Quantum Brownian Motion in a Bath of Parametric Oscillators: A model for system-field interactions

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Abstract

The quantum Brownian motion paradigm provides a unified framework where one can see the interconnection of some basic quantum statistical processes like decoherence, dissipation, particle creation, noise and fluctuation. The present paper continues the investigation into these issues begun in two earlier papers by Hu, Paz and Zhang on the quantum Brownian motion in a general environment via the influence functional formalism. Here, the Brownian particle is coupled linearly to a bath of the most general time dependent quadratic oscillators. This bath of parametric oscillators mimics a scalar field, while the motion of the Brownian particle modeled by a single oscillator could be used to depict the behavior of a particle detector, a quantum field mode or the scale factor of the universe. An important result of this paper is the derivation of the influence functional encompassing the noise and dissipation kernels in terms of the Bogolubov coefficients, thus setting the stage for the influence functional formalism treatment of problems in quantum field theory in curved spacetime. This method enables one to trace the source of statistical processes like decoherence and dissipation to vacuum fluctuations and particle creation, and in turn impart a statistical mechanical interpretation of quantum field processes. With this result we discuss the statistical mechanical origin of quantum noise and thermal radiance from black holes and from uniformly-accelerated observers in Minkowski space as well as from the de Sitter universe discovered by Hawking, Unruh and Gibbons-Hawking. We also derive the exact evolution operator and master equation for the reduced density matrix of the system interacting with a parametric oscillator bath in an initial squeezed thermal state. These results are useful for decoherence and backreaction studies for systems and processes of interest in semiclassical cosmology and gravity. Our model and results are also expected to be useful for related problems in quantum optics.

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1 Introduction

In two earlier papers, called Paper I, II henceforth \[1, 2\], Paz, Zhang and one of us began a systematic study of the celebrated problem of quantum Brownian motion (QBM) in a general environment using the Feynman-Vernon influence functional (IF) formalism \[3, 4, 5\]. The special features associated with a nonohmic bath, or ohmic bath at low temperatures are the appearance of colored noise and nonlocal dissipation. The motivation for this study was amply explained there. What prompted them to this undertaking was the belief that a correct and deepened understanding of many interesting quantum statistical processes in the early universe and black holes \[6\] requires an extension of the existing framework of quantum field theory in curved spacetime \[7\] to statistical and stochastic fields in the framework of quantum open systems \[8\] represented by the QBM \[4\]. This viewpoint and methodology have indeed been applied to the analysis of some basic issues in quantum cosmology \[10, 11, 12, 13, 14, 15\], effective field theory \[16, 17\], and the foundation of quantum mechanics, such as the uncertainty principle \[18, 19\] and, most significantly, decoherence \[20, 21, 22, 23, 24\] in the quantum to classical transition problem. (See the recent reviews of \[25, 26, 27\] and references therein and in Papers I, II for the standard literature on this topic). QBM is one of the two major paradigms of non-equilibrium statistical mechanics (the other being Boltzmann’s kinetic theory) which is also amenable to detailed analysis. The study of many problems mentioned above which have nonlinear and nonlocal characteristics typical of quantum processes in gravitation and cosmology necessitates a closer scrutiny of this model beyond the ordinary limited conditions.

As stated in the Introductions of Papers I and II, in order to make it useful for addressing issues in semiclassical gravity and quantum cosmology, a theory of quantum open systems has to be developed for quantum fields in curved spacetime. Noticeable effort has been put into this direction. Hu, Paz and Zhang \[28\] constructed a stochastic field theory based on the QBM model and described how thermal field theory can be obtained as the equilibrium limit. As a tool for the study of the quantum origin of noise, fluctuations and structure formation in cosmology, they \[28\] have extended the result of Paper II to quantum fields in Minkowski, Robertson-Walker and de Sitter spacetime. The nature and origin of quantum noise from particle-field interaction were discussed in \[30\] where a statistical field-theoretical derivation of thermal radiance in the Hawking \[31, 32\] and Unruh effects \[33\] were given. For semiclassical gravity Kuo and Ford \[34\] have studied the fluctuations of quantum fields on the Einstein equations. Calzetta and Hu \[35\], and the present authors \[36\] have analyzed the nature of noise, fluctuations, particle creation and backreaction for quantum fields in cosmological spacetimes and proposed an Einstein-Langevin equation as the centerpiece of a generalized theory of semiclassical gravity. For quantum cosmology, Sinha and Hu \[37\] had used the coarse-grained \[38\] Schwinger-Keldysh effective action \[39\] to analyze the validity of the minisuperspace approximation in quantum cosmology. Paz and Sinha \[13\] had used the influence functional method to discuss the transition from quantum to semiclassical gravity, and Calzetta and Hu \[40\] have studied dissipation problem in quantum cosmology. However, except for the few cases mentioned above, none of these earlier work made use of the master
or Langevin equation approach characteristic of the QBM study, which is necessary to probe into the noise, fluctuation \[35, 36\], instability and phase transition \[17\] aspects of quantum fields and spacetime.

The present paper is an intermediate step in that direction. It is a generalization of Papers I and II in that the oscillators which make up the system and bath are now the most general time-dependent quadratic oscillators. This bath of parametric oscillators (as the number of modes goes to infinity) is identical to a scalar field, while the motion of the Brownian particle modeled by a single oscillator could be used to depict the behavior of a particle detector (with zero spring constant, as in e.g., \[33\]), the scale factor of the universe, (with a negative kinetic energy term, as is seen in Eq.(2.2) of \[37\]) or the homogeneous or inhomogeneous (density fluctuation) modes of the inflaton field in an early universe \[11, 12, 13, 14, 15, 16, 17\]. Indeed the results obtained here can be taken over directly for the description of scalar fields in cosmological spacetimes, as our examples will demonstrate. Parametric amplification of the bath oscillator quanta gives rise to particle creation, as was pointed out by Parker and Zel’dovich \[48\], which can be depicted by the Bogolubov transformation between the creation and annihilation operators of the Fock spaces defined at different times. The averaged effect of the bath on the system is described by the influence functional, which, in the statistical field-theory context measures the backreaction of quantum processes associated with the field like particle creation on the dynamics of the background spacetime \[13, 50\]. There are two components in the influence functional, a noise kernel and a dissipation kernel. The noise kernel governs the decoherence process and also limits the degree of attainment of classicality \[23\]. It also depicts the effect of fluctuations (in particle number) \[33\]. The dissipation kernel which appears in the effective equation of motion depicts the effect of particle creation on the dynamics of the system. The QBM paradigm thus provides a unified framework where one can see the interconnection of the basic quantum statistical processes like decoherence, dissipation, particle creation, noise and fluctuation. The necessity of analyzing these processes on the same footing was emphasized earlier in \[11\].

An important result of this paper is the derivation of the influence functional and thus the noise and dissipation kernels in terms of the Bogolubov coefficients. This enables one to trace the source of statistical processes like decoherence and dissipation to vacuum fluctuations and particle creation, and in turn impart a statistical mechanical interpretation of quantum field processes. With this we discuss the well-known results by Unruh \[33\], Hawking \[31\] and Gibbons-Hawking \[32\] on thermal radiance from uniformly-accelerated observers \[31\], black holes and for comoving observers in de Sitter spacetime. From the explicit form of the noise and dissipation kernels we derived, one can see clearly the interplay of different factors which contribute to making the spectrum of particle creation in these situations thermal, and, more interestingly, what makes them nonthermal, as in the more general non-equilibrium situations. This is where the capability of the statistical field-theoretical interpretation supercedes the geometric interpretation (in terms of event horizons). We will discuss the implications of this point later.

Although we have used examples from quantum and semiclassical cosmology to illustrate
the physical relevance of the QBM model with parametric bath, the range of applicability of this model goes beyond. An important area where parametric amplification plays a central role is in quantum optics. Here the properties of baths prepared in squeezed initial states (rigged reservoirs) are of interest \cite{52, 53}. Squeezed baths are capable of processing optical signals (attenuation or amplification) while retaining their quantum features. It has also been shown that an appropriately squeezed bath is capable of greatly increasing the decoherence timescale \cite{54}. The description of these processes is based on the quantum optical master equation generalised to include squeezing in the initial state. It is an approximate equation derived under the rotating wave, Born and Markov approximations. Since our formalism is exact it is capable of a more accurate description of non-equilibrium quantum statistical processes in quantum optics. It also allows for the squeezing to be generated dynamically rather than imposed as an initial condition.

The effect of the bath on the system is studied here, as in the previous two QBM papers, by means of the influence functional formalism. We will derive exactly the evolution operator for the reduced density matrix, the influence functional, and the master equation for a time-dependent system and bath, using a slightly different method and language from Paper I. We adopt the language of squeeze and rotation operators \cite{55, 56, 57} for describing the evolution of the system. In Sec. 2 we define the model and mention its relevance in problems in quantum optics, quantum and semiclassical cosmology, and quantum field theory in curved spacetimes. We then derive an analytic expression for the influence functional of a system linearly coupled to a bath of parametric oscillators in terms of the Bogolubov coefficients. In Sec. 3 we derive the exact evolution operator for the reduced density matrix and adopt the simpler method introduced by Paz \cite{58} and used in \cite{28} for the derivation of the master equation. We consider the general case when the bath is initially in a squeezed thermal state, which includes the common cases of a thermal state and a squeezed vacuum. We indicate how it is different from the model with a bath of time independent oscillators. The diffusion coefficients of this equation can be analyzed for decoherence studies, as is done in Papers I, Refs. \cite{29} and \cite{20}. The relation of decoherence and particle creation was also discussed in the field theory context by Calzetta and Mazzitelli \cite{60} and in the quantum cosmology context by Paz and Sinha \cite{13}. Here we aim not at the decoherence or the dissipation processes, but focus on the definition and nature of noise associated with quantum fields and use them to depict some well-known processes such as the Hawking effect in gravitation and cosmology.

In Sec. 4 we give a few simple examples of a system interacting with a bath of parametric oscillator, first treating the case with constant frequency, but with an initial squeezed thermal state and then the case of inverted oscillators which can be used to model amplifiers in quantum optics and electronics \cite{53}. In Sec. 5, we derive the noise kernels for four cases: the accelerated observer, a two-dimensional black hole and a massless, conformally and minimally coupled scalar field in the de Sitter universe. In the de Sitter universe case the parametric oscillator bath can serve as a relatively simple model of the environment for homogenous and inhomogenous (density fluctuation) modes of the inflaton field in the early universe. We show the factors entering into the determination of the spectrum, and indicate how one can
understand the Hawking and Unruh effects in a purely statistical-mechanical sense without recourse to geometric notions (like the event horizon). We will discuss the fluctuation-dissipation relation approach \[61, 62, 63\] to understanding backreaction in semiclassical gravity in later work \[16, 64, 35, 36\]. In Sec. 6 we summarize our results and suggest further problems in cosmology and gravitation where our results can be usefully applied. The details of derivation in Sec. 2 are recorded in the Appendices.

2 Influence Functional

Our system, the Brownian particle, is modeled by a parametric oscillator with mass \(M(s)\), cross term \(B(s)\) and natural (bare) frequency \(\Omega(s)\). The environment (bath) is also modeled by a set of parametric oscillators with mass \(m_n(s)\), cross term \(b_n(s)\) and natural frequency \(\omega_n(s)\). The system is coupled to the bath through an arbitrary function \(F(x)\) of the system variable and linear in the bath variables \(q_n\) with coupling strength \(c_n(s)\) in each oscillator. The action of the combined system + environment is

\[
S[x, q] = S[x] + S_E[q] + S_{int}[x, q] = \int_0^t ds \left[ \frac{1}{2} M(s) (\dot{x}^2 + B(s)x\dot{x} - \Omega^2(s)x^2) \right. \\
+ \left. \sum_n \left\{ \frac{1}{2} m_n(s) (\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2) \right\} + \sum_n (-c_n(s)F(x)q_n) \right]
\]

(2.1)

where \(x\) and \(q_n\) are the coordinates of the particle and the oscillators. The bare frequency \(\Omega\) is different from the physical frequency \(\Omega_p\) due to its interaction with the bath, which depends on the cutoff frequency. We will discuss this point in more detail in Sec. 4.1. For problems discussed here we are interested in how the environment affects the system in some averaged way. The quantity containing this information is the reduced density matrix of the system obtained from the full density operator of the system + environment by tracing out the environmental degrees of freedom. The evolution operator is responsible for the time evolution of the reduced density matrix. The equation of motion governing this reduced density matrix is the master equation. Our central task is to derive the evolution operator and the master equation for the Brownian particle in a general environment.

We will briefly review here the Feynman-Vernon influence functional method for deriving the evolution operator. Readers who are familiar with it can skip this subsection. The method provides an easy way to obtain a functional representation for the evolution operator for the reduced density matrix \(J\). Let us start first with the evolution operator for the full density matrix \(\hat{J}\) defined by

\[
\hat{\rho}(t) = J(t, t_i)\hat{\rho}(t_i).
\]

(2.2)

As the full density matrix \(\hat{\rho}\) evolves unitarily under the action of (2.1), the evolution
operator $\mathcal{J}$ has a simple path integral representation. In the position basis, the matrix elements of the evolution operator are given by

$$
\mathcal{J}(x, \mathbf{q}, x', \mathbf{q}', t \mid x_i, \mathbf{q}_i, x_i', \mathbf{q}_i', t_i) = \mathcal{K}(x, \mathbf{q}, t \mid x_i, \mathbf{q}_i, t_i) \mathcal{K}^*(x', \mathbf{q}', t \mid x_i', \mathbf{q}_i', t_i)
$$

$$
= \int_{x_i}^{x} Dx \int_{\mathbf{q}_i}^{\mathbf{q}} D\mathbf{q} \ \exp \frac{i}{\hbar} S[x, \mathbf{q}] \int_{x_i'}^{x'} Dx' \int_{\mathbf{q}_i'}^{\mathbf{q}'} D\mathbf{q}' \ \exp -\frac{i}{\hbar} S[x', \mathbf{q}']
$$

(2.3)

where the operator $\mathcal{K}$ is the evolution operator for the wave functions. In the second equation, the path integrals are over all histories compatible with the boundary conditions. We have used $\mathbf{q}$ to represent the full set of environmental coordinates $q_n$ and the subscript $i$ to denote the initial variables.

