AMALGAMATED PRODUCTS AND PROPERLY 3-REALIZABLE GROUPS

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Abstract. In this paper, we show that the class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups. We recall that $G$ is said to be properly 3-realizable if there exists a compact 2-polyhedron $K$ with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a 3-manifold (with boundary).

1. Introduction

We are concerned about the behavior of the property of being properly 3-realizable (for finitely presented groups) with respect to the basic constructions in Combinatorial Group Theory; namely, amalgamated free products and HNN-extensions. Recall that a finitely presented group $G$ is said to be properly 3-realizable if there exists a compact 2-polyhedron $K$ with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a 3-manifold. It is worth mentioning that the property of being properly 3-realizable has implications in the theory of cohomology of groups, in the sense that if $G$ is properly 3-realizable then for some (equivalently any) compact 2-polyhedron $K$ with $\pi_1(K) \cong G$ we have $H^2_c(\tilde{K}; \mathbb{Z})$ free abelian (by manifold duality arguments), and hence so is $H^2_c(G; \mathbb{Z}G)$ (see [9]). It is a long standing conjecture that $H^2_c(G; \mathbb{Z}G)$ be free abelian for every finitely presented group $G$. In [11] it was shown that the property of being properly 3-realizable is preserved under amalgamated free products (HNN-extensions) over finite cyclic groups. See also [3, 4, 7] to learn more about properly 3-realizable groups and related topics. In this paper, we continue in the line of [11]. Our main result is:

Theorem 1.1. The class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups.

This generalizes to show that the fundamental group of a finite graph of groups with properly 3-realizable vertex groups and finite edge groups is properly 3-realizable, since such a group can be expressed as a combination of amalgamated free products and HNN-extensions of the vertex groups over the edge groups.

Recall that, given a finitely presented group $G$ and a compact 2-polyhedron $K$ with $\pi_1(K) \cong G$ and $\tilde{K}$ as universal cover, the number of ends of $G$ is the number of ends of $\tilde{K}$ which equals 0, 1, 2 or $\infty$ (see also [8, 13]). The 0-ended groups are the finite groups and the 2-ended groups are those having an infinite cyclic subgroup of finite index, and they are all known to be properly 3-realizable (see [11]). Note that Stallings’ Structure Theorem [12] characterizes those groups $G$ with more than one end as those which split as an amalgamated free product (or an HNN-extension)
over a finite group (see also [13, 8]). In addition, Dunwoody [8] showed that this process of further splitting \( G \) must terminate after finitely many steps.

**Corollary 1.2.** In order to show whether or not all finitely presented groups are properly 3-realizable it suffices to look among those groups which are 1-ended.

## 2. Main result

The purpose of this section is to prove Theorem 1.1. We will make use of the following result:

**Proposition 2.1 (II, Prop. 3.1).** Let \( M \) be a manifold of the same proper homotopy type of a locally compact polyhedron \( K \) with \( \dim(K) < \dim(M) \). Then, any Freudenthal end \( \epsilon \in F(M) \) can be represented by a sequence of points in \( \partial M \).

