On Fractional Tempered Stable Motion

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Abstract

Fractional tempered stable motion (fTSm) is defined and studied. FTSm has the same covariance structure as fractional Brownian motion, while having tails heavier than Gaussian but lighter than stable. Moreover, in short time it is close to fractional stable Lévy motion, while it is approximately fractional Brownian motion in long time. A series representation of fTSm is derived and used for simulation and to study some of its sample path properties.

1 Introduction

Fractional Brownian motion (fBm) and its various extensions are not only rich mathematical objects but have also been extensively used in application to model asset price dynamics, data traffic in telecommunication network, daily hydrological series, and turbulence, to mention but a few topics. We recall that standard fBm \( \{B^H_t : t \in \mathbb{R}\} \), \( H \in (0, 1] \), is a centered Gaussian process with continuous paths and with the following covariance structure:

\[
\text{Cov}(B^H_t, B^H_s) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad t, s \in \mathbb{R}.
\]

However, modeling drawbacks of fBm and of some of its extensions have also been discussed in the literature. For example, although Gaussianity provides analytical tractability, its light tails are often inadequate for modeling higher variability observed in various natural phenomena. On the other hand, non-Gaussian stable generalizations immediately lead to infinite second moment and to a lack of closed form for the density, resulting...
in significant analytical difficulties. Moreover, the selfsimilar and stationary increments properties of fBm are sometimes unrealistic in practical modeling.

In order to remove these drawbacks, we introduce and study fractional tempered stable motion (fTSm), which has the following properties:

(i) Its marginals have tails heavier than Gaussian but lighter than (non-Gaussian) stable (Proposition 2.5).
(ii) It has the same second order structure as fBm (Proposition 3.1).
(iii) In long time it behaves like fBm, while in short time it is more akin to fractional stable motions. (Theorem 6.4).

We present its series representation (Proposition 4.1), which is potentially useful for simulation and which is also used to study sample path properties. When \(H \in (1/\alpha, 1/\alpha + 1/2)\), it has a.s. Hölder continuous sample paths with exponent \((0, H - 1/\alpha)\) (Proposition 5.2) and is not a semimartingale (Proposition 5.3). In contrast, the sample paths of fTSm become nowhere bounded as soon as \(H < 1/\alpha\) (Proposition 5.1).

Let us close this section by introducing some notations and definitions which will be used throughout the text. \(\mathbb{R}^d\) is the \(d\)-dimensional Euclidean space with the norm \(\| \cdot \|\). \(\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}\), and \(\mathcal{B}(\mathbb{R}_0^d)\) is the Borel \(\sigma\)-field of \(\mathbb{R}_0^d\). \((\Omega, \mathcal{F}, \mathbb{P})\) is our underlying probability space. \(\mathcal{L}(Y)\) is the law of the random vector \(Y\), while \(\overset{d}{=}\) denotes equality in law, or equality of the finite dimensional distributions when stochastic processes are considered. Similarly, \(\overset{L}{\longrightarrow}\) is used for convergence in law, or of the finite dimensional distributions, while \(\overset{d}{\longrightarrow}\) denotes the weak convergence of stochastic processes in the space \(D([0, \infty), \mathbb{R})\) of càdlàg functions from \([0, \infty)\) into \(\mathbb{R}\) equipped with the Skorohod topology. \(C([0, \infty), \mathbb{R})\) is the space of continuous functions from \([0, \infty)\) to \(\mathbb{R}\) endowed with the uniform metric. A sequence of stochastic processes \(\{X^n_t : t \geq 0\}_{n \in \mathbb{N}}\) in \(C([0, \infty), \mathbb{R})\) is said to be tight if for each compact set \(K \subset [0, \infty)\) and each \(\epsilon > 0\),

\[
\lim_{n \to \infty} \limsup_{\delta \to 0} \mathbb{P} \left( \sup_{t, s \in K, |t-s| \leq \delta} |X^n_t - X^n_s| > \epsilon \right) = 0.
\]

A sequence of stochastic processes \(\{X^n_t : t \geq 0\}_{n \in \mathbb{N}}\) is said to converge uniformly on compacts in probability (ucp) to a stochastic process \(\{X_t : t \geq 0\}\) if for each compact set \(K \subset [0, \infty)\) and each \(\epsilon > 0\),

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in K} |X^n_t - X_t| > \epsilon \right) = 0.
\]

This last convergence will be denoted by \(\overset{\text{ucp}}{\longrightarrow}\) \(X^n \overset{\text{ucp}}{\longrightarrow} X\). Finally, \(\ln^+ a = \ln a\) if \(a \geq 1\), \(\ln^+ a = 0\) otherwise.

2 Definition of Fractional Tempered Stable Motion

We begin by briefly reviewing tempered stable distributions and processes (Rosinski [14]) which are building blocks for the fTSm defined below. Let \(\mu\) be an infinitely divisible
probability measure, without Gaussian component, on $\mathbb{R}^d$. Then, $\mu$ is called tempered stable if its Lévy measure has the form

$$
\nu(B) = \int_{\mathbb{R}_0^d} 1_B(sx)s^{-\alpha}e^{-s}ds\rho(dx), \quad B \in \mathcal{B}(\mathbb{R}_0^d),
$$

where $\alpha \in (0, 2)$ and where $\rho$, the inner measure, is such that

$$
\int_{\mathbb{R}_0^d} \|x\|^\alpha \rho(dx) < \infty. \quad (2)
$$

The two parameters $\alpha$ and $\rho$ above uniquely identify the Lévy measure of tempered stable distributions. Under the additional condition

$$
\begin{cases}
\int_{\mathbb{R}_0^d} \|x\|\rho(dx) < \infty, & \text{if } \alpha \in (0, 1), \\
\int_{\mathbb{R}_0^d} \|x\|(1 + \ln^+ \|x\|)\rho(dx) < \infty, & \text{if } \alpha = 1,
\end{cases}
$$

the characteristic function of $\mu$ has a closed form expression given by

$$
\hat{\mu}(y) = \exp \left[ i\langle y, b \rangle + \int_{\mathbb{R}_0^d} \phi_\alpha(\langle y, x \rangle)\rho(dx) \right], \quad (3)
$$

for some $b \in \mathbb{R}^d$ and where, for $s \in \mathbb{R}$,

$$
\phi_\alpha(s) = \begin{cases}
\Gamma(-\alpha)((1 - is)^\alpha - 1 + i\alpha s), & \text{if } \alpha \in (0, 1) \cup (1, 2), \\
(1 - is)\ln(1 - is) + is, & \text{if } \alpha = 1.
\end{cases} \quad (4)
$$

Below, we write $\mu \sim TS(\alpha, \rho; b)$ if $\hat{\mu}$ is given by (3) and denote by $\{X_{t}^{TS} : t \geq 0\}$ a tempered stable Lévy process in $\mathbb{R}$ such that $X_{t}^{TS} \sim TS(\alpha, \rho; b)$. Setting $b = 0$ gives $\mathbb{E}[X_{t}^{TS}] = 0$ for every $t \geq 0$, and then $\{X_{t}^{TS} : t \geq 0\}$ is a martingale.

From now on, we always assume that for any $t > 0$,

$$
\mathbb{E}[X_{t}^{TS}] = 0,
$$

and further that

$$
\int_{\mathbb{R}_0} |x|^2 \rho(dx) < \infty, \quad (5)
$$

so that for any $t > 0$, $\mathbb{E}[(X_{t}^{TS})^2] < +\infty$.

We will define $fTS_m$ as a process of stochastic integral with respect to the tempered stable process, i.e., $\{\int_{S} f(t, s)dX_{s}^{TS} : t \in \mathcal{T}\}$, where $f : \mathcal{T} \times \mathcal{S} \rightarrow \mathbb{R}$ is a deterministic function. Various such representations of fBm has been introduced in the literature. The moving-average representation (Mandelbrot and Van Ness [7]) and the harmonizable representation are the most commonly used. These have also been extended to non-Gaussian stable marginals. (See Chapter 7 of Samorodnitsky and Taqqu [16].) A lesser
known representation of fBm involves a Volterra kernel and is due to Decreusefond and Üstünel [23]. Recall that a Volterra kernel is a function $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $K(t, s) = 0$ for $s > t$. In the present paper, we will use the Volterra kernel $K_{H,\alpha} : [0, \infty) \times [0, \infty) \to [0, \infty)$, given by

$$K_{H,\alpha}(t, s) := c_{H,\alpha} \left[ \left( \frac{t}{s} \right)^{H-1/\alpha} (t-s)^{H-1/\alpha} ight.$$ \n
$$- \left( H - \frac{1}{\alpha} \right) s^{1/\alpha-H} \int_s^t u^{H-1/\alpha-1}(u-s)^{H-1/\alpha-1}du \right] 1_{[0,t]}(s),$$

where $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)\setminus(0,2)$, and

$$c_{H,\alpha} = \left( \frac{G(1-2G)\Gamma(1/2-G)}{\Gamma(2-2G)\Gamma(G+1/2)} \right)^{1/2},$$

with (throughout) $G := H - 1/\alpha + 1/2$. Clearly, $K_{1/\alpha,\alpha}(t, s) = 1_{[0,t]}(s)$. Note that when $H \in (1/\alpha, 1/\alpha + 1/2)$, we also have

$$K_{H,\alpha}(t, s) = c_{H,\alpha}(H-1/\alpha)s^{1/\alpha-H} \int_s^t (u-s)^{H-1/\alpha-1}u^{H-1/\alpha}du 1_{[0,t]}(s).$$

Despite its complex structure, there are two main advantages to using this kernel as an integrand:

(i) It is defined only on $[0, t]$, $t > 0$, while the domain of definition of the moving-average kernel $(t-s)^{H-1/\alpha} - (-s)^{H-1/\alpha}$ is the whole real line. (A moving-average fractional Lévy motion has recently been studied in Benassi et al. [11].) In general, however, it is impossible to generate background driving stochastic processes defined on $\mathbb{R}$.

