The role of the Pauli–Lubański vector for the Dirac, Weyl, Proca, Maxwell and Fierz–Pauli equations

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Abstract
We analyze basic relativistic wave equations for the classical fields, such as Dirac’s equation, Weyl’s two-component equation for massless neutrinos and the Proca, Maxwell and Fierz–Pauli equations, from the viewpoint of the Pauli–Lubański vector and the Casimir operators of the Poincaré group. In general, in this group-theoretical approach, the above wave equations arise in certain overdetermined forms, which can be reduced to the conventional ones by a Gaussian elimination. A connection between the spin of a particle/field and consistency of the corresponding overdetermined system is emphasized in the massless case.

Keywords: Poincaré group, Pauli–Lubański vector, Dirac equation, Weyl equation, Proca equation, Maxwell equations

1. Introduction

All physically interesting representations of the proper orthochronous inhomogeneous Lorentz group (known nowadays as the Poincaré group) were classified by Wigner [70] and, since then, this approach has been utilized for the mathematical description of mass and spin of an elementary particle. To this end, the Pauli–Lubański pseudo-vector is introduced,

\[ w_{\mu} = \frac{1}{2} \varepsilon_{\mu
u
\rho
\tau} p^{\nu} M^{\rho\tau}, \quad p_{\mu} W^{\mu} = 0, \quad (1.1) \]

where \( p_{\mu} \) is the relativistic linear momentum operator and \( M^{\rho\tau} \) are the angular momentum operators, or generators of the proper orthochronous Lorentz group, with summation over repeated indices. (We use Einstein summation convention unless stated otherwise.) The mass and spin of a particle are defined in terms of two quadratic invariants (Casimir operators of the Poincaré group) as follows

\[ p^{2} = p_{\mu} p^{\mu} = m^{2}, \quad w^{2} = w_{\mu} w^{\mu} = -m^{2}s(s+1), \quad (1.2) \]

when \( m > 0 \) (see, for example, [3–5, 10, 36–38, 54–56, 58, 60] and the references therein; throughout the article we use the standard notations in the Minkowski space–time \( \mathbb{R}^{4} \) and the natural units \( c = \hbar = 1 \)).

For the massless fields, when \( m = 0 \), one gets \( w^{2} = p^{2} = pw = 0 \), and the Pauli–Lubański vector should be proportional to \( p^{2} \)

\[ w_{\mu} = \lambda p_{\mu}, \quad (1.3) \]

(acting on common eigenstates \([45, 55]\)). The number \( \lambda \) is sometimes called the helicity of the representation and the value \( s = |\lambda| \) is called the spin of a particle with zero mass \([10, 55, 56, 58]\). Although the concept of helicity is discussed in most textbooks on quantum field theory, a practical implementation of this definition of the spin of a massless particle deserves a certain clarification. As is shown in our previous article [32], in the case of the electromagnetic field

2 This assumption was made by Bargmann and Wigner [4] for the massless limit of the spinor wave equation for particles with an arbitrary integer or half-integer spin proposed by Dirac [13] (see also [17, 18, 31, 48] and the references therein). The pseudo-vector (1.1) was introduced, in a slightly different notation, by equations (4.a) and (4.b) of [4].
in vacuum, the constant $\lambda$ in the latter equation is fixed, otherwise violating the classical Maxwell equations. Thus, for the photon, or a harmonic circular classical electromagnetic wave\footnote{Multiple meanings of the word ‘photon’ are analyzed in [30].}, the latter equation allows one to introduce the field equations and spin, but not the helicity, when a certain choice of eigenstates is required. A similar situation occurs in the case of Weyl’s equation for massless neutrinos. (In [32], we do not discuss the equation for a graviton, another massless spin-2 particle; it will be analyzed elsewhere; see [28, 42, 44, 49] and our discussion in section 7.)

The theory of relativistic-invariant wave equations is studied, from different perspectives, in numerous classical accounts [4, 5, 7, 12, 13, 17, 18, 23, 31, 39, 48, 52, 68] (see also [8, 36–38] and the references therein). Nonetheless, in our opinion, the importance of the Pauli–Lubański vector for conventional relativistic equations, which allows one to derive all of them directly from the postulated transformation law of the corresponding classical field in pure group-theoretical terms, is not fully appreciated. In this article, we would like to start from Dirac’s relativistic electron, or any free relativistic particle with a non-zero mass and spin-1/2, which can be described by a bispinor wave function. Our analysis shows that an analog of the linear operator relation (1.3) takes the form,

$$w_{\mu} = \frac{1}{2}(p_{\mu} + m\gamma_0)\gamma_5$$

(1.4)

provided that the Dirac equation, $(\gamma^\mu p_\mu - m)\psi = 0$, holds, when the corresponding overdetermined system of equations is consistent. This automatically implies that $s = 1/2$, in the covariant form, by definition (1.2). (We were not able to find the operator relation (1.4) in the extensive literature on Dirac’s equation.)

In the rest of the article, a similar program is utilized, in a systematic way and from first principles, for other familiar relativistic wave equations. Once again, we postulate the transformation law of the field in question (a law of nature) and, with the help of the corresponding Lorentz generators, evaluate the action of the Pauli–Lubański vector on the field in order to compute, eventually, not one, but both Casimir operators (1.2). If a linear relation, similar to (1.3) or (1.4), does exist, one obtains an overdetermined PDE system, which can be reduced to the corresponding relativistic wave equation by a matrix version of Gaussian elimination [22]. We show that this approach allows one to derive equations of motion for the most useful classical fields, including the Weyl, Proca, Fierz–Pauli and Maxwell equations in vacuum, as a statement of consistency for the original overdetermined systems. At the moment, we shall not discuss relativistic wave equations for particles with an arbitrary spin, such as the Bargmann–Wigner equations, Majorana equations and/or the (first order) Duffin–Kemmer equations which also describe spin-0 and spin-1 fields (see, for example [5, 9, 24, 59] for more details; the case of the Klein–Gordon equation, or the relativistic Schrödinger equation [57], is, of course, obvious).

We have also entirely concentrated on four-dimensions. A more general group-theoretical approach to the relativistic wave equations, which allows one to include higher dimensions and spins, is formulated in [5] with the help of induced representations of the semi-direct products of separable, locally compact groups. Spinors in arbitrary dimensions are also discussed in [60].

2. Dirac equation

In this section, for the reader’s convenience, we summarize basic facts about Dirac’s equation and then discuss its relation with the Pauli–Lubański vector (1.4). To this end, a familiar bispinor representation of the proper orthochronous Lorentz group $SO(1, 3)$ is used.

2.1. Gamma matrices, bispinors and transformation laws

We shall use the following Dirac matrices: $\gamma^\mu = (\gamma^0, \gamma)$, $\gamma_0 = g_{\mu\nu}\gamma^\nu = (\gamma^0, -\gamma)$, and $\gamma_5 = -\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, where

$$\gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(2.1)

and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the standard $2 \times 2$ Pauli matrices [47, 62]. The familiar anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g_{\mu\nu}, \quad \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0$$

(2.2)

hold. (Most of the results here will not depend on a particular choice of gamma matrices, but it is always useful to have an example in mind.) The four-vector notation, $x^\mu = (t, \mathbf{r})$, $\partial_\mu = \partial/\partial x^\mu$, and $\partial^\alpha = g^{\alpha\mu}\partial_\mu$ in natural units $c = \hbar = 1$ with the standard metric, $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, in the Minkowski space–time $\mathbb{R}^4$ are utilized throughout the article [6, 9, 10, 42, 59].

In this notation, the transformation law of a bispinor wave function\footnote{The relativistic wave equation for a massive spin-1/2 particle was proposed by Dirac [12], when only tensor representations of the Lorentz group were known. Thus, the problem of covariance of Dirac’s equation gave rise to a new class of representations of the Lorentz group, namely, the spinor representations [5].}

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4,$$

(2.3)

under a proper Lorentz transformation, is given by

$$\psi'(x') = S_L\psi(x), \quad x' = Lx,$$

(2.4)

together with the rule

$$S_L^{-1}\gamma^\mu S_L = L^\mu\gamma^\nu,$$

(2.5)

for the sake of covariance of the celebrated Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0.$$
As is well known, a general solution of the latter matrix equation has the form
\[
S = S_\lambda = \exp\left(-\frac{1}{4} \theta_{\mu
u} \Sigma^{\mu\nu}\right), \quad \theta_{\mu\nu} = -\theta_{\nu\mu},
\]
\[
\Sigma^{\mu\nu} = (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/2
\]
(2.7)
with summation over every two repeated indices; see, for example, [27, 43]) and, in turn
\[
S^{-1}_\lambda \Sigma^{\mu\nu} S_\lambda = \Lambda^\mu_\nu \Lambda^\nu_\rho \Sigma^{\rho\sigma}.
\]
(2.8)
In explicit form
\[
\Sigma^{\mu\nu} = \frac{1}{2} \left( \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) = \left( \begin{array}{cccc} 0 & \alpha_q & 0 & 0 \\ -\alpha_p & -i e_{pqr} \Sigma^r & 0 & i e_{pqr} \Sigma^r \\ 0 & -i \Sigma_3 & 0 & -i \Sigma_4 \\ -i \Sigma_3 & i \Sigma_4 & 0 & 0 \end{array} \right),
\]
(2.9)
where, by definition
\[
\Sigma = \left( \begin{array}{cc} 0 & \sigma \\ \sigma & 0 \end{array} \right), \quad \alpha = \left( \begin{array}{c} 0 \\ \sigma \end{array} \right)
\]
(2.10)
Their familiar product identities
\[
\Sigma_p \Sigma_q = i e_{pqr} \Sigma^r + \delta_{pq}, \quad \alpha_p \alpha_q = i e_{pqr} \alpha^r + \delta_{pq},
\]
\[
\alpha_p \Sigma_q = \Sigma_p \alpha_q = i e_{pqr} \alpha^r + \delta_{pq} \gamma^5,
\]
(2.11)
hold.

