Distributional Soft Actor Critic for Risk Sensitive Learning

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Abstract

Most of reinforcement learning (RL) algorithms aim at maximizing the expectation of accumulated discounted returns. Since the accumulated discounted return is a random variable, its distribution includes more information than its expectation. Meanwhile, entropy of policy indicates its diversity and it can help improve the exploration capability of algorithms. In this paper, we present a new RL algorithm named Distributional Soft Actor Critic (DSAC), combining distributional RL and maximum entropy RL together. Taking the randomness both in action and discounted return into consideration, DSAC over performs the state-of-the-art baselines with more stability in several continuous control benchmarks. Moreover, distributional information of returns can also be used to measure metrics other than expectation, such as risk-related metrics. With a fully parameterized quantile function, DSAC is easily adopted to optimize policy under different risk preferences. Our experiments demonstrate that with distribution modeling in RL the agent performs better both for risk-averse and risk-seeking control tasks.

1. Introduction

In the past few years, model-free deep reinforcement learning (Sutton & Barto, 2018; François-Lavet et al., 2018) has emerged to be an important and powerful paradigm that has been increasingly applied in several application domains (Mnih et al., 2013; Silver et al., 2016; Gu et al., 2017; Haarnoja et al., 2018b). Such well-founded optimism stems from the large performance gains that can be potentially unlocked by using deep neural networks for function approximation in RL—either to parameterize value functions or policies—and to harness the representation power therefore brought forth. However, simply throwing in “deep function approximators” is far from sufficient, and building efficient and effective RL algorithms often requires sophisticated thought, one that integrates function approximation well with the unique aspects of RL.

One important such aspect is exploration: since current actions influence the future state and hence the future rewards of the underlying Markov decision process (Puterman, 1994; Bertsekas, 1995), effective exploration is a key aspect of RL algorithms. In the flourishing literature that has been developed for model-free RL, randomness in action space is widely employed as a primitive to balance exploration and exploitation. On-policy algorithms, such as A3C (Mnih et al., 2016), TRPO (Schulman et al., 2015) and PPO (Schulman et al., 2017), use stochastic action space and optimize the parameters by policy gradient. However, on-policy algorithms are not data-efficient as they require new experience for policy evaluation. On the other hand, off-policy methods can reuse past experience and hence improve data efficiency. However, most off-policy methods, such as DQN (Mnih et al., 2015), DDPG (Van Hasselt et al., 2016a) and TD3 (Fujimoto et al., 2018), inherit the simple ϵ-greedy strategy from Q-learning by adding a small noise to a deterministic policy for exploration, where the scale of action perturbation is hard to choose in practice. To improve robustness of off-policy methods, SVG (Heess et al., 2015) introduces re-parameterized stochastic policy, while its objective is still standard maximum expected discounted returns.

Recently, Haarnoja et al. (2018a) take a principled approach for encouraging the randomness (and hence diversity) of actions by formally formulating the entropy of a policy into the objective. This maximum entropy approach is founded on theoretical principles and has been applied to many other contexts as well, such as in inverse reinforcement learning (Ziebart et al., 2008; Zhou et al., 2018) and optimal control (Todorov, 2008; Rawlik et al., 2013). Haarnoja et al. (2018a) propose a novel continuous action space algorithm called soft actor-critic (SAC), which is more robust than former methods and has achieved the state-of-the-art performance in many continuous action control tasks. More specifically, SAC computes an optimal policy by minimizing the KL-divergence between the action distribution and the
exponential form of the soft action-value function. From the view of probabilistic graphical models, the objective function of maximum entropy RL can be derived by minimizing the KL-divergence between experience distribution of the current policy and the optimal path distribution (Levine, 2018).

Concurrently, randomness in RL has been harnessed in other ways to improve performance. In particular, distributional RL (Jaquette et al., 1973; Sobel, 1982; White, 1988; Morimura et al., 2010; Bellemare et al., 2017), another approach that has gained momentum recently, proposes to take into account the whole distribution of value functions, rather than just the expectation. Since the whole distribution contains much more information beyond the first moment, one can leverage it to make more informed decisions that ultimately lead to superior rewards. Incidentally, latest experiments show that similar encouraging mechanisms also exist in human brains (Dabney et al., 2020). An important challenge in this approach lies in approximating the distribution of value functions well. To address this challenge, a series of recent works has studied this problem and provided progressively more effective ways to achieve the goal. Categorical DQN (C51) (Bellemare et al., 2017) uses a set of discrete fixed values to estimate possibilities and outperforms DQN in Atar 2600 games. QR-DQN (Dabney et al., 2018b) replaces fixed values with fixed possibilities, and introduced quantile regression for learning quantile values. Further, Dabney et al. extend QR-DQN with sampled quantile fractions in IQN (2018a), which is able to approximate the full quantile function with infinite samples. To avoid random samples, FQF (Yang et al., 2019) parameterizes both the quantile values and quantile fractions, which provides a fully-parameterized tool for distribution approximation. In continuous action setting, D4PG (Barth-Maron et al., 2018) combines DDPG (Van Hasselt et al., 2016a) and C51 with distributed training, showing the potential of distributional RL in continuous action tasks.

These two approaches–SAC and distributional RL–when considered together, reveal the components that each misses: SAC cannot take advantage of the information beyond the first moment of values and distributional RL does not effectively encourage the diversity of actions for efficient exploration. With the benefit of hindsight, it is natural to raise the following question: Could randomness be exploited at a deeper level that simultaneously encourages action diversity for better exploration and leverages the distributional information of value function? It turns out, as we show in this paper, that the answer is in the affirmative.

