The space of traces in symmetric monoidal infinity categories

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Abstract

We define a tracelike transformation to be a natural family of conjugation invariant maps $T_{x,C}: \text{hom}_C(x,x) \to \text{hom}_C(1,1)$ for all dualisable objects $x$ in any symmetric monoidal $\infty$-category $C$. This generalises the trace from linear algebra that assigns a scalar $\text{Tr}(f) \in k$ to any endomorphism $f: V \to V$ of a finite-dimensional $k$-vector space. Our main theorem computes the moduli space of tracelike transformations using the one-dimensional cobordism hypothesis with singularities.

As a consequence we show that the trace $\text{Tr}$ can be uniquely extended to a tracelike transformation up to a contractible space of choices. This allows us to give several model-independent characterisations of the $\infty$-categorical trace. Restricting our notion of tracelike transformations from endomorphisms to automorphisms we in particular recover a theorem of Toën and Vezzosi.

Other examples of tracelike transformations are for instance given by $f \mapsto \text{Tr}(f^n)$. Unlike for $\text{Tr}$ the relevant connected component of the moduli space is not contractible, but rather equivalent to $B\mathbb{Z}/n\mathbb{Z}$ or $BS^1$ for $n = 0$. As a result we obtain a $\mathbb{Z}/n\mathbb{Z}$-action on $\text{Tr}(f^n)$ as well as a circle action on $\text{Tr}(\text{id}_x)$.

1 Introduction

The trace, originally defined in the context of linear algebra, admits a well-known generalisation to arbitrary symmetric monoidal categories. Often the trace can be thought of as measuring ‘fixed-points’ of an endomorphism, most prominently in the case of the stable homotopy category, where it computes the Lefschetz fixed-point number, as we explain below. Recently this has been vastly generalised to indexed, relative, and equivariant settings using bicategories. We refer the reader to Ponto and Shulman’s paper [PS14] for an introduction to traces from this perspective and further references.

For bordism categories the trace behaves like the ‘closing up’ operation that sends a braid to its associated link. In the context of topological quantum field theories this corresponds to taking the so-called ‘state-sum’, see for example Stolz and Teichner’s work [ST12]. The example of bordism categories plays an important role later since, by the cobordism hypothesis, bordism categories are the universal setting for taking traces. This generalises the strategy of Toën and Vezzosi [TV15].

Another example, which highlights the importance of treating the trace $\infty$-categorically, is the derived Morita category of ring spectra and bimodules. Here the trace is given by topological Hochschild homology (THH). We will return to this example in the final section where we show that in this case Theorem [A] induces a circle action on $\text{THH}(R)$ and a $\mathbb{Z}/n\mathbb{Z}$-action on $\text{THH}(R; M \otimes_R \ldots \otimes_R M)$. 
Traces in symmetric monoidal 1-categories

Before we consider \(\infty\)-categories, let us recall the trace in 1-categories. The trace of an endomorphism \(f : V \to V\) on a finite-dimensional \(k\)-vector space \(V\) may be defined as

\[
\text{Tr}(f) := \sum_{i \in I} \beta^i(f(b_i)) \in k,
\]

where \((b_i)_{i \in I}\) is a basis of \(V\) and \((\beta^i)_{i \in I}\) is the dual basis of \(V^* := \text{hom}_{\text{Vect}_k}(V, k)\) uniquely characterised by \(\beta^i(b_j) = \delta_{ij}\). Let us define the evaluation and coevaluation of \(V\) by

\[
e : V^* \otimes V \to k, (\alpha, v) \mapsto \alpha(v) \quad \text{and} \quad c : k \to V \otimes V^*, \lambda \mapsto \lambda \sum b_i \otimes \beta^i
\]

and rewrite the trace as the composition

\[
\text{Tr}(f) : k \xrightarrow{\beta} V \otimes V^* \xrightarrow{f \otimes \text{id}_{V^*}} V \otimes V^* \xrightarrow{\text{swap}_{V^*}} V^* \otimes V \xrightarrow{e} k.
\]

This definition generalises to all symmetric monoidal categories \((\mathcal{C}, \otimes, 1)\): an object \(x \in \mathcal{C}\) is called dualisable if there is a dual \(y \in \mathcal{C}\) with an evaluation \(e : y \otimes x \to 1\) and a coevaluation \(c : 1 \to x \otimes y\) satisfying:

\[
id_x = (id_x \otimes e) \circ (c \otimes id_x) \quad \text{and} \quad id_y = (e \otimes id_y) \circ (id_y \otimes c).
\]

The data \((y, e, c)\) is essentially unique, if it exists. For such \(x\) the trace of \(f : x \to x\) is defined as:

\[
\text{Tr}_x(f) := e \circ \text{swap}_{x,y} \circ (f \otimes id_y) \circ c \in \text{hom}_\mathcal{C}(1, 1).
\]

Definition 1.1. In analogy to the category of \(k\)-vector spaces we will refer to endomorphisms of the unit object as scalars. They form a commutative monoid and taking scalars defines a functor:

\[
\text{Sc} : \text{Cat}^{\otimes} \to \text{CMon}, \quad \mathcal{C} \mapsto \text{Sc}(\mathcal{C}) := \text{hom}_\mathcal{C}(1, 1).
\]

To a symmetric monoidal functor \(F : \mathcal{C} \to \mathcal{D}\) with unitor \(\lambda_F : F(1_\mathcal{C}) \cong 1_\mathcal{D}\) this assigns the map \(\text{Sc}(\mathcal{C}) \to \text{Sc}(\mathcal{D})\) sending \(f\) to \(\lambda_F \circ F(f) \circ \lambda_F^{-1}\).

This generalisation of the trace retains the cyclicity of the trace that is well-known in the context of linear algebra [PS14, Proposition 2.4]: when \(f : x \to z\) and \(g : z \to x\) are morphisms between dualisable objects, then

\[
\text{Tr}_x(g \circ f) = \text{Tr}_y(f \circ g).
\]

The main advantage of having such a general definition is that we are now able to compare traces in different symmetric monoidal categories [PS14, Proposition 6.2]. Let \(F : \mathcal{C} \to \mathcal{D}\) be a symmetric monoidal functor, \(x \in \mathcal{C}\) dualisable, and \(f : x \to x\) an endomorphism. Then \(F(x)\) is dualisable and the trace of \(F(f)\) is

\[
\text{Tr}_{D,F(x)}(F(f)) = F(\text{Tr}_{C,x}(f)).
\]

An interesting example is the stable homotopy category \(\mathcal{C} = h_1\text{Sp}\) with the smash-product as symmetric monoidal structure. Its scalars are homotopy classes of maps \(S \to S\) and these are classified by integers. For a finite CW-complex \(X\) the suspension spectrum \(\Sigma^\infty_+ X\) is a dualisable object of \(h_1\text{Sp}\) with dual the Spanier-Whitehead dual of \(X\). A continuous map \(f : X \to X\) induces an endomorphism \(\Sigma^\infty_+ f\); its trace \(\text{Tr}(\Sigma^\infty_+ f) \in \text{Hom}_{h_1\text{Sp}}(S, S) \cong \mathbb{Z}\) is the so-called Lefschetz-number \(\Lambda(f)\) of \(f\), which can be understood in terms of fixed-points of \(f\).

The functor that takes a spectrum to its cohomology with coefficients in a field is symmetric monoidal by the Künneth-theorem. The naturality [2] implies that we can understand \(\Lambda(f)\) in terms of the action \(f\) has on the homology of \(X\). This is the Lefschetz fixed-point theorem.
Axiomatic description: tracelike transformations

Our axiomatisation of the properties of the trace is loosely based on Kelly and Laplaza’s notion of a ‘trace function’ from $[KL80]$. We discuss how they are related below. The main difference is that we will study compatible choices of traces for all symmetric monoidal categories at the same time:

**Definition 1.2.** A tracelike transformation $T$ is a family of maps

$$T_{C,x} : \text{End}_C(x) \to Sc(C) = \text{End}_C(1)$$

for all symmetric monoidal categories $C$ and dualisable objects $x \in C$ satisfying the following axioms:

- **Conjugation invariance:** $T_y(\varphi \circ e \circ \varphi^{-1}) = T_x(e)$ for all endomorphisms $e : x \to x$ and isomorphisms $\varphi : x \cong y$.
- **Naturality:** $T_{D,F(x)}(F(e)) = \lambda_F \circ F(T_{C,x}(e)) \circ \lambda_F^{-1}$ for every symmetric monoidal functor $F : C \to D$ with unitor $\lambda_F : F(1_C) \cong 1_D$, dualisable object $x \in C$, and endomorphism $e : x \to x$.

A restricted tracelike transformation is a family of maps $T_{C,x} : \text{Aut}_C(x) \to Sc(C)$ satisfying the same axioms. We denote the set of tracelike transformations by $\mathcal{T}$ and the set of restricted tracelike transformations by $\mathcal{T}^r$.

Since $Sc(C)$ is a commutative monoid for all symmetric monoidal categories $C$ the sets $\mathcal{T}$ and $\mathcal{T}^r$ inherit commutative monoid structures defined by multiplying tracelike transformations pointwise. Moreover, the multiplicative monoid $(\mathbb{N}, \cdot)$ acts on these monoids by mapping $P^n : \mathcal{T} \to \mathcal{T}$ defined as $P^n(T)(e) := T(e^n)$. Starting from the classical trace $\text{Tr} \in \mathcal{T}$ that we discussed above, we can therefore construct, for every finite sequence of natural numbers $k_1, \ldots, k_n \in \mathbb{N}$, a tracelike transformation:

$$\Theta_{k_1, \ldots, k_n}^x(e) = \text{Tr}_x(e^{k_1}) \circ \cdots \circ \text{Tr}_x(e^{k_n}).$$

In fact, if we only want $\Theta$ to be a restricted tracelike transformation, we can allow the $k_i$ to be integers.

We now describe an equivalent definition of tracelike transformations: Let $\text{Cat}_1^{\otimes}$ denote the category with objects symmetric monoidal categories and morphisms natural isomorphism classes of symmetric monoidal functors. We define a functor $E^{\otimes} : \text{Cat}_1^{\otimes} \to \text{Set}$ by

$$E^{\otimes}(C) = \bigcap_{x \in C \text{ dualisable}} \text{End}_C(x) / \sim$$

where $(a : x \to x) \sim (b : y \to y)$ whenever there is an isomorphism $\varphi : x \to y$ with $\varphi \circ a \circ \varphi^{-1} = b$. Similarly, define $\mathcal{A}^{\otimes}(C) \subset E^{\otimes}(C)$ as the subset of conjugacy classes represented by automorphisms. By construction, a tracelike transformation is simply a natural transformation $E^{\otimes} \Rightarrow Sc$ of functors $\text{Cat}_1^{\otimes} \to \text{Set}$. Similarly, a restricted tracelike transformation is a natural transformation $A^{\otimes} \Rightarrow Sc$.

Note that our $E^{\otimes}(C)$ is similar to Kelly and Laplaza’s set of cycles $[\mathcal{D}]$, which they define in $[KL80]$ for any category $\mathcal{D}$. If we let $C^{\otimes} \subset C$ denote the full subcategory on the dualisable objects and $(C^{\otimes})^\sim \subset C^{\otimes}$ the maximal subgroupoid, then there is a canonical surjection $E^{\otimes}(C) \to [C^{\otimes}]$, and a canonical bijection $A^{\otimes}(C) \to [(C^{\otimes})^\sim]$. The crucial difference is that they require a cyclicity condition $T(f \circ g) = T(g \circ f)$ for any two morphisms $f : x \to y$ and $g : y \to x$, whereas this only follows from our conjugation invariance if one of $f$ and $g$ is an isomorphism.
1-categorical classification of tracelike transformations

We will show that the $\Theta^{k_1, \ldots, k_n}$ are indeed the only examples of tracelike transformations: a variant of the cobordism hypothesis in dimension 1 implies that certain cobordism categories $\text{Cob}_1^N \subset \text{Cob}_1^Z$ are freely generated by a dualisable object $*_+$ and $\alpha : *_+ \to *$ an endomorphism or automorphism, respectively. Hence there is a bijection between the set of tracelike transformations and the scalars of $\text{Cob}_1^N$. These can be described as $\text{Mfd}_1^N/\text{Diff}^+$: diffeomorphism classes of closed 1-dimensional manifolds labelled by natural numbers. This in turn is in bijection with $\mathbb{N}[x]$, the free commutative monoid on the set $\{1, x, x^2, \ldots\}$. In summary this leads to:

**Theorem (see [4.6])**. There are compatible isomorphisms of commutative monoids

$$
\begin{array}{c}
\mathcal{T} \xrightarrow{\cong} \text{Sc}(\text{Cob}_1^N) \xrightarrow{\cong} \text{Mfd}_1^N/\text{Diff}^+ \xrightarrow{\cong} \mathbb{N}[x]
\end{array}
$$

and the tracelike transformation $\Theta^{k_1, \ldots, k_n}$ is sent to $\sum_{i=1}^n x^{k_i}$.

**Traces in $\infty$-categories**

If $\mathcal{C}$ is a symmetric monoidal $\infty$-category then its scalars $\text{Sc}(\mathcal{C}) = \text{hom}_{\mathcal{C}}(1, 1)$ are no longer just a set, but a space. In fact, they naturally carry the structure of an $E_\infty$-space. Consider for instance the $\infty$-category of spectra $\mathcal{C} = \text{Sp}$: the endomorphisms of the unit object are: $\text{Sc}(\mathbf{S}) = \text{hom}_{\text{Sp}}(S, S) = \Omega^\infty S$. For a finite CW-complex $X$ the 1-categorical trace on the homotopy category $h_1\mathbf{Sp}$ can be promoted to a map of spaces:

$$
\text{Tr} : \text{hom}_{\text{Sp}}(\Sigma^\infty_+ X, \Sigma^\infty_+ X) \to \Omega^\infty \mathbf{S}.
$$

In definition 2.16 and 2.17 we will construct $\infty$-functors $\text{Sc}$, $\mathcal{E}^{fd}$, and $\mathcal{A}^{fd}$ from the $\infty$-category of symmetric monoidal $\infty$-categories $\text{Cat}_{\infty}^\otimes$ to the $\infty$-category of spaces $\text{Spc}$. For every symmetric monoidal $\infty$-category $\mathcal{C}$ and dualisable object $x \in \mathcal{C}$ there are compatible maps $\text{hom}_{\mathcal{C}}(x, x) \to \mathcal{E}^{fd}(\mathcal{C})$ and $h\text{Aut}_\mathcal{C}(x) \to \mathcal{A}^{fd}(\mathcal{C})$.

**Definition 1.3.** An $\infty$-categorical tracelike transformation is a natural transformation $T : \mathcal{E}^{fd} \Rightarrow \text{Sc}$ of $\infty$-functors $\text{Cat}_{\infty}^\otimes \to \text{Spc}$. Similarly, a restricted tracelike transformation is a natural transformation $T : \mathcal{A}^{fd} \Rightarrow \text{Sc}$. We denote the space of (restricted) $\infty$-categorical tracelike transformations by $\mathcal{T}_{\infty}(r)$.

