Free particle on noncommutative plane – a coherent state path integral approach

Sunandan Gangopadhyay\textsuperscript{a}\textsuperscript{*}, Frederik G Scholtz \textsuperscript{a,b}\textsuperscript{†}

\textsuperscript{a}National Institute for Theoretical Physics (NITheP), Stellenbosch 7600, South Africa
\textsuperscript{b}Institute of Theoretical Physics, University of Stellenbosch, Stellenbosch 7600, South Africa

(Dated: December 19, 2008)

We formulate the coherent state path integral on a two dimensional noncommutative plane using the fact that noncommutative quantum mechanics can be viewed as a quantum system on the Hilbert space of Hilbert-Schmidt operators acting on noncommutative configuration space. The propagation kernel for the free particle shows ultra-violet cutoff which agrees with the earlier investigations made in the literature but the approach differs substantially from the earlier studies.

PACS numbers: 11.10.Nx

Introduction:

The idea of noncommutative spacetime was first formally introduced by Snyder in \textsuperscript{[1]} as an attempt to regulate the divergences of quantum field theories. These ideas were taken seriously when considerable evidence came from string theory \textsuperscript{[2]} to the issues of quantum gravity that suggests that attempts to unify gravity and quantum mechanics will ultimately lead to a noncommutative geometry of spacetime. Thereafter, despite a number of investigations into the possible physical consequences of noncommutativity in quantum mechanics and quantum mechanical many-body systems \textsuperscript{[3]-[11]}, quantum electrodynamics \textsuperscript{[12]-[14]}, the standard model \textsuperscript{[15]} and cosmology \textsuperscript{[16], [17]}, our understanding of the physical implications of noncommutativity is still far from being complete. The difficulty in having a through understanding of the physical implications of noncommutativity, is the lack of a systematic formulation and interpretational framework of noncommutative quantum mechanics. The difficulty persists even in the path integral formulation of noncommutative quantum mechanics and there seems to be a lot of disagreement in the results obtained in the literature.

The path integral formulation, in general, attempt to evaluate noncommutative analogues of the Feynman kernel:

\[
K(q,t;q_0,t_0) = \langle q | \hat{U}(t,t_0) | q_0 \rangle
\]  \hspace{1cm} (1)

where, \( \hat{U} \) is the unitary time evolution operator and \( |q\rangle = |q_1,...,q_d\rangle \) are the position eigenkets in \( d \) dimensions. Noncommutative geometry implies the absence of common position eigenkets. This problem was circumvented in \textsuperscript{[18]} by taking coherent states to define the propagation kernel. The coherent states being the eigenstates of complex combinations of the position operators act as a meaningful replacement for the position eigenstates admissible only in the commutative theory. The observation made in this paper is that the free particle propagator turns out to exhibit an ultra-violet cutoff induced by the noncommutative parameter \( \theta \). In another paper \textsuperscript{[19]}, the same trick of using coherent states to define the propagation kernel was employed, however, the final expressions for the propagator and the resulting physics were quite different from \textsuperscript{[18]}.

In this note, we develop an unambiguous formulation of the Feynman path integral representation for the free particle propagator using the ideas of a series of very recent papers by one of the authors \textsuperscript{[20, 21]} where a full fledged formulation and interpretation of noncommutative quantum mechanics have been carried out explicitly. The result indeed shows the presence of a damping exponential term induced by the parameter of noncommutativity \( \theta \) in the free particle propagator. However, in contrast to \textsuperscript{[18]}, the star product enters automatically in the computation once the completeness relation for the states living in the quantum Hilbert space are introduced. This is the new point in our paper which says that a systematic formulation of noncommutative quantum mechanics automatically brings in the star product into the game and furthermore, it is even possible to make exact analytical computations using the star product.

