An iterative approach for amplitude amplification with nonorthogonal measurements

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Abstract

Using three coupled harmonic oscillators, we present an amplitude-amplification method for factorization of an integer. We generalize the method in [arXiv:1007.4338] by employing non-orthogonal measurements on the harmonic oscillator. This method can increase the probability of obtaining the factors by repeatedly using the nonlinear interactions between the oscillators and non-orthogonal measurements. However, this approach requires an exponential amount of resources for implementation. Thus, this method cannot provide a speed-up over classical algorithms unless its limitations are resolved.

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I. INTRODUCTION

Quantum computing based on qubits has attracted considerable attention (see, e.g., [1-7]). There are several candidates to realize quantum computers, such as using nuclear spins in molecules, photons, trapped ions, superconducting circuit and quantum dots (see, e.g., [7, 8]). However, it is still a great challenge to build a large-scale quantum computer.

Quantum computers can significantly outperform classical computers in doing some specific tasks [3-5, 9]. For example, two important quantum algorithms are Shor’s [10] and Grover’s [11]. Shor’s algorithm [10] can factorize a large integer in polynomial time, offering an exponential speed-up over classical computation. Grover’s algorithm [11] gives a quadratic speed-up in searching database. This search algorithm has been found to be very useful in other related problems [4, 5, 9]. To date, the study of quantum algorithms is a very active area of research (see, e.g., [9]).

Using three coupled harmonic oscillators, we have recently proposed [12] an alternative approach for factoring integers. We consider these three harmonic oscillators to be coupled together via nonlinear interactions [12]. To factorize an integer \( N \), this approach involves only three steps: initialization, time evolution, and conditional measurement. In this approach, the states of the first two harmonic oscillators are prepared in a number-state basis, while the state of the third oscillator is prepared in a coherent state. The states of the first two harmonic oscillators encode the trial factors of the number \( N \). The nonlinear interactions between the oscillators produce coherent states that simultaneously rotate in phase space with different effective frequencies, which are proportional to the product of two trial factors [12]. In this way, all possible products of any two trial factors can be simultaneously computed, and then they are “written” to the rotation frequencies of the coherent states in a single step. The resulting state of the first two oscillators is the factors’ state [12] by performing a conditional measurement of a coherent state rotating with an effective frequency which is proportional to \( N \). However, the probability of obtaining this coherent state becomes low when \( N \) is large. In this paper, we can circumvent this limitation by using an iterative method for increasing the chance of finding the states of the factors. This amplitude-amplification method involves a number of iterations, where each iteration is very similar to the factoring approach we recently proposed [12].

Now we briefly describe this amplitude-amplification method for factorization using three...
coupled harmonic oscillators. Let us now consider the first step of our approach. Initially, the first two harmonic oscillators are in a number-state basis and the third oscillator is in a coherent state. Let the three coupled harmonic oscillators evolve for a period of time. The detection is then conditioned on a coherent state with a rotation frequency being proportional to $N$. The probability of finding this coherent state can be adjusted by choosing both an appropriate period of time evolution and magnitude of the coherent state. Here we find that this probability is not small. Indeed, the probability of finding the factors’ state can be increased by a factor which is the reciprocal of the probability of obtaining this coherent state. But the word ”probability” in these sentences is different to the total probability of obtaining the factors.

The resulting states of the first two oscillators, after the first step, are used as new input states in the second step of our approach. Also, the state of the third oscillator is now prepared as a coherent state with the same, or higher, magnitude. By repeating the same procedure described in the first step, we can obtain the states of the factors with a much higher probability. We then iterate these procedures $L \sim (\log_2 N)$ times, until the probability of finding the factors’ state is close to one.

As an example of how this method works, we show how to factorize the integer $1,030,189 = 1009 \times 1021$. Here the probabilities of obtaining coherent states, with rotation frequencies proportional to $N$, are larger than 0.1 in each iteration. The probability of finding the factors can reach nearly one after 12 iterations. By comparing with the examples $N = 101,617$ and $10,961$, we can show that the required number of iterations for factoring logarithmically scales with $N$.

However, this approach requires an exponential amount of resources for its implementation. First, it requires an exponential amount of energy to encode a number $N$ onto the state of a harmonic oscillator. The input energy scales with the number $N$. Therefore, the required energy becomes enormous if the number $N$ is large. Second, it is necessary to prepare an exponential size $O(N)$ of ensemble of oscillators for this iterative approach. This is because a large number of oscillators are abandoned after a conditional measurement. A number at least of order of $O(N)$ oscillators are thus required to be prepared for the input size $N$. Therefore, this approach does not provide any speed-up for factorization compared to classical algorithms.

This paper is organized as follows: In section II, we introduce a system of coupled har-
monic oscillators. In section III, we study the quantum dynamics of the coupled harmonic oscillators starting with a product state of number states and a coherent state. In section IV, we propose an amplitude-amplification method to factorize an integer using three coupled harmonic oscillators. We discuss the convergence and performance of this factoring algorithm. For example, we show how to factor the number 1,030,189 using this approach. In section V, we discuss the problems and limitations of this approach. Finally, we make a summary in section VI.

