Quantum scalar field in $D$-dimensional static black hole space-times.

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An Euclidean approach for investigating quantum aspects of a scalar field living on a class of $D$-dimensional static black hole space-times, including the extremal ones, is reviewed. The method makes use of a near horizon approximation of the metric and $\zeta$-function formalism for evaluating the partition function and the expectation value of the field fluctuations $\langle \phi^2(x) \rangle$. After a review of the non-extreme black hole case, the extreme one is considered in some details. In this case, there is no conical singularity, but the finite imaginary time compactification introduces a cusp singularity. It is found that the $\zeta$-function regularized partition function can be defined, and the quantum fluctuations are finite on the horizon, as soon as the cusp singularity is absent, and the corresponding temperature is $T = 0$.

I. INTRODUCTION.

The issue of determining the equilibrium (Unruh-Hawking) temperature of a black hole, is important. In fact, one can extract thermodynamical informations from its knowledge: for example the Bekenstein-Hawking entropy (i.e. the tree-level contribution to the entropy) can be defined as the response of the free energy of the black hole to the change of this equilibrium temperature. Furthermore, it defines the admissible temperatures of thermal states of free scalar fields in a static and globally hyperbolic space-time region with horizons.

As is well known, there exist several methods for evaluating the possible equilibrium temperature of a stationary black hole. Whithin the simplest of these methods, one has to make a Wick rotation of the time coordinate (passing in this way to the Euclidean time $\tau = it$), and eliminate all the metric (conical) singularities connected to the horizon by an opportune choice of the time periodicity $\beta_M$. Then, one has to impose the KMS condition for thermal states $\mathbb{B}$, i.e. to impose the periodicity condition on the imaginary time dependence of the thermal Wightman functions, and interprete the common period $\beta_T$ as the inverse of the temperature $T$ of the state. Although this procedure determines the correct Unruh-Hawking temperature in the case of a non-extreme black hole, it does not apply to the extreme case (for example to the case of an extreme Reissner-Nordström black hole), since one is unable to determine the time periodicity of the manifold $\beta_M$.

Later, a more sophisticated Lorentzian method was introduced in $\mathbb{B}$ and successively developed in $\mathbb{C}$ and $\mathbb{D}$. Without entering in the details of this approach, we only recall that the method is connected to the well known Hadamard expansion of the two-point Green functions in a curved background and in the limit of coincidence of the arguments. Basically in $\mathbb{B}$ was proved that assuming fairly standard axioms of quantum (quasi-free) field theory (such as local definitness and local stability in a stationary space-time region), then, when the distance between the two arguments vanishes, the thermal Wightman functions in the interior of this region will transform into non-thermal and massless Wightman functions in Minkowski space-time. This point coincidence behavior of the Wightman functions, must hold for any physically sensible state (thermal or not), and, in the case that the space-time region one is dealing with is just a part of the whole manifold separated by event horizons, it must hold on the horizons. This constraint actually selects the correct temperatures $T = \beta_T^{-1} = \beta_M^{-1}$, in the case of Rindler and Schwarzschild space-times. Then these results have been generalized in $\mathbb{C}$ to a large class of space-times admitting an appropriate reflection isometry.

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Both Haag’s method \[4\] and Kay-Wald approach \[5\], which, as they stand, work only for space-times with an intersection between past and future horizons, were extended in \[6,8\] for working in more physical situations than the eternal black holes one, including the extreme case.

However, all of these “Lorentzian” methods involve a certain amount of calculations (for example the procedure developed in \[8\], requires the evaluation of all the possible geodesics which start from the horizon); for this fact, even if it is not difficult to foresee a possible generalization to, say, $D$-dimensional extreme black holes, the concrete computation does not appear an easy task.

In this paper, making use of Euclidean methods, we would like to obtain some informations about the equilibrium temperature of a class of static, $D$-dimensional black holes, evaluating quantities such as the field fluctuations and the one-loop partition function. The inverse of the temperature is formally introduced as the period of the compactified imaginary time $0 \leq \tau \leq \beta$.

In order to deal with explicit calculations, we will make use of a near horizon approximation of the metric. This approximation may also be justified observing that only near the horizon interesting physical effects are supposed to be relevant.

First, the case of non-extreme black holes is reconsidered. Here, as is well known, a conical singularity is present. We will show that its presence leads to divergences of the field fluctuations on the horizon. However in this case it is known that as soon as the smoothness of the manifold is required, the Hawking temperature, as well as the absence of the divergences, is recovered.

