Spontaneous symmetry breaking for long-wave gravitons in the early Universe

A. A. Grib, V. Yu. Dorofeev*

A. A. Friedmann laboratory for theoretical physics
Moika 48, St. Petersburg, Russia.

Abstract

It is shown that nonlinear terms in equations of gravitons on the background of curved space-time of the expanding Universe can solve the problem of the negative square of the effective mass formally arising in linear approximation for gravitons. Similar to well known spontaneous breaking of symmetry in Goldstone model one must take another vacuum so that nonzero vacuum expectation value of the quantized graviton field leads to change of spectrum for gravitons. There appears two graviton fields, one with the positive mass, another with the zero mass. Energy density and the density of particles created by gravitation of the expanding Universe are calculated for some special cases of the scale factor. Numerical results are obtained for the dust universe case.

Introduction

There is a problem in quantum theory of gravitons created from vacuum in the expanding Universe with nonzero scalar curvature $R$ (inflation, dust, etc.) concerning the long wave graviton modes. In linearized theory of quantum gravitons in curved space-time of the isotropic homogeneous Universe one obtains after separation of variables in the wave equation the equation for the function dependent only on time. After conformal (Weyl) transformation to stationary metric this equation can be understood as equation in flat stationary metric with time dependent mass. It occurs that for long waves this effective mass squared is negative. All this occurs due to conformal noninvariance of the graviton theory for nonzero $R$ leading to tachyonic behaviour of long waves modes. The necessity of going from nonstationary metric to the stationary one is motivated by finite results for particle creation [1]. In the end one surely must return to the original space-time where the obtained results are still finite.

In some papers (see [2] and references there) it was proposed to consider these modes as classical excitations of the field growing in time, so that one must

*e-mail: Andrei_Grib@mail.ru, dor@vd8186.spb.edu, friedlab@mail.ru
quantize only modes with momentum with the square larger than the negative square of the effective mass. However one knows from the quantum field theory that tachyonic behaviour disappears if one takes into account nonlinear terms neglected in linearized theory. This is typical in theories of spontaneous breaking of symmetry due to redefinition of the vacuum leading to its noninvariance to this or that transformation of the Lagrangian. In quantum theory based on a new vacuum one gets new masses for the redefined quantum field so that there is no negative mass square.

In this paper the analogous program is made for gravitons. It occurs that if one is going from the linear theory of gravitons taking into account the next order of nonlinearity one gets the redefinition of vacuum solving the problem for long wave gravitons. In the result one gets gravitons with zero and positive effective mass.

Differently from the situation in theory of weak interactions where Higgs potential with nonlinear term is taken by hand in our case the nonlinear term naturally appears as the second order in Einstein equation.

In the end of the paper the expressions for the particle density and the energy density are obtained for gravitons created in expanding Universe with metric which has some special dependence of the scale factor on time.

1 Getting the graviton equation from Einstein equation

Einstein equations in presence of matter have the form

\[ R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik} \tag{1} \]

or

\[ R^i_k = \kappa(T^i_k - \frac{1}{2}\delta^i_k T) \tag{2} \]

Let us consider the case when matter is homogeneous isotropic liquid filling the Universe. Then

\[ T_{ik} = (\varepsilon + p)u_i u_k - g_{ik}p \tag{3} \]

where \( u_i \) is the four velocity, \( p \) – the pressure and \( \varepsilon \) – the energy density of the liquid.

The problem of creation of gravitons in the early Universe was discussed in literature with gravitons considered as quantized small term in the metrical tensor. Due to absence of exact quantum gravity usually one deals with linearized analogy with quantization of other quantum fields. First let us obtain equations for classical small perturbations of the metrical tensor and then do quantization. Consider the graviton perturbations-the gravitational waves as small term added to the background metric So there are

\[ g_{ik} = g^{(c)}_{ik} + h_{ik} \tag{4} \]
if $h_{ik} = 0$ the \((o)\) $g_{ik}$ is the solution of Einstein’s equation of the form

\[
R^i_k = \kappa (\mathbb{T}^i_k - \frac{1}{2} \delta^i_k \mathbb{T})
\]  

\[(5)\]

