Transformation of the Eilenberger Equations of Superconductivity to a Scalar Riccati Equation

Nils Schopohl

Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

Abstract. A new parametrization of the Eilenberger equations of superconductivity in terms of the solutions to a scalar differential equation of the Riccati type is introduced. It is shown that the quasiclassical propagator, and in particular the local density of states, may be reconstructed, without explicit knowledge of any eigenfunctions and eigenvalues, by solving a simple initial value problem for the linearized Bogoliubov-de Gennes’ equations. The Riccati parametrization of the quasiclassical propagator leads to a stable and fast numerical method to solve the Eilenberger equations.

1 Introduction

According to the BCS theory of superconductivity the quasiparticle excitations above the Cooper pairing groundstate depend on spin (↑ or ↓) and also on a particle-hole index (+ or −) which indicates the flight direction of a quasiparticle (parallel or antiparallel to the Fermi velocity $v_F$). Coherent superpositions of such excitations form wave packets that transport energy, momentum, charge and spin inside a superconductor.

In metals and alloys of interest to technical applications of superconductivity the Cooper pairs display an even parity symmetry (spin singlet), and often the influence of paramagnetic effects (Zeeman splitting, Pauli limiting, spin-orbit coupling etc.) may be ignored. Then spin and particle-hole indices may be identified. As a result the $4 \times 4$-matrix equations of superconductivity may be simplified to $2 \times 2$-matrix equations.

In the following we use notation such that $\mathbf{r}$ refers to a point in position space (center of mass of a Cooper pair), and $\mathbf{p}_F = \hbar \mathbf{k}_F$ denotes a point on the Fermi surface $F S$.

It is known that the characteristic length to heal a local (static) perturbation of the Cooper pairing amplitude $\Delta(\mathbf{r}, \mathbf{p}_F)$ in a superconductor (due to the presence of an impurity, a vortex line, an interface etc.) is approximately $\xi = \frac{\hbar v_F}{\Delta_\infty}$, and often the quasiclassical condition $k_F \xi \gg 1$ is fulfilled.

Then, as first shown by Eilenberger[1] and Larkin and Ovchinnikov[2], the relevant part of the physical information coded in quantum mechanical expectation values (for example the charge density, the current, the pressure functional etc.) may be calculated more efficiently with the help of the quasiclassical propagator

$$\bar{g}(\mathbf{r}; \mathbf{p}_F, i\epsilon_n) = \begin{pmatrix} g(\mathbf{r}; \mathbf{p}_F, i\epsilon_n) & f(\mathbf{r}; \mathbf{p}_F, i\epsilon_n) \\ f(\mathbf{r}; \mathbf{p}_F, i\epsilon_n) & \bar{g}(\mathbf{r}; \mathbf{p}_F, i\epsilon_n) \end{pmatrix}$$
The quasiclassical propagator is, by definition, just the Green’s function of the Gorkov theory of superconductivity in a form where it has been integrated with respect to the kinetic energy of the quasiparticles\[4\]. Remarkably, $\hat{g}(\mathbf{r}; p_F, i\varepsilon_n)$ may be also calculated directly solving a transport type system of ordinary differential equations (the right hand side is a commutator):

$$-i\hbar v_F \cdot \nabla \hat{g}(\mathbf{r}; p_F, i\varepsilon_n) = \left[ \left( i\varepsilon_n + v_F \cdot \hat{\mathbf{A}}(\mathbf{r}) \right) \begin{pmatrix} -\Delta(\mathbf{r}, p_F) & \Delta^\dagger(\mathbf{r}, p_F) \\ \Delta^\dagger(\mathbf{r}, p_F) & -i\varepsilon_n - v_F \cdot \hat{\mathbf{A}}(\mathbf{r}) \end{pmatrix}, \quad \hat{g}(\mathbf{r}; p_F, i\varepsilon_n) \right]$$

The physical solution to this equation must also fulfill a normalisation condition:

$$\hat{g}(\mathbf{r}; p_F, i\varepsilon_n) \cdot \hat{g}(\mathbf{r}; p_F, i\varepsilon_n) = -\pi^2 \cdot \mathbb{1}$$

General symmetries of the Gorkov Green’s functions imply corresponding symmetries of the quasiclassical propagator:

$$\bar{f}(\mathbf{r}; p_F, i\varepsilon_n) = -f^*(\mathbf{r}; p_F, -i\varepsilon_n)$$
$$\bar{g}(\mathbf{r}; p_F, i\varepsilon_n) = g(\mathbf{r}; -p_F, -i\varepsilon_n)$$
$$f(\mathbf{r}; -p_F, -i\varepsilon_n) = f(\mathbf{r}; p_F, i\varepsilon_n)$$
$$g(\mathbf{r}; p_F, i\varepsilon_n) = g^*(\mathbf{r}; p_F, -i\varepsilon_n)$$

In equilibrium the quasiclassical propagator also displays a particle-hole symmetry:

$$\bar{g}(\mathbf{r}; p_F, i\varepsilon_n) = -g(\mathbf{r}; p_F, i\varepsilon_n)$$

This means that the trace of $\hat{g}(\mathbf{r}; p_F, i\varepsilon_n)$ vanishes. As a traceless $2 \times 2$-matrix the square of $\hat{g}$ should be equal to a multiple of unity:

$$\hat{g}(\mathbf{r}; p_F, i\varepsilon_n) \cdot \hat{g}(\mathbf{r}; p_F, i\varepsilon_n) = C \cdot \mathbb{1}$$

Using the fact, that $\hat{g}^2$ is a solution to the Eilenberger equations (provided $\hat{g}$ is a solution), it follows that $-i\hbar v_F \cdot \nabla C = 0$, i.e. the scalar $C$ is necessarily a constant along a straight line orientated parallel to the Fermi velocity $v_F$. But $C$ could still be a function of the form $C = C(\mathbf{r} \land v_F; p_F, i\varepsilon_n)$. The normalisation condition Eq.(3) fixes $C$ such that $\hat{g}^2 = -\pi^2 \cdot \mathbb{1}$ for all straight lines orientated parallel to $v_F$, and this for all Fermi momenta $p_F$ on the Fermi surface and also for all Matsubara frequency $i\varepsilon_n$. The particular value $C = -\pi^2$ is chosen in order to achieve consistency with the functional form of the quasiclassical propagator in the bulk.

