AUTOMATIC GROUPS AND KNUTH–BENDIX WITH INFINITELY MANY RULES

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Abstract. It is shown how to use a small finite state automaton in two variables in order to carry out part of the Knuth–Bendix process for rewriting words in a group. The main objective is to provide a substitute for the most space-demanding module of the existing software which attempts to find a shortlex-automatic structure for a group. The two-variable automaton can be used to store an infinite set of rules and to carry out fast reduction of arbitrary words using this infinite set. We introduce a new operation, which we call welding, which applies to an arbitrary finite state automaton. In our context this operation is vital. We point out a small potential improvement in the subset algorithm for making a non-deterministic automaton deterministic.

1. Introduction

A celebrated result of Novikov and Boone asserts that the word problem for finitely presented groups is, in general, unsolvable. This means that a finite presentation of a group has been written down, with the property that there is no algorithm whose input is a word in the generators, and whose output states whether or not the word is trivial. So, given a presentation of a group which one is unable to analyze, can any help at all be given by brute force methods, using a computer?

The answer is that some help can be given with the kind of presentation that arises naturally in the work of many mathematicians, even though one can formally prove that there is no procedure that will always help.

There are two general techniques for trying to determine, with the help of a computer, whether two words in a group are equal or not. One is the Todd–Coxeter coset enumeration process and the other is the Knuth–Bendix process. Todd-Coxeter is more adapted to finite groups which are not too large. We are mostly interested in groups which arise in the study of low dimensional topology. In particular they are infinite groups, and the number of words of length $n$ rises exponentially with $n$. For this reason, Todd–Coxeter is not much use in practice. Well before Todd–Coxeter has had time to work out the structure of a large enough neighbourhood of the identity in the Cayley graph to be helpful, the computer is out of space.

On the other hand, the Knuth–Bendix process is much better adapted to this task, and it has been used quite extensively, particularly by Sims, for example in connection with computer investigations into problems related to the Burnside
problem. It has also been used to good effect by Holt and Rees in their automated searching for isomorphisms and homomorphisms between two given finitely presented groups (see \cite{8}). In connection with searching for a shortlex-automatic structure on a group (we say what this means in Section 4), Holt was the first person to realize that the Knuth–Bendix process might be the right direction to choose (see \cite{3}). However, Knuth–Bendix will run for ever on even the most innocuous hyperbolic triangle groups, which are perfectly easy to understand. Holt’s successful plan was to use Knuth–Bendix for a certain amount of time, decided heuristically, and then to interrupt Knuth–Bendix and use axiom-checking, a part of automatic group theory (see \cite{3}, Chapter 6), to find an automatic structure on the group. Thus, using the concept of an automatic group as a mechanism for bringing Knuth–Bendix to a halt has been one of the philosophical bases for the work done at Warwick in this field almost from the beginning. In addition to the works already cited in this paragraph, the reader may wish to look at \cite{8} and \cite{6}.

For a shortlex-automatic group, a minimal set of Knuth–Bendix rules may be infinite, but it is always a regular language, and therefore can be encoded by a finite state machine. In this paper, we carry this philosophical approach further, attempting to compute this finite state machine directly, and to carry out as much of the Knuth–Bendix process as possible using only approximations to this machine.

Thus, we describe a setup that can handle an infinite regular set of Knuth–Bendix rewrite rules. For our setup to be effective, we need to make several assumptions. Most important is the assumption that we are dealing with a group, rather than with a monoid. Secondly, our procedures are perhaps unlikely to be of much help unless the group actually is shortlex-automatic.

As a computer science byproduct of our work, we produce a new operation on automata, which we call welding. Although this is an operation which makes sense on the level of abstract languages, we do not see any use for it apart from those indicated in this paper, which is concerned very much with equations in groups. Another computer science byproduct is a small improvement which one can sometimes make in the process of determinizing a finite state automaton. Since determinization is potentially exponential, even a small improvement can be useful.

Previous computer implementations of the semi-decision procedure to find the shortlex-automatic structure on a group are essentially specializations of the Knuth–Bendix procedure \cite{3} to a string rewriting context together with fast, but space-consuming, automaton-based methods of performing word reduction relative to a finite set of shortlex-reducing rewrite rules. Since shortlex-automaticity of a given finite presentation is, in general, undecidable, space-efficient approaches to the Knuth–Bendix procedure are desirable. In this paper we present a new algorithm which performs a Knuth–Bendix type procedure relative to a possibly infinite regular set of shortlex-reducing rewrite rules, together with a companion word reduction algorithm which has been designed with space considerations in mind.

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2. String rewriting

In this section we review some standard material on string-rewriting, with the object of making this paper reasonably self-contained. Later sections will review standard material on automata and automatic groups.
Definition 2.1. Let $A$ be a finite set (usually called the alphabet). We define $A^*$ to be the set of strings of symbols in $A$. In other words, $A^*$ is the free monoid generated by $A$, with multiplication defined by concatenation. The identity element in this monoid is the empty string, denoted by $\varepsilon$.

Definition 2.2. Given a finite alphabet $A$, a subset $R$ of $A^* \times A^*$ is known as a rewrite system on $A^*$. The elements of $R$ are known as rewrite rules.

Definition 2.3. Associated with a rewrite system $R$ we define three relations $\to_R$, $\to_R^*$ and $\leftrightarrow_R^*$ on $A^*$. For $u, v \in A^*$ we write $u \to_R v$ (and say that $u$ rewrites to $v$) if there are strings $x, y \in A^*$ and a rewrite rule $(\lambda, \rho) \in R$ such that $u = x\lambda y$ and $v = x\rho y$. This is also called an elementary reduction. The relation $\to_R^*$ is the reflexive, transitive closure of $\to_R$ and the relation $\leftrightarrow_R^*$ is the reflexive, symmetric, transitive closure of $\to_R$. The congruence $\leftrightarrow_R^*$ is called the Thue congruence generated by $R$ and we denote the congruence class of a string $w \in A^*$ by $[w]_R$.

If there is no infinite sequence $u_1 \to_R u_2 \to_R \cdots$ of rewrites we say that $R$ is Noetherian. In such a system each congruence class contains at least one irreducible string, i.e., an element $w \in A^*$ which contains no substring equal to the left-hand side of any rewrite rule. In a Noetherian rewrite system any string $w$ is reduced to an irreducible element of $[w]_R$ by a finite sequence of rewrites. If each congruence class of a Noetherian system contains a unique irreducible then the word problem in the quotient monoid $A^*/\leftrightarrow_R^*$ is solved by rewriting.

A rewrite system $R$ is called confluent if whenever $u, v, w \in A^*$ with $u \to_R v$ and $u \to_R w$, there exists some string $z \in A^*$ with $v \to_R^* z$ and $w \to_R^* z$. Confluence can easily be proved necessary and sufficient for each congruence class in a Noetherian rewrite system to contain a unique irreducible. In this case elements of the monoid $A^*/\leftrightarrow_R^*$ can be defined by juxtaposition and reduction.

2.4. Critical pair analysis. For a finite Noetherian rewrite system $R$ the question of unique irreducibles entails only a finite computation known as critical pair analysis. If a finite Noetherian $R$ is not confluent one can easily prove that the property must fail at one of a finite number of triples $(u, v, w)$. Such triples are obtained by considering pairs of rules in $R$ whose left-hand sides have a non-trivial overlap. For such a pair of rules $(\lambda_1, \rho_1), (\lambda_2, \rho_2)$ there are two types of overlap. First, a non-empty string $z$ may be a suffix of $\lambda_1 = s_1z$ and a prefix of $\lambda_2 = zs_2$ (or vice versa). Second, $\lambda_2$ may be a substring of $\lambda_1$ (or vice versa) and we write $\lambda_1 = s_1\lambda_2s_2$.

These cases are not disjoint. In particular, if one of $s_1$ and $s_2$ is trivial in the second case, it can equally well be treated under the first case with $z$ equal either to $\lambda_1$ or to $\lambda_2$.

2.5. First case of critical pair analysis. In the first case, the triple $(u, v, w) = (s_1zs_2, \rho_1s_2, s_1\rho_2)$. There are two ways of starting to reduce $u = s_1zs_2$, namely to $v = \rho_1s_2$ and to $w = s_1\rho_2$. Further reduction to irreducibles either gives the same irreducible for each of the two computations, or else gives us distinct irreducibles $v'$ and $w'$. In the latter situation we can augment $R$ either with the rule $(v', w')$ or with $(w', v')$ provided the system obtained remains Noetherian. (If it doesn’t remain Noetherian with either choice, we will almost certainly have to give up on
the whole process.) Since \(v'\) was previously congruent to \(w'\), the congruence on \(A^*\) is unaltered by introducing such a rule.

Note that it is important to allow \((\lambda_1, \rho_1) = (\lambda_2, \rho_2)\) in the first case, provided there is a \(z\) which is both a proper suffix and a proper prefix of \(\lambda_1 = \lambda_2\).

2.6. Second case of critical pair analysis. In the second case, the triple \((u, v, w) = (\lambda_1, \rho_1, s_1\rho_2s_2)\). If \(\rho_1\) and \(s_1\rho_2s_2\) do not reduce to the same irreducible, we augment \(R\) with a new rule \((v', w')\) or with \((w', v')\), provided the system remains Noetherian.

2.7. Omitting rules. In practice, it is important to remove rules which are redundant, as well as to add rules which are essential. Omitting rules is unnecessary in theory, provided that we have unlimited time and space at our disposal. In practice, if we don’t omit rules, we are liable to be overwhelmed by unnecessary computation. Moreover, nearly all programs in computational group theory suffer from excessive demands for space. Indeed this is one of the reasons for developing the algorithms and programs discussed in this paper. So it is important to throw away information that is not needed and doesn’t help.

For this reason, in Knuth–Bendix programs one normally looks from time to time at each rule \((\lambda, \rho)\) to see if it can and should be omitted. If a proper substring of the left-hand side can be reduced, then we are in the situation of 2.6. If the two reductions mentioned in 2.6 lead to the same irreducible, we omit \((\lambda, \rho)\) from the set of rules. If the two reductions lead to different irreducibles, then we augment the set of rules as described in 2.6 and again omit \((\lambda, \rho)\).

We also investigate whether the right-hand side \(\rho\) of a rule \((\lambda, \rho)\) is reducible to \(\rho'\). If so, we can omit \((\lambda, \rho)\) from \(R\) and replace it with the rule \((\lambda, \rho')\) without changing the congruence on \(A^*\).

2.8. Maintaining the congruence. The insertion of a rule \((\lambda, \rho)\), for which we already know that \(\lambda\) and \(\rho\) are congruent, clearly does not change the congruence.

Suppose, on the other hand, that \((\lambda, \rho)\) is a rule where \(\lambda\) is known to be congruent to \(\mu\) using only rules other than \((\lambda, \rho)\). Then \((\lambda, \rho)\) can be omitted without changing the congruence.

The process of analysing critical pairs and augmenting or diminishing the rule set without changing the congruence on \(A^*\) is known as the Knuth–Bendix Process. If this terminates, it gives a finite confluent rewriting system for the congruence. Usually it does not terminate and it produces new rules ad infinitum. At each augmentation we have to choose one of two new rules to insert and we have to ensure that the augmented system is still Noetherian in order for the procedure to continue. The choice is generally made using some total ordering on the elements of \(A^*\).

2.9. Ordering. Given an alphabet \(A\), a reduction ordering on \(A^*\) is a well-ordering which is invariant under left and right multiplication. Suppose we have a rewriting system \(R\) on \(A^*\), such that, for each \((\lambda, \rho) \in R\), we have \(\rho < \lambda\). Then we say that \(R\) is consistent with the reduction ordering.

If \(R\) is consistent with a reduction ordering, then it is Noetherian. Moreover, to augment the rule set during the Knuth–Bendix Completion Procedure without destroying the Noetherian property, we have to choose either \((v', w')\), or \((w', v')\), using the notation introduced above where \(v'\) and \(w'\) are certain elements in \(A^*\).
We use \((v', w')\) if \(v' > w'\) and \((w', v')\) if \(w' > v'\). For more information on Knuth–Bendix and rewriting see \([2]\).

Given an ordering \(\prec\) on a finite alphabet \(A\) we can extend this to a well-ordering on \(A^*\) by defining \(u \prec v\) to mean either 1) \(|u| < |v|\) or 2a) \(|u| = |v|\) and 2b) \(u < v\) in the lexicographic order on \(A^*\) induced by the ordering on \(A\). This is clearly a reduction ordering on \(A^*\) and is termed the shortlex-ordering induced by \(\prec\). It is the only ordering we will use in this paper.

We order pairs \((\lambda, \rho) \in A^* \times A^*\) by setting \((\lambda', \rho') > (\lambda, \rho)\) if and only if either \(\lambda' > \lambda\) or \(\lambda' = \lambda\) and \(\rho' > \rho\). This is clearly a well-ordering on the set of pairs.

### 2.10. Infinite sets of rules.

In theory, critical pair analysis can be undertaken even on an infinite set of rules \(R\), provided we are working with a reduction ordering. We set \(R_1 = R\). In the \(n\)-th step, we do critical pair analysis on all rules of \(R_n\) such that the sum of the lengths of left-hand side and right-hand side is at most \(n\). The effect on \(R_n\) is to delete some rules, namely those such that either the right-hand side has been shown to be reducible or the left-hand side is reducible without using the rule itself, and to insert others, namely those that arise in the critical pair analysis. The resulting set of rules is \(R_{n+1}\). We can form \(S = \bigcap_m \bigcup_{n \geq m} R_n\). The congruence on \(A^*\) induced by \(R\) is the same as the congruence induced by \(R_n\) for each \(n\). It can also be proved to be the same as the congruence induced by \(S\). Moreover \(S\) can be proved to be a confluent system. The \(S\)-irreducibles are in one-to-one correspondence with the elements of the monoid \(A^*/\leftrightarrow^*_R\).

If \(R_1\) is finite, then \(R_n\) is finite for each \(n\).

Unfortunately, \(S\) is sometimes not a recursive set, even if the \(R_n\) are all finite, so that it cannot be computed by a Turing machine. \(S\) consists of exactly those rules with irreducible right-hand sides and reducible left-hand sides, such that any proper substring of the left-hand side is irreducible.

In our treatment, we will be dealing with an infinite set of rules defined implicitly by a finite state automaton. However, we will not attempt to perform Knuth–Bendix directly on this infinite set.

### 2.11. Knuth–Bendix pass.

One procedure for carrying out the Knuth–Bendix process is to divide the finite set \(S\) of rules found so far into three disjoint subsets. The first subset, called Considered, is the set of rules whose left-hand sides have been compared with each other and with themselves for overlaps. The second set of rules, called This, is the set of rules waiting to be compared with those in Considered. The third set, called New, consists of those rules most recently found. Here we only sketch the process. Full details are provided in Section 12.

The Knuth–Bendix process proceeds in phases, each of which is called a Knuth–Bendix pass. Each pass starts by looking at each rule in Considered and seeing whether it can be deleted as in 2.7. Consideration of an existing rule in Considered can lead to a new rule, in which case the new rule is added to New.

Next, we look at each rule \(r\) in New to see if it is redundant. If it is redundant it is replaced by a non-redundant rule. The details will be given in 2.7.4. The non-redundant version of the rule is moved into This.

We then look at each rule in This. Its left-hand side is compared with itself and with all the left-hand sides of rules in Considered, looking for overlaps as in 2.5. Any
new rules found are added to New. Then \( r \) is moved into Considered. Eventually this becomes empty.

We then proceed to the next pass.

3. Automata and operations on them

This section is devoted to standard material.

**Definition 3.1.** A non-deterministic finite state automaton (abbreviated NFA) is defined to be a quintuple \((S, A, \mu, F, S_0)\), where \( S \) is a finite set called the set of states, \( A \) is a finite set called the alphabet, \( \mu \) is a set of triples, called the set of arrows, of the form \((s, x, t)\) with \( s, t \in S \) and \( x \in A \) or \( x = \epsilon \), where \( \epsilon \) is defined in 2.1. \( F \subset S \) is called the set of final states and \( S_0 \subset S \) is called the set of initial states. The source of an arrow \((s, x, t)\) is defined to be \( s \) and the target is defined to be \( t \). Final states are sometimes called accept states, initial states are sometimes called start states and arrows are sometimes called transitions.

We define a path of arrows in a non-deterministic automaton \( M = (S, A, \mu, F, S_0) \) to be a finite sequence of the form \((u_0, x_1, u_1, \ldots, x_n, u_n)\), where \( n \geq 0 \) and, for each \( i \) with \( 1 \leq i \leq n \), \((u_{i-1}, x_i, u_i) \in \mu \). The length of the path is \( n \). The label associated to a path is the element \( x_1 \ldots x_n \in A^* \). If \( n = 0 \), the label is \( \epsilon \in A^* \).

