Level statistics of one-dimensional Schrödinger operators with random decaying potential

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Abstract

We study the level statistics of one-dimensional Schrödinger operator with random potential decaying like $x^{-\alpha}$ at infinity. We consider the point process $\xi_L$ consisting of the rescaled eigenvalues and show that: (i) (ac spectrum case) for $\alpha > \frac{1}{2}$, $\xi_L$ converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii) (critical case) for $\alpha = \frac{1}{2}$, $\xi_L$ converges to the limit of the circular $\beta$-ensemble.

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1 Introduction

1.1 Background

In this paper, we study the following Schrödinger operator

$$H := -\frac{d^2}{dt^2} + a(t) F(X_t) \quad \text{on } L^2(\mathbb{R})$$

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where \( a \in C^\infty \) is real valued, \( a(-t) = a(t) \), non-increasing for \( t \geq 0 \), and satisfies
\[
C_1 t^{-\alpha} \leq a(t) \leq C_2 t^{-\alpha}, \quad t \geq 1
\]
for some positive constants \( C_1, C_2 \) and \( \alpha > 0 \). \( F \) is a real-valued, smooth, and non-constant function on a compact Riemannian manifold \( M \) such that
\[
\langle F \rangle := \int_M F(x) \, dx = 0.
\]
\( \{X_t\} \) is a Brownian motion on \( M \). Since the potential \( a(t)F(X_t) \) is \(-\frac{d^2}{dt^2}\)-compact, we have \( \sigma_{\text{ess}}(H) = [0, \infty) \). Kotani-Ushiroya\[3\] proved that the spectrum of \( H \) in \([0, \infty)\) is
1. for \( \alpha < \frac{1}{2} \) : pure point with exponentially decaying eigenfunctions,
2. for \( \alpha = \frac{1}{2} \) : pure point on \([0, E_c]\) and purely singular continuous on \([E_c, \infty)\) with some explicitly computable \( E_c \),
3. for \( \alpha > \frac{1}{2} \) : purely absolutely continuous.

In this paper we study the level statistics of this operator. For that purpose, let \( H_L := H|_{[0,L]} \) be the local Hamiltonian with Dirichlet boundary condition and let \( \{E_n(L)\}_{n=1}^\infty \) be its eigenvalues in increasing order. Let \( n(L) \in \mathbb{N} \) be s.t. \( \{E_n(L)\}_{n \geq n(L)} \) coincides with the set of positive eigenvalues of \( H_L \). We arbitrary take the reference energy \( E_0 > 0 \) and consider the following point process
\[
\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}
\]
in order to study the local fluctuation of eigenvalues near \( E_0 \). Our aim is to identify the limit of \( \xi_L \) as \( L \to \infty \). Here we consider the scaling of \( \sqrt{E_n(L)} \)'s instead of \( E_n(L) \)'s. This corresponds to the unfolding with respect to the density of states.

This problem was first studied by Molchanov\[6\]. He proved that, when \( a(t) \) is constant, \( \xi_L \) converges to the Poisson process. It was extended to the multidimensional Anderson model by Minami \[7\]. Killip-Stoiciu \[2\] studied the CMV matrices whose matrix elements decay like \( n^{-\alpha} \). They showed that, \( \xi_L \) converges to (i) \( \alpha > \frac{1}{2} \) : the clock process, (ii) \( \alpha = \frac{1}{2} \) : the limit of the circular \( \beta \)-ensemble, (iii) \( 0 < \alpha < \frac{1}{2} \) : the Poisson process. Krchevski-Valko-Virag\[5\] studied the one-dimensional discrete Schrödinger operator with the
random potential decaying like $n^{-1/2}$, and proved that $\xi_L$ converges to the Sine$_\beta$-process.

The aim of our work is to do the analogue of that by Killip-Stoiciu\cite{2} for the one-dimensional Schrödinger operator in the continuum.

In subsection 1.2 (resp. subsection 1.3), we state our results for ac-case : $\alpha > \frac{1}{2}$ (resp. critical-case : $\alpha = \frac{1}{2}$). We have not obtained results for pp-case : $\alpha < \frac{1}{2}$.

### 1.2 AC-case

**Definition 1.1** Let $\mu$ be a probability measure on $[0, \pi)$. We say that $\xi$ is the clock process with spacing $\pi$ with respect to $\mu$ if and only if
\[
E[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp \left( -\sum_{n \in \mathbb{Z}} f(n\pi - \phi) \right)
\]
where $f \in C_c(\mathbb{R})$ and $\xi(f) := \int_{\mathbb{R}} f d\xi$.

We set
\[
(x)_{\pi \mathbb{Z}} := x - [x]_{\pi \mathbb{Z}}, \quad [x]_{\pi \mathbb{Z}} := \max\{y \in \pi \mathbb{Z} | y \leq x\}.
\]

We study the limit of $\xi_L$ under the following assumption

**A**

1. $\alpha > \frac{1}{2}$,
2. A sequence $\{L_j\}_{j=1}^\infty$ satisfies $\lim_{j \to \infty} L_j = \infty$ and
\[
(\sqrt{E_0} L_j)_{\pi \mathbb{Z}} = \beta + o(1), \quad j \to \infty
\]
for some $\beta \in [0, \pi)$.

The condition A(2) is set to guarantee the convergence of $\xi_L$ to a point process. If $a \equiv 0$ for instance, A(2) is indeed necessary.

**Theorem 1.1** Assume (A). Then $\xi_{L_j}$ converges in distribution to the clock process with spacing $\pi$ with respect to a probability measure $\mu_\beta$ on $[0, \pi)$.
Remark 1.1 Let $x_t$ be the solution to the eigenvalue equation: $H L x_t = \kappa^2 x_t$ ($\kappa > 0$). If we set

\[
\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t \\ r_t \cos \theta_t \end{pmatrix}, \quad \theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa),
\]

then $\tilde{\theta}_t(\kappa)$ has a limit as $t$ goes to infinity \([3]\): \(\lim_{t \to \infty} \tilde{\theta}_t(\kappa) = \tilde{\theta}_\infty(\kappa)\), a.s.; $\mu_\beta$ is the distribution of the random variable $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi Z}$. In some special cases, we can show that $(\tilde{\theta}_\infty(\sqrt{E_0}))_{\pi Z}$ is not uniformly distributed on $[0, \pi)$ for large $E_0$, implying that $\mu_\beta$ really depends on $\beta$.

Remark 1.2 We can consider point processes with respect to two reference energies $E_0, E'_0$ $(E_0 \neq E'_0)$ simultaneously: suppose a sequence $\{L_j\}_{j=1}^\infty$ satisfies

\[
(\sqrt{E_0 L_j})_{\pi Z} = \beta + o(1), \quad (\sqrt{E'_0 L_j})_{\pi Z} = \beta' + o(1), \quad j \to \infty
\]

for some $\beta, \beta' \in [0, \pi)$. We set

\[
\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)}-\sqrt{E_0})}, \quad \xi'_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)}-\sqrt{E'_0})}.
\]

Then the joint distribution of $\xi_{L_j}, \xi'_{L_j}$ converges, for $f, g \in C_c(\mathbb{R})$,

\[
\lim_{j \to \infty} \mathbb{E} \left[ \exp \left( -\xi_{L_j}(f) - \xi_{L_j}(g) \right) \right] = \int_0^\pi d\mu(\phi, \phi') \exp \left( -\sum_{n \in \mathbb{Z}} (f(n\pi - \phi) + g(n\pi - \phi')) \right)
\]

where $\mu(\phi, \phi')$ is the joint distribution of $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi Z}$ and $(\beta' + \tilde{\theta}_\infty(\sqrt{E'_0}))_{\pi Z}$. We are unable to identify $\mu(\phi, \phi')$ but it may be possible that $\phi$ and $\phi'$ are correlated.

Remark 1.3 Suppose we renumber the eigenvalues near the reference energy $E_0$ so that

\[
\cdots < E'_2(L) < E'_1(L) < E_0 \leq E'_0(L) < E'_1(L) < E'_2(L) < \cdots
\]

Then an argument similar to the proof of Theorem 2.4 in \([4]\) proves the following fact: for any $n \in \mathbb{Z}$ we have

\[
\lim_{L \to \infty} L(\sqrt{E'_{n+1}(L)} - \sqrt{E'_n(L)}) = \pi, \quad \text{a.s.} \tag{1.1}
\]
which is called the strong clock behavior [1]. We note that the integrated density of states is equal to $\sqrt{E}/\pi$.

We next study the finer structure of the eigenvalue spacing, under the following assumption.

\begin{enumerate}[(B)]
  \item $\frac{1}{2} < \alpha < 1$, 
  \item A sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j \to \infty} L_j = \infty$ and 
    \[ \sqrt{E_0} L_j = m_j \pi + \beta + \epsilon_j, \quad j \to \infty \]
    for some $\{m_j\}_{j=1}^{\infty} \subset \mathbb{N}$, $\beta \in [0, \pi)$ and $\{\epsilon_j\}_{j=1}^{\infty}$ with $\lim_{j \to \infty} \epsilon_j = 0$. 
  \item $a(t) = t^{-\alpha}(1 + o(1)), \quad t \to \infty$.
\end{enumerate}

Roughly speaking, $E_{m_j}(L_j)$ is the eigenvalue closest to $E_0$. In view of (1.1), we set 
\[ X_j(n) := \left\{ \left( \sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n}(L_j)} \right) L_j - \pi \right\} L_j^{\alpha - \frac{1}{2}}, \quad n \in \mathbb{Z}. \]

**Theorem 1.2** Assume (B). Then $\{X_j(n)\}_{n \in \mathbb{Z}}$ converges in distribution to the Gaussian system with covariance 
\[ C(n, n') = \frac{C(E_0)}{8E_0} \Re \int_0^1 s^{-2\alpha} e^{2i(n-n')\pi s} 2(1 - \cos 2\pi s) ds, \quad n, n' \in \mathbb{Z}, \]
where $C(E) := \int_M \left| \nabla (L + 2i\sqrt{E})^{-1} \right|^2 dx$ and $L$ is the generator of $(X_t)$.

**Remark 1.4** Lemma 2.1 in [3] and Lemma 4.1 in section 4 imply that 
\[ \sqrt{E_{m_j}(L_j)} = \sqrt{E_0} - \frac{\beta + \tilde{\theta}_\infty(\sqrt{E_0})}{L_j} + Y_j \]
where $Y_j = O(L_j^{-\alpha - \frac{1}{2} + \epsilon}) + O(\epsilon_j L_j^{-1})$, a.s. for any $\epsilon > 0$. Furthermore by the definition of $\{X_j(n)\}$ we have 
\[ \sqrt{E_{m_j+n}(L_j)} = \begin{cases}
\sqrt{E_{m_j}(L_j)} + \frac{n \pi}{L_j} + \frac{1}{L_j^{\alpha + \frac{1}{2}}} \sum_{l=0}^{n-1} X_j(l) & (n \geq 1) \\
\sqrt{E_{m_j}(L_j)} + \frac{n \pi}{L_j} - \frac{1}{L_j^{\alpha + \frac{1}{2}}} \sum_{l=n}^{-1} X_j(l) & (n \leq -1)
\end{cases} \]
and Theorem 1.2 thus describes the behavior of eigenvalues near $E_{m_j}(L_j)$ in the second order.

Remark 1.5 Suppose we consider two reference energies $E_0, E'_0 (E_0 \neq E'_0)$ simultaneously and suppose a sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j \to \infty} L_j = \infty$ and

$$\sqrt{E_0} L_j = m_j \pi + \beta + o(1), \quad \sqrt{E'_0} L_j = m'_j \pi + \beta' + o(1), \quad j \to \infty$$

for some $m_j, m'_j \in \mathbb{N}$, and $\beta, \beta' \in [0, \pi)$. Then $\{X_j(n)\}_n$ and $\{X'_j(n)\}_n$ converge jointly to the mutually independent Gaussian systems.

1.3 Critical Case
We set the following assumption.

$$(C) \quad a(t) = t^{-\frac{1}{2}}(1 + o(1)), \quad t \to \infty.$$ 

Theorem 1.3 Assume (C). Then

$$\lim_{L \to \infty} \mathbb{E}[e^{-\xi L(f)}] = \mathbb{E} \left[ \int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left( - \sum_{n \in \mathbb{Z}} f(\Psi^{-1}_1(2n\pi + \theta)) \right) \right]$$

where $\{\Psi_t(\cdot)\}_{t \geq 0}$ is the strictly-increasing function valued process such that for any $c_1, \cdots, c_m \in \mathbb{R}$, $\{\Psi_t(c_j)\}_{j=1}^m$ is the unique solution of the following SDE:

$$d\Psi_t(c_j) = 2c_j dt + D \text{Re} \left\{ (e^{i\Psi_t(c_j)} - 1) \frac{dZ_t}{\sqrt{t}} \right\}$$

$$\Psi_0(c_j) = 0, \quad j = 1, 2, \cdots, m$$

where $C(E_0) := \int_M |\nabla (L + 2i\sqrt{E_0})^{-1} F|^2 dx$, $D := \sqrt{\frac{C(E_0)}{2E_0}}$ and $Z_t$ is a complex Browninan motion.

Definition 1.2 For $\beta > 0$, the circular $\beta$-ensemble with $n$-points is given by

$$\mathbb{E}_n^\beta[G] := \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} G(\theta_1, \cdots, \theta_n) |\Delta(e^{i\theta_1}, \cdots, e^{i\theta_n})|^\beta$$

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where \( Z_{n,\beta} \) is the normalization constant, \( G \in C(T^n) \) is bounded and \( \Delta \) is the Vandermonde determinant. The limit \( \xi_\beta \) of the circular \( \beta \)-ensemble is defined

\[
E[e^{-\xi_\beta(f)}] = \lim_{n \to \infty} E_n^\beta \left[ \exp \left( -\sum_{j=1}^n f(n\theta_j) \right) \right], \quad f \in C_c^+(\mathbb{R})
\]

whose existence and characterization is given by [2]. The result in [2] together with Theorem 1.3 imply the limit of \( \xi_L \) coincides with that of the circular \( \beta \)-ensemble modulo a scaling.

**Corollary 1.4** Assume (C). Writing \( \xi_\beta = \sum_n \delta_\lambda_n \), let \( \xi'_\beta := \sum_n \delta_{\lambda_n/2} \). Then \( \xi_L \overset{d}{\to} \xi'_\beta \) with \( \beta = \beta(E_0) := \frac{8E_0}{C(E_0)} \).

