ON DYNAMICS OF QUASI-GRAH MAPS

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ABSTRACT. In this paper, we study dynamics of maps on quasi-graphs characterizing their invariant measures. In particular, we prove that every invariant measure of quasi-graph map with zero topological entropy has discrete spectrum. We also obtain an analog of Llibre-Misiurewicz’s result relating positive topological entropy with existence of topological horseshoes.

By these results, Sarnak’s Möbius Disjointness Conjecture restricted to the class of quasi-graph maps with zero topological entropy is reduced to already known cases. We prove however, that answering the conjecture for all maps on dendrites with zero topological entropy is equivalent to solving it for all dynamical systems with zero topological entropy.

1. INTRODUCTION

By a topological dynamical system or just dynamical system, we mean a pair $(X, T)$, where $X$ is a compact metric space with a metric $d$ and $T : X \rightarrow X$ is a continuous map. Denote by $C(X, \mathbb{C})$ the set of all continuous $\mathbb{C}$-valued functions over $X$.

In [31], Sarnak proposed the following conjecture (where $\mu$ is the Möbius function):

Möbius Disjointness Conjecture. The Möbius function $\mu(n)$ is linearly disjoint from any dynamical system $(X, T)$ with zero topological entropy, that is,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\varphi(T^n(x)) = 0.$$ 

Though this conjecture remains still open, many interesting cases have been verified. We refer the reader to a recent survey [13] for the current state of the art (for details of these cases see also the references therein). Recent results (see [18, Theorem 1.2]; cf. also [17] and [13]) show that the conjecture holds (in the class of homeomorphism), provided that all invariant measures have discrete spectrum. While this condition is elegant, it applicability in practice is quite limited, since beyond simplest cases not much is known about spectrum of invariant measures of maps. On the other hand earlier works prove with simple arguments that for zero entropy maps on: interval [19], circle [8] or even topological graphs [23] (see also [12]) the conjecture holds. While discrete spectrum of ergodic measures of these maps can be deduced from classical characterizations of dynamics of topological graph maps due to Blokh [5], the structure of the simplex of all invariant measures seems complicated, since clearly there may be uncountably many ergodic measures (e.g. see Example 2.4). On the other hand, it is natural to expect that all
invariant measures have discrete spectrum at least for these maps. Then the main question of our paper is the following:

**Question 1.1.** What one-dimension continua \( X \) have the property that if \((X, f)\) is dynamical system with zero entropy, then every invariant measure of \((X, f)\) has discrete spectrum?

In the above question we ask about connected spaces, because on the Cantor set many zero entropy systems have ergodic measures with continuous spectrum. Also in dimension two it is relatively easy to provide an example, e.g. see Remark 2.2; by the same example discrete spectrum of ergodic measures is not enough to induce discrete spectrum of all invariant measures.

To deal with Question 1.1 we start with a class of continua called quasi-graphs as introduced in [27], which is to some extent similar to topological graphs (in the same way as the Warsaw circle is “similar” to the unit circle; they seem to be a special case of a slightly more general notion of generalized \( \sin(1/x) \)-type continua of [15]). While it is quite easy to see that the structure of \( \omega \)-limit sets of maps on these spaces can be much richer than that was possible on topological graphs (recall from [6], that we have full characterization of admissible types of \( \omega \)-limit sets for topological graph maps), still some similarities exist. First of all, we are able to characterize all invariant measures of quasi-graph maps as convex combinations of finitely many invariant measures for some topological graph maps. We show that every ergodic invariant measure of a map on a quasi-graph is isomorphic to an ergodic invariant measure of a map on some topological graph and we also obtain an analog of Llibre-Misiurewicz’s result relating positive topological entropy with existence of topological horseshoes.

A partial answer to Question 1.1 is provided by the following theorem.

**Theorem 1.2.** Let \( X \) be a quasi-graph and let \( f : X \to X \) be a continuous map with zero topological entropy. Then every invariant measure of \((X, f)\) has discrete spectrum.

Then the following corollary is a simple consequence of the above and known results.

**Corollary 1.3.** The Möbius function is linearly disjoint from all quasi-graph maps with zero topological entropy.

In the view of Theorem 1.2 the reader may naively think that it can be extended to all one-dimensional continua, or at least expect that all spaces with structure “similar” to the interval can support only examples of maps with zero entropy whose invariant measures have discrete spectrum. The most natural class of the spaces that generalize interval maps (or even trees, which are their essential subclass) are dendrites. These continua share many properties similar to topological graphs without circuits (e.g., there exists a unique arc joining any two points in any dendrite), however usually they allow richer dynamical structure (e.g., see [15]). As shown by the following theorem, this generalization is not a good choice, because considering the conjecture in this class of continua is the same hard as solving it in full generality. Therefore, it may be an overestimated statement to say that dimension one makes things “easy”. Recall that a dynamical system \((X, T)\) is invertible if \( T : X \to X \) is a homeomorphism and surjective if \( T \) is a surjection.

**Theorem 1.4.** The Möbius function is linearly disjoint from all dynamical systems with zero topological entropy if and only if it is so for all surjective dynamical systems over the Gehman dendrite with zero topological entropy.
From the above it is quite obvious that the Gehman dendrite can support zero entropy maps with invariant measure beyond the class of these with discrete spectrum. Therefore continua in Question 1.1 must be in some sense “simpler” than these dendrites. Furthermore, it is well known that if a dendrite has an uncountable set of endpoints then it contains a copy of the Gehman dendrite (e.g., see [29, Corollary 2]) and if \( X \subset Y \) are dendrites, then there exists a standard retraction \( \pi: Y \to X \) (so-called first point map). Then any continuous map \( f: X \to X \) defines by \( g = f \circ \pi \) a continuous map \( g: Y \to Y \) with \( g|_X = f \). Thus as a direct corollary of Theorem 1.4 one has:

**Corollary 1.5.** The Möbius function is linearly disjoint from all dynamical systems with zero topological entropy if and only if so is for all dynamical systems with zero topological entropy over a dendrite with an uncountable set of endpoints.

It was shown recently that every group action (in particular, every homeomorphism) on a dendrite is tame (cf. [14, Theorem 1.1]), and then by [18, Theorem 1.8] the Möbius function is linearly disjoint from it. Thus the above Corollary 1.5 reflects a huge difference between invertible and non-invertible maps on dendrites.

Note that recently in [1] the authors proved that if the set \( \text{End}(X) \) of endpoints of some dendrite \( X \) is closed with finitely many accumulation points then the Möbius function is linearly disjoint from any dynamical system on \( X \) with zero topological entropy. At the end of the present paper we will provide another proof for it.

Since the Cantor set contains a countable closed set with infinitely many accumulation points (and so in the Gehman dendrite we can select a subdendrite with set of endpoints equal to it), there are dendrites covered neither by Corollary 1.5 nor by results of [1]. There are many other examples not covered by these results, e.g., obtained by attaching an arc to an endpoint.

Up to our knowledge, the following question remains open even in the case of closed \( \text{End}(X) \) with finitely many accumulation points, since arguments of [1] rely on local structure of these maps, rather than spectral properties of all invariant measures. This motivated the following question. Let us recall that \( X \) contains a copy of the Gehman dendrite if \( \text{End}(X) \) is uncountable.

**Question 1.6.** Assume that \( f: X \to X \) is a continuous map with zero topological entropy acting on a dendrite \( X \) with \( \text{End}(X) \) countable. Does every invariant measure of \((X, f)\) have discrete spectrum?

The paper is organized as follows. In Section 2 we explore maximal \( \omega \)-limit sets of topological graph maps with zero topological entropy and show that every invariant measure on a topological graph map with zero topological entropy has discrete spectrum. In Section 3 we characterize all invariant measures of quasi-graph maps as convex combinations of finitely many invariant measures for some topological graph maps. Consequently, we show that every ergodic invariant measure of a quasi-graph map is isomorphic to an ergodic invariant measure of some topological graph map. We also obtain an analog of Llibre-Misiurewicz’s result relating positive topological entropy with existence of topological horseshoes. Finally, in Section 4 we present proofs of Theorem 1.4 and Corollary 1.3 together with some related results.
2. Graph maps with zero topological entropy

In this section, we discuss invariant measures and discrete spectrum for general dynamical systems and then study the structure of $\omega$-limits of topological graph maps and show that if a topological graph map has zero topological entropy then every invariant measure has discrete spectrum.

Recall that a topological space $X$ is a continuum if it is a compact, connected metric space. An arc is a continuum homeomorphic to the closed interval $[0, 1]$. A topological graph or just graph for short is a continuum which is a union of finitely many arcs, any two of which are either disjoint or intersect in at most one common endpoint. We say that a graph $S$ is an $n$-star with center $v \in S$ if there is a continuous injection $\varphi: S \to \mathbb{C}$ such that $\varphi(v) = 0$ and $\varphi(S) = \{r \exp(\frac{2k\pi i}{n}) : r \in [0, 1], k = 1, 2, \ldots, n\}$.

