Explicit Calabi-Yau metrics are derived that are argued to map to the Maldacena-Nuñez $AdS$ solutions of M-theory and IIB under geometric transitions. In each case the metrics are singular where a $H^2$ Kähler two-cycle degenerates but are otherwise smooth. They are derived as the most general Calabi-Yau solutions of an ansatz for the supergravity description of branes wrapped on Kähler two-cycles. The ansatz is inspired by re-writing the $AdS$ solutions, and the structure defined by half their Killing spinors, in this form. The world-volume theories of fractional branes wrapped at the singularities of these metrics are proposed as the duals of the $AdS$ solutions. The existence of supergravity solutions interpolating between the $AdS$ and Calabi-Yau metrics is conjectured and their boundary conditions briefly discussed.
1 Introduction and main idea

The AdS/CFT correspondence [1] is best understood for D3 branes at the apex of a Calabi-Yau cone. There are two ways in which we know how to think about this system. One is in terms of open string theory and probe $D3$ branes on the singular Calabi-Yau; at low energies, one gets a four-dimensional conformal field theory, at weak ’t Hooft coupling, on the brane worldvolume. The other is in terms of closed string theory on the product of $AdS_5$ with a Sasaki-Einstein manifold; by the AdS/CFT correspondence, this is the same as the CFT at strong ’t Hooft coupling. The classical link between the two geometries is a smooth supergravity solution, preserving half their supersymmetries, that interpolates between them; the Calabi-Yau singularity is excised and replaced with an $AdS$ horizon at infinite proper distance. In this sense the branes are said to induce a geometric transition: they resolve (rather, remove to infinity) the singularity of the Calabi-Yau manifold. The geometrical data of both the Calabi-Yau and the Sasaki-Einstein manifold are encoded in the CFT (at weak and strong coupling, respectively), so interpolating the ’t Hooft coupling in the CFT gives a quantum definition of the geometric transition. The dictionary - encoding and decoding Calabi-Yau and Sasaki-Einstein data in the CFT at weak and strong coupling, respectively - has been worked out in detail in beautiful work for $R^6$, the conifold, and the $Y^{p,q}$ metrics [2]-[10].

Since the work of Maldacena and Nuñez [11], we know that there are many other ways in which Anti-de Sitter geometries can be related to special holonomy manifolds and conformal quantum theories. In [11], three $AdS$ solutions of M- and string theory were constructed: two $AdS_5$ solutions in eleven dimensions, with respectively sixteen and eight Killing spinors, and an $AdS_3$ solution admitting eight Killing spinors in IIB\(^1\). These were interpreted as arising, in the near-horizon limit, from branes wrapping $H^2$ Kähler two-cycles in, respectively, Calabi-Yau two-, three-, and three-folds. The dual conformal field theories are $\mathcal{N} = 2$ and $\mathcal{N} = 1$ in four dimensions, and $\mathcal{N} = (2, 2)$ in three dimensions. Since this work, it has been found that there exist $AdS$ solutions associated to all types of calibrated cycles in all types of special holonomy manifold of dimension ten or less; for example, [12]-[17]. The CFTs dual to $AdS$ manifolds of this type define quantum gravity theories for calibrated geometries. In line with the intuition gained from branes at conical singularities, one would expect that the CFTs could be realised, at weak coupling, as the world-volume theories of fractional probe branes, wrapped on degenerating calibrated cycles in singular special holonomy manifolds. Such a system is likewise expected to undergo a geometric transition, with the singularity excised and replaced with an $AdS$ region. Classically, there should be a supergravity solution interpolating between the Calabi-Yau and $AdS$ geometries.

Our understanding of AdS/CFT for wrapped branes is much more rudimentary than for branes at conical singularities. Chief among the obstacles has been the inability to move beyond the near-\(^1\)These solutions will be denoted by MN(I), MN(II) and MN(III) respectively.
horizon limit; typically, only the $AdS$ geometries are known. The lesson from branes on cones is that in order to get a real handle on field theory *dynamics* - to write down the particle content and superpotential for a dual of a specific $AdS$ solution - the associated Calabi-Yau geometry must be known. The main point of this paper is to give a way of associating a special holonomy metric to an $AdS$ metric, illustrated for the Maldacena-Nunez solutions. The main assumption of this paper is the existence of a supersymmetric supergravity solution interpolating between a special holonomy manifold and an $AdS$ spacetime when there exists an AdS/CFT dual. Roughly, an interpolating solution should be a metric and a flux admitting two distinct limits in which the supersymmetry doubles, with the metric becoming Calabi-Yau in one limit and $AdS$ in the other. More formally, we can think of the metric and flux of an interpolating solution as providing a smooth and smoothly-invertible map $f$

$$f : \text{Special Holonomy} \rightarrow AdS.$$ (1.1)

We take this as a definition of what is meant in this paper by a geometric transition. It is a purely classical definition; in more physical terms, such a map gives the full supergravity description of a wrapped brane. But if a CFT dual can be identified, the map can be promoted to the quantum level; the CFT itself provides the map, with the ’t Hooft coupling the interpolating parameter.