1.1. Our Contributions

Our contributions are threefold. First, we design a novel algorithm, called distributional soft actor critic (DSAC), that combines the maximum entropy RL approach with the distributional RL approach. DSAC can be simultaneously understood as an approach that leverages distributional information of value functions in SAC and as an approach that encourages action randomness through the entropy surrogate in distributional RL. Our algorithm consists of several aspects. We define a distributional soft Bellman operator, and modify FQF to estimate the soft discounted return for continuous action tasks. With a fully parameterized quantile function, we update a reparameterized policy with a risk distorted expectation of action-value distribution. The full architecture, together with how efficient training can be performed, is discussed in detail in Section 3.

Second, we move beyond the traditional RL objective of minimizing the sum of expected future (discounted) rewards and consider risk-sensitive learning, which can also influence the algorithms exploration. By adapting DSAC to this setting, we provide a unified framework that can simultaneously handle several risk functionals–CPW, CVaR, Wang, among others (see Section 4 for more risk metrics and details)—that reflect the decision maker’s varied risk preferences and learn the corresponding policies. Risk-sensitive learning is an important topic that has been explored in the literature: risk measures, such as variance (Sobel, 1982; Xia, 2016; Prashanth & Fu, 2018), CVaR (Chow et al., 2015) and CPT (Tversky & Kahneman, 1992), all involve higher moments of the underlying value distribution, which does not satisfy the Bellman equation. However, prior works can only handle single risk measures and only on small scales, while, we provide a unified framework that can perform risk-sensitive learning under various metrics on large-scale RL problems.

Third, we perform extensive experimental evaluations (Section 5), comparing our algorithm to the state-of-the-art model-free RL algorithms (on continuous action space). Our algorithm demonstrates superior performance in standard benchmark tasks (using MuJoCo with OpenAI Gym) and outperforms the current best algorithm SAC, thereby pushing the frontier of the performance of model-free RL.

2. Background and Related Work

We consider a standard Markov decision process (MDP) modeled by the tuple \((\mathcal{S}, \mathcal{A}, R, \mathcal{P}, \gamma)\) (Puterman, 1994), where \(\mathcal{S}\) and \(\mathcal{A}\) denote state space and continuous action space, \(\mathcal{P} : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \rightarrow [0, \infty)\) represents the unknown transition probability density of next state given current state and action, \(R : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}\) denotes the reward function and \(\gamma \in (0, 1)\) denotes the discount factor. Let policy \(\pi\) denote a mapping from each state to a probability distribution over \(\mathcal{A}\) and \(\Pi\) denote the set of policies.

The standard reinforcement learning aims at maximizing
the expectation of discounted rewards

\[ J(\pi) = \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \right], \tag{1} \]

given the initial state \( s_0 \) obeying a distribution \( \xi \). For a policy \( \pi \in \Pi \), action-value function \( Q^\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R} \) is defined as the expectation of discounted rewards,

\[ Q^\pi(s, a) := \mathbb{E}_\xi \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) \right], \]
\[ a_t \sim \pi(\cdot|s_t), s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t), s_0 = s, a_0 = a. \]

The Bellman operator \( T^\pi \) and Bellman optimality operator \( T^* \) are defined as

\[ T^\pi Q(s, a) := \mathbb{E}[R(s, a) + \gamma \mathbb{E}_{\mathcal{P},\pi}[Q(s', a')]], \]
\[ T^* Q(s, a) := \mathbb{E}[R(s, a) + \gamma \max_{a'} \mathbb{E}_{\mathcal{P}}[Q(s', a')]]. \tag{2} \]

Repeated application of either operator from some initial \( Q_0 \) converges exponentially to its fixed point \( Q^\pi \) or \( Q^* \) as both operators are contraction (Bertsekas & Tsitsiklis, 1995).

### 2.1. Maximum Entropy Reinforcement Learning

Different from the standard RL, maximum entropy RL maximizes the sum of rewards while keeping the entropy of policy as large as possible. The objective function of maximum entropy RL is

\[ J(\pi) = \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) + \alpha \mathcal{H}(\pi(\cdot|s_t)) \right]. \tag{3} \]

SAC Haarnoja et al. (2018a) proposed soft actor critic (SAC) for learning the policy in continuous action space with maximum entropy objective function. By adding entropy to the standard Bellman operator \( T^* \), the authors defined the soft Bellman operator \( T_S^* \) as

\[ T_S^* Q(s, a) := \mathbb{E}[R(s, a) + \gamma \mathbb{E}_{\mathcal{P},\pi}[Q(s', a') - \alpha \log \pi(a'|s')]]. \tag{4} \]

SAC proceeds by repeatedly applying soft policy evaluation, followed by soft policy improvement, until convergence occurs. Similar to the standard policy evaluation, the soft policy evaluation is completed by repeating \( T_S^* \). The soft policy improvement is achieved by minimizing the Kullback-Leibler divergence between the policy distribution and exponential form of soft action-value function,

\[ \pi_{\text{new}} = \arg \min_{\pi' \in \mathcal{H}} D_{\text{KL}} \left( \pi' (\cdot|s_t) \parallel \frac{1}{Z_{\pi_{\text{old}}}} \exp \left( \frac{Q_{\pi_{\text{old}}}(s_t, \cdot)}{\Delta_{\pi_{\text{old}}}(s_t)} \right) \right), \tag{5} \]

where \( \Delta_{\pi_{\text{old}}} \) is the partition function that normalizes the distribution.

