Isotropic radiative transfer as a phase space process: Lorentz covariant Green’s functions and first-passage times

VINCENT ROSSETTO

Univ. Grenoble Alpes, CNRS, LPMMC - 38000 Grenoble, France

received 12 November 2021; accepted in final form 30 May 2022
published online 21 July 2022

Abstract – The solutions of the radiative transfer equation, known for the energy density, do not satisfy the fundamental transitivity property for Green’s functions expressed by Chapman-Kolmogorov’s relation. I show that this property is retrieved by considering the radiance distribution in phase space. Exact solutions are obtained in one and two dimensions as probability density functions of continuous-time persistent random walks, the Fokker-Planck equation of which is the radiative transfer equation. The expected property of Lorentz covariance is verified. I also discuss the measured signal from a pulse source in one dimension, which is a first-passage time distribution, and unveil an effective random delay when the pulse is emitted away from the observer.

Copyright © 2022 EPLA

Introduction. – The physics of waves propagating in a cloud of scatterers, generically called radiative transfer, is a topic of intense research in astrophysics [1], meteorology [2], medical imaging [3], seismology [4] and computer graphics [5]. While specific questions are addressed in these domains of application, some fundamental aspects apply to all of them such as the spreading of energy into space, which is governed by Chandrasekhar’s radiative transfer equation [6]. Chandrasekhar’s equation describes the spatial and temporal evolution of the radiance (or the luminance) \( \phi \) as a function of the position \( r \), the direction of propagation \( \mathbf{u} \) and time. The radiance (luminance) is the (luminous) energy flux density. In a non-absorbing medium, the radiative transfer equation is

\[
\partial_t \phi + c \mathbf{u} \cdot \nabla \phi + \frac{c}{\ell} \phi = \frac{c}{\ell} \mathcal{I} \phi + S(t, r, \mathbf{u}),
\]

where \( c \) is the speed of light, \( \ell \) is the mean free path, and \( S \) is a source term. The integral operator \( \mathcal{I} \) is the emission term which in the case of isotropic scattering is the average of \( \phi \) with respect to \( \mathbf{u} \). Only isotropic scattering is considered in this letter. I will also neglect absorption and coherent effects such as interferences.

The solutions of the radiative transfer equation (1) strongly depend on the spatial dimension. In one dimension, the radiative transfer equation reduces to the telegrapher’s equation, a second-order linear partial differential equation arising from electric transport theory, dating back to Thomson [7]. The telegrapher’s equation has a wide range of applications [8]. It was solved by Hemmer [9] as he studied a modified version of Smoluchowski’s diffusion equation [10,11]. The solution of the radiative transfer equation in two dimensions was obtained by seismologists Shang and Gao [4,12]. In three dimensions no exact solutions have been derived despite the important efforts put to the task. Several approximations have nonetheless been obtained, the most notable by Paasschens [13].

As in all transport phenomena, it seems natural that the elementary solutions to the radiative transfer equation have the property of transitivity, which means that the spatial distribution of energy at a time \( t_2 \) can be deduced from the energy distribution at a previous time \( t_1 < t_2 \) using the distribution of energy from a pulse after a time \( t_2 - t_1 \), but this is not the case [14]; in other words, as will be shown in this letter (section “Chapman-Kolmogorov’s relation”), the elementary solutions from Hemmer, and Shang and Gao are not Green’s functions.

In this letter, I show that Green’s functions in one and two dimensions should be expressed in phase space rather than in position space. I show that these solutions are Lorentz covariant and have the transitivity property. The following section introduces continuous-time persistent random walks (CTPRW) and their relation to the radiative transfer equation. In the section “Solution in one dimension”, I compute the mean free path in one dimension, in a frame in relative motion with the cloud of scatterers, and I show that it depends on the direction of propagation.
This asymmetry suggests a reinterpretation of the phase space solution for asymmetric scattering as the solution for the symmetric case transformed by a Lorentz boost. In the section “Chapman-Kolmogorov’s relation”, I show that the CTPRW as a phase space process has the Markov property and discuss why the solution expressed in position space does not satisfy Chapman-Kolmogorov’s relation. The following section is dedicated to the first-passage time properties of the CTPRW: I show that an effective new “flip” process arises, of which I provide elementary properties. In the section “Lorentz covariance in two dimensions”, I show that the already known phase space solution in two dimensions is Lorentz covariant, a fact that was not hitherto known. Lastly, before concluding, I discuss the Brownian limit $c \to \infty$ of the investigated random walks and show that the phase space Green’s functions asymptotically approach a Gaussian in this limit, which only depends on position.

