Dynamics of Cournot duopoly games with quadratic costs and distinct rationality degrees

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Abstract

In this discussion draft, we explore different duopoly games of players with quadratic costs, where the market is supposed to have the isoelastic demand. Different from the usual approaches based on numerical computations, the methods used in the present work are built on symbolic computations, which can produce analytical and rigorous results. Our investigation shows that the stability regions are enlarged for the games considered in this work compared to their counterparts with linear costs.

1 Models

Let us consider a market served by two firms producing homogeneous products. We use $q_i(t)$ to denote the output of firm $i$ at period $t$. Moreover, the cost function of firm $i$ is supposed to be quadratic, i.e., $C_i(q_i) = c_i q_i^2$ with $c_i > 0$. At each period $t$, firm $i$ first estimates the possible price $p^e_i(t)$ of the product, then the expected profit of firm $i$ would be

$$\Pi^e_i(t) = p^e_i(t)q_i(t) - c_i q_i^2(t), \quad i = 1, 2.$$ 

In order to maximize the expected profit, at period $t$ each firm would decide the quantity of the output by solving

$$q_i(t) = \arg \max_{q_i(t)} \Pi^e_i(t) = \arg \max_{q_i(t)} [p^e_i(t)q_i(t) - c_i q_i^2(t)], \quad i = 1, 2.$$ 

Furthermore, assume that the demand function of the market is isoelastic, which is founded on the hypothesis that the consumers have the Cobb-Douglas utility function. Hence, the real (not expected) price of the product should be

$$p(Q) = \frac{1}{Q} = \frac{1}{q_1 + q_2},$$

where $Q = q_1 + q_2$ is the total supply.

Three types of players with distinct rationality degrees are involved in this draft, which are described as follows. A rational player not only knows clearly the form of the price...
function, but also has complete information of the decision of its rival. If firm \( i \) is a rational player, at period \( t + 1 \) we have
\[
p^i_t(t + 1) = \frac{1}{q_i(t + 1) + q^c_{-i}(t + 1)},
\]
where \( q^c_{-i}(t + 1) \) is the expectation of the output of the rival. Due to the assumption of complete information, which means that \( q^c_{-i}(t + 1) = q_{-i}(t + 1) \), it is acquired that the expected profit of firm \( i \) would be
\[
\Pi^i_t(t + 1) = \frac{q_i(t + 1)}{q_i(t + 1) + q_{-i}(t)} - c_iq^2_i(t + 1).
\]
The first order condition for profit maximization gives rise to a cubic polynomial equation. To be exact, the condition for the reaction function of firm \( i \) would be
\[
F_i(q_i(t + 1), q_{-i}(t + 1)) = 0,
\]
where
\[
F_i(x, y) = y - 2c_i x (x + y)^2.
\]
The player could maximize its profit by solving the above equation. It is easy to verify that there exists only one positive solution for \( q_i(t + 1) \) if solving (1), but the expression could be quite complex. To tackle this difficulty, we temporarily denote this solution by \( q_i(t + 1) = R_i(q_{-i}(t + 1)) \), where \( R_i \) is called the reaction function of firm \( i \).

A **boundedly rational** player knows the form of the price function, but do not know the rival’s decision of the production. If firm \( i \) is a boundedly rational player, then it naively expects its competitor to produce the same quantity of output as the last period, i.e., \( q^c_{-i}(t + 1) = q_{-i}(t) \). Thus,
\[
\Pi^i_t(t + 1) = \frac{q_i(t + 1)}{q_i(t + 1) + q_{-i}(t)} - c_iq^2_i(t + 1).
\]
Then the best response for firm \( i \) would be \( q_i(t + 1) = R_i(q_{-i}(t)) \).