The $\infty$-functors $\text{Sc}$, $\mathcal{E}^{fd}$, and $\mathcal{A}^{fd}$ recover their 1-categorical analogues on the homotopy category $h_1\mathcal{C}$ of $\mathcal{C}$ in the sense that $\pi_0\text{Sc}(\mathcal{C}) \cong \text{Sc}(h_1\mathcal{C})$, $\pi_0\mathcal{E}^{fd}(\mathcal{C}) \cong \mathcal{E}^{fd}(h_1\mathcal{C})$, and $\pi_0\mathcal{A}^{fd}(\mathcal{C}) \cong \mathcal{A}^{fd}(h_1\mathcal{C})$, see [2.21]. For any tracelike transformation $T$, symmetric monoidal $\infty$-category $\mathcal{C}$ and dualisable object $x \in \mathcal{C}^{fd}$ naturality with respect to the functor $\mathcal{C} \to h_1\mathcal{C}$ yields the following commutative diagram:

$$
\begin{array}{ccc}
\text{hom}_{\mathcal{C}}(x, x) & \xrightarrow{T_c} & \mathcal{E}^{fd}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\text{hom}_{h_1\mathcal{C}}(x, x) & \xrightarrow{T_{h_1\mathcal{C}}} & \mathcal{E}^{fd}(h_1\mathcal{C})
\end{array}
$$

Here we implicitly used the map $\mathcal{T}_{\infty} \to \mathcal{T}$ defined by sending an $\infty$-categorical tracelike transformation to its restriction to 1-categories. This map encodes how an $\infty$-categorical tracelike transformation behaves on the level of connected components.
Our main theorem describes the homotopy-type of $\mathcal{F}_\infty$ and the map to the discrete set $\mathcal{F}$. Generalising the commutative monoid structure on $\mathcal{F}^{(r)}$ there is an $E_\infty$-algebra structure $\mathcal{F}^{(r)}$. For simplicity, we will only identify the underlying space. To state the theorem, let $\text{Free}_{E_\infty}(X)$ denote underlying space of the free $E_\infty$-algebra on $X$.

**Theorem A** (5.11). There is a commutative diagram of spaces:

$$
\begin{array}{ccc}
\mathcal{F}_\infty & \xrightarrow{\simeq} & \text{Free}_{E_\infty}(BS^1 \coprod \coprod_{k \geq 1} B\mathbb{Z}/k\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathcal{F} & \xrightarrow{\simeq} & \mathcal{F}^{(r)} \\
\downarrow & & \downarrow \\
\mathcal{F}^{(r)} & \xrightarrow{\simeq} & \text{Free}_{E_\infty}(S^1 \times BS^1 \coprod \coprod_{k \in \mathbb{Z}\setminus\{0\}} B\mathbb{Z}/k\mathbb{Z}) \\
\downarrow & & \downarrow \\
& \xrightarrow{\simeq} & \mathbb{N}[x] \\
\end{array}
$$

where the horizontal maps are equivalences. Moreover, the vertical maps identify the sets in the bottom layer as the set of connected components of the top layer: $\mathcal{F} \simeq \pi_0 \mathcal{F}_\infty$ and $\mathcal{F}^{(r)} \simeq \pi_0 \mathcal{F}^{(r)}$.

**Warning 1.4.** The reader should be warned that in identifying $\mathcal{F}_\infty$ we use the cobordism hypothesis with singularities, which was sketched in [Lur09c], but a full proof of which has not yet appeared in the literature. We will therefore be treating it as a conjecture and all our statements about $\mathcal{F}_\infty$ are dependent on this conjecture. Note, however, that the analogous statements about the space of restricted tracelike transformations $\mathcal{F}^{(r)}$ only use the standard cobordism hypothesis in dimension 1, which has been proved in detail. (See [Lur09c] and [Har12].)

Of particular interest is the homotopy-type of the moduli space of those $\infty$-categorical tracelike transformations that recover the classical trace on homotopy categories. In their work on the derived Chern-character [TV15] Toën and Vezzosi show that for restricted tracelike transformations the space is contractible and that therefore there is an essentially unique $\infty$-categorical generalisation of the trace when applied to automorphisms.

In theorem A we use methods similar to theirs to compute the full homotopy-type of $\mathcal{F}^{(r)}$. Their result can now be read off by considering the fibre of $\mathcal{F}^{(r)} \to \mathcal{F}$ over the classical trace. Since we study $\mathcal{F}_\infty$ as well we can now generalise their result, removing the artificial restriction to automorphisms.

**Corollary B** (Generalising [TV15, Théorème 3.18], 5.12). The space of (restricted) $\infty$-categorical tracelike transformations that act as the 1-categorical trace $\text{Tr}$ on homotopy categories is contractible.

Using our knowledge of the structure of $\pi_0 \mathcal{F}_\infty$ it is in fact not hard to see that to uniquely characterise the $\infty$-categorical trace we only need to specify its behaviour on the category of vector spaces.

**Corollary C** (5.13). Any $\infty$-categorical tracelike transformation whose value on the category of complex vector spaces agrees with the trace from linear algebra is canonically equivalent to the $\infty$-categorical trace.

More informally, there is a unique extension of the trace from linear algebra to a family of maps

$$
\text{Tr}_{(x,\mathcal{C})} : \text{End}_{\mathcal{C}}(x) \to \text{hom}_{\mathcal{C}}(I, I)
$$

for any symmetric monoidal $\infty$-category $\mathcal{C}$ and any dualisable object $x \in \mathcal{C}$ while preserving the conjugation invariance of the trace and its naturality with respect to symmetric monoidal functors.

We can also give an alternative characterisation of $\text{Tr}$ as the unique generating tracelike transformation. This characterisation is purely categorical and does not require one to first define a trace for vector spaces.
Definition 1.5. The monoid \((\mathbb{N}, \cdot)\) acts on \(\mathcal{I}\) and \(\mathcal{I}_\infty\) by taking powers of any morphism before applying the tracelike transformation \(P^n(T)(a) := T(a^n)\). We call a tracelike transformation \(T \in \mathcal{I}_\infty\) generating if the monoid \(\pi_0 \mathcal{I}_\infty\) is generated by the orbit of \(T\) under the \(\mathbb{N}\)-action.

In other words, \(T\) is generating if every other tracelike transformation \(S \in \mathcal{I}_\infty\) is equivalent to one of the form \(P^{k_1}(T) \circ \cdots \circ P^{k_n}(T)\).

Corollary D \(5.19\). The space of generating tracelike transformations in \(\mathcal{I}_\infty\) is contractible and its image in \(\mathcal{I}\) is the usual trace.

Notation

We will assume that the reader has a convenient model of \((\infty, 1)\)-categories at hand. In this paper we will be working in the context of Joyal’s quasicategories, but really any equivalent Cartesian closed \(\infty\)-cosmos in the sense of Riehl and Verity will do. We will refer to these \((\infty, 1)\)-categories as \(\infty\)-categories and to morphisms between them as \(\infty\)-functors, or sometimes just as functors.

We write \(\text{Cat}_{\infty}\) for the \(\infty\)-category of \(\infty\)-categories and \(\text{Spc} \subset \text{Cat}_{\infty}\) for the full subcategory of \(\infty\)-groupoids, which we will refer to synonymously as ‘spaces’. The 1-category of topological spaces will be denoted by \(\text{Top}\). There is a functor \(\text{Top} \to \text{Spc}\) that ‘forgets the point-set information’, it sends a topological space to its \(\infty\)-groupoid of paths.

For \(\infty\)-categories \(C, D, E\) we denote the \(\infty\)-category of functors from \(C\) to \(D\) by \(\text{Fun}_{\infty}(C, D)\) and the maximal subgroupoid of \(E\) by \(E^\sim \in \text{Spc}\). For objects \(a, b \in E\) the space of morphisms from \(a\) to \(b\) is \(\text{hom}_{\infty}(a, b)\). In the case \(E = \text{Cat}_{\infty}\) the space of functors from \(C\) to \(D\) is \(\text{hom}_{\text{Cat}_{\infty}}(C, D) \simeq (\text{Fun}_{\infty}(C, D))^\sim\).

The \(\infty\)-category of (simplicial) presheaves on \(C\) is \(\mathbb{P}(C) := \text{Fun}_{\infty}(C^{op}, \text{Spc})\). The Yoneda embedding will be denoted by \(Y : C \to \mathbb{P}(C)\).

Structure of the paper

We begin by recalling complete Segal spaces and other \(\infty\)-categorical tools in section 2, where we also define the functors \(\mathcal{E}^{id}\) and \(\mathcal{A}^{id}\). Then, in section 3, we define concrete models of the one-dimensional bordism category with and without marked points and show that they define symmetric monoidal \(\infty\)-categories. Using these we formulate the cobordism hypothesis, as well as a variant with singularities. In section 4 we complete the proof of the classification result for \(1\)-categorical tracelike transformations. Section 5 contains the homotopy-theoretic computations and the proofs of the main theorems. In the final section we discuss how non-contractible connected components of the moduli space \(\mathcal{I}_\infty\) induce group actions on the trace.

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2 Modelling symmetric monoidal \(\infty\)-categories

Much of the literature on the \(\infty\)-categorical structure of the cobordism category, in particular Lurie’s work on the cobordism hypothesis \([\text{Lur09c}]\), is formulated in terms of \(\Gamma\)-objects in
complete Segal spaces. In this section we recall how to relate this approach to the more standard theory of quasicategories, by giving a model independent description of complete Segal spaces in the language of \( \infty \)-categories. Once this is established we make precise the functors \( \mathcal{L}^{\text{fd}}, \mathcal{A}^{\text{fd}}, \) and \( S^c \) from the introduction.

2.1 Complete Segal spaces

2.1. We let \( \text{Cat}_1 \) denote the 1-category of 1-categories. It admits a functor to the \( \infty \)-category of \( \infty \)-categories \( \text{Cat}_1 \to \text{Cat}_\infty \); in the quasicategory model this sends a category to its nerve. Write \( \text{Cat}_1 \subset \text{Cat}_\infty \) for the essential image of the inclusion. We will think of this as the \( \infty \)-category of 1-categories. This in fact is also a (2, 1)-category: the hom-spaces of \( \text{Cat}_1 \) are the 1-groupoids of 1-functors and natural isomorphisms:

\[
\text{hom}_{\text{Cat}_1}(\mathcal{C}, \mathcal{D}) \cong (\text{Fun}_1(\mathcal{C}, \mathcal{D}))^\sim.
\]

The inclusion \( \text{Cat}_1 \to \text{Cat}_\infty \) has a left-adjoint, the homotopy-category functor

\[
h_1 : \text{Cat}_\infty \to \text{Cat}_1
\]

and in this sense \( \text{Cat}_1 \) is a localisation of \( \text{Cat}_\infty \).

2.2. We define the simplex category as the full subcategory \( \Delta \subset \text{Cat}_1 \) generated by the partially ordered sets \( [n] = \{0 \leq \cdots \leq n\} \) thought of as categories, for \( n \geq 0 \). Using the embedding \( I : \Delta \to \text{Cat}_\infty \) we obtain

\[
N : \text{Cat}_\infty \xrightarrow{Y} \text{P}(\text{Cat}_\infty) \xrightarrow{I^*} \text{P}(\Delta).
\]

Theorem 2.7 states that this functor is fully faithful. This is a way of saying that the objects \( [n] \) generate \( \text{Cat}_\infty \) strongly under colimits. The essential image of \( N \) are the complete Segal spaces:

**Definition 2.3.** A simplicial space \( X \in \text{P}(\Delta) \) satisfies the *Segal condition* if for all \( n \geq 2 \) the map

\[
(\lambda^n_0, \ldots, \lambda^n_{n-1}) : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

induced by \( \lambda_i : [1] \to [n] \) with \( \lambda_i(k) = i + k \) is an equivalence.

**Definition 2.4.** Let \( \mathbb{I} \in \text{Cat}_1 \) be the contractible groupoid with two objects and \( * \in \text{Cat}_1 \) the discrete category with one object. Write \( N(\mathbb{I}) \) and \( N(*) \) for the simplicial sets that are the nerves of these categories and interpret them as simplicial spaces that are discrete in every layer.\(^1\) A simplicial space \( X \in \text{P}(\Delta) \) is called *complete* if the natural map

\[
X_0 \cong \text{hom}_{\text{P}(\Delta)}(N(*), X) \to \text{hom}_{\text{P}(\Delta)}(N(\mathbb{I}), X)
\]

coming from the 1-functor \( \mathbb{I} \to * \) is an equivalence.

**Definition 2.5.** The \( \infty \)-category of *complete Segal spaces* is defined as the full subcategory \( \text{CSS} \subset \text{P}(\Delta) \) spanned by the objects that are complete and satisfy the Segal condition.

**Remark 2.6.** Given a Segal space \( X \) there also is a simpler characterisation of the completeness condition due to Rezk. By [Rez01, Theorem 6.2] the space \( \text{hom}_{\text{P}(\Delta)}(N(\mathbb{I}), X) \) is equivalent to the subspace \( X_1^{eq} \subset X_1 \) on those 1-simplices that represent an isomorphism in the homotopy category \( h_1 X \), which we describe in 2.9. Moreover, \( X_1^{eq} \subset X_1 \) is always a union of connected components. Therefore a Segal space is complete if and only if \( s_0 : X_0 \to X_1^{eq} \) is an equivalence.

\(^{1}\)This construction defines a functor \( N : \text{Cat}_1 \to \text{P}(\Delta) \), but this functor does not preserve categorical equivalences: \( \mathbb{I} \) and \( * \) are equivalent, but \( N(\mathbb{I}) \) and \( N(*) \) are not. The functor \( N : \text{Cat}_\infty \to \text{P}(\Delta) \) preserves equivalences by construction, and simplicial spaces in its image will be complete as they ‘cannot see the difference between \( \mathbb{I} \) and \( * \)."
Theorem 2.7 ([Ber07]). The functor \(N\) takes values in complete Segal spaces and induces an equivalence

\[ N : \text{Cat}_\infty \xrightarrow{\simeq} \text{CSS}. \]

Proof by citation. It is not difficult to see that for an \(\infty\)-category \(\mathcal{C}\) the nerve \(N\mathcal{C}\) indeed satisfies the Segal and the completeness condition. To prove that \(N\) induces an equivalence is more difficult.

The first result of this type was [Ber07], but there the model used for \(\text{Cat}_\infty\) was simplicial categories. Joyal and Tierney show in [JT07] that the model categories of quasicategories and complete Segal spaces are Quillen-equivalent. The \(\infty\)-categorical statement is an immediate consequence of their result, see [Lur09b, Corollary 4.3.16].

2.8. It is important to understand how standard constructions in \(\text{Cat}_\infty\) change under the equivalence to \(\text{CSS}\). Let \(X = N(\mathcal{C})\) be the complete Segal space of some \(\infty\)-category \(\mathcal{C}\). By definition, the levels of \(X\) are of the form

\[ X_n = \text{hom}_{\text{Cat}_\infty}([n], \mathcal{C}) = (\text{Fun}_\infty([n], \mathcal{C}))^\sim. \]

In particular \(X_0\) is the maximal subgroupoid \(\mathcal{C}^\sim\) of \(\mathcal{C}\) and \(X_1\) is the maximal subgroupoid of the arrow category \(\mathcal{C}^{[1]}\).

