Two simple finite element methods for Reissner–Mindlin plates with clamped boundary condition

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Abstract

We present two simple finite element methods for the discretization of Reissner–Mindlin plate equations with clamped boundary condition. These finite element methods are based on discrete Lagrange multiplier spaces from mortar finite element techniques. We prove optimal a priori error estimates for both methods.

Key words Reissner–Mindlin plate, finite element, Lagrange multiplier, biorthogonality, a priori error estimates

AMS subject classification. 65N30, 74K20

1 Introduction

There has been an extensive research effort to design finite element methods for the Reissner–Mindlin plate equations over the last three decades. A finite element discretization of Reissner–Mindlin plate is a challenging task as a standard discretization locks when the plate thickness becomes

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too small. So the main difficulty is to avoid locking when the plate thickness becomes really small. There are now many locking-free finite element techniques with sound mathematical analysis for these equations [4, 10, 2, 12, 17, 5, 11, 7, 3, 11, 18]. However, most of these finite element techniques are too complicated or expensive. In this paper, we present two very simple finite element methods for Reissner–Mindlin plate equations with clamped boundary condition. These finite element methods are based on a finite element method described in [2] for Reissner–Mindlin plate equations with simply supported boundary condition. We combine the idea of mortar finite elements with the finite element method proposed in [2] to modify the discrete Lagrange multiplier space leading to optimal and efficient finite element schemes for Reissner–Mindlin plate equations with clamped boundary condition. We propose two Lagrange multiplier spaces: one is based on a standard Lagrange multiplier space for the mortar finite element proposed in [9], and the other is based on a dual Lagrange multiplier space proposed in [14]. The first one gives a continuous Lagrange multiplier, whereas the second one yields a discontinuous Lagrange multiplier. The stability and optimal approximation properties are shown for both approaches. We note that the second approach with the discontinuous Lagrange multiplier space for Reissner–Mindlin plate equations with simply supported boundary condition has not been presented before, where boundary modification is not necessary. However, we only focus on the clamped case as it is the most difficult case of the boundary condition in plate theory. Moreover, the second choice of the Lagrange multiplier space allows an efficient static condensation of the Lagrange multiplier leading to a positive definite system. Hence this approach is more efficient from the computational point of view. We note that we use finite element spaces with equal dimension for the Lagrange multiplier and the rotation of the transverse normal vector.

The rest of the paper is planned as follows. The next section briefly recalls the Reissner–Mindlin plate equations in a modified form as given in [2]. Section 3 is the main part of the paper, where we describe our finite element methods and show the construction of discrete Lagrange multiplier spaces. Finally, a conclusion is drawn in the last section.

2 A mixed formulation of Reissner–Mindlin plate

Let $\Omega \subset \mathbb{R}^2$ be a bounded region with polygonal boundary. We need the following Sobolev spaces for the variational formulation of the Reissner–
Mindlin plate with the plate thickness \( t \):

\[
\mathbb{H}^1(\Omega) = [H^1(\Omega)]^2, \quad \mathbb{H}_0^1(\Omega) = [H_0^1(\Omega)]^2, \quad \text{and} \quad \mathbb{L}^2(\Omega) = [L^2(\Omega)]^2.
\]

We consider the following modified mixed formulation of Reissner–Mindlin plate with clamped boundary condition proposed in [2]. The mixed formulation is to find \((\phi, u, \zeta) \in \mathbb{H}_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{L}^2(\Omega)\) such that

\[
a(\phi, u; \psi, v) + b(\psi, v; \zeta) = \ell(v), \quad (\psi, v) \in \mathbb{H}_0^1(\Omega) \times H_0^1(\Omega),
\]

\[
b(\phi, u; \eta) - \frac{t^2}{\lambda(1-t^2)}(\zeta, \eta) = 0, \quad \eta \in \mathbb{L}^2(\Omega),
\]

where \( \lambda \) is a material constant depending on Young’s modulus \( E \) and Poisson ratio \( \nu \), and

\[
a(\phi, u; \psi, v) = \int_{\Omega} C \varepsilon(\phi) : \varepsilon(\psi) \, dx + \lambda \int_{\Omega} (\phi - \nabla u) \cdot (\psi - \nabla v) \, dx, \quad (1)
\]

\[
b(\psi, v; \eta) = \int_{\Omega} (\psi - \nabla v) \cdot \eta \, dx, \quad \ell(v) = \int_{\Omega} g v \, dx. \quad (2)
\]

