Phase-locking dynamics of heterogeneous oscillator arrays

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Abstract

We consider an array of nearest-neighbor coupled nonlinear autonomous oscillators with quenched random frequencies and purely conservative coupling. We show that global phase-locked states emerge in finite lattices and study numerically their destruction. Upon change of model parameters, such states are found to become unstable with the generation of localized periodic and chaotic oscillations. For weak nonlinear frequency dispersion, metastability occur akin to the case of almost-conservative systems. We also compare the results with the phase-approximation in which the amplitude dynamics is adiabatically eliminated.

Keywords: Ginzburg-Landau lattice, Disorder, Localized chaos, Reactive coupling

1. Introduction

Nonlinear dynamics of systems driven out of equilibrium by external (possibly non-conservative) forces is a fascinating research subject. The aim of this vast, interdisciplinary research is to illustrate generic and universal features by simple paradigmatic models. Among them, systems of classical coupled oscillators are of particular interest as they represent a large variety of different physical problems like atomic vibrations in crystals and molecules or field modes in optics or acoustics.

Among a large number of nonlinear effects, synchronization of coupled autonomous oscillators, first empirically observed by Huygens, is one of the most remarkable. The basic theory has been advanced considerably \cite{1} but still attracts considerable attention, as demonstrated by the huge number of papers published in the last decade.

A notable source of complexity in nonlinear systems is heterogeneity of individual elements. For ensembles of linear, heterogeneous oscillators, static disorder is known to lead to Anderson localization of matter, light, and sound waves. Generically, the normal models (eigenstates) display a localized pattern that usually hinders energy propagation \cite{2}.

As a matter of fact, in presence of disorder, nonlinearity, and interactions a whole new set of phenomena may arise. The interplay between disorder and nonlinearity, their
localizing and delocalizing effects on lattice waves is also an intriguing and challenging current research theme. For conservative systems, it has been recognized that the addition of nonlinearity causes interaction among the eigenmodes, which results in a slow wave diffusion [3, 4, 5]. For instance, two different regimes of destruction of Anderson localization (asymptotic weak chaos, and intermediate strong chaos), separated by a crossover condition on densities have been reported [6]. External driving and dissipation can lead to even richer scenarios. For instance, disordered nonlinear array with forcing and dissipation admit Anderson attractors [7]—stationary multipeak patterns composed by a set of interacting, excited Anderson modes. Such attractors emerge by joint effect of the pumping-induced mode excitation, nonlinearity-induced mode interactions, and stabilization by dissipation.

Besides intrinsic heterogeneity of oscillators, another source of disorder may arise from topology of connections, as it occurs for coupled nonlinear oscillator on networks (see e.g. ref.8 and references therein). From the experimental point of view, we mention active optical systems, where joint effect of disorder (both in frequencies and cavity topology) and nonlinearity is relevant for random lasers [8] and lasing networks [10, 11].

If coupling of periodic self-sustained oscillators is weak, the dynamics can be described in terms of the phase approximation, where only a variation of oscillator phases are considered. For two coupled oscillators this leads to an Adler-type equation [1]. The corresponding phase models are extensively used to study oscillator lattices and globally coupled ensembles [12]. Much less studied is the case in which the amplitude dynamics enters into play.

A related important issue is the possibility of observing states with a non-trivial collective dynamics. In the context of coupled phase oscillators, the case of short-ranged coupling (e.g. to nearest-neighbor on a lattice) has been less considered than the globally coupled one. A general question regards the distinction between synchronization and frequency entrainment and the mapping to the diffusion equation [13]. The linear theory predicts frequency entrainment in any dimension, for any non-zero coupling but the role of nonlinear terms is still to be understood. It is known [14] that up to certain values of disorder in natural frequencies of phase oscillators, one can find synchronous states. Also, it has been proved that the probability of frequency entrainment is generically exponentially small in the number of oscillators [15].

In the present work we report a numerical study of a one-dimensional lattice of self-sustained heterogeneous oscillators described by their phase and amplitudes with purely reactive (i.e. conservative) coupling. More precisely, we consider a one-dimensional Ginzburg-Landau lattice [16] where each site has a different (random) natural frequency. In the literature, the case of purely diffusive coupling and global coupling it is mostly considered, see e.g. [17, 18, 19]. An example of global reactive coupling is in Ref. [20].

