Some asymptotic expansions for a semilinear reaction-diffusion problem in a sector*

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Abstract

The semilinear reaction-diffusion equation \(-\varepsilon^2 \Delta u + b(x, u) = 0\) with Dirichlet boundary conditions is considered in a convex unbounded sector. The diffusion parameter \(\varepsilon^2\) is arbitrarily small, and the "reduced equation" \(b(x, u_0(x)) = 0\) may have multiple solutions. A formal asymptotic expansion for a possible solution \(u\) is constructed that involves boundary and corner layer functions. For this asymptotic expansion, we establish certain inequalities that are used in [1] to construct sharp sub- and super-solutions and then establish the existence of a solution to a similar nonlinear elliptic problem in a convex polygon.

1 Introduction

In this note we consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

\[
Fu \equiv -\varepsilon^2 \Delta u + b(x, u) = 0, \quad x = (x_1, x_2) \in S \subset \mathbb{R}^2, \quad u(x) = g(x), \quad x \in \partial S.
\]

in a convex sector \(S\) with vertex \(O\) and sides \(\Gamma\) and \(\Gamma^-\). Our purpose in this note is to establish some asymptotic expansions and related inequalities for a possible solution to the problem. These are needed in [1] to construct sharp sub- and super-solutions and then establish the existence of a solution to a similar nonlinear elliptic problem in a convex polygon. The proofs involve lengthy formal calculations, and is the purpose of this paper.

The "reduced problem" associated with (1.1) is defined by formally setting \(\varepsilon = 0\) in (1.1b), i.e.

\[
b(x, u_0(x)) = 0 \quad \text{for} \quad x \in \bar{S}.
\]

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It is assumed that (1.2) has a smooth solution $u_0$ that is stable in a sense to be described below. The hypotheses on $b$ are such as to include the possibility of multiple solutions to (1.2) and therefore to (1.1). Since it may happen that $u_0 \neq g$ on $\partial S$, the solutions may exhibit boundary layer behavior near $\partial S$. We shall assume that the function $b$ is smooth and that $g$ is smooth on each $\Gamma$ and $\Gamma^-$ and continuous at the vertex $O$. Furthermore, we assume that $u_0(x)$, $g(x)$, and, for each fixed $s$, the function $b(x,s)$, as well as their derivatives, are bounded as $|x| \to \infty$.

In addition we make the following assumptions.

**A1** (stable reduced solution) There is a number $\gamma > 0$ such that

$$b_u(x,u_0(x)) > \gamma^2 > 0$$

for all $x \in S$.

**A2** (boundary condition) The boundary data $g(x)$ from (1.1b) satisfy

$$\int_{u_0(x)}^{v} b(x,s) \, ds > 0 \quad \text{for all} \quad v \in (u_0(x),g(x)], \quad x \in \partial S.$$

Here the notation $(a,b]'$ is defined to be $(a,b]$ when $a < b$ and $[b,a)$ when $a > b$, while $(a,b]' = \emptyset$ when $a = b$.

**A3** (corner condition) If $g(O) \neq u_0(O)$, then

$$\frac{b(O,g(O))}{g(O) - u_0(O)} > 0.$$

**A4** Only to simplify our presentation, we make a further assumption that

$$u_0(x) < g(x) \quad \text{for all} \quad x \in \partial S.$$

Using A4, we can simplify A3 to $b(O,g(O)) > 0$.

Note that if $g(x) \approx u_0(x)$, then A2 follows from A1 combined with (1.2), while if $g(x) = u_0(x)$ at some point $x \in \partial S$, then A2 does not impose any restriction on $g$ at this point. Similarly, if $g(O) \approx u_0(O)$, then A3 follows from A1 combined with (1.2), while if $g(O) = u_0(O)$ at some vertex $O$, then A3 does not impose any restriction on $g$ at this point. Assumption A1 is local and permits the construction of multiple solutions to (1.2) and therefore to (1.1). Assumptions A2 and A3 guarantee existence of boundary and corner layer ingredients, respectively in an asymptotic expansion for problem (1.1).

The note is organized as follows. Section 2 defines some boundary layer functions associated with each side of the sector $S$ and some corner layer functions associated with the vertex of $S$. The boundary layer functions are defined as solutions of some ordinary differential equations in a stretched
independent variable. The corner layer functions are solutions of some elliptic partial differential equations in stretched independent variables. In Sections 3 and 4, these boundary and corner functions are assembled into a formal first-order asymptotic expansion and a perturbed asymptotic expansion, respectively, and then certain properties of the unperturbed and perturbed asymptotic expansions are established, that are used in [1]. The proofs involve much computation, and is the purpose of this note.

**Notation.** Throughout the paper we let $C, \bar{C}, c, c'$ denote generic positive constants that may take different values in different formulas, but are always independent of $\varepsilon$ ($\bar{C}$ is usually used for a sufficiently large constant). A subscripted $C$ (e.g., $C_1$) denotes a positive constant that is independent of $\varepsilon$ and takes a fixed value. For any two quantities $w_1$ and $w_2$, the notation $w_1 = O(w_2)$ means $|w_1| \leq C|w_2|$.

### 2 Boundary and corner layer functions

This section defines some boundary layer functions associated with each side of the sector $S$ and some corner layer functions associated with the vertex of $S$. The boundary layer functions are defined as solutions of some ordinary differential equations in a stretched independent variable. The corner layer functions are solutions of some elliptic partial differential equations in stretched independent variables. The existence and properties of the corner layer functions are established in [1, Section 3].

We use the functions

$$B(x, t) = b(x, u_0(x) + t), \quad \tilde{B}(x, t; p) = b(x, u_0(x) + t) - pt.$$  \hfill (2.1)

The perturbed version $\tilde{B}$ of the function $B$ is used, with $|p|$ sufficiently small, in the construction of sub- and super-solutions. In the constructions that follow, a tilde will always denote a perturbed function. The perturbed functions always depend on the parameter $p$, but we will sometimes not show the explicit dependence. Thus, we will sometimes write $\tilde{B}(x, t)$ for $\tilde{B}(x, t; p)$.

We need a notation for the derivatives of $\tilde{B}$. For derivatives with respect to the first argument, we write $\nabla_x \tilde{B}$, $\nabla_x^2 \tilde{B}$, etc., for the vector, matrix of second derivatives, etc., with respect to $x$. We write $\tilde{B}_t$, $\tilde{B}_{tt}$, etc., for derivatives with respect to $t$. Note also that $\tilde{B}(x, 0) = 0$, so $\nabla_x^k \tilde{B}(x, 0) = 0$ for $k = 1, 2, \cdots$.

$$|\nabla_x^k \tilde{B}(x, t)| \leq C|t| \quad \text{for } k = 0, 1, 2, \cdots. \hfill (2.2)$$

We will occasionally use, for any function $f$, the notations

$$f_{[a}^b = f(b) - f(a), \quad f|_{a; b}^c = f(c) - f(b) - f(a). \hfill (2.3)$$
Since \( f^{(a+b)} + f(0) = abf''(t) \), we see that \( f(0) = 0 \) implies \( f^{(a+b)} = O(|ab|) \) and therefore \( f^{(a+b+c)} = O(|c| + |ab|) \). In view of (2.2), we thus have

\[
\nabla_{x}B(x, \cdot)\mid_{a;b} = O(|c| + |ab|). \tag{2.4}
\]

We shall now define functions needed to assemble a first-order asymptotic expansion and its perturbed version. The following two subsections deal respectively with a side \( \Gamma \) of \( S \), and with the vertex \( O \) of \( S \).

