Superradiant instabilities of the Kerr-like black holes in a dark matter halo

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ABSTRACT: The quasinormal modes (QNM) and quasibound states (QBS) are the characteristic “sound” for a black hole which can provide us with a new method to verify black holes in our universe. Furthermore, superradiant instability is closely related to the existence of QBS, which may have significant astrophysical implications. Based on these interesting physical background, we use a semianalytical method, continued fraction method, to study the QNM and QBS of the Kerr-like black holes with a scalar field immersed in a dark matter halo (by considering cold dark matter (CDM) model and scalar field dark matter (SFDM) model), and verify the existence of the superradiant instability. We have shown the instabilities of these two models at the state $l = 1, m = 1$. For CDM model, the most instability occurs at mass parameter $M\mu \lesssim 0.1$. The maximum instability occurs at $M\mu \approx 0.0875$ and the maximum growth rate is approximately $\tau^{-1} \approx 8.89 \times 10^{-10}(GM/c^3)^{-1}$. For SFDM model, the most instability occurs at mass parameter $M\mu \lesssim 0.4$. The maximum instability occurs at $M\mu \approx 0.375$ and the maximum growth rate is approximately $\tau^{-1} \approx 1.19 \times 10^{-7}(GM/c^3)^{-1}$.

KEYWORDS: Dark Matter, Superradiant Instability, Quasibound State, Quasinormal Mode

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1 Introduction

Now, there is a large number of observational data indicating the existence of dark matter (DM), such as the cosmic microwave background radiation (CMBR), the rotation curves (RC) of the galaxies and the large-scale structure of the universe. Based on these observational data, astronomers have proposed lots of dark matter models. Among all these dark matter models, cold dark matter (CDM) model [1, 2] and scalar field dark matter (SFDM) model have received much attention. In these dark matter models, the most important part is to obtain the spatial density distribution function of dark matter. At present, the distribution of dark matter in the large-scale structure of galaxies is clear [3]. However, the distribution of dark matter around supermassive black holes (BHs) is unclear [4]. Therefore, studying the distribution of dark matter around black holes is a very significant project. It is generally believed that the distribution of dark matter around black holes is a big "spike" [5–7]. Fortunately, with our efforts, we have derived black hole (BH) spacetime metrics both in a dark matter halo [8] and a dark matter spike [9], respectively, and generalized them to the case of rotation. In addition, through the black hole photos released by the Event Horizon Telescope (EHT) [10] and the observation of gravitational waves by the Laser Interferometer Gravitational Wave Observatory Scientific Collaboration and Virgo Collaboration (LIGO Scientific/Virgo) [11–14], the existence of black holes can be basically determined.

On the other hand, the term “black hole” was originally coined by Wheeler [15]. For a black hole, he proposed the famous “No-hair theorem” [16]. The object described by the No-hair theorem is a static isolated black hole. However, in our universe, it is hard to find an isolated black hole, suggesting that there may be various complex matter fields around the black hole, causing the black hole to be in a perturbed state. When a black hole is
perturbed by the matter field, its initial perturbation can be represented by a complex frequency of excited oscillation mode, which is so-called quasinormal mode (QNM) [17]. This mode is divided into three stages, the second stage is the main mode. This main mode contains the oscillation frequency of a black hole. The real part of this frequency represents the oscillation frequency of the black hole when perturbed, while the imaginary part represents the rate of oscillation, also known as damping [18]. There are different methods in calculating QNM, such as WKB method [19, 20], Pöschl-Teller potential approximation [21–23] and continued fraction method [24, 25]. The quasinormal mode is a characteristic “sound” of a black hole which can provide us with a new method to verify a black hole in our universe [18]. On the other hand, according to the fact that no matter can escape from a black hole, the solution at the event horizon of the black hole is a pure incoming wave. However, for the behavior of the solution of the equation at infinity, it can be divided into two types. One of types is QNM we introducing before and another is the quasibound states (QBS). The solution of QNM is a pure outgoing wave when it is far away from the black hole, while QBS is only shown as an exponential decay when it is far away from the black hole [25]. According to the superradiance phenomenon, a rotating black hole produces wave amplification, and when the perturbation field is large, the superradiance phenomenon can cause instability [26]. This superradiative instability is closely related to the existence of QBS, which may have important astrophysical implications.

In recent years, the QBS of black holes [27–33] and the QNM [34–45] have been extensively studied. In the nearest reaserch, the following questions are interesting and important. In Ref.[25], they study the instabilities of rotating black holes in the massive scalar field. M. Richartz et al. use the scalar field perturbation to study the eigenfrequencies of the Kerr-like black holes [26]. Besides, with the release of black hole photos, people are also increasingly concerned about the interaction of other celestial bodies (or matter) around the black hole [46–50]. C. Zhang et al. are interested in the physics related to dark matter around a black hole [51]. Cardoso et al. introduce an exact solution for a black hole immersed in a galactic-like distribution of matter and use gravitational perturbations to study the quasinormal modes of this black hole [52]. Based on this, Konoplya studies the matter field perturbations, the greybody factors and the Unruh temperature [53] of the exact solution of this black hole. These works of them are interesting and important.

In this paper, we mainly use the scalar field perturbation to study QNM and QBS of the Kerr-like black holes in a dark matter halo, and use the continued fraction method to calculate the frequencies of QNM and QBS. Besides, we will also analyze and verify the existence of the superradiant instability. This work is an in-depth study based on our previous work [54–56]. Although these previous works are on spherically symmetric black holes, these valuable experiences give us enough confidence to solve the case of axisymmetric black holes.

This paper is organized as follows. In Section 2, we briefly introduce the spacetime properties of the Kerr-like black holes in a dark matter halo, and the discussion of extremal parameter. In Section 3, we introduce scalar field perturbation of rotating black holes, and show that how to use ansatz to decompose the complex perturbation equations into radial and angular equations. In Section 4, we mainly introduce the continued fraction method,
including the asymptotic solution of the black hole oscillation behavior, the derivation of the 3-term recurrence formula of the radial and angular equations and the continued fraction equation. In Section 5, we mainly use the continued fraction method to give the frequencies of QNM and QBS in CDM and SFDM models, and verify the existence of superradiant instability. Besides, we also calculate the maximum instability. Finally, Section 6 is our conclusions and discussions. In this paper, we use mostly the units ($\hbar = G = c = 1$).