In thermal equilibrium the pair potential $\Delta(\mathbf{r}, p_F)$, the electrical current $\mathbf{J}(\mathbf{r})$ associated with a (stationary) flow of quasiparticles, the local density of states $N(\mathbf{r}, E)$, the Gibbs free energy $G_S$ of the superconducting state for weak coupling\[8\], and other observables may be directly calculated using the quasiclassical
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propagator:

\[ \Delta(r, p_F) = \int_{FS} dp_{F}' N_{FS}(p_{F}') V(p_F, p_{F}') \cdot k_B T \sum_{|\epsilon_n|<\omega_c} f(r, p_{F}', i\epsilon_n) \]  

\[ J(r) = \frac{k_B T}{\hbar} \sum_{\epsilon_n} \int_{FS} dp_{F}' N(p_{F}') v_F g(r, p_{F}', i\epsilon_n) \]  

\[ N(r, E) = -\frac{1}{\pi} \int_{FS} dp_{F}' N(p_{F}') Img(r, p_{F}', i\epsilon_n \to E + i0^+) \]  

\[ G_S(T) = \int dr \begin{bmatrix} -2T \cdot \int_{-\infty}^{\infty} dE N(r, E) \cdot \ln \left( e^{\frac{E}{T}} + e^{-\frac{E}{T}} \right) \\ + \int_{FS} dp_F \int_{FS} dp_{F}' \Delta \hat{g}(r, p_F) \circ \left( V^{-1} \right)_{p_F, p_{F}'} \circ \Delta(r, p_{F}') \\ + \frac{1}{8\pi} (\nabla \times A(r) - B_{ext}(r))^2 \end{bmatrix} \]  

In these expressions the function \( N_{FS}(p_F) \) denotes the (angle resolved) density of states in the normal phase at the Fermi level. This function typically enters as a weight function into Fermi surface integrals (\( FS \) denotes the Fermi surface) of the Eilenberger propagator. In the isotropic case \( N_{FS}(p_F) \) simplifies to the usual constant \( N(0) \).

The calculation of Fermi surface integrals of the Eilenberger propagator becomes comparatively simple in the bulk, where the pair potential, \( \Delta(p_F) \), is independent on position \( r \), and where the quasiclassical propagator assumes the form:

\[ \hat{g}(p_F, i\epsilon_n) = \frac{-\pi}{\sqrt{\epsilon_n^2 + |\Delta(p_F)|^2}} \begin{pmatrix} i\epsilon_n & -\Delta(p_F) \\ -\Delta(p_F)^\dagger & -i\epsilon_n \end{pmatrix} \]  

A considerably more complicated problem is posed when the pair potential depends on position \( r \), for instance near a surface, in the vicinity of an implanted impurity or ion, or around a flux line in a type-II superconductor.

Usually the solution \( \hat{g}(r; p_F, i\epsilon_n) \) of the Eilenberger equations must be found numerically. But the task is more difficult then just solving a differential equation. To determine the pair potential \( \Delta(r, p_F) \) and the magnetic field \( B(r) = \nabla \times A(r) \) from the (magnetostatic) Maxwell Equation \( \nabla \times B(r) = \frac{4\pi}{c} J(r) \), one needs to solve a (nonlinear) selfconsistency problem, since \( J(r) \) and \( \Delta(r, p_F) \) depend themselves on \( \hat{g}(r; p_F, i\epsilon_n) \).

2 Eilenberger Equations along a Characteristic Line

First we consider a layered material (normal axis parallel to \( \hat{c} \)) assuming, for example, a Fermi velocity \( v_F \) that is orientated predominantly within the \( ab-\)
plane (Fermi circle). Let the triade \( \{ \mathbf{\hat{a}}, \mathbf{\hat{b}}, \mathbf{\hat{c}} \} \) span an orthonormal basis in the lab frame, while \( \theta \) denotes the angle the Fermi velocity \( \mathbf{v}_F \) makes with the \( \mathbf{\hat{a}} \)-axis.

Clearly, along a straight line
\[
\mathbf{r}(x) = x \mathbf{\hat{v}} + y \mathbf{\hat{u}}
\]
\[
\equiv r_a(x) \mathbf{\hat{a}} + r_b(x) \mathbf{\hat{b}}
\]
\(-\infty < x < \infty\)

with \( \mathbf{\hat{v}} \) and \( \mathbf{\hat{u}} \) denoting unit vectors (orientated parallel and orthogonal to \( \mathbf{v}_F \), respectively),

\[\mathbf{\hat{v}} = \cos(\theta) \mathbf{\hat{a}} + \sin(\theta) \mathbf{\hat{b}} \]
\[\mathbf{\hat{u}} = -\sin(\theta) \mathbf{\hat{a}} + \cos(\theta) \mathbf{\hat{b}} ,\]

the directional derivative \( \mathbf{v}_F \cdot \nabla \) in the Eilenberger Equation Eq.(2) is equivalent to an ordinary derivative:
\[
\widehat{h} \mathbf{v}_F \cdot \nabla \widehat{g}(\mathbf{r}; \mathbf{p}_F, i\varepsilon_n) = \widehat{h} v_F \frac{\partial}{\partial x} \widehat{g}[\mathbf{r}(x); \mathbf{p}_F, i\varepsilon_n] \]

The \( \theta \)-dependent parameter \( y \) associated with such a characteristic line \( \mathbf{r}(x) \) (see Eq.(15)) has the natural meaning of an impact parameter. The straight line \( \mathbf{r}(x) \) intersects with a fixed position point
\[
\mathbf{r} = r_a \mathbf{\hat{a}} + r_b \mathbf{\hat{b}}
\]
(there where the solution \( \widehat{g}(\mathbf{r}; \mathbf{p}_F, i\varepsilon_n) \) is sought) at the particular parameter value \( x = x_P \). Introducing polar coordinates,
\[
r_a + ir_b = \sqrt{r_a^2 + r_b^2} e^{i\phi}
\]
it is evident that
\[
r_a(x) + ir_b(x) = (x + iy)e^{i\theta}
\]
and
\[
x_P + iy = \sqrt{r_a^2 + r_b^2} e^{i(\phi - \theta)}
\]

The extension to 3-dimensions is straightforward. For instance, for a spherical Fermi surface the unit vectors \( \mathbf{\hat{v}} \) and \( \mathbf{\hat{u}} \) are parametrised by two angles, the azimuthal angle \( \theta \in [0, 2\pi) \) and the polar angle \( \chi \in [0, \pi) \), respectively:

\[\mathbf{\hat{v}} = \sin(\chi) \left[ \cos(\theta) \mathbf{\hat{a}} + \sin(\theta) \mathbf{\hat{b}} \right] + \cos(\chi) \mathbf{\hat{c}}
\]
\[\mathbf{\hat{u}} = \sin(\chi) \left[ -\sin(\theta) \mathbf{\hat{a}} + \cos(\theta) \mathbf{\hat{b}} \right] + \frac{\partial}{\partial \theta} \mathbf{\hat{v}}
\]