Here \( g \) is the body force, \( u \) is the transverse displacement or normal deflection of the mid-plane section of \( \Omega \), \( \phi \) is the rotation of the transverse normal vector, \( \zeta \) is the Lagrange multiplier, \( C \) is the fourth order tensor, and \( \varepsilon(\phi) \) is the symmetric part of the gradient of \( \phi \). In fact, \( \zeta \) is the scaled shear stress defined by

\[
\zeta = \frac{\lambda(1-t^2)}{t^2} (\phi - \nabla u).
\]

### 3 Finite element discretization

We consider a quasi-uniform triangulation \( \mathcal{T}_h \) of the polygonal domain \( \Omega \) with mesh-size \( h \), where \( \mathcal{T}_h \) consists of triangles. Now we introduce the standard linear finite element space \( K_h \subset H^1(\Omega) \) defined on the triangulation \( \mathcal{T}_h \)

\[
K_h := \{ v \in H^1(\Omega) : v|_T \in \mathcal{P}_1(T), \ T \in \mathcal{T}_h \},
\]

and the space of bubble functions

\[
B_h := \{ b_T \in \mathcal{P}_3(T) : b_T|_T = 0, \ \text{and} \ \int_T b_T \, dx > 0, \ T \in \mathcal{T}_h \},
\]

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where $\mathcal{P}_n(T)$ is the space of polynomials of degree $n$ in $T$ for $n \in \mathbb{N}$. The bubble function on an element $T$ can be defined as $b_T(x) = c_b \prod_{i=1}^3 \lambda_{T_i}(x)$, where $\lambda_{T_i}(x)$ are the barycentric coordinates of the element $T$ associated with vertices $x_{T_i}$ of $T$, $i = 1, \cdots, 3$, and the constant $c_b$ is chosen in such a way that the value of $b_T$ at the barycenter of $T$ is one.

Let $S_h = H_0^1(\Omega) \cap K_h$. A finite element method for the simply supported Reissner–Mindlin plate is proposed in [2] using the finite element spaces $W_h := S_h \oplus B_h$ to discretize the transverse displacement, $V_h := [S_h]^2$ to discretize the rotation, and $M_h := [K_h]^2$ to discretize the Lagrange multiplier space. This is the lowest order case for the transverse displacement and rotation using the continuous piecewise linear shear approximation in [2]. Hence the discretization uses equal order interpolation for the rotation and the transversal displacement, and is one of the simplest finite element methods. However, for the clamped boundary condition we need to have $V_h \subset H_0^1(\Omega)$, and hence the stability condition is violated if we use $M_h = [K_h]^2$ to discretize the Lagrange multiplier space, and if we use $M_h = [S_h]^2$ to discretize the Lagrange multiplier space, the approximation property of the scheme is lost as the the Lagrange multiplier $\zeta$ is not assumed to satisfy the zero boundary condition. Indeed, the discrete space $M_h \subset L^2(\Omega)$ for the Lagrange multiplier space should have the following approximation property

$$\inf_{\mu_h \in M_h} \| \phi - \mu_h \|_{L^2(\Omega)} \leq C h |\phi|_{1,\Omega}, \quad \phi \in H^1(\Omega).$$

Our goal in this paper is to introduce two discrete spaces for the Lagrange multiplier space so that the resulting scheme is stable and has the optimal approximation property for the clamped plate. We also introduce a scheme where the Lagrange multiplier can be statically condensed out from the system leading to a positive definite formulation.

