an intersection of balls, and \( B' \subset B \) to be the intersection of \( B \) with \( M(\frac{\epsilon}{2}) \). We intend to use \( 4.5 \) in order to show that under some certain conditions, \( B' \) is contractible. It will be natural to use the star contraction to the Chebyshev center of the centers of the associated balls - The deformation retract which makes any point of \( B \) flow along the geodesic segment which connects it to the required Chebyshev center, at constant velocity \( s \), where \( s \) equals the initial distance. In order to do this, we need the following:

**Proposition 4.7** (Defining the constant \( \delta \)). There exists \( \delta \) \((0 < \delta < \frac{\epsilon}{2(b+1)})\) such that for any point \( x \in S \), any \( \frac{\epsilon}{2} \)-ball \( C \) which contains \( x \) on its boundary sphere, and any \( m \) points \( y_1, ..., y_m \in B_{(b+1)\delta}(x) \setminus (C)_{\delta} \), the inner product of the external normal vector of \( C \) at \( x \) with the tangent at \( x \) to the geodesic segment \([x, \sigma(y_1, ..., y_m)]\) is positive.
Proof. Assume the contrary. Then there is a sequence $\delta_n \to 0$, a corresponding sequence $(C_n)$ of $\frac{\epsilon}{2}$-balls tangent to some $x_0 \in S$ (which we may assume converge to some fixed such ball), and a corresponding sequence of $m$-tuples of points $y_n^1, \ldots, y_n^m \in B_{(b+1)\delta_n}(x_0) \setminus (C_n)\delta_n$, such that the inner product of the tangent at $x_0$ to the geodesic segment $[x_0, \sigma(y_1^0, \ldots, y_m^0)]$ with the outer normal vector of the corresponding sphere $\partial C_n$ is non-positive (we may fix $x = x_0$ since $G$ acts transitively).

Rescaling the metric each time we can assume that $\delta_n$ is fixed and equals 1. We then get a sequence of Riemannian metrics converging, on the ball of radius $b + 1$ around $x_0$, to the Euclidean metric on the ball of radius $b + 1$. More precisely, we look at the ball of radius $b + 1$ in the tangent space $T_{x_0} S$. We identify it each time with the ball of radius $(b + 1)\delta_n$ around $x_0$ in $S$ via the map $X \to \exp_{x_0} (\delta_n X)$, and we rescale the metric there to

$$d_n (X, Y) = \frac{1}{\delta_n} d \left( \exp_{x_0} (\delta_n X), \exp_{x_0} (\delta_n Y) \right).$$

(All these metrics induce the same topology.)

Now, in the rescaled metrics our tangent balls tend to a half space (since $\epsilon/\delta_n \to \infty$), and we may as well assume that our $m$-tuples also converge. In the limit, we get an $m$-tuple of points in a Euclidean space at distance at least 1 (and at most $b + 1$) from a half space, for which the inner product of the external normal vector to this half space with the vector $\overrightarrow{\nu}$, pointing from some $x_0$ on the boundary hyper-plane, towards the Chebyshev center of this $m$-tuple is non-positive. This is an absurd. \hfill $\Box$

Finally, we claim

**Proposition 4.8.** Let $C$ be a collection of balls of radius $(b + 1)\delta$, for which the set of centers $C'$ form a maximal $\delta$-discrete subset of $\Lambda^+$. Then $C$, i.e. the restrictions of its sets to $\Lambda^+$, is a good cover of $\Lambda^+$.

**Proof.** By corollary 4.3, $C$ is a cover of $\Lambda^+$.

Let $B$ be the intersection of $m$ (not necessarily different) balls of $C$, with centers $y_1, \ldots, y_m \in C$.

Proposition 4.4 implies that for any $\bar{x} \in \partial \Lambda^+$ and for any $X$, with $\bar{x} \in \partial (X)\overline{\nu}$ there is an $\frac{\epsilon}{2}$-ball, tangent to $\partial (X)\overline{\nu}$ at $\bar{x}$, which is contained in $(X)\overline{\nu} \subset (\bar{X})\overline{\nu}$. Thus, if in addition the image $x$ of $\bar{x}$ belongs to $B$, then Proposition 4.7 implies that the geodesic segment $[x, \sigma(y_1, \ldots, y_m)]$ lies inside $B' = B \cap \Lambda^+$. Let us explain this point as follows. Let $c(t)$, $t \in [0, 1]$ be a parameterization of the geodesic segment $[x, \sigma(y_1, \ldots, y_m)]$. Proposition 4.4 and 4.7 imply that $c(t) \notin (X)\overline{\nu}$ for all sufficiently small $t > 0$. Assume that $c(t) \in \partial (X)\overline{\nu}$ for some $t_0$, $0 < t_0 < 1$; (and let $t_0$ be the first such time). Then Propositions 4.4 and 4.7 applied to $c(t_0)$ imply that $c(t_0 - \Delta t) \in (X)\overline{\nu}$ for
small $\Delta t$. But this is a contradiction. We conclude that if $B$ is not empty then $\sigma(y_1, \ldots, y_m) \in B'$ and the star-contraction from $B$ to $\sigma(y_1, \ldots, y_m)$ induces a contraction of $B'$. Hence the set $B'$ is non-empty and contractible. This means that $\{C' : C \in \mathcal{C}\}$ is a good cover of $\mathcal{M}'(\frac{1}{2})$, where $C' := C \cap \mathcal{M}'(\frac{1}{2})$. □

We conclude that $\mathcal{M}'(\frac{1}{2})$, and therefore $\mathcal{M}'$, is homotopically equivalent to the simplicial complex $\mathcal{R}$ which corresponds to the nerve of $\mathcal{C}$. Since the collection of centers $\mathcal{C}'$ is $\delta$-discrete, and therefore the $\frac{1}{2}$-balls with the same centers are disjoint, we get that $|\mathcal{C}'|$, the number of the vertices of $\mathcal{R}$, is at most $\frac{\text{vol}(B) \cdot \frac{1}{2}}{\text{vol}(B \cdot 2)}$. Since the sets of $\mathcal{C}$ are subsets of $(b + 1)\delta$-balls, each of them intersects at most $\frac{\text{vol}(B \cdot 2)}{(b + 1)\delta}$ of the others. Hence, the degree of any vertex in $\mathcal{R}$ is at most $d := \frac{\text{vol}(B \cdot 2)}{(b + 1)\delta \cdot \frac{1}{2}}$. This completes the proof of Lemma 4.1.

5. An arithmetic variant of the Margulis lemma

The classical Margulis lemma yields information on the algebraic structure of a discrete group of isometries which is generated by “small elements”. If, in addition, this discrete group lies inside an arithmetic group then we can say a little more. In this section we shall explain this idea in the non-uniform case.

The Lie group of isometries $G = \text{Isom}(S)$ is center-free, semi-simple, without compact factors and with finitely many connected components. Its identity component $G^0$ coincides with the connected component of the group of real points $G(\mathbb{R})^0$ of some $\mathbb{Q}$-algebraic group $G$. We shall identify $G^0$ with its adjoint group $\text{Ad}(G^0) \leq \text{GL}(g)$ and think of it as a group of matrices. We will denote by $\mu$ a fixed Haar measure on $G$.

**Lemma 5.1 (An arithmetic variant of the Margulis lemma).** There are constants $\epsilon = \epsilon(S) > 0$ and $m = m(S) \in \mathbb{N}$, such that if $\Gamma \leq G$ is a non-uniform torsion-free arithmetic lattice, then for any $x \in S$, the group of real points of the Zariski closure $(\overline{\Gamma_\epsilon(x)})_{\mathbb{R}}$ of the group

$$\Gamma_\epsilon(x) = \langle \gamma \in \Gamma : d_\epsilon(x) \leq \epsilon \rangle$$

has at most $m$ connected components, and its identity component is a unipotent subgroup.

The following claims (5.2, 5.3, 5.4) are well known (c.f. [21], chapter IX section 4).

**Lemma 5.2.** For any non-uniform arithmetic lattice $\Delta \leq G^0$, there is a rational structure on $G^0$ (coming from a rational structure on the vector space $g$) with respect to which $\Delta$ is contained in $G^0(\mathbb{Q})$ and commensurable
to $G^0(\mathbb{Z})$. Conjugating by an element $g \in G^0(\mathbb{Q})$ we can assume that $\Delta \subset G^0(\mathbb{Z})$.

**Explanation.** In general, if $\Delta$ is an arithmetic lattice in $G$, there is a compact extension $H$ of $G$, a rational structure on $H$, and a subgroup $\Delta' \leq H_{\mathbb{Q}}$ commensurable to $H_{\mathbb{Z}}$ whose projection to $G$ is $\Delta$. However, when $\Delta$ is non-uniform we can always take $H = G$. This could be deduced, for example, from the fact that $\Delta$ is almost generated by unipotent elements, and that compact groups contain no non-trivial unipotent element.

For a given rational structure, the group $G_{\mathbb{Z}}$ is defined only up to commensurability. A subgroup $\Delta \leq G_{\mathbb{Q}}$ which is commensurable to $G_{\mathbb{Z}}$ is always conjugate, by an element of $G_{\mathbb{Q}}$, to a subgroup of $G_{\mathbb{Z}}$. □

If $g \in G$ is close to $1 \in G$, then its characteristic polynomial (i.e. the characteristic polynomial of the endomorphism $\text{Ad}(g)$) is close to $(\lambda - 1)^n$ where $n = \dim(g)$. In particular:

**Proposition 5.3.** There is an identity neighborhood $\Omega_1 \subset G$ such that, if $g \in \Omega_1$, and the characteristic polynomial of $g$ has integral coefficients, then $g$ is unipotent.

**5.2 and 5.3 implies:**

**Corollary 5.4 (21 4.21).** For any non-uniform arithmetic lattice $\Delta \leq G$, the intersection $\Omega_1 \cap \Delta$ consists of unipotent elements only.

Recall also the following theorem of Zassenhaus and Kazhdan-Margulis (see [27] theorem 8.16):

**Theorem 5.5 (Zassenhaus, Kazhdan-Margulis).** There exists an identity neighborhood $\Omega_2 \subset G$, called a Zassenhaus neighborhood, so that for any discrete subgroup $\Sigma \leq G$, the intersection $\Sigma \cap \Omega_2$ is contained in a connected nilpotent Lie subgroup of $G$.

**Proof of Lemma 5.1.** Let $\Omega \subset G$ be a relatively compact symmetric identity neighborhood which satisfies $\Omega^2 \subset \Omega_1 \cap \Omega_2$.

Fix an integer $m$,

$$m > \inf_{h \in G} \frac{\mu(\{g \in G : d_g(x) \leq 1\} \cdot h \Omega h^{-1})}{\mu(\Omega)},$$

and

$$\epsilon = \frac{1}{m}.$$

As $G$ is unimodular, $m$ can be chosen independently of $x$. Replacing $\Omega$ by some conjugate $h \Omega h^{-1}$, if needed, we can assume that

$$m > \frac{\mu(\{g \in G : d_g(x) \leq 1\} \cdot \Omega)}{\mu(\Omega)}.$$
Let 
\[ \Gamma_{\Omega} = \langle \Gamma \cap \Omega^2 \rangle. \]

Then
\[ [\Gamma_\epsilon(x) : \Gamma_\epsilon(x) \cap \Gamma_{\Omega}] \leq m. \]

To see this, assume for a moment that this index was \( \geq m + 1 \). Then we could find \( m + 1 \) representatives \( \gamma_1, \gamma_2, \ldots, \gamma_{m+1} \in \Gamma_\epsilon(x) \) for different cosets of \( \Gamma_\epsilon(x) \cap \Gamma_{\Omega} \) in the ball of radius \( m \) in \( \Gamma_\epsilon(x) \) according to the word metric with respect to the generating set \( \{ \gamma \in \Gamma_\epsilon(x) : d_\gamma(x) \leq \epsilon \} \). As they belong to different cosets, \( \gamma_i \Omega \cap \gamma_j \Omega = \emptyset \) for any \( 1 \leq i < j \leq m + 1 \). Since \( d_{\gamma_i}(x) \leq m \cdot \epsilon = 1 \) these \( \gamma_i \)'s are all inside \( \{ g \in G : d_g(x) \leq 1 \} \). This contradicts the assumption \( m \cdot \mu(\Omega) > \mu(\{ g \in G : d_g(x) \leq 1 \} \cdot \Omega) \).

It follows from the Zassenhaus-Kazhdan-Margulis theorem that \( \Gamma_{\Omega} \) is contained in a connected nilpotent Lie subgroup of \( G \), and therefore, by Lie’s theorem, \( \Gamma_{\Omega} \) is triangulable over \( \mathbb{C} \). As \( \Gamma_{\Omega} \) is generated by unipotent elements, it follows that \( \Gamma_{\Omega} \) is a group of unipotent elements. Thus the Zariski closure \( \overline{\Gamma_{\Omega}} \) is a unipotent algebraic group, and hence, the group of its real points \( (\overline{\Gamma_{\Omega}})^\mathbb{R} \) is connected in the real topology. Similarly, its subgroup \( (\overline{\Gamma_\epsilon(x) \cap \Gamma_{\Omega}})^\mathbb{R} \) is connected. Clearly, \( (\overline{\Gamma_\epsilon(x) \cap \Gamma_{\Omega}})^\mathbb{R} \) is the identity connected component of \( (\overline{\Gamma_\epsilon(x)})^\mathbb{R} \), and its index is at most \( m \).

\[ \square \]

**Remark 5.6.** Although \( \epsilon \) could be taken to be \( 1/m \), we use different letters for them because they play different roles. In the sequel we will assume that the above lemma is satisfied with \( \epsilon \) replaced by \( 10\epsilon \).

**Remark 5.7.** It follows from Lemma 5.1 that if \( \gamma \) is an element of a non-uniform arithmetic lattice of \( G \) and \( \inf d_\gamma < \epsilon \), then \( \gamma^j \) is unipotent for some \( j \leq m \). This implies that in a non-compact arithmetic \( S \)-manifold there are no closed geodesics of length \( \leq \epsilon \), and that in the \( \epsilon \)-thick thin decomposition, the thin part has no compact connected component. For example, the \( \epsilon \)-thin part of any non-compact arithmetic hyperbolic surface (or more generally of any rank-1 manifold) consists only of cusps.