The analysis then extended to the extreme black holes case. In this case, no conical singularity is present, but the compactification of the imaginary time and the related periodic identification, induces an isometry containing parabolic elements (translation in $\tau$), so that a cusp singularity appears. Its presence leads to the following features:

1. The field fluctuation have divergences on the horizon;

2. The global $\zeta$-function, besides the horizon divergences, does not exist, and requires a further regularization.

These undesired features disappear as soon as the imaginary time period is taken to be $\infty$, namely the associated temperature is to be $T = 0$, in agreement with the four-dimensional results obtained in \[7,8\].

The main objection to the approach proposed here could be the use of a near horizon approximation of the metric. However, we stress that in \[4\] only the limit form of the metric near the horizon was used in order to obtain the Unruh-Hawking temperature; moreover also the results in \[7,8\] were derived in a near horizon approximation contest.

The paper is organized as follows. In Sec. II we review the evaluation of the field fluctuations within the $\zeta$-function regularization procedure, while in Sec. III we will derive a near horizon approximation of the generic line element describing a non-extreme and an extreme black hole. Then in Sec. IV and V we will discuss in detail these two cases, taking advantage of the approximation done. The paper ends with some concluding remarks in Sec. VI.

### II. EVALUATING THE FIELD FLUCTUATIONS WITH THE $\zeta$-FUNCTION PROCEDURE.

As anticipated in the Introduction, we need to evaluate the expectation value of the field fluctuations $\langle \phi^2(x) \rangle$, in the framework of the $\zeta$-function regularization procedure. Within this approach (see \[9\] for an exhaustive discussion) one has \[10,11\]

$$\langle \phi^2(x) \rangle = -\frac{2}{\sqrt{g(x)}} \frac{\delta S_{\text{eff}}}{\delta J(x)} \bigg|_{J(x) \equiv 0} = \frac{1}{\sqrt{g(x)}} \frac{d}{ds} \left[ \frac{\delta \zeta(s; A\ell^2)}{\delta J(x)} \right]_{s=J(x)=0}, \quad (2.1)$$

where $J(x)$ is a classical source, and $\ell$ is the usual arbitrary parameter (with the dimension of mass$^{-1}$) necessary from dimensional considerations.

By a direct calculation, it follows that the $\zeta$-regularized field fluctuations turns out to be

$$\langle \phi^2(x) \rangle_{\text{ren}} = \ell^2 \frac{d}{ds} \left[ s \zeta(s+1; x|A\ell^2) \right]_{s=0}, \quad (2.2)$$

with the $\zeta$-function evaluated when the source $J(x)$ vanishes.

By making use of the the Laurent expansion of $\zeta(s+1; x|A\ell^2)$, and extracting from the $\zeta$-function the $\ell^2$ dependence, we can rewrite \[2.2\] as
\[ \langle \phi^2(x) \rangle_{\text{ren}} = \lim_{s \to 0} \left[ \zeta(s + 1; x|A) - \frac{1}{s} \text{Res} \zeta(s + 1; x|A) - \text{Res} \zeta(s + 1; x|A) \ln \ell^2 \right]. \tag{2.3} \]

Notice that when the manifold is smooth, the meromorphic structure of the \( \zeta \)-function is known (Seeley’s Theorem). In particular for a differential elliptic operator of the second order (Laplacian) one has

\[ \Gamma(z) \zeta(z; \vec{x}|L_N) = \sum_{r=0}^{\infty} \frac{A_r(\vec{x}|L_N)}{z + r - \frac{N}{2}} + \text{analytic part}, \quad A_r(\vec{x}|L_N) = \frac{a_r(\vec{x}|L_N)}{(4\pi)^{\frac{N}{2}}}. \tag{2.4} \]

The spectral coefficients \( a_r(\vec{x}|L_N) \) are computable functions known as the Seeley-de Witt coefficients.

As a consequence, if the dimension of the smooth manifold is odd, the \( \zeta \)-function is regular at \( z = 1 \) \((s = 0)\) and the dependence on the scale parameter \( \ell \) disappears and one gets

\[ \langle \phi^2(x) \rangle_{\text{ren}} = \zeta(1; x|A). \tag{2.5} \]

On the other hand, if the dimension is even, there is a simple pole at \( z = 1 \), and the \( \ell \) ambiguity will be present.

### III. NEAR HORIZON APPROXIMATION OF THE METRIC.

The metric for a general static spherically symmetric \( D \)-dimensional space-time, analytically continued into the Euclidean space, reads

\[ ds^2 = f(r)d\tau^2 + \frac{1}{h(r)}dr^2 + r^2d\Sigma_N^2, \quad x = (\tau, r, \vec{x}), \tag{3.1} \]

where \( \tau = it \) is the Euclidean time, \( f \) and \( h \) are arbitrary functions of \( r \) (which are constant in the \( r \to \infty \) limit, if the space-time has to be asymptotically flat), and \( d\Sigma_N \) represent the line element of a smooth \( N \)-dimensional (transverse) manifold without boundary (\( \vec{x} \) are the transverse coordinates).

