Bi-orthogonal harmonics for the decomposition of gravitational radiation II: applications for extreme and comparable mass-ratio black hole binaries

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The estimation of a physical system’s normal modes is a fundamental problem in physics. The quasi-normal modes of perturbed Kerr black holes, with their related spheroidal harmonics, are key examples, and have diverse applications in gravitational wave theory and data analysis. Recently, it has been shown that adjoint-spheroidal harmonics and the related spheroidal multipole moments may be used to estimate the radiative modes of arbitrary sources. In this paper, we investigate whether spheroidal multipole moments, relative to their spherical harmonic counterparts, better approximate the underlying modes of binary black hole spacetimes. We begin with a brief introduction to adjoint-spheroidal harmonics. We then detail a rudimentary kind of spheroidal harmonic decomposition, as well as its generalization which simultaneously estimates pro- and retrograde moments. Example applications to numerical waveforms from comparable and extreme mass-ratio binary black hole coalescences are provided. We discuss the morphology of related spheroidal moments during inspiral, merger, and ringdown. We conclude by discussing potential applications in gravitational wave theory and signal modeling.

Introduction — Each new binary black hole (BBH) detection provides new opportunities to compare the predictions of General Relativity (GR) with astrophysical data [1–3]. For current and future gravitational wave detectors such as LIGO, KAGRA, Virgo and LISA, such comparisons are essential to the objectives of gravitational wave science, from signal detection, to the estimation of source parameters, tests of GR, and astrophysical interpretation [4]. In practice, each scientific objective is underpinned by our ability to accurately and efficiently represent the multipole moments of gravitational radiation [5, 6]. While there is no unique definition of a BBH system’s radiative multipole moments, some definitions result in moments that closely resemble the system’s natural modes, potentially simplifying signal representation, modeling and analysis [7–11].

A motivation of Ref. [7] (hereafter “Paper I”) was to review, and perhaps improve upon, a common method for defining gravitational wave multipole moments: decomposition with spin weighted $−2$ spherical harmonics. While the usefulness of the spin weighted spherical harmonics is derived from their completeness (i.e. their ability to exactly represent arbitrary gravitational wave signals), and their orthogonality (meaning their multipole moments may be computed by projection, rather than e.g. fitting), their primary deficit is well known [9, 11–13]. Spherical harmonics are closely related to the natural modes of gravitational wave sources that have zero angular momentum. However, it is unclear if nature provides a mechanism to construct zero angular momentum objects and, to date, they have not been observed [1, 2, 14].

For this reason, Paper I considered the simplest generalization of spherical harmonics to spacetimes with angular momentum, namely, the spin weighted spheroidal harmonics. There, it was shown that, despite solving a non-classical1 and non-hermitian eigen-problem, spheroidal harmonics are capable of exactly representing arbitrary gravitational wave signals. It was also shown that spheroidal harmonics possess a non-trivial form of bi-orthogonality: rather than being orthogonal with themselves, a new sequence of special functions is required. In Paper I, these functions are referred to as adjoint-spheroidal harmonics because they are closely related to the eigenfunctions of the spheroidal differential equation’s formal adjoint.

For each spheroidal harmonic, there exists exactly one adjoint-spheroidal harmonic. In the absence of angular momentum, the adjoint-spheroidal harmonics reduce to spherical harmonics, meaning that they can be efficiently and non-perturbatively computed using invertible linear operators. A central result of Paper I is that spheroidal harmonics’ use in defining gravitational wave multipole moments causes the nullification of dominant nonphysical mode-mixing effects, resulting in a necessarily simpler representation of each moment’s amplitude and phase.

In this Letter, we survey potential applications of spheroidal harmonic decomposition to gravitational wave theory. We begin with a brief review of the adjoint-spheroidal harmonics, followed by a description of how a standard change-of-basis approach allows existing sets of spherical harmonic moments to be directly converted into spheroidal ones. Here, we treat the angular momentum of the system in a very crude way: for BBHs we define the spheroidal harmonics using the angular momentum of the remnant black hole (BH) [11, 16]. This simple framing allows us to also consider a powerful generalization of spheroidal decomposition that simultaneously estimates pro- and retrograde mode content [17]. We then consider two numerical applications: first, the spheroidal harmonic decomposition of gravitational waves from a non-precessing extreme mass-ratio BBH merger, and second, estimation of pro- and retrograde moments from a precessing comparable mass-ratio BBH merger. We conclude with a brief discussion of possible future directions.

