Notes on schematic finite spaces

F. Sancho and P. Sancho

February 19, 2021

Abstract

The schematic finite spaces are those finite ringed spaces where a theory of quasi-coherent modules can be developed with minimal natural conditions. We give various characterizations of these spaces and their natural morphisms. We show that schematic finite spaces are strongly related to quasi-compact quasi-separated schemes.

Introduction

In Topology, Differential Geometry and Algebraic Geometry, it is usual to study their geometric objects considering suitable finite open coverings and studying the associated finite ringed spaces. Let us remember how these finite ringed spaces are constructed:

1. Let $S$ be a topological space and let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite open covering of $S$. For each $s \in S$ define $U_s := \bigcap_{i \in I} U_i$. Observe that the topology generated by $\mathcal{U}$ is equal to the topology generated by $\{U_s\}_{s \in S}$. We shall say that $\mathcal{U}$ is a minimal open covering of $S$ if $U_i \neq U_j$ if $i \neq j$ and $U_i = U_s$ for some $s \in S$, for every $i$. Define the following equivalence relation on $S$: $s \sim s'$ if the covering $\mathcal{U}$ does not distinguish them, i.e., if $U_s = U_s'$. Consider on $S$ the topology generated by the covering $\mathcal{U}$ and let $X := S/\sim$ be the quotient topological space. $X$ is a finite $T_0$-topological space, then it is a finite poset as follows: $[s] \leq [s']$ if $U_{s'} \subseteq U_s$. Let $\pi: S \rightarrow X$, $s \mapsto [s]$ be the quotient morphism and let $U_{[s]} = \{[s'] \in S : [s'] \geq [s]\}$ be the minimal open neighborhood of $[s]$. One has that $\pi^{-1}(U_{[s]}) = U_s$. Suppose now that $(S, O_S)$ is a ringed space (a scheme, a differentiable manifold, an analytic space, etc.). We have then a sheaf of rings on $X$, namely $O := \pi_* O_S$, so that $\pi: (S, O_S) \rightarrow (X, O)$ is a morphism of ringed spaces. We shall say that $(X, O)$ is the ringed finite space associated with the finite covering $\mathcal{U}$. Observe that $O_{[s]} = O(U_{[s]}) = O_S(U_s)$.

To fix ideas, suppose that $S$ is a quasi-compact quasi-separated scheme (see [5] 1.2.1). There exists a minimal affine open covering $\mathcal{U} = \{U_{s_1}, \ldots, U_{s_n}\}$ of $S$ (see [9] 3.13). Consider the associated ringed finite space $X$. It is easy to prove that the functors $M \mapsto \pi_* M, N \mapsto \pi^* N$
establish an equivalence of categories between the category of quasi-coherent $O_S$-modules and the category of quasi-coherent $O_X$-modules. Besides, $H^i(S, \mathcal{M}) = H^i(X, \pi_*\mathcal{M})$ for any quasi-coherent $O_S$-module $\mathcal{M}$. Observe that for every $U_{sj}, U_{sj'} \subseteq U_{sj}$:

- The restriction morphism $O_S(U_{sj}) \rightarrow O_S(U_{sj})$ is a flat morphism, since the morphism $O_S(U_{sj}) \rightarrow (O_S(U_{sj}) \setminus \mathfrak{p})^{-1}O_S(U_{sj}) = O_{S, \mathfrak{p}} = (O_S(U_{sj}) \setminus \mathfrak{p})^{-1}O_S(U_{sj})$ is flat, for any $\mathfrak{p} \in U_{sj} = \text{Spec} O_S(U_{sj})$.

- The natural morphism $O(U_{sj}) \otimes_{O_S(U_{sj})} O_S(U_{sj}) \rightarrow O_S(U_{sj} \cap U_{sj'})$ is an isomorphism, because $U_{sj} \times_{U_{sj'}} U_{sj'} = U_{sj} \cap U_{sj'}$.

- The morphism $O_S(U_{sj} \cap U_{sj'}) \rightarrow \prod_{U_{sk} \subseteq U_{sj} \cap U_{sj'}} O_S(U_{sk})$ is faithfully flat: it is flat because $U_{sj} \cap U_{sj'}$ and $U_k$ are affine and it is faithfully flat because $\prod_{U_{sk} \subseteq U_{sj} \cap U_{sj'}} U_k \rightarrow U_{sj} \cap U_{sj'}$ is a surjective map.

Therefore, for any points $x_j, x_{j'} \geq x_i$ in $X$:

a. The natural morphism $O_{x_i} \rightarrow O_{x_j}$ is flat.

b. The natural morphism $O_{x_j} \otimes_{O_{x_j}} O_{x_{j'}} \rightarrow O(U_{x_j} \cap U_{x_{j'}})$ is an isomorphism.

c. The morphism $O(U_{x_j} \cap U_{x_{j'}}) \rightarrow \prod_{x_k \geq x_i, x_{j'}} O_{x_k}$ is faithfully flat.

2. We shall say that a ringed finite space is schematic if it satisfies a., b. and c. In [8] 4.4, 4.11, it is proved that a finite ringed space $X$ is schematic iff $X$ satisfies a. and $R^n i_! O_U$ is a quasi-coherent module for any $n$, where $\delta: X \rightarrow X \times X$ is the diagonal morphism. In [10] 4.5, it is proved that $X$ is schematic iff $X$ satisfies a. and $R^n i_! O_U$ is quasi-coherent for any open subset $U \subseteq X$ and for $n \in \mathbb{N}$. In Algebraic Geometry, it is usual to approach the study of schemes and their morphisms through the category of quasi-coherent modules, for example, the theory of intersection can be studied with the $K$-theory of quasi-coherent modules. We shall denote by $\text{Qc-Mod}_X$ the category of quasi-coherent $O_X$-modules. We prove that a finite ringed space $X$ is schematic iff $\text{Qc-Mod}_X$ satisfies minimal conditions:

- A ringed finite space $X$ is schematic iff for any morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ of quasi-coherent $O_X$-modules $\text{Ker} f$ is quasi-coherent, and $\delta_*(\mathcal{M}) \in \text{Qc-Mod}_{X \times X}$, for any $\mathcal{M} \in \text{Qc-Mod}_X$, where $\delta: X \rightarrow X \times X, \delta(x) = (x, x)$ is the diagonal morphism.

- A ringed finite space $X$ is schematic iff for any morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ of quasi-coherent $O_X$-modules $\text{Ker} f$ is quasi-coherent, and $i_*(\mathcal{M}) \in \text{Qc-Mod}_{U}$ for any $\mathcal{M} \in \text{Qc-Mod}_U$ and any open subset $U \subseteq X$. 

Likewise, we study and characterize affine finite spaces. Let us use the previous notations. It can be proved that $S$ is an affine scheme iff the morphisms $O(S) \rightarrow \prod_i O_{s_i}$ and $O(U_{s_i}) \otimes_{O(S)} O(U_{s_j}) \rightarrow \prod_{U_{s_i} \subset U_{s_j} \cap U_{s_j}} O(U_{s_k})$ are faithfully flat, for any $i, j$. We say that a finite ringed space $X$ is affine if

- The morphism $O(X) \rightarrow \prod_{x \in X} O_x$ is faithfully flat.

- The morphism $O_x \otimes_{O(X)} O_{x'} \rightarrow \prod_{x \in U_{x'}} \cap U_{x_{x'}} O_{x_{x'}}$ is faithfully flat, for any $x, x' \in X$.

Affine finite spaces are schematic and a finite ringed space $X$ is schematic iff $U_x$ is affine, for any $x \in X$. We prove that $X$ is affine iff $O(X) \rightarrow O_x$ is flat, $O_x(U_x \cap U_y) = O_x \otimes_{O(X)} O_y$, and $X$ and $U_x \cap U_y$ are acyclic, for any $x, y \in X$. A schematic space $X$ is affine iff $H^1(X, M) = 0$ for any quasi-coherent module $M$ (see [8] 5.11), which is equivalent to saying that the functor $M \rightarrow \Gamma(X, M)$ is exact (see Corollary 7.5).

3. Next, we study the morphisms between schematic finite spaces. Let $f: X \rightarrow Y$ be a morphism of ringed spaces between schematic finite spaces. We say that $f$ is affine if $f_* O_X$ is a quasi-coherent module and $f^{-1}(U)$ is affine, for any affine open subset $U \subset Y$. We prove that $f$ is affine if and only if $f_*$ preserves quasi-coherence and it is an exact functor.

We say that $f$ is schematic if $f_* O_X$ is quasi-coherent and the morphism $U_x \rightarrow U_{f(x)}$, $x' \mapsto f(x')$ is affine, for any $x \in X$. We prove that the following statements are equivalent:

- $f$ is schematic.

- $f_*$ preserves quasi-coherence.

- The natural flat morphism $O_x \otimes_{O_{f(x)}} O_{y} \rightarrow \prod_{x \in U_{x'} \cap f^{-1}(U_y)} O_{x'}$ is faithfully flat, for any $x \in X$ and $y \geq f(x)$.

- $R^n \Gamma_{f_*} O_X \in \text{Qc-Mod}_{X \times Y}$ for any $n$, where $\Gamma_f: X \rightarrow X \times Y$, $\Gamma_f(x) = (x, f(x))$ is the graph of $f$.

- $R^n (f \circ i)_* O_U \in \text{Qc-Mod}_X$, for any open subset $U \rightarrow X$ and any $n \geq 0$.

4. Now, we ask ourselves whether schematic finite spaces are determined by the category of their quasi-coherent modules. A morphism of schemes $F: S \rightarrow T$ between quasi-compact quasi-separated schemes is an isomorphism if and only if the functors

$$
\text{Qc-Mod}_S \xrightarrow{F_*} \text{Qc-Mod}_T
$$

are mutually inverse. Given a schematic morphism $f: X \rightarrow Y$, we prove that the following statements are equivalent
1. Finite ringed spaces: basic notions

- The functors $\mathbf{Qc-Mod}_X \xrightarrow{f^*} \mathbf{Qc-Mod}_Y$ are mutually inverse.

- $O_Y = f_*O_X$ and $f$ is affine.

- Essentially, $f$ is the quotient morphism defined on $X$ by a minimal affine open covering.

- The cylinder $C(f) = X \sqcup Y$ of $f$ is a schematic space and $f$ is a faithfully flat morphism.

A morphism that satisfies any of these statements will be called quasi-isomorphism.

5. Let us talk less accurately. Given a schematic finite space $X$, consider the ringed space

$$\tilde{X} := \lim_{\to} \text{Spec} \ O_x.$$ 

Spec $O_x$ is a subspace of $\tilde{X}$ via the natural morphism $\tilde{i}_x : \text{Spec} \ O_x \to \tilde{X}$ and $\tilde{X} = \bigcup_{x \in X} \text{Spec} \ O_x$. $\tilde{X}$ is quasi-compact, the set of its quasi-compact open subsets is a basis of its topology and the intersection of two quasi-compact open subsets is quasi-compact. Given a quasi-coherent $O_X$-module $M$, let $\tilde{M}_x$ be the $O_{\text{Spec} \ O_x}$-module of localizations of $M_x$ and consider the $O_{\tilde{X}}$-module $\tilde{M} := \lim_{i_x} \tilde{M}_x$. We prove that $H^n(X, M) = H^n(\tilde{X}, \tilde{M})$, for any $n \geq 0$, and that the category of quasi-coherent modules of $X$ is equivalent to the category of quasi-coherent modules of $\tilde{X}$. Let $X$ and $Y$ be schematic finite spaces. We say that a morphism of ringed spaces $f : \tilde{X} \to \tilde{Y}$ is schematic if $f_*$ preserves quasi-coherence. Let $C_W$ be the category of schematic finite spaces localized by the quasi-isomorphisms. We prove that

$$\text{Hom}_{C_W}(X, Y) = \text{Hom}_{sch}(\tilde{X}, \tilde{Y}).$$

1 Finite ringed spaces: basic notions

Let $X$ be a finite set. It is well known (1) that giving a topology on $X$ is equivalent to giving a preorder relation on $X$:

$$x \leq y \iff \bar{x} \subseteq \bar{y}, \quad \text{where} \ \bar{x}, \bar{y} \text{ are the closures of } x \text{ and } y.$$ 

In addition, the topology is $T_0$ if and only if the preorder is a partial order (i.e. it satisfies antisymmetric property).

Let $X$ be a finite topological space. For each point $p \in X$, let us denote

$$U_x = \text{smallest open subset containing } x,$$
that is, \( U_x = \{ y \in X : y \geq x \} \). Then, \( x \leq y \iff U_y \subseteq U_x \). The family of open subsets \( \{ U_x \}_{x \in X} \) constitutes a minimal basis of open subsets of \( X \) (any other base contains this one).

A map \( f : X \to Y \) between finite topological spaces is continuous if and only if it is monotone (i.e. \( x \leq y \) implies \( f(x) \leq f(y) \)).

Let \( X \) be a finite topological space and let \( F \) be a sheaf of abelian groups (resp. rings, etc.) on \( X \). The stalk of \( F \) at \( x \in X \), \( F_x \), is a ringed space \( (X, \mathcal{O}_X) \) is a ringed space \( (X, \mathcal{O}_X) \), which satisfies: \( r_{xy} : F_x \to F_y \) is just the restriction morphism \( F(U_x) \to F(U_y) \), which satisfies: \( r_{xx} = \text{Id} \) for any \( x \), and \( r_{yz} \circ r_{xy} = r_{xz} \) for any \( x \leq y \leq z \).

Conversely, consider the following data:
- An abelian group (resp. a ring, etc) \( F_x \) for each \( x \in X \).
- A morphism of groups (resp. rings, etc) \( r_{xy} : F_x \to F_y \) for each \( x \leq y \), satisfying: \( r_{xx} = \text{Id} \) for any \( x \), and \( r_{yz} \circ r_{xy} = r_{xz} \) for any \( x \leq y \leq z \).

Let \( \mathcal{F} \) be the following presheaf of groups (resp. rings, etc.): For each open subset \( U \subset X \), \( \mathcal{F}(U) : = \lim_{\rightarrow \ x \in U} F_x \). It is easy to prove that \( \mathcal{F}_x = F_x \) and that \( \mathcal{F} \) is a sheaf.

1. **Definition**: A ringed space is a pair \( (X, \mathcal{O}) \), where \( X \) is a topological space and \( \mathcal{O} \) is a sheaf of (commutative with unit) rings on \( X \). A morphism of ringed spaces \( (X, \mathcal{O}) \to (X', \mathcal{O}') \) is a pair \( (f, f^\#) \), where \( f : X \to X' \) is a continuous map and \( f^\# : \mathcal{O}' \to f_* \mathcal{O} \) is a morphism of sheaves of rings (equivalently, a morphism of sheaves of rings \( f^{-1} \mathcal{O}' \to \mathcal{O} \)). A finite ringed space is a ringed space \( (X, \mathcal{O}) \) whose underlying topological space \( X \) is finite.

A morphism of ringed spaces \( (X, \mathcal{O}) \to (X', \mathcal{O}') \) between two finite ringed spaces is equivalent to the following data:
- a continuous (i.e. monotone) map \( f : X \to X' \),
- for each \( x \in X \), a ring homomorphism \( f^\#_x : \mathcal{O}'_{f(x)} \to \mathcal{O}_x \), such that, for any \( x \leq y \), the diagram

\[
\begin{array}{ccc}
\mathcal{O}'_{f(x)} & \xrightarrow{f^\#_x} & \mathcal{O}_x \\
\downarrow r_{f(x)f(y)} & & \downarrow r_{xy} \\
\mathcal{O}'_{f(y)} & \xrightarrow{f^\#_y} & \mathcal{O}_y 
\end{array}
\]

is commutative. We denote by \( \text{Hom}(X,Y) \) the set of morphisms of ringed spaces between two ringed spaces \( X \) and \( Y \).

2. **Example**: Let \( \{ * \} \) be the topological space with one element. We denote by \( (*, R) \) the finite ringed space whose underlying topological space is \( \{ * \} \) and the sheaf of rings is a ring \( O_* = R \). For any ringed space \( (X, \mathcal{O}) \) there is a natural morphism of ringed spaces \( (X, \mathcal{O}) \to (*, O(X)) \).

Let \( (X, \mathcal{O}) \) be a finite ringed space. A sheaf \( \mathcal{M} \) of \( \mathcal{O} \)-modules (or \( \mathcal{O} \)-module) is equivalent
to these data: an \( \mathcal{O}_x \)-module \( \mathcal{M}_x \) for each \( x \in X \), and a morphism of \( \mathcal{O}_x \)-modules \( r_{xy} : \mathcal{M}_x \to \mathcal{M}_y \) for each \( x \leq y \), such that \( r_{xx} = \text{Id} \) and \( r_{xz} = r_{yz} \circ r_{xy} \) for any \( x \leq y \leq z \). Again, one has that \( \mathcal{M}_x = \text{stalk of } \mathcal{M} \text{ at } x = \mathcal{M}(U_x) \) and \( r_{xy} \) is the restriction morphism \( \mathcal{M}(U_x) \to \mathcal{M}(U_y) \).

For each \( x \leq y \) the morphism \( r_{xy} \) induces a morphism of \( \mathcal{O}_y \)-modules \( \overline{r}_{xy} : \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_y \to \mathcal{M}_y \).

An \( \mathcal{O} \)-module \( \mathcal{M} \) is said to be quasi-coherent if for any \( x \in X \) there exist an open neighbourhood \( U \) of \( x \) and an exact sequence of \( \mathcal{O}_|U| \)-modules

\[
\mathcal{O}_i|U| \to \mathcal{O}_j|U| \to \mathcal{M}|U| \to 0.
\]

3. **Theorem ([9] 3.6):** Let \((X, \mathcal{O})\) be a finite ringed space. An \( \mathcal{O} \)-module \( \mathcal{M} \) is quasi-coherent if and only if for any \( x \leq y \) the morphism

\[\overline{r}_{xy} : \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_y \to \mathcal{M}_y\]

is an isomorphism.

**Proof.** \( \Rightarrow \) Let \( U \) be an open neighbourhood of \( p \) such that there exists an exact sequence

\[
\mathcal{O}_i|U| \to \mathcal{O}_j|U| \to \mathcal{M}|U| \to 0.
\]

We can suppose \( X = U \). Obviously, \((\mathcal{O}_i, \mathcal{O}_j) = (\mathcal{O}_x, \mathcal{O}_y) = (\mathcal{O}_x \otimes_{\mathcal{O}_x} \mathcal{O}_y) = (\mathcal{O}_y) \) and \((\mathcal{O}_i, \mathcal{O}_j) = (\mathcal{O}_x, \mathcal{O}_y) = (\mathcal{O}_y) \), then \( \mathcal{M}_x \otimes_{\mathcal{O}_x} \mathcal{O}_y = \mathcal{M}_y \).

\( \Leftarrow \) Given \( x \in X \), consider an exact sequence of \( \mathcal{O}_x \) modules \( \mathcal{O}_i \to \mathcal{O}_j \to \mathcal{M}_i \to 0 \). Tensoring by \( \otimes_{\mathcal{O}_x} \mathcal{O}_y \), for any \( x \leq y \), one has the exact sequence \( \mathcal{O}_y \to \mathcal{O}_y \to \mathcal{M}_y \to 0 \). Then, one has a sequence of morphisms

\[
\mathcal{O}_i|U_x| \to \mathcal{O}_j|U_x| \to \mathcal{M}|U_x| \to 0
\]

which is exact since it is exact on stalks at \( y \), for any \( y \in U_x \). Therefore, \( \mathcal{M} \) is quasi-coherent. \( \square \)

We shall denote by \( \mathbf{Mod}_X \) the category of \( \mathcal{O} \)-modules on a ringed space \((X, \mathcal{O})\) and by \( \mathbf{Qc-Mod}_X \) the subcategory of quasi-coherent \( \mathcal{O} \)-modules. Also, for any ring \( R \), we shall denote by \( \mathbf{Mod}_R \) the category of \( R \)-modules.
4. Remarks: a) If \( f : X \to Y \) is a morphism of ringed spaces and \( N \) is a quasi-coherent \( O_Y \)-module, then \( f^* N := f^{-1} N \otimes_{f^{-1} O_Y} O_X \) is a quasi-coherent \( O_X \)-module. In particular, this is true for morphisms between finite ringed spaces.
b) If \( f : M \to N \) is a morphism of \( O_X \)-modules where \( M \) and \( N \) are quasi-coherent, then \( \text{Coker} \ f \) is quasi-coherent. However, it is not always true that \( \text{Ker} \ f \) is quasi-coherent.

5. Example: Let \((X, O)\) be a finite ringed space and \( \pi : (X, O) \to (\ast, O(X)) \) the natural morphism of ring. If \( M \) is an \( O(X) \)-module, then \( \pi^* M \) is a quasi-coherent \( O_X \)-module, which we denote \( \tilde{M} \). We say that \( \tilde{M} \) is the quasi-coherent module associated with \( M \) and we have a functor \( \pi^* : \text{Mod}_R \to \text{Qc-Mod}_X, M \mapsto \tilde{M} \). Note that \( \tilde{M}_x = M \otimes_{O(X)} O_x \), for each \( x \in X \).

6. Definition: A finite ringed space \((X, O)\) is a finite flat-restriction space (or finite fr-space) if the restriction morphisms \( r_{xy} : O_x \to O_y \) are flat, for any \( x \leq y \).

7. Proposition: Let \((X, O)\) be a finite ringed space.

\((X, O)\) is a finite fr-space \( \iff \) For any open subset \( U \) (resp. \( U_x \)) of \( X \) and any morphism \( f : M \to N \) of quasi-coherent \( O_U \)-modules (resp. \( O_{U_x} \)-modules), \( \text{Ker} \ f \) is quasi-coherent.

Proof. \( \Rightarrow \) Let \( f : M \to N \) be a morphism of quasi-coherent \( O_U \)-modules. We have to prove that, for each \( x \leq y \in U \), the morphism

\[ \tilde{r}_{xy} : (\text{Ker} \ f)_x \otimes_{O_x} O_y \to (\text{Ker} \ f)_y \]

is an isomorphism. This follows from the next commutative diagram of exact rows

\[
\begin{array}{ccc}
0 & \xrightarrow{} & (\text{Ker} \ f)_x \otimes_{O_x} O_y & \xrightarrow{r_{xy}} & M_x \otimes_{O_x} O_y & \xrightarrow{f_x} & N_x \otimes_{O_x} O_y & \xrightarrow{r_{xy}} & 0 \\
& & \downarrow{r_{xy}} & & \downarrow{r_{xy}} & & \downarrow{r_{xy}} & & \\
0 & \xrightarrow{} & (\text{Ker} \ f)_y & \xrightarrow{f_y} & M_y & \xrightarrow{f_y} & N_y & \\
\end{array}
\]

in which the first row is exact because \( O_x \to O_y \) is a flat morphism and the second and the third vertical morphisms are isomorphisms because \( M \) and \( N \) are quasi-coherent \( O_U \)-modules.

\( \Leftarrow \) Let \( x \leq y \in X \). Let \( f_x : M_x \hookrightarrow N_x \) be an injective morphism of \( O_x \)-modules. We have to prove that \( f_x \otimes 1 : M_x \otimes_{O_x} O_y \to N_x \otimes_{O_x} O_y \) is still injective. Consider the open subset \( U_x \) of \( X \) and the functor \( \text{Mod}_{O_x} \to \text{Qc-Mod}_{U_x}, M_x \mapsto \tilde{M}_x \). Then, the morphism \( f_x \) gives us a morphism \( \tilde{f}_x : \tilde{M}_x \to \tilde{N}_x \) of quasi-coherent \( O_{U_x} \)-modules. Note that \((\tilde{f}_x)_y = f_x \otimes 1 \). Since by hypothesis \( \text{Ker} \ \tilde{f}_x \) is quasi-coherent, we have that:

\[ \text{Ker}(f_x \otimes 1) = (\text{Ker} \ \tilde{f}_x)_y = \text{Ker} \ f_x \otimes_{O_x} O_y = 0 \otimes_{O_x} O_y = 0, \]

so we conclude that \( f_x \otimes 1 \) is injective. \( \square \)
8. **Remark**: Let \((X, O)\) be a finite ringed space. The proposition above says that \(\text{Qc-Mod}_U\) is an abelian category for each open subset \(U\) of \(X\) if and only if \((X, O)\) is a finite flat-restriction space. It is also true that in this case \(\text{Qc-Mod}_U\) is a Grothendieck category (see [3]).

2. **Affine finite spaces**

1. **Notation**: Let \((X, O)\) be a finite ringed space. For each \(x, y \in X\), let us denote \(U_{xy} = U_x \cap U_y\) and \(O_{xy} = O(U_{xy})\). If \(M\) is an \(O\)-module, we denote \(M_{xy} = M(U_{xy})\).

2. **Definition**: A finite ringed space \((X, O)\) is called an affine (schematic) finite space if it satisfies the following conditions:
   
   1. \(O(X) \to \prod_{x \in X} O_x\) is faithfully flat, for any \(x \in X\).
   2. \(O_x \otimes_{O(X)} O_y = O_{xy}\), for any \(x, y \in X\).
   3. \(O_{xy} \to \prod_{z \in U_{xy}} O_z\) is faithfully flat, for any \(x, y \in X\).

3. **Proposition**: If \((X, O)\) is an affine finite space, then it is a finite fr-space.

   **Proof.** By condition 3. of the definition above, \(O_{xx} = O_x \to \prod_{z \in U_x} O_z\) is faithfully flat. Therefore, \(O_x \to O_z\) is flat, for any \(z \geq x\). \(\square\)

4. **Proposition ([8] 4.12)**: Let \((X, O)\) be a ringed finite space. \(X\) is affine iff
   
   1. The morphism \(O(X) \to \prod_{x \in X} O_x\) is faithfully flat.
   2. The morphism \(O_y \otimes_{O(X)} O_{y'} \to \prod_{z \in U_{y'y'}} O_z\) is faithfully flat, for any \(y, y' \in X\).

   **Proof.** \(\Rightarrow\) It follows immediately from the definition.

   \(\Leftarrow\) In first place, note that for any \(x \leq u, u'\), the morphism \(O_u \otimes_{O(X)} O_{u'} \to O_u \otimes_{O_x} O_{u'}\) is an epimorphism and the composite morphism

   \[
   O_u \otimes_{O(X)} O_{u'} \to O_u \otimes_{O_x} O_{u'} \to \prod_{z \in U_{u'u'}} O_z
   \]

   is injective, because it is faithfully flat. Therefore, \(O_u \otimes_{O(X)} O_{u'} = O_u \otimes_{O_x} O_{u'}\).

   We only have to prove that

   \[
   O_y \otimes_{O(X)} O_{y'} = O_{yy'}
   \]

   Let us prove it by reduction to absurdity. Let \(y, y' \in X\) be maximal such that the morphism \(O_y \otimes_{O(X)} O_{y'} \to O_{yy'}\) is not an isomorphism.
First, if \( y \leq y' \), then \( U_{yy'} = U_{y'} \) and the epimorphism \( O_y \otimes_{O(X)} O_{y'} \to O_{y'} \) is faithfully flat, by 2. Therefore, \( O_y \otimes_{O(X)} O_{y'} = O_{y'} \). So, neither \( y \leq y' \), nor \( y' \leq y \). The morphism \( B := O_y \otimes_{O(X)} O_{y'} \to \prod_{z \in U_{yy'}} O_z \rightleftharpoons C \) is faithfully flat. Thus, the sequence of morphisms

\[
(*) \quad B \to C \xrightarrow{\pi} C \otimes_B C
\]

is exact. By the maximality of \( y \) and \( y' \), given \( z, z' \in U_{yy'} \), \( O_z \otimes_{O(X)} O_{z'} = O_z \otimes_{O_x} O_{z'} = O_{zz'} \). The natural morphism \( O_z \otimes_{O(X)} O_{z'} \to O_z \otimes_B O_{z'} \) is surjective. The composite morphism \( O_z \otimes_{O(X)} O_{z'} \to O_z \otimes_B O_{z'} \) is an isomorphism, then \( C \otimes_B \otimes_B \otimes_B \otimes_B C \) is exact. By the maximality of \( y \) and \( y' \), we have come to contradiction. \( \Box \)

5. **Corollary**: A finite ringed space \((U_x, O)\) is affine iff the morphism \( O_y \otimes_{O_x} O_y \to \prod_{z \in U_{yy'}} O_z \) is faithfully flat, for any \( y, y' \geq x \).

**Proof.** It follows easily from the proposition above. \( \Box \)

6. **Proposition**: Let \( X \) be an affine finite space and \( U \subseteq X \) an open set. Then, \( U \) is affine iff \( O(U) \to \prod_{q \in U} O_q \) is a faithfully flat morphism.

**Proof.** \( \Rightarrow \) \( O(U) \to \prod_{q \in U} O_q \) is a faithfully flat morphism by definition of affine finite space.

\( \Leftarrow \) We have to check that \( U \) satisfies the conditions 2. and 3. of Definition 2.2. Condition 3. is clear, because \( X \) is affine. Now, let us check 2.: for each \( x, y \in U \), the morphism \( O_x \otimes_{O(X)} O_y \to O_x \otimes_{O(U)} O_y \) is surjective and the composite morphism

\[
O_x \otimes_{O(X)} O_y \to O_y \otimes_{O(U)} O_y \to O_{xy}
\]

is an isomorphism, thus \( O_x \otimes_{O(U)} O_y \otimes_{O_x} O_{xy} \). \( \Box \)

7. **Corollary**: If \( X \) is an affine finite space, then \( U_{xy} \) is affine for every \( x, y \in X \).

**Proof.** It follows from condition 3. of Definition 2.2 and the proposition above. \( \Box \)

8. **Proposition**: Let \( X \) be an affine finite space and \( M \) a quasi-coherent \( O_X \)-module. The natural morphism

\[
M(V) \otimes_{O(X)} O(U) \to M(U \cap V)
\]

is an isomorphism, for any open set \( V \subseteq X \) and any affine open set \( U \subseteq X \).

**Proof.** 1. The morphism \( O(X) \to \prod_{x \in X} O_x =: B \) is faithfully flat. The sequence of morphisms

\[
O(X) \to B = \coprod_{x \in X} O_x \xrightarrow{\pi} B \otimes_{O(X)} B = \prod_{x, y \in X} O_{xy}
\]

is a split sequence of morphisms under a faithfully flat base change \((O(X) \to B)\), then this sequence of morphisms is universally exact, i.e., if we tensor the sequence of morphisms
by $M \otimes_C -$ (where $C$ is a commutative ring, $M$ is a $C$-module and $O(X)$ a $C$-algebra) then we obtain an exact sequence of morphisms. In particular, $O_{xy} \to \prod_{z \in U_{xy}} O_z$ is universally injective and the sequence of morphisms

\[(*) \quad O(X) \to \prod_{x \in X} O_x \longrightarrow \prod_{x,y \in X, z \notin U_{xy}} O_z\]

is universally exact.

