On reflection representations of Coxeter groups over non-commutative rings

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ABSTRACT
In this paper, we consider representations of Coxeter groups over a path algebra, $R$, introduced by Dyer in the study of root systems over non-commutative rings. We answer a question posed by Dyer about the multiplicative properties of $R$, showing that it is “almost a domain.” We also show that $R$ can be embedded in a matrix ring over a free product of extension fields of the rational numbers and rings of Laurent polynomials.

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Introduction

Coxeter groups admit representations as certain discrete reflection groups associated to root systems in real vector spaces [1, 2, 19, 24]. In many situations, these have an associated analogue of a Cartan matrix, describing the pairings between simple roots in the reflection representation, and simple coroots in the “dual” reflection representation. There are also reflection representations and root systems of corresponding Iwahori-Hecke algebras [3]. Reflection representations and root systems in modules over commutative rings have been considered in [9, 11, 12, 18].

In [13], similar notions of reflection representation of Coxeter groups and Hecke algebras are considered over non-commutative rings. They are associated to a certain class of matrices (non-commutative Cartan matrices, NCMs) with entries in non-commutative rings, defined by identities related to Chebyshev polynomials on the $2 \times 2$-principal submatrices. We call these “lax” reflection representations in this paper. They have the property that if $(W', S')$ is another Coxeter system and there is a group homomorphism $f : W \to W'$ which restricts to a bijection $S \to S'$, then lax reflection representations for $(W', S')$ pull back along $f$ to lax reflection representations of $(W, S)$.

An important role in the theory is played by a certain ring (the lax universal coefficient ring) with specified NCM, denoted in this paper as $\tilde{R} = \tilde{R}_{W,S}$, which gives (roughly speaking) an initial object in a category of rings with a specified NCM for $(W, S)$. This ring, which is of considerable interest, may be defined as quotient of the path algebra of the Coxeter digraph by certain rank two relations. It contains a family $\{[r]_{r \in S}\}$ of non-zero orthogonal idempotents (the images in the quotient of length 0 paths) such that $\tilde{R} = \bigoplus_{r,s \in S} [r] \tilde{R}[s]$, and is unital if and only if $S$ is finite. The ring $\tilde{R}$ is free as $\mathbb{Z}$-module and admits commuting left and right lax reflection $W$-actions, with the left (resp., right) action being by our conventions right (resp., left) $\tilde{R}$-linear. It is known see [8, 13, 23] that this ring is essentially the subregular $J$-ring of the Coxeter system that is, the two-sided ideal $J_1$ of Lusztig's asymptotic Hecke algebra [21] spanned by its standard basis elements parameterized by elements of a-value 1 in the Coxeter group (see these references for precise statements).
This paper concerns a subclass of the lax reflection representations, called here strict reflection representations, which is defined by strict NCMs, satisfying stronger conditions than NCMs, and for which the above pullback property no longer holds. There is again a universal ring with strict NCM, with analogous properties to those of $\tilde{R}$ mentioned above. We denote this ring as $R = R_{W,S}$.

We state a result which partly motivates the definition of $R$ and clarifies the relationship between lax and strict reflection representations, and reflection representations in real vector spaces. For standard dual reflection representations of $(W, S)$ on real vector spaces $V$ and $V^\vee$ with bases of simple roots $\Pi$ and simple coroots $\Pi^\vee$, there is an identification of $V^\vee \otimes_{\mathbb{R}} V$ with the ring $B$ of $S \times S$ real matrices with only finitely many non-zero entries (so that the matrix units are $\alpha \otimes \beta$ for $\alpha \in \Pi^\vee$ and $\beta \in \Pi$) and a ring homomorphism $R \rightarrow B$ which is equivariant for the left and right $W$-actions and which induces a bi-equivariant ring surjection $R' = R \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow B$. Here, we regard $W$ as acting on the left (resp., right) of $V^\vee$ (resp., $V$). This would be true with $R$ replaced by $\tilde{R}$, but that is of less interest since $R$ is a quotient (as ring with two-sided $W$-actions) of $\tilde{R}$. For an irreducible finite Weyl group, $R' \rightarrow B$ is an isomorphism. In this paper, we do not discuss more delicate results of a similar nature to the last one for other classes of Coxeter systems, or extensions of the above facts to Hecke algebras.

Although the class of strict reflection representations was not considered explicitly in [13], both rings $R$ and $\tilde{R}$ are constructed there. The main result of this paper answers affirmatively a basic question, raised in [13, 3.24], of whether the strict universal coefficient ring $R$ is nearly a domain, in the sense that if $r, s, t \in S$ and $a \in \mathbb{Z}[R][s]$ and $b \in \mathbb{Z}[R][t]$, then $ab = 0$ implies $a = 0$ or $b = 0$. The analogous statement for $\tilde{R}$ fails whenever $\tilde{R} \neq R$. One may hope that its validity for $R$ will make the $W$-actions on $R$ better behaved in some respects than those on $\tilde{R}$. We prove the main result by reducing to the case of finite $S$ and then embedding $R$ as a subring of a ring $\text{Mat}_{S \times S}(A)$ of $S' \times S'$-matrices, where $S' := S \cup \{\bullet\}$ and $A$ is a $\mathbb{Q}$-algebra, depending on the Coxeter matrix, which is a free product (in the sense of Cohn [4]), of certain real cyclotomic number fields, Laurent polynomial rings and polynomial rings over $\mathbb{Q}$.

Since the techniques used come from diverse fields, some of which may be unfamiliar to the reader, we include basic definitions and cite basic results from the various fields. In Section 1, we present basic definitions and notation relating to rings and modules used throughout the paper. In Section 2, we introduce Serre’s notation and terminology for graphs [22] and in Section 3, we give the basic definitions of path algebras and cite a number of basic results on Gröbner bases of ideals in that context from Green [16]. In Section 4, we introduce the path algebras related to Coxeter systems and their quotients, $\tilde{R}_{W,S}$ and $R_{W,S}$, appearing in Dyer [13]. In that section, we also prove some basic results about these algebras, which allow us to restrict our attention to finite rank Coxeter groups and to the ring $\tilde{R}_{W,S} = R_{W,S} \otimes_{\mathbb{Z}} \mathbb{Q}$ when proving the main result. In Section 5, we provide for motivation some unpublished results of Dyer extending material from [13] on the lax and strict reflection representations associated to these rings and how they relate to reflection representations on real vector spaces. We thank Matthew Dyer for his permission to include these results. In Section 6, we give the definition and results on free products of rings from Cohn [4–6] and we show that certain rings related to the edges of the underlying path algebra are free products in the sense of Cohn. In Section 7, we show how the ring $\tilde{R}_{W,S}$ can be imbedded in a matrix rig over a free product of rings, We conclude with a proof that the ring $R_{W,S}$ is nearly a domain, as described above, in Section 8.

1. General notation and definitions

1.1. Sets, groups and rings

The notation $X = \{a, b, c\}$ will be used to denote that $X$ is a set with elements $a$, $b$, and $c$. The set $X \setminus Y$ will be the set of all elements in $X$ but not in $Y$ and the set $X \cup Z$ will denote the disjoint union of the sets $X$ and $Z$.

Rings considered may or may not have an identity. If $R$ is a unital ring (has an identity), we will denote the identity of $R$ by $1_R$ in situations where the ring in question may not be clear from the context. If $\phi : R_1 \rightarrow R_2$ is a homomorphism of rings and $S \subset R_1$ is a subring of $R_1$, we let $\phi|_S$ denote the restriction
of $\phi$ to $S$. If $\psi : R_2 \to R_3$ is a homomorphism from $R_2$ to the ring $R_3$, we let $\psi \circ \phi$ denote the composition map. We will use the notation $R_1 \cong R_2$ to indicate that the rings $R_1$ and $R_2$ are isomorphic. In some cases we consider morphisms between rings with identity in the category of non-unital rings, we will clarify this where necessary.

If $X$ is a subset of a ring $R$, the two sided ideal of $R$ generated by the set $X$ will be denoted by $(X)$. If $I$ is an ideal of a ring $R$, we write $r_1 \equiv r_2 \mod I$ when $r_1 - r_2 \in I$. We denote the ring of $n \times n$ matrices over a ring $R$ by $M_n(R)$. We will also use the notation $(x \in X | r \in R)$ to denote a group with generators in the set $X$ and relations in the set $R$. In cases where the meaning of the notation is not clear from the context, we will add clarification.

The symbol $\mathbb{N}$ will be used to denote the non-negative integers, $\{0, 1, 2, 3, \ldots\}$, and the symbol $\mathbb{N}_{\geq k}$ will be used to denote the subset of $\mathbb{N}$ consisting of numbers greater than or equal to $k$. We let $[k]$ denote the set $\{1, \ldots, k\}$. The integers will be denoted by $\mathbb{Z}$, and the rational numbers by $\mathbb{Q}$. Polynomial rings $\mathbb{Q}\{\{x_i\}_{i \in I}\}$ will have non-commuting variables $\{x_i\}_{i \in I}$ unless stated otherwise.

**Definition 1.2.** If $R$ is a ring and $a \in R, a \neq 0$, we say that $a$ is a zero divisor in $R$ if there exists $b \in R, b \neq 0$ such that $ab = 0$ or $ba = 0$. We say that $R$ is a domain if $R$ has no zero divisors.

Let $A$ be a commutative unital ring. An (associative, possibly non-unital) $A$-algebra is a (possibly non-unital, associative) ring $B$ with an additional structure of unital $A$-module for which the multiplication map $B \times B \to B$ is $A$-bilinear (i.e. $(\alpha_1 b_1 + \alpha_2 b_2)(\alpha_3 b_3 + \alpha_4 b_4) = \sum_{i=1}^{2} \sum_{j=1}^{2} \alpha_i \alpha_j b_i b_j$, for all $\alpha_i \in A, b_j \in B$).

Morphisms of $A$-algebras are $A$-linear ring homomorphisms. We call $A$ the coefficient ring of the $A$-algebra $B$. We regard rings as $\mathbb{Z}$-algebras.

A module $M$ for the $A$-algebra $B$ is a module for the ring $B$ together with a unital $A$-module structure on $M$ such that the multiplication map $B \times M \to M$ is $A$-bilinear. Morphisms of modules for the algebra $B$ are $B$-linear maps which are also $A$-linear. We write $\text{End}_B(M)$ for the unital $A$-algebra of endomorphisms of $M$ and $\text{Aut}_B(M)$ for its unit group. For a bimodule $M$ for a pair $(B, B')$ of $A$-algebras, it is required that the $A$-module structures from $B$ and $B'$ coincide.

### 2. Graphs

We use the notation and terminology of Serre [22] for graphs and trees.

**Definition 2.1.** A graph $\Gamma$ consists of a set of vertices $S = \text{Vert } \Gamma$, a set of edges $Y = \text{Edge } \Gamma$ and two maps

$$Y \to S \times S, \quad y \mapsto (o(y), t(y))$$

and

$$Y \to Y, \quad y \mapsto \bar{y},$$

which satisfy the following condition: for each $y \in Y$, we have $\bar{\bar{y}} = y$, $\bar{y} \neq y$ and $o(y) = t(\bar{y})$.

Each $s \in S$ is called a vertex of $\Gamma$ and each $y \in Y$ is called an edge of $\Gamma$. If $y \in Y$, $o(y)$ is called the origin of $y$ and $t(y)$ is called the terminus of $y$, together $o(y)$ and $t(y)$ are called the extremities of $y$. We say that two vertices are adjacent if they are the extremities of some edge.

**Definition 2.2.** An orientation of a graph $\Gamma$ is a subset $Y_+ \subset Y = \text{Edge } \Gamma$ such that $Y$ is the disjoint union $Y = Y_+ \cup \bar{Y}_+$, where $\bar{Y}_+ = \{\bar{y} | y \in Y_+\}$. 
2.3. Paths

A path of length \( n \geq 1 \) in a graph \( \Gamma \) is a sequence of edges \( p = (y_1, \ldots, y_n) \) such that \( t(y_i) = o(y_{i+1}) \), \( 1 \leq i \leq n-1 \). We will denote such a path by \( p = y_1 \ldots y_n \). Paths of length zero are just single vertices and will be denoted by \( p = [s] \) for \( s \in S \). The length of a path \( p \) will be denoted by \( \ell(p) \).

Let \( \mathcal{P} \) denote the set of all paths in \( \Gamma \), we can extend the maps \( o \) and \( t \) to \( \mathcal{P} \) by letting \( o(y_1 \ldots y_n) = o(y_1) \) and \( t(y_1 \ldots y_n) = t(y_n) \) and \( o([s]) = s = t([s]) \). If \( p = y_1 \ldots y_n \) with \( s_i = t(y_i) \) and \( s_{i-1} = o(y_i) \), we say that \( p \) is a path from \( s_0 \) to \( s_n \) and that \( o(y_1) = s_0 \) and \( t(y_n) = s_n \) are the extremities of the path.

A graph is connected if any two vertices are the extremities of at least one path. If \( p = y_1 \ldots y_n \) and \( y_{i+1} = y_i \) for some \( 1 \leq i \leq n-1 \), then the pair \( (y_i, y_{i+1}) \) is called a backtracking. A path \( p \) is a circuit if it is a path without backtracking such that \( o(p) = t(p) \) and \( \ell(p) \geq 1 \). A circuit of length one is called a loop.

A graph \( \Gamma \) is called combinatorial if it has no circuit of length less than or equal to two. If a graph \( \Gamma \) is combinatorial, a path \( p = y_1 \ldots y_n \) is determined by the extremities of its edges and can be characterized and denoted by its vertices as

\[
p = y_1 \ldots y_n = [s_0 s_1 \ldots s_n], \quad \text{where} \quad s_i = t(y_i), s_{i-1} = o(y_i).
\]

Since our graphs of interest in this paper will be combinatorial, we will make frequent use of both notations for paths. In what follows, an edge may be denoted by its label \( y \) or it may be represented by its extremities as \( [o(y), t(y)] \).

A geometric edge of a combinatorial graph, \( \Gamma \), is a set \( \{s_1, s_2\} \) of extremities of an edge in \( \Gamma \). Each such geometric edge corresponds to a pair of edges \( \{y, \bar{y}\} \) in the edge set of \( \Gamma \). A combinatorial graph is determined by its vertices and geometric edges.

**Definition 2.4.** Let \( \Gamma_1 = (S_1, Y_1) \) and \( \Gamma_2 = (S_2, Y_2) \) be two graphs. A graph homomorphism \( \phi : \Gamma_1 \rightarrow \Gamma_2 \) is a pair of maps \( \phi_S : S_1 \rightarrow S_2 \) and \( \phi_Y : Y_1 \rightarrow Y_2 \) such that \( o(\phi_Y(y)) = \phi_S(o(y)) \) and \( t(\phi_Y(y)) = \phi_S(t(y)) \) for every \( y \in Y_1 \).

**Example 2.5.** Below we see a representation of the combinatorial graph \( \Gamma \), with vertices, \( S = \{r, s, t, u, v\} \), and edges, \( Y = Y_+ \cup Y_- \), where

\[
Y_+ = \{y_{\alpha} = [rs], y_e = [st], y_{\beta} = [ut], y_{\ell} = [tv], y_6 = [uv], y_{\gamma} = [ru]\}
\]

We show only the directed edges in \( Y_+ \) on the graph.

\[
\begin{array}{c}
s \\ \downarrow y_e \\ t \\ \downarrow y_{\ell}
\end{array}
\begin{array}{c}
y_{\alpha} \\ \downarrow y_{\beta} \\ \downarrow y_6
\end{array}
\begin{array}{c}
\gamma \\ \downarrow y_{\gamma}
\end{array}
\begin{array}{c}
r \\ \downarrow y_e \\ u
\end{array}
\]

The above example will serve as our running example throughout the paper.

3. Path algebras

In this section, we present the definition of a path algebra along with a summary of some related results on Gröbner bases from Green [16], which will prove useful in later sections.

3.1. The path algebra associated to a graph

Let \( \Gamma = (S, Y) \) be a combinatorial graph, with vertices \( S = \{s_i\}_{i \in I} \) and edges \( Y = \{y_{j}\}_{j \in J} \). Let \( \mathcal{P} \) denote the set of paths in \( \Gamma \) of finite length. Given a commutative ring \( A \) with identity, the path algebra of \( \Gamma \) over \( A \), denoted \( A\Gamma \), is an associative \( A \)-algebra, with an \( A \)-basis of paths in \( \mathcal{P} \). Multiplication is given
by concatenation;

$$[s_0s_1 \ldots s_n][s'_0s'_1 \ldots s'_m] = \begin{cases} 0 & \text{if } s_n \neq s'_0 \\ [s_0s_1 \ldots s_ns'_1 \ldots s'_m] & \text{if } s_n = s'_0 \end{cases}$$

Paths of length 0 form a set of orthogonal idempotents with this multiplication, and if $S$ is finite, $A\Gamma$ has an identity given by

$$1 = \sum_{s \in S} [s].$$

### 3.2. Admissible orders on $\mathfrak{P}$

For the remainder of this section, we will restrict our discussion to the case where the base ring of our path algebra is a field, $K$, and the case where the graph has finitely many vertices and edges, as in Green [16] and Green et. al. in [17]. We will combine results from Green [16] presented below with direct limits in subsequent sections to ensure that we can restrict our attention to the finite case when proving our main theorem.

A well order on the basis $\mathfrak{P}$ of $K\Gamma$ is a total order on $\mathfrak{P}$, with the property that every nonempty subset of $\mathfrak{P}$ has a minimal element, or equivalently, for every descending chain of elements of $\mathfrak{P}$, $p_1 \geq p_2 \geq p_3 \geq \ldots$, there exists some $N > 0$ for which $p_N = p_{N+1} = p_{N+2} = \cdots$.

