Nonlocal approaches for multilane traffic models

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Abstract

We present a multilane traffic model based on balance laws, where the nonlocal source term is used to describe the lane changing rate. The modelling framework includes the consideration of local and nonlocal flux functions. Based on a Godunov type numerical scheme, we provide BV estimates and a discrete entropy inequality. Together with the $L^1$-contractivity property, we prove existence and uniqueness of weak solutions. Numerical examples show the nonlocal impact compared to local flux functions and local sources.

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1 Introduction

The progress in autonomous driving brings new challenges for the modelling of traffic flow. Classical approaches such as the well-established Lighthill-Whitham-Richards (LWR) model \cite{Lighthill1955, Whitham1955} have been recently extended to include more information on the surrounding traffic, see for example \cite{Bando1995, Goetz2014, Goettlich2017, Goettlich2020, Goettlich2021}. Therefore, we distinguish between \textit{local} traffic flow models governed by conservation laws, where the fundamental diagram gives the relation between flux and density, and \textit{nonlocal} models with flux functions depending on an integral evaluation of the density or velocity through a convolution product. In case of autonomous vehicles the nonlocal models allow for an interpretation as the connection radius.

Nonlocal traffic flow models have been introduced in \cite{Bando1995} and since then have been studied regarding existence and well-posedness, e.g. \cite{Bando1995, Goetz2014, Goettlich2020}, numerical schemes \cite{Bando1995, Goetz2014, Goettlich2017, Goettlich2020, Goettlich2021}, convergence to local conservation laws e.g. \cite{Goettlich2020} (even if this question is still an open research problem), microscopic modelling approaches \cite{Goetz2014, Goettlich2017, Goettlich2020, Goettlich2021}, second order models \cite{Goetz2014}, multi-class models \cite{Goettlich2017}, time delay models \cite{Goettlich2021} and network formulations \cite{Goetz2014, Goettlich2020}.

The aim of this paper is to study a multilane model with local and nonlocal flux combined with a source term that also incorporates a nonlocality. Here, the nonlocal source term describes the lane changing rate depending on a (nonlinear) evaluation of the velocity. In this context, we refer to \cite{Goettlich2020}, where a nonlocal source term is used to describe the lane change. However, the modelling of our source term is inspired by \cite{Goettlich2021}. We would also like to mention

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that a similar multilane model with nonlocal flux and source has been recently introduced in [2]. Therein, well-posedness and uniqueness are proven based on Banach’s fixed point theorem using the method of characteristics. In contrast to the contributions [2, 11, 19], we do not investigate the model on the continuous description and present a Godunov type numerical scheme instead that can be used to show existence and uniqueness of approximate solutions.

From a modelling point of view, the key differences to [2] are that the nonlocal terms in the flux and in the source do not have necessarily the same kernels. More precisely, in [2] only forward looking and decreasing kernels can be considered within the convolution product. It seems that our approach is more flexible, since also back- and forward looking kernels can be considered.

The paper is organised as follows: In Section 2 we present the model with local flux and nonlocal source, while in Section 3 the Godunov type discretization is addressed to show the existence of a solution to the model as the limit of a sequence of approximate solutions. The uniqueness result is then discussed in Section 4. The extension to the model with nonlocal flux and source is given in Section 5, with particular focus on differences to the model with local flux only. In Section 6 a collection of numerical experiments is carried out.

2 A multilane model with nonlocal source term

In [19] the authors exploit the traditional LWR model to study vehicular traffic on a road with multiple lanes. The key feature of the model in [19] is that drivers tend to change to a neighbouring lane proportionally to the difference in the (local) velocity between the lanes.

However, as it is already well known, the use of nonlocal terms may lead to other dynamical behaviour, see e.g. [3]. In this paper, we aim to extend the multilane model [19] to account for a nonlocal evaluation of the velocity influencing the lane changing rate. The idea is that at position $x$ the flow between neighbouring lanes is governed by the difference in the velocity evaluated on the average density around position $x$, e.g. on the interval $[x-\nu, x+\nu]$, $\nu > 0$. This modelling hypothesis is motivated by a feature typical of drivers behaviour: when driving on a multilane road, at the moment of changing lane, the driver checks what is happening behind and in front of him/her, both on his/her lane and on the neighbouring one(s).

Recall the model introduced in [19] for a road with $M$ lanes:

$\begin{align*}
\partial_t \rho_j + \partial_x (\rho_j v_j(\rho_j)) &= S_{j-1}(\rho_{j-1}, \rho_1) - S_j(\rho_j, \rho_{j+1}) \quad j = 1, \ldots, M, \\
\rho_j(0, x) &= \rho_{o,j}(x) \quad j = 1, \ldots, M,
\end{align*}$

with

$S_j(\rho_j, \rho_{j+1}) = K (v_{j+1}(\rho_{j+1}) - v_j(\rho_j)) \begin{cases} 
\rho_j & \text{if } v_{j+1}(\rho_{j+1}) \geq v_j(\rho_j), \\
\rho_{j+1} & \text{if } v_{j+1}(\rho_{j+1}) < v_j(\rho_j),
\end{cases}$

and the boundary conditions

$S_0(\rho_0, \rho_1) = S_M(\rho_M, \rho_{M+1}) = 0,$

$K$ being a dimensional constant $(1/m)$. The modelling idea behind the term $S_j(\rho_j, \rho_{j+1})$ lies in the assumption that drivers prefer to be in the faster lane, and that the lane changing rate is proportional to the difference in the (local) velocity.
In contrast, our modelling approach accounts for a nonlocal evaluation of the velocity influencing the lane changing rate. Therefore, we introduce a kernel function \( w_\nu \in C^0([-\nu, \nu], \mathbb{R}_+) \), with \( \nu > 0 \) and \( \int_\mathbb{R} w_\nu(x) \, dx = 1 \), and define the flow from lane \( j \) to lane \( j+1 \) as follows: for \( j = 1, \ldots, M-1 \)

\[
S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) = K (v_{j+1}(R_{j+1}) - v_j(R_j)) \begin{cases} 
\rho_j (1 - \rho_{j+1}) & \text{if } v_{j+1}(R_{j+1}) \geq v_j(R_j), \\
\rho_{j+1} (1 - \rho_j) & \text{if } v_{j+1}(R_{j+1}) < v_j(R_j),
\end{cases}
\]

where \( (s)^+ = \max\{s, 0\} \), \( (s)^- = -\min\{s, 0\} \). Conversely, the flow from lane \( j+1 \) to lane \( j \) equals \(-S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1})\). Here, \( K \) is still a dimensional constant \((1/m)\). For simplicity, in the following time and space are scaled so that \( K = 1 \). The model we study is thus

\[
\begin{cases}
\partial_t \rho_j + \partial_x (\rho_j v_j(\rho_j)) = S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}, R_j) - S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) & j = 1, \ldots, M, \\
\rho_j(0, x) = \rho_{o,j}(x) & j = 1, \ldots, M,
\end{cases}
\]

with boundary conditions

\[
S_0(\rho_0, \rho_1, R_0, R_1) = S_M(\rho_M, \rho_{M+1}, R_M, R_{M+1}) = 0.
\]

The meaning of the source term defined by (2.1) is the following: similarly to the model studied in [19], the lane changing rate is proportional to the difference in the velocity between two adjacent lanes, but the velocities are now evaluated nonlocally, i.e. in a neighbourhood of the current position. Moreover, this rate is now proportional also to the density in the receiving lane, meaning that, if that lane is crowded, only a few vehicles can actually change lane. We remark that including this latter factor allows to prove the invariance of the set \([0, 1]^M\) for model (2.3), see Section 3.1. We emphasize that this is not necessary for the local model [19].

In a next step, we define a weak solution to (2.3) and present the key result of this paper for the existence and uniqueness of the solution. As in [19], we assume that the velocity functions \( v_i \) are strictly decreasing, positive and scaled such that \( v_i(1) = 0, i = 1, \ldots, M \). For simplicity, space and time are scaled so that \( K = 1 \). We assume that each map \( f_j(u) = u v_j(u) \) admits a unique global maximum in the interval \([0, 1]\) attained at \( u = \vartheta_j \).

