Quasiparticle properties of a single alpha particle in cold neutron matter

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Abstract

Light clusters such as alpha particles and deuterons are predicted to occur in hot nuclear matter as encountered in intermediate-energy heavy-ion collisions and protoneutron stars. To examine the in-medium properties of such light clusters, we consider a much simplified system in which like an impurity, a single alpha particle is embedded in a zero-temperature, dilute gas of non-interacting neutrons. By adopting a non-selfconsistent ladder approximation for the effective interaction between the impurity and the gas, which is often used for analyses of Fermi polarons in a gas of ultracold atoms, we calculate the quasiparticle properties of the impurity, i.e., the energy shift, effective mass, quasiparticle residue, and damping rate.

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I. INTRODUCTION

To understand the properties of hot nuclear matter as encountered in supernova cores is essential in describing various phenomena such as neutrino bursts, nucleosynthesis, and formation of neutron stars [1]. Remarkably, this matter can be opaque even to neutrinos, which in turn play a role in carrying a released gravitational energy of order $10^{53}$ erg during stellar collapse by diffusing out of supernova cores and at the same time in depositing a sufficient energy onto the material to cause a supernova explosion at which the total kinetic energy is of order $10^{51}$ erg. Here it is significant to note that at finite temperature, a nonzero number of light clusters such as alpha particles, deuterons, tritons, and $^3$He nuclei appear even in chemical equilibrium. These clusters, if sufficiently present in supernova cores, can play a role in scattering or absorbing the outgoing neutrinos. For example, even an $^{56}$Fe nucleus, one of the most stable nuclear configurations in vacuum, can decompose into thirteen alpha particles and four neutrons by absorbing a gamma-ray photon of energy in excess of $-Q = 124$ MeV. According to Saha’s arguments, the mass fraction of alpha particles is dominated by a factor of $e^{Q/13k_BT}$, with the temperature $T$. In estimating such a fraction, however, no in-medium modification of nuclear masses except the Coulomb corrections is normally considered. To deal with the in-medium modification, we will focus on a polaron picture, namely, a light cluster dressed by excitations in the medium.

The earliest theoretical investigations of hot nuclear matter are based on liquid-drop models [2]. The key ingredient of these models is mass formula for neutron-rich nuclei. Typically, in the presence of trapped electron neutrinos, nuclei in such matter (so-called supernova matter) are not extremely neutron-rich but too neutron-rich for their masses to be measured, which requires an extrapolation from empirical mass data. In the presence of internuclear Coulomb coupling, the Wigner-Seitz approximation for a lattice of nuclei embedded in a neutralizing background of electrons is often utilized. This approximation is known to give a good estimate of the lattice energy. Concerning the mass distribution, furthermore, a single species approximation in which only the nuclide that minimizes the system energy at fixed baryon density and lepton fraction occurs is often adopted for simplicity. This is good at sufficiently low temperatures. For better estimates of the mass distribution at temperatures relevant for supernova cores, the presence of alpha particles etc. has been allowed for or even the nuclear statistical equilibrium has been imposed in some cases [3].
The most important upgrade has been made on how to calculate the nuclear mass itself as an extrapolation from the empirical data. The nucleon density profile in a Wigner-Seitz cell can be better predicted if one implements the Thomas-Fermi or Hartree-Fock theory [4–7]. Even in such predictions, the result still depends on uncertainties in the adopted equation of state of asymmetric nuclear matter, especially the density dependence of the symmetry energy.

Experimentally, there can be two ways of studying properties of hot nuclear matter: heavy ion collisions at intermediate energy and quantum simulations with ultracold atoms. Data for such heavy ion collisions have already given some evidence for a liquid-gas phase transition of nuclear matter [8]. It is, however, important to note that the deduced temperature and density of finite matter (primary fragments) created in the collisions have some uncertainties, while finite-size corrections due to the Coulomb and surface effects have to be taken into consideration to deduce the critical point of bulk nuclear matter from the empirical data for the excitation energy, mass, and temperature of the primary fragments. Anyway, this is a direct way of probing nuclear matter. On the other hand, ultracold atoms provide us with an indirect way of probing nuclear matter. This is mainly because low density neutron matter, which is dominated by s-wave scattering with negative and large scattering length, is similar to trapped ultracold Fermi atoms near a Feshbach resonance [9, 10]. It is thus expected that various superfluid properties of neutron matter such as the pairing gap, the BCS-BEC crossover, and vortices under rotation could be deduced from laboratory experiments, although the neutron-neutron effective range is fairly large as compared with the case of atoms. Moreover, one can add impurity atoms to a system of ultracold Fermi atoms of a single species [11]. In the presence of interspecies interactions, these impurity atoms manifest themselves as polarons. The resultant atomic matter looks like hot nuclear matter of interest here.