2. Let $W \subset U_x$ be an affine open set. Consider the universally exact sequence of morphisms

$$O(W) \to \prod_{z \in W} O_z \longrightarrow \prod_{z, z' \in W, z' \notin U_{z'}} O_{z'}.$$ Tensoring by $M_x \otimes_{O_x} -$, we obtain the exact sequence of morphisms

$$M_x \otimes_{O_x} O(W) \to \prod_{z \in W} M_z \longrightarrow \prod_{z, z' \in W, z' \notin U_{z'}} M_{z'},$$

which shows that $M_x \otimes_{O_x} O(W) = M(W)$. Therefore (using Corollary 2.7),

$$M_x \otimes_{O(X)} O_y = M_x \otimes_{O_x} O_x \otimes_{O(O(X))} O_y = M_x \otimes_{O_x} O_{xy} = M_{xy}.$$

3. Consider the exact sequence of morphisms

$$M(V) \to \prod_{y \in V} M_y \longrightarrow \prod_{y, y' \in V, z \notin U_{y'y'}} M_z,$$

Tensoring by $\otimes_{O(X)} O_x$, we obtain the exact sequence of morphisms

$$M(V) \otimes_{O(X)} O_x \to \prod_{y \in V} M_{xy} \longrightarrow \prod_{y, y' \in V, z \notin U_{y'y'}} M_{xz}.$$ which shows that $M(V) \otimes_{O(X)} O_x = M(V \cap U_x)$.

4. Consider the universally exact sequence $(*)$, where $X = U$. Tensoring by $M(V) \otimes_{O(X)}$, we obtain the exact sequence of morphisms

$$M(V) \otimes_{O(X)} O(U) \to \prod_{x \in U} M(V \cap U_x) \longrightarrow \prod_{x, y \in U_x, z \notin U_{xy}} M(V \cap U_z),$$

which shows that $M(V) \otimes_{O(X)} O(U) = M(V \cap U)$. □
9. **Theorem (8, 2.5, 4.12):** Let \((X, O)\) be an affine finite space. Consider the canonical morphism
\[
\pi: (X, O) \to (\ast, O(X)), \quad \pi(x) = \ast, \text{ for any } x \in X.
\]
The functors
\[
\begin{align*}
\Qc\text{-Mod}_X & \xrightarrow{\pi_*} \text{Mod}_{O(X)}, & M \xmapsto{\pi_* M = M(X)} \\
\text{Mod}_{O(X)} & \xrightarrow{\pi^*} \Qc\text{-Mod}_X, & M \xmapsto{\pi^* M = \tilde{M}}
\end{align*}
\]
establish an equivalence between the category of quasi-coherent \(O_X\)-modules and the category of \(O(X)\)-modules.

**Proof.** The natural morphism \(\pi^* \pi_* M \to M\) is an isomorphism because this morphism on stalks at \(x\) is the morphism \(M(X) \otimes_{O(X)} O_x \to M_x\), which is an isomorphism by Proposition [2.8].

The natural morphism \(M \to \pi^* \pi_* M = (\pi^* M)(X)\) is an isomorphism: Tensoring the exact sequence of morphisms \((\ast)\), in the proof of Proposition [2.8] by \(M \otimes_{O(X)} -\) we obtain the exact sequence of morphisms
\[
M \otimes_{O(X)} O(X) \to \prod_{x \in X} (\pi^* M)_x \xrightarrow{\sim} \prod_{x, y \in X, z \in U_{x, y}} (\pi^* M)_z
\]
which shows that \(M = M \otimes_{O(X)} O(X) = (\pi^* M)(X)\).

\[
\square
\]

10. **Lemma:** Let \(A \to B\) and \(A' \to B'\) be flat (resp. faithfully flat) morphisms of commutative \(C\)-algebras. Then, \(A \otimes_C A' \to B \otimes_C B'\) is a flat morphism (resp. faithfully flat).

**Proof.** It follows from the equality \(M \otimes_{A \otimes_C A'} (B \otimes_C B') = (M \otimes_A B) \otimes_{A'} B'\).

\[
\square
\]

11. **Proposition:** The intersection of two affine open sets of an affine finite space is affine.

**Proof.** Let \(U\) and \(U'\) be two affine open sets of the affine finite space \(X\). Consider the faithfully flat morphisms \(O(U) \to \prod_{x \in U} O_x, \quad O(U') \to \prod_{x' \in U'} O_{x'}\). The composition of the faithfully flat morphisms (recall Lemma [2.10])
\[
O(U' \cap U) \xrightarrow{[2.8]} O(U) \otimes_{O(X)} O(U') \to \prod_{(x, x') \in U \times U'} O_{x, x'} \xrightarrow{\sim} \prod_{(x, x') \in U \times U', z \in U_{x, x'}} O_z,
\]
is faithfully flat, hence \(O(U' \cap U) \to \prod_{z \in U \cap U'} O_z\) is faithfully flat. By Proposition [2.6], \(U \cap U'\) is affine.

\[
\square
\]
2.1 Some commutative algebra results.

Let $R$ be a commutative ring with a unit. A finite $R$-ringed space is a finite ringed space $(X, \mathcal{O})$ such that $\mathcal{O}$ is a sheaf of $R$-algebras; that is, for any $x \in X$, $\mathcal{O}_x$ is an $R$-algebra and for any $x \leq x'$, $r_{x',x} : \mathcal{O}_x \to \mathcal{O}_{x'}$ is a morphism of $R$-algebras.

Let $X$ and $Y$ be two finite $R$-ringed spaces. The direct product $X \times_R Y$ is the finite $R$-ringed space $(X \times Y, \mathcal{O}_{X \times Y})$, where $(\mathcal{O}_{X \times Y})(x,y) := \mathcal{O}_x \otimes_R \mathcal{O}_y$, for each $(x, y) \in X \times Y$ and the morphisms of restriction are the obvious ones.

12. **Proposition ([10] 5.27):** Let $X$ and $Y$ be affine finite $R$-ringed spaces. Then, $X \times_R Y$ is an affine finite space and $\mathcal{O}(X \times_R Y) = \mathcal{O}(X) \otimes_R \mathcal{O}(Y)$.

**Proof.** Consider the universally exact sequence $\mathcal{O}(X) \to \prod_{x \in X} \mathcal{O}_x \otimes_{\mathcal{O}_{x'}} \mathcal{O}_{x'}$. Tensoring by $\otimes_R \mathcal{O}_y$ we obtain the exact sequence

$$\mathcal{O}(X) \otimes_R \mathcal{O}_y \to \prod_{x \in X} \mathcal{O}(U_x \times_R U_y) \otimes_{\mathcal{O}_{x'} \otimes_R \mathcal{O}_{y'}} \mathcal{O}_{x'} \mathcal{O}_{y'}.$$

Hence, $\mathcal{O}(X) \otimes_R \mathcal{O}_y = \mathcal{O}(X \times_R U_y)$. Consider the universally exact sequence $\mathcal{O}(Y) \to \prod_{y \in Y} \mathcal{O}_{x} \otimes_{\mathcal{O}_{y'}} \mathcal{O}_{y'}$. Tensoring by $\mathcal{O}(X) \otimes_R$ we obtain the exact sequence

$$\mathcal{O}(X) \otimes_R \mathcal{O}(Y) \to \prod_{y \in Y} \mathcal{O}(X \times_R U_y) \otimes_{\mathcal{O}_{x'} \otimes_R \mathcal{O}_{y'}} \mathcal{O}_{x'} \mathcal{O}_{y'}.$$

Hence, $\mathcal{O}(X) \otimes_R \mathcal{O}(Y) = \mathcal{O}(X \times_R Y)$. In particular, $\mathcal{O}_{x' \times} \otimes_R \mathcal{O}_{y' \times} = \mathcal{O}_{(x,y)(x',y')}$, for any $x, x' \in X$ and $y, y' \in Y$. By Lemma [2.10] the morphism

$$\mathcal{O}(X \times_R Y) = \mathcal{O}(X) \otimes_R \mathcal{O}(Y) \to \prod_{x \in X} \mathcal{O}_x \otimes_{\mathcal{O}_{y}} \mathcal{O}_y \mathcal{O}_{(x,y)}$$

is faithfully flat. By Lemma [2.10], the morphism

$$\mathcal{O}_{(x,y)(x',y')} = \mathcal{O}_{x' \times} \otimes_R \mathcal{O}_{y' \times} \to \prod_{z \in U_{x' \times}} \mathcal{O}_z \otimes_{\mathcal{O}_{y' \times}} \mathcal{O}_{z'} = \prod_{(z,z') \in U_{x' \times} \times U_{y' \times}} \mathcal{O}_{(z,z')} \mathcal{O}_{(z,z')}$$

is faithfully flat. Therefore, $X \times_R Y$ is affine.

\[\square\]

2.1 Some commutative algebra results.

If $X$ is an affine finite space then, for each $x \leq y \in X$, the morphism $\mathcal{O}_x \to \mathcal{O}_y$ is flat and $\mathcal{O}_y \otimes_{\mathcal{O}_x} \mathcal{O}_y = \mathcal{O}_{y_\times} = \mathcal{O}_y$. In this subsection we study this kind of morphisms. In this paper, we use well-known properties of flat morphisms and faithfully flat morphisms, that can be found in [7].
13. **Notation**: Given $p \in \text{Spec } R$ and an $R$-module $M$, we denote $M_p := (R \setminus p)^{-1} \cdot M$.

14. **Proposition**: Let $f: A \to B$ be a morphism of rings and $f^*: \text{Spec } B \to \text{Spec } A$ the induced morphism. The following conditions are equivalent:

1. $A \to B$ is a flat morphism and $B \otimes_A B = B$.
2. $A_{f^*(p)} = B_{f^*(p)}$ for all $p \in \text{Spec } B$.
3. The morphism $f^*$: Spec $B \to \text{Spec } A$ is injective and $A_{f^*(p)} = B_p$ for any $p \in \text{Spec } B$.

**Proof.**

1. $\Rightarrow$ 2. The morphism $A_{f^*(p)} \to B_{f^*(p)}$ is faithful flat, for any $p$. Besides, $B_{f^*(p)} = A_{f^*(p)} \otimes_A B = A_{f^*(p)} \otimes_A (B \otimes_A B) = B_{f^*(p)} \otimes_{A_{f^*(p)}} B_{f^*(p)}$, then $A_{f^*(p)} = B_{f^*(p)}$.

2. $\Rightarrow$ 3. If $A_{f^*(p)} = B_{f^*(p)}$, then $B_{f^*(p)} = B_p$ and $f^{-1}(f^*(p)) = \{p\}$, then $f^*$ is injective.

3. $\Rightarrow$ 1. The morphism $A \to B$ is flat: Given an injective morphism $N \hookrightarrow M$ of $A$-modules, $N_{f^*(p)} \to M_{f^*(p)}$ is injective, for any $p$. Then, $N \otimes_A B_p \to M \otimes_A B_p$ is injective, for any $p$ and $N \otimes_A B \to M \otimes_A B$ is injective.

   Spec $B_p \subseteq \text{Spec } B_{f^*(p)} \subseteq \text{Spec } A_{f^*(p)}$ and Spec $B_{f^*(p)} = \text{Spec } A_{f^*(p)}$. Hence, Spec $B_p = \text{Spec } B_{f^*(p)}$ and $B_{f^*(p)} = B_p$. Then, $(B \otimes_A B)_p = (B \otimes_A B) \otimes_B B_p = (B \otimes_A B) \otimes_B B_{f^*(p)} = (B \otimes_A B) \otimes_A A_{f^*(p)} B_{f^*(p)} = B_p$, for any $p \in \text{Spec } B$. Therefore, $B \otimes_A B = B$.

\[ \square \]

15. **Notation**: Given a morphism $f: A \to B$ and an ideal $I \subseteq B$ denote $A \cap I := f^{-1}(I)$. Denote $(I)_0 = \{p \in \text{Spec } B: I \subseteq p\}$.

16. **Proposition**: Let $A \to B$ be a flat morphism of rings such that $B \otimes_A B = B$. Then,

1. $(I \cap A) \cdot B = I$, for any ideal $I \subseteq B$.
2. $\text{Spec } B$ is a topological subspace of $\text{Spec } A$, with their Zariski topologies.
3. Let $q \in \text{Spec } A$.
   
   (a) If $q \notin \text{Spec } B$, then $q \cdot B = B$.
   
   (b) If $q \in \text{Spec } B$, then $q \cdot B \subseteq B$ is a prime ideal and $(q \cdot B) \cap A = q$.

4. Spec $B = \bigcap_{\text{Spec } B \subseteq \text{open set } U \subseteq \text{Spec } A} U$.

**Proof.**

1. Let $p \in \text{Spec } B$, $q := A \cap p$ and $M$ a $B$-module. By Proposition 2.14, $M_p = M \otimes_B B_p = M \otimes_B B_q = M_q$. Then,

   $$[ (I \cap A) \cdot B ]_q = [ (I \cap A) \cdot B ]_q = (I_q \cap A_q) \cdot B_q = I_q = I_p.$$ 

Hence, $(I \cap A) \cdot B = I$. 

2. By Proposition 2.14 we can think Spec $B$ as a subset of Spec $A$. Given an ideal $I \subseteq B$, observe that $(I)_0 = ((I \cap A) \cdot B)_0 = (I \cap A)_0 \cap \text{Spec } B$.

3. (a) Suppose that there exists a prime ideal $p \subseteq B$ that contains to $q \cdot B$. Denote $p' = p \cap A$. Then, $q \in \text{Spec } A_{p'} = \text{Spec } B_p \subseteq \text{Spec } B$, which is contradictory. (b) Let $p \in \text{Spec } B$ be a prime ideal such that $p \cap A = q$. Then, $p = (p \cap A) \cdot B = q \cdot B$.

4. If $q \in \text{Spec } A \setminus \text{Spec } B$, then $(q)_0 \cap \text{Spec } B = (q \cdot B)_0 = (B)_0 = \emptyset$. Then, Spec $B$ is equal to the intersection of the open sets $U \subseteq \text{Spec } A$ such that $\text{Spec } B \subseteq U$.

\[ \Box \]

3 Schematic finite spaces

3.1 Definition, examples and first characterizations

1. Definition: We say that a finite ringed space $(X, O)$ is a schematic finite space if it is locally affine; i.e. if there exists an open covering $\{U_i\}_{i \in I}$ on $X$, such that $U_i$ is an affine finite space, for each $i \in I$.

2. Proposition: Let $X$ be a finite ringed space. $X$ is a schematic finite space iff the open subsets $U_x$ are affine finite spaces for all $x \in X$.

Proof. $\Rightarrow$ Let $\{U_i\}_{i \in I}$ be an affine open covering of $X$. For each $x \in X$, $U_x$ is an open subset of one of the affine finite spaces $U_i$. So, it follows from Corollary 2.7 that $U_x$ is also affine.

$\Leftarrow$ It is clear, since $\{U_x\}_{x \in X}$ is an open covering of $X$. $\Box$

3. Remarks:

1. All schematic finite spaces are finite fr-spaces.
2. Affine finite spaces are schematic.
3. If $X = U_x$, then $X$ schematic if and only if it is affine.

The finite ringed space associated with a minimal affine finite covering $\mathcal{U}$ of a quasi-compact and quasi-separated scheme is a schematic finite space, by Paragraph 0.1.

4. Examples: Let us give some examples of schematic finite spaces (below we indicate the ringed space constructed in Paragraph 0.5):
3.1 Definition, examples and first characterizations

| Example                  | Description |
|-------------------------|-------------|
| 1. Projective line      | $k[x] \rightarrow k[x, 1/x] \rightarrow k[1/x]$ |
| 2. Projective plane     | $k[x, y] \rightarrow k[x, y, 1/y]$ |
| 3. Affine line with a double point | $k[x] \rightarrow k[x, 1/x] \rightarrow k[1/x]$ |
| 4. Two lines glued at the generic point | $k[x] \rightarrow k(x)$ |

It can be proved that the first three examples are finite models of the schemes we indicate, but the fourth it is not the model of any scheme. Also note that none of these examples are affine finite spaces.

If $X$ and $Y$ are schematic finite $R$-ringed spaces, then $X \times_R Y$ is an schematic finite space, by Proposition 2.12.

5. Proposition ([8] 4.11): Let $X$ be a finite ringed space. $X$ is a schematic finite space if and only if it satisfies the following two conditions:

1. The natural morphism $O_y \otimes O_x \rightarrow O_{yy'}$ is an isomorphism for any $y, y' \geq x$.

2. The natural morphism $O_{yy'} \rightarrow \prod_{z \in U_{yy'}} O_z$ is faithfully flat, for any $y, y' \in X$ for which there is an element $x \in X$ such that $y \geq x$ and $y' \geq x$.

Proof. It follows easily from the definition of affine finite space that the open subsets $\{U_x\}_{x \in X}$ are affine finite spaces iff the conditions 1. and 2. above are satisfied.

6. Proposition ([8] 4.11): Let $(X, O_X)$ be a ringed finite space. $X$ is a schematic finite space iff the morphism

$$O_y \otimes O_{y'} \rightarrow \prod_{z \in U_{yy'}} O_z$$

is faithfully flat, for any $x \leq y, y' \in X$.

Proof. It follows directly from Corollary 2.5.
3.2 More characterizations of schematic finite spaces

In this section, we see that schematic finite spaces can be characterized by the good behavior of their quasi-coherent modules.

7. Proposition: Let $X$ be a schematic finite space, $U \subseteq X$ an open subset and $N$ a quasi-coherent $O_U$-module. Then, $i_*N$ is a quasi-coherent $O_X$-module.

Proof. Let $x \leq y \in X$. We have to see that the morphism $(i_*N)_x \otimes_{O_x} O_y \to (i_*N)_y$ is an isomorphism. This morphism is equal to the morphism

$$N(U \cap U_x) \otimes_{O(U_x)} O(U_y) \to N(U \cap U_y),$$

which is an isomorphism by Proposition 2.8. \qed

8. Theorem: Let $X$ be a finite ringed space. $X$ is a schematic finite space if and only if it satisfies the next two conditions:

1. $\text{Ker} f$ is quasi-coherent, for any morphism $f: M \to N$ of quasi-coherent $O_X$-modules.

2. For any open subset $i: U_x \hookrightarrow X$ and any quasi-coherent $O_{U_x}$-module $M$, the $O_X$-module $i_*M$ is quasi-coherent.

Proof. $\Rightarrow$ We know that schematic finite spaces are finite fr-spaces. By Proposition 1.7, $\text{Ker} f$ is quasi-coherent. The second condition follows from the proposition above.

$\Leftarrow$ First, let us prove that $X$ is an fr-space. Let $i: U_x \hookrightarrow X$ be an open subset and $M \to N$ a morphism of quasi-coherent $O_{U_x}$-modules. $\text{Ker}[M \to N] = \text{Ker}[i_*M \to i_*N]|_{U_x}$, then it is quasi-coherent. By Proposition 1.7, $X$ is an fr-space.

If $X$ is an fr-space and satisfies condition 2., then it is schematic:

Consider $x \leq x'$, let $j: U_{x'} \hookrightarrow U_x$ be the inclusion morphism and $N$ a quasi-coherent $O_{U_{x'}}$-module. Since condition 2. is satisfied, the $O_{U_x}$-module $j_*N = i'((i \circ j)_*)N$ is quasi-coherent. It follows from this result that we can suppose $X = U_x$ (because being finite fr-space and schematic are local conditions).

Now, by Corollary 2.5, we only have to prove that, for each $y, y' \geq x$, the morphism

$$O_y \otimes_{O_x} O_{y'} \to \prod_{z \in U_{y'}} O_z$$

is faithfully flat.

Consider the open subset $i: U_y \hookrightarrow X = U_x$. Since $i_*O_{U_y}$ is quasi-coherent,

$$O_{yy'} = (i_*O_{U_y})(U_{y'}) = (i_*O_{U_y})(U_x) \otimes_{O_x} O_{y'} = O_y \otimes_{O_x} O_{y'}.$$
In particular, \( O_x = O_y = O_y \otimes_{O_x} O_y \). The morphism \( O_y \to O_z \) is flat, for any \( z \geq y, y' \), then the morphism \( O_y \otimes_{O_x} O_y' \to O_z \otimes_{O_x} O_y' = O_{y'} = O_z \) is flat.

If the morphism \( O_y \otimes_{O_x} O_y' \to \prod_{z \in U_y'} O_z \) is not faithfully flat, there exists an ideal \( I \subset O_y \otimes_{O_x} O_y' \) such that \( I \cdot \prod_{z \in U_y'} O_z = \prod_{z \in U_y'} O_z \). Observe that the morphism \( O_y \to O_y \otimes_{O_x} O_y' \) is flat since \( O_x \to O_y \) is flat. Besides,

\[
(O_y \otimes_{O_x} O_y') \otimes_{O_y} (O_y \otimes_{O_x} O_y') = O_y \otimes_{O_x} (O_y' \otimes_{O_x} O_y') = O_y \otimes_{O_x} O_y'.
\]

By Proposition 2.16, there exists an ideal \( J \subset O_y \) such that \( J \cdot (O_y \otimes_{O_x} O_y') = I \). Let \( M \) be the quasi-coherent \( O_{y'} \)-module associated with the \( O_y \)-module \( O_y/J \). Then, \( i_* M \) is the quasi-coherent \( O_X \)-module associated with the \( O_y \)-module \( O_y/J \) and

\[
M(U_{yy'}) = (i_* M)(U_{y'}) = (O_y/J) \otimes_{O_y} O_y' = (O_y \otimes_{O_x} O_y')/J \cdot (O_y \otimes_{O_x} O_y') = (O_y \otimes_{O_x} O_y')/I \neq 0.
\]

However, \( M_{U_{yy'}} = 0 \), since \( M_{U} = (O_y/J) \otimes_{O_y} O_z = O_y/J \cdot O_z = O_y/J \cdot O_z = 0 \), for any \( z \in U_{yy'} \). So we have a contradiction; therefore, the morphism \( O_y \otimes_{O_x} O_y' \to \prod_{z \in U_{y'}} O_z \) is faithfully flat.

\[ \square \]

9. Theorem: Let \((X, O)\) be a finite ringed space. Let \( \delta: X \to X \times X \), \( \delta(x) = (x, x) \) be the diagonal morphism. Then, \( X \) is schematic iff it satisfies these two conditions:

1. \( \ker f \) is quasi-coherent, for any morphism \( f: M \to N \) of quasi-coherent \( O_X \)-modules.

2. \( \delta_* N \) is a quasi-coherent \( O_{X \times X} \)-module for any quasi-coherent \( O_X \)-module \( N \).

Proof. \( \Rightarrow \) For any \((x, y) \leq (x', y')\), we have

\[
(\delta_* N)_{(x,y)} \otimes_{O_{(x,y)}} O_{(x',y')} = N_{x'y'} \otimes_{O_{x'y'}} (O_{x'} \otimes_{O_{x'}} O_{y'}) = N_{x'y'} \otimes_{O_{x'y'}} O_{x'} \otimes_{O_{x'}} O_{y'} = N_{x'y'} \otimes_{O_{x'y'}} O_{x'y'} = (\delta_* N)_{(x',y')}.
\]

\( \Leftarrow \) First, note that for any \( x \in X \) and any \( x' \leq x'' \),

\[
O_{xx'} \otimes_{O_{xx'}} O_{x''} = (\delta_* O)_{(x,x')} \otimes_{O_{xx'}} O_{x'} \otimes_{O_{x'}} O_{x''} = (\delta_* O)_{(x,x') \otimes_{O_{(x,x')}} O_{(x,x'')}} = (\delta_* O)_{(x,x'')} = O_{xx''}.
\]

In consequence, for any open subset \( i: U_x \hookrightarrow X \) and any quasi-coherent \( O_{U_x} \)-module \( M \) there exists a quasi-coherent \( O_X \)-module \( N \) such that \( N_{|U_x} \cong M \): define \( N_{x'} := M_x \otimes_{O_x} O_{xx'} \), for any \( x' \in X \). \( N \) is quasi-coherent since for any \( x' \leq x'' \),

\[
N_{x'} \otimes_{O_{x'}} O_{x''} = M_x \otimes_{O_x} O_{xx'} \otimes_{O_{xx'}} O_{x''} = M_x \otimes_{O_x} O_{xx'} = N_{x''}.
\]

Besides, \( N_x = M_x \), so \( N_{|U_x} = M \).
By Theorem 3.8 we have to prove that \((i_\ast M)_y \otimes_{O_y} O_y\)'s quasi-coherent \(O_X\)-module. That is, we have to prove that \((i_\ast M)_y \otimes_{O_y} O_y\)'s quasi-coherent \(O_X\)-module. That is, we have to prove that \((i_\ast M)_y \otimes_{O_y} O_y\)'s quasi-coherent \(O_X\)-module. That is, we have to prove that \((i_\ast M)_y \otimes_{O_y} O_y\)'s quasi-coherent \(O_X\)-module.

\[(i_\ast M)_y = M(U_{y',x}) = N(U_{y',x}) = (\delta, N)(U_{y',x} \times U_x) \otimes_{O_y \times O_x} O_y \otimes O_x = N(U_{y',x}) \otimes_{O_y} O_y = (i_\ast M)_y \otimes_{O_y} O_y.\]

\[\square\]

### Proposition 11.1

Let \(X\) be a finite \(fr\)-space and \(N\) a quasi-coherent \(O_X\)-module. Then, \(N_{pq} \otimes_{O_p} O_{p'} = N_{p'q}\), for any \(p \leq p' \in X\) and for any \(q \in X\).

\[\Rightarrow\) Let \(U \subseteq U_p\) be an open set and \(p \leq p'\). Consider the exact sequence of morphisms

\[N(U) \rightarrow \prod_{x \in U} N_x \rightarrow \prod_{z \geq x \in U} N_z.\]

Tensoring by \(O_{p'}\) we obtain the exact sequence of morphisms

\[N(U) \otimes_{O_p} O_{p'} \rightarrow \prod_{x \in U} N_p' \rightarrow \prod_{z \geq x \in U} N_p'z,\]

which shows that \(N(U) \otimes_{O_p} O_{p'} = N(U \cap U_{p'})\). In particular,

\[N_{pq} \otimes_{O_p} O_{p'} = N_{p'q}.\]

\[\square\]

### Proposition 11.2

Let \(X\) be a finite \(fr\)-space. \(X\) is schematic iff it is a finite \(fr\)-space and for any quasi-coherent \(O_X\)-module \(N\) and any \(x \leq x', x'' \in X\), \(N_{x'} \otimes_{O_x} O_{x''} = N_{x'x''}\).

\[\Rightarrow\) It follows directly from Theorem 3.9 and Proposition above.

\[\square\]
4 Affine morphisms

1. Definition: Let $X$ and $Y$ be schematic finite spaces. A morphism $f : X \to Y$ of ringed spaces is said to be an affine morphism if $f_*O_X$ is a quasi-coherent $O_Y$-module and the preimage of any affine open subspace of $Y$ is an affine open subspace of $X$.

2. Examples: 1. A schematic finite space $X$ is affine iff $(X, O) \to (\ast, O(X))$ is an affine morphism.

2. If $X$ is an affine finite space and $U \subseteq X$ an affine open subset, the inclusion morphism $i : U \hookrightarrow X$ is an affine morphism: $i_*O_U$ is quasi-coherent by Proposition 2.8, and for any affine open subset $V \subseteq Y$, $i^{-1}(V) = V \cap U$ is affine by Proposition 2.11.

3. Let $X$ be a schematic finite space. Given $x, x' \in X$, we shall say that $x \sim x'$ if $x \leq x'$ and $x' \leq x$. Let $\bar{X} := X/\sim$ be the Kolmogorov quotient of $X$ and define $O_{\bar{x}} := O_x$, for any $[x] \in \bar{X}$. Then, $\bar{X}$ is a schematic space, the quotient morphism $\pi : \bar{X} \to X$, $\pi(x) := [x]$ is affine and $\pi_*O_X = O_{\bar{X}}$.

3. Proposition: Let $X$ and $Y$ be affine finite spaces and $f : X \to Y$ an affine morphism. Let $M$ be an $O(X)$-module (therefore, an $O(Y)$-module). Then,

$$f_*M = \tilde{M}.$$ 

Proof. For any open set $U_y$,

$$(f_*\tilde{M})(U_y) = \tilde{M}(f^{-1}(U_y)) = \tilde{M}(X) \otimes_{O(X)} O_X(f^{-1}(U_y)) = M \otimes_{O(X)} O_X(f^{-1}(U_y))$$

$$= M \otimes_{O(X)} (f_*O_X)(U_y) = M \otimes_{O(X)} O_X \otimes_{O(Y)} O_Y(U_y) = M \otimes_{O(Y)} O_Y(U_y)$$

$$= \tilde{M}(Y) \otimes_{O(Y)} O_Y(U_y) = \tilde{M}(U_y).$$ 

□

4. Proposition: Let $f : X \to Y$ be an affine morphism and $M$ a quasi-coherent $O_X$-module. Then, $f_*M$ is a quasi-coherent $O_Y$-module.

Proof. Being $f_*M$ a quasi-coherent $O_Y$-module is a local property. We can suppose that $Y$ is affine. Then $X$ is affine. The proof is completed by the previous proposition. □

5. Proposition: The composition of affine morphisms is affine

Proof. It is obvious. □

6. Proposition: Let $X$ and $Y$ be schematic finite spaces. A morphism of ringed spaces $f : X \to Y$ is affine iff $f_*O_X$ is quasi-coherent and $f^{-1}(U_y)$ is affine for any $y \in Y$. 

Proof. \( \Rightarrow \) It is obvious.
\( \Leftarrow \) Let us proceed by induction on \( \#Y \). If \( \#Y = 1 \), it is obvious. We can suppose that
\( Y \) is affine and we only have to prove that \( X \) is affine. The morphism \( \mathcal{O}(Y) \to \prod_{y \in Y} \mathcal{O}_y \) is faithfully flat, then the morphism
\[
\mathcal{O}(X) \to \prod_{y \in Y} \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_y = \prod_{y \in Y} (f_* \mathcal{O}_X)(y) \otimes_{\mathcal{O}(Y)} \mathcal{O}_y \cong \prod_{y \in Y} \mathcal{O}(f^{-1}(U_y))
\]
is faithfully flat. Since \( f^{-1}(U_y) \) is affine, the morphism \( \mathcal{O}(f^{-1}(U_y)) \to \prod_{y \in f^{-1}(U_y)} \mathcal{O}_x \) is faithfully flat. The composition of faithfully flat morphisms is faithfully flat, then \( \mathcal{O}(X) \to \prod_{y \in Y, x \in f^{-1}(U_y)} \mathcal{O}_x \) is faithfully flat. Therefore, the morphism
\[
\mathcal{O}(X) \to \prod_{x \in X} \mathcal{O}_x
\]
is faithfully flat.

Let \( x, x' \in X \). Given an open set \( V \subseteq Y \) denote \( \tilde{V} := f^{-1}(V) \). \( \bar{U}_{f(x)f(x')} \) is an affine open subset of \( U_{f(x)} \) (by induction hypothesis), then \( \bar{U}_{f(x)f(x')} \cap U_x \) is affine, and it is included in \( \bar{U}_{f(x')} \), hence \( \bar{U}_{f(x)f(x')} \cap U_x \cap U_{x'} \) is affine. Then,
\[
U_{xx'} = \bar{U}_{f(x)f(x')} \cap U_x \cap U_{x'}
\]
is affine and the morphism \( \mathcal{O}_{xx'} \to \prod_{z \in U_{xx'}} \mathcal{O}_z \) is faithfully flat. Since \( f_* \mathcal{O}_x \) is quasi-coherent (and Proposition [2,8]),
\[
\mathcal{O}(\bar{U}_{f(x)}) = f_* \mathcal{O}_X(U_{f(x)}) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x)}, \quad \mathcal{O}(\bar{U}_{f(x')}) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x')}.
\]
\[
\mathcal{O}(\bar{U}_{f(x)f(x')}) = f_* \mathcal{O}_X(U_{f(x)f(x')}) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x)f(x')} = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x)} \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x')} \quad (*)
\]
\[
= (\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x)}) \otimes_{\mathcal{O}(X)} (\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}_{f(x')}) = \mathcal{O}((\bar{U}_{f(x)}) \otimes (\bar{U}_{f(x')})).
\]

Now it is easy to prove that
\[
\mathcal{O}_{xx'} = \mathcal{O}(\bar{U}_{f(x)f(x')} \cap U_x \cap U_{x'}) \cong (\mathcal{O}(\bar{U}_{f(x)f(x')})) \otimes_{\mathcal{O}(\bar{U}_{f(x)})} \mathcal{O}_x \otimes_{\mathcal{O}(\bar{U}_{f(x')})} \mathcal{O}_{x'} \cong (\mathcal{O}_x \otimes_{\mathcal{O}(X)} \mathcal{O}_{x'}). ^{(e)}
\]
Therefore, \( X \) is affine.

