Infrared behaviour of the fermion propagator in unquenched QED$_3$ with finite threshold effects

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Abstract

To remove the linear infrared divergences in quenched approximation we include the massive fermion loop to the photon spectral function. Spectral function of fermion has no one particle singularity if we fix the anomalous dimension to be unity. In the case of $N$ flavour, $N$ dependence of order parameter is mild which may be due to screening effects.
I. INTRODUCTION

To study the infrared behavior of the propagator in QED_{2+1}, we evaluated the spectral function which is known as the Bloch-Nordsieck approximation in four dimension. In quenched case linear and logarithmic infrared divergences appear in the order \( e^2 \) spectral function \( F \). Exponentiation of \( F \) yields full propagator with all order of infrared divergences. In four dimension anomalous dimension modifies the short distance singularity at least for weak coupling, which leads cut structure near the mass shell. For fixed infrared cut-off we expand the function \( F \) and find a mass shift, its log correction and anomalous dimension of wave function. To avoid the infrared divergences we choose the gauge \( d = -1 \) and include effects of massive fermion loops to the photon spectral function. The results have no infrared divergence but the effects of finite threshold seems to make oscillation of
the propagator. If we set the anomalous dimension to be unity short distance singularity disappears, chiral symmetry breaks dynamically, and the spectral function has \(-\delta'(p^2/m^2 - 1)\) like singularity which shows the absence of one particle state and vanishment of \(Z_2^{-1}\).

II. SPECTRAL REPRESENTATION OF THE PROPAGATOR

A. Fermion

In this section we show how to evaluate the fermion propagator non perturbatively by the spectral representation which preserves unitarity and analyticity [1,2,3]. Assuming parity conservation we adopt 4-component spinor. The spectral function of the fermion in (2+1) dimension is defined

\[
\langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle = i \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} \int_0^\infty ds \frac{\gamma \cdot p \rho_1(s) + \rho_2(s)}{p^2 - s + i\epsilon},
\]

\[
\rho(p) = -\frac{1}{\pi} \text{Im} S_F(p) = \gamma \cdot p \rho_1(p) + \rho_2(p)
\]

\[
= (2\pi)^2 \sum_n \delta^{(3)}(p - p_n) \langle 0 | \psi(x) | n \rangle \langle n | \bar{\psi}(0) | 0 \rangle.
\]

In the quenched approximation the state \(|n>\) stands for a fermion and arbitrary numbers of photons,

\[
|n> = |r; k_1, ..., k_n >, r^2 = m^2.
\]

In deriving the matrix element \(\langle 0 | \psi(x) | n \rangle\) we must take into account the soft photon emission vertex which is written in the textbook for the scattering of charged particle by external electromagnetic field or collision of charged particles. Based on low-energy theorem the most singular contribution for the matrix element \(T_n = \langle \Omega | \psi | r; k_1, ..., k_n \rangle\) is known as the soft photons attached to external line. Ward-Takahashi identity

\[
T_n = \epsilon^n \mu T^n\mu,
\]

\[
k_\mu^n T_\mu^n(r; k_1, ..., k_n) = \epsilon T_{n-1}(r; k_1, ..., k_{n-1}), k_\mu^{n2} \neq 0
\]

is proved to be satisfied with the use of LSZ reduction formula[2]. We have an approximate solution of (4)

\[
T_n|_{k_\mu^n = 0} = \epsilon \frac{\gamma \cdot \epsilon}{\gamma \cdot (r + k_n) - m} T_{n-1}.
\]
From this relation the \( n \)-photon matrix element is replaced by the products of \( T \)
\[
T_n T_n^+ \gamma_0 \rightarrow \prod_{j=1}^{n} T_1(k_j) T_1^+(k_j) \gamma_0. \tag{6}
\]

