DYNAMICAL BLUEPRINTS FOR GALAXIES

LAWRENCE M. WIDROW
Department of Physics, Engineering Physics, and Astronomy, Queen’s University, Kingston,
ON K7L 3N6, Canada; widrow@astro.queensu.ca

BRENT PYM
Department of Mathematics, University of Toronto, 40 St. George Street, Toronto,
ON M5S 2E4, Canada; bpym@math.toronto.edu

AND

JOHN DUBINSKI
Department of Astronomy and Astrophysics, University of Toronto, 60 St. George Street, Toronto,
ON M5S 3H8, Canada; dubinski@astro.utoronto.ca

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ABSTRACT

We present an axisymmetric, equilibrium model for late-type galaxies which consists of an exponential disk, a Sérsic bulge, and a cuspy dark halo. The model is specified by a phase-space distribution function which, in turn, depends on the integrals of motion. Bayesian statistics and the Markov chain Monte Carlo method are used to tailor the model to satisfy observational data and theoretical constraints. By way of example, we construct a chain of $10^5$ models for the Milky Way designed to fit a wide range of photometric and kinematic observations. From this chain, we calculate the probability distribution function of important Galactic parameters such as the Sérsic index of the bulge, the disk scale length, and the disk, bulge, and halo masses. We also calculate the probability distribution function of the local dark matter velocity dispersion and density, two quantities of paramount significance for terrestrial dark matter detection experiments. Although the Milky Way models in our chain all satisfy the prescribed observational constraints, they vary considerably in key structural parameters and therefore respond differently to nonaxisymmetric perturbations. We simulate the evolution of 25 models which have different Toomre $Q$ and Goldreich-Tremaine $X$ parameters. Virtually all of these models form a bar, although some more quickly than others. The bar pattern speeds are $\sim 40–50$ km s$^{-1}$ kpc$^{-1}$ at the time when they form and then decrease, presumably due to coupling of the bar with the halo. Since the Galactic bar has a pattern speed $\sim 50$ km s$^{-1}$ kpc$^{-1}$, we conclude that it must have formed recently.

Subject headings: dark matter — Galaxy: kinematics and dynamics — methods: $n$-body simulations — methods: statistical

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1. INTRODUCTION

Dynamical galactic models serve a variety of purposes. They may be used to interpret the structural and kinematical observables of galaxies—surface brightness profiles, rotation curves, and velocity dispersion profiles—in terms of intrinsic three-dimensional density and velocity distributions. Dynamical models also provide a starting point for controlled simulations of complicated processes such as the formation of bars and spiral structure. In short, galactic modeling provides the essential interface between observations and detailed theories of galaxy formation.

In this paper, we introduce a new dynamical model for late-type galaxies which comprises a disk, bulge, and dark halo. The model is derived from equilibrium solutions to the collisionless Boltzmann and Poisson equations. It is extremely flexible and may be tailored to satisfy observational data and theoretical constraints. We use Bayesian statistics and the Markov chain Monte Carlo (MCMC) method to implement these constraints and to determine the probability distribution function (PDF) of the model in the full multidimensional parameter space.

Our model builds on earlier work by Kuijken & Dubinski (1995) and Widrow & Dubinski (2005). The original Kuijken and Dubinski model consists of an exponential disk, a King-model bulge, and a lowered Evans-model halo and has the attractive feature that the phase-space distribution function (DF) is built from analytic functions of the integrals of motion. No additional assumptions about the velocity-space distribution are made. By contrast, the widely used approach described in Hernquist (1993; see also Springel & White 1999) assumes that the local velocity distributions of the halo and bulge particles are Gaussian with dispersions estimated from the Jeans equations. This approach leads to models which are slightly out of equilibrium. When used as initial conditions in $N$-body experiments, they readjust to a different state from the one proposed (see, e.g., Kazantzidis et al. 2004).

There are two main disadvantages of the Kuijken and Dubinski models. First, the bulge and halo have constant density (or weakly cuspy) centers, whereas actual bulges and dark halos may have central density cusps. Second, the structure of the bulge and halo are determined implicitly by the model parameters. (By construction, the disk’s structural parameters, namely, its radial and vertical scale lengths, its mass, and its truncation radius, are determined explicitly by the model parameters.) Widrow & Dubinski (2005) built a galactic model with $r^{-1}$ density cusps for both the bulge and halo, but again with a DF that determines the structure of the bulge and halo implicitly.

For our new model, the closed-form DF is abandoned in favor of a numerical DF which is designed to yield, to a very good approximation, user-specified density profiles for the bulge and halo. That is, the density profiles of the bulge and halo are now explicit...
functions of the model parameters. The present version of the model allows for a Sersic bulge and a halo with $\rho \propto r^{-\gamma}$ as $r \to 0$, where $\gamma$ is between 0 and 2.

Our model is specified in terms of 15 or so parameters. How are these parameters selected? One approach is to choose models at random and identify the ones that satisfy certain general constraints (e.g., the Tully-Fisher and size-luminosity relations). The result would be a catalog of disk-bulge-halo systems which could be used to study kinematical and dynamical trends such as the circular speed–central velocity dispersion ($V_c$, $\sigma_0$) relation (see, e.g., Couteau et al. 2007). A catalog of this type could also be used for N-body studies of mergers or interactions between galaxies in a cosmological environment. A second approach, and the one pursued here, is to build models for specific galaxies. Following Kuijkjen & Dubinski (1995) and Widrow & Dubinski (2005) we use the Milky Way as our illustrative example.

Modeling the Milky Way is a time-honored endeavor; notable examples include Innanen (1973), Clutton-Brock et al. (1977), Bahcall & Soneira (1980), Caldwell & Ostriker (1981), Kuijkjen & Gilmore (1991), Rohlfs & Kreitschmann (1988), Malhotra (1995), Kochanek (1996), Evans & Wilkinson (1999), and Klypin et al. (2002). Our construction of dynamical Milky Way models is in the spirit of the mass model survey by Dehnen & Binney (1998). Their models comprise a multicomponent disk, a bulge, and a halo and are characterized by 10 parameters. Twenty-two examples are presented, each the result of a maximum likelihood analysis in which some parameters are held fixed while others are allowed to vary. The Dehnen and Binney likelihood function is constructed by comparing model predictions with six sets of observational data: the inner and outer Galactic rotation curves, the Oort constants, the mass at large radii, the local vertical force, and the line-of-sight velocity dispersion in Baade’s window. The advantage of our models is that they not only describe the potential-density pair for the Galaxy, but also the underlying DF. We therefore have the ability to examine the stability of our Galactic models using N-body simulations, an issue that is often ignored (but see Sellwood 1985; Fux 1997).

For the most part, we adopt Dehnen & Binney’s choice of observational data, although we include more complete observations of the line-of-sight dispersion in the bulge region as well as photometric data from the COBE satellite. We also present what we believe to be a more balanced treatment of the likelihood function. Most significantly, we bring to the problem the powerful tools of Bayesian statistics and MCMC. These tools allow us to map out PDFs of both input parameters and derived quantities.

Although our model represents an axisymmetric, equilibrium system, it is susceptible to nonaxisymmetric instabilities and therefore provides a natural starting point for numerical studies of galactic dynamics. An N-body realization of the model can be easily generated from the DF and then used as the initial conditions for a numerical simulation.

The Milky Way models in our MCMC series all satisfy the observational constraints but vary considerably in their structural properties. We simulate a selection of 25 models which span a wide range in Toomre $Q$ (Toomre 1964) and Goldreich-Tremaine $X$ (Goldreich & Tremaine 1978, 1979) parameters and find that a bar develops in virtually all of the cases. The onset of the bar instability can occur immediately or after several Gyr, depending on the model.

We present the model in § 2, review the observational constraints in § 3, and provide a summary of the essentials of Bayesian statistics and the MCMC method in § 4. We discuss some preliminaries including our choice of prior probabilities, in § 5. We present the results of our MCMC analysis in § 6, and the results of our bar formation simulations in § 7. In § 8 we summarize our main conclusions and speculate on how we might improve on and extend the models and MCMC analysis.

2. GALACTIC MODELS

We consider axisymmetric, collisionless systems whose DF is of the form

$$f(\mathcal{E}, L_z, E_z) = f_d(\mathcal{E}, L_z, E_z) + f_h(\mathcal{E}) + f_b(\mathcal{E}),$$

where $\mathcal{E} \equiv -E$ is the relative energy, $L_z$ is the angular momentum about the symmetry axis, and $E_z$ is the energy associated with vertical motions of stars in the disk (Kuijkjen & Dubinski 1995; Widrow & Dubinski 2005). For time-independent, axiymmetric systems $\mathcal{E}$ and $L_z$ are integrals of motion, while $E_z$ is an approximate integral of motion for disk stars on nearly circular orbits. Jeans theorem implies that a system generated by equation (1) will be in approximate equilibrium.

Integrating equation (1) over all velocities yields the density in terms of the gravitational potential, $\Phi$, and the cylindrical coordinates $R$ and $z$:

$$\rho(R, z, \Psi) = \rho_d(R, z, \Psi) + \rho_h(\Psi) + \rho_b(\Psi),$$

where $\Psi \equiv -\Phi$ is the relative potential. (Note that implicit in eq. [1] is the assumption that the bulge and halo velocity dispersions are isotropic.) Self-consistency requires that $\rho$ and $\Psi$ satisfy Poisson’s equation:

$$\nabla^2 \Psi = -4\pi \rho(R, z, \Psi),$$

which is accomplished, in practice, through an iterative scheme. (Note that here and throughout, we set Newton’s constant $G = 1$.)