Let $X$ be a compact arcwise connected metric space and $v \in X$. The valence of $v$ in $X$, denoted by $\text{val}(v)$, is the number

$$\sup\{n \in \mathbb{N} : \text{there exists an n-star with center v contained in X}\}.$$ 

Note that the valence of $v$ may be $\infty$. The point $v$ is called an endpoint of $X$ if $\text{val}(v) = 1$, and a branching point of $X$ if $\text{val}(v) \geq 3$. The collections of all endpoints and branching points of $X$ are denoted by $\text{End}(X)$ and $\text{Br}(X)$, respectively.

2.1. Invariant measures and discrete spectrum. Let $(X, f)$ be a dynamical system and $x \in X$. The orbit of $x$, denoted by $\text{Orb}_f(x)$, is the set $\{f^n x : n \geq 0\}$; and the $\omega$-limit set of $x$, denoted by $\omega_f(x)$, is defined as $\cap_{n \geq 0} \{f^m x : m \geq n\}$. It is easy to check that $\omega_f(x)$ is closed and strongly $f$-invariant, i.e., $f(\omega_f(x)) = \omega_f(x)$.

Let $(X, f)$ be a dynamical system. The set of all Borel probability measures over $X$ is denoted by $M(X)$, and $M_f(X) \subset M(X)$ denotes the set of all invariant elements of $M(X)$. The set of all ergodic elements of $M_f(X)$ is denoted by $M_f^e(X)$. It is well known that $M(X)$ endowed with weak-* topology is a compact metric space and that $M_f^e(X)$ is its closed subset. We say that $\mu \in M_f^e(X)$ has discrete spectrum, if the linear span of the eigenfunctions of $U_f$ in $L^2(\mu)$ is dense in $L^2(\mu)$, where as usual $U_f$ denotes the Koopman operator: $U_f(\varphi) = \varphi \circ f$ for every $\varphi \in L^2(\mu)$. By a classical result by Kušnirenko, an invariant measure has discrete spectrum if and only if it has zero measure-theoretic sequence entropy [21]. We refer the reader to the textbook [32] on ergodic theory.

Remark 2.1. Let $\mu$ be an invariant measure for a dynamical system $(X, f)$. Assume that $X = \bigcup_{n=1}^{\infty} X_n$, where each $X_n$ is a Borel set and each $X_n \cap \text{supp}(\mu)$ is positively $f$-invariant (i.e., $f(X_n \cap \text{supp}(\mu)) \subset X_n \cap \text{supp}(\mu)$, here $\text{supp}(\mu)$ denotes the support of the measure $\mu$). Then $\mu$ has discrete spectrum, provided that the normalized invariant measure of each measure $\mu |_{X_n}$ has discrete spectrum.

It is shown in [7] that every invariant measure on an interval map with zero topological entropy has zero measure-theoretic sequence entropy. We shall generalize this result in next subsection to graph maps with zero topological entropy, for details see Theorem 2.11. In [23], we have showed that every ergodic invariant measure on a graph map with zero topological entropy has discrete spectrum. The following remark show that this still can not guarantee any invariant measure has discrete spectrum in general.
Remark 2.2. By [21], the Lebesgue measure is an invariant measure of the map \((x, y) \mapsto (x, y + x)\) on torus which does not have discrete spectrum, while ergodic invariant measures are the rotations of the circle which have discrete spectrum.

It is well known that if a dynamical system has only countably many ergodic invariant measures and each of them has discrete spectrum then any invariant measure also has discrete spectrum. We can generalize this result a little bit.

Theorem 2.3. Let \((X, f)\) be a dynamical system and \(\mu \in M_f(X)\). Let \(\mu = \int M_f(X) \nu d\rho(\nu)\) be the ergodic decomposition of \(\mu\). Assume that there are at most countably many non-atomic ergodic invariant measures in the decomposition (i.e., we can rewrite \(\mu\) as \(\mu = \int Q \nu d\rho(\nu)\) with set \(Q \subset M_f^s(X)\) containing at most countably many non-atomic measures) and each of them has discrete spectrum. Then \(\mu\) also has discrete spectrum.

Proof. Denote \(X_n = \{x \in X : f^n(x) = x\}\) and \(\{x \in X : f^i(x) \neq x\text{ for all } i = 1, 2, \ldots, n - 1\}\) for each \(n \in \mathbb{N}\) and \(X_\infty = X \setminus \bigcup_{n=1}^\infty X_n\). Clearly these \(X_n, n \in \mathbb{N} \cup \{\infty\}\) are disjoint \(f\)-invariant Borel subsets and then (it makes sense, no matter if \(\mu|_{X_n}, n \in \mathbb{N} \cup \{\infty\}\) is trivial or not) \[L^2(\mu) = \bigoplus_{n \in \mathbb{N}} L^2(\mu|_{X_n}) \oplus L^2(\mu|_{X_\infty}).\]

Now applying Remark 2.1 it is enough to show that after restriction to any of these sets \(\mu\) has discrete spectrum. For each \(n \in \mathbb{N}\), since \(f^n|_{X_n}\) is the identity, \(U_f^n\) is the identity as well on \(L^2(\mu|_{X_n})\), and then the spectrum of \(U_f\) restricted onto \(X_n\) consists of exactly \(n\)-th roots of 1, thus the normalized invariant measure of \(\mu|_{X_n}\) has discrete spectrum.

But by the assumption \(X_\infty\) has positive measure for at most countably many ergodic invariant measures from \(Q\), all of which are pairwise singular, so similarly we have \[L^2(\mu|_{X_\infty}) = \bigoplus_{\nu \in Q'} L^2(\mu|_{X_\nu}),\]
where \(Q'\) denotes the countable set of all non-atomic ergodic invariant measures of \((X, f)\) from \(Q\) and for each \(\nu \in Q'\) the set \(X_\nu\) is a Borel set such that \(\nu(X_\nu) = 1\) and \(\eta(X_\nu) = 0\) for all \(\eta \in M^s_f(f) \setminus \{\nu\}\), e.g., we can take \(X_\nu\) to be the set of all generic points of \(\nu\). Combining again Remark 2.1 with the assumption one has readily that the normalized invariant measure of \(\mu|_{X_\infty}\) has discrete spectrum. This finishes the proof.

The following example reveals that there exists an interval map with zero topological entropy which has uncountably many non-atomic ergodic measures. So we can not apply Theorem 2.3 directly.

Example 2.4. Figure 1(a) presents an example of Delahaye in [9] with odometer as a maximal \(\omega\)-limit set. Its simple modification Figure 1(b) leads to a map with uncountably many maximal \(\omega\)-limit sets which are odometers, and consequently, uncountably many non-atomic ergodic measures. The idea here is very simple. In original Delahaye’s example, the interval \([0, 1/3]\) is 2-periodic with image \([2/3, 1]\) and over \([2/3, 1]\) the same scheme is repeated for \(f^2\). In modified example, again \([0, 1/3]\) is periodic, but now it contains two 2-periodic subintervals with disjoint orbits, namely \([2/3, 7/9]\) and \([8/9, 1]\). Since at each step of construction \(2^n\)-periodic interval splits into two periodic intervals with disjoint orbits, at the end we will have uncountably many invariant sets, which by the construction are odometers.
Let \((X, f)\) be a dynamical system. Recall that a pair \((x, y)\) is proximal if
\[
\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0,
\]
asympotic if
\[
\lim_{n \to \infty} d(f^n(x), f^n(y)) = 0,
\]
and a Li-Yorke pair (or scrambled pair) if it is proximal but not asymptotic. Given two dynamical systems \((X, f)\) and \((Y, g)\), by a factor map we mean \(\pi: (X, f) \to (Y, g)\) where \(\pi: X \to Y\) a continuous surjection satisfying \(\pi \circ f = g \circ \pi\), in this case, \((X, f)\) is called an extension of \((Y, g)\) and \((Y, g)\) is called a factor of \((X, f)\).

We will use the following result which is essentially contained in the proof of [24, Theorem 3.8] (see also proofs of [11, Proposition 2.5] and [16, Theorem 4.4]).

**Lemma 2.5.** Let \(\pi: (X, f) \to (Y, g)\) be a factor map, and set \(\Delta_X = \{(x, x) : x \in X\}\) and \(R_\pi = \{(x_1, x_2) \in X^2 : \pi(x_1) = \pi(x_2)\}\). Assume that \(\Delta_X\) has full measure for each invariant measure of \((R_\pi, f \times f)\). Then every invariant measure of \((X, f)\) is measure-theoretically isomorphic to some invariant measure of \((Y, g)\).

### 2.2. The structure of \(\omega\)-limits of graph maps with zero topological entropy.

In this subsection we fix a topological graph \(G\) and a continuous map \(f: G \to G\) with zero topological entropy.

