ON MINIMAL FREE RESOLUTIONS
AND THE METHOD OF SHIFTED PARTIAL DERIVATIVES IN
COMPLEXITY THEORY

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Abstract. The minimal free resolution of the Jacobian ideals of the determinant polynomial were computed by Lascoux [22], and it is an active area of research to understand the Jacobian ideals of the permanent, see e.g., [23, 18]. As a step in this direction we compute several new cases and completely determine the linear strands of the minimal free resolutions of the ideals generated by sub-permanents.

Our motivation is an exploration of the utility and limits of the method of shifted partial derivatives introduced in [17, 13]. The method of shifted partial derivatives amounts to computing Hilbert functions of Jacobian ideals, and the Hilbert functions are in turn the Euler characteristics of the minimal free resolutions of the Jacobian ideals. We compute several such Hilbert functions relevant for complexity theory. We show that the method of shifted partial derivatives alone cannot prove the padded permanent \( \ell^{n-m} \perm_m \) cannot be realized inside the \( GL_n \)-orbit closure of the determinant \( \det_n \) when \( m < 1.5n^2 \).

1. Introduction

This article is directed at two different audiences. Its purpose for algebra is easily described: to study homological properties of permanental ideals: ideals generated by the \( \kappa \times \kappa \) sub-permanents of a generic \( n \times n \) matrix. We focus on the Hilbert series and minimal free resolution of such ideals. It turns out that there is a close connection to determinantal ideals, as well as to ideals generated by all square-free monomials in \( n \) variables. Our approach uses algebra, combinatorics, and representation theory.

Our second intended audience (and the motivation for the work) comes from complexity theory. In [13] the method of shifted partial derivatives was introduced as a potential technique for separating \( \text{VP} \) from \( \text{VNP} \), where \( \text{VP}, \text{VNP} \) respectively denote the complexity classes of sequences of polynomials that admit an “easy” evaluation (resp. an “easy” description of their coefficients) - for precise definitions of these and the other classes and complexity terms used in this introduction, see e.g. [4]. The passage is via circuits of depth four.

Let \( V = \mathbb{C}^N \), let \( S^n V = \mathbb{C}[x_1, \ldots, x_N]_n \) denote the space of homogeneous polynomials of degree \( n \) on \( V^* \), and let \( \text{Sym}(S^n V)^* \) denote the space of all polynomials on \( S^n V \). The method of shifted partial derivatives (implicitly) introduces a new class of modules of polynomials \( \text{Eqns}_n \subset \text{Sym}(S^n V)^* \). The initial hope was that for each \( n \), there would be an element of \( \text{Eqns}_n \) that vanished on any polynomial \( P \in S^n V \) admitting a sufficiently small depth four circuit, but which did not vanish on the (padded) permanent of the appropriate size. A substantial amount of literature, e.g., [5, 34, 14, 29, 7] examines the utility of shifted partial derivatives towards separating \( \text{VP} \) from \( \text{VNP} \) via depth three and four circuits.

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In complexity it is necessary to deal with inhomogeneous circuits. In algebraic language this means one needs to deal with inhomogeneous polynomials in intermediate steps, even though the polynomials one wants to evaluate tend to be homogeneous. However, in geometry it is preferable to deal with homogeneous polynomials. To fix this problem, when comparing two polynomials, one may “pad” the lower degree polynomial by multiplying it by an appropriate power of a linear form, which allows one to avoid inhomogeneous circuits, see [19] for details.

Any technique for separating VNP from VP would also have to separate the padded permanent from the determinant since the sequence (detₙ) lies in VP and (permₙ) is VNP-complete.

Our aim in this note is to examine the module \( \text{Eqns}_n \) directly on \( \ell^{n-m} \text{perm}_m \) and \( \text{det}_n \). Here \( \text{det}_n \in S^n \mathbb{C}^{n^2} \) is the determinant, \( \text{perm}_m \in S^m \mathbb{C}^{m^2} \) is the permanent, \( \ell \) is a linear function on \( \mathbb{C} \) and \( \ell^{n-m} \text{perm}_m \in S^n (\mathbb{C}^{m^2} \oplus \mathbb{C}) \subset S^n V \) is the padded permanent. The hope again is that there exists an element of \( \text{Eqns}_n \) that vanishes on \( \text{det}_n \) and does not vanish on \( \ell^{n-m} \text{perm}_m \) unless \( m \) is sufficiently small. To prove \( \text{VNP} \neq \text{VP} \), or more precisely \( \text{VNP} \neq \text{VP}_{ws} \), “sufficiently small” would mean \( m^C < n \) for any \( C > 0 \). Our goal is more modest: to determine if one can prove any result from these techniques. Unfortunately, our main result in this direction is negative, see Theorem 1.1.1.

1.1. A geometric description of the technique. Let \( V = \mathbb{C}^N \), let \( P \in S^n V \) be a homogeneous polynomial of degree \( n \), consider the ideal generated by the partial derivatives of \( P \) of order \( n - \kappa \). Call this the \( (n - \kappa) \)-th Jacobian ideal, and denote it by \( \mathcal{I}^{P, \kappa} \). It is generated in degree \( \kappa \).

Geometric Complexity Theory [27, 28, 4] addresses the question: given \( P, Q \in S^n V \), is \( Q \in GL(V) \cdot P \)? An insight of [13] is that a necessary condition for containment is \( \dim \mathcal{I}^{P, \kappa} \geq \dim \mathcal{I}^{Q, \kappa} \) for all \( \kappa, t \), where \( \mathcal{I}_t \) is the degree \( t \) component of \( \mathcal{I} \). We now explain why this is a closed condition.

First we review the method of partial derivatives: The space \( S^{n-\kappa} V^* \) may be interpreted as the space of linear homogeneous differential operators of order \( n - \kappa \) on \( \text{Sym}(V) \), so given \( P \in S^n V \), we obtain a linear map

\[
P_{n-\kappa, \kappa} : S^{n-\kappa} V^* \rightarrow S^\kappa V
\]

\[
D \mapsto D(P).
\]

The study of the rank of this map goes under the name the method of partial derivatives in the complexity theory literature (e.g. [30]) and catalecticants, or flattenings in the geometry literature. It dates back to Sylvester [33], who observed that if \( P = x^n \), then the map has rank one, so if \( \text{rank}(Q_{n-\kappa, \kappa}) = r \), to write \( Q \) as a sum of \( n \)-th powers one needs at least \( r \) terms. In complexity theory one does not usually restrict to homogeneous polynomials and derivatives of a fixed order as we do here, but the resulting bounds will be essentially the same: allowing padded polynomials has the effect of giving homogeneous circuits the computing power of arbitrary circuits, see [19]. Note that \( \mathcal{I}^{P, \kappa}_\kappa \subset S^n V \) is the image of \( P_{n-\kappa, \kappa} \).

Now consider the map

\[
P_{n-\kappa, \kappa[\tau]} : S^{n-\kappa} V^* \otimes S^\tau V \rightarrow S^{\kappa + \tau} V
\]

\[
D \otimes Q \mapsto D(P)Q,
\]

which is \( P_{n-\kappa, \kappa} \) composed with the multiplication map of polynomials. Its image is \( \mathcal{I}^{P, \kappa}_{\kappa + \tau} \subset S^{\tau + \kappa} V \). The minors of size \( r + 1 \) of \( P_{n-\kappa, \kappa[\tau]} \), viewed as polynomials on the coefficients of \( P \), furnish the above-mentioned module of equations \( \text{Eqns}_n \).

Theorem 1.1.1. Let \( P^m \in S^m \mathbb{C}^{m^2} \) be a sequence such that \( \dim P^m_{\kappa, m-\kappa} (S^{n-\kappa} \mathbb{C}^{n^2}) = (m^2)^2 \). If \( n \leq 1.5 \text{m}^2 \), then for all sufficiently large \( m \),

\[
\dim(\ell^{n-m} P^m_{(\kappa, n-\kappa)[\tau]}(S^{n-\kappa} \mathbb{C}^{n^2})) < \dim(\text{det}_{(\kappa, n-\kappa)[\tau]}(S^{n-\kappa} \mathbb{C}^{n^2}))
\]
for all $\kappa, \tau$.

Theorem 1.1.1 implies that the method of shifted partial derivatives cannot be used to separate the determinant $\det_n$ from the padded permanent $\perm_{m}^{n-m}$ when $n \geq 1.5m^2$. The best known separating lower bound is $n \geq 0.5m^2$ [21], so it is possible the method could be used to improve the current state of the art.

As will become clear in the proof, the difficulty lies not in the permanent or the determinant, but in the padding. As observed in [30], inhomogeneous circuits/padded polynomials are needed to prove meaningful complexity bounds. In the language of [16], the modules of equations obtained from shifted partial derivatives alone are not useful for GCT because the first part of the associated partitions is too small.

1.2. Hilbert functions and minimal free resolutions. For an ideal $\mathcal{I} \subset \text{Sym}(V)$, the function $t \mapsto \dim \mathcal{I}_t$ is called the Hilbert function of $\mathcal{I}$. The method of shifted partial derivatives is a comparison of the Hilbert functions of the $(n-\kappa)$-th Jacobian ideals of two polynomials. There is a substantial literature computing Hilbert functions and minimal free resolutions. A minimal free resolution of $\mathcal{I}$ is an exact sequence of free $\text{Sym}(V)$-modules

$$0 \to F_q \to F_{q-1} \to \cdots \to F_0 \to \mathcal{I} \to 0,$$

with image $(F_i) \subseteq \mathfrak{m}F_{i-1}$ where $\mathfrak{m}$ denotes the maximal ideal in $\text{Sym}(V)$ generated by the linear forms $V$. Each $F_j = \text{Sym}(V) \cdot M_j$ for some graded $\text{Sym}(V)$-module $M_j$, which may be taken to also be a $G$-module if $\mathcal{I}$ is invariant under $G \subset \text{GL}(V)$. Then $\dim \mathcal{I}_t = \sum_{j=0}^{q} (-1)^j \dim F_{j+t}$. The module $M_1$ is the space of generators of $\mathcal{I}$ and the module $M_2$ is called the space of syzygies of $\mathcal{I}$. Especially important for our study is Lascoux’s computation of the minimal free resolution of $\mathcal{I}^{\det n, \kappa}$. In the case of permanent ideals, very little is known: in [23], Laubenbacher and Swanson determine a Gröbner basis for the $2 \times 2$ sub-permanents, as well as the radical and primary decomposition of this ideal. In the case of $3 \times 3$ sub-permanents, Kirkup [18] describes the structure of the minimal primes. Interestingly, the motivation for the work comes from the Alon-Jaeger-Tarsi conjecture on matrices over a finite field [2].

Let $\mathfrak{S}_n$ denote the permutation group on $n$ elements. It acts on basis vectors of $\mathbb{C}^n$ (the Weyl group action), and if we write $E = \mathbb{C}^n$, we will write $\mathfrak{S}_E$ to denote this action. We take $E, F = \mathbb{C}^n$. For example, $\perm_n \in S^n(E \otimes F)$ is acted on trivially by $\mathfrak{S}_E \times \mathfrak{S}_F$, so the generating modules in the ideal of the minimal free resolution of its Jacobian ideals will be $\mathfrak{S}_E \times \mathfrak{S}_F$-modules. If $H \subset G$ is a subgroup of a finite group $G$, and $W$ an $H$-module, we let $\text{Ind}^G_H W := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ denote the induced $G$-module, see, e.g. [9, §3.3] for details. Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of $n$. If $\pi$ is such a partition, we let $[\pi]$ denote the corresponding $\mathfrak{S}_n$-module. For the next result, give $E = \mathbb{C}^n$ a basis $e_1, \ldots, e_n$ and let $E_q = \text{span}(e_1, \ldots, e_q)$, and similarly for $F = \mathbb{C}^n$.