We now start with finite element spaces for the transverse displacement $u$ and the rotation $\phi$ as $W_h := S_h \oplus B_h$ and $V_h := [S_h]^2$, respectively. Let

$$\{ \varphi_1, \varphi_2, \cdots, \varphi_m, \varphi_{m+1}, \cdots, \varphi_n \}$$

be the standard finite element basis for $K_h$, where $n > m$ and $\{ \varphi_1, \varphi_2, \cdots, \varphi_m \}$ is a basis of $S_h$. Note that the basis functions $\{ \varphi_{m+1}, \cdots, \varphi_n \}$ are associated with the boundary. We use the idea of boundary modification of Lagrange multiplier spaces in mortar finite element methods [6, 14, 15] to construct a discrete Lagrange multiplier space for Reissner–Mindlin plate equations.

Let $M_h \subset L^2(\Omega)$ be a piecewise polynomial space with respect to the mesh $\mathcal{T}_h$ to be specified later which satisfies the following assumptions:
Assumption 1.  

(i) \( \dim M_h = \dim S_h \).

(ii) There is a constant \( \beta > 0 \) independent of the triangulation \( T_h \) such that

\[
\| \phi_h \|_{L^2(\Omega)} \leq \beta \sup_{\mu_h \in M_h \setminus \{0\}} \frac{\int_\Omega \mu_h \phi_h \, dx}{\| \mu_h \|_{L^2(\Omega)}}, \quad \phi_h \in S_h. \quad (3)
\]

(iii) The space \( M_h \) has the approximation property:

\[
\inf_{\mu_h \in M_h} \| \mu - \mu_h \|_{L^2(\Omega)} \leq C h |\mu|_{H^1(\Omega)}, \quad \mu \in H^1(\Omega). \quad (4)
\]

(iv) There exist two bounded linear projectors \( Q_h : H^1_0(\Omega) \to V_h \) and \( R_h : H^1_0(\Omega) \to W_h \) for which

\[
b(Q_h \psi, R_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in [M_h]^2.
\]

If these assumptions are satisfied, we obtain an optimal error estimate for the finite element approximation, see [2]. Then the discrete space for the Lagrange multiplier space is defined as

\[
M_h = [M_h]^2 \subset L^2(\Omega),
\]

and the discrete saddle point formulation is to find \((\phi_h, u_h, \zeta_h) \in V_h \times W_h \times M_h\) such that

\[
a(\phi_h, u_h; \psi_h, v_h) + b(\psi_h, v_h; \zeta_h) = (g, v_h), \quad (\psi_h, v_h) \in V_h \times S_h,
\]

\[
b(\phi_h, u_h; \eta_h) - \frac{t^2}{\lambda(1-t^2)}(\zeta_h, \eta_h) = 0, \quad \eta_h \in M_h. \quad (5)
\]

We now show two examples of discrete Lagrange multiplier spaces satisfying above properties. The first example is based on the standard Lagrange multiplier space for three-dimensional mortar finite elements proposed in [9], and the second example is based on a dual Lagrange multiplier space proposed in [14]. As the dual Lagrange multiplier space satisfies a biorthogonality relation with the finite element space \( S_h \) leading to a diagonal Gram matrix, it allows an efficient solution technique. In fact, the Lagrange multiplier can be statically condensed out from the global system leading to a reduced linear system in this case. This reduced linear system can be solved more efficiently than the global saddle point system. The Lagrange multiplier can easily be recovered just by inverting a diagonal matrix. One
important factor in the construction of a Lagrange multiplier space for the *clamped* boundary condition case is the boundary modification so that Assumptions 1(i)–(iii) are satisfied. In order to satisfy Assumption 1(iii), the discrete Lagrange multiplier space should contain constants in $\Omega$. Therefore, it is not possible to take $M_h = S_h$. Here we follow closely [9] for the construction and boundary modification of the discrete Lagrange multiplier space.

