NONLINEAR $n$-TERM APPROXIMATION OF HARMONIC FUNCTIONS FROMhifts OF THE NEWTONIAN KERNEL

KAMEN G. IVANOV AND PENCHO PETRUSHEV

Abstract. A basic building block in Classical Potential Theory is the fundamental solution of the Laplace equation in $\mathbb{R}^d$ (Newtonian kernel). The main goal of this article is to study the rates of nonlinear $n$-term approximation of harmonic functions on the unit ball $B^d$ from shifts of the Newtonian kernel with poles outside $\mathbb{R}^d$ in the harmonic Hardy spaces. Optimal rates of approximation are obtained in terms of harmonic Besov spaces. The main vehicle in establishing these results is the construction of highly localized frames for Besov and Triebel-Lizorkin spaces on the sphere whose elements are linear combinations of a fixed number of shifts of the Newtonian kernel.

Contents

1. Introduction 2
2. Background and technical groundwork 6
2.1. Basic notation and simple inequalities 6
2.2. Spherical harmonics 7
2.3. Maximal $\delta$-nets and cubature formulas on the sphere 8
2.4. Maximal operator 9
2.5. Inner products of zonal functions 10
2.6. Inner products of localized functions 10
3. Spaces of functions and distributions on the ball and sphere 11
3.1. Harmonic Besov and Triebel-Lizorkin spaces on $B^d$ 11
3.2. Besov and Triebel-Lizorkin spaces on $S^{d-1}$ 12
3.3. Identification of harmonic Besov and Triebel-Lizorkin spaces 13
3.4. Harmonic Hardy spaces 14
3.5. Frame decomposition of distribution spaces on $S^{d-1}$ 16
4. Construction of frames by small perturbation 19
4.1. Setting and conditions on the old frame 19
4.2. Construction of new frames 20
4.3. Almost diagonal operators 23
5. Space localization of needlets and Newtonian kernels 24
5.1. Properties of the needlets 24
5.2. Highly localized kernels in terms of shifts of the Newtonian kernel 28
6. Frames in terms of shifts of the Newtonian kernel 30

2010 Mathematics Subject Classification. 41A17, 41A25, 42C15, 42C40, 42B35, 42B30.

Key words and phrases. Nonlinear approximation, harmonic functions, Newtonian kernel, Hardy spaces, Besov spaces, frame decomposition.

The first author has been supported by Grant DN 02/14 of the Fund for Scientific Research of the Bulgarian Ministry of Education and Science. The second author has been supported by NSF Grant DMS-1714369.
6.1. The main technical step in the construction of the new frame on $\mathbb{S}^{d-1}$
6.2. Completion of the construction of new frames on $\mathbb{S}^{d-1}$
6.3. Frames on $B^d$ in terms of shifts of the Newtonian kernel
7. Nonlinear approximation from shifts of the Newtonian kernel
7.1. Nonlinear $n$-term frame approximation on $\mathbb{S}^{d-1}$
7.2. Nonlinear $n$-term approximation of harmonic functions on $B^d$
8. Approximation of harmonic functions on $\mathbb{R}^d \setminus \overline{B^d}$ and $\mathbb{R}^d_+$
9. Proofs
9.1. Proofs of Propositions 2.3, 2.4, and 2.5
9.2. Proof of Theorem 4.6
References

1. Introduction

The fundamental solution of the Laplace equation $\frac{1}{|x|^{d-2}}$ in dimension $d > 2$ or $\ln \frac{1}{|x|}$ if $d = 2$ with $|x|$ being the Euclidean norm of $x \in \mathbb{R}^d$ is a basic building block in Potential theory. As is customary, we shall term the harmonic function $\frac{1}{|x|^{d-2}}$ or $\ln \frac{1}{|x|}$ “Newtonian kernel”.

The main purpose of this article is to study the nonlinear $n$-term approximation of harmonic functions on the unit ball $B^d$ in $\mathbb{R}^d$ from linear combinations of shifts of the Newtonian kernel. More explicitly, the problem is for a given harmonic function $U$ on $B^d$ and $n \geq 1$ to find $n$ locations $\{y_j\}$ in $\mathbb{R}^d \setminus \overline{B^d}$ and coefficients $\{a_j\}$ in $\mathbb{C}$ so that

$$a_0 + \sum_{j=1}^{n} \frac{a_j}{|x - y_j|^{d-2}} \quad \text{if} \quad d > 2 \quad \text{or} \quad a_0 + \sum_{j=1}^{n} a_j \ln \frac{1}{|x - y_j|} \quad \text{if} \quad d = 2$$

approximates $U$ with an optimal rate (near best) in the harmonic Hardy space $\mathcal{H}^p(B^d)$, $0 < p < \infty$. Denote by $N_n$ the set of all harmonic functions on $B^d$ that can be represented in the form (1.1). Here the points $\{y_j\}$ are allowed to vary with the function and hence $N_n$ is nonlinear. Given $U \in \mathcal{H}^p(B^d)$ we denote

$$E_n(U)_{\mathcal{H}^p} := \inf_{G \in N_n} \|U - G\|_{\mathcal{H}^p(B^d)}.$$

We shall term $E_n(U)_{\mathcal{H}^p}$ the best $n$-term approximation of $U$ in the harmonic Hardy space $\mathcal{H}^p(B^d)$ from shifts of the Newtonian kernel as in (1.1). Our goal is to study the rate of convergence of $\{E_n(U)_{\mathcal{H}^p}\}$ and the smoothness spaces that govern this approximation process. The same approximation problem is also important in the case when the function $U$ to be approximated is harmonic on $\mathbb{R}^d \setminus \overline{B^d}$ and the poles $\{y_j\}$ are in $B^d$ or $U$ is harmonic on $\mathbb{R}^d_+$ and the poles $\{y_j\}$ are in $\mathbb{R}^d_+$.

The results of A. Pekarski [27, 28] on rational approximation of holomorphic functions on the unit disc in $\mathbb{C}$ and also the results in [21] served as an inspiration and motivation for the development in this article. An important motivation to us also comes from some applications of Potential theory. In Geodesy people consider approximation of the gravitational (disturbing) potential using the potential of $n$ point masses. A given potential $U$ is approximated by the potential of $n$ point charges in Electrostatics or by the potential of $n$ magnetic poles in Magnetism. There is also a great deal of work done on the Method of Fundamental Solutions for
the Dirichlet problem of the Laplace equation in Numerical Analysis. This research is directly related to the problems we consider here. The multipole method of V. Rokhlin and his collaborators (e.g. [5, 13]) is also relevant to our undertaking.

We refer the reader to [2, 14, 19] for the basics of Potential theory.

The focus of this article is on the establishment of a direct (Jackson type) estimate for nonlinear $n$-term approximation of functions in the harmonic Hardy space $H^p(B^d)$, $0 < p < \infty$, from shifts of the Newtonian kernel. As one can expect the harmonic Besov spaces on the ball

$$B^r_{\tau}(\mathcal{H}) \quad \text{with} \quad 1/\tau = s/(d-1) + 1/p, \ s > 0,$$

will be naturally involved in the approximation process.

The poor localization of the Newtonian kernel is the first obstacle to overcome in approximating from linear combinations of its shifts. An important step forward in solving this approximation problem is the construction in [16] of highly localized summability kernels on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ that are restrictions to the sphere of linear combinations of a fixed number of shifts of the Newtonian kernel just as in (1.1). Note that the harmonic functions by their nature cannot be well localized in an open subset of $\mathbb{R}^d$, but they can be well localized on the boundary of such a set; typical examples are $S^{d-1}$ and $\mathbb{R}^{d-1}$.

To obtain our approximation result we proceed as follows: We first use the result from [16] to construct a pair of dual frames $\{\theta_\xi\}, \{\tilde{\theta}_\xi\}$ for all spaces of interest on $S^{d-1}$ whose elements $\{\theta_\xi\}$ are linear combinations of a fixed number of shifts of the Newtonian kernel and are well localized. Armed with these frames we apply an intermediate nonlinear $n$-term approximation from $\{\theta_\xi\}$ to the boundary value function/distribution $f_U$ of the harmonic function $U$ to be approximated. This leads us to the desired estimate by harmonic extension to $B^d$ of the approxiamant and using the fact that each $\theta_\xi$ is a finite linear combination of shifts of the Newtonian kernel.

Thus a major step in our development is to construct such a pair of dual frames. More precisely, one of our main goals is to construct (see Theorem 6.9) a pair of frames $\{\theta_\xi\}, \{\tilde{\theta}_\xi\}$ for the Besov and Triebel-Lizorkin spaces $B_p^q(S^{d-1})$ and $F_p^q(S^{d-1})$ with parameters $(s,p,q)$ in the range

$$Q = Q(A) := \{(s,p,q) : |s| \leq A, \ A^{-1} \leq p \leq A, \ \text{and} \ A^{-1} \leq q < \infty\},$$

where $A > 1$ is a fixed constant. This construction employs the small perturbation method for construction of frames developed in [9] and relies on the kernels from [16]. While the basic ideas behind the construction of the frames $\{\theta_\xi\}, \{\tilde{\theta}_\xi\}$ is relatively simple, some of the details become technical when applied to the specific case of this article. For example, the requirement that $\{\theta_\xi\}, \{\tilde{\theta}_\xi\}$ are frames for the class of Besov and Triebel-Lizorkin spaces $B_p^q(S^{d-1})$ and $F_p^q(S^{d-1})$ with parameters $(s,p,q) \in Q(A)$ compels us to carefully trace the constants appearing in all relevant estimates.

The next several remarks will perhaps clarify some of the issues arising in our construction of the pair of frames $\{\theta_\xi\}, \{\tilde{\theta}_\xi\}$ described above:

1. In applying the small perturbation method from [9] we use as a backbone a frame $\{\psi_\xi\}$ on $S^{d-1}$ from [25], which can characterize the Besov and Triebel-Lizorkin spaces $B_p^q(S^{d-1})$ and $F_p^q(S^{d-1})$ with complete range of parameters $(s,p,q)$, i.e. $s \in \mathbb{R}, \ 0 < p, q < \infty$. With the restriction that each frame element
{θξ} is a linear combination of a fixed number of shifts of the Newtonian kernel comes the natural limitation that the new frames {θξ}, {θ̃ξ} can characterize the Besov and Triebel-Lizorkin spaces \( B^q_p(\mathbb{S}^{d-1}) \) and \( F^q_p(\mathbb{S}^{d-1}) \) with parameters from \( Q(A) \) (see [14]). Here \( A > 1 \) can be arbitrarily large but is fixed and the number of shifts depends on \( A \).

(2) If the old frame \( \{ψξ\} \) is a basis, then the new frame \( \{θξ\} \) is also a basis. This is the case in dimension \( d = 2 \), where we use Meyer’s periodic wavelet basis on \( \mathbb{S}^1 \). As for now there are no convenient bases on \( \mathbb{S}^{d-1} \) when \( d > 2 \). For this reason we work with frames, which are completely satisfactory for our purposes.

(3) The rotation group on \( \mathbb{S}^{d-1} \) is not commutative in dimensions \( d \geq 3 \), which is a major difference from the translation group in \( \mathbb{R}^{d-1} \). This is an essential obstacle in constructing highly localized linear combinations of a fixed number of shifts of the Newtonian kernel with vanishing moments on \( \mathbb{S}^{d-1} \). In order to overcome this difficulty we replace the vanishing moment conditions on the \( ϕ \)-transform of Frazier and Jawerth with small moment conditions, see e.g. Propositions 2.3–2.5 and (6.15) in Theorem 6.1. In general, the vanishing moment conditions are not valid for \( θξ \).

(4) We restrict the parameters to \( p, q < \infty \) for several reasons. First, whenever \( p, q < \infty \) the Besov and Triebel-Lizorkin spaces \( B^q_p(\mathbb{S}^{d-1}) \) and \( F^q_p(\mathbb{S}^{d-1}) \) are separable and the finite sequences are dense in the respective Besov and Triebel-Lizorkin sequence spaces. Also, the respective frame representations converge unconditionally. These facts are important in the construction and utilization of the frames \( \{θξ\}, \{θ̃ξ\} \). Furthermore, as is well known, in general, nonlinear \( n \)-term approximation from frames or bases in \( L^∞ \) or as in our case \( \mathcal{F}_∞ \) is not quite natural. Just as in Harmonic analysis one should work in BMO instead.

The intimate relation between the harmonic Hardy and Besov spaces on \( B^d \) on the one hand and the Triebel-Lizorkin and Besov spaces of functions/distributions on \( \mathbb{S}^{d-1} \) on the other will play a critical role in our development. Harmonic Besov and Triebel-Lizorkin spaces on \( B^d \) and Besov and Triebel-Lizorkin spaces of distributions on \( \mathbb{S}^{d-1} \) with full range of parameters are treated in [15]. In particular, the equivalence of these spaces on \( B^d \) and on its boundary \( \mathbb{S}^{d-1} \) is established in [15]. These equivalences enable us to mediate between spaces and frames on \( B^d \) and on \( \mathbb{S}^{d-1} \). For example, it allows to transfer the constructed frame \( \{θξ\} \) on \( \mathbb{S}^{d-1} \) to a frame on \( B^d \) and approximation results from \( \mathbb{S}^{d-1} \) to \( B^d \).

Our main result in Theorem 7.1 asserts that if \( U ∈ B^s_p(\mathcal{H}) \) with \( s > 0 \) and \( 1/τ = s/(d-1) + 1/p \), then \( U ∈ \mathcal{F}^p(B^d) \) and

\[
E_n(U)_{\mathcal{F}^p} ≤ cn^{-s/(d-1)}\|U\|_{B^s_p(\mathcal{H})}, \quad n ≥ 1.
\]

We derive this estimate from a respective estimate for nonlinear \( n \)-term approximation of functions/distributions from \( \{θξ\} \) on the unit sphere \( \mathbb{S}^{d-1} \).

Denote by \( σ_n(f)_{F^p_0} \) the best \( n \)-term approximation of \( f \) in the Triebel-Lizorkin space \( F^p_0(\mathbb{S}^{d-1}) \) from the frame \( \{θξ\} \) mentioned above. We show that whenever \( f ∈ B^s_p(\mathcal{H}) \) with \( 1/τ = s/(d-1) + 1/p, 0 < p < ∞ \), then \( f ∈ F^p_0(\mathbb{S}^{d-1}) \) and

\[
σ_n(f)_{F^p_0(\mathbb{S}^{d-1})} ≤ cn^{-s/(d-1)}\|f\|_{B^s_p(\mathcal{H})}, \quad n ≥ 1.
\]

As is well known the harmonic Hardy space \( \mathcal{H}^p(B^d) \), \( 0 < p < ∞ \), can be identified with the Triebel-Lizorkin space \( F^p_0(\mathbb{S}^{d-1}) \) of functions/distributions on
NONLINEAR APPROXIMATION OF HARMONIC FUNCTIONS

$S^{d-1}$, and hence (1.5) implies that for any harmonic function $U \in B^s_{\infty}(\mathcal{H})$

\begin{equation}
\sigma_n(U)_{\mathcal{H}_p} \leq cn^{-s/(d-1)}\|U\|_{B^s_{\infty}(\mathcal{H})}, \quad n \geq 1,
\end{equation}

where $\sigma_n(U)_{\mathcal{H}_p}$ stands for the best $n$-term approximation of $U$ in $\mathcal{H}_p(B^d)$ from the harmonic extension to $B^d$ of $\{\theta_\xi\}$. Finally, estimate (1.6) yields (1.4) taking into account that each frame element $\theta_\xi$ is a linear combination of a fixed number of shifts of the Newtonian kernel.

It is insightful to study the nonlinear approximation from functions as in (1.1) in the norms of the closely related to $\mathcal{H}_p(B^d)$ harmonic Triebel-Lizorkin and Besov spaces $F^0_p(B^d)$ and $B^0_p(B^d)$. As shown in Theorem 7.3 the nonlinear $n$-term approximations in $F^0_p(B^d)$ have the optimal rate $O(n^{-s/(d-1)})$ for any $0 < q < \infty$, while the nonlinear $n$-term approximation in $B^0_p(B^d)$ achieves this optimal order only for $p \leq q < \infty$, see Theorems 7.10.

**Bernstein inequality: Conjecture.** We conjecture that the following Bernstein type inequality is valid: Let $1 < p < \infty$, $s > 0$, and $1/\tau = s/(d-1) + 1/p$. Then

\begin{equation}
\|G\|_{B^s_{\infty}(\mathcal{H})} \leq cn^{-s/(d-1)}\|G\|_{\mathcal{H}_p(\mathcal{H}_p)}, \quad \forall G \in N_n.
\end{equation}

If valid this estimate along with the Jackson estimate (1.4) would lead to a complete characterization of the rates of approximation (approximation spaces) of nonlinear $n$-term approximation in $\mathcal{H}_p(B^d)$, $1 < p < \infty$, from shifts of the Newtonian kernel.

It is natural to pose the question whether the approximation results of this paper hold when $p = \infty$. We think that just as in the case of rational approximation analogues of these results are valid if $\mathcal{H}_\infty(B^d)$ is replaced by the harmonic BMO space on $B^d$. We shall not pursue this line of research in the present article.

**Organization.** The outline of the paper is as follows. In Section 2 we introduce some basic notation and assemble background material about the maximal operator, spherical harmonics, and maximal $\delta$-nets; we also give some technical estimates on inner products of localized functions on the sphere. Section 3 presents some basic facts about harmonic Besov and Triebel-Lizorkin spaces on $B^d$ and $S^{d-1}$ developed in [15]; it also recalls the construction of frames for Besov and Triebel-Lizorkin spaces on $S^{d-1}$ and their frame decomposition developed in [25]. Section 4 presents and somewhat refines the small perturbation method for construction of frames developed in [9]. Section 5 deals with localization properties of the frame elements from [3] and highly localized kernels induced by shifts of the Newtonian kernel, developed in [16]. Section 6 contains the construction of a pair of frames for Besov and Triebel-Lizorkin spaces whose elements are finite linear combinations of shifts of the Newtonian kernel. Section 7 is devoted to nonlinear $n$-term approximation of functions in the harmonic Hardy spaces from shifts of the Newtonian kernel. Section 8 deals with nonlinear $n$-term approximation in the exterior of the unit ball in $\mathbb{R}^d$ and in the upper half space in $\mathbb{R}^d$ from shifts of the Newtonian kernel. Proofs of key estimates supporting our main results are given in Section 9.

**Notation.** Throughout this article the constants $d$, $M$, and $K$ will appear frequently. Here $d \in \mathbb{N}$ is the dimension of the space $\mathbb{R}^d$, $M > 0$ determines decay rates, and $K \in \mathbb{N}$ is a parameter determining the upper bound of the order of derivatives required from some functions. Positive constants will be denoted by $c$ and they may vary at every occurrence. Most of these constants will depend only on $d$, $K$, $M$. By $C$’s we denote numbers (constants) that also depend on parameters different from $d$, $K$, $M$. When we would like to trace the dependence of a constant $c$
on these parameters we use indexing, e.g. $c_1, c_2$, etc. or indicate the dependence on parameters in parenthesis. These indexed constants preserve their values throughout the article. The relation $a \sim b$ means that there exists a constant $c \geq 1$ such that $c^{-1} a \leq b \leq c a$.

2. Background and technical ground work

2.1. Basic notation and simple inequalities. In this article we use standard notation. Thus $\mathbb{R}^d$ stands for the $d$-dimensional Euclidean space. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y = \sum_{k=1}^{d} x_k y_k$ and the Euclidean norm of $x$ by $|x| = \sqrt{x \cdot x}$. We write $B(x_0, r) := \{ x : |x - x_0| < r \}$ and set $B^d := B(0, 1)$, the open unit ball in $\mathbb{R}^d$.

As usual $\mathbb{N}_0$ stands for the set of non-negative integers. For $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ the monomial $x^{\beta}$ is defined by $x^{\beta} := x_1^{\beta_1} \cdots x_d^{\beta_d}$ and its degree is $|\beta| := \beta_1 + \cdots + \beta_d$. The set of all polynomials in $\mathbb{R}^d$ of total degree $n$ is denoted by $\mathcal{P}_n^d$. We denote $\partial_k := \partial/\partial x_k$ and then $\partial^{\beta} := \partial^{\beta_1}_1 \cdots \partial^{\beta_d}_d$ is a differential operator of order $|\beta|$, the gradient operator is $\nabla := (\partial_1, \ldots, \partial_d)$, and $\Delta := \partial^2_1 + \cdots + \partial^2_d$ stands for the Laplacian. When necessary we indicate the variable of differentiation by a subscript, e.g. $\partial^2_x$.

The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1} := \{ x : |x| = 1 \}$. We denote by $\rho(x, y)$ the geodesic distance between $x, y \in S^{d-1}$, that is, $\rho(x, y) := \arccos(x \cdot y)$. The open spherical cap (ball on the sphere) centred at $\eta \in S^{d-1}$ of radius $r$ is denoted by $B(\eta, r) = \{ x \in S^{d-1} : \rho(\eta, x) < r \}$. We denote by $\Delta_0$ the Laplace-Beltrami operator on $S^{d-1}$. As is well known (e.g. [S] Theorem 1.8.2) $\Delta_0$ has the decomposition

\begin{equation}
\Delta_0 = \sum_{1 \leq i < \ell \leq d} D^2_{i,\ell}, \quad D_{i,\ell} = x_i \partial_\ell - x_\ell \partial_i, \quad x \in S^{d-1}.
\end{equation}

For any function $g$ on $S^{d-1}$ we denote by $\tilde{g}$ its standard extension, defined by

\begin{equation}
\tilde{g}(x) := g(x/|x|) \quad \text{for} \quad x \in \mathbb{R}^d \setminus \{0\}.
\end{equation}

As is well known (e.g. [S] Corollary 1.4.3) or [29]) for any $g \in C^2(S^{d-1})$

\begin{equation}
\Delta \tilde{g}(x) = \Delta_0 g(x), \quad x \in S^{d-1}.
\end{equation}

By definition $g \in W^K(S^{d-1}), K \in \mathbb{N}_0$, if $||\partial^\beta \tilde{g}||_{L^\infty(B(0, 2) \setminus B(0, 1/2))} \leq C, \forall |\beta| \leq K$.

The Lebesgue measure on $S^{d-1}$ is denoted by $\sigma$ and we set $|E| := \sigma(E)$ for a measurable set $E \subset S^{d-1}$. Thus, $\omega_d := |S^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$.

The inner product of $f, g \in L^2(S^{d-1})$ is given by

\[ (f, g) := \int_{S^{d-1}} f(y) g(y) \, d\sigma(y). \]

The nonstandard convolution of functions $F \in L^\infty[-1, 1]$ and $g \in L(S^{d-1})$ is defined by

\begin{equation}
F \ast g(x) := \langle F(x \cdot \cdot), \tilde{g} \rangle = \int_{S^{d-1}} F(x \cdot y) g(y) \, d\sigma(y).
\end{equation}

We say that a function $f$ defined on $S^{d-1}$ is localized around $\eta \in S^{d-1}$ with dilation factor $N$ and decay rate $M > 0$ if the estimate

\begin{equation}
|f(x)| \leq \kappa N^{d-1} (1 + N \rho(\eta, x))^{-M}, \quad x \in S^{d-1},
\end{equation}
holds for some constant $\kappa > 0$ independent of $N, x, \eta$. The multiplier $N^{d-1}$ is used as part of the decay function in (2.5) in order to have $\|f\|_{L^1(\mathbb{S}^{d-1})} \leq c$. Namely, for $M > d - 1$ we have

\[ \int_{\mathbb{S}^{d-1}} \frac{N^{d-1}}{(1 + N \rho(\eta, y))^{M}} d\sigma(y) \leq c_0, \quad \forall \eta \in \mathbb{S}^{d-1}, \quad \forall N \geq 1, \]

where $c_0 = c(d)/(M - d + 1)$ depends only on $d$ and $M$. The weight function in the right-hand side of (2.5) also has the property: For any $\eta_1, \eta_2 \in \mathbb{S}^{d-1}$ with $\rho(\eta_1, \eta_2) \leq N^{-1}$

\[ (1 + N \rho(\eta_2, x))^{-1} \leq 2(1 + N \rho(\eta_1, x))^{-1}, \quad \forall x \in \mathbb{S}^{d-1}. \]

Indeed, $1 + N \rho(\eta_1, x) \leq 1 + N(\rho(\eta_1, \eta_2) + \rho(\eta_2, x)) \leq 2 + N \rho(\eta_2, x)$, which implies (2.7).

Another simple inequality that will be useful is:

\[ |\Delta_0^{K/2}x^\beta| \leq c_6, \quad \forall x \in \mathbb{S}^{d-1}, \quad 0 \leq |\beta| \leq K, \]

where $c_6$ depends only on $d$ and $K$.

2.2. Spherical harmonics. The spherical harmonics will be our main vehicle in dealing with harmonic functions on the unit ball $B^d$ in $\mathbb{R}^d$.

Denote by $\mathcal{H}_k$ the space of all spherical harmonics of degree $k$ on $\mathbb{S}^{d-1}$. As is well known the dimension of $\mathcal{H}_k$ is $N(k, d) = \frac{2k + d + 1}{k} \binom{k}{d} \sim k^{d-1}$. Furthermore, the spaces $\mathcal{H}_k, k = 0, 1, \ldots$, are orthogonal and $L^2(\mathbb{S}^{d-1}) = \bigoplus_{k \geq 0} \mathcal{H}_k$.

Let $\{Y_{kv} : v = 1, \ldots, N(k, d)\}$ be a real-valued orthonormal basis for $\mathcal{H}_k$. Then the kernel of the orthogonal projector onto $\mathcal{H}_k$ is given by

\[ Z_k(x \cdot y) = \sum_{v=1}^{N(k,d)} Y_{kv}(x)Y_{kv}(y), \quad x, y \in \mathbb{S}^{d-1}. \]

As is well known (see e.g. [3, Theorem 1.2.6])

\[ Z_k(x \cdot y) = \frac{k + \mu}{\mu \omega_d} C_k^\mu(x \cdot y), \quad x, y \in \mathbb{S}^{d-1}, \quad \mu := \frac{d - 2}{2}, \quad d > 2. \]

Here $C_k^\mu$ is the Gegenbauer (ultraspherical) polynomial of degree $k$ normalized by $C_k^\mu(1) = \binom{k+2\mu-1}{k}$. The Gegenbauer polynomials are usually defined by the following generating function

\[ (1 - 2uz + z^2)^{-\mu} = \sum_{k=0}^{\infty} C_k^\mu(u)z^k, \quad |z| < 1, \quad |u| < 1. \]

The polynomials $C_k^\mu, k = 0, 1, \ldots$, are orthogonal in the space $L^2([-1, 1], w)$ with weight $w(u) := (1 - u^2)^{\mu - 1/2}$, see [23, p. 80, (4.7.1)] or [26, Table 18.3.1]. In the case $d = 2$ the kernel of the orthogonal projector onto $\mathcal{H}_k$ takes the form

\[ Z_0(x \cdot y) = \frac{1}{2\pi}, \quad Z_k(x \cdot y) = \frac{1}{\pi} T_k(x \cdot y), \quad k \geq 1, \]

where $T_k(u) := \cos(n \arccos u)$ is the $k$-th degree Chebyshev polynomial of the first kind. We refer the reader to [23, 51] for the basics of spherical harmonics.

As is well known (see e.g. [3, Theorem 1.4.5]) the spherical harmonics are eigenfunctions of the Laplace-Beltrami operator $\Delta_0$ on $\mathbb{S}^{d-1}$, namely,

\[ -\Delta_0 Y(x) = k(k + d - 2) Y(x), \quad x \in \mathbb{S}^{d-1}, \quad \forall Y \in \mathcal{H}_k. \]
The set of all band-limited functions (i.e. spherical polynomials) on $S^{d-1}$ of degree $\leq N$ will be denoted by $\Pi_N$, i.e. $\Pi_N := \bigoplus_{k=0}^N \mathcal{H}_k$.

The Poisson kernel on the unit ball $B^d$ is given by

$$P(y, x) := \sum_{k=0}^{\infty} |x|^k Z_k \left( \frac{x}{|x|} \cdot y \right) = \frac{1}{\omega_d} \frac{1 - |x|^2}{|x - y|^d}, \quad |x| < 1, \ y \in S^{d-1}. \tag{2.12}$$

Kernels on the sphere $S^{d-1}$ of the form

$$A_N(x \cdot y) := \sum_{k=0}^{\infty} \lambda(k/N) Z_k(x \cdot y), \quad x, y \in S^{d-1}, \ N \geq 1, \tag{2.13}$$

where $\lambda \in C^\infty(0, \infty)$ is compactly supported, will play a key role in this article. Observe that in this case

$$A_N(u) := \sum_{k=0}^{\infty} \lambda(k/N) Z_k(u), \quad u \in [-1, 1], \tag{2.14}$$

is simply a polynomial kernel. The localization of this kernel is given in the following

**Theorem 2.1.** Let $\nu \geq 0$ and $M \in \mathbb{N}$. Assume $\lambda \in C^\infty(0, \infty)$, $||\lambda^{(m)}||_\infty \leq \kappa$ for $0 \leq m \leq M$ and either $\sup \lambda \subset [1/4, 1]$ or $\sup \lambda \subset [0, 2]$ and $\lambda(t) = 1$ for $t \in [0, 1]$. Then there exists a constant $c > 0$ depending only on $M, \nu,$ and $d$ such that for any $N \geq 1$ the kernel $A_N$ from (2.13) – (2.14) obeys

$$|\Lambda_N^{(\nu)}(\cos \theta)| \leq c \frac{N^{d-1+2\nu}}{(1 + N|\theta|)^M}, \quad |\theta| \leq \pi, \tag{2.15}$$

and hence

$$|\Lambda_N^{(\nu)}(x \cdot y)| \leq c \frac{N^{d-1+2\nu}}{(1 + N \rho(x, y))^M}, \quad x, y \in S^{d-1}. \tag{2.16}$$

Furthermore, for $x, y, z \in S^{d-1}$

$$|\Lambda_N(x \cdot z) - \Lambda_N(y \cdot z)| \leq c \frac{\rho(x, y) N^d}{(1 + N \rho(x, z))^M}, \quad \text{if } \rho(x, y) \leq N^{-1}. \tag{2.17}$$

For a proof, see [24, Theorem 3.5] and [25, Lemmas 2.4, 2.6], also [17].

### 2.3. Maximal $\delta$-nets and cubature formulas on the sphere.

For discretization of integrals and construction of frames on $S^{d-1}$ we shall need cubature formulas, which are naturally constructed using maximal $\delta$-nets on $S^{d-1}$.

**Definition.** Given $\delta > 0$ we say that a finite set $Z \subset S^{d-1}$ is a maximal $\delta$-net on $S^{d-1}$ if (i) $\rho(\zeta_1, \zeta_2) \geq \delta$ for all $\zeta_1, \zeta_2 \in Z$, $\zeta_1 \neq \zeta_2$, and (ii) $\bigcup_{\zeta \in Z} B(\zeta, \delta) = S^{d-1}$.