By one-photon matrix element[4]
\[
T_1 = -ie \gamma \cdot (r + k) + m \frac{1}{(r + k)^2 - m^2 + i\epsilon} \gamma_\mu \epsilon_\lambda^\mu(k) U_S(r), \tag{7}
\]
and the function
\[
F = \int \frac{d^3k}{(2\pi)^2} \exp(i k \cdot x) \theta(k_0) \delta(k^2) \sum_{\lambda, S} T_1 T_1^+,
\]
where polarization sum we have
\[
\Pi^{\mu\nu}(k) = \sum_\lambda \epsilon_\lambda^\mu(k) \epsilon_\lambda^\nu(k) = -(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}) - \frac{d}{k^2}.
\tag{8}
\]
The infinite sum \( \sum_{n=0} T_n T_n^+ \gamma_0/n! \) leads an \( \exp(F) \). In this way we obtain a dressed fermion propagator with soft photon
\[
S_F(x) = -(i\gamma \cdot \partial + 1) \int \frac{md^2r}{(2\pi)^2 \sqrt{r^2 + m^2}} \exp( ir \cdot x ) \exp(F),
\tag{10}
\]
where \( F \) is known as model independent
\[
F = -\frac{e^2}{2} \left( \frac{\gamma \cdot r + m}{m} \right) \int \frac{d^3k}{(2\pi)^2} \exp(i k \cdot x) \theta(k_0) \delta(k^2) \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{1}{k^2} \right],
\tag{11}
\]
here we used covariant \( d \) gauge photon propagator and \( \delta(k^2) \) is read as the imaginary part of the free photon propagator at on shell. In our approximation two kinds of spectral function satisfy \( \rho_2 = m\rho_1 \).

**B. Photon**

For unquenched case we use the dressed photon with massive fermion loop with \( N \) flavours. Spectral functions for dressed photon are given by vacuum polarization [5,6]
\[
\Pi^{\mu\nu}(k) \equiv ie \int \frac{d^3p}{(2\pi)^3} \text{Tr} \left( \gamma_\mu \frac{1}{p - m + \gamma_\nu \gamma \cdot (p - k) - m} \right)
= -\frac{e^2}{8\pi} \frac{T^{\mu\nu}_{\mu\nu}}{(\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}}) \ln(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}})} - 4m, \tag{12}
\]
\[
T^{\mu\nu}_{\mu\nu} = (g^{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k^2}), \quad d^3p = \frac{d^3p}{(2\pi)^3},
\]
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\( D^{-1}_{\mu\nu}(k) \equiv -(T_{\mu\nu}k^2 + d\kappa_k\kappa_\nu) + \Pi_{\mu\nu}(k). \) \hfill (13)

Polarization function \( \Pi(k) \) is
\[
\Pi(k) = -\frac{e^2}{8\pi}[(\sqrt{-k^2} + 4m^2)\ln(2m + \sqrt{-k^2}) - 4m],
\]
\[
= -\frac{e^2}{8}i\sqrt{-k^2}(-k^2 > 0, m = 0),
\] \hfill (14)
\[
= \frac{e^2}{6\pi m} + O(k^4)(-k^2/m \ll 1).
\] \hfill (15)

Fermion mass is assumed to be generated dynamically. In quenched case it is shown that \( m \) is proportional to \( e^2 \). For massless case or high-energy limit we have for number of \( N \) fermion flavour
\[
\rho^D_\gamma(k) = \frac{1}{\pi} \Im D_F(k) = \frac{c\sqrt{k^2}}{k^2(c^2 + c^2)}, c = \frac{e^2N}{8},
\] \hfill (16)
\[
D_{\mu\nu}(k) = -T_{\mu\nu} \int_0^\infty \frac{\rho_\gamma(\mu^2)d\mu^2}{k^2 - \mu^2 + i\epsilon} - \frac{\mu\kappa_k\kappa_\nu}{(k^2 - i\epsilon)^2}. \]
\hfill (17)

If we include finite threshold effects of massive fermion pair we have
\[
\rho^F_\gamma(k) = \frac{1}{\pi} \Im D_F(k) = \delta(k^2) + \frac{1}{\pi} \Im \left( \frac{1}{(-k^2 - \Pi(-k^2))} \right) \theta(-k^2 - 4m^2).
\] \hfill (18)

III. ANALYSIS IN Position SPACE

To evaluate the function \( F \) it is helpful to use the exponential cut-off (infrared cut-off) [2,3]. Using the bare photon propagator with bare mass \( \mu \); \( D_F^{(0)}(x) = \exp(-\mu|x|)/8\pi i|x| \). The function \( F \) is written in the following form
\[
F = ie^2m^2 \int_0^\infty d\alpha D_F(x + \alpha r) - e^2 \int_0^\infty d\alpha D_F(x + \alpha r) - i(d - 1)e^2 \frac{\partial}{\partial \mu^2} D_F(x, \mu^2). \] \hfill (19)

The above formulae are derived by the parameter tric
\[
\lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(-\alpha(\epsilon - ik \cdot r)) = \frac{i}{k \cdot r},
\]
\[
\lim_{\epsilon \to 0} \int_0^\infty d\alpha \exp(-\alpha(\epsilon - ik \cdot r)) = -\frac{1}{(k \cdot r)^2}. \] \hfill (20)

