Binary and Polytomous Responses Modeling: Multiple-Campaign Ad-Targeting Without Personal User Information

Paolo D’Alberto *

January 12, 2015

Abstract

We present a vertical introduction to campaign optimization; that is, the ability to predict the user response to an ad campaign without any users’ profiles on average and for each exposed ad. In practice, we present an approach to build a polytomous model, multi response, composed by several hundred binary models using generalized linear models. The theory has been introduced twenty years ago and it has been applied in different fields since then. Here, we show how we optimize hundreds campaigns and how this large number of campaigns may overcome a few characteristic caveats of single campaign optimization. We discuss the problem and solution of training and calibration at scale. We present statistical performance as coverage, precision and recall used in classification. We present also a discussion about the potential performance as throughput: how many decisions can be done per second streaming the bid auctions also by using dedicated hardware.

1 Introduction

Free contents and targeted advertising are Janus-faced as beginning and end of a cycle. We start by watching original contents and exposing our likes to the contents providers. In turn, our preferences are used to provide audience targeting for advertisers, which at the end pay for the contents.

The advertising should be tailored to the audience in such a way that the ads are relevant, engaging, and with high precision in order to minimize the costs and to maximize the return to the original investment.

Collecting personal history is a common approach used to understand the audience and to find ways to reach them. Every time we log into our mail account(s), social network(s), search(es) and for anything we do afterwords, these events help greatly to picture who we are and what we do. These profiles attract a large spectrum of interests as the like of advertisers, recruiters, prospects, or soliciting email.

Considering that a social network can reach billions of people, collecting and classifying these volume of data is not an easy chore. Nonetheless, the classification changes little in time. For example, we like Italian Opera and we will, likely, for life; other likes may be more flaky.

Now let us put the previous problem in the context of avail ads. For example, there are ad-exchanges where an ad-company can go, bid, and buy impressions (ad-space): by counting the largest exchanges, we could bid up to 10 Billions impressions every day in US alone, reaching 150 Million people, and serving thousands of campaigns. If we use profiles, we need to have information only about the 150 Millions. If we do not have such information, we may have to make a new a different decision for each impression: the problem space becomes larger and thus the problem harder.

In practice, not everyone wants to collect personal information and not every one can or will be able to. One day, cookies associated to browsers may be gone and thus the main mean to follow us is to log in or tracking the location of our devices. In this work, we do not collect personal information: we care about the collective and anonymous response. We shall clarify what we collect in Section 5.

How can we tailor the response of several campaigns? Campaigns are now being designed for an interaction with the audience: Clicking to redeem a coupon, responding to a trivia question, playing a game and others. These interactions set apart these impressions: campaigns and advertisers want to respond to the feedback in order to focus more to who really show interests.

Click Through Rate (CTR) is a measure of performance for a campaign. By construction, CTR is a ratio defined as the number of clicks over the total number of impressions delivered. Assume we reach all members of our audience and the people really interested to the campaign clicked to their satisfaction. If we have to redo the same campaign, we would be parsimonious and reduce the number of impressions maintaining the same interests, same number of clicks, and thus larger CTR.

Problem. For example, given an impression $x$, with its features from a pool of $n$ impressions, we would like to compute the probability $P_{C_i}(x)$ of a click for the campaign $C_i$, where we have $1 \leq i \leq N$. In practice, we model this problem by estimating each binomial distribution:

$$P_{C_i}[X = k] = \binom{n}{k} p_{C_i}^k (1 - p_{C_i})^{n-k}, \quad k \in [0, n]$$

We are after the parameter $p_{C_i}$ and its connection with $x$; that is, $p_{C_i} = g(x)$.
We organize the paper as follows. In Section 2, we introduce the basic applied statistic method used to estimate the connection between any impression \( x \) and its probability as in Equation 1.1. In practice, we dwell with a problem that has a small number of positive cases and a very large number of negative ones. In Section 3, we present our approach to select retrospective sampling, ex post, and in particular how to select the negative cases (no real ones) and how to scale to provide prospective estimate, ex ante. We explain also our approach to calibrate the system and choose thresholds to mimic a binary response. In Section 4, we present how we explore the features space. In Section 5, we put everything together and we present measure of quality such a precision. Eventually, the computation of \( P_{C}(x) \) has to be efficient and performed at run time; in Section 6, we provide throughout using state of the art systems. We wrap up in Section 8 and we tip our hats to who helped us in Section 9.