Clearly, a maximal $\delta$-net on $S^{d-1}$ exists for any $\delta > 0$. For every maximal $\delta$-net $Z$ it is easy to construct (see [3, Proposition 2.5]) a disjoint partition $\{A_\zeta\}_{\zeta \in Z}$ of $S^{d-1}$ consisting of measurable sets such that

$$B(\zeta, \delta/2) \subset A_\zeta \subset B(\zeta, \delta), \quad \zeta \in Z. \tag{2.18}$$

Two kinds of cubature formulas on $S^{d-1}$ will be utilized.

**Simple cubature formulas on $S^{d-1}$.** Let $0 < \gamma \leq 1$ be a parameter to be selected. Let $Z_j \subset S^{d-1}$ ($j \in \mathbb{N}$) be a maximal $\delta_j$-net on $S^{d-1}$ with $\delta_j := \gamma 2^{-j+1}$. We shall use the cubature formula

$$\int_{S^{d-1}} f(x) d\sigma(x) \approx \sum_{\zeta \in Z_j} w_\zeta f(\zeta), \quad w_\zeta := |A_\zeta|, \tag{2.19}$$
where \( A_\zeta \) is from (2.18) with \( Z = Z_j, \delta = \delta_j \). The cubature (2.19) is apparently exact for all constants. Evidently,

\[
(2.20) \quad w_\zeta = |A_\zeta| \sim (\gamma 2^{-j+1})^{d-1}
\]

with constants of equivalence depending only on \( \delta \). Note that (2.18) implies that the number of elements in \( Z_j \) is \( \leq c(d)(\gamma^{-1}2^j)^{d-1} \).

Further, given \( j \in \mathbb{N} \) we define a map \( \zeta \) from \( S^{d-1} \) to \( Z_j \) as follows: For every \( y \in S^{d-1} \) we set \( \zeta(y) := \eta \in Z_j \) if \( y \in A_\eta \). We shall use this map in Lemmas 6.3 and 6.4 below.

**Nontrivial cubature formulas on** \( S^{d-1} \). Let \( X_j \subseteq S^{d-1} \) be a maximal \( \delta_j \)-net on \( S^{d-1} \) with \( \delta_j := \gamma 2^{-j+1}, 0 < \gamma < 1, j \geq 1 \). In [24] Theorem 4.3 it is shown that there exist \( \gamma \) (0 < \( \gamma \) < 1), depending only on \( d \), and weights \( \{ \tilde{w}_\xi \}_{\xi \in X_j} \), satisfying

\[
(2.21) \quad c_\gamma^{-1}2^{-j(d-1)} \leq \tilde{w}_\xi \leq c_\gamma 2^{-j(d-1)}, \quad \xi \in X_j
\]

with constant \( c_\gamma \) depending only on \( d \), such that the cubature formula

\[
(2.22) \quad \int_{S^{d-1}} f(x) d\sigma(x) \approx \sum_{\xi \in X_j} \tilde{w}_\xi f(\xi)
\]

is exact for all spherical harmonics \( f \) of degree \( \leq 2j+1, j \geq 1 \).

As before, the number of nodes in \( X_j \) is \( \leq c(d)(\gamma^{-1}2^j)^{d-1} \), since \( X_j \) is a maximal \( \delta_j \)-net on \( S^{d-1} \). Also the disjoint partition \( \{ A_\xi \}_{\xi \in X_j} \) of \( S^{d-1} \) exists with \( B(\xi, \delta_j/2) \subset A_\xi \subset B(\xi, \delta_j) \), but the equality \( \tilde{w}_\xi = |A_\xi| \) does not hold in general.

2.4. **Maximal operator.** The maximal operator is an important technical tool when dealing with Besov and Triebel-Lizorkin spaces. We shall use the following version of the Hardy-Littlewood maximal operator:

\[
(2.23) \quad M_tf(x) := \sup_{B \ni x} \left( \frac{1}{|B|} \int_B |f|^t \, d\sigma \right)^{1/t}, \quad x \in S^{d-1}, t > 0,
\]

where the sup is over all spherical caps \( B \subset S^{d-1} \) such that \( x \in B \).

The Fefferman-Stein vector-valued maximal inequality (see [30] Ch. II (13), p. 56) can be written in the form: If \( 0 < p < \infty, 0 < q \leq \infty, \) and \( 0 < t < \min\{p, q\} \), then for any sequence of measurable functions \( \{ f_\nu \} \) on \( S^{d-1} \)

\[
(2.24) \quad \left\| \left( \sum_\nu |M_t f_\nu(\cdot)|^q \right)^{1/q} \right\|_{L^p} \leq \tilde{c}_1 \left\| \left( \sum_\nu |f_\nu(\cdot)|^q \right)^{1/q} \right\|_{L^p}.
\]

From Theorem 2.1 in [12] it follows that the constant \( \tilde{c}_1 \) above can be written in the form

\[
(2.25) \quad \tilde{c}_1 = \left( c^* \max \left\{ \frac{p}{t}, \left( \frac{p}{t} - 1 \right)^{-1} \right\} \max \left\{ 1, \left( \frac{q}{t} - 1 \right)^{-1} \right\} \right)^{1/t},
\]

where \( c^* > 0 \) is a constant depending only on \( d \).

Note that the area/volume of a spherical cap \( B(x, r) \) on \( S^{d-1}, d \geq 2 \), is given by

\[
|B(x, r)| = \omega_{d-1} \int_0^r \sin^{d-2} v \, dv.
\]

Hence

\[
(2.26) \quad |B(x_1, r_1)|/|B(x_2, r_2)| \leq (r_1/r_2)^{d-1}, \quad 0 < r_2 \leq r_1 \leq \pi, \quad x_1, x_2 \in S^{d-1},
\]

\[
(2.27) \quad 1/\tilde{c}_2 \leq |B(x, r)|/r^{d-1} \leq \tilde{c}_2, \quad 0 < r \leq \pi, \quad x \in S^{d-1},
\]
where $\tilde{c}_2$ is a constant depending only on $d$.

2.5. Inner products of zonal functions. A function $f$ on $\mathbb{S}^{d-1}$ is zonal if it is invariant under rotation about a fixed axis. If this axis is in the direction of $\eta \in \mathbb{S}^{d-1}$, then $f$ can be represented as $f(x) = F(\eta \cdot x)$, $x \in \mathbb{S}^{d-1}$, for an appropriate function $F : [-1, 1] \to \mathbb{R}$.

**Lemma 2.2.** Let $F, G \in L^\infty[-1, 1]$. Then there exists $H \in C[-1, 1]$ such that

$$H(x \cdot z) = \int_{\mathbb{S}^{d-1}} F(x \cdot y)G(y \cdot z) \, d\sigma(y), \quad \forall x, z \in \mathbb{S}^{d-1}. \tag{2.28}$$

**Proof.** Assume first that $F$ and $G$ are algebraic polynomials of degree $m$. Then we can expand them in Gegenbauer polynomials to obtain $F = \sum_{k=0}^m \tilde{F}_k Z_k$ and $G = \sum_{k=0}^m \tilde{G}_k Z_k$. Using that $Z_k(x \cdot y)$ is the kernel of the orthogonal projector onto $H_k$ we have $\int_{\mathbb{S}^{d-1}} Z_k(x \cdot y)Z_k(y \cdot z) \, d\sigma(y) = Z_k(x \cdot z)$ (see (2.19)). Therefore,

$$\int_{\mathbb{S}^{d-1}} F(x \cdot y)G(y \cdot z) \, d\sigma(y) = \sum_{k=0}^m \tilde{F}_k \tilde{G}_k = H(x \cdot z),$$

where $H$ is an algebraic polynomial of degree $m$. Thus (2.28) holds for polynomials. Finally, a limiting argument implies that (2.28) is valid in general. \qed

From Lemma 2.2 it follows that for any $F, G \in L^\infty[-1, 1]$

$$\int_{\mathbb{S}^{d-1}} F(x \cdot y)G(y \cdot z) \, d\sigma(y) = \int_{\mathbb{S}^{d-1}} F(z \cdot y)G(y \cdot x) \, d\sigma(y), \quad \forall x, z \in \mathbb{S}^{d-1}. \tag{2.29}$$

2.6. Inner products of localized functions. The estimation of the inner products of well localized functions and functions with small moments on the sphere will play a key role in our further development. The following proposition is an analogue of [10] Lemma B1. We replace the vanishing moment condition used in [10] by the weaker “small moments” condition (2.32).

**Proposition 2.3.** Let $K \in \mathbb{N}$, $M > K + d - 1$, $N_2 \geq N_1 \geq 1$ ($N_1, N_2 \in \mathbb{R}$), and $\kappa_1, \kappa_2 > 0$. Assume $f \in L^\infty(\mathbb{S}^{d-1})$ and $g \in W^K(\mathbb{S}^{d-1})$, see (2.1). Furthermore, assume that for some $x_1, x_2 \in \mathbb{S}^{d-1}$

$$|\partial^\beta \tilde{g}(y)| \leq \frac{\kappa_1 N_1^{(|\beta|+d-1)}}{(1+N_1 \rho(x_1, y))^{dM}}, \quad \forall y \in \mathbb{S}^{d-1}, \quad 0 \leq |\beta| \leq K, \tag{2.30}$$

$$|f(y)| \leq \frac{\kappa_2 N_2^{-d-1}}{(1+N_2 \rho(x_2, y))^{dM}}, \quad \forall y \in \mathbb{S}^{d-1}, \quad \text{and} \tag{2.31}$$

$$|\int_{\mathbb{S}^{d-1}} y^\beta f(y) \, d\sigma(y)| \leq \kappa_2 N_2^{-K}, \quad 0 \leq |\beta| \leq K - 1. \tag{2.32}$$

Then

$$|\langle g, f \rangle| = \left| \int_{\mathbb{S}^{d-1}} g(y)f(y) \, d\sigma(y) \right| \leq c_1 \frac{\kappa_1 \kappa_2 (N_1/N_2)^K N_1^{d-1}}{(1+N_1 \rho(x_1, x_2))^{dM}}, \tag{2.33}$$

where $c_1$ depends only on $d, K, \text{and} \ M$. Above $\tilde{g}(y) := g(y/|y|)$ for $y \in \mathbb{R}^d \setminus \{0\}$ just as in (2.2).

For cases where condition (2.32) may not be satisfied we modify Proposition 2.3 as follows.
Proposition 2.4. Let $M > d$, $N_2 \geq N_1 \geq 1$, and $\kappa_1, \kappa_2 > 0$. Let $f \in L^\infty(S^{d-1})$ and $g \in W^1_{\infty}(S^{d-1})$. Also, assume that for some $x_1, x_2 \in S^{d-1}$

\begin{equation}
|\partial^\alpha g(y)| \leq \frac{\kappa_1 N_1^d}{(1 + N_1 \rho(x_1, y))^M}, \quad \forall y \in S^{d-1}, \quad |\alpha| = 1,
\end{equation}

\begin{equation}
|f(y)| \leq \frac{\kappa_2 N_2^{d-1}}{(1 + N_2 \rho(x_2, y))^M}, \quad \forall y \in S^{d-1}.
\end{equation}

Then

\begin{equation}
\left| \int_{S^{d-1}} g(y) \overline{f(y)} \, d\sigma(y) - g(x_2) \int_{S^{d-1}} \overline{f(y)} \, d\sigma(y) \right| \leq c_2 \frac{\kappa_1 \kappa_2 (N_1/N_2) N_1^{d-1}}{(1 + N_1 \rho(x_1, x_2))^M},
\end{equation}

where $c_2$ depends only on $d$ and $M$.

In cases where only the function localizations are known/matter we use

Proposition 2.5. Let $M > d - 1$, $N_2 \geq N_1 \geq 1$, and $\kappa_1, \kappa_2 > 0$. Let $f, g \in L^\infty(S^{d-1})$, and for some $x_1, x_2 \in S^{d-1}$

\begin{equation}
|g(y)| \leq \frac{\kappa_1 N_1^{d-1}}{(1 + N_1 \rho(x_1, y))^M}, \quad \forall y \in S^{d-1},
\end{equation}

\begin{equation}
|f(y)| \leq \frac{\kappa_2 N_2^{d-1}}{(1 + N_2 \rho(x_2, y))^M}, \quad \forall y \in S^{d-1}.
\end{equation}

Then

\begin{equation}
|\langle g, f \rangle| = \left| \int_{S^{d-1}} g(y) \overline{f(y)} \, d\sigma(y) \right| \leq c_3 \frac{\kappa_1 \kappa_2 N_1^{d-1}}{(1 + N_1 \rho(x_1, x_2))^M},
\end{equation}

where $c_3$ depends only on $d$ and $M$.

To streamline our presentation we defer the proofs of Propositions 2.3, 2.4, and 2.5 to Section 9.

3. Spaces of functions and distributions on the ball and sphere

The theory of harmonic Besov and Triebel-Lizorkin spaces on $B^d$, and their relation with the respective Besov and Triebel-Lizorkin spaces of distributions on $S^{d-1}$ is developed in [13]. In this section we review all definitions and results that will be needed from [13].

Denote by $\mathcal{H}(B^d)$ the set of all harmonic functions on the unit ball $B^d$ in $\mathbb{R}^d$.

3.1. Harmonic Besov and Triebel-Lizorkin spaces on $B^d$. It is convenient to define the harmonic Besov and Triebel-Lizorkin spaces on $B^d$ by using their expansion in solid spherical harmonics. As in §2.2 let $\{Y_{kj} : j = 1, \ldots, N(k, d)\}$ be a real-valued orthonormal basis for $\mathcal{H}_k$. The harmonic coefficients of $U \in \mathcal{H}(B^d)$ are defined by

\begin{equation}
b_{kj}(U) := \frac{1}{a^k} \int_{S^{d-1}} U(a\eta)Y_{kj}(\eta) d\sigma(\eta)
\end{equation}

for some $0 < a < 1$. It is an important observation that the coefficients are independent of $a$ for all $0 < a < 1$. This implies the representation

\begin{equation}
U(r\xi) = \sum_{k=0}^{\infty} \sum_{r=1}^{N(k, d)} b_{kj}(U) r^k Y_{kj}(\xi), \quad 0 \leq r < 1, \quad \xi \in S^{d-1},
\end{equation}
where the convergence is absolute and uniform on every compact subset of $B^d$.

For $U \in \mathcal{H}(B^d)$ and $\beta \in \mathbb{R}$ we define

$$J^\beta U(r\xi) := \sum_{k=0}^{\infty} r^k (k+1)^{-\beta} \sum_{\nu=1}^{N(k,d)} b_{k\nu}(U) Y_{k\nu}(\xi), \quad 0 \leq r < 1, \; \xi \in S^{d-1}.$$  \hspace{1cm} (3.3)

The above series converges absolutely and uniformly on every compact subset of $B^d$ and hence $J^\beta U$ is a well defined harmonic function on $B^d$.

Definition 3.1. Let $s \in \mathbb{R}$, $0 < q \leq \infty$, and $\beta := s + 1$.

(a) The harmonic Besov space $B^s_p(\mathcal{H})$, $0 < p \leq \infty$, is defined as the set of all $U \in \mathcal{H}(B^d)$ such that

$$\|U\|_{B^s_p(\mathcal{H})} := \left( \int_0^1 (1-r)^{(\beta-s)q} \|J^{-\beta} U(r\cdot)\|^q_{L^p(S^{d-1})} \frac{dr}{1-r} \right)^{1/q} < \infty \quad \text{if } q \neq \infty$$

and

$$\|U\|_{B^s_\infty(\mathcal{H})} := \sup_{0 < r < 1} (1-r)^{\beta-s} \|J^{-\beta} U(r\cdot)\|_{L^p(S^{d-1})} < \infty.$$  \hspace{1cm} (3.4)

(b) The harmonic Triebel-Lizorkin space $F^s_p(\mathcal{H})$, $0 < p < \infty$, is defined as the set of all $U \in \mathcal{H}(B^d)$ such that

$$\|U\|_{F^s_p(\mathcal{H})} := \left( \int_0^1 (1-r)^{(\beta-s)q} \|J^{-\beta} U(r\cdot)\|^q_{L^p(S^{d-1})} \frac{dr}{1-r} \right)^{1/q} < \infty \quad \text{if } q \neq \infty$$

and

$$\|U\|_{F^s_\infty(\mathcal{H})} := \sup_{0 < r < 1} (1-r)^{\beta-s} \|J^{-\beta} U(r\cdot)\|_{L^p(S^{d-1})} < \infty.$$  \hspace{1cm} (3.5)

Choosing an arbitrary $\beta > s$ above results in equivalent quasi-norms for the spaces $B^s_p(\mathcal{H})$ and $F^s_p(\mathcal{H})$.

3.2. Besov and Triebel-Lizorkin spaces on $S^{d-1}$. The Besov and Triebel-Lizorkin spaces on $S^{d-1}$ in general are spaces of distributions. As test functions we use the class $S := C^\infty(S^{d-1})$ of all functions $\phi$ on $S^{d-1}$ such that

$$\|Z_k * \phi\|_2 \leq c(\phi, m)(1+k)^{-m}, \quad \forall k, m \geq 0.$$

Recall that the convolution $Z_k * \phi$ is defined in (2.4). The topology on $S$ is defined by the sequence of norms

$$P_m(\phi) := \sum_{k=0}^{\infty} (k+1)^m \|Z_k * \phi\|_2 = \sum_{k=0}^{\infty} (k+1)^m \left( \sum_{\nu=1}^{N(k,d)} \|\langle \phi, Y_{k\nu} \rangle\|^2 \right)^{1/2}.$$  \hspace{1cm} (3.6)

$S$ is complete in this topology.

Observe that all $Y_{k\nu} \in S$ and hence by (2.9) $Z_k(x \cdot y) \in S$ as a function of $x$ for every fixed $y$ and as function of $y$ for every fixed $x$.

The space $S' := S'(S^{d-1})$ of distributions on $S^{d-1}$ is defined as the space of all continuous linear functionals on $S$. The pairing of $f \in S'$ and $\phi \in S$ will be denoted by $\langle f, \phi \rangle := f(\hat{\phi})$, which is consistent with the inner product on $L^2(S^{d-1})$. More precisely, $S'$ consists of all linear functionals $f$ on $S$ for which there exist constants $c > 0$ and $m \in \mathbb{N}_0$ such that

$$\|f\|_S \leq c P_m(\phi), \quad \forall \phi \in S.$$  \hspace{1cm} (3.7)

For any $f \in S'$ we define $Z_k * f$ by

$$Z_k * f(x) := \langle f, Z_k(x \cdot \bullet) \rangle = \langle f, Z_k(x \cdot \bullet) \rangle.$$  \hspace{1cm} (3.8)
where on the right $f$ is acting on $Z_k(x \cdot y) = Z_k(x)$ as a function of $y$ ($Z_k$ is real-valued).

Observe that the representation

$$f = \sum_{k=0}^{\infty} Z_k * f, \quad \forall f \in S'$$

holds with convergence in distributional sense.

**Definition 3.2.** Let $s \in \mathbb{R}$, $0 < q \leq \infty$, and $\varphi$ satisfy the conditions: $\varphi \in C^\infty(\mathbb{R}_+)$, supp $\varphi \subset [1/2, 2]$, and $|\varphi(u)| \geq c > 0$ for $u \in [3/5, 5/3]$. For a distribution $f \in S'$ set

$$\Phi_0 * f = Z_0 * f, \quad \Phi_j * f = \sum_{k=0}^{\infty} \varphi\left(\frac{k}{2^{j-1}}\right) Z_k * f, \quad j \geq 1,$$

where $Z_k * f$ is defined in (3.6).

(a) The Besov space $B_p^s := B_p^s(S^{d-1})$, $0 < p \leq \infty$, is defined as the set of all distributions $f \in S'$ such that

$$\|f\|_{B_p^s} := \left(\sum_{j=0}^{\infty} \left(2^{sj} \|\Phi_j * f\|_{L_p(S^{d-1})}\right)^q \right)^{1/q} < \infty,$$

where the $\ell^q$-norm is replaced by the sup-norm if $q = \infty$.

(b) The Triebel-Lizorkin space $F_p^s := F_p^s(S^{d-1})$, $0 < p \leq \infty$, is defined as the set of all distributions $f \in S'$ such that

$$\|f\|_{F_p^s} := \left\|(\sum_{j=0}^{\infty} \left(2^{sj} |\Phi_j * f(\cdot)|^q \right)^{1/q}\right\|_{L_p(S^{d-1})} < \infty,$$

where the $\ell^q$-norm is replaced by the sup-norm if $q = \infty$.

Note that the definitions of the Besov and Triebel-Lizorkin spaces above are independent of the particular selection of the function $\varphi$ with the required properties, that is, different $\varphi$'s produce equivalent quasi-norms.

### 3.3. Identification of harmonic Besov and Triebel-Lizorkin spaces.

We are interested in harmonic functions $U \in \mathcal{H}(B^d)$ with coefficients of at most polynomial growth:

$$|b_{k\nu}(U)| \leq c(k + 1)^\gamma, \quad \nu = 1, \ldots, N(k, d), \quad k = 0, 1, \ldots,$$

for some constants $\gamma, c > 0$. The functions in the harmonic Besov and Triebel-Lizorkin spaces have this property.

The relationship between harmonic functions on $B^d$ and distributions on $S^{d-1}$ is clarified by the following

**Proposition 3.3.** (a) To any $U \in \mathcal{H}(B^d)$ represented by (3.2) with coefficients satisfying (3.11) there corresponds a distribution $f \in S'$, $f = f_U$, (the boundary value function/distribution of $U$) defined by

$$f := \sum_{k=0}^{\infty} \sum_{\nu=1}^{N(k,d)} b_{k\nu}(U) Y_{k\nu} \quad (convergence \ in \ S')$$

with coefficients $b_{k\nu}(U) \in (f, Y_{k\nu})$. 

This proposition provides a bridge between the harmonic functions and their distributional counterparts, allowing for a deeper understanding of the approximation properties of harmonic functions in various spaces.
coefficients the same as the coefficients of $U_{f}$ valid. Consequently, there exists a distribution
Theorem 3.4.
Let $f \in \mathcal{S}'$ with coefficients $B_{k\nu}$ with coefficients $B_{k\nu}$ obeying (3.11), where the series converges uni-
formly on every compact subset of $B^{d}$.
(c) For every $U \in \mathcal{H}(B^{d})$ we have $U_{f_{U}} = U$ and for every $f \in \mathcal{S}'$ we have $f_{U_{f}} = f$.

The principle results of this subsection are:

**Theorem 3.4.** Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. A harmonic function $U \in F_{p}^{sq}(\mathcal{H})$ if and only if its boundary value distribution $f = f_{U}$ defined by (3.12) belongs to $\mathcal{F}_{p}^{sq}(S^{d-1})$, moreover $\|U\|_{F_{p}^{sq}} \sim \|f\|_{\mathcal{F}_{p}^{sq}}$.

**Theorem 3.5.** Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. A harmonic function $U \in B_{p}^{sq}(\mathcal{H})$ if and only if its boundary value distribution $f = f_{U}$ defined by (3.12) belongs to $B_{p}^{sq}(S^{d-1})$, moreover $\|U\|_{B_{p}^{sq}} \sim \|f\|_{B_{p}^{sq}}$.

3.4. Harmonic Hardy spaces. Here we consider the harmonic Hardy spaces $\mathcal{H}^{p}(B^{d})$ on the ball (usually denoted by $h^{p}(B^{d})$).

**Definition 3.6.** The space $\mathcal{H}^{p} := \mathcal{H}^{p}(B^{d})$, $0 < p \leq \infty$, is defined as the set of all harmonic functions $U \in \mathcal{H}(B^{d})$ such that

$$
\|U\|_{\mathcal{H}^{p}} := \|\sup_{0 \leq r < 1} |U(r)\|_{L^{p}(S^{d-1})} < \infty.
$$

The following identification of harmonic Hardy spaces holds.

**Theorem 3.7.** A harmonic function $U \in \mathcal{H}^{p}(B^{d})$, $0 < p < \infty$, if and only if its boundary distribution $f_{U} \in \mathcal{F}_{p}^{sq}(S^{d-1})$ and

$$
\|U\|_{\mathcal{H}^{p}} \sim \|U\|_{F_{p}^{sq}(\mathcal{H})} \sim \|f_{U}\|_{F_{p}^{sq}(S^{d-1})}.
$$

Furthermore, $U \in \mathcal{H}^{p}(B^{d})$, $1 < p < \infty$, if and only if $f_{U} \in L^{p}(S^{d-1})$ and

$$
\|U\|_{\mathcal{H}^{p}} \sim \|f_{U}\|_{L^{p}(S^{d-1})}.
$$

In addition, for any $U \in \mathcal{H}^{p}(B^{d})$, $1 < p < \infty$,

$$
\|U\|_{\mathcal{H}^{p}} \sim \sup_{0 \leq r < 1} \|U(r)\|_{L^{p}(S^{d-1})}
$$

and the right-hand side quantity is sometimes used to define $\|U\|_{\mathcal{H}^{p}}$ for $p > 1$.

To prove this theorem we shall need the following

**Lemma 3.8.** If $U \in \mathcal{H}^{p}(B^{d})$, $0 < p \leq \infty$, then

$$
|b_{k\nu}(U)| \leq c(k + 1)^{\gamma} \|U\|_{\mathcal{H}^{p}}, \quad \nu = 1, \ldots, N(k,d), \quad k = 0,1, \ldots,
$$

for some constants $\gamma, c > 0$, depending only on $d$ and $p$, i.e. inequalities (3.11) are valid. Consequently, there exists a distribution $f_{U} \in \mathcal{S}'$ with spherical harmonic coefficients the same as the coefficients of $U$, which in turn leads to

$$
U = P * f_{U}
$$
with $P(y,x)$ being the Poisson kernel, see (2.12). Here $P * f_U$ is defined by

$$P * f_U(x) := \langle f_U, P(\cdot, x) \rangle = \langle f_U, P(\cdot, x) \rangle,$$

where $f_U$ acts on $P(y,x) = P(y,x)$ as a function of $y$ ($P(y,x)$ is real-valued).

Proof. To prove (3.18) we invoke Proposition 4.2 from [15] which, in particular, asserts that for any $U \in B^s_{p,q}(H)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$,

$$|b_{k}\nu(U)| \leq c(k+1)^\gamma \|U\|_{H^\nu(B^s_p(H)), \nu = 1, \ldots, N(k,d), \; k = 0, 1, \ldots},$$

where the constants $\gamma, c > 0$ depend only on $d, s, p, q$.

If $U \in H^p(B^d)$ then by Definition 3.1 with $\beta = 0$

$$\|U\|_{H^p(B^{-1,1}(\mathbb{H}))} = \int_{0}^{1} \|U(\cdot, r)\|_{p, \mathbb{R}} dr \leq \sup_{0 \leq r < 1} \|U(\cdot, r)\|_{p} = \|U\|_{2\mathbb{R}^p},$$

which implies that $H^p(B^d)$ is continuously embedded in the harmonic Besov space $B^{-1,1}(\mathbb{H})$. Now, the above and (3.19) imply (3.18).

We set

$$f_U := \sum_{k \geq 0} \sum_{\nu=1}^{N(k,d)} b_{k}\nu(U)Y_{k}\nu.$$

Inequalities (3.18) and Proposition 3.3 lead to the conclusion that the series in (3.20) converge in $\mathcal{S}'$ and defines a distribution $f_U \in \mathcal{S}'$ with coefficients $b_{k}\nu(f_U) := b_{k}\nu(U)$. In turn, this implies that

$$\sum_{\nu=1}^{N(k,d)} b_{k}\nu(U)Y_{k}\nu\left(\frac{X}{|X|}\right) = Z_k * f_U\left(\frac{X}{|X|}\right), \; |X| < 1,$$

and hence

$$U(x) = \sum_{k = 0}^{\infty} |x|^k Z_k * f_U\left(\frac{X}{|X|}\right) = \sum_{k = 0}^{\infty} |x|^k \left\langle f_U, Z_k\left(\cdot, \frac{X}{|X|}\right) \right\rangle$$

$$= \lim_{m \to \infty} \left\langle f_U, \sum_{k = 0}^{m} |x|^k Z_k\left(\cdot, \frac{X}{|X|}\right) \right\rangle = \left\langle f_U, P(\cdot, x) \right\rangle = P * f_U(x).$$

Here we used the obvious fact that for any $|X| < 1$

$$P(y, x) = \lim_{m \to \infty} \sum_{k = 0}^{m} |x|^k Z_k\left(y, \frac{X}{|X|}\right) \quad \text{(convergence in } \mathcal{S}).$$

The proof is complete. □

Proof of Theorem 3.7. The Hardy space $H^p(S^{d-1})$, $0 < p < \infty$, on the sphere is defined as the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{H^p(S^{d-1})} := \sup_{0 \leq r < 1} |P * f(\cdot, r)|_{L^p(S^{d-1})} < \infty.$$

A frame characterization of the Triebel-Lizorkin spaces on $S^{d-1}$ has been established in [25, Theorem 4.5] (see Theorem 3.12(b) below), which along with the same frame characterization of the Hardy spaces $H^p(S^{d-1})$ from [7, Theorem 1.1] implies that $H^p(S^{d-1}) = \mathcal{F}_p^Q(S^{d-1})$, $0 < p < \infty$, with equivalent quasi-norms.
We set $X$. It is easy to see that $f$ only if $\psi$.

$$f = \psi$$

This along with the above observation and Theorem 3.4 implies

$$\|U\|_{\mathcal{H}^p(B^d)} = \|fU\|_{\mathcal{H}^p(B^d)} = \|fU\|_{\mathcal{H}^p(B^d)} = \|U\|_{\mathcal{H}^p(B^d)}$$

which confirms (3.15).

The equivalence $\|U\|_{\mathcal{H}^p} \sim \|fU\|_{L^p(S^{d-1})}$, when $1 < p < \infty$, follows by Lemma 3.8 and the maximal inequality just as in the case of Hardy spaces on $\mathbb{R}$, see [30, §1.2.1, p. 91]. For the equivalence $\sup_{0 \leq \xi < 1} \|U(\xi)\|_{L^p(S^{d-1})} \sim \|fU\|_{L^p(S^{d-1})}$, $1 < p < \infty$, see [31, Chapter 6]. The proof is complete.

3.5. Frame decomposition of distribution spaces on $S^{d-1}$. We next recall the construction of the frame (needlets) on $S^{d-1}$ from [25]. Note that in dimension $d = 2$ the Meyer’s periodic wavelets (see [22]) form a basis with the desired properties.

The first step in the construction of needlets on $S^{d-1}$, $d > 2$, is the selection of a real-valued function $\varphi \in C^\infty(\mathbb{R}_+)$ with the properties: $\text{supp} \varphi \subset [1/2, 2]$, $0 < \varphi \leq 1$, $\varphi(u) \geq c > 0$ for $u \in [3/5, 5/3]$, $\varphi^2(u) + \varphi^2(u/2) = 1$ for $u \in [1, 2]$, and hence $\sum_{\nu=0}^\infty \varphi(2^{-\nu} u) = 1$ for $u \in [1, \infty)$. Set

$$\Psi_0 := Z_0, \quad \text{and} \quad \Psi_j := \sum_{k=0}^\infty \varphi\left(\frac{k}{2^{j-1}}\right) Z_k, \quad j \geq 1.$$ 

It is easy to see that $f = \sum_{j=0}^\infty \Psi_j * \Psi_j * f$ for every $f \in S'$ (convergence in $S'$).