7. Corollary: Let \( X \) and \( Y \) be schematic finite spaces and let \( f : X \to Y \) be a morphism of ringed spaces. Then, being \( f \) affine is a local property on \( Y \).

8. Example: Let \( (X, \mathcal{O}) \) be a schematic finite space and \( \mathcal{O} \to \mathcal{O}' \) a morphism of sheaves of rings, such that \( \mathcal{O}' \) is a quasi-coherent \( \mathcal{O} \)-module. \( (X, \mathcal{O}') \) is a schematic finite space: Given \( x \leq y, y' \), the morphism \( \mathcal{O}_y \otimes_{\mathcal{O}_x} \mathcal{O}_{y'} \to \prod_{z \in U_{yy'}} \mathcal{O}_z \) is faithfully flat, by Proposition [3,6]. Tensoring by \( \otimes_{\mathcal{O}_x} \mathcal{O}' \) we obtain the faithfully flat morphism \( \mathcal{O}_y \otimes_{\mathcal{O}_x} \mathcal{O}_{y'} \to \prod_{z \in U_{yy'}} \mathcal{O}'_z \). Hence, \( (X, \mathcal{O}') \) is a schematic finite space by Proposition [3,6]. The obvious morphism \( \text{Id} : (X, \mathcal{O}) \to (X, \mathcal{O}) \) is affine.
5 Schematic morphisms

1. Definition: Let $X, Y$ be schematic finite spaces. A morphism of ringed spaces $f: X \to Y$ is said to be a schematic morphism if for any $x \in X$ the morphism $f_x: U_x \to U_{f(x)}$, $f_x(x') := f(x')$ is affine.

2. Example: If $X$ is a schematic finite space, $X \to (\ast, O(X))$ is a schematic morphism.

3. Example: If $U$ is an open subspace of a schematic finite space $X$, then the inclusion morphism $U \hookrightarrow X$ is schematic.

4. Remark: Let $X$ and $Y$ be schematic finite spaces and $f: X \to Y$ be a morphism of ringed spaces. Then, being $f$ schematic is a local property on $Y$ and on $X$.

5. Proposition: The composition of schematic morphisms is schematic.

Proof. It is a consequence of Proposition 4.5. □

6. Proposition: Affine morphisms between schematic finite spaces are schematic morphisms.

Proof. Let $f: X \to Y$ be an affine morphism. Then, $f^{-1}(U_{f(x)})$ is affine, $U_x \hookrightarrow f^{-1}(U_{f(x)})$ is an affine morphism and $f^{-1}(U_{f(x)}) \to U_{f(x)}$ is affine, by Corollary 4.7. The composition $U_x \hookrightarrow f^{-1}(U_{f(x)}) \to U_{f(x)}$ is affine, by Proposition 4.5. Hence, $f$ is a schematic morphism. □

7. Proposition: Let $f: X \to Y$ be a schematic morphism and $\mathcal{M}$ a quasi-coherent $O_X$-module. Then, $f_*\mathcal{M}$ is a quasi-coherent $O_Y$-module.

Proof. We can suppose that $Y$ is affine. Consider an open set $U_x \subseteq X$ and denote $\mathcal{M}_{U_x} = i_*\mathcal{M}_{U_x}$. Observe that $f_*\mathcal{M}_{U_x} = (f \circ i)_*\mathcal{M}_{U_x}$ is a quasi-coherent $O_Y$-module, because the composite morphism $f \circ i: U_x \to U_{f(x)} \hookrightarrow Y$ is affine and by Proposition 4.4.

Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be an open covering of $X$ and $\{U_{x_i}\}_{i}$ an open covering of $U_{x_i} \cap U_{x_j}$, for each $i, j$. Consider the exact sequence of morphisms

$$\mathcal{M} \to \prod_i \mathcal{M}_{U_{x_i}} \hookrightarrow \prod_{i,j} \mathcal{M}_{U_{x_i} \cap U_{x_j}}$$

Taking $f_*$, we obtain an exact sequence of morphisms, then $f_*\mathcal{M}$ is a quasi-coherent $O_Y$-module. □

8. Corollary: Let $X$ be a schematic finite space and $U \hookrightarrow X$ an open subset. Given a quasi-coherent $O_U$-module $\mathcal{N}$, there exists a quasi-coherent $O_X$-module $\mathcal{M}$, such that $\mathcal{M}|_U \cong \mathcal{N}$.
5. Schematic morphisms

Proof. Define $M := i_*N$. □

9. Lemma: Let $X$ be an affine finite space and $U \subset X$ an open set. Then, $U$ is affine iff $U \cap U_x$ is affine, for any $x \in X$.

Proof. If $U$ is affine, then $U \cap U_x$ is affine, for any $x \in X$, by Proposition [2.11]. Let us prove the converse implication. The inclusion morphism $i: U \hookrightarrow X$ is an affine morphism, by Proposition [4.6]. Hence, $U = i^{-1}(X)$ is affine. □

10. Proposition: A morphism of ringed spaces $f: X \rightarrow Y$ between affine finite spaces is affine iff it is a schematic morphism.

Proof. $\Rightarrow$ It is known (see Proposition [5.6]).

$\Leftarrow$ By Proposition [5.7], we only have to prove that $f^{-1}(U)$ is affine, for any affine open subset $U \subseteq Y$. By the previous lemma, we only have to prove that $f^{-1}(U) \cap U_x$ is affine. The composition of affine morphisms is affine, then $U_x \rightarrow U_{f(x)} \hookrightarrow Y$ is affine. Hence, $f^{-1}(U) \cap U_x$ is affine. □

11. Corollary: Let $f: X \rightarrow Y$ be a schematic morphism. Then, $f$ is affine iff there exists an affine open covering of $Y$, $\{U_i\}$, such that $f^{-1}(U_i)$ is affine, for any $i$.

Proof. Recall that being $f$ affine is a local property on $Y$. □

In [10] 5.6, it is proved that a morphism of ringed spaces $f: X \rightarrow Y$ is schematic iff $R^if_*M$ is quasi-coherent for any quasi-coherent module $M$ and any $i$.

12. Theorem: Let $X$ and $Y$ be schematic finite spaces and $f: X \rightarrow Y$ a morphism of ringed spaces. Then, $f$ is a schematic morphism iff $f_*M$ is quasi-coherent, for any quasi-coherent $O_X$-module $M$.

Proof. $\Rightarrow$ Recall Proposition [5.7].

$\Leftarrow$ We can suppose that $X$ and $Y$ are affine. We only have to prove that $f^{-1}(U_y)$ is affine, for any $y \in Y$, by Proposition [4.6]. By Proposition [2.11] we only have to prove that the morphism $O_X(f^{-1}(U_y)) \rightarrow \prod_{x \in f^{-1}(U_y)} O_x$ is faithfully flat. Observe that

$$O_X(f^{-1}(U_y)) = (f_*O_X)(U_y) = (f_*O_X)(Y) \otimes_{O_Y(Y)} O_y = O_x(X) \otimes_{O_Y(Y)} O_y$$

and for any $x \in f^{-1}(U_y)$

$$O_x \otimes_{O_Y(Y)} O_y = O_x \otimes_{O_y} O_y \otimes_{O_Y(Y)} O_y = O_x \otimes_{O_y} O_y = O_x.$$
The morphism $O_X(X) \rightarrow O$, is flat, then tensoring by $\otimes_{O_Y}O$ the morphism $O_X(f^{-1}(U_y)) \rightarrow O_x$ is flat, for any $x \in f^{-1}(U_y)$.

If the morphism $O_X(f^{-1}(U_y)) \rightarrow \prod_{x \in f^{-1}(U_y)}O_x$ is not faithfully flat, there exists an ideal $J \subseteq O_X(f^{-1}(U_y))$ such that $J \cdot \prod_{x \in f^{-1}(U_y)}O_x = \prod_{x \in f^{-1}(U_y)}O_x$. Observe that the morphism $O_X(X) \rightarrow O_X(X) \otimes_{O_Y}O_y = O_X(f^{-1}(U_y))$ is flat since $O_Y(Y) \rightarrow O_y$ is flat. Besides,

$$O_X(f^{-1}(U_y)) \otimes_{O_X(X)}O_X(f^{-1}(U_y)) = O_X(X) \otimes_{O_Y}O_y \otimes_{O_X(X)}O_X(X) \otimes_{O_Y}O_y$$

$$= O_X(X) \otimes_{O_Y}O_y \otimes_{O_Y}O_y = O_X(X) \otimes_{O_Y}O_y = O_X(f^{-1}(U_y)).$$

By Proposition 2.16, there exists an ideal $J \subseteq O_X(X)$ such that $J \cdot O_X(f^{-1}(U_y)) = 1$. Let $M$ be the quasi-coherent $O_X$-module associated with the $O_X(X)$-module $O_X(X)/J$. Then, $f_*M$ is the quasi-coherent $O_Y$-module associated with the $O_Y(Y)$-module $O_X(X)/J$ and

$$M(f^{-1}(U_y)) = f_*M(U_y) = f_*M(Y) \otimes_{O_Y(Y)}O_y = (O_X(X)/J) \otimes_{O_Y}O_y$$

$$= (O_X(X) \otimes_{O_Y}O_y)/J \cdot (O_X(X) \otimes_{O_Y}O_y) = O_X(f^{-1}(U_y))/J \neq 0.$$

However, $M_{f^{-1}(U_y)} = 0$ since $M_x = O_x/J \cdot O_x = O_x/I \cdot O_x = 0$, for any $x \in f^{-1}(U_y)$. This is contradictory, then the morphism $O_X(f^{-1}(U_y)) \rightarrow \prod_{x \in f^{-1}(U_y)}O_x$ is faithfully flat.

\[\Box\]

13. **Notation**: Let $f : X \rightarrow Y$ be a morphism of ringed spaces, between ringed finite spaces, $x \in X$ and $y \in Y$. We shall denote $U_{xy} := U_x \cap f^{-1}(U_y)$ and $O_{xy} := O(U_x \cap f^{-1}(U_y))$.

14. **Proposition**: A morphism of ringed spaces $f : X \rightarrow Y$ between schematic finite spaces is schematic iff for any $x \in X$ and $y \geq f(x)$

1. $U_{xy}$ is affine.
2. $O_{xy} = O_x \otimes_{O_{f(x)}}O_y$.

**Proof.** Consider the morphism $f_x : U_x \rightarrow U_{f(x)}$. Then, $f_*O_Ux$ is a quasi-coherent module iff condition 2. is satisfied. By Proposition 4.6, $f_x$ is affine iff the conditions 1. and 2. are satisfied. Then, $f$ is schematic iff the conditions 1. and 2. are satisfied. \[\Box\]

15. **Theorem**: Let $f : X \rightarrow Y$ be a morphism of ringed spaces between schematic finite spaces. Then, $f$ is schematic iff the induced morphism on spectra by the morphism of rings $O_x \otimes_{O_{f(x)}}O_y \rightarrow \prod_{z \in U_{xy}}O_z$ is surjective, for any $x$ and $y \geq f(x)$.

**Proof.** $\Rightarrow$ By Proposition 5.14, $U_{xy}$ is affine and $O_x \otimes_{O_{f(x)}}O_y = O_{xy}$. Therefore, the morphism

$$O_x \otimes_{O_{f(x)}}O_y = O_{xy} \rightarrow \prod_{z \in U_{xy}}O_z$$

\[\Box\]
is faithfully flat, and the induced morphism on spectra is surjective.

\( \iff \) Let \( z \in U_{xy} \). Since \( O_x \to O_z \) is a flat morphism, the morphism

\[
O_x \otimes_{O_{f(z)}} O_y \to O_z \otimes_{O_{f(z)}} O_y = O_z \otimes_{O_{f(z)}} O_{f(z)} = O_z
\]

is flat. Denote \( B = O_x \otimes_{O_{f(z)}} O_y \) and \( C = \prod_{z \in U_{xy}} O_z \). The morphism \( B \to C \) is faithfully flat.

Let \( z, z' \in U_{xy} \). The morphism \( O_z \otimes_{O_x} O_{z'} \to O_z \otimes_B O_{z'} \) is surjective and the composite morphism

\[
O_z \otimes_{O_x} O_{z'} \to O_z \otimes_B O_{z'} \to O_{zz'}
\]

is an isomorphism. Therefore, \( O_z \otimes_B O_{z'} = O_{zz'} \). The exact sequence of morphisms

\[
B \to C = \prod_{z \in U_{xy}} O_z \otimes_B C = \prod_{z \in U_{xy}} O_{zz'}
\]

shows that \( B = O_{xy} \). Therefore the morphism \( O_{xy} \to \prod_{z \in U_{xy}} O_z \) is faithfully flat. By Proposition 2.6, \( U_{xy} \) is affine. By Proposition 5.14, \( f \) is schematic. \( \square \)

### 6 Removable points. Minimal schematic space

1. **Proposition**: Let \( X \) be a schematic finite space. Let \( p \in X \) be a point such that the morphism \( O_p \to \prod_{q \neq p} O_q \) is faithfully flat. Consider the ringed subspace \( Y = X - \{p\} \) of \( X \) \( (O_{Y,y} := O_{X,y} \) for any \( y \in Y \)). Then, \( Y \) is a schematic finite space, the inclusion map \( i: Y \hookrightarrow X \) is an affine morphism and \( i_* O_Y = O_X \).

**Proof.** \( Y \) is a schematic finite space by Proposition 3.6 because \( X \) is a schematic finite space. Let us prove that \( i: Y \hookrightarrow X \) is affine and \( i_* O_Y = O_X \): Consider \( U \subset X \). If \( x \neq p \), then \( i^{-1}(U_x) = U_x \) and \( (i_* O_Y)(U_x) = O_{X,x} = O_X(U_x) \). If \( x = p \) denote \( U := U_p - \{p\} \). Observe that \( O_{X|U} = O_{Y|U} \). The morphism \( A = O_p \to \prod_{x \in U} O_x = B \) is faithfully flat. The exact sequence of morphisms

\[
A \to B \otimes_A B \otimes_A B
\]

and the equality \( B \otimes_A B = \prod_{x,y \in U} O_{xy} \) show that \( A = O_X(U) = O_Y(U) \). By Proposition 2.6, \( U \) is affine. Then, \( i^{-1}(U_p) = U \) is affine and \( (i_* O_Y)(U_p) = O_Y(U) = A = O_X(U_p) \).

\( \square \)

2. **Remark**: If \( U \) is affine then the morphism \( O_U \to \prod_{q \in U} O_q \) is faithfully flat. We have proved that \( O_p \to \prod_{q \neq p} O_q \) is faithfully flat iff \( U := U_p - \{p\} \) is affine and \( O_p = O(U) \).

3. **Lemma**: Let \( X \) be an affine finite space and \( U \subset X \) an affine open subset. Then, the restriction morphism \( O(X) \to O(U) \) is flat.

---

1. If \( I := \{q \in X : q > p\} = \emptyset \), define \( \prod_{q \in I} O_q := \{0\} \).
5. **Definition:** Let \( O(U) \rightarrow \prod_{x \in U} O_x \) be faithfully flat and the composite morphism \( O(X) \rightarrow O(U) \rightarrow \prod_{x \in U} O_x \) is flat. Then, the morphism \( O(X) \rightarrow O(U) \) is flat.

\[ \Box \]

\[ 2.11 \]

6. **Proposition:** Let \( X \) be a schematic finite space. Let \( p \in X \) be a point such that the morphism \( O_p \rightarrow \prod_{q \succ p} O_q \) is faithfully flat and let \( Y := X - \{ p \} \). An open set \( V \subseteq X \) is affine iff \( V \cap Y \) is affine.

**Proof.** We only have to prove the converse implication. We can suppose that \( V = Y \) and we have to prove that \( X \) is affine.

By Lemma 6.3, the morphism \( O_y(Y) = O_y(U \cap Y) = O_x(U_x) = O_x \) is flat, for any \( x \in X \). Then, the morphism \( O(X) = O(Y) \rightarrow \prod_{y \in Y} O_Y = \prod_{y \in Y} O_{x,y} \) is faithfully flat. Therefore the morphism \( O(X) \rightarrow \prod_{x \in X} O_{x, x} \) is faithfully flat. Likewise, given \( U_x, U_{x'} \subseteq X \), the morphism \( O_x(U_x \cap U_{x'}) \rightarrow \prod_{x' \in U_x \cap U_{x'}} O_{x, x'} \) is faithfully flat. Besides,

\[ O_x \otimes_{O(x)} O_{x'} = O_y(U_x \cap Y) \otimes_{O(Y)} O_y(U_{x'} \cap Y) \Rightarrow O_y(U_x \cap Y \cap U_{x'} \cap Y) = O_x(U_x \cap U_{x'}) \]

Therefore, \( X \) is affine.

\[ \Box \]

5. **Definition:** Let \( X \) be a schematic finite space. We shall say that \( x \in X \) is removable if \( O_x \rightarrow \prod_{x' \neq x} O_{x'} \) is faithfully flat.

If \( O_x = 0 \) obviously \( x \) is a removable point.

6. **Proposition:** Let \( X \) be a schematic finite space and let \( p, p' \in X \) be two points. Then, \( p, p' \) are removable points of \( X \) iff \( p \) is a removable point of \( X \) and \( p' \) is a removable point of \( X - p \).

**Proof.** It is immediate.

\[ \Box \]

7. **Proposition:** Let \( U' \subseteq U \) be affine open subsets of a schematic finite space \( X \) and suppose that the morphism \( O(U) \rightarrow O(U') \) is faithfully flat. Then, \( U - U' \) is a set of removable points of \( X \).

**Proof.** \( O(U) = O(U') \) because the morphism \( O(U) \rightarrow O(U') \) is faithfully flat and

\[ O(U) \otimes_{O(U)} O(U') = O(U') \otimes_{O(U)} O(U') \]

Let \( x \in U - U' \), then \( O_x = O(U) \otimes_{O(U)} O_x = O(U') \otimes_{O(U)} O_x \Rightarrow O(U' \cap U_x) \). By Proposition 2.11, \( U' \cap U_x \) is affine, then the morphism \( O_x = O(U' \cap U_x) \rightarrow \prod_{x' \in U' \cap U_x} O_x \) is faithfully flat. Hence, \( O_x \rightarrow \prod_{y \neq x} O_y \) is faithfully flat, and \( x \) is a removable point.

\[ \Box \]

8. **Remark:** In addition, we have proved that \( O(U) = O(U') \).
9. **Definition**: A schematic finite space $X$ is said to be minimal if there are no removable points in $X$ and it is $T_0$. Let $\tilde{X}$ the Kolmogorov space of $X$ and $P$ be the set of all the removable points of $\tilde{X}$, we shall denote $X_M := \tilde{X}\setminus P$.

By Proposition 6.1 [X_M is a schematic finite space, the natural morphism $X_M \hookrightarrow \tilde{X}$ is affine and $O_{\tilde{X}} = i_*O_{X_M}$.

10. **Proposition**: Let $f : X \rightarrow Y$ be a schematic morphism. If $x \in X$ is not a removable point, then $f(x) \in Y$ is not a removable point. Then, we have the commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & \tilde{X} \\
\downarrow f & & \downarrow f_M \\
Y & \rightarrow & Y_M
\end{array}
$$

where $\tilde{X}$ and $\tilde{Y}$ are the Kolmogorov spaces of $X$ and $Y$ respectively, and $\tilde{f}$ and $f_M$ are the induced morphisms.

Proof. Consider the affine morphism $f_x : U_x \rightarrow U_{f(x)}$, $f_x(x') := f(x')$. Since $f_x_*O_{U_x}$ is a quasi-coherent $O_{U_{f(x)}}$-module, then $O_x \otimes_{O_{f(x)}} O_y = O(f_x^{-1}(U_y))$, for any $y \in U_{f(x)}$. If $f(x)$ is a removable point, then $O_{f(x)} \rightarrow \prod_{y>f(x)} O_y$ is faithfully flat. Tensoring by $O_y \otimes_{O_{f(x)}}$, one has the faithfully flat morphism $O_x \rightarrow \prod_{y>f(x)} O(f_x^{-1}(U_y))$. The open sets $f_x^{-1}(U_y)$ are affine, because $f_x : U_x \rightarrow U_{f(x)}$ is affine. Then, the morphisms $O(f_x^{-1}(U_y)) \rightarrow \prod_{x'<f(x)} O_{x'}$ are faithfully flat. Hence, the morphism $O_x \rightarrow \prod_{x'>f(x)} O_{x'}$ is faithfully flat and $x$ is a removable point.

\]

11. **Proposition**: Let $p \in Y$ be a removable point, $i : Y \setminus \{p\} \rightarrow Y$ be the inclusion morphism and $f : X \rightarrow Y$ a schematic morphism. If $f(X) \subseteq Y \setminus \{p\}$ and $g : X \rightarrow Y \setminus \{p\}$ is the morphism of ringed spaces such that $f = i \circ g$, then $g$ is a schematic morphism.

Therefore, $f_{u'} : X_M \rightarrow Y_M$ is a schematic morphism.

Proof. It is an immediate consequence of Theorem 5.15

\]

12. **Proposition**: Let $X$ be a schematic finite space and $U \subset X$ an affine open subspace. Let $X' := X \coprod \{u\}$ be the ringed finite space defined by

1. The preorder on $X' \subset X'$ is the pre-established preorder. Given $x \in X$, then $u < x$ if $x \in U$, and $x < u$ if $x \leq x'$, for any $x' \in U$.

2. $O_{X',x} := O_{X,x}$ for any $x \in X$, and $O_{X',u} := O_X(U)$. The restriction morphisms are the obvious morphisms.
Then, $X'$ is a schematic finite space and $u$ is a removable point of $X'$.

Proof. Let us denote $U_u := U \subset X$ and $\tilde{U}_x := \{y \in X': y \geq x\}$, for any $x' \in X'$. By Proposition 2.11 the morphism

$$O_{X',y} \otimes_{O_{X',x'}^\sim} O_{X',y'} = O_X(U_y \cap U_{y'}) \to \prod_{x \in U_y \cap U_{y'}} O_{X,x} = \prod_{x = U_y \cap U_{y'}} O_{X',x}$$

is faithfully flat, for any $y, y' \geq x'$. If $U \subseteq U_y \cap U_{y'}$, the morphism $O_X(U_y \cap U_{y'}) \to O_X(U)$ is flat, by Lemma 6.3. Hence, the morphism $O_{X',y} \otimes_{O_{X',x'}^\sim} O_{X',y'} \to \prod_{x \in U_y \cap U_{y'}} O_{X',x}$ is faithfully flat. By Proposition 3.6, $X'$ is a schematic finite space.

The morphism

$$O_{X',u} = O_X(U) \to \prod_{x \in U} O_{X,x} = \prod_{x' > u} O_{X',x'}$$

is faithfully flat, because $U$ is affine. Hence, $u$ is removable.

\[\square\]

7 Serre Theorem

Let $X$ be a finite topological space and $F$ a sheaf of abelian groups on $X$.

1. Proposition: If $X$ is a finite topological space with a minimum, then $H^i(X, F) = 0$ for any sheaf $F$ and any $i > 0$. In particular, for any finite topological space one has

$$H^i(U_p, F) = 0$$

for any $p \in X$, any sheaf $F$ and any $i > 0$.

Proof. Let $p$ be the minimum of $X$. Then $U_p = X$ and, for any sheaf $F$, one has $\Gamma(X, F) = F_p$; thus, taking global sections is the same as taking the stalk at $p$, which is an exact functor. \[\square\]

Let $f : X \to Y$ a continuous map between finite topological spaces and $F$ a sheaf on $X$. The i-th higher direct image $R^i f_* F$ is the sheaf on $Y$ given by:

$$[R^i f_* F]_y = H^i(f^{-1}(U_y), F).$$

Let $F$ be a sheaf on a finite topological space $X$. We define $C^n F$ as the sheaf on $X$ whose sections on an open subset $U$ are

$$(C^n F)(U) = \prod_{U \ni x_0 < \cdots < x_n} F_{x_n}$$
and whose restriction morphisms \((C^n F)(U) \to (C^n F)(V)\) for any \(V \subseteq U\) are the natural projections.

One has morphisms \(d : C^n F \to C^{n+1} F\), \(a = (a_{x_0 < \cdots < x_n}) \mapsto d(a) = (d(a)_{x_0 < \cdots < x_{n+1}})\) defined in each open subset \(U\) by the formula

\[
(d a)_{x_0 < \cdots < x_{n+1}} = \sum_{0 \leq i \leq n} (-1)^i a_{x_0 < \cdots \hat{x}_i < \cdots < x_{n+1}} + (-1)^{n+1} \bar{a}_{x_0 < \cdots < x_n}
\]

where \(\bar{a}_{x_0 < \cdots < x_p}\) denotes the image of \(a_{x_0 < \cdots < x_p}\) under the morphism \(F_{x_n} \to F_{x_{n+1}}\). There is also a natural morphism \(d : F \to C^0 F\). One easily checks that \(d^2 = 0\).

2. **Theorem ([10] 2.15):** \(C^* F\) is a finite and flasque resolution of \(F\) (in fact, it is the Gode-ment resolution of \(F\)).

**Proof.** By definition, \(C^n F = 0\) for \(n > \dim X\). It is also clear that \(C^n F\) are flasque. Let us see that

\[
0 \to F \to C^0 F \to \cdots \to C^{\dim X} F \to 0
\]

is an exact sequence. We have to prove that \((C^* F)(U_p)\) is a resolution of \(F(U_p)\). One has a decomposition

\[
(C^n F)(U_p) = \prod_{p = x_0 < \cdots < x_n} F_{x_n} \times \prod_{p < x_0 < \cdots < x_n} F_{x_n} = (C^{n-1} F)(U_p^*) \times (C^n F)(U_p^*)
\]

with \(U_p^* := U_p - \{p\};\) via this decomposition, the differential \(d : (C^n F)(U_p) \to (C^{n+1} F)(U_p)\) becomes:

\[
d(a, b) = (b - d^* a, d^* b)
\]

with \(d^*\) the differential of \((C^* F)(U_p^*)\). If \(d(a, b) = 0\), then \(b = d^* a\) and \(d(0, a) = (a, b)\).

It is immediate now that every cycle is a boundary.

This theorem, together with De Rham’s theorem ([4], Thm. 4.7.1), yields that the cohomology groups of a sheaf can be computed with the standard resolution, i.e., \(H^i(U, F) = H^i\Gamma(U, C^* F)\), for any open subset \(U\) of \(X\) and any sheaf \(F\) of abelian groups on \(X\).

3. **Theorem ([8] 4.3, 4.12):** Every quasi-coherent module of an affine finite space is acyclic.

**Proof.** Let \(X\) be an affine finite space.

Proceed by induction over the order of \(X\). The open sets \(U_{xy}\) are affine by Corollary[2,7]. If \(X = U_{xy}\), then \(X = U_x\) and every sheaf is acyclic. We can suppose that every quasi-coherent module on \(U_{xy}\) is acyclic, by induction hypothesis. Let us prove that any quasi-coherent \(O_X\)-module \(\mathcal{M}\) is acyclic. We have to prove that the sequence of morphisms

\[
\mathcal{M}(X) \to \prod_{x_1 \in X} \mathcal{M}_{x_1} \to \prod_{x_1 < x_2} \mathcal{M}_{x_2} \to \cdots
\]
is exact. It is sufficient to check that tensoring the previous sequence by $\otimes_{O(X)}O_z$, for any $z \in X$, the sequence of morphisms

$$M_z \longrightarrow \prod_{z \in x_1} M_{x_1} \times \prod_{z \in x_2} M_{x_2} \longrightarrow \prod_{z \in x_1 < x_2} M_{x_1} \times \prod_{z \in x_1 < x_2} M_{x_2} \longrightarrow \prod_{z \in x_1 < x_2} M_{x_2} \longrightarrow \cdots$$

is exact. That is, we have to prove that the sequence of morphisms $(S)$

$$M(U_z) \longrightarrow C^0(U_z, M) \times \prod_{z \in x_1} M(U_{x_1 z}) \longrightarrow C^1(U_z, M) \times \prod_{z \in x_1} C^0(U_{x_1 z}, M) \times \prod_{z \in x_2} M(U_{x_2 z}) \longrightarrow \cdots$$

is exact. Let $D'_r := \oplus_{i \leq r} (M(U_{x_i z}) \oplus C^r(U_{x_i z}, M))[-r]$ and let $d_r$ be the differential such that over each direct summand $(M(U_{x_i z}) \oplus C^r(U_{x_i z}, M))[-r]$ is the known differential of $M(U_{x_i z}) \oplus C^r(U_{x_i z}, M)$ multiplied by $(-1)^i$. $H^i(D'_r) = 0$ for any $i \geq 0$. The sequence of morphisms $(S)$ is equal to the differential complex $D := D'_0 \oplus D'_1 \oplus \cdots \oplus D'_n$ with the differential

$$d = \begin{pmatrix}
  d_0 & 0 & 0 & \cdots & 0 & 0 \\
  - d_1 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  - & - & \cdots & d_{n-1} & 0 \\
  - & - & \cdots & d_n & 
\end{pmatrix}$$

Let $D'_{>0} = \oplus_{i > 0} D'_i$. Consider the exact sequence of morphisms of complexes

$$0 \rightarrow D'_{>0} \rightarrow D' \rightarrow D'_0 \rightarrow 0.$$ 

Then, $H^i(D') = H^i(D'_{>0})$, for any $i \geq 0$. Let $D'_{>1} = \oplus_{i > 1} D'_i$. Consider the exact sequence of morphisms $0 \rightarrow D'_{>1} \rightarrow D'_{>0} \rightarrow D'_1 \rightarrow 0$. Then, $H^i(D') = H^i(D'_{>0}) = H^i(D'_{>1})$. Recursively $H^i(D') = H^i(D'_{>n}) = 0$ for any $i \geq 0$, and the sequence of morphisms $(S)$ is exact.

Let $R$ and $R'$ be commutative rings and $R \rightarrow R'$ a flat morphism of rings. Let $(X, O)$ be an $R$-ringed finite space. Let $O \otimes_R R'$ be the sheaf of rings on $X$ defined by $(O \otimes_R R')(U) := O(U) \otimes_R R'$. Consider the obvious morphism $\pi : (X, O \otimes_R R') \rightarrow (X, O)$, $\pi(x) = x$. Let $M$ be a sheaf of $O$-modules. Then,

$$H^i(X, \pi^* M) = H^i(X, M) \otimes_R R'.$$
If \( \mathcal{N} \) is a quasi-coherent \( O \otimes_R R' \)-module, then \( \pi_*\mathcal{N} = \mathcal{N} \) is a quasi-coherent \( O \)-module.

