Duality of the alternating integral for quasi-linear differential games

Ikrom Iskanadjiev
Tashkent Technical University
University str. 2, Tashkent, Uzbekistan
iskan1960@mail.ru

Received: 25 April 2011 / Revised: 26 February 2012 / Published online: 21 May 2012

Abstract. The concept of alternated integrals proved to be useful in differential games and in control theory under the conditions of uncertainty. In this article the connection between upper and lower alternating integrals for quasi-linear differential games is established and its applications to the problem of pursuit is studied.

Keywords: differential games, alternating integral, multi-valued mapping, pursuer, evader, strategy.

1 Introduction

To solve the problem of pursuit in linear differential games, L.S. Pontryagin suggested two direct methods [1, 2]. Pontryagin’s second direct method, based on concept of the alternating integral, which has no analogs in integration of real function. In definition of alternating integral participate operations of integration of set-valued mappings and geometric difference (Minkovski difference) of sets. These operations make difficulties for computation of alternating integral. A simplified schemes for constructing of alternating integral were proposed in [3, 4]. An evaluation error of the numerical constructing of the alternating integral has been described in [5]. This elegant construction admits a generalization, which makes possible to define a differential and an integral of a set-valued mapping in such way that these two operations become mutually reverse. Such a generalization based on quasi-affine mapping has been proposed in [6, 7]. Numerical algorithms for evaluation of a generalized alternating integral were proposed in [8, 9]. An important facts has been established in [10, 11]: A generalized alternating integral describes the epigraph of a function, which is the viscosity solution to a Hamilton–Jacobi equation.

Another set-valued integration procedure unifying Riemann-type integral, Auman’s integral and Pontryagin’s alternating integral were proposed in [12,13]. In these works the following important observation has been made: an alternating integral may serve cross-
cuts of “bridge of Krasovski” and a level set of value functions for related Hamilton–Jacobi–Bellman–Isaacs equations. Some aspects of computation alternating integral basic on ellipsoidal calculation is studied. This procedure allowed to apply an alternating integral to the control synthesis problem.

Pontryagin’s second direct method has played the great role in development of the differential games and control theory. Therefore, many works are devoted to investigation of this method (see [14–22]). In particular, a lower analogue of the Pontryagin alternating integral for linear differential games was introduced in [14] (to complete the symmetry the alternating integral defined in [2] was called the upper Pontryagin alternating integral). The lower alternating integral has been found useful in solving problems of pursuit under certain information discrimination of the pursuer compared to the evader. In [14], a connection was also established between these concepts, and it was applied to the problem of informality [4]. In this article a connection between the upper and lower alternating integrals for quasi-linear differential games and its applications to the problem of pursuit are established.

We shall use the following notations: \( I = [0, \tau] \) is the fixed segment of time; \( \Delta \) is a subsegment of \( I \); \( |\Delta| \) is the length of \( \Delta \); \( \text{cl}(\mathbb{R}^d) \) (\( \text{Ccl}(\mathbb{R}^d) \), respectively) is the collection of all nonempty closed (convex closed, respectively) subsets of \( \mathbb{R}^d \); \( \text{cm}(\mathbb{R}^d) \) (\( \text{Ccm}(\mathbb{R}^d) \), respectively) is the collection of all nonempty compact (convex compact, respectively) subsets of \( \mathbb{R}^d \); \( H = \{ z \in \mathbb{R}^d \mid |z| \leq 1 \} \) is the unit closed ball in \( \mathbb{R}^d \); \( h(A, B) = \min\{r \geq 0 \mid A \subset B + rH, B \subset A + rH\} \) – Hausdorff metric; \( \omega_n = \{0, \delta, 2\delta, \ldots, n\delta = \tau\} \) is a uniform partition of the segment \([0, \tau]\); \( \int_{\cdot}^{\tau} \) is an integral by the segment \([i-1]\delta, i\delta\], and \( X(i) \) is the collection of all measurable functions \( x(\cdot) : [(i-1)\delta, i\delta] \to X, X \subset \mathbb{R}^d \).

A differential game described by the equation

\[
\dot{z} = -f(t, u, v)
\]  

is considered, where \( z \in \mathbb{R}^d \), \( t \in I \), \( u \) and \( v \) are control parameters, \( u \) is the pursuer parameter, \( v \) is the evader parameter, \( u \in P \in \text{Ccm}(\mathbb{R}^d) \), \( v \in Q \in \text{Ccm}(\mathbb{R}^d) \), and \( f : I \times P \times Q \to \mathbb{R}^d \) is a continuous function. The terminal set is \( M, M \subset \mathbb{R}^d \). An arbitrary measurable function \( u(\cdot) \in P(I) (v(\cdot) \in Q(I)) \) is called admissible control parameter of the pursuer (evader).

