WEAK KAM THEORY FOR GENERAL HAMILTON-JACOBI EQUATIONS II: THE FUNDAMENTAL SOLUTION UNDER LIPSCHITZ CONDITIONS

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ABSTRACT. We consider the following evolutionary Hamilton-Jacobi equation with initial condition:

\[
\begin{cases}
\partial_t u(x, t) + H(x, u(x, t), \partial_x u(x, t)) = 0, \\
u(x, 0) = \phi(x),
\end{cases}
\]

where \( \phi(x) \in C(M, \mathbb{R}) \). Under some assumptions on the convexity of \( H(x, u, p) \) with respect to \( p \) and the uniform Lipschitz of \( H(x, u, p) \) with respect to \( u \), we establish a variational principle and provide an intrinsic relation between viscosity solutions and certain minimal characteristics. By introducing an implicitly defined fundamental solution, we obtain a variational representation formula of the viscosity solution of the evolutionary Hamilton-Jacobi equation. Moreover, we discuss the large time behavior of the viscosity solution of the evolutionary Hamilton-Jacobi equation and provide a dynamical representation formula of the viscosity solution of the stationary Hamilton-Jacobi equation with strictly increasing \( H(x, u, p) \) with respect to \( u \).

Key words. fundamental solution, Hamilton-Jacobi equation, viscosity solution

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1. Introduction and main results

Let $M$ be a closed manifold and $H$ be a $C^r$ ($r \geq 2$) function called a Hamiltonian. We consider the following Hamilton-Jacobi equation:

$$\partial_t u(x, t) + H(x, u(x, t), \partial_x u(x, t)) = 0,$$

with the initial condition

$$u(x, 0) = \phi(x),$$

where $(x, t) \in M \times [0, T]$, $T$ is a positive constant. The characteristics of (1.1) satisfies the following equation:

$$\begin{cases}
\dot{x} = \frac{\partial H}{\partial p}, \\
\dot{p} = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial u} p, \\
\dot{u} = \frac{\partial H}{\partial p} p - H.
\end{cases}$$

To avoid the ambiguity, we denote the solution of (1.2) (the characteristics of (1.1)) by $(X(t), U(t), P(t))$.

In 1983, M. Crandall and P. L. Lions introduced a notion of weak solution named viscosity solution for overcoming the lack of uniqueness of the solution due to the crossing of characteristics (see [1, 16]). Owing to the notion itself, the uniqueness of the viscosity solution can be followed from comparison principle (see [4, 5, 12–14, 16] for instance). During the same period, S. Aubry and J. Mather developed a seminar work so called Aubry-Mather theory on global action minimizing orbits for area-preserving twist maps (see [2, 3, 28–31] for instance). Moreover, it was generalized to positive definite Lagrangian systems with multi-degrees of freedom in [32].

There is a close connection between viscosity solutions and Aubry-Mather theory. Roughly speaking, the global minimizing orbits used in Aubry-Mather theory can be embedded into the characteristic fields of PDEs. The similar ideas were reflected in pioneering papers [18] and [20] respectively. In [18], W. E was concerned with certain weak solutions of Burgers equation. In [20], A. Fathi provided a weak solution named weak KAM solution and implied that the weak KAM solution is a viscosity solution, which initiated so called weak KAM theory. Later, it was obtained the equivalence between weak KAM solutions and viscosity solutions for the Hamiltonian $H(x, p)$ without the unknown function $u$ under strict convexity and superlinear growth with respect to $p$. Moreover, based on the relations between weak KAM solutions and viscosity solutions, the regularity of global subsolutions was improved (see [8, 23]). A systematic introduction to weak KAM theory can be found in [22].

Due to the lack of the variational principle for more general Hamilton-Jacobi equations, the weak KAM theory had been limited to Hamilton-Jacobi equations without the unknown function $u$ explicitly. In [37], the authors made an attempt on the Hamilton-Jacobi equation formed as (1.1) by a dynamical approach and extended Fathi’s weak KAM theory to more general Hamilton-Jacobi equations under the monotonicity (non-decreasing) and Lipschitz of $H$ with respect to $u$. Roughly speaking, the weak KAM theory for the Hamilton-Jacobi equations with the unknown function $u$ explicitly is a cornerstone to handle weakly coupled systems and second order equations by a dynamical approach.

In this paper, the monotonicity (non-decreasing) assumption is dropped, which makes a further step to enlarge the scope of the weak KAM theory. More precisely,
we establish a variational principle and provide an intrinsic relation between viscosity solutions and certain minimal characteristics. By introducing an implicitly defined fundamental solution, we obtain a representation formula of the viscosity solution of (1.1). Moreover, we discuss the large time behavior of the viscosity solution of the evolutionary Hamilton-Jacobi equation and provide a dynamical representation formula of the viscosity solution of the stationary Hamilton-Jacobi equation with strictly increasing \( H(x,u,p) \) with respect to \( u \). Precisely speaking, we are concerned with a \( C^r \) \((r \geq 2)\) Hamiltonian \( H(x,u,p) \) satisfying the following conditions:

- **(H1) Positive Definiteness:** \( H(x,u,p) \) is strictly convex with respect to \( p \);
- **(H2) Superlinearity in the Fibers:** For every compact set \( I \) and any \( u \in I \), \( H(x,u,p) \) is uniformly superlinear growth with respect to \( p \);
- **(H3) Completeness of the Flow:** The flows of (1.2) generated by \( H(x,u,p) \) are complete;
- **(H4) Uniform Lipschitz:** \( H(x,u,p) \) is uniformly Lipschitz with respect to \( u \).

We use \( \mathcal{L} : T^*M \to TM \) to denote the Legendre transformation. Let \( \bar{\mathcal{L}} := (\mathcal{L}, \text{Id}) \), where \( \text{Id} \) denotes the identity map from \( \mathbb{R} \) to \( \mathbb{R} \). Then \( \bar{\mathcal{L}} \) denote a diffeomorphism from \( T^*M \times \mathbb{R} \) to \( TM \times \mathbb{R} \). By \( \bar{\mathcal{L}} \), the Lagrangian \( L(x,u,\dot{x}) \) associated to \( H(x,u,p) \) can be denoted by

\[
L(x,u,\dot{x}) := \sup_{p} \{ (\dot{x}, p) - H(x,u,p) \}.
\]

Let \( \Psi_t \) denote the flows of (1.2) generated by \( H(x,u,p) \). The flows generated by \( L(x,u,\dot{x}) \) can be denoted by \( \Phi_t := \bar{\mathcal{L}} \circ \Psi_t \circ \bar{\mathcal{L}}^{-1} \). Based on (H1)-(H4), it follows from \( \bar{\mathcal{L}} \) that the Lagrangian \( L(x,u,\dot{x}) \) satisfies:

- **(L1) Positive Definiteness:** \( L(x,u,\dot{x}) \) is strictly convex with respect to \( \dot{x} \);
- **(L2) Superlinearity in the Fibers:** For every compact set \( I \) and any \( u \in I \), \( L(x,u,\dot{x}) \) is uniformly superlinear growth with respect to \( \dot{x} \);
- **(L3) Completeness of the Flow:** The flows generated by \( L(x,u,\dot{x}) \) are complete;
- **(L4) Uniform Lipschitz:** \( L(x,u,\dot{x}) \) is uniformly Lipschitz with respect to \( u \).

If a Hamiltonian \( H(x,u,p) \) satisfies (H1)-(H4), then we obtain the following theorem:

**Theorem 1.1** For given \( x_0, x \in M, u_0 \in \mathbb{R} \) and \( t \in (0,T] \), there exists a unique \( h_{x_0,u_0}(x,t) \) satisfying

\[
h_{x_0,u_0}(x,t) = u_0 + \inf_{\gamma(t)=x,\gamma(0)=x_0} \int_0^t L(\gamma(\tau), h_{x_0,u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,
\]

where the infimums are taken among the absolutely continuous curves \( \gamma : [0,t] \to M \). In particular, the infimums are attained at the characteristics of (1.1). Moreover, let \( S_{x_0,u_0}^x \) denote the set of characteristics \((X(t),U(t),P(t))\) satisfying \( X(0) = x_0, X(t) = x \) and \( U(0) = u_0 \), then we have

\[
h_{x_0,u_0}(x,t) = \inf \{ U(t) : (X(t), U(t), P(t)) \in S_{x_0,u_0}^x \}.
\]
Theorem 1.1 provides a general variational principle, which builds a bridge between Hamilton-Jacobi equations under (H1)-(H4) and Hamiltonian dynamical systems. As an application, we will obtain a dynamical representation of the viscosity solution of (1.1). By analogy with the notion of weak KAM solution of the Hamilton-Jacobi equation without $u$ (see [22]). We define another weak solution of (1.1) with initial condition called a variational solution (see Definition 2.4). Based on Theorem 1.1, we construct a variational solution of (1.1) with initial condition. Following [22], we show that the variational solution of (1.1) is the unique viscosity solution of (1.1). More precisely, we have the following theorem:

**Theorem 1.2** There exists a unique viscosity solution $u(x,t)$ of (1.1) with initial condition $u(x,0) = \phi(x)$. Moreover, $u(x,t)$ can be represented as

$$u(x,t) = \inf_{y \in M} h_{y,\phi(y)}(x,t).$$

Theorem 1.1 and Theorem 1.2 implies the following theorem directly:

**Theorem 1.3** For $(x,t) \in M \times [0,T]$, the viscosity solution $u(x,t)$ of (1.1) with initial condition $u(x,0) = \phi(x)$ is determined by the minimal characteristic curve. More precisely, we have

$$u(x,t) = \inf_{y \in M} \inf \left\{ U(t) : (X(t),U(t),P(t)) \in S_{y,\phi(y)}^x \right\},$$

where $S_{y,\phi(y)}^x$ denotes the set of characteristics $(X(t),U(t),P(t))$ satisfying $X(0) = y$, $X(t) = x$ and $U(0) = \phi(y)$.

A similar result corresponding to the viscosity solutions of Hamilton-Jacobi equations without the unknown function $u$ was well known (see Theorem 6.4.6 in [10] for instance). Theorem 1.3 implies the relation between the viscosity solutions and the minimal characteristics still holds for more general Hamilton-Jacobi equations. Roughly speaking, the notion of viscosity solution was invented to avoid the lack of uniqueness owing to the crossing of characteristics. Based on Theorem 1.3, the reason why the notion of viscosity solution results in the fact without crossing is that the properties of viscosity solutions are determined by certain minimal characteristics.

**Theorem 1.4** There exists an implicitly defined semigroup denoted by $T_t$ such that

$$u(x,t) = T_t \phi(x), \quad h_{y,\phi(y)}(x,s + t) = T_t h_{y,\phi(y)}(x,s),$$

where $y \in M$, $s > 0$, $t \geq 0$ and

$$T_t \phi(x) = \inf_{\gamma(t)=x} \left\{ \phi(\gamma(0)) + \int_0^t L(\gamma(\tau),T_{\tau} \phi(\gamma(\tau)),\dot{\gamma}(\tau))d\tau \right\}.$$ 

Moreover, for any $\phi(x), \psi(x) \in C(M, \mathbb{R})$ and $t \in [0,T]$, the solution semigroup $T_t$ has following properties:

I. for $\phi \leq \psi$, $T_t \phi \leq T_t \psi$,

II. $\|T_t \phi - T_t \psi\|_{\infty} \leq e^{\lambda t}\|\phi - \psi\|_{\infty}$,
where $\lambda > 0$ is the Lipschitz constant of $L$.

For $c \in \mathbb{R}$, we denote $L_c := L + c$. For given $x_0, u_0, x, t$ where $t \in (0, +\infty)$, we define

$$h^{c}_{x_0, u_0}(x, t) = u_0 + \sup_{\gamma(t) = x, \gamma(0) = x_0} \int_{0}^{t} L_c(\gamma(\tau), h^{c}_{x_0, u_0}(\gamma(\tau), \tau)) d\tau,$$

where the supremums are taken among the absolutely continuous curves $\gamma : [0, t] \to M$.

**Definition 1.5** $c$ is called a critical value if for any $x_0 \in M$, $u_0 \in \mathbb{R}$ and $t \geq \delta$, it is contained in the following set

$$C = \{c : |h^{c}_{x_0, u_0}(x, t)| \leq K(u_0)\},$$

where $K(u_0)$ is a positive constant depending on $u_0$.

$C \neq \emptyset$ will be verified in Section 6. For $a \in \mathbb{R}$, we use $c(L(x, a, \dot{x}))$ to denote Mañé critical value of $L(x, a, \dot{x})$. By [11], we have

$$c(L(x, a, \dot{x})) = \inf_{u \in C^{1}(M, \mathbb{R})} \sup_{x \in M} H(x, a, \partial_x u).$$

Without ambiguity, we still use $L$ instead of $L_c$ to denote $L + c$ for $c \in C$. The same to $H$ and $T_t$. By inspiration of [21], the large time behavior of viscosity solutions of Hamilton-Jacobi equations with Hamiltonian independent of $u$ was explored comprehensively based on both dynamical and PDE approaches (see [17, 22, 36, 38] for instance). Recently, some results on the large time behavior of viscosity solutions of special weakly coupled systems were also obtained (see [9, 34, 35]). By Theorem 1.3, $u(x, t) := T_t \phi(x)$ is the unique viscosity solution of $\{1.1\}$ with initial condition $u(x, 0) = \phi(x)$. The following theorem implies the relation between the viscosity solution of $\{1.1\}$ and the one of the stationary equation:

$$H(x, u(x), \partial_x u(x)) = 0.$$

More precisely, there holds

**Theorem 1.6** For any $\phi(x) \in C(M, \mathbb{R})$, $\lim_{t \to \infty} T_t \phi(x)$ exists. Moreover, let

$$\underline{u}(x) := \liminf_{t \to \infty} T_t \phi(x),$$

then $\underline{u}$ is a weak KAM solution of $\{1.1\}$.

Let

$$h_{x_0, u_0}(x, \infty) := \liminf_{t \to \infty} h_{x_0, u_0}(x, t),$$

where $x_0 \in M, u_0 \in \mathbb{R}$. Based on Theorem 1.6, $h_{x_0, u_0}(x, \infty)$ is well-posed. We denote

$$B(x, u; y) := h_{x_0, u_0}(y, \infty) - u.$$

$B(x, u; y)$ can be referred as the barrier function dented by $h^{\infty}(x, y)$ in Mather-Fathi theory. Moreover, we define an invariant set called a projected Aubry set as follows;

$$A := \{(x, u) \in M \times \mathbb{R} \mid B(x, u; x) = 0\}.$$

We use $\pi : M \times \mathbb{R} \to M$ to denote the standard projection via $(x, u) \to x$. 