To solve the minimization problem in Equation 5, SAC used a reparameterized policy neural network \( f_{\phi}(s, \epsilon) \), where \( \phi \) is parameters of the network and \( \epsilon \) is a noise vector sampled from any fixed distribution like standard spherical Gaussian. Thus the original problem can be solved by gradient descent with the objective function,

\[ J_\phi(\phi) = \mathbb{E}_{s_t \sim D} \left[ \log \pi_{\phi}(s_t, \epsilon_t | s_t) - Q_\phi(s_t, f_{\phi}(s_t, \epsilon_t)) \right], \tag{6} \]

where \( Q_\theta \) is a differential function parameterized with \( \theta \) and \( D \) is the transitions replay buffer.

### 2.2. Distributional Reinforcement Learning

While maximum entropy RL takes the randomness in action space into consideration, distributional RL captures another important nature of MDP, which is the randomness in accumulated discounted returns. The objective function of distributional RL is aligned with traditional RL to maximizing the discounted return in Equation 1.

To measure the distance between value distributions, \( \rho \)-Wasserstein distance is introduced as

\[ W_p(U, V) = \left( \int_0^1 |F_U^{-1}(\omega) - F_V^{-1}(\omega)|^p \, d\omega \right)^{1/p}, \tag{7} \]

where \( U, V \) are two random variables and \( F_U, F_V \) are their cumulative distribution functions for \( p < \infty \). Let \( Z \) denote the value distribution space. For \( Z_1, Z_2 \in Z \), a metric \( d_p \) over value distributions is defined as

\[ d_p(Z_1, Z_2) = \sup_{s, a} W_p(Z_1(s, a), Z_2(s, a)). \tag{8} \]

The distributional Bellman operator \( T_D^* \) is defined as

\[ T_D^* Z(s) := R(s, a) + \gamma Z(s' + \epsilon' - \alpha \log \pi(a'|s')), \tag{9} \]

where \( s' \sim \mathcal{P}(\cdot|s, a), \epsilon' \sim \mathcal{P}(\cdot|s') \). This operator is a \( \gamma \)-contraction in \( d_p \). While this operator is no longer a contraction of any distributional metric in control setting, convergence to the optimal action-state value \( Q^* \) can still be achieved by taking expectation \( \mathbb{E}[Z] \) (Bellemare et al., 2017).

While the distributional RL has good convergence properties, approximation of value distribution is a main challenge in practice. Below is a sequence of recent works that presents different approaches for approximating the value distribution.

C51 Bellemare et al. (2017) first proposed the distributional view on RL and established the theory of distributional RL. They replaced the outputs of DQN with \( N \) fixed value atoms for approximation of value distribution and proposed Categorical DQN. The 51-atom version, named as C51, outperformed DQN and attained state-of-the-art performance in Atari games.

IQN To improve the distribution approximation, quantile regression DQN (QR-DQN) (Dabney et al., 2018b) replaced
value atoms in C51 with quantile regression, which means using the fixed quantile fractions instead of fixed values for better approximation of value distribution. The quantile network is trained by Huber quantile regression loss (Huber, 1964) with threshold $\kappa$,

$$
\rho^\kappa_i(\delta_{ij}) = |\tau - \mathbb{I}\{\delta_{ij} < 0\}| \frac{\mathcal{L}_\kappa(\delta_{ij})}{\kappa}, \quad \text{with}
$$

$$
\mathcal{L}_\kappa(\delta_{ij}) = \begin{cases} 
\frac{1}{2} \delta_{ij}, & \text{if } |\delta_{ij}| \leq \kappa \\
\kappa (|\delta_{ij}| - \frac{1}{2}\kappa), & \text{otherwise.}
\end{cases}
$$

(10)

With $F_Z^{-1}(\tau)$ denoting the quantile function at fraction $\tau \in [0, 1]$ for the value random variable $Z$, the pairwise TD-errors between two quantile fractions are calculated by

$$
\delta_{ij} = \tau + \gamma F_Z^{-1}(\tau_i) - F_Z^{-1}(\tau_j).
$$

(11)

Dabney et al. (2018a) extended the fixed quantile fractions to uniformed samples and proposed implicit quantile value network (IQN). With infinite sampled quantile fractions, IQN is able to approximate the full quantile function.

FQF Instead of sampling the quantile fractions uniformly, Yang et al. (2019) proposed the fully parameterized quantile function (FQF) to parameterize both the quantile values and quantile fractions. Similar to IQN, FQF updated the quantile value network with quantile regression,

$$
J_{F_Z^{-1}}(\theta) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \rho^\kappa_i(\delta_{ij}),
$$

(12)

where $\delta_{ij}$ is the TD-error with $\hat{\tau}_i$ and $\hat{\tau}_j$. In addition to a quantile value network $F_Z^{-1}(\tau)$, FQF used a fraction proposal network $P_{\theta}(s,a)$ to propose the most suitable fractions. They gave the gradient of $W_1$ with respect to $\tau$,

$$
\frac{\partial W_1}{\partial \tau_i} = 2 F_Z^{-1}(\tau_i) - F_Z^{-1}(\hat{\tau}_i) - F_Z^{-1}(\hat{\tau}_{i-1}),
$$

(13)

where $W_1$ denotes 1-Wasserstein error between approximated quantile function and the true quantile function and $\hat{\tau}_i = \frac{\tau_i + \tau_{i+1}}{2}$.

3. A Distributional Soft Critic Approach

In this section, we present our algorithm as Distributional Soft Actor Critic (DSAC). We start with theoretical analysis to combine maximum entropy RL and distributional RL together, and propose an abstract algorithm framework as soft distributional policy iteration. Next we discuss DSAC implementation with neural networks, and introduce a modified version of FQF for soft action-value distribution approximation. Three networks are used in our algorithm: quantile value network $F_Z^{-1}(\theta)$, fraction proposal network $P_{\phi}(s,a)$ and policy network $\pi_{\phi}(s)$. Networks architecture is shown in Figure 1 and full algorithm is shown in Algorithm 1.