### 3.1 Standard Lagrange multiplier space $M^1_h$

In the following, we assume that each triangle has at least one interior vertex. A necessary modification for the case where a triangle has all its vertices on the boundary is given in [9].

Let $\mathcal{N}$, $\mathcal{N}_0$ and $\partial \mathcal{N}$ be the set of all vertices of $\mathcal{T}_h$, the vertices of $\mathcal{T}_h$ interior to $\Omega$, and the vertices of $\mathcal{T}_h$ on the boundary of $\Omega$, respectively. We define the set of all vertices which share a common edge with the vertex $i \in \mathcal{N}$ as

$$ S_i = \{j : i \text{ and } j \text{ share a common edge}\}, $$

and the set of neighbouring vertices of $i \in \mathcal{N}_0$ as

$$ I_i = \{j \in \mathcal{N}_0 : j \in S_i\}. $$

Then the set of all those interior vertices which have a neighbour on the boundary of $\Omega$ is defined as

$$ \mathcal{I} = \bigcup_{i \in \partial \mathcal{N}} I_i, \quad \text{See Figure 3.1} $$

The finite element basis functions $\{\phi_1, \phi_2, \ldots, \phi_m\}$ for $M^1_h$ are defined as

$$ \phi_i = \begin{cases} \varphi_i, & i \in \mathcal{N}_0 \setminus \mathcal{I} \\ \varphi_i + \sum_{j \in \partial \mathcal{N} \cap S_i} A_{j,i} \varphi_j, & A_{j,i} \geq 0, & i \in \mathcal{I} \end{cases}. $$

We can immediately see that all the basis functions of $M^1_h$ are continuous, and $\dim M^1_h = \dim S_h$. Moreover, if the coefficients $A_{i,j}$ are chosen to satisfy

$$ \sum_{j \in S_i} A_{i,j} = 1, \quad i \in \mathcal{I}, $$

Assumptions 1(ii) and 1(iii) are also satisfied, see [9] for a proof. The vector Lagrange multiplier space is defined as $M^1_h = [M^1_h]^2$. 

Lemma 1. There exist two bounded linear projectors $Q_h : H^1_0(\Omega) \to V_h$ and $R_h : H^1_0(\Omega) \to W_h$ for which

$$b(Q_h \psi, R_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M^1_h.$$  

Proof. We first define two operators $Z_h : H^1_0(\Omega) \to B_h$ and $Q_h : H^1_0(\Omega) \to S_h$ as

$$\int_T (v - Z_h v) \, dx = 0, \quad T \in \mathcal{T}_h,$$

and

$$\int_\Omega (v - Q_h v) \, \eta_h \, dx = 0, \quad \eta_h \in M^1_h,$$

respectively. The first operator $Z_h$ is well-defined as we can have a bubble function $b_T \in B_h$ with

$$\int_T b_T \, dx = \int_T v \, dx.$$  

The second operator $Q_h$ is well-defined and stable in $L^2$ and $H^1$ -norms due to Assumption 1(i)-(ii), see, e.g., [9, 14]. Let $P_h$ be the $L^2$-orthogonal projection onto $S_h$. Now we define the operator $R_h : H^1_0(\Omega) \to W_h$ with

$$R_h v = P_h v + Z_h(v - P_h v). \quad (6)$$

Note that $R_h$ is a bounded linear projector onto $W_h = S_h \oplus B_h$ with the property [3, 2, 8]

$$\int_T (R_h v - v) \, dx = 0, \quad T \in \mathcal{T}_h.$$
Let $Q_h : \mathbb{H}^1_0(\Omega) \rightarrow \mathbf{V}_h$ be the vector version of $Q_h$. That means $Q_h u = (Q_h u_1, Q_h u_2)$ for $u = (u_1, u_2) \in \mathbb{H}^1_0(\Omega)$. Then