**Proof of Theorem 1.1.** Let \( G_0, G_1 \) be properly 3-realizable groups and \( F \) be a finite group with presentation \( \langle a_1, \ldots, a_N; r_1, \ldots, r_M \rangle \). Consider monomorphisms \( \varphi_i : F \to G_i (i = 0, 1) \), and denote by \( G_0 *_F G_1 = \langle G_0, G_1; \varphi_0(a_i) = \varphi_1(a_i), 1 \leq i \leq N \rangle \) the corresponding amalgamated free product. Let \( X_0, X_1 \) be compact 2-polyhedra with \( \pi_1(X_i) \cong G_i \) and such that their universal covers have the proper homotopy type of \( 3 \)-manifolds \( M_0, M_1 \) respectively. Let \( L = \bigvee_{i=1}^N S^1 \) and \( f_i : L \to X_i \) \((i = 0, 1)\) be cellular maps such that \( \text{Im} f_i \subseteq \pi_1(X_i) \) corresponds to the subgroup \( \text{Im} \varphi_i \subseteq G_i \). We take the standard 2-dimensional CW-complex \( Y' \) associated to the above presentation of \( F \), i.e., \( \text{Y}' \) has one 1-cell \( e_i \) for each generator \( a_i \) \((1 \leq i \leq N)\), all of them sharing the only vertex in \( \text{Y}' \), and one 2-cell \( d_j \) for each relation \( r_j \) \((1 \leq j \leq M) \) attached via a map \( S^1 \to \bigvee_{i=1}^N e_i \) which “spells” the relation \( r_j \). Consider the adjunction spaces \( Y = (\bigvee_{i=1}^N e_i) \times I \cup (\bigvee_{i=1}^N e_i) \times \{1\} \) \((\text{homotopy equivalent to } \text{Y}' \)) and \( Z = Y \cup f_0 \times \{0\} \cup f_1 \times \{1\} (X_0 \cup X_1) \). By van Kampen’s Theorem, \( Z \) is a compact 2-polyhedra with \( \pi_1(Z) \cong G_0 *_F G_1 \). Let \( \tilde{Z} \) be the universal cover of \( Z \) with covering map \( p : \tilde{Z} \to Z \). Then, \( p^{-1}(X_i) \) consists of a disjoint union of copies of the universal cover \( \tilde{X}_i \) of \( X_i \), since the inclusion \( X_i \to Z \) induces a monomorphism \( G_i \to G_0 *_F G_1 \) between the fundamental groups, \( i = 0, 1 \) (see [10]). On the other hand, let \( \Gamma \) be a connected component of \( p^{-1}(\bigvee_{i=1}^N e_i) \subseteq p^{-1}(\text{Y}') \) and \( \tilde{Y}' \) be the connected component of \( p^{-1}(\text{Y}') \) containing \( \Gamma \). Observe that \( \tilde{Y}' \) is a copy of the universal cover of \( \text{Y}' \) \((\text{which is compact})\), so the inclusion \( \text{Y}' \to Z \) induces a monomorphism \( F \to G_0 *_F G_1 \). Then, it is easy to see that \( p^{-1}(Y) \) consists of a disjoint union of copies of the compact CW-complex \( K = (\Gamma \times I) \cup _{\Gamma \times \{1\}} \tilde{Y}' \). Thus, \( \tilde{Z} \) comes together with the following data (see [13]):

\[(a)\] The disjoint unions \( \bigsqcup_{p \in \mathbb{N}} \tilde{X}_{0,p} \) and \( \bigsqcup_{r \in \mathbb{N}} \tilde{X}_{1,r} \) of copies of \( \tilde{X}_0 \) and \( \tilde{X}_1 \) respectively;

\[(b)\] a disjoint union \( \bigsqcup_{p \in \mathbb{N}} K_{p,q} \) of copies of \( K \); and

\[(c)\] a bijective function \( \varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}, (p, q) \mapsto (r, s) \) \((\text{given by the group action of } G_0 *_F G_1 \text{ on } \tilde{Z})\), so that for each \( p, q \in \mathbb{N}, \Gamma \times \{0\} \subset K_{p,q} \) is being glued to \( \tilde{X}_{0,p} \) via a lift \( \tilde{f}_{0,p} : \Gamma \times \{0\} \to \tilde{X}_{0,p} \) of the map \( f_0 \), and \( \Gamma \times \{1\} \subset K_{p,q} \) is being glued to \( \tilde{X}_{1,r} \) via a lift \( \tilde{f}_{r,s} : \Gamma \times \{1\} \to \tilde{X}_{1,r} \) of the map \( f_1 \).