Given a well order $>_{\mathfrak{P}}$ on $\mathfrak{P}$, and $w \neq 0 \in K\Gamma$, with $w = \sum_{i=1}^{n} a_i p_i$, $a_i \neq 0$, $a_i \in K$, $p_i \in \mathfrak{P}$, we say that $p_i$ is the tip of $w$, denoted $T(w)$, if $p_i \geq p_j$ for $1 \leq j \leq n$. If $X$ is a subset of $K\Gamma$, the tip of $X$ is

$$T(X) = \{ p \in \mathfrak{P} | p = T(x) \text{ for some } x \neq 0, x \in X \}.$$

We denote the set of non tips of $X$ by

$$NT(X) = \mathfrak{P} \setminus T(X).$$

Note both $T(X)$ and $NT(X)$ depend on the order chosen.

If $W$ is a subspace of $K\Gamma$. By Green [16], Theorem 2.1, we have

$$K\Gamma = W \oplus \text{Span}(NT(W)). \quad (3.2.1)$$

We say that an order on $\mathfrak{P}$ is admissible if it is a well order and satisfies the following conditions for $r, s, t, u \in \mathfrak{P}$:

1. if $r < s$ then $rt < st$ if both $rt \neq 0$ and $st \neq 0$.
2. if $r < s$ then $ur < us$ if both $ur \neq 0$ and $us \neq 0$.
3. if $r = st \neq 0$ then $r \geq s$ and $r \geq t$.

A number of admissible well orders can be imposed on the paths in a path algebra, see Green [16]. We give one example of such an admissible ordering below.

**Example 3.3.** The (left) length lexicographic order on $\mathfrak{P}$:

Let $\Gamma = (S, Y)$ be a graph, with vertices $S = \{s_i\}_{i \in [n]}$ and edges $Y = \{y_j\}_{j \in [m]}$. Let $\mathfrak{P}$ denote the set of paths in $\Gamma$ of finite length. We choose an ordering on paths of length $\leq 1$, in such a way that each element of $S$ is less than each element of $Y$.

$$[s_1] < [s_2] < \cdots < [s_n] < y_1 < y_2 < \cdots < y_m.$$ 

If $p$ and $q$ are paths of length at least one, $p < q$ if $l(p) < l(q)$, or if $l(p) = l(q)$ and $p = y_1y_2\cdots y_iy_{i+1}\cdots y_m$ and $q = y_1y_2\cdots y_iy'_{i+1}\cdots y'_m$ with $y_{i+1} < y'_{i+1}$. It is not difficult to check that this is in fact a well order and conditions (1), (2), and (3) in the definition of an admissible well order hold.
3.4. Gröbner basis of an ideal

If $>$ is an admissible order on $\mathbb{P}$, and $I$ an ideal in $K\Gamma$ ($K$ a field), we say that a subset $\mathfrak{G} \subset I$ is a Gröbner basis for the ideal $I$, with respect to $>$, if the two-sided ideal generated by $T(\mathfrak{G})$ is equal to the two-sided ideal generated by $T(I)$. Equivalently, $\mathfrak{G} \subset I$ is a Gröbner basis for $I$ if for every $b \in T(I)$ there is some $g \in \mathfrak{G}$ such that $b = pT(g)q$ for some $p, q \in \mathbb{P}$.

Given a Gröbner basis, $\mathfrak{G}$, for an ideal $I$ of $K\Gamma$ with respect to an admissible order $>$, we have, by Equation (3.2.1), that as a vector space over $K$,

$$K\Gamma = I \oplus \text{Span}(NT(I)) = \langle \mathfrak{G} \rangle \oplus \text{Span}(NT(I)).$$

Equation (3.4.1) allows us to use a Gröbner basis, $\mathfrak{G}$, to determine ideal membership for $I$. Given a Gröbner basis, $\mathfrak{G}$, for an ideal $I$ of $K\Gamma$, each $y \in K\Gamma$ has a unique normal form $N(y) \in \text{Span}(NT(I))$ with respect to the order $>$, where $y = y_1 + N(y)$ and $y_1 \in I$. Since $\langle T(\mathfrak{G}) \rangle = \langle T(I) \rangle$, we have that $\text{Span}(NT(I))$ has a basis consisting of those paths in $\mathbb{P}$ which are not divisible by $T(g)$ for some $g \in \mathfrak{G}$ (i.e. the paths in $K\Gamma$ which are not of the form $pT(g)q$ for some $g \in \mathfrak{G}, p, q \in \mathbb{P}$). The following is special case of Proposition 2.9, Green [16], which follows easily from Equation (3.4.1) and makes explicit the relationship between $K\Gamma/I$ and $\text{Span}(NT(I))$:

**Proposition 3.5.** (See Proposition 2.9, Green [16]) Let $\Gamma$ be a graph with finitely many vertices and edges. Let $K\Gamma$ be the corresponding path algebra with basis of paths, $\mathbb{P}$, admitting an admissible order $>$. Let $I$ be an ideal of $K\Gamma$ with Gröbner basis $\mathfrak{G}$ and let $x, y \in K\Gamma$ with equivalence classes $x + I$ and $y + I$ respectively in the quotient algebra $K\Gamma/I$. Then

1. $x + I = y + I$ if and only if $N(x) = N(y)$.
2. $x + I = N(x) + I$.
3. The map $\sigma : K\Gamma/I \rightarrow K\Gamma$ with $\sigma(x + I) = N(x)$ is a vector space splitting to the canonical surjection $\pi : K\Gamma \rightarrow K\Gamma/I$.
4. $\sigma$ induces a $K$-linear isomorphism between $K\Gamma/I$ and $\text{Span}(NT(I))$.
5. Identifying $K\Gamma/I$ with $\text{Span}(NT(I))$, then $NT(I)$ is a $K$-basis of $K\Gamma/I$ contained in $\mathbb{P}$.

3.6. Overlap relations

Let $>$ be an admissible order on $\mathbb{P}$. If $f, g \in K\Gamma, f, g \neq 0$, and $p, q \in \mathbb{P}$ such that

1. $T(f)q = pT(g)$.
2. $T(f)$ does not divide $p$ and $T(g)$ does not divide $q$.

The overlap relation of $f$ and $g$ by $p, q$ is

$$o(f, g, p, q) = (1/C(T(f)))pq - (1/C(T(g)))pq,$$

where $C(T(h))$ denotes the coefficient of $T(h)$ in $h \in K\Gamma$. Note that $T(o(f, g, p, q)) < T(f)q = pT(g)$.

We say the elements of a set $\mathfrak{G}$ is *tip reduced* if for distinct elements $g_1, g_2 \in \mathfrak{G}$, $T(g_1)$ does not divide $T(g_2)$. An element of the path algebra $K\Gamma, \sum_{i=1}^{n} a_ip_i, a_i \in A, a_i \neq 0, p_i \in \mathbb{P}$, is said to be *left uniform* if for each $p \in \mathbb{P}$, either $pp_i = 0$ for all $i$ or $pp_i \neq 0$ for all $i$. Right uniform elements are defined similarly and an element is *uniform* if it is both left and right uniform. An element of the path algebra $K\Gamma, \sum_{i=1}^{n} a_ip_i, a_i \in K, a_i \neq 0, p_i \in \mathbb{P}$ is uniform if and only if there are vertices $v, w$ such that for $1 \leq i \leq n, o(p_i) = v$ and $t(p_i) = w$. The following result, limited for simplicity to a special case of Theorem 2.3, Green [16] which is sufficient for our needs, can be used to identify when a set of polynomials form a Gröbner basis for an ideal in a path algebra:
Theorem 3.7. (See Green [16], Theorem 2.3) Let $\Gamma$ be a graph with finitely many vertices and edges. Suppose that $\mathcal{G}$ is a set of uniform, tip reduced elements of $K\Gamma$. If for every overlap relation, $o(g_1, g_2, p, q)$, with $g_1, g_2 \in \mathcal{G}$, $p, q \in \mathcal{P}$, we have,

$$o(g_1, g_2, p, q) = 0,$$

then $\mathcal{G}$ is a Gröbner basis for $(\mathcal{G})$.

4. Coxeter systems and associated path algebras

In this section, we introduce the Path Algebras related to Coxeter systems and the rings $R = R_{W,S}$ and $\tilde{R} = \tilde{R}_{W,S}$ from Dyer [13], discussed in the introduction. We also prove a number of lemmas which will be used in later sections to simplify the proof of our main result.

4.1. Path algebras associated to coxeter systems

A Coxeter matrix is a matrix $(m_{ij})_{i,j \in I}$, where $I$ is a set of indices, with $m_{ii} = 1$ for all $i \in I$ and $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $i, j \in I$, $i \neq j$. If $S = \{s_i \mid i \in I\}$ is a set indexed by $I$, the corresponding Coxeter group is the group with presentation

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} = 1 \text{ for } s_k, s_l \in S \text{ with } m_{kl} \neq \infty \rangle.$$  

The pair $(W, S)$ is called a Coxeter system with Coxeter matrix $(m_{ij})_{i,j \in I}$. We say that $(W, S)$ has finite rank if $S$ is finite.

We associate to a Coxeter system $(W, S)$ a graph $\Gamma_{W,S}$ with vertices $s_i \in S$ and edges $Y_{W,S} = \{y_{ij} = [s_is_j] \mid i \neq j \text{ and } m_{ij} \geq 3\}$. Let $\mathcal{P}_{W,S}$ denote the set of paths in $\Gamma_{W,S}$. We let $P_{W,S}$ denote the path algebra $\mathbb{Z}\Gamma_{W,S}$ and let $\hat{P}_{W,S}$ denote the path algebra $\mathbb{Q}\Gamma_{W,S}$.

For $m \in \mathbb{Z}$, $m \geq 3$, let $C_m(t)$ denote the minimum polynomial of $4 \cos^2 \frac{\pi}{m}$ in $\mathbb{Z}[t]$. We have

$$C_3(t) = t - 1, \quad C_4(t) = t - 2, \quad C_5(t) = t^2 - 3t + 1, \quad C_6(t) = t - 3, \ldots$$

To each edge $y_{ij} = [s_is_j]$ of $\Gamma_{W,S}$ where $3 \leq m_{ij} < \infty$, we associate an element $C_{y_{ij}}$ of the subring $[s_i]P_{W,S}[s_i] \subset P_{W,S}$ (resp. $[s_i]\hat{P}_{W,S}[s_i] \subset \hat{P}_{W,S}$), where the imbedding is non-unital. Notice that the ring $[s_i]P_{W,S}[s_i]$ (resp. $[s_i]\hat{P}_{W,S}[s_i]$) has identity $[s_i]$, making evaluation of the above polynomials possible over the ring. We let

$$C_{y_{ij}} = C_{m_{ij}}([s_is_j]) = C_{m_{ij}}(\bar{y}_{ij})$$

with $C_{m_{ij}}$ viewed as a polynomial over the ring $[s_i]P_{W,S}[s_i]$ (resp. $[s_i]\hat{P}_{W,S}[s_i]$). Let $I_{W,S}$, (respectively $\hat{I}_{W,S}$), be the ideal of $P_{W,S}$ (resp. $\hat{P}_{W,S}$) generated by the set $\mathcal{E}_{W,S} = \{C_y \mid y = [s_is_j] \in Y_{W,S}, m_{ij} < \infty\}$, and let $R_{W,S} = P_{W,S}/I_{W,S}$ (resp. $\hat{R}_{W,S} = \hat{P}_{W,S}/\hat{I}_{W,S}$) denote the quotient algebras.

Lemma 4.2. Let $\Gamma_{W,S}$ be a graph associated to a finite rank Coxeter System $(W, S)$ with Coxeter matrix $(m_{ij})_{(i,j) \in I}$. Then $\mathcal{E}_{W,S}$ is a Gröbner basis for the ideal $I_{W,S}$ in the path algebra $P_{W,S}$.

Proof. Regardless of the admissible ordering used, $T(C_{y_{ij}}) = (y_{ij}\bar{y}_{ij})^{d_{ij}}$, where degree $C_{m_{ij}} = d_{ij}$, for all $y_{ij} \in Y_{W,S}$ with $m_{ij} < \infty$. Clearly the set $\mathcal{E}_{W,S}$ is tip reduced and uniform, therefore, by Theorem 3.7, we need only verify that all overlap relations $o(C_{y_{ij}}, C_{y_{kl}}, p, q)$, $y_{ij}, y_{kl} \in Y, m_{ij}, m_{kl} < \infty, p, q \in \mathcal{P}_{W,S}$ are zero. Consider such an overlap relation, $T(C_{y_{ij}})q = pT(C_{y_{kl}})$, for $p, q \in \mathcal{P}$, where $T(C_{y_{ij}})$ does not divide $q$ and $T(C_{y_{ij}})$ does not divide $p$. If $T(C_{y_{ij}})q = pT(C_{y_{kl}}) = 0$, then the uniformity of the set $\mathcal{E}_{W,S}$ guarantees that the overlap relation equals zero, hence we can restrict our attention to the situation where $T(C_{y_{ij}})q = pT(C_{y_{kl}}) \neq 0$. Since $T(C_{y_{ij}}) = (y_{ij}\bar{y}_{ij})^{d_{ij}}$ and $T(C_{y_{kl}}) = (y_{kl}\bar{y}_{kl})^{d_{kl}}$, and $T(C_{y_{ij}})$ does not divide $q$, we must have $T(C_{y_{kl}}) = aq$ for some $a \in \mathcal{P}_{W,S}$ where $\ell(a) \geq 1$. Hence

$$T(C_{y_{ij}})q = (y_{ij}\bar{y}_{ij})^{d_{ij}}q = pT(C_{y_{kl}}) = paq = p(y_{kl}\bar{y}_{kl})^{d_{kl}},$$
and since $a$ divides both $(y_j y_i)^{d_j}$ and $(y_k y_l)^{d_k}$, we must have that either $y_{kl} = y_{ij}$ or $y_{kl} = y_{ij}$. If $y_{kl} = y_{ij}$, then $(y_j y_i)^{d_j} q = p(y_j y_i)^{d_j}$ and by comparing lengths and edges, we see that $p = q = (y_j y_i)^k$ for some $k$. Thus $C_{y_j} p - q C_{y_i} = 0$ in this case. If $y_{kl} = y_{ij}$, then $(y_j y_i)^{d_j} q = p(y_j y_i)^{d_j}$. In this case, a comparison of lengths and edges shows that $p = q = (y_j y_i)^k y_{ij}$ for some $k$ and it is clear that $C_{y_i} p - q C_{y_j} = 0$. This completes our proof.

**4.3. Path algebras associated to the standard parabolic subgroups**

Let $J \subset I$ and $S_j = \{s_j | j \in J \}$. Let $(W_J, S_J)$ denote the Coxeter system with Coxeter matrix $(m_{ij})_{i,j \in J}$. The group $W_J$ has generators and relations

$$W = \langle s_j (j \in J) | (s_k s_l)^{m_{kl}} = 1 \text{ for } s_k, s_l \in S_J \text{ with } m_{kl} \neq \infty \rangle.$$

and is called a *standard parabolic subgroup* of $W$. If $J_1 \subseteq J_2 \subseteq I$ we have an inclusion map of graphs from the vertices $S_J$ and edges $Y_{W_J, S_J}$ in $\Gamma_{W_{J_2}, S_{J_2}}$ to the corresponding vertices and edges in $\Gamma_{W_{J_1}, S_{J_1}}$. This extends to an inclusion map $i_{J_1, J_2} : \mathcal{P}_{W_{J_1}, S_{J_1}} \rightarrow \mathcal{P}_{W_{J_2}, S_{J_2}}$, which in turn gives us a non-unital monomorphism of $\mathbb{Q}$-algebras; $\hat{i}_{J_1, J_2} : \hat{\mathcal{P}}_{W_{J_1}, S_{J_1}} \rightarrow \hat{\mathcal{P}}_{W_{J_2}, S_{J_2}}$. Since $\hat{i}_{J_1, J_2}(\mathcal{C}_{W_{J_1}, S_{J_1}}) \subseteq \mathcal{C}_{W_{J_2}, S_{J_2}}$, we get a well-defined induced non-unital homomorphism of quotient algebras:

$$\hat{i}_{J_1, J_2} : \hat{\mathcal{R}}_{W_{J_1}, S_{J_1}} \rightarrow \hat{\mathcal{R}}_{W_{J_2}, S_{J_2}}.$$

**Lemma 4.4.** Let $(W, S = \{s_i \}_{i \in I})$ be a Coxeter system. Let $J_1, J_2$ be finite subsets of $I$ with $J_1 \subseteq J_2$, then the map $\hat{i}_{J_1, J_2} : \hat{\mathcal{R}}_{W_{J_1}, S_{J_1}} \rightarrow \hat{\mathcal{R}}_{W_{J_2}, S_{J_2}}$ is a monomorphism of non-unital $\mathbb{Q}$-algebras.