In the present study, under the motivation mentioned above, we evaluate in-medium modifications of an alpha particle that interacts with surrounding neutrons, by calculating its quasiparticle properties: the interaction energy, residue, effective mass, dispersion relation, and decay width. To this end, for simplicity, we employ an ideal situation where a single alpha particle is immersed in pure neutron matter that is in a normal state at zero temperature, rather than hot nuclear matter. Furthermore, we assume that the alpha particle is mobile with a small momentum and that the neutron matter is dilute enough, so
that we can employ the low-energy treatment, i.e., the alpha particle is point like, while the interaction between the alpha particle and a surrounding neutron is described only by the $s$-wave scattering length.

Note that such an alpha particle is not always stable. In fact, stability analysis of an alpha particle in neutron matter would require microscopic calculations, which will be addressed in the near future. We also remark that even before the experimental realization of trapped ultracold atoms, Kutschera and Wojcik [12] used to consider a proton impurity in neutron matter by sticking with the original polaron picture based on an electron-phonon system, which is different from the modern polaron picture based on a minority atom - majority atom system involving contact interactions.

We also note that alpha-nucleon interactions in hot nuclear matter were considered in terms of excluded-volume effects [13] and the virial expansion [14]. Essentially, the alpha-nucleon interactions considered in these approaches tend to be too repulsive and too attractive, respectively, when applied to a cold system of interest here. This is because in the former approach, no attraction is included, while in the latter approach, the second virial coefficient is related to the two-body phase shift dominated by the $p$-wave resonance.

It is interesting to note the possible relevance of the present polaron picture to alpha clustering, i.e., manifestation of alpha particle like configurations, in atomic heavy nuclei. Although there is no direct evidence for the presence of such alpha clustering, it is expected that the four-nucleon correlation responsible for the alpha clustering plays an essential role in describing the surface region of various heavy nuclei [13, 16]. In such a dilute neutron-rich situation, two minority particles (protons) may tend to form an alpha particle by picking two neutrons out of the medium. Consequently, the system may look like pure neutron matter containing alpha particles as impurities. This situation may help us describe the neutron skin structure of heavy nuclei. The energy and decay rate of such alpha particles, if being known experimentally, could give us an opportunity of probing the bulk properties of the neutron medium.

This paper is organized as follows: In Sec. II we present the effective Hamiltonian for a single alpha particle embedded in neutron matter, and give numerical values to the parameters included therein. In Sec. III we employ a variational method to obtain the energy of the alpha particle, which is equivalent to the self-energy calculation from the ladder type approximation. In Sec. IV we discuss the numerical results for various quasiparticle proper-
ties as functions of neutron density, and also of the scattering length for general arguments. Section V is devoted to summary, physical consequences, and outlook.

II. FORMULATION

We consider a single alpha particle that is assumed to be an inert point particle and to be immersed in normal neutron matter at zero temperature. The Hamiltonian of this system is described by

$$H(x) = \sum_s \int dr^3 \frac{\nabla^2}{2m_s} \psi_s(r) + \frac{1}{2} \sum_{s,t,s',t'} \int dr^3 dr' \psi^\dagger_{s'}(r') \psi^\dagger_s(r) V_{st's't}(r-r') \psi_s(r) \psi_t(r')$$

$$- \frac{\nabla^2}{2M} + g \sum_s \int dr \psi_s^\dagger(r) \psi_s(r) \delta(r-x)$$