Let \( S \subset R \) be a multiplicative system, \( R' = S^{-1} R \) and \( \mathcal{N} \) a quasi-coherent \( O \otimes_R R' \)-module. Then, \( \pi_*\mathcal{N} \) is a quasi-coherent \( O \)-module, \( \mathcal{N} = \pi_*\mathcal{N} \) and \( H^i(X, \mathcal{N}) = H^i(X, \pi_*\mathcal{N}) \).

4. **Serre Theorem (\cite{8} 5.11)**: Let \( X \) be a schematic finite space. \( X \) is affine iff every quasi-coherent \( O_X \)-module \( \mathcal{M} \) is acyclic (or \( H^1(X, \mathcal{M}) = 0 \)).

**Proof.** \( \Leftarrow \) Let \( R := O(X) \). Recall Notation \([2,13]\) Given \( \mathfrak{p} \in \text{Spec} R \), consider the sheaf of rings on \( X \), \( O \otimes_R R_{\mathfrak{p}} \). Obviously, \((X,O \otimes_R R_{\mathfrak{p}}) \) is a schematic finite space and \((X,O) \) is affine iff \((X,O \otimes_R R_{\mathfrak{p}}) \) is affine for any \( \mathfrak{p} \). Hence, we can suppose \( R \) is a local ring. We can suppose that \( X \) is minimal. Let \( X' \) be the set of the closed points of \( X \). Let \( x' \in X' \). The morphism \( O_{x'} \rightarrow \prod_{x \in X'} O_x \) is flat but it is not faithfully flat, then there exists a prime ideal \( I_{x'} \subset O_{x'} \) such that \( I_{x'} \cdot \prod_{x \in X'} O_x = \prod_{x \in X'} O_x \). Let \( \mathfrak{p} \) be the quasicoherent ideal defined by \( \mathfrak{p}_{x'} := I_{x'} \) if \( x' \in X' \) and \( \mathfrak{p}_{x} := O_x \) if \( x \notin X' \). Observe that \((O_{X}/\mathfrak{p}_{x'}) = 0 \), for any \( x \in X \setminus X' \), then \((O_{X}/\mathfrak{p})(X) = \prod_{x' \in X'} O_x/I_{x'} \). Consider the exact sequence of morphisms

\[
0 \rightarrow \mathfrak{p} \rightarrow O \rightarrow O_{X}/\mathfrak{p} \rightarrow 0
\]

The morphism \( R = O(X) \rightarrow (O_{X}/\mathfrak{p})(X) = \prod_{x \in X'} O_x/I_{x'} \) is surjective, because \( H^1(X, \mathfrak{p}) = 0 \). \( R \) is a local ring, then \( \prod_{x \in X'} O_x/I_{x'} \) is a local ring, hence \( X' = \{x'\} \). Therefore, \( X = U_{x'} \), which is affine.

\[ \square \]

This theorem yields the usual Serre’s criterion on algebraic varieties (see \cite{10} 4.13 and \cite{11}).

5. **Corollary**: A schematic finite space \( X \) is affine iff the functor

\[
\Gamma : \text{Qc-Mod}_X \rightarrow \text{Mod}_{O(X)}, \quad \mathcal{M} \mapsto \Gamma(X, \mathcal{M})
\]

is exact.

**Proof.** \( \Rightarrow \) By the Serre Theorem \( H^1(X, \mathcal{M}) = 0 \), for any quasi-coherent module \( \mathcal{M} \). Hence \( \Gamma \) is exact.

\( \Leftarrow \) It has been proved in the proof of Serre Theorem.

\[ \square \]

6. **Corollary**: A schematic finite space \( X \) is affine iff \( H^1(X, \mathcal{I}) = 0 \) for any quasi-coherent ideal \( \mathcal{I} \subseteq O \).

**Proof.** \( \Leftarrow \) Let \( R = O(X) \). We have just proved this implication when \( R \) is a local ring, in the proof of the Serre Theorem. Let \( \mathfrak{p} \in \text{Spec} R \) and let \( \mathcal{I}^\mathfrak{p} \subset O \otimes_R R_{\mathfrak{p}} \) be a quasi-coherent ideal. Consider the obvious morphism \( O \rightarrow O \otimes_R R_{\mathfrak{p}}, \mathcal{J} := O \otimes_{O \otimes_R R_{\mathfrak{p}}} \mathcal{I}^\mathfrak{p} \) is a quasi-coherent ideal of \( O \) and \( \pi^*\mathcal{J} = \mathcal{I}^\mathfrak{p} \) (where \( \pi : (X,O \otimes_R R_{\mathfrak{p}}) \rightarrow (X,O) \) is defined by \( \pi(x) := x \)). Then,
H^1(X, I^p) = H^1(X, \mathcal{F}) \otimes_R R_p = 0. \text{ Then, } (X, O \otimes_R R_p) \text{ is affine, for any } p. \text{ Therefore, } X \text{ is affine.} \square

7. **Corollary:** Let X be a minimal affine finite space and suppose that O(X) is a local ring. Then, there exists a point p ∈ X such that X = U_p.

8. **Theorem:** Let X and Y be schematic finite spaces. A ringed space morphism f : X → Y is affine iff f_*M is quasi-coherent and R^1 f_*M = 0, for any quasi-coherent O_X-module M.

Proof. ⇒) It is obvious.
⇐) Let U ⊆ Y be an affine open subspace. By Corollary [5.8] any quasi-coherent module M on f^{-1}(U) is the restriction of a quasi-coherent module on X. H^1(f^{-1}(U), M) = H^1(U, f_*M) = 0, for any quasi-coherent O_X-module M. By Serre Theorem [7.4] f^{-1}(U) is affine. Hence, f is affine. \square

9. **Theorem:** Let f : X → Y be a schematic morphism. The functor

\[ f_* : \text{Qc-Mod}_X \to \text{Qc-Mod}_Y, \quad M \mapsto f_*M \]

is exact iff f is affine.

Proof. ⇒) 1. Let U ⊆ Y be an open subset, V = f^{-1}(U) and f|_V : V → U, f|_V(x) := f(x). Then f|_V* is exact: Any short exact sequence of quasi-coherent modules N^• on V is a restriction of a short exact sequence of quasi-coherent modules M^• on X; and f|_V^*N^• = (f_*M^•)|_U.

2. We can suppose that Y is affine. We can suppose that Y = (*, A). We can suppose that A = O_X(X).

3. We have to prove that X is affine. The functor,

\[ \text{Qc-Mod}_X \to \text{Mod}_{O_X}, \quad M \mapsto f_*M = \Gamma(X, M) \]

is exact. By Corollary [7.5] X is affine.
⇐) By Theorem [7.8] R^1 f_*M = 0, for any quasi-coherent module M. Then, f_* is exact. \square

### 8 Cohom. characterization of schematic finite spaces

We say that an R-ringed space (X, O_X) is a flat R-ringed space if the morphism R → O_x is flat, for any x ∈ X.
1. **Theorem:** Let \((X, O)\) be a flat \(R\)-ringed finite space and \(M\) an \(R\)-module. If \(H^i(X, O)\) is a flat \(R\)-module, for any \(i > 0\), then \(O(X)\) is a flat \(R\)-module and 

\[
H^i(X, \tilde{M}) = H^i(X, O) \otimes_R M, \quad \forall i.
\]

**Proof.** Let \(C^i := \text{Ker}[C^i(X, O) \to C^{i+1}(X, O)]\) and \(B^i := \text{Im}[C^{i-1}(X, O) \to C^i(X, O)]\). Let \(n = \dim X\). \(C^n = C^n(X, O)\) is \(R\)-flat. The sequence \(0 \to B^n \to C^n \to H^n(X, O) \to 0\) is exact, then \(B^n\) is flat. The sequence \(0 \to C^{n-1} \to C^{n-1}(X, O) \to B^n \to 0\) is exact then \(C^{n-1}\) is \(R\)-flat. Recursively, \(C^i\) and \(B^i\) are \(R\)-flat for any \(i\). Hence, \(O(X) = C^0\) is \(R\)-flat and 

\[
H^i(X, \tilde{M}) = H^i(X, C^i O \otimes_R M) = H^i(X, C^i O) \otimes_R M = H^i(X, O_x) \otimes_R M.
\]

\(\Box\)

2. **Corollary:** Let \(X\) be a flat \(O(X)\)-ringed finite space. Assume \(H^i(X, O)\) is a flat \(O(X)\)-module, for any \(i > 0\). Then, the morphism \(\tilde{O}(X) = \prod_{x \in X} O_x\) is faithfully flat.

**Proof.** Let \(M\) be an \(\tilde{O}(X)\)-module. By Theorem 8.1 \(\tilde{M}(X) = M\). Then, the morphism 

\[
M = \tilde{M}(X) \hookrightarrow \prod_{x \in X} \tilde{M}_x = M \otimes_{\tilde{O}(X)} \prod_{x \in X} O_x
\]

is injective. Therefore, the flat morphism \(\tilde{O}(X) = \prod_{x \in X} O_x\) is faithfully flat. \(\Box\)

3. **Theorem:** Let \(X\) be a flat \(O(X)\)-ringed finite space. Then, \(X\) is affine iff 

1. \(X\) is acyclic.
2. \(O_x \otimes_{O(X)} O_y = O_{xy}\), for any \(x, y\).
3. \(U_{xy}\) is acyclic, for any \(x, y\).

**Proof.** \(\Rightarrow\) \(U_{xy}\) is affine by Corollary 2.7. By Theorem 7.3 \(X\) and \(U_{xy}\) are acyclic.

\(\Leftarrow\) Let \(z \in U_{xy}\). The morphism 

\[
O_{xy} \to O_z \otimes_{O(X)} O_{xy} = O_z \otimes_{O(X)} O_x \otimes_{O(X)} O_y = O_{2x} \otimes_{O(X)} O_y = O_z \otimes_{O(X)} O_y = O_{zy} = O_z
\]

is flat, since the morphism \(O(X) \to O_z\) is flat. By Corollary 8.2 the morphisms \(O(X) \to \prod_{x \in X} O_x\) and \(O(U_{xy}) \to \prod_{z \in U_{xy}} O_z\) are faithfully flat. Hence, \(X\) is affine. \(\Box\)

4. **Theorem:** A finite fr-space \(X\) is schematic iff for any \(x \leq y, y'\),
1. \( O_y \otimes_{O_x} O_y = O_{xy} \).

2. \( U_{yy} \) is acyclic.

Proof. \( \Rightarrow \) \( U_x \) is an affine finite space. By Theorem 8.3, we are done.

\( \Leftarrow \) \( U_x \) is an affine finite space by Theorem 8.3, then \( X \) is schematic.

\[ \square \]

5. Proposition: Let \( X \) be an affine finite space. An open subset \( U \subseteq X \) is affine iff it is acyclic.

Proof. \( \Leftarrow \) \( U \) satisfies 1′. and 3′ of Theorem 8.3. The composite morphism of the epimorphism \( O_x \otimes_{O(X)} O_y \to O_x \otimes_{O(U)} O_y \) and the morphism \( O_x \otimes_{O(U)} O_y \to O_{xy} \) is an isomorphism, then \( O_x \otimes_{O(U)} O_y \to O_{xy} \) is an isomorphism.

Besides, the morphism \( O(U) = O(X) \otimes_{O(X)} O(U) \to O_x \otimes_{O(X)} O(U) \) is \( O_x \) is flat.

\[ \square \]

6. Proposition: Let \( f: X \to Y \) be a schematic morphism and \( M \) a quasi-coherent \( O_X \)-module. Then, \( R^i f_* M \) is a quasi-coherent \( O_Y \)-module, for any \( i \geq 0 \). If \( Y \) is affine, \( R^i f_* M = H^i(X, M) \).

Proof. We can suppose that \( Y \) is affine. Given \( y \in Y \), the inclusion morphism \( f^{-1}(U_y) \hookrightarrow X \) is affine: Let \( U \subset X \) be an affine subset and \( i \) be the composite morphism \( U \hookrightarrow X \overset{f}{\to} Y \), which is an affine morphism. Then, \( f^{-1}(U) = f^{-1}(U_y) \cap U = i^{-1}(U_y) \) is affine.

Observe that \( R^n f_* (j_*N) = R^n (f \circ j)_* N = 0 \) for any \( n > 0 \) and any quasi-coherent module \( N \), since \( j \) and \( f \circ j \) are affine morphisms. Likewise, \( H^n(X, j_*N) = H^n(f^{-1}(U_y), N) = H^n(Y, (f \circ j)_* N) = 0 \), for any \( n > 0 \).

Denote \( M_{f^{-1}(U_y)} = j_* M f^{-1}(U_y) \) and consider the obvious exact sequence of morphisms

\[ 0 \to M \to \bigoplus_{y \in Y} M_{f^{-1}(U_y)} \overset{\pi}{\to} M' \to 0. \]

Then, \( H^1(X, M) = \text{Coker} \pi_X \) and \( H^{n-1}(X, M') = H^n(X, M) \) for any \( n > 1 \). Besides, \( R^1 f_* M = \text{Coker} \pi \), which is quasi-coherent, and \( R^n f_* M = R^{n-1} f_* M' \) for any \( n > 1 \). Therefore, \( R^1 f_* M = H^1(X, M) \) since \( Y \) is affine. Hence, \( R^1 f_* M' = H^1(X, M') \), since \( M' \) is quasi-coherent. By induction on \( n \),

\[ R^n f_* M = R^{n-1} f_* M' = H^{n-1}(X, M') = H^n(X, M) \]

for any \( n > 1 \).

\[ \square \]

This proposition, Theorem 5.12 and 10 Theorem 5.6 show that the definitions of schematic morphism given in this paper and in [10] are equivalent.
7. **Lemma**: Let $X$ be an $fr$-space and $\delta: X \to X \times X$, $\delta(x) := (x, x)$ be the diagonal morphism. Let $M$ be an $O_X$-module. $R^i\delta_* M$ is quasi-coherent iff

$$H^i(U_{pq}, M) \otimes_{O_p} O_{p'} = H^i(U_{p'q}, M),$$

for any $p \leq p'$ and for any $q$.

**Proof.** $(R^i\delta_* M)_{(q,q')} = H^i(U_{qq'}, M)$ and

$$(R^i\delta_* M)_{(p,p')} \otimes_{O_p} O_{p'} \otimes_{O_{p'}} \otimes_{O_q} O_{q'} = (H^i(U_{pp'}, M) \otimes_{O_p} O_q) \otimes_{O_{p'}} O_{q'},$$

for any $(p, p') \leq (q, q')$. Now, the proof is easily completed. □

8. **Proposition**: Let $X$ be an $fr$-space and $\delta: X \to X \times X$ the diagonal morphism. Let $M$ be an $O_X$-module. Then, $\delta_* M$ is a quasi-coherent $O_{X \times X}$-module iff $M_{p'} \otimes_{O_{p'}} O_{p''} = M_{p'p''}$, for any $p \leq p'$, $p''$.

**Proof.** It is a consequence of Lemma 8.7 and Proposition 3.11. □

9. **Cohomological characterization of schematic finite spaces** (8 4.7, 4.4): Let $X$ be an $fr$-space and $\delta: X \to X \times X$ the diagonal morphism. $X$ is a schematic finite space iff $R^i\delta_* O_X$ is a quasi-coherent module, for any $i \geq 0$.

**Proof.** $\Leftarrow$ We have to prove that $U_p$ is affine, $U_p$ is acyclic, and satisfies the property $2'$ of Theorem 8.3, by the previous proposition. We only need to prove that $U_{qq'}$ is acyclic, for any $q, q' \in U_p$:

$$0 = H^i(U_q, O) \otimes_{O_p} O_{q'} = H^i(U_{pq}, O) \otimes_{O_p} O_{q'} \overset{8.7}{=} H^i(U_{q'q}, O).$$

$\Rightarrow$ The diagonal morphism $\delta$ is schematic by Theorem 3.9 and Theorem 5.12. By Proposition 8.6, we are done. □

10. **Corollary** ([10] 4.5): An $fr$-space $X$ is schematic iff for any open set $j: U_q \hookrightarrow X$, $R^i j_* O_{U_q}$ is a quasi-coherent $O_X$-module, for any $i$.

**Proof.** Let $\delta: X \to X \times X$ be the diagonal morphism. Then, $X$ is a schematic finite space iff $R^i\delta_* O_X$ is a quasi-coherent module, for any $i$, which is equivalent to say that $H^i(U_{pq}, O) \otimes_{O_p} O_{p'} = H^i(U_{p'q}, O)$, for any $p \leq p'$, and any $q$, that is to say, $R^i j_* O_{U_q}$ is a quasi-coherent $O_X$-module, for any $i$ and any open set $j: U_q \hookrightarrow X$. □
A scheme is said to be a semiseparated scheme if the intersection of two affine open sets is affine. For example, the line with a double point is a semiseparated scheme (but it is not separated). The plane with a double point is not semiseparated, but it is quasi-separated.

11. Definition: A ringed finite space $X$ is said to be semiseparated if the open sets $U_{pq}$ are acyclic, for any $p, q \in X$.

12. Proposition: Let $X$ be a ringed finite space and let $\delta: X \to X \times X$ be the diagonal morphism. $X$ is semiseparated iff $R^i \delta_* O_X = 0$, for any $i > 0$.

Proof. $(R^i \delta_* O_X)_{(p,q)} = H^i(U_{pq}, O)$, and $R^i \delta_* O_X = 0$ iff $(R^i \delta_* O_X)_{(p,q)} = 0$ for any $p, q \in X$. Hence, $X$ is semiseparated iff $R^i \delta_* O_X = 0$, for any $i > 0$. □

13. Theorem: Let $X$ be an $fr$-space and $\delta: X \to X \times X$ be the diagonal morphism. $X$ is a semiseparated schematic finite space iff $R^i \delta_* O_X = 0$, for any $i > 0$ and $\delta_* O_X$ is a quasi-coherent module.

Proof. $\Rightarrow$) By 8.9, $\delta_* O_X$ is quasi-coherent, and $R^i \delta_* O_X = 0$ by the previous proposition.

$\Leftarrow$) $X$ is a schematic finite space, by 8.9 and it is semiseparated by the previous proposition. □

14. Proposition: A schematic finite space is semiseparated iff it satisfies any of the following equivalent conditions:

1. The intersection of any two affine open subspaces is affine.

2. There exists an affine open covering of $X$, $\mathcal{U} = \{U_1, \ldots, U_n\}$ such that $U_i \cap U_j$ is affine for any $i, j$.

Proof. Assume $X$ is a semiseparated schematic finite space. Let $\delta: X \to X \times X$, $\delta(x) = (x, x)$ be the diagonal morphism and $U, U'$ two affine open subspaces. Since $R^i \delta_* O_X = 0$, for any $i > 0$, $H^i(U \cap U', O) = H^i(U \times U', \delta_* O) = 0$. By 8.5 $U \cap U'$ is affine.

Assume that there exists an affine open covering of $X$, $\mathcal{U} = \{U_1, \ldots, U_n\}$ such that $U_i \cap U_j$ is affine for any $i, j$. $R^i \delta_* O_X$ is quasi-coherent, by 8.9 $R^i \delta_* O_X(U_i \times U_j) = H^i(U_i \cap U_j, O_X) = 0$, then $R^i \delta_* O_X = 0$ and $X$ is semiseparated. □

All the examples in Examples 3.4 are semiseparated finite spaces.

Finally, let us give some cohomological characterizations of schematic morphisms.

15. Proposition: Let $X$ be an affine finite space and $Y$ a schematic finite space. A morphism of ringed spaces $f: X \to Y$ is affine iff $f_* O_X$ is quasi-coherent and $R^i f_* O_X = 0$, for any $i > 0$. 

Proof. $\Rightarrow$ Affine finite spaces are acyclic, then $(R^i f_* O_X)_y = H^i(f^{-1}(U_y), O_X) = 0$, for any $i > 0$ and any $y \in Y$. Hence, $R^i f_* O_X = 0$, for any $i > 0$.

$\Leftarrow$ Let $U \subseteq Y$ be an affine open subspace. $H^i(f^{-1}(U), O_X) = H^i(U, f_* O_X) = 0$, for any $i > 0$, then $f^{-1}(U)$ is acyclic, therefore it is affine by Proposition 8.5.

16. Proposition: A morphism of ringed spaces $f : X \to Y$ between schematic finite spaces is affine iff $f_* O_X$ is quasi-coherent, $R^i f_* O_X = 0$ for any $i > 0$ and there exists an open covering $\{U_i\}$ of $Y$ such that $f^{-1}(U_i)$ is affine, for any $i$.

Proof. $\Rightarrow$ $(R^i f_* O_X)_y = H^i(f^{-1}(U_y), O_X) = 0$, for any $i > 0$ and any $y \in Y$. Hence, $R^i f_* O_X = 0$, for any $i > 0$.

$\Leftarrow$ The morphisms $f^{-1}(U_i) \to U_i$ are affine, by the previous proposition. Then, $f$ is affine.

17. Proposition: Let $f : X \to Y$ be a morphism of ringed spaces between schematic finite spaces. Then, $f$ is schematic iff $\Gamma_f : X \to X \times_Z Y$, $\Gamma_f(x) = (x, f(x))$ is schematic.

Proof. $\Leftarrow$ It is easy to check that $\pi_2 : X \times_Z Y \to Y$, $\pi_2(x, y) = y$ is schematic. Then, $f$ is schematic because $f = \pi_2 \circ \Gamma_f$ and $\pi_2$ and $\Gamma_f$ are schematic.

$\Rightarrow$ Let $x \in X$ and $(x', y) \in X \times_Z Y$ (where $(x, f(x)) \leq (x', y)$). $U_{x(x', y)} = U_{x'} \cap U_{xy}$ is affine because it is the intersection of two affine open subsets of the affine finite space $U_x$. Observe that

$$O_x \otimes_{O_{x(x', y)}} O_{(x', y)} = O_x \otimes_{O_{x'} \otimes O_{(x', y)}} O_{x'} \otimes_{O_y} O_y \overset{8.14}{=} O_{x'} = O_{x(x', y)}$$

Then, $\Gamma_f$ is schematic, by Proposition 5.14.

18. Theorem: A morphism of ringed spaces $f : X \to Y$ between schematic finite spaces is schematic iff $R^i \Gamma_f_* O_X$ is a quasi-coherent $O_{X \times Y}$-module, for any $i \geq 0$.

Proof. $\Rightarrow$ By Proposition 8.6, $R^i \Gamma_f_* O_X$ is a quasi-coherent $O_{X \times Y}$-module, for any $i \geq 0$.

$\Leftarrow$ $R^i \Gamma_f_* O_X$ is a quasi-coherent $O_{X \times Y}$-module, for any $i \geq 0$. Then,

$$H^0(U_x, O_X) \otimes_{f(x)} O_y = H^0(U_{xf(x)}, O_X) \otimes_{f(y)} O_y = H^0(U_{xf(x)}, O_X) \otimes_{O_{(x,y)}} O_{(x,y)} = H^0(U_{xy}, O_X),$$

for any $x$ and $y \geq f(x)$. Therefore, $O_x \otimes_{O_{f(x)}} O_y = O_{xy}$. Besides,

$$0 = H^i(U_x, O_X) \otimes_{O_{f(x)}} O_y = H^i(U_{xf(x)}, O_X) \otimes_{O_{f(y)}} O_y = H^i(U_{xf(x)}, O_X) \otimes_{O_{(x,y)}} O_{(x,y)} = H^i(U_{xy}, O_X),$$

for any $i > 0$. Then, the open subsets $U_{xy}$ are acyclic, hence $f$ is schematic by Proposition 5.14.
19. **Theorem**: Let \( f : X \to Y \) be a morphism of ringed spaces between schematic finite spaces. Let \( x \in X \) and let \( f_{U_x} \) be the composite morphism \( U_x \hookrightarrow X \to Y \). Then, \( f \) is schematic iff \( R^i f_{U_x}^* O_{U_x} \) is a quasi-coherent \( O_Y \)-module, for any \( i \geq 0 \) and any \( x \in X \).

**Proof.** \( \Rightarrow \) If \( f \) is schematic, \( f_{U_x} \) is schematic and \( R^i f_{U_x}^* O_{U_x} \) is a quasi-coherent \( O_Y \)-module, for any \( i \geq 0 \), by Proposition 8.6.

\( \Leftarrow \) \( R^i f_{U_x}^* O_{U_x} \) is a quasi-coherent \( O_Y \)-module. Then,

\[
H^0(U_x f(x), O_X) \otimes_{O_Y} O_y = H^0(U_x f^*(O_X), O_Y),
\]

for any \( x \) and \( y \geq f(x) \). Therefore, \( O_x \otimes_{O_Y} O_y = O_{xy} \). Besides,

\[
0 = H^i(U_x, O_X) \otimes_{O_Y} O_y = H^i(U_x f(x), O_X) \otimes_{O_Y} O_y = H^i(U_x f^*(O_X), O_Y),
\]

for any \( i > 0 \).

Then, the open sets \( U_{xy} \) are acyclic. By Proposition 5.14, \( f \) is schematic. \( \square \)

9. **Quasi-isomorphisms**

1. **Definition**: A schematic morphism \( f : X \to Y \) is said to be a quasi-isomorphism if

   1. \( f \) is affine.
   2. \( f^* O_X = O_Y \).

   If \( f : X \to Y \) is a quasi-isomorphism we shall say that \( X \) is quasi-isomorphic to \( Y \).

2. **Examples**: 1. If \( X \) is an affine finite space, the morphism \( X \to (\ast, O(X)) \) is a quasi-isomorphism.

2. Let \( X \) be a schematic finite space and let \( \hat{X} \) be the Kolmogorov quotient of \( X \). The quotient morphism \( \pi : X \to \hat{X} \) is a quasi-isomorphism (see Example 4.2.3.).

3. If \( X \) is a schematic finite \( T_0 \)-topological space, \( X_M \hookrightarrow X \) is a quasi-isomorphism.

4. Let \( f : X \to Y \) be a schematic morphism. \((Y, f^* O_X)\) is a schematic finite space by Example 4.8. Let us prove that the obvious ringed morphism \( f' : X \to (Y, f^* O_X) \), \( f'(x) = f(x) \) is schematic. By Theorem 5.15 the morphism

\[
O_x \otimes_{(f^* O_X)_x} (f^* O_X)_y = O_x \otimes_{(f^* O_X)_x} (f^* O_X)_{f(x)} \otimes_{O_Y} O_y = O_x \otimes_{O_Y} O_y \to \prod_{z \in U_y} O_z
\]
is surjective on spectra. Again by Theorem 5.15, \( f' \) is schematic. We have the obvious commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f'} & & \downarrow{\text{Id}} \\
(Y, f, O_X) & & \\
\end{array}
\]

\( \text{Id} \) is affine. If \( f \) is affine, then \( f' \) is a quasi-isomorphism.

3. Definition: Let \( f: X \to Y \) be a schematic morphism. We shall say that \( f \) is flat if the morphism \( O_{Y, f(x)} \to O_{X, x} \) is flat, for any \( x \in X \). We shall say that \( f \) is faithfully flat if the morphism \( O_{Y, y} \to \prod_{x \in f^{-1}(U_i)} O_{X, x} \) is faithfully flat, for any \( y \in Y \).

If \( \{U_i\} \) is an open covering of \( X \), the natural morphism \( \coprod_i U_i \to X \) is faithfully flat.

4. Remark: Quasi-isomorphisms are faithfully flat morphisms: Given \( y \in Y \), \( f^{-1}(U_y) \) is affine, then the morphism

\[
O_y = (f_* O_X)_y = O_X(f^{-1}(U_y)) \to \prod_{x \in f^{-1}(U_i)} O_x
\]

is faithfully flat.

5. Proposition: Let \( X \) be a schematic finite space, \( \mathcal{U} = \{U_1, \ldots, U_n\} \) a minimal affine open covering of \( X \) and \( Y \) the ringed finite space associated with \( \mathcal{U} \). Then, \( Y \) is a schematic finite space and the quotient morphism \( \pi: X \to Y \) is a quasi-isomorphism.

Proof. \( Y = \{y_1, \ldots, y_n\} \), where \( \pi^{-1}(U_{y_i}) = U_i \). Recall that \( \pi_* O_X = O_Y \). Let \( y_1 \leq y_2 \), then

\[
O_{y_1} = O_X(U_{y_1}) \to O_X(U_{y_2}) = O_{y_2}
\]

is a flat morphism by Lemma 6.3. Let \( y_i, y_j \geq y_k \), then

\[
O_{y_i} \otimes_{O_{y_k}} O_{y_j} = O_X(U_i) \otimes_{O_X(U_k)} O_X(U_j) = O_X(U_i \cap U_j) = O_X(\pi^{-1}(U_{y_i} \cap U_{y_j})) = O_Y(U_{y_i} \cap U_{y_j}) = O_{y_i y_j}.
\]

\( U_i \cap U_j \) is an affine finite space by Proposition 2.11. \( U_i \cap U_j = \bigcup_{U_k \subseteq U_i \cap U_j} U_k \). The morphisms

\[
O(U_k) \to \prod_{x \in U_k} O_x, \quad O(U_i \cap U_j) \to \prod_{U_k \subseteq U_i \cap U_j, x \in U_k} O_x
\]

are faithfully flat. Then, the morphism

\[
O(U_i \cap U_j) \to \prod_{U_k \subseteq U_i \cap U_j} O(U_k)
\]

is faithfully flat. Therefore, the morphism \( O_{y_i y_j} \to \prod_{y_k \in U_{y_i} \cap U_{y_j}} O_{y_k} \) is faithfully flat. Then, \( Y \) is a schematic finite space.

Finally, \( \pi \) is affine by Proposition 4.6.

\[\square\]
6. Proposition: The composition of quasi-isomorphisms is a quasi-isomorphism.

7. Theorem: Let \( f : X \to Y \) be a schematic morphism. The functors

\[
\begin{align*}
  f_* : \text{Qc-Mod}_X & \to \text{Qc-Mod}_Y \\
  f^* : \text{Qc-Mod}_Y & \to \text{Qc-Mod}_X
\end{align*}
\]

are mutually inverse (i.e., the natural morphisms \( M \to f_*(f^* M) \), \( f^* f_* N \to N \) are isomorphisms) iff \( f \) is a quasi-isomorphism.

Proof. \( \Leftarrow \) The morphism \( f^* f_* M \to M \) is an isomorphism: It is a local property on \( Y \). We can suppose that \( Y \) is an affine finite space (then \( X \) is affine). Consider a free presentation of \( M, \oplus_I O_X \to \oplus_J O_X \to M \to 0 \). Taking \( f_* \), which is an exact functor because \( R^1 f_* = 0 \), one has the exact sequence of morphisms \( \oplus_I O_Y \to \oplus_J O_Y \to f_* M \to 0 \). Taking \( f^* \), one has the exact sequence of morphisms \( \oplus_I O_X \to \oplus_J O_X \to f^* f_* M \to 0 \), then \( f^* f_* M = M \).

Likewise, \( f_* f^* N = N \).

\( \Rightarrow \) \( O_Y = f_* f^* O_Y = f_* O_X \). Obviously, \( f_* \) is an exact functor. By Theorem 7.9, \( f \) is affine. \( \square \)

8. Corollary: Let \( f : X \to Y \) be a quasi-isomorphism. \( Y \) is affine iff \( X \) is affine.

Proof. \( \Leftarrow \) For any quasi-coherent \( O_Y \)-module \( N, N = f_* f^* N \), then

\[
H^1(Y, N) = H^1(X, f^* N) = 0.
\]

By the Serre Theorem \( Y \) is affine. \( \square \)

9. Corollary: Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be schematic morphisms and assume \( g \circ f \) is a quasi-isomorphism. Then,

1. If \( g \) is a quasi-isomorphism, then \( f \) is a quasi-isomorphism.
2. If \( f \) is a quasi-isomorphism, then \( g \) is a quasi-isomorphism.

