Explicit Bound for the Prime Ideal Theorem in Residue Classes

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Abstract. We give explicit numerical estimates for the generalized Chebyshev functions. Explicit results of this kind are useful for estimating of computational complexity of algorithms which generates special primes. Such primes are needed to construct an elliptic curve over prime field using complex multiplication method.

1 Introduction

Let $K$ denote any fixed totally imaginary field of the discriminant $\Delta = \Delta(K)$ and degree $[K : \mathbb{Q}] = 2r_2$, where $2r_2$ is the number of complex-conjugate fields of $K$. Denote by $f$ a given non-zero integral ideal of the ring of algebraic integers $\mathcal{O}_K$ and by $H \pmod{f}$ any ideal class mod $f$ in the "narrow" sense. Let $h_1^+(K)$ be the number of elements of $H$. Let $\chi(H)$ be a character of the abelian group of ideal classes $H \pmod{f}$, and let $\chi(a)$ be the usual extension of $\chi(H)$. Let $s = \sigma + it$. The Hecke–Landau zeta-functions associated to $\chi$, are defined by

$$\zeta(s, \chi) = \sum_{a \in \mathcal{O}_K} \frac{\chi(a)}{(Na)^s}, \quad \sigma > 1,$$

where $a$ runs through integral ideals and $Na$ is the norm of $a$. Throughout, $\chi_0$ denote the principal character modulo $f$. Let

$$E_0 = E_0(\chi) = \begin{cases} 1 & \text{for } \chi = \chi_0 \\ 0 & \text{for } \chi \neq \chi_0 \end{cases}$$

If $\chi$ is a primitive character, then $\zeta(s, \chi)$ satisfies the functional equation

$$\Phi(s, \chi) = W(\chi)\Phi(1 - s, \overline{\chi}), \quad |W(\chi)| = 1,$$

where

$$\Phi(s, \chi) = A(f)^s \Gamma(s)^2 \zeta(s, \chi)$$

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and
\[ A(f) = (2\pi)^{-r_2} \sqrt{|\Delta|Nf}. \]
Let \( \Lambda(a) \) be the generalized Mangoldt function. Fix \( X \mod f \in H \). We define,
\[ \Psi(x, X) = \sum_{x \leq N \leq 2x} \Lambda(a) = \sum_{x \leq Np^m \leq 2x} \log Np, \]
where \( p \) runs through prime ideals of \( \mathcal{O}_K \). The aim of this paper is to prove the following theorem.

**Theorem 1.** Let \( K, \Delta, f, \zeta(s, \chi) \) denote respectively any algebraic number field of degree \([K : \mathbb{Q}] = 2r_2\), the discriminant of \( K \), any ideal in \( K \) and any Hecke-Landau zeta function with a character \( \chi \) modulo \( f \). Fix \( 0 < \varepsilon < 1 \). If \(|\Delta| \geq 9\) and there is no zero in the region
\[ \sigma \geq 1 - 0.0795 \left( \log |\Delta| + 0.7761 \log \left( (|t| + 1)^{2r_2} (Nf)^{1-E_0} \right) \right)^{-1}, \]
then
\[ \Psi(x, X) \geq \frac{x(1 - \varepsilon)}{h_f^*(K)}, \]
for
\[ \log x \geq \left( 23.148 \sqrt{r_2} \left( 1 + \left( 2 \log \left( \frac{c_1 \sqrt{r_2}}{0.117\varepsilon} \right) \right)^{\frac{1}{2}} + \frac{2}{3} \log \left( \frac{c_1 \sqrt{r_2}}{0.117\varepsilon} \right) \right)^2. \]
where
\[ c_1 = (36997.123|\Delta|^{\frac{1.933}{r_2}} + 19064.499|\Delta|^{\frac{1.885}{r_2}} (Nf)^{\frac{1}{r_2}} h_f^*(K))r_2^2 \log(|\Delta|Nf). \]

**Remark 1.** For real \( \chi \) (mod \( f \)) may exist in \( \{1\} \) one zero of \( \zeta(s, \chi) \), which may be real and simple. However, we can check numerically that a Hecke-Landau \( \zeta \) function has a simple real zero in \( \{1\} \) using scripts for computing zeros zeta functions associated to characters of finite order \( \{1\} \).

Explicit results of this kind are useful for estimating of computational complexity of algorithms which generates special primes. Such primes can be used in computational number theory and cryptography. In order to calculate exactly the time of the algorithms one need an explicit bound for the number of desired primes from the interval \([x, 2x], x \geq x_0\), where \( x_0 \) is computed explicitly. We give an example of such an algorithm. For this reason we recall the definition \( \{5\} \).

**Definition 1.** Let \( p, q \) be a pair of primes and \( \Delta < 0 \). The primes \( p, q \) are defined to be CM-primes with respect to \( \Delta \) if there exist integers \( f \) and \( t \) such that
\[ |t| \leq 2\sqrt{p}, \quad q | p + 1 - t, \quad 4p - t^2 = \Delta f^2. \]
If CM-primes $p$ and $q$ with respect to $\Delta$ and integers $f,t$ are given, then an ordinary elliptic curve $E$ over $\mathbb{F}_p$ of cardinality $p+1-t$ can be constructed using complex multiplication method \cite{1,2}. Let $E(\mathbb{F}_p)$ be the group of points on $E$ over $\mathbb{F}_p$, and let $|E(\mathbb{F}_p)|$ be the order of $E(\mathbb{F}_p)$. The group $E(\mathbb{F}_p)$ can be used to implement public key cryptographic systems, based on intractability of the discrete logarithm problem (DLP). To make the DLP in $E(\mathbb{F}_p)$ intractable, it is essential to generate a large prime $p$, and a curve $E$ defined over $\mathbb{F}_p$, such that $|E(\mathbb{F}_p)|$ has a large prime factor $q$. In \cite{6} an algorithmic method for constructing a pair $(E, p)$ such that $|E(\mathbb{F}_p)|$ has a large prime factor $q$ is given. Fix $K$ an imaginary quadratic number field, and positive integers $m, n, (m, n) = 1$. Then the algorithm generates $\alpha \in \mathcal{O}_K$ such that $q = N_{\mathcal{O}/\mathcal{O}_K}(\alpha) \equiv m \pmod{n}$ is a prime, and $x \leq q \leq 2x$ for sufficiently large $x \geq x_0$. Given $\alpha, q$ a prime $p$, $x < p < x^{\frac{1}{2\varepsilon}}$, is constructed, where $0 < \varepsilon < \frac{2}{7}$. For more algorithms of this kind we refer the reader to \cite{7,8}.

Let $x \in \mathbb{R}$, and let $W(x)$ be the Lambert $W$ function such that $W(x) e^{W(x)} = x$. If $-e^{-1} \leq x \leq 0$, then there are two possible real values of $W(x)$. We denote the branch satisfying $-1 < W(x)$ by $W_0(x)$ and the branch satisfying $W(x) \leq -1$ by $W_1(x)$. Fix $X$ (mod $\ell$) $\in H$. We define

$$\psi(x, X) := \sum_{Np^{m} \leq x} \log Np,$$

where $p$ runs through prime ideals of $\mathcal{O}_K$. Theorem \cite{1} follows from Theorem \cite{2}.