The next step is to discretize $\Psi_j * \Psi_j$ for $j \geq 1$ by using the cubature formula on $S^{d-1}$ from (2.22), where $\mathcal{X}_j$ is a maximal $\delta_j$-net with $\delta_j = 2^{-j+1}$, $0 < \gamma < 1$. In addition, for $j = 0$ we set $\mathcal{X}_0 := \{e_1\}$ with $e_1 := (1, 0, \ldots, 0)$, and $\tilde{w}_{e_1} := \omega_d$.

Since the cubature formula (2.22) is exact for spherical harmonics of degree $\leq 2^{j+1}$ we have

$$\Psi_j * \Psi_j(x \cdot y) = \int_{S^{d-1}} \Psi_j(x \cdot \eta) \Psi_j(\eta \cdot y) d\sigma(\eta) = \sum_{\xi \in \mathcal{X}_j} \tilde{w}_\xi \Psi_j(x \cdot \xi) \Psi_j(\xi \cdot y),$$

which allows to discretize $f = \sum_{j=0}^\infty \Psi_j * \Psi_j * f$ and obtain

$$f = \sum_{j=0}^\infty \sum_{\xi \in \mathcal{X}_j} \langle f, \psi_\xi \rangle \psi_\xi, \quad \forall f \in S' \quad (\text{convergence in } S'),$$

$$\psi_\xi(x) := \tilde{w}_\xi^{1/2} \Psi_j(\xi \cdot x), \quad \xi \in \mathcal{X}_j, \quad j \geq 0.$$ We set $\mathcal{X} := \cup_{j \geq 0} \mathcal{X}_j$, assuming that equal points from different sets $\mathcal{X}_j$ are distinct points in $\mathcal{X}$ so that $\mathcal{X}$ can be used as an index set. This completes the construction of the system $\Psi = \{\psi_\xi\}_{\xi \in \mathcal{X}}$.

Observe that the frame elements $\{\psi_\xi\}$ are not only band limited, but also have excellent localization on $S^{d-1}$. From the properties of $\varphi$ and Theorem 2.1 and (2.21) it follows that (see also [24, 25]) for any $M > 0$

$$|\psi_\xi(x)| \leq c_7 2^{j(1+(d-1)/2)} (1 + 2^{j-1} \rho(x, \xi))^{-M}, \quad x \in S^{d-1}, \quad \xi \in \mathcal{X}_j, \quad j \geq 0,$$

where $c_7 > 0$ is a constant depending only on $d$, $M$, $c_7$ and $\varphi$. Moreover, the localization of $\psi_\xi$ can be improved to sub-exponential as shown in [17, Theorem 5.1].
The normalization factor $\bar{\omega}_d^{-1/2}$ in (3.24) makes all $\psi_\xi$ essentially normalized in $L^2(S^{d-1})$, i.e. $\|\psi_\xi\|_{L^2(S^{d-1})} \sim 1$. In what follows we only need the lower bound estimate
\begin{equation}
\|\psi_\xi\|_{L^2(S^{d-1})} \geq \hat{c}_5, \quad \forall \xi \in \mathcal{X}_j, \ j \geq 0,
\end{equation}
with a constant $\hat{c}_5$ depending only on $d$, $M$, $c_7$ and $\varphi$. Inequality (3.26) follows from (3.24), (2.24), (3.22), the properties of $\varphi$, and $\int_{S^{d-1}} Z_k^\varphi(\xi \cdot x) d\sigma(x) = Z_k(1) \sim k^{d-2}$.

We next define the Besov and Triebel-Lizorkin sequence spaces $B^{sq}_p$ and $F^{sq}_p$ associated to $\mathcal{X}$.

**Definition 3.9.** Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. Then $B^{sq}_p := B^{sq}_p(\mathcal{X})$ is defined as the space of all complex-valued sequences $h := \{h_\xi\}_{\xi \in \mathcal{X}}$ such that
\begin{equation}
\|h\|_{B^{sq}_p} := \left( \sum_{j=0}^{\infty} 2^{js+(d-1)(1/2-1/p)} \left( \sum_{\xi \in \mathcal{X}_j} |h_\xi|^p \right)^{1/p} \right)^{1/q} < \infty
\end{equation}
with the usual modification when $p = \infty$ or $q = \infty$.

**Definition 3.10.** Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$. Then $F^{sq}_p := F^{sq}_p(\mathcal{X})$ is defined as the space of all complex-valued sequences $h := \{h_\xi\}_{\xi \in \mathcal{X}}$ such that
\begin{equation}
\|h\|_{F^{sq}_p} := \left\| \left( \sum_{\xi \in \mathcal{X}} |B_\xi|^{-s/(d-1)-1/2} |h_\xi| \mathbb{1}_{B_\xi}() \right)^{1/q} \right\|_{L^p} < \infty
\end{equation}
with the usual modification for $q = \infty$. Here $B_\xi := B(\xi, \gamma 2^{-j+1})$, $\xi \in \mathcal{X}_j$, where $\gamma$ is used in the selection of $\mathcal{X}_j$, $|B_\xi|$ is the measure of $B_\xi$ and $\mathbb{1}_{B_\xi}$ is the characteristic function of $B_\xi$.

**Remark 3.11.** The replacement of $B_\xi = B(\xi, \gamma 2^{-j+1})$ in Definition 3.10 with $B(\xi, \gamma 2^{-j})$ or with the disjoint partition sets $A_\xi$ produces equivalent quasi-norms. This immediately follows from the vector-valued maximal inequality as observed in [10, Proposition 2.7].

The main result here asserts that $\{\psi_\xi\}_{\xi \in \mathcal{X}}$ is a self-dual real-valued frame for Besov and Triebel-Lizorkin spaces on the sphere. To state this result we introduce the following **analysis** and **synthesis** operators:
\begin{equation}
S_\psi : f \mapsto \{\langle f, \psi_\xi \rangle\}_{\xi \in \mathcal{X}}, \quad T_\psi : \{h_\xi\}_{\xi \in \mathcal{X}} \mapsto \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi.
\end{equation}

**Theorem 3.12.** Let $s \in \mathbb{R}$ and $0 < p, q < \infty$.

(a) The operators $S_\psi : B^{sq}_p \to B^{sq}_p$ and $T_\psi : B^{sq}_p \to B^{sq}_p$ are bounded, and $T_\psi \circ S_\psi = I$ on $B^{sq}_p$. Hence, if $f \in S'$, then $f \in B^{sq}_p$ if and only if $\{\langle f, \psi_\xi \rangle\}_{\xi \in \mathcal{X}} \in B^{sq}_p$, and
\begin{equation}
f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{and} \quad \|f\|_{B^{sq}_p} \sim \|\{\langle f, \psi_\xi \rangle\}\|_{B^{sq}_p}.
\end{equation}

(b) The operators $S_\psi : F^{sq}_p \to F^{sq}_p$ and $T_\psi : F^{sq}_p \to F^{sq}_p$ are bounded, and $T_\psi \circ S_\psi = I$ on $F^{sq}_p$. Hence, if $f \in S'$, then $f \in F^{sq}_p$ if and only if $\{\langle f, \psi_\xi \rangle\}_{\xi \in \mathcal{X}} \in F^{sq}_p$, and
\begin{equation}
f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi \quad \text{and} \quad \|f\|_{F^{sq}_p} \sim \|\{\langle f, \psi_\xi \rangle\}\|_{F^{sq}_p}.
\end{equation}

The convergence in (3.30) and (3.31) is unconditional in $B^{sq}_p$ and $F^{sq}_p$, respectively.
For details and proofs, see [25] Theorems 4.5 and 5.5.

**Remark 3.13.** A careful examination of the proofs in [25] shows that the operators $S_\phi$ and $T_\phi$ are uniformly bounded on the respective spaces with parameters
\[(s, p, q) \in Q(A), \quad \text{for fixed } A > 1,
\]
where $Q(A)$ is the index set defined in (1.3), that is, all constants that appear in the equivalences in Theorem 3.12 depend only on $A, d,$ and $\phi,$ if $(s, p, q) \in Q(A)$.

In fact, the only nontrivial source of constants is the maximal inequality (2.24), however, as seen in (2.25) these constant are compatible with the definition of $Q(A)$ in (1.3).

The above observation will be needed for the construction of new frames below.

**Remark 3.14.** In general, one normally constructs and works with a pair of dual frames $\{\psi_\xi\}_{\xi \in X}, \{\tilde{\psi}_\xi\}_{\xi \in X}$ on $\mathbb{S}^{d-1}$, see [26]. In the construction presented above we consider the case when $\tilde{\psi}_\xi = \psi_\xi$ for simplicity.

Some embeddings between Besov or Triebel-Lizorkin spaces will be needed.

**Proposition 3.15.** Assume $s, s_0, s_1 \in \mathbb{R}$ and let $0 < p, p_0, p_1, q, q_0, q_1 \leq \infty$ in the case of Besov spaces and $0 < p, p_0, p_1 < \infty, 0 < q, q_0, q_1 \leq \infty$ in the case of Triebel-Lizorkin spaces. The following continuous embeddings are valid:

1. \[B_p^{s_0q_0} \subset B_p^{s_1q_1}, \quad F_p^{s_0q_0} \subset F_p^{s_1q_1}, \quad \text{if } s_0 = s_1, \quad q_0 \leq q_1 \quad \text{or} \quad s_0 > s_1, \quad \forall q_0, q_1; \tag{3.32}\]
2. \[B_{p_0}^{s_0q} \subset B_{p_1}^{s_1q}, \quad F_{p_0}^{s_0q} \subset F_{p_1}^{s_1q}, \quad \text{if } p_0 \geq p_1; \tag{3.33}\]
3. \[B_{p_0}^{s_0q} \subset B_{p_1}^{s_1q}, \quad \text{if } s_0 \geq s_1, \quad s_0 - \frac{d - 1}{p_0} = s_1 - \frac{d - 1}{p_1}; \tag{3.34}\]
4. \[F_{p_0}^{s_0q} \subset F_{p_1}^{s_1q}, \quad \text{if } s_0 > s_1, \quad s_0 - \frac{d - 1}{p_0} = s_1 - \frac{d - 1}{p_1}, \quad \forall q_0, q_1; \tag{3.35}\]
5. \[B_p^{s_0q} \subset F_p^{s_0q} \subset F_p^{s_p} = B_p^{s_p}, \quad \text{if } q < p; \tag{3.36}\]
6. \[B_p^{s_p} = F_p^{s_p} \subset F_p^{s_0q} \subset B_p^{s_0q}, \quad \text{if } p < q. \tag{3.37}\]

**Proof.** The proofs of embeddings (3.32), (3.33), (3.35), and (3.37) are easy and will be omitted.

Embedding (3.34) is an immediate consequence of the Nikolski inequality for spherical polynomials. Indeed, by Definition 3.2 it follows that $\Phi_j * f$ is a spherical polynomial of degree $\leq 2^j$, i.e. $\Phi_j * f \in \Pi_{2^j}$. Then by the Nikolski inequality, see e.g. [8] Theorem 5.5.1,

\[\|\Phi_j * f\|_{L^{p_1}} \leq c2^{j(1/p_0 - 1/p_1)(d-1)}\|\Phi_j * f\|_{L^{p_0}}, \quad p_0 \leq p_1, \tag{3.38}\]

and (3.33) follows readily.

The proof of embedding (3.35) relies on the Nikolski inequality (3.38) and can be carried out along the lines of the proof of the same embedding result in the classical case on $\mathbb{R}^n$ from [18] Theorem 2.1, see also [32] Theorem 2.7.1. We omit the details. \qed
4. CONSTRUCTION OF FRAMES BY SMALL PERTURBATION

Here we present the small perturbation method for construction of frames, developed in [9]. Special attention is paid to the dependence of the numerous constants on the parameters of the distribution spaces involved.

4.1. Setting and conditions on the old frame. As in Section 3.2 we denote by $S := C^\infty(\mathbb{R}^{d-1})$ the set of all test functions on $\mathbb{R}^{d-1}$ and let $S'$ be its dual. We assume that $Y$ is a collection of quasi-Banach spaces $B = B(\mathbb{R}^{d-1}) \subset S'$ of distributions on $\mathbb{R}^{d-1}$ with quasi-norms $\| \cdot \|_B$, which are continuously embedded in $S'$, i.e. there exist $m = m(B) \in \mathbb{N}$ and $C = C(B) > 0$ such that $|\langle f, \phi \rangle| \leq C\|f\|_B P_m(\phi)$ for all $f \in B$, $\phi \in S$. Also we assume that $S$ is a dense subset of each $B \in Y$.

Furthermore, we assume that there exists a collection $Y_d$ of quasi-Banach complex-valued sequence spaces $b = b(\mathcal{X})$ with quasi-norms $\| \cdot \|_b$, such that every $B \in Y$ is associated with a space $b \in Y_d$. We assume that the constants in the quasi-triangle inequalities for the quasi-Banach spaces in $Y$ and $Y_d$ are uniformly bounded, i.e. there exists a constant $C_1 = C_1(Y, Y_d)$ such that

$$
\|f_1 + f_2\|_b \leq C_1(\|f_1\|_b + \|f_2\|_b), \quad \forall f_1, f_2 \in B, \quad \forall B \in Y;
$$

$$
\|h_1 + h_2\|_b \leq C_1(\|h_1\|_b + \|h_2\|_b), \quad \forall h_1, h_2 \in b, \quad \forall b \in Y_d.
$$

A popular version of the Aoki-Rolewicz theorem states (see e.g. [4, Lemma 3.10.1]) that for any quasi-Banach space $B$ with a quasi-norm $\| \cdot \|_B$ satisfying the quasi-triangle inequalities with constant $C_1$ there exists a norm $\| \cdot \|_B^*$ on $B$, such that

$$
\|f\|_B^* \leq \|f\|_B \leq 2\|f\|_B^*, \quad \forall f \in B, \quad \text{where } \tau = \ln 2/(\ln 2 + \ln C_1) \leq 1.
$$

Targeted application of this construction is to the Besov and Triebel-Lizorkin function spaces introduced in Section 3.2 and the corresponding sequence spaces introduced in Section 3.5. For the Besov spaces the sets $Y$ and $Y_d$ are given by

$$
Y = \{B^p_{s,q}(\mathbb{R}^{d-1}) : (s, p, q) \in \mathcal{Q}(A)\} \quad \text{and} \quad Y_d = \{b^p_{s,q}(\mathcal{X}) : (s, p, q) \in \mathcal{Q}(A)\},
$$

where $A > 1$ is fixed and $\mathcal{Q}(A)$ is introduced in 1.3. A similar observation is valid for the Triebel-Lizorkin spaces $F^p_{s,q}(\mathbb{R}^{d-1})$ and $f^p_{s,q}(\mathcal{X})$.

**The old frame.** We stipulate the existence of a pair of dual frames $\{\psi_\xi\}_{\xi \in \mathcal{X}}$, $\{\hat{\psi}_\xi\}_{\xi \in \mathcal{X}}$ for all $B \in Y$ such that $\psi_\xi, \hat{\psi}_\xi \in S$, where $\mathcal{X}$ is a countable index set, with the following properties:

**A1.** The analysis and synthesis operators $S_\psi, S_{\hat{\psi}}, T_\psi, T_{\hat{\psi}}$ from (3.39) have the properties:

(a) The operators $S_\psi, S_{\hat{\psi}} : B \mapsto b$ are bounded.

(b) For any sequence $h = \{h_\xi\}_{\xi \in \mathcal{X}} \in b$ the series $\sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$ and $\sum_{\xi \in \mathcal{X}} h_\xi \hat{\psi}_\xi$ converge unconditionally in $B$ and $T_\psi, T_{\hat{\psi}} : b \mapsto B$ are bounded.

It is assumed that the norms of the operators $S_\psi, S_{\hat{\psi}}$ and $T_\psi, T_{\hat{\psi}}$ are uniformly bounded relative to $B \in Y$ and $b \in Y_d$ by a constant $C_2 = C_2(Y, Y_d, \{\psi_\xi\}) > 1$. Thus, for any $B \in Y$ and $f \in B$ we have

$$
C_2^{-1}\|f\|_B \leq \|S_\psi f\|_b = \|\{\langle f_\xi, \psi_\xi \rangle \}\|_b \leq C_2\|f\|_B,
$$

$$
C_2^{-1}\|f\|_B \leq \|S_{\hat{\psi}} f\|_b = \|\{\langle f_\xi, \hat{\psi}_\xi \rangle \}\|_b \leq C_2\|f\|_B.
$$
A2. We have $T_\psi S_\psi = T_\psi S_\psi = I$ in $B$, i.e. for any $f \in \mathcal{B}$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \tilde{\psi}_\xi,$$

where the two series converge unconditionally in $\mathcal{B}$ and hence in $S'$.

Note that the compositions $S_\psi T_\psi, S_\psi T_\psi$ are projectors due to

$$(S_\psi T_\psi)^2 = S_\psi (T_\psi S_\psi) T_\psi = S_\psi I T_\psi = S_\psi T_\psi.$$

A3. In addition, we assume that each $b \in \mathcal{Y}_d$ obeys the conditions:

(a) For any sequence $\{h_\xi\} \in \mathcal{B}$ one has $\|\{h_\xi\}\|_b = \|\{h_\xi\}\|_b$.
(b) If the sequences $\{h_\xi\}, \{g_\xi\} \in \mathcal{B}$ and $|h_\xi| \leq |g_\xi|$ for $\xi \in \mathcal{X}$, then
   $$\|\{h_\xi\}\|_b \leq \|\{g_\xi\}\|_b.$$  
(c) Compactly supported sequences belong to $b$ and are dense in $b$.

Note that conditions A3 (b)-(c) imply condition A3 (ii) in [9].

As a consequence of A1 we obtain that the operator $A := S_\psi T_\psi$ with matrix

$$A := \{a_{\xi,\eta}\}_{\xi,\eta \in \mathcal{X}}, \quad a_{\xi,\eta} := \langle \psi_\eta, \psi_\xi \rangle$$

is uniformly bounded on the sequence spaces $b \in \mathcal{Y}_d$, i.e.

$$\|A\|_{b \to b} \leq C_3 := C_2^2, \quad \forall b \in \mathcal{Y}_d.$$

4.2. Construction of new frames. We next construct a pair of dual frames

$$\{\theta_\xi\} \subset \mathcal{B} \subset \mathcal{Y}$$

for all spaces $\mathcal{B} \subset \mathcal{Y}$, where $\mathcal{X}$ is the index set from above.

For a system $\{\theta_\xi\} \subset \mathcal{B}$, we define the matrices

$$\mathcal{B} := \{b_{\xi,\eta}\}_{\xi,\eta \in \mathcal{X}}, \quad b_{\xi,\eta} := \langle \theta_\eta, \psi_\xi \rangle,$$

$$\mathcal{D} := \{d_{\xi,\eta}\}_{\xi,\eta \in \mathcal{X}}, \quad d_{\xi,\eta} := \langle \psi_\eta - \theta_\eta, \psi_\xi \rangle.$$

The only condition that we require when constructing $\{\theta_\xi\}_{\xi \in \mathcal{X}}$ is that the operator

$$D = S_\psi T_\psi - S_\psi T_\theta, \quad D : b \to b,$$

with matrix $\mathcal{D}$, defined by $(Dh_\xi) = \sum_{\eta \in \mathcal{X}} d_{\xi,\eta} h_\eta$, has a sufficiently small norm uniformly for all $b \in \mathcal{Y}_d$. More precisely we assume that

$$\|D\|_{b \to b} \leq \epsilon := \frac{(1 - 2^{-\tau})^{1/\tau}}{2C_1C_22^{1/\tau}}, \quad \forall b \in \mathcal{Y}_d,$$

with $\tau$ given in (4.2), where $C_1$ is the constant from (4.1) and $C_2$ is the constant in (4.3). For the operator $B$ with matrix $\mathcal{B}$ we have $B = A - D = S_\psi T_\theta$ and hence by (4.1), (4.5), and (4.6) it follows that $B$ is uniformly bounded on $\mathcal{Y}_d$, more precisely,

$$\|B\|_{b \to b} \leq C_4, \quad \forall b \in \mathcal{Y}_d,$$

with constant $C_4 = C_1(C_3 + \epsilon)$.

Condition (4.6) will be sufficient to show that $\{\theta_\xi\}_{\xi \in \mathcal{X}}$ is a frame for all spaces $\mathcal{B} \subset \mathcal{Y}$ and to construct its dual frame $\{\hat{\theta}_\xi\}_{\xi \in \mathcal{X}}$. To this end we introduce the operator:

$$Tf := \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\psi}_\xi \rangle \theta_\xi, \quad f \in \mathcal{B}.$$

The next three lemmas will be instrumental in the construction of $\{\hat{\theta}_\xi\}_{\xi \in \mathcal{X}}$. They are direct adaptations of Lemmas 3.1 - 3.3 in [9]; we omit their proofs.
Lemma 4.1. The operators $T_b$, defined in (3.29) with $\theta_\xi$ in the place of $\psi_\xi$, and $T$ are well defined and uniformly bounded, that is,

\[ \|T_bh\|_{\mathfrak{B}} \leq C_2C_4\|h\|_{\mathfrak{B}}, \quad \forall h \in \mathfrak{b}, \forall b \in \mathfrak{y}_d; \]

(4.9) \[ \|Tf\|_{\mathfrak{B}} \leq C_2^2C_4\|f\|_{\mathfrak{B}}, \quad \forall f \in \mathfrak{B}, \forall \mathfrak{B} \in \mathfrak{y}; \]

where $C_2$ is from (3.3) and $C_4$ is from (4.7). Furthermore, the series in (3.29) and (4.8) converge unconditionally in $\mathfrak{B}$ and hence in $\mathfrak{S}'$.

The fact the operator $T$ is invertible plays a key role in this construction.

Lemma 4.2. If (4.6) is satisfied, then

\[ \|I - T\|_{\mathfrak{B} \to \mathfrak{B}} \leq C_5 := \frac{2^{1/\tau}}{(1 - 2^{-\tau})^{1/\tau}}, \quad \forall \mathfrak{B} \in \mathfrak{y}, \]

and hence $T^{-1}$ exists and

(4.11) \[ \|T^{-1}\|_{\mathfrak{B} \to \mathfrak{B}} \leq C_5 := \frac{2^{1/\tau}}{(1 - 2^{-\tau})^{1/\tau}}, \quad \forall \mathfrak{B} \in \mathfrak{y}, \]

where $\tau$ is from (4.2).

Lemma 4.3. Assume (4.6) holds. Then the operator $H$ with matrix $H := \{(T^{-1}\psi_\eta, \tilde{\psi}_\xi)\}_{\xi,\eta \in \mathcal{X}}$ is uniformly bounded on $\mathfrak{b} \in \mathfrak{y}_d$, i.e.

(4.12) \[ \|H\|_{\mathfrak{b} \to \mathfrak{b}} \leq C_6 := C_2^2C_5, \quad \forall \mathfrak{b} \in \mathfrak{y}_d. \]

The operators from the previous three lemmas can be written as $T_0 = T_0^\psi B$, $T = T_0 S_\psi - T_0^\psi B S_\psi$, $I - T = T_0 D S_\psi$, $H = S_\psi T^{-1} T_\psi$.

Construction of the dual frame $\{\tilde{\theta}_\xi\}$. For any $\xi \in \mathcal{X}$ we define the linear functional $\tilde{\theta}_\xi$ by

(4.13) \[ \tilde{\theta}_\xi(f) = \langle f, \tilde{\theta}_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle T^{-1}\psi_\eta, \tilde{\psi}_\xi \rangle \langle f, \psi_\eta \rangle \quad \text{for} \ f \in \mathfrak{B}, \ \mathfrak{B} \in \mathfrak{y}. \]

Lemma 4.3 and A1 imply that for any $f \in \mathfrak{B}, \mathfrak{B} \in \mathfrak{y},$

(4.14) \[ \|\{\langle f, \tilde{\theta}_\xi \rangle\}\|_{\mathfrak{b}} \leq \|H\|_{\mathfrak{b} \to \mathfrak{b}}} \|\{\langle f, \psi_\eta \rangle\}\|_{\mathfrak{b}} \leq C_6 C_2 \|f\|_{\mathfrak{B}}. \]

Denote $1_\xi := \{\delta_{\xi n}\}_{n \in \mathcal{X}}$. Then $1_\xi \in \mathfrak{b}$ by A3 (c) and $\|1_\xi\|_{\mathfrak{b}} > 0$ because $\mathfrak{b}$ is a quasi-normed space. Now, condition A3 (b) and inequality (4.14) imply

\[ |\tilde{\theta}_\xi(f)| = |\langle f, \tilde{\theta}_\xi \rangle| = \frac{1}{\|1_\xi\|_{\mathfrak{b}}} \|\langle f, \tilde{\theta}_\xi \rangle 1_\xi\|_{\mathfrak{b}} \leq \frac{1}{\|1_\xi\|_{\mathfrak{b}}} \|\langle f, \tilde{\theta}_\xi \rangle\|_{\mathfrak{b}} \leq \frac{C_6 C_2 \|f\|_{\mathfrak{B}}}{\|1_\xi\|_{\mathfrak{b}}}, \]

i.e. $\tilde{\theta}_\xi (\xi \in \mathcal{X})$ is a bounded linear functional on every $\mathfrak{B} \in \mathfrak{y}$.

Also, for any $f \in \mathfrak{B}$ by Lemma 4.2 $T^{-1} f \in \mathfrak{B}$ and using Lemma 4.1

(4.15) \[ f = T(T^{-1} f) = \sum_{\xi \in \mathcal{X}} \langle T^{-1} f, \tilde{\psi}_\xi \rangle \theta_\xi. \]

Furthermore, from the fact that $T^{-1}$ is a bounded operator on $\mathfrak{B}$ and (4.4) it follows that for any $f \in \mathfrak{B}$

\[ T^{-1} f = \sum_{\eta \in \mathcal{X}} \langle f, \tilde{\psi}_\eta \rangle T^{-1} \psi_\eta, \]
where the series converges unconditionally in $\mathcal{B}$ and hence in $\mathcal{S}'$. This and the fact that $\psi_\xi \in \mathcal{S}$ imply

\begin{equation}
(T^{-1}f, \tilde{\psi}_\xi) = \sum_{\eta \in \mathcal{X}} \langle T^{-1}\psi_\eta, \tilde{\psi}_\xi \rangle \langle f, \psi_\eta \rangle = \langle f, \tilde{\theta}_\xi \rangle.
\end{equation}

Here the series converges unconditionally and hence absolutely because of the unconditional convergence of the former series. From (4.15)–(4.16) it follows that

\begin{equation}
f = \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\theta}_\xi \rangle \theta_\xi, \quad f \in \mathcal{B},
\end{equation}

where $\langle f, \tilde{\theta}_\xi \rangle$ is defined in (4.13) and the convergence is unconditional in $\mathcal{B}$.

The following theorem shows that $\{\theta_\xi\}$, $\{\tilde{\theta}_\xi\}$ is a pair of dual frames for all spaces $\mathcal{B} \in \mathcal{Y}$ if $e$ from (4.6) is sufficiently small.

**Theorem 4.4.** Let $\{\psi_\xi\}$, $\{\tilde{\psi}_\xi\}$ be a pair of dual old frames for all $\mathcal{B} \in \mathcal{Y}$ satisfying conditions A1–A3 in Subsection 4.4. Assume $\{\theta_\xi\}_{\xi \in \mathcal{X}} \subset \cap \{\mathcal{B} : \mathcal{B} \in \mathcal{Y}\}$ satisfies (4.0) and $\{\tilde{\theta}_\xi\}$ is defined as in (4.13). Then the analysis operator $S_\theta : \mathcal{B} \to \mathcal{b}$ and the synthesis operator $T_\theta : \mathcal{b} \to \mathcal{B}$ are uniformly bounded for $\mathcal{B} \in \mathcal{Y}$, $\mathcal{b} \in \mathcal{y}_d$. Furthermore, $T_\theta S_\theta = I$ on $\mathcal{B}$, i.e. for any $f \in \mathcal{B}$, $\mathcal{B} \in \mathcal{Y}$, we have

\begin{equation}
f = \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\theta}_\xi \rangle \theta_\xi,
\end{equation}

where the convergence is unconditional in $\mathcal{B}$, and

\begin{equation}
\|f\|_{\mathcal{B}} \leq C_7 \|\{\langle f, \tilde{\theta}_\xi \rangle\}\|_{\mathcal{b}}, \quad \||f, \tilde{\theta}_\xi\rangle\|_{\mathcal{b}} \leq C_8 \|f\|_{\mathcal{B}}
\end{equation}

with $C_7 = 2C_3C_2$ and $C_8 = C_2C_5$.

For a proof of Theorem 4.4 see the proof of Theorem 3.5 in [9].

The main assumption in constructing the frames $\{\theta_\xi\}_{\xi \in \mathcal{X}}$, $\{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}$ is the operator norm condition (4.6). A standard tool for evaluating the operator norms in sequence spaces is the following monotonicity lemma.

**Lemma 4.5.** Assume the quasi-norm in $\mathcal{b}(\mathcal{X})$ satisfies conditions A3 (a)–(b) in Subsection 4.4. If the entries of two matrices $F = \{f_{\xi, \eta}\}_{\xi, \eta \in \mathcal{X}}, G = \{g_{\xi, \eta}\}_{\xi, \eta \in \mathcal{X}}$ are related by $|f_{\xi, \eta}| \leq g_{\xi, \eta}$, $\xi, \eta \in \mathcal{X}$, then the respective operators $F$, $G$ are related by

\begin{equation}
\|F\|_{\mathcal{b} \to \mathcal{b}} \leq \|G\|_{\mathcal{b} \to \mathcal{b}}.
\end{equation}

**Proof.** The relation between $F$ and $G$ implies

\[|(Fh)_\xi| = \left| \sum_{\eta \in \mathcal{X}} h_\eta f_{\xi, \eta} \right| \leq \sum_{\eta \in \mathcal{X}} |h_\eta| |f_{\xi, \eta}| \leq \sum_{\eta \in \mathcal{X}} |h_\eta| g_{\xi, \eta} = (G|h)_\xi.\]

Now, (b) and (a) of A3 imply that for every $h \in \mathcal{b}(\mathcal{X})$

\[\|Fh\|_{\mathcal{b}} \leq \|G|h\|_{\mathcal{b}} \leq \|G\|_{\mathcal{b} \to \mathcal{b}} \|h\|_{\mathcal{b}} = \|G\|_{\mathcal{b} \to \mathcal{b}} \|h\|_{\mathcal{b}},\]

which implies (4.20).
4.3. Almost diagonal operators. In the next two sections we shall apply the small perturbation method described above for construction of new frames for the Besov spaces from

$$\mathcal{Y} = \{B_p^s(S_d^{d-1}) : (s, p, q) \in \mathcal{Q}(A)\}$$

for an arbitrary fixed $A > 1$ as well as for the respective collection of Triebel-Lizorkin spaces $F_p^s(S_d^{d-1})$. All spaces from $\mathcal{Y}$ satisfy (4.1) with $C_1 = C_1(A)$. On account of Theorem 3.12 and Remark 3.13 the frame $\Psi = \{\psi_\xi\}_{\xi \in \chi}$ is real-valued, self-dual, i.e. $\tilde{\psi}_\xi = \psi_\xi$, and conditions A1–A2 in Subsection 4.1 are satisfied with a constant $C_2 = C_2(d, A, \{\psi_\xi\})$. Conditions A3 are trivially satisfied for the sequence spaces $b_p^s, f_p^q$ with $0 < p, q < \infty$, $s \in \mathbb{R}$, and for $\ell^p$, $0 < p < \infty$, as well. It remains to establish sufficient conditions for verifying the operator norm bound (4.21). To this end, using Lemma 4.5 one can compare the operator matrix elements with the elements of an appropriate almost diagonal matrix (cf. [10, 20]).