### 2.1 Solution near a side

In this subsection we construct boundary layer functions associated with the side \( \Gamma \) of \( \partial S \). An analogous construction can be made for the side \( \Gamma^- \). Throughout the subsection, \( \Gamma \) denotes the line that extends the ray \( \Gamma \). Extend \( u_0 \) and \( b \) to smooth functions, also denoted \( u_0 \) and \( b \), on \( \mathbb{R}^2 \) and \( \mathbb{R}^2 \times \mathbb{R} \), respectively, so that (1.2) and A1 hold true for all \( x \in \mathbb{R}^2 \). Furthermore, extend \( g \) defined on the ray \( \Gamma \) to a smooth function, also denoted \( g \), on the line \( \Gamma \), which satisfies the extended form of A2 and A4 for all \( x \in \Gamma \).

Let \( e_s \) denote the unit vector pointing in the direction of \( \Gamma \). Let \( e_r \) be the unit vector perpendicular to \( e_s \) and oriented to point into \( S \). Let \( s \) denote the signed distance along \( \Gamma \) with \( s = 0 \) at \( O \) and \( s > 0 \) on the ray \( \Gamma \). For \( x \in \mathbb{R}^2 \) write \( x = O + se_s + re_r \). Then \( \bar{x} = O + se_s \) is the point on \( \Gamma \) which is closest to \( x \) and \( r \) is the signed distance from \( \bar{x} \) to \( x \), with \( r > 0 \) if \( x \in \Omega \). (\( e_s, e_r, x \) and \( \bar{x} \) are shown in Figure 1).

Let \( \tilde{v}_0(\xi, s; p) \) be the solution to the nonlinear autonomous two point boundary value problem

\[
-\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + \tilde{B}(\bar{x}, \tilde{v}_0; p) = 0, \tag{2.5a}
\]

\[
\tilde{v}_0(0, s; p) = g(\bar{x}) - u_0(\bar{x}), \quad \tilde{v}_0(\infty, s; p) = 0. \tag{2.5b}
\]

The geometric meaning of the variable \( \xi \) is given by the formula \( \xi = r/\varepsilon \). The variables \( p \) and \( s \) appear as parameters in the problem (2.5). The parameter \( p \) satisfies \( |p| < \gamma^2 \) and in general will be close to zero. We sometimes omit the explicit dependence of \( \tilde{v}_0 \) on \( p \) and write \( \tilde{v}_0(\xi, s; p) = \tilde{v}_0(\xi, s) \).

We set \( v_0(\xi, s) = \tilde{v}_0(\xi, s; 0) \). The function \( v_0 \) appears in the asymptotic expansion of the solution near the side \( \Gamma \). With \( v_0 \) defined, we define a function \( v_1(\xi, s) \) to be the solution to the linear two point boundary value problem

\[
-\frac{\partial^2 v_1}{\partial \xi^2} + v_1 B_t(\bar{x}, v_0) = -\xi \cdot e_r \cdot \nabla_x B(\bar{x}, v_0), \tag{2.6a}
\]

\[
v_1(0, s) = v_1(\infty, s) = 0. \tag{2.6b}
\]
Note that $v_1$ is not a perturbed function as it does not depend on $p$. We also define

$$
\begin{align*}
\hat{v}_0(\xi; p) &= \hat{v}_0(\xi, 0; p), \\
\hat{v}_0(\xi) &= v_0(\xi, 0), \\
\hat{v}_1(\xi) &= v_1(\xi, 0), \\
\hat{v} = \hat{v}_0 + \varepsilon v_1, \\
v = v_0 + \varepsilon v_1, \\
\hat{v} = \hat{v}_0 + \varepsilon \hat{v}_1, \\
v = v_0 + \varepsilon v_1.
\end{align*}
$$

(2.7)

In our notation, a small circle above a function name indicates that in the argument of the function we have set $s = 0$.

For the solvability and properties of problems (2.5) and (2.6) we cite a result from [1, Lemma 2.1].

**Lemma 2.1.** There is $p_0 \in (0, \gamma^2)$ such that for all $|p| \leq p_0$ there exist functions $\hat{v}_0$ and $v_1$ that satisfy (2.5), (2.6). For the function $\tilde{v}_0 = \hat{v}_0(\xi, s; p)$ we have

$$
\tilde{v}_0 \geq 0, \quad \frac{\partial \tilde{v}_0}{\partial p} \geq 0.
$$

(2.8)

Furthermore, for any $k \geq 0$ and arbitrarily small but fixed $\delta$, there is a $C > 0$ such that for $0 < \xi < \infty$, $s \in \mathbb{R}$ and $k = 0, 1, \ldots$,

$$
\left| \frac{\partial^k \hat{v}_0}{\partial \xi^k} \right| + \left| \frac{\partial^k \hat{v}_0}{\partial s^k} \right| + \left| \frac{\partial^k v_1}{\partial \xi^k} \right| + \left| \frac{\partial v_0}{\partial p} \right| + \left| \frac{\partial^2 \hat{v}_0}{\partial p \partial s} \right| \leq C e^{-(\gamma - \sqrt{|p| - \delta}) \xi}.
$$

For later purposes we shall now obtain an estimate for $\tilde{v}_0 - v_0$.

**Lemma 2.2.** We have, for $|p|$ sufficiently small,

$$
- \varepsilon^2 \Delta (\tilde{v}_0 - v_0) = -B(x, \cdot)\hat{v}_0 + pv_0 + O(\varepsilon^2 + p^2).
$$

(2.9)

**Proof.** It follows from Lemma 2.1 that $|\tilde{v}_0 - v_0| \leq Cpe^{-\varepsilon \xi}$ and $|\frac{\partial^2}{\partial \xi^2} \tilde{v}_0| \leq C$. The latter estimate yields $\varepsilon^2 \Delta (\tilde{v}_0 - v_0) \leq \frac{\partial^2}{\partial \xi^2} (\tilde{v}_0 - v_0) + O(\varepsilon^2)$. Furthermore, invoking (2.5a), (2.1) and the estimate for $\tilde{v}_0 - v_0$, we get

$$
- \frac{\partial^2}{\partial \xi^2} (\tilde{v}_0 - v_0) = -B(\bar{x}, \cdot)\hat{v}_0 + pv_0 = -G(\bar{x}, 0) + pv_0 + O(p^2),
$$

where we use the auxiliary function $G(x, t) = B(x, \cdot)|_{v_0 + t}$. Noting that $B(x, \cdot)|_{v_0 + \varepsilon v_1} = G(x, \varepsilon v_1)$, it remains to establish the estimate $G(x, \varepsilon v_1) - G(\bar{x}, 0) = O(\varepsilon^2 + p^2)$. Indeed, we have

$$
G(x, \varepsilon v_1) - G(\bar{x}, 0) = (x - \bar{x}) \cdot \nabla_s G^* + \varepsilon v_1 G^*_t = O(\varepsilon^2 + p^2),
$$

(2.10)

where the asterisks indicate that the derivatives are evaluated at an intermediate point. In the last step in (2.10) we combined $|G^*_t| = |G_t(x^*, t^*)| = |B_t(x^*, \varepsilon v_0 + t^*) - B_t(x^*, v_0 + t^*)| \leq C|\varepsilon v_0 - v_0|$, and a similar estimate $|\nabla_s G^*| \leq C|\varepsilon v_0 - v_0|$ with $|x - \bar{x}| = \varepsilon \xi$, $|v_1| \leq C$ and $|\varepsilon v_0 - v_0| \leq Cpe^{-\varepsilon \xi}$. ■
2.2 Solution near a vertex

In this subsection we construct corner layer functions associated with the vertex \( O \). Some notation is required for the constructions. Let \( s \) denote the distance along \( \Gamma \), measured from \( O \), and let \( r \) denote the perpendicular distance to a point \( x \in S \). Thus, \( x \to (s,r) \) is a linear orthogonal map. We also let \( e_s \) and \( e_r \) denote the unit vectors along \( \Gamma \) and orthogonal to \( \Gamma \) respectively, so \( x = r e_r + s e_s \). We denote by \( \bar{x} = s e_s \) the point of \( \Gamma \) that is closest to \( x \). In a similar manner, we define variables \( (s^-,r^-) \), so \( x = r^- e_r^- + s^- e_s^- \), and \( \bar{x}^- = s^- e_s^- \) associated with the side \( \Gamma^- \). The variable \( s^- \) denotes the distance along \( \Gamma^- \), measured from \( O \). We will also need stretched variables. We set \( \eta = x/\varepsilon \), \( \xi = r/\varepsilon \), \( \sigma = s/\varepsilon \), \( \xi^- = r^-/\varepsilon \), \( \sigma^- = s^-/\varepsilon \). These variables are shown in Figure 1.