Let us go from the up to low indices and vice verse by using the background metric \((o)\) $g_{ik} = g^{in} n_{ik}$ and expand equations \[(2)\] in a series in $h^i_k$:

\[
R^i_k + \delta R^i_k = \kappa (\mathbb{T}^i_k - \frac{1}{2} \delta^i_k \mathbb{T} + \delta \mathbb{T})
\]  

\[(6)\]

from which due to \[(6)\] the perturbations $h_{ik}$ satisfy equations

\[
\delta R^i_k = \kappa (\delta \mathbb{T})
\]  

\[(7)\]

Using the notation \((1 + h)^{-1}\) for the matrix inverse to \((1 + h)\) with small $h^i_k$ (small in the sense that all eigenvalues of the matrix \((1 + h)\) are smaller than the unit) one obtains

\[(1 + h)^{-1} = \delta^i_k - h^i_k + h^i_k h^n_l - \ldots\]

\[(8)\]

Write the Ricci tensor and the curvature tensor as

\[
R^i_k = (1 + h)^{-1} \nu^{-1} \nu \left( h^i_{ik} R^k + \frac{1}{2} (1 + h)^{-1} \nu^{-1} \nu \nu (h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l) \right)
\]  

\[
+ \frac{1}{4} (1 + h)^{-1} \nu^{-1} \nu \nu (h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l)
\]  

\[
\cdot (h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l)) \right) + \nu \left( R^i_k \right)
\]  

\[(9)\]

\[
\nu \left( h^i_{ik} R^k + \frac{1}{2} (1 + h)^{-1} \nu^{-1} \nu \nu \left( h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l) \right) \right)
\]  

\[
+ \frac{1}{4} (1 + h)^{-1} \nu^{-1} \nu \nu \left( h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l) \right) \right) + \nu \left( R^i_k \right)
\]  

\[(10)\]

\[
R = (1 + h)^{-1} \nu^{-1} \nu \left( h^i_{ik} R^k + \frac{1}{2} (1 + h)^{-1} \nu^{-1} \nu \nu (h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l) \right)
\]  

\[
+ \frac{1}{4} (1 + h)^{-1} \nu^{-1} \nu \nu \left( h^i_{ik} h^m_l m^i_l - 2h^i_{ik} h^m_l m^i_l) \right) \right) + \nu \left( R^i_k \right)
\]  

\[(11)\]

Here \((,)^{\nu}\) means the covariant derivative in background metric \((o)\) $g_{ik}$. Considering in \[(10)\] only first degree in $h_{ik}$ one obtains the linearized equations for $h_{ik}$ in \[(11)\] as

\[
h^m_{ik;n} + h_{i;k;n} - h^m_{i;k;n} - h^m_{i;k;n} \quad g_{ik} (h^m_{i;n} - h^m_{i;n} - h_{ik}) = \nu \left( R^i_k \right)
\]  

\[(12)\]
Let us consider the background as homogeneous isotropic nonstationary space-time

\[
ds^2 = dt^2 - a^2(t)\tilde{d}t^2
\]

where \( \eta \) is the conformal time. Here the Latin indices take the values 0, 1, 2, 3 and the Greek \( -1, 2, 3 \). Then write for the scalar curvature and the Ricci tensor

\[
\delta R = (1 + h)^{-1} \frac{k}{\nu} ((1 + h) - 1 \frac{k^{i'k'}}{\nu} - \frac{1}{a^2} h_{k_0,0}^{i'} - \frac{3a'}{a^3} h_{k_0}^{i'})
\]

\[
+ \frac{2\epsilon}{a^2} h_k^{i'} + \frac{1}{4} (1 + h)^{-1} \frac{k^{i'}\nu}{\nu} (1 + h)^{-1} \frac{n^{i'}n'}{n'} (4h_{\nu n}^{i'}(h_k^{k'k'}) - h_k^{k'k'}) + 3h_{k,n}^{i'} h_{n^{i'}k'}
\]

\[
- 2h_{k,n}^{i'} h_{n^{i'}k'} - h_{k_0}^{i'} h_{k_0}^{i'}
\]

\[
+ \frac{1}{4} (1 + h)^{-1} \frac{k^{i'}\nu}{\nu} h_{k_0,0}^{i'} - \frac{1}{2}(1 + h)^{-1} \frac{n^{i'}n'}{n'} h_{k_0,0}^{i'} h_{n^{i'}n'}
\]

\[
\delta R_\alpha = (1 + h)^{-1} \frac{h_{\alpha}^{i'\nu}}{\nu} (h_{\nu\beta}^{i'} + h_{\nu}^{i'\beta} - h_{\beta}^{i'\nu})
\]

