Interval valued Hadamard-Fejér and Pachpatte Type inequalities pertaining to a new fractional integral operator with exponential kernel

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Abstract: The aim of this research is to combine the concept of inequalities with fractional integral operators, which are the focus of attention due to their properties and frequency of usage. By using a novel fractional integral operator that has an exponential function in its kernel, we establish a new Hermite-Hadamard type integral inequality for an LR-convex interval-valued function. We also prove new fractional-order variants of the Fejér type inequalities and the Pachpatte type inequalities in the setting of pseudo-order relations. By showing several numerical examples, we further validate the accuracy of the results that we have derived in this study. We believe that the results, presented in this article are novel and that they will be beneficial in encouraging future research in this field.

Keywords: LR-convex interval-valued function; fractional integral operator; Hermite–Hadamard type inequality; Pachpatte type inequality; Fejér type inequality; exponential kernel

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1. Introduction

Fractional calculus has become increasingly important in domains such as inequality theory, applied mathematics, sciences, and engineering in recent years. Fractional integrals are used to describe the attributes of many physical processes, such as those in physics and medicine [1], epidemiology [2], fluid mechanics [3], nanotechnology [4], economy [5], bioengineering [6], and so on.

It is worth emphasizing that the concept of fractional calculus was first put forward by Leibniz and L’Hôpital (1695). Nevertheless, additional contributions to the topic of fractional calculus and its numerous applications were made by such other mathematicians as Riemann, Liouville, Grunwald, Letnikov, Erdélyi, Kober, and others. Fractional calculus has drawn the attention of many physical and engineering professionals due to its behaviour and capacity to tackle a wide range of real-world problems (see, for details, [7]; see also the recent survey-cum-expository review articles [8, 9]).

For fractional integrals, various integral inequalities have been found. These integrals are useful for generalizing important and well-known integral inequalities. One kind of integral inequalities is the Hermite-Hadamard integral inequality. It is widely used in the literature, and it specifies both the necessary and sufficient conditions for a function to be convex. Sarikaya et al. [10] generalized the Hermite-Hadamard inequalities using Riemann-Liouville fractional integrals. The conclusions by Sarikaya et al. [10] were also extended to Hermite-Hadamard-Fejér type inequalities by Iscan [11]. Chen [12] used the methodology of Sarikaya et al. [10] and established fractional Hermite-Hadamard type integral inequalities by using the product of two convex functions. Guessab [13] proposed Jensen type inequalities on convex polytopes and investigated the error in convex function approximation. Guessab et al. [14] discovered a Korovkin-type theorem that shows that a series of operators does not have to have an identity limit. Guessab also worked on the concepts such as higher order convexity [15] and bivariate Hermite interpolant (see [16]). The presentation of several well-known integral inequalities by employing various novel concepts of fractional integral operators has grown increasingly popular among mathematicians in recent years. In this regard, one can consult the results described in [10, 17–22].

Interval analysis is subdivided into the set-valued analysis. There is no doubting the importance of interval analysis in both pure and applied research. One of the early uses of interval analysis was the error limits of numerical solutions of finite state machines. However, in recent times, interval analysis has been a crucial component of mathematical and computer models for addressing interval uncertainty.

To generate more generalised results, we used a new fractional integral operator in this study. This is due to the fact that this fractional operator has an exponential kernel. The aforementioned fractional analogues of functional inequalities do not follow from our conclusions, which is a distinction between our results and existing generalisations. Many experts have presented extensions of the Hermite-Hadamard inequality with various fractional integral operators, but there is no exponential characteristic in their results. This research sparked an interest in creating more generalised fractional inequalities with an exponential function as the kernel. Moreover, the application of interval valued analysis to the main findings brings a new direction to the field of inequalities. We have incorporated the concepts of a Pseudo-order relation with Interval valued analysis to present new inequalities. Although we can find many studies on the growth of fractional order integral inequalities involving convex functions, there are still numerous gaps to be filled for fractional integral inequalities involving...
various types of convex functions. As a result, the primary goal of this study is to establish novel Hermite-Hadamard, Pachppate and Fejér type inequalities for LR-convex interval-valued function utilizing fractional integral operators.

2. Preliminaries

We denote by $\mathcal{M}_C$ the collection of all closed and bounded intervals of $\mathbb{R}$, given by

$$\mathcal{M}_C = \{ [\Lambda, \Lambda^*] : \Lambda, \Lambda^* \in \mathbb{R} \text{ and } \Lambda \leq \Lambda^* \}.$$ 

If $\Lambda \geq 0$, then we say the interval $[\Lambda, \Lambda^*]$ is positive interval. The set of all positive intervals is denoted by $\mathcal{M}_C^+$ and is given as

$$\mathcal{M}_C^+ = \{ [\Lambda, \Lambda^*] : \Lambda, \Lambda^* \in \mathcal{M}_C \text{ and } \Lambda \geq 0 \}.$$ 

Let us now study how the intervals behave under some arithmetic operations such as addition, multiplication and scalar multiplication. If $[\Lambda, \Lambda^*], [\Lambda, \Lambda^*] \in \mathcal{M}_C$ and $\kappa \in \mathbb{R}$, then

$$[\Lambda, \Lambda^*] + [\Lambda, \Lambda^*] = [\Lambda + \Lambda, \Lambda^* + \Lambda^*]$$

$$[\Lambda, \Lambda^*] \times [\Lambda, \Lambda^*] = \{ \min \{\Lambda, \Lambda^*, \Lambda^* \Lambda, \Lambda, \Lambda^* \Lambda^* \}, \max \{\Lambda, \Lambda^*, \Lambda^*, \Lambda^* \Lambda \} \}$$

$$\kappa \cdot [\Lambda, \Lambda^*] = \begin{cases} 
[\kappa \Lambda, \kappa \Lambda^*] & (\kappa > 0) \\
[0] & (\kappa = 0) \\
[\kappa \Lambda^*, \kappa \Lambda_*] & (\kappa < 0), 
\end{cases}$$

respectively.

For $[\Lambda, \Lambda^*], [\Lambda, \Lambda^*] \in \mathcal{M}_C$, the inclusion "$\subseteq" is given as follows

$[\Lambda, \Lambda^*] \subseteq [\Lambda, \Lambda^*]$, if and only if, $\Lambda \leq \Lambda$, $\Lambda^* \leq \Lambda^*$.

Khan et al. [23] presented the following properties of the newly introduced idea (LR-convex interval valued functions).

**Remark 2.1** (see [23]). 1. The pseudo order relation "$\leq_p$" defined on $\mathcal{M}_C$ by $[\Lambda, \Lambda^*] \leq_p [\Lambda, \Lambda^*]$ holds true if and only if $\Lambda \leq \Lambda$, $\Lambda^* \leq \Lambda^*$, for all $[\Lambda, \Lambda^*], [\Lambda, \Lambda^*] \in \mathcal{M}_C$. The relation $[\Lambda, \Lambda^*] \leq_p [\Lambda, \Lambda^*]$ similar to $[\Lambda, \Lambda^*] \leq_p [\Lambda, \Lambda^*]$ on $\mathcal{M}_C$.

2. It is easily seen that "$\leq_p$" looks same as that of “left and right” on the real line $\mathbb{R}$, and hence can also be called as “LR” order.

Moore [24] is credited for introducing the concept of the Riemann integral for interval valued functions, which is defined as follows:
Theorem 2.1. Let \( S : [g, p] \subset \mathbb{R} \to \mathcal{M}_C \) be an I-V-F on such that
\[
S(\upsilon) = [S_* (\upsilon), S^* (\upsilon)].
\]

Then \( S \) is Riemann integrable over \([g, p] \) iff both \( S_* \) and \( S^* \) are Riemann integrable over \([g, p] \).

\[
\int_g^p S(\upsilon) \, d\upsilon = \int_g^p S_* (\upsilon) \, d\upsilon, \quad \int_g^p S^* (\upsilon) \, d\upsilon.
\]

Let us denote the Riemann integrable functions and Riemann integrable interval valued functions by \( R_{[g, p]} \) and \( IR_{[g, p]} \), respectively.