Theorem 1.2.1. Let $M_j$ denote the module generating the $j$-th term of the minimal free resolution of $\mathcal{I}^{\perm_{n, \kappa}}$, the ideal generated by size $\kappa$-sub-permanents of an $n \times n$ matrix with variable entries, and let $M_{j, \kappa+j-1}$ denote its linear component. Then $\dim M_{j, \kappa+j-1} = \binom{n}{\kappa+j-1}^2 \binom{2(\kappa+j-2)}{j-1}$.

As an $\mathfrak{S}_E \times \mathfrak{S}_F$-module,

$$M_{j, \kappa+j-1} = \text{Ind}^{\mathfrak{S}_E \times \mathfrak{S}_F}_{\mathfrak{S}_{E_{\kappa+j-1}} \times \mathfrak{S}_{F_{\kappa+j-1}}} ( \bigoplus_{a+b \leq j-1} [\kappa + b, 1^a]_{E_{\kappa+j-1}} \otimes [\kappa + a, 1^b]_{F_{\kappa+j-1}} ).$$
In (4) we slightly abuse notation, the module should really be written
\[
M_{j,\kappa+j-1} = \text{Ind}_{S_{\kappa+j-1}}^{S_{\kappa+j}} \times \text{Ind}_{S_{\kappa+j}}^{S_{\kappa+j+1}} \times (S_{\kappa+j-1} \times (S_{\kappa+j} \times S_{\kappa+j+1} \times S_{\kappa+j+2} \times S_{\kappa+j+3})).
\]

Compare Theorem 1.2.1 with Theorem 1.3.1 below.

**Theorem 1.2.2.** Let \( I_t^{\text{perm}_{n,2}} \) denote the degree \( t \) component of the ideal generated by the size two sub-permanents of an \( n \times n \) matrix. Then \( \dim I_2^{\text{perm}_{n,2}} = \binom{n}{2}^2 \). For \( 3 \leq t \leq n \):

\[
\dim I_t^{\text{perm}_{n,2}} = \binom{n^2 + t - 1}{t} - \frac{1}{2} \left( \binom{n^2}{2} + n^2 + (t - 1) \left( \binom{n^2}{2} - \binom{n}{2} \right) + 2 \binom{t - 1}{2} \binom{n}{2} + n \binom{n}{3} \right) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j} \binom{n}{j+1},
\]

and for \( t > n \):

\[
\dim I_t^{\text{perm}_{n,2}} = \binom{n^2 + t - 1}{t} - \frac{1}{2} \left( \binom{n^2}{2} + n^2 + (t - 1) \left( \binom{n^2}{2} - \binom{n}{2} \right) + 2 \binom{t - 1}{2} \binom{n}{2} + n \binom{n}{3} \right) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j} \binom{n}{j+1}.
\]

The latter formula is \( \dim S^t \mathbb{C}^{n^2} \) minus the value of the Hilbert polynomial at \( t \).

Information from the minimal free resolution might lead to more modules of polynomials that one could use in complexity theory beyond the shifted partial derivatives.

1.3. The sum-product polynomial and depth three circuits. Another polynomial that arises in complexity theory is the so-called sum-product polynomial \( \text{SP}_n^r := \sum_{j=1}^r x_j^1 \cdots x_j^n \in S^n(\mathbb{C}^r) \). In complexity theory, \( \text{SP}_n^r \) goes under the name the sum-product polynomial and depth three circuits. Theorem 1.3.1 overlaps with the results of [3], as explained in [19, \S 8], this polynomial can be used to separate \( \text{VP} \) from \( \text{VNP} \), however one pays the price of leaving polynomial bounds. Explicitly, if \( \ell^{n-m} \text{perm}_{m} \not\in \text{GL}_{n^r} \cdot \text{SP}_n^{r^2} \) when \( r n < 2^{\text{mog}(m)} \omega(1) \), then \( \text{VP} \neq \text{VNP} \). In geometry, the variety \( \text{GL}_{n^r} \cdot \text{SP}_n^{r^2} \) goes under the name the r-th secant variety of the Chow variety, \( \sigma_r(\text{Ch}_n(\mathbb{C}^n)) \). The variety \( \text{Ch}_n(\mathbb{C}^n) \) has been studied extensively, see [19, \S 7.1] for a history. The Chow variety \( \text{Ch}_n(V) \subset \mathbb{P}^S \mathbb{C}^n \) is the set of polynomials \( P \in \mathbb{C}^n \) such that \( P \) is a product of \( n \) linear factors.

Note that \( \text{rank}(x_1 \cdots x_n)_{k,n-k} = \binom{n}{k} \), \( \text{rank}(\text{SP}_n^{r})_{k,n-k} = r \binom{n}{k} \), and \( \text{rank}(\text{perm}_n)_{k,n-k} = \binom{n}{k}^2 \).

**Theorem 1.3.1.** Let \( I_{x_1 \cdots x_n}^{\kappa} \) denote the ideal generated by the derivatives of order \( n - \kappa \) of the polynomial \( x_1 \cdots x_n \), i.e., \( \kappa \) is the ideal generated by the set of square free monomials of degree \( \kappa \) in \( n \) variables. The associated coordinate ring \( \text{Sym}(V)/I_{x_1 \cdots x_n}^{\kappa} \) is Cohen-Macaulay and its minimal free resolution is linear. As an \( S_n \)-module the \( j \)-th term in the minimal free resolution of \( I_{x_1 \cdots x_n}^{\kappa} \) is

\[
M_j = M_{j,\kappa+j-1} = \text{Ind}_{S_{\kappa+j}}^{S_{\kappa+j+1}} \left[ \kappa, 1^{j-1} \right],
\]

which has dimension \( \binom{\kappa+j-1}{j} \binom{n}{\kappa+j} \).

**Remark 1.3.2.** Theorem 1.3.1 overlaps with the results of [3], as the ideals generated by square-free monomials are a special case of the DeConcini-Procesi ideals of hooks discussed in [3], but Theorem 1.3.1 gives more precise information for this special case.
1.4. **Young flattenings.** The maps (1), (2) fit into a general theory of Young flattenings developed in [20], which is a general method for finding determinantal equations on spaces of polynomials invariant under a group action. The motivation in [20] was to obtain lower bounds for symmetric tensor border rank, that is for the expression of a polynomial as a sum of $n$-th powers of linear forms. Since the ideal generated by the partial derivatives of $\sum_{j=1}^{r} x_j^n \in S^n C^r$ grows at the maximal rate (see the discussion in §2), Hilbert functions should not give a better lower bound in that situation. Here the situation is different, as indeed the ideal generated by sub-determinants has more syzygies than the ideal generated by sub-permanents. In future work we plan to explore the extent that more general Young flattenings can prove circuit lower bounds.

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2. **On the growth of Hilbert functions**

Let $V = \mathbb{C}^N$. We will be concerned with ideals $\mathcal{I} \subset \text{Sym}(V)$ generated in a single degree $\kappa$, say by $x$ generators.

The maximum possible growth of $\mathcal{I} \subset \text{Sym}(\mathbb{C}^N)$ is given by Fröberg [8]: Let $\mathcal{I}$ be generated in degree $\kappa$, with $\dim \mathcal{I}_\kappa = x$. Then

$$\dim \mathcal{I}_{\kappa + t} \leq x \binom{N + t - 1}{t} - \binom{x}{2} \binom{N + t - \kappa - 1}{t - \kappa} + \binom{x}{3} \binom{N + t - 2\kappa - 1}{t - 2\kappa} + \ldots$$

where the alternating sum continues until one of the two binomial coefficients is zero.

Fröberg also conjectured equality is achieved when $\mathcal{I}$ is generic. Iarrobino [15] conjectured further that equality can even be achieved when each generator is of the form $t^\ell$ for some $\ell \in V$.

If $p, q \in \mathcal{I}$, then $hp @ q - hq @ p \in F_2$ for any polynomial $h$, where $F_2$ is as in Equation (3). These tautological syzygies and their analogs at each step in the resolution are called Koszul syzygies. Fröberg’s conjecture says that for all $N, \kappa, x$, there exist ideals where the only syzygies are the Koszul syzygies.

Both conjectures hold when $x \leq N$, because this corresponds to the growth of the ideal of a complete intersection.

For the minimum possible growth we have the results of Macaulay (see, e.g., [12]): Say $\mathcal{I}$ is generated in degree at most $\kappa$, and $\dim S^\kappa V / \mathcal{I}_\kappa = q$. Write

$$q = \binom{a_\kappa}{\kappa} + \binom{a_{\kappa - 1}}{\kappa - 1} + \ldots + \binom{a_\delta}{\delta}$$

with $a_\kappa > a_{\kappa - 1} > \ldots > a_\delta$ (such an expression exists and is unique), then

$$\dim \mathcal{I}_{\kappa + t} \geq \binom{N + \kappa + t - 1}{\kappa + t} - \left[ \binom{a_\kappa + t}{\kappa + t} + \binom{a_{\kappa - 1} + t}{\kappa + t - 1} + \ldots + \binom{a_\delta + t}{\delta + t} \right]$$

Equality is achieved for all $t$ if equality holds for $t = 1$ [11]. Ideals satisfying this minimal growth exist, they are lex-segment ideals, see [11].

3. **Shifted partial derivatives and permanent v. determinant**

In [13] lower bounds on the dimension of the space of shifted partial derivatives of both the determinant and permanent are obtained by studying the Hilbert function of a common degeneration of their spaces of partial derivatives.
Proposition 3.0.1. Let \( \mathcal{I}_n, \mathcal{I}_m \subset S^n V \) be subspaces such that \( \mathcal{I}_n \) is a degeneration of \( \mathcal{I}_m \) in the sense that if \( P_1, \ldots, P_q \) is a basis of \( \mathcal{I}_n \) and \( Q_1, \ldots, Q_m \) is a basis of \( \mathcal{I}_m \), then \( Q_j = \lim_{\epsilon \to 0} g(\epsilon) \cdot \sum\nolimits_{i=1}^q c_i^j P_i \) for some curve \( g(\epsilon) \subset GL(V) \) and constants \( c_i^j \). Let \( \mathcal{I}^1, \mathcal{I}^2 \) denote the corresponding ideals that they generate, then \( \dim \mathcal{I}_1^t \leq \dim \mathcal{I}_2^t \) for all \( t \).

The proof is left to the reader.

In [13] they consider the size \( \kappa \) and the ideal it generates \( \mathcal{D}_1^t \mathcal{I}_n^\kappa \). It is in \( \text{End}(\mathbb{C}^{n^2}) \cdot \mathcal{I}_n^{\det \kappa} \) as well as \( \text{End}(\mathbb{C}^{n^2}) \cdot \mathcal{I}_n^{\text{perm}_n \kappa} \). They show [13, Cor. 16] that, setting \( k = n - \kappa \),

\[
\dim \mathcal{I}_1^{\kappa + \kappa} \geq \binom{n + k}{2k} \left( \frac{n^2 + \tau - 2k}{\tau} \right).
\]

Proof of Theorem 1.1.1. Set \( k = n - \kappa \) so we are taking \( k \) derivatives. We may assume \( k \leq \frac{m}{2} \), otherwise there is no gain for \( \ell^{n-m} P \) but there is for \( \det_n \). We trivially have

\[
\dim(\ell^{n-m} P_{(k, n-k)[\tau]}(S^{n^2-k}\mathbb{C}^n)) < k \binom{m}{m-k} \left( \frac{n^2 + \tau - 1}{\tau} \right)
\]

the first two terms because the largest shifted partial has dimension \( \left( \frac{m}{m-k} \right)^2 \) and there are \( k \) different shifts (depending on how many derivatives one places on \( \ell \)), and the third factor is what would appear if there were no syzygies whatsoever.