The choice of a purely conservative and local coupling is motivated by the experiments on lasing networks, where the individual elements (non-identical active and passive fibers) are coupled by non-dissipative optical couplers [21] but applies to more general nonlinear multimode photonic networks [22]. A similar coupling has been recently realized for nanoelectromechanical oscillators [23]. We will illustrate that non-trivial state exist, including frequency-entrained domains and localized chaotic states. The paper is organized as follows. In Section 2 we define the model and in Section 3 we describe some of its phenomenology. Section 4 deals with average properties with respect to the disordered realizations. A brief discussion on the phase approximation and its continuum limit is
2. Model

In the present work we considered a specific paradigmatic model namely the \textit{discrete Ginzburg-Landau} (or locally coupled \textit{Stuart-Landau}) array of oscillators with \textit{random intrinsic frequencies}

\[
\dot{\psi}_n = [i\omega_n + \gamma + (i\alpha - \beta)|\psi_n|^2] \psi_n + (c_2 + ic_3)(\psi_{n-1} + \psi_{n+1}) \tag{1}
\]

where \(\psi_n\) is the complex oscillation amplitude. Frequencies \(\omega_n\) are chosen as independent random numbers uniformly distributed in \([-w/2, w/2]\), they produce \textit{quenched disorder}.

The index \(n = 1, \ldots, N\) labels the lattice sites and periodic boundary conditions are assumed. The parameter \(\alpha\) describes nonlinear frequency shift while \(\beta > 0\) is the nonlinear dissipation which is necessary to counterbalance the growth term \(\gamma > 0\).

We focus on the \textit{purely reactive coupling} case namely we set \(c_2 = 0\) and \(c_3 \equiv c \neq 0\).

The model has two interesting limits. For vanishing \(\alpha, \beta, \gamma\) it reduced to the well-known one-dimensional Anderson model (linear disordered lattice). For \(\gamma = \beta = 0\) and \(\omega_n\) non-random, it is the so-called \textit{Discrete Nonlinear Schrödinger equation} \cite{24}, which has been widely investigated in many different nonequilibrium conditions \cite{22, 26, 27}. The DNSE with random frequencies has been considered in \cite{3, 4}.

For later reference, we also write the equations for amplitude and phases of the complex field \(\psi_n = a_ne^{i\theta_n}\). Introducing the phase differences \(\varphi_n = \theta_{n+1} - \theta_n\) and the relative detunings \(\delta_n = \omega_{n+1} - \omega_n\) between neighbouring oscillators, we obtain the exact equations

\[
\dot{a}_n = (\gamma - \beta a_n^2)a_n + c[a_{n+1}\sin \varphi_n - a_{n-1}\sin \varphi_{n-1}]
\]

\[
\dot{\varphi}_n = \delta_n + \alpha(a_{n+1}^2 - a_n^2) + c\left[\frac{a_{n+2}}{a_{n+1}}\cos \varphi_{n+1} + \left(\frac{a_{n+1}}{a_{n+1}} - \frac{a_n}{a_n}\right)\cos \varphi_n - \frac{a_{n-1}}{a_n}\cos \varphi_{n-1}\right] \tag{2}
\]

Note that the \(\delta_n\) are random (with uniform distribution) but for periodic boundary they must satisfy the constraint \(\sum \delta_n = 0\) (only the first \(N-1\) differences are indeed independent). Correspondingly, the number of equations for the phase differences \(\varphi_n\) is \(N - 1\). Those equation will be the basis of some approximate treatment below.

The model has five control parameters but, upon rescaling, one may fix two to unity. In the following, we choose to leave the gain parameters \(\gamma\) free, and we set \(c = 1, \beta = 1\) so that \(w\) and \(\gamma\) are in units of \(c\).