Using these notations, Section 2.1 gives functions \( \tilde{v}_0(\xi,s;p) \) and \( v_1(\xi,s) \) associated with the side \( \Gamma \) and functions \( \tilde{v}_0^-(\xi^-,s^-;p) \) and \( v_1^-(\xi^-,s^-) \) associated with the side \( \Gamma^- \). We also recall the notations in (2.7) and use corresponding notations for the side \( \Gamma^- \). The function \( \tilde{v} \) matches the disparity between the boundary conditions of (1.1b) and the value of \( u_0 \) on \( \partial S \). We also set \( \tilde{z}_0 = A := g(O) - u_0(O) \) on \( \partial S \).

The function \( \tilde{z}_0 \) is defined to be a bounded solution of the autonomous nonlinear elliptic boundary value problem

\[
-\triangle_\eta \tilde{z}_0 + \tilde{B}(O,\tilde{z}_0;p) = 0 \quad \text{in } S, \\
\tilde{z}_0 = A := g(O) - u_0(O) \quad \text{on } \partial S. \tag{2.11}
\]

Here we have \( A > 0 \), by our assumption A4 at the point \( O \). We also set
Finally there is a constant $C > 0$ which is an increasing function of $\eta$. Under this notation, the boundary conditions in (2.14) become

$$0 < \max\{\tilde{v}_0, \tilde{v}_0^-\} \leq \tilde{z}_0(\eta; \cdot) \leq \max\{\hat{v}_0, \hat{v}_0^-\} + C|\eta|^{-1},$$

(2.12)

and which is an increasing function of $p$. Also, $|\nabla \tilde{z}_0|$ is bounded in $S$. Finally there is a constant $C > 0$ such that

$$\tilde{z}_0(\eta) \leq C \left(e^{-\gamma \xi} + e^{-\gamma \xi^-}\right).$$

(2.13)

We also consider a function $z_1(\eta)$ which satisfies the linear elliptic boundary value problem

$$\Delta z_1 + z_1 B_0(O, z_0) = -\eta \cdot \nabla z B(O, z_0) \text{ in } S,$$

$$z_1 = \sigma \frac{\partial}{\partial S}(g - u_0)|_{x=0} \text{ on } \Gamma, \quad z_1 = \sigma^{-} \frac{\partial}{\partial S}(g^+ - u_0)|_{x=0} \text{ on } \Gamma^-,$$

(2.14)

The functions $\tilde{z}_0$ and $z_1$ form a correction $\tilde{z}_0 + \varepsilon z_1$ to the reduced solution $u_0$ in close proximity of the vertex $O$. To extend it further away from $O$, the corrections $\tilde{v}_0 + \varepsilon v_1$ and $\tilde{v}_0^- + \varepsilon v_1^-$ to $u_0$ near the sides $\Gamma$ and $\Gamma^-$ are to be invoked as follows. We use the corner functions $\tilde{z}_0$ and $z_1$ together with the boundary functions $\tilde{v}_0, v_1, \tilde{v}_0^-, v_1^-$ to define a related pair of corner functions $\tilde{q}_0$ and $q_1$, which, rather than $\tilde{z}_0$ and $z_1$, will appear in a formal asymptotic expansion of the solution of (1.1) in the entire $S$; see Sections 3, 4 below.

We shall use the following notation. Pick a point $\eta S$. Having chosen $\eta$, the formulas

$$\eta = \xi e_r + \sigma e_s = \xi^- e_r^- + \sigma^- e_s^-$$

(2.15)

determine numbers $\xi, \sigma, \xi^-, \sigma^-;$ see Figure 1. With this notation, and using the functions $\tilde{z}_0, z_1$ and $\tilde{v}_0, \tilde{v}_0^-, \tilde{v}_1, \tilde{v}_1^-$ of (2.5), (2.6), (2.7), we define

$$\tilde{q}_0(\eta; p) = \tilde{z}_0(\eta; p) - \tilde{v}_0(\xi; p) - \tilde{v}_0^-(\xi^-; p),$$

(2.16a)

$$q_1(\eta) = z_1(\eta) - [\tilde{v}_1(\xi) + \sigma \tilde{v}_0, s(\xi)] - [\tilde{v}_1^-(\xi^-) + \sigma^- \tilde{v}_0, s^-(\xi^-)],$$

(2.16b)

and furthermore,

$$\tilde{q}(\eta; p) = \tilde{q}_0(\eta; p) + \varepsilon q_1(\eta), \quad q_0(\eta) = \tilde{q}_0(\eta; 0), \quad q(\eta) = q_0(\eta) + \varepsilon q_1(\eta).$$

(2.16c)

In these formulas, following the notational conventions of (2.7), we mean

$$\tilde{v}_0, s(\xi) = \left. \frac{\partial}{\partial S} \tilde{v}_0(\xi, s)\right|_{s=0}, \quad \tilde{v}_0^- = \left. \frac{\partial}{\partial S} \tilde{v}_0^-(\xi^-, s^-)\right|_{s^-=0}.$$

(2.16d)

Under this notation, the boundary conditions in (2.14) become

$$z_1 = \sigma \tilde{v}_0, s^- \text{ on } \Gamma, \quad \tilde{z}_0 = \sigma^{-} \tilde{v}_0, s^- \text{ on } \Gamma^-.$$

(2.17)
From the above formulas, noting that $\Delta_\eta \tilde{q}_0 = \Delta_\eta \tilde{q}_0 - \frac{\partial^2}{\partial x^2} \tilde{v}_0 - \frac{\sigma^2}{(\sigma^2 - z^2)} \tilde{v}_0^2$, and using (2.16d), (2.11), we derive a nonlinear boundary value problem satisfied by $\tilde{q}_0$:

$$\Delta_\eta \tilde{q}_0 = \tilde{B}(O, \tilde{q}_0 + \tilde{v}_0 + \tilde{v}_0^\tau) - \tilde{B}(O, \tilde{v}_0) - \tilde{B}(O, \tilde{v}_0^\tau), \quad (2.18a)$$

$$\tilde{q}_0 = -\tilde{v}_0^- \text{ on } \Gamma, \quad \tilde{q}_0 = -\tilde{v}_0^\tau \text{ on } \Gamma^- . \quad (2.18b)$$