Proof. 1. Considering the diagram

\[
\begin{array}{c}
\text{Qc-Mod}_X \\ \xrightarrow{f^*} \\
\xrightarrow{g^*} \\
\text{Qc-Mod}_Y \\ \xrightarrow{(g \circ f)^*} \\
\text{Qc-Mod}_Z
\end{array}
\]

it is easy to prove that \( f_* \) and \( f^* \) are mutually inverse functors.

2. Proceed likewise. \( \square \)
10. **Corollary**: Let $X$ and $Y$ be affine finite spaces. Then, a schematic morphism $f : X \to Y$ is a quasi-isomorphism iff $O_Y(Y) = O_X(X)$.

**Proof.** $\Leftarrow$) Observe that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
(\ast, O_X(X)) & \rightarrow & (\ast, O_Y(Y))
\end{array}
$$

is commutative, $(X, O_X)$ is quasi-isomorphic to $(\ast, O_X(X))$ and $(Y, O_Y)$ is quasi-isomorphic to $(\ast, O_Y(Y))$.

\[\square\]

11. **Corollary**: Let $f : X \to Y$ be a schematic morphism and let $f_M : X_M \to Y_M$ be the induced morphism. Then, $f$ is a quasi-isomorphism iff $f_M$ is a quasi-isomorphism.

**Proof.** It is an immediate consequence of Corollary 9.9. \[\square\]

12. **Corollary**: Let $f : X \to X'$ be a quasi-isomorphism, $X''$ a schematic finite space and $g : X' \to X''$ a morphism of ringed spaces. Then, $g$ is schematic (resp. affine) iff $g \circ f$ is schematic (resp. affine).

**Proof.** Recall Theorem 5.12 (resp. Theorem 7.9). \[\square\]

13. **Proposition**: Let $f : X \to X'$ be an affine morphism of schematic spaces. Assume that $X'$ is $T_0$. Let $U := \{f^{-1}(U_{x'})\}_{x' \in X'}$, let $X/\sim$ be the schematic space associated with the open covering $U$, and $\pi : X \to X/\sim$ the quotient morphism. The morphism $f' : X/\sim \to X'$, $f'([x]) = f(x)$, induced by $f$, is affine and $Y$ is homeomorphic to $\text{Im } f$.

**Proof.** By Corollary 9.12, we only have to prove that $Y$ is homeomorphic to $\text{Im } f$. The morphism $f' : X/\sim \to \text{Im } f$ is clearly bijective and continuous. Given $[x], [x'] \in X/\sim$, if $f'([x]) \leq f'([x'])$, then $f(x) \leq f(x')$ and $U_{f(x')} \subseteq U_{f(x)}$. Hence, $f^{-1}(U_{f(x')}) \subseteq f^{-1}(U_{f(x)})$ and $U_{[x']} \subseteq U_{[x]}$. Therefore, $[x] \leq [x']$. That is, $f'$ is a homeomorphism. \[\square\]

14. **Lemma**: Let $h : X \to Y$ be a quasi-isomorphism. Then, $Y \backslash h(X)$ is a set of removable points of $Y$. 
10. Change of base and flat schematic morphisms

**Proof.** Let \( y \in Y \setminus h(X) \). Since \( h^{-1}(U_y) \) is affine, the morphism

\[
O_Y(U_y) = O_X(h^{-1}(U_y)) \to \prod_{x' \in h^{-1}(U_y)} O_{X,x'}
\]

is faithfully flat. This morphism factors through the morphism

\[
O_Y(U_y) \to \prod_{h(x') \in U_y} O_{Y,h(x')},
\]

then this last morphism is faithfully flat. Hence, \( y \) is a removable point of \( Y \). \( \Box \)

15. **Theorem:** Let \( f : X \to Y \) be a quasi-isomorphism. Assume that \( Y \) is \( T_0 \). Consider the affine open covering of \( X \), \( \{f^{-1}(U_y)\}_{y \in Y} \) and let \( X/\sim \) be the associated schematic finite space. Then, \( f \) is the composition of the quotient morphism \( \pi : X \to X/\sim \) and an isomorphism \( f' : X/\sim \to Y \setminus P, f'([x]) = f(x) \), where \( P \) is a set of removable points of \( Y \). Therefore, if \( f : X \to Y \) is a quasi-isomorphism and \( Y \) is minimal, \( f \) is the composition of the quotient morphism \( X \to X/\sim \) and an isomorphism \( X/\sim \cong Y \).

**Proof.** By Lemma 9.14, we can suppose that \( f \) is surjective. By Proposition 9.13, \( f' : Y' \to Y \) is a homeomorphism and it is affine. Finally, \( O_{Y',[x]} = O_Y([x]) \) for any \( [x] \).

10. **Change of base and flat schematic morphisms**

1. **Proposition ([10] 5.27):** Let \( X, X' \) and \( Y \) schematic finite spaces and \( f : X \to Y \) and \( f' : X' \to Y \) schematic morphisms. Then,

1. \( X \times_Y X' \) is a schematic finite space.
2. If \( X, X' \) and \( Y \) are affine, then \( O(X \times_Y X') = O(X) \otimes_{O(Y)} O(X') \) and \( X \times_Y X' \) is affine.
3. Given a commutative diagram of schematic morphisms

\[
\begin{array}{ccc}
U & \xrightarrow{g} & X \\
\downarrow{h} & & \downarrow{f} \\
V & \xrightarrow{h'} & X' \\
\end{array}
\]

the morphism \( h \times h' : U \times_Y U' \to X \times_Y X' \), \( h \times h'(u,u') := (h(u), h'(u')) \) is schematic.
4. \( \pi : X \times_Y X' \to X, \pi(x,x') = x \), is schematic.
5. If \(h : X \rightarrow X'\) is a schematic \(Y\)-morphism, then \(\Gamma_h : X \rightarrow X \times_Y X', \Gamma_h(x) := (x, h(x))\) is schematic.

6. The diagonal morphism \(\delta : X \rightarrow X \times_Y X, \delta(x) = (x, x)\) is schematic.

**Proof.**

1. We only need to prove 2.

2. \(X \times_{O(Y)} X'\) is an affine schematic space and \(O(X \times_{O(Y)} X') = O(X) \otimes_{O(Y)} O(X')\), by Proposition 2.12. Let \((x, x') \in X \times_{O(Y)} X'\). If \(f(x) = f'(x')\), then

\[
O_x \otimes_{O(Y)} O_{x'} = O_x \otimes_{O(f(x))} (O_{f(x)} \otimes_{O(Y)} O_{f(x)}) \otimes_{O(f(x))} O_{x'} \overset{2.12}{=} O_x \otimes_{O(f(x))} O_{f(x)} \otimes_{O(f(x))} O_{x'} = O_x \otimes_{O(f(x))} O_{x'}
\]

If \(f(x) \neq f'(x')\), then \((x, x') \in X \times_{O(Y)} X'\) is a removable point: Consider the morphism \(f_x : U_x \rightarrow Y, f_x(z) := f(z)\). Then, \(O_{xy} = (f_x O_{U_x}(y)) \otimes_{O(Y)} O_y = O_x \otimes_{O(Y)} O_y\). Observe that \(U_x f(x) \subseteq X'\) and it is affine since \(f_x' : U_x' \rightarrow Y, f_x'(z) := f'(z)\) is affine. The morphism

\[
O_x \otimes_{O(Y)} O_{x'} = O_{xf(x)} \otimes_{O(Y)} O_{x'} = O_x \otimes_{O(f(x))} O_{f(x)} \otimes_{O(Y)} O_{x'} = O_x \otimes_{O(Y)} O_{x'} \rightarrow \prod_{z \in U_x f(x)} O_z \otimes_{O(Y)} O_z
\]

is faithfully flat. Therefore, \((x, x')\) is removable. In conclusion, \(X \times_Y X' = (X \times_{O(Y)} X') \setminus \{A\}\) set of removable points, then \(X \times_Y X'\) is affine and \(O(X \times_Y X') = O(X) \otimes_{O(Y)} O(X')\).

3. By Proposition 5.14 given \((u, u') \in U \times_{U'} U', (x, x') \in X \times_Y X'\) (where \((x, x') \geq (h(u), h'(u'))\)), we have to prove that \(U_{(u, u')(x, x')}: U_{(u, u')(x, x')}\) is acyclic and \(O_{(u, u')(h(u), h'(u'))} \otimes_{O(h(u), h'(u'))} O_{(x, x')} = O_{(u, u')(x, x')}: U_{(u, u')(x, x')} = U_{U_{(u, u')(x, x')}} U_{u'u'}\), which is affine (then acyclic) and

\[
O_{(u, u')} \otimes_{O(h(u), h'(u'))} O_{(x, x')} = (O_u \otimes_{O(h(u))} O_{u'}) \otimes_{O(h(u), O(h'(u}))} O_{(x, x')} = O_x \otimes_{O(f(x))} O_{x'} \overset{5.14}{=} O_{(u, u')(x, x')} U_{(u, u')(x, x')}
\]

4., 5. and 6. are particular cases of 3.

\[\square\]

Obviously,

\[
\text{Hom}_Y(Z, X \times_Y X') = \text{Hom}_Y(Z, X) \times \text{Hom}_Y(Z, X')
\]

for any schematic finite \(Y\)-spaces \(Z, X, X'\).

**2. Proposition:** Affine morphisms and quasi-isomorphisms are stable by base change.

**Proof.** Let \(f : X \rightarrow Y\) be an affine morphism and \(Y' \rightarrow Y\) a schematic morphism. In order to prove that the schematic morphism \(X \times_Y Y' \rightarrow Y'\) is affine, it is sufficient to prove that \(X \times_Y U_{y'}\) is affine, for any \(y' \in Y'\). Observe that \(X \times_Y U_{y'} = f^{-1}(U_{f(y')}) \times_{U_{f(y')}} U_{y'}\), which is affine because \(f^{-1}(U_{f(y')})\) is affine.
Let \( f \) be a quasi-isomorphism. We only have to prove that \( \mathcal{O}(X \times_Y U_{y'}) = \mathcal{O}_{y'} \):

\[
\mathcal{O}(X \times_Y U_{y'}) = \mathcal{O}(f^{-1}(U_{f(y')})) \times_{U_{f(y')}} U_{y'} = \mathcal{O}(f^{-1}(U_{f(y')})) \otimes_{\mathcal{O}_{f(y')}} \mathcal{O}_{y'} = \mathcal{O}_{f(y')} \otimes_{\mathcal{O}_{f(y')}} \mathcal{O}_{y'} = \mathcal{O}_{y'}.
\]

\[\square\]

3. **Lemma**: Let \( f : X \to Y \) and \( g : Y' \to Y \) be schematic morphisms and let \( g' : X \times_Y Y' \to X \) be defined by \( g'(x, y') := x \). Let \( \mathcal{M} \) be a quasi-coherent \( \mathcal{O}_X \)-module. If \( X, Y \) and \( Y' \) are affine, then

\[ \Gamma(X \times_Y Y', g'' \mathcal{M}) = \Gamma(X, \mathcal{M}) \otimes_{\mathcal{O}(Y')} \mathcal{O}(Y'). \]

**Proof.** Consider an exact sequence of \( \mathcal{O}_X \)-modules \( \bigoplus_i \mathcal{O}_X \to \bigoplus_j \mathcal{O}_X \to \mathcal{M} \to 0 \).

1. Taking \( g'' \), \( \bigoplus_i \mathcal{O}_{X_y} \to \bigoplus_j \mathcal{O}_{X_y} \to g'' \mathcal{M} \to 0 \) is exact. Taking sections, the sequence \( \bigoplus_i \mathcal{O}(X \times_Y Y') \to \bigoplus_j \mathcal{O}(X \times_Y Y') \to g'' \mathcal{M}(X \times_Y Y') \to 0 \) is exact. By Proposition \[10.1\]

\( \mathcal{O}(X \times_Y Y') = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \). Hence, the sequence of morphisms

\[ \bigoplus_i \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to \bigoplus_j \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to g'' \mathcal{M}(X \times_Y Y') \to 0 \]

is exact.

2. The sequence of \( \mathcal{O}(X) \)-modules \( \bigoplus_i \mathcal{O}(X) \to \bigoplus_j \mathcal{O}(X) \to \mathcal{M}(X) \to 0 \) is exact. Hence, the sequence \( \bigoplus_i \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to \bigoplus_j \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to \mathcal{M}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to 0 \) is exact.

3. Therefore, the natural morphism \( \Gamma(X, \mathcal{M}) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Y') \to \Gamma(X \times_Y Y', g'' \mathcal{M}) \) is an isomorphism.

\[\square\]

4. **Theorem**: Let \( f : X \to Y \) be a schematic morphism and \( g : Y' \to Y \) a flat schematic morphism. Denote \( f' : X \times_Y Y' \to Y' \), \( f'(x, y') = y' \), \( g' : X \times_Y Y' \to X \), \( g'(x, y') = x \) the induced morphisms. Then, the natural morphism

\[ g^* R^i f_* \mathcal{M} \to R^i f'_* (g'' \mathcal{M}) \]

is an isomorphism.

**Proof.** We have to prove that the morphism is an isomorphism on stalks at \( z \), for any \( z \in Y' \). That is to say, we have to prove that the morphism

\[ H^i(f^{-1}(U_{g(z)}), \mathcal{M}) \otimes_{\mathcal{O}_{g(z)}} \mathcal{O}_z \to H^i(f^{-1}(U_{g(z)}) \times_{U_{g(z)}} U_z, g'' \mathcal{M}) \]
is an isomorphism. We can suppose that \( Y' = U_z, Y = U_{g(z)} \) and \( X = f^{-1}(U_{g(z)}) \). Then, \( g' \) is an affine morphism and \( H^i(X \times_Y Y', g'' M) = H^i(X, g'_* M) \). By Lemma 10.3,

\[
(g'_* M)_x = \Gamma(U_x \times_Y Y', g'' M) = M_x \otimes_{O(Y)} O(Y').
\]

Hence, \( C(g'_* M) = (C' M) \otimes_{O(Y')} O(Y') \) and \( H^i(X, g'_* M) = H^i(X, M) \otimes_{O(Y')} O(Y'). \) That is,

\[
H^i(X \times_Y Y', g'' M) = H^i(X, g'_* M) = H^i(X, M) \otimes_{O(Y')} O(Y').
\]

\[\square\]

5. Proposition: Let \( X \) and \( Y \) be schematic finite spaces and \( f: X \to Y \) a schematic morphism. Then, \( f \) is flat (resp. faithfully flat) iff the functor

\[f^*: \text{Qc-Mod}_Y \to \text{Qc-Mod}_X, \ M \mapsto f^* M\]

is exact (resp. faithfully exact).

Proof. Obviously, if \( f \) is flat then \( f^* \) is exact. If \( f \) is faithfully flat then \( f^* \) is faithfully exact: We only have to check that \( M = 0 \) if \( f^* M = 0 \). Given \( y \in Y \), the morphism \( O_y \twoheadrightarrow \prod x \in f^{-1}(U), O_x \) is faithfully flat. Tensoring by \( M_y \otimes_{O_y} \) one has the injective morphism

\[
M_y \twoheadrightarrow \prod x \in f^{-1}(U), M_y \otimes_{O_y} O_x = \prod x \in f^{-1}(U), M_y \otimes_{O_y} O_x \otimes_{O_f(x)} O_x = \prod \prod M_{f(x)} \otimes_{O_f(x)} O_x = \prod M_{f(x)} \otimes_{O_f(x)} O_x = (f^* M)_x = 0.
\]

Hence, \( M = 0 \).

If \( f^* \) is exact \( f \) is flat: Let \( f(x) \in Y \). Consider an ideal \( I \subseteq O_{f(x)} \) and let \( \bar{I} \subseteq O_{f^{-1}(U)} \) be the quasi-coherent \( O_{U_{f(x)}} \)-module associated with \( I \). Consider the inclusion morphism \( i: U_f \to Y \). The morphism \( i, \bar{I} \hookrightarrow i, O_{f^{-1}(U)} \) is injective, then \( f^* i, \bar{I} \hookrightarrow f^* i, O_{f^{-1}(U)} \) is injective. Hence, \( i, O_{f^{-1}(U)} \otimes_{O_{f^{-1}(U)}} O_x = (f^* i, O_{f^{-1}(U)})_x = O_x \) is injective. Therefore, the morphism \( O_{f(x)} \to O_x \) is flat. Finally, if \( f^* \) is faithfully exact \( f \) is faithfully flat: Let \( y \in Y \) be a maximal point of \( Y \), if it exists, such that the flat morphism \( O_y \to \prod x \in f^{-1}(U), O_x \) is not faithfully flat. Then, there exists an ideal \( I \subseteq O_y \) such that \( I \cdot \prod x \in f^{-1}(U), O_x = \prod x \in f^{-1}(U), O_x \). Let \( M \) be the quasi-coherent \( O_{f^{-1}(U)} \)-module associated with \( O_y / I \) and let \( f': f^{-1}(U) \to U_y \) be the morphism defined by \( f'(x) := f(x) \). Obviously, \( f'^* M = 0 \). Let \( i: U_y \hookrightarrow Y \) and \( \bar{i}: f^{-1}(U) \hookrightarrow X \) be the inclusion morphisms. By Theorem 10.4, \( 0 = \bar{i}, f'^* M = f^* i, M, \) then \( i, M = 0 \). Hence, \( 0 = i, f^* M = M \) which is contradictory. 

\[\square\]
6. **Proposition**: Let $f : X \to Y$ be a schematic morphism. Then, $f$ is a quasi-isomorphism iff it is faithfully flat and the natural morphism $f^* f_* \mathcal{M} \to \mathcal{M}$ is an isomorphism, for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$.

**Proof.** $\Rightarrow$ It is Remark 9.4 and Theorem 9.7.

$\Leftarrow$) $f_*$ is an exact functor since $f^*$ is faithfully exact and $\text{Id} = f^* f_*$. By Theorem 7.9, $f$ is affine. Finally, the morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism, since $f^* \mathcal{O}_Y = \mathcal{O}_X \to f^* f_* \mathcal{O}_X$ is an isomorphism.

\[\square\]

7. **Proposition**: Let $f : X \to Y$ be a schematic morphism. Then, $f$ is faithfully flat iff $f_M : X_M \to Y_M$ is surjective and flat.

**Proof.** Let $g : X' \to X$ be a quasi-isomorphism. Then, $f$ is faithfully flat iff $f \circ g$ is faithfully flat, since $f^*$ is faithfully exact iff $g^* \circ f^*$ is faithfully exact. Let $g : Y \to Y'$ be a quasi-isomorphism. Likewise, $f$ is faithfully flat iff $g \circ f$ is faithfully flat.

Therefore, we have to prove that $f_M$ is a faithfully flat iff is surjective and flat. The converse implication is obvious. Let us prove the direct implication. If $f$ is not surjective, let $y \in Y_M$ be maximal satisfying $f^{-1}(y) = \emptyset$. Consider the commutative diagram of obvious morphisms

\[
\begin{array}{ccc}
\mathcal{O}_Y & \xrightarrow{i_1} & \prod_{f_M(x) \geq y} \mathcal{O}_x \\
\downarrow{i_2} & & \downarrow{i_4} \\
\prod_{y' \geq y} \mathcal{O}_{y'} & \xleftarrow{i_3} & \prod_{y' \geq y} \prod_{f_M(x) \geq y} \mathcal{O}_x \\
\end{array}
\]

The morphisms $i_1$ and $i_3$ are faithfully flat since $f_M$ is faithfully flat, $i_4$ is obviously faithfully flat, hence $i_2$ is faithfully flat and $y$ is a removable point, which is contradictory.

\[\square\]

8. **Proposition**: Let $f : X \to Y$ be a schematic morphism and $g : Y' \to Y$ a faithfully flat schematic morphism. Let $f' : X \times_Y Y' \to Y'$ be the morphism defined by $f'(x, y') = y$. Then,

1. $f$ is affine iff $f'$ is affine.

2. $f$ is a quasi-isomorphism iff $f'$ is a quasi-isomorphism.

**Proof.** We can suppose that $X, Y$ and $Y'$ are minimal schematic spaces. The morphism $g' : X \times_Y Y' \to X$, $g'(x, y') := x$ is faithfully flat since it is flat and surjective.

1. $\Leftarrow$) The functor $g^* f_* = f'_* g'^*$ is exact since $f'_*$ and $g'^*$ are exact. Hence, $f_*$ is exact since $g^*$ is faithfully exact and $f$ is affine.
2. \(\Leftarrow\) We only have to prove that the morphism \(O_Y \to f_*O_X\) is an isomorphism. Taking \(g^*\), we obtain the isomorphism \(O_X \to g^*f_*O_X = f'_*g^*O_X = f'_*O_{X \times_Y Y'} = O_X\). Hence, \(O_Y \to f_*O_X\) is an isomorphism. \(\square\)

11 Quasi-open immersions

1. Definition: We shall say that a schematic morphism \(f: X \to Y\) is a quasi-open immersion if it is flat and the diagonal morphism \(X \to X \times_Y X\) is a quasi-isomorphism.

2. Example: If \(X\) is a schematic finite space and \(U \subseteq X\) an open subset, then the inclusion morphism \(U \hookrightarrow X\) is a quasi-open immersion.

3. Proposition: If \(f: X \to Y\) is a quasi-isomorphism, then it is a quasi-open immersion

Proof. Quasi-isomorphisms are faithfully flat morphisms, by Observation \(\text{[9.4]}\). The morphism \(X \times_Y X \to X\) is a quasi-isomorphism by Proposition \(\text{[10.2]}\). The composite morphism \(X \to X \times_Y X \to X\) is the identity morphism, then \(X \to X \times_Y X\) is a quasi-isomorphism, by Corollary \(\text{[9.9]}\). \(\square\)

4. Proposition: If \(f: X \to Y\) is a quasi-open immersion and \(Y' \to Y\) a schematic morphism, then \(X \times_Y Y' \to Y'\) is a quasi-open immersion.

Proof. The morphism \(X \to Y\) is flat. Taking \(\times_Y Y'\), the morphism \(X \times_Y Y' \to Y'\) is flat. The morphism \(X \to X \times_Y X\) is a quasi-isomorphism. Taking \(\times_Y Y'\), the morphism \(X \times_Y Y' \to (X \times_Y Y') \times_Y (X \times_Y Y')\) is a quasi-isomorphism, by Proposition \(\text{[10.2]}\). Hence, \(X \times_Y Y' \to Y'\) is a quasi-open immersion. \(\square\)

5. Proposition: Let \(f: X \to Y\) be a schematic morphism. Let \(Y' \to Y\) be a faithfully flat schematic morphism and \(f': X \times_Y Y' \to Y'\), \(f'(x, y') := f(x)\) the induced morphism. Then \(f\) is a quasi-open immersion iff \(f'\) is a quasi-open immersion.

6. Proposition: The composition of two quasi-open immersions is a quasi-open immersion.

Proof. The composition of two flat morphisms is flat. Let \(f: X \to Y\), \(g: Y \to Z\) be quasi-open immersions. Consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\delta_X} & X \times_Y X \\
& & \downarrow \text{Id} \\
Y & \xrightarrow{\delta_Y} & Y \times_Z Y
\end{array}
\]

\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\text{Id} \times \text{Id}} & X \times_Z X \\
& & \downarrow f \times f \\
Y \times_Y Y' & & Y \times_Z Y'
\end{array}
\]
Observe that $X \times_Y X = Y \times_{Y \times_Y Y} (X \times_Z X)$ and $\delta_Y$ is a quasi-isomorphism. Then, $\text{Id} \times \text{Id}$ is a quasi-isomorphism by Proposition 11.4. The morphism $(\text{Id} \times \text{Id}) \circ \delta_X$ is a quasi-isomorphism by Proposition 9.6, hence $g \circ f$ is a quasi-open immersion.

7. Proposition: Let $f : X \to Y$, $g : Y \to Z$ be schematic morphisms and suppose $g \circ f$ is a quasi-open immersion.

1. If $g$ is a quasi-open immersion, then $f$ is a quasi-open immersion.

2. If $f$ is a quasi-isomorphism, then $g$ is a quasi-open immersion.

Proof. 1. Consider the commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\delta_X} & X \times_Y X & \xrightarrow{\text{Id} \times \text{Id}} & X \times_Z X \\
\downarrow & & \downarrow & & \downarrow f \times f \\
Y & \xrightarrow{\delta_Y} & Y \times_Z Y
\end{array}
$$

$\text{Id} \times \text{Id}$ is a quasi-isomorphism since $\delta_Y$ is a quasi-isomorphism. $(\text{Id} \times \text{Id}) \circ \delta_X$ is a quasi-isomorphism, since $g \circ f$ is a quasi-open immersion. Hence, $\delta_X$ is a quasi-isomorphism, by Corollary 9.9 that is, $f$ is a quasi-open immersion.

2. The obvious morphism $X \times_Z X \to Y \times_Z Y$ is a quasi-isomorphism, since is the composition of the quasi-isomorphisms $X \times_Z X \to X \times_Z Y$, $X \times_Z Y \to Y \times_Z Y$. Consider the commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\delta_X} & X \times_Z X \\
f & & \downarrow f \times f \\
Y & \xrightarrow{\delta_Y} & Y \times_Z Y
\end{array}
$$

Then, $\delta_Y$ is a quasi-isomorphism since $f$, $\delta_X$ and $f \times f$ are quasi-isomorphisms. That is, $g$ is a quasi-open immersion.

8. Definition: Let $X$ and $Y$ be ringed finite spaces and $f : X \to Y$ a morphism of ringed spaces. $C(f) := X \downarrow Y$ is a finite ringed space as follows: the order relation on $X$ and on $Y$ is the pre-established order relation, and given $x \in X$ and $y \in Y$ we shall say that $x > y$ if $f(x) \geq y$; $O_{C(f), x} := O_{X,x}$ for any $x \in X$, $O_{C(f), y} := O_{Y,y}$ for any $y \in Y$; the morphisms between the stalks of $O_{C(f)}$ are defined in the obvious way.
Observe that $X$ is an open subset of $C(f)$ and $F: C(f) \to Y$, $F(x) := f(x)$, for any $x \in X$ and $F(y) := y$, for any $y \in Y$ is a morphism of ringed spaces. $F_*O_{C(f)} = O_Y$ because

$$(F_*O_{C(f)})_y = O_{C(f),y} = O_{Y,y}$$

Besides, $R^iF_*O_{C(f)} = 0$ for any $i > 0$, because

$$(R^iF_*O_{C(f)})_y = H^i(U_y, O_{C(f)}) = 0.$$ 

9. **Theorem:** Let $f: X \to Y$ be a schematic morphism. Then, $f$ is a quasi-open immersion if $C(f) = X \sqcup Y$ is a schematic finite space. If $f$ is a quasi-open immersion, then it is the composition of the open inclusion $X \hookrightarrow C(f)$ and the quasi-isomorphism $F: C(f) \to Y$.

**Proof.** $\Rightarrow$ Given, $x \geq x' \in X \subset C(f)$, the morphism

$$O_{C(f),x'} = O_{X,x'} \to O_{X,x} = O_{C(f),x}$$

is flat. Given, $y \leq y' \in Y \subset C(f)$, the morphism

$$O_{C(f),y} = O_{Y,y} \to O_{Y,y'} = O_{C(f),y'}$$

is flat. Given $x \in X \subset C(f)$ and $f(x) \geq y \in Y \subset C(f)$, the morphism

$$O_{C(f),y} \to O_{C(f),f(x)} = O_{Y,f(x)} \to O_{X,x} = O_{C(f),x}$$

is flat.

Given $c \in C(f)$, we shall denote $\tilde{U}_c := \{z \in C(f) : z \geq c\}$. We have to prove that $\tilde{U}_c$ is affine. Recall Theorem 8.3

a. If $c = x \in X$, then $\tilde{U}_c = U_x \subseteq X$ is affine. If $c = y \in Y$, $\tilde{U}_y$ is acyclic.

b. Given $x, x' \in \tilde{U}_y \cap X$, $\tilde{U}_x \cap \tilde{U}_{x'} = U_x \cap U_{x'}$ which is quasi-isomorphic to $U_x \times_{U_y} U_{x'}$, then $\tilde{U}_x \cap \tilde{U}_{x'}$ is acyclic and

$$O_{C(f),x',x'} = O_{x',x'} = O_x \otimes_{O_y} O_{x'} = O_{C(f),x} \otimes_{O_{C(f),y}} O_{C(f),x'}$$

c. Given, $y', y'' \in \tilde{U}_y \cap Y$, $\tilde{U}_{y'} \cap \tilde{U}_{y''} = F^{-1}(U_{y'} \cap U_{y''})$, which is acyclic because $U_{y'} \cap U_{y''}$ is acyclic, and

$$O_{C(f),y',y''} = O_{y',y''} = O_{y'} \otimes_{O_y} O_{y''} = O_{C(f),y'} \otimes_{O_{C(f),y}} O_{C(f),y''}$$

d. Given $x, y' \in \tilde{U}_y$, where $x \in X$ and $y' \in Y$. Observe that $\tilde{U}_x \cap \tilde{U}_{y'} = U_{xy'}$ and $U_{xy'} = f_{xy}^{-1}(U_{f(x)} \cap U_{y'})$, which is affine since $f_{xy}: U_x \to U_{f(x)}$ is affine. and $U_{f(x)} \cap U_{y'} \subset U_y$ is affine. Finally,

$$O_{C(f),xy'} = O_{xy'} \otimes_{O_y} O_{y'} = O_{C(f),x} \otimes_{O_{C(f),y}} O_{C(f),y'}$$
(*) observe that $U_x \times U_y U_{xy} = U_{xy}$.
Therefore $C(f)$ is schematic.

If $\mathcal{M}$ is a $O_{C(f)}$-quasi-coherent module, $F, \mathcal{M}$ is a quasi-coherent $O_Y$-module since

$$(F, \mathcal{M})_y \otimes_{O_{Y_y}} O_{Y_y'} = \mathcal{M}_y \otimes_{O_{C(f), y'}} O_{C(f), y'} = (F, \mathcal{M})_{y'}.$$

By Theorem 5.12, $F$ is schematic. $F$ is an isomorphism since $F_* O_{C(f)} = O_Y$ and $F^{-1}(U_y) = \bar{U}_y$, for any $y \in Y$.

$\Rightarrow$ The morphism $f$ is the composition of the open immersion $X \hookrightarrow C(f)$ and the quasi-isomorphism $F : C(f) \to Y$, hence $f$ is a quasi-open immersion.

$\square$

10. Proposition: Let $f : X \to Y$ be a schematic morphism. Then, $f$ is a quasi-isomorphism iff it is a faithfully flat quasi-open immersion.

Proof. $\Rightarrow$ It is Remark 9.4 and Proposition 11.3

$\Leftarrow$ If $y \in Y$, then $y$ is a removable point of $C(f)$, since the morphism

$$O_{C(f), y} = O_{Y_y} \to \prod_{x \in f^{-1}(U_y)} O_{x, x} = \prod_{x, y \in X} O_{C(f), x}$$

is faithfully flat. The morphism $X \to C(f)$ is a quasi-isomorphism, since $X = C(f) - Y$ and $C(f) - Y$ is quasi-isomorphic to $C(f)$. Finally, $X$ is quasi-isomorphic to $Y$, since $C(f)$ is quasi-isomorphic to $Y$.