**Theorem 2.** Let $K, \Delta, f, \zeta(s, \chi)$ denote respectively any algebraic number field of degree $[K : \mathbb{Q}] = 2r_2$, the discriminant of $K$, any ideal in $K$ and any Hecke-Landau zeta function with a character $\chi$ modulo $\ell$. Let $A_0 \geq 0.7761$. If $|\Delta| \geq 9$ and there is no zero in the region

$$\sigma \geq 1 - 0.0795 \left( \log |\Delta| + A_0 \log \left( (|\ell| + 1)^{2r_2} (N\ell)^{1-E_0(\chi)} \right) \right)^{-1},$$

then

$$\psi(x, X) \geq \frac{x}{h_{\ell}^2(K)} \cdot \frac{c_2 x}{h_{\ell}^2(K)} \left( \log x \right)^{\frac{1}{2}} e^{-0.0432r_2^{-1/2} \sqrt{\log x}},$$

and

$$\psi(x, X) \leq \frac{x}{h_{\ell}^2(K)} \cdot \frac{c_3 x}{h_{\ell}^2(K)} \left( \log x \right)^{\frac{1}{2}} e^{-0.0459r_2^{-1/2} \sqrt{\log x}}$$

for $x \geq \exp \left( 116r_2 \log \left( 2|\Delta|^{\frac{1}{2r_2}} (N\ell)^{\frac{1}{2r_2}} \right) \right)$, where

$$c_2 = (10756.659|\Delta|^{\frac{1}{2r_2}} + 5541.374|\Delta|^{\frac{1}{2r_2}} (N\ell)^{\frac{1}{2r_2}} h_{\ell}^2(K))r_2^2 \log(|\Delta|N\ell),$$

$$c_3 = (14665.542|\Delta|^{\frac{1}{2r_2}} + 7555.065|\Delta|^{\frac{1}{2r_2}} (N\ell)^{\frac{1}{2r_2}} h_{\ell}^2(K))r_2^2 \log(|\Delta|N\ell).$$

**Proof.** See Section \cite{2}.
We are now in a position to prove Theorem 1

Proof. By Theorem 2 we have

$$\psi(2x, X) - \psi(x, X) \geq \frac{x}{h_f(K)} - \frac{c_1 x}{h_f(K)} (\log x)^{\frac{1}{2}} e^{-0.0432 r_2^{-1/2} \sqrt{\log x}},$$

where

$$c_1 = \left(2c_2 \left(1 + \frac{\log 2}{\log x}\right) + c_3\right) \leq 2.077c_2 + c_3$$

for \(x \geq \exp \left(116r_2 \log \left(2/|\Delta|^{1/2} (Nf)^{1/2}\right)\right)\). Fix \(0 < \varepsilon < 1\). If

$$c_1 (\log x)^{\frac{1}{2}} e^{-0.0432 r_2^{-1/2} \sqrt{\log x}} \leq \varepsilon,$$

then

$$0.0432 r_2 \left(\log x\right)^{\frac{1}{2}} \geq -W_{-1} \left(\frac{-0.0432 \varepsilon}{c_1 \sqrt{r_2}}\right).$$

By [2, Theorem 1]

$$\log x \geq \left(23.148 \sqrt{r_2} \left(1 + \left(2 \log \left(\frac{c_1 \sqrt{r_2}}{0.117\varepsilon}\right)\right)^{\frac{1}{2}} + \frac{2}{3} \log \left(\frac{c_1 \sqrt{r_2}}{0.117\varepsilon}\right)\right)\right)^{2}.$$

This finishes the proof.

2 The proof of Theorem 2

The proof of Theorem 2 rests on the following lemmas and theorems.

**Theorem 3.** Let \(K, f, \zeta(s, \chi)\) denote respectively any algebraic number field of degree \(n \geq 2\), any ideal in \(K\) and any Hecke-Landau zeta function with a character \(\chi\) modulo \(f\). Let further

$$L(t) = \log |\Delta| + A_0 \log \left((|t| + 1)^n (Nf)^{1 - E_0}\right) \geq 2.097. \quad (3)$$

Then in the case of the complex \(\chi\) in the region

$$\sigma \geq 1 - \frac{A_1}{L(t)} \geq 1 - 0.037911 = 0.962088 = A_2 \quad (4)$$

there is no zero of \(\zeta(s, \chi)\), where \(A_0 = 0.7761, A_1 = 0.0795\). For real \(\chi \pmod{f}\) they may exist in \(A\) one zero of \(\zeta(s, \chi)\), which may be real and simple.

Proof. See [5, Th. 2].
Lemma 1. Let $s = \sigma + it$, $0 < \eta \leq \frac{1}{4}$, $A_3 = 75.472$, $A_4 = 0.010$ and $|\Delta| \geq 9$. Assume that there is no exceptional zero in the region $\{3\}$ Then in the strip $1 - \frac{A_3}{6L(t)} \leq \sigma \leq 3$ we have.

$$\left| \frac{\zeta'(s, \chi_0)}{\zeta(s, \chi_0)} + \frac{1}{s-1} \right| \leq \phi_0(t, r_2, \eta, \Delta, f),$$

where

$$\phi_0(t, r_2, \eta, \Delta, f) = 32 \log \left( L(t)(|t| + 4)(|t| + 2)^{2r_2(1+\eta)} (1 + A_3 L(t))^{2r_2} \right)
+ 32 \log \left( A_3(|\Delta|Nf) \frac{1+\eta}{2} \zeta(1 + \eta)^{2r_2} \right) + 8A_3r_2L(t) + \frac{A_4r_2}{L(t)},$$

and

$$\left| \frac{\zeta'(s, \chi)}{\zeta(s, \chi)} \right| \leq \phi(t, r_2, \eta, \Delta, f),$$

where

$$\phi(t, r_2, \eta, \Delta, f) = 32 \log \left( (1 + A_3 L(t))^{2r_2} (|t| + 2)^{2r_2(1+2\eta)} \right)
+ 32 \log \left( 1.4(1 + \varepsilon_\chi)A(f)^{1+2\eta} \zeta(1 + \eta)^{2r_2} \right) + 4A_3r_2L(t) + \frac{A_4r_2}{L(t)}$$

for any character $\chi \neq \chi_0$ modulo $f$, where $\varepsilon_\chi = 0$ or $1$ to accordingly whether $\chi$ is primitive or not.

Proof. See Section 3

Lemma 2. Let $\phi_0$, $\phi$ be functions defined in Lemma 1. Let $T \geq 1$, $w \geq 1$, $|\Delta| \geq 9$, $c_4 = \sqrt{2wr_2}$ and

$$c_0 = c_0(\Delta, f, r_2, E_0) = |\Delta|^{-\frac{1}{2r_2^2}} (Nf)^{-\frac{1-E_0}{2r_2}}$$

If

$$T + 1 = c_0 \exp \left( c_4 \sqrt{\log x} \right),$$

then

$$\phi(T, r_2, \eta, \Delta, f) \leq 230.911r_2^\frac{3}{2} \log(|\Delta|Nf)(\log x)^\frac{1}{2}$$
$$\phi_0(T, r_2, \eta, \Delta, f) \leq 412.531r_2^\frac{3}{2} \log(|\Delta|Nf)(\log x)^\frac{1}{2}$$

for $x \geq \exp \left( (c_4^{-1} \log(2c_0^{-1}))^2 \right)$. 