2 The spacetime of the Kerr-like BHs in a dark matter halo

In this section, we will review the Kerr-like black hole metrics we obtained in a dark matter halo [8]. Both of them have the following form in four-dimensional coordinates,

$$
\begin{align*}
&ds^2 = -\left(1 - \frac{r^2 + 2Mr - r^2f(r)}{\Sigma^2}\right)dt^2 + \frac{\Sigma^2}{\Delta}dr^2 + \Sigma^2d\theta^2 + \frac{A\sin^2\theta}{\Sigma^2}d\phi^2 \\
&- 2\frac{(r^2 + 2Mr - r^2f(r))a\sin^2\theta}{\Sigma^2}d\phi dt,
\end{align*}
$$

with

$$
\Delta = r^2f(r) - 2Mr + a^2, \quad \Sigma^2 = r^2 + a^2\cos(\theta)^2, \quad A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad (2.2)
$$

where, $M$ is the mass of a black hole and $a$ is the rotation parameter. $f(r)$ represents the factor term for considering dark matter. For the cold dark matter (CDM) model, $f(r)$ has the form of the following,

$$
f_c(r) = \left(1 + \frac{r}{R_c}\right)^{-\frac{8\rho_cR_c^3}{\pi r}}, \quad (2.3)
$$

and for the scalar field dark matter (SFDM) model, $f(r)$ has the form of the following,

$$
f_s(r) = \exp\left(-\frac{8\rho_sR_s^2}{\pi} \sin(\pi r/R_s)\left(\frac{\pi r}{R_s}\right)\right), \quad (2.4)
$$

here, the parameter $\rho$ is the density of the universe when a dark matter halo collapses, and $R$ means its characteristic radius in this halo. The corresponding dark matter parameters can be selected from the galaxy ESO 1200211 in these Refs. [57, 58]. If the effect of dark matter on black holes is not considered, that is, $\rho = 0$, then the dark matter term will become $f(r) = 1$. The black hole metric in a dark matter halo will degenerate into the Kerr metric. Besides, in these two black holes, the location of the event horizons can be obtained from Eq. (2.2),

$$
\Delta = r^2f(r) - 2Mr + a^2 = (r - r_+)(r - r_-) = 0 \quad (2.5)
$$

where, $r_+$ and $r_-$ are the outer and inner horizon of this rotating black hole, respectively. Here, we also present pictures of the root of the $\Delta$ function for these three types of event horizon in Fig. 1. We find that with the rotation parameter $a$ increases, the roots of the $\Delta$ function gradually change from two to one, and then to zero. This process indicates the transition from a rotation black hole to an extremal black hole. For the case of extremal
Figure 1. The functional image of $\Delta$ function in the CDM model (left panel), the SFDM model and the Kerr spacetime (right panel) vary with different $a$ respectively. And the calculation parameters are $M = 1, R_c = 5.7, \rho_c = 0.00245, R_s = 2.92, \rho_s = 0.01366$.

black hole, it appears as the coincidence of the inner and outer horizons, and thermodynamically, it appears as Hawking temperature equals to zero [59]. Now, let’s calculate the extremal value of the rotation parameter $a$ of the extremal black hole. Using the following ansatz, $a^2$ can be rewritten as a function of $r$,  
\[ a^2(r) = -r^2 f(r) + 2Mr, \]  
then, the extremal value of the rotation parameter can be transformed into the maximal value of Eq. (2.6). At this time, we only need to solve its first derivative, that is, $da^2(r)/dr = 0$. According to the definition of quadratic equation, the highest order term is negative, so the functional image of Eq. (2.6) opens downwards and this function has a maximum value. For the CDM model, the rotation parameter obtained by the numerical calculation is $a \approx 2.11$ and for the SFDM model, $a \approx 1.11$ by defining $M = 1$. The value of them are bigger than the Kerr black hole ($a = 1$).

Next, to make our expression more compact, we continue to use $f(r)$ to represent the dark matter term. Then, we can obtain a covariant metric tensor $g_{\mu\nu}$ from the Eq.(2.1),  
\[ g_{\mu\nu} = \begin{pmatrix} - (1 - r^2 + 2Mr - r^2 f(r)) & 0 & 0 & - a \sin^2(\theta)(r^2 + 2Mr - r^2 f(r)) \\ 0 & \Sigma^2 & 0 & 0 \\ 0 & 0 & \Sigma^2 & 0 \\ - a \sin^2(\theta)(r^2 + 2Mr - r^2 f(r)) & 0 & 0 & \Sigma^2 \end{pmatrix}. \]  
With the Eq.(2.7), we can calculate the determinant of this metric,  
\[ g = det(g_{\mu\nu}) = -\Sigma^4 \sin^4(\theta). \]  
From Eqs.(2.7) and (2.8), we get the contravariant form of the metric,  
\[ g^{\mu\nu} = \begin{pmatrix} - \frac{A}{\Delta \Sigma^2} & 0 & 0 & \frac{ar(rf(r) - 2Mr - r^2)}{\Delta \Sigma^2} \\ 0 & \frac{\Delta}{\Sigma^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma^2} & 0 \\ \frac{ar(rf(r) - 2Mr - r^2 + \Sigma^2)}{\Delta \Sigma^2} & 0 & 0 & \frac{(r^2 f(r) - 2Mr - r^2 + \Sigma^2)}{\Delta \Sigma^2 \sin^2(\theta)} \end{pmatrix}. \]
3 Scalar field perturbation of the Kerr-like BHs in a dark matter halo

In this section, we will study the scalar field perturbation of Kerr-like black hole and derive the radial and angular equations of scalar particles in a dark matter halo. In a curved spacetime, the equation of the motion of scalar particles can be described by the Klein-Gordon (K-G) equation,

$$
\frac{1}{\sqrt{-g}} \partial \sigma (\sqrt{-g} g^{\mu \nu} \partial \nu \Psi) = \mu^2 \Psi,
$$