**Persistent random walks.** – In 1951, Goldstein remarked that the telegrapher’s equation is also the Fokker-Planck equation of a persistent random walk [15]. Therefore, the probability density function of a persistent random walk obeys the telegrapher’s equation, in the same manner that the probability density function of the Brownian motion obeys the diffusion equation. This correspondence extends to higher dimensions for continuous-time persistent random walks (CTPRW) as introduced by Masoliver, Lindenberg and Weiss [16]. Persistent random walks in any dimensions can thus serve as stochastic models from which the properties of radiative transfer are obtained. In this work, I use these models as processes in phase space and discuss the solution for the radiance $\phi(t, \mathbf{r}, \mathbf{u})$ when the source emits a pulse at $t = 0$.

**Persistent random walks.** The first mention of a persistent random walk dates back to the works by Fürth [17] and that of Taylor [18], who defined a one-dimensional persistent random walk as a sequence of steps of a fixed length $b$, occurring at a regular time pace. In modern language, the random process described by Fürth and Taylor is a Markov chain of states $(x_n, u_n)$ where $x_n$ is the position coordinate of the walker after $n$ steps and $u_n = \pm 1$ is the direction of the next displacement,

$$x_{n+1} = x_n + b_n u_n,$$

where $b_n > 0$ is a sequence of step lengths (constant equal to $b$ in the case of Fürth and Taylor’s definition) and $u_{n+1} = -u_n$ with probability $p$, or $u_{n+1} = u_n$ with probability $1 - p$. Fürth and Taylor’s process is characterized by the transport mean free path $\ell^* = \lim_{n\to\infty} \langle u_0 x_n \rangle - u_0 x_0 = b/2p$, which is the average length travelled without flipping direction.

Using this representation of the photon’s state, the collision (scattering event) of a photon with a scatterer is represented by a change of its state from $(x, u)$ into $(x, u')$. The walk alternates between steps, where the position is updated by eq. (2), and such collisions.

**Continuous-time persistent random walks.** Inspired by seminal remarks from Kac [19] and the works of DeWitt-Morette and Foong [20] about the telegrapher’s equation, Masoliver, Lindenberg and Weiss [16] introduced the continuous-time persistent random walk (CTPRW). In one dimension, this random walk evolves according to eq. (2), with randomly distributed and independent step lengths $b_n \sim \mathcal{E}(\ell)$, which means that each length is a random variable with an exponential probability distribution function (PDF) $\mathcal{E}(\ell)$: $PDF(b_n) = \frac{1}{\ell} \exp\left(-\frac{b}{\ell}\right)$.

In one dimension, the emission term is simply $I[\phi] = (1-p)\phi(t, x, u) + p\phi(t, x, -u)$ and the radiative transfer equation is

$$\left(\partial_t + cu\partial_x + p\frac{\partial}{\partial \mathbf{T}}\right)\phi(t, x, u) = p\frac{\partial}{\partial \mathbf{T}}\phi(t, x, -u) + S(t, x, u).$$

Remarkably, in this equation, $\ell$ only appears associated with $p$ in $\ell/p$, one deduces that the statistics of the generalized coordinate $\mathbf{X} = (x, u)$ as a function of time $t$ only depend on the transport mean free path $\ell^* = \ell/2p$ and therefore that combinations of $\ell$ and $p$ yielding the same transport mean free path $\ell^*$ are physically indistinguishable. In this work, I use $p = 1$, which has the benefit of simplifying the algebra without losing generality, such that $\ell$ is the only physical parameter of the process and the sequence $u_n$ is simply

$$u_n = (-1)^n u_0.$$ (3)

Thanks to eq. (3), the analytic solution can easily be interpreted in terms of parity of the number of scattering events [21,22].