The almost diagonal matrices we shall use are $\Omega_{K,M} := \{\omega^{(K,M)}_{\xi,\eta}\}_{\xi,\eta \in \chi}$ with entries

$$\omega^{(K,M)}_{\xi,\eta} := \frac{1}{\left(1 + \min\{N_\xi, N_\eta\} \rho(\xi, \eta)\right)^M} \frac{\rho(\xi, \eta)^{K+(d-1)/2}}{\max\{N_\xi, N_\eta\}^{K+(d-1)/2}},$$

where $N_\xi := 2^{j-1}$ for $\xi \in \chi_j$, $j \geq 0$. Other (non-symmetric) examples of almost diagonal matrices with index set $\chi$ are given in [20, Definition 3.9].

We next show that under appropriate conditions on $K$ and $M$ the operator $\Omega$ with matrix $\Omega_{K,M}$ is bounded on $b_p^s$ and $f_p^q$. In the following, we shall use the notation $J := (d-1)/\min\{1, p\}$ in the case of $b$-spaces and $J := (d-1)/\min\{1, p, q\}$ in the case of $f$-spaces.

**Theorem 4.6.** Let $s \in \mathbb{R}$, $0 < p, q < \infty$. For a fixed $\delta \in (0, 1]$ assume that $K, M \in \mathbb{N}$ satisfy

$$K \geq \max\{s, J - s - d + 1\} + \delta \quad \text{and} \quad M \geq J + \delta.$$  

Then the operator $\Omega$ with matrix $\Omega_{K,M}$ is bounded on $b_p^s$ and on $f_p^q$. More precisely, there exists a constant $C_9 > 0$ such that

$$\|\Omega h\|_{b_p^s} \leq C_9 \|h\|_{b_p^s}, \quad \forall h \in b_p^s; \quad \|\Omega h\|_{f_p^q} \leq C_9 \|h\|_{f_p^q}, \quad \forall h \in f_p^q.$$  

Here in the case of $b$-spaces the constant $C_9$ can be written in the form

$$C_9 = \left(\frac{c_1}{\delta^2} + \left(\frac{c_1}{\delta p}\right)^{2/p-1}\right) \left(\frac{c_0}{\delta} + \left(\frac{c_0}{\delta q}\right)^{1/q}\right)^{1/2},$$

and in the case of $f$-spaces in the form

$$C_9 = c_1^{d/J+2/s/(d-1)+1/2} \left(\frac{c_0}{\delta} + \left(\frac{c_0}{\delta q}\right)^{1/q}\right) \tilde{c}_1,$$

where $c_1$ is a constant depending only on $d$, $c_\ast$ is an absolute constant, and $\tilde{c}_1$ is the constant from the maximal inequality (2.25) with $1/t = 1/\min\{1, p, q\} + \delta/(d-1)$.

Observe that if $(s, p, q) \in \mathcal{Q}(A)$ for some fixed $A > 1$ and $\delta = 1$ then Theorem 4.6 holds with $C_9$ depending only on $d$ and $A$.

To streamline our presentation we defer the long tedious proof of Theorem 4.6 to Section 9.

In light of Theorem 4.6 we next use Proposition 2.5 and the localization property (3.25) to show that the scalar products of the elements of the needlet system $\{\psi_\xi\}$ from Subsection 3.3 are majorized by the entries of an almost diagonal matrix.
Proposition 4.7. For any $K \in \mathbb{N}_0$ and $M > d - 1$ the needlet system $\{\psi_j\}$ from Subsection 3.3 satisfies
\[
\langle \psi_j, \psi_k \rangle \leq C_{10} \omega_{\xi, \eta}^{(K, M)}, \quad \forall \xi, \eta \in X,
\]
where $\omega_{\xi, \eta}^{(K, M)}$ is defined in (4.21) and $C_{10} = 2^K c_3 \tilde{c}_4^2$ with $c_3$ from Proposition 2.7 and $\tilde{c}_4$ from (3.25).

Proof. Let $\xi \in X_j$ and $\eta \in X_k$. From (3.22) and (3.24) it readily follows that $\langle \psi_j, \psi_k \rangle = 0$ if $|j - k| \geq 2$. The symmetry of $\omega_{\xi, \eta}^{(K, M)}$ implies that it suffices to consider only the cases $k = j$ and $k = j + 1$.

On account of (3.28) condition (2.37) is satisfied for $g = \psi_k$ with $x_1 = \xi$, $N_1 = N_\xi$, and $\kappa_1 = \tilde{c}_4 N_{\xi}^{-(d-1)/2}$ and condition (2.38) is satisfied for $f = \psi_\eta$ with $x_2 = \eta$, $N_2 = N_\eta$, and $\kappa_2 = \tilde{c}_4 N_{\eta}^{-(d-1)/2}$. Then Proposition 2.5 and Proposition 4.7 with $N_\eta = N_\xi$ for $k = j$ or $N_\eta = 2N_\xi$ for $k = j + 1$ show that (4.23) is satisfied with $C_{10} = c_3 \tilde{c}_4^2$ or $C_{10} = 2^K c_3 \tilde{c}_4^2$, respectively. \qed

Note that a combination of Proposition 4.7, Lemma 4.5, and Theorem 4.6 readily yields another proof of (4.23).

Corollary 4.8. Let $b = b_{p,q}$ or $b = f_{p,q}$ with $(s, p, q) \in Q(A)$ for a fixed $A > 1$. Then $\{\psi_j\}$ satisfies (1.5) with a constant $C_3$ depending only on $d$, $A$, and $\varphi$, namely $C_3 = C_0 C_{10}$, where $C_0$ is from Theorem 4.6 with $K = M = \lfloor A d \rfloor$ and $C_{10}$ is from Proposition 4.7 with the same $K$ and $M$.

Proof. As $K$ and $M$ satisfy assumption (4.22) of Theorem 4.6 with $\delta = 1$, Proposition 4.7, Lemma 4.5, and Theorem 4.6 imply that (4.5) is valid with $C_3 = C_0 C_{10}$. \qed

We shall also apply Theorem 4.6 in Section 6 to show that condition (4.6) holds for the constructed new Newtonian kernel frame.

5. Space localization of needlets and Newtonian kernels

The basic localization property of the needlets is given in (3.25). In this section we establish some additional localization properties of the needlets introduced in (3.3). We also introduce the localized kernels developed in [16]. These kernels are linear combinations of shifts of the Newtonian kernel and will be the building blocks in the construction of Newtonian kernel frames in Section 6.

5.1. Properties of the needlets. Let $N := 2^j - 1$, $j \in \mathbb{N}$, and assume that the integer parameter $K$ is even, i.e. $K \in 2\mathbb{N}$. Let $\varphi$ be the $C^\infty[0, \infty)$ function introduced in Subsection 3.3. We define (cf. (3.22))
\[
K_N(u) := \| \psi_j(u) \| = \sum_{k=0}^{\infty} \varphi \left( \frac{k}{N} \right) Z_k(u) = \sum_{N/2 < k < 2N} \varphi \left( \frac{k}{N} \right) Z_k(u)
\]
and
\[
\Lambda_N(u) := (-1)^{K/2} \sum_{N/2 < k < 2N} \varphi \left( \frac{k}{N} \right) [k(k + d - 2)]^{-K/2} Z_k(u),
\]
where $Z_k$ is from (2.10). By (2.11) it follows that
\[-\Delta_0 Z_k(\eta \cdot x) = k(k + d - 2) Z_k(\eta \cdot x),\]
implying

\[ \Delta_0^{K/2} \Lambda_N(\eta \cdot x) = K_N(\eta \cdot x), \quad \eta, x \in \mathbb{S}^{d-1}. \]

Here \( \Delta_0 \) is the Laplace-Beltrami operator on \( \mathbb{S}^{d-1} \) (see Subsection 2.1).

Given \( \eta \in \mathbb{S}^{d-1} \) we extend \( \Lambda_N(\eta \cdot x) \) and \( K_N(\eta \cdot x) \) by

\[ \Lambda_N(\eta; x) := \Lambda_N(\frac{\eta \cdot x}{|x|}), \quad K_N(\eta; x) := K_N\left(\frac{\eta \cdot x}{|x|}\right), \quad x \in \mathbb{R}^d \setminus \{0\}. \]

In light of (2.3) and (5.3) this implies

\[ \Delta_0^{K/2} \Lambda_N(\eta; x) = \Delta_0^{K/2} \Lambda_N(\eta \cdot x) = K_N(\eta \cdot x) = K_N(\eta; x), \quad x \in \mathbb{S}^{d-1}. \]

We shall need the following simple claim.

**Lemma 5.1.** Let \( W(x, y) := \frac{x \cdot y}{|x||y|} \) for \( x, y \in \mathbb{R}^d \setminus \{0\} \).

(a) For any \( j = 1, 2, \ldots, d \) we have

\[ \frac{\partial}{\partial x_j} W(x, y) = P_j(x, y)|x|^{-3}|y|^{-1}, \quad x, y \in \mathbb{R}^d \setminus \{0\}, \]

where \( P_j \) is a homogeneous polynomial of degree 2 in \( x \) and a homogeneous polynomial of degree 1 in \( y \) and

\[ \left| \frac{\partial}{\partial x_j} W(x, y) \right| \leq 2\rho(x, y), \quad \left| \frac{\partial}{\partial y_j} W(x, y) \right| \leq 2\rho(x, y), \quad x, y \in \mathbb{S}^{d-1}. \]

(b) For any multi-index \( \beta \) with \( |\beta| \geq 1 \) and any \( G \in C^{[\beta]}[-1, 1] \) we have the representation

\[ \partial_\beta^G(W(x, y)) = \sum_{1 \leq \nu \leq |\beta|} G^{(\nu)}(W(x, y)) R_{\beta, \nu}(x, y)|x|^{-\nu - 2|\beta|}|y|^{-\nu} \]

with

\[ R_{\beta, \nu}(x, y) = \sum_{|\mu| = 2\nu - |\beta|} \prod_{k=1}^d \left( |x|^3 |y| \frac{\partial}{\partial x_j} W(x, y) \right)^{\mu_k} Q_{\beta, \nu, \mu}(x, y), \quad \frac{|\beta|}{2} < \nu \leq |\beta|, \]

where \( x, y \in \mathbb{R}^d \setminus \{0\}, \) \( R_{\beta, \nu}, 1 \leq \nu \leq |\beta|, \) is a homogeneous polynomial of degree \( |\beta| + \nu \) in \( x \) and a homogeneous polynomial of degree \( \nu \) in \( y \), and \( Q_{\beta, \nu, \mu}, |\beta|/2 < \nu \leq |\beta|, \) is a homogeneous polynomial of degree \( 3|\beta| - 3\nu \) in \( x \) and a homogeneous polynomial of degree \( |\beta| - \nu \) in \( y \). The coefficients of \( R_{\beta, \nu} \) and \( Q_{\beta, \nu, \mu} \) are independent of \( G \) and some of the polynomials \( Q_{\beta, \nu, \mu} \) are identically equal to zero.

(c) For any multi indices \( \alpha, \beta \) with \( |\alpha| = 1, |\beta| \geq 0 \) and any \( G \in C^{[|\beta|+1]}[-1, 1] \) we have the representation

\[ \partial_\alpha^\beta \partial_\gamma \partial_\delta^G(W(x, y)) = \sum_{0 \leq \nu \leq |\beta|} G^{(\nu + 1)}(W(x, y))(\partial_\gamma \partial_\delta W(x, y)) R_{\beta, \nu}(x, y)|x|^{-\nu - 2|\beta|}|y|^{-\nu} + \sum_{1 \leq \nu \leq |\beta|} G^{(\nu)}(W(x, y)) \left| y^2 \partial_\gamma R_{\beta, \nu}(x, y) - \nu y^\alpha R_{\beta, \nu}(x, y) \right| |x|^{-\nu - 2|\beta|}|y|^{-\nu}, \]

where \( x, y \in \mathbb{R}^d \setminus \{0\}, R_{0,0,0} = 1 \) and \( R_{\beta, \nu}, 1 \leq \nu \leq |\beta|, \) are from part (b).
Proof. Clearly,
\[
\frac{\partial}{\partial x_j} W(x, y) = |x|^{-3} |y|^{-1} \sum_{\nu \neq j} x_\nu (y_j x_\nu - y_\nu x_j) = |x|^{-3} |y|^{-1} \sum_{\nu \neq j} x_\nu [y_\nu (x_\nu - y_\nu) - y_\nu (x_j - y_j)]
\]
and the first inequality of part (a) follows using the Cauchy-Schwarz inequality and that \(|x - y| \leq \rho(x, y)\) for \(x, y \in S^{d-1}\). The second inequality of part (a) follows from the first one by symmetry.

Part (b) follows by induction on \(|\beta|\). Note that \(R_{\beta,0}\) is defined recursively by \(R_{0,0}(x,y) = 1, R_{\beta,0}(x,y) = 0\) for \(|\beta| \geq 1\), \(R_{\beta,|\beta|+1}(x,y) = 0\) for \(|\beta| \geq 0\, and for \(\alpha = 1, |\beta| \geq 0\)
\[
R_{\beta+\alpha,\nu}(x,y) = (|x|^3 |y| |\partial_\nu^\alpha W(x,y)|) R_{\beta,\nu-1}(x,y) + |x|^2 \partial_\nu^\alpha R_{\beta,\nu}(x,y) - (\nu + 2|\beta|) x^\alpha R_{\beta,\nu}(x,y), \quad 1 \leq \nu \leq |\beta| + 1.
\]
Part (c) follows from part (b) by differentiating (5.7) with respect to \(y\) for \(|\beta| \geq 1\) or trivially for \(|\beta| = 0\).

From Lemma 5.2 one easily derives localization estimates for zonal functions.

Lemma 5.2. Let \(K, d \in \mathbb{N}, G \in C^K[-1,1]\). Assume that for some \(N \geq 1\) and \(M > 0\)
\[
\left| G^{(\nu)}(u) \right| \leq \frac{\kappa N^{2\nu}}{(1 + N \arccos u)^M}, \quad \forall u \in [-1,1], \ 0 \leq \nu \leq K,
\]
where \(\kappa > 0\) is a constant depending on \(K, d, M,\) and \(N\). Then for all \(x, y \in S^{d-1}\) we have
\[
\left| \partial_\nu^\alpha G\left( \frac{y \cdot x}{|y||x|} \right) \right| \leq c \frac{\kappa N^{(|\beta|)}}{(1 + N \rho(y,x))^M}, \quad 0 \leq |\beta| \leq K,
\]
\[
\left| \partial_\nu^\alpha \partial_\nu^\beta G\left( \frac{y \cdot x}{|y||x|} \right) \right| \leq c \frac{\kappa N^{(|\beta|+1)}}{(1 + N \rho(y,x))^M}, \quad |\alpha| = 1, \ 0 \leq |\beta| \leq K - 1,
\]
where \(c > 0\) is a constant depending only on \(K\) and \(d\).

Proof. For \(|\beta| = 0\) (5.10) with \(\nu = 0\) coincides with (5.11) with \(c = 1\).

Let \(1 \leq |\beta| \leq K\). From (5.8) and (5.5) we get \(|R_{\beta,\nu}(x,y)| \leq c \rho(y,x)^{2\nu - |\beta|-1} x, y \in S^{d-1}\). Using this estimate and (5.10) with \(u = y \cdot x = \cos \rho(y,x)\), \(x, y \in S^{d-1}\), in (5.7) we get
\[
\left| \partial_\nu^\beta G\left( \frac{y \cdot x}{|y||x|} \right) \right| \leq c \sum_{1 \leq \nu \leq |\beta|} |G^{(\nu)}(y \cdot x)| \rho(y,x)^{2\nu - |\beta|-1}
\]
\[
\leq c \sum_{1 \leq \nu \leq |\beta|} \kappa N^{2\nu} (1 + N \rho(y,x))^{-M-\nu} \rho(y,x)^{(2\nu - |\beta|)+}
\]
\[
\leq c \sum_{1 \leq \nu \leq |\beta|} \kappa N^{2\nu} (1 + N \rho(y,x))^{-M-\nu+(2\nu - |\beta|)+}
\]
\[
\leq c \kappa N^{\beta}(1 + N \rho(y,x))^{-M},
\]
which confirms (5.11).
For the proof of (5.12) with the help of (5.6) and (5.8) we estimate the quantities in (5.9) for $x, y \in \mathbb{S}^{d-1}$ as follows

$$|(\partial_y \alpha W(x, y)) R_{\beta, \nu}(x, y)| \leq c \rho(x, y)^{1+(2\nu - |\beta|)} + |y|^2 \partial_y \alpha R_{\beta, \nu}(x, y) - \nu y^\alpha R_{\beta, \nu}(x, y)| \leq c \rho(x, y)^{(2\nu - |\beta| - 1)} + .$$

Now (5.9) implies for $x, y \in \mathbb{S}^{d-1}$

$$\left|\partial_y^\alpha \partial_y^\beta G\left(\frac{y \cdot x}{|y| |x|}\right)\right| \leq c \sum_{0 \leq \nu \leq |\beta|} |C^{(\nu+1)}(y \cdot x)| \rho(y, x)^{1+(2\nu - |\beta|)} +
+ c \sum_{1 \leq \nu \leq |\beta|} |C^{(\nu)}(y \cdot x)| \rho(y, x)^{(2\nu - |\beta| - 1)} +$$

and one completes the proof of (5.12) along the lines of proof of (5.11). $\square$

For $\tilde{A}_N, \tilde{K}_N$ and their partial derivatives we have the following estimates:

**Proposition 5.3.** For any $N \geq 1$, $K \in \mathbb{N}$, $M > 0$ and $x, \eta \in \mathbb{S}^{d-1}$ we have

(5.13) \[ |\partial^\alpha \tilde{A}_N(\eta; x)| \leq c_4 \frac{N^{-K+|\beta|+d-1}}{(1 + N \rho(\eta, x))^M}, \quad 0 \leq |\beta| \leq K + 1, \]

(5.14) \[ |\partial_y^\alpha \partial_y^\beta \tilde{A}_N\left(\frac{\eta \cdot x}{|\eta||x|}\right)| \leq c_4 \frac{N^{-K+|\beta|+d}}{(1 + N \rho(\eta, x))^M}, \quad |\alpha| = 1, 0 \leq |\beta| \leq K, \]

and

(5.15) \[ |\partial^\beta \tilde{K}_N(\eta; x)| \leq c_5 \frac{N^{|\beta|+d-1}}{(1 + N \rho(\eta, x))^M}, \quad 0 \leq |\beta| \leq K + 1, \]

(5.16) \[ |\partial_y^\alpha \partial_y^\beta \tilde{K}_N\left(\frac{\eta \cdot x}{|\eta||x|}\right)| \leq c_5 \frac{N^{|\beta|+d}}{(1 + N \rho(\eta, x))^M}, \quad |\alpha| = 1, 0 \leq |\beta| \leq K, \]

where $c_4, c_5$ depend only on $d, K, M$, and $\varphi$.

**Proof.** Let $\lambda(t) := (-1)^{K/2} N^{-K} (t^d + (d-1)/N)^{-K/2} \varphi(t)$, $t \in [0, \infty)$. Clearly,

$$\Lambda_N(u) = \sum_{N/2 < k < 2N} \lambda\left(\frac{k}{N}\right) Z_k(u), \quad \lambda \in C^\infty(0, \infty), \quad \supp \lambda \subset [1/2, 2],$$

It is readily seen that $||\lambda^{(m)}||_\infty \leq c N^{-K}$ for each $m \geq 0$ with $c = c(d, m, K, \varphi)$. Then for any $M > 0$ by Theorem 2.1 with $M + K + 1$ instead of $M$ we have

(5.17) \[ |\Lambda_N^{(\nu)}(z \cdot \theta)| \leq \frac{c N^{-K+d-1} N^{2\nu}}{(1 + N |\theta|)^M K+1}, \quad |\theta| \leq \pi, 0 \leq \nu \leq K + 1, \]

where $c = c(d, M, K, \varphi)$. Applying Lemma 5.2 with $\kappa = c N^{-K+d-1}$, $K$ replaced by $K + 1$, and (5.17) with $|\theta| = \rho(\eta, x)$ we get (5.13) and (5.14).

For the localization of $\tilde{K}_N$ we use (5.1) and the fact that $||\varphi^{(m)}||_\infty \leq c$ for each $m \geq 0$ with $c = c(d, m, \varphi)$. Thus for any $M > 0$ by Theorem 2.1 with $M$ replaced by $M + K + 1$ we obtain

$$|\tilde{K}_N^{(\nu)}(z \cdot \theta)| \leq \frac{c N^{d-1} N^{2\nu}}{(1 + N |\theta|)^M K+1}, \quad |\theta| \leq \pi, 0 \leq \nu \leq K + 1.$$ 

This estimate along with Lemma 5.2 with $\kappa = c N^{-d-1}$ and $K$ replaced by $K + 1$ imply (5.15) and (5.16). $\square$
For $\xi \in \mathcal{X}_j$, $j \in \mathbb{N}_0$, we set $N_\xi := 2^j - 1$. The elements of the needlet frame $\Psi = \{ \psi_\xi(x) : \xi \in \mathcal{X} \}$, defined in (3.24), can be represented in terms of the kernels $\mathcal{K}_N$ as follows:

\begin{equation}
\psi_\xi(x) := \mathcal{C}_\xi \psi_\xi^\circ(x), \quad \psi_\xi^\circ(x) := \mathcal{K}_{N_\xi}(\xi \cdot x) = \mathcal{Y}_j(\xi \cdot x), \quad \mathcal{C}_\xi := \tilde{w}_\xi^{1/2}
\end{equation}

for $x \in \mathbb{S}^{d-1}$, $\xi \in \mathcal{X}_j$, $j \in \mathbb{N}$, and the coefficients $\mathcal{C}_\xi$ satisfy

\begin{equation}
\mathcal{C}_\xi \leq c_9 N_\xi^{-(d-1)/2}, \quad \xi \in \mathcal{X},
\end{equation}

with $c_9 = 2^{(1-d)/2} c_7^{1/2}$ depending only on $d$ (cf. (2.21)).

In the following sections we assume that the needlet frame $\Psi$ is fixed; the dependence of some of the constants on $\varphi$ will not be indicated explicitly.

5.2. Highly localized kernels in terms of shifts of the Newtonian kernel.

As already explained in the introduction our tool for approximation of harmonic functions on the ball will consist of linear combinations of shifts of the Newtonian kernel:

$$\frac{1}{|x|^{d-2}} \text{ in dimension } d > 2 \text{ or } \ln \frac{1}{|x|} \text{ if } d = 2,$$

just as in (1.1). The poor localization of the Newtonian kernel, however, creates problems. Its directional derivatives achieve much better localization and are well approximated by finite differences. However, as explained in [16] they do not have either the right localization in the sense of (2.5) or $L^1(\mathbb{S}^{d-1})$ normalization.

We next invoke Theorem 3.1 from [16] to show (see Corollary 5.5 below) the existence of highly localized summability kernels that are linear combinations of finitely many directional derivatives of the Newtonian kernel. Consequently, they will be arbitrarily well approximated by linear combinations of a fixed number of shifts of the Newtonian kernel.

**Theorem 5.4** (Theorem 3.1 in [16]). Let $d \geq 2$, $M > d - 2$, and $0 < \varepsilon \leq 1$. Set $a := 1 + \varepsilon$, $\delta := 1 - a^{-2}$ and

\begin{equation}
m := \left\lceil (M - d + 2)/2 \right\rceil.
\end{equation}

Consider the function

\begin{equation}
\mathcal{F}_\varepsilon(u) := \varepsilon^{2m-1}(a^2 + 1 - 2au)^{-d/2+1-m}, \quad u \in [-1,1].
\end{equation}

The function $\mathcal{F}_\varepsilon$ has these properties:

\begin{equation}
\mathcal{F}_\varepsilon(x \cdot \eta) = \varepsilon^{2m-1}|x - a\eta|^{-d+2-2m}, \quad \forall x, \eta \in \mathbb{S}^{d-1},
\end{equation}

\begin{equation}
0 < \mathcal{F}_\varepsilon(x \cdot \eta) \leq \frac{c_1^\# \varepsilon^{-d+1}}{(1 + \varepsilon^{-1} \rho(x, \eta))^M}, \quad \forall x, \eta \in \mathbb{S}^{d-1},
\end{equation}

and

\begin{equation}
\int_{\mathbb{S}^{d-1}} \mathcal{F}_\varepsilon(x \cdot \eta) d\sigma(x) \geq c_2^\# > 0, \quad \forall \eta \in \mathbb{S}^{d-1},
\end{equation}

where $c_1^\#, c_2^\# > 0$ are constants depending only on $m$ and $d$. Furthermore, there exist real numbers $b_0, b_1, \ldots, b_m$ depending only on $\varepsilon$, $m$, and $d$ such that for every
\( \eta \in S^{d-1} \) the function \( F_\varepsilon(x \cdot \eta) \) is the restriction on \( S^{d-1} \) of the harmonic function \( \mathcal{F}_{\varepsilon,m} \) defined on \( \mathbb{R}^d \setminus \{a\eta\} \) by

\[
(5.25) \quad \mathcal{F}_{\varepsilon,m}(a\eta, x) := \sum_{\ell=0}^{m} b_\ell(\eta \cdot \nabla)^\ell |x - a\eta|^{2-d} \quad \text{if } d \geq 3,
\]
or

\[
(5.26) \quad \mathcal{F}_{\varepsilon,m}(a\eta, x) := b_0 + \sum_{\ell=1}^{m} b_\ell(\eta \cdot \nabla)^\ell \ln \frac{1}{|x - a\eta|} \quad \text{if } d = 2.
\]

We define the univariate function

\[
(5.27) \quad F_\varepsilon(u) := \kappa_{\varepsilon,m,d} F_\varepsilon(u), \quad u \in [-1, 1],
\]

where \( F_\varepsilon \) is defined in (5.21) and

\[
(5.28) \quad \kappa_{\varepsilon,m,d} := \left( \int_{S^{d-1}} F_\varepsilon(x \cdot \eta) d\sigma(\eta) \right)^{-1}, \quad \forall \eta \in S^{d-1}.
\]

Note that \( \kappa_{\varepsilon,m,d} \) is independent of \( \eta \) and (5.24) implies

\[
\kappa_{\varepsilon,m,d} \leq 1/c_2^d, \quad \forall 0 < \varepsilon \leq 1.
\]

Given \( \eta \in S^{d-1} \) we extend \( F_\varepsilon(\eta \cdot x) \) just as \( K_N(\eta \cdot x) \) in (5.4) by

\[
(5.29) \quad \tilde{F}_\varepsilon(\eta; x) = F_\varepsilon(\eta \cdot x), \quad x \in \mathbb{R}^d \setminus \{0\}.
\]

In this case \( 2.3 \) takes the form

\[
(5.30) \quad \Delta \tilde{F}_\varepsilon(\eta; x) = \Delta_0 F_\varepsilon(\eta \cdot x), \quad x \in S^{d-1}.
\]

We use \( \tilde{F}_\varepsilon \) to bound the derivatives of \( F_\varepsilon(\eta \cdot x) \) for \( x \in S^{d-1} \).

**Corollary 5.5.** Let \( d \geq 2, M > d-2, K \in \mathbb{N} \). Let \( 0 < \varepsilon \leq 1 \) and let \( F_\varepsilon \) be defined by (5.27)–(5.28). Then for all \( x, \eta \in S^{d-1} \) we have

\[
(5.31) \quad F_\varepsilon(\eta \cdot x) = \kappa_{\varepsilon,m,d} \mathcal{F}_{\varepsilon,m}(a\eta, x),
\]

\[
(5.32) \quad \int_{S^{d-1}} F_\varepsilon(\eta \cdot y) d\sigma(y) = 1,
\]

\[
(5.33) \quad |\partial^\beta \tilde{F}_\varepsilon(\eta; x)| \leq c_8 \frac{(\varepsilon^{-1})|\beta|+d-1}{(1+\varepsilon^{-1}\rho(\eta, x))^{M}}, \quad 0 \leq |\beta| \leq 2K + 1,
\]

\[
(5.34) \quad |\partial_\eta^|\alpha| \partial_x^|\beta| \tilde{F}_\varepsilon(\eta; x)| \leq c_8 \frac{(\varepsilon^{-1})|\beta|+d}{(1+\varepsilon^{-1}\rho(\eta, x))^{M}}, \quad |\alpha| = 1, \ 0 \leq |\beta| \leq 2K,
\]

where \( \tilde{F}_\varepsilon \) is defined by (5.29) and \( c_8 \) depends only on \( d, K, M \).

**Proof.** Identity (5.31) follows from (5.27) and the second part of Theorem 5.4 while (5.32) follows from (5.27) and (5.28). From (5.27) and (5.21) it follows that for any \( \nu \in \mathbb{N}_0 \)

\[
F_\varepsilon^{(\nu)}(u) = \kappa_{\varepsilon,m,d} a^\nu \varepsilon^{2m-1}(a^2 + 1 - 2au)^{-d/2+1-m-\nu} \prod_{k=0}^{\nu-1} (2m + d - 2 + 2k).
\]
On the other hand, using that \( a = 1 + \varepsilon \) it is easy to show that

\[
\frac{1}{5} \leq \frac{(a^2 + 1 - 2au)^{1/2}}{\varepsilon (1 + \varepsilon^{-1} \arccos u)} \leq 2, \quad u \in [-1, 1],
\]

see the proof of inequalities (3.7) in [16]. The above, (5.20), and (5.24) yield

\[
G = \varepsilon^{-2(\nu + d - 1)} \leq \frac{c(1 + \varepsilon^{-1} \arccos u)^{d-2+2m+2\nu}}{(1 + \varepsilon^{-1} \arccos u)^{M+\nu}}
\]

with \( c \) depending on \( d, m, \) and \( \nu \). In turn, this estimate and Lemma 5.2 with \( G = F_\varepsilon, N = \varepsilon^{-1}, \kappa = cN^{d-1}, \) and \( K \) replaced by \( 2K + 1 \) imply (5.33) and (5.34).

Note that the extension \( \tilde{F}_\varepsilon(\eta; x) \) of \( F_\varepsilon(\eta \cdot x) \) from (5.20) is different from its harmonic extension \( \kappa_{\varepsilon, m, d} \mathcal{F}_{\varepsilon, m}(\eta; x) \) given in (6.23) of the form (5.26) - (5.20).