Setting \( n = \{e_1, e_2, e_3\} \) for each of the unit vectors in the directions of the mutually orthogonal coordinate axes, one can write in compact form
\[
S_R = e^{i(n \cdot \Sigma)/2} = \cos \frac{\vartheta}{2} + i(n \cdot \Sigma) \sin \frac{\vartheta}{2},
\]
(\( n \cdot \Sigma \))^2 = I

(2.12)
and
\[
S_L = e^{-i(n \cdot \alpha)/2} = \cosh \frac{\vartheta}{2} - i(n \cdot \alpha) \sinh \frac{\vartheta}{2},
\]
(\( n \cdot \alpha \))^2 = I

(2.13)
with \( \vartheta = \nu \), in the cases of rotations and boosts, respectively [5, 43].

The important dual four-tensor identities
\[
i e_{\mu\sigma\tau\rho} \Sigma^{\rho\sigma\tau} = 2 \gamma_5 \Sigma_{\mu\rho}, \quad i e_{\mu\sigma\tau\rho} \gamma_5 \Sigma^{\rho\sigma\tau} = 2 \Sigma^{\rho\sigma\tau},
\]
(2.14)
can be directly verified. Here, \( e_{\mu\sigma\tau\rho} = -e_{\rho\tau\sigma\mu} \) and \( e_{0123} = +1 \) is the Levi–Civita symbol [10, 21].

For the conjugate bispinor
\[
\bar{\psi}(x) = \psi^\dagger(x) \gamma^0, \quad \bar{\psi}'(x') = \bar{\psi}(x)x', \quad x' = \Lambda x,
\]
(2.15)
the Dirac equation (2.6) takes the form
\[
i \not{\partial} \bar{\psi} \gamma^\mu + m \bar{\psi} = 0.
\]
(2.16)
(For more details see classical accounts [1, 6, 9, 16, 27, 43, 50, 55, 58, 59, 66].)

**Examples.** In particular, for the boost in the plane \((x^0, x^1)\),
\[
S_L = e^{-i(\vartheta/2) \Sigma^{01}} = e^{-i(\vartheta/2) \alpha_1} = \cosh \frac{\vartheta}{2} - \alpha_1 \sinh \frac{\vartheta}{2}
\]
(2.17)
with \( \tanh \vartheta = \nu \), one can easily verify by matrix multiplication that
\[
\begin{pmatrix}
\gamma^0 \\
\gamma^1 \\
\gamma^2 \\
\gamma^3
\end{pmatrix}
\]
\[
e^{-i(\vartheta/2) \alpha_1} =
\begin{pmatrix}
\cosh \vartheta & -\sinh \vartheta \\
-\sinh \vartheta & \cosh \vartheta \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
(2.18)
In a similar fashion, for the rotation in the plane \((x^1, x^2)\):
\[
S_R = e^{-i(\vartheta/2) \Sigma^{12}} = e^{i(\vartheta/2) \Sigma^1} = \cos \frac{\vartheta}{2} + i \Sigma_3 \sin \frac{\vartheta}{2}
\]
(2.19)
and
\[
\begin{pmatrix}
\gamma^0 \\
\gamma^1 \\
\gamma^2 \\
\gamma^3
\end{pmatrix}
\]
\[
e^{i(\vartheta/2) \alpha_1} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \vartheta & \sin \vartheta & 0 \\
0 & -\sin \vartheta & \cos \vartheta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
(2.20)
(See [5, 43] for further details.)

### 2.2. Generators and commutators

In the fundamental representation of the proper orthochronous Lorentz group, we shall choose the following six \( 4 \times 4 \) real-valued matrices \((\alpha, \beta = 0, 1, 2, 3)\) fixed with no summation
\[
\Lambda(\theta, \alpha) = \exp(-\theta \alpha \cdot m^{\alpha\beta}), \quad m^{\alpha\beta} = -m^{\beta\alpha},
\]
\[
(\alpha^{0\beta})^\mu = g^{0\mu} \delta^\beta_\nu - g^{\beta\nu} \delta^\alpha_\mu
\]
(2.21)
for the corresponding one-parameter subgroups of rotations and boosts [10, 32, 42, 58]. Then, differentiation of a particular expression (for the corresponding tensor operator [5])
\[
e^{i(\vartheta/2) \Sigma^{01}/2} \gamma^\mu e^{-i(\vartheta/2) \Sigma^{01}/2} = (e^{-i(\vartheta/2) \mu \nu})^\gamma_\nu \gamma^\mu,
\]
(2.22)
at \( \theta = 0 \) results in
\[
[\Sigma^{0\beta}, \gamma^\mu] = \Sigma^{0\beta} \gamma^\mu - \gamma^\mu \Sigma^{0\beta} = 2 (g^{0\mu} \gamma^\beta - g^{\beta\mu} \gamma^0),
\]
(2.23)
which can be independently verified with the help of (2.2).

In a similar fashion, the action of four-angular momentum operators
\[
M^{\alpha\beta} = x^\beta \partial^\alpha - x^\alpha \partial^\beta, \quad \partial^0 = g^{\alpha0} \partial_\alpha,
\]
(2.24)
5 We follow [32]. Traditionally, \( M^{\alpha\beta} \rightarrow -i M^{\alpha\beta} \); see, for example [10].
on Dirac’s bispinors (2.3) can be derived directly from the transformation law as follows

\[
M^{\alpha \beta} \psi := - \left( \frac{d}{d\theta_{\alpha \beta}} \right) \psi \left( (\lambda_\mu^\nu (\theta_{\alpha \beta}) x^\nu) \right) \left| _{\theta_{\alpha \beta} = 0} \right.
\]

\[
= \left( \frac{d}{d\theta_{\alpha \beta}} \right) e^{-\theta_{\alpha \beta} \sum_{\beta}} \psi(x) = \frac{1}{2} \sum_{\beta} \delta^{\alpha \beta} \psi\left| _{\theta_{\alpha \beta} = 0} \right.
\]

(2.25)

(see also [27] for a slightly different derivation). A familiar commutator

\[
[\Sigma^{\alpha \beta}, \Sigma^{\rho \tau}] = 2(g^{3\rho \tau} \Sigma^{\alpha \beta} - g^{3\beta \tau} \Sigma^{\alpha \rho} + g^{3\alpha \rho} \Sigma^{\beta \tau} - g^{3\tau \rho} \Sigma^{\alpha \beta}),
\]

(2.26)

can be readily verified with the help of (2.23) and/or independently derived from (2.8). These results are independent of our choice of the gamma matrices representation.

2.3. Balance conditions and energy–momentum tensors

We shall use a familiar notation for the partial derivatives [10]

\[
D^\rho \psi(x) = \frac{\partial \rho \psi}{\partial x^\rho}, \quad D^\rho \psi(x) := \psi(x)
\]

(2.27)

where \( p = (p_0, p_1, p_2, p_3) \) is an ordered set of non-negative integers. It follows from the Dirac equations (2.6) and (2.16) that

\[
\partial_\mu \left[ (D^\mu \psi(x)) \gamma^\mu ((D^\rho \psi(x)) \right] = 0,
\]

(2.28)

or, for a finite multi-sum

\[
\partial_\mu \left[ \sum_{p,q} \epsilon_{p,q} D^\mu \gamma^\mu D^\rho \psi \right] = 0,
\]

(2.29)

which can be thought of as a ‘master’ differential balance condition set.

Indeed, in view of \( D^\rho \partial_\rho = \partial_\rho D^\rho \), one gets

\[
i \gamma^\mu \partial_\mu (D^\rho \psi) = \gamma^\rho (D^\rho \psi), \quad i \partial_\mu (D^\mu \gamma^\rho) = -m(D^\rho \gamma^\rho).
\]

(2.30)

Let us multiply the first (second) equation by \( D^\rho \psi \) (\( D^\rho \psi \)) from the left (right) and add the results. Then

\[
i \partial_\mu (D^\rho \gamma^\mu D^\rho \psi) = i \partial_\mu (D^\rho \gamma^\rho) \gamma^\mu D^\rho \psi + i D^\rho \gamma^\mu \partial_\mu (D^\rho \psi)
\]

\[
= m \left( -D^\rho \psi D^\rho \psi + D^\rho \gamma^\rho D^\rho \psi \right) \equiv 0.
\]

Among important special cases of (2.28) are the following important identities

\[
\partial_\mu j^\mu(x) = 0, \quad j^\mu(x) = \overline{\psi}(x) \gamma^\mu \psi(x),
\]

(2.31)

corresponding to the total charge conservation and

\[
\partial_\mu [\overline{\psi} (x) \gamma^\mu \partial_\mu \psi(x)] = 0, \quad \partial_\mu [\overline{\partial_\mu \psi}(x) \gamma^\mu \psi(x)] = 0.
\]

(2.32)

Therefore, one can introduce the energy–momentum tensor, such that \( \partial_\rho T^\rho_\mu(x) = 0 \), in two different forms

\[
T^\rho_\mu = i \overline{\psi} \gamma^\mu \partial_\rho \psi, \quad T^\rho_\mu = m \overline{\psi} \gamma^\mu \psi
\]

(2.33)

and/or

\[
T^\rho_\mu = i \overline{\psi} \gamma^\mu (\partial_\rho \psi) - (\partial_\rho \overline{\psi}) \gamma^\mu \psi, \quad T^\rho_\mu = m \overline{\psi} \gamma^\mu \psi.
\]

(2.34)

As is well-known, all quantities of physical interest can be derived from the energy–momentum tensor [9, 58].