The language \( L(M) \) accepted by \( M \) is defined to be the set of all labels of paths of arrows starting with some \( u_0 \in S_0 \) and ending with some \( u_n \in F \). If a subset of \( A^* \) is equal to \( L(M) \) for some non-deterministic automaton \( M \), then the subset is called a regular language.

**Definition 3.2.** A partially deterministic finite state automaton (abbreviated PDFA) is defined to be an NFA \( M = (S, A, \mu, F, S_0) \) which contains exactly one initial state, has no \( \epsilon \)-arrows and where for each \( s \in S \) and \( x \in A \) there is at most one arrow of the form \((s, x, t)\).

**Definition 3.3.** A deterministic finite state automaton (abbreviated DFA) is defined to be an NFA \( (S, A, \mu, F, S_0) \), in which there are no \( \epsilon \)-arrows, \( S_0 \) is a singleton whose unique element \( s_0 \) is called the initial state, and such that, for each \( s \in S \) and \( x \in A \), there is exactly one arrow of the form \((s, x, t)\).

Given a non-deterministic automaton \( M \) and a subset \( T \) of the set of states \( S \) we define the \( \epsilon \)-closure \( E(T) \) of \( T \) to be the subset of \( S \) which one can reach from some element of \( T \) by following a path of \( \epsilon \)-arrows. A non-deterministic automaton can be converted into a deterministic automaton accepting the same language as follows. The states of the new automaton are the \( \epsilon \)-closed subsets of \( S \) (one of these states is the nullset). Given an \( \epsilon \)-closed set \( T \) and \( x \in A \), we define an arrow \((T, x, P)\), where \( P \) is obtained by taking the set of targets of all \( x \)-arrows with source in \( T \) and then taking its \( \epsilon \)-closure. The initial state is the \( \epsilon \)-closure of \( S_0 \). A state of the new deterministic automaton is final if and only if it contains a final state of \( M \).

This proves the following standard theorem.

**Theorem 3.4.** For any NFA \( M \) there is a DFA \( N \) with \( L(N) = L(M) \).

**Note 3.5.** We will use the abbreviation FSA to denote an automaton which is a DFA or an NFA or a PDFA.
Computationally, the procedure of finding the \( \epsilon \)-closed subsets and the arrows between them is known as the \textit{subset construction} and this is central to our word reduction algorithm. There is a theoretical exponential blow-up in the subset construction which is known to be unavoidable in general. In the cases which come up in practice in our work, the subset construction can certainly be a problem, but is often not as bad as the worst case analysis seems to suggest. The implementation need only construct those \( \epsilon \)-closed subsets which can be reached from \( S_0 \). The space and time demands of the procedure are proportional to the number of such subsets. We will also use lazy evaluation to reduce the worst effects of this exponential blow-up. This will be described later.

For a general DFA \( M \), the process of finding a DFA \( M' \) with \( L(M') = L(M) \), such that the number of states of \( M' \) is minimal, is known as \textit{minimization}. The existence and uniqueness (up to isomorphism) of such an automaton is known as the Myhill–Nerode theorem and many practical algorithms exist to find \( M' \) given \( M \)—for a detailed survey and comparisons see [12].

In order to define what is meant by an automatic group we need to first formalize what it means for an automaton to accept pairs of strings over an alphabet \( A \).

Consider, for example, the pair of strings \((abb, ccd)\). We regard this pair as a string \((a, c)(b, c)(b, d)\) over the product alphabet \( A \times A \). If the pair of strings is \((abb, ccdc)\), then we have to \textit{pad} the shorter of the two strings to make them the same length, regarding this pair as the string of length four \((a, c)(b, c)(b, d)(\$ , c)\). In general, given an arbitrary pair of strings \((u, v) \in A^* \times A^*\), we regard this instead as a string of pairs by adjoining a \textit{padding symbol} \( \$ \) to \( A \) and then “padding” the shorter of \( u \) and \( v \) so that both strings have the same length. We obtain a string over \( A \cup \{\$\} \times A \cup \{\$\} \). The alphabet \( A \cup \{\$\} \) is denoted \( A^+ \) and is called the \textit{padded extension} of \( A \). The result of padding an arbitrary pair \((u, v) \in A^* \times A^*\) is denoted \((u, v)^+\). A string \( w \in (A^+)^* \times (A^+)^* \) is called \textit{padded} if there exists \( u, v \in A^* \) with \( w = (u, v)^+ \) (in other words, at most one of the two components of \( w \) ends with a padding symbol).

A set of pairs of strings over \( A \) is called \textit{regular} if the corresponding set of padded strings is a regular language over the product alphabet \( A^+ \times A^+ \).

We will need two standard definitions when dealing with finite state automata.

\textbf{Definition 3.6.} Let \( M \) be an FSA. We define its \textit{reversal} \( \text{Rev}(M) \) to be the FSA obtained from \( M \) by taking the same set of states, interchanging the subsets of initial and final states, and then reversing the direction of all arrows. The reversal of a DFA is in general an NFA rather than a DFA.

\textbf{Definition 3.7.} An FSA is called \textit{trim} if each state has an accepted path of arrows passing through it.

\section{A Modified Determinization Algorithm}

In this section we discuss a modification to the usual determinization algorithm for turning an NFA into a DFA. Let \( N \) be an NFA. The proof that \( N \) can be determinized is discussed just before the statement of Theorem 3.4. Let \( M \) be the corresponding determinized automaton, so that a state of \( M \) is a subset of states of \( N \). In practice, to find \( M \), we start with the \( \epsilon \)-closure of the set of initial states of \( N \) and proceed inductively. If we have found a state \( s \) of \( M \) as a subset of the set of states of \( N \), we fix some \( x \in A \), and apply \( x \) in all possible ways to all \( t \in s \),
where \( t \) is a state of \( N \). We then follow with \( \epsilon \)-arrows to form an \( \epsilon \)-closed subset of states of \( N \). This gives us the result of applying \( x \) to \( t \). The modification we wish to make to the usual subset construction is now explained and justified.

We will denote by \( M' \) the modified version of \( M \) thus obtained. \( M' \) is a DFA which accepts the same language as \( M \) and \( N \), but the structure of \( M' \) might be a little simpler than that of \( M \).

Suppose \( p \) is a state of the NFA \( N \). Let \( N_p \) be the same automaton as \( N \), except that the only initial state is \( p \). Suppose \( p \) and \( q \) are distinct states of \( N \) and that \( L(N_p) \subseteq L(N_q) \). Suppose also that the \( \epsilon \)-closure of \( q \) does not include \( p \). Under these circumstances, we can modify the subset construction as follows. As before, we start with the \( \epsilon \)-closure of the set of initial states of \( N \). We follow the same procedure for defining the arrows and states of \( M' \) as for \( M \), except that, whenever we construct a subset containing both \( p \) and \( q \), we change the subset by omitting \( p \).

The situation can be generalized. Suppose that, for \( 1 \leq i \leq k \), \( p_i \) and \( q_i \) are states of \( N \). We assume that all \( 2k \) states are distinct from each other and that, for each pair \((i, j)\), the \( \epsilon \)-closure of \( q_i \) does not include \( p_j \). Suppose further that, for each \( i \), \( \epsilon \) \( \epsilon \) \( \epsilon \)-closure of \( N_{p_i} \) \( \subseteq \) \( \subseteq L(N_{q_i}) \). We follow the same procedure for defining the arrows and states of \( M' \) as for \( M \), except that, whenever we construct a subset containing both \( p_i \) and \( q_i \), we change the subset by omitting \( p_i \).

**Theorem 4.1.** Under the above hypotheses, \( L(M') = L(M) \).

**Proof.** Consider a string \( w = x_1 \cdots x_n \in A^* \) which is accepted by \( N \) via the path of arrows in \( N \)

\[
(v_0, \epsilon^*, u_1, x_1, v_1, \cdots, v_{n-1}, \epsilon^*, u_n, x_n, v_n, \epsilon^*, u_{n+1}).
\]

This means that, for each \( i \) with \( 0 \leq i \leq n \), there is an \( x_i \)-arrow in \( N \) from \( u_i \) to \( v_i \) and \( u_{i+1} \) is in the \( \epsilon \)-closure of \( v_i \). Moreover \( v_0 \) is an initial state and \( u_{n+1} \) is a final state.

Suppose inductively that after reading \( x_1 \cdots x_{i-1} \), \( M' \) is in state \( s_{i-1} \). We assume inductively that we have a path of arrows in \( N \)

\[
(u_{j}, x_i, v_{j}, \epsilon^*, u_{j+1}, \cdots, u_{n}, x_n, v_n, \epsilon^*, u_{n+1}),
\]

such that \( u_{j} \in s_{i-1} \) and \( u_{n+1} \) is a final state.

The induction starts with \( i = 1 \) and \( s_0 \) the initial state of \( M' \). We form \( s_0 \) by taking all initial states of \( N \), and taking their \( \epsilon \)-closure. If this subset of states of \( N \) contains both \( p_j \) and \( q_j \), then \( p_j \) is omitted from \( s_0 \), the initial state of \( M' \).

If \( u_1 \notin s_0 \), then we must have \( u_1 = p_j \) for some \( j \), with \( q_j \in s_0 \). Now \( w \in L(N_{p_j}) \subseteq L(N_{q_j}) \). It follows that we can take \( u_1 \) in the \( \epsilon \)-closure of \( q_j \) and then define the rest of the path of arrows for the case \( i = 1 \). Since \( q_j \in s_0 \) and \( u_1 \) is in the \( \epsilon \)-closure of \( q_j \), \( u_1 \) is not equal to any of the \( p_r \). So \( u_1 \in s_0 \) and the induction can start.

Now suppose we have a path of arrows

\[
(u_{j}, x_i, v_{j}, \epsilon^*, u_{j+1}, \cdots, u_{n}, x_n, v_n, \epsilon^*, u_{n+1}),
\]

in \( N \) such that \( u_{j} \in s_{i-1} \) and \( u_{n+1} \) is a final state of \( N \). We define \( s_i \) from \( s_{i-1} \) in the usual way, applying \( x_i \), in all possible ways to all states in \( s_{i-1} \), obtaining in particular \( v_i \), and then taking the \( \epsilon \)-closure, obtaining in particular \( u_{i+1} \). Finally, if, for some \( r \), \( s_i \) contains both \( p_r \) and \( q_r \), then \( p_r \) is deleted from \( s_i \) before it becomes a state of \( M' \).
It now follows that either \( u_{i+1}^i \in s_i \), or else, for some \( r \) with \( 1 \leq r \leq k \), \( u_{i+1}^i = p_r \), \( q_r \in s_i \) and \( p_r \not\in s_i \). In the first case we define \( u_{i+1}^{i+1} = u_i^i \) and \( v_i^{i+1} = v_i^i \) for \( j > i \) and the induction step is complete. In the second case, using the fact that \( x_{i+1} \cdots x_n \in L(N_{p_r}) \subseteq L(N_{q_r}) \), we see that we can take \( u_{i+1}^{i+1} \) in the \( \epsilon \)-closure of \( q_r \) and then define the rest of the path of arrows. Since \( q_r \in s_i \) and \( u_{i+1}^{i+1} \) is in the \( \epsilon \)-closure of \( q_r \), \( u_{i+1}^{i+1} \) is not equal to any other \( p_q \) and so \( u_{i+1}^{i+1} \in s_i \). This completes the induction step.

At the end of the induction, \( M' \) has read all of \( w \) and is in state \( s_n \). We also have the final state \( u_{n+1}^{n+1} \in s_n \), so that \( w \) is accepted by \( M' \).

Conversely, suppose \( w \) is accepted by \( M' \). It follows easily by induction that if \( M' \) is in state \( s_i \) after reading the prefix \( x_1 \cdots x_i \) of \( w \), then each state \( u \in s_i \) can be reached from some initial state of \( N \) by a sequence of arrows labelled successively \( x_1, \ldots, x_i \), possibly interspersed with \( \epsilon \)-arrows. Now \( s_n \) must contain a final state, and so \( w \) is accepted by \( N \).

\[ \square \]

**Remark 4.2.** The practical usage of this theorem clearly depends on having an efficient way of determining when the condition \( L(N_p) \subseteq L(N_q) \) is satisfied. Later we will see examples of such tests which cost virtually nothing to implement but have the potential to save an appreciable amount of both space and time.

## 5. Automatic groups

**Definition 5.1.** A group \( G \) is called **automatic** if there exists a finite inverse closed set \( A \) of monoid generators of \( G \) and a regular language \( L \) over \( A \) satisfying the following two properties.

1. The natural monoid epimorphism \( \gamma : A^* \to G \) remains surjective when restricted to \( L \).
2. The set
   \[
   \{(u,v) : u, v \in L \text{ and } (ux)\gamma = v\gamma \text{ for some } x \in A \cup \{\epsilon\}\}
   \]

   is regular.

An FSA \( W \) with \( L(W) = L \) is called a **word acceptor** for \( G \). A word acceptor together with an FSA accepting the language \( (\) \( ) \) is called an **automatic structure** for \( G \) relative to \( A \).

This definition is succinct but suppresses the geometry which lies behind the importance of this class of groups. Given a group \( G \) generated by a finite subset \( A \), the **Cayley graph** \( \mathcal{C}(G,A) \) is the graph whose vertices are the elements of \( G \) and where an edge joins two vertices \( g, h \) if and only if there is a generator \( a \in A \) with \( ga = h \). Denoting images under the natural epimorphism \( A^* \to G \) by overscores and the length of a string \( w \in A^* \) by \( |w| \), we define a metric on \( \mathcal{C}(G,A) \) by letting

\[
d(g,h) = \min\{|w| : w \in A^* \text{ with } \overline{w} = g^{-1}h\}.
\]

This is termed the **word metric**. We get the same metric by taking each edge of the Cayley graph and giving it length one. This makes the Cayley graph into a geodesic space. For \( w \in A^* \) and \( i \in \mathbb{N} \), we denote by \( w(i) \) the prefix of \( w \) of length \( i \) (for \( i \geq |w| \) this equals \( w \)).
Given two words \( u, v \in A^* \) and a positive real number \( k \), we say that \( u \) and \( v \) fellow-travel with constant \( k \) in \( C(G, A) \) if the group elements

\[
WD(G, A, u, v) = \left\{ u(i)^{-1}v(i) : i \in \mathbb{N} \right\}
\]

lie in the ball of radius \( k \) around the identity element of the group. We then have the following geometrical characterization of an automatic group.

**Theorem 5.2** ([3](Theorem 2.3.5)). Let \( G \) be a group generated by a finite inverse closed set of monoid generators \( A \), and \( L \) a regular language over \( A \) mapping onto \( G \) under the restriction of the natural epimorphism \( A^* \rightarrow G \). Then \( L \) satisfies property 2 of Definition 5.1 if and only if there exists a constant \( k > 0 \) such that for any \( u, v \in L \), if \( d(u, v) \leq 1 \) in \( C(G, A) \) then \( u \) and \( v \) fellow-travel with constant \( k \) in \( C(G, A) \).

It follows immediately that, for an automatic group, the union of the sets \( WD(G, A, u, v) \) taken over all pairs \((u, v)\) with \( u, v \in L \) and \( d(u, v) \leq 1 \), is a finite set. Here \( d \) is the distance in the Cayley graph. This is also the minimal length of \( u^{-1}v \) as a word over \( A \).

The union of this finite set with the set \( A \) of generators plus the identity of \( G \) is called the set of word differences \( WD(G, A) \) of the automatic structure. \( WD(G, A) \) can be regarded as a PDFA over the alphabet \( A^+ \times A^+ \) where an arrow labelled \((x, y)\) goes from the word difference \( w_1 \) to the word difference \( w_2 \) if and only if \( x^{-1}w_1y = w_2 \). We extend the domain of the epimorphism \( A^* \rightarrow G \) to \((A^+)^* \) by sending the padding symbol to the identity of \( G \). The initial state is the identity of \( G \) and this is also the only final state. The resulting automaton is known as the word difference automaton of the automatic structure. The goal of our main algorithm is to calculate this automaton starting from a finite presentation of a shortlex-automatic group (defined below). For definitive information on automatic groups see the book [3].

We can now give the definition of the class of groups we are chiefly interested in.

**Definition 5.3.** A group \( G \) is called shortlex-automatic with respect to a finite inverse closed well-ordered set of monoid generators \((A, <)\) if

1. \( G \) is automatic with respect to \( A \).
2. A string \( w \in A^* \) is accepted by the word acceptor if and only if \( w \) is the least element under the induced shortlex-ordering of \( \{ v : v \in A^* \text{ and } \overline{v} = \overline{w} \} \).

6. **Welding**

In this section we describe an operation, which we call welding, on FSAs which is central to our Knuth–Bendix procedure. The motivation for this operation is postponed to Section 7.

**Definition 6.1.** An FSA is called welded if it is partially deterministic, trim and has a (partially) deterministic reversal. These conditions imply that, given \( x \in A \) and a state \( t \), there is at most one \( x \)-arrow with target \( t \) and also that there is exactly one initial state and one final state.