**Proof.** Let $>_{W_{J_2}, S_{J_2}}$ be an admissible ordering on $\mathcal{P}_{W_{J_2}, S_{J_2}}$. The restriction of $>_{W_{J_2}, S_{J_2}}$ to $i_{J_1, J_2}(\mathcal{P}_{W_{J_1}, S_{J_1}})$, gives us an admissible ordering on $\mathcal{P}_{W_{J_1}, S_{J_1}}$ which, for the sake of simplicity, we will also denote by $>$. By Proposition 3.5, $\hat{\mathcal{R}}_{W_{J_1}, S_{J_1}}$ can be identified as a vector space over $\mathbb{Q}$ with $\text{Span}(\mathcal{B}(\hat{\Gamma}_{W_{J_1}, S_{J_1}}))$, $i \in \{1, 2\}$ (with respect to the ordering $>_{W_{J_2}, S_{J_2}}$). If $p \in \mathcal{N}(\hat{\Gamma}_{W_{J_1}, S_{J_1}})$, then $p$ is not divisible by $T(C_y)$ for any $C_y \in \mathcal{C}_{W_{J_1}, S_{J_1}}$. Since $p \in \mathcal{P}_{W_{J_1}, S_{J_1}}$, $p$ cannot be divisible by $T(C_y)$ for any $C_y \in \mathcal{C}_{W_{J_2}, S_{J_2}} \setminus \mathcal{C}_{W_{J_1}, S_{J_1}}$ and thus $i_{J_1, J_2}(p) \in \mathcal{N}(\hat{\Gamma}_{W_{J_2}, S_{J_2}})$. Thus, the map $i_{J_1, J_2} : \mathcal{P}_{W_{J_1}, S_{J_1}} \rightarrow \mathcal{P}_{W_{J_2}, S_{J_2}}$ restricts to a one-to-one map $i_{J_1, J_2} : \mathcal{N}(\hat{\Gamma}_{W_{J_1}, S_{J_1}}) \rightarrow \mathcal{N}(\hat{\Gamma}_{W_{J_2}, S_{J_2}})$. Using the vector space identification above, we see that $i_{J_1, J_2} : \hat{\mathcal{R}}_{W_{J_1}, S_{J_1}} \rightarrow \hat{\mathcal{R}}_{W_{J_2}, S_{J_2}}$ is a monomorphism as desired.

**4.5. Direct systems**

We shall prove some of our results involving rings attached to infinite rank Coxeter groups by regarding them as direct limits of rings attached to the finite rank standard parabolic subgroups.

**Definition 4.6.** A *directed poset* is a poset $(P, \leq)$, $P \neq \emptyset$, such that for any $p_1, p_2 \in P$, there is an element $p_3 \in P$ such that $p_1 \leq p_3$ and $p_2 \leq p_3$.

**Definition 4.7.** A *direct system* $\{P, \{R_p\}, \{i_{p_1, p_2}\}\}$ in the category of non-unital $\mathbb{Q}$-algebras consists of a directed poset, $P$, a collection of non-unital $\mathbb{Q}$-algebras, $\{R_p\}_{p \in P}$, and a set of morphisms of $\mathbb{Q}$-algebras, $\{i_{p_1, p_2} : R_{p_1} \rightarrow R_{p_2} | p_1, p_2 \in P, p_1 \leq p_2 \}$ such that

1. $i_{p, p}$ acts as the identity on $R_p$.

2. $i_{p_2, p_3} \circ i_{p_1, p_2} = i_{p_1, p_3}$ for $p_1, p_2, p_3 \in P$, $p_1 \leq p_2 \leq p_3$.

**Note.** When considering direct systems in the category of non-unital $\mathbb{Q}$-algebras throughout the remainder of this section, all associated morphisms considered will be morphisms in that category.
Definition 4.8. Let \( \{ P, \{ R_p \}, \{ ip_{p_1, p_2} \} \} \) be a direct system of non-unital \( \mathbb{Q} \) algebras. A \( \mathbb{Q} \) algebra \( R \) is called the direct limit of the direct system, denoted \( \lim \longrightarrow \) \( R_p \), if there exist homomorphisms, \( ip : R_p \to R \), \( p \in P \) such that

1. For any \( p_1, p_2 \in P \), \( p_1 \leq p_2 \), the following diagram commutes:

\[
\begin{array}{ccc}
R_{p_1} & \xrightarrow{ip_{p_1, p_2}} & R_{p_2} \\
\downarrow{ip_{p_1, p_2}} & \downarrow{ip_{p_1, p_2}} & \downarrow{ip_{p_1, p_2}} \\
R_p & \xrightarrow{ip_{p}} & R \\
\end{array}
\]

2. Given a non-unital \( \mathbb{Q} \) algebra, \( Q \), and a set of homomorphisms \( \phi_p : R_p \to Q \), \( p \in P \), such that

\[
\begin{array}{ccc}
R_{p_1} & \xrightarrow{ip_{p_1}} & R_{p_2} \\
\downarrow{ip_{p_1}} & \downarrow{ip_{p_2}} & \downarrow{ip_{p_2}} \\
R_p & \xrightarrow{ip_{p}} & Q \\
\end{array}
\]

commutes for all \( p_1, p_2 \in P \), \( p_1 \leq p_2 \), then there exists a unique homomorphism \( \phi : R \to Q \) such that the following diagram commutes for all \( p \in P \):

\[
\begin{array}{ccc}
R_p & \xrightarrow{ip} & R \\
\downarrow{ip} & \downarrow{ip} & \downarrow{ip} \\
Q & \xrightarrow{\phi} & Q \\
\end{array}
\]

Lemma 4.9. Let \( \{ P, \{ R_p \}, \{ ip_{p_1, p_2} \} \} \) be a direct system of non-unital \( \mathbb{Q} \) algebras. If \( ip_{p_1, p_2} \) is a monomorphism for all \( p_1, p_2 \in P \), then the homomorphisms \( ip : R_p \to \lim \longrightarrow R_p \) making the diagram below commute, are also monomorphisms:

\[
\begin{array}{ccc}
R_{p_1} & \xrightarrow{ip_{p_1, p_2}} & R_{p_2} \\
\downarrow{ip_{p_1, p_2}} & \downarrow{ip_{p_1, p_2}} & \downarrow{ip_{p_1, p_2}} \\
R_p & \xrightarrow{ip_{p}} & \lim \longrightarrow R_p \\
\end{array}
\]

Proof. One can construct the direct limit of non-unital \( \mathbb{Q} \) algebras as follows. Let

\[ R = \lim \longrightarrow R_p = \bigcup_{p \in P} R_p / \sim \]

where \( \sim \) is the equivalence relation on \( \bigcup_{p \in P} R_p \) defined by

\[ r_1 \sim r_2, r_1 \in R_{p_1}, r_2 \in R_{p_2} \text{ iff. } \begin{aligned} \text{ \( r_1, r_2 \in R_{p_1} \), } \text{ \( r_1 \leq r_2 \), } \text{ \( \exists p_3 > p_1, p_2 \) such that } \\
\text{ \( r_1 = i_{p_1, p_3}(r_{p_1}) \) and } \text{ \( r_2 = i_{p_2, p_3}(r_{p_2}) \) } \end{aligned} \]

The ring and algebra operations on the equivalence classes of \( R \), denoted by square brackets \([ \cdot ]\), are defined as follows. For each \( \alpha \in \mathbb{Q} \) and each \( [r] \in R \), let \( \alpha[r] = [\alpha r] \). Given \( [r_1], [r_2] \in R \) such that \( r_1 \in R_{p_1} \) and \( r_2 \in R_{p_2} \), then for \( p_3 > p_1, p_2 \), we define \( [r_1] + [r_2] = [i_{p_1, p_3}(r_{p_1}) + i_{p_2, p_3}(r_{p_2})] \) and \( [r_1][r_2] = [i_{p_1, p_3}(r_{p_1})i_{p_2, p_3}(r_{p_2})] \). The equivalence class \( [0] \) gives a zero element for the ring \( R \). It can be checked easily that \( R \) along with the homomorphisms \( ip : R_p \to R_p, p \in P \) given by \( ip(r) = [r] \) for \( r \in R_p \) form a non-unital \( \mathbb{Q} \) algebra making the diagram (4.9.1) above commute. Given a non-unital \( \mathbb{Q} \) algebra \( Q \) and a set of morphisms \( \phi_p : R_p \to Q, p \in P \), such that \( \phi_{p_2} \circ i_{p_1, p_2} = \phi_{p_1} \), we define \( \phi : R \to Q \) as \( \phi([r_p]) = \phi_p(r_p) \) for \( r_p \in R_p \). One can easily check that \( \phi \) is a well-defined homomorphism with
the property that $\phi \circ i_p = \phi_p$ for all $p \in P$. It is unique since any homomorphism $\psi : R \to Q$ with 
$\psi \circ i_p = \phi_p$ must agree with $\phi$.

By the universal mapping property, the direct limit is unique up to isomorphism, hence it is enough
to show that the homomorphisms $i_p : R_p \to R$ are monomorphism for all $p \in P$ for the non-unital $\mathbb{Q}$
algebra $R$ constructed above. Now it is clear that if $i_p(r) = [0]$ for some $r \in R_p, p \in P$, then we must
have $i_{p_1}i_{p_2}(r) = 0 \in R_{p_2}$ for some $p_2 \in P$. Thus if all of the maps $i_{p_1}, p_1, p_2 \in P$ are monomorphisms,
we must have that each $i_p, p \in P$ is also a monomorphism. 

\[4.10. \hat{R}_{W,S} \text{ as a direct limit}\]

For the Coxeter system $(W, S = \{s_i\}_{i \in I})$, let $\mathcal{P}(I)$ denote the set of finite subsets of $I$. Then $\mathcal{P}(I)$
is a directed poset with inclusion as a partial ordering. It is not hard to verify that the quotient
algebras associated to the family of parabolic subgroups,
\begin{equation}
\hat{R}_{W_{I,J}} \to \hat{R}_{W,S}, I \subseteq J,
\end{equation}
form a direct system, $\left\{\mathcal{P}(I), \{\hat{R}_{W_{I,J}}\}, \{i_{I,J}\}\right\}$ in the category of non-unital
algebras over $\mathbb{Q}$.

**Lemma 4.11.** Let $(W, S = \{s_i\}_{i \in I})$, be a Coxeter system and $\left\{\mathcal{P}(I), \{\hat{R}_{W_{I,J}}\}, \{i_{I,J}\}\right\}$ the direct system of
quotient algebras associated to the parabolic subgroups of $W$ of finite rank. Then

1. Given $x \in \hat{R}_{W,S}$, there exists $J \in \mathcal{P}(I)$ such that $x = \tilde{i}_{J,J'}(x')$ for some $x' \in \hat{R}_{W_{I,J}}$.

2. The homomorphisms $\tilde{i}_{J,J'} : \hat{R}_{W_{I,J}} \to \hat{R}_{W,S}$ have the property that $\tilde{i}_{J,J'} \circ \tilde{i}_{J,J''} = \tilde{i}_{J,J''}$ for all $J, J' \in \mathcal{P}(I), J \subseteq J'$.

3. Given a non-unital $\mathbb{Q}$ algebra $Q$ and a family of homomorphisms $\{\phi_I : \hat{R}_{W_{I,J}} \to Q| I \in \mathcal{P}(I)\}$ such
that the following diagram commutes for each $J_1 \subseteq J_2 \subseteq I, J_1$ and $J_2$ finite:

\[\begin{array}{ccc}
\hat{R}_{W_{I,J_1}} & \xrightarrow{\phi_{I,J_1}} & \hat{R}_{W_{I,J_2}} \\
\tilde{i}_{J_1,J_2} & \downarrow & \phi_{I,J_2} \\
\hat{R}_{W_{I,J_2}} & \xrightarrow{\phi_{I,J_2}} & Q
\end{array}\]

then there exists a homomorphism $\phi$ such that the following diagram commutes for all finite subsets $J$ of $I$:

\[\begin{array}{ccc}
\hat{R}_{W_{I,J}} & \xrightarrow{\tilde{i}_{J,J'}} & \hat{R}_{W,S} \\
\phi_{I,J} & \downarrow & \phi \\
Q & \xrightarrow{\phi} & Q
\end{array}\]

4. $\hat{R}_{W,S} = \lim_{J \to \infty} \hat{R}_{W_{I,J}}$ in the category of non-unital $\mathbb{Q}$ algebras.

5. The maps $\tilde{i}_{J,J'} : \hat{R}_{W_{I,J}} \to \hat{R}_{W,S}$ are injective.

**Proof.** Given an $x = p + \hat{I}_{W,S} \in \hat{R}_{W,S}$, where $p \in \hat{P}_{W,S}$, then $p = \sum_{i=1}^{n} \alpha_i p_i$ where $p_i \in \mathbb{Q}_{W,S}$. Since
each path $p_i$ is a product of only a finite number of edges, we have $x = \tilde{i}_{J,J'}(x')$ for some $J \in \mathcal{P}(I)$ and
$x' \in \hat{R}_{W_{I,J}}$. This proves (1).

Since, $(\tilde{i}_{J_2,J} \circ \tilde{i}_{J_1,J})(p + \hat{I}_{W_{I,J_1}}) = \tilde{i}_{J_2,J}(p)$ for all $J_1, J_2 \in \mathcal{P}(I), J_1 \subseteq J_2$, and all $p \in \mathbb{Q}_{W_{I,J_1}}$, (2) follows.
Assume now that we are given a non-unital $\mathbb{Q}$ algebra $Q$ and a family of homomorphisms $\{\phi_I : \hat{R}_{W_I,S_I} \to Q|I \in \mathcal{P}(I)\}$ making Diagram 4.11.1 commute. Let $x = p + \hat{I}_{W,S} \in \hat{R}_{W,S}$. By (1) $x = \tilde{i}_{ij}(x')$ for some $J \in \mathcal{P}(I)$ and $x' \in \hat{R}_{W_{ij},S_{ij}}$. We let $\phi(x) = \phi_I(x')$. By (2) and the commutativity of Diagram 4.11.1, $\phi(x)$ is independent of the choice of $J$. For each $C_y \in \mathcal{C}_{W,S}, \phi(C_y + \hat{I}_{W,S}) = \phi(\tilde{i}_{ij}(C_y + \hat{I}_{W_{ij},S_{ij}})) = \phi(\tilde{i}_{ij}(0)) = 0$, where $J = \{i,j\}o(y) = s_i,t(y) = s_j$. Thus $\phi$ is well-defined. That the diagram 4.11.2 commutes follows from the definition of $\phi$, and since any homomorphism making 4.11.2 commute must agree with $\phi$, we see it is the unique homomorphism with this property. This establishes (3).

From Definition 4.8, we see that (4) follows from (2) and (3). By Lemma 4.4, the maps $\tilde{i}_{ij} : J$ in our direct system are monomorphisms. Therefore (5) follows from (2), (3), (4) and Lemma 4.9. This completes the proof. □

Lemma 4.12. Let $(W,S)$ be a Coxeter system and $\Gamma_{W,S}$ be the associated graph. Let $\mathcal{H}$ denote the quotient map $\hat{P}_{W,S}$ to $\hat{R}_{W,S}$. Let $\mathcal{H} = \{x \in \mathcal{W}_{W,S}|x \neq ptq, t \in T(\mathcal{C}_{W,S}), p,q \in \mathcal{P}_{W,S}\}. \quad \text{The set } \mathcal{H}(\mathcal{H}) \text{ forms a basis for } \hat{R}_{W,S} \text{ as a vector space over } \mathcal{Q}.

Proof. Let $x \in \hat{R}_{W,S}$ by Lemma 4.11 (1), we have $x = \tilde{i}_{ij}(r)$ for some finite $I \subseteq I$ and $r \in \hat{R}_{W_{ij},S_{ij}}$. Hence $x \in \text{Span}(\tilde{i}_{ij}(\mathcal{N}(\hat{I}_{W_{ij},S_{ij}}))) \subseteq \text{Span}(\mathcal{H}(\mathcal{H}))$. Let $x_1,x_2,,x_n$ be a subset of $\mathcal{H}(\mathcal{H})$ such that $\sum_{i=1}^{n}a_ix_i = 0$ for some $a_1,a_2,,a_n \in \mathcal{Q}$, then we have $x_1 = \tilde{i}_{ij}(p_1 + \hat{I}_{W_{ij},S_{ij}})$ for all $i$, for some finite $I \subseteq I$ with each $p_1 \in \mathcal{N}(\hat{I}_{W_{ij},S_{ij}})$. Thus, since $\tilde{i}_{ij}$ is a monomorphism by Lemma 4.11 (5), we have $\sum_{i=1}^{n}a_ip_i + \hat{I}_{W_{ij},S_{ij}} = 0$ in $\hat{R}_{W_{ij},S_{ij}}$ and since the cosets corresponding to $\mathcal{N}(\hat{I}_{W_{ij},S_{ij}})$ form a vector space basis in $\hat{R}_{W_{ij},S_{ij}}$, we see that $a_i = 0$ for $1 \leq i \leq n$. Hence the elements of $\mathcal{H}(\mathcal{H})$ span $\hat{R}_{W,S}$ and are linearly independent over $\mathcal{Q}$ and thus form a basis for $\hat{R}_{W,S}$ as a vector space over $\mathcal{Q}$. □

Lemma 4.13. Let $(W,S)$ be a Coxeter system and $\Gamma_{W,S}$ be the associated graph. The inclusion map $i : P_{W,S} \to \hat{P}_{W,S}$ which identifies the basis of paths $\mathcal{P}_{W,S}$ in both algebras, induces a monomorphism $i : R_{W,S} \to \hat{R}_{W,S}$. $\hat{P}_{W,S}$

Proof. It is clear that $\tilde{i}$ is a well-defined homorphism of rings, since $\mathcal{C}_{W,S}$ is a generating set for both $I$ and $\hat{I}$. We let $\pi$ and $\hat{\pi}$ denote the quotient maps from $P_{W,S}$ to $R_{W,S}$ and $\hat{P}_{W,S}$ to $\hat{R}_{W,S}$ respectively. We let $\mathcal{H} = \{x \in \mathcal{P}_{W,S}|x \neq ptq, t \in T(\mathcal{C}_{W,S}), p,q \in \mathcal{P}_{W,S}\}. \quad \text{By Lemma 4.12, we have that } \hat{\pi}(\mathcal{H}) \text{ is a basis of } \hat{R}_{W,S} \text{ as a vector space over } \mathcal{Q}. \text{ Letting } M \text{ denote the } \mathcal{Z} \text{-submodule of } P_{W,S} \text{ generated by } \mathcal{H}, \text{ and } V = \text{Span}(\mathcal{H}) \text{ the subspace of } \hat{P}_{W,S} \text{ as a } \mathcal{Q} \text{ vector space, we get a commutative diagram}$.

$$\begin{align*}
\pi : P_{W,S} &\xrightarrow{i} \hat{P}_{W,S} \\
M &\xrightarrow{i|_M} V = \text{Span}(\mathcal{H}) \\
R_{W,S} &\xrightarrow{i|_{R_{W,S}}} \hat{R}_{W,S} \\
\hat{\pi} : \hat{P}_{W,S} &\xrightarrow{i|_V} \hat{R}_{W,S}
\end{align*}$$

where $\pi|_M$ and $\hat{\pi}|_V$ denote the restrictions of the ring homomorphisms $\pi$ and $\hat{\pi}$ as $\mathcal{Z}$ module and vector space homomorphism respectively. By Lemma 4.12, $\hat{\pi}|_V$ is an isomorphism of vector spaces. In particular $\hat{\pi}|_V$ is a monomorphism.