6. Frames in terms of shifts of the Newtonian kernel

We now come to the most technical part of our development – the construction of a frame whose elements are finite linear combinations of shifts of the Newtonian kernel. We shall carry out this construction in several steps.

6.1. **The main technical step in the construction of the new frame on \( \mathbb{S}^{d-1} \).**

We now focus on the construction of highly localized frame elements \( \{ \theta_\xi : \xi \in \mathcal{X} \} \) of the form

\[
\theta_\xi(x) = \sum_{\nu=1}^{\hat{n}} \frac{a_\nu}{|x - y_\nu|^{d-2}}, \quad \text{if} \quad d > 2,
\]

or

\[
\theta_\xi(x) = \sum_{\nu=1}^{\hat{n}} a_\nu \ln \frac{1}{|x - y_\nu|}, \quad \text{if} \quad d = 2.
\]

Here \( y_\nu \in \mathbb{R}^d \) with \( |y_\nu| > 1 \), \( a_\nu \in \mathbb{R} \), and \( \{y_\nu\}_{\nu=1}^{\hat{n}} \) and \( \{a_\nu\}_{\nu=1}^{\hat{n}} \) may vary with \( \xi \in \mathcal{X} \), but \( \hat{n} \) is fixed.

Assume that \( \Psi = \{\psi_\xi : \xi \in \mathcal{X} \}, \mathcal{X} = \cup_{j \geq 0} \mathcal{X}_j \), is the existing frame, described in (5.5). For the construction of the new frame elements \( \{\theta_\xi\} \) we utilize the small perturbation method, described in [4]. In applying this scheme the main step is to construct frame elements \( \theta_\xi, \xi \in \mathcal{X} \), of the form (6.1) - (6.2) so that

\[
|\langle \psi_\eta - \theta_\eta, \psi_\xi \rangle| \leq \gamma_0 \omega_{\xi, \eta}, \quad \xi, \eta \in \mathcal{X},
\]

and

\[
|\psi_\xi(x) - \theta_\xi(x)| \leq \frac{\gamma_0 N_\xi^{(d-1)/2}}{(1 + N_\xi p(x, \xi))^{M}}, \quad x \in \mathbb{S}^{d-1}, \xi \in \mathcal{X}.
\]

Here \( N_\xi = 2^{j-1} \) for \( \xi \in \mathcal{X}_j \), \( \gamma_0 > 0 \) is a small parameter, \( \omega_{\xi, \eta} \) are the entries of an almost diagonal matrix like \( \omega^{(K, M)} \) from (4.21), and \( M > 0 \) is sufficiently large. The result of this construction will be a frame \( \{\theta_\xi\}_{\xi \in \mathcal{X}} \) for the Besov and Triebel-Lizorkin spaces of interest.

It will be convenient to us to approximate the essentially \( L^1 \)-normalized frame elements \( \psi_\xi(x) := \mathcal{K}_N_\xi(\xi \cdot x) \) defined in (7.13) by essentially \( L^1 \)-normalized new frame elements \( \theta_\xi \). Then multiplication by constants \( C_\xi^0 \) (see (5.18)) will complete the construction of \( L^2 \)-normalized frame elements.
The construction of the new frame elements $\theta_{\xi}^N$ will be carried out in four steps:

(a) Approximation of $K_{N\xi}(\xi \cdot x)$, $\xi \in X$, by convolving $K_{N\xi}$ with the kernel $F_{\varepsilon}(y \cdot x)$ from (6.31).

(b) Discretization of the convolutions by using the cubature formula from (2.19).

(c) Truncation of the resulting sums.

(d) Approximation of the truncated sums by discrete versions of the operators involved.

These approximation steps will be governed by four small parameters (constants): $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$. The relations between these parameters and all involved constants will be carefully traced.

We next introduce some convenient notation and set up the approximation steps described above. For the only index $\xi \in X_0$ we set $\theta_{\xi}(x) := \psi_{\xi}(x) \equiv 1$. In the remaining part of this subsection we consider $\xi \in X \setminus X_0 = \bigcup_{j=1}^{\infty} X_j$.

Given $0 < \gamma_1 \leq 1$ (to be selected), we set

$$\varepsilon := \gamma_1/N\xi$$

and define

$$g_1(\xi; x) := \int_{\mathbb{R}^{d-1}} \Lambda_{N\xi}(\xi \cdot y) F_{\varepsilon}(y \cdot x) \, d\sigma(y), \quad x \in \mathbb{R}^{d-1},$$

where $\Lambda_{N\xi}$ is defined in (5.22), $F_{\varepsilon}(y \cdot x)$ is the kernel from (6.31) with $\varepsilon$ from (6.5), and $m$ from (5.20).

Given $0 < \gamma_2 \leq \gamma_1$ (to be selected), we let $Z_j \subset \mathbb{R}^{d-1}$ be a fixed maximal $\delta$-net with $\delta := \gamma_2^{-j+1}$ and let $\{A_{\xi}\}_{\xi \in Z_j}$ be the associated partition of $\mathbb{R}^{d-1}$ (see Subsection 2.3). Applying cubature formula (2.19) with nodes $\xi \in Z_j$ and weights $w_{\xi} = |A_{\xi}|$ to (6.6) we arrive at

$$g_2(\xi; x) := \sum_{\xi \in Z_j} w_{\xi}\Lambda_{N\xi}(\xi \cdot \xi) F_{\varepsilon}(\xi \cdot x), \quad x \in \mathbb{R}^{d-1}.$$

Observe that there is no connection between the nodal sets $X_j$ and $Z_j$ ($j \in \mathbb{N}$).

In particular, the cubature $\sum_{\xi \in Z_j} w_{\xi} f(\xi)$ from (2.19) has to be exact only for constants, while the cubature $\sum_{\xi \in X_j} w_{\xi} f(\xi)$ from (2.22) is required to be exact for all spherical harmonics of degree $\leq j+1$.

Given $0 < \gamma_3 \leq 1$ (to be determined), we truncate the sum in (6.7) by including only the nodes within distance $r_\xi := (\gamma_3 N\xi)^{-1}$ from $\xi$ to obtain

$$g_3(\xi; x) := \sum_{\xi \in Z_j \cap X_j} w_{\xi}\Lambda_{N\xi}(\xi \cdot \xi) F_{\varepsilon}(\xi \cdot x), \quad x \in \mathbb{R}^{d-1}.$$

The functions $g_1(\xi; x)$, $g_2(\xi; x)$, and $g_3(\xi; x)$ should be viewed as consecutive approximations of $\Lambda_{N\xi}(\xi \cdot \xi)$.

We obtain consecutive approximations to $K_{N\xi}(\xi \cdot x)$ by applying $\Delta_{0}^{K/2}$ to each of the functions $g_1$, $g_2$, $g_3$ in (6.6), (6.7), and (6.8). We set

$$h_1(\xi; x) := \Delta_{0}^{K/2} g_1(\xi; x) = \Delta_{0}^{K/2} \int_{\mathbb{R}^{d-1}} \Lambda_{N\xi}(x \cdot y) F_{\varepsilon}(y \cdot \xi) \, d\sigma(y)$$

$$= \int_{\mathbb{R}^{d-1}} \Delta_{0}^{K/2} \Lambda_{N\xi}(x \cdot y) F_{\varepsilon}(y \cdot \xi) \, d\sigma(y) = \int_{\mathbb{R}^{d-1}} K_{N\xi}(x \cdot y) F_{\varepsilon}(y \cdot \xi) \, d\sigma(y),$$
\begin{align}
\tag{6.10} h_2(\xi; x) &:= \Delta_0^{K/2} g_2(\xi; x) = \sum_{\zeta \in \mathbb{Z}} w_\zeta \Lambda_{N_i}(\xi \cdot \zeta) \Delta_0^{K/2} F_\varepsilon(\zeta \cdot x), \\
\tag{6.11} h_3(\xi; x) &:= \Delta_0^{K/2} g_3(\xi; x) = \sum_{\zeta \in \mathbb{Z}, \rho(\zeta, \xi) \leq r_\varepsilon} w_\zeta \Lambda_{N_i}(\xi \cdot \zeta) \Delta_0^{K/2} F_\varepsilon(\zeta \cdot x).
\end{align}

Above in (6.9) we first used the commutativity of the inner product of zonal functions (2.29) in the definition of \( g_i \) followed by (5.3) in the last equality.

Observe that \( h_3(\xi; x) \) is a linear combination of finitely many (independent of \( \xi \)) terms of the form

\[ \Delta_0^{K/2} F_\varepsilon(\zeta \cdot x) = \kappa_{\varepsilon, m, d} \sum_{\ell=0}^{m} b_\ell \Delta_0^{K/2} (\zeta \cdot \nabla)^\ell |x - a\zeta|^{2-d}, \quad \text{if } d \geq 3, \]

see (5.25), (5.28), and (5.31). We have a similar representation of \( h_3(\xi; x) \) in dimension \( d = 2 \). Replacing the differential operator \( \Delta_0^{K/2} (\zeta \cdot \nabla)^\ell \) in (6.11) by its discrete counterpart \( \mathcal{D}_r^{K/2} \mathcal{D}_r(\zeta) \) with an appropriate small \( t = t_j > 0 \) (to be specified) we arrive at the following definition of \( \theta_\varepsilon^2, \xi \in \mathcal{X}'_j, j \in \mathbb{N} \), in dimension \( d \geq 3 \)

\begin{align}
\tag{6.12} \theta_\varepsilon^2(x) &:= \kappa_{\varepsilon, m, d} \sum_{\zeta \in \mathbb{Z}, \rho(\zeta, \xi) \leq r_\varepsilon} w_\zeta \Lambda_{N_i}(\xi \cdot \zeta) \sum_{\ell=0}^{m} b_\ell \Delta_0^{K/2} \mathcal{D}_r^{\ell}\mathcal{D}_r(\zeta)|x - a\zeta|^{2-d}. \\
\text{If } d = 2 \text{ we set} \\
\tag{6.13} \theta_\varepsilon^2(x) &:= \kappa_{\varepsilon, m, 2} \sum_{\zeta \in \mathbb{Z}, \rho(\zeta, \xi) \leq r_\varepsilon} w_\zeta \Lambda_{N_i}(\xi \cdot \zeta) \sum_{\ell=1}^{m} b_\ell \Delta_0^{K/2} \mathcal{D}_r^{\ell}\mathcal{D}_r(\zeta) \ln \frac{1}{|x - a\zeta|}. 
\end{align}

Several remarks are in order:

(1) The finite difference operator \( \mathcal{D}_r^{\ell}(\zeta) := t^{-\ell} \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} T(\zeta, kt) \) is defined by the translation operator (in \( \mathbb{R}^d \)) in direction \( \zeta \in \mathbb{S}^{d-1} \) with step \( t \) given by \( T(\zeta, t) f(x) = f(x + t\zeta) \) for \( x \in \mathbb{R}^d \).

(2) Following [3] p. 23 or (4.2.1) on p. 81] the rotation \( Q_{1,2,\ell} \in SO(d) \) is given by

\[ Q_{1,2,\ell}(\varsigma_1, \zeta_2, \ldots, \varsigma_d) := (\varsigma_1 \cos t + \varsigma_2 \sin t, -\varsigma_1 \sin t + \varsigma_2 \cos t, \varsigma_3, \ldots, \varsigma_d), \quad \varsigma \in \mathbb{S}^{d-1}, \]

and \( Q_{i,\ell,\varsigma} \) is defined similarly for any \( 1 \leq i < \ell \leq d \). The translation operator corresponding to the rotation \( Q_{i,\ell,\varsigma}, 1 \leq i < \ell \leq d \), is given by

\[ T(Q_{i,\ell,\varsigma}) f(\varsigma) := f(Q_{i,\ell,\varsigma}^{-1} \varsigma) = f(Q_{i,\ell,\varsigma}^{-1} \varsigma). \]

The operator

\[ \mathcal{S}_t f(\varsigma) := t^{-2} \sum_{1 \leq i < \ell \leq d} (T(Q_{i,\ell,\varsigma}) + T(Q_{i,\ell,\varsigma}^{-1}) - 2I) f(\varsigma), \]

where \( I \) stands for the identity, approximates well \( \Delta_0 f(\varsigma) \) for small \( t \); the powers of \( \mathcal{S}_t \) are defined as usual by \( \mathcal{S}_t^k := \mathcal{S}_t \mathcal{S}_t^{k-1} \).

(3) The numbers \( a, \delta, m, \) and \( b_\ell, \ell = 0, 1, \ldots, m, \) are determined in Theorem 5.3 as functions of \( \varepsilon \) from (6.8) and \( M \).

We now come to the first main assertion in this section.
Theorem 6.1. Let \( d \geq 2, K \in 2\mathbb{N}, M > K + d - 1, \) and \( \gamma_0 > 0. \) Then there exist constants \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0 \) depending only on \( d, K, M, \gamma_0, \) and for every \( j \in \mathbb{N} \) there exists \( t_j > 0 \) depending only on \( d, K, M, \gamma_0, \) and \( j \) such that for every \( \xi \in \mathcal{X} \) the element \( \hat{\theta}_\xi^2 \) from (6.12) or (6.18) obeys

\[
|\partial^\beta \left[ \hat{\psi}_\xi^2(x) - \hat{\theta}_\xi^2(x) \right]| \leq \frac{\gamma_0 N_\xi^{1/|\beta|+d-1}}{(1 + N_\xi \rho(\xi, x))^M}, \quad \forall x \in \mathbb{S}^{d-1}, \quad 0 \leq |\beta| \leq K,
\]

\[
\left| \int_{\mathbb{S}^{d-1}} y^\beta \left[ \hat{\psi}_\xi^2(y) - \hat{\theta}_\xi^2(y) \right] d\sigma(y) \right| \leq \gamma_0 N_\xi^{-K}, \quad 0 \leq |\beta| \leq K - 1.
\]

Here \( \hat{\psi}_\xi(x) := \hat{\theta}_\xi^2(x/|x|), \quad x \in \mathbb{R}^d \setminus \{0\}, \) and \( \hat{\psi}_\xi(x) := \hat{\kappa}_{N_\xi}(\xi; x), \) see (5.4), (5.18).

The proof of Theorem 6.1 relies on four lemmas, which establish estimates similar to estimates (6.14) and (6.18) for the differences \( \hat{\psi}_\xi^2 - h_1(x; \cdot), \ h_1(\xi; \cdot) - h_2(\xi; \cdot), \ h_2(\xi; \cdot) - h_3(\xi; \cdot), \) and \( h_3(\xi; \cdot) - \hat{\theta}_\xi^2. \) The values of the parameters \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \) used in these four lemmas will be selected in the proof of Theorem 6.1 (see (6.11)).

In light of the last integral in (6.9) we define

\[
\hat{h}_1(\xi; x) := \int_{\mathbb{S}^{d-1}} \hat{\kappa}_{N_\xi}(y; x) F_{\varepsilon}(y; \xi) d\sigma(y), \quad x \in \mathbb{R}^d \setminus \{0\},
\]

where \( \hat{\kappa}_{N_\xi}(y; x) \) is defined in (5.4) and \( \varepsilon \) is from (6.5).

Lemma 6.2. Let \( \xi \in \mathcal{X}_j, \quad j \in \mathbb{N}, \) \( K \in 2\mathbb{N}, \) and \( M > K + d - 1. \) If \( 0 < \gamma_1 \leq 1, \) \( \varepsilon \) is from (6.5) and \( F_\varepsilon \) is from (5.27), then:

(a) For any \( \beta, \) \( 0 \leq |\beta| \leq K, \)

\[
|\partial^\beta \left[ \hat{\kappa}_{N_\xi}(\xi; x) - \hat{h}_1(\xi; x) \right]| \leq c_{10} \frac{\gamma_1 N_\xi^{1/|\beta|+d-1}}{(1 + N_\xi \rho(\xi, x))^M}, \quad \forall x \in \mathbb{S}^{d-1};
\]

(b) For any \( \beta, \) \( 0 \leq |\beta| \leq K - 1, \)

\[
\left| \int_{\mathbb{S}^{d-1}} y^\beta \left[ \hat{\kappa}_{N_\xi}(\xi; y) - h_1(\xi; y) \right] d\sigma(y) \right| \leq c_{10} \gamma_1 N_\xi^{-K},
\]

where \( c_{10} \) depends only on \( d, K, M. \)

Proof. Let \( 0 \leq |\beta| \leq K. \) We apply Proposition 2.4 with \( g(y) = \partial^\beta \hat{\kappa}_{N_\xi}(y; x), \ x_1 = x, \)

\( N_1 = N_\xi, \ \kappa_1 = c_5 N_\xi^{|\beta|} \) on account of (5.10), and with \( f(y) = F_\varepsilon(\xi; y), \ x_2 = \xi, \)

\( N_2 = 1/\varepsilon, \ \varepsilon = \gamma_1/N_\xi, \ \kappa_2 = c_8 \) on account of (5.33) with \( \beta = 0. \) Observe that \( N_2 = N_\xi/\gamma_1 \geq N_1. \) Hence, because of (5.32), inequality (2.30) implies

\[
\left| \partial^\beta \hat{h}_1(\xi; x) - \partial^\beta \hat{\kappa}_{N_\xi}(\xi; x) \right| \leq c_{2} c_5 c_8 \frac{\gamma_1 N_\xi^{1/|\beta|+d-1}}{(1 + N_\xi \rho(\xi, x))^M}, \quad x \in \mathbb{S}^{d-1}.
\]

For the proof of (6.17) we apply Proposition 2.4 with \( x_1 = \xi, \ N_1 = N_\xi, \ g(y) = \Lambda_{N_\xi}(\xi; y), \ k_1 = c_4 N_\xi^{-K} \) in view of (6.13) with \( |\beta| = 1, f(y) = F_\varepsilon(x; y) \) with any fixed \( x \in \mathbb{S}^{d-1}, \)

\( N_2 = 1/\varepsilon, \ \varepsilon = \gamma_1/N_\xi, \ \kappa_2 = c_8 \) in view of (5.33) with \( \beta = 0. \) Consequently, because of (5.32), inequality (2.30) implies

\[
\left| \Lambda_{N_\xi}(\xi; x) - g_1(\xi; x) \right| \leq c_{11} \frac{\gamma_1 N_\xi^{-K+d-1}}{(1 + N_\xi \rho(\xi, x))^M}, \quad x \in \mathbb{S}^{d-1},
\]
with \( c_{11} := c_2 c_5 c_8 \). Now, for \( 0 \leq |\beta| \leq K - 1 \) we apply consecutively (6.23), (6.29), the fact that the operator \( \Delta_0 \) is symmetric, (6.19) with \( y \) in place of \( x \), (2.8), and (6.20) to obtain

\[
\int_{S^{d-1}} y^{\beta} \left[ \mathcal{K}_{N_\xi} (\xi \cdot y) - h_1 (\xi; y) \right] d\sigma(y)
\]

Finally, (6.18) and (6.20) imply (6.16) and (6.17) with \( c_{10} := \max\{ c_2 c_5 c_8, c_{11} c_0 c_6 \} \). The proof is complete.

The first integral in (6.21), (2.29), and identity (5.30) give another representation of \( h_1 (\xi; x) \), namely,

\[
h_1 (\xi; x) = \int_{S^{d-1}} \Lambda_{N_\xi} (\xi \cdot y) \Delta^{K/2} F (y; x) d\sigma(y), \quad x \in \mathbb{R}^d \setminus \{0\},
\]

with \( F (y; x) \) defined in (6.29). Using (6.11) and (5.30) we also set for \( \xi \in \mathcal{X}_j \)

\[
h_2 (\xi; x) := \sum_{\zeta \in \mathcal{Z}_j} w_\zeta \Lambda_{N_\xi} (\xi \cdot \zeta) \Delta^{K/2} F (\zeta; x), \quad x \in \mathbb{R}^d \setminus \{0\}.
\]

**Lemma 6.3.** Assume \( \xi \in \mathcal{X}_j, j \in \mathbb{N}, K \in \mathbb{N}, M > K + d - 1, \) and let \( \gamma_1, \varepsilon, F \) be as in Lemma 6.2. If \( 0 < \gamma_2 \leq \gamma_1 \) and \( F \) is from (6.29), then:

(a) For any \( \beta, 0 \leq |\beta| \leq K \),

\[
\left| \partial^\beta [h_1 (\xi; x) - h_2 (\xi; x)] \right| \leq c_{20} \gamma_2^{-2K-1} N_\xi^{\beta + d - 1} (1 + N_\xi \rho (\xi; x))^M, \quad \forall x \in S^{d-1};
\]

(b) For any \( \beta, 0 \leq |\beta| \leq K - 1 \),

\[
\left| \int_{S^{d-1}} y^{\beta} [h_1 (\xi; y) - h_2 (\xi; y)] d\sigma(y) \right| \leq c_{20} \gamma_2^{-2K-1} N_\xi^{-K},
\]

where \( c_{20} \) depends only on \( d, K, M \).

**Proof.** Let \( 0 \leq |\beta| \leq K \). From (6.21) and (6.22) we get

\[
\left| \partial^\beta [h_1 (\xi; x) - h_2 (\xi; x)] \right|
\]

\[
= \left| \int_{S^{d-1}} \Lambda_{N_\xi} (\xi \cdot y) \partial^\beta \Delta^{K/2} F (y; x) d\sigma(y) - \sum_{\zeta \in \mathcal{Z}_j} w_\zeta \Lambda_{N_\xi} (\xi \cdot \zeta) \partial^\beta \Delta^{K/2} F (\zeta; x) \right|
\]

\[
= \left| \int_{S^{d-1}} [\Lambda_{N_\xi} (\xi \cdot y) \partial^\beta \Delta^{K/2} F (y; x) - \Lambda_{N_\xi} (\xi \cdot \zeta) \partial^\beta \Delta^{K/2} F (\zeta (y); x)] d\sigma(y) \right|
\]

\[
+ \left| \int_{S^{d-1}} [\partial^\beta \Delta^{K/2} F (y; x) - \partial^\beta \Delta^{K/2} F (\zeta (y); x)] d\sigma(y) \right|
\]

\[
\leq \left| \int_{S^{d-1}} [\Lambda_{N_\xi} (\xi \cdot y) - \Lambda_{N_\xi} (\xi \cdot \zeta) \partial^\beta \Delta^{K/2} F (\zeta ; x)] d\sigma(y) \right|
\]

\[
+ \left| \int_{S^{d-1}} [\partial^\beta \Delta^{K/2} F (y; x) - \partial^\beta \Delta^{K/2} F (\zeta (y); x)] d\sigma(y) \right|,
\]
where ζ(y) is defined in (2.3). Let η = η(s), s ∈ [0, ρ], ρ = ρ(y, ζ(y)), be the geodesic line on S^d−1 such that η(0) = y and η(ρ) = ζ(y). Then
\[\partial^β Δ^{K/2} F_ε(y; x) - \partial^β Δ^{K/2} F_ε(ζ(y); x) = \int_0^ρ \nabla_η \partial^β Δ_x^{K/2} F_x\left(\frac{η · x}{η ||x||}\right) |_{η=η(s)} · η'(s) \, ds.\]

Using the above representation (5.34), (2.7), ρ we get (6.24) with (6.26) and get
\[\partial^β Δ^{K/2} F_ε(ζ(y); x) = \int_0^ρ \nabla_η \partial^β Δ_x^{K/2} F_x\left(\frac{η · x}{η ||x||}\right) |_{η=η(s)} · η'(s) \, ds.\]

Now, for 0 ≤ |β| ≤ K − 1 we apply consecutively (6.9), (6.10), the fact that the operator Δ_0 is symmetric, (6.26), (2.6), and (2.8) to obtain
\[|g_1(ξ; x) - g_2(ξ; x)| ≤ c_{24} \frac{γ_1^{-1} N_ε^{-K+1}}{(1 + N_ε ρ(ξ, x)) M}, \quad x ∈ S^d−1.\]

Substituting the above two estimates in (6.23) we get (6.24) with c_{20} ≥ c_{23} := c_{22} + c_3 c_4 c_{21}.

For the proof of (6.24) we repeat the arguments applied for the proof of (6.23) and get
\[|g_1(ξ; x) - g_2(ξ; x)| ≤ c_{24} \frac{γ_1^{-1} N_ε^{-K+1}}{(1 + N_ε ρ(ξ, x)) M}.\]

Now, for 0 ≤ |β| ≤ K − 1 we apply consecutively (6.9), (6.10), the fact that the operator Δ_0 is symmetric, (6.26), (2.6), and (2.8) to obtain
\[|\int_{S^d−1} y^β [h_1(ξ; y) - h_2(ξ; y)] dσ(η)| = \left| \int_{S^d−1} y^β Δ_0^{K/2} [g_1(ξ; y) - g_2(ξ; y)] dσ(η) \right| \leq c_0 c_6 c_24 γ_2^{-1} N_ε^{-K}.\]

which yields (6.24) with c_{20} = max\{c_{23}, c_0 c_6 c_{24}\} in view of γ_1 ≤ 1.
The estimates on \( h_2(\xi; \cdot) - h_3(\xi; \cdot) \) are given in the following lemma, where \( \tilde{h}_2(\xi; x) \) is as in Lemma 6.3 and for \( \xi \in \mathcal{X}_j \) we set
\[
\tilde{h}_3(\xi; x) = \sum_{\zeta \in \mathbb{Z}/d \cdot \rho(\xi) \leq r_\xi} w_{\zeta}\Lambda_{N_\xi}(\xi \cdot \zeta)\Delta^{K/2}\tilde{F}_\xi(\zeta; x).
\]

**Lemma 6.4.** Let \( \xi \in \mathcal{X}_j, j \in \mathbb{N}, K \in 2\mathbb{N}, M > K + d - 1 \), and let \( \gamma_1, \gamma_2, \varepsilon, \tilde{F}_\xi \) be as in Lemma 6.3. If \( 0 < \gamma_3 \leq 1 \) and \( r_\xi = (\gamma_3 N_\xi)^{-1} \), then:

(a) For any \( \beta, 0 \leq |\beta| \leq K \), we have

\[
|\partial^\beta [h_2(\xi; x) - \tilde{h}_3(\xi; x)]| \leq c_{30} \gamma_3\gamma_1^{-2K} N_\xi^{|\beta|+d-1} \frac{1}{(1 + N_\xi r_\xi(x))^M}, \quad \forall x \in \mathbb{S}^{d-1},
\]

(b) For any \( \beta, 0 \leq |\beta| \leq K - 1 \), we have

\[
|\int_{\mathbb{S}^{d-1}} \partial^\beta [h_2(\xi; y) - h_3(\xi; y)] d\sigma(y)| \leq c_{30} \gamma_3\gamma_1^{-2K} N_\xi^{-K},
\]

where \( c_{30} \) depends only on \( d, K, M \).

**Proof.** Let \( \xi \in \mathcal{X}_j \). Let \( \Lambda_{N_\xi}(u) := \Lambda_{N_\xi}(u)\mathbb{1}_{[r_\xi, \pi]}(u) \). Then for \( 0 \leq |\beta| \leq K \) we get from (6.10) and (6.11)
\[
\partial^\beta [h_2(\xi; x) - \tilde{h}_3(\xi; x)] = \sum_{\zeta \in \mathbb{Z}/d \cdot \rho(\xi) \leq r_\xi} w_{\zeta}\Lambda_{N_\xi}(\xi \cdot \zeta)\Delta^2\tilde{F}_\xi(\zeta; x)
\]

\[
= \int_{\mathbb{S}^{d-1}} \Lambda_{N_\xi}(\xi \cdot \zeta(y))\Delta^2\tilde{F}_\xi(\zeta(y); x) d\sigma(y),
\]

where \( \zeta(y) \) is defined in (2.3). For \( \Lambda_{N_\xi}(\xi \cdot x) \) estimate (5.13) with \( \beta = 0 \) and \( M \) replaced by \( M + 1 \) yields
\[
|\Lambda_{N_\xi}(\xi \cdot x)| \leq c_3 \frac{N_\xi^{-K+d-1}}{(1 + N_\xi r_\xi(x))^{M+1}}, \quad x \in \mathbb{S}^{d-1}.
\]

This estimate and the inequality \( (1 + N_\xi r_\xi(x))^{-M-1} \leq \gamma_3(1 + N_\xi r_\xi(x))^{-M} \), if \( r_\xi(x) > r_\xi \), yield
\[
|\Lambda_{N_\xi}(\xi \cdot x)| \leq c_3 \frac{\gamma_3 N_\xi^{-K+d-1}}{(1 + N_\xi r_\xi(x))^{M}}, \quad x \in \mathbb{S}^{d-1},
\]
and hence

\[
|\Lambda_{N_\xi}(\xi \cdot \zeta(y))| \leq c_{31} \frac{\gamma_3 N_\xi^{-K+d-1}}{(1 + N_\xi r_\xi(y))^{M}}, \quad y \in \mathbb{S}^{d-1},
\]

with \( c_{31} = 2Mc_3^2 \) on account of \( r_\xi(y, \zeta(y)) \leq \gamma_2 N_\xi^{-1} \leq N_\xi^{-1} \) and (2.7). Using (2.7) again we get from (6.30)
\[
|\partial^\beta \Delta^{K/2}\tilde{F}_\xi(\zeta(y); x)| \leq c_{32} \frac{\varepsilon^{-(K+|\beta|+d-1)}}{(1 + \varepsilon^{-1} r_\xi(y, x))^{M}},
\]

with \( c_{32} = 2Md^{K/2}c_8 \). We now apply Proposition 2.5 to the integral in (6.29) with \( x_1 = \xi, g(y) = \Lambda_{N_\xi}(\xi \cdot \zeta(y)), N_1 = N_\xi, \kappa_1 = c_{31}\gamma_3 N_\xi^{-K} \) from (6.30), \( x_2 = x \), \( f(y) = \partial^\beta \Delta^{K/2}\tilde{F}_\xi(\zeta(y); x), \varepsilon = \gamma_1/N_\xi, N_2 = 1/\varepsilon \geq N_1, \kappa_2 = c_{32}(\varepsilon^{-1})^{K+|\beta|} \) from (6.31), and get (6.27) with \( c_{30} \geq c_{30}c_3c_{32} \).
For the proof of (6.28) we repeat the argument from the proof of (6.27) and get
(6.32) \[ |g_2(\xi; x) - g_3(\xi; x)| \]
\[ = \left| \int_{\mathbb{S}^{d-1}} \Lambda^*_N(\xi \cdot \zeta(y)) F_{\varepsilon}(\zeta(y) \cdot x) d\sigma(y) \right| \leq c_{35} \frac{\gamma_3 N_{\xi}^{-K+d-1}}{(1 + N_{\xi} \rho(\xi; x))^M}. \]

Now, for \( 0 \leq |\beta| \leq K - 1 \) we apply consecutively (6.10), (6.11), the fact that the operator \( \Delta_0 \) is symmetric, (6.32), (2.6), and (2.8) to obtain
\[
\left| \int_{\mathbb{S}^{d-1}} y^\beta [h_2(\xi; y) - h_3(\xi; y)] d\sigma(y) \right| = \left| \int_{\mathbb{S}^{d-1}} y^\beta \Delta_0^{K/2} [g_2(\xi; y) - g_3(\xi; y)] d\sigma(y) \right| \leq c_0 c_6 c_{33} \gamma_3 N_{\xi}^{-K},
\]
which yields (6.28) with \( c_{30} \geq c_0 c_6 c_{33} \) because of \( \gamma_1 \leq 1 \). Finally, we set \( c_{30} := \max\{c_3 c_{31} c_{32}, c_0 c_6 c_{33}\}. \]

**Lemma 6.5.** Assume \( \xi \in \mathcal{X}_j, j \in \mathbb{N}, K \in 2\mathbb{N}, M > K + d - 1 \), and let \( \gamma_1, \gamma_2, \gamma_3, \varepsilon, r_\xi, h_\xi \) be as in Lemma 6.4. Let \( \bar{\theta}_\xi(x) \) be defined by (6.12) if \( d \geq 3 \) or by (6.13) if \( d = 2 \) with \( x/|x| \) in the place of \( x \). Then for any \( \gamma_4 > 0 \) there exists \( t_j > 0 \) such that
(6.33) \[ \left| \partial^\beta \left[ h_\xi(\xi; x) - \bar{\theta}_\xi(x) \right] \right| \leq \frac{\gamma_4 N_{\xi}^{|\beta|+d-1}}{(1 + N_{\xi} \pi)^M}, \quad \forall x \in \mathbb{S}^{d-1}, \quad 0 \leq |\beta| \leq K,
\]
(6.34) \[ \left| \int_{\mathbb{S}^{d-1}} y^\beta [h_\xi(\xi; y) - \theta_\xi(y)] d\sigma(y) \right| \leq \gamma_4 N_{\xi}^{-M}, \quad 0 \leq |\beta| \leq K - 1.
\]

**Proof.** Inequalities (6.33) and (6.34) follow from (6.11)–(6.13) by approximating the operator \( \Delta_0^{K/2} (\zeta \cdot \nabla)^m \) by \( \bar{\Delta}_0^{K/2} \mathcal{D}_\zeta^m(\zeta) \) as \( t \to 0 \) and the infinite smoothness of \( |(1 + \varepsilon)\zeta - x|^{-d+2} \) and \( \log 1/(1 + \varepsilon)\zeta - x| \) on the compact \( \mathbb{S}^{d-1} \).

