Non-autonomous bright–dark solitons and Rabi oscillations in multi-component Bose–Einstein condensates

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Abstract

We study the dynamics of non-autonomous bright–dark matter-wave solitons in two- and three-component Bose–Einstein condensates. Our setting includes a time-dependent parabolic potential and scattering length, as well as Rabi coupling of the separate hyperfine states. By means of a similarity transformation, we transform the non-autonomous coupled Gross–Pitaevskii equations into the completely integrable Manakov model with defocusing nonlinearity, and construct the explicit form of the non-autonomous soliton solutions. The propagation characteristics for the one-soliton state, and collision scenarios for multiple soliton states are discussed in detail for two types of time-dependent nonlinearities: a kink-like one and a periodically modulated one, with appropriate time-dependence of the trapping potential. We find that in the two-component condensates the nature of soliton propagation is determined predominantly by the nature of the nonlinearity, as well as the temporal modulation of the harmonic potential; switching in this setting is essentially due to Rabi coupling. We also perform direct numerical simulation of the non-autonomous two-component coupled Gross–Pitaevskii equations to corroborate our analytical predictions. More interestingly, in the case of the three-component condensates, we find that the solitons can lead to collision-induced energy switching (energy sharing collision), that can be profitably used to control Rabi switching or vice versa. An interesting possibility of reversal

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of the nature of the constituent soliton, i.e., bright (dark) into dark (bright) due to Rabi coupling is demonstrated in the three-component setting.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Studies on atomic Bose–Einstein condensates (BECs) have received considerable attention in recent years [1]. In this context, it has been shown that nonlinear effects in matter-waves can emerge, with this particular direction having attracted much interest both in theory and in experiments [2, 3]. From a theoretical viewpoint, this interest arises from the fact that many of such nonlinear effects can be understood in the framework of lowest-order mean-field theory, namely by means of the so-called Gross–Pitaevskii equation (GPE), which is nothing but the ubiquitous nonlinear Schrödinger (NLS) equation with external potential [1–3]. The GPE is a nonlinear evolution equation (with the nonlinearity originating from the interatomic interactions) and, as such, it permits the study of a variety of interesting purely nonlinear phenomena. The latter have primarily been studied by treating the condensate as a purely nonlinear coherent matter-wave, i.e., from the viewpoint of the dynamics of nonlinear waves. Relevant studies have already been summarized in various books [2] and reviews; see, e.g., [4] for bright solitons, [5] for dark solitons, [6] for vortices, [7] for dynamical instabilities in BECs, and so on.

The formation and dynamical properties of solitons in BECs are determined by the nature of their two-body atomic interactions, i.e., the sign of the $s$-wave scattering length which may be positive (negative) for repulsive (attractive) interatomic interactions. For instance, atomic dark (bright) solitons are formed in condensates with repulsive (attractive) interactions [2–5] (note that bright gap solitons are also possible in repulsive BECs, but those are critically formed due to the presence of a periodic—optical lattice—potential [8]). On the other hand, it should be stressed that the above conditions can dramatically change in the case of multi-component BECs, composed by two or more different hyperfine states of the same atom species (e.g., $^{87}\text{Rb}$) [9, 10]: in such a case, e.g., binary condensates with repulsive inter- and intra-species interactions can support mixed bright–dark (BD) solitons. This type of vector soliton consists of a dark soliton in one component coupled to a bright soliton in the second component. These solitons are usually referred to as ‘symbiotic’ solitons, because the bright component cannot be supported in a stand-alone fashion (it is only supported as such in attractive BECs [4]; see also [5]), and is only sustained because of the presence of its dark counterpart, which acts as an effective external trapping potential. Such atomic BD solitons were predicted in theory [11] and observed in experiments from different experimental groups [12–14]. Notice that solitons in three-component condensates (e.g., spinor $F = 1$ BECs) have been studied too [15–17] and mixed, BD solitons were predicted to occur as well [18].

On the other hand, coming back to the role of the sign and magnitude of the scattering length, it is important to note that they can be adjusted over a relatively large range by employing external magnetic, electric or optical fields near Feshbach resonances [19]. This possibility has given rise to many theoretical and experimental studies, with a prominent example being the formation of bright matter-wave solitons and soliton trains in attractive condensates [20], by switching the interatomic interactions from repulsive to attractive. Many theoretical works studied the BEC dynamics under temporal modulation of the nonlinearity. In particular, the
application of such a ‘Feshbach resonance management’ (FRM) technique [21] can be used to stabilize attractive higher-dimensional BEC against collapse [22], to create robust matter-wave breathers in the effectively one-dimensional (1D) condensate [21, 23], or compress bright solitons in the presence of expulsive potentials [24]. Mathematically speaking, if the scattering length is time dependent then so does the nonlinearity coefficient in the GPE, and the latter becomes non-autonomous. In the pioneering works [25], it has been shown that the single-component non-autonomous GPE can be transformed into the standard integrable NLS equation. The procedure developed in [25] can be generalized to two-component condensates [26] and the corresponding non-autonomous matter-wave solitons were studied extensively with aid of the explicit soliton solutions reported in [27–29].

This work deals with the dynamics of multi-component BECs, particularly two- and three-component BECs, with temporally modulated scattering length and trapping potential, and in the presence of Rabi coupling; the latter is accounted for by a linear coupling between separate wave functions induced by a radio frequency [9] (see also [30–32] for a theoretical analysis and applications). Recently, coherent many body Rabi oscillations have been observed experimentally and theoretically discussed in [33]. Using two successive transformations, with one constraint equation being in the form of a Ricatti equation, we reduce the original multi-component non-autonomous GPEs to the integrable defocusing vector NLS equation, known as the Manakov model [34]. These transformations can be viewed as a rotation followed by a similarity transformation. This way, we find non-autonomous mixed (BD) solitons and analyze their dynamics and interactions in detail, in both two- and three-component settings. We find that in the two-component system, soliton propagation is determined predominantly by the nature of the nonlinearity, as well as the temporal modulation of the harmonic potential; switching in this setting is essentially due to Rabi coupling. Our analytical predictions are corroborated with results from direct simulations; very good agreement between the two is found. Then by going one step further in the case of the three-component system, we find that the solitons can lead to collision-induced energy switching (energy sharing collision) that can be profitably used to control Rabi switching or vice versa. An interesting possibility of reversal of the nature of the constituent soliton, i.e., bright (dark) into dark (bright) induced by the Rabi coupling is demonstrated in the three-component setting.

The remaining part of this paper is arranged as follows. In section 2, we introduce the generalized non-autonomous coupled GPEs, for two-component condensates, with time-varying interaction strength and Rabi coupling. Also, we construct the one- and two-soliton solutions to the 1D coupled non-autonomous GPEs and analyze their propagation characteristics. In section 3, we consider the three-component non-autonomous GPEs with Rabi coupling, and explore the interesting energy exchange collisional features of non-autonomous matter-wave solitons. Finally, in section 4, we present our conclusions.

2. Non-autonomous BD solitons in two-component BECs and Rabi oscillations

In the mean-field approximation, the dynamics of two-component condensates can be well described by a pair of two coupled three-dimensional (3D) GPEs (see, e.g., [35]). In the physically relevant case of highly anisotropic (cigar-shaped) traps, these GPEs can be reduced to an effectively 1D system (see, e.g., [3]), which is of the following dimensionless form:

\[
i \psi_{j,t} = -\frac{1}{2} \psi_{j,xx} + \sum_{l=1}^{2} g_{jl}(t)|\psi_l|^2 \psi_j + \sum_{l=1, (l \neq j)}^{2} \sigma_l \psi_l + V_j(x,t) \psi_j, \quad j = 1, 2. \tag{1}
\]

Here, \( \psi_j \) is the macroscopic wave function of the \( j \)th component, time \( t \) and spatial coordinate \( x \) are respectively measured in units of the inverse transversal frequency \( \omega_{\perp}^{-1} \) and the transverse...
harmomnic oscillator length $a_0 = \sqrt{\hbar/m\omega^2}$ (m denotes atomic mass). In equation (1) the nonlinearity coefficients $g_{jl}$ are proportional to the s-wave scattering lengths $a_{jl}$, namely $g_{jl}\phi = 2a_{jl}/a_B$ (where $a_B$ is the Bohr radius), and are assumed to be subject to the FRM technique (see below); furthermore $V_j(x, t)$ are the external potentials (for $V(x, t) < 0$ ($V(x, t) > 0$) the potential is expulsive (confining)), and the linear cross coupling parameters $\sigma_l$ represent the Rabi coupling: in fact these coefficients are proportional to the Rabi frequency, and are responsible for the transfer of atoms of component $\psi_1$ to component $\psi_2$ and vice versa. In our case, considering two different hyperfine states of the same atom, the external time-varying potentials $V_j$ are equal, i.e., $V_1 = V_2 = V = (1/2)\Omega^2(t)x^2$. Here, the strength of the parabolic trap is given by $\Omega^2(t) = \omega_0^2(t)/\omega_0^2$, where $\omega_0(t)$ is the temporally modulated axial trap frequency. Finally, the macroscopic wave functions $\psi_j$ are normalized such that $\int_{-\infty}^{\infty} |\psi_j|^2 dx = N_j$ ($j = 1, 2$), where $N_j$ is the number of atoms in the $j$th component.

