DIFFERENTIAL CALCULUS ON THE QUANTUM SPHERE AND DEFORMED SELF-DUALITY EQUATION

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Abstract

We discuss the left-covariant 3-dimensional differential calculus on the quantum sphere $SU_q(2)/U(1)$. The $SU_q(2)$-spinor harmonics are treated as coordinates of the quantum sphere. We consider the gauge theory for the quantum group $SU_q(2) \times U(1)$ on the deformed Euclidean space $E_q(4)$. A $q$-generalization of the harmonic-gauge-field formalism is suggested. This formalism is applied for the harmonic (twistor) interpretation of the quantum-group self-duality equation (QGSDE). We consider the zero-curvature representation and the general construction of QGSDE-solutions in terms of the analytic prepotential.

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1 Introduction

The 2-dimensional sphere $S^2$ is the simplest example of homogeneous space and can be treated as $SU(2)/U(1)$ coset space. $S^2$ plays an important role in the twistor program of Penrose [1] and, particularly, in the twistor interpretation of self-duality equation [2]-[4]. The harmonic approach [3],[4] is a specific version of the twistor formalism based on using the spinor harmonics as coordinates on $S^2$.

In the present talk, we make an attempt to construct a $q$-deformed harmonic formalism in the framework of the quantum-group concept [6],[7]. Noncommutative geometry of quantum spheres has been considered in Refs[7]-[9]. We shall use the left-invariant 3D differential calculus on the quantum group $SU_q(2)$ [10], [11] to study geometry on the quantum sphere $SU_q(2)/U(1) = S^2_q$. Global functions on $S^2_q$ can be defined as the subset of $SU_q(2)$-functions with a zero $U(1)$-charge; so we shall consider the $SU_q(2) \times U(1)$-covariant relations for the basic geometrical objects on $S^2_q$.

Quantum harmonics will be considered as matrix elements $u^i_{\pm}$ of the $SU_q(2)$-matrix $u$. An operator of external derivation $d_u$ on $SU_q(2)$ can be decomposed in terms of three invariant operators corresponding to the different generators of a deformed Lie algebra. We discuss the analogous decomposition of Maurer-Cartan equations on $SU_q(2)$.

The deformed harmonic formalism can be used for analysis of the self-duality equation on the quantum Euclidean space $E_q(4)$. The noncommutative coordinates $x$ of $E_q(4)$ satisfy the $SU_q^L(2) \times SU_q^R(2)$-covariant commutation relations. In this approach, quantum harmonics are connected with the left $SU_q(2)$-group.

We use the noncommutative algebra of differential complexes [12]- [14] as a basis of the quantum-group gauge theory. The quantum-group self-duality equation (QGSDE) on $E_q(4)$ can be formulated with the help of a duality operation on the curvature 2-form. We present the deformed analog of the classical BPST-instanton solution.

Quantum harmonics allow us to interpret QGSDE as a zero-curvature equation for some harmonic decomposition of the connection form. We discuss harmonic solutions of QGSDE by analogy with the classical harmonic formalism [3],[4].

2 Quantum harmonics and 3D-differential calculus on the quantum group $SU_q(2)$

We shall use the $R$-matrix approach [5] for definition of the unitary quantum group $U_q(2) = SU_q(2) \times U(1)$ where $q$ is a real deformation parameter. Let $T^i_k \ (i, k = 1, 2)$ be elements of a quantum matrix $T$ satisfying the standard $RTT$-relations ( in the notations of Ref[14] )

$$RTT^\prime = TT^\prime R$$
$$\ (T)^{ik}_{\ell m} = T^{ij}_{\delta m} \ (T^\prime)^{ik}_{\ell m} = \delta^i_j T^k_m$$

The symmetrical $R, \bar{R}$ and $P(\pm)$ matrices obey the following relations

$$R^2 = I + \lambda R, \quad \bar{R}R = I, \quad \bar{R} = R - \lambda I$$
$$P^{(+)} + P^{(-)} = I, \quad P^{(a)} P^{(b)} = \delta^{ab} P^{(b)}, \quad R = q P^{(+)} - q^{-1} P^{(-)}$$

where $\lambda = q - q^{-1}, \ a, b = +, -$. 