2 Generalized Linear Models, GLM

Suppose that we want to have a response \( Y_{i} \) that can take only two possible values: \( Y_{i} = 1 \) and \( Y_{i} = 0 \). For example, the former represents the response of a click or success and the latter the response to a non click or failure. We write

\[
P(Y_{i} = 0) = 1 - \pi_{i}, \quad P(Y_{i} = 1) = \pi_{i}
\]

We describe this as \( Y_{i} \sim B(1, \pi_{i}) \). This is a single trial. If we have \( m \) independent trials, and there is a common probability \( \pi \) of success and probability \( 1 - \pi \) of failure, we say that \( Y \sim B(m, \pi) \) and \( P(Y = k) = \binom{m}{k} \pi^{k}(1 - \pi)^{m - k} \), for any integer \( 1 < k < m \). The probability distribution is counting the number of \( k \) possible successes over a pool of \( m \) trials. The Binomial distribution is one of the oldest to be studied and it was derived by Jacob Bernoulli [1]. Our notations and references are from [2]. Chapter 2.

In general, we cannot assure homogeneity and we should consider a process such as \( Y = \sum_{i=1}^{m} Y_{i} \) where \( Y_{i} \sim B(1, \pi_{i}) \), that is the summation of non homogeneous Binomial. Also, the values of \( \pi_{1}, \ldots, \pi_{m} \) are often unknown and we will eventually compute an average \( Y_{i} \sim B(1, \pi) \) and \( Y \sim B(m, \pi) \) where \( \pi = \frac{1}{m} \sum_{i=1}^{m} \pi_{i} \).

It should be clear now that \( Y \) is a response to an event. The event is represented by a vector of explanatory variables \( x = (x_{1}, \ldots, x_{p}) \). The principal objective of a statistical analysis is to investigate the relationship between the response probability \( \pi \) and the explanatory \( x \), that is to find \( g(\pi) \sim x \).

What follows, especially the notations and the meaning behind the notation is from [3] Chapter 2 and 4. Linear models play an important role in applied and theoretical work. We suppose there is a linear dependence

\[
g(\pi_{i}) = \eta_{i} = \sum_{j=1}^{p} x_{i,j} \beta_{j}
\]

for to be computed \( \beta_{1}, \ldots, \beta_{p} \). The function \( g() \) is a transformation that makes possible to map the range \([0, 1]\), which is the range of probability, to a more appropriate space \((−\infty, +\infty)\), which is appropriate for a linear function.

In this work we use the logistic function

\[
\log_{e}\left(\frac{\pi_{i}}{1 - \pi_{i}}\right) = \beta_{0} + \sum_{j=1}^{p} x_{i,j} \beta_{j}
\]

where the fraction \( \frac{\pi_{i}}{1 - \pi_{i}} \) has range in the interval \([0, +\infty)\) and it also know as odd ratio (e.g., used for hypothesis testing and sequential analysis [6]). In Equation 2.4 we single out the parameter \( \beta_{0} \), which is the intercept. As a note, the equation 2.4 is for every trial and our ability of estimating \( \pi_{i} \) however the unknown \( \beta \)s are determined for all.

2.1 Computing \( \beta \); Maximum likelihood. The responses \( y_{1}, \ldots, y_{n} = y \) are observed from the independent binomial variables \( Y_{1}, \ldots, Y_{n} = Y \) such that \( Y_{i} \sim B(m_{i}, \pi_{i}) \) and we could use the expression \( Y \sim B(m, \pi) \).

The log likelihood may be written as:

\[
l(\pi, y) = \sum_{i=1}^{n} \left[ y_{i} \log\left(\frac{\pi_{i}}{1 - \pi_{i}}\right) + m_{i} \log(1 - \pi_{i}) \right]
\]

where we omit the term \( \sum \log\left(\binom{m_{i}}{y_{i}}\right) \), which is a constant independent of \( \pi \). If we substitute the linear logistic model in Equation 2.4 we have

\[
l(\beta, y) = \sum_{i} \sum_{j} y_{i} x_{i,j} \beta_{j} - \sum_{i} m_{i} \log(1 + e^{\sum_{j} x_{i,j} \beta_{j}}),
\]

or in matrix form

\[
l(\beta, y) = y^{T}X\beta - \sum_{i} m_{i} \log(1 + e^{x_{i}^{T}\beta})
\]

We identify with \( ^{\hat{}} \beta \) the value of \( \beta \) that maximizes Equation 2.6. The appealing of this formulation is that log likelihood depends on \( y \) only through the linear combinations \( y^{T}X \) and \( y^{T}X = E[Y^{T}X; ^{\hat{}} \beta] \). The details of the computation of \( ^{\hat{}} \beta \) are available [3] Section 4.4.2. It is an iterative solver where at every iteration involves a matrix factorization. The matrix factorization may be prohibitive as the matrix \( X \) will become larger.