$$b(Q_h \psi, R_h v; \eta_h) = \int_{\Omega} (Q_h \psi - \nabla R_h v) \cdot \eta_h \, dx$$

$$= \int_{\Omega} Q_h \psi \cdot \eta_h \, dx - \int_{\Omega} \nabla R_h v \cdot \eta_h \, dx$$

$$= \int_{\Omega} \psi \cdot \eta_h \, dx + \int_{\Omega} R_h v \nabla \cdot \eta_h \, dx,$$

where we use the divergence theorem and the fact that $\eta_h$ is continuous. Since $\eta_h$ is a continuous function and $\nabla \cdot \eta_h$ is a piecewise constant with respect to the mesh $T_h$, we have

$$b(Q_h \psi, R_h v; \eta_h) = \int_{\Omega} \psi \cdot \eta_h \, dx + \int_{\Omega} v \nabla \cdot \eta_h \, dx$$

$$= \int_{\Omega} (\psi - \nabla v) \cdot \eta_h \, dx = b(\psi, v; \eta_h).$$

The boundedness of $R_h$ in $H^1$-norm can be shown as in [3].

Thus we have the following theorem from the theory of saddle point problem [10, 8].

**Theorem 2.** There exists a constant $C$ independent of $t$ and $h$ such that

$$\| \phi - \phi_h \|_{H^1(\Omega)} + \| u - u_h \|_{H^1(\Omega)} + \| | \zeta - \zeta_h | \|_t \leq C \left( \inf_{\psi_h \in \mathbf{V}_h} \| \phi - \psi_h \|_{H^1(\Omega)} + \inf_{v_h \in \mathbf{W}_h} \| u - v_h \|_{H^1(\Omega)} + \inf_{\eta_h \in \mathbf{M}^1_h} \| | \zeta - \eta_h | \|_t \right),$$

where the norm $\| \cdot \|_t$ is defined as

$$\| | \zeta | \|_t = \| \zeta \|_{H^{-1}(\Omega)} + \| \nabla \cdot \zeta \|_{H^{-1}(\Omega)} + t \| \zeta \|_{L^2(\Omega)}.$$

Moreover, if $\phi \in \mathbb{H}^2(\Omega)$, $u \in H^2(\Omega)$ and $\zeta \in \mathbb{H}^1(\Omega)$, the approximation properties of $\mathbf{V}_h$, $\mathbf{W}_h$ and $\mathbf{M}^1_h$ imply that

$$\| \phi - \phi_h \|_{H^1(\Omega)} + \| u - u_h \|_{H^1(\Omega)} + \| | \zeta - \zeta_h | \|_t \leq Ch \left( \| \phi \|_{H^2(\Omega)} + \| u \|_{H^2(\Omega)} + \| \zeta \|_{H^1(\Omega)} \right).$$
3.2 Dual Lagrange multiplier space $M^2_h$

Another possibility of a discrete Lagrange multiplier space is to consider the dual Lagrange multiplier space proposed for mortar finite elements in [14, 19]. Interestingly, the boundary modification can be exactly done as in the case of standard Lagrange multiplier space. We start with the basis for the dual Lagrange multiplier space $\tilde{M}_h$ including the degree of freedom on the boundary of $\Omega$. Let $\{\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m, \tilde{\mu}_{m+1}, \ldots, \tilde{\mu}_n\}$ be the basis for $\tilde{M}_h$, which is biorthogonal to the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_m, \varphi_{m+1}, \ldots, \varphi_n\}$ of $K_h$ so that these basis functions satisfy the biorthogonality relation

$$\int_\Omega \tilde{\mu}_i \varphi_j \, dx = c_j \delta_{ij}, \quad c_j \neq 0, \quad 1 \leq i, j \leq n, \quad (7)$$

where $n := \dim \tilde{M}_h = \dim K_h$, $\delta_{ij}$ is the Kronecker symbol, and $c_j$ a scaling factor. In fact, we can construct local basis functions for $\tilde{M}_h$ on the reference triangle $\hat{T}$ so that the global basis functions for $\tilde{M}_h$ are constructed by gluing these local basis functions together. This means that we can use a standard assembling routine for the functions in $\tilde{M}_h$, and the basis functions of $\tilde{M}_h$ are also associated with the finite element vertices as the basis functions of $K_h$. For the reference triangle $\hat{T} := \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}$, we have

$$\hat{\mu}_1 := 3 - 4x - 4y, \quad \hat{\mu}_2 := 4x - 1, \quad \text{and} \quad \hat{\mu}_3 := 4y - 1.$$