Preliminaries — Gravitational wave theory is centrally interested in describing physical observables. Chief among them are the gravitational wave polarizations, $h_{+}$ and $h_{\times}$ [18]. It is common to consider both polarizations simultaneously by

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2 By “non-classical”, we mean that the spheroidal harmonics relevant for e.g. Kerr black holes correspond to not one, but a countably infinite set of differential systems [7]. This trait is shared by non-classical polynomials [15].
defining a complex valued strain, \( h \), as
\[
h = h_+ - i h_\times .
\] (1)

In Paper I, it was shown that gravitational wave polarizations from an arbitrary signal within GR may be exactly equated with its spheroidal harmonic decomposition,
\[
h(r,t,\theta,\phi) = \frac{1}{r} \sum_{\ell,m} h^S_{\ell m}(t) Y_{\ell m}(\theta,\phi) e^{i m \phi} ,
\] (2)
where \( h^S_{\ell m} \) are the spheroidal multipole moments, \( r \) is the gravitational wave’s luminosity distance, \( t \) is the observer’s time coordinate, \( \theta \) and \( \phi \) are the usual spherical polar and azimuthal angles, defined in a source centered frame where the total angular momentum is along the \( z \)-direction, and \( -2S_{\ell m} \) are spin-weighted spheroidal harmonics. In Eq. (2), \( \ell \) and \( m \) are the usual spherical harmonic indices, and \( \gamma_{\ell m} \) is an oblateness parameter [19]. If the source has mass \( M \), and angular momentum \( \vec{J} \), then the oblateness parameter is \( \gamma_{\ell m} = a\omega_{\ell m} \), where \( a = |\vec{J}|/M \), and \( \omega_{\ell m} \) is an \( M \) and \( a \) dependent Quasi-Normal Mode (QNM) frequency of the spacetime [20–22]. In what follows, we will focus on Kerr BHs, for which \( \omega_{\ell m} \) may be defined in terms of the fundamental prograde QNM frequencies\(^2\). \( \gamma_{\ell m} = a\omega_{\ell m} \).

The choice of oblateness values, \( \gamma_{\ell m} \), fixes the decomposition space according to \( -2S_{\ell m}(\theta,\gamma_{\ell m}) \). In turn, this determines the adjoint-spherical functions if: \( \mathcal{T} \) is the linear invertible operator that transforms spherical harmonics into spheroidal harmonics, and we represent \( -2S_{\ell m} \) and \( S_{\ell m} \) by their respective kets (\( |Y_{\ell m}\rangle \) and \(|S_{\ell m}\rangle \)), then it can be shown that
\[
\mathcal{T} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} |Y_{\ell m}\rangle \langle S_{\ell m}| .
\] (3)
In Eq. (3), inner products are integrals over the usual solid angle. Equation (3) may be used to non-perturbatively define the adjoint spherical harmonics in terms to their spherical harmonic expansion,
\[
|S_{\ell m}\rangle = \mathcal{T}^{-1} |Y_{\ell m}\rangle .
\] (4)
In Eq. (4), \( \dagger \) denotes adjugation. Given Eqs. (3-4), \( -2S_{\ell m} \) may be computed to machine precision via a finite-dimensional truncation of \( \mathcal{T} \) [7]. Given \( -2S_{\ell m} \), the spheroidal harmonic multipole moment of Eq. (2) is defined via projection,
\[
\hat{h}^S_{\ell m} = \langle S_{\ell m} | h \rangle = \int_0^{2\pi} \int_0^\pi -2S_{\ell m}(\theta,\gamma_{\ell m}) e^{-im\phi} \times h(r,t,\theta,\phi) \sin(\theta) d\theta d\phi .
\] (5)
To gain intuition about the adjoint-spherical harmonics, it is illustrative to consider their perturbative expansion to linear order in \( a \). Directly applying the usual spheroidal harmonic perturbative expansion
\[
-2S_{\ell m} \approx Y_{\ell m} + a\omega_{\ell m} c^{\ell-1}_{\ell m} Y_{\ell-1,m} + a\omega_{\ell m} c^{\ell+1}_{\ell m} Y_{\ell+1,m} .
\] (6)
to Eqs. (3-4) yields,
\[
-2\hat{S}_{\ell m} \approx Y_{\ell m} - a\omega^\ell_{\ell-1,m} c^{\ell-1}_{\ell-1,m} Y_{\ell-1,m} - a\omega^\ell_{\ell+1,m} c^{\ell+1}_{\ell+1,m} Y_{\ell+1,m} .
\] (7)
In Eqs. (6) and (7), \( c^{\ell \pm 1}_{\ell m} \) are positive constants defined in Eqs. C13 and C14 of Ref. [25], and \( * \) denotes complex conjugation.