II. SYSTEM

We consider a system of \((A + B)\) coupled harmonic oscillators. The Hamiltonian of the \(j\)-th harmonic oscillator is written as

\[
H_{\text{osc}}^j = \frac{P_j^2}{2m_j} + \frac{m_j \omega_j X_j^2}{2},
\]

where \(\omega_j\) is the frequency of the harmonic oscillator, \(m_j\) is the mass of the particle, and \(j = 1, \ldots, A+B\). The operators \(X_j\) and \(P_j\) are the position and momentum operators, which satisfy the commutation relation, \([X_j, P_j] = i\hbar\). The annihilation and creation operators of the \(j\)-th harmonic oscillator are defined as

\[
a_j = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left( X_j + \frac{iP_j}{m_j \omega_j} \right),
\]

\[
a_j^\dagger = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left( X_j - \frac{iP_j}{m_j \omega_j} \right).
\]

The operators \(a_j\) and \(a_j^\dagger\) obey the commutator \([a_j, a_j^\dagger] = 1\). The Hamiltonian of the three harmonic oscillators can be expressed in terms of the annihilation and creation operators:

\[
H_0 = \hbar \sum_{j=1}^{A+B} \omega_j a_j^\dagger a_j,
\]

Here we have ignored the constant term.

We consider the harmonic oscillators coupled to each other via nonlinear interactions \[^12^\text{].}\] Such nonlinear interactions can be described by the Hamiltonian \(H_I\) as

\[
H_I = \hbar \sum_{k=A+1}^{A+B} f_k(n_1, \ldots, n_A) a_k^\dagger a_k
\]
where $f_k(n_1, \ldots, n_A)$ are linear or nonlinear-operator functions (excluding divisions) of the number operators $n_j = a_j^\dagger a_j$, for $j = 1, \ldots, A$, and $k = A + 1, \ldots, A + B$.

The total Hamiltonian $H = H_0 + H_I$ can be written as

$$H = \hbar \sum_{j=1}^{A+B} \omega_j a_j^\dagger a_j + \hbar \sum_{k=A+1}^{A+B} f_k(n_1, \ldots, n_A) a_k^\dagger a_k.$$ (6)

The Hamiltonians $H_0$ and $H_I$ commute with each other, i.e., $[H_0, H_I] = 0$. The total Hamiltonian $H$ in Eq. (6) is exactly solvable. The eigenstate $|E_{m_1,\ldots,m_{A+B}}\rangle$ of the Hamiltonian $H$ is a product state of number states $|m_j\rangle_j$ of the harmonic oscillators, i.e.,

$$|E_{m_1,\ldots,m_{A+B}}\rangle = \prod_{j=1}^{A+B} |m_j\rangle_j,$$ (7)

corresponding to an eigenvalue $E_{m_1,\ldots,m_{A+B}}$

$$E_{m_1,\ldots,m_{A+B}} = \sum_{j=1}^{A+B} \omega_j m_j + \sum_{k=A+1}^{A+B} f_k(m_1, \ldots, m_A) m_k.$$ (8)

### III. QUANTUM DYNAMICS IN PHASE SPACE

We study the time evolution of the $k$-th harmonic oscillator in phase space starting with a state $\prod_j |m_j\rangle_j \otimes |\alpha\rangle_k$, where $\prod_j |m_j\rangle_j$ is the product state of the number states of the harmonic oscillators, and $|\alpha\rangle_k$ is the coherent state of the $k$-th harmonic oscillator, for $j = 1, \ldots, A$, and $k$ is a number from $A+1$ to $A+B$. By applying the time-evolution operator $U(t) = \exp(-iHt)$ [$H$ is the Hamiltonian in Eq. (6)] to the initial state $\prod_j |m_j\rangle_j |\alpha\rangle_k$, it becomes

$$U(t) \prod_j |m_j\rangle_j |\alpha\rangle_k$$

$$= \exp \left[ -i \left( \sum_j \omega_j m_j \right) t \right] \prod_j |m_j\rangle_j$$

$$\times \exp \left\{ -i[\omega_k + f_k(m_1, \ldots, m_A)] a_k^\dagger a_k t \right\} |\alpha\rangle_k,$$

$$= \exp \left[ -i \left( \sum_j \omega_j m_j \right) t \right] \prod_j |m_j\rangle_j |\alpha_{m_1,\ldots,m_A}(t)\rangle_k,$$ (9)

where $\alpha_{m_1,\ldots,m_A}(t)$ is a complex function, i.e.,

$$\alpha_{m_1,\ldots,m_A}(t) = \exp[-i\Omega_{m_1,\ldots,m_A} t] \alpha,$$ (10)
FIG. 1. (Color online) Schematic diagram of the time evolution in phase space of the coherent state of the harmonic oscillator $k$, corresponding to the product of number states $\prod_j |m_j\rangle_j$, where $j = 1, \ldots, A$. The coherent state with complex amplitude $\alpha$ is here depicted as a light blue circle in phase space $(X_k, P_k)$, where $X_k$ and $P_k$ represent the position and the momentum of the harmonic oscillator $k$. After a time $t$, the coherent state rotates about the origin with an angle $t\Omega_{m_1, \ldots, m_A}$.

and $\Omega_{m_1, \ldots, m_A}$ is an effective rotation frequency of the coherent state in phase space

$$\Omega_{m_1, \ldots, m_A} = \omega_k + f_k(m_1, \ldots, m_A). \quad (11)$$

In Eq. (9), we have used the relation

$$\exp(-i\vartheta a_k^\dagger a_k) |\alpha\rangle_k = |\alpha \exp(-i\vartheta)\rangle_k, \quad (12)$$

where $\vartheta$ is a phase factor.