The reduced density matrix is defined as

$$
\rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, \mathbf{q}; x', \mathbf{q}') \delta(\mathbf{q} - \mathbf{q}')
$$

(2.4)

and is propagated in time by the evolution operator $\mathcal{J}_r$

$$
\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx_i' \mathcal{J}_r(x, x', t \mid x_i, x_i', t_i) \rho_r(x_i, x_i', t_i).
$$

(2.5)

By using the functional representation of the full density matrix evolution operator given in (2.3), we can also represent $\mathcal{J}_r$ in path integral form. In general, the expression is very complicated since the evolution operator $\mathcal{J}_r$ depends on the initial state. If we assume that at a given time $t = t_i$ the system and the environment are uncorrelated

$$
\hat{\rho}(t = t_i) = \hat{\rho}_s(t_i) \times \hat{\rho}_b(t_i),
$$

(2.6)

then the evolution operator for the reduced density matrix does not depend on the initial state of the system and can be written as

$$
\mathcal{J}_r(x_f, x_f', t \mid x_i, x_i', t_i) = \int_{x_i}^{x_f} Dx \int_{x_i'}^{x_f'} Dx' \ \exp \frac{i}{\hbar} \left\{ S[x] - S[x'] \right\} \mathcal{F}[x, x']
$$

$$
= \int_{x_i}^{x_f} Dx \int_{x_i'}^{x_f'} Dx' \ \exp \frac{i}{\hbar} A[x, x']
$$

(2.7)

where the subscript $f$ denotes final variables, and $A[x, x']$ is the effective action for the open
quantum system. The factor $\mathcal{F}[x, x']$, called the ‘influence functional’, is defined as

$$
\mathcal{F}[x, x'] = \int_{-\infty}^{+\infty} dq f_{+\infty}^{i} \int_{-\infty}^{+\infty} dq' f_{+\infty}^{i} \int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dq D(q) D(q') \times \exp \frac{i}{\hbar} \{ S_b[q] + S_{int}[x, q] - S_b[q'] - S_{int}[x', q'] \} \rho_b(q, q', t_i)
$$

where $\delta \mathcal{A}[x, x']$ is the influence action. Thus $\mathcal{A}[x, x'] = S[x] - S[x'] + \delta \mathcal{A}[x, x']$.

It is not difficult to show that (2.8) has the representation independent form

$$
\mathcal{F}[x, x'] = \text{Tr}[ \hat{U}(t, t_i) \hat{\rho}_b(t_i) \hat{U}'(t, t_i)]
$$

where $\hat{U}(t)$ and $\hat{U}'(t)$ are the quantum propagators for the actions $S_E[q] + S_{int}[x(s), q]$ and $S_E[q] + S_{int}[x'(s), q]$ and $x(s)$ and $x'(s)$ are treated as time dependent classical forcing terms. We have found this form to be much more convenient for deriving the influence functional.

It is obvious from its definition that if the interaction term is zero, the influence functional is equal to unity and the influence action is zero. In general, the influence functional is a highly non–local object. Not only does it depend on the time history, but –and this is the more important property– it also irreducibly mixes the two sets of histories in the path integral of (2.7). Note that the histories $x$ and $x'$ could be interpreted as moving forward and backward in time respectively. Viewed in this way, one can see the similarity of the influence functional [3] and the generating functional in the closed-time-path (CTP or Schwinger-Keldysh) integral formalism [39]. The Feynman rules derived in the CTP method are very useful for computing the IF. We shall treat the field theoretic problems in later papers.

In those cases where the initial decoupling condition (2.6) is satisfied, the influence functional depends only on the initial state of the environment. The influence functional method can be extended to more general conditions, such as thermal equilibrium between the system and the environment [43], or correlated initial states [5, 4].

We now proceed to derive the influence functional for the model (2.1). From its definition it is clear that the influence functional is independent on the choice of system but only on the coupling of the system to the environment. Since our method is quite general we have been able to include, in Appendix A, the influence functional for the most general coupling linear in the bath variable. However in the body of the paper we only consider the position-position coupling in (2.1). For the case of a squeezed thermal initial state (to be defined later) we
find that for the model (2.1) the influence functional has the form

\[ F[x, x'] = \exp \left\{ -\frac{i}{\hbar} \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \left[ F(x(s)) - F(x'(s)) \right] \mu(s,s') \left[ F(x(s')) + F(x'(s')) \right] \right\} \]

\[ -\frac{1}{\hbar} \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \left[ F(x(s)) - F(x'(s)) \right] \nu(s,s') \left[ F(x(s')) - F(x'(s')) \right] \right\} \}

The functions \( \mu(s, s') \) and \( \nu(s, s') \) contain the effects of the environment on the system. They are known respectively as the dissipation and noise kernels. The reason for these names becomes clear in the semi-classical regime of the open system generated by (2.10).

To find the appropriate semiclassical limit of this open quantum system we must find an action which generates the same influence functional as (2.10). Consider the action

\[ S[a(s)] = \int_{t_i}^{t} ds \left( L(x, \dot{x}, s) + F(x) \xi(s) \right) \]

where \( \xi(s) \) is a gaussian stochastic force with a non-zero mean. This system generates the influence functional

\[ \mathcal{F}[\Sigma, \Delta] = \left\langle \exp \left[ \frac{i}{\hbar} \int_{t_i}^{t} \xi(s) \Delta(s) ds \right] \right\rangle \]

where \( \Sigma \) and \( \Delta \) are given by

\[ \Sigma(s) = \frac{1}{2} \left( F(x(s)) + F(x'(s)) \right), \quad \Delta(s) = F(x(s)) - F(x'(s)) \]

and the average is understood as a functional integral over \( \xi(s) \) multiplied by a normalised gaussian probability density functional \( P[\xi(s); \Sigma(s)] \). The probability density functional is a functional of \( \Sigma(s) \) if we allow the statistical properties of \( \xi \) to depend on the system history. The averaging can be performed to give \[ 36 \]

\[ \mathcal{F}[x, x'] = \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t} ds \Delta(s) \langle \xi(s) \rangle - \frac{1}{\hbar^2} \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \Delta(s) \Delta(s') C_2(s, s') \right\} \]

where \( C_2(s, s') \) is the second cumulant of the force \( \xi \). The equation of motion generated by the action (2.11) is

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{\partial F(x)}{\partial x} \langle \xi(t) \rangle = -\frac{\partial F(x)}{\partial x} \langle \xi(t) \rangle \]

where \( \tilde{\xi}(t) \) is a zero mean gaussian stochastic force with \( \langle \tilde{\xi}(t) \tilde{\xi}(t') \rangle = C_2(s, s') \). Now by comparing (2.14) and (2.10) we see that

\[ \langle \xi(s) \rangle \equiv -2 \int_{t_i}^{s} ds' \mu(s, s') \Sigma(s'), \quad C_2(s, s') \equiv \hbar \nu(s, s'). \]
Therefore the semiclassical equation for the system described by the influence functional (2.10) is

\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - 2 \frac{\partial F(x)}{\partial x} \left. \int_{t_i}^{t} \mu(t, s) F(x(s)) ds \right|_{t} = -\frac{\partial F(x)}{\partial x} \tilde{\xi}(t) \tag{2.17}
\]

where \( \langle \tilde{\xi}(t) \tilde{\xi}(t') \rangle = h\nu(t, t') \). Under special circumstances \( \mu \) tends to the derivative of a delta function which generates local dissipation. More generally we see that in the semiclassical limit \( \mu \) generates non-local dissipation while \( h\nu \) is the correlator of a zero mean gaussian stochastic force.

We find that the dissipation and noise kernels take the form

\[
\mu(s, s') = i \int_0^\infty d\omega I(\omega, s, s') \left[ \{ \alpha_\omega(s) + \beta_\omega(s) \}^* \{ \alpha_\omega(s') + \beta_\omega(s') \} \right. \\
- \{ \alpha_\omega(s) + \beta_\omega(s) \} \{ \alpha_\omega(s') + \beta_\omega(s') \}^* \tag{2.18}
\]

\[
\nu(s, s') = \frac{1}{2} \int_0^\infty d\omega I(\omega, s, s') \coth \left( \frac{h\omega(t_i)}{2k_B T} \right) \\
\times \left[ \cosh 2r(\omega) \{ \alpha_\omega(s) + \beta_\omega(s) \}^* \{ \alpha_\omega(s') + \beta_\omega(s') \} \right. \\
+ \cosh 2r(\omega) \{ \alpha_\omega(s) + \beta_\omega(s) \} \{ \alpha_\omega(s') + \beta_\omega(s') \}^* \\
- \sinh 2r(\omega) e^{-2i\phi(\omega)} \{ \alpha_\omega(s) + \beta_\omega(s) \}^* \{ \alpha_\omega(s') + \beta_\omega(s') \}^* \\
- \sinh 2r(\omega) e^{2i\phi(\omega)} \{ \alpha_\omega(s) + \beta_\omega(s) \} \{ \alpha_\omega(s') + \beta_\omega(s') \} \right]. \tag{2.19}
\]

The quantities in these kernels describe three different properties of the environment.

A) The \( \alpha \) and \( \beta \), known as the Bogolubov coefficients, are complex numbers that contain all the information about the quantum dynamics of the bath parametric oscillators. They are derived from two coupled first order equations

\[
\dot{\alpha}_n = -if_n^* \beta_n - ih_n \alpha_n \\
\dot{\beta}_n = ih_n \beta + if_n \alpha \tag{2.20}
\]

where the time dependent coefficients are given by

\[
f_n(t) = \frac{1}{2} \left( \frac{m_n(t) \omega_n^2(t)}{\kappa_n} + \frac{m_n(t) b_n^2(t)}{4\kappa_n} - \frac{\kappa_n}{m_n(t)} + ib_n(t) \right) \\
h_n(t) = \frac{1}{2} \left( \frac{\kappa_n}{m_n(t)} + \frac{m_n(t) \omega_n^2(t)}{\kappa_n} + \frac{m_n(t) b_n^2(t)}{4\kappa_n} \right). \tag{2.21}
\]

These equations are a by product of finding the quantum propagator for a parametric oscillator which is done in appendix B. We will usually choose \( \kappa_n \) (defined by (A.6)) so that \( f(t_i) = 0 \). Thus if \( b_n = 0 \) we will usually have \( \kappa_n = m_n(t_i) \omega_n(t_i) \). Eq’s (2.21) must satisfy
the initial conditions $\alpha(t_i) = 1, \beta(t_i) = 0$. Note that the mode label $\omega$ in the kernels is equivalent to $n$ in the continuous limit.

If we assume $b = 0$ and $m = 1$ we can show using (2.20) that

$$\ddot{X}_n + \omega_n^2(t) X_n = 0$$

(2.22)

where $X_n(t) = \alpha_n(t) + \beta_n(t)$. The solution of (2.22) must satisfy $X_n(t_i) = 1$. In this case the noise and dissipation kernels become

$$\mu(s, s') = \frac{i}{2} \int_0^\infty d\omega I(\omega, s, s') \left[ X^*_\omega(s)X_\omega(s') - X_\omega(s)X^*_\omega(s') \right]$$

(2.23)

$$\nu(s, s') = \frac{1}{2} \int_0^\infty d\omega I(\omega, s, s') \coth \left( \frac{\hbar \omega(t_i)}{2 k_B T} \right) \left[ \cosh 2r(\omega) \left[ X^*_\omega(s)X_\omega(s') + X_\omega(s)X^*_\omega(s') \right] - \sinh 2r(\omega) \left[ e^{-2i\phi(\omega)} X^*_\omega(s)X^*_\omega(s') + e^{2i\phi(\omega)} X_\omega(s)X_\omega(s') \right] \right].$$

(2.24)

Note that we can always write

$$X_n(t) = C_n(t) - i\omega_n(t_i)S_n(t)$$

(2.25)

where $C_n$ and $S_n$ are subject to the boundary conditions $C_n(t_i) = \dot{S}_n(t_i) = 1$ and $S_n(t_i) = \dot{C}_n(t_i) = 0$. If the kernels are written in this notation we can show that for a thermal initial state (2.17) reduces to the classical Langevin equation in the high temperature limit [59].

B) The spectral density, $I(\omega, s, s')$ defined formally by

$$I(\omega, s, s') = \sum_n \delta(\omega - \omega_n) \frac{c_n(s)c_n(s')}{2\kappa_n}$$

(2.26)

is obtained in the continuum limit. It contains information about the environmental mode density and coupling strength as a function of frequency. Different environments are classified according to the functional form of the spectral density $I(\omega)$. On physical grounds, one expects the spectral density to go to zero for very high frequencies. Let us introduce a certain cutoff frequency $\Lambda$ (a property of the environment) such that $I(\omega) \rightarrow 0$ for $\omega > \Lambda$.