Comparing Eq. (7) to Eq. (6) reveals that unlike the spheroidal harmonics, the adjoint-spheroidal harmonics depend on not one, but multiple QNM frequencies. This reflects the fact that the each spheroidal harmonic with oblateness \( \gamma_{\ell m} \) is the eigenfunction of a differential operator, \( \mathcal{L}(\gamma_{\ell m}) \), where
\[
\mathcal{L}(\gamma_{\ell m}) = u^2 y^{\ell 2}_{\ell m} - 2 s u y_{\ell m} - \frac{(m + s u)^2}{1 - u^2} + \partial_u (1 - u^2) \partial_u .
\] (8)
In Eq. (8), \( s = -2 \) and \( u = \cos(\theta) \).

As there are countably infinite QNM oblatenesses, \( \gamma_{\ell m} \), there are countably infinite operators \( \mathcal{L}(\gamma_{\ell m}) \) that must be taken into account when considering the orthogonality properties of the spheroidal harmonics. This concept underpins Eq. (3), and a key consequence of Eqs. (3-4) is that \( -2\hat{S}_{\ell m} \) and \( -2\hat{S}_{\ell m} \) are bi-orthogonal, meaning
\[
\langle S_{\ell m} | S_{\ell' m'} \rangle = \delta_{\ell\ell'} / 2\pi .
\] (9)
Spheroidal decomposition via change of basis – It is common for the output of numerical simulations of BBH mergers to store gravitational wave data using spherical harmonic multipole moments, \( \langle Y_{\ell m} | h \rangle \). Thus, it is useful to note that, if given a set of spherical harmonic multipole moments, then Eq. (5) need not be evaluated in order to compute the spheroidal ones. Instead, one may use Eqs. (2-4) to show that the matrix, \( \hat{T} \), whose elements are, \( \langle Y_{\ell m} | S_{\ell' m'} \rangle \), transforms vectors of spherical harmonic, \( \hat{h}^S_{\ell m} \), multipole moments into spheroidal ones, \( \hat{h}^Y_{\ell m} \),
\[
\hat{h}^Y_{\ell m} = \hat{T} \hat{h}^S_{\ell m} .
\] (10)
In Eq. (10), \( \hat{h}^Y_{\ell m} \) has elements \( \langle Y_{\ell m} | h \rangle \) where \( m \) is fixed (due to the orthogonality of the complex exponentials in Eq. (2)) and so only \( \ell \) varies. Similarly, \( \hat{h}^S_{\ell m} \) has elements \( \langle S_{\ell m} | h \rangle \). Thus the spheroidal harmonic multipole moments may be estimated via matrix inverse,
\[
\hat{h}^S_{\ell m} = \hat{T}^{-1} \hat{h}^Y_{\ell m} .
\] (11)
Note that for the oblateness values considered here, \( \hat{T} \) may be computed semi-analytically [7, 20, 25, 26].

Equations (5) and (11) may be used to estimate the spheroidal harmonic moments of general gravitational wave signals. As described in Paper I, when applied to non-precessing BBH systems, \( \hat{h}^S_{\ell m} \) will have negligible mixing of different spheroidal modes during post-merger. In that setting, \( \hat{h}^S_{\ell m} \) would, in principle, contain information about multiple overtone modes. For precessing systems, \( \hat{h}^S_{\ell m} \) will also contain information about different pro- and retrograde multipole moments, \( \hat{h}^{S3}_{\ell m} \) and \( \hat{h}^{S3}_{\ell m} \) respectively. During ringdown,
Figure 1. Example of spheroidal harmonic decomposition: radiative multipole moments from numerical binary black hole with $m_1/m_2 = 10000$, and a spin of $a_s = 0.9$ on the larger body. Spherical moments are defined by $\langle S_{\ell m} \rangle$ (see Eq. 5), and spherical ones by $\langle Y_{\ell m} \rangle$. Top panels compare multipole amplitudes. The perturbative predictions (thin dashed blue) are fits to the numerical data, where the long-linear slope is determined by single BH perturbation theory (i.e. the dominant quasi-normal mode). The bottom panels compare multipole phase derivatives, $\omega_{\ell m} = \partial t/\partial t h_{\ell m}$. Spheroidal harmonic multipole moments (black) are shown along side spherical harmonic moments (dashed grey), for the $(\ell, m) = (3, 2)$ (left) and $(\ell, m) = (4, 3)$ (right) moments. Here, $\omega_{\text{QNM}}$ is determined only by BH perturbation theory. Late time agreement between perturbative predictions and spheroidal moments signals that the correct physical modes are being estimated.

these moments correspond to perturbations that are pro- and retrograde with respect to the BH horizon frequency.

