The Maximum Tension Principle in General Relativity

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Abstract

I suggest that classical General Relativity in four spacetime dimensions incorporates a Principal of Maximal Tension and give arguments to show that the value of the maximal tension is $\frac{c^4}{4G}$. The relation of this principle to other, possibly deeper, maximal principles is discussed, in particular the relation to the tension in string theory. In that case it leads to a purely classical relation between $G$ and the classical string coupling constant $\alpha'$ and the velocity of light $c$ which does not involve Planck’s constant.
1 Introduction

Jacob Bekenstein has always been interested in simple Physical Principles (capitals intentional) and so it seems appropriate to celebrate the 30th anniversary of his work on black hole thermodynamics with an account of a simple, but perhaps un-noticed, principle in classical general relativity which, like the Christadoulous’s idea of irreducible mass and Hawking’s area increase theorem, also seems to point to to deeper things. The principle is one of those “impossibility statements” that impose an upper (or lower) bound on some physical quantity. The most obvious example is the upper bound on velocity in Special Relativity. Another example is the lower bound on temperature first noticed in kinetic theory and now accepted as universal and embedded into the more general framework of Statistical Mechanics. In like fashion one can contemplate, and I shall, a more general framework including General Relativity as a special limit in which the Maximum Tension Principle is embedded in a fundamental way. A striking feature of the Principle is that while it has an analogue in higher dimensions it takes on its most simple and natural form in four spacetime dimensions. Specifically I propose:

The Principle of Maximum Tension The tension or force between two bodies cannot exceed

\[ F_g = \frac{c^4}{4G}. \]  

(1)

The number 4 seems to be correct from the examples I am about to give but it may be subject to revision in the light of future developments. Numerically

\[ F_g \approx 3.25 \times 10^{43} \text{ Newtons}, \]  

(2)

which is about \( 3 \times 10^{39} \) Tonnes. In support of my contention that it has gone relatively unknown among the relativity community, it is interesting to note that \( \frac{c^4}{G} \) does not appear among the “useful combinations” of the two fundamental constants, the velocity of \( c \) and Newton’s constant of gravitation \( G \) in the well known textbook [1].

At the Newtonian level of course the tension in the gravitational field is unbounded. In the language developed by Maxwell, the Newtonian stress tensor has the opposite properties from those in electrostatics. It is given by

\[ T_{ij} = -\frac{1}{4\pi G} \left[ \partial_i U \partial_j U - \frac{1}{2} \delta_{ij} \partial U \partial U \right], \]  

(3)

and the gravitational force per unit volume \( F_i = \partial_j T_{ij} \). The force per unit area along the field lines, assumed to point along the 1 direction for example, is repulsive

\[ T_{11} = -\frac{1}{8\pi G} \partial U \partial U \]  

(4)
while in the transverse direction there are tensions, not pressures:

\[ T_{22} = T_{33} = + \frac{1}{8\pi G} |\partial U|^2. \]  

(5)

Maxwell and those that followed him, for example [2], found this paradoxical: the more so because the forces and tensions between heavenly bodies, which were at that time thought to be exerted through the ether, are so large compared with what is encountered in an ordinary terrestrial medium. In General Relativity these stresses remain just as large (very large) for ordinary celestial bodies but they cannot become unbounded. There is a natural limit because of the phenomenon of gravitational collapse and black hole formation.

My initial qualitative argument is too crude to deliver a precise upper bound, merely an order of magnitude, but more sophisticated calculations do. Consider two bodies (possibly black holes, but not necessarily so) of positive masses \( M_1 \) and \( M_2 \) separated a distance \( D \) apart, According to Newtonian theory, The gravitational force between them is

\[ F = \frac{GM_1 M_2}{D^2} = \left( \frac{G M_1}{c^2 D} \right) \left( \frac{G M_2}{c^2 D} \right) \frac{c^4}{G}. \]  

(6)

However \( M_1 M_2 \) cannot exceed \( \frac{1}{4} (M_1 + M_2)^2 \) and therefore

\[ F \leq \left( \frac{M_1 + M_2}{c^2 D} \right)^2 \frac{c^4}{4G}. \]  

(7)

Now if there are to be two bodies rather than a single black hole hole it must be true that \( M_1 + M_2 < c^2 D \) and so the Principle holds in this case. Obviously I have been a little cavalier with factors in the last sentence and so it is good to have some more exact arguments. The point is that to keep the bodies apart we need to pull them away from each other with some sort of strings. In the axisymmetric case at least, and in the thin string limit, these may be approximated by conical defects running off to infinity along the two portions of the axis on the outer side of each body. The deficit angle \( \delta \) given by

\[ \delta = \frac{8\pi G}{c^4} F, \]  

(8)

where the force \( F \) may identified (at least in four spacetime dimensions) with the tension or energy per unit length. Because the deficit angle \( \delta \) cannot exceed \( 2\pi \) we again see that the Principle holds. Note that this upper bound for the tension should be contrasted with a Bogomol’nyi style lower bound established in [3].