We will say that an object of \(\mathcal{C}\) is a functor \(\ast \to \mathcal{C}\), where \(\ast\) is the terminal category. For two such objects \(a, b : \ast \to \mathcal{C}\) one can reconstruct the hom-space \(\text{hom}_\mathcal{C}(a, b)\) from the complete Segal space \(X\) as:

\[ \text{hom}_\mathcal{C}(a, b) \simeq \{a\} \times_{\mathcal{C}^\sim} (\text{Fun}_\infty([1], \mathcal{C}))^\sim \times_{\mathcal{C}^\sim} \{b\} = \{a\} \times_{X_0} X_1 \times_{X_0} \{b\}. \]

We can recover the composition, up to inverting the weak equivalence \(X_2 \to X_1 \times_{X_0} X_1\):

\[ \text{hom}_\mathcal{C}(a, b) \times \text{hom}_\mathcal{C}(b, c) = (\{a\} \times_{X_0} X_1 \times_{X_0} \{b\}) \times (\{b\} \times_{X_0} X_1 \times_{X_0} \{c\}) \]
\[ \to \{a\} \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \{c\} \leftarrow \{a\} \times_{X_0} X_2 \times_{X_0} \{c\} \]
\[ \overset{d_1}{\to} \{a\} \times_{X_0} X_1 \times_{X_0} \{c\} = \text{hom}_\mathcal{C}(a, c). \]

2.9. The above suffices to reconstruct the homotopy category \(h_1\mathcal{C}\) of \(\mathcal{C}\) up to categorical equivalence: we define the set of objects as \(O := \pi_0 X_0\) and pick a section \(o : \pi_0 X_0 \to X_0\) to interpret them in the above sense. For two \(a, b \in O\) we define the morphism set as

\[ \text{Hom}_\mathcal{C}(a, b) := \pi_0(\{o(a)\} \times_{X_0} X_1 \times_{X_0} \{o(b)\}). \]

The composition in \(\mathcal{C}\) is constructed by taking \(\pi_0\) of what we did earlier. Note that this erases the ambiguity coming from inverting the equivalence.

Remark 2.10. The construction given above has the disadvantage that it is not functorial in \(X\): we need to make the unnatural choice of a section \(o : \pi_0 X_0 \to X_0\). In fact, we should not expect there to be a 1-categorical description since the \(\infty\)-functor \(h_1 : \text{Cat}_\infty \to \text{Cat}_1\) does not factor through the 1-category of 1-categories \(\text{Cat}_1\).

2.2 Dualisability

We recall the necessary definitions to talk about dualisable objects:

Definition 2.11 ([Seg74]). Segal’s category \(\Gamma\) is defined as a skeleton of the opposite category of the category of finite pointed sets. The objects are \((k) := \{*, 1, \ldots, k\}\) for \(k \geq 0\) and morphisms in \(\Gamma^{\text{op}}\) are basepoint preserving maps. A functor \(X : \Gamma^{\text{op}} \to \mathcal{C}\) is called a special \(\Gamma\)-object in
Proposition 2.4.2.5. Let $\text{CSS}^\circ$ denote the $\infty$-category of special $\Gamma$-objects in $\text{CSS}$. This can be thought of as a full subcategory of $\mathbf{P}(\Delta \times \Gamma)$. We write $X_n^{(k)}$ for the value of $X : \Delta^{op} \times \Gamma^{op} \to \text{Spc}$ on $([n],[k])$.

Theorem 2.12. The $\infty$-category of special $\Gamma$-objects in $\mathcal{C}$ is a model for the commutative monoid objects in $\mathcal{C}$. In particular, the nerve functor $N : \mathbf{Cat}_\infty \simeq \text{CSS}$ lifts to an equivalence $\mathbf{Cat}_\infty^\circ \simeq \text{CSS}^\circ$ between the $\infty$-category of symmetric monoidal $\infty$-categories and the $\infty$-category of special $\Gamma$-objects in $\text{CSS}$.

Proof by citation. Commutative monoids in $\mathcal{C}$ are by definition $E_\infty$-algebras in $\mathcal{C}$ with respect to the symmetric monoidal structure coming from the Cartesian product. That these are the same as functors $\Gamma^{op} \to \mathcal{C}$ satisfying the ‘specialness condition’ is for instance shown in [Lur18, Proposition 2.4.2.5].

Corollary 2.13. The localisation-adjunction $h_1 \dashv I$ between $\mathbf{Cat}_\infty$ and $\mathbf{Cat}_1$ lifts to a localisation-adjunction $h_1^\circ : \mathbf{Cat}_\infty^\circ \rightleftarrows \mathbf{Cat}_1^\circ : I^\circ$.

Proof. There is an adjunction on the functor categories

$$(h_1)_* : \text{Fun}_\infty(\Gamma^{op}, \mathbf{Cat}_\infty) \rightleftarrows \text{Fun}_\infty(\Gamma^{op}, \mathbf{Cat}_1) : I_*$$

and since both functors preserve the ‘specialness’ of $\Gamma$-objects this adjunction restricts to the full subcategories $\mathbf{Cat}_\infty^\circ$ and $\mathbf{Cat}_1^\circ$. The functor $I_*$ is fully faithful because $I$ was and hence $h_1^\circ$ is a localisation.

2.14. Note that here $\mathbf{Cat}_1^\circ$ is by definition the $\infty$-category of special $\Gamma$-objects in $\mathbf{Cat}_1$. It is a folklore theorem that this is equivalent to the $(2,1)$-category of symmetric monoidal categories. For the readers convenience we will use this theorem and from now on think of $h_1 X$ as a symmetric monoidal category. However, it is worth remarking that we could equally well work with (special) $\Gamma$-categories.

Definition 2.15. An object $x$ in a symmetric monoidal $\infty$-category $\mathcal{C}$ is called dualisable if $x$ is dualisable as an object of the symmetric monoidal 1-category $h_1 \mathcal{C}$. The 1-categorical definition was given in the introduction. If all objects of $\mathcal{C}$ have duals we say that $\mathcal{C}$ has duals. Define $\mathcal{C}^{fd}$ to be the maximal (full) subcategory of $\mathcal{C}$ that has duals.

2.3 Some functors of interest

Definition 2.16. For a complete Segal space $X \in \text{CSS}$ we define its space of objects as $\text{obj}(X) := X_0$. Its space of endomorphisms is defined as $\mathcal{E}(X) := X_0 \times_{X_0 \times X_0} X_1$, where the two maps are the diagonal $\Delta : X_0 \to X_0 \times X_0$ and the source-target map $(d_0,d_1) : X_1 \to X_0 \times X_0$. The space of automorphisms of $X$ is defined as $\mathcal{A}(X) := X_0 \times_{X_0 \times X_0} X_1^{eq}$. Here $X_1^{eq} \subset X_1$ is the subspace of those connected components that represent invertible morphisms in the homotopy category.

Definition 2.17. For a special $\Gamma$-object in complete Segal spaces $X \in \text{CSS}^\circ$ we let $\text{obj}^{fd}(X) \subset \text{obj}(X) = X_0^{(1)}$ denote the union of those connected components that correspond to dualisable objects in the homotopy category. Accordingly, we let $\mathcal{E}^{fd}(X) \subset \mathcal{E}(X)$ and $\mathcal{A}^{fd}(X) \subset \mathcal{A}(X)$ be the subspaces supported at the dualisable objects. Finally, we set

$$\text{Sc}(X) := X_0^{(0)} \times_{(X_0^{(1)}) \times (X_0^{(1)})} X_1^{(1)}.$$  

Note that this agrees with the functor $\text{End}(\mathbf{1})$ defined above [TV15, Proposition 2.7].
**Remark** 2.18. By construction there is a natural transformation $A \to E$ such that for each $X$ the map $A(X) \to E(X)$ is an equivalence onto the connected components it hits.

**Lemma 2.19.** For any complete Segal space represented by a simplicial topological space $X_*$ we can compute $E(X)$ as the space of tuples $(\gamma, f)$ where $f \in X_1$ and $\gamma$ is a path from $d_0 f$ to $d_1 f$. The space $A(X)$ is the subspace of those $(\gamma, f)$ where $f \in X_1^{eq}$, i.e. $f$ is invertible in the homotopy category. Moreover, $A(X)$ is equivalent to the free loop space $\Lambda(obj(X)) = Map(S^1, X_0)$.

**Proof.** In order to compute the homotopy pullback $A(X) = X_0 \times_{X_0 \times X_0} X_1^{eq}$ we replace the diagonal $X_0 \to X_0 \times X_0$ by the path fibration $P(X_0) \to X_0 \times X_0$ and obtain:

$$E(X) \simeq P(X_0) \times_{X_0 \times X_0} X_1.$$

Note that the right-hand space is indeed the space of tuples $(\gamma, f)$ as described in the lemma. For the second claim, recall from remark 2.6 that for a complete Segal space the degeneracy map $s_0 : X_0 \to X_1$ induces an equivalence $X_0 \simeq X_1^{eq}$. We therefore have another equivalence

$$A(X) \simeq P(X_0) \times_{X_0 \times X_0} X_1^{eq} \simeq P(X_0) \times_{X_0 \times X_0} X_0.$$

The latter space is the space of paths in $X_0$ whose start and end-point agree, i.e. the space of free loops $\Lambda(X_0) = \text{Map}(S^1, X_0)$.

**Lemma 2.20.** Write $X \in \text{CSS}^\circ$ as $X \simeq NC$ for some symmetric monoidal $\infty$-category $C$. Then the above functors can be described as follows:

- $\text{obj}(X)$ is the maximal subgroupoid $C^\sim$ of $C$ interpreted as a space,
- $\text{obj}^{fd}(X)$ is the $\infty$-groupoid $(C^{fd})^\sim$ interpreted as a space,
- $\text{Sc}(X)$ is the endomorphism space $\text{hom}_C(I, I)$ of the unit object $I \in C$,
- $A^{fd}(X)$ is equivalent to $\text{Map}(S^1, \text{obj}^{fd}(X))$, which is the functor $LI$ in [TV15, 498].

**Proof.** For $\text{obj}(X)$ this is a direct consequence of the definition on $NC$, as discussed in 2.8. Since the notion of dualisability is defined via the homotopy-category the description of $\text{obj}^{fd}$ follows immediately.

The definition of $\text{Sc}(X)$ is somewhat cryptic, but in fact not much is happening: the space $X_0^{(0)}$ is contractible since the $\Gamma$-object is special. We may hence replace it by the terminal space $\ast$. The functor $\ast \to X_0^{(0)} \to X_0^{(1)} = C^\sim$ picks out the unit object $I$ of $C$ and so the definition is equivalent to

$$\text{Sc}(X) \simeq \{I\} \times_{X_0^{(1)}} X_1^{(1)} \times_{X_0^{(1)}} \{I\} = \{I\} \times_{C^\sim} (C^{[1]})^\sim \times_{C^\sim} \{I\} \simeq \text{hom}_C(I, I)$$

as claimed.

Finally, the claim that $A^{fd}(X) \simeq \text{Map}(S^1, \text{obj}^{fd}(X))$ follows by the same argument as in lemma 2.19 now restricted to the full subcategory on dualisable objects.

We also show that these definitions are indeed compatible with the one-categorical definitions given in the introduction:

**Lemma 2.21.** For $C \in \text{CSS}^\circ$ there are natural bijections $\pi_0 \text{Sc}(C) \cong \text{Sc}(h_1 C)$, $\pi_0 C^{fd}(C) \cong E^{fd}(h_1 C)$, and $\pi_0 A^{fd}(C) \cong A^{fd}(h_1 C)$.
Proof. In lemma \[\text{2.20}\] we provided a natural equivalence \(Sc(C) \simeq \text{hom}_C(1, I)\). After applying \(\pi_0\) this becomes

\[
\pi_0Sc(C) \cong \pi_0\text{hom}_C(1, I) = \text{Hom}_{h_1C}(1, I) = Sc(h_1C).
\]

For the second part let us assume that every object in \(C\) is dualisable so that \(E^{fd}(C) = E(C)\). Since duals are computed on the level of homotopy categories this will not cause a problem.

The homotopy fibre of the map \(E(C) \to N_0C\) at an object \(x \in C\) is equivalent to \(\text{hom}_C(x, x)\), so the connected components of the fibre are \(\pi_0\text{hom}_C(x, x) = \text{End}_{h_1C}(x)\). The fundamental group of \(N_0C\) at \(x\) is \(\text{Aut}_{h_1C}(x)\). Therefore, choosing representatives \(x_i\) for the isomorphism classes in \(C\), we can write \(\pi_0E(C)\) as the disjoint union of quotients: \(\coprod_{i \in \pi_0N_0C} \text{End}_{h_1C}(x_i)/\text{Aut}_{h_1C}(x_i)\). Here the action is by conjugation and the resulting set is canonically isomorphic to \(E(h_1C)\). This isomorphism restricts to a bijection between \(A(C)\) and \(A(h_1C)\).

\[\square\]

### 3 The bordism category and the cobordism hypothesis

In order to apply the cobordism hypothesis, which is a key step in the proof of the main theorem, we need to first establish a concrete model for the one dimensional bordism category. In this section we take the model for \(\text{Bord}^q_\ast(X)\) from \[\text{SP17}\] and show that it is a special \(\Gamma\)-object in complete Segal spaces. Moreover, we define a “marked” version of the bordism category and show that it too is a special \(\Gamma\)-object in complete Segal spaces. Using this model we can then formulate a special case of the conjectural cobordism hypothesis with singularities.

#### 3.1 The bordism category as a complete Segal space

We are going to define a symmetric monoidal \(\infty\)-category \(\text{Bord}^q_\ast(X)\) by constructing a 1-functor \(F : \Delta^{op} \times \Gamma^{op} \to \text{Top}\) based on \[\text{SP17}\] and then showing in \[3.4\] that after composing with \(\text{Top} \to \text{Spc}\) it satisfies the required conditions to be in \(\text{CSS}^\circ\) as in theorem \[2.12\].

In what follows we let \(\mathbb{R}^{\infty}\) denote the countably dimensional vector space \(\mathbb{R}^{\infty} = \text{colim}_{n \to \infty} \mathbb{R}^n\) and we let \(\mathbb{R}^{1+\infty}\) denote the product \(\mathbb{R} \times \mathbb{R}^{\infty}\).

**Definition 3.1 (\[\text{SP17}\] Definition 5.7).** For a submanifold \(W \subset \mathbb{R}^{1+\infty}\) and an interval \(U \subset \mathbb{R}\) we write \(W_U := W \cap (U \times \mathbb{R}^{\infty})\). \(W\) is called cylindrical over \(U\) if there is a closed \((d-1)\)-dimensional submanifold \(N \subset \mathbb{R}^{\infty}\) such that \(W_U = U \times N\).

For \([n] \in \Delta\) we let \(\mathbb{R}[[n]]\) be the topological space of monotone maps \([n] \to \mathbb{R}\). We say that a submanifold \(W \subset \mathbb{R}^{1+\infty}\) is admissible with respect to \(t \in \mathbb{R}[[n]]\) if \(W\) is closed as a subset, the projection \(\pi : W \to \mathbb{R}\) to the first coordinate is proper, and there is an \(\varepsilon > 0\) such that \(W\) is cylindrical over each of the intervals \((t_i - \varepsilon, t_i + \varepsilon)\) for \(i = 0, \ldots, n\).

To topologise the space of bordisms we recall the plot-topologies from \[\text{SP17}\].

**Definition 3.2.** For a space \(X\) we let \(\Psi_d(X)\) denote the set of all tuples \((W, \varphi)\) where \(W \subset \mathbb{R}^{\infty}\) is an oriented \(d\)-dimensional submanifold that is closed as a subset and \(\varphi : W \to X\) is a continuous map.