**Remark 1.6** The corresponding \( \beta = \beta(E_0) = \frac{8E_0}{C(E_0)} \) depends on the reference energy \( E_0 \), so that the spacing distribution may change if we look at the different region in the spectrum. To see how \( \beta \) changes, we recall some results in [3]. Let \( \sigma_F(\lambda) \) be the spectral measure of the generator \( L \) of \( \{X_t\} \) with respect to \( F \). Then

\[
\gamma(E) := -\frac{1}{4E} \int_{-\infty}^0 \frac{\lambda}{\lambda^2 + 4E} d\sigma_F(\lambda), \quad E > 0
\]

is the Lyapunov exponent in the sense that any generalized eigenfunction \( \psi_E \) of \( H \) satisfies

\[
\lim_{|t| \to \infty} (\log t)^{-1} \log \left\{ \psi_E^2 + \psi'_E(t)^2 \right\}^{1/2} = -\gamma(E), \quad \text{a.s.}
\]

Moreover \( E < E_c \) (resp. \( E > E_c \)) if and only if \( \gamma(E) > \frac{1}{2} \) (resp. \( \gamma(E) < \frac{1}{2} \)) and \( \gamma(E_c) = \frac{1}{2} \). Since \( C(E) = 8E \cdot \gamma(E) \), we have \( \beta(E) = \frac{1}{\gamma(E)} \). It then follows that \( E < E_c \) (resp. \( E > E_c \)) if and only if \( \beta(E) < 2 \) (resp. \( \beta(E) > 2 \)) and \( \beta(E_c) = 2 \) (Figure 1.). Similar statement also holds for discrete Hamiltonian and CMV matrices. This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong). We also note that if \( \beta = 2 \), the circular \( \beta \)-ensemble with \( n \)-points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on \( U(n) \).
Remark 1.7 If we consider two reference energies \( E_0, E'_0 (E_0 \neq E'_0) \), then the corresponding point process \( \xi_L, \xi'_L \) converges jointly to the independent \( \xi_\beta, \xi'_\beta \).

In later sections, we prove theorems mentioned above based on the argument in [2, 3, 4]: The main ingredient of the proof is to study the limiting behavior of the relative Prüfer phase. In section 2 we prepare some notations and basic facts. In sections 3, 4, we consider the ac-case and prove Theorems 1.1, 1.2. In sections 6-9, we consider the critical case and prove Theorem 1.3 which is outlined in section 5. In what follows, \( C \) denotes general positive constant which is subject to change from line to line in each argument.

2 Preliminaries

Let \( x_t \) be the solution to the equation \( H_L x_t = \kappa^2 x_t \) \((\kappa > 0)\) which we set in the following form

\[
\begin{pmatrix}
x_t \\
x'_t / \kappa
\end{pmatrix} = r_t \begin{pmatrix}
\sin \theta_t \\
\cos \theta_t
\end{pmatrix}, \quad \theta_0 = 0.
\]

We define \( \tilde{\theta}_t(\kappa) \) by

\[
\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa).
\]

Then it follows that

\[
\begin{align*}
r_t(\kappa) &= \exp \left( \frac{1}{2\kappa} \int_0^t a(s) F(X_s) e^{2i\theta_s(\kappa)} ds \right) \quad (2.1) \\
\tilde{\theta}_t(\kappa) &= \frac{1}{2\kappa} \int_0^t \text{Re}(e^{2i\theta_s(\kappa)} - 1) a(s) F(X_s) ds \quad (2.2) \\
\frac{\partial \theta_t(\kappa)}{\partial \kappa} &= \int_0^t \frac{r_s^2}{r_t^2} ds + \frac{1}{2\kappa^2} \int_0^t \frac{r_s^2}{r_t^2} a(s) F(X_s)(1 - \text{Re} e^{2i\theta_s(\kappa)}) ds. \quad (2.3)
\end{align*}
\]
Since $\frac{\partial \theta}{\partial \kappa} > 0$, $\theta_t(\kappa)$ is increasing as a function of $\kappa$. Here and henceforth, for simplicity, we say $f$ is increasing if and only if $x < y$ implies $f(x) < f(y)$. Set

$$\theta_L(\sqrt{E_0}) = m(E_0, L)\pi + \phi(E_0, L)$$

$$m(E_0, L)\pi := [\theta_L(\sqrt{E_0})]_\pi \mathbb{Z}, \quad \phi(E_0, L) := (\theta_L(\sqrt{E_0}))_\pi \mathbb{Z} \in [0, \pi).$$

Moreover we define “the relative Prüfer phase”

$$\Psi_L(x) = \theta_L(\sqrt{E_0} + \frac{x}{L}) - \theta_L(\sqrt{E_0}).$$

As is done in [2] we use the following representation of the Laplace transform of $\xi_L$ in terms of $\Psi_L$.

**Lemma 2.1** For $f \in C^+_c(\mathbb{R})$ we have

$$E[e^{-\xi_L(f)}] = E \left[ \exp \left( - \sum_{n=n(L)-m(E_0, L)}^{\infty} f\left( \Psi_L^{-1}(n\pi - \phi(E_0, L)) \right) \right) \right].$$

**Proof.** Let

$$x_n(L) := L(\sqrt{E_n(L)} - \sqrt{E_0}), \quad n \geq n(L)$$

be the atoms of $\xi_L$. Since $\theta_L(\sqrt{E_n(L)}) = \theta_L(\sqrt{E_0} + \frac{x_n}{L}) = n\pi$ we have

$$\Psi_L(x_n) = (n - m(E_0, L))\pi - \phi(E_0, L).$$

Here we note that $\Psi_L(x)$ is continuous and increasing, and thus has the inverse $\Psi_L^{-1}$. $\square$

### 3 Convergence to a clock process

In what follows we set

$$\kappa := \sqrt{E_0}$$

for simplicity.
3.1 The behavior of $\Psi_L$

**Proposition 3.1** If $\alpha > \frac{1}{2}$, following fact holds for a.s. :

$$\lim_{L \to \infty} \Psi_L(x) = x$$

pointwise and this holds compact uniformly with respect to $\kappa$.

**Proof.** By (2.2) we have

$$\Psi_L(x) = x + \left( \frac{1}{2(\kappa + \frac{x}{L})} - \frac{1}{2\kappa} \right) \int_0^L \text{Re} \left( e^{2i\theta_s(\kappa+\frac{x}{L})} - 1 \right) a(s) F(X_s) ds$$

$$+ \frac{1}{2\kappa} \int_0^L a(s) F(X_s) \left( e^{2i\theta_s(\kappa+\frac{x}{L})} - e^{2i\theta_s(\kappa)} \right) ds$$

$$=: x + I + II.$$  

Since $a(s) = O(x^{-\alpha})$, $|x| \to \infty$,

$$I = -\frac{1}{2} \cdot \frac{\frac{x}{L}}{\kappa(\kappa + \frac{x}{L})} \int_0^L \text{Re} \left( e^{2i\theta_s(\kappa+\frac{x}{L})} - 1 \right) a(s) F(X_s) ds = O(L^{-\alpha}).$$

We next set

$$A_t(\kappa, \beta) := \int_0^t a(s) F(X_s) e^{i\beta \theta_s(\kappa)} ds.$$  

Take $\delta > 0$ such that

$$\int_0^\infty a(s)^2 s^\delta ds < \infty,$$

then by Lemma 2.2 $A_t(\kappa, \beta)$ has a limit as $t \to \infty$ : $\lim_{t \to \infty} A_t(\kappa, \beta) = A_\infty(\kappa, \beta)$ compact uniformly w.r.t. $\kappa$. Moreover, for any compact set $K \subset (0, \infty)$ and for any $\epsilon < \frac{\delta}{2}$, $\beta \in \mathbb{R}$ we have

$$\sup_{t \geq 0, \kappa, \kappa_1 \in K} \frac{|A_t(\kappa, \beta) - A_t(\kappa_1, \beta)|}{|\kappa - \kappa_1|^\epsilon} < \infty, \text{ a.s.}.$$  

Due to this fact we have, for fixed $x$,

$$II = \frac{1}{2\kappa} \text{Re} \left( A_L(\kappa + \frac{x}{L}, 2) - A_L(\kappa, 2) \right) = O(L^{-\epsilon}), \text{ a.s..}$$

It then suffices to use Lemma 3.2 stated below.

**Lemma 3.2** Let $\Phi_1, \Phi_2, \ldots$, and $\Phi$ be the non-decreasing functions s.t. $\Phi_n(x) \to \Phi(x)$ for $x \in \mathbb{Q}$. Then $\Phi_n(x) \to \Phi(x)$ for any continuity point $x \in \mathbb{R}$ of $\Phi$. 

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3.2 Proof of Theorem 1.1

We sometimes use the following elementary lemma.

**Lemma 3.3** Let $\Psi_n, n = 1, 2, \cdots$, and $\Psi$ are continuous and increasing functions such that $\lim_{n \to \infty} \Psi_n(x) = \Psi(x)$ pointwise. If $y_n \in \text{Ran} \Psi_n$, $y \in \text{Ran} \Psi$ and $y_n \to y$, then it holds that

$$\Psi^{-1}_n(y_n) \xrightarrow{n \to \infty} \Psi^{-1}(y).$$

**Proof of Theorem 1.1**

$\tilde{\theta}_t(\kappa) \xrightarrow{t \to \infty} \tilde{\theta}_\infty(\kappa) + o(1)$ by [3] Proposition 2.1 and $\kappa L_j = m_j \pi + \beta + o(1)$ for some $m_j \in \mathbb{N}$ by Assumption (A). Thus we have $\theta_{L_j}(\kappa) = m_j \pi + \beta + \tilde{\theta}_\infty(\kappa) + o(1)$ and hence

$$\lim_{j \to \infty} \phi(\kappa^2, L_j) = \phi_{\beta} := \left(\tilde{\theta}_\infty(\kappa) + \beta\right)_{\pi \mathbb{Z}}, \quad \text{a.s..} \quad (3.1)$$

By (3.1) and Proposition 3.1, the assumption for Lemma 3.3 is satisfied. Thus we use Lemma 2.1 and the fact that $\lim_{L \to \infty} (n(L) - m(\kappa^2, L)) = -\infty$ to conclude that

$$E[e^{-\xi_{L_j}(f)}] = E \left[ \exp \left( - \sum_{n=n(L_j)-m(\kappa^2,L_j)}^{\infty} f(\Psi^{-1}_{L_j}(n\pi - \phi(\kappa^2, L_j))) \right) \right]$$

$$j \to \infty \quad E \left[ \exp \left( - \sum_{n \in \mathbb{Z}} f(n\pi - \phi_{\beta}) \right) \right]$$

$$= \int d\mu_{\beta}(\phi) \exp \left( - \sum_{n \in \mathbb{Z}} f(n\pi - \phi) \right)$$

for $f \in C_c^1(\mathbb{R})$ where $\mu_{\beta}$ is the distribution of $\phi_{\beta}$. □

4 Second Limit Theorem

4.1 Behavior of eigenvalues near $E_0$

**Lemma 4.1** Assume (B) and let $n \in \mathbb{Z}$. Then for $j \to \infty$ we have

1. $\sqrt{E_{m_j+n}(L_j)} = \kappa + o(1)$
2. $\sqrt{E_{m_j+n}(L_j)} = \kappa + \frac{n\pi - \beta - \tilde{\theta}_\infty(\kappa)}{L_j} + o(L_j^{-1})$. 

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Proof. (1) This easily follows from the two equations given below.

\[(m_j + n)\pi = \theta_{L_j}(\sqrt{E_{m_j+n}(L_j)}) = \sqrt{E_{m_j+n}(L_j)L_j + \theta_{L_j}(\sqrt{E_{m_j+n}(L_j)})} \quad (4.1)\]

\[(m_j + n)\pi = \kappa L_j - \beta + n\pi + o(1). \quad (4.2)\]

(2) We substitute Lemma 4.1(1) into the last term in the RHS of (4.1). Since the convergence \(\tilde{\theta}_t(\kappa) \to \theta_\infty(\kappa)\) holds compact uniformly with respect to \(\kappa\) we have

\[(m_j + n)\pi = \sqrt{E_{m_j+n}(L_j)L_j + \theta_\infty(\kappa) + o(1)}. \quad (4.3)\]

Lemma 4.1(2) follows from (4.2) and (4.3).

By taking the difference between

\[(m_j + n + 1)\pi = \sqrt{E_{m_j+n+1}(L_j)} \cdot L_j + \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n+1}(L_j)})\]

\[(m_j + n)\pi = \sqrt{E_{m_j+n}(L_j)} \cdot L_j + \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n}(L_j)})\]

we see that

\[X_j(n) = -L_j^{\alpha - \frac{1}{2}} \left( \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n+1}(L_j)}) - \tilde{\theta}_{L_j}(\sqrt{E_{m_j+n}(L_j)}) \right). \]

By Lemma 4.1(2)

\[\sqrt{E_{m_j+n+1}(L_j)} = \kappa + c_1/L_j, \quad \sqrt{E_{m_j+n}(L_j)} = \kappa + c_2/L_j\]

\[c_1 = (n + 1)\pi - \beta - \tilde{\theta}_\infty(\kappa) + o(1)\]

\[c_2 = n\pi - \beta - \tilde{\theta}_\infty(\kappa) + o(1), \quad j \to \infty.\]

We set

\[\Theta_t^{(n)}(c_1, c_2) := \left( \tilde{\theta}_{nt}(\kappa + c_1/n) - \tilde{\theta}_{nt}(\kappa + c_2/n) \right) n^{\alpha - \frac{1}{2}}\]

\[l_t((c_1, c_2), (c_1', c_2')) := \frac{C(\kappa^2)}{8\kappa^2} \int_0^t s^{-2\alpha} |\text{Re}\left(e^{2ic_1s} - e^{2ic_2s}\right)| (e^{2ic_1's} - e^{2ic_2's}) ds.\]

When \(c_1, c_2\) are constant, the following fact is proved in [1] Lemma 3.1.

**Proposition 4.2** \(\{\Theta_t^{(n)}(c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbb{R}} \overset{d}{\to} \{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbb{R}}\) as \(n \to \infty\) where \(\{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \in \mathbb{R}}\) is the Gaussian system with covariance \(l_t(\nu((c_1, c_2), (c_1', c_2'))).\)
4.2 Independence of the limits

To finish the proof of Theorem 1.2, it is then sufficient to prove that $(\tilde{\theta}_{nt}(\kappa), (\Theta_t^{(n)}(c_1, c_2))_{c_1, c_2})$ converges jointly to the independent ones. Let $0 < \kappa_1 < \kappa_2$ and $I := [\kappa_1, \kappa_2]$. In the following lemma, we regard $\tilde{\theta}_t, \tilde{\theta}_\infty$ are $C(I)$-valued random elements.

**Lemma 4.3** For $t > 0$ fixed, we have

$$(\tilde{\theta}_{nt}, \{\Theta_t^{(n)}(c_1, c_2)\}_{c_1, c_2}) \xrightarrow{d} (\tilde{\theta}_\infty, \{Z(t, c_1, c_2)\}_{c_1, c_2})$$

as $n \to \infty$ where $\tilde{\theta}_\infty$ and $\{Z(t, c_1, c_2)\}_{c_1, c_2}$ are independent.

**Proof.** Let $A(\subset C(I))$ be a $\tilde{\theta}_\infty$-continuity set (i.e., $P(\tilde{\theta}_\infty \in \partial A) = 0$) and set $A_{\epsilon} := \{f \in C(I) \mid d(f, A) < \epsilon\}$. Since $\theta_t(\kappa)$ a.s. $\to \tilde{\theta}_\infty(\kappa)$ compact uniformly in $\kappa$, for any $\epsilon > 0$

$$P(\tilde{\theta}_{nt} \in A, \tilde{\theta}_T \notin A_{\epsilon}) = o(1)$$

for sufficiently large $T, n$.