The environment is classified as ohmic [4, 5] if in the physical range of frequencies ($\omega < \Lambda$) the spectral density is such that $I(\omega) \sim \omega$, as supra-ohmic if $I(\omega) \sim \omega^n, n > 1$ or as sub-ohmic if $n < 1$. The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of the system. We will show this in section 4.1.

C) The initial state of the bath is a squeezed thermal state. It has the form

$$\hat{\rho}_b(t_i) = \prod_n \hat{S}_n(r(n), \phi(n))\hat{\rho}_{th} \hat{S}^\dagger_n(r(n), \phi(n))$$

(2.27)
where $\hat{\rho}_{th}$ is a thermal density matrix of temperature $T$ defined by (A.18) and $\hat{S}(r, \phi)$ is a squeeze operator defined by (B.12). Since a squeezed thermal state is still gaussian it is a tractable generalisation of the usual thermal initial state that is of interest in quantum optics [34]. For the case of zero temperature we have a squeezed vacuum initial state.

Physically the term squeezing arises because the phase space noise distribution of a squeezed vacuum is an ellipse squeezed from a circle (coherent state) by $r$ and rotated by angle $\phi$ with respect to the $x$ and $p$ axes. Thus, for the squeezed vacuum [35]

\[
\begin{align*}
\langle q^2 \rangle &= \frac{1}{2\kappa} \left[ \cosh 2r - \sinh 2r \cos 2\phi \right] \\
\langle p^2 \rangle &= \frac{1}{2\kappa} \left[ \cosh 2r + \sinh 2r \cos 2\phi \right].
\end{align*}
\]

(2.28)

Note that the dissipation kernel is independent of the bath initial state.

For the case of no initial squeezing we see that the noise and dissipation kernels are built out of symmetric and anti-symmetric combinations of identical Bogolubov factors. Thus the two kernels are intimately linked. For the case when the bath is a standard harmonic oscillator this interrelationship can be written as a generalised fluctuation-dissipation relation [2].

3 Evolution Operator and Master Equation

In this section our goal is to calculate the evolution operator for the reduced density matrix and the master equation. The master equation is the evolution equation for the reduced density matrix. It provides a transparent means for sifting out the different physical processes caused by the bath on the system. First we must calculate the evolution operator $\rho_r$ in (2.7), which contains all the dynamical information about the open system. From this point on we shall put $F(x) = x$.

The influence functional (2.10) and the corresponding influence action (2.8) can be written in a compact way

\[
\begin{align*}
\mathcal{A}[x, x'] &= S[x] - S[x'] + \delta \mathcal{A}(x, x'), \\
\delta \mathcal{A}[x, x'] &= -2 \int_{t_i}^t ds \int_{t_i}^s ds' \Delta(s)\mu(s, s')\Sigma(s') + i \int_{t_i}^t ds \int_{t_i}^s ds' \Delta(s)\nu(s, s')\Delta(s') \\
S[x] - S[x'] &= \int_{t_i}^t ds \{ M(s)\dot{\Sigma}(s)\dot{\Delta}(s) + \frac{1}{2} M(s)B(s)[\Sigma(s)\dot{\Delta}(s) + \Delta(s)\dot{\Sigma}(s)] \\
&\quad - M(s)\Omega^2(s)\Sigma(s)\Delta(s) \}
\end{align*}
\]

(3.1)

(3.2)

with the use of the ‘center of mass’ and ‘relative’ coordinates defined earlier in (2.13).

As pointed out by many authors [3, 4, 5], and in Sec. 2, the real and imaginary parts of $\mathcal{A}[x, x']$ can be interpreted [3] as being responsible for dissipation and noise respectively. The imaginary part of (3.1) is determined by $\nu(s)$, the noise (or fluctuation) kernel. The
name becomes apparent when we realize that this term can be interpreted as coming from
the interaction between the system and a stochastic force $\xi$ that is linearly coupled to the
system and has a probability density given by $P[\xi] = \exp\{-\xi(\hbar\nu)^{-1}\xi\}$. On the other hand,
the kernel $\mu(s)$ in (3.1) is known as the dissipation kernel. The motivation for the name
comes from the fact [9] that it introduces a modification in the real saddle point trajectories
of the path integral in (2.7). Strictly speaking only the non-symmetric part of the $\mu$ kernel
should be associated with dissipation. Thus, all the symmetric part can be absorbed in a
non-local potential (that does not contribute to the mixing of the $x$ and $x'$ histories). There
is no such symmetric part in the $\mu$–kernel of our problem although it does appear in other
cases [2].

3.1 Evolution Operator

The evolution operator given in equation (2.7) generates a non–Markovian dynamics since
it fails in general to satisfy the relation

$$J_r(t_2, t_1) = J_r(t_2, t_i) J_r(t_1, t_i)$$

for the reason that the operator $J_r(t_2, t_1)$ depends on the state of the system at time $t_1$,
unless that time is the one for which the system and the environment were decoupled. The
non–Markovian behavior is, in fact, a direct consequence of the non–locality of the influence
functional.

Our task is to compute the evolution operator

$$J_r(\Sigma_f, \Delta_f, t \mid \Sigma_i, \Delta_i, t_i) = \int_{\Sigma_i}^{\Sigma_f} D\Sigma \int_{\Delta_i}^{\Delta_f} D\Delta \exp \left[ \frac{i}{\hbar} A[\Sigma(s), \Delta(s)] \right]. \quad (3.3)$$

Let us schematically describe how to compute the path integral in (3.3). We start by
reparametrizing the paths, writing

$$\Sigma(s) = x_+(s) + \Sigma_{cl}(s)$$
$$\Delta(s) = x_-(s) + \Delta_{cl}(s) \quad (3.4)$$

where the “classical paths” $(\Sigma_{cl}, \Delta_{cl})$ are solutions to the equations of motion derived from
the real part of $A[\Sigma, \Delta]$, and $x_\pm$ are the deviations from the classical paths. The equations
governing these functions are

$$\ddot{\Sigma}_{cl}(s) + \frac{\dot{M}(s)}{M(s)} \Sigma_{cl}(s) + \left( \Omega^2(s) + \frac{\dot{B}(s)}{2} + \frac{\dot{B}(s)B(s)}{2M(s)} \right) \Sigma_{cl}(s) + \frac{2}{M(s)} \int_{t_i}^{s} ds' \mu(s, s') \Sigma_{cl}(s') = 0$$
$$\Sigma_{cl}(t_i) = \Sigma_i, \quad \text{and} \quad \Sigma_{cl}(t) = \Sigma_f \quad (3.5)$$
\[ \ddot{\Delta}_cl(s) + \frac{\dot{M}(s)}{M(s)} \dot{\Delta}_cl(s) + \left( \Omega^2(s) + \frac{\dot{B}(s)}{2} + \frac{\dot{M}(s)B(s)}{2M(s)} \right) \Delta_cl(s) + \frac{2}{M(s)} \int_s^t ds' \mu(s', s) \Delta_cl(s') = 0 \]

\[ \Delta_cl(t_i) = \Delta_i, \quad \text{and} \quad \Delta_cl(t) = \Delta_f. \]  

(3.6)

After the path-reparametrization, (3.3) can be rewritten as

\[ J_r(\Sigma_f, \Delta_f, t | \Sigma_i, \Delta_i, t_i) = Z(t, t_i) \exp \left[ \frac{i}{\hbar} A[\Sigma_{cl}(s), \Delta_{cl}(s)] \right] \]  

(3.7)

where

\[ Z(t, t_i) = \int_{t_i; x_i = 0}^{t; x = 0} Dx^+ \int_{t_i; x_i = 0}^{t; x = 0} Dx^- \exp \left[ \frac{i}{\hbar} A[x^+(s), x^-(s)] \right] \]

\[ = \frac{1}{\hbar} \int_{t_i}^t ds \int_{t_i}^t ds' [x^-(s) \Delta_{cl}(s') \nu(s, s')] . \]  

(3.8)

We can write the classical solutions \( \Sigma_{cl} \) and \( \Delta_{cl} \) in terms of the elementary functions

\[ \Sigma_{cl}(s) = \Sigma_i u_1(s) + \Sigma_f u_2(s) \]  

(3.9a)

\[ \Delta_{cl}(s) = \Delta_i v_1(s) + \Delta_f v_2(s) \]  

(3.9b)

which satisfy the boundary conditions

\[ u_1(s = t_i) = 1 = u_2(s = t) \]  

(3.10a)

\[ u_1(s = t) = 0 = u_2(s = t_i) \]  

(3.10a)

\[ v_1(s = t_i) = 1 = v_2(s = t) \]  

(3.10b)

\[ v_1(s = t) = 0 = v_2(s = t_i). \]

Now setting

\[ b_1(t, t_i) = M(t) \dot{u}_2(t) + \frac{M(t)B(t)}{2}, \quad \dot{b}_2(t, t_i) = M(t_i) \dot{u}_2(t_i) \]

\[ b_3(t, t_i) = M(t) \dot{u}_1(t), \quad \dot{b}_4(t, t_i) = M(t_i) \dot{u}_1(t_i) + \frac{M(t_i)B(t_i)}{2} \]  

(3.11)

where the dot denotes the derivative with respect to \( s \) and

\[ a_{ij}(t, t_i) = \frac{1}{1 + \delta_{ij}} \int_{t_i}^t ds \int_{t_i}^s ds' v_i(s) \nu(s, s') v_j(s') \]  

(3.12)

we get

\[ J_r(x_f, x'_f, t | x_i, x'_i, t_i) = Z(t, t_i) \exp \left[ \frac{i}{\hbar} \left\{ b_1 \Sigma_f \Delta_f - b_2 \Sigma_f \Delta_i + b_3 \Sigma_i \Delta_f - b_4 \Sigma_i \Delta_i \right\} \right] \]

\[ \times \exp \left[ -\frac{1}{\hbar} \left\{ a_{11} \Delta_i^2 + a_{12} \Delta_i \Delta_f + a_{22} \Delta_f^2 \right\} \right] . \]  

(3.13)
The evolution operator (3.13) must preserve the normalisation of the density matrix. By requiring that $\text{Tr}(\rho) = 1$, (2.5) implies
\[
\int_{-\infty}^{\infty} dx J_r(x, x, t | x_i, x_i') = \delta(x_i - x_i').
\]
We therefore find that
\[
Z(t, t_i) = \frac{b_2(t, t_i)}{2\pi \hbar}.
\] (3.14)

### 3.2 Master Equation

We now proceed with the derivation of the master equation from the evolution operator (3.13) using the simplified method of Paz [58]. We first take the time derivative of both sides of (3.13), multiply both sides by $\rho_r(\Sigma_i, \Delta_i, t_i)$ and integrate over $\Sigma_i, \Delta_i$ to obtain
\[
\dot{\rho}_r(\Sigma_f, \Delta_f, t) = \left[ \frac{\dot{Z}}{Z} + \frac{i}{\hbar} b_1 \Sigma_f \Delta_f - \frac{\Delta_f^2}{\hbar} \right] \rho_r(\Sigma_f, \Delta_f, t)
\]
\[
+ \frac{i}{\hbar} \Delta_f b_3 \int d\Delta_i d\Sigma_i \rho_r(\Sigma_i, \Delta_i, t_i)
\]
\[
- \frac{1}{\hbar} (ib_2 \Sigma_f + \dot{a}_{12} \Delta_f) \int d\Delta_i d\Sigma_i \rho_r(\Sigma_i, \Delta_i, t_i)
\]
\[
- \frac{i}{\hbar} b_4 \int d\Delta_i d\Sigma_i \rho_r(\Sigma_i, \Delta_i, t_i)
\]
\[
- \frac{\dot{a}_{11}}{\hbar} \int d\Delta_i d\Sigma_i \Delta_i^2 \rho_r(\Sigma_i, \Delta_i, t_i).
\] (3.15)

Here the dot denotes derivative with respect to $t$. We can perform the integrals in (3.15) by using
\[
\Delta_i J_r = \frac{i\hbar}{b_3} \frac{\partial J_r}{\partial \Delta_f} + \frac{b_1 \Delta_f}{b_3} J_r
\] (3.16a)
\[
\Sigma_i J_r = -\frac{i}{b_3} \left[ \frac{\partial J_r}{\partial \Sigma_f} + (\Delta_i a_{12} + 2 \Delta_f a_{22}) J_r \right] - \frac{b_1}{b_3} \Sigma_f J_r
\] (3.16b)
\[
\Sigma_i \Delta_i J_r = -\left( \frac{i\hbar}{b_3} \frac{\partial}{\partial \Sigma_f} + \frac{b_1 \Delta_f}{b_3} \right) \frac{\partial}{\partial \Delta_f} \left( \frac{i\hbar}{b_3} \frac{\partial}{\partial \Sigma_f} + \frac{i}{b_3} [\Delta_i a_{12} + 2 \Delta_f a_{22}] + \frac{b_1 \Sigma_f}{b_3} \right) J_r + \left( \frac{i\hbar}{b_3} \frac{\partial}{\partial \Delta_f} + \frac{i}{b_3} [\Delta_i a_{12} + 2 \Delta_f a_{22}] + \frac{b_1 \Sigma_f}{b_3} \right) J_r.
\] (3.16c)