2.1. Non-autonomous BD one-soliton solution

Based on experimental results pertaining to two-component $^{87}$Rb BECs (see, e.g., [13]), we can assume that the scattering length ratios are almost equal to one, and also they can be tuned through Feshbach resonance (as it was demonstrated in the experimental works [36, 37]). Hence, we consider the case of equal interaction strengths, i.e., $g_{jl} = \rho(t)$ in equation (1), and choose the linear coupling coefficients $\sigma_l$ as $\sigma_1 = \sigma_2 = \sigma$, with $\sigma > 0$.

In such a case, as shown in appendix A, we can use two successive transformations to map the non-autonomous system (1) into an integrable autonomous nonlinear system. This is possible mainly due to the temporal dependence of the nonlinearity and external harmonic potential; in the absence of such dependence these transformations are not possible, in general. Notice, however, that in the absence of potential (homogeneous system), one can transform equation (1) into an integrable coupled nonlinear Schrödinger (CNLS) system even in the absence of inhomogeneities. The above mentioned transformations are: (i) a unitary transformation (see, e.g., [30, 32]) to reduce equation (1) to a non-autonomous system of equations without the Rabi coupling term; (ii) a similarity transformation, which reduces the aforementioned system to the following integrable defocusing CNLS equations [38, 39]:

$$iq_{j,t} + q_{j,xx} - 2 \sum_{l=1}^{2} |q_l|^2 q_j = 0, \quad j = 1, 2. \tag{2}$$

Equation (2) is the completely integrable Manakov system and admits $N$-soliton solutions of dark–dark and mixed (BD) types. Here, we focus only on mixed type soliton solutions due to their recent experimental observations and for their special dynamical features [12, 13].

The mixed one-soliton solution of equation (2) can be obtained by Hirota’s direct method [41]. In standard form, the mixed one-soliton solution of the two-component defocusing CNLS equations with the bright (dark) part appearing in the $q_1$ ($q_2$) component (see [27, 39]) is given in appendix B.1. Using the analytical form of this type of soliton, as well as the transformations of appendix A, we find an exact analytical non-autonomous soliton solution of equation (1) in the form:

$$\psi_1 = \cos(\sigma t)\phi_1 - i \sin(\sigma t)\phi_2, \quad \psi_2 = \cos(\sigma t)\phi_2 - i \sin(\sigma t)\phi_1, \tag{3}$$

where $\phi_1$ and $\phi_2$ are given by:

$$\phi_1(x, t) = \xi_1 \sqrt{2\rho}\left(|c_1|^2 \cos^2 \phi_1 - k_{1R}^2\right)^{1/4} e^{i(\xi_1 + \theta + \phi)} \times \text{sech}\left( k_{1R} \left( \sqrt{2\xi_1} \left( \rho \phi - 2\xi_2 \xi_1^2 \int_0^t \rho^2 \, dt \right) - 2k_{1R} \xi_1 \int_0^t \rho^2 \, dt \right) + R/2 \right). \tag{4}$$
Figure 1. Typical form of the kink-like nonlinearity $\rho(t)$ (a) and corresponding strength of external harmonic potential (b).

$$\phi_2(x, t) = -c_1 \xi_1 \sqrt{2} \rho e^{i(\tilde{\nu} + \varphi_1 + \delta)} \left( \cos \nu_1 \tanh \left[ k_1 R \left( \sqrt{2} \xi_1 \int_0^t \rho^2 \, dt \right) \right] + \text{isin} \varphi_1 \right),$$

(5)

where $\tilde{\nu}_1 = k_1 \sqrt{2} \xi_1 (\rho x - 2 \xi_2 \xi_1^0 \int_0^t \rho^2 \, dt) - (k_1^2 - k_1^2 - 2 |c_1|^2) \xi_1^0 \int_0^t \rho^2 \, dt$, $\tilde{\xi}_1 = -(b_1^2 + 2 |c_1|^2) \xi_1^0 \int_0^t \rho^2 \, dt + b_1 \sqrt{2} \xi_1 (\rho x - 2 \xi_2 \xi_1^0 \int_0^t \rho^2 \, dt)$ and the other parameters and functions involved in the above equations are defined in appendices A and B.

The above mixed soliton bears resemblance to the recently observed, so-called beating dark–dark soliton [14, 42], where the bright and dark parts co-exist in the same component with a constant asymptotic value as $|x| \to \infty$; however, the form of this BD soliton still depends on the Rabi coupling and the time-dependent nonlinearity and parabolic potential. Notice that due to the presence of Rabi coupling, there will be a periodic switching between the two components along with oscillating background. Another important observation for the integrable two-component case is the dependence of the existence of e.g., the bright part of the mixed soliton on its dark counterpart (see equations (B.1) and (B.2) and relevant discussion in appendix B.1). Thus, it is impossible to make any one of the constituents (bright/dark) of the mixed soliton to vanish completely. In fact, this will result in singularities in the other soliton part. This holds even in the presence of the Rabi coupling for the two-component system.

To elucidate the dynamics of such special type of mixed soliton we consider the following two examples.

**Example 1. Kink-like nonlinearity.** First we note that a fast time modulated nonlinearity displays interesting dynamics in BECs [43]. A relatively sudden jump in the nonlinearity coefficient can be well represented by a kink-like nonlinearity of the form:

$$\rho(t) = 1 + \tanh(\omega t + \delta),$$

(6a)

with the associated atomic scattering length being $a_s(t) = \frac{1}{2} a_B [1 + \tanh(\omega t + \delta)]$; here, $\omega$ denotes the time scale characterizing the jump, and $\delta$ is an arbitrary constant.

This is shown in figure 1(a) and the corresponding external harmonic potential is displayed in figure 1(b). The time-dependent nonlinearity $\rho(t)$ admitting the above form can be realized for the following scattering length [25],

$$\frac{a_s(t)}{a_{bg}} = \left( 1 + \frac{\Delta}{B_0 - B(t)} \right),$$

(6b)

where $a_{bg}$ is the value of scattering length far from the Feshbach resonance, $B(t)$ is the applied time-varying magnetic field, $B_0$ is the resonant value of the magnetic field, and $\Delta$
Figure 2. Two-component one-soliton solution of GPEs (1) with kink-like nonlinearity in the absence of Rabi coupling. Top panels show the intensity plots of exact analytical results (upper left panel: bright component, and upper right panel: dark component of the mixed soliton), while bottom panels show numerical results (bottom left panel: bright component, and bottom right panel: dark component of the mixed soliton). The parameter values are $k_1 = 0.5 + 0.02i$, $b_1 = 0.2$, $c_1 = 2$, $\xi_1 = 0.5$, $\xi_2 = 0$, $\alpha_1^{(1)} = 0.5$, $\delta = -4$ and $\omega = 0.7$.

is the resonance width in the presence of the magnetic field. The corresponding strength of the time-dependent magnetic trap is determined by a Ricatti equation (see equation (A.8) in appendix A) as follows:

$$\Omega^2(t) = -\left(\frac{4\omega^2}{1 + e^{2(\omega t + \delta)}}\right).$$  \hspace{1cm} (6c)

The propagation of the non-autonomous mixed soliton in the absence and presence of Rabi coupling are depicted in the top panels of figures 2 and 3, respectively. We have also solved the system (1) numerically by means of the split-step Fourier method, and the resulting numerical plots for the above choice of the time-dependent nonlinearity and potential strength (6c) are given in the bottom panel.

