DIRECT IMAGE OF PARABOLIC LINE BUNDLES

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ABSTRACT. Given a vector bundle $E$, on an irreducible projective variety $X$, we give a necessary and sufficient criterion for $E$ to be a direct image of a line bundle under a surjective étale morphism. The criterion in question is the existence of a Cartan subalgebra bundle of the endomorphism bundle $\text{End}(E)$. As a corollary, a criterion is obtained for $E$ to be the direct image of the structure sheaf under an étale morphism. The direct image of a parabolic line bundle under any ramified covering map has a natural parabolic structure. Given a parabolic vector bundle, we give a similar criterion for it to be the direct image of a parabolic line bundle under a ramified covering map.

1. INTRODUCTION

This work was inspired by [DP], and also by [Bo1], [Bo2]. In [DP], the authors address the following question: Given a vector bundle $E$ on a smooth projective curve $X$ over a field of characteristic zero, is there a branched covering of $X$ and a line bundle $L$ on $X$ such that $E \otimes L$ is the direct image of the structure sheaf under the covering map? They answer this question affirmatively.

Let $X$ and $Y$ be smooth projective curves defined over an algebraically closed field $k$, where $X$ is irreducible but $Y$ need not be, and let $f : Y \to X$ be a finite separable morphism. Then for any parabolic line bundle $L_*$ on $Y$, the direct image $f_* L_*$ has a natural parabolic structure.

Here we address the following questions:

Given a parabolic vector bundle $E_*$ on $X$, when there is a pair $(Y, f)$ as above such that

1. $E_*$ is isomorphic to the parabolic vector bundle $f_* \mathcal{O}_Y$ (the parabolic structure on $\mathcal{O}_Y$ is the trivial one), and more generally,
2. $E_*$ is isomorphic to the parabolic vector bundle $f_* L_*$, where $L_*$ is a parabolic line bundle on $Y$?

Under the assumption that the characteristic of $k$ is zero, the first question is answered in Corollary 3.2 and the second question is answered in Theorem 3.1. When the characteristic of $k$ is positive, these results remain valid if the parabolic structure on $E_*$ is tame; see Section 3.2.

To understand the criteria in Corollary 3.2 and Theorem 3.1 the key step is to consider the étale case. More precisely, consider the following questions:

1. Given a vector bundle $E$ on $X$, when there is an étale covering $f : Y \to X$, and a line bundle $L$ on $Y$, such that $f_* L = E$. 

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With $X$ and $E$ as above, when there is an étale covering $f : Y \rightarrow X$ such that $f_*\mathcal{O}_Y = E$.

We prove the following (see Theorem 2.4 and Corollary 2.6):

**Theorem 1.1.** Let $X$ be an irreducible projective variety defined over an algebraically closed field $k$. Given a vector bundle $E$ on $X$ of rank $d$, the following two are equivalent:

1. There is an étale covering of degree $d$
   
   $f : Y \rightarrow X$

   and a line bundle $L$ on $Y$, such that $f_*L = E$.

2. There is a subbundle $\mathcal{A} \subset \text{End}(E)$ of rank $d$ such that for each closed point $x \in X$, the subspace $\mathcal{A}_x \subset \text{End}(E)_x = \text{End}(E_x)$ is a Cartan subalgebra.

If there is a subbundle $\mathcal{A} \subset \text{End}(E)$ as in the second statement, then $(Y, f)$ in the first statement can be so chosen that $\mathcal{A} = f_*\mathcal{O}_Y$. In that case, the number of connected components of the scheme $Y$ coincides with $\dim H^0(X, \mathcal{A})$.

**Corollary 1.2.** Given $X$ and $E$ as in Theorem 1.1, the following two are equivalent:

1. There is an étale covering
   
   $f : Y \rightarrow X$

   such that $f_*\mathcal{O}_Y = E$.

2. There is a fiberwise injective homomorphism $\alpha : E \rightarrow \text{End}(E)$ such that for each closed point $x \in X$, the subspace $\alpha(E_x) \subset \text{End}(E)_x = \text{End}(E_x)$ is a Cartan subalgebra.

When these hold, the number of connected components of the scheme $Y$ coincides with $\dim H^0(X, E)$.

In Section 4 we describe, in terms of the above criterion, when an étale cover factors through another given étale covering.

2. Direct image of line bundles by étale coverings

2.1. **Construction of homomorphism of vector bundles from direct image.** Let $k$ be an algebraically closed field. Let $X$ be an irreducible projective variety defined over $k$. Take any pair $(Y, f)$, where $Y$ is a projective scheme and

$f : Y \rightarrow X$

is an étale covering of degree $d$. We do not assume that $Y$ is connected. Take any line bundle $L$ over $Y$. The direct image

$E := f_*L \rightarrow X$ \hspace{1cm} (2.1)

is a vector bundle of rank $d$. The direct image $f_*\mathcal{O}_Y$ of the structure sheaf will be denoted by $W$.

We have the natural homomorphism $E \otimes W \rightarrow E$ which on any open set $U \subseteq X$ is induced by the bilinear map

$\mathcal{O}_Y(f^{-1}U) \times L(f^{-1}U) \rightarrow L(f^{-1}U), \quad (y, l) \mapsto y \cdot l,$
from the universal property of tensor-product. This homomorphism defines a homomorphism

$$\varphi : W \rightarrow \text{End}(E) = E \otimes E^*,$$

where $E$ is the vector bundle in (2.1).

The fibers of $\text{End}(E)$ are reductive Lie algebras isomorphic to $\text{Lie}(\text{GL}(d,k)) = M(d,k)$ (the $d \times d$ matrices with entries in $k$). We recall that a Lie subalgebra $A$ of $M(d,k)$ is a Cartan subalgebra if

- $\dim A = d$, and
- there is an element $T \in \text{GL}(d,k)$ such that the conjugation of $M(d,k)$ by $T$ takes $A$ into the space of diagonal matrices.