\[
+ \frac{1}{4} (1 + h)^{-1} \frac{n^{i'}n'}{n'} (h_{\nu\beta}^{i'} h_{n^{i'}n'} - 2h_{\nu}^{i'} - h_{\beta}^{i'} h_{\nu\beta}^{i'})
\]

\[
- 2h_{\nu\beta}^{i'} h_{n^{i'}n'} - h_{\nu}^{i'} h_{\beta}^{i'} - 2h_{\nu}^{i'} h_{\beta}^{i'} h_{n^{i'}n'}
\]

\[
- \frac{1}{2a^2} h_{\nu}^{i'} h_{\nu}^{i'} - \frac{a'}{a^3} h_{\nu}^{i'} h_{\nu}^{i'} + \frac{2\epsilon}{a^2} h_{\nu}^{i'} - \frac{a'}{a^3} S_{\nu}^{\beta}(1 + h)^{-1} \frac{k^{i'}}{\nu} h_{\nu}^{i'}
\]

(14)

where \( \epsilon = \pm 1 \) for the closed, open and flat Universe. The sign "\( \nu \)" is used for the covariant derivative in space part of the metric and "\( \nu \)" or comma for the derivative in conformal time \( \eta \). Metric \( g_{ik} \) is defined up to arbitrary coordinate transformations so one can put some auxiliary conditions.

Solutions (6) for \( h_{ik} \) can be written as

\[
h_{ik}^{i'} = S_{ik}^{i'} + V_{ik}^{i'} + D_{ik}^{i'}
\]

where \( S_{ik}, V_{ik}, B_{ik} \) are irreducible scalar, vector and tensor components of the tensor satisfying the conditions (3):

\[
\begin{align*}
\tag{c} g_{ik} S^{ik} &= S 
eq 0, & \tag{c} g_{ik} V^{ik} &= 0, & \tag{c} g_{ik} B^{ik} &= 0
\end{align*}
\]

Considering only the gravitational waves exclude the scalar and vector parts by putting the gauge conditions

\[
h = 0, \quad h_{k_0}^{i'} = 0
\]

(15)
In this case the linearized equations for perturbations \( h^i_k \) take the form

\[
h^\alpha_{\beta,0,0} + \frac{2a'}{a} h^\alpha_{\beta,0} + \frac{2\epsilon}{a^2} h^\alpha_{\beta,\gamma} + h^{\alpha\gamma}_{\beta,0} + 2 a^2 h^\alpha_{\beta} = 0 \tag{16}
\]

Go from the variables \( h^i_k \) to new variables \( \mu^i_k = a(\eta) h^i_k \) and make the conformal transformation

\[
\tilde{g}_{ik} = g_{ik}/a^2(\eta) \tag{17}
\]

Then one obtains the equation for the field with spin 2 in Minkowsky flat space in some external effective field (\( \epsilon = 0 \))

\[
\mu^\alpha_{\beta,0,0} + \mu^\alpha_{\beta,\gamma} - \frac{a''}{a} \mu^\alpha_{\beta} = 0 \tag{18}
\]

After separation of variables and Fourier representation \( \mu^\alpha_{\beta}(x) \)

\[
\mu^\alpha_{\beta}(x) = \int d^3k (g^\alpha_k(\eta)e^{i\vec{k}\cdot\vec{x}}a^\alpha_{\beta} + g^\alpha_k(\eta)e^{-i\vec{k}\cdot\vec{x}}a^\alpha_{\beta}^*) \tag{19}
\]

one obtains for the time dependent \( g^\alpha_k(\eta) \) function the equation

\[
g''_k(\eta) + (k^2 - \frac{a''}{a}) g_k(\eta) = 0 \tag{20}
\]

which formally has the negative square of the effective mass \( m^2a^2 = -a''/a \).

Differently from the situation in theory of weak interactions where Higgs potential with nonlinear term is taken by hand in our case the nonlinear term naturally appears as the second order in Einstein equation.

Equation (16) was considered in the paper of L. P. Grishuk [5]. Calculations of gravitational excitations based on eq. (18) were made in [6], where absence of the infrared divergence in this method was shown. Calculation of the energy density and pressure of created gravitons was made in the papers of A. Starobinsky [8] and V. Sahni [7]. For small \( k \) the Fourier transform of the solution (16) was obtained as:

\[
h^\alpha_{\beta,k}(\vec{k}) = a^\alpha_{\beta}(\vec{k}) + b^\alpha_{\beta}(\vec{k}) \int \frac{d\eta}{a^2(\eta)} \tag{21}
\]

Note however that solutions of the form (21) for small \( k \) cannot be interpreted in terms of usual particles, that is why L. P. Grishuk [2] considered them as "frozen" modes forming some condensed classical state.