Kuijkjen & Dubinski (1995) chose $f_b$ to be the King model DF (King 1966) and $f_h$ to be the lowered Evans model DF from Kuijkjen & Dubinski (1994). (The latter depends on $L_z$ as well as $\mathcal{E}$, thereby allowing for flattened halos.) Their models have two main shortcomings. First, bulges and halos may have central density cusps, whereas the King and lowered Evans DFs yield density profiles with constant-density cores. Second, the relationship between the model parameters and the density profiles of the bulge and halo is implicit rather than explicit and not particularly intuitive.

Widrow & Dubinski (2005) built galactic models with cuspy ($\rho \propto r^{-1}$ as $r \to 0$) bulges and halos. Specifically, they chose the Hernquist (1990) DF for the bulge and a DF from Widrow (2000) for the halo. The latter was constructed to yield the so-called NFW profile (Navarro et al. 1996).

$$\rho_{\text{NFW}}(r) = \frac{\rho_h}{(r/a_h)(1 + r/a_h)^2}.$$
1948). The bulges of late-type galaxies are found to have surface brightness profiles which follow the more general Sérsic law,

$$\Sigma(r) = \Sigma_0 e^{-b(r/R_h)^{1/n}},$$  \hspace{1cm} (5)

with Sérsic index $n$ between 0.6 and 2 (Andredaki et al. 1995; Courteau et al. 1996). Likewise, dark matter halos may have density profiles more general than the NFW form. Since the work of Navarro et al. (1996), there has been considerable debate over the actual form of halo density profiles. Moore et al. (1999) find evidence in their simulations for steeper cusps ($\rho \propto r^{-3}$). More recently, Navarro et al. (2004) concluded that the logarithmic slope of the halo density profiles decreases steadily with radius, although their results are still consistent with equation (4). It is interesting to note that Navarro et al. [2004] model the density profiles using the Sérsic law applied to the space density rather than the projected density. For a detailed discussion, see Merritt et al. [2006], and references therein.) On the observational side, the rotation curves of dark matter-dominated low surface brightness galaxies appear to favor constant density cores (Moore 1994; Flores & Primack 1994; McGaugh & de Blok 1998; van den Bosch et al. 2000), although this interpretation of the data is being challenged on a number of fronts.

For our new models, we begin by choosing target density profiles, $\rho_0$ and $\tilde{\rho}_h$, for the bulge and halo. Assume, for the moment, that the system is spherically symmetric. Through the Abel integral transform (Binney & Tremaine 1987),

$$f_i(\xi) = \frac{1}{\sqrt{8\pi^3}} \int_0^\xi d\eta \frac{\partial \Psi_i}{\partial \Psi_{\text{tot}}} \frac{d\Psi_{\text{tot}}}{\sqrt{\xi - \Psi_{\text{tot}}}}, \quad i = b, h,$$  \hspace{1cm} (6)

we can construct bulge and halo DFs which yield the target density profiles. In the case of an isolated halo or bulge, $\Psi_{\text{tot}}$ is the potential derived from $\rho_0$ or $\tilde{\rho}_h$ and equation (6) reduces to the usual expression for the DF of a spherically symmetric system with isotropic velocities. The DF for a system following the Sérsic law was found with this method by Ciotti (1991). DFs for NFW halos were found by Zhao (1997), Widrow (2000), and Lokas & Mamom (2001). For a composite system or one with an external potential, one simply replaces the $\Psi_i$ derived from $\rho_0$ with the total gravitational potential. Tremaine et al. (2002) used this method to derive DFs for bulges with central black holes by setting $\Psi_{\text{tot}} = \Psi_h + GM_{\text{BH}}/r$.

Equation (6) is only valid for spherically symmetric systems, a condition violated once a disk is included. Our approach is to use a spherical approximation (essentially, the monopole term of a spherical harmonics expansion) for the disk potential in evaluating $\Psi_{\text{tot}}$. We stress that equation (6) is used to construct $f_b(\xi)$ and $f_h(\xi)$, not to solve for $\Psi(R, z)$ and $\rho(R, z)$. We can use $f_b(\xi)$ and $f_h(\xi)$ in equation (1) even though the composite system is not spherically symmetric; A DF of the form $f = f_b(\xi)$ yields an equilibrium system in any time-independent potential regardless of the spatial symmetries of the potential.\footnote{A self-consistent system with a DF that depends only on the energy must be spherically symmetric (Binney & Tremaine 1987). The statement in the text applies to systems in which there is an external potential that does not necessarily respect spherical symmetry. Here the potential due to the disk plays the role of an external potential to the halo and bulge.} The bulge and halo of the final model are axisymmetric (but not spherically symmetric), since isodensity surfaces follow isopotential surfaces (eq. [2]), the latter being flattened by the disk. As we demonstrate below, the spherically averaged density profiles of the bulge and halo are very close to the target profiles.

### 2.1. Target Density Profiles

We choose the target density profile for the bulge to be

$$\tilde{\rho}_b(r) = \rho_0 \left( \frac{r}{R_h} \right)^{-p} e^{-b(r/R_h)^{1/n}},$$  \hspace{1cm} (7)

which yields the Sérsic law (eq. [5]) for the projected surface density profile provided one sets $p = 1 - 0.6097n + 0.05563/n^2$ (Prugniel & Siminen 1997; Terzić & Graham 2005). Note that here and in equation (5), $\Sigma_0$, $R_e$, and $n$ are free parameters while the constant $b$ is adjusted so that $R_e$ encloses half the total projected light or mass. In our models, we use

$$\sigma_b \equiv \left\{ 4\pi n b^n (p-2)^{-1} n(2-p) \right\}^{1/2},$$  \hspace{1cm} (8)

rather than $\rho_0$, to parameterize the overall density scale of the bulge models. With this definition, $\sigma_b^2$ corresponds to the depth of the gravitational potential associated with the bulge.

We choose the target density profile of the halo to be

$$\tilde{\rho}_h = \frac{2^{2-\gamma} \sigma_h^2}{4\pi a_h^2 (r/a_h)^{3-\gamma}} \frac{1}{1 + r/a_h} C(r; r_h, \sigma_h),$$  \hspace{1cm} (9)

where $C$ is a truncation function that smoothly goes from unity to zero at $r = r_h$ over a width $br_h$. We use the function

$$C(r; r_h, \sigma_h) = \frac{1}{2} \text{erfc} \left( \frac{r - r_h}{\sqrt{2br_h}} \right).$$  \hspace{1cm} (10)

The halo profile is therefore characterized by five parameters: $r_h$, $\sigma_h$, the halo scale length, $a_h$, the velocity scale, $\sigma_h$, and the central “cusp strength,” $\gamma$. For $r < r_h$, the mass interior to radius $r$ is given by

$$M(r) = 2^{2-\gamma} \sigma_h^2 a_h \left[ \frac{1}{1 + r/a_h^2} + \log \left( \frac{1 + r/a_h^2}{1 + r/a_h} \right) \right].$$  \hspace{1cm} (11)

Following Kuijken & Dubinski (1995), we adjust the disk’s DF so that its space density falls off approximately exponentially in $R$ and as $\text{sech}^2$ in $z$ with radial and vertical scale lengths $R_d$ and $z_d$, respectively. The disk is truncated at a radius $R_B$ with a truncation sharpness parameter $\delta R_d$. In addition, we choose a DF where the radial dispersion profile is approximately exponential:

$$\sigma^2_R(R) = \sigma_{R0}^2 \exp \left( -R/R_c \right).$$  \hspace{1cm} (12)

For simplicity, we set $R_c = R_d$ in accord with observations by Bottema (1993).

### 3. OBSERVATIONAL CONSTRAINTS

We use nine sets of observational data as constraints on our Milky Way models. Five of these data sets—the inner and outer rotation curves, the Oort constants, the vertical force in the solar neighborhood, and the total mass at large radii—are largely taken from Dehnen & Binney (1998, and references therein). We incorporate measurements of the line-of-sight bulge dispersion from the compilation of data by Tremaine et al. (2002) as well as estimates of the local velocity ellipsoid from Binney & Merrifield.
(1998). We also use dust-corrected near-infrared DIRBE data from the COBE satellite (Binney et al. 1997).

**Inner rotation curve.**—Inside the solar circle, the Galactic rotation curve is usually presented in terms of the “terminal velocity”, the peak velocity along a given line of sight at Galactic coordinates $b = 0$ and $|l| < \pi/2$. Assuming that the Galaxy is axisymmetric, $v_{\text{term}}$ is given by

$$v_{\text{term}} = v_c(R) - v_c(R_0) \sin l, \quad (13)$$

where $R_0$ is the distance from the Sun to the Galactic center and $v_c$ is the circular speed (see, e.g., Binney & Merrifield 1998). Following Dehnen & Binney (1998) we use data from the H I emission observations by Malhotra (1995) restricted to the range $l \geq 0.3$ so as to avoid distortions from the bar.

**Outer rotation curve.**—The radial velocity of an object relative to the local standard of rest, $v_{\text{lsr}}$, is related to the circular rotation curve through the equation

$$v_{\text{lsr}} = \frac{R_0}{R} v_c(R) - v_c(R_0) \cos b \sin l, \quad (14)$$

where $R = (d^2 \cos^2 b + R_0^2 - 2 R_0 d \cos b \sin l)^{1/2}$, $(l, b)$ are the Galactic coordinates, and $d$ is the distance to the object. Measurements of $v_{\text{lsr}}$ and $d$ are compared to model estimates for $W(R)$ and $d(R)$, where $W(R) \equiv (R_0/R) v_c(R) - v_c(R_0) \equiv v_{\text{lsr}}/\cos b \sin l$. We adjust the $R$ to minimize the quantity

$$\chi^2_l = \left[ \frac{W(R) - W_l}{\Delta W_l} \right]^2 + \left[ \frac{d(R) - d_l}{\Delta d_l} \right]^2, \quad (15)$$

where $W_l \equiv v_{\text{lsr}}/\cos b \sin l$. In what follows, we use data from Brand & Blitz (1993) with the same restrictions as in Dehnen & Binney (1998); i.e., $l \leq 155^\circ$ or $l > 205^\circ$, $d > 1 \text{ kpc}$, and $W < 0$ so as to avoid contamination from noncircular motions.  