Consider the ratio:

\[
\frac{\binom{n+k}{2k} \left( \frac{n^2 + \tau - 2k}{\tau} \right)}{k \left( \frac{m}{m-k} \right)^2 \left( \frac{n^2 + \tau - 1}{\tau} \right)}
\]

Assume \( n = \delta m^2 \) for some constant \( \delta \) and \( k < \frac{m}{2} \) is increasing as a function of \( m \). Then, since \( \ln(!) = q \ln(q) - q + O(\ln(q)) \), we have:

\[
\ln \left( \frac{(n+k)}{k \left( \frac{m}{m-k} \right)^2} \right) = (\delta m^2 + k) \ln(\delta m^2 + k) - 2k \ln(2k) - (\delta m^2 - k) \ln(\delta m^2 - k)
\]

\[
- 2m \ln m + 2(m-k) \ln(m-k) + 2k \ln k - \ln(k) + O(\ln(\frac{(\delta m^2 + k)2(m-k)}{(\delta m^2 - k)(2m)}))
\]

\[
= k \ln \frac{\delta^2 m^4 - k^2}{4(m-k)^2} + \delta m^2 \ln \frac{\delta m^2 + k}{\delta m^2 - k} + 2m \ln(1 - \frac{k}{m}) - \ln(k) + O(\ln(\frac{(\delta m^2 + k)2(m-k)}{(\delta m^2 - k)(2m)})).
\]

Note that

\[
\frac{\delta m^2 \ln \frac{\delta m^2 + k}{\delta m^2 - k}}{\delta m^2 - k} = \ln(1 + \frac{k}{\delta m^2}) \frac{\delta m^2}{\delta m^2} - \ln(1 - \frac{k}{\delta m^2}) \frac{\delta m^2}{\delta m^2},
\]

which tends to \( 2k \ln e \) and

\[
k \ln \frac{\delta^2 m^4 - k^2}{4(m-k)^2} = 2k \ln \frac{\delta m}{2(1 - \frac{k}{m})} - 2k \ln(1 - \frac{k^2}{\delta^2 m^2}.
\]
Since the second term tends to zero, we obtain
\[
\ln \left( \frac{\binom{n+k}{2k}}{k^{\binom{m}{k}}} \right)^2 = 2k \ln \left( \frac{\delta m e}{2(1 - \frac{k}{m})} \right) + 2m \ln \left( 1 - \frac{k}{m} \right) - \ln(k) - O(1).
\]

Regarding the rest of the ratio,
\[
\ln \frac{\binom{n^2 + \tau - 2k}{\tau}}{(n^2 + \tau - 1)} = (n^2 + \tau - 2k) \ln(n^2 + \tau - 2k) - (n^2 - 2k) \ln(n^2 - 2k) - (n^2 + \tau - 1) \ln(n^2 + \tau - 1) + \ln \left( \frac{n^2 + \tau - 2k}{(n^2 - 2k)(n^2 + \tau - 1)} \right)
\]
\[
= 2k \ln \left( \frac{n^2 - 2k}{n^2 + \tau - 2k} \right) + O \left( \ln \left( \frac{n^2 + \tau - 2k}{(n^2 - 2k)(n^2 + \tau - 1)} \right) \right)
\]
\[
= -2k \ln \left( \frac{n^2 - 2k}{n^2 + \tau - 2k} \right) + O \left( \ln \left( \frac{n^2 + \tau - 2k}{(n^2 - 2k)(n^2 + \tau - 1)} \right) \right)
\]
and thus, if \( \tau \geq n^2 \),
\[
\ln \left( \frac{\binom{n^2 + \tau - 2k}{\tau}}{(n^2 + \tau - 1)} \right) = -2k \ln \left( \frac{\tau}{n^2} + 1 \right) + O(1).
\]

If \( \tau < n^2 \) it limits to \( -k \ln(C) + O(1) \) for some constant \( C \).

Thus \( \dim \mathcal{T}_{\tau+k}^{D_{2,k}} > \dim \mathcal{T}_{\tau+k}^{P,k} \) as long as
\[
\frac{\delta m e}{2} \left[ (1 - \frac{k}{m})^{\frac{k}{k-1}} k^{\frac{1}{k}} \right] > \frac{\tau}{n^2} + 1
\]
and since \( \lim_{m \to \infty} \left[ (1 - \frac{k}{m})^{\frac{k}{k-1}} k^{\frac{1}{k}} \right] \geq \frac{1}{\tau} \), \( \dim \mathcal{T}_{\tau+k}^{D_{2,k}} > \dim \mathcal{T}_{\tau+k}^{P,k} \) for sufficiently large \( m \) as long as
\[
\tau < n^2 \left( \frac{\delta m}{2} - 1 \right).
\]

When \( k \) is constant, we have
\[
\frac{\binom{n+k}{2k}}{k^{\binom{m}{k}}} = \frac{(n+k)(n+k-1)\cdots(n-k)(k)!}{(2k)!(m(m-1)\cdots(m-k))^2}
\]
and
\[
\lim_{m \to \infty} \left[ \frac{\binom{n+k}{2k}}{k^{\binom{m}{k}}} \right]^2 = 1
\]
Taking logs, we get \( 2k \log \left( \frac{n}{m} \right) + O(1) \). On the other hand
\[
\frac{\binom{n^2 + \tau - 2k}{\tau}}{(n^2 + \tau - 1)} = \frac{(n^2 + \tau - 1)\cdots(n^2 + \tau - 2k + 1)}{(n^2 - 1)\cdots(n^2 - 2k + 1)}
\]
and thus
\[
\lim_{m \to \infty} \left[ \frac{\binom{n^2 + \tau - 2k}{\tau}}{(n^2 + \tau - 1)} \right]^2 = 1
\]
Taking logs gives \( (2k - 1) \log \left( \frac{n^2 + \tau}{m^2} \right) \), so \( \dim \mathcal{T}_{\tau+k}^{D_{2,k}} > \dim \mathcal{T}_{\tau+k}^{P,k} \) as long as \( \frac{m}{n} > \frac{n^2 + \tau}{2k - 1} \), i.e.,
\[
\tau < n^2 \left( \delta m - 1 \right).
\]
For larger values of $\tau$, inspired by Iarrobino-Fröberg, we first use that $\mathcal{I}_\kappa^{\det_n, \kappa}$ can degenerate to $\mathcal{I}_\kappa^2 := \operatorname{span}\{e^\kappa_1, e^\kappa_2\}$.

More generally, if $\kappa > k$, then $\mathcal{I}_\kappa^{\det_n, \kappa}$ can degenerate to $\operatorname{span}\{e^\kappa_1, \ldots, e^\kappa_k\}$ by setting all entries of a matrix to zero except those in the upper $\kappa \times \kappa$ and lower $k \times k$ block. In the upper block, set all entries above the diagonal to zero, and set all entries on the lower diagonals equal. In the lower left block, put $e^\kappa_\ell$ along the diagonal as many times as necessary and fill the rest with zeros, then $e^\kappa_{k-1}$ along the next band etc., e.g. when $\kappa = 4$ and $k = 3$

$$
\begin{pmatrix}
\ell_1 & \ell_2 & \ell_1 \\
\ell_3 & \ell_2 & \ell_1 \\
\ell_4 & \ell_3 & \ell_2 & \ell_1
\end{pmatrix}.
$$

The the size $\kappa$-minors of this matrix will include $e^\kappa_1, \ldots, e^\kappa_k$. Note that even when $k = 1$, one can still degenerate to $\operatorname{span}\{e^\kappa_1, e^\kappa_2\}$.

Since $\mathcal{I}_\kappa^2$ is a complete intersection Equation (5) applies. Set $N = n^2 + 1$. Let $\mathcal{I}^2$ denote the ideal generated by $\mathcal{I}_\kappa^2$,

$$
\dim \mathcal{I}^2_t = 2 \left( \binom{N + t - \kappa}{t - \kappa} - \binom{N + t - 2\kappa}{t - 2\kappa} \right).
$$

On the other hand, for the padded polynomial, inspired by Macaulay, we use that

$$
\dim \mathcal{I}_t^{\ell_n - m \text{ perm}_n, \kappa} \leq \dim \mathcal{I}_t^1
$$

where $\mathcal{I}_1$ is generated in degree $q := n - k - m$ by $e^{n-k-m}$, so in particular $\mathcal{I}_\kappa^{\ell_n - m \text{ perm}_n, \kappa} \subset \dim \mathcal{I}_\kappa^1$.

We have

$$
\dim \mathcal{I}_t^1 = \binom{N + t - q}{t - q}.
$$

Thus

$$
\dim \mathcal{I}^2_t - \dim \mathcal{I}_t^1 = 2 \left( \binom{N + t - \kappa}{t - \kappa} - \binom{N + t - q}{t - q} - \binom{N + t - 2\kappa}{t - 2\kappa} \right)
$$

$$
= \left( \frac{N + t - \kappa}{t - \kappa} \right) \left[ 2 - \frac{(N + t - q)(N + t - q - 1)\cdots(N + t - k + 1)}{(t - q)(t - q - 1)\cdots(t - k + 1)} \right] - \frac{(t - \kappa)\cdots(t - 2\kappa + 1)}{(N + t - \kappa)\cdots(N + t - 2\kappa + 1)}
$$

$$
= \left( \frac{N + t - \kappa}{t - \kappa} \right) \left[ 2 - (1 + \frac{N}{t - q})\cdots(1 + \frac{N}{t - k + 1}) - [(1 + \frac{N}{t - \kappa})\cdots(1 + \frac{N}{t - 2\kappa + 1})]^{-1} \right]
$$

Note that $m = \kappa - q$. The second term in the brackets limits to a quantity bounded above by $e^{-Nm \kappa / \kappa - 1}$ and the third limits to a quantity bounded above by $e^{-Nm \kappa / \kappa - 1}$. Write $t = \alpha N m$, so we need, since we are assuming $\frac{\kappa}{Nm} \to 0$, and $\kappa = n - k > \delta m^2 - \frac{m}{2}$, in the limit $e^{\frac{\alpha^2}{1}} + e^{-\frac{\alpha m}{2}} < 2$, but the second term goes to zero, so we simply need $e^{\frac{\alpha^2}{1}} < 2$, i.e., $t > \frac{1}{\ln(9)} m N$.

We see that for the two estimates to overlap for sufficiently large $m$, if $k$ is constant it suffices that

$$
\frac{m(n^2 + 1)}{\ln(2)} \leq \delta mn^2 - 1,
$$

i.e., that $\delta > \frac{1}{\ln(2)} \approx 1.4427$. Take $\delta = 1.5$. Then our estimates up until now cover all ranges of $t$ when $k$ is constant and the ranges $t < \frac{1.5mn^2}{2}$ and $t > \frac{1}{\ln(2)} m N$ when $k$ is non-constant, so now...
assume $k$ is non-constant and $\frac{1}{\ln(2)} nN < t < \frac{15m^2}{2}$. (In fact, the argument below will show that for any constant $\delta$, if $k$ is non-constant, we must have $n \leq \delta m^2$ for the possibility of a non-trivial bound.)