**Lemma 2.1.** The homomorphism $\varphi$ in (2.2) satisfies the condition that for every closed point $x \in X$, the image $\varphi(x)(W_x)$ is a Cartan subalgebra of $\text{End}(E)_x = \text{End}(E_x)$.

**Proof.** Fix an ordering of the elements of the set $f^{-1}(x)$ of cardinality $d$. Let $\{y_1, \cdots, y_d\}$ be this ordered set $f^{-1}(x)$. Fix a nonzero element $v_i \in L_{y_i}$ for each $1 \leq i \leq d$. Since

$$E_x = \bigoplus_{i=1}^d L_{y_i},$$

the collection $\{v_1, \cdots, v_d\}$ defines an ordered basis of the vector space $E_x$. Similarly, we have

$$W_x = \bigoplus_{i=1}^d k_i,$$

where $k_i$ is a copy of $k$. For any $c \in k_i \subset W_x$, the endomorphism $\varphi(c) \in \text{End}(E_x)$ sends the basis element

- $v_i$ to $c \cdot v_i$, and
- $v_j, j \neq i$, to 0.

Therefore, the image $\varphi(W_x)$ is the space of all diagonal matrices with respect to the above basis $\{v_1, \cdots, v_d\}$. \qed

Since $H^0(Y, \mathcal{O}_Y) = H^0(X, f_*\mathcal{O}_Y)$, and the dimension of $H^0(Y, \mathcal{O}_Y)$ coincides with the number of connected components of the scheme $Y$, Lemma 2.1 has the following corollary:

**Corollary 2.2.** The number of connected components of the scheme $Y$ coincides with the dimension of $H^0(X, W)$.

### 2.2. Criterion for direct images under étale maps

Let $F$ and $V$ be vector bundles on $X$ of same rank $d$. Let

$$\varphi : V \rightarrow \text{End}(F) = F \otimes F^*$$

be an $\mathcal{O}_X$–linear homomorphism.

**Proposition 2.3.** Assume that $\varphi$ in (2.3) satisfies the condition that for every closed point $x \in X$, the image $\varphi(V_x)$ is a Cartan subalgebra of the Lie algebra $\text{End}(F)_x = \text{End}(F_x)$. Then there is an étale covering

$$f : Y \rightarrow X$$

and a line bundle $L$ over $Y$, such that

$$f_\ast L = F \quad \text{and} \quad f_\ast O_Y = V.$$  

Furthermore, the homomorphism $\varphi$ coincides with the homomorphism in (2.2) corresponding to the above triple $(Y, f, L)$.

**Proof.** For any closed point $x \in X$, consider the Cartan subalgebra $\varphi(V_x) \subset \text{End}(F_x)$. It produces an unordered set of $d$ lines in $F_x$

$$\{l^x_t\}_{t \in B_x}, \quad \# B_x = d \tag{2.4}$$

such that

- the $d$ lines $\{l^x_t\}_{t \in B_x}$ together generate the fiber $F_x$, and
- for each $t \in B_x$, there is a unique functional $\mu^x_t \in V_x^*$ with the property that for all $v \in V_x$, we have

$$\varphi(v)(w) = \begin{cases} \mu^x_t(v) \cdot w & \text{if } w \in l^x_t \\ 0 & \text{if } w \in l^x_s, s \neq t. \end{cases}$$

In (2.4) we use the notation $\{l^x_t\}_{t \in B_x}$ instead of $\{l^x_1, \cdots, l^x_d\}$ in order to emphasize that in general these $d$ lines do not have any ordering which can be chosen uniformly over $X$. We note that the $d$ elements $\{\mu^x_t\}_{t \in B_x}$ of $V_x^*$ are distinct. In fact, $\{\mu^x_t\}_{t \in B_x}$ is a basis of the dual vector space $V_x^*$.

The quasiprojective variety defined by the total space of the dual vector bundle $V^*$ will be denoted by $\mathbb{V}^*$. The locus in $\mathbb{V}^*$ of the above collection $\{\mu^x_t\}_{t \in B_x, x \in X}$ defines a reduced subscheme $Y \subset \mathbb{V}^*$. Let

$$f : Y \longrightarrow X \tag{2.5}$$

be the restriction to $Y$ of the natural projection $\mathbb{V}^* \longrightarrow X$. This map $f$ defines an étale covering of $X$ of degree $d$ because the $d$ lines $\{l^x_t\}_{t \in B_x}$ in (2.4) can be uniformly ordered over suitable étale open subsets of $X$.

Consider the pulled back vector bundle $f^\ast F \longrightarrow Y$. It has a line subbundle $L$ whose fiber over any closed point $\mu^x_t \in Y$ is the line $l^x_t$ contained in $F_x$.

The projection formula gives that $f_\ast f^\ast F = F \otimes f_\ast O_Y$. This and the trace homomorphism $f_\ast O_Y \longrightarrow f_\ast O_X$ together give a homomorphism $f_\ast f^\ast F \longrightarrow F$. This homomorphism and the inclusion map $L \hookrightarrow f^\ast F$ together produce a homomorphism

$$\eta : f_\ast L \longrightarrow F. \tag{2.6}$$

This $\eta$ is an isomorphism because the $d$ lines $\{l^x_t\}_{t \in B_x}$ in (2.4) generate $F_x$.

Now, as constructed in (2.2), we have a fiberwise injective homomorphism

$$\widehat{\varphi} : f_\ast O_Y \longrightarrow \text{End}(f_\ast L) = \text{End}(F).$$

It is straight-forward to check that the image of $\widehat{\varphi}$ coincides with the image of $\varphi$. Hence the composition $(\widehat{\varphi}^{-1})_{|\varphi(V)} \circ \varphi$ is an isomorphism from $V$ to $f_\ast O_Y$.