**Definition 2.1.** Let \( \rho_{o,j} \in (L^1 \cap BV)(\mathbb{R}; [0, 1]) \), for \( j = 1, \ldots, M \). We say that \( \rho_j \in C^0([0, T]; L^1(\mathbb{R}; [0, 1])) \), with \( \rho_j(t, \cdot) \in BV(\mathbb{R}; [0, 1]) \) for \( t \in [0, T] \), is a weak solution to (2.3) with initial datum \( \rho_{o,j} \) if for any \( \varphi \in C^1_c([0, T] \times \mathbb{R}; \mathbb{R}) \) and for all \( j = 1, \ldots, M \)

\[
\begin{aligned}
\int_0^T \int_\mathbb{R} \left( \rho_j \partial_t \varphi + \rho_j v_j(\rho_j) \partial_x \varphi + (S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}, R_j) - S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1})) \varphi \right) \, dx \, dt \\
+ \int_\mathbb{R} \rho_{o,j} \varphi(0, x) \, dx = 0,
\end{aligned}
\]
with \( S_j \) as in (2.1) and \( R_j = R_j(t, x) = (\rho_j(t) \ast w_o)(x) \). The solution \( \rho_j \) is an entropy solution if for any \( \varphi \in C^1_c([0, T] \times \mathbb{R}; \mathbb{R}_+) \), for all convex entropy-entropy flux pairs \((\eta, q)\) and for all \( j = 1, \ldots, M \)

\[
\int_0^T \int_{\mathbb{R}} (\eta(\rho_j) \partial_t \varphi + q(\rho_j) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \eta(\rho_{o,j}) \varphi(0, x) \, dx \geq \int_0^T \int_{\mathbb{R}} \eta_j'(\rho_j) \left( S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) - S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}, R_j) \right) \varphi \, dx \, dt.
\]

In the following it will be convenient to use the notation \( \rho = (\rho_1, \ldots, \rho_M) \) to denote the vector of component \( \rho_j, j = 1, \ldots, M \). The initial datum to problem (2.3) is then \( \rho_o \).

**Theorem 2.2.** Let \( \rho_o \in (L^1 \cap BV)(\mathbb{R}; [0, 1]^M) \). Then, for all \( T > 0 \), problem (2.3) has a unique solution \( \rho \in C^0([0, T]; L^1(\mathbb{R}; [0, 1]^M)) \) in the sense of Definition 2.1. Moreover, the following estimates hold: for any \( t \in [0, T] \)

\[
\|\rho(t)\|_{L^1(\mathbb{R})} = \sum_{j=1}^M \|\rho_j(t)\|_{L^1(\mathbb{R})} = \|\rho_o\|_{L^1(\mathbb{R})},
\]

for \( j = 1, \ldots, M : 0 \leq \rho_j(t, x) \leq 1 \),

\[
\sum_{j=1}^M TV(\rho_j(t)) \leq \sum_{j=1}^M TV(\rho_{o,j}).
\]

Existence of solutions to problem (2.3) is ensured by the convergence of a sequence of approximate solutions, constructed through a Godunov scheme, see Section 3.5. Uniqueness follows from the \( L^1 \)-contractivity property for the whole solution to (2.3), see Section 4. The \( L^1 \) and \( L^\infty \) bounds follow from the convergence of the scheme, while the total variation bound is a consequence of the \( L^1 \)-contractivity, see Corollary 4.2.

### 3 Numerical discretization: a Godunov type scheme

To prove several properties of the model (2.4), and in particular Theorem 2.2, we introduce a uniform space mesh of width \( \Delta x \) and a time step \( \Delta t \), subject to a CFL condition to be detailed later on. For any \( k \in \mathbb{Z} \) denote the centre of the \( k \)-th cell by \( x_k \) and its interfaces by \( x_{k \pm 1/2} \):

\[
x_k = \left( k + \frac{1}{2} \right) \Delta x, \quad x_{k-1/2} = k \Delta x.
\]

Set \( N_T = \lfloor T/\Delta t \rfloor \) and define the time mesh as \( t^n = n \Delta t, n = 0, \ldots, N_T \). Set \( \lambda = \Delta t/\Delta x \).

The initial data are approximated as follows: for \( j = 1, \ldots, M \) and \( k \in \mathbb{Z} \),

\[
\rho^0_{j,k} = \frac{1}{\Delta x} \int_{x_{k-1/2}}^{x_{k+1/2}} \rho_{o,j}(x) \, dx.
\]

We construct an approximate solution \( \rho_{\Delta} \) to (2.3) as follows: for \( j = 1, \ldots, M \) set

\[
\rho_{j,\Delta}(t, x) = \rho^n_{j,k} \quad \text{for} \quad \begin{cases} t \in [t^n, t^{n+1}], & \text{with} \quad n = 0, \ldots, N_T - 1, \\ x \in [x_{k-1/2}, x_{k+1/2}], & k \in \mathbb{Z}. \end{cases}
\]
The approximate solution $\rho_\Delta$ is obtained via a Godunov type scheme together with operator splitting, to account for the source terms, see Algorithm 3.1.

**Algorithm 3.1.**

$$F_j(u,w) = \min \left\{ f_j \left( \min \{ u, \vartheta_j \} \right), f_j \left( \max \{ w, \vartheta_j \} \right) \right\} \quad j = 1, \ldots, M \quad (3.2)$$

For $n = 0, \ldots, N_T - 1$:

for $j = 1, \ldots, M$ and $k \in \mathbb{Z}$:

$$\rho_{j,k}^{n+1/2} = \rho_{j,k}^n - \frac{\lambda}{\Delta t} \left[ F_j(\rho_{j,k}^n, \rho_{j,k+1}^n) - F_j(\rho_{j,k-1}^n, \rho_{j,k}^n) \right] \quad (3.3)$$

$$\rho_{j,k}^{n+1} = \rho_{j,k}^{n+1/2} + \Delta t S_{j-1}(\rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2})$$

$$- \Delta t S_j(\rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2}) \quad (3.4)$$

Above, $R_{j,k}^{n+1/2}$, for $j = 1, \ldots, M$, $k \in \mathbb{Z}$ and $n = 0, \ldots, N_T - 1$, denotes the discrete convolution operator, which is defined in the Lemma below.

**Lemma 3.2.** Let $w_\nu \in C^0([\nu, \nu]; \mathbb{R}^+)$ be such that $\int_{\mathbb{R}} w_\nu = 1$. Define the set

$$\mathcal{H} = \left\{ h \in \mathbb{Z} : \frac{\inf \text{spt } w_\nu}{\Delta x} \leq h \leq \frac{\sup \text{spt } w_\nu}{\Delta x} - 1 \right\} \quad (3.5)$$

and for all $h \in \mathcal{H}$ set

$$\gamma_h := \int_{x_{h-1/2}}^{x_{h+1/2}} w_\nu(y-x) \, dy.$$ 

Given $r(x) = r_k \chi_{[x_{k-1/2},x_{k+1/2}]}(x)$, with $r_k \in [0,1]$ and $k \in \mathbb{Z}$, the discrete convolution operator defined for all $k \in \mathbb{Z}$ as

$$R_k = \sum_{h \in \mathcal{H}} \gamma_h r_{k+h+1} \quad (3.6)$$

satisfies the following properties:

$$R_k \in [0,1] \quad \text{for all } k \in \mathbb{Z}, \quad (3.7)$$

$$\sum_{k \in \mathbb{Z}} |R_{k+1} - R_k| \leq \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k|. \quad (3.8)$$

Given $\tilde{r}(x) = \tilde{r}_k \chi_{[x_{k-1/2},x_{k+1/2}]}(x)$, with $\tilde{r}_k \in [0,1]$ and $k \in \mathbb{Z}$, and $\tilde{R}_k$ defined accordingly to (3.6), then

$$\sum_{k \in \mathbb{Z}} |R_k - \tilde{R}_k| \leq \sum_{k \in \mathbb{Z}} |r_k - \tilde{r}_k|. \quad (3.9)$$

**Proof.** It is immediate to see that $\gamma_h \in [0,1]$ for all $h \in \mathcal{H}$, due to the properties of $w_\nu$. Hence, for all $k \in \mathbb{Z}$, we clearly have $R_k \geq 0$ and

$$R_k = \sum_{h \in \mathcal{H}} \gamma_h r_{k+h+1} \leq \sum_{h \in \mathcal{H}} \gamma_h = \int_{\text{spt } w_\nu} w_\nu(y-x) \, dy = 1,$$

since each $r_k \in [0,1]$. 

5
Pass now to (3.8): rearranging the indexes yields
\[
\sum_{k \in \mathbb{Z}} |R_{k+1} - R_k| \leq \sum_{k \in \mathbb{Z}} \sum_{h \in \mathcal{H}} \gamma_h |r_{k+h+2} - r_{k+h+1}|
\]
\[
= \left( \sum_{h \in \mathcal{H}} \gamma_h \right) \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k| = \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k|.
\]

The proof of (3.9) is entirely analogous. \(\square\)

**Remark 3.3.** According to the support of the kernel function \(w_\nu\), the discrete convolution operator defined by (3.6) has one of the following two forms:

- **Forward looking kernel:** if \(\text{spt} w_\nu \subseteq [0, \nu]\), then \(\mathcal{H} = [0, \lfloor \nu/\Delta x \rfloor - 1]\), so that
  \[
  R_{n+1/2}^{j,k} = \sum_{h=0}^{\lfloor \nu/\Delta x \rfloor - 1} \gamma_h \rho_{j,k+h+1}^{n+1/2}.
  \] (3.10)

- **Back- and forward looking kernel:** if \(\text{spt} w_\nu \subseteq [-\nu, \nu]\), \(\mathcal{H} = [-\lfloor \nu/\Delta x \rfloor, \lfloor \nu/\Delta x \rfloor - 1]\), so that
  \[
  R_{n+1/2}^{j,k} = \sum_{h=-\lfloor \nu/\Delta x \rfloor}^{\lfloor \nu/\Delta x \rfloor - 1} \gamma_h \rho_{j,k+h+1}^{n+1/2}.
  \] (3.11)

### 3.1 Invariance of the set \([0, 1]^M\)

Under a suitable CFL condition, if each component of the initial datum takes values in the interval \([0, 1]\), then also the components of the approximate solution constructed via Algorithm 3.1 attain values in the same interval \([0, 1]\): the set \([0, 1]^M\) is thus invariant for problem (2.3).