Note that the product state of the number states $\prod_j |m_j\rangle_j$ is an invariant; namely it does not change with time. Nonlinear interactions, described by the Hamiltonian $H_I$ in Eq. (5), cause the coherent state of oscillator $k$ to rotate about the origin with a frequency $\Omega_{m_1, \ldots, m_A}$ in phase space. From Eq. (11), the frequency $\Omega_{m_1, \ldots, m_A}$ depends on the number states $\prod_j |m_j\rangle_j$ of the $A$ oscillators. A schematic diagram of the time evolution of the coherent state of the harmonic oscillator $k$ in phase space is shown in Fig. 1.
IV. FACTORIZATION

Extending our proposal in Ref. [12], we now present an amplitude-amplification method to factor any positive integer $N$. For example, let us consider three coupled harmonic oscillators for factorization by setting $A = 2$ and $B = 1$. Using this method, the probability of finding the factors’ state can reach nearly one after $L$ iterations. Simplified schematic diagrams of the factorization approach are shown in Figs. 2 and 3.

FIG. 2. (Color online) Factoring algorithm using three coupled harmonic oscillators. Initially, the state $\rho_r^{(1)}$ of the harmonic oscillators 1 and 2, is prepared in a number-state basis. By repeatedly applying $L$ times the iterations $I_l$ (as shown in Fig. 3), the resulting state of the harmonic oscillators 1 and 2 becomes the state of the factors of $N$. Finally, the factors of the number $N$ can be obtained by measuring the state of the oscillators 1 and 2.

FIG. 3. (Color online) The $l$-th iteration, $I_l$, in factorization. In each iteration, the third harmonic oscillator is prepared in a coherent state $|\alpha^{(l)}_N\rangle_3$ with a magnitude $|\alpha^{(l)}|$. Then, a unitary operator $U_f(t_l)$ is applied. By performing a conditional measurement of the coherent state $|\alpha^{(l)}_N(t_l)\rangle_3$, the reduced density matrix $\rho_r^{(l+1)}$ can be obtained.
A. Algorithm

Now we consider the total Hamiltonian $H_f$ of the system

$$H_f = \hbar \sum_{j=1}^{3} \omega_j a_j^\dagger a_j + \hbar f_{\text{factor}}(n_1, n_2)a_3^\dagger a_3,$$

where the nonlinear-operator function $f_{\text{factor}}(n_1, n_2)$ for computing the product of any two trial factors is written as,

$$f_{\text{factor}}(n_1, n_2) = \sum_{k=1}^{K} g_k (n_1 n_2)^k.$$ (14)

Here the parameters $g_k$ are the coupling strengths of the nonlinear interactions $(n_1 n_2)^k a_3^\dagger a_3$, for $k = 1, \ldots, K$. This operator function $f_{\text{factor}}$ will output an eigenvalue which is a power series of the product $n \times m$, for the product state $|n, m\rangle = |n\rangle_1 |m\rangle_2$.

The total state of the first two harmonic oscillators, in the number state basis, is initially prepared, i.e.,

$$\rho^{(1)}_r = \sum_{n,m,n',m'} p^{n'm'}_{nm} |n,m\rangle\langle n', m'|,$$ (15)

where the $p^{n'm'}_{nm}$ are the probabilities of the states of the first and second oscillators, while $n, n', m, m' = 2, \ldots, N/2$. The states of the harmonic oscillators 1 and 2 can be prepared in arbitrary states, including pure states or mixed states [12]. The states of oscillators 1 and 2 encode all trial factors of the number $N$. Each number state $|n\rangle_j$ represents each trial factor $n$, for $j = 1, 2$.

1. First iteration: $I_1$

In the first iteration, the state of the third harmonic oscillator is prepared in a coherent state $|\alpha^{(1)}\rangle_3$. By applying the time-evolution operator $U_f(t_1) = \exp(-iH_f t_1)$ to the initial state $\rho^{(1)}(0) = \rho^{(1)}_r \otimes |\alpha^{(1)}\rangle_3 \langle \alpha^{(1)}|$, it becomes

$$\rho^{(1)}(t_1) = U_f(t_1)\rho^{(1)}_r \otimes |\alpha^{(1)}\rangle_3 \langle \alpha^{(1)}|U_f^\dagger(t_1),$$

$$\rho^{(1)}(t_1) = \sum_{n,m,n',m'} \tilde{p}^{n'm'}_{nm} |n,m\rangle\langle n', m'|\alpha^{(1)}_n(t_1)\rangle_3 \langle \alpha^{(1)}_{m'}(t_1)|, \langle n', m'|$$ (16)

$$\tilde{p}^{n'm'}_{nm} (t_1) = \exp\{i[(n' - n)\omega_1 + (m' - m)\omega_2]t_1\} p^{n'm'}_{nm}. \quad \text{(18)}$$
Here $\alpha_{nm}^{(l)}(t)$, for $l = 1$, in Eq. (17) is a complex function,

$$\alpha_{nm}^{(l)}(t) = \exp(-i\Omega_{nm}t)\alpha^{(l)}, \quad (19)$$

and $\Omega_{nm}$ is an effective rotation frequency of the coherent state in phase space

$$\Omega_{nm} = \omega_3 + \sum_{k=1}^{K} g_k(nm)^k. \quad (20)$$

Note that the product of the two factors of $r$ and $s$ is equal to $N$: $r \times s = N$. The rotation frequency $\Omega_N$ of the coherent state is

$$\Omega_N = \omega_3 + \sum_{k=1}^{K} g_k N^k. \quad (21)$$

If the product of any two numbers is not equal to $N$, then the frequencies $\Omega_{nm}$ are different to the frequency $\Omega_N$. Thus, the state of the harmonic oscillator 3 can act as a “marker” for the states of factors and non-factors [12].