(ii) It can be written as a Riemann-Liouville fractional integral, whose inverse function has a closed form which is also a Riemann-Liouville fractional derivative. This is important for the prediction of the sample paths of fTSm, problem of consequence in financial modeling. This will be presented in a subsequent paper [23].

Below, we derive a necessary and sufficient condition on $H$ so that for each $t > 0$ and $p \geq 2$, the kernel is in $L^p([0,t])$.

**Lemma 2.1.** Let $t > 0$, let $\alpha \in (0, 2)$, and let $p \geq 2$. Then, $K_{H,\alpha}(t, \cdot) \in L^p([0,t])$ if and only if $H \in (1/\alpha - 1/p, 1/\alpha + 1/p)$. In particular, $K_{H,\alpha}(t, \cdot) \in L^2([0,t])$. Moreover, when $K_{H,\alpha}(t, \cdot) \in L^p([0,t])$,

$$\int_0^t K_{H,\alpha}(t, s)s^pds = C_{H,\alpha,p} t^{p(H-1/\alpha)+1},$$

where

$$C_{H,\alpha,p} = c_{H,\alpha}^p \int_0^1 v^p\left(\frac{1}{\alpha} - H\right) - \left( H - \frac{1}{\alpha} \right) \int_0^1 w^{H-1/\alpha-1} \left( w - v \right)^{H-1/\alpha}dw \right] ^p dv.$$
Proof. The case $H = 1/\alpha$ is trivial since $K_{H,\alpha}(t,s) = 1_{[0,t]}(s)$. If $H > 1/\alpha$, then $K_{H,\alpha}(t,s) \geq 0$, $K_{H,\alpha}(t,s)$ is decreasing in $s$, and $K_{H,\alpha}(t,s) \sim C's^{1/\alpha-H}$ as $s \downarrow 0$ for some constant $C'$. Hence, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if and only if $p(1/\alpha - H) > -1$, i.e., $H < 1/\alpha + 1/p$. When $H < 1/\alpha$, $K_{H,\alpha}(\cdot,s)$ explodes at $s = 0$ and $s = t$. In fact, $K_{H,\alpha}(t,s) \sim C's^{H-1/\alpha}$ as $s \downarrow 0$ and $K_{H,\alpha}(t,s) \sim C''(t-s)^{H-1/\alpha}$ as $s \uparrow t$ for some constants $C''$ and $C'''$. Thus, $K_{H,\alpha}(t,\cdot) \in L^p([0,t])$ if and only if $p(H - 1/\alpha) > -1$, i.e., $H > 1/\alpha - 1/p$, which proves the first claim. The last claim follows from elementary computation. 

Remark 2.2. Above, we only considered the case $p \geq 2$ because of the moment condition (5) and since $H \in (1/\alpha - 1/2, 1/\alpha + 1/2)$ in (6). However, it is easily seen that the results above remain true for arbitrary $p > 0$, provided that the kernel is defined for arbitrary $H$. In particular, we have $K_{H,\alpha}(t,\cdot) \in L^\alpha([0,t])$ since $(0,2/\alpha) \supset (1/\alpha - 1/2, 1/\alpha + 1/2)$.

Let us state two other known properties of the kernel. The proof of (ii) below can be found in, e.g., Decreusefond and Üstünel [3], or Nualart [8], while (i) is immediate.

Lemma 2.3. (i) For each $h > 0$,

$$K_{H,\alpha}(ht,s) = h^{H-1/\alpha}K_{H,\alpha}(t,s/h).$$

(ii) For $t,s > 0$,

$$\int_0^{t\wedge s} K_{H,\alpha}(t,u)K_{H,\alpha}(s,u)du = \frac{1}{2}(t^{2G} + s^{2G} - |t-s|^{2G}),$$

where $G = H - 1/\alpha + 1/2$.

We are now in a position to define fTSm.

Definition 2.4. Fractional tempered stable motion $\{L_t^H : t \geq 0\}$ in $\mathbb{R}$ is given by

$$L_t^H := \int_0^t K_{H,\alpha}(t,s)dX_s^{TS}, \quad t \geq 0,$$

where the integral is defined in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-sense.

The integral above is well defined by the moment condition (5), Lemma 2.3 (ii) and the help of the Wiener-Itô isometry. (See also the proof of Proposition 3.1 below.) For convenience, we will henceforth write $\{L_t^H : t \geq 0\} \sim fTSm(H,\alpha,\rho)$ when $\{L_t^H : t \geq 0\}$ is defined as (10). We note that $L_{1/\alpha}^1 = X^{TS}$, which is a Lévy process, since $K_{1/\alpha,\alpha}(t,s) = 1_{[0,t]}(s)$.

The following is an important result on the marginals of fTSm.

Proposition 2.5. The finite dimensional distributions of fTSm are tempered stable with finite second moment.
Proof. Let $k \in \mathbb{N}$. It suffices to show that for any real sequence \( \{a_i\}_{i=1}^k \) and any non-negative nondecreasing real sequence \( \{t_i\}_{i=1}^k \), the random variable $\sum_{i=1}^k a_i L_{t_i}^H$ is tempered stable. First, observe that

$$
\sum_{i=1}^k a_i L_{t_i}^H = \int_0^{t_k} \left( \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right) dX_s^{TS}.
$$

Then, by Proposition 35 of Rocha-Arteaga and Sato [10], we get

$$
\mathbb{E}[e^{iy\sum_{i=1}^k a_i L_{t_i}^H}] = \exp \left[ \int_0^{t_k} \int_{\mathbb{R}_0} \phi_\alpha(yx \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s)) \rho(dx)ds \right],
$$

where $\phi_\alpha$ is given by (4) and where $\eta = M \circ J$ with $M(dx, ds) = \rho(dx)ds$ and

$$
J(B) = \left\{ (x, s) \in \mathbb{R}_0 \times [0, t_k] : x \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \in B \right\}, \quad B \in \mathcal{B}(\mathbb{R}_0).
$$

The measure $\eta$ is well defined as an inner measure with finite second moment since for each $i$, $K_{H,\alpha}(t_i, \cdot) \in L^2([0, t_i])$ and

$$
\int_{\mathbb{R}_0} |x|^2 \eta(dx) = \int_{\mathbb{R}_0} |x|^2 \rho(dx) \int_0^{t_k} \left( \sum_{i=1}^k a_i K_{H,\alpha}(t_i, s) \right)^2 ds < \infty,
$$

which concludes the proof.

It is worth noting the one dimensional marginal result as a corollary.

**Corollary 2.6.** Let $\{L_t^H : t \geq 0\} \sim \text{fTSm}(H, \alpha, \rho)$ and let $\phi_\alpha$ be given by (4). For each $t > 0$,

$$
\mathbb{E}[e^{iyL_t^H}] = \exp \left[ \int_0^t \int_{\mathbb{R}_0} \phi_\alpha(yx K_{H,\alpha}(t, s)) \rho(dx)ds \right],
$$

and thus

$$
L_t^H \sim TS(\alpha, \eta_t; 0)
$$

where $\eta_t = M \circ J_t$ with $M(dx, ds) = \rho(dx)ds$ and $J_t(B) = \{ (x, s) \in \mathbb{R}_0 \times [0, t] : x K_{H,\alpha}(t, s) \in B \}, \quad B \in \mathcal{B}(\mathbb{R}_0)$.

### 3 Covariance Structure and Long-range Dependence

Let us first describe the covariance structure of fTSm.
Proposition 3.1. Let \( \{ L_t^H : t \geq 0 \} \sim fTSm(H, \alpha, \rho) \). Then, \( \mathbb{E}[L_t^H] = 0 \), and

\[
\text{Cov}(L_t^H, L_s^H) = \frac{1}{2} (t^{2G} + s^{2G} - |t - s|^{2G}) \mathbb{E}[(X_1^{TS})^2], \quad s, t > 0. \tag{13}
\]

Proof. Recall that \( \{ X_t^{TS} : t \geq 0 \} \) is a square-integrable centered martingale. The first claim is thus trivial. For the second claim, observe that for \( s \in [0, t] \),

\[
\text{Cov}(L_t^H, L_s^H) = \mathbb{E}[L_t^H L_s^H] = \mathbb{E} \left[ \int_0^t K_{H,\alpha}(t, u) dX_u^{TS} \int_0^s K_{H,\alpha}(s, u) dX_u^{TS} \right]
\]

\[
= \mathbb{E} \left[ \int_0^{s \wedge t} K_{H,\alpha}(t, u) dX_u^{TS} \int_0^{s \wedge t} K_{H,\alpha}(s, u) dX_u^{TS} \right]
\]

\[
= \mathbb{E}[(X_1^{TS})^2] \int_0^{s \wedge t} K_{H,\alpha}(t, u) K_{H,\alpha}(s, u) du,
\]

where the last equality holds by the Wiener-Itô isometry. Lemma 2.3 (ii) then gives the result. \( \square \)

Let us state some immediate consequences of the previous result.

