Graph-Theoretic Optimization for Edge Consensus

Mathias Hudoba de Badyn∗ Dillon R. Foight∗∗
Daniel Calderone** Mehran Mesbahi∗∗ Roy S. Smith∗

∗Automatic Control Laboratory, ETH Zürich, Switzerland (e-mails: {mbadyn, rsmith}@control.ee.ethz.ch)
∗∗William E. Boeing Department of Aeronautics & Astronautics, University of Washington, Seattle WA, 98109 USA (e-mails: {dfoight, djcal, mesbahi}@uw.edu)

Abstract: We consider network structures that optimize the $\mathcal{H}_2$-norm of weighted, time scaled consensus networks, under a minimal representation of such consensus networks described by the edge Laplacian. We show that a greedy algorithm can be used to find the minimum-$\mathcal{H}_2$ norm spanning tree, as well as how to choose edges to optimize the $\mathcal{H}_2$ norm when edges are added back to a spanning tree. In the case of edge consensus with a measurement model considering all edges in the graph, we show that adding edges between slow nodes in the graph provides the smallest increase in the $\mathcal{H}_2$ norm.

Keywords: Multi-agent systems, consensus, distributed control, $\mathcal{H}_2$ control

1. INTRODUCTION

Many natural and synthetic systems are distributed among agents in a network. One popular information-sharing protocol distributed over a network is consensus, used in many applications ranging from robotics, as in Joordens and Jamshidi (2009); sensor networks, as in Olfati-Saber (2005); and multi-agent systems, as in Olfati-Saber and Murray (2004), and Tanner et al. (2004). Much work on studying networked dynamical systems has focused on how the physical structure of the network affects its dynamics. For example, network symmetries have been shown to be related to controllability of consensus, as discussed by Rahmani et al. (2009) and Alemzadeh et al. (2017).

In this work, we are interested in studying system-theoretic measures of performance, and how to optimize graph structures for these system-theoretic measures. In particular, we examine the $\mathcal{H}_2$ norm, which can be interpreted as a measure of noise attenuation over the network, as discussed by Siami and Motee (2016), and has a rich history of use in the literature. For consensus networks with leaders, the $\mathcal{H}_2$ norm has an interpretation in terms of effective resistance across the network, as discussed in Chapman and Mesbahi (2013), and in Hudoba de Badyn and Mesbahi (2019, 2021), a fast method of computing the $\mathcal{H}_2$ norm for series-parallel networks exploited this interpretation. Bamieh et al. (2012) and Patterson and Bamieh (2010, 2014) have utilized the related concept of coherence to consider local feedback laws and leader selection to promote coherence.

A minimal representation of consensus networks can be obtained by looking at a system where only relative agent states across connections are considered. Such a system, represented by the edge Laplacian rather than the graph Laplacian, was considered for estimation and control in relative sensing networks in Sandhu et al. (2005, 2009), and for formation flight in Smith and Hadaegh (2005). When considering weighted consensus networks operating with agents that have non-homogeneous time scales, it was shown in Foight et al. (2019) that edge consensus allows explicit calculations of the $\mathcal{H}_2$ norm of the system. This latter formulation is the setting of the present work. In this paper, we focus on the optimization of graph structures in the setting of edge-weighted, and time scaled networks with the $\mathcal{H}_2$ norm of edge consensus as our optimization measure. The contributions of the paper are as follows. First, we show that a greedy algorithm can be used to find the minimum-$\mathcal{H}_2$ norm spanning tree. This has applications to systems where resilience to noise is desired, but a minimum number of communication links is also desired, such as in stealthy UAV systems. We then discuss how to add weighted edges back to the graph to improve the $\mathcal{H}_2$ norm. In the case of edge consensus with a measurement model considering all edges in the graph, we show that adding edges between slow nodes in the graph provides the smallest increase in the $\mathcal{H}_2$ norm.

The organization of this paper is as follows. In §2, we outline our notation, the mathematical preliminaries on graph and edge Laplacians, and describe our problem statement in the context of the $\mathcal{H}_2$ performance of edge consensus. Our main results are outlined in §3, and examples are shown in §4. The paper is concluded in §5.

2. PRELIMINARIES

In this section, we outline notation and the mathematical preliminaries on graph and systems theory, and then lay out the setting of the problem statement.
2.1 Preliminaries on Graphs & Consensus

We denote the real numbers as \( \mathbb{R} \), the non-negative reals as \( \mathbb{R}_+ \), the positive reals as \( \mathbb{R}_{++} \), and the real \( n \)-dimensional Euclidean vector space as \( \mathbb{R}^n \). Vectors in \( \mathbb{R}^n \) are written in lower-case \( x, y, z, \) etc., and matrices in \( \mathbb{R}^{n \times m} \) are written in capital-case \( A, B, C \). \( I \in \mathbb{R}^n \) denotes the length-\( n \) vector of ones, and \( 0 \) denotes a matrix of zeros of comfortable dimensions.

A graph \( G \) with \( n \) vertices (or nodes) is a triple of sets \( (V, \mathcal{E}, W) \), where \( V = [n] \) is a set of labeled vertices, \( \mathcal{E} \subseteq V \times V \) is a set of \( m \) edges denoting connections between vertices, and \( W \in \mathbb{R}^{m} \) is a set of edge weights, denoting some notion of ‘strength’ of the corresponding edge. A graph \( G \) is undirected if edge \( ij = ji \), and directed otherwise. The neighbourhood of vertex \( i \) in \( G \) is \( N_i = \{ j : ij \in \mathcal{E} \} \), and the unweighted degree of \( i \) is \( \deg(i) = |N(i)| \).

A path \( P \) is a set of edges \( \{i_1i_2, i_2i_3, \ldots, i_{n-1}i_n \} \) where \( i_1, \ldots, i_n \) are distinct, and a cycle \( C \) is a path, except that the first vertex and the last vertex are the same, i.e., \( i_n = i_1 \). An connected graph \( G \) is one in which there is an undirected path between any two vertices in \( G \). A tree \( T \) is a connected graph with no cycles, and a spanning tree of a connected graph \( G \) is a tree on the same vertex set as \( G \) (i.e., \( \mathcal{V}(T) = \mathcal{V}(G) \)), with edges \( \mathcal{E}(T) \subseteq \mathcal{E}(G) \). If \( T \) is a spanning tree of \( G \), then we can write \( T \subseteq G \), where ‘\( \subseteq \)’ is to be interpreted setwise. Note that a tree \( T \) with \( n \) vertices must have exactly \( n-1 \) edges.