Now we define a measurement operator $\mathcal{M}_l$ which can be written as [17]

$$\mathcal{M}_l \rho^{(l)}(t_l) = \mathcal{J}(E_l)\rho^{(l)}(t_l) = \sum_l E_l \rho^{(l)}(t_l) E_l^\dagger, \quad (22)$$

$$E_l = |\alpha_N^{(l)}(t_l)\rangle_3 \langle \alpha_N^{(l)}(t_l)|, \quad (23)$$

where $\alpha_N^{(l)}(t_l) = \alpha^{(l)} \exp(-i\Omega_N t_l)$ and $l$ is the number of iterations. A conditional measurement $\mathcal{M}_1$ is performed on the third oscillator at the time $t_1$. The probability of obtaining this coherent state $|\alpha_N^{(1)}(t_1)\rangle_3$ becomes

$$\Pr(E_1) = \text{Tr}[\mathcal{M}_1 \rho^{(1)}(t_1)], \quad (24)$$

$$= \sum_{n,m} p_{nm}^{\text{un}} |\epsilon_{nm}^{(1)}|^2, \quad (25)$$

where the coefficient $\epsilon_{nm}^{(l)} = 3 \langle \alpha_N^{(l)}(t_l)|\alpha_{nm}^{(l)}(t_l)\rangle_3$, for $l = 1$, and

$$\epsilon_{nm}^{(l)} = \exp\{-|\alpha^{(l)}|^2[1 - \exp(i\Omega_N t_l - i\Omega_{nm} t_l)]\}, \quad \text{for} \quad l \geq 1, \quad (26)$$

is the overlap between the two coherent states $|\alpha_N^{(l)}(t_l)\rangle_3$ and $|\alpha_{nm}^{(l)}(t_l)\rangle_3$, respectively. Note that the value of the probability $\Pr(E_1)$ can be adjusted by appropriately choosing the evolution time and the magnitude $|\alpha^{(l)}|$. In practice, this probability $\Pr(E_1)$ cannot be adjusted to be extremely small.
Recall that a state is conditioned when this state is conditional on the measurement of a certain state \([17]\). After the measurement of \(|\alpha^{(1)}_{N}(t_1)\rangle_3\), the density matrix of the conditioned state can be written as \([17]\):

\[
\rho^{(1)}_c = \frac{\mathcal{M}_1 \rho^{(1)}_{(t_1)}}{\text{Tr}[\mathcal{M}_1 \rho^{(1)}_{(t_1)}]},
\]

\[
= \frac{1}{C_1} \sum_{n,m,n',m'} \tilde{p}^{n'm'}_{nm}(t_1) \epsilon^{(1)}_{nm} \epsilon^{(1)*}_{n'm'} |n, m\rangle \langle n', m' | \otimes |\alpha^{(1)}_{N}(t_1)\rangle_3 \langle \alpha^{(1)}_{N}(t_1)|,
\]

where \(C_1\) is a normalization constant

\[
C_1 = \sum_{n,m} p^{nm}_{nm} |\epsilon^{(1)}_{nm}|^2.
\]

Note that the trace of this density matrix \(\text{Tr}[\rho^{(1)}]\) is equal to one. After the first iteration, the probability of finding the factors of the number \(N\) is increased by a factor \(C_1^{-1}\), which is the inverse of the probability \(\text{Pr}(E_1)\) as seen from Eqs. (25) and (28). The probability amplification [see also Eqs. (53) and (54)] is thus inversely proportional to the probability of obtaining the coherent state \(\text{Pr}(E_1)\).

2. Second iteration: \(I_2\)

After the first iteration, we now obtain the reduced density matrix \(\rho^{(2)}_r\) of the first two harmonic oscillators as

\[
\rho^{(2)}_r = \frac{1}{C_1} \sum_{n,m,n',m'} \tilde{p}^{n'm'}_{nm}(\tilde{t}_2) \epsilon^{(1)}_{nm} \epsilon^{(1)*}_{n'm'} |n, m\rangle \langle n', m' |.
\]

We consider the state \(\rho^{(2)}_r\) in Eq. (30) of the oscillators 1 and 2 as an input state for the second iteration. The coherent state of the third harmonic oscillator is prepared in a coherent state \(|\alpha^{(2)}\rangle_3\), with a magnitude \(|\alpha^{(2)}|\). The nonlinear interactions between the three harmonic oscillators are then turned on for a time \(t_2\). The state evolves as

\[
\rho^{(2)}(t_2) = \frac{1}{C_1} \sum_{n,m,n',m'} \tilde{p}^{n'm'}_{nm}(\tilde{t}_2) \epsilon^{(1)}_{nm} \epsilon^{(1)*}_{n'm'} |n, m\rangle \langle n', m' | \otimes |\alpha^{(2)}_{n'm'}(t_2)\rangle_3 \langle \alpha^{(2)}_{n'm'}(t_2)|,
\]

where \(\tilde{t}_2 = t_1 + t_2\).

