Analysis of internal delamination in multilayered rod in torsion

V Rizov

Department of Technical Mechanics, University of Architecture, Civil Engineering and Geodesy, 1046 - Sofia, Bulgaria

E-mail: v_rizov_fhe@uacg.bg

Abstract. A multilayered inhomogeneous rod with an internal delamination crack loaded in torsion is analyzed. The crack has a circular cross-section. The delamination is located symmetrically with respect to the middle of the rod. The layers exhibit continuous material inhomogeneity in radial and length directions. The material properties are distributed symmetrically with respect to the middle of the rod. In order to obtain the torsion moments in the crack arms, the rod is treated as a structure with one degree of internal static indeterminacy. The delamination is studied in terms of the strain energy release rate by using the compliance method. Effects of material inhomogeneity, length and location of the crack on the delamination behaviour are investigated.

1. Introduction
The multilayered inhomogeneous structural members and components are made of adhesively bonded layers of different materials. The use of multilayered structures in various load-bearing applications in different areas of practical engineering has been attempted mainly to achieve a substantial weight-reduction. Since multilayered structures have relatively low interlaminar strength, delamination fracture is one of the most common failure modes of these structures when they are subjected to various loading conditions [1, 2, 3, 4].

The present paper is concerned with a delamination fracture analysis of a multilayered inhomogeneous rod of circular cross-section loaded in torsion. It should be noted that in contrast to previous papers which deal with a delamination crack located in the end of rod [5, 6], the present paper studies an internal delamination.

2. Theoretical model of a multilayered rod with internal delamination
A multilayered inhomogeneous rod of circular cross-section of radius, $R$, is shown in figure 1. The length of the rod is $2l$. The rod is loaded by two torsion moments, $T$, applied at its ends. The rod is made of an arbitrary number of concentric layers. A delamination crack of length, $2a$, is located arbitrary between layers as shown in figure 1. The delamination is located in rod portion, $l - a \leq x \leq l + a$. In portions, $0 \leq x \leq l - a$ and $l + a \leq x \leq 2l$, the rod is un-cracked. It should be noted that the internal delamination crack arm is treated as a rod of length, $2a$, and radius of the cross-
section, $R_I$. The external delamination crack arm is treated as a ring-shaped rod of radiiuses of the internal and external surfaces, $R_I$ and $R_E$, and length, $2a$.

Each layer of the rod exhibits continuous material inhomogeneity in both radial and length directions. The distribution of the shear modulus, $G_i$, of the $i$-th layer in radial direction is written as

$$G_i = G_{Bi} + \frac{G_{Di} - G_{Bi}}{(R_{i+1} - R_i)^{m_i}}(R - R_i)^{m_i},$$  

(1)

where $G_{Bi}$ and $G_{Di}$ are the values of the shear modulus in the internal and external surfaces of the layer, $R_i$ and $R_{i+1}$ are the radiiuses of the internal and external surface of the layer, $m_i$ is a material property that controls the material inhomogeneity in radial direction.

![Figure 1](image-url)

**Figure 1.** Geometry and loading of a multilayered inhomogeneous rod with an internal delamination crack.

The distribution of $G_{Bi}$ and $G_{Di}$ along the rod length is expressed as

$$G_{Bi} = G_{NBi} + \frac{G_{SBi} - G_{NBi}}{l^q}x^q, \quad G_{Di} = G_{NDi} + \frac{G_{SDi} - G_{NDi}}{l^q}x^q, \quad 0 \leq x \leq l,$$

(2)

$$G_{Bi} = G_{NBi} + \frac{G_{SBi} - G_{NBi}}{l^q}(2l - x)^q, \quad G_{Di} = G_{NDi} + \frac{G_{SDi} - G_{NDi}}{l^q}(2l - x)^q, \quad l \leq x \leq 2l,$$

(3)

where $G_{NBi}$ and $G_{NDi}$ are the values of $G_{Bi}$ and $G_{Di}$ in the two ends of the rod, $G_{SBi}$ and $G_{SDi}$ are the values of $G_{Bi}$ and $G_{Di}$ in the mid-span, $x = l$, $p_i$ and $q_i$ are material properties which control the material inhomogeneity along the rod length. Formulae (2) and (3) indicate that $G_{Bi}$ and $G_{Di}$ are distributed symmetrically with respect to the middle of the rod.