2.4. Variants of Dirac’s equation

In view of (2.2) and (2.6)

\[
i \gamma^\mu \partial_\mu \psi = m \gamma^0 \psi,
\]

(2.35)

\[
\gamma^\mu \gamma^\nu = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) + \frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \Sigma^{\mu \nu} + g^{\mu \nu}
\]

(2.36)

and, as a result, we obtain an overdetermined but very convenient form of Dirac’s system

\[
i (\Sigma^{\mu \nu} + g^{\mu \nu}) \partial_\mu \psi = m \gamma^0 \psi.
\]

(2.37)

On the one hand, in a 3D ‘vector’ form

\[
i \partial_\rho \psi = (-i \alpha \gamma^\rho + m \beta) \psi,
\]

(2.38)

\[
\beta = \gamma^0 (h = c = 1), \quad \partial_0 = \partial / \partial t
\]

and

\[
-i \partial_\rho \alpha \gamma^\rho \psi + (\nabla \times \Sigma) \psi = (i \nabla + m \gamma) \psi.
\]

(2.39)

It is worth noting that the latter vectorial equation in our overdetermined system (2.37) and (2.38) can be obtained by matrix multiplication, from the first one, in view of familiar relations

\[
\alpha_p \alpha_q = i e_{pqr} \Sigma_r + \delta_{pq}, \quad \alpha \beta = -\gamma.
\]

(2.40)

On the other hand, by letting \( p_\mu = i \partial_\mu \), one gets

\[
(\Sigma^{\mu \nu} + g^{\mu \nu}) p_\mu \psi = m \gamma^\mu \psi
\]

(2.41)

and applying the momentum operator \( p_\mu \) to the both sides:

\[
p^2 \psi = (\Sigma^{\mu \nu} + g^{\mu \nu}) p_\mu p_\nu \psi = m \gamma^\mu p_\mu \psi,
\]

(2.42)

In view of \( \Sigma^{\mu \nu} p_\mu p_\nu \psi = 0 \). If \( p^2 \psi = m^2 \psi \), we derive, once again, that \( \gamma p_\mu \psi = mp \psi \). Therefore both forms of the Dirac system, (2.6) and (2.36), are equivalent and every component of the bispinor (2.3) does satisfy the d’Alembert equation,

\[
(\partial^\rho \partial_\rho + m^2 \psi) \psi = (\partial_\rho^2 + \Delta + m^2 \psi) = (\Box + m^2 \psi) = 0,
\]

(2.43)

as required by (1.2).

In a similar fashion, for the conjugate bispinor (2.15) one can obtain

\[
i \partial_\mu (\Sigma^{\mu \nu} - g^{\mu \nu}) = m \overline{\psi} \gamma^\mu
\]
and our equations (2.36) and (2.43) results in the following balance relation
\[ i\partial_\nu (\nabla^\mu \psi) + i[\nabla^\mu, \psi] = 2m\nabla^\nu \psi, \] (2.44)

which, in turn, implies a familiar conservation law, \( \partial_\mu (x) = 0, \) for the four-current, \( J^\mu (x) = \nabla^\mu \psi(x), \) in view of \( \nabla^\mu \nabla_\mu \psi \equiv 0 \) and \( \nabla_\mu [\nabla^\mu \psi - (\nabla^\mu \psi)] \equiv 0. \) (The latter equation gives a differential balance condition on its own.)

The following identity
\[ i\partial_\nu (\nabla^\mu \psi) = 2i\nabla^\mu \psi - (\nabla^\mu \psi) \] (2.45)
can be obtained with the help of (2.23), (2.36) and (2.43). The latter shows how the difference between two forms of the energy–momentum tensor (2.33) and (2.34) can be written as the four-divergence of a given tensor. Moreover, one can write
\[ T^\mu_\nu = i\nabla^\nu \partial_\mu \psi = mg^\mu_\nu \nabla^\nu \psi + m\nabla^\mu \psi - i\nabla^\mu \nabla^\nu \psi \] (2.46)

and
\[ T^\mu_\nu = i/2 [\nabla^\nu (\nabla^\mu \psi) - (\nabla^\mu \nabla^\nu \psi)] = mg^\mu_\nu \nabla^\nu \psi - i/2 [\nabla^\nu \nabla^\mu \nabla^\nu \psi + (\nabla^\mu \nabla^\nu \psi)] \] (2.47)
in view of (2.36) and (2.43).

2.5. Covariance and transformation of generators

The relativistic invariance of the Dirac equation is a fundamental consequence of (2.4)–(2.6); see for example, [6, 9, 27, 43, 58, 59]. Covariance of system (2.36) can be derived, in a similar fashion, by invoking (2.8). The details are left to the reader (see also section 5.3).

It is worth noting that from the four-tensor character of \( \Sigma^\mu_\nu, \) in (2.8), follow the transformation laws for the generators of rotations and boosts. Let us assume that the velocity vector \( v, \) for going over to a moving frame of reference, has the direction of one of the coordinate axes, say \( [e_1, e_2, e_3]. \) Consider also ‘orthogonal decompositions’, \( \Sigma = \Sigma_\perp + \Sigma_\parallel \) and \( \alpha = \alpha_\parallel + \alpha_\perp, \) in the corresponding parallel and perpendicular directions, respectively. Thus, under the Lorentz transformation,
\[ S^\dagger_\perp \Sigma_\parallel S_\parallel = \Sigma_\perp, \quad S^\dagger_\parallel \alpha_\parallel S_\parallel = \alpha_\parallel \] (2.48)

and
\[ S^\dagger_\parallel \Sigma_\parallel S_\parallel = \frac{\Sigma_\parallel - i(v \times \alpha)}{\sqrt{1 - v^2}}, \quad S^\dagger_\parallel \alpha_\parallel S_\parallel = \frac{\alpha_\parallel - i(v \times \Sigma)}{\sqrt{1 - v^2}}, \] (2.49)

by analogy with the transformations of electromagnetic fields in classical electrodynamics [32, 33, 59, 61].

The corresponding invariants [43] are given by
\[ I_1 = \Sigma_\mu_\nu \Sigma^\mu_\nu = -2(\alpha^2 + \Sigma^2) = -12 \] (2.50)

and
\[ I_2 = \epsilon^\mu_\nu_\alpha_\beta \Sigma^\mu_\nu \Sigma_\alpha_\beta = -8i\alpha \cdot \Sigma = -2i\Sigma_\alpha_\beta \Sigma^{\alpha_\beta} \gamma_5 = -2iv_5 \gamma_5 = 24iv_5, \] (2.51)
in view of the first identity (2.14). The invariant nature of the helicity and relativistic rotation of the particle spin [43] can be naturally explained from these transformations.

Example. If \( v = ve_1, \) we set \( \Sigma_i = \Sigma_1, \Sigma_\parallel = \Sigma_2, \Sigma_\perp = \Sigma_3 \) and \( \alpha_\parallel = \alpha_1, \alpha_\perp = \{\alpha_2, \alpha_3\} \) for the boost \( S_L \) given by (2.17). By the transformation law (2.48), one gets \( S^\dagger_\perp \Sigma_1 S_\perp = \Sigma_1, \) \( S^\dagger_\perp \alpha_1 S_\perp = \alpha_1, \) which is evident. In view of (2.49), the following matrix identities,
\[ S^\dagger_\perp \Sigma_2 S = \frac{\Sigma_2 + iv\alpha_3}{\sqrt{1 - v^2}}, \quad S^\dagger_\perp \alpha_2 S = \frac{\alpha_2 - iv\Sigma_2}{\sqrt{1 - v^2}}, \] (2.52)

and
\[ S^\dagger_\perp \alpha_3 S = \frac{\alpha_3 - iv\Sigma_2}{\sqrt{1 - v^2}}, \] (2.53)

hold. Indeed, in the first relation,
\[ S^\dagger_\perp \Sigma_2 S = \left( \cos \frac{\theta}{2} + \alpha_1 \sinh \frac{\theta}{2} \right) \left( \cos \frac{\theta}{2} - \alpha_1 \sinh \frac{\theta}{2} \right) = \Sigma_2 \cos^2 \frac{\theta}{2} + \alpha_1 \Sigma_2 - \Sigma_2 \alpha_1 \cosh \frac{\theta}{2} - \alpha_1 \Sigma_2 \alpha_1 \sinh \frac{\theta}{2} = \Sigma_2 \cosh \theta + i\Sigma_2 \sinh \theta = \frac{\Sigma_2 + iv\alpha_3}{\sqrt{1 - v^2}}, \] provided \( \tanh \theta = v. \) Verifications of the remaining identities are similar.

2.6. Hamiltonian and energy balance

The energy–momentum (density) four-vector is given by \( T^\mu_\nu, \) and, in view of (2.37) and (2.46), one gets
\[ T^\mu_\nu = i\nabla^\mu \partial_\nu \psi = m\nabla^\mu \psi - i\nabla^\mu \gamma_5 \nabla^\nu \partial_\nu \psi = \psi (-i\alpha \cdot \nabla + m\beta) \psi = \psi H \psi. \] (2.54)

Here,
\[ i\partial_\nu \psi = H \psi, \quad H = -i\alpha \cdot \nabla + m\beta, \] (2.55)

which presents a familiar Hamiltonian form of the Dirac equation. The differential balance equation take the form
\[ \partial_\nu (\psi H \psi) + \text{div} (\psi \alpha H \psi) = 0, \] (2.56)

where \( \alpha H = \nabla \times \Sigma - i\nabla \times \gamma_5 \) in view of (2.39).

2.7. The Pauli–Lubański vector and Dirac’s equation for a free particle

One can easily see that equation (2.36) is related to the Pauli–Lubański vector in view of the dual identities (2.14). Indeed
\[ \Sigma^\mu_\nu = \frac{i}{2} \epsilon^{\mu_\nu_\alpha_\beta} (\gamma_5 \Sigma_\alpha_\beta), \quad \gamma_5 \Sigma_\alpha_\beta = \Sigma_\alpha_\beta \gamma_5 \] (2.57)
and
\[ \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \gamma^5 \Sigma_{\sigma\tau} p_\nu \psi = -g^{\mu\nu} (p_\nu - m \gamma_0) \psi. \]

By ‘index manipulations’
\[ \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} (i \Sigma^{\sigma\tau}) \psi = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \partial^\nu \Sigma^{\sigma\tau} \psi \]
\[ = \gamma_5 (i \partial_\nu - m \gamma_0) \psi = (i \partial_\nu + m \gamma_0) \gamma_5 \psi \]
with the help of familiar properties of the gamma matrices, namely, \( \gamma_5^2 = I \) and \( \gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5 \). As a result, we arrive at the following equations
\[ \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \partial^\nu (\gamma^\sigma \gamma^\tau) \psi = (i \partial_\nu + m \gamma_0) \gamma_5 \psi, \]
with summation over any two repeated indices.

