The Photon Wavefunction: a covariant formulation and equivalence with QED

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We discuss the limits of the photon wavefunction (PWF) formalism, which is experiencing a revival in these days from the new practical applications in photonics and quantum optics. We build a Dirac-like equation for the PWF written in a manifestly covariant form and show that, in presence of charged matter fields, it reproduces the standard formulation of (classical) Electrodynamics. This shows the inconsistency of the attempts to construct a quantum theory of interacting photons, based on the so called photon wavefunction approach, alternative to standard QED. PWF formalism can then be used to provide an easier description of the propagation of free photons, when the photon number remains fixed in time.

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INTRODUCTION

The problem of writing a wavefunction for the photon takes its origins from the first attempts of quantizing the electromagnetic field, since the birth of Quantum Mechanics. Because of the localization problem for the photon, the definition and even the existence of a wavefunction for the photon is still controversial, as it cannot always give a complete description of the system, like that provided by Schrödinger equation for non-relativistic massive particles. The first attempts can be found in the unpublished notes by Majorana [1], where the quantum states of the electromagnetic field were tentatively described by using the language of first quantization in the form of a Dirac–like equation, obtained from the Riemann–Silberstein (RS) formulation of Maxwell equations [2, 3, 4]. Dirac equation was formulated to describe the evolution of the relativistic electron, a particle with non–zero rest mass, $\hbar/2$ spin, and elementary charge $e$. Weyl equations instead describe massless neutral spinors. Finally Majorana extended the Dirac equation also to particles with arbitrary spin, in a more general infinite–spin component formalism [5]. This approach for the quantization of the Electromagnetic field in the first quantization language is justified by the fact that Maxwell equations present an intrinsic mathematical structure similar to that of a quantum wave function in relativistic theory and, conversely, the same procedure followed by Dirac to write the relativistic equation for the electron can be used to derive Maxwell equations. New recent experiments with single, double and many–photon sources and also with entangled states, where the photon number is small and remains fixed, renewed the interest in the wavefunction of the photon [6–9, 10, 11, 12]. This revival of interest raised some, in our opinion too optimistic, hopes that this approach could open new perspectives for alternative quantum descriptions of photons, even if not free [13]. The Majorana-like equation for the PWF can be considered as a consistent Quantum Mechanics of a free photon only, even if modulo the well known difficulty of its non localization [14]. For a system of relativistic particles in interaction (and in particular of interacting photons) Quantum Mechanics cannot be used since it implies action-at-distance forces, incompatible with Relativity. Quantum Field Theories are, in fact, mandatory.

In this letter, after rewriting the Majorana-like equation of the photon wavefunction in a manifest covariant form, we discuss the lagrangian that reproduces this equation, to be used as the starting point for getting a Quantum Field Theory for (interacting) photons. Then we show that this lagrangian is equivalent to that of classical Electrodynamics so that after quantization it leads to standard QED.

PWF AND COVARIANT FORMULATION

Following Majorana formulation, without loosing in generality for a particular choice of helicity state, or of multiplicative constants, one defines the Riemann–Silberstein vector

$$F = \frac{E}{c} \pm iB$$

and Maxwell Equations in the vacuum become

$$\nabla \cdot F = 0, \quad i\nabla \times F = \pm \frac{1}{c} \frac{\partial F}{\partial t} \tag{2}$$

By using the correspondence principle, $p \leftrightarrow \hat{p} = -i\hbar \nabla$ ($i = 1, 2, 3$), $F$ here represents the wavefunction of the photon, leading to the wave equation $\pm i\hbar \frac{\partial F}{\partial t} + i\hat{p} \times F = 0$, while $\nabla \cdot F = 0$ is the transversality of the fields with respect to the propagation direction, namely $\hat{p} \cdot F = 0$. By introducing the $3 \times 3$ complex matrices

$$\hat{s}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{s}_y = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{s}_z = \begin{pmatrix} 0 & 0 & -i \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

one obtains a Dirac–like equation

$$i\hbar \frac{\partial}{\partial t} F = \hat{H} F, \tag{3}$$
with $\hat{H} = \pm c \mathbf{s} \cdot \mathbf{p}$ and $\mathbf{s} = (\hat{s}_x, \hat{s}_y, \hat{s}_z)$. This Hamiltonian has eigenvalues $\pm cp$. The eigenvalue 0 is forbidden by the transversality condition.