For this metric representing a black hole, one demands the presence, at \( r = r_+ \), of a zero in both \( f \) and \( h \), so that, according to the nature of this zero, one finds the following two interesting cases.

#### A. Non-extremal case.

In the case of a non-extremal black hole, one has a simple zero at \( r = r_+ \), so that the functions \( f \) and \( h \) can be expanded as [12]

\[ f(r) \simeq f'(r_+)(r - r_+), \quad h(r) \simeq h'(r_+)(r - r_+). \tag{3.2} \]

Thus, after changing to the coordinates \((\rho, \theta, \vec{x})\) by means of

\[ \rho^2 = \frac{4}{h'(r_+)}(r - r_+), \quad \theta = \frac{1}{2} \sqrt{f'(r_+)h'(r_+)}\tau, \tag{3.3} \]

the geometry near the event horizon is described by the approximated line element

\[ ds^2 \simeq d\rho^2 + \rho^2 d\theta^2 + r_+^2 d\Sigma_N^2. \tag{3.4} \]

We may generalize the argument to black hole solutions in semiclassical gravity. In this case, near the horizon, one has

\[ f(r) \simeq C_f(r - r_+)^{c_1}, \quad h(r) \simeq C_h(r - r_+)^{c_2}. \tag{3.5} \]

with the constants \( C_f > 0, C_h > 0, c_1 > 0, 0 < c_2 < 2 \). Here \( c_1 \) may be less than one, and the first derivative may not exist at the horizon. However, if \( c_1 = 2 - c_2 \), it is easy to show that by means of the following coordinates transformation

\[ (r - r_+)^{c_1/2} = \frac{c_1}{2} \sqrt{C_h} \rho, \quad \theta = \frac{c_1}{2} \sqrt{C_h C_f} \tau. \tag{3.6} \]
the line black hole element reduces again to the line element (3.4). For example, the previous case corresponds to \( c_1 = c_2 = 1 \), and very recently, in [13] the case \( c_1 = 1/2 \) and \( c_2 = 3/2 \) have been considered; in any case notice that since \( c_2 < 2 \), the proper radial distance to the horizon is finite.

Now, finite temperature effects are assumed to arise when the Euclidean time \( \tau \) (correspondingly \( \theta \)) is compactified requiring \( 0 \leq \tau \leq \beta \) \((0 \leq \theta \leq \gamma)\), with \( \beta \) the inverse of the temperature. So, for arbitrary \( \beta \) \((\gamma)\), the manifold \( \mathcal{M}^D \) shows, near the horizon, the topology of \( \mathcal{C}_\gamma \times \Sigma^N \), \( \mathcal{C}_\gamma \) being the simple two-dimensional flat cone with deficit angle \( 2\pi - \gamma \).

In such a space-time, one usually determines the temperature of the black hole, by requiring the absence of the conical singularity [1]: the manifold, in fact, is not smooth, showing a conical singularity at \( \rho = 0 \) unless \( \gamma = 2\pi \). In this way the temperature is found to be

\[
T = \frac{\sqrt{f'(r_+)}h'(r_+) \pi}{4}, \quad T = \frac{c_1}{4\pi} \sqrt{C_h C_f},
\]

respectively, which are the Unruh-Hawking temperatures of the black holes.

We will show that \( \gamma = 2\pi \) is the only possible requirements for having a well-behaved \( \langle \phi^2(x) \rangle \) on the horizon. Now, finite temperature effects are assumed to arise when the Euclidean time \( \tau \) (correspondingly \( \theta \)) is compactified requiring \( 0 \leq \tau \leq \beta \) \((0 \leq \theta \leq \gamma)\), with \( \beta \) the inverse of the temperature. So, for arbitrary \( \beta \) \((\gamma)\), the manifold \( \mathcal{M}^D \) shows, near the horizon, the topology of \( \mathcal{C}_\gamma \times \Sigma^N \), \( \mathcal{C}_\gamma \) being the simple two-dimensional flat cone with deficit angle \( 2\pi - \gamma \).

B. Extreme case.

In the case of an extreme black hole, one has a double zero at \( r = r_+ \), so that the behavior of \( f \) and \( h \) near the horizon, is [12]

\[
f(r) \simeq \frac{1}{2} f''(r_+)(r - r_+)^2, \quad h(r) \simeq \frac{1}{2} h''(r_+)(r - r_+)^2.
\]

Thus, if we define the new coordinates \((\rho, \theta, \bar{z})\) by means of

\[
\rho = \sqrt{\frac{2}{f''(r_+)}(r - r_+)^{-1}}, \quad \theta = \sqrt{\frac{h''(r_+)}{2} \tau} = \frac{\tau}{b},
\]

we get the approximated line element

\[
ds^2 \simeq \frac{b^2}{\rho^2} (d\rho^2 + d\theta^2) + r_+^2 d\Sigma_N^2.
\]

So, once the compactification in the Euclidean time is carried over, the manifold shows the topology \( \mathbb{H}^2/\Gamma \times \Sigma^N \), \( \mathbb{H}^2 \) being the two-dimensional hyperbolic space, and \( \Gamma \) being the (discontinuous and fixed-point-free) group of isometry induced by the identification \( \theta \sim \theta + n \gamma \).