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It is convenient to use a covariant expression for the $q$-generalization of an antisymmetrical symbol

$$\varepsilon_{ik}(q) = \sqrt{q(ik)} \varepsilon_{ik} = -q(ik)\varepsilon_{ki}(q)$$

$$q(12) = [q(21)]^{-1} = q, \quad q(11) = q(22) = 1$$

where $\varepsilon_{ik}$ is an ordinary antisymmetrical symbol ($\varepsilon_{ik} = \varepsilon_{ki}$).

$R$-matrix elements can be written in terms of $\delta$ and $\varepsilon(q)$ symbols

$$R_{lm}^{ik} = q\delta_i^l\delta_m^k + \varepsilon_{ki}(q)\varepsilon_{ml}(q)$$

Eq(1) for the $U_q(2)$ group is equivalent to the following relations:

$$\varepsilon_{ml}(q) T_j^l T_m^m = \varepsilon_{nj}(q) D(T)$$
$$\varepsilon^{ml}(q) T_l^i T_k^m = \varepsilon^{ki}(q) D(T)$$

where $D(T) = \text{Det}_q(T)$ is the quantum determinant

$$D(T) = -\frac{q}{1+q^2}\varepsilon_{ki}(q)\varepsilon^{ml}(q)T_l^i T_m^k$$

Write also the covariant relations for the inverse quantum matrix $S(T) = T^{-1}$

$$S(T_i^j) = S_k^i = \varepsilon_{kl}(q) T_j^l \varepsilon^{ji}(q) D^{-1}(T)$$
$$S_i^j S_j^m = \delta_k^i$$

$$T_i^l D_l^m(q) S_k^m = \mathcal{D}_i^k(q) = -\varepsilon_{ji}(q)\varepsilon^{jk}(q)$$

where the notation $\mathcal{D}$ and $\mathcal{D}^{-1}$ for $SU_q(2)$-metrics is introduced.

The unitarity condition for the matrix $T$ can be formulated with the help of involution

$$T_i^k \rightarrow \overline{T_k^i} = S_i^k$$

The condition $D(T) = 1$ corresponds to the case of $SU_q(2)$. Let us define quantum harmonics as matrix elements of the $SU_q(2)$-matrix $u_a^i$. We shall distinguish the upper $SU_q(2)$ index $i = 1, 2$ and low $U(1)$-index $a = +, -$. $SU_L^q(2) \times U(1)$ co-transformations of the harmonics have the following form:

$$u_{i}^{\pm} \rightarrow l_k^i u_{i}^{\pm} \exp(\pm i\alpha)$$

where $\alpha$ is the $U(1)$ parameter and $l$ is the $SU_L^q(2)$-matrix.

Eqs(2.7) for the matrix elements $u_a^i$ are equivalent to the basic relations

$$\varepsilon_{kl}(q) u_{i}^{\pm} u_{i}^{\pm} = 0$$
$$\varepsilon_{kl}(q) u_{a}^{i} u_{b}^{k} = \varepsilon_{ba}(q) ,$$
$$\varepsilon^{ba}(q) u_{a}^{i} u_{b}^{k} = \varepsilon^{ki}(q)$$
We shall use the left-covariant 3-dimensional differential calculus \([10], [11]\) for the quantum harmonics. Consider the \(q\)-traceless left-invariant 1-forms satisfying the Maurer-Cartan equations

\[
\theta^a_b = \bar{u}^a_i du^i_b \\
\text{Tr}_q \theta = q^{\theta^+} + q^{-1} \theta^- = 0 \\
d\theta^a_b = -\theta^c_a \theta^b_c 
\]

where \(\bar{u}^a_i\) are components of the inverse \(SU_q(2)\)-harmonics.