To each vertex \( i \in V \), we assign a state \( x_i \in \mathbb{R} \). The dynamics of each vertex state \( x_i \) are assumed, unless otherwise stated, to be single-integrator dynamics \( \dot{x}_i = u_i \). By setting \( u_i \) to a weighted average of the states of its neighbours, we arrive at the consensus dynamics:

\[
\dot{x}_i = \sum_{j \in N(i)} w_{ij} (x_j - x_i). \tag{1}
\]

One can also define the incidence matrix \( D_G \in \mathbb{R}^{m \times n} \) of an undirected graph \( G \), where each column \( d_{ij} \) of \( D_G \) corresponds to an edge \( ij \in \mathcal{E} \). The column \( d_{ij} \) has 1 in the \( i \)-th position, and −1 in the \( j \)-th position; the choice of setting \( i = 1 \) or \( j = 1 \) is arbitrary. By setting the matrix \( W_G \in \mathbb{R}^{m \times m} \) to be the diagonal matrix containing the edge weights \( w_{ii} \), one can write the (undirected) graph Laplacian as \( \mathcal{L}_G = D_G W_G D_G^\top \).

By appropriately stacking \( x_i \) into a vector \( x \in \mathbb{R}^n \), the consensus dynamics in (1) can be written in vector form as

\[
\dot{x} = -\mathcal{L}_G x.
\]

A time-scaled graph \( G \) is a quadruple of sets \( (\mathcal{V}, \mathcal{E}, W, S) \), where \( \mathcal{V}, \mathcal{E}, \) & \( W \) are as before, and \( S \in \mathbb{R}^+ \) is a set of time scales \( \{s_{ij} \}_{i,j=1}^n \) on the vertices. We can then define the time scaled consensus dynamics as

\[
\epsilon_i \dot{x}_i = \sum_{j \in N(i)} w_{ij} (x_j - x_i) \leftrightarrow E_G \dot{x}_i = -\mathcal{L}_G x,
\]

where \( E_G = \mathrm{diag}\{\epsilon_i\}_{i=1}^n \).

2.2 Edge Consensus and Problem Configuration

The problem considered in this paper is derived from the preceding consensus dynamics, but with process and measurement noise. Consider a graph \( G \) of \( n \) multi-scale integrators, and zero-mean Gaussian process noise \( \omega_i(t) \),

\[
\epsilon_i \dot{x}_i = u_i(t) + \omega_i(t), \tag{2}
\]

where \( \mathbb{E}[\omega(t)\omega(t)\top] = \sigma^2 \Omega^2 \), \( x_i \) is the (scalar) state of the \( i \)-th agent, \( \epsilon_i \) is node \( i \)'s time scale parameter, and \( u_i \) is the control input. A weighted, noisy, decentralized feedback controller that seeks to bring agents to consensus is given by,

\[
u_i(t) = \sum_{j \in N(i)} [w_{ij}(x_j(t) - x_i(t)) + v_{ij}(t)]
\]

\[
u(t) = D_G W_G D_G^\top \dot{x}(t) + D_G v(t), \tag{3}
\]

where the noise over the edge \( ij \) is \( v_{ij}(t) \), with covariance \( \mathbb{E} [v(t)v(t)\top] = \sigma^2 (\Omega^2) \) for all \((i,j) \in \mathcal{E} \). \( W_G \triangleq \mathbb{E}[D_G] \) is the matrix of edge weights, and \( v(t), u(t) \) are the stacked vectors of measurement noises and control inputs. Applying (3) to the stacked-vector version of (2) gives a general, time scaled and weighted consensus dynamics with process and measurement noise,

\[
\Sigma_G \triangleq \begin{cases}
\dot{x}(t) = & E_G^{-1} \mathcal{L}_G x(t) \\
& + [E_G^{-1} E_G^{-1} D_G] \mathbb{E}[\omega(t)]/v(t)
\end{cases}. \tag{4}
\]

The scaled Laplacian \( E_G^{-1} \mathcal{L}_G \) is rank-deficient (it possesses a zero eigenvalue for each connected component of \( G \)), which complicates reasoning about the \( H_2 \) performance of \( \Sigma_G \). Solutions to this problem in the literature include studying the \( H_2 \) norm of the Dirichlet Laplacian, as in Chapman et al. (2015); Hudoba de Badyn and Mesbahi (2019), or as considered in this paper, a minimal representation of \( \Sigma_G \) obtained by studying edge consensus, seen in Zelazo et al. (2013); Zelazo and Mesbahi (2011); Foight et al. (2019). In the case of a time scaled graph, the edge Laplacian is defined by \( \mathcal{L}_T \triangleq D_T^\top E_G^{-1} D_T \). We denote the edge Laplacian of a graph \( G \) with a superscript: \( \mathcal{L}^T \), and the graph Laplacian with a subscript: \( \mathcal{L}_G \).

In Zelazo and Mesbahi (2011), it was shown that the graph and edge Laplacians are related by a similarity transformation; Foight et al. (2019) extended the transformation to time-scaled and matrix-weighted graphs. This similarity transformation is constructed by choosing a spanning tree \( T \subseteq G \), and noting that the cycles can be considered as linear combinations of the spanning tree edges in the following sense. Let \( D_T \) denote the incidence matrix of the spanning tree \( T \), and let \( D_{G\setminus T} \) denote the incidence matrix of the remaining edges that correspond to the edges completing the cycles in \( G \). Define the Tucker representation matrix \( R_G \triangleq [I \ T_\perp] \), where

\[
T_\perp = [D_T^\top D_T]^{-1} D_T^\top D_{G\setminus T}.
\]

We refer to the matrix \( T_\perp \) as the cycle representation matrix. Note that if the graph is a tree, i.e., \( G = T \), then the only choice of spanning tree for \( R_G \) is in fact \( T \), and \( T_\perp \) is the empty matrix (\( R_G = I \) in this case).