A given initial point \( z_0 \) and a pair of controlling parameters \( u(\cdot) \in P(I), v(\cdot) \in Q(I) \) give rise to a unique trajectory \( z(t) = z(t, z_0, u(\cdot), v(\cdot)), t \geq 0 \), of the system (1) (precisely definition of a trajectory is given in Section 4). Pursuit starts from a point \( z_0 \in \mathbb{R}^d \setminus M \) and it is considered to be ended, when the phase point hits the set \( M \). In other words, the pursuer aims to realize the inclusion \( z(\tau) \in M \). Then, we say that pursuit from a point \( z_0 \) is completed at the time \( \tau \) in the game (1).

Naturally, there is a question: From which initial points \( z_0 \) pursuit can be completed at the time \( \tau \) in the game (1)?

To solve this problem L.S. Pontryagin has introduced the second method of pursuit in a linear differential game. The second method of pursuit is formulated in terms of an alternating integral.
To each partition \( \omega_n \) of the interval \( I \) we assign upper and lower alternating sums
\( S^+(M, \omega_n) \) and \( S^-(M, \omega_n) \) as described below:

\[
S^+ = \bigcup_{\nu(i) \in Q(i)} \bigcap_{\omega(i) \in P(i)} \left[ S^i + \int f(t, u(t), v(t)) \, dt \right], \quad S^-(M, \omega_n) = S^0,
\]

\[
S^- = \bigcup_{\nu(i) \in Q(i)} \bigcap_{\omega(i) \in P(i)} \left[ S^i + \int f(t, u(t), v(t)) \, dt \right], \quad S^-(M, \omega_n) = S^0.
\]

The set \( W^+(M) = \bigcap_{\omega_n} S^+(M, \omega_n) \) is said to be the upper Pontryagin alternating integral [1, 2, 16]. The set \( W^-(M) = \bigcup_{\omega_n} S^-(M, \omega_n) \) is called the lower Pontryagin alternating integral [14, 15].

Further, if it will be necessary, we shall indicate in notations, the dependence of sums and integrals not only of \( \omega \) or \( \tau \), but also of other initial data.

A concepts of the upper and lower alternating integrals have the following role in a quasi-linear differential games: For points \( z_0 \) with \( z_0 \in W^+(M) \) \( (z_0 \in W^-(M)) \) the pursuit can be completed at the time \( \tau \) with (without) discriminating against the evader controls [1, 2, 14–16].

Now we formulate the basic results. The following theorems establish connection between the upper and lower alternating integrals.

**Theorem 1.** Let \( M \in \text{Ccl}(\mathbb{R}^d) \) and the set \( f(t, P, v) \) is convex for any \( t \in I \) and \( v \in Q \). Then the equality

\[
W^+(M) = \bigcap_{\varepsilon > 0} W_\tau(M + \varepsilon H)
\]

holds.

**Theorem 2.** Let the set \( M \subset \mathbb{R}^d \) has the convex closed complement, and \( f(t, u, Q) \) is convex for any \( t \in I \) and \( u \in P \). Then the equality

\[
W_\tau(M) = \bigcup_{\varepsilon > 0} W^+(M \preceq \varepsilon H)
\]

holds. (Here \( A \preceq B = \{ z \in \mathbb{R}^d \mid z + B \subset A \} \).)

**Remark 1.** Note that if the set \( M \) is open (closed), then the sets \( S_n, W_\tau \) \( (S^0, W^+) \) are open (closed) [14, 19].

**Remark 2.** It should be noted (see proof of the Theorem 1 and Theorem 2), for any set \( M \subset \mathbb{R}^d \), the following relations are valid:

\[
W^+(M) \subset \bigcap_{\varepsilon > 0} W_\tau(M + \varepsilon H) \subset W^+(\text{cl}M),
\]

\[
W_\tau((\text{cl}(\mathbb{R}^d \setminus M))^c) \subset \bigcap_{\varepsilon > 0} W_\tau(M \preceq \varepsilon H) \subset \bigcap_{\varepsilon > 0} W_\tau(M),
\]

where \( A^c \) is a complement of a set \( A \).
2 Preliminary lemmas

In this section we give some preliminary facts which are necessary for proving basic theorems.