Introduce the simple \(U(1)\) notation

\[
\theta_0 = \theta^+_+, \quad \theta_{(2)} = \theta^+_-, \quad \theta_{(-2)} = \theta^-_+ 
\]

Consider the left-covariant bilinear relations between harmonics and \(\theta\)-forms

\[
q^{\pm 2} \theta_0 u^i_\pm = u^i_\pm \theta_0 \\
q^{\pm 1} \theta_{(p)} u^i_\pm = u^i_\pm \theta_{(p)}, \quad p \neq 0
\]

These formulas are consistent with Eqs (2.14)-(2.16). Using the standard Leibniz rules for the operator \(d\) one can obtain the relations for the \(\theta\)-forms

\[
\theta^2_{(p)} = 0, \quad \theta_{(2)} \theta_{(-2)} = -q^2 \theta_{(-2)} \theta_{(2)} \\
\theta_{(\pm 2)} \theta_0 = -q^{\pm 4} \theta_0 \theta_{(\pm 2)}
\]

Consider the \(SU_q(2) \times U(1)\) invariant decomposition of the harmonic external derivative

\[
d_u = \delta_0 + \delta + \bar{\delta} \\
\delta_0 = \theta_0 D_0, \quad \delta = \theta_{(-2)} D_{(2)}, \quad \bar{\delta} = \theta_{(+2)} D_{(-2)}
\]

where \(D_0\) and \(D_{(\pm 2)}\) are left-invariant differential operators. Note that the \(D\)-operators are generators of the \(q\)-deformed Lie algebra \([11]\)

\[
q^2 D_{(2)} D_{(-2)} - D_{(-2)} D_{(2)} = D_0 \\
D_0 D_{(2)} - q^4 D_{(2)} D_0 = q^2 (1 + q^2) D_{(2)} \\
D_{(-2)} D_0 - q^4 D_0 D_{(-2)} = q^2 (1 + q^2) D_{(-2)}
\]

The standard basis of the universal enveloping algebra \(U_q[SU(2)]\) \([6]\) can be obtained by the nonlinear substitution \([11]\)

\[
D_0 = \frac{q^2}{1 - q^4} (1 - q^{2H}) \\
D_{(\pm 2)} = q^H X^{(\pm)}
\]

The operators \(\delta_0, \delta\) and \(\bar{\delta}\) are nilpotent and obey the additional condition

\[
\{\delta_0, \delta\} + \{\delta_0, \bar{\delta}\} + \{\delta, \bar{\delta}\} = 0
\]
Define the manifest expressions for the action of these operators on quantum harmonics

\[
[\delta_0, u^i_+] = u^i_+\theta_0, \quad [\delta, u^i_+] = 0, \quad \bar{\delta} u^i_+ = u^i_+\theta_{(+2)}
\] (2.24)

\[
[\delta_0, u^i_-] = -\theta_0 u^i_-, \quad [\delta, u^i_-] = u^i_+\theta_{(-2)}, \quad [\bar{\delta}, u^i_-] = 0
\]

An invariant decomposition of the Maurer-Cartan equations on \( SU_q(2)/U(1) \) has the following form:

\[
d_u\theta_0 = 2\{\delta, \theta_0\} = 2\{\bar{\delta}, \theta_0\} = -\theta_{(-2)}\theta_{(+2)}
\] (2.25)

\[
d_u\theta_{(+2)} = 2\{\delta_0, \theta_{(+2)}\} = 2\{\bar{\delta}, \theta_{(+2)}\} = q^2(1 + q^2)\theta_0\theta_{(+2)}
\]

\[
d_u\theta_{(-2)} = 2\{\delta_0, \theta_{(-2)}\} = 2\{\bar{\delta}, \theta_{(-2)}\} = q^2(1 + q^2)\theta_{(-2)}\theta_0
\]

Global functions on the quantum sphere \( S^2_q = SU_q(2)/U(1) \) satisfy the invariant condition

\[
[\delta_0, f(u)] = \theta_0 D_0 f(u) = 0
\] (2.26)

We shall consider also the \( U(1) \)-charged functions of the harmonics \( f_{(p)}(u) \)

\[
[H, f_{(p)}(u)] = pf_{(p)}(u)
\] (2.27)

where \( p \) is an integer number.