In terms of the above isomorphisms $V \simto f_\ast O_Y$ and $\eta$ in (2.6), the homomorphism $\varphi$ in the statement of the proposition coincides with the homomorphism in (2.2). \qed
Note that the condition in Proposition 2.3 that $\varphi(V_x)$ is a Cartan subalgebra of $\text{End}(F_x)$ for every $x \in X$ implies that the homomorphism $\varphi$ is fiberwise injective.

Combining Lemma 2.1, Corollary 2.2 and Proposition 2.3 we have the following:

**Theorem 2.4.** Given a vector bundle $E$ on $X$ of rank $d$, the following two are equivalent:

1. There is an étale covering
   
   $$ f : Y \longrightarrow X, $$

   where $Y$ is a projective scheme, and a line bundle $L$ on $Y$, such that $f_*L = E$.

2. There is a subbundle $A \subset \text{End}(E)$ of rank $d$ such that for each closed point $x \in X$, the subspace $A_x \subset \text{End}(E)_x = \text{End}(E_x)$ is a Cartan subalgebra.

If there is a subbundle $A \subset \text{End}(E)$ as in the second statement, then $(Y, f)$ in the first statement can be so chosen that $A = f_*O_Y$. In that case, the number of connected components of $Y$ coincides with $\dim H^0(X, A)$.

**Corollary 2.5.** A vector bundle $F$ on $X$ of rank $d$ splits into a direct sum of $d$ line bundles if and only if there is a trivial subbundle of rank $d$

$$ \iota : O_X^{\oplus d} \hookrightarrow \text{End}(F) $$

such that for each closed point $x \in X$, the subspace

$$ \text{image}(\iota(x)) \subset \text{End}(F)_x = \text{End}(F_x) $$

is a Cartan subalgebra.

**Proof.** If $F = \bigoplus_{i=1}^d L_i$, then the homomorphism

$$ O_X^{\oplus d} = \bigoplus_{i=1}^d \text{End}(L_i) \hookrightarrow \text{End}(F) $$

satisfies the condition in the statement of the corollary.

To prove the converse, assume that there is a trivial subbundle of rank $d$

$$ \iota : O_X^{\oplus d} \hookrightarrow \text{End}(F) $$

satisfying the condition in the corollary. Now the unordered set in (2.4) becomes uniformed ordered over entire $X$. Therefore, the covering $Y$ in (2.5) becomes a disjoint union of $d$ copies of $X$. Consequently, the vector bundle $f_*L$ in (2.6) is a direct sum of line bundles. Since $\eta$ in (2.0) is an isomorphism, the vector bundle $F$ splits into a direct sum of $d$ line bundles. \(\square\)

Setting $L = O_Y$ in Theorem 2.3 we have the following:

**Corollary 2.6.** Given a vector bundle $E$ on $X$, the following two are equivalent:

1. There is an étale covering
   
   $$ f : Y \longrightarrow X $$

   such that $f_*O_Y = E$.

2. There is a fiberwise injective homomorphism $\alpha : E \longrightarrow \text{End}(E)$ such that for each closed point $x \in X$, the subspace $\alpha(E_x) \subset \text{End}(E)_x = \text{End}(E_x)$ is a Cartan subalgebra.
When these statements hold, the number of connected components of the scheme $Y$ coincides with $\dim H^0(X, E)$.

**Remark 2.7.** Let $E$ be a vector bundle on $X$ along with two subbundles $A, B \subseteq \text{End}(E)$ such that for each closed point $x \in X$, both $A_x$ and $B_x$ are Cartan subalgebras of $\text{End}(E_x)$. Let $f : Y \rightarrow X$ (respectively, $g : Z \rightarrow X$) be the étale covering and $L \rightarrow Y$ (respectively, $L' \rightarrow Z$) be the line bundle associated to $A$ (respectively, $B$). We note that if $T$ is an automorphism of $E$ such that $B = T^{-1}AT$, then there exists an isomorphism of $X$–schemes $h : Y \rightarrow Z$ such $h^*L' = L$. The converse is also true.

2.3. **An example.** Let $E$ be a vector bundle on $X$ such that there are two Cartan subalgebras $A$ and $B$ of $\text{End}(E)$. It is natural to ask whether there is an automorphism $T$ of $E$ such that $B = T^{-1}AT$. We give an example where there is no such $T$.

Let $X$ be a smooth projective elliptic curve defined over $\mathbb{C}$. It has exactly three non-trivial line bundles of order two. Let $L$ and $M$ be two distinct nontrivial line bundles on $X$ of order two. Therefore, the third nontrivial line bundle of order two is $L \otimes M$.

A theorem of Atiyah says that there are stable vector bundles on $X$ of rank two and degree one, and any two of them differ by tensoring with a holomorphic line bundle of degree zero [At p. 433, Theorem 6], [At p. 434, Theorem 7] (see [BB] p. 70, Theorem 4.6, [BB] p. 70–71, Theorem 4.7 for an exposition). Take a stable vector bundle $E$ on $X$ of rank two and degree one. If $N$ is a holomorphic line bundle on $X$, then

$$\text{End}(E) = E \otimes E^* = (E \otimes N) \otimes (E \otimes N)^* = \text{End}(E \otimes N).$$

Therefore, the endomorphism bundle $\text{End}(E)$ does not depend on the choice of the stable vector bundle $E$ of rank two and degree one. Since $E$ is stable, the vector bundle $\text{End}(E)$ is polystable. From the classification of vector bundles on $X$ we know that any polystable vector bundle on $X$ of degree zero is a direct sum of holomorphic line bundles [At p. 433, Theorem 6] (see also [BB] p. 70, Theorem 4.6]). It is known that

$$\text{End}(E) = \mathcal{O}_X \oplus L \oplus M \oplus (L \otimes M). \quad (2.7)$$

This can also be seen as follows. Let $f : Y \rightarrow X$ be the unramified double cover corresponding to $L$. Then there is a holomorphic line bundle $\xi$ on $Y$ of degree one such that $E = f_*\xi$. Since $f_*\mathcal{O}_Y = L \oplus \mathcal{O}_X$, this implies that $L$ is a subbundle of $\text{End}(E)$. Hence $L$ is a direct summand of $\text{End}(E)$ because $\text{End}(E)$ is polystable of degree zero. Similarly, $M$ and $M \otimes L$ are also direct summands of $\text{End}(E)$. Therefore, it follows that $\text{End}(E)$ decomposes as in (2.7).