Again, along a straight line, \( \mathbf{r}(x) = x \mathbf{\hat{v}} + y \mathbf{\hat{u}} + z \mathbf{\hat{v}} \wedge \mathbf{\hat{u}} \), the directional derivative in the Eilenberger equations becomes just an ordinary derivative. Making the identification \( \mathbf{r} \equiv r_a \mathbf{\hat{a}} + r_b \mathbf{\hat{b}} + r_c \mathbf{\hat{c}} = \mathbf{r}(x_P) \) explicit expressions for \( x_P \) and both
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'impact' parameters $y$ and $z$ in terms of $\theta, \chi$ and the cartesian coordinates $r_a, r_b, r_c$ of the fixed point $r$ in position space are easily derived.

Finally we simplify our notation by dropping the functional dependence of $\Delta, A$ and $\hat{g}$ on arguments that stay constant as $x$ varies from $-\infty$ to $\infty$:

\[
\Delta(x) = \Delta[r(x), p_F] \\
i\varepsilon_n(x) = i\varepsilon_n + v_F \cdot \frac{e}{c} A[r(x)] \\
\hat{g}(x) = \hat{g}[r(x); p_F, i\varepsilon_n]
\] (26)

3 Riccati Parametrisation of Eilenberger Propagator

Any traceless $2 \times 2$-matrix may be expanded into the basis

\[
\hat{K}_3 = \frac{1}{2} \hat{\tau}_3 \\
\hat{K}_\pm = -\frac{i}{2} \cdot (\hat{\tau}_1 \pm i\hat{\tau}_2)
\] (29)

(\(\hat{\tau}_1, \hat{\tau}_2 \), and \(\hat{\tau}_3\) are standard $2 \times 2$-Pauli matrices). We note that

\[
\begin{bmatrix}
\hat{K}_+, \hat{K}_-
\end{bmatrix} = -2\hat{K}_3 \\
\begin{bmatrix}
\hat{K}_3, \hat{K}_\pm
\end{bmatrix} = \pm\hat{K}_\pm
\] (31)

The Eilenberger equations may then be rewritten along a characteristic line $r(x)$ orientated parallel to the Fermi velocity $v_F$ in the form:

\[
\hbar v_F \frac{\partial}{\partial x} \hat{g}(x) = \begin{bmatrix} -2\varepsilon_n(x)\hat{K}_3 + \Delta(x)\hat{K}_+ - \Delta^\dagger(x)\hat{K}_- \end{bmatrix} \hat{g}(x)
\] (33)

Let us consider the following $2 \times 2$ system of ordinary differential equations for an auxiliary propagator $\hat{Y}(x)$ (fundamental system):

\[
\hbar v_F \frac{\partial}{\partial x} \hat{Y}(x) = \begin{bmatrix} -2\varepsilon_n(x)\hat{K}_3 + \Delta(x)\hat{K}_+ - \Delta^\dagger(x)\hat{K}_- \end{bmatrix} \hat{Y}(x) \\
\hat{Y}(0) = \hat{Y}_0
\] (34)

The initial values for $\hat{Y}(x)$ at $x = 0$ may be prescribed in terms of a (yet unknown) constant $2 \times 2$ matrix $\hat{Y}_0$ of rank 2. We may reconstruct the physical propagator $\hat{g}$, the one that solves the Eilenberger equations and respects the normalization condition, $\hat{g}(x) \cdot \hat{\bar{g}}(x) = -\pi^2 \cdot \hat{1}$, from the fundamental system $\hat{Y}(x)$:

\[
\hat{g}(x) = -\pi i \cdot \hat{Y}(x) \cdot 2\hat{K}_3 \cdot \hat{Y}^{-1}(x)
\] (36)

By putting $x$ at the end of the calculations to the particular value $x_p$, the physical propagator (i.e. the input into the selfconsistency equations) is recovered.
One finds from the differential equation for \( \hat{g} \) in particle-hole space. The physical propagator, Eq.(36), assumes then the form in terms of three unknown functions \( a \):

\[
\hat{g}(x_F) = \hat{g}[r(x_F); p_F, i\varepsilon_n] = \hat{g}[r; p_F, i\varepsilon_n]
\]  
(37)

The commutator in the Eilenberger equations implies the existence of several invariants along the characteristic line \( r(x) \). For example, if the normalization condition Eq.(3) is fulfilled at a particular fixed point \( r(x_0) \), it will be fulfilled everywhere along the line \( r(x) \). Likewise, the determinant \( \det \hat{g}(x) \) and the trace \( \text{tr}\hat{g}(x) \) remain constant for \( -\infty < x < \infty \).

Next we parametrize the \( 2 \times 2 \) matrix \( \tilde{Y}(x) \) in the form

\[
\tilde{Y} = \exp(a_+ \hat{K}_+) \exp(a_3 \hat{K}_3) \exp(a_- \hat{K}_-)
\]
(38)
in terms of three unknown functions \( a_3(x) \), \( a_+(x) \) and \( a_-(x) \) (Euler like 'angles' in particle-hole space). The physical propagator, Eq.(36), assumes then the form

\[
\hat{g}(x) = -\pi i \cdot \begin{bmatrix}
1 - 2a_-(x) a_+(x) \exp(-a_3(x)) \cdot 2\hat{K}_3 \\
+ a_+(x) \cdot [a_-(x) a_+(x) \exp(-a_3(x)) - 1] \cdot 2\hat{K}_+ \\
+a_-(x) \cdot \exp[-a_3(x)] \cdot 2\hat{K}_-
\end{bmatrix}
\]
(39)

One finds from the differential equation for \( \tilde{Y}(x) \) a set of three coupled differential equations for \( a_3(x) \), \( a_+(x) \) and \( a_-(x) \):

\[
\dot{a}_3 - 2a_+ \exp(-a_3) \dot{a}_- = -\frac{2\tilde{\varepsilon}_n}{h v_F}
\]
(40)
\[
\exp(-a_3) \dot{a}_- = -\frac{\Delta^+}{h v_F}
\]
(41)
\[
\dot{a}_+ - a_+ \dot{a}_3 + a_+^2 \exp(-a_3) \dot{a}_- = \frac{\Delta}{h v_F}
\]
(42)