A subgraph \(K\) of \(G\) is called periodic if there is a positive integer \(k\) such that \(K, f(K), \ldots, f^{k-1}(K)\) are pairwise disjoint and \(f^k(K) = K\). In such a case, \(k\) is called the period of \(K\) and \(\bigcup_{i=0}^{k-1} f^i(K)\) is called a cycle of graphs. For an infinite \(\omega\)-limit set \(\omega_f(x)\) we let
\[
C(x) = \{X : X \subset G \text{ is a cycle of graphs and } \omega_f(x) \subset X\}.
\]

By [5, Lemma1] (see also [30, Lemma 10 (i)]) the family \(C(x)\) is never empty. If periods of cycles of graphs in \(C(x)\) are unbounded then \(\omega_f(x)\) is called solenoid (for \((G, f)\)). We recall [5, Theorem 1] (see also [30, Lemma 11]).

**Lemma 2.6.** If \(\omega_f(x)\) is a solenoid, then there exists a sequence of cycles of graphs \((X_n)_{n \geq 1}\) with strictly increasing periods \(k_n\) such that

(1) for every \(n \geq 1\), \(k_{n+1}\) is a multiple of \(k_n\);
(2) for every $n \geq 1$, $X_{n+1} \subset X_n$;
(3) for every $n \geq 1$, each connected component of $X_n$ contains the same number of connected components of $X_{n+1}$; and
(4) $\omega_f(x) \subset \cap_{n \geq 1} X_n$.

Furthermore, $\omega_f(x)$ does not contain any periodic points.

Note that the set of all $\omega$-limit sets of a graph map is closed under Hausdorff metric [26], and then each $\omega$-limit set is contained in a maximal one by Zorn Lemma. The following three lemmas are implicitly contained in [5], here we provide proofs for completeness.

**Lemma 2.7.** Any two different solenoids for maximal $\omega$-limit sets are disjoint.

*Proof.* Assume that $\omega_f(x)$, $\omega_f(y)$ are two solenoids which are both maximal $\omega$-limit sets. As $G$ has only finite many branching points, by Lemma 2.6 there exists a positive integer $n$ such that $k_n$ is large than the number of branching points of $G$ and a connected component $I$ of $X_n$ which does not contain any branching points of $G$. Then $I$ is an arc in $G$ and $f^{k_n}(I) = I$. It is clear that $\omega_f(x) \cap I$ is uncountable, so $f^s(y) \in I$ for some $s > 0$. Then $\omega_{f^{k_n}}(f^s(x)) \subset I$ is a solenoid for $(I, f^{k_n})$. Furthermore, it is not hard to see that $\omega_{f^{k_n}}(f^s(x))$ is a maximal $\omega$-limit set for $(I, f^{k_n})$. In fact, if let $\omega_{f^{k_n}}(\xi) \supset \omega_{f^{k_n}}(f^s(x))$ be another $\omega$-limit set for $(I, f^{k_n})$, then $\omega_f(\xi) \supset \omega_f(x)$ will be a $\omega$-limit set for $(G, f)$ and hence $\omega_f(\xi) = \omega_f(x)$, finally one has $\omega_{f^{k_n}}(\xi) = \omega_{f^{k_n}}(f^s(x))$ since $\omega_{f^{k_n}}(\xi) = \omega_f(\xi) \cap I$ and $\omega_{f^{k_n}}(f^s(x)) = \omega_f(x) \cap I$ by the above construction.

Now suppose that there is $z \in \omega_f(y) \cap \omega_f(x)$. Observing invariance of $\omega_f(x) \cap \omega_f(y)$ and its periodic structure provided by Lemma 2.6 we may assume that $z$ belongs to the interior of $I$. Similar to the above arguments, there is $r > 0$ such that $f^r(y) \in I$ and then $\omega_{f^r}(f^r(y))$ is another maximal $\omega$-limit set for $(I, f^{k_n})$. Note that $z$ belongs to the intersection of $\omega_{f^{k_n}}(f^s(x))$ and $\omega_{f^{k_n}}(f^r(y))$. But now we are on the interval $I$ and clearly $(I, f^{k_n})$ has zero topological entropy, by [22, Lemma 3.4] one has $\omega_{f^{k_n}}(f^s(x)) = \omega_{f^{k_n}}(f^r(y))$, which leads to $\omega_f(x) = \omega_f(y)$. This finishes the proof. \qed

**Lemma 2.8.** Let $\omega_f(x)$ be a solenoid, and assume $\omega_f(x) \subset \cap_{n \geq 1} X_n$ where $(X_n)_{n \geq 1}$ is a sequence of cycles of graphs with increasing periods provided by Lemma 2.6. Then:

(1) the set $P(x) := \cap_{n \geq 1} X_n$ is closed, strongly invariant and does not contain any periodic points;
(2) $P(x)$ has uncountably many connected components and at most countably many of them can be non-degenerate;
(3) each connected component of $P(x)$ intersects $\omega_f(x)$;
(4) if $J$ is a non-degenerated connected component of $P(x)$, then the interior of $J$ is wandering, i.e., $f^i(\text{int}(J)) \cap \text{int}(J) = \emptyset$ for all $i \geq 1$;
(5) $P(x)$ depends only on $\omega_f(x)$, not on the choice of $(X_n)_{n \geq 1}$;
(6) for any $y \in P(x)$, $\omega_f(y) \setminus \omega_f(x)$ is at most countable;
(7) for every $z \in G$, $\omega_f(z) \cap \omega_f(x) \neq \emptyset$ implies that $\omega_f(z)$ is also a solenoid contained in $P(x)$ and $\omega_f(z) \setminus \omega_f(x)$ is at most countable; and
(8) there exists a unique maximal $\omega$-limit set containing $\omega_f(x)$ and this maximal $\omega$-limit is a solenoid which is contained in $P(x)$. 


Proof. (1)–(4) can be easily obtained from Lemma 2.6 and the construction of $P(x)$.

(5) Let $(Y_n)_{n \geq 1}$ be another sequence of cycles of graphs as in Lemma 2.6. Note that $\bigcap_{n \geq 1} Y_n$ also satisfies (1)–(4). In particular, $\bigcap_{n \geq 1} Y_n$ has uncountably many degenerate components and each of them is contained in $\omega_f(x)$. For every large enough integer $m$ there is a connected component $J_m$ of $X_m$ which does not contain branching points of $G$ such that $J_m$ contains at least three degenerate components of $\bigcap_{n \geq 1} Y_n$. But then by the assumption there exists a positive integer $s$ and a connected component $I_s$ of $Y_s$ such that $I_s \subset J_m$, which implies that $Y_s \subset X_m$ and hence $\bigcap_{n \geq 1} Y_n \subset \bigcap_{n \geq 1} X_n$. Finally by duality we obtain $\bigcap_{n \geq 1} Y_n = \bigcap_{n \geq 1} X_n$.

(6) Let $y \in P(x)$. Clearly $\omega_f(y) \subset P(x)$. By (1), $\omega_f(y)$ does not contain any periodic points, then $\omega_f(y)$ is uncountable. By (4), for every non-degenerated connected component $J$ of $P(x)$, $J \cap \omega_f(y)$ is contained in the boundary of $J$ and then is finite. So there are at most countable many points in $\omega_f(y)$ which are in the non-degenerated connected components $J$ of $P(x)$. By (3), other points in $\omega_f(y)$ must be contained in $\omega_f(x)$. Then $\omega_f(y) \setminus \omega_f(x)$ is at most countable.

(7) First note that $\omega_f(z) \cap \omega_f(x)$ is uncountable. For any positive integer $n$, as $\omega_f(x) \subset X_n$ and $X_n$ has only finite boundary points, there exists a positive integer $i$ such that $f^i(z) \in X_n$ and then $\omega_f(z) \subset X_n$. Then $\omega_f(z)$ is also a solenoid and $\omega_f(z) \subset P(x)$. Similar to the proof of (6), $\omega_f(z) \setminus \omega_f(x)$ is at most countable.

(8) Let $\omega_f(y)$ be a maximal $\omega$-limit set containing $\omega_f(x)$. In particular, $\omega_f(y) \cap \omega_f(x) \neq \emptyset$, and hence $\omega_f(y)$ must be a solenoid contained in $P(x)$ by (7). If $\omega_f(z)$ is another maximal $\omega$-limit set containing $\omega_f(x)$. Clearly $\omega_f(y) \cap \omega_f(z) \cap \omega_f(x) \neq \emptyset$. Then $\omega_f(y) = \omega_f(z)$ follows from Lemma 2.7.

\textbf{Lemma 2.9.} Let $\omega_f(x)$ and $\omega_f(y)$ be two different solenoids for maximal $\omega$-limit sets and assume that $P(x)$ and $P(y)$ are as in Lemma 2.8. Then $P(x)$ and $P(y)$ are disjoint.

\textbf{Proof.} If $P(x) \cap P(y) \neq \emptyset$, pick $z \in P(x) \cap P(y)$. By Lemma 2.8, $\omega_f(z)$ is uncountable, and both $\omega_f(z) \setminus \omega_f(x)$ and $\omega_f(z) \setminus \omega_f(y)$ are at most countable. Then $\omega_f(x) \cap \omega_f(y) \neq \emptyset$, and hence $\omega_f(x) = \omega_f(y)$ by Lemma 2.7. A contradiction. Thus, $P(x) \cap P(y) = \emptyset$. \hfill $\Box$

Thus we can obtain the following result which will be used later.