In higher dimensions, the persistent random walk is a sequence of steps of independent lengths $b_n \sim \mathcal{E}(\ell)$, and independent directions $\mathbf{u}_n$ uniformly distributed on the unit sphere. In all dimensions, the Fokker-Planck equation of these processes [23] is Chandrasekhar’s radiative transfer equation (1). Therefore, a CTPRW process microscopically describes photons in a cloud of scatterers, observed in a frame $\mathcal{R}$ where the cloud is at rest, while the radiative transfer equation describes the phenomenon macroscopically.

**Solution in one dimension.** – Let me consider the propagation in one dimension of a pulse of photons emitted in the direction $u_0$ at $t = 0$ observed from a moving frame $\mathcal{R}'$, in uniform translation with respect to $\mathcal{R}$. I denote by $v = \beta c$ the velocity of $\mathcal{R}$ observed in $\mathcal{R}'$. Geometrical quantities in $\mathcal{R}'$ are labelled with a prime and the Lorentz factor is denoted by $\gamma = 1/\sqrt{1-\beta^2}$.

**Asymmetry induced by relative motion.** I consider a photon moving toward a scatterer located at a distance $r'$ in $\mathcal{R}'$: the scatterer is reached by the photon after a time $r'/(c - u v)$. In $\mathcal{R}$, the initial distance is $r = \gamma^{-1}r'$ and the travel time is $t = \gamma^{-1}r'/c$, such that, when $r'$ is the distance between two scattering events, its average is the mean free path in $\mathcal{R}'$ and depends on the direction of
propagation of the photon. Writing \( \ell'_u = \ell'_\pm = \ell_\pm \) gives

\[
\frac{1}{\ell'_\pm} = \frac{c \mp v \gamma}{c} = (1 \mp \beta) \frac{\gamma}{\ell},
\]

and consequently \( \ell'_u \ell'_\pm = \ell^2 \) is Lorentz invariant.

As observed from the moving frame \( R' \), the photon’s random walk in the cloud of scatterers is an asymmetric persistent random walk, which is a CTPRW with different transport mean free paths depending on the direction of propagation \( u_n \):

\[
b_n \sim \mathcal{E}(\ell'_u).
\]

**General solution.** The solution of the asymmetric persistent random walk has been recently obtained without invoking special relativity [24]. The solution follows entirely from eqs. (2), (3), and (4), and is given in the cited reference in terms of the asymmetry factors \( \kappa \) and \( \mu \),

\[
\kappa = \frac{1}{2} \left( \frac{1}{\ell'_-} - \frac{1}{\ell'_+} \right) = \frac{\beta \gamma}{\ell}, \quad \mu = \frac{c}{2} \left( \frac{1}{\ell'_+} + \frac{1}{\ell'_-} \right) = \frac{\gamma}{\ell}.
\]

I here reproduce the solution written in relativistic form:

\[
\text{see eq. (5) above}
\]

where \( \xi = \sqrt{c^2 t^2 - \mathbf{x}^2 \ell} \), \( I_n \) is Bessel’s modified function of the first kind and \( n \)-th order, and \( \Theta \) is Heaviside’s unit step function.

The variable \( \xi \) is Lorentz invariant and the terms in the exponents of eq. (5b) correspond to the change of coordinates between \( R \) and \( R' \). It follows that global Lorentz covariance is satisfied by \( g dx \) thanks to the invariance of \( (ct' + u_0 (x' - x'_0)) dx' = (ct + u_0 (x - x_0)) dx \). This change of coordinates shows that the solutions of the symmetric persistent random walk in \( R \) and the asymmetric persistent random walk in \( R' \) are related by a Lorentz transformation.

**Chapman-Kolmogorov’s relation.** As \( g \) is a probability density function, it is normalized according to

\[
\int_{x'} g(t; \mathbf{x}' | \mathbf{x}'_0) = 1,
\]

where the symbolic integral \( \int_{x'} \) is a shortcut for \( \int_{x'} \sum_{u=\pm1} \) expressing the integration on the whole phase space. Note that the normalization depends on \( u_0 \).