3. Simulations

We have first performed a series numerical simulations for different gain parameters \(\gamma\) and \(N = 200\), and fixed \(w, \alpha\). Several lattice sizes have been considered with similar outcomes. The monitored quantities are the time averages (over the simulation time \(T\)) of the amplitudes \(\langle |\psi_n|^2 \rangle\) and the mean frequencies

\[
\langle \dot{\theta}_n \rangle = \frac{1}{T} [\theta_n(T) - \theta_n(0)]
\]
Figure 1: Space-time plots of $|\psi_n|^2$ for different $\gamma$ and $w = 1.5 \alpha = 2$, chain length $N = 200$, after discarding a transient of $10^4$ time units. At large $\gamma = 1$ a stationary state is observed. At $\gamma = 0.74$, periodic modulation close to $n = 170$ occurs. At $\gamma = 0.46$, several domains with periodic and chaotic modulations are observed. This state can be described as a coexistence of phase-locked clusters separated by localized chaotic or periodic oscillations. Finally, at $\gamma = 0.3$, the regime is rather turbulent.
Figure 2: Spatial configurations for $\gamma = 0.74$ (upper panel) and $\gamma = 0.46$ (lower panel) of the simulations presented in figure 1. In each panel the average amplitudes and their fluctuations, the average frequencies and their fluctuations. For $\gamma = 0.74$ there is only one irregular region; for $\gamma = 0.46$ there are four such domains.
Figure 3: Time evolution of amplitudes and phase velocities at the site $n = 166$ of the chain for the simulation reported above. Upon decreasing $\gamma$ there is a bifurcation from periodic to chaotic. Parameters are the same as in fig.1.

Figure 4: Localized chaos for $\alpha = 2$ and $w = 1.5$, $\gamma = 0.46$, chain length $N = 200$, calculated after discarding a transient of $10^4$ time units. Left: Space-time plot of the Lyapunov vector displaying a localized structure around site $n = 100$. Right: time traces of the oscillator amplitudes on two sites belonging to the chaotic region.
Figure 5: Space-time plots of $|\psi_n|^2$ showing multistability for four different initial conditions, at $\alpha = 0.03$ and $w = 1.5, \gamma = 2$, chain length $N = 200$. A transient of $10^4$ time units has been discarded.
as well as fluctuations of these quantities \(\langle \dot{\theta}_n^2 \rangle - \langle \dot{\theta}_n \rangle^2\) and \(\langle |\psi_n|^4 \rangle - \langle |\psi_n|^2 \rangle^2\), respectively.

The main phenomenology is exemplified by Figures 1, 2 and 3 for a particular realization of the disorder. The first observation is that, for relatively large \(\gamma\), the system approach a fully synchronized regime whereby all the oscillators have the same common frequency but different amplitudes \(a_n\) and constant phase shifts \(\varphi_n\). Such a state depends on the specific disorder realization and should correspond to a stable stationary solution of eqs. (2) (a uniformly rotating solution of eqs. (1)).

Upon decreasing \(\gamma\) the global-phase locked state destabilizes gradually. There is the spontaneous creation of a “defect” at some random location, where the amplitudes at a few neighboring sites start to oscillate in time (see the case \(\gamma = 0.74\) in figure 1 where the position of the defect is around \(n = 170\)). Further decrease of \(\gamma\) leads to appearance of new defects and to more complex dynamics at each defect (figure 2). As a result, a regime appears where patches of phase-entrained oscillators (frequency plateaus), are separated by defects that may oscillate periodically or chaotically (see the labeled arrows in figure 1).

An heuristic explanation of the existence of the localized instability can be given by the following argument. Let denote by \(\psi_n = u_n \exp(-i\mu t)\) the global phase-entrained solution (existing at large values of \(\gamma\)), with \(\mu\) being its frequency. The linear stability analysis is performed by letting \(\psi_n = (u_n + \chi_n) \exp(-i\mu t)\) with \(\chi_n\) complex, and linearizing the equation of motion to obtain

\[
\dot{\chi}_n = [i(\omega_n - \mu) + \gamma]\chi_n + ic[\chi_{n+1} + \chi_{n-1}] + (i\alpha - \beta)(2|u_n|^2\chi_n + u_n^2\chi_n^*). \tag{3}
\]

The above eigenvalue problem contains random terms (quenched disorder). It is thus akin to a random matrix problem where we expect to have localized eigenvectors. Also, it is not self-adjoint so we expect complex eigenvalues leading to oscillatory instability and thus localized modulations of the solution.

The localized nature of chaotic states is illustrated in the left panel of Fig. 4. We show the space-time evolution of the Lyapunov vector associated with the largest Lyapunov exponent, computed by evolving the linearized equation (same as eq. 3), but on top of a chaotic solution \(\psi_n(t)\). The appearance of chaotic oscillations in usually associated to the fact that amplitudes tend to become small (see figure 3 and the right panels in fig. 4). So it is evident that the amplitude dynamics plays a crucial role in this regime.

