More Connections on Valuated Binary Tree and Their Applications in Factoring Odd Integers

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper continues investigating connections on a valuated binary tree. By defining three types of new connections, the paper derives several new properties for the new connections, and proves that odd integers matching to the new cases can be easily and rapidly factorized. Proofs are presented for the new properties and conclusions with detail mathematical reasoning and numerical experiments are made with Maple software to demonstrate the fast factorization by factoring big odd composite integers that are of the length from 101 to 105 decimal digits. Source codes of Maple programs are also list for readers to test the experiments.

Keywords: Integer factorization; valuated binary tree; geometric relationship; connection.

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1 Introduction

By means of defining parallelism, connection and penetration, paper [1] investigated geometric relationships among nodes on a valuated binary tree, and it proved several properties about the connections and the

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penetrations as well as some significant corollaries for fast factorization of special kind of big odd integers. The results together with the previous results obtained in the bibliographies [2] to [7] exhibit that the valuated binary tree method is a new systematic approach to analyze odd integers.

This paper follows the study of paper [1], continues the investigation on the connections and their applications in factorization of odd composite integers. By defining three new types of the connections, the paper reasons and obtains several new properties and corollaries in analyzing the odd integers.

2 Preliminaries

The terms related with the valuated binary tree, subtree, root, node, son, father and ancestors as well as symbols used in this paper can be referred in [1]. Some cited lemmas were also seen in [1].

2.1 New lemmas

Lemma 1 ([8]). Let \( N \) be an odd integer on a tree; then \( N \)'s direct ancestor that is \( \alpha \) levels away from \( N \) is calculated by

\[
A_{\alpha} = \begin{cases} 
N - 1 & \text{if} \; f_{\alpha} \text{ is even} \\
N & \text{if} \; f_{\alpha} \text{ is odd}
\end{cases}
\]

where \( f_{\alpha} = \left\lfloor \frac{N - 1}{2^{\alpha - 1}} \right\rfloor \).

Lemma 2. Let \( i \geq 0 \), \( j \geq 0 \) and \( \omega \geq 0 \) be integers; the equality \( \left\lfloor \frac{j}{2^{i+1}} \right\rfloor = \omega \) holds for either \( \left\lfloor \frac{j}{2^{i+1}} \right\rfloor = 2\omega \) or \( \left\lfloor \frac{j}{2^{i+1}} \right\rfloor = 2\omega + 1 \).

Proof. See the following reasoning.

\[
\left\lfloor \frac{j}{2^{i+1}} \right\rfloor = 2\omega \Rightarrow j = 2\omega r + r, 0 \leq r < 2^{i+1} \Rightarrow \left\lfloor \frac{j}{2} \right\rfloor = \omega + \left\lfloor \frac{r}{2} \right\rfloor = \omega
\]

\[
\left\lfloor \frac{j}{2^{i+1}} \right\rfloor = 2\omega + 1 \Rightarrow j = 2\omega r + r, 2^{i+1} \leq r < 2^{i+2} \Rightarrow \left\lfloor \frac{j}{2} \right\rfloor = \omega + \left\lfloor \frac{r}{2} \right\rfloor = \omega
\]

3 Connections Parallel to Borders

The concept of the connection was introduced in [1]. This section mainly studies the connections parallel to the borders of a subtree.

3.1 Three types of connections parallel to the borders

Let \( T_A \) be a valuated binary tree, \( X \) and \( Y \) be nodes of \( T_A \); assume \( d_i \) is the distance from \( X \) (or \( Y \)) to the left border \( B_i^L \) of \( T_A \) and \( d_i \) is the distance from \( X \) (or \( Y \)) to the right border \( B_i^R \) of \( T_A \), as depicted with Fig. 1. The connection \( P_i^X \) starting from \( X \) and formed with the nodes that are \( d_i \) away from \( B_i^L \) is called a connection parallel to \( B_i^L \) and the connection \( P_i^Y \) starting from \( Y \) and formed with the nodes that are \( d_i \) away from \( B_i^R \) is called a connection parallel to \( B_i^R \). Likewise, so are the connections \( P_i^X \) and \( P_i^Y \) defined. Since \( A \) is an ancestor of \( X \) and \( Y \), connections parallel to the borders of \( T_A \) are said to be type-1 connections.

There is another kind of connections that are said to be type-2 ones. Seen in Fig. 2, \( X \) and \( Y \) are two nodes on the same level of \( T_A \). Then there is a connection, denoted by \( P_{i_{ih}}^X \), that is starting from \( X \) and parallel to the right border of \( T_Y \); the distance from \( P_{i_{ih}}^X \) to the right border of \( T_Y \) is the same as that from \( X \) to \( Y \). There is also a connection, denoted by \( P_{i_{ih}}^Y \), that is starting from \( Y \) and parallel to the left border of \( T_X \).
The type-3 connections are related with two nodes X and Y that lie on different levels of $T_A$, as seen in Fig. 3. This kind of connections looks like the type-2 ones but they are different from the distance defined from X to the right border of $T_Y$ or from Y to the left border of $T_X$. This time, the lower level is set to be a reference to calculate the distances. If $X = N_{(k,l)}^X$ and $Y = N_{(0,\delta)}^Y$ with $k > 0$ and $l - k = \delta > 0$, then the distance $d_i$ from Y to $N_{(0,\delta)}^X$ is defined to be the distance from Y to the left border of $T_X$, and the distance $d_i$ from Y to $N_{(k,l)}^X$ is defined to be the distance from Y to the right border of $T_X$. Of course, some other definitions might be given if only they could simplify the calculations. For convenience, the connection is simply called a type-3 connection starting from X or Y, respectively.
Fig. 3. Type-3 connections parallel to borders

### 3.2 Properties and proofs

**Property 1.** Let $X = A_{k,j}^l$ be a node of $T_A$. Denote $P_{k}^l$ and $P_{k}^r$ to be the two type-1 connections starting from $X$ and parallel to the left and the right borders of $T_A$, respectively. Assume $n^l_i \in P_{k}^l$ and $n^r_i \in P_{k}^r$ are respectively the $i$th nodes counted from $X$, where $i \geq 0$; then

\[
    n^l_i = N_{(k,0)}^l = 2^{k+i}(A-1) + 2j + 1
\]

and

\[
    n^r_i = N_{(k,2)_{-j}}^r = 2^{k+i}(A-1) + 2(2^{k+i} - 2^j + j) + 1 = 2^{k+i}(A+1) - 2(2^j - j) + 1.
\]