**Local circular speed.**—Further constraint from the rotation curve of the Galaxy comes from estimates of the circular speed at the solar radius, $v_c(R_0)$. Here, we adopt the estimate of Reid et al. (1999), who carried out VLBA observations of Sgr A*.

**Vertical force above the disk.**—Kuijken & Gilmore (1991) use K dwarf stars as tracers of the gravitational potential above the Galactic plane, thereby placing a constraint on the total surface density in the solar neighborhood. They find

$$v_c(R_0) = (219 \pm 20 \text{ km s}^{-1}) \left( \frac{R_0}{8 \text{ kpc}} \right), \quad (16)$$

which is consistent with other estimates (Sackett 1997).

**Mass at large radii.**—The observed velocity distribution of the Milky Way satellite system and the dynamics of the Magellanic Stream, together with measurements of the high-velocity tail of the local stellar velocity distribution, provide constraints on the large-scale mass distribution of the Galactic halo. Following Dehnen & Binney (1998), who base their arguments on analyses by Kochanek (1996) and Lin et al. (1995), we adopt

$$M(r < 100 \text{ kpc}) = (7 \pm 2.5) \times 10^{11} M_\odot, \quad (21)$$

in excellent agreement with estimates of known matter in the solar neighborhood. We adopt equation (17) as the constraint on the vertical force at $(R, z) = (R_0, 1.1 \text{ kpc})$ and equation (18) the constraint on the surface density of the disk at $R = R_0$.  

**Oort constants.**—The Oort constants,

$$A \equiv \frac{1}{2} \left( \frac{v_c}{R} \frac{\partial v_c}{\partial R} \right)_{R=R_0}, \quad B \equiv \frac{1}{2} \left( \frac{v_c}{R} + \frac{\partial v_c}{\partial R} \right)_{R=R_0}, \quad (19)$$

measure, respectively, the local shear and vorticity in the Galactic disk. Here we adopt the constraints

$$A = 14.8 \pm 0.8 \text{ km s}^{-1} \text{ kpc}^{-1}, \quad B = 12.4 \pm 0.6 \text{ km s}^{-1} \text{ kpc}^{-1}, \quad (20)$$

from the Feast & Whitelock (1997) analysis of Cepheid proper motion measurements by the Hipparcos satellite.  

**Local velocity ellipsoid.**—The kinematics of stars in the solar neighborhood provide important constraints on the structure of the Milky Way. The observation that $v^2_\phi \neq v^2_\theta$ already tells us that the disk DF necessarily involves a third integral of motion (Binney 1987). Our constraints for the local velocity ellipsoid are taken from Table 10.4 of Binney & Merrifield (1998), which in turn were derived from Edvardsson et al. (1993). Binney & Merrifield (1998) give separate values for the thin and thick disks. Since our models assume a single disk component, we use a mass-weighted average (Widrow & Dubinski 2005).

**Bulge dispersion.**—Observations of the line-of-sight velocity dispersion in the direction of the bulge provide important constraints on the bulge parameter and, to a lesser extent, the parameters of the other components. We use measurements of the line-of-sight dispersion between 4 and 1300 pc from the compilation by Tremaine et al. (2002). Since the bulge is triaxial, the measured line-of-sight dispersion depends on the observer’s orientation to its principal axes. Our line of sight to the Galactic center is approximately 20° from the long axis of the bulge (Binney et al. 1997), and therefore the measured line-of-sight dispersion will be systematically higher than the values one would obtain assuming a spherical bulge. Following Tremaine et al. (2002), we adjust the measured dispersions downward by a factor 1.07 to account for this effect.  

**Surface photometry.**—The distribution of stars in the Milky Way is most easily determined from observations in the near infrared, where stellar emission dominates over dust emission. Although dust is more transparent at these wavelengths than in the optical, extinction due to dust is still significant toward the Galactic center. Spitzer et al. (1996) produced extinction-corrected
maps of the inner Galaxy based on the DIRBE data set and a three-dimensional dust model (see also Freudenreich 1998). Binney et al. (1997) constructed three-dimensional models for the light distribution of the disk and bulge based on these maps with the aim of constraining the structural parameters of the Galactic bar.

Our initial goal is to construct axisymmetric models for the Galaxy. Toward this end, we use the surface brightness as a function of $l$ at mid–Galactic latitudes ($3^\circ < |b| < 4.5^\circ$) from Binney et al. (1997; their Fig. 2, lower panel) where the effects of the bar are not so pronounced (i.e., where their axisymmetric model adequately reproduces the observed surface brightness profile).

The mass model survey of Dehnen & Binney (1998) employs a maximum likelihood analysis where the likelihood function is $\exp(-\chi^2_{DB})$ with

$$\chi^2_{DB} = \frac{W_{in}}{N_{in}} \chi^2_{in} + \frac{W_{out}}{N_{out}} \chi^2_{out} + \frac{W_{other}}{N_{other}} \chi^2_{other}. \quad (22)$$

The subscripts “in,” “out,” and “other” refer to the inner and outer rotation curve constraints and the other constraints (e.g., Oort constants, vertical force), respectively. The $N_i$ are the numbers of data points actually used while $W_i$ are weights introduced by Dehnen & Binney (1998) to account for the “number of really independent constraints.” That is, the $W_i$ are meant to compensate for the fact that a quantity such as the Oort constant $A$ has been obtained from a large number of data points. Dehnen & Binney (1998) choose $W_{in} = W_{out} = W_{other} = 6$, although they admit that the choice of $W_i$ are “subject to ones prejudices.”

In our view, the likelihood function should be $\exp(-\chi^2_{tot}/2)$, where

$$\chi^2_{tot} = \chi^2_{in} + \chi^2_{out} + \chi^2_{other}. \quad (23)$$

The fact that the Oort constant constraints are obtained from a large number of (raw) data points is already accounted for by the small quoted errors. Dividing $\chi^2_{in}$ by $N_{in}$ unfairly shortchanges the rotation curve data in favor of the Oort constant constraints, and so forth.

To survey the model parameter space, Dehnen & Binney (1998) adopt the following approach: fix certain parameters and maximize the likelihood function by allowing the remaining parameters to vary. The procedure is then repeated with the fixed parameters set to different values, or different subsets of parameters held fixed. The result is a rather uneven survey of the full parameter space. A similar exercise was carried out by Widrow et al. (2003) for M31 using the original Kuijken and Dubinski models together with rotation curve, velocity dispersion, and surface brightness data. This procedure was also used for both M31 and the Milky Way in Widrow & Dubinski (2005). Bayesian statistics and MCMC provide a more complete picture of the model parameter space as we now demonstrate.

4. BAYESIAN ANALYSIS AND MCMC

Our aim is to calculate the posterior probability density function, $p(M|D, I)$, of a Galactic model $M$ given data $D$ and prior information $I$. The model is specified by the fifteen model parameters as well as additional astronomical parameters—here $R_0$ and the mass-to-light ratios of the disk and bulge, $(M/L)_d$ and $(M/L)_b$. We collect the parameters into a vector $A$ with components $A^j$ where $j = 1 \ldots N$ and $N$ is the total number of parameters. From Bayes’ theorem

$$p(M|D, I) = \frac{p(M|I)p(D|M, I)}{p(D|I)}, \quad (24)$$

where $p(M|I)$ is the prior probability density, $p(D|M, I)$ is the likelihood function, and $p(D|I) \equiv \int p(M|D, I) dA$ is a normalization factor. Our choice of priors is described in § 5.

MCMC is an efficient means of calculating $p(M|D, I)$ whereby one constructs a sequence or “chain” of models whose density in parameter space is proportional to the posterior PDF provided the chain is long enough to have fully explored all “important” regions of parameter space. Marginalization—that is, integration over a subset of parameters—is trivial; simply project the chain onto the appropriate subspace and compute the density of points. Likewise, the PDF of any model-dependent quantity is obtained by making a histogram of the quantity over the chain of models.

Our Markov chain is constructed via the Metropolis-Hastings algorithm (Metropolis et al. 1953; Hastings 1970) as outlined in Gregory (2005). The chain of models is described by the sequence $A_n, i = 0, 1, 2, \ldots$. We begin with a starting point $A_0$. A candidate for $A_1$ is chosen according to the “jumping rule” (also known as the “proposal distribution”), $q(A_1|A_0)$. The candidate $A_1$ is accepted with probability equal to $\min\{1, r\}$, where

$$r = \frac{p(A_1|D, I) q(A_0|A_1)}{p(A_0|D, I) q(A_1|A_0)}. \quad (25)$$

Otherwise, $A_1$ is set equal to $A_0$. The process is then repeated for $A_2$.

The success of an MCMC analysis rests, by and large, on choosing a suitable jumping rule. If the step size from $A_n$ to a candidate for $A_{n+1}$ is too small, the chain will move slowly through parameter space. On the other hand, if the step size is too large, most attempts to find a new point in parameter space will fail. In either case, exploration of parameter space, often referred to as mixing, can require a prohibitively large amount of computing resources. Ideally, the jumping rule is shaped like the PDF but scaled by a factor $2.4/N^{1/2}$ (Gelman et al. 1995), which explains why it is often referred to as the proposal distribution.