The equations that interpolating solutions should satisfy are known, through various symmetry arguments. An important property of these solutions is that they should admit a global reduction of their frame bundle, to a sub-bundle of the appropriate structure [18], [19]. For example, in the supergravity description of M5-branes wrapped on Kähler two-cycles in Calabi-Yau two-folds - maps $f : CY_2 \rightarrow AdS_5$ - the global structure of an interpolating solution is $SU(2)$. The structure is defined by eight Killing spinors, or alternatively, an almost complex structure $J$ and a $(2,0)$ form $\Omega$. The truncation of eleven-dimensional supergravity to this frame bundle was first worked out by Fayyazuddin and Smith [20] (see also [21], [18]). The metric and flux are

$$ds^2 = L^{-1} ds^2(\mathbb{R}^{1,3}) + ds^2(\mathcal{M}_4) + L^2[dt^2 + t^2 ds^2(S^2)],$$

$$\star_7 F = L^2 d(L^{-2} J).$$ (1.2)

Here and throughout we follow all conventions and orientations of [18]. The Minkowski isometries are isometries of the full solution, and $\mathcal{M}_4$ admits a globally-defined $SU(2)$ structure. The structure is constrained by the Fayyazuddin-Smith equations:

$$d(L^{-1/2} \Omega) = 0,$$

$$dt \wedge \text{Vol}[S^2] \wedge d(LJ) = 0.$$ (1.3)

Eleven-dimensional supergravity, in this truncation, reduces to the torsion conditions (1.3) and the four-form Bianchi identity.
To the knowledge of the author, no interpolating solutions of these equations, or their analogues in other contexts, are known. However in recent work [18], [22], [19], it has been shown how the supersymmetry conditions for general classes of supersymmetric AdS solutions of M-theory (including all known examples) can be derived from such equations. In particular, in [18] it was shown that the conditions of Lin, Lunin and Maldacena [23] for half-BPS AdS_5 solutions can be derived from the Fayyazuddin-Smith equations. It follows that any solution of the LLM conditions can be re-written as a solution of the Fayyazuddin-Smith equations; and similarly for every other AdS solution covered by [18], [22], [19]. Applying this procedure to the MN(I) solution, we will see in the next section that it may be re-written in the form

\[ ds^2 = L^{-1} \left[ ds^2(\mathbb{R}^{1,3}) + \frac{F}{2} ds^2(H^2) \right] + L^2 \left[ F^{-1} \left( du^2 + u^2 (d\psi - P)^2 \right) + dt^2 + t^2 ds^2(S^2) \right], \]
\[ dP = \text{Vol}[H^2], \] (1.4)

for particular determined functions \( F(u, t), L(u, t) \) and a particular choice of frame which will be discussed in detail. We use this form of the AdS solution as a guide to what the inverse geometric transition \( f^{-1} : \text{MN}(I) \to \text{CY}_2 \) should be. Clearly, it should respect the topological structure of MN(I); the simplest choice, which we make, is that \( f \) is given by a solution \( F(u, t), L(u, t) \), of the Fayyazuddin-Smith equations. With this metric and the frame of section 2, they reduce to

\[ \frac{1}{t^2} \partial_t \left( t^2 \partial_t F \right) = -u \partial_u \left( \frac{F}{u} \partial_u F \right), \]
\[ L^3 = -\frac{1}{4u} \partial_u (F^2). \] (1.5)

An interpolating solution of these equations has not been found. However, assuming one exists, the general Calabi-Yau solution of (1.5) is the image of MN(I) under \( f^{-1} \). Up to an overall scale, the general Calabi-Yau solution is \( L = 1 \) and

\[ ds_4^2 = \frac{dR^2}{\left( \frac{1}{R^4} - 1 \right)} + \frac{R^2}{4} \left[ ds^2(H^2) + \left( \frac{1}{R^4} - 1 \right) (d\psi - P)^2 \right]. \] (1.6)

The range of \( R \) is \([-1, 0) \) or \((0, 1]\). As expected, the metric is singular, where the Kähler two-cycle \( H^2 \) degenerates. The singularity, at \( R = 0 \), is at finite proper distance. The metric is non-singular at the \( H^2 \) bolt as \( R^4 \to 1 \), if \( \psi \) has period \( 2\pi \); we will see in the next section that this is precisely the periodicity that is inherited through \( f^{-1} \) from MN(I). Some additional evidence that this Calabi-Yau is a sensible candidate comes from the following. Every AdS_5 solution of the LLM conditions, including MN(I), is completely determined by a solution of the three-dimensional continuous Toda equation. There also exists a class of Calabi-Yau two-folds that is completely determined by a solution of the three-dimensional continuous Toda equation. This is such a Calabi-Yau metric, and furthermore it is given by the same solution of the Toda equation as MN(I). Toda-Calabi-Yau...
metrics have been obtained in this context before as scaling limits of the 1/2-BPS \( AdS_5 \) metrics \[23], \[24]. Here this metric is obtained in a different way, as a solution of the 1/4-BPS Fayyazuddin-Smith equations. It will be interesting to see how these procedures are related.