**Lemma 1.** (See [14].) Let a sequence \( X_k \in \text{cl}(\mathbb{R}^d) \) decreases monotonically by inclusion and \( Y \in \text{cm}(\mathbb{R}^d) \). Then the equality
\[
\left( \bigcap_{k=1}^{\infty} X_k \right) + Y = \bigcap_{k=1}^{\infty} (X_k + Y)
\] (4)
is valid.

It should be noted, for any family \( X_\alpha \) and a set \( Y \subset \mathbb{R}^d \), the following relations
\[
\bigcap \alpha X_\alpha + Y \subset \bigcap \alpha (X_\alpha + Y), \quad \bigcup \alpha (X_\alpha + Y) = \left( \bigcup \alpha X_\alpha \right) + Y
\] (5)
hold.

**Lemma 2.** (See [14].) If \( M \in \text{Ccl}(\mathbb{R}^d) \), then \( W_\tau(M) \subset W_\tau(M) \).

Further, we assume that the set \( f(t,P,v) \) is convex for any \( t \in I \) and \( v \in Q \).

One can easily verify the validity of the following

**Lemma 3.** If \( f : I \times P \times Q \to \mathbb{R}^d \) is continuous, then the multi-valued mapping \( F(t,v) = f(t,P,v) \) is also continuous on \( I \times Q \) in metric Hausdorff.

Let \( \Gamma(\delta) = \max \{ h(F(t_1,v_1), F(t_2,v_2)), |t_1 - t_2| < \delta, |v_1 - v_2| < \delta, t_1, t_2 \in I, v_1, v_2 \in Q \} \) is the module of continuity for mappings \( F(t,v) = f(t,P,v) \).

If \( \xi \in \Delta \subset I, v \in Q \), then
\[
\int_\Delta f(t,P,v) dt \subset \delta f(\xi,P,v) + \delta \Gamma(\delta), \quad \delta f(\xi,P,v) \subset \int_\Delta f(t,P,v) dt + \delta \Gamma(\delta). \] (6)

**Lemma 4.** Let \( L \) is arbitrary subset of \( \mathbb{R}^d \), and \( \Delta \subset I \). Then
\[
\bigcap v(\cdot) \in Q(\Delta) \bigcup u(\cdot) \in P(\Delta) \left[ L + \int_\Delta f(t,u(t),v(t)) dt \right] \subset \bigcap \xi \in \Delta \bigcup v(\cdot) \in Q \bigcup u(\cdot) \in P \left[ L + \Gamma(\delta)H + \delta f(\xi,u,v) \right]. \] (7)

**Proof.** It is obvious,
\[
\bigcap v(\cdot) \in Q(\Delta) \bigcup u(\cdot) \in P(\Delta) \left[ L + \int_\Delta f(t,u(t),v(t)) dt \right] \subset \bigcap v(\cdot) \in Q \left[ L + \int_\Delta f(t,P,v) dt \right]. \] (8)
Applying the inequality (6) to the right side of (8), we obtain
\[ \bigcap_{v \in Q} \left[ L + \int_{\Delta} f(t, P, v) \, dt \right] \subset \bigcap_{v \in Q} \left[ L + \Gamma(\delta) \delta H + \delta f(\xi, P, v) \right]. \]

By virtue of (8), we get from here (7).

Let \( \omega_n \in \Omega, \xi_i \in \Delta_n, \) and
\[ B^n = M, \quad B^{i-1} = \bigcap_{v \in Q} \bigcup_{u \in P} \left[ B^i + \delta f(\xi_i, u, v) \right], \quad B^\tau(M, \omega_n) = B^0. \]

**Lemma 5.** For any \( \varepsilon > 0, \) there exists a positive integer \( N \) such that for all \( n \geq N, \) the inclusion
\[ S^\tau(M, \omega_n) \subset B^\tau(M + \frac{\varepsilon}{3} H, \omega_n) \]
takes place.

**Proof.** Let the partition \( \omega_n \) satisfies condition \( \Gamma(\delta) < \varepsilon/(3\tau). \) By definition,
\[ S^{n-1} = \bigcap_{v \in Q(n)} \bigcup_{u \in P(n)} \left[ S^n + \int_{n} f(t, u(t), v(t)) \, dt \right]. \]
Applying Lemma 4 to the right side of the last equality, we obtain
\[ S^{n-1} \subset \bigcap_{v \in Q} \bigcup_{u \in P} \left[ S^n + \int_{n} f(t, u(t), v(t)) \, dt \right], \]
where \( \xi_n \) is arbitrary point from the segment \( \Delta_n. \)