Next, a conditional measurement \(\mathcal{M}_2\) is applied to the system at the time \(t_2\). The probability of obtaining the coherent state \(|\alpha^{(2)}_{N}(t_2)\rangle_3\) becomes

\[
\text{Pr}(E_2) = \text{Tr}[\mathcal{M}_2 \rho^{(2)}(t_2)],
\]

\[
= \frac{1}{C_1} \sum_{n,m} p^{nm}_{nm} |\epsilon^{(1)}_{nm}|^2 |\epsilon^{(2)}_{nm}|^2.
\]
The conditioned state can be written as
\[ \rho_c^{(2)} = \frac{\mathcal{M}_2 \rho^{(2)}(t_2)}{\text{Tr}[\mathcal{M}_2 \rho^{(2)}(t_2)]}, \quad (34) \]
\[ = \frac{1}{C_2} \sum_{n,m,n',m'} \tilde{p}_{nm}^{n'm'}(\tilde{t}_2) \epsilon_{n,n'm'}^{(1)} \epsilon_{n,m}^{(2)*} |n,m\rangle \langle n',m'| \otimes |\alpha_N^{(2)}(t_2)\rangle_3 |\alpha_N^{(2)}(t_2)|, \quad (35) \]
where \( C_2 \) is the normalization constant,
\[ C_2 = \sum_{n,m} p_{nm}^{(1)} |\epsilon_{nm}^{(1)}|^2 |\epsilon_{nm}^{(2)}|^2. \quad (36) \]
The probability of finding the factors is enhanced by a factor \( C_2^{-1} \) after the second iteration.

The coefficients \( |\epsilon_{nm}^{(2)}| \) is less than one for any product \( n \times m \neq N \). From Eqs. (29) and (36), it can be seen that \( C_2 < C_1 \). Therefore, the probability of finding the factors is now higher after one additional iteration.

3. \( L \)-th iteration: \( I_L \)

Similarly, we now iterate the procedure \( (L-1) \) times. After \( (L-1) \) iterations, the reduced density matrix of the oscillators 1 and 2 can be written as
\[ \rho_r^{(L)}(t_L) = \frac{1}{C_{L-1}} \sum_{n,m,n',m'} \tilde{p}_{nm}^{n'm'}(\tilde{t}_{L-1}) \prod_{l=1}^{L-1} \epsilon_{nm}^{(l)} \epsilon_{n'm'}^{(l)*} |n,m\rangle \langle n',m'|, \quad (37) \]
where
\[ C_{L-1} = \sum_{n,m} p_{nm}^{(L)} \prod_{l=1}^{L-1} |\epsilon_{nm}^{(l)}|^2, \quad (38) \]
\[ \tilde{t}_{L-1} = \sum_{l=1}^{L-1} t_l. \quad (39) \]
The state of the third harmonic oscillator is now prepared in a coherent state \( |\alpha^{(L)}(t_L)\rangle_3 \), with a magnitude \( |\alpha^{(L)}| \). Let the three coupled harmonic oscillators evolve for a time \( t_L \). This gives
\[ \rho^{(L)}(t_L) = \frac{1}{C_{L-1}} \sum_{n,m,n',m'} \tilde{p}_{nm}^{n'm'}(\tilde{t}_L) \prod_{l=1}^{L-1} \epsilon_{nm}^{(l)} \epsilon_{n'm'}^{(l)*} |n,m\rangle \langle n',m'| \otimes |\alpha_N^{(L)}(t_L)\rangle_3 |\alpha_N^{(L)}(t_L)|. \quad (40) \]
By performing a conditional measurement \( |\alpha_N^{(L)}(t_L)\rangle_3 \), the state becomes
\[ \rho_c^{(L)} = \frac{\mathcal{M}_L \rho^{(L)}(t_L)}{\text{Tr}[\mathcal{M}_L \rho^{(L)}(t_L)]}, \quad (41) \]
\[ = \frac{1}{C_L} \sum_{n,m,n',m'} \tilde{p}_{nm}^{n'm'}(\tilde{t}_L) \prod_{l=1}^{L-1} \epsilon_{nm}^{(l)} \epsilon_{n'm'}^{(l)*} |n,m\rangle \langle n',m'| \otimes |\alpha_N^{(L)}(t_L)\rangle_3 |\alpha_N^{(L)}(t_L)|. \quad (42) \]
After the $L$-th step, the probability of finding the factors is increased by a factor $C_L^{-1}$. From Eq. (38), the probability of obtaining the coherent state $|\alpha_N^{(L)}(t_L)\rangle_3$ can be written as

$$\Pr(E_L) = \frac{1}{C_{L-1}} \sum_{n,m} p_{nm}^L \prod_{l=1}^L |\epsilon_{nm}^{(l)}|^2,$$

$$= \frac{C_L}{C_{L-1}}. \quad (44)$$

The entire iterative procedure is now completed. The convergence and performance of this method will be discussed in the following subsections.