The finite element basis functions $\{\mu_1, \mu_2, \ldots, \mu_m\}$ for $M^2_h$ are defined as

$$\mu_i = \begin{cases} \hat{\mu}_i, & i \in \mathcal{N}_0 \backslash \mathcal{I} \\ \hat{\mu}_i + \sum_{j \in \partial \mathcal{N} \cap \mathcal{S}_i} A_{i,j} \hat{\mu}_j, & A_{i,j} \geq 0, \quad i \in \mathcal{I}. \end{cases}$$

If the coefficients $A_{i,j}$ are chosen to satisfy

$$\sum_{j \in \mathcal{S}_i} A_{i,j} = 1, \quad i \in \mathcal{I},$$

Assumptions 1(ii) and 1(iii) are also satisfied exactly as in the case of the standard Lagrange multiplier space. The proof of Assumption 1(ii) is much easier than in [9] due to the biorthogonality relation. In fact, if we set
\[ \phi_h = \sum_{k=1}^n a_k \phi_k \in S_h \quad \text{and} \quad \mu_h = \sum_{k=1}^n a_k \mu_k \in M_h^2, \]

the biorthogonality relation (7) and the quasi-uniformity assumption imply that

\[ \int_{\Omega} \varphi_h \mu_h \, dx = \sum_{i,j=1}^n a_i a_j \int_{\Omega} \varphi_i \mu_j \, dx = \sum_{i=1}^n a_i^2 c_i \geq C \sum_{i=1}^n a_i^2 h_i^d \geq C \| \varphi_h \|_{L^2(\Omega)}^2, \]

where \( h_i \) denotes the mesh-size at the \( i \)th vertex. Taking into account the fact that \( \| \varphi_h \|_{L^2(\Omega)}^2 \equiv \| \mu_h \|_{L^2(\Omega)}^2 \equiv \sum_{i=1}^n a_i^2 h_i^d \), we find that Assumption 1(ii) is satisfied. Since the sum of the local basis functions of \( M_h^2 \) is one, Assumption 1(iii) can be proved as in [14].

The vector Lagrange multiplier space is defined as before

\[ M_h^2 = \left[ M_h^2 \right]^2. \]

Although the condition \( \dim M_h^2 = \dim S_h \) is satisfied as before, the basis functions for \( M_h^2 \) are not continuous, and we cannot show the existence of a bounded linear operator \( R_h \) as in Lemma 1. We need to use an alternative method. As before we need to prove the following theorem to show the well-posedness of the discrete problem:

**Theorem 3.** There exist two bounded linear projectors \( Q_h : H^1_0(\Omega) \to V_h \) and \( R_h : H^1_0(\Omega) \to W_h \) for which

\[ b(Q_h \psi, R_h v; \eta_h) = b(\psi, v; \eta_h), \quad \eta_h \in M_h^2. \]

We define the projector \( Q_h : H^1_0(\Omega) \to S_h \) as

\[ \int_{\Omega} (v - Q_h v) \eta_h \, dx = 0, \quad \eta_h \in M_h^2. \]

Here \( Q_h \) is well-defined and bounded in \( L^2 \) and \( H^1 \)- norms due to Assumption 1(i)-(ii), see [15]. Our task is now to show the existence of the operator \( R_h : H^1_0(\Omega) \to W_h \) satisfying

\[ \int_{\Omega} \nabla R_h v \cdot \eta_h \, dx = \int_{\Omega} \nabla v \cdot \eta_h \, dx, \quad \eta_h \in M_h^2. \]

