Relativistic Topological Insulator Model

J. Gamboa$^1$ and F. Méndez$^1$

$^1$Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile

Abstract

A relativistic topological insulator model in three spatial dimensions which is a non trivial extension of the non-abelian Landau problem is proposed. The model is exactly soluble and energy levels have both a discrete and a continuous degeneracy. The chromomagnetic field is strong and the fermions are confined in a plane and the physical effects that appear reflects the $\mathbb{Z}_2$ symmetry.
The physics of a topological insulator is an interesting field of research not only because of its interest in condensed matter physics and statistical mechanics but also because of its potential applications in other fields of physics [1].

In general, a topological insulator resembles the FQHE but does not require an external magnetic field which, formally, is hidden in matter by a topological protection mechanism [2]. The technical way to implement these properties is due to Kane and Mele who proposed to add a spin-orbit term to the Hamiltonian in order to impose reversal-time invariance [3, 4].

In \(2 + 1\)-dimensions this procedure was also studied by Benevig and Zhang who explicitly showed how a topological insulator mimics as two Landau problems with opposite magnetic fields thus providing the \(Z_2\)-symmetry [5].

Under these conditions it seems natural to ask how this viewpoint works in the relativistic domain and how these ideas could provide a new approach to problems in high energy physics and mathematical physics [6].

In this letter we would like to propose an example of a relativistic system that contains all the properties of a topological insulator, is also exactly soluble and it can be the starting point for modeling dark matter problems.

Although the model we propose looks like Moshinsky’s Dirac oscillator [7] from both a physical and a mathematical point of view, it is very different. Some properties that the model contains are, on the one hand, a closed connection with the QHE and non-commutative geometry and on the other, it contains all the physical properties of a true topological insulator.

In order to formulate the model let us consider the non-trivial modification of the Dirac oscillator where, instead of making the substitution \(p_i \rightarrow \pi_i = p_i + im\beta \omega x_i\) as in [7], we analyze

\[
i \frac{\partial \psi}{\partial t} = H \psi,
\]

for the Hamiltonian

\[
H = \begin{pmatrix}
m & \sigma \cdot \pi \\
\sigma \cdot \pi & -m
\end{pmatrix},
\]

with \(\pi\) defined as

\[
\pi_i = \begin{cases}
(p_i + B\epsilon_{ij}x_j \otimes \sigma_3), & \{i, j\} = 1, 2, \\
p_3, & i = 3.
\end{cases}
\]
Here $B$ is a constant with canonical dimension $+2$ and $\sigma_3$ is defined in the color space. Note that the products with identity has been omitted, as for example $m \otimes 1$ and $p_3 \otimes 1$.

Note that the operators $x_i$ and $\pi_i$ satisfy

$$[x_i, \pi_j] = i\delta_{ij},$$
$$[\pi_i, \pi_j] = 2iB\epsilon_{ij}\sigma_3,$$  \hspace{1cm} (3)

and zero in all other cases.

The quantity $A^a_i = \epsilon_{ij}Bx_j\sigma^a$ with $a \in \{1, 2, 3\}$ is an element of the $SU(2)$ algebra, and then, the commutator (3) indicates that in the RHS, $A^3_i = \epsilon_{ij}Bx_j\sigma_3$ is a component of the $SU(2)$ gauge potential in the (internal) direction $a = 3$ equivalent to a constant chromomagnetic field [8]. Relation (3) furnishes an example of a deformed commutator as those appearing in non commutative quantum field theory, graphene or in non-commutative geometry.

The Hamiltonian in (1) can be written as follows

$$H = \begin{pmatrix}
  m \mathbb{1} \otimes \mathbb{1} & \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \\
  \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 & -m \mathbb{1} \otimes \mathbb{1}
\end{pmatrix},$$  \hspace{1cm} (4)

Since $H$ is independent of time we look for solutions with the form

$$\psi(x, t) = e^{-iEt} \begin{pmatrix}
  \Phi_1(x) \\
  \Phi_2(x)
\end{pmatrix},$$  \hspace{1cm} (5)

with

$$\Phi_1 = \phi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \phi_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi_2 = \chi_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \chi_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$  \hspace{1cm} (6)

and $\{\phi_i, \chi_j\}_{i,j \in \{1,2\}}$, two-components spinors.