Given a trim non-empty NFA \( M \), we can form a welded automaton from it as follows. Given any \( \epsilon \)-arrow \((s, \epsilon, t)\), we may identify \( s \) with \( t \). Given distinct initial states \( s_1 \) and \( s_2 \), we may identify \( s_1 \) with \( s_2 \). Given distinct final states \( t_1 \) and \( t_2 \), we may identify \( t_1 \) with \( t_2 \). Given distinct arrows \((s, x, t_1)\) and \((s, x, t_2)\), we may
identify \( t_1 \) with \( t_2 \). Given distinct arrows \((s_1, x, t)\) and \((s_2, x, t)\), we may identify \( s_1 \) with \( s_2 \). Immediately after any identification of two states, we change the set of arrows accordingly, omitting any \( \epsilon \)-arrow from a state to itself. Since the number of states continually decreases, this process must come to an end, and at this point the automaton is welded.

**Theorem 6.2.** Given a trim non-empty NFA \( M \), all welded automata obtained from it by a process like that described above are isomorphic to each other; that is the welded automaton \( Q \) is independent of the order in which identifications are made. Moreover \( Q \) depends only on the language \( L(M) \). \( Q \) is the minimal PDFA accepting \( L(Q) \). It follows that welding can be regarded as an operation on regular languages.

**Proof.** For each \( x \in A \), let \( x^{-1} \) be its formal inverse and let \( A^{-1} \) be the set of these formal inverses. We form from \( M \) an automaton over \( A \cup A^{-1} \) by adjoining an arrow of the form \((t, x^{-1}, s)\) for each arrow \((s, x, t)\) of \( M \), and adjoining an arrow \((t, \epsilon, s)\) for each arrow \((s, \epsilon, t)\). We also adjoin \((s_1, \epsilon, s_2)\) if \( s_1 \) and \( s_2 \) are either both initial states or both final states. We denote this new automaton by \( N \).

Let \( F \) be the free group generated by \( A \). We define a relation on the set of states by \( s \sim t \) if there is a path of arrows from \( s \) to \( t \) in \( N \) whose label gives the identity element of \( F \). This is clearly an equivalence relation. Let \( Q \) be the quotient automaton, each of whose states is one of the equivalence classes above, with arrows inherited from \( M \), not from \( N \). All \( \epsilon \)-arrows are omitted from \( Q \). It is easy to see that \( Q \) is welded.

If \( M \) starts out by being welded, then it is easy to see that \( Q = M \). Consider the identifications of states made during welding (see the passage following Definition 6.1). It is easy to see that the equivalence classes of states used in the definition of \( Q \) are unaltered by one of these identifications. It follows that the automaton \( Q \) remains unaltered during the entire welding process. When no more identifications can be made, we have \( Q \) itself. This shows that \( Q \) is independent of the order in which the identifications are carried out. In fact \( Q \) can be characterized as the largest welded quotient of \( M \).

We claim that every element of \( L(Q) \) arises as follows. Let \((w_1, w_2, \ldots, w_{2k+1})\) be a \( 2k+1 \)-tuple of elements of \( L(M) \), where \( k \geq 0 \). Now consider \( w_1 w_2^{-1} \ldots w_{2k}^{-1} w_{2k+1} \in F \), and write it in reduced form, that is, cancel adjacent formal inverse letters wherever possible. If the result is in \( A^* \), that is, if after cancellation there are no inverse elements, then it is in \( L(Q) \). Moreover, any element of \( A^* \) obtained in this way is in \( L(Q) \). This is straightforward to prove. We leave the details to the reader because we do not need the result. The proof uses the fact that \( M \) is trim.

A welded automaton is minimal. For let \( s \) and \( t \) be distinct states, and let \( u \) and \( v \) be strings over \( A \) which lead from \( s \) and \( t \) respectively to the unique final state. Then \( u \) does not lead from \( t \) to the final state and \( v \) does not lead from \( s \) to the final state (otherwise \( s \) and \( t \) would be equal). It follows that \( s \) and \( t \) remain distinct in the minimized automaton.

If \( M \) is a non-empty trim NFA, we denote by \( \text{Weld}(M) \) the PDFA obtained from it by welding. To compute \( \text{Weld}(M) \) efficiently, we first add “backward arrows” to \( M \). That is, for each arrow \((s, x, t)\) in \( M \), including \( \epsilon \)-arrows, we add the arrow \((t, x', s)\), where \( x' \) represents a backwards version of \( x \). We also add \( \epsilon \)-arrows to connect the initial states, and \( \epsilon \)-arrows to connect the final states. We then make
use of a slightly modified version of the coincidence procedure of Sims given in [11, 4.6]. When this stops we have a welded automaton.

In practice, in the automata which we want to weld, backward arrows are needed in any case for some of our algorithms. The procedure described in the preceding paragraph therefore fits our needs particularly well.

7. A motivating example of welding

We will look at some particular examples to see what can happen during the Knuth–Bendix process on words in a group, and these examples will, we hope, convince the reader of the significance of welding as introduced in the previous section.

We will use the standard generators \( x, y \), and their inverses \( X \) and \( Y \) for the free abelian group on two generators. Using different orderings on this set of four generators, we will see how welding works and why we want to use it.

Consider the alphabet \( A = \{ x, X, y, Y \} \) with the ordering \( x < X < y < Y \), and denote the identity of \( A^* \) by \( \epsilon \). Let \( R \) be the rewriting system on \( A^* \) defined by the set of rules

\[
\{(xX, \epsilon), (Xx, \epsilon), (yY, \epsilon), (Yy, \epsilon), (yx, xy), (yX, Xy), (Yx, xY), (YX, XY)\}.
\]

Using the shortlex-ordering on \( A^* \), it is straightforward to see that \( R \) is a confluent system.

We now change the ordering of the set of generators to \( x < y < X < Y \) and interchange the sides of the sixth rule (to get an order reducing and therefore Noetherian system). Once again the rules define the free abelian group on two generators. But this time there can be no finite confluent set of rewrite rules defining the same congruence. To see this, we consider the set of strings \( \{ xy^nX : n \in \mathbb{N} \} \). None of these is shortlex-least within its \( \leftrightarrow^*_R \)-equivalence class. Therefore each of these strings is reducible relative to any confluent set of rewrite rules which defines \( \leftrightarrow^*_R \). On the other hand, each proper substring of one of the strings \( xy^nX \) is clearly shortlex-least within its \( \leftrightarrow^*_R \)-equivalence class, and is therefore irreducible. It follows that a confluent set of rewrite rules must contain each of the strings \( xy^nX \) as a left-hand side. Hence, in this situation, the Knuth–Bendix procedure will never terminate.

We will show how to generate, after only a few steps, the automaton giving the required infinite confluent set of rewrite rules.

We consider the rule \( r_n = (xy^nX, y^n) \) for some \( n \in \mathbb{N} \). The corresponding padded string \( r_n^* \) gives rise to an \((n + 3)\)-state PDFA \( M(r_n) \) whose accepted language consists solely of the rule \( r_n \). For \( n > 2 \) this PDFA is shown in Figure 1.

![Figure 1. The PDFA \( M(r_n) \) for \( n > 2 \).](image-url)
(final) states for the various $M(r_i)$. If $n > 1$ then $\text{Weld}(M_n)$ is isomorphic to the PDFA given in Figure 2 and the accepted language of this PDFA is the set of rules $\{r_i : i \in \mathbb{N}\}$. This is independent of $n$ if $n > 1$.

So in this example, after only two steps, the welding procedure provides us with a PDFA whose accepted language consists of an infinite set of rules, each of which is a valid identity in the group $A^*/\leftrightarrow_{R}$.

Moreover, by using this PDFA to define a suitable reduction procedure, each of the strings $xy^nX$ with $n \in \mathbb{N}$ can be reduced to the shortlex-least representative of its $\leftrightarrow_{R}$-equivalence class.

For this group with the given ordering on the generators, it is not hard to show that by welding the original defining rules for the group together with the 4 rules $\{(xyX, y), (xy^2X, y^2), (yXY, X), (yX^2Y, X^2)\}$, we obtain a PDFA whose accepted language is a confluent set of rules. The reduction procedure, which we will describe later, corresponding to this PDFA will reduce any string to its shortlex-least representative.

8. Rule automata

For the welding procedure to be used in a general Knuth–Bendix situation, we need to show that any rules obtained are valid identities in the corresponding monoid. We now show that if the monoid is a group (the situation we are interested in), any rules obtained are valid identities.

Definition 8.1. Let $A$ be a finite inverse closed set of monoid generators for a group $G$ and, as before, denote images under the epimorphism $(A^+)^* \to G$ by overscores. A rule automaton for $G$ is an NFA $M = (S, A^+ \times A^+, \mu, F, S_0)$ together with a function $\phi_M : S \to G$ satisfying

1. $F, S_0 \neq \emptyset$.
2. If $s$ is an initial or final state then $\phi_M(s) = 1_G$.
3. For any $s, t \in S$ and $(x, y) \in A^+ \times A^+$ with $(s, (x, y), t) \in \mu$ we have $\phi_M(t) = \overline{x^{-1}\phi_M(s)\overline{y}}$.
4. For any $s, t \in S$ with $(s, \epsilon, t) \in \mu$ we have $\phi_M(s) = \phi_M(t)$.

Here is an equivalent way to look at the definition of a rule automaton. We regard $G$ as the set of states of an automaton with alphabet $A^+ \times A^+$ and with an arrow $(g, (x, y), h)$ if and only if $x^{-1}gy = h$ in $G$. Since $G$ might be infinite, this would mean considering automata with an infinite number of states, and we would have to generalize our definitions. (Automata with an infinite number of states are fairly standard objects in the literature.) In this approach, we next define what we mean by a morphism of automata. A morphism sends states to states and arrows to arrows, but preserves labels on arrows. A rule automaton is then a two-variable automaton with a morphism into the two-variable automaton $G$. We leave the straightforward details to the reader.
Example 8.2. If \( A \) is a finite inverse closed set of monoid generators for a group \( G \) and \( r = (u, v) \in A^* \times A^* \) satisfies \( \rho = \tau \) then, as in Figure 1, writing \( r^+ \) as a string \((u_1, v_1) \cdots (u_n, v_n) \in (A^+ \times A^+)^n\), we obtain an \((n+1)\)-state rule automaton \( M(r) = (\{s_0, \ldots, s_n\}, A^+, A^+, \mu, \{s_0\}, \{s_n\}) \) for \( G \) where the arrows are given by

\[
\mu(s_i, (u_{i+1}, v_{i+1})) = s_{i+1}, \quad 0 \leq i \leq n-1.
\]

The function \( \phi = \phi_M(r) \) assigning group elements to states is defined inductively by \( \phi(s_0) = 1_G \) and \( \phi(s_i) = \overline{u_i}^{-1}\phi(s_{i-1})\overline{v_i} \) for \( 1 \leq i \leq n \). As usual, the padding symbol is sent to \( 1_G \). The fact that \( \rho = \tau \) ensures that condition 2 of Definition 8.1 is satisfied.

Remark 8.3. For a two-variable NFA \( M \) which is a rule automaton, the PDFA \( P \) obtained by applying the subset construction to the (non-empty) set of initial states of \( M \) (and the sets that arise), is also a rule automaton for \( G \) where the map \( \phi_P \) is induced from \( \phi_M \). The fact that this map is well-defined follows from conditions 2, 3 and 4 of Definition 8.1 and the fact that \( P \) is connected (by construction).

The same remark applies to the modified subset construction described in Section 4.

Proposition 8.4. Let \( A \) be a finite inverse closed set of monoid generators for a group \( G \) and suppose that \( M \) is a rule automaton for \( G \). Then

1. Every accepted rule of \( M \) is a valid identity in \( G \).
2. \( \text{Weld}(M) \) is a rule automaton for \( G \).

Consequently every accepted rule of \( \text{Weld}(M) \) is a valid identity in \( G \).

Proof. Let \( r = (u, v) \in A^* \times A^* \) be an accepted rule of \( M \) and write the padded string \((u, v)^+ \) as \((u_1, v_1) \cdots (u_n, v_n) \) where \( n = \max\{|u|, |v|\} \). Then in the PDFA \( P \) obtained from \( M \) (as in Remark 8.3), there exists a sequence of states \( s_0, \ldots, s_n \); also, for each \( i, 1 \leq i \leq n \), there is a arrow from \( s_{i-1} \) to \( s_i \) labelled by \((u_i, v_i)\). Hence, from condition 3 of Definition 8.1, we have

\[
\phi_P(s_i) = \overline{u_i}^{-1} \cdots \overline{u_1}^{-1} v_1 \cdots v_i, \quad \text{for all } i \text{ with } 0 \leq i \leq n.
\]

In particular, \( u_1 \cdots u_m = \overline{u_1}^{-1} \cdots \overline{u_m}^{-1} v_1 \cdots v_m \) and therefore the rule \( r \) is valid in \( G \).

To prove 2, we need only show that when any of the operations described just after Definition 6.1 is applied to a rule automaton \( M \), we continue to have a rule automaton. This is obvious. The final statement is now immediate.

Corollary 8.5. Let \( A \) be a finite inverse closed set of monoid generators for a group \( G \) and suppose that \( r_1, \ldots, r_m \in A^* \times A^* \) are valid identities in \( G \). Then any rule accepted by \( \text{Weld}(M(r_1), \ldots, M(r_m)) \) is also a valid identity in \( G \).

Proof. For \( 1 \leq k \leq m \) let \( M(r_k) \) be the rule automaton for \( G \) as in Example 8.2Then the disjoint union \( U(M(r_1), \ldots, M(r_m)) \) is also a rule automaton for \( G \) and so the result follows by Proposition 8.4.

Remark 8.6. Given a rule automaton \( M \) for a group \( G \), the map \( \phi_M \) may not be injective. However, if \( \phi_M(s) = \phi_M(t) \) and we can somehow determine that this is the case, then we can connect \( s \) to \( t \) by an \( \epsilon \)-arrow, and we still have a rule automaton. If we then weld, \( s \) and \( t \) will be identified. So we can hope to make \( \phi_M \) injective. However, even if \( \phi_M \) is not injective, the rule automaton \( M \) can still be
useful for finding equalities in the group $G$. $M$ may not tell the whole truth, but it does tell nothing but the truth.

9. Which words are reducible?

Suppose $G$ is a group with a finite, inverse closed and ordered set of generators $(A, <)$. In this section, we will work with a fixed two-variable automaton $Rules$. The automaton $Rules$ arises in our work by welding together appropriate rules found so far in the Knuth–Bendix process. However we will not make use of the specific way in which $Rules$ has been constructed. Instead we will write down a list of properties of this automaton—when we come to construct the automaton, it will be easy to see that the properties are either already satisfied or that it can be arranged for them to be satisfied.

9.1. Properties of the rule automaton.

1. $Rules$ is a trim rule automaton.
2. $Rules$ has one initial state and one final state and they are equal.
3. $Rules$ and its reversal $Rev(Rules)$ are both partially deterministic. These conditions imply that $Rules$ is welded.
4. Any arrow labelled $(x, x)$, with source the initial state, also has target the initial state. Any arrow labelled $(x, x)$, with target the initial state, also has source the initial state. If either of these conditions are not fulfilled, we can identify the source and target of the appropriate $(x, x)$-arrows, and then weld. We will still have a rule automaton. Later on (see Lemmas 9.4 and 9.5) we will show that (after any necessary identifications and welding) we can omit such arrows without loss, and, in fact, with a gain given by improved computational efficiency. After proving these lemmas, we will assume there are no arrows labelled $(x, x)$ with source or target the initial state of $Rules$.

Since $Rules$ is a rule automaton, Proposition 8.4 shows that each accepted pair $(u, v) \in L(Rules)$ gives a valid identity $\bar{u} = \bar{v}$ in $G$.

The automaton $Rules$ may accept pairs $(u, v)$ such that $u$ is shorter than $v$. We cannot consider such a pair as a rule and so we want to exclude it. To this end we introduce the automaton $SL2$. This is a five state automaton, depicted in Figure 3, which accepts pairs $(u, v) \in A^* \times A^*$, such that $u$ and $v$ have no common prefix, $u$ is shortlex-greater than $v$ and $|v| \leq |u| \leq |v| + 2$. By combining $SL2$ with $Rules$, we obtain a regular set of rules $Set(Rules)$, which is possibly infinite, namely $L(Rules) \cap L(SL2)$. An automaton accepting this set can be constructed as follows. Its states are pairs $(s, t)$, where $s$ is a state of $Rules$ and $t$ is a state of $SL2$. Its unique initial state is the pair of initial states in $Rules$ and $SL2$. A final state is any state $(s, t)$ such that both $s$ and $t$ are final states. Its arrows are labelled by $(x, y)$, where $x \in A$ and $y \in A^+$. Such an arrow corresponds to a pair of arrows, each labelled with $(x, y)$, the first from $Rules$ and the second from $SL2$.