The latter equation can also be obtained in view of (2.25), by letting \( p_\nu = i \partial_\nu \) in operator relation (1.4). Once again, our goal is to emphasize that both overdetermined systems (2.36) and (2.59), which are related to the Pauli–Lubanski vector, are equivalent to Dirac’s equation in vacuum (2.6).

In components, by (2.59), for the rotations \( \Sigma \) and boosts \( \alpha \) the following standard equations hold
\[ i(\nabla \cdot \Sigma) \psi = (i \partial_0 + m \gamma_0) \gamma_5 \psi \]
and
\[ i \partial_0 \Sigma \psi - (\nabla \times \alpha) \psi = (i \nabla - m \gamma_5) \gamma_5 \psi, \]
respectively. Once again, this system is overdetermined and by a proper matrix multiplication, each of the four equations (2.60) and (2.61) can be reduced to a single Dirac’s equation. We leave the details to the reader.

2.8. Relativistic definition of spin for Dirac particles

In view of (1.2) and (1.4), one gets
\[ 4w_i w^\mu = (p_\mu + m \gamma_0) \gamma_5 (p^\mu + m^\mu) \gamma_5 \]
\[ = (p_\mu + m \gamma_0) \gamma_5^2 (p^\mu + m \gamma_5) \]
\[ = p_\mu p^\mu - m^2 \gamma_5 \gamma_\mu = -3 m^2, \]
where \( s(s + 1) = 3/4 \), or \( s = 1/2 \), in covariant form.

On the other hand, introducing the familiar generators \( M = (i/2) \Sigma \) and \( N = (1/2) \alpha \) for the rotations and boosts, respectively, one gets
\[ N^2 = -M^2 = 3/4, \quad M \cdot N = i(3/4) \gamma_5. \]

In view of \( \Sigma \gamma_5 = \gamma_5 \Sigma = \alpha \), in the complex space of the bispinors under consideration, we arrive at
\[ M \pm iN = \frac{1}{2} (1 \mp \gamma_5) \Sigma \]
and the Casimir operators of the proper orthochronous Lorentz group are given by \( (M \pm iN)^2/4 = -3(1 \mp \gamma_5)^2/16 = -3(1 \mp \gamma_0)/8 \). Here, \( M^2 = -s(s + 1) = -3/4 \), which implies, once again, that the spin is equal to 1/2 (we have chosen real-valued boost generators; see also [2, 4, 5, 10, 23, 54, 58, 66] for more details on the Lorentz group representations).

Miscellaneous. In addition
\[ \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} \partial^\nu (\gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau) = 3i \gamma^5 \gamma_\mu \partial^\nu \psi = 3m \gamma_5 \psi, \]
in view of familiar relations
\[ \gamma^5 = \frac{i}{4!} \epsilon_{\mu\nu\sigma\tau} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau, \]
\[ \gamma^5 \gamma_\mu = \frac{i}{3!} \epsilon_{\mu\nu\sigma\tau} \gamma^\nu \gamma^\sigma \gamma^\tau \]
(see, for example, [43]).

2.9. Massless limit

Under the transformation,
\[ \gamma^\mu \rightarrow \gamma^\mu = U \gamma^\mu U^{-1}, \quad \psi \rightarrow \psi' = (\phi) = U \psi, \]
\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = U^{-1} \]
Dirac’s equation (2.6) takes a familiar block form
\[ i \begin{pmatrix} 0 & \partial_0 - \sigma \cdot \nabla \\ \partial_0 + \sigma \cdot \nabla & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = m \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]
(see, for example [16, 43, 50]. As \( m \rightarrow 0 \), this system decouples,
\[ \partial_0 \phi + (\sigma \cdot \nabla) \phi = 0, \quad \partial_0 \chi - (\sigma \cdot \nabla) \chi = 0, \]
resulting in Weyl’s two-component equations for massless neutrinos.

3. Weyl equation for massless neutrinos

The complex unimodular matrix group \( SL(2, \mathbb{C}) \) is a two-fold universal covering group of the Lorentz group \( SO(1, 3) \) (see, for example, [5, 10, 23, 54]). In this section, we shall use this connection in order to analyze the two-component spinor field associated with Weyl’s equation.

3.1. Rotations, boosts and their generators

Let us consider the fundamental representation of \( SL(2, \mathbb{C}) \), namely, we take a spinor field
\[ \phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathbb{C}^2, \]
and postulate the transformation law under the proper orthochronous Lorentz group as follows
\[ \phi'(x') = S_\Lambda \phi(x), \quad x' = \Lambda x, \quad S_\Lambda = \exp \left( \frac{i}{4} \theta_{\mu \nu} \Sigma^{\mu \nu} \right). \]

Explicitly, these transformations include rotations
\[ S_R = e^{i \theta (n \cdot \sigma)/2} = \cos \frac{\theta}{2} + i (n \cdot \sigma) \sin \frac{\theta}{2}, \]
\[ (n \cdot \sigma)^2 = I. \]
about the coordinate axes \( n = \{e_1, e_2, e_3\} \), and boosts
\[
S_k = e^{-i(n \cdot \sigma)/2} = \cosh \frac{\theta}{2} - (n \cdot \sigma) \sinh \frac{\theta}{2}, \quad n = \frac{v}{v}
\]
in the directions \( n = \{e_1, e_2, e_3\} \), respectively, when the familiar relations
\[
v = \tanh \theta, \quad \cosh \theta = \frac{1}{\sqrt{1 - v^2}}, \quad \sinh \theta = \frac{v}{\sqrt{1 - v^2}} \quad (c = 1),
\]
hold. (Here, \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices with the products
\[
\sigma_\rho \sigma_\xi = \epsilon_{\rho\xi\sigma} \sigma_\sigma + \delta_{\rho\xi} \quad \text{and} \quad [e_\rho]_{\rho=1,2,3} \text{ is an orthonormal basis in } \mathbb{R}^3.
\]

The action of the generators \( M^\alpha_\beta = \chi^\beta \partial^\alpha - \chi^\alpha \partial^\beta \) takes the form
\[
M^\alpha_\beta \phi(x) = -\left( \frac{d}{d\theta_{\alpha_\beta}} \phi'(Ax) \right) \bigg|_{\theta_{\alpha_\beta} = 0}
= \left( -\frac{d\phi}{d\theta_{\alpha_\beta}} \right) \bigg|_{\theta_{\alpha_\beta} = 0}
(\phi(x) = -\frac{1}{2} \Sigma^{\alpha_\beta}(\phi(x))
\]
for the corresponding one-parameter subgroups:
\[
S_\alpha = \exp(\theta_{\alpha_\beta} \Sigma^{\alpha_\beta}/2) \quad (\alpha, \beta = 0, 1, 2, 3 \text{ are fixed with no summation}).
\]

In block form, the generators of this spinor representation are given by
\[
\Sigma^{\alpha_\beta} = -\Sigma^{\beta\alpha} = \begin{pmatrix} 0 & -\sigma_q & & \\ \sigma_p & i \epsilon_{pqr} \sigma_r & & \\ & & 0 & -\sigma_q \\ & & \sigma_p & i \epsilon_{pqr} \sigma_r \end{pmatrix}
\]
and the following self-duality identity holds
\[
\epsilon_{\mu\nu\sigma} \Sigma^{\sigma\tau} = 2i \Sigma_{\mu\nu} = 2ig_{\mu\nu} g_{\sigma\tau} \Sigma^{\sigma\tau},
\]
where
\[
\Sigma_{\mu\nu} = g_{\sigma\tau} g_{\rho\xi} \Sigma^{\mu\nu} = \begin{pmatrix} 0 & \sigma_q & & \\ -\sigma_p & i \epsilon_{pqr} \sigma_r & & \\ & & 0 & \sigma_q \\ & & -\sigma_p & i \epsilon_{pqr} \sigma_r \end{pmatrix}
\]
and
\[
\begin{pmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ -\sigma_2 & \sigma_3 & 0 & \sigma_1 \\ -\sigma_3 & \sigma_2 & \sigma_1 & 0 \end{pmatrix}
\]
\[
\text{stated here for the reader’s convenience.}
\]

### 3.2. The Pauli–Lubański vector and Weyl’s equation

Letting \( \lambda = -1/2 \) in (1.3), one gets
\[
\epsilon_{\mu\nu\sigma} (i\partial\phi)(-i\Sigma^{\sigma\tau}) \phi = -\frac{1}{2} \epsilon_{\mu\nu\sigma} \partial^\mu \Sigma^{\sigma\tau} \phi = -i\partial^\rho \phi
\]
by (3.6). With the help of (3.8), we finally arrive at the following overdetermined system
\[
\Sigma^{\mu\nu} \partial_\nu \phi = \partial^\mu \phi,
\]
which takes the explicit form
\[
(\sigma \cdot \nabla) \phi = \partial_0 \phi,
\]
(3.11)

The latter vectorial equation can be obtained from the first one by matrix multiplication with the help of familiar products of the Pauli matrices. Moreover, in (1.3), only the value \( \lambda = -1/2 \) results in a consistent system, defining the spin as \( s = |\lambda| = 1/2 \).

Thus, the relativistic two-component Weyl equations for a massless particle with the spin-1/2, namely, \( \partial_0 \phi + (\sigma \cdot \nabla) \phi = 0 \), can be derived from the representation theory of the Poincaré group with the aid of the Pauli–Lubański vector.