Similarly (see Lemma 2.4 below for details), using (2.5), (2.11), and (2.14), we formally derive a linear boundary value problem satisfied by $\tilde{q}_0$ and using (2.5), (2.11), we derive a nonlinear boundary value problem satisfied by $q_1$:

$$-\Delta_\eta q_1 + q_1 \tilde{B}_t(O, z_0) = -\eta \cdot \nabla_x B(O, \cdot) \bigg|_{\tilde{v}_0}^{z_0} - (\tilde{v}_1 + \sigma \tilde{v}_0, s) \tilde{B}_t(O, \cdot) \bigg|_{\tilde{v}_0}^{z_0} , \quad (2.19a)$$

$$q_1 = -(\tilde{v}_1 + \sigma \tilde{v}_0^\tau, s) \text{ on } \Gamma, \quad q_1 = -(\tilde{v}_1 + \sigma \tilde{v}_0, s) \text{ on } \Gamma^- . \quad (2.19b)$$

where we used the notation (2.3). Finally, by formally differentiating relation (2.16a) and problem (2.11) (or the equivalent problem (2.18)) with respect to $p$ and invoking (2.1), we formally derive a boundary value problem that is satisfied by $\tilde{q}_0, q_1$:

$$-\Delta_\eta \tilde{q}_0 + \tilde{q}_0 \tilde{B}_t(O, z_0) = \tilde{q}_0 - \tilde{v}_0, \tilde{B}_t(O, \cdot) \bigg|_{\tilde{v}_0}^{z_0} - \tilde{v}_0, \tilde{B}_t(O, \cdot) \bigg|_{\tilde{v}_0}^{z_0} , \quad \tilde{q}_0 = -\tilde{v}_0^\tau \text{ on } \Gamma, \quad \tilde{q}_0 = -\tilde{v}_0 \text{ on } \Gamma^- . \quad (2.20)$$

It is shown in [1, Lemmas 3.6, 3.16, 3.17] that the functions $\tilde{q}_0, q_1$ and $\tilde{q}_0, p$ exist and are exponentially decaying in $S$, i.e. there are constants $C_1$ and $c_1$ such that

$$|\tilde{q}_0| + |q_1| + |\tilde{q}_0, p| \leq C_1 e^{-c_1|\eta|} \text{ in } S. \quad (2.21)$$

In view of (2.16b), the existence of $q_1$ immediately implies existence of $z_1$. Similarly, having proved the existence of the solution to (2.20), an integration is used to show that this solution is in fact the derivative of $\tilde{q}_0$ with respect to $p$.

We now derive the boundary value problem satisfied by the function $q_1$.

**Lemma 2.4.** The function $q_1$ defined by (2.16b) satisfies problem (2.19).

**Proof.** To prove (2.19a), note that

$$\Delta_\eta q_1 = \Delta_\eta z_1 - (\Delta_\eta \tilde{v}_1 + \sigma \Delta_\eta \tilde{v}_0, s) - (\Delta_\eta \tilde{v}_1 + \sigma \Delta_\eta \tilde{v}_0, s^-) . \quad (2.22)$$

Next, using (2.16d) and then (2.5a), we calculate

$$\Delta_\eta \tilde{v}_0, s = \frac{\partial}{\partial s} v_0, x |_{s=0} = \frac{\partial}{\partial s} B(s e_s, v_0) |_{s=0} = e_s \cdot \nabla_x B(O, \tilde{v}_0) + \tilde{v}_0, s B_t(O, \tilde{v}_0).$$


Combining this with (2.6a), yields
\[
\triangle_\eta \hat{v}_1 + \sigma \triangle_\eta \hat{v}_{0,s} = [\hat{v}_1 B_t(O, \hat{v}_0) + \xi \mathbf{e}_r \cdot \nabla_x B(O, \hat{v}_0)] \\
+ \sigma[\mathbf{e}_s \cdot \nabla_x B(O, \hat{v}_0) + \hat{v}_{0,s} B_t(O, \hat{v}_0)] \\
= (\hat{v}_1 + \sigma \hat{v}_{0,s}) B_t(O, \hat{v}_0) + \eta \cdot \nabla_x B(O, \hat{v}_0),
\] (2.23)
where we used \( \xi \mathbf{e}_r + \sigma \mathbf{e}_s = \eta \) from (2.15). Similarly, one gets
\[
\triangle_\eta \hat{v}_1^- + \sigma^- \triangle_\eta \hat{v}_{0,s}^- = (\hat{v}_1^- + \sigma^- \hat{v}_{0,s}^-) B_t(O, \hat{v}_0^-) + \eta \cdot \nabla_x B(O, \hat{v}_0^-),
\] (2.24)
Recalling that, by (2.14), we have \( \triangle_\eta \hat{z}_1 = \hat{z}_1 B_t(O, \hat{z}_0) + \eta \nabla \cdot B(O, \hat{z}_0) \), where in the right-hand side \( \hat{z}_1 \) is replaced by \( \hat{v}_1 + \sigma \hat{v}_{0,s} + (\hat{v}_1^- + \sigma^- \hat{v}_{0,s}^-) \), and combining this with (2.22), (2.23), (2.24), yields (2.19). Finally, noting that \( \hat{v}_1 = 0 \) on \( \Gamma \) and \( \hat{v}_1^- = 0 \) on \( \Gamma^- \), and then comparing (2.16b) and (2.17), we immediately get (2.19b).

### 3 Asymptotic expansion

In Section 2.1 we have defined boundary layer functions \( \hat{v} = \hat{v}_0 + \varepsilon v_1 \) and \( \hat{v}^- = \hat{v}_0^- + \varepsilon v_1^- \) associated, respectively, with the sides \( \Gamma \) and \( \Gamma^- \) of \( S \), and in Section 2.2 we have defined corner layer functions \( \hat{q} = \hat{q}_0 + \varepsilon q_1 \) associated with the vertex \( O \) on \( S \). In the present and next sections, these functions are used to assemble a formal first-order asymptotic expansion and then a perturbed asymptotic expansion for the problem (1.1). We establish certain properties of the unperturbed and perturbed asymptotic expansions that are used in \( \Pi \). The proofs involve lengthy formal calculations, and is the purpose of this paper.

The asymptotic expansion \( u_{as,S} \) is defined as follows:
\[
u_{as,S}(x) = u_0(x) + v(\xi, s) + v^- (\xi^-, s^-) + q(\eta).
\] (3.1)
The next lemma shows that the differential equation applied to this asymptotic expansion is \( O(\varepsilon^2) \).

**Lemma 3.1.** For the asymptotic expansion \( u_{as,S}(x) \) of (3.1) one has
\[
Fu_{as,S} = O(\varepsilon^2), \quad u_{as,S}(x) = g(x) + O(\varepsilon^2) \text{ for } x \in \partial S.
\] (3.2a)
(3.2b)

**Proof.** (i) We start by establishing
\[
-\varepsilon^2 \Delta v + B(x, v) = O(\varepsilon^2), \quad -\varepsilon^2 \Delta v^- + B(x, v^-) = O(\varepsilon^2).
\] (3.3)
The first bound here is obtained estimating \( B(x, v) \) as follows. Fix \( \xi \) and \( s \); then \( B(x, v) \) is a function of \( \varepsilon \), i.e. \( B(x, v) = G(\varepsilon) \), where
\[
G(\varepsilon) = B(\bar{x} + \varepsilon \xi \mathbf{e}_r, v_0 + \varepsilon v_1) \quad \text{with } \bar{x} = \mathbf{e}_s, \quad v_0 = v_0(\xi, s), \quad v_1 = v_1(\xi, s).
\]

\[\begin{align*}
\]
Expand $G$ in a Taylor series around $\varepsilon = 0$ to obtain

\[ G(\varepsilon) = G(0) + \varepsilon G'(0) + \frac{1}{2} \varepsilon^2 G''(\varepsilon^*) \]  

(3.4)

with $0 < \varepsilon^* < \varepsilon$. A calculation shows that

\[ G'(\varepsilon) = \xi e_r \cdot \nabla x B(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1) + v_1 B_t(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1), \]

\[ G''(\varepsilon) = \xi^2 e^T \nabla^2 x B(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1)e_r \]

\[ + 2v_1 \xi e_r \cdot \nabla x B_t(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1) + v_1^2 B_{tt}(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1). \]