$$= \sum_{s,p} \frac{p^2}{2M} a^\dagger_{s,p} a_{s,p} + \frac{1}{2} \sum_{q,q',p,s,t,s',t'} a^\dagger_{q',q'-p} a^\dagger_{s',q+p} a_{s,q} a_{t,q'} \tilde{V}_{s't's't}(p)$$

$$- \frac{\nabla^2}{2M} + \sum_{p,q,s} g a^\dagger_{s,p} a_{s,p} e^{-i(p-q)x},$$

where we have used the first quantization for the single alpha particle in the coordinate representation by $x$, and expanded the neutron field operator as $\psi_s(r) = \frac{1}{\sqrt{V}} \sum_p e^{ipr} a_{s,p}$ with the canonical relation $\{a^\dagger_{s,p}, a_{t,q}\} = \delta_{p,q} \delta_{s,t}$, where the subscript $s$ and $p$ represent the neutron spin and momentum. It is noted that the bare coupling constant $g$ for the alpha-neutron interaction is related to the scattering length $a$ via the Lippmann-Schwinger equation in the low-energy limit:

$$g^{-1} = \frac{m_r}{2\pi \hbar^2 a} - \sum_p \frac{2m_r}{p^2},$$

where $m_r^{-1} = m^{-1} + M^{-1}$ is the reduced mass. We use the natural units in which $\hbar = c = 1$.

Now we implement a gauge transformation [17],

$$S(x) := e^{ix\hat{P}}, \quad \text{with} \quad \hat{P} = \sum_{s,p} p a^\dagger_{s,p} a_{s,p},$$

which sends a gas of neutrons to the comoving frame of the impurity alpha particle. By
using $S(x)a_{s,p}S^{-1}(x) = a_{s,p}e^{-iqx}$, the Hamiltonian (2) can be transformed to

$$SH(x)S^{-1} = \sum_{s,p} \frac{\hbar^2}{2m} a_{s,p}^\dagger a_{s,p} + \frac{1}{2} \sum_{q,q',p,s,t} a_{t,q'}^\dagger a_{s,q+p} a_{s,q} a_{t,q'} \tilde{V}_{s,t}(p)$$

$$+ \frac{\left(-i\nabla_x - \hat{P}\right)^2}{2M} a_{s,p}^\dagger a_{s,p} + \sum_{p,q,s} g_{s,p,q} a_{s,p}^\dagger a_{s,p},$$

which satisfies the commutation relation $[SH(x)S^{-1}, -i\nabla_x] = 0$, implying that after the transformation the momentum of the alpha particle represents the total momentum of the system. Therefore, we can replace the momentum operator by a $c$-number vector, i.e., $-i\nabla_x \rightarrow P$ in the transformed Hamiltonian,

$$SH(x)S^{-1} \rightarrow H_{\text{eff}}(P) = \sum_{s,p} \varepsilon_{s,p} a_{s,p}^\dagger a_{s,p} + \frac{(P - \hat{P})^2}{2M} + \sum_{p,q,s} g_{s,p,q} a_{s,p}^\dagger a_{s,p},$$

where we have furthermore dropped the $\tilde{V}_{s,t}$ by assuming that the neutron self-interaction effects are incorporated in the single-particle energy $\varepsilon_{s,p} = \frac{\hbar^2}{2m} + U$. Here, $U$ is the density dependent interaction energy per particle, which, e.g., can be deduced from a Thomas-Fermi approach [18] but is not relevant for the present study. We will hereafter use the above Hamiltonian (6) in the following calculations.

III. SINGLE PARTICLE-HOLE PAIR APPROXIMATION

To describe excitations accompanied by the impurity alpha particle, we implement a variational method in which the variational state is spanned by a single particle-hole ($p$-$h$) pair excitation near the neutron Fermi surface [19, 20]:

$$|\Psi\rangle = F_0|\psi_0\rangle + \sum_{k>p<s} F_{k,p,s} a_{s,k}^\dagger a_{s,p}|\psi_0\rangle,$$