Next, for each copy of \( \tilde{X}_i, i = 0, 1, \) in \( \tilde{Z} \) \((\text{written as } \tilde{X}_{0,p} \text{ or } \tilde{X}_{1,r})\), we take one of the maps \( \tilde{f}_{r,s} : \Gamma \times \{1\} \to \tilde{X}_i \) and observe that this map is nullhomotopic so
we can replace it (up to homotopy) with a constant map $g_{\lambda,\mu}^i : \Gamma \times \{i\} \to \hat{X}_i$ with $Im \ g_{\lambda,\mu}^i \subset Im \ \hat{f}_{\lambda,\mu}^i$, and we do this equivariantly using the group action of $G_i$ on $\hat{X}_i$. Since this action is properly discontinuous, the collection of all these homotopies gives rise to a proper homotopy equivalence between $\hat{Z}$ and a new 2-dimensional CW-complex $W$ obtained from a collection of copies of $K$ and a collection of copies of $\check{X}_0$ and $\check{X}_1$ by gluing each copy of $\Gamma \times \{i\}$ to the corresponding copy of $\hat{X}_i$ via the bijection $\varphi$ and the new maps $g_{\lambda,\mu}^i$, $i = 0, 1$.

We will now manipulate the CW-complex $K$ as follows. First, let $K'$ be the CW-complex obtained from $K$ by shrinking to a point $v \times \{i\}$ each copy $T \times \{i\}$ ($i \in I$) of a maximal tree $T \subset \check{Y}' \subset K$. Next, we take $K''$ to be the CW-complex obtained from $K'$ by identifying the subcomplexes $\Gamma \times \{i\}/T \times \{i\}$, $i = 0, 1$, to a (different) point which we will denote by $[v \times \{0\}]$ and $[v \times \{1\}]$. Note that $K''$ has a copy of $Y'/T$ as a subcomplex. Since $Y'/T$ is compact and simply connected, it follows from ([1], Prop. 3.3) that $Y'/T$ is homotopy equivalent to a finite bouquet of 2-spheres $\vee_{a \in A} S^2$ (which we may regard as a connected 2-dimensional CW-complex with no 1-cells). Moreover, we may assume that this homotopy equivalence is given by a cellular map $\hat{Y}'/T \to \vee_{a \in A} S^2$ so that the 1-skeleton $\Gamma/T$ of $Y'/T$ is mapped to the wedge point. Finally, taking into account this homotopy equivalence, it is not difficult to see that $K''$ is homotopy equivalent to the CW-complex $\hat{K}$ obtained from the disjoint union of a finite bouquet $\vee_{a \in A \cup B} S^2$ (where $Card(B) = 2 \ rank(\pi_1(\Gamma))$ and the unit interval $I$ by identifying $\frac{1}{2} \in I$ with the wedge point, so that $I \subset \hat{K}$ would correspond to the subcomplex $v \times I \subset K'$ and $0, 1 \in I$ would correspond to $[v \times \{0\}], [v \times \{1\}] \in K''$. Notice that $\hat{K}$ thickens to a 3-manifold $P \setminus \hat{K}$ containing 3-dimensional 1-handles $H$ and $H'$ (with a free end face of each of them) corresponding to the edges $[0, \frac{1}{2}], [\frac{1}{2}, 1] \subset I \subset \hat{K}$ respectively.

According to the above, one can see that the CW-complex $W$ (proper homotopy equivalent to $\hat{Z}$) is in turn proper homotopy equivalent to the quotient space obtained from the following data:

(a) A disjoint union $\bigsqcup_{p \in \N} \check{X}_{0,p}$ of copies of $\check{X}_0$ together with a locally finite sequence of points $\{x^p_q\}_{q \in \N} \subset \check{X}_{0,p}$, for each $p \in \N$, corresponding to the images of the constant maps $g^0_{p,q} : \Gamma \times \{0\} \to \check{X}_{0,p}$ considered above in the construction of $W$;

(b) a disjoint union $\bigsqcup_{r \in \N} \check{X}_{1,r}$ of copies of $\check{X}_1$ together with a locally finite sequence of points $\{y^r_s\}_{s \in \N} \subset \check{X}_{1,r}$, for each $r \in \N$, corresponding to the images of the constant maps $g^1_{r,s} : \Gamma \times \{1\} \to \check{X}_{1,r}$ from the construction of $W$;

(c) a disjoint union $\bigsqcup_{p,q \in \N} \hat{K}_{p,q}$ of copies of $\hat{K}$; and

(d) the bijective function $\varphi : \N \times \N \to \N \times \N, (p, q) \mapsto (r, s)$, so that $0 \in I \subset \hat{K}_{p,q}$ is being identified with $x^p_q \in \check{X}_{0,p}$ and $1 \in I \subset \hat{K}_{p,q}$ is being identified with $y^r_s \in \check{X}_{1,r}$ ($(r, s) = \varphi(p, q)$), for each $p, q \in \N$.