Comparing the top and bottom panels, one can see that there is a very good agreement between numerical and analytical results. We find that, in the absence of the Rabi coupling, there are no oscillations for the solitons and the background. However, the shape, width and velocity of solitons are modulated. Then, the introduction of Rabi term leads to an exchange of number of atoms between the components, which ultimately results in beating oscillations, as shown in figure 3. For small values of $\sigma$, the background just starts to oscillate (see figure 3). However, by increasing the parameter $\sigma$, one can observe rapid oscillations of the background also. To illustrate this, the contour plots of $|\psi_1|^2$ for $\sigma = 0.1$ and $\sigma = 0.2$ are shown in figure 4. Similar observations can be made for the second component, $|\psi_2|^2$, as well. These oscillations of the background are due to the exchange of condensates between the soliton and the background.

Another important observation following from figures 3 and 4, as well as from expressions (3)–(5) is that the increase of the value of Rabi coupling leads a significant portion of the dark part of mixed soliton to appear in the first component ($\psi_1$), along with the bright part
Figure 3. Two-component mixed one-soliton solution with co-existing BD parts for the kink-like nonlinearity in the presence of Rabi coupling: analytical results (upper left panel: \( \psi_1 \) component, and upper right panel: \( \psi_2 \) component of the mixed soliton). Numerical results (bottom left panel: \( \psi_1 \) component, and bottom right panel: \( \psi_2 \) component of the mixed soliton). The parameter values are \( k_1 = 0.5 + 0.02i, b_1 = 0.2, c_1 = 2, \xi_1 = 0.5, \xi_2 = 0, \alpha_1^{(1)} = 0.5, \delta = -4, \omega = 0.7 \) and \( \sigma = 0.02 \).

Figure 4. Mixed one-soliton solution for the kink-like nonlinearity in left panel for \( \sigma = 0.1 \) and right panel for \( \sigma = 0.2 \). The other parameters are fixed as \( k_1 = 0.5 + 0.02i, b_1 = 0.2, \xi_1 = 0.5, \xi_2 = 0, \alpha_1^{(1)} = 0.5, \delta = -4 \) and \( \omega = 0.7 \).

accompanied by beating effects. Similar effects take place in component \( \psi_2 \) too. Thus, in a given component one can have co-existing oscillating BD soliton.

Enlarging figure 3 in the region \( t = 0 \) to 5, we observe that the non-autonomous mixed soliton is absent in this region. This is due to the form of the nonlinearity \( \rho(t) \), which becomes zero in the range \( t = 0 \) to 5 and reaches a saturation for large positive \( t \) values. Thus, even though the Rabi coupling leads to an exchange of atoms among the two components leading to oscillations, the nature of soliton propagation is predominantly determined by the nature of time-dependent nonlinearity.

**Example 2. Periodically modulated nonlinearity.**

Next, we consider another physically interesting example, namely the case of a periodic time-varying nonlinearity of the form:
Figure 5. Periodically modulated nonlinearity $\rho(t)$ (a) and corresponding strength of the external harmonic potential (b).

\begin{equation}
\rho(t) = 1 + \varepsilon \cos(\omega t + \delta),
\end{equation}

where $\varepsilon$ is an arbitrary real constant, $\omega$ is the characteristic frequency, and $\delta$ is a real constant parameter. In this case, the form of atomic scattering length is $a_i(t) = \frac{1}{2} a_B [1 + \varepsilon \cos(\omega t + \delta)]$. Figure 5(a) represents the form of such a time-varying nonlinearity. The expression for the pertinent time-dependent external magnetic field $B(t)$ required to achieve the above form of the nonlinearity can be determined from the formula (6b). The corresponding strength of the magnetic trap admits the form

\begin{equation}
\Omega^2(t) = \omega^2 \varepsilon \left( -\frac{3\varepsilon - 2 \cos(\omega t + \delta) + \varepsilon \cos[2(\omega t + \delta)]}{2[1 + \varepsilon \cos(\omega t + \delta)]^2} \right),
\end{equation}

in order to satisfy equation (A.8). Note that the potential is sign reversible and can support the same type of soliton for both positive and negative signs; see figure 5(b).

It is apparent from the periodic nature of the nonlinearity, and the time modulation of the strength of the external potential, that the condensates also execute periodic oscillations in both the components even in the absence of linear coupling as shown in figure 6. The numerical results, obtained by a direct integration of equation (1) are shown in the bottom panel of figure 6 in the absence of Rabi coupling. Next, we introduce the Rabi coupling between the two components; then, as expected, there will be an exchange of atoms between the components, as well as between the soliton and the background. This exchange induces oscillations in the density of condensates which, in turn, modulate the oscillations due to the periodic nature of $\rho(t)$ and $\Omega^2(t)$ and result in soliton beating in both components. This becomes clear in figure 7, from the plots showing the non-autonomous soliton (3) for the choice (7a) (upper panel) and the corresponding numerical results (bottom panel).

As in the previous example, here also both parts of the mixed soliton co-exist in a given component due to Rabi switching. The left panel of figure 8 displays the propagation of mixed soliton for a small value of $\sigma$ (= 0.1). It is observed that the soliton only oscillates periodically, but the background is not oscillating. The right panel of figure 8 displays that for larger value of $\sigma$ (say $\sigma$ = 0.2), the background oscillates rapidly and significant switching of dark and bright parts among the components occurs, thereby resulting in the co-existence of both dark and bright parts in the same component. By increasing the values of $\omega$, we also observe ‘creeping’ soliton propagation with beating effects, which are not presented here. Similar type of creeping soliton appears in the presence of inhomogeneities too.

From the above two examples, we can conclude that in the non-autonomous two-component GPEs (1) the nature of soliton propagation is determined predominantly by the
Figure 6. Two-component mixed one-soliton solution of equation (1) for the periodically modulated nonlinearity in the absence of Rabi coupling. Top panels show the exact analytical results (upper left panel: bright component, and upper right panel: dark component) and bottom panels show numerical results (bottom left panel: bright component, and bottom right panel: dark component). The parameter values are $k_1 = 0.5 + 0.1i$, $b_1 = 0.2$, $a_1^{1/2} = 1.5$, $c_1 = 1$, $\xi_1 = 0.6$, $\xi_2 = 0$, $\epsilon = 0.2$, $\omega = 1.2$ and $\delta = 0$.

temporal dependence of $\rho(t)$ and $\Omega_2(t)$ while the switching of condensates is completely dependent on the Rabi coupling.

2.2. Two-soliton solution of non-autonomous two-component GPEs

We now turn our attention to the case of mixed two-soliton solution of the integrable 2-CNLS system (2). The explicit form of this solution, given in appendix B.1, contains all the information regarding the collision of two solitons. We will employ this autonomous soliton solution to construct the exact two-soliton solution of the corresponding non-autonomous system (1) in detail.

The asymptotic analysis of the two-soliton solution of the two-component GPEs system (2) given in appendix C.1, shows that the two solitons undergo standard elastic collision as $|A^{i+}_l| = |A^{i-}_l|$, $i, l = 1, 2$, where $A^i_l$ represents the amplitude of $l$th soliton in $i$th component. Here and in the following, the superscript (subscript) of $A$ (or $\psi$) represents the number of soliton (component) while $-(+)$ appearing in the corresponding quantities indicates their form before (after) collision. Thus the amplitude and speed of the bright and dark components are preserved after the collision, except for a phase-shift ($\Phi = R_1 - R_2 - R_3$), where $R_1$, $R_2$ and $R_3$ are defined in appendix B.1. This elastic collision of solitons is shown in figure 9.