We shall treat harmonic functions as formal expansions on irreducible harmonic polynomials. The \( q \)-symmetrized product of \( r \) harmonics \( u^i_+ \) and \( s \) harmonics \( u^i_- \) is the basis of the irreducible \( SU_q(2) \)-representation with the \( U(1) \)-charge \( p = r - s \)

\[
\Phi^{(r,s)}(u) = \Phi^{(i_1,\ldots,i_r,-i_{r+s})}(u) = u^{i_1}_+u^{i_2}_+\ldots u^{i_r}_+u^{i_{r+1}}_-\ldots u^{i_{r+s}}_- = (u^+_r)^r(u^-_s)^s
\] (2.28)

where \( (r, s) = I \) is the \( q \)-symmetrized multiindex

\[
P_{k,k+1}^{(+)}\Phi^{(r,s)} = q^{-1}R_{k,k+1}\Phi^{(r,s)} = \Phi^{(r,s)}
\] (2.29)

Here the \( R \)-matrix and the projectional operator \( P^{(+)} \) act on the indices \( i_k \) and \( i_{k+1} \).

The monomials \( \Phi^{(r,s)} \) obey complicated commutation relations depending on the values \( r, s \), so the polynomials \( f_{(p)}(u) \) with complex numerical coefficients have not covariant commutation properties. It is useful to extend the algebra of harmonics by adding the set of noncommuting coefficients \( C^{(r,s)} \). These coefficients are the components of the covariant neutral harmonic polynomials (covariant \( q \)-harmonic fields)

\[
F(u) = \sum C^{(r,r)}\Phi^{(r,r)}(u) = \sum C_I\Phi^I
\] (2.30)

The bilinear commutation relations between \( C_I \) and \( u \) follow from the requirement of harmonic commutativity:

\[
[u^i_{\pm}, F(u)] = 0
\] (2.31)

Relations between different coefficients \( C_I \) can be obtained, for instance, from the additional assumption of commutativity for the monomials in Eq[2.30]. If one has a matrix harmonic field \( F^a_b(u) \) satisfying the bilinear relations, then new relations for the corresponding coefficients arise too.

A construction of the differential calculus on covariant harmonic fields includes the relations for the harmonic external derivatives (2.21) and \( C_I \)

\[
[\delta_0, C_I] = [\delta, C_I] = [\bar{\delta}, C_I] = 0
\] (2.32)
3 Quantum Euclidean space and quantum self-duality equation

Quantum deformations of the Minkowski and Euclidean 4-dimensional spaces have been considered in Refs[16]-[19]. We shall use the coordinates $x^{i \alpha}$ of $q$-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra covariant under the coaction of the quantum group $G_q(4) = SU^L_q(2) \times SU^R_q(2)$

$$x^{i \alpha} \rightarrow (lxr)^{i \alpha} = l^i_k x^{k \beta} \otimes x^{i \alpha}_\beta$$

(3.1)

where $l$ and $r$ are quantum matrices of the left and right $SU^q(2)$ groups:

$$R_{lm} x^{i \alpha} x^{m \beta} = x^{i \gamma} x^{\gamma \delta} R_{\alpha \beta}$$

(3.2)

$$R r r' = r r' R, \quad R l l' = l l' R$$

(3.3)

$[r, l'] = [r, x'] = [l, x'] = 0, \quad \text{Det}_q(l) = 1 = \text{Det}_q(r)$

We use two identical copies of $R$-matrices for $SU^L_q(2)$ and $SU^R_q(2)$.

The $q$-deformed central Euclidean interval $\tau$ can be constructed by analogy with the quantum determinant

$$\tau(x) = -\frac{q}{1 + q^2} \varepsilon^{\beta \alpha}(q) \varepsilon_{\alpha \beta}(q) x^{i \alpha} x^{i \beta}$$

(3.4)

We do not consider the quantum-group structure on $E_q(4)$ but we shall apply the standard formula (2.10) for a definition of the inverse matrix $S(x)$.