3.1. Distributional Soft Policy Iteration

In DSAC, we optimize the policy to maximize objective function in Equation 3, aligned with maximum entropy RL. For a policy $\pi \in \Pi$, we define the soft action-value distribution $Z^\pi : S \times A \rightarrow \mathbb{Z}$ as

$$
Z^\pi(s,a) := \sum_{t=0}^{\infty} \gamma^t [R(s_t, a_t) - \alpha \log \pi(a_{t+1}|s_{t+1})],
$$

(14)

$$
a_t \sim \pi(\cdot|s_t), s_{t+1} \sim \mathcal{P}(\cdot|s_t, a_t), s_0 = s, a_0 = a.
$$

We define the distributional soft Bellman operator $\mathcal{T}^\pi_{DS}$ as

$$
\mathcal{T}^\pi_{DS}Z(s,a) := R(s,a) + \gamma \left[Z(\cdot|s,a') - \alpha \log \pi(\cdot|s')\right].
$$

(15)

where $s' \sim \mathcal{P}(\cdot|s,a)$, $a' \sim \pi(\cdot|s')$. With the straightforward combination of Equation 4 and Equation 9, the new defined operator inherits the excellent properties from original operators.

**Lemma 1** $\mathcal{T}^\pi_{DS} : \mathbb{Z} \rightarrow \mathbb{Z}$ is a $\gamma$-contraction in $\hat{d}_p$.

With the new defined operator, an abstract algorithm similar to soft policy iteration can be obtained, which we name distributional soft policy iteration. The algorithm consists of two parts: distributional soft policy evaluation and distributional soft policy improvement. For any policy $\pi$, soft action-value distribution $Z^\pi$ is obtained by repeatedly applying $\mathcal{T}^\pi_{DS}$.

**Lemma 2 (Distributional Soft Policy Evaluation)** For any mapping $Z^k : S \times A \rightarrow \mathbb{Z}$, we define $Z^{k+1} = \mathcal{T}^\pi_{DS}Z^k$. The sequence $Z^k$ will converge to $Z^\pi$ as $k \rightarrow \infty$.

With a well evaluated value distribution, policy improvement is achieved by minimizing Equation 5.

**Lemma 3 (Distributional Soft Policy Improvement)** For all $(s,a) \in S \times A$, let $Q^\pi(s,a) = \mathbb{E}[Z^\pi(s,a)]$. With $\pi_{\text{old}} \in \Pi$ and $\pi_{\text{new}}$ as the solution of problem defined in Equation 5, we have $Q^\pi_{\text{old}}(s,a) \leq Q^\pi_{\text{new}}(s,a)$.

By repeatedly applying distributional soft policy evaluation and distributional soft policy improvement, the policy will finally converge to the optimum. Since the policy improvement is similar to SAC, how to estimate the soft action-value distribution becomes the main challenge.

See the supplementary material for proofs of above lemmas.

3.2. Training Networks

In DSAC, we use the similar idea of FQF to approximate the soft action-value distribution. Let $F_Z^{-1}$ denote the quantile function (Müller, 1997) for random variable $Z$, which represents the inverse function of cumulative distribution function $F_Z(z) = \mathbb{P}(Z < z)$. By definition we have

$$
F_Z^{-1}(\tau) := \inf\{z \in \mathbb{R} : \tau \leq F_Z(z)\},
$$

(16)

where $\tau \in [0, 1]$. Networks architecture is shown in Figure 1 and full algorithm is shown in Algorithm 1.
where $\tau$ denotes the quantile fraction.

FQF parameterizes a quantile function with $N$ adjustable quantile values and $N$ adjustable quantile fractions. The distribution of the random return is approximated by a weighted mixture of $N$ Diracs,

$$Z_{\phi, \tau}(s, a) := \sum_{i=0}^{N-1} (\tau_{i+1} - \tau_i) \delta_{\phi_i(s, a, \tau_i)},$$  \hspace{1cm} (17)

where $\phi_i$ and $\tau_i$, $i = 0, \ldots, N - 1$ denote quantile values and quantile fractions respectively. The approximated quantile function is represented as a staircase function supported by $\phi$ and $\tau$,

$$F_{\phi, \tau}^{-1}(\omega) = \phi_0 + \sum_{i=0}^{N-1} (\phi_{i+1} - \phi_i) H(\omega - \tau_{i+1}),$$

where $H$ denotes the Heaviside step function.

The quantile fractions $\tau_i \in [0, 1], i = 0, \ldots, N$ are given by a fraction proposal network $P(s, a)$ parameterized by $\phi$, satisfying $\tau_0 = 0, \tau_N = 1$ and $\tau_i < \tau_j$ for $\forall i < j$. $P(s, a)$ takes $s, a$ as the input and outputs $q_i \in (0, 1), i = 0, \ldots, N - 1$ after a softmax layer. Let $\tau_i = \sum_{j=0}^{i-1} q_j, i = 1, \ldots, N$ and all constraints are satisfied. $P(s, a)$ is trained by minimizing the 1-Wasserstein error between the continuous quantile function $F_{\phi, \tau}^{-1}$ and the Heaviside step function $F_{\phi, \tau}^{-1}$.

$$J_F(\phi) = W_1(Z, \tau) = \sum_{i=0}^{N-1} \int_{\tau_i}^{\tau_{i+1}} |F_{\phi, \tau}^{-1}(\omega) - F_{\phi, \tau}^{-1}(\tau_i)| \, d\omega,$$

For training stability, we introduce target networks to calculate target value in Equation 18 and soft-update the parameters towards current networks.