The derivation of the master equation simplifies greatly with the use of the following relations
\[
u_1(s) = w_1(s) - w_2(s)\frac{w_1(t)}{w_2(t)}, \quad \nu_2(s) = \frac{w_2(s)}{w_2(t)}.
\] (3.17)
In order to satisfy the boundary conditions, (3.10a), we require

\[ w_1(t_i) = \dot{w}_2(t_i) = 1, \quad w_2(0) = \dot{w}_1(0) = 0. \]  

(3.18)

In this representation we can show that

\[ \frac{\dot{b}_1}{b_2b_3} = -\frac{1}{M(t)}, \quad b_1 = -M(t)\frac{\dot{b}_2}{b_2} + M(t)\frac{B(t)}{2}, \quad \dot{a}_{11} = -\dot{v}_1(t)a_{12}. \]  

(3.19)

With these relations the master equation reduces to

\[
i\hbar \frac{\partial}{\partial t} \rho_r(x, x', t) = \left\{ -\frac{\hbar^2}{2M(t)} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) + \frac{i\hbar}{2} B(t) \left( \frac{x}{x} + x' \frac{\partial}{\partial x'} \right) 
\right. 
\left. + \frac{M(t)}{2} \Omega_{ren}^2(t, t_i)(x^2 - x'^2) + i\hbar B(t) \right\} \rho_r(x, x', t)
\]

\[- i\hbar \Gamma(t, t_i)(x - x')(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}) \rho_r(x, x', t)
\]

\[ + iD_{pp}(t, t_i)(x - x')^2 \rho_r(x, x', t)
\]

\[ - \hbar \left( D_{xp}(t, t_i) + D_{px}(t, t_i) \right) (x - x')(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}) \rho_r(x, x', t)
\]

\[ - i\hbar^2 D_{xx}(t, t_i) \frac{\partial^2}{(\partial x + \partial x')^2} \rho_r(x, x', t)
\]

where

\[ \Omega_{ren}^2(t, t_i) = \frac{b_1b_3}{M(t)b_3} - \frac{\dot{b}_1}{b_1} + \frac{B^2(t)}{4} - \frac{\dot{b}_2B(t)}{2b_2} \]

(3.21)

\[ \Gamma(t, t_i) = -\frac{1}{2} \left( \frac{b_3}{b_3} - \frac{\dot{b}_2}{b_2} \right) \]

(3.22)

\[ D_{pp}(t, t_i) = \frac{b_1^2}{b_2} \left( \frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2} \right) + \frac{2b_1}{b_2} a_{22} - \dot{a}_{22} + \frac{2\dot{b}_3}{b_3} a_{22} + a_{12} \frac{\dot{b}_3}{b_2b_3} + \frac{\dot{a}_{12}}{b_2} \frac{b_1}{b_2} \]

(3.23)

\[ D_{xp}(t, t_i) = D_{px}(t, t_i) = -\frac{1}{2} \left[ \frac{\dot{a}_{12}}{b_2} - 2 \frac{a_{22}}{M(t)} - \frac{\dot{b}_3a_{12}}{b_3b_2} - \frac{2b_1}{b_2} \left( \frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2} \right) \right] \]

(3.24)

\[ D_{xx}(t, t_i) = \frac{1}{b_2} \left( \frac{a_{12}}{M(t)} - \frac{\dot{a}_{11}}{b_2} \right) \]

(3.25)

The dot in these equations denotes the derivative with respect to \( t \). We can rewrite the master equation in the operator form which may be easier for physical interpretation. We find that it becomes

\[
i\hbar \frac{\partial}{\partial t} \hat{\rho}_r(t) = [\hat{H}_{ren}, \hat{\rho}] + iD_{pp} [\hat{x}, [\hat{x}, \hat{\rho}]] + iD_{xx} [\hat{p}, [\hat{p}, \hat{\rho}]]
\]

\[ + iD_{xp} [\hat{x}, [\hat{p}, \hat{\rho}]] + iD_{px} [\hat{p}, [\hat{x}, \hat{\rho}]] + \Gamma [\hat{x}, \{\hat{p}, \hat{\rho}\}] \]

(3.26)
where

\[ \hat{H}_{\text{ren}} = \frac{\hat{p}^2}{2M(t)} - \frac{B(t)}{4}(\hat{p}\hat{x} + \hat{x}\hat{p}) + \frac{M(t)}{2}\Omega_{\text{ren}}(t)\hat{x}^2. \] (3.27)

From the master equation we know that \( D_{xx} \) and \( D_{pp} \) generate decoherence in \( p \) and \( x \) respectively and \( \Gamma \) gives dissipation. The master equation differs from Paper I by more than changing the kernels. The factor \( a_{12}/M(t) - \dot{a}_{11}/b_2 \) vanishes only when the dissipation kernel is stationary (i.e. a function of \( s - s' \)) and also when the system is a time independent harmonic oscillator. When this happens \( v_1(s) = u_2(t - s) \) and we have \( \dot{v}_1(t) = -b_2/M(t) \).

We see from (3.19) that the factor \( a_{12}/M(t) - \dot{a}_{11}/b_2 \) is zero in this case. All the diffusion coefficients contain this factor and \( D_{xx} \) depends solely on it. Thus \( D_{xx} \) arises purely from non-stationarity in the dissipation kernel and system.

The coefficients \( D_{xx}, D_{pp}, D_{xp} \) and \( D_{px} \) promote diffusion in the variables \( p^2, x^2 \) and \( xp + px \) respectively. This can be seen by going from the master equation to the Fokker-Planck equation for the Wigner function \[1, 67\]. The Wigner function is defined by

\[ F_W(\Sigma, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i\Delta/\hbar}\langle\Sigma - \frac{\Delta}{2}|\hat{\rho}|\Sigma + \frac{\Delta}{2}\rangle d\Delta. \] (3.28)

where \( \Sigma, \Delta \) are defined in (2.13). We can show that the Wigner distribution function from the master equation (3.26-7) (with \( B(t) = 0 \)) obeys the following Fokker-Planck type equation \[67\]

\[ \frac{\partial}{\partial t} F_W(\Sigma, p, t) = \left[ -\frac{p}{M(t)} \frac{\partial}{\partial \Sigma} + \frac{M(t)}{2} \frac{\Omega_{\text{ren}}^2(t)}{\partial p} + \Gamma(t) \frac{\partial}{\partial p} p - 2D_{pp}(t) \frac{\partial^2}{\partial p^2} 
- \hbar D_{xx}(t) \frac{\partial^2}{\partial \Sigma^2} + 2(D_{xp}(t) + D_{px}(t)) \frac{\partial^2}{\partial \Sigma \partial p} \right] F_W(\Sigma, p, t). \] (3.29)

\section{Simple Examples}

\subsection{Squeezed Thermal Bath of Static Harmonic Oscillators}

This is the simplest case treated before in Paper I and II. In this case the bath modes have time independent coupling constants with the Lagrangian

\[ L(t) = \frac{1}{2}[\dot{q}^2 - \omega^2 q^2]. \] (4.1)

From (2.1) \( m_n = 1, b_n = 0 \) and \( \omega_n^2 = \omega^2 \). Substituting these into (2.21) and solving (2.20) (with \( \kappa = \omega \)) one obtains

\[ \alpha = e^{-i\omega t}, \quad \beta = 0 \] (4.2)

where \( \alpha = 1 \) at the initial time \( t = 0 \). Substituting these into (2.18-19) one obtains

\[ \mu(s, s') = -\int_0^\infty d\omega I(\omega) \sin \omega(s - s'). \] (4.3)
\[\nu(s, s') = \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) I(\omega) \left[ \cosh 2r(\omega) \cos[\omega(s - s')] \right.\]
\[\left. - \sinh 2r(\omega) \cos[2\phi(\omega) - \omega(s + s')] \right]. \tag{4.4}\]

This is a simple generalisation of previous studies in that we have a squeezed thermal initial state \[54\] rather than a thermal state. There are two distinct contributions to the noise kernel for an initially squeezed bath. The first term represents a change in the spectrum of the stationary vacuum noise. The second term has a new feature which is a non-stationary contribution to the noise kernel. This is expected since the fluctuations of a squeezed vacuum mode oscillate between conjugate observables.

As \((s + s') \to \infty\) the second term in (4.4) tends to zero. Thus the nonstationarity in the noise kernel is a transient effect for the initial squeezed bath. For an initial squeezed bath with thermal spectrum the late time noise kernel would tend to that of the usual thermal state. This is because at late times, the noise kernel \(\nu\) loses track of the initial phase distribution \(\theta(\omega)\). This is, however, not true for the master equation diffusion coefficients. Equations (3.23-25) show that the diffusion coefficients depend on the noise kernel in a non-local way in time. It may be interesting to compare the timescales in which the semi-classical system and the full quantum system forget the \(\phi(\omega)\) initial condition in the bath.

Although we have considered only single mode squeezed initial states our results can be easily extended to two-mode squeezed initial states \[55\]. This will change the noise kernel (4.4) but not the dissipation kernel (4.3) which remains independent of the initial state. Since the influence functional (2.10) is unchanged the exact forms for the evolution operator and master equations in Sec. 3 will stay. These results could then be used for an accurate description of systems coupled to an initially squeezed bath\[52, 54\].

If we set the initial squeezing to zero we regain the results of Paper I. For completeness we will summarise the simple analytical results obtained previously. In this case the noise and dissipation kernels are functions only of \(s - s'\). They can always be related by some integral equation known as the fluctuation–dissipation relation (FDR) \[1\]:

\[\nu(s) = \int_{-\infty}^{+\infty} ds' K(s - s') \gamma(s') \tag{4.5}\]

where the kernel \(K(s)\) is

\[K(s) = \int_0^\infty \frac{d\omega}{\pi} \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) \frac{1}{2} \beta \hbar \omega \cos \omega s \tag{4.6}\]

and \(\mu(s) = \frac{d}{ds} \gamma(s)\). In the classical or high temperature limit, the kernel \(K\) is proportional to the delta function \(K(s) = 2k_B T \delta(s)\) and the FDR is equivalent to the well known Einstein formula.
An interesting case is an environment which generates an ohmic spectral density

\[ I(\omega) = \frac{2}{\pi} \gamma_0 M \omega. \quad (4.7) \]

With a discrete high frequency cutoff \( \Lambda \),

\[ \mu(s) = \frac{2}{\pi} \gamma_0 M \frac{d}{ds} \left( \frac{\sin \Lambda s}{s} \right) \]

\[ \rightarrow 2 \gamma_0 M \frac{d}{ds} \delta(s), \quad \text{as } \Lambda \rightarrow \infty. \quad (4.8) \]

In this case for a harmonic oscillator system (2.17) becomes

\[ \ddot{X}(s) + 2 \gamma_0 \dot{X} + \Omega_r^2 X = -\xi(t) \quad (4.9) \]

where \( \Omega_r = \Omega - \frac{4}{\pi} \gamma_0 \Lambda \). We see that the ohmic environment is special in that it gives local dissipation in the infinite cutoff limit.

Theoretically, the meaning of renormalization can be understood as follows [1]: We can rewrite the action as

\[ S = \int_0^t ds \left\{ \frac{1}{2} M (\dot{x}^2 - \Omega^2 x^2) + \sum_n \left[ \frac{1}{2} m_n q_n^2 - \frac{1}{2} m_n \omega_n^2 (q + \frac{c_n}{m_n \omega_n^2} x)^2 + \frac{1}{2} \frac{c_n^2}{m_n \omega_n^2} x^2 \right] \right\}. \quad (4.10) \]

The last term can be viewed as a frequency counter term \( \Omega_c^2 \) arising from the interaction of the Brownian particle with the bath oscillators

\[ \Omega_c^2 = -\frac{1}{2M} \sum_n \frac{c_n^2}{m_n \omega_n^2} = -\int d\omega \frac{I(\omega)}{\omega}. \quad (4.11) \]

The bare frequency \( \Omega^2 \) is thus modified into a renormalized frequency \( \Omega_r^2 \) given by

\[ \Omega_r^2 = \Omega^2 + \Omega_c^2. \quad (4.12) \]

Another interesting case is the high temperature limit. If we consider the temperature to be such that \( \frac{\hbar}{k_B T} \ll \Lambda^{-1} \) and then let \( \Lambda \rightarrow \infty \) (the order in which we take the limits is important), the noise kernel (4.4) is simplified to

\[ \nu(s) = \frac{4Mk_B T \gamma_0}{\hbar} \delta(s). \quad (4.13) \]

In this case we see that the noise is white with an amplitude \( 4\gamma_0 M k_B T \), and (4.9) reduces to the Nyquist formula. In the ohmic, high temperature and infinite cutoff limit the master equation coefficients can be calculated. Using (3.5) we find that, for a time independent harmonic oscillator system, \( u_1 \) and \( u_2 \) must satisfy

\[ \ddot{u}(s) + 2 \gamma_0 \dot{u}(s) + \Omega_r^2 u(s) = -4 \gamma_0 \delta(s) u(0). \quad (4.14) \]
The solutions satisfying the appropriate boundary conditions (with \( t_i = 0 \)) are

\[
\begin{align*}
  u_1(s) &= -\frac{\sin[\tilde{\Omega}(s-t)]e^{-\gamma_0 s}}{\sin \tilde{\Omega} t},
  & u_2(s) &= \frac{\sin[\tilde{\Omega}s]e^{-\gamma_0(s-t)}}{\sin \tilde{\Omega} t} \tag{4.15}
\end{align*}
\]

where \( \tilde{\Omega}^2 = \Omega_r^2 - \gamma_0^2 \). Applying these to (3.11) we find

\[
\begin{align*}
  b_2(t) &= \frac{M\tilde{\Omega}e^{\gamma_0 t}}{\sin \tilde{\Omega} t},
  & b_3(t) &= -\frac{M\tilde{\Omega}e^{-\gamma_0 t}}{\sin \tilde{\Omega} t} \tag{4.16}
  \\
  b_4(t) &= -b_1(t) = M(\gamma_0 - \tilde{\Omega} \cot \tilde{\Omega} t). \tag{4.17}
\end{align*}
\]

Since \( b_4 \) is discontinuous before and after \( t = 0 \) (due to the kick) we have taken the average.