The delamination is analyzed in terms of the strain energy release rate, $G$. In order to derive the strain energy release rate, first, the torsion moments in the internal and external crack arms are obtained (the torsion moments are needed to calculate the shear strains which are used to determine the strain energy). The rod is treated as a structure with one degree of internal static indeterminacy. Thus, the torsion moment in the external crack arm, $T_E$, is obtained by the theorem of Menabrea

$$\frac{\partial U}{\partial T_E} = 0,$$

(4)

where $U$ is the strain energy. Due to the symmetry, only half of the rod, $l \leq x \leq 2l$, is analyzed.
Thus, the strain energy is written as

$$ U = U_I + U_E + U_U, $$

where $U_I$, $U_E$ and $U_U$ are the strain energies in the internal and external crack arms, and the uncracked portion of the rod. The strain energy in the internal crack arm is expressed as

$$ U_I = \sum_{i=1}^{n_I} \int_{r_i}^{r_{i+1}} u_{i0} 2\pi R dR, $$

where $n_I$ is the number of layers in the internal crack arm. The strain energy density in the $i$-th layer, $u_{i0}$, is obtained as

$$ u_{i0} = \frac{1}{2} G_I \gamma_i^2. $$

Since in the present paper rods of high length to diameter ratio are considered, the distribution of shear strain, $\gamma_H$, is treated by applying the Bernoulli’s hypothesis. Thus, the distribution of $\gamma_H$ is written as

$$ \gamma_H = \frac{\gamma_I}{R_I}, $$

where the shear strain at the periphery of the internal crack arm, $\gamma_I$, is obtained from the equation of equilibrium of the elementary forces in the cross-section of the internal crack arm

$$ \sum_{i=1}^{n_I} \int_{r_i}^{r_{i+1}} G_I \gamma_H 2\pi R^2 dR = T_I, $$

where $T_I$ is the torsion moment in the internal crack arm. It is obvious that (figure 1)

$$ T_I = T - T_E. $$

After substituting of (1), (8) and (10) in (9), the equation for equilibrium is solved with respect to $\gamma_I$. In this way, $\gamma_I$ is expressed as a function of $T_E$.

The strain energy in the external crack arm is written as

$$ U_E = \sum_{i=1}^{n_E} \int_{r_i}^{r_{i+1}} u_{E0} 2\pi R dR, $$

where $n_E$ is the number of layers in the external crack arm. The strain energy density in the $i$-th layer of the external crack arm, $u_{E0}$, is obtained by replacing of $\gamma_H$ with $\gamma_{EE}$ in (7). The distribution of the shear strain, $\gamma_{EE}$, in the cross-section of the external crack arm is expressed by replacing of $\gamma_I$ with $\gamma_E$ in (8). The shear strain, $\gamma_E$, at the periphery of the external crack arm is obtained as a function of $T_E$ from the following equation of equilibrium of the cross-section of the external crack arm:

$$ \sum_{i=1}^{n_E} \int_{r_i}^{r_{i+1}} G_I \gamma_{EE} 2\pi R^2 dR = T_E. $$

After substituting of $G_I$ and $\gamma_{EE}$ in (12), the equation for equilibrium is solved with respect to $\gamma_E$. The strain energy in the un-cracked rod portion is written as

$$ U_U = \sum_{i=1}^{n_U} \int_{r_i}^{r_{i+1}} u_{U0} 2\pi R dR, $$
where \( n \) is the number of layers in the rod. The strain energy density in the \( i \)-th layer, \( u_{U0i} \), is obtained by formula (7). For this purpose, \( \gamma_H \) is replaced with \( \gamma_{UU} \). The distribution of the shear strain, \( \gamma_{UU} \), in the cross-section of the un-cracked rod portion is found by replacing of \( \gamma_I \) and \( R_I \) with \( \gamma_E \) and \( R_E \) in (8). The shear strain at the periphery of the rod, \( \gamma_E \), is obtained as a function of \( T \) by equation of equilibrium (13). For this purpose, \( n \), \( \gamma_{UU} \) and \( T \), and then the equation of equilibrium is solved with respect to \( \gamma_E \).

After substituting of (5), (6), (7), (11) and (13) in (4), the equation is solved with respect to \( T_E \). The strain energy release rate is obtained by applying the compliance method

\[
G = 2 \frac{T^2}{2l_{cf}} \frac{dC}{da},
\]

where the delamination crack front length, \( l_{cf} \), is written as

\[
l_{cf} = 2\pi R_I. \tag{15}
\]

The compliance, \( C \), is obtained as

\[
C = \frac{\varphi}{T}. \tag{16}
\]

![Figure 2](image.png)

**Figure 2.** Two three-layered rods with a delamination crack located (a) between layers 2 and 3, and (b) between layers 1 and 2.