The iterative approach uses weights to give more importance to specific dimensions. The weight may adapt at each iterations and it may require a computation of a different factorization \( QR \). There are suggestions where the factorization can be done once and being reused (circumventing one of the most expensive step) because the weights will change the spectrum of \( X \sim \sqrt{w^{T}X} \) but will not change the spectrum of \( Q \).
2.2 Ex Post versus Ex Ante: different Intercept. In Section 4.3.3, there is a full explanation but this property should strike a chord to anyone using models for predictions. In general, we use the past responses (i.e., ex post) to compute  \( \tilde{\beta} \). We consider the past clicks and non-clicks for the impressions we have delivered. If we are in the process of completion we have not seen all clicks and, most importantly, we have not seen all impressions yet. As matter of fact, the pool of available impressions is even larger than the one we are going to deliver (i.e., ex ante).

The ex post building model approach is practical. The ex ante is not. If we could build both they will differ only by their intercept \( \beta_0 \). The difference can be computed if we have at least an estimate of the difference in scale of the ex-post (training) and ex-ante (total) sets.

This mathematical adjustment is simple to explain and to use. But it has an even more important ramifications: if we have multiple models for different campaigns, then we can argue we can compare their probability estimates of success and choose accordingly based on they ex ante, which is common, instead of their ex post, which is small and specific. It will become really a prediction and not just a classification. We will come back to this subject in Section 6.

3 Positive and Negative Set
Events like clicks or any call to action are rare. If you consider the training set as the composition of positive events and negative events, the choice of positives is clearly defined. The choice and the quantity of negatives is quite a different problem: clicks to other campaigns, non-clicked impressions but delivered to the same campaign, to other campaigns, and impressions that are not even selected for bidding. The list is long and the number of impression in it very large. To give a quantitative measure, we may have 1 Million click, 3.5 Billion delivered impressions, and 2 Trillion available impressions every year. Common practice would be to choose 1 Million negatives, but where and what are tricky questions: the choice of the negative affects any model.

At first, given a campaign and its clicks, say one thousand, we thought we could choose another one thousand from the delivered impressions. Then we could build a model \( \tilde{\beta} \). Having the model, we could score all delivered impressions and create a distribution. Then we could bin the distribution in 100 bins and sample 10 impression per bin in order to find a second negative set. Then compute \( \beta \).

There is one practical problem: we need to score all impressions and this is to retrieve a suitable sample.

From a practical point of view, we used clicked impressions only: Given an active campaign from time \( t_A \) to \( t_B \) with \( t_A < t_B \), we collected all its clicks, these are the positives. The clicks to the other campaigns in the same interval of time are the negatives. In practice, we split the click space so that each model and campaign will bring forth its unique features and all may cover the available clicks. We shall show the models so created will actually cover this space.

This is the training set upon we are going to build the model \( \beta \). Training and calibrations shall be described in the following section. Once the model is built, we will have a better understanding what features are important and how they affect the model. Also, we may have a clear understanding what features determine a clear rejection: Thus we can estimate a realistic size of available impressions (ex ante) and thus scale the model accordingly.

4 Binomial Training and Calibration
For every campaign, we associate a model. A campaign has positives in the interval of time \( t_A \) to \( t_C \) with \( t_A < t_C \). We take all other clicks in the same period of time as negative. We are bound to create a training set and a calibration set. We have two options: either we split the set by time or by size.

If the campaign has been running for some time and often there are campaign running for years, we could choose an instant of time \( t_A < t_B < t_C \) so that \( \frac{t_B-t_A}{t_C-t_A} = \frac{2}{3} \). The division ratio of \( \frac{2}{3} \) is arbitrary. The training \( T \) is based on the interval \([t_A,t_B]\) and the calibration \( C \) is based on the interval \([t_B,t_C]\).

However, often campaigns are short spanning a few weeks. The calibration period would be of only few days of a week and covering the end of delivery (e.g., less active because we have already reached our audience or more active because we reach critical mass of delivery). We could consider the events during the interval \( t_A \) and \( t_C \) as a set and choose \( T \) and \( C \) randomly so that \( \frac{|T|}{|C|} = 3 \). This latter choice is our default. The training should have enough information and the calibration should provide an independent validation.