**Proof.** Assume $d_l$ and $d_r$ are the distances from $X$ to the left and the right borders of $T_A$, respectively. Since

\[
    N_{(k,0)}^l = 2^k(A-1) + 1,
\]

\[
    N_{(k,2)_{-j}}^r = 2^k(A-1) + 2(2^k - 1) + 1
\]

and

\[
    N_{(k,j)} = 2^k(A-1) + 2j + 1,
\]

it follows

\[
    d_l = \frac{N_{(k,2)_{-j}}^r - N_{(k,j)}^l}{2} + 1 = 2^k - j,
\]

and

\[
    d_r = \frac{N_{(k,j)} - N_{(k,0)}^l}{2} + 1 = j + 1.
\]

Consider the case $n^l_i \in P_{k}^l$. Since $X$ lies on level $k$ of $T_A$, $n^l_i$ is on level $k + i$ of $T_A$ and it follows
\[ n^i = N^{A}_{(i-1,i)} + 2(d_i - 1) = 2^{i+1}(A-1) + 2j + 1 = N^{A}_{(i,j+1)} . \]

Similarly, it yields
\[
\begin{align*}
d_i &= \frac{N^{A}_{(i+1,i+3j-3)} - n^i}{2} + 1 \Rightarrow n^i = N^{A}_{(i+1,i+3j-3)} - 2(d_i - 1) = N^{A}_{(i+1,i+3j-3)} - 2(2^i - j - 1) \\
&= 2^{i+1}(A-1) + 2(2^{i+1} - 2^i + j) + 1 = N^{A}_{(i+1,3j+i-2+2^j)} 
\end{align*}
\]

**Example 1.** Taking in \( T_{21} \) the node 83, it is seen that, \( d_i = 2 \) (from 81 to 83 ) and \( d_i = 3 \) (from 83 to 87). The connection starting from 83 and parallel to the left and the right borders of \( T_{21} \) are respectively
\[
P^L_{83} = \{83, 163, 323, \ldots \} \text{ and } P^R_{83} = \{83, 171, 347, \ldots \}
\]

Because
\[
\begin{align*}
n^L_0 &= N^{T_{21}}_{(0,0)} = 2^2(21-1) + 2 + 1 = 83 \\
n^L_1 &= N^{T_{21}}_{(1,1)} = 2^2(21-1) + 2 + 1 = 163 \\
n^L_2 &= N^{T_{21}}_{(2,2)} = 2^2(21-1) + 2 + 1 = 323 \\
n^R_0 &= N^{T_{21}}_{(2+i,2^i-2^j+i)} = 2^2(21-1) + 2(2^i - 2^j + 1) + 1 = 83 \\
n^R_1 &= N^{T_{21}}_{(2+i,2^i-2^j+i)} = 2^2(21-1) + 2(2^i - 2^j + 1) + 1 = 171 \\
n^R_2 &= N^{T_{21}}_{(2+i,2^i-2^j+i)} = 2^2(21-1) + 2(2^i - 2^j + 1) + 1 = 347 
\end{align*}
\]

![Fig. 4. Example of connections parallel to the two borders of \( T_{21} \)](image)

**Proposition 1.** Let \( X = N^{A}_{(i,j)} \) with \( k > 0 \) be a node of \( T_{A} \). \( P^L_x \) and \( P^R_x \) be defined as those in Property 1; assume \( n^L_i \in P^L_x \) and \( n^R_i \in P^R_x \) are respectively the \( i \)-th nodes counted from \( X \), where \( i \geq 0 \); then it holds
\[
\begin{align*}
n^R_i - n^L_i &= 2(2^{i+1} - 2^i) = 2^{i+1}(2^i - 1) \\
n^L_i &= X + 2^i(2^i - 1)(A-1) \\
and
n^R_i &= X + 2^i(2^i - 1)(A+1) 
\end{align*}
\]

**Proof.** Direct calculation by Property 1 immediately yields
Now by $n^L_i = 2^{i+1}(A - 1) + 2j + 1$ it follows

$$n^L_i = 2^{i+1}(A - 1) + 2j + 1$$

$$\Rightarrow$$

$$n^L_{i+1} = 2^{i+1}(A - 1) + 2j + 1$$

$$\Rightarrow$$

$$n^L_{i+1} - n^L_i = 2^{i+1}(A - 1)$$

$$\Rightarrow$$

$$n^L_i - n^L_0 = 2^i(A - 1)$$

$$n^L_i - n^L_{i-1} = 2^{i+1}(A - 1)$$

$$\ldots$$

$$n^L_i - n^L_{i-1} = 2^{i+1}(A - 1)$$

$$\Rightarrow$$

$$n^L_i = n^L_0 + 2^i(2^i - 1)(A - 1)$$

Since $n^L_i = 2^i(A - 1) + 2j + 1 = N^A_{i,0} = X$, it yields

$$n^L_i = X + 2^i(2^i - 1)(A - 1), s \geq 0$$

Likewise, it yields

$$n^R_i = X + 2^i(2^i - 1)(A + 1), s \geq 0$$

**Remark 1.** Proposition 1 shows that, the distance from $n^L_i \in P^X_i$ to $n^R_i \in P^X_i$ is

$$d_i = \frac{n^R_i - n^L_i}{2} = 2^{i+1} - 2^i + 1$$

This is a quantity that merely depends on the level where $X$ lies and the level where $n^L_i$ lies.

**Proposition 2.** Let $X = N^A_{i,0}$ with $k > 0$ be a node on the left border of $T_A$; denote $T_{X_i,0}, T_{X_i,1}, \ldots, T_{X_i,j}$ to be the $2^j$ subtrees whose roots are respectively the $2^i$ nodes on level $k$ of $T_A$. Assume $n^R_i \in P^X_i$ is the $i$th node counted from $X$, where $i \geq 0$ and $P^X_i$ is as defined in Property 1; then

$$n^R_i = N^A_{(k+i,2^{i+1}-2^i)} \in \begin{cases} T_{X_i,j-i-1}, 0 \leq i \leq k \\ T_{X_i,k}, i > k \end{cases}$$

**Proof.** Taking $j = 0$ in Property 1 immediately yields

$$n^R_i = N^A_{(k+i,2^{i+1}-2^i)}$$

Now consider the ancestor of $N^A_{(k+i,2^{i+1}-2^i)}$, when $0 \leq i \leq k$ direct calculation shows
and when \( i > k \) it follows

\[
\left[ \frac{N^A_{(k+i,2^k-2^i)}}{2^l} - 1 \right] = \left[ \frac{2^k(A - 1) + 2(2^k - 2^i)}{2^i} \right] = 2^i (A - 1) + 2(2^k - 2^i)
\]

By Lemma 1, on level \( k \) of \( T_A \), the ancestor of \( N^A_{(k+i,2^k-2^i)} \) is \( N^A_{(k,2^k-2^i)} \) when \( 0 \leq i \leq k \), whereas it is \( N^A_{(k,2^k-2^i)} \) when \( i > k \).