Theorem 1.7 Let $H(x,u,p)$ is strictly increasing with respect to $u$ for a given $(x,p) \in T^*M$, then there exists a unique viscosity solution $u(x)$ of (1.11). Moreover,

$$u(x) = \inf_{y \in \pi A} h_{y,u(y)}(x, \infty).$$

2. Preliminaries

In this section, we recall the definitions of the weak KAM solution and the viscosity solution of (1.1) (see [12, 16, 22]). In addition, we provide some aspects of Mather-Fathi theory for the sake of completeness.

2.1. Weak KAM solutions and viscosity solutions

A function $H : TM \to \mathbb{R}$ called a Tonelli Hamiltonian if $H$ satisfies (H1)-(H2). For the autonomous Hamiltonian systems, the assumption (H3) holds obviously from the compactness of $M$. The associated Lagrangian is denoted by $L$ via the Legendre transformation. In [19], Fathi introduced the definition of the weak KAM solution of negative type of the following Hamilton-Jacobi equation:

$$(2.1) \quad H(x, \partial_x u(x)) = 0, \quad x \in M,$$

where $H$ is a Tonelli Hamiltonian.

Definition 2.1 A function $u \in C(M, \mathbb{R})$ is called a weak KAM solution of negative type of (2.7) if

(i) for each continuous piecewise $C^1$ curve $\gamma : [t_1, t_2] \to M$ where $t_2 > t_1$, we have

$$(2.2) \quad u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau;$$

(ii) for any $x \in M$, there exists a $C^1$ curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ such that for any $t \in (-\infty, 0]$, we have

$$(2.3) \quad u(x) - u(\gamma(t)) = \int_{t}^{0} L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau.$$

By analogy of the definition above, it is easy to define the weak KAM solution of negative type of more general Hamilton-Jacobi equation as follows:

$$(2.4) \quad H(x, u(x), \partial_x u(x)) = 0, \quad x \in M.$$

Definition 2.2 A function $u \in C(M, \mathbb{R})$ is called a weak KAM solution of negative type of (2.7) if

(i) for each continuous piecewise $C^1$ curve $\gamma : [t_1, t_2] \to M$ where $t_2 > t_1$, we have

$$(2.5) \quad u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(\tau), \dot{\gamma}(\tau), u(\gamma(\tau))) d\tau;$$
(ii) for any \( x \in M \), there exists a \( C^1 \) curve \( \gamma : (-\infty, 0] \to M \) with \( \gamma(0) = x \) such that for any \( t \in (-\infty, 0] \), we have

\[
(2.6) \quad u(x) - u(\gamma(t)) = \int_t^0 L(\gamma(\tau), u(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau.
\]

Following from [12, 16, 22], a viscosity solution of (1.1) can be defined as follows:

**Definition 2.3** Let \( V \) be an open subset \( V \subset M \),

(i) A function \( u : V \times [0, T] \to \mathbb{R} \) is a subsolution of (1.1), if for every \( C^1 \) function \( \phi : V \times [0, T] \to \mathbb{R} \) and every point \( (x_0, t_0) \in V \times [0, T] \) such that \( u - \phi \) has a maximum at \( (x_0, t_0) \), we have

\[
(2.7) \quad \partial_t \phi(x_0, t_0) + H(x_0, u(x_0, t_0), \partial_x \phi(x_0, t_0)) \leq 0;
\]

(ii) A function \( u : V \times [0, T] \to \mathbb{R} \) is a supersolution of (1.1), if for every \( C^1 \) function \( \psi : V \times [0, T] \to \mathbb{R} \) and every point \( (x_0, t_0) \in V \times [0, T] \) such that \( u - \psi \) has a minimum at \( (x_0, t_0) \), we have

\[
(2.8) \quad \partial_t \psi(x_0, t_0) + H(x_0, u(x_0, t_0), \partial_x \psi(x_0, t_0)) \geq 0;
\]

(iii) A function \( u : V \times [0, T] \to \mathbb{R} \) is a viscosity solution of (1.1) on the open subset \( V \subset M \), if it is both a subsolution and a supersolution.

Under the assumptions (H1)-(H4), it follows from the comparison theorem that the viscosity solution of (1.1) with initial condition is unique (see [16]).

Both of Definition 2.1 and Definition 2.2 are concerned with the weak KAM solutions defined on \( M \times \mathbb{R} \), while the viscosity solutions of (1.1) are defined on \( M \times [0, T] \). As a bridge connecting them, we give the definition of another weak solution of (1.1) with initial condition called a variational solution.

**Definition 2.4** For a given \( T > 0 \), a variational solution of (1.1) with initial condition is a function \( u : M \times [0, T] \to \mathbb{R} \) for which the following are satisfied:

(i) for each continuous piecewise \( C^1 \) curve \( \gamma : [t_1, t_2] \to M \) where \( 0 \leq t_1 < t_2 \leq T \), we have

\[
(2.9) \quad u(\gamma(t_2), t_2) - u(\gamma(t_1), t_1) \leq \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau;
\]

(ii) for any \( 0 \leq t_1 < t_2 \leq T \) and \( x \in M \), there exists a \( C^1 \) curve \( \gamma : [t_1, t_2] \to M \) with \( \gamma(t_2) = x \) such that

\[
(2.10) \quad u(x, t_2) - u(\gamma(t_1), t_1) = \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau.
\]

The existence of the variational solutions will be verified in Section 4.
2.2. The minimal action and the fundamental solution

Let \( L : TM \to \mathbb{R} \) be a Tonelli Lagrangian. We define the function \( h_t : M \times M \to \mathbb{R} \) by

\[
h_t(x, y) = \inf_{\gamma(0) = x, \gamma(t) = y} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau,
\]

where the infimums are taken among the absolutely continuous curves \( \gamma : [0, t] \to M \). By Tonelli theorem (see \([22, 32]\)), the infimums in (2.11) can be achieved. Let \( \bar{\gamma} \) where the infimums are taken among the absolutely continuous curves is achieved at \( \bar{\gamma} \), where \( \bar{\gamma} \) is a minimal curve with \( \bar{\gamma}(0) = x \) and \( \bar{\gamma}(t) = y \) such that the infimum is achieved at \( \bar{\gamma} \). Then \( \bar{\gamma} \) is called a minimal curve. By \([32]\), the minimal curves satisfy the Euler-Lagrange equation generated by \( L \). The quantity \( h_t(x, y) \) is called a minimal action. From the definition of \( h_t(x, y) \), it follows that for each \( x, y, z \in M \) and each \( t, t' > 0 \), we have

\[
h_{t+t'}(x, z) \leq h_t(x, y) + h_{t'}(y, z).
\]

In particular, we have

\[
h_{t+t'}(x, y) = h_t(x, \bar{\gamma}(t)) + h_{t'}(\bar{\gamma}(t), y),
\]

where \( \bar{\gamma} \) is a minimal curve with \( \bar{\gamma}(0) = x \) and \( \bar{\gamma}(t + t') = y \).

Consider the following Hamilton-Jacobi equation:

\[
\begin{cases}
\partial_t u(x, t) + H(x, \partial_x u(x, t)) = 0, \\
u(x, 0) = \phi(x),
\end{cases}
\]

where \( \phi(x) \in C(M) \). By \([22]\), a viscosity solution of (2.14) can be represented as

\[
u(x, t) := \inf_{y \in M} \{ \phi(y) + h^t(y, x) \}.
\]

The right side of (2.15) is also called inf-convolution of \( \phi \), due to the formal analogy with the usual convolution (see \([10]\)). Moreover, the minimal action \( h^t(y, x) \) can be viewed as a fundamental solution of (2.14) (see \([24]\)).

The following conception is crucial in our context.

**Definition 2.5** For \( u(x, t) \in C(M \times [0, T], \mathbb{R}) \), a curve \( \gamma : I \to M \) is called a calibrated curve of \( u \) if for every \( t_1, t_2 \in I \) with \( 0 \leq t_1 < t_2 \), we have

\[
u(\gamma(t_2), t_2) = u(\gamma(t_1), t_1) + \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau.
\]

We are devoted to detecting the viscosity solution of (1.1) from a dynamical view. For given \( x_0, x \in M \), \( u_0 \in \mathbb{R} \) and \( t \in (0, T] \), we define formally:

\[
h_{x_0, u_0}(x, t) = u_0 + \inf_{\gamma(0) = x_0} \int_0^t L(\gamma(\tau), h_{x_0, u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,
\]

where the infimums are taken among the absolutely continuous curves \( \gamma : [0, t] \to M \). It is easy to see that the curve achieving the infimum in the right side of (2.16) is a calibrated curve of \( h_{x_0, u_0}(x, t) \). To fix the notions, we call \( h_{x_0, u_0}(x, t) \) the fundamental solution of (1.1). In next section, we will show the well-posedness of \( h_{x_0, u_0}(x, t) \) under the assumptions (L1)-(L4).
3. Variational principle

In this section, we are devoted to proving Theorem 1.1. The proof will be proceeded by four steps. In the first step, we will prove the existence and uniqueness of \( h_{x_0,u_0}(x,t) \). In the second step, we will verify \( h_{x_0,u_0}(x,t) \) to satisfy a triangle inequality. In the third step, we will show that the relation between calibrated curves and characteristics. Based on the preliminaries in former steps, we will give a relation between \( h_{x_0,u_0}(x,t) \) and \( U(t) \) belonging to a characteristic curve \((X(t), U(t), P(t))\) in the last step.

3.1. Existence and uniqueness of the fundamental solution

In this step, we are concerned with the existence and uniqueness of \( h_{x_0,u_0}(x,t) \). We use \( C^{ac}([0,t], M) \) to denote the set of all absolutely continuous curves \( \gamma : [0,t] \to M \). First of all, we verify the existence of \( h_{x_0,u_0}(x,t) \).

Lemma 3.1 There exists \( h_{x_0,u_0}(x,t) \in C(M \times (0,T], \mathbb{R}) \) such that

\[
(3.1) \quad h_{x_0,u_0}(x,t) = u_0 + \inf_{\gamma(0)=x_0} \int_0^t L(\gamma(\tau), h_{x_0,u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau,
\]

where \( \gamma \in C^{ac}([0,t], M) \).

Proof For the simplicity of notations, without ambiguity, we drop the subscripts \( x_0 \) and \( u_0 \) of \( h_{x_0,u_0}(x,t) \). We consider a sequence generated by the following iteration:

\[
(3.2) \quad h_{i+1}(x,t) = u_0 + \inf_{\gamma(0)=x_0} \int_0^t L(\gamma(\tau), h_i(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau,
\]

where \( i = 0, 1, 2, \ldots \) and \( h_0(x,t) = u_0 \). By means of a simple modification of Tonelli’s theorem (see [22] and [22]), we have that for a given \( h_i(x,t) \in C(M \times (0,T], \mathbb{R}) \) there exists an absolutely continuous curve \( \gamma_i : [0,t] \to M \) satisfying \( \gamma_i(0) = x_0 \) and \( \gamma_i(t) = x \) such that the infimum in (2.10) can be achieved. To fix the notions, \( \gamma_i \) is called a minimal curve of \( h_i \).

Let \( \tilde{\gamma} : [0,t] \to M \) be an absolutely continuous curve satisfying \( \tilde{\gamma}(0) = x_0 \) and \( \tilde{\gamma}(t) = x \). By the construction of \( h_i \), there holds for \( s \in [0,t] \),

\[
(3.3) \quad h_1(\gamma(s), s) = u_0 + \inf_{\gamma(s)=\gamma(0)} \int_0^s L(\gamma(\tau), u_0, \dot{\gamma}(\tau))d\tau,
\]

It is easy to see that

\[
(3.4) \quad |h_1(\gamma(s), s) - u_0| \leq K,
\]

where \( K \) is a positive constant independent of \( s \). Let \( \gamma_2 : [0, t] \to M \) be a minimal curve of \( h_2 \) with \( \gamma_2(0) = x_0 \) and \( \gamma_2(t) = x \). Let \( \gamma_1 : [0, t] \to M \) be a minimal curve of \( h_1 \) with \( \gamma_1(0) = x_0 \) and \( \gamma_1(s) = \gamma_2(s) \). By (L4), we have

\[
(3.5) \quad |L(x, u, \dot{x}) - L(x, v, \dot{x})| \leq \lambda|u - v|.
\]
Then for \( s \in [0, t] \), we have

\[
\begin{align*}
& h_2(\gamma_2(s), s) - h_1(\gamma_2(s), s) \\
& \leq \int_0^s L(\gamma_1(\tau), h_1(\gamma_1(\tau), \tau), \dot{\gamma}_1(\tau))d\tau - \int_0^s L(\gamma_1(\tau), u_0, \dot{\gamma}_1(\tau))d\tau, \\
& \leq \int_0^s |L(\gamma_1(\tau), h_1(\gamma_1(\tau), \tau), \dot{\gamma}_1(\tau)) - L(\gamma_1(\tau), u_0, \dot{\gamma}_1(\tau))|d\tau, \\
& \leq \lambda \int_0^s |h_1(\gamma_1(\tau), \tau) - u_0|d\tau,
\end{align*}
\]

which together with (3.4) implies

(3.6) \hspace{1cm} h_2(\gamma_2(s), s) - h_1(\gamma_2(s), s) \leq C\lambda s.

By a similar argument, we have

(3.7) \hspace{1cm} h_2(\gamma_2(s), s) - h_1(\gamma_2(s), s) \geq -C\lambda s.

In particular, there holds for \((x, t) \in M \times (0, T)\),

\[
|h_2(x, t) - h_1(x, t)| \leq C\lambda t.
\]

Let \( \gamma_3 : [0, t] \to M \) be a minimal curve of \( h_3 \) with \( \gamma_3(0) = x_0 \) and \( \gamma_3(t) = x \).

Let \( \gamma_2 : [0, t] \to M \) be a minimal curve of \( h_2 \) with \( \gamma_2(0) = x_0 \) and \( \gamma_2(s) = \gamma_3(s) \).