To see the bound operating in more detail, particularly in the case of black holes, we consider static axi-symmetric vacuum metrics and use what may be called

2 The Method of Newtonian Rods

This arose out a paper of Einstein and Rosen [4] and was applied to black holes in [5, 6, 7]. The method has been nicely reviewed recently in [8] and so I
will not give many mathematical details here. We start by following Weyl and expressing a static axisymmetric vacuum metric as

\[ ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}(dz^2 + d\rho^2) + \rho^2 d\phi^2\right\}, \]

where \( U \) is an axisymmetric harmonic function on flat three dimensional Euclidean space \( \mathbb{E}^3 \) with cylindrical coordinates \((z, \rho, \phi)\) and the axisymmetric function \( k \) is given in terms of \( U \) by a line integral whose contour-independence is a consequence of Laplace’s equation for \( U \).

To obtain a single black hole we choose for \( U \) the Newtonian potential of a uniform rod of mass \( M \) length \( L \) placed on the axis of symmetry, where

\[ \frac{2M}{c^2 L} = 1. \]

In this representation, the event horizon corresponds to the interval of the axis of symmetry occupied by rod and the mass per unit length condition guarantees that the horizon is non-singular and that there is a smooth extension through it. To obtain Rindler spacetime, that is flat space in accelerating coordinates, one takes a semi-infinite rod which is now the acceleration horizon. These two case are non-singular because there the portion of the axis not occupied by the rod has no deficit angle. The situation changes if one considers two disjoint rods. Deficit angles are now inevitable and one may take them to be along the portions of the axes running off to infinity. Using the contour integral for the metric function \( k \) one finds \[9\] that the values are related to the net Newtonian force on one rod due to the other. Another case is to take one semi-infinite rod and one finite rod. This gives the C-metric. Examination of the deficit angles shows that to provide the acceleration one must pull the black hole with a string whose tension cannot exceed the limit. Note that according to classical General Relativity, which incorporates Newton’s Second Law, there is no upper bound on the acceleration since we may apply a given force to an arbitrary small mass. Quantum gravity may impose an upper bound on acceleration and probably does. This we will turn to briefly later.

3 Melvin solution

Another illustration of the principle may be obtained by considering the Melvin flux tube in Einstein Maxwell theory. The bundle of flux lines is on the verge of collapsing, being kept up by Maxwell stresses. One can imagine charges at either end of the flux tube and one then needs to calculate the force between them.

The solution is

\[ ds^2 = -(1 + \frac{\rho^2}{a^2})^2\left\{-dt^2 + dz^2 + dr^2 + \frac{\rho^2}{(1 + \frac{\rho^2}{a^2})^4} d\phi^2\right\}, \]

\[ (11) \]
with

\[ F = B_0 \frac{r \, dr \wedge d\phi}{(1 + \frac{r^2}{a^2})^2}, \]  

(12)

and \( a = \frac{1}{B_0 \sqrt{\pi G}} \). For convenience I have set \( c = 1 \) in (11). The total magnetic flux is \( \Phi_m = G B_0 \). The integrated stress across the flux tube is proportional to \( a^2 B_0^2 \) which, restoring units, is proportional to the maximal tension \( \frac{c^4}{\pi G} \). The exact factor of proportionality depends on how one defines the total force. This is not completely obvious in this very non-linear context. It is possible that with an appropriate definition even the factor would come out right.