For any \(k\)-dimensional manifold \(U\) we say that a map \(f : U \to \Psi_d(X)\) is smooth if the graph

\[
\Gamma(f) = \{(u, v) \in U \times \mathbb{R}^{\infty} \mid v \in f(u)\}
\]

is a smooth submanifold of \(U \times \mathbb{R}^{\infty}\) and the map \(\varphi f : \Gamma(f) \to X\) is continuous. The plot topology on \(\Psi_d(X)\) is the finest topology such that all smooth maps \(U \to \Psi_d(X)\) are continuous.

We are now ready to define the simplicial space that will give rise to the bordism category. Our model is almost identical to the one defined in \[\text{SP17}\] Definition 5.8\] with the only difference being that we chose to encode the symmetric monoidal structure by using \(\Gamma\)-spaces whereas Schommer-Pries uses \(E^{\infty}\)-algebras.
Definition 3.3. For any topological space $X$, we define $\operatorname{Bord}^d_f(X)$ by the functor $\Delta^{op} \times \Gamma^{op} \to \operatorname{Top}$ that sends $([n], [k])$ to the topological space of tuples $(t, (W_1, \varphi_1), \ldots, (W_k, \varphi_k))$ where $t \in \mathbb{R}^n$ and the $W_i$ are pairwise disjoint $d$-dimensional oriented submanifolds of $\mathbb{R} \times (-1, 1)^\infty$ admissible with respect to $t$. The $\varphi_i$ are continuous maps $\varphi_i : W_i \to X$. This is topologised as a subspace of $\mathbb{R}^{[n]} \times (\Psi_d(X))^k$. Functoriality in the $\Gamma$-direction is defined as follows. For $\lambda : [k] \to [l]$ we set

$$
\lambda_* (t, (W_1, \varphi_1), \ldots, (W_k, \varphi_k)) = \left( t, \prod_{\lambda(i_1) = 1} (W_{i_1}, \varphi_{i_1}), \ldots, \prod_{\lambda(i_l) = l} (W_{i_l}, \varphi_{i_l}) \right).
$$

In the $\Delta$-direction a morphism $\rho : [n] \to [m]$ acts by reindexing the $t_i : (\rho^* t)_j = t_{\rho(j)}$.

We now proceed to show that the above satisfies the Segal and completeness condition, so that it lies in $\mathbb{CSS}^\otimes \subset \mathbf{Fun}_{\infty} (\Delta^{op} \times \Gamma^{op}, \mathbf{Spc})$ and gives rise to a symmetric monoidal infinity category via theorem 2.12.

Theorem 3.4. For any $d \geq 0$ and $X \in \operatorname{Top}$ the simplicial space $\operatorname{Bord}^d_f(X)^{(1)}$ is a Segal space and the functor $\operatorname{Bord}^d_f(X) : \Delta^{op} \times \Gamma^{op} \to \mathbf{Spc}$ is a special $\Gamma$-object in Segal spaces. If $d \leq 2$, then $\operatorname{Bord}^d_f(X)$ is also complete and defines a special $\Gamma$-object in complete Segal spaces.

Proof. We begin with the Segal condition for $\operatorname{Bord}^d_f(X)^{(1)}$. Let $B_n = \{(t \in \mathbb{R}^n, (W, \varphi)) \mid \ldots \}$ be the simplicial topological space from definition 3.3 that represents $\operatorname{Bord}^d_f(X)^{(1)}$.

For any $n$ we let $B'_n \subset B_n$ be the subspace of those $(t, (W, \varphi))$ satisfying that $t_i = i$, that $W$ is cylindrical over $(-\infty, 0]$ and $[n, \infty)$, and that the two restrictions $\varphi : W_{[0], n} \to X$ and $\varphi : W_{[n], \infty} \to X$ factor through the projection to $W_{[0]}$ and $W_{[n]}$, respectively. These conditions ensure that $(t, (W, \varphi))$ is uniquely determined by the restriction $(W_{[0], n}, \varphi_{|W_{[0], n}})$. This encodes exactly the data of $n$ composable bordisms of length 1. Put into formulas we have a homeomorphism:

$$
B'_n \xrightarrow{\simeq} B'_1 \times B'_0 \cdots \times B'_0 B'_1
$$

defined by sending $(W, \varphi)$ to the tuple $(W_{[i, i+1]}, \varphi_{|W_{[i, i+1]}})_{i=0}^{n-1}$. To be precise we should actually translate $W_{[i, i+1]}$ by $i$ to the left and then extend it cylindrically over $(-\infty, 0]$ and $[1, \infty)$, then it is a well-defined point in $B'_1 \subset B_1$.

The above homeomorphism already indicates that $B'_n$ satisfies some type of Segal condition. However, one should note that $B'_n \subset B_n$ is not closed under face or degeneracy operators, so the $B'_n$ do not actually define a simplicial space. Still, this will be very useful as we have the following homotopy commutative diagram:

$$
\begin{array}{ccc}
B'_n & \xrightarrow{i_n} & B_n \\
\downarrow \cong & & \downarrow S \\
B'_1 \times B'_0 \cdots \times B'_0 B'_1 & \xrightarrow{q} & B'_1 \times B'_0 \cdots \times B'_0 B'_1 & \xrightarrow{j} & B_1 \times B_0 \cdots \times B_0 B_1
\end{array}
$$

Our goal is to show that the map $S$ is an equivalence – this is the Segal condition. The map $i_n$ is the canonical inclusion map $i_n : B'_n \to B_n$. It has a homotopy inverse $B_n \to B'_n$ defined by piece-wise linearly rescaling the first coordinate direction of $\mathbb{R} \times (-1, 1)^\infty$ to that $t_i = i$ and then pushing off everything outside of $[0, n]$ to infinity. This is a standard type of argument and we refer the reader to [GRW10] Proof of 3.9 for details. This also implies that $j$ is an equivalence as it is a homotopy pullback of $i_1$ and $i_0$.

It remains to check that the map $q$ that compares the strict pullback with the homotopy pullback is an equivalence. For this it will suffice to show that the maps $B'_1 \xrightarrow{\simeq} B'_0$ involved in the pullback are Serre fibrations. But now observe that $B'_1$ is in fact homeomorphic to
the space $\coprod_{[W]} M^X(W)$ discussed in the start of \cite[section 3.1]{ERW19}. This is true because the bordism category from \cite{ERW19} agrees with the one from \cite{GRW10} after removing units and \cite[Theorem A.2]{SP17} shows that their topology in turn is homeomorphic to the plot topology we used. We can therefore cite \cite[Proposition 8.20]{ERW19}, which tells us that the source and target maps $s,t: B'_1 \to B'_0$ are Serre fibrations. This shows that $q$ is an equivalence and hence the map $S$ also has to be, and $B_\bullet$ is a Segal space for any $d$ and $X$ as claimed.

The next claim is that $B_{\bullet}$ is complete for $d \leq 2$. As in remark 2.6 we let $B_{1}^{eq} \subset B_1$ denote the subspace of those 1-simplices that represent invertible morphisms in the homotopy category $h(B_1)$. This is always a union of connected components. By \cite[Theorem 6.2]{Rez01} $B_\bullet$ is complete if and only if $d_0 : B_{1}^{eq} \to B_1$ is an equivalence. To prove this, we use that $h(B_\bullet)$ is equivalent to $\text{Cob}_{d}(X)$ by lemma 3.8. Since $d \leq 2$ it is not difficult to see that if a bordism $W : M \to N$ is invertible in $\text{Cob}_{d}(X)$, then it is diffeomorphic to $M \times [0,1]$. (See remark 3.6 for why this does not work for general $d$.) As before we use \cite[section 3.1]{ERW19} to see that $B_{1}^{eq} \subset B_1$ is equivalent to $\coprod_{[M]} \text{Map}(M \times [0,1], X)/\text{Diff}^+(M \times [0,1])$ and the space of objects $B_0$ is equivalent to $\coprod_{[M]} \text{Map}(M, X)/\text{Diff}^+(M)$. In both cases the coproduct runs over representatives for diffeomorphism classes of closed oriented $(d-1)$-manifolds. The map $B_0 \to B_{1}^{eq}$ is an equivalence for $d \leq 2$ because $\text{Diff}^+(M \times [0,1]) \simeq \text{Diff}^+(M)$ holds for all $(\leq 1)$-manifolds $M$, but it is not true for general $d$ as the pseudo-isotopy space is generally non-trivial.

Finally, it remains to show that $\text{Bord}_{d}^{qr}(X)$ satisfies the specialness condition as a $\Gamma$-object in simplicial spaces. This can be checked in each level separately, and relies on the idea that $(-1,1)_{\infty}$ is "large enough" for us to make any two submanifolds disjoint, canonically up to a contractible space of choices. We refer the reader to \cite[Proposition 7.2]{CS19} for details.

**Remark** 3.5. The first model for a bordism category satisfying the Segal condition was constructed in \cite{Sch14}. There the completeness condition is enforced by replacing $\text{Bord}_{d}^{qr}$ with its completion. This is not necessary for $d \leq 2$ as the more naive construction is already complete – see \cite[Remark 5.25]{CS19}. Since we have not been able to find a proof for this claim in the literature, we give the one above.

**Remark** 3.6. Note that for $d > 2$ the simplicial space $\text{Bord}_{d}^{qr}(X)^{(1)}$ is still a Segal space, but the completeness condition is not automatic. In fact, we used in the proof that every invertible bordism $W$ is of the form $M \times [0,1]$. For general dimension $d$ two objects $M,N \in \text{Cob}_{d}$ are isomorphic if and only if $M \times \mathbb{R}$ and $N \times \mathbb{R}$ are diffeomorphic, see \cite[Proposition 3.3]{HJ18}. When $d \neq 4$ an isomorphism $M \to N$ in $\text{Cob}_{d}$ is exactly an $h$-cobordism \cite[Proposition 3.11]{HJ18}. Therefore when $d \geq 5$ the existence of non-trivial $h$-cobordisms implies that $B_\bullet$ is not complete.

We now show that $\text{Bord}_{d}^{qr}(X)$ indeed recovers the desired 1-category $\text{Cob}_{d}(X)$ as its homotopy category. The lemma is based on \cite[Proposition 8.20]{CS19}, which shows the same in a different model for $\text{Bord}_{d}^{qr}(X)$.

**Definition** 3.7. The 1-category $\text{Cob}_{d}(X)$ has as objects tuples $(M,\varphi)$ where $M$ is a closed oriented $(d-1)$-manifold and $\varphi : M \to X$ is a continuous map. A morphism $(M,\varphi) \to (N,\psi)$ is represented by a triple $(W,i,\chi)$ where $W$ is a compact oriented $d$-manifold, $i : M^- \amalg N \cong \partial W$ is an orientation-preserving diffeomorphism, and $\chi : W \to X$ is a continuous map such that $\chi \circ i = \varphi \amalg \psi$. Two such triples $(W,i,\chi)$ and $(W',i',\chi')$ represent the same morphism if there is an orientation preserving diffeomorphism $f : W \cong W'$ such that $f|_{\partial W} \circ i = i'$ and that $\chi' \circ f$ is homotopic to $\chi$ relative to $\partial W$. Composition is defined by gluing cobordisms. This category has a symmetric monoidal structure induced by disjoint union.

**Lemma** 3.8. For all $X$ the homotopy category $h_1 \text{Bord}_{d}^{qr}(X)$ is canonically equivalent to the symmetric monoidal category $\text{Cob}_{d}(X)$ described in definition 3.7.

**Proof.** As discussed in 2.3 we are free to choose representatives for $\pi_0 \text{Bord}_{d}^{qr}(X)_0$. In fact choosing multiple representatives per connected component yields an equivalent category, so we
can simply set the objects of \( h_1\text{Bord}_d^\text{fr}(X) \) to be tuples \((\mathbb{R} \times M, \varphi \circ \text{pr}_M)\) where \( M \subset (-1, 1)^\infty \) is a closed oriented \((d-1)\)-manifold and \( \varphi : M \to X \) is a continuous map. This maps to the objects of \( \text{Cob}_d(X) \) by forgetting the embedding.

The set of morphisms \((M, \varphi) \to (N, \psi)\) can be computed as \( \pi_0 \) of the homotopy pullback

\[
\{(M, \varphi) \} \times_{\text{Bord}_d^\text{fr}(X)} \text{Bord}_d^\text{fr}(X)_1 \times_{\text{Bord}_d^\text{fr}(X)} \{(N, \psi)\}.
\]

Up to equivalence we may replace the middle space with the space \( B_1^d \) of bordism of length 1 from the proof of theorem 3.3. So we are interested in the homotopy fibre of the map \( B_1^d \to (B_0^d)^2 \) sending \( W \) to \((W_0, W_1)\). By \cite{ERW19} Proposition 3.2.4(ii) this map is a fibration, so we can consider the strict fibre instead. From the description in \cite{ERW19} section 3.1 it follows that this is given by:

\[
\text{Hom}_{h_0\text{Bord}_d^\text{fr}(X)}((M, \varphi), (N, \psi)) = \coprod_{[W, \partial W = M^{-1}\Pi N]} \pi_0 \text{Map}(W_{[0,1]}, X; \varphi \Pi \psi)/\text{Diff}^+(W).
\]

Here \( W \) only runs over equivalence classes of abstract bordisms that go from \( M \) to \( N \) and the maps \( W \to X \) are required to restrict to \( \varphi \) and \( \psi \), respectively. There is a canonical functor \( h_1\text{Bord}_d^\text{fr}(X) \to \text{Cob}_d(X) \) defined by forgetting the embedding on objects and sending bordisms to their equivalence class. It follows from the above that this functor is fully faithful. Moreover, it is essentially surjective because every closed oriented \((d-1)\)-manifold can be embedded into \((-1, 1)^\infty\).

Moreover, if we define a \( \Gamma \)-structure on \( \text{Cob}_d(X) \) the same way we did on \( \text{Bord}_d^\text{fr}(X) \), then this is an equivalence of \( \Gamma \)-categories, and the \( \Gamma \)-structure on \( \text{Cob}_d(X) \) corresponds to the standard symmetric monoidal structure defined by disjoint union. We refer the reader to the proof of \cite{CS19} proposition 8.20 for more details on how this is compatible with the \( \Gamma \)-structure.

### 3.2 The cobordism hypothesis

**Lemma 3.9.** For all \( X \) the symmetric monoidal \( \infty \)-category \( \text{Bord}_d^\text{fr}(X) \) has duals.

**Proof.** By definition 2.15 we need to show that every object in the homotopy category \( h_1\text{Bord}_d^\text{fr}(X) \) is dualisable. Because of lemma 3.8 we can equivalently construct duals for the category \( \text{Cob}_d(X) \). This is well-known to those familiar with the cobordism hypothesis, but we include a brief description in the interest of completeness.

Let \((M, \varphi)\) be some object of \( \text{Cob}_d(M) \). Then we will show that \((M^-, \varphi)\) is the dual. As evaluation \( \text{ev} : (M^- \Pi M, \varphi \Pi \varphi) \to (\emptyset, \emptyset) \) we take the manifold \( W = M \times [0,1] \) with the boundary identification \( \partial W \cong (M^- \Pi M)^- \Pi \emptyset \). This will be equipped with the map \( \varphi \circ \text{pr}_M : M \times [0,1] \to M \to X \). Similarly, the coevaluation \( \text{co} : (\emptyset, \emptyset) \to (M \Pi M^-, \varphi \Pi \varphi) \) is \( W \), but with the boundary identification \( \partial W \cong \emptyset \Pi (M \Pi M^-) \). The two compositions \( (\text{id}_M \text{lev}) \circ (\text{co} \Pi \text{id}_M) \) and \( (\text{ev} \Pi \text{id}_{M^-}) \circ (\text{id}_{M^-} \Pi \text{co}) \) are equivalent to the identity bordisms \( M \times [0,1] \) and \( M^- \times [0,1] \), respectively. This is best checked by drawing out the bordisms as illustrated in figure 1.

