Hidden Depths in a Black Hole: Surface Area Information Encoded in the \((r,t)\) Sector

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Based on an investigation into the near-horizon geometrical description of black hole spacetimes (the so-called \((r,t)\) sector), we find that the surface area of the event horizon of a black hole is mirrored in the area of a newly-defined surface, which naturally emerges from studying the intrinsic curvature of the \((r,t)\) sector at the horizon. We define this new, abstract surface for a range of different black holes and show that, in each case, the surface encodes event horizon information, despite its derivation relying purely on the \((r,t)\) sector of the metrical description. This is a very surprising finding as this sector is orthogonal to the sector explicitly describing the horizon geometry.

Our results provide new evidence supporting the conjecture that black holes are, in some sense, fundamentally two-dimensional. As black hole entropy is known to be proportional to horizon area, a novel two-dimensional interpretation of this entropy may also be possible.

Introduction.— Over the past few decades, the counterintuitive idea of regarding black holes as thermodynamical systems has been very fruitful and has led to several deep results \([1,6]\). It was predicted by Hawking \([1]\) in the 1970s that black holes must emit thermal radiation (hence they are not quite “black”) and, based on the efforts of Bekenstein and Wheeler, they were also found to contribute to the total entropy of the Universe. This latter discovery leads to arguably the most interesting concept in black hole physics: a black hole has an entropy and the value of this entropy is proportional to the surface area \(A\) of the event horizon \([2,3]\). This concept has led to profound ideas, such as the holographic principle \([7,9]\), the AdS/CFT correspondence \([10]\), and it provides deep insights into the role played by information theory in fundamental physics \([11,13]\). Indeed, black hole thermodynamics has been thought of for several decades as a possible window into a consistent theory of quantum gravity \([13,15]\).

One of the most striking consequences of the thermodynamical approach is the realisation that almost all of the information pertaining to a black hole is contained (or encoded) in a two-dimensional sector of its geometrical description: the “\((r,t)\) sector” \([16,17]\). It is known that this sector is sufficient to study Hawking radiation emission, whether one uses anomalies \([18,19]\), tunnelling \([20,21]\), topological techniques \([22]\), or the standard approach enforcing the absence of conical singularities in Euclidean spacetimes \([23]\).

Less well known is the fact that the entropy of a black hole relies fundamentally on the properties of its \((r,t)\) sector: after a Wick rotation to imaginary time, in fact, the entropy becomes directly proportional to the value of a topological invariant (the Euler characteristic \(\chi\)) of this two-dimensional sector \([16,17]\), i.e., \(S = \chi A/4\), where \(A\) is the surface area of the event horizon. The \((r,t)\) sector, therefore, plays a fundamental role in determining the thermodynamical properties of a black hole. However, the surface area \(A\) is still calculated by considering the whole four-dimensional structure of the black hole and, to the best of our knowledge, it has yet to be linked to the \((r,t)\) sector. This is understandable as the surface area of a given black hole is explicitly described by the orthogonal sector, rather than the \((r,t)\) sector \([11]\).

In recent years, a major advance in understanding the two-dimensional nature of black hole entropy has been made using conformal field theory techniques \([24,25]\). It has been shown that it is possible to derive the Bekenstein entropy by considering an algebra of surface deformations on the black hole horizon \([25]\); in this theory, the central charge of a subalgebra is found to be proportional to the surface area \(A\) of the horizon. Yet, in all cases, the area \(A\) is derived by integrating over the horizon geometry, i.e., by implicitly taking information from the orthogonal sector, rather than the \((r,t)\) sector.

In this Letter, we show that the horizon area information of a black hole is encoded entirely in the \((r,t)\) sector. To do this, we introduce a new surface, described locally by the near-horizon geometry, which effectively mirrors the event horizon, with an altered geometry but equal area: we term this new surface the \emph{lethesurface} (from the ancient Greek word \emph{lethe}, meaning “concealment”, as the surface defined in this Letter is somewhat hidden due to its definition relying on a single point of the \((r,t)\) sector). We show that the concept applies for a variety of different black hole spacetimes. This is highly unexpected as the horizon area \(A\) is described explicitly by the orthogonal sector. Our results show that a dimensional reduction of a black hole description leaves rich and, importantly, complete geometrical information behind. These results chime with the growing evidence (see \([26]\) for a review) that spacetime at small distances is fundamentally two-dimensional. The importance of these results for describing black hole entropy is also discussed.