**B. Convergence**

We now study the convergence of this iterative method. We first consider the magnitude of the coherent state $|\alpha^{(l)}|$ for each iteration as

$$|\alpha^{(1)}| \leq |\alpha^{(2)}| \leq \ldots \leq |\alpha^{(l)}| \leq \ldots \leq |\alpha^{(L)}|. \quad (45)$$

Thus, we have

$$1 > |\epsilon_{nm}^{(1)}|^2 \geq |\epsilon_{nm}^{(2)}|^2 \geq \ldots \geq |\epsilon_{nm}^{(l)}|^2 \geq \ldots \geq |\epsilon_{nm}^{(L)}|^2. \quad (46)$$

For any product of $n$ and $m$ being not equal to $N$, the coefficient $|\epsilon_{nm}^{(l)}|^2$ is less than one and decreasing for higher $l$, and the evolution time $t_l$ is non-zero and appropriately chosen. When the number of iterations $L$ tends to infinity, the product of the coefficients $|\epsilon_{nm}^{(l)}|^2$ tends to zero,

$$\lim_{L \to \infty} \prod_{l=1}^L |\epsilon_{nm}^{(l)}|^2 \to 0. \quad (47)$$

The coefficients $|\epsilon_{rs}^{(l)}|^2$ are equal to one for any product of two factors $r$ and $s$ being equal to $N$, i.e., when $r \times s = N$.

Now we consider the probability of finding a pair of factors, $r$ and $s$, after the $l$-th iteration, which is

$$\Pr_l(|r, s\rangle) = \frac{p_{r,s}^l}{C_l}. \quad (48)$$

Since the coefficient $|\epsilon_{nm}^{(l)}|^2$ is less than one, the normalization constant $C_l$ in Eq. (38) is decreasing, i.e., $C_l < C_{l-1}$. Therefore, the probability of finding a pair of factors increases after an additional iteration. This shows that this iterative method is convergent.
From Eqs. (38) and (47), it is very easy to show that

$$\lim_{L \to \infty} C_L = \lim_{L \to \infty} \sum_{n,m}^{L} \prod_{l=1}^{L} |r_{nm}(l)|^2$$

$$= \sum_{r,s} p_{rs}^L$$

(49)

In the limit of large number of iterations $L$, we can obtain the state $\rho_f$ of the factors

$$\lim_{L \to \infty} \rho^{(L)}_r \to \rho_f,$$

(51)

where

$$\rho_f = \frac{1}{C_f^*} \sum_{r,s,r',s'} p_{rs} r' s' |r, s\rangle \langle r', s'|,$$

(52)

and $C_f^* = \sum_{r,s} p_{rs}^L$, with $r, s, r', s'$ are factors of $N$ ($r \times s = r' \times s' = N$). This shows that the state of the factors can be achieved by employing this iterative method, if a sufficiently large number of iterations is used.

C. Performance

We can now estimate the number $L$ of the iterations required to achieve a probability of order of one for factoring $N$. We investigate the amplification ratio $\lambda_l$ of the two probabilities of finding the factors $\text{Pr}_{l-1}(|r, s\rangle)$ and $\text{Pr}_l(|r, s\rangle)$ after the $(l-1)$-th and the $l$-th iterations. From Eq. (48), we have

$$\lambda_l = \frac{\text{Pr}_l(|r, s\rangle)}{\text{Pr}_{l-1}(|r, s\rangle)} = \frac{C_{l-1}}{C_l}.$$  

(53)

(54)

Note that this ratio $\lambda_l$ is just the reciprocal $\text{Pr}^{-1}(E_l)$ in Eq. (41) of the probability of obtaining the coherent state. Practically, this probability $\text{Pr}(E_l)$ cannot be too small. For example, let the amplification ratio $\lambda_l$ be roughly equal to $\lambda \sim O(10^{-1})$ for each iteration, and let the probability of the factors’ state $\text{Pr}_0(|r, s\rangle)$ before the iterations be of order $O(N^{-z})$, where $z$ is a positive number. After $L$ iterations, the probability of finding the states of the factors can be increased by a factor $\lambda^L$. Here we require

$$O(\lambda^L) \sim O(N^z).$$

(55)
Therefore, we obtain that the number $L$ of necessary iterations is of order of $O(\log_2 N)$. In the limit of large $L$, the probabilities $\Pr(E_l)$ in Eq. (44) tend to one because $C_L$ approaches the sum of probabilities of the factors’ states in Eq. (50). The probability of finding the factors will be slowly increased after the number $L \sim O(\log_2 N)$ of iterations is reached.

D. Example: Initial pure states

In this section, we study how to factorize an integer $N$ with an initial pure state using this factoring algorithm. We consider the initial state of the first two harmonic oscillators as the superposition of number states, i.e.,

$$|\Psi(0)\rangle = \left( \frac{1}{D_1} \sum_{n=3}^{\lceil \sqrt{N} \rceil} |n\rangle_1 \right) \left( \frac{1}{D_2} \sum_{m=\lceil \sqrt{N+1} \rceil}^{\lceil N/3 \rceil} |m\rangle_2 \right),$$

(56)

where

$$D_1 = (\lceil \sqrt{N} \rceil - 2)^{1/2},$$

(57)

$$D_2 = (\lceil N/3 \rceil - \lceil \sqrt{N+1} \rceil + 1)^{1/2},$$

(58)

are two normalization constants. Here we consider trial factors from 3, \ldots, $N/3$. The probability of finding the product of two factors is of order of $O(N^{-3/2})$.

For example, now we show how to factor the integer $N = 1,030,189 = 1009 \times 1021$. For simplicity, we now take $K = 1$, which is the lowest order of nonlinearity. The Hamiltonian can be written as

$$H_1 = \hbar \sum_{j=1}^{3} \omega_j a_j^\dagger a_j + \hbar g a_1^\dagger a_1 a_2^\dagger a_2 a_3^\dagger a_3,$$

(59)

where $g$ is the nonlinear strength. The stronger nonlinear strengths and high-order nonlinearity can significantly shorten the required time evolution of the system [12]. But the role of nonlinearity is not directly relevant to the number $L$ of the required iterations for the amplitude amplification.