Hence

\[ G(0) = B(\bar{x}, v_0) = \frac{\partial^2}{\partial \xi^2} v_0, \]

\[ G'(0) = \xi e_r \cdot \nabla x B(\bar{x}, v_0) + v_1 B_t(\bar{x}, v_0) = \frac{\partial^2}{\partial \xi^2} v_1, \]

(3.5)

where we also used (2.5a) and (2.6a). Applying (2.2), we get

\[ |\nabla^2 x B(\bar{x} + \varepsilon \xi e_r, v_0 + \varepsilon v_1)| \leq C|v_0 + \varepsilon v_1|. \]

By Lemma 2.1, $v_0$ and $v_1$ are exponentially decaying in $\xi$, which yields $\xi^2|v_0 + \varepsilon v_1| \leq C$. Hence the first term in the formula for $G''(\varepsilon)$ is bounded. The other 2 terms are bounded for a similar reason, so $|G''(\varepsilon)| \leq C$, where $C$ is independent of $\xi$ and $s$. Combining this with (3.4) and (3.5), yields

\[ B(x, v) = G(\varepsilon) = \frac{\partial^2}{\partial \xi^2} (v_0 + \varepsilon v_1) + O(\varepsilon^2) = \frac{\partial^2}{\partial \xi^2} v + O(\varepsilon^2). \]

As $\varepsilon^2 \Delta v = \frac{\partial^2}{\partial \xi^2} v + O(\varepsilon^2)$, the first bound in (3.3) is established. The second bound is obtained similarly.

(ii) To show (3.2a), we calculate

\[ F_{u_{\text{as}, S}} = -\varepsilon^2 \Delta [u_0 + v + v^- + q] + b(x, u_0 + v + v^- + q) \]

\[ = B(x, \cdot) v_{\xi \sigma}^+ v^- + \Delta q + O(\varepsilon^2), \]

(3.6)

where we used (2.1) and (3.3), and also the notation (2.3).

Fix a point $\eta \in S$; then the first term in (3.6) is a function of $\varepsilon$, which we denote $F(\varepsilon)$. To be more precise, having chosen $\eta$, the formulas (2.15) determine fixed numbers $\xi, \sigma, \xi^-, \sigma^-$ . With the understanding that

\[ v = v(\varepsilon) = v_0(\xi, \varepsilon \sigma) + \varepsilon v_1(\xi, \varepsilon \sigma), \]

\[ v^- = v^- (\varepsilon) = v^-_0(\xi^-, \varepsilon \sigma^-) + \varepsilon v^-_1(\xi^-, \varepsilon \sigma^-), \]

(3.7a)

\[ q = q(\varepsilon) = q_0(\eta) + \varepsilon q_1(\eta), \]

define a function $F(\varepsilon)$ by

\[ F(\varepsilon) = B(\varepsilon \eta, \cdot) v_{\xi \sigma}^+ v^- + \frac{\partial}{\partial v^-} . \]

(3.7b)
In view of (3.6), to prove (3.2a), we need to show that
\[ F(\varepsilon) - \Delta \eta [q_0 + \varepsilon q_1] = O(\varepsilon^2) \text{ in } S. \]

Thus we must show that there is a number \( C \), independent of \( \eta \), such that
\[ F(0) = \Delta \eta q_0, \quad (3.8a) \]
\[ F'(0) = \Delta \eta q_1, \quad (3.8b) \]
\[ |F''(\varepsilon)| \leq C. \quad (3.8c) \]

From the definition (3.7) of \( F \), we have
\[ \frac{\partial}{\partial \varepsilon} v \bigg|_{\varepsilon=0} = \dot{v}_0, \quad v \bigg|_{\varepsilon=0} = \dot{v}_0, \]

so (2.18a) gives (3.8a).

A calculation using (3.7a) yields
\[ \frac{dv}{d\varepsilon} = v_1 + \sigma v_{0,s} + \varepsilon \sigma v_{1,s}, \quad \frac{d^2 v}{d\varepsilon^2} = 2\sigma v_{1,s} + \sigma^2 v_{0,ss} + \varepsilon \sigma^2 v_{1,ss}, \quad (3.9) \]
similar relations for \( v^- \), and also
\[ \frac{dq}{d\varepsilon} = q_1, \quad \frac{d^2 q}{d\varepsilon^2} = 0. \]

As \( |\sigma| \leq |\eta| \) and \( |\sigma^-| \leq |\eta| \), invoking Lemma 2.1 and (2.21), for \( k = 0, 1, 2 \) we get
\[ \left| \frac{d^k v}{d\varepsilon^k} \right| \leq C(1 + |\eta|^k)e^{-c\varepsilon}, \quad \left| \frac{d^k v^-}{d\varepsilon^k} \right| \leq C(1 + |\eta|^k)e^{-c\varepsilon}, \quad \left| \frac{d^k q}{d\varepsilon^k} \right| \leq Ce^{-c|\eta|}. \quad (3.10) \]

We now calculate
\[ F'(\varepsilon) = \eta \cdot \nabla_x B(\varepsilon \eta, \cdot) \bigg|_{v=v^-}^{v+v^-+q} \]
\[ + \frac{dq}{d\varepsilon} B_t(\varepsilon \eta, \cdot) \bigg|_{v=v^-+q} + \frac{d^2 v}{d\varepsilon^2} B_t(\varepsilon \eta, \cdot) \bigg|_{v=v^-+q} + \frac{d^2 v^-}{d\varepsilon^2} B_t(\varepsilon \eta, \cdot) \bigg|_{v=v^-}. \]

Hence, using the first relation in (3.9) and its analogue for \( v^- \), we get
\[ F'(0) = \eta \cdot \nabla_x B(O, \cdot) \bigg|_{\dot{v}_0 + \dot{v}_0^- + q_0} \]
\[ + q_1 B_t(O, \dot{v}_0 + \dot{v}_0^- + q_0) \]
\[ + (\dot{v}_1 + \sigma \dot{v}_0,s) B_t(O, \cdot) \bigg|_{\dot{v}_0} + (\dot{v}_1^- + \sigma^- \dot{v}_0,s) B_t(O, \cdot) \bigg|_{\dot{v}_0^-}. \]

Recalling that \( \dot{v}_0 + \dot{v}_0^- + q_0 = z_0 \) and inspecting (2.19a), we see that (3.8b) holds.
A formula for the quantity $F''(\varepsilon)$ is obtained by a lengthy but straightforward computation, which gives

$$F''(\varepsilon) = \eta^T \left( \nabla^2_B(\varepsilon\eta, \cdot) \right)_{v^{v+v+q}} \eta + \eta \cdot \left( \frac{\partial q}{\partial \varepsilon} \nabla_x B_t(\varepsilon\eta, \cdot) \right)_{v^{v+v+q}} + \frac{\partial v}{\partial \varepsilon}\nabla_x B_t(\varepsilon\eta, \cdot)_{v^{v+v+q}} + \frac{\partial v}{\partial \varepsilon}\nabla_x B_t(\varepsilon\eta, \cdot)_{v^{v+v+q}}$$

From inspection of this formula it is seen that each term in the formula is of one of three types, which we refer to as type I, type II, or type III. We shall invoke (3.10) to estimate them.

The only term of type I is in the first line of this formula and is clearly $|\eta|^2O(|q| + |v|)$, by (2.4), and thus $O(1)$ by (3.10).