Then the Dirac equation reads

$$\begin{pmatrix}
  \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \\
  \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3
\end{pmatrix} \Phi_2 = (E - m) \mathbb{1} \otimes \mathbb{1} \Phi_1,$$  \hspace{1cm} (7)

$$\begin{pmatrix}
  \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \\
  \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3
\end{pmatrix} \Phi_1 = (E + m) \mathbb{1} \otimes \mathbb{1} \Phi_2.$$  \hspace{1cm} (8)

This equation can be decoupled by using standard methods (see supplementary material in the appendix) to get

$$\begin{pmatrix}
  (p_1^2 + p_3^2 + B^2 x_3^2) \otimes \mathbb{1} - 2B(L_3 + \sigma_3) \otimes \sigma_3 \\
  (p_1^2 + p_3^2 + B^2 x_3^2) \otimes \mathbb{1} - 2B(L_3 + \sigma_3) \otimes \sigma_3
\end{pmatrix} \Phi_1 = (E^2 - m^2) \mathbb{1} \otimes \mathbb{1} \Phi_1,$$  \hspace{1cm} (9)
and similar expression for $\Phi_2$. Here where the index $\perp$ denotes quantities in the plane $x_1 - x_2$ and the angular momentum along the axis $x_3$ is $L_3 = x_1p_2 - x_2p_1$. From now on, we will focus only in the solution for $\Phi_1$.

The term $-2BL_3\sigma_3$ is the spin orbit coupling containing the up and down projections of the magnetic field and $B\sigma_3$ is a constant that in the end will only contribute to the energy spectrum.

Operator in (9) is diagonal and commutes with $p_3 \otimes \mathbb{1}$. Therefore we look for solutions with the form

$$\Phi_1 = e^{ip_3x_3} \begin{pmatrix} \varphi_1(x_1, x_2) \\ \varphi_2(x_1, x_2) \end{pmatrix}. $$

Equivalently $\phi_i = e^{ip_3x_3} \varphi_i$ in (6), and then spinors $\varphi_i$ satisfy

$$\begin{align*}
(p_1^2 + B^2x_1^2 - 2B(L_3 + \sigma_3)) \varphi_1 &= (E^2 - m^2 - p_2^2)\varphi_1, \\
(p_1^2 + B^2x_1^2 + 2B(L_3 + \sigma_3)) \varphi_2 &= (E^2 - m^2 - p_2^2)\varphi_2,
\end{align*}$$

Equations (10) and (11) are related through the transformation $B \to -B$. Therefore, the two independent equations are

$$\begin{align*}
(p_1^2 + B^2x_1^2 - 2BL_3) \varphi_\pm &= (E^2 - m^2 - p_2^2 \pm 2B) \varphi_\pm.
\end{align*}$$

Here $\varphi_\pm$ denotes the two components of the spinor $\varphi_1$, and the two components of $\varphi_2$ once the $B \to -B$ transformation has been performed.

For convenience let us define (12) in terms of the dimensionless variables

$$\bar{x} = \sqrt{B} x_\perp,$$

so (12) becomes

$$\begin{align*}
(p^2 + \bar{x}^2 - 2L_3) \varphi_\pm &= \left(\frac{E^2 - m^2 - p_2^2}{B} \pm 2\right) \varphi_\pm.
\end{align*}$$

with $p^2 = -\nabla_\bar{x}^2$.

In order to find the explicit solutions, let us note that the effective movement takes place in the plane $x_1 - x_2$ and it is enough to solve the equation (12) in polar coordinates.

We look for solutions with the form $\varphi_\pm \sim R_\pm(r) e^{i\ell\theta}$, where $\ell \in \mathbb{Z}$, $\theta$ is the polar angle and $r^2 = \bar{x}_1^2 + \bar{x}_2^2$. The equation (12) becomes

$$
R''_\pm + \frac{R'_\pm}{r} + \left[\mathcal{E}^2_\pm - r^2 - \frac{\ell^2}{r^2}\right] R_\pm = 0,
$$

(14)
with $\mathcal{E}_\pm^2 = \frac{E^2-m^2-p^2}{B} + 2(\ell \pm 1)$.