9.2. Restrictions on relative lengths. The restriction $|u| \leq |v| + 2$ needs some explanation. The point is that if we have a rule with $|u| > |v| + 2$, then we have an equality $\bar{u} = \bar{v}$ in $G$. We write $u = u'x$, where $x \in A$. The formal inverse $X$ of $x$ is also an element of $A$. We therefore have a pair $(u', vX)$ which represent equal elements in $G$. If our set of rules were to contain such a rule, then $u = u'x$ would reduce to $vXx$, and this reduces to $v$, making the rule $(u, v)$ redundant. This leads
to an obvious technique for transforming any rule we find into a new and better rule with $|v| \leq |u| \leq |v| + 2$. Since we take this into account when constructing the automaton $Rules$, we are justified in making the restriction.

This analysis can be carried further. Let $u = u_1 \cdots u_{r+2} = u'u_{r+2} = u'u''$ and let $v = v_1 \cdots v_r$. If $u_1 > v_1$, then the rule $(u, v)$ can be replaced by the better rule $(u', vu_{r+1})$. If $u_2 > u_1^{-1}$, then $(u, v)$ can be replaced by $(u'', u_1^{-1}v)$. We do in fact carry out these steps when installing new rules, but we have not so far tried adjusting the finite state automaton $SL2$ accordingly to see what effect this would have on the whole process.

9.3. Rules for which no prefix or suffix is a rule. At the moment, it is possible for an element $(u, v)^+$ of $\text{Set}(Rules)$ to have a prefix or suffix which is also a rule. This is undesirable because it makes the computations we will have to do bigger and longer without any compensating gain.

Recall that the automaton recognizing $\text{Set}(Rules)$ is the product of $Rules$ with $SL2$, the initial state being the product of initial states and the set of final states being any product of final states.

We remove from $Rules$ any arrow labelled $(x, x)$ from the initial state to itself. We then form the product automaton, as described above, with two restrictions. Firstly, we omit any arrow whose source is a product of final states. Secondly, we omit completely the state and all arrows whose source or target is the state with first component equal to $s_0$, the initial state of $Rules$, and second component equal to state 3 of $SL2$ (see Figure 3). We call the resulting automaton $Rules'$.

Lemma 9.4. The language accepted by $Rules'$ is the set of labels of accepted paths in the product automaton, starting from the product of initial states and ending at a product of final states, such that the only states along the path with first component equal to $s_0$ are at the beginning and end of the path.

Proof. First consider an accepted path $\alpha$ in $Rules'$. The only arrows in $Rules'$ with source having first component $s_0$ are those with source the product of initial states.

Figure 3. The automaton $SL2$. Solid dots represent final states. Roman letters represent arbitrary letters from the alphabet $A$ and the labels on the arrows indicate multiple arrows. For example, from state 2 to itself there is one arrow for each pair in $A \times A$. 
In $SL2$ it is not possible to return to the initial state. It follows that $\alpha$ has the required form.

Conversely any such path in the product automaton also lies in $Rules'$ because it avoids all omitted arrows.

**Lemma 9.5.** The language accepted by $Rules'$ is the subset of $\text{Set}(Rules)$ which has no proper suffix or proper prefix in $\text{Set}(Rules)$.

**Proof.** If $\alpha$ is an accepted path in $Rules'$, then it is clearly in $\text{Set}(Rules)$. Moreover if it had a proper suffix or proper prefix which was in $\text{Set}(Rules)$, there would be a state in the middle of $\alpha$ with first component $s_0$. We have seen that this is impossible in Lemma 9.4.

Conversely, we must show that if $\alpha$ is an accepted path in the product automaton such that no proper prefix and no proper suffix of $\alpha$ would be accepted by the product automaton, then no state met by $\alpha$, apart from its two ends, has $s_0$ as a first component. Let $\alpha = ((s_0, 1), (u_1, v_1), q_1, \ldots, (u_n, v_n), q_n)$.

First suppose $u_1 < v_1$. Since $\alpha$ is accepted by $SL2$, we must have $v_n = \$. Let $r < n$ be chosen as large as possible so that the first component of $q_r$ is $s_0$. Then $(u_{r+1}, v_{r+1}), \ldots, (u_n, v_n)$ will be accepted by $Rules$ and will be accepted by $SL2$ because $v_n = \$. Since this cannot be a proper suffix of $\alpha$ by assumption, we must have $r = 0$. Hence $q_1$ has a first component equal to $s_0$ if and only if $i = 0$ or $i = n$.

Next note that we cannot have $u_1 = v_1$. This is because there is no arrow labelled $(u_1, u_1)$ in $SL2$ with source the initial state, so $\alpha$ would not be accepted by the product automaton.

Now suppose that $u_1 > v_1$ and let $r > 0$ be chosen as small as possible, so that the first component of $q_r$ is $s_0$. Since $u_1 > v_1$, the second component of $q_r$ will be a final state (see Figure 3). Since $\alpha$ has no accepted proper prefix, we must have $r = n$. Hence $q_i$ has a first component equal to $s_0$ if and only if $i = 0$ or $i = n$.

So we have proved the required result for each of the three possibilities.

Let $w = x_1 \cdot \ldots \cdot x_n \in A^*$ be a string which we wish to reduce to a $\text{Set}(Rules)$-irreducible. It is important for this to be done quickly, as it has been observed by many people that the Knuth–Bendix process for strings spends most of its time reducing. Reduction needs to be carried out during critical pair analysis.

Reduction with respect to $\text{Set}(Rules)$ is done in a number of steps. First we find the shortest reducible prefix of $w$, if this exists. Then we find the shortest suffix of that which is reducible. This is a left-hand side of some rule in $\text{Set}(Rules)$. Then we find the corresponding right-hand side and substitute this for the left-hand side which we have found in $w$. This reduces $w$ in the shortlex-order. We then repeat the operation until we obtain an irreducible string. The process will be described in detail in this section and in the subsequent two sections. There is an outline of the reduction process in 11.1.

Our objective in this section is to find the shortest reducible prefix of $w$, if this exists. To achieve this, we must determine whether $w$ contains a substring which is the left-hand side of a rule belonging to $\text{Set}(Rules)$.

Let $Rules''$ be the automaton obtained from $Rules'$ (see Lemmas 9.4 and 9.5) by adding arrows labelled $(x, x)$ from the initial state to the initial state.

We construct an NFA $Rble_N(Rules)$ in one variable by replacing each label of the form $(x, y)$ on an arrow of $Rules''$ by $x$. Here $x \in A$ and $y \in A^+$. The name of the automaton $Rble_N(Rules)$ refers to the fact that the automaton accepts reducible
strings, and does so non-deterministically. We obtain an NFA with no \( \epsilon \)-arrows. However there may be many arrows labelled \( x \) with a given source. Let \( LHS(Rules) \) be the regular language of left-hand sides of rules in \( \text{Set}(Rules) \) such that no proper prefix or proper suffix of the rule is itself a rule.

**Lemma 9.6.** \( A^* \cdot LHS(Rules) = L(Rble_N(Rules)) \).

**Proof.** Because of the extra arrows labelled \((x, x)\) from initial state to initial state, inserted into \( Rules'' \), the inclusion \( A^* \cdot LHS(Rules) \subset L(Rble_N(Rules)) \) is clear.

If \( u \) is accepted by \( Rble_N(Rules) \), there is a corresponding pair \((u, v)\) accepted by \( Rules'' \). We find a maximal common prefix \( p \) of \( u \) and \( v \), so that \( u = pu' \) and \( v = pv' \). \( Rules'' \) remains in the initial state while reading \((p, p)\). Since the initial state of \( SL2 \) is not a final state, \((u', v')\) must be non-empty. Since there is no way of returning to the initial state of \( SL2 \), once \( Rules'' \) starts reading \((u', v')\), it can never return to the initial state, and therefore \((u', v')\) must be accepted by \( Rules' \). Therefore \( u' \in LHS(Rules) \), as claimed.

To find the shortest reducible prefix of a given string \( w \) we could feed \( w \) into the FSA \( Rble_N(Rules) \). However, reading a string with a non-deterministic automaton is very time-consuming, as all possible alternative paths need to be followed.

For this reason, it may at first sight seem sensible to determinize the automaton. However, determinizing a non-deterministic automaton potentially leads to an exponential increase in size. The states of the determinized automaton are subsets of the non-deterministic automaton, and there are potentially \( 2^n \) of them if there were \( n \) states in the non-deterministic automaton. By trying examples, we have observed that the theoretical exponential blow-up in this construction is sometimes a practical reality for the automaton \( Rble_N(Rules) \).

For this reason, we use a lazy state-evaluation form of the subset construction. The lazy evaluation strategy (common in compiler design—see for example [1]), calculates the arrows and subsets as and when they are needed, so that a gradually increasing portion \( P(Rules) \) of the determinized version \( Rble_D(Rules) \) of \( Rble_N(Rules) \) is all that exists at any particular time.

Lazy evaluation is not automatically an advantage. For example, if in the end one has to construct virtually the whole determinized automaton \( Rble_D(Rules) \) in any case, then nothing would be lost by doing this immediately. In our special situation, lazy evaluation is an advantage for two reasons. First, during a single pass of the Knuth–Bendix process (see Paragraph 2.11), only a comparatively small part of the determinized one-variable automaton \( Rble_D(Rules) \) needs to be constructed. In practice, this phenomenon is particularly marked in the early stages of the computation, when the automata are far from being the “right” ones. Second, this approach gives us the opportunity to abort a pass of Knuth–Bendix, recalculate on the basis of what has been discovered so far in this pass, and then restart the pass. If an abort seems advantageous early in the pass, very little work will have been done in making the structure of the determinized version of \( Rble_D(Rules) \) explicit.

We now describe the details of this strategy.

At the start of a Knuth–Bendix pass (see Paragraph 2.11) we let \( P(Rules) \) be the one-variable automaton consisting of a single non-final state containing only the ordered pair of initial states of \( Rules \) and \( SL2 \). At a subsequent time during the pass, \( P(Rules) \) may have increased, but it will always be a portion of \( Rble_D(Rules) \).
Suppose now that we wish to find the shortest prefix of the string \( w = x_1 \cdots x_n \in A^* \) which is \( \text{Set}(\text{Rules}) \)-reducible. Suppose that \( s_0, s_1, \ldots, s_k \) are states of \( P(\text{Rules}) \), where \( 0 \leq k \leq n - 1 \), that \( s_0 \) is the start state of \( P(\text{Rules}) \), and that, for each \( i \) with \( 1 \leq i \leq k \), we have \( \mu(s_{i-1}, x_i) = s_i \). Suppose that the target of the arrow \( \mu(s_k, x_{k+1}) \) is not yet defined.

By definition, the subset construction applied to the state \( s_k \) of \( P(\text{Rules}) \) under the alphabet symbol \( x_{k+1} \) yields the set \( \mu_1(s_k, x_{k+1}) \) as follows. For each \( (s', t') \in s_k \), we look for all arrows in \( \text{Rble}_N(\text{Rules}) \) labelled \( x_{k+1} \) with source \( (s', t') \). If \( (s, t) \) is the target of such an arrow, then \( (s, t) \) is an element of \( \mu_1(s_k, x_{k+1}) \). Note that this subset is always non-empty, because the initial state of \( \text{Rble}_N(\text{Rules}) \) is an element of each \( s_i \).

In the standard determinization procedure one would now look to see whether there is already a state \( s_{k+1} \) of \( P(\text{Rules}) \) which is equal to \( \mu_1(s_k, x_{k+1}) \). If not one would create such a state \( s_{k+1} \). One would then insert an arrow labelled \( x_{i+1} \) from \( s_k \) to \( s_{k+1} \), if there wasn’t already such an arrow. A new state is defined to be a final state of \( P(\text{Rules}) \) if and only if the subset contains a final state of \( \text{Rble}_N(\text{Rules}) \).

Of course, one does not need to determine the subset \( \mu_1(s_k, x_{k+1}) \) if there is already an arrow in \( P(\text{Rules}) \) labelled \( x_{k+1} \) with source \( s_k \), because in that case the subset is already computed and stored.

In our procedure we improve on the procedure just described. The point is that \( \mu_1(s_k, x_{k+1}) \) may contain pairs which are not needed and can be removed. From a practical point of view this has the advantage of saving space and reducing the amount of computation involved when calculating subsequent arrows. Specifically, we remove a pair \( (p, q') \) from \( \mu_1(s_k, x_{k+1}) \) if \( q' \) is state 3 of SL2 (see Figure 3) and \( \mu_1(s_k, x_{k+1}) \) also contains the pair \( (p, q) \) where \( q \) is state 2 of SL(5, A). Removing all such pairs \( (p, q') \) yields the set \( \mu_0(s_k, x_{k+1}) \) and we add the corresponding arrow and state to \( P(\text{Rules}) \), creating a new state if necessary. We make the state a final state if the subset contains a final state of \( \text{Rble}_N(\text{Rules}) \). The validity of this modification follows from Theorem 4.1 and we see that some prefix of \( w \) arrives at a final state of \( P(\text{Rules}) \) if and only if \( w \) is \( \text{Set}(\text{Rules}) \)-reducible.

When finding the corresponding left-hand side of a rule inside \( w \), we need never compute beyond a final state of \( P(\text{Rules}) \). As a space-saving and time-saving measure our implementation therefore replaces each final state of \( P(\text{Rules}) \), as soon as it is found, by the empty set of states. As remarked above, the standard determinization of \( \text{Rble}_N(\text{Rules}) \) never produces an empty set of states, and so there is no possibility of confusion.

Reading \( w \) can be quite slow if many states need to be added to \( P(\text{Rules}) \) while it is being read. However, reading \( w \) is fast when no states need to be built. In practice, fairly soon after a Knuth–Bendix pass starts, reading becomes rapid, that is, linear with a very small constant.

10. Finding the left-hand side in a string

We retain the hypotheses of Section 9. Namely, we have a two-variable automaton \( \text{Rules} \) satisfying the conditions of Paragraph 9.1. We are given a word \( w = x_1 \cdots x_n \), and we wish to reduce it. In the previous section we showed how to find the minimal reducible prefix \( w' = x_1 \cdots x_m \) of \( w \) with respect to the rules implicitly specified by \( \text{Rules} \). We now wish to find the minimal suffix of \( w' \) which is
a left-hand side of some rule in $\text{Set}(\text{Rules})$. The procedure is quite similar to that of the previous section.

We form the two-variable automaton $\text{Rev}(\text{Rules})$, which we combine with $\text{Rev}(\text{SL2})$. The first automaton is, by hypothesis, partially deterministic. If we determinize the second automaton, we obtain another PDFA. Figure 4 shows the determinization of $\text{Rev}(\text{SL2})$, where the subsets of states of $\text{SL2}$ are explicitly recorded.

![Diagram](image)

**Figure 4.** This PDFA arises by applying the accessible subset construction to $\text{Rev}(\text{SL2})$ in the case where the base alphabet has more than one element. Each state is a subset of the state set of $\text{Rev}(\text{SL2})$ and final states have a double border. This PDFA, when reading a pair $(u, v)$ from right to left, keeps track of whether $u$ is longer than $v$ or not, which it discovers immediately since padding symbols if any must occur at the right-hand end of $v$. Note that this automaton is minimized.

We take the product of the two automata $\text{Rev}(\text{Rules})$ and $\text{Rev}(\text{SL2})$. A new state is a pair of old states. An arrow is a pair of arrows with the same label $(x, y)$. The initial state in the product is the unique pair of initial states. A final state in the product is a pair of final states.

To form the one-variable non-deterministic automaton $\text{Rev}_X(\text{LHS}(\text{Rules}))$ without $\epsilon$-arrows, we use the same states and arrows as in the product automaton, but replace each label of the form $(x, y)$ in the product automaton by the label $x$. The deterministic one-variable automaton $\text{Rev}_D(\text{LHS}(\text{Rules}))$ can then be constructed using the subset construction.

We have gone through the above description to give the reader a theoretical understanding of what is going on before going into details. Also, our procedure may be adaptable to related situations which are not identical to this one. In fact, we use not the construction just described, but a related construction which we describe below. The point of what we do may not become fully apparent until we get to Section 11.

10.1. **Reversing the rules.** We first describe a two-variable PDFA $M$ which accepts exactly the reverse of each rule $(\lambda, \rho)^+$ in $\text{Set}(\text{Rules})$ such that no proper suffix
and no proper prefix of $(\lambda, \rho)^+$ is in $\text{Set}(\text{Rules})$ (cf. Lemma 9.5). We assume that we have a two-variable automaton Rules satisfying the conditions of Paragraph 9.1.

A state of $M$ is a triple $(s, i, j)$, where $s$ is a state of $\text{Rev}(\text{Rules})$, $i \in \{0, 1, 2\}$ and $j \in \{+, -, \}$, $M$ has a unique initial state $(s_0, 0, +)$ and $s_0$ is the unique initial state of $\text{Rev}(\text{Rules})$. In addition, $M$ has three final states $f_0 = (s_0, 0, -), f_1 = (s_0, 1, -)$ and $f_2 = (s_0, 2, -)$. We do not allow states of $M$ of the form $(s_0, i, j)$, except for the initial state and the three final states just mentioned. We will construct the arrows of $M$ to ensure that any path of arrows accepted by $M$ has first component equal to $s_0$ for its initial state and its final state and for no other states. (Compare this with Lemma 9.4.)