In FQF, the objective function of quantile values is given by Equation 12, which is a straightforward variant of IQN. We argue that FQF ignores the fact that the target distribution is approximated by proposed fractions $\tau_i$, no longer by uniform samples. Thus, we modify the objective function as

$$J_{F_{\phi, \tau}}(\phi) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (\tau_{i+1} - \tau_i) \rho_{\phi_j}(\delta_{ij}),$$

where each $\rho_{\phi_j}$ is weighted by the target distribution fractions $\tau_{i+1} - \tau_i$.

With the parameterized quantile function, it is easy to adapt SAC to train the policy network. In DSAC, we train a reparameterized policy network $\pi_\phi(a|s)$ by solving Equation 5 with minimizing Equation 6. $Q$ function in Equation 6 is
We extend the idea of double learning to DSAC. The most
$
Q(s, a) = \sum_{i=0}^{N-1} (\tau_{i+1} - \tau_i) F_{Z, \theta}^{-1}(\hat{\tau}_i). \tag{20}
$

3.3. Double Learning

According to the maximization operation when calculating
the target value with Bellman optimal equation, RL
algorithms based on Q-learning share the same problem
of overestimation (Van Hasselt et al., 2016b). In the con-
tinuous action control algorithms, maximization bias still
exists without explicit maximization operations in the update
equations (Fujimoto et al., 2018). Double learning suggests
overcome the overestimation with two action-value func-
tions. In TD3 and SAC, they use two Q networks and use
the minimization of the outputs as the target,

$y_t = r_t + \gamma \min_{k=1,2} Q_k(s_{t+1}, \mu(s_{t+1}))$, \tag{21}

where $\mu$ is a deterministic policy. Note that the idea of
double learning could also be used in the update of policy
network.

We extend the idea of double learning to DSAC. The most
straightforward way of double learning is that replacing
the target distribution $F_{Z, \theta}^{-1}(\hat{\tau})$ in Equation 18 with
the minimum of two quantile value networks outputs
$\min_{k=1,2} F_{Z, \theta_k}^{-1}(\hat{\tau})$. However, this way is too strict to slow
down training. Instead, we are aligned with TD3 and SAC
to keep the expectation of two output distributions as the
minimum of them. For two quantile fractions $\hat{\tau}_i$ and $\hat{\tau}_j$, the
TD-error is redefined by

$y_{t,k} = F_{Z, \theta_k}^{-1}(\hat{\tau}_i) - \left( Q_k - \min_{k=1,2} Q_k \right) + \gamma \min_{k=1,2} Q_k(s_{t+1}, \mu(s_{t+1}))$,

$\delta_{ij} = r_t + \gamma (y_{t,k} - \alpha \log \pi(a_{t+1}|s_{t+1})) - F_{Z, \theta_k}^{-1}(\hat{\tau}_j)$,

where $Q_k := Q_k(s_{t+1}, a_{t+1})$ is calculated by Equation 20.

3.4. Implementation Details

In order to embed quantile fraction $\tau$ into the quantile value
network, we borrow the idea of implicit representations from
IQN. With the element-wise (Hadamard) product (denoted
as $\odot$) of state feature $\nu(s, a)$ and embedding $\varphi(\tau)$, the
quantile values are given by $F_{Z, \theta}^{-1}(\tau) = F_{Z, \theta}^{-1}(\nu(s, a) \odot \varphi(\tau))$.

After studying different ways for embedding $\tau$, IQN suggests
an embedding formula with $\cos(\cdot)$,

$\varphi_j(\tau) := g \left( \sum_{i=1}^{n} \cos(i\pi\tau)w_{ij} + b_j \right)$, \tag{22}

where $w_{ij}, b_j$ are network parameters, and $g$ is the nonlinear
activation function which is ReLU in IQN and FQF. In our
practice, we find that using the activate function ReLU will
make training unstable as the activation value $\nu \odot \varphi$ may
contain too many zeros after multiplication. Therefore, we
replace ReLU in Equation 22 with LeakyReLU to avoid
gradient vanishing while keeping the activation values more
meaningful. Note that we also change activation function
LeakyReLU in other parts of networks, including the
quantile fraction network and the policy network. Moreover,
we implement layer normalization (Jimmy Lei Ba, 2016) to
the hidden activation of all networks.

4. Risk-sensitive Learning

The distribution of returns contains more information than
the expectation, which gives the ability to optimize policy
under different measures, especially risk metrics. In this
section, we extend the basic DSAC to optimize policy under
risk measures. By introducing some distortion functions for
quantile fractions, DSAC can train the policy for any risk
appetite flexibly.

4.1. Policy Learning under Risk Measures

Ability of optimizing policy under different risk appetites is
one of the most attractive parts of distributional RL. Most
of the risk measures are hard to estimated directly, but with
distribution of return, it is able to obtain value defined under
risk by taking the distorted expectation.

With $\beta : [0, 1] \rightarrow [0, 1]$ denoting a distortion risk function,
the distorted expectation of $Z$ is defined as

