Abstract
In this paper, we consider the Lamé operator $-\Delta^*$ and study resolvent estimate, uniform Sobolev estimate, and Carleman estimate for $-\Delta^*$. First, we obtain sharp $L^p-L^q$ resolvent estimates for $-\Delta^*$ for admissible $p, q$. This extends the particular case $q = \frac{p}{p-1}$ due to Barceló et al. [4] and Cossetti [8]. Secondly, we show failure of uniform Sobolev estimate and Carleman estimate for $-\Delta^*$. For this purpose we directly analyze the Fourier multiplier of the resolvent. This allows us to prove not only the upper bound but also the lower bound on the resolvent, so we get the sharp $L^p-L^q$ bounds for the resolvent of $-\Delta^*$. Strikingly, the relevant uniform Sobolev and Carleman estimates turn out to be false for the Lamé operator $-\Delta^*$ even though the uniform resolvent estimates for $-\Delta^*$ are valid for certain range of $p, q$. This contrasts with the classical result regarding the Laplacian $\Delta$ due to Kenig, Ruiz, and Sogge [23] in which the uniform resolvent estimate plays a crucial role in proving the uniform Sobolev and Carleman estimates for $\Delta$. We also describe locations of the $L^q$-eigenvalues of $-\Delta^* + V$ with complex potential $V$ by making use of the sharp $L^p-L^q$ resolvent estimates for $-\Delta^*$.

Keywords Lamé operator · Resolvent estimate
1 Introduction

Let \(-\Delta^*\) be the Lamé operator acting on \(S(\mathbb{R}^d)^d\) which is given by

\[
-\Delta^* u := -\mu \Delta u - (\lambda + \mu) \nabla \text{div} u, \quad u \in S(\mathbb{R}^d)^d.
\]

Here, the Lamé coefficients \(\lambda\) and \(\mu\) are real numbers satisfying

\[
\mu > 0, \quad \lambda + 2\mu > 0,
\]

and \(S(\mathbb{R}^d)^d\) denotes the space of all \(d\)-tuples of complex-valued Schwartz functions on \(\mathbb{R}^d\). When \(d = 3\), the Lamé operator has significant role in describing a linear homogeneous and isotropic elastic medium, and in such case \(u\) denotes the displacement field of the medium. For more about physical and mathematical backgrounds of the operator, see, for example, [14, pp. 1023–1033], [26], [30], and [31].

In this paper, we are concerned with the following \(L^p-L^q\) resolvent estimates for \(-\Delta^*\);

\[
\|(-\Delta^* - z)^{-1} f\|_{L^q(\mathbb{R}^d)^d} \leq C \kappa_{p,q}(z) \|f\|_{L^p(\mathbb{R}^d)^d}, \quad \forall f \in S(\mathbb{R}^d)^d,
\]

for admissible pairs \(p, q\) and spectral parameters \(z\) in the resolvent set \(\rho(-\Delta^*) := \mathbb{C} \setminus \sigma(-\Delta^*) = \mathbb{C} \setminus [0, \infty)\). Here, \(\kappa_{p,q}: \rho(-\Delta^*) \to \mathbb{R}\) is a positive function and \(C\) is a constant independent of \(z \in \rho(-\Delta^*)\). Moreover, we also show the sharpness of the bound \(\kappa_{p,q}(z)\) in (1.1) up to a multiplicative constant.

As is to be seen below, the bound in (1.1) not only superficially resembles the \(L^p-L^q\) resolvent estimate for the Laplacian \(-\Delta\) but also share similar characteristics with that of the Laplacian. So, we begin with a brief discussion of the resolvent estimates for \(-\Delta\) and relevant previous results including the uniform Sobolev and Carleman estimates.

Resolvent estimate for the Laplacian The \(L^p-L^q\) resolvent estimate for the Laplacian \(-\Delta\) is referred to the following form of \textit{a priori} inequality

\[
\|(-\Delta - z)^{-1}\|_{p \to q} \leq C \kappa_{p,q}(z),
\]

where \(\|T\|_{p \to q}\) denotes

\[
\|T\|_{p \to q} := \inf \left\{ K : \|Tf\|_{L^q(\mathbb{R}^d)} \leq K \|f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in S(\mathbb{R}^d) \right\},
\]

\(\kappa_{p,q}\) is a positive function defined on the resolvent set \(\rho(-\Delta) := \mathbb{C} \setminus \sigma(-\Delta) = \mathbb{C} \setminus [0, \infty)\), and \(C = C_{p,q,d}\) is a constant independent of \(z \in \rho(-\Delta)\). The first result on (1.2) goes back to the seminal work of Kenig, Ruiz, and Sogge [23] in which they used the estimate for study of unique continuation. They proved that the estimate (1.2) holds with \(\kappa_{p,q}(z) \equiv 1\) if and only if \(\frac{1}{p} - \frac{1}{q} = \frac{2}{d}\) and \(\frac{2d}{d+3} < p < \frac{2d}{d+1}\),
which is equivalent to the condition that \((\frac{1}{p}, \frac{1}{q}) \in (A, A')\). See Figures 1 and 2, and Definition 1.1 below. Later, the range of \(p, q\) was extended by Gutiérrez [16] to \(\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d} - \frac{1}{p} > \frac{d+1}{2d}\) and \(\frac{1}{q} < \frac{d-1}{2d}\) (the range \(\mathcal{R}_1\)). In this range, the bound takes the form \(\kappa_{p,q}(z) = |z|^{-1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}\), and the bound is independent of the distance between \(z\) and the spectrum \(\sigma(-\Delta)\).

Recently, two of the authors [25] extended the resolvent estimate outside the uniform boundedness range. More precisely, they proved (1.2) for general pairs of \((\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}_1 \cup \tilde{\mathcal{R}}_2 \cup \tilde{\mathcal{R}}_3 \cup \tilde{\mathcal{R}}'_3\) (see Definition 1.1 and Figures 1 and 2 below for precise description) with

\[
\kappa_{p,q}(z) := |z|^{-1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \text{dist}(z/|z|, [0, \infty))^{-\gamma_{p,q}} \tag{1.4}
\]

and

\[
\gamma_{p,q} := \max \left\{ 0, \; 1 - \frac{d + 1}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \; \frac{d + 1}{2} - \frac{d}{p}, \; \frac{d}{q} - \frac{d - 1}{2} \right\}. \tag{1.5}
\]

We also refer to [13, Proposition 2.1] for \(L^p-L^{p'}\) estimates (in a dual form) which can be deduced by interpolation between the trivial \(L^2-L^2\) estimate and the \(L^{\frac{2(d+1)}{d+3}}L^{\frac{2(d+1)}{d-1}}\) estimate due to Kenig, Ruiz, and Sogge [23]. Outside the range \(\mathcal{R}_1\), the bound \(\kappa_{p,q}(z)\) depends not only on \(|z|\) but also on \(\text{dist}(z, [0, \infty))\), and it exhibits singular behavior as the spectral parameter \(z\) approaches to the spectrum \(\sigma(-\Delta) = [0, \infty)\). Moreover, it is also proven in [25] that the estimate (1.2) is sharp in the sense that the inequality is reversed when \(C\) is replaced with some smaller constant. The sharp resolvent estimate (1.2) was used to characterize various profiles of the spectral region which contains \(L^q\)-eigenvalues of the non-self-adjoint operators \(-\Delta + V\) with complex-valued potential \(V\).
Uniform Sobolev inequality and Carleman estimate for $-\Delta$ In [23], making use of the uniform resolvent estimate, the authors established the uniform Sobolev inequality

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \|(-\Delta + a \cdot \nabla + b)u\|_{L^p(\mathbb{R}^d)}$$

with $C$ independent of $(a, b) \in \mathbb{C}^d \times \mathbb{C}$ whenever $(\frac{1}{p}, \frac{1}{q}) \in (A, A')$. This immediately gives the following type of Carleman estimate

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C \|e^{v \cdot x}(-\Delta)e^{-v \cdot x}u\|_{L^p(\mathbb{R}^d)}$$

(1.7)

with $C$ independent of $v \in \mathbb{R}^d$ for the same range of $p, q$. This type of Carleman estimate (1.7) was used to obtain unique continuation property of the differential inequality $|\Delta u| \leq |Vu|$ for $V \in L^\frac{d}{2}(\mathbb{R}^d)$ (see [23, Theorem 3.1 and Corollaries 3.1 and 3.2]). Also, see [6, 19, 20, 33, 34] for related results.

In [20] it was shown that, for $d \geq 3$, and $1 < p, q < \infty$, the Carleman estimate (1.7) holds if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \quad \frac{d^2 - 4}{2d(d-1)} \leq \frac{1}{p} \leq \frac{d + 2}{2(d - 1)}.$$ 

(1.8)

The range of $p, q$ in (1.8), which properly contains that of $(1/p, 1/q) \in (A, A')$, is optimal for the Carleman estimate (1.7) when $d \geq 4$ ([20, Theorem 1.1]). This exhibits different natures of the uniform resolvent estimates and the Carleman estimates. Such difference in the boundedness is attributed to different size of sets which carry singularities of the relevant Fourier multipliers. See [20] for more details.

**Notations** In order to facilitate the statement of our results, we introduce some notations. We use the following norms in the vector-valued setting: For a vector-valued
function $u = (u_1, \ldots, u_d)$ let us set
\[
\|u\|_{L^q(\mathbb{R}^d)} = \left\{ \begin{array}{ll}
\left( \sum_{j=1}^d \|u_j\|_{L^q(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\
\max_{1 \leq j \leq d} \|u_j\|_{L^\infty(\mathbb{R}^d)}, & q = \infty,
\end{array} \right.
\]
and define the Lorentz norm $\|u\|_{L^{q,r}(\mathbb{R}^d)}$ similarly. For $T = (-\Delta^* - z)^{-1}$ we define $\|T\|_{p \to q}$ in the same way as in (1.3) replacing $L^p(\mathbb{R}^d)$ with $L^p(\mathbb{R}^d)^d$. Next, we recall the notations from [25]. We record them below for the reader’s convenience and to make this article self-contained. For the cases $d = 3, 4$, referring to Figures 1 and 2 can be helpful for the reader to follow the definitions and notations below (see [25, Figs. 3 and 4] for $d = 2$ and $d \geq 5$). Also, the interested readers are encouraged to refer to [25] for details regarding $P_0, P_\ast$ and the regions $\tilde{\mathcal{R}}_2, \tilde{\mathcal{R}}_3$ which are defined below.

**Definition 1.1** Let $I^2 = \{(x, y) \in \mathbb{R}^2; 0 \leq x, y \leq 1\}$. For each $(x, y) \in I^2$ we set $(x, y)' = (1 - y, 1 - x)$. Similarly, for $\mathcal{R} \subset I^2$ we define $\mathcal{R}' \subset I^2$ by $\mathcal{R}' = \{(x, y) \in I^2; (x, y)' \in \mathcal{R}\}$. For $X_1, \ldots, X_m \in I^2$, we denote by $[X_1, \ldots, X_m]$ the convex hull of the points $X_1, \ldots, X_m$. In particular, if $X, Y \in I^2$, $[X, Y]$ denotes the closed line segment connecting $X$ and $Y$ in $I^2$. We also denote by $(X, Y)$ and $[X, Y]$ the open interval $[X, Y] \setminus \{X, Y\}$ and the half-open interval $[X, Y] \setminus \{Y\}$, respectively.