Definition 2.1 (see [10]; see also [7, 25]). Let \( S \in \mathcal{L}([g, p], \mathcal{M}_C^+) \). Then the left- and right- interval Riemann-Liouville fractional integral of order \( \alpha > 0 \) is defined by
\[
I^\alpha_g S(\upsilon) = \frac{1}{\Gamma(\alpha)} \int_g^\upsilon (\upsilon - r)^{\alpha-1} S(r) \, dr, \quad (\upsilon > g)
\]
and
\[
I^\alpha_p S(\upsilon) = \frac{1}{\Gamma(\alpha)} \int_\upsilon^p (r - \upsilon)^{\alpha-1} S(r) \, dr, \quad (\upsilon < p),
\]
respectively, where
\[
\Gamma(\upsilon) = \int_0^\infty r^{\upsilon-1} e^{-r} \, dr,
\]
is the Euler gamma function.

We now recall a potentially useful fractional integral operator, introduced by Ahmad et al. [26], that has an exponential function in its kernel.

Definition 2.2 (see, for details, [26]). Let \( S \in \mathcal{L}([g, p], \mathcal{M}_C^+) \). Then the fractional integrals \( I^\alpha_g \) and \( I^\alpha_p \) of order \( \alpha > 0 \) are defined as
\[
I^\alpha_g S(\upsilon) := \frac{1}{\alpha} \int_g^\upsilon e^{-\frac{1}{\alpha}(\upsilon - r)} S(r) \, dr \quad (0 \leq g < \upsilon < p),
\]
and
\[
I^\alpha_p S(\upsilon) := \frac{1}{\alpha} \int_\upsilon^p e^{-\frac{1}{\alpha}(r - \upsilon)} S(r) \, dr \quad (0 \leq g < \upsilon < p),
\]
respectively.

Definition 2.3 (see [27]). The interval valued function \( S : J \to \mathcal{M}_C^+ \) is said to be Left-Right-convex interval valued function on convex set \( J \) iff
\[
S(rg + (1 - r)p) \leq_p rS(g) + (1 - r)S(p), \quad (2.1)
\]
holds true for all \( g, p \in J \) and \( r \in [0, 1] \).

\( S \) is said to be LR-concave on \( J \), if inequality (2.1) is reversed, and if \( S \) is both LR-convex and LR-concave function, then it is said to be affine.
Theorem 2.2 (see [27]). Let \( J \) be a convex set and \( S : J \to M_1^c \) be an interval valued function such that

\[
S(v) = [S_+(v), S_-(v)], \quad \forall v \in J,
\]

for all \( v \in J \). Then \( S \) is said to be left-right-convex interval valued function on \( J \), if and only if both \( S_+(v) \) and \( S_-(v) \) are convex functions.

Budak et al. [25] examined the Hermite-Hadamard and Pachpatte type inequalities for interval-valued convex functions via fractional integrals. The following are some connected outcomes:

**Theorem 2.3.** Let \( S : [g, p] \to \mathbb{R}_1^+ \) be an interval-valued convex function with

\[
S(v) = [S_+(v), S_-(v)],
\]

Then, the fractional-order H-H integral inequality of order \( \alpha > 0 \) for interval-valued functions is given by

\[
S\left(\frac{g+p}{2}\right) \geq \frac{\Gamma(\alpha+1)}{2(p-g)\alpha} \left[ \int_{(g)}^p S(v)dv + \int_{(p)}^g S(g)dv \right] \geq \frac{S(g) + S(p)}{2}.
\]

**Theorem 2.4.** If \( S, Y : [g, p] \to \mathbb{R}_1^+ \) are two interval-valued convex functions with

\[
S(v) = [S_+(v), S_-(v)],
\]

and

\[
Y(v) = [Y_+(v), Y_-(v)],
\]

then the fractional-order H-H type inequality for \( \alpha > 0 \) holds true as follows:

\[
\frac{\Gamma(\alpha+1)}{2(p-g)\alpha} \left[ \int_{(g)}^p S(v)Y(v)dv + \int_{(p)}^g S(g)Y(g)dv \right] \\
\geq \left[ 1 - \frac{\alpha}{\alpha+2} \right] \Psi(g, v) + \left[ \frac{\alpha}{\alpha+2} \right] \Omega(g, v),
\]

where

\[
\Psi(g, v) = [S(g)Y(g) + S(v)Y(v)],
\]

and

\[
\Omega(g, v) = [S(g)Y(v) + S(p)Y(g)].
\]

**Theorem 2.5.** Let \( S, Y : [g, p] \to \mathbb{R}_1^+ \) be two interval-valued convex functions with

\[
S(v) = [S_+(v), S_-(v)],
\]

and

\[
Y(v) = [Y_+(v), Y_-(v)].
\]

Then the fractional-order H-H type inequality for \( \alpha > 0 \) is given as

\[
2S\left(\frac{g+p}{2}\right)Y\left(\frac{g+p}{2}\right) \geq \frac{\Gamma(\alpha+1)}{2(p-g)\alpha} \left[ \int_{(g)}^p S(v)Y(v)dv + \int_{(p)}^g S(g)Y(g)dv \right] \\
+ \left[ 1 - \frac{\alpha}{\alpha+2} \right] \Omega(g, v) + \left[ \frac{\alpha}{\alpha+2} \right] \Psi(g, v).
\]
Many scientists have linked integral inequalities with interval-valued functions (IVFs) in recent decades, giving numerous important findings. Costa [28] postulated Opial-type inequalities as a basis for IVF inequalities. The generalized Hukuhara derivative was used by Chalco-Cano [29] to look into Ostrowski type inequalities for interval-valued functions. In [30], Roman-Flores deduced the Minkowski type inequalities and Beckenbach’s type inequalities for interval-valued functions. Zhao et al. [31] recently improved the concept by introducing H – H type inequalities for interval-valued coordinated functions. The idea of interval-valued analysis was applied to reinforce this inequality via some innovative concepts such as coordinated convexity [33, 34], and preinvex functions [35–37]. Recently, Lai et al. [38] extended the concept of interval valued preinvex functions to interval valued coordinated preinvex function. Kalsoom et al. [39] proved some Hermite-Hadamard-Fejér type fractional inequalities for h-convex and Harmonically h-convex interval valued functions. Shi et al. [40] investigated some related inequalities employing coordinated log-h-convex interval valued functions.

Khan and his colleagues recently extended this (apparently new) concept to include LR-convex interval valued functions and fuzzy convex interval-valued functions, both of which take a pseudo-order relation into account. To illustrate inequalities of the Hermite-Hadamard, Hermite-Hadamard-Fejér, and Pachpatte types, his team utilised LR-h-convex interval-valued functions (see [41]), LR-χ-preinvex functions (see [42]), and LR-(h1, h2)-convex interval-valued functions (see [43]). For various recent achievements related to the notion of fuzzy interval-valued analysis of some well-known integral inequalities, we refer to Khan et al. [44–46].

Motivated by the above articles, we generalize the Hermite-Hadamard inequality for LR-convex interval valued functions and product of two LR-convex interval valued functions. Moreover, we also prove Hermite-Hadamard-Fejér type inequalities. We believe that our findings will motivate more researchers to work in this subject, particularly in the area of interval valued concepts.