We claim that $\pi|_{i|_M}$ maps $M$ onto $R_{W,S}$. We show that $P_{W,S} = M + I_{W,S}$. First, we note that the paths of length zero, $\{|s_i|s_i \in S\}$ are in $\mathcal{H} \subseteq M$. Next, we show that given any $p \in \mathcal{H}$, and any $y_{ij} \in Y$, with $s_j = o(p)$, then $y_{ij}p = q + I_{W,S}$ for some $q \in M$. If $y_{ij}p \in \mathcal{H}$, then $q = y_{ij}p$ and we are done. Otherwise, $p \in \mathcal{H}$ and $y_{ij}p \not\in \mathcal{H}$. Thus, $p$ is not divisible by any element of $T(\mathcal{C}_{W,S})$ and $y_{ij}p$ is divisible.
by some element of $T(C,W)$. Comparing edges along the paths $p$ and $y_{ij}p$, we see that $y_{ij}p = T(C_{y_{ij}})p_1$, for some $p_1 \in \mathfrak{Y}$. Thus, since $C_{y_{ij}}$ is a monic polynomial over $\mathbb{Z}$ in $y_{ij}y_{ij}$, we have $y_{ij}p - C_{y_{ij}}p_1$ is a $\mathbb{Z}$-linear combination of paths which divide $p$. Since paths which divide $p$ are already in normal form and are therefore in $\mathfrak{N}$, we have $y_{ij}p - C_{y_{ij}}p_1 \in M$ and $y_{ij}p \in M + I_{W,S}$ as asserted. To prove our claim, we note that because $[s_i] \in \mathfrak{N}$ for all $s_i \in S$, and $y_{ij}p + M + I_{W,S}$ for all $y_{ij} \in Y$ and $p \in \mathfrak{N}$ such that $o(p) = s_j$, we have $q + I_{W,S} \in M + I_{W,S}$ for any $q \in \mathfrak{Y}$. Therefore as a $\mathbb{Z}$ module, $P_{W,S} = M + I_{W,S}$ and thus $\pi |_M$ maps $M$ onto $R_{W,S}$, proving our claim.

Now since $\hat{P}_{W,S} \cong \mathbb{Q} \otimes_\mathbb{Z} P_{W,S}$ and $P_{W,S}$ is a free $\mathbb{Z}$ module, we have that $i$ is a monomorphism. Thus its restriction $i|_M$ is also a monomorphism. Because $\pi |_M$ is an epimorphism and $i|_M$ and $\tilde{\pi}|_V$ are both monomorphisms, we can conclude from our diagram 4.13.1 that $i$ is a monomorphism as desired. □

**Corollary 4.14.** Let $(W, S)$ be a Coxeter system and $\Gamma_{W,S}$ be the associated graph. Let $\pi$ denote the quotient map $P_{W,S} \to R_{W,S}$. Let $\mathfrak{N} = \{x \in \mathfrak{Y}|x \neq ptq, t \in T(C,W), p, q \in \mathfrak{Y}\}$. The elements of the set $\pi(\mathfrak{N})$ are pairwise distinct and form a basis for $R_{W,S}$ as a $\mathbb{Z}$-module.

**Proof.** As in the proof of Lemma 4.13, let $M$ be the $\mathbb{Z}$-submodule of $P_{W,S}$ spanned by $\mathfrak{N}$. Since $\pi |_M$ is onto, we get that $\pi(\mathfrak{N})$ spans $R_{W,S}$ as a $\mathbb{Z}$-module. From the proof of Lemma 4.13, we have a commutative diagram:

$$
\begin{array}{ccc}
P_{W,S} & \xrightarrow{i} & \hat{P}_{W,S} \\
\pi & \downarrow & \tilde{\pi} \\
M & \xrightarrow{i|_M} & V = \text{Span}(\mathfrak{N}) \\
\pi |_M & \downarrow & \tilde{\pi}|_V \\
R_{W,S} & \xrightarrow{i} & \hat{R}_{W,S}
\end{array}
$$

(4.14.1)

where $\tilde{\pi}|_V \circ i|_M$ is a monomorphism. Thus $\tilde{i} \circ \pi |_M$ is a monomorphism and since $\tilde{i} \circ \pi |_M(\mathfrak{N})$ forms a basis over $\mathbb{Q}$ for $\hat{R}_{W,S}$ by Lemma 4.12, we must have that the elements of the set $\pi(\mathfrak{N})$ are linearly independent over $\mathbb{Z}$ in $R_{W,S}$. □

**Example 4.15.** Let $(W, S)$ be the Coxeter system with Coxeter matrix

$$
\begin{pmatrix}
1 & 3 & 2 & 4 & 2 \\
3 & 1 & 5 & 2 & 2 \\
2 & 5 & 1 & 6 & 5 \\
4 & 2 & 6 & 1 & \infty \\
2 & 2 & 5 & \infty & 1
\end{pmatrix},
$$

then $(W, S)$ be a Coxeter system with generators $S = \{s_1 = r, s_2 = s, s_3 = t, s_4 = u, s_5 = v\}$ and relations $r^2 = s^2 = t^2 = u^2 = v^2 = 1 = (rs)^3 = (st)^5 = (tu)^6 = (tv)^5 = (ru)^4 = (rt)^2 = (su)^2 = (vr)^2 = (vs)^2$.

Let $\Gamma_{W,S}$ be the associated graph. On the left below, we show the edges in $Y_{W,S,+}$ as before on the right, we label each geometric edge $(s_is_j), s_i, s_j \in S$, in the corresponding graph with $m_{ij}$, if $(s_is_j)^{m_{ij}} = 1$ is among the above relations and $m_{ij} \neq 2$. The remaining edges are labeled with infinity.

We have $\hat{P}_{W,S} = \mathbb{Q}\Gamma_{W,S}$ and $\hat{I}_{W,S}$ is the ideal of $\hat{P}_{W,S}$ with the following generators:
\[ C_{y}\alpha = [r\alpha] - [\alpha], \quad C_{y}\beta = [r\beta] - 2[\alpha], \quad C_{y}\gamma = [\alpha u] - 3[\alpha], \]
\[ C_{y}\alpha = [s\alpha] - [\alpha], \quad C_{y}\beta = [u\beta] - 2[\alpha], \quad C_{y}\gamma = [\alpha t] - 3[\alpha], \]
\[ C_{y}\alpha = [s\alpha st] - 3[\alpha st] + [\alpha], \quad C_{y}\beta = [s\alpha tvt] - 3[\alpha tvt] + [\alpha], \]
\[ C_{y}\alpha = [s\alpha st] - 3[\alpha st] + [\alpha], \quad C_{y}\beta = [s\alpha tvt] - 3[\alpha tvt] + [\alpha]. \]

4.16. The Ring \( \tilde{R} \)

In this section, we give the definition of the ring \( \tilde{R} = \tilde{R}_{W,S} \) for a Coxeter System. The ring has relations given by polynomials satisfying a recurrence relation. To facilitate a discussion of the representation theory motivating the theory in the next section, we give the definition of the polynomials in the more general setting of a ring with idempotents.

4.17. Ring elements \( c_n \)

Let \( B \) be a (possibly non-unital) ring. For any idempotents \( e, f \in B \) and elements \( a \in \text{fBe}, \ b \in \text{eBf}, \) define elements \( c_n(a, b, e, f) \in B \) for \( n \in \mathbb{Z} \) by the recurrence formulæ

\[ c_{2n+2}(a, b, e, f) = ac_{2n+1}(a, b, e, f) - c_{2n}(a, b, e, f), \quad c_{2n+1}(a, b, e, f) = bc_{2n}(a, b, e, f) - c_{2n-1}(a, b, e, f) \]

for \( n \in \mathbb{Z} \) and initial conditions

\[ c_0(a, b, e, f) = 0, \quad c_1(a, b, e, f) = e. \]

We have

\[ c_2(a, b, e, f) = a, \quad c_3(a, b, e, f) = ba - e, \quad c_4(a, b, e, f) = aba - 2a, \quad c_5(a, b, e, f) = baba - 3ba + e, \ldots \]

Define also elements \( C_n(a, b, e, f) \in B \) for \( n \in \mathbb{N}_{\geq 2} \) as follows. Let \( C_2(a, b, e, f) = a. \) For \( n \geq 3, \) let \( C_n(a, b, e, f) := \rho_{B,\epsilon,a,b}(C_n(t)) \) where \( C_n(t) \in \mathbb{Z}[t] \) is the minimal polynomial over \( \mathbb{Q} \) of (the algebraic integer) \( 4 \cos^2 \frac{\pi}{n} \), and \( \rho_{B,\epsilon,a,b} \) denotes the ring homomorphism \( \rho_{B,\epsilon,a,b} : \mathbb{Z}[t] \to e\text{Be} \) which maps \( 1 \mapsto e \) and \( t \mapsto ba \) (note that the ring \( e\text{Be} \) is \( \epsilon \) and \( ba \) has identity element \( e \)). We see that

\[ C_3(a, b, e, f) = ba - e, \quad C_4(a, b, e, f) = ba - 2e, \quad C_5(a, b, e, f) = baba - 3ba + e, \quad C_6(a, b, e, f) = ba - 3e, \ldots \]

We write \( c_n = c_{n,B} \) and \( C_n = C_{n,B} \) if it is necessary to indicate dependence on the ring \( B. \) Then for a ring homomorphism \( \theta : B \to B', \) one has \( \theta(c_{n,B}(a, b, e, f)) = c_{n,B'}(\theta(a), \theta(b), \theta(e), \theta(f)) \) and similarly with \( c_n \) replaced by \( C_n \) if \( n \geq 2. \)

4.18. Factorization of \( c_n \)

By [13], \( c_n(a, b, e, f) = \prod_{N \in \mathbb{N}_{\geq 2}, N \neq a} C_N(a, b, e, f) \) for all \( n \in \mathbb{N}_{\geq 2} \), where \( N \mid n \) means \( N \) is a divisor of \( n \) in \( \mathbb{Z} \) and the order of factors in the product on the right is immaterial except that \( C_2(a, b, e, f) \) must be the leftmost factor if \( n \) is even. In fact, if the pairwise distinct positive, non-unit integer divisors, of \( n \) are \( N_1, \ldots, N_k, \) then \( c_n(a, b, e, f) = C_{N_1} \ldots C_{N_k} \) where \( C_N := C_N(a, b, e, f) \) unless \( n \) is even and \( N = N_i \) where \( 1 \leq i < j \leq k \) and \( N_j = 2, \) in which case \( C_N := C_N(b, a, f, e). \) The equivalence of these various factorizations is trivial on noting that all \( C_n(a, b, e, f) \) with \( n \geq 3 \) lie in the commutative unital subring of \( B \) generated by \( e \) and \( ba, \) with \( aC_n(a, b, e, f) = C_n(b, a, f, e)a. \)
4.19. A path algebra quotient for a coxeter system

We now take notation as in 4.1. Thus, $P_{W,S}$ is the path algebra over $\mathbb{Z}$ of $\Gamma_{W,S}$, $I_{W,S}$ is an ideal of $P_{W,S}$ and $R_{W,S}$ is the quotient ring $R_{W,S} := P_{W,S}/I_{W,S}$.

By definition, $I_{W,S}$ is the two-sided ideal of $P_{W,S}$ generated by elements $c_{[sr]} = c_{m_{rs}}([rs]) = c_{m_{rs}}([rs],[sr],[s],[r])$ of 4.17, for $(r,s) \in S \times S$ such that $2 < m_{rs} < \infty$. Let $\tilde{I}_{W,S}$ denote the two-sided ideal of $P_{W,S}$ generated by the elements $c_{m_{rs}}([rs],[sr],[s],[r])$ for $(r,s) \in S \times S$ such that $2 < m_{rs} < \infty$. From 4.17, we have $\tilde{I}_{W,S} \subseteq I_{W,S}$. Define the quotient ring $\tilde{R}_{W,S} := P_{W,S}/\tilde{I}_{W,S}$ of $P_{W,S}$. The above gives a canonical surjective ring homomorphism $\tilde{R}_{W,S} \rightarrow R_{W,S}$.

4.20. Parameterizing bases of $R_{W,S}$ and $\tilde{R}_{W,S}$.

Recall from Tits solution to the word problem for $(W,S)$ that a sequence $(s_1, \ldots , s_n)$ in $S$ is the unique reduced expression of the corresponding product $s_1 \cdots s_n \in W$ if and only if it contains no consecutive subsequence $(s_i, \ldots , s_j)$, where $1 \leq i < j \leq n$, of length $j-i+1$ which either is of the form $(s,s)$ for some $s \in S$ or is an alternating sequence $(s,r,s,\ldots )$ where $r,s \in S$ are distinct with $m_{rs} = j-i+1$. Let $W_1$ be the set of all such sequences with $n \geq 1$.

Let $\varphi : \mathbb{N}_{\geq 1} \rightarrow \mathbb{N}$ denote the Euler totient function (that is, $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$ where $(\mathbb{Z}/n\mathbb{Z})^*$ indicates the unit group. Note that if $n \geq 3$, then $\varphi(n)/2$ is the degree of $C_n(t) \in \mathbb{Z}[t]$.

Let $W'_1$ be the set of all sequences $(s_1, \ldots , s_n)$ in $S$, with $n \geq 1$, which contain no consecutive subsequence $(s_i, \ldots , s_j)$ which is of the form $(s,r,s,\ldots )$ where $r,s \in S$, $m_{rs} < \infty$ and $j-i+1 = 1 + \varphi(m_{rs})$. Note that $W'_1 \subseteq W_1$.

It is known from [13] that the elements $[s_1 \ldots s_n]$ of $\tilde{R}_{W,S}$ for $(s_1, \ldots , s_n) \in W'_1$ are pairwise distinct and form a $\mathbb{Z}$-module basis of $\tilde{R}_{W,S}$.

Theorem 4.21. (1) The elements $[s_1 \ldots s_n]$ of $\tilde{R}_{W,S}$ for $(s_1, \ldots , s_n) \in W'_1$ are pairwise distinct and form a basis of $\tilde{R}_{W,S}$ as vector space over $\mathbb{Q}$.

(2) The elements $[s_1 \ldots s_n]$ of $R_{W,S}$ for $(s_1, \ldots , s_n) \in W'_1$ are pairwise distinct and form a $\mathbb{Z}$-module basis of $R_{W,S}$.

Proof. Part (1) follows from Lemmas 4.12 and Part (2) follows from Corollary 4.14.

Example 4.22. (1) Suppose that $m_{rs} \in \{2,3,4,6\}$ for all $r \neq s$ in $S$. Since $\varphi(m) = 2$ for $m = 3,4,6$, the sequences $(s_1, \ldots , s_n) \in W'_1$ are precisely the sequences of successive vertices of non-backtracking paths in $\Gamma_{W,S}$. If, further, the Coxeter graph of $(W,S)$ is a tree, then for any $r,s \in S$, there is a unique non-backtracking path in $\Gamma_{W,S}$ from $r$ to $s$ and it follows that

$$\dim_{\mathbb{Q}}([r]\tilde{R}_{W,S}[s]) = \text{rank}_{\mathbb{Z}}([r]R_{W,S}[s]) = 1,$$

and so

$$\dim_{\mathbb{Q}}(\tilde{R}_{W,S}) = \text{rank}_{\mathbb{Z}}(R_{W,S}) = |S|^2.$$

In particular, these facts all hold if $(W,S)$ is an irreducible Weyl group or an irreducible affine Weyl group which is not of type $\tilde{A}_n$ for any $n \in \mathbb{N}_{\geq 1}$.

(2) Suppose that for each $r \neq s$ in $S$, $m_{rs}$ is either $\infty$ or a prime integer. Then $I_{W,S} = \tilde{I}_{W,S}$ and hence $R_{W,S} = \tilde{R}_{W,S}$.

$^1$A different basis for $\tilde{R}$, with some very favorable properties, is also considered in [13].
5. Reflection representations

In this section, we describe results of Dyer on the relation between standard reflection representations of Coxeter systems in real vector spaces and lax and strict reflection representations over non-commutative rings.

5.1. Group actions

Let $A$ be a commutative ring. Let $B$ be an $A$-algebra, and $M$ a left $A$-algebra module for $B$, as defined in Section 1. For a group $G$, a representation of $G$ on the $B$-module $M$ is by definition a group homomorphism $\theta : G \to \text{Aut}_B(M)$. We usually write $gm := (\theta(g))(m)$ and say simply that $G$ acts $B$-linearly on $M$ (leaving tacit the condition that the action is also $A$-linear). Similar conventions apply to right $G$ actions and right modules.