The same type of scenario is observed upon changing \(\alpha\) at fixed \(\gamma\) (data not reported). A novel feature however emerges in the limit of very small \(\alpha\) when the steady state depends on the initial conditions (see figure 5). The observed final state entails phase-entrained cluster with chaotic ones that resemble spatio-temporal intermittency. Multistable behavior is confirmed by computing the Lyapunov exponent \(\lambda\). As seen in figure 6 regular trajectories with \(\lambda \approx 0\) coexist with chaotic ones for the same value of the parameter \(\alpha\) suggesting coexistence of regular and chaotic attractors.

The above results have been obtained for an uniform and thus bounded distribution of frequencies. Actually, the choice of the distribution is relevant. For instance, a set of simulations performed with a Gaussian distribution with the same width \(w\) indicate that it is somehow easier to find unlocked states there.
Figure 6: (a) The largest Lyapunov exponent versus $\gamma$ for the same disorder realization and parameters as in previous figures. (b) The largest Lyapunov exponent versus $\alpha$ for four different initial conditions showing the multistable behavior $N = 200, w = 1.5, \gamma = 2$. The error on the exponent is roughly of the size of the symbols.

4. Ensemble properties

In the section above we have focused on a specific realization of the random frequencies $\omega_n$. For a statistical characterization, we measured the length of the phase-locked clusters $l_i$ defined by the condition that the mean frequency difference is less than a given threshold. Let us denote by $M$ their number and by $N_s$ the size of the largest one. In fig. 7 we report two series of simulations averaged over disorder realizations. The data show a significant dependence on the lattice size: full phase-locking is achieved provided that $\gamma$ or $\alpha$ are large enough but the value required increases systematically with $N$. On the other hand, the number of clusters $M$ seems to be extensive in the size. This also means that the average density of localized (regular or chaotic) defects is roughly constant but vanishes for large $\gamma$ or $\alpha$.

To appreciate the role of the disorder strength and fluctuations, we report in Figure 8 the number $M$ of phase-locked clusters as a function of $w$. To have an idea of the sample-to-sample statistics, we draw $M$ for each disorder realization. There is a clear transition from full entrainment ($M = 1$) upon increasing $w$ that for smaller $\alpha$ appears to be more abrupt.

5. Phase approximation

An interesting question is to what extent the dynamics can be described in term of the phase approximation, where the amplitude evolution is neglected. This approach allows also to rationalize the phenomenology observed in the above model.

For weak coupling and large enough gains, the system can indeed be approximated in terms of an effective equation for the relative phases $\varphi_n$ between neighboring oscillators.
If the amplitude relaxes quickly enough and $c$ is small, one can perform a straightforward adiabatic elimination of the amplitudes in eqs. (2): letting first $a_n = \sqrt{\gamma/\beta}(1 + r_n)$ we approximate eqs. (2) as

$$
\dot{r}_n \approx -2\gamma r_n + c \left[ \sin \varphi_n - \sin \varphi_{n-1} \right]
$$

$$
\dot{\varphi}_n \approx \delta_n + 2\alpha \frac{\gamma}{\beta}(r_{n+1} - r_n) + c \left[ \cos \varphi_{n+1} - \cos \varphi_{n-1} \right]
$$

Eliminating the first equation adiabatically, we get $r_n = c [\sin \varphi_n - \sin \varphi_{n-1}] / (2\gamma)$ and

$$
\dot{\varphi}_n = \delta_n + \frac{\alpha c}{\beta}(\sin \varphi_{n+1} + \sin \varphi_{n-1} - 2 \sin \varphi_n) + c \left[ \cos \varphi_{n+1} - \cos \varphi_{n-1} \right]
$$

which is akin to the Kuramoto-Sagacuchi model \cite{28, 29}. In the case with no disorder $\delta_n = 0$, this equation has been studied in detail in ref. \cite{30}. As discussed there, the terms in $\sin \varphi$ (dissipative couplings) favor synchronization while those in $\cos \varphi$ (dispersive or conservative couplings) hinders it. This is best understood by considering the “conservative limit” where the frequency dispersion is small $\alpha \to 0$. The phase equation is further approximated as

$$
\dot{\varphi}_n \approx \delta_n + c \left[ \cos \varphi_{n+1} - \cos \varphi_{n-1} \right]
$$

which has vanishing phase-space divergence, meaning that the dynamics is similar to a (disordered) conservative system. The physical interpretation is that in this limit, the amplitude and the phase dynamics decouple from the beginning, and the phase dynamics is purely conservative because the coupling is purely reactive in eq. (1). Since the phase space is compact (variables are angles), a small dissipation will stabilize motion on tori and we may expect some form of multistability as in the case of Hamiltonian systems.
with weak dissipation \[31\]. This qualitatively fits with the phenomenology observations made in the previous sections, see again figures 5 and 6.