Suppose
\[ S^{n-k}(M) \subset B^{n-k}(M + k\delta \Gamma(\delta) H). \]
We shall show that
\[ S^{n-(k+1)}(M) \subset B^{n-(k+1)}(M + (k + 1)\delta \Gamma(\delta) H). \]

By definition,
\[ S^{k-1} = \bigcap_{v(\cdot) \in Q(k-1)} \bigcup_{u(\cdot) \in P(k-1)} \left[ S^k + \int_{n-k} f(t, u(t), v(t)) \, dt \right]. \]
Using Lemma 4, we have
\[ S^{n-(k+1)} \subset \bigcap_{v \in Q} \bigcup_{u \in P} \left[ S^{n-k} + \delta \Gamma(\delta) H + \delta f(\xi_{n-k}, u, v) \right], \]
where \( \xi_{n-k} \) is arbitrary point from the segment \( \Delta_{n-k}. \)
By virtue of assumption, 
\[ S^{n-(k+1)} \subset \bigcap_{v \in Q} \bigcup_{u \in P} \left[ B^{n-k} \left( M + k\delta \Gamma(\delta)H \right) + \delta \Gamma(\delta)H + \delta f(\xi_{n-k}, u, v) \right]. \]

Now, using (5), we obtain
\[ S^{n-(k+1)} \subset \bigcap_{v \in Q} \bigcup_{u \in P} \left[ B^{n-k} \left( M + (k+1)\delta \Gamma(\delta)H \right) + \delta f(\xi_{n-k}, u, v) \right] \]
\[ = B^{n-(k+1)} \left( M + (k+1)\delta \Gamma(\delta)H \right). \]

This implies
\[ S^0(M) \subset B^0 \left( M + \tau \Gamma(\delta)H \right). \]

Since \( \Gamma(\delta) < \varepsilon/(3\tau) \) by condition, we have
\[ S^\tau(M, \omega_n) \subset B^\tau \left( M + \frac{\varepsilon}{3}H, \omega_n \right). \]

Let now
\[ B_n = M, \quad B_{i-1} = \bigcup_{u \in P} \bigcap_{v \in Q} \left[ B^i + \delta f(\xi_i, u, v) \right], \quad B_\tau(M, \omega_n) = B_0. \]

**Lemma 6.** For any \( \varepsilon > 0 \), there exists a positive integer \( N \) such that for all \( n \geq N \) the relation
\[ B^\tau(M, \omega_n) \subset B^\tau \left( M + \frac{\varepsilon}{3}H, \omega_n \right) \]
holds. Here \( \gamma(\cdot) \) is the continuity module of a function \( f(t, u, v) \).

**Proof.** Let the partition \( \omega_n \) satisfies the following conditions:
1. \( |\delta f(\xi, u, v)| < \varepsilon/9 \) for any \( \xi \in I, u \in P, v \in Q \); 
2. \( \tau \gamma(2\delta) < \varepsilon/9 \).

Then by virtue of (5) we have
\[ B^\tau(M, \omega_n) \]
\[ = \bigcap_{v \in Q} \bigcup_{u \in P} \cdots \bigcap_{v \in Q} \bigcup_{u \in P} \left[ \bigcap_{v \in Q} \bigcup_{u \in P} \left( M + \delta f(\xi_n, u, v) \right) + \delta f(\xi_{n-1}, u, v) \right] \]
\[ + \cdots + \delta f(\xi_2, u, v) \right] + \delta f(\xi_1, u, v) \right] \]
\[ \subset \bigcup_{u \in P} \bigcap_{v \in Q} \cdots \bigcup_{u \in P} \bigcap_{v \in Q} \left[ \bigcup_{u \in P} \bigcap_{v \in Q} \left( M + \delta f(\xi_n, u, v) \right) \right] \]
\[ + \delta f(\xi_{n-1}, u, v) \right] + \delta f(\xi_{n-3}, u, v) \right] + \cdots + \delta f(\xi_1, u, v) \right] + \delta f(\xi_1, u, v) \right] + \frac{\varepsilon}{9}H \right]. \]
Lemma 7. If

By given \( \varepsilon > 0 \), Obviously, the left side of (9) is contained in its right side. Prove the opposite.

Proof. By choose of partitions we have \( \tau \gamma (2 \delta) < \varepsilon / 9 \). Therefore,

Further, we denote the collection of piecewise constant functions \( \hat{v}(\cdot) : \Delta \rightarrow Q \) as \( \hat{Q}(\Delta) \).