#### 3.3. Covariance

Equations (3.11) are covariant under a proper Lorentz transformation. In view of the laws (3.2) one gets
\[
S_\lambda ^{-1} \Sigma^{\sigma\tau} S_\lambda = \Lambda^\sigma_\mu \Lambda^\tau_\nu \Sigma^{\mu\nu}
\]
and
\[
\frac{1}{2} \left[ \Sigma^{\alpha_\beta}, \Sigma^{\sigma\tau} \right] = g^{\alpha\delta} \Sigma^{\delta\beta} - g^{\alpha\beta} \Sigma^{\delta\gamma} + g^{\beta\gamma} \Sigma^{\alpha\delta} - g^{\beta\delta} \Sigma^{\alpha\gamma},
\]
(3.13)

which can be readily verified with the aid of
\[
(\Lambda^{-1})^\alpha_\nu \Lambda^\gamma_\nu = \delta^\alpha_\gamma, \quad (\Lambda^{-1})^\alpha_\nu = g^{\alpha\delta} \Lambda^\delta_\nu g_{\sigma\tau} \Sigma^{\sigma\tau},
\]
(3.14)

As a result
\[
(\text{det} \Lambda) e_{\lambda\nu\rho} \partial^\sigma \Sigma^{\mu\nu} \phi = i \partial_\lambda \phi,
\]
(3.15)

in view of the following determinant identity [43]
\[
e_{\mu\nu\rho} \Lambda^\mu_\lambda \Lambda^\nu_\rho \Lambda^\tau_\chi = (\text{det} \Lambda) e_{\lambda\nu\rho}.
\]
(3.16)

This consideration reveals the pseudo-vector character of the Pauli–Lubański operator.

---

6 Another choice of the generators in the transformation laws (3.2)–(3.4), corresponds to \( \theta \to -\theta \); see, for example, [58].
Note. Weyl’s equation (3.12) was originally introduced in [68] and then quickly rejected [48], being ‘resurrected’ only after the discovery of parity violation in beta decay [53, 72]. (The experimentally detected oscillation among the different flavors of neutrinos leads us to believe that they are not massless after all [29, 40, 51].)

Let \( \mathbf{v} \), the velocity vector of the moving frame of reference, lie along one of the coordinate axes, say \( (e_x)_k = 1, 2, 3 \). Also, let \( \sigma = \sigma_\parallel + \sigma_\perp \) be the corresponding ‘orthogonal decomposition’ in parallel and perpendicular directions, respectively. These components transform as
\[
S_{\Lambda}^{-1} \sigma_\parallel S_{\Lambda} = \sigma_\parallel, \quad S_{\Lambda}^{-1} \sigma_\perp S_{\Lambda} = \frac{\sigma_\perp - i(\mathbf{v} \times \sigma)}{\sqrt{1 - v^2}}, \tag{3.20}
\]
under a Lorentz transformation, thus resembling the transformations of electromagnetic fields in classical electrodynamics [32, 33, 41, 59, 61].

**Example.** Let \( \mathbf{v} = \mathbf{v}_1 \). Then \( \sigma_\parallel = \sigma_1, \sigma_\perp = (\sigma_2, \sigma_3) \) and
\[
S_{\parallel} = e^{-i(\phi/2)v_1} = \cosh \frac{\phi}{2} - \sigma_1 \sinh \frac{\phi}{2},
S_{\perp} = e^{i(\phi/2)v_1} = \cosh \frac{\phi}{2} + \sigma_1 \sinh \frac{\phi}{2}. \tag{3.21}
\]
By the transformation law (3.20), one should get \( S_{\Lambda}^{-1} \sigma_1 S_{\Lambda} = \sigma_1 \), which is obvious and
\[
S_{\Lambda}^{-1} \sigma_2 S_{\Lambda} = \frac{\sigma_2 + i\sigma_3 v}{\sqrt{1 - v^2}}, \quad S_{\Lambda}^{-1} \sigma_3 S_{\Lambda} = \frac{\sigma_3 - i\sigma_2 v}{\sqrt{1 - v^2}} \tag{3.22}
\]
for the corresponding Lorentz boost. Let us directly verify, for instance, the first relation. Indeed
\[
S_{\Lambda}^{-1} \sigma_2 S_{\Lambda} = \left( \cosh \frac{\phi}{2} + \sigma_1 \sinh \frac{\phi}{2} \right) \sigma_2 \left( \cosh \frac{\phi}{2} - \sigma_1 \sinh \frac{\phi}{2} \right) = \sigma_2 \cosh \frac{\phi}{2} + (\sigma_1 \sigma_2 - \sigma_1 \sigma_2) \sigma_1 \cosh \frac{\phi}{2} \frac{\sigma_2}{2} - \sigma_1 \sigma_2 \sigma_1 \sinh^2 \frac{\phi}{2} = \sigma_2 \cosh \frac{\phi}{2} + i\sigma_3 \sinh \frac{\phi}{2},
\]
provided that \( \tanh \frac{\phi}{2} = \mathbf{v} \). The proof of the last identity is similar.

As is well-known, under spatial rotations the set of three Pauli matrices \( \sigma \) transform as a 3D vector. For instance
\[
e^{-i(\phi/2)v_1} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \tag{3.23}
\]

3.4. An alternative derivation

Denoting, \( \sigma^\mu = (\sigma_0 = I, \sigma_1, \sigma_2, \sigma_3) \), one can rewrite Weyl’s equation in a more familiar form [50, 58]
\[
\sigma^\mu \partial_\mu \phi = 0, \tag{3.24}
\]
Then, under the Lorentz transformations,
\[
S_{\Lambda}^{-1} \sigma^\mu S_{\Lambda} = \Lambda^\mu_\nu \sigma^\nu \tag{3.25}
\]
and
\[
(\Sigma^\alpha^\beta)^\nu \sigma^\mu + \sigma^\mu (\Sigma^\alpha^\beta) = 2(g^{\beta\gamma} \sigma^\alpha - g^{\alpha\beta} \sigma^\gamma), \tag{3.26}
\]
which also implies the covariance.

**Examples.** In particular, one can easily verify that
\[
e^{-i(\phi/2)v_1} = \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \tag{3.27}
\]
with \( \tanh \phi = \mathbf{v} \) and
\[
e^{-i(\phi/2)v_1} = \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \tag{3.28}
\]
for the corresponding boost and rotation, respectively.

4. Proca equation

The fundamental four-vector representation of the proper orthochronous Lorentz group \( SO(1, 3) \) is related to the relativistic wave equation for a massive particle with spin-1.

4.1. Massive vector field

The relativistic equation of motion for a real or complex four-vector field \( A^\mu = (A^0, \mathbf{A}) \) with a positive mass \( m > 0 \) can be derived in a natural way with the help of the Pauli–Lubański vector. By definition,
\[
w_{\mu\nu}A^\nu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho M^{\sigma\nu} A^\nu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho ((mT)^\nu_\rho A^\nu) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho (g^{\rho\nu} A^\nu - g^{\nu\sigma} A^\nu) = -g^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} \partial^\sigma A^\rho, \tag{4.1}
\]
where the matrix form of the generators (2.21) has been used. The standard decomposition
\[
\partial^\mu A^\nu = \frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2} (\partial^\mu A^\nu + \partial^\nu A^\mu),
\]
followed by the dual tensor relation
\[
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \epsilon_{\mu\nu\rho\sigma} F^{\sigma\nu} = -2G_{\mu\rho}, \tag{4.2}
\]
gives the explicit action of the Pauli–Lubański operator on the four-vector field
\[
w_{\mu} A^\nu = g^{\mu\nu} G_{\mu\nu} = g_{\mu\nu} G^{\alpha\beta}, \quad w_{\mu} A_\alpha = G_{\mu\alpha}. \tag{4.3}
\]
It is worth noting that this results in a second rank four-tensor.

8 Relations of this spinor representation with Maxwell’s equations are discussed in section 6.2.
In a similar fashion, for the squared operator
\[ w^2A^\alpha = w^\mu (w_\mu A^\alpha) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\mu (M^{\mu\nu}G^{\rho\sigma}). \] (4.4)

But, in view of (4.7) of [32]
\[ M^{\mu\nu}G^{\rho\sigma} = g^{\rho\sigma}G^{\mu\nu} - g^{\rho\nu}G^{\mu\sigma} + g^{\rho\mu}G^{\sigma\nu} - g^{\sigma\nu}G^{\rho\mu}, \] (4.5)
and with the help of a companion dual tensor identity, \( \epsilon^{\mu\nu\rho\sigma}G_{\alpha\beta} = 2F_{\mu\alpha}, \) one gets
\[ w^2A^\alpha = g^{\rho\alpha}\partial^\rho (e_{\sigma\mu\nu}G^{\mu\nu}) = 2g^{\rho\alpha}\partial^\nu(F_{\sigma\nu}) = 2\partial_\nu F^{\nu\alpha} - 2m^2A^\alpha \]
as a consequence of (1.2). As a result, we have arrived at the Proca equation for a vector particle with a finite mass [52]
\[ \partial_\nu F^{\nu\alpha} + m^2A^\alpha = 0, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \] (4.6)
directly from the representation theory of the Poincar\'e group. In view of \( w^2 = -m^2(s + 1) = -2m^2, \) one concludes that the spin of the particle is equal to one. Moreover,
\[ \partial_\nu A^\nu = 0, \quad \Box A^\mu = -m^2A^\mu, \quad m > 0, \] (4.7)
in view of
\[ 0 \equiv \partial_\mu \partial_\nu F^{\nu\mu} = -m^2\partial_\mu A^\mu \] (4.8)
and
\[ \partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\nu \partial^\mu A^\nu = -m^2A^\mu. \] (4.9)
The massless case of the Proca equation, \( m = 0, \) reveals a gauge invariance. Indeed, if \( A_\mu \rightarrow A_\mu' = A_\mu + \partial_\mu f, \) then \( F_{\mu\nu}' = \partial_\mu (A_\nu + \partial_\nu f) - \partial_\nu (A_\mu + \partial_\mu f) = F_{\mu\nu}. \)