There has been a debate in the literature on the interpretation of the negative energy state with eigenvalue $-cp$. The interpretation of the states with eigenvalues $\pm cp$ as states of positive energy and helicity $\pm 1$ is discussed in $[3]$. A simple way to understand this point is the following: since in our case the observables of Energy and Helicity commute, we can interpret the generator of translations in time, $\hat{H}$ in $[3]$ (after diagonalization) as the product of the Hamiltonian $H = cp$ with the helicity operator $\lambda$. The wave equation now has energy eigenvalues always positive and the negative eigenvalue of $\hat{H}$ is due to the negative helicity value. As for the transversality condition $\nabla \cdot \mathbf{F} = 0$, let us recall that it is at the origin of the non-localization of the PWF. Indeed the Hilbert space where $F(x)$ lives, is the space of modulo, square functions $\phi(x)$ that satisfy the condition $\nabla \cdot \phi = 0$ and therefore, the position operator $X \phi(x) = x \phi(x)$ is not an operator of this space, since $x \phi(x)$ does not satisfy the transversality condition.

The first step to a covariant formulation for $\mathbf{F}$ is the isomorphism between the algebras of the group $SL(2C)$, of unimodular $2 \times 2$ matrices in the complex field and the (proper orthocrons) Lorentz group, $SO(1,3)$, of $4 \times 4$ (pseudo-)orthogonal real matrices that leave invariant the Minkowsky metrics $g_{\mu \nu} = \text{diag}(1,-1, -1, -1)$. Since the two algebras are isomorphic, the two groups satisfy a local isomorphism that, extended to a global one, becomes a $2 \rightarrow 1$ homomorphism. Consider the matrix

$$\mathbf{\tau} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$$

built with the space-time coordinates, $x_0, x_1, x_2, x_3$ and transform $\mathbf{\tau}$ by an $SL(2C)$ transformation as $\mathbf{\tau}' = A^{-1} \mathbf{\tau} A$, where $A \in SL(2C)$, so that $det(\mathbf{\tau}') = det(\mathbf{\tau})$. Since $det(\mathbf{\tau}) = x_0^2 - |x|^2 = g_{\mu \nu} x_\mu x_\nu$, the transformation leaves invariant the four dimensional interval $s^2 = g_{\mu \nu} x_\mu x_\nu$ and therefore induce a Lorentz transformation $A$ on $x^\mu$, but to both $\pm A$ correspond the same Lorentz transformation (i.e. the homomorphism is $2 \rightarrow 1$). The group $SL(2C)$ has two inequivalent, fundamental representations called $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ that can be also considered as two spinorial representations of $SO(1,3)$ (chiral and antichiral Weyl spinors).

The elements of the vector space on which these representations operate are $\phi_\alpha(x)$, $(\alpha = 1, 2)$ for the $(\frac{1}{2}, 0)$ and $\mathbf{\tau}_\alpha \equiv \epsilon^{\alpha \delta} \phi_\delta$, $(\delta, \alpha = 1, 2)$ for $(0, \frac{1}{2})$, such that $\phi_\alpha(x') = A_{\alpha}^\beta \phi_\beta(x)$ and $\mathbf{\tau}_\alpha(x') = (A^{-1})^\alpha \beta \mathbf{\tau}_\beta(x)$ where $A, A^{-1} \in SL(2C)$