- **Points and lines in $I^2$:** Let $\mathcal{L} = \{(x, y) \in I^2; y = \frac{d}{d+1}(1 - x)\}$. For $d \geq 3$ let us denote by $A \in I^2$ the intersection of the lines $\mathcal{L}$ and $x - y = \frac{2}{d}$, that is, $A = (\frac{d+1}{2d}, \frac{d-3}{2d})$. For $d \geq 2$ let $B \in I^2$ be the intersection of the lines $\mathcal{L}$ and $x - y = \frac{2}{d+1}$, that is, $B = (\frac{d+1}{2d}, \frac{(d-1)^2}{2(d+1)})$. The point $D = (d+1, 2d+1)$ is the intersection of the diagonal $x = y$ and $\mathcal{L}$, and the point $E = (\frac{d+1}{2d}, 0)$ is the projection of $A$ (or $B$) onto the $x$-axis. Also, we set $H = (\frac{1}{2}, \frac{1}{2})$, and define the points $P_\ast = (\frac{1}{p_\ast}, \frac{1}{p_\ast})$ and $P_0 = (\frac{1}{p_0}, \frac{1}{q_0})$ by
\[
\frac{1}{p_\ast} = \begin{cases} 
\frac{3(d-1)}{2(3d+1)} & \text{if } d \text{ is odd,} \\
\frac{3d-2}{2(3d+2)} & \text{if } d \text{ is even,}
\end{cases} \quad \left(\frac{1}{p_0}, \frac{1}{q_0}\right) = \begin{cases} 
\frac{(d+5)(d-1)}{2(d^2+4d-1)} & \text{if } d \text{ is odd,} \\
\frac{d^2+3d-6}{2(d^2+3d-2)} & \text{if } d \text{ is even.}
\end{cases}
\]

- **Regions in $I^2$:** $\tilde{\mathcal{R}}_2 = \{B, B', P_\ast, H, P_0\} \setminus ((P_0, H) \cup (P_\ast, H) \cup \{B, B'\})$.

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1. Consequently, the line segments $[A, A']$ and $[B, B']$ are contained in the lines $x - y = \frac{2}{d}$ and $x - y = \frac{2}{d+1}$, respectively.

2. The number $p_\ast$ relates to the range of the oscillatory integral of Carleson–Sjölin type with elliptic phase ([15, Theorem 1.2]). Being combined with Tao’s bilinear restriction theorem ([37]) and the bilinear argument in [7,27], this is one of main ingredients for the results in [25, Theorem 1.4]. The point $P_\ast$ is the intersection of $\mathcal{L}$ and the line connecting $P_\ast$ and $(\frac{1}{2}, \frac{d}{2(d+2)})$; see [25, Sects. 2 and 3] for details. If $d = 2$ then $P_\ast = P_0 = D = (1/4, 1/4)$. 

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\[ \mathcal{R}_1 = \begin{cases} (B, E, B', E', (1, 0)) \setminus ((B, E) \cup (B', E') \cup \{(1,0)\}) & \text{if } d = 2, \\ (A, B, A', B') \setminus ((A, B) \cup (A', B')) & \text{if } d \geq 3, \end{cases} \]

\[ \tilde{\mathcal{R}}_3 = \begin{cases} (1(0, 0), E, B, D) \setminus ([B, D] \cup [B, E]) & \text{if } d = 2, \\ (1(0, 0), \frac{2}{\pi}, A, B, P_\circ, P_\circ) \setminus ([A, B] \cup [A', B']) & \text{if } d \geq 3. \end{cases} \]

- **Spectral regions**: For \( p, q \) satisfying \( \left( \frac{1}{p^\ast}, \frac{1}{q^\ast} \right) \in \mathcal{R}_1 \cup \left( \bigcup_{i=2}^{3} \tilde{\mathcal{R}}_i \right) \cup \tilde{\mathcal{R}}_3' \), and \( \ell > 0 \) we define the region \( \mathcal{Z}_{p, q}(\ell) \) in the complex plane by

\[ \mathcal{Z}_{p, q}(\ell) = \{ z \in \mathbb{C} \setminus [0, \infty) : \kappa_{p, q}(z) \leq \ell \}. \]

Various profiles of \( \mathcal{Z}_{p, q}(\ell) \) depending on \( p, q, d \), and \( \ell \) can be found in [25].

**Resolvent estimate for the Lamé operator**: If \( d = 1 \) the Lamé operator is just a constant times the Laplacian, that is, \( \Delta^\ast = (\lambda+2\mu) \frac{d^2}{dx^2} \). Also, in every dimension, if \( \mu = \lambda + 2\mu \) then \( \Delta^\ast = \mu(\Delta, \ldots, \Delta) \). In these cases, the resolvent estimate (1.1) trivially follows from the estimate (1.2) regarding the Laplacian. Hence, throughout the paper, we shall assume that \( d \geq 2 \) and \( \mu \neq \lambda + 2\mu \).

Barceló, Folch-Gabayet, Pérez-Esteva, Ruiz, and Vilela [4, Theorem 1.1] showed that \( (\Delta^\ast - z)^{-1} \|_{p \to q} \lesssim |z|^{-1+\frac{\nu}{d}} \left( \frac{1}{p} \right)^{\frac{1}{2}} \left( \frac{2}{q} \right)^{\frac{1}{2}} \) for \( p, q \) satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d} \), that is, when \( \left( \frac{1}{p}, \frac{1}{q} \right) \) lies on the closed line segment of which endpoints are the midpoints of \( [A, A'] \) and \( [B, B'] \). Also, see [8, Theorem 2.3]. They utilized the Leray projection \( \Pi \) defined via the Fourier transform by

\[ \widehat{\Pi f}(\xi) = \left( \hat{f}(\xi) \cdot \frac{\xi}{|\xi|} \right) \frac{\xi}{|\xi|} \]

to decompose the Lamé resolvent as follows:

\[ (\Delta^\ast - z)^{-1} = (\mu\Delta - z)^{-1}(I_d - \Pi) + (\lambda + 2\mu\Delta - z)^{-1}\Pi, \quad (1.9) \]

where \( I_d \) denotes the \( d \times d \) identity matrix. The estimate for \( (\Delta^\ast - z)^{-1} \) is now a simple consequence of the corresponding estimate for \( (\Delta - z)^{-1} \) and boundedness of the Riesz transforms. Indeed, applying the known estimate (1.2) ([25]) one can get the upper bounds on \( \| (\Delta^\ast - z)^{-1} \|_{p \to q} \) for \( p, q \) contained in a range which is wider than that of [4,8]. See Theorem 1.3.

This immediately leads us to a couple of related questions which are already known to be true for the Laplacian. First, one may ask whether these bounds are sharp. Secondly, in point of view of the above (1.9) it seems likely that the uniform Sobolev inequality (1.14) and Carleman estimate (1.15) are also possible since (1.9) and the known estimates for \( (\Delta - z)^{-1} \) give uniform resolvent estimate (1.1) for \( p, q \) satisfying \( \left( \frac{1}{p}, \frac{1}{q} \right) \in (A, A') \) (see Theorem 1.3 below).

Now we begin stating our results on the sharp resolvent estimates for \( -\Delta^\ast \). By the following proposition, we cannot expect the resolvent estimate (1.1) when \( \left( \frac{1}{p}, \frac{1}{q} \right) \) lies...
outside the admissible range:

\[ \mathcal{R}_0 := \begin{cases} \{(x, y) \in \mathbb{R}^2 : 0 \leq x - y < 1\} & \text{if } d = 2, \\ \{(x, y) \in \mathbb{R}^2 : 0 \leq x - y \leq \frac{2}{d}\} \setminus \{(1, \frac{d-2}{d}), \left(\frac{2}{d}, 0\right)\} & \text{if } d \geq 3. \end{cases} \]

**Proposition 1.2** Let \( d \geq 2 \) and let \( 1 \leq p, q \leq \infty \). If \( \left(\frac{1}{p}, \frac{1}{q}\right) \not\in \mathcal{R}_0 \), then for any \( z \in \mathbb{C} \setminus [0, \infty) \), \( \|(-\Delta^* - z)^{-1}\|_{p \rightarrow q} = \infty \).

In what follows, we characterize \( L^p - L^q \) resolvent estimates for the Lamé operator for a large set of the admissible \( \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{R}_0 \).

**Theorem 1.3** Let \( d \geq 2 \), \( z \in \mathbb{C} \setminus [0, \infty) \), \( 1 < p \leq q < \infty \). If \( \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{R}_0 \cup \widehat{\mathcal{R}}_2 \cup \widehat{\mathcal{R}}_3 \cup \widehat{\mathcal{R}}_3', \) we have

\[ C^{-1} \kappa_{p,q}(z) \leq \|(-\Delta^* - z)^{-1}\|_{p \rightarrow q} \leq C \kappa_{p,q}(z), \tag{1.10} \]

where the constant \( C \) may depend on \( p, q, d, \lambda, \) and \( \mu \), but is independent of \( z \in \mathbb{C} \setminus [0, \infty) \). Furthermore, we have the following weak type and restricted weak type estimates in the critical cases:

\[ \|(-\Delta^* - z)^{-1} f\|_{L^{q,\infty}(\mathbb{R}^d)^d} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^d)^d} \quad \text{if} \quad (1/p, 1/q) = B \text{ or } B', \tag{1.11} \]

and

\[ \|(-\Delta^* - z)^{-1} f\|_{L^{q,\infty}(\mathbb{R}^d)^d} \lesssim \|f\|_{L^{p}(\mathbb{R}^d)^d} \quad \text{if} \quad \begin{cases} (1/p, 1/q) \in (B', E'), \quad \text{when } d = 2, \\ (1/p, 1/q) \in (B', A'), \quad \text{when } d \geq 3. \end{cases} \tag{1.12} \]

Throughout the paper, \( A \lesssim B \) denotes \( A \leq CB \) for some constant \( C > 0 \), and \( A \approx B \) denotes \( A \lesssim B \lesssim A \). The lower bound in (1.10) also holds for all \( 1 \leq p, q \leq \infty \). See Sect. 2 for the details. In contrast to the case of the Laplacian ( [25]), the cases \( p = 1 \) and \( p = \infty \) are excluded in the theorem. This is due to the failure of the \( L^1 - L^1 \) and \( L^\infty - L^\infty \) estimates for the Riesz transform.

**Eigenvalues of** \(-\Delta^* + V\)** The sharp resolvent estimates (1.10) can be used to specify the location of eigenvalues of the perturbed Lamé operator \(-\Delta^* + V\) acting in \( L^q(\mathbb{R}^d)^d, 1 < q < \infty \), for a matrix-valued potential \( V: \mathbb{R}^d \to \mathcal{M}_d \times d(\mathbb{C}) \). It does not seem that the Birman–Schwinger principle is applicable as in [8,12] because \( q \neq 2 \) in general.