Since $E = f_*\xi$, we know that $f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L$ is a Cartan subalgebra of $\text{End}(E)$. Similarly, $\mathcal{O}_X \oplus M$ and $\mathcal{O}_X \oplus (M \otimes L)$ are also Cartan subalgebra bundles of $\text{End}(E)$.

On the other hand $H^0(X, \text{End}(E)) = \mathbb{C}$ because $E$ is stable; note that this also follows from (2.7). Hence the automorphisms of $E$ act trivially on $\text{End}(E)$. Therefore, the above Cartan subalgebra bundles $\mathcal{O}_X \oplus L$, $\mathcal{O}_X \oplus M$ and $\mathcal{O}_X \oplus (M \otimes L)$ are not related by automorphism of $E$.

### 3. Ramified coverings of curves

Throughout this section we assume that $X$ is an irreducible smooth projective curve defined over an algebraically closed field $k$. 

A quasiparabolic structure on a vector bundle $E$ over $X$ consists of the following:

- a finite set of reduced distinct closed points $S = \{x_1, \cdots, x_n\} \subset X$, and
- for each $x_i \in S$, a filtration of subspaces
  \[ 0 \subsetneq F^i_1 \subsetneq \cdots \subsetneq F^i_{\ell_i} = E_{x_i} \]
  of the fiber $E_{x_i}$.

The subset $S$ is called the parabolic divisor. A quasiparabolic bundle is a vector bundle equipped with a quasiparabolic structure. A parabolic vector bundle is a quasiparabolic bundle $(E, S, \{F^i_j\})$ as above together with real numbers $\lambda^i_j, 1 \leq i \leq n, 1 \leq j \leq \ell_i$, such that

\[ 1 > \lambda^i_1 > \lambda^i_2 > \cdots > \lambda^i_{\ell_i} \geq 0. \]

These numbers $\lambda^i_j$ are called parabolic weights. For notational convenience, a parabolic vector bundle $(E, S, \{F^i_j\}, \{\lambda^i_j\})$ is abbreviated as $E_*$. See [MS] for more on parabolic vector bundles.

We will consider only rational parabolic weights. Henceforth, we will assume that all the parabolic weights are rational numbers.

Let $Y$ be a smooth projective curve, which need not be irreducible, and let

\[ f: Y \longrightarrow X \]

be a finite separable morphism. Let $L_*$ be a parabolic line bundle on $Y$ with $L$ being the underlying line bundle. The direct image $E := f_*L$ has a natural parabolic structure which will be described below.

Let $R \subset Y$ be the set of points where $f$ is ramified. Let $P \subset Y$ be the parabolic divisor for $L_*$. The parabolic divisor for the parabolic structure on $E$ is the image $f(R \cup P)$. Take a point $x \in f(R \cup P) \setminus f(R)$ in the complement of $f(R)$. Then

\[ (f_*L)_x = \bigoplus_{y \in f^{-1}(x)} L_y. \]

The quasiparabolic filtration of $(f_*L)_x$ is constructed using this decomposition. The parabolic weight of the line $L_y \subset (f_*L)_x$ is the parabolic weight of $L_*$ at the point $y$. If $y$ is not a parabolic point of $L_*$, then the parabolic weight of the line $L_y \subset (f_*L)_x$ is taken to be zero. Combining these we get a parabolic structure on $E_x$.

Now take any $x \in f(R)$. Let $\{y_1, \cdots, y_m\}$ be the reduced inverse image $f^{-1}(x)_{\text{red}}$. The multiplicity of $f$ at $y_i$ will be denoted by $b_i$; so $f^{-1}(x) = \sum_{i=1}^m b_i y_i$. For every $1 \leq i \leq m$, let $V_i \subset E_x$ be the image in the fiber $E_x$ for the natural homomorphism

\[ f_* (L \otimes \mathcal{O}_Y (- \sum_{j=1, j \neq i}^m b_j y_j)) \longrightarrow f_*L = E. \]

We have $\dim V_i = b_i$, and

\[ E_x = \bigoplus_{i=1}^m V_i. \quad (3.1) \]

We will construct a weighted filtration on each $V_i$; these combined together will give the weighted filtration of $E_x$ using (3.1). For each $0 \leq \ell \leq b_i$, let $F^i_{\ell} \subset E_x$ be the image for
the natural homomorphism

\[ f_*(L \otimes \mathcal{O}_Y(-\ell y_i - \sum_{j=1, j \neq i}^m b_j y_j)) \longrightarrow f_* L. \]

Note that \( F_{b_i}^i = 0 \) and \( F_0^i = V_i \), in particular, \( F_\ell^i \subset V_i \) for all \( \ell \). It is easy to see that \( \dim F_\ell^i = b_i - \ell \), so \( \{F_\ell^i\}_{\ell=0}^b \) is a complete flag of subspaces of \( V_i \). The weight of the subspace \( F_\ell^i \subset V_i, 0 \leq \ell < b_i \), is \((\ell + \lambda_y)/b_i\), where \( \lambda_y \) is the parabolic weight of \( L_* \) at \( y_i \); if \( y_i \) is not a parabolic point of \( L_* \), then \( \lambda_y \) is taken to be zero. Note that \( 0 \leq (\ell + \lambda_y)/b_i < 1 \).