Consider now a CTPRW after a travel time \( t_1 \): The probability that \( t_1 \) coincides with a scattering event is zero such that \( u(t_1) \) is well defined. Moreover the length of the step being performed at time \( t_1 \) that remains to be traveled is distributed according to \( \mathcal{E}(\ell'_u(t_1)) \) exactly as if this was the first step of a new CTPRW starting at time \( t_1 \). This last property is called the memorylessness of the exponential distribution. These observations imply that the generalized coordinate \( X = (x', u) \) of the CTPRW follows a Markov process. Chapman-Kolmogorov’s relation results from the Markovian nature of the CTPRW process:

\[
\int_{x'} g(t_1; X | X'_0) g(t_2; X' | X_0) = g(t_1 + t_2; X | X_0).
\]

It follows immediately that if the source is the distribution \( S = \delta(t) \phi_0(X_0) \) then the radiance is the superposition

\[
\phi(t, X) = \int_{X_0} g(t; X | X_0) \phi_0(X_0),
\]

and so, the solution (5) has the properties of Green’s functions.

In the relation (6), relativistic random walkers are characterized not only by their position, but also their momentum. Classical random walks are usually Markov point processes, in which the available transitions and their probabilities only depend on the current spatial position of the walker. The relation (6) suggests that relativistic random walks should be represented as processes in phase space. If the source is not localized on a single point in phase space then the process is a superposition of two independent Markov processes, it therefore cannot be represented as a single point in phase space. As an example, consider the irradiance solution for an isotropic source

\[
h(t; x | x_0) = \frac{1}{2} \sum_{u_0=\pm1} g(t; x, u | x_0, u_0),
\]

which appears in refs. [9,14,15,19] and [25] (p. 865). The convolution \( \int_{x'} h(t_1; x | x') h(t_2; x' | x_0) \) contains superfluous terms such as

\[
\int_{x'} g(t_1; x', +1 | x', +1) g(t_2; x', -1 | x_0, +1),
\]

that are non-physical because the directions of propagation at position \( x' \) in the two Green’s functions (underlined in the above equation) do not match. This is the reason why \( h \) does not satisfy Chapman-Kolmogorov’s relation.
the probability of passage at the origin \( O \) is equal to 1 and the average first-passage time is finite. (b) The case \( \beta > 0 \). In this case \( \ell_+ > \ell_- \), and the photon has a finite probability of never visiting the origin \( O \).

**First-passage time statistics.** — Numerous applications of random walks in Physics involve properties like local time, integrated area and first-passage times [26,27]. In the case of radiative transfer, the first-passage time distribution is the signal measured after the emission of a pulse localized on a single point in phase space. A device recording such a pulse indeed measures a signal proportional to the first-passage time distribution at its position. Let me then consider the first-passage problem at the origin \( O \) in \( \mathcal{R}' \) when the CTPRW starts at \( x'_0 > 0 \). As a consequence of the asymmetry of mean free path, a photon observed from \( \mathcal{R}' \) experiences an average drift. Figure 1 displays an illustration of the two possible cases. In the first case \((\beta < 0, \text{fig. 1(a)})\), the photon is drifted toward \( O' \); the distribution of the first-passage time at the origin \( P_1(t' \mid X'_0) \) has finite moments, as opposed to the case where \( \beta \geq 0 \). In the second case \((\beta > 0, \text{fig. 1(b)})\), the photon is drifted away from \( O' \); the probability \( R(X'_0) = \int_0^{\infty} P_1(t' \mid X'_0)dt' \) that the photon starting at position \( x'_0 \) ever reaches \( O' \) depart from 1 as opposed to the case where \( \beta \leq 0 \). The expressions of \( P_1 \) and \( R \) are given in table 1.

An effective “flip” process. — As shown in ref. [24], the Laplace transforms \( \hat{P}_1 \) of the first-passage time distributions for \( u_0 = \pm 1 \) are related by

\[
\hat{P}_1(s \mid x'_0, +1) = \hat{P}_1(s \mid x'_0, -1)\tilde{f}(s),
\]

where \( \tilde{f}(s) = \frac{(1 - \beta)\gamma}{s + \gamma + (\xi + \gamma)^2 - 1} \) (8).