**Corollary 1.4** Let \( d \geq 2 \), \( \left(\frac{1}{p}, \frac{1}{q}\right) \in \mathcal{R}_0 \cup \widehat{\mathcal{R}}_2 \cup \widehat{\mathcal{R}}_3 \cup \widehat{\mathcal{R}}_3' \), and let \( C \) be the constant in (1.10). Fix a positive number \( \ell > 0 \) (we choose \( \ell \geq 1 \) if \( 1/p - 1/q = 2/d \)). Suppose that, for some \( t \in (0, 1) \),

\[ \|V\|_{L^{p,q}(\mathbb{R}^d)^d} \leq \frac{t}{C \ell d^{1 - \frac{1}{p} + \frac{1}{q}}}. \tag{1.13} \]
If \( E \in \mathbb{C} \setminus [0, \infty) \) is an eigenvalue of \(-\Delta^* + V\) acting in \( L^q(\mathbb{R}^d)^d \), then \( E \) must lie in \( \mathbb{C} \setminus \mathcal{Z}_{p,q}(\ell) \).

From the above we can deduce properties of the complex eigenvalues of the operator \(-\Delta^* + V\) which depend on the potential \( V \). In this regards we make a couple of remarks.

**Remark 1** If \((1/p, 1/q)\) lies in the range

\[
\left\{ (x, y) \in \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_3' : \frac{d - 1}{d} < x + y < \frac{d + 1}{d}, \ (x, y) \neq \left( \frac{1}{2}, \frac{1}{2} \right) \right\},
\]

then the region \( \mathbb{C} \setminus \mathcal{Z}_{p,q}(\ell) \) is a neighborhood of \([0, \infty)\) which shrinks along the positive real line (see Figure 8(b,c,d,e) and Figure 9(e) in [25]). Hence, for any sequence of eigenvalues \( \{E_j\} \) of \(-\Delta^* + V\) acting in \( L^q(\mathbb{R}^d)^d \) such that \( \text{Re} \ E_j \to \infty \), the sequence \( \{\text{Im} \ E_j\} \) converges to zero provided that \( V \) satisfies (1.13). Concerning analogous results for the Laplacian see [13, p.220, Remark (1)] and [25, Remark 3].

**Remark 2** If \((1/p - 1/q = 2/d)\) and (1.13) is satisfied with some \( \ell \geq 1 \) and \( t \in (0, 1) \), the region \( \mathbb{C} \setminus \mathcal{Z}_{p,q}(\ell) \) is a conic region which is contained in the right half plane and its apex is at the origin (see [25, Figure 10]). Thus, we deduce from Corollary 1.4 that there is no eigenvalue of the operator \(-\Delta^* + V\) acting in \( L^q(\mathbb{R}^d)^d \) which has negative real part. We refer the reader to the recent papers [5,8] for descriptions of bounds for eigenvalues of \(-\Delta^* + V\) acting in \( L^2(\mathbb{R}^d)^d \) in terms of size of \( V \) measured in the Lebesgue, Morrey-Campanato, or Kerman-Sawyer spaces.

**Uniform Sobolev and Carleman estimates for \(-\Delta^*\)** In view of Theorem 1.3, the \( L^p - L^q \) estimate for the resolvent of the Lamé operator displays similar behavior as that for the resolvent of the Laplacian. In particular, we have uniform estimates for the resolvent of the Lamé operator when \((\frac{1}{p}, \frac{1}{q}) \in (A, A')\). Thus, in perspective of the results in [23], it is natural to expect that the following uniform Sobolev inequality holds:

\[
\|u\|_{L^q(\mathbb{R}^d)^d} \leq C\|(-\Delta^* + M_1\nabla + M_2)u\|_{L^p(\mathbb{R}^d)^d},
\]

where \( C \) is independent of all complex valued \((d \times d)\)-matrices \( M_1, M_2 \). If this were true, then we could particularly deduce the following form of Carleman estimate

\[
\|u\|_{L^q(\mathbb{R}^d)^d} \leq C\|e^{v \cdot x}(-\Delta^*)e^{-v \cdot x}u\|_{L^p(\mathbb{R}^d)^d}
\]

with \( C \) independent of \( v \in \mathbb{R}^d \setminus \{0\} \), which would imply the unique continuation property for the differential inequality \( |\Delta^* u| \leq |Vu| \). As was already mentioned before, if \( \lambda + 2\mu = \mu \), then \(-\Delta^* = -\mu(\Delta, \ldots, \Delta)\), and it follows from the Laplacian case ((1.7), (1.8)) that (1.15) holds if and only if \( p, q \) satisfy the condition (1.8). However, contrary to the seemingly natural expectation, the Carleman estimate (1.15) fails whenever \( \lambda + 2\mu \neq \mu \). Hence, the uniform Sobolev inequality (1.14) also fails.

**Theorem 1.5** Let \( d \geq 2 \) and \( \lambda + 2\mu \neq \mu \). Then for any \( 1 \leq p, q \leq \infty \), both the uniform Sobolev inequality (1.14) and the Carleman estimate (1.15) fail.

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The failure contrasts with the previous result concerning the Carleman estimate (1.7) for the Laplacian ([20,23]). While the relevant multiplier for the estimate (1.7) is \((|\xi|^2 - 1 + 2i\xi_1)^{-1}\), the estimate (1.15) implies \(L^p - L^q\) boundedness of the multiplier operator given by the multiplier \((\xi_1 + i)\xi_2(|\xi|^2 - 1 + 2i\xi_1)^{-2}\) (see (4.3) below). Compared with \((|\xi|^2 - 1 + 2i\xi_1)^{-1}\) the function \((|\xi|^2 - 1 + 2i\xi_1)^{-2}\) exhibits more singular behavior near the sphere \(\{0\} \times S^{d-2} := \{\xi: |\xi| = 1, \xi_1 = 0\}\) and this leads to failure of the estimate (1.15).

We close the introduction with a couple of remarks.

**Remark 3** Concerning the strong unique continuation property of Schrödinger operators, Jerison, and Kenig [21,22] proved the following type of Carleman estimate

\[
\|u\|_{L^q(\mathbb{R}^d)} \leq C\||x|^{-\tau} \Delta |x|^\tau u\|_{L^p(\mathbb{R}^d)}
\]  

(1.16)

with \(p = \frac{2d}{d+2}, q = \frac{p}{p-1}\), and with \(C\) independent of \(\tau \in \mathbb{R}\) such that \(\text{dist}(\tau, \mathbb{Z} + \frac{d}{q}) > 0\). Later, this estimate was extended for general off-dual pairs of \(p, q\) in [24,35,36]. In [39, Proposition 2.5], Wolff made a simple observation that (1.16) implies (1.7). By the same argument, we can easily deduce the false estimate (1.15) from the following type of Carleman estimate

\[
\|u\|_{L^q(\mathbb{R}^d)^d} \leq C\||x|^{-\tau} \Delta |x|^\tau u\|_{L^p(\mathbb{R}^d)^d},
\]  

(1.17)

where the constant \(C > 0\) is independent of a sequence \(\tau \to \infty\). Therefore, by Theorem 1.5, we conclude that the estimate (1.17) is also impossible for any \(p, q\).

**Remark 4** A large body of literature (e.g., [1,2,9–11,28,38]) is available regarding unique continuation for the Lamé system

\[
\mu \Delta u + (\lambda + \mu) \nabla \text{div} u + \nabla \lambda \text{div} u + (\nabla u + (\nabla u)^t) \nabla \mu + \rho u = 0.
\]  

(1.18)

Among others, Lin, Nakamura, and Wang [29] proved the strong unique continuation property for (1.18) whenever \(\lambda, \mu \in C^0[1], \min\{\mu, \lambda + 2\mu\} \geq \delta_0 > 0,\) and \(\rho \in L^\infty\). To the authors’ knowledge, it seems that there is no result on unique continuation for the system when \(\rho\) is unbounded, e.g., \(\rho \in L^p\) for \(p \neq \infty\). From the typical viewpoint of applying Carleman estimates to unique continuation problem for unbounded potentials, \(L^p - L^q\) Carleman estimate such as (1.15) need to be developed. However, Theorem 1.5 alludes negatively to the approach in this direction.

### 2 Resolvent Estimates

In this section, we prove Proposition 1.2, Theorem 1.3, and Corollary 1.4. For the purpose we make use of the identity (1.9). For the sake of completeness we provide a Fourier-analytic proof of (1.9).

For a function \(f: \mathbb{R}^d \to \mathbb{C}\), we denote its Fourier transform by \(\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,\) and inverse Fourier transform by \(\mathcal{F}^{-1} f(x)\). If \(f = (f_1, \ldots, f_d)\)
if vector-valued, then we also write \( \hat{f} = (\hat{f}_1, \ldots, \hat{f}_d) \). For a matrix \( A \), \( [A]_{jk} \) denotes the \((j, k)\)-component of \( A \).

**Lemma 2.1** Let \( d \geq 2 \) and \( z \in \mathbb{C} \setminus [0, \infty) \). For every \( f = (f_1, \ldots, f_d) \in S(\mathbb{R}^d)^d \) and \( 1 \leq j \leq d \), we have

\[
[(\Delta^* - z)^{-1} f]_j = (-\mu \Delta - z)^{-1} \sum_{k=1}^d (\delta_{jk} + R_j R_k) f_k
\]

\[
-(-\lambda + 2\mu \Delta - z)^{-1} \sum_{k=1}^d R_j R_k f_k,
\]

(2.1)

where \( \delta_{jk} \) is the Kronecker delta and \( R_j \) is the Riesz transform defined by \( \hat{R}_j f_k(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}_k(\xi) \).

**Proof** Let us formally write \( u := (-\Delta^* - z)^{-1} f \) and take the Fourier transform on the system \( f = (-\Delta^* - z)u \). Then we see that

\[
(\mu |\xi|^2 - z)\hat{u}_j(\xi) + (\lambda + \mu) \left( \sum_{k=1}^d \xi_k \hat{u}_k(\xi) \right) \xi_j = \hat{f}_j(\xi), \quad 1 \leq j \leq d.
\]

If we regard every \( d \)-dimensional vector as a \((d \times 1)\)-matrix, the system of equations is written as follows:

\[
\hat{f}(\xi) = \left[ (\mu |\xi|^2 - z) I_d + (\lambda + \mu)(\xi \xi^t) \right] \hat{u}(\xi) =: L_z(\xi) \hat{u}(\xi).
\]

(2.2)

To obtain (2.1) we need to invert the matrix \( L_z(\xi) \). It is sufficient to show that, for \( \xi \in \mathbb{R}^d \setminus \{0\} \) and \( 1 \leq j \leq d \),

\[
\hat{u}_j(\xi) = \frac{-\hat{f}_j(\xi)}{\mu |\xi|^2 - z} + \sum_{k=1}^d \left( \frac{1}{(\lambda + 2\mu)|\xi|^2 - z} - \frac{1}{\mu |\xi|^2 - z} \right) \frac{\xi_j \xi_k \hat{f}_k(\xi)}{|\xi|^2},
\]

(2.3)

which gives (2.1) by the Fourier inversion formula.