The world-volume theory of fractional M5-branes wrapped at the singularity of this metric (whatever it might be) is proposed as the quantum dual of MN(I). Though the geometry is non-compact, this is not necessarily problematic, as the field theory should only encode oscillations in the directions transverse to the brane, purely in the fibre; and the fibre has finite proper volume. The cycle may in any event be rendered compact by taking a freely-acting quotient by a discrete subgroup of its isometry group. The Calabi-Yau will still be noncompact, because of the singularity.

In a similar vein, we obtain candidate Calabi-Yaus for inverse geometric transitions from MN(II) and MN(III). For MN(II), to be discussed in detail in section 3, the first step is to use the results of \[4], \[18] to write it in the form

\[
\begin{align*}
\frac{L}{4} &= L_1^2 \left[ \frac{d^2 s^2(\mathbb{R}^{1,3})}{1 + \sin^2 \xi} + \frac{F_1 F_2}{3} d^2 (H^2) \right] \\
&+ L_2^2 \left[ F_1^{-1} \left( \frac{u^2}{4} (d \psi + P - P')^2 \right) + F_2^{-1} \frac{u^2}{4} d^2 (S^2) + dt^2 \right].
\end{align*}
\] (1.7)

with \( dP = \text{Vol}[S^2] \) and \( dP' = \text{Vol}[H^2] \). Then, letting \( L, F_1, F_2 \) be arbitrary functions of \( u, t \), the general Calabi-Yau three-fold solution\(^2\) is, up to an overall scale,

\[
\begin{align*}
\frac{L}{4} &= \frac{1}{2} (1 + \sin \xi) d^2 (H^2) + \frac{\cos^2 \xi}{2 (1 + \sin \xi)} d^2 (S^2) + \frac{1}{\cos^2 \xi} \left( dR^2 + R^2 (d \psi + P - P')^2 \right),
\end{align*}
\] (1.8)

where \( \sin \xi \) is a root of the cubic equation

\[
-1 \sin^3 \xi + \sin \xi = \frac{2}{3} - R^2.
\] (1.9)

The metric is singular, as expected, at \( \xi = -\pi/2, R = 2/\sqrt{3} \), where the \( H^2 \) cycle degenerates. The metric is smooth at \( \xi = \pi/2 \), which coincides with \( R = 0 \); there an \( S^3 \) smoothly degenerates, provided that \( \psi \) has the \( 4\pi \) periodicity it inherits from MN(II) under \( f^{-1} \). The quantum dual of MN(II) is proposed to be the worldvolume theory of fractional M5s wrapped at the singularity of this metric.

The MN(III) solution comes from D3 branes wrapped on a Kähler \( H^2 \) cycle in a Calabi-Yau three-fold; the geometric transition is \( CY_3 \rightarrow AdS_3 \) in IIB. The discussion in this case proceeds along very much the same lines as for MN(II), and will be reported in detail elsewhere [25]. The image of MN(III) under the inverse transition is again the Calabi-Yau (1.8). The dual field theory is proposed to be the world-volume theory of D3 branes wrapped at the singularity. It seems that

\(^2\)With suitable regularity properties, to be discussed in detail in section 3.
both M5 branes and D3 branes can probe the singularity of this manifold; the quantum descriptions are respectively four- and two-dimensional conformal theories. In the IIB description, given the metric, it might be possible to construct the field theory with existing techniques.

The remainder of this paper is organised as follows. Section 2 is devoted to MN(I) and its Calabi-Yau image. Section 3 repeats the analysis of section 2 for MN(II). Section four contains conclusions and outlook, and also some discussion of the boundary conditions for interpolating solutions for the MN/Calabi-Yau pairs.

2 The $\mathcal{N} = 2$ M-theory solution

To begin, we will review the $\mathcal{N} = 2$ $AdS_5$ geometry of [11] in some detail. The metric is given by

$$\begin{aligned}
ds^2 &= \frac{1}{\lambda} \left[ ds^2(AdS_5) + \frac{1}{2} ds^2(H^2) + (1 - \lambda^3 \rho^2)(d\psi - P)^2 + \frac{\lambda^3}{4} \left( \frac{d\rho^2}{1 - \lambda^3 \rho^2} + \rho^2 ds^2(S^2) \right) \right],
\end{aligned}$$

(2.1)

where

$$\begin{aligned}
\lambda^3 &= \frac{8}{1 + 4\rho^2}; \\
dP &= \text{Vol}[H^2].
\end{aligned}$$

Here and throughout we denote by $ds^2(M)$ the metric of unit radius of curvature on $\mathcal{M}$. The range of the coordinate $\rho$ is either $\rho \in [-1/2, 0]$ or $\rho \in [0, 1/2]$. At $\rho = 0$, in either branch, the R-symmetry $S^2$ smoothly degenerates$. As $\rho \to \pm 1/2$, the R-symmetry $U(1)$, with coordinate $\psi$, smoothly degenerates, provided that $\psi$ is identified with period $2\pi$. Henceforth we will take $\rho$ to be non-negative.