The exact dynamics of the non-autonomous two mixed solitons can be understood after transforming the two-soliton solution (see equations (B.3)–(B.13)) by using the transformations given in appendix A (see equations (A.4) and (A.1)). In the following analysis, the collision dynamics for two interesting forms of the nonlinearity coefficient $\rho(t)$, namely kink-like and the periodically modulated nonlinearity are considered in detail.
Collision dynamics in the presence of kink-like modulated nonlinearity. Figure 10 shows the two-soliton collision in the non-autonomous two-component GPEs (1) with kink-like nonlinearity, whose form is given by equation (6a) in the absence of Rabi terms, and the corresponding strength of the magnetic trap is given by equation (6c). The figure shows that the two solitons undergo a collision around $t = 80$ and get well separated for larger values of $t$.

Next, we include the Rabi coupling and plot the non-autonomous two-soliton collision in figure 11. We notice from figures 10 and 11 that the nature of soliton collision in the presence of Rabi term looks alike the soliton collision in the absence of Rabi term. Additionally, in the present case, there is an oscillating exchange of atoms between the
two components. The background oscillates in a periodic manner, in which the oscillations in a given component is maximum while it is minimum in the other.

Here we present the asymptotic analysis of the non-autonomous two-soliton solution of two-component GPEs with kink-like nonlinearity, obtained by making use of (B.3)–(B.13) and the transformations (A.1) and (A.4). For the choice \( k_{1R} < k_{2R}, k_{1I} > k_{2I}, \) the asymptotic forms of the non-autonomous solitons \((S_l, l = 1, 2)\) well before and after collision can be expressed as below:

**Before collision.**

\[
\psi_{l+} = \xi_l \sqrt{2 \rho} [A_{l+}^1 \cos(\sigma t) \text{sech} (\eta_{lR} + R_l/2) e^{i(\eta_{lI} + \delta)} \\
- i A_{l+}^2 \sin(\sigma t) (\cos \psi \tanh (\eta_{lR} + R_l/2) + i \sin \psi)].
\]

**Figure 9.** Two-soliton collision of system (2) (see equation (B.4)). The soliton parameters are fixed as \( k_1 = -1 + i, k_2 = 1 - i, c_1 = 3, b_1 = 0.2 \) and \( \alpha_1 = \alpha_2 = 0.02. \)

**Figure 10.** Interaction of non-autonomous solitons in two-component GPEs for the kink-like nonlinearity in the absence of Rabi coupling. Top panels show the analytical results (upper left panel: bright component, and upper right panel: dark component) and bottom panels show the numerical results (bottom left panel: bright component, and bottom right panel: dark component). The soliton parameters are \( k_1 = -1 + i, k_2 = 1 - i, c_1 = 3, b_1 = 0.2, \) \( \alpha_1 = \alpha_2 = 0.03, \delta = -4.5, \xi_1 = 0.2, \xi_2 = 0 \) and \( \omega = 0.75. \)
computing the densities \( (\sigma\text{ term and the inhomogeneity, which are defined in appendix C.1. We notice that in j }\psi\text{ and Rabi term. BD solitons and also in the background along with a modulation due to nonlinearity |\psi|^2}.\) Here, \( A_j^- \) \( (A_j^+) \) is the amplitude of the soliton \( S_j \) in the \( j \)th component, \( l, j = 1, 2 \), before (after) collision, in the absence of Rabi term and the inhomogeneity, which are defined in appendix C.1. We notice that in Rabi term (\( \sigma = 0 \)), there is no oscillatory terms as expected. By computing the densities \( (|\psi_j^l|^2, j, l = 1, 2) \) before and after collision and noticing \( |\psi_j^l|^2 = |\psi_j^l|^2 \) (since \( A_j^- \) \( = A_j^+ \)), see equations (C.1)–(C.4)), we identify that the nature of collision is elastic except for a phase-shift. This phase-shift is same for both solitons \( S_1 \) and \( S_2 \) that can be found as \( -\frac{R_1 - R_2 - R_3}{2} \). There will be oscillations in BD solitons and also in the background along with a modulation due to nonlinearity and Rabi term.
(ii) Collision dynamics in the presence of periodically modulated nonlinearity. The collision of two solitons in the two-component GPEs with periodically modulated nonlinearity \( \rho(t) \) in the absence and presence of Rabi coupling are depicted in figures 12 and 13 respectively. For this case the strength of the parabolic trap is given by equation (7b).

The role of time modulated scattering length is to introduce periodic modulations in the soliton profile before and after collision uniformly. Meanwhile, the Rabi term leads to a periodic exchange of condensates between the components along with oscillations in the background. As in the previous example, here also the Rabi coupling makes it feasible to have both dark and bright parts of the mixed soliton in first and second components. We find that the solitons exhibit elastic collision even in the presence of Rabi coupling, though there is an exchange of atoms between the components. This example shows that the Rabi coupling does not affect the elastic nature of the collision in the two-component GPEs. An asymptotic analysis similar to that of kink-like nonlinearity can be carried out for this case too.

3. Non-autonomous BD solitons in three-component BECs and Rabi oscillations

Following our considerations for the binary BEC case, we now proceed with the investigation of the three-component system. The dynamics of three-component BECs in 1D with equal time-dependent interaction strengths (i.e., \( g_{jl} = \rho(t), j,l = 1,2,3 \)), and in the presence of external time-dependent harmonic potential \( V_j(x,t) (\equiv V) \), is governed by the following dimensionless non-autonomous three-coupled GPEs (see, e.g., [3]):

\[
i\psi_{j,t} = -\frac{1}{2}\psi_{j,xx} + \rho(t) \sum_{l=1}^{3} |\psi_j|^2 \psi_j + \sum_{l=1, l\neq j}^{3} \sigma l \psi_l + V(x,t) \psi_j, \quad j = 1, 2, 3, \tag{12}\]
where $\psi_j(x,t)$ are the wave functions of the condensates. As before, coefficients $\sigma_l$ account for the (linear) Rabi coupling and are chosen to be equal (i.e., $\sigma_l = \sigma$, $l = 1, 2, 3$); furthermore, the strength of the nonlinear coupling is given by $\rho(t)$.

As in the two-component case, we employ a rotation transformation and a similarity transformation (see details in appendix A) and reduce equation (12) in the form:

$$\begin{align*}
 iq_{jx} + q_{jxx} - 2 \sum_{l=1}^{3} |q_l|^2 q_j &= 0, \quad j = 1, 2, 3.
\end{align*}$$

The above model, three-coupled NLS system (13), is also a completely integrable system (the three-component generalization of Manakov system with defocusing nonlinearity) and the soliton solutions can be obtained by various methods, e.g., the inverse scattering transform method, the Hirota’s direct method, the Bäcklund transformation method, etc.

### 3.1. Non-autonomous mixed (bright–bright–dark) one-soliton solution

The explicit form of the mixed one-soliton solution of equation (13), in the form of a bright–bright–dark soliton, can be obtained by means of Hirota’s method [40, 41]. Here, we consider a mixed soliton solution, with the bright part appearing in $q_1$ and $q_2$ components, and with the dark part in $q_3$ component. The exact analytical form of this type of mixed one- and two-solitons are given in appendix B.2. Then we make use of the transformations, corresponding to three-component case given in appendix A, for obtaining the exact non-autonomous soliton solutions of (12), as in the two-component GPE system.

We again consider the same two forms for $\rho(t)$ which were discussed in the previous section during our analysis of two-component condensates.
Figure 14. Intensity plot of the exact three-component non-autonomous mixed one-soliton solution of equation (12) for the kink-like nonlinearity. Top and bottom panels show mixed non-autonomous one soliton in the absence and in the presence of Rabi coupling, respectively. The parameters are $k_1 = 0.5 + 0.02i, b_1 = 0.2, a_1^{(1)} = 0.2, a_1^{(2)} = 0.2, e_1 = 2, \delta = -5, \xi_1 = 0.4, \xi_2 = 0, \omega = 2.5$ and $\sigma = 0.02$.

(a) Kink-like nonlinearity.

Figure 14 shows the mixed one-soliton solution of the non-autonomous three-component GPEs (12) with kink-like nonlinearity. In this case, the shape of the soliton is affected significantly by increasing the value of $\omega$ as in the two-component case.