Let \( \xi \neq 0 \). Choosing \( R \in \text{SO}(d) \) such that \( R\xi = |\xi|e_1 \), it is clear that

\[
\det L_z = \det (RL_z R^t) = \det \left( (\mu |\xi|^2 - z) I_d + (\lambda + \mu)|\xi|^2 e_1 e_1^t \right)
\]

\[
= ((\lambda + 2\mu)|\xi|^2 - z)(\mu |\xi|^2 - z)^{d-1}.
\]

Writing \( \xi' = (\xi_2, \ldots, \xi_d)^t \), the \((1, 1)\)-minor \( M_{11} \) of \( L_z \) can be computed in a similar manner and we get

\[
M_{11} = \det \left( (\mu |\xi|^2 - z) I_{d-1} + (\lambda + \mu)\xi_1' \xi_1'' \right)
\]

\[
= ((\lambda + 2\mu)|\xi|^2 - (\lambda + \mu)\xi_1'^2 - z)(\mu |\xi|^2 - z)^{d-2}.
\]
These also can be checked without difficulty by applying elementary column (row) operations and utilizing properties of determinant. Similarly,

\[
M_{jj} = ((\lambda + 2\mu)|\xi|^2 - (\lambda + \mu)\xi_j^2 - z)(\mu|\xi|^2 - z)^{d-2}, \quad 1 \leq j \leq d.
\]

If \( j \neq k \), column (or row) operations give

\[
M_{jk} = (-1)^{j+k-1}(\lambda + \mu)\xi_j\xi_k(\mu|\xi|^2 - z)^{d-2}.
\]

By Cramer’s rule we see that

\[
[L_z(\xi)^{-1}]_{jk} = \frac{(-1)^{j+k}M_{kj}}{\det L_z} = \frac{\delta_{jk}}{\mu|\xi|^2 - z} + \left(\frac{1}{(\lambda + 2\mu)|\xi|^2 - z} - \frac{1}{\mu|\xi|^2 - z}\right)\xi_j\xi_k.
\]

Since \( \hat{u}(\xi) = L_z(\xi)^{-1}\hat{f}(\xi) \) we have (2.3), from which (2.1) follows. \( \square \)

**Proof of Theorem 1.3** Once we have Lemma 2.1, the \( L^p-L^q \) resolvent estimates for the Lamé operator can be deduced by making use of those for the Laplacian in [25].

*Upper bound in (1.10)* First, we recall from [25, Theorem 1.4] the estimate

\[
\| (-\Delta - z)^{-1} \|_{p\to q} \lesssim \kappa_{p,q}(z), \quad (2.4)
\]

where \( p, q \) are given as in Theorem 1.3. By Lemma 2.1 and (2.4) we then have that for every \( j = 1, \ldots, d \),

\[
\| (-\Delta^* - z)^{-1} f_j \|_{L^q_d} \lesssim \frac{1}{\mu} \kappa_{p,q}\left(\frac{z}{\mu}\right)\|f_j\|_{L^p_d} + \sum_{k=1}^{d} \left(\frac{1}{\lambda + 2\mu}\kappa_{p,q}\left(\frac{z}{\lambda + 2\mu}\right) + \frac{1}{\mu}\kappa_{p,q}\left(\frac{z}{\mu}\right)\right)\|R_j R_k f_k\|_{L^p_d} \leq \left[\mu^{-d\frac{1}{2}}(\frac{1}{p} - \frac{1}{d}) + \left(\lambda + 2\mu\right)^{-\frac{1}{2}}(\frac{1}{p} - \frac{1}{d}) + \mu^{-d\frac{1}{2}}(\frac{1}{p} - \frac{1}{d})\right] \tan^2\left(\frac{\pi}{2\min[p, p']\mu}\right)\kappa_{p,q}(z)\|f\|_{L^p_d}.
\]

since \( \kappa_{p,q}(z/\mu) = \mu^{1-d\frac{1}{2}}(\frac{1}{p} - \frac{1}{d})\kappa_{p,q}(z) \), and the Riesz transforms are bounded on \( L^p(\mathbb{R}^d), 1 < p < \infty \), with norm \( \|R_j\|_{p\to p} = \tan(\frac{\pi}{2\min[p, p']\mu}) \). (See [3, Theorem 3]). This proves the upper bound in (1.10).

*Proof of (1.11) and (1.12)* For the restricted weak type bound (1.11) we argue similarly using the restricted weak type \((p, q)\) bound for the Laplacian resolvent ( [25, Theorem 1.4]) and the \( L^{p,1}-L^{p,1} \) estimate of the Riesz transforms (see [32, Theorem 1.1]). Indeed,

\[
\| (-\Delta^* - z)^{-1} f_j \|_{L^{q,\infty}_d} \lesssim \frac{1}{\mu} \|f_j\|_{L^{p,1}_d} + \left(\frac{1}{\lambda + 2\mu} + \frac{1}{\mu}\right)\sum_{k=1}^{d} \|R_j R_k f_k\|_{L^{p,1}_d} \lesssim \|f\|_{L^{p,1}_d}.
\]
The weak type bound (1.12) can also be shown in the similar way using the weak type \((p, q)\) bound for \((-\Delta - z)^{-1}\) in [25, Theorem 1.4].

**Lower bound in (1.10)** On the other hand, the lower bound in (1.10) can be obtained by considering functions whose Fourier transform is supported near the sphere \(S^{d-1}\) as it was done in [25, Lemma 5.1]. However, as each component of the multiplier is not rotationally symmetric, we cannot directly use the well-known asymptotic of the Bessel function anymore as in [25]. In order to get around this difficulty, we instead apply the stationary phase method to obtain the asymptotic of oscillatory integrals.

We now show the lower bound in (1.10). It is enough to show, taking \(f = (f_1, 0, \ldots, 0)\) and \(j = 1\) in (2.1), that

\[
\left\| \left(-\mu \Delta - z\right)^{-1}\left(1 + R_1^2\right) - \left(-\left(\lambda + 2\mu\right) \Delta - z\right)^{-1} R_1^2 \right\|_{p \to q} \gtrsim \kappa_{p,q}(z).
\]

Multiplying this by \(\mu\) and then replace \(z\) with \(\mu z\), this is equivalent to

\[
\left\| \left(-\Delta - z\right)^{-1}\left(1 + R_1^2\right) - \left(-\frac{\lambda + 2\mu}{\mu} \Delta - z\right)^{-1} R_1^2 \right\|_{p \to q} \gtrsim \mu \kappa_{p,q}(\mu z) \approx \kappa_{p,q}(z),
\]

which is again identical with

\[
\left\| \left(-\Delta - \frac{z}{|z|}\right)^{-1}\left(1 + R_1^2\right) - \left(-\frac{\lambda + 2\mu}{\mu} \Delta - \frac{z}{|z|}\right)^{-1} R_1^2 \right\|_{p \to q} \gtrsim \text{dist}(\frac{z}{|z|}, [0, \infty])^{-\gamma_{p,q}}
\]

since \(\left\| (-\Delta - z)^{-1}\right\|_{p \to q} = |z|^{-1+\frac{d}{p}-\frac{1}{q}} \|(-\Delta - \frac{z}{|z|})^{-1}\|_{p \to q}\) and the multiplier of the Riesz transform \(R_1\) is of homogeneous of degree zero. (The identity can be readily checked by the Fourier inversion formula and scaling; \(x \to |z|^{-\frac{1}{2}} x\) for the space variable and \(\xi \to |z|^{\frac{1}{2}} \xi\) for the frequency variable.) Hence, recalling the definition of \(\gamma_{p,q}\) it is enough to show that, for \(0 < \delta \ll 1\),

\[
\left\| \left(-\Delta - (1 + i\delta)\right)^{-1}\left(1 + R_1^2\right) - \left(-\rho \Delta - (1 + i\delta)\right)^{-1} R_1^2 \right\|_{p \to q} \gtrsim \max\left\{1, \delta^{1+\frac{d-1}{2}}(1 + \frac{1}{p} - \frac{1}{q}), \delta^{-\frac{1}{2}}, \frac{d-1}{2}, \delta \frac{d}{p}, \delta^{-\frac{d+1}{2}}\right\}
\]

under the assumption \(\rho := \frac{\lambda + 2\mu}{\mu} \neq 1\).

In the above (2.5) the first lower bound is clear because the operator is nontrivial. And the last lower bound follows from the third if we replace \(1 + i\delta\) with \(1 - i\delta\), by the following general principle of duality (for every linear operator \(T\) and its adjoint \(T^*\))

\[
\|T^*\|_{p \to q} = \|T\|_{q' \to p'}.
\]

Hence it suffices to show the second and third. If we denote the multiplier in (2.5) by

\[
a_\delta(\xi) = \frac{1}{|\xi|^2 - (1 + i\delta)} \frac{|\xi'|^2}{|\xi|^2} + \frac{1}{\rho|\xi|^2 - (1 + i\delta)} \frac{\xi_1^2}{|\xi|^2}, \quad \xi' = (\xi_2, \ldots, \xi_d) \in \mathbb{R}^{d-1},
\]
and use the standard notation \( a_\delta(D) f = F^{-1}(a_\delta \hat{f}) \) (which is of course equal to the operator in (2.5)), then it is easy to see that \( \|a_\delta(D)\|_{p \to q} = \|\overline{a_\delta(D)}\|_{p \to q} \) since \( a_\delta(\xi) \) is invariant under the reflection \( \xi \to -\xi \). Consequently, we have

\[
2\| a_\delta(D) \|_{p \to q} \geq \| a_\delta(D)\pm \overline{a_\delta(D)}\|_{p \to q}.
\]

Thus, to show (2.5) it is enough to consider the imaginary part \( I_\delta := \text{Im} a_\delta \), that is,

\[
I_\delta(\xi) = \frac{\delta}{(|\xi|^2 - 1)^2} \frac{|\xi'|^2}{|\xi|^2} + \frac{\delta}{(\rho|\xi|^2 - 1)^2} \frac{\xi_1^2}{|\xi|^2} =: M_\delta(\xi) + m_\delta(\xi).
\]

Thus, for the lower bound (2.5), it suffices to show the following proposition, which completes the proof of Theorem 1.3. Let us set \( D = -i\nabla \) and, for any bounded measurable function \( m \) on \( \mathbb{R}^d \), let \( m(D) \) denote the Fourier multiplier with multiplier (symbol) \( m \), i.e., \( m(D) f = F^{-1}(m \hat{f}) \).

**Proposition 2.2** Let \( d \geq 2 \), \( 1 \leq p, q \leq \infty \) and let \( 0 < \delta \ll 1 \). Then

\[
\| I_\delta(D) \|_{p \to q} \gtrsim \delta^{-1} + \frac{d-1}{2p} (\frac{1}{p} - \frac{1}{q}), \quad (2.7)
\]

\[
\| I_\delta(D) \|_{p \to q} \gtrsim \delta^{-\frac{d-1}{2}} - \frac{d}{q}. \quad (2.8)
\]

Proving Proposition 2.2 is a messy affair and we shall therefore hold off doing so until the next section. We close this section with the proofs of Proposition 1.2 and Corollary 1.4.