The following is a breakdown of our current investigation. After reviewing the prerequisite and relevant facts regarding the related inequalities and the interval-valued analysis in Section 2, we derive some new versions of the interval-valued H-H type inequalities and Pachpatte type inequalities in Section 3. Inequalities of the Hermite-Hadamard-Fejér type in the frame of LR interval-valued functions are also established in Section 4. Some examples are also reviewed to determine whether the established consequences are useful. In Section 5, a brief conclusion is offered, as well as potential areas for future research that are related to the findings presented in this paper.

3. **Left and Right interval valued fractional integral inequalities**

This section is devoted towards the main results of our manuscript, where different new versions of the Hermite-Hadamard type inequalities incorporated with LR-convex interval valued functions are discussed.

For brevity we will denote \( \mu = \frac{1-a}{a}(\varphi - g) \).

**Theorem 3.1.** Let \( S : [g, p] \to \mathcal{M}^+ \) be an LR-convex I-V-F on \([g, p]\), which is given by

\[
S(\nu) = [S_+(\nu), S^-(\nu)],
\]

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for all \( v \in \mathfrak{g}, p \). If \( S \in \mathcal{L}(\mathfrak{g}, p, M_C^+) \), then
\[
S\left(\frac{\mathfrak{g} + p}{2}\right) \leq_p \frac{1 - \alpha}{2(1 - e^{-\mu})} \left[ I_{\mathfrak{g}}^\alpha S(p) + I_p^\alpha S(\mathfrak{g}) \right] \leq_p \frac{S(\mathfrak{g}) + S(p)}{2}.
\]

**Proof.** Since \( S \) is an LR-convex I-V-F, we have
\[
2S\left(\frac{\mathfrak{g} + p}{2}\right) \leq_p S(\mathfrak{g} + (1 - r)p) + S((1 - r)\mathfrak{g} + rp).
\]
Therefore, we have
\[
2S_\ast\left(\frac{\mathfrak{g} + p}{2}\right) \leq S_\ast(\mathfrak{g} + (1 - r)p) + S_\ast((1 - r)\mathfrak{g} + rp), \tag{3.1}
\]
and
\[
2S^\ast\left(\frac{\mathfrak{g} + p}{2}\right) \leq S^\ast(\mathfrak{g} + (1 - r)p) + S^\ast((1 - r)\mathfrak{g} + rp). \tag{3.2}
\]
Multiplying both sides of the equations (3.1) and (3.2) by \( e^{-\frac{1 - \alpha}{\mu}(p - q)r} \) and then integrating with respect to \( r \) over \([0, 1]\), we obtain
\[
2 \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S_\ast\left(\frac{\mathfrak{g} + p}{2}\right) dr \\
\leq \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S_\ast(\mathfrak{g} + (1 - r)p) dr + \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S_\ast((1 - r)\mathfrak{g} + rp) dr,
\]
and
\[
2 \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S^\ast\left(\frac{\mathfrak{g} + p}{2}\right) dr \\
\leq \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S^\ast(\mathfrak{g} + (1 - r)p) dr + \int_0^1 e^{-\frac{1 - \alpha}{\mu}(p - q)r} S^\ast((1 - r)\mathfrak{g} + rp) dr,
\]
respectively.

Now, if we let \( \tau = \mathfrak{g} + (1 - r)p \) and \( \nu = (1 - r)\mathfrak{g} + rp \), then we have
\[
\frac{1 - e^{-\mu}}{\mu} S_\ast\left(\frac{\mathfrak{g} + p}{2}\right) \leq \frac{1}{2(p - q)} \int_\mathfrak{g} e^{-\frac{1 - \alpha}{\mu}(p - \tau)} S_\ast(\tau)d\tau + \frac{1}{2(p - q)} \int_\mathfrak{g} e^{-\frac{1 - \alpha}{\mu}(p - \nu)} S_\ast(\nu)d\nu \\
= \frac{\alpha}{2(p - q)} \left[ I_{\mathfrak{g}}^\alpha S_\ast(p) + I_p^\alpha S_\ast(\mathfrak{g}) \right] ,
\]
and
\[
\frac{1 - e^{-\mu}}{\mu} S^\ast\left(\frac{\mathfrak{g} + p}{2}\right) \leq \frac{1}{2(p - q)} \int_\mathfrak{g} e^{-\frac{1 - \alpha}{\mu}(p - \tau)} S^\ast(\tau)d\tau + \frac{1}{2(p - q)} \int_\mathfrak{g} e^{-\frac{1 - \alpha}{\mu}(p - \nu)} S^\ast(\nu)d\nu \\
= \frac{\alpha}{2(p - q)} \left[ I_{\mathfrak{g}}^\alpha S^\ast(p) + I_p^\alpha S^\ast(\mathfrak{g}) \right].
\]

**AIMS Mathematics**

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That is
\[
\frac{1 - e^{-\mu}}{\mu} \left[ S_s \left( \frac{g + p}{2} \right), S^c \left( \frac{g + p}{2} \right) \right] \leq p \frac{\alpha}{2(p - g)} \left[ I^\alpha_{g^s} S_s(p) + I^\alpha_{p^s} S_s(g) \right],
\]
which readily yields
\[
\frac{1 - e^{-\mu}}{\mu} S \left( \frac{g + p}{2} \right) \leq p \frac{\alpha}{2(p - g)} \left[ I^\alpha_{g^s} S(p) + I^\alpha_{p^s} S(g) \right]. \tag{3.3}
\]
Similarly, we have
\[
\frac{\alpha}{2(p - g)} \left[ I^\alpha_{g^s} S(p) + I^\alpha_{p^s} S(g) \right] \leq p \frac{S(g) + S(p)}{2}. \tag{3.4}
\]
From the Eqs (3.3) and (3.4), we have
\[
\frac{1 - e^{-\mu}}{\mu} S \left( \frac{g + p}{2} \right) \leq p \frac{\alpha}{2(1 - e^{-\mu})} \left[ I^\alpha_{g^s} S(p) + I^\alpha_{p^s} S(g) \right] \leq p \frac{S(g) + S(p)}{2}.
\]
This concludes the proof of Theorem 3.1. \(\square\)

**Corollary 3.1.** Let \( S : [g, p] \to M_C^\alpha \) be an LR-concave I-V-F on \([g, p]\), which is given by
\[
S(v) = [S_s(v), S^c(v)],
\]
for all \( v \in [g, p] \). If \( S \in L \left([g, p], M_C^\alpha\right) \), then
\[
S \left( \frac{g + p}{2} \right) \geq p \frac{1 - \alpha}{2(1 - e^{-\mu})} \left[ I^\alpha_{g^s} S(p) + I^\alpha_{p^s} S(g) \right] \geq p \frac{S(g) + S(p)}{2}.
\]

**Remark 3.1.** If we choose \( \alpha \to 1 \), then Theorem 3.1 reduces to the following inequality for LR-convex interval valued function given in [47]:
\[
S \left( \frac{g + p}{2} \right) \leq p \frac{1 - \alpha}{p - g} \int_g^p S(v) \, dv \leq p \frac{S(g) + S(p)}{2}.
\]

If we choose \( S_s(v) = S^c(v) \) in Theorem 3.1, we recapture the the following fractional integral Hermite-Hadamard type inequality, presented by Ahmad et al. (see [26]).
\[
S \left( \frac{g + p}{2} \right) \leq \frac{1 - \alpha}{2(1 - e^{-\mu})} \left[ I^\alpha_{g^s} S(p) + I^\alpha_{p^s} S(g) \right] \leq \frac{S(g) + S(p)}{2}.
\]

If we choose \( \alpha \to 1 \) and \( S_s(v) = S^c(v) \) in Theorem 3.1. Then the classical Hermite-Hadamard type inequality is recovered and is given as follows:
\[
S \left( \frac{g + p}{2} \right) \leq \frac{1}{p - g} \int_g^p S(v) \, dv \leq \frac{S(g) + S(p)}{2}.
\]
Example 3.1. If we take $\alpha = \frac{1}{2}$, $\nu \in [0, 2]$ and the following interval valued function $S(\nu) = [1, 2] \left( \nu^2 \right)$.