\textbf{Proposition 2.10.} There exists a continuous map $g$ acting on a graph $Y$ without Li-Yorke pairs and a factor map $\pi : (G, f) \to (Y, g)$ such that the pair $(p, q)$ is asymptotic whenever $p, q \in \pi^{-1}(y)$ for some $y \in Y$.

\textbf{Proof.} For every maximal $\omega$-limit set which is a solenoid, denote by $P$ the unique set obtained from Lemma 2.8. Note that there are at most countably many sets $P$ which have non-degenerate connected components, as we are working on a graph $G$. Denote by $(P_n)_{n \in \Lambda}$ the sequence (finite or not) of these sets, where the index set $\Lambda$ is at most countable. By Lemma 2.9, $P_n \cap P_m = \emptyset$ whenever $m \neq n$. Let $R$ be the relation on $G \times G$ given by $x \sim y$ if and only if $x = y$ or there exists an $n \in \Lambda$ and a connected component $C$ of $P_n$ such that $x, y \in C$. In particular, if $\Lambda = \emptyset$ then $(x, y) \in R$ if and only if $x = y$. Thus $R$ is positively $f \times f$-invariant, i.e., $(f \times f)(R) \subset R$. Since for every $\varepsilon > 0$ there are at most finitely many disjoint connected subsets of diameter at least $\varepsilon$, we immediately obtain that the subset $R$ is closed. It is also clear that equivalence classes of $R$ are connected (and so arc-wise connected). We have readily that the quotient space $Y = G/R$ is a graph.
In the following let us check that the induced maps \(g = f/R\) and \(\pi: (G, f) \rightarrow (Y, g)\) have desired properties. Firstly let \(y \in Y\) and \(p, q \in \pi^{-1}(y)\). Say \(p, q \in P_n\) for some \(n \in \Lambda\). Note that \(p, q\) are contained in a connected component of \(P_n\), by Lemma 2.8 it is easy to see that the pair \((p, q)\) is asymptotic. Now it is sufficient to show that \((Y, g)\) contains no Li-Yorke pairs. By the argument in the proof of [30, Theorem 3], we only need to check that for every solenoid \(\omega_f(y)\) in \(Y\), \(P(y)\) does not contain any non-degenerate connected components. Pick a point \(x \in G\) with \(\pi(x) = y\). Then \(\pi(\omega_f(x)) = \omega_f(y)\). As \(\pi\) is monotone (i.e., pre-images of connected sets are connected), \(\omega_f(x)\) is also a solenoid and \(\pi(P(x)) = P(y)\). By the construction of \(Y\), we have collapsed all the non-degenerated connected components of \(P(x)\), so \(P(y)\) does not contain any non-degenerate connected components. This finishes our proof.

Now we can prove the main result of this section.

**Theorem 2.11.** Let \(f: G \rightarrow G\) be a graph map with zero topological entropy. Then every invariant measure of \((G, f)\) has discrete spectrum.

**Proof.** Let \(\pi: (G, f) \rightarrow (Y, g)\) be the factor map provided by Proposition 2.10. Since the graph map \((Y, g)\) does not contain Li-Yorke pairs, by [23, Theorem 1.5] it has zero topological sequence entropy and then each invariant measure has zero sequence entropy. By [21, Theorem 4], each invariant measure of \((Y, g)\) has discrete spectrum. Note that every pair in \(R_\pi \setminus \Delta_G\) is asymptotic. It is easy to check that \(\Delta_G\) has full measure for each invariant measure of \((R_\pi, f \times f)\), and so by Lemma 2.5 every invariant measure of \((G, f)\) is measure-theoretically isomorphic to some invariant measure of \((Y, g)\) and hence also has discrete spectrum.

### 3. Dynamics on quasi-graphs

In this section, we characterize all invariant measures of quasi-graph maps as convex combinations of finitely many invariant measures for some graph maps. Consequently, we show that every invariant measure of a quasi-graph map with zero topological entropy has discrete spectrum and every ergodic invariant measure of a quasi-graph map is essentially an ergodic invariant measure of some graph map. We also obtain an analog of Llibre-Misnerewicz’s result relating positive topological entropy with existence of topological horseshoes.

#### 3.1. Preliminaries for quasi-graph maps

Let \(X\) be a compact metric space and let \(L\) be an arcwise connected subset of \(X\). If there exists a continuous bijection \(\varphi: \mathbb{R}_+ \rightarrow L\), then we say that \(L\) is a quasi-arc with the parameterization \(\varphi\). The point \(\varphi(0)\) is called an endpoint of \(L\) and the \(\omega\)-limit set of \(L\) is the set \(\omega(L) = \bigcap_{m \in \mathbb{N}} \varphi([m, \infty))\). A quasi-arc \(L\) is called oscillatory if \(\omega(L)\) contains more than one point. Note that the endpoint and the \(\omega\)-limit set of \(L\) are dependent on the parameterization \(\varphi\), however if \(L\) is an oscillatory quasi-arc then the endpoint is uniquely determined and the parameterization is unique up to a topological conjugacy (cf. [27, Propositions 2.17 and 2.20]).

A quasi-graph is a non-degenerate, compact, arcwise connected metric space \(X\) satisfying that there is a positive integer \(N\) such that \(\overline{Y} \setminus Y\) has at most \(N\) arcwise connected components for every arcwise connected subset \(Y \subset X\). The following fact from [27, Theorem 2.24] is an important characterization of quasi-graphs. The case of \(n = 0\) is the simplest situation, when a quasi-graph is in fact a graph.
Theorem 3.1. A continuum $X$ is a quasi-graph if and only if there is a graph $G$ and pairwise disjoint oscillatory quasi-arcs $L_1, \ldots, L_n$ (with $n \geq 0$) in $X$ such that:

1. $X = G \cup \bigcup_{i=1}^{n} L_i$ and $\text{End}(X) \cup \text{Br}(X) \subset G$;
2. for each $1 \leq i \leq n$, $L_i \cap G = \{a_i\}$ where $a_i$ is the endpoint of $L_i$;
3. $\omega(L_i) \subset G \cup \bigcup_{j=1}^{i-1} L_j$ for each $1 \leq i \leq n$, and
4. if $\omega(L_i) \cap L_j \neq \emptyset$ for some $1 \leq i, j \leq n$, then $\omega(L_i) \supset L_j$.

First we have the following useful observation.

Proposition 3.2. Let $X$ be a quasi-graph with $G$ and $L_1, \ldots, L_n$ as in Theorem 3.1. Then:

for every two different points $a, b \in X$ there are only finite many different arcs in $X$ with endpoints $a$ and $b$, Furthermore, if $a$ and $b$ are in the same quasi-arc $L_i$ then there is a unique arc with endpoints $a$ and $b$.

Proof. By Theorem 3.1, all the endpoints and branching points of $X$ are in $G$. As $G$ is a graph, the sum of valences of all branching points is finite. Then for every two different points $a, b \in X$ there are only finite many different arcs in $X$ with endpoints $a$ and $b$.

Now assume that $a$ and $b$ are in the same quasi-arc $L_i$, and let $\varphi: \mathbb{R}_+ \to L_i$ be a parameterization of $L_i$. By Theorem 3.1, $\omega(L_i) \cap L_i = \emptyset$, and so $\varphi$ is a homeomorphism. Pick $s, t \in \mathbb{R}_+$ such that $\varphi(s) = a$ and $\varphi(t) = b$. Without loss of generality, assume that $s < t$. Then $\varphi|_{[s, t]}$ is an arc with $\varphi(s) = a$ and $\varphi(t) = b$. Let $\alpha: [0, 1] \to X$ be another arc (different from $\varphi$) with endpoints $a$ and $b$, and say $\alpha(0) = a$ and $\alpha(1) = b$. As every point in $\varphi((s, t))$ has valence 2, $\alpha((0, 1)) \cap \varphi((s, t)) = \emptyset$. Then there exists $c \in (0, 1)$ such that $\alpha((c, 1)) = \varphi((t, \infty))$. But this implies that $\omega(L_i) = \{\alpha(c)\}$, which is in contradiction to the assumption in Theorem 3.1 that $L_i$ is an oscillatory quasi-arc.

Thus any non-degenerate arcwise connected closed set $H$ of a quasi-graph $G$ is again a quasi-graph, but the positive integer $N$ in the definition can increase as some path in $G$ may not belong to $H$.

Proposition 3.3. Let $X$ be a quasi-graph and let $f: X \to X$ be a continuous map. Then $\bigcap_{n=0}^{\infty} f^n(X)$ is also a quasi-graph if it is non-degenerate.