Corollary 3.2. Let \( \{ L_t^H : t \geq 0 \} \sim fTSm(H, \alpha, \rho) \). For each \( t > 0 \) and each \( h > 0 \),

\[
\mathbb{E}[(L_t^H)^2] = h^{2G} \mathbb{E}[(L_t^H)^2],\tag{14}
\]

and for \( s, t > 0 \),

\[
\mathbb{E}[(L_t^H - L_s^H)^2] = \mathbb{E}[(L_{t-s}^H)^2] = |t - s|^{2G} \mathbb{E}[(X_1^{TS})^2].\tag{15}
\]

The property (14) is sometimes called second-order selfsimilarity. Moreover, (15) says that \( fTSm \) has second-order stationary increments, which clearly implies its continuity in probability.

We are now in a position to discuss the long-range dependence of \( fTSm \). The definition of long-range dependence is often ambiguous and varies among authors. In the present paper, we will follow Samorodnitsky and Taqqu \[16\]; the increments of a second-order stochastic process \( \{ X_t : t \geq 0 \} \) exhibit long-range dependence if for each \( h > 0 \),

\[
\sum_{n=1}^{\infty} |\text{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| = \infty,
\]

or short-range dependence, if each \( h > 0 \),

\[
\sum_{n=1}^{\infty} |\text{Cov}(X_h - X_0, X_{nh} - X_{(n-1)h})| < \infty.
\]
Proposition 3.3. The increments of fTSm exhibit long-range dependence when \( H \in (1/\alpha, 1/\alpha + 1/2) \), and short-range dependence when \( H \in (1/\alpha - 1/2, 1/\alpha] \).

Proof. By Lemma 3.1, we have for each \( h > 0 \),

\[
\text{Cov}(L^H_t, L^H_{t+h} - L^H_t) = \frac{1}{2} t^{2G} (1 + h/t)^{2G} - 2 + (1 - h/t)^{2G}
\]

\[
\sim \frac{1}{2} t^{2(G-1)} G(2G-1) h^2,
\]

as \( t \to \infty \). The claim then holds since \( 2(G-1) > -1 \) for \( H \in (1/\alpha, 1/\alpha + 1/2) \), while \( 2(G-1) \leq -1 \) for \( H \in (1/\alpha - 1/2, 1/\alpha] \).

In relation to the second moment, we will consider higher moments of fTSm.

Proposition 3.4. Let \( \{L^H_t : t \geq 0\} \sim fTSm(H, \alpha, \rho) \). Then, for each \( p > 2 \) and each \( t > 0 \),

\[
\mathbb{E}[|L^H_t|^p] < \infty \text{ if and only if } H \in (1/\alpha - 1/p, 1/\alpha + 1/p) \text{ and } \int_{|x|>1} |x|^p \rho(dx) < \infty.
\]

Proof. By Corollary 2.6, for each \( t \geq 0 \), \( L^H_t \) is tempered stable. By Proposition 2.3 (iii) of Rosiński [14], \( \mathbb{E}[|L^H_t|^p] < \infty \) if and only if \( \int_{|x|>1} |x|^p \eta_t(dx) < \infty \), where \( \eta_t \) is the inner measure of \( L^H_t \) given as in Corollary 2.6, that is,

\[
\int \int_{|x|>1} (|x|K_{H,\alpha}(t, s))^p \rho(dx) ds < \infty.
\]

The left hand side of the above can be decomposed into two terms;

\[
\int_0^t K_{H,\alpha}(t, s)^p \int_{1/\eta_{H,\alpha}(t, s) \vee 1}^{1/\eta_{H,\alpha}(t, s)} |x|^p \rho(dx) ds,
\]

and

\[
\int_0^t K_{H,\alpha}(t, s)^p \int_{|x|>1/\eta_{H,\alpha}(t, s) \vee 1} |x|^p \rho(dx) ds.
\]

The first term is equivalent to \( \int_0^t K_{H,\alpha}(t, s)^p ds \) due to the moment condition (5) on \( \rho \), while the second terms is clearly equivalent to \( \int_0^t K_{H,\alpha}(t, s)^p ds \int_{|x|>1} |x|^p \rho(dx) \). Then, Lemma 2.1 concludes the proof.

Remark 3.5. It is shown in Proposition 2.3 (iv) of Rosiński [14] that a tempered stable distribution has exponential moment of certain order if and only if its inner measure has a compact support. Unfortunately, the tempered stable marginal of fTSm cannot have exponential moment since, with the notation of the preceding proposition, for any \( \epsilon > 0 \),

\[
\eta_t(\{x \in \mathbb{R}_0 : |x| > \epsilon\}) = \int_{|x|>\eta_{H,\alpha}(t, t) \vee \epsilon} |x|^p \rho(dx) ds > 0,
\]

due to the unboundedness of the kernel \( K_{H,\alpha} \).
4 Series Representation

In this section, we derive a series representation of fTSm, which is inherited from the one of tempered stable processes obtained by Rosiński [14]. This representation can also be used for simulation. (See Figure 11) Moreover, we will make use of its structure to derive some sample path properties in Section 5.

Let \( \{T_i\}_{i \geq 1} \) be arrival times of a standard Poisson process, let \( \{E_i\}_{i \geq 1} \) be a sequence of iid exponential random variables with parameter 1, let \( \{U_i\}_{i \geq 1} \) be a sequence of iid uniform random variables on \([0, 1]\), let \( \{V_i\}_{i \geq 1} \) be a sequence of iid random variables in \( \mathbb{R}_0 \) with common distribution

\[
\frac{|x|^\alpha \rho(dx)}{m(\rho)^\alpha},
\]

and let \( \{T_i\}_{i \geq 1} \) be a sequence of iid uniform random variables on \([0, T]\). Also let \( m(\rho)^\alpha, k', \) and \( z_T \) be constants given by

\[
m(\rho)^\alpha = \int_{\mathbb{R}_0} |x|^\alpha \rho(dx), \quad k' = m(\rho)^{-\alpha} \int_{\mathbb{R}_0} x |x|^{\alpha-1} \rho(dx),
\]

and

\[
z_T = \begin{cases} m(\rho)(\alpha/T)^{-1/\alpha} \zeta(1/\alpha) k' T^{-1} + |\Gamma(1-\alpha)| \int_{\mathbb{R}_0} x \rho(dx), & \text{if } \alpha \neq 1, \\ (\ln(m(\rho) T) + 2 \gamma) \int_{\mathbb{R}_0} x \rho(dx) - \int_{\mathbb{R}_0} x \ln |x| \rho(dx), & \text{if } \alpha = 1, \end{cases}
\]

where \( \zeta \) denotes the Riemann zeta function and \( \gamma (= 0.5772...) \) is the Euler constant.

Then, Theorem 5.4 of Rosiński [14] tells us that

\[
\sum_{i=1}^{\infty} \left[ \left( m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} \right) \frac{V_i}{|V_i|} \mathbb{1}(T_i \leq t) - m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} k' \frac{t}{T} \right] + z_T t
\]

converges a.s. uniformly in \( t \in [0, T] \) to a tempered stable process with TS\((\alpha, \rho; 0)\). This series representation can easily be extended to fTSm as follows.

**Proposition 4.1.** Let \( \{L_i^H : t \geq 0\} \sim fTSm(H, \alpha, \rho) \) and let \( T > 0 \). Then, \( \{L_i^H : t \in [0, T]\} \)

\[
\frac{\sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) - m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} k' \frac{t}{T} \} + z_T C_{H,\alpha,1} \frac{t^{H-1/\alpha + 1}}{T} + z_T C_{H,\alpha,1} \frac{t^{H-1/\alpha + 1}}{T}: t \in [0, T]
\]

where \( C_{H,\alpha,1} \) is the constant defined by (3). If \( \rho \) is symmetric, then

\[
\{L_i^H : t \in [0, T]\} \leq \frac{\sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) : t \in [0, T]\}
\]