The language accepted by $M$ is the set of reversals of rules $(\lambda, \rho)^+ \in \text{Set}(\text{Rules})$. Moreover, there are no proper prefix of $(\lambda, \rho)^+$ in $\text{Set}(\text{Rules})$.

The proof of this lemma is much the same as the proofs of Lemmas 9.4 and 9.5. We therefore omit it.

Using the above description of $M$, we now describe how to obtain a non-deterministic one-variable automaton $\text{Rev}_N(LHS(\text{Rules}))$ from $M$ in an analogous manner to that used to obtain $\text{RLbe}_N(\text{Rules})$ from $\text{Rules}^\prime$ in Section 4. $\text{Rev}_N(LHS(\text{Rules}))$ accepts reversed left-hand sides of rules in $\text{Set}(\text{Rules})$ which do not have a proper prefix or a proper suffix which is in $\text{Set}(\text{Rules})$. $\text{Rev}_N(LHS(\text{Rules}))$ has the same set of states as $M$ and the same set of arrows. However, the label $(x, y)$ with $x \in A$ and
sets

\{ the initial state of \{ the singleton \\
Set \}\ in \ x \ Q \ has been read so far and that as a result the current state of \\
\text{Note 10.3.} In order to construct states and arrows in \text{Set} \algorithm that finds the shortest such suffix.

\text{Rev} to have access to \text{explicitly constructed.} \\
variable PDFA \text{Rev} \ Q \ modified subset construction, using lazy evaluation. \text{Rev} whose reversals are accepted by \text{Rev}. Hence \text{two automata,} \\
\text{N} \ y \ ∈ \ A^+ \ and we know it has a suffix which is the left-hand side of some rule \\
in \text{Set} \algorithm. Suppose no proper prefix of \ x \ \text{has this property.} We give an \\
algorithm that finds the shortest such suffix.

We read the string from right to left, starting with \ x \ n. We assume that \ x_{k+1}x_{k+2} \ · \ · \ · \ x_n \\
has been read so far and that as a result the current state of \ Q \algorithm is \ S_k, where \\
\ S_k \ is a state of \ Q \algorithm \ (so \ S_k \ is a subset of the set of states of \ Rev \text{Rev} \ Q \algorithm)). \\
We start the algorithm with \ k = n \ and the current state of \ Q \algorithm equal to \\
the singleton \ \{(s_0, 0, +)\} whose only element is the initial state of \ M, where \ s_0 \ is \\
the initial state of \ Rev \text{Rev} \ Q \algorithm. \ Q \algorithm has three final states, namely the singleton \\
sets \ \{f_i\} for \ i = 0, 1, 2.

The steps of the algorithm are as follows:

1. Record the current state as the \ k-th entry in an array of size \ n, where \ n \ is \\
the length of the input string.
2. If the current state is not a final state, go to \ Step \[10.4.3\]. If the current state \\
is a final state, then stop. Note that the initial state of \ Q \algorithm is not a final \\
state, so this step does not apply at the beginning of the algorithm. If the \\
current state is a final state, then the shortest suffix of \ x_1 \ · \ · \ · \ x_n \ which is the \\
left-hand side of a rule in \text{Set} \algorithm \text{can then be proved to be} \ x_{k+1}x_{k+2} \ · \ · \ · \ x_n.
3. If the arrow labelled \ x_k \ with source the current state is already defined, then \\
redefine the current state to be the target of this arrow and decrease \ k \ by one.
4. If the preceding step does not apply, we have to compute the target \ T \ of the \\
arrow labelled \ x_k \ with source the current state \ S_k. \ We do this by looking \\
for all arrows labelled \ x_k \ in \ Rev \text{Rev} \ (LHS(\text{Rules})) with source in \ S_k, and define \\
\ T \ to be the set of all targets. Note that this set of targets cannot be empty since \\
we know that some suffix of \ x_1 \ · \ · \ · \ x_n \ is accepted by \ Rev \text{Rev} \ LHS(\text{Rules}). \\
5. There are two modifications which we can make to the previous step.
   (a) Firstly, if the set of targets contains some final state \ f_j, then we look for 
       the largest value of \ i = 0, 1, 2 \ such that \ f_i \ ∈ \ T \ and redefine \ T \ to be \ \{f_i\}. 
       We then insert into \ Q(\text{Rules}) \ an arrow labelled \ x_k \ from \ S_k \ to this final 
       state. If we have found that \ T \ is a final state, we set \ S_{k-1} \ equal to \ T, 
       decrease \ k \ by one, and go to \ Step \[0.4.3\].
   (b) Secondly, if, while calculating the set \ T, we find that a state \ s \ of \ Rev(\text{Rules}) 
       occurs in more than one triple \ (s, i, j), then we only include the triple with
the largest value of $i$. For this to be well-defined, we need to know that
$(s, i, +)$ and $(s, i, -)$ cannot both come up as potential elements of $T$—
this is addressed in the proof of Theorem 10.3 along with justifications of
the other modifications.

6. Having found $T$, see if it is equal to some state $T'$ of $Q(Rules)$ which has
already been constructed. If so, define an arrow labelled $x_k$ from $S$ to $T'$.

7. If $T$ has not already been constructed, define a new state of $Q(Rules)$ equal
to $T$ and define an arrow labelled $x_k$ from $S$ to $T$.

8. Set the current state equal to $T$ and decrease $k$ by one. Then go to Step 10.4.1.

Theorem 10.5. Suppose $x_1 \cdots x_n$ has a suffix which is the left-hand side of a rule
in $Set(Rules)$ and suppose no prefix of $x_1 \cdots x_n$ has this property. Then the above
algorithm correctly computes the shortest such suffix.

Proof. We first show that the modification in Step 10.4.5.b is well-defined in the
sense that triples $(s, i, +)$ and $(s, i, -)$ cannot both occur while calculating $T$. The
reason for this is that the third component can only be $+$ if either non of $x_1 \cdots x_n$
has been read, in which case the only relevant state is $(s_0, 0, +)$, or else only $x_n$
has been read, in which case the possible relevant states are $(f, 1, -)$, $(s, 1, +)$ with
$s \neq f$, and $(s, 0, -)$. So a state of the form $(s, i, j)$ with a given $s$ occurs at most
once in a fixed subset with the maximum possible value of $i$.

The effect of Step 10.4.5.a in the above algorithm is to ensure that termi-
nation occurs as soon as a final state of $Rev(Rules)$ appears in a calculated triple. Since we
know that $x_1 \cdots x_n$ contains a left-hand side of a rule in $Set(Rules)$ as a suffix we
need only show that the introduction of Step 10.4.5.b does not affect the accepted
language of the constructed automaton. This will be a consequence of Theorem 4.1,
as we now proceed to show.

Consider a triple $t = (s, i, j)$ arising during the calculation of a subset $T$, and
suppose that $s$ is a non-final state of $Rev(Rules)$. If $j = +$ then $T$ cannot contain
both $(s, 0, +)$ and $(s, 1, +)$ and so $t$ will not be removed from $T$ as a result of
Step 10.4.5.a. Therefore we only need to consider the case $j = -$. For $k = 0, 1, 2,$
let $L_k \subseteq A^* \times A^*$ be the language obtained by making $(s, k, -)$ the only initial
state of $M$, and observe that there can be no padded arrows in any path of arrows
from $(s, k, -)$ to a final state of $M$. Now by considering the definition of the non-
padded transitions in $M$ given in 10.1.4, it is straightforward to see that $L_0 \subseteq
L_1 = L_2$. Therefore, since $Rev_N(LHS(Rules))$ has no $-\epsilon$-arrows, we have just shown
that the hypotheses of Theorem 4.1 apply to Step 10.4.5.b. Hence the omission in
Step 10.4.5.b does not affect the accepted language of $Q(Rules)$.

As with $P(Rules)$, reading a word into $Q(Rules)$ from right to left can be slow in
the initial stages of a Knuth–Bendix pass, but soon speeds up to being linear with
a small constant.

11. Finding the right-hand side of a rule

We retain the hypotheses of Section 9.1. Namely, we have a two-variable rule
automaton $Rules$ which is welded and satisfies various other minor conditions. We
are given a word $w = x_1 \cdots x_n$, and we wish to reduce it relative to the rules
implicitly contained in $Rules$. So far we have located a left-hand side $\lambda$ which
is a substring of $w$. In this section we show how to construct the corresponding
right-hand side.
We first go into more detail as to how we propose to reduce \( w \). In outline we proceed as follows.

11.1. Outline of the reduction process.

1. Feed \( w \) one symbol at a time into the one-variable automaton \( P(Rules) \) described in Section 2, storing the history of states reached on a stack.
2. If a final state is reached after some prefix \( u \) of \( w \) has been read by \( P(Rules) \), then \( u \) has some suffix which is a left-hand side. Moreover, this procedure finds the shortest such prefix.
3. Feed \( u \) from right to left into \( Q(Rules) \). A final state is reached as soon as \( Q(Rules) \) has read the shortest suffix \( \lambda \) of \( u \) such that there is a rule \((\lambda, \rho) \in \text{Set}(Rules)\). We now have \( u = p\lambda \) and \( w = p\lambda q \), where \( p, q \in A^* \), every proper prefix of \( p\lambda \) and every proper suffix of \( \lambda \) is \( \text{Set}(Rules) \)-irreducible.
4. Find \( \rho \), the smallest string such that there is a rule \((\lambda, \rho) \in S \) (see 2.11). If there is no such rule in \( S \), find \( \rho \) by a method to be described in this section, such that \( \rho \) is the smallest string such that \((\lambda, \rho) \in \text{Set}(Rules)\).
5. If \((\lambda, \rho) \) is not already in \( S \), insert it into the part of \( S \) called \( \text{New} \).
6. Replace \( \lambda \) with \( \rho \) in \( w \) and pop \(|\lambda|\) levels from the stack so that the stack represents the history as it was immediately after feeding \( p \) into \( P(Rules) \).
7. Redefine \( w \) to be \( ppq \). Restart at Step 1 as though \( p \) has just been read and the next letter to be read is the first letter of \( \rho \). The history stack enables one to do this.

Note that other strategies might lead to finding first some left-hand side in \( w \) other than \( \lambda \). Moreover, there may be several different right-hand sides \( \rho \) with \((\lambda, \rho) \in \text{Set}(Rules)\). A rule \((\lambda, \rho) \) in \( \text{Set}(Rules) \) gives rise to paths in \( Rules, SL2 \) and \( \text{Rev}_{D}(SL2) \). We will find the path for which right-hand side \( \rho \) is shortlex-least, given that the left-hand side is equal to \( \lambda \).

Let \( \lambda = y_1 \cdots y_m \). Recall that a state of the one-variable automaton \( Q(Rules) \) used to find \( \lambda \) is a set of states of the form \((s, i, j)\), where \( s \) is a state of \( Rules \), \( i \in \{0, 1, 2\} \) and \( j \in \{+, -\} \). When finding \( \lambda \) we kept the history of states of \( Q(Rules) \) which were visited—see Step 0.4.1. Let \( Q_k \) be the set of triples \((s, i, j)\) comprising the state of \( Q(Rules) \) after reading the string \( y_{k+1} \cdots y_m \) from right to left. \( Q_0 = \{f_1\} = \{(s_0, i, -)\} \) where \( s_0 \) is the unique initial and final state of \( Rules \), and \( i \) is the difference in length between \( \lambda \) and the \( \rho \) that we are looking for.

11.2. Right-hand side routine. Inductively, after reading \( y_1 \cdots y_k \) we will have determined \( z_1 \cdots z_k \), the prefix of \( \rho \). Inductively we also have a triple \((s_k, i_k, j_k)\), where \( s \) is a state of \( Rules \), \( i_k \) is 0 or 1 or 2 and \( j_k \) is + or -. Note that we always have \( m - k \geq i_k \).

1. If \( m - k = i_k \), then we have found \( \rho = z_1 \cdots z_k \) and we stop. So from now on we assume that \( m > i_k + k \). This means that the next symbol \((y_{k+1}, z_{k+1})\) of \((\lambda, \rho) \) does not have a padding symbol in its right-hand component.
2. We now try to find \( z_{k+1} \) by running through each element \( z \in A \) in increasing order. Set \( z \) equal to the least element of \( A \).
3. If \( k = 0 \) and \( i_0 = 0 \), then \( \lambda \) and \( \rho \) will be of equal length, so the first symbol of \((\lambda, \rho) \) must be \((y_1, z_1)\), where \( y_1 > z_1 \). So at this stage we can prove that we have \( y_1 > z \), since we know that there must be some right-hand side corresponding to our given left-hand side.
If $k = 0$ and $i_0 > 0$, then the first symbol of $(\lambda, \rho)^+$ is $(y_1, z_1)$ with $z_1 \in A$ and $y_1 \neq z_1$. If $k = 0$, $i_0 > 0$ and $y_1 = z$, we increase $z$ to the next element of $A$.

4. Here we are trying out a particular value of $z$ to see whether it allows us to get further. We look in Rules to see if $s_k^{(y_{k+1}, z)} = s_{k+1}$ is defined. If it is not defined, we increase $z$ to the next element of $A$ and go to Step 11.2.3.

5. If $s_{k+1}$ is defined in Step 11.2.4, we look in $Q_{k+1}$ for a triple $(s_{k+1}, i_{k+1}, j_{k+1})$ which is the source of an arrow labelled $(y_{k+1}, z)$ in the automaton $M$. Recall that $M$ was defined in Section 10. Note that, by the proof of Theorem 10.5, $Q_{k+1}$ contains at most one element whose first coordinate is $s_{k+1}$. As a result, the search can be quick.

6. If $(s_{k+1}, i_{k+1}, j_{k+1})$ is not found in Step 11.2.5, increase $z$ to the next element of $A$ and go to Step 11.2.3.

7. If $(s_{k+1}, i_{k+1}, j_{k+1})$ is found in Step 11.2.5, set $z_{k+1} = z$, increase $k$ and go to Step 11.2.1.

The above algorithm will not hang, because each triple $(s, i, j)$ that we use does come from a path of arrows in $M$ which starts at the initial state of $M$ and ends at the first possible final state of $M$. Therefore all possible right-hand sides $\rho$ such that $(\lambda, \rho) \in \text{Set}(\text{Rules})$, are implicitly computed when we record the states of $Q(\text{Rules})$ (see Step 10.4.1). Since $i_k$ does not vary during our search, we will always find the shortest possible $\rho$, with $|\lambda| - |\rho|$ being equal to this constant value of $i_k$. Since we always look for $z$ in increasing order, we are bound to find the lexicographically least $\rho$.

We remind the reader that an overview of the entire reduction process for a given string $w$ is given in 11.1.

12. OUR VERSION OF KNUTH–BENDIX.

For finite Noetherian rewriting systems the question of confluence is decidable by the critical pair analysis described in Section 2. However, for infinite Noetherian rewriting systems the confluence question is, in general, undecidable. Examples exhibiting undecidability are given in [10] and are length-reducing rewriting systems $R$ which are regular in the sense that $R$ contains only a finite number of right-hand sides and for each right-hand side $r$, the set $\{l : (l, r) \in R\}$ is a regular language. These examples are in the context of rewriting for monoids. As far as we know, there is no known example of undecidability if we add to the hypothesis that the monoid defined by $R$ is in fact a group.

In this section we consider a rewriting system which is the accepted language of a rule automaton for some finitely presented group. We describe a Knuth–Bendix type algorithm for such a system. In light of the undecidability result mentioned above, our algorithm does not provide a test for confluence. We can however use our algorithm together with other algorithms for dealing with shortlex-automatic groups, to prove confluence by an indirect route if the group is shortlex-automatic. Details of the theory of how this is done can be found in [3]. The practical details are carried out in programs by Derek Holt—see [5].

Suppose throughout that $G$ is a monoid given by a finite presentation $\langle A/R \rangle$, where $A$ is a set of generators for $G$ with a fixed total ordering $<$ and $R$ is a finite set of equalities. The monoid is defined by the congruence generated by these equalities. We will assume that there is an involution $i : A \to A$ (which will send
each generator to its formal inverse) such that, for each element \( x \in A \), there are equalities in \( R \) of the form \( \alpha(x) = \epsilon \) and \( \delta(x) = \epsilon \). This implies that \( G \) is a group. The equalities in \( R \) can be regarded as a finite set of rules which define \( G \).