Here we recall eq.(3.3) in [4].

$$\Theta_t^{(n)}(c_1, c_2) = T_t^{(n)}(c_1, c_2) + O(n^{\frac{1}{2} - \alpha})$$

where $T_t^{(n)}(c_1, c_2) := n^{\alpha - \frac{1}{2}} \text{Re} \left( S_t^{(n)}(\kappa + \frac{c_1}{n}) - S_t^{(n)}(\kappa + \frac{c_2}{n}) \right)$

$$S_t^{(n)}(\kappa) := \frac{1}{2\kappa} \int_0^{nt} a(s) e^{2i\tilde{\theta}_s(\kappa)} dM_s(\kappa)$$

$M_s(\kappa)$ is the complex martingale defined in subsection 6.2.

Let $m \in \mathbb{N}$. For $c_1 = (c_1^{(1)}, \ldots, c_1^{(m)})$, $c_2 = (c_2^{(1)}, \ldots, c_2^{(m)})$, we use the following convention: $\Theta_t^{(n)}(c_1, c_2) = (\Theta_t^{(n)}(c_1^{(1)}, c_2^{(1)}), \ldots, \Theta_t^{(n)}(c_1^{(m)}, c_2^{(m)}))$ and similarly for $T_t^{(n)}(c_1, c_2)$ and $Z(t, c_1, c_2)$. Let $B \in \mathcal{B}(\mathbb{R}^m)$ be a $Z(t, c_1, c_2)$-continuity set and let $B_{\epsilon} := \{x \in \mathbb{R}^m \mid d(x, B) < \epsilon\}$. Writing $\Theta_t^{(n)} = \Theta_t^{(n)}(c_1, c_2)$, $T_t^{(n)} = T_t^{(n)}(c_1, c_2)$ we have, for sufficiently large $n$,

$$P(\tilde{\theta}_{nt} \in A, \Theta_t^{(n)} \in B) \leq P(\tilde{\theta}_T \in A_{\epsilon}, T_t^{(n)} \in B_{\epsilon}) + o(1)$$

$$= P(\tilde{\theta}_T \in A_{\epsilon}, T_t^{(n)} - T_{T/n}^{(n)} + T_{T/n}^{(n)} \in B_{\epsilon}) + o(1)$$

$$= P(\tilde{\theta}_T \in A_{\epsilon}, T_t^{(n)} - T_{T/n}^{(n)} \in B_{2\epsilon}) + o(1).$$
Here we used $T_{T/n}^{(n)} \xrightarrow{P} 0$. By the Markov property
\[ = \mathbb{E} \left[ 1_{\{\tilde{\theta}_T \in A\}} \mathbb{E}_{X_T} \left[ 1_{\{\tilde{T}_T^{(n)} \in B_{2\epsilon}\}} \right] \right] + o(1) \]
where $\tilde{T}_T^{(n)}$ is the suitable “time-shift” of $T_T^{(n)}$. Because $\tilde{T}_T^{(n)}$ converges in distribution to the Gaussian $Z(t, c_1, c_2)$ as $n \to \infty$ being irrespective of $X_T$,
\[ = \mathbb{P} \left( \tilde{\theta}_T \in A \right) \mathbb{P} \left( Z(t, c_1, c_2) \in B_{2\epsilon} \right) + o(1) \]
\[ \leq \mathbb{P} \left( \tilde{\theta}_\infty \in A_{2\epsilon} \right) \mathbb{P} \left( Z(t, c_1, c_2) \in B_{2\epsilon} \right) + o(1). \]
Since $A$ is a $\tilde{\theta}_\infty$-continuity set and $B \in \mathcal{B}(\mathbb{R}^m)$ is a $Z(t, c_1, c_2)$-continuity set,
\[ \limsup_{n \to \infty} \mathbb{P} \left( \tilde{\theta}_{nt} \in A, \Theta_T^{(n)} \in B \right) \leq \mathbb{P} (\tilde{\theta}_\infty \in A) \mathbb{P} (Z(t, c_1, c_2) \in B). \]
The opposite inequality can be proved similarly. \(\square\)

5 SC-case : outline of proof of Theorem 1.3

In this section we overview the proof of Theorem 1.3. First of all, set
\[ (x)_{2\pi \mathbb{Z}} := x - \lfloor x \rfloor_{2\pi \mathbb{Z}}, \quad [x]_{2\pi \mathbb{Z}} := \max\{y \in 2\pi \mathbb{Z} | y \leq x\} \]
\[ 2m(\kappa^2, L)\pi := [2\theta_L(\kappa)]_{2\pi \mathbb{Z}} \]
\[ \phi(\kappa^2, L) := (2\theta_L(\kappa))_{2\pi \mathbb{Z}} \in [0, 2\pi) \]
We also set “the relative Prüfer phase” by
\[ \Phi_L(x) := 2\theta_L(\kappa + \frac{x}{L}) - 2\theta_L(\kappa). \]
Then we have a variant of Lemma 2.1.

Lemma 5.1 For $f \in C_c^+(\mathbb{R})$
\[ \mathbb{E} \left[ e^{-\xi_L(f)} \right] = \mathbb{E} \left[ \exp \left( - \sum_{n=n(L)-m(\kappa^2, L)}^{\infty} f \left( \Phi_L^{-1}(2n\pi - \phi(\kappa^2, L)) \right) \right) \right]. \]

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So our task is to study the limit of the joint distribution of \((\Phi_L, \phi(\kappa^2, L))\) as \(L \to \infty\). Following [2] we consider
\[
\Psi_t(n)(x) := 2\theta_n(t)\left(\kappa + \frac{x}{n}\right) - 2\theta_n(t),
\]
regard it as an increasing function-valued process, and find a process \(\Psi_t(x)\) such that for any fixed \(c_1, \ldots, c_m \in \mathbb{R}\) \(\{\Psi_t(n)(c_j)\}_{j=1}^m \overset{d}{\to} \{\Psi_t(c_j)\}_{j=1}^m\) (Theorem 6.10). \(\Psi_t\) is characterized as the unique solution to the following SDE.
\[
d\Psi_t(c) = 2cdt + \frac{2}{\sqrt{\beta t}} Re [(e^{i\Psi_t(c)} - 1)dZ_t] \tag{5.1}
\]
\[
\Psi_0(c) = 0, \quad \beta = \frac{8\kappa^2}{C(\kappa^2)}.
\]
Moreover, \(\Psi_t(c)\) is continuous and increasing with respect to \(c\) (Lemma 6.11).

On the other hand we have \(\{\Psi_t(n)(c_j)\}_{j=1}^m, \phi(\kappa^2, n) \overset{d}{\to} \{\Psi_t(c_j)\}_{j=1}^m, \phi_1\) jointly, where \(\phi_1\) is uniformly distributed on \([0, 2\pi]\) and independent of \(\Psi_1\) (Proposition 9.1). Moreover \(\Psi_t(n)\) converges to \(\Psi_t\) also as a sequence of increasing function-valued process (Lemma 9.3), so that we can find a coupling such that for a.s. \((\Psi_1(n)^{-1}(x), \phi(\kappa^2, n)) \to (\Psi_1^{-1}(x), \phi_1)\) for any \(x \in \mathbb{R}\) (Proposition 9.2). Therefore
\[
\lim_{L \to \infty} \mathbb{E}[e^{-\xi_L(f)}] = \mathbb{E} \left[ \int_0^{2\pi} \frac{d\phi_1}{2\pi} \exp \left( - \sum_{n \in \mathbb{Z}} f \left( \Psi_1^{-1}(2n\pi - \phi_1) \right) \right) \right]
\]
which coincides with what is derived in [2] (except that the drift term \(cdt\) in [2] is replaced by \(2cdt\) that is why we need to consider a scaling : \(\xi^t = \sum_{\lambda_n/2}^{n}\) thereby identifying the limit of \(\xi_L\) with that of the circular \(\beta\)-ensemble.

6  Convergence of \(\Psi\)

6.1  Preliminaries

For \(f \in C^{\infty}(M)\) let \(R_\beta f := (L + i\beta)^{-1}f (\beta > 0), Rf := L^{-1}(f - \langle f \rangle)\). Then by Ito’s formula,
\[
\int_0^t e^{i\beta s}f(X_s)ds = [e^{i\beta s}(R_\beta f)(X_s)]_0^t + \int_0^t e^{i\beta s}dM_s(f, \beta)
\]
\[
\int_0^t f(X_s)ds = \langle f \rangle t + [(Rf)(X_s)]_0^t + M_t(f, 0)
\]
where \( M_s(f, \beta), M_s(f, 0) \) are the complex martingales whose variational process satisfy

\[
\langle M(f, \beta), M(f, \beta) \rangle_t = \int_0^t [R_\beta f, R_\beta f](X_s)ds,
\]
\[
\langle M(f, \beta), M(f, \beta) \rangle_t = \int_0^t [\overline{R_\beta f}, R_\beta f](X_s)ds
\]
\[
\langle M(f, 0), M(f, 0) \rangle_t = \int_0^t [Rf, Rf](X_s)ds,
\]
\[
\langle M(f, 0), M(f, 0) \rangle_t = \int_0^t [\overline{Rf}, Rf](X_s)ds
\]

where

\[
[f_1, f_2](x) := L(f_1 f_2)(x) - (Lf_1)(x)f_2(x) - f_1(x)(Lf_2)(x)
= (\nabla f_1, \nabla f_2)(x).
\]

Then the integration by parts gives us the following formulas to be used frequently.

**Lemma 6.1**

1. \[
\int_0^t b(s)e^{i\gamma \theta_s} f(X_s)ds
\]
   \[
   = \left[ b(s)e^{i\gamma \theta_s} e^{i\beta s}(R_\beta f)(X_s) \right]_0^t - \int_0^t b'(s)e^{i\gamma \theta_s} e^{i\beta s}(R_\beta f)(X_s)ds
   \]
   \[
   - \frac{i\gamma}{2\kappa} \int_0^t b(s)a(s)Re(e^{2\theta_s} - 1)e^{i\gamma \theta_s} e^{i\beta s} F(X_s)(R_\beta f)(X_s)ds
   \]
   \[
   + \int_0^t b(s)e^{i\beta s} e^{i\gamma \theta_s} dM_s(f, \beta).
   \]

2. \[
\int_0^t b(s)e^{i\gamma \theta_s} f(X_s)ds
\]
   \[
   = \langle f \rangle \int_0^t b(s)e^{i\gamma \theta_s} ds
   \]
   \[
   + \left[ b(s)e^{i\gamma \theta_s}(Rf)(X_s) \right]_0^t - \int_0^t b'(s)e^{i\gamma \theta_s}(Rf)(X_s)ds
   \]

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\[-\frac{i\gamma}{2\kappa} \int_0^t a(s)b(s)\text{Re}(e^{2i\theta_s} - 1)e^{i\tilde{\theta}_s}F(X_s)(R_f)(X_s)ds + \int_0^t b(s)e^{i\tilde{\theta}_s}dM_s(f,0)\].

We will also use following notation for simplicity.

\[g_\kappa := (L + 2i\kappa)^{-1}F, \quad g := L^{-1}(F - \langle F \rangle),\]
\[M_s(\kappa) := M_s(F, 2\kappa), \quad M_s := M_s(F, 0).\]

6.2 A priori estimates

In this section we derive a priori estimate for the following quantity

\[\Psi_{t}^{(n)}(c) := 2\theta_{nt}(\kappa + \frac{c}{n}) - 2\theta_{nt}(\kappa).\]

By \((2.2)\) we have

\[\Psi_t^{(n)}(c) = 2ct + \frac{1}{(\kappa + \frac{c}{n})}\int_0^{nt} \text{Re}(e^{2i\theta_s(\kappa + \frac{c}{n})} - 1)a(s)F(X_s)ds - \frac{1}{\kappa}\int_0^{nt} \text{Re}(e^{2i\theta_s(\kappa)} - 1)a(s)F(X_s)ds = 2ct - \frac{\frac{c}{n}}{\kappa(\kappa + \frac{c}{n})}\int_0^{nt} \text{Re}(e^{2i\theta_s(\kappa + \frac{c}{n})} - 1)a(s)F(X_s)ds + \frac{1}{\kappa}Re \int_0^{nt} (e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)})a(s)F(X_s)ds \quad (6.1)\]

and the third term in the RHS will be dominant.

**Lemma 6.2** Suppose

\[\int_0^\infty a(s)^3ds < \infty\]

we then have

(1)

\[-\frac{i}{2\kappa} \int_0^t a(s)^2g_\kappa(X_s)F(X_s)ds = -\frac{i}{\kappa} \int_0^t a(s)^2g_\kappa(X_s)F(X_s)ds + Y_t(\kappa) + \delta_t(\kappa)\]
where

\[ Y_t(\kappa) := \int_0^t a(s)e^{2i\theta_s(\kappa)}dM_s(\kappa) \]

\[ \delta_t(\kappa) := [a(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s)]_0^t - \int_0^t a'(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s)ds \]

\[ - \frac{i}{\kappa} \int_0^t a(s)^2e^{2i\theta_s(\kappa)} \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) g_\kappa(X_s)F(X_s)ds. \]

(2) For a.s., \( \delta_t(\kappa) \) has the limit as \( t \to \infty \) \( F \lim_{t \to \infty} \delta_t(\kappa) = \delta_\infty(\kappa) \), a.s.

(3) For any \( 0 < T < \infty \), we have

\[ \mathbb{E} \left[ \max_{0 \leq t \leq T} \left| \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right|^2 \right] \xrightarrow{n \to \infty} 0. \]

Proof. (1) By Lemma 6.1 (1)

\[ \int_0^t a(s)e^{2i\theta_s(\kappa)}F(X_s)ds \]

\[ = \left[ a(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s) \right]_0^t - \int_0^t a'(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s)ds \]

\[ - \frac{2i}{2\kappa} \int_0^t a(s)^2 \text{Re}(e^{2i\theta_s(\kappa)} - 1)e^{2i\theta_s(\kappa)}g_\kappa(X_s)F(X_s)ds \]

\[ + \int_0^t a(s)e^{2i\theta_s(\kappa)}dM_s(\kappa) \]

which we decompose into “non-oscillating” term + martingale-term + remainder.

\[ = - \frac{i}{2\kappa} \int_0^t a(s)^2F(X_s)g_\kappa(X_s)ds + Y_t(\kappa) + \delta_t(\kappa) \]

where the remainder term \( \delta_t(\kappa) \) is further decomposed for later use

\[ \delta_t(\kappa) = \delta_t^{(1)}(\kappa) + \delta_t^{(2)}(\kappa) \]

\[ \delta_t^{(1)}(\kappa) := [a(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s)]_0^t - \int_0^t a'(s)e^{2i\theta_s(\kappa)}g_\kappa(X_s)ds \]

\[ \delta_t^{(2)}(\kappa) := - \frac{i}{\kappa} \int_0^t a(s)^2 \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) e^{2i\theta_s(\kappa)}g_\kappa(X_s)F(X_s)ds. \]
Lemma 6.2(1) is proved.

