ON AN EXPLICIT SET OF COMPLETE EIGENFUNCTIONS
FOR THE CALOGERO-SUTHERLAND MODEL

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Abstract

An operator method is provided to generate an explicit set of complete eigenfunctions for the Calogero-Sutherland model, obtained earlier by Vacek, Okiji and Kawakami, through a special ansatz. We find the connection of the above basis set with the general eigenfunctions of this model and explicitly show that these states describe only the center of mass motion.

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Calogero-Sutherland (CS) model,\textsuperscript{1} and its generalizations to spin-chains,\textsuperscript{2} have attracted wide attention in the recent literature because of their relevance to many branches of physics.\textsuperscript{3} This quantum mechanical model in one dimension represents a system of $N$-identical particles, simultaneously subjected to a harmonic confinement and a pairwise inverse square potential. The correlated ground state of this model was first analysed by Sutherland\textsuperscript{1} in the context of thermodynamics, who also pointed out its connection with the random matrices.\textsuperscript{4} The algebraic structure of the CS model, studied earlier by Perelomov through group theoretical means,\textsuperscript{5} has recently been further investigated by a number of authors, using the $S_N$-extended Heisenberg algebra.\textsuperscript{6} Much akin to the harmonic oscillator case, the excited state wavefunctions are constructed by operating appropriately symmetrized monomials of the raising operators on the vacuum, which is annihilated by the lowering operators. However, in practice, the construction of an explicit basis set has not proceeded much further than a few particle case,\textsuperscript{7} due to the complicated sums appearing in the definition of the creation and annihilation operators.

Recently, a special set of complete eigenfunctions spanning all energy levels of the CS Hamiltonian ($\hbar = \omega = m = 1$)

\[
H = -\frac{1}{2} \sum_{i} \partial_{i}^2 + \frac{1}{2} \sum_{i} x_{i}^2 + \frac{g^2}{2} \sum_{i\neq j} \frac{1}{(x_{i} - x_{j})^2},
\]

was constructed by Vacek, Okiji and Kawakami (VOK)\textsuperscript{8} through a Jastrow-type ansatz. The states are characterized by an integer $I$:

\[
\psi_I = \left[ \prod_{i<j}^{N} |x_{i} - x_{j}|^{\gamma}(x_{i} - x_{j})^{\gamma} \right] \exp \left[ -\frac{1}{2} \sum_{i} x_{i}^2 \sum_{m_i=I}^{N} \frac{H_{m_i}(x_{i})}{m_i!} \right],
\]

with an eigenvalue $E_I = E_0 + I$. Here, $\gamma = 0$ ($\gamma = 1$) and $g^2 = (\lambda + \gamma)(\lambda + \gamma - 1)$ for the bosonic (fermionic) case and $E_0 = N^2 + \frac{1}{2}(\lambda + \gamma)N(N - 1)$ is the ground state energy which depends on the interaction parameter $\lambda$.

A thorough analysis of these eigenfunctions is of relevance, since a basis set is crucial for the computation of the correlation functions (CF). In the case of CS model on a circle, the CF are calculated by using the properties of Jack polynomials.\textsuperscript{9} Furthermore, the basis
functions may clarify various intriguing aspects of this model, one of which is the occurrence of quasi-particles with the so called exclusion statistics.\(^{10}\)

In this paper, we first provide an operator method based on the \(S_N\)-extended Heisenberg algebra, to generate the above mentioned eigenstates. This is done by introducing suitable raising and lowering operators, keeping in mind the identical nature of the particles and the symmetry of the wavefunctions under transposition of the particle coordinates. Powers of this creation operator acting on the ground state produce the desired states given by VOK. It is then found, using the variable separation method, that these eigenfunctions are a special, albeit complete subset of the most general wavefunctions of the CS model. Furthermore, it is shown that these states describe only the center of mass motion.