**Proposition 2.** Let \( X = N^A_{(k,2^k-2^i)} \) with \( k > 0 \) be a node on the right border of \( T_A \); denote \( T_{k_0}, T_{k_1}, \ldots, T_{k_{2^k-2^i}} \) to be the \( 2^i \) subtrees whose roots are respectively the \( 2^i \) nodes on level \( k \) of \( T_A \). Assume \( n^L_i = P^k_i \) is the \( i \)th node counted from \( X \), where \( i \geq 0 \) and \( P^k_i \) is as defined in Property 1; then

\[
n^L_i = \begin{cases} 
T_{X_k}^{k+i,1} & 0 \leq i \leq k \\
T_{X_0}^{k+i,1} & i > k
\end{cases}
\]

**Proof.** Taking \( j = 2^i - 1 \) in Property 1 immediately yields

\[
n^L_i = N^A_{(k+i,2^k-2^i)}
\]

Note that

\[
\left[ \frac{N^A_{(k+i,2^k-2^i)}}{2^i} - 1 \right] = \left[ \frac{2^k(A - 1) + 2(2^i - 1)}{2^i} \right] = 2^i (A - 1) + 2(2^i - 1) - 1
\]

\[
= 2^i (A - 1) + 2(2^i - 1) - 1, i = 0
\]

\[
= 2^i (A - 1) + 2(2^i - 1) + 1, 1 \leq i \leq k
\]

\[
= 2^i (A - 1), i > k
\]

it is known by Lemma 1 that, on level \( k \) of \( T_A \), the ancestor of \( N^A_{(k+i,2^k-2^i)} \) is \( N^A_{(k,2^k-2^i)} \) when \( 0 \leq i \leq k \) whereas it is \( N^A_{(k,2^k-2^i)} \) when \( i > k \).

**Example 2.** Again taking in \( T_{21} \) as an example. Take \( k = 2 \) and \( j = 0 \); then

\[
n^A_0 = N^A_{(0,2^2-2^0)} = N^A_{(0,0)} = 81 \in T_{X_0}
\]

\[
n^A_0 = N^A_{(2,2^2-2^1)} = N^A_{(2,1)} = 169 \in T_{X_0}\]

\[
n^A_0 = N^A_{(2,2^2-2^1)} = N^A_{(4,1)} = 345 \in T_{X_0}\]

\[
n^A_0 = N^A_{(2,2^2-2^1)} = N^A_{(3,0)} = 697 \in T_{X_0}
\]

Take \( k = 2 \) and \( j = 3 \); then
Remark 2. It can be seen that, by the identity 
\[ n^L_k = n^{21}_{(2+1,2'-1)} = n^{21}_{(2+1,2') = 167 \in T_{X_1'}} = T_{X_1}, \]
\[ n^R_k = n^{21}_{(2+2,2'-1)} = n^{21}_{(2+2,2') = 327 \in T_{X_1'}} = T_{X_1}, \]
\[ n^L_k = n^{21}_{(2+2,2') = 87 \in T_{X_1'}} = T_{X_1}, \]
\[ n^R_k = n^{21}_{(2+3,2') = 645 \in T_{X_1'}} = T_{X_1}. \]

Proposition 3. Let \( X = N^A_{k,i,j} \) with \( k > 0 \) be a node of \( T_A \), \( P^L_k \) and \( P^R_k \) be defined as those in Property 1; assume \( n^L_k \in P^L_k \) and \( n^R_k \in P^R_k \) are respectively the \( i \)th nodes counted from \( X \), where \( i \geq 0 \); then
\[ n^L_k \in T_{X^L} \text{ and } n^R_k \in T_{X^R}, \]
where \( \omega = \left\lfloor \frac{j}{2^i} \right\rfloor \) and symbol \( X_\omega \) means \( N^A_{k,i,j}. \)

Proof. Consider on level \( k \) of \( T_A \) the ancestors of \( n^L_k \) and \( n^R_k \), respectively. By Property 1, it follows
\[ \frac{n^L_k - 1}{2^i} = 2^i(A-1) + \frac{j}{2^{i-1}} \Rightarrow \left\lfloor \frac{n^L_k - 1}{2^i} \right\rfloor = 2^i(A-1) + \left\lfloor \frac{j}{2^{i-1}} \right\rfloor \]
and
\[ n^R_k = 2^{i+1}(A-1) + 2(2^{i+1} - 2^i + j) + 1 \Rightarrow \left\lfloor \frac{n^R_k - 1}{2^i} \right\rfloor = 2^i(A-1) + 2(2^i - 2^{i-1}) + \left\lfloor \frac{j}{2^{i-1}} \right\rfloor. \]

Thus the quantity \( \left\lfloor \frac{j}{2^{i-1}} \right\rfloor \) is the key to determine the ancestors. By Lemma 2, whether \( \left\lfloor \frac{j}{2^{i-1}} \right\rfloor \) is even, say \( \left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega \), or it is odd, say \( \left\lfloor \frac{j}{2^{i-1}} \right\rfloor = 2\omega + 1 \), where \( \omega \geq 0 \) is an integer, it results in
\[ A'_{\omega} = 2^i(A-1) + 2\omega + 1 \]
and

![Fig. 5. Symmetric connections parallel to the borders of \( T_2 \)](image-url)
\[ A'_k = 2^j (A - 1) + 2(2^j - 2^{k-j} + \omega) + 1 \]

Referring to Lemma 2, it follows
\[ A'_k = 2^j (A - 1) + 2 \left\lfloor \frac{j}{2} \right\rfloor + 1 \]
and
\[ A'_k = 2^j (A - 1) + 2(2^j - 2^{k-j} + \left\lfloor \frac{j}{2} \right\rfloor) + 1 \]

Since \( A'_k \) and \( A'_k \) are roots of subtrees from level \( k \), the proposition is established.

Remark 3. It is seen that \( \omega = 0 \) when \( i > \lfloor \log_2 j \rfloor \). There is always an \( i_n \) such that \( i > i_n \) leads to \( n^i \in T_{k_a} \) and \( n^i \in T_{k_a} \), where \( X_0 = N_{(0,0)}^k \) and \( X_{2^{-i}} = N_{(2,2^{-i})}^k \).