\textbf{Proof.} Since \( T + 1 \geq 2, \) \( \log x \geq 2r_2w \left( \log(2c_0^{-1}) \right)^2 \geq 2 \left( \log(2 \cdot 9^{1/w}) \right)^2 = 8.892, \) and hence \( x > e^{8.892}. \) By (3), (5) we obtain
\[
L(T) = \frac{A_0\sqrt{2r_2}}{\sqrt{w}} \left( \log x \right)^{\frac{1}{2}}, \quad L(T) \geq 2.097. \tag{9}
\]
Let \( |\Delta| \geq 9, \) \( x > e^{8.892} \) and \( \eta = \frac{1}{4}. \) We have \( \zeta \left( \frac{5}{4} \right) \leq 4.596, \)
\[
32 \log(1 + A_3 L(T))^{2r_2} \leq 32r_2 \log x + 64r_2 \log(A_0\sqrt{2r_2}(A_3 + \frac{1}{2.097})) \leq 111.419r_2^\frac{3}{2} \log \log x.
\]
\[
32r_2(1 + 2\eta) \log(T + 2) \leq \frac{32(1 + 2\eta)\sqrt{r_2}}{\sqrt{2}} \left( \log x \right)^{\frac{1}{2}} \leq 37.336r_2^\frac{3}{2} \left( \log x \right)^{\frac{1}{2}},
\]
\[
4A_3r_2L(T) \leq 4\sqrt{2}A_0A_3r_2^\frac{3}{2} \left( \log x \right)^{\frac{1}{2}} \leq 331.334r_2^\frac{3}{2} \left( \log x \right)^{\frac{1}{2}},
\]
\[
32 \log (1.4(1 + \varepsilon \chi)A(f)^{1+2\eta}\zeta(1 + \eta)^{2r_2}) + \frac{A_4r_2}{L(T)} \leq 32 \log (2.8(1 + \eta)^{2r_2}) + 16(1 + 2\eta) \log(|\Delta|Nf) + \frac{A_4r_2}{2.097} \leq 73.419r_2 \log(|\Delta|Nf).
\]
By the above and (6) we obtain
\[
\phi(T, r_2, \eta, \Delta, f) \leq 230.911r_2^\frac{3}{2} \log(|\Delta|Nf)(\log x)^{\frac{1}{2}},
\]
Similarly,
\[
32 \log L(T) \leq 16 \log \log x + 32 \log(\sqrt{2r_2A_0}) \leq 24.686r_2^\frac{3}{2} \log \log x,
\]
\[
32 \log(T + 4)^{r_2(1+\eta)+1} \leq 56.3r_2(\log x)^{\frac{1}{2}},
\]
and
\[
32 \log \left( A_3(|\Delta|Nf)^{1+\varepsilon \chi}(1 + \eta)^{2r_2} \right) + \frac{A_4r_2}{L(t)} \leq 84.361r_2 \log(|\Delta|Nf).
\]
By the above and (5) we obtain
\[
\phi_0(T, r_2, \eta, \Delta, f) \leq 412.531r_2^\frac{3}{2} \log(|\Delta|Nf)(\log x)^{\frac{1}{2}}.
\]
This finishes the proof.
Lemma 3. Let $T \geq 1$, $w \geq 1$, $|\Delta| \geq 9$, and let $k \geq 1$. Let $c_0$ be the constant appearing in (7). If

$$T + 1 = c_0 \exp \left( \sqrt{\frac{\log x}{2wr^2}} \right),$$

then

$$\frac{1}{T^k} \leq 2^k c_0^{-k} e^{-kc_4\sqrt{\log x}} \quad \text{for} \quad \log x \geq (c_4^{-1} \log(2c_0^{-1}))^2, \quad (11)$$

$$\log(e(T + k)) \leq c_4\sqrt{\log x} + \log \left( e \left( \frac{k+1}{2} \right) \right). \quad (12)$$

Proof. By (10) we have

$$\frac{1}{T^k} = \exp(-k \log(c_0 e^{c_4 \sqrt{\log x}} (1 - (c_0 e^{c_4 \log x} - 1)))) \leq \exp(-k \log(\frac{1}{2} c_0 e^{c_4 \sqrt{\log x}})), \quad (10)$$

for $\log x \geq (c_4^{-1} \log(2c_0^{-1}))^2$. The proof of (12) is left to the reader. This finishes the proof.

Lemma 4. For $T \geq 1$ we have

$$\int_T^\infty t^{-2} dt \leq T^{-1}, \quad \int_T^\infty t^{-2} \log(t+4) dt \leq T^{-1} \log(e(T + 4))$$

Proof. The proof is left to the reader.

Lemma 5. Let $L(t)$ be the function which occur in (3). For $T \geq 1$ we have

$$\int_T^\infty t^{-2} L(t) dt \leq c_5 T^{-1} \log(e(T + 4)),$$

where $c_5 = 1.09r_2 \log \left( |\Delta| (N\hat{f})^{A_6(1-E_0)} \right)$.

Proof. We have

$$\int_T^\infty t^{-2} L(t) dt \leq 2r_2 A_0 \int_T^\infty t^{-2} \log(t+4) dt + \log \left( |\Delta| (N\hat{f})^{A_6(1-E_0)} \right) \int_T^\infty t^{-2} dt.$$

The Lemma 5 follows from Lemma 4. This finishes the proof.

Lemma 6. Let $L(t)$ be the function which occur in (3). For $T \geq 1$ we have

$$\int_T^\infty t^{-2} \log(1 + A_3 L(t))^{2r_2} dt \leq c_6 T^{-1} \log(e(T + 4)),$$

where $c_6 = 11.605r_2^2 \log \left( |\Delta| (N\hat{f})^{A_6(1-E_0)} \right)$, and $A_3$ is the constant appearing in Lemma 7.
Proof. By (3) we have
\[ \int_{T}^{\infty} t^{-2} \log (1 + A_3 L(t))^{2r_2} dt \leq 2r_2 c_7 \int_{T}^{\infty} t^{-2} dt + r_2 \int_{T}^{\infty} t^{-2} L(t) dt, \]
where \( c_7 = \log \left( A_3 \left( 1 + \frac{1}{2.097 A_3} \right) \right) \). The Lemma 6 follows from Lemma 4 and Lemma 5. This finishes the proof.

Lemma 7. Let \( \phi_0 \) be the function which occur in (5). For \( T \geq 1 \) we have
\[ \int_{T}^{\infty} \phi_0(t, r_2, \eta, \Delta, f) t^{-2} dt \leq c_8 T^{-1} \log(e(T + 4)), \]
where \( c_8 = 1138.428 r_2^2 \log(|\Delta|(Nf)^\frac{3}{2}). \)

Proof. By (3) and Lemma 1 with \( \eta = \frac{1}{4} \) we have
\[ \int_{T}^{\infty} \phi_0(t, r_2, \eta, \Delta, f) t^{-2} dt \leq (40r_2 + 32) \int_{T}^{\infty} t^{-2} \log(t + 4) dt \]
\[ + \left( 32 \log(A_3(|\Delta|Nf)^\frac{3}{2}) + 64r_2 \log \left( \frac{5}{4} \right) + \frac{A_4 r_2}{2.097} \right) \int_{T}^{\infty} t^{-2} dt \]
\[ + (32 + 8A_3 r_2) \int_{T}^{\infty} t^{-2} L(t) dt + 32 \int_{T}^{\infty} t^{-2} \log (1 + A_3 L(t))^{2r_2} dt \]
The Lemma 7 follows from Lemmas 4, 5 and 6. This finishes the proof.

Lemma 8. Let \( \phi \) be the function which occur in (6). For \( T \geq 1 \) we have
\[ \int_{T}^{\infty} \phi(t, r_2, \eta, \Delta, f) t^{-2} dt \leq c_9 T^{-1} \log(e(T + 4)), \]
where \( c_9 = 821.212 r_2^2 \log(|\Delta|Nf). \)

Proof. By (3) and Lemma 1 with \( \eta = \frac{1}{4} \) we have
\[ \int_{T}^{\infty} \phi(t, r_2, \eta, \Delta, f) t^{-2} dt \leq \left( 32 \log \left( 2.8A(f)^\frac{3}{2} \zeta(\frac{5}{4})^{2r_2} \right) + \frac{A_4 r_2}{2.097} \right) \int_{T}^{\infty} t^{-2} dt \]
\[ + 32 \int_{T}^{\infty} t^{-2} \log (1 + A_3 L(t))^{2r_2} dt + 16r_2 \int_{T}^{\infty} t^{-2} \log(t + 4) dt \]
\[ + 4A_3 r_2 \int_{T}^{\infty} L(t) t^{-2} dt. \]
The Lemma 8 follows from Lemmas 4, 5 and 6. This finishes the proof.
We are now in a position to prove Theorem 2.