Here we continue this research considering what changes in the form of the condensed state (or the new vacuum) are introduced by next orders in Einstein’s equations.

\section{Third order equations for gravitational waves}

Let us consider the right hand side of Einstein’s equations. For [3]

\[
\delta T^i_k = (\delta \epsilon + \delta p) u^i u_k - \delta^i_k \delta p + (\epsilon + p)(\delta u^i u_k + u^i \delta u_k + \delta u^i \delta u_k) \tag{22}
\]
One has for the four velocities \( u_i \) and \((^o u)_i\) the conditions \( g_{ik} u^i u^k = 1 \) and \((^o g)_{ik} (^o u)^i (^o u)^k = 1\). So
\[
2 (^o u)_k \delta u^k + (^o g)_{ik} \delta u^i \delta u^k = 0 \tag{23}
\]

Note that \( \delta u^i \) due to constraints \((15)\) can depend only on \( h^i_{\ell \beta} h_{\ell \beta} \) \ldots so \( \delta u^\alpha \delta u^\beta \) depends on the squares of these terms. But we shall neglect the fourth and higher orders. In synchronous reference system \( u^0 = 1/a, u^\alpha = 0 \) so from \((23)\) one has \( \delta u^0 = 0 \) and
\[
\delta R^0 = \frac{1}{2} \kappa (\delta \varepsilon + 3 \delta p), \quad \delta R^\alpha = \frac{1}{2} \kappa \delta^\beta_\alpha (\delta \varepsilon - \delta \varepsilon) \tag{24}
\]

Consider the case of flat space \( \varepsilon = 0 \). Then \( h^\alpha_{\beta \gamma} = h^\alpha_{\beta \gamma}, h^\beta_{\alpha \gamma} = \frac{1}{a^2} h^\beta_{\beta \gamma} \) where Greek indices are put up and below by use of the Minkowsky metric. One can look on eqs. \((24)\) as on Euler Lagrange equations for the fields \( h_{ik} \). Then due to constraints \((15)\) up to terms of the divergence form one can obtain that not only \( h^i_{k \ell} = 0 \) but \( h^i_{k \ell} h_{k \ell} = 0 \ldots \) so that \((14)\) can be transformed to
\[
\frac{1}{2} \kappa (\delta \varepsilon + 3 \delta p)
\]

From \((24)\) follows that one can take instead of \((7)\) the equations
\[
(1 + h) \delta R^\alpha = \frac{1}{2} \kappa (1 + h) \delta \varepsilon \tag{25}
\]

Multiply the last equation on \( \nu - 2a^2 \), then from \((24), (25)\) one obtains
\[
\begin{align*}
\delta R^\alpha & = \frac{1}{2} \kappa (1 + h) \delta \varepsilon \\
\frac{2a'}{a} h^\alpha_{\beta,0} + \frac{a'}{a} (1 + h) h^\alpha_{\beta,0} & = \frac{1}{2} \kappa (1 + h) \delta \varepsilon
\end{align*}
\tag{26}
\]

Consider first three orders in \( h^i_{k \ell} \) in equations \((27)\).
\[
\begin{align*}
\delta R^\alpha & = \frac{1}{2} \kappa (1 + h) \delta \varepsilon \\
\frac{2a'}{a} h^\alpha_{\beta,0} + \frac{1}{2} \delta_\beta^\alpha & = \frac{1}{2} \kappa (1 + h) \delta \varepsilon
\end{align*}
\tag{27}
\]
So account for fixed nonzero components of the tensor after simple transformations one obtains

\[
2h_{\alpha,0} + \frac{2a'}{a} h_{\alpha,0} + h_{n,0} h_{\alpha,0} - h_{l,0} h_{\alpha,0} - h_{l,n} h_{\alpha,n} - \frac{1}{2} h_{l,0} h_{\alpha,0} - h_{l,0} h_{\alpha,0}
\]

or

\[
\frac{1}{2} h_{\alpha,0} + \frac{2a'}{a} h_{\alpha,0} + h_{n,0} h_{\alpha,0} = \frac{1}{2} h_{\alpha,0} h_{n,0} + \frac{2a'}{a} h_{\alpha,0} h_{n,0} = h_{\alpha,0} h_{n,0} = 0
\]