Since, the left and right end points $S_*(\nu) = \nu^2$, $S^*(\nu) = 2\nu^2$ are LR convex interval valued functions, then $S(\nu)$ is LR-convex I-V-F. Thus, we obtain

$$S_* \left( \frac{\theta + \nu}{2} \right) = 1,$$

$$S^* \left( \frac{\theta + \nu}{2} \right) = 2,$$

$$\frac{S_*(\theta) + S_*(\nu)}{2} = 2,$$

and

$$\frac{S^*(\theta) + S^*(\nu)}{2} = 4.$$

We also note that

$$\frac{1 - \alpha}{2(1 - e^{-\mu})} \left[ I^\alpha_{\theta} S_*(\nu) + I^\alpha_{\nu} S_*(\theta) \right] \approx 1.373,$$

and

$$\frac{1 - \alpha}{2(1 - e^{-\mu})} \left[ I^\alpha_{\theta} S^*(\nu) + I^\alpha_{\nu} S^*(\theta) \right] \approx 2.746.$$

Therefore, we have

$$[1, 2] \preceq_p [1.373, 2.746] \preceq_p [2, 4].$$

This evidently verifies Theorem 3.1.

Pachpatte type fractional inequalities in interval valued settings

In the next two theorems, we establish Hermite-Hadamard related inequalities employing product of two LR-convex interval valued functions.

Theorem 3.2. Let $S, Y : [\theta, \nu] \to \mathcal{M}_C^I$ be two LR-convex I-V-Fs on $[\theta, \nu]$, such that

$$S(\nu) = [S_*(\nu), S^*(\nu)] \quad \text{and} \quad Y(\nu) = [Y_*(\nu), Y^*(\nu)],$$

for all $\nu \in [\theta, \nu]$. If $S, Y \in \mathcal{L} \left( \left[ \theta, \nu \right], \mathcal{M}_C^I \right)$, then

$$\frac{\alpha}{(\nu - \theta)} \left[ I^\alpha_{\theta} S(\nu) Y(\nu) + I^\alpha_{\nu} S(\theta) Y(\theta) \right] \leq_p \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{\mu^3} N(\theta, \nu) + \frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^3} \nabla(\theta, \nu),$$

where

$$N(\theta, \nu) = S(\theta) Y(\nu) + S(\nu) Y(\theta),$$

$$\nabla(\theta, \nu) = S(\theta) Y(\nu) + S(\nu) Y(\theta),$$

and

$$N(\theta, \nu) = [N_*(\theta, \nu), N^*(\theta, \nu)],$$

$$\nabla(\theta, \nu) = [\nabla_*(\theta, \nu), \nabla^*(\theta, \nu)].$$
\textbf{Proof.} Considering \(S\) and \(Y\) as LR-convex interval valued functions, we have
\[
S_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \leq rS_y(g) + (1 - r) S_y(p),
\]
\[
S'_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \leq rS'_y(g) + (1 - r) S'_y(p),
\]
and
\[
Y_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \leq rY_y(g) + (1 - r) Y_y(p),
\]

It follows from the definition of LR convex function that \(0 \leq \rho S(y)\) and \(0 \leq \rho Y(y)\), which implies
\[
S_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) Y_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \leq [rS_y(g) + (1 - r) S_y(p)][rY_y(g) + (1 - r) Y_y(p)]
\]
\[
= r^2S_y(g)Y_y(g) + (1 - r)^2 S_y(p)Y_y(p) + r(1 - r) S_y(g)Y_y(p) + r(1 - r) S_y(p)Y_y(g),
\]

and
\[
S'_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) Y'_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \leq [rS'_y(g) + (1 - r) S'_y(p)][rY'_y(g) + (1 - r) Y'_y(p)]
\]
\[
= r^2S'_y(g)Y'_y(g) + (1 - r)^2 S'_y(p)Y'_y(p) + r(1 - r) S'_y(g)Y'_y(p) + r(1 - r) S'_y(p)Y'_y(g).
\]

Similarly, we have
\[
S_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) Y_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) \leq (1 - r)^2 S_y(g)Y_y(g) + r^2S_y(p)Y_y(p) + r(1 - r) S_y(g)Y_y(p) + r(1 - r) S_y(p)Y_y(g),
\]

and
\[
S'_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) Y'_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) \leq (1 - r)^2 S'_y(g)Y'_y(g) + r^2S'_y(p)Y'_y(p) + r(1 - r) S'_y(g)Y'_y(p) + r(1 - r) S'_y(p)Y'_y(g).
\]

It follows from the above developments that
\[
S_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) Y_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) + S_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) Y_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) \leq [r^2 + (1 - r)^2] \left[ S_y(g)Y_y(g) + S_y(p)Y_y(p) \right] + 2r(1 - r) \left[ S_y(g)Y_y(p) + S_y(p)Y_y(g) \right],
\]  \hspace{1cm} (3.5)

and
\[
S'_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) Y'_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) + S'_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) Y'_y((1 - \mathfrak{r}) g + \mathfrak{r} \mathfrak{p}) \leq [r^2 + (1 - r)^2] \left[ S'_y(g)Y'_y(g) + S'_y(p)Y'_y(p) \right] + 2r(1 - r) \left[ S'_y(g)Y'_y(p) + S'_y(p)Y'_y(g) \right].
\]  \hspace{1cm} (3.6)

Multiplying both the Eqs (3.5) and (3.6) by \(e^{- \frac{1}{a}(p - \varrho \mathfrak{r})} \) and then integrating with respect to \( \mathfrak{r} \) over [0,1], we obtain
\[
\int_0^1 e^{- \frac{1}{a}(p - \varrho \mathfrak{r})} S_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) Y_y(\mathfrak{r}_g + (1 - \mathfrak{r}) \mathfrak{p}) \mathfrak{d}\mathfrak{r}
\]
This concludes the proof of Theorem 3.2.

Consequently, we have

\[ \int_0^1 e^{-\frac{1-n}{2(p-q)}S_\alpha ((1-r) g + r p) Y_\alpha ((1-r) g + r p)} dr \]

\[ \leq \mathcal{N}_\alpha (g, p) \int_0^1 e^{-\frac{1-n}{2(p-q)}r (1-r)} dr + 2 \mathcal{N}_\alpha (g, p) \int_0^1 e^{-\frac{1-n}{2(p-q)}r (1-r)} dr , \]

and

\[ \int_0^1 e^{-\frac{1-n}{2(p-q)}S_\alpha (r g + (1-r) p) Y_\alpha (r g + (1-r) p)} dr \]

\[ + \int_0^1 e^{-\frac{1-n}{2(p-q)}S_\alpha ((1-r) g + r p) Y_\alpha ((1-r) g + r p)} dr \]

\[ \leq \mathcal{N}_\alpha (g, p) \int_0^1 e^{-\frac{1-n}{2(p-q)}r (1-r)} dr + 2 \mathcal{N}_\alpha (g, p) \int_0^1 e^{-\frac{1-n}{2(p-q)}r (1-r)} dr . \]