Notice that in this case it is not possible to determine the temperature by using the method of the conical singularity, since no conical singularity is present. It is anyway not correct to deduce from this fact that such a manifold admits any temperature [4], since it is known [3] that, \( T = 0 \) is the only physical temperature admissible in the case of a four-dimensional extreme Reissner-Nordström black hole (which can be recovered by setting \( d\Sigma_N^2 = d\Omega_2 \) in (3.10)). Futher evidence in favour of this fact comes from the absence of the Hawking radiation in the extreme Reissner-Nordström black hole [13].

Again we will see that this is the only temperature which gives a well behaved \( \langle \phi^2(x) \rangle \) on the horizon, and a smooth manifold near it.

Without loss of generality, we set \( b^2 \) as well as \( r_+ \) equal to 1.

IV. FIRST CASE: \( \mathcal{M}^D = \mathcal{C}_\gamma \times \Sigma^N \).

As we mentioned in the Introduction, our method requires the evaluation of the expectation value of the squared of the scalar field, using the \( \zeta \)-function regularization tecnique. We can so start by reviewing the evaluation of the heat kernel and the local \( \zeta \)-function on \( \mathcal{M}^D = \mathcal{C}_\gamma \times \Sigma^N \) (for a complete discussion see [16]).
We consider a massless and minimally coupled scalar field on $C_\gamma$, so that the associated operator is the pure Laplacian $L_\gamma = -\nabla_\gamma^2 + \frac{2}{r^2} \partial_\rho + \frac{1}{r^2} \partial_\rho^2$; the spectral properties of this operator are well known, and, in fact, a complete set of normalized eigenfunctions is easily found to be

$$\psi_{n\lambda} = \frac{1}{\sqrt{\gamma}} e^{\frac{2\pi i n}{\gamma} \theta} J_{\nu_n}(\lambda \rho), \quad \nu_n = \frac{2\pi |n|}{\gamma}, \quad n \in \mathbb{Z},$$

(4.1)

together with its complex conjugate.

Here $\lambda^2$ ($\lambda \geq 0$) is the eigenvalue corresponding to $\psi$ and $\psi^*$, while $J_{\nu}$ is the regular Bessel function. So, using the standard separation of variables, it is easy to get the spectrum and eigenfunctions of the operator $L_D = -\nabla_\gamma + L_N$ on $\mathcal{M}^D = C_\gamma \times \Sigma^N$, $L_N$ being a Laplace-like operator on $\Sigma^N$ including, eventually, a mass and a scalar curvature coupling term. Moreover, since we suppose $\Sigma^N$ an arbitrary smooth manifold without boundary, all known results concerning the heat kernel and the $\zeta$-function for $L_N$ on $\Sigma^N$ (which we assume to be known) are applicable.

In particular the heat kernel has the usual asymptotic expansion (see also Sec. II)

$$K(t; \vec{x}|L_N) \simeq \sum_{r=0}^{\infty} A_r(\vec{x}|L_N) t^{-\frac{r}{2}},$$

(4.2)

and the meromorphic structure of the local $\zeta$-function reads

$$\Gamma(s)\zeta(s; \vec{x}|L_N) = \sum_{r=0}^{\infty} \frac{A_r(\vec{x}|L_N)}{s + r - \frac{1}{2}} + J(s; \vec{x}|L_N),$$

(4.3)

where $J(s; \vec{x}|L_N)$ is the (generally unknown) analytic part. Here we have supposed the absence of zero modes, but one can easily take them into account with a simple modification of the formulas.

We can now derive the meromorphic structure of $\zeta_\gamma(s; \vec{x}|L_D)$ on $\mathcal{M}^D = C_\gamma \times \Sigma^N$. To this aim, one can use the factorization property of the heat kernel

$$K_\gamma(t; x|L_D) = K(t; \theta, \rho|L_\gamma)K(t; \vec{x}|L_D),$$

(4.4)

in which the heat kernels of the Laplace-like operators on $\mathcal{M}^D$, $C_\gamma$, and $\Sigma^N$, respectively appear.

By taking the Mellin transform of (4.4), one usually gets the Dikii-Gelfand representation of the $\zeta$-function, from which the meromorphic structure can be deduced.