It should be noted that the right-hand side of (14) is doubled in view of the symmetry (figure 1). By using the integrals of Maxwell-Mohr, the angle of twist, \( \varphi \), of the end section of the rod is found as

\[
\varphi = \int_{l}^{l+a} \frac{\gamma_I}{R_I} \ dx + \int_{l+a}^{2l} \frac{\gamma_{UU}}{R_E} \ dx. \tag{17}
\]

By substituting of (15), (16) and (17) in (14), one derives the following expression for the strain energy release rate:
In order to verify (18), the strain energy release rate is obtained also by differentiating the strain energy with respect to the delamination crack area

\[ G = \frac{2}{l_d} \frac{dU}{da}, \]  

(19)

where \( da \) is an elementary increase of the delamination crack length.

**Figure 3.** The strain energy release rate in non-dimensional form plotted against \( \frac{G_{NA}}{G_{NB}} \) ratio (curve 1 - for the three-layered rod configuration with a delamination between layers 1 and 2, curve 2 - for the three-layered rod configuration with a delamination between layers 2 and 3).

The right-hand side of (19) is doubled in view of the symmetry (figure 1). It should be noted that the strain energy release rate calculated by (19) is exact match of that obtained by (18). This fact is a verification of the strain energy release rate analysis developed in the present paper.

3. **Parametric analysis**

The solution to the strain energy release rate (18) is applied to evaluate the influence of material inhomogeneity, the delamination crack location and the delamination length on the delamination fracture behaviour. The strain energy release rate is presented in non-dimensional form by using the formula \( G_N = G \left( \frac{G_{NB} R_E}{E} \right) \). Two three-layered rod configurations are analyzed (figure 2) in order to evaluate the effect of delamination crack location in radial direction on the delamination behaviour. A delamination crack is located between layers 2 and 3 in the rod shown in figure 2a. A rod with delamination between layers 1 and 2 is also investigated (figure 2b). The thickness of each layer is \( t \).

It is assumed that \( l = 0.3 \text{ m}, t = 0.001 \text{ m}, m_i = p_i = q_i = 0.8 \) and \( T = 15 \text{ kNm} \).

The material inhomogeneity in radial direction is characterized by \( \frac{G_{NA}}{G_{NB}} \) ratio. The strain energy release rate in non-dimensional form is plotted against \( \frac{G_{NA}}{G_{NB}} \) ratio in figure 3 for the two rod configurations shown in figure 2. One can observe in figure 3 that the strain energy release rate decreases with increasing of \( \frac{G_{NA}}{G_{NB}} \) ratio. The curves in figure 3 indicate also that the strain energy release rate is lower when the crack is located between layers 2 and 3.
Figure 4. The strain energy release rate in non-dimensional form plotted against $G_{SH}/G_{NB}$ ratio (curve 1 - at $a/l = 0.2$, curve 2 - at $a/l = 0.4$, curve 3 - at $a/l = 0.6$).

The effect of material inhomogeneity in the length direction of the rod on the delamination behaviour is illustrated in figure 4 where the strain energy release rate in non-dimensional form is plotted against $G_{SH}/G_{NB}$ ratio at three $a/l$ ratios. The curves in figure 4 show that the strain energy release rate decreases with increasing of $G_{SH}/G_{NB}$ ratio. The increase of $a/l$ ratio leads to increase of the strain energy release rate (figure 4).

4. Conclusions

The delamination behaviour of a multilayered inhomogeneous rod with an internal delamination crack is analyzed. The crack is located symmetrically with respect to the middle of the rod. The rod is loaded in torsion. The layers of the rod exhibit continuous material inhomogeneity in both radial and length directions. The rod has a circular cross-section. In order to obtain the torsion moments in the delamination crack arms, the rod is treated as a structure with one degree of internal static indeterminacy by applying the theorem of Menabrea. The strain energy release rate is derived by using the compliance method. The strain energy release rate is obtained also by differentiating the strain energy for verification. Parametric analyses are carried-out in order to investigate influences of the material inhomogeneity, the location and the length of the crack on the strain energy release rate. Two three-layered rods with different location of the delamination crack are considered to assess the influence of the crack location in radial direction. It is found that the strain energy release rate is lower when the delamination is located near the periphery of the rod. The analysis reveals that the strain energy release rate increases with increasing of the delamination crack length. The strain energy release rate decreases also with increasing of $G_{ND}/G_{NB}$ and $G_{SH}/G_{NB}$ ratios.

References

[1] Wang J and Zhang C 2007 Composites science and technology 67 3323-3330
[2] Klingbeil NW and Beuth JI 1997 Eng. Fract. Mech. 56 113-126
[3] Yang W and Shih CF 1994 Int. J. Solids Structures 31 985-1002
[4] Ogasawara T, Yokozeki T and Aoki T 2008 Compos. Sci. Technol. 68 760-767
[5] Rizov V 2020 IOP Conf. Ser.: Mater. Sci. Eng. 739 012003
[6] Rizov V 2018 Multidiscipline Modelling in Materials and Structures 15 156-169