We now follow an argument similar to the proof of ([1], Lemma 3.2). Fix proper homotopy equivalences $h : \check{X}_0 \to M$ and $h' : \check{X}_1 \to N$, where we now denote $M_0$ by $M$ and $M_1$ by $N$. Given the above data, we set $A = \N \times \N$ and consider
maps \( i : A \longrightarrow \bigcup_{p \in \mathbb{N}} \hat{X}_{0,p} \), \( i' : A \longrightarrow \bigcup_{r \in \mathbb{N}} \hat{X}_{1,r} \) given by \( i(p, q) = x_p^r \) and \( i'(p, q) = y_s^r \), where \((r, s) = \varphi(p, q)\). It is easy to check that \( i \) and \( i' \) are proper cofibrations, as the corresponding sequences of points are locally finite. Next, we take exhaustive sequences \( \{A_p^m\}_{m \in \mathbb{N}} \) and \( \{B_r^n\}_{r \in \mathbb{N}} \) of copies of \( M_p \) and \( N_r \) of the 3-manifolds \( M \) and \( N \) respectively by compact submanifolds, and define proper cofibrations \( j : A \longrightarrow \bigcup_{p \in \mathbb{N}} M_p \), \( j' : A \longrightarrow \bigcup_{r \in \mathbb{N}} N_r \) as follows. Given \((p, q) \in A\) and the proper homotopy equivalences \( h_p = h : \hat{X}_{0,p} \longrightarrow M_p \), \( h'_p = h' : \hat{X}_{1,r} \longrightarrow N_r \) (with \((r, s) = \varphi(p, q)\)), we take \( m(q), n(s) \in \mathbb{N} \) to be the least natural numbers such that \( h_p \circ i(p, q) \notin A_{m(q)}^p \subset M_p \) and \( h'_p \circ i'(p, q) \notin B_{n(s)}^r \subset N_r \). Then, using Proposition 2.1, we define \( j(p, q) \) and \( j'(p, q) \) to be points \( j(p, q) = a_p,q \in \partial M_p - A_{m(q)}^p \) and \( j'(p, q) = b_{r,s} = \partial N_r - B_{n(s)}^r \) so that \((i)\) \( j, j' \) are one-to-one maps (note that \( h, h' \) need not be one-to-one); and \((ii)\) \( a_{p,q} \) and \( h_p \circ i(p, q) \) (resp. \( b_{r,s} \) and \( h'_p \circ i'(p, q) \)) are in the same path component of \( M_p - A_{m(q)}^p \) (resp. \( N_r - B_{n(s)}^r \)). Notice that \( j \) and \( j' \) are proper maps by construction. Consider now maps \( G, H \) extend to proper maps

\[
G : \left( \bigcup_{p \in \mathbb{N}} \hat{X}_{0,p} \right) \times \{0\} \cup (i(A) \times I) \longrightarrow \bigcup_{p \in \mathbb{N}} M_p
\]

\[
H : \left( \bigcup_{r \in \mathbb{N}} \hat{X}_{1,r} \right) \times \{0\} \cup (i'(A) \times I) \longrightarrow \bigcup_{r \in \mathbb{N}} N_r
\]

with \( G|_{\hat{X}_{0,p} \times \{0\}} = h_p = h \) and \( H|_{\hat{X}_{1,r} \times \{0\}} = h'_p = h' \) \((r, s) \in \mathbb{N})\), and so that \( \alpha_{p,q} = G|_{i(p,q) \times I} \) (resp. \( \alpha_{r,s} = H|_{i'(p,q) \times I} \)) is a path in \( M_p - A_{m(q)}^p \) from \( h_p \circ i(p, q) \) to \( a_{p,q} \) (resp. a path in \( N_r - B_{n(s)}^r \) from \( h'_p \circ i'(p, q) \) to \( b_{r,s} \)). Observe that \( G \) and \( H \) are proper maps, since \( h, h', j \) and \( j' \) are proper. By the Homotopy Extension Property, the maps \( G, H \) extend to proper maps