The terms of type II involve, for $l = 0, 1$ and $k = 1, 2$, the quantities

$$\nabla_x \left( \frac{\partial^k}{\partial x^k} B(\varepsilon\eta, \cdot) \right)_{v^{v+v+q}} = O(v^{-q}) = O(e^{-\xi^+}),$$

$$\nabla_x \left( \frac{\partial^k}{\partial x^k} B(\varepsilon\eta, \cdot) \right)_{v^{-v-q}} = O(v^q) = O(e^{-\xi^-}),$$

which are always multiplied by $(1 + |\eta|^2)O(e^{-\xi^+})$ or $(1 + |\eta|^2)O(e^{-\xi^-})$, respectively. Thus the terms of type II are $O(1)$.

Finally the terms of type III involve, for $l = 0, 1$ and $k = 1, 2$, the quantity

$$\nabla_x \left( \frac{\partial^k}{\partial x^k} B(\varepsilon\eta, \cdot) \right)_{v^{v+v+q}} = O(1),$$

which is always multiplied by $(1 + |\eta|^2)O(e^{-\xi^+}e^{-\xi^-})$. As above, one sees that terms of type III are bounded. This completes the proof of (3.8), and therefore (3.2a).

(iii) It is sufficient to prove (3.2b) for $x = \bar{x} \in \Gamma$, as the other case of $x \in \Gamma^-$ is similar. Let $\bar{x} \in \Gamma$ be given. Define $s, \xi^-$ and $s^-$ by the formulas

$$\bar{x} = s e_s = \varepsilon \xi^- e_r^- + s^- e_{s^-}.$$

By (2.5b), (2.6b), we have $v_0(0, s) = g(\bar{x}) - u_0(\bar{x})$ and $v_1(0, s) = 0$; therefore

$$(u_0 + v)|_{\bar{x}} = u_0(\bar{x}) + [v_0(0, s) + \varepsilon v_1(0, s)] = g(\bar{x}).$$
Thus it remains to show that \((v^+ + q)|_x = O(\varepsilon^2)\). Indeed, by (2.18b) (2.19b), we have
\[
v^+ + q = [v_0^+ - \dot{v}_0^-] + \varepsilon[v_1^+ - (\dot{v}_1^- + \sigma^- \dot{v}_0^+,s^-)] = O(\varepsilon^2).
\]
In the last step here we have invoked the formulas
\[
|v_0^+ - \dot{v}_0^- - \varepsilon \sigma^- \dot{v}_0^+,s^-| = |v_0^+ (\xi^-,s^-) - v_0^- (\xi^-,0) - \varepsilon \sigma^- v_0^+,s^- (\xi^-,0)|
\]
\[
= \frac{1}{2} \varepsilon^2 (\sigma^-)^2 |v_0^+,s^- - (\xi^-,\dot{s}^-)| = O(\varepsilon^2),
\]
\[
|v_1^- - \dot{v}_1^-| = |v_1^- (\xi^-,s^-) - v_1^- (\xi^-,0)|
\]
\[
= \varepsilon \sigma^- |v_1^-,s^- (\xi^-,\dot{s}^-)| = O(\varepsilon),
\]
which are obtained using the exponential decay of \(v_0^-\) and \(v_1^-\) in \(\xi^-\) and noting that \(\sigma^- = (\cot \omega)\xi^-\) on the side \(\Gamma\), where \(\omega\) is the angle at the apex.

\section{Perturbed asymptotic expansion}

The perturbed version \(\beta_S\) of the asymptotic expansion \(u_{as,S}\) of (3.1) is defined as follows:
\[
\beta_S(x;p) = u_0(x) + \bar{v}(\xi, s; p) + \bar{v}^- (\xi^-, s^-; p) + \bar{q}(\eta; p) + \theta p,
\]
where a value for the positive parameter \(\theta\) and a range of values for \(p\) will be chosen below. Comparing (4.1) with (3.1), yields \(\beta_S(x;0) = u_{as,S}(x)\) and, furthermore, an alternative equivalent representation
\[
\beta_S(x;p) = u_{as,S}(x) + V(x, s; p) + V^- (\xi^-, s^-; p) + Q(\eta; p) + \theta p,
\]
where \(V = \bar{v} - v\), \(V^- = \bar{v}^--v^-\), \(Q = \bar{q} - q\), and therefore
\[
V = \bar{v}_0 - v_0, \quad V^- = \bar{v}_0^- - v_0^-, \quad Q = \bar{q}_0 - q_0.
\]
Note that for \(V\), \(V^-\) and \(Q\) here, by the exponential-decay estimates for \(\frac{\partial}{\partial p} \bar{v}_0\) and \(\frac{\partial}{\partial p} \bar{v}_0\) from Lemma 2.1 and 2.21, we have
\[
(1 + |\xi|)|V| \leq Cp, \quad (1 + |\xi^-|)|V^-| \leq Cp, \quad (1 + |\eta|)|Q| \leq Cpe^{-c|\eta|} \leq Cp.
\]
Furthermore, since \(|\eta| \leq C(\xi + \xi^-)\), invoking the exponential-decay estimates for \(\frac{\partial}{\partial p} \bar{v}_0\) and \(\frac{\partial^2}{\partial p^2} \bar{v}_0\) from Lemma 2.1 yields a more elaborate estimate
\[
(1 + |\eta|e^{-c\xi^-}) (|V| + |\frac{\partial V}{\partial \eta}|) \leq Cp,
\]
and a similar estimate involving \(V^-\).

In the remainder of this section we establish some inequalities that involve the perturbed asymptotic expansions \(\beta_S\). In particular, the inequalities of Lemmas 4.1 and 4.4 are used in [11] to construct sub- and super-solutions to our nonlinear boundary value problem.
Lemma 4.1. For the function $\beta_S$ of (4.1) we have $\beta_S = u_{as,S} + O(p)$. Furthermore, for some sufficiently small $\varepsilon^* > 0$, if $p \geq 0$ and $\varepsilon \leq \varepsilon^*$, then for all $x \in S$ we have

$$\beta_S(x; -p) \leq u_{as,S}(x) - \frac{1}{2} \theta p, \quad u_{as,S}(x) + \frac{1}{2} \theta p \leq \beta_S(x; p). \quad (4.5)$$

Proof. The assertion $\beta_S = u_{as,S} + O(p)$ immediately follows from (4.2). Furthermore, by (4.2), the bound for $\beta_S(x; p)$ in the remaining assertion (4.5) can be rewritten as

$$V + V^- + Q + \frac{1}{2} \theta p \geq 0 \quad \text{for } p \geq 0. \quad (4.6)$$

By (4.3), there is a sufficiently large $\bar{C} = \bar{C}(\theta)$ such that if $|\eta| \geq \bar{C}$, then $|Q| \leq pC e^{-c|\eta|} \leq \frac{1}{2} \theta p$. Combining this with $V \geq 0$ and $V^- \geq 0$, which follow from monotonicity of $\hat{v}_0$ and $\check{v}_0$ in $p$, established in (2.8), we get (4.6). Now let $|\eta| < \bar{C}$. Then $|s|, |s^-| < \varepsilon \bar{C}$. Invoking (2.1a), we have

$$Q = \tilde{q}_0 - q_0 = (\hat{z}_0 - z_0) - (\hat{v}_0 - \hat{v}_0) - (\check{v}_0 - \check{v}_0) = (\hat{z}_0 - z_0) - V - V^-,$$

and therefore $V + V^- + Q = (\hat{z}_0 - z_0) + I + I^-$, where $I = V - \hat{V}$ and $I^- = V^- - \check{V}$, and, as usual, a small circle above a function name indicates that in the argument of the function we have set $s = 0$. For $I$, using (4.4), we get $|I| = |s \frac{\partial V}{\partial x}| \leq C|s|p \leq C\bar{C}\varepsilon p \leq \frac{1}{4} \theta p$. Similarly, $|I^-| \leq \frac{1}{4} \theta p$. As, by Theorem 2.3 we also have $\tilde{z}_0 - z_0 \geq 0$, then again $V + V^- + Q \geq -\frac{1}{2} \theta p$. Thus we have obtained (4.6), and therefore the bound for $\beta_S(x; p)$ in (4.5). The bound for $\beta_S(x; -p)$ in (4.5) is obtained similarly. $\blacksquare$

Next, to estimate $F \beta_S$, we prepare two lemmas.