**Proof of Theorem 6.1.** This proof follows at once by Lemmas 6.2, 6.3, 6.4 and 6.5 with the following selection of parameters:
(6.35) \[
\gamma_1 := \min\{\gamma_0/(4c_{10}), 1\}, \quad \gamma_2 := \min\{\gamma_0 \gamma_1^{2K+1}/(4c_{20}), \gamma_1\}, \\
\gamma_3 := \min\{\gamma_0 \gamma_1^{2K}/(4c_{30}), 1\}, \quad \gamma_4 := \gamma_0/4.
\]

\[ \square \]

### 6.2. Completion of the construction of new frames on \( \mathbb{S}^{d-1} \)

We use the scheme from Section 3 to complete the construction of a pair of dual frames \( \{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\bar{\theta}_\xi\}_{\xi \in \mathcal{X}} \) on \( \mathbb{S}^{d-1} \), where each frame element \( \theta_\xi \) is a linear combination of a fixed number of shifts of the Newtonian kernel.

Following the definition \( \psi_\xi(x) := C_\xi^2 \psi_\xi^2(x) \) of the elements of old frame \( \Psi \) given in (6.18), we similarly construct the elements
\[
\theta_\xi(x) := C_\xi^2 \bar{\theta}_\xi(x), \quad \xi \in \mathcal{X}_j, \quad j \geq 1.
\]
of the new frame $\Theta = \{\theta_\xi\}_{\xi \in \mathcal{X}}$. In light of (6.12) and (6.13) we have for $j \geq 1$

$$
(6.36) \quad \theta_\xi(x) := C_\xi^2 \kappa_{\epsilon,m,d} \sum_{\substack{\xi \in \mathbb{Z}^d \setminus \rho(\xi,\xi) \leq r_\xi}} w_{\xi} \Lambda_{N_\xi}(\xi \cdot \zeta) \sum_{\ell=0}^{m} b_{\ell} \Delta_j^{K/2} D_j^\ell(\zeta) |x - a\zeta|^{2-d}, \quad d \geq 3,
$$

$$
(6.37) \quad \theta_\xi(x) := C_\xi^2 \kappa_{\epsilon,m,2} \sum_{\substack{\xi \in \mathbb{Z}^d \setminus \rho(\xi,\xi) \leq r_\xi}} w_{\xi} \Lambda_{N_\xi}(\xi \cdot \zeta) \sum_{\ell=1}^{m} b_{\ell} \Delta_j^{K/2} D_j^\ell(\zeta) \ln \frac{1}{|x - a\zeta|}, \quad d = 2.
$$

The only frame element excluded from this definition is the constant function corresponding to $\xi \in \mathcal{X}_0$. For $\xi \in \mathcal{X}_0$ we set $\theta_\xi(x) := \psi_\xi(x) \equiv 1, \ x \in \mathbb{S}^{d-1}$.

In Theorem 6.7 below we collect some important properties of the new frame $\Theta$. Its proof is based on Theorem 6.1 and the following lemma.

**Lemma 6.6.** Let $\eta \in \mathcal{X}$, $K \in 2\mathbb{N}$, and $M > K + d - 1$.

(a) For any $\beta$, $0 \leq |\beta| \leq K$, we have

$$
(6.38) \quad |\partial^\beta \tilde{K}_{N_\eta}(\eta; x)| \leq c_{40} \frac{N_\eta^{\beta + d - 1}}{(1 + N_\eta \rho(\eta, x))^M}, \quad \forall x \in \mathbb{S}^{d-1},
$$

with $\tilde{K}_{N_\eta}(\eta; x)$ given by (5.4).

(b) For any $\beta$, $0 \leq |\beta| \leq K - 1$, we have

$$
(6.39) \quad \left| \int_{\mathbb{S}^{d-1}} y^\beta \tilde{K}_{N_\eta}(\eta \cdot y) d\sigma(y) \right| \leq c_{40} N_\eta^{-K}.
$$

**Proof.** For $\eta \in \mathcal{X}_0$ we have $\tilde{K}_{N_\eta}(\eta; x) \equiv 1/\omega_d$, $N_\eta = 2^{-1}$, and inequalities (6.38), (6.39) are trivial.

Let $\eta \in \mathcal{X} \setminus \mathcal{X}_0$. Inequality (5.13) with $N = N_\eta$ yields (6.38) with $c_{40} \geq c_5$. For any multi-index $\beta$, $0 \leq |\beta| \leq K - 1$, we get from (5.3), the fact that the operator $\Delta_0$ is symmetric, (5.13) with $\beta = 0$ and $N = N_\eta$, (2.6), and (2.8) that

$$
\left| \int_{\mathbb{S}^{d-1}} y^\beta \tilde{K}_{N_\eta}(\eta \cdot y) d\sigma(y) \right| = \left| \int_{\mathbb{S}^{d-1}} y^\beta \Delta_0^{K/2} \Lambda_{N_\eta}(\eta \cdot y) d\sigma(y) \right| = \left| \int_{\mathbb{S}^{d-1}} \Lambda_{N_\eta}(\eta \cdot y) \Delta_0^{K/2} y^\beta d\sigma(y) \right| \leq c_0 c_6 c_4 N_\eta^{-K},
$$

which proves the lemma with $c_{40} := \max\{c_5, c_0 c_6 c_4\}$. \hfill \Box

**Theorem 6.7.** Let $d \geq 2$, $K \in 2\mathbb{N}$, $M > K + d - 1$, and $0 < \gamma_0 \leq 1$. Then there exist constants $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4 > 0$ depending only on $d, K, M, \gamma_0$, and for every $\xi \in \mathcal{X}_j$, $j \in \mathbb{N}$, there exists $t_j > 0$ depending only on $d, K, M, \gamma_0$, and $j$ such that:

(a) The new frame $\Theta = \{\theta_\xi\}_{\xi \in \mathcal{X}}$ is real-valued and satisfies

$$
(6.40) \quad |\partial^\beta \theta_\xi(x)| \leq c_{41} \frac{N_\eta^{\beta + (d-1)/2}}{(1 + N_\eta \rho(\xi, x))^M}, \quad \forall x \in \mathbb{S}^{d-1}, \ \forall \xi \in \mathcal{X}, \ \forall \beta, \ 0 \leq |\beta| \leq K,
$$

$$
(6.41) \quad |\langle \psi_\eta, \psi_\xi - \theta_\xi \rangle| \leq c_{42} \gamma_0 \omega^{(K,M)}_{\xi,\eta}, \quad \forall \xi, \eta \in \mathcal{X},
$$

with $c_{41} = \max\{2^{d-1}/(1 + \pi/2)^M, c_9(c_{40} + \gamma_0)\}$, $c_{42} = c_1 c_{40} c_3^2$. 

(b) Every frame element $\theta_\xi$, $\xi \in X \setminus X_0$, is a linear combination of at most $\tilde{n}$ shifts of Newtonian kernels, where $\tilde{n} \leq c_{43}\gamma_0^{(4K+3)(1-d)}$ with $c_{43}$ depending only on $d, K, M$.

(c) Moreover, if

$$\gamma_0 \leq \frac{c_{42}^2}{4c_{42}},$$

where $c_{42}$ is from (6.11) and $\tilde{c}_{5}$ is from (6.26), then

$$\|\theta_\xi\|_{L^p(\mathbb{R}^{d-1})} \sim N(\xi)^{(d-1)(1/2-1/p)}, \quad \frac{d-1}{M} < p \leq \infty, \quad \forall \xi \in X,$$

with uniformly bounded constants of equivalence for $p \geq \frac{d-1+\delta}{M}$, $\delta > 0$.

**Proof.** For $\xi \in X_0$, we have $\theta_\xi = 1$, $N_\xi = 2^{-1}$, and (6.40) is satisfied with $c_{41} \geq 2^{(d-1)/2(1+\pi/2)}$. Also $\psi_\xi - \theta_\xi = 0$ and inequality (6.41) is obvious for all $\eta \in X$.

Let $\xi \in X \setminus X_0$. Then (6.14), (6.38) with $\xi$ in the place of $\eta$, and (5.19) imply (6.40) with $c_{41} \geq c_9(c_{40} + \gamma_0)$.

For the proof of (6.11) first assume that $N_\xi \geq N_\eta$. We apply Proposition 2.3 with $g = \psi_\eta = C^0_\eta \psi_\eta$, $x_1 = \eta$, $f = \psi_\xi - \theta_\xi = C^0_\xi (\psi_\xi - \theta_\xi)$, $x_2 = \xi$. Lemma 6.6 implies that (2.33) is satisfied with $N_1 = N_\eta$, $\kappa_1 = c_{40} C^0_\eta$, and Theorem 6.1 implies that (2.31), (2.32) are satisfied with $N_2 = N_\xi$, $\kappa_2 = \gamma_0 C^0_\xi$. Now, (2.33) and (5.19) give

$$|\langle \psi_\eta, \psi_\xi - \theta_\xi \rangle| \leq c_{41} c_{40} C^0_\eta \frac{N_\eta^{d-1}}{N_\xi^{d-1}} (N_\eta/N_\xi)^K N_\eta^{d-1} \leq c_{42} \gamma_0 \omega_\xi^{(K,M)},$$

with $c_{42} := c_1 c_{40} c_{43}^2$, which establishes (6.31) in this case.

Second, assume that $N_\xi \leq N_\eta$. Here, we apply Proposition 2.3 with $g = \psi_\xi - \theta_\xi$, $f = \psi_\eta$, and then (6.31) follows similarly as above. This completes the proof of (a).

The number of $\xi \in Z_\ell$ in (6.30) (or (6.37)) can be estimated as follows. From

$$\bigcup_{\xi \in Z_\ell, \rho(\xi, \zeta) \leq r_\zeta} A_\xi \subset B(\xi, r_\xi + \gamma_2/N_\xi)$$

we find that the total volume covered by $A_\xi$ does not exceed $c((\gamma_3^{-1} + \gamma_2)/N_\xi)^{d-1}$. From this estimate and (2.20) with $\gamma = \gamma_2$ we get that the number of $\xi \in Z_\ell$ in (6.36) is at most $c_{43} \gamma_0^{(4K+3)(1-d)}$ in light of (6.35).

Clearly, the number of translation terms in $A^K_\ell$ is at most $(d(d-1)/2)^K/2 + 1$. The number of translation terms in $B^\ell_\ell(\zeta)$ is $\ell + 1$, $\ell = 0, 1, \ldots, m$, and every such term is also a term for $B^\ell_\ell(\zeta)$. This leads to the estimate $\tilde{n} \leq c_{43}\gamma_0^{(4K+3)(1-d)}$ with $c_{43} := c_{44} [(d(d-1)/2)^K/2 + 1](m + 1)$ for the number of shifts of Newtonian kernels used in (6.36) or in (6.37). Thus, the proof of (b) is complete.

We now establish (c). Inequality (6.40) with $|\beta| = 0$ along with (2.6) imply

$$\|\theta_\xi\|_{L^p} \leq c_0^{1/p} c_{41} N_\xi^{(d-1)(1/2-1/p)}$$

with $c_0 = c(d)/\delta$.

To prove the estimate in the other direction:

$$\|\theta_\xi\|_{L^p} \geq c_{45} N_\xi^{(d-1)(1/2-1/p)}$$
we first consider the case \( p = 2 \). From
\[
\langle \theta_\xi, \theta_\zeta \rangle = \langle \psi_\xi, \psi_\xi \rangle - 2\langle \psi_\xi, \psi_\zeta - \theta_\xi \rangle + \langle \psi_\zeta - \theta_\xi, \psi_\zeta - \theta_\zeta \rangle,
\]
we get
\[
\langle \theta_\xi, \theta_\zeta \rangle \geq \langle \psi_\xi, \psi_\xi \rangle - 2|\langle \psi_\xi, \psi_\zeta - \theta_\xi \rangle| \geq \tilde{c}^2_3 - 2\gamma_0 c_{42} \omega^{(K,M)}_{\xi,\zeta} \geq \tilde{c}^2_3/2,
\]
due to \( \omega^{(K,M)}_{\xi,\zeta} = 1 \) (see (4.21)). This gives (6.45) for \( p = 2 \) with \( c_{45} = \tilde{c}_3/\sqrt{2} \).

Now, consider the case \( p < 2 \). Using (6.45) with \( p = 2 \) and (6.44) with \( p = \infty \) we obtain
\[
\tilde{c}^2_3/2 \leq \|\theta_\xi\|_{L^2}^2 \leq \|\theta_\xi\|_{L^\infty}^{2-p}\|\theta_\xi\|_{L^p}^p \leq \left(c_{41} N^{(d-1)/2}_\xi \right)^{2-p} \|\theta_\xi\|_{L^p}^p,
\]
which proves (6.45) for \( p < 2 \) with \( c_{45} = (c_{41}^{-2} \tilde{c}^2_3/2)^{1/p} \).

Finally, consider the case \( 2 < p \leq \infty \). Using Hölder's inequality, (6.45) with \( p = 2 \) and (6.44) with \( 1 \leq p' < 2 \) we obtain
\[
\tilde{c}^2_3/2 \leq \|\theta_\xi\|_{L^2}^2 \leq \|\theta_\xi\|_{L^{p'}} \|\theta_\xi\|_{L^p} \leq c_{12}^{1/p'} c_{41} N^{(d-1)(1/2 - 1/p')} \|\theta_\xi\|_{L^p},
\]
which proves (6.45) for \( p > 2 \) with \( c_{45} = c_{12}^{-1+p} \tilde{c}^2_3/2 \). This proves (6.45) for all \( p \) and completes the proof of the theorem.

**Remark 6.8.** All poles of the Newtonian kernels in (6.36) and in (6.37) are placed on \( m + 1 \) concentric spheres of radii \( 1 + \gamma_1 N_\xi^{-1} + k t_j \), \( k = 0, 1, \ldots, m \). On every such sphere the poles are located in the spherical cap of radius \( (\gamma_3 N_\xi^{-1} + t_j K)/2 \) centred at \( (1 + \gamma_1 N_\xi^{-1} + k t_j)\xi \).

Our next step is to show that the above defined system \( \Theta \) coupled with the dual system \( \tilde{\Theta} = \{ \tilde{\theta}_\xi \}_{\xi \in \mathcal{X}} \) constructed by the scheme from (4.2) form a pair of frames for all Besov and Triebel-Lizorkin space \( \mathcal{B}^p_{s,q}, \mathcal{F}^p_{s,q} \) with parameters \( (s,p,q) \in \mathbb{Q}(A) \) for a fixed \( A > 1 \) with \( \mathbb{Q}(A) \) defined in (4.3).

**Theorem 6.9.** Assume \( d \geq 2 \), \( A > 1 \), and let \( \Theta = \{ \theta_\xi \}_{\xi \in \mathcal{X}} \) be the real-valued system constructed in (6.36) or (6.37), where
\[
K \geq \lceil Ad \rceil, K \in 2\mathbb{N}, \quad M = K + d.
\]
If the constant \( \gamma_0 \) in the construction of \( \{ \theta_\xi \}_{\xi \in \mathcal{X}} \) is sufficiently small, namely,
\[
\gamma_0 \leq \frac{\epsilon}{c_{42} C_9},
\]
where \( \epsilon \) is from (4.3), \( c_{42} \) is from (6.44), and \( C_9 \) is from Theorem 4.6, then:

(a) The synthesis operator \( \mathcal{T}_\theta \) defined by \( \mathcal{T}_\theta h := \sum_{\xi \in \mathcal{X}} h_\xi \theta_\xi \) on sequences of complex numbers \( h = \{ h_\xi \}_{\xi \in \mathcal{X}} \) is bounded as a map \( \mathcal{T}_\theta : \mathcal{B}^p_{s,q} \rightarrow \mathcal{B}^p_{s,q} \), uniformly with respect to to \( (s,p,q) \in \mathbb{Q}(A) \).

(b) The operator
\[
 Tf := \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \theta_\xi = T\theta S\psi f,
\]
is invertible on \( \mathcal{B}^p_{s,q} \) and \( T, T^{-1} \) are bounded on \( \mathcal{B}^p_{s,q} \), uniformly with respect to \( (s,p,q) \in \mathbb{Q}(A) \).
(c) For \((s,p,q) \in Q(A)\) the dual system \(\tilde{\Theta} = \{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}\) consists of bounded linear functionals on \(B_p^{sq}\) defined by
\[
(6.49) \quad \tilde{\theta}_\xi(f) = \langle f, \tilde{\theta}_\xi \rangle = \sum_{\eta \in \mathcal{X}} \langle T^{-1}\psi_\eta, \psi_\xi \rangle \langle f, \psi_\eta \rangle \quad \text{for} \quad f \in B_p^{sq},
\]
with the series converging absolutely. Also, the analysis operator
\[
S_\theta : B_p^{sq} \rightarrow b_p^{sq}, \quad S_\tilde{\theta} = S_\psi T^{-1}S_\psi = S_\psi T^{-1},
\]
is uniformly bounded with respect to \((s,p,q) \in Q(A)\). Moreover, \(\{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}\) form a pair of dual frames for \(B_p^{sq}\) in the following sense: For any \(f \in B_p^{sq}\)
\[
(6.50) \quad f = \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\theta}_\xi \rangle \theta_\xi \quad \text{and} \quad \|f\|_{B_p^{sq}} \sim \|\{\langle f, \tilde{\theta}_\xi \rangle\}\|_{b_p^{sq}},
\]
where the convergence is unconditional in \(B_p^{sq}\).

Furthermore, (a), (b), and (c) hold true when \(B_p^{sq}, b_p^{sq}\) are replaced by \(F_p^{sq}, \ell_p^{sq}\), respectively.

Proof. The parameters \(K\) and \(M\) from (6.46) satisfy (4.22) with \(\delta = 1\) for all \((s,p,q) \in Q(A)\) and we can apply Theorem 4.6. From estimate (6.41) in Theorem 6.7, Lemma 4.5, and Theorem 4.6 we obtain that \(\{\theta_\xi\}_{\xi \in \mathcal{X}}\) satisfies (4.6) due to \(\gamma_0 c_2 C_0 \leq \epsilon\). Also, all conditions on the old frame laid in Subsection 4.1 are satisfied as shown in Subsection 4.1. Now, we apply Lemma 4.1 and Lemma 4.2 to get (a) and (b). Finally, Theorem 4.3 implies (c). \(\square\)

6.3. Frames on \(B^d\) in terms of shifts of the Newtonian kernel. Note that by the fact that each frame element \(\theta_\xi, \xi \in \mathcal{X}\), is represented as a finite linear combination of shifts of the Newtonian kernel it readily follows that \(\theta_\xi(x)\) defined in (6.36) or (6.37) as a function of \(x \in B^d\) is harmonic on \(B^d\). This leads immediately to the conclusion that \(\{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}\) is a pair of dual frames for harmonic Besov and Triebel-Lizorkin spaces in the sense of the following

**Theorem 6.10.** Under the hypothesis of Theorem 6.9 let \(\{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}\) be the frames from Theorem 6.9. Then for any \((s,p,q) \in Q(A)\) and \(U \in B_p^{sq}(\mathcal{H})\)
\[
(6.51) \quad U(x) = \sum_{\xi \in \mathcal{X}} \langle f_U, \tilde{\theta}_\xi \rangle \theta_\xi(x), \quad x \in B^d, \quad \text{and} \quad \|U\|_{B_p^{sq}} \sim \|\{\langle f_U, \tilde{\theta}_\xi \rangle\}\|_{b_p^{sq}}.
\]

Here \(f_U\) is the boundary value of \(U\) (see Proposition 3.3) and the series converges uniformly on every compact subset of \(B^d\). Furthermore, the above holds true when \(B_p^{sq}(\mathcal{H}), b_p^{sq}\) are replaced by \(F_p^{sq}(\mathcal{H}), \ell_p^{sq}\), respectively.

This theorem follows immediately by Theorems 6.9, 3.4, and 3.5

7. Nonlinear approximation from shifts of the Newtonian kernel

The primary goal of this article is to establish a Jackson type estimate for nonlinear \(n\)-term approximation of harmonic functions on \(B^d\) from shifts of the Newtonian kernel in the harmonic Hardy space \(H^0(B^d)\). For any \(n \geq 1\) write
\[
(7.1) \quad N_n := \left\{ G : G(x) = a_0 + \sum_{\nu=1}^n \frac{a_\nu}{|x - y_\nu|^{d-2}}, \ |y_\nu| > 1, \ a_\nu \in \mathbb{C} \right\}, \ \text{if} \ d > 2,
\]
Theorem 3.7. where the constant $c > 0$ then $f_n$ establishing a Jackson estimate for nonlinear $E_0$ and $E_1$ of the form $\sum_{n=1}^{\infty} a_n \ln \frac{1}{|x - y_n|}, \quad |y_n| > 1, a_n \in \mathbb{C}$, if $d = 2$.

Observe that the points $\{y_n\}$ above may vary with $G$ and hence $N_n$ is nonlinear.

Let $\mathcal{B}$ be one of the spaces $\mathcal{H}^p(B^d)$, $B_p^q(\mathcal{H})$, or $F_p^q(\mathcal{H})$, $0 < p, q < \infty$. Given $U \in \mathcal{B}$ we define

$$(7.3) \quad E_n(U)_\mathcal{B} := \inf_{G \in N_n} \|U - G\|_\mathcal{B}.$$ 

We call $E_n(U)_\mathcal{B}$ the best nonlinear $n$-term approximation of $U$ from shifts of the Newtonian kernel in the harmonic space $\mathcal{B}$.

We now come to the main result in this article.

**Theorem 7.1.** Let $s > 0$, $0 < p < \infty$, and $1/\tau = s/(d - 1) + 1/p$. If $U \in B_\tau^\tau(\mathcal{H})$, then $U \in \mathcal{H}^p(B^d)$ and

$$(7.4) \quad E_n(U)_\mathcal{H} \leq cn^{-s/(d-1)}|U|_{B_\tau^\tau(\mathcal{H})}, \quad n \geq 1,$$

where the constant $c > 0$ depends only on $p, s, d$.

Theorem 7.1 is an immediate consequence of Theorem 7.9 below with $q = 2$ and Theorem 6.7.

Our approach to approximating a harmonic function $U$ on $B^d$ amounts to first establishing a Jackson estimate for nonlinear $n$-term approximation of its boundary value function $f_U$ on $\mathbb{S}^{d-1}$ from the frame elements $\{\theta_\xi\}_{\xi \in X}$ constructed in Section 6 and then considering the harmonic extension to $B^d$ of the approximant. The gist of our approximation method is that each frame element $\theta_\xi$ is a linear combination of a fixed number of shifts of the Newtonian kernel.

**7.1. Nonlinear $n$-term frame approximation on $\mathbb{S}^{d-1}$**. Let $\{\theta_\xi\}_{\xi \in X}$ be the frame constructed in Section 6 with parameters to be specified. Denote by $\Sigma_n$ the set of all functions $g$ on $\mathbb{S}^{d-1}$ of the form

$$g = \sum_{\xi \in Y_n} a_\xi \theta_\xi, \quad a_\xi \in \mathbb{C},$$

where $Y_n \subset X$ is an index set such that $\#Y_n \leq n$. Define

$$(7.5) \quad \sigma_n(f)_\mathcal{B} := \inf_{g \in \Sigma_n} \|f - g\|_\mathcal{B},$$

where $\mathcal{B}$ is one of the spaces $L^p(\mathbb{S}^{d-1})$, $B_p^q(\mathbb{S}^{d-1})$, or $F_p^q(\mathbb{S}^{d-1})$, $0 < p, q < \infty$.

As one can expect the smoothness spaces on $\mathbb{S}^{d-1}$ governing this kind of approximation should be the Besov spaces $B_\tau^\tau(\mathbb{S}^{d-1})$ with $s$ and $\tau$ as in Theorem 7.1. For this to be true, however, $\{\theta_\xi\}_{\xi \in X}$, $\{\tilde{\theta}_\xi\}_{\xi \in X}$ have to provide frame decomposition of all spaces involved just as in Theorem 6.6.

**Assumptions:** The construction of the frames $\{\theta_\xi\}_{\xi \in X}$, $\{\tilde{\theta}_\xi\}_{\xi \in X}$ in Section 6 depends on the parameters $A, K, M,$ and $\gamma_0$. The main parameter is $A$. In light of Theorems 6.7 and 6.9 we require that

$$0 < s \leq A, \quad A^{-1} - 1 \leq p \leq A, \quad A^{-1} \leq q < \infty, \quad A^{-1} \leq s/(d - 1) + 1/p \leq A, \quad A > 1,$$

which reduces to the following principle conditions:

$$(7.6) \quad 0 < s \leq A, \quad 0 < p \leq A, \quad A^{-1} \leq q < \infty, \quad s/(d - 1) + 1/p \leq A, \quad A > 1.$$
The parameters $K$, $M$ are secondary and are defined as

\[(7.7)\quad K := 2 \left\lfloor \frac{Ad}{2} \right\rfloor, \quad M := K + d.\]

Further, $\gamma_0 > 0$ is a “small” parameter and using the notation from Theorems 6.7 and 6.9 we fix it as

\[(7.8)\quad \gamma_0 := \min \left\{ \frac{\epsilon}{c_{42} C_9}, \frac{c_5^2}{4 c_{42}} \right\}.\]

It is easy to see that $\gamma_0$ depends only on $A$, $d$, and the “old” frame $\{\psi_\xi\}_{\xi \in \mathcal{X}}$, described in § 3.5.

Conditions (7.6) can be viewed in two ways:

(a) Given $A > 1$ consider (7.6) as conditions on $s, p, q$;

(b) Given $s, p, q$ consider (7.6) as conditions on $A$. For Theorem 7.1 we take $A$ to be the smallest number satisfying (7.6).

Either way conditions (7.6) coupled with (7.7)–(7.8) imply that the hypotheses of Theorems 6.7 and 6.9 are obeyed and hence the conclusions of these theorems are valid for the frames $\{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\delta_\xi\}_{\xi \in \mathcal{X}}$ and the spaces $\mathcal{B}^0_q(S^{d-1}), \mathcal{F}^0_q(S^{d-1})$, and $\mathcal{B}^\tau_q(S^{d-1})$.

Although we are mainly interested in approximation of functions in the harmonic Hardy space $\mathcal{H}^0(B^d)$ or their boundary values in $\mathcal{F}^0_q(S^{d-1})$, to put it in perspective we shall examine the approximation process at hand in the slightly more general spaces $\mathcal{F}^0_q(S^{d-1})$ and $\mathcal{B}^0_q(S^{d-1})$.

### 7.1.1 Nonlinear n-term frame approximation in Triebel-Lizorkin spaces on $S^{d-1}$.

**Theorem 7.2.** Assume $A > 1$ and let the parameters $K$, $M$, and $\gamma_0$ be defined as in (7.7)–(7.8). Let $s > 0, 0 < p, q < \infty$, and $1/\tau = s/(d-1) + 1/p$, and assume that $s, p, q$ satisfy conditions (7.6). If $f \in \mathcal{B}^\tau_q(S^{d-1}) = \mathcal{F}^\tau_q(S^{d-1})$, then $f \in \mathcal{F}^0_q(S^{d-1})$ and with the notation from (7.5)

\[(7.9)\quad \sigma_n(f)_{\mathcal{F}^0_q(S^{d-1})} \leq c n^{-s/(d-1)} \|f\|_{\mathcal{B}^\tau_q(S^{d-1})}, \quad n \geq 1,
\]

where the constant $c > 0$ depends only on $A, d$.

The proof of this theorem depends on the following simple

**Lemma 7.3.** For any $0 < p < \infty$ and any finite subset $Y$ of $\mathcal{X}$ we have

\[(7.10)\quad \left\| \sum_{\xi \in Y} |B_\xi|^{-1/p} 1_{B_\xi} \right\|_{L^p} \leq c(\#Y)^{1/p}
\]

with $B_\xi = B(\xi, \gamma N_\xi^{-1})$ from Definition 3.10.

**Proof.** First, we observe that for any $j \in \mathbb{N}$ and any $x \in S^{d-1}$ the number of points $\eta \in \mathcal{X}_j$ such that $x \in B_\eta = B(\eta, \delta_j)$ does not exceed a constant $c(d)$. Indeed, the spherical caps $\{B(\eta, \delta_j/2)\}_{\eta \in \mathcal{X}_j}$ are mutually disjoint and $x \in B_\eta$ implies $B(\eta, \delta_j/2) \subset B(\eta, \delta_j/2)$, which together with (2.20) justifies the observation.