Moreover, if \( H \in [1/\alpha, 1/\alpha + 1/2) \), then the series converges almost surely uniformly in \( t \) to fTSm\((H, \alpha, \rho)\).
Proof. We will only consider the asymmetric case. By arguments as in Theorem 5.4 of Rosiński [14], we get
\[ \sum_{i=1}^{\infty} \left[ m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} k'C_{H,\alpha,1} \frac{t^{H-1/\alpha+1}}{T} - c_i(T) E[K_{H,\alpha}(t, T_1)] \right] = z_T C_{H,\alpha,1} t^{H-1/\alpha+1}, \]
uniformly in \( t \), where
\[ c_i(T) := \int_{i-1}^{i} \mathbb{E} \left[ \left( m(\rho) \left( \frac{\alpha r}{T} \right)^{-1/\alpha} \wedge E_1 U_1^{1/\alpha} \right) \frac{V_i}{|V_1|} \right] dr. \]
Hence, the right hand side of (16) can be rewritten as
\[ \sum_{i=1}^{\infty} \left[ \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} \right) \frac{V_i}{|V_1|} K_{H,\alpha}(t, T_i) - c_i(T) E[K_{H,\alpha}(t, T_1)] \right] := Z_t. \]
(17)
Next, we need to analyze the finite dimensional distributions of \( \{Z_t : t \in [0, T]\} \). Let \( k \in \mathbb{N} \), let \( \{a_j\}_{j=1}^{k} \) be a real sequence, and let \( \{t_j\}_{j=1}^{k} \) be a nondecreasing sequence taking values in \( [0, T] \). We will show that the random variable \( \sum_{j=1}^{k} a_j Z_{t_j} \) has the same law as \( \sum_{j=1}^{k} a_j L_{t_j}^{H} \). In view of Proposition 2.5, we have
\[ \mathbb{E} \left[ e^{iy \sum_{j=1}^{k} a_j L_{t_j}^{H}} \right] = \exp \left[ \int_{0}^{T} \int_{\mathbb{R}_{0}} \phi_{\alpha}(yx \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, s)) \rho(dx) ds \right], \]
where \( \phi_{\alpha} \) is given by (4). Also, observe that
\[ \sum_{j=1}^{k} a_j Z_{t_j} = \sum_{i=1}^{\infty} \left[ \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} \right) \frac{V_i}{|V_1|} \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, T_i) \right. \]
\[ \left. -c_i(T) \mathbb{E} \left[ \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, T_1) \right] \right]. \]
This series representation is induced by the Lévy measure
\[ \nu(B) \]
\[ = \int_{0}^{T} \int_{\mathbb{R}_{0}} \int_{0}^{\infty} \int_{0}^{\infty} 1_{B}(H(r/T, u, s, x) \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, v)) dr du e^{-s} ds \rho_1(dx) \frac{dv}{T} \]
\[ = \int_{0}^{T} \int_{\mathbb{R}_{0}} \int_{0}^{\infty} 1_{B}(sx \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, v)) s^{-\alpha-1} e^{-s} ds \rho(dx) dv, \]
where \( H(r, u, s, x) = (m(\rho)(\alpha r)^{-1/\alpha} \wedge s u^{1/\alpha} x) |x|/|x| \) and \( \rho_1(dx) = m(\rho)^{-\alpha} |x|^{\alpha} \rho(dx) \). In fact, the measure \( \nu \) is well defined as a Lévy measure since \( K_{H,\alpha}(t_j, \cdot) \in L^2([0, t_j]) \) for
each \( j \). Therefore, by Theorem 4.1 (B) of Rosiński [13],
\[
\mathbb{E}[e^{iy \sum_{j=1}^{k} a_j z_{t_j}}] = \exp \left[ \int_{\mathbb{R}_0} (e^{iyz} - 1 - iyz) \nu(dz) \right]
\]
\[
= \exp \left[ \int_0^T \int_{\mathbb{R}_0} \phi_{\alpha}(yx \sum_{j=1}^{k} a_j K_{H,\alpha}(t_j, s)) \rho(dx)ds \right],
\]
which proves the equality of all finite dimensional distributions.

Next, let \( H \in [1/\alpha, 1/\alpha + 1/2] \). Define, for \( t \in [0,T] \) and \( s \in [0,\infty) \),
\[
Z_{t,s} := \sum_{\{i: \Gamma_i \leq s\}} \left[ \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} |V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) \right.
\]
\[
- c_i(T) \mathbb{E}[K_{H,\alpha}(t, T_i)] \].
\]
Notice that \( \{Z_{t,s} : t \in [0,T], s \in [0,\infty)\} \) has independent increments in \( s \) (not in \( t \), of course.) Arguments as in the proof of Theorem 5.1 of [13] give the a.s. convergence of the series uniformly on \([0,T]\). \( \square \)

5 Sample Path Properties

In this section, we investigate sample path properties of fTSm. Let us begin with the case \( H \in (1/\alpha - 1/2, 1/\alpha) \).

Proposition 5.1. When \( H \in (1/\alpha - 1/2, 1/\alpha) \), fTSm is a.s. unbounded on every interval of positive length.

Proof. Let \( T > 0 \). For each \( t \in (0,T] \), \( \lim_{s \uparrow t} K_{H,\alpha}(t, s) = +\infty \). Hence, in the series representation given in Proposition 4.1, \( K_{H,\alpha}(T_i, T_i) = +\infty \), for all \( i \in \mathbb{N} \), and so none of the summands are well defined. This implies that sup\( t \in [0,T] \, |L^H_t| = \infty \) a.s. \( \square \)

Unfortunately, the above sample path property makes fTSm with short-range dependence of little practical use. We notice that for any \( H \in (1/\alpha - 1/2, 1/\alpha + 1/2) \), \( K_{H,\alpha}(t, 0) = +\infty \). But, this turns out to be irrelevant to the unboundedness of sample paths since \( T_i \neq 0 \) a.s., \( i \in \mathbb{N} \).

FTSm with long-range dependence has better sample path properties. In particular, it has a Hölder continuous version with exponent \( \gamma \in (0, H - 1/\alpha) \).

Proposition 5.2. If \( H \in (1/\alpha, 1/\alpha + 1/2) \), there exists a continuous modification of fTSm, which is a.s. locally Hölder continuous with exponent \( \gamma \) for every \( \gamma \in (0, H - 1/\alpha) \).

Proof. By Corollary 3.2, we have \( \mathbb{E}[|L^H_t - L^H_s|^2] = |t - s|^{2G} \mathbb{E}[(X^T)^2] \). If \( H > 1/\alpha \), then \( 2G > 1 \), and thus the Kolmogorov–Čentsov Theorem (see, for example, Theorem 3.23 of Kallenberg [5]) directly applies, giving the result. \( \square \)
We will henceforth always assume that when $H \in (1/\alpha, 1/\alpha + 1/2)$, we are using such a continuous version of fTSm.

**Proposition 5.3.** Let $\{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho)$ with $H \in (1/\alpha, 1/\alpha + 1/2)$.

(i) 
\[
\lim_{N \to \infty} \mathbb{E} \left[ \sum_{n=0}^{N-1} \left| L_{n+1}^H - L_n^H \right|^2 \right] = 0.
\]

(ii) FTSm is a.s. of infinite variation on every interval of positive length.

(iii) FTSm is not semimartingale.

**Proof.** (i) Immediate from
\[
\mathbb{E} \left[ \sum_{n=0}^{N-1} \left| L_{n+1}^H - L_n^H \right|^2 \right] = (N/T)^{2(1/\alpha - H)}.
\]

(ii) Let $T > 0$. For each $s \in [0, T]$, we have
\[
\limsup_{t_1, t_2 \downarrow s} \frac{|K_{H,\alpha}(t_1, s) - K_{H,\alpha}(t_2, s)|}{|t_1 - t_2|} = +\infty,
\]
which implies that for each $s \in [0, T]$, $K_{H,\alpha}(\cdot, s)$ is of infinite variation. By Theorem 4 of Rosiński [11] and a symmetrization argument given there, fTSm is of infinite variation with positive probability. FTSm is selfdecomposable and hence by Corollary 3 of Rosiński [12], it obeys a zero-one law. This gives the result.

(iii) The convergence in (i) implies convergence in probability, and together with (ii), the claim follows from the same argument in Lin [6].

**Remark 5.4.** In view of (iii) above, the stochastic integration for fTSm cannot be defined in the classical semimartingale sense. However, a slight modification of the kernel $K_{H,\alpha}$ induces a corresponding semimartingale, which can be arbitrarily close to fTSm. Recall the simpler expression of the kernel for $H \in (1/\alpha, 1/\alpha + 1/2)$ given by (7). Observe that
\[
\frac{\partial}{\partial t} K_{H,\alpha}(t, s) = c_{H,\alpha}(H - 1/\alpha)(t - s)^{H-1/\alpha-1} \left( \frac{t}{s} \right)^{H-1/\alpha} 1_{[0,t]}(s),
\]
and that
\[
K_{H,\alpha}(t, s) = \int_s^t \frac{\partial}{\partial u} K_{H,\alpha}(u, s) du 1_{[0,t]}(s).
\]
Therefore,
\[
L_t^H = \int_0^t K_{H,\alpha}(t, s) dX_s^{TS} = \int_0^t \left( \int_s^t \frac{\partial}{\partial u} K_{H,\alpha}(u, s) du \right) dX_s^{TS}.
\]
If the two integrals could be interchanged, then fTSm would be of finite variation, i.e., it would be a semimartingale; we have seen that this is not the case. On the other hand,
the integrability condition of the stochastic Fubini’s theorem (Theorem 46 of Protter [9]) can be achieved by slightly modifying the kernel $K_{H,\alpha}$. Set

$$K^n_{H,\alpha}(t, s) := c_{H,\alpha}(H - 1/\alpha) s^{1/\alpha - H} \int_s^t \left(u + \frac{1}{n} - s\right)^{H-1/\alpha-1} u^{H-1/\alpha} du \mathbb{1}_{[0,t]}(s), \quad n \in \mathbb{N},$$

and so we have

$$\frac{\partial}{\partial t} K^n_{H,\alpha}(t, s) = c_{H,\alpha}(H - 1/\alpha) \left(t + \frac{1}{n} - s\right)^{H-1/\alpha-1} \left(\frac{t}{s}\right)^{H-1/\alpha} \mathbb{1}_{[0,t]}(s).$$

