On the filtration of a free algebra by its associative lower central series.

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Abstract

This paper concerns the associative lower central series ideals $M_i$ of the free algebra $A_n$ on $n$ generators. Namely, we study the successive quotients $N_i = M_i/M_{i+1}$, which admit an action of the Lie algebra $W_n$ of vector fields on $\mathbb{C}^n$. We bound the degree $|\lambda|$ of tensor field modules $F_\lambda$ appearing in the Jordan-H"older series of each $N_i$, confirming a recent conjecture of Arbesfeld and Jordan. As an application, we compute these decompositions for small $n$ and $i$.

1 Introduction

Let $A_n = \mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle$ be the algebra over $\mathbb{C}$ of noncommutative polynomials with generators $x_1, x_2, \ldots, x_n$. We consider the lower central series of Lie ideals $L_i$ defined inductively by $L_1 = A_n$ and $L_{i+1} = [A_n, L_i]$. We denote by $M_i$ the two-sided ideal in $A_n$ generated by $L_i$, $M_i := A_nL_iA_n$. This is the same as the left-sided ideal $A_nL_i$. This follows from the identity below where $a, c \in A_n$ and $b \in L_{i-1}$:

$$[a, b]c = -a[b, c] + [ac, b].$$

In this paper, we study the Jordan-Hölder series of $N_i = M_i/M_{i+1}$. The Jordan-Hölder series give decompositions (in the Groethendieck group) of $N_i$ into sums of irreducible $W_n$-modules of $F_\lambda$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ for $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are non-negative integers; $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Here $W_n$, the Lie algebra of polynomial vector fields, acts on the $N_i$. The Jordan-Hölder constituents are $F_\lambda$.

We prove the following conjecture of Arbesfeld and Jordan on the upper bound of $|\lambda|$. 


Theorem 1.1. For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $N_m$ we have

$$|\lambda| \leq 2m - 2 + 2^{\frac{n-2}{2}}$$

For $m$ odd, this can be improved to

$$|\lambda| \leq 2m - 2.$$

We apply similar techniques to those of [AJ], [BJ], who studied the Lie quotients $B_i = L_i/L_{i+1}$. The proof of Theorem 1.1 depends on the following:

Theorem 1.2. $N_i = V \cdot L_i/(A \cdot L_{i+1} \cap V \cdot L_i)$ where $V$ is spanned by elements of $A_n$ of degree $\leq 1$.

Feigin and Shoikhet [FS] introduced the quotients $B_i = L_i/L_{i+1}$. They were further studied by Dobrovolska, Etingof, Kim and Ma [DE] [DKM] [EKM], as well as Arbesfeld and Jordan ([AJ]). In [DE] and [AJ], bounds $|\lambda|$ for the Jordan-Hölder series of $B_i$ were produced, and checked for small $i$ and $n$ by computer. We prove an analogous bound on $|\lambda|$ for the Jordan-Hölder series of $N_i$ and show that it grows linearly with $i$.

The structure of this paper is as follows. The following two subsections 1.1 and 1.2 contain a review of the representation theory of the Lie algebra of polynomial vector fields and the tensor field modules over $W_n$. Section 2 presents a proof of the main result. In Section 3 the Jordan-Hölder series for $N_m(A_n)$ for small $n$ and $m$ are computed.

1.1 Representation theory of the Lie algebra of polynomial vector fields

Let $W_n$ denote the Lie algebra of polynomial vector fields. As a vector space, we have:

$$W_n = \bigoplus_i \mathbb{C}[x_1, x_2, \ldots, x_n]\partial_i.$$ 

The Lie bracket is given by:

$$[p\partial_i, q\partial_j] = p \frac{\partial q}{\partial x_i} \partial_j - q \frac{\partial p}{\partial x_j} \partial_i.$$
According to [EKM], $W_n$ acts on each $N_i$, and as a $W_n$-module, $N_i$ has a Jordan-Hölder series whose simple quotients are of the form $F_\lambda$ (see section 2.3 below for the definition). Let $h_M$ be the Hilbert series for a graded vector space $M$. Then we have:

$$h_{F_\lambda} = \frac{p}{(1 - t_1)(1 - t_2) \cdots (1 - t_n)},$$

where $\deg p = \| \lambda \|$.

We will bound $h_{N_i} \prod_i (1 - t_i)$. This bound and the knowledge of the Hilbert series of $F_\lambda$ as above will allow us to control decompositions of the $N_i$ into the $F_\lambda$.