**Lemma 3.4.** Let \(\rho_o \in L^\infty(\mathbb{R}; [0, 1]^M)\). Assume that
\[
\lambda \mathcal{V} \leq \frac{1}{2},
\] (3.12)

where
\[
\mathcal{V} = \|v\|_{C^1([0,1];\mathbb{R}^M)} = V_{\max} + V'_{\max},
\] (3.13)
\[
V_{\max} = \|v\|_{C^0([0,1];\mathbb{R}^M)} = \max_{j=1,\ldots,M} \|v_j\|_{L^\infty([0,1];\mathbb{R})},
\] (3.14)
\[
V'_{\max} = \|v'\|_{C^0([0,1];\mathbb{R}^M)} = \max_{j=1,\ldots,M} \|v'_j\|_{L^\infty([0,1];\mathbb{R})}.
\] (3.15)

Then, for all \(t > 0\) and \(x \in \mathbb{R}\), the piece-wise constant approximate solution \(\rho_\Delta\) constructed through Algorithm 3.1 attains value in the set \([0, 1]^M\), i.e.

\[0 \leq \rho_{j,\Delta}(t, x) \leq 1 \quad \text{for all} \ j = 1, \ldots, M.\]
The properties above will be used in order to exploit the following elementary inequality:

There are four possibilities, according to the signs of the differences in the velocity:

\[ \rho_{n+1} = \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \]

\[ \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) - \rho_{2,k}(1 - \rho_{1,k}) \]

\[ - \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) + \rho_{2,k}(1 - \rho_{1,k}) - \rho_{3,k}(1 - \rho_{2,k}) \]

There are four possibilities, according to the signs of the differences in the velocity:

| \( v_2(R_{3,k}) \geq v_2(R_{2,k}) \) | \( v_2(R_{2,k}) \geq v_1(R_{1,k}) \) | \( v_2(R_{2,k}) < v_1(R_{1,k}) \) |
|---------------------------------|---------------------------------|---------------------------------|
| Case A                         | Case B                          | Case C                          |
| Case D                          |                                 |                                 |

We analyse them in detail below. Observe first that the following facts hold true:

(i) Whenever \( v_j(R_{j,k}) \geq v_j(R_{j,k}) \), \( j, \ell \in \{1, \ldots, M\} \), \( j \neq \ell \), then

\[ 1 - \Delta t (v_j(R_{\ell,k}) - v_j(R_{j,k})) \rho_{j,k} \geq 0. \]

Indeed, thanks to the CFL condition (3.12) and to the fact that \( \Delta x < 1 \), we have

\[ 1 - \Delta t v_j(R_{\ell,k}) \rho_{j,k} + \Delta t v_j(R_{j,k}) \rho_{j,k} \geq 1 - \Delta t v_j(R_{\ell,k}) \rho_{j,k} \geq 1 - \Delta t V_{\text{max}} \geq 0. \]

(ii) Whenever \( v_{j+1}(R_{j+1,k}) < v_j(R_{j,k}) \) and \( v_j(R_{j,k}) \geq v_{j-1}(R_{j-1,k}) \), \( j \in \{2, \ldots, M-1\} \), then

\[ 1 - \Delta t (v_j(R_{j,k}) - v_{j-1}(R_{j-1,k})) \rho_{j-1,k} - (v_{j+1}(R_{j+1,k}) - v_j(R_{j,k})) \rho_{j+1,k} \geq 0. \]

Indeed, thanks to the CFL condition (3.12), to the fact that \( \Delta x < 1 \) and that \( \rho_{j,k} \leq 1 \), we have

\[ 1 - \Delta t (v_j(R_{j,k}) - v_{j-1}(R_{j-1,k})) \rho_{j-1,k} - (v_{j+1}(R_{j+1,k}) - v_j(R_{j,k})) \rho_{j+1,k} \]

\[ \geq 1 - \Delta t (v_j(R_{j,k}) \rho_{j-1,k} + v_j(R_{j,k}) \rho_{j+1,k}) \]

\[ \geq 1 - 2 \Delta t v_j(R_{j,k}) \]

\[ \geq 1 - 2 \Delta t V_{\text{max}} \geq 0. \]

The properties above will be used in order to exploit the following elementary inequality:

\[ 0 \leq \rho \leq 1 \quad \text{and} \quad A \geq 0 \quad \implies \quad \rho A \leq A. \]
Case A. Here (3.16) reads
\[
\rho_{2,k}^{n+1} = \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k}(1 - \rho_{2,k}) - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{2,k}(1 - \rho_{3,k}).
\]

Thus, aiming for the bound from above, thanks to (i) and since 0 ≤ \( \rho_{2,k} \leq 1 \), we get
\[
\rho_{2,k}^{n+1} \leq \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k}(1 - \rho_{2,k}) \\
= \rho_{2,k} \left( 1 - \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} \right) + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} \\
\leq 1 - \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} \\
= 1.
\]

Pass now to the positivity: thanks to (i) we obtain
\[
\rho_{2,k}^{n+1} \geq \rho_{2,k} - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{2,k}(1 - \rho_{3,k}) \\
\geq \rho_{2,k} - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{2,k} \\
\geq 0.
\]

Case B. In this case (3.16) reads
\[
\rho_{2,k}^{n+1} = \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{2,k}(1 - \rho_{1,k}) - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{2,k}(1 - \rho_{3,k}).
\]

Since \( v_2(R_{2,k}) - v_1(R_{1,k}) < 0 \) and \( v_3(R_{3,k}) - v_2(R_{2,k}) \geq 0 \), it is immediate to prove that \( \rho_{2,k}^{n+1} \leq \rho_{2,k}^{n+1/2} \) and thus \( \rho_{2,k}^{n+1} \) is bounded by 1 from above. Moreover, by the CFL condition (3.12),
\[
\rho_{2,k}^{n+1} \geq \rho_{2,k} + \Delta t \left[ \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{2,k} - \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \right] \\
\geq \rho_{2,k} \left( 1 - \Delta t \left( v_1(R_{1,k}) + v_3(R_{3,k}) \right) \right) \\
\geq \rho_{2,k}(1 - 2 \Delta t V_{\text{max}}) \\
\geq 0.
\]

Case C. Here (3.16) reads
\[
\rho_{2,k}^{n+1} = \rho_{2,k} + \Delta t \left[ \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k}(1 - \rho_{2,k}) - \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k}(1 - \rho_{2,k}) \right].
\]

The positivity of \( \rho_{2,k}^{n+1} \) follows immediately, since \( v_2(R_{2,k}) - v_1(R_{1,k}) \geq 0 \) and \( v_3(R_{3,k}) - v_2(R_{2,k}) < 0 \). On the other hand, thanks to (ii)
\[
\rho_{2,k}^{n+1} = \rho_{2,k} \left( 1 - \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} + \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} \right) \\
+ \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} - \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} \\
\leq 1 - \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} + \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} \\
+ \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{1,k} - \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} \\
\leq 1.
\]
Case D. In this latter case \(3.16\) reads
\[
\rho_{2,k}^{n+1} = \rho_{2,k} + \Delta t \left[ \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{2,k}(1 - \rho_{1,k}) - \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k}(1 - \rho_{2,k}) \right].
\]
By \(3.4\) since \(0 \leq \rho_{2,k} \leq 1\), we get
\[
\rho_{2,k}^{n+1} \leq \rho_{2,k} - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k}(1 - \rho_{2,k})
= \rho_{2,k} \left( 1 + \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} \right) - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k}
\leq 1 + \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k} - \Delta t \left( v_3(R_{3,k}) - v_2(R_{2,k}) \right) \rho_{3,k}
\leq 1.
\]
The positivity of \(\rho_{2,k}^{n+1}\) follows from the CFL condition \(3.12\):
\[
\rho_{2,k}^{n+1} \geq \rho_{2,k} + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \rho_{2,k}(1 - \rho_{1,k})
\geq \rho_{2,k} \left( 1 + \Delta t \left( v_2(R_{2,k}) - v_1(R_{1,k}) \right) \right)
\geq \rho_{2,k} \left( 1 - \Delta t v_1(R_{1,k}) \right)
\geq 0.
\]
The proof is completed. \(\Box\)

3.2 Conservation of total mass

When considering an initial datum \(\rho_o\) with finite total mass, that is \(\sum_{j=1}^{M} \|\rho_{o,j}\|_{L^1(\mathbb{R})} < +\infty\), it is possible to prove that the corresponding solution preserves this norm. Clearly, because of lane changing, the \(L^1\)-norm is not preserved in each lane, but only in the whole.