$\square$

11. Theorem: Let $f : X \to Y$ be a schematic morphism. Then, $f$ is a quasi-open immersion iff $f$ is flat and the morphism $f^* f_* \mathcal{M} \to \mathcal{M}$ is an isomorphism for any quasi-coherent $O_X$-module.

Proof. $\Rightarrow$ The diagonal morphism $\delta : X \to X \times_Y X$ is a quasi-isomorphism. Then, $\delta$ is affine. Consider the projections, $\pi_1, \pi_2 : X \times_Y X \to X$. The morphism $f$ is flat, then $f^* f_* \mathcal{M} = \pi_2^* \pi_1^* \mathcal{M}$, by Theorem 10.3. Observe that

$$(\pi_2^* \pi_1^* \mathcal{M})_x = \Gamma(X \times_Y U_x, \pi_1^* \mathcal{M}) = \Gamma(X \times_Y U_x, \delta_* \delta^* \pi_1^* \mathcal{M}) = \Gamma(X \times_Y U_x, \delta_* \mathcal{M})$$

$$= \Gamma(U_x, \mathcal{M}) = \mathcal{M}_x,$$

for any $x \in X$. Therefore, the morphism $f^* f_* \mathcal{M} \to \mathcal{M}$ is an isomorphism.

$\Leftarrow$ Let $i : U_x \to X$ be the obvious inclusion and denote $i_* \mathcal{M}|_{U_x} = \mathcal{M}|_{U_x}$. The natural morphism $f^* f_* \mathcal{M}|_{U_x} \to \mathcal{M}|_{U_x}$ is an isomorphism. Then,

$$(\pi_1^* \mathcal{M})(x, x') = \mathcal{M}_x \otimes_{O_{U_x}} O_{x'} = (f_* \mathcal{M}_{U_x})_{f(x)} \otimes_{O_{f(x)}} O_{x'} = (f^* f_* \mathcal{M}|_{U_x})(x') = (\mathcal{M}|_{U_x})(x')$$

is

$$= \mathcal{M}|_{U_{xx'}} = (\delta_* \mathcal{M})(x, x'),$$
Besides, $\delta$ is an exact functor, since $\pi_X$ is flat. By Theorem 7.9, $\delta$ is an affine morphism. Besides, $O_{X \times_Y X} = \pi_X^* O_X = \delta_* O_X$. Hence, $\delta$ is a quasi-isomorphism and $f$ is a quasi-open immersion.

\[\square\]

12. Lemma: Let $f : X \to Y$ be a schematic morphism and suppose that $X$ is affine and $f_* O_X = O_Y$. Then, $f$ is a quasi-open immersion.

Proof. We have to prove that $\mathcal{C}(f)$ is a schematic finite space. By Proposition 3.6, we have to prove that $O_c \otimes_{O_X} O_{z_2} \to \prod_{w \in U_{z_2}} O_w$ is faithfully flat, for any $z \leq z_1, z_2 \in C(f)$.

Suppose that $z_1, z_2 \in X$. The epimorphism $O_{z_1} \otimes_{O_Y} O_{z_2} \to O_{z_1} \otimes_{O_Y} O_{z_2}$ is an isomorphism, since the composite morphism $O_{z_1} \otimes_{O_Y} O_{z_2} \to O_{z_1} \otimes_{O_Y} O_{z_2} \to O_{z_2}$ is an isomorphism. Besides, the morphism $O_{z_1} \otimes_{O_{z_1}} O_{z_2} \to \prod_{w \in U_{z_2}} O_w$ is faithfully flat, since $X$ is affine.

Suppose that $z_1 \in X$ and that $z_2 \in Y$ (then $z \in Y$). Observe that $U_{z_1} \cap f^{-1}(U_{z_2}) = U_{z_1} \cap \{c \in C(f) : c \geq z_1, z_2\}$. Then,

$$O_{z_1} \otimes_{O_{z_1}} O_{z_2} = O_{z_1} \otimes_{O_{z_1}} O_{z_2} = O_{z_1} \otimes_{O_Y} (O_{z_1} \otimes_{O_{z_2}} O_{z_2}) = O_{z_1} \otimes_{O_{z_1}} O_{z_2} = O_{z_1} \otimes_{O_{z_1}} O_{z_2} = O_{z_1} \otimes_{O_{z_1}} O_{z_2}$$

is faithfully flat, since $U_{z_1} \subset X$ is affine, by Proposition 5.14.

Suppose that $z_1, z_2 \in Y$. Observe that $U_{C(f) \times_{z_2}} = U_{Y_{z_2}} \prod f^{-1}(U_{Y_{z_2}})$ and $O_Y(U_{Y_{z_2}}) = O_X(f^{-1}(U_{Y_{z_2}}))$. The morphism $O_{z_1} \otimes_{O_Y} O_{z_2} = O_{Y_{z_2}} \to \prod_{w \in U_{Y_{z_2}}} O_w$ is faithfully flat, since $U_{Y_{z_2}}$ is affine. For any open subset $V \subset X$ and any $x \in V$, the morphism $O_X(V) = O_X \otimes_{O_X} O(V) \to O_{y \in O_X(Y)} O(V) \otimes_{O_X} O_x$ is flat. Hence, the morphism $O_{z_1} \otimes_{O_Y} O_{z_2} = O_{Y_{z_2}} = O_X(f^{-1}(U_{Y_{z_2}})) \to \prod_{x \in f^{-1}(U_{Y_{z_2}})} O_x$ is flat. Therefore, the morphism

$$O_{z_1} \otimes_{O_Y} O_{z_2} \to \prod_{w \in U_{C(f) \times_{z_2}}} O_w = \prod_{w \in U_{Y_{z_2}}} O_w \times \prod_{x \in f^{-1}(U_{Y_{z_2}})} O_x$$

is faithfully flat.

\[\square\]

13. Proposition: Let $f : X \to Y$ be a schematic morphism and suppose that $X$ is affine. Then, there exist an open immersion $i : X \to Z$ such that $i_* X = O_Z$ and an affine morphism $g : Z \to Y$ such that $f = g \circ i$.

Proof. The obvious morphism $f' : X \to (Y, f_* O_X)$, $f'(x) = f(x)$ is a quasi-open immersion by the lemma above and Example 9.2. Let $i : X \to C(f')$, $\pi : C(f') \to (Y, f_* O_X)$ and let $\text{Id} : (Y, f_* O_X) \to Y$ be the obvious morphism. Observe that $i$ is an open immersion, $g := \text{Id} \circ \pi$ is affine, since $\pi$ is a quasi-isomorphism and $\text{Id}$ is affine, and

$$f = \text{Id} \circ f' = \text{Id} \circ \pi \circ i = g \circ i.$$

\[\square\]
12 Quasi-closed immersions

Let $I \subset O_X$ be a quasi-coherent ideal and $(I)_0 := \{x \in X: (O_X/I)_x \neq 0\}$, which is a closed subspace of $X$. Consider the schematic space $(X, O_X/I)$ and observe that $x \in X \setminus Y$ iff $(O_X/I)_x = 0$. Hence, $X \setminus Y$ is a set of removable points of $(X, O_X/I)$ and the obvious morphism $((I)_0, O_X/I) \to (X, O_X/I)$ is a quasi-isomorphism. We shall say that the composition of the morphisms

$$((I)_0, O_X/I) \to (X, O_X/I) \to (X, O_X)$$

is a closed immersion.

Let $f: X' \to X$ be a schematic morphism. Let $I = \text{Ker}[O_X \to f_*O_{X'}]$. The obvious morphism $f': X' \to (X, O_X/I)$, $f'(x) = f(x)$ is schematic since $f'^*M = f_*M$ is a quasi-coherent $O_X/I$-module, any quasi-coherent $O_X/I$-module $M$, because it is a quasi-coherent $O_X$-module. Obviously, $f$ is the composition of the morphisms $X' \to (X, O_X/I) \to (X, O_X)$. Assume that $O_{X', x'} \neq 0$ for any $x' \in X'$ (recall that if $O_{X', x'} = 0$ then $x'$ is a removable point). The closure of $f(X')$ in $X$ is $(I)_0$: $x \in X \setminus (I)_0$ iff $(O_X/I)_x = 0$, which is equivalent to saying that $(f_*O_{X'})_x = 0$ since the morphism of sheaves (of rings) $O_X/I \to f_*O_{X'}$ is injective. $(f_*O_{X'})_x = O_{X'}(f^{-1}(U_x)) = 0$ iff $f^{-1}(U_x) = \emptyset$. The morphism $X' \to ((I)_0, O_X/I)$, $x \mapsto f(x)$ is schematic and $f'$ is the composition of the morphisms $X' \to ((I)_0, O_X/I) \to (X, O_X/I)$ (see Proposition 6.11).

1. **Definition**: Let $f: X' \to X$ be a schematic morphism. We shall say that $f$ is a quasi-closed immersion if it is affine and the morphism $O_X \to f_*O_{X'}$ is an epimorphism.

Suppose that $O_{X', x'} \neq 0$, for any $x' \in X'$, let $f: X' \to X$ be a quasi-closed immersion and $I = \text{Ker}[O_X \to f_*O_{X'}]$. Then $f$ is the composition of a quasi-isomorphism $X' \to ((I)_0, O_X/I)$ and a closed immersion $((I)_0, O_X/I) \to (X, O_X)$.

13 Spec $O_X$

Let $\{X_i, f_{ij}\}_{i,j \in I}$ (where $|I| < \infty$) be a direct system of morphisms of ringed spaces. Let $\lim_{\rightarrow i} X_i$ be the direct limit of the topological spaces $X_i$: $\lim_{\rightarrow i} X_i = \bigsqcup_{i} X_i/\sim$, where $\sim$ is the equivalence relation generated by the relation $x_i \sim f_{ij}(x_i)$, and $U \subseteq \lim_{\rightarrow i} X_i$ is an open subset iff $f_{ij}^{-1}(U)$ is an open subset for any $j \in I$, where $f_j: X_j \to \lim_{\rightarrow i} X_i$ is the natural map. Define $O_{\lim_{\rightarrow i} X_i}(V) := \lim_{\rightarrow i} O_X(f_{ij}^{-1}(V))$, for any open set $V \subseteq \lim_{\rightarrow i} X_i$. It is well known that
4. Proposition: Let $X$ be a schematic finite space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal. The ideal $\operatorname{rad}(\mathcal{I}) \subseteq \mathcal{O}_X$, defined by $\operatorname{rad}(\mathcal{I})_x := \operatorname{rad}(\mathcal{I})_x$ for any $x \in X$, is a quasi-coherent ideal of $\mathcal{O}_X$.

Proof. We only have to prove that given a flat morphism $A \to B$ such that $B \otimes_A B = B$ and an ideal $I \subseteq A$, then $\operatorname{rad}(I) \cdot B = \operatorname{rad}(I \cdot B)$. This is a consequence of Lemma 13.3. 

13. Spec $\mathcal{O}_X$

$(\lim_{i} X_i, \mathcal{O}_{\lim X_i})$ is the direct limit of the direct system of morphisms $\{X_i, f_{ij}\}$ in the category of ringed spaces.

1. Definition: Given a schematic finite space $X$ we shall denote

$$\operatorname{Spec} \mathcal{O}_X := \lim_{x \in X} \operatorname{Spec} \mathcal{O}_x.$$ 

(the sheaf of rings considered on $\operatorname{Spec} \mathcal{O}_x$ is the sheaf of localizations of $\mathcal{O}_x$, $\mathcal{O}_x$)

Observe that $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_x}(\operatorname{Spec} \mathcal{O}_x) = \lim_{x \in X} \mathcal{O}_x = \mathcal{O}_X(X)$.

2. Example: Obviously, if $X = U_x$, then $\operatorname{Spec} \mathcal{O}_X = \operatorname{Spec} \mathcal{O}_x$ and $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_x} = \mathcal{O}_x$.

Consider the following relation on $\bigsqcup \operatorname{Spec} \mathcal{O}_x$: Given $p \in \operatorname{Spec} \mathcal{O}_x$ and $q \in \operatorname{Spec} \mathcal{O}_y$ we shall say that $p \equiv q$ if there exist $u \geq x, y$ and $r \in \operatorname{Spec} \mathcal{O}_u$ such that the given morphisms $\operatorname{Spec} \mathcal{O}_u \hookrightarrow \operatorname{Spec} \mathcal{O}_x$ and $\operatorname{Spec} \mathcal{O}_u \hookrightarrow \operatorname{Spec} \mathcal{O}_y$ map $r$ to $p$ and $r$ to $q$, respectively (recall Proposition 2.14). Let us prove that $\equiv$ is an equivalence relation: Let $p \equiv q$, $r \mapsto p, q$ and $q \equiv q'$ ($q' \in \operatorname{Spec} \mathcal{O}_z$, there exist $u' \geq y, z$ and $r' \in \operatorname{Spec} \mathcal{O}_{u'}$ such that $r' \mapsto q, q'$). Recall that $\mathcal{O}_{u'} = \mathcal{O}_u \otimes_{\mathcal{O}_x} \mathcal{O}_{u'}$ and that $\mathcal{O}_{u'} \to \bigsqcup_{w \in U_{u'}} \mathcal{O}_w$ is faithfully flat. Then,

$$\operatorname{Spec} \mathcal{O}_u \cap \operatorname{Spec} \mathcal{O}_{u'} = \operatorname{Spec} \mathcal{O}_{u'} = \bigcup_{w \in U_{u'}} \operatorname{Spec} \mathcal{O}_w$$

Since, $q = r = r' \in \operatorname{Spec} \mathcal{O}_u \cap \operatorname{Spec} \mathcal{O}_{u'}$ there exists $v \in U_{u'}$ and $r'' \in \operatorname{Spec} \mathcal{O}_v$ such that $r'' = r, r'$. Then, $r'' \mapsto p, q'$ and $p \equiv q'$.

Observe that $\operatorname{Spec} \mathcal{O}_X = \bigsqcup_{x \in X} \operatorname{Spec} \mathcal{O}_x/\equiv$ as topological spaces. Besides, the morphisms $\operatorname{Spec} \mathcal{O}_x \to \operatorname{Spec} \mathcal{O}_x$ are injective, $\operatorname{Spec} \mathcal{O}_x = \bigcup_{x \in X} \operatorname{Spec} \mathcal{O}_x$ as topological spaces ($U \subset \operatorname{Spec} \mathcal{O}_X$ is an open set iff $U \cap \operatorname{Spec} \mathcal{O}_x$ is an open set, for any $x \in X$).

3. Lemma: Let $A \to B$ be a flat morphism and assume $B \otimes_A B = B$. If $I \subseteq A$ is a radical ideal, then $I \cdot B$ is a radical ideal of $B$.

Proof. Let $p \in \operatorname{Spec} B \subseteq \operatorname{Spec} A$ and recall Notation 2.13 Then,

$$(\operatorname{rad}(I \cdot B))_p = \operatorname{rad}(I \cdot B)_p = \operatorname{rad}(I \cdot A)_p = \operatorname{rad}(I) \cdot A_p = I \cdot A_p = \bigcup_{w \in U_{u'}} \mathcal{O}_w$$

Therefore, $\operatorname{rad}(I \cdot B) = I \cdot B$. 

4. Proposition: Let $X$ be a schematic finite space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal. The ideal $\operatorname{rad}(\mathcal{I}) \subset \mathcal{O}_X$, defined by $\operatorname{rad}(\mathcal{I})_x := \operatorname{rad}(\mathcal{I})_x$, for any $x \in X$, is a quasi-coherent ideal of $\mathcal{O}_X$.

Proof. We only have to prove that given a flat morphism $A \to B$ such that $B \otimes_A B = B$ and an ideal $I \subseteq A$, then $\operatorname{rad}(I) \cdot B = \operatorname{rad}(I \cdot B)$. This is a consequence of Lemma 13.3. 

□
5. **Remark:** \((\text{rad } I)(U) = \text{rad}(I(U))\), for any open subset \(U \subset X\):

\[
(\text{rad } I)(U) = \lim_{x \in U} (\text{rad } I)_x = \lim_{x \in U} (\text{rad } I_x) = \text{rad} \lim_{x \in U} I_x = \text{rad } I(U).
\]

6. **Definition:** Let \(X\) be a schematic finite space. We shall say that a quasi-coherent ideal \(I \subseteq O_X\) is radical if \(I = \text{rad}(I)\).

7. **Notation:** Let \(X\) be a schematic finite space. Given a quasi-coherent ideal \(I \subset O_X\), we shall denote

\[
(I)_0 := \bigcup_{x \in X} \{p \in \text{Spec } O_x : I_x \subseteq p\} \subseteq \text{Spec } O_X
\]

Given a closed subset \(C \subset \text{Spec } O_X\), let \(I_C \subset O_X\) be the radical quasi-coherent ideal defined by \(I_{C,x} := \bigcap_{p' \in C \cap \text{Spec } O_x} p' \subset O_x\) for any \(x \in X\).

8. **Proposition:** The maps

\[
\{\text{Closed subspaces of } \text{Spec } O_X\} \longleftrightarrow \{\text{Radical quasi-coherent ideals of } O_X\}
\]

\[
C \longleftrightarrow I_C
\]

\[
(I)_0 \longleftrightarrow I
\]

are mutually inverse.

9. **Notation:** Given a ring \(B\) and \(b \in B\) we denote \(B_b = \{1, b^{-1}, b^{-2}, \ldots\} \cdot B\).

10. **Proposition:** If \(X\) is an affine finite space, then \(\text{Spec } O_X = \text{Spec } O(X)\).

**Proof.** The morphism \(O(X) \to O_x\) is flat and \(O_x \otimes_{O(X)} O_x \cong O_x\), then \(\text{Spec } O_x \hookrightarrow \text{Spec } O(X)\) is a subspace. The morphism \(O(X) \to \prod_{x \in X} O_x\) is faithfully flat, then the induced morphism \(\text{Spec } O_x \to \text{Spec } O(X)\) is surjective. The sequence of morphisms

\[
O(X) \to \prod_{x \in X} O_x \to \prod_{x \leq x' \in X} O_{x'}
\]

is exact. Then, the natural morphism \(f : \text{Spec } O_X \to \text{Spec } O(X)\) is continuous and bijective. Given a closed set \(C \subset \text{Spec } O_X\), let \(I_C\) be the radical quasi-coherent ideal of \(O_X\) associated with \(C\). \(I_C = I_{C,x}(X)\) since \(X\) is affine. Recall Notation \([2,15]\).

Then, \(C \cap \text{Spec } O_x = (I_{C,x})_0 = (I_C(X) \cdot O_x)_0\). Hence \(f(C) = (I_C(X))_0\) and \(f\) is a homeomorphism.

Also observe that \((\lim_{x \in X} O_x)_{(a_i)} = \lim_{x \in X} O_{x,a_i}\), for any \((a_i) \in \lim_{x \in X} O_x \subseteq \prod_{x \in X} O_x\), hence

\[
O_{\lim_{x \in X} \text{Spec } O_x} = \overline{O}(X).
\]

\[\text{If } C \cap \text{Spec } O_x = \emptyset, \text{ then } \bigcap_{p' \in C \cap \text{Spec } O_x} p' := O_X.\]
11. **Definition:** Let \( f: X \to Y \) be a schematic morphism. Consider the morphisms \( O_{f(x)} \to O_x \), which induce the scheme morphisms \( \text{Spec} O_x \to \text{Spec} O_{f(x)} \), which induce a morphism of ringed spaces

\[ \tilde{f}: \text{Spec} O_X \to \text{Spec} O_Y. \]

We shall say that \( \tilde{f} \) is the morphism induced by \( f \).

12. **Proposition:** Let \( f: X \to Y \) be a quasi-isomorphism. Then, the morphism induced by \( f \), \( \tilde{f}: \text{Spec} O_X \to \text{Spec} O_Y \), is an isomorphism.

**Proof.** Observe that \( \text{Spec} O_X = \lim_{\to x \in X} \text{Spec} O_x = \lim_{\to y \in Y} \lim_{\to x \in f^{-1}(U_y)} \text{Spec} O_x = \lim_{\to y \in Y} \text{Spec} O_Y(U_y) = \text{Spec} O_Y. \)

\[ \square \]

13. **Proposition:** Let \( X \) be a schematic finite space, \( U \subset X \) an open subset and \( I \subset O_U \) a quasi-coherent ideal. Then, there exists a quasi-coherent ideal \( J \subset O_X \) such that \( J|_U = I \).

**Proof.** \( J := \text{Ker}[O_X \to i_*(O_U/I)] \) holds \( J|_U = I \).

\[ \square \]

14. **Notation:** Given a schematic finite space \( X \) we shall denote \( \tilde{X} = \text{Spec} O_X \).

15. **Proposition:** Let \( X \) be a schematic finite space and \( U \subset X \) an open subset. Then,

1. \( \tilde{U} \) is a topological subspace of \( \tilde{X} \).

2. \( \tilde{U} = \bigcap_{U \subseteq \text{open subset } \tilde{V} \subseteq \tilde{X}} \tilde{V} \).

**Proof.** 1. Given a closed set \( C \subset \tilde{U} \), let \( I_C \subset O_U \) be the radical quasi-coherent ideal associated. Let \( J \subset O_X \) be the quasi-coherent ideal such that \( J|_U = I \). Then, the closed subset \( D = (J)_0 = (\text{rad} J)_0 \) of \( \tilde{X} \) holds that \( D \cap \tilde{U} = C \).

2. Let \( p \in \tilde{X} - \tilde{U} \). Let \( P \subset O_X \) be the sheaf of ideals defined by \( P_x = p \subset O_x \) if \( p \in \text{Spec} O_x \) and \( P_x = O_x \) if \( p \notin \text{Spec} O_x \). By Proposition 2.16, \( P \) is quasi-coherent. \( (P)_0 \subset \text{Spec} O_X \) is the closure of \( p \) and \( (P)_0 \cap \tilde{U}_x = \emptyset \), for any \( \tilde{U}_x \subset \tilde{U} \), hence \( (P)_0 \cap \tilde{U} = \emptyset \). Then, \( \tilde{U} \) is equal to the intersection of the open subsets \( \tilde{V} \subseteq \tilde{X} \), such that \( \tilde{U} \subseteq \tilde{V} \).

\[ \square \]

16. **Definition:** Let \( X \) be a schematic finite space. We shall say that a quasi-coherent \( O_X \)-module \( M \) is finitely generated if \( M_x \) is a finitely generated \( O_x \)-module, for any \( x \in X \).
17. **Proposition**: Let $X$ be an affine finite space and $M$ a quasi-coherent $O_X$-module. Then, $M$ is finitely generated iff $M(X)$ is a finitely generated $O(X)$-module.

**Proof.** $\Rightarrow$) Given $x \in X$, $M_x = M(X) \otimes_{O(X)} O_x$. Let $N^x \subset M(X)$ be a finitely generated $O(X)$-submodule such that $N^x \otimes_{O(X)} O_x = M_x$ and $N := \sum_{x \in X} N^x$. Then $N = M(X)$, since $N \otimes_{O(X)} O_x = M_x$ for any $x \in X$.

$\Leftarrow$) $M_x = M(X) \otimes_{O(X)} O_x$ is a finitely generated $O_x$-module, for any $x \in X$. \hfill $\Box$

18. **Proposition**: Let $X$ be a schematic finite space. Any quasi-coherent $O_X$-module is the direct limit of its finitely generated submodules.

**Proof.** Let $M$ be a quasi-coherent $O_X$-module. Let us fix $x_1 \in X$ and a finitely generated submodule $N_1 \subset M_{x_1}$. Consider the inclusion morphism $i_1 : U_{x_1} \hookrightarrow X$ and let $M_1 := \text{Ker}[M \to i_1,(M_{|U_{x_1}}/\tilde{N}_1)]$. Observe that $M_1 \subset M$ and $M_{|U_{x_1}} = \tilde{N}_1$. Given $x_2 \in X$, let $N_2 \subset M_{x_1,x_2}$ be a finitely generated submodule such that $N_2 \otimes_{O_{x_2}} O_y = M_{1,y}$, for any $y \in U_{x_1} \cap U_{x_2}$. Let $U_2 = U_{x_1} \cup U_{x_2}$ and let $N_2 \subset M_{|U_{x_1}}$ be the finitely generated $O_{x_2}$-module such that $N_2 \otimes_{O_{x_2}} O_y = \tilde{N}_1$ and $N_2 \otimes_{O_{x_2}} O_y = \tilde{N}_2$. Consider the inclusion morphism $i_2 : U_2 \hookrightarrow X$ and let $M_2 := \text{Ker}[M_1 \to i_2,(M_{1|U_{x_2}}/\tilde{N}_2)]$. Observe that $M_2|_{U_{x_2}} = \tilde{N}_2$. Given $x_3 \in X$, let $N_3 \subset M_{x_1,x_2} \otimes_{O_{x_3}} O_y = M_{2,y}$, for any $y \in U_{x_1} \cap U_{x_2}$. Let $U_3 := U_2 \cup U_{x_3}$ and let $N_3 \subset M_{2|U_{x_3}}$ be the finitely generated $O_{x_3}$-module such that $N_3 \otimes_{O_{x_3}} O_y = \tilde{N}_2$ and $N_3 \otimes_{O_{x_3}} O_y = \tilde{N}_3$. Consider the inclusion morphism $i_3 : U_3 \hookrightarrow X$ and let $M_3 := \text{Ker}[M_2 \to i_3,(M_{2|U_{x_3}}/\tilde{N}_3)]$. Observe that $M_3|_{U_{x_3}} = \tilde{N}_3$. So on we shall get a finitely quasi-coherent $O_X$-submodule $M_n \subset M$ such that $M_{n,x_1} = \tilde{N}_1$. Now it is easy to prove this proposition. \hfill $\Box$

19. **Corollary**: Let $X$ be a schematic finite space. Any quasi-coherent ideal $I \subset O_X$ is the direct limit of its finitely generated ideals $I_t \subset I$.

20. **Lemma**: Let $X$ be a schematic finite space, $\tilde{U} \subset \tilde{X}$ an open subset and $C = \tilde{X} - \tilde{U}$. Then, $\tilde{U}$ is quasi-compact iff there exists a finitely generated ideal $I \subset O_X$ such that $(I)_0 = C$.

**Proof.** $\Rightarrow$) Consider the quasi-coherent ideal $I_C \subset O_X$. Let $J = \{I_j\}_{j \in J}$ the set of finitely generated ideals of $O_X$ contained in $I_C$. By Corollary [13.19], $I_C = \lim_{\rightarrow \infty} I_j$. Then, $C = (I_C)_0 = (\lim_{\rightarrow \infty} I_j)_0 = \cap_{j \in J}(I_j)_0$. $\tilde{U} = \tilde{X} - (I_j)_0$, since $\tilde{U}$ is quasi-compact. Hence, $C = (I_j)_0$.

$\Leftarrow$) Let $x \in X$, then $I_x = (a_1, \ldots, a_n) \subset O_x$ is finitely generated. $C \cap I_x = (I_x)_0 = \cap_i (a_i)_0$, then $\tilde{U} \cap \tilde{X}_x = \cup_i \text{Spec } O_{x,a_i}$ is quasi-compact. Therefore, $\tilde{U} = \cup_i (\tilde{U} \cap \tilde{X}_x)$ is quasi-compact. \hfill $\Box$

21. **Proposition**: Let $X$ be a schematic finite space. Then,
1. The intersection of two quasi-compact open subsets of $\tilde{X}$ is quasi-compact.

2. The family of quasi-compact open subsets of $\tilde{X}$ is a basis for the topology of $\tilde{X}$.

3. If $\tilde{V} \subseteq \tilde{X}$ is a quasi-compact open subset then $\tilde{V} \cap \tilde{U}$ is quasi-compact, for any open subset $U \subset X$.

Proof. 1. Let $\tilde{U}_1, \tilde{U}_2 \subset \tilde{X}$ be two quasi-compact open subsets, $C_1 := \tilde{X} - \tilde{U}_1, C_2 := \tilde{X} - \tilde{U}_2$, and $I_1, I_2 \subset O_X$ two finitely generated ideals such that $C_1 = (I_1)_0$ and $C_2 = (I_1)_0$. Then, $C_1 \cup C_2 = (I_1)_0 \cup (I_2)_0 = (I_1 \cdot I_2)_0$ and $\tilde{U}_1 \cap \tilde{U}_2 = \tilde{X} - (I_1 \cdot I_2)_0$. By Lemma 13.20 $\tilde{U}_1 \cap \tilde{U}_2$ is quasi-compact.

2. Let $\tilde{U} \subset \tilde{X}$ be an open subset and $C = \tilde{X} - \tilde{U}$. $I_C = \lim_{j \in J} I_j$, where $(I_j)_{j \in J}$ is the set of the finitely generated of $O_X$ contained in $I_C$. Then, $C = (I_C)_0 = (\lim_{j \in J} I_j)_0 = \cap_{j \in J} (I_j)_0$ and $\tilde{U} = \cup_{j \in J}(\tilde{X} - (I_j)_0)$, where the open subsets $\tilde{X} - (I_j)_0$ are quasi-compact by Lemma 13.20

3. Let $C = \tilde{X} - \tilde{V}$ and let $I \subset O_X$ be a finitely generated ideal such that $C = (I)_0$. Then, $C \cap \tilde{U} = (I_{\tilde{U}})_0$ and $\tilde{V} \cap \tilde{U} = \tilde{U} - (I_{\tilde{U}})_0$. By Lemma 13.20 $\tilde{V} \cap \tilde{U}$ is quasi-compact.

$\square$

22. Corollary: Let $X$ be a schematic finite space, $U \subseteq X$ an open subset and $\tilde{U} \subset \tilde{U}$ a quasi-compact open subset. Then,

1. There exists a quasi-compact open subset $\tilde{W} \subset \tilde{X}$, such that, $\tilde{W} \cap \tilde{U} = \tilde{U}$.

2. $\tilde{U}$ is equal to the intersection of the quasi-compact open subsets of $\tilde{X}$ which contain it.

Proof. 1. By Proposition 13.15 there exists an open subset $\tilde{W}' \subseteq \tilde{X}$ such that $\tilde{W}' \cap \tilde{U} = \tilde{U}$. Given $p \in \tilde{U}$, there exists a quasi-compact open subset $\tilde{W}_p \subset \tilde{W}'$ such that $p \in \tilde{W}_p$. There exist $p_1, \ldots, p_n \in \tilde{U}$ such that $\tilde{U} \subset \cup_{i=1}^n \tilde{W}_{p_i} \subset \tilde{W}'$. Hence, $\tilde{W} := \cup_{i=1}^n \tilde{W}_{p_i}$ holds $\tilde{W} \cap \tilde{U} = \tilde{U}$.

2. Given an open subset $\tilde{V} \subset \tilde{X}$ such that $\tilde{U} \subset \tilde{V}$, there exists a quasi-compact open subset $\tilde{V}' \subset \tilde{X}$ such that $\tilde{U} \subset \tilde{V}' \subset \tilde{V}$. By Proposition 13.15 we are done.