\emph{Schwarzschild Black Hole}— The first system we analyse is the simplest: the Schwarzschild black hole. The
Schwarzschild metric describes the spacetime outside of an uncharged, nonrotating black hole (or generally any compact spherical body) of mass $M$, and is given in natural units by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$  \hspace{1cm} (1)

where $f(r) = 1 - 2M/r$ is the emblackening factor of a Schwarzschild black hole, and $d\Omega^2$ is the line element on the two-sphere. The intrinsic curvature of this spacetime, measured by the Ricci scalar $R$, is identically zero everywhere. However, the intrinsic curvatures of different planes cut through the spacetime are not necessarily vanishing. For example, taking a time slice at $t = 0$ (any time slice is equivalent here) and then fixing the radial coordinate $r$ at the event horizon position $r_H = 2M$ leaves only the two-sphere geometry of the horizon. The Ricci scalar of this sector is then a measure of the intrinsic curvature of a spherical surface of radius $r_H$, given by $R^{(t,r)}|_{r_H} = 2/r_H^2$.

A more interesting spacetime slice is found instead by fixing the angular coordinates $\theta$ and $\phi$. This leaves the line element of the so-called $(r,t)$ sector, i.e.,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2,$$  \hspace{1cm} (2)

which describes the spacetime curvature felt by a radially-moving observer outside of the black hole. More generally, this sector is what is felt by fields in the vicinity of the Schwarzschild black hole horizon at any angular coordinate values [27]. The Ricci scalar of this sector is $R^{(r,t)} = 4M/r^3$. Setting the radial coordinate to $r = r_H$ then yields the intrinsic curvature of the $(r,t)$ sector at the horizon. After substituting in the value $M = r_H/2$, the intrinsic curvature of this sector is found to be

$$R^{(t,r)}|_{r_H} = \frac{2}{r_H^2},$$  \hspace{1cm} (3)

which mirrors the intrinsic curvature of the orthogonal, purely spatial sector, derived above, at the horizon.

Eq. (3) signifies that the intrinsic curvature at the horizon of the $(r,t)$ sector is the same as that at a point on the surface of a sphere of radius $r_H$. We extrapolate from (3) by saying that it describes a point on a new, abstract surface; the most natural, symmetric choice being a surface having the constant curvature everywhere, i.e., that of a sphere of area $A = 4\pi r_H^2$. This is a clear indication that the $(r,t)$ sector contains all of the information required to determine the event horizon area. Equation (3) indicates that the geometry at the horizon describes the local region of a hidden surface, that we name the letthesurface. The presence of this surface is only revealed once the geometry of the $(r,t)$ sector at the horizon is studied in detail. The letthesurface for a Schwarzschild black hole is then simply a two-sphere with an area equal to that of the event horizon itself (as shown in Fig. [1]). This proves that the horizon area $A$ is encoded in the $(r,t)$ sector, despite the sector not explicitly describing the horizon geometry and it containing one temporal dimension. This is the first result of our Letter.

At first glance, this result may appear trivial due to the spherical symmetry of the Schwarzschild spacetime. However, note that this mirroring of the $(\theta,\phi)$ sector Ricci scalar does not apply for a general spherically symmetric compact object. For example, using the Schwarzschild metric (1) to describe the vacuum spacetime exterior to a star, one can see that the Ricci scalars of the two sectors at the surface of the star never match – its radius would have to decrease to the Schwarzschild radius in order for this to occur: something impossible without the star itself becoming a black hole.