**Remark 3.10.** One can also equip bordism categories with more general tangential structures. In the context of the cobordism hypothesis one usually uses the framed bordism category \( \text{Bord}_d^\text{fr} \) where each \( W \) comes with a framing. There always is a functor \( \text{Bord}_d^\text{fr}(X) \to \text{Bord}_d^\text{fr}(X) \) that forgets the framing and only remembers the induced orientation. For \( d = 1 \) the space of framings on a fixed \( W \) that are compatible with a preferred orientation is contractible. Therefore \( \text{Bord}_d^\text{fr}(X) \to \text{Bord}_d^\text{fr}(X) \) is a level-wise equivalence of Segal spaces and the associated symmetric monoidal \( \infty \)-categories are equivalent. As we will only need the one-dimensional cobordism hypothesis we can therefore state it using the oriented bordism category \( \text{Bord}_d^\text{fr}(X) \) rather than \( \text{Bord}_d^\text{fr}(X) \).
Corollary 3.14.  \( \text{The object } \psi \) from hypothesis 3.12 and lemma 2.20 we have equivalences.

**Proof.** Consider the full subcategory \( \mathcal{M} \) of \( \text{Bord}^{d+1}_d \) for the case \( d = 1 \) and \( M = *_+ \). The general case can be shown by taking the product of the above with an arbitrary manifold \( M \).

3.11. To formulate the cobordism hypothesis in dimension 1 we need to construct a natural map \( X \to \text{obj}^{fd}(\text{Bord}^{d+1}_d(X)) \) for any space \( X \). As we saw above all objects in \( \text{Bord}^{d+1}_d(X) \) are dualisable, and so the space of dualisable objects is simply \( \text{Bord}^{d+1}_d(X)_0 \). We can define the map \( f : X \to \text{Bord}^{d+1}_d(X)_0 \) by sending \( x \in X \) to the (positively oriented) manifold \( W := \mathbb{R} \times \{0\} \subset \mathbb{R} \times (-1,1) \) equipped with the constant map \( \varphi_x : W \to \{x\} \to X \). This is natural in \( X \) and induces a map:

\[
\text{hom}_{\text{Cat}_{\infty}^\otimes}(\text{Bord}^{d+1}_d(X), C) \to \text{hom}_{\text{Sp}}(\text{obj}^{fd}(\text{Bord}^{d+1}_d(X)), \text{obj}^{fd}(C)) \to \text{hom}_{\text{Sp}}(X, \text{obj}^{fd}(C)).
\]

**Theorem 3.12** (Cobordism hypothesis in dimension 1, Lur09c, for more details see Har12). The map constructed above is an equivalence for all \( X \in \text{Top} \) and \( C \in \text{Cat}_{\infty}^\otimes \):

\[
\text{hom}_{\text{Cat}_{\infty}^\otimes}(\text{Bord}^{d+1}_d(X), C) \simeq \text{hom}_{\text{Sp}}(X, \text{obj}^{fd}(C)).
\]

We will be particularly interested in the case \( X = S^1 \). We now choose the positively oriented point \( *_+ \in \text{Bord}^{d+1}_d(S^1) \) and a preferred endomorphism \( \alpha : *_+ \to *_+ \) as follows:

**Definition 3.13.** The object \( *_+ \) is defined as \( (t = 0, \mathbb{R} \times \{0\}, \varphi) \in (\text{Bord}^{d+1}_d(S^1))^{(1)}_0 \) where \( \mathbb{R} \) is given the standard orientation and \( \varphi : \mathbb{R} \to S^1 \) is the constant map that sends everything to the base point \( 0 \in \mathbb{R}/\mathbb{Z} =: S^1 \). The endomorphism \( \alpha : *_+ \to *_+ \) has as underlying bordism the trivial bordism \( ((t_0 = 0, t_1 = 1), W = \mathbb{R} \times \{0\}) \) of length 1. We equip this with the labelling \( \psi : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z} \) defined as \( \psi(x) = [x] \) for \( x \in [0,1] \) and \( \psi(x) = [0] \) otherwise.

Since \( \alpha \) represents an invertible morphism in the homotopy category this defines a point \( (*_+, \alpha) \in \mathcal{A}(\text{Bord}^{d+1}_d) \) in the space of automorphisms.

**Corollary 3.14.** For every \( Y \in \text{CSS}^\otimes \) evaluating on \( (*_+, \alpha) \in \mathcal{A}^{fd}(\text{Bord}^{d+1}_d(S^1)) \) yields an equivalence

\[
\text{hom}_{\text{CSS}^\otimes}(\text{Bord}^{d+1}_d(S^1), Y) \simeq \mathcal{A}^{fd}(Y).
\]

**Proof.** Consider the full subcategory \( Y^{fd} \subset Y \) on the dualisable objects. By the cobordism hypothesis and lemma 2.20, we have equivalences

\[
\text{hom}_{\text{CSS}^\otimes}(\text{Bord}^{d+1}_d(S^1), Y) \simeq \text{hom}_{\text{Sp}}(S^1, (Y^{fd})^{(1)}_0) \simeq \mathcal{A}^{fd}(Y)
\]

This equivalence corresponds to the evaluation at the point \( \mathcal{A}(\text{Bord}^{d+1}_d(S^1)) \), which is represented by the free loop \( \gamma : S^1 \to \text{obj}(\text{Bord}^{d+1}_d(S^1)) \) sending \( x \) to the manifold \( *_+ \) labeled by \( i : * \to \{x\} \subset S^1 \). Following through the equivalence in the proof of lemma 2.19 we see that this is in the same connected component as the preferred endomorphism \( (\alpha : *_+ \to *_+) \in \mathcal{A}(\text{Bord}^{d+1}_d(S^1)) \). Therefore the natural transformation described in the claim is an equivalence as well. □
3.3 The cobordism category with marked points

We now introduce a variant of the bordism category where morphisms $W : M \to N$ come equipped with a finite subset $A \subset W \setminus \partial W$ of marked points. Then we show that it is a complete Segal space and use it to formulate a special case of Lurie’s cobordism hypothesis with singularities in our setting.

Definition 3.15. For $d \geq 0$ we let $\Psi^m_d$ denote the space of oriented marked $d$-dimensional submanifolds of $\mathbb{R}^\infty$; a point in $\Psi^m_d$ is a tuple $(W, A)$ where $W \subset \mathbb{R}^{1+\infty}$ is a $d$-dimensional oriented submanifold and $A \subset W$ is a 0-dimensional submanifold. We topologise this with a plot topology as in definition 3.2 except that now both the graph obtained from $W$ and the graph obtained from $A$ have to be smooth.

Note that $\Psi^m$ can also be thought as the subspace of $\Psi_d \times \Psi_0$ containing precisely those tuples $(W, A)$ where $A \subset W$.

Definition 3.16. We define a simplicial $\Gamma$-space $\text{Bord}^{\text{or}, m}_d \in \text{P}(\Delta \times \Gamma)$ by the functor $\Delta^{op} \times \Gamma^{op} \to \text{Top}$ that sends $([n], \langle k \rangle)$ to the topological space of tuples

$$(t, (W_1, A_1), \ldots, (W_k, A_k))$$

where $t \in \mathbb{R}^{[n]}$ and the $W_i$ are pairwise disjoint $d$-dimensional oriented submanifolds of $\mathbb{R} \times (-1,1)^\infty$, each of which is admissible with respect to $t$. The $A_i$ are finite subsets of $W_i$ such that $\pi(A_i) \subset \bigcup_{j=1}^n (t_{j-1}, t_j)$. This is topologised as a subspace of $\mathbb{R}^{[n]} \times (\Psi^m)^k$. Functoriality in $([n], \langle k \rangle)$ is as in definition 3.3 except that we also need to forget all points of $A_i$ that lie outside of $(t_0, t_n)$ after applying a face operator.

We have an analogue of 3.4.

Proposition 3.17. For any $d \geq 0$ the simplicial space $(\text{Bord}^{\text{or}, m}_d)^{(1)}$ is a Segal space and the functor $\text{Bord}^{\text{or}, m}_d : \Delta^{op} \times \Gamma^{op} \to \text{Spc}$ is a special $\Gamma$-object in Segal spaces. When $d \leq 2$ this Segal space is moreover complete.

Proof. We will show how the different parts of the proof of theorem 3.4 generalise to the marked case. Let us write $B_\bullet := (\text{Bord}^{\text{or}, m}_d)^{(1)}$ and $B^m_\bullet := (\text{Bord}^{\text{or}, m}_d)^{(1)}$. We let $B'_n \subset B_n$ and $B'^{m}_n \subset B^n_n$ be the length 1 versions as in the proof of theorem 3.4.

Our proof of the Segal property relied on the fact that the maps $s, t : B'_1 \to B'_0$ that send a bordism of length one to its source or target are fibrations. In the next paragraph we will argue that the map $p : B'^{m}_1 \to B'^{m}_1$ that forgets the markings is a locally trivial fiber bundle whose fiber at a point $W \in B'_1$ is the unordered configuration space $\text{Conf}_*(W_{(0,1)})$. In particular $p$ is a Serre fibration. Since $B'_0 = B^m_0$ it follows that the source and target maps of $B^m_\bullet$ are composites of Serre fibrations $B'^{m}_1 \to B'^1_1 \to B'_0 = B^m_0$. Therefore the same proof as in theorem 3.4 applies and $B^m_\bullet$ is a Segal space.

To see that $p : B'^{m}_1 \to B'^1_1$ is a fiber bundle let $W \in B'_1$ any point in the base. We choose a tubular neighbourhood $N \subset \mathbb{R} \times (-1, 1)^\infty$ of $W$ and an identification of $N$ with the normal bundle $\nu_W$. For any smooth section of the normal bundle $f \in \Gamma(\nu_W)$ the image $f(W) \subset N \subset \mathbb{R} \times (-1, 1)^\infty$ is a smooth submanifold and for certain admissible $f$ it is an element $f(W) \in B'_1$: let $U_W \subset \Gamma(\nu_W)$ denote the subset of such $f$. This is in bijection with the set $U'_W \subset B'_1$ of those $V \in B'_1$ such that there is an $f \in \Gamma(\nu_W)$ for which $f(W) = V$. It follows from [CS19, Lemma A.5] that $U_W$ and $U'_W$ are homeomorphic and that $U'_W$ is a neighbourhood of $W \in B'_1$. (For this it is important to note that any $V \in B'_1$ is uniquely determined by its intersection with the compact subspace $[0, 1]^{1+\infty} \subset \mathbb{R}^{1+\infty}$.) All that is left to do is to trivialise $p : B'^{m}_1 \to B'^1_1$ over $U'_W$. Indeed we have the following homeomorphism:

$$U_W \times \text{Conf}_*(W_{(0,1)}) \cong p^{-1}(U_W), \quad (f, A \subset W_{(0,1)}) \mapsto (f(W), f(A)).$$
Next, we check the completeness condition for $B_m$. For this it is useful to think of $B_m$ as the subspace of $B_m$ given by the manifolds with empty configurations. From the definition of the topology we can see that we cannot change the number of points in a configuration by a continuous path and so $B_0 \subseteq B_m$ is a union of connected components for each layer $n$. Moreover, note that for a morphism to be homotopy invertible in $B_m$ both it and its inverse have to be labeled by an empty configuration: this is true because composition adds the number of points in the configuration and the identity morphisms have empty configurations. Therefore, the spaces of equivalences agree $(B_m^1)_{eq} = B_0$. Since we also have $B_0 = B_0$ this means that $B_m$ is complete if and only if $B_m$ is. We have argued why this is true for $d \leq 2$ in 3.4.

Finally, the specialness condition for the $\Gamma$-direction is again standard. □

We pick the positively oriented point $*_+$ as a preferred object in $\text{Bord}_{1}^{\text{or},m}$ and construct an endomorphism $\beta : *_+ \to *_+$. 

**Definition 3.18.** The endomorphism $\beta = (t,W,A) : *_+ \to *_+$ is defined by $(t_0,t_1) = (0,1)$, $W = \mathbb{R} \times \{0\}$, and $A = \{1\} \times \{0\} \subset W$. This defines an element $(*_+,\beta) \in \mathcal{E}(\text{Bord}_{1}^{\text{or},m})$ in the space of endomorphisms from definition 2.16.

We will now construct a functor $L : \text{Bord}_{1}^{\text{or},m} \to \text{Bord}_{1}^{\text{or}}(S^1)$ that sends a marked bordism $(W,A) : M \to N$ to $(W,l) : M \to N$ where the labeling $l : W \to S^1$ maps most of $W$ to the basepoint, except for a small neighbourhood of $A$ where it loops around the circle $S^1$ once per point in $A$.

**Construction 3.19.** To define the functor $L$ we first need to enhance the $\infty$-category $\text{Bord}_{1}^{\text{or},m}$ to contain the (contractible) data of disjoint small $\epsilon$-balls around the marked points. For any oriented 1-manifold $W \subset \mathbb{R}^{1+\infty}$ we can define a signed distance function $d_W : W \times W \to \mathbb{R}\{\infty\}$ by setting $d(x,y) = \pm l$ where $l$ is the length of the shortest path from $x$ to $y$ and the sign depends on whether that path agrees with the orientation of $W$. Let $B$ be the simplicial $\Gamma$-space defined just like $\text{Bord}_{1}^{\text{or},m}$ except that each $(W_i,A_i)$ comes with a function $\epsilon_i : A_i \to (0,\infty)$ satisfying, for all $a \in A_i$:

$$2\epsilon_i(a) < \min\{|d_W(a,b)| \mid b \in A_i \setminus \{a\}\} \quad \text{and} \quad \epsilon_i(a) < \min\{|\pi(a) - t_j| \mid j \in \{0,\ldots,n\}\}.$$  

Because the space of possible choices for $\epsilon_i$ is convex, the forgetful map $B \to \text{Bord}_{1}^{\text{or},m}$ is a level-wise equivalence. As a result $B$ is a complete Segal space.

We will now construct a functor $L : B \to \text{Bord}_{1}^{\text{or}}(S^1)$. Let $(t_i(M_1,A_1,\epsilon_1),\ldots,(W_k,A_k,\epsilon_k))$ be a point in $B_n^{(\text{or})}$. Then we define $l_i : W_i \to S^1 = \mathbb{R}/\mathbb{Z}$ as follows:

$$l_i(x) = \begin{cases} \frac{1}{2} + \frac{d(a,x)}{2\epsilon_i(a)} & \text{if there is } a \in A_i \text{ with } |d(a,x)| \leq \epsilon_i(a) \\ 0 & \text{otherwise.} \end{cases}$$

One checks that this is continuous and loops around $S^1$ once in every $\epsilon_i(a)$-ball.

We can now state our interpretation of Lurie’s cobordism hypothesis with singularities in the one-dimensional case. As explained in warning 1.4 some of our theorems are conditional on this conjecture.

