Control of tunneling in a triple-well system: an atomtronic switching device

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We study a model of bosons confined to three coupled wells. The model describes interactions between bosons, tunneling of bosons between adjacent wells, and the effect of an external field. We consider both integrable and non-integrable regimes within the model. These regimes are first analyzed by means of the energy level spacing distribution. We then conduct a study of the quantum dynamics of the system to probe the conditions under which switching behavior can occur. Through variation of an external field on one well, we demonstrate how the system can be controlled between various “switched-on” and “switched-off” configurations.

The two-well Bose-Hubbard Hamiltonian has been very successful in modeling quantum tunneling in ultracold physical systems \cite{1, 2}. This model displays two principal dynamical scenarios, Josephson tunneling and self-trapping, which have been experimentally observed \cite{3, 4}. A three-well system opens up a wider range of possibilities for physical behaviors, most notably as an ultra-cold physical systems \cite{1, 2}. This model displays two strongly broken, yet the chaoticity as predicted by the Brody distribution remains low. In these regimes it will also be seen that it is often the case that the system can be manipulated, by application of an external field, with remarkable control. The result can be understood through restriction to a two-well subsystem which mimics an integrable system, even though the entire three-well system is not integrable.

The model – We begin with a Hamiltonian for a triple-well system introduced in \cite{14} for quantum tunneling in dipolar systems. It was subsequently established to be a member of an integrable family of multi-well tunneling models \cite{15}:

\begin{equation}
H_0 = U(N_1 - N_2 + N_3)^2 + J_1(a_1^\dagger a_2 + a_1 a_2^\dagger) + J_3(a_3^\dagger a_3 + a_2 a_3^\dagger),
\end{equation}

where the coupling $U > 0$ parametrizes inter-well and intra-well interactions between bosons, and $J_i$, $i = 1, 3$ are the coupling constants for the tunneling. Units are adopted throughout such that $\hbar = 1$. The canonical creation and annihilation operators, $a_i^\dagger$ and $a_i$, $i = 1, 2, 3$, represent the three degrees of freedom in the model, and $N_i = a_i^\dagger a_i$, $i = 1, 2, 3$ is the number operator for each well. Note that $\hat{1}$ commutes with the total number operator $N = N_1 + N_2 + N_3$, and the change of the indexes 1 and 3 leaves the Hamiltonian invariant. The Hamiltonian acts on the Fock space spanned by the normalized vectors $|N_1, N_2, N_3\rangle = C^{-1}(a_1^\dagger)^{N_1}(a_2^\dagger)^{N_2}(a_3^\dagger)^{N_3}|0\rangle$, where $C = \sqrt{N_1!N_2!N_3!}$ and $|0\rangle \equiv |0, 0, 0\rangle$ is the Fock vacuum. The Hamiltonian $\hat{1}$ may be interpreted as describing tunneling between nearest neighbor wells of an interacting Bose-Einstein condensate.

The Hamiltonian has, beyond the energy and the total number of particles $N$, another independent conserved quantity expressed through the operator $\hat{Q}$:

\begin{equation}
\hat{Q} = J_2^2 N_3 + J_3^2 N_1 - J_1 J_3(a_3^\dagger a_3 + a_2^\dagger a_1).
\end{equation}

This conserved operator can alternatively be interpreted as a tunneling Hamiltonian for a two-well subsystem containing only wells 1 and 3. Because $\hat{Q}$ admits the factorization $\hat{Q} = \Omega \hat{\Omega}$, where $\Omega = J_1 a_3 - J_3 a_1$, the dynamical...
evolution governed by $Q$ is harmonic for any initial state. Later, it will be shown that $Q$ assumes a fundamental role in the analysis of resonant [24] quantum dynamics of the system [4]. This arises due to an unexpected connection with virtual processes (see Supplemental Material [25]).

As the model has three degrees of freedom and three independent conserved quantities satisfying the commutation relations

$$[H_0, N] = 0, \quad [H_0, Q] = 0, \quad [N, Q] = 0,$$

the model is integrable. Further details about the integrability, and relation to exact solvability through the Yang-Baxter equation and associated Bethe Ansatz methods, are presented in [15].