Lemma 3.5. Let \(\rho_o \in L^1(\mathbb{R}; [0, 1]^M)\). Under the CFL condition \(3.12\), the piece-wise constant approximate solution \(\rho_\Delta\) constructed through Algorithm 3.2 preserves the \(L^1\)-norm, in the sense that for all \(t > 0\)
\[
\|\rho_\Delta(t)\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \|\rho_{j,\Delta}(t)\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \|\rho_{o,j}\|_{L^1(\mathbb{R})} = \|\rho_o\|_{L^1(\mathbb{R})}.
\]

Proof. The proof is done by induction. Since the Godunov type scheme \(3.3\) is conservative \([24\text{ Chapter 13}]\), we have
\[
\sum_{j=1}^{M} \left\| \rho_j^{n+1/2} \right\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \|\rho_{o,j}\|_{L^1(\mathbb{R})}.
\]
The positivity of \(\rho_\Delta\) and the fact that the source terms sum up to 0 when considering the relaxation step in \(3.3\) yields the thesis:
\[
\sum_{j=1}^{M} \left\| \rho_j^{n+1} \right\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \left\| \rho_j^{n+1/2} \right\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \|\rho_{o,j}\|_{L^1(\mathbb{R})}.
\]
\(\Box\)
3.3 BV estimates

We first prove the Lipschitz continuity of the source term (2.1) in each of its argument.

**Lemma 3.6.** For all \( j = 1, \ldots, M \), the map \( S_j \) defined in (2.1) is Lipschitz continuous in each argument with Lipschitz constant

\[
K = \max\{V_{\text{max}}, 2V'_{\text{max}}\},
\]

where \( V_{\text{max}} \) and \( V'_{\text{max}} \) are defined in (3.14) and (3.14) respectively.

**Proof.** For \( j \in \{1, \ldots, M - 1\} \) we have

\[
\begin{align*}
|S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) - S_j(\tilde{\rho}_j, \tilde{\rho}_{j+1}, \tilde{R}_j, \tilde{R}_{j+1})| \\
\leq |S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) - S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1})| \\
+ |S_j(\tilde{\rho}_j, \tilde{\rho}_{j+1}, R_j, R_{j+1}) - S_j(\tilde{\rho}_j, \tilde{\rho}_{j+1}, \tilde{R}_j, \tilde{R}_{j+1})| \\
+ |S_j(\tilde{\rho}_j, \tilde{\rho}_{j+1}, \tilde{R}_j, \tilde{R}_{j+1}) - S_j(\rho_j, \rho_{j+1}, \tilde{R}_j, \tilde{R}_{j+1})|.
\end{align*}
\]

By the definition of the source term (2.1) we have

\[
|3.18| = \left|(v_{j+1}(R_{j+1}) - v_j(R_j))^+ (1 - \rho_{j+1})(\rho_j - \tilde{\rho}_j) - (v_{j+1}(R_{j+1}) - v_j(R_j))^- \rho_{j+1}(\tilde{\rho}_j - \rho_j) \right| \\
\leq V_{\text{max}}|\rho_j - \tilde{\rho}_j|,
\]

\[
|3.19| \leq V_{\text{max}}|\rho_{j+1} - \tilde{\rho}_{j+1}|.
\]

Pass now to (3.20):

\[
|3.20| = \left| \left((v_{j+1}(R_{j+1}) - v_j(R_j))^+ - (v_{j+1}(R_{j+1}) - v_j(R_j))^+ \right) \tilde{\rho}_j (1 - \tilde{\rho}_{j+1}) \\
- \left((v_{j+1}(R_{j+1}) - v_j(R_j))^+ - (v_{j+1}(R_{j+1}) - v_j(R_j))^+ \right) \tilde{\rho}_{j+1} (1 - \tilde{\rho}_j) \right|.
\]

We distinguish the following cases:

| Case A | Case B |
|------------------------|------------------------|
| \( v_{j+1}(R_{j+1}) \geq v_j(R_j) \) | \( v_j(R_{j+1}) < v_j(R_j) \) |
| \( v_{j+1}(R_{j+1}) < v_j(R_j) \) | Case C | Case D |

We analyse in detail cases A and B, the others being entirely similar.

**Case A.** We have

\[
|3.20| = \left| (v_j(R_j) - v_j(R_j)) \tilde{\rho}_j (1 - \tilde{\rho}_{j+1}) \right| \leq V'_{\text{max}}|R_j - \tilde{R}_j|.
\]
Case B. Add and subtract \((v_{j+1}(R_j) - v_j(R_j))\tilde{\rho}_{j+1}(1 - \tilde{\rho}_j)\) inside the absolute value in (3.20) to obtain

\[
(3.20) = \left| (v_j(R_j) - v_j(R_j))\tilde{\rho}_{j+1}(1 - \tilde{\rho}_j) + (v_{j+1}(R_{j+1}) - v_j(R_j))(\tilde{\rho}_{j+1}(1 - \tilde{\rho}_j) - \tilde{\rho}_j(1 - \tilde{\rho}_{j+1})) \right|
\]

\[
\leq V'_\text{max} \left| R_j - \tilde{R}_j \right| + (v_{j+1}(R_{j+1}) - v_j(R_j))
\]

\[
< V'_\text{max} \left| R_j - \tilde{R}_j \right| + (v_j(R_j) - v_j(\tilde{R}_j))
\]

\[
\leq 2 V'_\text{max} \left| R_j - \tilde{R}_j \right|
\]

since \(v_{j+1}(R_{j+1}) < v_j(R_j)\) and \(|\tilde{\rho}_{j+1} - \tilde{\rho}_j| \leq 1\), with \(\tilde{\rho}_j, \tilde{\rho}_{j+1} \in [0, 1]\).

Cases D and C are treated similarly to Case A and Case B, respectively. Therefore we have

\[
(3.20) \leq 2 V'_\text{max} \left| R_j - \tilde{R}_j \right|.
\]

The term (3.21) is treated analogously to (3.20), leading to

\[
(3.21) \leq 2 V'_\text{max} \left| R_{j+1} - \tilde{R}_{j+1} \right|.
\]

The proof is completed. \(\square\)

The Lipschitz continuity of the source term proved in Lemma 3.6 is one of the key ingredients in order to prove the following total variation bound on the numerical approximation.

**Proposition 3.7 (BV estimate in space).** Let \(\rho_j \in (L^1 \cap BV)(\mathbb{R}; [0, 1]^M)\). Assume that the CFL condition (3.12) holds. Then, for \(n = 0, \ldots, N_T - 1\) the following estimate holds

\[
\sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n| \leq e^{8t^nK} \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^0 - \rho_{j,k}^0| = e^{8t^nK} \sum_{j=1}^{M} \text{TV}(\rho_j^0). \quad (3.22)
\]

**Proof.** By (3.11), for \(j = 1, \ldots, M\) we have

\[
\rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} = \rho_{j,k+1}^{n+1/2} - \rho_{j,k}^{n+1/2}
\]

\[
+ \Delta t \left[ S_{j-1} \left( \rho_{j-1,k+1}^{n+1/2}, \rho_{j,k+1}^{n+1/2}, R_{j-1,k+1}^{n+1/2}, R_{j,k+1}^{n+1/2} \right) - S_{j-1} \left( \rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) \right]
\]

\[
- \Delta t \left[ S_j \left( \rho_{j,k+1}^{n+1/2}, \rho_{j+1,k+1}^{n+1/2}, R_{j,k+1}^{n+1/2}, R_{j+1,k+1}^{n+1/2} \right) - S_j \left( \rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2} \right) \right].
\]

By the Lipschitz continuity of the maps in the source term, see Lemma 3.6 and the properties of the discrete convolution operator, see Lemma 3.2, we obtain

\[
\sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1}|
\]

\[
\leq \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} (1 + 4 \Delta tK) |\rho_{j,k+1}^{n+1/2} - \rho_{j,k}^{n+1/2}| + 4 \Delta tK \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} |R_{j,k+1}^{n+1/2} - R_{j,k}^{n+1/2}|
\]

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(1 + 8 \Delta t K) \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1/2} - \rho_{j,k}^{n+1/2} \right|.