Here \( \dot{a}(x) = \frac{\partial}{\partial x} a(x) \). It is readily seen that the three equations decouple, and that \( a_- \) and \( a_3 \) may be expressed in terms of \( a_+ \) only:

\[
a_3(x) = -\frac{2}{h v_F} \left[ \tilde{\varepsilon}_n x + \int_0^x ds \Delta^s(s) a_+(s) \right] + a_3^{(0)}
\]
(43)

\[
a_-(x) = -\frac{1}{h v_F} \int_0^x ds \Delta^s(s) \exp[a_3(s)] + a_-^{(0)}
\]
(44)

The differential equation that remains to be solved for \( a_+(x) \) is a Riccati equation:

\[
h v_F \frac{\partial}{\partial x} a_+(x) + \left[ 2\tilde{\varepsilon}_n + \Delta^s(s) a_+(x) \right] a_+(x) - \Delta(x) = 0
\]
(45)

However, the accurate numerical calculation of the nested integral for \( a_-(x) \) is time consuming (even on a fast computer). To overcome this difficulty we use a trick.

Let \( \hat{g}_A(x) \) and \( \hat{g}_B(x) \) be two different solutions of the Eilenberger equations. Then not only the linear combination \( c_A \hat{g}_A(x) + c_B \hat{g}_B(x) \) is a solution, but the
products \( \hat{g}_B(x) \cdot \hat{g}_A(x) \) and \( \hat{g}_A(x) \cdot \hat{g}_B(x) \) are solutions as well. For example, the linear combination \( \hat{g}_B(x) \cdot \hat{g}_A(x) - \hat{g}_A(x) \cdot \hat{g}_B(x) \) solves the Eilenberger equations and fulfills the necessary condition \( \text{tr}\hat{g}(x) = 0 \).

Let us construct two particular zero trace solutions to the Eilenberger equations:

\[
\hat{g}_A(x) = \hat{Y}_A(x) \cdot \hat{K}_- \cdot \left[ \hat{Y}_A(x) \right]^{-1}
\]

\[
\hat{g}_B(x) = \hat{Y}_B(x) \cdot \hat{K}_+ \cdot \left[ \hat{Y}_B(x) \right]^{-1}
\]

with

\[
\hat{Y}_A = \exp(a_+ \hat{K}_+) \exp(a_3 \hat{K}_3) \exp(a_- \hat{K}_-)
\]

\[
\hat{Y}_B = \exp(b_- \hat{K}_- \exp(b_3 \hat{K}_3) \exp(b_+ \hat{K}_+)
\]

denoting two equivalent fundamental systems \( \hat{Y}_A(x) \) and \( \hat{Y}_B(x) \). The different order of factors in the defining expressions for \( \hat{Y}_A \) and \( \hat{Y}_B \) serves the purpose to avoid the difficult terms \( a_- \) and \( b_+ \) in the expressions for \( \hat{g}_A \) and \( \hat{g}_B \). The evaluation of nested integrals, see Eq.(43), is then not necessary.

The set of equations fulfilled by \( b_3(x) \) and \( b_\pm(x) \) is only slightly different from the one for \( a_3(x) \) and \( a_\pm(x) \):

\[
\dot{b}_3 + 2b_- \exp(b_3) \dot{b}_+ = -\frac{2\tilde{\varepsilon}_n}{\hbar v_F}
\]

\[
\exp(b_3) \dot{b}_+ = \frac{\Delta}{\hbar v_F}
\]

\[
\dot{b}_- + b_+ \dot{b}_3 + b_-^2 \exp(b_3) \dot{b}_+ = -\frac{\Delta^\dagger}{\hbar v_F}
\]

Here \( \dot{b} \equiv \frac{\partial}{\partial x} b(x) \). It is readily seen that the three equations decouple, and that \( b_+(x) \) and \( b_3(x) \) may be expresses in terms of \( b_-(x) \) only:

\[
b_3(x) = -\frac{2}{\hbar v_F} \left[ \tilde{\varepsilon}_n x + \int_0^x ds \Delta(s) b_-(s) \right] + b_3^{(0)}
\]

\[
b_+(x) = \frac{1}{\hbar v_F} \int_0^x ds \Delta(s) \exp[-b_3(s)] + b_+^{(0)}
\]

The differential equation to be solved for \( b_-(x) \) is also a Riccati equation:

\[
\hbar v_F \frac{\partial}{\partial x} b_-(x) - [2\tilde{\varepsilon}_n + \Delta(x) b_-(x)] b_-(x) + \Delta^\dagger(x) = 0
\]

We observe that any solution of this differential equation is related to the Riccati equation Eq.(55) via a reciprocity relation:
If \( a_+(x) \) solves Eq.(45), then
\[
b_-(x) = -\frac{1}{a_+(x)}
\]
solves Eq.(55).

We continue now our construction of the physical propagator. From the defining equations, Eq.(46) and Eq.(47), we find the following explicit expressions:
\[
\hat{g}_A = \exp(-a_3) \left( \hat{K}_- - 2a_+ \hat{K}_3 + a_+^2 \hat{K}_+ \right)
\]
\[
\hat{g}_B = \exp(b_3) \left( \hat{K}_+ + 2b_- \hat{K}_3 + b_-^2 \hat{K}_- \right)
\]
Note that the square of \( \hat{g}_A(x) \) and \( \hat{g}_B(x) \) vanishes identically,
\[
\hat{g}_A \cdot \hat{g}_A = \hat{g}_B \cdot \hat{g}_B = 0
\]
because \( \hat{K}_+^2 \equiv 0 \). For \( x \to \pm \infty \) the propagators \( \hat{g}_A \) and \( \hat{g}_B \) 'explode', i.e.
\[
\hat{g}_{A,B} \sim \exp(\pm \frac{2 \pi}{\hbar v_F} \sqrt{\frac{\varepsilon^2}{e_n} + |\Delta|^2})
\]
On the other hand, the commutator \([\hat{g}_A(x), \hat{g}_B(x)]\) remains bounded in the limit \( x \to \pm \infty \). The observation that a bounded solution to the Eilenberger equations may be constructed using the commutator of two unbounded solutions \( \hat{g}_A(x) \) and \( \hat{g}_B(x) \) is the well known 'explosion' trick [5].