(2) It is easy to see
\[
\lim_{t \to \infty} \delta_t^{(1)}(\kappa) = \delta_{\infty}^{(1)}(\kappa), \ a.s.
\]

To see the convergence of \(\delta_t^{(2)}(\kappa)\) we write
\[
\delta_t^{(2)}(\kappa) = -\frac{i}{2\kappa} D_t^{(4)}(\kappa) + \frac{i}{\kappa} D_t^{(2)}(\kappa)
\]

\[
D_t^{(\beta)}(\kappa) := \int_0^t a(s)^2 e^{i\beta\theta_s(k)} F(X_s) g_\kappa(X_s) ds, \ \beta = 2, 4.
\]

We use Lemma 6.1(1) to decompose \(D_t^{(\beta)}(\kappa)\) into martingale part and the remainder: Setting \(h_{\kappa, \beta} = R_{\kappa, \beta}(F g_\kappa)\) and \(\tilde{M}_s^{(\beta)}(\kappa) = M_s(F g_\kappa, \beta \kappa)\), we have
\[
D_t^{(\beta)}(\kappa) = I_t^{(\beta)}(\kappa) + N_t^{(\beta)}(\kappa)
\]

\[
I_t^{(\beta)}(\kappa) := [a(s)^2 e^{i\beta\theta_s(k)} h_{\kappa, \beta}(X_s)]_0^t - \int_0^t (a(s)^2)' e^{i\beta\theta_s(k)} h_{\kappa, \beta}(X_s) ds
\]

\[
- \frac{i\beta}{2\kappa} \int_0^t a(s)^3 Re(e^{2i\theta_s(k)} - 1)e^{i\beta\theta_s(k)} F(X_s) h_{\kappa, \beta}(X_s) ds
\]

\[
N_t^{(\beta)}(\kappa) := \int_0^t a(s)^2 e^{i\beta\theta_s(k)} d\tilde{M}_s^{(\beta)}(\kappa).
\]

\(I_t^{(\beta)}(\kappa)\) is easily seen to be convergent: \(\lim_{t \to \infty} I_t^{(\beta)}(\kappa) = I_{\infty}^{(\beta)}(\kappa), \ a.s.\). Since

\[
|\langle N^{(\beta)}, N^{(\beta)} \rangle_t|, \quad |\langle N^{(\beta)}, \overline{N^{(\beta)}} \rangle_t| \leq (\text{const.}) \int_0^t a^4(s) ds < \infty
\]

Re \(N\), Im \(N\) can be represented by the time-change of a Brownian motion and thus have limit a.s..

(3) We consider \(\delta_t^{(1)}(\kappa), \delta_t^{(2)}(\kappa)\) separately. For \(\delta_t^{(1)}(\kappa)\), we have
\[
\delta_{nt}^{(1)}(\kappa + \frac{c}{n}) - \delta_{nt}^{(1)}(\kappa)
\]

\[
= a(nt) \left( e^{2i\theta_{nt}(\kappa + \frac{c}{n})} - e^{2i\theta_{nt}(\kappa)} \right) g_{\kappa + \frac{c}{n}}(X_{nt})
\]

\[
- \int_0^{nt} a'(s) \left( e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)} \right) g_{\kappa + \frac{c}{n}}(X_s) ds + O(n^{-1})
\]

by (5.3). The second term is \(o(1)\) as \(n \to \infty\) due to the Lebesgue’s dominated convergence theorem. Thus the following equation will give us \(E[|\delta_{nt}^{(1)}(\kappa +
$$c_n \to 0 \quad \text{as} \quad n \to \infty,$$

$$\delta_{nt}(\kappa)^2 \to 0.$$ 

\begin{equation}
\max_{0 \leq t \leq T} a(nt)|e^{2i\theta_{nt}(\kappa + c_n)} - e^{2i\theta_{nt}(\kappa)}| \to 0. \tag{6.6}
\end{equation}

Take any $M > 0$.

$$\max_{0 \leq t \leq M} a(nt)|e^{2i\theta_{nt}(\kappa + c_n)} - e^{2i\theta_{nt}(\kappa)}|$$

$$= \max_{0 \leq t \leq M/n} a(nt)|e^{2i\theta_{nt}(\kappa + c_n)} - e^{2i\theta_{nt}(\kappa)}| \vee \max_{M/n \leq t \leq T} a(nt)|e^{2i\theta_{nt}(\kappa + c_n)} - e^{2i\theta_{nt}(\kappa)}|$$

$$\leq C M \max_{0 \leq t \leq M} |e^{2i\theta_{t}(\kappa + c_n)} - e^{2i\theta_{t}(\kappa)}| \vee 2a(M).$$

By (2.1)-(2.3) we have

$$\max_{0 \leq t \leq M} |e^{2i\theta_{t}(\kappa + c_n)} - e^{2i\theta_{t}(\kappa)}| \leq C M/n. \tag{6.7}$$

for some positive constant $C_M$ depending on $M$. Hence

$$\limsup_{n \to \infty} \max_{0 \leq t \leq T} a(nt)|e^{2i\theta_{nt}(\kappa + c_n)} - e^{2i\theta_{nt}(\kappa)}| \leq 2a(M).$$

Since $M$ is arbitrary, we obtain (6.6).

Similar argument shows $\max_{0 \leq t \leq T} \left| I_{nt}^{(\beta)}(\kappa + c_n) - I_{nt}^{(\beta)}(\kappa) \right| \to 0$ so that we have only to show

$$\mathbb{E} \left[ \max_{0 \leq t \leq T} \left| N_{nt}^{(\beta)}(\kappa + c_n) - N_{nt}^{(\beta)}(\kappa) \right|^2 \right] \to 0, \quad \beta = 2, 4$$

to finish the proof of Lemma 6.2(3). By the martingale inequality,

$$\mathbb{E} \left[ \max_{0 \leq t \leq T} \left| N_{nt}^{(\beta)}(\kappa + c_n) - N_{nt}^{(\beta)}(\kappa) \right|^2 \right]$$

$$\leq C \mathbb{E} \left[ \int_0^{nt} a(s)^4 \left[ e^{i\beta\theta_s(\kappa + c_n)} h_{\beta,\kappa + c_n} - e^{i\beta\theta_s(\kappa)} h_{\beta,\kappa} + e^{i\beta\theta_s(\kappa + c_n)} h_{\beta,\kappa + c_n} - e^{i\beta\theta_s(\kappa)} h_{\beta,\kappa} \right] ds \right]$$

which converges to 0 due to the fact that $\int_0^{\infty} a(s)^4 ds < \infty$ and Lebesgue’s theorem.

We next set

$$V_{t}^{(n)}(c) := Y_{nt}(\kappa + c) - Y_{nt}(\kappa)$$
and assume in what follows
\[ a(t) = t^{-1/2}(1 + o(1)). \]

**Lemma 6.3**

\[ \Psi_t^{(n)}(c) = 2ct + \text{Re} \epsilon_t^{(n)} + \frac{1}{\kappa} \text{Re} \ V_t^{(n)}(c) + \frac{1}{\kappa} \text{Re} \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right) \]  

(6.8)

for some \( \epsilon_t^{(n)} \) satisfying

\[ |\epsilon_t^{(n)}| \leq Ct + C \sqrt{\frac{t}{n}}. \]

**Proof.** We compute the third term of (6.1) by using Lemma 6.2

\[
\int_0^{nt} \left( e^{2i\theta_s(\kappa + \frac{c}{n})} - e^{2i\theta_s(\kappa)} \right) a(s)F(X_s) ds
\]

\[ = \frac{i}{2} \cdot \frac{c}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} a(s)^2 g_{\kappa + \frac{c}{n}}(X_s)F(X_s) ds \]

\[ + \frac{i}{2\kappa} \int_0^{nt} a(s)^2 (g_{\kappa}(X_s) - g_{\kappa + \frac{c}{n}}(X_s))F(X_s) ds \]

\[ + V_t^{(n)}(c) + \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa). \]

Therefore

\[ \Psi_t^{(n)}(c) = 2ct + \text{Re} \epsilon_t^{(n)} + \frac{1}{\kappa} \text{Re} \ V_t^{(n)}(c) + \frac{1}{\kappa} \text{Re} \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right) \]

where

\[ \epsilon_t^{(n)} := -\frac{c}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} \left( e^{2i\theta_s(\kappa)} - 1 \right) a(s)F(X_s) ds \]

\[ + \frac{1}{\kappa} \left\{ \frac{i}{2} \cdot \frac{c}{\kappa(\kappa + \frac{c}{n})} \int_0^{nt} a(s)^2 g_{\kappa + \frac{c}{n}}(X_s)F(X_s) ds \right\} \]

\[ + \frac{i}{2\kappa} \int_0^{nt} a(s)^2 \left( g_{\kappa}(X_s) - g_{\kappa + \frac{c}{n}}(X_s) \right) F(X_s) ds \}

It then suffices to see

\[ |\epsilon_t^{(n)}| \leq \frac{C}{n} \int_0^{nt} a(s) ds \leq Ct + C \sqrt{\frac{t}{n}}. \]
Lemma 6.4

\[ \mathbb{E}[|\Psi_t^{(n)}(c)|] \leq C \left( t + \sqrt{\frac{t}{n}} + \frac{1}{\sqrt{n}} \right), \quad t \geq 0, \quad n > 0. \]

Proof. We decompose \( \delta_t(\kappa) \) as is done in (6.4) to estimate \( \delta_t(\kappa) \) further.

\( \delta_t(\kappa) = \delta_t^{(1)}(\kappa) + \delta_t^{(2)}(\kappa). \)  \hspace{1cm} (6.9)

Let

\[ \Lambda_t^{(n)}(c) := e^{2i\theta_t(\kappa + \frac{c}{n})} - e^{2i\theta_t(\kappa)} \]

then

\( \delta_t^{(1)}(\kappa + \frac{c}{n}) - \delta_t^{(1)}(\kappa) = \Lambda_t^{(n)}(c) a(nt) g_{\kappa + \frac{c}{n}}(X_{nt}) \]

\[ - \int_0^{nt} a'(s) g_{\kappa + \frac{c}{n}}(X_s) \Lambda_s^{(n)}(c) ds + O(n^{-1}). \]  \hspace{1cm} (6.10)

\( \delta_t^{(2)} \) is also decomposed, as in (6.4), (6.5)

\[ \delta_t^{(2)}(\kappa) = -\frac{i}{2\kappa} D_t^{(4)}(\kappa) + \frac{i}{\kappa} D_t^{(2)}(\kappa) \]  \hspace{1cm} (6.11)

\[ D_t^{(\beta)}(\kappa) = I_t^{(\beta)}(\kappa) + N_t^{(\beta)}(\kappa), \quad \beta = 2, 4. \] \hspace{1cm} (6.12)

The \( I_t^{(\beta)} \)-term can be written as

\[ I_t^{(\beta)}(\kappa + \frac{c}{n}) - I_t^{(\beta)}(\kappa) = a(nt)^2 h_{\kappa, n}(nt) \Lambda_t^{(n)}(c) \]

\[ - \int_0^{nt} (a(s)^2) f_{\kappa, n}(s) \Lambda_s^{(n)}(c) ds - \int_0^{nt} a(s)^3 g_{\kappa, n}(s) \Lambda_s^{(n)}(c) ds \] \hspace{1cm} (6.13)

for some bounded functions \( f_{\kappa, n}, g_{\kappa, n}, h_{\kappa, n} \). Substituting (6.10)-(6.13) into (6.9) we have

\[ \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) = \Lambda_{nt}^{(n)}(c) \left( a(nt) g_{\kappa + \frac{c}{n}}(X_{nt}) + a(nt)^2 h_{\kappa, n}(nt) \right) \]

\[ + \int_0^{nt} \Lambda_s^{(n)}(c) b_{\kappa, n}(s) ds + N_{nt}(\kappa + \frac{c}{n}) - N_{nt}(\kappa) + O(n^{-1}) \]

for some bounded functions \( h_{\kappa, n}, b_{\kappa, n} \) and a martingale \( N_t \). \( b_{\kappa, n}(s) \) is a linear combination of \( a'(s) g_{\kappa + \frac{c}{n}}, (a(s)^2) f_{\kappa, n}, \) and \( a(s)^3 g_{\kappa, n} \), so that it is integrable.
\[ \int_0^\infty b_{\kappa,n}(s) ds < \infty. \] Taking expectations, the martingale terms vanish and it follows that
\[
E \left[ \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right] = E \left[ \Lambda^{(n)}_{nt}(c) \left( a(nt)g_{\kappa,\frac{c}{n}}(X_{nt}) + a(nt)^2 h_{\kappa,n}(nt) \right) \right] \\
+ \int_0^{nt} E \left[ \Lambda^{(n)}_{s/n}(c)b_{\kappa,n}(s) \right] ds + O(n^{-1}).
\]

Therefore we can find a non-random function
\[ b(s) = C(a'(s) + (a(s)^2)' + a(s)^2) \]
for some \( C > 0 \) such that \( \int_0^\infty b(s) ds < \infty \) and
\[
|E \left[ \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right]| \leq Ca(nt)E[|\Lambda^{(n)}_{nt}(c)|] + \int_0^{nt} E[|\Lambda^{(n)}_{s/n}(c)||b_{\kappa,n}(s)|] ds + \frac{C}{n}.
\]

Here without loss of generality, we may suppose \( c \geq 0 \). We use \( \Psi^{(n)}_{t}(c) \geq 0 \) for \( c \geq 0 \) and take expectation in (6.8).
\[
E[|\Psi^{(n)}_{t}(c)|] = E[\Psi^{(n)}_{t}(c)] \\
= 2ct + E[Re \epsilon^{(n)}_{t}] + \frac{1}{\kappa} E \left[ Re \left( \delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa) \right) \right] \\
\leq Ct + C \sqrt{\frac{t}{n}} + Ca(nt)E[|\Lambda^{(n)}_{t}(c)|] \\
+ C \int_0^{nt} E \left[ |\Lambda^{(n)}_{s/n}(c)| b(s) \right] ds + \frac{C}{n}.
\]

Let
\[ \rho_n(t) := C \left( t + \sqrt{\frac{t}{n}} + \frac{1}{n} \right). \]

Since \( |\Lambda^{(n)}_{t}(c)| \leq |\Psi^{(n)}_{t}(c)| \), we have
\[
E \left[ |\Psi^{(n)}_{t}(c)| \right] \\
\leq \rho_n(t) + Ca(nt)E \left[ |\Psi^{(n)}_{t}(c)| \right] + C \int_0^{nt} E \left[ |\Psi^{(n)}_{s/n}(c)| b(s) \right] ds.
\]
Fix $M > 0$ arbitrary. We may suppose $nt > M$ since otherwise Lemma 6.4 holds true by (6.7). (6.7) also implies
\[
\int_0^M E \left[ \left| \Psi_{s/n}^{(n)} \right| b(s) ds \right] \leq 2 \int_0^M E \left[ \max_{0 \leq s \leq M} \left| \theta_s(\kappa + \frac{c}{n}) - \theta_s(\kappa) \right| \right] b(s) ds
\]
which gives us
\[
E \left[ \left| \Psi_t^{(n)}(c) \right| \right] 
\leq \rho_n(t) + Ca(M) E \left[ \left| \Psi_t^{(n)}(c) \right| \right] + C \int_M^{nt} E \left[ \left| \Psi_{s/n}^{(n)}(c) \right| \right] b(s) ds + \frac{C}{n}.
\]
Take $M$ large enough such that $Ca(M) < 1$ and renew the positive constant $C$ in the definition of $\rho_n(t)$. Then we have
\[
E \left[ \left| \Psi_t^{(n)}(c) \right| \right] \leq \rho_n(t) + C \int_{M/n}^t E \left[ \left| \Psi_{s/n}^{(n)}(c) \right| \right] nb(ns) ds.
\]
By Grownwall’s inequality,
\[
E \left[ \left| \Psi_t^{(n)}(c) \right| \right] \leq \rho_n(t) + C \int_{M/n}^t \rho_n(s)nb(ns) \exp \left( C \int_s^t nb(nu) du \right) ds.
\]
Since $b$ is integrable, $\exp \left( C \int_s^t nb(nu) du \right)$ is bounded so that
\[
E \left[ \left| \Psi_t^{(n)}(c) \right| \right] \leq \rho_n(t) + C \int_{M/n}^t \rho_n(s)nb(ns) ds.
\] 
(6.14)
Substituting
\[
\int_{M/n}^t \rho_n(s)nb(ns) ds = C \int_M^{nt} \left( \frac{s}{n} + \sqrt{\frac{s}{n^2} + \frac{1}{n}} \right) b(s) ds
\]
\[
\leq C \sqrt{n}
\]
into (6.14) yields the conclusion. □
Lemma 6.5 For $t > 0$, we have

$$E\left[ \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \right] \leq C t + o(1)$$

as $n \to \infty$. In particular, $\sup_n E\left[ \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \right] < \infty$.