In our algorithm, we keep two sets of rules. One set, which we call \( S \), is a finite set of rules. The other is a possibly infinite set of rules which is kept implicitly in a rule automaton called \( Rules \). When we want to specify that we are working with the \( Rules \) automaton during the \( n \)th Knuth-Bendix pass (see 2.11 for the definition of a Knuth-Bendix pass), we will use the notation \( Rules[n] \). We extract explicit rules from \( Rules[n] \) by taking elements of the intersection \( Set(Rules[n]) = L(Rules[n]) \cap L(SL2) \). The two-variable automaton \( SL2 \) was defined in Section 9 and is depicted in Figure 3.

\( S \) will change almost continually, while \( Rules \) is constant during a Knuth-Bendix pass. We do in fact need to change \( Rules \) from time to time, and we do so as the last step of each Knuth–Bendix pass. We will perform the Knuth–Bendix process, using the rules in \( S \) for critical pair analysis, as described in 2.4.

12.1. **Rapid reduction.** A difference between our situation and that of classical Knuth–Bendix is that reduction is not carried out by applying the rules of \( S \). When running Knuth–Bendix, one of the most time-consuming aspects is reduction. This is partly because there is a lot of reduction to be done and partly because one normally has to spend a long time looking through a long list of rules to see if the string one is trying to reduce contains a left-hand side of some rule. Much of the effort in producing new Knuth–Bendix algorithms, like the algorithm described in this paper, goes on finding methods of locating relevant rules quickly. In the past this has involved using structures which use a lot of space. In our procedure we use the method described in [11.1] to find relevant rules quickly without using an inordinate amount of space. We refer to this as \( R \)-reduction. We also use the terms \( R \)-reduce and its various derivatives. \( R \) stands for “relation”, for “reduction” and for “rapid”.

**Note 12.2.** Note that a string is \( R \)-reducible at one point in a Knuth–Bendix pass if and only if it is \( R \)-reducible at another point in the same pass. However, as we shall see, the result of \( R \)-reduction may change during a pass, because \( S \) changes.

12.3. **The basic structures.** The basic structures used in our procedure are:

1. A two-variable automaton \( Rules \) satisfying the conditions laid down in [9.1].
2. A finite set \( S \) of rules, which is the disjoint union of several subsets of rules : \( Considered \), \( This \), \( New \) and \( Delete \).
3. \( Considered \) is a subset of \( S \) such that each rule has already been compared with each other rule in \( Considered \), including with itself, to see whether left-hand sides overlap. The consequent critical pair analysis has also been carried out for pairs of rules in \( Considered \). Such rules do not need to be compared with each other again.
4. \( This \) is a subset of \( S \) containing rules which we plan to use during this pass to compare for overlaps with the rules in \( Considered \), as in [2.4]. These rules have been minimized during the current pass (see [12.7]) and so should not be minimized again.
5. \( New \) is a subset of \( S \) containing new rules which have been found during the current pass, other than those which are output by the minimization routine
AUTOMATIC GROUPS AND KNUTH–BENDIX (see 12.7). Non-trivial rules which are the final output of the minimization routine are added to This.

6. **Delete** is a subset of $S$ containing rules which are to be deleted at the beginning of the next pass.

7. A two-variable automaton $WDiff$ which contains all the states and arrows of $Rules[n]$, and possibly other states and arrows. It satisfies the conditions of 9.1.

### 12.4. Initial arrangements

Before describing the main Knuth–Bendix process, we explain how the data structures are initially set up. Recall that $R$ (which should be distinguished from $R$) is the original set of defining relations together with special rules of the form $(x\iota(x), \varepsilon)$ and $(\iota(x)x, \varepsilon)$ which make the formal inverse $\iota(x)$ into the actual inverse of $x$.

We rewrite each non-special element of $R$ in the form of a relator, which we cyclically reduce in the free group. Since $\iota(x)$ is the formal inverse of the letter $x$, we are able to write down the formal inverse of any string in $A^*$. We may therefore assume that each relator has the form $lr^{-1}$, where $l$ and $r$ are elements of $A^*$ and $(l, r)$ is accepted by $SL2$.

For each rule $(l, r)$, including the special rules, we form a rule automaton, as explained in Example 8.2. These automata are then welded together to form the two-variable rule automaton $WDiff$ satisfying the conditions of 9.1. Each state and arrow of $WDiff$ is marked as needed. (At certain well-chosen moments we will delete from $WDiff$ states and arrows that are not needed). Each of these rules is inserted into This. Considered, New and Delete are initially empty. Set $Rules[1] = WDiff$.

### 12.5. The main loop—a Knuth–Bendix pass

A significant proportion of the time in a Knuth–Bendix pass is spent in applying a procedure which we term minimization. Each rule encountered during the pass is input to this procedure and the output is called a minimal rule. The exact details of this process are given in sections 12.6 and 12.9, but we point out that minimization often results in rules being added to and/or deleted from $S$. Any rules added to $S$ during the minimization of a rule $(\lambda, \rho)$ are strictly smaller than $(\lambda, \rho)$ in the ordering of 2.9.

1. At this point, This is empty. If $n > 0$, save space by deleting previously defined automata $P(Rules[n]), Q(Rules[n])$ and $Rules[n]$. Increment $n$. The integer $n$ records which Knuth–Bendix pass we are currently working on.

2. Delete the rules in Delete.

3. For each rule $(\lambda, \rho)$ in Considered, minimize $(\lambda, \rho)$ as in 12.7 and handle the output as in 12.9.

4. For each rule $(\lambda, \rho)$ in New, minimize $(\lambda, \rho)$ as in 12.7 and handle the output as in 12.9.

   Since rules added to New during minimization are always strictly smaller than the rule being minimized, it follows that eventually each rule in New will be processed; that is, the process of examining rules in New does not continue indefinitely.

5. For each rule $(\lambda, \rho)$ in This:
   (a) Delete the rule from This and add it to Considered.
   (b) For each rule $(\lambda_1, \rho_1)$ in Considered:
Look for overlaps between $\lambda$ and $\lambda_1$. That is we have to find each suffix of $\lambda$ which is a prefix of $\lambda_1$ and each suffix of $\lambda_1$ which is a prefix of $\lambda$. Note that we may have to allow $\lambda = \lambda_1$ in order to deal with the case where two different rules have the same left-hand side. In this case, both the prefix and suffix of both left-hand sides is equal to $\lambda = \lambda_1$. Then $R$-reduce in two different ways as in 2.7, obtaining a pair of strings $(u, v)$. If they differ then rearrange them so that $u > v$ and insert the result into $\text{New}$, unless it is already in $S$.

6. Delete from $WDiff$ all arrows and states which are not marked as needed. Copy $WDiff$ into $Rules[n+1]$ and mark all arrows and states of $WDiff$ as not needed. The details of this step are given in 12.10.

7. This ends the description of a Knuth–Bendix pass. Now we decide whether to terminate the Knuth–Bendix process. Since we know of no procedure to decide confluence of an infinite system of rules (indeed, it is probably undecidable), this decision is taken on heuristic grounds. In our context, a decision to terminate could be taken simply on the grounds that $WDiff$ and $Rules[n]$ have the same states and arrows. In other words, no new word-differences or arrows between word-differences has been found during this pass. If the Knuth–Bendix process is not terminated, go to 12.5.1.

12.6. Minimizing a rule. We now provide the details of the minimization routine.

**Definition 12.7.** Let $(u, v) \in A^* \times A^*$ and let $u = u_1 \cdots u_p$ and $v = v_1 \cdots v_q$, where $u_i, v_j \in A$. We say that $(u, v)$ is a minimal rule if $u \neq v, \bar{u} = \bar{v}$ in $G$ and the following procedure does not change $(u, v)$. The procedure is called minimizing a rule or the minimization routine. We always start the minimization routine with $u >_{SL} v$, though this condition is not necessarily maintained as $u$ and $v$ change during the routine.

1. $R$-reduce the maximal proper prefix $u_1 \cdots u_{p-1}$ of $u$ obtaining $u'$. Reduction may result in rules being added to $\text{New}$ as described in 11.1. If $u \neq u' u_p$, change $u$ to $u' u_p$ and go to Step 12.7.3.
2. $R$-reduce the maximal proper suffix $u_2 \cdots u_p$ of $u$ obtaining $u''$. Reduction may result in new rules being added to $\text{New}$. Replace $u$ by $u_1 u''$.
3. If $u$ has changed since the original input to the minimization routine, then $R$-reduce $u$. This may result in new rules being added to $\text{New}$.
4. If $p > q + 2$ or if $p = q + 2$ and $u_1 > v_1$, replace $(u, v)$ by $(u_1 \cdots u_{p-1}, v_1 \cdots v_q i(u_p))$ and repeat this step until we can go no further.
5. If $p = q + 2$ and $u_2 > v(u_1)$, replace $(u, v)$ by $(u_2 \cdots u_p, i(u_1) v_1 \cdots v_q)$.
6. If $q > 0$ and $u_1 = v_1$, cancel the first letter from $u$ and from $v$ and repeat this step.
7. If $q > 0$ and $u_p = v_q$, cancel the last letter from $u$ and from $v$ and repeat this step.
8. $R$-reduce $v$ as explained in 11.1. This may result in rules being added to $\text{New}$ as described in 11.1.3.
9. If $v > u$, interchange $u$ and $v$.
10. If $(u, v)$ has changed since the last time Step 12.7.4 was executed, go to Step 12.7.4.
11. Output $(u, v)$ and stop.
From Note 12.2, we see that if a rule is minimal at one time during a Knuth–Bendix pass then it is minimal at all later times during the same pass.

Note that the output could be $(\epsilon, \epsilon)$, which means that the rule is redundant. Otherwise we have output $(u, v)$ with $u > v$. Note that the minimization procedure keeps on decreasing $(u, v)$ in the ordering given by 2.9. Since this is a well-ordering, the minimization procedure has to stop. Also any rules added to \texttt{New} as a result of $R$-reduction during minimization are smaller than $(u, v)$.

Lemma 12.8. Let $(\lambda_1, \rho_1)$ be the output from minimizing $(\lambda, \rho)$. If $\lambda$ has no proper $R$-reducible substrings, then $\lambda_1$ is a non-trivial substring of $\lambda$.

Proof. Under the hypotheses, the successive steps of minimization change $\lambda$ and $\rho$, while maintaining the inequality $\lambda > \rho$. As a result the left-hand and right-hand sides are never interchanged. It follows that $\lambda_1 > \rho_1$, so $\lambda_1$ is non-trivial. It is easy to see that $\lambda_1$ is a substring of $\lambda$.

12.9. Handling minimization output. Suppose the input to minimization is $(\lambda, \rho)$ and its output is $(\lambda_1, \rho_1)$. We now describe how the rule $(\lambda_1, \rho_1)$ is treated.

1. In order to avoid unnecessary subsequent work of minimization, mark $(\lambda, \rho)$ as minimized and $(\lambda_1, \rho_1)$ as minimal for this pass.

2. If $(\lambda_1, \rho_1) \neq (\epsilon, \epsilon)$, incorporate $(\lambda_1, \rho_1)$ into the language accepted by \texttt{WDiff}, using the method described in 12.10.

3. If $(\lambda_1, \rho_1) = (\lambda, \rho)$, that is, if $(\lambda, \rho)$ was already minimal, then:
   (a) If $(\lambda, \rho)$ was in \texttt{Considered}, do nothing.
   (b) If $(\lambda, \rho)$ was in \texttt{New}, move it to \texttt{This}.
   (c) We saw in 12.7, that the above two situations are the only ones in which minimization is carried out.

4. If $(\lambda, \rho) \neq (\lambda_1, \rho_1) \neq (\epsilon, \epsilon)$, then:
   (a) If $(\lambda_1, \rho_1)$ is already in \texttt{S}, mark the copy in \texttt{S} as minimal.
   (b) If $(\lambda_1, \rho_1)$ is not already in \texttt{S}, insert it in \texttt{This}.

5. If $(\lambda_1, \rho_1) = (\epsilon, \epsilon)$, do nothing.

6. If $\lambda$ was affected by the minimization routine before Step 12.7.4, that is, if some proper substring of $\lambda$ was $R$-reducible, then delete $(\lambda, \rho)$.

7. If, at the time of minimization, all proper substrings of $\lambda$ were $R$-irreducible and if $(\lambda, \rho)$ was not minimal, move $(\lambda, \rho)$ to the \texttt{Delete} list. The reason for this possibly surprising policy of not deleting immediately is that further reduction during this pass may once again produce $\lambda$ as a left-hand side by the methods of Sections 9 and 10. We want to avoid the work involved in finding the right-hand side by the method of Section 11. For this, we need to have a rule in \texttt{S} with left-hand side equal to $\lambda$—see 11.1.3.

12.10. Details on the structure of \texttt{WDiff}. At the beginning of Step 12.5.6, each state $s$ of \texttt{WDiff} has an associated string $w_s \in A^*$ which is irreducible with respect to $\texttt{Set}(\text{Rules}[n])$. \texttt{WDiff} is a rule automaton: we associate the element $w_s$ to the state $s$. These state labels are calculated as and when new states and arrows are added to \texttt{WDiff} during a Knuth–Bendix pass (see 12.11).

At the end of the $n$th Knuth–Bendix pass, \texttt{WDiff} is an automaton which represents the word-differences and arrows between them encountered during that pass. At this stage the string attached to each state is irreducible with respect to the rules
in \(\text{Set(Rules}[n])\) but not necessarily with respect to the rules implicitly contained in \(\text{WDiff}\). Before starting the next pass, we \(\mathcal{R}\)-reduce the state labels of \(\text{WDiff}\) with respect to \(\text{Set(WDiff)}\). If \(\text{WDiff}\) now contains distinct states labelled by the same string we connect them by epsilon arrows and replace \(\text{WDiff}\) by \(\text{Weld}(\text{WDiff})\) (see remark \(8.6\)). We then repeat this procedure until all states are labelled by distinct strings which are irreducible with respect to \(\text{Set(WDiff)}\). If during this procedure a state or arrow marked as \textit{needed} is identified with one not marked as \textit{needed}, the resulting state or arrow is marked as \textit{needed}.

Whenever a minimal rule \(r\) is encountered during the \(n\)th pass, it is adjoined to the accepted language of \(\text{WDiff}\). One method of doing this is to form the rule automaton \(M(r)\) as given in example \(5.2\), and replace \(\text{WDiff}\) by the result of welding the union of the two rule automata \(\text{WDiff}\) and \(M(r)\). However, there is a more efficient way to proceed which eliminates the need to construct \(M(r)\) and possibly avoids the necessity for welding. We call this procedure \textit{sewing}.

12.11. The sewing procedure. Suppose we have a rule \(r = (u_1, v_1) \cdots (u_n, v_n)\), with each \(u_i \in A\) and each \(v_i \in A^+\) which we want to add to the language accepted by \(\text{WDiff}\). We read the rule into \(\text{WDiff}\) from left to right, starting at the initial state of \(\text{WDiff}\). Suppose it is possible to read in \((u_1, v_1) \cdots (u_k, v_k)\), arriving at a state \(s_k\), where \(k\) is chosen as large as possible, subject to the condition that \(k \leq n\). If \(k < n\), then the arrow labelled \((u_{k+1}, v_{k+1})\) is undefined on \(s_k\). We now read \((u_{k+1}, v_{k+1}) \cdots (u_n, v_n)\) into \(\text{WDiff}\), starting from the initial and final state \(s_0 = t_n\), and reading from right to left, arriving at a state \(t_r\) with \(r \geq k\), on which the backward arrow labelled \((u_r, v_r)\) is undefined. We mark all states and arrows encountered as \textit{needed}.

We now proceed as follows:

1. If \(k = r\) and the states \(s_k\) and \(t_r\) do not coincide, join them by an epsilon-arrow, replace \(\text{WDiff}\) by \(\text{Weld}(\text{WDiff})\), and then stop. When identifying states with different labels during the welding procedure, we choose the shortlex-least label for the amalgamated state.
2. If \(k < r\), let \(w_k\) be the label for \(s_k\). Reduce \(u_{k+1} w_k v_{k+1}\), obtaining \(w_{k+1}\).
3. If \(w_{k+1}\) is the label of an existing state \(t\) of \(\text{WDiff}\), set \(s_{k+1} = t\).
4. If \(w_{k+1}\) is not the label of an existing state, create a new state \(s_{k+1}\) with label \(w_{k+1}\).
5. Create an arrow labelled \((u_{k+1}, v_{k+1})\) from \(s_k\) to \(s_{k+1}\).
6. Mark the state \(w_{k+1}\) and the arrow \((u_{k+1}, v_{k+1})\) as \textit{needed}.
7. Increment \(k\) and go to Step [12.11.1].

Note that the automaton obtained by sewing is a rule automaton.

13. Correctness of our Knuth–Bendix Procedure

In this section we will prove that the procedure set out in Section 12 does what we expect it to do.

Definition 13.1. For a discrete time \(t\), we denote by \(\mathcal{S}(t)\) the rules in \(\mathcal{S}\) at time \(t\) in our Knuth—Bendix procedure. We can take \(t\) to be the number of machine operations since the program started, or any similar discrete measure.