$Q_{\beta}(s, a) = \int_{0}^{1} F_{Z}^{-1}(\tau) d\beta(\tau)$, \tag{23}

which is estimated by sampling in IQN (Dabney et al.,

Algorithm 2 DSAC update under risk measure

| Parameter: $N, \kappa$ |
|-----------------------|
| Input: $s, a, r, s', \gamma \in (0, 1)$ |
| $\tau \leftarrow P_\phi(s, a), a' \sim \pi_\theta(\cdot|s'), r' \leftarrow P_\psi(s', a')$ |
| # Compute quantile regression loss |
| for $i = 0$ to $N - 1$ do |
| for $j = 0$ to $N - 1$ do |
| Calculate TD-error $\delta_{ij}$ for $\hat{\tau}_i, \hat{\tau}_j$ with Equation 18 |
| end for |
| end for |
| $J_{F_{Z, \theta}^{-1}}(\theta) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} (\tau_{i+1} - \tau_i) p_{\tau}^\kappa(\delta_{ij})$ |
| $\frac{\partial W_{\kappa}}{\partial \tau} = 2 F_{Z}^{-1}(\tau_i) - F_{Z}^{-1}(\hat{\tau}_i) - F_{Z}^{-1}(\hat{\tau}_{i-1})$ |
| $a \sim \pi_\theta(\cdot|s), \tau \leftarrow P_\psi(s, a)$ |
| # Compute risk distorted expectation of value distribution |
| $Q_{\beta}(s, a) = \sum_{i=0}^{N-1} (\beta^{-1}(\tau_{i+1}) - \beta^{-1}(\tau_i)) F_{Z, \theta}^{-1}(\hat{\tau}_i)$ |
| $J_{\pi}(\phi) = \log(\pi_\theta(a|s)) - Q_{\beta}(s, a)$ |
| Update $\theta$ with $\nabla J_{F_{Z, \theta}^{-1}}(\theta)$ |
| Update $\phi$ with $\nabla J_{\pi}(\phi)$ |
| Update $\psi$ with $\nabla W_{\kappa}$ |
Since DSAC has a powerful approximation of value distribution, we avoid random samples and estimate distorted expectation with the full parameterized quantile function,

\[ Q_\beta(s,a) := \mathbb{E}_{\tau \sim U(0,1)} \left[ F_{Z,\theta}^{-1}(\tau|s,a) \right]. \quad (24) \]

Thus the expectation of the soft action-value distribution defined in Equation 20 can be viewed as risk-neutral distorted expectation with \( \beta(\tau) = \tau \). We keep the proposed fractions unchanged and distort the weights to make sure quantile values well estimated. Note that as we take the distortion expectation over soft action-value distribution which contains not only reward but also entropy, we change the original meaning of risk measures. Algorithm 2 shows the full algorithm workflow.

4.2. Distortion Functions

We collect a number of risk measures to evaluate the performance of DSAC. Risk measures and corresponding distortion function \( \beta \) with suggested parameters are shown in Figure 2.

**CPW** Cumulative probability weighting parameterization proposed in cumulative prospect theory (Tversky & Kahneman, 1992),

\[ \beta(\tau) = \frac{\tau^\eta}{(\tau^\eta + (1 - \tau)^\eta)^{\frac{1}{\eta}}} . \quad (26) \]

Tversky & Kahneman found \( \eta = 0.71 \) matches most closely human subjects. It is interesting that CPW behaves as risk-seeking near two ends when \( \tau \to 0 \) or \( \tau \to 1 \) and behaves as risk-averse in the middle part.

**Wang** A distortion risk measure proposed by Wang (2000),

\[ \beta(\tau) = \Phi(\Phi^{-1}(\tau) + \eta) . \quad (27) \]

where \( \Phi \) and \( \Phi^{-1} \) are the standard Normal cumulative distribution function and its inverse, \( \eta > 0 \) for risk-seeking and \( \eta < 0 \) for risk-averse.

**CVaR** Conditional Value at Risk. Corresponding distortion function is defined as

\[ \beta(\tau) = \eta \tau, \quad (28) \]

where \( \eta \in (0, 1) \) denotes confidence level. It is a common risk measure in financial engineering, which means the losses in worst cases. While CVaR has been widely studied in RL (Chow et al., 2015; Prashanth & Fu, 2018), estimation of CVaR is hard and data-inefficient, as only \( \eta \) of data could affect value of CVaR.

**Pow** We consider a simple power function proposed in IQN,

\[ \beta(\tau) = \tau^{\eta}, \quad (29) \]

where \( \eta < 0 \) for risk-averse polices. Pow has a similar curve shape with Wang but more aggressive when \( \tau \to 0 \).

Some risk distortion functions mentioned above are meaningful either at \( \eta > 0 \) or at \( \eta < 0 \). Furthermore, we also consider their dual distortion function \( \tilde{\beta}(\tau, \eta) = 1 - \beta(1 - \tau, \eta) \) (Balbás et al., 2009). For simplicity, we denote \( \beta(\tau, -\eta) = \tilde{\beta}(\tau, \eta) \) when \( -\eta \) has no definition. For example, \( CVaR(\tau, -\eta) = 1 - CVaR(1 - \tau, \eta) \) improves the best cases for risk-seeking while original definition \( CVaR(\tau, \eta) \) is focused on the worst cases for risk-averse.

5. Experiments

To evaluate our algorithm performance, we design a series of experiments to compare DSAC with current state of the art for continuous control RL algorithms and test DSAC in different risk scenarios. We implement our algorithm based on rlkit (Pong et al., 2019), a well-developed PyTorch (Paszke et al., 2019) RL toolkit. All experiments are performed on NVIDIA Tesla P100 16GB graphics cards.

5.1. Comparison with State-of-the-art

We evaluate our algorithm with a physics engine suite MuJoCo (Todorov et al., 2012) with OpenAI Gym (Brockman et al., 2016) as the interface. We compare with two baseline methods viewed as the state-of-the-art for continuous control tasks: twin delayed deep deterministic policy gradient algorithm (TD3) (Fujimoto et al., 2018) and soft actor critic (SAC) (Haarnoja et al., 2018a). We do not compare with another distributional RL algorithm D4PG (Barth-Maron et al., 2018), as its performance improvement comes from distributed training and prioritized replay. We run 4 different tasks over 6 random seeds and show the results in Figure 3. DSAC outperforms other baselines and shows more stability over different random seeds in training. In the experiments, we use our implementation of TD3 and SAC based on their published versions. Except for different random seeds and
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![Learning curves for the continuous control benchmarks in MuJoCo.](image)

Figure 3. Learning curves for the continuous control benchmarks in MuJoCo. Each learning curve is averaged over 6 different random seeds and shaded by the half of their variance. All curves are smoothed for better visibility.