\[
\hat{G} : \left( \bigcup_{p \in \mathbb{N}} \hat{X}_{0,p} \right) \times I \longrightarrow \bigcup_{p \in \mathbb{N}} M_p , \quad \hat{H} : \left( \bigcup_{r \in \mathbb{N}} \hat{X}_{1,r} \right) \times I \longrightarrow \bigcup_{r \in \mathbb{N}} N_r
\]

which yield commutative diagrams

\[
\begin{array}{ccc}
\bigcup_{p \in \mathbb{N}} \hat{X}_{0,p} & \xrightarrow{i} & \bigcup_{p \in \mathbb{N}} M_p \\
\bigcup_{r \in \mathbb{N}} \hat{X}_{1,r} & \xrightarrow{j} & \bigcup_{r \in \mathbb{N}} N_r
\end{array}
\]

\[
\begin{array}{ccc}
\bigcup_{p \in \mathbb{N}} \hat{X}_{0,p} & \xrightarrow{k} & \bigcup_{p \in \mathbb{N}} M_p \\
\bigcup_{r \in \mathbb{N}} \hat{X}_{1,r} & \xrightarrow{\hat{h}} & \bigcup_{r \in \mathbb{N}} N_r
\end{array}
\]

where \( \hat{h} = \hat{G}|_{(\bigcup_{p \in \mathbb{N}} \hat{X}_{0,p}) \times \{1\}} \) and \( \hat{h}' = \hat{H}|_{(\bigcup_{r \in \mathbb{N}} \hat{X}_{1,r}) \times \{1\}} \) are proper homotopy equivalences. Moreover, \( \hat{h} \) and \( \hat{h}' \) are proper homotopy equivalences under \( A \), by \([2, \text{Prop. 4.16}]\) (compare with \([11, \text{Chap. 6, §5}]\). Hence, they induce a proper homotopy equivalence between the quotient space described above (proper homotopy equivalent to \( W \)) and the following 3-manifold obtained as the quotient space given by the data:

(a) The disjoint unions \( \bigcup_{p \in \mathbb{N}} M_p \) and \( \bigcup_{r \in \mathbb{N}} N_r \) of copies of the 3-manifolds \( M \) and \( N \) respectively;
(b) a disjoint union \[ \bigsqcup_{p,q \in \mathbb{N}} P_{p,q} \] of copies of the compact 3-manifold \( P \setminus \hat{K} \); and

(c) the bijective function \( \varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}, (p, q) \mapsto (r, s) \), so that for each \( p, q \in \mathbb{N} \), the free ends of the corresponding 3-dimensional 1-handles \( H_{p,q}, H'_{p,q} \subset P_{p,q} \) considered above are being identified homeomorphically with small disks \( D_{p,q} \subset \partial M_p \) and \( D'_{r,s} \subset \partial N_r \) about the points \( a_{p,q} \) and \( b_{r,s} \) respectively.

In the case of an HNN-extension \( G \rtimes F = \langle G, t ; t^{-1} \psi_0(a_i) t = \psi_1(a_i), 1 \leq i \leq N \rangle \) (with monomorphisms \( \psi_i : F \to G, i = 0,1 \)), let \( X \) be a compact 2-polyhedron with \( \pi_1(X) \cong G \) and whose universal cover has the proper homotopy type of a 3-manifold, and let \( f_i : \bigvee_{i=1}^N S^1 \to X \) \( (i = 0,1) \) be cellular maps so that \( \text{Im } f_i \subseteq \pi_1(X) \) corresponds to the subgroup \( \text{Im } \psi_i \subseteq G \). Let \( Y \) be the 2-dimensional CW-complex constructed as above and consider the adjunction space \( Z = Y \cup f_0 \times \{0\} \cup f_1 \times \{1\} \times X \), with \( \pi_1(Z) \cong G \rtimes F \). Then, the proof goes just as the one given above for the amalgamated free product. \( \square \)

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