Let $I^\kappa \subset \text{Sym}(\mathbb{C}^N)$ be generated by $\ell^\kappa_1, \ldots, \ell^\kappa_n$, so $\dim I^t_{\det, \kappa} = \dim I^\kappa$.

Since the next calculation may be of interest beyond the case at hand, we present it with slightly more general parameters:

As before, $I^1 \subset \text{Sym}(\mathbb{C}^N)$ is generated by $\ell^t$ with $q < \kappa$.

We have

$$\dim I^\kappa - \dim I^1 = \binom{N + t - \kappa}{t - \kappa} - \binom{N + t - q}{t - q} + \sum_{j=2}^k (-1)^{j+1} \binom{k}{j} \binom{N + t - j \kappa}{t - j \kappa}$$

$$= \binom{N + t - \kappa}{t - \kappa} \left[ k \cdot \frac{(N + t - q) \cdots (N + t - \kappa + 1)}{(t - q) \cdots (t - \kappa + 1)} + \sum_{j=2}^k (-1)^{j+1} \binom{k}{j} \frac{(t - \kappa) \cdots (t - j \kappa + 1)}{(N + t - \kappa)(N + t - \kappa + 1) \cdots (N + t - j \kappa)} \right]$$

$$\geq \binom{N + t - \kappa}{t - \kappa} \left[ k (1 + \frac{N}{t - \kappa})^m - \sum_{j=2}^k \binom{k}{j} (1 + \frac{N}{t - \kappa})^{-j \kappa} \right]$$

so

$$\lim_{m \to \infty} \frac{\dim I^\kappa - \dim I^1}{\binom{N + t - \kappa}{t - \kappa}} \geq \left( \frac{N + t - \kappa}{t - \kappa} \right) \left[ k - e^{\ln(k)} \sum_{j=2}^k \frac{1}{j \kappa} \right]$$

but now if

$$\frac{1.5mN}{2} < t < \frac{mN}{\ln(2)}$$

the term $\sum_{j=2}^k e^{\ln(k)} \frac{1}{j \kappa}$, which is bounded above by $e^{\ln(k)} \frac{N}{m}$, goes to zero and the second term limits to $e^{-\frac{N}{\ln(mN)}} \leq C'$, where $c, C'$ are constants. Since we assume $k$ is growing, the quantity in brackets is eventually positive.

**Remark 3.0.2.** In [13] they use the degeneration $D_{2, \kappa}$ to show the determinant (and permanent) requires a homogeneous depth four arithmetic circuit with bottom fanin bounded by $\sqrt{n}$ must be of size $2^{\Omega(\sqrt{n})}$.

4. The minimal free resolution of the ideal generated by minors of size $\kappa$

This section, except for §4.3, is expository. The results are due to Lascoux [22]. The results in §4.3 were known in slightly different language, but to our knowledge are only available in an unpublished manuscript of Roberts [31]. For the other subsections, we follow the presentation in [35].

4.1. Statement of the result. Let $E, F = \mathbb{C}^n$, give $E \otimes F$ coordinates $(x^i_j)$, with $1 \leq i, j \leq n$. The weight (under $GL(E) \times GL(F)$) of a monomial $x^i_1 \cdots x^i_q \in S^q(E \otimes F)$ is given by a pair of $n$-tuples $((w^E_1, \ldots, w^E_n), (w^F_1, \ldots, w^F_n))$ where $w^E_i$ is the number of $i$s equal to $s$ and $w^F_i$ is the number of $j$s equal to $t$. A vector is a weight vector of weight $((w^E_1, \ldots, w^E_n), (w^F_1, \ldots, w^F_n))$ if it can be written as a sum of monomials of weight $((w^E_1, \ldots, w^E_n), (w^F_1, \ldots, w^F_n))$. Any $GL(E) \times GL(F)$-module has a basis of weight vectors, and any irreducible module has a unique highest weight which (if the representation is polynomial) is a pair of partitions, $(\pi, \mu) = ((p_1, \ldots, p_n), (m_1, \ldots, m_n))$, where we allow a string of zeros to be added to a partition to make it of length $n$. The corresponding $GL(E) \times GL(F)$-module is denoted $S_{\pi} E \otimes S_{\mu} F$. 
Set \( r = \kappa - 1 \). Let \( \hat{\sigma}_r = \hat{\sigma}_r(\text{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) \subset \mathbb{C}^n \otimes \mathbb{C}^n = E^* \otimes F^* \) denote the variety of \( n \times n \) matrices of rank at most \( r \). By “degree \( S_rE^* \), we mean \( |\pi| = p_1 + \cdots + p_n \). Write \( \ell(\pi) \) for the largest \( j \) such that \( p_j > 0 \). Write \( \pi + \pi' = (p_1 + p'_1, \ldots, p_n + p'_n) \).

**Theorem 4.1.1.** [22] Let \( 0 \to F_N \to \cdots \to F_1 \to \text{Sym}(E \otimes F) = F_0 \to \mathbb{C}[\hat{\sigma}_r] \to 0 \) denote the minimal free resolution of \( \hat{\sigma}_r \). Then

1. \( N = (n-r)^2 \), i.e., \( \hat{\sigma}_r \) is arithmetically Cohen-Macaulay.
2. \( \hat{\sigma}_r \) is Gorenstein, i.e., \( F_N = \text{Sym}(E \otimes F) \), generated by \( S_{(n-r)n}E \otimes S_{(n-r)n}F \). In particular \( F_{N-j} \simeq F_j \) as \( SL(E) \times SL(F) \)-modules, although they are not isomorphic as \( GL(E) \times GL(F) \)-modules.
3. For \( 1 \leq j \leq N-1 \), the space \( F_j \) has generating modules of degree \( sr + j \) where \( 1 \leq s \leq \lfloor \sqrt{j} \rfloor \). The modules of degree \( r + j \) form the generators of the linear strand of the minimal free resolution.
4. The generating module of \( F_j \) is multiplicity free.
5. Let \( \alpha, \beta \) be (possibly zero) partitions such that \( \ell(\alpha), \ell(\beta) \leq s \). Independent of the lengths (even if they are zero), write \( \alpha = (\alpha_1, \ldots, \alpha_s) \), \( \beta = (\beta_1, \ldots, \beta_s) \). The degree \( sr + j \) generators of \( F_j \), for \( 1 \leq j \leq N \) are

\[
M_{j,rs+j} = \bigoplus_{a \geq 1} \bigoplus_{|\alpha|+|\beta|=s+2} S_s^{(r+s+(\alpha_0,0',\beta')} E \otimes S_s^{(r+s+(\beta,0',\alpha')} F.
\]

The Young diagrams of the modules are depicted in Figure 1 below.

![Figure 1](image.png)

**Figure 1.** Partition \( \pi \) and pairs of partitions \((s)^{r+s} + (\alpha_0,0',\beta') = w \cdot \pi \) and \((s)^{r+s} + (\beta,0',\alpha') = \pi' \) it gives rise to in the resolution (see §4.4 for explanations).

6. In particular the generator of the linear component of \( F_j \) is

\[
M_{j,j+r} = \bigoplus_{a+b=j-1} = S_{a+1,1} E \otimes S_{b+1,1} F.
\]

This module admits a basis as follows: form a size \( r+j \) submatrix using \( r+b+1 \) distinct rows, repeating a subset of \( a \) rows to have the correct number of rows and \( r+a+1 \)
distinct columns, repeating a subset of \( b \) columns, and then performing a “tensor Laplace expansion” as described below.

**Remark 4.1.2.** Our \( \beta \) is \( \beta' \) in [35].

### 4.2. The Koszul resolution

If \( I = \text{Sym}(V) \), the minimal free resolution is given by the exact complex

\[
\cdots \rightarrow S^{q-1}V \otimes \Lambda^{p+2}V \rightarrow S^qV \otimes \Lambda^{p+1}V \rightarrow S^{q+1}V \otimes \Lambda^pV \rightarrow \cdots
\]

The maps are given by the transpose of exterior derivative (Koszul) map \( d_{p,q} : S^qV \otimes \Lambda^{p+1}V^* \rightarrow S^{q-1}V^* \otimes \Lambda^{p+2}V^* \). Write \( d^T_{p,q} : S^{q-1}V \otimes \Lambda^{p+2}V \rightarrow S^qV \otimes \Lambda^{p+1}V \). We have the \( GL(V) \)-decomposition

\[
S^qV \otimes \Lambda^{p+1}V = S_{q,1}V \otimes S_{q+1,1}V,
\]

so the kernel of \( d^T_{p,q} \) is the first module, which also is the image of \( d^T_{p+1,q-1} \).

Explicitly, \( d^T_{p,q} \) is the composition of polarization \( (\Lambda^{p+2}V \rightarrow \Lambda^{p+1}V \otimes V) \) and multiplication:

\[
S^qV \otimes \Lambda^{p+2}V \rightarrow S^{q-1}V \otimes \Lambda^{p+1}V \otimes V \rightarrow S^qV \otimes \Lambda^{p+1}V.
\]

For the minimal free resolution of any ideal, the linear strand will embed inside (11).

Throughout this article, we will view \( S_{q+1,1}V \) as a submodule of \( S^qV \otimes \Lambda^{p-1}V \), \( GL(V) \)-complementary to \( d^T_{p,q}(S^{q-1,1}V) \).

For \( T \in S^qV \otimes V^{\otimes 3} \), and \( P \in S^4V \), introduce notation for multiplication on the first factor, \( T \cdot P \in S^{q+4}V \otimes V^{\otimes 2} \). Write \( F_j = M_j \cdot \text{Sym}(V) \). As always, \( M_0 = \mathbb{C} \).

### 4.3. Geometric interpretations of the terms in the linear strand

(10). First note that \( F_1 = M_1 \cdot \text{Sym}(E \otimes F) \), where \( M_1 = M_{1,r+1} = \Lambda^{r+1}E \otimes \Lambda^{r+1}F \), the size \( r + 1 \) minors which generate the ideal. The syzygies among these equations are generated by

\[
M_{2,r+2} := S_{1,r+2}E \otimes S_{2,r+1}F \oplus S_{2,r+1}E \otimes S_{1,r+2}F \subset T^T_{r+2} \otimes V
\]

(i.e., \( F_2 = M_2 \cdot \text{Sym}(E \otimes F) \)), where elements in the first module may be obtained by choosing \( r + 1 \) rows and \( r + 2 \) columns, forming a size \( r + 2 \) square matrix by repeating one of the rows, then doing a ‘tensor Laplace expansion’ that we now describe:

In the case \( r = 1 \) we have highest weight vector

\[
S_{123}^{1|2} := (x_2^2 x_3^2 - x_2^2 x_3) \otimes x_1^1 - (x_1^1 x_3^3 - x_1^1 x_3^1) \otimes x_2^1 + (x_1^1 x_2^2 - x_2^1 x_1^1) \otimes x_3^1
\]

\[
= M_{12}^1 \otimes x_1^1 - M_{13}^1 \otimes x_2^1 + M_{12}^1 \otimes x_3^1
\]

where in general \( M_I^J \) will denote the minor obtained from the submatrix with indices \( I, J \). This corresponds to the Young tableau pair:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3
\end{array}
\]

To see (12) is indeed a highest weight vector, first observe that it has the correct weights in both \( E \) and \( F \), and that in the \( F \)-indices \( \{1, 2, 3\} \) it is skew and that in the first two \( E \) indices it is also skew. Finally to see it is a highest weight vector note that any raising operator sends it to zero. Also note that under the multiplication map \( S^2V \otimes V \rightarrow S^3V \) the element maps to zero, because the map corresponds to converting a tensor Laplace expansion to an actual one, but the determinant of a matrix with a repeated row is zero.