6. The proof of 1.5(1)

In this section we shall prove:

**Theorem 6.1 (Theorem 1.5(1) of the introduction).** Let \( S \) be a symmetric space of non-compact type. Then there are constants \( \alpha \) and \( d \) (depending only on \( S \)) such that any non-compact arithmetic \( S \)-manifold \( M \) is homotopically equivalent some \( (d, \alpha \cdot \vol(M)) \)-simplicial complex.
Fix $\epsilon' = \epsilon'(S)$, $m = m(S)$, such that Lemma 5.1 is satisfied with $10\epsilon'$, and $m$. Assume that $M = \Gamma \backslash S$ is a given non-compact arithmetic $S$-manifold. Denote by $\Gamma^u$ the set of unipotent elements in $\Gamma$,
\[ \Gamma^u = \{ \gamma \in \Gamma : \gamma \text{ is unipotent} \}, \]
and define
\[ \tilde{X} = \bigcup_{\gamma \in \Gamma^u \setminus \{1\}} \{ d_\gamma < \epsilon' \}, \]
and
\[ \tilde{M}' = \tilde{M} \setminus \tilde{X}, \]
and let $X \subset M$ and $M' \subset M$ be the images of $\tilde{X}$ and $\tilde{M}'$ under the universal covering map.

In order to prove Theorem 6.1 we shall show:

1. There is a deformation retract from $M$ to $M'$. In particular $M$ is homotopically equivalent to $M'$.
2. $M'$ satisfies the conditions of Lemma 4.1 with $\epsilon = \epsilon'/m$.

Let us start with some preliminaries. Let $W \subset S(\infty)$ be a Weyl chamber of the Tits spherical building. Then $W$ is isometric to a spherical simplex. We define its center of mass $z$ to be
\[ z = \frac{\int_W x d\mu(x)}{\| \int_W x d\mu(x) \|} \]
where $\mu$ is the Lebesgue measure on the sphere. Since $W$ is contained in a half sphere (see [2] appendix 3), we get $\| \int_W x d\mu(x) \| \neq 0$. Since $W$ is convex (in the spherical metric), and since its interior is non-empty and is exactly the set of regular points in $W$, it follows that $z$ is regular and contained in $W$.

We shall also use:

Lemma 6.2. Assume that $g$ is a parabolic isometry (i.e. $\min(g) = \emptyset$), $c(t)$ is a regular geodesic, and $g$ stabilizes $c(-\infty)$. Then $\frac{d}{dt}d_g(c(t)) \neq 0$ for any $t \in \mathbb{R}$.

Proof. Let $c(t)$ be a regular geodesic in $S$, and let $g$ be an isometry which stabilizes $c(-\infty)$. Assume that $\frac{d}{dt}|_{t=0}d_g(c(t)) = 0$, then the function $d_g(c(t))$, being analytic and convex must be constant, since it doesn’t tend to $\infty$ as $t \to -\infty$. This means that $g \cdot c$ is a geodesic parallel to $c$. We conclude that $g$ preserves the unique flat which contains $c$, since this flat is exactly the set of points through which there is a geodesic parallel to $c$. But this implies that $g$ is semisimple, a contradiction. \[ \square \]

Furthermore, if $g$ is unipotent, then $d_g(c(t)) \to 0$ as $t \to -\infty$ (see [2] appendix 5, section 4).
The proof of (1). We will show that the conditions of Lemma 3.2 are satisfied with \( L = Y = S \) and the \( \Gamma \)-invariant family of functions \( \mathcal{F} = \{ d_\gamma \}_{\gamma \in \Gamma \setminus \{1\}} \). Then \( \tilde{M}' = \mathcal{F}_{\geq \epsilon} \).

The finiteness of the sets

\[ \Psi_{x,\tau} = \{ \gamma \in \Gamma \setminus \{1\} : d_\gamma(x) \leq \tau \} \]

follows from the compactness of \( \{ g \in G : d_g(x) \leq \tau \} \) together with the discreteness of \( \Gamma \). All the functions \( \{ d_\gamma \}_{\gamma \in \Gamma \setminus \{1\}} \) are convex, strictly positive, without critical values. We need to define the continuous function \( \beta : (0, 3\epsilon) \to \mathbb{R}^+ \), and the appropriate direction \( \hat{n}(x) \in T_x(S) \) for any \( x \in S \) with \( \Psi_{x,3\epsilon} \neq \emptyset \).

Fix \( x \in S \) with \( \Psi_{x,3\epsilon} \neq \emptyset \) and let \( \Psi_{x,3\epsilon} = \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} \). By Lemma 5.1 the Zariski closure \( \overline{\Delta}^z \) of the group \( \Delta = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle \) has a unipotent identity component, and \( \leq m \) connected components. Since \( \gamma_i \) is unipotent, it is contained in the Zariski closure of the cyclic group generated by any power of \( \gamma_i \). As \( \gamma_i \) belongs to the identity component \( (\overline{\Delta}^z)^0 \) for some \( j \leq m \), also \( \gamma_i \in (\overline{\Delta}^z)^0 \). Since this holds for any generator \( \gamma_i \), we get that \( \overline{\Delta}^z = (\overline{\Delta}^z)^0 \) and hence the group of its real points \( (\overline{\Delta}^z)_\mathbb{R} \) is a connected unipotent group. Hence \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are contained in a connected unipotent group.

Let \( N \leq G \) be a maximal connected unipotent subgroup which contains \( \gamma_1, \gamma_2, \ldots, \gamma_k \). Let \( W \leq S(\infty) \) be the Weyl chamber of the Tits boundary of \( S \) which corresponds to \( N \). Let \( c(t) = c_x(t) \) be the geodesic line with \( c(0) = x \) for which \( c(-\infty) \) is the center of mass of \( W \).

By Lemma 6.2

\[ \left. \frac{d}{dt} \right|_{t=0} \left( d_g(c(t)) \right) > 0 \]

for any \( g \in N \setminus \{1\} \). In addition, the continuous function

\[ h(g) = \dot{c}(0) \cdot \nabla d_g(x) \]

attains a minimum on the compact set

\[ \{ g \in N : d_g(x) = \tau \} \]

We define \( \beta(\tau) \) to be this minimum (then \( \beta(\tau) \) is defined for any \( \tau > 0 \)). By definition, \( \dot{c}(0) \cdot \nabla d_{\gamma_i}(x) \geq \beta(d_{\gamma_i}(x)) \). Moreover, it is easy to see that \( \beta \) is a continuous positive function. Since \( G \) acts transitively on the set of couples \((W, x)\) of a Weyl chamber \( W \subset S(\infty) \) and a point \( x \in S \), \( \beta \) is independent of \( N \) and of \( x \). The conditions of Lemma 3.2 are satisfied with the tangent vector \( \hat{n}(x) = \dot{c}_x(0) \).

\[ \square \]

Remark 6.3. We used the existence and uniqueness of the center of mass of \( W \) in order to define \( \beta \) in a canonical way. However, as any two Weyl chambers are isometric, we could choose arbitrarily a regular point in one
Weyl chambers, and translate it to any other Weyl chambers, and to use these points when defining \( c(\infty) \) and then \( c, \hat{n}(x) \) and \( \beta \) in our proof.

**Remark 6.4.** A slight modification of the argument above, yields an elementary proof of the fact that \( M = \Gamma \backslash S \) is homotopic to a compact manifold with boundary, and hence that \( \Gamma \) is finitely presented, when \( \Gamma \leq G \) is any torsion free (non-uniform) lattice. We used the arithmeticity of \( \Gamma \) in order to get a uniform estimate for all \( \Gamma \)'s. However, for a fixed \( \Gamma \), we could have used corollary 11.18 from [27] instead of corollary 5.4 above, and by this to avoid the assumption that \( \Gamma \) is arithmetic. Moreover, we don’t really have to assume that \( \Gamma \) is torsion free (when proving just the finiteness without explicit estimate). The same argument shows that in general, any \( S \)-orbifold \( \Gamma \backslash S \) is homotopic to a compact orbifold with boundary. In particular, our method gives a quite elementary proof of the finite presentability of lattices. Furthermore, changing appropriately the factor \( \sqrt{\cdot} \) in the vector field which induces the deformation retract, so that it will decay more slowly, we can get a simple proof that \( \Gamma \backslash G \) is diffeomorphic to the interior of a compact manifold with boundary.

**The proof of (2).** In order to check that the conditions of Lemma \ref{lem:main} are satisfied, we will show that

- \( M' \subset M_{\geq \epsilon} \) (where \( \epsilon = \frac{\epsilon'}{m} \)).
- \( \tilde{X} \) is a locally finite union of convex open sets with smooth boundary.
- \( M' \) is homotopically equivalent to its \( \frac{\epsilon'}{2m} \)-shrinking \( M'(\frac{\epsilon'}{2m}) \).

And that there are

- \( b = b(S) > 0 \) and
- \( \hat{n}(x) \in T_x(S) \) for any \( x \in (\tilde{X})_{\frac{\epsilon'}{2m}} \backslash (\tilde{X}) \)

such that

\[
\hat{n}(x) \cdot \nabla D_{\{\gamma \leq \epsilon'\}}(x) > \frac{1}{b}
\]

for any \( \gamma \in \Gamma_\backslash \{1\} \) with \( D_{\{\gamma \leq \epsilon'\}}(x) \leq \frac{\epsilon'}{2m} \) (this is stronger than the condition on the inner products required in Lemma \ref{lem:main}).

If \( x \in S \) and \( d_\gamma(x) \leq \frac{\epsilon'}{m} \leq \epsilon' \) for some \( \gamma \in \Gamma \), then by Lemma \ref{lem:unipotent} \( \gamma^j \) is unipotent for some \( j \leq m \), and since \( d_\gamma(x) \leq \frac{j\epsilon'}{m} \leq \epsilon' \), we obtain that \( x \in \tilde{X} \). This proves that \( M' \) is contained in the \( \frac{\epsilon'}{m} \)-thick part \( M_{\geq \epsilon'} \).

Since for \( \gamma \in \Gamma_\backslash \{1\} \) the displacement function is convex and analytic and \( \inf d_\gamma = 0 \), and since \( \Gamma \) is discrete, we have that \( \tilde{X} = \cup_{\gamma \in \Gamma_\backslash \{1\}} \{d_\gamma < \epsilon\} \).
is a locally finite union of convex open sets with smooth boundary.

We shall prove that there is a deformation retract from \( M' \) to its shrinking \( M'(\epsilon' \cdot 2m) \) by showing that the conditions of Lemma 3.2 are satisfied with \( \mathcal{F} = \{ D_{\{d_i \leq \epsilon'\}} \}_{\gamma \in \Gamma^u \setminus \{1\}} \), \( Y = S \), \( L = \mathcal{F}_{\geq 0} = \tilde{M}' \), and \( \mathcal{F}_{\geq \epsilon' \cdot 2m} = \tilde{M}'(\epsilon' \cdot 2m) \).

(In this proof \( \epsilon' \cdot 2m \) plays the role of \( \epsilon \) in Lemma 3.2.)

The finiteness of the sets

\[
\Psi_{x, \tau} = \{ \gamma \in \Gamma^u \setminus \{1\} : D_{\{d_i \leq \epsilon'\}}(x) \leq \tau \}
\]

follows from the discreteness of \( \Gamma \) together with the compactness of

\[
\{ g \in G : D_{\{d_i \leq \epsilon\}}(x) \leq \tau \}.
\]

We shall define the direction \( \hat{n}(x) \in T_x(S) \) analogously to the way it is done in the proof of (1), and shall show that the conditions of Lemma 3.2 are satisfied with the constant function \( \beta(\epsilon') \cdot 2 \), where \( \beta \) is the function defined in the proof of (1).

Fix \( x \in L = \mathcal{F}_{\geq 0} \), with \( \Psi_{x, \epsilon' \cdot 2m} \neq \emptyset \), and denote \( \Psi_{x, \epsilon' \cdot 2m} = \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} \).

Since \( \epsilon' + 2 \epsilon' \cdot 2m = 4 \epsilon' \) we have \( \{ d_{\gamma_i} \leq \epsilon' \} \cdot \epsilon' \cdot 2m \subset \{ d_{\gamma_i} \leq 4 \epsilon' \} \) which implies that \( \cap \{ d_{\gamma_i} \leq 4 \epsilon' \} \neq \emptyset \) and hence \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are contained in a connected unipotent group. As in the proof of (1), let \( N \) be a maximal connected unipotent group which contains \( \gamma_1, \gamma_2, \ldots, \gamma_k \), let \( W \subset S(\infty) \) be the Weyl chamber which corresponds to \( N \) in the Tits boundary. Let \( c(t) = c_x(t) \) be the geodesic line with \( c(0) = x \) and \( c(-\infty) \) the center of mass of \( W \). Since \( \gamma_i \in N, d(c(t), \gamma_i \cdot c(t)) \) tends to 0 as \( t \to -\infty \), and since \( d_{\gamma_i}(x) > \epsilon' \) we have \( d_{\gamma_i}(c(t_0)) = \epsilon'(t_0) \) for some negative \( t_0 \).

The function \( d_{\gamma_i}(c(t)) \) is convex and hence has a non-decreasing derivative, thus, taking \( \hat{n}(x) = \hat{c}(0) \), we have:

\[
\hat{n}(x) \cdot \nabla D_{\{d_{\gamma_i} \leq \epsilon'\}}(x) = \hat{c}(0) \cdot \nabla D_{\{d_{\gamma_i} \leq \epsilon'\}}(x)
\]

\[
= \hat{c}(0) \cdot \frac{\nabla d_{\gamma_i}(c(0))}{\| \nabla d_{\gamma_i}(c(0)) \|}
\]

\[
\geq \frac{1}{2} \cdot \hat{c}(0) \cdot \| \nabla d_{\gamma_i}(c(0)) \| \cdot \epsilon(\epsilon') \cdot 2
\]

\[
= \frac{1}{2} \cdot \hat{c}(0) \cdot \| \nabla d_{\gamma_i}(c(t_0)) \| \cdot \epsilon(\epsilon') \cdot 2
\]

\[
= \frac{1}{2} \cdot \hat{c}(0) \cdot \| \nabla d_{\gamma_i}(c(t_0)) \| \geq \beta(\epsilon') \cdot 2.
\]
where $\beta$ is the function defined in the proof of \ref{lem:main}. In the above computation we made use of the facts that $\|\nabla D_{\{d_\gamma \leq \epsilon'\}}\| = 1$ everywhere outside $\{d_\gamma < \epsilon'\}$, and that $d_\gamma$ is 2-Lipschitz and hence $\|\nabla d_\gamma\| \leq 2$.