Moreover,

\[
\begin{align*}
& h_3(\gamma_3(s), s) - h_2(\gamma_3(s), s) \\
& \leq \int_0^s L(\gamma_2(\tau), h_2(\gamma_2(\tau), \tau), \dot{\gamma}_2(\tau))d\tau - \int_0^s L(\gamma_2(\tau), h_1(\gamma_2(\tau), \tau), \dot{\gamma}_2(\tau))d\tau, \\
& \leq \int_0^s |L(\gamma_2(\tau), h_2(\gamma_2(\tau), \tau), \dot{\gamma}_2(\tau)) - L(\gamma_2(\tau), h_1(\gamma_2(\tau), \tau), \dot{\gamma}_2(\tau))|d\tau, \\
& \leq \lambda \int_0^s |h_2(\gamma_2(\tau), \tau)) - h_1(\gamma_2(\tau), \tau)|d\tau \leq \lambda^2C \int_0^s C\lambda^2 d\tau = \frac{1}{2}C(\lambda s)^2.
\end{align*}
\]

By a similar argument, we have

(3.8) \hspace{1cm} h_3(\gamma_3(s), s) - h_2(\gamma_3(s), s) \geq -\frac{1}{2}C(\lambda s)^2.

In particular, we have

(3.9) \hspace{1cm} |h_3(x, t) - h_2(x, t)| \leq \frac{1}{2}C(\lambda t)^2.

Repeating the argument above \( n \) times, we have

(3.10) \hspace{1cm} |h_{n+1}(x, t) - h_n(x, t)| \leq \frac{1}{n}C(\lambda t)^n.

It follows from (3.10) that as \( n \to \infty \),

(3.11) \hspace{1cm} |h_{n+1}(x, t) - h_n(x, t)| \to 0,
which implies that \( \{h_n\} \) is a Cauchy sequence, hence there exists \( \bar{h}(x, t) \in C(M \times (0, T], \mathbb{R}) \) such that

\[
\lim_{n \to \infty} h_n(x, t) = \bar{h}(x, t),
\]

where \( \bar{h}(x, t) \) satisfies \( 2.16 \). This finishes the proof of Lemma 3.1.

Lemma 3.1 implies that there exists \( h_{x_0, u_0}(x, t) \in C(M \times (0, T], \mathbb{R}) \) such that

\[
\lim_{n \to \infty} h_n(x, t) = \bar{h}(x, t),
\]

where \( \bar{h}(x, t) \) satisfies \( 2.16 \). This finishes the proof of Lemma 3.1.

\[
(3.13) \quad h_{x_0, u_0}(x, t) = u_0 + \inf_{\gamma(0) = x_0, \gamma(t) = x} \int_0^t L(\gamma(\tau), h_{x_0, u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau.
\]

In particular, the infimum can be achieved at an absolutely continuous curve denoted by \( \bar{\gamma} \). By Definition 2.5, \( \bar{\gamma} \) is a calibrated curve.

The following lemma implies the uniqueness of \( h_{x_0, u_0}(x, t) \).

**Lemma 3.2** If both \( h_{x_0, u_0}(x, t) \) and \( g_{x_0, u_0}(x, t) \) satisfy \( 2.16 \), then \( h_{x_0, u_0}(x, t) = g_{x_0, u_0}(x, t) \) for \( (x, t) \in M \times (0, T] \).

The proof of Lemma 3.2 depends on a useful inequality as follows.

**Gronwall’s inequality:** Let \( F : [0, t] \to \mathbb{R} \) be continuous and nonnegative. Suppose \( C \geq 0 \) and \( \lambda \geq 0 \) are such that for any \( s \in [0, t] \),

\[
(3.14) \quad F(s) \leq C + \int_0^s \lambda F(\tau)d\tau.
\]

Then, for any \( s \in [0, t] \),

\[
(3.15) \quad F(s) \leq Ce^{\lambda s}.
\]

Taking \( C = 0 \), we have the following lemma.

**Lemma 3.3** Let \( F : [0, t] \to \mathbb{R} \) be continuous with \( F(0) = 0 \) and \( F(s) > 0 \) for \( s \in (0, t] \). Then for a given \( \lambda \geq 0 \), there exists \( s_0 \in (0, t] \) such that

\[
(3.16) \quad F(s_0) > \int_0^{s_0} \lambda F(\tau)d\tau.
\]

**Proof of Lemma 3.2:** The same as the notations in Lemma 3.1, we denote \( h_{x_0, u_0}(x, t) \) and \( g_{x_0, u_0}(x, t) \) by \( h(x, t) \) and \( g(x, t) \) respectively.

On one hand, we will prove

\[
(3.17) \quad h(x, t) \leq g(x, t).
\]

By contradiction, we assume \( h(x, t) > g(x, t) \). Let \( \gamma_g \) be a calibrated curve of \( g \) with \( \gamma_g(0) = x_0, \gamma_g(t) = x \). We denote

\[
(3.18) \quad F(\tau) = h(\gamma_g(\tau), \tau) - g(\gamma_g(\tau), \tau),
\]

where \( \tau \in [0, t] \). By \( 2.16 \), we have \( F(0) = 0 \). The assumption \( h(x, t) > g(x, t) \) implies \( F(t) > 0 \). Hence, there exists \( \tau_0 \in (0, t] \) such that \( F(\tau_0) = 0 \) and \( F(\tau) > 0 \).
for $\tau > \tau_0$. Let $\gamma_h$ be a calibrated curve of $h$ with $\gamma_h(0) = x_0$, $\gamma_h(\tau_0) = \gamma_g(\tau_0)$. For $s \in [\tau_0, t]$, we construct $\gamma_s : [0, s] \to M$ as follows:

$$\gamma_s(\tau) = \begin{cases} \gamma_h(\tau), & \tau \in [0, \tau_0], \\ \gamma_g(\tau), & \tau \in (\tau_0, s]. \end{cases}$$

(3.19)

Based on the definition of $h(x, t)$ (see (2.16)), we have

$$h(\gamma_g(s), s) = u_0 + \inf_{\gamma(0) = x_0} \int_0^s L(\gamma(\tau), h(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,$$

$$\leq u_0 + \int_0^s L(\gamma_s(\tau), h(\gamma_s(\tau), \tau), \dot{\gamma}_s(\tau)) d\tau,$$

$$= h(\gamma_g(\tau_0), \tau_0) + \int_{\tau_0}^s L(\gamma_g(\tau), h(\gamma_g(\tau), \tau), \dot{\gamma}_g(\tau)) d\tau.$$\hspace{1cm}\text{(3.20)}

Similarly, for $g(x, t)$, we have

$$g(\gamma_g(s), s) = g(\gamma_g(\tau_0), \tau_0) + \int_{\tau_0}^s L(\gamma_g(\tau), g(\gamma_g(\tau), \tau), \dot{\gamma}_g(\tau)) d\tau.$$\hspace{1cm}\text{(3.21)}

Since $h(\gamma_g(\tau_0), \tau_0) - g(\gamma_g(\tau_0), \tau_0) = F(\tau_0) = 0$, then we have

$$h(\gamma_g(s), s) - g(\gamma_g(s), s)$$

$$\leq \int_{\tau_0}^s L(\gamma_g(\tau), h(\gamma_g(\tau), \tau), \dot{\gamma}_g(\tau)) - L(\gamma_g(\tau), g(\gamma_g(\tau), \tau), \dot{\gamma}_g(\tau)) d\tau,$$

$$\leq \int_{\tau_0}^s \lambda|h(\gamma_g(\tau), \tau) - g(\gamma_g(\tau), \tau)| d\tau,$$

$$= \int_{\tau_0}^s \lambda(h(\gamma_g(\tau), \tau) - g(\gamma_g(\tau), \tau)) d\tau,$$

where the second inequality is owing to (A2). It follows that for any $s \in (\tau_0, t]$

$$F(s) \leq \int_{\tau_0}^s \lambda F(\tau) d\tau,$$

which is in contradiction with Lemma 3.3. Thus, we obtain $h(x, t) \geq g(x, t)$.

On the other hand, it follows from a similar argument that $h(x, t) \geq g(x, t)$. So far, we have shown that $h(x, t) = g(x, t)$ for $(x, t) \in M \times (0, T]$, which finishes the proof of Lemma 3.2. \hspace{1cm}\Box

### 3.2. A triangle inequality

Lemma 3.1 and Lemma 3.2 imply the well definiteness of $h_{x_0, u_0}(x, t)$. For the simplicity of notations, we drop the subscripts $x_0$ and $u_0$ of $h_{x_0, u_0}(x, t)$. The following lemma implies that $h(x, t)$ satisfies a triangle inequality.

**Lemma 3.4**

$$h(x, t + s) = \inf_{y \in M} h_{y, h(x, t)}(x, s).$$

(3.22)
Proof} On one hand, we will prove \( h(x, t + s) \geq \inf_{y \in M} h_{y, h(y, t)}(x, s) \). Let \( \gamma_1 : [0, t + s] \rightarrow M \) be a calibrated curve of \( h \) with \( \gamma_1(0) = x_0 \) and \( \gamma_1(t + s) = x \). Consider \( y \in \gamma_1 \) with \( \gamma_1(t) = y \). It suffices to show
\[
(3.23) \quad h(x, t + s) \geq h_{y, h(y, t)}(x, s).
\]
By contradiction, we assume \( h(x, t + s) < h_{y, h(y, t)}(x, s) \). By the definition of \( h(x, t + s) \), we have
\[
h(x, t + s) = u_0 + \int_0^{t+s} L(\gamma_1(\tau), h(\gamma_1(\tau), \tau), \dot{\gamma}_1(\tau))d\tau,
\]
\[
= h(y, t) + \int_t^{t+s} L(\gamma_1(\tau), h(\gamma_1(\tau), \tau), \dot{\gamma}_1(\tau))d\tau,
\]
\[
= h(y, t) + \int_0^s L(\gamma_1(\sigma + t), h(\gamma_1(\sigma + t), \sigma + t), \dot{\gamma}_1(\sigma + t))d\sigma.
\]
By the definition of \( h_{y, h(y, t)}(x, s) \), we have
\[
h_{y, h(y, t)}(x, s) = h(y, t) + \inf_{\gamma(0) = y} \int_0^s L(\gamma(\tau), h_{y, h(y, t)}(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau,
\]
\[
\leq h(y, t) + \int_0^s L(\gamma_1(\sigma + t), h_{y, h(y, t)}(\gamma_1(\sigma + t), \sigma), \dot{\gamma}(\sigma + t))d\sigma.
\]
For the simplicity of notations, we denote \( u(\sigma) := h(\gamma_1(\sigma + t), \sigma + t) \) and \( v(\sigma) := h_{y, h(y, t)}(\gamma_1(\sigma + t), \sigma) \). In particular, we have \( h(x, t + s) = u(\sigma) \) and \( h_{y, h(y, t)}(x, s) = v(\sigma) \). Let
\[
(3.24) \quad F(\sigma) := v(\sigma) - u(\sigma),
\]
where \( \sigma \in [0, s] \). It is easy to see that \( u(0) = h(y, t) = v(0) \). Then we have \( F(0) = 0 \). The assumption \( h(x, t + s) < h_{y, h(y, t)}(x, s) \) implies \( F(s) > 0 \). Hence, there exists \( \sigma_0 \in [0, s] \) such that \( F(\sigma_0) = 0 \) and \( F(\sigma) > 0 \) for \( \sigma > \sigma_0 \). Moreover, for any \( \tau \in (\sigma_0, s] \), we have
\[
(3.25) \quad u(\tau) = u(\sigma_0) + \int_{\sigma_0}^{\tau} L(\gamma_1(\sigma + t), u(\sigma), \dot{\gamma}_1(\sigma + t))d\sigma.
\]
Let \( \gamma_2 \) be a calibrated curve of \( h \) with \( \gamma_2(0) = y \), \( \gamma_2(\sigma_0) = \gamma_1(\sigma_0 + t) \). For \( \sigma \in [\sigma_0, \tau] \), we construct \( \gamma_\tau : [0, \tau] \rightarrow M \) as follows:
\[
(3.26) \quad \gamma_\tau(\sigma) = \begin{cases} 
\gamma_2(\sigma), & \sigma \in [0, \sigma_0], \\
\gamma_1(\sigma + t), & \sigma \in (\sigma_0, \tau].
\end{cases}
\]
Moreover, for any \( \tau \in (0, s] \), we have
\[
v(\tau) \leq h(y, t) + \int_0^{\tau} L(\gamma_\tau(\sigma + t), h_{y, h(y, t)}(\gamma_\tau(\sigma), \sigma), \dot{\gamma}_\tau(\sigma))d\sigma,
\]
\[
= v(\sigma_0) + \int_{\sigma_0}^{\tau} L(\gamma_1(\sigma + t), v(\sigma), \dot{\gamma}_1(\sigma + t))d\sigma.
\]
Since \( v(\sigma_0) - u(\sigma_0) = F(\sigma_0) = 0 \), it follows from (A2) that
\[
(3.27) \quad v(\tau) - u(\tau) \leq \int_{\sigma_0}^{\tau} \lambda(v(\sigma) - u(\sigma))d\sigma.
\]
It yields that for any $\tau \in (\sigma_0, s]$,

\[ F(\tau) \leq \int_{\sigma_0}^{\tau} \lambda F(\sigma)d\sigma, \]

which is in contradiction with Lemma 3.3. Hence,

\[ h(x, t + s) \geq \inf_{y \in M} h_{y, h(y, t)}(x, s). \]

On the other hand, we will prove $h(x, t + s) \leq \inf_{y \in M} h_{y, h(y, t)}(x, s)$. Due to the compactness of $M$, there exists $\tilde{y}$ such that $\inf_{y \in M} h_{y, h(y, t)}(x, s) = h_{\tilde{y}, h(\tilde{y}, t)}(x, s)$. Let $\gamma_3 : [0, t] \to M$ be a calibrated curve of $h$ with $\gamma_3(0) = x_0$ and $\gamma_3(t) = \tilde{y}$. Let $\gamma_4 : [0, s] \to M$ be a calibrated curve of $h_{\tilde{y}, h(\tilde{y}, t)}$ with $\gamma_4(0) = \tilde{y}$ and $\gamma_4(s) = x$. By a similar argument as (3.23)-(3.29), we have

\[ h(x, t + s) \leq \inf_{y \in M} h_{y, h(y, t)}(x, s), \]

which together with (3.24) implies

\[ h(x, t + s) = \inf_{y \in M} h_{y, h(y, t)}(x, s). \]

This completes the proof of Lemma 3.4.