**Conjecture 3.20.** For any $\infty$-category $\mathcal{C}$ evaluating on the morphism $(W,A) : *_+ \to *_+$ yields an equivalence

$$\text{Fun}_\infty(\text{Bord}_{1}^{\text{or},m},\mathcal{C}) \simeq \mathcal{E}(\mathcal{C}^{(\text{or})}).$$

**Remark 3.21** (Comparison to Lurie’s conjecture). We will now informally derive our formulation from Lurie’s more general cobordism hypothesis with one type of singularity as stated in [Lur09c, Proposition 4.3.1].

17
In the general setting one fixes a $d - 1$-dimensional manifold $Y$ and then allows bordisms to have singularities of the form of a cone on $Y$. We will only need this in dimension 1 and for the case $Y = S^0$. Of course a cone on $S^0$ is diffeomorphic to $D^1$ and so instead of introducing actual singularities in a bordism $W$ it suffices to keep track of the cone points. This is the set of marked points $A \subset W$. A small ball around each marked point $a \in A$ is then is a cone on $S^0$. The cobordism hypothesis with singularities says that there is an equivalence between functors $\text{Bord}_{1,\text{or}} \to \mathcal{C}$ and functors $\mathcal{Z} : \text{Bord}_{1,\text{or}} \to \mathcal{C}$ together with a choice of morphism $\alpha : \mathbb{I} \to \mathcal{Z}(S^0)$.

By the cobordism hypothesis without singularities $\text{Fun}^\otimes_{\infty}(\text{Bord}_{1,\text{or}}^!, \mathcal{C})$ is equivalent to $(\mathcal{C}^\text{fd})^\simeq$ via the evaluation on the positively oriented point. Let $x \in \mathcal{C}$ denote $\mathcal{Z}(*)$. Then the value of $\mathcal{Z}$ on $S^0$ is $\mathcal{Z}(S^0) \cong \mathcal{Z}(\star) \cong \mathcal{Z}(\star) \otimes \mathcal{Z}(\star) \cong x \otimes x$. By duality $\text{hom}(\mathbb{I}, x \otimes x)$ is equivalent to $\text{hom}(x, x)$, so instead of $\alpha : \mathbb{I} \to \mathcal{Z}(S^0) = x \otimes x$ we may equivalently choose $\alpha^* : x \to x$.

In summary, the data of a symmetric monoidal functor $\text{Bord}_{1,\text{or}} \to \mathcal{C}$ is equivalent to that of a dualisable object $x \in \mathcal{C}^\text{fd}$ together with an endomorphism $\alpha^* : x \to x$. In other words, to a point $(x, \alpha^*) \in \mathcal{E}(\mathcal{C}^\text{fd})$.

3.22. By construction the functor $L : \text{Bord}_{1,\text{or}} \to \text{Bord}_{1,\text{or}}(S^1)$ sends the preferred endomorphism $\beta : \star \to \star$ in $\text{Bord}_{1,\text{or}}$ to the preferred automorphism $\alpha : \star \to \star$ in $\text{Bord}_{1,\text{or}}(S^1)$. Therefore the following diagram commutes:

$$\text{Fun}^\otimes_{\infty}(\text{Bord}_{1,\text{or}}(S^1), \mathcal{C}) \xrightarrow{\sim} \mathcal{A}(\mathcal{C}^\text{fd})$$

Moreover by the two variants of the cobordism hypothesis the horizontal morphisms in this diagram are equivalences for all $\mathcal{C}$. This means that via the Yoneda lemma the functor $L$ corresponds to the natural transformation $\mathcal{A}^\text{fd} \Rightarrow \mathcal{E}^\text{fd}$.

4 1-categorical classification

In this section we complete the proof of the classification of 1-categorical tracerlike transformations sketched in the introduction.

**Definition 4.1.** The category $\text{Cot}_{1}^Z$ has as objects finite sets $M$ equipped with an orientation $M \to \{+, -\}$. A morphism $X : M \to N$ is a diffeomorphism class of 1-dimensional oriented bordisms $X$ equipped with $\partial X \cong M - \Pi N$ and a relative integral cohomology class $\xi \in H^1(X, M \Pi N)$. The composition of two morphisms $(X, \xi) : M \to N$ and $(Y, \zeta) : N \to L$ is the morphism $(X \Pi_N Y, \chi)$ where $\chi$ is defined as the image of $(\xi, \zeta)$ under

$$H^1(X, M \Pi N) \oplus H^1(Y, N \Pi L) \cong H^1(X \Pi_N Y, M \Pi N \Pi L) \to H^1(X \Pi_N Y, M \Pi L).$$

The symmetric monoidal structure is defined by taking disjoint unions, the unit is the empty set.

4.2. Since the bordisms $X : M \to N$ are oriented 1-manifolds we can canonically identify $H^1(X, M \Pi N)$ with $\mathbb{Z}^{\tau_0 X}$. Therefore choosing $\xi$ is equivalent to labelling every connected component of $X$ by an integer. The composition adds integers of connected components that are joint in the process of glueing bordisms. Graphically this can be described as:

```
2
+ 0
+ 1
+ 3
+ 1
+ -1
```

18
We write $*_{+}$ for the object defined by one positively oriented point and $\alpha : *_{+} \to *_{+}$ for its automorphism defined by the trivial bordism labelled by the integer 1.

**Definition 4.3.** Let $\text{Cob}_{1}^{N} \subset \text{Cob}_{1}^{Z}$ denote the subcategory that contains all objects, but only those morphisms that are labelled by non-negative integers under the identification in [4.2](#).

**4.4.** We will now show that $\text{Cob}_{1}^{Z}$ is equivalent to the homotopy category of $\text{Bord}_{1}^{or}(S^{1})$ and that $\text{Cob}_{1}^{N}$ is equivalent to the homotopy category of $\text{Bord}_{1}^{or,m}$. Recall that in the process of defining $h_{1}X$ for a complete Segal space $X : \Delta^{op} \to \text{Spc}$ we had to choose a section $o : \pi_{0}X_{0} \to X_{0}$. For $\text{Bord}_{1}^{or}(S^{1})$ we may choose $o$ to take values $(t,(M,\varphi))$ such that $\varphi : M \to S^{1}$ only hits the base-point of $S^{1}$.

**Lemma 4.5.** There is a commutative diagram of symmetric monoidal functors

$$
\begin{array}{ccc}
\text{h}_{1}\text{Bord}_{1}^{or,m} & \xrightarrow{\text{h}_{1}\text{L}} & \text{h}_{1}\text{Bord}_{1}^{or}(S^{1}) \\
\downarrow \text{G} & & \downarrow \text{G} \\
\text{Cob}_{1}^{N} & \xrightarrow{\text{F}} & \text{Cob}_{1}^{Z} \\
\end{array}
$$

where the vertical functors are equivalences.

**Proof.** The rightmost vertical functor is the symmetric monoidal equivalence from [3.8](#). To prove the the lemma we have to provide compatible lifts $G$ and $G'$.

For $G$ this means that we have to give, for each $[X,\varphi] : (M,\varphi|_{M}) \to (N,\varphi|_{N})$ a class $\alpha \in H^{1}(X,M\amalg N)$. Because of the choice we made in 4.4 we may assume that both $\varphi|_{M}$ and $\varphi|_{N}$ are the constant maps to the basepoint $* \in S^{1}$. Therefore, the pullback of the canonical generator $[S^{1}] \in H^{1}(S^{1},*)$ gives a well-defined class $\alpha := \varphi^{*}(S^{1}) \in H^{1}(X,M\amalg N)$. This construction is compatible with gluing and disjoint union and therefore yields a symmetric monoidal functor $G$. Moreover, the assignment $[\varphi] \mapsto \varphi^{*}[S^{1}]$ defines a bijection between $\pi_{0}\text{Map}((X,M\amalg N),(S^{1},*))$ and $H^{1}(X,M\amalg N)$ for any bordism $X$ and hence $G$ is an equivalence of categories.

To obtain the functor $G'$, first consider the composite $G \circ \text{h}_{1}\text{L}$. Concretely, let $(W,A) : M \to N$ be some marked bordism. Then $L(W,A) = (W,\varphi)$ is equipped with a labelling such that $\varphi : W \to S^{1}$ winds around the circle once for each marking. This is done compatibly with the orientation and hence every connected component $W_{0} \subset W$ will be labeled in $(G \circ \text{h}_{1}\text{L})(W,A)$ by the number of elements of $W_{0} \cap A$. Since this is non-negative it lies in the subcategory $\text{Cob}_{1}^{N}$ and the functor $G \circ \text{h}_{1}\text{L}$ can be factored through a unique symmetric monoidal functor $G' : \text{h}_{1}\text{Bord}_{1}^{or,m} \to \text{Cob}_{1}^{N}$. This functor is an equivalence of categories because up to diffeomorphism the marking of a bordism is uniquely determined by the number of marked points in each connected components.

Note that warning [1.3](#) applies to the first line of the following proposition as we use the conjectural cobordism hypothesis with singularities.

**Proposition 4.6.** There are compatible isomorphisms of commutative monoids

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\cong} & \text{Sc}(\text{Cob}_{1}^{N}) \\
\downarrow & & \downarrow \\
\mathcal{T}^{r} & \xrightarrow{\cong} & \text{Sc}(\text{Cob}_{1}^{Z}) \\
\end{array}
\begin{array}{ccc}
\xrightarrow{\cong} & \xrightarrow{\cong} & \xrightarrow{\cong} \\
\text{Mfd}_{1}^{N}/\text{Diff}^{+} & \to & \text{N}[x] \\
\downarrow & & \downarrow \\
\text{Mfd}_{1}^{Z}/\text{Diff}^{+} & \to & \text{N}[x^{\pm 1}] \\
\end{array}
$$

under which the tracelike transformation $\Theta^{k_{1},\ldots,k_{n}}$ is sent to $\sum_{i=1}^{n} x^{k_{i}}$.  

19
\textbf{Proof.} We begin by constructing the isomorphisms in the second line, the first line is then obtained similarly. As a consequence of lemma 4.5, the $h_1$-adjunction 2.13, the cobordism hypothesis 3.14 and lemma 2.21, we have isomorphisms natural in $C \in \text{Cat}_1^{\otimes}$:

$$\pi_0 \text{Fun}_1^\otimes(Cob_1^\otimes, Z) \cong \pi_0 \text{Fun}_1^\otimes(h_1 \text{Bord}_1^\otimes(S^1), Z) \cong \pi_0 \text{Fun}_\infty^\otimes(\text{Bord}_1^\otimes(S^1), \text{NC}) \\ \cong \pi_0 A^{id}(\text{NC}) \cong A^{id}(C).$$

The object $*_+$ and automorphism $\alpha$ from 4.2 define an element of $A^{id}(\text{Cob}_1^\otimes)$; this is precisely the element that corresponds to $\text{id}_{\text{Cob}_1^\otimes}$ when we apply the above bijection in the case $C = \text{Cob}_1^\otimes$.

We now apply the 1-categorical coYoneda lemma for the 1-category $h_1 \text{Cat}_1^\otimes$. (See theorem 5.2 where we recall the $\infty$-categorical version.) Recall that in this category objects are symmetric monoidal 1-categories and morphisms are isomorphism classes of symmetric monoidal functors. So the $A^{id}(C)$ can be described as $\text{Hom}_{h_1 \text{Cat}_1^\otimes}(\text{Cob}_1^\otimes, C)$ in this category. Therefore the coYoneda lemma implies that sending $T$ to $T_{\text{Cob}_1^\otimes}(*_+, \alpha)$ defines a bijection:

$$\mathcal{T}^r = \text{Hom}_{\text{Fun}(h_1 \text{Cat}_1^\otimes, \text{Set})}(A^{id}, S_c) \cong \text{Hom}_{\text{Fun}(h_1 \text{Cat}_1^\otimes, \text{Set})}(\text{Hom}_{\text{Cat}_1^\otimes}(\text{Cob}_1^\otimes, -), S_c) \cong S_c(\text{Cob}_1^\otimes).$$

This is an isomorphism of monoids since the monoid structure on $\mathcal{T}^r$ is defined by pointwise multiplication.

The scalars of $\text{Cob}_1^\otimes$ are diffeomorphism classes of closed oriented 1-manifolds labelled by integers, this is precisely how the set $\text{Mfd}_1^\otimes/\text{Diff}^+$ was defined in the introduction. Recall that $\mathbb{N}[x^{\pm 1}]$ is the free commutative monoid on the set $\{\ldots, x^{-1}, x^0, x^1, x^2, \ldots\} \cong \mathbb{Z}$. We define a homomorphism $\mathbb{N}[x^{\pm 1}] \to \text{Mfd}_1^\otimes/\text{Diff}^+$ by sending $x^k$ to the circle labelled by $k$. This is a bijection since all closed 1-manifolds can be written as the disjoint union of circles.

This establishes the isomorphisms in the second line of the proposition. The first line is obtained similarly, except that we use the cobordism hypothesis with singularities 3.20 instead of 3.14.

To complete the proof we need to compute the value of the tracelike transformation $\Theta^{k_1, \ldots, k_n}$ on $(*_+, \alpha)$. In other words, we have to compute $\text{Tr}(\alpha^{k_1}) \circ \cdots \circ \text{Tr}(\alpha^{k_n})$ using the classical definition of the trace. Using the evaluation and coevaluation provided in 3.9 one sees that the trace of a bordism $(X, \partial X \cong M \amalg M^\sim)$ is given by gluing the two boundaries of $X$. See also ST12 for a different perspective on this. In the case at hand we compute:

$$0 \begin{array}{ccc} + & + & k_1 \end{array} 0 \begin{array}{ccc} + & + & k_n \end{array} 0 \cdots 0$$

This verifies that $\Theta^{k_1, \ldots, k_n}(*_+, \alpha)$ is a disjoint union of $n$ circles labelled by the integers $k_1, \ldots, k_n$. Hence $\Theta^{k_1, \ldots, k_n}$ is sent to the polynomial $x^{k_1} + \cdots + x^{k_n}$ as claimed. \hfill \Box

\section{$\infty$-categorical classification}

In this section we combine the cobordism hypothesis with the Yoneda lemma and make the necessary computations to prove the main theorems.

\subsection{Applying the Yoneda lemma and the cobordism hypothesis}

\textbf{5.1.} We begin by recalling the ‘coYoneda lemma’: using the Yoneda embedding $Y_C : C \to P(C)$ we can think of $Y_C(x) = \text{hom}_C(\_, x)$ as a functor from $C^{\text{op}}$ to $\text{Spc}$. For $\mathcal{D} := C^{\text{op}}$ we then have $\text{hom}_D(x, \_) : \mathcal{D} \to \text{Spc}$. The coYoneda lemma tells us how to compute natural transformations
out of this functor. Recall that for two $\infty$-categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$ the space of natural transformations $F \Rightarrow G$ is

$$\text{hom}_{\text{Fun}_\infty(\mathcal{C}, \mathcal{D})}(F, G).$$

Evaluating at a certain object $x \in \mathcal{C}$ gives a functor $ev_x : \text{Fun}_\infty(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ and consequently a map $\text{hom}_{\text{Fun}_\infty(\mathcal{C}, \mathcal{D})}(F, G) \to \text{hom}_\mathcal{D}(F(x), G(x))$.