Lemma 4.2. For $Q = Q(\eta; p) = \tilde{q}_0 - q_0$ we have

$$\varepsilon^2 \Delta Q = B(x, \cdot)\hat{q}^{+ + \hat{v}}}_{\hat{v}} - B(x, \cdot)\check{q}^{+ + \check{v}}}_{\check{v}} - pq_0 + O(\varepsilon^2 + p^2). \quad (4.7)$$

Proof. From (2.1a), also using (2.1) and $\tilde{q}_0 = q_0 + Q = q_0 + O(p)$, we get

$$\varepsilon^2 \Delta Q = \Delta_\eta (\tilde{q}_0 - q_0) = B(O, \cdot)|_{\check{q}_0 + \hat{v}}^{\check{q}_0 + \hat{v}}_{\check{v}} - B(O, \cdot)|_{\hat{q}_0 + \check{v}}^{\hat{q}_0 + \check{v}}_{\hat{v}} - pq_0 + O(p^2). \quad (4.8)$$

Recalling the definitions (4.2b), introduce the function

$$\mathcal{H}(\varepsilon, \tau) := B(\varepsilon \eta, \cdot)|_{\tau + V, \tau + V}^{\tau + V, \tau + V} + \tau Q + V + V^-, \quad (4.9)$$

in which we write $v_k = v_k(\xi, \varepsilon \sigma)$ and $v^{-}_k = v^{-}_k(\xi, \varepsilon \sigma^-)$ for $k = 0, 1$, and also $V = V(\xi, \varepsilon \sigma), V^- = V^- (\xi, \varepsilon \sigma^-), Q = Q(\eta)$. The function $\mathcal{H}$ is defined so that, using (4.8),

$$\varepsilon^2 \Delta Q = \mathcal{H}(0, 1) - \mathcal{H}(0, 0) - pq_0 + O(p^2), \quad (4.9)$$
and the assertion (4.7) may be written as
\[ \varepsilon^2 \nabla Q = \mathcal{H}(\varepsilon, 1) - \mathcal{H}(\varepsilon, 0) - pq_0 + O(\varepsilon^2 + p^2). \] (4.10)

To check the formulas (4.9) and (4.10) we calculate
\[ \mathcal{H}(0, 0) = B(O, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^-}, \]
\[ \mathcal{H}(0, 1) = B(O, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + [Q + \hat{V} + \hat{V}^-]} = B(O, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^-}, \]
\[ \mathcal{H}(\varepsilon, 0) = B(\varepsilon \eta, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + \varepsilon[q_1 + v_1 + v_1^-]} = B(\varepsilon \eta, \cdot)|_{q + v + v^-}, \]
\[ \mathcal{H}(\varepsilon, 1) = B(\varepsilon \eta, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + \varepsilon[q_1 + v_1 + v_1^-] + [Q + \hat{V} + \hat{V}^-]} = B(\varepsilon \eta, \cdot)|_{q + v + v^-}. \]

Here, as usual in our notation, a small circle above a function name indicates that in the argument of the function we have set \( s = 0 \); in particular, \( \hat{V} = \hat{v}_0 - \hat{v}_0\).

Hence (4.9) is indeed equivalent to (4.8) and (4.10) is equivalent to (4.7). To show that (4.9) implies (4.10) we use the mean value theorem for the second difference and write the discrepancy between these two formulas as
\[ \mathcal{H}(\varepsilon, 1) - \mathcal{H}(\varepsilon, 0) - \mathcal{H}(0, 1) + \mathcal{H}(0, 0) = \varepsilon \frac{\partial^2 \mathcal{H}}{\partial \varepsilon \partial \tau}(\varepsilon^*, \tau^*). \]

Now it suffices to show that \( |\frac{\partial^2 \mathcal{H}}{\partial \varepsilon \partial \tau}| \leq C \varepsilon \). Then the discrepancy between the two formulas for \( \varepsilon^2 \Delta Q \) is bounded by \( C \varepsilon |p| \), which yields (4.10), and therefore (4.7).

To get the desired estimate for \( \frac{\partial^2 \mathcal{H}}{\partial \varepsilon \partial \tau} \), we first evaluate
\[ \frac{\partial \mathcal{H}}{\partial \tau}(\varepsilon, \tau) = QA + VB + V^- B^-, \]
where
\[ A = B_t(\varepsilon \eta, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + \varepsilon[q_1 + v_1 + v_1^-] + \tau[Q + V + V^-]}, \]
\[ B = B_t(\varepsilon \eta, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + \varepsilon[q_1 + v_1 + v_1^-] + \tau[Q + V + V^-]}, \]
\[ B^- = B_t(\varepsilon \eta, \cdot)|_{q_0 + \hat{v}_0 + \hat{v}^- + \varepsilon[q_1 + v_1 + v_1^-] + \tau[Q + V + V^-]}. \]

To estimate \( \frac{\partial^2 \mathcal{H}}{\partial \varepsilon \partial \tau} \), we show that each of \( \frac{\partial(QA)}{\partial \varepsilon}, \frac{\partial(VB)}{\partial \varepsilon} \) and \( \frac{\partial(V^-B^-)}{\partial \varepsilon} \) is \( O(p) \).

For \( \frac{\partial(QA)}{\partial \varepsilon} \) we have \( \frac{\partial(QA)}{\partial \varepsilon} = QA \frac{\partial Q}{\partial \varepsilon} \). A calculation then shows that
\[ \frac{\partial A}{\partial \varepsilon} = \eta \cdot \nabla_x B_t \]
\[ + \{[q_1 + v_1 + v_1^-] + \sigma \frac{\partial}{\partial \varepsilon}(v_0 + \varepsilon v_1 + \tau V) + \sigma \frac{\partial}{\partial \varepsilon}[-v_0^- + \varepsilon v_1^- + \tau V^-] \} B_t, \]

15
where the terms $\nabla_x B_t$ and $B_t$ are computed at the point $(\varepsilon \eta, q_0 + v_0 + v_0^- + \varepsilon [q_1 + v_1 + v_1^-] + \tau [Q + V + V^-])$. Recalling that $|\sigma| \leq |\eta|$ and $|\sigma^-| \leq |\eta|$, we get $|\partial Q A| \leq C(1 + |\eta|)|Q| \leq C_p$, where we also used (4.3).

To estimate $\frac{\partial B}{\partial \varepsilon}$, another tedious calculation gives

$$
\frac{\partial B}{\partial \varepsilon} = \eta \cdot \nabla_x B_t(\varepsilon \eta, \cdot)|q_0 + v_0 + \varepsilon [q_1 + v_1 + v_1^-] + \tau [Q + V + V^-] + \{v_1 + \sigma \frac{\partial}{\partial \varepsilon}(v_0 + \varepsilon v_1 + \tau V)\} B_t(\varepsilon \eta, \cdot)|q_0 + v_0 + \varepsilon [q_1 + v_1 + v_1^-] + \tau [Q + V + V^-] + \{[q_1 + v_1^-] + \sigma \frac{\partial}{\partial \varepsilon}(v_0^- + \varepsilon v_1^- + \tau V^-)\} B_t(\varepsilon \eta, \cdot)|q_0 + v_0 + \varepsilon [q_1 + v_1 + v_1^-] + \tau [Q + V + V^-].
$$

Now invoking Lemma 2.1 and (2.21), we observe that $|B| \leq C e^{-c \varepsilon}$ and $|\frac{\partial B}{\partial \varepsilon}| \leq C(1 + |\eta|) e^{-c \varepsilon}$. Combining this with $\frac{\partial V}{\partial \varepsilon} = \sigma \frac{\partial V}{\partial \varepsilon}$, where $|\sigma| \leq |\eta|$, we have $|\frac{\partial (V B)}{\partial \varepsilon}| \leq C(1 + |\eta|) e^{-c \varepsilon} (|V| + |\frac{\partial V}{\partial \varepsilon}|)$. By (4.4), this yields the desired estimate $|\frac{\partial (V B)}{\partial \varepsilon}| \leq C \varepsilon$. A similar argument gives $|\frac{\partial^2 H}{\partial \varepsilon \partial \sigma}| \leq C \varepsilon$.