From Lemma 3.4, we obtain

\[ h(x, t + s) - u_0 \leq (h_{y, h(y, t)}(x, s) - h(y, t)) + (h(y, t) - u_0), \]

which can be seen as a triangle inequality. In particular, the equality holds if and only if $y$ belongs to the calibrated curve $\gamma$ of $h$ with $\gamma(0) = x_0$ and $\gamma(t + s) = x$.

### 3.3. Calibrated curves and characteristics

In Step 1, we obtain that there exists a unique $h(x, t) \in C(M \times (0, T], \mathbb{R})$ satisfying (3.13) and a calibrated curve $\gamma$ of $h$. In this step, we will show that the relation between calibrated curves and characteristics. More precisely, we have the following lemma:

**Lemma 3.5** Let $\bar{\gamma} : [0, t] \to M$ be a calibrated curve of $h$, then $\bar{\gamma}$ is $C^1$ and for $\tau \in (0, t)$, $(\bar{\gamma}(\tau), u(\tau), p(\tau))$ satisfies the characteristics equation (1.2) where

\[ u(\tau) = h(\bar{\gamma}(\tau), \tau) \quad \text{and} \quad p(\tau) = \frac{\partial L}{\partial x}(\bar{\gamma}(\tau), h(\bar{\gamma}(\tau), \tau), \dot{\bar{\gamma}}(\tau)). \]

**Proof** Since $\bar{\gamma} \in C^{ac}([0, t], M)$, then the derivative $\dot{\bar{\gamma}}(\tau)$ exists almost everywhere for $\tau \in [0, t]$. Let $t_0 \in (0, t)$ be a differentiate point of $\bar{\gamma}(\tau)$. Without loss of generality, we assume $0 < t_0 < t$. For the simplicity of notations and without ambiguity, we denote

\[ (x_0, u_0, v_0) := (\bar{\gamma}(t_0), h(\bar{\gamma}(t_0), t_0), \dot{\bar{\gamma}}(t_0)). \]

First of all, we will construct a classical solution on a cone-like region (see (3.33) below). Let $k := |v_0|$ and

\[ B(0, 2k) := \{ v : |v| < 2k, \; v \in T_{x_0}M \}. \]
We use $B^*(0, 2k)$ to denote the image of $B(0, 2k)$ via the Legendre transformation $\mathcal{L}^{-1} : TM \rightarrow T^*M$. That is

$$B^*(0, 2k) := \left\{ p : p = \frac{\partial L}{\partial v}(x_0, u_0, v), \; v \in B(0, 2k) \right\}.$$ 

Let $\Psi_t : T^*M \times \mathbb{R} \rightarrow T^*M \times \mathbb{R}$ denote the flow generated by the characteristics equation. Let $\pi$ be a projection from $T^*M \times \mathbb{R}$ to $T^*M$ via $(x, p, u) \rightarrow (x, p)$ and let $B^*_t(0, 2k) := \pi \circ \Psi_{t-\tau}(B^*(0, 2k), u_0)$. We denote

$$\Pi_t : B^*_t(0, 2k) \rightarrow M.$$ 

Since the Legendre transformation $\mathcal{L}$ is a diffeomorphism, then for given $\epsilon > 0$ small and $\tau \in [t_0, t_0 + \epsilon]$, $\Pi_\tau$ is a diffeomorphism onto the image denoted by $\Omega_{\tau} := \Pi_\tau(B^*_\tau(0, 2k))$. We use $\Omega^\epsilon$ to denote the following cone-like region:

$$(3.35) \quad \Omega^\epsilon := \{ (\tau, x) : \tau \in (t_0, t_0 + \epsilon), \; x \in \Omega_{\tau} \}.$$ 

Then for any $(\tau, x) \in \Omega^\epsilon$, there exists a unique $p_0 \in B^*(0, 2k)$ such that $X(\tau) = x$ where

$$(X(\tau), U(\tau), P(\tau)) := \Phi_\tau(x_0, u_0, p_0).$$ 

Hence, for any $(\tau, x) \in \Omega^\epsilon$, one can define a $C^1$ function by $S(x, \tau) = U(\tau)$. In particular, we have $S(x, t_0) = u_0$. Moreover, it follows from the method of characteristics (see [22, 27] for instance) that $S(x, \tau)$ is a solution of the following equation:

$$(3.36) \quad \partial_\tau S(x, \tau) + H(x, S(x, \tau), \partial_x S(x, \tau)) = 0$$ 

with $\partial_x S(x_0, t_0) = \partial_x L(x_0, S(x_0, t_0), v_0)$, where $L$ denotes Lagrangian via the Legendre transformation associated to the Hamiltonian $H$. Fix $\tau \in [t_0, t_0 + \epsilon]$ and let $S_\tau(x) := S(x, \tau)$. We denote

$$(3.37) \quad \text{grad}_L S_\tau(x) := \frac{\partial H}{\partial p}(x, S_\tau(x), p),$$ 

where $p = \partial_x S_\tau(x)$. In particular, we have $v_0 = \text{grad}_L S_{t_0}(x_0)$. It is easy to see that $\text{grad}_L S_\tau(x)$ gives rise to a vector field on $M$. Let $\Omega$ be the Legendre transformation of $\Omega^\epsilon$. Moreover, we have the following claim:

**Claim:** Let $\gamma$ be an absolutely continuous curve with $(\tau, \gamma(\tau)) \in \Omega$ for $\tau \in [a, b] \subset [t_0, t_0 + \epsilon]$, we have

$$(3.38) \quad S(\gamma(b), b) - S(\gamma(a), a) \leq \int_a^b L(\gamma(\tau), S(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,$$

where the equality holds if and only if $\gamma$ is a trajectory of the vector field $\text{grad}_L S_\tau(x)$.

**Proof** From the regularity of $S(x, \tau)$, it follows that

$$(3.39) \quad S(\gamma(b), b) - S(\gamma(a), a) = \int_a^b \left\{ \frac{\partial S}{\partial t}(\gamma(\tau), \tau) + \langle \frac{\partial S}{\partial x}(\gamma(\tau), \tau), \dot{\gamma}(\tau) \rangle \right\} d\tau.$$ 

By virtue of Fenchel inequality, for each $\tau$ where $\dot{\gamma}(\tau)$ exists, we have

$$\langle \frac{\partial S}{\partial x}(\gamma(\tau), \tau), \dot{\gamma}(\tau) \rangle \leq H(\gamma(\tau), S(\gamma(\tau), \tau), \frac{\partial S}{\partial x}(\gamma(\tau), \tau)) + L(\gamma(\tau), S(\gamma(\tau), \tau), \dot{\gamma}(\tau)).$$
It follows from (3.39) that for almost every $\tau \in [a, b]$

\[(3.40) \quad \frac{\partial S}{\partial t}(\gamma(\tau), \tau) + \langle \frac{\partial S}{\partial x}(\gamma(\tau), \tau), \dot{\gamma}(\tau) \rangle \leq L(\gamma(\tau), S(\gamma(\tau), \tau), \dot{\gamma}(\tau)).\]

By integration, it follows from (3.39) that

\[(3.41) \quad S(\gamma(b), b) - S(\gamma(a), a) \leq \int_{a}^{b} L(\gamma(\tau), S(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau.\]

We have equality in (3.41) if and only if the equality holds in the Fenchel inequality, i.e. $\dot{\gamma}(\tau) = \text{grad}_{\tau} L S_{\tau}(x)$ which means that $\gamma$ is a trajectory of the vector field $\text{grad}_{\tau} L S_{\tau}(x)$.

Based on the construction of $\Omega'$, we have $(\tau, \bar{\gamma}(\tau)) \in \Omega'$ for $\tau \in [t_0, t_0 + \epsilon]$. By virtue of a similar argument as the one in the proof of Lemma 3.2, we have for any $\tau \in [t_0, t_0 + \epsilon]$,

\[(3.42) \quad S(\bar{\gamma}(\tau), \tau) = h(\bar{\gamma}(\tau), \tau),\]

where $\bar{\gamma}$ is a calibrated curve of $h$ with $\bar{\gamma}(t_0) = x_0$. From the definition of $h_{x_0, u_0}$ (see (2.16)), it follows that

\[S(\bar{\gamma}(t_0 + \epsilon), t_0 + \epsilon) = S(\bar{\gamma}(t_0), t_0) + \int_{t_0}^{t_0+\epsilon} L(\bar{\gamma}(\tau), S(\bar{\gamma}(\tau), \tau), \dot{\bar{\gamma}}(\tau))d\tau,\]

which implies $\dot{\bar{\gamma}}(\tau)$ is a trajectory of the vector field $\text{grad}_{\tau} L S_{\tau}(x)$. Let

\[(3.43) \quad u(\tau) := h(\bar{\gamma}(\tau), \tau) \quad \text{and} \quad p(\tau) := \frac{\partial L}{\partial x}(\gamma(\tau), h(\gamma(\tau), \tau), \dot{\gamma}(\tau)).\]

Then $(\dot{\bar{\gamma}}(\tau), u(\tau), p(\tau))$ is $C^1$ and satisfies the characteristics equation (L2).

By (L3) and Lemma 3.4, a standard argument (see [22, 32]) shows that the differentiability of $\bar{\gamma}(\tau)$ for $\tau \in [t_0, t_0 + \epsilon]$ can be extended to the whole interval $(0, t)$. So far, we complete the proof of Lemma 3.5.

### 3.4. Characteristics and fundamental solutions

In this step, we will prove a relation between $h_{x_0, u_0}(x, t)$ and $U(t)$, where $U(t)$ belongs to a characteristic curve $(X(t), U(t), P(t))$. More precisely, we have the following lemma:

**Lemma 3.6** For $t \in (0, T]$, let $(X(t), U(t), P(t))$ denote a characteristic curve and $S_{x_0, u_0}^x$ denote the set of $(X(t), U(t), P(t))$ satisfying $X(0) = x_0$, $X(t) = x$ and $U(0) = u_0$, then we have

\[(3.44) \quad h_{x_0, u_0}(x, t) = \inf \left\{ U(t) : (X(t), U(t), P(t)) \in S_{x_0, u_0}^x \right\}.\]

**Proof** For the simplicity of notations, we use $C_i$ to denote the constants only depending on $t$. First of all, we prove that the infimum on the right side of (3.44) can be achieved. More precisely, there exists $(\bar{X}(t), \bar{U}(t), \bar{P}(t))$ such that

\[(3.45) \quad \bar{U}(t) = \inf \left\{ U(t) : (X(t), U(t), P(t)) \in S_{x_0, u_0}^x \right\}.\]
In terms of the characteristic equation \(1.2\), it follows from the Legendre transformation that

\[(3.46) \quad \dot{U}(t) = L(X(t), U(t), \dot{X}(t)).\]

By the assumptions (L2) and (L4), there exists a constant \(C_1\) such that for any \(s \in [0, t]\),

\[L(X(s), U(s), \dot{X}(s)) \geq C_1,\]

hence \(\dot{U}(s) \geq C_1\). Moreover, a simple calculation implies

\[(3.47) \quad U(s) \geq u_0 - |C_1| t.\]

Therefore, \(\inf_{S\times x_0, u_0} U(t)\) exists, which is denoted by \(\bar{u}\). Then, one can find a sequence \((X_n(t), U_n(t), \dot{X}_n(t))\) such that (extracting a subsequence if necessary) \(U_n(t) \rightarrow \bar{u}\) as \(n \rightarrow \infty\), hence, for \(n\) large enough, we have

\[(3.48) \quad U_n(t) \leq \bar{u} + 1.\]

From \(\dot{U}(s) \geq C_1\), it follows that for any \(s \in [0, t]\),

\[U(t) - U(s) \geq C_1(t - s),\]

which together with (3.48) implies

\[U_n(s) \leq \bar{u} + 1 + |C_1| t.\]

It follows that for any \(s \in [0, t]\), \(|U_n(s)| \leq C_2\). Hence, according to (L2), it follows that for \(n\) large enough, there exists \(s_n \in [\frac{t}{2}, t]\) such that \(|\dot{X}_n(s_n)| \leq C_3\). Based on the compactness of \(M \times [0, t]\), we have (extracting a subsequence if necessary) as \(n \rightarrow \infty\)

\[(3.49) \quad (X_n(s_n), U_n(s_n), \dot{X}_n(s_n)) \rightarrow (\bar{x}, \bar{u}, \bar{x}).\]

By virtue of continuous dependence of solutions of \(1.2\) on initial conditions, it follows that there exists a characteristic curve \((\bar{X}(t), \bar{U}(t), \bar{X}(t))\) via Legendre transformation such that

\[(3.50) \quad \bar{X}(0) = x_0, \quad \bar{X}(s) = \bar{x}, \quad \bar{X}(t) = x, \quad \bar{U}(0) = u_0, \quad \bar{U}(s) = \bar{u}, \quad \bar{U}(t) = \bar{u}.\]

Therefore, there exists \((\bar{X}(t), \bar{U}(t), \bar{P}(t))\) such that

\[(3.51) \quad \bar{U}(t) = \inf \left\{U(t) : (X(t), U(t), P(t)) \in S_{x_0, u_0}^x\right\}.\]

In the following, we will prove

\[(3.52) \quad h_{x_0, u_0}(x, t) = \bar{U}(t).\]

By virtue of (3.46), we have

\[(3.53) \quad \bar{U}(t) = u_0 + \int_0^t L(\bar{X}(\tau), \bar{U}(\tau), \dot{X}(\tau))d\tau.\]
By Lemma 3.1 and Lemma 3.5 it follows that there exists a characteristic curve (via Legendre transformation) \((\bar{X}(t), \bar{U}(t), \dot{\bar{X}}(t))\) such that \(\bar{U}(t) = h_{x_0, u_0}(x, t)\) and

\[
(3.54) \quad h_{x_0, u_0}(x, t) = u_0 + \int_0^t L(\bar{X}(\tau), h_{x_0, u_0}(\bar{X}(\tau), \tau), \dot{\bar{X}}(\tau)) d\tau.
\]

By contradiction, we assume \(\bar{U}(t) \neq h_{x_0, u_0}(x, t)\). From (3.54), it follows that \(\bar{U}(t) < h_{x_0, u_0}(x, t)\). Based on Lemma 3.5, for given \(\bar{x}, \bar{u}\) and \(t\), there exist \(\epsilon, \delta > 0\) small enough such that for any \(x \in B(\bar{x}, \epsilon)\) and \(s \in (\bar{t} - \delta, \bar{t} + \delta)\), we have

\[
(3.55) \quad h_{\bar{x}, \bar{u}}(x, s) = S(x, s),
\]

where \(S(x, t)\) is a classical solution of (1.1) satisfying \(S(\bar{x}, \bar{t}) = \bar{u}\). Since \((\bar{X}(t), \bar{U}(t), \dot{\bar{X}}(t))\) is a \(C^1\) characteristic curve, then it is easy to see that there exist \(N > 0\) and a partition as follows:

\[
(3.56) \quad \{(\bar{X}(s_i), s_i) : i = 0, \ldots, N\},
\]

such that \(s_{i+1} \in (s_i - \delta, s_i + \delta)\) and \(\bar{X}(s_{i+1}) \in B(\bar{X}(s_i), \epsilon)\). In particular, we let \(s_0 = 0\) and \(s_N = t\). By (3.55), it yields

\[
(3.57) \quad h_{\bar{X}(s_i), \bar{U}(s_i)}(\bar{X}(s_{i+1}), s_{i+1}) = \bar{U}(s_{i+1}).
\]

From (3.32), it follows that

\[
\begin{aligned}
h_{x_0, u_0}(x, t) - u_0 &\leq \sum_{i=0}^N h_{\bar{X}(s_i), \bar{U}(s_i)}(\bar{X}(s_{i+1}), s_{i+1}) - \bar{U}(s_i) \\
&= \bar{U}(t) - u_0,
\end{aligned}
\]

which implies \(h_{x_0, u_0}(x, t) \leq \bar{U}(t)\). It contradicts the assumption \(\bar{U}(t) < h_{x_0, u_0}(x, t)\). This finishes the proof of Lemma 3.6. \(\square\)

So far, we have finished the proof of Theorem 1.1.