(3.1)

where \( \mu \) is the mass of the scalar particle. With Eqs. (2.8) and (2.9), we can get the following form,

$$
\begin{align*}
& -\frac{A}{\Delta \Sigma^2} \partial_t^2 \Psi + \frac{ar(-2M - r + rf(r))}{\Delta \Sigma^2} \partial_t \partial_r \Psi + \frac{1}{\Sigma^2 \sin^2(\theta)} \partial_\theta (\sin(\theta) \partial_\theta \Psi) + \frac{1}{\Sigma^2} \partial_\varphi (\Delta \partial_\varphi \Psi) \\
& + \frac{ar(-2M - r + rf(r))}{\Delta \Sigma^2} \partial_\varphi \partial_t \Psi + \frac{-2Mr - r^2 + \Sigma^2 + r^2 f(r)}{\Delta \Sigma^2 \sin^2(\theta)} \partial_\varphi^2 \Psi = \mu^2 \Psi.
\end{align*}
$$

(3.2)

Eq. (3.2) is a complex second-order partial differential equation, but it can be separated by using the following ansatz,

$$
\Psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta),
$$

(3.3)

where, \( m \) is the azimuthal quantum number and \( \omega \) is the frequency of this system. Therefore, we can get ordinary differential equations about angular and radial parts. If we introduce the variable \( x = \cos \theta \), with \( x \in [-1, 1] \), the angular part can be written as

$$
(1 - x^2) \frac{d^2 S(x)}{dx^2} - 2x \frac{dS(x)}{dx} + \left( \Lambda_{lm} + a^2 k^2 x^2 - \frac{m^2}{1 - x^2} \right) S(x) = 0,
$$

(3.4)

where, \( k^2 = \omega^2 - \mu^2 \) and \( \Lambda_{lm} \) is a separation constant. Eq. (3.4) is also known as the spheroidal equation, and \( \Lambda_{lm} \) is the eigenvalue of this equation [60]. Normally, the eigenvalue \( \Lambda_{lm} \) has no analytical expression in this case. A simple method is to calculate its value by calling the SpheroidalEigenvalue command in Mathematica. However, in the case of non-rotating limit, the spheroidal function can be reduced to a spherical harmonic function, that is, \( S_{lm} \rightarrow Y_{lm} \), and \( \Lambda_{lm} = l(l+1) \). The eigenvalue at this time can be uniquely determined, and it is related to the angular quantum number \( l \).

Another is radial equation, and it can be written as

$$
\Delta^2 \frac{d^2 R(r)}{dr^2} + \Delta \frac{d\Delta}{dr} \frac{dR(r)}{dr} + \left( K^2(r) - \left( \lambda + \mu^2 r^2 \right) \Delta \right) R(r) = 0
$$

(3.5)

where, \( \Delta = (r - r_+)(r - r_-) \), \( K(r) = \omega(r^2 + a^2) - am \) and \( \lambda = \Lambda_{lm} + a^2 \omega^2 - 2am\omega \). There are two eigenvalues \( \Lambda_{lm} \), \( \omega \) in this equation. Among them, the eigenvalue \( \omega \) is the frequency of this system, which is a complex number. The real part of the frequency represents the oscillation frequency and its imaginary part represents the decay rate. The relationship between decay rate and time scale is \( \tau^{-1} = M \times \text{Im}(\omega) \), that is growth rate. To solve the radial equation, we need to first determine the separation constant \( \Lambda_{lmp} \) in the angular equation. In the following section, instead of using the Mathematica function, we will demonstrate how to use the numerical method to determine the eigenvalues both in the angular and radial equations.
4 Numerical method

In this section, we will introduce the numerical method, that is, the continued fraction method. The rotating black hole is different from the general spherically symmetric black hole because there are two eigenvalues \( \Lambda_{lm}, \omega \) unclear in the radial equation. The continued fraction method is considered to be one of the most direct and effective methods to solve the QNM and QBS of rotating black holes. This method was first used by Leaver to study the QNM of the Kerr black holes [24]. Next, we will strictly follow the continued fraction method used in these Refs. [24, 25, 28] and generalize it to apply to our Kerr-like black holes. Firstly, from the angular equation (3.4), we find that there are two regular singular points \((x = -1, x = 1)\) and one irregular singular point \((r = \infty)\). Therefore, their asymptotic behaviors at the location of the singular points can be written as

\[
\lim_{x \to -1} S(x) \sim (1 + x)^{|m|/2}, \quad \lim_{x \to 1} S(x) \sim (1 - x)^{|m|/2}.
\]

(4.1)

Taking Eq.(4.1) into account, this eigenfunction \( S(x) \) to Eq.(3.4) has the following series solution,

\[
S(x) = \exp(akx)(1 - x)^{|m|/2}(1 + x)^{-|m|/2} \sum_{n=0}^{\infty} b_n (1 + x)^n
\]

(4.2)

Putting Eq.(4.2) into Eq.(3.4) for calculation, it can be found that \( b_n \) must satisfy the following 3-term recurrence relation,

\[
\begin{aligned}
\alpha_0 b_1 + \beta_0 b_0 &= 0, \\
\alpha_n b_{n+1} + \beta_n b_n + \gamma_n b_{n-1} &= 0, \quad n \geq 1
\end{aligned}
\]

(4.3)

where, \( b_0 = 1 \) and the coefficients \( \alpha_n, \beta_n, \gamma_n \) are as follows,

\[
\begin{aligned}
\alpha_n &= -2(n + 1)(|m| + n + 1), \\
\beta_n &= -\Lambda_{lm} + |m|(-2ak + 2n + 1) - ak(ak + 2) + m^2 + n^2 - 4akn + n, \\
\gamma_n &= 2ak(|m| + n).
\end{aligned}
\]

(4.4)

If we define the convergence series \( R_n = b_n/b_{n-1} \), the recurrence relation can be rewritten as

\[
R_n = \frac{-\gamma_n}{\beta_n + \alpha_n R_{n+1}}.
\]

(4.5)

With Eq.(4.3), we also have \( R_1 = b_1/b_0 = -\beta_0/\alpha_0 \). So, we can get the main equation of the continued fraction method,

\[
0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \cdots}}},
\]

(4.6)

Given \( a, k, m, \), these coefficients only depend on the eigenvalue \( \Lambda_{lm} \). So, the infinite continued fraction is an equation for \( \Lambda_{lm} \), and \( \Lambda_{lm} \) is a root of the continued fraction equation or any of its inversions [24, 60].