Let \( v(\cdot) \) be arbitrary function from \( Q(\Delta) \). Then there exists a sequence of piecewise constant functions \( v_k(\cdot) \in \hat{Q}(\Delta) \) convergent almost everywhere to \( v(\cdot) \) on \( \Delta \). Then

By given \( \varepsilon > 0 \), we find \( \eta > 0 \) such that

\( t \in \Delta, u \in P, v_1, v_2 \in Q, |v_1 - v_2| < \eta \). By the Egorov theorem, there exists an open subset \( \Delta_1 \subset \Delta \) with the measure \( \mu(\Delta_1) < \varepsilon \) such that \( v_k(\cdot) \rightarrow v(\cdot) \) uniformly on the closed set \( \Delta \setminus \Delta_1 \). Then \( |v_k(t) - v(t)| < \eta \) for all \( t \in \Delta \setminus \Delta_1 \) and \( k > N \) if \( N \) is sufficiently large. Hence, by virtue of (11) we have

for all \( t \in \Delta \setminus \Delta_1 \) and \( k > N \).

Let \( \lambda = \max \{|f(t, u, v)|, t \in \Delta, u \in P, v \in Q\} \), and |\( \Delta \)| is the length of the segment \( \Delta \). Then by the inclusion (12) we have

\[
\int_{\Delta} f(t, u, \hat{v}_k(t)) \, dt \subset \int_{\Delta \setminus \Delta_1} f(t, u, v(t)) \, dt + \int_{\Delta_1} f(t, u, \hat{v}_k(t)) \, dt + \varepsilon |\Delta| H.
\]
Moreover,
\[
\int_{\Delta} f(t, u, \hat{v}_k(t)) \, dt \subset \int_{\Delta \setminus \Delta_1} f(t, u, v(t)) \, dt + (\lambda + |\Delta|) \varepsilon H. \tag{13}
\]

Note, \(\int_{\Delta_1} f(t, u, v(t)) \, dt \subset \lambda \varepsilon H\). Therefore
\[
0 \in \int_{\Delta_1} f(t, u, v(t)) \, dt + \lambda \varepsilon H. \tag{14}
\]

Adding (13) and (14), we obtain
\[
\int_{\Delta} f(t, u, \hat{v}_k(t)) \, dt \subset \int_{\Delta} f(t, u, v(t)) \, dt + (2\lambda + |\Delta|) \varepsilon H.
\]

Thus, by virtue of (10), we have
\[
x \in L + \int_{\Delta} f(t, u, v(t)) \, dt.
\]

Lemma 8. If \(L \in \text{Ccl}(\mathbb{R}^d)\), then
\[
\bigcup_{u \in P} \bigcap_{v \in Q} \left[ L + \int_{\Delta} f(t, u, v) \, dt \right] \subset \bigcup_{u \in P} \bigcap_{v \in Q(\Delta)} \left[ L + \int_{\Delta} f(t, u, v(t)) \, dt + 2\delta \gamma(\delta) H \right].
\]

Proof. By Lemma 7, it is sufficient to prove the following inclusion
\[
\bigcup_{u \in P} \bigcap_{v \in Q} \left[ L + \int_{\Delta} f(t, u, v) \, dt \right] \subset \bigcup_{u \in P} \bigcap_{v \in Q(\Delta)} \left[ L + \int_{\Delta} f(t, u, \hat{v}(t)) \, dt + 2\delta \gamma(\delta) \right]. \tag{16}
\]

Let \(x\) be arbitrary element from the left side of (16). Then
\[
x \in \bigcap_{v \in Q} \left[ L + \int_{\Delta} f(t, u, v) \, dt \right] \tag{17}
\]
for some \(u \in P\). Choose arbitrary piecewise constant function \(\hat{v}(t)\) from the collection \(\hat{Q}(\Delta)\). We shall consider \(\hat{v}(t) = v_j\) at \([t_{j-1}, t_j]\), where \(t_0 < t_1 < \cdots < t_m\), \(\Delta = [t_0, t_m]\).
Set \( \delta = |\Delta|, \delta_j = |t_j - t_{j-1}| \). Obviously, \( \sum_{j=1}^{n} \delta_j = \delta \). From each \([t_{j-1}, t_j]\) we choose point \( \xi_j \) and apply twice the following inequalities

\[
\int_{\Delta} f(t, u, v) \, dt \subset \delta f(\xi, u, v) + \delta \gamma(\delta) H,
\]

at first to the segment \( \Delta \), then to the segment \([t_{j-1}, t_j]\). Then we get