Also, we notice that the oscillations of the background are rapid as compared to that of two-component case, even for the same value of Rabi coupling $\sigma$. This can be inferred by comparing the bottom panels of and figures 3 and 14. Thus, an increase in the number of components can make the exchange of atoms between soliton and its background is too faster.

(b) Periodically modulated nonlinearity.

The propagation of three-component non-autonomous mixed one-soliton in the presence of periodically modulated nonlinearity having the form of Mathieu function (see equation (7a)) is shown in figure 15. As in the two-component case, here also occurs oscillatory transfer of atoms among the components accompanied by oscillations of the background due to Rabi coupling.

The above two examples show that the role of Rabi coupling on three-component non-autonomous BECs is similar to that of two-component BECs except for an increase in the oscillations due to the additional component. Apart from this, we would like to give emphasis to a particular dynamical feature of the three-component system, in the presence of Rabi coupling, which is not possible in the two-component case. It can be recalled from the study on two-component non-autonomous case that it is impossible to make any component (bright/dark) of the mixed soliton to vanish completely, in the presence—as well as in the
Figure 15. Exact three-component one-soliton solution of equation (10) for the periodically modulated nonlinearity in the absence (top panels, $\psi_1$, $\psi_2$ and $\psi_3$ components) and in the presence of Rabi switch (bottom panels, $\psi_1$, $\psi_2$ and $\psi_3$ components). The parameters are chosen as $k_1 = 0.7 + 0.1i$, $b_1 = 0.2$, $\alpha_1^{(1)} = 1.5$, $\alpha_1^{(2)} = 1$, $c_1 = 1$, $\xi_1 = 1$, $\xi_2 = 0$, $\omega = 0.9$, $\delta = 0$, $\epsilon = 0.2$ and $\sigma = 0.2$.

Figure 16. Three-component one-soliton solution with periodically modulated nonlinearity in the absence of Rabi coupling. The parameters are $k_1 = 0.5 + 0.1i$, $b_1 = 0.2$, $\alpha_1^{(1)} = 1$, $\alpha_1^{(2)} = 0$, $\xi_1 = 1$, $c_1 = 1$, $\xi_2 = 0$, $\epsilon = 0.2$ and $\omega = 0.8$.

absence—of Rabi coupling. In contrast, in the three-component system, it is possible to make any one of the bright parts of the mixed soliton to vanish completely before the introduction of Rabi coupling term. The additional component brings an additional arbitrariness to the non-autonomous GPE system (12), which allows to make the density of any one of the bright parts of the mixed soliton to be zero before the incorporation of the Rabi term. The role of the Rabi term is to switch the condensates between all three components. So, in the presence of Rabi term, some amount of field will be transformed to the component where there was no field before its introduction. Thus, in three-component condensates one can transfer a significant part of the condensates to a component which admits no field at all before the introduction of Rabi coupling from the other components. This is shown in figures 16 and 17, in which the condensate is completely absent in the second component in figure 16 but, after introducing the Rabi coupling, appreciable portion of condensate appears in the same component as shown in figure 17.

3.2. Non-autonomous mixed (bright–bright–dark) two-soliton collision

The explicit form of mixed two-soliton solution of (13) with the bright parts appearing in components $(q_1, q_2)$ and the dark part in $q_3$, obtained by Hirota’s bilinearization method is given in appendix B.2 and the corresponding asymptotic analysis is presented in appendix C.2.
shown in figure 18. In figure 18, the bright part of the mixed soliton is enhanced in the presence of Rabi coupling. The parameters are \( k_1 = 0.5 + 0.1i, k_2 = 1.1 - i, b_1 = 0.2, c_1 = 3, \alpha_1^{(1)} = 0.02, \alpha_1^{(2)} = 0.01 + 0.025i \) and \( \alpha_2^{(1)} = 0.02 \).

![Figure 17](image17.png)

![Figure 18](image18.png)

From the asymptotic expressions (C.5) and (C.7) one can find that the amplitudes (condensate densities) of the colliding solitons before and after collision can be related

\[
|A_j^{(m)}|^2 = |T_j^{(m)}|^2 |A_j^{(n)}|^2, \quad j, l = 1, 2,
\]

where \( |T_j^{(m)}|^2 = \frac{2\delta_{j1} - \delta_{j2} + (\delta_{j1} - \delta_{j2}) \xi_j}{\omega_j^2} \) and \( |T_j^{(n)}|^2 = \frac{2\delta_{j1} - \delta_{j2} - (\delta_{j1} - \delta_{j2}) \xi_j}{\omega_j^2} \), \( j = 1, 2 \), where \( \delta_{j1}, \delta_{j2}, R_1, R_2 \) and \( R_3 \) are defined in appendix B.2. It is instructive to notice that \( |T_j^{(m)}|, j, l = 1, 2 \), are not unimodular, in general, and also depend on dark soliton parameters \( c_1 \) and \( b_1 \) in addition to the bright soliton parameters. This will result in the energy sharing collision displaying suppression (enhancement) of condensate density in the bright part of a given soliton with commensurate changes in the bright part of the other colliding soliton. However, one can have standard elastic collision for the choice \( \frac{\alpha_1^{(1)}}{\alpha_1^{(2)}} = \frac{\alpha_2^{(1)}}{\alpha_2^{(2)}} \), for which \( |T_j^{(m)}|, j, l = 1, 2 \), become unimodular. Additionally, the colliding solitons also experience a phase-shift \( \Phi_1 = \Phi_2 = \frac{\langle R_s - R_j^1 - R_j^2 \rangle}{\omega_j^2} \).

It can also be inferred from the asymptotic expressions (C.6) and (C.8) that the dark solitons in the \( q_3 \) component always exhibit elastic collision as \( |A_q^{(m)}|^2 = |A_q^{(n)}|^2, l = 1, 2 \). These dark solitons also undergo a phase-shift, same as that of bright solitons.

This study can be straightforwardly extended to integrable \( N \)-component CNLS system with defocusing nonlinearity, for \( N > 3 \). It is interesting to note that the above kind of energy sharing collision in such \( N \)-component CNLS system can take place only if the bright part of the mixed soliton appears at least in two components.

Such fascinating energy sharing collision in the three-component GPE system (13), is shown in figure 18. In figure 18, the bright part of the mixed soliton \( S_1 \) is enhanced in the \( q_1 \) component whereas it is suppressed in the \( q_2 \) component after its collision with the soliton \( S_2 \). The reverse scenario takes place for the bright parts of the mixed soliton \( S_2 \). Notice that the dark parts of mixed solitons \( S_1 \) and \( S_2 \) are unaffected by the collision. The energy sharing collision is characterized by an exchange of condensates among the bright parts of the colliding mixed solitons \( S_1 \) and \( S_2 \), leaving the dark part unaltered after collision. This type of energy
Figure 19. Energy sharing collision of mixed solitons in the non-autonomous three-component GP equation (12) with kink-like nonlinearity in the absence (top panels) and in the presence (bottom panels) of Rabi coupling. The soliton parameters are $k_1 = -1 + i, k_2 = 1 - i, b_1 = 0.2, c_1 = 3, a_1^{(1)} = 0.02, a_2^{(1)} = 0.01 + 0.025i, a_1^{(2)} = a_2^{(2)} = 0.02, \xi_1 = 2, \xi_2 = 0, \delta = -4, \omega = 2$ and $\sigma = 0.02$.

sharing collision in different integrable CNLS systems appearing in nonlinear optics has been extensively studied in [28, 29, 39, 40].

The non-autonomous two-soliton solution can be expressed as

$$\psi_j = \frac{1}{3} \xi_j \sqrt{2\rho} e^{i\tilde{\theta}} \left[ e^{-2i\sigma t} \sum_{l=1}^{3} q_l + e^{i\sigma t} \left( 2q_j - \sum_{l=1, l\neq j}^{3} q_l \right) \right], \quad j = 1, 2, 3,$$

where $\rho, \tilde{\theta}$ and $\xi_j$ are defined in appendix A and $q_j$ are given in equations (B.18)–(B.20), in which $X$ and $T$ are defined by equations (A.6) and (A.7), respectively. Now, it is of further interest to study whether the above discussed energy sharing collision still prevails in the presence of time-varying nonlinearity and Rabi coupling. For illustrative purpose, again we consider the two types of the time-dependent nonlinearities discussed in the two-component case.