Similarly, a solution \( R(r) \) of the radial equation (3.5) can also be found since it is a
spherical wave equation with similar boundary conditions as Eq. (3.4). For the perturbation theory of the black hole, the essence of the radial equation can be simplified as a wave equation. The solution to the wave equation is directly related to the location of the boundary conditions of this system, which at the event horizon and at infinity. For the behavior away from the black hole, it can be divided into two types, namely QNM and QBS. The solution of this equation is generally related to the oscillating mode of the matter field. QNM and QBS are the modes with complex frequencies. Its real part is the oscillation frequency of a black hole and the imaginary part is the decay rate of this oscillation. For the QNM, its solution is usually represented by pure incoming waves at the event horizon and pure outgoing waves at infinity. So, we require that

\[ \lim_{r \to r_+} R(r) \sim (r - r_+)^{-i\alpha}, \quad \alpha = \frac{r_+^2 + a^2}{r_+ - r_-}(\omega - m\Omega), \]

\[ \lim_{r \to \infty} R(r) \sim \exp(-\overline{k}r)r^{\beta-1}, \quad \beta = \frac{(r_+ + r_-)(\mu^2 - 2\omega^2)}{2\overline{k}}, \quad \Omega = a/(r_+^2 + a^2), \quad \overline{k} = \pm \sqrt{\mu^2 - \omega^2}. \]

For QNM, its behavior is pure outgoing waves far away from the black hole (Re(\overline{k}) < 0) and for QBS, its behavior becomes exponentially decay away from the black hole (Im(\overline{k}) > 0) [25, 26].

Now, back to Eq.(3.5), there are two regular singular points (\( r = r_+ \), \( r = r_- \)) and one irregular singular point (\( r = \infty \)) in this equation. Meanwhile, taking Eq.(4.7) into account, the appropriate series solution has the following form

\[ R(r) = \exp(-\overline{k}r)(r - r_+)^{-i\alpha}(r - r_-)^{\beta-1} \sum_{n=0}^{\infty} a_n (r - r_+) \gamma^n, \quad (4.8) \]

Putting Eq.(4.8) into Eq.(3.5), it can be found that \( a_n \) must satisfy the following 3-term recurrence relation,

\[ \begin{cases} 
\alpha_0 a_1 + \beta_0 a_0 = 0, \\
\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n \geq 1
\end{cases} \quad (4.9) \]

where, \( a_0 = 1 \) and the coefficients \( \alpha_n, \beta_n, \gamma_n \) are depend on these two eigenvalues \( \Lambda_{lm} \) and \( \omega \). Since these coefficients are too complex, we do not show them here. The benefit of Eq. (4.8) is that it guarantees that there are no other singular points than the event horizon. So, to find the eigenvalues \( \Lambda_{lm} \) and \( \omega \), we have to solve two continued fraction equations as before

\[ \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \cdots}}} = 0, \quad (4.10) \]

Finally, this infinite continued fraction equation \( R_n \) needs to be truncated at some order \( n \in N \) to ensure the convergence of this equation [27]. Regarding the equation of convergence, Nollert had proposed some solutions to guarantee the convergence rate of this method [61, 62]. From the recurrence relation (4.9), it can be found that the convergence series \( a_{n+1}/a_n \) satisfies the following relation at the order \( n^{-1} \),

\[ \frac{a_{n+1}}{a_n} = 1 - \sqrt{\frac{-2k(r_+ - r_-)}{n}} - \left[ \frac{3}{4} + \frac{(r_+ + r_-)k}{2} - 2r_+ \overline{k} + \frac{\omega^2(r_+ + r_-)}{2\overline{k}} \right] \frac{1}{n} \quad (4.11) \]
5 Numerical results

In this section, we will use the continued fraction method introduced in the previous section to calculate the QNM and QBS of the Kerr-like black holes immersed in a dark matter halo under the scalar field. Besides, we will also analyze and verify the existence of the superradiant instability, that is an unstable modes \( \text{Im}(\omega) > 0 \). For the angular equation, given \( a, k, m \), the continued fraction equation only depend on the eigenvalue \( \Lambda_{lm} \). Similarly, for the radial equation, given \( M, a, \mu \) and \( m \), the continued fraction equation will depend on eigenfrequency \( \omega \). In other words, the eigenvalues \( \omega \) and \( \Lambda_{lm} \) are the roots of these two continued fraction equations. Now, these two equations are converted into two infinite continued fraction equations. What needs to be done in the next step is to make a truncation at the appropriate position and ensure the convergence of the equation. Here, we will truncate the continued fraction equation at \( N \) terms and use a root finding algorithm to solve the equation. However, this method also need to give the initial guess value of \( \omega \) and \( \Lambda_{lm} \). S. R. Dolan has shown that, in the nonrelativistic limit, the frequency of a Schwarzschild black hole of the massive scalar field bound has the following form \[25, 63\],

\[
\hbar \omega_n \approx (1 - \frac{M^2 \mu^2}{2\bar{n}^2}) \mu c^2, \tag{5.1}
\]

where, \( \bar{n} = n + l + 1 \) and \( \bar{n} \) is the principal quantum number of this system. In this paper, we choose the Schwarzschild fundamental mode and the slowly rotating parameter \( (a = 0.0001) \) as our initial guess and then use a root-finding algorithm for these eigenvalues. Finally, we set the terms \( N = 100 \) and all the results of frequencies in the tables are kept 6 significant figures.

5.1 Quasinormal modes

To ensure the accuracy of our method, it is necessary for us to check the QNM frequencies for Kerr black hole. Without loss of generality, we calculate the QNM frequencies of the Kerr black hole with angular quantum \( l = 1 \) both in the massless scalar field and the massive scalar field. Note that when the angle quantum \( l = 1 \), the value of the azimuthal quantum number \( m \) can be \( m = -1, 0, 1 \). We implemented the continued fraction algorithm and our results about the massless scalar field giving in Table 1 are in good agreement with the data recorded in Refs.\[24, 25, 28\]. Among them, they all used the continued fraction method to calculate the QNM frequencies of rotating black holes. These evidences we obtained show that our method can provide an important guarantee for the calculation of the QNM frequencies in a dark matter halo. Besides, we also give the QNM frequencies for the states \( l = 0, m = 0 \) and \( l = 1, m = 0 \) in Table 1.