Proof. Fix $T \geq 1$, and let $c = 1 + \frac{1}{\log x}$. Fix $X \ (\text{mod } f)$. We define

$$
\psi_1(x, X) := \int_0^x \psi(t, X) dt,
$$

(13)

and

$$
\gamma(n) = \sum_{Np^m = n} \log Np.
$$

Hence,

$$
\psi(x, X) = \sum_{n \leq x} \gamma(n)
$$

By partial summation we obtain

$$
\sum_{n \leq x} (x - n) \gamma(n) = \int_0^x \psi(t, X) dt.
$$

Now, we write

$$
f(s, \chi) = \frac{x^{s-1}}{s(s+1)} \left[ -\frac{\zeta'}{\zeta}(s, \chi) \right].
$$

By Theorem B [9, see p. 31] and the orthogonality properties of $\chi \ (\text{mod } f)$ we deduce the formula

$$
\sum_{n \leq x} (x - n) \gamma(n) = \frac{x^2}{2\pi i h_1(K)} \sum_{\chi} \chi(X) \int_{c-i\infty}^{c+i\infty} f(s, \chi) ds,
$$

(14)

where $c > 1$. Let $A_1$ be the constant appearing in [4], and let $B = \frac{A_1}{6} = 0.0133$. We define the contour $C$ consisting of the following parts:

$$
C_1 : s = c + it, \text{ where } -T \leq t \leq T,
$$

(15)

$$
C_2 : s = \sigma + iT, \text{ where } 1 - \frac{B}{L(T)} \leq \sigma \leq c,
$$

$$
C_3 : s = 1 - \frac{B}{L(t)} + it, \text{ where } -T \leq t \leq T.
$$

and of $C'_2$ situated symmetrically to $C_2$. If $\chi = \chi_0$, then $\frac{\zeta'}{\zeta}(s, \chi)$ has a first order pole of residue $-1$ at $s = 1$. From the Cauchy formula we get

$$
\frac{1}{2\pi i} \int_{C_1} f(s, \chi) ds = \frac{\delta(\chi)}{2} - \frac{1}{2\pi i} \int_{C_2 + C_3 + C'_2} f(s, \chi) ds,
$$

(16)
where
\[ \delta(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases} \]

From (13), (14) and (16) we obtain
\[ \left| \psi_1(x, X) - \frac{x^2}{2h_f^*(K)} \right| \leq \frac{x^2(I_1 + I_2 + I_3)}{h_f^*(K)} + \frac{x^2(J_1 + J_2 + J_3)}{h_f^*(K)}, \quad (17) \]

where
\[ I_1 + I_2 + I_3 = \left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s, \chi_0) ds \right| + \left| \frac{1}{2\pi i} \int_{c_2}^{c+i\infty} f(s, \chi_0) ds \right| + \left| \frac{1}{2\pi i} \int_{c+\infty}^{c+iT} f(s, \chi_0) ds \right|, \]
\[ J_1 + J_2 + J_3 = \sum_{\chi \neq \chi_0} \lambda(X) \left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s, \chi) ds \right| + \sum_{\chi \neq \chi_0} \lambda(X) \left| \frac{1}{2\pi i} \int_{c_2+c_1+c_2}^{c+i\infty} f(s, \chi) ds \right| + \sum_{\chi \neq \chi_0} \lambda(X) \left| \frac{1}{2\pi i} \int_{c+\infty}^{c+iT} f(s, \chi) ds \right|. \]

We define
\[ h_0(s, \chi_0) = \left[ -\frac{\zeta'(s, \chi_0)}{\zeta(s, \chi_0)} - \frac{1}{s - 1} \right] \frac{x^{s-1}}{s(s + 1)}, \quad h_1(s) = \frac{x^{s-1}}{s(s + 1)}. \quad (18) \]

We estimate the above integrals. Let \( T \geq 1, x \geq e^{8.892}, 1 < c \leq 1 + \frac{1}{\log x} \leq 1.12 \).

We write We need to consider the following cases:
1. Bound over \( C_2 \) and \( C'_2 \), case \( \chi = \chi_0 \). From Lemmas 2 and 3 we obtain
\[ \left| \frac{1}{2\pi i} \int_{c_2}^{c+iT} f(\sigma + iT, \chi_0) d\sigma \right| \leq \frac{e}{2\pi T^2 \log x} \phi_0(T, r_2, \eta, \Delta, f) + \frac{e}{2\pi T^3 \log x} \]
\[ \leq c_0^{-3} r_2^{\frac{3}{2}} \log(|\Delta|Nf)(\log x)^{-\frac{1}{2}} \left( \frac{412.531 \co e}{\pi} + \frac{4coe}{\log x^{\frac{3}{2}}} \right) e^{-2c_1\sqrt{\log x}} \]
\[ \leq c_0 \log x^{-\frac{1}{2}} e^{-2c_1\sqrt{\log x}}, \]
where \( c_0 = 360.992 |\Delta|^\frac{3}{2} r_2^3 \log(|\Delta|Nf) \). The same bound holds with \( \int_{C'_2} \) in place of \( \int_{C_2} \).
2. Bound over \( C_3 \), case \( \chi = \chi_0 \). Lemmas 2 and 3 shows that
\[ \left| \frac{1}{2\pi i} \int_{C_3}^{T} h_0 \left( 1 - \frac{B}{L(T)} + it, \chi_0 \right) dt \right| \leq \frac{1}{\pi} x^{-\frac{1}{2}} \phi_0(T, r_2, \eta, \Delta, f) \int_{0}^{T} \frac{dt}{\left( 1 - \frac{B}{L(T)} \right)^2 + t^2} \]
\[ \leq \frac{1}{\pi} 2.01 e^{-c_1\sqrt{\log x}} \phi_0(T, r_2, \eta, \Delta, f) \leq 263.939 r_2^{\frac{3}{2}} \log(|\Delta|Nf)(\log x)^{\frac{3}{2}} e^{-c_1\sqrt{\log x}} \]
where \( c_{18} = \frac{B \sqrt{w}}{\log \sqrt{\Delta w}} \). Indeed, \( 1 - \frac{B}{L(T)} \geq 1 - \frac{0.0133}{2 \log w} = 0.993 \), and

\[
\int_0^T \frac{dt}{(1 - \frac{B}{L(T)})^2 + t^2} = \frac{1}{\pi} \int_0^T \frac{dt}{(1 - \frac{B}{L(T)})^2 + t^2} + \frac{1}{\pi} \int_1^T \frac{dt}{(1 - \frac{B}{L(T)})^2 + t^2} \\
\leq \frac{1}{\pi} \int_0^1 \frac{dt}{(0.993)^2} + \frac{1}{\pi} \int_1^T \frac{dt}{(0.993)^2} + 1 = 2.01.
\]

Moreover,

\[
\left| \frac{1}{2 \pi i} \int_{C_3} h_1 \left( 1 - \frac{B}{L(T)} + it \right) ds \right| \leq \frac{1}{\pi} \int_0^T \frac{dt}{1 - \frac{B}{L(T)} + t} \left| 2 - \frac{B}{L(T)} + t \right| - \frac{B}{L(T)} + t \\
\leq \frac{c_{11}}{\pi} e^{-c_{18} \sqrt{\log x}},
\]

where \( c_{11} = \frac{0.993(0.993 - 1)(0.993 + 1)}{1} \). By the above and (18),

\[
\left| \frac{1}{2 \pi i} \int_{C_3} f \left( 1 - \frac{B}{L(T)} + it, \chi_0 \right) dt \right| \leq c_{12} (\log x)^{\frac{1}{2}} e^{-c_{18} \sqrt{\log x}},
\]

where \( c_{12} = 267.495 r_2^\frac{3}{2} \log(\Delta |Nf|) \). Hence,

\[
I_2 \leq \frac{\Delta}{(\log x)^{\frac{3}{2} - \frac{3}{4}}} \frac{r_2^\frac{3}{2}}{\log x} \log(\Delta |Nf|) (\log x)^{\frac{1}{2}} e^{-c_{18} \sqrt{\log x}} \\
\cdot \left( 267.495 + \frac{2 \cdot 360.992}{\log x} e^{-(2c_4 - c_{18}) \sqrt{\log x}} \right) \leq c_{13} (\log x)^{\frac{1}{2}} e^{-c_{18} \sqrt{\log x}}, \quad (19)
\]

if \( w < \frac{244}{255} = 117.14 \), where \( c_{13} = 348.69 \frac{\Delta^{\frac{3}{4}}}{\log x} r_2^\frac{3}{2} \log(\Delta |Nf|) \).