To make our argument more convincing, and to show that it is possible to define a letthesurface for a more complicated black hole having less symmetry, in the remainder of this Letter we discuss the properties of letthesurfaces for both charged and rotating black holes. In particular, we will show that each black hole has associated to it a letthesurface (in some cases, more than one) and that, strikingly, in general, the letthesurface and the event horizon have vastly different geometries yet always equal area.

**Charged Black Hole** — The spacetime exterior to a black hole with electric charge $Q$ can be described by the metric in Eq. (1), with the emblackening factor

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right),$$  \hspace{1cm} (4)

which describes the so-called Reissner-Nordström metric. Dimensionally reducing the metric as before gives the $(r,t)$ sector of a charged black hole, namely

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2,$$  \hspace{1cm} (5)

with Ricci scalar at the horizon given by

$$R^{(t,r)}|_{r_H} = \frac{2}{r_H^2} - \frac{4Q^2}{r_H^2},$$  \hspace{1cm} (6)

where $r_H = M + \sqrt{M^2 - Q^2}$ is the outer horizon of the Reissner-Nordström black hole [28].

The local geometry described by Eq. (6) is very interesting, in particular due to the unusual form of the Ricci scalar. To see this, let us first consider the two limiting cases of vanishing charge and extremal black hole. In the uncharged limit, i.e., $Q \to 0$, the Ricci scalar above is again that of a point on the surface of a sphere of area equal to the Schwarzschild horizon area, as one would expect. In the maximally charged (extremal) limit $Q \to M$, the geometry locally describes a pseudosphere, with a
surface area equal to that of the extremal charged black hole’s horizon.

For the intermediate case $Q < M$, on the other hand, the intrinsic curvature given by Eq. (4) has two competing curvature terms, one positive and one negative. Depending on the magnitude of the charge $Q$, the total curvature can be positive, negative, or zero. Clearly, this geometry is highly nontrivial. Remarkably, this curvature again corresponds locally to a surface with the same area as the event horizon. The global geometry of this lethesurface can be described by the line element

$$ds^2 = g(\alpha, Q) da^2 + \frac{r_+^4}{g(\alpha, Q)} \sin^2 \alpha d\beta^2,$$

where $\alpha \in [0, \pi]$, $\beta \in [0, 2\pi)$, and $g(\alpha, Q) = \left[ r_+^4 - \left( \frac{Q^2}{4} \right) \sin^2 \alpha \right]$.

Note that there is no possibility of a lethesurface being defined for the charged black hole with such a high degree of symmetry as in the Schwarzschild case treated above, without the surface behaving pathologically as the parameter $Q$ evolves. For example, a spherically symmetric lethesurface transitioning from positive to zero curvature as the charge increased to $Q = 2\sqrt{2} M/3$ would have to be a sphere growing to infinite radius (and, of course, infinite area); continuing to enforce a high degree of symmetry as the charge increased further, the infinitely large sphere would then be forced to discontinuously transition to a surface with constant negative curvature. By far the most natural choice of Reissner-Nordström lethesurface is then that described by Eq. (5), without any discontinuous transitions as the value of $Q$ evolves. As discussed later on, the surface (7) is in fact familiar from black hole theory, as it has the same structure as the outer event horizon of a Kerr black hole [29], as can be proven through a simple reparameterisation.

The line element in Eq. (7) describes a family of geometries which, depending on the magnitude of $Q$, covers the surface of a sphere ($Q = 0$), the surface of a spheroid ($0 < Q < 2\sqrt{2} M/3$), or, for $2\sqrt{2} M/3 < Q \leq M$, an intermediate shape resembling a spheroid near the equator, but with local negative curvature at the poles. The lethesurface for a charged black hole is visualised in Fig. 2. The area of all these surfaces can be directly calculated from Eq. (7) and is given by $A = 4\pi r_+^2$. Moreover, the Ricci scalar for the surface described by Eq. (7) at either pole is equivalent to Eq. (6). Despite its unusual shape, the area of the lethesurface exactly corresponds to the horizon area of the charged black hole for any physical value of $Q$.