It suffices to measure the relative states of the nodes \( x_i - x_j \) over the edges \( ij \) of a spanning tree \( T \subseteq G \) to capture consensus: if \( x_i - x_j = 0 \) for all \( ij \in T \), then we know that the network is at consensus. This is the motivation for the edge consensus model, formalized below. We define the edge states for a graph weighted and time scaled graph
as \( x_e \triangleq S_e^{-1}x_e \), where the similarity transformation and corresponding state matrix are,

\[
S_e = \begin{bmatrix} E_c^{-1}D_T [D_T^T E_c^{-1}D_T]^{-1} 1 \\
S_e^{-1}L_g S_e = \begin{bmatrix} \mathcal{L}^T R_g W_g R_g^T 0 \\
0 0 \end{bmatrix},
\]

where \( \mathcal{L}^T = D_T^T E_c^{-1}D_T \).

The dynamics of the edge states \( x_e \) corresponding to (4) are then given by

\[
\dot{x}_e = \begin{bmatrix} -\mathcal{L}^T R_g W_g R_g^T 0 \\
0 0 \end{bmatrix} x_e + \begin{bmatrix} D_T E_c^{-1} \omega(t) 0 \\
0 \nu(t) \end{bmatrix} (t).
\]

The form of the system matrix in (5) suggests a partitioning of \( x_e \) into the edge states of the chosen spanning tree \( T \) and, and the consensus subspace, i.e. \( x_e = [x_T^T \; x_\chi]^T \).

Finally, we arrive at two models of the spanning tree edge states, differing in their choice of output:

\[
\Sigma^T[G] \triangleq \begin{cases} \dot{x}_T = -\mathcal{L}^T R_g W_g R_g^T x_T + D_T E_c^{-1} \omega \\
z = R_T^T x_T \end{cases}
\]

\[
\hat{\Sigma}^T[G] \triangleq \begin{cases} \dot{x}_T = -\mathcal{L}^T R_g W_g R_g^T x_T + D_T E_c^{-1} \omega \\
z = -\mathcal{L}^T R_g v \end{cases}
\]

There are multiple motivations for each of the above models, and the edge consensus formulation in general. As we can see in (5), the edge state dynamics naturally exclude the zero eigenvalue of \( L_g \). Also, as proposed in Zelazo and Mesbah (2011), it suffices to measure the \( n-1 \) relative states across the edges of a spanning tree \( T \) to determine if the states of the \( n \) nodes are at consensus. Therefore, a satisfactory output measurement of the network is precisely the edge states of that spanning tree as in (7). Of course, the remaining edges that are not in \( T \) also influence the dynamics, and to study the \( \mathcal{H}_2 \) norm of the entire system one can take the output of these edge states into account as well. The model in (6) has output \( z = R_T^T x_T \), which reconstructs the states of the remaining edges in \( \hat{G} \setminus T \) using the Tucker representation matrix, and thus considers all edges in the output.

In the next section, we discuss the \( \mathcal{H}_2 \) norm as a system-theoretic metric of performance of these models. For brevity, we may drop the argument of \( \Sigma^T[G] \).

2.3 Systems Theory and \( \mathcal{H}_2 \) Performance

The input-output excitation properties of a linear system \( \dot{x} = Ax + Bu, \; y = Cx \) with transfer function \( G(s) = C(sI - A)^{-1}B \) can be described using the \( \mathcal{H}_2 \) norm,

\[
\mathcal{H}_2^2(G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr} \{G(j\omega)^*G(j\omega)\} \, d\omega.
\]

The \( \mathcal{H}_2 \) norm measures the steady-state covariance of the output of the system under zero-mean, unit-covariance white noise inputs, or equivalently the root-mean-square of the impulse response of the system.

As discussed in Foight et al. (2019), the \( \mathcal{H}_2 \) norms of the models \( \Sigma^T[G], \hat{\Sigma}^T[G] \) in (6) and (7) are obtained by solving the Lyapunov equation

\[
-L^T R_g W_g R_g^T X - XR_g W_g R_g^T L^T + D_T E_c^{-1} \Omega \mathcal{T} E_c^{-1} D_T + \mathcal{L}^T R_g \mathcal{T} \mathcal{R}^T R_g^T \mathcal{L} = 0.
\]

An analytic solution to Equation (8) can be obtained with a convenient choice of the error covariances \( \Omega \) and \( \Gamma \), specifically \( \Omega = \sigma_\omega^2 E_c^{-1} \) and \( \Gamma = \omega_s W_g^{-1} \). In this case, the solution of (8) is given by

\[
X^* = \frac{1}{2} \left[ \sigma_\omega^2 (R_g W_g R_g^T)^{-1} + \sigma_\omega^2 L^T \right],
\]

and so the \( \mathcal{H}_2 \) norms of \( \Sigma^T[G], \hat{\Sigma}^T[G] \) are, respectively,

\[
\mathcal{H}_2^2(\Sigma^T[G]) = \text{tr} \left[ R_T^T X^* R_T \right] = \frac{\sigma_\omega^2}{2} \text{tr} \left[ R_T^T (R_g W_g R_g^T)^{-1} R_T \right] + \frac{\sigma_\omega^2}{2} \text{tr} \left[ R_T^T \mathcal{L} R_g \right] \quad (9)
\]

\[
\mathcal{H}_2^2(\hat{\Sigma}^T[G]) = \text{tr} \left[ X^* \right] = \frac{\sigma_\omega^2}{2} \text{tr} \left[ (R_g W_g R_g^T)^{-1} \right] + \frac{\sigma_\omega^2}{2} \text{tr} \left[ L^T \right]. \quad (10)
\]

Of course, the choice of the error covariances causes a loss of generality in the applicability of the analysis of this model. In Foight et al. (2019, 2020), it was shown that this choice, plus the true error covariances can be combined with an ordering in the semi-definite cone of covariances to reasonably bound the true performance, suggesting that this model is a convenient proxy for general noise models.