A **local monopolistic approximation** (LMA) player, however, even do not know the exact form of the price function. If firm \( i \) is an LMA player, then it just can observe the current market price \( p(t) \) and the corresponding total supply \( Q(t) \), and is able to correctly estimate the slope \( p'(Q(t)) \) of the price function around the point \( (p(t), Q(t)) \). Then firm \( i \) uses such information to conjecture the demand function and expect the price at period \( t + 1 \) to be
\[
p^i_t(t + 1) = p(Q(t)) + p'(Q(t))(Q^c_i(t + 1) - Q(t)),
\]
where \( Q^c_i(t + 1) = q_i(t + 1) + q^c_{-i}(t + 1) \) represents the expected aggregate production of firm \( i \) at period \( t + 1 \). Moreover, an LMA player do not know the decision of its rival either, and is assumed to to use the naive expectation, i.e., \( q^c_{-i}(t + 1) = q_{-i}(t) \). Thus,
\[
p^i_t(t + 1) = \frac{1}{Q(t)} - \frac{1}{Q^2(t)}(q_i(t + 1) - q_i(t)).
\]
The expected profit would be
\[
\Pi^i_t(t + 1) = q_i(t + 1) \left[ \frac{1}{Q(t)} - \frac{1}{Q^2(t)}(q_i(t + 1) - q_i(t)) \right] - c_iq^2_i(t + 1).
\]
By solving the first order condition, the best response for firm \( i \) would be
\[
q_i(t + 1) = \frac{2q_i(t) + q_{-i}(t)}{2(1 + c_i(q_i(t) + q_{-i}(t))^2)}.
\]
For simplicity, we denote the above map as \( q_i(t + 1) = S_i(q_i(t), q_{-i}(t)) \).
1.1 Model BB

We consider a duopoly where two boundedly rational firms simultaneously supply the market. To be exact, we consider a game modeled as

\[ T_{BB}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = R_1(q_2(t)), \\
q_2(t + 1) = R_2(q_1(t)). 
\end{cases} \] (2)

1.2 Model BR

Model BR is similar to Model BB. The only difference is that the second player in the former could know exactly the rival’s output at the present period. Thus, the model is described as

\[ T_{BR}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = R_1(q_2(t)), \\
q_2(t + 1) = R_2(q_1(t + 1)). 
\end{cases} \] (3)

This could be transformed into a one-dimensional system

\[ T_{BR}(q_1) : q_1(t + 1) = R_1(R_2(q_1(t))). \] (4)

1.3 Model LR

If replacing the first player in Model BR with an LMA player, we get a new model named LR, which is described as

\[ T_{LR}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = S_1(q_1(t), q_2(t)), \\
q_2(t + 1) = R_2(q_1(t + 1)). 
\end{cases} \] (5)

This could be transformed to the following system.

\[ T_{LR}(q_1) : q_1(t + 1) = S_1(q_1(t), R_2(q_1(t))). \] (6)

1.4 Model LB

Similarly, Model LB is described as

\[ T_{LB}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = S_1(q_1(t), q_2(t)), \\
q_2(t + 1) = R_2(q_1(t)). 
\end{cases} \] (7)

1.5 Model LL

In the end, we have the following model named LL.

\[ T_{LL}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = S_1(q_1(t), q_2(t)), \\
q_2(t + 1) = S_2(q_1(t), q_2(t)). 
\end{cases} \] (8)

2 Local Stability

2.1 Model LL

The stability analysis of Model LL was first given in [1], which is restated herein.
**Proposition 1.** For Model LL, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable for all feasible parameter values, i.e., such parameter values that \( c_1, c_2 > 0 \).

To illustrate the basic idea of the symbolic method used in this draft, we reprove the above proposition in a computational style. In order to acquire the equilibria, we set \( q_1(t+1) = q_1(t) = q_1 \) and \( q_2(t+1) = q_2(t) = q_2 \) in (8). Moreover, we focus only on such equilibria that \( q_1, q_2 > 0 \), then the following system is obtained.

\[
\begin{align*}
q_1 - S_1(q_1, q_2) &= 0, \\
q_2 - S_2(q_1, q_2) &= 0, \\
q_1 > 0, & q_2 > 0, \\
c_1 > 0, & c_2 > 0,
\end{align*}
\]

(9)

where \( c_1 > 0, c_2 > 0 \) are obvious according to the economic constraints of the parameters. The problem of counting non-vanishing equilibria is equivalent to counting real roots of system (9). Using the approach proposed by the author [5], this problem can be solved in 4 steps as follows.