Putting away the divergence of \( h_{\alpha,0} h_{n,0} \) and taking into account for fixed nonzero components of the tensor \( h_{\alpha,0} \) the condition \( h_{\alpha,0} h_{l,0} = 0 \) after simple transformations one obtains

\[
h_{\alpha,0} + \frac{2a'}{a} h_{\alpha,0} + h_{n,0} h_{\alpha,0} + h_{l,0} h_{\alpha,0}
\]

Now let us go from variables \( h_{\alpha,0} \) to variables \( \mu_{\alpha} = a(\eta) h_{\alpha,0} \) and make the conformal transformation

\[
\bar{g}_{ik} = g_{ik}/a^2(\eta)
\]

Then we obtain the equation in flat Minkowsky space with some effective external field

\[
\mu_{\alpha,0,0} + \mu_{\alpha,0} - \frac{a'}{a} \mu_{\alpha} + \frac{2a'}{a} (\mu_{\alpha,0} - \mu_{\alpha,0} \mu_{\alpha,0} - \frac{a'}{a} \mu_{\alpha,0} \mu_{\alpha,0})
\]

or

\[
\frac{1}{a^2} h_{\alpha,0} (\frac{1}{2} h_{\alpha,0} h_{\alpha,0} + h_{\alpha,0} h_{\alpha,0} + h_{\alpha,0} h_{\alpha,0} + h_{\alpha,0} h_{\alpha,0}) = 0
\]
3 Spontaneous breaking of symmetry for gravitons

Let us consider vacuum solution \(32\) depending only on time. In quantum field theory this means dependence of vacuum on time. Then

\[
\mu_{\beta,0,0}^\alpha = \frac{a''}{a^2} \mu_{\beta}^\alpha - \frac{a'^2}{8a^4} \mu_{\alpha}^\beta \mu_{\beta}^\alpha \tag{33}
\]

Taking into account constraints \(15\) in variables \(\mu_1 = -\mu_2, \quad \mu_2 = \mu_1^2\) one gets the potential corresponding to \(33\) as

\[
V = -\frac{a''}{2a} \mu_{\beta}^\alpha \mu_{\alpha}^\beta + \frac{a'^2}{8a^4} (\mu_{\beta}^\alpha \mu_{\alpha}^\beta)^2 \tag{34}
\]

Write the field \(\mu_{\beta}^\alpha\) close to the minimum of the potential energy

\[
\frac{a''}{a} = \frac{a'^2}{2a^2} \mu_{\beta}^\alpha(0) \mu_{\alpha}^\beta(0), \quad \mu_{\beta}^\alpha = \mu_{\beta}^\alpha(0) + \xi_{\alpha}^\beta \tag{35}
\]

One must note that the condition \(35\) on \(\mu_{\beta}^\alpha(0)\) is the condition of minimal energy at some fixed moment \(t_0\).

This is the basic idea. Instead of dealing with time dependent \(m, \lambda\) we put the initial conditions at some \(t_0\). This corresponds to the principle of minimal energy at this moment. Surely \(m, \lambda\) at this moment are numbers.

Take the solution \(36\) as

\[
\mu_1(0) = \mu_1^2(0) = 0, \quad \mu_1(0) = -\mu_2^2(0) = \mu_0 = \sqrt{\frac{a''}{2a^2}} \tag{36}
\]

Then the Lagrangian

\[
L = \frac{1}{2} \mu_{\beta}^\alpha \mu_{\alpha}^\beta - \frac{a'^2}{8a^4} (\mu_{\beta}^\alpha \mu_{\alpha}^\beta)^2
\]

can be written as

\[
L = \frac{1}{2}(\mu_{\beta}^\alpha(0) + \xi_{\beta}^\alpha)(\mu_{\alpha}^\beta(0) + \xi_{\alpha}^\beta) + \frac{a''}{8a^4} (\mu_{\beta}^\alpha(0) + \mu_{\alpha}^\beta(0)) (\mu_{\alpha}^\beta(0) + \mu_{\alpha}^\beta(0))
\]

\[
= \mu_{0,0}^2 + \frac{1}{2} \xi_{\alpha}^\alpha \xi_{\alpha}^\beta \mu_{\alpha,0} + \mu_{0,0}(\xi_{1,1}^\beta - \xi_{2,2}^\beta) + \frac{a''}{2a}(2\mu_{0}^\beta + 2\mu_{0}^\beta(\xi_{1}^\beta - \xi_{2}^\beta))
\]