Proof. First note that $f(X)$ is a quasi-graph since it is arcwise connected as an image of an arcwise connected set. Furthermore $f^n(X)$ is also a quasi-graph for every $n \geq 1$. Let $X_0 = \bigcap_{n=0}^{\infty} f^n(X)$. It is sufficient to show that $X_0$ is arcwise connected. Fix any two different points $p, q \in X_0$ (if exist). For every $n \geq 1$, as $f^n(X)$ is arcwise connected, there exists an arc $J_n \subset X$ with endpoints $p, q$ such that $J_n \subset f^n(X)$. By Proposition 3.2, there are only finitely many different arcs in $X$ connecting $p$ and $q$. Note that since $f^n(X)$ is a nested sequence, there exists an arc $J \subset X$ with endpoints $p, q$ such that $J \subset f^n(X)$ for all $n \geq 0$. Then $J \subset X_0$, which implies that $X_0$ is arcwise connected.

A non-oscillatory quasi-arc in a compact metric space $X$ is called a 0-order oscillatory quasi-arc. An oscillatory quasi-arc $L$ is called a $k$-order oscillatory quasi-arc for some $k > 0$ if $\omega(L)$ contains at least one $(k - 1)$-order oscillatory quasi-arc, and $\omega(K)$ contains no any $(k - 1)$-order oscillatory quasi-arc for every quasi-arc $K$ in $\omega(L)$. It is not hard to see that the $\omega$-limit set $\omega(L)$ of an oscillatory quasi-arc $L$ of order $k$ contains at least one oscillatory quasi-arc $K_i \subset \omega(L)$ of order $i$ for each $i = 0, 1, \ldots, k - 1$, and does not contain any quasi-arc of order $n \geq k$. The following lemma combines [27, Lemma 3.1, Corollaries 3.2 and 3.3 and Proposition 3.4].
Lemma 3.4. Let $X$ be a quasi-graph and let $f : X \to X$ be a continuous map.

1. If $G \subset X$ is a graph, then $f(G)$ contains no oscillatory quasi-arcs.
2. If $L$ and $K$ are two oscillatory quasi-arcs in $X$ with $L \subset f(K)$, then the order of $L$ is not bigger than the order of $K$ and $\omega(L) \subset f(\omega(K))$.

Let $L$ be an oscillatory quasi-arc in a quasi-graph $X$. For $t \in \mathbb{R}_+$, we will use $L[t, \infty)$ to denote $\varphi([t, \infty))$ with respect to a given parameterization $\varphi : \mathbb{R}_+ \to L$. The following result is proved as [27, Proposition 3.5].

Proposition 3.5. Let $X$ be a quasi-graph and let $f : X \to X$ be a continuous map. Suppose that $L$ and $K$ are $k$-order oscillatory quasi-arcs in $X$ for some $k \geq 1$ and $\varphi : \mathbb{R}_+ \to L$ and $\phi : \mathbb{R}_+ \to K$ are parameterizations of $L$ and $K$ respectively. If $L \subset f(K)$, then $f(\omega(K)) = \omega(L)$ and $f(\varphi([s, \infty))) = L[r, \infty)$ for some $r,s \in \mathbb{R}_+$.

Before proceeding, firstly we extend [27, Proposition 3.5] as follows. Two quasi-arcs $L, K$, with parameterizations $\varphi, \phi : \mathbb{R}_+ \to X$ respectively, are called eventually the same if there are $s,t \geq 0$ such that $\varphi([s, \infty)) = \phi([t, \infty))$.

Lemma 3.6. Let $X$ be a quasi-graph, $f : X \to X$ a continuous surjection and $L \subset X$ a k-order (with $k \geq 1$) oscillatory quasi-arc with a parameterization $\varphi : \mathbb{R}_+ \to L$. Then:

1. There is an oscillatory quasi-arc $K$ in $X$ such that $L[a, \infty) \subset f(K)$ for some $a \in \mathbb{R}_+$.
2. If $K$ is an oscillatory quasi-arc in $X$ such that $L[a, \infty) \subset f(K)$ for some $a \in \mathbb{R}_+$, then the order of $K$ is exactly $k$, $f(K)$ and $L$ are eventually the same and $\omega(L) = f(\omega(K))$.
3. $f(L)$ contains a $k$-order oscillatory quasi-arc.

Proof. (1) Since $f$ is surjective, there is a sequence of points $x_n \in X$ such that $f(x_n) = \varphi(n)$. Write $X = G \cup \bigcup_{j=1}^{N} L_j$ as in Theorem 3.1. By Lemma 3.4, $f(G)$ does not contain oscillatory quasi-arcs and so there is $r_1 > 0$ such that $f(G) \cap \varphi([r_1, \infty)) = \emptyset$, in particular, $x_n \notin G$ for all sufficiently large $n$. Similarly, there is $s_1 > 0$ such that $G \cap \varphi([s_1, \infty)) = \emptyset$. But then we must have $x_n \in L_i$ for some $i$ and infinitely many $n$. By Proposition 3.2, there exists $a \geq 0$ such that $L[a, \infty) \subset f(L_i)$. This finishes the proof of (1).

(2) Now assume that $K$ is an oscillatory quasi-arc in $X$ such that $A = L[a, \infty) \subset f(K)$ for some $a \in \mathbb{R}_+$. In the following we shall prove that the order of $K$ is exactly $k$, and then obtain the conclusion by applying Proposition 3.5.

By Lemma 3.4 the order of $K$ cannot be smaller than the order of $A$. Note that the quasi-arcs $A$ and $L$ are eventually the same and then they have the same order, thus the order of $K$ is at least $k$. It suffices to prove that the order of $K$ is at most $k$.

Denote by $m$ the maximal order among oscillatory quasi-arcs of $X$. Observe that each oscillatory quasi-arc in $X$ must be eventually the same to $L_{\ell}$ for some $\ell \in \{1, \ldots, N\}$ by Proposition 3.2, and then we have $m \geq k$. Now let us prove the conclusion by induction.

First we consider the case of $k = m$. Clearly the order of $K$ is at most $k$ by the definition of $m$, as by Proposition 3.2 each oscillatory quasi-arc in $X$ has its order at most $m$.

Next let $n \geq 0$ be such that the result holds for all quasi-arcs in $X$ with its order among $m-n, m-n+1, \ldots, m$ and assume that $k = m-n-1 \geq 1$.

Fix any $j \in \{m-n, \ldots, m\}$ and any oscillatory quasi-arc $Q$ in $X$ with its order $j$. Now it suffices to prove $L \nsubseteq f(Q)$. By (1) and our inductive assumptions, using Proposition 3.5 we can construct in $X$ a finite sequence of oscillatory quasi-arcs $A_{-N-1}, A_{-N}, \ldots, A_{-1}, A_0$.
of order \( j \) such that \( f(A_i) = A_{i+1} \) for all \( i = -N - 1, -N, \ldots, -1 \) and \( A_0 = Q[r_0, \infty) \) for some \( r_0 > 0 \). But by Proposition 3.2 each oscillatory quasi-arc in \( X \) must be eventually the same to \( L_\ell \) for some \( \ell \in \{1, \ldots, N\} \), hence we must have that \( A_p \) and \( A_q \) are eventually the same for some indexes \( p < q \). Applying Proposition 3.5 again we obtain that oscillatory quasi-arcs \( A_{p+1} \) and \( A_{q+1} \), as images of oscillatory quasi-arcs \( A_p \) and \( A_q \) respectively, are eventually the same, and then by induction, there is \( t > 0 \) such that \( A_{-t} \) and \( A_0 \) are eventually the same. Note that \( f(A_{-t}) = A_{-t+1} \) and so there exists \( r_1 \in \mathbb{R}_+ \) such that \( f(Q[r_1, \infty)) \) and \( A_{-t+1} \) are eventually the same. But then \( f(Q[r_1, \infty)) \) is eventually the same to \( L_{\ell_1} \) for some \( \ell_1 \in \{1, \ldots, N\} \), and \( L_{\ell_1} \) has order \( j \). As \( L \) has order \( k \) and \( \{L_{\ell_i}\}_{i=1}^N \) are pairwise disjoint, \( L \) is eventually the same \( L_{\ell_2} \) for some \( \ell_2 \in \{1, \ldots, N\} \setminus \{\ell_1\} \). Then by Proposition 3.5, \( L \not\subset f(Q[r_1, \infty)) \) for any \( r \in \mathbb{R}_+ \). This proves that \( K \) must have order \( k \).

3.2. **Invariant measures and topological entropy for quasi-graph maps.** Now we are ready to show that every invariant measure of a map acting on a quasi-graph is isomorphic to a finite convex combination of invariant measures on graphs. Recall that for a dynamical system \((X, f)\) a point \( x \in X \) is **recurrent** if \( \liminf_{n \to \infty} d(f^n x, x) = 0 \).

**Lemma 3.7.** Let \( X \) be a quasi-graph, \( f: X \to X \) be a continuous map, and \( k > 1 \) be the maximal order among oscillatory quasi-arcs of \( X \). Then there exists a quasi-graph map \((Y, f_1)\) and a graph map \((G, f_2)\) such that

1. the maximal order among oscillatory quasi-arcs of \( Y \) is at most \( k - 1 \), and
2. if the system \((X, f)\) has zero topological entropy then both \((Y, f_1)\) and \((G, f_2)\) have zero topological entropy.