A simple generalization of Eq. (11) allows $h_{\ell m}^S$ and $h_{\ell m}^S$ to be estimated simultaneously [17]. However, this expansion of utility requires two additional assumptions that, unlike Eq. (11), result in a form of least squares fitting that is not underpinned by the adjoint-spheroidal harmonics. The first assumption is that each spheroidal moment is well modeled by

$$h_{\ell m}^S = A_{\ell m} \exp(i\omega_{\ell m} t) \quad \text{and} \quad h_{\ell m}^S = A_{\ell m}' \exp(i\omega'_{\ell m} t).$$

(12)

In Eq. (12), $A_{\ell m}$ and $A_{\ell m}'$ are complex valued, time dependent amplitudes, and $\omega_{\ell m}$ and $\omega'_{\ell m}$ are complex valued pro- and retrograde QNM frequencies. The second assumption is that $A_{\ell m}(t)$ always evolves slowly relative to $i\omega_{\ell m} t$, such that it may be treated as constant. Each assumption is explicitly compatible with late QNM ringdown where one mode dominates. Outside of ringdown, one may always find some $A_{\ell m}(t)$ such that the first assumption holds. The second assumption may, in general, break down due to e.g. very strong precession; however, departing from Ref. [17], we note that the assumption is valid when the implicitly rotating source picture of BBH merger holds [27]. Therefore the following generalization of Eq. (11) is particularly relevant for BH ringdown, but also expected to provide non-pathological results for most BBH configurations. At the time of publication, what follows is to our knowledge the only known method for estimating pro- and retrograde multipole moments from inspiral and merger.

The key idea is that to estimate both $h_{\ell m}^S$ and $h_{\ell m}^S$, one requires twice as much information as needed in spheroidal decomposition (Eq. 11). One robust way of obtaining this information is to consider $h_{\ell m}^S$, along with its first time derivative, $\partial_t h_{\ell m}^S$. With Eqs. (2) and (11) in mind, we must also explicitly consider the derivatives of each spheroidal moment. If we choose to treat the combined lists of $h_{\ell m}^S$ and $h_{\ell m}^S$ as we did $h_{\ell m}^S$ in Eqs. (2) and (11), then the resulting algebraic system has the following schematic form,

$$\begin{align*}
\begin{pmatrix}
\frac{\partial h_{\ell m}^S}{\partial t}
\end{pmatrix}
&= \begin{pmatrix}
(Y_{\ell m}^S | S_{\ell m}^S)
\end{pmatrix}
\begin{pmatrix}
\omega_{\ell m}^S
\end{pmatrix}
\begin{pmatrix}
(Y_{\ell m}^S | S_{\ell m}^S)
\end{pmatrix}
\begin{pmatrix}
S_{\ell m}^S
\end{pmatrix}
\begin{pmatrix}
\omega_{\ell m}^S
\end{pmatrix}
\begin{pmatrix}
\omega_{\ell m}^S
\end{pmatrix}
\end{align*}
$$

(13)

In Eq. (13), each cell should be understood as the element of a vector or matrix whose rows and/or columns span values of $\ell$ (and/or $\ell'$). Each grouping of cells should be understood as vector and/or matrix concatenation, and each under-brace denotes the symbol we will use to represent each vector or matrix quantity. In the right-hand-side of Eq. (13), $S_{\ell m}^S$ are the spin weighted spheroidal harmonics defined with retrograde QNM frequencies.

Equation (13) is the direct generalization of Eq. (10). We note that Eq. (13) does not correspond to decomposition because, e.g., unlike Eq. (10), in the limit of zero spacetime angular momentum, $\hat{U}$ does not reduce to the identity. While in Eq. (10), we wish to solve for $\hat{S}$, in Eq. (13) we wish to solve for $\hat{S}$. Just as in Eq. (11), $\hat{S}$ (i.e. $h_{\ell m}^S$ and $h_{\ell m}^S$) may be determined by matrix inversion,

$$\hat{S} = \hat{U}^{-1} \hat{Y}.$$

(14)

In this way, Eqs. (12-14) allow the simultaneous estimation of pro- and retrograde moments through a kind of matrix least-
squares fitting. Note that neither Eq. (11) nor Eq. (14) makes any assumption about the time domain behavior of spheroidal moments.

Example application to extreme mass-ratio non-precessing binary black hole system – In Fig. 1, we show example spheroidal moments that result from applying Eq. (11) to a fiducial extreme mass-ratio binary black hole coalescence ($G = c = M = 1$) [28]. The primary BH of this system has a dimensionless spin of $a/M = 0.9$, and the system’s effective mass-ratio is 10,000 : 1. The secondary BH is non-spinning, and follows an equatorial quasi-circular orbit. The purely prograde inspiral of this system means that only the prograde moment is excited.

