MAXIMAL GREEN SEQUENCES FOR CLUSTER ALGEBRAS ASSOCIATED TO THE ORIENTABLE SURFACE OF GENUS $n$ WITH ARBITRARY PUNCTURES

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Abstract. All cluster algebras of finite mutation type can be classified by looking at cluster algebras from surfaces, and a few additional exotic cases. In this paper we will provide a triangulation for orientable surfaces of genus $n$ with an empty boundary component and an at least 3 marked interior points, whose corresponding quiver has a maximal green sequence. With previous work being done on cluster algebras from surfaces with non-empty boundary, this result will mean for any surface we can answer the question of whether it can be associated to a cluster algebra which exhibits a maximal green sequence.

1. Introduction

The study of cluster algebras is an elegant intersection of many areas of mathematics. They combine combinatorics, algebra, geometry, and topology. The notion of a cluster algebra was formulated and defined by Fomin and Zelevinsky in 2003 [7]. At first they were designed to attribute combinatorial structure to algebraic objects, but since 2003 they have had an important role throughout other areas of math. In this paper we will concentrate on their connection to topology and triangulations of surfaces.

It is well known that a cluster algebra can be produced from any orientable surface by looking at the possible triangulations of that surface. The process of “flipping” diagonals in the triangulation correlates to a cluster mutation. With that notion in mind, one can produce a cluster algebra. Gekhtman, Shapiro, and Vainshtein established this construction in [9] and it was introduced in a more general setting by Fock and Goncharov in [5]. All cluster algebras of finite mutation type can be realized as a cluster algebra which arises from some surface using this construction, with a few exceptional cases. This means that addressing the question of whether these cluster algebras have maximal green sequences or not, is important in addressing the question of whether finite mutation type cluster algebras have maximal green sequences. In general since the work of Alim, et al [1] gives an explicit description of a maximal green sequence for cluster algebras arising from surfaces with non-empty boundary, the next place to look is if we can find a maximal green sequence for cluster algebras from surfaces with empty boundary component. Previous work by the first author [3] constructs maximal green sequences for cluster algebras which arise from surfaces with empty boundary component and two interior marked points (called punctures). In this paper, we construct them for surfaces with more than 2 punctures.

This notion of maximal green sequences was first looked at to study Donaldson-Thomas invariants by Keller [11]. Since their first inception, they have been connected to the computation of BPS state of gauge theories by Alim, et. al. in [1].

Exhaustive methods are not necessarily effective when addressing the existence of maximal green sequences because there are initial choices of mutation that can lead to a non-terminal sequence of...
mutations. This means that more theoretical methods often must be employed. Though difficult in general to work with, there has been much progress made in the area. The existence of maximal
green sequences for cluster algebras of finite type was proven by Brüstle, Dupont, and Perotin in [2]. Yakimov proved the existence of maximal green sequences for the Berenstein-Fomin-Zelevinsky cluster algebras on all double Bruhat cells in Kac-Moody groups in [15]. In addition to the work by Alim, et. al. on cluster algebras from surfaces with non-empty boundary, Garver and Musiker constructed a sequence for any type A quiver [8].

In this paper we prove the existence of a maximal green sequence for cluster algebras which arise from surfaces with empty boundary component and at least three punctures. This is an infinite family of cluster algebras for which we explicitly find the maximal green sequence. Dealing with these surfaces can be difficult because the lack of boundary component introduces a large amount of cycles into the quiver. In our construction we start with an $n$-torus and use a modified version of the triangulation that was used in [3]. We will give the triangulation in section 3. This triangulation creates a large amount of symmetry in the quiver which we will utilize to break the quiver into smaller sub quivers. After the construction of the triangulation we look at the associated quiver which we denote $Q^n_p$, where $p$ is the number of punctures on the surface. By adding punctures to the structure we add a ladder structure which we denote $P_p$. Increasing the number of punctures lengthens the ladder, but its connection to the rest of the quiver is unaffected. In section 4 we give the proof of our main result. The proof is done by inducting on the number of punctures. We will construct a maximal green sequence for the subquiver $P_p$ and then show how to utilize that sequence to get a maximal green sequence for the larger quiver $Q^n_p$.