Given $Y \subset \mathcal{X}$ we set $\Omega := \bigcup_{\xi \in Y} B_\xi$ and $h(x) := \min \{|B_\xi| : x \in B_\xi, \xi \in Y\}$ if $x \in \Omega$. Clearly, if $x \in B_\xi$ for some $\xi \in Y$, then the above observation and (2.20) imply
We may assume $N$ are fulfilled and from (6.43) and (2.27) we get

$$\bigcup B_{d}$$

with constants of equivalence depending only on (7.12)

From (7.11) and (3.27)–(3.28) we obtain

$$p/q$$ replaced by (d

implies $$(d + 1 + \delta)/M \leq p$$ with $\delta = 1$. Hence the assumptions of Theorem 6.7 (c) are fulfilled and from (6.43) and (2.27) we get

$$\|\theta_{\xi}\|_{L^{p}} := \|\theta_{\xi}\|_{L^{p}({\mathbb S}^{d-1})} \sim N_{\xi}^{(d-1)(1/2-1/p)} \sim \|B_{\xi}\|^{1/p-1/2}, \quad \xi \in X,$$

with constants of equivalence depending only on $d$ and $A$. Set $a_{\xi} := (\hat{\theta}_{\xi}), \xi \in X$. From (7.11) and (5.27)–(4.28) we obtain

$$\|a\|_{L^{p}({\mathbb S}^{d-1})} \sim \|a\|_{\dot{F}^{s}_{\theta}({\mathbb S}^{d-1})} \sim \left(\sum_{\xi \in X} \|\theta_{\xi}\|_{L^{p}}\right)^{1/\tau} =: N(f).$$

We may assume $N(f) > 0$. Denote

$$\mathcal{Y}_{\nu} := \left\{\xi \in X : 2^{-\nu}N(f) < \|a_{\xi}\|_{L^{p}} \leq 2^{-\nu+1}N(f)\right\}, \quad \nu \in \mathbb{N}.$$

Then

$$\bigcup_{\nu \leq \mu} \mathcal{Y}_{\nu} = \left\{\xi : 2^{-\mu}N(f) < \|a_{\xi}\|_{L^{p}}\right\}, \quad \mu \in \mathbb{N},$$

and hence

$$\# \mathcal{Y}_{\mu} \leq \sum_{\nu \leq \mu} \# \mathcal{Y}_{\nu} = \# \left(\bigcup_{\nu \leq \mu} \mathcal{Y}_{\nu}\right) \leq 2^\mu.$$

Set $F_{\mu} := \sum_{\xi \in \mathcal{Y}_{\mu}} |a_{\xi}|B_{\xi}^{1-q/2}B_{\xi}$. We next show that for $m \geq 0$

$$\left\|\left(\sum_{\mu \geq m+1} F_{\mu}\right)^{1/q}\right\|_{L^{p}({\mathbb S}^{d-1})} \leq c2^{-m\tau s/(d-1)}N(f).$$

To this end we first estimate $\|F_{\mu}\|_{L^{p/q}}$. Using (4.13), (4.11), Lemma (2.39) with $p$ replaced by $p/q$, and (7.14) we obtain

$$\|F_{\mu}\|_{L^{p/q}} = \left\|\sum_{\xi \in \mathcal{Y}_{\mu}} \left|a_{\xi}\right|B_{\xi}^{1-p-1/2}B_{\xi}\right\|_{L^{p/q}} \leq c2^{-q(\mu+1)}N(f)^{q}$$

$$\leq c2^{-q(\mu+1)}N(f)^{q} \left\|\sum_{\xi \in \mathcal{Y}_{\mu}} B_{\xi}^{1-q/p}B_{\xi}\right\|_{L^{p/q}} \leq c2^{-q\mu}N(f)^{q}(\# \mathcal{Y}_{\mu})^{q/p}$$

$$\leq c2^{-q(\mu+1-\tau/p)}N(f)^{q} = c2^{-q\mu\tau s/(d-1)}N(f)^{q}.$$
To prove (7.15) we consider two cases. If \( q \leq p \), then using (7.10)

\[
\left\| \left( \sum_{\mu \geq m+1} F_{\mu} \right) \right\|_{L^p}^{1/q} \leq \left\| \sum_{\mu \geq m+1} F_{\mu} \right\|_{L^{p/q}} \leq \sum_{\mu \geq m+1} \left\| F_{\mu} \right\|_{L^{p/q}}
\]

\[
\leq c \sum_{\mu \geq m+1} 2^{-\mu q r/(d-1)} N(f)^q \leq c 2^{-\mu q r/(d-1)} N(f)^q,
\]

which implies (7.14). In the case \( q > p \) using that \( p/q < 1 \) and (7.10), we have

\[
\left\| \left( \sum_{\mu \geq m+1} F_{\mu} \right) \right\|_{L^p}^{1/q} \leq \left\| \sum_{\mu \geq m+1} F_{\mu} \right\|_{L^{p/q}} \leq \sum_{\mu \geq m+1} \left\| F_{\mu} \right\|_{L^{p/q}}
\]

\[
\leq c \sum_{\mu \geq m+1} 2^{-\mu q r/(d-1)} N(f)^p \leq c 2^{-\mu q r/(d-1)} N(f)^p,
\]

yielding again (7.15).

Choose \( m \geq 0 \) so that \( 2^{m r} \leq n < 2^{(m+1)r} \) and denote \( Z_m := \bigcup_{\nu \leq n} X_\nu \). Also, set \( a_\xi := a_\xi \) if \( \xi \in X \setminus Z_m \) and \( a_\xi := 0 \) if \( \xi \in Z_m \). By (7.14) it follows that \( \# Z_m \leq 2^{m r} \leq n \). This, the frame representation (6.5) for \( f \in \mathcal{F}^{02}_{p,q}(S^{d-1}) \), and the boundedness of the synthesis operator \( T_0 \) from Theorem 6.9 (b) yield

\[
\sigma_n(f)_{\mathcal{F}^{02}_{p,q}} \leq \left\| f - \sum_{\xi \in Z_m} a_\xi \theta_\xi \right\|_{\mathcal{F}^{02}_{p,q}} = \left\| \sum_{\xi \in X \setminus Z_m} a_\xi \theta_\xi \right\|_{\mathcal{F}^{02}_{p,q}} \leq c \left\| a^* \right\|_{\mathcal{F}^{02}_{p,q}}
\]

\[
= c \left\| \left( \sum_{\xi \in X \setminus Z_m} |a_\xi|^q |B_\xi|^{-q/2} \right)^{1/q} \right\|_{L^p} \leq c \left\| \left( \sum_{\mu \geq m+1} F_{\mu} \right)^{1/q} \right\|_{L^p}.
\]

Finally, we use (7.15), (7.12), and Theorem 6.9 (c) to obtain

\[
\sigma_n(f)_{\mathcal{F}^{02}_{p,q}} \leq c 2^{-m r s/(d-1)} N(f) \leq c n^{-s/(d-1)} \|a^*\|_{B^r_{s,d-1}} \leq c n^{-s/(d-1)} \|f\|_{B^r_{s,d-1}},
\]

which confirms (7.9).

From Theorem 7.3 and the equivalence \( \mathcal{F}^{02}_{p,s}(S^{d-1}) \sim L^p(S^{d-1}) \) for \( 1 < p < \infty \) we immediately get

**Corollary 7.4.** Assume \( A > 1 \) and let the parameters \( K, M, \) and \( \gamma_0 \) be defined as in (7.7)–(7.8). Let \( s > 0, 1 < p < \infty, \) and \( \gamma_0 \) be such that \( s, p \) satisfy conditions (7.6) with \( q = 2 \). If \( f \in B^{r}_{s,d-1} \), then \( f \in L^p(S^{d-1}) \) and

\[
\sigma_n(f)_{L^p(S^{d-1})} \leq c n^{-s/(d-1)} \|f\|_{B^{r}_{s,d-1}}, \quad n \geq 1,
\]

where the constant \( c > 0 \) depends only on \( A, d \).

**7.1.2. Nonlinear \( n \)-term frame approximation in Besov spaces on \( S^{d-1} \).**

**Theorem 7.5.** Assume \( A > 1 \) and let the parameters \( K, M, \) and \( \gamma_0 \) be as in (7.7)–(7.8). Let \( s > 0, 0 < p, q < \infty, 1/\tau = s/(d-1) + 1/p, \) and \( q \geq \tau \), and assume that \( s, p, q \) satisfy conditions (7.6). If \( f \in B^{r}_{s,d-1} \), then \( f \in B^{\tau}_{p,q}(S^{d-1}) \) and for every \( n \geq 1 \) we have

\[
\sigma_n(f)_{B^{\tau}_{p,q}(S^{d-1})} \leq c n^{-s/(d-1)} \|f\|_{B^{r}_{s,d-1}}, \quad p \leq q,
\]

\[
\sigma_n(f)_{B^{\tau}_{p,q}(S^{d-1})} = o(n^{1/q-1/\tau}) \|f\|_{B^{r}_{s,d-1}}, \quad \tau \leq q < p,
\]

where the constant \( c > 0 \) depends only on \( A, d \).
For the proof of this theorem we shall utilize inequality (6.7) from [25] given in the following

**Lemma 7.6.** Let \(0 < \tau < p < \infty\) and \(x_1 \geq x_2 \geq \cdots \geq 0\). Then for every \(n \in \mathbb{N}\) we have

\[
\left( \sum_{k=n+1}^{\infty} x_k^p \right)^{1/p} \leq n^{1/p-1/\tau} \left( \sum_{k=1}^{\infty} x_k^\tau \right)^{1/\tau}.
\]

**Proof of Theorem 7.7.** Assume that the hypotheses of Theorem 7.7 are obeyed and let \(f \in B_\tau^p(S^{d-1})\). Embeddings (3.31) and (3.32) imply \(f \in B_\tau^0(S^{d-1})\).

As above the frames \(\{\theta_\xi\}_{\xi \in \mathcal{X}}\), \(\{\theta_\xi\}_{\xi \in \mathcal{X}}\) are well defined and the conclusions of Theorems 6.7 and 6.9 are valid for the spaces \(B_\tau^0(S^{d-1})\) and \(B_\tau^p(S^{d-1})\). Denote \(a_\xi := \langle f, \theta_\xi \rangle, \xi \in \mathcal{X}\). Recall the equivalence (7.11) which holds in this case.

Let \(\|a_\eta, \theta_\eta, \|_{L^p} \rangle_{\xi \in \mathcal{X}}, \) i.e.

\[
\|a_\eta, \theta_\eta, \|_{L^p} \geq \|a_{\eta_2}, \theta_{\eta_2}, \|_{L^p} \geq \cdots
\]

Consider first the case \(p \leq q\). Fix \(n \geq 1\) and set \(a_{\eta_k}^* := a_{\eta_k}\) if \(k > n\) and \(a_{\eta_k}^* := 0\) if \(k \leq n\). Note that from (3.27) and (7.11) it follows that

\[
\|a^*\|_{B^p_{\eta_k}} \sim \left( \sum_{\xi \in \mathcal{X}} \|a_{\eta_k}^* \theta_\xi\|_{L^p} \right)^{1/p} = c \left( \sum_{k=n+1}^{\infty} \|a_{\eta_k}, \theta_{\eta_k}, \|_{L^p} \right)^{1/p}.
\]

Using this, embedding (3.32), and the boundedness of the synthesis operator \(T_{\eta}\) from Theorem 6.9(b) we get

\[
\sigma_n(f)_{B^p_{\tau}(S^{d-1})} \leq \left\| \sum_{\eta \in \mathcal{X}} a_{\eta} \theta_{\eta} - \sum_{k=1}^{n} a_{\eta_k} \theta_{\eta_k} \right\|_{B^p_{\tau}} \leq c \sum_{k=n+1}^{\infty} a_{\eta_k} \theta_{\eta_k} \|_{B^p_{\tau}}
\]

\[
\leq c\|a^*\|_{B^p_{\eta_k}} \leq c \left( \sum_{k=n+1}^{\infty} \|a_{\eta_k}, \theta_{\eta_k}, \|_{L^p} \right)^{1/p}.
\]

Further, we apply the inequality of Lemma 7.6 with \(x_k = \|a_{\eta_k}, \theta_{\eta_k}, \|_{L^p}\), (7.11) and Theorem 6.9(c) to obtain

\[
\sigma_n(f)_{B^p_{\tau}(S^{d-1})} \leq cn^{1/p-1/\tau} \left( \sum_{k=1}^{\infty} \|a_{\eta_k}, \theta_{\eta_k}, \|_{L^p} \right)^{1/\tau}
\]

\[
\leq cn^{-s/(d-1)} \|a\|_{B^\tau_{S^{d-1}}} \leq cn^{-s/(d-1)} \|f\|_{B_\tau^\tau(S^{d-1})},
\]

which confirms (7.13).

In the case \(q = \tau\), we use the embedding \(B_{S^{d-1}}^\tau(S^{d-1}) \subset B_\tau^0(S^{d-1})\) (see (3.34)) to obtain \(\|f-g\|_{B_\tau^\tau} = o(1)\|f\|_{B_\tau^\tau}\) with \(g = \sum_{k=1}^{\infty} a_{\eta_k} \theta_{\eta_k}\). Combining this estimate with estimate (7.13) with \(q = p\) and applying Hölder’s inequality we obtain in the case \(\tau \leq q < p\)

\[
\|f-g\|_{B_\tau^\tau} \leq \|f-g\|_{B_\tau^p}^{(p-q)/((p-\tau)q)} \|f-g\|_{B_\tau^p}^{(q-\tau)p/((p-\tau)q)}
\]

\[
= o(1)\|f\|_{B_\tau^\tau}^{(p-q)/((p-\tau)q)} \|f\|_{B_\tau^p}^{(q-\tau)p/((p-\tau)q)} = o(n^{1/q-1/\tau}) \|f\|_{B_\tau^\tau}.
\]

This proves (7.19) and completes the proof of the theorem. \(\square\)
Remark 7.7. In comparing Theorem 7.2 and Theorem 7.5 we see that the optimal rate $n^{s/(d-1)}$ holds for approximation in $F^0_p(S^{d-1})$ for every $q > 0$ but only for $q = p$ for approximation in $B^0_p(S^{d-1})$. Theorem 7.5 cannot be extended for $q < \tau$ because $B^0_p(S^{d-1}) \setminus B^0_q(S^{d-1}) \neq \emptyset$ if $q < \tau$.

Remark 7.8. Note that in both Theorem 7.2 and Theorem 7.5 we form the next use Theorems 7.2 and 7.5 to establish respective Jackson estimates for non-linear $n$-term approximation of harmonic functions on $B^d$. We next use Theorems 7.2 and 7.5 to establish respective Jackson estimates for non-linear $n$-term approximation of harmonic functions on $B^d$ from shifts of Newtonian kernel.

Theorem 7.9. Let $s > 0$, $0 < p, q < \infty$, and $1/\tau = s/(d-1) + 1/p$. If the harmonic function $U \in B^0_p(S^d) = F^\tau_p(S^d)$, then $U \in F^q_p(S^d)$ and

$$E_n(U)_{F^q_p(S^d)} \leq cn^{-s/(d-1)}\|U\|_{B^0_p(S^d)}, \quad n \geq 1,$$

where the constant $c > 0$ depends only on $s, p, q, d$.

Proof. By Theorem 6.7 it follows that the boundary value function $f_U$ of $U$ given in Proposition 5.3 belongs to $B^0_p(S^d) = F^\tau_p(S^d)$ and $\|f_U\|_{B^0_p(S^d)} \sim \|U\|_{B^0_p(S^d)}$. Then embedding (5.3) of Proposition 5.1 implies that $f_U \in \mathcal{F}^q_p(S^d)$ and in turn Theorem 3.4 yields $U \in F^q_p(S^d)$.

With $s, p, q, \tau$ already fixed, we choose $A := \max\{s, p, q^{-1}, \tau^{-1}, 2\}$. Then conditions (7.1) are satisfied. Pick the parameters $K, M,$ and $\gamma_0$ as in (7.7)–(7.8). Then the frames $\{\theta_\xi\}_{\xi \in \mathcal{X}}, \{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}$ are well defined. Appealing to Theorem 7.2 we conclude that for any $n \geq 1$ there exist $\xi_1, \ldots, \xi_n \in \mathcal{X}$ and coefficients $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$\left\|f_U - \sum_{j=1}^n a_j\theta_{\xi_j}\right\|_{\mathcal{F}^q_p(S^d)} \leq cn^{-s/(d-1)}\|f_U\|_{B^0_p(S^d)} \leq cn^{-s/(d-1)}\|U\|_{B^0_p(S^d)}.$$

Write $G_n(x) := \sum_{j=1}^n a_j\theta_{\xi_j}(x), \ x \in B^d$. From above by harmonic extension using Theorem 3.4 we obtain

$$\|U - G_n\|_{\mathcal{F}^q_p(S^d)} \leq cn^{-s/(d-1)}\|U\|_{B^0_p(S^d)}.$$

However, by Theorem 6.7 (b) we know that for every $\xi \in \mathcal{X}\setminus\mathcal{X}_0$ the frame element $\theta_\xi$ is a linear combination of $\leq \hat{n}$ shifts of the Newtonian kernel, where $\hat{n}$ is a constant. Therefore, $G_n \in \mathcal{N}_\hat{n}$ and then estimate (7.20) follows readily by (7.21). \qed

Theorem 7.10. Let $s > 0$, $0 < p, q < \infty$, $1/\tau = s/(d-1) + 1/p$, and $q \geq \tau$. If the harmonic function $U \in B^0_p(S^d) = F^\tau_p(S^d)$, then $U \in B^0_q(S^d)$ and for every $n \geq 1$ we have

$$E_n(U)_{B^0_q(S^d)} \leq cn^{-s/(d-1)}\|U\|_{B^0_p(S^d)}, \quad p \leq q,$$

$$E_n(U)_{B^0_q(S^d)} = o(n^{1/q-1/\tau})\|U\|_{B^0_p(S^d)}, \quad \tau \leq q < p,$$

where the constant $c > 0$ depends only on $A, d$.

The proof of Theorem 7.10 goes along the lines of the proof of Theorem 7.2 with Theorem 7.2 replaced by Theorem 7.5. We omit it.
8. Approximation of harmonic functions on $\mathbb{R}^d \setminus \bar{B}^d$ and $\mathbb{R}^d_+$

The results in Section 7 have their analogues for approximation of harmonic functions on $\mathbb{R}^d \setminus \bar{B}^d$ or $\mathbb{R}^d_+$. In the following we established the analogue of the main result (Theorem 7.1) on $\mathbb{R}^d \setminus \bar{B}^d$ and explain briefly its analogue on $\mathbb{R}^d_+$.

In analogy to the set $\mathcal{N}_n$ from (7.1)–(7.2) we denote by $\overline{\mathcal{N}}_n$ the set of all linear combinations of shifts of the Newtonian kernel as in (7.1)–(7.2) with the requirement that the poles $y_{\nu} \in B^d$.

The approximation will take place in the harmonic Hardy space $\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)$. Let $\mathcal{H}(\mathbb{R}^d \setminus \bar{B}^d)$ denote the set of all harmonic functions $U$ on $\mathbb{R}^d \setminus \bar{B}^d$ such that $\lim_{|x| \to +\infty} U(x) = 0$ if $d > 2$ or $\lim_{|x| \to \infty} U(x) = \text{const}$ if $d = 2$. The harmonic Hardy space $\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)$, $0 < p < \infty$, is defined as the set of all harmonic functions $U \in \mathcal{H}(\mathbb{R}^d \setminus \bar{B}^d)$ such that

$$\|U\|_{\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)} := \|\sup_{r>1} r^{d-2}|U(r)|\|_{L^p(\mathbb{R}^{d-1})} < \infty.$$  \hfill (8.1)

Given $U \in \mathcal{H}(\mathbb{R}^d \setminus \bar{B}^d)$ we define

$$E_n(U)_{\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)} := \inf_{G \in \overline{\mathcal{N}}_n} \|U - G\|_{\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)}.$$  \hfill (8.2)

Denote by $\mathring{B}_p^s(\mathcal{H})$ the harmonic Besov spaces on $\mathbb{R}^d \setminus \bar{B}^d$ (see [15] Section 8).

As one can expect the following Jackson type theorem is valid:

**Theorem 8.1.** Let $s > 0$, $0 < p < \infty$, and $1/\tau = s/(d-1) + 1/p$. If $U \in \mathring{B}_p^\tau(\mathcal{H})$, then $U \in \mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)$ and

$$E_n(U)_{\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)} \leq c n^{-s/(d-1)} \|U\|_{\mathring{B}_p^s(\mathcal{H})}, \quad n \geq 1,$$

where the constant $c > 0$ depends only on $p, s, d$.

**Proof.** As is well known the Kelvin transform $KU(x) := |x|^{2-d}U(x/|x|^2)$ maps one-to-one $\mathcal{H}(B^d)$ onto $\mathcal{H}(\mathbb{R}^d \setminus \bar{B}^d)$ and $K^{-1} = K$. It is easy to see that the Kelvin transform is an isometric isomorphism of $\mathcal{H}^p(\mathbb{R}^d \setminus \bar{B}^d)$ onto $\mathcal{H}^p(B^d)$. Also, as shown in [15] Section 8 the Kelvin transform is an isometric isomorphism between the harmonic Besov spaces $\mathring{B}_p^s(\mathcal{H})$ on $\mathbb{R}^d \setminus \bar{B}^d$ and the harmonic Besov spaces $B_p^s(\mathcal{H})$ on $B^d$. Furthermore, it is readily seen by the symmetry lemma that for a fixed $y \in \mathbb{R}^d$, $y \neq 0$,

$$K\left(\frac{1}{|x-y|^{d-2}}\right)(x) = \frac{|y|^{2-d}}{|x-y|/|y|^{2(d-2)}}, \quad d > 2,$$

and

$$K\left(\ln \left|\frac{1}{x-y}\right|\right)(x) = \ln \frac{1}{|y|} + \ln |x| + \ln \frac{1}{|x-y|/|y|^2}, \quad d = 2.$$  \hfill (8.3)

Assuming that $U \in \mathring{B}_p^s(\mathcal{H})$ we apply estimate (7.1) to $KU$ and use all of the above to conclude that estimate (8.3) holds true. \hfill $\Box$

**Approximation of harmonic functions on $\mathbb{R}^d_+$.** Closely related to the approximation problem considered above is the problem for nonlinear $n$-term approximation of functions in the harmonic Hardy spaces $\mathcal{H}^p(\mathbb{R}^d_+)$, $0 < p < \infty$, from linear combinations of shifts of the Newtonian kernel with poles in $\mathbb{R}^d$. This problem should be regarded as a limiting case of the same problem on $B(0, R) \subset \mathbb{R}^d$ as
$R \to \infty$. For lack of space we do not elaborate on this sort of approximation. We would like to observe only that all definitions and statements in this article have analogues in the more common setting on $\mathbb{R}^d$ from Harmonic analysis point of view, in particular, our main Jackson estimate (7.4) is valid.

9. Proofs

9.1. Proofs of Propositions 2.3, 2.4, and 2.5. For the proofs of Proposition 2.3 we need the following simple

Lemma 9.1. Let $K \in \mathbb{N}$, $x_0 \in \mathbb{S}^{d-1}$, $g \in W^K(\mathbb{S}^{d-1})$ and $\tilde{g}(x) := g(x/|x|)$ for $x \in \mathbb{R}^d \setminus \{0\}$. Then for every $x \in \mathbb{S}^{d-1}$ with $\rho(x, x_0) \leq 1$ we have

$$
\left| \tilde{g}(x) - \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_0)}{\beta!} (x-x_0)^\beta \right| \leq c \rho(x, x_0)^K \sup_{\rho(z, x_0) \leq \rho(x, x_0)} \max_{|\beta| = K} |\partial^\beta \tilde{g}(z)|
$$

with $c$ depending only on $d$ and $K$.

Proof. Assuming $x \neq x_0$, we set $\eta := (x-x_0)/|x-x_0| \in \mathbb{S}^{d-1}$. Then from Taylor's theorem there exists $\lambda \in (0, 1)$ such that

$$
\left| \tilde{g}(x) - \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_0)}{\beta!} (x-x_0)^\beta \right| = \frac{|x-x_0|^K}{K!} |(\eta \cdot \nabla)^K \tilde{g}(x_\lambda)| = \frac{|x-x_0|^K}{|x_\lambda|^K K!} |(\eta \cdot \nabla)^K \tilde{g}(\frac{x_\lambda}{|x_\lambda|})|,
$$

where $x_\lambda := x_0 + \lambda(x-x_0)$ and the definition of $\tilde{g}$ is used for the last equality. Now, we use that $|x-x_0| \leq \rho(x, x_0)$, $|x_\lambda| \geq \cos 1/2$ for $\lambda \in (0, 1)$, and

$$
|(|\eta \cdot \nabla)^K \tilde{g}(y)| \leq c \max_{|\beta| = K} |\partial^\beta \tilde{g}(y)|, \quad y \in \mathbb{S}^{d-1},
$$

to complete the proof. \hfill \Box

Proof of Proposition 2.3. We represent $(g, f)$ in the form

$$(g, f) = S_1 + S_2,$$

(9.1)

$$
S_1 := \int_{\mathbb{S}^{d-1}} \left( \tilde{g}(y) - \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_2)}{\beta!} (y-x_2)^\beta \right) f(y) d\sigma(y),
$$

$$
S_2 := \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_2)}{\beta!} \int_{\mathbb{S}^{d-1}} (y-x_2)^\beta f(y) d\sigma(y).
$$

From (2.32) we get

$$
\left| \int_{\mathbb{S}^{d-1}} (y-x_2)^\beta f(y) d\sigma(y) \right| \leq c \kappa_2 N_2^{-K}, \quad 0 \leq |\beta| \leq K-1
$$

and using (2.31)

$$
|S_2| \leq \sum_{|\beta| \leq K-1} \frac{\kappa_1 \lambda_1^{|eta|+d-1}}{\beta!(1+N_1 \rho(x_1, x_2))^M \kappa_2 N_2^{-K}} \leq \frac{\kappa_1 \kappa_2^d (N_1/N_2)^K N_2^{d-1}}{(1+N_1 \rho(x_1, x_2))^M},
$$
We bound $S_1$ by

$$|S_1| \leq \int_{\mathbb{S}^{d-1}} \left| \tilde{g}(y) - \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_2)}{\beta!} (y - x_2)^\beta \right| d\sigma(y) =: \int_{A_1} + \int_{A_2} + \int_{A_3},$$

where

$$A_1 = \{ y \in \mathbb{S}^{d-1} : \rho(x_2, y) \leq N_1^{-1} \},$$

$$A_2 = \{ y \in \mathbb{S}^{d-1} : \rho(x_2, y) > N_1^{-1}, \rho(x_1, y) \leq \rho(x_1, x_2)/2 \},$$

$$A_3 = \{ y \in \mathbb{S}^{d-1} : \rho(x_2, y) > N_1^{-1}, \rho(x_1, y) > \rho(x_1, x_2)/2 \}.$$

For $y \in A_1$, Lemma 2.11 and (2.30) imply

$$\left| \tilde{g}(y) - \sum_{|\beta| \leq K-1} \frac{\partial^\beta \tilde{g}(x_2)}{\beta!} (y - x_2)^\beta \right| \leq c \sup_{z \in A_1} \left( \partial^\beta \tilde{g}(z) \right) \leq c \left( 1 + N_1 \rho(x_1, x_2) \right)^M \rho(y, x_2)^K \leq c \frac{\rho(y, x_2)^K}{(1 + N_1 \rho(x_1, x_2))^M}$$

due to (2.7). Using the above estimate, (2.31), and (2.6) we see that

$$\int_{A_1} \rho(y, x_2)^K \frac{\rho(y, x_2)^d}{(1 + N_1 \rho(x_1, x_2))^M} d\sigma(y) \leq c \frac{\kappa_1 N_1^{K+d-1}}{(1 + N_1 \rho(x_1, x_2))^M} \int_{\mathbb{S}^{d-1}} \rho(y, x_2)^K (1 + N_2 \rho(y, x_2))^{d-M} d\sigma(y) \leq c \frac{\kappa_1 \kappa_2 (N_1/N_2)^K N_1^{d-1}}{(1 + N_1 \rho(x_1, x_2))^M}.$$

For $y \in A_2$ we have $\rho(x_1, x_2)/2 \leq \rho(x_2, y) \leq 3 \rho(x_1, x_2)/2$, and hence

$$(1 + N_2 \rho(y, x_2))^{-M} \leq (N_2/\rho(y, x_2))^{-M} \leq 3^M (N_1/N_2)^M (1 + N_1 \rho(x_1, x_2))^{-M}.$$

This combined with (2.30) and (2.31) implies

$$\int_{A_2} \rho(y, x_2)^K \frac{\rho(y, x_2)^d}{(1 + N_1 \rho(x_1, x_2))^M} d\sigma(y) \leq c \frac{\kappa_1 N_1^{K+d-1}}{(1 + N_1 \rho(x_1, x_2))^M} \int_{\mathbb{S}^{d-1}} \rho(y, x_2)^K (1 + N_2 \rho(y, x_2))^{d-M} d\sigma(y) \leq c \frac{\kappa_1 \kappa_2 (N_1/N_2)^K N_1^{d-1}}{(1 + N_1 \rho(x_1, x_2))^M}.$$
For \( y \in A_3 \), we have \((1 + N_1 \rho(x_1, x_2))/2 \leq 1 + N_1 \rho(x_1, y)\) and \(\rho(x_2, y) > N_1^{-1}\). Therefore,
\[
\int_{A_3} \frac{\kappa_1 N_{N_1}^d}{(1 + N_1 \rho(x_1, y))^M} \frac{\kappa_2 N_{N_2}^d}{(1 + N_2 \rho(y, x_2))^M} \ d\sigma(y) \\
+ \int_{A_3} \sum_{|\beta| \leq K - 1} \frac{\kappa_1 N_{|\beta| + d - 1} \rho(y, x_2)^{|\beta|}}{(1 + N_1 \rho(x_1, x_2))^M} \frac{\kappa_2 N_{N_2}^d}{(1 + N_2 \rho(y, x_2))^M} \ d\sigma(y)
\]
\[
\leq c \frac{\kappa_1 \kappa_2 N_{N_1}^d}{(1 + N_1 \rho(x_1, x_2))^M} \int_{A_3} \frac{(N_1 \rho(y, x_2))^K N_{N_2}^d}{(1 + N_2 \rho(y, x_2))^M} \ d\sigma(y)
\]
\[
\leq c \frac{\kappa_1 \kappa_2 (N_1/N_2)^K N_{N_2}^d}{(1 + N_1 \rho(x_1, x_2))^M},
\]
using that \(M > K\). Instead of estimating \(S_2\) in (9.1), we move it to the left-hand side. Only the localization of the first derivatives of \(g\), but not of \(g\) itself, is needed here.

**Proof of Proposition 2.4.** This proof is the same as the proof of Proposition 2.3 for \(K = 1\). Instead of estimating \(S_2\) in (9.1), we move it to the left-hand side. Only the localization of the first derivatives of \(g\), but not of \(g\) itself, is needed here.

**Proof of Proposition 2.5.** This proof follows along the lines of the proof of Proposition 2.3 with \(K = 0\). Of course, in this case the Taylor series is missing from the definitions of both \(S_1\) and \(S_2\) in (9.1), i.e. \(S_2 \equiv 0\). Lemma 9.1 is also not used in the proof.

**9.2. Proof of Theorem 4.6.** This proof depends on the next three lemmas.

**Lemma 9.2.** Let \(j, m \geq 0, 0 < \beta \leq 1, x \in S^{d-1} \) and \(\xi \in \mathcal{X}_j\). Then
\[
\sum_{\eta \in \mathcal{X}_j+m} \frac{1}{(1 + N_{\xi} \rho(x, \eta))^{d-1+\beta}} \leq c_1^* 2^{m(d-1)}
\]
with \(c_1^* = c(d)\beta^{-1}\).