The change of variables

$$R_\pm(r) \propto e^{-\frac{r^2}{2}} r^{|\ell|} g_\pm(r),$$

(15)

yields to

$$g''_\pm(r) + \left( \frac{2|\ell| + 1}{r} - 2r \right) g'_\pm(r) + \left( \mathcal{E}_\pm^2 - 2(|\ell| + 1) \right) g_\pm(r) = 0.$$  

(16)

Finally making $\xi = r^2$ in (16), we obtain the confluent hypergeometric equation

$$\xi g''_\pm(\xi) + (|\ell| + 1 - \xi) g'_\pm(\xi) + \left( \frac{\mathcal{E}_\pm^2}{4} - \frac{|\ell| + 1}{2} \right) g_\pm(\xi) = 0,$$  

(17)

The requirement to have a square integrable solution for $\xi \in [0, \infty)$ imposes

$$\frac{\mathcal{E}_\pm^2}{4} - \frac{|\ell| + 1}{2} = n_\pm,$$  

(18)

where $n_\pm \in \mathbb{Z}$, implying the spectrum

$$E_\pm^2 = m^2 + p_z^2 + 2B \left( 2n_\pm + |\ell| - \ell \pm 1 \right).$$  

(19)

There is an apparent inconsistency in the spectrum (19), however since particle-antiparticle symmetry implies that $n_- \leftrightarrow n_+$ and $\ell \leftrightarrow -\ell$, the condition $E_- = E_+$ yields

$$n_- - n_+ = 1.$$  

(20)

The last equation shows the complete symmetry of the particles and antiparticles spectrum.

The convergent solution of (17) is the hypergeometric function $F(n_\pm, |\ell| + 1, r^2)$ –except by a normalization constant– which coincides with the associated Laguerre polynomials $L_{n_\pm}(|\ell|) r^2$.

Finally, restoring the original variables, the component $\Phi_1$ (see (5) and (6)) turn out to be

$$\left( \Phi_1 \right)_n^\ell(x) = e^{i p_z x_3 + i \ell \theta} e^{\frac{Bp_z^2}{2}} \rho^{|\ell|} \left[ \begin{array}{c} C_+ L_{n+1}^{|\ell|}(B \rho^2) \\ C_- L_{n+1}^{|\ell|}(B \rho^2) \end{array} \right] \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \left( D_+ L_{n}^{|\ell|}(B \rho^2) \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$  

(21)

where $C_\pm$ and $D_\pm$ are normalization constants and $\rho^2 = x_1^2 + x_2^2$.  

5
The other component, namely $\Phi_2$, has the same form since it is the solution of the same operator appearing in (9). However, it is not independent of $\Phi_1$ due to the relation (8). In other words, given $\Phi_1$ in (21), the second component $\Phi_2$ in (5) is

$$
(\Phi_2)_n^\ell(x) = (E + m)^{-1} \left[ \sigma \cdot p \otimes 1 + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \right] (\Phi_1)_n^\ell
$$

It is worth mentioning that even though in the model discussed in this paper the chromo-magnetic field is strong and the fermions are confined in a plane, the $\mp B$ effect that appears in (12) reflects the $Z_2$ symmetry of the model from Benevig-Zhang [5].

The model discussed in this paper can be extended in various directions such as spin non-commutativity [9], graphene in the sense discussed in [10]. However, the most interesting idea is the potential applications to dark matter and the analogies that can be established with topological insulators.

We will discuss these ideas in a forthcoming paper.

ACKNOWLEDGEMENTS

It is my pleasure to thank Prof. A. P. Balachandran for the discussions and comments. One of us (J.G.) thanks the Alexander von Humboldt Foundation by support. This research was supported by DICYT 042131GR (J.G.) and 041931MF (F.M.).