This manifold, as a solution of eleven-dimensional supergravity, admits sixteen Killing spinors. The Killing spinors may be used to define an identity structure - a preferred frame associated to them. The structure is discussed in detail in [18]. We choose coordinates for the preferred frame according to

$$\begin{aligned}
e^1 + ie^2 &= \frac{1}{\sqrt{2\lambda}} e^{i\psi}(d\mu + i \sinh \mu d\beta), \\
e^3 &= \sqrt{\frac{1 - \lambda^3 \rho^2}{\lambda}}(d\psi - \cosh \mu d\beta), \\
n \rho &= \frac{\lambda d\rho}{2 \sqrt{1 - \lambda^3 \rho^2}}, \\
n \hat{r} &= \lambda^{-1/2}dr,
\end{aligned}$$

(2.3)

$^3$The R-symmetry of the dual theory is $SU(2) \times U(1)$. 


where we have chosen Poincaré coordinates on $AdS$,
\[
ds^2(AdS_5) = e^{-2r} ds^2(\mathbb{R}^{1,3}) + dr^2.
\]
(2.4)

The remaining directions play no rôle in the rest of the discussion.

The MN(I) solution is a particular case of a broader class of half-BPS $AdS_5$ solutions which are completely determined by a solution of the three-dimensional continuous Toda equation\textsuperscript{4}. The Toda equation,
\[
\nabla^2_{\mathbb{R}^2} D + \partial^2 \rho e^D = 0,
\]
(2.5)
may be viewed as a three-dimensional Laplace equation,
\[
\nabla^2_3 D = 0,
\]
(2.6)
on a three-manifold with metric
\[
ds^2 = d\rho^2 + e^D ds^2(\mathbb{R}^2).
\]
(2.7)

The metric on every half-BPS $AdS_5$ solution of eleven dimensional supergravity determined by the Toda equation may be written as follows [23]:
\[
ds^2 = \frac{1}{\lambda} \left[ ds^2(AdS_5) + (1 - \lambda^3 \rho^2)(d\psi + V)^2 + \frac{\lambda^3}{4} \left( \frac{1}{1 - \lambda^3 \rho^2} [d\rho^2 + e^D ds^2(\mathbb{R}^2)] + \rho^2 ds^2(S^2) \right) \right],
\]
(2.8)
where
\[
\lambda^3 = \frac{-\partial_\rho D}{\rho(1 - \rho \partial_\rho D)},
\]
(2.9)
\[
V = \frac{1}{2} \ast_2 d_2 D,
\]
(2.10)
where $d_2$ is the exterior derivative restricted to $\mathbb{R}^2$, and $D$ solves (2.5). The MN(I) solution is given by
\[
e^D = \frac{1}{4x_1^4} (1 - 4\rho^2).
\]
(2.11)

In order to make the relationship between the MN(I) geometry and wrapped branes more concrete, we now want to exhibit it as a solution of the Fayazzuddin-Smith equations. The essential point is that in addition to its identity structure, MN(I) also admits an $SU(2)$ structure, defined by half its Killing spinors, which indeed solves the Fayyazuddin-Smith equations. In [18] it was

\textsuperscript{4}It is strongly believed, at least by the author, that all half-BPS $AdS_5$ solutions of M-theory are of this form.
shown how to obtain an arbitrary Toda-AdS\(_5\) manifold as a 1/4 BPS solution of the wrapped brane conditions by constructing its SU(2) structure. Here we will apply these general results to the specific case of interest.

The canonical frame of the identity structure is related to the canonical frame of the SU(2) structure by a local rotation. If we define the Minkowski frame \(e^a, e^4, \hat{t} = Ldt, a = 1, 2, 3\), with \(e^a, e^4\) a basis for \(\mathcal{M}_4\), the relationship between the “AdS” and “Minkowski” frames is given by

\[
\begin{align*}
    e^a_{\text{Mink}} &= e^a_{\text{AdS}}, \\
    e^4 &= \cos \theta \hat{\rho} + \sin \theta \hat{r}, \\
    \hat{t} &= -\sin \theta \hat{\rho} + \cos \theta \hat{r}.
\end{align*}
\] (2.12)

For more details of this procedure, which seems to be a universally applicable way of writing AdS manifolds in a wrapped brane form, the reader is referred to [18], [22], [19]. In the case at hand, the rotation angle is related to the AdS warp factor and the coordinate \(\rho\) by

\[
\cos \theta = \lambda^{3/2} \rho,
\] (2.13)

and also the warp factors are related by

\[
L = \lambda e^{2r}.
\] (2.14)