**Theorem 5.2** (coYoneda lemma, [Lur09, Lemma 5.5.2.1]). For any functor $F : \mathcal{D} \to \text{Spc}$ and object $x \in \mathcal{C}$ the following composition of evaluations is a natural equivalence

$$\text{hom}_{\text{Fun}_\infty(\mathcal{D}, \text{Spc})}(\text{hom}_\mathcal{D}(x, \_), F) \xrightarrow{ev_x} \text{hom}_{\text{Spc}}(\text{hom}_\mathcal{D}(x, x), F(x)) \xrightarrow{\text{ev}_{F(x)}} F(x).$$

We can now use the coYoneda lemma in conjunction with the one-dimensional cobordism hypothesis to express the moduli spaces $\text{Bord}_1^{or}(S^1)$ and $\text{Bord}_1^{or,m}$, respectively. Note, however, that to compute $\mathcal{I}_\infty$ we use the conjectural cobordism hypothesis with singularities, so warning 1.4 applies.

**Corollary 5.3.** There are equivalences of spaces

$$\mathcal{I}_0 \simeq \text{Sc}(\text{Bord}_1^{or}(S^1)) \quad \text{and} \quad \mathcal{I}_\infty \simeq \text{Sc}(\text{Bord}_1^{or,m}).$$

**Proof.** By corollary 3.14 of the cobordism hypothesis the functor $\mathcal{A}^{fd}$ is naturally equivalent to the corepresented functor $\text{hom}_{\text{Cat}_\infty^\Gamma}(\text{Bord}_1^{fd}(S^1), \_).$ Then first equivalence now follows from the coYoneda lemma for $\mathcal{D} = \text{Cat}_\infty^\Gamma$, $x = \text{Bord}_1^{fd}(S^1)$, and $F = \text{Sc}$.

Similarly, the cobordism hypothesis, with singularities in dimension 1 says that $\mathcal{E}^{fd}$ is naturally equivalent to the corepresented functor $\text{hom}_{\text{Cat}_\infty^\Gamma}(\text{Bord}_1^{or,m}, \_).$ Hence the second equivalence also follows from the coYoneda lemma.

### 5.2 Identification of the homotopy-type of the scalars of the bordism category

We give a more geometric description of $\text{Sc}(\text{Bord}_1^{or}(S^1))$:

**Definition 5.4.** For a topological space $X$, let $\text{Mfd}_d^X \subset \Psi_d(X)$ be the space of of closed $d$-manifolds submanifolds $M \subset (-1, 1)^\infty$ equipped with an $X$-structure $\varphi : M \to X$.

**Lemma 5.5.** There is an equivalence $\text{Sc}(\text{Bord}_1^{or}(X)) \simeq \text{Mfd}_d^X$.

**Proof.** The monodial unit of $\text{Bord}_1^{or}(X)$ is given by the empty manifold, so by lemma 2.20 the space of scalars is equivalent to $\text{hom}_{\text{Bord}_1^{or}(X)}(\emptyset, \emptyset)$, which is indeed equivalent to $\text{Mfd}_d^X$. □

**Definition 5.6.** We write $\mathbb{T}$ for $S^1$ with the usual group structure, i.e. $\mathbb{T} \cong U_1 \cong SO_2$. For a space $X \in \text{Spc}$ we denote its free loop space by

$$\Lambda X := \text{Map}(\mathbb{T}, X).$$

The group $\mathbb{T}$ acts on this by precomposition and we denote the homotopy-orbits by

$$(\Lambda X)_{h\mathbb{T}} := (ET \times \Lambda X)/\mathbb{T}.$$  

The space of closed manifolds is best understood as a free $E_\infty$-algebra on the space of connected closed manifolds. Recall the following:

**Theorem 5.7.** For a topological space $X$ the underlying space of the free $E_\infty$-algebra on $X$ is given by:

$$\text{Free}_{E_\infty}(X) \simeq \coprod_{n \geq 0} X^{n}_{h\Sigma_n}.$$
Proof. This is a classical theorem by Segal: his proof of the Barratt-Priddy-Quillen theorem in [Seg74, Proposition 3.5 and 3.6] in fact shows that the unordered configuration space $Conf_n(\mathbb{R}^\infty; X)$ is a model for the free $\Gamma$-space on $X$. A modern reference is [Lur18 Proposition 3.1.3.13].

The following is fairly standard, see for instance [Gia19, Lemma 7.1], but we give a proof for completeness sake.

Lemma 5.8. For any topological space $X$ the space of closed 1-manifolds with map to $X$ is:

$$\text{Mfd}^X_1 \simeq \text{Free}_{E_\infty}((\Lambda X)_{hT})$$

Here, as in theorem 5.7, $\text{Free}_{E_\infty}(Y)$ denotes the free $E_\infty$-algebra on a space $Y$.

Proof. The space $\text{Mfd}_1$ has a natural $\mathbb{N}$-grading induced by the number of connected components. All closed 1-manifolds are disjoint unions of circles, so the $n$th level of the grading is space of $1$-submanifolds of $(-1,1)^\infty$ that are abstractly diffeomorphic to $(S^1)^{\{1\}}$. In the case $X = *$ this space is equivalent to the classifying space $B\text{Diff}^+(S^1)^{\{1\}}$ by [SP17, Corollary B.5]. A similar argument shows that for general $X$ it is the homotopy-orbits of $\text{Map}((S^1)^{\{1\}}, X)$ with respect to the action of $\text{Diff}^+(S^1)^{\{1\}}$:

$$\text{Map}((S^1)^{\{1\}}, X)_{\text{Diff}^+(S^1)^{\{1\}}}$$

The group $\text{Diff}^+(S^1)^{\{1\}}$ can be decomposed as a wreath product $\Sigma_n \text{Diff}^+(S^1)$ acting component-wise $\text{Map}(S^1, X)^n$. Since we are working with homotopy actions, we may replace the group $\text{Diff}^+(S^1)$ by the equivalent group $T$. Homotopy-orbits with respect to the action of a wreath product can be computed as

$$(\text{Map}(S^1, X)^n)_{h(\Sigma_n \times T)} \simeq \left((\text{Map}(S^1, X)_{hT})^n\right)_{h\Sigma_n}.$$  

Putting all the parts of the $\mathbb{N}$-grading back together we get

$$\text{Mfd}^X_1 \simeq \coprod_{n\geq 0} \left((\text{Map}(S^1, X)_{hT})^n\right)_{h\Sigma_n} \simeq \text{Free}_{E_\infty}(\text{Map}(S^1, X)_{hT})$$

as claimed. Here the last equivalence is as in theorem 5.7.

Lemma 5.9. In the case $X = S^1$ that is most relevant to us, we compute:

$$\Lambda(S^1)_{hT} \simeq (S^1 \times BS^1) \amalg \coprod_{k \in \mathbb{Z}\setminus\{0\}} B(\mathbb{Z}/k\mathbb{Z}).$$

Proof. The space $\text{Map}(S^1, S^1)$ has as connected components the spaces $\text{Map}^k(S^1, S^1)$ of maps with winding number $k$ for $k \in \mathbb{Z}$. We need to compute $\text{Map}^k(S^1, S^1)_{hT}$ for all $k \in \mathbb{Z}$. Let $X_k$ be the space $S^1 \subset \mathbb{R}^2$ with the action of $T$ defined by $\lambda \cdot \zeta := \lambda^k \cdot \zeta$. There is an $T$-equivariant embedding $\iota : X_k \to \text{Map}^k(S^1, S^1)$ that identifies $X_k$ with the space of degree $k$ maps $S^1 \to S^1$ of constant speed. Non-equivariantly $\iota$ is a homotopy equivalence with the inverse $\text{Map}^k(S^1, S^1) \to S^1$ given by evaluation on the base-point. Therefore $\iota$ is a Borel weak equivalence and induces an equivalence on the homotopy-orbits:

$$\text{Map}(S^1, S^1)_{hT} \simeq \coprod_{k \in \mathbb{Z}} (X_k)_{hT}.$$  

To compute $(X_k)_{hT}$ for $k \neq 0$ observe that $X_k$ can be thought of as the quotient $S^1/(\mathbb{Z}/k\mathbb{Z})$ with $T$-action induced from the standard action of $T$ on $S^1$. Therefore

$$(X_k)_{hT} \simeq (X_k \times ET)/T \cong ((S^1/(\mathbb{Z}/k\mathbb{Z}) \times ET)/T \cong ((S^1 \times ET)/T)/(\mathbb{Z}/k\mathbb{Z})$$
The space \((S^1 \times ET)/T\) is homeomorphic to \(ET\) and therefore contractible. The action of \(\mathbb{Z}/k\mathbb{Z}\) is free and hence \((X_k)_{hT}\) is a model for \(B(\mathbb{Z}/k\mathbb{Z})\).

The remaining case \(k = 0\) is easy: the space \(X_0\) is \(S^1\) with the trivial \(T\)-action, therefore the homotopy fixed-points decompose as \((X_0)_{hT} \simeq S^1 \times BT\).

5.3 The space of marked 1-manifolds

Lemma 5.10. There is an equivalence:

\[
\text{Sc}(\text{Bord}_1^{or,m}) \simeq \text{Free}_{E_\infty}\left( BS^1 \amalg \prod_{k=1}^{\infty} B(\mathbb{Z}/k\mathbb{Z}) \right).
\]

Proof. Just like in lemma 5.5 we see that the space of scalars \(\text{Sc}(\text{Bord}_1^{or,m})\) is equivalent to the space of closed marked oriented 1-dimensional submanifolds of \((-1,1)^\infty\). By an argument as in proposition 3.17 this space is homeomorphic to a Borel construction:

\[
\text{Mfd}_1^{or,m} \simeq \prod_{n=0}^{\infty} \left( \text{Emb}((S^1)^{1n}, (-1,1)^\infty) \times \text{Conf}_*((S^1)^{1n}) \right) / \text{Diff}^+((S^1)^{1n})
\]

Here the diffeomorphism group \(\text{Diff}^+(M)\) acts diagonally on \(\text{Conf}_*_i(M) \subset \prod_{k \geq 0} M^k/\Sigma_k\). The configuration space has the property that there is a canonical homeomorphism \(\text{Conf}_*_i(M \amalg M') \cong \text{Conf}_*_i(M) \times \text{Conf}_*_i(M')\) for any two manifolds \(M\) and \(M'\). In particular we have \(\text{Conf}_*_i((S^1)^{1n}) \cong \prod_{i=1}^{n} \text{Conf}_*(S^1)\). We can therefore argue just like in lemma 5.8 to see that \(\text{Mfd}_1^{or,m}\) is equivalent to the underlying space of a free \(E_\infty\)-algebra:

\[
\text{Mfd}_1^{or,m} \simeq \text{Free}_{E_\infty}(\text{Conf}_*(S^1)_{hT})
\]

We are now left to compute the homotopy \(T\)-orbits of \(\text{Conf}_*(S^1)\). To do so, we can decompose the configuration space by the cardinality of the configuration. In the \(k = 0\) case we have only the empty configuration, so \(\text{Conf}_0(S^1) = \{\emptyset\}\) is a point and the homotopy orbits are \(BT\). For \(k > 0\) let, as before, \(X_k\) be the space \(S^1 \subset C\) with the action of \(T\) defined by \(\lambda \cdot \zeta := \lambda^k \cdot \zeta\). There is an equivariant map \(i : X_k \to \text{Conf}_0(S^1)\) which sends \(\zeta\) to the subset \(\{\xi \mid \xi^k = \zeta\} \subset S^1\). This map identifies \(X_k\) with the subspace of ‘equally spaced’ configurations of \(k\) points on the circle and therefore is an equivalence. As in lemma 5.9 we have that \((X_k)_{hT} \simeq B\mathbb{Z}/k\mathbb{Z}\) and hence the result follows.

5.4 Proof of theorem A

Theorem 5.11 (Theorem A). There is a commutative diagram of spaces:

\[
\begin{array}{ccc}
\mathcal{F}_\infty & \simeq & \text{Free}_{E_\infty}(BS^1 \amalg \prod_{k\geq 1} B(\mathbb{Z}/k\mathbb{Z})) \\
\downarrow & & \downarrow \\
\mathcal{F}_0 & \simeq & \text{Free}_{E_\infty}(S^1 \times BS^1 \amalg \prod_{k\in\mathbb{Z}\setminus\{0\}} B(\mathbb{Z}/k\mathbb{Z})) \\
\downarrow & & \downarrow \\
\mathcal{F} & \simeq & \mathbb{N}[x] \\
\downarrow & & \downarrow \\
\mathcal{F}^r & \simeq & \mathbb{N}[x^{\pm 1}]
\end{array}
\]

where the horizontal maps are equivalences. Moreover, the vertical maps identify the sets in the bottom layer as the set of connected components of the top layer: \(\mathcal{F} \cong \pi_0\mathcal{F}_\infty\) and \(\mathcal{F}^r \cong \pi_0\mathcal{F}_0^r\).

Proof. The bijections in the bottom row are discussed in proposition 4.6 about the classification of the 1-categorical trace-like transformations. For the top row corollary 5.3 identifies \(\mathcal{F}_\infty^r\)
and $\mathcal{T}_\infty$ with $\text{Sc}(\text{Bord}_1^\text{or}(S^1))$ and $\text{Sc}(\text{Bord}_1^\text{or,m}(S^1))$, respectively. These spaces of scalars are then computed in lemmas 5.5, 5.8, 5.9, and 5.10.

Since the $\infty$-categorical classification followed the same steps as the 1-categorical one, the left-hand square commutes. By 3.22 the top-right map can be understood as $\text{Sc}(L) : \text{Sc}(\text{Bord}_1^\text{or}(S^1)) \to \text{Sc}(\text{Bord}_1^\text{or,m}(S^1))$. Combining the definition of $L$ with the equivalences in lemmas 5.8, 5.9, and 5.10 it follows that it is the inclusion $\mathcal{T}^\infty_1 \to \mathcal{T}^\infty_1$ on the first component and the identity on $B\mathbb{Z}/k \to B\mathbb{Z}/k$ for $k > 0$. The right-most vertical map is $\text{Sc}(\text{Bord}_1^\text{or}(S^1)) \to \text{Sc}(\text{Cob}_1^\infty)$, which is a bijection on connected components because $\text{Cob}_1^\infty \simeq h_1 \text{Bord}_1^\text{or}(S^1)$. This implies that $\pi_0 \mathcal{T}^\infty_\infty \cong \mathcal{T}$ and by the same argument $\pi_0 \mathcal{T}^r_\infty \simeq \mathcal{T}^r$.

Note that the vertical map sends $S^1 \times BS^1 \to \{x^0\}$ and $B\mathbb{Z}/k\mathbb{Z}$ to $\{x^k\}$.

**Corollary 5.12** (Corollary 3 [generalising [TV15 Théorème 3.18])]. The space of (restricted) $\infty$-categorical tracelike transformations that act as the 1-categorical trace $\text{Tr}$ on homotopy categories is contractible.

**Proof.** Under the equivalences of theorem A, the preimage of $\text{Tr}$ under $\mathcal{T}_\infty \to \mathcal{T}$ is the connected component
\[
B(\mathbb{Z}/1\mathbb{Z}) \subset B\mathbb{T} \amalg \prod_{k=1}^\infty B\mathbb{Z}/k\mathbb{Z} \subset \text{Free}_{E\mathbb{Z}} \left( B\mathbb{T} \amalg \prod_{k=1}^\infty B\mathbb{Z}/k\mathbb{Z} \right).
\]

But $\mathbb{Z}/1\mathbb{Z}$ is the trivial group so this connected component is contractible. The same argument applies to the restricted case $\mathcal{T}^r_\infty \to \mathcal{T}^r$, recovering Toën and Vezzosi’s [TV15 Théorème 3.18]. (Note that by lemma 2.20 the definition of $\mathcal{A}$ really recovers that of Toën and Vezzosi.)