Proof. A straightforward computation using Lemma 6.1(2) yields

$$\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t = \int_0^{nt} a(s)^2 |e^{2i(\theta_s(\kappa + \frac{\pi}{n}) - \theta_s(\kappa))} - 1|^2 \langle g_\kappa, g_\kappa \rangle(X_s)ds + o(1)$$

as $n \to \infty$. We take expectations and use Lemma 6.4

$$E\left[ \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \right] = Cn \int_0^t a(ns)^2 E \left[ |e^{i\Psi^{(n)}(c)} - 1|^2 \right] ds + o(1)$$

$$\leq Cn \int_0^t a(ns)^2 E \left[ |\Psi^{(n)}_s(c)| \right] ds + o(1)$$

$$\leq Cn \int_0^t a(ns)^2 \left( s + \sqrt{\frac{s}{n}} + \frac{1}{\sqrt{n}} \right) ds + o(1)$$

$$\leq C \left( t + \sqrt{\frac{t}{n}} + \frac{\log(nt)}{\sqrt{n}} \right) + o(1).$$

\[\square\]

Lemma 6.6 For each $c > 0$, $T > 0$ fixed we have

$$E \left[ \sup_{0 \leq t \leq T} \Psi^{(n)}_t(c) \right] \leq C \left( T + \sqrt{\frac{T}{n}} \right) + CT^{\frac{1}{2}} + o(1) + C E \left[ \max_{0 \leq t \leq T} |\delta_{nt}(\kappa + \frac{c}{n}) - \delta_{nt}(\kappa)| \right].$$

as $n \to \infty$. 

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Proof. We estimate the third term of (6.8) by the martingale inequality and use Lemma 6.5:

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |V_t^{(n)}(\kappa)| \right] \leq C \mathbb{E} \left[ |V_T^{(n)}(\kappa)| \right]^{1/2} \\
\leq C (T + o(1))^{1/2}.
\]

\[\square\]

Lemma 6.7 For each \(0 < t_0 < t_1 < \infty\), we can find \(C = C(t_0, t_1)\) such that for large \(n\), we have

\[
\mathbb{E} \left[ |V_t^{(n)}(c) - V_s^{(n)}(c)|^4 \right] \leq C (t - s)^2
\]

for any \(s, t \in [t_0, t_1]\).

Proof. By martingale inequality,

\[
\mathbb{E} \left[ |V_t^{(n)}(c) - V_s^{(n)}(c)|^4 \right] \leq C \mathbb{E} \left[ |V_t^{(n)}(c) - V_s^{(n)}(c)|^2 \right]^2 \\
\leq C \mathbb{E} \left[ \int_{n\kappa}^{nt} a(u)^2 \left[ e^{2i\theta_u(\kappa + \frac{\omega}{n})} g_{\kappa + \frac{\omega}{n}} - e^{2i\theta_u(\kappa)} g_{\kappa} e^{2i\theta_u(\kappa + \frac{\omega}{n})} g_{\kappa + \frac{\omega}{n}} - e^{2i\theta_u(\kappa)} g_{\kappa} \right] (X_u)du \right]^2 \\
\leq C \left( \int_{n\kappa}^{nt} a(u)^2 du \right)^2.
\]

We can find \(N = N(t_0)\) such that for \(n \geq N\)

\[
C \left( \int_{n\kappa}^{nt} a(u)^2 du \right)^2 \leq C \log \left( 1 + \frac{t - s}{t_0} \right)^2 \leq C (t - s)^2.
\]

\[\square\]

6.3 Tightness of \(\Psi\)

Lemma 6.8 For any \(c = (c_1, c_2, \cdots, c_m) \in \mathbb{R}^m\), the sequence of \(\mathbb{R}^m\)-valued process \(\{\Psi_t^{(n)}(c)\}_{n \geq 1} = \{(\Psi_t^{(n)}(c_1), \cdots, \Psi_t^{(n)}(c_m))\}_{n \geq 1}\) is tight as a family in \(C([0, T] \to \mathbb{R}^m)\).
Proof. It is sufficient to show

\( \lim_{A \to \infty} \sup_n P( |\Psi_t^{(n)}(c)| \geq A) = 0 \)

\( \lim \limsup_{\delta \downarrow 0} \limsup_{n \to \infty} P \left( \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) = 0, \quad T, \rho > 0. \)

(1) follows from Lemma 6.4. To prove (2), we fix \( M > 0 \) arbitrary and decompose

\[
P \left( \sup_{0 \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right)
\leq P \left( \sup_{0 \leq s, t \leq M, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) + P \left( \sup_{M \leq s, t \leq T, |t-s| < \delta} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right)
=: I + II.
\]

Since \( \Psi_0^{(n)}(c) = 0 \) we have

\[
I \leq P \left( \sup_{t \leq M} |\Psi_t^{(n)}(c)| > \frac{\rho}{2} \right) + P \left( \sup_{s \leq M} |\Psi_s^{(n)}(c)| > \frac{\rho}{2} \right)
\]

and we use Lemma 6.6

\[
P \left( \sup_{t \leq M} |\Psi_t^{(n)}(c)| > \frac{\rho}{2} \right)
\leq \frac{2}{\rho} E \left[ \sup_{0 \leq t \leq M} |\Psi_t^{(n)}(c)| \right]
\leq C \left( M + \sqrt{\frac{M}{n}} \right) + CM^{1/2} + o(1)
\]

as \( n \to \infty. \) By Lemma 6.2(3) the third term vanishes as \( n \to \infty \) and it holds that

\[
\limsup_{n \to \infty} I \leq CM^{1/2}. \quad (6.15)
\]
Thus following estimate will be sufficient

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \leq CM^{1/2}
\]

because (6.15), (6.16) would imply

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{0 \leq s, t \leq T, |t-s| < \delta} \left| V_t^{(n)}(c) - V_s^{(n)}(c) \right| > \rho = 0
\]

and the arbitrariness of \( M > 0 \) will yield the conclusion. By Lemmas 6.2, 6.3, eq. (6.16) will follow from the following equation

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{M \leq t, s \leq T, |t-s| < \delta} \left| W_t^{(c_j)} - W_s^{(c_j)} \right| > \rho = 0
\]

which, in turn, follows from Lemma 6.7 and Kolmogorov’s theorem.

6.4 SDE satisfied by \( \Psi \)

In this subsection we show that \( \Psi^{(n)} \) has a limit \( \Psi \) which satisfies (5.1).

**Lemma 6.9** For any \( c_1, \cdots, c_m \in \mathbb{R} \), the solution of the following martingale problem is unique:

\[
W_t(c_j) = \Psi_t(c_j) - 2c_j t, \quad j = 1, 2, \cdots, m
\]

are martingales whose variational process satisfy

\[
\langle W(c_i), W(c_j) \rangle_t = D^2 \int_0^t s^{-1} Re \left\{ (e^{i\Psi_s(c_i)} - 1) (e^{-i\Psi_s(c_j)} - 1) \right\} ds.
\]

Moreover \( \Psi_t(c_j) \) can be characterized by the unique solution of the following SDE.

\[
d\Psi_t = 2c_j dt + D \sqrt{t} Re \left[ (e^{i\Psi_t(c_j)} - 1) \right] dZ_t, \quad \Theta_0(c_j) = 0.
\]

**Theorem 6.10** For any \( c_1, \cdots, c_m \in \mathbb{R} \), \( (\Psi_t^{(n)}(c_1), \cdots, \Psi_t^{(n)}(c_m)) \to (\Psi_t(c_1), \cdots, \Psi_t(c_m)) \). \( \Psi_t(c_j) \) satisfies

\[
d\Psi_t(c_j) = 2c_j dt + D \sqrt{t} \left[ (e^{i\Psi_t(c_j)} - 1) \right] dZ_t \quad (6.17)
\]

\[
\Psi_0(c_j) = 0, \quad D := \sqrt{\frac{\langle g_\kappa, g_\kappa \rangle}{2\kappa}}.
\]
Proof. By Lemma 6.8, the sequence \( \{ (\Psi_t^{(n)}(c_1), \ldots, \Psi_t^{(n)}(c_m)) \}_{n \geq 1} \) has a limit point \((\Psi_t(c_1), \ldots, \Psi_t(c_m))\). Since Lemmas 6.2, 6.3 imply 
\[
\Psi_t^{(n)}(c) = 2ct + \frac{1}{\kappa} \text{Re} \ V_t^{(n)}(c) + o(1)
\]
in probability, we study \( V_t^{(n)}(c) \). By a computation using Lemma 6.1,
\[
\langle V_t^{(n)}(c), V_t^{(n)}(c') \rangle_t \xrightarrow{n \to \infty} 0
\]
in mean square. Similarly,
\[
\langle V_t^{(n)}(c), V_t^{(n)}(c') \rangle_t = \langle [g_\kappa, g_{\kappa}] \rangle \int_0^t a(s) \left( e^{2i(\theta_s(c) + \frac{c}{n})} - 1 \right) \left( e^{2i(\theta_s(c') + \frac{c'}{n})} - 1 \right) ds + o(1)
\]
\[
= \langle [g_\kappa, g_{\kappa}] \rangle \int_0^t na(nu)^2 \left( e^{i\Psi_u^{(n)}(c)} - 1 \right) \left( e^{i\Psi_u^{(n)}(c')} - 1 \right) du + o(1).
\]
By Skorohod’s theorem, we can suppose
\[
\Psi_t^{(n)}(c) \to \Psi_t(c)
\]
compact uniformly with respect to \( t \). Hence for \( 0 < s < t \),
\[
\langle V_t^{(n)}(c), V_t^{(n)}(c') \rangle_t - \langle V_s^{(n)}(c), V_s^{(n)}(c') \rangle_s
\]
\[
= \langle [g_\kappa, g_{\kappa}] \rangle \int_s^t na(nu)^2 \left( e^{i\Psi_u^{(n)}(c)} - 1 \right) \left( e^{i\Psi_u^{(n)}(c')} - 1 \right) du + o(1)
\]
\[
\xrightarrow{n \to \infty} \langle [g_\kappa, g_{\kappa}] \rangle \int_s^t u^{-1} \left( e^{i\Psi_u(c)} - 1 \right) \left( e^{i\Psi_u(c')} - 1 \right) du.
\]
On the other hand by Lemma 6.4 we have
\[
\int_0^t \mathbb{E} \left[ |e^{i\Psi_s(c)} - 1|^2 \right] \frac{ds}{s} \leq C \int_0^t \mathbb{E} \| \Psi_s(c) \| \frac{ds}{s} < \infty
\]
so that \( V_t(c) = \lim_{n \to \infty} V_t^{(n)}(c) \) is a square integrable continuous martingale whose variational process satisfy
\[
\langle V(c), V'(c) \rangle_t = 0
\]
\[
\langle V(c), V'(c) \rangle_t = \langle [g_\kappa, g_{\kappa}] \rangle \int_0^t \left( e^{i\Psi_s(c)} - 1 \right) \left( e^{i\Psi_s(c')} - 1 \right) \frac{ds}{s}.
\]
Therefore

\[ W_t(c) = \Psi_t(c) - 2ct = \frac{1}{\kappa} Re V_t(c) \]

is a square integrable continuous martingale whose variational process is equal to

\[ \langle W(c), W'(c) \rangle_t = \left\langle \left[ g_{\kappa}, \overline{g_{\kappa}} \right] \right\rangle \int_0^t Re \left[ (e^{i\Psi_s(c)} - 1)(e^{-i\Psi_s(c')} - 1) \right] \frac{ds}{s}. \]

Lemma 6.9 yields the conclusion. \( \square \)

**Lemma 6.11** For a.s., \( \Psi_t(c) \) is continuous on \([0, \infty) \times \mathbb{R} \) and is increasing with respect to \( c \).

**Proof.** We shall show the following inequality: for \( p > 1 \) sufficiently close to 1,

\[ E[|\Psi_t(c_1) - \Psi_t(c_2)|^p] \leq \frac{2^p(c_1 - c_2)^p}{1 - \frac{1}{2} (p - 1) D^2 t^p}. \quad (6.18) \]

Hence by Kolmogorov’s theorem, for any fixed \( t > 0 \), \( \Psi_t(c) \) has a continuous version with respect to \( c \in \mathbb{R} \) a.s.. We first note that \( \Psi_t(c) \) satisfies

\[ d\Psi_t(c) = 2cdt + \frac{D}{2\sqrt{t}} \left\{ (e^{i\Psi_t} + e^{-i\Psi_t} - 2)dB_t^1 + i(e^{i\Psi_t} - e^{-i\Psi_t})dB_t^2 \right\}. \]

Here we note that if \( c_1 > c_2 \) then \( \Psi_t(c_1) > \Psi_t(c_2) \) by the comparison theorem of SDE which proves the desired monotonicity of \( \Psi_t(c) \). We set

\[ \Gamma_t := \Psi_t(c_1) - \Psi_t(c_2) \]
\[ \Xi_t := e^{i\Psi_t(c_1)} - e^{i\Psi_t(c_2)}. \]

For \( c_1 > c_2 \), we see

\[ d\Gamma_t = 2(c_1 - c_2)dt + \frac{D}{2\sqrt{t}} \left\{ (\Xi_t + \overline{\Xi_t})dB_t^1 + i(\Xi_t - \overline{\Xi_t})dB_t^2 \right\}. \]

Hence

\[ (d\Gamma_t)^2 = \frac{D^2}{4t} \left\{ (\Xi_t + \overline{\Xi_t})^2 - (\Xi_t - \overline{\Xi_t})^2 \right\} dt = \frac{D^2}{t} |\Xi_t|^2 dt. \]
Then for $p > 1$

$$d\Gamma_t^p = p\Gamma_t^{p-1}d\Gamma_t + \frac{p(p-1)}{2}\Gamma_t^{p-2}(d\Gamma_t)^2$$

$$= 2(c_1 - c_2)p\Gamma_t^{p-1}dt + \frac{p(p-1)}{2}\Gamma_t^{p-2}D^2t|\Xi_t|^2dt$$

$$+ p\Gamma_t^{p-1}D \frac{t}{2\sqrt{t}} \{(\Xi_t + \Xi_t)dB^1_t + i(\Xi_t - \Xi_t)dB^2_t\}.$$

Taking expectation yields

$$E[\Gamma_t^p] = 2(c_1 - c_2)p \int_0^t E[\Gamma_s^{p-1}]ds + \frac{p(p-1)}{2}D^2 \int_0^t E[\Gamma_s^{p-2}]|\Xi_s|^2\frac{ds}{s}. \quad (6.19)$$

We have

$$|\Xi_t|^2 \leq C\Gamma_t^\gamma, \quad 0 < \gamma < 2$$

for some positive constant $C$ and some $0 < \gamma < 2$. Then

$$\int_0^t E[\Gamma_s^{p-2}]|\Xi_s|^2\frac{ds}{s} \leq C \int_0^t E[\Gamma_s^{p-2+\gamma}]\frac{ds}{s}.$$

We use $E[|X|^r] \leq E[|X|^r]$ for $r \leq 1$ and the fact that $E[\Psi_t(c)] = 2ct$. Assuming $p - 1 \leq 1$ and $0 < p - 2 + \gamma \leq 1$ yields

$$E[\Gamma_t^p] \leq 2(c_1 - c_2)p \int_0^t E[\Gamma_s]^{p-1}ds + C \int_0^t E[\Gamma_s]^{p-2+\gamma}\frac{ds}{s}$$

$$= 2^p(c_1 - c_2)^p t^p + C(c_1 - c_2)^{p-2+\gamma}t^{p-2+\gamma}$$

so that for $0 \leq t \leq T$ we have

$$f(t) := E[\Gamma_t^p] \leq C_T t^{p-2+\gamma}$$

and hence

$$h(t) := \int_0^t \frac{f(s)}{s}ds \leq C_T t^{p-2+\gamma}.$$

Thus for any $p > 1$ sufficiently close to 1, we take $\gamma$ satisfying $1 + (p - 1)(\frac{p}{2}D^2 - 1) < \gamma \leq 3 - p$ so that

$$h(t) \leq C t^{\frac{p}{2}(p-1)D^2+\delta} \quad (6.20)$$
for some $\delta > 0$.