where $|\psi_0\rangle$ is the state occupied by neutrons up to the Fermi momentum $k_F$, $k > (p <)$ denotes $|k| > k_F (|p| < k_F)$, and $F_0$ and $F_{k,p,s}$ are variational parameters. In fact, $F_{k,p,s}$ serves as the wave function of the $p$-$h$ pair of the corresponding momentum and spin. We remark in passing that even such a lowest-order form of the variational state can well reproduce the empirical energy and effective mass of an impurity that repulsively interacts with medium fermions in the vicinity of the unitarity limit, i.e., $|a| \rightarrow \infty$ and zero effective range [11].
The expectation value of the transformed Hamiltonian (6) with respect to the state \( \varepsilon_{s,q} \) gives

\[
\langle H_{\text{eff}}(P) \rangle = \sum_{s,q} \varepsilon_{s,q} \langle a_{s,q}^\dagger a_{s,q} \rangle + \frac{\langle (P - \hat{P})^2 \rangle}{2M} + g \sum_{q,q',s} \langle a_{s,q}^\dagger a_{s,q'} \rangle,
\]

where \( \langle \cdots \rangle = \langle \Psi | \cdots | \Psi \rangle \), and each term in the right side is given, respectively, by

\[
\sum_{q} \varepsilon_{s,q} \langle a_{s,q}^\dagger a_{s,q} \rangle = \sum_{q<} \left( |F_0|^2 + \sum_{k>,p<,t} |F_{k,p}^t|^2 \right) + \sum_{k>,p<} (\varepsilon_{s,k} - \varepsilon_{s,p}) |F_{k,p}^s|^2,
\]

\[
\frac{\langle (P - \hat{P})^2 \rangle}{2M} = P^2|F_0|^2 + \sum_{k>,p<,s} |F_{k,p}^s|^2 \left[ P^2 - 2P \cdot (k - p) + (k - p)^2 \right],
\]

and

\[
\sum_{q,q'} \langle a_{s,q}^\dagger a_{s,q'} \rangle = \sum_{q<} \left( |F_0|^2 + \sum_{k>,p<,t} |F_{k,p}^t|^2 \right) + \sum_{k>,p<} (F_0 F_{k,p}^{s*} + F_0^{s*} F_{k,p}^s)
+ \sum_{q>,q'>,p<} F_{q,p}^{s*} F_{q',p}^s - \sum_{k>,q<,q'<} F_{k,q}^{s*} F_{k,q'}^s.
\]

A. Quasiparticle energy

We impose the normalization condition by using a Lagrange multiplier \( \mu \) that turns out to be the ground state energy \( E_P \) of the system with momentum \( P \). In fact, \( E_P \) represents the alpha particle dispersion relation up to the Fermi energy of the medium neutrons. The variational condition \( \delta \langle H_{\text{eff}} - \mu \rangle = 0 \) leads to a set of eigenvalue equations,

\[
\frac{P^2}{2M} F_0 + \sum_{q<,s} g \left( F_0 + \sum_{k>} F_{k,q}^s \right) = \omega F_0,
\]

\[
\Omega_{k,p,P}^s F_{k,p}^s + g \left( F_0 + \sum_{q'>p<} F_{q',p}^s - \sum_{q'<} F_{k,q'}^s \right) = \omega F_{k,p}^s,
\]

where \( \omega = \mu - \sum_{q<,s} \varepsilon_{s,q} \), and

\[
\Omega_{k,p,P}^s := \varepsilon_{s,k} - \varepsilon_{s,p} + \frac{P^2 - 2P \cdot (k - p) + (k - p)^2}{2M} + g \sum_{q<}.
\]

* One can also obtain the same result from a time-dependent variational approach to the Dirac type effective action, i.e., \( \delta \langle \Psi(t)|i\partial_t - H_{\text{eff}}|\Psi(t) \rangle = 0 \), if one assumes \( |\Psi(t)\rangle \sim e^{-i\mu t} \). Note that this \( |\Psi(t)\rangle \) includes excited states, as well as the ground state.
From the Lippmann-Schwinger equation \[3\], the bare coupling constant \(g\) turns out to be vanishingly small. As in the renormalization procedure in terms of the scattering length \[19\], therefore, we will drop the terms \(g \sum_{q'} F_{k,q'}^s\) in Eq. \[13\] and \(g \sum_{q<} \Omega_{k,p}^s\).