The integrability condition is then satisfied; for every $u \in [0, t]$,

$$\int_0^u \left(\frac{\partial}{\partial u} K^n_{H,\alpha}(u, s)\right)^2 ds = c^2_{H,\alpha}(H - 1/\alpha)^2 \int_0^u \left(\frac{1}{n} - s\right)^{2(H-1/\alpha-1)} \left(\frac{u}{s}\right)^{2(H-1/\alpha)} ds \leq c^2_{H,\alpha}(H - 1/\alpha)^2 (1 - 2(H - 1/\alpha))^{-1} n^{-2(H-1/\alpha-1)} < \infty,$$

and thus the stochastic Fubini’s theorem applies. Therefore, we get

$$\int_0^t K^n_{H,\alpha}(t, s) dX^{TS}_s = \int_0^t \left( \int_0^u \frac{\partial}{\partial u} K^n_{H,\alpha}(u, t) dX^{TS}_s \right) du,$$

which is clearly of finite variation. For financial modeling, it is of interest to further modify the above to get an infinite variation semimartingale. This can be done as follows. For $\epsilon > 0$, set $K_{H,\alpha}^{n,\epsilon}(t, s) := K^n_{H,\alpha}(t, s) + \epsilon$. Since $\frac{\partial}{\partial \alpha} K^n_{H,\alpha}(t, s) = \frac{\partial}{\partial \alpha} K^n_{H,\alpha}(t, s)$, the stochastic Fubini’s theorem again applies and thus we get

$$\int_0^t K_{H,\alpha}^{n,\epsilon}(t, s) dX^{TS}_s = \int_0^t (\epsilon + K^n_{H,\alpha}(t, s)) dX^{TS}_s = \epsilon X^{TS}_t + \int_0^t \left( \int_0^u \frac{\partial}{\partial u} K^n_{H,\alpha}(u, s) dX^{TS}_s \right) du,$$

which exactly follows the definition of the canonical decomposition of semimartingales, i.e. a martingale plus a finite variation process.

6 Short and Long Time Behavior

In this section, we obtain the short and long time behavior of fTSm, which are also inherited from the background driving tempered stable processes obtained by Rosiński [14]. In short time, fTSm is asymptotically fractional stable motion (to be defined below), while in long time it is approximately fBm.
Let us begin by briefly reviewing the corresponding behaviors of tempered stable processes, proved in Theorem 3.1 of [14].

(i) **Short time behavior:** Let \( \{X_t^{TS} : t \geq 0\} \sim TS(\alpha, \rho; 0) \) and let
\[
b_{h,\alpha} = \begin{cases} 
  h\Gamma(1 - \alpha) \int_{\mathbb{R}_0} x \rho(dx), & \text{if } \alpha \in (0, 1), \\
  -(1 + \ln h) \int_{\mathbb{R}_0} x \rho(dx), & \text{if } \alpha = 1, \\
  0, & \text{if } \alpha \in (1, 2). 
\end{cases}
\]
(20)

Then,
\[
\left\{ h^{-1/\alpha} (X_{ht}^{TS} + b_{h,\alpha} t) : t \geq 0 \right\} \overset{d}{\to} \{X_t^\alpha : t \geq 0\}, \quad \text{as } h \to 0,
\]
where \( \{X_t^\alpha : t \geq 0\} \) is an \( \alpha \)-stable process in \( \mathbb{R} \) such that
\[
\mathbb{E}[e^{iuX_t^\alpha}] = \exp \left[ t \int_{\mathbb{R}_0} \varphi_\alpha(yx) \rho(dx) \right],
\]
where
\[
\varphi_\alpha(s) = \begin{cases} 
  -\Gamma(-\alpha) \cos \frac{\pi\alpha}{2}|s|^\alpha(1 - i \tan \frac{\pi\alpha}{2} \text{sgn}(s)), & \text{if } \alpha \in (0, 1) \cup (1, 2), \\
  -(\frac{\pi}{2}|s| + is \ln |s|) + is, & \text{if } \alpha = 1.
\end{cases}
\]
(21)

(ii) **Long time behavior:** Let \( \{X_t^{TS} : t \geq 0\} \sim TS(\alpha, \rho; 0) \). Then,
\[
\left\{ h^{-1/2}X_{ht}^{TS} : t \geq 0 \right\} \overset{d}{\to} \{cW_t : t \geq 0\}, \quad \text{as } h \to \infty,
\]
where \( \{W_t : t \geq 0\} \) is a standard (centered) Brownian motion and
\[
c^2 = \Gamma(2 - \alpha) \int_{\mathbb{R}_0} x^2 \rho(dx).
\]
(22)

We will say that the limiting stable process \( \{X_t^\alpha : t \geq 0\} \) given above is **associated to** the tempered stable process \( \{X_t^{TS} : t \geq 0\} \).

Let us now define *fractional stable motions* (fSm), which turns out to be a short time limit of fTSm. Below, for \( \alpha \neq 1 \), the integral in (23) is well defined in probability since \( K_{H,\alpha}(t, \cdot) \in L^\alpha([0, t]) \), \( t > 0 \). On the other hand, when \( \alpha = 1 \), the extra symmetry assumption on \( \rho \) also ensures that it is well defined in probability. (See Remark 2.2 and Samorodnitsky and Taqqu [16].)

**Definition 6.1.** Let \( \{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho) \), where when \( \alpha = 1 \), \( \rho \) is additionally assumed to be symmetric. Fractional stable motion (fSm) \( \{L_t^{H,\alpha} : t \geq 0\} \) associated to fTSm \( \{L_t^H : t \geq 0\} \) is given via
\[
L_t^{H,\alpha} := \int_0^t K_{H,\alpha}(t, s) dX_s^\alpha, \quad t \geq 0.
\]
(23)
Let us derive some basic properties of fSm.

**Lemma 6.2.** Let \( \{L_t^{H,\alpha} : t \geq 0\} \) be fSm associated to \( \{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho) \).

(i) The finite dimensional distributions of fSm are stable.

(ii) For each \( t > 0 \), the characteristic function of \( L_t^{H,\alpha} \) is given by

\[
\mathbb{E}[e^{i\lambda L_t^{H,\alpha}}] = \exp \left( C_{H,\alpha,\alpha} t^{\alpha H} \int_{\mathbb{R}_0} \tilde{\varphi}_\alpha(yx) \rho(dx) \right),
\]  

where \( C_{H,\alpha,\alpha} \) is the constant given by (9) and where

\[
\tilde{\varphi}_\alpha(s) = \begin{cases} 
\varphi_\alpha(s), & \text{if } \alpha \in (0, 1) \cup (1, 2), \\
-\frac{\pi}{2}|s|, & \text{if } \alpha = 1,
\end{cases}
\]

where \( \varphi_\alpha \) is defined by (21).

(iii) FSm is selfsimilar; \( \{h^{-H}L_{ht}^{H,\alpha} : t \geq 0\} \overset{d}{=} \{L_t^{H,\alpha} : t \geq 0\} \).

(iv) FSm has (strictly) stationary increments; for each \( t > s > 0 \), \( L_t^{H,\alpha} - L_s^{H,\alpha} \overset{d}{=} L_{t-s}^{H,\alpha} \).

(v) When \( H \in (1/\alpha, 1/\alpha + 1/2) \), fSm has a continuous version, while when \( H \in (1/\alpha - 1/2, 1/\alpha) \), it is unbounded on every interval of positive length.

(vi) When \( H = 1/\alpha \), fSm is an \( \alpha \)-stable (Lévy) process.

(vii) With the notation in Section 4, if \( \alpha \in (1, 2) \), then

\[
\{L_t^{H,\alpha} : t \in [0, T]\}
\]

\[
\overset{d}{=} \left\{ \sum_{i=1}^{\infty} m(\rho) \left( \frac{\alpha T_i}{|V_i|} \right)^{-1/\alpha} V_i K_{H,\alpha}(t, T_i) - m(\rho) \left( \frac{\alpha i}{T} \right)^{-1/\alpha} k'C_{H,\alpha,1} \frac{t^{H-1/\alpha+1}}{T} + m(\rho) \left( \frac{\alpha}{T} \right)^{-1/\alpha} \zeta(1/\alpha) k'C_{H,\alpha,1} \frac{t^{H-1/\alpha+1}}{T} : t \in [0, T] \right\},
\]

while if \( \alpha \in (0, 1) \), or if \( \alpha \in (1, 2) \) and \( \rho \) is symmetric, then

\[
\{L_t^{H,\alpha} : t \in [0, T]\}
\]

\[
\overset{d}{=} \left\{ \sum_{i=1}^{\infty} m(\rho) \left( \frac{\alpha T_i}{|V_i|} \right)^{-1/\alpha} V_i K_{H,\alpha}(t, T_i) : t \in [0, T] \right\}.
\]

Moreover, if \( H \in [1/\alpha, 1/\alpha + 1/2) \), \( \alpha \in (0, 2) \), the above series converges almost surely uniformly in \( t \in [0, T] \) to a version of \( \{L_t^{H,\alpha} : t \in [0, T]\} \).
Proof. With the notation of Proposition 2.5 for $\alpha \neq 1$,

$$\mathbb{E}[e^{iy\sum_{i=1}^{k} a_i L_i^{H,\alpha}}]$$

$$= \exp \left[ -\Gamma(-\alpha) \cos \frac{\pi \alpha}{2} \int_0^t \int_{\mathbb{R}_0} \left| \sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, s)yx \right|^\alpha \left( 1 - i \tan \frac{\pi \alpha}{2} \text{sgn} \left( \sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, s) yx \right) \right) \rho(dx)ds \right]$$

$$= \exp \left[ -\Gamma(-\alpha) \cos \frac{\pi \alpha}{2} \int_0^t \left| \sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, s) \right|^\alpha ds \int_{\mathbb{R}_0} |yx|^\alpha \left( 1 - i \tan \frac{\pi \alpha}{2} \text{sgn}(yx) \right) \rho(dx) \right].$$