**Breaking the integrability**—In order to break the integrability we add to the integrable Hamiltonian the operator $H_1 = \mu N_3$, which acts as an external potential for the well labeled 3. We emphasize that the breaking of integrability is in no way unique. However, the inclusion of this term is one of the simplest cases for integrability breaking.

The non-integrable Hamiltonian is

$$H = H_0 + H_1$$

$$H = U(N_1 - N_2 + N_3)^2 + J_1(a_1^\dagger a_2 + a_2^\dagger a_1) + J_3(a_2^\dagger a_3 + a_3^\dagger a_2) + \mu N_3,$$

which is schematically shown in Fig. 1. It is important to observe the above Hamiltonian still commutes with the operator $N$. However, the operator $Q$ is not conserved because the commutator

$$[H, Q] = \mu J_1 J_3 (a_1^\dagger a_3 - a_3^\dagger a_1),$$

is non-zero when the parameters $\mu$, $J_1$ and $J_3$ are all non-zero.

![FIG. 1. Schematic representation of the Hamiltonian](image)

*FIG. 1. Schematic representation of the Hamiltonian (4). The arrows $J_1$ and $J_3$ represent the tunneling couplings between the wells, $U$ characterizes inter-well and intra-well interaction between bosons, while $\mu$ is the coupling strength for an external potential applied to the well labeled 3.*

**Energy level spacing distribution**—For quantum systems there have been many studies relating integrability, or the absence of it, with the statistics of energy level spacing distributions. It was argued by Berry and Tabor [19] that a Poisson distribution holds for integrable systems when the number of degrees of freedom is greater than one. There are several known examples of non-integrable systems where the Wigner-Dyson distribution [20] [21] [29] applies [27] [30]. It is possible to interpolate between these scenarios through the Brody distribution, which is given by [23]

$$P_q(s) = \alpha(q + 1)s^q \exp(-\alpha s^{q+1}),$$

where $\alpha = \Gamma((q + 2)/(q + 1))^{q+1}$, $\Gamma$ denotes the Gamma function, and $s$ is the energy level spacing. The Wigner-Dyson distribution corresponds to $q = 1$ while the Poisson distribution is obtained by setting $q = 0$. In this sense the value of $q$ in the Brody distribution provides a measure of *chaoticity* in the system. Fig. 2 displays the energy level spacing distribution for three values of $\mu$ going from Poisson distribution ($\mu = 0, q = 0$), passing through the Brody distribution ($\mu = 0.4, q = 0.35$) towards the Wigner-Dyson distribution; ($\mu = 1, q = 1$), before returning to the Poissonian case ($\mu = 4, q = 0$).

The results of Fig. 2 might be initially surprising, in that sufficiently increasing the value of $\mu$ drives the system back towards a Poissonian distribution, although integrability remains broken. However a different perspective is that in the large $\mu$ limit the system approaches an integrable setting, with $H = H_1$ to leading order in $\mu$ and $H_0$ seen as the perturbing term. For such a system $N_1$ and $N_2$ play the role of conserved operators in...
the $\mu \to \infty$ limit. Fundamentally, we will be interested in observing how the quantum dynamics are influenced as the system passes through these different degrees of chaoticity, characterized by the value of $q$ in fitting the Brody distribution \cite{6}.

Quantum dynamics– The time evolution of the expectation values for the number operators are computed using

$$\langle N_i \rangle = \langle \Psi(t)|N_i|\Psi(t)\rangle, \quad i = 1, 2, 3$$  (7)

where $|\Psi(t)\rangle = \exp(-i\mathbf{H}t)|\phi\rangle$ and $|\phi\rangle$ represents the initial state configuration in Fock space. We will adopt a protocol for which $|\phi\rangle = |N,0,0\rangle$. So, we consider the well labeled 1 to be the source, the well labeled 2 to be the gate, and the well labeled 3 to be the drain.