In the next section, we briefly review the formalism developed in \textsuperscript{[20, 21]} to deal with noncommutative quantum systems and then derive the completeness relations for the momentum and position eigenstates living in the quantum Hilbert space. Using this, we move on to construct the path integral representation for the propagator of the free particle in the two dimensional noncommutative space. In the rest of the paper, we shall work with natural units \( \hbar = c = 1 \).

\textsuperscript{*} e-mail: sunandan.gangopadhyay@gmail.com, sunandan@sun.ac.za
\textsuperscript{†} e-mail: fgs@sun.ac.za
Before constructing the path integral representation of the free particle propagation kernel on noncommutative space, we present a brief review of the formalism of noncommutative quantum mechanics developed recently in \[20,21\]. This formalism have been developed in complete analogy with commutative quantum mechanics.

We start by giving precise meaning to the concepts of the classical configuration space and the Hilbert space of a noncommutative quantum system. The first step is to define classical configuration space. In two dimensions, the coordinates of noncommutative configuration space satisfy the commutation relation

\[ [\hat{x},\hat{y}] = i\theta \]  

(2)

where without loss of generality it is assumed that \( \theta \) is a real positive parameter. Using this, it is convenient to define the creation and annihilation operators

\[ b = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{y}) \quad , \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x} - i\hat{y}) \]

that satisfy the Fock algebra \([b,b^\dagger] = 1\). The noncommutative configuration space is then isomorphic to the boson Fock space

\[ \mathcal{H}_c = \text{span}\{ |n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle \}_{n=0}^{\infty} \]

(3)

where the span is take over the field of complex numbers.

The next step is to introduce the Hilbert space of the noncommutative quantum system. We consider the set of Hilbert-Schmidt operators acting on noncommutative configuration space

\[ \mathcal{H}_q = \{ \psi(\hat{x},\hat{y}) : \psi(\hat{x},\hat{y}) \in \mathcal{B}(\mathcal{H}_c), \text{tr}_c(\psi^\dagger(\hat{x},\hat{y})\psi(\hat{x},\hat{y})) < \infty \} . \]

(4)

Here \( \text{tr}_c \) denotes the trace over noncommutative configuration space and \( \mathcal{B}(\mathcal{H}_c) \) the set of bounded operators on \( \mathcal{H}_c \). This space has a natural inner product and norm

\[ (\phi(\hat{x},\hat{y}),\psi(\hat{x},\hat{y})) = \text{tr}_c(\phi(\hat{x},\hat{y})^\dagger\psi(\hat{x},\hat{y})) \]

(5)

and forms a Hilbert space \( \mathcal{H}_q \). This space is the analog of the space of square integrable wave functions of commutative quantum mechanics and to distinguish it from the noncommutative configuration space \( \mathcal{H}_c \), which is also a Hilbert space, we shall refer to it as quantum Hilbert space and use the subscripts \( c \) and \( q \) to make this distinction. Furthermore, we denote states in the noncommutative configuration space by \( |\cdot\rangle \) and states in the quantum Hilbert space by \( \psi(\hat{x},\hat{y}) = |\hat{\psi}\rangle \) and the elements of its dual (linear functionals) by \( \langle \cdot | \psi \rangle \), which maps elements of \( \mathcal{H}_q \) onto complex numbers by \( \langle \phi | \psi \rangle = \langle \phi | \psi \rangle = \text{tr}_c(\phi(\hat{x},\hat{y})^\dagger\psi(\hat{x},\hat{y})) \). We also need to be careful when denoting hermitian conjugation to distinguish between these two spaces. We use the notation \( \dagger \) to denote hermitian conjugation on noncommutative configuration space and the notation \( \dagger \) for hermitian conjugation on quantum Hilbert space.