For the $A$-algebra $B$, the left (resp., right) regular $B$-module is $B$ as $A$-module, with $B$-action by left (resp., right) multiplication. If $B$ is unital and $M$ is its left regular module, then $\text{End}_B(M) \cong B^{op}$, where $B^{op}$ is the opposite algebra, and $\text{Aut}_B(M)$ identifies with the group of units of $B^{op}$. If $B$ is commutative unital, we often write $\text{GL}_B(M) := \text{Aut}_B(M)$.

5.2. Coxeter group actions on paired modules

Let $M$ be a left $B$-module, $M^\vee$ be a right $B$-module and $\langle -,- \rangle : M \times M^\vee \to B$ be a $B$-bilinear map. That is, for all $r, r' \in B$, we require $\langle rm + r' m', m'' \rangle = r\langle m, m'' \rangle + r'\langle m', m'' \rangle$ for $m, m' \in M$ and $m'' \in M^\vee$, and $\langle m'', mr + mr' \rangle = \langle m'', m \rangle r + \langle m'', m' \rangle r'$ for $m'' \in M$ and $m, m' \in M$. We also require these conditions to hold with $r$ and $r'$ in the coefficient ring, $A$, of $B$. Suppose given families $(\alpha_s)_{s \in S}$ in $M$, $(\alpha^s)_{s \in S}$ in $M^\vee$ and $(e_s)_{s \in S}$ in $B$ such that for all $s \in S$, $e_s^2 = e_s, e_s \alpha_s = \alpha_s, \alpha^s e_s = \alpha^s$ and $\langle \alpha_s, \alpha^s \rangle = 2e_s$. Define the $S \times S$ matrix $A = (a_{rs})_{r,s \in S}$ with entries in $B$ by $a_{rs} := (\alpha_r, \alpha^s) e_s \in e_sBe_s$.

Define $B$-module endomorphisms $\phi_s \in \text{End}_B(M)$ of $M$ for $s \in S$ by $\phi_s(m) = m - \langle m, \alpha_s^\vee \rangle \alpha_s$ for all $m \in M$. It is easy to check that $\phi_s^2 = 1d_M$ for all $s \in S$, so $\phi_s \in \text{Aut}_B(M)$.

The following is a main result in Dyer [13]:

Proposition 5.3. (Dyer [13]) Suppose notation is as above and that for all $(r,s) \in S \times S$ with $r \neq s$ and $m_{r,s} \neq \infty$, one has $c_{m_{r,s}}(a_{rs}, a_{rs}, e_r, e_s) = 0$. Then there is a left $B$-linear $W$-action on $M$ determined by $sm = m - \langle m, \alpha^s \rangle \alpha_s$ for all $m \in M$ and $s \in S$ and a right $B$-linear left $W$-action on $M^\vee$ determined by $sm' = m' - \alpha^s \langle \alpha_s, m' \rangle$ for all $m' \in M^\vee$. These satisfy $\langle wm, wm' \rangle = \langle m, m' \rangle$ for all $m, m' \in M$ and $w \in W$.

We indicate the ideas involved in the proof that the given formula defines a left $W$-action on $M$; the analogous assertion for $M^\vee$ follows by symmetry, and the formula $\langle wm, wm' \rangle = \langle m, m' \rangle$ then follows by induction on the length of $w \in W$ from the special case $w \in S$, which is checked by simple calculation. The non-trivial point is to show that for distinct $r, s \in S$ with $m_{r,s}$ finite, one has $(rsr\cdots)(m) = (srs\cdots)m$ for $m \in M$, where each alternating product of $r$ and $s$ has $m_{r,s}$ factors.

In the case of standard reflection representations in real vector spaces, one checks this by simple geometric arguments if $m$ is in the $\mathbb{R}$-linear span of $\alpha_r$ and $\alpha_s$, and then notes that $M$ is the vector space direct sum of that span and the intersection of the kernels of $\alpha^r$ and $\alpha^s$, on which $r$ and $s$ act trivially (see [19, 5.3]). The idea may be summarized as “reduction to the case of a dihedral Coxeter system.”

The basic idea in general is similar, but since there is no analogue of the direct sum decomposition, one proceeds as follows. For any sequence $r_1, \ldots, r_n \in S$, one has an identity

$$m - s_{a_{r_1}} \cdots s_{a_{r_n}} m = \sum_{i=1}^n s_{a_{r_1}} \cdots s_{a_{r_{i-1}}} (m - s_{a_{r_i}} m) = \sum_{i=1}^n (m, \alpha^s_{a_{r_i}}) s_{a_{r_1}} \cdots s_{a_{r_{i-1}}} (\alpha_{r_i}).$$
This identity is well known in the case of standard real reflection representations (see [20, Ex 3.12]). One applies the identity in turn with \( r_1, \ldots, r_n \) equal to the alternating sequences \( r, s, r, \ldots \) and \( s, r, s, \ldots \) of length \( m_{r,s} \). It suffices to show that the rightmost terms in these two applications are equal. In either application, the “dihedral” roots \( s_0 r_1 \cdots s_{n-1} (a_{n}) \) appearing may be explicitly expressed as \( B \)-linear combinations of \( \alpha_r \) and \( \alpha_s \), in which the coefficients are of the form \( c_k (a_{r,s}, a_{r,s}, e_r, e_s) \) or \( c_k (a_{r,s}, a_{s,r}, e_s, e_r) \) for certain integers \( k \), as follows readily by induction on \( i \) using the defining recurrence for the \( c_k \). The desired equality of rightmost terms follows readily from these explicit expressions and the assumption that

\[
c_{m_{r,s}} (a_{s,r}, a_{r,s}, e_r, e_s) = 0 = c_{m_{r,s}} (a_{r,s}, a_{s,r}, e_s, e_r).
\]

### 5.4. Root systems and non-commutative Cartan matrices

We call the matrix \( (a_{r,s})_{r,s \in S} \) a non-commutative generalized Cartan matrix (NCM) when the assumptions in 5.3 hold. In that case, the left \( W \)-action on \( M \) gives a representation \( \phi: W \to Aut_B(M) \), and similarly for \( M^\vee \). We call these lax reflection representations of \((W, S)\) on \( M \) and \( M^\vee \). If, more strongly, one has \( C_{m_{r,s}} (a_{s,r}, a_{r,s}, e_r, e_s) = 0 \) for all \((r, s) \in S \times S\) with \( r \neq s \) and \( m_{r,s} \neq \infty \), we say these representations are strict reflection representations and say that \( A \) is a strict NCM.

In either case, the subset \( \Phi := \{ \omega S | w \in W, s \in S \} \) of \( M \) is called the (corresponding) root system of \((W, S)\) in \( M \), and \( \Phi^\vee := \{ \omega S^\vee | w \in W, s \in S \} \) is called the coroot system of \((W, S)\) in \( M^\vee \). For some basic properties of lax reflection representations and their root systems, see [13].

**Example 5.5.** Suppose above that \( B = \mathbb{R} \), \( M = V \) and \( M^\vee = V^\vee \) are \( \mathbb{R} \)-vector spaces (unital \( B \)-modules) and \( e_r = 1_B \) for all \( r \in S \). Let \( a_{r,s} := \langle \alpha_r, \alpha_s \rangle \). The proposition produces lax reflection representations \( W \to GL_{\mathbb{R}}(V) \) and \( W \to GL_{\mathbb{R}}(V^\vee) \) provided that \( a_{s,s} = 2 \) for all \( s \in S \), and, for all \( r \neq s \in S \),\( a_{r,s} = 0 \) if \( m_{r,s} = 2 \) and \( a_{r,s} a_{s,r} = 4 \cos^2 \frac{k_{r,s} \pi}{m_{r,s}} \) for some \( 1 \leq k_{r,s} \leq m_{r,s} / 2 \) if \( 3 \leq m_{r,s} < \infty \). These reflection representations are strict if and only if \( k_{r,s} \) is relatively prime to \( m_{r,s} \) whenever \( 3 \leq m_{r,s} < \infty \). We call the matrix \( A := (a_{r,s})_{r,s \in S} \) a strict (resp., lax) real reflection matrix, abbreviated SRRM (resp., LRRM) for \((W, S)\) if it arises in this way from a strict (resp., lax) reflection representation.

**Remark.** (1) Representations as in the preceding example are mostly studied under conditions which ensure they are strict and that the representation and its root system have standard (convexity and positivity) properties. Essentially the most general such conditions are the following:

1. \( a_{r,s} = 2 \) for all \( r \in S \).
2. \( a_{r,s} \leq 0 \) for all \( r \neq s \) in \( S \).
3. \( a_{r,s} = 0 \) if \( r, s \in S \) and \( m_{r,s} = 2 \).
4. \( a_{r,s} a_{s,r} = 4 \cos^2 \frac{\pi}{m_{r,s}} \) if \( r, s \in S \) and \( 3 \leq m_{r,s} < \infty \).
5. \( a_{r,s} a_{s,r} \geq 4 \) if \( r, s \in S \) and \( m_{r,s} = \infty \).
6. \( \Pi := \{ \omega S | s \in S \} \) and \( \Pi^\vee := \{ \omega S^\vee | s \in S \} \) are positively independent.

Here, a family \( (\beta_i)_{i \in I} \) of elements in a real vector space is said to be positively independent if \( \sum_{i \in I} a_i \beta_i = 0 \) for non-negative real scalars \( a_i \), almost all zero, implies \( a_i = 0 \) for all \( i \in I \). In particular, linearly independent vectors are positively independent. When (1)-(5) hold, we say that \( A := (a_{r,s})_{r,s \in S} \) is a (possibly) non-crystallographic generalized Cartan matrix (NGCM) for \((W, S)\). The finite NGCMs with
integral entries are essentially the Generalized Cartan Matrices (GCMs) appearing in the study of Kac-Moody Lie algebras [20] and elsewhere. Note that NGCMs are SRRMs. See [15], [14], [10] and [13] for further details on reflection representations attached to NGCMs.

(2) In some very special cases including finite irreducible Coxeter systems, real reflection representations attached to SRRMs may be viewed as “Galois twists” of those attached to NGCMs, but this is not the case in general (a precise discussion of this would involve reflection representations defined over subfields of \( \mathbb{R} \)).

(3) For any Coxeter system \((W', S)\) for which there is a surjective group homomorphism \( W \to W' \) which is the identity on \( S \), a SRRM for \((W', S')\) is a LRRM for \((W, S)\).

The following proposition summarizes part of the above discussion which is used subsequently.

**Proposition 5.6.** Let \( A = (a_{rs})_{r,s \in S} \) be a LRRM for \((W, S)\) and \( V = V_A, V^\vee = V_A^\vee \) be real vector spaces with bases \( \{ \alpha_s \mid s \in S \} \) and \( \{ \alpha^\vee_s \mid s \in S \} \) respectively. Then there are \( \mathbb{R} \)-linear lax reflection representations of \( W \) on \( V \) and \( V^\vee \) such that for \( s, r \in S \), one has \( s \alpha_r = \alpha_r - a_{rs} \alpha_s \) and \( s \alpha^\vee_r = \alpha^\vee_r - a_{rs}^\vee \alpha^\vee_s \). We call \((V_A, V_A^\vee)\) the standard paired (lax) real reflection representations of \((W, S)\) with LRRM \( A \). If \( A \) is a SRRM, these representations are strict.

### 5.7. Left and right actions

Let \( A \) be a LRRM for \((W, S)\) and let \((V, V^\vee) = (V_A, V_A^\vee)\) be the standard paired real reflection representations of \((W, S)\) with LRRM \( A \). As constructed above, \( V \) and \( V^\vee \) have left \( W \)-actions, but we shall find it convenient to regard \( W \) as instead acting on the right of \( V \), by defining \( vw := w^{-1}v \) for \( v \in V \) and \( w \in W \). We still regard the \( W \)-action on \( V^\vee \) as a left action. Then \( V^\vee \otimes_\mathbb{R} V \) has natural commuting left and right \( W \)-actions, defined by

\[
w(v^r \otimes v) = (w^r) \otimes v, \quad (v^r \otimes v)w = v^r \otimes (vw), \quad \text{if } v^r \in V^\vee, v \in V, w \in W.
\]

One has

\[
s(\alpha^r_r \otimes \alpha_t) = \alpha^r_r \otimes \alpha_t - a_{rt} \alpha^r_s \otimes \alpha_t, \quad (\alpha^r_r \otimes \alpha_t)s = \alpha^r_r \otimes \alpha_t - a_{rt} \alpha^r_r \otimes \alpha_s, \quad \text{for } r, s, t \in S.
\]

### 5.8. Matrix rings

Let \( B := \text{Mat}_{S \times S}(\mathbb{R}) \) denote the \( \mathbb{R} \)-algebra of real \( S \times S \)-matrices with only finitely many non-zero entries. Note that \( B \) has a \( \mathbb{R} \)-basis \((e_{rs})_{r,s \in S}\) where \( e_{rs} \) is the matrix unit (the matrix with \((r', s')\)-entry \( \delta_{r', r} \delta_{s', s} \) where \( \delta_{j,k} \) denotes the Kronecker delta in \( \mathbb{R} \)).

We shall also regard \( B \) as a \((B,B)\)-bimodule in the natural way. Identify \( V^\vee \otimes_\mathbb{R} V = B \) as \( \mathbb{R} \)-vector space so that \( \alpha^r_r \otimes \alpha_s = e_{rs} \). This endows \( B \) with structure of \( \mathbb{R} \)-algebra and with left and right \( W \)-actions which commute, such that the left (resp., right) \( W \)-action is right (resp., left) \( B \)-linear.

### 5.9. Quotients of the path algebra

Consider any ring \( R' := P_{W,S}/I \) where \( I \) is a two-sided ideal of \( P_{W,S} \) such that \( \bar{I}_{W,S} \subseteq I \). The two cases of greatest interest are (1) \( I = I_{W,S} \) and \( R' = R_{W,S} \), and (2) \( I = I_{W,S} \) and \( R' = R_{W,S} \). For any element \( x \) of \( P_{W,S} \), we denote its image in \( R' \) under the canonical surjection \( P_{W,S} \to R' \) still by \( x \). In particular, this defines \( \{x_1, \ldots, x_n\} \in R' \) if \( n > 0 \) and \( x_1, \ldots, x_n \in S \) are the successive vertices of a path in \( \Gamma_{W,S} \). We also set \( \{x_1, \ldots, x_n\} := 0_{R'} \) if \( n > 0 \) and \( x_1, \ldots, x_n \in S \) are not the successive vertices of a path in \( \Gamma_{W,S} \) (that is, if for some \( 1 \leq i \leq n-1, m_{x_i, x_{i+1}} \leq 2 \)). Note that the elements \([s] \) for \( s \in S \) are pairwise orthogonal idempotents in \( R' \) such that \( R' = \bigoplus_{s \in S} [s] R' [s] \) as abelian group.
Proposition 5.10. Define the quotient ring $R' = P_{W,S}/I$ where $I \supseteq \tilde{I}_{W,S}$ is a two-sided ideal of $P_{W,S}$. Let $M$ (resp., $M^\vee$) denote the left regular $R'$-module $R'$.

1. There is a (unique) left $R'$-linear reflexion representation of $W$ on $M$ such that
$$s[r] = \begin{cases} -[s], & \text{if } r = s \\ [r] + [rs], & \text{if } r \neq s. \end{cases} \quad \text{for } r, s \in S,$$

and a (unique) right $R'$-linear reflexion representation of $W$ on $M^\vee$ such that
$$s[r] = \begin{cases} -[s], & \text{if } r = s \\ [r] + [sr], & \text{if } r \neq s. \end{cases} \quad \text{for } r, s \in S.$$

2. Base change $- \otimes \mathbb{R}$ gives a (possibly non-unital) $\mathbb{R}$-algebra $R' \otimes \mathbb{R}$ and left (resp., right) $R' \otimes \mathbb{R}$-linear reflexion actions of $W$ on the left and right regular $R' \otimes \mathbb{R}$-modules $M \otimes \mathbb{R}$ and $M^\vee \otimes \mathbb{R}$ respectively.

3. If $I \supseteq I_{W,S}$, then the reflexion representations in (1)–(2) are strict.

Proof. We prove (1). Define families $(e_s)_{s \in S}$ in $R'$, $(\alpha_s)_{s \in S}$ in $M$ and $(\alpha_s^\vee)_{s \in S}$ in $M^\vee$ by setting $e_s = \alpha_s = \alpha_s^\vee := [s]$ for all $s \in S$. A $R'$-bilinear map $\langle -,- \rangle : M \times M^\vee \to R'$ is uniquely determined by the values $\langle [r],[s] \rangle$ for $r,s \in S$, which may be arbitrarily assigned subject only to the conditions that
$$\langle [r],[s] \rangle \in e_r R' e_s \quad \text{for all } r,s \in S.$$ Hence, there is a unique $R'$-bilinear map $\langle -,- \rangle : M \times M^\vee \to R'$ such that
$$\langle \alpha_r, \alpha_s^\vee \rangle = \begin{cases} 2[r], & \text{if } r = s \\ -[rs], & \text{if } r \neq s. \end{cases}$$

Note that $\langle \alpha_r, \alpha_s^\vee \rangle = 0$ if $m_{rs} = 2$ by our conventions. The conditions of Proposition 5.3 are satisfied, by definition of $I_{W,S}$ and the assumption that $I_{W,S} \subseteq I$. Hence the proposition gives reflexion representations as required in (1). Part (2) follows easily by base change and (3) is immediate from the definitions.