In Fig. 3 the time evolution of the expectation value of the number operators is displayed for the same set of parameter values that are used in Fig. 1. It is apparent that increasing the value of $\mu$ suppresses the tunneling of particles into the drain, while increasing the time-average value of $\langle N_2 \rangle$. For $\mu = 4$ this suppression of tunneling into the drain is strong enough that its number expectation value is negligible, i.e. tunneling into the drain has been switched-off. The resulting dynamics resembles that of the Josephson tunneling regime for a subsystem comprising wells 1 and 2 \cite{31}. A two-well system, with just two degrees of freedom, is an integrable model with Poissonian energy level distribution \cite{23}. From this perspective, the distribution displayed in Fig. 2(d) is entirely consistent with the dynamical behavior displayed in Fig. 3(d).

The next step is to investigate how the tunneling between the source and the drain, via virtual processes mediated by the gate, can be manipulated. We return to the integrable model \cite{1} and first consider variations in the interaction parameter $U$ to manipulate the tunneling across the wells. In Fig. 1 we present results using four choices for the interacting controlling parameter. The dynamics typically display collapse and revival of oscillations for $U << J_i N_i$, $i = 1, 3$. On increasing $U$, the period of collapse and revival increases, while the time-average of $\langle N_2 \rangle$ decreases. Furthermore, the dynamics between wells 1 and 3 approach harmonic oscillations with $\langle N_1 \rangle + \langle N_3 \rangle \approx N$ and $\langle N_2 \rangle \approx 0$.

In this latter regime, these dynamical features can be understood by observing that the integrable Hamiltonian possesses a hidden two-well algebraic structure. Defining $J = \sqrt{J_1^2 + J_3^2}$ and setting $N_{1,3} = N_1 + N_3$, $a_{1,3}^\dagger = J^{-1}(J_1 a_1^\dagger + J_3 a_3^\dagger)$ and $a_{1,3} = J^{-1}(J_1 a_1 + J_3 a_3)$ realizes the Heisenberg algebra \cite{25}. We may then write that $H_0 = U(N_{1,3}^2 - N_2^2) + J(a_{1,3}^\dagger a_2 + h.c.)$, with an effective well given by the combined source and drain. As is well known \cite{1 2 31 32}, the self-trapping regime is expected to occur in the two-well model when $UN > J$.

To be more precise, $\langle N_2 \rangle/N < 0$ when $UN > J/2\sqrt{\epsilon - \epsilon^2}$ if well 2 is initially empty. Thus, for $UN >> J$ we find $\langle N_2 \rangle/N \approx 0$, and almost all bosons must be distributed between the source and the drain if only a small fraction of bosons are initially in the gate \cite{25}.

On the other hand, due the conservation of $N$ the three-mode integrable Hamiltonian takes the form $H_0 = -4UN_2(N_1 + N_3)$+ tunneling terms, up to an $N$-dependent constant. It was pointed out in \cite{14} that for isotropic tunneling $J_1 = J_3$ the source and the drain can form an effective non-interacting two-well system, by second-order processes \cite{33 35} through the gate, such that $\langle N_2 \rangle \approx 0$. For general tunneling, we find the remarkable result that the effective Hamiltonian is simply given by $H_{\text{eff}} = -\lambda Q$, where $Q$ is the conserved charge \cite{2}, and $\lambda^{-1} = 4U(N - 1)$ \cite{25}. This produces an effective tunneling coupling given by $J_{\text{eff}} = \lambda J_1 J_3$, which decreases with increasing $N$, and therefore will only be observed in mesoscopic samples \cite{14}. In view of the above observations, concerning the hidden two-well structure and the conserved charge serving as the effective Hamiltonian, we formally identify the resonant tunneling regime to be that for which $UN >> J$.

Control of resonant tunneling – In Fig. 4(d) the dynamics is seen to be remarkably close to being harmonic over sufficiently short time scales, with the period monotonically increasing with interaction coupling $U$. This behavior supports the conclusion that the effective Hamiltonian for the resonant tunneling regime is simply related to the conserved charge $Q$. The frequency of oscillation in this regime is given by $\omega = \lambda J^2$, with the amplitude also being $U$-dependent. The maximum amplitude of os-
oscillation is obtained when $J_1 = J_3$. If the initial state is $|N, 0, 0\rangle$ or $|0, 0, N\rangle$, the oscillations between the source and drain are coherent, i.e. a superfluid-like regime, with tunneling to the gate switched-off. On the other hand, if the initial state is $|0, N, 0\rangle$ the system will remain trapped in this initial state configuration, i.e. an insulator regime, with tunneling from the gate switched-off.