(i) Energy sharing collision of mixed solitons in non-autonomous three-component GPE.

(a) Kink-like nonlinearity. The soliton collision for a kink-like modulated nonlinearity in the context of the non-autonomous three-component GPEs (12), in the presence of Rabi term, is depicted in figure 19. The density of the bright part of the mixed soliton $S_1$ gets enhanced and $S_2$ is suppressed after the collision in the $\psi_1$ component, whereas the component $\psi_2$ experiences reverse effects for $S_1$ and $S_2$, and the dark parts of mixed solitons in $\psi_3$ component remain unaltered. Thus, the switching nature of energy sharing collision in the autonomous system (13) is unaffected by the presence of Rabi term in the non-autonomous GP system, for this choice of time-dependent nonlinearity and external potential. This shows that the energy sharing collision of bright parts of the mixed solitons can exist for a wide range of time-dependence of nonlinearity, external potential and Rabi coupling, for which the non-autonomous system (12) is integrable. An asymptotic analysis of the non-autonomous two-soliton solution (14) can be carried out as in the two-component case. One can notice that the energy sharing collision still prevails as $|A_j^+| \neq |A_j^-|, j, l = 1, 2$, in general,
Figure 20. Energy sharing collision of mixed soliton in the non-autonomous three-component GPEs with periodically modulated nonlinearity in the absence (top panels) and in the presence (bottom panels) of Rabi coupling. The choice of soliton parameters are $k_1 = -1 + i, k_2 = 1 - i, b_1 = 0.2, c_1 = 2, \alpha_1 = 0.02, \alpha_1' = 0.01 + 0.025 i, \alpha_2 = 0.02, \delta = 0, \xi_1 = 0.4, \xi_2 = 0.2, \omega = 2, \varepsilon = 0.2$ and $\sigma = 0.02$.

which ultimately leads to $|\psi_{j+}^+| \neq |\psi_{j-}^-|$. The phase-shift can be found to be the same for both the solitons and is given by $(R_3 - R_2 - R_1)$, where $R_1, R_2$ and $R_3$ are defined in appendix B.

In the two-component case, we observe a gradual increase in the density of condensates in the bright parts in the region $t = 2$ to 6 and it remains constant for $t > 6$, due to the nature of nonlinearity (figure 10) and onset of Rabi oscillations due to Rabi coupling (see figure 11). Note that, there the collision is elastic in the absence as well as in the presence of Rabi coupling and is not an energy sharing one. Thus, in two-component BECs the growth of condensate is purely due to the nature of $\rho(t)$.

However, in the present case, for the mixed soliton $S_2$ there is a suppression in the density of its bright part in $\psi_1$ after interaction due to the energy sharing collision. This suppression balances the enhancement in its density in this regime due to the nature of the time-varying nonlinearity, and will result in standard soliton ($S_2$) with constant amplitude after interaction (see figure 19). Also, the Rabi term leads to beating effects in this soliton $S_2$ in $\psi_1$, but with a smaller period as compared to that of the two-component case. However, the opposite effect takes place for $S_2$ in the second component $\psi_2$. This shows that energy sharing collision can be profitably used in altering the exchange of condensates through Rabi switching.

(b) Periodic modulated nonlinearity. Energy sharing collision of mixed solitons in the three-component GPEs with periodically modulated nonlinearity is shown in figure 20. We observe that in this case also the energy sharing collision for the non-autonomous system (1) takes place as that of the autonomous system (2) (see figure 18).

We observe another dramatic switching of condensates during collision of two solitons in the non-autonomous three-component GPEs (12) with periodically modulated time-dependent nonlinearity, for smaller values of Rabi term $\sigma$. Here, before collision, we have an oscillating bright soliton part ($S_2$) in the $\psi_1$ component, which completely transforms to an oscillating mixed (BD) soliton in the same component after collision. In the second component, $\psi_2$, the amplitude of bright soliton part ($S_2$) gets enhanced
after the collision. For the soliton $S_1$, in $\psi_1$ ($\psi_2$) component enhancement (suppression) with periodic oscillations takes place. In the component $\psi_3$ the solitons undergo elastic collision and there occurs only switching of condensates due to Rabi coupling. Notably, in $\psi_3$ component the pure bright part of $S_1$ is transformed to mixed (BD) part after collision (see $\psi_3$ component in the bottom panel of figure 20). This interesting collision is a consequence of the combined effects of time-dependent nonlinearity and external potential, Rabi term and the exchange of condensates due to energy sharing collision. Another noticeable effect arising due to the Rabi coupling, particularly for this type of periodic nonlinearity, is an increase in the relative separation distance between the solitons well before and after collision. Furthermore, the collision takes place faster in the presence of Rabi term.

4. Conclusions

In this work, we have studied the dynamics of non-autonomous mixed bright–dark matter-wave solitons in two- and three-component Bose–Einstein condensates (BECs). Our setting included a time-dependent parabolic potential and scattering length, as well as Rabi coupling between separate hyperfine states. We have transformed the non-autonomous two- and three-component Gross–Pitaevskii equations (GPEs), into a set of integrable defocusing two- and three-component Manakov autonomous systems by means of two successive transformations. These transformations can be viewed as a rotation followed by a similarity transformation. Then, with the aid of the two- and three-component soliton solutions of defocusing Manakov systems, and by inverting the transformations, we have studied the dynamics of single- and multiple-non-autonomous matter-wave solitons. Our considerations involved two different choices of nonlinearities namely, a kink-like and Mathieu-like ones, together with their corresponding time-varying harmonic potentials.

Our study on the two-component case has shown that the nature of soliton propagation depends significantly on the nature of nonlinearity and the associated potential. The switching depends purely on the Rabi coupling. Here, the bright–dark matter-wave solitons exhibit elastic collisions along with oscillations due to Rabi coupling, and profile modulation due to the nonlinearity $\rho(t)$ and strength of the potential. In this two-component non-autonomous case, the dark–bright solitons exist as ‘symbiotic’ structures, i.e., the bright component exist only due to the presence of its dark counterpart. Hence it is impossible to make any part (either bright or dark) to be zero. Also, Rabi coupling makes it possible to have a special type of mixed soliton in which bright and dark parts can co-exist in the same component. Such structures are reminiscent to the so-called beating dark–dark solitons, which have recently been observed in autonomous binary BECs [14, 42]. The present study shows the possibility of observing such structures, with co-existing bright and dark parts in the same component, also in non-autonomous multi-component systems in the presence of Rabi coupling. Our analytical results of the two-component non-autonomous GPE system are found to be in very good agreement with results of direct numerical simulations.

We have also studied non-autonomous three-component BECs. An important observation in this case is that the bright part of the mixed soliton can be absent in any of the three-component condensate in the absence of Rabi coupling. The introduction of Rabi coupling makes it feasible to have some portion of mixed soliton in that component. Also, in the three-component BECs the number of oscillations increases as compared to that of two-component case and ultimately the exchange of atoms between the soliton and background also increases in a periodic manner.
Finally, we considered the collision of non-autonomous solitons in the three-component condensate. In this case, the non-autonomous matter-wave solitons undergo interesting energy sharing collision, leading to an exchange of atoms, in addition to Rabi coupling, in the bright ($\psi_1$ and $\psi_2$) components; notice that in the third component ($\psi_3$) the solitons undergo elastic collision. The matter-wave mixed (bright–bright–dark) soliton collision in non-autonomous three-component condensates also displays another fascinating feature, namely the reversal of the type of mixed-soliton part (i.e., bright type to dark type or vice versa).

Our study can be extended to other types of forms for nonlinearity modulation by identifying the corresponding modulation of the harmonic potential using the Riccati equation (A.8) and vice versa. The agreement between the numerical and analytical results indicates that the presented exact non-autonomous solution can well be utilized for the purpose of studying non-integrable multi-component GPE type systems with Rabi coupling. It is of future interest to investigate the effect of spatial and spatio-temporal modulations of nonlinearity. Works along these directions are in progress. This study may have ramifications in the design of matter-wave devices (e.g., switches) and also in experiments involving growth and oscillations of condensates.