In view of the above developments, we have

\[ \frac{\alpha}{(p-g)} \left[ I_\theta^{\alpha} S_\alpha (p) Y_\alpha (p) + I_\theta^{\alpha} S_\alpha (g) Y_\alpha (g) \right] \]

\[ \leq \mathcal{N}_\alpha (g, p) \frac{\mu^2 - 2 \mu + 4 - (\mu^2 + 2 \mu + 4) e^{-\mu}}{\mu^3} + \nabla_\alpha (g, p) \frac{2 \mu - 4 + (2 \mu + 4) e^{-\mu}}{\mu^3} , \] (3.7)

and

\[ \frac{\alpha}{(p-g)} \left[ I_\theta^{\alpha} S_\alpha (p) Y_\alpha (p) + I_\theta^{\alpha} S_\alpha (g) Y_\alpha (g) \right] \]

\[ \leq \mathcal{N}_\alpha (g, p) \frac{\mu^2 - 2 \mu + 4 - (\mu^2 + 2 \mu + 4) e^{-\mu}}{\mu^3} + \nabla_\alpha (g, p) \frac{2 \mu - 4 + (2 \mu + 4) e^{-\mu}}{\mu^3} . \] (3.8)

It follows from the above Eqs (3.7) and (3.8) that

\[ \frac{\alpha}{(p-g)} \left[ I_\theta^{\alpha} S_\alpha (p) Y_\alpha (p) + I_\theta^{\alpha} S_\alpha (g) Y_\alpha (g), I_\theta^{\alpha} S_\alpha (p) Y_\alpha (p) + I_\theta^{\alpha} S_\alpha (g) Y_\alpha (g) \right] \]

\[ \leq \frac{\mu^2 - 2 \mu + 4 - (\mu^2 + 2 \mu + 4) e^{-\mu}}{\mu^3} \left[ \mathcal{N}_\alpha (g, p), \nabla_\alpha (g, p) \right] + \frac{2 \mu - 4 + (2 \mu + 4) e^{-\mu}}{\mu^3} \left[ \nabla_\alpha (g, p), \nabla_\alpha (g, p) \right] .

Consequently, we have

\[ \frac{\alpha}{(p-g)} \left[ I_\theta^{\alpha} S_\alpha (p) Y_\alpha (p) + I_\theta^{\alpha} S_\alpha (g) Y_\alpha (g) \right] \]

\[ \leq \frac{\mu^2 - 2 \mu + 4 - (\mu^2 + 2 \mu + 4) e^{-\mu}}{\mu^3} \mathcal{N}_\alpha (g, p) + \frac{2 \mu - 4 + (2 \mu + 4) e^{-\mu}}{\mu^3} \nabla_\alpha (g, p) .

This concludes the proof of Theorem 3.2. \[\square\]
Remark 3.2. If we choose $S_{*}(v) = S^{*}(v)$ in Theorem 3.2, we have the following fractional integral inequality for LR-convex interval valued functions as given in [26].

$$\frac{\alpha}{(p - \varrho)} \left[I_{\varrho}^{\alpha} S(p)Y(p) + I_{p}^{\alpha} S(\varrho)Y(\varrho)\right] \leq \frac{\mu^{2} - 2\mu + 4 - (\mu^{2} + 2\mu + 4)e^{-\mu}}{\mu^{3}} N(\varrho, p) + \frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^{3}} \nabla(\varrho, p),$$

where $N(\varrho, p)$ and $\nabla(\varrho, p)$ are defined as Theorem 3.2.

Example 3.2. If we let, $[\varrho, p] = [0, 2]$, $\alpha = \frac{1}{2}$. Also, let the interval valued functions be given as

$$S(v) = [v, 2v] \text{ and } Y(v) = [v^{2}, 2v^{2}].$$

Since, the left and the right end point functions $S_{*}(v) = v$, $S_{*}(v) = 2v$, $Y_{*}(v) = v^{2}$ and $Y_{*}(v) = 2v^{2}$ are LR-convex functions, both $S(v)$ and $Y(v)$ are LR-convex interval valued functions. Then, we have $S(v)Y(v) \in L([\varrho, p], M_{\infty}^{3})$ and

$$\frac{(\alpha)}{p - \varrho} \left[I_{\varrho}^{\alpha} S_{*}(v)Y_{*}(v) + I_{p}^{\alpha} S_{*}(\varrho)Y_{*}(\varrho)\right] \approx 1.8346.$$  

$$\frac{(\alpha)}{p - \varrho} \left[I_{\varrho}^{\alpha} S^{*}(v)Y^{*}(v) + I_{p}^{\alpha} S^{*}(\varrho)Y^{*}(\varrho)\right] \approx 7.338.$$  

Also, note that

$$\left(\frac{\mu^{2} - 2\mu + 4 - (\mu^{2} + 2\mu + 4)e^{-\mu}}{\mu^{3}}\right) N_{*}(\varrho, p) = [S_{*}(\varrho)Y_{*}(\varrho) + S_{*}(v)Y_{*}(v)] = 13.556,$$

$$\left(\frac{\mu^{2} - 2\mu + 4 - (\mu^{2} + 2\mu + 4)e^{-\mu}}{\mu^{3}}\right) N^{*}(\varrho, p) = [S^{*}(\varrho)Y^{*}(\varrho) + S^{*}(v)Y^{*}(v)] = 54.224,$$

$$\left(\frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^{3}}\right) \nabla_{*}(\varrho, p) = [S_{*}(\varrho)Y_{*}(\varrho) + S_{*}(v)Y_{*}(v)] = 0,$$

$$\left(\frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^{3}}\right) \nabla^{*}(\varrho, p) = [S^{*}(\varrho)Y^{*}(\varrho) + S^{*}(v)Y^{*}(v)] = 0.$$  

Therefore, we have

$$\left(\frac{\mu^{2} - 2\mu + 4 - (\mu^{2} + 2\mu + 4)e^{-\mu}}{\mu^{3}}\right) N(\varrho, p) + \left(\frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^{3}}\right) \nabla(\varrho, p) = [13.556, 54.224]$$

It follows from the above developments that

$$[1.8346, 7.338] \leq_{p} [13.556, 54.224],$$

and this evidently verifies Theorem 3.2.
Theorem 3.3. Let $S, Y : [g, v] \to \mathcal{M}^{+}_{c}$ be two LR-convex interval valued functions, such that $S(v) = [S_{+}(v), S^{-}(v)]$ and $Y(v) = [Y_{+}(v), Y^{-}(v)]$ for all $v \in [g, p]$. If $SY \in L([g, v], \mathcal{M}^{+}_{c})$, then

$$2S\left(\frac{g + p}{2}\right)Y\left(\frac{g + p}{2}\right) \leq \frac{(1 - \alpha)}{2(1 - e^{-\mu})} \left[ I_{g}^{+} S(p)Y(p) + I_{p}^{+} S(g)Y(g) \right] + \frac{\mu^{2} - 2\mu + 4 - (\mu^{2} + 2\mu + 4)e^{-\mu}}{2\mu^{2}(1 - e^{-\mu})} \nabla (g, p) + \frac{\mu - 2 + (\mu + 2)e^{-\mu}}{\mu^{2}(1 - e^{-\mu})} N(g, p), \quad (3.9)$$

where

$$N(g, p) = S(g)Y(g) + S(p)Y(p), \quad \nabla (g, p) = S(g)Y(p) + S(p)Y(g),$$

and

$$N(g, p) = [N_{+}(g, p), N^{-}(g, p)], \quad \nabla (g, p) = [\nabla_{+}(g, p), \nabla^{-}(g, p)].$$