**Proof of Proposition 1.2** The proof is similar to that of the necessary part of [25, Proposition 1.1]. It is well-known that the condition \( p \leq q \) is necessary for the resolvent estimate since every Fourier multiplier on \( \mathbb{R}^d \) is translation invariant ([17, Theorem 1.1]). For the other necessary conditions, let us first assume that \( d \geq 3 \). In view of (2.5), it is enough to show that, for every \( \rho > 0 \) with \( \rho \neq 1 \) and every \( z \in S^1 \setminus \{1\} \),

\[
\|(-\Delta - z)^{-1}(1 + R_1^2) - (-\rho \Delta - z)^{-1} R_1^2 \|_{p \to q} < \infty \quad (2.9)
\]

holds only if

\[
\frac{1}{p} - \frac{1}{q} \leq 2/d, \quad (p, q) \neq (d/2, \infty), \quad \text{and} \quad (p, q) \neq (1, d/(d-2)). \quad (2.10)
\]

By duality (2.6), we need only to consider the first and second conditions. Let \( \phi \in C_c^\infty(\mathbb{R}) \) be the standard Littlewood–Paley bump function supported on the interval \([1/2, 2]\) satisfying

\[
\sum_{k \geq 0} \phi(2^{-k} t) = 1, \quad t \geq 1,
\]

and let \( \phi_0(t) := 1 - \sum_{k \geq 0} \phi(2^k t) \). Let us also define projection operators \( P_k \) and \( P_0 \) by \( \overline{P_k f}(\xi) = \phi(2^{-k} |\xi|) \hat{f}(\xi) \) and \( \overline{P_0 f}(\xi) = \phi_0(|\xi|) \hat{f}(\xi) \), respectively. Testing the
bound (2.9) with \( P_k f \) and scaling \( \xi \to 2^k \xi \) give the inequality

\[
\left\| \int_{\mathbb{R}^d} e^{ix \cdot \xi} \left( \frac{1}{|\xi|^2 - 2^{-2k_2} z} \frac{|\xi|^2}{|\xi|^2} + \frac{1}{\rho|\xi|^2 - 2^{-2k_2} |\xi|^2} \right) \phi(|\xi|) \hat{g}(\xi) d\xi \right\|_{L^q_p(\mathbb{R}^d)} \lesssim 2^{k(2+\frac{d}{2})} \|g\|_{L^p(\mathbb{R}^d)}
\]

for all \( g \in L^p(\mathbb{R}^d) \). If we consider the limit \( k \to \infty \), the left side converges to a nonzero value provided that \( g \neq 0 \) is suitably chosen, while the right side converges to zero whenever \( \frac{1}{p} - \frac{1}{q} > \frac{2}{d} \). This is contradiction to the assumption (2.9), which shows that the first condition of (2.10) is necessary for (2.9).

Now, let us assume that (2.9) is true for \((p, q) = (d/2, \infty)\). Denoting the multiplier of (2.9) by \( m_z(\xi) \) we then have

\[
\|D|^{-2}(1 - \phi_0(|D|)) f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|m_z(D)^{-1} |D|^{-2}(1 - \phi_0(|D|)) f\|_{L^d(\mathbb{R}^d)}.
\]

Since the multiplier \( m_z(\xi)^{-1} |\xi|^{-2}(1 - \phi_0(|\xi|)) \) satisfies the Mikhlin’s condition, it follows from Mikhlin’s multiplier theorem that

\[
\|D|^{-2}(1 - \phi_0(|D|)) f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^d(\mathbb{R}^d)},
\]

which, by scaling, implies

\[
\|D|^{-2}(1 - \phi_0(\varepsilon |D|)) f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^d(\mathbb{R}^d)}, \quad \forall \varepsilon > 0.
\]

By Fatou’s lemma this estimate in turn implies the following inequality

\[
\|(-\Delta)^{-1} f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^d(\mathbb{R}^d)},
\]

which is false. Thus, we see that the second condition in (2.10) is necessary for (2.9).

Finally, we assume \( d = 2 \) and show that the estimate (2.9) fails when \((p, q) = (1, \infty)\). Note that the multiplier operator in (2.9) is written

\[
T_\varepsilon h(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \left( \frac{1}{|\xi|^2 - \varepsilon^2} \frac{\xi_2^2}{|\xi|^2} + \frac{1}{\rho|\xi|^2 - \varepsilon^2} \xi_1^2 \right) \hat{h}(\xi) d\xi.
\]

For every \( \varepsilon > 0 \), let us define \( h_\varepsilon \in \mathcal{S}(\mathbb{R}^2) \) by

\[
\hat{h}_\varepsilon(\xi) = \phi_0(\varepsilon^2 \xi) - \phi_0(\varepsilon \xi).
\]

By the definition of \( \phi_0 \) it is clear that \( 0 \leq \hat{h}_\varepsilon \leq 1 \),

\[
\text{supp} \hat{h}_\varepsilon \subset \{ \xi \in \mathbb{R}^2 : 1/2\varepsilon \leq |\xi| \leq 1/\varepsilon^2 \}, \quad \hat{h}_\varepsilon(\xi) = 1 \quad \text{if} \quad 1/\varepsilon \leq |\xi| \leq 1/2\varepsilon^2,
\]

\( \varepsilon \)
and
\[
\|h_\varepsilon\|_{L^1(\mathbb{R}^2)} \leq 2\|\mathcal{F}^{-1}\phi_0\|_{L^1(\mathbb{R}^2)} \lesssim 1. \tag{2.14}
\]

If the estimate (2.9) were true for \((p, q) = (1, \infty)\), then it would also be true that
\[
|\text{Re} \, T\!^*_{\varepsilon} h_\varepsilon(0)| \leq |T\!^*_{\varepsilon} h_\varepsilon(0)| \leq \|T\!^*_{\varepsilon} h_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \|h_\varepsilon\|_{L^1(\mathbb{R}^2)} \lesssim 1 \tag{2.15}
\]
with all the inequalities uniform in \(\varepsilon > 0\). Let us now examine \(|\text{Re} \, T\!^*_{\varepsilon} h_\varepsilon(0)|\). If we write \(z = a + ib \in S^1 \setminus \{1\}\), then
\[
\text{Re} \, T\!^*_{\varepsilon} h_\varepsilon(0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left( \frac{|\xi|^2 - a}{(|\xi|^2 - a)^2 + b^2 |\xi|^2} + \frac{\rho|\xi|^2 - a}{(\rho|\xi|^2 - a)^2 + b^2 |\xi|^2} \right) \hat{h}_\varepsilon(\xi) d\xi. \tag{2.16}
\]
Since \(a^2 + b^2 = 1\) it is easy to see that, for \(|\xi| \geq 2\) satisfying \(|\xi_2| \geq |\xi_1|\),
\[
\frac{|\xi|^2 - a}{(|\xi|^2 - a)^2 + b^2 |\xi|^2} \geq \frac{|\xi|^2 - 1}{2(|\xi|^2 + 1)^2} \gtrsim \frac{1}{|\xi|^2}.
\]
Similarly, if \(|\xi| \geq \frac{2}{\sqrt{\rho}}\) and \(|\xi_2| \leq |\xi_1|\), then
\[
\frac{\rho|\xi|^2 - a}{(\rho|\xi|^2 - a)^2 + b^2 |\xi|^2} \geq \frac{\rho|\xi|^2 - 1}{2(\rho|\xi|^2 + 1)^2} \gtrsim \frac{1}{\rho|\xi|^2}.
\]
For every \(\varepsilon \leq \frac{1}{2} \min\{1, \sqrt{\rho}\}\), it follows from (2.13) and (2.16) that
\[
|\text{Re} \, T\!^*_{\varepsilon} h_\varepsilon(0)| \gtrsim \int_{1/\varepsilon \leq |\xi| \leq 1/2\varepsilon^2} |\xi|^{-2} 1_{\{\xi : |\xi_2| \geq |\xi_1|\}}(\xi) d\xi + (\rho|\xi|)^{-2} 1_{\{\xi : |\xi_2| \leq |\xi_1|\}}(\xi) d\xi
\gtrsim \int_{1/\varepsilon \leq |\xi| \leq 1/2\varepsilon^2} |\xi|^{-2} d\xi \approx \log \frac{1}{\varepsilon},
\]
which diverges to infinity as \(\varepsilon \to 0\). This is contradiction to (2.15). Thus, the estimate (2.9) fails with \((p, q) = (1, \infty)\) when \(d = 2\).

**Proof of Corollary 1.4** Since \(E\) is an \(L^q\)-eigenvalue of \(-\Delta^* + V\), there exists a nonzero \(u \in L^q(\mathbb{R}^d)^d\) such that \((-\Delta^* + V)u = Eu\). Assume that \(E\) lies in \(Z_{p,q}(\ell)\). Since \(q \geq p\) Hölder's and Minkowski's inequalities give
\[
\|Vu\|_{L^p(\mathbb{R}^d)^d} \leq \left( \sum_{j=1}^d \left( \sum_{k=1}^d \|V_{jk}\|_{L^{p/q'}(\mathbb{R}^d)^d} \right)^{\frac{p}{q'}} \right)^{\frac{1}{q'}} \|u\|_{L^q(\mathbb{R}^d)^d} =: \|V_{jk}\|_{L^{p/q'}(\mathbb{R}^d)^d} \|u\|_{L^q(\mathbb{R}^d)^d}.\]
Now we recall the inequality
\[
\left( \sum_{i=1}^{d} a_i \right)^{\theta} \leq d^{\theta - 1} \sum_{i=1}^{d} a_i^{\theta}
\]  
(2.17)
which holds for every $\theta \geq 1$ and $a_i \geq 0$. Applying this inequality with $(i = j$ and $a_j = \left( \sum_{k=1}^{d} \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \right)^{\frac{p}{q}}$ and $\theta = \frac{q}{q-p}$

we see that
\[
\| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \leq \left( \sum_{j=1}^{d} \left( \sum_{k=1}^{d} \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \right)^{\frac{p}{q}} \right)^{\frac{q-p}{q}} d^{\frac{q}{q-p} - 1} \left( \sum_{j=1}^{d} \left( \sum_{k=1}^{d} \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \right)^{\frac{p}{q}} \right)^{\frac{q-p}{q}}
\]

Similarly, applying (2.17) with $(i = k$ and $a_k = \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)}$ and $\theta = \frac{p(q-1)}{q-p}$

we have
\[
\| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \leq \left( \sum_{j=1}^{d} \left( \sum_{k=1}^{d} \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)} \right)^{\frac{p}{q}} \right)^{\frac{q-p}{p}} d^{\frac{q}{q-p} + \frac{p(q-1)}{q-p} - 2} \sum_{j=1}^{d} \sum_{k=1}^{d} \| V_{jk} \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)}^{\frac{p}{q}} \| V \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)^d}.
\]

-From the resolvent estimate (1.10) and the assumptions $E \in \mathcal{Z}_{p,q} (\ell)$, $(-\Delta^* + V)u = Eu$, and (1.13), it follows that

\[
\| u \|_{L^q (\mathbb{R}^d)} \leq C \kappa_{p,q} (E) \| (-\Delta^* - E) u \|_{L^p (\mathbb{R}^d)} d \leq C \ell \left( \| (-\Delta^* + V)u \|_{L^p (\mathbb{R}^d)} + \| Vu \|_{L^p (\mathbb{R}^d)} \right) d \leq C \ell d^{1 - \frac{1}{p} + \frac{1}{q}} \| V \|_{L^{\frac{pq}{q-p}} (\mathbb{R}^d)^d} \| u \|_{L^q (\mathbb{R}^d)} d \leq C t \| u \|_{L^q (\mathbb{R}^d)} d.
\]

which forces $u$ to be identically zero since $t \in (0,1)$. This contradicts that $u$ is nonzero. Therefore, $E$ must lie in $\mathbb{C} \setminus \mathcal{Z}_{p,q} (\ell)$.