Proposition 4. Let \( X = N^k_{(0,j)} \) and \( Y = N^k_{(0,2-j)} \) with \( k > 0 \) be two symmetric nodes on level \( k \) of \( T_A \). Let \( P^Y_k \) and \( P^Y_k \) be the two type-1 connections starting from \( X \) and parallel respectively to the left and the right borders of \( T_A \). Assume \( n^{iX} \in P^Y_k \), \( n^{iX} \in P^Y_k \), \( n^{iY} \in P^Y_k \) and \( n^{iY} \in P^Y_k \) are nodes on the connections, where \( i \geq 0 \); then on level \( k+i \) of \( T_A \), \( n^{iX} \) is symmetric to \( n^{iY} \) and \( n^{iY} \) is symmetric to \( n^{iY} \).

Proof. By Property 1, it follows
\[ n^{iX} = N^k_{(2i,2^{-i-2}+j)} = 2^{k+i}(A-1) + 2(2^{k+i} - 2^j + j) + 1 \]
and
\[ n^{iY} = N^k_{(2i+1,2^{-i-1})} = 2^{k+i}(A-1) + 2(2^j - 1 - j) + 1 \]

Let \( \omega = 2^j - 1 - j \); then \( 2^{k+i} - 1 - \omega = 2^{k+i} - 2^j + j \). This immediately shows \( n^{iX} \) is symmetric to \( n^{iY} \). Likewise, it can be proved that \( n^{iY} \) is symmetric to \( n^{iY} \).

Property 2. Let \( X = N^k_{(0,j)} \) and \( Y = N^k_{(0,2-j)} \) with \( k > 0 \) and \( X < Y \) be two symmetric nodes on level \( k \) of \( T_A \); let \( P^Y_k \) be the type-2 connection starting from \( X \) and parallel to the right border of \( T_A \). \( P^Y_k \) be the type-2 connection starting from \( Y \) and parallel to the left border of \( T_A \). Assume \( n^{iX} \in P^Y_k \) and \( n^{iX} \in P^Y_k \); then
\[ n^{iX} = 2^i (X - 1) + 2(2^i - 2j) - 1 \]
and
\[ n^{iY} = 2^i (Y + 1) - 2(2^i - 2j) + 1 \]

Proof. Let \( d \) be the distance from \( X \) to \( Y \); then
\[ d = \frac{Y - X}{2} + 1 = 2^i - 2j \]
For integer $i \geq 0$, the node on level $i$ and on the left border of $T_X$ is

$$N^X_{(i,0)} = 2^i (X - 1) + 1$$

The node on level $i$ and on the right border of $T_Y$ is

$$N^Y_{(i,2^i - 1)} = 2^i (Y + 1) - 1$$

There by, $n^\text{ll}_i$ and $n^\text{rr}_i$ are necessary to satisfy

$$n^\text{ll}_i = N^X_{(i,0)} + 2(d - 1) = 2^i (X - 1) + 1 + 2(2^i - 2j - 1)$$

and

$$n^\text{rr}_i = N^Y_{(i,2^i - 1)} - 2(d - 1) = 2^i (Y + 1) - 1 - 2(2^i - 2j - 1)$$

**Proposition 5.** Let $X = N^A_{(k,j)}$ and $Y = N^A_{(k',2^k - 1)}$ with $k > 0$ and $X < Y$ be two symmetric nodes on level $k$ of $T_A$; let $P^*_k$ and $P^*_s$ be the connections as defined in Property 2. Assume $n^\text{ll}_i \in P^*_k$ and $n^\text{rr}_i \in P^*_s$; where $i \geq 0$; then

1. $n^\text{ll}_i$ and $n^\text{rr}_i$ are symmetric nodes in $T_A$.
2. $n^\text{ll}_i \in T_k$ and $n^\text{rr}_i \in T_s$ when $s \geq k$.
3. $n^\text{ll}_i$ and $n^\text{rr}_i$ are alternatively calculated by

$$n^\text{ll}_i = Y + (2^i - 1)(X - 1)$$

and

$$n^\text{rr}_i = X + (2^i - 1)(Y + 1)$$

**Proof.** By Property 2, it holds

$$n^\text{ll}_i = 2^i (X - 1) + 1 + 2(2^i - 2j - 1)$$

and

$$n^\text{rr}_i = 2^i (Y - 1) + 2(2^i - 2^j + 2j) + 1$$

Since $X$ and $Y$ are symmetric, it follows $0 \leq j \leq 2^i - 1$, $1 \leq 2^i - 2j - 1 \leq 2^i - 1$ and $2^i - 2^j \leq 2^i - 2^j + 2j \leq 2^i - 2$. It is sure that $n^\text{ll}_i \in T_k$ and $n^\text{rr}_i \in T_s$ when $i \geq k$, which validates the conclusion (2).

Note that

$$X = N^A_{(k,j)} = 2^i (A - 1) + 2j + 1$$

and

$$Y = N^A_{(k,2^k - 1)} = 2^i (A - 1) + 2(2^i - 1 - j) + 1$$

Substituting these two into the expressions of $n^\text{ll}_i$ and $n^\text{rr}_i$ yields
\[ n_{yi}^{IL} = 2^{k_i} (A - 1) + 2(2^k + 2^j + 2j - 2j - 1) + 1 \]

and

\[ n_{yi}^{IR} = 2^{k_i} (A - 1) + 2(2^{k_i} - 2^j - 2^j + 2j + 1) + 1 \]

Let \( \omega = 2^j + 2^j - 2^j - 1 \); then \( 2^{k_i} - 2^j + 2j = 2^{k_i} - 1 - (2^j + 2^j - 2j - 1) = 2^{k_i} - 1 - \omega \) and it yields

\[ n_{yi}^{IL} = 2^{k_i} (A - 1) + 2\omega + 1 \]

and

\[ n_{yi}^{IR} = 2^{k_i} (A - 1) + 2(2^{k_i} - 1 - \omega) + 1 \]

Obviously, \( n_{yi}^{IL} \in T_A \) and \( n_{yi}^{IR} \in T_A \) if \( 0 \leq \omega \leq 2^{k_i} - 1 \). In fact, direct calculation shows \( \omega = 2^k - 1 - j \) and \( 2^{k_i} - 1 - \omega = j \) when \( i = 0 \). This means \( n_{yi}^{IL} \in T_A \) and \( n_{yi}^{IR} \in T_A \). When \( i > 0 \), it yields by \( 0 \leq j \leq 2^{k_i} - 1 \)

\[ 0 \leq j \leq 2^{k_i} - 1 \Rightarrow 0 \leq 2^j \leq 2^{k_i - 1} - 2^i \]

\[ \Rightarrow 1 \leq \omega \leq 2^{k_i - 1} - 2^i + 2^i - 1 < 2^{k_i} - 1 \]

and accordingly,

\[ n_{yi}^{IL} = N_A^{(k_i, \omega)} \in T_A \]

and

\[ n_{yi}^{IR} = N_A^{(k_i, 2^{k_i - 1} - \omega)} \in T_A \]