Anyway in dealing with the conical manifold one has a convergence obstruction, in the meaning that there are no value of $s$ for which the Mellin transform of (4.4) is a finite quantity. The solution to this problem has been suggested by Cheeger [17], and simply consist in a separation between higher and lower eigenvalues. In practice we split the sum which appears in the heat kernel (and in the related $\zeta$-function) in two sums, the first over the lower eigenvalues, and the second over the higher ones; then, after the analytic continuation is performed, one may define the full $\zeta$-function by summing up the two contributions obtained in this way (of course such a definition has all the requested properties and coincides with the usual one if the manifold is smooth).

So we set

$$\zeta_<(s; x|L_D) = \int_0^{\infty} dt t^{s-1} K_<(t; \theta, \rho|L_\gamma)K(t; \vec{x}|L_N),$$

(4.5)

$$\zeta_>(s; x|L_D) = \int_0^{\infty} dt t^{s-1} K_>(t; \theta, \rho|L_\gamma)K(t; \vec{x}|L_N),$$

(4.6)

where $K_<(t; \theta, \rho|L_\gamma)$ and $K_>(t; \theta, \rho|L_\gamma)$ are, respectively, the “lower” and the “higher” heat kernels, which are related to the corresponding $\zeta$-function by the relations

$$K_<(t; \theta, \rho|L_\gamma) = \frac{1}{2\pi i} \int_{\frac{1}{2} < \text{Re} (s) < 1} ds t^{-s} \Gamma(s) \zeta_<(s; \theta, \rho|L_\gamma),$$

(4.7)

$$K_>(t; \theta, \rho|L_\gamma) = \frac{1}{2\pi i} \int_{\frac{1}{2} < \text{Re} (s) < 1+\nu_1} ds t^{-s} \Gamma(s) \zeta_>(s; \theta, \rho|L_\gamma),$$

(4.8)

$$\zeta_<(s; \theta, \rho|L_\gamma) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} K_<(t; \theta, \rho|L_\gamma), \quad \frac{1}{2} < \text{Re} (s) < 1,$$

(4.9)

$$\zeta_>(s; \theta, \rho|L_\gamma) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} K_>(t; \theta, \rho|L_\gamma), \quad \frac{1}{2} < \text{Re} (s) < 1 + \nu_1,$$

(4.10)
and, by definition,
\[
K_\gamma(t; \theta, \rho|L_\gamma) = K_<\gamma(t; \theta, \rho|L_\gamma) + K_>\gamma(t; \theta, \rho|L_\gamma),
\]
(4.11)
\[
\zeta_\gamma(s; \theta, \rho|L_\gamma) = \zeta_<\gamma(s; \theta, \rho|L_\gamma) + \zeta_>\gamma(s; \theta, \rho|L_\gamma).
\]
(4.12)

Now, making use of the Mellin-Parseval identity and paying attention to the range of convergence, one gets, for \( \text{Re}(s) > 1 + \frac{N}{2} \), the following representation [13]
\[
\zeta_\gamma(s; \bar{x}|L_D) \simeq \frac{\zeta(s - 1; \bar{x}|L_N)}{2\gamma(s - 1)} + \frac{1}{\gamma \Gamma(s)} \sum_{r=0}^{P} A_r(\bar{x}|L_N)I_\gamma(s + r - \frac{N}{2})\rho^{2r - D} + O(\rho^{2s + 2p - D}),
\]
(4.13)
where \( P \) is an arbitrary large integer, while
\[
I_\gamma(s) = \frac{\Gamma(s - \frac{1}{2})}{\sqrt{\pi}} [G_\gamma(s) + G_{2\pi}(s)].
\]
(4.14)

For \( \text{Re}(s) > 1 \),
\[
G_\gamma(s) = \sum_{n=1}^{\infty} \frac{\Gamma(\nu_n - s + 1)}{\Gamma(\nu_n + s)} , \quad G_{2\pi} = -\frac{\Gamma(1 - s)}{2\Gamma(s)}.
\]
(4.15)

It is possible to show that \( G_\gamma(s) \) admits an analytical continuation, and the properties of \( I_\gamma \) and \( G_\gamma \) on the whole complex plane are studied in detail in the Appendix of [16]; an important property is that the analytical continued \( I_\gamma \) as well as \( G_\gamma \), has only a simple pole at \( s = 1 \), with residue
\[
\text{Res} I_\gamma(s)|_{s=1} = \frac{1}{2} \left( \frac{\gamma}{2\pi} - 1 \right).
\]
(4.16)

Having found the meromorphic structure of the \( \zeta \)-function on our manifold, we can determine the vacuum expectation value of the fluctuation of a scalar field. With regard to this, it is convenient to distinguish between odd- and even-dimensional space-times.