Furthermore, if \( \mu \) is an invariant measure of \((X, f)\), then there exist:

3. invariant measures \( \mu_1, \mu_2 \) on \((X, f)\) and \( \alpha \in [0, 1] \) such that \( \mu = \alpha \mu_1 + (1 - \alpha) \mu_2 \), additionally, if \( \alpha \in (0, 1) \) then \( \mu_1 \) and \( \mu_2 \) are singular;
4. an invariant measure \( \nu_1 \) of \((Y, f_1)\) such that \((X, \mu_1, f)\) and \((Y, \nu_1, f_1)\) are measure-theoretically isomorphic; and
5. an invariant measure \( \nu_2 \) of \((G, f_2)\) such that \((X, \mu_2, f)\) and \((G, \nu_2, f_2)\) are measure-theoretically isomorphic once \( \alpha \in [0, 1) \).

**Proof.** Since we are dealing with invariant measures, by Proposition 3.3 we may assume that \( f \) is surjective (replacing \( X \) by \( \bigcap_{n=0}^\infty f^n(X) \) if necessary). Fix a presentation \( X = G^* \cup \bigcup_{j=1}^n L_j \) provided by Theorem 3.1. As each oscillatory quasi-arc in \( X \) must be eventually the same to \( L_\ell \) for some \( \ell \in \{1, \ldots, n\} \), by Lemma 3.6 we know that the closed set \( Q := \bigcup_{j=1}^n \omega(L_j) \) is \( f \)-invariant, i.e., \( f(Q) = Q \).

**Claim.** There is a quasi-graph \( Y \) obtained by adding some arcs to \( Q \) (and so \( Y \supset Q \)) and a continuous surjection \( f_1: Y \to Y \) such that \( f_1|_Q = f|_Q \).

**Proof of Claim.** By [27, Proposition 2.31], every compact connected set in \( X \) has at most \( n + 1 \) arcwise connected components. Each \( \omega(L_j) \) is connected, then \( Q \) has finitely many arcwise connected components, which we enumerate as \( X_0, X_1, \ldots, X_{m-1} \). Note that the
continuous image of every arcwise connected set is itself arcwise connected, therefore by
the condition \( f(Q) = Q \), there is a permutation \( \tau \) on \( \{0, 1, \ldots, m-1\} \) such that \( f(X_i) = X_{\tau(i)} \) for all \( i = 0, 1, \ldots, m-1 \). Assume first that \( f(X_i) = X_{i+1} \mod m \). Fix a point \( x \in X_0 \).

Then \( f^i(x) \in X_i \) for all \( i = 1, 2, \ldots, m-1 \) and \( f^m(x) \in X_0 \). We construct \( Y \) in the following way: add to \( X_0 \) an exterior point \( z \) and then connect \( z \) with \( f^i(x) \) by an arc \( J_i \) for every \( i = 0, 1, \ldots, m-1 \) in such a way that \( J_i \cap J_j = \{ z \} \) and \( J_i \cap X_i = \{ f^i(x) \} \). It is clear that \( Y \) is arcwise connected and then a quasi-graph. We define a map \( f_1 : Y \to Y \) as follows.

First we define \( f_1 \) as \( f \) on \( Q \). For \( i = 0, 1, \ldots, m-2 \), \( g \) maps the arc \( J_i \) homeomorphically onto the arc \( J_{i+1} \) with \( f_1(z) = z \) and \( f_1(f^i(x)) = f^{i+1}(x) \). If \( f^m(x) = x \), \( g \) maps the arc \( J_{m-1} \) homeomorphically onto the arc \( J_0 \) with \( f_1(z) = z \) and \( g(f^{m-1}(x)) = x \). If \( f^m(x) \neq x \), \( X_0 \) is arcwise connected, we pick an arc \( K \subset X_0 \) with endpoints \( x \) and \( f^m(x) \). Then we define \( f \) to map the arc \( J_{m-1} \) homeomorphically onto the arc \( J_0 \cup K \) in such a way that \( f_1(z) = z \) and \( f_1(f^{m-1}(x)) = f^m(x) \). Then \( f_1 \) is well-defined and continuous. Since we can arrange the above homeomorphisms arbitrarily, we can require that \( f_1 \) is strictly monotone on each \( J_i \), that is, if \( x_s, f^m(x_s) \in Y \setminus Q \) and \( x_s \neq z \) then \( x_s \neq f^m(x_s) \) and the shortest arc \([z, x_s]\) connecting \( z, x_s \) is contained in the arc \([z, f^m(x_s)]\). In particular, \( z \) is the only recurrent point in \( Y \setminus Q \).

If \( \tau \) has more than one cycle, we construct an appropriate quasi-graph for each cycle independently, and then combine them into one quasi-graph by identifying the fixed point \( z \) in all of these independent quasi-graphs.

Next we define a relation \( \sim \) on \( X \) by putting \( a \sim b \) provided that \( a = b \) or \( a \in \omega(L_i) \) and \( b \in \omega(L_j) \) for some indexes \( i, j \). The relation \( \sim \) is clearly a closed equivalence relation. Let \( G = X/\sim \) and \( \pi : X \to G \) be the associated quotient mapping. As the relation \( \sim \) is invariant under \( f \times f \), we can naturally define a continuous map \( f_2 = f/\sim \) on \( G = X/\sim \) such that \( \pi \circ f = f_2 \circ \pi \). Since we collapsed \( \omega \)-limit sets of all oscillating quasi-arcs of \( X \) to a single point, it is not hard to see that \( G \) is a graph.

In the proof of Claim, we add only some arcs to \( Q \) and then obtain the quasi-graph \( Y \). So the maximal order of quasi-arcs in \( Y \) is at most \( k - 1 \). Moreover, it is easy to see from the above construction that if \( (X, f) \) has zero topological entropy then both \( (Y, f_1) \) and \( (G, f_2) \) have zero topological entropy. Furthermore, any invariant measure supported on \( Q \) can be regarded as an \( f_1 \)-invariant measure on \( Y \) (by observing \( f^i|Q = f_1|Q \)) and any invariant measure supported on \( X \setminus Q \) can be regarded as an \( f_2 \)-invariant measure on \( G \) via the factor map \( \pi \) (by observing that \( \pi|X\setminus Q \) is an one-to-one map).

Now assume that \( \mu \) is an invariant measure of \( (X, f) \), and we put \( \alpha = \mu(Q) \). If \( \alpha = 1 \), then set \( \mu_1 = \mu = \mu_2 \). If \( 0 < \alpha < 1 \), then set \( \mu_1 = \frac{1}{\alpha} \mu|Q \) and \( \mu_2 = (1 - \frac{1}{\alpha}) \mu|X \setminus Q \). If \( \alpha = 0 \), then let \( \mu_1 \) be any invariant measure supported on \( Q \) and set \( \mu_2 = \mu \). Then the invariant measures \( \nu_1 \) and \( \nu_2 \) are defined naturally. It is easy to show that these invariant measures \( \mu_1, \mu_2, \nu_1, \nu_2 \) satisfy the required properties.

If we inductively apply Lemma 3.7 to the maximal order of oscillatory quasi-arcs of a quasi-graph map, then we have the following main result of this section.

**Theorem 3.8.** Let \( X \) be a quasi-graph and let \( f : X \to X \) be a continuous map. Then there exist graph maps \( (G_1, f_1), \ldots, (G_k, f_k) \) for some \( k \in \mathbb{N} \) such that

1. each invariant measure on \( (X, f) \) is measure-theoretically isomorphic to a finite convex combination of invariant measures on these graph maps,
(2) each ergodic invariant measure on \((X, f)\) is measure-theoretically isomorphic to an ergodic invariant measure on \((G_i, f_i)\) for some \(i = 1, \ldots, k\), and
(3) if the system \((X, f)\) has zero topological entropy then all \((G_1, f_1), \ldots, (G_k, f_k)\) have zero topological entropy.

Similar to Theorem 2.3, a finite convex combination of invariant measures which have discrete spectrum also has discrete spectrum.

**Proof of Theorem 1.2.** It is enough to combine Theorems 2.11 and 3.8. □

Let \(X\) be a quasi-graph and let \(s \geq 2\). An \(s\)-horseshoe for \(f: X \to X\) is a closed arc \(I \subset X\) and closed subarcs \(J_1, \ldots, J_s \subset I\) with pairwise disjoint interiors, such that \(f(J_j) = I\) for all \(j = 1, \ldots, s\). We shall denote this horseshoe by \((I; J_1, \ldots, J_s)\). An \(s\)-horseshoe is strong if in addition the intervals \(J_j\) are contained in the interior of \(I\) and are pairwise disjoint.

Liibre and Misuurewicz proved the following result relating positive topological entropy and the existence of horseshoes on graph maps (cf. [25, Theorem B]).