Equation (8) shows that the first-passage time process of a CTPRW with initial direction \( u_0 = +1 \) is the convolution of the first-passage time process of a CTPRW with \( u_0 = -1 \) with a new process \( F \) that I call a “flip process”.

A CTPRW with initial direction \( u_0 = +1 \) has the following effective behavior: it first spends a certain time to “flip” (that is: to set itself in the same initial conditions as a CTPRW process with \( u_0 = -1 \)) then it follows the course of a regular CTPRW with \( u_0 = -1 \) until it reaches the origin. As \( \tilde{f} \) is independent of \( x'_0 \), the “flip” is completely independent of the initial distance to the origin and its effect cannot be interpreted as, or reproduced by, a shift of the starting position \( x'_0 \).

The photons emitted away from \( O' \) (with \( u_0 = +1 \)) are thus observed by the moving operator in \( O' \) with a random supplementary delay corresponding to the “flip time”.

**Statistics of the “flip” process.** — The inverse Laplace transform of \( \tilde{f} \) is the distribution of time until the “flip” occurs. Using ref. [28] (formulæ 29.3.52 and 29.3.53), one obtains from eq. (9)

\[
f(t) = (1 - \beta)\gamma \frac{I_1(ct/\ell)}{t} e^{-\gamma ct/\ell}. \tag{10}\]

This distribution is displayed for some values of \( \beta \) in fig. 2.

Up to my knowledge, this expression is not a referenced probability distribution. Note that \( f(t) = P_1(t \mid 0, +1) \) so that, when \( x'_0 = 0 \), the “flip” time is also the time of return at \( O' \). For \( \beta < 0 \), the mean “flip” time and its variance are

\[
E[F] = \frac{1}{|\beta|\gamma}, \quad \text{Var}(F) = \frac{1}{|\beta|^3} \left( \frac{\ell}{\gamma} \right)^2. \tag{11}\]

More properties of \( F \) are given in the appendix.

**Lorentz covariance in two dimensions.** — As previously suggested in the case of the one-dimensional persistent random walk, relativistic random walks should be constructed as Markov processes in phase space. In two dimensions, a point in the photon’s phase space is a pair \((r, \hat{u})\) where \( r \) is the two-dimensional position and \( \hat{u} \) is the two-dimensional unit vector of the direction of propagation. The exact solution of the two-dimensional CTPRW in the reference frame where the cloud of scatterers is at rest, \( \mathcal{R} \), with initial conditions \( X_0 = (r_0 = 0, \hat{u}_0) \) at \( t = 0 \) was recently published [23]. It essentially depends on the Lorentz invariant \( c^2t^2 - r^2 \) and on the variable \( X \):

\[
X^2 = \frac{2}{\ell^2} \frac{(ct - r \cdot \hat{u}_0)(ct - r \cdot \hat{u}_0) - \xi^2}{1 - \hat{u}_0 \cdot \hat{u}_0} - \xi^2 \tag{12}\]

\((\xi = \sqrt{c^2t^2 - r^2}/\ell)\) and it decomposes as the sum \( g = g_0 + g_1 + g_\infty \), where \( g_0 \) is the unscattered contribution (Dirac
Table 1: Table of main results for the first-passage time statistics. \( P_1(t' \mid x_0') \) is the distribution of the first passage time at the origin \( O' \) and \( R(X_0') \) is the probability that the photon will reach the origin in a finite time. \( R(X_0') \) is equal to 1 whenever \( \beta \leq 0 \).

| \( u_0 \) | \( \beta < 0 \) | \( \beta > 0 \) |
|---|---|---|
| \( u_0 = -1 \) | \( P_1(t' \mid x_0', -1) = 2x_0' g_0(x_0' - x_0', t') + \delta(c t' - x_0') \exp\left(-\frac{(1 + \beta)\gamma x_0'/\ell}{\ell}\right) \) | \( R(x_0', -1) = \exp\left(-\frac{2\beta\gamma x_0'/\ell}{1 + \beta}\right) \) |
| \( u_0 = +1 \) | \( P_1(t' \mid x_0', +1) = \frac{2(1 - \beta)\gamma}{\ell(c t' + x_0')} (x_0' g_0(x_0' - x_0', t') + \delta(c t' - x_0') g_0(-x_0', t')) \) | \( R(x_0', +1) = \frac{1 - \beta}{1 + \beta} \exp\left(-\frac{2\beta\gamma x_0'/\ell}{1 + \beta}\right) \) |