Using the auxiliary field \(\chi^s_p = F_0 + \sum_{q'} F_{q',p}^s\) in Eqs. \[12\]–\[13\], we obtain the following equation:

\[
\omega = \frac{P^2}{2M} + \Sigma (\omega, P),
\]

with \(\Sigma (\omega, P) = \sum_{p<,s} \frac{m_r}{2\pi a_s} - \sum_k \frac{1}{(\omega - \Omega_{k,p}^s + \frac{2m_r}{k^2}) - \sum_{q<} \frac{2m_r}{q^2}}
\]

from which the eigenvalues of \(\omega\) can be determined. In this equation, \(\Sigma (\omega, P)\) can be interpreted as the self-energy obtained from the non-selfconsistent ladder approximation for the alpha-neutron scattering amplitude \[20\]. In the case of a repulsive interaction \(a > 0\) of interest here, the resulting positive energy state is characterized by the outgoing scattering amplitude. The real part of the corresponding quasiparticle energy \(E_P\) can thus be obtained from the spectral peak as

\[
E_P = \frac{P^2}{2M} + \text{Re} \Sigma (E_P + i0, P),
\]

where the analytic continuation to the upper half plane \(\omega \rightarrow \omega + i0\) has been made; it traces back to the (retarded) propagator of the alpha particle that undergoes multiple scattering.

### B. Quasiparticle residue, width, and effective mass

Validity of the quasiparticle picture for the alpha particle requires a finite residue and a relatively small width (long lifetime) compared with the real part of quasiparticle energy, both of which imply that the propagator of the alpha particle near its pole behaves as

\[
G^R(\omega, P) = \frac{1}{\omega + i0 - \frac{P^2}{2M} - \Sigma (\omega + i0, P)} \sim \frac{Z_P}{\omega - E_P + i\Gamma_P},
\]

where the width is given approximately by the imaginary part, \(\Gamma_P = -Z_P \text{Im} \Sigma (E_P + i0, P)\), and the quasiparticle residue is defined by

\[
Z_P = \left[ 1 - \text{Re} \left( \frac{\partial \Sigma (\omega + i0, P)}{\partial \omega} \right) \bigg|_{\omega=E_P} \right]^{-1}.
\]
The effective mass $M^*$, which characterizes the mobility of the alpha particle in the medium, is defined in the momentum expansion around the pole as

$$M^* := \left( \frac{d^2 E_P}{dP^2} \bigg|_{P=0} \right)^{-1} = \frac{M}{Z} \left[ 1 + M \frac{\partial^2 \text{Re} \Sigma(\omega + i0, P)}{\partial P^2} \bigg|_{\omega=E, P=0} \right]^{-1},$$

(20)

where $E = E_{P=0}$ and $Z = Z_{P=0}$. As will be shown by numerical results below, the quasiparticle picture is indeed valid for the parameter region that corresponds to the low density and large isospin asymmetric situation considered in this study. The quasiparticle dispersion of the alpha particle can thus be well approximated by

$$E_P \simeq E + \frac{P^2}{2M^*}$$

(21)

near the low energy limit.

IV. NUMERICAL RESULTS AND DISCUSSION

Before exhibiting numerical results for the quantities described in the previous section, we give typical reference values of the scattering length and the neutron density as

$$a_{\text{ref}} = 2.64 \text{ fm},$$

$$\rho_{\text{ref}} = 0.01 \rho_0, \quad \rightarrow \quad k_{F\text{ref}} = 0.36 \text{ fm}^{-1},$$

(22)

(23)

where $\rho_0 = 0.16 \text{ fm}^{-3}$ is the normal nuclear density. These values lead to the dimensionless coupling parameter $a_{\text{ref}}k_{F\text{ref}} = 0.95 \simeq 1$, which marginally correspond to the strong coupling regime. The value of $\rho_{\text{ref}}$ is supposed to be of order the typical density above which neutrinos are trapped in supernova cores [1], but lower than the typical density in the neutron skin of heavy nuclei [21]. The value of the scattering length has been determined from the alpha-neutron potential [22] that reproduces the experimental phase shift for low-energy alpha-neutron scattering cross section. In the course of this determination, the effective range has been simultaneously obtained as $r_0 = 1.43 \text{ fm}$. This value gives $r_0k_{F\text{ref}} = 0.51 < 1$.