5.11. Some commuting $W$-actions

Regard the left $R'$-linear $W$-action just defined on $M$ as a right action by setting $mw := w^{-1}m$ for $w \in W$ and $m \in M$, and henceforward write $M$ and $M^\vee$ just as $R'$. Then $R'$ admits a right $R'$-linear left reflexion action of $W$, and a left $R'$-linear right reflexion action of $W$. Moreover, these actions commute, as one can see by direct computation or from the following lemma.

Lemma 5.12. Let $C$ be any idempotent ring (i.e., $C^2 = C$). Let $\theta : C \to C$ (resp., $\tau : C \to C$) be an endomorphism of the left (right) regular $C$ module. Then $\tau$ and $\theta$ commute in $\text{End}_C(C)$.

Proof. Let $c \in C$. Since $c \in C = C^2$, we may write $c = \sum_{i=1}^n c_id_i$ for some $n \in \mathbb{N}$ and $c_i, d_i \in C$. Then
$$\tau(\theta(c)) = \tau(\sum_{i=1}^n c_i \theta(d_i)) = \sum_{i=1}^n \tau(c_i) \theta(d_i) = \sum_{i=1}^n \tau(c_i) d_i = \theta(\sum_{i=1}^n c_i d_i) = \theta(\tau(c)).$$

5.13. Some homomorphisms from the path algebra

Let $A := (a_{rs})_{r,s \in S}$ be an $S \times S$-indexed family in $\mathbb{R}$. Then there is a unique ring homomorphism $\theta'_{A} : P_{W,S} \to B$ such that for any path in $\Gamma_{W,S}$ with successive vertices $s_1, \ldots, s_m$, one has $\theta'_{A}([s_1 \cdots s_m]) = (-1)^{m-1} a_{s_1,s_2} \cdots a_{s_{m-1},s_m} e_{s_1,s_m}$ (where here and below $a_{s_1,s_2} \cdots a_{s_{m-1},s_m} = 1_{\mathbb{R}}$ by convention if $n = 1$). Note that $\theta'_{A}$ doesn’t depend on the values $a_{rs}$ with $m_{rs} \leq 2$. 
Proposition 5.14. Let $A$ and $\theta_A'$ be as above.

1. If $A$ is a LRRM for $(W, S)$, then $\theta_A'$ factors as $P_{W,S} \xrightarrow{\tilde{\theta}_{A,R}} \tilde{R}_{W,S} \xrightarrow{\theta_{A,R}} B$ for a unique ring homomorphism $\theta_{A,R}$.

2. If $A$ is a SRRM for $(W, S)$, then $\theta_A$ factors as $\tilde{R}_{W,S} \xrightarrow{\theta_{A,R}} R_{W,S} \xrightarrow{\theta_{A,R}} B$ for a unique ring homomorphism $\theta_{A,R}$.

3. Suppose $A$ is a LRRM. Let $(S_i)_{i \in I}$ be the equivalence classes for the finest equivalence relation $\sim$ on $S$ such that $r \sim s$ if $a_{r,s} \neq 0$. Then $\text{Span}(\text{Im}g(\theta_{A,R})) = \bigoplus_{i \in I} \text{Mat}_{S_i \times S_i}(\mathbb{R})$ where $(W_i, S_i)_{i \in I}$ are the irreducible components of the Coxeter system $(W, S)$.

Proof. We prove (1). Let $A$ be a LRRM. Since $\tilde{R}_{W,S} = P_{W,S}/\tilde{I}_{W,S}$, it will suffice to show that $\ker \theta_A' \supseteq \tilde{I}_{W,S}$. Recall that $\tilde{I}_{W,S}$ is generated by elements $c_m((rs), [sr], [s],[r]) \in P_{W,S}$ where $r,s \in S$ with $3 \leq m := m_{r,s} < \infty$. Let $x := e_{s,t} \in B$ if $m$ is odd and $x := e_{r,s} \in B$ if $m$ is even. It is easily seen that

$$\theta_A'(c_{m,p}([rs],[sr],[s],[r])) = c_{m,B}(-a_{r,s}e_{r,s} - a_{s,t}e_{s,t}, e_{s,t}, e_{r,s}) = c_{m,S}(-a_{r,s}, -a_{s,t}, 1_R, 1_R)x.$$ 

Since $A$ is a LRRM, we may write $a_{r,s}, a_{s,t} = 4 \cos^2 \frac{k\pi}{m} = 4 \cos^2 \frac{k\pi}{n}$ where $k \in \mathbb{N}$ and $n \in \mathbb{N}_{\geq 2}$ are relatively prime, $n|m$ and $1 \leq k < n$. We have $c_{m,S}(-a_{r,s} - a_{s,t}, 1_R, 1_R) = C_n \left(4 \cos^2 \frac{k\pi}{n}\right) = 0$. So $c_{m,S}(-a_{r,s} - a_{s,t}, 1_R, 1_R) = 0$ by 4.18, since $n|m$, and hence $\theta_A'(c_{m,p}([rs],[sr],[s],[r])) = 0$ as required.

The proof of (2) is similar but simpler. Let $A$ be a SRRM. Since $R_{W,S} = P_{W,S}/I_{W,S}$, it will suffice to show that $\ker \theta_A' \supseteq I_{W,S}$. Recall that $I_{W,S}$ is generated by elements $c_m((rs), [sr], [s],[r]) \in P_{W,S}$ where $r,s \in S$ with $3 \leq m := m_{r,s} < \infty$. Since $\varphi(m)$ is even, we have

$$\theta_A'(c_{m,p}([rs],[sr],[s],[r])) = c_{m,B}(-a_{r,s}e_{r,s} - a_{s,t}e_{s,t}, e_{s,t}, e_{r,s}) = c_{m,S}(-a_{r,s} - a_{s,t}, 1_R, 1_R)e_{r,s}.$$ 

Since $A$ is a SRRM, we may write $a_{r,s}, a_{s,t} = 4 \cos^2 \frac{k\pi}{m}$ where the integer $k$ is relatively prime to $m$ and $1 \leq k < m$. We then have $c_{m,S}(-a_{r,s} - a_{s,t}, 1_R, 1_R) = C_m \left(4 \cos^2 \frac{k\pi}{m}\right) = 0$ and hence $\theta_A'(c_{m,p}([rs],[sr],[s],[r])) = 0$ as required.

In (3)–(4), we regard $\text{Mat}_{S_i \times S_i}(\mathbb{R})$, for $J \subseteq S$, as a $\mathbb{R}$-subalgebra of $B$ in the natural way (identifying its matrix units $e_{j,k}$, for $j,k \in J$, with $e_{j,k} \in B$). To prove (3), observe that by (1), $\text{Im}(\theta_{A,R}) = \text{Im}(\theta_{A})$. This image has the same $\mathbb{R}$-linear span as the set of matrix units $e_{r,s}$ such that there is some path in $\Gamma_{W,S}$ with successive vertices $r = s_1, \ldots, s_t = s$ such that $a_{s_i,s_{i+1}} \neq 0$ for $i = 1, \ldots, n - 1$. Since $A$ is a LRRM, for any $t \neq u \in S$, we have $a_{t,u} \neq 0 \iff a_{u,t} \neq 0 \iff m_{t,u} \neq 2$, and (3) follows.

Part (4) follows from (3) since if $A$ is a SRRM, then $\text{Im}(\theta_{A,R}) = \text{Im}(\theta_{A,R})$ and for distinct $t, u \in S$, we have $a_{t,u} \neq 0 \iff a_{u,t} \neq 0 \iff m_{t,u} \neq 2$. 

5.15. Base change to the real numbers

By base change $\otimes_{\mathbb{Z}} \mathbb{R}$, we obtain from the ring $\tilde{R}_{W,S}$ and ring homomorphism $\theta_{A,R}: \tilde{R}_{W,S} \rightarrow B$ into the $\mathbb{R}$-algebra $B$, a $\mathbb{R}$-algebra $\tilde{R}' := \tilde{R}_{W,S} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{R}$-algebra homomorphism $\theta_{A,R}' : \tilde{R}' \rightarrow B$. Similarly, define the $\mathbb{R}$-algebra $R' := R_{W,S} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{R}$-algebra homomorphism $\theta_{A,R}' : \tilde{R}' \rightarrow B$. We have natural inclusion homomorphisms $\tilde{R}_{W,S} \rightarrow \tilde{R}'$ and $R_{W,S} \rightarrow R'$ of rings, given in each case by $r \rightarrow r \otimes 1$, since $\tilde{R}_{W,S}$ and $R_{W,S}$ are free as $\mathbb{Z}$-modules.

Note that if $C$ denotes one of the rings (or $\mathbb{R}$-algebras) $\tilde{R}_{W,S}, \tilde{R}', R_{W,S}, R'$ or $B$, we have defined on $C$ commuting left and right $W$-actions, such that the left (resp., right) $W$-action is right (resp., left) $C$-linear. We regard each such $C$ as a $W \times W^{\text{op}}$-set in the natural way. The two inclusion homomorphisms
\[ \tilde{R}_{W,S} \twoheadrightarrow \tilde{R}' \text{ and } R_{W,S} \twoheadrightarrow R', \text{ and the canonical surjective homomorphisms } \tilde{R}_{W,S} \rightarrow R \text{ and } \tilde{R}_{W,S} \rightarrow R' \text{ are obviously } W \times W^{\text{op}}\text{-equivariant.} \]

**Theorem 5.16.** Let notations and assumptions be as above.

(1) If \( A \) is a LRRM for \((W,S)\), the ring homomorphism \( \theta_{A,R}: \tilde{R}_{W,S} \rightarrow B \) and \( \mathbb{R}\)-algebra homomorphism \( \theta_{A,R}: \tilde{R}' \rightarrow B \) are \( W \times W^{\text{op}}\)-equivariant.

(2) If \( A \) is a SRRM for \((W,S)\), the ring homomorphism \( \theta_{A,R}: R_{W,S} \rightarrow B \) and \( \mathbb{R}\)-algebra homomorphism \( \theta_{A,R}: R' \rightarrow B \) are \( W \times W^{\text{op}}\)-equivariant.

(3) Suppose that \((W,S)\) is irreducible and \( A \) is a SRRM. Then \( \theta_{A,R}' \) is surjective.

(4) The \( W \times W^{\text{op}} \) actions on \( R_{W,S}, R', \tilde{R}_{W,S}, \text{ and } \tilde{R}' \) are all faithful.

**Proof.** For all the various left and right \( W \)-sets \( X \) involved in this proof, we write the various action maps \( W \times X \rightarrow X \) as \((w,x) \mapsto wx\) and \( X \times W \rightarrow X \) as \((x,w) \mapsto xw\) for notational simplicity. To prove (1), it will suffice to prove the assertion for \( \theta_{A,R} \), as that for \( \theta_{A,R}' \) then follows by base change. We show first that for \( r, s \in S \), we have \( \theta_{A,R}([r]s) = (\theta_{A,R}([r])) s \). If \( s = r \), then

\[
\theta_{A,R}(\{r\}s) = \theta_{A,R}([r] + [rs]) = e_{r,r} = e_{r,r} r = \theta_{A,R}(\{r\}s),
\]

while if \( s \neq r \), then

\[
\theta_{A,R}(\{r\}s) = \theta_{A,R}([r] + [rs]) = e_{r,r} - a_{r,s} e_{r,s} = e_{r,s} = \theta_{A,R}(\{r\}s),
\]

as required. Now for any \( x \in R \) and \( r, s \in S \) we have

\[
\theta_{A,R}(x \cdot [r]s) = \theta_{A,R}(x \cdot (\{r\}s)) = \theta_{A,R}(x) \cdot \theta_{A,R}(\{r\}s) = \theta_{A,R}(x) \cdot (\theta_{A,R}(\{r\}) s)
\]

\[
= (\theta_{A,R}(x) \cdot \theta_{A,R}(\{r\})) s = (\theta_{A,R}(x \cdot [r])) s.
\]

Since \( x = \sum_{r \in R} x \cdot [r] \) where \( x \cdot [r] = 0 \) for almost all \( r \in S \), this implies

\[
\theta_{A,R}(xs) = \theta_{A,R} \left( \sum_{r \in S} (x \cdot [r]) s \right) = \sum_{r \in S} \theta_{A,R} \left( (x \cdot [r]) s \right) = \sum_{r \in S} \theta_{A,R}(x \cdot [r]s) = \theta_{A,R}(xs).
\]

Finally, by induction on the length of \( w \in W \), it follows \( \theta_{A,R}(xw) = \theta_{A,R}(x) w \) for all \( x \in \tilde{R}_{W,S} \) and \( w \in W \). Similarly or by symmetry, \( \theta_{A,R}(wx) = w \theta_{A,R}(x) \) for all \( x \in \tilde{R}_{W,S} \) and \( w \in W \), completing the proof of (1).

Part (2) follows from (1) since if \( A \) is a SRRM, we have the factorization of \( \theta_{A,R} \) in **Proposition 5.14** where the canonical surjection \( \tilde{R}_{W,S} \rightarrow R_{W,S} \) is \( W \times W^{\text{op}}\)-equivariant. Part (3) follows from **Proposition 5.14** (4).

We prove (4). There is no loss of generality in taking \((W,S)\) to be irreducible. Let \( A \) be a NGCM for \((W,S)\). It is known that the \( W \) actions on \( V_A \) and \( V_A^\vee \) are faithful, so the induced \( W \times W^{\text{op}}\) action on \( V_A \otimes \mathbb{R} V_A \cong B \) is faithful. But \( B \) is a quotient of \( R' \), so the \( W \times W^{\text{op}}\)-action on \( R' \) is faithful. Since that action arises by base change from that on \( R_{W,S} \), the action on \( R_{W,S} \) is faithful. Since \( R_{W,S} \) (resp., \( R' \)) is a quotient of \( \tilde{R}_{W,S} \) (resp., \( \tilde{R}' \)), all the required \( W \times W^{\text{op}}\)-actions are faithful. \( \square \)

The above theorem has many consequences for special classes of Coxeter groups, such as finite Coxeter groups and affine Weyl groups, amongst which we observe here only the following.
Corollary 5.17. Suppose that \((W, S)\) is an irreducible finite Weyl group with Cartan matrix \(A\). Then \(R'\) is \(W \times W^{op}\)-equivariantly isomorphic to the matrix ring \(B = \text{Mat}_{S \times S}(\mathbb{R})\) as \(\mathbb{R}\)-algebra, where \(B\) may be \(W \times W^{op}\)-equivariantly identified with \(V_A^{\vee} \otimes_{\mathbb{R}} V_A\) so that \(\alpha^r \otimes \alpha_s = \epsilon_{rs}\).

Proof. We prove this under the more general hypotheses that \((W, S)\) is of finite rank, that the Coxeter graph of \((W, S)\) is a tree and that \(mr_{rs} \in \{2, 3, 4, 6\}\) for all \(r \neq s\) in \(S\).\(^2\) By (2) and Example 4.22(1), \(\theta_{A,R'} : R' \to B = \text{Mat}_{S \times S}(\mathbb{R})\) is a surjective, \(W \times W^{op}\)-equivariant homomorphism of \(\mathbb{R}\)-algebras of the same finite dimension \(|S|^2\). Hence it is an isomorphism as asserted. \(\square\)

6. Free products of rings

In this section, we give a summary of definitions and results on the free product of rings from Cohn \([4-6]\). We apply the results to show that a ring associated to an edge of the graph of a Coxeter system is a free product of rings.

6.1. Background on free products of rings

Let \(B\) be an associative ring with a unit element and let \(A\) be a (not necessarily commutative) subring of \(B\) containing the unit element, then we say \(B\) is an \(A\) ring. (Note that if \(A\) is commutative and \(B\) is a unital \(A\)-algebra in which \(A\) embeds by \(a \mapsto a1_B\), then \(B\) is an \(A\) ring). A homomorphism of \(A\) rings \(f : B \to C\) is an \(A\)-bimodule homomorphism that sends the unit element of \(B\) to the unit element of \(C\).

Definition 6.2. (Cohn [4], Section 3) The \(A\) ring \(R\) is said to be the free product of the \(A\) rings \(\{R_i | i \in I\}\) (over \(A\)) if \(\{R_i | i \in I\}\) forms a family of subrings of \(R\) such that

(i) \(R_i \cap R_j = A\) for \(i \neq j\),

(ii) if \(X_i\) is a set of generators of \(R_i\), \(i \in I\), then \(\bigcup_I X_i\) is a set of generators of \(R\),

(iii) if \(C_i, i \in I\) is a set of defining relations of \(R_i\), \(i \in I\) (in terms of the set of generators \(X_i\)) then \(\bigcup_I C_i\) is a set of defining relations of \(R\) (in terms of the set of generators \(\bigcup_I X_i\)).

The free product of \(A\) rings does not always exist, but when it does it coincides with the universal product which is always guaranteed to exist, see Cohn [4], Theorem 3.1. When the free product of the \(A\)-rings \(\{R_i | i \in I\}\) exists, we will denote it by \(\ast_A R_i\) or \(R_1 \ast_A R_2\) if \(I = \{1, 2\}\).

Definition 6.3. An \(A\) ring \(U\) is called the universal product of the \(A\) rings \(\{R_i | i \in I\}\) if there exist homomorphisms \(\phi_i : R_i \to U\) such that

(i) \(U\) is generated by the subrings \(\phi_i(R_i)\),

(ii) given any \(A\) ring, \(C\), and any family of homomorphisms \(\psi_i : R_i \to C\), there exists a homomorphism \(\psi : U \to C\) satisfying \(\psi \circ \phi_i = \psi_i\) for each \(i \in I\).