The former, the distinction between \( T \) and \( C \) by time, would show the model predictive capability with the assumption that \( T \) is representative. The latter would show the classification prowess and calibration is an independent validation.

4.1 Receiver Operating Characteristic, ROC. Assume we can build any model using the training set. We measure its quality using the Calibration set by its ROC. This is a graphical and quantitative measure. For each element in \( \mathbf{x} \in C \), compute its probability of success using the model computed using Equation 2.7 \( P[Y_i = 1] \) where \( Y_i \sim x \). We know the estimated probability by the model and we know whether or not it was a true click. Considering \( C \), we can compute all probabilities and sort them from the largest to the smallest: \( P = \{ P[Y_i = 1]\}_{i \in C} \).

Then, we compute for each \( p \in P \) the number of true-click events \( Y_i \) such that \( P[Y_i = 1] \geq p \) over the total number true clicks: True Positive Rate (TP). Also, we
compute the number of true-no-click events $Y_i$ such that $P[Y_i = 1] \geq p$ over the total number of no-clicks: False Positive Rate (FP).

For each probability $p$ above, we have two rates $TP(p)$ and $FP(p)$ where $0 \leq FP(p), TP(p) \leq 1$. These represent a curve (i.e., the ROC curve) where the abscissa is $FP$ and the ordinate is $TP$. In practice, this curve is embedded into a square with unitary side and left-bottom vertex on the coordinate $(0, 0)$ and right-top vertex on the coordinate $(1, 1)$.

If we draw a straight line from $(0, 0)$ to $(1, 1)$, this is a ROC curve with a specific meaning: If a model has such a curve, it means that for every $p$ we have $TP(p) = FP(p)$ and thus we have a constant probability $1/2$ to guess an event right. This looks like a fair coin flip. In practice, any useful model should provide more information than a coin toss and its curve must be above this straight line. The area between these represents the quality of a model. Given a model $M$ and a calibration set $C$, we represent this area as $ROC(M)_C$.

Thus if we have two models $M_0$ and $M_1$ built from the same training set $T$ and validate of the same calibration set $C$, we say that model $M_0$ is better than $M_1$, when $ROC(M_0)_C > ROC(M_1)_C$.

### 5 Binomial Dimensions/Features Exploration

In practice, we assume that the explanatory variable-vector $x$ as in Equation 2.3 will be able to bring forward the features necessary to create a binomial model. We need to explore and choose these explanatory variables and thus quantify their explanatory power.

So far, Training is used to build the model and Calibration is used to validate the model. We have formalized a quantitative measure of model quality. To explore the feature space, we choose different spaces and build models, then we compare the models.

We have available the following feature space:

1. **Ad-Exchange**: e.g., AppNexus, MoPub. This represents the set of available publishers at our disposal, the different devices and different bidding engagements.

2. **Hour of the day**: e.g., 17 PM. Evening hours in the west coast have more users than early hours in the east coast.

3. **Day of the week**: e.g., Wednesday. Working days are often more engaging than week ends.

4. **Ad format**: e.g., video, banner, which represents also the location of the ad. Ad Video have more clicks because are more difficult to turn off or pause.

5. **Ad size**: the real estate size of the ad space. Larger ads capture more attention and they are more expensive.

6. **Domains**: the sites where the ads are distributed (user interests). Most of user targeting is based on the sites we visit or particular pages of a site. For example, yahoo.com and finance.yahoo.com are two different domains, google.com and google.com/finance are not. In general, the domain provide a signal (but not always).

7. **Geographical distribution by ZIP**: e.g., 95131 (user location). There are different levels of precision: IP, lat-long, parcel, City, ZIP-4, ZIP, DMA, State. We use ZIP because is an intermediary location and it is coarse enough for our purpose. At the beginning of this project we considered ZIP and City together, we dropped the City because it is a feature harder to compute at run time.

In practice, the first five dimensions describe a limited feature space: there are only 24 hours in a day. The last two dimensions are different: there are thousands of ZIPs and there can be thousands of domains and changing during the year. We need to explore subset of domains, subset of ZIPs, and we need to understand if there is correlation. Even if this scenario is specific to our problem, the properties above cover a wider spectrum of applications.

Before we present what we do, this is what we do not do: we could take all possible dimensions and cross correlation and train a single model. If convergence is possible, we could then manually remove betas: we could start with removing betas associated to rejection, then removing betas with little contribution. The latter could be achieved by minimizing the so called $L_1$ (max error) instead of $L_2$ (variance error) to naturally suppress dimensions. Dimension suppression will change the overall response of the model and thus the ROC. We measure the quality of a model by its overall ROC: thus, we would like to compare their un-altered ROC curves.