† Unlike unitary cold atoms there is no bound state between an alpha particle and a neutron. In fact, a resonant $^5\text{He}(3/2^-$), if appearing, would be very unstable to $p$-wave dissociation into an alpha particle and a neutron, while in the $s$-wave channel there is no positive scattering length state that can be obtained continuously from the negative scattering length state (attractive branch). In the absence of decay to the attractive branch, therefore, the present positive energy state is not an excited one.
which is not so small but still implies the validity of the present low-energy treatment of the alpha-neutron interaction in terms of the zero-range potential characterized by the scattering length (3) alone. We remark that the values of $a_{\text{ref}}$ and $r_0$ obtained here are consistent with the earlier results [23]. Finally, we set the bare mass ratio of the alpha particle as

$$\frac{M}{m} = 4.$$

(24)

For these reference values of the parameters given above, the numerical results for quasiparticle properties of the alpha particle at $P = 0$ are given as follows:

$$E_{\text{ref}} = 0.467 \varepsilon_{F,\text{ref}} = 1.26 \text{ MeV},$$

(25)

$$Z_{\text{ref}} = 0.650,$$

(26)

$$M^*_{\text{ref}} = 1.217 M,$$

(27)

and $\Gamma_{\text{ref}} = 0.032 \varepsilon_{F,\text{ref}} = 0.086 \text{ MeV},$

(28)

where $\varepsilon_{F,\text{ref}} = \frac{(\hbar c)^2 k_{F,\text{ref}}^2}{2mc^2} = 2.69 \text{ MeV}$. The interaction energy of the alpha particle at rest, $E_{\text{ref}}$, is positive, a feature that reflects the positive scattering length corresponding to repulsion, and is much larger than the decay width. As can be interpreted from the diagrammatic representation of the present self-energy, the decay width comes solely from the process in which the alpha particle, a quasiparticle dressed by a cloud of $p$-$h$ bubbles, decays into a bare alpha particle and a neutron in the $s$-wave scattering state. In addition, the quasiparticle residue obtained here is fairly close to unity. All these results support the quasiparticle picture presumed in this study. The effective mass increases by about 20% from its vacuum value. Such increase seems like a general feature of various impurity-medium combinations, irrespective of whether the impurity-medium interaction is repulsive or attractive. While the scattering length (22) employed here is an empirical one, for more general discussion, we extend our calculations and present in Fig. 1 the results as functions of the dimensionless coupling. For weak coupling the quasiparticle picture works obviously. It is also found from the figure that while even in the strong coupling regime $1/ak_F \ll 1$, the relation $E \gg \Gamma$ still holds, in such a regime the quasiparticle is less identifiable due to the smallness of the residue.

Now we show in Fig. 2 the full dispersion relation $E_P$ obtained from Eq. (17) with the reference values of the parameters, together with the approximate one (21) expressed by the effective mass. We observe that the appreciable $P$ dependence of the self-energy $\text{Re} \Sigma$, also
FIG. 1: Energy, residue, and mass ratio of an alpha particle calculated as functions of $1/ak_F$ at $P = 0$.

FIG. 2: Dispersion relations of an alpha particle calculated in units of $\varepsilon_F$ at $ak_F = 1.0 \approx a_{ref}k_{F_{ref}}$. Plotted in the figure, accounts for the deviation between the full and the approximate dispersion relations for finite $P$. The deviation is nevertheless small enough that the approximate one seems to work for a relatively wide range of the momentum.

Finally we show how the quasiparticle properties depend on the neutron density. Since the increase in the density extends the phase space available for the $p$-$h$ fluctuations, the interaction energy and the decay width increase with density, as can be seen in Fig.3. On the
FIG. 3: Neutron density dependence of the energy, residue, effective mass, and width of an alpha particle calculated at $a k_F = 1.0$.

other hand, the relation $E \gg \Gamma$ is kept, and eventually at $\rho/\rho_{ref} = 3$, the kinetic energy of the background neutron gas reaches $\varepsilon_F = \varepsilon_{Fref} \left( \frac{\rho}{\rho_{ref}} \right)^{2/3} = 5.6$ MeV, which is comparable with the binding energy of the alpha particle per nucleon $\sim 7$ MeV and thus makes the stability of the alpha particle itself doubtful. Moreover, at this density, the dimensionless effective range becomes $r_0 k_F = 0.8$, which is so close to unity that the present low-energy treatment is doubtful. Thus, the quasiparticle picture looks even worse at still higher densities; at $\rho = 6 \rho_{ref}$, $r_0 k_F = 1$ is reached.