\[
\int_{\Delta} f(t, u, v_j) \, dt \subset \delta f(\xi_j, u, v_j) + \delta \gamma(\delta) H \subset \frac{\delta}{\delta_j} \int_{\Delta_j} f(t, u, v_j) \, dt + 2 \delta \gamma(\delta) H,
\]

from which

\[
x \in L + \frac{\delta}{\delta_j} \int_{\Delta_j} f(t, u, v_j) \, dt + 2 \delta \gamma(\delta) H
\]

follows. Now we multiply these inclusions by \( \delta_j/\delta \) and add term by term. Since, the set \( L \) is convex and by virtue of (17) we have

\[
x \in L + \int_{\Delta} f(t, u, \bar{v}(t)) \, dt + 2 \delta \gamma(\delta) H.
\]

Let \( \omega_n \in \Omega \) and

\[
C_n = M, \quad C_{i-1} = co \bigcup_{u \in P} \bigcap_{v \in Q(\Delta)} \left[ \text{cl} C_i + \int_i f(t, u, v) \, dt \right], \quad C_T(M, \omega_n) = C_0.
\]

Applying Lemma 8 sequentially to the partial sums \( B^i, i = n, n - 1, \ldots, 1 \), we get the following

**Lemma 9.** If \( M \in C_c(\mathbb{R}^d) \), then for any \( \varepsilon > 0 \), there exists a positive integer \( N \) such that the inclusion

\[
B_T(M, \omega_n) \subset C_T \left( M + \frac{\varepsilon}{6} H \right)
\]

is valid at all \( n \geq N \).

**Lemma 10.** If \( L \) is a convex subset of \( \mathbb{R}^d \), then the set

\[
\bigcup_{u(\cdot) \in P(\Delta)} \bigcap_{v(\cdot) \in Q(\Delta)} \left[ L + \int_{\Delta} f(t, u(t), v(t)) \, dt \right]
\]

is convex.
Proof of Lemma 10 is obvious.\hfill \square

By virtue of Lemma 10, Lemma 9 implies the following

**Lemma 11.** Let $M \in \text{Ccl}(\mathbb{R}^d)$. Then for any $\varepsilon > 0$, there exists a positive integer $N$ such that the relation

$$B_\varepsilon(M, \omega_n) \subset S_\varepsilon(M + \frac{\varepsilon}{3}H, \omega_n)$$

holds at all $n \geq N$.

Now Lemmas 5, 6, and 11 imply

**Lemma 12.** If $M \in \text{Ccl}(\mathbb{R}^d)$, then for any $\varepsilon > 0$, there exists a positive integer $N$ such that the inclusion

$$S_\varepsilon(M, \omega_n) \subset S_\varepsilon(M + \varepsilon H, \omega_n)$$

takes place for all $n \geq N$.

3 Proof of basic theorems

**Proof of Theorem 1.** It follows from Lemma 12,

$$W_\tau(M) \subset \bigcap_{\varepsilon > 0} W_\varepsilon(M + \varepsilon H).$$

On the other hand, we have on the base of Lemmas 2 and 1

$$\bigcap_{\varepsilon > 0} W_\varepsilon(M + \varepsilon H) \subset \bigcap_{\varepsilon > 0} W_\tau(M + \varepsilon H) = \bigcap_{\varepsilon > 0} \bigcap_{n} S_\varepsilon(M + \varepsilon H, \omega_n)$$

$$= \bigcap_{n} \bigcap_{\varepsilon > 0} S_\tau(M + \varepsilon H, \omega_n) = \bigcap_{n} S_\tau(M, \omega_n) = W_\tau(M).$$

Theorem 1 is proved.\hfill \square

Let $L^c$ be a notation of the complement for the set $L \subset \mathbb{R}^d$.

From the duality of operations of intersection and join it follows that

$$\bigcap_{v(\cdot) \in Q(\Delta)} \bigcup_{u(\cdot) \in P(\Delta)} \left[ L + \int_{\Delta} f(t, u(t), v(t)) \, dt \right]$$

$$= \bigcup_{v(\cdot) \in Q(\Delta)} \bigcap_{u(\cdot) \in P(\Delta)} \left[ L^c + \int_{\Delta} f(t, u(t), v(t)) \, dt \right]. \quad (18)$$

Applying formulas (18) to the upper alternating sum, we obtain

$$(S_\tau(M, P, Q, \omega_n))^c = S_\tau(M^c, Q, P, \omega_n). \quad (19)$$

Analogously on can verify

$$(S_\tau(M, P, Q, \omega_n))^c = S_\tau(M^c, Q, P, \omega_n). \quad (20)$$

Now, relations (19) and (20) imply
Lemma 13. The following equalities
\[ [W_\tau(M, P, Q)]^c = W_\tau(M^c, Q, P), \quad (21) \]
\[ [W_\tau(M, P, Q)]^c = W_\tau(M^c, Q, P), \quad (22) \]
take place.