Tensorial products of the fundamental representations of the Lorentz group give rise to higher dimensional representations, divided in two main classes: Tensorial representations that derive from the product of even times the fundamental representations $(\frac{1}{2}, 0)$ and/or $(0, \frac{1}{2})$, and Spinorial representations coming from the product of odd ones. The simplest tensorial representations are $\psi_{\alpha}^\beta \equiv \phi_\alpha \otimes \mathbf{\tau}_\beta$, $\psi_{\alpha \beta} \equiv \phi_\alpha \otimes \chi_\beta$, $\psi_{\alpha \beta} \equiv \bar{\phi}_\alpha \otimes \mathbf{\tau}_\beta$, with $\phi_{\alpha}$, $\chi_{\beta} \in (\frac{1}{2}, 0)$, $\bar{\phi}_{\alpha}$, $\mathbf{\tau}_{\beta} \in (0, \frac{1}{2})$. Written in its symmetric and antisymmetric parts, $\psi_{\alpha \beta} = \psi_{[\alpha \beta]} + \psi_{(\alpha \beta)}$, then, $\psi_{[\alpha \beta]} = \lambda \epsilon_{\alpha \beta}$, has the only degree of freedom $\lambda$, where $\epsilon_{\alpha \beta}$ is the $2 \times 2$ completely antisymmetric tensor. The symmetric part has instead 3 independent components. A similar decomposition holds for $\psi_{\alpha \beta}$. The constant tensors $\epsilon_{\alpha \beta}$, $\epsilon_{\alpha \beta}$ and their inverses $\epsilon^{\alpha \beta}$, $\epsilon^{\alpha \beta}$ can be used to rise and lower the spinorial indices. Let us define the two by two matrices $(\sigma^{\mu \nu}_\alpha, \epsilon^{\alpha \beta}_\mu)$ and $(\bar{\sigma}^{\mu \nu}_\alpha, \bar{\epsilon}^{\alpha \beta}_\mu)$ where $\bar{\sigma}^{0 \mu} = \sigma^{0 \mu} = 1$, $\bar{\sigma} = -\sigma$ and $\sigma$ are the Pauli matrices. The tensorial representations $\psi_{\alpha}^\beta$, $\psi_{\alpha \beta}$ and $\psi_{\alpha \beta}$ can be expressed in terms of these matrices,

$$\psi_{\alpha}^\beta = (\sigma^{\mu \nu}_\alpha)_{\beta} \psi_{\mu \nu}$$

(5)

$$\psi_{\alpha \beta} = (\sigma^{\mu \nu}_\alpha)_{(\alpha \beta)} \psi_{\mu \nu}$$

(6)

$$\psi_{\alpha \beta} = (\bar{\sigma}^{\mu \nu}_\alpha)_{(\alpha \beta)} \psi_{\mu \nu}$$

(7)

but the first of these equations, that defines a four-vector $\psi_{\mu \nu}$. Similarly,

$$\psi_{\mu \nu}^\pm = \psi_{\mu \nu} \pm \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \psi^{\rho \sigma}$$

(8)

Both $\psi_{\mu \nu}^+$ and $\psi_{\mu \nu}^-$ have 3 independent components and $\psi_{\mu \nu}^+ = \psi_{\mu \nu}^+$, $\psi_{\mu \nu}^- = -\psi_{\mu \nu}$, where $\psi_{\mu \nu}^D = \frac{i}{2} \epsilon_{\mu \nu \rho \sigma} \psi^{\rho \sigma}$ is the dual tensor of $\psi_{\mu \nu}$.

One can verify easily that $\psi^-$ and $\psi^+$ do not contribute to $\psi_{\alpha \beta}$ and to $\psi_{\alpha \beta}$ respectively so that

$$\psi_{\alpha \beta} = (\sigma^{\mu \nu}_\alpha)_{\alpha \beta}$$

(9)

$$\psi_{\alpha \beta} = (\bar{\sigma}^{\mu \nu}_\alpha)_{\alpha \beta}$$

(10)

Therefore $\psi_{\alpha \beta}$ describes the self-dual part, $\psi_{\mu \nu}^+$ and $\psi_{\alpha \beta}$ the antiself-dual part, $\psi_{\mu \nu}^-$ of a six-components double–antisymmetric tensor $\psi_{\mu \nu} \in (\frac{1}{2}, \frac{3}{2}) \oplus (0, \frac{1}{2})$.