When $\alpha = 1$, the symmetry of $\rho$ yields $b_{h,1} = 0$, and so

$$\mathbb{E}[e^{iy\sum_{i=1}^{k} a_i t_i^{H,1}}] = \exp \left[ -\pi \int_0^t \left| \sum_{i=1}^{k} a_i K_{H,1}(t_i, s)yx \right|^\alpha ds \int_{\mathbb{R}_0} |yx| \rho(dx) \right].$$

For each $\alpha \in (0, 2)$, $\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, \cdot) \in L^{\alpha}([0, t_k])$ since $a_i K_{H,\alpha}(t_i, \cdot) \in L^{\alpha}([0, t_i])$ for each $i$. Thus, (i) holds. Clearly, (ii) is a direct consequence of (i). (iii) and (iv) follow from the selfsimilarity and stationary increments properties of $\{X(t) : t \geq 0\}$ with the help of Lemma 2.3. By (iii) and (iv), $E[|L_t^{H,\alpha} - L_s^{H,\alpha}|^p] = |t - s|^p E[|L_1^{H,\alpha}|^p]$, $0 < p < \alpha$. Hence, when $H \in (1/\alpha, 1/\alpha + 1/2)$, the continuity follows from the Kolmogorov-Čentsov Theorem. Next, let $H \in (1/\alpha - 1/2, 1/\alpha)$. For each $T > 0$, $\sup_{t \in [0, T]} |K_{H,\alpha}(t, s)| = +\infty$, $s \in [0, T]$. The nowhere boundedness follows from (i) and Theorem 4 of Rosiński [11] as well as the zero-one law for stable processes and a symmetrization argument given there. Hence, (v) holds. (vi) is immediate from $K_{1/\alpha,\alpha}(t, s) = 1_{[0,T]}(s)$. Finally, (vii) follows from arguments as in Proposition 5.5 of Rosiński [14] and from Proposition 4.1.

We will henceforth always assume that when $H \in (1/\alpha, 1/\alpha + 1/2)$, we are using a continuous version of fSm.

Remark 6.3. Many extensions of fBm are available in the stable literature; for example, linear fractional stable motion, log-fractional stable motion, harmonizable fractional stable motion. (See, e.g., Samorodnitsky and Taqqu [16].) These various extensions are not necessarily identical in law since their marginals are determined by kernels in the stochastic integral representations. Indeed, fSm defined above is still different from any of them.

We are now in a position to present the main result of this section.
Theorem 6.4. Let \( \{L_t^H : t \geq 0\} \sim fTSm(H, \alpha, \rho) \) with \( H \neq 1/\alpha \).

(i) Short time behavior: Let

\[
b = \begin{cases} 
\Gamma(1 - \alpha) \int_{\mathbb{R}_0} x \rho(dx), & \text{if } \alpha \in (0, 1), \\
0, & \text{if } \alpha \in [1, 2), 
\end{cases}
\]

and let \( k_t = \int_0^t K_{H,\alpha}(t, s)ds \). Then,

\[
\{h^{-H}L_t^H + h^{1-1/\alpha}bk_t : t \geq 0\} \xrightarrow{L} \{L_t^{H,\alpha} : t \geq 0\}
\]

as \( h \to 0 \),

where \( \{L_t^{H,\alpha} : t \geq 0\} \) is fSm associated to \( \{L_t^H : t \geq 0\} \).

(ii) Long time behavior:

\[
\{h^{-G}L_t^H : t \geq 0\} \xrightarrow{L} \{cB_t^G : t \geq 0\}
\]

as \( h \to \infty \),

where \( \{B_t^G : t \geq 0\} \) is a standard fBm with parameter \( G \) and where \( c \) is the constant given by \( (22) \).

(iii) When \( H \in (1/\alpha, 1/\alpha + 1/2) \), the convergence in (i) and (ii) can be strengthened to the weak convergence in \( C([0, \infty), \mathbb{R}) \).

Proof of (i) and (ii). (i) Observe that for each \( t \geq 0 \),

\[
h^{-H}L_t^H + h^{1-1/\alpha}bk_t = \int_0^t K_{H,\alpha}(t, s)h^{-1/\alpha}d(X_{hs}^T + bhs) := Y_t^h.
\]

It thus suffices to show that for any real sequence \( \{a_i\}_{i=1}^k \) and nonnegative nondecreasing real sequence \( \{t_i\}_{i=1}^k \), \( k \in \mathbb{N} \), the random variable \( \sum_{i=1}^k a_iY_{t_i}^h \) converges in law to \( \sum_{i=1}^k a_iL_{t_i}^{H,\alpha} \), as \( h \to 0 \). Since

\[
\sum_{i=1}^k a_iY_{t_i}^h = \int_0^{t_k} \left( \sum_{i=1}^k a_iK_{H,\alpha}(t_i, s) \right) h^{-1/\alpha}d(X_{hs}^T + bhs),
\]

we have by Proposition \ref{prop:conv} that

\[
\mathbb{E}\left[e^{iy\sum_{i=1}^k a_iY_{t_i}^h} \right] = \exp\left[ \int_0^{t_k} \int_{\mathbb{R}_0} h\psi_\alpha(yxh^{-1/\alpha} \sum_{i=1}^k a_iK_{H,\alpha}(t_i, s)) \rho(dx)ds \right],
\]

where

\[
\psi_\alpha(s) = \begin{cases} 
\Gamma(-\alpha)((1 - is)^\alpha - 1), & \text{if } \alpha \in (0, 1), \\
\frac{1}{2}\ln(1 + s^2) - s \tan^{-1}s, & \text{if } \alpha = 1, \\
\Gamma(-\alpha)((1 - is)^\alpha - 1 + i\alpha s), & \text{if } \alpha \in (1, 2). 
\end{cases}
\]

Note that \( \psi_1 \) is obtained via the symmetry of \( \rho \). (See Proposition 2.8 of Rosiński \cite{rosinski}.) We then want to show that \( (25) \) tends to the characteristic function of the random variable \( \sum_{i=1}^k a_iL_{t_i}^{H,\alpha} \), as \( h \to 0 \).
The proof of Theorem 3.1 (i) of Rosiński [14] shows that for $\alpha \neq 1$, 

$$\lim_{h \to 0} h \psi_\alpha(h^{-1/\alpha}s) = \varphi_\alpha(s),$$

where $\varphi_\alpha$ is given by (21) and

$$|h \psi_\alpha(h^{-1/\alpha}s)| \leq z_\alpha |s|^{\alpha},$$

where $z_\alpha$ is some constant depending only on $\alpha (\neq 1)$. When $\alpha = 1$, we have

$$\lim_{h \to 0} h \psi_1(h^{-1}s) = -\frac{\pi}{2} |s|,$$

and the uniform boundedness (in $h > 0$) of $|h \psi_1(h^{-1}s)|$ can be shown as

$$|h \psi_1(h^{-1}s)| \leq |h \ln \sqrt{1 + h^{-2}s^2} + |s \tan^{-1}(h^{-1}s)| \leq |h \ln(1 + h^{-1}|s|)| + \frac{\pi}{2} |s| \leq \left(1 + \frac{\pi}{2}\right) |s|.$$

Clearly, $\sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, s) \in L^\alpha([0, t_k])$ since $a_i K_{H,\alpha}(t_i, s)$ are in $L^\alpha([0, t_i])$. Together with the moment condition (2) on $\rho$, the passage to the limit in (25) is justified. Hence,

$$\lim_{h \to 0} \mathbb{E} \left[ e^{iy \sum_{i=1}^{k} a_i Y_h^{t_i}} \right] = \exp \left[ \int_0^{t_k} \int_{\mathbb{R}_0} \varphi_\alpha(yx \sum_{i=1}^{k} a_i K_{H,\alpha}(t_i, s)) \rho(dx)ds \right],$$

which is the characteristic function of $\sum_{i=1}^{k} a_i L_{t_i}^{H,\alpha}$.

(ii) We have that for each $h > 0$,

$$\text{Cov}(h^{-G} L_{ht}^H, h^{-G} L_{hs}^H) = \frac{1}{2} \left(t^{2G} + s^{2G} - (t - s)^{2G}\right) \mathbb{E}[\langle X_1^{TS}\rangle^2], \quad s \in [0, t].$$

Hence, for the convergence of all finite dimensional distributions, we only need to show that the marginal law at any time of $\{h^{-G} L_{ht}^H : t \geq 0\}$ converges to Gaussian. Without loss of generality, let $t = 1$. By Lemma 2.3 (i),

$$\mathbb{E}[e^{ih^{-G} L_h^H}] = \exp \left[ \int_0^{h} \int_{\mathbb{R}_0} \vartheta_\alpha(h^{-G} yx K_{H,\alpha}(h, s)) \rho(dx)ds \right] = \exp \left[ \int_0^{1} \int_{\mathbb{R}_0} \vartheta_\alpha(h^{-1/2} yx K_{H,\alpha}(1, s)) \rho(dx)ds \right],$$

where $\vartheta_\alpha(u) = \int_0^\infty (e^{ius} - 1 - ius)s^{-\alpha - 1}e^{-s}ds$. As in the proof of Theorem 3.1 (ii) of Rosiński [14], it follows that

$$|\vartheta_\alpha(yx K_{H,\alpha}(1, s))| \leq (yx)^2 \Gamma(2 - \alpha) \int_0^{1} K_{H,\alpha}(1, s)^2 ds,$$
which justifies the passage to the limit below

$$\lim_{h \to \infty} \mathbb{E}[e^{ih h^{-G} L_h^H}] = \exp \left[ -\frac{y^2}{2} \int_0^1 K_{H, 0}(1, s)^2 ds \Gamma(2 - \alpha) \int_{\mathbb{R}_0} x^2 \rho(dx) \right].$$

This shows the convergence to a Gaussian law, which concludes the proof. \qed

To proof (iii), let us present a technical lemma.