3. Bound over \( C_2 \) and \( C'_2 \), case \( \chi \neq \chi_0 \). From Lemmas 2 and 3 we obtain

\[
\left| \frac{1}{2 \pi i} \int_{C_2} f(\sigma + iT, \chi_0) ds \right| \leq c_{14} (\log x)^{\frac{1}{2}} e^{-2c_4 \sqrt{\log x}},
\]

where \( c_{14} = 399.594 \frac{\Delta^{\frac{3}{4}}}{\log x} (Nf)^{\frac{3}{2}} r_2^\frac{3}{2} \log(\Delta |Nf|) \). The same bound holds with \( \int_{C'_2} \) in place of \( \int_{C_2} \).

4. Bound over \( C_3 \), case \( \chi \neq \chi_0 \). Lemmas 2 and 3 shows that

\[
\left| \frac{1}{2 \pi i} \int_{C_3} f \left( 1 - \frac{B}{L(T)} + it, \chi_0 \right) ds \right| \leq \frac{1}{\pi} \int_0^T e^{-\frac{B}{L(T)}} \phi(T, r_2, \eta, \Delta, f) \int_0^T \frac{dt}{(1 - \frac{B}{L(T)})^2 + t^2} \\
\leq \frac{1}{\pi} 2.01 e^{-c_{18} \sqrt{\log x}} \phi(T, r_2, \eta, \Delta, f) \leq c_{15} (\log x)^{\frac{1}{2}} e^{-c_{18} \sqrt{\log x}},
\]
where \( c_{15} = 147.738 r_2^2 \log(|\Delta|Nf) \). Hence, by the above
\[
J_2 \leq 2c_{14}(\log x)\frac{\pi}{2} e^{-2c_{14}\sqrt{\log x}} + c_{15}(\log x)\frac{\pi}{2} e^{-c_{15}\sqrt{\log x}} \\
\leq c_{16}(\log x)\frac{\pi}{2} e^{-c_{16}\sqrt{\log x}}, \tag{20}
\]
if \( w < \frac{2A_B}{B} = 117.14 \), where \( c_{16} = 237.616|\Delta|^{\frac{2}{3}\sqrt{r_2^2}} (Nf)^{\frac{3}{4}} r_2^2 \log(|\Delta|Nf) \). 5. Bound for \( \int_{c-iT}^{c+iT} \), case \( \chi = \chi_0 \). By (18) and Lemmas 3 and 7 we obtain
\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s, \chi_0) ds \right| \leq \left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} h_0(s, \chi_0) ds \right| + \left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} h_1(s) ds \right| \\
\leq \frac{e}{2\pi} \int_{c-iT}^{c+iT} \phi_0(t, r_2, \eta, \Delta, f) t^{-2} dt + \frac{\pi}{2\pi} \int_{c-iT}^{c+iT} t^{-3} dt \leq \frac{e}{2\pi} c_0 \log(e(T + 4)) \frac{\log(e(T + 4))}{T} \\
+ \frac{\pi}{4\pi T^2} \leq \frac{e c_0}{\pi c_0} (\log x)\frac{\pi}{2} \left( \frac{c_0}{\sqrt{2}w} + \frac{\frac{1.17 c_0}{\log x}}{\sqrt{2}w} + \frac{c_0}{2c_0 (\log x)\frac{\pi}{2}} \right) e^{-c_4\sqrt{\log x}} \\
\leq c_{17}(\log x)\frac{\pi}{2} e^{-c_4\sqrt{\log x}}
\]
for \( \log x \geq (c_4^{-1} \log(2c_0^{-1}))^2 \geq 8.892 \), with \( (\log x)\frac{\pi}{2} \geq 2.98 \), \( c_0 \leq 1 \), \( w = 58 \). where \( c_{17} = 724.845|\Delta|^{\frac{2}{3}\sqrt{r_2^2}} \log(|\Delta|Nf) \). The same bound holds with \( \int_{c-i\infty}^{c+i\infty} \) in place of \( \int_{c-iT}^{c+iT} \). Hence,
\[
I_1 + I_3 \leq 2c_{17}(\log x)\frac{\pi}{2} e^{-c_4\sqrt{\log x}}. \tag{21}
\]
6. Bound for \( \int_{c-i\infty}^{c+i\infty} \), case \( \chi \neq \chi_0 \). Lemmas 9 and 8 shows that
\[
\left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s, \chi) ds \right| \leq \frac{e}{2\pi} \int_{c-i\infty}^{c+i\infty} \phi(t, r_2, \eta, \Delta, f) t^{-2} dt \leq \frac{e}{2\pi} c_0 \log(e(T + 4)) \frac{\log(e(T + 4))}{T} \\
\leq \frac{e c_0}{\pi c_0} (\log x)\frac{\pi}{2} \left( \frac{1}{\sqrt{2}w} + \frac{\frac{1.17 c_0}{\log x}}{\sqrt{2}w} \right) e^{-c_4\sqrt{\log x}} \leq c_{19}(\log x)\frac{\pi}{2} e^{-c_4\sqrt{\log x}}
\]
where \( c_{19} = 522.77|\Delta|^{\frac{2}{3}\sqrt{r_2^2}} (Nf)^{\frac{3}{4}} r_2^2 \log(|\Delta|Nf) \), and \( w = 58 \). The same bound holds with \( \int_{c-i\infty}^{c+i\infty} \) in place of \( \int_{c-i\infty}^{c+i\infty} \). Hence,
\[
J_1 + J_3 \leq 2c_{19}(\log x)\frac{\pi}{2} e^{-c_4\sqrt{\log x}}. \tag{22}
\]
By (19), (21) we have
\[
I_1 + I_2 + I_3 \leq c_{20}(\log x)\frac{\pi}{2} e^{-c_{20}\sqrt{\log x}}, \tag{23}
\]
where \( c_{20} = 3585.536|\Delta|^{\frac{2}{3}\sqrt{r_2^2}} r_2^2 \log(|\Delta|Nf) \), for \( 1 \leq w < \frac{2A_B}{B} = 58.57 \). From (20), (22) we obtain
\[
J_1 + J_2 + J_3 \leq c_{21}(\log x)\frac{\pi}{2} e^{-c_{21}\sqrt{\log x}}, \tag{24}
\]
where $c_{21} = 1847.116h^*_f(K)|\Delta|^{-1/\sqrt{2}}(N_f)^{-1/2}r_f^2 \log(|\Delta|N_f)$ for $1 \leq w < \frac{4v}{B} = 58.57$. Now, by [17], [23], [24] we obtain

\[
\left| \psi_1(x, X) - \frac{x^2}{2h^*_f(K)} \right| \leq \frac{x^2}{h^*_f(K)}c_{22}(\log x)^{1/2}e^{-c_{18}\sqrt{\log x}}
\]

where $c_{22} = c_{20} + c_{21}$. Now, let $x > 2$, and $h$ be a function of $x$ satisfying $0 < h < \frac{1}{2}x$. Let $W(x) = c_{22}(\log x)^{3}e^{-c_{18}\sqrt{\log x}}$. Since $\psi(t, X)$ is an increasing function

\[
\psi(x, X) \geq \frac{1}{h} \int_{x-h}^{x} \psi(t, X)dt = \frac{\psi_1(x, X) - \psi_1(x-h, X)}{h}
\]

\[
\geq \frac{x}{h^*_f(K)} - \frac{x^2}{h^*_f(K)h^*_f(K)}W(x) - \frac{h}{2h^*_f(K)} - \frac{x^2 + h^2}{h^*_f(K)h^*_f(K)}W(x-h).
\]