**Theorem 3.9.** Let \(G\) be a graph and let \(f: G \to G\) be a continuous map. Assume \(h_{\text{top}}(f) > 0\), where \(h_{\text{top}}(f)\) denotes the topological entropy of \((G, f)\). Then there exist strictly increasing sequences \(s_n, k_n\) of positive integers such that each \(f^{k_n}\) has an \(s_n\)-horseshoe and \(\lim_{n \to \infty} \frac{1}{k_n} \log(s_n) = h_{\text{top}}(f)\).

We show that this result also holds for quasi-graph maps.

**Theorem 3.10.** Let \(X\) be a quasi-graph and let \(f: X \to X\) be a continuous map. Assume \(h_{\text{top}}(f) > 0\). Then there exist strictly increasing sequences \(s_n, k_n\) of positive integers such that each \(f^{k_n}\) has an \(s_n\)-horseshoe and \(\lim_{n \to \infty} \frac{1}{k_n} \log(s_n) = h_{\text{top}}(f)\).

**Proof.** It suffices to construct strictly increasing sequences \(s_n, k_n\) of positive integers such that each \(f^{k_n}\) has an \(s_n\)-horseshoe and \(\liminf_{n \to \infty} \frac{1}{k_n} \log(s_n) \geq h_{\text{top}}(f)\). We shall prove the conclusion by performing induction on the maximal order of quasi-arcs in \(X\).

If the maximal order of quasi-arcs in \(X\) is zero, i.e., \(X\) is a graph, then the result is just Theorem 3.9. Now assume that the result holds for all quasi-graphs with quasi-arcs of order at most \(k\). Let \(X\) be a quasi-graph with \(k + 1\) the maximal order of quasi-arcs in \(X\). Fix a presentation \(X = G^* \cup \bigcup_{j=1}^n L_j\) provided by Theorem 3.1. Fix a continuous map \(f: X \to X\) satisfying \(h_{\text{top}}(f) > 0\). Since topological entropy of \(f\) over \(X\) and that restricted to the closure of all recurrent points (so-called Birkhoff center) are the same (e.g., see [2]) and by Proposition 3.3 the set \(\cap_{n=0}^{\infty} f^n(X)\) is also a quasi-graph, without loss of generality we may assume that \(X = \bigcap_{n=0}^{\infty} f^n(X)\), in particular, \(f\) is surjective.

Fix any \(h_{\text{top}}(f) > \varepsilon > 0\). By the variational principle concerning entropy, there exists an ergodic invariant measure \(\mu\) on \((X, f)\) such that \(h_{\text{top}}(f) - \varepsilon < h_{\mu}(f)\), where \(h_{\mu}(f)\) denotes the measure-theoretic \(\mu\)-entropy of \(f\). Set \(Q = \bigcup_{j=1}^n \omega(L_j)\). Then there are two possibilities, either \(\mu(Q) = 0\) or \(\mu(Q) = 1\).

Let us consider firstly the case of \(\mu(Q) = 0\). Following the proof of Lemma 3.7, we collapse the \(\omega\)-limit set \(Q\) to a single point \(p\), and get a factor map \(\pi: (X, f) \to (G, g)\) by setting \(\pi(Q) = \{p\}\). Let \(\nu = \pi(\mu)\), which in fact is measure-theoretically isomorphic to \(\mu\). Then \(h_{\text{top}}(g) \geq h_{\nu}(g) = h_{\mu}(f)\). As \(G\) is a graph, by Theorem 3.9, there exist large positive integers \(s\) and \(k\) such that \(g^k\) has an \(s\)-horseshoe \((I; J_1, \ldots, J_s)\),

\[
\frac{\log s}{k} > h_{\text{top}}(g) - \varepsilon > h_{\text{top}}(f) - 2\varepsilon \quad \text{and} \quad \frac{\log 2}{k} < \varepsilon.
\]
The point $p$ is contained in at most two intervals $J_i$. Removing those intervals and shortening $I$ if necessary, we get an $r$-horseshoe $(I';J_1',...,J_r')$ for $g^k$ such that $p \not\in I'$ and $r \geq \frac{s-2}{k}$. Since the restricted factor map $\pi: X \setminus Q \to G \setminus \{p\}$ is invertible, we may view $(I';J_1',...,J_r')$ as an $r$-horseshoe for $f^k$. Furthermore,
\[
\frac{\log r}{k} \geq \frac{\log(s-2)}{k} - \frac{2}{k} > \frac{\log s}{k} - \epsilon > h_{\text{top}}(g) - 2\epsilon > h_{\text{top}}(f) - 3\epsilon.
\]

Now we consider the second case of $\mu(Q) = 1$. Following the Claim in the proof of Lemma 3.7, we get a quasi-graph $Y$ and a continuous map $g: Y \to Y$ extending $f$, where $Y$ was obtained by adding some arcs to $Q$. It is clear that no point from $Q$ can map to interiors of these new arcs. By the definition of the map $g$, we know that $Y \setminus Q$ contains a fixed point $z$, which is the unique recurrent point of $g$ in $Y \setminus Q$. Note that $f|_Q = g|_Q$ and since topological entropy of a system is focused on the closure of its all recurrent points (cf. [2]), we have $h_{\text{top}}(g) = h_{\text{top}}(Q) \geq h_{\mu}(f)$, where $h_{\text{top}}(Q,f)$ denotes the topological entropy of $f$ restricted to $Q$. As the maximal order of quasi-arcs of $Y$ is at most $k$, applying our inductive assumption we can choose integers $s$ and $m$ large enough such that $g^m$ has an $s$-horseshoe and $\frac{\log s}{m} > h_{\text{top}}(g) - \epsilon \geq h_{\mu}(f) - \epsilon > h_{\text{top}}(f) - 2\epsilon$. We shall finish the proof by showing $I \subset Q$ once $(I;J_1,...,J_s)$ is an $s$-horseshoe for $g^m$, as in this case $(I;J_1,...,J_s)$ is also an $s$-horseshoe for $f^m$ (here recall again that $f|_Q = g|_Q$). We show firstly that the arc $J_1$ is contained in $Q$. Assume the contrary that $J_1 \setminus Q \neq \emptyset$. Then $z \in J_1$, as otherwise by the strict monotonicity of the map $g$ we have $J_1 \setminus g^m(J_1) \neq \emptyset$ once $k$ is large enough, which is impossible by the definition of a horseshoe. Furthermore, by the construction of the quasi-graph $Y$ and the definition of the map $g$, it is not hard to show that $z$ will not be an ending point of the arc $J_1$, that is, $z$ belongs to the interior of $J_1$. But then $z \not\in J_2$ and so $z \not\in g^m(J_2)$ (by the definition of the map $g$), which is impossible since $J_1 \subset g^m(J_2)$ by the definition of a horseshoe. Therefore the only possibility is that $J_1 \subset Q$. But then $g^m(J_1) \subset Q$ (by the invariance of $Q$), and so $I \subset Q$, which finishes the proof. 

**Remark 3.11.** In [25], Llibre and Misiolewikwicz also proved that the topological entropy, as a function of a continuous map of a given graph is lower-semicontinuous (cf. [25, Theorem C]). In fact, as shown by the proof of Theorem 3.10 the same result also holds for quasi-graph maps, because if a quasi-graph map $f$ has a strong $s$-horseshoe with a closed arc $I$ then by the definition of the supremum metric, all sufficiently small perturbations of $f$ also have an $s$-horseshoe (see [25] for a detailed proof).

4. MöBIUS DISJOINTNESS AND MAPS ON DENDRITES

In this section we will prove Theorem 1.4 and Corollary 1.3. We also obtain some related results. Recall that the natural extension of a surjective dynamical system $(X,T)$ is the invertible dynamical system $(\lim(X,T),\hat{T})$ given by
\[
\lim(X,T) = \left\{(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X : T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\right\},
\]
\[
\hat{T} : (x_1,x_2,...,x_n,...) \mapsto (T(x_1),T(x_2),...,T(x_n),...).
\]
Let $\pi_i : \lim(X,T) \to X$ be the $i$-th coordinate projection for each $i \in \mathbb{N}$. By the well-known Stone-Weierstrass Theorem, $\{f \circ \pi_i : f \in C(X,\mathbb{C}), i \in \mathbb{N}\}$ is dense in $C(\lim(X,T),\mathbb{C})$. So we have the following easy fact.
Lemma 4.1. Let \((X, T)\) be a surjective dynamical system. Then the Möbius function is linearly disjoint from the system \((X, T)\) if and only if it is so for its natural extension.

To show that the Möbius Disjointness Conjecture, it only need to consider its all orbit closures. If the map restricted on a orbit closure is not surjective, then it can be embedded into a surjective dynamical system which does not increase the entropy or the sequence entropy. For a surjective dynamical system, we can consider its natural extension which is an invertible system. So we obtain the following basic observation.

Lemma 4.2. The Möbius function is linearly disjoint from all dynamical systems with zero topological entropy if and only if it is so for all invertible dynamical systems with zero topological entropy.

It was proved in [17] that the Möbius function is linear disjoint from a dynamical system if it admits only countably many ergodic invariant measures such that each of them has discrete spectrum, and then in [18] that the Möbius function is linear disjoint from an invertible dynamical system if each of its invariant measure has discrete spectrum. By the arguments above Lemma 4.2, we have the following result.