This work completes the classification of which cluster algebras arising from surfaces have a maximal green sequence. It is known that a maximal green sequence cannot be constructed for a cluster algebra from a surface with empty boundary component and only one puncture [13]. With our result and the results of those mentioned above, given any marked surface with empty boundary component we now know whether there exists a triangulation whose corresponding quiver has a maximal green sequence. In general combining this with the results on cluster algebras arising from surfaces with nonempty boundary component, given any marked surface the question will be answered as to whether we can construct a cluster algebra with a maximal green sequence which corresponds to that surface.

2. Preliminaries

We will follow the notation laid out by Brüstle, Dupont, and Perotin [2].

Definition 2.1. A quiver, $Q$, is a directed graph possibly containing 2-cycles, but no loops. A cluster quiver is a quiver with no 2-cycles. We are only concerned with cluster quivers in this paper and will write quiver when we mean cluster quiver.

Let $Q_0 \subset \mathbb{N}$ denote the set of vertices of $Q$. Also, let $Q_1$ denote the set of edges of $Q$, which are referred to as arrows.

Definition 2.2. An ice quiver is a pair $(Q, F)$ where $Q$ is a quiver as described above and $F \subset Q_0$ is a subset of vertices called frozen vertices; such that there are no arrows between them. For simplicity, we always assume that $Q_0 = \{1, 2, 3, \ldots, n+m\}$ and that $F = \{n+1, n+2, \ldots, n+m\}$ for some integers $n, m \geq 0$. If $F$ is empty we write $(Q, \emptyset)$ for the ice quiver.

We will be working with a process called quiver mutation throughout this paper. Mutation is a process of obtaining a new ice quiver from an existing one.

Definition 2.3. Let $(Q, F)$ be an ice quiver and $k \in Q_0$ a non-frozen vertex. The mutation of a quiver $(Q, F)$ at a vertex $k$ is denoted $\mu_k$, and produces a new ice quiver $(\mu_k(Q), F)$. The vertices
of \((\mu_k(Q), F)\) are the same vertices from \((Q, F)\). The arrows of the new quiver are obtained by performing the following 3 steps:

1. For every 2-path \(i \to k \to j\), adjoin a new arrow \(i \to j\).
2. Reverse the direction of all arrows incident to \(k\).
3. Delete any 2-cycles created during the first two steps as well as any arrows created between frozen vertices.

We should note that we do not allow mutation at a frozen vertex. We describe all non-frozen vertices as mutable. We will let \(\text{Mut}((Q, F))\) denote the set of all quivers which can be obtained from \(Q\) by some sequence of mutations. When no confusion arises we will write \(\text{Mut}(Q)\) instead of \(\text{Mut}((Q, F))\).

In this paper we are concerned with ice quivers that have a very specific set of frozen vertices. These quivers are referred to as the framed and coframed quivers associated to \(Q\).

**Definition 2.4.** The **framed quiver** associated with a quiver \(Q\) is the ice quiver \((\hat{Q}, Q_0')\) such that:

\[
Q_0' = \{i' \mid i \in Q_0\}, \quad \hat{Q}_0 = Q_0 \sqcup Q_0' \\
\hat{Q}_1 = Q_1 \sqcup \{i \to i' \mid i \in Q_0\}
\]

Since the frozen variables of the framed are so natural we will simplify the notation and just write \(\hat{Q}\). Now we must discuss what is meant by red and green vertices.

**Definition 2.5.** Let \(R \in \text{Mut}(\hat{Q})\).
A mutable vertex \(i \in R_0\) is called **green** if

\[
\{j' \in Q_0' \mid \exists j' \to i \in R_1\} = \emptyset.
\]

It is called **red** if

\[
\{j' \in Q_0' \mid \exists j' \leftarrow i \in R_1\} = \emptyset.
\]

The result that every mutable vertex in \(R_0\) is either red or green is due to [4] for cluster quivers. The result was later shown for general quivers in [10]. This idea is what motivates our work in this paper. We will phrase it as a question of green sequences.

**Definition 2.6.** A **green sequence** for \(Q\) is a sequence \(i = (i_1, \ldots, i_l) \subset Q_0\) such that \(i_1\) is green in \(\hat{Q}\) and for any \(2 \leq k \leq l\), the vertex \(i_k\) is green in \(\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q})\). The integer \(l\) is called the length of the sequence \(i\) and is denoted by \(l(i)\).