Taking $h = xe^{-\frac{1}{2}c_{18}\sqrt{\log x}}$ and $x > \left(\frac{2\log 2}{c_{18}}\right)^{2}$, we get

\[
\psi(x, X) \geq \frac{x}{h^*_f(K)} - \frac{x}{h^*_f(K)c_{22}(\log x)^{1/2}e^{-\frac{1}{2}c_{18}\sqrt{\log x}}} - \frac{1}{2h^*_f(K)}xe^{-\frac{1}{2}c_{18}\sqrt{\log x}}
\]

\[
- \frac{x}{h^*_f(K)c_{22}(\log x)^{1/2}e^{-c_{18}(c_{23}+0.5)\sqrt{\log x}}} + \frac{x}{h^*_f(K)c_{22}(\log x)^{1/2}e^{-c_{18}(c_{23}+0.5)\sqrt{\log x}}}
\]

\[
\geq \frac{x}{h^*_f(K)} - \frac{x}{h^*_f(K)c_{22}(\log x)^{1/2}e^{-0.47c_{18}\sqrt{\log x}}(3 + c_{24})}
\]

\[
\geq \frac{x}{h^*_f(K)} - \frac{c_{22}x}{h^*_f(K)}(\log x)^{1/2}e^{-0.47c_{18}\sqrt{\log x}},
\]

(25)

where $c_{23} = (1 - \frac{\log 2}{\log x})^{1/2}, 0.97 \leq c_{23} \leq 0.98$, $c_{24} = \frac{1}{2c_{22}}(\log x)^{-1/2} \leq 0.0001$, and $c_{2} = c_{22}(3 + c_{24})$. On the other hand,

\[
\psi(x, X) \leq \frac{1}{h} \int_{x}^{x+h} \psi(t, X)dt = \frac{\psi_1(x+h, X) - \psi_1(x, X)}{h}
\]

\[
\leq \frac{x}{h^*_f(K)} + \frac{h}{2h^*_f(K)} + \frac{(x+h)^2}{h^*_f(K)h^*_f(K)}W(x+h) + \frac{x^2}{h^*_f(K)h^*_f(K)}W(x)
\]

\[
\leq \frac{x}{h^*_f(K)} + \frac{x}{h^*_f(K)c_{22}(\log x)^{1/2}e^{-\frac{1}{2}c_{18}\sqrt{\log x}}(c_{25} + 5c_{1})}
\]

\[
\leq \frac{x}{h^*_f(K)} + \frac{c_{22}x}{h^*_f(K)(\log x)^{1/2}e^{-\frac{1}{2}c_{18}\sqrt{\log x}}}
\]

where $c_{25} = \frac{1}{2c_{22}c_{26}}(\log x)^{-1/2} \leq 0.001$, $c_{26} = \left(1 + \frac{\log 2}{\log x}\right)^{1/2} \leq 1.013$, $c_{3} = c_{22}(c_{25} + 5c_{26})$. Putting $c_{18} = \frac{B_0}{A_0\sqrt{2\pi}} = 0.0919\sqrt{\pi}$ we obtain the result. This finishes the proof.
3 Proof of Lemma \[1\]

The proof of Lemma \[1\] rests on the following lemmas.

**Lemma 9.** Let \(|K : Q| = 2r_2\) and \(0 < \eta \leq \frac{1}{4}\). In the region \(-\eta \leq \sigma \leq 3\) we have the estimate

\[
|\zeta(\sigma + it, \chi)| \leq 1.4^{r_2}(1 + \varepsilon_\chi)A(f)^{1+2\eta}\zeta(1 + \eta)^{2r_2}(|t| + 1)^{r_2(1+2\eta)}
\]

for any character \(\chi \neq \chi_0\) modulo \(f\), where \(\varepsilon_\chi = 0\) or \(1\) to accordingly whether \(\chi\) is primitive or not.

**Proof.** Consider

\[
g(s, \chi) = \frac{\zeta(s, \chi)}{\zeta(1 - s, \overline{\chi})},
\]

where \(\chi\) is a primitive character modulo \(f\). From the functional equation for \(\zeta(s, \chi)\) it follows that

\[
g(s, \chi) = W(\chi)A(f)^{1-2s}\left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2}.
\] (26)

We estimate \(g(s, \chi)\) on the line \(s = -\eta + it\), \(0 \leq \eta \leq \frac{1}{4}\) using the following inequality (see [4], p. 58)

\[
\left|\frac{\Gamma(1-s)}{\Gamma(s)}\right| \leq 1.4\max(1, |s|^{1+2\eta})
\] (27)

From (26) and (27) we obtain

\[
|g(-\eta + it, \chi)| \leq 1.4^{r_2}A(f)^{1+2\eta}(\max(1, |-\eta + it|^{(1+2\eta)}))^{r_2}
\] (28)

for \(-\infty < t < \infty\). Write

\[
G(s, \chi) = \frac{\zeta(s, \chi)}{(s+1)^{r_2(1+2\eta)}}.
\] (29)

From (28) we have

\[
|G(-\eta + it, \chi)| \leq 1.4^{r_2}A(f)^{1+2\eta}|\zeta(1 + \eta, \overline{\chi})| \leq 1.4^{r_2}A(f)^{1+2\eta}|\zeta(1 + \eta)|^{2r_2}
\] (30)

for \(\chi \neq \chi_0\). If \(\chi\) is not a primitive character, then there is an ideal \(f_0\) which divides \(f\), and there is a primitive character \(\psi\) (mod \(f_0\)) such that

\[
\zeta(s, \chi) = \zeta(s, \psi) \prod_{p|f, p|f_0} \left(1 - \frac{\psi(p)}{(Np)^s}\right).
\]
Write $f = f_0 f_1$. From [4] see p. 60] we get
\[ \prod_{p|f: p|f_0} \left( 1 - \frac{\psi(p)}{(Np)^s} \right) \leq 2(Nf_1)^{1/2 + \eta}. \]

Hence,
\[ |G(-\eta + it, \chi)| \leq 1.4^{r_2}(1 + \varepsilon_\chi)A(f)^{1 + 2\eta}|\zeta(1 + \eta, \chi)| \]
\[ \leq 1.4^{r_2} A(f)^{1 + 2\eta}\zeta(1 + \eta)^{2r_2} \]
for any character $\chi \neq \chi_0$ modulo $f$, where $\varepsilon_\chi = 0$ or 1 to accordingly whether $\chi$ is primitive or not. On the other hand,
\[ |G(3 + it, \chi)| \leq \frac{|\zeta(3 + \eta, \chi)|}{(4 + it)^{r_2(1 + 2\eta)}} \leq \frac{1}{4^{r_2}} \zeta(3 + \eta)^{2r_2}. \]

Using the estimate
\[ |\zeta(s, \chi)| \leq A_1 e^{A_2|t|}, \]
which is valid in the strip $-\eta \leq \sigma \leq 3$, where $A_1, A_2$ depends on $K, \chi,$ and $f$, we get
\[ |G(s, \chi)| = O(e^{A_3|t|}) \]
for $-\eta \leq \sigma \leq 3$. From (31)-(33) and the well-known theorem of Phragmen-Lindelöf we obtain
\[ |G(s, \chi)| \leq \frac{1.4^{r_2}(1 + \varepsilon_\chi)A(f)^{1 + 2\eta}\zeta(1 + \eta)^{2r_2}}{(|t| + 1)^{r_2(1 + 2\eta)}} \]
in the strip $-\eta \leq \sigma \leq 3$. From (31), (29)
\[ |\zeta(s, \chi)| \leq 1.4^{r_2}(1 + \varepsilon_\chi)A(f)^{1 + 2\eta}\zeta(1 + \eta)^{2r_2} (|t| + 1)^{r_2(1 + 2\eta)} \]
for any character $\chi \neq \chi_0$ modulo $f$, where $\varepsilon_\chi = 0$ or 1 to accordingly whether $\chi$ is primitive or not. This finishes the proof.