In order to show the existence of this operator, we use the following result proved in [16]:

**Lemma 4.** Let \( W_h = [W_h]^2 \), and

\[ \tilde{M}_h^0 = \left\{ \mu_h = \sum_{i=1}^n a_i \tilde{\mu}_i, \int_{\Omega} \mu_h \, dx = 0 \right\}. \]

Then there exists a constant \( \tilde{\beta} > 0 \) independent of mesh-size \( h \) such that

\[ \sup_{u_h \in W_h} \frac{\int_{\Omega} \nabla \cdot u_h \mu_h \, dx}{\| u_h \|_{H^1(\Omega)}} \geq \tilde{\beta} \| \mu_h \|_{L^2(\Omega)}, \quad \mu_h \in \tilde{M}_h^0. \]
Let
\[ L^2_0(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} u \, dx = 0 \right\}. \]

Using Lemma 4 and the fact that the two spaces \( H^1_0(\Omega) \) and \( L^2_0(\Omega) \) satisfy the inf-sup condition
\[ \sup_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} \nabla \cdot u \, \mu \, dx}{\|u\|_{H^1(\Omega)}}, \geq \beta \|\mu\|_{L^2(\Omega)} \]
we can show the existence of a bounded linear projector \( \Pi_h : H^1_0(\Omega) \rightarrow W_h \) as in [10, 8] such that
\[ \int_{\Omega} \nabla \cdot \Pi_h u \, \mu_h \, dx = \int_{\Omega} \nabla \cdot u \, \mu_h \, dx, \quad \mu_h \in \tilde{M}^0_h. \]

Let \( \Pi_h : H^1_0(\Omega) \rightarrow W_h \) be the scalar version of \( \Pi_h \) meaning that \( \Pi_h u = (\Pi_h u_1, \Pi_h u_2) \) for the vector \( u = (u_1, u_2) \in H^1_0(\Omega) \). Since \( \Pi_h \) is bounded in \( H^1 \)-norm, \( \Pi_h \) is also bounded in \( H^1 \)-norm. Since \( \Pi_h u \) and \( u \) both satisfy homogeneous boundary condition, we even have
\[ \int_{\Omega} \nabla \cdot \Pi_h u \, \mu_h \, dx = \int_{\Omega} \nabla \cdot u \, \mu_h \, dx, \quad \mu_h \in \tilde{M}^0_h. \]

If we use the function \( u = (v, 0)^T \in H^1_0(\Omega) \) in the above equation, we get
\[ \int_{\Omega} \left( \frac{\partial \Pi_h v}{\partial x} - \frac{\partial v}{\partial x} \right) \mu_h \, dx = 0, \quad \mu_h \in \tilde{M}_h, \]
and similarly for \( v \in H^1_0(\Omega) \), we have
\[ \int_{\Omega} \left( \frac{\partial \Pi_h v}{\partial y} - \frac{\partial v}{\partial y} \right) \mu_h \, dx = 0, \quad \mu_h \in \tilde{M}_h. \]

Since \( M_h^2 \subset \tilde{M}_h \), we have the following result.

**Lemma 5.** There exists a bounded linear projector \( \Pi_h : H^1_0(\Omega) \rightarrow W_h \) such that for \( v \in H^1_0(\Omega) \), we have
\[ \int_{\Omega} \left( \frac{\partial \Pi_h v}{\partial x} - \frac{\partial v}{\partial x} \right) \mu_h \, dx = 0, \quad \mu_h \in M_h^2, \]
and
\[ \int_{\Omega} \left( \frac{\partial \Pi_h v}{\partial y} - \frac{\partial v}{\partial y} \right) \mu_h \, dx = 0, \quad \mu_h \in M_h^2. \]
Theorem 6. The interpolation operator $\Pi_h$ defined in Lemma 5 satisfies

$$\int_{\Omega} \nabla \Pi_h v \cdot \eta_h \, dx = \int_{\Omega} \nabla v \cdot \eta_h \, dx, \quad \eta_h \in M^2_h.$$  