The conclusion (3) is simply proved by the following reasoning

\[ n_{yi+1} - n_{yi}^{IL} = 2^i (X - 1) \]
\[ n_{yi+1}^{IL} - Y = 2^i (X - 1) \]
\[ n_{yi}^{IR} - n_{yi}^{IL} = 2(X - 1) \]
\[ n_{yi}^{IR} - n_{yi+1}^{IL} = 2^i (X - 1) \]

\[ \Rightarrow n_{yi}^{IL} = Y + (2^i - 1)(X - 1) \]

\[ n_{yi+1} - n_{yi}^{IR} = 2^i (Y + 1) \]
\[ n_{yi}^{IR} - X = 2^i (Y + 1) \]
\[ n_{yi}^{IR} - n_{yi+1}^{IR} = 2^i (Y + 1) \]

\[ \Rightarrow n_{yi}^{IR} = X + (2^i - 1)(Y + 1) \]
Property 3. Let $X = N_{k,l}^{N}$ and $Y = N_{l,k}^{N}$ with $k > 0$ and $l - k = \delta > 0$ be two nodes of $T_x$; assume $Y \not\in T_x$ and $Y$ is to the right of $T_x$. Let $P_{18}^{\text{ARJOM}}$ be the type-3 connection starting from $X$ and parallel to the right border of $T_x$, $P_{18}^{\text{ARJOM}}$ be the type-3 connection starting from $Y$ and parallel to the left border of $T_x$, as depicted in Fig. 3. Assume $n_{ik}^{X} \in P_{18}^{X}$ and $n_{i \delta}^{Y} \in P_{18}^{Y}$; then for given $i \geq 0$

\[ n_{ik}^{X} = Y + 2^{i}(2l - 1)(X - 1) \]

and

\[ n_{i \delta}^{Y} = (2^{l} - 1)(Y + 1) + 2^{i}(X + 1) - 1 \]

Proof. Let $d_{l}$ be the distance from $Y$ to the left border of $T_x$, and $d_{r}$ be the distance from $Y$ to the right border of $T_x$; then

\[ d_{l} = \frac{Y - 2^{l}(X - 1) - 1}{2} + 1 \Rightarrow 2(d_{l} - 1) = Y - 2^{l}(X - 1) - 1 \]

and

\[ d_{r} = \frac{Y - 2^{l}(X + 1) + 1}{2} + 1 \Rightarrow 2(d_{r} - 1) = Y - 2^{l}(X + 1) + 1 \]

It follows

\[ n_{ik}^{X} = 2^{i}(Y + 1) - 2(d_{l} - 1) = (2^{l} - 1)(Y + 1) + 2^{i}(X + 1) - 1 \]

Example 3. Again taking in $T_{21}$ as an example. Take $X = 41$ and $Y = 85$; then

\[ n_{11}^{X} = 85 + 2^{1}(2^{1} - 1)(41 - 1) = 165 \]
\[ n_{2}^{X} = 85 + 2^{2}(2^{2} - 1)(41 - 1) = 325 \]
\[ n_{3}^{X} = 85 + 2^{3}(2^{3} - 1)(41 - 1) = 645 \]
\[ n_{1}^{Y} = (2^{1} - 1)(85 + 1) + 2^{1}(41 + 1) - 1 = 169 \]
\[ n_{2}^{Y} = (2^{2} - 1)(85 + 1) + 2^{1}(41 + 1) - 1 = 341 \]
\[ n_{3}^{Y} = (2^{3} - 1)(85 + 1) + 2^{1}(41 + 1) - 1 = 685 \]

Fig. 6. Type-3 connections in $T_{21}$
**Property 3**.* Let \( X = N_{ik, l}^A \) and \( Y = N_{i, l-k}^A \) with \( k > 0 \) and \( l - k = \delta > 0 \) be two nodes of \( T_A \); assume \( Y \notin T_X \) and \( Y \) is to the left of \( T_X \). Let \( P^X_{il} \) be the type-3 connection starting from \( X \) and parallel to the left border of \( T_Y \). Let \( P^X_{ir} \) be the type-3 connection starting from \( Y \) and parallel to the right border of \( T_X \), as depicted in Fig. 7. Assume \( n^X_{il} \in P^X_{il} \) and \( n^X_{ir} \in P^X_{ir} \); then for given an \( i \geq 0 \)

\[
n^X_{il} = (2^i - 1)(Y - 1) + 2^i(X - 1) + 1
\]

and

\[
n^X_{ir} = Y + 2^i(2^i - 1)(X + 1)
\]

**Proof.** Let \( d_l \) be the distance from \( Y \) to the left border of \( T_X \) and \( d_r \) be the distance from \( Y \) to the right border of \( T_X \); then

\[
d_l = \frac{2^i(X - 1) + 1 - Y}{2} + 1 \Rightarrow 2(d_l - 1) = 2^i(X - 1) + 1 - Y
\]

and

\[
d_r = \frac{2^i(X + 1) - 1 - Y}{2} + 1 \Rightarrow 2(d_r - 1) = 2^i(X + 1) - 1 - Y
\]

It follows

\[
n^X_{il} = 2^i(Y - 1) + 1 + 2(d_l - 1) = (2^i - 1)(Y - 1) + 2^i(X - 1) + 1
\]

and

\[
n^X_{ir} = 2^i - 1 - 2(d_r - 1) = Y + 2^i(2^i - 1)(X + 1)
\]

**Proposition 6.** Let \( X = N_{ik, l}^A \) and \( Y = N_{i, l-k}^A \) with \( k > 0 \) and \( l - k = \delta > 0 \) be two nodes of \( T_A \); assume \( Y \notin T_X \) and \( Y \) is to the right of \( T_X \). Let \( P^X_{is} \) be the type-3 connection starting from \( X \) and parallel to the right border of \( T_Y \). Let \( P^X_{is} \) be the type-3 connection starting from \( Y \) and parallel to the left border of \( T_X \). Assume \( n^X_{is} \in P^X_{is} \) and \( n^X_{ir} \in P^X_{ir} \); then \( n^X_{is} \in T_X \) with \( i > \max(k, \delta) \) while \( n^X_{ir} \in T_Y \) with \( s \geq l \); and it holds