4 Higher Dimensions

One may apply the heuristic argument in the introduction in \( n \) spacetime dimensions but one gets a bound on \( F/D^{n-4} \). The method of rods breaks down because there is no analogue of the Weyl metrics and the C-metric in higher dimensions. Moreover the idea of deficit angles does not go over in a nice way. What are natural sources for distributional Ricci curvature are not strings (with two-dimensional world sheets) but rather \((n-2)\)-branes with co-dimension two world volumes. A closely related point is that one cannot think of the Regge calculus in terms of a network of strings in higher dimensions. One may see this mis-match by applying elementary dimensional analysis to the classical Einstein equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8 \pi G}{c^4} T_{\mu\nu}. \]  

(13)

The left hand side has the dimensions of \( L^{-2} \). The stress tensor \( T_{\mu\nu} \) has the dimensions of stress, i.e. \( FL^{-(n-2)} \) or, equivalently, energy per unit \((n-1)\)-volume. Thus \( \frac{c^4}{\pi G} \) has dimensions \( FL^{-(n-4)} \) and no statement can be made about fundamental limits on forces or indeed lengths but just this combination.

5 String Theory

It is a striking fact that in classical string theory one is given a natural unit of force or tension, the energy per unit length of the string \([11]\).

\[ F_s = \frac{1}{2\pi \alpha'}, \]  

(14)

where \( \alpha' \) is called the Regge slope parameter. In fact not everyone defines \( \alpha' \) in this way. Some prefer to insert a factors of \(\hbar c \) so that \( \sqrt{\alpha'} \) is a fundamental length. However from the classical point of view, it is more natural to use the tension \( F_s \) rather than \( \alpha' \) as the fundamental constant of string theory. We shall see shortly how the Regge slope parameter enters the quantum theory. It seems clear, and this is without benefit of quantum mechanics, that if gravity
and string theory are related then these two tensions must be proportional, that is purely classically one must have

$$F_g = \frac{1}{4} c^4 \propto F_s = \frac{1}{2\pi\alpha'}.$$  \hspace{1cm} (15)

Note that Planck’s constant does not enter (15). The two theories may be related purely classically. Related observations have been made by Veneziano [11] and are subject to a debate in [12] on the meaning of “fundamental constants”. The view taken here is an operational one. Thus a possible unit of velocity is that of the universal upper limit to the speed of propagation of closed string states. One could use as a unit the upper limit to the speed of one’s favourite race-horse but that would lack the appealing feature of universality. It is a non-trivial fact about the world, i.e. a law of nature, that such a universal limit exists (at least for closed, as opposed to open string states). Moreover this law of nature, unlike that of Galilean physics, is not invariant under independent re-scalings of any well defined units of length \(L\) and time \(T\).

In like fashion, I claim that it is a non-trivial fact about the world that there is a natural upper bound for the tension or force in the macroscopic world of general relativity and we could use that as a natural unit, or fundamental constant. It then becomes an interesting question of how this is related to the natural unit of tension in the classical micro-world of strings. The precise relation must involve the string coupling constant \(g_s = e^\Phi\), where \(\Phi\) is the classical value of the dilaton. Since I have nothing new to say about it, factors of \(g_s\) will be ignored in what follows.

At the purely classical level, the invariance of the equations of classical general relativity without sources under rescaling

$$g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu},$$ \hspace{1cm} (16)

with \(\lambda\) a constant, makes it clear that in that theory there can be no upper bound to masses, distances or times and hence no natural or fundamental unit of mass, length or time, since under (16) all three scale as \(\lambda\)

$$M \rightarrow \lambda M \quad L \rightarrow \lambda L \quad T \rightarrow \lambda T.$$ \hspace{1cm} (17)

The fundamental constants \(c\) and \(G\) are invariant as is the combination \(\frac{c^4}{G}\) because they have the correct dimensions, for example

$$\left[ \frac{c^4}{G} \right] = MLT^{-2}.$$ \hspace{1cm} (18)

Quantities like angular momentum \(J\) and action \(S\) satisfy

$$\left[ J \right] = \left[ S \right] = ML^2T^{-1},$$ \hspace{1cm} (19)

and so general relativity provides no complete set of natural units, for that we need Quantum Mechanics and Planck’s constant. In fact one may regard (16)
as the residual invariance that is left from the two parameter groups of scalings of the laws of special relativity under which

\[ M \rightarrow \lambda M \quad L \rightarrow \mu L \quad T \rightarrow \mu T, \]

(20)

and the laws of Newtonian Gravity under which

\[ M \rightarrow \lambda M \quad L \rightarrow \mu L \quad T \rightarrow \mu N \lambda^{-\frac{1}{2}}, \]

(21)

Note that the non-relativistic scaling (21) is just the statement of Kepler’s Third Law.

To return to classical string theory: the Nambu-Goto action \( S \) of a classical string is given by

\[ S = \frac{1}{\pi \alpha'} \int_{\Sigma} dx dt, \]

(22)

where \( \Sigma \) is the world sheet of the string.