We completed the verification that the conditions of Lemma \ref{lem:shrinking} are satisfied, and hence proved that $M'$ is homotopic to its $\frac{\epsilon'}{2m}$-shrinking.

Finally, observe that we can take also the constant $b$ to be $b = \frac{2}{\beta(\epsilon')}$, and the unit tangent vector $\hat{n}(x)$ from \ref{lem:shrinking} to be the same $\hat{n}(x)$ used above. Then all the conditions of Lemma \ref{lem:boundary} are satisfied, and the proof of (2) is completed. $\square$

7. Estimating angles at corners of the boundary

The next 3 sections are devoted to the remaining cases (2), (3) of Theorem \ref{thm:main}. The main result on this section is Theorem \ref{thm:angles}.

Our method is to replace a manifold $M$ by a submanifold with boundary $M'$, in which the injectivity radius is uniformly bounded from below, and then to apply Lemma \ref{lem:boundary}. This requires some information on the boundary. The pre-image $\tilde{X}$ of the complement $X = M \setminus M'$ in the universal covering is required to be a locally finite union of convex open sets with smooth boundary, and what we need is a control on the angles of the corners - where the boundaries of two or more such sets intersect. In the non-uniform case we used the presence of many unipotents, and the nice actions of unipotent groups on the boundary at infinity. In the compact case there are no unipotents, so different tools are required.

Let $A, B \subset S$ be convex bodies with smooth boundary and with a common interior point in their intersection. The angle between the boundaries at a common point $x \in \partial A \cap \partial B$ is measured by $\pi$ minus the angle between the external normal vectors $\hat{n}_A(x), \hat{n}_B(x)$

$$\phi_x(\partial A, \partial B) = \pi - \angle(\hat{n}_A(x), \hat{n}_B(x)).$$

Thus, a big angle between the boundaries corresponds to a small angle between the normals. The following lemma states that when $A$ and $B$ are sub-level sets of commuting isometries, these angles are $\geq \frac{\pi}{2}$. A similar statement appeared in \cite{2}.

For an isometry $\gamma$, and for $x \in S$ with $d_\gamma(x) = a > \min d_\gamma$ we denote by $\hat{n}_\gamma(x)$ the external (with respect to $\{d_\gamma \leq a\}$) normal to $\{d_\gamma = a\} = \partial \{d_\gamma \leq a\}$.

Lemma 7.1. (Commutativity implies big angles): If the isometries $\alpha$ and $\beta$ commute then $\hat{n}_\alpha(x) \cdot \hat{n}_\beta(x) \geq 0$. 

Proof. Let \( a_\alpha = d_\alpha(x), a_\beta = d_\beta(x) \). Since \{d_\alpha = a_\alpha\} is a level set for \( d_\alpha \), we see that \( \hat{n}_\alpha(x) \) is the direction of the gradient \((\nabla \cdot d_\alpha)(x)\). Thus if \( c(t) \) is the geodesic line through \( x \) with \( \dot{c}(0) = \hat{n}_\beta(x) \) then it is enough to show that

\[
(\nabla \cdot d_\alpha)(x) \cdot \hat{n}_\beta(x) = \frac{d}{dt}
|_{t=0}\{d_\alpha(c(t))\} \geq 0.
\]

Let \( p \) denote the projection on the convex set \( \{d_\beta \leq a_\beta\} \). Since \( \alpha \) and \( \beta \) commute, the set \( \{d_\beta \leq a_\beta\} \) is \( \alpha \)-invariant and therefore \( \alpha \) commute with \( p \). Since \( p \) decreases distances and since the geodesic lines \( c(t), \alpha \cdot c(t) \) are both orthogonal to \( \partial\{d_\beta \leq a_\beta\} \) one sees that \( d(c(t), \alpha \cdot c(t)) \) is a non-decreasing function of \( t \). Thus

\[
(\nabla \cdot d_\alpha)(x) \cdot \hat{n}_\beta(x) = \frac{d}{dt}
|_{t=0}\{d_\alpha(c(t))\} = \frac{d}{dt}
|_{t=0}\{d(c(t), \alpha \cdot c(t))\} \geq 0.
\]

\( \square \)

We shall need similar information when \( A, B \) are replaced by their \( \epsilon \)-neighborhoods \( (A)_\epsilon, (B)_\epsilon \). The following lemma explains that the situation then only improves.

Define

\[
\varphi_t = \sup_{x \in \partial(A) \cap \partial(B)_t} \angle \left( \hat{n}_{(A)_t}(x), \hat{n}_{(B)_t}(x) \right),
\]

then we have

Lemma 7.2. (Monotonicity of angles): \( \varphi_t \) is a non-increasing function of \( t \).

Proof. We need to show that if \( t_1 > t_2 \geq 0 \) then \( \varphi_{t_1} \leq \varphi_{t_2} \). If \( \varphi_t \) vanishes at some point \( t = t_0 \) then, as it is easy to verify, the union \( A_{t_0} \cup B_{t_0} \) is convex, and \( \varphi_t = 0 \) for any \( t > t_0 \). We may therefore assume that \( \varphi_t \) is strictly positive in our segment \([t_2, t_1] \).

Fix a common interior point

\[
y \in \text{int}(A) \cap \text{int}(B).
\]

Now

\[
\varphi_t = \sup_{x \in \partial(A) \cap \partial(B)_t} \angle \left( \hat{n}_{(A)_t}(x), \hat{n}_{(B)_t}(x) \right) = \sup_{R \subseteq S} \max_{x \in R \cap \partial(A)_t \cap \partial(B)_t} \angle \left( \hat{n}_{(A)_t}(x), \hat{n}_{(B)_t}(x) \right)
\]

where \( R \) runs over the (compact) balls centered at \( y \). It is therefore enough to show that

\[
\varphi_{t_2} \geq \max_{x \in R \cap \partial(A)_{t_1} \cap \partial(B)_{t_1}} \angle \left( \hat{n}_{(A)_{t_1}}(x), \hat{n}_{(B)_{t_1}}(x) \right)
\]

for any such \( R \).
Fix $R$ large enough (so that the intersection $R \cap \partial(A)_{t_1} \cap \partial(B)_{t_1}$ is not empty) and let

$$\tilde{\varphi}_t = \max_{x \in R \cap \partial(A)_{t_1} \cap \partial(B)_{t_1}} \angle(\hat{n}(A)_t(x), \hat{n}(B)_t(x)).$$

Since $\tilde{\varphi}_t$ is obviously continuous, it is enough to show that for any $t_2 < t \leq t_1$ we have $\tilde{\varphi}_{t_2} \geq \tilde{\varphi}_t$ whenever $\Delta t$ is small enough. Fix $t$, and let $\Delta t$ be sufficiently small so that the argument below holds.

To simplify notation we replace $A$ (resp. $B$) by $(A)_{t_1} \cap \partial(B)_{t_1}$ and assume that $t - \Delta t = 0$ (this is just a matter of changing names after $t$ and $\Delta t$ are fixed). Let $x_t \in R \cap \partial(A)_{t_1} \cap \partial(B)_{t}$ be such that

$$\angle(\hat{n}(A)_t(x_t), \hat{n}(B)_t(x_t)) = \tilde{\varphi}_t.$$

It is enough to show that there is $x_0 \in R \cap \partial(A) \cap \partial(B)$ with

$$\angle(\hat{n}(A)(x_0), \hat{n}(B)(x_0)) \geq \angle(\hat{n}(A)_t(x_t), \hat{n}(B)_t(x_t)).$$

Let $x_0 \in \partial(A) \cap \partial(B)$ be a point at the minimal possible distance from $x_t$. Denote by $\hat{u}_t$ (resp. $\hat{u}_0$) the tangent to the geodesic line $x_0, x_t$ at $x_0$ (resp. at $x_t$).

Since $\Delta t$ is arbitrarily small, $\hat{u}_t$ is roughly in the direction of the bisector of the angle between $\hat{n}(A), (x_t)$ and $\hat{n}(B), (x_t)$, and $x_0$ is closer to $y$ than $x_t$. In particular $x_0 \in R$.

By the Lagrange multipliers theorem $\hat{u}_0$ is a linear combination of $\hat{n}(A)(x_0)$ and $\hat{n}(B)(x_0)$ and since $\Delta t$ is arbitrarily small, we can assume that $\hat{u}_0$ is in the convex cone spanned by $\hat{n}(A)(x_0)$ and $\hat{n}(B)(x_0)$ (again, in the limit case $\hat{u}_0$ is the direction of the bisector of the angle between $\hat{n}(A)(x_0), \hat{n}(B)(x_0)$). Thus

$$\angle(\hat{n}(A)(x_0), \hat{n}(B)(x_0)) = \angle(\hat{n}(A)(x_0), \hat{u}_0) + \angle(\hat{u}_0, \hat{n}(B)(x_0)),$$

and since (by the triangle inequality for angles)

$$\angle(\hat{n}(A)_t(x_t), \hat{n}(B)_t(x_t)) \leq \angle(\hat{n}(A)_t(x_t), \hat{u}_t) + \angle(\hat{u}_t, \hat{n}(B)_t(x_t)),$$

it is enough to show that

$$\angle(\hat{n}(A)(x_0), \hat{u}_0) \geq \angle(\hat{n}(A)_t(x_t), \hat{u}_t)$$

(and the analogous inequality for $B$ instead of $A$ whose proof is the same).

Let $c(s)$ be the geodesic line of unit speed with $c(0) = x_0, c(0) = \hat{u}_0$, and let $c(s) = x_t$ (i.e. $s_0 = d(x_0, x_t)$). Then the above inequality follows from the following

$$\hat{u}_0 \cdot \hat{n}(A)(x_0) = \frac{d}{ds}|_{s=0} D_A(c(s)) \leq \frac{d}{ds}|_{s=s_0} D_A(c(s)) = \hat{u}_t \cdot \hat{n}(A)_t(x_t)$$

and this follows from the convexity of the function $D_A$. \qed
The following lemma, explains how we can use the information obtained above, in order to find a direction with respect to which the directional derivatives of all the corresponding distance functions are large.

**Lemma 7.3. (Existence of a good direction):** There is a constant $b(d) > 0$ such that for any set of unit vectors $\{\hat{n}_i\}_{i \in I} \subset \mathbb{R}^d$, that satisfy the condition $\hat{n}_i \cdot \hat{n}_j \geq 0$ for any $i, j \in I$, there is a unit vector $\hat{f}$ such that $\hat{f} \cdot \hat{n}_i > \frac{1}{b(d)}$ for any $i \in I$.

**Proof.** Let $\Delta$ be a maximal 1-discrete subset of $\{\hat{n}_i\}_{i \in I}$. Then there are at most $\frac{b(d)}{2}$ elements in $\Delta$, where $b(d)$ is some constant. For any $\hat{n}_i \in \{\hat{n}_i\}_{i \in I}$ there is $\hat{n}_{i_0} \in \Delta$ with $\|\hat{n}_i - \hat{n}_{i_0}\| < 1$, which implies $\hat{n}_i \cdot \hat{n}_{i_0} > 1/2$. Let $\hat{f} = \frac{\sum_{\hat{n}_j \in \Delta} \hat{n}_j}{\|\sum_{\hat{n}_j \in \Delta} \hat{n}_j\|}$, then

$$\hat{n}_i \cdot \hat{f} > \hat{n}_i \cdot \frac{\sum_{\hat{n}_j \in \Delta} \hat{n}_j}{\frac{b(d)}{2}} \geq \frac{2}{b(d)} \hat{n}_i \cdot \hat{n}_{i_0} \geq \frac{1}{b(d)}.$$

$\square$

Summarizing the above discussion together with the arguments of the previous section, we conclude the following less general but more practical version of Lemma 4.1.

**Theorem 7.4.** Let $S$ be a symmetric space of non-compact type, $M = \Gamma \backslash S$ an $S$-manifold, and $\epsilon' \geq \epsilon > 0$. Assume that:

- $M' \subset M_{\geq \epsilon}$ is a submanifold with boundary for which the complement of the pre-image in the universal covering is given by $\hat{X} = S \setminus \hat{M}' = \cup_{\gamma \in \Gamma'} \{d_\gamma < \epsilon'\}$, where $\Gamma'$ is a subset of $\Gamma$ which is invariant under conjugation by elements of $\Gamma$.
- $M'$ is homotopically equivalent to its $\hat{\epsilon}$-shrinking $M'(\hat{\epsilon})$.
- For any $x \in S$ the group $\langle \gamma \in \Gamma' : d_\gamma(x) \leq 3\epsilon' \rangle$ is either unipotent or commutative.

Then there are positive constants $d, \alpha$ which depend on $S$, $\epsilon, \epsilon'$, such that $M'$ is homotopically equivalent to some $(d, \alpha \cdot \text{vol}(M'))$-simplicial complex.

**Proof.** We shall show that the conditions of Lemma 4.1 are satisfied. The sets $\{d_\gamma < \epsilon'\}$ are convex and open with smooth boundary and $\hat{X}$ is the locally finite union of them.
It is given that \( M' \) is contained in the \( \epsilon \)-thick part and that it is homotopic to its \( \frac{\epsilon}{2} \)-shrinking.

We should indicate how to define the constant \( b \) and the direction \( \hat{n}(x) \), from the conditions of Lemma 4.1.

We define

\[
    b = \max\{ \frac{2}{\beta(\epsilon')}, b(d) \}
\]

where \( \beta \) is the function defined in the proof of Theorem 6.1 in the previous section, and \( b(d) \) is the constant defined in Lemma 7.3 for \( d = \dim S \).