Near \(\rho = 0\), the AdS radial direction aligns with \(\pm e^4\). Near \(\rho = 1/2\), it aligns with \(\hat{t}\). Since we know everything on the right-hand side of (2.12), we can see that

\[
\begin{align*}
    e^4 &= L e^{-r} \left(-\sqrt{1 - 4\rho^2/8} e^{-r}\right), \\
    \hat{t} &= L d \left(-\frac{\rho}{2} e^{-2r}\right).
\end{align*}
\] (2.15)

Therefore defining the Minkowski-frame coordinates

\[
\begin{align*}
    u &= -\sqrt{1 - 4\rho^2/8} e^{-r}, \\
    t &= -\frac{\rho}{2} e^{-2r},
\end{align*}
\] (2.16)

we can re-write the AdS\(_5\) solution as

\[
\begin{align*}
    ds^2 &= L^{-1} \left[ ds^2(\mathbb{R}^{1,3}) + \frac{F}{2} ds^2(H^2) \right] + L^2 \left[ F^{-1} \left( du^2 + u^2 (d\psi - P)^2 \right) + dt^2 + t^2 ds^2(S^2) \right],
\end{align*}
\] (2.17)

\[\text{The frame rotation, as a way of deriving warped AdS}_{d+2} \text{ supersymmetry conditions from warped } \mathbb{R}^{1,d} \text{ supersymmetry conditions, was first employed in [4].}\]
where \( F = e^{2r} \) is determined by a root of the quadratic
\[
2t^2 e^{4r} + u^2 e^{2r} - \frac{1}{8} = 0. \tag{2.18}
\]
One of these roots is always negative, so we choose the other, which is always positive:
\[
F = \frac{u^2}{4t^2} \left( -1 + \sqrt{1 + t^2/u^4} \right). \tag{2.19}
\]
The warp-factor in the Minkowski frame is
\[
L^3 = \frac{u^2}{\sqrt{1 + t^2/u^4}} \left( -1 + \sqrt{1 + t^2/u^4} \right)^2. \tag{2.20}
\]
The canonical frame for the \( SU(2) \) structure, re-written in terms of the new coordinates, is
\[
e^1 + ie^2 = \sqrt{F} \frac{e^{i\psi}}{2L} (d\mu + i \sinh \mu d\beta),
\]
\[
e^3 = \frac{Lu}{\sqrt{F}} (d\psi - \cosh \mu d\beta),
\]
\[
e^4 = \frac{L}{\sqrt{F}} du,
\]
\[
\hat{t} = L dt,
\]
with a minus sign in the second equation because of the definition of \( u \). The \( SU(2) \) structure then takes the standard form:
\[
J = e^{12} + e^{34},
\]
\[
\Omega = (e^1 + ie^2)(e^3 + ie^4), \tag{2.22}
\]
and it may now be verified by explicit computation that it satisfies the Fayyazudin-Smith equations. This was, of course, guaranteed by the construction, but it serves as a consistency check. Having obtained the \( SU(2) \) structure of MN(I), it is now an obvious thing to use it as an ansatz for further, topologically related, solutions of the Fayyazuddin-Smith equations. To this end, we let \( F \) and \( L \) be arbitrary functions of \( u, t \), and insert the frame (2.21) into the \( SU(2) \) torsion conditions and Bianchi identity. They reduce to the single non-linear second order pde for \( F \):
\[
\frac{1}{t^2} \partial_t \left( t^2 \partial_t F \right) = -u \partial_u \left( \frac{F}{u} \partial_u F \right). \tag{2.23}
\]
Given a solution of this equation, \( L \) is then determined by
\[
L^3 = -\frac{1}{4u} \partial_u (F^2). \tag{2.24}
\]
As a purely mathematical aside, we observe that the other root of the quadratic (2.18) is also a solution of these equations. But of particular interest is the most general Calabi-Yau solution of this system. It may be most easily determined by imposing \( L = \text{constant} \) and closure of \( J \). The \((2, 0)\) form \( L^{-1/2} \Omega \) is always closed with this ansatz. The general Calabi-Yau solution is

\[
F^2 = a + bu^2. \tag{2.25}
\]

For a metric of the right signature, we must have \( a > 0, \ b < 0 \). By rescaling, we can set \( b = -2 \), so that \( L = 1 \) (up to an overall scale in the eleven-dimensional metric). This Calabi-Yau is diffeomorphic to a Toda-Calabi-Yau, as may be seen by performing the coordinate transformation

\[
16U^2 = a - 2u^2. \tag{2.26}
\]

Defining \( A^2 = a/4 \), the metric becomes

\[
ds^2 = \frac{4}{\partial u}D(d\alpha + V)^2 + \partial uD(du^2 + e^D ds^2(R^2)), \tag{2.27}
\]

where

\[
e^D = \frac{1}{4x_1^2}(A^2 - 4U^2), \tag{2.28}
\]

which, modulo the constant, is the same solution of the Toda equation as that determining \( \text{MN(I)} \). An alternative form of the metric, reminiscent of Eguchi-Hanson, is given by choosing the coordinate

\[
R^2 = \frac{1}{a^{1/4}} \sqrt{2a - 4u^2}. \tag{2.29}
\]

Up to an overall scale the metric becomes

\[
ds^2 = \frac{dR^2}{(R^2 - 1)} + \frac{R^2}{4} \left[ ds^2(H^2) + \left( \frac{1}{R^4 - 1} \right) (d\psi - P)^2 \right], \tag{2.30}
\]

which is the form given in the introduction.