$\square$

23. Lemma: Let $X$ be a schematic finite space, $U_1, U_2 \subset X$ open subsets, $\tilde{V}_1 \subset \tilde{U}_1$ and $\tilde{V}_2 \subset \tilde{U}_2$ quasi-compact open subsets and $\tilde{W} \subset \tilde{X}$ an open subset such that $\tilde{V}_1 \cap \tilde{V}_2 \subseteq \tilde{W}$. Then, there exist open subsets $\tilde{W}_1, \tilde{W}_2 \subset \tilde{X}$ such that $\tilde{V}_1 \subset \tilde{W}_1$, $\tilde{V}_2 \subset \tilde{W}_2$ and $\tilde{W}_1 \cap \tilde{W}_2 \subseteq \tilde{W}$.

Proof. By the quasi-compactness of $\tilde{V}_1$ and $\tilde{V}_2$, to prove this theorem we can easily reduce ourselves to the case in which $\tilde{V}_1 = \text{Spec } O_{x_1, a_1} \subset \tilde{U}_1 (a_1 \in O_{x_1})$ and $\tilde{V}_2 = \text{Spec } O_{x_2, a_2} \subset \tilde{U}_2 (a_2 \in O_{x_2})$. 

1. Suppose that $\bar{V}_1 \cap \bar{V}_2 = \emptyset$. Let $O_{U_{x_1}a_1}$ be the quasi-coherent $O_{U_{x_1}}$-module defined by $O_{U_{x_1}}(U_z) = O_{z,a_1}$, for any $z \in U_{x_1}$. Let $i_{x_1} : U_{x_1} \subset X$ be the inclusion morphism. Let $I_1$ be the kernel of the natural morphism $O_X \rightarrow i_{x_1}, O_{U_{x_1}a_1}$. Likewise, define $O_{U_{x_2}a_2}, i_{x_2}$ and $I_2$.

Observe that $i_{x_1}, O_{U_{x_1}a_1}(U_z) = O_{x_1,z,a_1}$ for any $z \in X$, and the natural morphism

$$i_{x_1}, O_{U_{x_1}a_1}(U_z) \rightarrow \bigcap_{y \in U_{x_1}z} O_{y,a_1}$$

is injective. Then, the sequence of morphisms

$$(***) \quad 0 \rightarrow I_{1,z} \rightarrow O_z \rightarrow \bigcap_{y \in U_{x_1}z} O_{y,a_1}$$

is exact. Observe that $\bar{V}_1 \cap \bar{U}_z = \bigcup_{y \in U_{x_1}z} \text{Spec } O_{x_1,a_1}$. Then, $(I_1)_0$ is equal to the closure $\text{Cl}(\bar{V}_1)$ of $\bar{V}_1$ in $\bar{X}$.

Let $z = x_2$, tensoring $(***)$ by $\otimes_{O_{x_2}} O_{x_2,a_2}$, we obtain the exact sequence

$$0 \rightarrow I_{1,x_2} \otimes_{O_{x_2}} O_{x_2,a_2} \rightarrow O_{x_2,a_2} \rightarrow \bigcap_{y \in U_{x_1}x_2} O_{y,a_1} \otimes_{O_{x_2}} O_{x_2,a_2}.$$  

$O_{x_1,a_1} \otimes_{O_{x_2}} O_{x_2,a_2} = 0$ since $\text{Spec } O_{x_1,a_1} \cap \text{Spec } O_{x_2,a_2} \subset \bar{V}_1 \cap \bar{V}_2 = \emptyset$. Then, $I_{1,x_2} \otimes_{O_{x_2}} O_{x_2,a_2} = O_{x_2,a_2}$, hence $I_{1,x_2} : O_{x_2,a_2} = O_{x_2,a_2}$. Therefore, $(I_1)_0 \cap \bar{V}_2 = \emptyset$, that is, $\text{Cl}(\bar{V}_1) \cap \bar{V}_2 = \emptyset$. Let $\mathcal{J}_1 \subset I_1$ be a finitely generated ideal such that $\mathcal{J}_{1,x_2} \cdot O_{x_2,a_2} = O_{x_2,a_2}$. Again, $(\mathcal{J}_1)_0 \cap \bar{V}_2 = \emptyset$ and $\bar{V}_1 \subset (\mathcal{J}_1)_0$. Likewise, define a finitely generated ideal $\mathcal{J}_2$ such that $(\mathcal{J}_2)_0 \cap \bar{V}_1 = \emptyset$ and $\bar{V}_2 \subset (\mathcal{J}_2)_0$.

Given a subset $Y \subset \bar{X}$ denote $Y^c := \bar{X} - Y$. Let $\bar{W}_1 := (\text{Cl}(\mathcal{J}_1 \cdot \mathcal{J}_2)_0) \cup (\mathcal{J}_1)_0$ and $\bar{W}_2 := (\text{Cl}(\mathcal{J}_1 \cdot \mathcal{J}_2)_0) \cup (\mathcal{J}_1)_0$. Obviously, $\bar{W}_1 \subset (\mathcal{J}_1 \cdot \mathcal{J}_2)_0$ and $\bar{W}_2 \subset (\mathcal{J}_2)_0$. Then, $\bar{W}_1 \cap \bar{W}_2 = \emptyset$. We only have to prove that $\bar{V}_1 \subset \bar{W}_1$. We know that $\bar{V}_1 \cap (\mathcal{J}_2)_0 = \emptyset$, it remains to prove that $\bar{V}_1 \cap \text{Cl}(\mathcal{J}_1 \cdot \mathcal{J}_2)_0 = \emptyset$. $\bar{V}_1 \cap (\mathcal{J}_1 \cdot \mathcal{J}_2)_0 = \emptyset$ and $(\mathcal{J}_1 \cdot \mathcal{J}_2)_0$ is the union of a finite set of subsets $\text{Spec } O_{y,b} \subset \bar{U}_y \subset \bar{X}$ (with $b \in O_y$ and $y \in X$). As we have proved above (†), $\bar{V}_1 \cap \text{Cl}(\text{Spec } O_{y,b}) = \emptyset$ since $\bar{V}_1 \cap \text{Spec } O_{y,b} = \emptyset$. Then, $\bar{V}_1 \cap \text{Cl}(\mathcal{J}_1 \cdot \mathcal{J}_2)_0 = \emptyset$.

2. Suppose that $V_1 \cap V_2 \neq \emptyset$. Let $I := I_{\bar{X} - \bar{W}} \subset O_X$. $\bar{Y} = \text{Spec } O_X/I$ is equal to $\bar{Y} := \text{Spec } O_X/I$. $\bar{V}_1 \cap \bar{Y} = \text{Spec } (O_X/I)_{x_1,a_1}$ and $\bar{V}_2 \cap \bar{Y} = \text{Spec } (O_X/I)_{x_2,a_2}$. By 1., there exist open subsets $\bar{W}_1', \bar{W}_2' \subset \bar{Y}$ such that $\bar{W}_1' \cap \bar{W}_2' = \emptyset$. $\bar{V}_1 \cap \bar{Y} \subset \bar{W}_1'$ and $\bar{V}_2 \cap \bar{Y} \subset \bar{W}_2'$. Then, $\bar{W}_1 = \bar{W} \cup \bar{W}_1'$ and $\bar{W}_2 = \bar{W} \cup \bar{W}_2'$ are the searched open subsets.

24. Lemma: Let $X$ be a schematic finite space and $B$ the family of quasi-compact open subsets of $\bar{X}$. Let $\mathcal{F}'$ be a presheaf on $\bar{X}$ and $\mathcal{F}$ the sheafification of $\mathcal{F}'$. If for any $\bar{V} \in B$ and
any finite open covering \(\{\tilde{V}_i \in B\}\) of \(\tilde{V}\) the sequence of morphisms

\[
\mathcal{F}'(\tilde{V}) \to \prod_i \mathcal{F}'(\tilde{V}_i) \to \prod_{ij} \mathcal{F}'(\tilde{V}_i \cap \tilde{V}_j)
\]

is exact, then \(\mathcal{F}'(\tilde{V}) = \mathcal{F}(\tilde{V})\).

Proof. It is well known. \(\square\)

25. Corollary: Let \(X\) be a schematic finite space and let \(\{F_i\}_i\) be a direct system of sheaves of abelian groups on \(\tilde{X}\). Then,

\[
H^n(\tilde{X}, \lim_{\longrightarrow i \in I} F_i) = \lim_{\longrightarrow i \in I} H^n(\tilde{X}, F_i),
\]

for any \(n \geq 0\).

26. Corollary: Let \(X\) be a schematic finite space, \(U \subset X\) an open subset, \(\tilde{V} \subset \tilde{U}\) a quasi-compact open subset and \(\mathcal{F}\) a sheaf of abelian groups on \(\tilde{X}\). Then,

\[
\mathcal{F}_{|\tilde{U}}(\tilde{V}) = \lim_{\longrightarrow \tilde{V} \subset \tilde{W}} \mathcal{F}(\tilde{W}).
\]

Therefore, \(H^n(\tilde{V}, \mathcal{F}_{|\tilde{U}}) = \lim_{\longrightarrow \tilde{V} \subset \tilde{W}} H^n(\tilde{W}, \mathcal{F})\), for any \(n \geq 0\).

Proof. Let \(\mathcal{G}\) be the presheaf on \(\tilde{U}\) defined by \(\mathcal{G}(\tilde{V}) := \lim_{\longrightarrow \tilde{V} \subset \tilde{W}} \mathcal{F}(\tilde{W})\). \(\mathcal{F}_{|\tilde{U}}\) is the sheafification of \(\mathcal{G}\). Let \(\{\tilde{V}_i\}\) a finite quasi-compact open covering of \(\tilde{V}\). Let \(I_i\) be the family of open subsets \(\tilde{W}_i \subset \tilde{X}\) such that \(\tilde{V}_i \subset \tilde{W}_i\) and \(I = \prod_i I_i\). The sequence of morphisms

\[
\mathcal{F}(\cup_i \tilde{W}_i) \to \prod_i \mathcal{F}(\tilde{W}_i) \to \prod_{ij} \mathcal{F}(\tilde{W}_i \cap \tilde{W}_j)
\]

is exact. Taking direct limits, we obtain the exact sequence

\[
\lim_{\longrightarrow (\tilde{W}_i) \in I} \mathcal{F}(\cup_i \tilde{W}_i) \to \lim_{\longrightarrow (\tilde{W}_i) \in I} \prod_i \mathcal{F}(\tilde{W}_i) \to \lim_{\longrightarrow (\tilde{W}_i) \in I} \prod_{ij} \mathcal{F}(\tilde{W}_i \cap \tilde{W}_j)
\]

Observe that \(\lim_{\longrightarrow (\tilde{W}_i) \in I} \mathcal{F}(\cup_i \tilde{W}_i) = \mathcal{G}(\cup_i \tilde{V}_i)\), \(\lim_{\longrightarrow (\tilde{W}_i) \in I} \prod_i \mathcal{F}(\tilde{W}_i) = \prod_i \mathcal{G}(\tilde{V}_i)\), and by Lemma 13.23,

\[
\lim_{\longrightarrow (\tilde{W}_i) \in I} \prod_{ij} \mathcal{F}(\tilde{W}_i \cap \tilde{W}_j) = \prod_{ij} \mathcal{G}(\tilde{V}_i \cap \tilde{V}_j).\]

Hence,

\[
\mathcal{G}(\cup_i \tilde{V}_i) \to \prod_i \mathcal{G}(\tilde{V}_i) \to \prod_{ij} \mathcal{G}(\tilde{V}_i \cap \tilde{V}_j)
\]
is exact. By Lemma \[13.24\], \( G(\bar{V}) = \mathcal{F}_{|\bar{U}}(\bar{V}) \).

Finally, let \( F \to C'\mathcal{F} \) be the Godement resolution, then

\[
H^n(\bar{V}, \mathcal{F}_{|\bar{U}}) = H^n\Gamma(\bar{V}, (C'\mathcal{F})_{|\bar{U}}) = \lim_{\bar{V} \subset \bar{W}} H^n\Gamma(\bar{W}, C'\mathcal{F}) = \lim_{\bar{V} \subset \bar{W}} H^n(\bar{W}, \mathcal{F}).
\]

\[\square\]

**27. Theorem:** Let \( X \) be a schematic finite space and \( U \subset X \) an open subset. Then,

\[ O_{\bar{X}|\bar{U}} = O_{\bar{V}}. \]

Let \( x \in X \) and \( v \in \bar{U}_x \subset \bar{X} \). Then, \( O_{\bar{X},\bar{v}} = O_{x,v} \).

**Proof.** Let \( i : U \hookrightarrow X \) be the inclusion morphism and \( \bar{i} : \bar{U} \hookrightarrow \bar{X} \) the induced morphism. The natural morphism \( O_{\bar{x}} \to \bar{i}_! O_{\bar{v}} \) defines by adjunction the morphism \( O_{\bar{X}|\bar{U}} \to O_{\bar{U}} \) and we have to prove that the morphism \( O_{\bar{X},\bar{v}} = O_{\bar{X}|\bar{U},\bar{v}} \to O_{\bar{U},\bar{v}} \) is an isomorphism, for any \( v \in \bar{U} \).

Let \( I := \{ (\bar{W}, \bar{V}) \}, \) where \( \bar{V} \) is any quasicompact open subset of \( \bar{U} \) such that \( v \in \bar{V} \) and \( \bar{W} \) is any open subset of \( \bar{X} \) such that \( \bar{V} \subset \bar{W} \). Then,

\[
O_{\bar{X},\bar{v}} = \lim_{(\bar{W},\bar{V}) \in I} O_{\bar{X}}(\bar{W}) = \lim_{v \in \bar{V}} \lim_{\bar{V} \subset \bar{W}} \lim_{v \in \bar{V}} O_{\bar{X}}(\bar{W}) = \lim_{v \in \bar{V}} O_{\bar{U}}(\bar{V}) = O_{\bar{U},\bar{v}}.
\]

Finally, \( O_{\bar{X},\bar{v}} = O_{\bar{U},\bar{v}} = O_{x,v} \).

\[\square\]

### 14 \( H^n(X, \mathcal{M}) = H^n(\bar{X}, \tilde{\mathcal{M}}) \)

1. **Notation:** Given an affine scheme \( \text{Spec} R \) and an \( R \)-module \( M \) we shall denote \( \tilde{M} \) the sheaf of localizations of the \( R \)-module \( M \).

2. **Definition:** Let \( X \) be a schematic finite space and \( \bar{X} = \text{Spec} O_X \). We shall say that an \( O_{\bar{X}} \)-module \( \tilde{\mathcal{M}} \) is quasi-coherent if \( \tilde{\mathcal{M}}_{\bar{U}_x} \) is a quasi-coherent \( O_{\bar{U}_x} \)-module for any \( x \in X \).

I warn the reader that this definition is not the usual definition of quasi-coherent module.

Let \( X \) be a schematic finite space and \( \tilde{\mathcal{M}} \) a quasi-coherent \( O_{\bar{X}} \)-module. Let \( \mathcal{M} \) be the \( O_X \)-module defined by \( \mathcal{M}_x = \tilde{\mathcal{M}}_{\bar{U}_x}(\bar{U}_x) \), then it easy to check that \( \mathcal{M} \) is a quasi-coherent \( O_X \)-module (see \[6\] II 5.1 (d) and 5.2 c.).

Let \( \mathcal{M} \) be a quasi-coherent \( O_X \)-module. Define \( \tilde{\mathcal{M}} := \lim_{\bar{x} \in X} \tilde{\mathcal{M}}_{\bar{x}}, \) where \( \bar{i}_x : \bar{U}_x \to \bar{X} \) is the morphism induced by the inclusion morphism \( i_x : U_x \hookrightarrow X \). Observe that \( \tilde{\mathcal{M}}(\bar{X}) = \lim_{\bar{x} \in X} \mathcal{M}_{\bar{x}} = \mathcal{M}(X). \)
3. Proposition: Let $X$ be an affine finite space and $M$ a quasi-coherent $O_X$-module. Then $\hat{M} = \hat{M}(X)$.

Proof. Observe that $\hat{i_x}_x M_x = \hat{M}_x$, then

$\hat{M} = \lim_{\leftarrow x \in X} \hat{i_x}_x M_x = \lim_{\leftarrow x \in X} \hat{M}_x = \hat{M}(X)$.

□

4. Proposition: Let $X$ be a schematic finite space, $U \subset X$ an open subset and $M$ a quasi-coherent $O_X$-module. Then, $\hat{M}_{\mid U} = \hat{M}_{\mid U}$. In particular, $\hat{M}$ is a quasi-coherent $O_X$-module.

Proof. Proceed like in the proof of Theorem 13.27. □

Let $M$ and $M'$ be quasi-coherent $O_X$-modules. Any morphism of $O_X$-modules $M \to M'$ induces a natural morphism $\hat{M} = \lim_{\leftarrow x \in X} \hat{i_x}_x M_x \to \lim_{\leftarrow x \in X} \hat{i_x}_x M'_x = \hat{M}'$.

Let $\hat{N}$ and $\check{N}$ be quasi-coherent $O_X$-modules. Any morphism of $O_X$-modules $\hat{M} \to \check{N}$ induces natural morphisms $M_x := \hat{M}_{\mid U_x}(\hat{U}_x) \to \check{N}_{\mid U_x}(\hat{U}_x) =: N_x$ and then a morphism $M \to N$.

5. Theorem: Let $X$ be a schematic finite space. The category of quasi-coherent $O_X$-modules is equivalent to the category of quasi-coherent $O_X$-modules.

Proof. The functors $\check{M} \mapsto \{\hat{M}_{\mid U_x}(\hat{U}_x)\}_{x \in X}$ and $M \mapsto \check{M}$ are mutually inverse. □

6. Proposition: Let $f: X \to Y$ be a schematic morphism and $\check{f}: \check{X} \to \check{Y}$ the induced morphism. Let $\hat{M}$ be a quasi-coherent $O_X$-module and $\check{N}$ a quasi-coherent $O_Y$-module. Then,

1. $\check{f}_* \hat{M} = \hat{f}_* \check{M}$.
2. $\check{f}^* \check{N} = f^* \hat{N}$.

Proof. Consider the obvious commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \check{Y} \\
\downarrow \hat{i}_x & & \downarrow \hat{i}_y \\
\hat{U}_x & \xrightarrow{f_{xy}} & \hat{U}_y \\
& \uparrow \check{i}_x & \\
\check{U}_x & \xrightarrow{\check{f}_{xy}} & \check{U}_y
\end{array}
$$

$(f(x) \geq y)$
1. Observe that \((f_*\mathcal{M})(U_\alpha) = \mathcal{M}(f^{-1}(U_\alpha)) = \lim_{x \in f^{-1}(U_\alpha)} \mathcal{M}_x\), then

\[
\tilde{f}_*\mathcal{M} = \lim_{y \in Y} \tilde{f}_y(\lim_{x \in f^{-1}(U_\alpha)} \mathcal{M}_x) = \lim_{y \in Y} \tilde{f}_y(\lim_{x \in f^{-1}(U_\alpha)} \mathcal{M}_x) = \lim_{y \in Y} \tilde{f}_y \lim_{x \in f^{-1}(U_\alpha)} \mathcal{M}_x = \lim_{x \in X} \mathcal{M}_x.
\]

2. Observe that \((f^*\mathcal{N}_x) = N_{f(x)} \otimes_{\mathcal{O}_{f(x)}} \mathcal{O}_x\), then \((f^*\mathcal{N})|_{U_\alpha} = N_{f(x)} \otimes_{\mathcal{O}_{f(x)}} \mathcal{O}_x\).

On the other hand, \((\tilde{f}^*\mathcal{N})|_{U_\alpha} = \tilde{f}^*_x(N_{f(x)}) = \tilde{f}^*_x(N_{f(x)}) = N_{f(x)} \otimes_{\mathcal{O}_{f(x)}} \mathcal{O}_x\).

7. **Lemma**: Let \(X\) be a schematic finite space and \(\mathcal{F}\) a sheaf of abelian groups on \(X\). Let \(U, V \subset X\) be two open subsets. Consider the obvious commutative diagram

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{i} & \tilde{X} \\
\downarrow{j} & & \downarrow{j} \\
U \cap V & \xrightarrow{\tilde{i}} & \tilde{V}
\end{array}
\]

Then, \(j^*(R^n i_* \mathcal{F}) = R^n i_*(j^* \mathcal{F})\), for any \(n \geq 0\).

**Proof**. Let \(p \in \tilde{V}\). Then,

\[
(f^*(R^n i_* \mathcal{F}))_p = (R^n i_* \mathcal{F})_p = \lim_{W \subset \tilde{X}} H^n(\tilde{W} \cap \tilde{U}, \mathcal{F}|_{\tilde{W}}) = \lim_{W' \subset \tilde{V}} H^n(\tilde{W}' \cap \tilde{U} \cap \tilde{V}, \mathcal{F}|_{\tilde{W}' \cap \tilde{U} \cap \tilde{V}}) = (R^n i_*(j^* \mathcal{F}))_p.
\]

\[\Box\]

8. **Theorem**: Let \(X\) be a semiseparated schematic finite space and \(\mathcal{M}\) a quasi-coherent \(\mathcal{O}_X\)-module. Then,

\[
H^n(X, \mathcal{M}) = H^n(\tilde{X}, \tilde{\mathcal{M}}),
\]

for any \(n \geq 0\).

**Proof**. Let \(\tilde{i}: \tilde{U}_x \hookrightarrow \tilde{X}\) be the inclusion morphism. The morphism \(\tilde{i}: \tilde{U}_y \cap \tilde{U}_x \hookrightarrow \tilde{U}_y, p \mapsto i(p)\), is an affine morphism of schemes since \(\tilde{U}_x \cap \tilde{U}_y\) is an affine scheme because \(X\) is semiseparated. Let \(\tilde{\mathcal{N}}\) be a quasi-coherent \(\mathcal{O}_{\tilde{U}_y}\)-module and \(p \in \tilde{U}_y\). By Lemma 14.7 \((R^n i_* \tilde{\mathcal{N}})_p = (R^n i_* \tilde{\mathcal{N}}|_{\tilde{U}_y \cap \tilde{U}_x})_p = 0\), for any \(n > 0\). Hence, \(R^n i_* \mathcal{N} = 0\) and \(H^n(\tilde{X}, \tilde{i}_* \mathcal{N}) = H^n(\tilde{U}_x, \mathcal{N}) = 0\), for any \(n > 0\). That is, \(i_* \mathcal{N}\) is acyclic.
Given a quasi-coherent $O_X$-module $M$ denote $\tilde{M}_{\tilde{U}_s} = \hat{i}_s^* \tilde{M}$. Observe that $\tilde{M}_{\tilde{U}_s}$ is acyclic and $\tilde{M}_{\tilde{U}_s}(\tilde{X}) = \tilde{M}_{\tilde{U}_s}(\tilde{U}_s) = \overline{M(U_s)} = M(U_s) = M_s$. The obvious sequence of morphisms

$$
\tilde{M} \to \prod_{x \in \tilde{X}} \tilde{M}_{\tilde{U}_s} \to \prod_{x_1 < x_2} \tilde{M}_{\tilde{U}_{x_1}} \to \prod_{x_1 < x_2 < x_3} \tilde{M}_{\tilde{U}_{x_1}} \to \cdots
$$

is exact. Denote this resolution $\tilde{M} \to \tilde{C} \to \tilde{M}$ and let $M \to C_s M$ be the standard resolution of $M$. Then,

$$
H^n(\tilde{X}, \tilde{M}) = H^n(\Gamma(\tilde{X}, \tilde{C} \to \tilde{M})) = H^n(\Gamma(X, C_s M)) = H^n(X, M).
$$

□

15 \ Hom_{sch}(\tilde{X}, \tilde{Y}) = \Hom_{[sch]}(X, Y)

1. **Proposition:** Let $f : X \to Y$ be a schematic morphism. Then, the induced morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ is quasi-compact, that is, $\tilde{f}^{-1}(\tilde{V})$ is quasi-compact for any quasi-compact open subset $V \subset \tilde{Y}$.

*Proof.* Any affine scheme morphism is quasi-compact. Given $x \in X$, denote $\tilde{f}_x : \tilde{U}_s \to \tilde{U}_{f(x)}$ the morphism induced by $f_x : U_s \to U_{f(x)}, f_x(x') := f(x')$. By Proposition 13.21.2., $V \cap \tilde{U}_{f(x)}$ is quasi-compact. Then,

$$
\tilde{f}^{-1}(\tilde{V}) = \bigcup_{x \in X} \tilde{f}_x^{-1}(\tilde{V}) \cap \tilde{U}_x = \bigcup_{x \in X} \tilde{f}_x^{-1}(\tilde{V} \cap \tilde{U}_{f(x)})
$$

is quasi-compact. □

2. **Proposition:** Let $f : X \to Y$ be a schematic morphism and $\tilde{f} : \tilde{X} \to \tilde{Y}$ the induced morphism. Then, $\tilde{f}^{-1}(\tilde{U}) = \overline{f^{-1}(U)}$, for any open subset $U \subset X$.

*Proof.* Given two open subsets $V, V'$ of a schematic finite space, observe that $\tilde{V} \cap \tilde{V}' = V \cap V'$ and $\tilde{V} \cup \tilde{V}' = \overline{V \cup V'}$.

Given $y \geq f(x)$, $\text{Spec} \prod_{z \in U_y} O_z \to \text{Spec}(O_x \otimes_{O_f(x)} O_y) = \text{Spec} O_x \times_{\text{Spec} O_f(x)} \text{Spec} O_y$ is surjective, by Theorem 5.15. Hence, $\tilde{U}_x \cap \tilde{f}^{-1}(\tilde{U}_y) = \cup_{z \in U_y} \tilde{U}_z = \tilde{U}_y = \tilde{U}_x \cap \tilde{f}^{-1}(\tilde{U}_y)$.

Obviously, $\overline{f^{-1}(U)} \subseteq \tilde{f}^{-1}(\tilde{U})$. Let $\mathfrak{p} \in \tilde{f}^{-1}(\tilde{U})$ and $x \in X$ such that $\mathfrak{p} \in \tilde{U}_x$. Then, $\tilde{f}(\mathfrak{p}) \in \tilde{U}_{f(x)} \cap \tilde{U}_x$. Let $y \in U_{f(x)} \cap U$ such that $\tilde{f}(\mathfrak{p}) \in \tilde{U}_y$. Then, $\mathfrak{p} \in \tilde{U}_x \cap \tilde{f}^{-1}(\tilde{U}_y) = U_x \cap \tilde{f}^{-1}(U_y) \subset \tilde{f}^{-1}(\tilde{U})$. Therefore, $f^{-1}(\tilde{U}) \subseteq \overline{f^{-1}(U)}$. □

3. **Definition:** Let $X$ and $Y$ be schematic finite spaces. We shall say that a morphism of ringed spaces $f^* : \tilde{X} \to \tilde{Y}$ is a schematic morphism if $f^* \tilde{M}$ is a quasi-coherent $O_Y$-module for any quasi-coherent $O_X$-module $\tilde{M}$. 
4. Example: If \( f: X \to Y \) is a schematic morphism, then \( \tilde{f}: \tilde{X} \to \tilde{Y} \) is a schematic morphism, by Proposition \([14.6]\).

5. Proposition: Let \( f: \text{Spec } B \to \text{Spec } A \), \( f': \tilde{A} \to f\tilde{B} \) be a morphism of ringed spaces. If \( f_i\tilde{M} \) is a quasi-coherent \( \tilde{A} \)-module for any \( B \)-module \( M \), then \( (f, f') \) is a morphism of schemes.

Proof. Let \( f'' : \text{Spec } B \to \text{Spec } A \) be the morphism defined on spectra by \( f'_\text{Spec } A: A \to B \). We only have to prove that \( f = f'' \).

Let \( p \in \text{Spec } B, a \in A \) and \( U_a = \text{Spec } A \backslash (a)_0 \). By the hypothesis, \( f_i\tilde{B}/p \) is a quasi-coherent \( \tilde{A} \)-module. Then,

\[
(B/p)_a = ((f_i\tilde{B}/p)(\text{Spec } A))_a = (f_i\tilde{B}/p)(U_a) = \tilde{B}/p(f^{-1}(U_a)).
\]

Then,

\[
f'_\text{Spec } A(a) \in p \iff (B/p)_a = 0 \iff \tilde{B}/p(f^{-1}(U_a)) = 0 \iff f^{-1}(U_a) \cap (p)_0 = \emptyset \iff p \not\in f^{-1}(U_a) \iff f(p) \not\in U_a \iff a \in f(p).
\]

Therefore, \( f'_\text{Spec } A^{-1}(p) = f(p) \), that is to say, \( f'' = f \). \( \square \)

6. Proposition: Any schematic morphism \( f': \tilde{X} \to \tilde{Y} \) is a morphism of locally ringed spaces.

Proof. Let \( p \in \tilde{U}_x \subset \tilde{X} \). Let \( g \) be the composition of the schematic morphisms

\[
\text{Spec } O_{x,p} \hookrightarrow \text{Spec } O_x \hookrightarrow \tilde{X} \to \tilde{Y}.
\]

Let \( y \in Y \) be a point such that \( f(p) \in \tilde{U}_y \), then \( g^{-1}(\tilde{U}_y) = \text{Spec } O_{x,p} \). Consider the continuous morphism \( h: \text{Spec } O_{x,p} \to \tilde{U}_y, q \mapsto g(q) \) and let \( i_y: \tilde{U}_y \hookrightarrow \tilde{Y} \) be the inclusion morphism. Consider the morphism \( O_{\tilde{Y}} \to g_*\tilde{O}_{x,p} \). Taking \( i^* \), we obtain a morphism \( \phi: O_{\tilde{U}_x} \to h_*\tilde{O}_{x,p} \).

The morphism of ringed spaces \((h, \phi)\) is schematic, since \( h_*\tilde{M} = \tilde{i}_y^*h_*\tilde{M} = \tilde{i}_y^*g_*\tilde{M} \) is quasi-coherent, for any quasi-coherent \( O_{\tilde{U}_x} \)-module \( \tilde{M} \). By Proposition \([15.5]\) \( h \) is a morphism of locally ringed spaces. We are done.

\( \square \)

7. Lemma: Let \( \tilde{M} \) be a finitely generated \( O_{\tilde{X}} \)-module. For any \( p \in \tilde{X} \), there exist an open neighbourhood \( \tilde{U} \) of \( p \) and an epimorphism of sheaves \( O'_p \to \tilde{M}_{|\tilde{U}} \).
15. Hom_{sch}(\tilde{X}, \tilde{Y}) = \text{Hom}_{(sch)}(X, Y)

\textbf{Proof.} \tilde{V} := \{ q \in \tilde{X} : \tilde{M}_q = 0 \} is an open subset of \tilde{X}, since \tilde{V} \cap \tilde{U}_x = \{ q \in \tilde{U}_x : M_{xq} = 0 \} is an open subset of \tilde{U}_x. Hence, given another quasi-coherent module \tilde{M}', and a morphism of \text{O}_{\tilde{X}}\text{-modules} \tilde{M}' \to \tilde{M}, if the morphism on stalks at \mathfrak{p}, \tilde{M}'_{\mathfrak{p}} \to \tilde{M}_{\mathfrak{p}} is an epimorphism then there exists an open neighbourhood \tilde{V} of \mathfrak{p} such that the morphism of sheaves \tilde{M}'_{\tilde{V}} \to \tilde{M}_{\tilde{V}} is an epimorphism.