Next, we maintain the system in the resonant tunneling regime $UN >> J$ and study the non-integrable dynamics using the parameter $\mu$ to control the behavior of the source and drain subsystem. The approach here, following the study above, is to choose the initial state $|N, 0, 0\rangle$ and investigate the ability to control the frequency and amplitude of the populations oscillating between the source and the drain.

In Fig. 5 the interaction coupling is fixed as $U = 0.17$, and results are shown for the expectation values of the populations using four choices for $\mu$. It is seen that the presence of the external potential does not significantly influence the gate, in the sense that it does not affect the negligible average population $\langle N_3 \rangle$. Fig. 5(d) shows how the amplitude decays while increasing the external potential, as well as the dependence of the frequency. The three points highlighted in the curves correspond to the values of the amplitude and frequency of Figs. 5(a) (cyan circle), 5(b) (lime triangle), and 5(c) (yellow diamond).

In this non-integrable regime the effective Hamiltonian is given by $H_{\text{eff}} = -\lambda Q + \mu N_3$. For short time scales the dynamics exhibits Josephson-like oscillation [4] with frequency $\omega = 2\lambda J_1 J_3 / \sqrt{2} \Delta n$, where $\Delta n = 1/(1 + \gamma^2)$ is the amplitude and $\gamma = (\lambda (J_2^2 - J_3) - \mu) / 2\lambda J_1 J_3$ (see SM [23]) for details). Increasing the external potential reduces the oscillation amplitude $\Delta n$ and the period between the source and the drain, until the amplitude of oscillation is completely suppressed i.e. all tunneling is switched-off. Through semi-classical analysis, one can obtain analytic expressions for the expectation values of the relative populations, $n_i = N_i / N$ ($i = 1, 3$), in the wells 1 and 3, given by $\langle n_1 \rangle = 1 - \langle n_3 \rangle$ and $\langle n_3 \rangle = \Delta n \sin^2(\omega t/2)$ (see SM [23]). In agreement with [30], the maximum amplitude oscillation is obtained when the external potential difference becomes small.

Conclusion – We studied a model for boson tunneling in a triple-well system, both in integrable and non-integrable cases, through variation of the system coupling parameters. The model draws an analogy with a transistor through identification of the wells as the source, gate, and drain, and our primary objective was to investigate how the system could be implemented as a switching device. A simple proof of concept was shown through Fig. 3 that an applied potential to the drain was able to switch-off tunneling to the drain.
In the integrable setting we then identified the resonant tunneling regime between the source and drain, for which expectation values of particle numbers in the gate are negligible, i.e. tunneling through the gate is switched-off. Moreover, it was found that a conserved operator of the integrable system acts as an effective Hamiltonian, which predicts harmonic oscillations. This is in agreement with observation from numerical calculations.

We then investigated the effects of breaking integrability in the resonant tunneling regime through the application of an external potential to the drain. It was found that application of the potential did not destroy the harmonic nature of the oscillations, but did influence the amplitude and frequency. Increasing the applied potential allowed for tuning the system from the switched-on configuration through to switched-off. Results from semi-classical analyses produced formulae for the amplitude and frequency, which proved to be remarkably accurate when compared to numerical calculations. This demonstrates the possibility to reliably and predictably control the harmonic dynamical behavior of the model in a particular regime. A surprising feature of this result that this precise level of control arises through the breaking of integrability in the system.

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Supplemental Material for “Control of tunneling in a triple-well system: an atomtronic switching device”

I. TWO-MODE STRUCTURE AND THE RESONANT TUNNELING REGIME

The integrable three-well model \( H_0 \) can be structured through two modes, as follows. From

\[
H_0 = U(N_1 - N_2 + N_3)^2 + J_1(a_1^\dagger a_2 + a_1 a_2^\dagger) + J_3(a_3^\dagger a_3 + a_2 a_1^\dagger).
\]