Since the Godunov scheme used in (5.2) is total variation diminishing [12, Proposition 3.1 (d)], we get

\[ \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} \right| \leq (1 + 8 \Delta t K) \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n} - \rho_{j,k}^{n} \right| \leq e^{8 \Delta t K} \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n} - \rho_{j,k}^{n} \right|, \]

which applied recursively yields the thesis. \( \square \)

**Proposition 3.8.** Let \( \rho_o \in (L^1 \cap BV)(\mathbb{R}; [0, 1]^M) \). Assume that the CFL condition (3.12) holds. Then, for \( n = 0, \ldots, N_T - 1 \),

\[ \Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} \right| \leq 2 \Delta t \left( 2 V_{\text{max}} \|\rho_o\|_{L^1(\mathbb{R})} + V e^{8 \Delta t K} \sum_{j=1}^{M} \text{TV} (\rho_j^0) \right), \]

with \( K \) as in (3.17) and \( V \) as in (3.13).

**Proof.** Observe that

\[ \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n} \right| \leq \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n+1/2} \right| + \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right|. \]

We then estimate separately each term on the right hand side of the inequality above.

By the relaxation step (3.4) we have

\[ \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n+1/2} \right| = \Delta t \left| S_j^{j-1} \left( \rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) - S_j \left( \rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2} \right) \right|. \]

It is easy to see that the numerical source term \( S_j \) (2.1)-(2.3) satisfies, for \( j = 1, \ldots, M \),

\[ \left| S_j \left( \rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2} \right) \right| \leq V_{\text{max}} \left( \rho_{j,k}^{n+1/2} + \rho_{j+1,k}^{n+1/2} \right). \]

Thus,

\[ \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n+1/2} \right| \leq \Delta t V_{\text{max}} \left( \rho_{j-1,k}^{n+1/2} + 2 \rho_{j+1,k}^{n+1/2} + \rho_{j-1,k}^{n+1/2} \right). \]

By the convective step (3.9), since the numerical flux defined in (5.2) is Lipschitz continuous in both arguments with Lipschitz constant \( V \) (3.13), we have

\[ \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right| = \lambda \left| F_j \left( \rho_{j,k}^{n+1/2}, \rho_{j,k+1}^{n+1/2} \right) - F_j \left( \rho_{j,k-1}, \rho_{j,k}^{n+1/2} \right) \right| \]

\[ \leq \lambda V \left( \rho_{j,k}^{n+1/2} - \rho_{j,k-1}^{n+1/2} + \rho_{j,k+1}^{n+1/2} \right). \]

Collecting together (3.26) and (3.27) and exploiting Lemma 3.5 and Proposition 3.7 yields

\[ \Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} \right| \leq \Delta t V_{\text{max}} \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1/2} \right|_{L^1(\mathbb{R})} + 2 \Delta t V \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n} - \rho_{j,k-1}^{n} \right| \]

\[ \leq 2 \Delta t \left( 2 V_{\text{max}} \|\rho_o\|_{L^1(\mathbb{R})} + V e^{8 \Delta t K} \sum_{j=1}^{M} \text{TV} (\rho_j^0) \right). \]

\( \square \)
Using the estimates provided by Propositions 3.7 and 3.8 we obtain the following BV estimate in space and time.

**Corollary 3.9 (BV estimate in space and time).** Let \( \rho_\circ \in (L^1 \cap BV)(\mathbb{R}; [0,1]^M) \). Assume that the CFL condition (3.12) holds. Then, for all \( n = 1, \ldots, N_T \), the following estimate holds

\[
\sum_{m=0}^{n-1} \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left( \Delta t \rho_{j,k+1}^m - \rho_{j,k}^m \right) + \Delta x \rho_{j,k+1}^{m+1} - \rho_{j,k}^{m+1} \right) \leq n \Delta t e^{8 \lambda^2 K} \left( 2(V+1) \sum_{j=1}^{M} \text{TV}(\rho_j^0) + 4V_{\max} \sum_{j=1}^{M} \| \rho_{o,j} \|_{L^1(\mathbb{R})} \right).
\]

### 3.4 Discrete entropy inequality

We derive a discrete entropy inequality for the approximate solution \( \rho_\Delta \) constructed through Algorithm 3.1. The proof is entirely similar to [15, Lemma 2.7], with the simplification that now the flux does not depend on the spatial variable.

Define, for each \( c \in [0,1] \) and \( j = 1, \ldots, M \), the Kružkov numerical entropy flux as

\[
\mathcal{F}_j^c (u,w) = F_j (u \lor c, w \lor c) - F_j (u \land c, w \land c),
\]

where \( a \lor b = \max \{a,b\} \) and \( a \land b = \min \{a,b\} \).

**Lemma 3.10.** Let \( \rho_\circ \in (L^1 \cap BV)(\mathbb{R}; [0,1]^M) \). Assume that the CFL condition (3.12) holds. Then, the approximate solution \( \rho_\Delta \) constructed by Algorithm 3.1 satisfies the following discrete entropy inequality: for all \( j = 1, \ldots, M \), for \( k \in \mathbb{Z} \), for \( n = 0, \ldots, N_T - 1 \) and for any \( c \in [0,1] \),

\[
\left| \rho_{j,k}^{n+1} - c \right| - \left| \rho_{j,k}^n - c \right| + \lambda \left( \mathcal{F}_j^c (\rho_{j,k}^n, \rho_{j,k+1}^n) - \mathcal{F}_j^c (\rho_{j,k-1}^n, \rho_{j,k}^n) \right) - \Delta t \text{sgn} (\rho_{j,k}^{n+1} - c) \left( S_{j-1} \left( \rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) \right. \\
\left. - S_{j} \left( \rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) \right) \leq 0.
\]

### 3.5 Convergence

The results obtained in the preceding sections, namely Lemma 3.4 for the invariance of the set \([0,1]^M \) and Corollary 3.9 for the total variation bound in space and time, allow to apply Helly’s compactness theorem, which ensures the existence of a subsequence of \( \rho_\Delta \) converging in \( L^1 \) to a function \( \rho \in L^\infty([0,T] \times \mathbb{R}; [0,1]^M) \), with the additional property of preserving the initial mass, that is \( \| \rho(t) \|_{L^1(\mathbb{R})} = \| \rho_\circ \|_{L^1(\mathbb{R})} \) for \( t \in [0,T] \). Moreover, Proposition 3.8 and in particular formula (3.24), imply that \( \rho \in C^0([0,T]; L^1(\mathbb{R}^d; [0,1]^M)) \).

The limit function \( \rho \) is a solution to problem (2.3) in the sense of Definition 2.1. Indeed, the weak formulation, i.e. the integral equality in the first part of Definition 2.1 follows from a Lax–Wendroff type calculation [24, Theorem 12.1], and the presence of the source terms does not add any difficulty in the proof.

Concerning the entropy inequality in the second part of Definition 2.1 rather standard computations starting from the discrete entropy inequality in Lemma 3.10 yield the desired result.

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4 Uniqueness of solutions: $\mathbf{L}^1$ contractivity

As for the local model [19], the special form of the source terms implies the $\mathbf{L}^1$-contractivity of the solution to (2.3). In particular, this result guarantees uniqueness of solutions to problem (2.3).

**Theorem 4.1.** Let $\rho$ and $\pi$ be two solutions to problem (2.3) in the sense of Definition 2.4 with initial data $\rho_o, \pi_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}; [0, 1]^M)$, respectively. Then, for a.e. $t \in [0, T]$, we denote by $H$ to (2.3), yields, for $\tau \in [0, T]$ and for any $j = 1, \ldots, M$:

$$\sum_{j=1}^{M} \left\| \rho_j(t) - \pi_j(t) \right\|_{\mathbf{L}^1(\mathbb{R})} \leq \sum_{j=1}^{M} \left\| \rho_{j,o} - \pi_{j,o} \right\|_{\mathbf{L}^1(\mathbb{R})}. \quad (4.1)$$

**Proof.** The proof follows the idea of [19, Theorem 3.3], with the main difference that now the source terms are nonlocal functions of the solution. We recall the proof briefly for completeness, focusing mainly on those parts where the nonlocality comes in.