Next we define the directions \( \hat{n}(x) \). Let \( x \in S \setminus \tilde{X} \) be a point with \( d(x, \tilde{X}) \leq \epsilon \). If the group \( \langle \gamma \in \Gamma' : d_{\gamma}(x) \leq 3\epsilon' \rangle \) is unipotent, then we define the direction \( \hat{n}(x) \) in the same way as it is done in the previous section. (It is shown in the previous section that in this case the conditions of Lemma 4.1 are satisfied.) Assume that the group \( \langle \gamma \in \Gamma' : d_{\gamma}(x) \leq 3\epsilon' \rangle \) is abelian. Let \( A(x) \subset \Gamma' \) be the set of elements \( \gamma \in \Gamma' \) with \( D(d_{\gamma}(x) < \epsilon')(x) = d(x, \tilde{X}) \). Then for \( \gamma \in A(x) \)

\[
    d_{\gamma}(x) \leq 2d(x, \tilde{X}) + \epsilon' \leq 3\epsilon',
\]

and hence \( A(x) \) is contained in \( \langle \gamma \in \Gamma' : d_{\gamma}(x) \leq 3\epsilon' \rangle \). Thus \( A(x) \) is abelian.

By Lemma 7.1 for any \( \alpha, \beta \in A(x) \) and any \( y \in \partial(d_{\alpha} < \epsilon') \cap \partial(d_{\beta} < \epsilon') \), the inner product of the corresponding external normal vectors at \( y \) is non-negative. By Lemma 7.2 the inner product at \( T_x(S) \) of the external normal vectors to \( \partial(d_{\alpha} < \epsilon')(x) \) and \( \partial(d_{\beta} < \epsilon')(x) \) at \( x \), is also non-negative. Therefore it follows from Lemma 7.3 there is a direction \( \hat{f} \in T_x(S) \) for which the inner product of \( \hat{f} \) with the external normal vector at \( x \) to \( \partial(d_{\gamma} < \epsilon')(x) \) is \( \geq \frac{1}{b(d)} \) for any \( \gamma \in A(x) \). Thus, we can take \( \hat{n}(x) = \hat{f} \).

\[\square\]

8. The proof of 1.5(2) in the rank-1 case

Theorem 1.5(2) can be proved, independently of the rank, using the argument that we shall present in section 9. For two reasons we chose to do the rank-1 case separately. The first reason is that in the remaining higher rank case all locally symmetric manifolds of finite volume are arithmetic. Since Theorem 1.5 already took care of the non-compact arithmetic case, we can assume compactness when considering higher rank manifolds. This would make the argument in section 10 simpler. The second reason is that in the rank-1 case there is another proof, which is in some sense more simple. This proof, which we shall present below, is basically the same as the one given in [9] for the hyperbolic case, and with the tools developed in sections 3,4,6 and 7, it could be applied to general rank-1 symmetric spaces which are not necessarily of constant curvature.
Remark 8.1. For \( S = \mathbb{H}^2 \) the statement of Conjecture 1.3 follows from the Gauss Bonnet theorem without the arithmeticity assumption. This is because the volume determines the genus and bound the possible number of cusps. For this reason we allow ourselves to ignore this case in this and in the following section.

Recall what we intend to prove:

**Theorem 8.2 (Theorem 1.5(2) for the rank-1 case).** If \( S \) is a rank-1 symmetric space of dimension \( \geq 4 \), then for some constants \( \alpha, d \), the fundamental group \( \pi_1(M) \) of any \( S \)-manifold \( M = \Gamma \setminus S \) is isomorphic to the fundamental group of some \((d, \alpha \cdot \text{vol}(M))\)-simplicial complex.

We shall use the ordinary thick-thin decomposition. Let \( \epsilon_s \) be the constant from the Margulis lemma, and let \( \epsilon = \epsilon_s^2 \).

**Theorem 8.3 (Thick-thin decomposition in rank-1).** (see [30] section 4.5, and [2] section 10) Assume that \( \text{rank}(S) = 1 \). Let \( M \) be an \( S \)-manifold of finite volume and \( M^0_{\leq \epsilon} \) an arbitrary connected component of \( M_{\leq \epsilon} \). Then \( M^0_{\leq \epsilon} \) belongs to either one of the following two types:

1. A tube, i.e. a tubular neighborhood of a short geodesic. In this case \( M^0_{\leq \epsilon} \) is topologically a ball-bundle over the circle. Its fundamental group \( \pi_1(M^0_{\leq \epsilon}) \) is infinite cyclic, and in particular abelian.

   or

2. A cusp. In which case \( M^0_{\leq \epsilon} \) is homeomorphic to \( \mathbb{R}_{\geq 0} \times \partial M^0_{\leq \epsilon} \), where \( \partial M^0_{\leq \epsilon} \) is topologically a sub-manifold of codimension 1. A connected component of its pre-image \( \tilde{M}^0_{\leq \epsilon} \) is given by \( \tilde{M}^0_{\leq \epsilon} = \bigcup_{\gamma \in \Gamma_0} \{ d \gamma \leq \epsilon \} \) where \( \Gamma_0 \leq \Gamma \) is a subgroup isomorphic to \( \pi_1(M^0_{\leq \epsilon}) \), and it is a “star-shaped” neighborhood of some point \( z \in S(\infty) \) (i.e. each geodesic line with \( c(\infty) = z \) enters once into \( M^0_{\leq \epsilon} \) and stays in it). \( \Gamma_0 \) consists of unipotent elements only, and each \( \gamma \in \Gamma_0 \) fixes \( z \) and leaves the horospheres around \( z \) invariant. Moreover \( \Gamma_0 \) is metabelian.

There are only finitely many connected components of \( M_{\leq \epsilon} \). Moreover, as \( \text{dim}(M) \geq 3 \) the boundary of each connected component of \( M_{\leq \epsilon} \) is connected, and hence the thick part \( M_{\geq \epsilon} \) is connected. Each cusp is homotopically equivalent to its boundary. As \( \text{dim}(M) \geq 4 \) each tube is a ball bundle over the circle, with fibers of dimension \( \geq 3 \), and hence, its boundary is a sphere bundle for a sphere of dimension \( \geq 2 \). Therefore, for each connected component of the thin part, the injection of the boundary into the component induces an isomorphism between the fundamental groups (which are both \( \cong \mathbb{Z} \)). Thus by Van-Kampen’s theorem we obtain:

**Corollary 8.4.**

\[ \pi_1(M) \cong \pi_1(M_{\geq \epsilon}). \]
Take $M' = M_{\geq \epsilon}$. Then the following claim finishes the proof of Theorem 8.2.

**Claim 8.5.** $M'$ satisfies the condition of Theorem 7.4 with respect to $\epsilon' = \epsilon$.

**Proof.** We only need to explain why $M'$ is homotopic to its $\frac{\epsilon}{2}$-shrinking, since the other conditions of Theorem 7.4 follow directly from the definition of $M'$ and from Theorem 8.3.

Since $\epsilon = \frac{\epsilon}{2}$, the $\frac{\epsilon}{2}$-neighborhoods of different connected components of $M_{\leq \epsilon}$ are still disjoint, and hence, we can prove the homotopy equivalence by showing that there is a deformation retract from $\overline{(M_{\leq \epsilon})_{\frac{\epsilon}{2}} \setminus M_{\leq \epsilon}^0}$ to $\partial(M_{\leq \epsilon})_{\frac{\epsilon}{2}}$, for each connected component $M_{\leq \epsilon}^0$ of $M_{\leq \epsilon}$.

If $M_{\leq \epsilon}^0$ is a cusp, then one can define such a deformation retract by letting each point flow (at constant speed = the initial distance) along the unique geodesic line which connects it to the unique end of the cusp, i.e., along the geodesic whose lifting in the universal covering converges, when $t \to -\infty$, to the unique limit point in $S(\infty)$ of a lifting of the cusp.

If $M_{\leq \epsilon}^0$ is a tube, then the lifting of $M_{\leq \epsilon}^0$ and its $\frac{\epsilon}{2}$-neighborhood in $S = \tilde{M}$ are both star-shaped with respect to the lifting $c$ of the short closed geodesic which lies inside $M_{\leq \epsilon}^0$. The generalized star-contraction (see 3.3) from $\overline{(M_{\leq \epsilon})_{\frac{\epsilon}{2}} \setminus M_{\leq \epsilon}^0}$ to $\partial(M_{\leq \epsilon})_{\frac{\epsilon}{2}}$ projects to a deformation retract of the corresponding subsets of $M$. \qed

9. The proofs of 1.5(2) and 1.5(3)

We shall use the fact that when a low dimensional submanifold is removed from a high dimensional manifold, it doesn’t change the low dimensional homotopy groups. More precisely:

**Lemma 9.1.** Let $M$ be a connected manifold and $N \subset M$ a closed (not necessarily connected) sub-manifold.

- If $\text{codim}_M(N) \geq 2$ then there is a surjective homomorphism
  \[ \pi_1(M \setminus N) \to \pi_1(M). \]

- If $\text{codim}_M(N) \geq 3$ then $\pi_1(M \setminus N) \cong \pi_1(M)$.

**Proof.** If $\text{codim}_M(N) \geq 2$ then any closed loop can be pushed to one which does not intersect $N$. Similarly, if $\text{codim}_M(N) \geq 3$ then any homotopy of loops can be pushed to $M \setminus N$. \qed

The submanifold we are about to remove consists of the union of all short closed geodesics of some certain type. Observe that the lifting to $S = \tilde{M}$ of a closed geodesic in $M = \Gamma \setminus S$ which corresponds to $\gamma \in \Gamma$ is an axis of $\gamma$. In particular, any two closed geodesics from the same homotopy class are parallel.
Lemma 9.2. Let $\gamma \in G^0$ be a hyperbolic isometry which projects non-trivially to each simple factor of $G^0$, and let $\bar{N} \subset S$ be the union of all geodesics which are axes of $\gamma$.

- If $S$ is not isometric to $\mathbb{H}^2$ then $\text{codim}_S(\bar{N}) \geq 2$.
- If additionally $S$ is neither isometric to $\mathbb{H}^3, \mathbb{H}^2 \times \mathbb{H}^2$ nor to $\text{PSL}_3(\mathbb{R})/\text{PSO}_3(\mathbb{R})$, then $\text{codim}_S(\bar{N}) \geq 3$.

Proof. Let $S^*$ be an irreducible factor of $S$. Denote by $\gamma^*$ the projection of $\gamma$ to the corresponding simple factor $G^*$ of $G^0$. We need to estimate $\text{codim}_{S^*}(\min(\gamma^*))$.

If $\gamma^*$ fixes a point then we just remark that since the projection $\gamma^*$ is non-trivial, $\text{codim}_{S^*}(\min(\gamma^*)) > 0$. Otherwise let $c$ be an axis of $\gamma^*$ in $S^*$ and let

$$G^*(c) = \{ g \in G^* : g \cdot c \text{ is parallel to } c \}.$$ 

An alternative way to define $G^*(c)$ is as follows. Pick two points $p, q \in c(\mathbb{R})$ and let $g_{p,q} = \sigma_p \cdot \sigma_q$ be the corresponding transvection, then $G^*(c)$ coincides with the centralizer group $C_{G^*}(g_{p,q}) \leq G^*$ of $g_{p,q}$. $G^*(c)$ is a closed reductive Lie subgroup of $G^*$ which acts transitively on $N^* = \text{the union of all geodesics parallel to } c \text{ in } S^*$. Observe that $N^*$ contains the projection of $\bar{N}$ to $S^*$.

Choose an Iwasawa decomposition of $G^*$ which induces an Iwasawa decomposition of $G^*(c)^0$, i.e. write the Iwasawa decompositions of $G^*$ and $G^*(c)$ simultaneously

$$G^* = K \cdot A \cdot U, \text{ and } G^*(c)^0 = K(c) \cdot A \cdot U(c),$$

such that $K(c) = K \cap G^*(c)^0$ and $U(c) = U \cap G^*(c)^0$. (This could be done by choosing a flat $F \supset c$ and defining the torus $A$ to be the subgroup of $G^*$ which acts on $F$ by translations, and then choosing an order on the dual of $\mathfrak{a} = \text{Lie}(A)$ such that if $c(t)$ is given by $c(t) = e^{tH}$ for $H \in \mathfrak{a}$, then $\alpha(H) \geq 0$ for any simple root $\alpha$, and then taking the Iwasawa decomposition which corresponds to $A$ with this ordering.)

Then $\text{dim}(S^*) = \text{dim}(A \cdot U)$ while $\text{dim}(N^*) = \text{dim}(A \cdot U(c))$, and hence

$$\text{codim}_{S^*}(N^*) = \text{codim}_U(U(c)).$$

Let $P_{c(-\infty)}$ be the parabolic subgroup of $G^*$ which corresponds to the point $c(-\infty) \in S^*(\infty)$, and let $P^-$ be the minimal parabolic opposite to $P^+ = AU$. Then, as it is easy to verify, the Lie algebra of $G^*$ is the direct sum

$$\text{Lie}(G^*) = \text{Lie}(U) + \text{Lie}(P^-),$$

while the Lie algebra of the parabolic $P_{c(-\infty)}$ is the direct sum

$$\text{Lie}(P_{c(-\infty)}) = \text{Lie}(U(c)) + \text{Lie}(P^-).$$

Hence $\text{codim}_U(U(c)) = \text{codim}_{G^*}(P_{c(-\infty)})$. The lemma follows from the following proposition by a standard case analysis.
Proposition 9.3 (see [1] lemma 3.4 and corollary 8). Let $H$ be a connected simple Lie group and $P \leq H$ a proper closed connected subgroup. Then $\text{codim}_H(P) \geq \text{rank}(H)$, and the equality can hold only if $H$ is locally isomorphic to $\text{SL}_n(\mathbb{R})$.

\[ \square \]

We shall now prove:

Theorem 9.4 (Theorem 1.5(2) from the introduction in the general case). Let $S$ be a symmetric space of non-compact type, not isometric to $H^2$, $H^3$, $H^2 \times H^2$, $\text{PSL}_3(\mathbb{R})/\text{PSO}_3(\mathbb{R})$. Then there are constants $\alpha, d$ such that the fundamental group of any irreducible $S$-manifold of finite volume $M$ is isomorphic to the fundamental group of some $(d, \alpha \cdot \text{vol}(M))$-simplicial complex.

Proof of Theorem 9.4. Let $G = \text{Isom}(S)$, let $r$ be the rank of $S$, i.e. the real rank of $G$, let $n_s$ be the index from the Margulis lemma and let $n = n_s!$. Fix $\epsilon$ to be one third of the $\epsilon_s$ from the Margulis lemma.