In general, a basis of \( S_\pi E \otimes S_\mu F \) is indexed by pairs of semi-standard Young tableau in \( \pi \) and \( \mu \). In the linear strand, all partitions appearing are hooks, a basis of \( S_{\alpha,1}E \) is given by two sequences of integers taken from \([n]\), one weakly increasing of length \( a \) and one strictly
increasing of length \( b \), where the first integer in the first sequence is at least the first integer in the second sequence.

A highest weight vector in \( S_{21r}E \otimes S_{1^r+2}F \) is
\[
S_{1^r+2}^{1 \cdot r+1} = M_{1^r+2}^{1 \cdot r+1} \otimes x_1^1 + \cdots + (−1)^r M_{1^r+2}^{1 \cdot r+1} \otimes x_r^1,
\]
and the same argument as above shows it has the desired properties. Other basis vectors are obtained by applying lowering operators to the highest weight vector, so their expressions will be more complicated.

Remark 4.3.1. If we chose a size \( r + 2 \) submatrix, and perform a tensor Laplace expansion of its determinant about two different rows, the difference of the two expressions corresponds to a linear syzygy, but these are in the span of \( M_2 \). These expressions are important for comparison with the permanent, as they are the only linear syzygies for the ideal generated by the size \( r + 1 \) sub-permanents, where one takes the permanent Laplace expansion.

Continuing, \( F_3 \) is generated by the module
\[
M_{3,r+3} = S_{1^r+3}E \otimes S_{3,1^r}F \oplus S_{2,1^r+1}E \otimes S_{2,1^r+1}F \oplus S_{3,1^r}E \otimes S_{1^r+3}F \subset M_2 \otimes V.
\]
These modules admit bases of double tensor Laplace type expansions of a square submatrix of size \( r + 3 \). In the first case, the highest weight vector is obtained from the submatrix whose rows are the first \( r + 3 \) rows of the original matrix, and whose columns are the first \( r \)-columns with the first column repeated three times. For the second module, the highest weight vector is obtained from the submatrix whose rows and columns are the first \( r + 2 \) such, with the first row/column repeated twice. A highest weight vector for \( S_{3,1^r}E \otimes S_{1^r+3}F \) is
\[
S_{1^r+3}^{1 \cdot r+1} = \sum_{1 \leq \beta_1 < \beta_2 \leq r+3} (−1)^{\beta_1 + \beta_2} M_{1^r+3}^{1 \cdot r+1} \otimes (x_{\beta_1}^1 \wedge x_{\beta_2}^1)
\]
\[
= \sum_{\beta = 1}^{r+3} (−1)^{\beta+1} S_{1^r+3}^{1 \cdot r+1} \otimes x_\beta^1.
\]
Here \( S_{1^r+3}^{1 \cdot r+1} \) is defined in the same way as the highest weight vector.

A highest weight vector for \( S_{2,1^r+1}E \otimes S_{2,1^r+1}F \) is
\[
S_{1^r+1}^{1 \cdot r+2} = \sum_{\alpha, \beta \geq 1} (−1)^{\alpha + \beta} M_{1^r+2}^{1 \cdot r+2} \otimes (x_\alpha^1 \wedge x_\beta^1)
\]
\[
= \sum_{\beta = 1}^{r+3} (−1)^{\beta+1} S_{1^r+2}^{1 \cdot r+2} \otimes x_\beta^1 - \sum_{\alpha = 1}^{r+3} (−1)^{\alpha+1} S_{1^r+2}^{1 \cdot r+2} \otimes x_\alpha^1.
\]
Here \( S_{1^r+2}^{1 \cdot r+2} \) are defined in the same way as the corresponding highest weight vectors.

**Proposition 4.3.2.** The highest weight vector of \( S_{p+1,1^r+q}E \otimes S_{q+1,1^r+p}F \subset M_{p+q+1,r+p+q+1} \) is
\[
S_{1^r+1}^{1 \cdot r+q+1} = \sum_{j \in \{r+1\}, j \notin \{p+1\}, j \notin \{p+q\}} (−1)^{[j]} M_{1^r \cdot i_1 \cdot i_2 \cdot \cdots \cdot i_r \cdot i_{r+q+1}} \otimes (x_{j_1}^1 \wedge \cdots \wedge x_{j_p}^1 \wedge x_{i_1}^1 \wedge \cdots \wedge x_{i_{r+q+1}}^1).
\]
A hatted index is one that is omitted from the summation.
Proof. It is clear the expression has the correct weight and is a highest weight vector, and that it lies in $S^{r+1}V \otimes \Lambda^{p+q} V$. We now show it maps to zero under the differential.

Under the map $a^T : S^{r+1}V \otimes \Lambda^{p+q} V \to S^r V \otimes \Lambda^{p+q+1} V$, the element $S^{1p[1,\ldots,r+q+1]}_{1q[1,\ldots,r+p]}$ maps to:

$$
\sum_{\ell \leq [r+q+1], |\ell|=q, j \leq [r+p+1], |j|=p} (-1)^{[\ell]+|j|} \left[ \sum_{\alpha \in I} (-1)^{p+\alpha} M_{1\ell_{j_1}\ldots j_{p+1}^{(r+q+1)}} \left( x^{i_1}_{j_1} \wedge \cdots \wedge x^{i_p}_{j_p} \wedge x^{i_{p+1}}_{j_{p+1}} \wedge \cdots \wedge x^{i_{q+r}}_1 \right) \right]
$$

$$
+ \sum_{\beta \in [r+p+1]\backslash J'} (-1)^{f(\beta,j')} M_{1\ell_{j_1}\ldots j_{p+1}^{(r+q+1)}} \left( x^{i_1}_{j_1} \wedge \cdots \wedge x^{i_p}_{j_p} \wedge x^{i_{p+1}}_{j_{p+1}} \wedge \cdots \wedge x^{i_{q+r}}_1 \right)
$$

where $f(\beta,j')$ equals the number of $j' \in J$ less than $\beta$. This term is the Laplace expansion of the determinant of a matrix of size $r+1$ which has its first row appearing twice, and is thus zero.

Notice that if $q, p > 0$, then $S^{1p[1,\ldots,r+q+1]}_{1q[1,\ldots,r+p]}$ is the sum of terms including $S^{1p[1,\ldots,r+q]}_{1q[1,\ldots,r+p]} \otimes x_1^{r+q+1}$ and $S^{1p[1,\ldots,r+q+1]}_{1q[1,\ldots,r+p]} \otimes x_1^{r+q+1}$. This implies the following corollary:

**Corollary 4.3.3 (Roberts [31]).** Each module $S_{a,1r} E \otimes S_{b,1r} F$, where $a + b = j$ that appears with multiplicity one in $F_{j,j+r}$, appears with multiplicity two in $F_{j-1,j+r}$ if $a, b > 0$, and multiplicity one if $a$ or $b$ is zero. The map $F_{j-1,j+r+1} \to F_{j-1,j+r+1}$ restricted to $S_{a,1r} E \otimes S_{b,1r} F$, maps non-zero to both $S_{a-1,1r} E \otimes S_{b,1r} F \cdot E \otimes F$ and $(S_{a,1r} E \otimes S_{b,1r} F) \cdot E \otimes F$.

**Proof.** The multiplicities and realizations come from applying the Pieri rule. (Note that if $a$ is zero the first module does not exist and if $b$ is zero the second module does not exist.) That the maps to each of these is non-zero follows from the remark above.

**Remark 4.3.4.** In [31] it is proven more generally that all the natural realizations of the irreducible modules in $M_j$ have non-zero maps onto every natural realization of the module in $F_{j-1}$. Moreover, the constants in all the maps are determined explicitly. The description of the maps is different than the one presented here.

### 4.4 Proof of Theorem 4.1.1.

This section is expository and less elementary than the rest of the paper. The variety $\hat{\sigma}_r$ admits a desingularization by the geometric method of [35], namely consider the Grassmannian $G(r, E^*)$ and the vector bundle $p : S \otimes F \to G(r, E^*)$ whose fiber over $x \in G(r, E^*)$ is $x \otimes F$. (Although we are breaking symmetry here, it will be restored in the end.) The total space admits the interpretation as the incidence variety

$$\{ (x, \phi) \in G(r, E^*) \times \text{Hom}(F, E^*) \mid \phi(F) \subseteq x \},$$

and the projection to $\text{Hom}(F, E^*) = E^* \otimes F^*$ has image $\hat{\sigma}_r$. One also has the exact sequence

$$0 \to S \otimes F^* \to E^* \otimes F^* \to Q \otimes F^* \to 0$$

where $E^* \otimes F^*$ denotes the trivial bundle with fiber $E^* \otimes F^*$ and $Q = E^*/S$ is the quotient bundle. As explained in [35], letting $q : S \otimes F^* \to E^* \otimes F^*$ denote the projection, $q$ is a desingularization of $\hat{\sigma}_r$, the higher direct images $R_i q^*(\mathcal{O}_S(-r))$ are zero for $i > 0$, and so by [35, Thm. 5.12.5.13] one concludes $F_i = M_i \cdot \text{Sym}(E \otimes F)$ where

$$M_i = \oplus_{r \geq 0} H^i(G(r, E^*), \Lambda^{i+j}(Q^* \otimes F))$$

$$= \oplus_{r \geq 0} \oplus_{|\pi|=i+j} H^i(G(r, E^*), S_{\pi, Q} \otimes S_{\pi, F})$$
One now uses the Bott-Borel-Weil theorem to compute these cohomology groups. An algorithm for this is given in [35, Rem. 4.1.5]: If \( \pi = (p_1, \ldots, p_n) \) (where we must have \( p_1 \leq n \) to have \( S_{\pi}F \) non-zero, and \( q \leq n - r \) as \( \text{rank} Q = n - r \)), then \( S_{\pi}Q \) is the vector bundle corresponding to the sequence

\[
(0^r, p_1, \ldots, p_n-r).
\]

The dotted Weyl action by \( \sigma_i = (i, i+1) \in \mathfrak{S}_n \) is

\[
\sigma_i \cdot (\alpha_1, \ldots, \alpha_n) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, \ldots, \alpha_n)
\]

and one applies simple reflections to try to transform \( \alpha \) to a partition until one either gets a partition after \( u \) simple reflections, in which case \( H^u \) is equal to the module associated to the partition one ends up with and all other cohomology groups are zero, or one ends up on a wall of the Weyl chamber, i.e., at one step one has \( (\beta_1, \ldots, \beta_n) \) with some \( \beta_{i+1} = \beta_i + 1 \), in which case there is no cohomology.

In our case, we need to move \( p_1 \) over to the first position in order to obtain a partition, which means we need \( p_1 \geq r + 1 \), and then if \( p_2 < 2 \) we are done, otherwise we need to move it etc...

The upshot is we can get cohomology only if there is an \( s \) such that \( p_s \geq r + s \) and \( p_{s+1} < s + 1 \), in which case we get

\[
S_{(p_1-r, \ldots, p_s-r, s', p_{s+1}, \ldots, p_n-r)} \mathcal{E} \otimes S_{\pi'}F
\]

contributing to \( H^{r+s} \). Say we are in this situation, then write \( (p_1 - r - s, \ldots, p_n - r - s) = \alpha, (p_{s+1}, \ldots, p_n) = \beta' \), so

\[
(p_1 - r, \ldots, p_s - r, s', p_{s+1}, \ldots, p_n - r) = (s^{r+s}) + (\alpha, 0^r, \beta')
\]

and moreover we may write

\[
\pi' = (s^{r+s}) + (\beta, 0^r, \alpha')
\]

proving Theorem 4.1.1. The case \( s = 1 \) gives the linear strand of the resolution.