Kružkov doubling of variables techniques, together with the fact that $\rho$ and $\pi$ are solutions to (2.3), yields, for $\tau \in [0, T]$ and for any $j = 1, \ldots, M$,

$$\int_{\mathbb{R}} (\rho_j(\tau) - \pi_j(\tau))^+ \, dx \leq \int_{\mathbb{R}} (\rho_j(0) - \pi_j(0))^+ \, dx + \int_0^\tau \int_{\mathbb{R}} H(\rho_j - \pi_j) \left( S(\rho, R_j) - S(\pi, P_j) \right) \, dx \, dt, \quad (4.2)$$

where $H$ is the Heaviside function,

$$S(u, U, j) = S_j-1(u_{j-1}, u_j, U_{j-1}, U_j) - S_j(u_j, u_{j+1}, U_j, U_{j+1}),$$

$$U_j(t, x) = (u_j(t) * w_{\nu})(x),$$

and we denote by $U$ the vector of components $U_j$, $j = 1, \ldots, M$. It can be easily verified that the map $S_j(u, w, U, W)$ defined in (2.1) is nondecreasing in the first and third variables and nonincreasing in the second and fourth variables, thus $\partial_u S_j, \partial_U S_j \geq 0$ and $\partial_w S_j, \partial_W S_j \leq 0$. Hence, if $\rho_j > \pi_j$, clearly $R_j > P_j$ and moreover

$$S(\rho, R, j) - S(\pi, P, j) = S_j-1(\rho_j-1, \rho_j, R_{j-1}, R_j) - S_j(\pi_j-1, \pi_j, P_{j-1}, P_j) - S_j(\rho_j, R_{j+1}, R_j) + S_j(\pi_j, \pi_{j+1}, P_{j+1}, P_j) + S_j(\rho_j, R_j, R_{j+1}) + S_j(\pi_j, \pi_j, R_j, P_{j+1}) \leq \partial_u S_j(\sigma_{j-1}, \pi_j, R_{j-1}, P_j) \left( \rho_j - \pi_j \right) + \partial_U S_j(\pi_j-1, \pi_j, T_{j-1}, P_j) \left( R_j-1 - P_{j-1} \right) - \partial_u S_j(\rho_j, \sigma_{j+1}, R_j, R_{j+1}) \left( \rho_{j+1} - \pi_{j+1} \right) - \partial_W S_j(\rho_j, \sigma_{j+1}, R_j, T_{j+1}) \left( R_{j+1} - P_{j+1} \right) \leq K \left( \left( \rho_j - \pi_j \right)^+ + \left( R_j - P_j \right)^+ \right),$$

where $\sigma_{j\pm 1}$ lies in the interval between $\rho_j$ and $\pi_j$, $T_{j\pm 1}$ lies in the interval between $R_j$ and $P_j$, and the Lipschitz constant $K$ of the map $S_j$ is in (3.17). Therefore

$$\sum_{j=1}^{M} H(\rho_j - \pi_j) \left( S(\rho, R, j) - S(\pi, P, j) \right) \leq 2K \sum_{j=1}^{M} (\rho_j - \pi_j)^+ + 2K \sum_{j=1}^{M} (R_j - P_j)^+. \quad (4.3)$$
Observe that \( \int_{\mathbb{R}} (g \ast w_{\nu})^+(x) \, dx = \int_{\mathbb{R}} g^+(x) \, dx \), thus, due to (3.9), when integrating (4.3) over \( \mathbb{R} \) we obtain
\[
\sum_{j=1}^{M} \int_{\mathbb{R}} H(\rho_j - \pi_j) \left( S(\rho, R, j) - S(\pi, P, j) \right) \, dx \leq 4K \sum_{j=1}^{M} \int_{\mathbb{R}} (\rho_j - \pi_j)^+ \, dx.
\] (4.4)

Define
\[
\Theta(t) = \sum_{j=1}^{M} \int_{\mathbb{R}} (\rho_j - \pi_j)^+ \, dx,
\]
so that, collecting together (4.2) and (4.4), we get
\[
\Theta(\tau) \leq \Theta(0) + 4K \int_{0}^{\tau} \Theta(t) \, dt.
\]

Gronwall’s inequality yields \( \Theta(t) \leq e^{4Kt} \Theta(0) \). If \( \Theta(0) = 0 \), that is \( \rho_{o,j}(x) \leq \pi_{o,j}(x) \) a.e. in \( \mathbb{R} \) for all \( j \), then \( \Theta(t) = 0 \) for \( t > 0 \), that is \( \rho_j(t, x) \leq \pi_j(t, x) \) a.e. in \( \mathbb{R} \) for all \( j \).

The proof of \( L^1 \)-contractivity is concluded by an application of the Crandall–Tartar lemma [18, Lemma 2.13]. \( \square \)

Following [19, Corollary 3.4], the \( L^1 \)-contractivity of the solution proved in Theorem 4.1 guarantees that the solution to problem (2.3) satisfies some \textit{a priori} estimates.

\[\textbf{Corollary 4.2.} \text{Let } \rho \text{ be a solution to problem (2.3) in the sense of Definition 2.1, with initial datum } \rho_o \in (L^1 \cap BV)(\mathbb{R}; [0, 1]^M). \text{ Then,}\]
\[
\sum_{j=1}^{M-1} \|\rho_{j+1}(t) - \rho_j(t)\|_{L^1(\mathbb{R})} \leq \sum_{j=1}^{M-1} \|\rho_{j+1,o} - \rho_{j,o}\|_{L^1(\mathbb{R})}, \quad (4.5)
\]
\[
\sum_{j=1}^{M} \text{TV} (\rho_j(t)) \leq \sum_{j=1}^{M} \text{TV} (\rho_{j,o}), \quad (4.6)
\]
\[
\sum_{j=1}^{M} \|\rho_j(t + h) - \rho_j(t)\|_{L^1(\mathbb{R})} \leq \sum_{j=1}^{M} \|\rho_j(h) - \rho_{j,o}\|_{L^1(\mathbb{R})}, \quad h \in \mathbb{R}. \quad (4.7)
\]

The proof relies solely on (4.1), together with the enforced boundary conditions \( \rho_0(t, x) = \rho_1(t, x), v_0(u) = v_1(u), \rho_{M+1}(t, x) = \rho_M(t, x), v_{M+1}(u) = v_M(u) \).

Notice that Corollary 4.2 provides better estimates than those coming from the approximate solution built in Section 3. Compare in particular (4.6) to the total variation in space provided by (3.22).

\[\textbf{5 A multilane model with nonlocal flux and nonlocal source term}\]

In the following, we consider a modification of problem (2.3) assuming additionally a \textit{nonlocal} velocity in the flux function. In particular, the treatment of the nonlocal flux in each lane is
inspired by [3]. The problem under consideration reads

\[
\begin{align*}
\partial_t \rho_j + \partial_x \left( \rho_j \, v_j \left( \rho_j * w_\iota \right) \right) &= S_j \left( \rho_{j-1} \rho_j, R_{j-1}, R_j \right) - S_j \left( \rho_j, \rho_{j+1}, R_j, R_{j+1} \right) & j = 1, \ldots, M, \\
\rho_j(0, x) &= \rho_{o,j}(x) & j = 1, \ldots, M.
\end{align*}
\]

(5.1)

In order to have a well defined model, we only consider kernel functions such that \( \text{spt } w_\iota \subseteq [0, \iota] \), meaning that drivers adapt their speed to the downstream traffic. In addition, we assume that the kernel \( w_\iota \in C^1([0, \iota]; \mathbb{R}_+) \) is non-increasing, i.e. \( w'_\iota \leq 0 \), and, as usual \( \int_{\mathbb{R}} w_\iota = 1 \). The convolution product is thus defined as

\[
R^\iota_j = R^\iota_j(t, x) = \left( \rho(t) * w_\iota \right)(x) := \int_x^{x+i} \int_{\mathbb{R}} w_\iota(y - x) \rho(t, y) dy.
\]

(5.2)

We remark that the additional assumptions on the kernel \( w_\iota \) in the flux are not needed for the kernel \( w_\nu \) in the source term, see Section 2. Moreover, when considering both nonlocal flux and nonlocal source, we underline that the kernels may differ. In the following, we denote the convolution products in the source by \( R^\nu_j \) (5.2) and those in the flux by \( R^\iota_j \), to emphasize the different kernels.

We underline that the kernel function \( w_\nu \) appearing in the source can look either only forward or both back- and forward, differently from the kernel function \( w_\iota \) appearing in the flux, which is assumed to be only forward-looking. As already mentioned in the introduction, these are the key points in which the proposed model (5.1) differs from the approach presented in [2]. Therein, the uniqueness is only shown for the same nonlocality in the flux and source term, such that both have to be forward looking with the same non-increasing kernel and the same nonlocal range. So the model (5.1) provides more flexibility in terms of modelling. However, using the same non-increasing, forward looking kernel and nonlocal range, the model (5.1) fits into the framework proposed in [2, Definition 1.1, Assumption 2.2, Assumption 3.1]. We also note that in [2] the authors use a different technique to show existence and uniqueness of solutions, which enables them to prove uniqueness without an entropy condition.

We proceed as in Section 3. We construct a sequence of approximate solutions to problem (5.1) and prove its convergence. The approximate solution \( \rho^N \) is defined as in (3.1) and it is constructed as in Algorithm 3.1, substituting the numerical flux in (3.2) by

\[
F_j \left( \rho^{n}_{j,k}, R^{e,n}_{j,k} \right) = v_j \left( R^{e,n}_{j,k} \right) \rho^{n}_{j,k},
\]

(5.3)

and the convective step (3.3) by

\[
\rho^{n+1/2}_{j,k} = \rho^n_{j,k} - \lambda \left[ F_j \left( \rho^n_{j,k}, R^{e,n}_{j,k} \right) - F_j \left( \rho^n_{j,k-1}, R^{e,n}_{j,k-1} \right) \right],
\]

(5.4)

where \( R^{e,n}_{j,k} \) is computed as in (3.6), and in particular as in (3.10), with \( w_\iota \) instead of \( w_\nu \). Due to the definition of the kernel \( w_\nu \), notice that the case (3.11) does not apply to the present setting. Accordingly, we rename the discrete convolution appearing in the source, defined by (3.6), as \( R^{e,n+1/2}_{j,k} \). The choice of the numerical flux (5.3) follows from [8, 14].