V. SUMMARY AND OUTLOOK

In this study we elucidate the quasiparticle properties of a single alpha particle immersed in a cold dilute neutron gas by evaluating the self-energy from the variational treatment equivalent to a non self-consistent ladder approximation that incorporates an empirical value of the alpha-neutron low-energy s-wave scattering length $[22]$. Our result shows that adding a single alpha particle into a dilute neutron gas costs at least the interaction energy $[25]$ that we calculated for the alpha particle at rest. Note that such interaction energy reads $E = 0.467 \varepsilon_F = \frac{2.23}{m} \rho^{2/3}$. This could be useful to deduce the fraction of alpha particles in supernova matter. It is also interesting to note that the effective mass calculated with
the same parameter set leads to an approximate dispersion relation of the alpha particle, 
\[ E_p = E + P^2/2M^* \], with the density-dependent interaction energy \( E \) given above. Applying 
the dispersion relation to the Bose distribution for a dilute alpha gas in neutron matter, we 
can estimate the transition temperature for possible Bose-Einstein condensation \[27\] to be 
\[ T_{\text{bec}} = \frac{2\pi\hbar^2}{M^*} \left( \frac{\rho_\alpha}{c(3)} \right)^{2/3} \text{MeV} \], which gives \( T_{\text{bec}} = 0.14 \text{ MeV at } \rho_\alpha = \rho_{\text{ref}}/10 \). This might have some relevance to alpha clustering in the surface of heavy nuclei.

In the present work the alpha particle in neutron matter is treated as an inert point particle, i.e., the inner structure is neglected from the low-energy point of view. To go beyond it, the alpha particle has to be regarded as a cluster of four nucleons. Then, diagrammatically needed is to take into account neutron exchange in and out of the cluster in scattering processes as well as deuteron-like pair correlations, which might eventually break up the alpha particle at sufficiently high densities \[24,26\]. For elaborate studies to clarify the stability of such alpha clustering in neutron matter, the wave function of the system is needed in terms of interacting nucleons with phenomenological potentials \[18\].

As discussed in the previous section, the present study is restricted to the dilute limit of the alpha density and to the cold, isospin asymmetric limit of the nuclear medium. Next it is interesting to consider the system of two alpha particles in neutron matter. The interaction between the alpha particles, which resembles that between two repulsive Fermi polarons in the context of ultracold atoms except for the Coulomb repulsion, has some possible relevance to neutrino scattering off light clusters in supernova matter. To make better estimates, it would be necessary to raise both the proton fraction and temperature in the nuclear medium and also consider screening corrections to the alpha-alpha Coulomb repulsion \[28\].

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Appendix A: Dimensionless expression for the self-energy

It is convenient to rewrite the self-energy in terms of dimensionless quantities (symbols with tilde) as well as $k_F$ and $\varepsilon_F$ as

$$\Sigma(\omega + i0, P) = \sum_{p<} \frac{1}{g_r^{-1} - G(\omega + i0, P) - \sum_{q<} \frac{2m_r}{q^2}}$$

$$= (2\pi)^2 \varepsilon_F \sum_{p<} \frac{\tilde{g}^{-1} - \text{Re}\tilde{G} + i\text{Im}\tilde{G}}{\left(\tilde{g}^{-1} - \text{Re}\tilde{G}\right)^2 + \text{Im}^2\tilde{G}},$$

(A1)

with

$$G(\omega, P; p) \equiv \sum_{k>} \left\{\left[\omega - \varepsilon_k + \varepsilon_p - \frac{P^2 - 2P \cdot (k - p) + (k - p)^2}{2M}\right]^{-1} + \left[\frac{k^2}{2m_r}\right]^{-1}\right\}.$$  