Now, let’s return to the discussion of QNM frequencies for the Kerr-like black holes in a dark matter halo. For a black hole, its oscillatory behavior is always related to dissipation, originating QNM. This oscillatory behavior usually appears as a pure incoming wave at the horizon and a pure outgoing wave at infinity. Therefore, the process of QNM is a stable mode. In this paper, we first study the oscillatory behavior of black holes in massless scalar field both in CDM and SFDM models, and use the continued fraction method to calculate
Table 1. The frequencies of quasinormal modes in massless scalar field in Kerr spacetime.

| l = 0, m = 0 | l = 1, m = −1 | l = 1, m = 0 | l = 1, m = 1 |
|-----|-----|-----|-----|
| a  | Re  | -Im | Re  | -Im | Re  | -Im | Re  | -Im |
| 0.00 | 0.110456 | 0.1048950 | 0.292938 | 0.0976593 | 0.292938 | 0.0976594 |
| 0.10 | 0.110534 | 0.1048010 | 0.285573 | 0.0976248 | 0.293129 | 0.0975786 |
| 0.30 | 0.111159 | 0.1040090 | 0.272639 | 0.0972250 | 0.294682 | 0.0969023 |
| 0.50 | 0.112380 | 0.1021850 | 0.261579 | 0.0964980 | 0.297933 | 0.0953636 |
| 0.70 | 0.113979 | 0.0986280 | 0.251936 | 0.0955236 | 0.303192 | 0.0924357 |
| 0.90 | 0.113828 | 0.0915484 | 0.243370 | 0.0944214 | 0.310779 | 0.0866559 |
| 0.99 | 0.111425 | 0.0886749 | 0.239830 | 0.0939634 | 0.314590 | 0.0822809 |

Table 2. The frequencies of quasinormal modes in massless scalar field in SFDM model.

| l = 0, m = 0 | l = 1, m = −1 | l = 1, m = 0 | l = 1, m = 1 |
|-----|-----|-----|-----|
| a  | Re  | -Im | Re  | -Im | Re  | -Im | Re  | -Im |
| 0.00 | 0.101040 | 0.0959169 | 0.267876 | 0.0892971 | 0.267883 | 0.0892960 | 0.267889 | 0.0892970 |
| 0.30 | 0.101165 | 0.0948323 | 0.249874 | 0.0885930 | 0.268247 | 0.0883457 | 0.289179 | 0.0881859 |
| 0.50 | 0.101455 | 0.0927444 | 0.239004 | 0.0873525 | 0.268914 | 0.0865332 | 0.306444 | 0.0858674 |
| 0.70 | 0.101673 | 0.0892242 | 0.228701 | 0.0856011 | 0.269945 | 0.0835156 | 0.328028 | 0.0813250 |
| 0.90 | 0.101142 | 0.0835144 | 0.218734 | 0.0833962 | 0.271304 | 0.0787340 | 0.358243 | 0.0719192 |
| 1.00 | 0.0995770 | 0.0795220 | 0.213834 | 0.0833962 | 0.271969 | 0.0753276 | 0.380763 | 0.0622460 |
| 1.10 | 0.0957679 | 0.0769970 | 0.208950 | 0.0807987 | 0.272290 | 0.0710090 | 0.419855 | 0.0351584 |
| 1.11 | 0.0954008 | 0.0767748 | 0.208491 | 0.0806425 | 0.272824 | 0.0705232 | 0.426706 | 0.0259618 |

Table 3. The frequencies of quasinormal modes in massless scalar field in CDM model.

| l = 0, m = 0 | l = 1, m = −1 | l = 1, m = 0 | l = 1, m = 1 |
|-----|-----|-----|-----|
| a  | Re  | -Im | Re  | -Im | Re  | -Im | Re  | -Im |
| 0.00 | 0.0304096 | 0.028876 | 0.0806380 | 0.0268922 | 0.0806381 | 0.0268922 | 0.0806386 | 0.0268922 |
| 0.40 | 0.0305079 | 0.0287890 | 0.0785797 | 0.0266335 | 0.0808900 | 0.0268162 | 0.0833195 | 0.0268024 |
| 0.70 | 0.0307182 | 0.0285943 | 0.0759104 | 0.0264686 | 0.0814060 | 0.0266653 | 0.0858685 | 0.0266250 |
| 1.00 | 0.0310550 | 0.0282590 | 0.073445 | 0.0265411 | 0.0822592 | 0.0264070 | 0.0890835 | 0.0262956 |
| 1.40 | 0.0317312 | 0.0275033 | 0.0732199 | 0.0261823 | 0.0840656 | 0.0258005 | 0.0950026 | 0.0254253 |
| 1.70 | 0.0323775 | 0.0264460 | 0.0748783 | 0.0258032 | 0.0861218 | 0.0249660 | 0.101692 | 0.0239718 |
| 2.00 | 0.0325283 | 0.0243916 | 0.0744333 | 0.0252911 | 0.0890258 | 0.0232953 | 0.104299 | 0.0236177 |
| 2.11 | 0.0318178 | 0.0239195 | 0.0743510 | 0.0250581 | 0.0902754 | 0.0222526 | 0.108882 | 0.0255046 |

their QNM frequencies for the angular quantum $l = 1$. We divide this QNM frequency into two parts, real and imaginary parts. where the real part represents the oscillation frequency of this system, and the imaginary part represents the decay rate, as a function of rotating parameter $a$. Then, we record the QNM frequencies of the black holes about these two models in Tables 2 and 3, respectively. From Table 2, we found that when the modes are $l = 0, m = 0$ and $l = 1, m = −1$, the real and imaginary parts of the QNM frequencies in the SFDM model both decrease with the increasing of the rotation parameter $a$. However, when the modes are $l = 1, m = 0$ and $l = 1, m = 1$, the real part of the QNM frequency
Table 4. The frequencies of quasinormal modes in massive scalar field in SFDM model.