**Lemma 6.5.** Let \(\{X_t^n : t \geq 0\}_{n \in \mathbb{N}}\) and \(\{Y_t^n : t \geq 0\}_{n \in \mathbb{N}}\) be sequences of stochastic processes in \(C([0, \infty), \mathbb{R})\). If the sequence \(\{X_t^n : t \geq 0\}_{n \in \mathbb{N}}\) is tight and if for each \(n \in \mathbb{N}\), there exists a version \(\tilde{Y}_t^n\) of \(Y_t^n\) defined on the same probability space as \(X_t^n\) and such that the sequence \(\{\tilde{Y}_t^n - X_t^n : t \geq 0\}\) converges \(\text{ucp}\) to zero, then the sequence \(\{Y_t^n : t \geq 0\}_{n \in \mathbb{N}}\) is tight.

**Proof.** For each compact set \(K \in [0, \infty)\) and for each \(\delta > 0\), we have

$$\mathbb{E} \left[ \sup_{t, s \in K, |t-s| \leq \delta} |Y_t^n - Y_s^n| \wedge 1 \right] \leq \mathbb{E} \left[ \sup_{t, s \in K, |t-s| \leq \delta} |\tilde{Y}_t^n - X_t^n| \wedge 1 \right] + \mathbb{E} \left[ \sup_{t, s \in K, |t-s| \leq \delta} |X_t^n - X_s^n| \wedge 1 \right]$$

$$+ \mathbb{E} \left[ \sup_{t, s \in K, |t-s| \leq \delta} |X_s^n - \tilde{Y}_s^n| \wedge 1 \right] = 2 \mathbb{E} \left[ \sup_{t \in K} |\tilde{Y}_t^n - X_t^n| \wedge 1 \right] + \mathbb{E} \left[ \sup_{t, s \in K, |t-s| \leq \delta} |X_t^n - X_s^n| \wedge 1 \right].$$

The first term in (27) tends to zero as \(n \to \infty\), since \(\tilde{Y}_t^n - X_t^n \xrightarrow{\text{ucp}} 0\). The second term also tends to zero as \(n \to \infty\) and \(\delta \to 0\) by the tightness of \(\{X_t^n : t \geq 0\}_{n \in \mathbb{N}}\) in \(C([0, \infty), \mathbb{R})\). The claimed result then follows. \qed

**Proof of Theorem 6.4 (iii).** By, for example, Lemma 16.2, Theorem 16.3 and 16.5 of Kallenberg [5], it suffices to show the tightness of the sequences \(\{h^{-H} L_{ht}^H + h^{1-\alpha} b_{kt} : t \geq 0\}\) (as \(h \downarrow 0\)) and \(\{h^{-G} L_{ht}^H : t \geq 0\}\) (as \(h \uparrow \infty\)) in \(C([0, \infty), \mathbb{R})\).

We begin with the short time behavior case. By Lemma 6.2 (iii) and (iv), for each \(p \in (0, \alpha)\),

$$E[|h^{-H} L_{ht}^{H, \alpha} - h^{-H} L_{ht}^{H, \alpha}|^p] = (t-s)^p E[|L_{ht}^{H, \alpha}|], \quad s \in [0, t].$$

By Corollary 16.9 of Kallenberg [5], the uniform boundedness in \(h\) seen in (28) implies the tightness of the sequence \(\{h^{-H} L_{ht}^{H, \alpha} : t \geq 0\}\) in \(C([0, \infty), \mathbb{R})\). Hence, by Lemma 6.5, it is enough to find \(h^{-H} L_{ht}^{H, \alpha}\) and \(h^{-H} L_{ht}^{H} + h^{1-\alpha} b_{kt}\) in \(C([0, T], \mathbb{R})\), \(T > 0\), defined on a common probability space and such that \(h^{-H} L_{ht}^{H, \alpha} - (h^{-H} L_{ht}^{H} + h^{1-\alpha} b_{kt}) \xrightarrow{\text{ucp}} 0\). To this end, we make use of their series representations. First, let \(\alpha \in (0, 1]\), or let \(\alpha \in (1, 2)\)
with a symmetric \( \rho \). With the notation of Section 4, it follows from Proposition \ref{prop:short_time_behavior} and Lemma \ref{lem:asymptotics} (vii) that the stochastic processes

\[
h^{-H} \sum_{i=1}^{\infty} m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \frac{V_i}{|V_i|} K_{H,\alpha}(h^t, h T_i) := h^{-H} \tilde{L}_{h,T}^H
\]

and

\[
h^{-H} \sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} |V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(h^t, h T_i) := h^{-H} \tilde{L}_{h,T}^H + h^{1-1/\alpha} b_k,
\]

converge almost surely uniformly on \([0, T]\), respectively, to versions of \( f_{Sm} L^H \) and \( f_{TSm} h^{-H} L_{h,T}^H + h^{1-1/\alpha} b_k \), defined on a common probability space by using the common random sequences \( \{\Gamma_i\}_{i \geq 1}, \{T_i\}_{i \geq 1}, \{V_i\}_{i \geq 1}, \{E_i\}_{i \geq 1} \) and \( \{U_i\}_{i \geq 1} \). Then, in view of Lemma \ref{lem:asymptotics} (i),

\[
h^{-H} \tilde{L}_{h,T}^H \sim (h^{-H} \tilde{L}_{h,T}^H + h^{1-1/\alpha} b_k) = \sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} - h^{-1/\alpha} E_i U_i^{1/\alpha} |V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i), \tag{29}
\]

which clearly converges ucp to zero as \( h \to 0 \). Also, in case \( \alpha \in (1, 2) \) and \( \rho \) is asymmetric, a similar argument yields (29). Hence, the sequence \( \{h^{-H} L_{h,T}^H + h^{1-1/\alpha} b_k : t \geq 0\} \) is tight in \( C([0, \infty), \mathbb{R}) \), which concludes the short time behavior case.

For the long time behavior case, Corollary \ref{cor:long_time_behavior} gives for \( h > 0 \),

\[
\mathbb{E}[((h^{-G} L_{h}^H - h^{-G} L_{h}^{H,T}))^2] = (t - s)^{2G} \mathbb{E}[(X_1 T^S)^2]. \tag{30}
\]

Again by Corollary 16.9 of Kallenberg \cite{Kallenberg}, the uniform boundedness in \( h \) seen in \( (30) \) implies the tightness of the sequence \( \{h^{-G} L_{h,T}^H : t \geq 0\} \) in \( C([0, \infty), \mathbb{R}) \), which completes the proof.

**Remark 6.6.** The short time behavior result can also be seen from the series representation. For simplicity, consider the symmetric case. With the notations of Theorem \ref{thm:short_time_behavior}, we have by Lemma \ref{lem:asymptotics} (i)

\[
h^{-H} L_{h,T}^H = h^{-H} \sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha \Gamma_i}{hT} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} |V_i| \right) \frac{V_i}{|V_i|} K_{H,\alpha}(h^t, h T_i)
\]

\[
= \sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \wedge E_i U_i^{1/\alpha} |V_i| h^{-1/\alpha} \right) \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i)
\]

\[
\to \sum_{i=1}^{\infty} m(\rho) \left( \frac{\alpha \Gamma_i}{T} \right)^{-1/\alpha} \frac{V_i}{|V_i|} K_{H,\alpha}(t, T_i) \text{ a.s., as } h \to 0. \tag{31}
\]
We have seen in Lemma 6.2 that the marginals of fSm are α-stable and so their covariance does not exist. An alternative notion is the one of covariation. For two jointly symmetric α-stable random variables X and Y with α > 1;

\[ \tau(X, Y) := \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X - Y\|_\alpha^\alpha, \]  

(32)

where the norm \( \| \cdot \| \) gives the scale of parameter, i.e. for \( Z \sim S_\alpha(\sigma, 0, 0) \), \( \|Z\|_\alpha = \sigma \). More generally, one can also define the codifference for any jointly infinitely divisible random variables X and Y as

\[ I(\theta_1, \theta_2; X, Y) := -\ln E[e^{i(\theta_1 X + \theta_2 Y)}] + \ln E[e^{i\theta_1 X}] + \ln E[e^{i\theta_2 Y}], \]  

(33)

for \( \theta_1, \theta_2 \in \mathbb{R} \). Clearly, (32) is a special case of (33). Let \( \{L_t^{H,\alpha} : t \geq 0\} \) be fSm associated to \( \{L_t^H : t \geq 0\} \sim fTSM(H, \alpha, \rho) \) where \( \rho \) is symmetric. Then,

\[ I(1, -1; L_t^{H,\alpha}, L_s^{H,\alpha}) = C(t^\alpha + s^\alpha - (t - s)^\alpha), \quad s \in [0, t], \]  

(34)

for some constant \( C \). In the Gaussian case, the codifference coincides with the covariance. For example, for a standard fBm \( \{B_t^G : t \geq 0\} \),

\[ \tau(B_t^G, B_s^G) = \frac{1}{2}(t^{2G} + s^{2G} - (t - s)^{2G}) = \text{Cov}(B_t^G, B_s^G), \quad s \in [0, t], \]

and

\[ \tau(B_{t+1}^G - B_t^G, B_{s+1}^G - B_s^G) = \text{Cov}(B_{t+1}^G - B_t^G, B_{s+1}^G - B_s^G) \sim Ct^{2(G-1)}, \]

as \( t \to \infty \). See Samorodnitsky and Taqqu for more details of the covariation and the codifference.