The Faraday electromagnetic tensor $F_{\mu \nu}$ such that $F_{\mu \nu}^{\alpha \beta} = -\frac{1}{2} E_\alpha$, and $F^{\alpha \beta \gamma \delta} = -e^{\alpha \beta \gamma \delta} B_\delta$ is a double antisymmetric tensor and its self-dual and anti self-dual parts $\left(F^{+}_{\mu \nu}, F^{-}_{\mu \nu}\right)$ can be written in covariant spinor notation

$$F_{\alpha \beta} = (\sigma^{\mu \nu}_\alpha) F_{\mu \nu}^{\alpha \beta}$$

(11)

$$F_{\alpha \beta} = (\bar{\sigma}^{\mu \nu}_\beta) F_{\mu \nu}^{\alpha \beta}$$

(12)

Where $F_{\alpha \beta} \in (\frac{1}{2}, \frac{1}{2})$, and $F_{\alpha \beta} \in (0, \frac{1}{2})$, but

$$F_{\alpha \beta} \propto \left((\sigma^{\nu \sigma \nu}_\alpha) F^{\alpha \beta}_{\mu \nu}\right) = (\sigma^{\nu \sigma \nu}_\alpha) F^{\alpha \beta}_{\mu \nu}$$

(13)

$$F_{\alpha \beta} \propto \left((\bar{\sigma}^{\nu \sigma \nu}_\beta) F^{\alpha \beta}_{\mu \nu}\right) = (\bar{\sigma}^{\nu \sigma \nu}_\beta) F^{\alpha \beta}_{\mu \nu}$$

(14)
and
\[ F_i^\pm = \frac{E_i}{c} \pm iB_i \]  
(15)
so that \( F_{(\alpha\beta)} \) and \( \overline{F}_{(\alpha\beta)} \) are the positive and negative helicity wavefunctions of the photon in covariant notations.

To write the Dirac-like equation for the photon wave function, consider
\[ (\overline{\sigma}^\mu \partial_\mu)_{\dot{\alpha}}^\beta F_{(\beta\alpha)} = (\overline{\sigma}^\mu \partial_\mu \sigma^\nu \sigma^\lambda)_{\dot{\alpha}}^\alpha F_{\alpha\lambda}^+ = 0 \]  
(16)
and its complex conjugate
\[ (\sigma^\mu \partial_\mu)_{\dot{\alpha}}^\beta \overline{F}_{(\beta\dot{\alpha})} = (\sigma^\mu \partial_\mu \sigma^\sigma \sigma^\lambda)_{\dot{\alpha}}^\alpha F_{\alpha\lambda}^- = 0. \]  
(17)
When eqn (16) is saturated with \( (\sigma^\tau)^{\dot{\alpha}\alpha} \) one gets, for \( \tau = 0 \), the first equation in (2) and, for \( \tau = i \), the second equation in (2) that is equation (3). The same results are obtained saturating (17) with \( (\sigma^\tau)^{\dot{\alpha}\alpha} \).

Equation (3), together with the transversality condition \( \nabla \cdot \mathbf{F} = 0 \), is equivalent to free Maxwell equations. This leads to speculate that this approach could be taken as the starting point for a new quantum description of, even not free, photons. However when it is rewritten in covariant form, (16), it becomes completely clear that it describes just free Maxwell equations in a different notation. That leaves little room to the speculations previously mentioned.

Let us add some further considerations to stress this point ever more. As noted in the introduction, a relativistic quantum theory with interactions must be necessarily a local QFT. The recipe to write the (free) classical field lagrangian density, to be quantized, is to look at the classic action that yields the Schroedinger equation of the Quantum Mechanics of the single particle (eventually supplemented with local interaction terms) and quantize that classical action according to the canonical rules. This procedure has been named, quite improperly, second quantization.