delta function), \( g_1 \) is the single scattering contribution and

\[
g_\infty(t, X | X_0) = \frac{1}{2\pi^D 1 - \hat{u} \cdot \hat{u}_0} \Theta(ct - ||r||) \times \ldots \Re \left[ E_i(iX) e^{iX} - E_i(iX - \xi) e^{iX} \right]
\]

is the multiple scattering contribution, with \( E_i \) the exponential integral function [28] (Chapt. 5).

As the process is Markovian, Chapman-Kolmogorov’s relation is fulfilled and the expression (6) is valid using the convention that, in two dimensions, the phase space integral \( \int_X \) denotes \( \int_{x_1} \int_{x_2} \). Again, Chapman-Kolmogorov’s relation is satisfied for all pulse initial conditions in phase space separately, but not necessarily for their superpositions.

The velocity of \( R \) in \( R' \) is denoted, without loss of generality, by \( v = v e_x \). The coordinates of \( r \) transform into the coordinates of \( r' \) by a Lorentz boost whereas the coordinates of \( \hat{u} \) and \( \hat{u}_0 \) transform according to the addition law of velocities. Using these relations and elementary algebra operations shows that the variable \( X' \) is Lorentz invariant (see the appendix for details). The Lorentz covariance of the term \( g_\infty \) follows and that of the terms \( g_0 \) and \( g_1 \) is straightforward.

The Brownian limit. – As mentioned in the introduction, the telegrapher’s equation also appears as a modification of Smoluchowski’s diffusion equation for Brownian motion [10,11]. The classical constructions of Brownian motion are, in various ways, mathematical limits of an underlying random walk performing small steps at a rate going to infinity (see for instance the works of Wiener [29] and Itô [30]). This limit implies that the instantaneous speed of a particle performing the random walk is infinite. Although the limit of infinite speed is legitimate in classical physics, it does not comply with special relativity. Arguably, a relativistic counterpart of the classical Brownian motion should involve particles moving at largest possible speed, the speed of light \( c \). Such particles then must be massless. The relativistic Brownian motion of massive particles has been extensively studied by Dunkel and Hänggi [14,31,32] using stochastic differential equations, where the limit of infinite rate pertains to the concept of noise.

The Brownian limit of the general solution (5) corresponds to taking the speed of light \( c \) to infinity and the mean free path \( \ell \) to zero, while keeping the product \( \ell c = 2D \) constant. It yields a Gaussian distribution in \( R' \) centered at \( \beta c t' = v t' \) and of variance \( 2D t' \). The contributions of the initial direction of propagation \( u_0 \) vanish in the Brownian limit of eq. (5). The two cases \( u_0 = \pm 1 \) in eq. (5a) are therefore indistinguishable such that the solution \( g(x', t') \) in \( R' \) is their sum. Lorentz invariance (and thus causality) also disappears in this limit.

In the limit \( c \to \infty \), the Lorentz factor is \( \gamma \approx 1 + \frac{1}{2} \beta^2 \). The expansion \( \xi \approx \frac{x'}{\beta c t'} - x^2/2\ell c \) and the asymptotic form \( I_0(x) \approx e^{x^2/\sqrt{2\ell c}} \) [28] (formula 9.7.1) give

\[
g(x', t') \approx \frac{1}{\sqrt{4\pi D t'}} \exp\left(-\frac{(x' - \beta c t')^2}{2\ell c t'}\right) \Theta(ct' - |x'|).
\]

The total solution is, in the limit \( c \to \infty \) and \( t' \to 0 \) with \( \ell c = 2D \) constant, equal to

\[
g(x', t') = \frac{1}{\sqrt{4\pi D t'}} \exp\left(-\frac{(x' - v t')^2}{4D t'}\right).
\]