| \( l = 0, m = 0 \) | \( \mu = 0.1 \) | \( \mu = 0.2 \) | \( \mu = 0.3 \) |
|----------------|----------------|----------------|----------------|
| \( a \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) |
| 0.00 | 0.103031 | 0.0871434 | 0.107117 | 0.0643718 | 0.115766 | 0.0319086 |
| 0.30 | 0.103268 | 0.0861315 | 0.107453 | 0.0635594 | 0.115190 | 0.0324617 |
| 0.50 | 0.103705 | 0.0841904 | 0.108202 | 0.0619034 | 0.113787 | 0.0305360 |
| 0.70 | 0.104209 | 0.0809201 | 0.109634 | 0.0595231 | 0.119358 | 0.0279494 |
| 0.90 | 0.104243 | 0.0756060 | 0.110521 | 0.0581034 | 0.118787 | 0.0249600 |
| 1.00 | 0.103291 | 0.0718147 | 0.110931 | 0.0533863 | 0.119289 | 0.0260292 |
| 1.10 | 0.100371 | 0.0688903 | 0.111321 | 0.0512764 | 0.122936 | 0.0255111 |
| 1.11 | 0.0996246 | 0.0686557 | 0.110603 | 0.0512764 | 0.123983 | 0.0192189 |

Table 5. The frequencies of quasinormal modes in massive scalar field in CDM model.

| \( l = m = 0 \) | \( \mu = 0.05 \) | \( \mu = 0.07 \) | \( \mu = 0.10 \) |
|----------------|----------------|----------------|----------------|
| \( a \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) |
| 0.00 | 0.0317098 | 0.0220103 | 0.0329370 | 0.0163456 | 0.0350249 | 0.00429413 |
| 0.40 | 0.0318799 | 0.0219881 | 0.0330963 | 0.0163764 | 0.0357392 | 0.00445587 |
| 0.70 | 0.0321447 | 0.0218891 | 0.0333845 | 0.0164515 | 0.0366924 | 0.00542001 |
| 1.00 | 0.0325666 | 0.0217209 | 0.0337304 | 0.0164703 | 0.0366317 | 0.00673906 |
| 1.40 | 0.0334808 | 0.0212908 | 0.0347922 | 0.0160658 | 0.0376629 | 0.00534245 |
| 1.70 | 0.0343361 | 0.0206117 | 0.0357727 | 0.0159752 | 0.0381296 | 0.00691071 |
| 2.00 | 0.0351817 | 0.0191139 | 0.0369464 | 0.0147428 | 0.0416327 | 0.00639188 |
| 2.11 | 0.0346944 | 0.0186670 | 0.0373529 | 0.0148740 | 0.0427671 | 0.00602062 |

Table 6. The frequencies of quasinormal modes in massive scalar field in SFDM model.

| \( l = 1, m = 0 \) | \( \mu = 0.1 \) | \( \mu = 0.2 \) | \( \mu = 0.3 \) |
|----------------|----------------|----------------|----------------|
| \( a \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) | \( \text{Re} \) | \( \text{-Im} \) |
| 0.00 | 0.264834 | 0.0907272 | 0.286539 | 0.0780796 | 0.312645 | 0.0606205 |
| 0.30 | 0.273107 | 0.0854443 | 0.287880 | 0.0763941 | 0.312726 | 0.0599135 |
| 0.50 | 0.273744 | 0.0837432 | 0.288361 | 0.0750108 | 0.312980 | 0.0590097 |
| 0.70 | 0.274702 | 0.0809107 | 0.289102 | 0.0727078 | 0.313352 | 0.0574899 |
| 0.90 | 0.275962 | 0.0764254 | 0.290070 | 0.0690668 | 0.313836 | 0.0550882 |
| 1.00 | 0.276577 | 0.0732350 | 0.290540 | 0.0664884 | 0.314065 | 0.0533933 |
| 1.10 | 0.276878 | 0.0691947 | 0.290776 | 0.0632316 | 0.314143 | 0.0512828 |
| 1.11 | 0.276874 | 0.0687376 | 0.290777 | 0.0628471 | 0.314155 | 0.0508289 |

increases with the increasing of the rotation parameter \( a \), and its imaginary part decreases with the increasing of the parameter \( a \). On the other hand, we found that when \( l = 1 \), the real part of the frequency of a black hole in the SFDM model increases with the increase of \( m \), and its imaginary part decrease with the increase of \( m \). Similarly, when \( m = 0 \), the real part of the frequency in the SFDM model increases with the increasing of \( l \), and its
imaginary part decreases with the increasing of $l$. From Table 3, the oscillating behavior of black holes in the CDM model is much the same as that in the SFDM model. Finally, according to Tables 1, 2 and 3, under the same mode ($a,l,m$ are uniquely determined), we found that the oscillation frequency and decay rate of Kerr black hole are greater than that of the SFDM model, while the SFDM model is greater than the CDM model. Besides, we
Figure 2. The quasinormal modes and quasibound states of the black holes in a dark matter halo (left panel is CDM model and the right is SFDM model). In these two pictures, three colors of red, green and blue represent three modes respectively. These points of the same color are given by the increasing value of the rotation parameter $a$, corresponding to the value $a$ from 0 to 2.11 (left panel) and 1.11 (right panel). The steps of $a$ are 0.15 (left panel) and 0.05 (right panel).

still consider the oscillation behavior of the black hole when the angular quantum $l = 0$ and $l = 1$ with the different mass $\mu$. For CDM model, we consider the scalar field when the mass $\mu$ is 0.05, 0.07 and 0.10, respectively. For SFDM model, we consider the scalar field when the mass $\mu$ is 0.10, 0.20 and 0.30, respectively. Then, we record the calculation results in Tables 4, 5, 6, 7, 8 and 9, respectively. From our results, we found that whether it is the CDM model or the SFDM model, the real part of their frequency of the QNM always increases with the increasing of the mass $\mu$, and its imaginary part decreases with the increasing of the mass $\mu$. Besides, under the same conditions ($a, l, m$ and $\mu$ are uniquely determined), the oscillation frequency and decay rate of the SFDM model are still greater than that of the CDM model. At the same time, from the data in these tables, we found that the imaginary part of the QNM are all negative numbers in a massive scalar field, which indicates that QNM are a stable mode in our chosen mass range. For a stable model, we show the QNM of $\mu = 0$ and $\mu \neq 0$ in Fig. 2. The decay rate of QNM with $\mu = 0$ is greater than that of QNM with $\mu \neq 0$ ($a, l, m$ and $\mu$ are uniquely determined) but the oscillation frequency of the QNM with $\mu = 0$ is less than the QNM with $\mu \neq 0$. Both in these two models, the decay rate of QNM with $\mu \neq 0$ decreases with the increasing of the mass $\mu$ and oscillation frequency increases with the increasing of mass $\mu$.