We report below the definition of solution to problem (5.1), analogous to Definition 2.1 and then recall the main results, analogous to those in Section 3. Only those parts of the proofs which are substantially different will be reported.
Definition 5.1. Let \( \rho_{o,j} \in (L^1 \cap BV)(\mathbb{R}; [0,1]) \), for \( j = 1, \ldots, M \). We say that \( \rho_j \in C^0([0,T]; L^1(\mathbb{R}; [0,1])) \), with \( \rho_j(t, \cdot) \in BV(\mathbb{R}; [0,1]) \) for \( t \in [0,T] \), is a weak solution to (5.1) with initial datum \( \rho_{o,j} \) if for any \( \varphi \in C^1_c([0,T] \times \mathbb{R}; \mathbb{R}) \) and for all \( j = 1, \ldots, M \)

\[
\int_0^T \int_\mathbb{R} \left( \rho_j \frac{\partial}{\partial t} \varphi + \rho_j V_j \frac{\partial}{\partial x} \varphi + \left( S_j - S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}^o, R_j^o) - S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) \right) \varphi \right) dx \, dt
\]

\[
+ \int_\mathbb{R} \rho_{o,j} \varphi(0,x) dx = 0,
\]

where \( V_j(t,x) = v_j \left( (\rho_j(t) * w_i)(x) \right) \), \( S_j \) is as in (2.1) and \( R_j^o = R_j^o(t,x) = (\rho_j(t) * w_v)(x) \). The solution \( \rho_j \) is an entropy solution if for any \( \varphi \in C^1_c([0,T] \times \mathbb{R}; \mathbb{R}) \), for all \( \kappa \in \mathbb{R} \) and for all \( j = 1, \ldots, M \)

\[
\int_0^T \int_\mathbb{R} \left( |\rho_j - \kappa| \frac{\partial}{\partial t} \varphi + |\rho_j - \kappa| V_j \frac{\partial}{\partial x} \varphi \right) dx \, dt + \int_\mathbb{R} |\rho_{o,j} - \kappa| \varphi(0,x) dx
\]

\[
\geq \int_0^T \int_\mathbb{R} \text{sgn}(\rho_j - \kappa) \left( S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) - S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}, R_j) + \kappa \frac{\partial}{\partial x} V_j \right) \varphi dx \, dt.
\]

In the following, whenever we refer to the modified Algorithm we mean Algorithm 3.1 with (3.2) and (3.3) substituted by (5.3) and (5.4), respectively. All the approximate solutions appearing in the results below are constructed via this modified Algorithm.

Lemma 5.2. Let \( \rho_o \in L^\infty(\mathbb{R}; [0,1]^M) \). Assume that the CFL condition (3.12) holds. Then, for all \( t > 0 \) and \( x \in \mathbb{R} \), the piece-wise constant approximate solution \( \rho_\Delta \) constructed through the modified Algorithm attains value in the set \( [0,1]^M \), i.e.

\[
0 \leq \rho_{j,\Delta}(t,x) \leq 1 \quad \text{for all} \quad j = 1, \ldots, M.
\]

Proof. Since the CFL condition (3.12) is more restrictive than that necessary for the convergence of the Godunov type scheme, see [14] Theorem 3.1, the convective step (3.3) still preserves the invariance of the set \( [0,1]^M \) and the rest of the proof of Lemma 8.3 can be applied. \( \square \)

Lemma 8.5 still holds, since the modified Algorithm preserves the \( L^1 \)-norm.

Proposition 5.3 (BV estimate in space). Let \( \rho_o \in (L^1 \cap BV)(\mathbb{R}; [0,1]^M) \). Assume that the CFL condition (3.12) holds. Then, for \( n = 0, \ldots, N_T - 1 \) the following estimate holds

\[
\sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n| \leq e^{n(8K+w_v(0)V)} \sum_{j=1}^M \text{TV}(\rho_j^n).
\]

Proof. The proof of Proposition 5.7 can be easily adapted. We just have to replace estimate (5.23), involving the convective step, since the scheme with the new numerical flux (5.3) is not total variation diminishing. Following [14] Theorem 3.2 we obtain

\[
\sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1}| \leq (1 + 8 \Delta t K)(1 + \Delta t w_t(0)V) \sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n|.
\]
Lipschitz continuous in both variables with Lipschitz constant.

Therefore, thanks to Lemma 3.5, we have

\[
\Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} |\rho_{j,k}^{n+1} - \rho_{j,k}^{n}| \leq 2 \Delta t \left( 2 \|\rho_{o}\|_{L^1(\mathbb{R})} + \mathcal{V} e^{t^n(8K+w_{o}(0))} \sum_{j=1}^{M} \text{TV} (\rho_{j}^{0}) \right),
\]

(5.6)

with \( K \) as in (3.17), \( \mathcal{V} \) as in (3.13) and \( \text{TV} \) as in (3.14).

**Proof.** Observe that

\[
\left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n} \right| = \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right| \leq \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right| + \left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right|.
\]

We then estimate each term on the right hand side separately.

By the relaxation step (3.4) and the bound (3.22) we have

\[
\left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right| = \Delta t \left| S_{j-1} \left( \rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) - S_{j} \left( \rho_{j,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2} \right) \right|
\leq \Delta t \text{TV} \left( \rho_{j,k}^{n+1/2} + 2 \rho_{j,k}^{n+1/2} + \rho_{j+1,k}^{n+1/2} \right).
\]

Therefore, thanks to Lemma 3.5,

\[
\Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n} \right| \leq \Delta t \text{TV} \sum_{j=1}^{M} \frac{1}{4} \left| \rho_{j}^{n+1/2} \right|_{L^1(\mathbb{R})} = 4 \Delta t \text{TV} \sum_{j=1}^{M} \left| \rho_{j,o} \right|_{L^1(\mathbb{R})},
\]

(5.7)

Exploiting the modified convective step (5.4), since the numerical flux defined in (5.3) is Lipschitz continuous in both variables with Lipschitz constant \( \mathcal{V} \) (3.13), we have

\[
\left| \rho_{j,k}^{n+1/2} - \rho_{j,k}^{n} \right| = \lambda \left| F_{j} \left( \rho_{j,k}^{n}, R_{j,k}^{n} \right) - F_{j} \left( \rho_{j,k-1}^{n}, R_{j,k-1}^{n} \right) \right|
\leq \lambda \mathcal{V} \left( \rho_{j,k}^{n} - \rho_{j,k-1}^{n} + R_{j,k}^{n} - R_{j,k-1}^{n} \right),
\]

Hence, using also (3.8) and the total variation bound provided by Proposition 5.3, we get

\[
\Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n} \right| \leq 2 \Delta t \mathcal{V} \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n} - \rho_{j,k-1}^{n} \right| \leq 2 \Delta t \mathcal{V} e^{t^n(8K+w_{o}(0))} \sum_{j=1}^{M} \text{TV} (\rho_{j}^{0}).
\]

(5.8)

Collecting together (5.7) and (5.8) yields the thesis

\[
\Delta x \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n+1} - \rho_{j,k}^{n} \right| \leq 2 \Delta t \left( 2 \|\rho_{o}\|_{L^1(\mathbb{R})} + \mathcal{V} e^{t^n(8K+w_{o}(0))} \sum_{j=1}^{M} \text{TV} (\rho_{j}^{0}) \right).
\]

\( \square \)
Lemma 4] for the treatment of the nonlocal flux, while the source terms are treated similarly.

Proof. The doubling of variables technique [23] allows to get the following estimate, see [9, Proposition 5.4] we obtain a BV estimate in space and time.

Analogously to Section 5.4, a discrete entropy inequality could be derived also in the case of nonlocal flux, see [1, Proposition 2.8]. Indeed, combining [14, Theorem 3.4] for the nonlocal flux and Lemma 3.10 for the treatment of the source terms we get the following result.