It was noted in Foight et al. (2019) that the contributions of the weights and time scales can be separated in (9) & (10) as,

\[
\mathcal{H}_2^2(\Sigma^T[G]) = \mathcal{H}_2^2(\Sigma^T[E]) + \mathcal{H}_2^2(\Sigma^T[W])
\]

\[
\mathcal{H}_2^2(\hat{\Sigma}^T[G]) = \mathcal{H}_2^2(\hat{\Sigma}^T[E]) + \mathcal{H}_2^2(\hat{\Sigma}^T[W])
\]

where

\[
\mathcal{H}_2^2(\Sigma^T[W]) \triangleq \frac{\sigma_w^2}{2} \text{tr} \left[ R_T^T (R_g W_g R_g^T)^{-1} R_T \right],
\]

\[
\mathcal{H}_2^2(\hat{\Sigma}^T[W]) \triangleq \frac{\sigma_w^2}{2} \text{tr} \left[ (R_g W_g R_g^T)^{-1} \right],
\]

\[
\mathcal{H}_2^2(\Sigma^T[E]) \triangleq \frac{\sigma_w^2}{2} \text{tr} \left[ R_T^T \mathcal{L} R_g \right],
\]

\[
\mathcal{H}_2^2(\hat{\Sigma}^T[E]) \triangleq \frac{\sigma_w^2}{2} \text{tr} \left[ \mathcal{L} \right].
\]

Furthermore, it was shown in Foight et al. (2019) that the \( \mathcal{H}_2 \) norm of \( \Sigma \) can be written in terms of the \( \mathcal{H}_2 \) norm of \( \Sigma \) as follows:

\[
\mathcal{H}_2^2(\Sigma[W]) = \mathcal{H}_2^2(\Sigma[E]) + \frac{\sigma_w^2}{2} \text{tr} \left[ T_T^T (R_g W_g R_g)^{-1} T_T \right] \quad (11)
\]

\[
\mathcal{H}_2^2(\Sigma[E]) = \mathcal{H}_2^2(\Sigma[W]) + \frac{\sigma_w^2}{2} \text{tr} \left[ T_T^T L_T T_T \right].
\]

We examine the behaviour of each of these components when an edge is added to a spanning tree – \( \mathcal{H}_2^2(\Sigma, W) \) and \( \mathcal{H}_2^2(\Sigma, E) \) are discussed in §3.2, and \( \mathcal{H}_2^2(\Sigma, W) \& \mathcal{H}_2^2(\Sigma, W) \) in §3.3 and §3.4 respectively.

3. MAIN RESULTS

3.1 Minimum-\( \mathcal{H}_2 \) Spanning Tree

In this section, we show how a minimum-\( \mathcal{H}_2 \) norm spanning tree of a graph \( \mathcal{G} \) can be found with a greedy algorithm. Let \( \mathcal{G} \) be a weighted, time scaled graph with edge consensus dynamics (7) and corresponding \( \mathcal{H}_2 \) norm (10).
Suppose that we want to remove communication links from \( \mathcal{G} \) until we are left with a minimally-connected graph — a tree. Such an operation can be used during a ‘stealth mode’ for networked systems, during which time having the minimum number of communication links while still being connected is desired. For example, a swarm of UAVs may be using consensus to agree on a formation heading, but may want to limit their chance of detection when entering a hostile area. In this section, we show that one can choose the spanning tree \( T \subseteq \mathcal{G} \) that minimizes the \( H_2 \) norm in (10) using a greedy algorithm. This allows the network to minimize the number of communication links used in the consensus algorithm, while maintaining optimal noise rejection properties. The main result of this section is summarized in Proposition 1.

Algorithm 1 Minimum-\( H_2 \) Spanning Tree

1. Input: \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}, \mathcal{S}) \)
2. Initialize:
3. Choose \( v \in \mathcal{V} \), set \( Q = \mathcal{V} \setminus \{v\}, \mathcal{P} = \{v\}, \mathcal{R} = \emptyset \)
4. While \( |Q| > 0 \) do
5. Set
   \[ N(\mathcal{P}) = \{ j \in Q : i \in \mathcal{P}, ij \in \mathcal{E} \} \]
6. Solve
   \[ (P1) \coloneqq \min_{j \in N(\mathcal{P})} \left\{ \sigma_v^2 (\epsilon_k^{-1} + \epsilon_j^{-1}) + \sigma_e^2 w_{jk} : jk \in \mathcal{E} \right\} \]
7. From \( \arg \min (P1) \coloneqq (j^*, j^* k) \), update
   \[ \mathcal{P} \mapsto \mathcal{P} \cup \{ j^* \} \]
   \[ \mathcal{R} \mapsto \mathcal{R} \cup \{ j^* k \} \]
   \[ Q \mapsto Q \setminus \{ j^* \} \]
8. End while
9. Output: \( T^* = (\mathcal{P}, \mathcal{R}, \mathcal{W}(\mathcal{R}), \mathcal{S}) \)

Proposition 1. Consider a graph \( \mathcal{G} \) with the edge consensus dynamics (7), and with corresponding \( H_2 \) norm (10). Any spanning tree \( T^* \subseteq \mathcal{G} \) has \( H_2 \) norm

\[
\frac{\| T \|_2^2}{2} = \sigma_v^2 \sum_{e \in T^*} w_e^{-1} + \frac{\sigma_e^2}{2} \sum_{i=1}^{n} \deg(i) \epsilon_i.
\]

Algorithm 1 returns a spanning tree \( T^* \subseteq \mathcal{G} \) that minimizes (12) out of all possible spanning trees \( T \subseteq \mathcal{G} \).

Proof. Define an auxiliary graph \( \mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{W}') \) on the same vertex and edge set as \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}, \mathcal{S}) \), (i.e., \( \mathcal{V}' = \mathcal{V}, \mathcal{E}' = \mathcal{E} \)) but with edge weights \( w'(\mathcal{G}') \in \mathcal{W}' \) defined by

\[
w'(\mathcal{G})_{ij} = \frac{\sigma_v^2}{\epsilon_i} + \frac{\sigma_v^2}{\epsilon_j} + \frac{\sigma_e^2}{w_{ij}}, \quad \forall i, j \in \mathcal{V}, ij \in \mathcal{E}.
\]

Then, define the total cost of the graph \( \mathcal{G}' \) as the sum of the edge weights:

\[
TC(\mathcal{G}') = \frac{1}{2} \sum_{e \in \mathcal{E}'} w'(\mathcal{G}')_e.
\]

It is clear from (13) that

\[
TC(\mathcal{G}') = \frac{1}{2} \sum_{e \in \mathcal{E}'} w'(\mathcal{G}')_e = \frac{1}{2} \sum_{ij \in \mathcal{E}} \left[ \frac{\sigma_v^2}{\epsilon_i} + \frac{\sigma_v^2}{\epsilon_j} + \frac{\sigma_e^2}{w_{ij}} \right].
\]

For every edge \( ij \) we get one term with \( \epsilon_i^{-1} \) in the sum of (15), and so summing over all edges yields \( \deg(i) \) terms

\[
\epsilon_i^{-1}. \text{ Hence, we can conclude that from (10), for any spanning tree } T' \subseteq \mathcal{G}, \text{ its total cost is exactly the } H_2 \text{ norm of the corresponding tree } T \text{ in } \mathcal{G}.
\]

\[ TC(T') = \frac{1}{2} \left[ \sum_{e \in T'} w_e + \sum_{i \in \mathcal{V}} \sigma_e^2 \deg(i) \epsilon_i \right] = H_2^2(\Sigma T'). \]

By construction, the minimal spanning tree \( T^* \) of \( \mathcal{G}' \), defined by

\[ T^* \coloneqq \arg \min \{ TC(T') : T' \text{ is a spanning tree of } \mathcal{G}' \} \]

corresponds precisely to the spanning tree \( T \) of \( \mathcal{G} \) minimizing the \( H_2 \) norm in (10).