The \(S_N\)-extended Heisenberg algebra is generated by the operators \(a_i, a_i^\dagger\) and \(K_{ij}\) which satisfy the following relations

\[
[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 , \\
[a_i, a_j^\dagger] = \delta_{ij}\{1 + (\lambda + \gamma)\sum_l K_{il}\} - (\lambda + \gamma)K_{ij} , \\
K_{ij} = K_{ji} ; \quad (K_{ij})^2 = 1 , \\
K_{ij}a_j = a_iK_{ij} ; \quad K_{ij}a_j^\dagger = a_i^\dagger K_{ij} \quad \text{(no summation over repeated indices)}, \\
K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij} , \quad \text{for } i \neq j, i \neq l, j \neq l , \\
K_{ij}K_{mn} = K_{mn}K_{ij} , \quad \text{for } i, j, m, n \text{ all different}. \tag{3}
\]

It can be checked, after a straightforward calculation, that the \(S_N\)-extended Heisenberg algebra follows with, \(a_i = \frac{1}{\sqrt{2}}(x_i + D_i)\) and \(a_i^\dagger = \frac{1}{\sqrt{2}}(x_i - D_i)\). Here,

\[
D_i = \partial_i + (\lambda + \gamma)\sum_{j \neq i}^N \frac{1}{(x_i - x_j)}(1 - K_{ij}) \tag{4}
\]

is the so called Dunkl derivative.\(^{11}\)

Based on this algebra, one can construct a Hamiltonian \(\overline{H}\), whose connection with the CS model will be established shortly, in the form given below,
\[ H = \frac{1}{2} \sum_i \{ a_i, a_i^\dagger \} = \left( -\frac{1}{2} \sum_i \partial_i^2 + \frac{1}{2} \sum_i x_i^2 - (\lambda + \gamma) \sum_{i,j} \frac{1}{(x_i - x_j)} \partial_i \right) + \left( \frac{\lambda + \gamma}{2} \sum_{i,j} \frac{1}{(x_i - x_j)^2} (1 - K_{ij}) \right) - \left( \frac{\lambda + \gamma}{2} \sum_{i,j,d} \frac{1}{(x_i - x_d)(1 - K_{ij})} \right). \]  

Making use of Eq. (3), the following commutation relationships (CR) can be worked out,

\[ [\bar{H}, a_i] = -a_i, \]
\[ [\bar{H}, a_i^\dagger] = a_i^\dagger. \]  

A generic excited eigenstate \( \bar{\psi} \) for \( \bar{H} \), can then be written as

\[ \bar{\psi} = \prod_i (a_i^\dagger)^{m_i} \psi_o. \]  

Here, \( \psi_o \) is the ground state wavefunction obtained from

\[ a_i \psi_o = 0. \]  

An eigenstate of \( \bar{H} \), satisfying \( K_{ij} \bar{\psi} = \bar{\psi}, \) i.e., symmetric under the transposition of the particle coordinates, is also an eigenstate of another related Hamiltonian \( H' \), where

\[ H' = \left( -\frac{1}{2} \sum_i \partial_i^2 + \frac{1}{2} \sum_i x_i^2 - (\lambda + \gamma) \sum_{i,j} \frac{1}{(x_i - x_j)} \partial_i \right). \]  

\( H' \) and the CS Hamiltonian \( H \) in Eq. (1) are related by a similarity transformation of the form

\[ H' = Z^{-(\lambda+\gamma)} \ H \ Z^{(\lambda+\gamma)}. \]  

where,

\[ Z = \prod_{i<j}(x_i - x_j). \]  

Noting the fact that, the eigenstates of \( H' \) are symmetric under the exchange of particle coordinates, we introduce the raising and lowering operators, respectively given by
\[ A = \frac{1}{\sqrt{N}} \sum_{i}^{N} a_i = \frac{1}{\sqrt{2N}} \left( \sum_{i}^{N} (x_i + \partial_i) + (\lambda + \gamma) \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)}(1 - K_{ij}) \right) \]
\[ = \frac{1}{\sqrt{2N}} \left( \sum_{i}^{N} (x_i + \partial_i) \right) \quad (11) \]

and
\[ A^\dagger = \frac{1}{\sqrt{N}} \sum_{i}^{N} a_i^\dagger = \frac{1}{\sqrt{2N}} \left( \sum_{i}^{N} (x_i - \partial_i) \right) \quad (12) \]

It is straightforward to check that \([A, A^\dagger] = 1\) and the CR of \(A\) and \(A^\dagger\) with \(\bar{H}\) are identical to the ones given in Eq. (6).