4. Representation of the viscosity solution

In this section, we are devoted to proving Theorem 1.2. First of all, we construct a variational solution of (1.1) with initial condition.

4.1. Fundamental solutions and variational solutions

Based on Theorem 1.1 it follows that under the assumptions (L1)-(L4), there exists a unique \(h_{y, \phi(y)}(x, t) \in C(M \times (0, T], \mathbb{R})\) such that

\[
(4.1) \quad h_{y, \phi(y)}(x, t) = \phi(y) + \inf_{\gamma(t)=x} \int_0^t L(\gamma(\tau), h_{y, \phi(y)}(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,
\]

where the infimums are taken among the absolutely continuous curves \(\gamma : [0, t] \to M\).
Lemma 4.1  Let

\[ u(x,t) := \inf_{y \in M} h_{y, \phi(y)}(x,t), \]

then

\[ u(x,t) = \inf_{\gamma(t) = x} \left\{ \phi(\gamma(0)) + \int_0^t L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau \right\}. \]

Proof  We denote the Lipschitz constant by \( \lambda \). The idea of the proof is similar to the one in Lemma \ref{lem:4.1}. By Lemma \ref{lem:4.2}, \( \inf_{y \in M} h_{y, \phi(y)}(x,t) \) and \( u(x,t) \) determined by (4.1) and (4.3) are unique. We will prove \( \inf_{y \in M} h_{y, \phi(y)}(x,t) = u(x,t) \) in the following.

On one hand, we will prove

\[ u(x,t) \geq \inf_{y \in M} h_{y, \phi(y)}(x,t). \]

Let \( \gamma_1 : [0,t] \to M \) be a calibrated curve of \( u \) with \( \gamma_1(t) = x \). Let \( \bar{y} := \gamma_1(0) \). It suffices to show

\[ u(x,t) \geq h_{\bar{y}, \phi(\bar{y})}(x,t). \]

By contradiction, we assume \( u(x,t) < h_{\bar{y}, \phi(\bar{y})}(x,t) \). By (4.1) and (4.3), we have

\[ h_{\bar{y}, \phi(\bar{y})}(x,t) \leq \phi(\bar{y}) + \int_0^t L(\gamma_1(\tau), u(\gamma_1(\tau), \tau), \dot{\gamma}_1(\tau))d\tau. \]

For any \( \sigma \in [0,t] \), we denote \( \bar{u}(\sigma) := u(\gamma_1(\sigma), \sigma) \) and \( \bar{h}(\sigma) := h_{\bar{y}, \phi(\bar{y})}(\gamma_1(\sigma), \sigma) \). In particular, we have \( \bar{u}(t) = u(x,t) \) and \( \bar{h}(t) = h_{\bar{y}, \phi(\bar{y})}(x,t) \). Let

\[ F(\sigma) := \bar{h}(\sigma) - \bar{u}(\sigma), \]

where \( \sigma \in [0,t] \). It is easy to see that \( \bar{u}(0) = \phi(\bar{y}) = \bar{h}(0) \). Then we have \( F(0) = 0 \). The assumption \( u(x,t) < h_{\bar{y}, \phi(\bar{y})}(x,t) \) implies \( F(t) > 0 \). Hence, there exists \( \sigma_0 \in [0,t] \) such that \( F(\sigma_0) = 0 \) and \( F(\sigma) > 0 \) for \( \sigma > \sigma_0 \). Moreover, for any \( \tau \in (\sigma_0, t] \), we have

\[ \bar{u}(\tau) = \bar{u}(\sigma_0) + \int_{\sigma_0}^\tau L(\gamma_1(\sigma), \bar{u}(\sigma), \dot{\gamma}_1(\sigma))d\sigma. \]

Let \( \gamma_2 \) be a calibrated curve of \( h_{\bar{y}, \phi(\bar{y})} \) with \( \gamma_2(0) = \bar{y}, \gamma_2(\sigma_0) = \gamma_1(\sigma_0) \). For \( \sigma \in [\sigma_0, \tau] \), we construct \( \gamma_\tau : [0, \tau] \to M \) as follows:

\[ \gamma_\tau(\sigma) = \begin{cases} \gamma_2(\sigma), & \sigma \in [0, \sigma_0], \\
\gamma_1(\sigma), & \sigma \in (\sigma_0, \tau]. \end{cases} \]

Moreover, for any \( \tau \in (\sigma_0, t] \), we have

\[ \bar{h}(\tau) \leq \bar{h}(\sigma_0) + \int_{\sigma_0}^\tau L(\gamma_1(\sigma), \bar{h}(\sigma), \dot{\gamma}_1(\sigma))d\sigma. \]
Since \( \bar{h}(\sigma_0) - \bar{u}(\sigma_0) = F(\sigma_0) = 0 \), a direct calculation implies
\[
\bar{h}(\tau) - \bar{u}(\tau) \leq \int_{\sigma_0}^{\tau} \lambda(\bar{h}(\sigma) - \bar{u}(\sigma))d\sigma.
\]
Hence, we have
\[
F(\tau) \leq \int_{\sigma_0}^{\tau} \lambda F(\sigma)d\sigma,
\]
which is in contradiction with Lemma 3.3. Hence, we have
\[
\gamma \leq \int_{\gamma_1(0)}^{\gamma_2(0)} L(h, u, \gamma_1(\tau), \gamma_2(\tau))d\tau.
\]

Lemma 4.2 u(x, t) determined by (4.3) is a variational solution of (1.1) with initial condition.

Proof Let \( \gamma : [t_1, t_2] \to M \) be a continuous and piecewise \( C^1 \) curve and Let \( \bar{\gamma} : [0, t_1] \to M \) be a calibrated curve of \( u \) satisfying \( \bar{\gamma}(t_1) = \gamma(t_1) \). We construct a curve \( \xi : [0, t_2] \to M \) defined as follows:
\[
\xi(t) = \begin{cases} \gamma(t), & t \in [0, t_1], \\ \gamma(t), & t \in (t_1, t_2]. \end{cases}
\]
From (4.3), it follows that
\[
\begin{align*}
u(\gamma(t_2), t_2) - u(\gamma(t_1), t_1) &= \inf_{\gamma_2(t_2) = \gamma(t_2)} \left\{ \phi(\gamma_2(0)) + \int_{0}^{t_2} L(\gamma_2(\tau), u(\gamma_2(\tau), \gamma_2(\tau))d\tau \right\} \\
&\quad - \inf_{\gamma_1(t_1) = \gamma(t_1)} \left\{ \phi(\gamma_1(0)) + \int_{0}^{t_1} L(\gamma_1(\tau), u(\gamma_1(\tau), \gamma_1(\tau))d\tau \right\} \\
&\leq \phi(\xi(0)) + \int_{0}^{t_2} L(\xi(\tau), u(\xi(\tau), \xi(\tau))d\tau \\
&\quad - \phi(\bar{\gamma}(0)) - \int_{0}^{t_1} L(\bar{\gamma}(\tau), u(\bar{\gamma}(\tau), \bar{\gamma}(\tau))d\tau,
\end{align*}
\]
which together with (4.13) gives rise to
\begin{equation}
(4.14) \quad u(\gamma(t_2), t_2) - u(\gamma(t_1), t_1) \leq \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau,
\end{equation}
which verifies (i) of Definition 2.4.

By means of Lemma 3.5 there exists a \( C^1 \) calibrated curve \( \gamma : [t_1, t_2] \to M \) with \( \gamma(t_2) = x \) such that
\begin{equation}
(4.15) \quad u(x, t_2) - u(\gamma(t_1), t_1) = \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau.
\end{equation}
which implies (ii) of Definition 2.4. This completes the proof of Lemma 4.2. \( \square \)

4.2. Variational solutions and viscosity solutions

In this subsection, we will prove the following lemma:

**Lemma 4.3** A variational solution of (1.1) with initial condition is a viscosity solution.

**Proof** Let \( u \) be a variational solution. Since \( u(x,0) = \phi(x) \), then it suffices to consider \( t \in (0,T] \). We use \( V \subset M \) to denote an open subset. Let \( \phi : V \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) test function such that \( u - \phi \) has a maximum at \( (x_0, t_0) \). This means \( \phi(x_0, t_0) - \phi(x, t) \leq u(x_0, t_0) - u(x, t) \). Fix \( v \in T_{x_0} M \) and for a given \( \delta > 0 \), we choose a \( C^1 \) curve \( \gamma : [t_0 - \delta, t_0 + \delta] \to M \) with \( \gamma(t_0) = x_0 \) and \( \dot{\gamma}(t_0) = \xi \). For \( t \in [t_0 - \delta, t_0] \), we have
\[
\phi(\gamma(t_0), t_0) - \phi(\gamma(t), t) \leq u(\gamma(t_0), t_0) - u(\gamma(t), t),
\]
where the second inequality is based (i) of Definition 2.4. Hence,
\begin{equation}
(4.16) \quad \frac{\phi(\gamma(t), t) - \phi(\gamma(t_0), t_0)}{t - t_0} \leq \frac{1}{t - t_0} \int_{t_0}^{t} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau.
\end{equation}
Let \( t \to t_0 \), we have
\[
\partial_t \phi(x_0, t_0) + \partial_x \phi(x_0, t_0) \cdot \xi \leq L(x_0, u(x_0, t_0), \xi),
\]
which together with Legendre transformation implies
\[
\partial_t \phi(x_0, t_0) + H(x_0, u(x_0, t_0), \partial_x \phi(x_0, t_0)) \leq 0,
\]
which shows that \( u \) is a viscosity subsolution.

To complete the proof of Theorem 4.3 it remains to show that \( u \) is a supersolution. \( \psi : V \times \mathbb{R} \to \mathbb{R} \) be a \( C^1 \) test function and \( u - \psi \) has a minimum at \( (x_0, t_0) \). We have \( \psi(x_0, t_0) - \psi(x, t) \geq u(x_0, t_0) - u(x, t) \). From (ii) of Definition 2.4 there exists a \( C^1 \) curve \( \gamma : [0, t_0] \to M \) with \( \gamma(t_0) = x_0 \) and \( \dot{\gamma}(t_0) = \eta \) such that for \( 0 \leq t < t_0 \), we have
\begin{equation}
(4.17) \quad u(\gamma(t_0), t_0) - u(\gamma(t), t) = \int_{t}^{t_0} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau)) d\tau.
\end{equation}
Hence
\[ \psi(x_0, t_0) - \psi(x, t) \geq \int_t^{t_0} L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau. \]
Moreover, we have
\[ \frac{\psi(\gamma(t), t) - \psi(\gamma(t_0), t_0)}{t - t_0} \geq \frac{1}{t - t_0} \int_{t_0}^t L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau. \]
Let \( t \) tend to \( t_0 \), it gives rise to
\[ \partial_t \psi(x_0, t_0) + \partial_x \psi(x_0, t_0) \cdot \eta \geq L(x_0, u(x_0, t_0), \eta), \]
which implies
\[ \partial_t \phi(x_0, t_0) + H(x_0, u(x_0, t_0), \partial_x \phi(x_0, t_0)) \geq 0. \]
This finishes the proof of Lemma 4.3.

By the comparison theorem (see [7] for instance), it yields that the viscosity solution of (1.1) is unique under the assumptions (H1)-(H4). So far, we have obtained that there exists a unique viscosity solution \( u(x, t) \) of (1.1) with initial condition \( u(x, 0) = \phi(x) \). So far, we complete the proof of Theorem 1.2.

5. Solution semigroup

In this section, we will prove Theorem 1.4. Let \( u(x, t) \) be the unique viscosity solution of (1.1) with initial condition \( u(x, 0) = \phi(x) \). We introduce an implicitly defined nonlinear operator \( T_t \) such that
\[
(5.1) \quad u(x, t) = T_t \phi(x).
\]
It follows from (5.1) that
\[
(5.2) \quad T_t \phi(x) = \inf_{\gamma(t) = x} \left\{ \phi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_t \phi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau \right\}.
\]
where the infimums are taken among absolutely continuous curves. In particular, the infimums are attained at the characteristics of (1.1). The following lemma implies \( T_t \) is a semigroup.

**Proposition 5.1** \{\( T_t \)\}_{t \geq 0} is a one-parameter semigroup of operators from \( C(M, \mathbb{R}) \) into itself.

**Proof** It is easy to see \( T_0 = Id \). It suffices to prove that \( T_{t+s} = T_t \circ T_s \) for any \( t, s \geq 0 \).