One can see that the universal product of \(A\) rings is determined up to isomorphism by the universal mapping property. It is not difficult to see that the free product, when it exists, has the same universal mapping property and thus must be isomorphic to the universal product. The following theorem gives the precise relationship between the two:

Theorem 6.4. (Cohn [4]) Let \(R_i, i \in I\) be a family of \(A\) rings and \(\{U, \phi_i : R_i \to U | i \in I\}\) their universal product. Then the free product of the \(R_i, i \in I\) exists if and only if

\(^2\)The finite rank assumption can also be omitted but we leave the additional arguments to show this to the reader.
(i) the canonical homomorphism $\Phi_i : R_i \rightarrow U$ is a monomorphism for each $i \in I$,
(ii) $\Phi_i(R_i) \cap \Phi_j(R_j) = A$ for $i \neq j$.

When these two conditions hold, $U$ is in fact the free product of the rings $R_i$.

In Cohn [6] (also see Cohn [5] and Serre [22], Exercise 1.2. for special cases), the construction of the free product of a family of $A$ rings, using tensor products of bimodules, is described. The free product exists when some conditions of flatness are satisfied, see Cohn [4], Theorem 4.5. In particular, when the base ring, $A$, is a field the free product of any family of $A$ rings exists, Cohn [4], Corollary to Theorem 4.7.

Using the construction of the free product via tensor products of bimodules, Cohn proves the following:

**Theorem 6.5.** (Cohn [6], Theorem 2.5) If $\{R_i | i \in I\}$ is any family of $K$-rings without zero divisors, where $K$ is a field, then their free product has no zero divisors.

We use Cohn’s results to gain insight into the structure of the following rings, which will in turn help us to understand the structure of the path algebras we have associated to Coxeter systems.

**Theorem 6.6.** Let $f(t)$ be a non-constant polynomial in $\mathbb{Q}[t]$ with non-zero constant term, and let $\mathbb{Q}[y, \bar{y}]$ be the ring of polynomials over $\mathbb{Q}$ in the non-commuting variables $y$ and $\bar{y}$. Let $R_f$ denote the quotient ring:

$$R_f = \mathbb{Q}[y, \bar{y}] / \langle f(y\bar{y}), f(\bar{y}y) \rangle.$$ 

Then $R_f$ is isomorphic to the ring

$$\mathbb{Q}[x_y, x_{\bar{y}}^{-1}] \ast_{\mathbb{Q}} K,$$

where $\mathbb{Q}[x_y, x_{\bar{y}}^{-1}]$ is the ring of Laurent polynomials in the variable $x_y$ over $\mathbb{Q}$ and $K$ is the ring extension $\mathbb{Q}[t] / \langle f(t) \rangle$ of $\mathbb{Q}$. Furthermore, if $f(t)$ is irreducible, then $R_f$ is a domain.

**Proof.** If $f(t)$ is irreducible, then $K$ is a field extension of $\mathbb{Q}$ and hence, is a domain. That the ring of Laurent polynomials, $\mathbb{Q}[x_y, x_{\bar{y}}^{-1}]$, is a domain is well known. By Theorem 6.5, the ring $\mathbb{Q}[x_y, x_{\bar{y}}^{-1}] \ast_{\mathbb{Q}} K$ does not have zero divisors, since both $\mathbb{Q}[x_y, x_{\bar{y}}^{-1}]$ and $K = \mathbb{Q}[t] / \langle f(t) \rangle$ are domains. Hence it is enough to prove that $R_f$ is isomorphic to the free product

$$\mathbb{Q}[x_y, x_{\bar{y}}^{-1}] \ast_{\mathbb{Q}} K.$$

First we claim that $y$ has an inverse in the ring $R_f = \mathbb{Q}[y, \bar{y}] / \langle f(y\bar{y}), f(\bar{y}y) \rangle$. Since $f$ is non-constant with a non-zero constant term, $f(t) = th(t) + \kappa = h(t)t + \kappa$ for some $\kappa \in \mathbb{Q}, \kappa \neq 0$ and some $h(t) \in \mathbb{Q}[t]$. Without loss of generality, we can assume that $\kappa = -1$. Thus,

$$y\bar{y}h(y\bar{y}) = 1_{R_f} = h(y\bar{y})y\bar{y},$$

and $(y\bar{y})^{-1} = h(\bar{y}y)$ in the ring $R_f$. By symmetry, $\bar{y}y$ has an inverse in $R_f$ and $(\bar{y}y)^{-1} = h(\bar{y}y)$. Using our expression for $(y\bar{y})^{-1}$, we see that $\bar{y}h(\bar{y}y)$ is a right inverse for $y$ and using our expression for $(\bar{y}y)^{-1}$, we see that

$$1_{R_f} = h(\bar{y}y)\bar{y}y = \bar{y}h(y\bar{y})y,$$

which shows that $y\bar{h}(y\bar{y})$ is also a left inverse for $y$ in $R_f$. This proves our claim.

Now let $\phi : \mathbb{Q}[y, \bar{y}] \rightarrow \mathbb{Q}[x_y, x_{\bar{y}}^{-1}] \ast_{\mathbb{Q}} K$ be the unique $\mathbb{Q}$ algebra homomorphism with $\phi(y) = x_y$ and $\phi(\bar{y}) = x_{\bar{y}}^{-1}t$. This induces a well-defined quotient homomorphism, $\bar{\phi} : R_f \rightarrow \mathbb{Q}[x_y, x_{\bar{y}}^{-1}] \ast_{\mathbb{Q}} K$ since $\bar{\phi}(f(y\bar{y})) = f(\phi(y\bar{y})) = f(x_yx_{\bar{y}}^{-1}t) = f(t) = 0$ and $\bar{\phi}(f(\bar{y}y)) = f(\phi(\bar{y}y)) = f(x_y^{-1}tx_y) = x_{\bar{y}}^{-1}f(t)x_y = 0$.

To show that the map $\phi$ defined above is an isomorphism, we construct its inverse. We let $\bar{\psi}_1 : \mathbb{Q}[t] \rightarrow R_f$ be the $\mathbb{Q}$ algebra homomorphism such that $\bar{\psi}_1(t) = y\bar{y}$. Since $\bar{\psi}_1(f(t)) = f(\bar{\psi}_1(t)) = f(y\bar{y}) = 0$, we
get a well-defined quotient homomorphism $\psi_1 : K \to R_f$. Let $\psi_2 : \mathbb{Q}[x_y, x_y^{-1}] \to R_f$ be the $\mathbb{Q}$ algebra homomorphism with $\psi_2(x_y) = y$ and $\psi_2(x_y^{-1}) = y h(y q)$. This is a well-defined homomorphism since $\psi_2(x_y^{-1}) = (\psi_2(x_y))^{-1}$. The universal mapping property of the free product gives a ring homomorphism $\psi : \mathbb{Q}[x_y, x_y^{-1}] \ast \mathbb{Q} K \to R_f$ which restricts to $\psi_1$ and $\psi_2$ on $K$ and $\mathbb{Q}[x_y, x_y^{-1}]$, respectively.

We finish the proof by showing that the homomorphism $\psi$ is the inverse of the homomorphism $\phi$. We need only verify that $\psi \circ \phi$ is the identity on the generators of $R_f$ and that $\phi \circ \psi$ is the identity on the generators of $\mathbb{Q}[x_y, x_y^{-1}] \ast \mathbb{Q} K$. We have $\psi(\phi(y)) = \psi(x_y) = y$ and

$$\psi(\phi(y)) = \psi(x_y^{-1} t) = y h(y q)(y q)^{-1} = y^{-1} y q = y.$$ 

Thus, $\psi \circ \phi$ is the identity map on $R_f$. On the other hand, $\phi(\psi(t)) = \phi(y q^{-1}) = x_y x_y^{-1} t = t$ and $\phi(\psi(x_y)) = \phi(y) = x_y$. Finally we have

$$\phi(\psi(x_y^{-1})) = \phi(y h(y q)) = x_y^{-1} t h(x_y x_y^{-1} t) = x_y^{-1} t h(t) = x_y^{-1},$$

since $f(t) = t h(t) - 1$ and hence $t h(t) = 1 \mathbb{Q}[x_y, x_y^{-1}] \ast \mathbb{Q} K$. This shows that the rings are isomorphic and completes our proof.

6.7. Free products associated to Coxeter systems

Consider now a graph $\Gamma_{W,S}$ associated to a Coxeter system $(W, S)$ with generators $S = \{s_1, s_2, \ldots, s_N\}$ and Coxeter matrix $(m_{ij})_{1 \leq i, j \leq N}$. Let $Y_{W,S} = Y_{W,S,+} \cup Y_{W,S,+}^{-1}$ be an orientation of $\Gamma_{W,S}$. To each $y_{ij} = [s_i s_j] \in Y_{W,S,+}$, we associate a quotient ring, $R_{y_{ij}}$, of the polynomial ring $\mathbb{Q}[x_{y_{ij}}, \bar{x}_{y_{ij}}]$ in the non-commuting variables $x_{y_{ij}}$ and $\bar{x}_{y_{ij}}$, as follows:

$$R_{y_{ij}} = \begin{cases} \mathbb{Q}[x_{y_{ij}}, \bar{x}_{y_{ij}}] & \text{if } m_{ij} = \infty \\ \mathbb{Q}[x_{y_{ij}}, \bar{x}_{y_{ij}}]/(C_{m_{ij}}(x_{y_{ij}}, \bar{x}_{y_{ij}}), C_{m_{ij}}(\bar{x}_{y_{ij}}, x_{y_{ij}})) & \text{if } m_{ij} < \infty \end{cases}$$

Corollary 6.8. Let $\Gamma_{W,S}$ be a graph associated to a Coxeter system $(W, S)$ with generators $S = \{s_1, s_2, \ldots, s_N\}$ and Coxeter matrix $(m_{ij})_{1 \leq i, j \leq N}$. Let $R_{y_{ij}}$ be the ring associated to the edge $y_{ij} = [s_i s_j] \in Y_{W,S,+}$. Then we have the following:

$$R_{y_{ij}} = \begin{cases} \mathbb{Q}[x_{y_{ij}}, \bar{x}_{y_{ij}}] & \text{if } m_{ij} = \infty \\ \mathbb{Q}[x_{y_{ij}}, x_{y_{ij}}^{-1}] \ast \mathbb{Q} K & \text{if } m_{ij} < \infty \end{cases}$$

where $\mathbb{Q}[x_{y_{ij}}, x_{y_{ij}}^{-1}]$ is the ring of Laurent polynomials in $x_{y_{ij}}$ over $\mathbb{Q}$ and $K \cong \mathbb{Q}[t]/(C_{m_{ij}}(t))$ is a field extension of $\mathbb{Q}$. Furthermore, $R_{y_{ij}}$ is a domain for all $y_{ij} \in Y_{W,S,+}$.

Proof. This follows directly from Theorem 6.6 and the fact that the ring $\mathbb{Q}[x_{y_{ij}}, \bar{x}_{y_{ij}}]$ in the non-commuting variables $x_{y_{ij}}$ and $\bar{x}_{y_{ij}}$ is a domain (see, for example, Cohn [7], Section 5.3).

7. A faithful representation of $\hat{R}_{W,S}$.

In this section, we show that the algebra $\hat{R}_{W,S}$ (defined in Section 4) associated to a finite rank Coxeter system is isomorphic to a subring of a matrix ring over a non-commutative ring.

7.1. Conventions for this section

Throughout the section $\Gamma_{W,S}$ denotes a graph associated to a Coxeter system $(W, S)$ with generators $S = \{s_1, \ldots, s_N\}$ and Coxeter matrix $(m_{ij})_{1 \leq i, j \leq N}$. We fix an orientation, $Y_{W,S} = Y_{W,S,+} \cup Y_{W,S,+}^{-1}$, for our graph $\Gamma_{W,S}$. We will show that the associated algebra $\hat{R}_{W,S}$ can be imbedded as a $\mathbb{Q}$ algebra into a matrix ring $M_{N+1}(Q)$, where $Q$ is a quotient of a non-commutative polynomial ring over $\mathbb{Q}$ in several variables. This in turn will give us some insight into the zero divisors in $\hat{R}_{W,S}$ and its subring $R_{W,S}$. 
7.2. A graph extension

Let \((W^*, S^*)\) denote the Coxeter system with Coxeter generators \(S^* = \{s_1, \ldots, s_N, s_{N+1}\}\) and Coxeter matrix \((m^*)_{ij} \leq i, j \leq N+1\), such that \(m^*_{ij} = m_{ij}, 1 \leq i, j \leq N, m^*_{N+1} = 3, 1 \leq i \leq N\) and \(m^*_{N+1} = 1\). (We denote \(N+1\) by \(N + 1\) in subscripts where confusion might be caused by ambiguity.)

Since \(W\) can be identified with the parabolic subgroup \(W^*_{[N]}\) of \(W^*\) corresponding to the Coxeter system \((W^*_{[N]}, S^*_{[N]})\), by Lemma 4.4, we have an inclusion of the graph \(\Gamma_{W,S}\) in the graph \(\Gamma^*_{W,S^*}\) and imbeddings of the associated non-unital \(Q\) algebras \(i_{[N],[N+1]} : \hat{P}_{W,S} \rightarrow \hat{P}_{W^*,S^*}\) and \(\tilde{i}_{[N],[N+1]} : \hat{R}_{W,S} \rightarrow \hat{R}_{W^*,S^*}\).

For simplicity of notation, we will identify the vertices and edges of the graph \(\Gamma_{W,S}\) with their images in the graph of \(\Gamma_{W^*,S^*}\). We can extend the orientation of \(\Gamma_{W,S}\) to an orientation for \(\Gamma_{W^*,S^*}\) by letting \(Y_{W^*,S^*,+} = Y_{W,S,+} \cup \{y_{N+i}|1 \leq i \leq N\}\).

Example 7.3. For our running example, let \(S^* = \{r, s, t, u, v, w\}\) with new vertex \(w\). We show the positively oriented edges, \(Y_{W^*,S^*,+}\), of the graph \(\Gamma^*_{W^*,S^*}\), below. The vertices and edges in \(\Gamma_{W^*,S^*}\) are shown in blue. Alongside, we show the new vertex and new geometric edges of \(\Gamma_{W^*,S^*}\) with their labels in blue.

7.4. Isomorphism of \(\hat{R}_{W^*,S^*}\) and a matrix ring

For \(y \in Y_{W^*,S^*,+}\), let \(R_y\) be the associated ring described at the end of Section 6. By Definition 6.2, the free product \(\{\ast \in \mathbb{Q}_y\}_{y \in Y_{W^*,S^*}}\) is isomorphic to the quotient ring, \(Q\), of the polynomial ring in the non-commuting variables \(\{x_{y_i}, \bar{x}_{y_i}| y \in Y_{W,S,+}\}\);

\[
Q = \mathbb{Q}\left[\left\{x_{y_i}, \bar{x}_{y_i}| y \in Y_{W,S,+}\right\}\right] / \left\langle \left\{C_{m_{ij}}(x_{y_i}\bar{x}_{y_j}), C_{m_{ij}}(\bar{x}_{y_i}x_{y_j})\right\}| y \in Y_{W,S,+}, m_{ij} < \infty\right\rangle.
\]

By Theorem 6.5 and Corollary 6.8, \(Q\) is a domain.

We will denote the coset, \(p + \hat{I}_{W^*,S^*}\), of \(p \in \hat{P}_{W^*,S^*}\) in \(\hat{R}_{W^*,S^*}\) by \(p^\sim\) in what follows. We first single out some elements of \(\hat{R}_{W^*,S^*}\) which will play a central role in defining a homomorphism from \(\hat{R}_{W^*,S^*}\) to \(M_{N+1}(Q)\). For \(1 \leq i \leq N + 1\), we let

\[
p_i^j = \begin{cases}
[s_i s_{N+1}]^\sim = (y_{[N+1]}^N y_{N+1})^\sim & \text{if } 1 \leq i, j \leq N \\
[s_i s_{N+1}]^\sim = (y_{N+1}^N)^\sim & \text{if } 1 \leq i \leq N, j = N + 1 \\
[s_{N+1} s_j]^\sim = y_{N+1}^N & \text{if } i = N + 1, 1 \leq j \leq N \\
[s_{N+1}]^\sim & \text{if } i = j = N + 1
\end{cases}
\]  

(7.4.1)

Note. In what follows we use the notation \(\hat{o}(y)\) and \(\hat{t}(y)\) to denote the index of \(o(y)\) and \(t(y)\) respectively in the set \([N + 1]\), for \(y \in Y_{W^*,S^*}\), i.e. if \(y = [s_i, s_j]\), then \(\hat{o}(y) = i\) and \(\hat{t}(y) = j\).
Lemma 7.5. We have the following identities in $\hat{R}_{W^*, S^*}$, where $s_i, s_j, s_k, s_l \in S^*$: 

$$
P_i^j = [s_j]^\sim
$$

$$
P_i^j P_k^j = 0^\sim \text{ if } j \neq k
$$

$$
P_i^j P_k^j = P_i^j \text{ if } j = k
$$

Proof. Note that $C_{ij}^- = [s_i s_j s_l]^- - [s_j]^- = 0^\sim$ if either $i$ or $j$ are equal to $N + 1$. Hence, if $i \neq N + 1$, then $P_i^j = [s_i s_{N+1} s_l]^- = [s_j]^-$. If $i = N + 1$, then $P_i^j = [s_{N+1}]^\sim$. This proves the first identity.