Now, the first model we compute is without domains and without ZIPs.

Given the training set, we compute the frequency of domains and ZIPs for the positive and for the negative cases. For example, we take the $K=10$ most frequent domains that are associated with the positives and we take the $K=10$ most frequent domains associated with the negatives. Then, we take their union ($K \leq 20$). We build a model and we compare with the best built so far using the Calibration set. We repeat the procedure only for the top $K=10$ ZIPs. We repeat the process with top $K=10$ ZIPs and domains. We record the best model which has the bigger ROC. We do not create a model with ZIP + Domain.

Now, we repeat the process for $K \in \{20, 50, 100, 200\}$. While the choice of $K$ is arbitrary, the idea is based on the incremental introduction of more attributes in order to check weather or not they provide more discriminating informa-
rare events have little signal. In practice, domain and ZIP are not correlated and features so that the model would have no more than one thousand ZIPs. We found reasonable to have a maximum of 800 completely the rare positives. However, for computational and procedure: We do not know if rare features determine computation. The exploration is mechanic and there is no early stop

5.2 The model. In practice, the final model is the composition of: the intercept \( \beta_0 \), the set of betas \( \{ \beta_i \}_{i>1} \) related to the explanatory features, and the threshold. Because the function \( g() \) is increasing and continuous, we do not need to compute the real probability.

We accept an impression if \( \beta_0 + \sum_{i>0} \beta_i > \text{threshold} \). The scoring boils down to a sum of betas, which can be done quickly. The matching of the impression dimensions with the model dimensions requires a little more work, but it can always be done by a binary search (or hashing).

6 Polytomous Response

In this section, we are going to present a few considerations on the application of these models to real campaigns, impressions, and what could be the performance at run time for these systems.

6.1 Coverage. In isolation, a single model will tend to reduce the number of acceptable impressions: if we are targeting a rare event, only few impression will be very likely and the majority will be at best disputable and rejected. In practice, one model would choke the delivery. What about 100 models?

We considered a few hundred campaigns deployed in the past and we modeled about 102 models. Then we took 279,560,699 of delivered impressions (e.g., one week say). Only 349,585 impressions are rejected by all models. This means that while each campaign may well starve, overall they do not. The models re-distribute the impressions already bought and delivered. 1 model will choke the delivery, 100 will have complete coverage. This means that the current rules used for the buying and delivery provide the variety and the quantity to serve all campaigns as a whole.

The models can be used after the decision of bid is taken assuring that delivery and pacing of campaigns.

7 Precision and Recall

Now, given all clicks can the 102 model recognize them back? In this section, an impression is taken from the set of clicks, thus we know what is the campaign associated with any impression.

Let us introduce the following common terminology: given an impression \( x \) and a model \( P_{C_j}(x) \) for campaign \( C_j \),
- A tp true positive case is when \( x \) is a click for campaign \( C_j \) and \( P_{C_j}(x) \sim 1 \) (a.k.a. above the threshold).
- A fp false positive case is when \( x \) is NOT a click for \( C_j \) and still \( P_{C_j}(x) \sim 1 \).
- A tn true negative case is when \( x \) is NOT a click and \( P_{C_j}(x) \sim 0 \).
- A fn false negative case is when \( x \) is a click for \( C_j \) and \( P_{C_j}(x) \sim 0 \).

We can then recall the following definition:

\[
\text{Precision} = \frac{\sum \text{tp}}{\sum \text{tp} + \sum \text{fp}} \tag{7.8}
\]

\[
\text{NegativeRate} = \frac{\sum \text{tn}}{\sum \text{tn} + \sum \text{fn}} \tag{7.9}
\]

\[
\text{Recall} = \frac{\sum \text{tp}}{\sum \text{tp} + \sum \text{fn}} \tag{7.10}
\]

\[
\text{Accuracy} = \frac{\sum \text{tp} + \sum \text{tn}}{\sum \text{tp} + \sum \text{fp} + \sum \text{tn} + \sum \text{fn}} \tag{7.11}
\]

Having multiple models, it may happen that one impression is vetted by multiple models. We could give it to the model with the top score, or we can provide at random to
any of the models with scores higher than their thresholds, this is like a set decision. Of course, the top and the set policies count precision and recall differently: we have different metrics. In Figure 1, we show a graphical representation of 4 measures used in the field. The figure is like a time series.