4.2. An alternative ‘bispinor’ derivation

Let us consider a second rank bispinor of the form \( Q = A_\chi \gamma_\chi = A_\lambda \gamma_\lambda, \) where \( \{\gamma_\lambda\}_{\lambda=0,1,2,3} \) are the standard gamma matrices. We shall use the following transformation law for a proper Lorentz transformation,
\[ Q'(x') = S_A Q(x) S_A^{-1}, \quad x' = \Lambda x, \] (4.10)
where the matrix \( S_A \) is given by (2.7), as in the case of Dirac’s equation. Then \( A_\mu \) must be a four-vector. Indeed,
\[ Q' = A_\lambda S_{\alpha\beta}(S_{\gamma^\sigma}S^{-1}) = (g^{\gamma^\lambda}A_\lambda)S_{\mu\nu}[A_{\sigma}(S_{\gamma^\sigma}S^{-1})] = A'^\nu, \]
\[ A'^\nu = \Lambda_{\nu}^\rho A^\rho, \] (4.11)
in view of an ‘inversion’ of (2.5), \( \Lambda_{\nu}^\rho (S_{\gamma^\sigma}S^{-1}) = \gamma^\rho, \) and the familiar property:
\[ A_\mu B^\mu = \text{invariant or} \]
\[ g_{\mu\nu}A^\mu A^\nu S_{\gamma^\sigma}S^{-1} = \delta^\chi_\chi. \] (4.12)

In this ‘bispinor representation’, the action of generators of the corresponding one-parameter subgroups is given by
\[ M^{\alpha\beta}Q = -\left( \frac{d}{d\theta^{\alpha\beta}} Q'(\Lambda x) \right) \bigg|_{\theta^{\alpha\beta}=0} = \frac{1}{2} (\Sigma^{\alpha\beta} Q - Q \Sigma^{\alpha\beta}). \] (4.13)
But by letting \( Q = \gamma^\lambda A_\lambda, \) one gets
\[ M^{\alpha\beta}Q = (g^{\beta\gamma}\gamma^\alpha - g^{\alpha\lambda}\gamma^\beta)A_\lambda, \]
in view of the familiar commutator (2.23). As a result, we obtain \( w^2Q = G_{\mu\nu}\gamma^\nu \) for the action of the Pauli–Lubański operator on the bispinor \( Q, \) which also follows from (4.3). One gets, in a similar fashion, that
\[ w^2Q = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial^\mu (M^{\mu\nu}G^{\rho\sigma}) \gamma_\lambda, \] (4.14)
where equation (4.5) holds, once again, in view of the transformation law of the four-tensor \( G^{\mu\lambda}. \) As a result, Proca’s equation follows.

4.3. Maxwell’s equations versus Proca equation

For the real-valued vector potential \( A^\nu \) and \( m = 0, \) the Proca equation (4.6) is reduced to the Maxwell equations in vacuum. Indeed, in view of the dual relation
\[ 6\partial_\nu G^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma} (\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho}) = 0, \] (4.15)
both pairs of Maxwell’s equations can be written together in the following complex form
\[ \partial_\nu Q^{\mu\nu} = 0, \quad Q^{\mu\nu} = F^{\mu\nu} - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \] (4.16)
with the help of a self-dual complex four-tensor [8, 32]
\[ 2iQ^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} Q_{\rho\sigma}, \quad \epsilon^{\mu\nu\rho\sigma} Q^{\rho\sigma} = 2iQ_{\mu\nu}. \] (4.17)
The corresponding overdetermined system of Maxwell’s equations in vacuum, which is related to the Pauli–Lubański vector, is investigated in [32]; see equation (5.2) there and section 5.5 below. (Two different spinor forms of Maxwell’s equation will be discussed in section 6.)

5. Complex vector field

Finally, we would like to discuss the fundamental representation for the complex orthogonal group \( SO(3, \mathbb{C}) \) in connection with Maxwell’s equations in vacuum.

5.1. Vector covariant form

As is well-known, the transformation laws of the complex electromagnetic field \( F(r, t) = E + iH \in \mathbb{C}^3 \) in vacuum can be written in terms of \( SO(3, \mathbb{C}) \) rotations. In addition to the standard rotations of the frame of reference, the Lorentz transformations are equivalent to rotations through imaginary angles thus preserving the relativistic invariant \( F^2 = E^2 + H^2 + 2iE \cdot H, \) which can be thought of as a ‘complex length’ of this vector [34, 41]. In these transformations, \( F'(x') = S_A F(x), \) \( x' = \Lambda x, \) (or \( F'_{\mu}(x') = \theta_{\mu\nu} F_{\nu}(x) \)
for a given complex orthogonal matrix), one can choose $S_R = e^{-i\psi_0}$ and $S_L = e^{-i\psi_0}$ for the rotations and boosts, respectively. Here, $n = \{e_1, e_2, e_3\}$, when $\{e_i, i=1,2,3\}$ is an orthonormal basis in $\mathbb{R}^3$, and $s_1, s_2, s_3$ are the real-valued spin matrices [62]:

$$
\begin{align*}
  s_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
  s_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\
  s_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
$$

(5.1)

such that $[s_p, s_q] = s_p s_q - s_q s_p = e_{pqr} s_r$ and $s_1^2 + s_2^2 + s_3^2 = -2$. (In this representation the matrices $M = s$ and $N = i$ obey the commutation law of the generators of the proper orthochronous Lorentz group.)

It can be directly verified that the corresponding generators,

$$
\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha} = \begin{pmatrix} 0 & i s_q \\ -i s_p & e_{pqr} s_r \end{pmatrix}
$$

form a self-dual four-tensor

$$
2\Sigma^{\mu\nu} = e^{\mu\tau} \Sigma_{\tau\sigma} e^{\nu \sigma}, \quad e_{\mu\sigma} \Sigma^{\tau\sigma} = 2i \Sigma_{\mu\nu}.
$$

(5.3)

As a result, in this realization, $M^{\alpha\beta} F = m^{\alpha\beta} F$ and, in view of (1.3) and (5.3), we arrive at the set of overdetermined equations

$$
(\Sigma^{\mu\nu} + g^{\mu\nu}) \partial_\nu F = 0,
$$

(5.4)

where

$$
S^{-1}_k \Sigma^{\mu\nu} S_k = \Lambda_k^\alpha \Lambda_k^{\nu\sigma}, \quad x^{\mu\nu} = \Lambda_k^\alpha x^{\sigma},
$$

$$
F'(x') = S_k F(x)
$$

(5.5)

under a proper Lorentz transformation. Once again, the latter equations, that, as we shall see later, determine the complete dynamics of the electromagnetic field in vacuum, are obtained here by a pure group-theoretical consideration up to an undetermined sign of the fixed constant in (1.3).

5.2. Commutators

For a one-parameter transformation in the plane $(\alpha, \beta)$, when

$$
\Lambda(\theta_{\alpha\beta}) = \exp(-\theta_{\alpha\beta} m^{\alpha\beta}), \quad (m^{\alpha\beta})' = g^{\alpha\mu} \delta^\beta_\nu - g^{\beta\nu} \delta^\alpha_\nu
$$

(6.6)

and

$$
S_k = \exp(-\theta_{\alpha\beta} \Sigma^{\alpha\beta}), \quad \theta_{\alpha\beta} = -\theta_{\beta\alpha}
$$

(5.7)

$(\alpha, \beta = 0, 1, 2, 3$ are fixed), one gets

$$
e^{\theta_{\alpha\beta} \Sigma^{\mu\nu}} e^{-\theta_{\beta\alpha} \Sigma^{\mu\nu}} = \Lambda_k^\alpha (\theta_{\alpha\beta}) \Lambda_k^{\nu\sigma} (\theta_{\beta\alpha}) \Sigma^{\tau\sigma}.
$$

(5.8)

The differentiation $(d/d\theta_{\alpha\beta}) |_{\theta_{\alpha\beta}=0}$, results in a familiar law

$$
[S^{\alpha\beta}, \Sigma^{\mu\nu}] = S^{\alpha\beta} \Sigma^{\mu\nu} - \Sigma^{\mu\nu} S^{\alpha\beta} = g^{\mu\nu} S^{\alpha\beta} - g^{\alpha\beta} S^{\mu\nu} - g^{\beta\nu} S^{\alpha\mu} - g^{\mu\alpha} S^{\beta\nu}.
$$

(5.9)

which can be directly verified in components with the help of commutators of the spin matrices (5.1) that form the four-tensor (5.2).

5.3. Lorentz invariance

With the help of (5.4) and (5.5), under a Lorentz transformation

$$
S^{-1} [(\Sigma^{\mu\nu} + g^{\mu\nu}) \partial_\nu F'(x')] = 0,
$$

(5.10)

where $\Sigma^{\mu\nu} = \Sigma_{\mu\nu}$ is inv and $g^{\mu\nu} = g_{\mu\nu}$ is inv. Then

$$
0 = [S^{-1} (\Sigma^{\mu\nu} + g^{\mu\nu}) S \partial_\nu F'(x')] = \Lambda_k^\alpha (\Sigma^{\sigma\tau} + g^{\sigma\tau}) S^{-1} [\Lambda_k^\nu \partial_\nu F'(x')]
$$

(5.11)

by $\partial_\nu \partial_\nu F'(x') = \partial_\nu F(x)$. Thus, our equation $(\Sigma^{\mu\nu} + g^{\mu\nu}) \partial_\nu F(x) = 0$ preserves its covariant form for all real and complex rotations $S \in SO(3, \mathbb{C})$.