Thus, we have shown that each of the three components in $\frac{\partial^2 H}{\partial \varepsilon \partial \sigma}$ is bounded by $C \varepsilon$, which completes the proof. \[\blacksquare\]

For $F \beta_S$ we get the following preliminary result.

**Lemma 4.3.** For the function $\beta_S$ of (4.1) we have

$$
F \beta_S = \theta p b_u(x, u_0) + p [1 + \theta \lambda(x)] (v_0 + v_0^- + q_0) + O(\varepsilon^2 + p^2),
$$

where $\lambda(x) := b_{uu}(x, u_0 + \vartheta [v_0 + v_0^- + q_0])$ with some $\vartheta = \vartheta(x) \in (0, 1)$.

**Proof.** As from Lemma 3.1 we have $F u_{as, S} = O(\varepsilon^2)$, in view of (4.2), (4.1) and (4.1), we get

$$
F \beta_S = F \beta_S - F u_{as, S} + O(\varepsilon^2)
$$

$$
= -\varepsilon^2 \Delta (\beta_S - u_{as, S}) + b(x, \cdot) \beta_S|_{u_{as, S}} + O(\varepsilon^2)
$$

$$
= -\varepsilon^2 \Delta (V + V^- + Q) + B(x, \cdot) \beta_S|_{v_{0} + v_{0}^- + q} + O(\varepsilon^2).
$$

By (2.9) and its analogue for $v^-$, we readily have

$$
-\varepsilon^2 \Delta (V + V^-) = -B(x, \cdot) \beta_S|_{v_{0} + v_{0}^- + q} + B(x, \cdot) \beta_S|_{v_{0} + v_{0}^-} + O(\varepsilon^2 + p^2).
$$

Since (4.7) can be rewritten as

$$
-\varepsilon^2 \Delta Q = -B(x, \cdot) \beta_S|_{v_{0} + v_{0}^-} + B(x, \cdot) \beta_S|_{v_{0} + v_{0}^-} + pq_0 + O(\varepsilon^2 + p^2),
$$

16
we now arrive at
\[
F \beta_S = -B(x, \cdot)\hat{q} + B(x, \cdot)(v_0 + v^-_0 + q_0) + O(\varepsilon^2 + p^2)
\]
\[
= B(x, \cdot)(v_0 + v^-_0 + q_0) + O(\varepsilon^2 + p^2).
\]

Note that (2.7), (4.3) imply that \( \hat{v} + \hat{v}^- + \hat{q} = v_0 + v^-_0 + q_0 + O(\varepsilon + p) \). Hence
\[
B(x, \cdot)(\hat{v} + \hat{v}^- + \hat{q}) = \theta p [B_t(x, v_0 + v^-_0 + q_0) + O(\varepsilon + p)]
\]
\[
= \theta p [B_t(x, 0) + \lambda(x)(v_0 + v^-_0 + q_0) + O(\varepsilon + p)].
\]

Here, by (2.1), one has \( B_t(x, 0) = b_u(x, u_0) \), and \( \lambda(x) = B_{tt}(x, \theta[v_0 + v^-_0 + q_0]) = b_{uu}(x, u_0 + \theta[v_0 + v^-_0 + q_0]) \), as in the statement of this lemma. Combining these formulas, we complete the proof. ■

We are now prepared to establish our main result for \( F \beta_S \).

**Lemma 4.4.** There are positive numbers \( \theta, \varepsilon^*, p^* \) and \( c_1 \) such that with \( \varepsilon \leq \varepsilon^* \) and \( |p| \leq p^* \), for the function \( \beta_S \) of (4.1) one has
\[
F \beta_S \geq \frac{1}{2} \theta^2 \varepsilon^2 - c_1 \varepsilon^2 \quad \text{for } p > 0,
\]
\[
F \beta_S \leq -\frac{1}{2} \theta^2 |p| + c_1 \varepsilon^2 \quad \text{for } p < 0.
\]

**Proof.** By (2.21), one has \( |q_0| \leq C e^{-c_1|\eta|} \). Since \( v_0 \geq 0 \) and \( v^-_0 \geq 0 \) it then follows that \( v_0 + v^-_0 + q_0 \geq -|q_0| \geq -Ce^{-c_1|\eta|} \) and therefore
\[
v_0 + v^-_0 + q_0 \geq -C^{-1} \varepsilon |\ln \varepsilon|, \tag{4.11}
\]
provided that \( |\eta| \geq c_1^{-1} |\ln \varepsilon| \) and \( \varepsilon^* < \varepsilon^{-1} \) so that \( |\ln \varepsilon| > 1 \). Furthermore, (4.11) also holds, with possibly a different constant \( C \), when \( |\eta| \leq c_1^{-1} |\ln \varepsilon| \).

Indeed, by (2.16a), (2.12), we have \( v_0 + v^-_0 + q_0 = \hat{v}_0 = (v_0 - \hat{v}_0) + (v^-_0 - \hat{v}_0) \), where \( \hat{v}_0 \geq 0 \) and, by (2.7), \( |v_0 - \hat{v}_0| \leq C |\sigma|, |v^-_0 - \hat{v}_0| \leq C |\sigma^-| \). Combining these observations with \( |s| + |s^-| = \varepsilon (|\sigma| + |\sigma^-|) \leq 2\varepsilon |\eta| \), we obtain (4.11) for \( |\eta| \leq c_1^{-1} |\ln \varepsilon| \). Thus we have (4.11) everywhere in \( S \).

Next, choose the parameter \( \theta \) in the definition (4.1) of \( \beta_S \) sufficiently small so that \( 0 < \theta \leq |\lambda(x)|^{-1} \), where \( \lambda(x) \) is from Lemma 4.3, and thus \( |1 + \theta \lambda| \geq 0 \). Now from Lemma 4.3 and (4.11) we obtain, for some constants \( C' \) and \( C'' \),
\[
F \beta_S \geq \theta p b_u(x, u_0) - C' \varepsilon |\ln \varepsilon| + C''(\varepsilon^2 + p^2).
\]

Since from our assumption A1 we have \( b_u(x, u_0) \geq \gamma^2 > 0 \), by choosing \( \varepsilon^* \) and \( p^* \) sufficiently small we get the assertion of the lemma in the case \( p > 0 \). The case \( p < 0 \) is similar. ■
Conclusion

In this note we have established four results, Lemmas 2.3, 3.1, 4.1 and 4.4, whose proofs involve lengthy calculations. These results are used in [1] to construct sub- and super-solutions to a nonlinear boundary value problem of type (1.1) posed in a polygonal domain.

References

[1] R.B. Kellogg and N. Kopteva, A singularly perturbed semilinear reaction-diffusion problem in a polygonal domain, J. Differential Equations (2009), doi:10.1016/j.jde.2009.08.020.