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Appendix A. Mapping non-autonomous GPEs to integrable CNLS systems
In this appendix, we explicitly show that the non-autonomous two- and three-component GP systems (cf equations (1) and (12)) with Rabi coupling can be transformed to integrable CNLS systems, i.e., the Manakov model and its three-component generalization, with defocusing nonlinearities; this will be done by means of two successive transformations. In particular, in the case of the two-component BEC, first we apply the following unitary transformation to equation (1) with $g_{jl} = \rho(t)$ and $\sigma_1 = \sigma_2 = \sigma$:

$$
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \begin{pmatrix}
\cos(\sigma t) & -\sin(\sigma t) \\
-\sin(\sigma t) & \cos(\sigma t)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}.
$$

(A.1)

In the case of the three-component BECs, a similar transformation is applied to equation (12); this transformation is of the form [30, 32]:

$$
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} = \frac{1}{3}
\begin{pmatrix}
(2e^{i\sigma t} + e^{-2i\sigma t}) & (e^{-2i\sigma t} - e^{i\sigma t}) & (e^{-2i\sigma t} - e^{i\sigma t}) \\
(2e^{i\sigma t} + e^{-2i\sigma t}) & (2e^{i\sigma t} + e^{-2i\sigma t}) & (e^{-2i\sigma t} - e^{i\sigma t}) \\
(e^{-2i\sigma t} - e^{i\sigma t}) & (e^{-2i\sigma t} - e^{i\sigma t}) & (2e^{i\sigma t} + e^{-2i\sigma t})
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}.
$$

(A.2)

This way, we obtain the following set of non-autonomous equations without the Rabi coupling term:

$$
i\phi_{j,t} = -\frac{1}{2}\phi_{j,xx} + \left(\rho(t)\sum_{l=1}^{N}|\phi_l|^2 + V(x, t)\right)\phi_j,
$$

(A.3)
Then, by performing the similarity transformation

$$\phi_j(x, t) = \xi_j \sqrt{2} \rho(t) e^{i\theta} q_j(X, T),$$

(A.4)

where

$$\theta = -\frac{1}{2} \int_0^t \left[ \frac{d}{dt} \ln(\rho) \right] x^2 + 2 \xi_2 \xi_1^2 \left( \rho x - \xi_2 \xi_1^2 \int_0^t \rho^2 \, dt \right).$$

(A.5)

$$X = \sqrt{2} \xi_1 \left( \rho x - 2 \xi_2 \xi_1^2 \int_0^t \rho^2 \, dt \right),$$

(A.6)

$$T = \xi_1^2 \int_0^t \rho^2 \, dt,$$

(A.7)

with $\xi_1$ and $\xi_2$ being arbitrary real constants, equation (A.3) can be transformed into the system of equation (2) for the two-component system, or to equation (13) for the three-component system, with the condition

$$\frac{d\Lambda}{dt} = \Lambda^2 - \Omega^2(t) = 0,$$

(A.8)

where $\Lambda = (\rho_i/\rho)$. Equation (A.8) is nothing but a Riccati-type equation. Notice that a similar type of transformation has been reported for two-component BECs [26] but in the absence of Rabi coupling.

**Appendix B. One- and two-soliton solutions of the integrable two- and three-component defocusing CNLS systems**

**B.1. Bright–dark solitons in the two-component system (2)**

The BD one-soliton solution of equation (2), with bright (dark) part appearing in the $q_1$ ($q_2$) component, obtained by Hirota’s bilinearization method [27, 39]:

$$q_1(X, T) = \sqrt{|c_1|^2 \cos^2 \varphi_1 - k_{1R}^2 \text{sech}^2[k_{1R}(X - 2k_{1I}T) + R/2]} e^{i(\eta_1 + \theta)}$$

(B.1)

$$q_2(X, T) = -c_1 e^{i(\xi_1 + \phi_1)} \left( \cos \varphi_1 \text{tanh}[k_{1R}(X - 2k_{1I}T) + R/2] + i \sin \varphi_1 \right),$$

(B.2)

where $\varphi_1 = \tan^{-1}\left( \frac{k_{2R} - b_1}{k_{1R}} \right)$, $\xi_1 = -(b_1^2 + 2|c_1|^2)T + b_1 X$, $\alpha_1^{(1)} = a_1^{(1)} + i \delta_1^{(1)}$, $\theta = \tan^{-1}\left( \frac{a_1}{a_2} \right)$, $\eta_1 = k_1 X + i(k_2^2 - 2|c_1|^2)T$ and $e^R = -\frac{1}{(k_1 + k_2)|c_1|^2} \left( 1 - \frac{|c_1|^2}{k_1 + k_2} \right)^{-1}$. The bright part of the mixed soliton is characterized by three complex parameters $k_1$, $\alpha_1$, and $c_1$, and one real parameter $b_1$. Note that $\alpha_1$ does not affect the amplitude of the soliton. The dark soliton part of the mixed soliton also influences the bright part through the parameters $c_1$ and $b_1$. Also, the solution becomes non-singular only for the choice $|c_1|^2 > |k_1 - ib_1|^2$. As a consequence of this, it is impossible to make either one of the soliton part (i.e., dark/bright) completely zero. Thus, the bright and the dark parts of the mixed soliton co-exist but appear in separate components due to the presence of the other and can be viewed as ‘symbiotic solitons’, as mentioned in the introduction.

On the other hand, the explicit form of mixed two-soliton solution of the integrable 2-CNLS system (2) can be written as [27]:

$$q_1(X, T) = \frac{1}{D} \left( \alpha_1^{(1)} e^{\eta_1} + \alpha_2^{(1)} e^{\eta_2} + e^{\eta_1 + \eta_2 + \delta_1^{(1)}} + e^{\eta_1 + \eta_2 + \delta_2^{(1)}} \right).$$

(B.3)
\[ q_2(X, T) = \frac{c_1 e^{it}}{D} [1 + e^{\eta_1 + \eta_2 + Q_{11}} + e^{\eta_1 + \eta_3 + Q_{12}} + e^{\eta_2 + \eta_3 + Q_{21}} + e^{\eta_1 + \eta_3 + \eta_2 + Q_{12}} + e^{\eta_1 + \eta_2 + \eta_3 + Q_{21}} + e^{\eta_1 + \eta_2 + \eta_3 + Q_{11}}], \]  

(B.4)

where

\[ D = 1 + e^{\eta_1 + \eta_1 + R_1} + e^{\eta_1 + \eta_2 + k_1} + e^{\eta_2 + \eta_3 + k_2} + e^{\eta_1 + \eta_3 + k_2} + e^{\eta_1 + \eta_2 + k_1 + \eta_3 + k_2}. \]  

(B.5)

The various parameters in the above equation are defined below:

\[ \eta_j = k_j X + i (k_j^2 - \lambda) T, \quad \lambda = 2 |c_1|^2, \quad \epsilon^{Q_{ij}} = - \left( \frac{k_i - ib_1}{k_j^* + ib_1} \right) \mu_{ij}, \quad i, j = 1, 2, \]  

(B.7)

\[ \epsilon^{R_{ij}} = |k_1 - k_2|^2 \mu_{11} \mu_{12} \mu_{21} \mu_{22} (X_{12} X_{21} - X_{11} X_{22}), \]  

(B.9)

\[ \mu_{il} = \frac{1}{(k_i + k_l)^2}, \quad i, l = 1, 2, \]  

(B.10)

\[ \epsilon^{(1)} = (k_2 - k_1) \mu_{11} \mu_{21} (a_2^{(1)} \chi_{21} - a_1^{(1)} \chi_{11}), \]  

(B.11)

\[ \epsilon^{(2)} = (k_2 - k_1) \mu_{11} \mu_{22} (a_2^{(1)} \chi_{22} - a_1^{(1)} \chi_{12}), \]  

(B.12)

\[ \chi_{il} = \left[ (k_i + k_l)^2 \left( \frac{|c_1|^2}{(k_i - ib_1)(k_j^* + ib_1)} \right) \right], \quad i, l = 1, 2. \]  