We first consider the case in which \( N \) (or, equivalently, \( D \)) is odd, so that \( I_\gamma \) is finite at \( s = 1 \). To begin with, notice that the first term in (4.13) depends only on the transverse coordinates and is finite on the horizon. As a result, making use of the meromorphic structure of \( \zeta(s; \bar{x}|L_N) \), we get
\[
\langle \phi^2(x) \rangle \simeq \frac{1}{2\gamma} \left[ \sum_{r=0}^{\infty} A_r(\bar{x}|L_N) \right] + \frac{1}{\gamma} \sum_{r=0}^{P} A_r(\bar{x}|L_N)I_\gamma(1 + r - \frac{N}{2})\rho^{2r - N} + O(\rho^{2p - N}).
\]
(4.17)

It is now easy to see that the above expression contains \( \left[ \frac{N}{2} \right] \) (where \( [\ ] \) means “integer part”) terms which are divergent as \( \rho^{2r - N} \) (\( r < \left[ \frac{N}{2} \right] \)) in the limit \( \rho \to 0 \), and so on the horizon (see (3.3)). Thus, if we want a good behavior on it, we must demand that all the \( I_\gamma \)'s vanish for \( r < \left[ \frac{N}{2} \right] \), i.e. \( \gamma = 2\pi \). In particular notice that within this value of \( \gamma \) all of the \( I_\gamma \)'s actually vanish.

We now come to the case in which \( N \) (\( D \)) is even. In this case, the \( \zeta \)-function (4.13) has a pole at \( s = 1 \), coming from the first and the \( I_\gamma \) term in (4.13). From (2.3) one gets
\[
\langle \phi^2(x) \rangle_{\text{ren}} \simeq \frac{1}{2\gamma} \left[ \sum_{r=0}^{\infty} A_r(\bar{x}|L_N) \right] - \frac{1}{4\pi} A_0(\bar{x}|L_N) \ln \mu^2
\]
\[
+ \frac{1}{\gamma} \sum_{r=0}^{P} A_r(\bar{x}|L_N)I_\gamma(1 + r - \frac{N}{2})\rho^{2r - N} + O(\rho^{2p - N}),
\]
(4.18)
where the ' in the sums means omission of the \( r = \frac{N}{2} \) term.

Again, as long as \( r < \left[ \frac{N}{2} \right] \), we get divergent terms on the horizon, unless we require the \( I_\gamma \)'s to vanish, i.e. \( \gamma = 2\pi \) as in the odd-dimensional case.

As a result, for a manifold \( \mathcal{M}^D \) whose near horizon geometry is described by \( \mathcal{C} \times \Sigma^N \), the requirement of having a well behaved \( \langle \phi^2(x) \rangle \) on the horizon, selects \( \gamma = 2\pi \), and so the Unruh-Hawking temperature, according to the conical singularity method.

We also remind that the choice \( \gamma = 2\pi \) makes the manifold smooth, getting rid of the conical singularity otherwise present in \( \rho = 0 \).

The computation of the partition function for arbitrary \( \gamma \) has been done in [10], and it as be used in order to discuss thermodynamic properties. Only the horizon divergences are present, and these are still present in the on-shell (\( \gamma = 2\pi \)) entropy.
V. SECOND CASE: \( \mathcal{M}^D = \mathbb{H}^2/\Gamma \times \Sigma^N \).

After having checked that our procedure works at least in the case of non-extreme black holes, we can tackle the case of the extreme ones, i.e. manifold whose topology near the horizon is described by \( \mathbb{H}^2/\Gamma \times \Sigma^N \).

Again, one can start by making use of the factorization property of the heat kernel, writing that

\[
K_\gamma(t; x|L_D) = K(t; \theta, \rho|L_\gamma)K(t; \bar{x}|L_N),
\]

where the heat kernels of the Laplace-like operators on \( \mathcal{M}^D, \mathbb{H}^2/\Gamma \) and \( \Sigma^N \), respectively appear.

As in the previous case, we suppose that \( L_\gamma \) is the operator associated to a massless and minimally coupled scalar field on \( \mathbb{H}^2 \), while \( L_N \) is a Laplace-like operator on \( \Sigma^N \) including eventually, mass and scalar curvature coupling term (so that the expansions (4.2) and (4.3) are still valid). In this way, \( L_\gamma = -\Delta_\gamma = -\rho^2 (\partial_\theta^2 + \partial_\rho^2) \), and a complete set of normalized eigenfunctions is easily found to be

\[
\psi_{\lambda k} = \sqrt{\frac{y}{2\pi}} e^{ik\theta} K_{1\lambda}(|k|y),
\]