5.4. Vector covariant form versus traditional form of Maxwell's equations

A vector form of (5.4) is given by

$$
(\nabla \cdot s) F = i \partial_0 F
$$

(5.12)

and

$$
\partial_0 s F + i (\nabla \times s) F = i \nabla F.
$$

(5.13)

As it has been pointed out in [32], equation (5.12) implies the complex Maxwell equation, curl $F = i \partial_0 F$. The first component of (5.13) is given by $\partial_0 s \mathbf{F} + i (\partial_2 s_3 - \partial_3 s_2) \mathbf{F} = i \partial_1 \mathbf{F}$, or

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} + i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = i \partial_1 \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}
$$

and

$$
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = i \partial_3 \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.
$$

(5.14)

As a result, div $\mathbf{F} = 0$ and (curl $\mathbf{F})_{2,3} = i \partial_0 (\mathbf{F})_{2,3}$. (Cyclic permutations of the spatial indices cover the two remaining cases.)
5.5. An alternative form of Maxwell’s equations

Complex covariant form of Maxwell’s equations, which was introduced in [35] (see also [32]), can be presented as follows

\[
(\partial_0, \partial_1, \partial_2, \partial_3) \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -iF_1 & iF_2 & -iF_3 \\
0 & iF_2 & -iF_3 & -iF_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
= (j_0, j_1, j_2, j_3),
\]

or \( \partial Q = j \), by matrix multiplication. Here, \( \partial = (\partial_0, \partial_1, \partial_2, \partial_3) \), \( j = (j_0, j_1, j_2, j_3) \), and \( J = (J_1, J_2, J_3) \) such that

\[
Q = F \cdot J = F_1 \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0
\end{pmatrix}
+ F_2 \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -i \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
+ F_3 \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\] (5.15)

These matrices are transformed as a dual complex vector in \( \mathbb{C}^3 \)

\[
\Lambda J_0^N = a_{pq}d_q,
\] (5.16)
under a proper Lorentz transformation. In infinitesimal form

\[
m^{a\beta}J + J(m^{a\beta})^T = (\Sigma^{a\beta})^T J,
\] (5.17)
which gives an alternative representation of the group \( SO(3, \mathbb{C}) \) in a subspace of complex 4 \( \times \) 4 matrices. (Details are left to the reader.)

As a result, \( \Lambda Q^N = Q' \) and, in ‘new’ coordinates, \( \partial'Q' = f' \), provided that \( \partial' = \partial(\Lambda^{-1}) \) and \( j' = jN \).

6. On spinor forms of Maxwell’s equations

In conclusion, the complex matrix group \( SL(2, \mathbb{C}) \) has a representation of the proper orthochronous Lorentz group \( SO_t(1, 3) \) by the second rank spinors.

6.1. Spinor covariant form

The complex electromagnetic field in vacuum, \( F = E + iH \), can also be written in a familiar form of the following 2 \( \times \) 2 matrix

\[
Q = \sigma \cdot F = \sigma_1 F_1 + \sigma_2 F_2 + \sigma_3 F_3 = \begin{pmatrix}
F_3 & F_1 - iF_2 \\
F_1 + iF_2 & -F_3
\end{pmatrix},
\] (6.1)
where \( \sigma_1, \sigma_2, \sigma_3 \) are the standard Pauli matrices. The corresponding transformation law

\[
Q'(x') = S_L Q(x) S_L^{-1}, \quad x' = \Lambda x,
\]

\[
S_L = \exp \left( \frac{1}{4} \theta_{\mu\nu} \Sigma^{\mu\nu} \right),
\] (6.2)
under a proper Lorentz transformation preserves two invariants \( tr Q = 0 \) and \( det Q = -F^2 \). Here, \( S_L \sigma_p S_L^{-1} = a_{pq} \sigma_q \) and \( F_p'(x') = a_{pq} F_q(x) \), with \( a_{pq} \delta_{pq} = \delta_3 \) for a given complex orthogonal \( 3 \times 3 \) matrix (see section 5.1).

In this ‘spinor’ representation, one gets

\[
M^{a\beta}Q = - \left( \frac{d}{d\theta_{\alpha\beta}} Q' (\Lambda x) \right)_{\theta_{\alpha\beta}=0} = -\frac{1}{2} (\Sigma^{a\beta}Q - Q\Sigma^{a\beta})
\] (6.3)
for generators of the one-parameter subgroups. Here, as in the case of Weyl’s equation, the matrices \( \Sigma^{a\beta} \) are given by (3.7), but now, in view of (6.3), equation (1.3) with \( \lambda = -1 \) takes the form

\[
\frac{1}{2} (\Sigma_{\mu\nu} \partial^{\mu} Q - \partial^{\mu} Q \Sigma_{\mu\nu}) = \partial_\nu Q,
\] (6.4)
when the self-duality property (3.8) is applied.

Equations (6.4), obtained with the aid of the Pauli-Lubański vector, are equivalent to the system of complex Maxwell equations in vacuum

\[
\text{div} F = 0, \quad \text{curl} F = i\partial F.
\] (6.5)

Indeed, when \( \mu = 0 \), with the help of (3.9) one gets

\[-\frac{1}{2} (\sigma_p \partial_\mu Q - \partial_\mu Q \sigma_p) = \partial_\mu Q,
\] or

\[-(\sigma_p \sigma_q - \sigma_q \sigma_p) \partial_\mu F_q = 2 \partial_\mu (\sigma_q F_q),
\]
which gives the second complex Maxwell equation (6.5) in view of the commutation relation \([\sigma_p, \sigma_q] = 2i\epsilon_{pqr} \sigma_r \).

When \( \mu = p = 1, 2, 3 \), in a similar fashion

\[2 \partial_\mu Q = \partial_\mu Q \sigma_p - \sigma_p \partial_\mu Q + i\epsilon_{pqr} (\partial_\mu Q \sigma_r - \sigma_r \partial_\mu Q)
\] (6.6)
and letting \( Q = F_\sigma \), we obtain

\[2 \partial_\mu (F_\sigma \sigma_r) = (\sigma_\sigma \sigma_p - \sigma_p \sigma_r) \partial_\mu F_\sigma
+ i\epsilon_{pqr} (\sigma_r \sigma_r - \sigma_r \sigma_r) \partial_\mu F_q.
\]

Evaluation of the commutators

\[\sigma_r \partial_\mu F_\sigma = i\epsilon_{pqr} \sigma_\sigma \partial_\mu F_q + \epsilon_{pqr} \epsilon_{\sigma\mu\nu} \sigma_\nu \partial_\mu F_q,
\] (6.7)
and a familiar identity

\[\epsilon_{pqr} \epsilon_{\sigma\mu\nu} \sigma_\nu = \binom{\delta_{\mu\nu} \delta_{pq} - \delta_{\mu\sigma} \delta_{pq} - \delta_{\mu\nu} \delta_{pq} + \delta_{\mu\nu} \delta_{pq}}{\delta_{\mu\nu} \delta_{pq} - \delta_{\mu\sigma} \delta_{pq} - \delta_{\mu\nu} \delta_{pq} + \delta_{\mu\nu} \delta_{pq}} = \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\nu\sigma},
\] (6.8)
result in the system of Maxwell’s equations (6.5).

6.2. Traditional spinor form of Maxwell’s equations

Equations (6.4), that are obtained here with the help of the Pauli–Lubański vector, give an alternative (vacuum) version of the spinor form of Maxwell’s equations

\[(\partial_0 + \sigma \cdot \nabla)(\sigma \cdot F) = j_0 + \sigma \cdot j,
\] (6.9)
or, explicitly,

\[
\begin{pmatrix}
\partial_0 + \partial_3 & \partial_1 - i\partial_2 \\
\partial_1 + i\partial_2 & \partial_0 - \partial_3
\end{pmatrix}
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix}
= \begin{pmatrix}
\hat{J}_1 + \hat{J}_3 \\
\hat{J}_1 - i\hat{J}_2
\end{pmatrix}
\begin{pmatrix}
F_1 + iF_2 \\
F_1 - iF_2
\end{pmatrix}
\]

so we obtain

\[S_{\lambda} = 0, \quad F_{\mu\nu} = F_{\mu\nu}^0 = \frac{2}{3} \mathrm{g}_{\alpha\beta} \partial_\mu F^{\alpha\beta}_{\gamma\delta} g_{\gamma\delta}\]

and, in view of (3.25), one gets

\[S_{\lambda} = 0, \quad F_{\mu\nu} = F_{\mu\nu}^0 = \frac{2}{3} \mathrm{g}_{\alpha\beta} \partial_\mu F^{\alpha\beta}_{\gamma\delta} g_{\gamma\delta}\]

or, \(S_{\lambda} = 0, \quad F_{\mu\nu} = F_{\mu\nu}^0 = \frac{2}{3} \mathrm{g}_{\alpha\beta} \partial_\mu F^{\alpha\beta}_{\gamma\delta} g_{\gamma\delta}\), as a short proof of the covariance of Maxwell’s equations.

7. Massive symmetric four-tensor field

7.1. Group-theoretical derivation

The relativistic wave equation for a massive particle of spin two, described by a real or complex symmetric four-tensor field \(A^\mu(x) = A^\mu(x)\) (see [17, 19, 20]), can be obtained in a way that is similar to our study of the Proca equation in section 4. Once again

\[w_\mu A^{\alpha\beta} = \frac{1}{2} \epsilon_{\mu\sigma\tau\nu} \partial_\nu (M^{\sigma\tau} A^{\alpha\beta})
= - \frac{2}{3} \epsilon_{\mu\sigma\tau\nu} \partial_\nu A^{\alpha\beta} - \frac{1}{3} \epsilon_{\mu\sigma\tau\nu} \partial_\nu A^{\alpha\beta}\]

and

\[\epsilon_{\mu\sigma\tau\nu} \partial_\nu A^{\alpha\beta} = \frac{1}{2} \epsilon_{\mu\sigma\tau\nu} \partial_\nu A^{\alpha\beta} = - G_{\mu\nu}^{\alpha\beta}\]

where, by definition, \(F^{\sigma\tau} = \partial_\sigma A^{\tau\alpha} - \partial_\tau A^{\sigma\alpha}\) and

\[\epsilon_{\mu\sigma\tau\nu} F^{\sigma\tau} = - 2G_{\mu\nu}^{\alpha\beta}\]