The general (particle-hole symmetric) solution to the Eilenberger equations (2) may be given in the form
\[
\hat{g}(x) = c_A \hat{g}_A(x) + c_B \hat{g}_B(x) + [\hat{g}_A(x), \hat{g}_B(x)]
\]
Here, \( c_A \) and \( c_B \) represent initial values, \( b_3(0) \) and \( a_3(0) \), to the functions \( b_3(x) \) and \( a_3(x) \). Of course, in an unbounded region exploding solutions must be forbidden. Then the physical propagator \( \hat{g} \) must be written, on either side of the turning point \( x = 0 \), as a superposition of a decaying solution and a bounded solution:
\[
\hat{g}(x) = \begin{cases} 
  c_B \hat{g}_B(x) + [\hat{g}_A(x), \hat{g}_B(x)] & \text{if } x > 0 \\
  c_A \hat{g}_A(x) + [\hat{g}_A(x), \hat{g}_B(x)] & \text{if } x < 0 
\end{cases}
\]
The square of \( \hat{g}(x) \) is in this case independent on the constants \( c_A \) and \( c_B \):
\[
\hat{g}(x) \cdot \hat{g}(x) = [\hat{g}_A(x), \hat{g}_B(x)] \cdot [\hat{g}_A(x), \hat{g}_B(x)] = -\pi^2 \cdot 1
\]
It is not difficult to show, that \( c_A = 0 = c_B \), provided the propagators \( \hat{g}_A(x) \) and \( \hat{g}_B(x) \) are continuous at \( x = 0 \).

In fact, let us assume the contrary: \( c_A \cdot c_B \neq 0 \). Continuity of \( \hat{g}(x) \) at \( x = 0 \) leads to
\[
c_B \hat{g}_B(0^+) + [\hat{g}_A(0^+), \hat{g}_B(0^+)] = \hat{g}(0) = c_A \hat{g}_A(0^-) + [\hat{g}_A(0^-), \hat{g}_B(0^-)]
\]
Both solutions, $\hat{g}_A(x)$ and $\hat{g}_B(x)$, are continuous at $x = 0$. Then it follows from Eq. (63) : $c_B \hat{g}_B(0) = c_A \hat{g}_A(0)$. This implies, in turn, a vanishing commutator, $[\hat{g}_A(0), \hat{g}_B(0)] = 0$, since $\hat{g}_B(0)$ and $\hat{g}_A(0)$ become proportional. Also, the physical solution $\hat{g}(0)$ at $x = 0$ must fulfill the normalization condition, i.e. $\hat{g}(0) \cdot \hat{g}(0) = -\pi^2 \cdot \hat{1}$. But $\hat{g}_B(0) \cdot \hat{g}_B(0) = 0 = \hat{g}_A(0) \cdot \hat{g}_A(0)$ according to Eqs. (59). This is a contradiction! Hence $c_A = 0 = c_B$.

The conclusion is that in an infinitely extended system the physical propagator $\hat{g}(x)$ is completely determined by the commutator of the 'exploding' solutions:

$$\hat{g}(x) = [\hat{g}_A(x), \hat{g}_B(x)] = \exp(b_3 - a_3) \cdot \left[ \begin{array}{cc} 1 - (a_+ b_-)^2 & 2ia_+ (1 + a_+ b_-) \\ -2ib_- (1 + a_+ b_-) & -1 + (a_+ b_-)^2 \end{array} \right]$$

Next we check the normalisation condition:

$$\hat{g}(x) \cdot \hat{g}(x) = [\hat{g}_A(x), \hat{g}_B(x)] \cdot [\hat{g}_A(x), \hat{g}_B(x)]$$
$$= [g_3(x) \cdot g_3(x) + b_+(x) \cdot b_-(x)] \cdot \hat{1}$$
$$= [1 + a_+(x) b_-(x)]^4 \cdot \exp[2b_3(x) - 2a_3(x)] \cdot \hat{1}$$
$$= C \cdot \hat{1}$$

Indeed, $C$ is a constant multiple of unity:

$$\frac{\partial}{\partial x} \left[ [1 + a_+(x) b_-(x)]^4 \cdot \exp[2b_3(x) - 2a_3(x)] \right] = 0$$

From the normalisation condition, $C = -\pi^2$, there follows for all $x$ (up to a sign ± that is chosen to coincide with the bulk propagator):

$$\exp[b_3(x) - a_3(x)] = -\pi i \frac{1}{[1 + a_+(x) b_-(x)]}$$

Then the Eilenberger propagator may be parametrised in the form:

$$\hat{g}(x) = -\frac{\pi i}{1 + a(x) \cdot b(x)} \cdot \left[ \begin{array}{cc} 1 - a(x) \cdot b(x) & 2i \cdot a(x) \\ -2i \cdot b(x) & -1 + a(x) \cdot b(x) \end{array} \right]$$

Here and in the following we use notation such that $b_-(x) \equiv b(x)$ and $a_+(x) \equiv a(x)$, since the other functions $a_-(x), a_3(x)$ and $b_+(x), b_3(x)$ are obsolete for the parametrisation of the Eilenberger propagator. It is remarkable that the solution to the Eilenberger equations (2) may be given a representation where it depends just on the solution of an initial value problem to a scalar differential equation of the Riccati type [15],[13].

To integrate the Riccati equations (45, 55) in a stable manner we need suitable initial values for the functions $b(x)$ and $a(x)$. For $i \epsilon \pi$ situated in the upper half of the complex plane the function $a(x)$ may be found in a stable manner integrating Eq. (45) as an initial value problem from $x = -\infty$ towards increasing $x$-values, while the function $b(x)$ may be found integrating Eq. (55) as an initial value problem.
problem from $x = +\infty$ backwards towards decreasing $x$-values. The initial values for $a(x)$ at $x = -\infty$ and $b(x)$ at $x = +\infty$ are

$$a(-\infty) = \frac{\Delta(-\infty)}{\varepsilon_n + \sqrt{\varepsilon_n^2 + |\Delta(-\infty)|^2}}$$  \hspace{1em} (69)$$

$$b(+\infty) = \frac{\Delta^\dagger(+\infty)}{\varepsilon_n + \sqrt{\varepsilon_n^2 + |\Delta(+\infty)|^2}}$$  \hspace{1em} (70)$$

provided $i\varepsilon_n$ is in the upper half of the complex plane.