It is worth pointing out that, although \(A\) and \(A^\dagger\) are identical to the lowering and raising operators of the harmonic oscillator algebra and have similar CR with \(\bar{H}\), the Hamiltonian in the present case can not be written as a bilinear of these operators.

Defining the ground state \(\phi_o\) as
\[ A\phi_o = \frac{1}{\sqrt{2N}} \sum_{i}^{N} (x_i + \partial_i)\phi_o = 0 \quad , \]

one finds,
\[ \phi_o = \exp \left( -\frac{1}{2} \sum_{i}^{N} x_i^2 \right) \quad . \quad (13) \]

Now, the excited states \(\phi_I's\), can be obtained by the repeated application of \(A^\dagger\) on \(\phi_o\). The \(I\)-th excited state is
\[ \phi_I = (A^\dagger)^I \phi_o = \frac{I!}{(N)^{I/2}} \sum_{m_i=I}^{N} \prod_{i} (a_i^\dagger)^{m_i} \phi_o \]
\[ = \frac{I!}{(N)^{I/2}} \exp \left[ -\frac{1}{2} \sum_{i}^{N} x_i^2 \right] \sum_{m_i=I}^{N} \prod_{i} \frac{H_{m_i}(x_i)}{m_i!} \quad . \quad (14) \]

These eigenstates are obviously symmetric under the exchange of particle coordinates and hence we conclude that the \(\phi_I's\) are also the eigenfunctions of \(H'\). An eigenfunction \(\psi_I\) of the CS Hamiltonian can then be written as,
\[ \psi_I = \frac{I!}{(N)^{I/2}} \left[ \prod_{i<j}^{N} (x_i - x_j)^\lambda (x_i - x_j)^\gamma \right] \exp \left[ -\frac{1}{2} \sum_{i}^{N} x_i^2 \right] \sum_{m_i=I}^{N} \prod_{i} \frac{H_{m_i}(x_i)}{m_i!} \quad . \quad (15) \]
These eigenfunctions, modulo an overall normalization factor, are identical to the one given in Eq. (2), obtained earlier by VOK, through a special ansatz. It should be noted that, unlike the group theoretical approach, this method does not account for the degeneracy of the energy eigenvalues.

Now, we show that the above set is a special, albeit complete, subset of the most general eigenfunctions of the CS model. Making use of the following identity for the Hermite polynomials\textsuperscript{12}

\[
\frac{\left(\sum_{k=1}^{r} \beta_k^2\right)^{1/2}}{I!} H_I \left[ \frac{\sum_{k=1}^{r} \beta_k x_k}{\left(\sum_{k=1}^{r} \beta_k^2\right)^{1/2}} \right] = \sum_{m=I}^{I} \prod_{k=1}^{r} \frac{\beta_k^{m_k}}{m_k!} H_{m_k}(x_k) ,
\]

Eq. (14) can be written as

\[
\phi_I = \exp\left(-\frac{1}{2} \sum_{i}^{N} x_i^2\right) H_I \left( \frac{1}{\sqrt{N}} \sum_{i}^{N} x_i \right) .
\]

Using the center of mass coordinate \( R = \frac{1}{N} \sum_{i}^{N} x_i \), the radial variable \( r = \sqrt{\frac{1}{N} \sum_{i<j}^{N} (x_i - x_j)^2} \) and the identity \( \sum_{i}^{N} x_i^2 = NR^2 + r^2 \), one finds

\[
\phi_I = \exp\left(-\frac{1}{2} NR^2\right) \exp\left(-\frac{1}{2} r^2\right) H_I(\sqrt{N} R) .
\]

As shown in Ref. 1, the Hamiltonian \( H' \) can be written in a separated form as

\[
H' = H_R + H_r + r^{-2} H_\Omega ,
\]

by making use of \( R, r, (N - 2) \) angular coordinates \( \Omega_i \) and the following identities

\[
\sum_{i}^{N} \partial_i^2 = \frac{1}{N} \partial^2 + \frac{1}{r^2} \partial_r^2 + \frac{(N - 2)}{r} \partial_r + \frac{1}{r^2} \hat{L} ,
\]

\[
\sum_{i,j,i \neq j}^{N} \frac{1}{(x_i - x_j)} \partial_i = \frac{1}{2} N(N - 1) \frac{1}{r} \partial_r + \frac{1}{r^2} \hat{M} .
\]