It is well-known that a minimal-weighted spanning tree of \( \mathcal{G}' \) (i.e., one that minimizes (14)) can be obtained by the Jarník-Dijkstra-Prim (JDP), or Kruskal algorithms; see Sedgewick and Wayne (2011). Algorithm 1 is precisely a JDP algorithm that builds the minimal-weighted spanning tree \( T^* \) by starting with a single node in \( T^* \), and at each iteration adding a new node to \( T^* \) by picking the neighboring vertex \( k \) to \( T^* \) (but \( k \) not already in \( T^* \)) with the smallest term \( \sigma_v^2 w_{jk}^{-1} + \sigma_e^2 (\epsilon_j^{-1} + \epsilon_k^{-1}) \) for \( j \in T^* \). By the above argument, this JDP algorithm on \( \mathcal{G}' \) with cost \( TC(\mathcal{G}') \) in (14) outputs the minimum-\( H_2 \) norm spanning tree of \( \mathcal{G} \). \( \square \)

Remark 2. For an edge-weighted graph where the cost is the sum of the edge weights, it is known that if the edge weights are distinct, then the minimal-weighted spanning tree is unique. The minimum-\( H_2 \) spanning tree \( T^* \) generated by Algorithm 1 is not unique, even if all the nodal time scales and edge weights are distinct. See §4.1 for an example.

Remark 3. Another practical algorithm would be a distributed version of Algorithm 1, which could be constructed by considering a distributed minimal-weighted spanning tree algorithm such as Awerbuch (1987); Gallager et al. (1983); Kutten and Peleg (1998). Such an algorithm would allow a consensus network to autonomously restructure itself to the \( H_2 \)-optimal tree configuration in a distributed manner.

3.2 Weighted Cycle Selection

In the previous section, we discussed how one may find the minimum-\( H_2 \) norm spanning tree from a connected graph \( \mathcal{G} \). In this section, we examine what happens to the \( H_2 \) norm when edges are added to a spanning tree. In particular, we show that adding weighted edges improve the \( H_2 \) norm, and discuss what choice of edge optimizes this improvement. This can be used in the previous motivating example of a stealthy system that wants to still minimize the number of communication links, but does not reject noise well enough with just the links in the spanning tree.

Zelazo et al. (2013) discussed how adding cycles to an unweighted, mono-scaled graph impacts the \( H_2 \) performance in the presence of unit covariance noise applied to the edges and nodes. In particular, they found that the biggest improvement in \( H_2 \) performance resulted from adding an edge to a tree \( T \) which maximized the length of the resulting cycle. Our first contribution is showing that in the presence of edge weights and time scales, the length of the cycle no longer matters, rather one should
add an edge to form a cycle that, roughly speaking, has small weights.

First, we define the weighted and unweighted length of a cycle $C$ as,

$$ l_w(C) := \sum_{e \in C} w_e^{-1}, \quad l(C) := \sum_{e \in C} 1. $$

**Proposition 4.** Consider a graph consisting of a weighted, time scaled tree $T$, and consider the dynamics $\Sigma T$, with the $H_2$ norm of $\Sigma T$ given by Equation (10). Consider the task of adding edges to $T$ to optimize the $H_2$ norm of $\Sigma T$.

Then, the following hold:

1. Adding an edge $e$ with weight $W_e$ to $T$ improves the $H_2$ performance in the following manner:

$$ H_2^2(\Sigma | T \cup \{e\}) = H_2^2(\Sigma | T) - \frac{\sigma_w^2}{2l_w(C)} \sum_{ij \in T \cup C} w_{ij}^{-2}, $$

where $C$ is the unique cycle formed by adding edge $e$ to $T$. Furthermore, adding an edge to $T$ to form a cycle always decreases the $H_2$ norm.

2. Adding $p$ edge-disjoint cycles $C_1, \ldots, C_p$ via edges $e_1, \ldots, e_p$ improves the $H_2$ performance in the following manner:

$$ H_2^2(\Sigma | T \cup \{e_i\}) = H_2^2(\Sigma | T) - \sum_{k=1}^p \left[ \frac{\sigma_w^2}{2l_w(C_k)} \sum_{ij \in T \cup C_k} w_{ij}^{-2} \right], $$

**Proof.** Recall that the $H_2$ norm of (7) for a given spanning tree $T \subseteq G$ is given by

$$ H_2^2(\Sigma T | G) = \frac{\sigma_w^2}{2} \text{tr} [(R_w G R_T^{-1})^{-1}] + \frac{\sigma_w^2}{2} \text{tr} [C^T T]. $$

Since the graph in question is an edge added to a tree, we have $G = T \cup e$. This introduces one cycle, $C$. Our choice of spanning tree in the cycle representation matrix $T$ is the original tree $T$ to which the edge $e$ is added. Consider the trace argument in the weight matrix term,

$$ (R_w G R_T^{-1})^{-1} = \left( [I \ T] \begin{bmatrix} W_T & 0 \\ 0 & W_C \end{bmatrix} [I \ T^T] \right)^{-1} = (W_T + T W_T^{-1} W_C)^{-1}, $$

where $W_T$ is the top-left block of $W_C$ corresponding to the weights of the edges in $T$, and $W_C$ is the bottom-right block of $W_C$ corresponding to the weight of the edge being added to $T$. In the last display, we made the substitution $T' \triangleq T W_C^{-1/2}$.