![Fig. 7. Type-3 connections parallel to borders](image-url)
\[ n_{i,i}^{XL} - n_{i,i}^{XR} = 2^{j} (X - 1) \]

and

\[ n_{i,i}^{YR} - n_{i,i}^{XR} = 2^{j} (Y + 1) \Rightarrow n_{j,j}^{YR} = N_{(d',d')-}^{Y} + (2^{j} - 1)(Y + 1), \ j \geq 0 \]

**Proof.** First, \( l - k = \delta > 0 \) implies \( 0 < \delta \leq l \) because it is contradictory that \( l - k > l \Rightarrow k < 0 \). Now by Property 3, it holds

\[ \frac{n_{i,i}^{XL} - 1}{2^{j}} = \frac{Y - 1 - 2^{j} (X - 1)}{2^{j}} + 2^{j} (X - 1) \]

and

\[ \frac{n_{i,i}^{YR} - 1}{2^{j}} = (Y + 1) - \frac{Y + 1 - 2^{j} (X + 1) + 2}{2^{j}} \]

Note that, referring to \( d_{j} \) and \( d' \), defined in the proof of Property 3, it follows

\[ 2(d_{j} - 1) = Y - 2^{j} (X - 1) - 1 \]

and

\[ 2(d'_{j} - 1) = Y + 1 - 2^{j} (X + 1) \]

As a result, it leads to

\[ \frac{n_{i,i}^{XL} - 1}{2^{j}} = \frac{d_{j} - 1}{2^{j-1}} + 2^{j} (X - 1) \]

and

\[ \frac{n_{i,i}^{YR} - 1}{2^{j}} = Y + 1 - \frac{d'_{j}}{2^{j-1}} \]

Since

\[ \frac{N_{(d,d')-}^{X} - N_{(d,0)0}^{X}}{2} + 1 \leq d_{j} \leq \frac{Y - N_{(d,0)0}^{X}}{2} + 1 \]

\[ \Rightarrow 2^{j} + 1 \leq d_{j} \leq \frac{2^{j} - 1 - 2^{j} (X - 1) - 1}{2} + 1 = 2^{j+1} - 2^{j} (X - 1) \]

\[ \Rightarrow 2^{j} + 1 \leq d_{j} \leq 2^{j+1} - 2^{j} (2^{j+1} + 1 - 1) = 2^{j+1} - 2^{j} = 2^{j} \]

\[ \Rightarrow 2^{j} \leq d_{j} \leq 2^{j+1} - 1 \leq 2^{j} \]

and
\[ 1 \leq d_r \leq \frac{Y-N^X_{\alpha, 2^{i-1}}}{2} + 1 \]
\[ \Rightarrow 1 \leq d_r \leq \frac{2^{i-1} - 2^X(2^{i-1} + 1) + 1}{2} = 2^{i-1} - 2^X(2^{i-1} + 1) + 1 \]
\[ \Rightarrow 1 \leq d_r \leq \frac{2^{i-1} - 2^X(2^{i-1} + 1) + 1}{2} = 2^{i-1} - 2^X - 2^{i-1} - 2^{i-1} + 1 = 2^i - 2^X + 1 \]
\[ \Rightarrow \frac{1}{2^{i-1}} \leq \frac{d_r}{2^{i-1}} \leq \frac{2^{i-1} - 2^X + 1}{2^{i-1}} \]
\[ \Rightarrow -2^{i-1} + 2^X + \frac{1}{2^{i-1}} \leq \frac{d_r}{2^{i-1}} \leq \frac{1}{2^{i-1}} \]

it yields
\[ 2^X(X - 1) + \frac{2^X}{2^{i-1}} \leq \frac{N^X_{\nu} - 1}{2} \leq 2^X(X - 1) + \frac{2^X - 1}{2^{i-1}} \]

and
\[ Y + 1 - \frac{2^i - 2^X}{2^{i-1}} \leq \frac{N^Y_{\nu} - 1}{2} = Y + 1 - \frac{1}{2^{i-1}} \]

Obviously, on level \( \delta \) of \( T_X \), the ancestor of \( n_{\nu}^{\alpha} \) is to the right of \( N_{(\delta, 0)}^X \) and it lies in \( T_X \) if \( i > \max(l - \delta, \delta) = \max(k, \delta) \). Similarly, the ancestor of \( n_{\nu}^{\alpha} \) is to the left of \( Y \) and it lies in \( T_Y \) if \( i \geq l \).

Next the proof of the two equalities is just a simple reasoning like those in the proof of Proposition 1.

**Proposition 6*. Let \( X = N_{X_{(k,i)}}^A \) and \( Y = N_{Y_{(k,i)}}^A \) with \( k > 0 \) and \( l - k = \delta > 0 \) be two nodes of \( T_A \); assume \( Y \notin T_X \) and \( Y \) is to the left of \( T_X \). Let \( P_{i}^{X} \) be the type-3 connection starting from \( X \) and parallel to the left border of \( T_X \). \( P_{i}^{Y} \) be the type-3 connection starting from \( Y \) and parallel to the right border of \( T_X \). Assume \( n_{\nu}^{\alpha} \in P_{i}^{X} \) and \( n_{\nu}^{\gamma} \in P_{i}^{Y} \); then \( n_{\nu}^{\alpha} \in T_Y \) with \( s \geq l \) while \( n_{\nu}^{\gamma} \in T_Y \) with \( i > \max(k, \delta) \); and it holds
\[ n_{i+1}^{\alpha} - n_{i}^{\alpha} = 2^X(X + 1) \]

and
\[ n_{i+1}^{\alpha} - n_{i}^{\alpha} = 2(Y - 1) \Rightarrow n_{i}^{\alpha} = N_{(\delta, 0)}^X + (2^{i} - 1)(Y - 1) \]

**Proof.** (Omitted)

### 4 Applications in Integer Factorization

Connections make an outer node of a subtree to be related with an inner node of the subtree; since it is easy for an inner node to trace up to reach its root, this thereby enables certain properties of the outer node to be associated with (transmitted to) the root of the subtree in the least searching steps. This section demonstrates such an operation.

#### 4.1 General rule

The propositions proven previously reveal that, a node \( c \) on a connection starting from a node \( o \) can penetrate into an \( r \)-rooted subtree, as shown in Fig. 8, and the nodes \( o, c \) and \( r \) are related with the following equality
\[ c = r + 2^X(2^\delta - 1) + o \]
This relationship can be said to be a triangle relationship of connections and it can derive many amazing results by means of evaluating different values to $r$ and $o$. This idea is the general rule of connecting different nodes in a valued binary tree.