It is convenient to use the following $E_q(4)$-involution:

$$x^{i \alpha} = \varepsilon_{ik}(q) x^{k \beta} \varepsilon^{\beta \alpha}(q) = \tau S^{i \alpha}_\beta(x)$$

(3.5)

$$\tau = \tau, \quad x^{i \alpha}_\beta = x^{i \alpha}$$

Let us consider the bicovariant differential calculus on the quantum group $U_q(2)$ [20]-[23]

$$TdT' = R dTT' R$$

(3.6)

$$D(T)dT = q^2 dT D(T)$$

(3.7)

$$\omega R \omega + \omega R \omega R = 0$$

(3.8)

$$T \omega' = R \omega RT$$

(3.9)

where $\omega_k^j(T) = dT^j_k S(T^k_j)$ are the right-invariant differential forms.

The quantum trace $\xi$ of the form $\omega$ plays an important role in this calculus

$$\xi(T) = D^k_i(q) \omega^j_k(T) \neq 0, \quad \xi^2 = 0, \quad d\xi = 0$$

(3.10)

$$dT = \omega T = (q^2 \lambda)^{-1} [T, \xi], \quad qdD(T) = \xi D(T)$$

(3.11)

$$d\omega = \omega^2 = -(q^2 \lambda)^{-1} \{\xi, \omega\}$$

(3.12)

All these formulae can be used for a construction of the $G_q(4)$-covariant differential calculus on $E_q(4)$ via the substitution

$$T \rightarrow x, \quad dT \rightarrow dx, \quad \omega(T) \rightarrow \omega(x) = dx \ S(x)$$

(3.13)
The noncommutative algebra of differential complexes [12]- [14] can be used for a consistent formulation of the $U_q(2)$ gauge theory on the quantum space $E_q(4)$. Consider the $U_q(2)$ gauge matrix $T_a^b$ defined on $E_q(4)$. Suppose that Eqs(2.2, 3.7 - 3.12) locally satisfy for each "point" $x$. Coaction of the gauge group $U_q(2)$ on the connection 1-form $A^a_0$ has the following form [12]-[14]:

$$A \to T(x) A S(T(x)) + dT(x) S(T(x)) = T A S + \omega(T)$$  \hspace{1cm} (3.14)

$$A^a_0 = dx^i_\alpha A^\alpha_{ab}(x)$$

The basic commutation relations for the form $A$ are covariant under the gauge transformation

$$A R A + R A R A R = 0$$  \hspace{1cm} (3.15)

Note that the general relation for $A$ contains a nontrivial right-hand side [14]. The restriction $\alpha = \text{Tr}_q A = 0$ is inconsistent with Eq(3.15), but we can choose the zero field-strength condition $d\alpha = \text{Tr}_q dA = 0$. This constraint for the $U(1)$-gauge field is gauge invariant.

The curvature 2-form is $q$-traceless for this model

$$F = dA - A^2 = dx^i_\alpha dx^k_\beta F^\alpha_{ki}(x)$$  \hspace{1cm} (3.16)

Basic 2-forms on $E_q(4)$ can be decomposed with the help of the projectional operators $P^{(\pm)}$ [23]

$$dx^i_\alpha dx^k_\beta = [P^{(-)} dx dx' P^{(+)} + P^{(+)} dx dx' P^{(-)}]_{\alpha\beta}^{ik} =$$

$$= \frac{q}{1+q^2} [\varepsilon_{ki}(q) d^2 x_{\alpha\beta} + \varepsilon_{\beta\alpha}(q) d^2 x^{ik}]$$  \hspace{1cm} (3.17)

By analogy with the classical case we can treat these two parts as self-dual and anti-self-dual 2-forms under the action of a duality operator $\ast$.

Let us consider the deformed anti-self-duality equation

$$\ast F = -F$$  \hspace{1cm} (3.18)

We can obtain a 5-parameter solution for the $q$-deformed anti-self-dual $U_q(2)$-connection [23]:

$$A^a_0 = dx^a_\alpha \varepsilon_{bk}(q) \hat{x}^k_\beta \varepsilon^{\beta\alpha}(q)(c + \hat{\tau})^{-1}$$  \hspace{1cm} (3.19)

$$\hat{x}^k_\beta = x^k_\beta - c^k_\beta, \quad d\hat{x} = dx, \quad dc = 0$$  \hspace{1cm} (3.20)

$$R \hat{x} \hat{x}' = \hat{x} \hat{x}' R, \quad R c c' = c c' R, \quad c x' = R x c' R$$

$$c dx' = R dx c' R, \quad [\hat{x}, \tau(\hat{x})] = 0$$

$$\tau(\hat{x}) dx = q^2 dx \tau(\hat{x})$$

where $c$ and $c^k_\beta$ are some "parameters" and a central function $\hat{\tau} = \tau(\hat{x})$ can be defined by substitution $x \to \hat{x}$ in Eq(3.4).