Proof. Consider $S, Y : [g, p] \to \mathcal{M}^{+}_{c}$ are LR-convex interval valued functions. Then we have

$$S\left(\frac{g + p}{2}\right)Y\left(\frac{g + p}{2}\right) \leq \frac{1}{4} \left[ S_{+}(r_{g} + (1 - r) p) Y_{+}(r_{g} + (1 - r) p) + S_{-}(r_{g} + (1 - r) p) Y_{-}(1 - r) g + r p) \right] + \frac{1}{4} \left[ S_{+}((1 - r) g + r p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] \leq \frac{1}{4} \left[ S_{+}(r_{g} + (1 - r) p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] + \frac{1}{4} \left[ (rS_{+}(g) + (1 - r) S_{-}(p)) ((1 - r) Y_{+}(g) + r Y_{-}(p)) \right] + ((1 - r) S_{+}(g) + r S_{-}(p)) (r Y_{+}(g) + (1 - r) Y_{-}(p)) \right] = \frac{1}{4} \left[ S_{+}(r_{g} + (1 - r) p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] + \frac{1}{4} \left[ (r^{2} + (1 - r)^{2}) \nabla_{+}(g, p) + (r (1 - r) + (1 - r) r) N_{+}(g, p) \right],$$

and

$$S^{-}\left(\frac{g + p}{2}\right)Y^{-}\left(\frac{g + p}{2}\right) \leq \frac{1}{4} \left[ S_{+}(r_{g} + (1 - r) p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] + \frac{1}{4} \left[ S_{+}((1 - r) g + r p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] \leq \frac{1}{4} \left[ S_{+}(r_{g} + (1 - r) p) Y_{+}(r_{g} + (1 - r) p) + S_{-}((1 - r) g + r p) Y_{-}(1 - r) g + r p) \right] + \frac{1}{4} \left[ (r S_{+}(g) + (1 - r) S_{-}(p)) ((1 - r) Y_{+}(g) + r Y_{-}(p)) \right] + ((1 - r) S_{+}(g) + r S_{-}(p)) (r Y_{+}(g) + (1 - r) Y_{-}(p)) \right]$$

AMS Mathematics

Volume 7, Issue 8, 15041–15063.
Upon multiplying the above developments by $e^{-\frac{1}{\mu}(\nu-p)x}$ and then integrating the obtained result over $[0, 1]$, we get

$$
\frac{1 - e^{-\mu} S}{\mu} S, \left( \frac{\beta + p}{2} \right) Y, \left( \frac{\beta + p}{2} \right)
\leq \frac{\alpha}{4(p - \beta)} \left[ I^\alpha_{p^+} S, (p) Y, (p) + I^\alpha_{p^-} S, (p) Y, (p) \right] + \nabla, (g, \nu) \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{4\mu^3} + \frac{\mu - 2 + (\mu + 2)e^{-\mu}}{2\mu^3} \mathcal{N}, (g, \nu) \quad (3.10)
$$

and

$$
\frac{1 - e^{-\mu} S}{\mu} S, \left( \frac{\beta + p}{2} \right) Y, \left( \frac{\beta + p}{2} \right)
\leq \frac{\alpha}{4(p - \beta)} \left[ I^\alpha_{p^+} S, (p) Y, (p) + I^\alpha_{p^-} S, (p) Y, (p) \right] + \nabla, (g, \nu) \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{4\mu^3} + \frac{\mu - 2 + (\mu + 2)e^{-\mu}}{2\mu^3} \mathcal{N}, (g, \nu) \quad (3.11)
$$

In view of the above Eqs (3.10) and (3.11), we have

$$
2S \left( \frac{\beta + p}{2} \right) Y \left( \frac{\beta + p}{2} \right) \leq_p \frac{(1 - \alpha)}{2(1 - e^{-\mu})} \left[ I^\alpha_{p^+} S, (p) Y, (p) + I^\alpha_{p^-} S, (p) Y, (p) \right] + \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{2\mu^2(1 - e^{-\mu})} \nabla, (g, \nu) + \frac{\mu - 2 + (\mu + 2)e^{-\mu}}{\mu^2(1 - e^{-\mu})} \mathcal{N}, (g, \nu) \quad (3.12)
$$

This concludes the proof of Theorem 3.3.

Remark 3.3. If $S, (\nu) = S^*(\nu)$, then, from Theorem 3.3 we obtain the following fractional integral inequality for LR-convex I-V-Fs as given in [26].

$$
2S \left( \frac{\beta + p}{2} \right) Y \left( \frac{\beta + p}{2} \right) \leq_p \frac{(1 - \alpha)}{2(1 - e^{-\mu})} \left[ I^\alpha_{p^+} S, (p) Y, (p) + I^\alpha_{p^-} S, (p) Y, (p) \right] + \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{2\mu^2(1 - e^{-\mu})} \nabla, (g, \nu) + \frac{\mu - 2 + (\mu + 2)e^{-\mu}}{\mu^2(1 - e^{-\mu})} \mathcal{N}, (g, \nu) \quad (3.12)
$$

where $\mathcal{N}, (g, \nu) = S, (g) Y, (g) + S, (p) Y, (p), \nabla, (g, \nu) = S, (g) Y, (g) + S, (p) Y, (p)$.

4. Interval-valued Fejér type fractional inequalities

Theorem 4.1. Let $S : [g, \nu] \rightarrow M^+_\mathcal{C}$ be an LR-convex interval valued function with $(g < \nu)$ and given by $S, (\nu) = \left[ S, (\nu), S^*(\nu) \right]$ for all $\nu \in [g, \nu]$. Let $S \in \mathcal{L}, [g, \nu] \rightarrow \mathbb{R}, Y, (\nu) \geq 0$, symmetric with respect to $\frac{g + p}{2}$. Then

$$
\left[ I^\alpha_{p^+} S, Y, (p) + I^\alpha_{p^-} S, Y, (g) \right] \leq_p \frac{S, (g) + S, (p)}{2} \left[ I^\alpha_{p^+} Y, (p) + I^\alpha_{p^-} Y, (g) \right] \quad (4.1)
$$
Proof. Since $S$ be an LR-convex interval valued function and

$$e^{-\frac{1}{\alpha}(p-r)\alpha}Y((1-r)g + rp) \geq 0.$$ 

We have

$$e^{-\frac{1}{\alpha}(p-r)\alpha}S_\alpha((1-r)g + rp)Y((1-r)g + rp) \leq e^{-\frac{1}{\alpha}(p-r)\alpha}(rS_\alpha(g) + (1-r)S_\alpha(p))Y((1-r)g + rp),$$

$$e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star((1-r)g + rp)Y((1-r)g + rp) \leq e^{-\frac{1}{\alpha}(p-r)\alpha}(rS^\star(g) + (1-r)S^\star(p))Y((1-r)g + rp),$$

$$e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star((1-r)g + rp)Y((1-r)g + rp) \leq e^{-\frac{1}{\alpha}(p-r)\alpha}((1-r)S_\alpha(g) + rS_\alpha(p))Y((1-r)g + rp),$$

and

$$e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star((1-r)g + rp)Y((1-r)g + rp) \leq e^{-\frac{1}{\alpha}(p-r)\alpha}((1-r)S^\star(g) + rS^\star(p))Y((1-r)g + rp).$$

Consequently, from the above developments, we have

$$\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S_\alpha((1-r)g + rp)Y((1-r)g + rp)dr$$

$$+ \int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S_\alpha((1-r)g + rp)Y((1-r)g + rp)dr$$

$$\leq \int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S_\alpha(g)\{rY((1-r)g + rp) + (1-r)Y((1-r)g + rp)\}$$

$$+ e^{-\frac{1}{\alpha}(p-r)\alpha}S_\alpha(p)\{(1-r)Y((1-r)g + rp) + rY((1-r)g + rp)\} dr$$

$$= S_\alpha(g)\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}Y((1-r)g + rp)dr + S_\alpha(p)\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}Y((1-r)g + rp)dr,$$

and

$$\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star((1-r)g + rp)Y((1-r)g + rp)dr$$

$$+ \int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star((1-r)g + rp)Y((1-r)g + rp)dr$$

$$\leq \int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star(g)\{rY((1-r)g + rp) + (1-r)Y((1-r)g + rp)\}$$

$$+ e^{-\frac{1}{\alpha}(p-r)\alpha}S^\star(p)\{(1-r)Y((1-r)g + rp) + rY((1-r)g + rp)\} dr$$

$$= S^\star(g)\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}Y((1-r)g + rp)dr + S^\star(p)\int_0^1 e^{-\frac{1}{\alpha}(p-r)\alpha}Y((1-r)g + rp)dr.$$