together with its complex conjugate. Here \( \lambda^2 (\lambda \geq 0) \) is the eigenvalue corresponding to \( \psi \) and \( \psi^* \), while \( K_\nu \) is the MacDonald function. Thus the spectral representation of the (off-diagonal) heat kernel associated to \( L_\gamma \) on \( \mathbb{H}^2 \) reads (see, for example \[3\])

\[
K_{\mathbb{H}^2}(t; \theta, \rho; \theta', \rho'|L_\gamma) = \frac{1}{2\pi} \int_0^\infty d\lambda \lambda \tanh(\pi \lambda) e^{-\left(\lambda^2 + \frac{4}{t}\right)} P_{i\lambda - \frac{1}{4}}(\cosh \sigma),
\]

where \( P \) is the associated Legendre function, while \( \sigma \) is the \( \mathbb{H}^2 \) geodesic distance between \((\theta, \rho)\) and \((\theta', \rho')\).

As previously remarked, for studying thermal effects, one has to deal with the quotient space \( \mathbb{H}^2/\Gamma \), with \( \Gamma \) the (discontinuous and fixed-point-free) group of isometry induced by the time compactification \( 0 \leq \theta \leq \gamma \). In our case we have translations, corresponding to parabolic elements. By applying the method of images, the diagonal heat kernel turns out to be

\[
K(t; \theta, \rho|L_\gamma) = \frac{1}{2\pi} \int_0^\infty d\lambda \lambda \tanh(\pi \lambda) e^{-\left(\lambda^2 + \frac{4}{t}\right)} t + \frac{1}{\pi} \sum_{n=1}^\infty \int_0^\infty d\lambda \lambda \tanh(\pi \lambda) e^{-\left(\lambda^2 + \frac{4}{t}\right)} P_{\lambda - \frac{1}{4}}(\cosh \sigma_n),
\]

where now

\[
cosh \sigma_n = 1 + \frac{n^2 \gamma^2}{2\rho^2}.
\]

Let us show that the partition function does not exist, and requires, besides the horizon divergence regularization, a further regularization. The partition function is proportional to the first derivative of the \( \zeta \)-function, which may be defined by the Mellin transform of the heat kernel trace. The latter may be obtained integrating over the manifold coordinates. As a result

\[
\zeta(s|L_\gamma) = \frac{\gamma}{4\pi \varepsilon} \zeta(s - 1|L_N + \frac{1}{4}) + \frac{\gamma}{2\pi \varepsilon} \int_0^\infty d\lambda \frac{\lambda}{1 + e^{2\pi \lambda}} \zeta(s|L_N + \lambda^2 + \frac{1}{4})
+ \frac{1}{\sqrt{\pi}} \zeta_{R}(1 + \delta) \zeta(s - 1|L_N + \frac{1}{4}) + \mathcal{O}(\delta),
\]

where \( \zeta_R \) is the Riemann \( \zeta \)-function, and we have introduced the horizon cutoff \( \varepsilon \), and the cusp regularization \( \delta > 0 \). It should be noticed the divergence for \( \delta = 0 \), which is usually present when one is dealing with parabolic elements 27.

As far as the field fluctuations are concerned, we only need the expression of the local \( \zeta \)-function near the horizon. Thus with regard to the sum over \( n \), we may apply the simplest version of the Euler-MacLaurin resummation formula, namely

\[
\sum_{n=1}^\infty f(n) = \int_1^\infty dx f(x) - \frac{1}{2} f(1) + \int_1^\infty dx \left( x - [x] - \frac{1}{2} \right) f'(x).
\]
As a result, for large ρ

\[ \sum_{n=1}^{\infty} P_{\lambda, \frac{1}{2}}(\cosh \sigma_n) = \frac{1}{2} \gamma \tanh(\pi \lambda) + C(\lambda) + O\left(\frac{\gamma}{\rho}\right), \]  

(5.8)

where \( C(\lambda) \) does not depend on \( \rho \).

Thus, the diagonal part of the heat-kernel may be rewritten as

\[ K(t; \theta, \rho|L_D) = \frac{1}{2\pi} \int_0^\infty d\lambda \lambda \tan(\pi \lambda) e^{-(\lambda^2 + \frac{1}{4})^t} + \frac{\rho}{2\sqrt{2\pi t}} e^{-\frac{\rho^2}{4t}} + O(1) + O\left(\frac{\gamma}{\rho}\right). \]  

(5.9)

As a result, the related local \( \zeta \)-function reads

\[ \zeta_D(s; x|L_D) = \zeta(s; L_D) + \frac{\rho}{2\sqrt{2\pi}} \gamma \left( \Gamma(s) \right) \left( \frac{s - \frac{1}{2}}{\Gamma(s)} \right) \zeta(s - \frac{1}{2}; x|L_D + \frac{1}{4}) + O(1) + O\left(\frac{\gamma}{\rho}\right). \]  