On the other hand by using $|\Xi_s|^2 \leq \Gamma_s^2$ in (6.19) we have

$$
E[\Gamma_t^p] \leq 2(c_1 - c_2)p \int_0^t 2^{p-1}(c_1 - c_2)^{p-1}s^{p-1}ds + \frac{D}{2}(p - 1)D^2 \int_0^t E[\Gamma_s^p] \frac{ds}{s}.
$$

Hence if $\frac{1}{2}(p - 1)D^2 < 1$, (6.20) and a Gronwall type argument give the desired inequality (6.18).

Having established the continuity of $\Psi_{t_0}(c)$ with respect to $c$, the joint continuity of $\Psi_t(c)$ on $[t_0, \infty) \times \mathbb{R}$ is valid due to the absence of singularity in this time domain. The continuity of $\Psi_t(c)$ at $t = 0$ follows from the monotonicity of $\Psi_t(c)$ with respect to $c$. \hfill \square

**Theorem 6.12** The limit process $\{\Psi_t(c)\}_{t \geq 0, c \in \mathbb{R}}$ satisfies the following two properties:

1. The process has invariance

$$
\{\Psi_t(c)\}_{t \geq 0, c \in \mathbb{R}} \overset{\text{law}}{=} \{\Psi_t(c + c_0) - \Psi_t(c_0)\}_{t \geq 0, c \in \mathbb{R}}
$$

for any $c_0 \in \mathbb{R}$.

2. For each fixed $c$ there exists a 1-D Brownian motion $\{B_t(c)\}$ such that

$$
\Phi_t = 2 \int_0^t \exp \left( \int_s^t \frac{D}{\sqrt{u}} dB_u - \int_s^t \frac{D^2}{2u} du \right) ds
$$

$\{B_t(c)\}$ are a family of martingales satisfying

$$
\langle B(c), B(c') \rangle_t = \int_0^t \cos (\Psi_s(c) - \Psi_s(c')) ds.
$$

**Proof.** (1) For any fixed $c_0 \in \mathbb{R}$, letting $\Phi_t(c) = \Psi_t(c + c_0) - \Psi_t(c_0)$ yields

$$
d\Phi_t(c) = 2cdt + DRe[(e^{i\Psi_t(c+c_0)} - e^{i\Psi_t(c_0)}) t^{-1/2}dZ_t].
$$

Here noting

$$
\tilde{Z}_t = \int_0^t e^{i\Psi_s(c_0)} dZ_s
$$
is a complex Brownian motion, we see
\[ d\Phi_t(c) = 2cdt + DRe[(e^{i\Phi_t(c)} - 1)t^{-1/2}d\tilde{Z}_t], \quad \Phi_0(c) = 0 \]

hence the uniqueness of the solutions gives us
\[ \{\Phi_t(c)\}_{t \geq 0, c \in \mathbb{R}} \overset{\text{law}}{=} \{\Psi_t(c)\}_{t \geq 0, c \in \mathbb{R}}. \]

(2) Set
\[ \Phi_t(c) := \frac{\partial \Psi_t(c)}{\partial c}, \quad B_t(c) := Re \int_0^t ie^{i\Psi_s(c)} dZ_s. \]

Then, for a fixed \( c, \{B_t(c)\}_{t \geq 0} \) is a 1D Brownian motion and
\[ d\Phi_t(c) = 2dt + D\Phi_t(c)Re[ie^{i\Psi_t(c)}t^{-1/2}d\tilde{Z}_t] = 2dt + Dt^{-1/2}\Phi_t(c)dB_t(c) \]
holds, hence
\[ \Phi_t = 2 \int_0^t \exp \left( \int_s^t \frac{D}{\sqrt{u}} dB_u - \int_s^t \frac{D^2}{2u} du \right) ds \]

The variational process for \( \{B_t(c)\} \) are
\[
\langle B_t(c), B_t(c') \rangle_t = \frac{1}{4} \left( \int_0^t ie^{i\Psi_s(c)} dZ_s - \int_0^t ie^{-i\Psi_s(c)} d\overline{Z}_s, \int_0^t ie^{i\Psi_s(c')} dZ_s - \int_0^t ie^{-i\Psi_s(c')} d\overline{Z}_s \right) \\
= \frac{1}{4} \int_0^t e^{i(\Psi_s(c') - \Psi_s(c))} 2ds + 1/4 \int_0^t e^{i[\Psi_s(c') - \Psi_s(c)]} 2ds \\
= \int_0^t \cos (\Psi_s(c) - \Psi_s(c')) ds.
\]

\[ \Box \]

7 Convergence of \( \theta_t(\kappa) \mod \pi \)

**Proposition 7.1** As \( t \to \infty \) \( (2\theta_t(\kappa))_{2\pi\mathbb{Z}} \) converges to the uniform distribution on \([0, 2\pi)\).
Proof. Letting
\[ \xi_t(\kappa) := e^{2m\tilde{\theta}_t(\kappa)}, \quad m \in \mathbb{Z} \]
it suffices to show
\[ \mathbb{E}[\xi_t(\kappa)] \xrightarrow{t \to \infty} 0, \quad m \neq 0. \]
We omit the \( \kappa \)-dependence of \( \theta_t \). By (2.2) we decompose
\[ \xi_t = 1 + \frac{mi}{2\kappa} \int_0^t e^{2i\kappa s + 2(m+1)i\tilde{\theta}_s} a(s) F(X_s) ds \]
\[ + \frac{mi}{2\kappa} \int_0^t e^{-2i\kappa s + 2(m-1)i\tilde{\theta}_s} a(s) F(X_s) ds \]
\[ - \frac{mi}{\kappa} \int_0^t e^{2mi\tilde{\theta}_s} a(s) F(X_s) ds \]
\[ =: 1 + I + II + III. \]
We use Lemma 6.1(1) and decompose \( I \) further into “non-oscillating”-term + martingale-term + remainder:
\[ I = \frac{mi}{2\kappa} \left( -\frac{2i(m+1)}{4\kappa} \int_0^t a(s)^2 e^{2mi\tilde{\theta}_s} F(X_s) g_{\kappa}(X_s) ds \right. \]
\[ + \int_0^t a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa) + \delta_{1,1}(t) \bigg). \] (7.1)
where
\[ \delta_{1,1}(t) := \left[ a(s) e^{2i(m+1)\tilde{\theta}_s} e^{2i\kappa s} g_{\kappa}(X_s) \right]_0^t \]
\[ - \int_0^t a'(s) e^{2i(m+1)\tilde{\theta}_s} e^{2i\kappa s} g_{\kappa}(X_s) ds \]
\[ - \frac{2i(m+1)}{2\kappa} \int_0^t a(s)^2 \left( e^{2(m+2)i\tilde{\theta}_s} e^{4i\kappa s} - e^{2(m+1)i\tilde{\theta}_s} e^{2i\kappa s} \right) F(X_s) g_{\kappa}(X_s) ds. \]
We further compute the third term of \( \delta_{1,1} \) by Lemma 6.1(1) and see that \( \delta_{1,1}(t) \) has a limit as \( t \to \infty \). Taking expectation, martingale term vanishes and we have
\[ \mathbb{E}[\delta_{1,1}(t)] - \mathbb{E}[\delta_{1,1}(\infty)] = O(a(t)), \quad t \to \infty. \] (7.2)
By Lemma 6.1(2), the first term of (7.1) satisfies
\[
\int_0^t a(s)^2 e^{2mi\hat{\theta}_s} F(X_s)g_\kappa(X_s)ds = \langle Fg_\kappa \rangle \int_0^t a(s)^2 e^{2mi\hat{\theta}_s}ds + \delta_{1,2}(t)
\]
where \( \delta_{1,2}(t) \) has a limit as \( t \to \infty \) and satisfies the same estimate as (7.2).

We substitute it into (7.1) and let \( \delta_{1} = \delta_{1,1} + \delta_{1,2} \). Then
\[
I = \frac{mi}{2\kappa} \left( -\frac{2i(m + 1)}{4\kappa} \langle Fg_\kappa \rangle \int_0^t a(s)^2 e^{2mi\hat{\theta}_s}ds + \int_0^t a(s)e^{2i\kappa s}e^{2i(m+1)\hat{\theta}_s}dM_s(\kappa) + \delta_1(t) \right). 
\]

Similarly, we have
\[
II = \frac{mi}{2\kappa} \left( -\frac{2(m - 1)i}{4\kappa} \langle Fg_{-\kappa} \rangle \int_0^t a(s)^2 e^{2mi\hat{\theta}_s}ds + \int_0^t a(s)e^{-2i\kappa s}e^{2i(m-1)\hat{\theta}_s}dM_s(-\kappa) + \delta_2(t) \right)
\]
and
\[
III = -\frac{mi}{\kappa} \left\{ \langle F \rangle \int_0^t a(s)e^{2mi\hat{\theta}_s}ds + \frac{2mi}{2\kappa} \langle Fg \rangle \int_0^t a(s)^2 e^{2mi\hat{\theta}_s}ds + \int_0^t a(s)e^{2mi\hat{\theta}_s}dM_s + \delta_3(t) \right\}.
\]

To summarize,
\[
\xi_t = 1 - \frac{mi}{\kappa} \langle F \rangle \int_0^t a(s)e^{2mi\hat{\theta}_s}ds + \langle G_m \rangle \int_0^t a(s)^2 e^{2mi\hat{\theta}_s}ds + N_t + \delta(t) 
\]
where
\[
G_m = \left( \frac{m(m + 1)}{4\kappa^2}g_\kappa + \frac{m(m - 1)}{4\kappa^2}g_{-\kappa} + \frac{m^2}{\kappa^2}g \right)F
\]
\[
N_t = \frac{mi}{2\kappa} \int_0^t a(s)e^{2i\kappa s}e^{2i(m+1)\hat{\theta}_s}dM_s(\kappa) + \frac{mi}{2\kappa} \int_0^t a(s)e^{-2i\kappa s}e^{2i(m-1)\hat{\theta}_s}dM_s(-\kappa) - \frac{mi}{\kappa} \int_0^t a(s)e^{2mi\hat{\theta}_s}dM_s
\]
where \( \delta(\infty) = \lim_{t \to \infty} \delta(t) \) exists a.s. and
\[
E[\delta(t)] - E[\delta(\infty)] = O(a(t)), \quad t \to \infty.
\]

Let \( \sigma_F(d\lambda) \) be the spectral measure of \( L \) with respect to \( F \). Then by noting
\[
Re\langle Fg \kappa \rangle = Re\langle Fg - \kappa \rangle = \int_{-\infty}^{0} \lambda \sigma_F(d\lambda) < 0, \quad Re\langle Fg \rangle = \int_{-\infty}^{0} \frac{\sigma_F(d\lambda)}{\lambda} < 0
\]
we have
\[
-\gamma := Re\langle G_m \rangle < 0.
\]

Set
\[
\rho(t) := E[\xi_t], \quad b(t) := -\frac{mi}{\kappa} \langle F \rangle a(t) + \langle G_m \rangle a(t)^2.
\]

Then (7.4) turns to
\[
\rho(t) = 1 + \int_0^t b(s) \rho(s) ds + E[\delta(t)]
\]
and hence
\[
\rho(t) = \exp \left( \int_0^t b(u) du \right) + E[\delta(t)] + \int_0^t E[\delta(s)] b(s) \exp \left( \int_s^t b(u) du \right) ds
\]
\[
= \exp \left( \int_0^t b(u) du \right) + E[\delta(t)] + \int_0^t b(s) \exp \left( \int_s^t b(u) du \right) ds
\]
\[
+ \int_0^t (E[\delta(s)] - E[\delta(\infty)]) b(s) \exp \left( \int_s^t b(u) du \right) ds
\]
\[
= I + II + III + IV.
\]

Noting \( Re b(t) = Re \langle G_m \rangle a(t)^2 = -\gamma a(t)^2 \), we compute \( I, III \)
\[
|I| \leq \exp \left( \int_0^t Re b(s) ds \right) \leq C \exp \left( -\gamma \int_1^t \frac{1}{s} ds \right) \xrightarrow{t \to \infty} 0
\]
\[
III = E[\delta(\infty)] \left(-1 + \exp \left( \int_0^t b(u) du \right) \right) \xrightarrow{t \to \infty} -E[\delta(\infty)].
\]

We further decompose \( IV \) :
\[
|IV| = \left| \int_0^t (E[\delta(s)] - E[\delta(\infty)]) b(s) \exp \left( \int_s^t b(u) du \right) ds \right|
\]
\[
\leq C \left( \int_0^M \int_s^t a(s) b(s) \exp \left( Re \int_s^t b(u) du \right) ds \right)
\]
\[
=: IV_1 + IV_2.
\]
It is easy to see that $IV_1 \overset{t \to \infty}{\to} 0$. For $IV_2$ we use $\langle F \rangle = 0$ and compute, for large $M$,

$$|IV_2| \leq C \int_{M}^{t} a(s)^3 \exp \left( \int_{s}^{t} \text{Re} b(u) du \right) ds$$

$$\leq C \int_{M}^{t} s^{-3/2} \left( \frac{t}{s} \right)^{-\gamma} ds$$

$$= \begin{cases} 
C t^{-\gamma} \log \frac{t}{M} & (\gamma = \frac{1}{2}) \\
C t^{-\gamma} \frac{t^{\frac{1}{2}} - M^{\frac{1}{2}}}{\gamma - \frac{1}{2}} & (\gamma \neq \frac{1}{2})
\end{cases} \overset{t \to \infty}{\to} 0.$$ 

8 Limiting behavior of $\tilde{\theta}_t$

To study the limiting behavior of $(2\tilde{\theta}_t)_{2\pi \mathbf{Z}}$ we set

$$\tilde{\xi}_t := e^{2i\tilde{\theta}_t(\kappa)}.$$

8.1 Estimate of integral equation

As in the proof of Proposition 7.1, we can show the following lemma.