**Lemma 10.** For $\sigma > 1$ we have
\[ \frac{1}{|\zeta(\sigma + it, \chi)|} \leq \zeta_K(\sigma). \]

**Proof.** See [?, Lemma 2.4].

**Lemma 11.** Let $|K : \mathbb{Q}| = 2r_2$ and $0 < \eta \leq \frac{1}{4}$. In the region $-\eta \leq \sigma \leq 1 + \eta$, $-\infty < t < \infty$ we have estimate
\[ |(s - 1)\zeta(s, \chi_0)| \leq (3 + |t|)(1 + |t|)^{r_2(1 + \eta - \sigma)}(|\Delta f| Nf)^{\frac{|s - 1 - \eta|}{2}} \zeta_K(1 + \eta). \]
Proof. See [4, (5.4)].

Lemma 12. Let \( f(s) \) be a function regular in the circle \( |s - s_0| \leq r \) and satisfying the inequality
\[
\left| \frac{f(s)}{f(s_0)} \right| \leq M.
\]
If \( f(s) \neq 0 \) in the region \( |s - s_0| \leq \frac{r}{2}, \Re(s - s_0) > 0 \), then
\[
\Re \frac{f'}{f}(s_0) \geq -\frac{A}{r} \log M.
\]

Proof. See [11], p. 384-385

Lemma 13. Let \( f(s) \) be a function regular in the circle \( |s - s_0| \leq R \) and satisfying the conditions
\[
\Re f(s) \leq M \quad \text{for} \quad |s - s_0| = R
\]
Then
\[
|f^{(k)}(s)| \leq 2k!(M - \Re f(s)) \frac{R}{(R - r)^{k+1}}, \quad k \geq 1.
\]
in the circle \( |s - s_0| \leq r < R \).

Proof. See [11], p. 384-385

We are in a position to prove Lemma [1].

Proof. Let \( B = \frac{A_1}{\eta} = 0.01325 \), where \( A_1 \) is the constant appearing in [4]. Let \( s_0 = \sigma_0 + it_0, t_0 \geq 0, \)
\[
\sigma_0 = 1 + \frac{B}{L(t_0)}.
\]
where \( L(t_0) \) is defined in [3]. We define the function
\[
H(s, \chi) = \log \frac{g(s, \chi)}{g(s_0, \chi)}, \quad g(s) = h(s, \chi) \prod_{\rho}(s - \rho)^{-1},
\]
where \( h(s, \chi) = \zeta(s, \chi) \) if \( \chi \neq \chi_0 \) and \( h(s, \chi_0) = (s - 1)\zeta(s, \chi_0) \), where \( \rho \) are zeros of the function \( h(s, \chi) \) in the circle \( |s - s_0| \leq \frac{1}{2} \). Firstly, we estimate \( \left| \frac{g(s)}{g(s_0)} \right| \).

Lemmas [9], [10] and [11] shows that in the circle \( |s - \sigma_0| \leq 1 \)
\[
\left| \frac{\zeta(s + it, \chi)}{\zeta(s_0, \chi)} \right| \leq 1.4(1 + \varepsilon_{\chi})A(f)^{1+2\eta} \zeta(1 + \eta)^{2\eta^2} \zeta_k(\sigma_0)(t_0 + 2)^{r_2(1+2\eta)}, \quad (36)
\]
for any character $\chi \neq \chi_0$ modulo $f$, where $\varepsilon = 0$ or 1 to accordingly whether $\chi$ is primitive or not, and

$$\left| \frac{\zeta(s, \chi)}{\zeta(\sigma_0, \chi_0)} \right| \leq \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{r_2(1+\eta)}(|\Delta|Nf)^{\frac{1-2s}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0).$$

(37)

On $|s - s_0| = 1$, $|s_0 - \rho| \leq \frac{1}{2}$ and $|s - \rho| \geq \frac{1}{2}$. From (36), (37) and the maximum principle we obtain

$$\left| \frac{\zeta(s, \chi)}{\zeta(s_0, \chi_0)} \prod_{\rho}(s_0 - \rho) \right| \leq \left| \frac{(s - 1)\zeta(s, \chi_0)}{(s_0 - 1)\zeta(s_0, \chi_0)} \prod_{\rho}(s_0 - \rho) \right| \leq \frac{L(t_0)}{B} \frac{(4 + |t_0|)(2 + |t_0|)^{r_2(1+\eta)}(|\Delta|Nf)^{\frac{1-2s}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0)}{(s_0 - 1)\zeta(s_0, \chi_0)}$$

(38)

and

$$\left| \frac{(s - 1)\zeta(s, \chi)}{(s_0 - 1)\zeta(s_0, \chi_0)} \prod_{\rho}(s_0 - \rho) \right| \leq \frac{L(t_0)}{B} \frac{(4 + |t_0|)(2 + |t_0|)^{r_2(1+\eta)}(|\Delta|Nf)^{\frac{1-2s}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0)}{(s_0 - 1)\zeta(s_0, \chi_0)} \prod_{\rho}(s_0 - \rho)$$

(39)

in the circle $|s - s_0| \leq 1$. Secondly, we apply Lemma [13] to the function $H(s, \chi)$ with $k = 1$, $R = \frac{t}{f}$ and $r = \frac{1}{2}$. The function $H(s, \chi)$ is regular in the circle $|s - s_0| \leq \frac{1}{2}$, therefore by [38], [38] we obtain

$$\Re H(s, \chi) = \log \left| \frac{g(s, \chi)}{g(s_0, \chi)} \right| \leq \log \left( (1 + \varepsilon_\chi)A(f)^{1+2zetac(1 + \eta)^2r_2\zeta_K(\sigma_0)(t_0 + 2)^{r_2(1+2\eta)}) \right),$$

if $\chi \neq \chi_0$

$$\log \left( \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{r_2(1+\eta)}(|\Delta|Nf)^{\frac{1-2s}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0) \right),$$

if $\chi = \chi_0$.

in the circle $|s - s_0| \leq \frac{1}{2}$. Therefore, in the circle $|s - s_0| \leq \frac{1}{2}$ we have

$$\left| \frac{\zeta'(s, \chi)}{\zeta(s_0, \chi_0)} - \sum_{\rho} \frac{1}{s - \rho} \right| \leq 16 \log \left( (1 + \varepsilon_\chi)A(f)^{1+2zetac(1 + \eta)^2r_2\zeta_K(\sigma_0)(t_0 + 2)^{r_2(1+2\eta)}) \right),$$

(40)

and

$$\left| \frac{\zeta'(s, \chi_0)}{\zeta(s_0, \chi_0)} + \frac{1}{s - 1} - \sum_{\rho} \frac{1}{s - \rho} \right| \leq 16 \log \left( \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{r_2(1+\eta)}(|\Delta|Nf)^{\frac{1-2s}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0) \right).$$

(41)
Finally, we estimate $|\sum_{\rho} \frac{1}{s_0 - \rho}|$ and $|\sum_{\rho} \frac{1}{2s_0 - \rho}|$. In [10] Israilov show that, if $1 < \sigma \leq 2$ then

$$-\frac{\zeta'(s)}{\zeta(s)} \leq \frac{1}{\sigma - 1} - \gamma + C_1(\sigma - 1),$$

where $C_1 = 0.1875463$. Hence, [33] shows

$$\left| \frac{\zeta'(s_0, \chi)}{\zeta(s_0, \chi)} \right| \leq 2r_2 \frac{L(t_0)}{B} + 2r_2 C_1 \frac{B}{L(t_0)}, \quad (42)$$

and

$$\left| \frac{\zeta'(s_0, \chi_0)}{\zeta(s_0, \chi_0)} + \frac{1}{s_0 - 1} \right| \leq 4r_2 \frac{L(t_0)}{B} + 2r_2 C_1 \frac{B}{L(t_0)}. \quad (43)$$