Example application to comparable mass-ratio precessing binary black hole system – In Fig. 2, we show example pro- and retrograde spheroidal moments that result from applying Eq. (14) to the dominant quadrupole emission of a precessing BBH coalescence [29]. The primary BH of this system has a dimensionless spin of $a/M = 0.8$, and the system’s mass-ratio is 8 : 1. The secondary BH is non-spinning, and follows an initially quasi-circular orbit. The angle between the initial binary’s orbital and spin angular momentum is $120^\circ$. The non-zero value of this angle means that, e.g., the system’s instantaneous orbital angular momentum precesses about $J_\ell$, which itself varies slowly in direction and magnitude. For consistency with the development of Eq. (14), this system’s spherical multipole moments were transformed into a frame where $\hat{J}(t)$ always points along the $z$-axis [29–31]. The low-level data product for this simulation is the Weyl scalar, $\psi_4 = \partial_\ell^2 h$ [30, 32]. In Fig. 2, we focus on this quantity to minimize the effect of data processing choices, and to more clearly display the pro- and retrograde spheroidal moments. While results for this case are representative of many recent simulations of BBH mergers, it was found that some older simulations (and multipole moments with $\ell > 2$) are of insufficient accuracy to yield spheroidal moments of the quality shown here.

Discussion – An ongoing challenge for gravitational wave astronomy is to model gravitational wave signals (numerically and analytically) in a way that is both precise and, ideally, explicitly encodes the physical principles particular to the astrophysical source and chosen theory of gravity. Here, we have introduced and provided examples for two tools that may facilitate these goals: (i) spheroidal harmonic decomposition, and (ii) the matrix least-squares estimation of pro- and retrograde spheroidal multipole moments.

Like spin-weighted spherical harmonics, the spin-weighted spheroidal harmonics may be used in both numerical and analytic relativity to represent gravitational waves from arbitrary sources. Unlike spherical harmonics, the spheroidal harmonics are closely related to the modes of stationary spacetimes with angular momentum. The techniques used to derive the adjoint-spheroidal harmonics may be applied to any scenario (within one theory, or across alternative theories of gravity) in which the field equations are separable [7, 33, 34].

As seen in Fig. 1, spheroidal harmonic decomposition may find use in representing gravitational waves from non-precessing BBH systems: For the non-quadrupolar moments, the decomposition results in (a) post-merger ($t/M > 0$) waveforms that are naturally consistent with BH perturbation theory, (b) merger waveforms (near $t/M = 0$) that are single-peaked, reminiscent of the quadrupole moments, and (c) late inspiral ($t/M < 0$) waveforms that appear to be qualitatively different from their spherical counterparts. Analytic and numerical analyses of each difference, (a)-(c), may be of future interest.

Like spherical harmonics, the spin weighted spheroidal harmonics cannot explicitly determine pro- and retrograde multipole moments. For this, versions of the matrix least-squares method encapsulated by Eq. (14) may find use in the analysis of simulated gravitational waves from precessing BBH systems. In Fig. 2 we see for the first time pro- and retrograde contributions to the multipole asymmetry in comparable mass-ratio gravitational wave data, implying that this generalization of spheroidal decomposition may find use in, e.g., gravitational wave signal modeling, where precession’s imprints on radiative multipole moments are still being refined for use in current and future detection scenarios [29].

Note that like spheroidal decomposition, this fitting method makes no strong assumption about the time domain behavior of gravitational wave signals; instead, it is assumed that the different multipole moments primarily depend on the inner-products within $U$ (Eq. 13). Thus agreement with expected time domain behavior evidences consistency between the different spheroidal moments that are encoded within each spherical one. This consistency is related to the known inner-product ratio test for BH ringdown [9, 12]. A matter of ongoing investigation is whether Eq. (14) may be further extended to self-consistently estimate the presence of BH overtones (see e.g. Refs. [9, 20, 21, 35]).

Finally, we note that the results presented here imply at least two salient questions for future work: (Q1) Here we have used a very rudimentary form of spheroidal decomposition were the spacetime angular momentum is held fixed – is it possible to generalize this to account for non-stationary angular mo-
momentum? And lastly, (Q2) given that the adjoint-spheroidal harmonics may be motivated using the angular Teukolsky equation (See Eq. 8 and e.g. Ref. [36]), do similar arguments apply to Teukolsky’s radial equation, given that they are both generalized spheroidal (Heun-type) equations [26, 37]?

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