Since we are adding a single edge $e$ to $T$, $W_C$ is a scalar and $T'$ is a vector. Therefore, $T' T'^T$ is a rank-one matrix, and we can apply the Sherman-Morrison-Woodbury update, yielding

$$ (R_w G R_T^{-1})^{-1} = W_T^{-1} - \frac{W_T^{-1} T W_T^{-1} W_C T^T}{1 + T W_T^{-1} T^T} = W_T^{-1} - \frac{W_T^{-1} T W_T^{-1} W_C}{1 + W_C (T W_T^{-1} T^T)^{-1}}. $$

We recall the following lemma.

**Lemma 5.** (Prop 1 & 2 in Zelazo et al. (2013)). Recall the definition of the cycle representation matrix:

$$ T = (D_T^T D_T)^{-1} D_T^T D_{G \setminus T} := [c_1 \cdots c_{|G \setminus T|}]. $$

Each column $c_i$ of $T$ represents a cycle $C_i$ of $G$. The matrices $T_T^T T_T$ and $T_T^T T_T^T$ encode the following information about the cycles of $G$:

1. $[T_T^T T_T]_{ii} = l(C_i) - 1$
2. $[T_T^T T_T]_{ij} = 0$ if and only if the cycles $c_i$ and $c_j$ are edge-disjoint.
3. $[T_T^T T_T]_{ee}$ is the number of times edge $e$ is used to construct the cycles of $G$.

Consider the graph in question, $G := T \cup e$. Adding this edge introduces a single cycle, and so by Lemma 5,

$$ T_T^T T_T = c_1^T c_1 = l(C) - 1. $$

This holds because $T_T^T T_T$ adds a ‘1’ to the sum for each edge in the tree that ends up in the cycle $C$, in other words we can write

$$ T_T^T T_T = \left[ \sum_{e \in T \cup C} (1) \right] - 1, $$

where the ‘-1’ comes from excluding the edge $e$ that is added to $T$. Similarly, it can be seen that

$$ (T_T) W_T^{-1} T_T = \sum_{ij \in T \cup C} w_{ij}^{-1} = \left[ \sum_{ij \in C} w_{ij}^{-1} \right] - W_C^{-1} = l_w(C) - W_C^{-1}. $$

From this, we can conclude that

$$ 1 + W_C (T_T) W_T^{-1} T_T = 1 + W_C (l_w(C) - W_C^{-1}) = W_C l_w(C), $$

and so from (17) we have that

$$ (R_w G R_T^{-1})^{-1} = W_T^{-1} - \frac{1}{l_w(C)} W_T^{-1} T T^T W_T^{-1}, $$

where the length of the cycle added by edge $e$ is given by

$$ l_w(C) = \sum_{e \in T \cup C} w_e^{-1} + W_C^{-1}. $$

Recall from Lemma 5 that $[T_T^T T_T]_{ee}$ is the number of times the edge $e \in T$ is used in the cycle $C$. Therefore,

$$ [W_T^{-1} T_T^T W_T^{-1}]_{ee} = w_e^{-2}. $$

Taking the trace of (18) and using (19) yields

$$ \text{tr} [(R_w G R_T^{-1})^{-1}] = \text{tr} [W_T^{-1}] + \frac{1}{l_w(C)} \text{tr} [W_T^{-1} T T^T W_T^{-1}] = \frac{1}{l_w(C)} \sum_{e \in T \cup C} w_e^{-2}. $$

For the second part, note that the improvement in $H_2$ performance after adding edge $e$ is

$$ -\frac{\sigma_w^2}{2l_w(C)} \sum_{e \in T \cup C} w_e^{-2} < 0, $$

since $l_w(C) > 0$ and $W_e^{-2} > 0$ for all $e \in E$, and so the $H_2$ norm always decreases.

In the case of adding $k$-edge disjoint cycles, each edge can only be in one cycle, and so by Lemma 5 we have that (19) still holds. The result follows by applying the preceding argument for each of the $k$ cycles. \qed
3.3 Contributions of Time Scales to $\mathcal{H}_2^2(\Sigma^T[T])$

As discussed in §2.3, the two models $\Sigma^T$, $\hat{\Sigma}^T$ differ in the outputs – the former considers only the edge states of the spanning tree, and the latter considers all edges of the graph. We noted in the proof of Proposition 1 that when adding an edge to a tree, the $\mathcal{H}_2$ norm of the dynamics $\Sigma^T$ is not affected by the time scales of the nodes on which the edge is placed. In this section, we show that these dynamics, adding an edge to slow nodes provides the smallest increase in $\mathcal{H}_2$ norm.

Recall that the $\mathcal{H}_2$ performance of $\Sigma^T$ can be written as,

$$\mathcal{H}_2^2(\Sigma^T) = \frac{\sigma^2}{2} \text{tr}[R_G^\top(R_GR_G)^{-1}R] + \frac{\sigma^2}{2} \text{tr}[R_G^\top L^T R_G]$$

$$:= \mathcal{H}_2^2(\Sigma^T;W) + \mathcal{H}_2^2(\Sigma^T;E). \tag{20}$$

Here we have separated the contributions of the weights and time scales into two terms. Consider the second term in (20), defined as

$$\mathcal{H}_2^2(\Sigma^T;E) := \frac{1}{2} \sigma^2 \text{tr} [R_G^\top L^T R_G].$$

We have the following proposition.

**Proposition 6.** There exists a separation of $\mathcal{H}_2(\Sigma^T)$ into two terms, one depending on the cycle states, and the other the spanning tree states. Namely,

$$\mathcal{H}_2^2(\Sigma;E) = \sigma^2 \left( \text{tr}[L^T] + \text{tr}[L^\top_\sigma T] \right).$$

**Proof.** By trace cyclicity, we can compute:

$$\mathcal{H}_2^2(\Sigma^T,E) = \text{tr}[R_G^\top L^T R_G]$$

$$= \text{tr} \left[ \left( D_T^\top \right)^T L^T \left[ I \ T_T \right] \right] = \text{tr}[L^T] + \text{tr}[D_T^\top L^T T_T].$$

Via trace cyclicity, we can manipulate the argument of the second trace term in the last display as follows:

$$\text{tr}[D_T^\top L^T T_T] =$$

$$\text{tr}[D_T^\top (D_T^\top D_T)^{-1} D_T^\top D_{G\setminus T} D_{G\setminus T}^\top D_T (D_T^\top D_T)^{-1} D_T^\top E_{G}^{-1}]$$