We now replace the abstract Heisenberg algebra by the noncommutative Heisenberg algebra. In two dimensions this reads

\[ [\hat{x},\hat{p}_x] = i \quad , \quad [\hat{y},\hat{p}_y] = i \theta \]

\[ [\hat{x},\hat{y}] = i\theta \]

\[ [\hat{p}_x,\hat{p}_y] = 0. \]

(6)

A unitary representation of this algebra in terms of operators \( \hat{X}, \hat{Y}, \hat{P}_x \) and \( \hat{P}_y \) acting on the states of the quantum Hilbert space \( \mathcal{H}_q \) with inner product \( \langle \cdot | \psi \rangle \), which is the analog of the Schrödinger representation of the Heisenberg algebra, is easily found to be

\[ \hat{X}\psi(\hat{x},\hat{y}) = \hat{x}\psi(\hat{x},\hat{y}) \]

\[ \hat{Y}\psi(\hat{x},\hat{y}) = \hat{y}\psi(\hat{x},\hat{y}) \]

\[ \hat{P}_x\psi(\hat{x},\hat{y}) = i\theta[\hat{y},\psi(\hat{x},\hat{y})] = -i\frac{\partial\psi(\hat{x},\hat{y})}{\partial\hat{x}} \]

\[ \hat{P}_y\psi(\hat{x},\hat{y}) = -i\theta[\hat{x},\psi(\hat{x},\hat{y})] = -i\frac{\partial\psi(\hat{x},\hat{y})}{\partial\hat{y}} . \]

(7)
Writing the trace in terms of coherent states (10) and using $|\langle \rangle\rangle = \psi(\hat{x}, \hat{y})$. These states also have the property

$$
B = \frac{1}{\sqrt{20}} \left( \hat{X} + i \hat{Y} \right)
$$

$$
B^\dagger = \frac{1}{\sqrt{20}} \left( \hat{X} - i \hat{Y} \right)
$$

$$
\hat{P} = \hat{P}_x + i \hat{P}_y
$$

$$
\hat{P}^\dagger = \hat{P}_x - i \hat{P}_y.
$$

We note that $\hat{P}^2 = \hat{P}_x^2 + \hat{P}_y^2 = P^t P = PP^t$. These operators act in the following way

$$
B\psi(\hat{x}, \hat{y}) = b\psi(\hat{x}, \hat{y})
$$

$$
B^\dagger\psi(\hat{x}, \hat{y}) = b^\dagger \psi(\hat{x}, \hat{y})
$$

$$
P\psi(\hat{x}, \hat{y}) = -i \sqrt{\frac{2}{\theta}} [b, \psi(\hat{x}, \hat{y})]
$$

$$
P^\dagger\psi(\hat{x}, \hat{y}) = i \sqrt{\frac{2}{\theta}} [b^\dagger, \psi(\hat{x}, \hat{y})].
$$

The operator $\psi(\hat{x}, \hat{y})$ is just a vector in the quantum Hilbert space and can also be denoted as $|\psi\rangle = \psi(\hat{x}, \hat{y})$. With the above notions in place, we now take the axioms of commutative quantum mechanics to apply with the simple replacement of $L^2$ by $\mathcal{H}_q$. Although this provides a consistent interpretational framework, the measurement of position needs more careful consideration. The problem with a measurement of position is not that the axioms above do not apply to the hermitian operators $\hat{X}$ and $\hat{Y}$, rather the problem is that these operators do not commute and thus a precise measurement of one of these observables leads to total uncertainty in the other. Yet, we would like to preserve the notion of position in the sense of a particle being localized around a certain point. The best that can be done in the noncommutative case is to construct a minimal uncertainty state in noncommutative configuration space and use that to give meaning to the notion of position. We now move on to describe this procedure.