For every \( \eta \in C(M, \mathbb{R}) \) and \( u \in C(M \times [0, T], \mathbb{R}) \), we define an operator \( A_\eta \) such that
\[
(5.3) \quad A_\eta[u](x, t) = \inf_{\gamma(t) = x} \left\{ \eta(\gamma(0)) + \int_0^t L(\gamma(\tau), u(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau \right\}.
\]
By virtue of Theorem 1.2 it follows that \( A_\eta \) has a unique fixed point.
By \[(5.2),\] we have
\[
T_t \circ T_s \phi(x) = \inf_{\gamma(t)=x} \left\{ T_s \phi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_\tau \circ T_s \phi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\}
\]
\[
= A^{T_t \phi} [T_t \circ T_s \phi](x).
\]

On the other hand,
\[
T_{t+s} \phi(x) = \inf_{\gamma(t+s)=x} \left\{ \phi(\gamma(0)) + \int_0^{t+s} L(\gamma(\tau), T_\tau \phi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\}
\]
\[
= \inf_{\gamma(t+s)=x} \left\{ \phi(\gamma(0)) + \left( \int_s^t + \int_s^{t+s} \right) L(\gamma(\tau), T_\tau \phi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\}
\]
\[
= \inf_{\gamma(t)=x} \left\{ T_s \phi(\gamma(s)) + \int_s^t L(\gamma(\tau), T_\tau \phi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\}
\]
\[
= \inf_{\gamma(t)=x} \left\{ T_s \phi(\gamma(0)) + \int_0^t L(\gamma(\tau), T_{\tau+s} \phi(\gamma(\tau)), \dot{\gamma}(\tau)) \, d\tau \right\}
\]
\[
= A^{T_t \phi} [T_{t+s} \phi](x).
\]

Hence, both \(T_t \circ T_s \phi\) and \(T_{t+s} \phi\) are fixed points of \(A^{T_t \phi}\), which together with the uniqueness of the fixed point of \(A^{T_t \phi}\) yields \(T_{t+s} = T_t \circ T_s\). This completes the proof of Proposition 5.1.

**Proposition 5.2** For given \(y \in M\) and \(s > 0\), we have
\[
h_{y,\phi(y)}(x, s+t) = T_t h_{y,\phi(y)}(x, s),
\]
where \(h_{y,\phi(y)}(\cdot, \cdot)\) is defined as \[(3.1)\].

**Proof** Fix \(y \in M\) and \(s > 0\), one can define a continuous function \(h^s_{y,\phi(y)} : M \to \mathbb{R}\) by \(h^s_{y,\phi(y)}(x) = h_{y,\phi(y)}(x, s)\). Based on Lemma 4.1 and \[(5.1)\], we have
\[
T_t h^s_{y,\phi(y)}(x) = \inf_{z \in M} h^s_{z, h^s_{y,\phi(y)}(z)}(x, t).
\]
It follows from Lemma 5.4 that
\[
h_{y,\phi(y)}(x, s+t) = \inf_{z \in M} h^s_{z, h^s_{y,\phi(y)}(z)}(x, t).
\]
Hence, we have
\[
h_{y,\phi(y)}(x, s+t) = T_t h_{y,\phi(y)}(x, s).
\]
This completes the proof of Proposition 5.2.

To fix the notion, we call \(T_t\) a solution semigroup. In the following subsections, we will prove some further properties of the solution semigroup \(T_t\).

First of all, it is easy to obtain the following proposition about the monotonicity of \(T_t\).

**Proposition 5.3** (Monotonicity) For given \(\phi, \psi \in C(M, \mathbb{R})\) and \(t \geq 0\), if \(\phi \leq \psi\), then \(T_t \phi \leq T_t \psi\).
Proof For given $\phi, \psi \in C(M, \mathbb{R})$ with $\phi \leq \psi$, by contradiction, we assume that there exist $t_1 > 0$ and $x_1 \in M$ such that $T_{t_1} \phi(x_1) > T_{t_1} \psi(x_1)$. Let $\gamma_\psi : [0, t_1] \to M$ be a calibrated curve of $T_{t_1} \psi$ with $\gamma_\psi(t_1) = x_1$. We denote

$$F(\tau) = T_\tau \phi(\gamma_\psi(\tau)) - T_\tau \psi(\gamma_\psi(\tau)).$$

It is easy to see that $F(\tau)$ is continuous and $F(t_1) > 0$. Since

$$F(0) = \phi(\gamma_\psi(0)) - \psi(\gamma_\psi(0)) \leq 0,$$

there exists $t_0 \in [0, t_1)$ such that $F(t_0) = 0$ and for any $\tau \in [t_0, t_1)$, $F(\tau) \geq 0$. It follows from (5.2) that for any $s \in [t_0, t_1)$, we have

$$T_s \phi(\gamma_\psi(s)) - T_s \psi(\gamma_\psi(s)) = \inf_{\gamma(s) = \gamma_{\psi}(s)} \left\{ T_{t_0} \phi(\gamma(t_0)) + \int_{t_0}^{s} L(\gamma(\tau), T_\tau \phi(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\} - \inf_{\gamma(s) = \gamma_{\psi}(s)} \left\{ T_{t_0} \psi(\gamma(t_0)) + \int_{t_0}^{s} L(\gamma(\tau), T_\tau \psi(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau \right\}.$$

(5.4)

This finishes the proof of Proposition 5.3. □

Proposition 5.3 can be viewed as a comparison principle for (1.1). By a similar argument as the one in Proposition 5.3, one can obtain the Lipschitz continuity of $T_t$. For $\phi \in C(M, \mathbb{R})$, we use $\|\phi\|_\infty$ to denote $C^0$-norm of $\phi$. We have the following proposition.

Proposition 5.4 (Lipschitz continuity) For given $\phi, \psi \in C(M, \mathbb{R})$ and $t \geq 0$, we have $\|T_t \phi - T_t \psi\|_\infty \leq e^M \|\phi - \psi\|_\infty$.

First of all, we prove the following lemma.

Lemma 5.5 For given $(x, t) \in M \times (0, T]$, $x_0 \in M$, $u, v \in \mathbb{R}$, we have

(5.6) $|h_{x_0, u}(x, t) - h_{x_0, v}(x, t)| \leq e^M |u - v|.$
Proof. A similar argument implies the monotonicity of \( h_{x_0,u}(x,t) \) with respect to \( u \). More precisely, if \( u \geq v \), then \( h_{x_0,u}(x,t) \geq h_{x_0,v}(x,t) \).

Since \( u,v \in \mathbb{R} \), then we have the dichotomy: a) \( u \leq v \), b) \( u > v \). For Case a), we have \( h_{x_0,u}(x,t) \leq h_{x_0,v}(x,t) \). Let \( \gamma_u \) be a calibrated curve of \( h_{x_0,u} \) with \( \gamma_u(0) = x_0 \) and \( \gamma_u(t) = x \). From the monotonicity of \( h_{x_0,u}(x,t) \), it follows that for any \( s \in (0,t] \),

\[
(5.7) \quad h_{x_0,u}(\gamma_u(s),s) \leq h_{x_0,v}(\gamma_u(s),s).
\]

In terms of the definition of \( h_{x_0,u}(x,t) \), we have

\[
\begin{align*}
& h_{x_0,u}(\gamma_u(s),s) - h_{x_0,u}(\gamma_u(s),s) \\
\leq & v - u + \int_0^s L(\gamma_u(\tau), h_{x_0,v}(\gamma_u(\tau),\tau), \dot{\gamma}_u(\tau)) - L(\gamma_u(\tau), h_{x_0,u}(\gamma_u(\tau),\tau), \dot{\gamma}_u(\tau))d\tau, \\
\leq & v - u + \int_0^s \lambda |h_{x_0,v}(\gamma_u(\tau),\tau) - h_{x_0,u}(\gamma_u(\tau),\tau)|d\tau.
\end{align*}
\]

Let \( F(\tau) := h_{x_0,v}(\gamma_u(\tau),\tau) - h_{x_0,u}(\gamma_u(\tau),\tau) \). It follows from (5.7) that \( F(\tau) \geq 0 \) for any \( \tau \in (0,t] \). Hence, we have

\[
F(s) \leq v - u + \int_0^s \lambda F(\tau)d\tau.
\]

By Gronwall’s inequality, it yields

\[
(5.8) \quad F(s) \leq (v - u)e^{\lambda s}.
\]

In particular, we verify Lemma 5.5 for Case a).

For Case b), we have \( h_{x_0,u}(x,t) \geq h_{x_0,v}(x,t) \). Let \( \gamma_v \) be a calibrated curve of \( h_{x_0,v} \) with \( \gamma_v(0) = x_0 \) and \( \gamma_v(t) = x \). Let \( G(\tau) := h_{x_0,u}(\gamma_u(\tau),\tau) - h_{x_0,v}(\gamma_u(\tau),\tau) \). By a similar argument as Case a), we have

\[
(5.9) \quad G(s) \leq (u - v)e^{\lambda s}.
\]

Therefore, we completes the proof of Lemma 5.5 for any \( u,v \in \mathbb{R} \).

Proof of Proposition 5.4. By Theorem 1.2 we have

\[
u(x,t) = \inf_{y \in M} h_{y,\phi(y)}(x,t),
\]

where \( u(x,t) \) is a viscosity solution of (1.1). It follows from (5.1) that \( u(x,t) = T_t \phi(x) \). Hence, we have

\[
(5.10) \quad T_t \phi(x) = \inf_{y \in M} h_{y,\phi(y)}(x,t).
\]

Similarly, we have

\[
(5.11) \quad T_t \psi(x) = \inf_{z \in M} h_{z,\psi(z)}(x,t).
\]

Lemma 5.5 implies that \( h_{y,\phi(y)}(x,t) \) is continuous with respect to \( y \). Based on the compactness of \( M \), the infimums in (5.10) and (5.11) can be attained at \( y_0 \) and \( z_0 \) respectively. On one hand, we have

\[
T_t \phi(x) - T_t \psi(x) \\
\leq h_{x_0,\phi(z_0)}(x,t) - h_{x_0,\psi(z_0)}(x,t), \\
\leq e^{M|\phi(z_0) - \psi(z_0)|}, \\
\leq e^{M||\phi(x) - \psi(x)||_{\infty}}.
\]
On the other hand, we have
\[ T_t \phi(x) - T_t \psi(x) \]
\[ \geq h_{y_0, \phi(y_0)}(x, t) - h_{y_0, \psi(y_0)}(x, t), \]
\[ \geq -e^{M} |\phi(y_0) - \psi(y_0)|, \]
\[ \geq -e^{M} \|\phi(x) - \psi(x)\|_{\infty}. \]

Hence,
\[ \|T_t \phi(x) - T_t \psi(x)\|_{\infty} \leq e^{M} \|\phi(x) - \psi(x)\|_{\infty}. \]

This completes the proof of Proposition 5.4.

6. Large time behavior of the viscosity solution

In this section, we will prove Theorem 1.6, which is concerned with the large time behavior of \( T_t \).

6.1. Critical values

For \( c \in \mathbb{R} \), we denote \( L_c := L + c \). For given \( x_0, u_0, x, t \) where \( t \in (0, +\infty) \), we denote
\[ h^c_{x_0, u_0}(x, t) = u_0 + \inf_{\gamma(t) = x} \int_0^t L_c(\gamma(\tau), h^c_{x_0, u_0}(\gamma(\tau), \tau), \dot{\gamma}(\tau))d\tau, \]
where the infimums are taken among the absolutely continuous curves \( \gamma : [0, t] \to M \).

The critical value set is defined as
\[ C = \{ c : |h^c_{x_0, u_0}(x, t)| \leq K(u_0) \}, \]
where \( K(u_0) \) is a positive constant depending on \( u_0 \). For \( a \in \mathbb{R} \), we use \( c(L(x, a, \dot{x})) \) to denote M\'an\’e critical value of \( L(x, a, \dot{x}) \). By (37), \( C \neq \emptyset \) if \( L(x, u, \dot{x}) \) is non-increasing with respect to \( u \). The following theorem gives the more general conditions under which \( C \) is not empty.

**Theorem 6.1** If there exist \( L_1(x, \dot{x}), L_2(x, \dot{x}) \) satisfying (L1) and (L2) such that
\[ \lim_{a \to +\infty} L(x, a, \dot{x}) \leq L_1(x, \dot{x}), \]
\[ \lim_{a \to -\infty} L(x, a, \dot{x}) \geq L_2(x, \dot{x}), \]
then for any \( c \in [c(L_2(x, \dot{x})), c(L_1(x, \dot{x}))] \), there exists \( K > 0 \) such that for \( t \geq \delta \)
\[ |h^c_{x_0, u_0}(x, t) - u_0| \leq K. \]

**Proof** For the simplicity of notations, we denote \( c_1 := c(L_1(x, \dot{x})) \) and \( c_2 := c(L_2(x, \dot{x})) \). The proof is divided into two steps.