5.2 Quasibound states and superradiant instabilities

In the previous subsection, we considered a scalar field and calculated the QNM of the black holes both in CDM and SFDM models. Here, we will continue to investigate the oscillatory behavior of black holes in these two models by considering a massive scalar field. The oscillation behavior at this time is called quasibound states (QBS). Unlike QNM, it behaves in a form of exponential decay as it is far away from the black hole. In other words, QBS are localized inside the potential well formed by the mass of the field. On the other hand, this oscillation frequency of the QBS usually satisfies the condition of superradiance,
which has the following form

\[ \mu < \text{Re}(\omega) < m\Omega. \]  

(5.2)

where, \( \Omega \) is the angular velocity of rotation. Similar to QNM, the imaginary part of QBS is less than zero \( \text{Im}(\omega) < 0 \), which corresponds to a stable mode. However, for an unstable mode, it corresponds to \( \text{Im}(\omega) > 0 \). In this case, superradiant instability emerges. If \( \text{Im}(\omega) = 0 \), it corresponds to the bound states of the so-called scalar clouds \([26, 64]\). As the same in the calculation of QNM, in the massive scalar field, we will give the real and imaginary parts of the QBS frequency, as a function of the mass \( \mu \). In order to further verify whether exist superradiant instability in these two models, we write the oscillation frequency and decay rate as \( \text{Re}(\omega)/\mu \) and \( \text{Im}(\omega)/\mu \), respectively, as a function of the mass parameter \( M\mu \). Then, we will test all the state for the angular quantum \( l = 1 \) in the case of the nearly extremal parameter \( a = 2.11 \) (CDM model) and \( a = 1.11 \) (SFDM model) and use the Eq.(5.1) as the initial guess. We show the results in Figs. 3 and 4. From these figures, we found that the imaginary part (right panel) of the QBS in these two models are negative numbers at the range of large mass, which indicates that QBS is a stable mode in the large mass. However, at the range of the low mass, the value of the decay rate is very close to 0 (may be greater than 0 or less than 0), and the former case may lead to the occurrence of superradiant instability. Next, we will discuss the oscillation behavior of black holes at the range of the low mass. Now, the evidences show that this superradiative instability most appear when the mode is \( l = 1, m = 1 \) \([65–67]\). So, in order to verify and analyze the unstable mode of the black hole in CDM and SFDM models, we will focus on the mode of \( l = 1, m = 1 \) in the following discussion. Here, the QBS frequencies of CDM and SFDM models for the state \( l = m = 1 \) with different rotation parameter \( a \) are given in Fig. 5 and 6, respectively. The results of these figures also reveal that superradiant instability can be be found at the range of low mass. For simplicity, we give the enlarged diagrams of the oscillation frequency and decay rate of extremal black holes of CDM and SFDM models for the state \( l = m = 1 \) in Figs. 7 and 8, respectively. From them, we find that for the CDM model, the superradiant instability occurs at \( M\mu \lesssim 0.1 \), and the SFDM model occurs at \( M\mu \lesssim 0.4 \). Besides, in Figs. 7 and 8, we also investigate the maximum instability of the CDM and SFDM models and use an iterative method to give the mass and growth rate corresponding to the maximum instability. For example, how to find the maximum instability of the state \( a = 1.11, l = 1, m = 1 \) in SFDM model. First of all, through the results in Fig. 3, we can roughly determine a range of mass corresponding to the maximum instability, and form an array of the mass and decay rate. Second, we reduce the mass stepwise and use the continued fraction method to determine the frequency of each step. where the initial guess of frequency is the frequency obtained in the previous step. Finally, we stop iterating as soon as a local maximum occurs among a series of frequencies. Following this procedure strictly, we obtain the maximum instability of extremal black holes for the state \( l = m = 1 \) in the CDM and SFDM models. For CDM model, the maximum instability occurs at \( M\mu \approx 0.0875 \) and the maximum growth rate is approximately \( \tau^{-1} = M\text{Im}(\omega) \approx 8.89 \times 10^{-10}(GM/c^2)^{-1} \). For SFDM model, the maximum instability occurs at \( M\mu \approx 0.375 \) and the maximum growth rate is approximately
Figure 3. The quasibound states frequencies for the $l = 1$ state with different $m$ at $a = 2.11$ in CDM model. The left panel is the oscillation frequency and the right panel is the decay rate. They are all the function of the mass parameter $M\mu$.

Figure 4. The quasibound state frequencies for the $l = 1$ state with different $m$ at $a = 1.11$ in SFDM model. The left panel is the oscillation frequency and the right panel is the decay rate. They are all the function of the mass parameter $M\mu$.

Table 10. The maximum instability growth rate for the state $l = m = 1$ in CDM model.

| $a$  | 1.7  | 1.8  | 1.9  | 2.0  | 2.1  | 2.11 |
|------|------|------|------|------|------|------|
| $\mu$ | 0.0400 | 0.0525 | 0.0575 | 0.0650 | 0.0825 | 0.0875 |
| $\tau^{-1}$ | $1.66 \times 10^{-13}$ | $8.40 \times 10^{-12}$ | $4.06 \times 10^{-11}$ | $1.26 \times 10^{-10}$ | $7.03 \times 10^{-10}$ | $8.89 \times 10^{-10}$ |

Table 11. The maximum instability growth rate for the state $l = m = 1$ in SFDM model.

| $a$  | 0.7  | 0.8  | 0.9  | 1.0  | 1.1  | 1.11 |
|------|------|------|------|------|------|------|
| $\mu$ | 0.145 | 0.175 | 0.205 | 0.255 | 0.350 | 0.375 |
| $\tau^{-1}$ | $8.14 \times 10^{-11}$ | $4.30 \times 10^{-10}$ | $2.15 \times 10^{-9}$ | $1.17 \times 10^{-8}$ | $9.49 \times 10^{-8}$ | $1.19 \times 10^{-7}$ |