In this work, we take $q$ to be a multivariate Gaussian, so that

$$A_{n+1} = A_n + D \cdot G, \quad (26)$$

where $G$ is a vector of length $N$ whose components are Gaussian random variables with unit variance and $D$ is a user-specified $N \times N$ matrix. Since neither $D$ nor $G$ depend on the model parameters, $q(A_{n+1}|A_n) = q(A_n|A_{n+1})$, and therefore $r = p(A_{n+1}|D, I) / p(A_n|D, I)$.

We begin with a simple Ansatz for the proposal distribution in which $D$ is a diagonal matrix whose nonzero elements are given by our best guess for the variance of each of the model parameters multiplied by $2.4/N^{1/2}$. From a short chain of a few thousand models we estimate the covariance matrix

$$C_{ij} = \frac{(A^i - \bar{A}^i)(A^j - \bar{A}^j)}{(A^i \bar{A}^j)}, \quad (27)$$

where the angle brackets denote an average along the chain. Our improved expression for the proposal distribution is given by equation (26) with $D^2 = (2.4^2/N)C$. 

Each “data point” carries with it a quoted error. Of course, the observer may have underestimated the error or there may be features in the data which cannot be explained by the model. Both situations can be handled by introducing an unknown error parameter for each data set which is added in quadrature to the quoted error (Gregory 2005). For the purpose of the MCMC calculation, these error parameters are simply incorporated into an expanded definition of $A$, that is, treated as model parameters.

5. PRELIMINARIES AND PRIORS

Simple arguments, based on general features of the Galaxy, provide preliminary estimates for the model parameters which in turn guide our choices of the prior probabilities used in the MCMC analysis. We assume that the priors for each of the model parameters are nonzero over a range somewhat larger than the range suggested by these estimates. For parameters that correspond to a physical scale (e.g., $R_e$, $M_d$, $v_{th}$), we assume a Jeffreys prior, essentially, equal probability in logarithmic intervals over the prescribed range. For dimensionless parameters, such as the halo cusp strength and Sersic index, we assume a flat prior. (See Gregory 2005, for a discussion.)

The projected velocity dispersion profile toward the Galactic bulge, $\sigma_p(R)$, reaches a peak value of $\sim 130$ km s$^{-1}$ at a radius $\sim 200$ pc (Tremaine et al. 2002). On the other hand, estimates of the half-light or effective radius of the bulge, $R_e$, range from 570 to 920 pc (see Tremaine et al. 2002, and references therein).

The projected velocity dispersion of the Prugniel & Simien (1997) profile exhibits a similar structure to that of the Milky Way: $\sigma_p(R)$ is nonzero at $R = 0$, rises to a peak value of $\sigma_{pk}$ at a radius $R_{pk}$ and then decreases with radius (see Fig. 10 of Prugniel & Simien 1997; as well as earlier work by Binney 1986; Ciotti 1991). The dimensionless ratios $R_{pk}/R_e$, $\sigma_{pk}/\sigma_b$, and $M_b/\sigma_b^2 R_e$ are functions of $n$ as shown in Figure 1. From the figure we deduce that for the Milky Way, $n$ is less than 2, $R_e$ is between 0.57 and 0.92 kpc, $\sigma_b$ is between 340 and 400 km s$^{-1}$, and $M_b$ is between $1 \times 10^{10} - 3.4 \times 10^{10} M_\odot$.

Binney et al. (1997) constructed a model for the luminosity density of the Galaxy to fit data from the DIRBE experiment. Their model consisted of a triaxial bulge and double exponential disk with bulge-to-total luminosity ratio of 0.16. Subsequently, Malhotra et al. (1996) derived a total $L$-band luminosity for the Milky Way of $7.1 \times 10^{10} L_\odot$ with $1.1 \times 10^{10} L_\odot$ attributed to the bulge.

Stellar population synthesis models provide estimates for the stellar mass-to-light ratios in different wave bands (Bell & de Jong 2001, and references therein). The $L$-band stellar mass-to-light ratio for the disk is expected to be in the range 0.5–0.65 in solar units (R. S. de Jong 2007, private communication) assuming a scaled Salpeter IMF (Bell & de Jong 2001) and the Pegasus population synthesis model. The mass-to-light ratio for the bulge could be somewhat higher. On the other hand, since our model does not include a separate gas disk, the effective mass-to-light ratio for the disk must account for any gas and should therefore be higher than the value for a pure stellar disk. The local stellar-to-gas ratio is $\sim 1.6$ (see Table 1 of Binney & Tremaine 1987) and therefore the effective $(M/L)_b$ might be closer to 1. Together with our estimate for the disk luminosity, we conclude that $M_d$ is between $3 \times 10^{10}$ and $6 \times 10^{10} M_\odot$.

Reid (1993) reviewed estimates of the distance from the Sun to the Galactic center and concluded that $R_0 = 8.0 \pm 0.5$ kpc. More recently Eisenhauer et al. (2003) observed the star S2 in orbit about the Galaxy’s massive central black hole using the ESO VLT and found $R_0 = 7.94 \pm 0.42$ kpc.

Sackett (1997) reviewed estimates of the radial scale length of the Galactic disk and found $R_d = 3.0 \pm 1$ kpc. More recent estimates show a similarly large spread in values. Zheng et al. (2001) found $R_d = 2.75 \pm 0.3$ kpc from HST observations of M dwarfs while Lopez-Corredoira et al. (2002) found $R_d = 3.3^{+0.5}_{-0.7}$ kpc from an analysis of old stellar populations using 2MASS survey data. As emphasized by Sackett (1997), the ratio $R_0/R_d$ is observationally more secure than $R_d$. The estimates collected in her review show $R_0/R_d$ between 2.7 and 3.5.

The disk scale height parameter, $h_d$, is more difficult to constrain, since the Galactic disk comprises at least three distinct components: the gas disk, the thin disk, and the thick disk, whereas our model has a single-component disk. Multicomponent disks will be incorporated into future versions of the model but for the time being, $h_d$ must represent the vertical mass distribution of all disk-like components. From Sackett (1997), we surmise that $h_d$ is between 0.2 and 1 kpc.

The radial velocity dispersion in the solar neighborhood is $36 \pm 5.4$ km s$^{-1}$ (Binney & Merrifield 1998). Together with estimates of the radial scale length of the disk and with equation (12), this translates into a range of possible values for $\sigma_{R_0}$.

We allow the halo parameters to vary over a wide range of values. For example, we assume that the prior probability distribution

![Figure 1](image-url)
of $\gamma$ is uniform between 0 and 1.5 and nonzero otherwise. An alternative approach is to use cosmological models of halo and galaxy formation to guide one’s choice of the halo parameters (see, e.g., Valenzuela & Klypin 2003), but given uncertainties in the exact nature of adiabatic compression, variations among halo profiles found in different simulations, and possible discrepancies between halo profiles as inferred from observations and those found in simulations, we take a more conservative approach. Furthermore, since the data do not probe the Galaxy beyond 100 kpc, we set $r_h = 100$ kpc and $\sigma_h = 5$ kpc, although no physical meaning should be ascribed to these values; $r_h$ can be increased without changing the fit to the data.

Our choices for the prior probabilities of the model parameters are found in Table 1. In addition to the parameters listed in Table 1, we include unknown error parameters (see §4) for the terminal velocity, outer rotation curve, bulge dispersion, and surface brightness profile data sets. Thus, our MCMC analysis is run with 17 parameters.

| Parameter | Prior | Lower Bound | Upper Bound |
|-----------|-------|-------------|-------------|
| $\sigma_h$ | Jeffreys | 2 km s$^{-1}$ | 6 km s$^{-1}$ |
| $a_b$ | Jeffreys | 2 kpc | 35 kpc |
| $\gamma$ | Uniform | 0 | 1.5 |
| $M_d$ | Jeffreys | $2 \times 10^{10} M_\odot$ | $7 \times 10^{10} M_\odot$ |
| $R_d$ | Jeffreys | 2 kpc | 3.5 kpc |
| $h_d$ | Jeffreys | 0.2 kpc | 1 kpc |
| $\sigma_R$ | Jeffreys | 80 km s$^{-1}$ | 300 km s$^{-1}$ |
| $n$ | Uniform | 0.6 | 2.0 |
| $\sigma_R$ | Jeffreys | 150 km s$^{-1}$ | 400 km s$^{-1}$ |
| $\langle M/L \rangle_d$ | Uniform | 0.6 | 1.2 |
| $\langle M/L \rangle_b$ | Uniform | 0.6 | 1.2 |
| $R_0$ | Jeffreys | 7 kpc | 9 kpc |

6. MCMC RESULTS

We generate five Markov chains of length $2 \times 10^4$. Each chain has roughly 3400 distinct members corresponding to an acceptance rate of 17%. We can calculate the PDFs for different quantities using the five chains individually and in combination. If mixing has been achieved then the results will be the same to within statistical uncertainties. In Figure 2 we illustrate that this is indeed the case by plotting the average values with 1$\sigma$ error bars for a selection of six model parameters calculated for each of the five chains. The values are normalized by dividing by the overall average.

6.1. Selected Models

In Figure 3 we show the rotation curve and density profile for a model from our MCMC series with $n \simeq 1$ and $\gamma \simeq 1$. Dashed line is for the bulge; dotted line is for the disk; long-dashed line is for the halo. Top: Solid black line shows the total rotation curve. Bottom: We plot $r^2 \rho$ for the bulge and halo and $k_r \rho$ for the disk, quantities proportional to the mass in radial bins. Also shown are the “target” density profiles for the bulge and halo (thin lines; eqs. [7] and [4], respectively). [See the electronic edition of the Journal for a color version of this figure.]
We choose models with $\gamma \simeq 1$ and Sérsic parameters $n \simeq 0.6, 1,$ and $2$. We also include a model with $n = 4$, that is, with essentially an $R^{1/4}$ law bulge. Since our MCMC run does not find any models with a Sérsic parameter this large, we generate this model by fixing $n = 4$ and allowing the remaining parameters to vary.