The results (4.16-17) are exact in the infinite cutoff limit of an ohmic environment. This is a local approximation which has been shown to be good for timescales greater than the inverse cutoff \cite{20}. Equations (4.16-17) depend only on the dissipation kernel which is unchanged by initial squeezing in the bath. Thus these equations can also be applied to more general situations.

Using the noise kernel (4.13) and the fact that \( v_1(s) = u_2(t-s), \ v_2(s) = u_1(t-s) \) we can calculate \( a_{ij} \) and find that the master equation coefficients to be

\[
\begin{align*}
  \Omega_{\text{ren}}(t) &= \Omega_r, \quad \Gamma(t) = \gamma_0, \quad D_{xp}(t) = D_{xx}(t) = 0, \quad D_{pp}(t) = -\frac{2\gamma_0 k_B T M}{\hbar}. \tag{4.18}
\end{align*}
\]

For decoherence studies under these and other environmental conditions see \cite{20}.

### 4.2 Bath of Upside Down Oscillators

This is the next simplest case. In this case the bath modes have the Lagrangian.

\[
L(t) = \frac{1}{2}[q^2 + \omega^2 q^2]. \tag{4.19}
\]

From (2.1) \( m_n = 1, b_n = 0 \) and \( \omega_n^2 = -\omega^2 \). Substituting these into (2.21) and solving (2.20) (with \( \kappa = \omega \)) we obtain

\[
\begin{align*}
  \alpha_\omega(t) &= \cosh \omega t, \quad \beta_\omega(t) = -i \sinh \omega t \tag{4.20}
\end{align*}
\]

where \( \alpha = 0 \) at \( t = 0 \) which is our initial time. Substituting these into (2.18-19) we obtain

\[
\mu(s,s') = -\int_0^\infty d\omega I(\omega) \sinh \omega(s-s') \tag{4.21}
\]

and

\[
\nu(s,s') = \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2 k_B T} \right) I(\omega) \left[ \cosh 2r(\omega) \cosh \omega(s+s') \right. \right.
\]

\[
\left. \left. - \sinh 2r(\omega) \cos 2\phi(\omega) \cosh \omega(s-s') \right) \right. \left. - \sinh 2r(\omega) \sin 2\phi(\omega) \sinh \omega(s+s') \right]. \tag{4.22}
\]

This case can be used as an amplifier model in quantum optics and electronics \cite{53}.
5 Particle Detector in a Scalar Field Bath

The formalism developed here can be used to study quantum statistical processes in cosmological and black hole spacetimes. The model (2.1) can be used to depict a particle detector in motion, or an observer near a black hole. It can also be used to describe the non-equilibrium dynamics of homogeneous and inhomogeneous modes (density fluctuations) of the inflaton field or gravity wave perturbations (which in the linear approximation obey the wave equation of a massless, minimally coupled scalar field) in the early universe.

In this section we will show how a general real scalar field in an expanding universe is reduced to a sum over quadratic time dependent Hamiltonians. The action for a free massive \((m)\) scalar field in a curved spacetime with metric \(g_{\mu\nu}\) and scalar curvature \(R\) is given by

\[
S = \int \mathcal{L}(x) d^4x = \int \sqrt{-g} \frac{1}{2} \, d^4x \left( g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - (m^2 + \xi_d R) \Phi^2 \right) \tag{5.1}
\]

where \(\nabla_\nu\) denotes covariant derivative, and \(\xi_d\) is the field coupling constant \((\xi_d = 0, 1/6 \text{ respectively for minimal and conformal coupling})\). In the spatially-flat Robertson-Walker (RW) metric

\[
\text{ds}^2 = a^2(\eta) \left[ d\eta^2 - \sum_i dx_i^2 \right] \tag{5.2}
\]

we can write

\[
\mathcal{L}(x) = \frac{1}{2} a^2(\eta) \left[ (\dot{\Phi})^2 - \sum_i (\Phi_{,i})^2 - \left( m^2 a^2 + 6 \xi_d \frac{\ddot{a}}{a} \right) \Phi^2 \right] \tag{5.3}
\]

where a dot denotes derivative taken with respect to conformal time \(\eta = \int dt/a\). If we rescale the field variable \(\chi = a\Phi\), this becomes

\[
\mathcal{L}(x) = \frac{1}{2} \left[ (\dot{\chi})^2 - \sum_i (\chi_{,i})^2 - \left( m^2 a^2 + (6 \xi_d - 1) \frac{\ddot{a}}{a} \right) \chi^2 - \frac{d}{d\eta} (\frac{\dot{a}}{a} \chi^2) \right] \tag{5.4}
\]

where the last term is a surface term.\(^1\)

If we confine the scalar field in a box of co-moving volume \(L^3\) (fixed coordinate volume), we can expand it in normal modes

\[
\chi(x) = \sqrt{\frac{2}{L^3}} \sum_k \left[ q^+_k \cos \vec{k} \cdot \vec{x} + q^-_k \sin \vec{k} \cdot \vec{x} \right] \tag{5.5}
\]

which leads to the Lagrangian

\[
L(\eta) = \frac{1}{2} \sum_{\sigma} \sum_k \left[ (q^+_{k\sigma})^2 - 2(1 - 6 \xi_d) \frac{\dot{a}}{a} q^+_{k\sigma} q^-_{k\sigma} - \left( k^2 + m^2 a^2 + (6 \xi_d - 1) \frac{\ddot{a}}{a} \right) q^{2\sigma}_{k\sigma} \right] \tag{5.6}
\]

\(^1\)The part of the surface term proportional to \(\xi_d\) has been added in by hand. The surface term ensures that the second derivative of the scale factor doesn’t appear in the Lagrangian.\(^{[17]}\). This is necessary to have a consistent variational theory when the scale factor is treated dynamically rather than kinematically.\(^{[68]}\).
where \( k = |\vec{k}| \) and \( L(\eta) = \int \mathcal{L}(x) d^3 \vec{x} \). Canonical momenta are
\[
p^\sigma_k = \frac{\partial L(\eta)}{\partial \dot{q}^\sigma_k} = \dot{q}^\sigma_k - (1 - 6\xi_d) \frac{\dot{a}}{a} q^\sigma_k.
\] (5.7)

Defining the canonical Hamiltonian the usual way we find
\[
H(\eta) = \frac{1}{2} \sum_{\sigma} \sum_{\vec{k} > 0} \left[ p^2_k + (1 - 6\xi_d) \frac{\dot{a}}{a} (p^\sigma_k q^\sigma_k + q^\sigma_k p^\sigma_k) + \left( k^2 + m^2 a^2 + 6\xi_d(6\xi_d - 1) \frac{\dot{a}^2}{a^2} \right) q^2_k \right]
\] (5.8)

where the sum is over positive \( \vec{k} \) only since we have an expansion over standing rather than travelling waves.

The system is quantized by promoting \((p^\sigma_k, q^\sigma_k), (p^\sigma_{\vec{k}}, q^\sigma_{\vec{k}})\) to operators obeying the usual harmonic oscillator commutation relation. Thus the amplitude functions of the normal modes behave like time-dependent harmonic oscillators. (The Hamiltonian is not unique but is a result of our time coordinate and choice of canonical variables.)

The above shows that a scalar field can be represented as a bath of parametric oscillators. In order to study the noise properties of the quantum field, we now introduce an interaction between the system, which can be a particle detector or a field mode, and the bath, the scalar field.

### 5.1 Spectral Density of a Scalar Field

Consider the general form of interaction between the system harmonic oscillator \( r \), and a scalar field \( \chi \) of the form
\[
\mathcal{L}_{int}(x) = -\epsilon r \chi(x) \delta(\vec{x}_0).
\] (5.9)

They are coupled at the spatial point \( \vec{x}_0 \) with coupling strength \( \epsilon \). We want to derive the spectral density function for this field \( I(\omega) = \sum \delta(\omega - \omega_n)c_n^2/2\kappa_n \). Integrating out the spatial variables we find that
\[
L_{int}(\eta) = \int \mathcal{L}_{int}(x) d^3 \vec{x} = -\epsilon r \chi(\vec{x}_0, \eta)
\] (5.10)

where
\[
\chi(\vec{x}_0, \eta) = \sqrt{\frac{2}{L^3}} \sum_{\vec{k}} [q^+_{\vec{k}} \cos \vec{k} \cdot \vec{x}_0 + q^-_{\vec{k}} \sin \vec{k} \cdot \vec{x}_0].
\] (5.11)

Comparing this with (2.1) we see that each set of modes has the effective coupling constants
\[
c^+_{\vec{k}} = \sqrt{\frac{2}{L^3}} \epsilon \cos \vec{k} \cdot \vec{x}_0, \quad c^-_{\vec{k}} = \sqrt{\frac{2}{L^3}} \epsilon \sin \vec{k} \cdot \vec{x}_0.
\] (5.12)

In the continuous limit the oscillator label \( n \) is replaced by \( \vec{k} \). Adding the spectral densities from both the \( \pm \) sets of modes we obtain
\[
I(k) = \frac{c^2}{L^3} \sum_{\vec{k}} \delta(k) \frac{1}{\kappa_{\vec{k}}}
\] (5.13)
where $\omega$ is replaced by $k$. In the continuous limit: $\sum_\mathbf{k} \to \left(\frac{1}{2\pi}\right)^3 \int d^3\mathbf{k}$. Writing $d^3\mathbf{k} = k^2 \sin \theta dk d\theta d\phi$ and integrating between the limits $\phi[2\pi, 0]$ and $\theta[\pi/2, 0]$ (remembering we only include half of the modes) $\sum_\mathbf{k} \to \frac{L^3}{(2\pi)^2} \int_0^\infty k^2 dk$, we get

$$I(k) = \frac{\epsilon^2 k^2}{(2\pi)^2 \kappa_k}. \quad (5.14)$$

For a two-dimensional scalar field we get

$$I(k) = \frac{\epsilon^2}{2\pi \kappa_k}.$$

### 5.2 Accelerated Observer

We consider a two dimensional massless scalar field $\Phi$ in flat space with mode decomposition

$$\Phi(x) = \sqrt{\frac{2}{L}} \sum_k [q_k^+ \cos kx + q_k^- \sin kx]. \quad (5.15)$$

The Lagrangian for the field can be expressed as a sum of coupled oscillators with amplitudes $q_k^\pm$ for each mode

$$L(s) = \frac{1}{2} \sum_\sigma \sum_k \left[ (\dot{q}_k^\sigma)^2 - k^2 q_k^\sigma^2 \right]. \quad (5.16)$$

Now consider an observer undergoing constant acceleration $a$ in this field with trajectory

$$x(\tau) = \frac{1}{a} \cosh a\tau, \quad s(\tau) = \frac{1}{a} \sinh a\tau. \quad (5.17)$$

We want to show via the influence functional method that the observer detects a thermal radiation. This effect was first proposed by Unruh [33], as inspired by the Hawking effect [31] for black holes. Let us see what the spectral density is. The particle-field interaction is

$$\mathcal{L}_{\text{int}}(x) = -\epsilon r \Phi(x) \delta(x(\tau)) \quad (5.18)$$

where they are coupled at the spatial point $x(\tau)$ with coupling strength $\epsilon$ and $r$ is the detector’s internal coordinate. Integrating out the spatial variables we find that

$$L_{\text{int}}(\tau) = \int \mathcal{L}_{\text{int}}(x) dx = -\epsilon r \Phi(x(\tau)). \quad (5.19)$$

Comparing (5.19) with (2.1) we see that the accelerated observer is coupled to the field with effective coupling constants

$$c_n^+(s) = \epsilon \sqrt{\frac{2}{L}} \cos kx(\tau), \quad c_n^-(s) = \epsilon \sqrt{\frac{2}{L}} \sin kx(\tau). \quad (5.20)$$
From (2.26) the spectral density in the discrete case is given by
\[ I(k, \tau, \tau') = \sum_{\sigma} \sum_{n} \frac{\delta(k - k_n) c^0_{\sigma}(\tau)c^0_{\sigma}(\tau')}{2\omega_n} \]  
(5.21)
where we have to include the sum over both sets of modes and we have put \( \kappa_n = \omega_n = |k_n| \). This ensures that \( f_n(s_i) = 0 \) in (2.21). Making use of (5.20) and \( \sum \rightarrow \frac{L}{2\pi} \int \frac{dk}{2\pi} \) we find that (5.21) becomes
\[ I(k, \tau, \tau') = I(k) \cos k[x(\tau) - x(\tau')] \]  
(5.22)
where \( I(k) = \frac{\omega^2}{2\pi \omega} \) is the spectral density of the (2-dim) scalar field seen by an inertial detector. From (4.3) and (4.4) we can write, using an initial vacuum state,
\[ \zeta(s(\tau), s(\tau')) = \nu(s, s') + i\mu(s, s') = \int_0^\infty dk I(k, \tau, \tau') e^{-i\omega[s(\tau) - s(\tau')].} \]  
(5.23)
We can rewrite this as
\[ \zeta(\tau, \tau') = \frac{1}{2} \int_0^\infty dk' I(k') \exp \left(-2ik'e^{a\Sigma} \sinh[a\Delta]/a\right) \]
(5.25)
which upon using (5.17) can be written as
\[ \zeta(\tau, \tau') = \frac{1}{2} \int_0^\infty dk G(k) \left[ \coth \left( \frac{\pi k}{a} \right) \cos k(\tau - \tau') - i \sin k(\tau - \tau') \right] \]  
(5.27)
where
\[ G(k) = \frac{2}{\pi a} \sinh(\pi k/a) \int_0^\infty dk' I(k') \left[ K_{2ik/a} \left( 2k'e^{a\Sigma}/a \right) \right] \]
(5.28)
In deriving this we have used the integral identity
\[ \int_0^\infty dx x^\mu K_\nu(ax) = 2^{\mu-1} a^{-\mu-1} \Gamma \left( \frac{1+\mu+\nu}{2} \right) \Gamma \left( \frac{1+\mu-\nu}{2} \right) \]  
(5.29)
and the properties of gamma functions. Comparing (5.27) with (4.3-4) we see that a thermal spectrum is detected at temperature
\[ k_B T = \frac{a}{2\pi}. \]  
(5.30)
This was first found by Unruh \[ \text{[33]} \] and stated in this form recently by Anglin \[ \text{[51]} \].
5.3 Hawking Radiation in Black Holes