\[
+ \xi_{\beta}^\beta \xi_{\alpha}^\alpha - \frac{a'^2}{8a^4} (4\mu_0^4 + (\xi_{\beta}^\alpha \xi_{\alpha}^\beta)^2 + 4\mu_0 \xi_{\beta}^\alpha \xi_{\alpha}^\beta (\xi_{1}^\beta - \xi_{2}^\beta) + 8\mu_0^2 (\xi_{1}^\beta - \xi_{2}^\beta))
\]

\[
+ 4\mu_0^2 (\xi_{\beta}^\alpha \xi_{\alpha}^\beta + 4\mu_0^2 ((\xi_{1}^\beta)^2 + (\xi_{2}^\beta)^2 - 2\xi_{1}^\beta \xi_{2}^\beta))
\]

\]

8
Consider the quadratic in $\xi^\beta_\alpha$ terms

$$L = \frac{1}{2} \xi^\alpha_\alpha, n \xi^\beta_\beta, n - \frac{a''}{2a^2} \xi^\alpha_\alpha \xi^\beta_\beta + \frac{a'}{2a} \mu_0^2 \xi^\alpha_\alpha \xi^\beta_\beta + (\xi^1_1)^2 + (\xi^2_2)^2 - 2\xi^1_1 \xi^2_2$$  \hspace{1cm} (37)

Taking into account the equality $\mu_0^2 \frac{a''}{a} = \frac{a''}{a}$ and the gauge $\xi^1_1 + \xi^2_2 = 0$ one obtains the Lagrangian for gravitons

$$L(\xi) = \frac{1}{2} \xi^\alpha_\alpha, n \xi^\beta_\beta, n - \frac{a''}{a} ((\xi^1_1)^2 + (\xi^2_2)^2)$$  \hspace{1cm} (38)

which is called by us the effective Lagrangian after the spontaneous breaking of symmetry.

Euler-Lagrange equations have the form

$$\xi^1_1, n + \frac{2a''}{a} \xi^1_1 = 0, \quad \xi^2_2, n + \frac{2a''}{a} \xi^2_2 = 0, \quad \xi^1_1, n = 0, \quad \xi^2_2, n = 0$$  \hspace{1cm} (39)

One sees that now in (39) the sign of the mass squared is a correct one. The components $\xi^1_1, \xi^2_2$ are massless while $\xi^2_1, \xi^1_2$ have the nonnegative mass squared $\frac{2a''}{a}$. The solutions for diagonal components $\xi^1_1, \xi^2_2$ (or $\xi^\alpha_\alpha, \alpha = 1, 2$) can be written as $\xi$ the Fourier integral

$$\xi(x) = \frac{1}{(2\pi)^3} \int d\vec{k} (c_\xi g_k^*(\eta) e^{i \vec{k} \vec{x}} + c_\xi g_k(\eta) e^{-i \vec{k} \vec{x}})$$  \hspace{1cm} (40)

And one has the equation

$$g_k'' + (k^2 + \frac{2a''}{a}) g_k = 0$$  \hspace{1cm} (41)

This equation is free from the problem of the negative square of the effective mass if $\frac{2a''}{a} > 0$ and one can construct the quantum theory of gravitons based on the new vacuum state (36). One can notice that the vacuum expectation value of the field $\mu(\eta)$ is close to the scale factor. For the scale factor $a(\eta) = \eta^p$ one obtains

$$\mu_0 = a(\eta) \sqrt{\frac{p-1}{2p}}$$

One sees that for all $p \in [-1; 0]$ (if $p \in (0; 1)$ then $m^2 = -2a''/a > 0$ and we leave vacuum as $\mu_0 = 0$) the dynamical perturbation $h^2_\beta \geq 1$ which contradicts the condition of the expansion of the curvature tensor into a series in this perturbation. The value $p = -1$ corresponds to inflation, so we cannot deal the inflation model here while other situations can be considered. However this case must be considered separately and it is not studied in this paper.
The Lagrange formalism for gravitons