Since we already proved the theorem in the rank one case, we can assume that $\text{rank}(S) \geq 2$. Then any $S$-manifold is arithmetic. Moreover, as Theorem 6.1 implies Theorem 9.4 for non-compact arithmetic manifold, we may assume that $M = \Gamma\backslash S$ is compact. In this case, we have the following strengthening of the Margulis lemma.

Lemma 9.5. For any $x \in S$, the group $\Gamma_{\epsilon_s}(x)$ contains an abelian subgroup of index $n_s$.

Proof. By the Margulis lemma $\Gamma_{\epsilon_s}(x)$ contains a subgroup $\Gamma^0_{\epsilon_s}(x)$ of index $n_s$ which is contained in a connected nilpotent Lie subgroup of $G$. By Lie’s theorem $\Gamma^0_{\epsilon_s}(x)$ is triangulable over $\mathbb{C}$. Therefore its commutator $[\Gamma^0_{\epsilon_s}(x), \Gamma^0_{\epsilon_s}(x)]$ contains only unipotents. As $\Gamma$ is cocompact it has no non-trivial unipotents. Thus $\Gamma^0_{\epsilon_s}(x)$ is abelian.

It follows that when $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\{d_{\gamma_1} \leq \epsilon\} \cap \{d_{\gamma_2} \leq \epsilon\} \neq \emptyset$ then $\gamma_1^n$ and $\gamma_2^n$ commute.

Consider an element $\gamma \in \Gamma$. Replacing $\gamma$ by $\gamma^{[G;G^0]}$, we may assume that $\gamma \in G^0$. As $\Gamma \cap G^0$ is a uniform lattice in $G^0$, $\gamma$ is semisimple. Consider the sequence of centralizers

$C_G(\gamma) \leq C_G(\gamma^n) \leq C_G(\gamma^{n^2}) \leq C_G(\gamma^{n^3}) \leq \ldots \leq C_G(\gamma^{n^r})$.

As the centralizer of a semisimple element is determined by the type of the singularity of the element and by a torus which contains it, two consecutive terms $C_G(\gamma^{ni})$ and $C_G(\gamma^{ni+1})$ in this sequence must coincide. Take $i$ to be the first time at which this happens and write $\gamma' = \gamma^{ni}$. In this way we
attach $\gamma'$ to each $\gamma \in \Gamma$. Notice that if $\gamma \in G^0$ then $\gamma' = \gamma^j$ for some $j \leq n^r$, and in general $\gamma' = \gamma^j$ for some $j \leq [G : G^0]n^r$. Set $m = [G : G^0]n^r$.

The reason we prefer to work with the $\gamma'$'s is the following. If

$$\{d_{\gamma_1'} \leq 3\epsilon\} \cap \{d_{\gamma_2'} \leq 3\epsilon\} \neq \emptyset,$$

then $\gamma_1'$ and $\gamma_2'$ commute. Of course, by the above lemma, the non-empty intersection implies that $\gamma_1'^m, \gamma_2'^m$ commute. Since $\gamma_1'$ has the same centralizer as $\gamma_1'^m$, this implies that $\gamma_1', \gamma_2'^m$ commute, and since $\gamma_2'$ has the same centralizer as $\gamma_2'^m$, this implies that $\gamma_1', \gamma_2'$ commute.

Let $N \subset M$ be the subset which consists of the union of all closed geodesics of length $\leq \epsilon$ which correspond to elements of the form $\gamma'$ for $\gamma \neq 1$. In other words, its pre-image in $S$ is given by

$$\tilde{N} = \bigcup \{\min(\gamma') : \gamma \in \Gamma \setminus \{1\}, \min(d_{\gamma'}) \leq \epsilon\}.$$

Then $N$ is a finite union of totally geodesic closed submanifolds. Since $\Gamma$ is irreducible and $G$ is center free, any $\gamma \in \Gamma \setminus \{1\}$ projects non-trivially to each factor of $G$. So it follows from Lemma 9.2 that $\dim_M(N) \geq 3$, and hence, by Lemma 9.1

$$\Gamma \cong \pi_1(M) \cong \pi_1(M \setminus N).$$

Define

$$\tilde{X} = \bigcup_{\gamma \in \Gamma \setminus \{1\}} \{d_{\gamma'} < \epsilon\}, \ \tilde{M}' = S \setminus \tilde{X},$$

and

$$X = \Gamma \setminus \tilde{X}, \ M' = \Gamma \setminus \tilde{M}' = M \setminus X.$$

Clearly $N \subset X$. In order to prove the theorem we shall show:

1. $M \setminus N$ and $M'$ are homotopically equivalent.
2. $M'$ satisfies the conditions of Theorem 7.4 with respect to $\epsilon$ and $\frac{\epsilon}{m}$ (corresponding to $\epsilon'$ and $\epsilon$ respectively in 7.4).

**Proof of (1):** We shall construct the desired homotopy in two steps. Since $M$ and $N$ are compact, and each connected component of $N$ is a finite union of totally geodesic submanifolds, there is small positive number $\eta > 0$, such that the $\eta$-neighborhoods of the components of $N$ are still disjoint and contained in $X$. It is easy to verify that if $\eta$ is small, $M \setminus N$ and $M \setminus \overline{(N)_{\eta}}$ are diffeomorphic.

Next, we claim that there is a deformation retract from $M \setminus \overline{(N)_{\eta}}$ to $M'$. In order to show this, we shall apply Lemma 3.2 to the set of functions

$$F = \{d_{\gamma'} : \gamma \in \Gamma \setminus \{1\}, \min(d_{\gamma'}) < \epsilon\}$$

on $Y = L = S \setminus \overline{(N)_{\eta}}$, and $M' = F_{\geq \epsilon}$. We have to indicate what are the directions $\hat{n}(x)$, and what is the continuous function $\beta$. 


Let \( x \in \bar{X} \setminus (N)_\eta \) and let \( \{d_{\gamma_1}, \ldots, d_{\gamma_k}\} \subset \mathcal{F} \) be the subset of functions satisfying \( d_{\gamma}(x) \leq 3\epsilon \). Then the group \( \langle \gamma_1, \ldots, \gamma_k \rangle \) is abelian. Thus for each \( j \leq k \) the convex set \( \cap_{i=1}^{j-1} \min(\gamma_i') \) is \( \gamma_j' \) invariant, and hence, by induction
\[
\cap_{i=1}^{j} \min(\gamma_i') \neq \emptyset.
\]
Pick arbitrarily \( y \in \cap_{i=1}^{k} \min(\gamma_i') \) and define \( \hat{n}(x) \) to be the tangent at \( x \) to the geodesic line which goes from \( y \) to \( x \).

For \( t \leq 3\epsilon \) we define \( \beta(t) \) to be the minimum of the directional derivative of \( d_{\gamma'} \) at \( x \) with respect to the tangent to the geodesics \( \overline{y,x} \), where this minimum is taken over all \( x \in \bar{X} \setminus (N)_\eta \), all \( d_{\gamma'} \in \mathcal{F} \) with \( d_{\gamma'}(x) = t \), and all \( z \in \min(\gamma') \). Since

- \( N \) and \( M \setminus (N)_\eta \) are compact,
- up to conjugations there are only finitely many \( \gamma' \)'s with \( \min(d_{\gamma'}) < \epsilon' \), and
- for any selection of \( x, z, \gamma' \) as above, the corresponding directional derivative is positive,

it follows that \( \beta \) is a well defined continuous positive function.

**Proof of (2):** We shall check that the conditions of Theorem 7.4 are satisfied.

Let \( x \in S \) and assume \( d_{\gamma}(x) \leq \frac{\epsilon}{m} \) for some \( \gamma \neq 1 \) in \( \Gamma \). As \( \gamma' = \gamma^j \) for some \( j \leq m \), it follows that
\[
d_{\gamma'}(x) = d_{\gamma^j}(x) \leq j \cdot d_{\gamma}(x) \leq \frac{j \cdot \epsilon}{m} \leq \epsilon.
\]
Thus \( x \in \bar{X} \). This shows that \( M' \) is contained in \( M_{\frac{\epsilon}{m}} \).

If \( \{d_{\gamma_1'} \leq 3\epsilon\} \cap \{d_{\gamma_2'} \leq 3\epsilon\} \) then \( \gamma_1' \) commutes with \( \gamma_2' \). Thus, the last condition of Theorem 7.4 is also satisfied.

We shall show that there is a deformation retract from \( M' \) to its \( \frac{\epsilon}{2m} \)-shrinkage, by an analogous way to the second step in the proof of (1) above. This time take
\[
\mathcal{F} = \{D_{\{d_{\gamma'} \leq \epsilon\}} : \gamma \in \Gamma \setminus \{1\}, \min(d_{\gamma'}) < \epsilon\}
\]
while \( \frac{\epsilon}{2m} \) plays the role of \( \epsilon \) in Lemma 3.2. We define the directions \( \hat{n}(x) \in T_x(S) \) to be the tangent to the geodesic \( \overline{y,x} \) for arbitrary \( y \in \cap \{\min(\gamma') : d_{\gamma'} \in \Psi_{x,3\epsilon}\} \). Additionally, we define \( \beta(t) \) to be the minimum of the directional derivative of \( D_{\{d_{\gamma'} \leq \epsilon\}} \) at \( x \) with respect to the tangent of the geodesics \( \overline{y,x} \), where the minimum is taken over all \( x \in S \), all \( D_{\{d_{\gamma'} \leq \epsilon\}} \in \mathcal{F} \) with \( D_{\{d_{\gamma'} \leq \epsilon\}}(x) = t \), and all \( z \in \min(\gamma') \). Since for \( t \geq 0 \) and for a relevant \( \gamma' \), the set \( \{D_{\{d_{\gamma'} \leq \epsilon\}} \leq t\} \) is a convex body with smooth boundary and since any \( z \) as above belongs to the interior of this set, the directional derivatives
mentioned above are always positive. By compactness we get that \( \beta \) is a continuous positive function defined for any \( 0 \leq t \leq \frac{3\epsilon}{2m} \). This finishes the proof of Theorem 9.4

Next we prove:

**Theorem 9.6 (Theorem 1.5(3) of the introduction).** For any \( S \), there are constants \( \alpha, d \), such that the fundamental group of any irreducible \( S \)-manifold \( M \) is isomorphic to a quotient of the fundamental group of some \((d, \alpha \cdot \text{vol}(M))\)-simplicial complex.

**Proof.** There are only 3 cases left to deal with. \( \text{PSL}_3(\mathbb{R})/\text{PSO}_3(\mathbb{R}), \mathbb{H}^2 \times \mathbb{H}^2 \) and \( \mathbb{H}^3 \).

The proof for the first two cases goes verbatim as the proof of Theorem 9.4, since these cases are of higher rank and we can assume compactness. The only difference is that in these cases, we have only \( \text{codim}_M(N) \geq 2 \) (instead of \( \geq 3 \)) by the first part of Lemma 9.2, so the result follows from the first part of Lemma 9.1.

For hyperbolic 3-manifolds, one should only throw out finitely many circles which are all the closed geodesics of length \( \leq \epsilon \) and then prove that what is left is homotopically equivalent to the \( \epsilon \)-thick part. This is done by deforming each cusp to its boundary as in section 8, and deforming each compact component minus a circle to its boundary, as it is done in the proof of 9.4. Then, again, Theorem 7.4 finishes the proof.

\[ \square \]

10. **Some remarks on Conjecture 1.3, Theorem 1.5 and their relations to algebraic number theory**

Let \( p(x) \in \mathbb{Z}[x] \) be an integral monic polynomial, and let

\[
p(x) = \prod_{i=1}^{k} (x - \alpha_i)
\]

be its factorization into linear factors over \( \mathbb{C} \). Denote by \( m(p) \) its **exponential Mahler measure**

\[
m(p) = \prod_{|\alpha_i| > 1} |\alpha_i|.
\]

The following is known as Lehmer’s conjecture.

**Conjecture 10.1.** There exists a constant \( \ell > 0 \) such that if \( p(x) \) is an integral monic polynomial with \( m(P) \neq 1 \), then \( m(p) > 1 + \ell \).

Denote by \( d(p) \) the number of roots \( \alpha_i \) with absolute value \( > 1 \)

\[
d(p) = \#\{\alpha_i : |\alpha_i| > 1\}.
\]

The following conjecture of Margulis is weaker than Lehmer’s conjecture.
Conjecture 10.2 (see [21] IX 4.21). There is a function $\ell : \mathbb{N} \to \mathbb{R}_{>0}$ such that $m(p) \geq 1 + \ell(d(p))$ for any non-cyclotomic monic polynomial $p(x) \in \mathbb{Z}[x]$.