Lemma 8.1 Let $0 < t_0 < t$. Then we have

$$\tilde{\xi}_t = \tilde{\xi}_{t_0} + \frac{1}{2\kappa^2} \langle F \cdot (g_{\kappa} + 2g) \rangle \int_{t_0}^{t} a(s)^2 e^{2i\tilde{\theta}_s} ds - \frac{i}{\kappa} \langle F \rangle \int_{t_0}^{t} a(s) e^{2i\tilde{\theta}_s} ds$$

$$+ \frac{i}{2\kappa} \left( Y_t + \tilde{Y}_t - 2\hat{Y}_t \right) + O(a(t_0)), \quad t_0 \to \infty.$$

where

$$Y_t := \int_{t_0}^{t} a(s) e^{2i\kappa s} dM_s(\kappa)$$

$$\tilde{Y}_t := \int_{t_0}^{t} a(s) e^{-2i\kappa s} dM_s(-\kappa)$$

$$\hat{Y}_t := \int_{t_0}^{t} a(s) e^{2i\tilde{\theta}_s} dM_s.$$
The variational process of $Y, \tilde{Y}$, and $\hat{Y}$ satisfy, as $t_0 \to \infty$,

$$\langle Y, Y \rangle_t = O(a(t_0))$$

$$\langle Y, \tilde{Y} \rangle_t = \langle [g_\kappa, \tilde{g}_\kappa] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0))$$

$$\langle \tilde{Y}, \tilde{Y} \rangle_t = O(a(t_0))$$

$$\langle \tilde{Y}, Y \rangle_t = \langle [g_\kappa, \tilde{g}_\kappa] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0))$$

$$\langle \tilde{Y}, \tilde{Y} \rangle_t = \langle [g, g] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0))$$

$$\langle \hat{Y}, \hat{Y} \rangle_t = \langle [g, g] \rangle \int_{t_0}^t a(s)^2 ds + O(a(t_0)).$$

8.2 Tightness of $\eta$

Let

$$\eta^{(n)}_t := \tilde{\xi}_nt = e^{2i\tilde{\theta}_nt(\kappa)}$$

$$U := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Lemma 8.2 \{$\eta^{(n)}_t\}_{n \geq 1}$ is tight as a family in $C((0, \infty) \to U)$.

Proof. It suffices to show, for any $t_0 > 0, \rho > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P \left( \sup_{t_0 < s < t, t - s < \delta} |\eta^{(n)}_t - \eta^{(n)}_s| > \rho \right) = 0.$$

Noting $\langle F \rangle = 0$, Lemma 8.1 implies

$$\xi_{nt} - \xi_{ns} = \frac{1}{2\kappa^2} \langle F \cdot (g_\kappa + 2g) \rangle \int_{ns}^{nt} a(u)^2 e^{2i\tilde{\theta}_u} du$$

$$+ \frac{i}{2\kappa} W_{t,s}^{(n)} + o(1), \quad n \to \infty \quad (8.1)$$

where

$$W_{t,s}^{(n)} := (Y_{nt} + \tilde{Y}_{nt} - 2\hat{Y}_{nt}) - (Y_{ns} + \tilde{Y}_{ns} - 2\hat{Y}_{ns}).$$

We note that $W_{t,s}^{(n)}$ satisfies the estimate in Lemma 6.7 and the rest of the argument is the same as that of Lemma 6.8.
8.3 Identification of $\eta_t$

Let $\eta_t$ be a limit point of $\eta_t^{(n)}$ which is uniformly distributed on $U$ for each fixed $t > 0$ by Lemma 7.1. In this subsection we show that the distribution of the process $\eta_t$ is uniquely determined.

Lemma 8.3

(1) For any $0 < t_0 < t$,

$$\lim_{n \to \infty} E[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})}|F_{nt_0}] = \left( \frac{t}{t_0} \right)^{\langle G_m \rangle}$$

where $F_t$ is the $\sigma$-algebra generated by $\{X_s\}_{0 \leq s \leq t}$.

(2) For any $0 < t_0 < t_1 < \cdots < t_k$, the family of random variables $\{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \cdots, \eta_{t_k}/\eta_{t_{k-1}}\}$ are independent.

Proof. (1) Let $m, m' \in \mathbb{Z}$. By an argument similar to deduce (7.4), we have

$$e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})} = 1 + \langle G_m \rangle \int_{nt_0}^{nt} a(s)^2 e^{2mi(\tilde{\theta}_s - \tilde{\theta}_{nt_0})} ds$$

$$+ N_{nt,nt_0} e^{-2mi\tilde{\theta}_{nt_0}} + \delta_n(t) e^{-2mi\tilde{\theta}_{nt_0}}$$

$$= 1 + \langle G_m \rangle \int_{t_0}^{t} n(a(nu)^2 e^{2mi(\tilde{\theta}_nu - \tilde{\theta}_{nt_0})} du$$

$$+ N_{nt,nt_0} e^{-2mi\tilde{\theta}_{nt_0}} + \delta_n(t) e^{-2mi\tilde{\theta}_{nt_0}}$$

where

$$G_m = \left( \frac{m(m+1)}{4\kappa^2} g_\kappa + \frac{m(m-1)}{4\kappa^2} g_{-\kappa} + \frac{m^2}{\kappa^2} g \right) F$$

$$N_{nt,nt_0} = \frac{mi}{2\kappa} \int_{nt_0}^{nt} a(s) e^{2i\kappa s} e^{2i(m+1)\tilde{\theta}_s} dM_s(\kappa)$$

$$+ \frac{mi}{2\kappa} \int_{nt_0}^{nt} a(s) e^{-2i\kappa s} e^{2i(m-1)\tilde{\theta}_s} dM_s(-\kappa)$$

$$- \frac{mi}{\kappa} \int_{nt_0}^{nt} a(s) e^{-2mi\tilde{\theta}_s} dM_s$$

$$E[\delta_n(t)|F_{nt_0}] \xrightarrow{n \to \infty} 0, \text{ a.s.}$$

Taking a conditional expectation and letting

$$\rho_n(t) := E[e^{2mi(\tilde{\theta}_{nt} - \tilde{\theta}_{nt_0})}|F_{nt_0}]$$
we have
\[ \rho_n(t) = 1 + \langle G_m \rangle \int_{t_0}^t n(a(u))^2 \rho_n(u)du + E[\delta_n(t)\big| F_{nt_0}] e^{-2mi\tilde{\eta}_{nt_0}}. \]

Therefore
\[ \rho_n(t) = \exp \left( \langle G_m \rangle \int_{t_0}^t n(a(u))^2 du \right) + E[\delta_n(t)\big| F_{nt_0}] e^{-2mi\tilde{\eta}_{nt_0}} \]
\[ + \int_{t_0}^t E[\delta_n(s)\big| F_{nt_0}] e^{-2mi\tilde{\eta}_{nt_0}} \langle G_m \rangle n(a(u))^2 \exp \left( \langle G_m \rangle \int_{s}^t n(a(u))^2 du \right) ds \]
\[ \xrightarrow{n \to \infty} \exp \left( \langle G_m \rangle \int_{t_0}^t \frac{du}{u} \right) = \left( \frac{t}{t_0} \right)^{\langle G_m \rangle}. \]

(2) The required independence easily follows from (1) and the fact that \( e^{2i\tilde{\eta}_{nt}} \) converges to the uniform distribution on \( U \) as \( n \to \infty \). In fact, for \( k = 1 \),
\[ E \left[ E[e^{2mi(\tilde{\eta}_{nt} - \tilde{\eta}_{nt_0})} \big| F_{nt_0}] e^{2mi'\tilde{\eta}_{nt_0}} \right] \]
\[ = E \left[ \left( E[e^{2mi(\tilde{\eta}_{nt} - \tilde{\eta}_{nt_0})} \big| F_{nt_0}] \left( \frac{t}{t_0} \right)^{\langle G_m \rangle} \right) e^{2mi'\tilde{\eta}_{nt_0}} \right] + E \left[ \left( \frac{t}{t_0} \right)^{\langle G_m \rangle} e^{2mi'\tilde{\eta}_{nt_0}} \right] \]
\[ \to \begin{cases} 0 & (m' \neq 0) \\ \left( \frac{1}{t_0} \right)^{\langle G_m \rangle} & (m' = 0) \end{cases} \]

For \( k \geq 2 \), the proof is similar. \( \square \)

**Lemma 8.4** For each fixed \( t_0 > 0 \), \( \eta_t \) satisfies the following SDE on \( t \geq t_0 \):
\[ d\eta_t = C_1 \frac{\eta_t}{t} dt + C_2 \frac{\eta_t}{\sqrt{t}} dB_t, \quad (8.2) \]
where \( C_1 := \langle (g_\kappa + 2g)F \rangle \), \( C_2 := \frac{i}{2\kappa} \sqrt{\langle 2[g_\kappa, \overline{g_\kappa}] + 4[g, g] \rangle} \).

**Proof.** Letting \( s = t_0 > 0 \) in (8.1) yields, as \( n \to \infty \),
\[ \int_{nt_0}^{nt} a(u)^2 e^{2i\tilde{\eta}_u} du \to \int_{t_0}^{t} \frac{\eta_u}{u} du \]
\[ \langle W^{(n)}_{s,t_0}, \overline{W^{(n)}_{s,t_0}} \rangle \to \langle (g_\kappa + 2g), \overline{g_\kappa} \rangle + 4\langle g, g \rangle \int_{t_0}^{t} \frac{\eta_u^2}{u} du \]
\[ \langle W^{(n)}_{s,t_0}, \overline{W^{(n)}_{s,t_0}} \rangle \to \langle (g_\kappa + 2g), \overline{g_\kappa} \rangle + 4\langle g, g \rangle \int_{t_0}^{t} \frac{du}{u}. \]

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We then proceed as in the proof of Theorem 6.10.

Remark 8.1 \( Z_t, B_t \) which appear in SDE’s (6.17, 8.2) of \( \Psi, \eta \) are not independent. In fact,

\[
\begin{align*}
    dW_t &= \sqrt{2\langle [g_\kappa, \overline{g}_\kappa] \rangle + 4[g,g]} \frac{\eta_t}{\sqrt{t}} dB_t \\
    dV_t &= \sqrt{\langle [g_\kappa, \overline{g}_\kappa] \rangle} \left( e^{i\Psi_t(c)} - 1 \right) \frac{dZ_t}{\sqrt{t}} \\
    d\langle W, V \rangle &= \langle [g_\kappa, \overline{g}_\kappa] \rangle \left( e^{i\Psi_t(c)} - 1 \right) \eta_t \frac{dt}{t} \\
    d\langle W, \overline{V} \rangle &= \langle [g_\kappa, \overline{g}_\kappa] \rangle \left( e^{-i\Psi_t(c)} - 1 \right) \eta_t \frac{dt}{t}
\end{align*}
\]

which imply

\[
dZ dB = \sqrt{\frac{\langle [g_\kappa, \overline{g}_\kappa] \rangle}{2\langle [g_\kappa, \overline{g}_\kappa] \rangle + 4[g,g]}} dt.
\]

Here we note the following fact. By the time change \( u = \log t, \ z_u := \log \eta_{e^u} \) satisfies the following SDE which is stationary in time.

\[
dz_u = iC_3 du + iC_4 d\tilde{B}_u \tag{8.3}
\]

where \( C_3 := -\frac{1}{\kappa} \langle |g_\kappa|^2 \rangle \in \mathbb{R}, \ C_4 := \frac{1}{2\kappa} \sqrt{2\langle [g_\kappa, \overline{g}_\kappa] \rangle + 4[g,g]} \in \mathbb{R} \).

To summarize, the following facts have been proved.

(i) For any \( t > 0, \ \eta_t \) has uniform distribution (Lemma 7.1).

(ii) For any \( 0 < t_0 < t_1 < t_2 < \cdots < t_n, \) random variables \( \{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \cdots, \eta_{t_n}/\eta_{t_{n-1}}\} \) are independent (Lemma 8.3).

(iii) For any \( t_0 > 0, \ x_t = \eta_t/\eta_{t_0} \) satisfies an SDE on \( t \geq t_0 \) (Lemma 8.4):

\[
dx_t = C_1 \frac{x_t}{t} dt + C_2 \frac{x_t}{\sqrt{t}} dB_t, \quad x_{t_0} = 1.
\]

These facts determine (in distribution) the process \( \eta_t \) uniquely. In fact, for any \( 0 < t_0 < t_1 < \cdots < t_n, \) the distribution of \( \{\eta_{t_0}, \eta_{t_1}, \cdots, \eta_{t_n}\} \) can be computed from that of \( \{\eta_{t_0}, \eta_{t_1}/\eta_{t_0}, \cdots, \eta_{t_n}/\eta_{t_{n-1}}\} \) and the latter distribution can be determined uniquely from (ii) and (iii). Therefore the distribution of \( \{\eta_t\} \) is characterized by the constants \( C_1, C_2. \) More concretely, if we prepare
1D Brownian motion \( \{B_t\}_{t \in \mathbb{R}} \) with \( B_0 = 0 \) and independent random variable \( X \in \mathbb{C} \) with uniform distribution on \( \mathbb{U} \), a process

\[
X \exp \left[ i(C_3 u + C_4 B_u) \right]
\]

has the same distribution as \( \{\eta_{i,u}\} \) by (8.2), (8.3).

9 Convergence of the joint distribution

We finish the proof of Theorem 1.3.

9.1 Behavior of the joint distribution

Proposition 9.1 For any \( c_1, \ldots, c_m \in \mathbb{R} \), \( t > 0 \),

\[
(\Psi_t^{(n)}(c_1), \ldots, \Psi_t^{(n)}(c_m), (\theta_{nt}(\kappa))_{2\pi \mathbb{Z}}) \xrightarrow{d} (\Psi_t(c_1), \ldots, \Psi_t(c_m), \phi_t), \quad (9.1)
\]

as \( n \to \infty \), where \((\Psi_t(c_1), \ldots, \Psi_t(c_m))\) and \( \phi_t \) are independent and \( \phi_t \) is uniformly distributed on \([0, 2\pi)\).

Proof. For simplicity, we use the following notation. \( \mathbf{c} := (c_1, \ldots, c_m) \), \( \Psi_t^{(n)}(\mathbf{c}) := (\Psi_t^{(n)}(c_1), \ldots, \Psi_t^{(n)}(c_m)) \), and \( \Psi_t(\mathbf{c}) := (\Psi_t(c_1), \ldots, \Psi_t(c_m)) \). It suffices to show (9.1) with \((\theta_{nt}(\kappa))_{2\pi \mathbb{Z}}\) being replaced by \((\tilde{\theta}_{nt}(\kappa))_{2\pi \mathbb{Z}}\), since \((\tilde{\theta}_{nt}(\kappa))_{2\pi \mathbb{Z}}\) converges to the uniform distribution by Lemma 7.1. By Lemmas 6.8, 8.2, for any fixed \( t_0 > 0 \), the process \( \{(\Psi_t^{(n)}(\mathbf{c}), \eta_t^{(n)})\}_{n \geq 1} \) on \([t_0, \infty)\) is a tight family. Hence we can assume \((\Psi_t^{(n)}(\mathbf{c}), \eta_t^{(n)})_{t > 0} \xrightarrow{d} (\Psi_t(\mathbf{c}), \eta_t)_{t > 0} \). By Lemma 8.3 \( \eta_1/n \) and \( \eta_t/\eta_1/n \) are independent.