We take $t^*_l$ as the evolution time for the $l$-th iteration,

$$t^*_l = \frac{2\pi}{g} r_l,$$

(60)

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TABLE I. This table shows the fidelities $F_l$, the probabilities $\Pr(E_l)$ of obtaining the coherent states $|\alpha^{(l)}_N(t^*_l)\rangle_3$ in Eq. (23), and the evolution time $t^*_l$ for the $l$ iterations. Here $E_l$ is the density matrix of the coherent state $|\alpha^{(l)}_N(t^*_l)\rangle_3$ in Eq. (23) and $t^*_l$ is measured in units of $g^{-1}$.

| Iterations | Fidelities $F_l$ | Probabilities $\Pr(E_l)$ | Evolution times $t^*_l$ |
|------------|-----------------|--------------------------|-------------------------|
| $l$         |                 |                          |                         |
| 1          | $2.010 \times 10^{-8}$ | 0.143                     | 1.704                   |
| 2          | $1.403 \times 10^{-7}$ | 0.143                     | 1.342                   |
| 3          | $9.782 \times 10^{-7}$ | 0.143                     | 5.000                   |
| 4          | $6.821 \times 10^{-6}$ | 0.143                     | 4.610                   |
| 5          | $4.739 \times 10^{-5}$ | 0.144                     | 0.732                   |
| 6          | $3.259 \times 10^{-4}$ | 0.145                     | 3.108                   |
| 7          | $2.172 \times 10^{-3}$ | 0.150                     | 1.635                   |
| 8          | $1.445 \times 10^{-2}$ | 0.150                     | 4.559                   |
| 9          | $1.045 \times 10^{-1}$ | 0.138                     | 4.222                   |
| 10         | $5.092 \times 10^{-1}$ | 0.205                     | 6.046                   |
| 11         | $8.506 \times 10^{-1}$ | 0.599                     | 2.434                   |
| 12         | $9.919 \times 10^{-1}$ | 0.858                     | 1.175                   |
| 13         | $9.985 \times 10^{-1}$ | 0.994                     | 5.089                   |
| 14         | $9.997 \times 10^{-1}$ | 0.999                     | 5.833                   |
| 15         | $1.000 \times 10^{-1}$ | 1.000                     | 0.708                   |

where $r_l$ is a uniformly distributed random number on the interval $[0, 1]$. We now evaluate the performance of this method by investigating the fidelity $F_l$ between the reduced density matrix $\rho_{r^{(l+1)}}$ and the factor’s states $\rho_f$ as

$$F_l = \left\{ \text{Tr} \left[ \left( \rho_f^{1/2} \rho_{r^{(l+1)}} \rho_f^{1/2} \right)^{1/2} \right] \right\}^2.$$  \hspace{1cm} (61)

Table I shows the relevant fidelity, the probability of obtaining the coherent state $|\alpha^{(l)}_N(t^*_l)\rangle_3$ with a magnitude $|\alpha^{(l)}| = 2$, and the evolution time $t^*_l$ for each iteration. Before starting the iteration, the initial fidelity $F_0$ is very low: $2.883 \times 10^{-9}$. In the first few steps, the probabilities, $\Pr(E_l)$, of obtaining the coherent states, are about $10^{-1}$. We emphasize that now the probabilities $\Pr(E_l)$ are not extremely small, even when $N$ is large. This
FIG. 4. (Color online) Bar chart: the average of the fidelities between the reduced density matrix $\rho_r^{(l)}$ and the factors’ states $\rho_f$ are plotted versus the number $l$ of iterations in (a) $N = 1,030,189$, (b) $N = 101,617$ and (c) $N = 10,961$. In each iteration, the fidelity $F_l$ is taken at the time $t_l^*$, which is a uniformly distributed random number ranging from 0 to $2\pi/g$. Here the sample size is equal to 100 fidelities. The bars, which indicate the mean values, are shown in light blue. The error bars, which indicate the standard deviations, are shown in black. The dashed lines indicate that the fidelities attain 0.9.

Resolves the limitation of our previous proposal [12]. The probability of finding the factor’s state can be increased by 10 after a single iteration. After ten iterations, the fidelity can exceed 0.5. The fidelity can reach nearly one after two more iterations, while the probabilities
\( P(E_l) \) approach one. Also, we have explicitly shown that this method can factor a number of order of 10^6 with 12 iterations.

Since the evolution time \( t^*_l \) in Eq. (60) is a uniformly distributed random number from 0 to 2\( \pi / g \), it is necessary to study the effect of this randomly chosen time \( t^*_l \) to the performance of factoring. Now we study the averages of the fidelities \( F_l \) and probabilities \( \Pr(E_l) \) of obtaining the coherent states. We also examine their standard deviations in each iteration.

In Fig. 4(a), we plot the average of the fidelities \( F_l \) versus the number \( l \) of iterations. Here we take the sample size to be equal to 100 fidelities in each iteration. As shown in Fig. 4, the average of the fidelities is about 0.5 at the 10-th iteration. The fidelity is greater than 0.9 at the 12-th iteration. After 12 iterations, the mean values of the fidelities reach nearly one. The error bars, which indicate the standard deviations, are shown in the same figure. We can see that, after the 12-th iteration, the standard deviations are relatively small compared to the mean values in each iteration.