![Learning curves for BidepalWalkerHardcore-v2.](image)

Figure 4. Learning curves for BidepalWalkerHardcore-v2. The yellow car should approach the green goal without collision with other blue cars. Risk-averse polices outperform the neutral one.

![Learning curves for summon-v0.](image)

Figure 4. Learning curves for summon-v0. The yellow car should approach the green goal without collision with other blue cars. Risk-averse polices outperform the neutral one.

We train our agents under different risk-averse measures, where each experiment is repeated 3 times with different random seeds. All curves are smoothed for better visibility.

different versions MuJoCo (current available version is v2 in Gym), we are all consistent with the original papers. Hyper-parameters are listed in the supplementary material.

Some important details of experiments has to be mentioned. Instead of using fixed the temperature parameter $\alpha$ to balance reward and entropy, SAC has a variant (Haarnoja et al., 2018b) which introduces a mechanism of fine-tuning $\alpha$ to achieve target entropy adaptively. While our algorithm does not conflict with this parameter adaptive method, we use fixed $\alpha$ suggested in original SAC paper in our experiments to reduce irrelevant factors.

5.2. Risk-averse Policies

We evaluate risk-averse polices with highway-env (Leurent, 2018), which contains a collection of environments for autonomous driving and tactical decision-making tasks. We test our algorithm in a reach-type task named Summon, where the agent observes their position and velocity and controls their acceleration and steering to reach a given goal. To reach the goal, the agent must get rid of collisions with other cars. Reward is given as the opposite number of distance between desired position and current position added a big penalty for collision. It is seen as finishing the task if reward is above $-0.12$.

We train our agents under different risk-averse measures, where each experiment is repeated 3 times with different random seeds. We show the training curves of maximum reward in Figure 4. As the figure shows all the risk-averse polices are better than the risk-neutral policy, which shows that the algorithm is effective in risk-averse scenarios.

5.3. Risk-seeking Policies

Distributional RL has studied to enhance exploration capability of policy in former work (Tang & Agrawal, 2018). We point out that stronger exploration capability can be obtained through risk-seeking strategies. We choose BidepalWalkerHardcore in gym (Brockman et al., 2016) for evaluation, which is a harder variant of BidepalWalker. The bipedal robot agent needs to learn not only how to stand up and walk forward, but also how to jump over obstacles like slopes, stairs, gaps and hurdles.

We run each experiments 3 times with different random seeds and plot the training curves in Figure 5. The results show risk-seeking polices greatly outperform the risk-neutral policy and achieve desired scores in the end. Unlike former solutions combining evolution strategy (Hämäläinen et al., 2018), implementing recurrent policy (Song et al., 2018) or using specific tricks for this task, we do not make any adjustment to the algorithm expect the risk-seeking measures. It shows that DSAC has great potential for more general applications.

Among all risk measures in testing, CVaR seems to perform worse in both risk-seeking and risk-averse cases. One potential reason is that CVaR only considers part of the quantile fractions. From this point, Wang and Pow are able to viewed as soft version of CVaR, which explains their better performances.

![Learning curves for BidepalWalkerHardcore-v2.](image)

Figure 5. Learning curves for BidepalWalkerHardcore-v2. The robot should learn to walk and pass the obstacles. Risk-seeking polices outperform the neutral one.
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A. Hyper-Parameter Setting

Hyper-parameters used in experiments are listed in Table 1 and some environment specific parameters are shown in Table 2.

| Hyper-parameter                        | DSAC | SAC | TD3 |
|----------------------------------------|------|-----|-----|
| Policy network learning rate           | 3e-4 | 3e-4| 3e-4|
| (Quantile) Value network learning rate | 3e-4 | 3e-4| 3e-4|
| Fraction proposal network learning rate| 1e-4 | None| None|
| Optimizer                              | Adam | Adam| Adam|
| Discount factor                        | 0.99 | 0.99| 0.99|
| Batch size                             | 256  | 256 | 256 |
| Replay buffer size                     | 10^6 | 10^6| 10^6|
| Target smoothing                       | 5e-3 | 5e-3| 5e-3|
| Entropy temperature                    | 1    | 1   | None|
| Number of proposed fractions           | 32   | None| None|
| Huber regression threshold             | 1    | None| None|

For SAC and TD3, we use two-layer fully connected networks with 256 hidden units for both value and policy network and ReLU as activation function. In DSAC, the same network structure is used in policy network $\pi_\phi(s, a)$ and fraction proposal network $P_\psi(s, a)$ with LeakyReLU as activation function. As for the quantile value network $F_{Z, \theta}^{-1}(\tau)$, we use two-layer 256-unit fully connected network for $\nu(s, a)$ and one-layer 64-unit fully connected network for $\varphi(\tau)$ Equation 22, followed with one-layer 128-unit fully connected network for their multiplied values $\nu(s, a) \odot \varphi(\tau)$.