Now the parabolic structure on \( E \) over \( x \) is given by these weighted filtrations using (3.1). More precisely, for any \( 0 \leq c < 1 \), if \( S_X^i(c) \subset V_i, 1 \leq i \leq m \), is the subspace of \( V_i \) of weight \( c \), then the subspace of \( E_x \) of parabolic weight \( c \) is the direct sum \( \bigoplus_{i=1}^m S^i_X(c) \).

The direct image \( f_* L \) equipped with the above parabolic structure will be denoted by \( f_* L_* \).

Since \( H^i(Y, L) = H^i(X, f_* L) \), using the Riemann–Roch theorem for \( L \) and \( f_* L \), we have

\[ \text{degree}(f_* L) = \text{degree}(L) - \text{genus}(Y) + 1 + \text{degree}(f)(\text{genus}(X) - 1), \]

where \( \text{genus}(Y) = \dim H^1(Y, \mathcal{O}_Y) \) (recall that \( Y \) need not be connected). On the other hand,

\[ 2(\text{genus}(Y) - 1) = \text{degree}(K_Y) = \text{degree}(K_X) + \sum_{y \in R} (b_y - 1) = 2(\text{genus}(X) - 1) + \sum_{y \in R} (b_y - 1), \]

where \( b_y \) is the multiplicity of \( f \) at \( y \) while \( K_X \) and \( K_Y \) are the canonical line bundles of \( X \) and \( Y \) respectively. From these it follows that

\[ \text{par-deg}(f_* L_*) = \text{par-deg}(L_*). \]

Generalizing the constructions of direct sum, tensor product and dual of vector bundles, there are direct sum, tensor product and dual of parabolic vector bundles [Y6, MY, B2]. It should be mentioned that for two parabolic vector bundles \( E_* \) and \( F_* \), while the underlying vector bundle for the parabolic direct sum \( E_* \oplus F_* \) is the direct sum of the vector bundles underlying \( E_* \) and \( F_* \), the underlying vector bundle for the parabolic tensor product \( E_* \otimes F_* \) is not necessarily the tensor product of the vector bundles underlying \( E_* \) and \( F_* \). Similarly, the underlying vector bundle for the parabolic dual \( E^*_* \) is different from the dual of the vector bundles underlying \( E_* \), unless the parabolic structure on \( E_* \) is trivial (meaning there are no nonzero parabolic weights).

The endomorphism bundle for a parabolic vector bundle \( E_* \) is defined to be

\[ \text{End}(E_*) := E_* \otimes E^*_*; \quad (3.2) \]

it should be emphasized that both the tensor product and dual in (3.2) are in the parabolic category.

3.1. **When the characteristic is zero.** In this subsection we assume that the characteristic of the base field \( k \) is zero.

Let \( E_* \) be a parabolic vector bundle on \( X \). Let \( S \subset X \) be the parabolic divisor for \( E_* \). The vector bundle underlying \( E_* \) will be denoted by \( E_0 \). Consider the endomorphism (parabolic) bundle \( \text{End}(E_*) \) defined in (3.2). The vector bundle underlying it will be
denoted by \( \text{End}(E_0) \). The two vector bundles \( \text{End}(E_0) \) and \( \text{End}(E_0) \) are identified over
the complement \( X \setminus S \). This isomorphism extends to a homomorphism
\[
\beta : \text{End}(E_0) \rightarrow \text{End}(E_0)
\]
over entire \( X \). For any point \( x \in S \), the subspace \( \beta(x)((\text{End}(E_0)x) \subset \text{End}(E_0)x \) coincides with the space of endomorphisms of the fiber \((E_0)x \) that preserve the quasiparabolic
filtration of \((E_0)x \).

A parabolic vector bundle on \( X \) can be expressed as the invariant direct image of an
equivariant vector bundle over a (ramified) Galois cover of \( X \) \([Bi1],[Bo1],[Bo2]\); recall the
assumption that all the parabolic weights are rational numbers. Let \( \tilde{X} \) be an irreducible
smooth projective curve,
\[
\gamma : \tilde{X} \rightarrow X
\]
a Galois covering which may be ramified, and \( \mathcal{E} \) a \( \Gamma \)-linearized vector bundle on \( \tilde{X} \), where
\( \Gamma := \text{Gal}(\gamma) \), such that \( E_0 \) corresponds to \( \mathcal{E} \). The vector bundle \( E \) underlying \( E_0 \) is
the invariant direct image \( (\gamma_\ast \mathcal{E})^1 \); note that the action of \( \Gamma \) on \( \mathcal{E} \) produces an action of \( \Gamma \)
on \( \gamma_\ast \mathcal{E} \). In particular, we have \( \text{rank}(E) = \text{rank}(\mathcal{E}) \). Consider the finite subset \( D' \) of \( \tilde{X} \)
consisting of all points \( y \in \tilde{X} \) satisfying the following two conditions:

- \( y \) has nontrivial isotropy for the action of \( \Gamma \) on \( \tilde{X} \), and
- the action of isotropy subgroup \( \Gamma_y \) for \( y \) on the fiber \( \mathcal{E}_y \) is nontrivial.

The image \( \gamma(D') \subset X \) is the subset of \( S \) consisting of all points over which \( E_0 \) has
nontrivial parabolic weight. The above isotropy subgroup \( \Gamma_y \) for \( y \) is cyclic; let \( m_y \) be the
order of \( \Gamma_y \). Fix a generator \( \nu \) of \( \Gamma_y \). A rational number \( 0 \leq c < 1 \) is a parabolic weight
for \( E_0 \) at \( \gamma(y) \) if and only if \( \exp(2\pi \sqrt{-1}c) \) is an eigenvalue for the action of \( \nu \) on \( \mathcal{E}_y \).
In particular, \( c \cdot m_y \) is an integer.