In this case we have
\[
F = \frac{e^2}{8\pi} \frac{\exp(-\mu|x|) - \mu|x|E_1(\mu|x|)}{\mu} - \frac{E_1(\mu|x|)}{m}
\]
\[
+ \frac{(d - 1)\exp(-\mu|x|)}{2\mu}, |x| = \sqrt{x^2}
\] \hfill (21)
where $\mu$ is a bare photon mass. Short distance behaviour of $F$ has the following form
\[
F \sim \frac{e^2(1 + d)}{16\pi\mu} + \frac{e^2}{8\pi m}(\gamma + (1 + m|x|)\ln(\mu |x|)) - \frac{(d + 1 - 2\gamma)e^2|x|}{16\pi}, (\mu |x| \ll 1). \tag{22}
\]

Long distance behaviour is given by the asymptotic expansion of $E_1(\mu |x|)$
\[
E_1(z) \sim \exp(-\frac{z}{2})(1 + \frac{z}{2} + \frac{z^2}{2} + \frac{z^3}{6} + \ldots), (|\text{arg } z| < \frac{3\pi}{2}), \tag{23}
\]
\[
F \sim -\frac{e^2}{8\pi}\frac{\exp(-\mu |x|)}{\mu^2|x|} + \frac{\exp(-\mu |x|)}{m\mu |x|} + \frac{\exp(\mu |x|)}{2\mu}, (\mu |x| \gg 1). \tag{24}
\]
where $\gamma$ is an Euler constant. In (20) linear term in $|x|$ is understood as the finite mass shift from the form of the propagator in position space (9,25) and $|x|\ln(\mu |x|)$ term is position dependent mass
\[
m = m_0 + \frac{e^2}{16\pi}(1 + d - 2\gamma) + \frac{e^2}{8\pi m}\mu, \tag{25}
\]
\[
m(x) = m - \frac{e^2}{8\pi}\ln(\mu |x|), \tag{26}
\]
which has mass changing effects at short distance and it will be discussed in section 4. These mass terms has different gauge dependence from that obtained by self-energy in the Dyson-Schwinger equation.

Here we notice that the propagator in position space can be written
\[
S_F(x) = -(i\gamma \cdot \partial + m)\frac{1}{4\pi|x|}\left(\begin{array}{c}
\exp(-m|x|)A(\mu |x|) e^{D+C|x|}(\mu |x| \ll 1) \\
\exp(-m|x|) \exp(\mu |x|)(\mu |x| \gg 1)
\end{array}\right),
\]
\[
= i\gamma \cdot \partial S_V(x) + S_S(x), \tag{27}
\]
where
\[
A = \exp\left(\frac{e^2(1 + d)}{16\pi\mu} + \frac{e^2\gamma}{8\pi m}\right), C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}. \tag{28}
\]

For the finitenes of the value $S_F(0)$ we imply $D = 1$. In this case the physical mass equals to $m = e^2/8\pi$ and fermion may be confined for finite $\mu$. Here we apply the spectral function of photon to evaluate the unquenched fermion propagator. We simply integrate the function $F(x, \mu)$ for quenched case which is given in (19), where $\mu$ is a photon mass. Spectral function of photon with massless fermion loop is given in (13) and we have
\[
\rho_\gamma^P(\mu) = \frac{c}{\pi\mu(\mu^2 + e^2)}, Z_\gamma^{-1} = \int_0^\infty 2\rho_\gamma^P(\mu)\mu d\mu = 1. \tag{29}
\]
In this case the spectral function of the fermion is given
\[
\tilde{\rho}(x) = \int_0^\infty 2\rho_\gamma^P(\mu) \exp(F(x,\mu))d\mu. \tag{30}
\]
In this way the short distance fermion propagator with \( N \) flavours is modified in the gauge \( d = -1 \)

\[
S_F(x) = -(i\gamma \cdot \partial + m) \frac{\exp(-m|x|)}{4\pi |x|} \tilde{\rho}(x).
\]

(31)

For the case of \( N \) fermion flavour we assume the physical mass \( m \) as \( m = c/N\pi \). In the whole region of \( |x| \) we evaluate \( \tilde{\rho}(x) \) numerically including finite threshold effect for the photon spectral function with massive fermion loop

\[
\tilde{\rho}(x) = \int_{4m^2}^{\infty} \exp(F(s, x))\rho^F(s)ds.
\]

(32)

The renormalization constant \( Z_3^{-1} \) is defined as the residue of pole

\[
Z_3^{-1} = 1 + \int_{4m^2}^{\infty} \rho^F(s)ds.
\]

(33)

We set the mass \( m = c/N\pi \) and see the \( N \) dependence of \( Z_3^{-1} \) for weak coupling \( c = 1/8 \) in Fig.1. In Fig.2 we see the screening effect leads infrared finite spectral function with dynamical fermion mass \( m = c/N\pi \) from \( N = 1 \) to \( 4 \) in the gauge \( d = -1, c = 1/8 \). In comparison with quenched case with finite cut-off \( \mu \) these are reduced from unity by screening effects at large distance.