For instance the lagrangian density of a free electron, derived from the Dirac equation, is \( \mathcal{L}_e = \overline{\Psi} (i \gamma^\mu \partial_\mu - mI_4) \Psi \), which is invariant under the global gauge transformation \( \Psi \rightarrow e^{i\lambda} \Psi \), where \( \lambda \) is the global gauge parameter. To extend this trasformation to a local one with gauge parameter \( \lambda(x) \) one must introduce a gauge field \( A_\mu \) that trasforms as \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \) and the lagrangian density becomes
\[ \mathcal{L}_e = \overline{\Psi} (i \gamma^\mu (\partial_\mu - icA_\mu) - mI_4) \Psi. \]  
(18)

Now we have to search for a lagrangian density that gives rise to (16) as its Eulero-Lagrange field equations. Since (16) transform covariantly as a four-vector one needs a four-vector, let say written in spinor notations, \( A^{\alpha\dot{\alpha}} = A_\mu (\sigma^\mu)^{\alpha\dot{\alpha}} \). A lagrangian density that reproduces (16) by varying \( A \) is \( \mathcal{L} = a(\overline{\sigma}^\mu \partial_\mu)^{\alpha\beta} F_{\alpha\beta} \) where \( a \) is a normalization constant. Notice that this lagrangian is invariant under the gauge trasformation \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \).

If one adds to the lagrangian \( \mathcal{L} \) the Dirac lagrangian \( \mathcal{L}_e \) it is right to identify, as anticipated with the notations, the real four-vector in \( A \) with the \( A_\mu \) in (13) since with this identification the field equations for \( A_\mu \) yield correctly the current term, \( j^\mu = e \overline{\Psi} \gamma^\mu \Psi \) in the right hand side of the first group of Maxwell equations (or equivalently a non linear term \( j^1 \) in the r.h.s. of (3) and a non linear term \( j^0 \) in the r.h.s. of \( \nabla \cdot \mathbf{F} = 0 \)). However the field equations obtained varying \( F_{(\alpha\beta)} \) are \( (\partial_\mu A_\nu)^\pm = 0 \) that imply \( A_\mu = 0 \) modulo a gauge transformation. But \( \mathcal{L} \) has a serious drawback: it is not real.

By adding to \( \mathcal{L} \) the complex conjugate counterpart, the action becomes
\[ I = \int \frac{a}{2} \left[ (\overline{A} \sigma^\mu \partial_\mu)^{\alpha\beta} F_{\alpha\beta} \right] + \int \mathcal{L}_e \]  
(19)
where \( \overline{A} = A_\lambda \sigma^\lambda \). Unfortunately (19) reproduces only the first group of Maxwell equations \( \partial^\mu F_{\mu\nu} = e j_\nu \) (with \( a = 3/4 \)).

A possible cure of this desease could be to allows \( A_\lambda \) to become complex that is \( A = (A_\lambda + iB_\lambda) \sigma^\lambda \), \( \overline{A} = (A_\lambda - iB_\lambda) \sigma^\lambda \) \( (A_\mu \) and \( B_\mu \) real). Now (16) (with \( \sigma_0 \)) are reproduced correctly but there is an unacceptable doubling of degree of freedom. Indeed now the action (19) becomes
\[ I = \int (\partial_\mu A_\mu F_{\mu\nu} + B_\mu e^{i\nu\rho\sigma} \partial_\nu F_{\rho\sigma} + \mathcal{L}_e) \]  
(20)
The second group of Maxwell equations \( e^{i\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \) implies \( F_{\mu\nu} = \frac{1}{2} (\partial_\nu A_\mu - \partial_\mu A_\nu) \). When this algebraic equation is used to remove \( F_{\mu\nu} \) in (20) the \( B_\mu \) field drops out and the action becomes \( I = \int (\partial_\mu A_\mu)^\pm \partial^\nu A_\nu + \mathcal{L}_e \) Even worse, if one defines \( A^{\pm}_\mu \) \( = \frac{1}{2} (A_\mu \pm \partial_\mu \lambda) \), then
\[ I = \int (\partial_\mu A^{\pm}_\mu)^\pm \partial^\nu A_\nu^{\pm\nu} - \partial_\mu A^{\pm}_\mu (-)^\partial^\nu A^{\pm\nu} + \mathcal{L}_e \]  
(21)
so that, after quantization, one of the two “photons” described by the gauge fields \( A^{\pm}_\mu \) has negative metric and therefore the action (21) is inconsistent.