Thus

\[w_\mu A^{\alpha\beta} = g^{\alpha\nu} G_{\mu\nu}^{\beta\alpha} + g^{\beta\nu} G_{\mu\nu}^{\alpha\beta}\]

or

\[w_\mu A^{\alpha\beta} = G^{\alpha\beta} + G^{\nu\alpha}, \quad w_\mu A^{\alpha\beta} = G_{\mu\alpha}^{\beta} + G_{\mu\beta}^{\alpha}\]

and

\[w_\mu A^{\alpha\beta} = w_\mu (w_\mu A^{\alpha\beta}) = w_\mu G^{\mu\alpha\beta} + w_\mu G^{\mu\alpha\beta}\]

In a similar fashion, one can show that

\[w_\mu G^{\mu\alpha\beta} = 2 \partial_\mu F^{\mu\alpha\beta} + \partial_\mu F^{\mu\alpha\beta}\]

subject to \(A = g_{\mu\nu} A^{\mu\nu} = \partial_\mu \partial_\nu A^{\mu\nu} = 0\). It is worth noting, once again, that we have derived these equations by using the Pauli–Lubański vector and the relativistic definition of mass and spin in terms of Casimir operators of the Poincaré group.
In the massless limit $m \to 0$, instead of (7.12), one has
\[ \partial^2 A^{\mu \nu} = 0 \] and $\partial^2 A = 0$ subject to
\[ \partial^2 \left( \partial \partial A^{\alpha \beta} - \frac{1}{4} \partial^\alpha A \right) + \partial^2 \left( \partial \partial A^{\alpha \beta} - \frac{1}{4} \partial^\beta A \right) = \frac{1}{2} g^\alpha \beta \partial \partial A^{\mu \nu}, \tag{7.13} \]
in a similar fashion. Moreover, if $m = 0$, equation (7.8) is invariant under a familiar gauge transformation $A_{\alpha \beta} \to A_{\alpha \beta} + \partial \partial f_{\alpha \beta} + \partial \partial f_{\alpha}$ provided that $\partial^2 (\partial \partial f_{\alpha}) = 0$ (Maxwell’s equations in vacuum).

### 7.2. An alternative gauge condition

In view of the following identity
\[ \partial^2 \left( \partial \partial A^{\alpha \beta} - \frac{1}{4} \partial^\alpha A \right) + \partial^2 \left( \partial \partial A^{\alpha \beta} - \frac{1}{4} \partial^\beta A \right) = \partial^2 \partial \partial A^{\alpha \beta} + \partial^2 \partial \partial A^{\alpha \beta} - \frac{1}{2} \partial^2 \partial \partial A^{\mu \nu}, \]
one can impose another condition
\[ 4 \partial \partial A^{\mu \nu} - \partial^2 A = 0, \tag{7.14} \]
in order to simplify (7.10)
\[ \partial^2 A^{\mu \nu} - \frac{1}{2} g^{\mu \nu} \partial^2 A = -m^2 A^{\mu \nu}. \tag{7.15} \]
Moreover, by contraction\(^{10}\)
\[ 0 \equiv \partial^2 A - \frac{1}{2} g^{\mu \nu} \partial^2 A = -m^2 A, \quad A = g^{\mu \nu} A^{\mu \nu} = 0, \]
if $m > 0$. As a result, we obtain
\[ \partial^2 A^{\mu \nu} + m^2 A^{\mu \nu} = 0, \quad \partial \partial A^{\mu \nu} = \partial \partial A^{\mu \nu} = 0. \tag{7.16} \]
These equations were originally introduced by Fierz and Pauli [17, 19, 20] with the help of a Lagrangian approach. It is worth noting that our equations (7.12) are necessary and sufficient with the relativistic definition of mass and spin-2 of the field in question.

### 7.3. Fierz–Pauli versus Maxwell’s equations

When $m = 0$, one gets
\[ \partial \partial F^{\mu \nu \alpha \beta} = \partial \partial (\partial \partial A^{\alpha \beta} - \partial \partial A^{\mu \nu}) = \partial \partial \partial (\partial \partial A^{\alpha \beta} - \partial \partial A^{\mu \nu}) = 0, \tag{7.17} \]
subject to (7.16). In addition,
\[ \partial \partial F^{\mu \nu \alpha \beta} + \partial \partial F^{\lambda \mu \alpha \beta} + \partial \partial F^{\lambda \mu \alpha \beta} = 0, \tag{7.18} \]
which follows from definition. These facts allow one to represent the massless Fierz–Pauli equations in terms of the third rank field tensor, somewhat similar to classical electrodynamics. Indeed, by analogy with Maxwell’s equations, we obtain
\[ \partial \partial F^{\mu \nu \alpha \beta} = 0, \quad \partial \partial G^{\mu \nu \alpha \beta} = 0 \tag{7.19} \]
\(^{10}\) Condition $A = 0$ is not required in the massless limit.

\[ \text{in view of } 2G^{\mu \nu \alpha} = -e^\mu \eta^{\sigma \tau} F^{\sigma \tau \alpha} \quad \text{and} \]
\[ \partial \partial G^{\mu \nu \alpha} = -\frac{1}{6} e \eta^{\sigma \tau} (\partial \partial F^{\sigma \tau \alpha} + \partial \partial F^{\tau \alpha \sigma} + \partial \partial F^{\sigma \tau \alpha}) = 0 \]
(for every fixed $\alpha = 0, 1, 2, 3$).

Finally, both pairs of these equations can be combined together in the following complex form
\[ \partial \partial Q^{\mu \nu \alpha} = 0, \quad Q^{\mu \nu \alpha} = F^{\mu \nu \alpha} - i \frac{1}{2} e \eta^{\sigma \tau} F^{\sigma \tau \alpha}, \tag{7.20} \]
with the help of a self-dual complex four-tensor
\[ 2i Q^{\mu \nu \alpha} = e^{\mu \eta \tau \sigma} Q^{\sigma \tau \alpha}, \quad e_{\mu \nu \sigma \tau} Q^{\sigma \tau \alpha} = 2i Q^{\mu \nu \alpha}. \tag{7.21} \]
The covariant field ‘energy–momentum’ tensor and the corresponding differential balance equation,
\[ \partial \partial (Q^{\mu \nu \alpha}) = 0, \tag{7.22} \]
can be derived in a complete analogy with complex electrodynamics [32, 33].

### 7.4. Fierz–Pauli versus linearized Einstein’s equations

In general relativity, the linearized equations for a weak gravitational field [14, 25], namely,
\[ \partial \partial h_{\nu \alpha} + \partial \partial h_{\nu \tau} \partial \partial h_{\tau \alpha} = \partial \partial h, \quad \partial \partial h_{\mu \nu} = \partial \partial h_{\mu \nu} \partial \partial h_{\tau \sigma} - \partial \partial h_{\mu \nu} \partial \partial h_{\tau \sigma} = 0 \tag{7.23} \]
describe small deviations from the flat Minkowski metric, $g_{\mu \nu} = \text{diag}(1, -1, -1, -1)$, on the pseudo-Riemannian manifold subject to a gauge condition
\[ 2 \partial \partial h_{\mu \nu} - \partial \partial h = 0, \quad h = g^{\sigma \tau} h_{\sigma \tau} \tag{7.24} \]
(see, for example, [11, 15, 20, 21, 26, 28, 34, 42, 46, 63–65, 67, 69, 71, 73] and the references therein for more details).

Our calculations have shown that linearized Einstein’s equation (7.23) do not coincide with the massless limit of the spin-2 particle wave equation (7.10). But they can be reduced to the massless case of the Fierz–Pauli equation (7.16) in view of an additional condition (7.24) on a certain solution set. In the literature, this fact is usually interpreted as spin-2 for a graviton although, from the group-theoretical point of view, this massless limit yet requires certain analysis of helicity, say similar to the one in electrodynamics [32], which will be discussed elsewhere.

### 8. Summary

In this article, we analyze kinematics of the fundamental relativistic wave equations, in a traditional way, from the viewpoint of the representation theory of the Poincaré group. In particular, the importance of the Pauli–Lubański pseudo-vector is emphasized here not only for the covariant definition of spin and helicity of a given field but also for the derivation of the corresponding equation of motion from first principles. In this consistent group-theoretical approach, the resulting
wave equations occur, in general, in certain overdetermined forms, which can be reduced to the standard ones by a matrix version of Gaussian elimination.

Although, mathematically, all representations of the Poincaré group are locally equivalent [4], their explicit realizations in conventional linear spaces of four-vectors and tensors, spinors and bispinors, etc. are quite different from the viewpoint of physics. This is why, as the reader can see in the table below, the corresponding relativistic wave equations are so different.

| Classical field | Transformation law (a law of nature) | Wave equation |
|-----------------|-------------------------------------|---------------|
| Bispinor (2.3)  | $\psi'(x') = S(x')\psi(x)$, $x' = \Lambda x$; see (2.7) | Dirac (2.6) and (2.21) |
| Spinor (3.1)    | $\psi'(x') = S_i(x')\psi(x)$, $x' = \Lambda x$; see (3.2) | Weyl (3.24) and (2.21) |
| Four-vector     | $\Lambda^\mu(x') = \Lambda^\mu\nu A^\nu(x)$; see (2.21) | Proca (4.6) |
| ‘Feynman slash’ | $Q'(x') = S_{iQ}(x)S_{iQ}^{-1}$; $x' = \Lambda x$; see (4.10) | Proca (4.6) |
| Four-tensor     | $Q^\mu(x') = \Lambda^\mu_{\nu\sigma}A^\nu_{\sigma}(x)$, $x' = \Lambda x$; see (2.21) | Maxwell (4.16), (5.14) |
| Complex 3D vector | $F'(x') = S_F F(x)$, $x' = \Lambda x$; see sect 5.1, (2.21) | Maxwell (6.5) |
| Complex matrix  | $Q'(x') = S_Q Q(x)S_{Q}^{-1}$; $x' = \Lambda x$; see (6.2) | Maxwell (6.5) |
| Symmetric four-tensor | $\Lambda^\mu(x') = \Lambda^\mu_{\nu\sigma}A^\nu_{\sigma}(x)$, $x' = \Lambda x$ | Fierz–Pauli (7.12) |

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