Fig.1 \( Z_3^{-1} - 1 \) for \( N = 1/2..10, c = 1/8 \). \( c = 1/8, N = 1(bottom), 2, 3, 4(top) \).
IV. ANALYSES IN MOMENTUM SPACE

A. structure in Euclid space

Now we turn to the fermion propagator in momentum space. The momentum space propagator at short distance is given by Fourier transform

$$ S_F(p) = - \int d^3x \exp(-ip \cdot x)(i\gamma \cdot \partial + m) \frac{\exp(-m|x|)}{4\pi |x|} \exp(F(x)), $$

where

$$ \exp(F(x)) = A \exp(D\gamma (\mu |x|)^D + C|x|), D = \frac{c}{N\pi m}, C = \frac{c}{N}. $$

First we examine the perturbative effects by expanding $F$ in powers of $e^2$ for quenched case at short distance

$$ \exp(F) = 1 + F(x) = 1 + A - B|x| + C|x| \ln(\mu |x|) + D \ln(\mu |x|), $$

$$ A = \exp(e^2(1 + d) \frac{1}{16\pi \mu} + e^2 \frac{\gamma}{8\pi m}), B = (d + 1 - 2\gamma) \frac{e^2}{16\pi}, C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}. $$

Terms proportional to $B, C$ represents dynamical mass generation correctly. On the other hand it is known that

$$ \int d^3x \exp(-ip \cdot x) \frac{\exp(-m|x|)}{4\pi |x|} (\mu |x|)^D = \frac{2m\mu}{(p^2 + m^2)^2} \text{ for } D = 1. $$

for Euclidean momentum $p^2 \geq 0$. However above formulæ are restricted for small $|x|$, we may include the finite range effect and test the high energy behaviour for small $\mu$

$$ \int_0^{1/\mu} \frac{\exp(-m|x|)(\mu|x|)}{|x|} \frac{\sin(p|x|)}{p|x|} x^2 dx $$

$$ \rightarrow -\frac{\mu \exp(-m/\mu) \cos(p/\mu)}{p^2} + \frac{\mu \exp(-m/\mu) \sin(p/\mu)(m - \mu)}{p^3} + O(1/p^4). $$

Above formula shows the oscillation and the propagator does not dump as $1/p^4$. For large $N$ this oscillation effects becomes small as $\mu \rightarrow 0$. Numerical solutions of the scalar part of the propagator $S_S(p)$ with various $N$ are shown in Fig.3-1, Fig.3-2, for $c = 1/8, 1, N = 1, 2, 3$ respectively. The boundary condition at $p = 0$ is not specified in our approximation. For weak coupling $c = 1/8$ and $c = 1$ zero momentum mass for $N = 1$ is the largest among them, which has been seen in the Dyson-Schwinger analysis[8].
B. structure in Minkowski space

Now we derive the propagator in Minkowski momentum region. To do this it is helpful in real time to derive the spectral function [1,9]. In Minkowski space we change the variable from $r = \sqrt{x^2}$ to $iT$. We may define

$$\rho(s^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i(s^2/m^2 - 1)T) \text{Im}(\exp(\tilde{F}(iT)))dT,$$

which is normalized to $\delta(s^2/m^2 - 1)$ for free case where

$$\exp(\tilde{F}(iT)) = \int_{4m^2}^{\infty} d\sigma \rho^F(\sigma) \exp(F(iT, \sqrt{\sigma})).$$

(39)

It is understood that the Fourier transformation is performed in $s^2$ by the following form

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dT \exp(-i(m - s)T) + \exp(-i(m + s)T) = 2|s|\delta(s^2 - m^2).$$

(40)

Taking the imaginary part of the function $\exp(F(\mu, iT))$ at short distance in the gauge $d = -1$ for quenched case for $D = 1$

$$\text{Im} \exp(\tilde{F}(iT)) \approx \exp(-\frac{\pi}{2}CT)\mu T \cos(CT \ln(\mu T)),$$

(41)
provided

\[(i\mu T)^{iT+D} = \exp\left(\frac{\pi}{2}(-CT + Di)(\mu T)^D(\cos(CT \ln(\mu T)) + i \sin(CT \ln(\mu T)))\right).\] (42)