Appendix A: Supplementary material

Consider (11) written in the form

$$
H = \begin{pmatrix}
    m \mathbb{1} \otimes \mathbb{1} & \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \\
    \sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 & -m \mathbb{1} \otimes \mathbb{1}
\end{pmatrix}, \quad (A1)
$$

with $\hat{z}$ the direction $i = 3$ in space. We look for solutions with the form

$$
\Psi = e^{-iEt} \begin{pmatrix}
    \Phi_1(x) \\
    \Phi_2(x)
\end{pmatrix},
$$

and therefore, equation of motion reads

$$
\begin{align}
\sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \Phi_2 &= (E - m) \mathbb{1} \otimes \mathbb{1} \Phi_1, \quad (A2) \\
\sigma \cdot p \otimes \mathbb{1} + B(\sigma \times x) \cdot \hat{z} \otimes \sigma_3 \Phi_1 &= (E + m) \mathbb{1} \otimes \mathbb{1} \Phi_2. \quad (A3)
\end{align}
$$
This system can be decoupled in the standard way. For example, if we multiply first equation by \((E + m) \mathbb{1} \otimes \mathbb{1}\) we obtain
\[
\left[ \mathbf{\sigma} \cdot \mathbf{p} \otimes \mathbb{1} + B(\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}} \otimes \sigma_3 \right] ((E + m) \mathbb{1} \otimes \mathbb{1}) \Phi_2 = (E^2 - m^2) \mathbb{1} \otimes \mathbb{1} \Phi_1,
\]
and we can replace now \(\Phi_2\) from the second equation to obtain
\[
\left[ \mathbf{\sigma} \cdot \mathbf{p} \otimes \mathbb{1} + B(\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}} \otimes \sigma_3 \right]^2 \Phi_1 = (E^2 - m^2) \mathbb{1} \otimes \mathbb{1} \Phi_1.\quad (A4)
\]
Let us evaluate all terms in the LHS in previous expression.
\[
(\mathbf{\sigma} \cdot \mathbf{p}) = \mathbf{p}^2,\quad (A5)
\]
since \(\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k\). On the other hand,
\[
[(\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}}]^2 = \epsilon_{ij3} \sigma_i x_j \epsilon_{mn3} \sigma_m x_n
\]
\[
= x_j x_n \epsilon_{ij3} \epsilon_{mn3} (\delta_{im} + i \epsilon_{imk} \sigma_k)
\]
\[
= x_j x_n (\epsilon_{ij3} \epsilon_{mn3} + i \sigma_k \epsilon_{ij3} \epsilon_{mn3} \epsilon_{imk})
\]
\[
= x_j x_n (\delta_{jn} \delta_{33} - \delta_{j3} \delta_{n3}) + x_j x_n i \sigma_k \epsilon_{mn3} (\delta_{jm} \delta_{3k} - \delta_{jk} \delta_{3m})
\]
\[
= \mathbf{x}^2 - (x_3)^2 + i(x_m x_n \epsilon_{mn3} \sigma_3 + \mathbf{x} \cdot \mathbf{\sigma} x_n \epsilon_{n33})
\]
\[
= x_1^2 + x_2^2
\]
\[
\equiv x_1^2.\quad (A6)
\]
The last term is (up to the constant factor \(B\))
\[
(\mathbf{\sigma} \cdot \mathbf{p})((\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}}) + ((\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}})(\mathbf{\sigma} \cdot \mathbf{p}) \equiv \{ \mathbf{\sigma} \cdot \mathbf{p}, (\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}} \},
\]
Let us write both terms separately
\[
(\mathbf{\sigma} \cdot \mathbf{p})((\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}}) = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)(\sigma_1 x_2 - \sigma_2 x_1)
\]
\[
= p_1 x_2 - p_2 x_1 - \sigma_1 \sigma_2 p_1 x_1 + \sigma_2 \sigma_1 p_2 x_2 + \sigma_3 \sigma_1 p_3 x_2 - \sigma_3 \sigma_2 p_3 x_1
\]
\[
((\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{p}})(\mathbf{\sigma} \cdot \mathbf{p}) = (\sigma_1 x_2 - \sigma_2 x_1)(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)
\]
\[
= p_1 x_2 - p_2 x_1 + \sigma_1 \sigma_2 x_2 p_2 - \sigma_2 \sigma_1 x_1 p_1 + \sigma_1 \sigma_3 x_2 p_3 - \sigma_2 \sigma_3 x_1 p_3,\quad (A7)
\]
therefore
\[
\{ \mathbf{\sigma} \cdot \mathbf{p}, (\mathbf{\sigma} \times \mathbf{x}) \cdot \hat{\mathbf{z}} \} = -2 L_3 + \sigma_1 \sigma_2 (x_2 p_2 - p_1 x_1) + \sigma_2 \sigma_1 (p_2 x_2 - x_1 p_1) + p_3 x_2 (\sigma_3 \sigma_1 + \sigma_1 \sigma_3) - x_1 p_3 (\sigma_3 \sigma_2 + \sigma_2 \sigma_3)
\]
\[
= -2 L_3 + \sigma_1 \sigma_2 (x_2 p_2 - p_2 x_2 + x_1 p_1 - p_1 x_1),\quad (A8)
\]