In Step One, we prove \( h^c_{x_0, u_0}(x, t) \) is upper bounded. By contradiction, we assume that for any \( K > 0 \), there exists \( t' \gg 0 \) such that \( h^c_{x_0, u_0}(x, t') = K + u_0 \). Let
$\gamma : [0,t'] \to M$ be a calibrated curve of $h^c_{x_0,u_0}(x,t)$ with $\gamma(0) = x_0$ and $\gamma(t') = x$. Since $h^c_{x_0,u_0}(\gamma(t),t) = U(t)$ is $C^1$ with respect to $t$ for $t \in (0,t')$, then one can find $t'' > \delta$ such that $h^c_{x_0,u_0}(\gamma(t''),t'') = K/2 + u_0$ and for any $\tau \in [t'',t']$,

$$K/2 \leq h^c_{x_0,u_0}(\gamma(\tau),\tau) - u_0 \leq K. \tag{6.5}$$

Based on the definition of $h^c_{x_0,u_0}(x,t)$, we have

$$h^c_{x_0,u_0}(x,t') = u_0 + \inf_{\gamma(t') = x} \int_0^{t'} L_c(\gamma(\tau),h^c_{x_0,u_0}(\gamma(\tau),\tau),\dot{\gamma}(\tau))d\tau. \tag{6.6}$$

Let $\gamma_1 : [t'',t] \to M$ be a minimal curve with $\gamma_1(t'') = \gamma(t'')$ and $\gamma_1(t') = x$. That is

$$\int_0^{t'} L_1(\gamma_1(\tau),\dot{\gamma}_1(\tau))d\tau = \inf_{\tilde{\gamma}(t'') = \gamma(t'')} \int_0^{t'} L_1(\tilde{\gamma}(\tau),\dot{\tilde{\gamma}}(\tau))d\tau. \tag{6.7}$$

Moreover, it follows that

$$h^c_{x_0,u_0}(x,t') = h^c_{x_0,u_0}(\gamma(t''),t'') + \int_{t''}^{t'} L_c(\gamma(\tau),h^c_{x_0,u_0}(\gamma(\tau),\tau),\dot{\gamma}(\tau))d\tau,$$

$$\leq K/2 + u_0 + \int_{t''}^{t'} L_c(\gamma_1(\tau),h^c_{x_0,u_0}(\gamma_1(\tau),\tau),\dot{\gamma}_1(\tau))d\tau,$$

$$\leq K/2 + u_0 + \int_{t''}^{t'} L_1(\gamma_1(\tau),\dot{\gamma}_1(\tau)) + cd\tau,$$

$$\leq K/2 + u_0 + \int_{t''}^{t'} L_1(\gamma_1(\tau),\dot{\gamma}_1(\tau)) + c_1d\tau,$$

$$= K/2 + u_0 + h_{c_1}^{t'-t''}(\gamma(t''),x),$$

where the third inequality is owing to (6.5) and the assumption

$$\lim_{a \to +\infty} L(x,a,\dot{x}) \leq L_1(x,\dot{x}).$$

$h_{c_1}^{t'-t''}(\gamma(t''),x)$ denotes the minimal action with respect to $L_1$ (see [22] for instance).

It is easy to see that $t' - t'' > 1$, then from the compactness of $M$, it follows that $h_{c_1}^{t'-t''}(\gamma(t''),x)$ has a bound denoted by $A$ independent of $t', t''$ and $x$. Hence, we have

$$K \leq \frac{K}{2} + A.$$

Since $K$ is large enough, then we have a contradiction if we take $K > 3A$.

In Step Two, we prove $h^c_{x_0,u_0}(x,t)$ is lower bounded. By contradiction, we assume that for any $-K < 0$, there exists $t' > 0$ such that $h^c_{x_0,u_0}(x,t') = -K + u_0$. Let $\gamma : [0,t'] \to M$ be a calibrated curve of $h^c_{x_0,u_0}(x,t)$ with $\gamma(0) = x_0$ and $\gamma(t') = x$. Hence, one can find $t'' > \delta$ such that $h^c_{x_0,u_0}(\gamma(t''),t'') = -K/2 + u_0$ and for any $\tau \in [t'',t']$,

$$-K \leq h^c_{x_0,u_0}(\gamma(\tau),\tau) - u_0 \leq -\frac{K}{2}. \tag{6.8}$$
Moreover, it follows that
\[
h_{x_0,u_0}(x,t') = h_{x_0,u_0}(\gamma(t''), t'') + \int_{t''}^{t'} L_c(\gamma(\tau), h_{x_0,u_0}(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau,
\]
\[
\geq -\frac{K}{2} + u_0 + \int_{t''}^{t'} L_2(\gamma(\tau), \dot{\gamma}(\tau)) + c_d d\tau,
\]
\[
\geq -\frac{K}{2} + u_0 + \int_{t''}^{t'} L_2(\gamma(\tau), \dot{\gamma}(\tau)) + c_d d\tau,
\]
\[
= -\frac{K}{2} + u_0 + h_{c_2 - t''}(\gamma(t''), x),
\]
where the second inequality is owing to \(6.8\) and the assumption
\[
lim_{a \to -\infty} L(x,a,\dot{x}) \geq L_2(x,\dot{x}).
\]
\(h_{c_2 - t''}(\gamma(t''), x)\) denotes the minimal action with respect to \(L_2\). It is easy to see that \(h_{c_2 - t''}(\gamma(t''), x) > 0\) (see [22]). Hence, we have
\[
-K \geq -\frac{K}{2},
\]
which contradicts the assumption \(K > 0\). So far, we have shown \(h_{x_0,u_0}(x,t)\) is uniformly bounded for \(t \geq \delta\). This finishes the proof of Theorem 6.1. \(\square\)

Unfortunately, we do not know whether the conditions in Theorem 6.1 is sharp or not. Generally, the critical value may not exist. For instance, we consider the Hamilton-Jacobi equation:

\[
\begin{align*}
\partial_t u - u + \frac{1}{2} |\partial_x u|^2 &= c, \\
u(x,0) &= \phi(x),
\end{align*}
\]
where the Hamiltonian \(H(x,u,p) = \frac{1}{2} |p|^2 - u\) which satisfies the assumptions (H1)-(H4). It is easy to see that for any non-constant function \(\phi(x) \in C(M, \mathbb{R})\) and \(c \in \mathbb{R}\), there holds \(u(x,t) \to \infty\) exponentially as \(t \to \infty\), which means \(C = \emptyset\). Conversely, we are concerned with

\[
\begin{align*}
\partial_t u + u + \frac{1}{2} |\partial_x u|^2 &= c, \\
u(x,0) &= \phi(x).
\end{align*}
\]
It follows that for any \(\phi(x) \in C(M, \mathbb{R})\) and \(c \in \mathbb{R}\), there holds \(u(x,t) \to c\) as \(t \to \infty\), for which \(C = \mathbb{R}\). Moreover, \(u(x,t)\) converges to \(u(x) \equiv c\) which is the unique viscosity solution of the stationary equation on \(M\):

\[
u + \frac{1}{2} |\partial_x u|^2 = c.\]

### 6.2. Large time behavior of the solution semigroup

In order to consider the large time behavior of \(T_t\), we need the following assumption:
(H5) **Non-emptiness:** The critical value set $C$ is not empty.

Without ambiguity, we still use $L$ instead of $L_c$ to denote $L + c$ for $c \in C$. The same to $H$ and $T_t$. Based on a similar argument as the one in [37], we have

**Proposition 6.2** For any $\phi(x) \in C(M, \mathbb{R})$, $x, y \in M$ and $t > \delta$, we have

I. there exists a positive constant $K$ independent of $t$ such that $\|h_{y,\phi(y)}(x, t)\|_{\infty} \leq K(\phi)$;

II. for $\delta > 0$, there exists a compact subset $K_\delta$ such that for every calibrated curve $\gamma$ of $h_{y,\phi(y)}(x, t)$ and any $t > \delta$, we have

$$(\gamma(t), h_{y,\phi(y)}(\gamma(t), t), \dot{\gamma}(t)) \in K_\delta.$$  

III. for $\delta > 0$, the family of functions $(x, t) \to h_{y,\phi(y)}(x, t)$ is equi-Lipschitz on $(x, t) \in M \times [\delta, +\infty)$.

We omit the proof of Proposition 6.2 here. See Theorem 1.3 in [37] for the details. By Theorem 1.4, there holds

$$T_t \phi(x) = \inf_{y \in M} h_{y,\phi(y)}(x, t).$$

Let

$$\underline{u}(x) := \liminf_{t \to \infty} T_t \phi(x).$$

Proposition 6.2 implies $\underline{u}(x)$ is a Lipschitz function.

**Lemma 6.3** For any $t \geq 0$, we have

$$T_t \underline{u}(x) = \underline{u}(x).$$

**Proof** We denote $u_s(x) := \inf_{t \geq s} T_t \phi(x)$, then $\underline{u}(x) = \lim_{s \to \infty} u_s(x)$. It suffices to prove $T_\delta \underline{u}(x) = \underline{u}(x)$ for any $\delta \geq 0$.

One one hand, we will prove $T_\delta \underline{u}(x) \leq \underline{u}(x)$. It is easy to see that

$$u_{\delta+s}(x) = \inf_{t \geq \delta+s} T_t \phi(x),$$

(6.11)

$$= \inf_{t-s \geq \delta} T_{t-s} \circ T_s \phi(x),$$

$$= \inf_{t \geq s} T_\delta \circ T_t \phi(x).$$

For $t \geq s$, we have $u_s(x) \leq T_t \phi(x)$. It follows from the monotonicity of $T_t$ that

$$T_\delta \circ T_t \phi(x) \geq T_\delta u_s(x).$$

Moreover, we have

$$\inf_{t \geq s} T_\delta \circ T_t \phi(x) \geq T_\delta u_s(x),$$

which together with (6.11) implies

$$u_{\delta+s}(x) \geq T_\delta u_s(x).$$
Taking the limit as \( s \to \infty \) in both sides, we have
\[
(6.12) \quad T_\delta u_s(x) \leq u(x).
\]

On the other hand, we have
\[
(6.13) \quad T_\delta u_s(x) = T_\delta \left( \inf_{t \geq s} T_t \phi(x) \right) = \inf_{y \in M} h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta).
\]

**Claim:**
\[
\inf_{y \in M} h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) = \inf_{t \geq s} \left( \inf_{y \in M} h_y, T_t \phi(y)(x, \delta) \right).
\]

**Proof** It is easy to see that \( h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) \) is continuous with respect to \( y \). Based on the compactness of \( M \), there exists \( y_0 \in M \) such that the infimum is attained. Hence,
\[
\inf_{y \in M} h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) = h_{y_0}, \inf_{t \geq s} T_t \phi(y_0)(x, \delta) \geq \inf_{t \geq s} \left( \inf_{y \in M} h_y, T_t \phi(y)(x, \delta) \right).
\]

On the other hand, it follows from the monotonicity of \( h_y, u(x, t) \) with respect to \( u \) that
\[
h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) \leq h_y, T_t \phi(y)(x, \delta),
\]
which yields
\[
\inf_{y \in M} h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) \leq \inf_{t \geq s} \left( \inf_{y \in M} h_y, T_t \phi(y)(x, \delta) \right).
\]
This completes the proof of the claim. \( \square \)

From (6.13), we have
\[
T_\delta u_s(x) = \inf_{y \in M} h_y, \inf_{t \geq s} T_t \phi(y)(x, \delta) = \inf_{t \geq s} \left( \inf_{y \in M} h_y, T_t \phi(y)(x, \delta) \right),
\]
\[
= \inf_{t \geq s} T_\delta \circ T_t \phi(x) = \inf_{t \geq \delta + s} T_t \phi(x),
\]
\[
= u_{\delta + s}(x) \geq u_s(x).
\]
Taking the limit as \( s \to \infty \) in both sides, it follows that
\[
(6.14) \quad T_\delta u(x) \geq u(x),
\]
which together with (6.12) completes the proof of Lemma 6.3. \( \square \)

**Lemma 6.4** \( T_t u(x) = u(x) \) for any \( t \geq 0 \) if and only if \( u(x) \) is a weak KAM solution of the following stationary equation:
\[
(6.15) \quad H(x, u(x), \partial_x u(x)) = 0.
\]
Proof. We suppose $T_t u(x) = u(x)$ for any $t \geq 0$. By virtue of a similar argument as Lemma 6.2, it yields that for each continuous piecewise $C^1$ curve $\gamma : [t_1, t_2] \to M$ where $0 \leq t_1 < t_2 \leq T$, we have

\[(6.16) \quad u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(\tau), u(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau,\]

which implies (i) of Definition 2.2. In addition, there exists a $C^1$ calibrated curve $\gamma_t : [-t, 0] \to M$ with $\gamma_t(0) = x$ such that for any $t' \in [-t, 0]$, we have

\[(6.17) \quad u(x) - u(\gamma(t')) = \int_{t'}^0 L(\gamma_t(\tau), u(\gamma_t(\tau)), \dot{\gamma}_t(\tau)) d\tau.\]

Based on the a priori compactness given by Lemma 6.2 II., for a given $\delta > 0$, there exists a compact subset $K_\delta$ such that for any $s > \delta$, we have

\[(\gamma_t(s), u(\gamma_t(s)), \dot{\gamma}_t(s)) \in K_\delta.\]

Since $\gamma_t$ is a calibrated curve, it follows from Lemma 3.3 that

\[ (\gamma_t(s), u(\gamma_t(s)), \dot{\gamma}_t(s)) = \Phi_s(\gamma_t(0), u(\gamma_t(0)), \dot{\gamma}_t(0)) = \Phi_s(x, u(x), \dot{\gamma}_t(0)). \]

The points $(\gamma_t(0), u(\gamma_t(0)), \dot{\gamma}_t(0))$ are contained in a compact subset, then one can find a sequence $t_n$ such that $(x, \gamma_{t_n}(0))$ tends to $(x, v_\infty)$ as $n \to \infty$. Fixing $t' \in (-\infty, 0]$, the function $s \mapsto \Phi_s(x, u(x), \dot{\gamma}_t(0))$ is defined on $[t', 0]$ for $n$ large enough. By the continuity of $\Phi_s$, the sequence converges uniformly on the compact interval $[t', 0]$ to the map $s \mapsto \Phi_s(x, v_\infty)$. Let

\[(\gamma_\infty(s), u(\gamma_\infty(s)), \dot{\gamma}_\infty(s)) := \Phi_s(x, v_\infty), \]

then for any $t' \in (-\infty, 0]$, we have

\[(6.18) \quad u(x) - u(\gamma_\infty(t')) = \int_{t'}^0 L(\gamma_\infty(\tau), u(\gamma_\infty(\tau)), \dot{\gamma}_\infty(\tau)) d\tau,\]

which implies (ii) of Definition 2.2. Hence, $u$ is a weak KAM solution of $(6.15)$. 

Conversely, we suppose $u$ is a weak KAM solution of $(6.15)$. By (i) of Definition 2.2, we have

\[ u \leq T_t u. \]

By (ii) of Definition 2.2, for any $x \in M$, there exists a $C^1$ curve $\bar{\gamma} : (-\infty, 0] \to M$ with $\bar{\gamma}(0) = x$ such that for any $t \in [0, +\infty)$,

\[(6.19) \quad u(x) - u(\bar{\gamma}(-t)) = \int_{-t}^0 L(\bar{\gamma}(\tau), u(\bar{\gamma}(\tau)), \dot{\bar{\gamma}}(\tau)) d\tau.\]

We define the curve $\gamma : [0, t] \to M$ by $\gamma(s) = \bar{\gamma}(s - t)$. There hold $\gamma(t) = x$ and

\[ u(x) = \int_0^t L(\gamma(\tau), u(\gamma(\tau)), \dot{\gamma}(\tau)) d\tau + u(\gamma(0)), \]

which implies

\[ T_t u \leq u. \]

This completes the proof of Lemma 6.4. \[\square\] 

So far, we have finished the proof of Theorem 1.6.
7. Projected Aubry set and the stationary equation

In this section, we will define the projected Aubry set with respect to the stationary Hamilton-Jacobi equation \((1.11)\). Moreover, we will prove Theorem 1.7. Without ambiguity, we still use \(L\) instead of \(L_c\) to denote \(L + c\) for \(c \in \mathcal{C}\). The same to \(H\) and \(T_t\).