By [10], [12], we obtain

$$\left| \sum_{\rho} \frac{1}{s_0 - \rho} \right| \leq 16 \log \left( \frac{1.4(1 + \varepsilon) A(f)^{1+2\eta} \zeta(1 + \eta) 2r_2 \zeta_K(s_0)(t_0 + 2)^{2(1+2\eta)}}{(1+2\eta)} \right)$$

$$+ 2r_2 \frac{L(t_0)}{B} + 2r_2 C_1 \frac{B}{L(t_0)}, \quad (44)$$

and [11], [13]

$$\left| \sum_{\rho} \frac{1}{s_0 - \rho} \right| \leq 16 \log \left( \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{2(1+\eta)}(\Delta |Nf|)^{\frac{1+\eta}{2}} \zeta_K(1 + \eta) \zeta_K(s_0) \right)$$

$$+ 4r_2 \frac{L(t_0)}{B} + 2r_2 C_1 \frac{B}{L(t_0)}. \quad (45)$$

in the circle $|s - s_0| \leq \frac{1}{4}$. Now, we define

$$r_1 = \frac{2B}{L(t_0)} < \frac{1}{4}. \quad$$

By Theorem 11 the function $\zeta(s, \chi) \neq 0$ in the region $|s - s_0| \leq r$, $\Re(s - s_0) > -2r_1$. Hence

$$|s_0 - \rho| \geq 2r_1, \quad |s_0 - \rho| \geq 2r_1, \quad \Re(s_0 - \rho) \geq 2r_1$$

for all zeros $\rho$ in the circle $|s - s_0| \leq \frac{1}{4}$, and for $s$ in the circle $|s - s_0| \leq r_1$. For $|s - s_0| \leq r_1$ we obtain

$$\left| \sum_{\rho} \frac{1}{s_0 - \rho} - \sum_{\rho} \frac{1}{s_0 - \rho} \right| \leq \sum_{\rho} \frac{|s - s_0|}{|s - \rho||s_0 - \rho|} \leq \sum_{\rho} \frac{r_1}{2|s_0 - \rho|^2}$$

$$\leq \sum_{\rho} \frac{\Re(s_0 - \rho)}{|s_0 - \rho|^2} \leq \sum_{\rho} \frac{\Re}{s_0 - \rho} \leq \left| \sum_{\rho} \frac{1}{s_0 - \rho} \right|. \quad$$
Thus,

$$\left| \sum_{\rho} \frac{1}{s - \rho} \right| \leq 2 \left| \sum_{\rho} \frac{1}{s_0 - \rho} \right|. \quad (46)$$

From (40), (44) and (46) we have

$$\left| \frac{\zeta'}{\zeta}(s, \chi) \right| \leq 4r_2 \frac{L(t_0)}{B} + 4r_2 C_1 \frac{B}{L(t_0)}$$

$$+ 32 \log \left( 1.4(1 + \varepsilon_{\chi})A(f)^{1+2\eta} \zeta(1 + \eta)^{2r_2} \zeta_K(\sigma_0) (t_0 + 2)^{r_2(1+2\eta)} \right).$$

and by (40), (45) and (46)

$$\left| \frac{\zeta'}{\zeta}(s, \chi) + \frac{1}{s - 1} \right| \leq 8r_2 \frac{L(t_0)}{B} + 4r_2 C_1 \frac{B}{L(t_0)}$$

$$+ 32 \log \left( \frac{L(t_0)}{B} (4 + |t_0|) (2 + |t_0|) r_2 (1 + \eta) (|\Delta|Nf)^{1+2\eta} \zeta_K(1 + \eta) \zeta_K(\sigma_0) \right)$$

in the circle $|s - s_0| \leq r_1$, and consequently in the strip

$$1 - \frac{B}{L(t)} = 1 - \frac{A_1}{6L(t)} < \sigma < 1 + \frac{3B}{L(t)} = 1 + \frac{A_1}{2L(t)}.$$

Put $s_0 = \sigma_0 + t_0$, $t_0 \geq 0$, and

$$1 + \frac{A_1}{2L(t_0)} \leq \sigma \leq 3.$$

Lemma 12 and (36), (37) shows that

$$-\frac{\zeta'}{\zeta}(\sigma_0, \chi) \leq 4 \log \left( 1.4(1 + \varepsilon_{\chi})A(f)^{1+2\eta} \zeta(1 + \eta)^{2r_2} \zeta_K(\sigma_0) (t_0 + 2)^{r_2(1+2\eta)} \right)$$

for any character $\chi \neq \chi_0$ modulo $f$, where $\varepsilon_{\chi} = 0$ or 1 to accordingly whether $\chi$ is primitive or not, and

$$-\frac{\zeta'}{\zeta}(\sigma_0, \chi_0) \leq \frac{1}{\sigma_0 - 1} +$$

$$+ 4 \log \left( \frac{L(t_0)}{B} (4 + |t_0|) (2 + |t_0|) r_2 (1 + \eta) (|\Delta|Nf)^{1+2\eta} \zeta_K(1 + \eta) \zeta_K(\sigma_0) \right).$$

Therefore,

$$\left| \frac{\zeta'}{\zeta}(\sigma_0, \chi) \right| \leq 4 \log \left( 1.4(1 + \varepsilon_{\chi})A(f)^{1+2\eta} \zeta(1 + \eta)^{2r_2} \zeta_K(\sigma_0) (t_0 + 2)^{r_2(1+2\eta)} \right)$$
for any character \( \chi \neq \chi_0 \) modulo \( f \), where \( \epsilon_\chi = 0 \) or \( 1 \) to accordingly whether \( \chi \) is primitive or not, and

\[
\left| \frac{\zeta'(\sigma_0, \chi)}{\zeta(\sigma_0, \chi)} \right| \leq \frac{2L(t_0)}{A_1} + 4 \log \left( \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{r_2(1+r)}(|\Delta|Nf)^{\frac{1+r}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0) \right).
\]

By the above we obtain

\[
\left| \frac{\zeta'(s_0, \chi)}{\zeta(s_0, \chi)} \right| \leq \left| \frac{\zeta'(\sigma_0, \chi)}{\zeta(\sigma_0, \chi)} \right| \leq 4 \log \left( 1.4(1 + \epsilon_\chi)A(f)^{1+2\eta}\zeta(1 + \eta)^{2r_2}\zeta_K(\sigma_0)(t_0 + 2)^{r_2(1+2\eta)} \right)
\]

for any character \( \chi \neq \chi_0 \) modulo \( f \), where \( \epsilon_\chi = 0 \) or \( 1 \) to accordingly whether \( \chi \) is primitive or not, and

\[
\left| \frac{\zeta'(s_0, \chi)}{\zeta(s_0, \chi)} + \frac{1}{s_0 - 1} \right| \leq \left| \frac{\zeta'(\sigma_0, \chi)}{\zeta(\sigma_0, \chi)} \right| + \frac{1}{\sigma_0 - 1} \leq \frac{4L(t_0)}{A_1} + 4 \log \left( \frac{L(t_0)}{B} (4 + |t_0|)(2 + |t_0|)^{r_2(1+r)}(|\Delta|Nf)^{\frac{1+r}{2}} \zeta_K(1 + \eta)\zeta_K(\sigma_0) \right).
\]

The proof is completed by applying

\[
\zeta_K(\sigma_0) \leq \zeta(\sigma_0)^{2r_2} \leq \left( 1 + \frac{6L(t_0)}{A_1} \right)^{2r_2}.
\]

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