5. The Minimal Free Resolution of the Ideal Generated by the Space of Square-Free Monomials

The space of shifted partial derivatives of \( x_1 \cdots x_n \) is the space of square-free monomials in \( S^{\kappa} \mathbb{C}^n \). While the ideal these generate has been well-studied, we were unable to find its minimal free resolution in the literature.

**Proposition 5.0.1.** The Hilbert function of \( \mathcal{I}^{x_1 \cdots x_n, \kappa} \) in degree \( \kappa + t \) is

\[
\dim \mathcal{I}^{(x_1 \cdots x_n, \kappa)}_{\kappa+t} = \sum_{j=0}^{n-\kappa} \binom{n}{\kappa-j} \binom{\kappa+t-1}{\kappa+j-1}
\]

**Proof.** The ideal in degree \( d = t + \kappa \) has a basis of the distinct monomials of degree \( d \) containing at least \( n - k \) distinct indices. When we divide such a basis vector by \( x_1 \cdots x_n \) the denominator will have degree at most \( k \). For each \( i \leq k \), the space of possible numerators with a denominator of degree \( i \) that is fixed, has dimension \( \dim S^{d-n+i} \mathbb{C}^{n-i} \), and there are \( \binom{n}{i} \) possible denominators. Summing over \( i \) gives the result. \( \square \)

For the Hilbert function of the coordinate ring, we have the following expression:

**Proposition 5.0.2.** The Hilbert function of \( \text{Sym}(\mathbb{C}^n)/\mathcal{I}^{x_1 \cdots x_n, \kappa} \) in degree \( t \) is

\[
\dim(\text{Sym}(\mathbb{C}^n)/\mathcal{I}^{x_1 \cdots x_n, \kappa})_t = \sum_{j=0}^{n-\kappa-2} \binom{n}{j+1} \binom{t-1}{j},
\]

if \( t \geq n - \kappa - 1 \), and \( \binom{n+t-1}{n-1} \) if \( t < n - \kappa - 1 \).
The expression (15) will be a consequence of the results of the next section.

5.1. The minimal free resolution.

**Definition 5.1.1.** [36] A simplicial complex $\Delta$ on a vertex set $V$ is a collection of subsets $\sigma$ of $V$, such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. If $|\sigma| = i + 1$ then $\sigma$ is called an $i$-face. Let $f_i(\Delta)$ denote the number of $i$-faces of $\Delta$, and define $\dim(\Delta) = \max\{i \mid f_i(\Delta) \neq 0\}$. If $\dim(\Delta) = n - 1$, we define $f_\Delta(t) = \sum_{i=0}^{n} f_{i-1} t^{n-i}$. The ordered list of coefficients of $f_\Delta(t)$ is called the $f$-vector of $\Delta$, and the coefficients of $h_\Delta(t) := f_\Delta(t-1)$ is called the $h$-vector of $\Delta$. The Alexander dual $\Delta^\vee$ of $\Delta$ [26] is the simplicial complex

$$\Delta^\vee = \{ \tau \mid \neg \tau \in \Delta \}$$

where $\neg$ denotes the complement $V \setminus \tau$.

For example, if $\Delta$ is the one skeleton of a three simplex, then $f(\Delta) = (1, 4, 6)$ and $h(\Delta) = (1, 2, 3)$, and $\Delta^\vee$ consists of the four vertices and the empty face.

**Definition 5.1.2.** Let $\Delta$ be a simplicial complex on vertices $\{x_1, \ldots, x_n\}$. The Stanley-Reisner ideal $I_\Delta$ is

$$I_\Delta = \langle x_{i_1} \cdots x_{i_j} \mid \{x_{i_1}, \ldots, x_{i_j}\} \text{ is not a face of } \Delta \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n],$$

and the Stanley-Reisner ring is $\mathbb{C}[x_1, \ldots, x_n]/I_\Delta$.

The Stanley-Reisner ideal $I_{\Delta^\vee}$ of $\Delta^\vee$ is obtained by monomializing the primary decomposition of $I_\Delta$: for each primary component in the primary decomposition (for a square free monomial ideal, these are just collections of variables), take the product of the terms in the component. So if

$$I_\Delta = \bigcap_j \langle x_{i_{j_1}}, \ldots, x_{i_{j_s}} \rangle,$$

then $x_{i_{j_1}} \cdots x_{i_{j_s}}$ is a minimal generator of $I_{\Delta^\vee}$, and all minimal generators arise this way. Of special interest to us is the ideal $I_{\Delta(n,k)}$ generated by all square-free monomials of degree $k$ in $n$ variables; it is the Stanley-Reisner ideal of the $k-2$ skeleton of an $n-1$ simplex.

**Lemma 5.1.3.** [3, Lem. 2.8] The quotient $\text{Sym}(V)/I_{\Delta(n,k)}$ is Cohen-Macaulay and has a minimal free resolution which is linear.

We include a proof along the lines of the above discussion.

**Proof.** The ideal $I_{\Delta(n,k-2)}$ is the Stanley Reisner ideal of the $n-k-2$ skeleton of the $n-1$ simplex $\Delta_{n-1}$. The primary decomposition of $I_{\Delta(n,k-2)}$ is [32, Thm 5.3.3]

$$I_{\Delta(n,k-2)} = \bigcap_{1 \leq j_1 < \cdots < j_{k+1} \leq n} \langle x_{j_1}, \ldots, x_{j_{k+1}} \rangle.$$

Thus, the Alexander dual ideal satisfies

$$I_{\Delta(n,k-2)^\vee} = I_{\Delta(n,k)}.$$

It follows that the Alexander dual of $I_{\Delta(n,k-2)}$ is the Stanley-Reisner ideal of the $k-2$ skeleton of $\Delta_{n-1}$, which is $I_{\Delta(n,k)}$. For all $k$, the $k$-skeleta of the simplex $\Delta_{n-1}$ are shellable [36, p.286], hence $I_{\Delta(n,k)}$ is Cohen-Macaulay [26, Thm 13.45]. The Eagon-Reiner theorem [6] now implies that $I_{\Delta(n,k-2)}$ has a linear minimal free resolution. Applying Alexander duality shows that $I_{\Delta(n,k)}$ also has a linear minimal free resolution and is Cohen-Macaulay.

If the minimal free resolution of an ideal $I$ has the $j$-th term $F_j$, the graded Betti numbers are defined to be $b_{j,u} := \dim F_{j,u}$. 

Proposition 5.1.4. The graded Betti numbers of $\mathcal{I}_{\Delta(n, \kappa)}$ are, writing $F_{j,q}$ for the degree $q$ term in the $j$-th term in the minimal free resolution of $\mathcal{I}_{\Delta(n, \kappa)}$,

$$\dim F_{j,j+\kappa} = \binom{n}{\kappa+j} \binom{\kappa-1+j}{j},$$

and the graded Betti numbers are zero in all degrees other than $j + \kappa$.

Proof. By Lemma 5.1.3, the minimal free resolution of $\text{Sym}(V)/\mathcal{I}_{\Delta(n, \kappa)}$ is linear. Hence, there can be no cancellation in the Hilbert series, and the dimensions of the graded Betti numbers may be read off from the numerator of the Hilbert series. As the numerator of the Hilbert series is the $h$-vector [26, Cor 1.15] and remarks following it) of the $\kappa - 2$ skeleton of $\Delta_{n-1}$, the result follows.

Example 5.1.5. Consider the ideal $\mathcal{I}_{\Delta(5,3)}$, consisting of the ten square-free cubic monomials in five variables. The graded Betti numbers $b_{i,j}$ are displayed in a Betti table; starting at position $(0,0)$, the entry (reading right and down) in position $(i,j)$ is $b_{i,i+j}$. The odd shift in the second index allows the Betti table to reflect the regularity of $M$, that is, the largest $j$ such the Betti table is non-zero in row $j$. The Betti table of the minimal free resolution of $\mathcal{I}_{\Delta(5,3)}$ is

| total | 10 | 15 | 6 |
|-------|----|----|---|
| 0     | -  | -  | - |
| 1     | -  | -  | - |
| 2     | -  | -  | - |
| 3     | 10 | 15 | 6 |

So, for example, $\dim F_{1,4}(\mathcal{I}_{\Delta(5,3)}) = 15$.

5.2. The minimal free resolution from a representation-theoretic perspective. First, to fix notation, the ideal is generated in degree $\kappa$ by $x_{i_1} \cdots x_{i_n}$, $I \subset [n]$. As an $\mathfrak{S}_n$-module the space of generators is

$$M_1 = \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_\kappa} [\kappa] = \bigoplus_{j=0}^{\min\{\kappa, n-\kappa\}} [n-j,j],$$

and it has dimension $\binom{n}{\kappa}$.

Proposition 5.2.1. The generator of the $j$-th term in the minimal free resolution of $\mathcal{I}^{x_1 \cdots x_n, \kappa}$, as an $\mathfrak{S}_n$-module, is

$$(16) \quad M_{j, \kappa+j-1} = \text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{\kappa+j-1}} [\kappa, 1^{j-1}],$$

which has dimension $\binom{\kappa+j-1}{j} \binom{n}{\kappa+j}$.

Proof. Let $I \subset [n]$ have cardinality $\kappa - 1$, and let $i,j \in [n]\setminus I$ be distinct. Then $M_2$ has generators $S_{I,i,j} := x_{i_1} \cdots x_{i_n} x_i \otimes x_j - x_{i_1} \cdots x_{i_n} x_j \otimes x_i$. That these map to zero and are linearly independent is clear, and since they span a space of the correct dimension, these must be the generators of $M_2$. The $\mathfrak{S}_{\kappa+1}$ action on $I \cup i \cup j$ is by $[\kappa, 1]$.

In general, $M_{j+1}$ has a basis

$$S_{I,u_1, \cdots, u_j} = \sum_{\alpha} (-1)^{\alpha+1} x_{i_1} \cdots x_{i_n} x_{u_\alpha} \otimes x_{u_1} \wedge \cdots \wedge x_{u_\alpha} \wedge \cdots \wedge x_{u_j}.$$ 

It is clear $S_{I,u_1, \cdots, u_j}$ is a syzygy, and for each fixed $I$ it has the desired $\mathfrak{S}_{\kappa+j}$-action, and the number of such equals the dimension of $M_j$. □
6. On the minimal free resolution of the ideal generated by sub-permanents

Let \( E, F = \mathbb{C}^n \), \( V = E \otimes F \), and let \( \mathcal{T}_\kappa^{\text{perm}, \kappa} \subset S^\kappa(E \otimes F) \) denote the span of the sub-permanents of size \( \kappa \) and let \( \mathcal{T}_\kappa^{\text{perm}} \subset \text{Sym}(E \otimes F) \) denote the ideal it generates. Note that \( \dim \mathcal{T}_\kappa^{\text{perm}, \kappa} = \binom{n}{\kappa}^2 \).

Fix complete flags \( 0 \subset E_1 \subset \cdots \subset E_n = E \) and \( 0 \subset F_1 \subset \cdots \subset F_n = F \). Write \( \mathcal{S}_{E_j} \) for the copy of \( \mathcal{S}_j \) acting on \( E_j \) and similarly for \( F_j \).