The only consistent way to cure these problems is to add to the lagrangian density in (19) the term
\[ \mathcal{L}_0 = -\frac{3}{32} (F^{\alpha\beta} F_{\beta\alpha} + \overline{F}^{\alpha\beta} \overline{F}_{\beta\alpha}) = \frac{1}{2} F^{\mu\nu} F_{\mu\nu}. \]  
(22)
to obtain
\[ I = \int \mathcal{L}_{tot} = -\int \partial^\mu A_\nu^{\pm\nu} F_{\mu\nu} - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \]  
(23)
\[ + \int \overline{\Psi} (i \gamma^\mu (\partial_\mu - icA_\mu) - mI_4) \Psi \]
The field equation for $F_{\mu\nu}$ identifies $F_{\mu\nu}$ with $\partial_\mu A_\nu$: 

\[ F_{\mu\nu} = -1/2(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu A_\nu. \]

Since this is an algebraic equation it can be used to replace $F_{\mu\nu}$ in the lagrangian with the following result,

\[ I = \frac{1}{2} \int \partial_\mu A_\nu \partial^{\mu} A^{\nu} + \int \overline{\Psi} (i\gamma^\mu (\partial_\mu - ieA_\mu) - mI) \Psi \tag{24} \]

eqn. (24) is the standard action of classical electrodynamics and after quantization it gives rise standard QED. In the presence of charged matter fields, this formulation reproduces only the standard formulation of (classical) Electrodynamics. This clearly shows the inconsistency of the attempts to construct a quantum theory of interacting photons, based on the so called photon wavefunction approach, alternative to standard QED. There is a perfect correspondence between PWF and QED only when photons are free, non interacting and when the photon number remains constant during the evolution of the field, with the problems of the photon localization.

**DISCUSSION AND CONCLUSIONS**

In this note we have presented and discussed the equivalence of the PWF formalism with that of standard Quantum Electrodynamics. PWF can only describe scenarios where the photons are free, non interacting and maintain a constant number during their evolution. No absorption and/or emission of photons can be directly described by the formalism of PWF. The equivalence is set by a manifestly covariant version of the so called PWF equation. Moreover, on the basis of this covariant formulation we have motivated the statement that the photon wave function approach, at the second quantization level, cannot give anything else than the standard Quantum ElectroDynamics.

The correspondence set between PWF and QED for free photons is useful to shed some light in the Orbital Angular Momentum (OAM) of the photon and the PWF formalism [13, 16, 17, 18]. OAM of light is deeply connected with the vorticity of the E-M field and with the creation of optical vortices. Recently the RS vector, which is the basis with which the PWF is built, was used to describe the E-M field vorticity [19, 21, 21, 22]. Riemann Silberstein vortices are defined by

\[ \mathbf{F}(r, t) \cdot \mathbf{F}(r, t) = 0. \tag{25} \]

The loci of points satisfying this condition are lines in space, the phase of the field is singular surrounded by zones where the phase gradient vector is circulating. Laguerre–Gaussian beams are particular cases in which the field has spatial symmetry and the RS vortex lines are stationary. Anyway exact solutions of electromagnetic waves carrying angular momentum have been recently described in by using the momentum representation and then were cast in terms of PWF in the RS formalism [22] that, by using the correspondence here discussed, is simply equivalent to the description obtained with QED by quantizing the field in paraxial approximation that, at the single photon level, represents the probability amplitude of finding a photon in a certain eigenstate of momentum, helicity and OAM [23], that is clearly not an intrinsic property of the photon [24]. The equivalence of the two formulations can be easily set by expressing the PWF in terms of the photon annihilation and creation operators, using the vector potential $A$ in the RS vector, but this goes beyond the purpose of this work.

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[24] By definition, the intrinsic properties of a particle are those that do not depend on the choice of a reference frame, i.e the rest mass, the electric charge and spin.