Notice that the term $D_T^\top D_T^{-1} D_T^\top D_{G\setminus T}$ is the projection operator on the range space of $D_T$. Since the cycles are linear combinations of the spanning tree edges, every column of $D_{G\setminus T}$ is in the range of $D_T$. Therefore, the projection of $D_{G\setminus T}$ onto the range space of $D_T$ is nothing but $D_{G\setminus T}$:

$$D_T^\top D_T^{-1} D_T^\top D_{G\setminus T} = D_{G\setminus T},$$

and so we can conclude that

$$\text{tr}[D_T^\top L^T T_T] = \text{tr}[D_{G\setminus T} D_{G\setminus T}^\top D_T (D_T^\top D_T)^{-1} D_T^\top E_{G}^{-1}]$$

$$= \text{tr}[D_{G\setminus T} E_{G}^{-1} D_{G\setminus T}] = \text{tr}[L^\top_\sigma T].$$

$\square$

From Proposition 6, we can conclude with the following corollary.

**Corollary 7.** Consider the dynamics (6) of $\Sigma^T$ on a graph $G$. Then, consider adding an edge $ij$ to $G$ to get $G \cup \{ij\}$. The time scale term of the $\mathcal{H}_2^2$ norm in (20) satisfies

$$\mathcal{H}_2^2(\Sigma^T[G \cup \{ij\}]; E) = \sigma^2 \left( \epsilon_i^{-1} + \epsilon_j^{-1} \right) + \mathcal{H}_2^2(\Sigma^T[G], E).$$

3.4 Contribution of Weights to $\mathcal{H}_2^2(\Sigma^T[T])$

In §3.2 we showed that adding an edge to a spanning tree always improves the $\mathcal{H}_2$ norm of $\Sigma$. In this section, we discuss what happens to $\mathcal{H}_2$ norm of $\Sigma$ by examining (11).

Recall from §2.3 that we can write

$$\mathcal{H}_2^2(\Sigma^T;W) = \mathcal{H}_2^2(\Sigma^T;W)$$

$$+ \frac{\sigma^2}{2} \text{tr} \left[ T_T^\top (R_G W_G R_G)^{-1} T_T \right]. \tag{21}$$

When $G = T$, we examined what happens to $\mathcal{H}_2^2(\Sigma^T;W)$ when an edge is added to $T$ in Proposition 4. When $G = T$, the second term in (21) is zero since $T_T$ is the empty matrix. In the next proposition, we show that this term is positive when an edge is added to $T$.

**Proposition 8.** Consider the edge consensus model (6) on a graph $T$ with $\mathcal{H}_2$ norm (9). When an edge $e$ with weight $W_G$ is added to $T$ forming a cycle $C$, the weight portion of the $\mathcal{H}_2$ norm satisfies

$$\mathcal{H}_2^2(\Sigma^T[T \cup \{ij\}]; W) = \mathcal{H}_2^2(\Sigma^T[T]; W)$$

$$+ \frac{\sigma^2}{2 W_G} - \frac{\sigma^2}{2 l(w(C))} \sum_{ij \in C} w_{ij}^2. \tag{22}$$

**Proof.** First, note that the term in (21) is zero when there are no cycles in $G$. Adding a single edge to the tree $T$ forms a cycle, and hence $T_T$ has one column. We can compute using the rank-one update in (17),

$$\frac{\sigma^2}{2} \text{tr} \left[ T_T^\top (R_G W_G R_G)^{-1} T_T \right]$$

$$= \frac{\sigma^2}{2} \text{tr} \left[ T_T^\top \left( W_T^{-1} - \frac{1}{l(w(C))} W_T^{-1} T_T T_T^\top W_T^{-1} \right) T_T \right]$$

$$= \frac{\sigma^2}{2} \left( \text{tr} \left[ T_T^\top W_T^{-1} T_T \right] - \frac{1}{l(w(C))} \text{tr} \left[ (T_T^\top W_T^{-1} T_T)^2 \right] \right)$$

$$= \frac{\sigma^2}{2} \left( l(w(C)) - 1 - \frac{(l(w(C)) - 1)^2}{l(w(C))} \right)$$

$$= \frac{\sigma^2}{2} \left( l(w(C)) W_G^{-1} - W_G^{-2} \right). \tag{23}$$

Summing (23) with the improvement of $\mathcal{H}_2^2(\Sigma;W)$ in (16) from Proposition 4 yields (22).

$\square$

4. EXAMPLES

4.1 Minimum $\mathcal{H}_2$-Norm Spanning Trees

In this section, we show examples of the minimum-$\mathcal{H}_2$-norm spanning trees from Algorithm 1. Three such graphs are shown in Figure 1; edges in the minimum-$\mathcal{H}_2$ spanning tree are highlighted in red, and the remaining rejected edges in blue. In each example, the edge weights $w_e$ and time scales $\epsilon_i$ are drawn from a uniform random distribution on $[0,1]$ — the width of each edge is proportional to $\log(w_e)$ for clarity, and the node size is proportional to $\epsilon_i$. The graph in Figure 1a is an example of a series-parallel network; the graph in Figure 1b is the molecular adjacency structure of the caffeine molecule, and the graph in Figure 1c is an Erdős-Rényi random graph. The random graph is generated on 40 nodes, and an edge between node $i$ and $j$ exists with probability 0.09.
Next, we show an example of a graph with distinct time scales on the nodes, and distinct edge weights, but no unique minimum-$H_2$ spanning tree. Consider the triangle in Figure 2 with the edge consensus model $\Sigma^T$, and the following time scales and edge weights:

$$\begin{align*}
\epsilon_1 &= 1, \ w_{23} = 1 \\
\epsilon_2 &= 2, \ w_{13} = 2 \\
\epsilon_3 &= 3, \ w_{12} = 3.
\end{align*}$$

Further suppose that the noise weightings satisfy $\sigma_\omega = \sigma_v = 1$. Then, the spanning tree $T_1$ of the triangle in Figure 2 on the top right has $H_2$ norm

$$H_2(\Sigma^{T_1}(G)) = \frac{1}{2} \left[ w_{12}^{-1} + w_{13}^{-1} + \epsilon_1^{-1} + \sum_{i=1}^{3} \epsilon_i^2 \right] = 1.83$$

and the spanning tree $T_2$ in Figure 2 on the bottom right has $H_2$ norm

$$H_2(\Sigma^{T_2}(G)) = \frac{1}{2} \left[ w_{12}^{-1} + w_{23}^{-1} + \epsilon_3^{-1} + \sum_{i=1}^{3} \epsilon_i^2 \right] = 1.83.$$

Remark 9. The reason the minimum-$H_2$ spanning trees in Figure 2 are not unique, despite having distinct edge weights and time scales in $G$, is because the edge weights of the auxiliary graph $G'$ defined in the proof of Proposition 1, given by

$$w'_{ij} = \sigma_2^2(\epsilon_i^{-1} + \epsilon_j^{-1}) + \sigma_\omega^2 w_{ij},$$

are not distinct in $G'$.