Proof of Theorem 2. Theorem 1 implies
\[ W_\tau(M^c, Q, P) = \bigcap_{\varepsilon > 0} W_\tau(M^c + \varepsilon H, Q, P). \]

Applying the operation of complement to the both parts of this equality, using relations (21) and (22), we obtain
\[ (W_\tau(M^c, Q, P))^c = \left( \bigcap_{\varepsilon > 0} W_\tau(M^c + \varepsilon H, Q, P) \right)^c. \]

Hence,
\[ W_\tau(M) = \bigcup_{\varepsilon > 0} W_\tau(M + \varepsilon H). \]

Theorem 2 is proved.

4 Application of duality of the alternating integral to differential games of pursuit

Applications of the upper and lower alternating integral to quasi-linear differential games are similarly to the linear case [3]. Therefore, in this section we turn our attention to definitions of basic concepts and restrict ourself to state basic results in connection with the system (1).

Let \( \theta > 0 \), \( X \subset \mathbb{R}^d \). We denote by \( X(\theta) \) the family of all measurable functions \( x(\cdot) : [0, \theta] \to X \).

Definition 1. The mapping \( V^*_\delta : \mathbb{R}^d \to Q(\delta) \) is said to be \( \delta \)-strategy of the evader in the upper game. The mapping \( U^*_\delta : \mathbb{R}^d \times Q(\delta) \to P(\delta) \) is said to be \( \delta \)-strategy of the pursuer in the upper game.

Definition 2. The mapping \( U^\delta_\delta : \mathbb{R}^d \to P(\delta) \) is said to be \( \delta \)-strategy of the pursuer in the lower game. The mapping \( V^\delta_\delta : \mathbb{R}^d \times P(\delta) \to Q(\delta) \) is said to be \( \delta \)-strategy of the evader in the lower game.

A given starting point \( z_0 \) and a pair of strategies \( U^*_\delta, V^*_\delta \) correspond to the unique absolutely continuous trajectory \( z(t) = z(t, z_0, U^*_\delta, V^*_\delta), t \geq 0 \), defined in a following way.
The trajectory $z(t)$ on the segment $[0, \delta]$ is defined as the solution of the Cauchy problem
\[
\dot{z}(t) = f(t, u_0(t), v_0(t)), \quad z(0) = z_0,
\]
where $v_0(\cdot) = V^*_\delta(z_0)$ and $u_0(\cdot) = U^*_\delta(z_0, v_0(\cdot))$.

The trajectory $z(t)$ is continued from the segment $[0, k\delta]$ on the segment $[0, (k + 1)\varepsilon]$ as the solution of the following Cauchy problem
\[
\dot{z}(t) = f(t, u_k(t), v_k(t)), \quad z(k\delta) = z_k,
\]
where $v_k(t) = v(t - k\delta)$, $v(\cdot) = V^*_\delta(z(k\delta))$ and $u_k(t) = U^*(z(k\delta), v(\cdot))(t - k\delta)$, $t \in [k, (k + 1)\delta]$.

The trajectory $z(t) = z(t, z_0, U^*_\delta, V^*_\delta)$ is defined similarly. It corresponds to the given starting point $z_0$ and the pair of strategies $U^*_\delta, V^*_\delta$.

**Definition 3.** Pursuit from a point $z_0$ can be completed at the time $\tau$ in the upper game if for any $\delta = \tau/n$, there exists $\delta$-strategy of the pursuer $U^*_\delta$ such that $z(\tau, z_0, U^*_\delta, V^*_\delta) \in M$ for any $\delta$-strategy of the evader $V^*_\delta$.

Concept of possibility to complete pursuit at the time $\tau$ in the lower game can be introduced similarly.

On the base of these definitions, Theorem 1 can be interpreted as follows. If $M \in \text{Ccl}(\mathbb{R}^d)$, then pursuit from a point $z_0$ can be completed at the time $\tau$ in the upper game if and only if the pursuer in the lower game can transfer the phase point from the starting state $z_0$ into any neighborhood of the terminal set at the time $\tau$ (see [2, 14, 18, 23–25]).

Theorem 2 admits similarly interpretation.