7
where, in the last line, we have used $\sigma_1\sigma_2 = -\sigma_2\sigma_1$ and $\sigma_3\sigma_2 = -\sigma_2\sigma_3$. Finally,

$$\{\sigma \cdot p, (\sigma \times x) \cdot \hat{z}\} = -2L_3 + \sigma_1\sigma_2([x_1, p_1] + [x_2, p_2])$$

$$= -2L_3 + 2i\sigma_1\sigma_2$$

$$= -2L_3 - 2\sigma_3 \quad (A9)$$

By replacing (A5), (A6) and (A9) in (A4) we obtain the equation for $\Phi_1$

$$\left(\begin{array}{c}
E^2 - m^2 - p_3^2 - p_1^2 - B^2x_1^2 + 2B(L_3 + \sigma_3) \\
0
\end{array}\right) \Phi_1 = (E^2 - m^2) \mathbb{1} \otimes \mathbb{1} \Phi_1 \quad (A10)$$

where the subindex $\perp$ denotes quantities in the plane $x_1 - x_2$ and the angular momentum along the axis $x_3$ is $L_3 = x_1p_2 - x_2p_1$.

Previous equation can be recast in matrix form as follows

$$\begin{pmatrix}
E^2 - m^2 - p_3^2 - p_1^2 - B^2x_1^2 + 2B(L_3 + \sigma_3) & 0 \\
0 & E^2 - m^2 - p_3^2 - p_1^2 - B^2x_1^2 - 2B(L_3 + \sigma_3)
\end{pmatrix}\Phi_1 = 0 \quad (A11)$$

Since $p_3$ is a conserved quantity, we look for solutions with the form

$$\Phi_1 = e^{ip_3x_3} \begin{pmatrix}
\varphi_1(x_1, x_2) \\
\varphi_2(x_1, x_2)
\end{pmatrix}.$$

The spinors $\varphi_i$ satisfy

$$\begin{array}{c}
(p_1^2 + B^2x_1^2 - 2B(L_3 + \sigma_3)) \varphi_1 = (E^2 - m^2 - p_1^2)\varphi_1, \\
(p_1^2 + B^2x_1^2 + 2B(L_3 + \sigma_3)) \varphi_2 = (E^2 - m^2 - p_1^2)\varphi_2,
\end{array} \quad (A12)$$

(A13)

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82 (2010), 3045; X. L. Qi and S. C. Zhang, Rev. Mod. Phys. 83 (2011) no.4, 1057-1110.

[2] G. Y. Cho and J. E. Moore, Annals Phys. 326 (2011), 1515-1535.

[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).

[4] L. Fu, C. L. Kane and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).

[5] B. A. Bennevig and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).
[6] J. Gamboa and F. Mendez, “Topological Insulators Quantum Mechanics,” [arXiv:2110.11455 [hep-th]].

[7] M. Moshinsky and A. Szczepaniak, J. Phys. 22, L817 (1989).

[8] L. S. Brown and W. I. Weisberger, Nucl. Phys. B 157 (1979), 285 [erratum: Nucl. Phys. B 172 (1980), 544].

[9] H. Falomir, J. Gamboa, M. Loewe, F. Mendez and J. C. Rojas, Phys. Rev. D 85 (2012), 025009; M. Gomes, V. G. Kupriyanov and A. J. da Silva, Phys. Rev. D 81 (2010), 085024.

[10] H. Falomir, J. Gamboa, M. Loewe and M. Nieto, J. Phys. A 45 (2012), 135308; D. Nath, M. Presilla, O. Panella and P. Roy, fields,” EPL 123 (2018) no.2, 20008; C. Bastos, O. Bertolami, N. Costa Dias and J. Nuno Prata, Int. J. Mod. Phys. A 28 (2013), 1350064.