While suitable models are found for $n$ between 0.6 and 2 this is not the case for $n = 4$. This result is in agreement with studies of bulges in late-type spiral galaxies (Andreadakis et al. 1995; Courteau et al. 1996; MacArthur et al. 2003) and suggests that the Galaxy has a pseudobulge rather than a classical bulge (see Kormendy & Kennicutt 2004, for a review).

The models clearly have a difficulty reproducing the shape of the line-of-sight velocity dispersion profile. In particular, the line-of-sight velocity minimum found in the data at $R \simeq 3$ pc is much deeper than is allowed by the models. The discrepancy may indicate that the density profile of the Galactic bulge is significantly different from the one proposed by Prugniel & Simien (1997) or that velocity-space anisotropy and deviations from spherical symmetry are necessary to model the dispersion profile in the innermost region of the bulge (Spergel et al. 1996; Fux 1997; Freudenreich 1998).

Table 2: Results for input parameters

| Parameter | Average |
|-----------|---------|
| $\sigma_{B}$ | $330^{+35}_{-32}$ |
| $a_{0}$ | $13.6^{+1.2}_{-1.7}$ |
| $\gamma$ | $0.8^{+0.6}_{-0.9}$ |
| $M_{d}$ | $4.1^{+0.33}_{-0.3}$ |
| $R_{d}$ | $2.8^{+0.12}_{-0.12}$ |
| $h_{d}$ | $0.36^{+0.04}_{-0.04}$ |
| $\sigma_{B}$ | $119^{+12}_{-11}$ |
| $n$ | $1.32^{+0.03}_{-0.33}$ |
| $\sigma_{B}$ | $272^{+25}_{-23}$ |
| $R_{e}$ | $0.64^{+0.09}_{-0.09}$ |
| $(M/L)_{d}$ | $0.96^{+0.11}_{-0.09}$ |
| $(M/L)_{d}$ | $0.69^{+0.09}_{-0.07}$ |
| $R_{0}$ | $7.94^{+0.2}_{-0.2}$ |

Note.—Units are km s$^{-1}$ for velocities, $10^{10} M_{\odot}$ for masses, and kpc for distances.

Fig. 4.—Surface brightness profile as a function of $l$ for four models and for data from Spergel et al. (1996). Three models are chosen from our main MCMC run: red curve, $(n, \gamma) \simeq (0.6, 1)$; magenta curve, $(n, \gamma) \simeq (1.1)$; blue curve, $(n, \gamma) \simeq (2.1)$. Also shown (green curve) is a model chosen from a run where $n$ is fixed to the “de Vaucouleurs” value, 4 and $\gamma = 0.86$. Top: Thin curves show the separate contributions of the disk and bulge. Bottom: Residuals between the models and the data.

Fig. 5.—Terminal velocity as a function of sin $l$ for four models and for data from Malhotra (1995). Line types and colors are the same as in Fig. 4.

Fig. 6.—Line-of-sight velocity dispersion toward the bulge as a function of projected radius from the Galactic center for models and for data from Tremaine et al. (2002). Line types and colors are the same as in Fig. 4.
The maximum a posteriori values and boundaries of the 68.3% credibility regions for the input parameters and calculated quantities are given in Tables 2 and 3, respectively. Not surprisingly, $a_h$ and $\gamma$ exhibit the largest fractional uncertainties. Most of the data used in this study pertain to the region of the Galaxy near and inside the Sun’s orbit about the Galactic center; equation (21) provides a rather weak constraint on the mass at large radii (or equivalently, the circular speed at these radii). We note that $a_h$ and $\gamma$ are correlated in the sense that models with larger values of $a_h$ have $\gamma$ closer to 1—large constant-density cores ($\gamma \simeq 0$ and $a_h \simeq 20–30$ kpc) are disfavored by the data.

In Figure 7 we show the PDFs for the disk and bulge masses, as well as the halo mass within 10 and 100 kpc. We also show the pseudolikelihood function for the 22 models considered in Dehnen & Binney (1998). Specifically, we plot

$$L_{DB} = e^{-\chi^2_{DB}},$$

(28)

where $\chi^2_{DB}$ is from their Table 3 with $\chi^2_{min}$ set to the value for their best-fit model. Our results for the disk, bulge, and halo masses are consistent with those of Dehnen & Binney (1998). Figure 7 illustrates the advantages of the MCMC method. The Dehnen and Binney analysis involves 22 separate maximum likelihood calculations characterized by the authors’ ad hoc choices of fixed and free parameters. MCMC, on the other hand, yields the full multidimensional posterior probability function from which PDFs for one or more parameters are easily obtained.

Figure 8 provides a contour plot of the PDF in the $R_0-R_d$ plane. Our results are consistent with previous estimates of these

| Parameter | Average |
|-----------|---------|
| $M_d$ | $4.23 \pm 0.52$ |
| $M_b$ | $0.96 \pm 0.12$ |
| $M_{10}$ | $4.23 \pm 0.48$ |
| $M_{25}$ | $12.6 \pm 2.8$ |
| $M_{50}$ | $24.9 \pm 5.27$ |
| $M_{100}$ | $40.0 \pm 9.16$ |
| $\rho_0$ | $0.0080 \pm 0.0004$ |
| $\sigma_0$ | $241 \pm 23$ |

Notes.—Units for $\rho_0$ are $M_\odot$ pc$^{-3}$. Other quantities use the following units: km s$^{-1}$ for velocities, $10^{10} M_\odot$ for masses, and kpc for distances.
two quantities. The plot also shows the general trend that models with higher values of \( R_d \) tend to favor higher values of \( R_0 \).

In Figure 9 we show the PDFs for \( n \) and \( \gamma \). As noted above, our analysis clearly favors bulges with surface brightness profiles closer to an exponential than to de Vaucouleurs \( R^{1.4} \) law. Our results allow for a dark halo with a wide range of inner logarithmic density slopes which includes the NFW value as well as steeper and shallower indices.

### 6.3. Comparison with Published Milky Way Models

In Figure 10, we compare values for the disk and bulge masses from our Markov chain analysis with those from a number of popular Milky Way models. One of the earliest mass models was constructed by Bahcall et al. (1983). While they focus on constraining the Galactic spheroid through star counts (see also Bahcall & Soneira 1980) they also fit the Galactic rotation curve by modeling the mass distribution of the disk, bulge, and dark halo. Their choice for the disk and bulge masses (\( M_d = 5.6 \times 10^{10} M_\odot \) and \( M_b = 1.1 \times 10^{10} M_\odot \)) is represented in Figure 10 by the filled triangle.

From Shuttle IRT observations, Kent et al. (1991) derived total \( K \)-band luminosities for the disk and bulge of \( L_d = 4.9 \times 10^{10} L_\odot \) and \( L_b = 1.1 \times 10^{10} L_\odot \), respectively. (Note that these values differ slightly from the values quoted in the original paper because a different value for the \( K \) magnitude of the Sun was used.) Kent (1992) went on to construct disk-bulge-halo mass models based on these results and was able to fit the Galactic rotation curve for three choices of the disk mass-to-light ratio: a maximal disk model \( (M/L)_d = 1.3 \), a high disk model \( (M/L)_d = 1.0 \), and a low disk model \( (M/L)_d = 0.68 \). In each case, the bulge mass-to-light ratio was \( (M/L)_b = 1.0 \). Kent’s models are shown in Figure 10 as blue, red, and green stars.

Klypin et al. (2002) constructed models for the Milky Way and Andromeda galaxies based on cosmologically motivated theories of disk formation. Their favored model for the Milky Way has \( M_d = 4.0 \times 10^{10} M_\odot \) and \( M_b = 0.8 \times 10^{10} M_\odot \) and is represented in Figure 10 by the solid square.

Since its discovery (Ibata et al. 1994; Ibata et al. 1995), the Sagittarius dwarf galaxy has held the promise of providing useful constraints on the size and shape of the Milky Way’s dark halo. The usual approach is to simulate the tidal disruption of Sagittarius as it passes through the Galactic potential and compare the distribution of tidal debris with photometric and kinematic observations of the observed tidal streams (see Law et al. 2005, for a recent example and references to earlier work). Johnston et al. (1999) introduced a model for the Galactic potential which has now become a standard for work in this area. The Johnston et al. (1999) values, \( M_d = 1.0 \times 10^{11} M_\odot \) and \( M_b = 3.4 \times 10^{10} M_\odot \), are shown in Figure 10 as an open square.

Our models occupy a smaller region of \( M_d-M_b \) parameter space than is spanned by popular models from the literature. The Johnston et al. (1999) choices for \( M_d \) and \( M_b \) are inconsistent with our results by factors of 2.5 and 4, respectively. More to the point, their choices fall well outside the region of acceptable models. The choices for disk and bulge masses in Bahcall et al. (1983) and Kent (1992) are consistent with our results as is the preferred model from Klypin et al. (2002).
Implications for Dark Matter Detection Experiments

Models of the Milky Way are an essential ingredient in the analysis of dark matter detection experiments. For example, microlensing experiments, which attempt to measure the distribution of massive compact halo objects (MACHOs) along various lines of sight through the halo, require a model for the MACHO component of the halo as well as the distribution of stars in the disk. In analyzing their 5.7 yr data set, the MACHO experiment considered a wide range of Galactic models (Alcock et al. 2000). Here we focus on their "standard model." This model includes both a thin and thick disk, each with radial scale length $R_d = 4$ kpc and total disk mass $M_d = 4.5 \times 10^{10} M_\odot$. The halo is modeled as a cored isothermal sphere with a density profile given by

$$\rho(r) = \frac{\rho_0 a^2 + R_0^2}{a^2 + r^2},$$

(29)

where, for their standard model, Alcock et al. (2000) assumed $R_0 = 8.5$ kpc, $a = 5$ kpc and $\rho_0 = 0.0079 M_\odot$ pc$^{-3}$. Although they did not model the bulge explicitly, we can infer its mass through the requirement that the circular rotation speed of the Galaxy at $r = R_0$ is $\sim 220$ km s$^{-1}$. Doing so yields $M_b \sim 2.6 \times 10^{10} M_\odot$ for their standard model (open triangle in Fig. 10). Evidently, the bulge mass is inconsistent with our results by a factor of 3.