We conclude our analysis of charged black holes by noticing two puzzling features. The first concerns the curvature described by the Ricci scalar in Eq. (6). When the black hole charge reaches the special value $|Q| = 2\sqrt{2} M/3$ (which is below the extremal value $|Q| = M$), the sign of the Ricci scalar changes from positive to negative. It is not clear whether this change of sign occurring at $|Q| = 2\sqrt{2} M/3$ corresponds to a physically important transition. The second feature concerns a seemingly hidden connection between charged and rotating black holes: the lethesurface of a charged black hole, described by Eq. (7), has the same geometrical structure as the event horizon of a Kerr black hole [30], as noted earlier. This unexpected link is very interesting and should be investigated further.

Kerr Black Hole— The Kerr black hole rotates around its axis with angular momentum per unit mass $a$, the rotation flattening its event horizon towards a spheroidal shape [28, 33]. Due to a lack of spherical symmetry, a useful geometrical description of this spacetime in terms of only radial and temporal coordinates would seem unlikely but, strikingly, a two-dimensional effective metrical description, valid near the event horizon $r_+ = M + \sqrt{M^2 - a^2}$, has been found [27]. The effective metric of the $(r, t)$ sector is given by

$$ds^2 = -K(r) dt^2 + \frac{1}{K(r)} dr^2,$$

with $K(r) = (r^2 - 2Mr + a^2)/(r_+^2 + a^2)$. The intrinsic curvature of this lower-dimensional sector provides complete information concerning the surface area of the horizon,
up to the extremal value of $a$. The Ricci scalar for the geometry defined above is

$$R^{(t,r)} = -\frac{2}{r_+^2 + a^2}.$$  (9)

Notice that, contrary to the other cases, the Ricci scalar for the effective $(r,t)$ sector of a Kerr black hole need not be evaluated at the event horizon position $r_+$, since the effective metric defined above is by construction only defined near the horizon. Most importantly, Eq. (9) describes an intrinsic curvature identical to that at a point on the surface of a pseudosphere of radius $(r_+^2 + a^2)^{1/2}$. The area of the lethesurface is then the area of such a pseudosphere, i.e., $A = 4\pi (r_+^2 + a^2)^{1/2}$. This is precisely the horizon area. For a rotating black hole, therefore, the lethesurface is a pseudosphere, a shape quite unlike that of its horizon – see Fig. 3. The horizon of a Kerr black hole, in fact, can be hyperbolic near the poles (and only near the poles) if the rotation is rapid enough. However, the horizon never becomes pseudospherical over its entire surface in any limit. This is another indication of the fact that, in general, the lethesurface and event horizon of a given black hole have different geometries yet equal areas.

**Uniqueness of the lethesurface**— There is an important point to make concerning the uniqueness of the lethesurface for a given black hole. To do this, we consider a different dimensional reduction of the Kerr black hole, using the so-called Boyer-Lindquist coordinates. This can be performed by fixing the $\theta$ variable at either pole ($\theta = 0$ or $\pi$). Notably, the Ricci scalar of the $(r,t)$ sector for this new geometry at the horizon again locally describes a surface with an area matching that of the horizon, but in this case not a pseudosphere. Furthermore, the curvature described by Eq. (9) can also describe the local geometry of a twisted pseudosphere (also known as Dini’s surface) with a much more complicated structure. Nevertheless, the form of the twisted pseudosphere (though not uniquely determined by its Ricci scalar) can be constructed to have an equal surface area to the Kerr horizon.

One must take care, therefore, to define the lethesurface unambiguously. It appears from our analysis that each black hole has at least one lethesurface which, in every case so far treated, has the same area as the black hole’s event horizon. (As demonstrated above, for example, at least three lethesurfaces can be found for the rotating black hole, each one with a different geometry.) In all those cases, there is no simple one-to-one mapping between horizon and lethesurface for a given black hole. It would seem, from the example of the Kerr spacetime, that if a series of lethesurfaces can be constructed for a particular black hole, one should choose the one with the highest symmetry. By doing that, the surface area is uniquely fixed and always matches the horizon area.