Let \{ m_1, \ldots, m_r \} be a generator system of the \text{O}_{\mathfrak{p}}\text{-module} \tilde{M}_{\mathfrak{p}}. Let \tilde{W} \subset \tilde{X} be an open neighbourhood of \mathfrak{p}, such that there exist \{ n_1, \ldots, n_r \} \in \tilde{M}(\tilde{W}) satisfying \{ n_i \} = m_i. The morphism of \text{O}_{\tilde{W}}\text{-modules} \text{O}_{\tilde{W}} \to \tilde{M}_{\tilde{W}}, (a_i) \mapsto \sum a_i \cdot n_i is an epimorphism on stalks at \mathfrak{p}. Hence, it is an epimorphism in an open neighbourhood \tilde{U} of \mathfrak{p}. \hfill \square

\textbf{8. Proposition:} Any schematic morphism \( f' : \tilde{X} \to \tilde{Y} \) is quasi-compact.

\textbf{Proof.} Let \( \tilde{V} \subset \tilde{Y} \) be a quasi-compact open subset, we have to prove that \( f'^{-1}(\tilde{V}) \) is quasi-compact. We only have to prove that \( \tilde{U}_x \cap f'^{-1}(\tilde{V}) \) is quasi-compact, for any \( x \in X \). Hence, we can suppose that \( \tilde{X} = \tilde{U}_x \).

Let \( C := \tilde{Y} - \tilde{V} \). By Lemma \textcolor{red}{13.20} there exists a finitely generated ideal \( I \subset \text{O}_Y \) such that \( (I)_0 = C \). Consider the exact sequence of morphisms \( 0 \to I \to \text{O}_{\tilde{Y}} \to \text{O}_{\tilde{Y}}/I \to 0 \). By Lemma \textcolor{red}{15.7} there exist an open covering \( \{ \tilde{U}_i \} \) of \( \tilde{Y} \) and epimorphisms \( \text{O}_{\tilde{U}_i}/I \to I|_{\tilde{V}} \). Taking \( f''' \), one has an exact sequence of morphisms

\[ f'''I \to \tilde{O}_{\tilde{X}} \to f'''(\text{O}_{\tilde{Y}}/I) \to 0 \]

and \( \mathcal{F} := \text{Im}[f'''I \to \tilde{O}_{\tilde{X}}] \) is a finitely generated quasi-coherent ideal of \( \tilde{O}_{\tilde{X}} \), since it is so locally (over \( f'^{-1}(\tilde{U}_i) \)). \( \mathcal{F}_0 = f^{-1}(C) \), therefore \( f'^{-1}(\tilde{V}) = \tilde{X} - (\mathcal{F}_0) \) is a quasi-compact open subset. \hfill \square

Let \( C_{sch} \) be the category of schematic finite spaces and \( W \) the family of quasi-isomorphisms. Let us construct the localization of \( C_{sch} \) by \( W, C_{sch}[W^{-1}] \).

\textbf{9. Definition:} A schematic pair of morphisms from \( X \) to \( Y \), \( X \rightarrow_{f} Y \) is a pair of schematic morphisms \((\phi', f') \)

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y \\
\phi' & \downarrow & \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \phi' \) is a quasi-isomorphism.

\textbf{10. Example:} A schematic morphism \( f : X \to Y \) can be considered as a schematic pair of morphisms: Consider the pair of morphisms \((I\text{d}_X, f) \)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \xrightarrow{I\text{d}_X} & \\
X & \xrightarrow{f} & Y
\end{array}
\]
11. **Definition:** Let $f = (\phi', f'): X \rightarrow Y$ and $g = (\varphi', g'): Y \rightarrow Z$ be two schematic pairs of morphisms, where $\phi': X' \rightarrow X$ is a quasi-isomorphism and $f': X' \rightarrow Y$ is a schematic morphism and $\varphi': Y' \rightarrow Y$ is a quasi-isomorphism and $g': Y' \rightarrow Z$ is a schematic morphism. Let $\pi_1, \pi_2: X' \times_Y Y' \rightarrow X', Y'$ be the two obvious projection maps (observe that $\pi_1$ is a quasi-isomorphism). We define $g \circ f := (\phi' \circ \pi_1, g' \circ \pi_2): X \rightarrow Z$

\[
\begin{array}{c}
\text{X} \xrightarrow{f} \text{Y} \\
\text{X' \times_Y Y'} \xrightarrow{\pi_1} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\pi_2} \text{Y'} \\
\text{X} \xrightarrow{\phi'} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\varphi'} \text{Y'} \xrightarrow{g'} \text{Z}
\end{array}
\]

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be schematic morphisms. Then,

$$(\text{Id}_X, f) \circ (\text{Id}_Y, g) = (\text{Id}_X, f \circ g).$$

12. **Definition:** Two schematic pairs of morphisms $(\phi', f'), (\phi'', f'')$: $X \rightarrow Y$

\[
\begin{array}{c}
\text{X} \xrightarrow{f} \text{Y} \\
\text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\phi'} \text{X'} \\
\text{X} \xrightarrow{f} \text{Y} \xrightarrow{\phi'} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\phi''} \text{X''} \xrightarrow{f''} \text{Y} \\
\text{X} \xrightarrow{\phi'} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\phi''} \text{X''} \xrightarrow{f''} \text{Y}
\end{array}
\]

are said to be equivalent, $(\phi', f') \equiv (\phi'', f'')$, if there exist a schematic space $T$ and two quasi-isomorphisms $\pi': T \rightarrow X', \pi'': T \rightarrow X''$ such that the diagram

\[
\begin{array}{c}
\text{T} \\
\text{X'} \xrightarrow{\phi'} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\phi''} \text{X''} \xrightarrow{f''} \text{Y} \\
\text{T} \xrightarrow{\pi'} \text{X} \xrightarrow{\phi'} \text{X'} \xrightarrow{f'} \text{Y'} \xrightarrow{\phi''} \text{X''} \xrightarrow{f''} \text{Y} \\
\end{array}
\]

is commutative.

In order to prove the associative property the reader should consider the following com-
15. **Hom}_{sch}(\tilde{X}, \tilde{Y}) = \text{Hom}_{\text{sch}}(X, Y)

mutative diagram (where the double arrows are quasi-isomorphisms)

13. **Definitions and notations:** Let \( f = (\phi, f') : X \rightarrow Y \) be a schematic pair of morphisms. The equivalence class of \( f \) (resp. \( (\phi, f') \)) will be denoted \([f]\) (resp. \([\phi, f']\)). We shall say that \([f]\) (or \([f] : X \rightarrow Y\)) is a [schematic] morphism from \( X \) to \( Y \).

Let \( f : X \rightarrow Y \) be a schematic morphism. The equivalence class of \((\text{Id}, f)\) will be denoted \([f]\).

14. **Proposition:** Let \((\phi', f') : X \rightarrow Y\) be a schematic pair of morphisms, where \( \phi' : X' \rightarrow X \) is a quasi-isomorphism and \( f' : X' \rightarrow Y \) is a schematic morphism. Let \( \varphi : X'' \rightarrow X' \) be a quasi-isomorphism. Then, \([\phi', f'] = [\phi' \circ \varphi, f' \circ \varphi]\).

**Proof.** Consider the commutative diagram

15. **Theorem:** Let \( f, F : X \rightarrow Y \) and \( g, G : Y \rightarrow Z \) be schematic pairs of morphisms. If \([f] = [F]\) and \([g] = [G]\), then \([g \circ f] = [G \circ F]\).

**Proof.** 1. Let us prove that \([g \circ f] = [g \circ F]\). Write \( f = (\phi, f'), \phi : X' \rightarrow X, f' : X' \rightarrow Y \). Let \( \varphi : X'' \rightarrow X' \) be a quasi-isomorphism and let \( F' := (\phi \circ \varphi, f' \circ \varphi) \). By Proposition \[15.14\]
[f] = [F']. Consider the commutative diagram

\[
\begin{array}{cccc}
X'' & \xrightarrow{\psi} & X' \times_Y Y' & \\
\downarrow & & \downarrow & \\
X' & \xrightarrow{f} & X' \times_Y Y' & \xrightarrow{Y'} \\
\downarrow & & \downarrow & \\
X & \xrightarrow{\phi} & Y & \xrightarrow{g} Z
\end{array}
\]

By Proposition\cite{15.14}, \([g \circ f] = [g \circ F'].\) Finally, since \([f] = [F],\) there exists \(F'\) such that \([g \circ f] = [g \circ F'] = [g \circ F].\)

2. Likewise, \([g \circ f] = [G \circ f].\)

3. The theorem is a consequence of 1. and 2. \(\square\)

Let \(f: X \rightarrow Y\) and \(g: Y \rightarrow Z\) be two schematic pairs of morphisms. We define \([g] \circ [f] := [g \circ f].\)

16. Proposition: Let \((\phi, f'): X \rightarrow Y\) be a schematic pair of morphisms. Then,

1. If \(f'\) is a quasi-isomorphism, then \([\phi, f']^{-1} = [f', \phi].\)

2. \([\phi, f'] = [f'] \circ [\phi]^{-1}.\)

Proof. We have the quasi-isomorphism \(\phi: X' \rightarrow X\) and the morphism \(f': X' \rightarrow Y.\) Let

\[\pi_1, \pi_2: X' \times_X X' \rightarrow X', \quad \pi_1(x_1', x_2') := x_1', \quad \pi_2(x_1', x_2') := x_2',\]

which are quasi-isomorphisms.

1. Let \(\delta: X' \rightarrow X' \times_X X'\) be the diagonal morphism, which is a quasi-isomorphism because \(\pi_1\) and \(\pi_1 \circ \delta = Id\) are quasi-isomorphisms. Then,

\[
[\phi, f'] \circ [f', \phi] = [f' \circ \pi_1, f' \circ \pi_2] = [f' \circ \pi_1 \circ \delta, f' \circ \pi_1 \circ \delta] = [f', f'] = [Id_Y, Id_Y] = [Id_Y, Id_Y]
\]

Likewise, \([f', \phi] \circ [\phi, f'] = [Id_X].\)

2. It is easy to check that \([f'] \circ [\phi]^{-1} = [f' \circ \pi_1, f' \circ \pi_2] \circ [\phi, f'] = [\phi, f'].\) \(\square\)

17. Proposition: Let \(X\) be a minimal schematic finite space. Let \(f, g: X \rightarrow Y\) be two schematic morphisms. Then, \([f] = [g]\) iff \(f = g.\)

Proof. \(\Rightarrow\) There exists a (surjective) quasi-isomorphism \(\pi: T \rightarrow X\) such that \(f \circ \pi = g \circ \pi.\) Then \(f\) and \(g\) are equal as continuous maps. Finally, the morphism \(O_Y \rightarrow f_\pi O_T = f_\pi O_X\) coincides with the morphism \(O_Y \rightarrow g_\pi O_T = g_\pi O_X.\) \(\square\)
18. Proposition: Let $X$ be a minimal schematic finite space. Let $f, g: X \to Y$ be two schematic morphisms and $\pi: Y \to Y'$ a quasi-isomorphism. Then, $\pi \circ f = \pi \circ g$ if and only if $f = g$.

Proof. $\Rightarrow$) Observe that $[f] = [g]$ since $[\pi] \circ [f] = [\pi \circ f] = [\pi \circ g] = [\pi] \circ [g]$. By Proposition 15.17, $f = g$. $\square$

19. Proposition: A schematic morphism $[\phi, f]: X \to Y$ is an isomorphism if and only if $f$ is a quasi-isomorphism.

Proof. $\Leftarrow$) $[\phi, f]^{-1} = [f, \phi]$, by Proposition 15.16.

$\Rightarrow$) If $[\phi, f] = [f] \circ [\phi]^{-1}$ is invertible, then $[f]$ is invertible. Put $f: Z \to Y$ and let $[\varphi, g]: Y \to Z$ be the inverse morphism of $[f]$, where $\varphi: T \to Y$ is a quasi-isomorphism (and we can assume that $T$ is minimal) and $g: T \to Z$ is a schematic morphism. Then, $[\text{Id}_Y] = [f] \circ [\varphi, g] = [f] \circ [g] \circ [\varphi]^{-1}$ and $[\varphi] = [f \circ g]$. Hence, $\varphi = f \circ g$, by Proposition 15.17. Besides, $[\text{Id}_Z] = [\varphi, g] \circ [f]$. If we consider the commutative diagram

\[
\begin{array}{ccc}
Z \times_Y T & \overset{\pi^2}{\longrightarrow} & T \\
\downarrow{\pi_1} & & \downarrow{\varphi} \\
Z & \underset{f}{\longrightarrow} & Y \\
\end{array}
\]

then $[\text{Id}_Z] = [\varphi, g] \circ [f] = [\pi_1, g \circ \pi_2]$. Let $i: (Z \times_Y T)_M \subseteq Z \times_Y T$ be the natural inclusion, $\pi'_1 = \pi_1 \circ i$ and $\pi'_2 = \pi_2 \circ i$. Then, $[\pi'_1, g \circ \pi'_2] = [\text{Id}_Z]$ and $[\pi'_1] = [g \circ \pi'_2]$. By Proposition 15.17, $\pi'_1 = g \circ \pi'_2$. Let $\tilde{g}: T \to (Z \times_Y T)_M$, $\tilde{g}(t) = (g(t), t)$. Observe that

$$\pi'_1 \circ \tilde{g} \circ \pi'_2 = g \circ \pi'_2 = \pi'_1.$$

By Proposition 15.18, $\tilde{g} \circ \pi'_2 = \text{Id}_{(Z \times_Y T)_M}$. Obviously, $\pi'_2 \circ \tilde{g} = \text{Id}_T$. Hence, $\pi'_2$ is an isomorphism and $\pi_2$ a quasi-isomorphism. Finally, $f$ is a quasi-isomorphism since $\pi_1, \phi$ and $\pi_2$ are quasi-isomorphisms and $f \circ \pi_1 = \phi \circ \pi_2$. $\square$

Given a schematic morphism $g = [\phi, f]: X \to Y$, consider the functors

\[
\begin{align*}
g_*: \text{QC-Mod}_X & \to \text{QC-Mod}_Y, \quad g_* M := f_* \phi^* M \\
g^* : \text{QC-Mod}_Y & \to \text{QC-Mod}_X, \quad g^* M := \phi_* f^* M
\end{align*}
\]

Recall $[\phi, f] = [\phi \circ \varphi, f \circ \varphi]$, where $\varphi$ is a quasi-isomorphism. Then we have canonical isomorphisms $f_* \phi^* M = f_* \varphi_* \varphi^* M = (f \circ \varphi)_* (\phi \circ \varphi)^* M$ and $\phi_* f^* M = \phi_* \varphi_* \varphi^* f^* M = (\phi \circ \varphi)_* (f \circ \varphi)^* M$.

20. Proposition: Let $g = [\phi, f]: X \to Y$ be a schematic morphism. The functors $g_*$ and $g^*$ are mutually inverse if $g$ is a schematic isomorphism.
15. \( \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y}) = \text{Hom}_{\text{sch}}(X, Y) \)

**Proof.** \( \Rightarrow \) If \( \text{Id} = g \circ g^* \) and \( \text{Id} = g^* g \), then \( \text{Id} = f, \phi \circ f^* = f, f^* \) and \( \text{Id} = \phi \circ f^* \circ \phi^* \), hence \( f^* f = \phi^* \phi = \text{Id} \). By Theorem \( \ref{thm:quasi-isomorphism} \) \( f \) is a quasi-isomorphism and \( g \) is invertible.

\( \Leftarrow \) By Proposition \( \ref{prop:schematic} \), \( f \) is a quasi-isomorphism, then \( g \circ g^* = f, \phi \circ f^* = f, f^* = \text{Id} \) and \( g^* g = \phi \circ f^* \circ \phi^* = \phi^* \phi = \text{Id} \).

\( \Box \)

21. **Notation:** Let \( X \) and \( Y \) be two schematic finite spaces. \( \text{Hom}_{\text{sch}}(X, Y) \) will denote the family of [schematic] morphisms from \( X \) to \( Y \). \( \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y}) \) will denote the set of schematic morphisms from \( \tilde{X} \) to \( \tilde{Y} \).

22. **Lemma:** Let \( g : Y' \to Y \) be a [schematic] isomorphism and \( X \) a schematic finite space. Then, the maps

\[
\text{Hom}_{\text{sch}}(X, Y') \to \text{Hom}_{\text{sch}}(X, Y), \quad [f] \mapsto [g] \circ [f]
\]

\[
\text{Hom}_{\text{sch}}(Y, X) \to \text{Hom}_{\text{sch}}(Y', X), \quad [f] \mapsto [f] \circ [g]
\]

are biyective.

23. **Proposition:** Let \( X \) be a schematic finite space and \( Y \) an affine finite space. Then,

\[
\text{Hom}_{\text{sch}}(X, Y) = \text{Hom}_{\text{ring}}(O(Y), O(X)).
\]

**Proof.** For any schematic finite space \( T \), \( \text{Hom}_{\text{sch}}(T, (\ast, A)) = \text{Hom}_{\text{ring}}(A, O(T)) \).

Consider the natural morphism \( \pi : Y \to (\ast, O(Y)) \), which is a quasi-isomorphism. Then,

\[
\text{Hom}_{\text{sch}}(X, Y) = \text{Hom}_{\text{sch}}(X, (\ast, O(Y))) = \text{Hom}_{\text{ring}}(O(Y), O(X)).
\]

\( \Box \)

Let \( [\phi, f] : X \to Y \) be a [schematic] morphism, where \( \phi : X' \to X \) is a quasi-isomorphism and \( f : X' \to Y \) a schematic morphism. Consider the morphisms

\[
\begin{array}{ccc}
\tilde{X}' & \xrightarrow{\tilde{\phi}} & \tilde{Y} \\
\downarrow f & & \\
\tilde{X} & \xrightarrow{\tilde{\phi}} & \tilde{Y}
\end{array}
\]

where \( \tilde{\phi} \) is an isomorphism, by Proposition \( \ref{prop:isomorphism} \). The morphism

\[
\text{Hom}_{\text{sch}}(X, Y) \to \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y}), \quad [\phi, f] \mapsto \tilde{f} \circ \tilde{\phi}^{-1}
\]

is well defined.

24. **Lemma:** Let \( X \) be a minimal schematic space, \( Y \) a schematic \( T_0 \)-space, \( f : X \to Y \) a schematic morphism and \( \tilde{f} : \tilde{X} \to \tilde{Y} \) the induced morphism. Given \( x \in X \), \( y = f(x) \) iff \( y \) is the greatest element of \( Y \) such that \( \tilde{f}(\tilde{U}_x) \subseteq \tilde{U}_y \),
26. **Definition:** Let \( f(U_x) \subset U_{f(x)} \) and \( \tilde{f}(\tilde{U}_x) \subset \tilde{U}_{f(x)} \). Observe that if \( f \) is injective, then \( \tilde{f} \) is injective. Hence, \( \tilde{U}_x = \tilde{U}_x \cap f^{-1}(U_{f(x)}) \). Therefore, \( x \in U_x \cap f^{-1}(U_{f(x)}) \) is a removable point. That is, \( f(x) \in U_{f(x)} \) and \( f(x) \geq y' \).

25. **Proposition:** Let \( X \) and \( Y \) be schematic finite spaces. The natural morphism

\[
\text{Hom}_{\text{sch}}(X, Y) \to \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y}), \quad [\phi, f] \mapsto \tilde{f} \circ \tilde{\phi}^{-1}
\]

is injective.

**Proof.** Let \([\phi, f], [\phi', f'] : X \to Y\) be [schematic] morphisms such that \( \tilde{f} \circ \tilde{\phi}^{-1} = \tilde{f}' \circ \tilde{\phi}'^{-1} \). We can suppose that \( \phi = \phi' \), then \( \tilde{f} = \tilde{f}' \). Let us say that \( f \) and \( f' \) are morphisms from \( X' \) to \( Y \). By Proposition 13.12 and Lemma 15.22 we can suppose that \( X' \) and \( Y \) are minimal schematic spaces.

By Lemma 15.24 the map \( f \) is determined by \( \tilde{f} \). The morphism of rings \( O_{f(x')} \to O_{x'} \) is determined by the morphism of schemes \( \text{Spec} O_{x'} \to \text{Spec} O_{f(x')} \). Therefore, \( f = f' \).

26. **Definition:** Let \( U \xrightarrow{i_1} U_1 \) and \( U \xrightarrow{i_2} U_2 \) be quasi-open immersions. We denote \( U_1 \cup U_2 := C(i_1) \big|_U C(i_2) \).

27. Observe that \( C(i_1) \) and \( C(i_2) \) are open subsets of \( U_1 \cup U_2 \), \( C(i_1) \cup C(i_2) = U_1 \cup U_2 \), and the natural morphisms \( C(i_j) \to U_j \) are quasi-isomorphisms, for \( j = 1, 2 \).

Let \( U_1, U_2 \subset X \) be open subsets. Then, the natural morphism \( U_1 \cup U_2 \to U_1 \cup U_2 \) is a quasi-isomorphism.

Let

\[
\begin{array}{ccc}
V_1 & \to & U_1 \\
\downarrow & & \downarrow \\
V & \to & U \\
\downarrow & & \downarrow \\
V_2 & \to & U_2
\end{array}
\]

be a commutative diagram of quasi-open immersions, where the arrows \( \tilde{\to} \) are quasi-isomorphisms. Then, the natural morphism \( V_1 \cup V_2 \to U_1 \cup U_2 \) is a quasi-isomorphism.

28. **Theorem:** Let \( U \xrightarrow{i_1} U_1 \) and \( U \xrightarrow{i_2} U_2 \) be quasi-open immersions. Then,

\[
\text{Hom}_{\text{sch}}(U_1 \cup U_2, Y) = \text{Hom}_{\text{sch}}(U_1, Y) \times_{\text{Hom}_{\text{sch}}(U, Y)} \text{Hom}_{\text{sch}}(U_2, Y).
\]

In other words (by 15.27), let \( U_1, U_2 \subset X \) be open subsets. Then,

\[
\text{Hom}_{\text{sch}}(U_1 \cup U_2, Y) = \text{Hom}_{\text{sch}}(U_1, Y) \times_{\text{Hom}_{\text{sch}}(U \cap U_2, Y)} \text{Hom}_{\text{sch}}(U_2, Y).
\]
By Proposition 15.17, let $X$ and $Y$ be schematic finite spaces and suppose that $Y$ is semi separated. We have to prove that

$$\text{Hom}_{sch}(X, Y) = \text{Hom}_{sch}(X, Y)$$

is bijective.

Let $[f], [g]$ such that $([f] \circ [j_1], [f] \circ [j_2]) = ([g] \circ [j_1], [g] \circ [j_2])$. There exist a minimal schematic finite space $W$, a quasi-isomorphism $\phi : W \to U_1 \cup U_2$ and morphisms $f', g' : W \to Y$ such that $[f] = [\phi, f']$ and $[g] = [\phi, g']$. Then,

$$\text{Hom}_{sch}(U_1 \cup U_2, Y) \to \text{Hom}_{sch}(U_1, Y) \times \text{Hom}_{sch}(U_1 \cap U_2, Y) \text{ Hom}_{sch}(U_2, Y)$$

is bijective.

Let $\phi : W \to U_1 \cup U_2$ and morphisms $f', g' : W \to Y$ such that $[f] = [\phi, f']$ and $[g] = [\phi, g']$. Then, $f' \circ [j_1] = [\phi, f'] \circ [j_1]$ and $g' \circ [j_1] = [\phi, g'] \circ [j_1]$.

By Proposition 15.17, $f'_{|_{\phi^{-1}(U_1)}} = g'_{|_{\phi^{-1}(U_1)}}$. Likewise, $f'_{|_{\phi^{-1}(U_2)}} = g'_{|_{\phi^{-1}(U_2)}}$. That is, $f' = g'$ and $[f] = [g]$.

Let $([f_1], [f_2]) \in \text{Hom}_{sch}(U_1, Y) \times \text{Hom}_{sch}(U_1 \cap U_2, Y)$. Write $[f_1] = [\phi_1, g_1]$ and $[f_2] = [\phi_2, g_2]$. Since $[f_1] \circ [i_1] = [f_2] \circ [i_1]$, there exist a schematic finite space $V$ and a commutative diagram

![Diagram](image)

(Where the arrows $\cdots \to$ are quasi-isomorphisms). Then, we have a schematic morphism $g : V_1 \cup V_2 \to Y$ and the composition $\phi$ of the quasi-isomorphisms $V_1 \cup V_2 \to U_1 \cup U_2 : U_1 \cup U_2 \to U_1 \cup U_2$. The reader can check that $[\phi, g] \mapsto ([f_1], [f_2])$.

**29. Theorem:** Let $X$ and $Y$ be schematic finite spaces and suppose that $Y$ is semiseparated. Then, the morphism

$$\text{Hom}_{sch}(X, Y) \to \text{Hom}_{sch}(X, Y), [\phi, f] \mapsto f \circ \phi^{-1}$$

is bijective.
15. \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y}) = \text{Hom}_{\text{sch}}(X, Y)

\textbf{Proof.} By Proposition \[15.25\] it is injective. Let \(f' \in \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y})\).

1. Assume that \(\tilde{X} = \text{Spec} \ A\) is affine. We can suppose that \(Y\) is a \(T_0\)-space. Observe that

\[f'_{*} \tilde{A} \text{ is a quasi-coherent } \mathcal{O}_Y\text{-module, then } f'_{*} \tilde{A} = \tilde{A}, \text{ where } \mathcal{A} \text{ is a quasi-coherent } \mathcal{O}_Y\text{-module and an } \mathcal{O}_Y\text{-algebra. Let us prove that } (Y, \mathcal{A}) \text{ is affine. We only have to prove that } (Y, \mathcal{A}_p)\]

is affine for any \(p \in \text{Spec} \ A\). Given an open subset \(U \subset Y\), let \(I_U := \{\text{quasi-compact open subsets } \tilde{V} \subset \tilde{Y} : \tilde{U} \subset \tilde{V}\}.\) Recall

\[\mathcal{A}(U) = \mathcal{A}_U(U) = \mathcal{A}_U(\tilde{U}) \cong \tilde{A}; \lim_{\tilde{V} \in I_U} \tilde{A}(\tilde{V}) = \lim_{\tilde{V} \in I_U} \tilde{A}(f'^{-1}(\tilde{V})).\]

Denote by \(f'_p\) the composition of the morphisms \(\text{Spec} \ A_p \hookrightarrow \text{Spec} \ A \xrightarrow{f'} \tilde{Y}\). Observe that

\[\mathcal{A}_p(U) := \mathcal{A}_U(U) = \lim_{\tilde{V} \in I_U} \tilde{A}(f'^{-1}(\tilde{V})) = \lim_{\tilde{V} \in I_U} \tilde{A}_p(f'^{-1}(\tilde{V})).\]

Then, we can suppose that \(A = A_p\). The obvious morphism \(\text{Id}: (Y, \mathcal{A}) \to (Y, \mathcal{O}_Y)\) is affine and \((U_{yy'}, \mathcal{O}_{y(U_{yy'})})\) is affine, for any \(y, y' \in Y\), since \(Y\) is semiseparated. Hence, \((U_{yy'}, \mathcal{A}_{U_{yy'}})\) is affine. Then, the morphism \(\mathcal{A}_{yy'} \to \prod_{k \in U_{yy'}} \mathcal{A}_k\) is faithfully flat. Let \(y\) be the greatest point of \(Y\) such that \(f'(y) \in U_y\) and let \(y' \in Y\) be another point. Observe that

\[\mathcal{A}_y = \lim_{\tilde{V} \in I_{yy'}} \tilde{A}(\tilde{V}) \cong \lim_{\tilde{V} \in I_{yy'}, \tilde{W} \in I_{yy'}} \tilde{A}(\tilde{V} \cap \tilde{W}) \cong \lim_{\tilde{W} \in I_{yy'}} \tilde{A}_{U_{yy'}}(\tilde{U}_{yy'}) = \mathcal{A}_{yy'}\]

(\(\cong\) since \(f'^{-1}(\tilde{V}) = f'^{-1}(\tilde{V} \cap \tilde{W})\)). Then, \(\mathcal{A}_y = \mathcal{A}_{yy'} \to \prod_{k \in U_{yy'}} \mathcal{A}_k\) is faithfully flat. Hence, \(y'\)

is a removable point of \((Y, \mathcal{A})\) if \(y \not\in y'\). Therefore, \((Y, \mathcal{A})\) is quasi-isomorphic to \((U_y, \mathcal{A}_U)\), then it is affine.

Finally, \(\text{Spec } \mathcal{A} = \text{Spec } \mathcal{A}(Y) = \text{Spec } \tilde{X} = \tilde{X}\) and the morphism \(\tilde{X} = \text{Spec } \mathcal{A} \to \text{Spec } \mathcal{O}_Y = \tilde{Y}\) induced by the obvious morphism \(\text{Id}: (Y, \mathcal{A}) \to (Y, \mathcal{O}_Y)\) is \(f'\). Therefore, \(\text{Hom}_{\text{sch}}(X, Y) = \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y})\).

2. Now, in general.

\[\text{Hom}_{\text{sch}}(X, Y) = \lim_{x \in X} \text{Hom}_{\text{sch}}(U_x, Y) \cong \lim_{x \in X} \text{Hom}_{\text{sch}}(U_x, Y) = \lim_{x \in X} \text{Hom}_{\text{sch}}(\tilde{U}_x, \tilde{Y}) \cong \text{Hom}_{\text{sch}}(\lim_{x \in X} \tilde{U}_x, \tilde{Y}) = \text{Hom}_{\text{sch}}(\tilde{X}, \tilde{Y})\]

(\(\cong\) given \(f_x \in \lim_{x \in X} \text{Hom}_{\text{sch}}(\tilde{U}_x, \tilde{Y})\) the induced morphism of ringed spaces \(f: \lim_{x \in X} \tilde{U}_x \to \tilde{Y}\) is schematic, since for any quasi-coherent \(\mathcal{O}_X\)-module \(\tilde{M} = \lim_{x \in X} \tilde{i}_{xs} \tilde{M}_s\), the \(\mathcal{O}_Y\)-module \(f_* \tilde{M} = \lim_{x \in X} f_* \tilde{i}_{xs} \tilde{M}_s = \lim_{x \in X} f_* \tilde{M}_s\) is quasi-coherent). \(\square\)
References

[1] Alexandroff, P.S., Diskrete Raume. Mathematische Sbornik, (N.S.) 2, 501-518, (1937).

[2] Eisenbud, D., Commutative Algebra with a View Toward Algebraic Geometry, GTM 150, Springer-Verlag, 1995.

[3] Enochs, E. and Estrada, S., Relative homological algebra in the category of quasi-coherent sheaves, Adv. Math. 194, no. 2, 284-295 (2005).

[4] Godement, R., Topologie algébrique et théorie des faisceaux, Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13 Hermann, Paris (1958), viii+283 pp.

[5] Grothendieck, A., and Dieudonné, J., Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Première partie, Publ. Math. IHÉS, (1964).

[6] Hartshorne, R., Algebraic Geometry, GTM 52, Springer (1997).

[7] Matsumura, H., Commutative algebra, W.A. Benjamin Co, New York (1970).

[8] Sancho, F. and Sancho, P., Affine ringed spaces and Serre’s criterion, Rocky Mountain J. Math. 47, 6, (2017).

[9] Sancho, F., Homotopy of finite ringed spaces, J. Homotopy Relat. Struct. 13, no. 3, 481-501 (2018).

[10] Sancho, F., Finite spaces and schemes, J. Geom. Phys. 122, 3-27 (2017).

[11] Serre, J.P., Sur la cohomologie des variétés algébriques, J. de Maths. Pures et Appl. 36, 1-16 (1957).