### 1.2 Tensor field modules over $W_n$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, where the $\lambda_i$ are nonnegative integers, let $V_\lambda$ be the irreducible representation of $\mathfrak{gl}_n$ of highest weight $\lambda$. We set $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Let $\mathcal{F}_\lambda$ be the space of polynomial tensor fields of type $V_\lambda$ on $\mathbb{C}^n$. As a vector space $\mathcal{F}_\lambda := \mathbb{C}[x_1, x_2, \ldots, x_n] \otimes V_\lambda$. It is known that $\mathcal{F}_\lambda$ is a representation of $W_n$ with action given by the standard Lie derivative formula for action of vector fields on covariant tensor fields (see [R]).

**Theorem 1.3.** [R] If $\lambda_1 \geq 2$, or if $\lambda = (1^n)$, then $\mathcal{F}_\lambda$ is irreducible. Otherwise, if $\lambda = (1^k, 0^{n-k})$, then $\mathcal{F}_\lambda$ is the space $\Omega^k = \Omega^k(\mathbb{C}^n)$ of polynomial differential $k$-forms on $\mathbb{C}^n$, and it contains a unique irreducible submodule which is the space of all closed differential $k$-forms.

Denote by $\mathcal{F}_\lambda$ the irreducible submodule of $\mathcal{F}_\lambda$, so that $\mathcal{F}_\lambda = \mathcal{F}_\lambda$ unless $\lambda = (1^k, 0^{n-k})$ for some $1 \leq k \leq n - 1$.

### 2 Proof of Conjecture 1.1

In this section we prove the bound of the Jordan-Hölder series of $N_i$ stated in Theorem 1.1.

We begin by proving Theorem 1.2. This result is also useful in simplifying computation of the Hilbert series of $N_i$. We fix $n$, and let $A$ denote $A_n$. For the proof of Theorem 1.2 we use the following Lemma:
Lemma 2.1. For \( b \in L_{i-1}, a, x, y \in A \), we have the following identity:

\[
yx[a, b] = x[ya, b] + y[xa, b] - [xya, b] \mod A L_{i+1}.
\]

Proof. Let \( b, a, x, y \) be as above. Then we have the following identities:

\[
[xa, b] = x[a, b] + a[x, b] \mod L_{i+1}, \tag{1}
\]

\[
[xya, b] = xy[a, b] + xa[y, b] + ya[x, b] \mod L_{i+1}. \tag{2}
\]

Multiplying (1) by \( y \), we get

\[
y[xa, b] = yx[a, b] + ya[x, b] \mod A L_{i+1}. \tag{3}
\]

Interchanging \( x \) and \( y \) in (3), we get

\[
x[ya, b] = xy[a, b] + xa[y, b] \mod A L_{i+1}. \tag{4}
\]

Now subtract (2) from (3) and (4). We get

\[
yx[a, b] = x[ya, b] + y[xa, b] - [xya, b] \mod A L_{i+1}. \tag{5}
\]

Taking \( y \) to be of degree one and applying the lemma repeatedly, we can reduce the term in front of the bracket of an arbitrary element of \( A \cdot L_i \) to something in \( V \cdot L_i \) by adding terms in \( A \cdot L_{i+1} \cap V \cdot L_i \), which is exactly Theorem 1.2.

2.1 Proof of Theorem 1.1

We first recall some definitions and results that we will need in the proof.

Definition 2.2. Let \( \bar{Z} \) be the image of \( A[A, [A, A]] \) in \( B_1 \), which was shown in \([\text{FS}]\) to be central in \( B \). We define \( \bar{B}_1 \) to be the quotient \( \bar{B}_1 = B / \bar{Z} \).

Now, recall the Feigin-Shoiket map \([\text{FS}]\).

Theorem 2.3. There is a unique isomorphism of algebras,

\[
\xi : \Omega^e_x \to A/A[A, [A, A]],
\]

\[
x_i \mapsto x_i.
\]

It restricts to an isomorphism \( \xi : \Omega^e_{*, \text{ex}} \to B_2 \), and descends to an isomorphism \( \xi : \Omega^e_x / \Omega^e_{*, \text{ex}} \to \bar{B}_1 \).

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For the second part of the proof of Theorem 1.1 we use some of the methods introduced in [AJ]. Recall the map from [AJ]:

**Theorem 2.4.** There is a surjective map $f_m : (\Omega^\text{ev})^\otimes m \to B_m$ such that

$$f_m(a_1, \ldots, a_m) = [\xi(a_1), [\xi(a_2), \ldots [\xi(a_{m-1}), \xi(a_m)]]],$$

where $\xi : \Omega^\text{ev} \to \hat{B}_1$ is the Feigin-Shoiket map from Theorem 2.3.