Consider the metric of a two-dimensional uncharged black hole with mass $m$
\[
d s^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \tag{5.31}
\]

In the Regge-Wheeler coordinates
\[
r^* = r + 2m \ln \left|\frac{r}{2m} - 1\right| \tag{5.32}
\]
this can be written as
\[
d s^2 = \left(1 - \frac{2m}{r}\right) (dt^2 - dr^{*2}). \tag{5.33}
\]

The Kruskal coordinates are defined by
\[
\bar{t} - \bar{r}^* = -4m \exp \left[\frac{r^* - t}{4m}\right], \quad \bar{t} + \bar{r}^* = 4m \exp \left[\frac{r^* + t}{4m}\right]. \tag{5.34}
\]

With this the metric becomes
\[
d s^2 = \frac{2m}{r} e^{-r/(2m)} (d\bar{t}^2 - d\bar{r}^{*2}). \tag{5.35}
\]

Since the metric (5.35) is conformal to flat space, the field theory is equivalent to that of flat space. Thus a detector at constant Kruskal position $\bar{r}^*$ will have an influence functional identical in form to that of an inertial detector in flat two-dimensional spacetime in Kruskal coordinates. However we are interested in a detector at constant $r^*$. In this case we see from (5.34) that constant $r^*$ is effectively an accelerating detector in Kruskal coordinates since
\[
\bar{r}^*(t) = 4me^{r^*/(4m)} \cosh[\frac{t}{(4m)}]. \tag{5.36}
\]

We also want to express the influence functional in cosmic time $t$ which from (5.34) is
\[
\bar{t}(t) = 4me^{r^*/(4m)} \sinh[\frac{t}{(4m)}] \tag{5.37}
\]
for the detector at constant $r^*$. This case is now similar to the accelerating observer and as in Sec. 5.2 the spectral density is
\[
I(k, t, t') = I(k) \cos k[\bar{r}^*(t) - \bar{r}^*(t')] \tag{5.38}
\]
where $I(k) = \frac{\omega^2}{2k^2}$ and $\omega = |k|$. With this spectral density we can write for a massless scalar field in a two-dimensional black hole spacetime
\[
\zeta(t, t') \equiv \nu(t, t') + i\mu(t, t') = \frac{1}{2} \int_0^{\infty} dk \ I(k)e^{-ik[\bar{r}^*(t) - \bar{r}^*(t') + \bar{t}(t) - \bar{t}(t')]}
+ \frac{1}{2} \int_0^{\infty} dk \ I(k)e^{-ik[\bar{r}^*(t') - \bar{r}^*(t) + \bar{t}(t) - \bar{t}(t')]} \tag{5.39}
\]
Comparing (5.39) and (5.24) we see that this case is identical to the accelerated observer if we identify $a \equiv 1/(4m)$. The factor involving $r^*$ can be absorbed into the definition of $k$. Hence we can rewrite (5.39) as

$$\zeta(t, t') = \int_0^\infty dk \ I(k) \left[ \coth(4\pi mk) \cos{k(t-t') - i \sin{k(t-t')}} \right]. \quad (5.40)$$

Comparing (5.40) with (4.3-4) we see that a thermal spectrum is detected by an observer at constant $r^*$ at temperature

$$k_B T = \frac{1}{8\pi m}. \quad (5.41)$$

In the two dimensional case the detector response is independent of its position $r^*$. This will not be the case in four dimensions.

## 5.4 Hawking Radiation in de Sitter Space

We now illustrate how the Gibbons-Hawking result [32] can be obtained from the influence functional method. These examples are also of practical use for describing the non-equilibrium dynamics of the homogenous and inhomogenous (density fluctuations) modes of the inflaton field in the early universe [11, 12, 13, 14, 15, 16, 17].

### 5.4.1 Massless conformally coupled field

Consider now a four-dimensional spatially-flat Robertson-Walker (RW) spacetime with metric

$$ds^2 = dt^2 - \sum_i a^2(t) dx_i^2. \quad (5.42)$$

For this metric the Lagrangian density of a massless conformally coupled scalar field, defined by (5.1), is

$$\mathcal{L}(x) = \frac{a^3}{2} \left[ (\dot{\Phi})^2 - \frac{1}{a^2} \sum_i (\Phi, i)^2 - \left( \frac{\ddot{a}^2}{a^2} - \frac{\dot{a}}{a} \right) \Phi^2 \right] \quad (5.43)$$

where a dot denotes a derivative with respect to $t$. Decomposing $\Phi$ in standing wave normal modes we find (after adding a surface term)

$$L(t) = \int \mathcal{L}(x) d^3\bar{x} = \sum_{\sigma} \sum_{\vec{k}} \frac{a^3}{2} \left[ (\dot{q}_k^\sigma)^2 + 2\frac{\dot{a}}{a} q_k^\sigma \dot{q}_k^\sigma - \left( \frac{k^2}{a^2} - \frac{\dot{a}^2}{a^2} \right) q_k^\sigma \right]^2 \quad (5.44)$$

where $k=|\vec{k}|$. If we wrote the Lagrangian in terms of conformal rather than cosmic time we see from (5.6) that we would have obtained a bath of stationary oscillators. Our kernels would then be (4.3) and (4.4) but written in conformal time. If we were to rewrite these kernels in cosmic time we would get the same kernels as those by starting with a Lagrangian written in cosmic time as we are doing here.
The detector-field interaction is of the same form as (5.9) (with $\chi$ replacing $\Phi$) and we find that with $\kappa_k = k$ (5.14) gives

$$I(k) = \left( \frac{\epsilon}{2\pi} \right)^2 k.$$

(5.45)

Using the Lagrangian (5.44) we find from (2.20) that the Bogolubov coefficients are

$$\alpha = \frac{1 + a^2}{2a} e^{-i\eta}, \quad \beta = \frac{1 - a^2}{2a} e^{-i\eta}$$

(5.46)

where $\eta = \int_{t_i}^t dt/a(t)$ with $a(t_i) = 1$. Using these we find that the noise and dissipation kernels (2.18-19) are, for an initial vacuum state

$$\zeta(t, t') = \nu(t, t') + i\mu(t, t') = \frac{1}{a(t)a(t')} \int_0^\infty dk I(k)e^{-ik(\eta-\eta')}.$$

(5.47)

We will now specialise to the de Sitter dynamics where, in the spatially-flat RW coordinatization [7], the scale factor has the time-dependence

$$a(t) = e^{Ht}.$$

(5.48)

In this case $\eta = -\frac{1}{H} e^{-Ht}$ with $t_i = 0$. If we define $\Delta = t - t'$, $2\Sigma = t + t'$ we find that (5.47) becomes

$$\zeta(t, t') = e^{-2H\Sigma} \int_0^\infty dk I(k) \exp \left[ -\frac{2ik}{H} e^{-H\Sigma} \sinh(H\Delta/2) \right].$$

(5.49)

Using (5.26) we find that

$$\zeta(t, t') = \int_0^\infty dk \ G(k) \left[ \coth \left( \frac{\pi k}{H} \right) \cos k(t - t') - i \sin k(t - t') \right].$$

(5.50)

where

$$G(k) = \frac{4 \sinh(\pi k/H)}{\pi He^{2\Sigma}} \int_0^\infty dk' I(k') K_{2ik'/H}(2k' e^{-H\Sigma}/H)$$

$$= \left( \frac{\epsilon}{2\pi} \right)^2 k = I(k).$$

(5.51)

We have again used the integral identity (5.29) and the properties of gamma functions. Comparing (5.50) with (4.3-4) we see that a thermal spectrum is detected by an inertial observer in de Sitter space at temperature

$$k_B T = \frac{H}{2\pi}.$$

(5.52)

Cornwall and Bruinsma [45] who considered the evolution of low momentum modes of an inflaton field coupled to a thermal bath in a de Sitter background also derived the influence
functional for a conformally coupled scalar field in de Sitter space. The noise and dissipation kernels they found in their Eq. (3.31) differs from ours since they did not add a surface term proportional to \( \xi \). As a result they got nonstationary kernels when written in conformal time. As we pointed out previously [47] a surface term is needed to give a consistent variational theory when the scale factor is treated as a dynamical variable. In this case we see from (5.6) that in conformal time conformal coupling with a bath of ordinary stationary oscillators gives the usual stationary kernels. In cosmic time these kernels lead to (5.50) which is still stationary, but shows the expected Gibbons-Hawking temperature.

5.4.2 Massless minimally coupled field

From (5.6) the Lagrangian for a minimally coupled massless field in de Sitter space is

\[
L(\eta) = \sum_\sigma \sum_\vec{k} \frac{1}{2} \left[ (q^\sigma_\vec{k})^2 + \frac{2}{\eta} q^\sigma_\vec{k} q^{\sigma*}_\vec{k} - \left( k^2 - \frac{1}{\eta} \right) q^\sigma_\vec{k} q^{\sigma*}_\vec{k} \right].
\]  

Solving (2.20) (with \( \kappa_n = k \)) we find that the Bogolubov coefficients are

\[
\alpha(\eta) = \left( 1 - \frac{i}{2k\eta} \right) e^{-ik\eta}, \quad \beta(\eta) = -\frac{i}{2k\eta} e^{-ik\eta}.
\]

Substituting these into (2.18-19) we find that

\[
\zeta(\eta, \eta') = \nu(\eta, \eta') + i\mu(\eta, \eta') = \int_0^\infty dk \, I(k) e^{-ik(\eta-\eta')} \left( 1 + \frac{k^2 \eta' + ik(\eta - \eta')}{k^2 \eta'} \right)
\]

where \( I(k) \) is given by (5.45). We want to write this in terms of cosmic time given by \( \eta = -\frac{1}{H} e^{-Ht} \). Following a similar procedure as before, we find

\[
\zeta(t, t') = \int_0^\infty dk \, G(k) \left[ \coth \left( \frac{\pi k}{H} \right) \cos k(t - t') - i \sin k(t - t') \right]
\]

where

\[
G(k) = I(k) \left[ 1 + \frac{H^2}{k^2} + 2i \frac{H}{k} \sinh \left( \frac{H(t - t')}{2} \right) \tanh \left( \frac{\pi k}{H} \right) \right]
\]

and we have ignored a factor \( e^{H(t+t')} \) which gets cancelled by changing the integration measure from \( \eta \) to \( t \) in the influence functional.

The imaginary part of (5.57) generates a contribution to the dissipation kernel of the form

\[
\mu_{im}(t - t') = \frac{\epsilon^2 H}{2\pi} \sinh \left( \frac{H(t - t')}{2} \right) \delta(t - t').
\]

Inserting this into the influence functional (2.10) we see that it leads to a vanishing contribution to the influence functional. Similarly the imaginary part of (5.57) generates a
contribution to the noise kernel of the form

\[
\nu_{im}(t - t') = \frac{2H\epsilon^2}{(2\pi)^2} \sinh \left( \frac{H(t - t')}{2} \right) \int_0^\infty dk \tan \left( \frac{\pi k}{H} \right) \sin k(t - t')
\]

\[
= \frac{2H\epsilon^2}{(2\pi)^2} \left[ -\sinh \left( \frac{H(t - t')}{2} \right) \cos \Lambda(t - t') \bigg|_{\Lambda \to \infty} + \frac{H}{2} \right]
\]

(5.59)

where we have first integrated by parts and then used a standard integral. The first term in (5.59) will generate a vanishing contribution to the influence functional (2.10) since it involves an integral over a term oscillating infinitely fast. The second term in (5.59) can also be ignored since it generates only a zero frequency contribution to the noise spectrum. Thus the imaginary part of (5.57) can be ignored leaving a thermal influence functional at the Gibbons-Hawking temperature but with an effective spectral density of the form

\[
G(k) = I(k) \left[ 1 + \frac{H^2}{k^2} \right].
\]

(5.60)

We see in this spectral density the well known infrared divergence associated with massless, minimally coupled fields in de Sitter spacetime.

Habib and Kandrup claimed [59], from a classical analysis, that a fluctuation-dissipation relation (FDR) would increasingly fail to hold as the physical period of oscillation increased over the expansion timescale of the universe. We suspect that the definition of FDR and its applicability in their work is more restricted than ours. We see that in both of these examples here the FDR (4.5) is exact despite the fact that the physical period of oscillation can be arbitrarily greater than the expansion timescale. This is consistent with the view of [16, 1, 2] that the FDR is a categorical relation as it is a description of the full backreaction of the environment on the system.