Theorem 4.3. Let \((X, T)\) be a dynamical system with zero topological entropy. If every invariant measure of \((X, T)\) has discrete spectrum, then the Möbius function is linearly disjoint from the system \((X, T)\).

Now Corollary 1.3 is a direct consequence of Corollary 1.2 and Theorem 4.3.

Recall that a dendrite is any locally connected continuum containing no simple closed curve, and the Gehman dendrite is the topologically unique dendrite whose set of all endpoints is homeomorphic to the Cantor set and whose branching points are all of order three (e.g., see [28]).

Proof of Theorem 1.4. Assume that the Möbius function is linear disjoint from all surjective dynamical systems over the Gehman dendrite with zero topological entropy, and let \((X, T)\) be a dynamical system with zero topological entropy. In the following we will show that the Möbius function is also linear disjoint from the system \((X, T)\).

It makes no any difference to assume that the system \((X, T)\) is an orbit closure. It is standard to take a surjective system \((Y, S)\) with zero topological entropy such that the system \((X, T)\) is embedded into \((Y, S)\). Let \((Z, R)\) be the natural extension of \((Y, S)\). Now it suffices to prove that the Möbius function is linear disjoint from the system \((Z, R)\).

As \((Z, R)\) is an invertible dynamical system with zero topological entropy, it admits an extension \((U, g)\) which is an invertible subshift with zero topological entropy (e.g., see [10, Theorem 6.9.9]). It makes no any difference to assume that \(U\) admits no isolated points (else, if necessary, we consider its product with any invertible and infinite minimal subshift with zero topological entropy, e.g., a Sturmian subshift), and that \(U\) is an invertible subshift over \(A\) consisting of at most \(2^n\) different symbols for some \(n \in \mathbb{N}\). We consider the one-sided subshift \((V, h)\) generated by \((U, g)\), that is,

\[ V = \{(w_0, w_1, w_2, \ldots) : (\ldots, w_{-1}, w_0, w_1, w_2, \ldots) \in U\}. \]

It is easy to check that \(V\) does not contain isolated points.

By standard technique, on the Gehman dendrite \(D_*\) we can construct a surjective map \(f_*\), such that there is a point \(c \in D_*\) (fixed by \(f_*\)) and an \(f_*\)-invariant set \(\Sigma \subset D_* \setminus \{c\}\) such
that \((\Sigma, f)\) is conjugated to the one-sided full shift on 2 symbols and for every \(x \in D_s \setminus \Sigma\) we have \((f_\ast)^m(x) = c\) for some \(m \in \mathbb{N}\) (e.g., see [20, Example 6]). Now over the Gehman dendrite \(D_s\), we consider the map \(f = (f_\ast)^n\). in \(D_s \setminus \{c\}\) there exists an \(f\)-invariant set \(\Sigma\) such that \((\Sigma, f)\) is conjugated to the one-sided full shift on \(2^n\) symbols (and hence we may view \((V, h)\) as a subsystem of \((\Sigma, f)\)), furthermore, \(f(c) = c\) and for every \(x \in D_s \setminus \Sigma\) we have \(f^m(x) = c\) for some \(m \in \mathbb{N}\).

As \(D_s\) is a dendrite, for any two different points \(x_1, x_2 \in D_s\) there exists a unique arc connecting them (denote it by \([x_1, x_2]\)). By above discussions we may view \(V \subset \Sigma\), and then we consider an arcwise connected \(f\)-invariant set \(D = \bigcup_{y \in V} [y, c]\), where \(f\) acts naturally over \(D\). As \(V\) does not contain \(f\)-isolated points, \(D\) is again the Gehman dendrite (see [3, Theorem 4.1]). Then \((D, f)\) contains \((V, h)\) as a subsystem, and is a surjective dynamical system over a Gehman dendrite, where \(f\) maps \([y, c]\) exactly to \([y, c]\) for every \(y \in V\). Furthermore, by the above construction we know that for every \(x \in D \setminus V\) we have \(f^m(x) = c\) for some \(m \in \mathbb{N}\), and then besides of ergodic invariant measures for \((V, h)\), the system \((D, f)\) admits only one more ergodic invariant measure whose support is the fixed point \(c\), hence topological entropy of \((D, f)\) is the same as that of \((V, h)\) which is zero.

By our assumptions, the Möbius function is linearly disjoint from the dynamical system \((D, f)\), hence the same is true for its subsystem \((V, h)\). Observe that the two-sided subshift \((U, g)\) and the natural extension of the system \((V, h)\) are conjugate. Then by Lemma 4.1, the Möbius function is linearly disjoint from the system \((U, g)\) and hence from its factor \((Z, R)\). This finishes the proof. 

Recently in [1] the authors showed that if the set \(\text{End}(X)\) of endpoints of some dendrite \(X\) is closed with finitely many accumulation points then the Möbius function is linearly disjoint from any dynamical system on \(X\) with zero topological entropy. The key point is the following fact which is one of the main results of [4].

**Lemma 4.4.** Let \(X\) be a dendrite such that \(\text{End}(X)\) is a closed set having finitely many accumulation points and let \(f : X \to X\) be a continuous map with zero topological entropy. If \(L = \omega_f(x)\) is an uncountable \(\omega\)-limit set for some \(x \in X\), then for every \(k \geq 1\) there is an \(f\)-periodic subdendrite \(D_k\) of \(X\) and an integer \(n_k \geq 2\) with the following properties:

1. \(D_k\) has period \(\alpha_k := n_1 n_2 \ldots n_k\) for every \(k \geq 1\),
2. \(\bigcup_{i=0}^{n_j-1} f^{i} D_{j-i} \cap D_{j-1}\) for every \(j \geq 2\),
3. \(L \subset \bigcup_{i=0}^{\alpha_k-1} f^i(D_k)\) for every \(k \geq 1\),
4. for every \(k \geq 1\) and each \(0 \leq i \leq \alpha_k - 1\), \(f(L \cap f^i(D_k)) = L \cap f^{i+1}(D_k)\) and in particular \(L \cap f^i(D_k) \neq \emptyset\), and
5. for every \(k \geq 1\) and all \(0 \leq i \neq j \leq \alpha_k\), \(f^{i}(D_k) \cap f^{j}(D_k)\) has an empty interior.

Recall that a dynamical system \((X, f)\) is mean equicontinuous if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for every \(x, y \in X\),

\[
d(x, y) < \delta \text{ implies } \limsup_{n \to \infty} \sum_{i=0}^{n-1} d(f^i(x), f^i(y)) < \varepsilon.
\]

It is shown in [24] (see also [11]) that if a system \((X, f)\) is mean equicontinuous then every orbit closure is uniquely ergodic and its unique invariant measure has discrete spectrum.
Proposition 4.5. Let $X$ be a dendrite such that $\text{End}(X)$ is a closed set having finitely many accumulation points and let $f : X \to X$ be a continuous map with zero topological entropy. Fix any point $x \in X$.

\begin{enumerate}
\item If $\omega_f(x)$ is at most countable, then every invariant measure on $(\text{Orb}_f(x), f)$ has discrete spectrum.
\item If $\omega_f(x)$ is uncountable, then $(\text{Orb}_f(x), f)$ is mean equicontinuous. In particular, $(\text{Orb}_f(x), f)$ is uniquely ergodic and its unique invariant measure has discrete spectrum.
\end{enumerate}

Proof. (1) If $\omega_f(x)$ is at most countable, then $\text{Orb}_f(x)$ is also at most countable. So every ergodic invariant measure on $(\text{Orb}_f(x), f)$ is an equidistributed measure on a periodic orbit. By Theorem 2.3, every invariant measure on $(\text{Orb}_f(x), f)$ has discrete spectrum.

(2) If $\omega_f(x)$ is uncountable, then by Lemma 4.4 the structure of $\omega_f(x)$ is similar to a solenoid of a graph map. By [28, Theorem 10.27], the dendrite $X$ can approximated by subtrees. Then for every $\varepsilon > 0$ and $\delta > 0$ there exists $N$ such that for any integer $n \geq N$ and any pairwise disjoint connected closed subsets $A_1, A_2, \ldots, A_n$ of $X$ there are at most $\delta n$ sets $A_i$ with diameter larger than $\varepsilon$ (simply, maximal number of such sets is bounded by some constant). Now the result follows from the proof of [23, Lemma 3.3]. \qed

Consequently, we provide another proof of [1, Theorem 4.10] by applying directly Theorem 4.3 and Proposition 4.5.

Corollary 4.6. Let $X$ be a dendrite such that $\text{End}(X)$ is a closed set having finitely many accumulation points and let $f : X \to X$ be a continuous map with zero topological entropy. Then the Möbius function is linearly disjoint from the system $(X, f)$.

Remark 4.7. In Proposition 4.5, it is interesting to know that whether every orbit closure is uniquely ergodic, in other words, if $\omega_f(x)$ is at most countable, does there exist only one periodic orbit in $\omega_f(x)$?

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