Write \( T_E \subset SL(E) \) for the maximal torus (diagonal matrices). By [25], the subgroup \( G_{\text{perm}} \) of \( GL(E \otimes F) \) preserving the permanent is \( [(T_E \times \mathcal{S}_E) \times (T_F \times \mathcal{S}_F)] \times \mathbb{Z}_2 \), divided by the image of the \( n \)-th roots of unity.

As an \( \mathcal{S}_{E_n} \times \mathcal{S}_{F_n} \)-module the space \( \mathcal{T}_\kappa^{\text{perm}, \kappa} \) decomposes as

\[
\text{Ind}_{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}}^{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}} [\kappa]_{E_n} \otimes [\kappa]_{F_n} = ([n]_E \oplus [n-1, 1]_E \oplus \cdots \oplus [n-\kappa, \kappa]_E) \otimes ([n]_F \oplus [n-1, 1]_F \oplus \cdots \oplus [n-\kappa, \kappa]_F).
\]

6.1. The linear strand.

Example 6.1.1. The space of linear syzygies \( M_{2, \kappa+1} := \ker (\mathcal{T}_\kappa^{\text{perm}, \kappa} \otimes V \rightarrow S^{\kappa+1} V) \) is the \( \mathcal{S}_{E_n} \times \mathcal{S}_{F_n} \)-module

\[
M_{2, \kappa+1} = \text{Ind}_{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}}^{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}} ([\kappa+1]_{E_n} \oplus [\kappa, 1]_{F_n} \oplus [\kappa, 1]_{E_n} \oplus [\kappa+1]_{F_n}).
\]

This module has dimension \( 2\kappa \binom{n}{\kappa+1}^2 \). A spanning set for it may be obtained geometrically as follows: for each size \( \kappa + 1 \) sub-matrix, perform the permanent “tensor Laplace expansion” along a row or column, then perform a second tensor Laplace expansion about a row or column and take the difference. An independent set of such for a given size \( \kappa + 1 \) sub-matrix may be obtained from the expansions along the first row minus the expansion along the \( j \)-th for \( j = 2, \ldots, \kappa + 1 \), and then from the expansion along the first column minus the expansion along the \( j \)-th, for \( j = 2, \ldots, \kappa + 1 \).

Remark 6.1.2. Compare this with the space of linear syzygies for the determinant, which has dimension \( \frac{2(\kappa+1)}{n-\kappa} \binom{n}{\kappa+1}^2 \). The ratio of their sizes is \( \frac{n+1}{n-\kappa} \), so, e.g., when \( \kappa \approx \frac{n}{2} \), the determinant has about twice as many linear syzygies, and if \( \kappa \) is close to \( n \), one gets nearly \( n \) times as many.

Theorem 6.1.3. \( \dim M_{j+1, \kappa+j} = \binom{n}{\kappa+j}^2 \). As an \( \mathcal{S}_n \times \mathcal{S}_n \)-module,

\[
M_{j+1, \kappa+j} = \text{Ind}_{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}}^{\mathcal{S}_{E_n} \times \mathcal{S}_{F_n}} \left( \bigoplus_{a+b=j} [\kappa+b, 1, t^a]_{E_n} \otimes [\kappa+a, 1, t^b]_{F_n} \right).
\]

The \( \binom{n}{\kappa+j}^2 \) is just the choice of a size \( \kappa + j \) submatrix, the \( \binom{2(\kappa+j-1)}{j} \) comes from choosing a set of \( j \) elements from the set of rows union columns. Naively there are \( \binom{2(\kappa+j)}{j} \) choices but there is redundancy as with the choices in the description of \( M_2 \).

Proof. The proof proceeds in two steps. We first get “for free” the minimal free resolution of the ideal generated by \( S^\kappa E \otimes S^\kappa F \). Write the generating modules of this resolution as \( \tilde{M}_j \). We then locate the generators of the linear strand of the minimal free resolution of our ideal, whose generators we denote \( M_{j+1, \kappa+j} \), inside \( \tilde{M}_{j+1, \kappa+j} \) and prove the assertion.

To obtain \( \tilde{M}_{j+1} \), we use the involution \( \omega \) on the space of symmetric functions (see, e.g. [24, §I.2]) that takes the Schur function \( s_{\pi} \) to \( s_{\pi'} \). This involution extends to an endofunctor of \( GL(V) \)-modules and hence of \( GL(E) \times GL(F) \)-modules, taking \( S_{\lambda} E \otimes S_{\mu} F \) to \( S_{\lambda'} E \otimes S_{\mu'} F \) (see [1, §2.4]). This is only true as long as the dimensions of the vector spaces are sufficiently large, so to properly define it one passes to countably infinite dimensional vector spaces.
Applying this functor to the resolution (9), one obtains the resolution of the ideal generated by \( S^m E \otimes S^n F \subset S^k(E \otimes F) \). The \( GL(E) \times GL(F) \)-modules generating the linear component of the \( j \)-th term in this resolution are:

\[
\tilde{M}_{j,j+k-1} = \bigoplus_{a+b=j-1} S_{(a,1^{b-1})} E \otimes S_{(b,1^{a-1})} F \\
\quad = \bigoplus_{a+b=j-1} S_{(k+b+1,1^{a-1})} E \otimes S_{(k+a+1,1^{b-1})} F.
\]

Moreover, by Corollary 4.3.3 and functoriality, the map from \( S_{(k+b+1,1^{a-1})} E \otimes S_{(k+a+1,1^{b-1})} F \) into \( M_{j,j+k-1} \) is non-zero to the copies of \( S_{(k+b+1,1^{a-1})} E \otimes S_{(k+a+1,1^{b-1})} F \) in \( (S_{k+b+1,1^{a-1}} E \otimes S_{k+a+1,1^{b-1}} F) \cdot (E \otimes F) \) and \( (S_{k+b+1,1^{a-1}} E \otimes S_{k+a+1,1^{b-1}} F) \cdot (E \otimes F) \), when \( a,b > 0 \).

Inside \( S^m E \otimes S^n F \) is the ideal generated by the sub-permanents (17) which consists of the weight spaces \( (p_1, \ldots, p_n) \times (q_1, \ldots, q_n) \), where all \( p_i, q_j \) are either zero or one. (Each sub-permanent has such a weight, and, given such a weight, there is a unique sub-permanent to which it corresponds.) Call such a weight space regular. Note that the set of regular vectors in any \( E^m \otimes F^n \) (where \( m \leq n \) to have any) is a \( \mathcal{G}_E \times \mathcal{G}_F \)-submodule.

The linear strand of the \( j \)-th term in the minimal free resolution of the ideal generated by (17) is thus a \( \mathcal{G}_E \times \mathcal{G}_F \)-submodule of \( \tilde{M}_{j,j+k-1} \). We claim this sub-module is the set of regular vectors. In other words:

**Lemma 6.1.4.** \( M_{j+1,k+j} = (\tilde{M}_{j+1,k+j})_{reg} \).

Assuming Lemma 6.1.4, Theorem 6.1.3 follows because if \( \pi \) is a partition of \( k+j \) then the weight \((1, \ldots, 1)\) subspace of \( S_{\pi} E_{\kappa+j} \), considered as an \( \mathcal{G}_{E_{\kappa+j}} \)-module, is \([\pi]\) (see, e.g., [10]), and the space of regular vectors in \( S_{\pi} E \otimes S_{\mu} F \) is \( \text{Ind}_{\mathcal{G}_{E_{\kappa+j}} \times \mathcal{G}_{F_{\kappa+j}}}^{\mathcal{G}_E \times \mathcal{G}_F} [\pi] \otimes [\mu] \).

Before proving Lemma 6.1.4 we establish conventions for the inclusions \( S_{q+1,1^p} E \subset S_{q+1,1^{p-1}} E \otimes E \) and \( S_{q+1,1^{p-1}} E \subset S_{q,1^p} E \otimes E \).

Let \( \Theta(p,q) : S_{q+1,1^p} E \rightarrow S_{q,1^{p-1}} E \otimes E \) be the \( GL(E) \)-module map defined such that the following diagram commutes:

\[
\begin{array}{ccc}
S^q E \otimes \Lambda^{p+1} E & \rightarrow & S_{q+1,1^p} E \\
\downarrow & & \downarrow \Theta(p,q) \\
S^q E \otimes E \otimes \Lambda^p E & \rightarrow & S_{q,1^{p-1}} E \otimes E
\end{array}
\]

where the left vertical map is the identity tensored with the polarization \( \Lambda^{p+1} E \rightarrow \Lambda^p E \otimes E \).

We define two \( GL(E) \)-module maps \( S^q E \otimes \Lambda^{p+1} E \rightarrow S^{q-1} E \otimes E \otimes \Lambda^{p+1} E \): \( \sigma_1 \), which is the identity on the second component and polarization on the first, i.e. \( S^q E \rightarrow S^{q-1} E \otimes E \), and \( \sigma_2 \), which is defined to be the composition of

\[
S^q E \otimes \Lambda^{p+1} E \rightarrow (S^{q-1} E \otimes E) \otimes (\Lambda^p E \otimes E) \rightarrow (S^{q-1} E \otimes E) \otimes (\Lambda^p E \otimes E) \rightarrow S^{q-1} E \otimes E \otimes \Lambda^{p+1} E
\]

where the first map is two polarizations, the second map swaps the two copies of \( E \) and the last is the identity times skew-symmetrization. Let \( \Sigma(p,q) : S_{q+1,1^p} E \rightarrow S_{q,1^{p-1}} E \otimes E \) denote the unique (up to scale) \( GL(E) \)-module inclusion (unique because \( S_{q+1,1^p} E \) has multiplicity one in \( S_{q,1^{p-1}} E \otimes E \)). A short calculation shows that the following diagram is commutative:

\[
\begin{array}{ccc}
S^q E \otimes \Lambda^{p+1} E & \rightarrow & S_{q+1,1^p} E \\
\sigma_2 - p\sigma_1 \downarrow & & \downarrow \Sigma(p,q) \\
S^{q-1} E \otimes E \otimes \Lambda^{p+1} E & \rightarrow & S_{q,1^p} E \otimes E
\end{array}
\]

**Proof of Lemma 6.1.4.** We work by induction, the case \( j = 1 \) was discussed above. Assume the result has been proven up to \( M_{j,k+j-1} \) and consider \( M_{j+1,k+j} \). It must be contained in
\[ M_{j,k+j-1} \otimes (E \otimes F) \], so all its weights are either regular, or such that one of the \( p_i \)’s is 2, and/or one of the \( q_i \)’s is 2, and all other \( p_a, q_a \) are zero or 1. Call such a weight sub-regular. It remains to show that no linear syzygy with a sub-regular weight can appear. To do this we show that no sub-regular weight vector in \( (M_{j,k+j})_{\text{sub-reg}} \) maps to zero in \( (M_{j-1,k+j-1})_{\text{reg}} \cdot (E \otimes F) \).

First consider the case where both the \( E \) and \( F \) weights are sub-regular, then (because the space is a \( \mathcal{S}_E \times \mathcal{S}_F \)-module), the weight \((2,1,\ldots,1,0,\ldots,0)\times(2,1,\ldots,1,0,\ldots,0)\) must appear in the syzygy. But the only way for this to appear is to have a term of the form \( T \cdot x_1 \), which cannot map to zero because, since \( x_1 \) is a non-zero divisor in \( \text{Sym}(V) \), our syzygy is a syzygy of degree zero multiplied by \( x_1 \). But by minimality no such syzygy exists.