6 Discussion

Many physical problems can be modeled by a quantum Brownian particle in a parametric oscillator bath. We mention quantum optics, quantum and semiclassical cosmology and gravity. This paper aims to accomplish two goals:

I. To derive the influence functional of a parametric oscillator bath, the evolution operator and the master equation for the reduced density matrix for explicit use in these problems.

II. To relate the quantum mechanics of oscillators to quantum fields, thus providing a bridge from quantum stochastical mechanics to quantum field theory. This connection can benefit the former with the well-developed techniques of field theory (e.g., use of Feynman diagrams [2, 37, 38]) and enrich the latter with imparting a statistical mechanics meanings to many quantum effects [6, 11, 16].
Two issues are discussed in this paper:
A. The nature and origin of noise and dissipation in quantum fields
B. The statistical mechanical meaning of quantum processes in the early universe and black holes.

On the first issue we have discussed these problems:
1) How to extract the statistical information of a quantum field. We couple a particle detector to the oscillator bath and study the detector’s response to the fluctuations of the field. We found that the characteristics of quantum noise vary with the nature of the field, the type of coupling between the field and the background spacetime, and the time-dependence of the scale factor of the universe.
2) How to relate noise to particle creation. Parametric amplification of vacuum fluctuations and backscattering of waves in the second-quantized formulation give rise to particle creation. By writing the influence functional in terms of the Bogolubov coefficients which determine the amount of particles produced, one can identify the origin of noise in this system to particle creation [30, 35, 36].

On the second issue, we have studied the problem of
3) Quantum noise and thermal radiance. We illustrate how a uniformly accelerating detector in Minkowski space, a static detector outside a black hole and a comoving observer in a de Sitter universe observes a thermal spectrum. The viewpoint of quantum open systems and the method of influence functionals can, in our opinion, lead to a deeper understanding of black hole thermodynamics and quantum processes in the early universe [6].

As further studies, the results obtained here can be useful for the following problems:
a) Decoherence. The transition of the system from quantum to classical requires the diminishing of coherence in the wave function. The noise kernel is found to be primarily responsible for this decoherence process. Decoherence can be studied by analyzing the magnitude of the diffusion coefficients in the master equation. The new result obtained here is useful for the analysis of decoherence where parametric excitation is present in the environment. This is the case when considering the quantum to classical transition of the wavefunction of the universe [13, 15], homogenous and inhomogenous modes (density fluctuations) of an inflaton field [16, 17, 21] or the primordial gravitational radiation background. For the case of density fluctuations we can expect decoherence, dissipation and diffusion to have important consequences for the amplitude and spectrum of density perturbations. The relation of particle creation and decoherence was one of the original physical motivations for this work. Indeed one of us has speculated [11] that in the early universe, vacuum particle creation and decoherence can be important at the same scale near the Planck time. We will address these issues at a later time.
b) Backreaction. The backreaction of these quantum field processes manifests as dissipation effect, which is described by the dissipation kernel [20]. In [38, 39] we outline a program for studying the backreaction of particle creation in semiclassical cosmology in the open system framework. We use a model where the quantum Brownian particle and the oscillator bath are coupled parametrically. The field parameters change in time through the time-dependence of
the scale factor of the universe, which is governed by the semiclassical Einstein equation. We can derive an expression for the influence functional in terms of the Bogolubov coefficients as a function of the scale factor. The equation of motion becomes an Einstein-Langevin equation, from which a new, extended theory of semiclassical gravity is obtained. This, in our opinion, is necessary for furthering the investigation of quantum and statistical processes in curved spacetimes. We are currently pursuing these investigations from this viewpoint.

c) A fluctuation-dissipation relation for non-equilibrium quantum fields. Sciama [62] first suggested that the thermal radiance in a uniformly accelerated observer (Unruh effect) and in black holes (Hawking effect) can be understood in terms of a fluctuation-dissipation relation. This relation was also later derived for de Sitter spacetime via linear response theory by Mottola [63]. These familiar cases all deal with spacetimes with event horizons and thermal particle creation. From earlier particle creation-backreaction studies in semiclassical gravity [49] a general FDR was conjectured by one of us [16] for quantum fields in curved spacetimes. It corresponds to a non-equilibrium generalization of Hawking-Unruh effect to general dynamical spacetimes without event horizons. Such a relation can in principle be identified from the results of this paper. The interpretation of backreaction processes in terms of fluctuation-dissipation relations will be explored further in [64, 36].

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A Influence Functional

Here we describe the calculation of the influence functional. From (2.9) the influence functional is

$$\mathcal{F}[x, x'] = Tr[\hat{U}(t, t_i)\hat{\rho}_b(t_i)\hat{U}'(t, t_i)]$$ (A.1)

where $\hat{U}(t)$ and $\hat{U}'(t)$ are the quantum propagators for the actions $S_E[q] + S_{int}[x(s), q]$ and $S_E[q] + S_{int}[x'(s), q]$, and $x(s)$ and $x'(s)$ are treated as time dependent classical forcing terms.

Our first task is to determine the propagator for the action

$$S_E[q] + S_{int}[x(s), q] = \int_{t_i}^t ds \left[ \sum_n \frac{1}{2} m_n(s)(\dot{q}_n^2 + b_n(s)q_n\dot{q}_n - \omega_n^2(s)q_n^2) \right]$$

$$+ \sum_n \left( -c_{1n}(s)F(x(s))q_n - c_{2n}(s)F(\dot{x}(s))q_n 
- c_{3n}(s)F(x(s))\dot{q}_n(s) - c_{4n}(s)F(\dot{x}(s))\dot{q}_n \right).$$ (A.2)

This interaction is the most general interaction possible which is linear in the bath. Dropping the $n$ subscript the Lagrangian for a mode takes the form

$$L(t) = \frac{1}{2} \dot{q}^2 + b(t)q\dot{q} - \omega^2(t)q^2 - q[c_1(t)F(x(t)) + c_2(t)F(\dot{x}(t))]
- q[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))].$$ (A.3)

Defining the canonical momenta the usual way we find that

$$p_c = \frac{\partial L(t)}{\partial \dot{q}} = m(t)\dot{q} + m(t)b(t)\frac{q}{2} - [c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))].$$ (A.4)

The Hamiltonian, $H(t) = p_c\dot{q} - L(t)$ then takes the form

$$H(t) = \frac{p_c^2}{2m(t)} - b(t)(p_cq + qp_c) + \frac{m(t)}{2} \left( \omega^2(t) + \frac{b^2(t)}{4} \right)q^2$$

$$+ \left[ c_1(t)F(x(t)) + c_2(t)F(\dot{x}(t)) - \frac{b(t)}{2}(c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))) \right]q$$

$$+ \frac{[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]}{m(t)}p_c + \frac{[c_3(t)F(x(t)) + c_4(t)F(\dot{x}(t))]^2}{2m(t)}.$$ (A.5)

The system is quantized by promoting $q, p_c$ to operators obeying $[\hat{q}, \hat{p}_c] = i\hbar$. Then writing

$$\hat{q} = \sqrt{\frac{\hbar}{2\kappa}}(\hat{\alpha} + \hat{\alpha}^\dagger), \quad \hat{p}_c = i\sqrt{\frac{\hbar\kappa}{2}}(\hat{\alpha}^\dagger - \hat{\alpha})$$ (A.6)
we find that (A.5) becomes
\[ \dot{H}(t) = f(t) \dot{A} + f^*(t) \dot{A}^\dagger + h(t) \dot{B} + d(t) \dot{a} + d^*(t) \dot{a}^\dagger + g(t) \]  
(A.7)
where \( \dot{A} \) and \( \dot{B} \) are defined in (B.2) and
\[ f(t) = \frac{\hbar}{2} \left( \frac{m(t) \omega^2(t)}{\kappa} + \frac{m(t) b^2(t)}{4\kappa} - \frac{\kappa}{m(t)} + ib(t) \right) \]  
(A.8)
\[ h(t) = \frac{\hbar}{2} \left( \frac{\kappa}{m(t)} + \frac{m(t) \omega^2(t)}{\kappa} + \frac{m(t) b^2(t)}{4\kappa} \right) \]  
(A.9)
\[ d(t) = \sqrt{\frac{\hbar}{2\kappa}} \left[ c_1(t) F(x(t)) + c_2(t) F(\dot{x}(t)) - \left( \frac{b(t)}{2} - i \frac{\kappa}{m(t)} \right) [c_3(t) F(x(t)) + c_4(t) F(\dot{x}(t))] \right] \]  
(A.10)
\[ g(t) = \frac{[c_3(t) F(x(t)) + c_4(t) F(\dot{x}(t))]^2}{2m(t)} \]  
(A.11)

In appendix B we have derived the evolutionary operator generated by the Hamiltonian of (A.7). It has the form
\[ \hat{U}(t, t_i) = \hat{S}(r, \phi) \hat{R}(\theta) \hat{D}(p) \exp \left[ -\frac{pp^*}{2} - \frac{i}{\hbar} \int_{t_i}^{t} g(s) ds + \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \hat{p}(s) \hat{p}^*(s') \right]. \]  
(A.12)

From the first two equations of (B.13) we see that the squeeze and rotation operators do not depend on \( x \). Thus \( \hat{S} = \hat{S}', \hat{R} = \hat{R}' \). Using this fact, the unitiary nature of the operators in the propagator, the cyclic trace rule and the identity
\[ \text{Tr}[\hat{D}(p) \hat{D}(p')] = \hat{D}(p + p') \exp \left[ \frac{1}{2}(pp^* - p^*p') \right] \]  
(A.13)
we find that (A.1) becomes
\[ \mathcal{F}[x, x'] = \text{Tr}[\hat{\rho}_b(t_i) \hat{D}(p - p')] \exp \left[ \frac{1}{2}(pp^* - p^*p' - pp^* - p^*p') \right] \times \exp \left[ \int_{t_i}^{t} ds \int_{t_i}^{s} ds' [\hat{p}(s) \hat{p}^*(s') + \hat{p}^*(s) \hat{p}'(s')] - i \int_{t_i}^{t} ds [g(s) - g'(s)] \right]. \]  
(A.14)
Making use of the integral identity
\[ \int_{b}^{a} g(t) dt \int_{b}^{a} h(t) dt = \int_{b}^{a} \int_{b}^{t} [g(t)h(t') + g(t')h(t)] dt' dt \]  
(A.15)
its possible to write

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\[ \mathcal{F}[x, x'] = Tr[\hat{\rho}_b(t_i) \hat{D}(p - p')] \exp \left[ -\frac{i}{\hbar} \int_{t_i}^{t} ds [g(s) - g'(s)] \right] \]
\[ \times \exp \left[ \frac{1}{2} \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \left( [\hat{p}(s) - \hat{p}'(s)][\hat{p}^*(s') + \hat{p}'^*(s')] + [\hat{p}(s') + \hat{p}'(s')][\hat{p}^*(s) - \hat{p}'^*(s)] \right) \right] \]
\[ = Tr[\hat{\rho}_{th}(t_i) \hat{S}^\dagger(r, \phi) \hat{S}(r, \phi)] \]  \hspace{1cm} \text{(A.16)}

We will now evaluate the influence functional for a squeezed thermal initial state. Our first task is to compute the trace in (A.16). Our initial state is of the form
\[ \hat{\rho}_b(t_i) = \hat{S}(r, \phi) \hat{\rho}_{th} \hat{S}^\dagger(r, \phi) \]  \hspace{1cm} \text{(A.17)}

where \( \hat{\rho}_{th} \) is a thermal density matrix of temperature \( T \) defined by the thermal density matrix takes the form
\[ \hat{\rho}_{th} = \left[ 1 - \exp \left( -\frac{\hbar \omega}{k_B T} \right) \right] \sum_n \exp \left( -n \frac{\hbar \omega}{k_B T} \right) |n\rangle \langle n| \]  \hspace{1cm} \text{(A.18)}

and \( \hat{S}(r, \phi) \) is a squeeze operator defined in (B.12).