Interestingly enough, as shown below, the codifference of the increments of fTSM has the same order of decay as the covariance (see Proposition 3.3).

**Proposition 6.7.** Let \( \{L_t^H : t \geq 0\} \sim fTSM(H, \alpha, \rho) \). Then,

\[ I(\theta_1, \theta_2; L_{t+1}^H - L_t^H, L_{s+1}^H - L_s^H) \sim C(\theta_1, \theta_2)t^{2(G-1)}, \]

as \( t \to \infty \), where

\[ C(\theta_1, \theta_2) = \frac{-ic_{H,\alpha}\theta_1\pi}{\Gamma(\alpha)\sin(\pi\alpha)} \int_{[0,1] \times \mathbb{R}_0} ((1 - ix\theta_2K_H(1, s))^{\alpha-1} - 1) xs^{1/\alpha - H} ds \rho(dx). \]

**Proof.** Observe that

\[ I(\theta_1, \theta_2; L_{t+1}^H - L_t^H, L_{s+1}^H - L_s^H) \]

\[ = \Gamma(-\alpha) \int_{\mathbb{R} \times \mathbb{R}_0} \left( - (1 - ix(\theta_1(K_{H,\alpha}(t + 1, s) - K_{H,\alpha}(t, s)) + \theta_2 K_{H,\alpha}(1, s)))^\alpha \right. 

\[ + (1 - ix(\theta_1(K_{H,\alpha}(t + 1, s) - K_{H,\alpha}(t, s)))^\alpha \]

\[ + (1 - ix\theta_2 K_{H,\alpha}(1, s))^{\alpha - 1}) ds \rho(dx), \]
and that
\[ K_{H,\alpha}(t + 1, s) - K_{H,\alpha}(t, s) \sim c_{H,\alpha}s^{1/2 - H}t^2(G - 1), \]
as \( t \to \infty \). Hence, for each \( s > 0 \),
\[-(1 - ix\theta_1(K_{H,\alpha}(t + 1, s) - K_{H,\alpha}(t, s)) + \theta_2K_{H,\alpha}(1, s))\alpha + (1 - ix\theta_2K_{H,\alpha}(1, s))\alpha - 1 \\
\sim i\alpha x\theta_1c_{H,\alpha}s^{1/\alpha - H}((1 - ix\theta_2K_{H,\alpha}(1, s))^{a-1} - 1)t^2(G - 1), \]
as \( t \to \infty \). The result then holds since \( \Gamma(-\alpha) = \frac{-\pi}{\alpha \Gamma(\alpha) \sin(\pi\alpha)} \).

7 Concluding Remarks

Willinger, Taqqu and Teverovsky \[17\] assert that a numerical analysis of stock price time series indicates long-range dependence with marginal tails heavier than Gaussian but lighter than stable. Moreover, it has been known that in shorter time, the asset price paths tend to lack higher moments, while they have a Gaussian behavior in long time. Indeed, fTSm achieves all those properties.

In Figure 1, we give typical sample paths of fTSm and of its background driving tempered stable processes, generated via the series representation presented in Proposition 4.1. We put the inner measures \( \rho_1 \) and \( \rho_2 \) as \( \rho_1(dx) = \delta_{-1,0}(dx) + \delta_{1,0}(dx) \) and \( \rho_2(dx) = 0.5^{-\alpha}\delta_{-0.5}(dx) + \delta_{1,0}(dx) \). (Tempered stable Lévy processes whose inner measure is discrete as above are studied in Carr, Geman, Madan and Yor \[2\] with emphasis on financial application and called CGMY processes.) Observe that sample paths of fTSm look like their background driving Lévy process as \( H \) is closer to \( 1/\alpha \), while the dependence range gets longer and fTSm paths behave milder for greater \( H \).

For the reader’s convenience for comparison, we draw in Figure 2 daily time series of TOYOTA shares on the Tokyo Stock Exchange, together with fTSm drawn in Figure 1. It is observed that in short time the time series looks like fTSm with \((0.8, 1.6, \rho_2)\), while behaving in a Gaussian manner in longer time.

To finish this study, let us mention that the long time behavior result provides yet another way to simulate sample paths of fBm. For simplicity, consider a symmetric inner measure with a very simple structure, e.g. \( \rho(dx) = 2^{-1}(\delta_{-1}(dx) + \delta_{1}(dx)) \), which reduces the random sequence \( \{V_i\}_{i \geq 1} \) to a sequence of iid Rademacher random variables \( \{\epsilon_i\}_{i \geq 1} \). Observe that
\[ h^{-G}\int_{0}^{H} \leq \sum_{i=1}^{\infty} \left( m(\rho) \left( \frac{\alpha T_i}{T} \right)^{-1/\alpha} h^{1/2 - 1/2} \right) \epsilon_i K_{H,\alpha}(t, T_i). \]
Clearly, for sufficiently large \( h \), the right hand side of the above behaves like
\[ h^{-1/2} \sum_{i=1}^{\infty} E_i U_i^{1/\alpha} \epsilon_i K_{H,\alpha}(t, T_i). \]
Figure 1: Typical sample paths of fTSm (thick line) and of its background driving TS process (thin line) generated via the series representation.

Figure 2: fTSm \((H, \alpha, \rho) = (0.8, 1.6, \rho_2)\) (left thick) and \((0.6, 1.9, \rho_1)\) (right thick) with (scaled) daily time series of TOYOTA shares on the Tokyo Stock Exchange; 247 days (left thin) and 493 days (right thin) up to 02/11/2005.
Theorem 6.4 (ii) tells us that this stochastic process (on $[0, T]$) approximates fBm. In order that its second moment is equal to that of a standard fBm, we set $h = \frac{2\alpha}{2+\alpha}N$ since then

$$E \left[ \left( h^{-1/2} \sum_{i=1}^{N} E_{i}U_{i}^{1/\alpha} \epsilon_{i}K_{H,\alpha}(t, T_{i}) \right)^{2} \right] = h^{-1}N \frac{2\alpha}{2+\alpha} t^{2G} = t^{2G}.$$ 

Therefore, for sufficiently large $N$, the stochastic process

$$\left\{ \left( \frac{2\alpha}{2+\alpha}N \right)^{-1/2} \sum_{i=1}^{N} E_{i}U_{i}^{1/\alpha} \epsilon_{i}K_{H,\alpha}(t, T_{i}) : t \in [0, T] \right\}$$

can be used for simulation of standard fBm.

**References**

[1] Benassi, A., Cohen, S., Istas, J. (2004) On roughness indices for fractional fields, *Bernoulli* **10**, 357-373.

[2] Carr, P., Geman, H., Madan, D., Yor, M. (2002) The fine structure of asset returns: an empirical investigation, *J. Business* **75**, 305-332.

[3] Decreusefond, L., Üstünel, A. (1997) Stochastic analysis of the fractional Brownian motion, *Potential Analysis* **10**, 177-214.

[4] Houdré, C., Kawai, R. (2005) An empirical study on the time dependence structure of assets price dynamics, *In preparation*.

[5] Kallenberg, O. (2001) *Foundations of Modern Probability* (2nd ed.), Springer.

[6] Lin, S.J. (1995) Stochastic analysis of fractional Brownian motions, *Stochastics Stochastics Rep.* **55**, 121-140.

[7] Mandelbrot, B., Van Ness, J.W. (1968) Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10**, 422-437.

[8] Nualart, D. (2003) Stochastic integration with respect to fractional Brownian motion and applications, *Contemporary Mathematics* **336**, 3-39.

[9] Protter, P. (1990) *Stochastic integration and differential equation*, Springer-Verlag.

[10] Rocha-Arteaga, A., Sato, K. (2003) *Topics in Infinitely Divisible Distributions and Lévy Processes*, Aportaciones Mathemáticas, Investigación **17**, Sociedad Matemática Mexicana.

[11] Rosiński, J. (1989) On path properties of certain infinitely divisible processes, *Stoch. Proc. Appl.* **33**, 73-87.
[12] Rosiński, J. (1990) An application of series representations to zero-one laws for infinitely divisible random vectors, In: Probability in Banach Spaces 7, Progress in Probability 25, Birkhäuser, 189-199.

[13] Rosiński, J. (2001) Series representations of Lévy processes from the perspective of point processes, In: Lévy Processes - Theory and Applications, Eds. Barndorff-Nielsen, O.E., Mikosch, T., Resnick, S.I., Birkhäuser, 401-415.

[14] Rosiński, J. (2004) Tempering stable processes, Preprint.

[15] Sato, K. (1999) Lévy processes and infinitely divisible distributions, Cambridge University Press.

[16] Samorodnitsky, G., Taqqu, M.S. (1994) Stable non-Gaussian random processes, Chapman & Hall.

[17] Willinger, W., Taqqu, M.S., Teverovsky, V. (1999) Stock market prices and long-range dependence, Finance Stochast. 3, 1-13.