**References**

1. L.S. Pontryagin, On linear differential games, *Dokl. Akad. Nauk SSSR*, 175(4), pp. 764–766, 1967 (in Russian).
2. L.S. Pontryagin, Linear differential games of pursuit, *Mat. Sb.*, 112(3), pp. 307–330, 1980 (in Russian).
3. M.S. Nikol’skii, On the Pontryagin alternated integral, *Mat. Sb.*, 116(1), pp. 136–144, 1981 (in Russian).
4. A. Azamov, Semistability and duality in the theory of the Pontryagin alternating integral, *Dokl. Akad. Nauk SSSR*, 299(2), pp. 265–268, 1988 (in Russian).
5. A.P. Ponomarev, N.Kh. Rozov, Stability and convergence of the Pontryagin alternating sums, *Vestn. Mosk. Univ., Ser. XV, 1*, pp. 82–90, 1978 (in Russian).
6. D.B. Silin, On set-valued differentiation and integration, *Set-Valued Anal.*, 5(2), pp. 107–146, 1997.
7. D.B. Silin, On set-valued differentiation and integration using quasi-affine mappings, *Dokl. Akad. Nauk, Ross. Akad. Nauk*, 340(2), pp. 164–166, 1995.
8. D.B. Silin, N.G. Trin’ko, A modification of Graham’s algorithm for the convexification of a positive-uniform function, *Comput. Math. Math. Phys.*, 34(4), pp. 545–548, 1994.

9. G.B. Orlova, D.B. Silin, Approximate calculation of the convex hull of a positively homogeneous function, *Mosc. Univ. Comput. Math. Cybern.*, 2, pp. 44–49, 1997.

10. D.B. Silin, Viscosity solutions via unbounded set-valued integration, *Nonlinear Anal., Theory Methods Appl.*, Ser. A, Theory Methods, 31(1–2), pp. 55–90, 1998.

11. D.B. Silin, Set-valued integration and viscosity solutions to the Hamilton–Jacobi equations, *Differ. Equ.*, 31(1), pp. 116–123, 1995.

12. A.B. Kurzhansky, N.B. Melnikov, On the problem of synthesis of controls: The Pontryagin alternating integral and the Hamilton–Jacobi equation, *Mat. Sb.* 191(6), pp. 69–100, 2000 (in Russian).

13. A.B. Kurzhansky, Pontryagin’s alternated integral in the theory of the control synthesis, *Tr. Mat. Inst. Steklova*, 224, pp. 234–248, 1999 (in Russian).

14. A. Azamov, On Pontryagin’s second method in linear differential games of pursuit, *Mat. Sb.*, 118(3), pp. 422–430, 1982 (in Russian).

15. M.C. Nikol’skii, On the lower Pontryagin alternating integral in linear differential games of pursuit, *Mat. Sb.*, 128(1), pp. 35–49, 1985 (in Russian).

16. E.F. Mishchenko, N. Satimov, An alternating integral in linear differential games with nonlinear controls, *Diff. Uravn.*, 10(12), pp. 2173–2178, 1974 (in Russian).

17. E.C. Polovinkin, Nonautonomous differential games, *Diff. Uravn.*, 15(6), pp. 1007–1017, 1979 (in Russian).

18. B.N. Pshenichny, M.I. Sagaydak, On differential games with fixed time, *Kibernetika*, 2, pp. 54–63, 1970 (in Russian).

19. P.B. Gusyatnikov, On the information available to players in a differential games, *Prikl. Mat. Mekh.*, 36(5), pp. 917–924, 1972 (in Russian).

20. B.N. Pshenichny, V.V. Ostapenko, *Differential Games*, Naukova Dumka, Kiev, 1992 (in Russian).

21. R.V. Konstantinov, Quasilinear pursuit differential game with a simple dynamics under phase constraints, *Mat. Zametki*, 69(4), pp. 581–590, 2001 (in Russian).

22. B.M. Mukanov, M.E. Mansurova, Pontryagin alternated integrals with mixed constraints, *Vychisl. Technol.*, 12(2), pp. 104–114, 2007 (in Russian).

23. A.I. Subbotin, A.G. Chentsov, *Optimization of a Guarantee in Control Problems*, Nauka, Moscow, 1981.

24. A.I. Subbotin, *Generalized Solutions of First-Order PDEs: The Dynamical Optimization Perspective*, Syst. Control Found. Appl., Birkhäuser, Boston, 1995.

25. N.N. Krasovskii, A.I. Subbotin, *Positional Differential Games*, Springer, 1988.