Step 1. Using the method of triangular decomposition [7, 11, 31, 60], we decompose

\[
P = [q_1 - S_1(q_1, q_2), q_2 - S_2(q_1, q_2)]
\]

into two triangular forms:

\[
T_1 = [q_1, q_2]
\]

and

\[
T_2 = [4c_1^3q_1^4 - 8c_1^2c_2q_1^4 + 4c_1c_2^2q_1^4 + 8c_1c_2q_1^2 - c_2, 2c_1^2q_1^3 + 2c_1c_2q_1^3 + 4c_1c_2q_1^2q_2 - c_2q_2^2].
\]

(10)

According to the properties of triangular decomposition, the zero set of \( P \) is the same as the union of zeros of \( T_1 \) and \( T_2 \). Of them, only \( T_2 \) should be considered as \( T_1 \) is corresponding to the original equilibrium \((0, 0)\). Furthermore, one may observe that the second polynomial in \( T_2 \) has degree 1 with respect to \( q_2 \), thus \( q_2 \) could be represented as

\[
q_2 = \frac{2c_1^2q_1^3 + 2c_1c_2q_1^3}{c_2 - 4c_1c_2q_1^2}.
\]

(11)

Step 2. Substitute (11) into the inequality \( q_2 > 0 \) in (9), we obtain

\[
\frac{2c_1^2q_1^3 + 2c_1c_2q_1^3}{c_2 - 4c_1c_2q_1^2} > 0,
\]

or equivalently

\[
(2c_1^2q_1^3 + 2c_1c_2q_1^3)(c_2 - 4c_1c_2q_1^2) > 0.
\]

Thus, system (9) is transformed to a univariate system

\[
\begin{align*}
4c_1^3q_1^4 - 8c_1^2c_2q_1^4 + 4c_1c_2^2q_1^4 + 8c_1c_2q_1^2 - c_2 &= 0, \\
q_1 > 0, & (2c_1^2q_1^3 + 2c_1c_2q_1^3)(c_2 - 4c_1c_2q_1^2) > 0, \\
c_1 > 0, & c_2 > 0.
\end{align*}
\]

(12)

\[\text{The method of triangular decomposition is extended from the Euclidean algorithm. The main ideas of both are to transform a system into a triangular form. However, the method of triangular decomposition is for polynomial systems, while the Euclidean algorithm is for linear systems.}\]
Step 3. Define the border polynomial of system (12) to be

\[ BP = A_0 \cdot \text{discr}(T) \cdot \text{res}(T, q_1) \cdot \text{res}(T, Q), \]

where \( \text{discr}(T) \) represents the discriminant of \( T \), \( \text{res}(T, Q) \) stands for the resultant of \( T \) with respect to \( Q \), with

\[ T = 4c_1^3q_1^4 - 8c_1^2c_2q_1^4 + 4c_1c_2^2q_1^4 + 8c_1c_2q_1^2 - c_2, \]
\[ Q = (2c_1^2q_1^3 + 2c_1c_2q_1^2)(c_2 - 4c_1c_2q_1^2), \]
\[ A_0 = 4c_1^3 - 8c_1^2c_2 + 4c_1c_2^2. \]

(13)

It is noted that \( A_0 \) is the leading coefficient of \( T \) with respect to \( q_1 \), i.e., the coefficient of \( q_1^4 \). We have

\[ BP = -67108864c_1^{11}c_2^{11}(c_1 - c_2)^6(c_1 + c_2)^{12}, \]

(14)

and its squarefree part is

\[ SP = c_1c_2(c_1 - c_2)(c_1 + c_2) \]

It is proved in [5] that \( BP = 0 \), or equivalently \( SP = 0 \), divides the parameter space into connected regions, and on each of them the number of distinct real solutions is invariant.

Step 4. From each region of our concerned parameter set with \( c_1, c_2 > 0 \), select\(^2\) one sample point:

\[ s_1 = (1, 1/2), \ s_2 = (1, 2). \]

At each sample point, count\(^3\) the number of distinct real solutions of system (12). We obtain that there exists exactly one real solution for each sample point. This means that

\(^2\)For a simple border polynomial as (14), the selection of sample points could be done by hand. Generally, however, the selection could be automated by using, e.g., the PCAD method [2].

\(^3\)The first thought is to count real solutions by solving the system numerically. However, numerical methods have several shortcomings: first, the numerical computation may encounter the problem of instability, which could make the results completely useless; second, floating-point errors may cause the problem that it is extremely hard to distinguish between real solutions and complex solutions with tiny imaginary parts; third, most numerical algorithms only search for a single equilibrium and are nearly infeasible for multiple equilibria. Thus we herein need symbolic methods, e.g., [8], which can be used to count the real solutions exactly.