One can see from (40-41) that gravitons in the expanding isotropic Universe are described by the effective scalar field $\vartheta(x)$ with the Lagrangian

$$L = \sqrt{-g}(\vartheta(x),\partial\vartheta(x)^n - \frac{1}{3}R\vartheta(x)\partial\vartheta(x))$$

where $g = \det(g_{ik})$ and $R$ the scalar curvature. There is no factor $\frac{1}{2}$ because one deals with two polarizations. Euler-Lagrange equation for the field is

$$\xi_1^1 = \xi_2^2 = \xi = \vartheta \cdot a.$$ Look for solutions of (41) in the form (40). The numbers $c_\vec{k}, c_\vec{k}^\dagger$ are changed on the operators $\hat{c}_\vec{k}, \hat{c}_\vec{k}^\dagger$ with commutation relations

$$[\hat{c}_\vec{k}, \hat{c}_\vec{k}^\dagger] = \delta(\vec{k} - \vec{k}'), \quad [\hat{c}_\vec{k}, \hat{c}_\vec{k}^\dagger] = [\hat{c}_\vec{k}^\dagger, \hat{c}_\vec{k}^\dagger] = 0$$

The Fock vacuum state $|in\rangle$ is defined as

$$\hat{c}_\vec{k}|in\rangle = 0, \quad <\text{in}|\text{in}\rangle = 1$$

Then for $\tilde{\vartheta}(x) = \frac{1}{a}\xi(x)$ one has

$$\tilde{\vartheta}(x) = \frac{1}{(2\pi)^3 a(\eta)} \int d\vec{k} (\hat{c}_\vec{k}^\dagger g_k^*(\eta)e^{i\vec{k}\vec{x}} + \hat{c}_\vec{k}g_k(\eta)e^{-i\vec{k}\vec{x}})$$

where $g_k(\eta)$ satisfy (41) written as

$$g_k'' + \omega^2_k(\eta)g_k = 0, \quad \omega^2_k(\eta) = \frac{2a''}{a} + k^2$$

with initial conditions

$$g_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \quad g_k'(\eta_0) = i\sqrt{\omega_k(\eta_0)}$$

The condition for the wronskian

$$g_k(\eta_0)g_{k'}^*(\eta_0) - g_{k'}^2(\eta_0)g_k'(\eta_0) = -2i$$

leads to existence of the full set of solutions of (41) in the sense of the indefinite scalar product

$$(\xi_1, \xi_2) = i \int d\vec{x}(\xi_1^\dagger \rightarrow \partial_0 \xi_2)$$

The Hamiltonian of the quantized field $\hat{\vartheta}(x)$ in the metric (13) has the form

$$\hat{H}(\eta) = \int d\vec{x}(\xi_1^\dagger \xi_2 + \frac{2a''}{a} \xi_1^\dagger \xi_1)$$

Putting the field $\tilde{\vartheta}(x)$ from (49) into (44) one obtains

$$\hat{H}(\eta) = \frac{1}{\pi^2 a^4(\eta)} \int_0^\infty k^2 dk \omega_k(\eta)(E_k(\eta)(\hat{c}_k^\dagger \hat{c}_k + \hat{c}_k \hat{c}_k^\dagger))$$
\[
+ F_k \hat{c}_k^+ \hat{c}_k + F_k^* \hat{c}_k \hat{c}_k^*
\]

(50)

where the coefficients \(E_k, F_k\) are expressed through the solutions of the

\[
E_k(\eta) = \frac{1}{2\omega}(|g_k|^2 + |g_k|^2) \quad F_k(\eta) = \frac{1}{2\omega}(g_k^2 + g_k^2)
\]

(51)

The corpuscular interpretation can be made in terms of creation and annihilation operators \(\hat{b}_k, \hat{b}_k^+\) diagonalizing the Hamiltonian. If

\[
\hat{c}_k = \alpha_k^*(\eta) \hat{b}_k - \beta_k(\eta) \hat{b}_k^+
\]

then the Hamiltonian is

\[
\hat{H}(\eta) = \frac{1}{\pi^2 a^4(\eta)} \int_0^\infty k^2 dk \omega_k(\eta)(E_k(\eta) - 1)(\hat{b}_k^+ \hat{b}_k + \hat{b}_k \hat{b}_k^+)
\]

(52)

The density of created particles and their energy density [1] can be found using formulas

\[
n(\eta) = \frac{1}{\pi^2 a^4(\eta)} \int_0^\infty k^2 dk |\beta_k|^2
\]

(53)