The minimal uncertainty states on noncommutative configuration space, which is isomorphic to boson Fock space, are well known to be the normalized coherent states (23)

$$
|z\rangle = e^{-z^2/2} e^{z b^\dagger} |0\rangle
$$

where, $z = \frac{1}{\sqrt{20}} (x + iy)$ is a dimensionless complex number. These states provide an overcomplete basis on the noncommutative configuration space. Corresponding to these states we can construct a state (operator) in quantum Hilbert space as follows

$$
|z, \bar{z}\rangle = \frac{1}{\sqrt{\theta}} |z\rangle \langle z|.
$$

These states also have the property

$$
B|z, \bar{z}\rangle = z|z, \bar{z}\rangle.
$$

Writing the trace in terms of coherent states (10) and using $|\langle z|w\rangle|^2 = e^{-|z-w|^2}$ it is easy to see that

$$
(z, \bar{z}|w, \bar{w}\rangle = \frac{1}{\theta} tr_c(|z\rangle \langle z|w\rangle \langle w|) = \frac{1}{\theta} |\langle z|w\rangle|^2 = \frac{1}{\theta} e^{-|z-w|^2}
$$

which shows that $|z, \bar{z}\rangle$ is indeed a Hilbert-Schmidt operator. We can now construct the 'position' representation of a state $|\psi\rangle = \psi(\hat{x}, \hat{y})$ as

$$
(z, \bar{z}|\psi\rangle = \frac{1}{\sqrt{\theta}} tr_c(|z\rangle \langle z| \psi(\hat{x}, \hat{y})) = \frac{1}{\sqrt{\theta}} (z|\psi(\hat{x}, \hat{y})|z\rangle.
$$

In particular, introducing momentum eigenstates

$$
|p\rangle = \sqrt{\frac{\theta}{2\pi}} e^{i\sqrt{\theta} ( pb + p^b)}
$$

(15)
To prove this, we use (18) and compute

\[ \psi_p(p) = p_x|p\rangle, \quad \psi_y(p) = p_y|p\rangle \]

(16)

we have

\[ (p'|p) = e^{-\frac{i}{\hbar}(\hat{p}p + \hat{p}'p')} e^{\frac{i}{\hbar}\hat{p}'\hat{p}} \delta(p - p') \]

(17)

We now proceed to obtain the completeness relations for the momentum and position eigenstates (|p⟩ and |z, \bar{z}\rangle) which are important ingredients in the construction of the path integral representation. To do this, using (18), we compute

\[ \int d^2p(w, \tilde{w})|p(z, \bar{z}) = \frac{1}{\theta} e^{-\frac{i}{\hbar}|w - \bar{z}|^2} \]

(19)

which implies that the momentum eigenstates |p⟩ satisfy the following completeness relation

\[ \int d^2p |p\rangle\langle p| = 1_Q . \]

(20)

The position eigenstates |z, \bar{z}\rangle, on the other hand, satisfy the following completeness relation

\[ \int \frac{d\theta}{2\pi} dz d\bar{z} |z, \bar{z}\rangle \langle z, \bar{z}| = 1_Q \]

(21)

where the star product between two functions \( f(z, \bar{z}) \) and \( g(z, \bar{z}) \) is defined as

\[ f(z, \bar{z}) \star g(z, \bar{z}) = f(z, \bar{z}) e^{\hbar \hat{\imath}_x \hat{\imath}_y} g(z, \bar{z}) . \]

(22)

To prove this, we use (18) and compute

\[ \int \frac{d\theta}{2\pi} dz d\bar{z} (p'|z, \bar{z}) \star (z, \bar{z}|p) = e^{-\frac{i}{\hbar}(\hat{p}'p + \hat{p}p')} e^{\frac{i}{\hbar}\hat{p}'\hat{p}} \delta(p - p') = (p'|p) . \]

(23)

Thus, the position representation of the noncommutative system maps quite naturally to the Moyal plane. With the above formalism and the completeness relations for the momentum and the position eigenstates (20, 21) in place, we now proceed to write down the path integral for the free particle propagation kernel on the two dimensional noncommutative space. This reads

\[ (z_f, t_f|z_0, 0) = \lim_{n \to \infty} \left\langle \frac{\theta}{2\pi} \right\rangle^n \prod_{j=1}^n dz_j d\bar{z}_j (z_f, t_f|z_n, t_n) \star n (z_n, t_n| \ldots |z_1, t_1) \star 1 (z_1, t_1|z_0, 0) \]