The subbundles of \( E_0 \) with the parabolic structure induced by \( E_0 \) correspond to sub-
bundles of \( \mathcal{E} \) preserved by the action of \( \Gamma \). The parabolic vector bundles \( E_0^* \) and \( \text{End}(E_0^*) \)
correspond to \( \mathcal{E}^* \) and \( \text{End}(\mathcal{E}) \) respectively; note that the \( \Gamma \)-linearization of \( \mathcal{E} \) induces
\( \Gamma \)-linearizations on both \( \mathcal{E}^* \) and \( \text{End}(\mathcal{E}) \).

A subbundle \( \mathcal{A} \subset \text{End}(E_0^*) \) will be called a Cartan subalgebra bundle if the following
two conditions hold:

1. For each closed point \( x \in X \setminus S \), the fiber \( A_x \subset (\text{End}(E_0^*)x = \text{End}(E_0)x \) is a
Cartan subalgebra of the Lie algebra \( \text{End}(E_0)x \).
2. For each point \( x \in S \), the fiber of the \( \Gamma \)-linearized subbundle of \( \text{End}(\mathcal{E}) \) corresponding to \( \mathcal{A} \) over a point \( y \in \gamma^{-1}(x) \) is a Cartan subalgebra of the Lie algebra
\( \text{End}(\mathcal{E}_y) \). (If this condition holds for one point of \( \gamma^{-1}(x) \) then it holds for all points
of \( \gamma^{-1}(x) \); this is because of the action of \( \Gamma \).)

The above definition of a Cartan subalgebra bundle of \( \text{End}(E_0) \) does not depend on
the choice of the covering \( \gamma \). To see this, if
\[
\gamma' : \tilde{X}' \rightarrow X
\]
is another such covering, then consider the normalization \( \mathcal{X}' \) of the fiber product \( \tilde{X} \times_X \tilde{X}' \).
This covering \( \mathcal{X}' \) of \( X \) also satisfies the conditions. If \( \mathcal{E}' \) is the equivariant vector bundle on
\( \tilde{X}' \) corresponding to \( E_0^* \), then the pullbacks of \( \mathcal{E} \) and \( \mathcal{E}' \) to \( \mathcal{X}' \) are equivaritantly isomorphic
to the equivariant bundle on \( \mathcal{X}' \) corresponding to \( E_0^* \). Hence a fiberwise decomposition of
$E$ produces a fiberwise decomposition of its pullback to $X$, which in turn descends to a fiberwise decomposition of $E'$.

Note that $A \subset \text{End}(E_*)_0$ is a Cartan subalgebra bundle if and only if the $\Gamma$–linearized subbundle $\tilde{A} \subset \text{End}(E)$ corresponding to $A$ has the property that for every $y \in \tilde{X}$, the subspace $\tilde{A}_y \subset \text{End}(E)_y$ is a Cartan subalgebra of the Lie algebra $\text{End}(E)_y$.

**Theorem 3.1.** Given a parabolic vector bundle $E_*$ on $X$, the following two are equivalent:

1. There is a finite surjective map $f : Y \to X$, where $Y$ is a smooth projective curve not necessarily connected, and a parabolic line bundle $L_*$ on $Y$, such that
   - $E_*$ has a nontrivial parabolic weight at $x \in X$ if and only if there is a point $y \in f^{-1}(x)$ satisfying the condition that either $y$ is a parabolic point for $L_*$ or $f$ is ramified at $y$ (or both), and
   - the parabolic vector bundle $f_*L_*$ is isomorphic to $E_*$. 
2. There is a Cartan subalgebra bundle $A$ of $\text{End}(E_*)_0$.

When there is a Cartan subalgebra bundle $A \subset \text{End}(E_*)_0$, the pair $(Y, f)$ in the first statement can be so chosen that the subbundle $A$ equipped with the parabolic structure induced by $\text{End}(E_*)$ is isomorphic to $f_*\mathcal{O}_Y$ equipped with the natural parabolic structure (the parabolic structure on $\mathcal{O}_Y$ is the trivial one, meaning it has no nonzero parabolic weight). In that case, the number of connected components of $Y$ coincides with $\dim H^0(X, A)$.

**Proof.** Fix a Galois covering $(\tilde{X}, \gamma)$ as in (3.3) with $\Gamma = \text{Gal}(\gamma)$ such that there is a $\Gamma$–linearized vector bundle $E$ on $\tilde{X}$ that corresponds to the parabolic vector bundle $E_*$. Assume that the first statement in the theorem holds. Take $(Y, f, L_*)$ as in the first statement. Let $\tilde{Y}$ denote the normalization of the fiber product $Y \times_X \tilde{X}$. Let $p_1 : \tilde{Y} \to Y$ and $p_2 : \tilde{Y} \to \tilde{X}$ be the natural projections. We note that the normalization of a fiber product has the following property: Consider two ramified coverings $\mathbb{A}^1 \to \mathbb{A}^1$ defined by $z \mapsto z^a$ and $z \mapsto z^b$ respectively; then the projection to the second factor of the normalization of the fiber product $\mathbb{A}^1 \times_{\mathbb{A}^1} \mathbb{A}^1$ is an étale covering of $\mathbb{A}^1$ if $b$ is a multiple of $a$. From this it follows that the above projection $p_2$ is an étale covering.

The action of $\Gamma$ on $\tilde{X}$ produces an action of $\Gamma$ on $Y \times_X \tilde{X}$, hence $\tilde{Y}$ is equipped with an action of $\Gamma$; the map $p_2$ intertwines the actions of $\Gamma$ on $\tilde{Y}$ and $\tilde{X}$. The above morphism $p_1$ is evidently $\Gamma$ invariant, and hence it produces a morphism $\tilde{Y}/\Gamma \to Y$; it is easy to see that this morphism is an isomorphism. There is a $\Gamma$–linearized line bundle $\tilde{L} \to \tilde{Y}$ that corresponds to the parabolic line bundle $L_*$ on $Y$.