Since $Y$ is symmetric, and using the fact that

$$I_{\alpha}^\alpha Y(p) = I_{\alpha}^\alpha Y(g) = \frac{1}{2}(I_{\alpha}^\alpha Y(p) + I_{\alpha}^\alpha Y(g)),$$
we have,
\[
S_\alpha(S) \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr + S_\alpha(p) \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr
\]
\[
= [S_\alpha(S) + S_\alpha(p)] \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr
\]
\[
= \frac{S_\alpha(S) + S_\alpha(p)}{2} \frac{\alpha}{p - g} \left[ I^\alpha_g Y(p) + I^\alpha_p Y(g) \right]
\]

and
\[
S^*(S) \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr + S^*(p) \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr.
\]
\[
= [S^*(S) + S^*(p)] \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) Y((1-r)g + rp)} dr.
\]
\[
= \frac{S^*(S) + S^*(p)}{2} \frac{\alpha}{p - g} \left[ I^\alpha_g Y(p) + I^\alpha_p Y(g) \right].
\]

We also have
\[
\int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) S_\alpha(rg + (1-r)p) Y((1-r)g + rp)} dr
\]
\[
+ \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) S_\alpha((1-r)g + rp) Y((1-r)g + rp)} dr
\]
\[
= \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha)(p-\alpha)} S_\alpha(g + p - \nu Y(\nu) d\nu + \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha) S_\alpha(\nu) Y(\nu) d\nu
\]
\[
= \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha)(p-\alpha)} S_\alpha(g + p - \nu Y(\nu) d\nu + \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha) S_\alpha(\nu) Y(\nu) d\nu
\]
\[
= \frac{\alpha}{p - g} \left[ I^\alpha_g S_\alpha Y(p) + I^\alpha_p S_\alpha Y(\nu) \right].
\]

and
\[
\int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) S^*(rg + (1-r)p) Y((1-r)g + rp)} dr
\]
\[
+ \int_0^1 e^{-\frac{1}{\alpha} (p-\alpha) S^*((1-r)g + rp) Y((1-r)g + rp)} dr
\]
\[
= \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha)(p-\alpha)} S^*(p + \nu Y(\nu) d\nu + \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha) S^*(\nu) Y(\nu) d\nu
\]
\[
= \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha)(p-\alpha)} S^*(p + \nu Y(\nu) d\nu + \frac{1}{p - g} \int_0^\alpha e^{-\frac{1}{\alpha} (p-\alpha) S^*(\nu) Y(\nu) d\nu
\]
\[
= \frac{\alpha}{p - g} \left[ I^\alpha_g S^* Y(p) + I^\alpha_p S^* Y(\nu) \right].
\]

It follows from the above developments that
\[
\frac{\alpha}{p - g} \left[ I^\alpha_g S_\alpha Y(p) + I^\alpha_p S_\alpha Y(\nu) \right] \leq \frac{S_\alpha(g) + S_\alpha(p)}{2} \frac{\alpha}{(p - g)} \left[ I^\alpha_g Y(p) + I^\alpha_p Y(\nu) \right],
\]

AIMS Mathematics

Volume 7, Issue 8, 15041–15063.
and
\[
\frac{\alpha}{(p-q)} \left[ I_\alpha^\nu S^*(Y(p)) + I_\alpha^\nu S^*(Y(g)) \right] \leq \frac{S^*(g) + S^*(p)}{2} \left[ I_\alpha^\nu Y(p) + I_\alpha^\nu Y(g) \right].
\]
Consequently,
\[
\left[ I_\alpha^\nu S, Y(p) + I_\alpha^\nu S, Y(g) \right] \leq_p \left[ S, (g) + S, (p) \right] \left[ I_\alpha^\nu Y(p) + I_\alpha^\nu Y(g) \right],
\]
which readily follows
\[
\left[ I_\alpha^\nu S, Y(p) + I_\alpha^\nu S, Y(g) \right] \leq_p \frac{S(g) + S(p)}{2} \left[ I_\alpha^\nu Y(p) + I_\alpha^\nu Y(g) \right].
\]
This concludes the proof of Theorem 4.1. □

**Corollary 4.1.** Let \( S : [g, p] \rightarrow M^\nu_C \) be an LR-concave function with \( q < p \), and \( S \in \mathcal{L}([g, p]) \) defined by \( S(v) = [S_\alpha(v), S^*(v)] \) for all \( v \in [g, p] \). If \( Y : [g, p] \rightarrow \mathbb{R}, Y(v) \geq 0 \), is symmetric with respect to \( \frac{q+p}{2} \). Then the following inequality holds true.
\[
\left[ I_\alpha^\nu S, Y(p) + I_\alpha^\nu S, Y(g) \right] \geq_p \frac{S(g) + S(p)}{2} \left[ I_\alpha^\nu Y(p) + I_\alpha^\nu Y(g) \right].
\]

**Theorem 4.2.** Let \( S : [g, p] \rightarrow M^\nu_C \) be an LR-convex interval valued function with \( q < p \), and defined by \( S(v) = [S_\alpha(v), S^*(v)] \) for all \( v \in [g, p] \). If \( S \in \mathcal{L}([g, p], M^\nu_C) \) and \( Y : [g, p] \rightarrow \mathbb{R}, Y(v) \geq 0 \), symmetric with respect to \( \frac{q+p}{2} \), then
\[
\left[ S \left( \frac{g+p}{2} \right) \left[ I_\alpha^\nu Y(p) + I_\alpha^\nu Y(g) \right] \right] \leq_p \left[ I_\alpha^\nu S, Y(p) + I_\alpha^\nu S, Y(g) \right]. \tag{4.2}
\]

**Proof.** If we utilize the definition of LR-convex interval valued function, then we have
\[
S_\alpha \left( \frac{g+p}{2} \right) \leq \frac{1}{2} \left( S_\alpha \left( r \alpha \right) + \left( 1 - r \right) \frac{p}{2} \right) + S_\alpha \left( \left( 1 - r \right) g + rv \right),
\]
and
\[
S^* \left( \frac{g+p}{2} \right) \leq \frac{1}{2} \left( S^* \left( r \alpha \right) + \left( 1 - r \right) \frac{p}{2} \right) + S^* \left( \left( 1 - r \right) g + rv \right). \tag{4.3}
\]
Since, \( Y \left( r \alpha \right) + \left( 1 - r \right) \frac{p}{2} \) = \( Y \left( 1 - r \right) g + rv \), then multiplying the above Eq (4.3) by \( e^{-\frac{p}{2}Y((1 - r) g + rv)} \) and integrate it with respect to \( r \) over \([0, 1]\), we obtain
\[
S_\alpha \left( \frac{g+p}{2} \right) \int_0^1 e^{-\frac{p}{2}Y((1 - r) g + rv)} dr 
\leq \frac{1}{2} \left( \int_0^1 e^{-\frac{p}{2}Y((1 - r) g + rv)} S_\alpha \left( r \alpha \right) + \left( 1 - r \right) \frac{p}{2} \right) \right) Y \left( 1 - r \right) g + rv \right) dr 
\]
\[
\int_0^1 e^{-\frac{p}{2}Y((1 - r) g + rv)} S_\alpha \left( \left( 1 - r \right) g + rv \right) Y \left( 1 - r \right) g + rv \right) dr,
\]