If Conjecture 10.2 is true, then for any symmetric space of non-compact type $S$, the minimal injectivity radius of any compact arithmetic $S$-manifold is bounded from below by some positive constant $r = r(S)$ (see also [21], page 322 for a similar statement). To see this, we argue as follows: Let $G^0$ be the identity component of $G = \text{Isom}(S)$. As $G^0$ is center-free we can identify it with its adjoint group $\text{Ad}(G^0) \leq \text{GL}(g)$. Let $\Gamma \leq G^0$ be a torsion-free uniform arithmetic lattice in $G^0$. We think of $\Gamma$ as the intersection of the fundamental group of some compact arithmetic $S$-manifold with $G^0$. Since $\Gamma$ is arithmetic, there is a compact extension $G^0 \times O$ of $G^0$ and a $\mathbb{Q}$-rational structure on the Lie algebra $g \times o$ of $G^0 \times O$, such that $\Gamma$ is the projection to $G^0 \times O$ of a lattice $\tilde{\Gamma}$, which is contained in $(G^0 \times O)_\mathbb{Q}$ and commensurable to the group of integral points $(G^0 \times O)_\mathbb{Z}$ with respect to some $\mathbb{Q}$-base of $(g \times o)_\mathbb{Q}$. By changing this $\mathbb{Q}$-base, we can assume that $\tilde{\Gamma}$ is in fact contained in $(G^0 \times O)_\mathbb{Z}$. This means that the characteristic polynomial $p_{\tilde{\gamma}}$ of any $\tilde{\gamma} \in \tilde{\Gamma}$ is a monic integral polynomial. As $\Gamma$ is discrete and torsion-free, $m(p_{\tilde{\gamma}}) > 1$ for any $\tilde{\gamma} \in \tilde{\Gamma}$ which projects to a non-trivial element in $\Gamma$. Since $O$ is compact, any eigenvalue of $\tilde{\gamma}$ with absolute value different from 1 is also an eigenvalue of its projection $\gamma \in G^0$. In particular

$$m(p_{\tilde{\gamma}}) = m(p_{\gamma}) \geq 1 + \min_{i \leq \dim G} \ell(i).$$

We conclude that $\gamma$ is outside the open set

$$U = \{ g \in G^0 : m(p_g) < 1 + \min_{i \leq \dim G} \ell(i) \}.$$

Clearly $U$ contains any compact subgroup of $G$, and contains a subset of the form

$$\{ g \in G^0 : g \text{ is semisimple and } \min d_g < \tilde{r}(S) \}$$

for some positive constant $\tilde{r}(S)$. Hence the minimal injectivity radius of $\Gamma \backslash S$ is $\geq \tilde{r}(S)$. Finally, if $\Gamma \leq G$ is a torsion free lattice which is not necessarily contained in $G^0$ then the minimal injectivity radius of $\Gamma \backslash S$ is at least $\frac{\tilde{r}(S)}{|G:G^0|} = r(S)$. This implies:

**Corollary 10.3.** Conjecture 10.2 implies Conjecture 1.3 (for compact arithmetic manifolds).

**Proof.** For a given $M$, choose a maximal $r$-discrete net $\mathcal{C}$, and take $\mathcal{R}$ to be the simplicial complex which corresponds to the nerve of the cover of $M$ by the $r$-balls whose centers form $\mathcal{C}$. Then $M$ is homotopic to $\mathcal{R}$ which is a $\left( \frac{\text{vol}(B_2 \cdot r)}{\text{vol}(B_r)} \cdot \frac{\text{vol}(M)}{\text{vol}(B_r)} \right)$-simplicial complex. \qed
For compact locally symmetric manifolds, the minimal injectivity radius equals half of the length of the shortest close geodesic. For non compact arithmetic $S$-manifolds we already know the absence of short closed geodesics (see Remark 5.7). Therefore Conjecture 10.2 implies

**Conjecture 10.4.** For any $S$, there exists a constant $l = l(S)$ such that no arithmetic $S$-manifold contains a closed geodesic of length $\leq l$.

A simple argument, which uses the fact that for non-cyclotomic monic integral polynomials $F(x)$ of a fixed degree $k = \dim(G) + \dim(O)$, $m(F)$ is bounded away from 1, shows that for all the compact arithmetic $S$-manifolds, which arise by constructions in which the compact extending group $O$ is fixed, the infimum on the minimal injectivity radius is positive, and the statement of Conjecture 10.4 holds, independently of the rational structure and of the specific choice of a manifold within a commensurability class. More precisely:

**Proposition 10.5.** Given a compact semi-simple Lie group $O$, there is a positive constant $r = r(S, O)$, such that the minimal injectivity radius is $\geq r$, for any compact manifold $M = \Gamma \setminus S$ such that $\Gamma \cap G^0$ is commensurable to the group $\pi_{G^0}((G^0 \times O)Z)$ for some $Z$-structure on $g \times o$. In particular, any such manifold is homotopically equivalent to a $(\frac{\vol(B_{2r,\tau})}{\vol(B_{2r,1})}, \frac{\vol(M)}{\vol(B_{r,\tau})})$-simplicial complex.

A real algebraic integer $\tau > 1$ is called a **Salem number** if all its conjugates in $\mathbb{C}$ have absolute value $\leq 1$. In Conjecture 10.2 it is not even known whether $\ell(1) > 0$, i.e. whether there is a positive gap between 1 and the set of Salem numbers. Sury [29] showed that the existence of such a gap is equivalent to the existence of an identity neighborhood in $SL_2(\mathbb{R})$ which intersects trivially any uniform arithmetic lattice. Therefore the existence of such a gap implies a positive infimum on the length of closed geodesics, when considering all the arithmetic surfaces. We shall now discuss the relation between this gap and manifolds locally isometric to $S = SL_3(\mathbb{R})/SO_3(\mathbb{R})$ or to $\mathbb{H}^2 \times \mathbb{H}^2$ (the two cases for which we could not prove the analog of Theorem 1.5(2)).

**Claim 10.6.** Assume there is a positive gap between 1 and the set of Salem numbers (i.e. $\ell(1) > 0$). Then the analog of Theorem 1.5(2) holds also for the symmetric space $S = SL_3(\mathbb{R})/SO_3(\mathbb{R})$.

**Proof.** Let $\tau > 1$, and assume that $\gamma = \text{diag}(\tau, \tau, \tau^{-2})$ is an element of a lattice $\Gamma \subseteq SL_3(\mathbb{R})$. One can easily compute the eigenvalues of $\text{Ad}(\gamma)$: they are $\tau^3, \tau^{-3}$ and 1. As $\Gamma$ is arithmetic, $\tau$ is a Salem number, and thus, by our assumption, is bounded away from 1. We conclude that the displacement functions $d_\gamma$ of such elements are uniformly bounded away from 0. Take

$$
\epsilon^* < \inf\{\min d_\gamma : \gamma = \text{diag}(\tau, \tau, \tau^{-2}), \tau \text{ is a Salem number}\}
$$
which is also smaller than $\epsilon_s$ from the Margulis lemma.

If $g$ is a hyperbolic element of an arithmetic lattice in $\text{SL}_3(\mathbb{R})$ with $\min(d_g) \leq \epsilon^*$, then $g$ has no real eigenvalues of multiplicity 2. It is easy to see that $\min(g)$ is then a flat or a single geodesic. In particular $\dim(\min(g)) \leq 2$.

As $\dim(S) = 5$, the submanifold $N$ which consists of the union of all closed geodesics of length $\leq \epsilon$ has codimension $\geq 3$. Therefore we can apply word by word the argument of the proof of Theorem 9.4 for $S$ with $\epsilon^*$.

If moreover $\ell(2) > 0$, then the same is true also for the second remaining case $S = \mathbb{H}^2 \times \mathbb{H}^2$.

**Claim 10.7.** Assume $\ell(1) \cdot \ell(2) > 0$. Then the analog of 1.5(2) holds also for $S = \mathbb{H}^2 \times \mathbb{H}^2$.

**Proof.** In contrast with the situation for $\text{SL}_3(\mathbb{R})$, our problem here arises only for regular elements $\gamma \in \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ (for otherwise $\min(\gamma)$ is a single geodesic and its codimension is 3). But such an element always has the form $\gamma = (\gamma_1, \gamma_2)$ where $\gamma_i \in \text{SL}_2(\mathbb{R})$ are diagonalizable over $\mathbb{R}$. If $\gamma_i$ is conjugate to $\text{diag}(\tau_i, \tau_i^{-1})$, $\tau_i > 1$ then the only possible non-trivial conjugate of $\tau_1$ outside the unit disk is $\tau_2$. Therefore, assuming $\ell(1), \ell(2) > 0$ we can choose $\epsilon^*$ small enough, so that all arithmetic elements with minimal displacement $\leq \epsilon^*$ are singular, and each has a unique axis. Then we can apply the argument of the proof of Theorem 9.4 to this case.

**Remark 10.8.** Similarly, $\ell(1) > 0$ implies also the analog of 1.5(2) for compact arithmetic 3-manifolds.

---

11. **Estimating the size of a minimal presentation**

If Conjecture 1.3 is true, then, given a symmetric space of non-compact type $S$, and an $S$-manifold $M$, such that either

- $S$ is not $\mathbb{H}^3$, or
- $M$ is arithmetic,

the minimal size of a presentation for the fundamental group should be bounded linearly by the volume. From Theorem 1.5 we deduce this for most cases:

**Definition 11.1.** We say that a presentation of a group $\langle \Sigma : W \rangle$ is **standard** if the length of each $w \in W$ is $\leq 3$.

**Theorem 11.2.** Assume that either

- $S$ is not isomorphic to $\mathbb{H}^3, \mathbb{H}^2 \times \mathbb{H}^2, \text{PSL}_3(\mathbb{R})/\text{PSO}_3(\mathbb{R})$, or
- $M$ is non-compact arithmetic.
Then for some constant $\eta = \eta(S)$, independent of $M$, the fundamental group $\pi_1(M)$ admits a standard presentation

$$\pi_1(M) \cong \langle \Sigma : W \rangle$$

with $|\Sigma|, |W| \leq \eta \cdot \operatorname{vol}(M)$.

**Proof.** Let $\mathcal{R}$ be the $(d, \alpha \cdot \operatorname{vol}(M))$-simplicial complex which corresponds to $M$ by Theorem 1.5 (1) or (2). Fix a spanning tree $T$ for $\mathcal{R}$, and take the generating set $\Sigma$ for $\pi_1(\mathcal{R}) \cong \pi_1(M)$ which consists of those closed loops which contain exactly one edge outside $T$. We thus obtain a generating set of size less than the number of edges of the 1-skeleton $\mathcal{R}^1$ which is at most $\alpha(S) \operatorname{vol}(M)d(S)$. In other words, we take for each edge of $\mathcal{R}^1 \setminus T$ the element of $\pi_1(\mathcal{R}^1)$ which corresponds to the unique cycle (with arbitrarily chosen orientation) which is obtained by adding this edge to $T$. Additionally, let the set of relations $W$ consist exactly of those words which are induced from 2-simplexes of $\mathcal{R}^2$ (we take one such relation for each 2-simplex). In this way we obtain a set of relations of size $\leq \alpha(S) \operatorname{vol}(M)d^2(S)$ which is a bound for the number of triangles in $\mathcal{R}^1$. Thus,

$$\eta = \alpha(S)d^2(S) = \max\{\alpha(S)d^2(S), \alpha(S)d(S)/2\}$$

will do. Finally, the length of each $w \in W$ is exactly the number of edges in the corresponding 2-simplex which lie outside $T$, and thus the presentation $\langle \Sigma : W \rangle$ is standard. \qed

**Remark 11.3.** Lower bounds for the size of any presentation, are known for hyperbolic 3-manifolds (see [13]). In this case, the upper bound obtained above (for non-compact arithmetic 3-manifolds) is tight.

For non-arithmetic hyperbolic 3-manifolds the analogous statement is evidently false (see Remark 11.6). Surprisingly the following result holds:

**Theorem 11.4.** There is a constant $\eta$ such that the fundamental group of any complete hyperbolic 3-manifold $M$ admits a presentation $\pi_1(M) \cong \langle \Sigma : W \rangle$ for which both $|\Sigma|$ and $|W|$ are $\leq \eta \cdot \operatorname{vol}(M)$.

**Lemma 11.5.** Let $S$ be a rank one symmetric space, and let $\epsilon = \frac{\epsilon_s}{2}$ where $\epsilon_s$ is the constant from the Margulis lemma. There is a constant $c = c(S)$ such that for every $S$-manifold $M$, the number of closed geodesics of length $\leq \epsilon$ in $M$ is at most $c \cdot \operatorname{vol}(M)$.

**Proof.** Write $M = \Gamma \backslash S$ and let $\alpha, \beta \in \Gamma$ be elements which correspond to two different closed geodesics in $M$ of length $\leq \epsilon$. Then the axes of $\alpha$ and $\beta$ are bounded away from each other, and hence, for large enough $m$, $\langle \alpha^m, \beta^m \rangle$ is a non-abelian free group. It follows that $\{d_\alpha < \epsilon_s\} \cap \{d_\beta < \epsilon_s\} = \emptyset$. 

---

42  TSACHIK GELANDER
Since \( d_\gamma(x) \leq \epsilon + 2D_{\{d_\gamma<\epsilon\}}(x) \), we see that

\[
\{D_{\{d_\alpha<\epsilon\}} < \epsilon\} \cap \{D_{\{d_\beta<\epsilon\}} < \epsilon\} = \emptyset.
\]

This implies that if we take, for each connected component \( M_{\leq \epsilon} \) of \( M_{\leq \epsilon} \), an \( \epsilon \)-ball \( B_\epsilon \), whose center lies on the boundary of \( M_{\leq \epsilon} \), then these balls are disjoint and injected. Thus the number of geodesics of length \( \leq \epsilon \) (which coincides with the number of connected components of \( M_{\leq \epsilon} \) and hence with the number of these \( \epsilon \)-balls) is \( \leq \text{vol}(M)/\text{vol}(B_\epsilon) \).

\[
\square
\]

**Proof of Theorem 11.4.** Let \( \epsilon = \frac{4\eta}{3} \), let \( M \) be a complete hyperbolic 3-manifold, and let \( N \subset M \) be the union of all closed geodesics in \( M \) of length \( \leq \epsilon \). Then \( N \) consists of \( \leq \frac{\text{vol}(M)}{\text{vol}(B_\epsilon)} \) circles. As in the proof of 9.6, \( \pi_1(M \setminus N) \) is isomorphic to \( \pi_1(R) \) for some \( (\alpha, d \cdot \text{vol}(M)) \)-simplicial complex \( R \). It follows that \( \pi_1(M \setminus N) \) admits a presentation with \( \leq \frac{\alpha d}{2} \cdot \text{vol}(M) \) generators, and \( \leq \alpha d^2 \cdot \text{vol}(M) \) relations. Van-Kampen’s theorem implies that when adding these circles one by one to \( M \setminus N \), we should add one relation for each (note that each circle has a neighborhood homeomorphic to a solid torus or a solid Klein bottle). Hence, we get a presentation of \( \pi_1(M) \) with the same number of generators and with at most \( \frac{\alpha d}{\text{vol}(B_\epsilon)} \cdot \text{vol}(M) \) additional relations.

\[
\square
\]

**Remark 11.6.** In contrast with Theorem 11.2, Theorem 11.4 does not yield bounds for the length of the relations. Since for \( v \) large enough, there are infinitely many complete hyperbolic 3-manifolds with volume \( \leq v \), the length of the relations in the above presentations can not be bound in terms of \( \text{vol}(M) \).