\[\square\]

3 Proof of Proposition 2.2

In this section, we prove Proposition 2.2.
Proof of (2.7) Let us choose $\phi, \psi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \phi, \psi \leq 1$, $\text{supp} \phi \subset [-1, 1]$, $\phi = 1$ on $[-1/2, 1/2]$, $\text{supp} \psi \subset [1/4, 1]$, and $\psi = 1$ on $[1/2, 3/4]$. Now we define $h_\delta \in \mathcal{S}(\mathbb{R}^d)$ by

$$\hat{h}_\delta(\xi) = \psi \left( \frac{\xi_d - 1}{\delta} \right) \prod_{j=1}^{d-1} \phi \left( \frac{\xi_j}{\sqrt{\delta}} \right).$$

It is obvious that

$$|\text{supp} \hat{h}_\delta| \approx |\{ \xi \in \mathbb{R}^d : \hat{h}_\delta(\xi) = 1 \}| \approx \delta^{1 + \frac{d-1}{2}}. \quad (3.1)$$

By the Fourier inversion formula it is also clear that

$$h_\delta(x) = e^{ix_d \delta} \mathcal{F}^{-1}(\psi)(\delta x_d) \prod_{j=1}^{d-1} \sqrt{\delta}(\mathcal{F}^{-1}(\phi)(\sqrt{\delta} x_j)),$$

from which it follows that

$$\|h_\delta\|_{L^p(\mathbb{R}^d)} \approx \delta^{\frac{d+1}{2} - \frac{d+1}{2p}}. \quad (3.2)$$

If $\xi \in \text{supp} \hat{h}_\delta$ then, writing $\xi = (\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}$, we see that

$$|\xi|^2 - 1 = |\eta|^2 + (\tau + 1)(\tau - 1) \approx \delta,$$

$$|\xi|'^2 = 1 + |\eta|^2 + (\tau + 1)(\tau - 1) \approx 1 + \delta \approx 1$$

since $\delta$ is small. Similarly, it is easy to check that $|\xi'|^2 \approx 1$ for $\xi \in \text{supp} \hat{h}_\delta$. Hence,

$$\delta^{-1} \geq M_\delta(\xi) := \frac{\delta}{(|\xi|^2 - 1)^2 + \delta^2} \frac{|\xi'|^2}{|\eta|^2} \approx \frac{\delta}{\delta^2 + \delta^2} \approx \delta^{-1}, \quad \xi \in \text{supp} \hat{h}_\delta.$$

On the other hand, since $\sqrt{\rho} \neq 1$,

$$|\rho|\xi|^2 - 1| = |\rho|\eta|^2 + (\sqrt{\rho} \tau + 1)(\sqrt{\rho}(\tau - 1) + \sqrt{\rho} - 1) | \approx 1, \quad \xi \in \text{supp} \hat{h}_\delta.$$

Hence $0 \leq m_\delta(\xi) \lesssim \delta$ on $\text{supp} \hat{h}_\delta$, and we have

$$I_\delta(\xi) = M_\delta(\xi) + m_\delta(\xi) \gtrsim \delta^{-1}, \quad \xi \in \text{supp} \hat{h}_\delta. \quad (3.3)$$

Let us set

$$A_\delta := \{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : |t| \leq (100\delta)^{-1}, |y_j| \leq (100d \sqrt{\delta})^{-1}, 1 \leq j \leq d - 1\}. $$
It is easy to see that if \( x = (y, t) \in A_\delta \) and \( \xi = (\eta, \tau) \in \text{supp} \hat{h}_\delta \), then
\[
|x \cdot (\xi - e_d)| \leq |y||\eta| + |t||\tau - 1| \leq \frac{\sqrt{d - 1}}{100d \sqrt{\delta}} \cdot \sqrt{(d - 1)\delta} + \frac{1}{100\delta} \cdot \delta \leq \frac{1}{50},
\]
and it follows that
\[
\cos(x \cdot (\xi - e_d)) \geq 1 - \frac{|x \cdot (\xi - e_d)|^2}{2} \gtrsim 1. \tag{3.4}
\]
Hence, by the estimates (3.1), (3.3), and (3.4), we have
\[
|\mathcal{I}_\delta(D)h_\delta(x)| = \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot (\xi - e_d)} \mathcal{I}_\delta(\xi) \hat{h}_\delta(\xi) d\xi \right| \\
\geq \frac{1}{(2\pi)^d} \int_{\{\xi \in \mathbb{R}^d : \hat{h}_\delta(\xi) = 1\}} \cos(x \cdot (\xi - e_d)) \mathcal{I}_\delta(\xi) d\xi \gtrsim \delta \frac{d - 1}{2}.
\]
Since \( |A_\delta| \approx \delta^{-\frac{d+1}{2}} \) this estimate gives \( \|\mathcal{I}_\delta(D)h_\delta\|_{L^q(A_\delta)} \gtrsim \delta \frac{d - 1}{2} - \frac{d + 1}{2q} \). Therefore, from (3.2) we conclude that
\[
\|\mathcal{I}_\delta(D)\|_{p \to q} := \sup_{f \neq 0} \frac{\|\mathcal{I}_\delta(D)f\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} \geq \frac{\|\mathcal{I}_\delta(D)h_\delta\|_{L^q(A_\delta)}}{\|h_\delta\|_{L^p(\mathbb{R}^d)}} \gtrsim \delta^{\frac{d - 1}{2} - \frac{d + 1}{2q}} = \delta^{-1 + \frac{d + 1}{2p} \left( \frac{1}{q} - \frac{1}{2} \right)},
\]
which shows the bound (2.7).

From now on, we write \( \xi = (\xi_1, \xi') = (\xi_1, \xi_2, \bar{\xi}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2} \). Also, we sometimes write \( \tau = \xi_1 \).

**Proof of (2.8)** By scaling, we note that \( \|\mathcal{I}_\delta(D)\|_{p \to q} = \rho^{-\frac{d}{2p} \left( \frac{1}{p} - \frac{1}{q} \right)} \|\mathcal{I}_\delta(D)\|_{p \to q} \), where
\[
\mathcal{I}_\delta(\xi) := \mathcal{I}_\delta(\rho^{-\frac{1}{d}}\xi) = \frac{\delta}{(\rho^{-1}|\xi|^2 - 1)^2 + \delta^2 |\xi|^2} + \frac{\delta}{(2+1)^2 + \delta^2 |\xi|^2} \frac{\xi^2}{\delta^2 |\xi|^2}.
\]
Thus, in order to prove (2.7), it is harmless to assume that \( \mathcal{I}_\delta = \mathcal{I}_\delta \),
\[
M_\delta(\xi) = \frac{\delta}{(\rho^{-1}|\xi|^2 - 1)^2 + \delta^2 |\xi|^2}, \text{ and } m_\delta(\xi) = \frac{\delta}{(|\xi|^2 - 1)^2 + \delta^2 |\xi|^2}.
\]

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If we put $\psi(\xi') := 1 - \sqrt{1 - |\xi'|^2}$ and apply change of variables via diffeomorphism $\xi \rightarrow (\xi_1 - 1 + \psi(\xi'), \xi')$, then we have

$$m_\delta(D)f(x) = \frac{e^{-ix_1}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix_1 \xi_1} \int_{\mathbb{R}^{d-1}} e^{i(x' \xi' + x_1 \psi(\xi'))} \frac{\delta}{\xi_1^2 (\xi_1 + 2 \psi(\xi') - 2)^2 + \delta^2} \frac{(\xi_1 + \psi(\xi') - 1)^2}{(\xi_1 + \psi(\xi') - 1)^2 + |\xi'|^2} \hat{f}(\xi_1 + \psi(\xi') - 1, \xi') d\xi' d\xi_1.$$ We then choose a function $f \in \mathcal{S}(\mathbb{R}^d)$ so that

$$\frac{(\xi_1 + \psi(\xi') - 1)^2}{(\xi_1 + \psi(\xi') - 1)^2 + |\xi'|^2} \hat{f}(\xi_1 + \psi(\xi') - 1, \xi') = \chi(\xi') \varphi(\xi_1), \quad (3.5)$$

where $\chi \in C_c^\infty(B_{d-1}(0, \frac{1}{10}))$ such that $0 \leq \chi \leq 1$ and $\chi(\xi') = 1$ if $|\xi'| \leq \frac{1}{20}$, and $\varphi \in C_c^\infty((-2\epsilon_0, 2\epsilon_0))$ satisfying $0 \leq \varphi \leq 1$ and $\varphi(t) = 1$ if $|t| \leq \epsilon_0$. Here, $\epsilon_0 > 0$ is a fixed small number to be determined depending on $\rho$ (see Lemma 3.1 below and its proof).

With the choice of $f$, $m_\delta(D)f$ is now written as the following favorable form:

$$m_\delta(D)f(x) = \frac{e^{-ix_1}}{(2\pi)^d} \int_{-2\epsilon_0}^{2\epsilon_0} e^{ix_1 \tau} I_\delta(x; \tau) \varphi(\tau) d\tau,$$

where we set

$$I_\delta(x; \tau) := \int e^{i(x' \xi' + x_1 \psi(\xi'))} a_\delta(\tau, \xi') \chi(\xi') d\xi', \quad a_\delta(\tau, \xi') := \frac{\delta}{\tau^2 (\tau + 2 \psi(\xi') - 2)^2 + \delta^2}.$$ By Lemma 3.2 below and the triangle inequality, if $x_1 \geq 1/2$ and $2^5 |x'| \leq x_1$, we have

$$|m_\delta(D)f(x)| \approx \int_{-2\epsilon_0}^{2\epsilon_0} e^{ix_1 \tau} \left| \sum_{j=0}^{N-1} x_1^{-\frac{d+1}{2} - j} D_j a_\delta(\tau, \xi')|_{\xi' = -\frac{x'}{|x'|}} + E_{\delta,N}(x; \tau) \right| \varphi(\tau) d\tau$$

$$\geq x_1^{-\frac{d+1}{2}} \int e^{ix_1 \tau} a_\delta(\tau, -\frac{x'}{|x'|}) \varphi(\tau) d\tau$$

$$- \sum_{j=1}^{N-1} x_1^{-\frac{d+1}{2} - j} \int |D_j a_\delta(\tau, \xi')|_{\xi' = -\frac{x'}{|x'|}} \varphi(\tau) d\tau - \int \left| E_{\delta,N}(x; \tau) \varphi(\tau) \right| d\tau \quad (3.6)$$

$$=: Q_0(x) - \sum_{j=1}^{N-1} Q_j(x) - R_N(x).$$

Now, for a large number $\nu > 0$ to be chosen shortly, let us define the set

$$B_\delta := \{ x \in \mathbb{R}^d : (20\nu\delta)^{-1} \leq x_1 \leq (10\nu\delta)^{-1}, x_1 \geq 2^5 |x'| \}. $$
Then we break the integral in the term $Q_0(x)$ as

$$
\int_{-\nu\delta}^{\nu\delta} e^{ix_1\tau}a_0(\tau, -\frac{x'}{|x|})\varphi(\tau)d\tau + \left(\int_{-\nu\delta}^{\nu\delta} + \int_{-2\epsilon_0}^{-\nu\delta} + \int_{\nu\delta}^{2\epsilon_0}\right) e^{ix_1\tau}a_0(\tau, -\frac{x'}{|x|})\varphi(\tau)d\tau =: \tilde{Q}_0(x) + \tilde{R}_0(x).
$$