Closing remarks. – In this letter, I have shown that the solutions of the radiative transfer equation are naturally Lorentz covariant and can be obtained from the solutions expressed in phase space. I proved that these solutions satisfy the transitivity of Green’s functions only in phase space, not in position space. In one dimension, I have unveiled a “flip process” that translates as a delay in the measured signal for photons emitted away from the observer. I also have shown that the moments of a signal measured from a pulse become finite in the case where the observer is moving toward the source (\( \beta < 0 \)). These results have been obtained thanks to the stochastic model of continuous-time persistent random walks (CTPRW), which are Markov processes in phase space. I showed that the Markov property of the CTPRW follows from the memorylessness of the exponential probability distribution and the phase space representation.

One should naturally expect that the same approach applies in three dimensions. The three-dimensional Green’s function would indeed be of interest in astrophysics and medical imaging, but no exact solutions are known in three dimensions, even for the energy density (the process in position space). It is however possible that a solution for the radiance, i.e., in phase space, exists. Such a solution could be expressed in terms of a Lorentz invariant variable such...
as \( \mathcal{X} \) from eq. (12), whose three-dimensional counterpart also is Lorentz invariant. In any event, I believe that this letter will trigger progress in this direction.

Using the limit \( c \to \infty \), I showed that the CTPRW is an extension of Brownian motion and therefore that radiative transfer is a natural extension of diffusion in special relativity, requiring a solution in phase space. Several works have already suggested to consider relativistic Brownian motion as a process in phase space [33,34]. A straightforward extension of the CTPRW for massive particles would, for instance, be a process selecting a random momentum according to the Jüttner-Maxwell distribution [35], and a random, exponentially distributed, free travel distance at each step. Such a construction could avoid the difficulties related to the interpretation of stochastic integrals in the above-mentioned works.

Data availability statement: No new data were created or analysed in this study.

Appendix A: Lorentz invariance of \( \mathcal{X} \). Here is a proof that the variable \( \mathcal{X} \) defined by eq. (12) is a relativistic invariant. In two dimensions, the space-time coordinates of \( r = (x, y) \) transform as

\[
x = \gamma x' - \beta \gamma ct', \quad y = y', \quad ct = \gamma ct' - \beta \gamma x',
\]

where the components of the direction of propagation \( \mathbf{u} = (u_x, u_y) \) transform through the velocity addition law:

\[
u_x = \frac{u_x' - \beta}{1 - \beta u_x'}, \quad u_y = \frac{u_y'}{\gamma(1 - \beta u_x')},
\]

Therefore we obtain

\[
ct - \mathbf{r} \cdot \mathbf{u} = \gamma ct' - \beta \gamma x' = \frac{\gamma (x' - \beta \gamma ct') (u_x' - \beta) + y' u_y' / \gamma}{1 - \beta u_x'}
\]

Moreover,

\[
\mathbf{u} \cdot \mathbf{u}_0 = \frac{u_x' - \beta}{1 - \beta u_x'} \frac{u_x' - \beta}{1 - \beta u_x'} + \frac{u_y'}{\gamma(1 - \beta u_x')} \frac{u_y'}{\gamma(1 - \beta u_x')},
\]

and so

\[
1 - \mathbf{u} \cdot \mathbf{u}_0 = \frac{1 - \beta^2}{(1 - \beta u_x')(1 - \beta u_x')} (1 - \mathbf{u}' \cdot \mathbf{u}_0').
\]

As a conclusion, the fraction in eq. (12) transforms as

\[
\frac{(ct - \mathbf{r} \cdot \mathbf{u})(ct - \mathbf{r} \cdot \mathbf{u}_0)}{1 - \mathbf{u} \cdot \mathbf{u}_0} = \frac{(ct - \mathbf{r}' \cdot \mathbf{u}')(ct - \mathbf{r}' \cdot \mathbf{u}_0')}{1 - \mathbf{u}' \cdot \mathbf{u}_0'}
\]

This proves the announced invariance of \( \mathcal{X} \).