Note that one can treat $c$ as a central periodical function which define a solution of the first-order finite-difference equation: $c(\tau) = c(q^2 \tau)$. This solution is a deformed analogue of Belavin-Polyakov-Schwarz-Tyupkin instanton. The multiparameter $q$-generalization of the ’t Hooft solution can be considered too.
4 Harmonic (twistor) interpretation of quantum-group self-duality equation

The QGSD-equation for the field strength has the following form:

\[
F^{\beta \alpha}_{ki} = [P(+)FP(-)]_{ik} = \varepsilon_{ki}(q)F^{\beta \alpha}
\]  

(4.1)

One can obtain the integrability condition multiplying this equation by the product of \( q \)-harmonics \( u^k_1 u^k_2 \).

Let us discuss the covariant formulation of this integrability condition using the deformed harmonic space. It is convenient to introduce new analytic coordinates \( x_{\alpha(\pm)} \) for \( E_\alpha(4) \otimes q S^2_q \). One should use the following commutation relations

\[
\partial^\alpha_i x^j_\beta = \delta^\alpha_\beta \delta^i_j + R_{ij}^{\beta \rho} R^{\alpha \rho}_{\beta \gamma} \partial^\gamma_\delta \partial^i_\delta
\]

(4.2)

\[
q \partial^\alpha_i u^i_a = R^{\alpha \rho}_{ik} u^i_a \partial^\rho_a
\]

(4.3)

\[
q u^i_a x^j_\beta = R^{\alpha \rho}_{lk} x^l_\beta u^m_a
\]

(4.4)

Define the charged analytical coordinates and derivatives and the corresponding commutation relations

\[
x_{aa} = \varepsilon_{ab}(q)x^b_a = \varepsilon_{ik}(q)x^k_i u^i_a = -q^2 \varepsilon_{ki}(q)u^i_a x^k_a
\]

(4.5)

\[
R^{\alpha \gamma}_{ab} x_{\alpha \beta} = R^{\rho \gamma}_{\beta \alpha} x_{\alpha \rho}
\]

(4.6)

\[
\partial^\alpha_a = u^i_a \partial^\alpha_i, \quad R^{\alpha \beta}_{ab} \partial^\beta_c = R^{\alpha \gamma}_{bc} \partial^\gamma_a
\]

(4.7)

Note that upper and low indices \( a, b \ldots \) have opposite \( U(1) \)-charges.

Consider the symmetrical decomposition of the external derivative \( d_x \) on \( E_\alpha(4) \)

\[
d_x = dx^i_a \partial^\alpha_i = \kappa^a_\alpha \partial^\alpha_a = d^a_\alpha = d_1 + d_2
\]

(4.8)

where \( \kappa^a_\alpha = \varepsilon^{ab}(q)\kappa^a_{ab} \) are the covariant analytic 1-forms:

\[
\kappa^a_{aa} = \varepsilon_{ab}(q)dx^i_a u^i_a = dx_{aa} - x_{ab} \theta^b_a
\]

(4.9)

\[
\{d_x, \kappa^a_\alpha\} = 0, \quad \kappa^a_\alpha \kappa^b_\beta = -R^{ba}_{\alpha \rho} \kappa_{\beta} \kappa^d_\rho R^{\alpha \rho}_{\gamma \beta}
\]

(4.10)

\[
\partial^\alpha_a \kappa^b_\beta = R^{\alpha \gamma}_{\beta \beta} R^{\gamma \rho}_{ba} \kappa^a_\rho \partial^\alpha_\beta
\]

It is not difficult to check the following relations:

\[
d_2^2 = 0, \quad d_2^2 + \{d_1, d_2\} = 0
\]

(4.11)

Stress that \( d_2^2 \to 0 \) in the limit \( q \to 1 \).