4.2 Adding Edges to a Tree

Consider the path graph in Figure 3 denoted by the solid edges, where all the edges in the path have weight $w_e = 1$. Suppose $W_1 = 10$ and $W_2 = 5$, and $\sigma_\omega = 1$. Then, the improvement of adding either edge $W_1$ to $G$ to create cycle $C_1$, or adding $W_2$ to $G$ to create cycle $C_2$ is

$$\begin{align*}
&\frac{1}{2l_w(C_1)} \left[ \sum_{e \in T \cap C_1} w_e^{-2} \right] \approx -0.9524 \\
&\frac{1}{2l_w(C_2)} \left[ \sum_{e \in T \cap C_2} w_e^{-2} \right] \approx -0.9375,
\end{align*}$$

and so the shorter (but more highly weighted) cycle $C_1$ offers the better improvement.

5. CONCLUSION

In this paper, we have considered various perspectives on optimizing weights and time scales for edge consensus. We
showed that a greedy algorithm can be used to find the minimum-$H_2$ norm spanning tree of a graph, and then discussed how the $H_2$ norm can be optimized by adding cycles for two output models of edge consensus.

Future work includes implementing a distributed version of Algorithm 1, as well as examining how to optimally choose more general graph structures. Choosing time scales to improve controllability or observability of edge consensus is also an open problem.

ACKNOWLEDGEMENTS

This work was supported by the Swiss Competence Centers for Energy Research FEEB&D project and the ETH Foundation.

REFERENCES

Alemzadeh, S., Hudoba de Badyn, M., and Mesbahi, M. (2017). Controllability and stabilizability analysis of signed consensus networks. In Proc. IEEE Conference on Control Technology and Applications, 55–60. Kohala Coast, USA.

Awerbuch, B. (1987). Optimal Distributed Algorithms for Minimum Weight Spanning Tree, Counting, Leader Election and Related Problems. Conference Proceedings of the Annual ACM Symposium on Theory of Computing, 230–240.

Bamieh, B., Jovanovic, M.R., Mitra, P., and Patterson, S. (2012). Coherence in Large-Scale Networks: Dimension-Dependent Limitations of Local Feedback. IEEE Transactions on Automatic Control, 57(9), 2235–2249.

Chapman, A. and Mesbahi, M. (2013). Semi-autonomous consensus: Network measures and adaptive trees. IEEE Transactions on Automatic Control, 58(1), 19–31.

Chapman, A., Schoof, E., and Mesbahi, M. (2015). Online Adaptive Network Design for Disturbance Rejection. In Principles of Cyber-Physical Systems. Cambridge University Press.

Foight, D.R., Hudoba de Badyn, M., and Mesbahi, M. (2019). Efficient Computation of Performance on Series-Parallel Networks. In Proc. American Control Conference, 3364–3369. Philadelphia, USA.

Hudoba de Badyn, M. and Mesbahi, M. (2021). H2 Performance of Series-Parallel Networks: A Compositional Perspective. IEEE Transactions on Automatic Control, In Press. doi:10.1109/TAC.2020.2979758.

Joordens, M.A. and Jamshidi, M. (2009). Underwater swarm robotics consensus control. In Proc. IEEE International Conference on Systems, Man and Cybernetics, October, 3163–3168. San Antonio, USA.

Kutten, S. and Peleg, D. (1998). Fast distributed construction of small k-dominating sets and applications. Journal of Algorithms, 28(1), 40–66.

Olfati-Saber, R. (2005). Distributed Kalman filter with embedded consensus filters. In Proceedings of the 44th IEEE Conference on Decision and Control and the European Control Conference, 8179–8184. Seville, Spain.

Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control, 49(9), 1520–1533.

Patterson, S. and Bamieh, B. (2010). Leader selection for optimal network coherence. Proceedings of the IEEE Conference on Decision and Control, 2692–2697.

Patterson, S. and Bamieh, B. (2014). Consensus and coherence in fractal networks. IEEE Transactions on Control of Network Systems, 1(4), 338–348.

Rahmani, A., Ji, M., Mesbahi, M., and Egerstedt, M. (2009). Controllability of multi-agent systems from a graph-theoretic perspective. SIAM Journal on Control and Optimization, 48(1), 162–186.

Sandhu, J., Mesbahi, M., and Tsukamaki, T. (2005). Relative sensing networks: Observability, estimation, and the control structure. In Proc. 44th IEEE Conference on Decision and Control and the European Control Conference., 6400–6405. Seville, Spain.

Sandhu, J., Mesbahi, M., and Tsukamaki, T. (2009). On the control and estimation over relative sensing networks. IEEE Transactions on Control of Network Systems, 54(12), 2859–2863.

Sedgewick, R. and Wayne, K.D. (2011). Algorithms. Addison-Wesley, 4th edition.

Siami, M. and Motee, N. (2016). Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks. IEEE Transaction on Automatic Control, 61(12), 4055–4062.

Smith, R.S. and Hadaegh, F.Y. (2005). Control of deep-space formation-flying spacecraft; relative sensing and switched information. Journal of Guidance, Control, and Dynamics, 28(1), 106–114.

Tanner, H.G., Pappas, G.J., and Kumar, V. (2004). Leader-to-formation stability. IEEE Transactions on Robotics and Automation, 20(3), 443–455.

Zelazo, D. and Mesbahi, M. (2011). Edge agreement: Graph-theoretic performance bounds and passivity analysis. IEEE Transactions on Automatic Control, 56(3), 544–555.

Zelazo, D., Schuler, S., and Allgöwer, F. (2013). Performance and design of cycles in consensus networks. Systems and Control Letters, 62(1), 85–96.