The differential equations to be solved are:

$$\hbar v_F \frac{\partial}{\partial x} a(x) + \left[ 2\varepsilon_n + \Delta^\dagger(x) \cdot a(x) \right] \cdot a(x) - \Delta(x) = 0$$  \hspace{1em} (71)$$

$$\hbar v_F \frac{\partial}{\partial x} b(x) - \left[ 2\varepsilon_n + \Delta(x) \cdot b(x) \right] \cdot b(x) + \Delta^\dagger(x) = 0$$  \hspace{1em} (72)$$

Sometimes knowledge of just one of the functions, say $a(x)$, along a line $r(x)$ (for $-\infty < x < +\infty$) suffices to fix the other function, $b(x)$, along the same line. An illustrative example is provided by a single cylindrically symmetric vortex line, orientated parallel to $\hat{c}$, and centered at the origin of the $ab$-plane, say at $R = 0$. Due to energetic reasons, of course, only a single quantum of circulation, $\frac{\hbar}{2m}$, is attached to the vortex. The corresponding pair potential becomes along the straight line $r(x) = r_a(x)\hat{a} + r_b(x)\hat{b}$ a function of $x$ (and also of the impact parameter $y$) of the form:

$$\Delta(r(x), p) = F(\sqrt{x^2 + y^2}, \theta) \cdot \frac{x + iy}{\sqrt{x^2 + y^2}} \cdot e^{i\theta}$$

The prefactor $F(\sqrt{x^2 + y^2}, \theta)$ is a suitable 'form factor' to shape the vortex core. We see from Eqs.(71,72) that in the presence of such a vortex line, $b(x)$ is related to $a(x)$ by symmetry:

$$b(x) = -a(-x)e^{-2i\theta}$$  \hspace{1em} (73)$$

Using Eq.(68) it follows that the corresponding Eilenberger propagator $\hat{g}$ for negative $x$ is related to the propagator for positive $x$ by the relation:

$$\hat{g}(-x) = -e^{i\theta\hat{\tau}_3} \cdot \hat{\tau}_2 \cdot \hat{g}(x) \cdot \hat{\tau}_2 \cdot e^{i\theta\hat{\tau}_3}$$  \hspace{1em} (74)$$

To determine the local density of states one needs the retarded and the advanced propagator of the quasiclassical theory. Actually, in equilibrium, only the retarded (or advanced) propagator is needed in the calculations, since both propagators are related to each other by complex conjugation. A convenient numerical method for the calculation of the retarded propagator $\hat{g}^{(ret)}(r, \theta, E)$ is to replace the discrete Matsubara frequency $i\varepsilon_n$ according to the prescription $i\varepsilon_n \rightarrow E + i0^+$, and to solve the Riccati equations, Eqs.(71,72), as functions of the energy $E$ and of the impact parameters $y$ and $z$. 
If the denominator of the quasiclassical propagator, $1 + a(x) \cdot b(x)$, becomes equal to zero at a point $r(x_0)$ for a characteristic energy $E = E_b$, it vanishes indeed for all $x$ along the trajectory $r(x)$:

$$[1 + a(x) \cdot b(x)] = \exp \left[ \frac{i}{\hbar v_F} \int_{x_0}^{x} ds \left( \Delta(s) b(s) - \Delta^\dagger(s) a(s) \right) \right] \cdot [1 + a(x_0) \cdot b(x_0)]$$

(75)

A simple proof of this relation uses differentiation with respect to $x$, and Eqs.(71, 72). So, if the denominator $1 + a(x_0) \cdot b(x_0)$ of the quasiclassical propagator Eq.(68), considered as a function of energy $E$, displays a simple zero at $E = E_b$, this zero, $E_b$, has a natural interpretation as a bound state energy, provided there exists a finite residue of the retarded propagator at $E = E_b + i0^+$. Since it is almost never possible to solve the equations of superconductivity exactly, one needs numerical methods. The alternative to a straightforward (but costly) numerical solution of the BdG-eigenvalue problem is the numerical solution of the Eilenberger equations, provided the fundamental condition of quasi classical theory, $k_F \cdot \xi \gg 1$, is valid. For example, using the quasiclassical approach of Eilenberger, it is comparatively easy to determine the deep lying bound states $E_b$ of localised vortex core fermions (attached to a single vortex line) as a function of the impact parameters $E_b = E_b(y, z)$. The summation over (exact) eigenenergies of the bound states obtained from solving the BdG-eigenvalue problem (see Ref.[7]) for a single vortex line becomes equivalent to integrating the quasiclassical propagator with respect to the impact parameter. In certain materials, for example cuprates, the Cooper pairs display (perhaps) an unconventional $d_{x^2-y^2}$-symmetry. Results of a quasiclassical calculation of the bound state spectrum of quasiparticles around a single flux line in a superconductor with $d_{x^2-y^2}$-pairing symmetry are published in Refs.[13], [14].

For superfluid $^3$He $- B$, a prominent system with unconventional $p$-wave pairing symmetry, our method can be extended to the $4 \times 4$-Eilenberger propagator for triplet pairing. A calculation of the spectrum of vortex core fermions around the $o$-vortex, the $v$-vortex and also the double core vortex is discussed in Refs. [16],[15].

We conclude that the quasiclassical propagator may be determined solving an initial value problem for a scalar differential equation of the Riccati type. For numerical calculations this method to solve the Eilenberger equations may be recommended for its intrinsic stability and speed. Also the Eilenberger approach is a suitable one for parallel computers.

4 Connection to Linearized Bogoliubov-de Gennes Equations

There exists an interesting connection between the solutions $u(x)$ and $v(x)$ to the linearized BdG-equations (often referred to as Andreev equations [3]), and the solutions to the Riccati equations. Along a characteristic line $r(x)$ orientated
parallel to the Fermi velocity we have:

\[-i\hbar v_F \frac{\partial}{\partial x} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} i\varepsilon_n(x) - \Delta(x) \\ \Delta^\dagger(x) - i\varepsilon_n(x) \end{bmatrix} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \]  

(76)

We observe that \(a(x)\), the solution to the Riccati Equation Eq.(45), may be represented as the ratio:

\[a(x) = i \cdot \frac{u(x)}{v(x)}\]  

(77)

This connection between \(a(x)\) and the amplitudes \(u(x)\) and \(v(x)\) is readily demonstrated:

\[-\hbar v_F \frac{\partial}{\partial x} a(x) = \frac{v(x) \left[ -i\hbar v_F \frac{\partial}{\partial x} u(x) \right] - u(x) \left[ -i\hbar v_F \frac{\partial}{\partial x} v(x) \right]}{[v(x)]^2}
\]

\[= \frac{v(x) [i\varepsilon_n(x)u(x) - \Delta(x)v(x)] - u(x) [\Delta^\dagger(x)u(x) - i\varepsilon_n(x)v(x)]}{[v(x)]^2}
\]

\[= [2\varepsilon_n + \Delta^\dagger(x) \cdot a(x)] \cdot a(x) - \Delta(x)\]  

(78)

The conclusion is, that any solution to the Eilenberger equations (for \(i\varepsilon_n(x)\) in the upper half of the complex plane for \(x \to -\infty\)) may be reconstructed from a solution to the linearized BdG-differential equations, provided the initial values for \(u(x)\) and \(v(x)\) are chosen in accordance with the known asymptotic behaviour of \(a(x)\) for \(x \to -\infty\) (see Eq.(69)).