5 Some models of graviton creation

Let us consider some matter filling the Universe with the equation of state \(p = \gamma \varepsilon\) where \(p\) is pressure and \(\varepsilon\) the energy density. One has for the homogeneous quasieuclidean isotropic Universe the equation [4]

\[
8\pi \kappa \frac{\varepsilon}{c^4} = \frac{3a'^2}{a^4}, \quad \text{then} \quad a(\eta) = C\eta^{\frac{2}{1+3\gamma}}
\]

(54)

Let us take \(a(\eta) = C\eta^p\) (take \(p > 1\)) and put it into (50), then one obtains

\[
g''(\eta) + (2p(p-1)\frac{1}{\eta^2} + k^2)g(\eta) = 0
\]

(55)

where

\[
m^2 = 2p(p-1) = \frac{4(1-3\gamma)}{(1+3\gamma)^2}, \quad a(\eta) = \frac{1}{\eta}
\]

So the results obtained for the scalar field in [1] are valid for gravitons for any scale factor with the scale factor of a given form. In [1] it was shown that for the density of created particles and the energy density defined by (52 - 53) one gets convergent integrals. Let us calculate them. Putting the notation \(x = k\eta\) one gets

\[
\frac{d^2g}{dx^2} + (1 + \frac{m^2}{x^2})g(x) = 0
\]

(56)

Then the energy density of created particles due to (56) is calculated as

\[
\varepsilon(\eta) = \langle 0|\hat{T}_0^0|0\rangle
\]
\begin{equation}
\frac{2}{\pi^2(a(\eta)\eta)^4} \int_0^\infty x^3 dx \omega(x) \left( \frac{1}{2\omega(x)} (|\frac{dg(x)}{dx}|^2 + \omega^2(x)|g(x)|^2) - 1 \right) \tag{57}
\end{equation}

The solutions of (56) are Bessel functions

\begin{equation}
g(x) = C_1 \sqrt{\frac{\pi}{2}} J_{\frac{1}{2}} \left( \frac{1}{2} \sqrt{1 - 4m^2}, x \right) + C_2 \sqrt{\frac{\pi}{2}} Y_{\frac{1}{2}} \left( \frac{1}{2} \sqrt{1 - 4m^2}, x \right) \tag{58}
\end{equation}

Then

\begin{equation}
\varepsilon(\eta) \approx \frac{2}{\pi^2(a(\eta)\eta)^3} 0.04 m^3 = \frac{1.5 \cdot 10^{-3} R^{3/2}}{a(\eta)\eta^4}, \quad 0 < m^2 < 4 \tag{59}
\end{equation}

For the density of created particles in the unit volume one gets

\begin{equation}
n(\eta) = \frac{1}{\pi^2(a(\eta)\eta)^3} \int_0^\infty x^2 dx \left( \frac{1}{2\omega(x)} (|\frac{dg(x)}{dx}|^2 + \omega^2(x)|g(x)|^2) - 1 \right) \tag{60}
\end{equation}

This integral is convergent [11]. For small \(m\) (\(0 < m < 0.5\)) \(n(\eta) \sim R\) and for large \(m\) (\(m > 0.5\)) \(n(\eta) \sim \sqrt{R}\).

Consider dust Universe with \(a(\eta) = C \eta^2\). Then

\begin{equation}
\varepsilon_{\text{gr.}} = \frac{4 \cdot 10^{-3}}{t^4} \tag{61}
\end{equation}

For the background classical matter one has

\begin{equation}
\varepsilon_{\text{ matt.}} = \frac{2 \cdot 10^{84}}{t^2} \tag{62}
\end{equation}

So for Planckean time (\(t_{pl} = 10^{-43}\) sek) the graviton energy density created from vacuum is some ten percent of the matter density while at the inflation time \(t_{inf} = 10^{-36}\) sek it is only \(10^{-14}\) of matter. These numbers are consistent with our approximation for the metric in perturbation theory. At the modern epoch one gets from (60) that the energy flow from the time of the end of inflation \(t_{inf} = 10^{-36}\) sek is

\begin{equation}
\varepsilon = 0.5 \cdot 10^{-12} \left( \frac{\text{erg}}{\text{sec} \cdot \text{cm}^2} \right) \tag{63}
\end{equation}

This can be compared with the flow from the Crab nebula [9]. One sees that it is much smaller.

\begin{equation}
\varepsilon_{\text{Crab}} = 10^{-8} \left( \frac{\text{erg}}{\text{sec} \cdot \text{cm}^2} \right) \tag{64}
\end{equation}

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