(A2)

Here we have factored out the dimensional coefficients as $G = \frac{k_F^2}{(2\pi)^2 \varepsilon_F} \tilde{G}$ and $g_r^{-1} = \frac{k_F^2}{(2\pi)^2 \varepsilon_F} \tilde{g}^{-1}$, and expressed the rest parts as

$$\tilde{G}(\omega, P; p - P) = \int_{1}^{\infty} \int_{-1}^{1} d\kappa d^2\rho \quad \text{d}x \quad \left\{\mathcal{E} - \kappa^2 + \rho^2 \cdot \left[\frac{\kappa^2 + \rho^2 - 2\kappa\rho x}{R}\right] + i0\right\}^{-1} + \left[\kappa^2(R^{-1} + 1)\right]^{-1},$$

(A3)

and

$$\tilde{g}^{-1} = \frac{2\pi^2}{R + 1} \left(\frac{1}{2\pi a k_F} - \sum_{\rho<} \frac{2}{\rho^2}\right),$$

(A4)

where $\bar{\rho} \equiv \sqrt{(p - P)^2}/k_F$, and we have introduced variables: $\mathcal{E} = \omega/\varepsilon_F$, $\mathcal{P} = P/k_F$, $R = M/m$, $|p|/k_F = \rho$, and $|k|/k_F = \kappa$, and shifted the momentum temporarily, $p \to p - P,$
for convenience. Performing the $x$ integration in Eq. (A3) first, we obtain
\[
\int_{-1}^{1} dx \left[ \mathcal{E} - \frac{R\kappa^2 - 2\rho^2 + \kappa^2 + \rho^2 - 2\kappa \rho x}{R} + i0 \right]^{-1}
\]
\[
= \frac{R}{2\kappa \rho} \left[ \ln \left| \frac{(\kappa - \kappa_-)(\kappa - \kappa_+)}{(\kappa + \kappa_-)(\kappa + \kappa_+)} \right| - i\pi \theta \left( 1 - \frac{R\mathcal{E} - R\kappa^2 + R\rho^2 - \kappa^2 - \rho^2}{2\kappa \rho} \right) \right], \quad (A5)
\]
where \( \theta(x) \) is the Heaviside function, and
\[
\kappa_{\pm} = \frac{\rho}{R + 1} \pm \frac{1}{R + 1} \sqrt{R(R + 1)\rho^2 - R\rho^2 + \mathcal{E}R(R + 1)}. \quad (A6)
\]
Note that \( \kappa_+ \geq 0 \) and \( \kappa_- \leq 0 \) for \( R > 1 \) and \( \mathcal{E} > 0 \). Then, the real and imaginary parts read respectively as
\[
\text{Re} \tilde{G}(\omega, P; p - P) = \int_{1}^{\infty} d\kappa \left\{ \frac{R\kappa}{2\rho} \ln \left| \frac{(\kappa - \kappa_-)(\kappa - \kappa_+)}{(\kappa + \kappa_-)(\kappa + \kappa_+)} \right| + 2 \frac{R}{R + 1} \right\}
\]
\[
= \frac{R}{4\rho} \left[ (1 - \kappa_-^2) \ln \left| \frac{1 + \kappa_+}{1 - \kappa_+} \right| + (1 - \kappa_+^2) \ln \left| \frac{1 + \kappa_-}{1 - \kappa_-} \right| \right] - \frac{R}{R + 1} \quad (A7)
\]
and
\[
\text{Im} \tilde{G}(\omega, P; p - P) = -\pi \int_{1}^{\infty} d\kappa \frac{R\kappa}{2\rho} \theta \left( 1 - \frac{R\mathcal{E} - R\kappa^2 + R\rho^2 - \kappa^2 - \rho^2}{2\kappa \rho} \right)
\]
\[
= -\frac{R\pi}{4\rho} \theta (\kappa_+ - 1) \left[ \kappa_+^2 - \kappa_-^2 \theta (-\kappa_- - 1) - \theta (1 + \kappa_-) \right]. \quad (A8)
\]
The $p$ integration in (A1) can be done numerically after shifting back $p - P \rightarrow p$. 

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