(24)

Now we compute the propagator over a small segment in the above path integral. With the help of (18) and (20), we have

\[ (z_{i+1}, t_{i+1}|z_i, t_i) = (z_{i+1}|e^{-iH\tau}|z_i) \]

(25)

where, \( N = \frac{m}{m\theta + i\tau} \), \( \beta = \frac{m}{2(m\theta + i\tau)} \) and \( H = \frac{\hat{p}_x^2}{2m} \) being the Hamiltonian for the free particle acting on the quantum Hilbert space. Using the above result, we now write down after some algebra, the following generic element in the above path integral

\[ \int dx_1dy_1 (z_{i+1}, t_{i+1}|z_i, t_i) \star 1 (z_i, t_i|z_0, t_0) = N_1N_2 \left( \frac{\pi}{\beta_1\Lambda} \right) \exp \left[ -\frac{\beta_1 \gamma}{\Lambda} (\bar{x}_{i+1} - \bar{x}_i)^2 \right] \]

(26)
where, \( \gamma = \beta_2/\beta_1 \) and \( \Lambda = 1 + \gamma - 2\theta\beta_2 \). It should be noted that the above computation has been carried out with

\[
(z_{i+1}, t_{i+1}| z_i, t_i) = N_1 \exp \left[ -\beta_1 (\vec{x}_{i+1} - \vec{x}_i)^2 \right] \\
(z_i, t_i| z_0, t_0) = N_2 \exp \left[ -\beta_2 (\vec{x}_i - \vec{x}_0)^2 \right]
\]

(27)

where, \( N_1, N_2 \neq N \) and \( \beta_1, \beta_2 \neq \beta \). The reason for doing this will become clear as we proceed further. We start by computing each of the integrals in (26) from the extreme right.

The first integral that we encounter on the extreme right of the path integral (24) is the following

\[
\int_{-\infty}^{+\infty} dx_1 dy_1 (z_2, t_2| z_1, t_1) \star_1 (z_1, t_1| z_0, t_0) . \tag{28}
\]

To compute this integral, we set \( i = 1 \) on the left hand side of (26) and set \( N_1 = N_2 = N \) and \( \beta_1 = \beta_2 = \beta \) on the right hand side of (26) which yields

\[
\int_{-\infty}^{+\infty} dx_1 dy_1 (z_2, t_2| z_1, t_1) \star_1 (z_1, t_1| z_0, t_0) = \frac{2\pi m}{\theta m + 2i\tau} \exp \left[ -\frac{\beta}{2(1 - \theta\beta)} (\vec{x}_2 - \vec{x}_0)^2 \right]. \tag{29}
\]

The first two integrals starting from the extreme right of (24) reads the following

\[
\int_{-\infty}^{+\infty} (dx_2 dy_2)(dx_1 dy_1) (z_3, t_3| z_2, t_2) \star_2 (z_2, t_2| z_1, t_1) \star_1 (z_1, t_1| z_0, t_0) . \tag{30}
\]

Using (29), we observe that the above integral (30) is of the form (26) with \( i = 2 \) and \( N_1 = N, N_2 = \frac{2\pi m}{\theta m + 2i\tau} \), \( \beta_1 = \beta \) and \( \beta_2 = \frac{\beta}{2(1 - \theta\beta)} \). Hence, we obtain

\[
\int_{-\infty}^{+\infty} (dx_2 dy_2)(dx_1 dy_1) (z_3, t_3| z_2, t_2) \star_2 (z_2, t_2| z_1, t_1) \star_1 (z_1, t_1| z_0, t_0) = \frac{(2\pi)^2 m}{\theta m + 3i\tau} \exp \left[ -\frac{\beta}{(3 - 4\theta\beta)} (\vec{x}_2 - \vec{x}_0)^2 \right]. \tag{31}
\]