We recall that the algebra of zero forms is in fact $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and will be referred to as $S$. We also use that $f_m$ is surjective when restricted to $Y := (\Omega^0)^\otimes m - 2 \otimes (\bigoplus_{j+k\leq \frac{n-2}{2}} \Omega^{2j} \otimes \Omega^{2k})$ as shown by Bapat and Jordan. Using Theorem 1.2, we define a similar map for $Z = S \otimes Y = \Omega^0 \otimes (\Omega^0)^{\otimes m-2} \otimes (\bigoplus_{j+k\leq \frac{n-2}{2}} \Omega^{2j} \otimes \Omega^{2k}).$

Since $f_m|_Y$ is surjective, by Theorem 1.2, so is $\tilde{f}_m : Z \to N_m$.

$$a \otimes b \mapsto af_m(b)$$

where $a \in \Omega^0$, $b \in \Omega^0$. Here, on the right hand side, by abuse of notation, $a$ is $\xi(a)$.

Surjectivity of $\tilde{f}_m$ implies that the Jordan-Hölder series of $Z$ dominates the Jordan-Hölder series of $N_m$. We seek a large $W_n$-submodule $\tilde{I} \subseteq \text{Ker} \tilde{f}_m$, so that the Jordan-Hölder series of $Z/\tilde{I}$ still dominates the Jordan-Hölder series of $N_m$. Then for the proof of Conjecture 1.1 it will be sufficient to show that all $F_\lambda$ occurring in the Jordan-Hölder series of $Z/\tilde{I}$ satisfy the bound on $|\lambda|$.

As in [AJ], we define $R$ as

$$R := \mathbb{C}[x_1, x_2, \ldots, x_n]^{\otimes m} = \mathbb{C}[x_{1,1}, x_{2,1}, \ldots, x_{n,1}, x_{1,2}, \ldots, x_{n,m}].$$

Let $R' = S \otimes R = \mathbb{C}[x_{1,0}, x_{2,0}, \ldots, x_{n,0}, x_{1,1}, \ldots, x_{n,m}]$. Let the ideals $J_j$ of $R'$ for $0 \leq j \leq m-1$ be generated by $X_{i,j} = x_{i,j} - x_{i,j+1}$.

Let $\tilde{I} = J_0^2 + \sum_{i=1}^{m-2} J_i^3 + J_{m-1}^2$. Let $J = \sum_{i=1}^{m-2} J_i^3 + J_{m-1}^2$. We will show that $\tilde{I} = \text{Ker} \tilde{f}_m$.

**Lemma 2.5.** The ideal $J_0^2Z$ is a subset of the Kernel of $\tilde{f}_m$
Proof. This is straightforward from Lemma 2.1. $J_0^2Z$ is spanned by elements of the form $(x \otimes 1 \otimes 1 - 1 \otimes x \otimes 1) * (y \otimes 1 \otimes 1 - 1 \otimes y \otimes 1) * (1 \otimes a \otimes b)$ where * is the Fedosov product. We have that
\[
(x \otimes 1 \otimes 1 - 1 \otimes x \otimes 1) * (y \otimes 1 \otimes 1 - 1 \otimes y \otimes 1) * (1 \otimes a \otimes b) = xy \otimes a \otimes b - x \otimes ya \otimes b - y \otimes xa \otimes b + 1 \otimes xy a \otimes b.
\]
Consider the image of the map $\tilde{f}_m$.
\[
\tilde{f}_m(x y \otimes a \otimes b - x \otimes ya \otimes b - y \otimes xa \otimes b + 1 \otimes xy a \otimes b) = y[x[a, b] - x[ya, b] + y[xa, b] - [xya, b]].
\]
By Lemma 2.1 this is zero in $N_m$ so the spanning set of $J_0^2Z$ maps to zero. □

Arbesfeld and Jordan showed that $JZ \subseteq \text{Ker} f_m$. Lemma 2.5 implies that $J_0^2 \cdot Z \subseteq \text{Ker} \tilde{f}_m$. Thus we have $Z/IZ \rightarrow N_m$.

We finish the proof analogously to [AJ].