An analyticity condition for the functions of \( x_{aa} \) and \( u^i_a \) has manifest solutions \( \Lambda \) depending on the analytical coordinate \( x_{\alpha(+)a} \)

\[
\partial^\alpha_a \Lambda = 0 \iff d_1 \Lambda = 0
\]

(4.12)
It should be remarked that the action of the harmonic derivatives $\delta_0$ and $\delta$ (2.25) conserves the analyticity
\[ \{\delta_0, d_1\} \Lambda = 0, \quad \{\delta, d_1\} \Lambda = 0 \] (4.13)

Consider a decomposition of the $U_q(2)$-connection in the central basis (CB) (3.13) $A = a_1 + a_2$ corresponding to the decomposition (1.9) where $a_1 = \kappa_+^x A_+^\alpha(x)$ is a connection for the derivative $d_1$. The quantum-group self-duality equation (4.3) is equivalent to the zero-curvature equation
\[ d_1 a_1 - a_1 d = 0 \] (4.14)

This equation has the following harmonic solution:
\[ a_1 = d_1 h S(h) = \omega(h, d_1 h) \] (4.15)

where $h(x, u)$ is a ”bridge” $U_q(2)$-matrix function. The matrix elements of $h$, $d_1 h$, $d_x h$ and $d_u h$ satisfy the relations analogous to Eqs(3.7-3.9). Additional harmonic conditions are
\[ \delta_0 h = 0, \quad d\text{Tr}_q \omega(h, dh) = 0 \] (4.16)

where $d$ is a nilpotent operator ($d_1$, $d_x$ or $d_u$).

The bridge solution possesses a nontrivial gauge freedom
\[ h \rightarrow T(x)h(x_+, u), \quad \delta_0 \Lambda = d_1 \Lambda = 0 \] (4.17)

where $\Lambda$ is an analytical $U_q(2)$ gauge matrix.

The matrix $h$ is a transition matrix from the central basis to the analytic basis (AB) where $d_1$ has no connection. Consider formally the decomposition $d = d_x + d_u$ in the CB equations (3.13-3.16) although the CB-harmonic connection is equal to zero ($d_u T = 0 = d_u A$). The bridge transform is a transition to a new $u$-dependent basis $A$ in the algebra of $U_q(2)$ differential complexes
\[ A = S(h)A_+ - S(h)dh = \tilde{A}_x + V \] (4.18)
\[ \tilde{A}_x = S(h)A_+ - S(h)d_x h = \kappa_+^{(-)} A_+^\alpha \] (4.19)
\[ V = v + \bar{v} = -S(h)d_u h, \quad v = \theta_{(-2)} V_{(+2)}, \quad \bar{v} = \theta_{(+2)} V_{(-2)} \] (4.20)

where $\tilde{A}_x, V, v$ and $\bar{v}$ are the AB-connection 1-forms for the operators $d_x, d_u, \delta$ and $\bar{\delta}$ correspondingly.

A general solution of QGSDE can be obtained as a solution of the basic harmonic gauge equation [4], [5]
\[ \delta h + hv = \theta_{(-2)} [D_{(+2)} h + hV_{(+2)}] = 0 \] (4.21)

where the connection $v$ contains the analytic prepotential $V_{(+2)}$.

We can discuss also the harmonic equations for the AB-gauge fields by analogy with Refs[24]
\[ \partial_+^x V_{(+2)} = 0, \quad A_+^\alpha = -q^{-2} \partial_+^x V_{(-2)} \] (4.22)
\[ [D_{(+2)} + V_{(+2)}] V_{(-2)} - q^{-2} [D_{(-2)} + V_{(-2)}] V_{(+2)} = 0 \] (4.23)

where $V_{(-2)}$ is the nonanalytic gauge field for $D_{(-2)}$. 

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One can obtain explicit or perturbative solutions of these equations by using the noncommutative generalizations of classical harmonic expansions and harmonic Green functions [4],[5],[24].

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