$\tau^{-1} = M\text{Im}(\omega) \approx 1.19 \times 10^{-7}(GM/c^3)^{-1}$. Similarly, we also list the maximum instability of CDM and SFDM models with different rotation parameters $a$ at the state $l = 1, m = 1$ in Tables. 10 and 11, respectively. From the data from Tables. 10 and 11, both in CDM and SFDM models, the maximum instability increases with the increasing of the rotation parameter $a$. The value of the maximum instability of the SFDM model is greater than that of the CDM model.
\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure5a}
\includegraphics[width=0.45\textwidth]{figure5b}
\caption{The quasibound state frequencies of the state $l = 1, m = 1$ with different rotation parameter $a$ in CDM model. The left panel is the oscillation frequency and the right panel is the decay rate. They are all the function of the mass parameter $M\mu$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure6a}
\includegraphics[width=0.45\textwidth]{figure6b}
\caption{The quasibound states frequencies of the state $l = 1, m = 1$ with different rotation parameter $a$ in SFDM model. The left panel is the oscillation frequency and the right panel is the decay rate. They are all the function of the mass parameter $M\mu$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure7a}
\includegraphics[width=0.45\textwidth]{figure7b}
\caption{The quasibound states frequencies of a nearly extremal black hole ($a=2.11$) for the state $l = 1, m = 1$ in CDM model. The decay rate (right panel) is positive at low mass ($M\mu \lesssim 0.1$), creating the superradiant instability. The maximum instability occurs at $M\mu \approx 0.0875$. The maximum growth rate is approximately $\tau^{-1} = M\Im(\omega) \approx 8.89 \times 10^{-10} (GM/c^3)^{-1}$.}
\end{figure}
Figure 8. The quasibound state frequencies of a nearly extremal black hole \((a=1.11)\) for the state \(l = 1, m = 1\) in SFDM model. The decay rate (right panel) is positive at low mass \((M\mu \lesssim 0.4)\), creating the superradiant instability. The maximum instability occurs at \(M\mu \approx 0.375\). The maximum growth rate is approximately \(\tau^{-1} = M\Im(\omega) \approx 1.19 \times 10^{-7}(GM/c^3)^{-1}\).

6 Conclusions and discussions

In this paper, we mainly use the scalar field perturbation to study QNM and QBS of the Kerr-like black holes in a dark matter halo, and use the continued fraction method to calculate the frequencies of QNM and QBS. Besides, we also analyze and verify the existence of the superradiant instability. Next, I will give our conclusions and discussions from the QBS, superradiant instability, and the QNM.

First of all, the QBS both in CDM and SFDM models in massive field are stable except the range of the low mass. At the range of low mass, superradiative instabilities are generated. In other words, the superradiant instability can occur both in CDM and SFDM models. Especially, we have shown that the superradiant instability can occur at the range of the low mass in Fig. 3 and 4 for the state \(l = 1\) with the different \(m\). At the same time, we also give the QBS frequencies with the different rotating parameter \(a\) at the state \(l = 1, m = 1\) in Fig. 5 and 6. Maximum instability increases with the increasing of the rotation parameter \(a\). The most unstable mode are occurred at the state \(l = 1, m = 1\).

In Fig. 7 and 8, we have shown the maximum instability of CDM and SFDM models in the massive field. Besides, we also list the other maximum instability of CDM and SFDM models with different rotation parameters \(a\) for the state \(l = 1, m = 1\) in Tables. 10 and 11, respectively. The value of the maximum growth rate is positively correlated with the values of rotation parameter \(a\) and mass \(\mu\). In other words, both in CDM and SFDM models, the maximum instability increases with the increasing of the rotation parameter \(a\). The value of the maximum instability of the SFDM model is greater than that of the CDM model. For CDM model, the maximum instability occurs at \(M\mu \approx 0.0875\) and the maximum growth rate is approximately \(\tau^{-1} \approx 8.89 \times 10^{-10}(GM/c^3)^{-1}\) at the state \(l = 1, m = 1, a = 2.11\). For SFDM model, the maximum instability occurs at \(M\mu \approx 0.375\) and the maximum growth rate is approximately \(\tau^{-1} \approx 1.19 \times 10^{-7}(GM/c^3)^{-1}\) at the state \(l = 1, m = 1, a = 1.11\).

Secondly, the QNM both in CDM and SFDM models in massless field are stable. From data in Tables 2 and 3, the oscillation frequencies of the CDM and SFDM models increase with the increasing of the rotation parameter \(a\), while the decay rate decreases with the
increasing of the rotation parameter $a$ at the state $l = 1$. Meanwhile, in the massive scalar field, the QNM is still stable mode. In this stable mode, the oscillation frequencies of the CDM and SFDM model increase with the increasing of rotation parameter $a$, while the imaginary part of them decrease with the increasing of rotation parameter $a$ in the states of $(l = 0, m = 0, l = 1, m = 0$ and $l = 1, m = 1)$ in Tables. 4, 5, 6, 7, 8 and 9. Under the same black hole parameters, both of the oscillation frequency and decay rate of SFDM model are higher than that of CDM model. At the same time, we also give the QNM frequencies for the states $l = 0, m = 0$ and $l = 1, m = 0$. Their oscillation frequency increases with the increasing of the rotation parameter $a$, while the decay rate decreases with the increasing of $a$.

At last, it is also worth mentioning that we are very interested in the exploration of the relevant physical processes of black holes around the dark matter. Based on this premise, we choose to study the QNM and QBS of a rotating black hole in a dark matter halo. The QNM and QBS of a black hole are the characteristic “sound” which can provide us with a new method to identify black holes in the universe. On the other hand, in some recent studies, we found some discussion about echoes[68–73]. These studies show that the echoes appear after the quasinormal mode. Echoes are one of the important means currently used to test gravitational waves. Our next plans are to study the physics of rotating black holes associated with echoes. About the study of echoes, we have already made some preliminary attempts[74–76]. To sum up, we also hope that our work can form a complete research system in the direction of interaction between dark matter and black holes.

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