**Remark 11.7.** For \( S = \mathbb{H}^2 \times \mathbb{H}^2 \) the analog of Theorem 11.4 holds. The proof is almost the same, except that in this case, each of the \( \leq \frac{\text{vol}(M)}{\text{vol}(B_{\epsilon/2})} \) connected components of the union \( N \) of all closed geodesics of length \( \leq \epsilon \) is either a circle or a two dimensional torus or a Klein bottle.

However for the symmetric space \( S = \text{SL}_3(\mathbb{R})/\text{SO}_3(\mathbb{R}) \) we have only the following result which follows directly from Theorem 9.6

**Proposition 11.8.** For any \( S \), there is a constant \( \eta(S) \) such that the fundamental group of any \( S \)-manifold \( M \) has a generating set of size \( \eta(S) \cdot \text{vol}(M) \).

12. **A quantitative version of Wang’s theorem**

Denote by \( \rho_S(v) \) the number of isometric classes of irreducible \( S \)-manifolds of volume \( \leq v \).

If Conjecture 1.3 is true then for any symmetric space, \( S \), of non-compact type of dimension \( \geq 4 \), there is some constant \( c = c(S) \), such that

\[
\rho_S(v) \leq v^{c \cdot v}
\]
for any \( v > 0 \). While in dimension 3, the validity of Conjecture 1.3 would yield an analogous upper bound for the growth of arithmetic 3-manifolds.

Theorems 1.5(2) implies:

**Theorem 12.1.** Assume that \( S \) is neither isometric to \( \mathbb{H}^2, \mathbb{H}^3, SL_3(\mathbb{R})/SO_3(\mathbb{R}) \) nor to \( \mathbb{H}^2 \times \mathbb{H}^2 \). Then we have

\[
\rho_S(v) \leq v^{c(S)\cdot v}.
\]

And 1.5(1) implies:

**Proposition 12.2.** The number of non-compact arithmetic hyperbolic 3-manifolds of volume \( \leq v \) is at most \( v^{cv} \) for some constant \( c \).

**Remark 12.3.** Similarly, for \( S = SL_3(\mathbb{R})/SO_3(\mathbb{R}) \) or \( \mathbb{H}^2 \times \mathbb{H}^2 \), the number of non-compact irreducible manifolds with volume \( \leq v \) is bounded by \( v^{cv} \) for some \( c \).

**Proof of 12.1, 12.2 and 12.3.** Since \( \dim(S) \geq 3 \), it follows from Mostow’s rigidity theorem that an irreducible \( S \)-manifold \( M \) is characterized by its fundamental group, which, by Theorem 11.2 has a presentation \( \pi_1(M) \cong \langle \Sigma, W \rangle \) with \( |\Sigma|, |W| \leq \eta(S)\text{vol}(M) \) in which all the relations has length \( \leq 3 \). A rough estimate of the number of groups admitting such a presentation yields 12.1.

**Remark 12.4.** It was shown in [9] that for \( \mathbb{H}^n \) when \( n \geq 4 \) this estimate is tight. However in the higher rank case it is very likely that \( \rho_S(v) \) grows much slower. It was guessed in [9] that when \( \text{rank}(S) \geq 2 \)

\[
\log \rho_S(v) \approx c(S)\frac{(\log V)^2}{\log \log V}.
\]

**Remark 12.5.** Since Mostow rigidity does not hold for surfaces, our method does not yield a quantitative version for Borel’s finiteness theorem for arithmetic hyperbolic surfaces of a given genus.

### 13. Some complements

13.1. **Extending some of the results to non-compact orbifolds.** In the previous sections we have considered \( S \)-manifolds of finite volume. It is natural to try to generalize the results obtained, to the larger family of \( S \)-orbifolds of finite volume. This amounts to consider general lattices in \( G \) instead of just torsion free lattices.

It turns out that some of the main statements could be generalized to non-compact \( S \)-orbifolds, i.e. to general non-uniform lattices \( \Gamma \leq G \). We remark that we do not know how to deal with general compact orbifolds. In the non-compact case, our generalizations rely on the following effective version of Selberg’s lemma:
Lemma 13.1. There is a constant \( i = i(G) \in \mathbb{N} \) such that any non-uniform arithmetic lattice \( \Gamma \leq G \) has a torsion free normal subgroup of index \( \leq i \).

Proof. Let \( \mathfrak{g} \) denote the Lie algebra of \( G \) and let \( n \) be its dimension. Replacing \( G \) by its identity component we can assume it is connected and center free, and therefore may be identified with its image under the adjoint representation \( \text{Ad}(G) \leq GL(\mathfrak{g}) \cong GL_{n}(\mathbb{R}) \). As follows from the proof of Margulis’ arithmeticity theorem, for any non-uniform arithmetic lattice \( \Gamma \leq G \) there is a base \( B \) for the vector space \( \mathfrak{g} \cong \mathbb{R}^{n} \) with respect to which \( \Gamma \leq GL_{n}(\mathbb{Q}) \) and is commensurable to \( GL_{n}(\mathbb{Z}) \). Replacing this base by a \( \mathbb{Z} \)-base for the \( \mathbb{Z} \)-span of \( \Gamma \cdot B \) (which is easily seen to be a \( \mathbb{Z} \)-lattice in \( \mathbb{R}^{n} \)), we can assume that \( \Gamma \) is contained in \( GL_{n}(\mathbb{Z}) \).

Let \( T \leq GL_{n}(\mathbb{Z}) \) be a fixed torsion-free congruence subgroup (which exists, for instance, by Selberg’s lemma) and let \( i = i(G) \) be its index

\[ i = [GL_{n}(\mathbb{Z}) : T]. \]

Clearly, \( \Gamma \cap T \) is torsion free and \( [\Gamma : \Gamma \cap T] \leq i \). \( \square \)

The following generalization of Theorem 11.2 follows immediately:

Theorem 13.2. There is a constant \( \eta(G) \) such that any non-uniform arithmetic lattice \( \Gamma \leq G \) has a presentation \( \Gamma \cong \langle \Sigma : W \rangle \) with

\[ |\Sigma|, |W| \leq \eta(G) \cdot \text{vol}(G/\Gamma). \]

Proof. Take a torsion free normal subgroup \( \Gamma_{1} \) of index \( \leq i \) in \( \Gamma \). Then \( \text{vol}(G/\Gamma_{1}) = \text{vol}(G/\Gamma)[\Gamma/\Gamma_{1}] \leq \text{vol}(G/\Gamma) \cdot i \), and \( \Gamma_{1} \), being torsion free, has a presentation with \( \leq \eta \cdot \text{vol}(G/\Gamma_{1}) \) generators and relations by 11.2. We should add at most \( |\Gamma/\Gamma_{1}| \leq i \) generators and \( |\Gamma/\Gamma_{1}|^{i} \) relations to get a presentation for \( \Gamma \). \( \square \)

However, unlike the case of 11.2, we do not know how to bound the lengths of the relations in \( W \).

The following extends the results of section 12.

Theorem 13.3. There is a constant \( c = c(G) \) such that for any \( v > 0 \), the number of conjugacy classes of non-uniform arithmetic lattices of covolume \( \leq v \) is at most \( v^{e^{c \cdot v}} \).

Proof. We already know (by section 12) that for any \( v > 0 \) there are at most \( v^{e^{c \cdot v}} \) conjugacy classes of torsion free lattices of covolume \( \leq v \), and therefore at most \( v^{e^{c \cdot i \cdot v}} \) conjugacy classes of torsion free lattices of covolume \( \leq i \cdot v \) where \( i = i(G) \) is the constant from Lemma 13.1.

Let \( v_{0} \) be the minimal covolume of a lattice \( \Gamma \leq G \). Let \( \Gamma \leq G \) be a lattice of covolume \( \leq v \). By Lemma 13.1 \( \Gamma \) contains a torsion free normal subgroup...
\( \Gamma' \) of index \( \leq i \). Let \( N_G(\Gamma') \) be the normalizer of \( \Gamma' \) in \( G \). Then \( N_G(\Gamma') \) is a lattice containing \( \Gamma \) whose covolume satisfies
\[
v_0 \leq \text{vol}(G/N_G(\Gamma')) \leq v.
\]

It follows from Theorem 13.2 that \( N_G(\Gamma') \) has a generating set of size \( \leq [\eta \cdot v] \). Thus, for any \( j \), the number of subgroups of \( N_G(\Gamma') \) of index \( j \) is no more than \( (j + [\eta \cdot v])!^2 \) (which is a trivial upper bound for the number of index \( j \) subgroups of the free group of rank \([\eta \cdot v] \)). The index \([N_G(\Gamma') : \Gamma]\) is at most \( v/v_0 \), so there are at most \( [v/v_0] \cdot ([v/v_0] + [\eta \cdot v])!^2 \leq v^{c'' . v} \) choices for \( \Gamma \) as a subgroup of index \( \leq [N_G(\Gamma') : \Gamma] \) of \( N_G(\Gamma') \).

Thus, the number of lattices \( \Gamma \) which contain the same \( \Gamma' \) as a normal subgroup of index \( \leq i \) is at most \( v^{c'' . v} \). Since there are at most \( v^{c'' . v} \) possibilities for \( \Gamma' \), it follows that the number of conjugacy classes of lattices of covolume \( \leq v \) is at most \( v^{c'' . v} \cdot v^{c'' . v} \leq v^{c''} \).

13.2. Commensurable growth. Let us now restrict our attention to a fixed commensurability class.

**Definition 13.4.** Two \( S \)-manifolds \( M, N \) are called commensurable if they have a common finite cover. I.e. \( \Gamma_1 \backslash S \) is commensurable to \( \Gamma_2 \backslash S \) iff \( \Gamma_1 \) is commensurable to some conjugate of \( \Gamma_2 \) in \( G = \text{Isom}(S) \).

The following definition is natural.

**Definition 13.5.** The **commensurable growth** \( \kappa_M(v) \) of a locally symmetric manifold \( M \), is the number of non-isometric manifolds commensurable to \( M \) with volume \( \leq v \). The **commensurable growth** \( \kappa_\Gamma(v) \) of a lattice \( \Gamma \leq G \) is the number of conjugacy classes of lattices commensurable to \( \Gamma \) with covolume \( \leq v \).

One can define the notion of commensurable growth for arbitrary subgroup \( \Gamma \leq G \), not necessarily a lattice, as follows: define the generalized index between commensurable subgroups \( \Gamma, \Gamma' \leq G \) to be the rational number
\[
[\Gamma : \Gamma'] = \frac{[\Gamma : \Gamma \cap \Gamma']}{[\Gamma' : \Gamma \cap \Gamma]},
\]
and use this concept instead of “covolume” in the above definition.

Clearly, for a locally symmetric manifold \( M = \Gamma \backslash S \), we have \( \kappa_M(v) \leq \kappa_\Gamma(v) \). It is natural to ask what is the relation between these functions. In particular, do they have the same asymptotic behavior?

Another interesting question is what is the relation between the commensurable growth and the congruence subgroup problem.
We shall now give upper bounds for the commensurable growth of locally symmetric manifolds and for its fundamental group when the dimension is $> 2$.

**Claim 13.6.** Let $M = \Gamma \backslash S$ be an irreducible locally symmetric manifold of dimension $> 2$ with finite volume. Then there is a constant $c = c(M)$ such that $\kappa_M(v) \leq v^c$.

*Proof.* If $M = \Gamma \backslash S$ is not arithmetic then, by Margulis’ criterion for arithmeticity, the commensurability class of $\Gamma$ admits a unique maximal element which contains all the others, and the result follows by considering the subgroup growth of this maximal element. If $M$ is arithmetic, this follows from 10.5 and from 6.1. \qed

When $\Gamma \leq G$ is a non-arithmetic lattice, the above proof applies also to $\kappa_\Gamma$ and gives $\kappa_\Gamma(v) \leq v^c$ as well. In the arithmetic case, as in 13.3, we obtain similar upper bounds by using the following lemma, which can be proved in the same way as Lemma 13.1.

**Lemma 13.7.** For any commensurability class $\mathcal{R}$ of arithmetic lattices in $G$, there is a constant $i = i(\mathcal{R})$ such that any $\Gamma \in \mathcal{R}$ contains a torsion free subgroup of index $\leq i$.

The following is immediate from 10.5 and 13.7.

**Lemma 13.8.** Given a commensurability class $\mathcal{R}$ of arithmetic lattices in $G$, there is a constant $\eta = \eta(\mathcal{R})$ such that any $\Gamma \in \mathcal{R}$ has a presentation $\Gamma \cong \langle \Sigma : W \rangle$ with $|\Sigma|, |W| \leq \eta(G) \cdot \text{vol}(G/\Gamma)$.

As in the proof of 13.3, these two lemmas imply:

**Proposition 13.9.** For any lattice $\Gamma \leq G$, $\kappa_\Gamma(v) \leq v^{c(\Gamma)v}$.

### 13.3. How to construct a simplicial complex for non-arithmetic manifolds.

We shall now explain how to attach simplicial complexes to non-arithmetic, or more generally to rank-1 manifolds, of dimension $\geq 4$. We are not trying to do it in the most economical way, but just to explain an idea of how this could be done.

The following proposition follows from a rough estimate for the diameter and the minimal injectivity radius of compact connected components of the thin part.

**Proposition 13.10.** There are positive constants $\alpha = \alpha(n)$ and $d = d(n)$ such that any compact rank-1 locally symmetric manifold $M$ of dimension $n \geq 4$ is homotopically equivalent to a $(d, \alpha \cdot \text{vol}(M)^{3n^2+1})$-simplicial complex.
Proof. Fix $n$, let $\epsilon(n)$ be the constant of the Margulis lemma, and let $M_{\leq \epsilon(n)}$ be the thin part of the ordinary $\epsilon(n)$ thick-thin decomposition.

It follows from [10] (see proposition 3.2 there) that for some constant $c$, the diameter of any compact connected component of $M_{\leq \epsilon(n)}$ is at most $3 \log (c \cdot \text{vol}(M))$. Applying formula 8.5 from [16] (page 381), which implies that the injectivity radius decreases at most exponentially as one moves along the manifold, we get that the injectivity radius at any point belonging to $M_{\leq \epsilon(n)}$, and therefore at any point of $M$, is at least $c' \cdot \text{vol}(M)^{-3n}$, for some positive constant $c'$.