### 3 The \( \mathcal{N} = 1 \) M-theory solution

Again, we begin with a review of the \( AdS \) geometry. The \( \text{MN(II)} \) metric is

\[
ds^2 = \frac{1}{\lambda} \left[ ds^2(AdS_5) + \frac{1}{3} ds^2(H^2) + \frac{1}{9} (1 - \lambda^3 \rho^2) \left( ds^2(S^2) + (d\psi + P - P')^2 \right) + \frac{\lambda^3}{4(1 - \lambda^3 \rho^2)} d\rho^2 \right], \tag{3.1}
\]
where now
\[
\lambda = \frac{4}{4 + \rho^2},
\]
\[
dP = \text{Vol}[S^2],
\]
\[
dP' = \text{Vol}[H^2].
\] (3.2)

This time, the range of \(\rho\) is \([-2/\sqrt{3}, 2/\sqrt{3}]\); at \(\rho = \pm 2/\sqrt{3}\), an \(S^3\) smoothly degenerates. This manifold admits eight Killing spinors, which collectively define an \(SU(2)\) structure. If we define the frame
\[
e^1 + ie^2 = \frac{1}{\sqrt{3}\lambda} e^{i\gamma\psi}(d\mu + i \sinh \mu d\beta),
\]
\[
e^3 + ie^4 = \frac{1}{3} \sqrt{\frac{1 - \lambda^3 \rho^2}{\lambda}} e^{i\delta\psi}(d\theta + i \sin \theta d\phi),
\]
\[
e^5 = \frac{1}{3} \sqrt{\frac{1 - \lambda^3 \rho^2}{\lambda}} (d\psi + P - P'),
\]
\[
\hat{\rho} = \frac{\lambda}{2 \sqrt{1 - \lambda^3 \rho^2}} d\rho,
\] (3.3)

where the constant phases \(\gamma, \delta\) sum to unity, then the \(SU(2)\) structure forms are given by
\[
J_4 = e^{12} + e^{34},
\]
\[
\Omega_4 = (e^1 + ie^2)(e^3 + ie^4).
\] (3.4)

It may be explicitly verified that this six-dimensional \(SU(2)\) structure satisfies the conditions of [4].

The MN(II) solution is interpreted as coming from M5 branes wrapping a \(H^2\) Kähler two-cycle in a Calabi-Yau three-fold. Again, we will make this more precise, by exhibiting MN(II) as a solution of the 1/8 BPS \(SU(3)\) analogue of the Fayazzuddin-Smith equations. In this case, half the Killing spinors of the \(AdS\) manifold define an \(SU(3)\) structure, with structure forms \(J_6, \Omega_6\). Then the supergravity description of 1/8 BPS M5 branes wrapping a Kähler two-cycle in a Calabi-Yau three-fold [26], [18] is as follows. The metric and flux are
\[
ds^2 = L^{-1} ds^2(\mathbb{R}^{1,3}) + ds^2(\mathcal{M}_6) + L^2 dt^2,
\]
\[
\star_7 F = -L^2 d(L^{-2} J_6),
\] (3.5)

where \(\mathcal{M}_6\) admits a globally-defined \(SU(3)\) structure, and again, all conventions and orientations follow [18]. The torsion conditions for the structure are
\[
dt \wedge d(L^{-1} J \wedge J) = 0,
\]
\[
d(L^{-3/2} \Omega) = 0.
\] (3.6)
These, together with the Bianchi identity, are sufficient to guarantee a solution of eleven-dimensional supergravity.

We now perform the frame rotation exactly as in the previous section. The relationship between the Minkowski and $AdS$ frames is

\begin{align}
 e^a_{\text{Mink}} &= e^a_{AdS}, \\
 e^6 &= \cos \theta \hat{\rho} + \sin \theta \hat{r}, \\
 \hat{t} &= -\sin \theta \hat{\rho} + \cos \theta \hat{r},
\end{align}

(3.7)

where now $a = 1, \ldots, 5$. Again, $\lambda^{3/2} \rho = \cos \theta$. Therefore, $\hat{r}$ is anti-aligned with $\hat{t}$ at $\rho = -2/\sqrt{3}$. It then rotates through an angle of $\pi$ as $\rho$ spans its range, so that it is aligned with $\hat{t}$ at $\rho = 2/\sqrt{3}$.