A green sequence \(i\) is called maximal if every mutable vertex in \(\mu_i(\hat{Q})\) is red where \(\mu_i = \mu_{i_l} \circ \cdots \circ \mu_{i_1}\). We denote the set of all maximal green sequences for \(Q\) by

\[
\text{MGS}(Q) = \{i \mid i \text{ is a maximal green sequence for } Q\}.
\]

In this paper we will construct a maximal green sequence for a specific infinite family of quivers which will be described in the following section. In essence what we want to show is that \(\text{MGS}(Q) \neq \emptyset\) for each quiver, \(Q\), in this family. One important observation that the proof of our main result relies on is that if \(Q_1\) and \(Q_2\) are quivers such that \(Q_1\) has a green sequence \(\lambda\), and \(Q_1\) is a full subquiver of \(Q_2\) consisting of only green vertices, then \(\lambda\) is a green sequence for \(Q_2\). Furthermore, if \(\lambda\) is maximal, then all the vertices in the copy of \(Q_1\) inside of \(Q_2\) will be red.
3. Constructing the Triangulation

Following the work done by Fomin, Shapiro, and Thurston in [6] we construct a quiver $Q^p_n$ associated to the genus $n$ surface with no boundary and $p \geq 3$ punctures. The triangulation of the surface with two punctures that was given in [3] (see Figure 1) gives rise to a very natural generalization. When $p \geq 3$ we can remove the arc $f_n$ and replace it with a $(p - 2)$-times punctured digon. We then triangulate the digon as shown on the right of Figure 2. The final triangulation is also given in Figure 2.

From this triangulation we can construct the quiver $Q^p_n$. As an example we give $Q^7_3$ on the left of Figure 3. Also we define the quiver $P^p_{p-3}$ for $p \geq 3$ to be the full subquiver of $Q^p_n$ consisting of the vertices $\{g_0, g_1^1, g_2^1, g_3^1, \ldots, g_1^{p-3}, g_2^{p-3}, g_3^{p-3}\}$. $P_4$ is given on the right of Figure 3. Note that by increasing the genus of the surface the cycle containing the $f$ vertices gets longer, and more handles are added to it. Increasing the number of punctures will increase the number of rows in the $P$ subquiver. The important thing to notice is that the fundamental shape of the quiver $Q^p_n$ doesn’t change.

4. Statement and Proof of Main Result

**Theorem 4.1.** Let $Q^p_n$ be the quiver obtained from our triangulation of a genus $n$ surface with no boundary and $p \geq 3$ punctures. Then $Q^p_n$ has a maximal green sequence given by

$$f_{n+1}f_{n+1} \cdots f_1 f_3 f_4 \cdots f_{n+1} f_{n+2} \sigma_n \cdots \sigma_1 \alpha_0 \alpha_1 \cdots \alpha_{p-3}$$
Proof. The sequence is easily checked for Lemma 4.2. Following diagram.

Figure 3. The quivers $Q^7_3$ (left) and $P_4$ (right).

$$f_{n+2}f_2f_1f_3f_4 \cdots f_nf_{n-2}f_{n-3} \cdots f_3f_1f_2f_{n+1}\beta_{p-3}\tau_1\tau_2\cdots\tau_n,$$

where $\sigma, \tau, \alpha,$ and $\beta$ are defined as follows:

$\sigma_i = e_idic_d eic_d eic_d eic_d,$ $\tau_i = e_1b_1a_1 e_1b_1a_1 e_1b_1a_1 e_1b_1a_1,$

$$\alpha_j = \begin{cases} g_0 & j = 0 \\ g_0^3g_1^1g_3^1g_0 & j = 1 \\ g_0^3g_1^1g_3^1g_0 & j = 2 \\ g_0^3g_1^1g_3^1g_0 & j \geq 3 \end{cases}$$

$$\beta_j = \begin{cases} \emptyset & j = 0 \\ g_1^1g_2^1g_3^1 & j = 1 \\ g_1^1g_2^1g_3^1 & j = 2 \\ g_1^{n-1}g_1^{n-1}g_3^{n-2}g_1^{n-3} \cdots g_1^3g_1^2g_1^1 & j \geq 3 \end{cases}$$