Here, the operators \( \hat{L} \) and \( \hat{M} \) are identical to the ones given by Calogero\textsuperscript{1} and act only on the angular coordinates.
Explicitly,

\[ H_R = -\frac{1}{2N} \frac{\partial^2}{\partial R^2} + \frac{1}{2} NR^2, \]  
\[ H_r = -\frac{1}{2} \frac{\partial^2}{\partial r^2} - \frac{[N - 2 + (\lambda + \gamma)N(N - 1)]}{2} \frac{\partial}{\partial r} + \frac{1}{2} r^2, \]  
\[ H_\Omega = \hat{L} + 2(\lambda + \gamma) \hat{M} . \]  

As is clear from above, \( H_R \) describes the center of mass degree of freedom.

It has been further established in Ref. 1 that, with a class of symmetrical homogeneous polynomials \( P_k(x) \) of degree \( k \), the function \( r^{-k} P_k(x) \) is an eigenfunction of \( H_\Omega \) with the eigenvalue \( E_\Omega = -k[k + N - 3 + N(N - 1)(\lambda + \gamma)] \). These eigenfunctions depend only on the \( N - 2 \) angular coordinates \( \Omega_i \) and are independent of both the radial and center of mass coordinates. The eigenfunctions for \( H_R \) and \( (H_r + r^{-2}E_\Omega) \) are respectively given by,

\[ \chi_{I}(R) = \exp\left(-\frac{1}{2} NR^2\right) H_I(\sqrt{NR}) , \] 

and

\[ \eta_{n,k}(r) = r^k \exp\left(-\frac{1}{2} r^2\right) L_n^{b+k}(r^2) . \] 

Here, \( b = \frac{1}{2}(N - 3) + \frac{1}{2} N(N - 1)(\lambda + \gamma) \).

The total wavefunction for Eq. (19) is

\[ \phi_{I,n,k} = \exp\left(-\frac{1}{2} NR^2\right) H_I(\sqrt{NR}) \exp\left(-\frac{1}{2} r^2\right) L_n^{b+k}(r^2) P_k(x) \] 

and the corresponding energy eigenvalue is given by

\[ E_{I,n,k} = I + 2n + k + E_o . \] 

For \( n = k = 0 \), Eq. (27) becomes

\[ \phi_{I,o,o} = \exp\left(-\frac{1}{2} NR^2\right) \exp\left(-\frac{1}{2} r^2\right) H_I(\sqrt{NR}) . \] 

which is exactly the same as the one given by VOK.
It is worth pointing out that the integers $n$ and $k$ originate from the radial and angular degrees of freedom respectively. For the special case of the Calogero Hamiltonian, where the harmonic potential is of the type $V = \frac{1}{2N} \sum_{i<j}(x_i - x_j)^2$, the center of mass motion can be eliminated completely. In that case, as is well known, the energy eigenvalues are $E_{n,k} = 2n + k + E_o$. The translation invariant wavefunctions for the above case can be obtained from Eq. (27) by putting $R = 0$. In the CS case, the presence of the center of mass motion brings in the quantum number $I$.

In conclusion, although the basis set given by VOK forms a complete set, it only describes the center of mass motion. The present method can also be extended to the wavefunctions given by the same authors for the spin Hamiltonians with identical result. It is of great interest to generate the full set of degenerate eigenfunctions by generalizing the algebraic procedure advocated here. It is also of interest to consider the algebra of the operators which will connect the degenerate wavefunctions, since the degeneracies have a deep connection with symmetries. It should be noted in this context that, recently various infinite dimensional algebras, e.g., the $W_\infty$-algebra, have manifested in the CS model. Construction and analysis of the coherent states is another direction worth exploring, since they will provide a better understanding of the semi-classical behavior of this quantum mechanical model. Whether these algebraic techniques can be applied to other $N$-body Hamiltonians is also an important question. These problems are currently under investigation and will be reported elsewhere.

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