We find that $e^6, \hat{t}$ are given by

\begin{align}
 e^6 &= Le^{-r/2}d\left(-\frac{1}{3}e^{-3r/2}\sqrt{4 - 3\rho^2}\right), \\
 \hat{t} &= Ld\left(-\frac{\rho}{2}e^{-2r}\right).
\end{align}

(3.8)

Defining the Minkowski-frame coordinates,

\begin{align}
 u &= -\frac{1}{3}e^{-3r/2}\sqrt{4 - 3\rho^2}, \\
 t &= \frac{\rho}{2}e^{-2r},
\end{align}

(3.9)

the metric in the Minkowski frame is given by

\begin{equation}
 ds^2 = L^{-1}\left[ds^2(\mathbb{R}^{1,3}) + \frac{F^2}{3} ds^2(H^2)\right] + L^2 \left[ F^{-1}\left(du^2 + \frac{u^2}{4}[ds^2(S^2) + (d\psi + P - P')^2]\right) + dt^2 \right],
\end{equation}

(3.10)

where $F = e^r$. This time, in order to determine $F$ in terms of the Minkowski-frame coordinates, we must find the roots of a quartic polynomial. The polynomial is

\begin{equation}
 12t^2e^{4r} + 9u^2e^{3r} - 4 = 0.
\end{equation}

(3.11)

The Minkowski frame is given by

\begin{align}
 e^1 + ie^2 &= \frac{F}{\sqrt{3}L}e^{i\gamma\psi}(d\mu + i \sin \mu d\beta), \\
 e^3 + ie^4 &= -\frac{Lu}{2\sqrt{F}}e^{i\delta\psi}(d\theta + i \sin \theta d\phi), \\
 e^5 &= -\frac{Lu}{2\sqrt{F}}(d\psi + P - P'), \\
 e^6 &= \frac{L}{\sqrt{F}}du, \\
 \hat{t} &= Ldt.
\end{align}

(3.12)
Again, the minus signs come from the definition of $u$. Then the $SU(3)$ structure of MN(II) is given by

$$J_6 = e^{12} + e^{34} + e^{56},$$
$$\Omega_6 = (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6).$$  \hfill (3.13)

At this point, repeating the analysis of the previous section directly, we would let $L, F$ become arbitrary functions of $u, t$, and then find the general Calabi-Yau solution. The torsion conditions and Bianchi identity reduce to

$$\partial_t^2 F + \frac{1}{u} \partial_u (uF \partial_u F) = 0,$$
$$L^3 + \frac{2F^2}{3u} \partial_u F = 0.$$  \hfill (3.14)

It follows from the construction of [4], [18] that the root of the quartic corresponding to the MN(II) solution solves these equations. It seems very likely that so do all the roots, though this has not been verified. However, it turns out that there is no Calabi-Yau solution. To find one, we must extend the ansatz, to

$$ds^2 = L^{-1} \left[ ds^2(\mathbb{R}^{1,3}) + \frac{F_1 F_2}{3} ds^2(H^2) \right] + L^2 \left[ F_1^{-1} \left( du^2 + \frac{u^2}{4} (d\psi + P - P')^2 \right) + F_2^{-1} \frac{u^2}{4} ds^2(S^2) + dt^2 \right].$$  \hfill (3.15)

This extension of the ansatz is not unnatural as it clearly contains MN(II) as the special case $F_1 = F_2$. Furthermore it leaves the (3,0) form $\Omega$ invariant\(^6\); it is a purely Kähler deformation of the $SU(3)$ structure. We also make the obvious modification of the frame ansatz. In general, the torsion conditions and Bianchi identity are rather complicated. However it is easy to determine the most general Calabi-Yau solution with this ansatz, imposing closure of $J$ and constancy of $L$. The Calabi-Yau condition reads

$$\partial_t F_1 = \partial_t F_2 = 0,$$
$$\frac{1}{3} \partial_u (F_1 F_2) + \frac{u}{2F_1} = 0,$$
$$\partial_u \left( \frac{u^2}{4F_2} \right) - \frac{u}{2F_1} = 0.$$  \hfill (3.16)

with general (positive signature) solution

$$F_1 = \frac{3a^4}{u^2} \cos^2 \xi,$$
$$F_2 = \frac{u^2}{2a^2} \frac{(1 + \sin \xi)}{\cos^2 \xi},$$  \hfill (3.17)