In principle, we could take different thresholds for each models and determine the configuration maximizing any measure. In practice and at run-time, the thresholds will change so that to throttle the delivery. This is a hard problem to solve, even to formulate.

Nonetheless, if we now take the definition of true/false positive/negative, we can consider to compute the Precision and Recall of the set of models as a single entity: the true positives are the sum of all models’ true positives.

We have: top Total Precision 0.2945 and Total Recall 0.2917; Set Total Precision 0.1300 and Total Recall 0.4103. In practice, the former has a better precision overall, it can recognize true positives, but it will increase the false negative. The latter will have fewer false negative.

We have a measure of the polytomous model, which is composed of binary models: in practice, every model will have a weight associated to the importance of the campaign (e.g., money or total number of impression to deliver).

Clearly the building of each model separately is appealing because we can turn them off without need to retraining the others.

7.1 Scoring Speed. The scoring in itself is the sum of betas. The sum can be done very quickly as soon as we know which betas to use. Given an impression \( x \), each model betas are different and some beta may not have the betas associated with \( x \)’s features.

How many impression we can score per second or QPS (query per second)?

We implemented a multithreaded library written in C for the scoring above. We chose C because we wanted to measure performance in two systems: Intel Phi coprocessor with 50 cores (with four thread each for a total of 200 cores) and 2 xeon Westemere processors each with 6 cores with two thread each for a total of 24 core.

A single core on the Westemere can provide 10,000 QPS (having all impressions in memory already). Both systems can provide steady 1 Million QPS making the scoring affordable at run time; note, we moved all impressions to the internal memory in the Phi system before measuring the throughput. Using in combination, we can achieve twice as much. This test is designed to compute the peak throughput and not the minimum or maximum latency, which is more common for real time bidding. Considering the highly parallel Phi that can achieve 3 TFLOPS sustained performance, with 300Watt consumption, 8 GB of memory and much more cache memory to assist the internal cores, we must admit that the server configuration with 2 xeon processors is a better choice (160
Watts and 64 GB memory). This is because the scoring function uses very little the deep pipeline avail in the Phi, which is the real reason of its high peak performance.

Unfortunately, we could not test the performance on GPUs such as Radeon 290 (available in the same system) because of time constraints and unavailable resource to export the library as OpenCL kernels.

8 Conclusions
Training polytomous models composed of binary models is appealing for campaign optimization. In this work, we show how to scale to hundreds of binary models (hundreds on independent responses). This is to show the applicability of the theory developed twenty years ago.

In practice, multiple campaigns can be optimized at the same time. Our ability to deploy multiple models circumvent a few critical issues about binary models and we can measure the quality of each campaign in isolation and as a collective. The collective set of models and each model can be modified at any time without affecting the others or the scoring speed.

Each model is built on top of the unique features that describe the campaign among the other campaigns.

9 Acknowledgments
The work presented here was mostly done while the author was at Brand.net.

We would like to thank several bright(er than us) people: Aram Campau, David Folk, Christofer Gilliard, Konstantin Bay, James Tsiao, and Yang Li. They helped starting and they nurtured this project. The inspiration to write our contributions stems from the work by Lee et al. [4] and their application in their real time bid optimizations [3].

References

[1] Jacok Bernoulli. Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola gallice scripta de ludo pilae reticularis. Basileae, impensis Thurnisiorum, fratru, 1713.

[2] Norman L. Johnson, Samuel Kotz, and Adrienne W. Kemp. Univariate Discrete Distributions (Wiley Series in Probability and Statistics). Wiley-Interscience, 2 edition, February 1993.

[3] Kuang-Chih Lee, Ali Jalali, and Ali Dasdan. Real time bid optimization with smooth budget delivery in online advertising. In Proceedings of the Seventh International Workshop on Data Mining for Online Advertising, ADKDD ’13, pages 1:1–1:9, New York, NY, USA, 2013. ACM.

[4] Kuang-chih Lee, Burkay Orten, Ali Dasdan, and Wentong Li. Estimating conversion rate in display advertising from past eformance data. In Proceedings of the 18th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, KDD ’12, pages 768–776, New York, NY, USA, 2012. ACM.

[5] P. McCullagh and John A. Nelder. Generalized Linear Models, Second Edition. Chapman & Hall/CRC Monographs on Statistics & Applied Probability, Taylor & Francis, 1989.

[6] Abraham Wald. Sequential Analysis. Advanced Mathematics. Dover, 1947.