7.1. Projected Aubry set

For a given \(s_0 > 0\), we take \(\phi(x) = h_{x_0,u_0}(x,s_0)\), where \(x_0 \in M, u_0 \in \mathbb{R}\). Then \(\phi(x) \in C(M, \mathbb{R})\). By virtue of Theorem 1.4, for \(t \geq 0\), there holds
\[
h_{x_0,u_0}(x,s_0 + t) = T_t h_{x_0,u_0}(x,s_0) = T_t \phi(x).
\]
Let
\[
h_{x_0,u_0}(x,\infty) := \liminf_{t \to \infty} h_{x_0,u_0}(x,t).
\]
It is easy to see that
\[
h_{x_0,u_0}(x,\infty) = \liminf_{t \to \infty} h_{x_0,u_0}(x,s_0 + t) = \liminf_{t \to \infty} T_t \phi(x).
\]
By Theorem 1.6, \(h_{x_0,u_0}(x,\infty)\) is a viscosity solution of \((1.11)\). Based on \([37]\), “\(\liminf\)” can be replaced with “\(\lim\)” if \(H(x,u,p)\) is non-decreasing with respect to \(u\).

We denote
\[
B(x,u;y) := h_{x,u}(y,\infty) - u.
\]

\(B(x,u;y)\) can be referred as the barrier function denoted by \(h_{\infty}(x,y)\) in Mather-Fathi theory. Moreover, we define the projected Aubry set as follows;
\[
\mathcal{A} := \{(x,u) \in M \times \mathbb{R} \mid B(x,u;x) = 0\}.
\]

**Proposition 7.1** \(\mathcal{A} \neq \emptyset\) under the assumptions \((H1)-(H5)\).

**Proof** For given \(x_0, y_0 \in M\) and \(u_0 \in \mathbb{R}\), let \(\gamma_n : [0,t_n] \to M\) be a calibrated curve of \(h_{x_0,u_0}\) with \(\gamma_n(0) = x_0\) and \(\gamma_n(t_n) = y_0\). Under the assumptions \((H1)-(H5)\), Proposition 6.2 implies that for \(\delta > 0\), there exists a compact subset \(\mathcal{K}_\delta \subset M \times \mathbb{R}\) such that for any \(\tau > \delta\), we have
\[
(\gamma_n(\tau), h_{x_0,u_0}(\gamma_n(\tau),\tau)) \in \mathcal{K}_\delta.
\]
Exchanging a subsequence if necessary, one can find \(s_n, \tau_n \in [0,t_n]\) satisfying \(s_n - \tau_n \to \infty\) as \(n \to \infty\) and
\[
d((\gamma_n(s_n), h_{x_0,u_0}(\gamma_n(s_n),s_n)), (\gamma_n(\tau_n), h_{x_0,u_0}(\gamma_n(\tau_n),\tau_n))) \leq \frac{1}{n},
\]
where \(d(\cdot,\cdot)\) denotes the distance induced by a Riemannian metric on \(M \times \mathbb{R}\). Hence, there exists \((x_\infty,u_\infty) \in M \times \mathbb{R}\) such that
\[
h_{x_\infty,u_\infty}(x_\infty,\infty) = u_\infty.
\]
This completes the proof of Proposition 7.1. \(\square\)
7.2. Projected Aubry set and the viscosity solution

In order to find finer properties of $\mathcal{A}$, we add the following assumption:

(H6) **Strict increase**: $H(x,u,p)$ is strictly increasing with respect to $u$ for a given $(x,p) \in T^*M$.

It is easy to see that (H5) holds under (H6) (see [37]). (H6) is equivalent to

(L6) **Strict decrease**: $L(x,u,\dot{x})$ is strictly decreasing with respect to $u$ for a given $(x,\dot{x}) \in TM$.

It is easy to obtain the following proposition about the contractibility of $T_t$. For $\phi \in C(M,\mathbb{R})$, we use $\|\phi\|_{\infty}$ to denote $C^0$-norm of $\phi$. We have the following proposition.

**Lemma 7.2 (Contractibility)** For given $\phi, \psi \in C(M,\mathbb{R})$ and $t \geq 0$, if there exists $\bar{x}$ such that $\phi(\bar{x}) \neq \psi(\bar{x})$, then $\|T_t \phi - T_t \psi\|_{\infty} < \|\phi - \psi\|_{\infty}$.

First of all, we prove the following lemma.

**Lemma 7.3** For given $x_0, x \in M$, $u, v \in \mathbb{R}$ and $t > 0$, if $u \neq v$, then we have

$$|h_{x_0,u}(x,t) - h_{x_0,u}(x,t)| < |u - v|.$$  

**Proof** A similar argument as Proposition [37] implies the monotonicity of $h_{x_0,u}(x,t)$ with respect to $u$. More precisely, if $u \geq v$, then $h_{x_0,u}(x,t) \geq h_{x_0,v}(x,t)$.

Since $u, v \in \mathbb{R}$ and $u \neq v$, then we have the dichotomy: a) $u < v$, b) $u > v$. For Case a), we have $h_{x_0,u}(x,t) \leq h_{x_0,v}(x,t)$. Let $\gamma_u$ be a calibrated curve of $h_{x_0,u}$ with $\gamma_u(0) = x_0$ and $\gamma_u(t) = x$. From the monotonicity of $h_{x_0,u}(x,t)$ with respect to $u$ and the continuity of $h_{x_0,u}(x,t)$ with respect to $t$, it follows that there exits $\delta > 0$ such that for any $s \in (0,\delta)$,

$$h_{x_0,u}(\gamma_u(s), s) < h_{x_0,v}(\gamma_u(s), s).$$

In terms of the definition of $h_{x_0,u}(x,t)$, we have

$$h_{x_0,v}(x,t) = v + \inf_{\gamma(0)=x_0} \int_0^t L(\gamma(\tau), h_{x_0,v}(\gamma(\tau), \dot{\gamma}(\tau) )d\tau,$$

$$\leq v + \int_0^t L(\gamma_u(\tau), h_{x_0,u}(\gamma_u(\tau), \dot{\gamma}_u(\tau) )d\tau,$$

$$< v + \int_0^t L(\gamma_u(\tau), h_{x_0,u}(\gamma_u(\tau), \dot{\gamma}_u(\tau) )d\tau,$$

$$= v - u + \int_0^t L(\gamma_u(\tau), h_{x_0,u}(\gamma_u(\tau), \dot{\gamma}_u(\tau) )d\tau, $$

$$= v - u + h_{x_0,u}(x,t),$$

where the third inequality is from (L6) and (7.4). Hence, there holds

$$0 \leq h_{x_0,v}(x,t) - h_{x_0,u}(x,t) < v - u.$$  

In particular, we verify Lemma [37] for Case a).
For Case b), we have $h_{x_0,u}(x,t) \geq h_{x_0,v}(x,t)$. Let $\gamma_v$ be a calibrated curve of $h_{x_0,v}$ with $\gamma_v(0) = x_0$ and $\gamma_v(t) = x$. It follows that there exists $\delta' > 0$ such that for any $s \in (0, \delta')$,

\[(7.4)\quad h_{x_0,v}(\gamma_v(s),s) < h_{x_0,u}(\gamma_v(s),s)\]

By a similar argument as Case a), we have

\[(7.5) \quad 0 \leq h_{x_0,u}(x,t) - h_{x_0,v}(x,t) < u - v.\]

Therefore, we completes the proof of Lemma 7.3 for any $u,v \in \mathbb{R}$ and $u \neq v$. □

**Proof of Lemma 7.2** By Theorem 1.2 and Theorem 1.4, we have

\[(7.6) \quad T_t \phi(x) = \inf_{y \in \mathcal{M}} h_{y,\phi(y)}(x,t), \quad T_t \psi(x) = \inf_{z \in \mathcal{M}} h_{z,\psi(z)}(x,t).\]

Based on the compactness of $\mathcal{M}$, the infimums in (7.6) can be attained at $y_0$ and $z_0$ respectively. If $\phi(z_0) = \psi(z_0)$, we have

\[T_t \phi(x) - T_t \psi(x) \leq h_{z_0,\phi(z_0)}(x,t) - h_{z_0,\psi(z_0)}(x,t) = 0,\]

\[<|\phi(x) - \psi(x)| \leq \|\phi(x) - \psi(x)\|_{\infty}.\]

If $\phi(z_0) \neq \psi(z_0)$, we have

\[T_t \phi(x) - T_t \psi(x) \leq h_{z_0,\phi(z_0)}(x,t) - h_{z_0,\psi(z_0)}(x,t),\]

\[<|\phi(x) - \psi(x)| \leq \|\phi(x) - \psi(x)\|_{\infty}.\]

Hence, there holds

\[(7.7) \quad T_t \phi(x) - T_t \psi(x) < \|\phi(x) - \psi(x)\|_{\infty}.\]

Similarly, we have

\[(7.8) \quad T_t \phi(x) - T_t \psi(x) > -\|\phi(x) - \psi(x)\|_{\infty},\]

which together with (7.7) implies

\[\|T_t \phi(x) - T_t \psi(x)\|_{\infty} < \|\phi(x) - \psi(x)\|_{\infty}.\]

This completes the proof of Lemma 7.2 □

**Lemma 7.4 (Uniqueness)** Let $H$ satisfy (H1)-(H4) and (H6), then there exists a unique viscosity solution satisfying the stationary equation

\[(7.9) \quad H(x,u(x),\partial_x u(x)) = 0.\]
**Proof** A similar argument as Theorem 7.6.2 in [22] implies the weak KAM solution is also equivalent to the viscosity solution for the stationary equation [19]. By Lemma 6.4, $u$ is a weak KAM solution if and only if $T_t u = u$ for any $t \geq 0$.

By contradiction, we assume there exist at least two weak KAM solutions $u(x), v(x) \in C(M, \mathbb{R})$ with $u(x_0) \neq v(x_0)$ for some $x_0 \in M$. Hence, there hold $T_t u = u$ and $T_t v = v$. It follows from Proposition 7.2 that

$$\|u - v\|_\infty = \|T_t u - T_t v\|_\infty < \|u - v\|_\infty,$$

which is a contradiction. This completes the proof of Lemma 7.4. \qed

We use $\pi : M \times \mathbb{R} \to M$ to denote the standard projection via $(x, u) \to x$.

**Lemma 7.5** Let $H$ satisfy $(H1)$-$(H4)$ and $(H6)$, for $x \in \pi \mathcal{A}$, there exists a unique $u_x$ such that $(x, u_x) \in \mathcal{A}$. Moreover, there holds $u(x) = u_x$ for any $x \in \pi \mathcal{A}$, where $u(x) \in C(M, \mathbb{R})$ is the unique viscosity solution.

**Proof** By contradiction, we assume for $x_0 \in \pi \mathcal{A}$, there exist $u_1, u_2 \in \mathbb{R}$ such that $h_{x_0, u_1}(x_0, \infty) = u_1$ and $h_{x_0, u_2}(x_0, \infty) = u_2$. By Theorem 1.6, $h_{x_0, u_1}(x, \infty)$ and $h_{x_0, u_2}(x, \infty)$ are viscosity solutions of (7.9). Lemma 7.4 implies

$$h_{x_0, u_1}(x, \infty) \equiv h_{x_0, u_2}(x, \infty).$$

In particular, we have

$$u_1 = h_{x_0, u_1}(x_0, \infty) = h_{x_0, u_2}(x_0, \infty) = u_2.$$

For $(y, u_y), (z, u_z) \in \mathcal{A}$, we denote

$$u_y(x) := h_{y, u_y}(x, \infty), \quad u_z(x) := h_{z, u_z}(x, \infty).$$

It follows from Lemma 7.4 that $u_y(x) \equiv u_z(x)$ denoted by $u(x)$. Then $u(x) = u_x$ for any $x \in \pi \mathcal{A}$. \qed

Based on Lemma 7.4, one can obtain the following lemma.

**Lemma 7.6 (Representation formula)** Let $u(x)$ be the viscosity solution of (7.9), then

$$u(x) = \inf_{y \in \pi \mathcal{A}} h_{y, u_y}(x, \infty). \tag{7.10}$$

**Proof** By virtue of the uniqueness of the viscosity solution, it follows that for given $(x_0, u_0) \in M \times \mathbb{R}$, $u(x) = h_{x_0, u_0}(x, \infty)$. Lemma 3.4 yields

$$h(x, t + s) = \inf_{y \in M} h_{y, h(y,t)}(x, s), \tag{7.11}$$

where the “$\inf$” can be attained at $y$ belonging to the calibrated curve $\gamma$ of $h$ with $\gamma(0) = x_0$ and $\gamma(t + s) = x$. Hence, we have

$$u(x) = h_{x_0, u_0}(x, \infty) = \inf_{y \in M} h_{y, h_{x_0, u_0}(y, \infty)}(x, \infty). \tag{7.12}$$

On one hand, it follows from Lemma 7.25 that

$$\inf_{y \in M} h_{y, h_{x_0, u_0}(y, \infty)}(x, \infty) \leq \inf_{y \in \pi \mathcal{A}} h_{y, h_{x_0, u_0}(y, \infty)}(x, \infty) = \inf_{y \in \pi \mathcal{A}} h_{y, u(y)}(x, \infty). \tag{7.13}$$
On the other hand, there exists $y_0 \in \pi A$ such that
\[
\inf_{y \in M} h_{y,h_{x_0,u_0}(y,\infty)}(x,\infty) = h_{y_0,h_{x_0,u_0}(y_0,\infty)}(x,\infty) = h_{y_0,u(y_0)}(x,\infty),
\]
which together with (7.12) and (7.13) yields
\[
u(x) = \inf_{y \in \pi A} h_{y,u(y)}(x,\infty).
\]
This completes the proof of Lemma 7.6. □

So far, we have finished the proof of Theorem 1.7.

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