**Proof.** Using that \(\mathcal{X}_{j+m}\) is a maximal \(\gamma 2^{-j-m+1}\) net with a fixed \(\gamma = c(d) \in (0, 1)\) (as stated in (2.3), (2.20), the inequality \((1 + \gamma)(1 + N_{\xi} \rho(x, \eta)) \geq 1 + N_{\xi} \rho(x, y)\) for any \(y \in A_3\), and (2.6) we obtain
\[
\sum_{\eta \in \mathcal{X}_j+m} \frac{1}{(1 + N_{\xi} \rho(x, \eta))^{d-1+\beta}} \leq c(d) N_{N_1}^d \sum_{\eta \in \mathcal{X}_j+m} \frac{|A_\eta|}{(1 + N_{\xi} \rho(x, \eta))^{d-1+\beta}}
\]
\[
\leq c(d) N_{N_1}^d \int_{S^{d-1}} \frac{d\sigma(y)}{(1 + N_{\xi} \rho(x, y))^{d-1+\beta}} \leq c(d) N_{N_1}^d c(d) \beta^{-1} N_{\xi}^{-d+1},
\]
which proves (9.2).

**Lemma 9.3.** Let \(0 < t \leq 1\) and \(M \geq (d-1)/t + \delta, 0 < \delta \leq 1\). Then for any sequence of complex numbers \(\{h_\eta\}_{\eta \in \mathcal{X}_m}, m \geq 0,\) and for any \(x \in B_{\xi} = B(\xi, \gamma 2^{-j})\),
ξ ∈ X_j, j ≥ 0, we have

\[(9.3) \sum_{\eta \in X_m} \frac{|h_\eta|}{(1 + \min\{N_\xi, N_\eta\} \rho(\xi, \eta))^M} \leq c^*_2 \max \{1, 2^{(m-j)(d-1)/t}\} M \left( \sum_{\eta \in X_m} |h_\eta| \mathbb{1}_{B_\eta}(x) \right),\]

where \(c^*_2 := (2/\ln 2)4^{(d-1)/t}(2/\gamma)^M \delta^{-1}\) with \(\gamma \in (0, 1)\) being the constant from the construction of the old frame \(\Psi\) in \|(3.3\).

**Proof.** Two cases present themselves here.

**Case 1:** \(m ≥ j\). Set \(Q_0 := \{\eta ∈ X_m : 2^{-1} \rho(\xi, \eta) < \gamma\}\) and

\[Q_\nu := \{\eta ∈ X_m : \gamma 2^{\nu - 1} ≤ 2^{j-1} \rho(\xi, \eta) < \gamma 2^\nu\}, \quad \nu ≥ 1.\]

Since \(0 < t \leq 1\) we have for \(\nu ≥ 1\)

\[\sum_{\eta \in Q_\nu} \frac{|h_\eta|}{(1 + 2^{j-1} \rho(\xi, \eta))^M} ≤ \left(\frac{2}{\gamma}\right)^M 2^{-\nu M} \sum_{\eta \in Q_\nu} |h_\eta| ≤ \left(\frac{2}{\gamma}\right)^M 2^{-\nu M} \left( \sum_{\eta \in Q_\nu} |h_\eta| \right)^{1/t}.\]

The same estimate holds trivially for \(\nu = 0\). Put

\[R_\nu := B(\xi, 2\gamma(2^{-m} + 2^{-j+\nu})), \quad \nu ≥ 0.\]

Clearly \(\bigcup_{\eta \in Q_\nu} B_\eta \subset R_\nu\). Using this, the fact that the sets \(\{B_\eta : \eta ∈ X_m\}\) are disjoint, and \(\|(2.26)\) we obtain for every \(x ∈ B_\xi \subset R_\nu\)

\[\sum_{\eta \in Q_\nu} |h_\eta|^t = \int_{B_\xi} \left( \sum_{\eta \in Q_\nu} |h_\eta| \mathbb{1}_{B_\eta}(y) \right)^t d\sigma(y) = \frac{|R_\nu|}{|B(\xi, \gamma 2^{-m})|^{1/t}} \int_{R_\nu} \left( \sum_{\eta \in Q_\nu} |h_\eta| \mathbb{1}_{B_\eta}(y) \right)^t d\sigma(y) \leq 4^{d-1} 2^{(m-j+\nu)(d-1)} \left[ M_t \left( \sum_{\eta \in X_m} |h_\eta| \mathbb{1}_{B_\eta}(x) \right) \right]^t.\]

Therefore, since \(M ≥ (d-1)/t + \delta\) we get for any \(x ∈ B_\xi\)

\[\sum_{\eta \in X_m} \frac{|h_\eta|}{(1 + 2^{j-1} \rho(\xi, \eta))^M} \leq \left(\frac{2}{\gamma}\right)^M 4^{(d-1)/t} M_t \left( \sum_{\eta \in X_m} |h_\eta| \mathbb{1}_{B_\eta}(x) \right) \sum_{\nu ≥ 0} 2^{-\nu M} 2^{(\nu-j+m)(d-1)/t} \leq \left(\frac{2}{\gamma}\right)^M 4^{(d-1)/t} 2^{(m-j)(d-1)/t} M_t \left( \sum_{\eta \in X_m} |h_\eta| \mathbb{1}_{B_\eta}(x) \right) \sum_{\nu ≥ 0} 2^{-\nu \delta} \leq \frac{2}{\delta \ln 2} \left(\frac{2}{\gamma}\right)^M 4^{(d-1)/t} 2^{(m-j)(d-1)/t} M_t \left( \sum_{\eta \in X_m} |h_\eta| \mathbb{1}_{B_\eta}(x) \right),\]

which confirms \(\|(3.3)\).

**Case 2:** \(m < j\). Set \(\tilde{Q}_0 := \{\eta ∈ X_m : 2^{m-1} \rho(\xi, \eta) < \gamma\}\) and

\[\tilde{Q}_\nu := \{\eta ∈ X_m : \gamma 2^{\nu - 1} ≤ 2^{m-1} \rho(\xi, \eta) < \gamma 2^\nu\}, \quad \nu ≥ 1.\]

Write

\[\tilde{R}_\nu := B(\xi, \gamma 2^{-m+1}(1 + 2^\nu)), \quad \nu ≥ 0.\]
We use that $0 < t \leq 1$ to obtain
\[
\sum_{\eta \in Q_v} \left| h_\eta \right|^t \left( \frac{1}{1 + 2^{m-1} \rho(\xi, \eta)} \right)^M \leq \left( \frac{2}{\gamma} \right)^M 2^{-\nu M} \sum_{\eta \in Q_v} \left| h_\eta \right| \leq \left( \frac{2}{\gamma} \right)^M 2^{-\nu M} \left( \sum_{\eta \in Q_v} \left| h_\eta \right|^t \right)^{1/t}.
\]

Just as in Case 1 we obtain for $x \in B_\xi \subset \mathcal{R}_\nu$
\[
\sum_{\eta \in Q_v} \left| h_\eta \right|^t = \int_{\mathbb{R}^{d-1}} \left( \sum_{\eta \in Q_v} \left| h_\eta \right| |B_\eta|^{-1/2} \mathbb{1}_{B_\eta}(y) \right)^t \, d\sigma(y)
\]
\[
\leq 4^{d-1} 2^{(d-1)/t} \left[ M_t \left( \sum_{\eta \in X_m} \left| h_\eta \right| \mathbb{1}_{B_\eta}(x) \right)^t \right].
\]

As before, since $M \geq (d-1)/t + \delta$ we get for any $x \in B_\xi$
\[
\sum_{\eta \in X_m} \left| h_\eta \right| \left( \frac{1}{1 + 2^{m-1} \rho(\xi, \eta)} \right)^M
\leq \left( \frac{2}{\gamma} \right)^M 4^{(d-1)/t} M_t \left( \sum_{\eta \in X_m} \left| h_\eta \right| \mathbb{1}_{B_\eta}(x) \right) \sum_{\nu \geq 0} 2^{-\nu (M-(d-1)/t)}
\leq \left( \frac{2}{\delta \ln 2} \right)^M 4^{(d-1)/t} M_t \left( \sum_{\eta \in X_m} \left| h_\eta \right| \mathbb{1}_{B_\eta}(x) \right),
\]
which verifies (9.3). The proof of the lemma is complete. \qed

In the next lemma we specify the constants in certain well known discrete Hardy inequalities that will be needed.

**Lemma 9.4.** Let $\beta > 0$, $0 < q < \infty$, and $a_m \geq 0$ for $m \geq 0$. Then

\[
(9.4) \quad \left( \sum_{j \geq 0} \left( \sum_{m \geq j} 2^{-(m-j)\beta} a_m \right)^q \right)^{1/q} \leq c^*_3 \left( \sum_{m \geq 0} a_m^q \right)^{1/q}
\]

and

\[
(9.5) \quad \left( \sum_{j \geq 0} \left( \sum_{m=0}^j 2^{-(j-m)\beta} a_m \right)^q \right)^{1/q} \leq c^*_3 \left( \sum_{m \geq 0} a_m^q \right)^{1/q}
\]

with
\[
c^*_3 = 2^\beta \max \left\{ \frac{1}{\beta \ln 2}, \frac{1}{(\beta q \ln 2)^{1/q}} \right\}.
\]

**Proof.** In the case $0 < q \leq 1$ inequalities (9.4)–(9.5) follow readily by applying the $q$-inequality and switching the order of summation. More precisely, we have
\[
\left( \sum_{j \geq 0} \left( \sum_{m \geq j} 2^{-(m-j)\beta} a_m \right)^q \right)^{1/q} \leq \left( \sum_{j \geq 0} \sum_{m \geq j} 2^{-(m-j)\beta q} a_m^q \right)^{1/q}
\leq \left( \sum_{\nu \geq 0} \sum_{m \geq 0} 2^{-\nu \beta q} a_m^q \right)^{1/q} \left( \sum_{m \geq 0} a_m^q \right)^{1/q}.
\]
which gives (9.4) with $c \in X_{\xi,\eta}$.

Proof of Theorem 4.6. We shall use the abbreviated notation $\omega_{\xi,\eta} := \omega_{\xi,\eta}^{(K,M)}$ for $\xi, \eta \in X$ (see (4.21)).

We first establish the result for the sequence Besov spaces $b^p_q$, that is,

$$\|\Omega h\|_{b^p_q} \leq C_5 \|h\|_{b^p_q}.$$

Set $p_* := \max\{p, 1\}$. We start with the proof of the estimate

$$\left( \sum_{\xi \in X_j} \left| \sum_{\eta \in X} \omega_{\xi,\eta} h_\eta \right|^p \right)^{1/p} \leq C_{11} \sum_{m=0}^{\infty} 2^{-m(K+(d-1)(1/2-1/p_*)-\delta)/2} \left( \sum_{\eta \in X_{j+m}} |h_\eta|^p \right)^{1/p}$$

(9.7)

for any $j \geq 0$. For $1 < p < \infty$ using (4.21) and the convexity of $u^p$ we obtain

$$\sum_{\xi \in X_j} \left| \sum_{\eta \in X} \omega_{\xi,\eta} h_\eta \right|^p \leq \sum_{\xi \in X_j} \left( \sum_{m=0}^{\infty} \frac{2^{-m(K+(d-1)/2)|h_\eta|^p}}{(1 + N_\xi \rho(\xi,\eta))^{M}} \right)^p$$

(9.8)

Applying in the first double sum in the right-hand side of (9.8) twice Hölder’s inequality, first in the summation on $m$ and then on $\eta$, and Lemma 9.2 with $\beta = \delta$
we get with $M_1 = (d - 1 + \delta)/p'$ and $M_2 = M - M_1$

\[
\begin{align*}
\text{(9.9)} & \quad \left( \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^M} \right)^p \\
& \leq \left( \sum_{m=0}^{\infty} \frac{2^{-mp'\delta/2}}{(1 + N_{\xi}(\xi, \eta))^M} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^M} \right)^p \\
& \leq C_{12}^p \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{1}{(1 + N_{\xi}(\xi, \eta))^M} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^M} \\
& \leq c_1^{p-1} C_{12}^p \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^M},
\end{align*}
\]

where $C_{12} := (1 - 2^{-p'\delta/2})^{-1/p'} \leq 5^{\delta-1/p'}$. Applying the same arguments to the second double sum in the right-hand side of (9.8) we obtain

\[
\text{(9.10)} & \quad \left( \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2)|h_{\eta}|}}{(1 + N_{\eta}(\xi, \eta))^M} \right)^p \\
& \leq c_1^{p-1} C_{12}^p \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)/2-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^M}.
\]

Note that (4.22) implies $M_{2p} \geq d - 1 + \delta$. Substituting (9.9) and (9.10) in (9.8) and using Lemma 9.2 with $\beta = \delta$ we get

\[
\text{(9.11)} & \quad \sum_{\xi \in \mathcal{X}_j} \left| \sum_{\eta \in \mathcal{X}_j} \omega_{\xi, \eta} h_{\eta} \right|^p \\
& \leq 2^{p-1} C_{12}^p \left( \sum_{m=0}^{\infty} \sum_{\eta \in \mathcal{X}_{j+m}} \frac{2^{-m(K+(d-1)(1/2-1/p')-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^{d-1+\delta}} \right)^{1/p} \\
& \quad + \sum_{m=1}^{j} \sum_{\eta \in \mathcal{X}_{j-m}} \frac{2^{-m(K+(d-1)/2-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^{d-1+\delta}} \\
& \leq 2^{p-1} C_{12}^p \left( \sum_{m=0}^{\infty} c_1^2 \frac{2^{-m(K+(d-1)(1/2-1/p')-\delta/2)|h_{\eta}|}}{(1 + N_{\xi}(\xi, \eta))^{d-1+\delta}} \right)^{1/p} \\
& \quad + \sum_{m=1}^{j} \frac{c_1^2}{2} \sum_{\eta \in \mathcal{X}_{j-m}} \frac{|h_{\eta}|}{(1 + N_{\xi}(\xi, \eta))^{d-1+\delta}}.
\]

Now, we raise both sides of (9.11) to the power $1/p < 1$ and apply the $1/p$-inequality to its right-hand side to obtain (9.7) for $1 < p < \infty$ with $C_{11} \geq C_{13} := 2^{1-1/p} c_1^2 C_{12}$. 
Let $0 < p \leq 1$. Using the $p$-inequality, observing that \([1.22]\) implies in this case $M p \geq d - 1 + \delta p$, and using Lemma 9.2 with $\beta = \delta p \leq 1$ we obtain

\[
(9.12) \quad \sum_{\substack{\xi \in \mathcal{X}_j \\eta \in \mathcal{X}}} |\sum_{\eta \in \mathcal{X}} \omega_{\xi,\eta} h_{\eta} |^p \leq \sum_{m=0}^{\infty} c_1^2 2^{-m \delta/2} 2^{-m(K+(d-1)/2-\delta/2)p} \sum_{\eta \in \mathcal{X}_{j+m}} |h_{\eta}|^p \\
+ \sum_{m=1}^{j} c_1^2 2^{-m \delta/2} 2^{-m(K+(d-1)(1/2-1/p)-\delta/2)p} \sum_{\eta \in \mathcal{X}_{j-m}} |h_{\eta}|^p.
\]

We now raise both sides of (9.12) to the power $1/p \geq 1$, use the convexity of $u^{1/p}$ to break the right-hand side to two terms and apply Hölder’s inequality with exponents $r = 1/(1-p)$ and $r' = 1/p$ in the summations on $m$ in order to get the $1/p$ power inside the sum and to prove \([3.27]\) for $0 < p \leq 1$ with $C_{11} := \max\{C_{13}, C_{15}\}$, where $C_{15} = 2^{1/p-1} c_1^{1/p} C_{14}$ with $c_1^p$ is for $\beta = \delta p$ and $C_{14} := (1 - 2^{-\delta p/(2(1-p))})^{(1-p)/p}$.

Finally, using \([3.27], [9.7], [11.22]\), and Lemma 9.3 with $\beta = \delta/2$ and $a_m = 2^{m[s+(d-1)(1/2-1/p)]} \left( \sum_{\eta \in \mathcal{X}_m} |h_{\eta}|^p \right)^{1/p}$ we obtain

\[
(9.13) \quad \|\Omega h\|_{b^{p,q}} = \left( \sum_{j=0}^{\infty} \left[ \sum_{m=0}^{\infty} 2^{j+(d-1)(1/2-1/p)} \left( \sum_{\substack{\xi \in \mathcal{X}_j \\eta \in \mathcal{X}}} |\sum_{\eta \in \mathcal{X}} \omega_{\xi,\eta} h_{\eta} |^p \right)^{1/p} q^{-1} \right]^{1/q} \right)^{1/p}
\]

\[
\leq C_{11} \left( \sum_{j=0}^{\infty} \left[ \sum_{m=0}^{\infty} 2^{j+(d-1)(1/2-1/p)-m(K+(d-1)(1/2-1/p)-\delta/2)} \left( \sum_{\eta \in \mathcal{X}_{j+m}} |h_{\eta}|^p \right)^{1/p} q^{-1} \right]^{1/q} \right)^{1/p}
\]

\[
+ \sum_{m=1}^{j} 2^{j+(d-1)(1/2-1/p)-m(K+(d-1)(1/2-1/p)-\delta/2)} \left( \sum_{\eta \in \mathcal{X}_{j-m}} |h_{\eta}|^p \right)^{1/p} q^{-1} \right]^{1/q}
\]

\[
= C_{11} \left( \sum_{j=0}^{\infty} \left[ \sum_{m=0}^{\infty} 2^{-m(K+s+(d-1)(1/p_1-1/p)-\delta/2)} d_{j+m} + \sum_{m=1}^{j} 2^{-m(K-s-\delta/2)} a_{j-m} \right] q^{-1} \right)^{1/q}
\]

\[
\leq C_{11} \left( \sum_{j=0}^{\infty} \left[ \sum_{m=0}^{\infty} 2^{-(m-j)\delta/2} a_{m} + \sum_{m=0}^{j-1} 2^{-(j-m)\delta/2} a_{m} \right] q^{-1} \right)^{1/q}
\]

\[
\leq 2^{1/q+1} C_{11} c_4^q \left( \sum_{m=0}^{\infty} a_{m}^q \right)^{1/q} = C_{9} \|h\|_{b^{p,q}}.
\]

Thus, \(9.10\) is established with a constant $C_{9} = 2^{1/q+1} C_{11} c_4^q$ of the claimed form.

We next prove the result for the sequence Triebel-Lizorkin spaces $T^{pq}_p$, that is,

\[
(9.14) \quad \|\Omega h\|_{T^{pq}_p} \leq C_{9} \|h\|_{T^{pq}_p}.
\]

Taking into account Remark 3.11 we chose the quasi-norm of $T^{pq}_p$ in Definition 3.10 to be defined with $B_\xi = B(\xi, \gamma 2^{-j})$ for $\xi \in \mathcal{X}_j$, $j \geq 1$. Thus $B_\xi \cap B_\eta = \emptyset$ for $\xi \neq \eta \in \mathcal{X}_j$. 


Let $h \in \ell^p_{\Omega}$. Then $(\Omega h)_{\xi} = \sum_{\eta \in \mathcal{X}} \omega_{\xi, \eta} h_{\eta}$, where the series converges absolutely (see proof below). Then by \eqref{eq:227}

\begin{equation}
(9.15) \quad \|\Omega h\|_{\ell^p_{\Omega}} := \left\| \left( \sum_{\xi \in \mathcal{X}} \left| B_{\xi} \right|^{-s/(d-1)-1/2} \| (\Omega h)_{\xi} \|_{L^p} \right)^q \right\|_{L^q_p}^{1/q} \leq C_{21} \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ N_{\xi}^{s+(d-1)/2} \sum_{\eta \in \mathcal{X} : N_{\eta} \geq N_{\xi}} \omega_{\xi, \eta} |h_{\eta}| \right] \right)^q \right\|_{L^p}^{1/q} \leq C_{21} 2^{1/p+1/q} (\Sigma_1 + \Sigma_2),
\end{equation}

where $C_{21} := \frac{1}{2} \left[ s/(d-1)+1/2 \right] (2/\gamma)^{s+(d-1)/2}$,

\begin{align*}
\Sigma_1 &:= \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ N_{\xi}^{s+(d-1)/2} \sum_{\eta \in \mathcal{X} : N_{\eta} \geq N_{\xi}} \omega_{\xi, \eta} |h_{\eta}| \right] \right)^q \right\|_{L^p}^{1/q}, \\
\Sigma_2 &:= \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ N_{\xi}^{s+(d-1)/2} \sum_{\eta \in \mathcal{X} : N_{\eta} \leq N_{\xi}} \omega_{\xi, \eta} |h_{\eta}| \right] \right)^q \right\|_{L^p}^{1/q}.
\end{align*}

Write $\lambda_{\xi} := N_{\xi}^{s+(d-1)/2} |h_{\xi}|$, $\xi \in \mathcal{X}$, and choose $t$ so that $(d-1)/t = J + \delta/2$. Then $0 < t < \min\{1, p, q\}$. If $N_{\eta} \geq N_{\xi}$, then

\[
\omega_{\xi, \eta} = \left( \frac{N_{\xi}}{N_{\eta}} \right)^{K+(d-1)/2} \left( 1 + N_{\xi} \rho(\xi, \eta) \right)^{-M}.
\]

Then we have

\[
\Sigma_1 \leq \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ \sum_{\eta \in \mathcal{X} : N_{\eta} \geq N_{\xi}} \left( \frac{N_{\xi}}{N_{\eta}} \right)^{J-s-(d-1)/2+\delta} (1 + N_{\xi} \rho(\xi, \eta))^{-J-\delta} |h_{\eta}| \lambda_{\xi} \right]^q \right) \right\|_{L^p}^{1/q}.
\]

Next, we apply Lemma \ref{lem:123} (with $(d-1)/t$ and $\delta/2$ in the place of $M$ and $\delta$) and the fact that the sets $\{B_{\xi} : \xi \in \mathcal{X}_j\}$ are mutually disjoint to obtain

\[
\Sigma_1 \leq c_2^2 \left\| \left( \sum_{j \geq 0} \sum_{\xi \in \mathcal{X}_j} \left[ \sum_{m \geq j} 2^{-(m-j)(J-s-d/4+\delta)} \sum_{\eta \in \mathcal{X}_m} (1 + N_{\xi} \rho(\xi, \eta))^{-J-\delta} |h_{\eta}| \lambda_{\xi} \right]^q \right) \right\|_{L^p}^{1/q}.
\]

The application of inequality \ref{eq:224}, the maximal inequality \ref{eq:224a}, the fact that the sets $\{B_{\eta} : \eta \in \mathcal{X}_j\}$ are mutually disjoint, and \ref{eq:227} leads to

\begin{equation}
(9.16) \quad \Sigma_1 \leq c_2^2 c_3 \left\| \left( \sum_{j \geq 0} \left[ \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} |h_{\eta}| \lambda_{\eta} \right) \right]^q \right) \right\|_{L^p}^{1/q} \leq c_2^2 c_3^2 \left\| \left( \sum_{j \geq 0} \left[ \sum_{\xi \in \mathcal{X}_j} \left[ N_{\xi}^{s+(d-1)/2} |h_{\xi}| \right] \right]^q \right) \right\|_{L^p}^{1/q} \leq c_2^2 c_3^2 C_{22} \left\| \left( \sum_{j \geq 0} \sum_{\xi \in \mathcal{X}_j} \left[ |B_{\xi}|^{-s/(d-1)-1/2} |h_{\xi}| \right] \right)^q \right\|_{L^p}^{1/q} = c \|f\|_{\ell^p_{\Omega}}
\end{equation}

with $C_{22} := c_2^2 [s/(d-1)+1/2] (\gamma/2)^{s+(d-1)/2}$. 

If $N_\eta < N_\xi$, then
\[
\omega_{\xi, \eta} = \left( \frac{N_\eta}{N_\xi} \right)^{K+ (d-1)/2} \left( 1 + N_\eta \rho(\xi, \eta) \right)^{-M}
\]
and hence
\[
\Sigma_2 \leq \left\| \left( \sum_{\xi \in \mathcal{X}} \left[ \sum_{\eta \in \mathcal{X} : N_\eta < N_\xi} \left( \frac{N_\eta}{N_\xi} \right)^{s+(d-1)/2+\delta} (1 + N_\eta \rho(\xi, \eta))^{-\beta} |\eta| |\lambda_\xi| \right)^q \right]^{\frac{1}{q}} \right\|_{L^p}
\]
\[
= \left\| \left( \sum_{j \geq 0} \sum_{\xi \in \mathcal{X}_j} \sum_{m < j} 2^{-(j-m)(s+(d-1)/2+\delta)} \sum_{\eta \in \mathcal{X}_m} (1 + N_\eta \rho(\xi, \eta))^{-\beta} |\eta| |\lambda_\xi| \right)^q \right\|^{\frac{1}{q}}_{L^p}
\]
As above employing Lemma 9.3, using the fact that the sets $\{B_\xi : \xi \in \mathcal{X}_j\}$ are mutually disjoint, applying \(9.5\), the maximal inequality \(2.24\), and \(2.27\) we obtain
\[
\Sigma_2 \leq c_2^* \left\| \left( \sum_{j \geq 0} \left[ \sum_{m < j} 2^{-(j-m)(s+(d-1)/2+\delta)} \sum_{\eta \in \mathcal{X}_m} |\eta| |\lambda_\xi| \right]^q \right]^{\frac{1}{q}} \right\|_{L^p}
\]
\[
\leq c_2^* c_3 \left\| \left( \sum_{j \geq 0} \left[ \sum_{\eta \in \mathcal{X}_j} |\eta| |\lambda_\xi| \right]^q \right]^{\frac{1}{q}} \right\|_{L^p}
\]
\[
\leq c_2^* c_3 c_4 \left\| \left( \sum_{j \geq 0} \left[ \sum_{\eta \in \mathcal{X}_j} N_\eta^{s+(d-1)/2} |\eta| |\lambda_\xi| \right]^q \right]^{\frac{1}{q}} \right\|_{L^p}
\]
\[
\leq c_2^* c_3 c_4 C_{22} \left\| \left( \sum_{j \geq 0} \sum_{\eta \in \mathcal{X}_j} \left[ |B_\eta|^{-(s/(d-1)-1)/2} |\eta| |\lambda_\xi| \right]^q \right]^{\frac{1}{q}} \right\|_{L^p} = c \| f \|_{f^q}
\]
where the constants $c_2^*, c_3, c_4, C_{22}$ are as above.

Finally, using estimates \(9.16\) and \(9.17\) in \(9.15\) we obtain \(9.14\) with $C_0 = C_{21} 2^{1/p+1/4} C_3^2 C_4 C_{22}$, which is of the claimed form. This completes the proof of Theorem 4.6.

\[\square\]

### References

1. G. E. Andrews, R. Askey, R. Roy, Special functions, Cambridge University Press, 1999.
2. D. H. Armitage, S. J. Gardiner, Classical potential theory, Springer, New York, 2001.
3. S. Axler, P. Bourdon, W. Ramey, Harmonic function theory, Springer, 2001.
4. J. Bergh, J. L"ofstr"om, Interpolation spaces. An introduction, Springer, Berlin – Heidelberg – New York, 1976.
5. H. Cheng, L. Greengard, V. Rokhlin, A fast adaptive algorithm in three dimensions, J. Comput. Phys. 155 (1999), no. 2, 468–498.
6. T. Coulhon, G. Kerkyacharian, P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces, J. Fourier Anal. Appl. 18 (2012), 995–1066.
7. Feng Dai, Characterizations of function spaces on the sphere using frames, Trans. Amer. Math. Soc. 359 (2007), no. 2, 567–589.
8. F. Dai, Y. Xu, Approximation theory and harmonic analysis on spheres and balls, Springer, 2015.
9. S. Dekel, G. Kerkyacharian, G. Kyriazis, and P. Petrushev, Compactly supported frames for spaces of distributions associated with nonnegative self-adjoint operators, Studia Math. 225 (2014), no. 2, 115–163.
10. M. Frazier, B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93 (1990), 34–170.
11. P. Flajolet, R. Sedgewick, Analytic combinatorics, Cambridge University Press, Cambridge, 2009.
[12] L. Grafakos, L. Liu, and D. Yang Vector-valued singular integrals and maximal functions on spaces of homogeneous type, Math. Scand. 104 (2009), 296–310.
[13] L. Greengard, V. Rokhlin, A new version of the fast multipole method for the Laplace equation in three dimensions, Acta numerica, 1997, 229–269, Acta Numer. 6, Cambridge Univ. Press, Cambridge, 1997.
[14] L. Helms, Potential theory, Second edition, Springer, London, 2014.
[15] K. G. Ivanov, P. Petrushev, Harmonic Besov and Triebel-Lizorkin spaces on the ball, J. Fourier Anal. Appl. 23 (2017), no. 5, 1062–1096.
[16] K. G. Ivanov, P. Petrushev, Highly localized kernels on the sphere induced by Newtonian kernels, manuscript.
[17] K. G. Ivanov, P. Petrushev, and Y. Xu, Sub-exponentially localized kernels and frames induced by orthogonal expansions, Math. Z. 264 (2010) 361–397.
[18] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces, Math. Scand. 40 (1977), 94–104.
[19] O. Kellogg, Foundations of potential theory, New York, Dover, 1954.
[20] G. Kyriazis, P. Petrushev, On the construction of frames for spaces of distributions, J. Funct. Anal. 257 (2009), 2159–2187.
[21] G. Kyriazis, P. Petrushev, Rational bases for spaces of holomorphic functions in the disc, J. Lond. Math. Soc. (2) 89 (2014), no. 2, 434–460.
[22] Y. Meyer, Ondelettes et opérateurs I: ondelettes, Hermann, Paris, 1990.
[23] C. Müller, Spherical harmonics, Lecture Notes in Mathematics, Vol. 17, Springer Verlag, Berlin, 1966.
[24] F. J. Narcowich, P. Petrushev, and J. D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal. 38 (2006), 574–594.
[25] F. J. Narcowich, P. Petrushev and J. D. Ward, Decomposition of Besov and Triebel-Lizorkin spaces on the sphere, J. Funct. Anal. 238 (2006), 530–564.
[26] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST Handbook of mathematical functions, Cambridge University Press, Cambridge, 2010.
[27] A. Pekarskii, Inequalities of Bernstein type for derivatives of rational functions, and inverse theorems of rational approximation, Mat. Sb. (N.S.) 124(166) (1984), no. 4, 571–588.
[28] A. Pekarskii, Classes of analytic functions defined by best rational approximations in $H_p$, Mat. Sb. (N.S.) 127(169) (1985), no. 1, 3–20.
[29] R. T. Seeley, Spherical harmonics, Amer. Math. Monthly, 73 (1966), 115–121.
[30] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.
[31] E. Stein, G. Weiss, Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, New Jersey, 1971.
[32] H. Triebel, Theory of function spaces, Monogr. Math. vol. 78, Birkhäuser, Basel, 1983.
[33] G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc. Providence, 1975.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

E-mail address: kamen@math.bas.bg

Department of Mathematics, University of South Carolina, Columbia, SC
E-mail address: pencho@math.sc.edu