In fact, the contribution to the gravitational action from a conical defect or cosmic string of the form we contemplated earlier is identical up to a factor if we make use of (15) [9]. This confirms the relation (15). Note finally that the scaling invariance (17) of classical general relativity is shared by classical relativistic string theory. If it weren’t, then (15) wouldn’t make sense.

6 Born-Infeld Theory

One sees the upper bound on the tension arising naturally in the Born-Infeld Lagrangian which is an effective theory arising from open strings. Actually what comes into the Born-Infeld action is the force on the end of the string because one adds a boundary term

\[ \int_{\partial \Sigma} e A_\mu dx^\mu \]

(23)

where \( e \) is called the charge carried by an end of the string. However, the effective action involves only the product \( e F_{\mu \nu} \) and is (ignoring a possible additive constant and factors of \( g_s \))

\[ - F_s^2 \sqrt{-\det (g_{\mu \nu} + 2\pi \alpha' F_{\mu \nu})} \]

(24)

Expanding out to lowest order in \( F_{\mu \nu} \) one gets the standard Maxwell Lagrangian but at the full non-linear level, the Born-Infeld action gives rise to an upper bound to the force or tension on the ends of the string:

\[ e E_s = \frac{1}{2\pi \alpha'} = F_s, \]

(25)

where \( E_s \) is the critical electric field.
7 Regge Trajectories

String theory arose in attempt to obtain models with Regge trajectories. Classically for an open relativistic string, one has the bound

\[ J \leq \frac{\alpha'}{c^3} M^2, \]

(26)

while for a closed relativistic string \[13\]

\[ J \leq \frac{1}{2} \frac{\alpha'}{c^3} M^2. \]

(27)

It is interesting to compare this with the classical cosmic censorship limit on Kerr black holes

\[ J \leq \frac{G}{c} M^2. \]

(28)

If, as many people have, one thought of the black hole as a rotating ring with a tension, one would obtain, up to a factor of proportionality, the relations \([13]\). This nice picture fails for higher dimensional rotating black holes\[10\] essentially because of the different dependence of the gravitational force on separation.

8 Quantum Mechanics

The introduction by Planck of a unit of action, or equivalently angular momentum, breaks the scaling invariance \([16]\) and makes possible a complete system of fundamental units \([14]\). Conventionally, one introduces the Planck length,

\[ l_p = \sqrt{\frac{G \hbar}{c^3}}, \]

(29)

which one believes may give a least length, although it is probably rather shorter than the string length

\[ l_s = \sqrt{\alpha' \hbar c}. \]

(30)

Another way to obtain "fundamental units" pre-dating Planck and avoiding the introduction of Planck’s constant \(\hbar\) is to follow Stoney, coiner of the term electron, and note that the existence in nature of a fundamental electric charge breaks the residual scaling invariance \([16], [13]\) because classically

\[ [e] = M^\frac{1}{2} L^\frac{1}{2} T^{-1}. \]

(31)

Of course the in quantum field theory one drops \(G\) but introduces \(\hbar\) and to get a natural unit of charge \(\sqrt{4\pi \hbar c}\) and obtains the usual dimensionless gauge coupling constant

\[ g_e = \frac{e}{\sqrt{4\pi \hbar c}}. \]
9 Maximal Acceleration and Temperature

In any special relativistic theory acceleration $a$ may be converted to an inverse length $ac^{-2}$ and in any theory one may think of temperature $T$ as an energy. In any quantum theory temperature $T$ and time are related via periodicity in imaginary time with period $\beta = \hbar T$. In a relativistic quantum theory one puts these two facts together to get the purely kinematic Unruh relation

$$T = \frac{\hbar a}{2\pi c}.$$  \hspace{1cm} (32)

It follows that a theory with a minimal length should have a maximal acceleration and a maximal temperature. By the same token, a theory with a maximal temperature should have a maximal acceleration. There is a clear mechanism here: if one tried to increase one’s acceleration one would absorb hotter and hotter Unruh radiation which would, from the point of view of an inertial observer, have to be emitted at the expense of the energy source causing the acceleration.

It is clear that a maximal acceleration or maximal temperature can come out of neither classical general relativity nor classical string theory. It clearly emerges in some form from classical string theory and quantum gravity. The Hagedorn temperature and the Hawking temperature of a Planck mass black hole will, up to factors give its magnitude. Whether there is a simple universal value however remains unclear. For some ideas about this with some references to earlier work the reader is directed to [16].

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