(8)

define \( J = \sqrt{J_1^2 + J_3^2} \) and the operators \( N_{1,3} = N_1 + N_3, a_{1,3} = J^{-1}(J_1 a_1 + J_3 a_3) \) and \( a_{1,3}^\dagger = J^{-1}(J_1 a_1^\dagger + J_3 a_3^\dagger) \) satisfying the Heisenberg algebra

\[
[N_{1,3}, a_{1,3}] = -a_{1,3}, \quad [N_{1,3}, a_{1,3}^\dagger] = a_{1,3}^\dagger, \quad [a_{1,3}, a_{1,3}^\dagger] = 1.
\]

Then

\[
H_0 = U(N_{1,3} - N_2)^2 + J(a_{1,3}^\dagger a_2 + a_2 a_{1,3}^\dagger),
\]

such that the modes of wells 1 and 3 are now represented by the single mode “1,3”.

The two-well model exhibits a self-trapping regime, onsetting around \( UN \approx J \) or \( \chi \approx UN/J \approx 1 \) \([1, 31, 32]\). This translates to a resonant tunneling regime for the triple-well model. Through semi-classical analysis this regime may be clearly identified. Using the usual number-phase correspondence, that is, \( a_2 = e^{i\theta_2}\sqrt{N_2}, a_{1,3} = e^{i\theta_{1,3}}\sqrt{N_{1,3}} \) and the conservation of boson number \( N_{1,3} + N_2 = N \), we find

\[
h = \frac{H_0}{N} = UN(1 - 2n_2)^2 + 2J\sqrt{(1 - n_2)n_2} \cos \phi,
\]

where \( n_2 = N_2/N \) and \( \phi = \theta_{1,3} - \theta_2 \). Consider the dynamics where the initial condition is \( n_2 = 0 \). In the initial time \( t = 0 \) the system has the energy \( h = UN \). By energy conservation at all times, we obtain the expression

\[
n_2 = \frac{1}{2} - \frac{\sqrt{\chi^2 - \cos^2 \phi}}{2\chi}.
\]

(9)

The conditions \( \chi > 1 \) and \( |\cos \phi| = 1 \) (maximum value) imply that \( 0 \leq n_2 \leq 0.5 \). From Eq. (9) we conclude that when \( \chi \gg |\cos \phi|, n_2 \to 0 \), and the bosons are distributed between the wells labeled 1 and 3, producing the resonant tunneling regime.

II. EFFECTIVE INTEGRABLE HAMILTONIAN IN THE RESONANT TUNNELING REGIME

In order to better understand the dynamics in the resonant tunneling regime, we first observe that the integrable Hamiltonian can be written, by using the conserved quantity \( N \), as an effective Hamiltonian without on-site interaction (up to a global constant \( UN^3 \)). Specifically \( H_0 = \tilde{H}_I + V \), where the interaction term \( H_I = -4UN_2(N_1 + N_3) \) has eigenstate and eigenvalues given by

\[
H_I|N_1, N_2, N_3\rangle = -4UN_2(N_1 + N_3)|N_1, N_2, N_3\rangle,
\]

and the tunneling term \( V = (J_1 a_1^\dagger + J_3 a_3^\dagger)a_2 + h.c. \) is treated as a perturbation. As observed in \([14]\) for the isotropic case \( J_1 = J_3 = J/\sqrt{2} \), since \( n_2 \approx 0 \) the interaction part is \( H_I \approx 0 \) and the wells 1 and 3 form an effective non-interacting two-system coupled through well 2 by a second-order process \([14]\) \([33, 35]\) with the effective Hamiltonian \( H_{\text{eff}} = V_{\text{eff}} = J_{\text{eff}}(a_1^\dagger a_3 + a_3^\dagger a_1) \). Recall that the transition rate from initial state \( |s\rangle \) to final state \( |k\rangle \) is expressed

\[
W^{(i)} = 2\pi|\langle k|V^{(i)}|s\rangle|^2 \delta(E_k - E_s), \quad i = 1, 2,
\]

where \( V^{(1)} = V \) for first-order transition (Fermi’s golden rule), and

\[
V^{(2)} = \sum_m \frac{V|m\rangle\langle m|V}{E_s - E_m}
\]

for second-order transition. Equating second-order transition and the matrix elements for the states \( |N,0,0\rangle \) and \( |N-1,0,1\rangle \), it is found that (see also \([14]\))

\[
J_{\text{eff}} = \frac{J_1 J_3}{4U(N - 1)}.
\]