Since $p_2$ is $\Gamma$–equivariant, the action of $\Gamma$ on $L$ produces an action of $\Gamma$ on the direct image $p_{2*}\tilde{L}$. The parabolic vector bundle $f_*L_*$ corresponds to this $\Gamma$–linearized vector bundle $p_{2*}\tilde{L}$. 
Since $p_2$ is étale, by Lemma 2.1 there is a homomorphism from the direct image
\[ \xi : p_2_* O_{\tilde{Y}} \longrightarrow \text{End}(p_2_* \tilde{L}) \]
whose fiberwise images are Cartan subalgebras. The action of $\Gamma$ on $\tilde{Y}$ produces a $\Gamma$–linearization on $O_{\tilde{Y}}$. Since $p_2$ is $\Gamma$–equivariant, the action of $\Gamma$ on $O_{\tilde{Y}}$ produces an action of $\Gamma$ on $p_2_* O_{\tilde{Y}}$. The above homomorphism $\xi$ is $\Gamma$–equivariant for the action of $\Gamma$ on $\text{End}(p_2_* \tilde{L})$ induced by the action of $\Gamma$ on $p_2_* \tilde{L}$ and the action of $\Gamma$ on $p_2_* O_{\tilde{Y}}$. Therefore, the second statement in the theorem holds.

Now assume that the second statement in the theorem holds. Let $\mathcal{A}$ be a Cartan subalgebra bundle of $\text{End}(E_*)_0$. The $\Gamma$–linearized vector bundle $\text{End}(E_*)$ corresponds to the parabolic vector bundle $\text{End}(E_*)$. Let $\mathcal{B}$ be the subbundle of $\text{End}(E_*)$ preserved by the action of $\Gamma$ such that $\mathcal{B}$ corresponds to the subbundle $\mathcal{A}$ of $\text{End}(E_*)_0$. So for any closed point $x \in \tilde{X}$, the subspace $\mathcal{B}_x \subset \text{End}(E_x)$ is a Cartan subalgebra.

By Proposition 2.3 there is an étale covering
\[ \phi : \tilde{Y} \longrightarrow \tilde{X} \]
and a line bundle $\mathcal{L}$ on $\tilde{Y}$ such that
\[ \phi_* \mathcal{L} = \mathcal{E}. \] (3.4)
Since $\mathcal{B}$ is preserved by the action of $\Gamma$ on $\text{End}(\mathcal{E})$ induced by the action of $\Gamma$ on $\mathcal{E}$, from the construction in Proposition 2.3 it follows that

- $\Gamma$ acts on $\tilde{Y}$,
- the map $\phi$ intertwines the actions of $\Gamma$ on $\tilde{Y}$ and $\tilde{X}$,
- the line bundle $\mathcal{L}$ is $\Gamma$–linearized, and
- the isomorphism in (3.4) is $\Gamma$–equivariant.

Since $\phi$ is $\Gamma$–equivariant, the composition
\[ \gamma \circ \phi : \tilde{Y} \longrightarrow X \]
factors through a map
\[ f : Y := \tilde{Y}/\Gamma \longrightarrow X. \] (3.5)

Let $q : \tilde{Y} \longrightarrow \tilde{Y}/\Gamma = Y$ be the quotient map. The parabolic line bundle on $Y$ corresponding to the $\Gamma$–linearized line bundle $\mathcal{L}$ on $\tilde{Y}$ will be denoted by $L_*$. For any vector bundle $W \longrightarrow \tilde{Y}$, there is a canonical isomorphism
\[ (f \circ q)_* W = f_* q_* W \quad \cong \quad \gamma_* \phi_* W = (\gamma \circ \phi)_* W. \] (3.6)
The action of $\Gamma$ on $\mathcal{L}$ produces actions of $\Gamma$ on both $(f \circ q)_* \mathcal{L}$ and $(\gamma \circ \phi)_* \mathcal{L}$. The isomorphism in (3.6) is $\Gamma$–equivariant for $W = \mathcal{L}$.

Since the isomorphism in (3.6) for $\mathcal{L}$ is $\Gamma$–equivariant, it can be deduced that for the above parabolic line bundle $L_*$ on $Y$, the parabolic vector bundle $f_* L_*$ on $X$, where $f$ is defined in (3.5), corresponds to the $\Gamma$–linearized vector bundle $\phi_* \mathcal{L}$ on $\tilde{X}$. Also, as the isomorphism in (3.4) for $\mathcal{L}$ is $\Gamma$–equivariant, and the parabolic vector bundle $E_*$ corresponds to the $\Gamma$–linearized vector bundle $\mathcal{E}$, from the above observation on (3.6) it also follows that the two parabolic vector bundle $f_* L_*$ and $E_*$ are isomorphic.
In view of the above proof, from Theorem 2.4 we conclude the following: If there is a Cartan subalgebra bundle \( \mathcal{A} \subset \text{End}(E^*_0) \), then the pair \((Y, f)\) in the first statement of the theorem can be so chosen that the subbundle \( \mathcal{A} \) equipped with the parabolic structure induced by \( \text{End}(E^*_0) \) is isomorphic to \( f_*\mathcal{O}_Y \) equipped with the natural parabolic structure (the parabolic structure on \( \mathcal{O}_Y \) is the trivial one). In that case, the number of connected components of \( Y \) coincides with \( \dim H^0(X, \mathcal{A}) \). \( \square \)

In Theorem 3.1 setting \( L = \mathcal{O}_Y \) equipped with the trivial parabolic structure, we have the following:

**Corollary 3.2.** Given a parabolic vector bundle \( E^*_0 \) on \( X \), the following two are equivalent:

1. There is a finite surjective map
   \[ f : Y \longrightarrow X, \]
   where \( Y \) is a smooth projective curve not necessarily connected, such that \( f_*\mathcal{O}_Y \) equipped with the natural parabolic structure is isomorphic to \( E^*_0 \) (the parabolic structure on \( \mathcal{O}_Y \) is the trivial one).
2. There is a homomorphism of parabolic bundles \( \alpha : E^*_0 \longrightarrow \text{End}(E^*_0) \) such that
   - \( \alpha \) is an isomorphism of \( E^*_0 \) with the image \( \alpha(E^*_0) \) equipped with the parabolic structure induced by the parabolic structure of \( \text{End}(E^*_0) \), and
   - \( \alpha(E^*_0) \) is a Cartan subalgebra bundle of \( \text{End}(E^*_0)_0 \), where \( E^*_0 \) is the vector bundle underlying \( E^*_0 \).