**Corollary 5.13** (Corollary C). Any $\infty$-categorical tracelike transformation whose value on the category of complex vector spaces agrees with the trace from linear algebra is canonically equivalent to the $\infty$-categorical trace.

**Proof.** By corollary 5.12 the connected component of $\text{Tr}$ in $\mathcal{T}_\infty$ is contractible. Since we have a full classification of tracelike transformations it is enough to show that for any $\mathbb{N}$-polynomial $\sum_{i=1}^n x^k_i$, the map
\[
\Theta_{\text{Vect}_k}^{k_1,\ldots,k_n} : E^d(\text{Vect}_\mathbb{C}) \to \text{Sc}(\text{Vect}_\mathbb{C}) \cong \mathbb{C}, \quad (V,f : V \to V) \mapsto \text{Tr}(f^{k_1}) \cdots \text{Tr}(f^{k_n})
\]
is the trace from linear algebra only if $n = 1$ and $k_1 = 1$.

For this purpose consider the $2 \times 2$ diagonal matrix $A_\lambda$ with entries 1 and $\lambda \in \mathbb{C}$:
\[
\Theta_{\text{Vect}_\mathbb{C}}^{k_1,\ldots,k_n}(\mathbb{C}^2, A_\lambda) = (1 + \lambda^{k_1}) \cdots (1 + \lambda^{k_n}).
\]
Comparing coefficients we see that this can only be equal to $\text{Tr}(A_\lambda) = 1 + \lambda$ for all $\lambda \in \mathbb{C}$ if $n = 1$ and $k_1 = 1$.

Note that by the same argument corollary C also holds for restricted tracelike transformations.

**Remark 5.14.** The statement of corollary C remains true if we replace $\mathbb{C}$ by any other infinite field. It then however is not sufficient to look only at the matrices $A_\lambda$. If, for instance, $k$ is of characteristic 3, then $\Theta_{\text{Vect}_\mathbb{F}_p}^{k_1,\ldots,k_n}(\mathbb{C}^2, A_\lambda) = (1 + \lambda) \cdot 2 \cdot 2 = 1 + \lambda$. This problem can be solved by considering larger matrices. Note that corollary C is not true for finite fields: over $\mathbb{F}_p$, the tracelike transformations $\Theta^p$ and $\Theta^1 = \text{Tr}$ agree.
5.5 The action of $\mathbb{N}$ and the proof of corollary [D]

We would like to give a purely categorical characterisation of the trace among the other tracelike transformations that does not rely on a pre-defined notion of trace. To define the notion of generating tracelike transformation we first need to specify the $\mathbb{N}$-action outlined in the introduction.

Definition 5.15. The multiplicative monoid $(\mathbb{N}, \cdot)$ acts on the sets $\mathcal{F}$ and $\mathcal{F}^r$ by $(P^nT)(e) := T(e^n)$.

It is clear that $P^n \circ P^m = P^{nm}$, but we need to briefly check that $P^nT$ is indeed still conjugation invariant:

$$(P^nT)(g^{-1} \circ f \circ g) = T((g^{-1} \circ f \circ g)^n) = T(g^{-1} \circ f^n \circ g) = T(f^n) = (P^nT)(f).$$

We will now lift the maps $P^n$ to maps $\mathcal{P}^n : \mathcal{F}^r_\infty \rightarrow \mathcal{F}^r_\infty$ of the moduli space of $\infty$-categorical tracelike transformations. Once could assemble these maps into a coherent action of $(\mathbb{N}, \cdot)$, but that is not necessary for our purposes. Instead we will just check that the $\mathcal{P}^n$ induce $P^n$ on $\pi_0 \mathcal{F}^r_\infty \cong \mathcal{F}^r$.

Construction 5.16. Fix $n \in \mathbb{N}$ and some complete Segal space $X_\bullet$. There is a diagonal map

$$\Delta^n_{X_1} : X_0 \times_{(X_0)^2} X_1 \rightarrow X_0 \times_{(X_0)^2} (X_1 \times_{X_0} \cdots \times_{X_0} X_1),$$

where as usual all pullbacks are computed in the infinity category of spaces. We can pick a homotopy inverse to the Segal map $X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$ and construct the composite

$$p^n_X : \mathcal{E}(X) = X_0 \times_{(X_0)^2} X_1 \xrightarrow{\Delta^n_{X_1}} X_0 \times_{(X_0)^2} (X_1 \times_{X_0} \cdots \times_{X_0} X_1) \xleftarrow{\triangleleft} X_0 \times_{(X_0)^2} X_n \xrightarrow{d} X_0 \times_{(X_0)^2} X_1.$$

Here $d : X_n \rightarrow X_1$ is the face operator coming from the long edge $\{0, n\} \subset [n]$. This defines a natural transformation $p^n : \mathcal{E} \Rightarrow \mathcal{E}$ that commutes with the projection $\mathcal{E}(X) \rightarrow X_0$. Given a symmetric monoidal structure on $X$ the transformation $p^n$ therefore preserves the subspace $\mathcal{E}^{d}(X) \subset \mathcal{E}(X)$. We can hence define a map

$$\mathcal{P}^n : \mathcal{F}_\infty \rightarrow \mathcal{F}_\infty, \quad (T : \mathcal{E} \Rightarrow \mathcal{E}) \mapsto (T \circ p^n : \mathcal{E}^{d} \Rightarrow \mathcal{E}^{d} \Rightarrow \mathcal{E}).$$

Lemma 5.17. For $T \in \mathcal{F}_\infty$ and $n \in \mathbb{N}$ the tracelike transformation $\mathcal{P}^nT$ acts as $(\mathcal{P}^nT)(e) = T(e^n)$ on the homotopy category.

Proof. The inverse of the Segal map composed with the face map:

$$X_1 \times_{X_0} \cdots \times_{X_0} X_1 \xleftarrow{\triangleleft} X_n \xrightarrow{d} X_1,$$

is given by $(f_1, \ldots, f_n) \mapsto f_n \circ \cdots \circ f_1$ on the homotopy category. Therefore $p^n_X : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$ yields

$$E(h_1X) \rightarrow E(h_1X), \quad (x, e) \mapsto (x, e^n).$$

Precomposing with this induces $P^n : \mathcal{F} \rightarrow \mathcal{F}$ as described in definition [5.15].

We can now make the definition from the introduction precise:

Definition 5.18. A tracelike transformation $T \in \mathcal{F}_\infty$ is generating if the monoid $\pi_0 \mathcal{F}_\infty$ is generated by the set $\{[\mathcal{P}^n(T)] \mid n \in \mathbb{N}\}$. Equivalently, $T$ is generating if for every other $S \in \mathcal{F}_\infty$ there are non-negative integers $k_1, \ldots, k_n$ such that $S$ is equivalent to $\mathcal{P}^{k_1}(T) \cdots \mathcal{P}^{k_n}(T)$.

Corollary 5.19 (Corollary [D]). The space of generating tracelike transformations is contractible and its image in $\mathcal{F}$ is the classical trace $\text{Tr}$.  

25
Proof. By theorem \[5.11\] we know that \(\pi_0\mathcal{T}_\infty \cong \mathcal{T}\), and by corollary \[5.12\] the fibre of \(\text{Tr}\) under the projection \(\mathcal{T}_\infty \to \mathcal{T}\) is contractible. It will therefore suffice to show that that \(\text{Tr} = \Theta^1\) is the unique generating \(1\)-categorical tracelike transformation in \(\mathcal{T}\).

The action of \(P^n\) on \(\Theta_{k_1,\ldots,k_n}\) is given by
\[
P^n(\Theta_{k_1,\ldots,k_n})(e) = \Theta_{k_1,\ldots,k_n}(e^m) = \text{Tr}((e^m)_{k_1}) \circ \cdots \circ \text{Tr}((e^m)_{k_n}) = \Theta_{mk_1,\ldots,mk_n}(e).
\]
Therefore, under the isomorphism \(\mathcal{T} \cong \mathbb{N}[x]\) of proposition \[4.6\] \(P^n\) acts on \(\mathbb{N}[x]\) as \(x^m \mapsto x^{nm}\). It is clear that \(x\) is the only element \(y \in \mathbb{N}[x]\) such that \(\{P^n(y) \mid n \in \mathbb{N}\}\) generates the monoid \(\mathbb{N}[x]\), and hence the same holds for \(\text{Tr} \in \mathcal{T}\). \(\square\)

6 Application: the cyclic group action on \(\text{Tr}(f^k)\) and \(\text{THH}\)

In this section we study the other connected components of the moduli space \(\mathcal{T}_\infty\) of tracelike transformations, which we computed in Theorem \[A\]. We will see that while \((e \mapsto \text{Tr}(e))\) uniquely lifts to a \(\infty\)-categorical tracelike transformation, \((e \mapsto \text{Tr}(e^k))\) has ‘\(B(\mathbb{Z}/k\mathbb{Z})\)-many lifts’. This induces a coherent \(\mathbb{Z}/k\mathbb{Z}\)-action on \(\text{Tr}(e^k)\). As an example we will look at the derived Morita category where the trace is given by topological Hochschild homology (\(\text{THH}\)).

Construction 6.1. For every symmetric monoidal \(\infty\)-category \(\mathcal{C}\) there is a canonical map \(\mathcal{T}_\infty \to \text{Map}(\mathcal{E}^{fd}(\mathcal{C}), \text{Sc}(\mathcal{C}))\) and by adjunction a map \(\mathcal{E}^{fd}(\mathcal{C}) \to \text{Map}(\mathcal{T}_\infty, \text{Sc}(\mathcal{C}))\). Given a dualisable object \(x \in \mathcal{C}\) we can moreover compose this with the canonical map \(\text{End}_{\mathcal{C}}(x) \to \mathcal{E}^{fd}(\mathcal{C})\) to obtain:
\[
\theta : \text{End}_{\mathcal{C}}(x) \to \text{Sc}(\mathcal{C})_{\mathcal{T}_\infty}.
\]
Under the equivalence of theorem \[A\] we in particular have
\[
\theta_0 : \text{End}_{\mathcal{C}}(x) \to \text{Sc}(\mathcal{C})^{BT} \quad \text{and} \quad \theta_k : \text{End}_{\mathcal{C}}(x) \to \text{Sc}(\mathcal{C})^{\mathbb{Z}/k\mathbb{Z}} \quad \text{for} \quad k > 0.
\]

6.2. As part of theorem \[A\] we know that the effect of \(\theta_k\) on the homotopy category is:
\[
\theta_k(x,e) = \text{Tr}(e^k).
\]
Since giving a map \(\mathbb{Z}/k\mathbb{Z} \to \mathcal{E}\) into an \(\infty\)-groupoid \(\mathcal{E}\) is equivalent to giving an object of \(\mathcal{E}\) with a coherent \(\mathbb{Z}/k\mathbb{Z}\)-action, the above construction shows that \(\text{Tr}(e^k)\) comes with a natural \(\mathbb{Z}/k\mathbb{Z}\)-action when thought of as an object in the \(\infty\)-groupoid \(\text{Sc}(\mathcal{C})\). For \(k = 0\) we have that \(\text{Tr}(e^0) = \text{Tr}(\text{id}_x)\) and the corresponding connected component of \(\mathcal{T}_\infty\) is \(\mathcal{B}T\), so \(\text{Tr}(\text{id}_x)\) has an action of the circle group \(T\).

Note that all of these actions are trivial when \(\mathcal{C}\) is a 1-category, since then \(\text{Sc}(\mathcal{C})\) is just a set and hence discrete as an \(\infty\)-groupoid. We can therefore not observe the group actions in the category of vector spaces. Instead we consider the following example:

Example 6.3. Consider the stable Morita category \textbf{Morita} as defined in \[4.8.4.9\] where objects are \(E_1\)-ring spectra \(A, B, C, \ldots\), morphisms \(A \to B\) are \((A,B)\)-bimodule spectra and composition of morphisms \(M : A \to B\) and \(N : B \to C\) is given by the relative tensor product \((M \otimes_B N) : A \to C\). This category is symmetric monoidal with respect to the smash product \(\otimes = \wedge\) on spectra.

In this category all objects \(A\) are dualisable with dual \(A^{op}\). Evaluation and coevaluation are given by \(A\) thought of as a \((A^{op} \otimes A, S)\) and \((S, A \otimes A^{op})\)-bimodule, respectively. The trace of an \((A,A)\)-bimodule \(M : A \to A\) is therefore given by the relative tensor product:
\[
\text{Tr}(M) \simeq A \otimes_{A \otimes A^{op}} M \otimes_{A \otimes A^{op}} A \simeq A \otimes_{A \otimes A^{op}} M.
\]
This, by definition, is the \textit{topological Hochschild homology} \(\text{THH}(A;M)\) of \(A\) with coefficients in \(M\).

The scalars of \textbf{Morita} form the \(\infty\)-groupoid of \((S,S)\)-bimodules, i.e. the maximal subgroupoid of the category of spectra: \(\text{Sc}(\text{Morita}) \simeq \text{Sp}^\infty\). Our construction of \(\theta_k\) induces a natural \(\mathbb{Z}/k\mathbb{Z}\)-action on \(\text{THH}(A;M \otimes_A \cdots \otimes_A M)\) and a natural \(T\)-action on \(\text{THH}(A) = \text{THH}(A;A)\). 26
**Remark 6.4.** There is a well-known $T$-action on $\text{THH}(A)$, which we expect to be equivalent to the above one. Let us briefly sketch how one could go about proving this:

Given a ring spectrum $A$, Scheimbauer in her thesis [Sch14] uses factorization homology to construct a symmetric monoidal functor $Z : \text{Bord}^\text{or}_1 \to \text{Morita}$ that sends the positive point to $A$. We can compose this with the functor that forgets markings to obtain $Z' : \text{Bord}^\text{or,m}_1 \to \text{Bord}^\text{or}_1 \to \text{Morita}$. By naturality, the $T$-action on $\theta_0(A)$ then has to be given by $Z' : BT \subset \text{Sc}(\text{Bord}^\text{or,m}_1) \to \text{Sc}(\text{Morita})$. From Scheimbauer’s construction we see that this is the usual action of $T$ on the factorization homology $\int_{S^1} A$ via $T \mapsto S^1$. The claim therefore follows from the folklore theorem that there is a $T$-equivariant equivalence $\int_{S^1} A \simeq \text{THH}(A)$, see for instance [AMR17].

**Remark 6.5.** Lindenstrauss and McCarthy construct a $\mathbb{Z}/n\mathbb{Z}$-equivariant spectrum $U^n(F,P)$ for every ring spectrum $F$ and bimodule $P$, see [LM12, Definition 4.1]. By [LM12, Lemma 2.6] this spectrum is non-equivariantly equivalent to $\text{THH}(F; P \otimes F \ldots \otimes F P)$. We expect that this induces the same $\mathbb{Z}/n$-action as one obtains form the trace-like transformation $e \mapsto \text{Tr}(e^n)$ as described above. One could also attempt to prove this in a similar way as indicated for $\theta_0$. However, one would now need a version of stratified factorization homology to construct the relevant functor $\text{Bord}^\text{or,m}_1 \to \text{Morita}$. 

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