If $x \in B_\delta$, we have $|\frac{x'}{|x|}| \leq \frac{1}{20}$. Also, since $|\tau| \leq 2\epsilon_0$ it is easy to see that $1 - 2\epsilon_0 \leq |\tau + 2\psi(-\frac{x'}{|x|}) - 2| \leq 2 + 2\epsilon_0$. Hence, if $\nu\delta \leq \epsilon_0$, for any $x \in B_\delta$,

$$
|\tilde{Q}_0(x)| \geq \int_{-\nu\delta}^{\nu\delta} \cos(x_1\tau)a_0(\tau, -\frac{x'}{|x|})d\tau \gtrsim \int_{-\nu\delta}^{\nu\delta} \frac{\delta}{(2 + 2\epsilon_0)^2\tau^2 + \delta^2}d\tau \gtrsim \int_{-\nu}^{\nu} \frac{1}{\tau^2 + 1}d\tau.
$$

Similarly, for any $x \in B_\delta$, it is clear that

$$
|\tilde{R}_0(x)| \lesssim \int_{-\nu\delta}^{\nu\delta} \frac{\delta}{(1 - 2\epsilon_0)^2\tau^2 + \delta^2}d\tau \lesssim \int_{-\nu}^{\nu} \frac{1}{\tau^2 + 1}d\tau.
$$

If we choose $\nu$ large enough, we have

$$
Q_0(x) \geq x_1^{d-1}(|\tilde{Q}_0(x)| - |\tilde{R}_0(x)|) \gtrsim \delta^{\frac{d-1}{2}}, \quad \forall x \in B_\delta. \quad (3.7)
$$

For $1 \leq j \leq N - 1$, by the estimate (3.12) below, we see that

$$
Q_j(x) \lesssim \delta^{\frac{d-1}{2} + j} \int \frac{\delta}{(1 - 2\epsilon_0)^2\tau^2 + \delta^2}d\tau \lesssim \delta^{\frac{d-1}{2} + j}, \quad \forall x \in B_\delta. \quad (3.8)
$$

Now we utilize the estimates (3.15) and (3.12) to obtain

$$
R_N(x) \lesssim \delta^N \int_{-2\epsilon_0}^{2\epsilon_0} \sup_{|\alpha| \leq 2N} |\partial_\alpha a_\delta(\tau, \xi')|d\tau \lesssim \delta^{N-1}, \quad \forall x \in B_\delta. \quad (3.9)
$$

Choosing $N$ large enough and combining all together the estimates (3.6), (3.7), (3.8), and (3.9), we conclude that

$$
\|m_\delta(D)f\|_{L^q(B_\delta)} \gtrsim \delta^{\frac{d-1}{2} - \frac{d}{q}}, \quad (3.10)
$$

On the other hand, the same change of variables as before in the frequency domain gives

$$
M_\delta(D)f(x) = \frac{e^{-ix_1}}{(2\pi)^d} \int_{-2\epsilon_0}^{2\epsilon_0} e^{ix_1\tau} J_\delta(x; \tau)\varphi(\tau)d\tau, \quad J_\delta(x; \tau) := \int e^{i(x'\cdot \xi' + x_1\psi(\xi'))} b_\delta(\tau, \xi')\chi(\xi')d\xi',
$$
where

$$b_\delta(\tau, \xi') := \frac{\rho^2 \delta}{[\tau(\tau + 2\psi(\xi') - 2) + 1 - \rho]^2 + (\rho\delta)^2} \cdot |\xi'|^2.$$

Since $\rho \neq 1$, unlike the previous case of $a_\delta$, the symbol $b_\delta$ is not singular on the support of the function $\chi(\xi')\varphi(\tau)$ provided that $\epsilon_\circ$ is small enough, and this admits the uniform bound (3.13) below. Making use of Lemma 3.2 and (3.13) we see that, for any $x \in B_\delta$,

$$|M_\delta(D)f(x)| \lesssim x_1^{-\frac{d-1}{2}} \int_{-2\epsilon_\circ}^{2\epsilon_\circ} x_1^{-j} \left| D_j b_\delta(\tau, \xi') \right|_{-\frac{|\xi'|}{|\tau|}} + |E_\delta, N(x; \tau)| d\tau \lesssim \delta^{\frac{d-1}{2}},$$

and for a fixed large number $N$ it follows that

$$\|M_\delta(D)f\|_{L^q(B_\delta)} \lesssim \delta^{\frac{d-1}{2} + 1}\|B_\delta\|_{\frac{1}{2}} \lesssim \delta^{\frac{d+1}{2} - \frac{d}{q}}.$$  (3.11)

Therefore, from (3.10), (3.11), and the choice of the function $f \in S(\mathbb{R}^d)$ in (3.5), we conclude that

$$\|\mathcal{I}(D)\|_{p\rightarrow q} \geq \frac{\|\mathcal{I}_\delta(D)f\|_{L^q(B_\delta)}}{\|f\|_{L^p(\mathbb{R}^d)}} \gtrsim \|m_\delta(D)f\|_{L^q(B_\delta)} - \|M_\delta(D)f\|_{L^q(B_\delta)} \gtrsim \delta^{\frac{d-1}{2} - \frac{d}{q}},$$

which completes the proof of (2.8). \hfill \Box

Now it remains to prove the following two lemmas.

**Lemma 3.1** Let $\rho \neq 1$ be a positive number and let $\psi$, $a_\delta$ and $b_\delta$ be as in the Proof of (2.8). For a fixed small number $\epsilon_\circ > 0$, the following hold true: For $|\tau| \leq 2\epsilon_\circ$, $|\xi'| \leq \frac{1}{10}$, and $0 < \delta \ll 1$, we have

$$|\partial^\alpha \xi a_\delta(\tau, \xi')| \lesssim \frac{\delta}{\tau^2(\tau + 2\psi(\xi') - 2)^2 + \delta^2}, \quad (3.12)$$

$$|\partial^\alpha \xi b_\delta(\tau, \xi')| \lesssim \frac{\delta}{[\tau(\tau + 2\psi(\xi') - 2) + 1 - \rho]^2 + (\rho\delta)^2} \lesssim \delta. \quad (3.13)$$

**Proof** The case $|\alpha| = 0$ is trivial, so let us consider the case $|\alpha| \geq 1$.

For $2 \leq j \leq d$,

$$\partial_j a(\tau, \xi') = \frac{-4\delta \tau^2(\tau + 2\psi - 2)}{(\tau^2(\tau + 2\psi - 2) + \delta^2)^2} \cdot \partial_j \psi.$$  (3.12)
Hence, the estimate (3.12) with \(|\alpha| = 1\) follows since \(\epsilon_0\) is small and \(|\tau + 2\psi - 2| \approx 1\). For \(2 \leq j, k \leq d\),

\[
\partial_{jk}a(\tau, \xi') = \frac{-4\delta\tau^2(2\partial_j\psi\partial_k\psi + (\tau + 2\psi - 2)\partial_{jk}\psi)}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^2} + \frac{32\delta\tau^4(\tau + 2\psi - 2)^2\partial_j\psi\partial_k\psi}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^3},
\]

and the estimate (3.12) with \(|\alpha| = 2\) follows. Next, for \(2 \leq j, k, l \leq d\),

\[
\partial_{jkl}a(\tau, \xi') = \frac{-4\delta\tau^2(2\partial_j\psi\partial_k\psi + 2\partial_{kl}\psi\partial_j\psi + 2\partial_j\psi\partial_k\psi + (\tau + 2\psi - 2)\partial_{jkl}\psi)}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^2}
\]
\[\quad + \frac{32\delta\tau^4(\tau + 2\psi - 2)(6\partial_j\psi\partial_k\psi\partial_l\psi + (\tau + 2\psi - 2)(\partial_{jkl}\psi\partial_j\psi + \partial_{kl}\psi\partial_j\psi + \partial_{j}\psi\partial_k\partial_l\psi))}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^3} - \frac{384\delta\tau^6(\tau + 2\psi - 2)^3\partial_j\psi\partial_k\psi\partial_l\psi}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^4},
\]

and this gives (3.12) for \(|\alpha| = 3\). Now, an easy induction argument shows that for any \(|\alpha| \geq 1\),

\[
\partial^\alpha_{\xi'}a(\tau, \xi') = \sum_{j=1}^{\lfloor |\alpha| \rfloor} \frac{\delta\tau^2j p_j(\nabla^{|\alpha|}\psi)}{(\tau^2(\tau + 2\psi - 2)^2 + \delta^2)^{j+1}},
\]

where \(\nabla^k\psi := \{\partial^\beta\psi : |\beta| \leq k\}\) for \(k \in \mathbb{N}\) and \(p_j\) is a polynomial with coefficients in \(\mathbb{Z} \cup \{\tau\}\). Therefore, the estimate (3.12) follows.

The first inequality in (3.13) can be proved in the same argument and we omit repetition. The second inequality in (3.13) holds since \(\rho \neq 1\) and

\[
|\tau(\tau + 2\psi(\xi') - 2) + 1 - \rho| \geq |1 - \rho| - 2\epsilon_0(2 + 2\epsilon_0),
\]

which is \(\geq 1\) if \(\epsilon_0\) is small enough depending on \(|1 - \rho|\). \(\square\)

We now invoke the stationary phase method (see [18, Chapter VII]) to obtain the asymptotic for the function \(x \mapsto I_3(x; \tau)\). Since \(\nabla\psi(0) = 0\) and \(\nabla\psi(\xi') = \xi' + O(|\xi'|^3)\), by the inverse function theorem, there exists a unique diffeomorphism \(g\) from the open ball \(B_{d-1}(0, 1/2)\) onto an open set \(U \subset \mathbb{R}^{d-1}\) such that \(g(0) = 0\) and

\[
\eta + \nabla\psi(g(\eta)) = 0.
\]

In fact, \(g\) can be computed explicitly and we have

\[
g(\eta) = \frac{-\eta}{\sqrt{1 + |\eta|^2}} \quad (3.14)
\]

with \(U = B_{d-1}(0, 1/\sqrt{5})\). For each \(\xi' \in \text{supp } \chi\), let us denote by \(K(\xi')\) the Gaussian curvature of the graph (of the unit sphere) \(\mathcal{G}(\psi) := \{\xi \in \mathbb{R}^d : \xi_1 = \psi(\xi') = 1 - \sqrt{1 - |\xi'|^2}, \xi' \in \text{supp } \chi\}\) at point \((\psi(\xi'), \xi')\). Hence, \(|K(\xi')| = 1\). \(\circ\)
The following lemma is now an immediate consequence of [18, Theorem 7.7.5 and Theorem 7.7.6]. Also, see [25, Lemma 2.7].