Terrestrial dark matter detection experiments aim to observe elementary particle dark matter candidates (e.g., WIMPs or axions) as they interact with a detector on Earth (see, e.g., Bertone et al. 2004). These experiments are therefore sensitive to the local density and velocity distribution of dark matter particles. Estimates for the local dark matter density range from 0.005 to 0.02 $M_\odot$ pc$^{-3}$ (0.2–0.8 GeV cm$^{-3}$; see Bahcall et al. 1983; Caldwell & Ostriker 1981; Turner 1986; Bergström et al. 1998). The range quoted above is from Bergström et al. (1998), where a variety of halo profiles were considered.

We find (Table 3) $\rho_{\text{loc}} = 0.0080 \pm 0.0014 M_\odot$ pc$^{-3}$ and $\sigma_{\text{loc}} = 240 \pm 23$ km s$^{-1}$. The PDFs for these two quantities are plotted in Figure 11. Our values for the mean and lower bound for $\rho_{\text{loc}}$ are consistent with the values found in Bergström et al. (1998), although our analysis suggests that their upper bound is too high by a factor of 2. Our mean value for $\sigma_{\text{loc}}$ is lower than the standard value by about 30 km s$^{-1}$, although the standard value still falls within the range of acceptable models. Our models tend to favor values for $M_{100}$ at the low end of the range found in equation (21). Inspection of a scatter plot of models in the $M_{100}$-$\sigma_{\text{loc}}$ plane reveals a clear correlation between the two quantities—models with values of $M_{100}$ closer to the central in equation (21) have values of $\sigma_{\text{loc}}$ closer to the standard value.

6.5. Connection with Cosmology

Klypin et al. (2002) construct models for the Milky Way based on standard galaxy formation theory. In particular, they use model
halos based on predictions from cosmological simulations with the further assumption that the halos undergo adiabatic contraction in response to the formation of the disk and bulge.

Although our MCMC analysis does not include cosmological constraints we can test whether our models are consistent with the standard cosmological paradigm a posteriori. To this end, we calculate the virial radius, $R_{\text{vir}}$, virial mass, $M_{\text{vir}}$, and concentration, $c_{\text{vir}}$, for all of the models in our MCMC series. By definition, the mean density inside $R_{\text{vir}}$ is a factor $\Delta_{\text{vir}}$ greater than the background density, $\rho_{\text{m}}$. That is,

$$M_{\text{vir}} \equiv M(R_{\text{vir}}) = \frac{4\pi}{3} \Delta_{\text{vir}} \rho_{\text{m}} R_{\text{vir}}^3,$$

where $M_{\text{vir}}$ is the mass interior to $R_{\text{vir}}$. The value of $\Delta_{\text{vir}}$ depends on the cosmological model; in what follows, we assume $\Delta_{\text{vir}} = 0.3$ where $\rho_{\text{crit}} \equiv 3H^2/8\pi$ is the critical density and $H = 70$ km s$^{-1}$ Mpc (Tegmark et al. 2004). With these values, $\Delta_{\text{vir}} \approx 340$ (Bryan & Norman 1998; see also Bullock et al. 2001; Wechsler et al. 2002). The concentration, $c_{\text{vir}}$, is defined as the ratio of $R_{\text{vir}}$ to the halo scale length $R_s$, the latter identified as the radius at which the logarithmic slope of the halo density profile equals $-2$. For a halo profile given by equation (9), $R_s = (2 - \gamma)\rho_s$.

In Figure 12, we show the PDF for our MCMC series projected onto the $M_{\text{vir}}$-$c_{\text{vir}}$ and $R_{\text{vir}}$-$c_{\text{vir}}$ planes. Also shown are the low concentration, high concentration, and favored models from Klypin et al. (2002). In the top panels, $R_{\text{vir}}$, $M_{\text{vir}}$, and $c_{\text{vir}}$ are calculated for the actual halos used in our models. The implication would seem to be that our models are more concentrated than the ones assumed in Klypin et al. (2002).

Recall, however, that Klypin et al. (2002) incorporate adiabatic contraction into their models. In order to compare our halos with theirs, we should assume they too have undergone adiabatic contraction. We should therefore adiabatically “de-contract” our halos and then calculate $c_{\text{vir}}$. We have done this using the usual assumptions (Blumenthal et al. 1986; Flores et al. 1993); the system is treated as being spherically symmetric. Initially, the baryons and dark matter are well mixed. The baryons cool and form a disk and bulge while the halo particles respond adiabatically to the changing gravitational potential. Moreover, halo particles are assumed to follow circular orbits which do not cross...
as their orbits shrink. Under these assumptions, the quantity \( r M'(r) \) remains constant and one can calculate the initial radii of the dark matter particles given the final structure of the disk, bulge, and halo.

The results of this analysis are shown in the bottom panels of Figure 12. Our models are now in excellent agreement with those found in Klypin et al. (2002).

Alam et al. (2002) proposed \( \Delta V/2 \), the mean density within the radius \( R_{V/2} \) in units of \( \rho_{\text{crit}} \), as a measure of the halo core densities. \( R_{V/2} \) is defined as the radius at which the halo contribution to the rotation curve reaches half its maximum value \( V_{\text{max}} \). The quantity \( (\Delta V/2) / 8\pi^2 \) is equal to the number of rotation periods per Hubble time at the radius \( R_{V/2} \). Alam et al. (2002) found values of \( \Delta V/2 \) in the range \( 5 \times 10^4 - 5 \times 10^6 \) for dark matter–dominated galaxies, considerable scatter and no correlation between \( \Delta V/2 \) and \( V_{\text{max}} \).

The PDF for our MCMC series in the \( \Delta V/2-V_{\text{max}} \) plane is shown in Figure 13. The range of values for \( \Delta V/2 \) is certainly consistent with those found in the Alam et al. (2002) survey, although some models have higher values—possibly reflecting that influence of baryons on the Milky Way’s dark halo. The considerable range indicates that \( \Delta V/2 \) is poorly constrained by the data used in our analysis.

7. BAR FORMATION

Near-IR photometry, gas and stellar kinematic measurements, and observations of gravitational microlensing events all strongly suggest that the Milky Way is a barred galaxy (see reviews by Kuijken 1996; Gerhard 1996). A bar represents a strong departure from axisymmetry and adds considerably to the challenge of modeling the Galaxy. A promising avenue is to use N-body simulations to follow the development of bars and spiral structure in an initially axisymmetric, equilibrium model. Ostriker & Peebles (1973) and Sellwood (1985) provide early examples of this approach. They were interested in stabilizing their Galactic models to avoid bar formation, the former proposing an unseen dark halo and the latter stressing the importance of the bulge. More recently, Fux (1997) attempted to construct self-consistent models for the Milky Way’s bar by evolving unstable axisymmetric models and comparing the results with observations from the DIRBE experiment (Dwek et al. 1995).

Bars, at least in idealized, initially axisymmetric disk galaxies, form through swing amplification (Toomre 1981). (Whether bars in real galaxies form through this process or through some more complicated process during the formation of the galaxy itself is another matter.) In swing amplification, leading waves propagate outward and are amplified into trailing waves (and contained by the outer Lindblad resonance). Trailing waves then wind up. Within linear theory, if there is an inner Lindblad resonance (ILR), it absorbs the tightly wound trailing waves and thereby acts as a barrier preventing further growth. In the absence of an ILR, the trailing waves propagate through the center of the system and transform into leading waves. This feedback loop can lead to growth of a barlike perturbation.

The Toomre \( Q \) (Toomre 1964) and Goldreich-Tremaine \( X \) (Goldreich & Tremaine 1978, 1979) parameters are the two most widely used diagnostic quantities for studying a galaxy’s resistance to the bar instability. \( Q \) measures the kinetic “temperature” of the disk; stellar disks with \( Q \ll 1 \) are unstable to local gravitational instabilities. The value of \( X \) indicates a disk’s susceptibility to global instabilities. (In general, \( X \) depends on the azimuthal mode number, \( m \), of the perturbation. Here, we take \( m = 2 \) since we are interested in bars.) For disks with \( X \approx 3 \), the gain of the swing amplifier is large and bars are more likely to form. In general, the greater the contribution of the disk to the gravitational force felt by disk particles, the smaller the value of \( X \).

Higher values of \( Q \) and \( X \) make a galaxy more resistant to the bar instability. The existence of an ILR barrier from a dense bulge or cuspy halo can also prevent the instability (Sellwood & Evans 2001). However, a bar can still form even if the galaxy initially has an ILR barrier. For example, interactions between halo substructure and the bulge or cusp may temporarily lower the ILR barrier and trigger bar formation in an otherwise stable galaxy (Gauthier et al. 2006). Nonaxisymmetric structure in the disk might also jostle the cusp and remove the ILR barrier, if only temporarily. In short, the notion of an ILR barrier assumes linear perturbation theory; nonlinear disturbances may be able to overcome or disrupt the barrier and initiate the bar instability.