Another interesting feature follows making a polar decomposition of the pair potential:

\[\Delta(x) = |\Delta(x)| e^{i\phi(x)}\]  

(79)

and making the transformation

\[a(x) = e^{i\eta(x)}\]  

(80)

in Eq.(45). It is readily shown that

\[v_F \frac{\partial}{\partial x} \eta(x) - 2i\varepsilon_n(x) + 2 |\Delta(x)| \sin [\eta(x) - \phi(x)] = 0\]  

(81)

It is interesting that this differential equation, which is equivalent to Eq.(45), was already derived by J. Bardeen et al. \[8\] in their so called WKBJ-approach to the BdG-equations.

So, the 'new' result is not the Riccati equation Eq.(45) itself (or any equivalent representation). New is the fact, that the full quasiclassical propagator \(\hat{g}(\mathbf{r}, \theta, i\varepsilon_n)\) (in equilibrium) is simply a rational function (see Eq.(68)) of the solutions to a scalar Riccati equation. In turn, solutions to these Riccati equations are related by Eq.(77) to the solutions of an initial value problem for the linearised Bogoliubov-de Gennes equations. This implies, that the standard procedure of wave mechanics to calculate the observables of superconductivity, namely first solving the Bogoliubov-de Gennes eigenvalue problem and afterwards doing
a summation over the eigenfunctions and eigenvalues to calculate the Green's function, is indeed obsolete and may be replaced by solving a scalar Riccati equation, or equivalently, by solving an initial value problem for the linearised BdG-equations, provided the quasiclassical condition $k_F \xi \gg 1$ is fulfilled.

5 Exact Solutions

In some cases we may calculate exact solutions to the Eilenberger equations for a given profile of the pair potential. First the term $\mathbf{v}_F \cdot \mathbf{A}(x)$ in the definition of $\tilde{\varepsilon}_n(x)$ (local Doppler shift) may be removed from the diagonal of Eqs. (76) making a gauge transformation, i.e. we may always assume $\tilde{\varepsilon}_n(x) \rightarrow \varepsilon_n$ independent on $x$. Then, after a decomposition of the (transformed) pair potential into real and imaginary parts, $\Delta(x) = \Delta_1(x) + i \Delta_2(x)$ (82)

the linearised BdG-equations (76) may be given the form:

$$\begin{bmatrix} \hat{\tau}_3 \cdot i \hbar v_F \frac{\partial}{\partial x} + \Delta_1(x) \cdot \hat{\tau}_1 - \Delta_2(x) \cdot \hat{\tau}_2 + i \varepsilon_n \cdot \hat{1} \end{bmatrix} \tilde{\psi}(x) = \hat{0}$$ (83)

where

$$\tilde{\psi}(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$ (84)

Introducing an auxiliary spinor $\tilde{\chi}(x)$ via

$$\tilde{\psi}(x) = \begin{bmatrix} \tilde{\tau}_3 \cdot i \hbar v_F \frac{\partial}{\partial x} + \Delta_1(x) \cdot \hat{\tau}_1 - \Delta_2(x) \cdot \hat{\tau}_2 - i \varepsilon_n \cdot \hat{1} \end{bmatrix} \tilde{\chi}(x)$$ (85)

the following 2$^\text{nd}$ order linear differential equation be derived:

$$\begin{bmatrix} -\hbar^2 v_F^2 \frac{\partial^2}{\partial x^2} + \Delta_1^2(x) + \Delta_2^2(x) + \varepsilon_n^2 \end{bmatrix} \tilde{\chi}(x) = \hat{0}$$ (86)

Provided an exact solution (in accordance with the initial conditions Eqs. (69), (70)) to Eq. (86) can be found, the quasiclassical propagator may be calculated exactly from Eqs. (77, 68). This procedure is a useful principle for the construction of exact solutions to the Eilenberger equations.

The major obstacle to decompose Eq. (86) just into two decoupled scalar differential equations is the spatial dephasing between imaginary and real parts of the gradient of the pair potential, $\frac{\partial \Delta_1(x)}{\partial x}$ and $\frac{\partial \Delta_2(x)}{\partial x}$, in Eq. (86). A simple case occurs, for example, if $\frac{\partial \Delta_2(x)}{\partial x} = 0$. In this case a rotation around the $\hat{\tau}_1$-axis by an angle $\pi/2$ leads to a diagonal matrix, i.e. the problem may be effectively decoupled into two scalar differential equations of 2$^\text{nd}$-order. The well known WKB method [9] in its standard guise may then be applied to construct an
(approximate) analytical solution in this case. If there is no dephasing between real and imaginary parts of the pair potential, some pair potentials \( \Delta(x) \) with model character allow the construction of exact solutions, as we show below.

On the other hand, if there exists dephasing between real and imaginary parts of the pair potential, the problem is more difficult. One way to proceed is stratification. This means one approximates the pair potential, \( \Delta(x) = \Delta_1(x) + i \cdot \Delta_2(x) \), by a sequence of strata along \( x \), such that \( \Delta_1(x) \) and \( \Delta_2(x) \) become continuous and piecewise linear functions of \( x \):

\[
\begin{align*}
\Delta_1(x) &= c_1 x + d_1 \\
\Delta_2(x) &= c_2 x + d_2
\end{align*}
\]

The real constants \( d_1, d_2, c_1, c_2 \) may change from one stratum to another stratum, like a (linear) spline function. Inside a fixed stratum the \( 2 \times 2 \) matrix

\[
\frac{\partial \Delta_1(x)}{\partial x} \cdot \hat{\tau}_2 + \frac{\partial \Delta_2(x)}{\partial x} \cdot \hat{\tau}_1 = c_1 \hat{\tau}_2 + c_2 \hat{\tau}_1
\]

may be diagonalised using a suitable, \( x \)-independent unitary transformation matrix \( \hat{U} \) of the form

\[
\hat{U} = \frac{1}{\sqrt{2}} \left[ \hat{\tau}_3 + \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cdot \hat{\tau}_2 + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \cdot \hat{\tau}_1 \right]
\]