When these hold, the number of connected components of \( Y \) coincides with \( \dim H^0(X, E^*_0) \).

3.2. **The case of positive characteristic.** Assume that the base field \( k \) is of positive characteristic. Let \( p \) denote the characteristic of \( k \).

The correspondence between parabolic vector bundles and equivariant vector bundles used extensively in Section 3.1 remains valid under the following tameness condition (see \([Bo1], [Bo2]\)):

If \( a/b \) is a parabolic weight, where \( a \) and \( b \) are nonzero coprime integers, then we assume that \( b \) is not a multiple of \( p \).

Once we impose the above condition on \( E^*_0 \), the proof of Theorem 3.1 goes through without any change. Similarly, Corollary 3.2 remains valid after the above tameness condition on \( E^*_0 \) is imposed.

**Remark 3.3.** Theorem 2.4 and Corollary 2.6 remain valid when \( X \) is a root-stack; see \([Ca], [Bo1], [Bo2]\) for root-stacks. When the parabolic divisor is a simple normal crossing divisor, quasiparabolic filtrations satisfy certain conditions and the parabolic weights are tame (in positive characteristic case), there is an equivalence between parabolic vector bundles over a smooth projective variety \( Y \) and vector bundles over smooth root-stack whose underlying coarse moduli space is \( Y \); see \([Bo1], [Bo2]\). Therefore, under the above assumptions on parabolic structure, Theorem 3.1 and Corollary 3.2 extend to higher dimensions.

4. **Factoring of covering maps**

In this section we assume that the characteristic of \( k \) is zero.
Let be a commutative diagram of étale coverings, and define \( V := h_*O_Z \) and \( W := f_*O_Y \).

Notice that the homomorphism \( O_Z \rightarrow g_*O_Y \) induces a homomorphism \( V \rightarrow W \)

which is fiberwise injective, and so \( V \) is a subbundle of \( W \). Notice moreover that \( V \) gives way to a Cartan subalgebra of \( \text{End}(V) \) and \( W \) produces a Cartan subalgebra of \( \text{End}(W) \).

Let \( V^* \) and \( W^* \) be the total spaces of \( V^* \) and \( W^* \), respectively. We have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
Y \\
\downarrow g \\
Z \\
\downarrow h \\
X
\end{array}
\end{array}
\end{array}
\]

where the map \( W^* \rightarrow V^* \) is the dual of the homomorphism in \((4.1)\), and \((g_*O_Y)^* \) is the restriction of this fiber bundle \( W^* \) to \( Z \subset V^* \). Over \( x \in X \), in \( W^* \) we have the functionals \( G_x := \{ \mu^x_{t,W} \}_{t \in B_x,W} \) that define the preimage of \( x \) in \( Y \) and in \( V^* \) we have the linear functionals \( H_x := \{ \mu^x_{t,V} \}_{t \in B_x,V} \) that define the preimage of \( x \) in \( Z \). We note that under the map \( W^* \rightarrow V^* \), \( G_x \) is taken to \( H_x \). This implies that for every \( v \in V \), there exists \( t' \in B_{x,W} \) such that for every \( v' \in \ell^x_{t,V} \),

\[ v(v') = \mu^x_{t',W}(v) \cdot v'. \]

Now notice that \( V \) is a direct summand of \( W \), and therefore we have homomorphisms \( i_V : V \rightarrow W \) and \( p_V : W \rightarrow V \) such that \( p_V i_V = \text{id}_V \). These induce a homomorphism \( \psi : \text{End}(W) \rightarrow \text{End}(V) \), \( f \mapsto p_V \circ f \circ i_V \). Now, the previous condition just means that if \( V_x \) is embedded in \( \text{End}(V_x) \) as a Cartan subalgebra, \( W_x \) is embedded in \( \text{End}(W_x) \) as a Cartan subalgebra, then the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
V_x \\
\downarrow i_{V,x} \\
\text{End}(V_x)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
W_x \\
\downarrow \psi_x \\
\text{End}(W_x)
\end{array}
\end{array}
\]

Indeed, we see that necessarily there is a subset of \( \{ \ell^x_t \}_{t \in B_{W,x}} \) that generates \( V_x \), and so the commutativity of the previous diagram just means that \( \mu^x_{t,W} \) is taken to a \( \mu^x_{t',V} \).

By retracing our steps, we have proved the following proposition:

**Proposition 4.1.** Let \( f : Y \rightarrow X \) be an étale covering. Then there is a bijection between intermediate étale coverings as in \((4.1)\) and direct summands \( V \) of \( f_*O_Y \) such that
$V_x$ has an embedding as a Cartan subalgebra of $\text{End}(V_x)$ for every $x \in X$ and such that the induced diagram

$$
\begin{array}{c}
V_x \\
\downarrow \\
\text{End}(V_x) \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
(f_\ast \mathcal{O}_Y)_x \\
\downarrow \\
\text{End}(f_\ast \mathcal{O}_Y)_x \\
\end{array}
$$

commutes.

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