Our models provide a natural starting point for investigations of bar formation. The models generated by our MCMC run all yield acceptable fits to the observational data yet have very different stability properties. Figure 14 shows a contour plot of the model distribution in the \( Q-X \) plane. (Of course, both \( Q \) and \( X \) are functions of radius. Here we use their minimum values.) Also shown is the distribution of models in the \( X'-X \) plane, where

\[
X' \equiv \frac{v_{\text{rot}}^2(R)}{v_d^2(R_{R=2R_d})},
\]

is a measure of the disk’s contribution to the gravitational force necessary to keep a particle in a circular orbit at a given radius. Here \( v_{\text{rot}} \) is the circular rotation speed at cylindrical radius \( R \) and \( v_d \) is the contribution to \( v_{\text{rot}} \) from the disk (Debattista & Sellwood 1998, 2000). The radius \( R = 2.2 R_d \) is where \( v_d \) reaches a maximum, assuming an exponential disk. The tight correlation between \( X \) and \( X' \) indicates these two quantities are interchangeable for most purposes.
Fig. 14.—Contour plots of probability distribution function of models in the $Q$-$X$ and $X'$-$X$ planes. Solid contours enclose 68% and 95% of the models. Filled circles correspond to models used in bar formation study in § 7. [See the electronic edition of the Journal for a color version of this figure.]

Fig. 15.—Rotation curves for the 25 models used in our bar formation simulations. Models are arranged so that $Q$ increases to the right and $X'$ increases from bottom to top. Shown are the total rotation curve (long-dashed line) and contributions to the rotation curve from the disk (solid curve), bulge (dotted curve), and halo (dashed curve).
We consider 25 models from our MCMC run which cover most of the area within the 95% likelihood contour in the $Q$-$X$ plane. The models in this study are depicted as filled circles in Figure 14, while their circular-speed curves are shown in Figure 15. In all cases, the circular speed reaches a peak value of approximately 220 km s$^{-1}$ at a radius between 5 and 8 kpc and then declines slowly.

Figure 16 shows the behavior of the functions $\Omega$ and $\omega \pm \kappa/2$, where $\Omega$ is the angular velocity and $\kappa$ is the epicyclic radial frequency. These functions should be compared with the pattern speed, $\Omega_p$, of an emerging bar or spiral density wave. In particular, $\Omega = \Omega_p$ indicates the position of corotation while $\Omega_p = \Omega - \kappa/2$ indicates the position of the ILR. As we will see shortly, all of the bars that form in our simulations have initial pattern speeds in the range of $40-50$ km s$^{-1}$ kpc$^{-1}$. Thus, about two-thirds of the models in this study have initial ILRs. As we will see shortly, all of the bars that form in our simulations have initial pattern speeds in the range of $40-50$ km s$^{-1}$ kpc$^{-1}$. Thus, about two-thirds of the models in this study have initial ILRs.

Each of the 25 models depicted in Figure 14 are evolved for 5 Gyr using a parallelized tree code (Dubinski 1996). Simulations have 800K disk, 200K bulge, and 1M halo particles. The particle Plummer softening length is $c = 25$ pc and the simulations are run for $10^4$ equal time steps of length $\Delta t = 0.5$ Myr. We generate surface density maps for the face on view of every tenth time step and determine the strength and pattern speed of the bars that form. In Figure 17 we plot the amplitude of the bar-strength parameter

$$A_n \equiv \frac{1}{N} \left| \sum_j e^{2i \phi_j} \right|,$$

as a function of time. Here $\phi_j$ is the usual azimuthal angle of the $j$th disk particle and $N$ is the total number of disk particles. All of the models form bars, with the possible exception of the $Q = 2$, $X = 4.5$ model (upper right-hand corner of the figures). Bars form almost immediately in models with low values of $Q$ and especially higher values of $X$.

$^2$ An animation depicting the evolution of the grid of simulations is available at http://www.cita.utoronto.ca/~dubinski/DynamicalBlueprints/.
Interestingly enough, the growth rate of bars does not vary smoothly across the grid of models. In particular, models in the central column of the grid ($Q \approx 1.5$) with higher values of $X$ quickly develop bars. In general, these models have less massive and less dense bulges and no ILRs (see Figs. 15 and 16). Hence, they are immediately vulnerable to Toomre’s swing amplification mechanism.

Virtually all of the models with ILRs eventually form bars although the onset of the instability is typically delayed. Two effects, one numerical and one physical, can lead to bar formation in the presence of an initial ILR. First, two-body relaxation may be important at radii near the ILR even with $10^6$ halo particles. (Recall that in some models, the radius of the ILR can be as small as 100 pc.) As the central halo relaxes, the peak of the $\frac{b}{a}$ curve can drop below the initial pattern speed. If this occurs, the ILR barrier disappears and the bar forms.

A second, more physical, explanation is that swing amplified spiral waves disturb the central region of the galaxy causing the ILR barrier to disappear, at least temporarily. Other nonlinear disturbances, such as interactions between the disk and halo substructure, may also cause the ILR barrier to disappear. Simply put, the theory of the ILR barrier may not apply to situations where nonlinear perturbations are present. We note that the importance of nonlinear effects for bar formation is discussed in Sellwood (1989).

The evolution of the bar length, $R_{\text{bar}}$, and pattern speed, $\Omega_p$, are shown in Figures 18 and 19. To estimate the bar length, we first construct the axis ratio profile of the isodensity contours for the disk. The axis ratio profile is defined to be the axis ratio, $b/a$, as a function of the semimajor axis, $a$. In general, $b/a$ goes through a minimum with a typical value of $(b/a)_{\text{min}} \approx 0.4$ before rising abruptly in the transition from the end of the bar to the outer disk. As a heuristic measure of $R_{\text{bar}}$ we use the major axis length of the isodensity contour beyond the $b/a$ minimum and where $b/a \approx 0.6$. The pattern speed is given by the time derivative of the phase angle $\phi_0 = \arctan \left( \frac{\text{Im}(A_2)}{\text{Re}(A_2)} \right)/2$. We ensure that $\phi_0$ is sampled with enough time resolution to avoid aliasing and to obtain a reasonable estimate of the pattern speed from $\Delta \phi/\Delta t$. In Figure 19, we show the pattern speed once the bar is easily detected above the noise.

Though the bars in our survey appear at different times, their initial pattern speeds are always in the range of $40\text{–}50$ km s$^{-1}$ kpc$^{-1}$. Angular momentum transfer to the halo causes $\Omega_p$ to decay to about $20$ km s$^{-1}$ kpc$^{-1}$ over a few Gyr with the decay rate being slightly larger in models with higher mass halos. As expected, the bar lengths are generally less than the corotation radius.

![Fig. 17.—Growth of the bar strength parameter, $A_2$, as a function of time.](image-url)
Fig. 18.—Bar length, $R_{bar}$, as a function of time.
Fig. 19.—Bar pattern speed $\Omega_p$ as a function of time. Bars are born with pattern speeds $\Omega_p \sim 50 \text{ km s}^{-1} \text{ kpc}^{-1}$, which immediately begin to decay as they transfer angular momentum to halos.
although transient long bars with $R_{\text{bar}} \approx 10$ kpc form in models with small $Q$ and $X$ before collapsing on themselves. For most models, bars grow monotonically in length as the pattern speed declines.

Bar formation alters the structure of the model, and it is therefore natural to ask whether the evolved models still satisfy the original observational constraints. Although we will leave the details of such an analysis for future work, we include, in Figure 20, the evolution of the surface density profiles for our 25 simulated systems. We see that the surface density profiles of models with low values of $Q$ and $X$ are dramatically deformed; the redistribution of mass is so violent that the models almost certainly do not satisfy our observational constraints. On the other hand, models where the bar forms relatively late in the simulation show little evolution of the surface density profile. Their structure, at least in an azimuthally averaged sense, remains largely unchanged.

The Galaxy is known to have a bar and estimates of its length and pattern speed can be compared with our results. Binney et al. (1997) find $R_{\text{bar}} = 3.5$ kpc and $\Omega_p \approx 60$–$70$ km s$^{-1}$ kpc$^{-1}$ in their analysis of DIRBE photometry while Dehnen (1999) find a similar bar length but lower pattern speed ($\Omega_p = 53 \pm 3$ km s$^{-1}$ kpc$^{-1}$) using the velocity distribution of solar neighborhood stars. Weiner & Sellwood (1999) model the gas kinematics of the inner galaxy and find $\Omega_p \approx 42$ km s$^{-1}$ kpc$^{-1}$. The fact that nearly all of the models in our study are bar unstable and have initial pattern speeds near the range of the inferred values is promising. We note that certain models can be excluded such as those with very small values of $Q$ and $X$. If we assume that the Galaxy’s bar has formed very recently, then many of the models have the correct combination of bar length and pattern speed to match the observations. Furthermore, for larger $Q$ and $X$, the change in the disk’s radial profile in response to the bar is small. These models may well provide a good barred model of the Galaxy. It should be noted that in all of these models, the pattern speed declines to 20–30 km s$^{-1}$ kpc$^{-1}$ within a few Gyr after the bar forms. If we take these models seriously as reasonable facsimiles of the Galaxy, then we must conclude that the Galactic bar formed within the last 1–2 Gyr.

8. CONCLUSIONS

We have introduced a dynamical model for late-type galaxies that incorporates our current understanding of disk-bulge-halo systems. In particular, the bulge has a Sérsic surface density profile and the halo has a central density cusp.
We have carried out an MCMC analysis of dynamical models for the Milky Way using a variety of kinematic and photometric constraints. The results are presented in the form of PDFs for both input parameters and derived quantities. The MCMC analysis provides a picture of the distribution of models in parameter space that is more complete than can be obtained by other approaches. Avoided is the awkward procedure of fixing a subset of parameters while allowing the remaining parameters to vary in some minimization scheme. Instead, a sequence of models is generated which contains all of the desired information. Marginalization over a subset of parameters is accomplished by simply projecting the model distribution onto the appropriate parameter subspace.