**Lemma 3.2** Let \( 0 < \delta \leq 1, -1 \leq \tau \leq 1 \), and let \( I_\delta \) be as in the Proof of (2.8). If \(|x_1| \geq 1/2 \) and \( 2^5 |x'| \leq |x_1| \), then for every \( N \in \mathbb{N} \) we have

\[
I_\delta(x_1; \tau) = \sum_{j=0}^{N-1} D_j a_\delta(x_1; \xi') |x_1|^{-d+1-j} + E_\delta, N(x_1; \tau),
\]

where \( c_d \) is a constant depending only on \( d \), \( D_0 a_\delta = a_\delta \) and, for each \( j \geq 1 \), \( D_j \) is a differential operator in \( \xi' \) of order \( 2j \) whose coefficients vary smoothly depending on \( (\partial_\alpha \xi') \circ g(x_1; \tau) \), \( 2 \leq |\alpha| \leq 2j + 2 \). For \( E_\delta, N(x_1; \tau) \) we have the estimate

\[
|E_\delta, N(x_1; \tau)| \leq |x_1|^{-N} \sum_{|\alpha| \leq 2N} \sup_{(\tau, \xi')} |\partial_\alpha^\alpha a_\delta(x_1; \xi')|
\]

(3.15)

with implicit constant independent of \( \delta \).

**4 Failure of Carleman Estimate: Proof of Theorem 1.5**

In this section, we prove Theorem 1.5. Scaling consideration shows that the estimate (1.15) is possible only if

\[
\frac{1}{p} - \frac{1}{q} = \frac{2}{d}.
\]

(4.1)

Hence, by homogeneity, we may assume that \(|v| = 1\) without loss of generality. Furthermore, it is sufficient to consider \( v = e_1 \) only, since the Lamé operator is rotationally symmetric and the estimate (1.15) is invariant under any rotation \( x \rightarrow Rx, R \in \text{SO}(d) \). Now we shall find another necessary condition for (1.15) (with \( v = e_1 \)) which cannot be true under the condition (4.1).

Setting \( f := e^{v \cdot x}(\Delta^*) e^{-v \cdot x} u, f = (f_1, \ldots, f_d)' \) and \( u = (u_1, \ldots, u_d)' \), direct calculation shows that, for \( 1 \leq j \leq d \),

\[
f_j = -\mu \left( \Delta - 2v \cdot \nabla + |v|^2 \right) u_j - (\lambda + \mu) \sum_{k=1}^{d} (\partial_j \partial_k - v_j \partial_k - v_k \partial_j + v_j v_k) u_k.
\]

Taking the Fourier transform we get the following identity:

\[
\hat{f}(\xi) = (\mu((\xi + iv)'(\xi + iv)) I_d + (\lambda + \mu)(\xi + iv)(\xi + iv)' \hat{u}(\xi).
\]
Setting \( \eta := \xi + iv \) we note that the matrix \( M_\eta := \mu(\eta')\eta I_d + (\lambda + \mu)\eta\eta' \) is of the form (2.2) in the proof of Lemma 2.1 by replacing \( \xi \rightarrow \eta \) and \( z \rightarrow 0 \). The inverse \( M_\eta^{-1} \) can be computed without difficulty by the same manner as in the proof of (2.3). Thus, we get

\[
\hat{u}_j(\xi) = \frac{\hat{f}_j(\xi) + \mu(\xi + ivj)(\xi_k + ivk)\hat{f}_k(\xi)}{\mu(\xi + ivj)(\xi + iv)^{\xi + iv2}}.
\]

(4.2)

Let us assume (1.15) and set \( f_k = 0 \) whenever \( k \neq 2 \). Since \( v = e_1 \), we have

\[
\hat{u}_1(\xi) = \frac{1}{\lambda + 2 \mu} \frac{(\xi_1 + iv_2)(\xi_1 + iv_2)}{(|\xi|^2 - 1 + 2i\xi_1)^2},
\]

and the inequality (1.15) implies

\[
\left\| \mathcal{F}^{-1} \frac{(\xi_1 + iv_2\hat{h}(\xi))}{(|\xi|^2 - 1 + 2i\xi_1)^2} \right\|_{L^q(\mathbb{R}^d)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}.
\]

(4.3)

Now we show that (4.3) is possible only if

\[
\frac{1}{p} - \frac{1}{q} \geq \frac{4}{d + 2}.
\]

(4.4)

To see this let us fix nonnegative functions \( \phi \in C^\infty_c((1/2, 2)) \) and \( \psi \in C^\infty_c([0, 2)) \) such that \( \phi = 1 \) on \([2/3, 3/2]\) and \( \psi = 1 \) on \([0, 1]\). Then for every \( \delta > 0 \) small enough, let us define \( h_\delta \in S(\mathbb{R}^d) \) by

\[
\hat{h}_\delta(\xi) = \phi \left( \frac{\xi_1}{\delta} \right) \phi \left( \frac{\xi_2 - 1}{\delta} \right) \psi \left( \frac{\xi_1}{\sqrt{\delta}} \right),
\]

where \( \xi := (\xi_3, \ldots, \xi_d) \in \mathbb{R}^{d-2} \). Note that on the support of \( \hat{h}_\delta \) we have

\[
||\xi||^2 - 1 + 2i\xi_1^2 = (\xi_1^2 + (\xi_2 + 1)(\xi_2 - 1) + |\xi|^2)^2 + 4\xi_1^2 \leq (4\delta^2 + 2\delta(2 + 2\delta) + 4\delta)^2 + 16\delta^2 \lesssim \delta^2.
\]

If we define the set \( A_\delta \) by

\[
A_\delta := \{ x \in \mathbb{R}^d : |x_1| \leq (100\delta)^{-1}, |x_2| \leq (100\delta)^{-1}, |\vec{x}| \leq (100\sqrt{\delta})^{-1} \},
\]

then the estimate (4.3) with \( h = h_\delta \) yields

\[
|A_\delta| \lesssim \|h_\delta\|_{L^p(\mathbb{R}^d)}
\]
with the implicit constant independent of \( \delta \ll 1 \). This implies \( \delta^{d - 2} \delta^{-\frac{d + 2}{2q}} \lesssim \delta^{\frac{d - 2}{2}} \delta^{-\frac{d + 2}{2p}} \) or, equivalently, \( \delta^{-2 + \frac{d + 2}{2} (\frac{1}{p} - \frac{1}{q})} \lesssim 1 \) which implies (4.4). We have shown that the estimate (1.15) implies both (4.1) and (4.4). Combining these results we conclude that (1.15) is impossible other than the case \((d, p, q) = (2, 1, \infty)\).

It remains to show the failure of (4.3) with \((d, p, q) = (2, 1, \infty)\). Let \( \nu = (1, 0) \) and let \((f_1, f_2) = (0, h_{\epsilon})\), where \(h_{\epsilon}\) is the function defined in (2.11). Then (4.2) gives

\[
(2\pi)^2 u_2(0) = \frac{1}{\mu} \int_{\mathbb{R}^2} |\xi|^2 + 2i\xi_1 - 1 \, d\xi + \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) \int_{\mathbb{R}^2} \frac{\xi_1^2 \hat{h}_\epsilon(\xi)}{|\xi|^2 + 2i\xi_1 - 1} d\xi.
\]

Since we want to disprove the \(L^1 - L^\infty\) bound and \(\{h_{\epsilon} : \epsilon > 0\}\) is uniformly bounded in \(L^1(\mathbb{R}^2)\) (see (2.14)), as in Proof of Proposition 1.2, it is sufficient to show that \(\text{Re} \, u_2(0) \to \infty\) as \(\epsilon \to 0\).

We note that

\[
(2\pi)^2 \text{Re} \, u_2(0) = \frac{1}{\mu} \int (\xi_1^2 - 1)[((|\xi|^2 - 1)^2 - (2\xi_1)^2)] + 8\xi_1^2 (|\xi|^2 - 1) \hat{h}_\epsilon(\xi) d\xi + \frac{1}{\lambda + 2\mu} \int \frac{\xi_1^2 ((|\xi|^2 - 1)^2 - (2\xi_1)^2)}{|(\xi|^2 - 1^2) + (2\xi_1)^2} \hat{h}_\epsilon(\xi) d\xi,
\]

which we rearrange as follows:

\[
\frac{1}{\mu} \int \frac{\xi_1^2 ((|\xi|^2 - 1)^2 - (2\xi_1)^2)}{|(\xi|^2 - 1^2) + (2\xi_1)^2} \hat{h}_\epsilon(\xi) d\xi + \frac{1}{\lambda + 2\mu} \int \frac{\xi_1^2 ((|\xi|^2 - 1)^2 - (2\xi_1)^2)}{|(\xi|^2 - 1^2) + (2\xi_1)^2} \hat{h}_\epsilon(\xi) d\xi
\]

\[
+ \frac{1}{\mu} \int \frac{8\xi_1^2 (|\xi|^2 - 1) - ((|\xi|^2 - 1)^2 - (2\xi_1)^2)}{|(\xi|^2 - 1^2) + (2\xi_1)^2} \hat{h}_\epsilon(\xi) d\xi.
\]

Easy computations show that for \(|\xi| \geq 4\),

\[
|\xi|^4 / 2 \leq (|\xi|^2 - 1)^2 - (2\xi_1)^2 \leq |\xi|^4,
\]

(4.5)

\[
|\xi|^4 / 2 \leq (|\xi|^2 - 1)^2 + (2\xi_1)^2 \leq 2|\xi|^4,
\]

(4.6)

and

\[
0 \leq 8\xi_1^2 (|\xi|^2 - 1) \leq 8|\xi|^4.
\]

(4.7)

Let \(m = \min\{\frac{1}{\mu}, \frac{1}{\lambda + 2\mu}\}\). Since \(\hat{h}_\epsilon\) is non-negative and satisfies (2.12) and (2.13), it follows from the triangle inequality and the estimates (4.5), (4.6), and (4.7) that if
\( \epsilon \leq 1/8 \) then

\[
|(2\pi)^2 \text{Re} u_2(0)| \geq m \int \frac{|\xi|^2[(|\xi|^2 - 1)^2 - (2\xi_1)^2]}{|(|\xi|^2 - 1)^2 + (2\xi_1)^2|^2} \hat{h}_\epsilon(\xi) d\xi
- \frac{1}{\mu} \int \frac{8\xi_1^2(|\xi|^2 - 1) - [|(|\xi|^2 - 1)^2 - (2\xi_1)^2]|}{|(|\xi|^2 - 1)^2 + (2\xi_1)^2|^2} \hat{h}_\epsilon(\xi) d\xi
\geq \int_{1/\epsilon \leq |\xi| \leq 1/2\epsilon^2} |\xi|^{-2} d\xi - \int_{1/2\epsilon \leq |\xi| \leq 1/\epsilon^2} |\xi|^{-4} d\xi
\geq \log \frac{1}{\epsilon} - \epsilon^2,
\]

which diverges to infinity as \( \epsilon \to 0 \).

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