We then plot the average of the fidelities \( F_l \) versus the number \( l \) of iterations for \( N = 10,1617 \) and \( 10,961 \) in Fig. 4(a) and (b), respectively. The fidelities exceed 0.9 after the 10-th iteration for \( N = 101,617 \) and the 8-th iteration for \( N = 10,961 \). The numerical results in Fig. 4 show that the required number of iterations increases logarithmically with \( N \) for which the fidelity is greater than 0.9.

In Fig. 5(a), the average of the probabilities \( \Pr(E_l) \) of obtaining the coherent states is plotted versus the number \( l \) of iterations. In the first ten iterations, the probabilities \( \Pr(E_l) \) are about 0.1. Then, it increases and saturates around one after 13 iterations. Moreover, these standard deviations are much smaller than the mean values of \( \Pr(E_l) \). This means that the statistical effect of the time \( t^*_l \) is small on the performance of quantum factorization.

In Fig. 5(b) and (c), the average of the probabilities \( \Pr(E_l) \) of obtaining the coherent states are plotted versus the number \( l \) of iterations for \( N = 101,617 \) and \( N = 10,961 \), respectively. The probabilities are greater than 0.9 after the 11-th iteration for \( N = 101,617 \) and the 9-th iteration for \( N = 10,961 \). Therefore, the results show that the number of iterations \( (\Pr(E_l) \geq 0.9) \) also logarithmically scales with \( N \).
FIG. 5. (Color online) Bar chart: the average of the probabilities of obtaining the coherent states $|\alpha_N^{(l)}(t^*_l)\rangle_3$ are plotted versus the number $l$ of iterations in (a) $N = 1,030,189$, (b) $N = 101,617$ and (c) $N = 10,961$. In each iteration, the probability is taken at the time $t^*_l$ which is a uniformly distributed random number from 0 to $2\pi/g$. Here the sample size is taken to be 100 probabilities. The bars, which indicate the mean values, are in red. The error bars, which indicate the standard deviations, are shown in black. The dashed lines indicate that the probabilities attain 0.9.

V. LIMITATIONS AND PROBLEMS

We have introduced an amplitude-amplification method for factoring integers by using three coupled harmonic oscillators. However, this method requires an exponential amount of resources for implementation. This means that the required resource is of order of $O(N)$,
where $N$ is the input size. This means that this approach cannot provide any speed-up compared to classical algorithms. We discuss these two limitations in the following subsections.

### A. Exponential energy resource

First, this approach requires an exponential amount of energy resource to encode a number $N$ onto the state of a single harmonic oscillator. For example, to factorize a number $N$, two harmonic oscillators are used for encoding $O(N^{3/2})$ possible states of trial factors. The energy $E \sim \hbar \omega O(N^{3/2})$ is thus required, where $\omega \approx \omega_{1,2}$ are the frequencies of the harmonic oscillators 1 and 2. The required energy scales with the size of the input number $N$. Therefore, the energy becomes enormous when the encoded number is large. This encoding method becomes impractical when a large input number is used. This problem may be resolved by using another encoding method. For example, the number could be encoded either onto qubit or qudit states. Thus, the required energy for encoding numbers could be reduced.

### B. Exponential size of the ensemble

Second, this method requires an exponential size of the ensemble of harmonic oscillators for an input $N$. In each iteration, it is necessary to abandon a large number of harmonic oscillators after each conditional measurement. The number of abandoned oscillators is proportional to the failure probability of obtaining conditional measurement of the coherent states in each step. Therefore, a number at least of order of $O(N)$ harmonic oscillators are needed in order to complete the entire procedure. This requires an exponential resource for preparing the ensemble of oscillators. This approach may be improved by employing efficient methods for preparing the ensembles.

### VI. CONCLUSIONS

We have presented an amplitude-amplification method by repeatedly using the nonlinear interactions between the harmonic oscillators and non-orthogonal measurements. We have shown that this approach can be used for factoring integer, and the factors of an integer $N$. 

can be obtained, with a high probability, by using a number of iterations \( L \sim O(\log_2 N) \). We have numerically studied an example for factoring \( N = 1,030,189 \), respectively. We have shown how to factorize an integer of order of \( O(10^6) \) within 12 iterations. In each iteration, the probability of obtaining this coherent state, with the rotation frequency being proportional to \( N \), is not less than 0.1. By comparing examples with \( N = 101,617 \) and \( N = 10,961 \), we have shown that the required number of iterations increases logarithmically with \( N \).

However, using coupled harmonic oscillators, this method requires the use of an exponential amount of resources, i.e., exponential energy and ensemble size. Thus, this approach becomes impractical for large input sizes. We hope that the resolutions could be proposed to overcome the problems of this approach.

Also, we stress that the nonlinear interactions between the coupled oscillators and conditional measurements are essential in this approach. By appropriately controlling nonlinear interactions between the coupled harmonic oscillators, the functions with integer inputs can be evaluated in a single operation. To implement this approach, it is necessary to engineer “many-body” interactions of the system of harmonic oscillators. For example, to perform quantum factorization, it is required to generate “three-body” interactions between the harmonic oscillators. We have briefly discussed the possible implementations in Ref. \[12\]. One of the promising candidates is neutral atoms or polar molecules trapped in optical lattices \[20–22\]. The “three-body” interactions can be tuned by external fields \[20, 21\].

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