Our analysis suggests that the Milky Way has a pseudobulge with a Sérsic index of $1.3 \pm 0.3$. Our results for the masses of the disk, bulge, and halo are consistent with those of Dehnen & Binney (1998) but call into question choices for these quantities in some popular models from the literature. For example, the disk and bulge masses in Johnston et al. (1999) are entirely inconsistent with our results. The inferred bulge mass for the standard model used by the MACHO collaboration (Alcock et al. 2000) is inconsistent with our findings by a factor of 2.5. On the other hand, the standard values used by terrestrial dark matter detection experiments for the local dark matter density and velocity dispersion are consistent with our results.

A weak point of our analysis is the inability to tightly constrain the halo mass at large radii. Planetary nebulae, globular clusters, and satellite galaxies may be used as tracers of the Galactic potential. Dynamical models for the tracer populations are required to properly model kinematic data. In principle, it is straightforward to construct such models, but there are subtle issues. Previous studies showed that velocity anisotropy in a tracer population can affect interpretation of kinematic data. Velocity anisotropy requires a DF that depends on at least one integral of motion in addition to the energy. The standard practice is to use the total angular momentum, $L$, but since our models include a disk, $L$ is not conserved. (Previous analyses side-stepped this issue by using a spherically symmetric Galactic potential.)

Our MCMC analysis provides an ideal starting point for studies of disk stability and bar formation in that we have some 10$^3$ models, each of which can serve as initial conditions for a numerical experiment. We have performed a suite of simulations which focuses on the susceptibility of the disk to bar formation as a function of the stability parameters $Q$ and $X$. Many of the models provide a good match to the inferred properties of the Galactic bar with the proviso that the bar has formed recently.

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REFERENCES

Alam, S. M. K., Bullock, J. S., & Weinberg, D. H. 2002, ApJ, 572, 34
Alcock, C., et al. 2000, ApJ, 542, 281
Andredakis, Y. C., Peletier, R. F., & Balcacells, M. 1995, MNRAS, 275, 874
Bahcall, J. N., Schmidt, M., & Soneira, R. M. 1983, ApJ, 265, 730
Bahcall, J. N., & Soneira, R. M. 1980, ApJS, 44, 73
Bell, E. F., & de Jong, R. S. 2001, ApJ, 550, 212
Bergström, L., Ullio, P., & Buckley, J. H. 1998, Astropart. Phys., 9, 137
Bertone, G., Hooper, D., & Silk, J. 2004, Phys. Rep., 405, 279
Binney, J. 1980, MNRAS, 200, 873
———. 1987, in The Galaxy, ed. G. Gilmore & B. Carswell (Dordrecht: Reidel), 399
———. 1995, in IAU Symp. 169, Unsolved Problems of the Milky Way, ed. L. Blitz & P. Teuben (Dordrecht: Kluwer), 76
Goldreich, P., & Tremaine, S. 1978, ApJ, 222, 850
Gregory, P. 2005, Bayesian Logical Data Analysis for the Physical Sciences (Cambridge: Cambridge Univ. Press)
Hastings, W. K. 1970, Biometrika, 57, 97
Hernquist, L. 1990, ApJ, 356, 359
———. 1993, ApJS, 86, 389
Ibata, R. A., Gilmore, G., & Irwin, M. J. 1994, Nature, 370, 194
———. 1995, MNRAS, 277, 781
Innanen, K. A. 1973, ApSS, 22, 393
Johnston, K. V., et al. 1999, AJ, 118, 1719
Kazantzidis, S., Magorrian, J., & Moore, B. 2004, ApJ, 601, 37
Kent, S. M. 1992, ApJ, 387, 181
———. 2000, ApJ, 543, 704
Dehnen, W. 1999, ApJ, 524, L35
Dehnen, W., & Binney, J. 1998, MNRAS, 294, 429
de Vaucouleurs, G. 1948, Ann. d’Astrophys., 11, 247
Dubinski, J. 1996, NewA, 1, 133
Dwek, E., et al. 1995, ApJ, 445, 716
Edvardsson, B., et al. 1993, A&A, 275, 101
Eisenhauer, F., et al. 2003, ApJ, 597, L121
Feast, M. W., & Whitelock, P. A. 1997, MNRAS, 291, 683
Flores, R., et al. 1993, ApJ, 412, 443
Flores, R. A., & Primack, J. R. 1994, ApJ, 427, L1
Freudenreich, H. T. 1998, ApJ, 492, 495
Fux, R. 1997, A&A, 327, 983
Gauthier, J.-R., Dubinski, J., & Widrow, L. M. 2006, ApJ, 653, 1180
Gelman, A., Carlin, J. B., Stern, H. S., & Rubin, D. B. 1995, Bayesian Data Analysis (London: Chapman & Hall)
Gerhard, O. E. 1996, in IAU Symp. 169, Unsolved Problems of the Milky Way, ed. L. Blitz & P. Teuben (Dordrecht: Kluwer), 76
Goldreich, P., & Tremaine, S. 1978, ApJ, 222, 850
Gregory, P. 2005, Bayesian Logical Data Analysis for the Physical Sciences (Cambridge: Cambridge Univ. Press)
Hastings, W. K. 1970, Biometrika, 57, 97
Hernquist, L. 1990, ApJ, 356, 359
———. 1993, ApJS, 86, 389
Ibata, R. A., Gilmore, G., & Irwin, M. J. 1994, Nature, 370, 194
———. 1995, MNRAS, 277, 781
Innanen, K. A. 1973, ApSS, 22, 393
Johnston, K. V., et al. 1999, AJ, 118, 1719
Kazantzidis, S., Magorrian, J., & Moore, B. 2004, ApJ, 601, 37
Kost, S. M. 1992, ApJ, 387, 181
King, I. R. 1966, AJ, 71, 64
Klypin, A., Zhao, H., & Somerville, R. S. 2002, ApJ, 573, 597
Kochanek, C. S. 1996, ApJ, 457, 228
Kormendy, J., & Kennicutt, R. C., Jr. 2004, ARA&A, 42, 603
Kuijken, K. 1996, in ASP Conf. Ser. 91, Barred Galaxies, ed. R. R. Rutan (San Francisco: ASP), 504
Kuijken, K., & Dubinski, J. 1994, MNRAS, 269, 13
———. 1995, MNRAS, 277, 1341
Kuijken, K., & Gilmore, G. 1991, ApJ, 367, L9
Law, D. R., Johnston, K. V., & Majewski, S. R. 2005, ApJ, 619, 807
Lin, D. N. C., Jones, B. F., & Klemola, A. R. 1995, ApJ, 439, 652
Lokas, E. E., & Mamon, G. A. 2001, MNRAS, 321, 155
Lopez-Corredoira, M., et al. 2002, A&A, 394, 883
Malhotra, S. 1995, ApJ, 448, 138
———. 1996, ApJ, 473, 687
———. 2000, ApJ, 543, 704
MacArthur, L. A., Courteau, S., & Holtzman, J. A. 2003, ApJ, 582, 689
McCaughey, S. S., & de Blok, W. J. G. 1998, ApJ, 499, 41
Merritt, D., et al. 2006, AJ, 132, 2685
Metropolis, N., et al. 1953, J. Chem. Phys., 21, 1087
Moore, B. 1994, Nature, 370, 629
Moore, B. et al. 1999, MNRAS, 310, 1147
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563
Navarro, J. F., et al. 2004, MNRAS, 349, 1039
Ostriker, J. P., & Peebles, P. J. E. 1973, ApJ, 186, 467
Prugniel, P., & Simien, F. 1997, A&A, 321, 111
Reid, M. J. 1993, ARA&A, 31, 345
Reid, M. J., et al. 1999, ApJ, 524, 816
Rohlfs, K., & Kreitschmann, J. 1988, A&A, 201, 51
Sackett, P. D. 1997, ApJ, 483, 103
Sellwood, J. A. 1985, MNRAS, 217, 127
———. 1989, MNRAS, 238, 115
Sellwood, J. A., & Evans, N. W. 2001, ApJ, 546, 176
Spergel, D. N., Malhotra, S., & Blitz, L. 1996, in Spiral Galaxies in the Near-
Infrared, ed. D. Minnitti & H.-W. Rix (Berlin: Springer), 128
Springel, V., & White, S. D. M. 1999, MNRAS, 307, 162
Tegmark, M., et al. 2004, Phys. Rev. D, 69, 103501
Terzić, B., & Graham, A. W. 2005, MNRAS, 362, 197

Toomre, A. 1964, ApJ, 139, 1217
———. 1981, in The Structure and Evolution of Normal Galaxies, ed. S. M.
Fall (Cambridge: Cambridge Univ. Press), 111
Tremaine, S., et al. 2002, ApJ, 574, 740
Turner, M. S. 1986, Phys. Rev. D, 33, 889
Valenzuela, O., & Klypin, A. 2003, MNRAS, 345, 406
van den Bosh, F. C., et al. 2000, AJ, 119, 1579
Wechsler, R. H., et al. 2002, ApJ, 568, 52
Weiner, B. J., & Sellwood, J. A. 1999, ApJ, 524, 112
Widrow, L. M. 2000, ApJS, 131, 39
Widrow, L. M., & Dubinski, J. 2005, ApJ, 631, 838
Widrow, L. M., Perrett, K. M., & Suyu, S. H. 2003, ApJ, 588, 311
Wilkinson, M. I., & Evans, N. W. 1999, MNRAS, 310, 645
Zhao, H. 1997, MNRAS, 287, 525
Zheng, Z., et al. 2001, ApJ, 555, 393