**Lemma 4.2.** $P_n$ has a maximal green sequence given by $\alpha_0\alpha_1 \cdots \alpha_n$.

*Proof.* The sequence is easily checked for $n = 0, 1, 2$. For $n=3$, apply $\alpha_0\alpha_1\alpha_2$ to $P_3$. We know that this is a green sequence for the $P_2$ subquiver of $P_3$. The current state of the quiver is given in the following diagram.
We now apply the first four mutations of $\alpha_3$ to the quiver above.
After these four mutations $g_1^3$ is the only remaining green vertex, and is the initial vertex in a 2-path through a frozen vertex for 6 vertices. However, the terminal vertex in these 2-paths form an equioriented affine subquiver with $g_1^3$ being the sink for this subquiver. The remaining mutations of $\alpha_3$ is just the mutation along the vertices of this subquiver. Note that at each step through this part of the sequence there is a unique green vertex with a unique edge with head at a mutable vertex and tail at the green vertex. Rearranging the vertices from our previous picture we obtain the following picture from which it is easy to see that the remaining mutations give a maximal green sequence.

Thus $\alpha_0\alpha_1\alpha_2\alpha_3$ is a maximal green sequence for $P_3$. Our claim follows from induction on $n$. Suppose for $1 \leq k < n$ $\alpha_0 \cdots \alpha_k$ gives a maximal green sequence for $\alpha_k$. Note that $Q_n$ has a subquiver of $P_{n-1}$ for which $\alpha_0\alpha_{n-1}$ is a green sequence. Note that the local configuration of "top nine" vertices $g_i^j \ i = 1, 2, 3 j = n - 2, n - 1, n$ have the exact same configuration as the "top nine" vertices of $P_3$ and the first four mutations of $\alpha_n$ exactly mimic that of $n = 3$ case. (Possibly need to show this in a lemma.) Therefore we get that after the green sequence $\alpha_0 \cdots \alpha_{n-1}g_3^j g_2^j g_1^j g_1^1$ we have the quiver:
Where again it is easy to check the remaining mutations will give us a maximal green sequence for $P_n$. 

**Lemma 4.3.** [3 Lemma 4.2] If $C$ is an oriented $n$-cycle with vertices labeled $c_i$ $i = 1, \ldots, n$, then $c_{n}c_{n-1}c_{n-2}\cdots c_1c_3c_4\cdots c_n$ is a maximal green sequence for $C$.

**Theorem 4.4.** [3 Theorem 4.1] The quiver $Q^2_n$ has a maximal green sequence of 

$$f_nf_{n-1}\cdots f_1f_3f_4\cdots f_n\sigma_0\sigma_0^{-1}\cdots \sigma_1f_3f_4\cdots f_nf_2f_1f_nf_{n-1}\cdots f_3\tau_n\tau_{n-1}\cdots \tau_1$$

*Proof of Theorem.* By Lemma 4.2 and the proof of Theorem 4.4 in [3] we know that

$$f_{n+2}f_{n+1}\cdots f_1f_3f_4\cdots f_{n+2}\sigma_n\cdots \sigma_1\alpha_0\alpha_1\cdots \alpha_{p-3}$$

is a green sequence. After performing this mutation sequence we have that all of the vertices are red except for the vertices $f_1\cdots f_{n+2}$. Furthermore, all of these vertices except for $f_{n+1}$ form an $(n + 1)$-cycle.

By Lemma 4.3 our next section of the maximal green sequence is a green sequence for this cycle.
After mutating at $f_{n+1}$ and $g_1^{p-4}$ we have a similar situation as we did at the end of the proof of Lemma 4.2. The remaining vertices that we mutate along in the $\beta$ subsequence form an equioriented affine subquiver. It is easy to follow that this is a green sequence that will make the $P_{p-3}$ subquiver of $Q_n$ red.

Finally, the only remaining green vertices are $e_i$ for $i = 1, \ldots, n$. By inspection of the local configuration of the frozen vertices we see that this is the exact same configuration as in the twice punctured case. Therefore it follows from the proof of Theorem 4.3 that $\tau_i$ is a green sequence for our quiver, and concluding the proof that our sequence is maximal.

\[ \Box \]

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