### 4.2 Factorization of odd integers

With the triangle relationship of the connections, this section shows a divisor of the form $2^r + 1, 2^r - 1$, $\gamma + 2^\alpha(2^\beta - 1)$ or $\gamma + 2^\alpha(2^\beta - 1)\lambda$ can be easily found out in a positive composite odd integer, as the following corollaries state.

**Corollary 1.** Let $m = pq > 3$ be an odd composite positive integer and the divisor $p$ is of the form $2^r - 1$ with integer $\alpha > 0$; then $p$ can be found out in $O(\log_2 p + 2)$ searching steps.

**Proof.** By Lemma 1, $m$ lies on level $k = \lceil \log_2 m \rceil - 1$ of $T_3$. If $m$ lies on the left border of $T_3$, construct a sequence $M_\alpha$ by

$$M_\alpha = \{m + 2^{i+1}(2^i - 1), m + 2^{i+1}(2^i - 1), \ldots, m + 2^{i+1}(2^i - 1), \ldots\}$$

If it lies on the right border of $T_3$, construct $M_\beta$ by

$$M_\beta = \{m + 2^{i+1}(2^i - 1), m + 2^{i+1}(2^i - 1), \ldots, m + 2^{i+1}(2^i - 1), \ldots\}$$

If it is an intermediate node, construct

$$M_\gamma = \{m + 2^{i+1}(2^i - 1), m + 2^{i+1}(2^i - 1), \ldots, m + 2^{i+1}(2^i - 1), \ldots\}$$

or

$$M_\delta = \{m + 2^{i+1}(2^i - 1), m + 2^{i+1}(2^i - 1), \ldots, m + 2^{i+1}(2^i - 1), \ldots\}$$
It can be seen by Proposition 1 that, for each case there must be an $\alpha$ such that $p = \gcd(m, m)$, where $m = m + 2^i(2^i - 1)$ or $m = m + 2^i(2^i - 1)$. Since $p = 2^d - 1$, it follows $\alpha = \lceil \log_2 p \rceil + 2$.

**Corollary 2.** Let $m = pq > 3$ be an odd composite positive integer and the divisor $p$ is of the form $2^a + 1$ with integer $\alpha > 0$; then $p$ can be found out in at most $O(\log_2 p)$ searching steps.

**Proof.** By Lemma 1, $m$ lies on level $\lceil \log_2 m \rceil - 1$ of $T_3$. Consider the case that it is an intermediate node. Refer to the proof of Corollary 1 and construct

$$M_j = \{m + 2(2^j - 1), \ldots, m + 2(2^{j+1} - 1), \ldots\}$$

it is seen that must be an $\alpha$ such that $p = \gcd(m, m)$, where $m = m + 2^i(2^i - 1)$ or $m = m + 2(2^i - 1)$.

**Remark 4.** Corollaries 1 and 2 look very trivial because they seem so elementary that any one might be able to think of the constructions. However, they are here derived theoretically from Proposition 1. In other words, they are rather theoretical results than something someone thinks of.

**Corollary 3.** Suppose the divisor $p$ of the positive composite odd $n = pq$ is of the form $\gamma + 2^a + b$, where $\alpha > 0$, $\beta > 0$ are integers, and $q > 1, \gamma > 1$ are odd integers; then $q$ can be found out in at least $O(\log_2 p)$ and at most $O((\log_2 p + 1)^2)$ searching steps.

**Proof.** Take the triangle formula $n^{\gamma Y} = Y + 2^a(2^b - 1)(X - 1)$ established in Property 3 as an example to have an analysis. Let $X = 2^a s$, where $\sigma > 0$ and $s$ is a positive integer. For convenience denote $n^{\gamma Y}$ by $n$; then $n = Y + 2^a s$.

Take an arbitrary positive odd integer $Y$ satisfying $Y = \gamma s$ and $Y > 2X + 1$; denote $\gamma + 2^a s + b = u$; then

$$n = (\gamma + 2^a s + b - 2^a s) u s$$

By Property 3, there is an $i$ such that $n = n^{\gamma Y} \in T_s$ when $i \geq i_0 > 0$. Consequently, $n$ is a descendant of $X$. Thus $X$ can be found by searching in the direct ancestors of $n$, and then $s$ can be found by $s = \frac{X - 1}{2^a}$. Note that, there are $\lambda = \lceil \log_2 n \rceil - \lceil \log_2 X \rceil$ levels from $n$ to $X$, and it takes $\sigma$ steps to calculate $s$ with $X$. It is sure that the total searching steps are

$$t = \lambda \sigma = \lceil \log_2 n \rceil - \lceil \log_2 X \rceil \sigma$$

Since

$$\lceil \log_2 X \rceil = \lceil \log_2 (2^a s + 1) \rceil = \sigma + \lceil \log_2 (s + \frac{1}{2^a}) \rceil = \sigma + \lceil \log_2 s \rceil$$

it follows

$$t = \sigma(\lceil \log_2 n \rceil - \lceil \log_2 s \rceil - \sigma) \leq \sigma(\log_2 n - \log_2 s + 1 - \sigma) = \sigma(\log_2 u + 1 - \sigma)$$

Next is to show $\sigma \leq \log_2 u$. This can be done with the proof by contradiction. Assume $\sigma > \log_2 u$; then it follows $\sigma > \log_2 u \Rightarrow u < 2^{\sigma} \Rightarrow X = 2^a s + 1 > us + 1 \Rightarrow X > n + 1$.
which is a contradiction. Consequently, under the condition $\sigma \leq \log_2 u$, it holds

$$\log_2 u \leq t \leq \left(\frac{\log_2 u + 1}{2}\right)^2$$

The corollary surely holds by substituting $s$ with $q$ and $u$ with $p$ in the above reasoning process.

**Corollary 4.** Suppose the divisor $p$ of the positive composite odd $n = pq$ is of the form $\gamma + 2\sigma(2^\beta - 1)\lambda$, where $\alpha > 0, \beta > 0$ are integers, and $q > 1, \gamma \geq 1, \lambda \geq 1$ are odd integers; then $q$ can found out in at most $O(\log_2 p)$ searching steps.

**Proof.** Take the triangle formula $n^{\mu} = Y + 2^\sigma(2^\beta - 1)\lambda = Y + \sigma s$, where integer $\sigma > 0$ and $\lambda > 1, \gamma > 1$ are odd integers; denote $n^{\mu}$ by $n$. Then

$$n = (\gamma + 2^{\sigma+\sigma}(2^\beta - 1)\lambda)s = us$$

where $u = \gamma + 2^{\sigma+\sigma}(2^\beta - 1)\lambda$.