Repeating this procedure \( n \) times, we finally obtain the free particle propagation kernel on the two dimensional noncommutative space

\[
(z_f, t_f| z_0, t_0) = \lim_{n \to \infty} \frac{m}{\theta m + i(n + 1)\tau} \exp \left[ -\frac{\beta}{(n + 1) - 2n\theta\beta} (\vec{x}_f - \vec{x}_0)^2 \right] \\
= \lim_{n \to \infty} \frac{m}{\theta m + i(n + 1)\tau} \exp \left[ -\frac{m}{2(i(n + 1)\tau + m\theta)} (\vec{x}_f - \vec{x}_0)^2 \right] \\
= \frac{m}{\theta m + iT} \exp \left[ -\frac{m}{2 iT + m\theta} (\vec{x}_f - \vec{x}_0)^2 \right] ; \quad (n + 1)\tau = T = t_f - t_0 . \tag{32}
\]

**Conclusion :**

In this paper, we have formulated the path integral representation of the free particle propagation kernel based on the newly established consistent formulation of noncommutative quantum mechanics. In contrast to the approach in [18], where the star product does not arise, we have shown that the star product plays an important role in our approach and interestingly, the star product still allows the exact computation of the free particle propagator to all orders in the noncommutative parameter \( \theta \). The result for the propagator exhibits the ultra-violet cutoff induced by the noncommutative parameter and is in conformity with the result obtained earlier in the literature [18].

**Acknowledgements :** This work was supported under a grant of the National Research Foundation of South Africa.

[1] H.S. Snyder, Phys. Rev. 71, 38.
[2] N. Seiberg and E. Witten, JHEP 9909 (1999) 032.
[3] C. Duval, P.A. Horvathy; Phys. Lett. B 479, (2000) 284.
[4] V.P. Nair, A.P. Polychronakos, Phys. Lett. B 505, (2001) 267.
[5] R. Banerjee, Mod. Phys. Lett. A 17: 631, 2002; [hep-th/0106280].
[6] B. Chakraborty, S. Gangopadhyay, A. Saha, Phys. Rev. D 70: 107707, 2004; [hep-th/0312292].
[7] F.G. Scholtz, B. Chakraborty, S. Gangopadhyay, A.G. Hazra, Phys. Rev. D 71: 085005, 2005; [hep-th/0502143].
[8] K. Li and S. Dulat, Eur. Phys. J. C 46 (2006) 825.
[9] R. Vilela Mendes, Eur. Phys. J. C 42 (2005) 445.
[10] F.S. Bemfica and H.O. Girotti, Jnl. Phys. A 38 (2005) L539.
[11] S. Khan, B. Chakraborty and F.G. Scholtz, Phys. Rev. D 78 (2008) 025024.
[12] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu1, Phys. Rev. Lett. 86 (2001) 2716.
[13] N. Chair and M. M. Sheikh-Jabbari, Phys. Lett. B 504 (2001) 141.
[14] Y. Liaoa, C. Dehneb, Eur. Phys. J. C 29 (2003)125.
[15] T. Ohl and J. Reuter, Phys. rev. D 70 (2004) 076007.
[16] H. Garca-Compen, O. Obregn and C. Ramirez, Phys. Rev. Lett. 88 (2002)161301.
[17] S. Alexander, R. Brandenberger and J. Magueijo, Phys. Rev. D 67 (2003) 081301.
[18] A. Smailagic, E. Spallucci, J. Phys. A 36, (2003) L467.
[19] H.S. Tan, J. Phys. A 39, (2006) 15299.
[20] F.G. Scholtz, B. Chakraborty, J. Govaerts, S. Vaidya, J. Phys. A 40, (2007) 14581.
[21] F.G. Scholtz, L. Gouba, A. Hafver, C.M. Rohwer, arXiv:0812.2803 [math-ph].
[22] A.S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North-Holland Publishing Company, Amsterdam, 1982) p79.
[23] J.R. Klauder, B. Skagerstam, Coherent states : Applications in Physics and Mathematical Physics (World Scientific, Singapore, 1985).