The main result of our Letter is that we have shown how it is possible to define a lethesurface for uncharged, charged, and rotating black holes. Importantly, this concept is valid for any physical value of system parameter, up to and including extremal values. The lethesurface of each black hole reflects the event horizon, although, in general, imperfectly, like an image reflected by a distorting mirror. Despite this distortion, the area of the lethesurface always matches the area of the event horizon.

**Extremal Kerr-Newman Black Holes**— An immediate application of our results is that they explain a previously mysterious feature of extremal black holes. Extremal Kerr-Newman black hole spacetimes on the polar axis (i.e., at $\theta = 0$ or $\pi$) are known to be AdS$_2$. As noted by Bardeen and Horowitz in Ref. 35: “the area of the [extremal] event horizon is related to the effective cosmological constant of AdS$_2$ in a universal way that is independent of whether the extreme black hole has only angular momentum, charge, or both”. The authors then speculate that this feature may play a role in the context of the AdS/CFT correspondence.

The explanation for this “universal” relation between the effective cosmological constant of AdS$_2$ spacetime and the extremal horizon area follows directly from our results. To prove this, we first notice that AdS$_2$ space is pseudospherical and that the AdS length scale $L$ is just the radius of said pseudosphere. The “universal” relation mentioned in Ref. 35 is then given as $4\pi L^2$. Now consider the fact that, near the horizon, and on the polar axis, the $(r,t)$ sector of the extremal Kerr-Newman black

FIG. 3: The Kerr lethesurface: a pseudosphere. The pseudosphere is a hyperbolic surface formed by rotating a tractrix around its asymptote – despite its infinite extent, a pseudosphere of radius $L$ (with Ricci scalar $R = -2/L^2$) has a finite surface area of $4\pi L^2$.
hole locally describes a pseudosphere, i.e., the extremal Kerr-Newman black hole has a pseudospherical lethesurface. The area of this lethesurface is then the area of a pseudosphere of radius $r_0$, i.e., $4\pi r_0^2$. Horowitz and Bardeen’s interesting “universal” relation can be then solely explained in terms of classical general relativity, using the concept of lethesurface.

**Lethesurface and Entropy** — The entropy of a black hole can be put in relation to its $(r, t)$ sector (via its topology) by the relation $S = \chi A_s/4$, i.e., the entropy is proportional to the area of the lethesurface ($A_s$) associated to the black hole. As has been articulated by Susskind, “information equals area” for a black hole [30]. This “area” of course refers to that of the event horizon. Motivated by our results, perhaps the definition of “area” in Susskind’s equation can be generalised such that it holds not only for horizon area, but also for lethesurface area. This is the second important result of our Letter. By introducing the lethesurface of a black hole, we can now completely determine its entropy purely by probing its $(r, t)$ sector, without the need to access information from the rest of the black hole metric.

**Conclusions**. — In this Letter, we have introduced the concept of lethesurface, i.e., an abstract surface associated to each black hole, which is shown to contain geometrical information concerning the event horizon, despite originating from an orthogonal metric sector. This is a completely new feature in black hole theory and is a direct result of only classical general relativistic considerations. By investigating the $(r, t)$ sector of each black hole spacetime, this previously undefined surface is found to have equal surface area to the corresponding black hole’s horizon for uncharged, charged, and rotating black holes, for any parameter value up to and including extremality. A previously unexplained feature of extremal Kerr-Newman spacetimes, namely a link between horizon area and an effective AdS$_2$ cosmological constant, is fully explained by our results. The encoding of surface area, and therefore entropy, at the horizon of the $(r, t)$ sector as established in this Letter adds strong evidence in favour of the intrinsic (1+1)-dimensional nature of black holes and perhaps spacetime in general. An extension of the results presented in this Letter to those spacetimes with nonzero cosmological constant, as well as further probing of the role of information theory, will be the subject of future work.

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[1] S. W. Hawking, Comm. Math. Phys. 43, 199 (1975).