Proof. Let $\eta_h = (\mu_h, \eta_h)^T \in M^2_h$, where $\mu_h, \eta_h \in M^2_h$. Then we need to satisfy

$$\int_{\Omega} \frac{\partial \Pi_h v}{\partial x} \mu_h \, dx + \int_{\Omega} \frac{\partial \Pi_h v}{\partial y} \eta_h \, dx = \int_{\Omega} \frac{\partial v}{\partial x} \mu_h \, dx + \int_{\Omega} \frac{\partial v}{\partial y} \eta_h \, dx,$$

which results in adding two equations of Lemma 5.

Thus Theorem 3 is proved with $R_h$ replaced by $\Pi_h$. Hence we get the same approximation as in Theorem 2.

Theorem 7. There exists a constant $c$ independent of $t$ and $h$ such that

$$\|\phi - \phi_h\|_{H^1(\Omega)} + \|u - u_h\|_{H^1(\Omega)} + ||\zeta - \zeta_h||_t \leq c \left( \inf_{\psi_h \in V_h} \|\phi - \psi_h\|_{H^1(\Omega)} + \inf_{v_h \in W_h} \|u - v_h\|_{H^1(\Omega)} + \inf_{\eta_h \in M^2_h} ||\zeta - \eta_h||_t \right).$$

Moreover, if $\phi \in H^2(\Omega)$, $u \in H^2(\Omega)$ and $\zeta \in H^1(\Omega)$, the approximation properties of $V_h$, $W_h$ and $M^2_h$ imply that

$$\|\phi - \phi_h\|_{H^1(\Omega)} + \|u - u_h\|_{H^1(\Omega)} + ||\zeta - \zeta_h||_t \leq Ch \left( \|\phi\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)} + \|\zeta\|_{H^1(\Omega)} \right).$$

3.3 Algebraic formulation

We want to write the algebraic system of the discrete formulation. In the following, we use the same notation for the vector representation of the solutions and the solutions as elements in $V_h$, $W_h$ and $M_h$. The algebraic formulation of the saddle point problem can be written as

$$\begin{bmatrix}
A & \lambda B^T & D \\
\lambda B & K & B \\
D & B^T & R
\end{bmatrix}
\begin{bmatrix}
\phi_h \\
u_h \\
\zeta_h
\end{bmatrix} =
\begin{bmatrix}
0 \\
l_h \\
0
\end{bmatrix},$$

where $A$, $B$, $K$, $D$ and $R$ are suitable matrices arising from the discretization of different bilinear forms, and $l_h$ is the vector form of discretization of the
linear form $\ell(\cdot)$. Note that $D$ is the Gram matrix between basis functions of $M_h$ and $V_h$. This matrix is a square matrix under Assumption \(1(i)\).

We now briefly discuss the advantage of using the dual Lagrange multiplier space. Working with the dual Lagrange multiplier space $D$ will be a diagonal matrix due to the biorthogonality relation between the bases of $M_h^2$ and $V_h$. The first equation of the algebraic system gives

$$A\phi_h + B^T u_h + D\zeta_h = 0.$$ 

This equation can be solved for $\zeta_h$ as

$$\zeta_h = -D^{-1} \left( A\phi_h + B^T u_h \right).$$

Thus we can statically condense out the Lagrange multiplier $\zeta_h$ from the saddle point system. This leads to a reduced and positive definite system. Hence an efficient solution technique can be applied to solve the arising linear system.

4 Conclusion

We have combined the idea of constructing discrete Lagrange multiplier spaces in mortar finite element techniques to construct discrete Lagrange multiplier spaces for the Reissner–Mindlin plate equations. Working with a dual Lagrange multiplier space results in a very efficient finite element method.

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