As in [AJ] $h_{Z/IZ} = h_{R'/I} \times h_{X'}$ where $h_{X'}$ is the Hilbert series of the generators over $R'$. Again by using the results from [BJ] we have
\[
h_{X'} = \sum_{j+k \leq 2\left\lfloor \frac{m-2}{2} \right\rfloor} \sigma_{2j} \times \sigma_{2k},
\]
where $\sigma_l = \sum_{i_1 \leq i_2 \leq \ldots \leq i_l} t_{i_1}t_{i_2} \ldots t_{i_l}$ are the elementary symmetric functions.

We can also compute
\[
h_{R'/I} = \frac{(1 + \sum t_i + \sum_{i \leq j} t_it_j)^{m-2}(1 + \sum t_i)^2}{(1 - t_1)(1 - t_2) \ldots (1 - t_n)}.
\]
We now use that
\[
h_{Z/IZ} = \frac{Q(t_1, t_2, \ldots, t_n)}{(1 - t_1)(1 - t_2) \ldots (1 - t_n)}.
\]
Thus $Q = h_{X'} \times (1 + \sum t_i + \sum_{i \leq j} t_it_j)^{m-2}(1 + \sum t_i)^2$. So the degree of $Q$ is $2m - 2 + 2\left\lfloor \frac{m-2}{2} \right\rfloor$.

For $m$ odd, we can improve this bound to $|\lambda| \leq 2m - 2$ using the following result of [BJ]:

**Theorem 2.6.** $M_jM_k \subset M_{j+k-1}$ whenever $j$ or $k$ is odd.
To apply this, we use the following argument suggested by Pavel Etingof. Notice that

\[ a[a_1, \ldots, [a_{m-1}, b[c, d]]] = a \sum_{S \subset [1, m-1]} (\prod_{i \in S} \text{ad} a_i)(b) \cdot (\prod_{i \notin S} \text{ad} a_i([c, d])) \]

If \( |S| = s \), then the corresponding term on the right hand side is in \( M_{s+1}M_{m-s+1} \). But one of the numbers \( s + 1 \) and \( m - s + 1 \) is odd, since their sum is \( m + 2 \) which is odd. So by theorem 2.6, all the terms on the right hand side are in \( M_{m+1} \), hence are zero in \( N_m \). So the left hand side is zero in \( M_m \). Now under the Feigin-Shoiket isomophism, \( A[A, A] \) corresponds to forms of degree 2 and higher, so in the proof of Theorem 2.4 we may replace \( \bigoplus_{j+k \leq \binom{m}{2}} \Omega^{2j} \otimes \Omega^{2k} \) by \( \Omega^0 \). We may then carry out the argument exactly as above, but with the dependence on \( n \) removed.

### 3 Conclusion

Now that we have found a lower bound of \( |\lambda| \) (Theorem 1.1), we may obtain the Jordan-Hölder series for several values of \( m, n \) for \( N_m(A_n) \) via MAGMA computation. For example, we have the following results, in which we will denote each instance of \( F_\lambda \) by the \( n \)-tuple \( \lambda \) for economy of notation.

**Theorem 3.1.** The Jordan-Hölder series for \( N_m(A_2) \) for \( 3 \leq m \leq 7 \) are:

- \( N_3 = (2, 1) + (2, 2) \).
- \( N_4 = (3, 1) + (3, 2) + (3, 3) \).
- \( N_5 = (4, 1) + (3, 2) + 2(4, 2) + (4, 3) + (4, 4) \).
- \( N_6 = (5, 1) + (4, 2) + (3, 3) + 2(5, 2) + 2(4, 3) + 2(5, 3) + (5, 4) + (5, 5) \).
- \( N_7 = (6, 1) + 2(5, 2) + 2(4, 3) + 3(6, 2) + 3(5, 3) + 3(4, 4) + 3(6, 3) + 2(5, 4) + 2(6, 4) + (6, 5) + (6, 6) \).

These decompositions were conjectured by Arbesfeld and are now theorems.

**Theorem 3.2.** The Jordan-Hölder series for \( N_m(A_3) \) for \( m = 3, 4 \) are:

- \( N_3 = (2, 1, 0) + (2, 2, 0) \).
Theorem 3.3. The Jordan-Hölder series for $N_m(A_4)$, $m = 3, 4$ are:

- $N_3 = (2, 1, 0, 0) + (2, 2, 0, 0)$.
- $N_4 = (3, 3, 0, 0) + (3, 2, 0, 0) + (3, 1, 1, 1) + (3, 1, 1, 0) + (3, 1, 0, 0) + (2, 2, 1, 1) + (2, 2, 0, 0) + (2, 1, 1, 1) + (2, 1, 1, 0)$.

Note that the decomposition of $N_3$ was also computed in [EKM].

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