Definition 13.2. A quintuple \((t, s_1, s_2, \lambda, \rho)\), where \(t\) is a time, and \(s_1, s_2, \lambda\) and \(\rho\) are elements of \(A^*\), is called an \textit{elementary} \(\mathcal{S}(t)\)-reduction \(u \rightarrow_{\mathcal{S}(t)} v\) from \(u\) to \(v\) if
(\(\lambda, \rho\)) is a rule in \(S(t)\), \(u = s_1\lambda s_2\) and \(v = s_1\rho s_2\). We call \((\lambda, \rho)\) the rule associated to the elementary reduction.

Definition 13.3. Let \(t \geq 0\). By a time-\(t\) Thue path between two strings \(w_1\) and \(w_2\), we mean a sequence of elementary \(S(t)\)-reductions and inverses of elementary \(S(t)\)-reductions connecting \(w_1\) to \(w_2\), such that none of the rules associated to the elementary reductions is in Delete at time \(t\). We talk of the strings which are the source or target of these elementary reductions as nodes. The path is considered as having a direction from \(w_1\) to \(w_2\). The elementary reductions will be consistent with this direction and will be called rightward elementary reductions. The inverses of elementary reductions will be in the opposite direction and will be called leftward elementary reductions.

Proposition 13.4. Let \(\langle A/R \rangle\) be the finite presentation at the start of the Knuth-Bendix procedure. Then if \((\lambda, \rho) \in S\) during the \(n\)-th Knuth-Bendix pass, we have \(\lambda \leftrightarrow^* R \rho\).

Proof. The proof of this is an easy induction on \(n\) using Corollary 8.3.

Proposition 13.5. Let \(t \geq 0\) and suppose that we have a Thue path from \(\alpha\) to \(\beta\) in \(S(t)\) with maximum node \(w\). Then for any time \(s \geq t\), there exists a time-\(s\) Thue path from \(\alpha\) to \(\beta\) with each node less than or equal to \(w\).

Proof. We show by induction on \(s\) that, if at some time \(t \leq s\) there is a Thue path between strings \(u\) and \(v\) with all nodes no bigger than \(u\) or \(v\), then there is also such a Thue path at time \(s\). So suppose that we have proved this statement for all times \(s' < s\).

We first consider the special case where \((u, v)\) is a rule being input to the minimization routine (see Definition 12.7) at time \(t\), and \(s\) is the time at end of the subsequent invocation of the minimization handling routine 12.9.

Each step of minimization takes an input string and outputs a possibly different string which is used as the input to the next step. The initial input is \((u, v)\) and the final output is either \((\epsilon, \epsilon)\) or a minimal rule \((u', v')\). Let \(r_1, r_2, \ldots, r_n\) be the sequence of outputs in the minimization of \((u, v)\), and let \(r_0 = (u, v)\). By considering each step of minimization in turn, we will show that for each \(i, 1 \leq i \leq n\), if there is a time-\(s\) Thue path between the two sides of \(r_i\) with maximum node no bigger than either side, then there is a time-\(s\) Thue path between the two sides of \(r_{i-1}\) with maximum node no bigger than either side. We then obtain the desired time-\(s\) Thue path between \(u\) and \(v\) by using descending induction on \(i\) given that the base case \(i = n\) is trivially true.

To make the task of checking the proof easier, we use the same numbering here as in Definition 12.7.

1. At the end of this step, there is a sequence of elementary reductions from \(u_1 \cdots u_{p-1}\) to \(u'\), but this may not constitute a Thue path since some of the associated rules may be in Delete. However, any such rule \((\lambda, \rho)\) will, at some time \(s' < s\), have been in \(S\) but not in Delete. Therefore by our induction on \(s\), at the end of this step there will be a Thue path from \(\lambda\) to \(\rho\) with maximum node \(\lambda\). Since no rule used in this Thue path is equal to \((u, v)\), this will still be a Thue path at time \(s\). Hence we can construct a time-\(s\) Thue path from \(u\) to \(v\) with maximum node \(u\).
2. This step is analogous to the previous step.
3. At the end of this step, the sequence of $R$-reductions of $u$ to the current left-hand side does not use the rule $(u, v)$ (hence the condition at the start of this step), and so the required Thue path exists.
4. Suppose that the input to this step is $(u'x, v)$. Then the output is either the same as the input or is equal to $(u', vx^{-1})$. In the first case there is nothing to prove. In the latter case, a time-$s$ Thue path from $u'$ to $vx^{-1}$ with maximum node $u'$ will give a time-$s$ Thue path from $u'x$ to $vx^{-1}x$ with maximum node $u'x$. Note that there have been no deletions of rules since this particular minimization was started. Induction on $s$ therefore gives us a specific Thue path from $x^{-1}x$ to $\epsilon$ at the end of this step. Moreover, no node along the Thue path is bigger than $x^{-1}x$. In particular, the input rule to the minimization is not used in this Thue path (this follows from either of the conditions at the start of the step being satisfied), and so there is a time-$s$ Thue path from $vx^{-1}x$ to $v$ with maximum node $vx^{-1}x$. Hence we obtain the required time-$s$ Thue path from $u'x$ to $v$.
5. This step is analogous to the previous step.
6. If the input to this step is $(xu', xv')$ then the output is $(u', v')$. A time-$s$ Thue path from $u'$ to $v'$ with maximum node $u'$ yields a time-$s$ Thue path from $xu'$ to $xv'$ with maximum node $xu'$.
7. This step is analogous to the previous step.
8. Let $v'$ be the $R$-reduction of $v$. Immediately after this step there is a Thue path from $v$ to $v'$ with maximum node $v$ which does not use the rule initially input. Using induction if necessary we have a time-$s$ Thue path from $v$ to $v'$ with maximum node $v$. Hence a time-$s$ Thue path from $u$ to $v'$ with maximum node either $u$ or $v'$ yields a time-$s$ Thue path from $u$ to $v$ with maximum node either $u$ or $v$.
9. If there is a Thue path from $u$ to $v$ with maximum node either $u$ or $v$, then the reverse of this path is a Thue path from $v$ to $u$.

This completes the induction step for the special case. Now consider the general case. The only reason why a Thue path at time $t < s$ between $u$ and $v$ will not work at time $s$ is if some elementary reduction used in this path has an associated rule $(\lambda, \rho)$ in $S(t)$ which is found to be non-minimal between $t$ and $s$. But in the proof of the special case we have seen that there is a time-$s$ Thue path between $\lambda$ and $\rho$ with no node bigger than $\lambda$. Therefore the time-$t$ Thue path can always be replaced by a time-$s$ Thue path without increasing the size of the nodes. \hfill $\square$

Lemma 13.6. If a string is $S(s)$-reducible, it is $S(t)$-reducible for all $t > s$.

Proof. If $u$ is $S(s)$-reducible, there is an elementary $S(s)$-reduction $u \rightarrow_{S(s)} v$. This means that $v < u$. By Proposition 13.3, for each time $t > s$, there is a Thue path from $u$ to $v$ with maximum node $u$. The first elementary reduction in this path has the form $u \rightarrow w$ at time $t$. This proves the result. \hfill $\square$

Lemma 13.7. If $(\lambda, \rho)$ is a rule in $S$ at some time during the $n$-th Knuth–Bendix pass but before the beginning of Step 12.5.5, then $\lambda$ will be $R$-reducible during all subsequent passes. If $\lambda$ is $R(s)$-reducible then $\lambda$ is $R(t)$-reducible for any $t > s$.

Proof. Let $(\lambda, \rho)$ be a rule as in the statement of the lemma. Then at some prior time, $(\lambda, \rho)$ will have been a rule in $S$ but not in Delete. Therefore for any $m \geq n$,
there will be a Thue path from $\lambda$ to $\rho$ with maximum node $\lambda$ at the beginning of Step 12.5.5 during the $m$-th Knuth-Bendix pass. Now at the beginning of Step 12.5.5, all rules in $S$ but not in $\text{Delete}$ will have been output by the minimization handling routine 12.9 at some prior time during that Knuth-Bendix pass. In particular, each of these minimal rules $(u,v)$ will have been sewn into $W\text{Diff}$. This does not, however, imply that $(u,v)$ will be accepted by $\text{Rules}$ at the start of the next pass since this rule may use or define an $(x,x)$ arrow in $W\text{Diff}$. Due to some collapsing in $W\text{Diff}$ caused by a welding operation, this may give rise to an $(x,x)$ arrow from the initial state to itself. Such an arrow will be removed so that $W\text{Diff}$ satisfies the properties 9.1. If this is the case then $(u,v)$ will still have some prefix or suffix accepted by $W\text{Diff}$ and hence by $\text{Rules}$ at the start of the next Knuth-Bendix pass. Therefore for any $m \geq n$, $\lambda$ will have a substring which is the left-hand side of a rule accepted by $\text{Rules}[m+1]$, and so $\lambda$ is $R$-reducible during pass $m+1$ which proves the first statement.

If $\lambda$ is $R$-reducible at any time during a pass, it is $R$-reducible at any later time in the same pass by Lemma 12.2. We have proved that $\lambda$ is $R$-reducible once the next pass starts. So this completes the proof of the last sentence in the statement of the lemma.

**Lemma 13.8.** At any time $t$, $S(t)$ contains no duplicates. If a rule is deleted from $S$, it will never be re-inserted.

**Proof.** The first statement follows by looking through 12.5 and checking where insertions of rules take place. We always take care not to insert a rule a second time if it is already present.

Let $(\lambda, \rho)$ be a rule which is deleted at time $s$. Deletion either takes place during Step 12.5.2 or during Step 12.9.1. In the latter case, some proper substring of $\lambda$ is the left-hand side of a rule in $S$, and this rule was present in $S$ at some time before the beginning of Step 12.5.5 of the Knuth-Bendix pass in which $(\lambda, \rho)$ was deleted. Therefore by Lemma 13.7, we see that this proper substring of $\lambda$ stays $R$-reducible. This means that no rule with left-hand side $\lambda$ will ever be re-inserted.

So we assume that deletion takes place during Step 12.5.2 of the $(n+1)$-th Knuth-Bendix pass. Then between the time of the $n$-th Knuth next pass when $(\lambda, \rho)$ is minimized, and the start of the next pass, it is in the subset $\text{Delete}$ of $S$. Therefore it cannot be re-inserted during pass $n$.

Now let $m > n$ and suppose $(\lambda, \rho)$ has not been re-inserted before the beginning of the $m$-th pass. We will prove that it cannot be re-inserted during the $m$-th pass.

Observe that no rules are minimized between the time $r$ at the beginning of Step 12.5.5 in the $(m-1)$-st pass and the time $t$ just defined. Therefore any time-$r$ Thue path between $\lambda$ and $\rho$ will also be a time-$t$ Thue path. In particular, the rule $(\lambda', \rho')$ associated to the elementary reduction of $\lambda$ is unaltered during this time. At time-$r$ all rules in $S$, except for those in $\text{Delete}$, are minimal. In particular, $(\lambda', \rho')$ was minimal and had been minimized at some prior point in the $(m-1)$-st pass. Therefore, $(\lambda', \rho')$ was sewn into $W\text{Diff}$ during the $(m-1)$-st pass. As in the proof of Lemma 13.7 it is possible that $(\lambda', \rho')$ is not accepted by $\text{Rules}[m]$, but a substring $\lambda''$ of $\lambda'$ (and hence $\lambda$), will be the left-hand side of an accepted rule. If $\lambda''$ is a proper substring of $\lambda$, then $\lambda$ cannot be the left-hand side of a rule inserted during the $m$-th Knuth-Bendix pass. So we need only examine the case when $\lambda = \lambda' = \lambda''$. In this situation, $\rho$ must have been $R$-reduced to a strictly
smaller string during the minimization of $\lambda$. Therefore some substring of $\rho$ was the
left-hand side of a rule in $S$ at that time. By Lemma 13.7, $\rho$ stays $R$-reducible.

Suppose then that $(\lambda, \rho)$ is re-inserted during the $m$-th Knuth–Bendix pass. This
can only happen as the result of Step 11.1.5. But for this to occur there could have
been no rule in $S$ with left-hand side equal to $\lambda$ at that time. Since we are assuming
that $\lambda$ is the left-hand side of some rule $(\lambda, \rho'')$ not on the Delete list at the start of
the $m$-th pass, it follows that $(\lambda, \rho'')$ must be deleted at some point during the $m$-th
pass. But this can only happen if some proper substring of $\lambda$ is $R$-reducible during
the $m$-th pass. By Lemma 12.2, this proper substring of $\lambda$ must be $R$-reducible at
the point of re-insertion of $(\lambda, \rho)$ which is a contradiction.

Definition 13.9. We say that a string $u$ is permanently irreducible if there are
arbitrarily large times $t$ for which $u$ is $S(t)$-irreducible. By Lemma 13.6 this is
equivalent to saying that $u$ is $S(t)$-irreducible at all times $t \geq 0$. A rule $(\lambda, \rho)$
in $S$ is said to be permanent if $\rho$ and every proper substring of $\lambda$ is permanently irreducible.

Lemma 13.10. A permanently irreducible string is permanently $R$-irreducible. A
permanent rule of $S$ is never deleted. A permanent rule is accepted by Rules $[n + 1]$
provided it is present in $S$ when the $n$-th Knuth–Bendix pass begins, (and is accepted
by Rules $[m]$ for all $m > n$).

Proof. Let $u$ be permanently irreducible. $R$-reduction of $u$ can only take place if,
immediately afterwards, some substring of $u$ is $S$-reducible. This is impossible by hypothesis.

A rule $(\lambda, \rho)$ is deleted only as a result of minimization. By Lemma 13.3, there
would have to be a Thue path from $\lambda$ to $\rho$ with largest node $\lambda$. The first elementary
reduction must therefore be rightward (see Definition 13.3) $\lambda \rightarrow S(t) \mu$. Since every
proper substring of $\lambda$ is permanently $R$-irreducible, this first elementary reduction
must be associated to a rule $(\lambda, \mu)$.

This is only possible if, when $(\lambda, \rho)$ was input to the minimization routine, $\rho$ was
$R$-reducible. However, it is permanently $R$-irreducible which is a contradiction.

It follows that if $(\lambda, \rho)$ is present at the start of the $n$-th Knuth–Bendix pass, it
will be sewn into $WDiff$ at some point during the $n$-th Knuth–Bendix pass. As in
the proof of Lemma 13.7, the only way $(\lambda, \rho)$ would not be accepted by $Rules[n + 1]$ is
if some proper prefix or suffix is accepted by $Rules[n + 1]$. But this would contradict
$(\lambda, \rho)$ being a permanent rule. Therefore, $(\lambda, \rho)$ is accepted by $Rules[m]$ for each
$m \geq n$.

Lemma 13.11. Let $u$ be a fixed string. Then there is a $t_0$ depending on $u$, such
that, for all $t \geq t_0$, each elementary $S(t)$-reduction of $u$ is associated to a permanent
rule. If all proper substrings of $u$ are permanently irreducible, then, for $t \geq t_0$, there
is at most one elementary reduction of $u$, and this is associated to a permanent rule
$(u, w)$.

Proof. There are only finitely many substrings of $u$. So we need only show that,
given any string $v$, there is a $t_0$ such that for all $t \geq t_0$, each rule in $S(t)$ with
left-hand side $v$ is permanent. If there is a proper substring of $v$ which is not
permanently irreducible, then at some time $s_0$ it becomes $S(s_0)$-reducible. By Lemma 13.3, it is $S(s)$-reducible for $s \geq s_0$. By Lemma 13.7 it becomes $R$-reducible
at the beginning of the next Knuth–Bendix pass after \( s_0 \). During this pass all rules with left-hand side \( v \) will be deleted. Also, since this proper substring of \( v \) is now permanently R-reducible, no rule with left-hand side equal to \( v \) will ever be inserted subsequently. In this case the lemma is true since ultimately there are no rules with left-hand side \( v \).

So we assume that each proper substring of \( v \) is permanently irreducible, and that \( v \) itself is S-reducible at some time \( t \). A rule \((v, w)\) will be permanent if \( w \) is permanently irreducible. Otherwise it will disappear as a result of minimization and, by Lemma 13.8, never reappear. There cannot be two permanent rules \((v, w_1)\) and \((v, w_2)\) with \( w_1 > w_2 \). For critical pair analysis would produce a new rule \((w_1, w_2)\) during the next Knuth–Bendix pass, and so \( w_1 \) would not be permanently irreducible.

\[ \square \]

**Theorem 13.12.** Let \( u \) be a fixed string in \( A^* \) and let \( v \) be the smallest element in its Thue congruence class. Then, for large enough times, there is a chain of elementary reductions from \( u \) to \( v \) each associated to a permanent rule. After enough time has elapsed, R-reduction of \( u \) always gives \( v \).