B. Proof

B.1. Lemma 1

**Lemma 1** $T_{\overline{D}}^\pi : Z \rightarrow Z$ is a $\gamma$-contraction in $\overline{d}_p$.

**Proof.** This lemma can be proved based on the definitions of $\gamma$-contraction mapping and the operator $T_{\overline{D}}^\pi$ (Bellemare et al., 2017). For any $(s, a) \in S \times A$, we have

$$
W_p\left(T_{\overline{D}}^\pi Z_1(s, a), T_{\overline{D}}^\pi Z_2(s, a)\right)
= W_p\left(R(s, a) + \gamma (Z_1(s, a) - \alpha \log \pi(a'|s')), R(s, a) + \gamma (Z_2(s, a) - \alpha \log \pi(a'|s'))\right), s' \sim P(\cdot|s, a), a' \sim \pi(\cdot|s')
\leq \gamma W_p(Z_1(s', a'), Z_2(s', a'))
\leq \gamma \sup_{s', a'} W_p(Z_1(s', a'), Z_2(s', a'))
$$

(30)
By definition, we have

\[ \bar{d}_p(T_{DS}^s Z_1, T_{DS}^s Z_2) = \sup_{s,a} W_p(T_{DS}^s Z_1(s, a), T_{DS}^s Z_2(s, a)) \]

\[ \leq \gamma \sup_{s',a'} W_p(Z_1(s', a'), Z_2(s', a')) \]

\[ = \gamma \bar{d}_p(Z_1, Z_2) \]  

(31)

B.2. Lemma 2

Lemma 2 (Distributional Soft Policy Evaluation) For any mapping \( Z^0 : S \times A \rightarrow Z \), we define \( Z^{k+1} = T_{DS}^s Z^k \). The sequence \( Z^k \) will converge to \( Z^* \) as \( k \rightarrow \infty \).

Proof. As we have proved that \( T_{DS}^s \) is a \( \gamma \)-contraction, policy evaluation can be obtained by repeatedly applying \( T_{DS}^s \).

B.3. Lemma 3

Lemma 3 (Distributional Soft Policy Improvement) For all \((s, a) \in S \times A\), let \( Q^\pi (s, a) = \mathbb{E}[Z^\pi (s, a)] \). With \( \pi_{\text{old}} \in \Pi \) and \( \pi_{\text{new}} \) as the solution of problem defined in Equation 5, we have \( Q^{\pi_{\text{old}}}(s, a) \leq Q^{\pi_{\text{new}}}(s, a) \).

Proof. The proof of this lemma has the similar idea to that of the soft policy improvement in (Haarnoja et al., 2018a).

For any policy \( \pi \in \Pi \) and its soft action-value distribution \( Z^\pi \), we define the soft action-value by taking the expectation,

\[ Q^\pi (s, a) = \mathbb{E}[Z^\pi (s, a)] = \mathbb{E}[R(s, a)] + \gamma \mathbb{E}_{s' \sim P(\cdot|s, a), a' \sim \pi(\cdot|s')} [Z^\pi (s', a') - \alpha \log \pi(a'|s')] . \]  

(32)

Let denote a policy \( \pi_{\text{old}} \in \Pi \) and its soft action-value \( Q^{\pi_{\text{old}}} \). We can obtain a new policy \( \pi_{\text{new}} \in \Pi \) by solving the minimization problem defined in Equation 5,

\[ \pi_{\text{new}} (\cdot|s) = \arg \min_{\pi \in \Pi} \lim_{t \rightarrow \infty} D_{\text{KL}} \left( \pi' (\cdot|s) \| \exp \left( \frac{1}{\alpha} Q^{\pi_{\text{old}}} (s, \cdot) - \log \Delta^{\pi_{\text{old}}} (s) \right) \right) . \]  

(33)

Since \( \pi_{\text{new}} \) is the solution to the minimization problem above, it must be

\[ \mathbb{E}_{a \sim \pi_{\text{new}}(\cdot|s)} \left[ \log \pi_{\text{new}} (a|s) - \frac{1}{\alpha} Q^{\pi_{\text{old}}} (s, a) + \log \Delta^{\pi_{\text{old}}} (s) \right] \leq \mathbb{E}_{a \sim \pi_{\text{old}}(\cdot|s)} \left[ \log \pi_{\text{old}} (a|s) - \frac{1}{\alpha} Q^{\pi_{\text{old}}} (s, a) + \log \Delta^{\pi_{\text{old}}} (s) \right] , \]  

(34)

which can be reduced as follows for partition function \( \log \Delta^{\pi_{\text{old}}} \) is only related to \( s \),

\[ \mathbb{E}_{a \sim \pi_{\text{new}}(\cdot|s)} \left[ \alpha \log \pi_{\text{new}} (a|s) - Q^{\pi_{\text{old}}} (s, a) \right] \leq \mathbb{E}_{a \sim \pi_{\text{old}}(\cdot|s)} \left[ \alpha \log \pi_{\text{old}} (a|s) - Q^{\pi_{\text{old}}} (s, a) \right] . \]  

(35)

Thus, we expand the soft Bellman equation and have

\[ Q^{\pi_{\text{old}}} (s_t, a_t) = \mathbb{E}[R(s_t, a_t)] + \gamma \mathbb{E}_{s_{t+1} \sim P(\cdot|s_t, a_t), a_{t+1} \sim \pi_{\text{old}}(\cdot|s_{t+1})} [Q^{\pi_{\text{old}}} (s_{t+1}, a_{t+1}) - \alpha \log \pi_{\text{old}} (a_{t+1}|s_{t+1})] \]

\[ \leq \mathbb{E}[R(s_t, a_t)] + \gamma \mathbb{E}_{s_{t+1} \sim P(\cdot|s_t, a_t), a_{t+1} \sim \pi_{\text{new}}(\cdot|s_{t+1})} [Q^{\pi_{\text{new}}} (s_{t+1}, a_{t+1}) - \alpha \log \pi_{\text{new}} (a_{t+1}|s_{t+1})] \]

\[ \vdots \]

\[ \leq Q^{\pi_{\text{new}}} (s_t, a_t) . \]  

(36)