An analysis of eq. (5) in presence of the random detunings \(\delta_n\) has been given recently in ref. \[29\]. It is there shown that the stationary synchronized solution persists upon increasing the level of disorder in the natural frequencies and that the phase shifts. Also, the domain of existence of the stable synchronous regime widens in comparison to the case of zero phase shift. This is quite counterintuitive, because a phase shift terms usually act against synchronization.

Note that equation (5) is, in this approximation, independent on \(\gamma\). We remark that in the case in which \(\gamma \to \gamma_n\) is not uniform, we expect the nearest-neighbor couplings to be site dependent, thus representing a further source of inhomogeneity in the phase equations. Moreover, as in the case of the well-know Adler equation, it gives a rough criterion: stationary solutions exist only if \(\delta_n\) is not too large. A necessary condition for locking would be \(w < 4 \alpha c\) as the frequencies are distributed uniformly.

To conclude this Section we add some remarks on the continuum limit of the derived phase equations. For small \(\varphi_n\) the equation is approximated to quadratic order as

\[
\dot{\varphi}_n \approx \Delta_n + \frac{\alpha c}{\beta} (\varphi_{n+1} - 2\varphi_n - \varphi_{n-1} - \frac{c}{2} (\varphi_{n+1}^2 - \varphi_{n-1}^2))
\]

(7)

the first term reminds of a diffusion equation with random sources. The nonlinear term is dispersive. Taking the continuum limit, letting \(D = \frac{\alpha c}{\beta}\) and denoting by \(\Delta(x)\) the random frequency differences, we have

\[
\dot{\varphi} = \Delta(x) + D\partial_x^2 \varphi - c\partial_x (\varphi^2)
\]

(8)

which is incidentally a Burgers equation with \textit{quenched} static disorder. As it is well know, the Burgers equation can be mapped onto the Kardar-Parisi-Zhang (KPZ) equation via the transformation \(\varphi = \partial_x \hat{h}\) yielding

\[
\dot{\hat{h}} = \omega(x) + D\partial_x^2 \hat{h} - c(\partial_x \hat{h})^2 + F
\]

(9)
where $F$ is an integration constant. The relation to coupled oscillator lattices with KPZ dynamics has been pointed out before [1] and has been invoked to explain phase dynamics in the noisy Kuramoto-Sakaguchi model with local couplings [32]. It should be emphasized that here we are dealing with the case of KPZ with quenched additive noise it usually referred to as columnar disorder in the literature [33].

Finally, an alternative formulation is obtained taking the time derivative of (8) and introduce the new field $v$

$$\dot{\phi} = v$$

$$\dot{v} = D\partial_x^2 v - 2\phi \partial_x (v\phi)$$

from which it is seen that $v = 0$, $\bar{\phi}(x)$ is the stationary frequency-entrained solution.

This introduces some form of inertia in the phase dynamics. This will not be discussed further but we remark that this equation is different from the one given in eqs. (36-37) of ref. [13]. This suggests that the model considered here may belong to a different class.

6. Conclusions

In summary, we have studied synchronization transitions in a disordered Ginzburg-Landau lattice of coupled limit-cycle oscillators. We have demonstrated that at large values of the activity parameter $\gamma$ a fully synchronous state in the lattice establishes. For smaller values of $\gamma$, transition to turbulence occurs through a sequence of appearing isolated periodic or chaotic “defects”. At the intermediate stage the patches between the defects remain synchronized, so that the dynamics can be described as a combination of several synchronous clusters having different frequencies, plus several localized regions with weak chaos. In the latter regions the dynamics of the amplitudes appears to be crucial. Thus, in the phase approximation, where the amplitudes are considered as slaved variables, the features of localized chaotic defects are not properly reproduced. We have also demonstrated how the number of different clusters in the lattice grows with the growing strength of disorder.

Declaration of Competing Interest

The Authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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