We find that the effective Hamiltonian for general tunneling, which includes the anisotropic case \( J_1 \neq J_3 \), is given by \( H_{\text{eff}} = -\lambda Q \), where \( Q = J_1^2 N_3 + J_3^2 N_1 - J_1 J_3(a_1 a_3^\dagger + a_3 a_1^\dagger) \) is the conserved charge and \( \lambda^{-1} = 4U(N - 1) \).
III. EFFECTIVE NON-INTEGRABLE HAMILTONIAN AND QUANTUM CONTROL

For the non-integrable case, the effective Hamiltonian in the resonant tunneling regime $\chi \gg 1$ is given by

$$H_{\text{eff}} = -\lambda Q + \mu N_3.$$  

Returning to a semiclassical analysis it is found that

$$h = H_{\text{eff}} N_1 = -(\lambda J_1^2 - \mu)(1 - n_1) - \lambda J_3^n 1 + 2\lambda J_1 J_3 \sqrt{n_1(1 - n_1)} \cos \varphi,$$

where $n_1 = N_1/N$ and $\varphi = \theta_1 - \theta_3$. For initial condition $n_1 = 1$ and $n_3 = 0$ we have $h = -\lambda J_3^n$, a constant. Applying energy conservation and the condition $\cos \varphi = \pm 1$, we find that the amplitude of oscillation $\Delta n = 1 - n$ (Fig. 5d)) is given by $\Delta n = 1/(1 + \gamma^2)$, where $\gamma = [\lambda(J_1^2 - J_3^2) - \mu]/2\lambda J_1 J_3$. Hamilton’s equation gives

$$\dot{N}_1 = -\frac{\partial H_{\text{eff}}}{\partial \theta_1} = 2\lambda J_1 J_3 \sqrt{N_1(N - N_1)} \sin(\theta_1 - \theta_3), \quad \Rightarrow \quad \dot{n}_1 = 2\lambda J_1 J_3 \sqrt{n_1(1 - n_1)} \sin \varphi.$$

We can easily verify that the above results provide analytic expressions for the expectation values

$$\langle n_1 \rangle = 1 - \Delta n \sin^2(\omega t/2), \quad \langle n_3 \rangle = \Delta n \sin^2(\omega t/2),$$

where $\omega$ is the frequency given by $\omega = 2\lambda J_1 J_3/\sqrt{\Delta n}$. These results are plotted in the Figs. 5(d) and 6. Similar types of investigation have been presented in [35] for the case of pair-tunneling between two wells.

Fig. 6(a) presents the period dependence on the external potential obtained from the analytic expressions. It compares three points obtained from the numerical analysis, giving support that the analytic expressions are in accordance with quantum dynamics. In Fig. 6(b), it is clearly shown how the temporal evolution of amplitude oscillation depends on the external potential for the resonant tunneling case.

FIG. 6. Period $(2\pi/\omega)$ vs. the external potential $\mu$. The blue line above shows the period function obtained from the analytic expressions, comparing with the numerical points marked in “x”. The other markers in the (a) correspond to the values of the amplitude oscillation and frequency of Figs. 5(a) (cyan circle), 5(b) (yellow triangle), and 5(c) (lime diamond) also obtained from the numerical analysis. The graph (b) shows the dynamic evolution of the fractional occupation in well 1 (source) as a function of time and external potential parameter $\mu$. The dynamics of $\mu = 0$ (cyan line), $\mu = 0.04$ (yellow line) and $\mu = 0.34$ (lime line) are highlighted and the dashed pink line is marking the amplitude function $\Delta n$. The configuration used here has $N = 60, U = 0.17, J_1 = J_3 = 1/\sqrt{2}$ and initial state $|60, 0, 0\rangle$, as in Fig. 5. Recall that the dynamics of well 3 (drain) has the same configuration, however it is out of phase by $\pi$, while $\langle n_2 \rangle = 0$. 

(a) Resonant tunneling regime: Period vs. $\mu$.  
(b) Resonant tunneling regime: Source dynamics vs. $\mu$. 

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