(5.10)

The first term on the r.h.s. of the above relation, is the local \( \zeta \)-function associated with the smooth manifold \( M^D = H^2 \times \Sigma^N \) and it depends only on the transverse coordinates \( \vec{x} \). The other terms are the asymptotic contribution for large \( \rho \). As a result, we have obtained the meromorphic structure of \( \zeta_D(s; x|L_D) \) via the meromorphic structure of \( \zeta_D(s; \vec{x}|L_D) \) and \( \zeta_D(s - \frac{1}{2}; \vec{x}|L_D + \frac{1}{4}) \). It should be noticed that the second term, which is the relic of the sum over masses, contains the shift \( s = 1/2 \). This means that, with regard to the evaluation of the field fluctuation, one has a simple pole at \( s = 1 \) for any \( D \), not only for \( D \) even. This violation of Seeley’s Theorem is related to the presence of parabolic elements, which make the manifold not smooth.

On the other side, no matter the dimension of the space-time, the field fluctuations contain terms proportional to \( \xi\) which are divergent on the horizon (\( \rho \to \infty \), see (3.9)), unless we demand \( \gamma \to \infty \), and so \( T = 0 \), according to the result obtained in [3] in the four-dimensional case. We finally notice that if \( \gamma \to \infty \), the partition function contains only the usual volume divergence associated to the non-compact nature of the Euclidean section.

VI. CONCLUSIONS

In this paper, making use of an Euclidean approach, we have evaluated the field fluctuations and the partition function of a scalar field in a \( D \)-dimensional static black hole. A near horizon approximation has lead to quite explicit expressions for the local \( \zeta \)-function related to such a field propagating in the Euclidean section of the black hole space-time. The period of the compactified imaginary time has been interpreted as the inverse of the temperature; further, making use of \( \zeta \)-function regularization, the field fluctuations have been evaluated. It is then stressed the fact that this quantity is finite on the horizon as soon as one selects a distinguished period of the imaginary time (temperature), which coincides with the Hawking-Unruh temperature in the non-extreme case, and with the zero temperature in the extreme one. With regard to the partition function, no problem exists in the conical singularity case, while in the presence of the cusp singularity, new divergences show up in the global \( \zeta \)-function, and, strictly speaking, the partition function does not exist. This drawback disappears if the cusp singularity is absent, namely if the temperature is again zero.

With regard to the local quantities, we also notice that the regular behavior of the fluctuations on the horizon, leads also to the regular behavior of the expectation value of the stress tensor. This can be verified by means of a direct calculation, starting from the local off-diagonal \( \zeta \)-function which can be obtained with our approach.

As far as the extreme black holes are concerned, all the properties we have been deriving, and the lack of the Hawking radiation [4], strongly suggest that the only admissible temperature is the zero one, in agreement with the 4-dimensional case studied in [4].

The only class of space-times for which our analysis seems to have no direct application is the one in which the double zero occurs at \( r_+ = 0 \). As an example, we may recall the so called massless ground state of the asymptotically AdS toroidal black holes [4, 22]. In fact these black holes have

\[ f(r) = \frac{1}{h(r)} = \left( \frac{l^2}{r^2} - \frac{C_D M}{r^{D-1}} \right), \]

(6.1)

where \( C_D \) is a constant, \( M \) is the mass of the black hole and the parameter \( l \) is related to the cosmological constant, namely \( \Lambda = -l^{-2} \). For \( D = 3 \), one recovers the celebrated BTZ black hole [23]. The ground state of this class of black holes is the zero mass solution, and the Euclidean metric becomes
ds^2 = \frac{r^2}{l^2} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 dT^2_N, \quad (6.2)

where $dT^2_N$ represents the metric of a $N$-dimensional torus. In the above metric, $r = 0$ is a naked coordinate singularity. If one compactifies the Euclidean time and make the coordinate transformation $r = l^2/\rho$, one gets

$$ds^2 = \frac{l^2}{\rho^2} \left[ d\rho^2 + d\tau^2 + l^2 dT^2_N \right]. \quad (6.3)$$

This metric describes locally the $D$-dimensional hyperbolic space $\mathbb{H}^D$.

For the zero temperature case and in $D = 3$ and $D = 4$, the field fluctuations as well as the expectation value of the stress tensor has been computed in [24,25] and [26] respectively, and divergences have been found as $\rho$ goes to infinity. In this case, our analysis does not select any distinguished temperature. However, it should be noticed that in this case, it is not reasonable to neglect the back-reaction effects. In fact, in [24–26] it has been shown that there is a quantum implementation of the Cosmic Censorship Principle due to the back-reaction on the metric.

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