AIMS Mathematics
Volume 7, Issue 8, 15041–15063.
and
\[
S^*\left(\frac{g + p}{2}\right) \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} Y ((1 - r) g + rv) \, dr
\]
\[
\leq \frac{1}{2} \left( \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} S^* \left( r g + (1 - r) v \right) Y ((1 - r) g + rv) \, dr 
+ \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} S^* ((1 - r) g + rv) Y ((1 - r) g + rv) \, dr \right).
\]

Let \( \nu = (1 - r) g + rv \). Then we have
\[
\int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} \, S^* (rg + (1 - r) v) Y ((1 - r) g + rv) \, dr
+ \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} \, S^* ((1 - r) g + rv) Y ((1 - r) g + rv) \, dr
\]
\[
= \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (g + p - \nu) Y (\nu) \, d\nu + \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (\nu) Y (\nu) \, d\nu
\]
\[
= \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (z) Y (g + p - z) \, dz + \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (\nu) Y (\nu) \, d\nu
\]
\[
= \frac{\alpha}{p - g} \left[ \mathcal{I}_{\alpha}^* S^* Y (p) + \mathcal{I}_p^p S^* Y (g) \right],
\]
and
\[
\int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} S^* (rg + (1 - r) v) Y ((1 - r) g + rv) \, dr
+ \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} S^* ((1 - r) g + rv) Y ((1 - r) g + rv) \, dr
\]
\[
= \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (g + p - \nu) Y (\nu) \, d\nu + \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (\nu) Y (\nu) \, d\nu
\]
\[
= \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (z) Y (g + p - z) \, dz + \frac{1}{p - g} \int_0^p e^{-\frac{1}{\alpha}(\nu - \alpha)} S^* (\nu) Y (\nu) \, d\nu
\]
\[
= \frac{\alpha}{p - g} \left[ \mathcal{I}_{\alpha}^* S^* Y (p) + \mathcal{I}_p^p S^* Y (g) \right].
\]

Also,
\[
S^* \left( \frac{g + p}{2} \right) \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} Y ((1 - r) g + rv) \, dr = \frac{\alpha}{(p - g)} S \left( \frac{g + p}{2} \right) \left[ \mathcal{I}_{\alpha}^* Y (p) + \mathcal{I}_p^p Y (g) \right],
\]
and
\[
S^* \left( \frac{g + p}{2} \right) \int_0^1 e^{-\frac{1}{\alpha}r(\nu-\alpha)} Y ((1 - r) g + rv) \, dr = \frac{\alpha}{(p - g)} S \left( \frac{g + p}{2} \right) \left[ \mathcal{I}_{\alpha}^* Y (p) + \mathcal{I}_p^p Y (g) \right].
\]

In view of the above developments, we have
\[
\left[ S^* \left( \frac{g + p}{2} \right), S^* \left( \frac{g + p}{2} \right) \right] \left[ \mathcal{I}_{\alpha}^* Y (p) + \mathcal{I}_p^p Y (g) \right]
\]
Hence, Theorem 4.2 is also verified.

It readily follows

\[ S\left(\frac{\beta + p}{\alpha}\right)\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] \leq_p \left[I^\alpha_{\beta} SY(p) + I^\alpha_{\nu} SY(\gamma)\right]. \]

This concludes the proof of Theorem 4.2.

\[ \square \]

**Corollary 4.2.** Let \( S: [g, p] \to \mathcal{M}_C \) be an LR-concave function with \( g < p \), and \( S \in \mathcal{L}([g, p]) \) defined by \( S(\nu) = [S_\nu(\nu), S'(\nu)] \) for all \( \nu \in [g, p] \). If \( Y: [g, p] \to \mathbb{R}, \ Y(\nu) \geq 0 \), is symmetric with respect to \( \frac{g + p}{2} \). Then the following inequalities hold true:

\[ S\left(\frac{\beta + p}{\alpha}\right)\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] \leq_p \left[I^\alpha_{\beta} SY(p) + I^\alpha_{\nu} SY(\gamma)\right]. \]

\[ (4.4) \]

**Remark 4.1.** If we choose \( S_\nu(\nu) = S'(\nu) \) in the above Theorem 4.1 and Theorem 4.2, we obtain the following fractional H-H-F inequality (see [26]).

\[ S\left(\frac{\beta + p}{\alpha}\right)\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] \leq_p \left[I^\alpha_{\beta} SY(p) + I^\alpha_{\nu} SY(\gamma)\right] \leq_p \frac{S(\nu) + S'(\nu)}{2}\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right]. \]

**Example 4.1.** If let \( \alpha = \frac{1}{2} \) and The LR-convex I-V-Fs \( S: [0, 2] \to \mathcal{M}_C \) be given by \( S(\nu) = [\nu^2, 2\nu^2] \). If

\[ Y(\nu) = \begin{cases} \nu, & \nu \in [0, 1], \\ 2 - \nu, & \nu \in (1, 2], \end{cases} \]

then \( Y(2 - \nu) = Y(\nu) \geq 0 \), for all \( \nu \in [0, 2] \). We clearly observe that:

\[ \left[I^\alpha_{\beta} SY(p) + I^\alpha_{\nu} SY(\gamma)\right] \leq_p \frac{S(\nu) + S'(\nu)}{2}\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right]. \]

\[ \frac{S_\nu(\nu) + S'(\nu)}{2}\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] \approx 3.1966, \]

\[ \frac{S'(\nu) + S'(\nu)}{2}\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] \approx 6.3932, \]

\[ \left[I^\alpha_{\beta} SY(p) + I^\alpha_{\nu} SY(\gamma)\right] \approx 1.8948, \]

\[ \left[I^\alpha_{\beta} S'Y(p) + I^\alpha_{\nu} S'Y(\gamma)\right] \approx 3.7896. \]

It follows from the above developments that \([1.8948, 3.7896] \leq_p [3.1966, 6.3932] \].

This evidently verifies Theorem 4.1.

Next, for Theorem 4.2, we have the following computations:

\[ S_\nu\left(\frac{\beta + p}{\alpha}\right)\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] = 1.5983, \]

and

\[ S'(\nu)\left(\frac{\beta + p}{\alpha}\right)\left[I^\alpha_{\beta} Y(p) + I^\alpha_{\nu} Y(\gamma)\right] = 3.1966, \]

which readily follows

\[ [1.5983, 3.1966] \leq_p [1.8948, 3.7896]. \]

Hence, Theorem 4.2 is also verified.
5. Conclusions

In this study, we have obtained the Hermite-Hadamard type fractional inclusions for LR-convex interval-valued functions. Following that, we have shown fractional-order Pachpatte type inclusions for the product of two LR-convex interval-valued functions, as well as the Hermite-Hadamard-Fejér type fractional-order inclusions for symmetric functions. We can look into the LR-convex interval-valued functions on coordinates and the quantum (or $q$-) calculus in the future. The new work is intended to inspire scholars in fractional calculus, interval analysis, and other relevant fields.

We choose to conclude our current investigation by commenting that, in numerous recent scientific articles, fractional-order analogues of many different well-known integral inequalities have been regularly investigated by utilizing some trivial or redundant parametric variations of some widely-and extensively-studied fractional integral and fractional derivative operators (see [9] for detailed observation about the triviality and inconsequential aspect of the so-called “post quantum” calculus involving a redundant or superficial forced-in parameter).

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Conflicts of interest

The authors declare they have no conflicts of interest.

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