Figure 1: The 2-dimensional \((c_1, c_2)\) parameter plane with \( SP = 0 \) and the selected sample points.

Step 4. From each region of our concerned parameter set with \( c_1, c_2 > 0 \), select\(^2\) one sample point:

\[ s_1 = (1, 1/2), \ s_2 = (1, 2). \]

At each sample point, count\(^3\) the number of distinct real solutions of system (12). We obtain that there exists exactly one real solution for each sample point. This means that
system (12), or equivalently (9), has one real solution for any feasible parameter value. Hence, we conclude that the dynamic system (8) has one unique equilibrium with \( q_1, q_2 > 0 \) for any \( c_1, c_2 > 0 \).

It is worth noting that all computations involved in the above 4 steps are symbolic and rigorous. This means our approach could acquire analytical results, thus could be used to discover and prove theorems of economic models.

In order to investigate the local stability of this unique equilibrium, the Jacobian matrix

\[
J_{LL} = \begin{bmatrix}
\frac{\partial S_1}{\partial q_1} & \frac{\partial S_1}{\partial q_2} \\
\frac{\partial S_2}{\partial q_1} & \frac{\partial S_2}{\partial q_2}
\end{bmatrix}
\]

plays an ambitious role. We use \( \text{Det}(J) \) and \( \text{Tr}(J) \) to denote the determinant and the trace of \( J \), respectively. According to the Jury’s criterion, an equilibrium \( E \) is locally stable provided that

\[
\begin{align*}
1 + \text{Tr}(J_{LL}) + \text{Det}(J_{LL}) &> 0, \\
1 - \text{Tr}(J_{LL}) + \text{Det}(J_{LL}) &> 0, \\
1 - \text{Det}(J_{LL}) &> 0.
\end{align*}
\]  

(15)

Combine these inequalities with system (9), transform the resulting system to a univariate system, and then compute its border polynomial likewise. It is obtained that the squarefree part of the border polynomial is

\[
SP^* = c_1c_2(c_1 - 4c_2)(c_1 - c_2)(c_1 + c_2)(c_1 - 1/4c_2)(c_1^2 - 7c_1c_2 + c_2^2).
\]

The selected sample points might be

\[
s_1 = (2, 1/8), \ s_2 = (3, 9/16), \ s_3 = (2, 1), \ s_4 = (1, 2), \ s_5 = (9/16, 3), \ s_6 = (1/8, 2).
\]

We could obtain that the system (9)+(15) has exactly one real solution at any of these sample points, which means the unique non-vanishing equilibrium of (8) is locally stable for all parameter values that satisfy \( c_1, c_2 > 0 \).

![Figure 2: The 2-dimensional \((c_1, c_2)\) parameter plane with \( SP^* = 0 \) and the selected sample points.](image-url)
2.2 Model LB

The Jacobian matrix is

\[ J_{LB} = \begin{bmatrix} \frac{\partial S_1}{\partial q_1} & \frac{\partial S_1}{\partial q_2} \\ \frac{dR_2}{dq_1} & 0 \end{bmatrix} \] \tag{16} \]

It might be difficult to directly calculate \( \frac{dR_2}{dq_1} \) as the analytical expression of \( R_2 \) is quite complicated. However, according to (1), it is known that

\[ q_1 - 2c_2R_2(q_1)(q_1 + R_2(q_1))^2 = 0, \tag{17} \]

We could calculate the derivative of the implicit function, if the derivative exists, using the method called implicit differentiation. It is acquired that

\[ \frac{dR_2}{dq_1} = -\frac{4c_2q_1q_2}{2c_2(q_1^2 + 4q_1q_2 + 3q_2^2)}. \]

Hence, the stable equilibria can be described by

\[
\begin{aligned}
q_1 - S_1(q_1, q_2) &= 0, \\
F_2(q_2, q_1) &= 0, \\
q_1 &> 0, \quad q_2 > 0, \\
1 + \text{Tr}(J_{LB}) + \text{Det}(J_{LB}) &> 0, \\
1 - \text{Tr}(J_{LB}) + \text{Det}(J_{LB}) &> 0, \\
1 - \text{Det}(J_{LB}) &> 0, \\
c_1 &> 0, \quad c_2 > 0.
\end{aligned}
\] \tag{18} 

Based on a series of computations, we finally obtain the following proposition.