We next consider a process \( \tilde{\Psi}_t^{(n)}(\mathbf{c}) \) which is defined on \([\frac{1}{n}, \infty)\) and is the solution to (6.17) with initial value \( \tilde{\Psi}_1^{(n)}(\mathbf{c}) = \mathbf{c}/n \). Proposition 4.5 proves the following fact

\[
\sup_{n^{-1} < t < n} \left| \tilde{\Psi}_t^{(n)}(\mathbf{c}) - \Psi_t(\mathbf{c}) \right| \xrightarrow{P} 0, \quad n \to \infty.
\]

Since \( \eta_1/n \) and \((\eta_t/\eta_1/n, \tilde{\Psi}_t^{(n)}(\mathbf{c}))\) are independent, by letting \( n \to \infty \), it follows that \( \eta_0 := \lim_{t \to 0} \eta_t \) and \((\eta_t/\eta_0, \Psi_t(\mathbf{c}))\) are independent. Since \( \eta_0 \) is uniformly distributed on \( \mathbb{U} \), \( \tilde{\phi}_t := \arg \eta_t = \arg \left( \eta_0 \cdot \frac{\mathbf{c}}{\eta_0} \right) \) and \( \Psi_t \) are independent. \( \square \)
9.2 Convergence of $\Psi_t^{(n)}$ as increasing functions

Proposition 9.2 Fix any $t > 0$. Then we can find a coupling such that the following statement is valid for a.s.

$$\lim_{n \to \infty} (\Psi_t^{(n)})^{-1}(x) = \Psi_t^{-1}(x), \quad \lim_{n \to \infty} (2\theta_{nt}(\kappa))_{2\pi Z} = \phi_t$$

for any $x \in \mathbb{R}$ where $\phi_t$ is uniformly distributed and independent of $\Psi_t$.

As is explained in section 5, Proposition 9.2 completes the proof of Theorem 1.3. To prove Proposition 9.2 we shall show below that the convergence $\Psi_t^{(n)} \to \Psi_t$ holds in the sense of increasing function-valued process.

Let $\mathcal{M}$ be the set of non-negative measures on $[a, b]$. Fix $\{f_j\}_{j \geq 1}$ a family of smooth functions on $[a, b]$ satisfying the property

for $\omega \in \mathcal{M}$ if $\int_a^b f_j(x)d\omega(x) = 0$ for any $j \geq 1 \Rightarrow \omega = 0$.

We define a metric $\rho$ on $\mathcal{M}$ by

$$\rho(\omega_1, \omega_2) := \sum_{j=1}^{\infty} \frac{1}{2^j} \left( \left| \int_a^b f_j(x)d(\omega_1(x) - \omega_2(x)) \right| \wedge 1 \right).$$

Let

$$\Omega := C([0, T] \to \mathcal{M})$$

for $T < \infty$. We further define for a smooth function $f$ on $[a, b]$ a map $\Phi_f : \Omega \to C([0, T] \to \mathbb{R})$ by

$$\Phi_f(\omega)(t) := \int_a^b f(x)d\omega_t(x)$$

$$= [f(x)\omega_t(x)]_a^b - \int_a^b f'(x)\omega_t(x)dx. \quad (9.2)$$

Lemma 9.3 Let $\{\mu_n\}_{n \geq 1}$ be a family of probability measures on $\Omega$. Suppose for each smooth function $f$ on $[a, b]$ a family of probability measures $\{\Phi_f^{-1}\mu_n\}_{n \geq 1}$ on $C([0, T] \to \mathbb{R})$ is tight. Assume further there exists a constant $C$ such that

$$E_{\mu_n} \left[ \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \right] \leq C \quad (9.3)$$

holds for any $n \geq 1$. Then $\{\mu_n\}_{n \geq 1}$ is tight.
Proof. From (9.3) we see that for any \( \epsilon > 0 \), there exists a \( M > 0 \) such that
\[
\mu_n \left( \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \leq M \right) \geq 1 - \epsilon.
\]
Set
\[
\Omega_0 := \left\{ \omega \in \Omega \mid \sup_{0 \leq t \leq T} \int_a^b d\omega_t(x) \leq M \right\}.
\]
Since \( \left\{ \Phi^{-1}_{f_j} \mu_n \right\}_{n \geq 1} \) is tight for each \( j \geq 1 \), there exists a compact set \( K_j \) in \( C([0, T] \to \mathbb{R}) \) such that
\[
\mu_n \left( \Phi^{-1}_{f_j}(K_j) \right) > 1 - \frac{\epsilon}{2^j}.
\]
Set
\[
K := \bigcap_{j=1}^{\infty} \Phi^{-1}_{f_j}(K_j) \cap \Omega_0 \subset \Omega.
\]
Then
\[
\mu_n(K^c) \leq \sum_{j=1}^{\infty} \mu_n \left( \Phi^{-1}_{f_j}(K^c_j) + \mu_n(\Omega_0^c) \right) \leq \epsilon + \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = 2\epsilon. \tag{9.4}
\]
We show \( K \) is compact in \( \Omega \). Let \( \{\omega_n\}_{n \geq 1} \) be a sequence in \( K \). Since \( K_1 \) is compact, there exists a subsequence \( \{n_1^j\} \) along which \( \Phi_{f_1} \left( \omega_{n_1^j} \right) \) is uniformly convergent in \( C([0, T] \to \mathbb{R}) \). Then, using the compactness of \( K_2 \) we can find a subsequence \( \{n_2^j\} \) of \( \{n_1^j\} \) along which \( \Phi_{f_2} \left( \omega_{n_2^j} \right) \) is uniformly convergent in \( C([0, T] \to \mathbb{R}) \). Continuing this procedure for each \( j \) we find a subsequence \( \{n_3^j\} \) of \( \{n_2^j\} \) along which \( \Phi_{f_3} \left( \omega_{n_3^j} \right) \) is uniformly convergent in \( C([0, T] \to \mathbb{R}) \). Let \( m_i = n_i^j \). Then for each \( j \geq 1 \), \( \Phi_{f_j} \left( \omega_{m_i} \right) \) converges uniformly in \( C([0, T] \to \mathbb{R}) \). Since, for any \( f \in C[a, b] \) and \( \epsilon' > 0 \), there exists a finite linear combination \( g \) of \( \{f_j\} \) such that
\[
\sup_{x \in [a, b]} |f(x) - g(x)| < \epsilon'.
\]
We easily have
\[
\sup_{t \in [0, T]} \left| \Phi_f(\omega_{m_i})(t) - \Phi_g(\omega_{m_i})(t) \right| \leq \epsilon' M
\]
where we have used \( \int_a^b d\omega_t(x) \leq M \) for any \( \omega \in \mathcal{K} \). Therefore we see that the limit
\[
\lim_{i \to \infty} \Phi_f(\omega_{m_i})(t)
\]
exists uniformly w.r.t. \( t \in [0,T] \), which implies that there exists a \( \omega \in \Omega \) satisfying
\[
\int_a^b d\omega_t(x) \leq M \quad \text{and} \quad \lim_{i \to \infty} \Phi_f(\omega_{m_i}) = \Phi_f(\omega)(t)
\]
for any \( t \in [0,T] \) and \( f \in C([a,b]) \). Consequently we have the compactness of \( \mathcal{K} \) which together with (9.4) shows the tightness of \( \{\mu_n\}_{n \geq 1} \).

We would like to check that the conditions for Lemma 9.3 are satisfied for \( \Psi_t^{(n)}(\cdot) \). The inequality (9.3) follows from Lemma 6.6. In view of (9.2), the required tightness is implied by the following lemma.

**Lemma 9.4** For \( f \in C^\infty(a,b) \) let
\[
g_n(t) := \int_a^b f(x)\Psi_t^{(n)}(x)dx.
\]
Then, as a family of probability measures on \( C([0,T] \to \mathbb{R}) \), \( \{g_n\}_{n \geq 1} \) is tight.

**Proof.** It is sufficient to show that following two equations.

(1) \( \lim_{A \to \infty} \sup_n P( |g_n(0)| \geq A ) = 0 \),

(2) For any \( \rho > 0 \),
\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup P \left( \sup_{|t-s| < \delta} |g_n(t) - g_n(s)| > \rho \right) = 0.
\]

By Lemma 6.6, (1) is clear. By bounding \( f \), the following equation implies (2).
\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} P \left[ \sup_{|t-s| < \delta} \left| \int_a^b |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx \right| > \rho \right] = 0.
\]
Here we borrow an argument in [2] Proposition 2.5: We divide $[a, b]$ into $N$-intervals

$$x_j = a + \frac{b - a}{N} x_j, \quad j = 0, 1, \ldots, N - 1,$$

and have

$$\int_a^b |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx. \quad (9.5)$$

Since $\Psi_t^{(n)}(x)$ is increasing with respect to $x$, for $x \in [x_j, x_{j+1}]$ the integrand is bounded from above by

$$|\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| \leq \Psi_t^{(n)}(x_{j+1}) - \Psi_t^{(n)}(x_j) + |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)| + \Psi_s^{(n)}(x_{j+1}) - \Psi_s^{(n)}(x_j).$$

Substituting it into (9.5) yields

$$J := \int_a^b |\Psi_t^{(n)}(x) - \Psi_s^{(n)}(x)| dx$$

$$\leq \sum_{j=0}^{N-1} \frac{1}{N} (b - a) \left( \Psi_t^{(n)}(x_{j+1}) - \Psi_t^{(n)}(x_j) \right) + (t \leftrightarrow s)$$

$$+ \sum_{j=0}^{N} \frac{1}{N} (b - a) |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)|$$

$$= \frac{1}{N} (b - a) \left( \Psi_t^{(n)}(b) - \Psi_t^{(n)}(a) \right) + (t \leftrightarrow s)$$

$$+ \sum_{j=0}^{N} \frac{1}{N} (b - a) |\Psi_t^{(n)}(x_j) - \Psi_s^{(n)}(x_j)|$$

$$=: I + II.$$

Thus we decompose the probability in question into two terms.

$$\Pr\left( \sup_{|t-s|<\delta} J > \rho \right) \leq \Pr\left( \sup_{|t-s|<\delta} I > \rho/2 \right) + \Pr\left( \sup_{|t-s|<\delta} II > \rho/2 \right)$$

$$=: III + IV.$$
The $III$-term can be estimated by Lemma 6.6,

$$III \leq \mathbb{P} \left( \frac{b - a}{N} \left( \Psi^{(n)}_t(b) - \Psi^{(n)}_t(a) \right) > \rho/4 \right) + (t \leftrightarrow s)$$

$$\leq \frac{4}{\rho} \cdot \frac{b - a}{N} \mathbb{E} \left[ \left( \Psi^{(n)}_t(b) - \Psi^{(n)}_t(a) \right) \right] + (t \leftrightarrow s)$$

$$\leq 2 \cdot \frac{4}{\rho} \cdot \frac{b - a}{N} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \Psi^{(n)}_t(b) \right) \right] \leq \frac{C}{N}.$$ 

Thus for any $\epsilon > 0$ we take $N$ large enough independently of $\delta$ to have

$$III < \frac{\epsilon}{2}.$$

For such fixed $N$, we have

$$IV \leq \sum_{j=0}^{N} \mathbb{P} \left( \frac{b - a}{N} \sup_{|t-s|<\delta} |\Psi^{(n)}_t(x_j) - \Psi^{(n)}_s(x_j)| > \frac{\rho}{2N} \right)$$

$$= \sum_{j=0}^{N} \mathbb{P} \left( \sup_{|t-s|<\delta} |\Psi^{(n)}_t(x_j) - \Psi^{(n)}_s(x_j)| > \frac{\rho}{2(b-a)} \right) .$$

Since $\{\Psi^{(n)}_t(x_j)\}_{j=0}^{N}$ is tight by Lemma 6.8 we can let $IV < \epsilon/2$ by taking $n$ large and then taking $\delta > 0$ small. □

We identify an element of $\mathcal{M}$ with a non-decreasing and right continuous function $\omega$ on $[a, b]$ satisfying $\omega(a) = 0$. Then $\omega_n$ converges to $\omega \in \Omega$ if and only if $\omega_n(x) \to \omega(x)$ at any point of continuity of $\omega$.

**Lemma 9.5** Suppose $\{\omega_n\}_{n \geq 1} \subset \mathcal{M}$ converges to $\omega$ of $\mathcal{M}$. Assume $\omega$ is continuous. Then the convergence is uniform.

**Proof.** Assume $\{\omega_n\}_{n \geq 1}$ does not converge to $\omega$ uniformly. Then there exists a sequence $n_1 < n_2 < \cdots$, $\{t_k\}_{k \geq 1}$ and a positive number $\epsilon_0$ such that

$$|\omega_{n_k}(t_k) - \omega(t_k)| \geq \epsilon_0 \quad (9.6)$$

is valid for any $k = 1, 2, \cdots$. We can assume $t_k \to t_0 \in [a, b]$ keeping $t_1 < t_2 < \cdots < t_0$. Then

$$\omega_{n_k}(t_l) - \omega(t_k) \leq \omega_{n_k}(t_k) - \omega(t_k) \leq \omega_{n_k}(t_0) - \omega(t_k)$$

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for any \( l < k \), hence letting \( k \to \infty \), we have

\[
\omega(t_l) - \omega(t_0) \leq \liminf_{k \to \infty} \left( \omega_{n_k}(t_k) - \omega(t_k) \right) \\
\leq \limsup_{k \to \infty} \left( \omega_{n_k}(t_k) - \omega(t_k) \right) \leq \omega(t_0) - \omega(t_0) = 0.
\]

Consequently, letting \( l \to \infty \), we see

\[
\lim_{k \to \infty} \left( \omega_{n_k}(t_k) - \omega(t_k) \right) = 0,
\]

which contradicts (9.6).

**Proof of Proposition 9.2**

By Lemma 9.3, the sequence of increasing function-valued process \( \{ \Psi_t^{(n)}(\cdot) \}_n \) is tight. Hence \( (\Psi_t^{(n_k)}(2\theta_{n_k}t)_{2\pi\mathbb{Z}}) \xrightarrow{d} (\Psi_t, \phi_t) \) for some subsequence \( \{ n_k \} \). By Skorohod’s theorem, we can suppose \( (\Psi_t^{(n_k)}(2\theta_{n_k}t)_{2\pi\mathbb{Z}}) \xrightarrow{a.s.} (\Psi_t, \phi_t) \). Hence in particular we fix any \( t > 0 \) and obtain

\[
\rho(\Psi_t^{(n_k)}, \Psi_t) = \sum_{j \geq 1} \frac{1}{2j} \left( \left| \int_a^b f_j(x)d(\Psi_t^{(n_k)}(x) - \Psi_t(x)) \right| \wedge 1 \right) \xrightarrow{n \to \infty} 0, \quad a.s.
\]

By Lemma 6.11, \( \Psi_t \) is continuous and increasing. Hence for a.s., \( \Psi_t^{(n_k)}(x) \to \Psi_t(x) \) holds for any \( x \). Moreover by Lemma 3.3, \( (\Psi_t^{(n_k)})^{-1} \xrightarrow{a.s.} \Psi_t^{-1} \). Therefore Proposition 9.2 is proved.

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