Finally consider the case where there is a vector of weight \((2,1^{j+k-2}) \times (1^j \kappa)\) appearing. Consider the set of vectors of this weight as a module for \( \mathcal{S}_{j+k-2} \times \mathcal{S}_{j+k} \). This module is

\[
\bigoplus_{a+b=j} [\kappa+a,1^b]/[2] \otimes [\kappa+b,1^a]
\]

Here

\[
[\kappa+a,1^b]/[2] = [\kappa+a-2,1^b] \oplus [\kappa+a-1,1^{b-1}]
\]

is called a skew Specht module.

By Howe-Young duality and Corollary 4.3.3 if \( a, b > 0 \), \( S_{\kappa+a,1^b} E \otimes S_{\kappa+b,1^a} F \subset M_{a+b+1,k+a+b} \) maps non-zero to the two distinguished copies of the same module in \( M_{a+b,k+a+b} \). This in turn implies that the two distinguished copies of \( S_{\kappa+a,1^b} E \otimes S_{\kappa+b,1^a} F \subset M_{a+b,k+a+b} \), each map non-zero to \( M_{a+b+1,k+a+b} \).

The module (20) will take image inside

\[
\bigoplus_{c+d=j-1} \text{Ind}_{\mathcal{S}_{j+k-2} \times \mathcal{S}_{j+k}}^{\mathcal{S}_{j+k-1} \times \mathcal{S}_{j+k+1}} ([\kappa+c,1^d]/[2] \otimes [1]) \otimes ([\kappa+d,1^c] \otimes [1]).
\]

Fix a term \([\kappa+b,1^a] \) on the right hand side and examine the map on the left hand side. It is a map

\[
[\kappa+a,1^b]/[2] \rightarrow \text{Ind}_{\mathcal{S}_{j+k-1} \times \mathcal{S}_{j+k}}^{\mathcal{S}_{j+k-2} \times \mathcal{S}_{j+k}} ([\kappa+a,1^{b-1}]/[2] \otimes [1]) \oplus \text{Ind}_{\mathcal{S}_{j+k-2} \times \mathcal{S}_{j+k}}^{\mathcal{S}_{j+k-1} \times \mathcal{S}_{j+k+1}} ([\kappa+a-1,1^b]/[2] \otimes [1])
\]

If \( b > 0 \), the map to the first summand is the restriction of the map \( \Theta(b,\kappa+a) : S_{\kappa+a+1,1^b} E \rightarrow S_{\kappa+a+1,1^{b-1}} E \otimes E \), and, due to the fact that it has to map to a sub-regular weight, there is no polarization because the basis vector \( e_1 \) has to stay on the left hand side. So the map is the identity, thus injective.

It remains to show that for \( b = 0 \), the map corresponding to the summand \( b = 0, a = j \) which is the restriction of the injective map \( \Sigma(0,\kappa+j-2) : S_{\kappa+j-1} E \rightarrow S_{\kappa+j-2} E \otimes E \) tensored with the map \( \Theta(j-1, \kappa) \) injects into the cokernel of the summand corresponding to \( c = 0, d = j-1 \) modulo the image of the map coming from the summand \( a = 1, b = j-1 \). Both modules consist of just two irreducible \( \mathcal{S}_{E_{j+k-1}} \times \mathcal{S}_{F_{j+k-1}} \)-modules and, using formulas for \( \Sigma \) and \( \Theta \), the map is injective. This concludes the proof.

\[\square\]

**Example 6.1.5.** For small \( n \) and \( \kappa \), computer computations show no additional first syzygies on the \( \kappa \times \kappa \) sub-permanents of a generic \( n \times n \) matrix (besides the linear syzygies) in degree less than the Koszul degree \( 2\kappa \). For example, for \( \kappa = 3 \) and \( n = 5 \), there are 100 cubic generators for the ideal and 5200 minimal first syzygies of degree six. There can be at most \({100 \choose 2} = 4950 \) Koszul syzygies, so there must be additional non-Koszul first syzygies.
6.2. The Hilbert function in the case $\kappa = 2$. First, the Hilbert polynomial:

**Theorem 6.2.1.** For the ideal $I^{\text{perm}_{n,2}}$ of $2 \times 2$ permanents of an $n \times n$ matrix, the Hilbert polynomial of $\text{Sym}(V)/I^{\text{perm}_{n,2}}$ is

\[
\sum_{i=0}^{n} f_i \binom{t-1}{i},
\]

where $f_i$ is the $i^{th}$ entry in the vector

\[
\left[ n^2, \left( \frac{n^2}{2} \right)^2, \left( \frac{n^2}{2} \right)^2 + 2n \left( \frac{n}{3} \right), 2n \left( \frac{n}{4} \right), \ldots, 2n \left( \frac{n}{n} \right) \right]
\]

**Proof.** [23, Thm. 3.2] gives a Gröbner basis for $\sqrt{I^{\text{perm}_{n,2}}}$, the radical of $I^{\text{perm}_{n,2}}$, and by [23, Thm. 3.3], $\sqrt{I^{\text{perm}_{n,2}}}/I^{\text{perm}_{n,2}}$ has finite length, so vanishes in high degree. The Hilbert polynomial only measures dimension asymptotically, so

\[ HP(\text{Sym}(V)/\sqrt{I^{\text{perm}_{n,2}}}, t) = HP(\text{Sym}(V)/I^{\text{perm}_{n,2}}, t). \]

By [23], for any diagonal term order, the Gröbner basis for $\sqrt{I^{\text{perm}_{n,2}}}$ is given by quadrics of the form

\[ x_{ij} x_{kl} + x_{kj} x_{il} \text{ with } i < k, j < l, \]

and five sets of cubic monomials

\[
\begin{align*}
&x_{i_{1}j_{1}}x_{i_{1}j_{2}}x_{i_{2}j_{3}} \quad i_{1} > i_{2} \quad j_{1} < j_{2} < j_{3} \\
&x_{i_{1}j_{1}}x_{i_{2}j_{2}}x_{i_{2}j_{3}} \quad i_{1} > i_{2} \quad j_{1} < j_{2} < j_{3} \\
&x_{i_{1}j_{1}}x_{i_{2}j_{1}}x_{i_{3}j_{2}} \quad i_{1} < i_{2} < i_{3} \quad j_{1} > j_{2} \\
&x_{i_{1}j_{1}}x_{i_{2}j_{1}}x_{i_{3}j_{2}} \quad i_{1} < i_{2} < i_{3} \quad j_{1} > j_{2} \\
&x_{i_{1}j_{1}}x_{i_{2}j_{1}}x_{i_{3}j_{3}} \quad i_{1} < i_{2} < i_{3} \quad j_{1} > j_{2} > j_{3}.
\end{align*}
\]

The key observation is that all the cubic monomials are square-free, as are the initial terms of the quadrics. Thus the initial ideal of $\sqrt{I^{\text{perm}_{n,2}}}$ is a square-free monomial ideal and corresponds to the Stanley-Reisner ideal of a simplicial complex $\Delta$. By [32, Lemma 5.2.5], the Hilbert polynomial is as in Equation (21), where $f_i$ is the number of $i$-dimensional faces of $\Delta$. As the vertex set of $\Delta$ corresponds to all lattice points $(i, j)$ with $1 \leq i, j \leq n$, it is immediate that $f_0 = n^2$.

Since $x_{ij} x_{kl}$ is a non-face if $i < k, j < l$, no edge connects a southwest lattice point to a northeast lattice point. Hence, the edges of $\Delta$ consist of all pairs $(i, j), (k, l)$ with $i \geq k$ and $j \geq l$, of which there are $\binom{n^2}{2} - \binom{n}{2}^2$.

Next, consider the triangles of $\Delta$. Equation (22) says there are no triangles in $\Delta$ of the types in Figure 2. Also, there are no triangles which contain an edge connecting vertices at positions $(i, j)$ and $(k, l)$ with $i < k, j < l$. Thus, the only triangles in $\Delta$ are right triangles, but with hypotenuse sloping from northwest to southeast. For a lattice point $v$ at position $(d, e)$ there are exactly $(d-1)(e-1)$ right triangles having $v$ as their unique north-most vertex. In the rightmost column $n$, there are no such triangles, in the next to last column $n-1$ there are $(n-1) + (n-2) + \cdots = \binom{n}{2}$ such triangles. Continuing this way yields a total count of

\[
(n-1)\binom{n}{2} + (n-2)\binom{n}{2} + \cdots + \binom{n}{2} + \binom{n}{2} = \binom{n}{2}^2
\]

such right triangles, and taking into account the right triangles for which $v$ is the unique south-most vertex doubles this number.
Figure 2. Non-triangles of $\Delta$ from Equation (22)

However, this count neglects thin triangles—those which have all vertices in the same row or column. Since the number of thin triangles is $2n(n^2)$, the final count for the triangles of $\Delta$ is

$$2n(n^2) + 2n(n^3).$$

For tetrahedra, the conditions of Equation (22) imply that there can only be thin tetrahedra, and an easy count gives $2n(n^4)$ such. The same holds for higher dimensional simplices, and concludes the proof.

\[\Box\]

Corollary 6.2.2. For the ideal $I_{\text{perm},2}$ of $2 \times 2$ permanents of an $n \times n$ matrix, the Hilbert function of $\text{Sym}(V)/I_{\text{perm},2}$ is, when $3 \leq t \leq n$,

\[(23) \quad HF(\text{Sym}(V)/I_{\text{perm},2}, t) = \binom{n}{t}^2 + HP(\text{Sym}(V)/I_{\text{perm},2}, t),\]

and it equals the Hilbert polynomial for $t > n$.

Proof. The Hilbert function of $\sqrt{I_{\text{perm},2}}/I_{\text{perm},2}$ in degree $t$ is $\binom{n}{t}^2$ by [23, Thm. 3.3]. The result follows by combining Theorem 21 with the short exact sequence

$$0 \rightarrow \sqrt{I_{\text{perm},2}}/I_{\text{perm},2} \rightarrow \text{Sym}(V)/I_{\text{perm},2} \rightarrow \text{Sym}(V)/\sqrt{I_{\text{perm},2}} \rightarrow 0,$$

and additivity of the Hilbert function.

\[\Box\]

For the purposes of comparing with other ideals, we rephrase this as:
Theorem 6.2.3. \( \dim \mathcal{I}_t^\text{perm}_{n,2} = {n \choose 2}^2 \). For \( 3 \leq t \leq n \):

\[
\dim \mathcal{I}_t^\text{perm}_{n,2} = \left( {n^2 + t - 1 \choose t} - \left( {n^2 \choose 2} + n^2 + (t-1)\left( {n^2 \choose 2} - {n \choose 2} \right) + 2\left( t-1 \right)\left( {n \choose 2} + n{\binom{n}{3}} \right) \right) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j}(n \choose j+1) \right)
\]

and for \( t > n \):

\[
\dim \mathcal{I}_t^\text{perm}_{n,2} = \left( {n^2 + t - 1 \choose t} - \left[ n^2 + (t-1)\left( {n^2 \choose 2} - {n \choose 2} \right) + 2\left( t-1 \right)\left( {n \choose 2} + n{\binom{n}{3}} \right) \right) + 2n \sum_{j=3}^{t-1} \binom{t-1}{j}(n \choose j+1) \right]
\]

and the latter formula is \( \dim S^t \mathcal{C} \mathbb{P}^n \) minus the Hilbert polynomial for all \( t \).

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