Appendix B: more properties of \( F \). The probability density of the “flip” time is, to my knowledge, not a referenced probability density function. I give here the expressions of the probability density \( \sigma_n \) of the sum of \( n \geq 1 \) independent “flip” processes \( F_1 + F_2 + \cdots + F_n = \Sigma_n \) and the moments of order \( k \) of \( \Sigma_n \). These are established using the relation \( \sigma_n = f^{\otimes n} \) and its Laplace transform \( \tilde{\sigma}_n = (\hat{f})^n \), where \( \otimes \) denotes convolution and \( \Sigma_n \) is Gauss’s hypergeometric function,

\[
\sigma_n(t) = n \left[ (1 - \beta) \gamma \right]^n \int_0^t \frac{e^{-\gamma \tau}}{\tau^{n+1}} d\tau,
\]

\[
\mathbb{E}[\Sigma_k^n] = \left( \frac{1 - \beta}{2\gamma} \right)^n \frac{k}{\gamma} \frac{(k + n + 1)!}{(n - 1)!} \times \left( \frac{2}{n + 1} \right) \left( \frac{1}{\gamma^2} \right).
\]

REFERENCES

[1] Ishimaru A., Wave Propagation and Scattering in Random Media (Academic Press, New York) 1978.
[2] Marshak A. and Davis A. B., 3D Radiative Transfer in Cloudy Atmospheres (Springer, Berlin) 2005.
[3] Asllanaj F., Contassot-Vivier S., Liemert A. and Kienle A., J. Biomed. Opt., 19 (2014) 015002.
[4] Shang T. and Gao L., Sci. Sin. B, 31 (1988) 1503.
[5] Jakob W., Arbree A., Moon J., Bala K. and Marschner S., ACM Trans. Graph., 29 (2010) 1.
[6] Chandrasekhar S., Radiative Transfer (Dover, New York) 1960.
[7] Thomson W., Proc. R. Soc. London, 7 (1855) 382.
[8] Weiss G. H., Physica A, 311 (2002) 381.
[9] Hélder P. C., Physica (Amsterdam), 27 (1961) 79.
[10] Brinkman H. C., Physica, 22 (1956) 29.
[11] Sack R. A., Physica, 22 (1956) 917.
[12] Sato H., Geophys. J. Int., 117 (1993) 487.
[13] Paasschens J. C. J., Phys. Rev. E, 56 (1997) 1135.
[14] Dunkel J. and Hänggi P., Phys. Rep., 471 (2009) 1.
[15] Goldstein S., Q. J. Mech. Appl. Math., 4 (1951) 129.
[16] Masoliver J., Lindenberg K and Weiss G. H., Physica A, 157 (1989) 891.
[17] Fürth R., Z. Phys., 2 (1920) 244.
[18] Taylor G. I., Proc. London Math. Soc., 20 (1922) 196.
[19] Kac M., Rocky Mt. J. Math., 4 (1974) 497.
[20] DeWitt-Morette C. and Foong S. K., Phys. Rev. Lett., 62 (1989) 2201.
[21] Foong S. K., Phys. Rev. A, 46 (1992) R707.
[22] Foong S. K. and Kanno S., Stoch. Process. Appl., 53 (1994) 147.
[23] Rossetto V., J. Phys. A: Math. Theor., 50 (2017) 165001.
[24] Rossetto V., J. Stat. Mech., 2018 (2018) 043204.
[25] Morse P. and Feshbach H., Methods of Theoretical Physics I (McGraw Hill, New York) 1953.
[26] Majumdar S., Carv. Sci., 89 (2005) 2076.
[27] Rossetto V., Phys. Rev. E, 88 (2013) 022103.
[28] Abramowitz M. and Stegun I. A., Handbook of Mathematical Functions, 10th edition (Dover, New York) 1972.
[29] Wiener N., J. Math. Phys., 2 (1923) 131.
[30] Ito K., Proc. Jpn. Acad., 26 (1950) 4.
[31] Dunkel J. and Hänggi P., Phys. Rev. E, 71 (2005) 015124.
[32] Dunkel J. and Hänggi P., Phys. Rev. E, 72 (2005) 036106.
[33] Dudley R. M., Ark. Mat., 6 (1966) 241.
[34] Hakim R., J. Math. Phys., 9 (1968) 1805.
[35] Jüttner F., Ann. Phys., 339 (1911) 856.