\(^6\)Observe that $L^{-3/2} \Omega$ is always closed with this frame ansatz.
where \(a^2, b\) are constants and \(\sin \xi\) is a root of the cubic equation
\[
-\frac{1}{3} \sin^3 \xi + \sin \xi = b - \frac{u^4}{12a^5}. \tag{3.18}
\]
This Calabi-Yau has two moduli. One, as usual, is just the overall scale. Defining
\[
R = \frac{u^2}{2\sqrt{3}a^3}, \tag{3.19}
\]
the metric is
\[
\begin{aligned}
ds^2 &= a^2 \left[ \frac{1}{2}(1 + \sin \xi) ds^2(H^2) + \frac{\cos^2 \xi}{2(1 + \sin \xi)} ds^2(S^2) + \frac{1}{\cos^2 \xi} \left( dR^2 + R^2(d\psi + P - P')^2 \right) \right], \\
-\frac{1}{3} \sin^3 \xi + \sin \xi &= b - R^2. \tag{3.20}
\end{aligned}
\]
The modulus \(b\) parameterises inequivalent metrics. The generic metric of this form has three degeneration points: \(R = 0\) and \(\xi = \pm \pi/2\). The point \(\xi = -\pi/2\) (where the \(H^2\) cycle degenerates) is necessarily and expectedly singular. This is where fractional branes are wrapped, in the probe picture. If \(R = 0\) and \(\xi = \pi/2\) do not coincide, the point \(R = 0\) is also singular, since there the \(U(1)\) degenerates with a \(4\pi\) periodicity inherited from the \(AdS\) frame. To analyse what happens near \(\xi = \pi/2\), we expand the cubic to fourth order in \(\xi\) in the vicinity of this point to find
\[
R^2 = \left( b - \frac{2}{3} \right) + \frac{\xi^4}{4}. \tag{3.21}
\]
Clearly we require \(b \geq 2/3\) (otherwise \(\xi = \pi/2\) is not part of the space). If \(b > 2/3\), then \(R^2\) goes to a fixed positive value at \(\xi = \pi/2\); the metric there is clearly singular when written in terms of \(\xi\). However the point in moduli space \(b = 2/3\) where the \(R = 0\) and \(\xi = \pi/2\) degeneration points of the metric coincide is special. With \(b = 2/3\) the metric near \(\xi = \pi/2\) becomes
\[
\begin{aligned}
ds^2 &= a^2 \left[ ds^2(H^2) + d\xi^2 + \frac{\xi^2}{4} \left( ds^2(S^2) + (d\psi + P - P')^2 \right) \right]. \tag{3.22}
\end{aligned}
\]
With the periodicity of \(\psi\) as inherited from the \(AdS\) frame, an \(S^3\) smoothly degenerates. The Calabi-Yau (3.20), with \(b = 2/3\), is interpreted as the image of \(MN(II)\) under an inverse geometric transition. Analysing the relationship between \(R\) and \(\xi\) near \(\xi = -\pi/2\), we can deduce that the range of \(R\) is is either \([-2/\sqrt{3}, 0)\) or \((0, 2/\sqrt{3}]\). In either branch, the singularity is at finite proper distance.
4 Conclusions and outlook

In this paper a way of mapping a supersymmetric $AdS$ manifold to a special holonomy manifold has been proposed. The main conclusion is that this procedure should be applicable to all known wrapped-brane $AdS$ solutions of string and M-theory; it will be very interesting to explore the the special holonomy metrics in each case. Given the metrics for the string theory solutions, it should be possible to make progress towards constructing the dual field theories.

The construction relies in an essential way on the existence of an interpolating solution. For MN(I) and (II), we can say a little about what the boundary conditions for an interpolation should be. For MN(I), the interpolating solution should be globally smooth, and should contain a neighbourhood where the metric is diffeomorphic to the limit of the metric (2.30) as $R \to 1$. It should also contain a neighbourhood where the metric is diffeomorphic to the limit as $\rho \to 0$ of (3.1). For MN(II), an interpolating solution should contain a neighbourhood diffeomorphic to the Calabi-Yau metric (3.20) near $\xi = \pi/2$. It should also contain a neighbourhood where the metric is diffeomorphic to (3.1) as $\rho \to \pm 2/\sqrt{3}$. Global topological considerations will be important in trying to construct an interpolation; for example, by a careful analysis it should be possible to fix the relative scales of the MN/CY metrics. This might be done, for example, by comparing the sizes of the $H^2$ bolts in the $AdS$ and Calabi-Yau metrics, at the point in each where the $U(1)$ or $S^3$ degenerates, for MN(I) and MN(II) respectively. However since an interpolating solution would necessarily be cohomogeneity two, and the governing equations are non-linear, finding one explicitly will be challenging. It might be worthwhile to perform a numerical analysis, if a better handle can be obtained on the boundary conditions.

There might be other, more complicated, Calabi-Yau manifolds that could be related to the Maldacena-Nuñez solutions. This would be analagous to the way in which conical Calabi-Yaus can be thought of as generic local models for a particular sort of singularity, in a manifold whose global structure could be much more complicated. Placing D-branes at the singularity is usually argued to produce an $AdS$ throat, which is insensitive to the global structure. It would be interesting to know if the Calabi-Yau metrics obtained here can be thought of in a similar way - as local models of a more generic type of singularity. The topology of the manifolds in this paper is more complicated, so it is not obvious yet whether or not this is true. In any event, for the purposes of constructing the dual, in the conical case only the geometry near the singularity - the conical metric - is required. Analagously, for the purposes of constructing the duals of the Maldacena-Nuñez solutions, the metrics of this paper are interpreted as the appropriate backgrounds.

There appears to be an intriguing link between solutions of the various non-linear equations we have encountered and roots of polynomials. This seems to suggest some underlying algebraic geometry which has not been properly appreciated. It will be interesting to explore this in more
detail; it appears to be a generic feature of how \textit{AdS} manifolds solve wrapped brane structure equations.

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