Since the constants \( c_1, c_2 \) may change from a given stratum to the neighbouring stratum, the unitary transformation matrix \( \hat{U} \) changes accordingly. Inside a fixed stratum Eqs.(86) may be decomposed into two decoupled scalar differential equations of 2nd-order for the components of the new spinor:

\[
\begin{align*}
\hat{U} \cdot \hat{\chi}(x) &\equiv \hat{\Phi}(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} \\
\hat{U}^2 &= \hat{1} \\
c_1 \hat{\tau}_2 + c_2 \hat{\tau}_1 &= \hat{U} \cdot \sqrt{c_1^2 + c_2^2} \hat{\tau}_3 \cdot \hat{U}
\end{align*}
\]

Exact solutions (inside a chosen stratum) may be presented to these differential equations using the linearly independent parabolic cylinder functions \( D_{\nu}(x) \) and \( D_{-\nu-1}(ix) \). The latter special functions are solutions to the differential equation of the harmonic oscillator:

\[
\left( \frac{\partial^2}{\partial x^2} + \nu + \frac{1}{2} - x^2 \right) D(x) = 0
\]
The components of \( \hat{\Phi}(x) \) are related to \( \hat{\chi}(x) \) via the relation
\[
\hat{\chi}(x) = \hat{U} \cdot \hat{\Phi}(x)
\]
i.e. the changes of the constants \( c_1 \) and \( c_2 \) from one stratum to the next stratum determine also the admixture of \( \Phi_1(x) \) and \( \Phi_2(x) \), and henceforth the spinor \( \hat{\chi}(x) \):

\[
\chi_1(x) = \frac{1}{\sqrt{2}} \left[ \Phi_1(x) + \frac{c_2 - ic_1}{\sqrt{c_1^2 + c_2^2}} \Phi_2(x) \right] \\
\chi_2(x) = \frac{1}{\sqrt{2}} \left[ \Phi_2(x) + \frac{c_2 + ic_1}{\sqrt{c_1^2 + c_2^2}} \Phi_1(x) \right]
\]

Finally, from Eq.(85) the ratio \( a(x) = i \cdot \frac{u(x)}{v(x)} \) may be calculated such that \( a(x) \) remains a smooth function of \( x \). Once \( b(x) \) has been found from a similar consideration (or from \( a(x) \) by a symmetry argument) the quasiclassical propagator follows from Eq.(68).

Closing this section we give three examples of pair potential profiles \( \Delta(x) \) that allow an exact solution to the Eilenberger equations. Although these pair potential are not selfconsistent they are helpful for a qualitative physical understanding:

\[
\Delta(x) = \Delta_\infty \cdot \frac{x + iy}{\xi} \quad (98)
\]
\[
\Delta(x) = \Delta_\infty \cdot \frac{x + iy}{\sqrt{x^2 + y^2}} \quad (99)
\]
\[
\Delta(x) = \Delta_\infty \cdot \tanh\left( \frac{x}{\xi} \right) \quad (100)
\]

The model Eq.(98) represents the inner core of a vortex, and an exact solution to the Eilenberger equations may be constructed in terms of the parabolic cylinder functions along the lines explained above[15].

The model Eq. (99) represents the outer core of a vortex (it describes a pure phase vortex), and may actually be solved exactly for the special case \( \varepsilon_n \rightarrow E = |\Delta_\infty| \).

\[
a(x) = \frac{(1 - 2iW)(y + \sqrt{x^2 + y^2}) + (1 + 2iW) \cdot ix}{(1 + 2iW)(y + \sqrt{x^2 + y^2}) - (1 - 2iW) \cdot ix} \quad (101)
\]
\[
W = \sqrt{\frac{1}{4} - \frac{y}{\xi}} \quad (102)
\]
\[
\xi = \frac{\hbar v_F}{|\Delta_\infty|} \quad (103)
\]

At the gap edge, \( E = |\Delta_\infty| \), the corresponding expression for the quasiclassical propagator, Eq. (68), displays algebraic decay for \( x \rightarrow \pm \infty \). Certainly, it would
be nice to know the exact solution also for other energies $E$, but this problem is still unsolved. Nevertheless, a qualitatively correct (but not exact) solution to Eq.(86) for the case of a vortex line may be found [15] using a method of Stueckelberg[10].

The third model, Eq.(100), is simpler than the previous model, Eq.(99), because the pair potential $\Delta(x)$ is a real function (no dephasing of components of spinor $\hat{\chi}$). The differential equation for $\hat{\chi}(x)$, Eq.(86), may be solved exactly in terms of the hypergeometric function[19]). A particularly simple looking analytic solution results if the size parameter $\xi$ of the domain wall described by Eq.(100) assumes the special value $\xi = \frac{\hbar v}{\Delta_\infty}$:

$$a(x) = \frac{\varepsilon_n - \sqrt{\varepsilon_n^2 + \Delta_\infty^2} + \Delta_\infty \tanh \frac{x}{\xi}}{\varepsilon_n - \sqrt{\varepsilon_n^2 + \Delta_\infty^2} - \Delta_\infty \tanh \frac{x}{\xi}}$$ (104)

$$\tilde{g}(x) = -\frac{i\pi}{\sqrt{\varepsilon_n^2 + \Delta_\infty^2}} \begin{bmatrix} \left(\varepsilon_n + \frac{\Delta_\infty^2}{2\varepsilon_n \cosh^2 \frac{x}{\xi}} \right) \cdot \hat{\tau}_3 \\ - \Delta_\infty \tanh \frac{x}{\xi} \cdot \hat{\tau}_2 \\ -\frac{i\Delta_\infty^2}{2\varepsilon_n \cosh^2 \frac{x}{\xi}} \cdot \hat{\tau}_1 \end{bmatrix}$$ (105)

We see that the explicit analytical expression for the quasiclassical propagator, $\tilde{g}(x)$, nicely reveals the existence of a mid gap state (a quasiparticle bound state with an excitation energy $E$ around zero), assuming we make the analytical continuation $i\varepsilon_n \rightarrow E+i0^+$. The existence of such a mid gap state is in agreement with our previous remarks associated with Eq.(75).

6 Outlook

In this article we have shown how to parametrise the $2 \times 2$-Eilenberger equations in terms of the solutions to a scalar Riccati equation, or equivalently, in terms of the solutions to an initial value problem for the linearised BdG-equations. The latter method for the reconstruction of the Eilenberger propagator may be extended [15] to the full $4 \times 4$-matrix equations of the quasiclassical theory of superconductivity and superfluidity (including paramagnetic effects). For a generalisation of the method to non equilibrium see [17],[20]). For an authoritative review of the quasiclassical theory see Ref.[4].

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