**Proposition 2.** For Model BL, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable for all feasible parameter values.

2.3 Model BB

The Jacobian matrix is

\[ J_{BB} = \begin{bmatrix} 0 & \frac{dR_1}{dq_2} \\ \frac{dR_2}{dq_1} & 0 \end{bmatrix} \] \tag{19} \]

where \( \frac{dR_1}{dq_2} \) and \( \frac{dR_2}{dq_1} \) could be computed similarly as for Model BL. The stable equilibria can be described by

\[
\begin{aligned}
F_1(q_1, q_2) &= 0, \\
F_2(q_2, q_1) &= 0, \\
q_1 &> 0, \quad q_2 > 0, \\
1 + \text{Tr}(J_{BB}) + \text{Det}(J_{BB}) &> 0, \\
1 - \text{Tr}(J_{BB}) + \text{Det}(J_{BB}) &> 0, \\
1 - \text{Det}(J_{BB}) &> 0, \\
c_1 &> 0, \quad c_2 > 0.
\end{aligned}
\] \tag{20} 

We finally obtain the following proposition.

**Proposition 3.** For Model BB, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable for all feasible parameter values.
2.4 Model BR

This model can be studied by means of the one-dimensional map (4). The derivative of \( q_1(t + 1) \) with respect to \( q_1(t) \) is critical, which is

\[
\frac{dq_1(t + 1)}{dq_1(t)} = \frac{d[R_1(R_2(q_1(t)))]}{dq_1(t)}.
\]

By the chain rule, we have

\[
\frac{d[R_1(R_2(q_1))]}{dq_1} = \frac{dR_1}{dq_2} \frac{dR_2}{dq_1} = \frac{4c_1q_1q_2 + 4q_1^2 - 1}{2c_1(q_2^2 + 4q_1q_2 + 3q_1^2)} - \frac{4c_2q_1q_2 + 4q_2^2 - 1}{2c_2(q_1^2 + 4q_1q_2 + 3q_2^2)}.
\]

(21)

For a one-dimensional dynamic system, an equilibrium \( E \) is stable provided that at \( E \)

\[
\left|\frac{d[R_1(R_2(q_1))]}{dq_1}\right| < 1.
\]

Therefore, the stable equilibria are the solutions of the following system.

\[
\begin{cases}
F_1(q_1, q_2) = 0, \\
F_2(q_2, q_1) = 0, \\
q_1 > 0, q_2 > 0, \\
1 + \frac{d[R_1(R_2(q_1))]}{dq_1} > 0, \\
1 - \frac{d[R_1(R_2(q_1))]}{dq_1} > 0, \\
c_1 > 0, c_2 > 0.
\end{cases}
\]

(22)

We finally obtain the following proposition.

**Proposition 4.** For Model BR, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable for all feasible parameter values.

2.5 Model LR

This model is similar as Model BR. According to (6), one knows that

\[
\frac{dq_1(t + 1)}{dq_1(t)} = \frac{d[S_1(q_1(t), R_2(q_1(t)))]}{dq_1(t)}.
\]

By the chain rule,

\[
\frac{d[S_1(q_1, R_2(q_1))]}{dq_1} = \frac{\partial S_1}{\partial q_1} + \frac{\partial S_1}{\partial q_2} \cdot \frac{dR_2}{dq_1}
\]

(23)

Therefore, the stable equilibria are the solutions of the following system.

\[
\begin{cases}
q_1 - S_1(q_1, q_2) = 0, \\
F_2(q_2, q_1) = 0, \\
q_1 > 0, q_2 > 0, \\
1 + \frac{d[S_1(q_1, R_2(q_1))]}{dq_1} > 0, \\
1 - \frac{d[S_1(q_1, R_2(q_1))]}{dq_1} > 0, \\
c_1 > 0, c_2 > 0.
\end{cases}
\]

(24)

We finally obtain the following proposition.
**Proposition 5.** For Model LR, there exists a unique equilibrium with $q_1, q_2 > 0$. Moreover, this equilibrium is locally stable for all feasible parameter values.

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