(B.13)

B.2. Bright–bright–dark solitons in the three-component system (13)

The one-soliton solution of equation (13), in the form of a bright–bright–dark soliton, obtained by Hirota’s bilinearization method is given below:

\[ q_j(X, T) = A_j \sqrt{|c_1|^2 \cos^2 \varphi_j - k_{1j}^2 \sech[k_{1j}(X - 2k_{1j} T) + R/2]} e^{i \eta_j}, \quad j = 1, 2, \]  

(B.14)

\[ q_j(X, T) = -c_1 e^{i(\zeta_j + \psi_j)} (\cos \varphi_1 \tanh[k_{1j}(X - 2k_{1j} T) + R/2] + i \sin \varphi_1), \]  

(B.15)

where

\[ \varphi_j = \tan^{-1} \left( \frac{k_{1j} - b_1}{k_{1j}} \right), \quad A_j = \left( \frac{\alpha_j^{(1)}}{\sqrt{|\alpha_j^{(1)}|^2 + |\alpha_j^{(2)}|^2}} \right), \quad j = 1, 2, \]  

(B.16)

\[ e_R = \frac{\sum_{j=1}^2 (\alpha_j^{(1)} \alpha_j^{(1)*})}{(k_1 + k_2)^2} \left( \frac{|c_1|^2}{|k_1 - ib_1|^2} - 1 \right)^{-1}. \]  

(B.17)

The above soliton solution is characterized by three complex parameters $c_1$, $\alpha_j^{(1)}$, $\alpha_j^{(2)}$, and one real parameter $b_1$ along with the non-singular condition $|c_1|^2 > |k_1 - ib_1|^2$. Parameters $A_j$ may be viewed as spin-polarization in the case of spinor condensates [15, 16].

On the other hand, the respective two-soliton solution of equation (13) can be obtained as follows:

\[ q_j(X, T) = \frac{1}{D} (\alpha_j^{(1)} e^{\eta_1} + \alpha_j^{(2)} e^{\eta_2} + e^{\eta_1 + \eta_1 + \eta_2 + b_1} + e^{\eta_2 + \eta_2 + \eta_1 + b_1}), \quad j = 1, 2. \]  

(B.18)
For the analysis, we choose \( k \) performing the asymptotic analysis of the two-soliton solution given by equations (B.3)–(B.5).

**C.1. Asymptotic analysis of two-soliton solution of the Manakov system (2)**

The various parameters in the above equation are defined below

\[
\zeta_1 = -(b_1^2 + \lambda)T + b_1X, \quad (B.21)
\]

\[
e^{\delta_{ij}} = (k_2 - k_1)\mu_{11}\mu_{21}(a_2^{(j)}X_{21} - \alpha_1^{(j)}X_{11}), \quad (B.22)
\]

\[
e^{\delta_{ij}} = (k_2 - k_1)\mu_{12}\mu_{22}(a_2^{(j)}X_{22} - \alpha_1^{(j)}X_{12}), \quad j = 1, 2, \quad (B.23)
\]

\[
\kappa_{il} = \frac{1}{(k_i + k_l^*\chi_{il}),} \quad (B.24)
\]

\[
\chi_{il} = -\left[\frac{(k_i + k_l^*)}{\sum_{j=1}^{2}(\alpha_i^{(j)}}\alpha_j^{(j)*}) \left(1 - \frac{|c_1|^2}{(k_i - ib_l)(k_l^* + ib_l)}\right)\right], \quad i, l = 1, 2, \quad (B.25)
\]

The other parameters \( R_1, R_2, R_3, \delta_0, \) and \( \delta_0^* \) in the above equation are similar to that of the mixed two-soliton solution for the two-component case, given after equation (B.5), with the above redefinition of \( \chi_{il} \).

**Appendix C. Asymptotic analysis**

**C.1. Asymptotic analysis of two-soliton solution of the Manakov system (2)**

In this appendix, we present the asymptotic forms of the two colliding solitons obtained by performing the asymptotic analysis of the two-soliton solution given by equations (B.3)–(B.5).

For the analysis, we choose \( k_{1R} < k_{2R} \) and \( k_{1I} > k_{2I} \), without loss of generality.

**Before collision**

\[
d_1^{-} = A_1^{-}\sech\left(\eta_{1R} + \frac{R_1}{2}\right)\e^{\i\eta_{1I}}, \quad (C.1)
\]

\[
d_2^{-} = A_2^{-}\left[\cos \varphi_1\tanh\left(\eta_{1R} + \frac{R_1}{2}\right) + \i \sin \varphi_1\right], \quad l = 1, 2, \quad (C.2)
\]

**After collision**

\[
d_1^{+} = A_1^{+}\sech\left(\eta_{1R} + \frac{R_1 - R_{3-l}}{2}\right)\e^{\i\eta_{1I}}, \quad (C.3)
\]

\[
d_2^{+} = A_2^{+}\left[\cos \varphi_1\tanh\left(\eta_{1R} + \frac{R_3 - R_{3-l}}{2}\right) + \i \sin \varphi_1\right], \quad l = 1, 2, \quad (C.4)
\]

where \( \eta_{1R} = k_{1R}(X - 2k_{1I}T), \eta_{1I} = k_{1I}(X - (k_1^2 - k_{1I}^2)T), A_1^{-} = \frac{a_1^{(i)}}{A_1}\e^{-R/2}, A_1^{+} = -c_1\e^{(l+\varphi_0)}A_1^{-}, A_2^{(i)} = \frac{1}{2}\e^{-(R_{1R} + R_{3-I})/2 + \i\eta_{1I}} \) and \( A_2^{+} = c_1\e^{-\i(l+2\varphi_0+\varphi)} \). Here and in the following, as mentioned in the text, the superscript (subscript) in \( q \) (or \( \psi \)) and in \( A \) represents the number of soliton (component) while \(-\) (\( + \)) appearing in the corresponding quantities indicates their form before (after) collision. All the quantities found in equations (C.1)–(C.4) are defined below equation (B.5). From the above expressions it can be easily verified that \( |A_1^{(i)}|^2 = |A_1^{(i)}|^2, l, l = 1, 2. \)
C.2. Asymptotic analysis of two-soliton solution of integrable three coupled NLS system (13)

In this appendix, we present the results of the asymptotic analysis of two-soliton solution of equation (13) (see equations (B.18) and (B.19)) briefly. As before, here also without loss of generality, we choose $k_{1R} < k_{2R}, k_{1I} > k_{2I}$. The asymptotic forms are given below:

Before collision

\[ q_j^- = A_j^- \tanh \left( \frac{\eta_{1R} + R_j}{2} \right) e^{i\eta_1 j}, \quad (C.5) \]

\[ q_j^+ = A_j^+ \left[ \cos \theta_j \tan \left( \frac{\eta_{1R} + R_j}{2} \right) + \sin \phi_j \right], \quad j, l = 1, 2, \quad (C.6) \]

After collision

\[ q_j^+ = A_j^+ \tanh \left( \frac{\eta_{1R} + R_j - R_{3-l}}{2} \right) e^{i\eta_1 j}, \quad (C.7) \]

\[ q_j^- = A_j^- \left[ \cos \theta_j \tan \left( \frac{\eta_{1R} + R_j - R_{3-l}}{2} \right) + \sin \phi_j \right], \quad j, l = 1, 2, \quad (C.8) \]

where $\eta_{1R} = k_{1R}(X - 2k_{1I} T), \eta_{1I} = k_{1I} X - (k_{2R} - k_{1R}) - 2 |c_1|^2 T, A_j^- = \frac{a_j^-}{2} e^{-\frac{x}{2}}$, $A_j^+ = -c_j e^{i(|c_j|^2 + |c_1|^2)T}$, and $A_3^- = c_1 e^{i(2|c_2| + 2|c_1| + |c_3|)T}$. The quantities $R_j, R_2, R_3$ and $\delta_{1j}, j, l = 1, 2$, appearing in the above expressions are defined below equation (B.5) and $\phi_j = \tan^{-1} \left( \frac{\Delta_j}{k_j \eta_{1R}} \right)$, $l = 1, 2$.

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