Consider now $P_i^j P_k^j$. It is clear that the product is $0^\sim$ if $j \neq k$. Suppose that $1 \leq i, j, k \leq N$, then

$$
P_i^j P_k^j = ([s_i s_{N+1} s_l] [s_j s_{N+1} s_k])^- = ([s_i s_{N+1} s_k] [s_{N+1} s_k])^- = [s_i s_{N+1} s_k]^- = P_i^k
$$

by our opening remark. The other cases follow similarly and are left to the reader. \qed

7.6. The homomorphisms $\phi_i$

For each $i \in \{1, 2, \ldots, N + 1\}$, we use the universal mapping property of the polynomial ring to define a ring homomorphism $\hat{\psi}_i : \hat{Q}[[x_i, \tilde{x}_i]_{y_i} \in Y_{W, S^*} \rightarrow [s_i]^- \hat{R}_{W^*, S^*}[s_i]^- \subset \hat{R}_{W^*, S^*}$, such that $\hat{\psi}_i(1_Q) = [s_i]^-$, $\hat{\psi}_i(x_i) = P_i^j y^\sim P_i^{(j)}$ and $\hat{\psi}_i(\tilde{x}_i) = P_i^j y^\sim P_i^{(j)}$ for $y \in Y_{W, S^*}$. Since $o(\tilde{y}) = t(y)$ and $t(\tilde{y}) = o(y)$, we have

$$
P_i^j y^\sim P_i^{(j)} y^\sim P_i^{(j)} = P_i^j (y^\sim P_i^{(j)}).
$$

Hence $\hat{\psi}_i((x_i, \tilde{x}_i, y_i)) = P_i^j ((y_i^\sim)^k) \sim P_i^{(j)}$, by Lemma 7.5. Thus, for $y_{ij} \in Y_{W, S^*}, m_{ij} < \infty$, we have

$$
\hat{\psi}_i (C_{m_{ij}}(x_{ij}, \tilde{x}_{ij})) = C_{m_{ij}} (P_i^j (y_{ij}^\sim) \sim P_i^{(j)}) = P_i^j C_{m_{ij}} ((y_{ij}^\sim) \sim P_i^{(j)}) = 0^\sim.
$$

Therefore, the homomorphism $\hat{\psi}_i$ gives us a well-defined quotient homomorphism

$$
\psi_i : Q \rightarrow [s_i]^- \hat{R}_{W^*, S^*}[s_i]^- \subset \hat{R}_{W^*, S^*}
$$

acting as follows on the generators of $Q$:

$$
\psi_i(1_Q) = [s_i]^- \\
\psi_i(x_i) = P_i^j y^\sim P_i^{(j)}, \quad y \in Y_{W, S^*}. \\
\psi_i(\tilde{x}_i) = P_i^j y^\sim P_i^{(j)}, \quad y \in Y_{W, S^*}.
$$

Lemma 7.7. Let $\psi_i : Q \rightarrow [s_i]^- \hat{R}_{W^*, S^*}[s_i]^- \subset \hat{R}_{W^*, S^*}$ and $\psi_j : Q \rightarrow [s_j]^- \hat{R}_{W^*, S^*}[s_j]^- \subset \hat{R}_{W^*, S^*}$, $1 \leq i, j \leq N + 1$ be as defined above. We have $P_i^j \psi_i(q) P_i^j = \psi_j(q)$ for all $q \in Q, 1 \leq i, j \leq N + 1$.

Proof. It suffices to show equality on the generators $1_Q, x_i, \tilde{x}_i, y \in Y_{W, S^*}$ of $Q$. Using the identities in Lemma 7.5, we get:

$$
P_i^j \psi_i(1_Q) P_i^j = P_i^j [s_i]^- \sim P_i^j = P_i^j [s_j]^- = \psi_j(1_Q),
$$

$$
P_i^j \psi_i(x_i) P_i^j = P_i^j P_i^j y^\sim P_i^{(j)} P_i^j = P_j^j y^\sim P_i^{(j)} = \psi_j(x_i),
$$

and

$$
P_i^j \psi_i(\tilde{x}_i) P_i^j = P_i^j P_i^j y^\sim P_i^{(j)} P_i^j = P_j^j y^\sim P_i^{(j)} = \psi_j(\tilde{x}_i).
$$

This proves the lemma. \qed
7.8. The ring isomorphism $\Phi$

We let $\Psi : M_{N+1}(Q) \to \hat{R}_{W^{*}\cdot S^{*}}$ be the homomorphism of $\mathbb{Q}$ vector spaces which acts on basis elements of $M_{N+1}(Q)$ of the form $q e_{ij}$ as follows:

$$\Psi(q e_{ij}) = \psi_{i}(q) P_{i}^{j},$$

where $q \in Q$ and $e_{ij}$ is the $(N + 1) \times (N + 1)$ matrix with 1 in the $(i,j)$ position and zeros elsewhere.

**Lemma 7.9.** $\Psi$ is a ring homomorphism.

**Proof.** To verify this, we need only show that $\Psi((q_{1}e_{ij})(q_{2}e_{kl})) = \Psi(q_{1}e_{ij})\Psi(q_{2}e_{kl})$ for $q_{1}, q_{2} \in Q$ and $1 \leq i,j,k,l \leq N + 1.$ If $j \neq k,$ it is easy to see that both sides are zero and are thus equal. If $j = k,$ then, using Lemma 7.7, we get

$$\Psi((q_{1}e_{ij})(q_{2}e_{jj})) = \Psi(q_{1}q_{2}e_{ii}) = \psi_{i}(q_{1})\psi_{i}(q_{2})P_{i}^{i}$$

$$= \psi_{i}(q_{1})P_{i}^{j}\psi_{i}(q_{2})P_{j}^{i}P_{i}^{i}$$

$$= \psi_{i}(q_{1})P_{i}^{j}\psi_{i}(q_{2})P_{j}^{j}$$

$$= \Psi(q_{1}e_{ij})\Psi(q_{2}e_{jj}).$$

We show that both rings are isomorphic by showing that $\Psi$ has an inverse. We define a homomorphism,

$$\phi : \hat{P}_{W^{*}\cdot S^{*}} \to M_{N+1}(Q),$$

which acts on the vertices and edges of $\Gamma_{W^{*}\cdot S^{*}}$ as follows:

$$\phi([s_{i}]) = e_{ii}, \quad 1 \leq i \leq N + 1,$$

$$\phi(y_{ij}) = x_{y_{j}}e_{ij}, \quad 1 \leq i, j \leq N, y_{ij} \in Y_{W^{*}\cdot S^{*}},$$

$$\phi(y_{ij}) = \check{x}_{y_{j}}e_{ij}, \quad 1 \leq i, j \leq N, y_{ij} \in \overline{Y}_{W^{*}\cdot S^{*}},$$

$$\phi(y_{ij}) = e_{ij}, \quad N + 1 \in [i,j], i \neq j.$$

The homomorphism extends to a well-defined homomorphism on $\hat{P}_{W^{*}\cdot S^{*}}$ since $\phi$ respects the multiplicative relations on the basis of paths $\mathcal{P}_{W^{*}\cdot S^{*}}$ in $\hat{P}_{W^{*}\cdot S^{*}}$ namely:

$$\phi(y_{ij})\phi(y_{kl}) = \phi(0) = 0 \quad \text{if } j \neq k, y_{ij}, y_{kl} \in Y_{W^{*}\cdot S^{*}}.$$  

$$\phi(y_{ij})\phi([s_{k}]) = \phi([s_{k}])\phi(y_{ij}) = \phi(0) = 0 \quad \text{if } j \neq k, y_{ij} \in Y_{W^{*}\cdot S^{*}}.$$  

$$\phi(y_{ij})\phi([s_{j}]) = \phi([s_{i}])\phi(y_{ij}) = \phi(y_{ij}) \quad \text{if } y_{ij} \in Y_{W^{*}\cdot S^{*}}.$$  

The following lemma shows that the quotient homomorphism $\Phi : \hat{R}_{W^{*}\cdot S^{*}} \to M_{N+1}(Q)$ given by

$$\Phi(p^{\sim}) = \phi(p), \quad p \in \hat{P}_{W^{*}\cdot S^{*}}$$

is well-defined:

**Lemma 7.10.** Let $\phi : \hat{P}_{W^{*}\cdot S^{*}} \to M_{N+1}(Q)$ be as defined above. Then, $\hat{I}_{W^{*}\cdot S^{*}} \subseteq \ker \phi.$

**Proof.** It suffices to prove that the generators of $\hat{I}_{W^{*}\cdot S^{*}}$ are in $\ker \phi.$ For $1 \leq i \neq j \leq N, y_{ij} \in Y_{W^{*}\cdot S^{*}},$ we have

$$\phi(y_{ij}y_{ji}) = \phi(y_{ij})\phi(y_{ji}) = x_{y_{j}}e_{ij}\check{x}_{y_{j}}e_{ji} = x_{y_{j}}\check{x}_{y_{j}}e_{ii},$$
and if \(m_{ij} < \infty\),
\[
\phi(C_{y_{ij}}) = C_{m_{ij}}(\phi(y_{ij})\phi(\tilde{y}_{ij})) = C_{m_{ij}}(\tilde{x}_{y_{ij}}x_{y_{ij}})e_{ii} = 0.
\]
Similarly, if \(1 \leq i \neq j \leq N, y_{ij} \in Y_{W^*, S^*, +}\), \(m_{ij} < \infty\), we have
\[
\phi(C_{\tilde{y}_{ij}}) = C_{m_{ij}}(\phi(\tilde{y}_{ij})\phi(y_{ij})) = C_{m_{ij}}(\tilde{x}_{y_{ij}}x_{y_{ij}})e_{ij} = 0.
\]
If \(1 \leq i \leq N \) and \(j = N + 1\), we have \(C_{y_{ij}} = y_{ij}\tilde{y}_{ij} - [s_i]\) and
\[
\phi(C_{y_{ij}}) = \phi(y_{ij})\phi(\tilde{y}_{ij}) - e_{ii} = e_{iN+1}e_{N+1} - e_{ii} = 0.
\]

Likewise, if \(i = N + 1\) and \(1 \leq j \leq N\), we have
\[
\phi(C_{y_{ij}}) = e_{iN+1}e_{N+1} - e_{N+1} = 0.
\]
Thus all generators of \(\hat{I}_{W^*, S^*}\) are in ker \(\phi\) and hence \(\phi(\hat{I}_{W^*, S^*}) = 0\).

**Lemma 7.11.** Let \(\Phi : \hat{R}_{W^*, S^*} \to M_{N+1}(Q)\) and \(\Psi : M_{N+1}(Q) \to \hat{R}_{W^*, S^*}\) be defined as above. Then \(\Psi \circ \Phi\) is the identity map on \(\hat{R}_{W^*, S^*}\).

**Proof.** It is enough to show that this is true on the generators \([s_i]\), \(s_i \in S\) and \(y_{ij}\), \(y_{ij} \in Y_{W^*, S^*}\). Let \(s_i \in S\), then
\[
\Psi \circ \Phi([s_i]) = \Psi(\phi([s_i])) = \Psi(e_{ii}) = \psi_i(1_Q)P_i^i = [s_i].
\]
If \(y_{ij} \in Y_{W^*, S^*, +}, 1 \leq i \neq j \leq N\), we have
\[
\Psi \circ \Phi(y_{ij}) = \Psi(\phi(y_{ij})) = \Psi(\tilde{x}_{y_{ij}}e_{ii}) = \psi_i(x_{y_{ij}})P_i^i = P_i^iP_i^jP_i^j = y_{ij}.
\]
If \(y_{ij} \in Y_{W^*, S^*, +}, 1 \leq i \neq j \leq N\), we have
\[
\Psi \circ \Phi(\tilde{y}_{ij}) = \Psi(\phi(\tilde{y}_{ij})) = \Psi(\tilde{x}_{y_{ij}}e_{ii}) = \psi_j(x_{y_{ij}})P_j^i = P_j^iP_j^iP_j^j = \tilde{y}_{ij}.
\]
Finally, if \(i = N + 1\) or \(j = N + 1\), then
\[
\Psi \circ \Phi(y_{ij}) = \Psi(\phi(y_{ij})) = \Psi(e_{ij}) = \psi_i(1_Q)P_i^j = [s_i] = y_{ij}.
\]

**Lemma 7.12.** Let \(\Phi : \hat{R}_{W^*, S^*} \to M_{N+1}(Q)\) and \(\Psi : M_{N+1}(Q) \to \hat{R}_{W^*, S^*}\) be defined as above. Then \(\Phi \circ \Psi\) is the identity map on \(M_{N+1}(Q)\).
Proof. It is enough to show that $\Phi \circ \Psi$ acts as the identity on the generators of $M_{N+1}(Q)$ as a ring; $\{x_je_{ij}, y_je_{ij}, e_{ij} | 1 \leq i, j \leq N + 1, y \in Y_{W,S,+}\}$. For $1 \leq i, j \leq N + 1$, we have

$$
\Phi(\Psi(e_{ij})) = \Phi(P^j_i) = \begin{cases} 
\Phi([s_i]) & \text{if } i = j \\
\Phi([y_{N+1}N_{N+1}]) & \text{if } 1 \leq i \neq j \leq N \\
\Phi([y_{N+1}N_{N+1}]) & \text{if } i = N + 1 \text{ and } 1 \leq j \leq N \\
\Phi([y_{N+1}N_{N+1}]) & \text{if } j = N + 1 \text{ and } 1 \leq i \leq N \\
\end{cases}
$$

Thus, $\Phi(\Psi(e_{ij})) = \Phi(P^j_i) = e_{ij}, 1 \leq i, j \leq N + 1$.

From the above calculation we have that $\Phi(P^j_i) = e_{ij}, 1 \leq i, j \leq N + 1$. Thus if $1 \leq i, j \leq N + 1, y \in Y_{W,S,+}$, we have

$$
\Phi \big( \Psi(x_je_{ij}) \big) = \Phi \big( \psi_1(x_j)P^j_i \big) = \Phi \big( P^{\hat{\gamma}(j)}i y^{-1}P^{\hat{\gamma}(j)}iP^j_i \big) = e_{\hat{\gamma}i} = e_{\hat{\gamma}i}\phi(y)e_{i}e_{ij} = e_{\hat{\gamma}i}x_je_{i}e_{ij} = x_je_{ij}.
$$

For $1 \leq i, j \leq N + 1, y \in Y_{W,S,+}$, we have

$$
\Phi \big( \Psi(y_je_{ij}) \big) = \Phi \big( \psi_1(y_j)P^j_i \big) = \Phi \big( P^{\hat{\gamma}(j)}y^{-1}P^{\hat{\gamma}(j)}iP^j_i \big) = e_{\hat{\gamma}i} = e_{\hat{\gamma}i}\phi(i)e_{ij} = e_{\hat{\gamma}i}x_je_{ij} = x_je_{ij}.
$$

This concludes our proof. \qed

8. Zero divisors in $R_{W,S}$

In this section, we put together the results from the previous sections, to get an understanding of zero divisors in the ring $R_{W,S}$, which is the main topic of interest of the paper.

8.1. Main result on zero divisors

We let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$, not necessarily finite, and Coxeter matrix $(m_{ij})_{i,j \in I}$. We use the notation established in Section 4.1 throughout. For $p \in \hat{R}_{W,S}$, we let $x^{-1} = p + \hat{1}_{W,S}$ denote the coset of $p$ in the quotient ring $\hat{R}_{W,S}$.

Although $\hat{R}_{W,S}$ is clearly not a domain, we have the following result:

Theorem 8.2. Let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$, and Coxeter matrix $(m_{ij})_{i,j \in I}$. Let $x_1 \in [s_i]\hat{R}_{W,S}[s_j]$ and $x_2 \in [s_j]\hat{R}_{W,S}[s_k]$, be elements of $\hat{R}_{W,S}$ such that $x_1x_2 = 0$, then either $x_1 = 0$ or $x_2 = 0$. 

Proof. If $S$ is infinite, then $x_1 = p_1\sim$ and $x_2 = p_2\sim$, where $p_1, p_2 \in \hat{P}_{W,S}$ and are linear combinations of paths involving only a finite number of vertices and edges. Thus, using the direct limit structure of $\hat{R}_{W,S}$ from Lemma 4.11, we have $x_1$ and $x_2$ are both in $i_{j,I}(\hat{R}_{W,j,S})$ for some parabolic subgroup $(W_j, S_j)$ where $J \subset I$ is finite. Since $i_{j,I}$ is a monomorphism, we can assume without loss of generality that $S$ is finite.

When $S$ is finite with cardinality $N$, by Lemmas 7.11 and 7.12, we have a monomorphism $\Psi : \hat{R}_{W,S} \rightarrow M_{N+1}(Q)$ which maps $[s_i]^{-\sim} \hat{R}_{W,S}[s_i]^{-\sim}$ into $Qe_{ij}$. Furthermore, $Q$ is a domain by Theorem 6.5. Thus, it is enough to show that if $(q_1e_{ij})(q_2e_{jk}) = 0$ where $q_1, q_2 \in Q$, then either $q_1 = 0$ or $q_2 = 0$. This is obvious since $Q$ is a domain.

Corollary 8.3. Let $(W, S)$ be a Coxeter system with $S = \{s_i\}_{i \in I}$, and Coxeter matrix $(m_{ij})_{i,j \in I}$. Let $x_1 \in [s_i]R_{W,S}[s_i]$ and $x_2 \in [s_j]R_{W,S}[s_k]$ be elements of $R_{W,S}$ such that $x_1x_2 = 0$, then either $x_1 = 0$ or $x_2 = 0$.

Proof. By Lemma 4.13, we have a monomorphism $\tilde{i} : R_{W,S} \rightarrow \hat{R}_{W,S}$ on the quotient algebras. Therefore, the result follows from Theorem 8.2.

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