The proposition follows by applying a good covering argument with ordinary balls of radius $\epsilon(c' \cdot \text{vol}(M))^{-3n}$. The needed number of balls in such a cover is $\leq c'' \cdot \text{vol}(M)^{3n^2+1}$. □

In fact, it is easy to obtain stronger results by means of elementary computations of volumes of neighborhoods of short closed geodesics. We shall demonstrate this in the real hyperbolic case. We remark that in all other rank-1 cases, one can obtain similar estimates by using the same means, and applying Rauch’s comparison theorems (see [11] 1.10). However, also the following estimate is probably not tight, and it should be possible (and not necessarily very hard) to obtain better estimates.

**Proposition 13.11.** For $n \geq 4$, there are constants $\alpha = \alpha(n), d = d(n)$, such that any compact hyperbolic $n$-manifold $M$ is homotopically equivalent to some $(d, \alpha \cdot \text{vol}(M)^{(1+n([n^2-1]+1))/2(n-2-[n^2-1]))}$. simplicial complex.

**Sketched proof for $n = 4$.** In this case any connected component of the thin part of the ordinary thick-thin decomposition (with respect to some fixed $\epsilon$) is a neighborhood of a short closed geodesic which is topologically a ball bundle over a circle. In order to understand the geometry of the thin components it is most convenient to look at the upper half-space model for the hyperbolic space $\mathbb{H}^4$ (see [3] for details). Lift the component so that the short closed geodesic is lifted to the line connecting $0$ to $\infty$. Then our lifted component is a cone, centered by this line. The intersection of this cone with a horosphere perpendicular to this line is a union of coaxial ellipsoids (with respect to the induced $(n-1)$-Euclidean structure on the horosphere).

Using the fact that any abelian subgroup of $\text{SO}_3(\mathbb{R})$ (isomorphic to the fixator group of this line) is contained in a 1-dimensional torus, one can show, by a simple pigeonhole argument on the powers of the corresponding isometry, that if the length of our short closed geodesic is $a << \epsilon$ then, for some constant $c$, the 3 dimensional Euclidean ball of radius $c a^{1/2}$ is contained in the union of the above ellipsoids. This implies that the volume of the component is at least $c' (1/a^{1/2})^3 \cdot a$. Thus $a \geq c'' \cdot \text{vol}(M)^{3/2}$.
Thus the injectivity radius at any point of $M$ is at least $\rho = c''\frac{1}{\text{vol}(M)}$, and one can construct, as above, a simplicial complex with at most $c'''\frac{\text{vol}(M)^9}{\rho^4} = \alpha \cdot \text{vol}(M)^9$ vertices, all of them of degree bounded by some constant $d$.

We remark that the proof for general dimension $n$ uses the fact that any abelian subgroup of $\text{SO}_{n-1}(\mathbb{R})$ is contained in some $\left[\frac{n-1}{2}\right]$-dimensional torus.

\begin{remark}
In order to obtain analogous estimates for non-compact rank-1 manifolds, one should use the thick-thin decomposition with $\epsilon = \epsilon(\text{vol}(M))$ as above, so that all the components of $M_{\leq \epsilon}$ would be cusps, and then estimate explicitly the function $\beta(\epsilon)$ which is defined in the proof of Theorem 6.1 and detect its influence on the determination of the constant $b = b(\epsilon)$ in Proposition 4.1 in order to finally calculate the resulting simplicial complex.
\end{remark}

13.4. Wang’s theorem for products of $\text{SL}_2$’s. This paragraph is not precisely a part of the main theme of this paper, but only a part of the same subject of mathematics. Moreover, we are not presenting any new result here, but only clarify things which are evidently known to some peoples. The author decided to write this paragraph because it might serve as a complement to Wang’s paper \cite{31}.

Wang’s theorem states that if $G$ is a connected semisimple Lie group without compact factors, and $G$ is not locally isomorphic to $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$, then for any $v > 0$, there are only finitely many conjugacy classes of irreducible lattices in $G$ of covolume $\leq v$.

In \cite{31} Wang didn’t consider the case where $G$ is of higher rank and has factors locally isomorphic to $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$. Margulis’ arithmeticity theorem implies that if $G$ has both a factor which is locally isomorphic to $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$ and a factor which is not locally isomorphic $\text{PSL}_2(\mathbb{R})$ or $\text{PSL}_2(\mathbb{C})$ then $G$ contains no irreducible lattices (see \cite{21} corollary 4.5, page 315). It was remarked by Borel (see \cite{4} 8.3 and 8.1) that Wang’s argument implies also the finiteness of the number of conjugacy classes of irreducible lattices for groups locally isometric to $G_{a,b} = \text{SL}_2(\mathbb{R})^a \times \text{SL}_2(\mathbb{C})^b$ when $(a,b) \neq (1,0), (0,1)$. Let as now explain this remark of Borel.

It follows from Margulis’ super rigidity theorem that irreducible higher rank lattices are locally rigid. Thus, the missing ingredient in Wang’s argument (\cite{31}, 8.1) when applied to groups locally isometric to $G_{a,b}$ is the following statement (which was also noted without proof in \cite{4}).

\begin{proposition}
Let $G$ be a semi-simple Lie group with no compact factors, and let $\Gamma_n \leq G$ be a sequence of irreducible lattices. Assume that $(\Gamma_n)$ converges to a lattice $\Delta$ in the topology of closed subgroups (Hausdorff convergence on compact sets). Then $\Delta$ is also irreducible.
\end{proposition}
Proof. Assume that $\Delta$ is reducible. Then we can write $G = G_1 \cdot G_2$ in such a way that $\Delta$ is commensurable with $\Delta_1 \cdot \Delta_2$ where $\Delta_i = \Delta \cap G_i$. Fix a finite generating set for $\Delta$, and for large $n$, denote by $f_n : \Delta \to \Gamma_n$ the homomorphism induced by sending each generator to the closest element in $\Gamma_n$. As explained in [31], since $\Delta$ is finitely presented, $f_n$ is a well defined homomorphism whenever $\Gamma_n$ is close enough to $\Delta$.

For any $\delta \in \Delta$, $f_n(\delta) \to \delta$. We will show that $f_n(\delta)$ is central for each non-central $\delta \in \Delta_1$, and for any large enough $n$. Since the center of $G$ is discrete, this will imply the desired contradiction.

Fix $\delta \in \Delta_1$ non-central. As $\Gamma_n$ is irreducible, we will show that $f_n(\delta)$ is central by showing that its projection to the second factor $\pi_2(f_n(\delta))$ is the unit element in $G_2$.

$\Delta_2$ is a lattice in $G_2$. Let $\{\delta_{2,1}, \delta_{2,2}, ..., \delta_{2,k}\} \in \Delta_2$ be a finite set of generators for $\Delta_2$. By Borel’s density theorem, $\{\text{Ad}(\delta_{2,1}), ..., \text{Ad}(\delta_{2,k})\}$ generates the algebra

$$\langle \text{Ad}(G_2) \rangle \leq \text{End}(g_2)$$

(here $g_2$ denotes the Lie algebra of $G_2$). Since this algebra is finite dimensional, it is generated by $\{\text{Ad}(\pi_2(f_n(\delta_{2,i})))\}_{i=1}^k$ whenever $n$ is large enough. Since $G_2$ is semi-simple, the adjoint representation $\text{Ad} : G_2 \to \text{GL}(g_2)$ has no invariant vectors.

Let $\epsilon_n = \pi_2(f_n(\delta))$. Since $f_n(\delta)$ is close to $\delta$, $\epsilon_n = \pi_2(f_n(\delta))$ is close to the identity of $G_2$. We can therefore assume that $\epsilon_n$ is contained in an identity neighborhood of $G_2$ where $\log = \exp^{-1} : G_2 \to g_2$ is a well defined diffeomorphism. As $\delta$ commutes with each $\delta_{2,i}$, $\epsilon_n$ commutes with each $\pi_2(f_n(\delta_{2,i}))$, and it follows that

$$\text{Ad}(\pi_2(f_n(\delta_{2,i}))) (\log \epsilon_n) = \log \epsilon_n,$$

which in turn implies $\log \epsilon_n = 0$, i.e. $\epsilon_n = 1$. \hfill $\square$

Together with Proposition [13.13], the original argument from [31] 8.1 gives:

**Theorem 13.14** (Wang’s theorem). *Let $G$ be a connected semi-simple Lie group without compact factors, which is not locally isomorphic to $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$. Then for any $v > 0$ there are only finitely many conjugacy classes of irreducible lattices in $G$ with covolume $\leq v$.***

**Remark 13.15.** In [6], Borel and Prasad established a very strong and general finiteness result, but they omitted the cases of $G = G_{a,b}$ by requiring absolute rank $\geq 2$. This requirement was used in their proof of the stronger finiteness statement, in which the ambient group $G$ can be varied. However, Prasad remarked to the author that, when $G$ is fixed, this requirement is unnecessary in their argument, and hence, the finiteness of the number of conjugacy classes of arithmetic lattices of covolume $\leq v$ in $G$ could be proved also by using their methods.
Remark 13.16. More generally, for any $G$, the finiteness statement holds for lattices (not necessarily arithmetic) which are irreducible with respect to the $\text{SL}_2$ factors of $G$. I.e. for the set of conjugacy classes of lattices in $G$ which project densely to any factor of $G$ which is locally isomorphic to $\text{SL}_2(\mathbb{R})$ or $\text{SL}_2(\mathbb{C})$.

Acknowledgments 13.17. I would like to thank Shahar Mozes (my Ph.D. adviser) for his guidance throughout the last few years and for insightful suggestions for this work, to Pierre Pansu for hours of helpful conversations and for suggestions which were essential to this research, to Alex Lubotzky for many discussions, suggestions and ideas, and to Uri Bader, Yair Glasner, and Yehuda Shalom for many discussions and clever suggestions. Proofs of some of the statements presented here were established during these discussions. I would also like to thank Emmanuel Breuillard, Assaf Naor and Gopal Prasad for some remarks concerning early versions of this paper. Finally, I would like to express my sincere gratitude to the anonymous referees for their careful reading of the manuscript and for many suggestions and corrections which tremendously improved the exposition of this paper.

References

[1] U. Bader, A. Nevo, Conformal actions of simple Lie groups on compact pseudo-Riemannian manifolds, J. Differential Geom. 60 (2002), no. 3, 355–387.
[2] W. Ballmann, M. Gromov, V. Schroeder, Manifolds of Nonpositive Curvature, Birkhauser, 1985.
[3] R. Benedetti, C. Petronio, Lectures on Hyperbolic Geometry, Springer-Verlag, 1992.
[4] A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds, Ann. Scuola Norm. Sup. Pisa, Ser. IV, 8 (1981) 1-33.
[5] A. Borel, Harish-Chandra, Arithmetic subgroups of algebraic groups. Ann. Math. (2) 75 (1962) 485-535.
[6] A. Borel, G. Prasad, Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups, Publ. Math. I.H.E.S. 69 (1989), 119-171.
[7] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, 1982.
[8] R. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer, 1999.
[9] M. Burger, T. Gelander, A. Lubotzky, S. Mozes, Counting hyperbolic manifolds, Geom. Funct. Anal. 12 (2002), no. 6, 1161–1173.
[10] M. Burger, V. Schroeder, Volume, diameter and the first eigenvalue of locally symmetric spaces of rank one, J. Differential Geometry 26 (1987), 273-284.
[11] J. Cheeger, D.G. Ebin, Comparison theorems in Riemannian geometry, North-Holland
[12] T. Chinburg, Volume of hyperbolic manifolds, J. Differential Geometry 18 (1983), 783-789.
[13] D. Cooper, The volume of a closed hyperbolic 3-manifold is bounded by pi times the length of any presentation of its fundamental group, Proc. Amer. Math. Soc. 127 (1999), 941-942.
[14] k. Corlette, Archimedean superrigidity and hyperbolic geometry. Ann. of Math. 135 (1992), no. 1, 165–182.
[15] H. Garland, M.S. Raghunathan, Fundamental domains for lattices in (R-)rank 1 Lie groups, Ann. of Math. 92 (1970), 279-326.

[16] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhauser, 1998.

[17] M. Gromov, Hyperbolic manifolds according to Thurston and Jorgensen, Semin. Bourbaki, 32e annee, vol. 1979/80, exp. 546, Lect. Notes Math. 842, (1981), 40-53.

[18] M. Gromov, R. Schoen, Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Inst. Hautes tudes Sci. Publ. Math. 76 (1992), 165–246.

[19] D.A. Kazhdan, Connection of the dual space of a group with the structure of its closed subgroups, Functional Analysis and Application 1 (1967), 63-65.

[20] A. Lubotzky, Subgroup growth and congruence subgroups, Invent. Math. Springer-Verlag 119 (1995), 267-295.

[21] G.A. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1990.

[22] G.A. Margulis, Arithmeticity of the irreducible lattices in the semi-simple groups of rank greater than 1, (Russian), Invent. Math. 76 (1984) 93-120.

[23] G.A. Margulis, Non-uniform lattices in semisimple algebraic groups. Lie groups and their representations (Proc. Summer School on Group Representations of the Bolyai Jnos Math. Soc., Budapest, 1971), Halsted, New York, (1975) 371–553.

[24] G.A. Margulis, J. Rohlfs, On the proportionality of covolumes of discrete subgroups. Math. Ann 275 (1986) 197-205

[25] V. Platonov, A. Rapinchuk, Algebraic Groups and Number Theory, Academic Press, 1994.

[26] G. Prasad, Volume of S-arithmetic quotients of semi-simple groups, Publ. Math. I.H.E.S. 69 (1989), 91-117.

[27] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, New York, 1972.

[28] E. Springer, Algebraic Topology, Springer-Verlag, 1966.

[29] B. Sury, Arithmetic groups and Salem numbers, Manuscripta Math 75 (1992), 97-102.

[30] W.P. Thurston, Three-Dimensional Geometry and Topology, Volume 1, Princeton univ. press, 1997.

[31] H.C. Wang, Topics on totally discontinuous groups, Symmetric Spaces, edited by W. Boothby and G. Weiss (1972), M. Dekker, 460-487.

[32] S.P. Wang, The dual space of semisimple Lie groups, Amer. J. Math 91 (1969), 921-937.

Tsachik Gelander, Institute of Mathematics, Hebrew university, Jerusalem, Israel
E-mail address: tsachik@math.huji.ac.il