Since $n \in T_n$ for some $i$, tracing up from $n$ surely reaches $X$ and thus $s$ is the common divisor of $X - 1$ and $n$. The searching step from $n$ to $X$ is $t = \left\lfloor \log_2 n \right\rfloor - \left\lfloor \log_2 X \right\rfloor \geq \left\lfloor \log_2 u \right\rfloor$. It can be proven $t \leq \left\lfloor \log_2 u \right\rfloor$. In fact, assumption of $t > \left\lfloor \log_2 u \right\rfloor$ yields

$$t = \left\lfloor \log_2 n \right\rfloor - \left\lfloor \log_2 X \right\rfloor > \left\lfloor \log_2 u \right\rfloor \Rightarrow \left\lfloor \log_2 n \right\rfloor - \left\lfloor \log_2 u \right\rfloor > \left\lfloor \log_2 X \right\rfloor \Rightarrow \left\lfloor \log_2 s \right\rfloor - \left\lfloor \log_2 X \right\rfloor + 1 > 0$$

It is contradictory because $X = 2^\sigma \lambda s + 1 \Rightarrow \left\lfloor \log_2 X \right\rfloor = \left\lfloor \log_2 (2^\sigma \lambda s + 1) \right\rfloor \geq \sigma + \left\lfloor \log_2 \lambda \right\rfloor + \left\lfloor \log_2 s \right\rfloor$.

Take the triangle formula $n^{\mu} = Y + 2^\sigma(2^\beta - 1)(X - 1)$ in Property 3* can also derive this corollary.

### 4.3 Numerical experiments

Experiments for testing Corollary 4 are made to factorize odd integers that are of the length from 101 to 105 decimal digits. Table 1 lists the experimental results. In the table, the column ‘Big Number $N$’ is the big odd composite number to be factorized, the column ‘nDigits’ is the number of decimal digits, the column ‘Found Divisor’ is the found divisor of $N$, the column ‘Tsteps’ is the number of searching steps calculated theoretically from the previous corollaries and the column ‘Rsteps’ is the real searching steps recorded by the computer. It can be seen that the real searching steps are within the bounds of the orifical searching steps in each case. For readers to know the algorithms more deeply, the Maple programs are list in the appendix section. Readers can test them with the programs.

Table 1. Experimental Results for Testing Corollary 4

| Big Number $N$ | nDigits | Found Divisor | Tsteps | Rsteps |
|----------------|---------|---------------|--------|--------|
| 38432692938461041576664924646972711 | 101 | 73777946855370122040579650 | 201 | 138 |
| 37497178448383674628658527 | 102 | 04755144379743 | 204 | 138 |
| 29479839898246579587923898718346875 | 103 | 11066692028305518306086947 | 204 | 138 |
| 3509413030690503381040945 | 104 | 507132716570077 | 204 | 138 |
### Big Number

| N      | nDigits | Found Divisor | Tsteps | Rsteps |
|--------|---------|---------------|--------|--------|
| 163836521093280851652408996506181118 49897 | 103     | 25822281399379542714202877 5166430053977757 | 205    | 140    |
| 209030087090003249475422808302558123 722275018413802635088341 9796076392339838633561706607281450 147127 | 104     | 472178859874367878105970976 0304329243474189 | 204    | 144    |
| 1920616751993233994904647629999390147 2266235295055415248292245 9958351626899969024668781364943956474 4978023 | 105     | 37774308789949502484776780 82434633948186509 | 206    | 147    |

### 5 Assessments, Conclusions and Future Work

#### 5.1 Assessments of the new results

Referring to paper [1], it is seen that three corollaries were proven and those corollaries were quite like the corollaries proven in 4.2. Nevertheless, there are differences. Comparing Corollary 1 of this paper to that Corollary 1, it can be seen that, the two corollaries state two different results in finding a divisor of a positive odd integer that has a divisor of the form $2^\alpha - 1$. For an odd composite positive integer $m = pq > 3$ whose divisor $p$ is of the form $2^\alpha - 1$, this Corollary 1 shows $p$ can be found out in $O(\log_2 p)$ searching steps, whereas that Corollary 1 showed $q$ could be found out within $O(1 + \log_2 p)$ under the condition $\alpha \geq \left\lceil \frac{\log_2 m}{2} \right\rceil + 1$. Obviously, this Corollary 1 is more flexible and applicable. It can also be seen that, this Corollary 3 is more flexible than that Corollary 3 although the two corollaries state the same topic in finding a divisor of a positive odd integer that has a divisor of the form $\gamma + 2^\eta \cdot (2^\alpha - 1)$. Finally, in paper [1] there was no Corollary 4, which is actually more extensive than Corollary 3. Consequently, it can be concluded that, the investigation in this paper is more subtle and beneficial for the researching purpose.

#### 5.2 Conclusions and expectations

The valuated binary tree method demonstrated more and more attractive and reliable results in analyzing odd integers. Especially the application of geometric means enables it easy and clear to set up kinds of relationships among subtrees and nodes. This derives a very simple and fast way to factorize the kind of odd composite integers. However, one thing should be told the readers here. That is that failures occur in factoring an $n = pq$ when both $p$ and $q$ are of the form $\gamma + 2^\eta \cdot (2^\alpha - 1)\lambda$. This remains further studies. Hope more young to join the work.

### Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Maple Source Codes

**SubRoutine** Father (Calculate the father of a node Son)

F\text{a\text{ther}}: =p\text{roc}(\text{Son})

local X, r;

r: =\text{modp}(\text{Son},4);

if \ r=1 \ then 
X: =\frac{(\text{Son}+1)}{2};

else 
X: =\frac{(\text{Son} -1)}{2};
fi

End proc

**MainRoutine** Findq (Calculate the divisor q of odd composite integer N)

F\text{indq}: =p\text{roc}(\text{N})

local X, T, AA, g, p, q, Tsteps, Rsteps:=0, len;

AA: =\text{Father}(\text{N});

g: =\text{gcd}(\text{AA}, \text{N});

while g=1 do

Rsteps: =Rsteps+1;

T: =AA;

X:=T-1;

q: =\text{gcd}(X, N);

AA: =\text{Father}(T);

od;

p: =N/q;

Tsteps: =\text{floor}(\text{evalf}(\log(p)))

lprint(“F\text{i\text{n\text{d} q}=”, q, “Tsteps=”, Tsteps, “Rsteps=”, Rsteps);

End proc

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