The trace in (A.16) becomes
\[ Tr[\hat{\rho}_b(t_i) \hat{D}(p - p')] = Tr[\hat{\rho}_{th} \hat{S}^\dagger(r, \phi) \hat{D}(p - p') \hat{S}(r, \phi)]. \]  \hspace{1cm} \text{(A.19)}

Making use of \[ 55 \]
\[ \hat{S}^\dagger(r, \phi) \hat{D}(p) \hat{S}(r, \phi) = \hat{D}(p \cosh r + p^* \sinh re^{2i\phi}) \]  \hspace{1cm} \text{(A.20)}
equation (B.16) and \[ 70 \]
\[ Tr[\hat{\rho}_{th} \exp(t \hat{a}^\dagger + u \hat{a})] = \exp \left[ \frac{tu}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right] \]  \hspace{1cm} \text{(A.21)}
we find that
\[ Tr[\hat{\rho}_b(t_i) \hat{D}(p - p')] = \exp \left[ -\frac{1}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) |(p - p') \cosh r + (p - p')^* \sinh re^{2i\phi}|^2 \right]. \]  \hspace{1cm} \text{(A.22)}
Making use of the integral identity (A.15) we can write
\[ Tr[\hat{\rho}_b(t_i) \hat{D}(p - p')] = \exp \left[ -\frac{1}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \int_{t_i}^{t} ds \int_{t_i}^{s} ds' \right. \]
\[ \times \left\{ [\hat{p}(s) - \hat{p}'(s)][\hat{p}(s') - \hat{p}'(s')]^* \cosh 2r \right. \]
\[ + [\hat{p}(s) - \hat{p}'(s)][\hat{p}(s') - \hat{p}'(s')] \cosh 2r \]
\[ + \sinh 2re^{-2i\phi}[\hat{p}(s) - \hat{p}'(s)][\hat{p}(s') - \hat{p}'(s')] \]
\[ + \sinh 2re^{2i\phi}[\hat{p}(s) - \hat{p}'(s)][\hat{p}(s') - \hat{p}'(s')]^* \right\} \]  \hspace{1cm} \text{(A.23)}

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Now put into (B.17)

\[ d = uF(x) + vF(\dot{x}) \]  

(A.24)

where from (A.10)

\[
\begin{align*}
    u &= \sqrt{\frac{\hbar}{2\kappa}} \left( c_1 - \frac{b c_3}{2} - i\kappa \frac{c_3}{m} \right), \\
    v &= \sqrt{\frac{\hbar}{2\kappa}} \left( c_2 - \frac{b c_4}{2} - i\kappa \frac{c_4}{m} \right)
\end{align*}
\]  

(A.25)

and further define

\[
U = u\beta^* + u^*\alpha^*, \quad V = v\beta^* + v^*\alpha^*.
\]  

(A.26)

Using (B.17,A.23) and (A.16) we find that the influence functional for a mode \( n \) takes the form

\[
F_n[x, x'] = \exp \left[ -\frac{2i}{\hbar} \int_{t_i}^{t_f} ds \int_{t_i}^{s'} ds' \left[ \Delta(s) \mu_{1n}(s, s')\Sigma(s') + \dot{\Delta}(s) \mu_{2n}(s, s')\Sigma(s') \right. \right.
\]

\[
\left. + \Delta(s) \mu_{3n}(s, s') \dot{\Sigma}(s') + \dot{\Delta}(s) \mu_{4n}(s, s') \dot{\Sigma}(s') \right]
\]

\[
\left. - \frac{1}{\hbar} \int_{t_i}^{t_f} ds \int_{t_i}^{s} ds' \left[ \Delta(s) \nu_{1n}(s, s')\Delta(s') + \Delta(s) \nu_{2n}(s, s')\dot{\Delta}(s') \right. \right.
\]

\[
\left. + \dot{\Delta}(s) \nu_{3n}(s, s')\Delta(s') + \dot{\Delta}(s) \nu_{4n}(s, s')\dot{\Delta}(s') \right] \right]
\]

\[ - \frac{i}{\hbar} \int_{t_i}^{t_f} ds [g(s) - g'(s)] \]

(A.27)

where

\[
\Delta(s) = [F(x(s)) - F(x'(s))], \quad 2\Sigma(s') = [F(x(s')) + F(x'(s'))]
\]

\[
\dot{\Delta}(s) = [F(\dot{x}(s)) - F(\dot{x}'(s))], \quad 2\dot{\Sigma}(s') = [F(\dot{x}(s')) + F(\dot{x}'(s'))]
\]

\[
\begin{align*}
\mu_{1n}(s, s') &= \frac{i}{2\hbar} [U(s)U^*(s') - U^*(s')U(s')], \\
\mu_{2n}(s, s') &= \frac{i}{2\hbar} [V(s)V^*(s') - V^*(s')V(s')], \\
\mu_{3n}(s, s') &= \frac{i}{2\hbar} [U(s)V^*(s') - U^*(s)V(s')], \\
\mu_{4n}(s, s') &= \frac{i}{2\hbar} [V(s)V^*(s') - V^*(s)V(s')]
\end{align*}
\]  

(A.28)

and

\[
\nu_{1n}(s, s') = \frac{1}{2\hbar} \coth \left( \frac{\hbar \omega}{2\kappa_B T} \right) \left[ \cosh 2r(U(s)U^*(s') + U^*(s)U(s')) \right.
\]

\[
\left. - \sinh 2re^{-2i\phi} U(s)U(s') - \sinh 2re^{2i\phi} U^*(s)U^*(s') \right]
\]  

(A.29)
ν₂ₙ(s, s') = \frac{1}{2\hbar} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \cosh 2r(U(s)V^*(s') + U^*(s)V(s')) - \sinh 2re^{-2i\phi}U(s)V(s') - \sinh 2re^{2i\phi}U^*(s)V^*(s') \right] \tag{A.30}

ν₃ₙ(s, s') = \frac{1}{2\hbar} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \cosh 2r(V(s)U^*(s') + V^*(s)U(s')) - \sinh 2re^{-2i\phi}V(s)U(s') - \sinh 2re^{2i\phi}V^*(s)U^*(s') \right] \tag{A.31}

ν₄ₙ(s, s') = \frac{1}{2\hbar} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \cosh 2r(V(s)V^*(s') + V^*(s)V(s')) - \sinh 2re^{-2i\phi}V(s)V(s') - \sinh 2re^{2i\phi}V^*(s)V^*(s') \right] \tag{A.32}

and

\[ g(s) = \frac{\left[ c_3(s)F(x(s)) + c_4(s)F(\dot{x}(s)) \right]^2}{2m(s)}. \tag{A.33} \]

Note that the g term in the influence functional can be absorbed into the system Lagrangian. Of course the total influence functional is an infinite product of influence functionals over n. Therefore if we define the spectral density as

\[ I(\omega, s, s') = \sum_n \delta(\omega - \omega_n) \frac{c_n(s)c_n(s')}{2\kappa_n} \] \tag{A.34}

we obtain the results of (2.18-19) where we have put c₂, c₃, c₄ = 0.

B Propagator

Consider the Hamiltonian

\[ \hat{H}(t) = f(t)\hat{A} + f^*(t)\hat{A}^\dagger + h(t)\hat{B} + d(t)\hat{a} + d^*(t)\hat{a}^\dagger + g(t) \] \tag{B.1}

where

\[ \hat{A} = \frac{\hat{a}^2}{2}, \quad \hat{A}^\dagger = \frac{\hat{a}^\dagger 2}{2}, \quad \hat{B} = \hat{a}^\dagger \hat{a} + 1/2. \tag{B.2} \]

and [\hat{a}, \hat{a}^\dagger] = 1. We want to find the propagator for this general time dependent system. We make the ansatz

\[ \hat{U}(t, t_i) = e^{\hat{x}(t)\hat{B}} e^{\hat{y}(t)\hat{A}^\dagger} e^{\hat{z}(t)\hat{A}^\dagger} e^{\hat{\phi}(t)\hat{a}^\dagger} e^{\hat{\rho}(t)\hat{a}^\dagger} e^{\hat{r}(t)}. \tag{B.3} \]

It has proved by Fernandez [39] that this is global. It must satisfy the evolution equation for the propagator

\[ \hat{H}(t)\hat{U}(t, t_i) = i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_i) \] \tag{B.4}
subject to the initial condition $\hat{U}(t_i, t_i) = 1$. We find that the operators $\hat{A}, \hat{A}^\dagger, \hat{B}, \hat{a}, \hat{a}^\dagger$ satisfy the following commutation relations

$$\begin{align*}
[\hat{A}, \hat{A}^\dagger] &= \hat{B} = \hat{B}^\dagger, \\
[\hat{A}, \hat{B}] &= \hat{A}, \\
[\hat{A}^\dagger, \hat{B}] &= -2\hat{A}^\dagger \\
[\hat{a}, \hat{A}^\dagger] &= \hat{a}^\dagger, \\
[\hat{a}^\dagger, \hat{A}] &= -\hat{a} \\
[\hat{a}, \hat{B}] &= \hat{a}, \\
[\hat{a}^\dagger, \hat{B}] &= -\hat{a}^\dagger \\
[\hat{a}, \hat{A}] &= [\hat{a}^\dagger, \hat{A}^\dagger] = 0.
\end{align*}$$

(B.5)

Making use of the commutation relations and the operator relation

$$e^{u\hat{O}}\hat{P}e^{-u\hat{O}} = \hat{P} + u[\hat{O}, \hat{P}] + \frac{u^2}{2!}[\hat{O}, [\hat{O}, \hat{P}]] + \ldots$$

(B.6)

we find

$$\begin{align*}
e^{q\hat{a}^\dagger} &= (\hat{a}^\dagger + q)e^{q\hat{a}} \\
e^{z\hat{A}^\dagger\hat{a}} &= (\hat{a} - \hat{a}^\dagger z)e^{z\hat{A}^\dagger} \\
e^{y\hat{A}^\dagger} &= (\hat{a}^\dagger + y\hat{a})e^{y\hat{A}} \\
e^{x\hat{B}^\dagger\hat{a}} &= e^{-x}\hat{a}e^{x\hat{B}} \\
e^{x\hat{B}^\dagger\hat{a}^\dagger} &= e^{-x}\hat{a}^\dagger e^{x\hat{B}} \\
e^{x\hat{B}} &= e^{-2x}\hat{A}e^{x\hat{B}} \\
e^{y\hat{A}^\dagger} &= (\hat{A}^\dagger + \hat{B}y + y^2\hat{A})e^{y\hat{A}} \\
e^{x\hat{B}^\dagger\hat{A}^\dagger} &= e^{2x}\hat{A}^\dagger e^{x\hat{B}}.
\end{align*}$$

(B.7)

Substituting (B.3) into (B.4) and using (B.7) we find

$$\begin{align*}
f &= \imath \hbar(\dot{y}e^{-2x} + \dot{y}y^2e^{-2x}) \\
f^* &= \imath \hbar(\dot{z}e^{2x}) \\
h &= \imath \hbar(\dot{x} + \dot{y}) \\
d &= \imath \hbar(\dot{q}(1 - yz)e^{-x} + \dot{p}ye^{-x}) \\
d^* &= \imath \hbar(\dot{p}e^x - \dot{q}ze^x) \\
g &= \imath \hbar(\dot{p}q + \dot{r}).
\end{align*}$$

(B.8)

Since the first three equations of (B.8) are independent of $d$ and $g$ the first three terms in the propagator (B.3) are independent of the last three. As it stands (B.3) is not necessarily unitary. Thus $x, y, z$ must satisfy some further restrictions. If we write

$$x = \ln \alpha, \quad y = -\beta \alpha, \quad z = \beta^*/\alpha$$

(B.9)
where
\[ \alpha = e^{-i\theta} \cosh r, \quad \beta = -e^{-2i\varphi} \sinh r \] (B.10)

then we can write (B.3) as (using relations in [55])
\[ \hat{U}(t, t_i) = \hat{S}(r, \phi) \hat{R}(\theta)e^{\hat{q}\hat{a}^\dagger}e^r \] (B.11)

where 2\phi = 2\varphi - \theta and
\[ \hat{R}(\theta) = e^{-i\theta \hat{B}}, \quad \hat{S}(r, \phi) = \exp[r(\hat{A}e^{-2i\phi} - \hat{A}^\dagger e^{2i\phi})]. \] (B.12)

\( \hat{S} \) and \( \hat{R} \) are called squeeze and rotation operators respectively [55]. They are both unitary as is required. Substituting (B.9) into (B.8) we find
\[ \hat{h} \dot{\alpha} = -if^* \beta - i\hbar \alpha \]
\[ \hat{h} \dot{\beta} = ih\beta + if \alpha \]
\[ \hat{h} \dot{p} = -i(d\beta^* + d^* \alpha^*) \] (B.13)
\[ \dot{q} = -\hat{p}^* \]
\[ \hat{h} \dot{r} = -i g - h \hat{p} q = -i g + h \hat{p}^* p^*. \]

The first 2 equations of (B.13) completely determine \( \alpha \) and \( \beta \). The last three determine \( p, q, r \). Making use of
\[ e^{\hat{F} + \hat{G}} = e^\hat{F} e^\hat{G} e^{-\frac{[\hat{F}, \hat{G}]}{2}} \] (B.14)

where \( \hat{F} \) and \( \hat{G} \) are any operators that satisfy \([\hat{F}, \hat{G}] = \text{constant}\), we find that (B.11) becomes
\[ \hat{U}(t, t_i) = \hat{S}(r, \phi) \hat{R}(\theta) \hat{D}(p) e^{-pp^*/2}e^r \] (B.15)

where
\[ \hat{D}(p) = \exp[p\hat{a}^\dagger - p^* \hat{a}] \] (B.16)

and
\[ p(t, t_i) = -\frac{i}{\hbar} \int_{t_i}^t dt [d(t)\beta^*(t) + d^*(t)\alpha^*(t)] \] (B.17)
\[ r(t, t_i) = -\frac{i}{\hbar} \int_{t_i}^t g(t) dt + \int_{t_i}^t \dot{p}(t) p^*(t, t_i) dt. \] (B.18)

If we define
\[ \dot{p}(t) = \dot{p}_1(t) + i\dot{p}_2(t) \] (B.19)

and use the identity, (A.15), we find that (B.15) becomes
\[ \hat{U}(t, t_i) = \hat{S}(r, \phi) \hat{R}(\theta) \hat{D}(p) \exp \left[i \int_{t_i}^t ds \int_{t_i}^s ds' [\dot{p}_2(s) \dot{p}_1(s') - \dot{p}_1(s) \dot{p}_2(s')] \right] \times \exp \left[-i \frac{h}{\hbar} \int_{t_i}^t g(s) ds \right]. \] (B.20)

This form shows explicitly that the propagator is unitary.
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