In this way we see the oscillation of the propagator in Minkowski space by the effects of position dependent mass as \(m = iCT \ln(iCT)\). In Fig.4 we see the profile of real and imaginary part of the function \(\exp(F(iT))\) for \(c = 1, N = 1\).

\[\text{Fig.4 } \Re(\exp(\tilde{F}(iT))), \Im(\exp(\tilde{F}(iT))_. \text{ Fig.5 } \rho(s), N = 1(bottom), 2, 3(top).\]

In Fig.5 the spectral function \(\rho(s)\) in unit of \(e^2\) looks symmetric about the point \(s = 1\). The function \(\exp(F(iT))\) is approximated as \(i\mu T\) for small \(T\). This fact suggests the singularity of \(\rho(s)\) such as the \(-\delta'(s - 1)\).

V. RENORMALIZATION CONSTANT AND ORDER PARAMETER

In this section we consider the renormalization constant in our model. It is easy to evaluate the renormalization constant and bare mass which are defined by assuming multiplicative renormalization

\[Z_2^{-1} = \frac{\gamma \cdot p - m_0}{S_F(p)}.\] (43)

\[Z_2^{-1} = \lim_{p \to \infty} \frac{1}{4} tr(\gamma \cdot p S_F(p))\]

\[= \lim_{p \to \infty} \int \frac{p^2 \rho(s)ds}{p^2 - s + i\epsilon} = \int \rho(s)ds,\] (44)
where $Z_2$ is defined for one particle state in the initial and final state in a weak sense

$$
\psi(x)_{t \rightarrow +\infty, -\infty} \rightarrow \sqrt{Z_2} \psi(x)_{\text{out,in}}
$$

$$
Z_2(2\pi)^{-2} \theta(p_0) \delta(p^2 - m^2) \equiv | \langle p| \psi(0)| 0 \rangle |^2 \frac{1}{p^2 = m^2}.
$$

(45)

However pole part is absent in our approximation and $Z_2$ vanishes for $D > 0$ case. For $D = 1$ we have

$$
Z_2^{-1} = \int_{0}^{\infty} ds \int_{-\infty}^{\infty} dt \frac{dt}{2\pi} \exp(-i(s - 1)t) \Im(\exp(\tilde{F}(it)))
$$

$$
= \int_{-\infty}^{\infty} dt \delta_+(t) \Im(\exp(\tilde{F}(it))) = 0,
$$

(46)

provided $\Im(\exp(\tilde{F}(it))) = 0$ in the limit $t_+ \rightarrow 0$ by eq(38). Order parameter for each flavour $\langle \bar{\psi}\psi \rangle$ is given

$$
\langle \bar{\psi}\psi \rangle = -tr S_F(x).
$$

(47)

The value of $\langle \bar{\psi}\psi \rangle$ is $3.1(1.5, 1.0) \times 10^{-3} e^4$ for $N = 1(2, 3)$ in $1/N$ which may be compared with $1.2(0.13, 0.0002) \times 10^{-3} e^4$ for $N = 1(2, 3)$ in the CP vertex with massless loop correction [8]. In our approximation $N$ dependence is mild. This may be understood as the large screening effect at small $N$. Recently we solved the Bethe-Salpeter equation in the ladder approximation for axial-scalar which corresponds the pseudoscalar in four dimension. For massless boundstate solution we find the $D = e^2/4\pi m = 1$ from the normalization condition at short distance [9].

VI. SUMMARY

We evaluated the fermion propagator in three dimensional QED with dressed photon by the dispersion method. In the evaluation of lowest order matrix element for fermion spectral function we obtain finite mass shift, wave function renormalization and gauge invariant position dependent mass with bare photon mass $\mu$ as an infrared cut-off. To remove linear infrared divergences we include finite threshold effects of massive fermion loop to the photon spectral function. We set the coupling constant mass ratio $c/N\pi m$ to be unity which is consistent with perturbative analysis of the mass at high energy which is proportional to $1/p^4$. In this case the order parameter $\langle \bar{\psi}\psi \rangle$ becomes finite. In our approximation vacuum expectation value and infrared mass are not sensitive to flavour number $N$ which may due
to the screening effects of $Z^{-1}_3$ for small $N$. In Minkowski space, the spectral function is not positive definite and has a form such as $-\delta'(p^2/m^2 - 1)$. In the strong coupling there seems to be no particle content in the model. In our analysis chiral symmetry breaks dynamically by anomalous dimension. Our results suggest the importance of anomalous dimension to understand confinement.

VII. AKNOWLEDGEMENT

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