**Proof.** We start by proving the first assertion. By hypothesis, we have, for each time \( t \), a time-\( t \) Thue path \( p_t \) from \( u \) to \( v \), and we can suppose that \( p_t \) contains no repeated nodes by cutting out part of the path if necessary. The only reason why we couldn't take \( p_{t+1} \) to be \( p_t \) is if some rule \((\lambda, \rho)\), used along the Thue path \( p_t \), is deleted at time \( t \). By Lemma 13.5, we may assume that each node of \( p_{t+1} \) is either already a node of \( p_t \) or is smaller than some node of \( p_t \).

Let \( h_0 \) be the largest node on \( p_0 \), and suppose that we have already proved the theorem for all pairs \( u \) and \( v \) which are connected by a Thue path with largest node smaller than \( h_0 \). By induction, using Proposition 13.3, we can assume that \( h_0 \) is the largest node on \( p_t \) for all time \( t \). If \( v = h_0 \) then since \( v \) is the smallest element in its congruence class, there are no elementary reductions starting from \( v \), and we must have \( u = v \) in this case.

By Lemma 13.11, we may assume that \( t_0 \) has been chosen with the property that, for all strings \( w \leq h_0 \) and for all \( t \geq t_0 \), all elementary \( S(t) \)-reductions of \( w \) are associated to permanent rules which are accepted by Rules[\( n \)] provided \( n \) is sufficiently large.

Let \( h_0 = \mu_1 \alpha_t \nu_t \to_{S(t)} \mu_t \beta_t \nu_t \) be the rightward elementary reduction of \( h_0 \) at time \( t \). The rule \((\alpha_t, \beta_t)\) is independent of \( t \) for large values of \( t \). Then \((\alpha_t, \beta_t)\) is permanent and \( \alpha_t \) is R-reducible for large enough \( t \). If \( u \neq h_0 \), the same argument applies to the unique elementary leftward reduction with source \( h_0 \) at time \( t \).

If \( h_0 = u \), let \( u \to_{S(t)} w \) be the first rightward elementary reduction for large values of \( t \). By our induction hypothesis, there is a Thue path of elementary reductions from \( w \) to \( v \), each associated to a permanent rule, and with no node larger than \( w \), and so we have the required Thue path from \( u \) to \( v \).

Suppose now that \( h_0 \neq u \), so that we get two permanent rules, associated to the leftward and rightward elementary reductions of \( h_0 \). If the two elementary reductions are identical, that is, if the two permanent rules are equal and if their left-hand sides occur in the same position in \( h_0 \), then \( p_t \) contains a repeated node which we are assuming not to be the case. So the two elementary reductions occur in different positions in \( h_0 \). Now choose \( t \) to be large enough so that the two rules
concerned have already been compared in a critical pair analysis in Step 12.5.5.b during some previous $n$-th Knuth–Bendix pass.

If these two rules have left-hand sides which are disjoint substrings of $h_0$, then we can interchange their order so as to obtain a Thue path from $u$ to $v$ where all nodes are strictly smaller than $h_0$—see Figure 5. The first assertion of the theorem then follows by the induction hypotheses in this particular case.

If the two left-hand sides do not correspond to disjoint substrings of $h_0$ then, by assumption, there is some time $t' < t$, such that a critical pair $(u', v', w')$ was considered. Here $u' \rightarrow_{S(t')} v'$ and $u' \rightarrow_{S(t')} w'$ are elementary $S(t')$-reductions given by the two rules, and $u'$ is a substring of $h_0$. After the critical pair analysis, at time $t'' \leq t$, the Thue paths illustrated in Figure 6 are possible. As a consequence of Lemma 13.3, it is straightforward to see that for all times $s \geq t''$, $v'$ and $w'$ can be connected by a time-$s$ Thue path in which all nodes are no larger than the largest of $v'$ and $w'$. In particular, this applies at time $t$ so that the targets of the two elementary $S(t)$-reductions from $h_0$ can be connected by a time-$t$ Thue path in which all nodes are smaller than $h_0$. This completes the inductive proof of the first assertion of the theorem.

We have arranged that $t$ is large enough so that, for all $w \leq u$, all elementary $S(t)$-reductions of $w$ are associated to permanent rules, and such a $w$ can be permanently $R$-reduced to the least element in its Thue congruence class. It follows that such a $w$ is $R$-irreducible if and only if it is minimal in its Thue class. In particular $R$-reduction of $u$ must give $v$.

**Corollary 13.13.** The set of permanent rules in $R$ is confluent. The set of such rules is equal to $P = \bigcap_t \bigcup_{s \geq t} S(s)$. A string $u$ is smallest in its Thue congruence class if and only if it is permanently irreducible.

**Proof:** The first and third statements are obvious from Theorem 13.12. For the second statement, each permanent rule is contained in $P$ by Lemma 13.10. Conversely, if we have a rule $r$ in $S$ which is not permanent, then for all sufficiently large times $s$ either its right-hand side or a proper substring of its left-hand side is $S(s)$-reducible. Theorem 13.12 ensures that this reducible string is $R(s)$-reducible.
Figure 6. When the leftward and rightward reductions from $h_0$ are obtained from rules $(\lambda_1, \rho_1)$ and $(\lambda_2, \rho_2)$ having overlapping left-hand sides, this diagram shows the time-$t''$ Thue paths that exist after the critical pair associated with the triple $(u'_1 u'_2 u'_3, \rho_1 u'_3, u'_1 \rho_2)$ has been resolved to the pair $(z_1, z_2)$. 

for all sufficiently large times $s$. Therefore $r$ will be minimized and deleted from $S$. Hence from Lemma 13.8 we see that $r$ is not contained in $P$. 

The next result is the main theorem of this paper.

**Theorem 13.14.** Let $G$ be a group with a given finite presentation and a given ordering of the generators and their inverses. Suppose that $G$ is shortlex-automatic. Then the procedure given in 12.5 will stabilize at some $n_0$ with $\text{Rules}[n + 1] = \text{Rules}[n]$ if $n \geq n_0$. $P$ is then the language of a certain two-variable finite state automaton and the automaton can be explicitly constructed.

**Proof.** $P$ consists of pairs of strings $(\lambda, \rho)$ giving a valid identity in $G$ (by Proposition 3.4), and where $\rho$ and all proper substrings of $\lambda$ are permanently irreducible. A string is permanently irreducible if and only if it is the unique shortlex representative of the corresponding group element. Since this structure is automatic, there are only finitely many word differences and arrows generated by the rules in $P$. If we therefore weld together the automata $M(r)$ corresponding to the rules $r$ in $P$, we obtain a finite rule automaton $\text{Rules}$. For each arrow in this automaton, we can pick a specific rule $r$ which makes use of the arrow when it is read into $\text{Rules}$.

These specific rules will eventually be generated by our Knuth–Bendix process. Such a rule is never deleted once it is generated, since it is permanent. So eventually $\text{Rules}[n]$ will contain $\text{Rules}$ as a sub-automaton. But once this has happened, $\text{Rules}$-irreducible will be equivalent to shortlex-minimal. Therefore all non-permanent rules will be removed during the next pass, and the redundant states and arrows of $\text{Rules}[n]$ will be removed. $\text{Rules}[m]$ is then constant for $m > n$. 

\[\square\]
Of course, the problem with the above result is that we do not currently have any method of knowing when we have reached $n_0$. It might be possible to prove that this question is undecidable if one varies over all shortlex presentations of shortlex-automatic groups. It might also be undecidable in one varies over all finite presentations of word hyperbolic groups.

14. Miscellaneous details

In this section we present a number of points which did not seem to fit elsewhere in this paper.

14.1. The structure of $S$. This set is given in a data structure arranged so that it is quick to find a rule in it, given only the left-hand side, quick to delete a specified rule and quick to add a rule. All these operations take place repeatedly in the Knuth–Bendix program. It is also an advantage to have a robust enough method for iterating through $S$, so that the process is not disrupted if rules are added or deleted while the iteration is proceeding. (We don’t mind if the iteration fails to catch the rules added during iteration.)

14.2. Aborting. It is possible that we come to a situation where the procedure is not noticing that certain strings are reducible, even though the necessary information to show that they are reducible is already in some sense known. It is also possible that reduction is being carried out inefficiently, with several steps being necessary, whereas in some sense the necessary information to do the reduction in one step is already known. An indication that our procedure could be improved is that $WDiff$ is constantly changing, with two states being identified and consequent welding, or with new states or arrows being added. In this case it might be advisable to abort the current Knuth–Bendix pass.

To see if abortion is advisable, we can record statistics about how much $WDiff$ has changed since the beginning of a pass. If the changes seem excessive, then the pass is aborted. A convenient place for the program to decide to do this is just before another rule from New is examined at Step 12.5.4.

If an abort is decided upon then all states and arrows of $WDiff$ are marked as needed. At this point the program jumps to Step 12.5.1.

14.3. Priority rules. A well-known phenomenon found when using Knuth–Bendix to look for automatic structures, is that rules associated with finding new word differences or new arrows in $WDiff$ should be used more intensively than other rules. Further aspects of the structure are then found more quickly. These observations are not the consequence of a theorem—they are observed when programs are run.

A new rule associated with new word differences or new arrows is marked as a priority rule. If a priority rule is minimized, the output is also marked as a priority rule. If a priority rule is added to one of the lists Considered, This or New, it is added to the front of the list, whereas rules are normally added to the end of the list. Just before deciding to add a priority rule to New, we check to see if the rule is minimal. If so, we add it to the front of This instead of to the front of New.

When a rule is taken from This at Step 12.5.5 during the main loop, it is normally compared with all rules in Considered, looking for overlaps between left-hand sides. In the case of a priority rule, we compare left-hand sides not only with rules in
Considered, but also with all rules in This. If a normal rule \((\lambda, \rho)\) is taken from This and comparison with a rule in Considered gives rise to a priority rule, then the rule \((\lambda, \rho)\) is also marked as a priority rule. It is then compared with all rules in This, once it has been compared with all rules in Considered.

Treating some rules as priority rules makes little difference unless there is a mechanism in place for aborting a Knuth–Bendix pass when \(WDiff\) has sufficiently changed. If there is such a mechanism, it can make a big difference.

14.4. **An efficiency consideration.** During reduction we often know have a state \(s\) in a two-variable automaton. We usually know \(x \in A\) and we are looking for an arrow labelled \((x, y)\) with certain properties, where \(y \in A^+\). It therefore makes a big difference if the arrows with source \(s\) are arranged so that we have rapid access to arrows labelled \((x, y)\) if \(x\) is given.

15. **The past and the future**

15.1. **A failed idea.** Our original idea was to avoid having an explicit finite set of rules \(S\). Instead we tried to attach extra information to the states and arrows of our automata so that the set of rules implicitly held included both a finite set, corresponding to our current \(S\), and the possibly infinite set held by the automata. The idea was to avoid using the considerable amount of space used by \(S\). This idea did not work and we now explain why.

The idea was that it didn’t matter too much if the finite set of rules held implicitly was too big. The logic of Knuth–Bendix only goes wrong if it is too small. However, if the extra information attached to states and arrows is not sufficiently explicit, there is often a huge growth from one pass to the next in the finite set of rules implicitly held. This growth is not caused by the Knuth–Bendix process itself, but is a by-product of the way we are using the extra information to specify the finite set of rules.

Another approach is then to attach much more information to states and arrows in an attempt to limit the unnecessary growth referred to in the previous paragraph. But this extra information itself requires a lot of space, more even than holding the rules separately! Moreover, it turned out not to be possible to conveniently limit the growth as much as was necessary. So holding more information in the finite state automata was worse on all counts, including the complexities of writing the code, than the simpler scheme of holding the rules separately.

15.2. **The present.** Many of the ideas in this paper have been implemented in C++ by the second author. But some of the ideas in this paper only occurred to us while the paper was being written, and the procedures and algorithms presented in this paper seem to us to be substantial improvements on what has been implemented so far. An unfortunate result of this is that we are unable to present experimental data to back up our ideas, although many of these ideas have been explored in depth with actual code. Our experimental work has been essential in enabling us to come to the better algorithms which are presented here.

15.3. **Comparison with kbmag.** Here we describe the differences between our ideas and the ideas in Derek Holt’s \(kbmag\) programs \([5]\). These programs try to
compute the shortlex-automatic structure on a group. Our program is a substitute only for the first program in the $kbmag$ suite of programs.

In $kbmag$, fast reduction is carried out using an automaton with a state for every prefix of every left-hand side. In our program we also keep every rule. However, the space required by a single character in our program is less by a constant multiple than the space required for a state in a finite state automaton. Moreover, compression techniques could be used in our situation so that less space is used, whereas compression is not available in the situation of $kbmag$.

The other large objects in our set-up are the automata $P(Rules[n])$ defined in Section 9 and $Q(Rules[n])$ defined in Section 10. In $kbmag$, there has also to be an automaton like $P(Rules[n])$, and it is possible to arrange that this automaton is only constructed after the Knuth–Bendix process is halted. This can avoid running out of space. In $kbmag$ there is no analogue of our $Q(Rules[n])$.

In $kbmag$, reduction is carried out extremely rapidly. However, as new rules are found, the automaton in $kbmag$ needs to be updated, and this is quite time-consuming. In our situation, updating the automata is quick, but reduction is slower because the string has to be read into two different automata. Moreover we sometimes need to use the method of Section 11 which is slower (by a constant factor) than simply reading a string into a deterministic finite state automaton.

In $kbmag$, there is a heuristic, which seems to be inevitably arbitrary, for deciding when to stop the Knuth–Bendix process. In our situation there is a sensible heuristic, namely we stop if we find $Rules[n+1] = Rules[n]$.

In the case of $kbmag$, there are occasional cases where the process of finding the set of word differences oscillates indefinitely. This is because redundant rules are sometimes unavoidably introduced into the set of rules, introducing unnecessary word differences. Later redundant rules are eliminated and also the corresponding word differences. This oscillation can continue indefinitely. Holt has tackled this problem in his programs by giving the user interactive modes of running them.

In our case, the results in Section 13 show that, given a shortlex automatic group, the automaton $Rules[n]$ will eventually stabilize, given enough time and space.

We believe that the main advantage of our approach will only become evident when looking at very large examples. We plan to carry out a systematic examination of shortlex-automatic groups generated by Jeff Weeks’ SnapPea program—see [13]—in order to carry out a systematic comparison.

15.4. Other situations. We should remark that our methods should apply with some modifications to certain other orderings, not only to the shortlex-ordering. The essential feature we need is that the set of pairs $(u, v)$, such that $u > v$, is a regular language. Other orderings than shortlex, for example an ordering called the wreath product ordering, have been useful in theoretical discussions [7]. The wreath product ordering is used in programs by Holt which look for coset automatic structures.

Bill Thurston has suggested that we generalize our programs to apply directly to a triangulated space rather than to a group. It should be straightforward to make this generalization in both the $kbmag$ programs and in ours.

References

[1] A.V. Aho, R. Sethi, and J.D. Ullman. *Compilers, Principles, Techniques, and Tools*. Addison-Wesley Publishing Company, 1986.
[2] B. Buchberger and R. Loos. Algebraic simplification. In B. Buchberger, G.E. Collins, and R. Loos, editors, *Computer Algebra, Symbolic and Algebraic Computation*, pages 11–43. Springer-Verlag, New York, second edition, 1982.

[3] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V. Levy, M.S. Paterson, and W.P. Thurston. *Word Processing in Groups*. Jones and Bartlett, 1992.

[4] D.B.A. Epstein, D.F. Holt, and S.E. Rees. The Use of Knuth-Bendix Methods to Solve the Word Problem in Automatic Groups. *J. Symbolic Comput.*, 12:397–414, 1991.

[5] D.F. Holt. KBMAG (Knuth-Bendix in Monoids and Groups), Version 2. Software package, 1996. Available by anonymous ftp from ftp.maths.warwick.ac.uk in directory people/dfh/kbmag2.

[6] D.F. Holt. The Warwick Automatic Groups Software. In *Geometrical and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994)*, volume 25 of *DIMACS Ser. Discrete Math, Theoret. Comput. Sci.*, pages 69–82. Amer. Math. Soc., Providence RI, 1996.

[7] D.F. Holt and D.F. Hurt. Computing automatic coset systems and subgroup presentations. to appear in J. Symbolic Comput.

[8] D.F. Holt and S.E. Rees. Software for automatic groups, isomorphism testing and finitely presented groups. In *Geometric group theory, Vol. 1 (Sussex 1991)*, volume 181 of *Londin Math. Soc. Lecture Note Ser.*, pages 120–125, Cambridge, 1993. Cambridge Univ. Press.

[9] D.E. Knuth and P.B. Bendix. Simple word problems in universal algebra. In *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, 1970.

[10] C. Ó’Dúnlaing. Infinite regular Thue systems. *Theoret. Comput. Sci.*, 25:171–192, 1983.

[11] Charles C. Sims. *Computation with finitely presented groups*. Cambridge University Press, 1994.

[12] B.W. Watson. *Taxonomies and Toolkits of Regular Language Algorithms*. PhD thesis, Eindhoven University of Technology, 1995.

[13] J.R. Weeks. SnapPea: a computer program for studying hyperbolic 3-manifolds. Freely available from www.geom.umn.edu.

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