Lemma 5.5. Let \( \rho \in L^1 \cap BV(\mathbb{R}; [0, 1]^M) \). Let the CFL condition (3.12) hold. Then the approximate solution \( \rho_\Delta \) constructed through the modified Algorithm satisfies the following discrete entropy inequality: for all \( j = 1, \ldots, M \), for \( k \in \mathbb{Z} \), for \( n = 0, \ldots, N_T - 1 \) and for any \( c \in [0, 1] \)

\[
\left| \rho_{j,k}^{n+1} - c \right| - \left| \rho_{j,k}^n - c \right| + \lambda \left( \mathcal{F}_j^c \left( \rho_{j,k}^n \right) - \mathcal{F}_j^c \left( \rho_{j,k}^n \right) \right) - \Delta t \text{sgn} \left( \rho_{j,k}^{n+1} - c \right) \left( S_{j-1} \left( \frac{n+1}{2}, \rho_{j-1,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) \right)
- S_j \left( \rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j,k}^{n+1/2} \right)
+ \lambda \text{sgn} \left( \rho_{j,k}^{n+1} - c \right) c \left( v_j \left( \frac{R_{j,k}^n}{R_{j,k}^n} \right) - v_j \left( \frac{R_{j,k}^n}{R_{j,k}^n} \right) \right) \leq 0,
\]

where \( R_{j,k}^{\nu,n+1/2} \) and \( R_{j,k}^{\nu,n} \) are defined accordingly to (3.6) and

\[
\mathcal{F}_j^c(u) = G_j(u \vee c) - G_j(u \wedge c), \quad \text{with } G_j(\rho_{j,k}) = \rho_{j,k}^n v_j \left( \frac{R_{j,k}^n}{R_{j,k}^n} \right).
\]

The results described in Section 5.5 hold analogously for the modified Algorithm, given the bounds obtained in the present section: this ensures the existence of solutions to (5.1).

Uniqueness of solution follows from the Lipschitz continuous dependence of the solution on the initial data. Differently from Theorem 3.1 in the case of nonlocal flux function the solution is not contractive in \( L^1 \).

Theorem 5.6. Let \( \rho \) and \( \pi \) be two solutions to problem (5.1) in the sense of Definition 5.1 with initial data \( \rho_o, \pi_o \in L^1 \cap BV(\mathbb{R}; [0, 1]^M) \) respectively. Assume \( v \in C^2([0, 1], \mathbb{R}) \). Then, for a.e. \( t \in [0, T] \),

\[
\sum_{j=1}^{M} \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \leq e^{Ct} \sum_{j=1}^{M} \left\| \rho_j(0) - \pi_j(0) \right\|_{L^1(\mathbb{R})},
\]

with \( C \) defined as in (5.16).

Proof. The doubling of variables technique [23] allows to get the following estimate, see [9, Lemma 4] for the treatment of the nonlocal flux, while the source terms are treated similarly to [19, Theorem 3.3]: any \( j = 1, \ldots, M \),

\[
\int_{\mathbb{R}} |\rho_j(\tau, x) - \pi_j(\tau, x)| \, dx \leq \int_{\mathbb{R}} |\rho_j(0, x) - \pi_j(0, x)| \, dx
+ \int_0^\tau \int_{\mathbb{R}} |S(\rho, R^\nu, j) - S(\pi, P^\nu, j)| \, dx \, dt \tag{5.9}
\]
thus

\[ \int_0^T \int_{\mathbb{R}} |v_j(R^i_j) - v_j(P^i_j)| \left| \partial_x \rho_j(t, x) \right| \, dx \, dt \quad (5.10) \]

\[ + \int_0^T \int_{\mathbb{R}} \left| \partial_x v_j(R^i_j) - \partial_x v_j(P^i_j) \right| \left| \rho_j(t, x) \right| \, dx \, dt, \quad (5.11) \]

where for the source terms we use the notation introduced in the proof of Theorem 4.1, emphasizing which kernel, \( w_t \) or \( w_{\nu} \), is used. We remark that \( \partial_x \rho \) should be understood in the sense of measures.

To bound the term in \((5.9)\), exploit the Lipschitz continuity of the map \( S_j \) \((2.1)\) in the source term:

\[ \int_0^T \int_{\mathbb{R}} |S(\rho, R^\nu, j) - S(\pi, P^\nu, j)| \, dx \, dt \leq K \int_0^T \left( \| \rho_{j-1}(t) - \pi_{j-1}(t) \|_{L^1(\mathbb{R})} \right. \]

\[ + 2 \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} + \| \rho_{j+1}(t) - \pi_{j+1}(t) \|_{L^1(\mathbb{R})} \]

\[ + \left\| R^\nu_{j-1}(t) - P^\nu_{j-1}(t) \right\|_{L^1(\mathbb{R})} + 2 \left\| R^\nu_j(t) - P^\nu_j(t) \right\|_{L^1(\mathbb{R})} \]

\[ + \left\| R^\nu_{j+1}(t) - P^\nu_{j+1}(t) \right\|_{L^1(\mathbb{R})} \right) \, dt. \]

Observe that for each \( j = 1, \ldots, M \)

\[ \left\| R^\nu_j(t) - P^\nu_j(t) \right\|_{L^1(\mathbb{R})} \leq \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})}, \]

since \( \int_\mathbb{R} w_{\nu} = 1 \). Therefore

\[ \sum_{j=1}^M \left| (5.9) \right| \leq 4K \sum_{j=1}^M \int_0^T \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt. \quad (5.12) \]

Concerning \((5.10)\), note that

\[ \left| v_j(R^i_j) - v_j(P^i_j) \right| \leq w_t(0) \left\| v_j \right\|_{L^\infty([0,1];\mathbb{R})} \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})}, \]

thus

\[ \sum_{j=1}^M \left| (5.10) \right| \leq w_t(0) V^\nu_{\max} \sum_{j=1}^M \int_0^T \text{TV} \left( \rho_j(t) \right) \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt \]

\[ \leq w_t(0) V^\nu_{\max} \left( \sum_{j=1}^M \sup_{t \in [0,\tau]} \text{TV} \left( \rho_j(t) \right) \right) \left( \sum_{j=1}^M \int_0^T \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt \right). \quad (5.13) \]

Pass now to \((5.11)\). Observe first that

\[ \left| \partial_x R^i_j(t, x) \right| = \left| \partial_x (\rho_j(t) * w_t)(x) \right| \]

\[ = \left| - \int_x^{x+i} \rho_j(t, y) w_t'(x-y) \, dy + \rho_j(t, x+i) w_t(i) - \rho_j(t, x) w_t(0) \right| \]

\[ \leq \int_0^\tau \rho_j(t, u + x) w_t'(u) \, du + \| \rho_j(t) \|_{L^\infty(\mathbb{R})} (w_t(i) + w_t(0)) \]

\[ \leq \left| \partial_x \rho_j(t, x) \right| \]
An application of Gronwall Lemma to (5.15) yields the desired result.

\[
\leq \left\| \rho_j(t) \right\|_{L^\infty(\mathbb{R})} \left( \int_0^t |w'_v(u)| \, du + w_v(t) + w_v(0) \right)
\]
\[
= \left\| \rho_j(t) \right\|_{L^\infty(\mathbb{R})} \left( - \int_0^t w'_v(u) \, du + w_v(t) + w_v(0) \right)
\]
\[
= 2 w_v(0) \left\| \rho_j(t) \right\|_{L^\infty(\mathbb{R})},
\]

since the kernel \( w_v \) is such that \( w'_v \leq 0 \). Hence,

\[
\left| \partial_x v_j(R^j) - \partial_x v_j(P^j) \right|
\leq \left| v'_j(R^j) - v'_j(P^j) \right| \left| \partial_x R^j - \partial_x P^j \right|
\leq \left\| v'_j \right\|_{L^\infty((0,1))} \left| R^j - P^j \right| 2 w_v(0) \left\| \rho_j(t) \right\|_{L^\infty(\mathbb{R})}
\]
\[
+ \left\| v'_j \right\|_{L^\infty((0,1))} \int_x^{x+\epsilon} (\pi_j - \rho_j)(t, y) w'_v(x-y) \, dy + (\rho_j - \pi_j)(t, x+\epsilon) w_v(t) - (\rho_j - \pi_j)(t, x) w_v(0)
\]
\[
\leq \left( 2 (w_v(0))^2 \left\| v''_j \right\|_{L^\infty((0,1))} + \left\| v''_j \right\|_{L^\infty((0,1))} \left\| w'_v \right\|_{L^\infty((0,1))} \right) \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})}
\]
\[
+ w_v(0) \left\| v'_j \right\|_{L^\infty((0,1))} \left| \left( \rho_j - \pi_j \right)(t, x+\epsilon) + \left( \rho_j - \pi_j \right)(t, x) \right|
\cdot
\]

Therefore, since the total mass is conserved and \( \rho_j(t, x) \in [0, 1] \) for all \( j = 1, \ldots, M \), \( t \in [0, \tau] \) and \( x \in \mathbb{R} \) by Lemma 5.2,

\[
\sum_{j=1}^M \left( 5.11 \right) \leq \left( 2 (w_v(0))^2 V''_{\text{max}} + V''_{\text{max}} \left\| w'_v \right\|_{L^\infty((0,1))} \right) \left\| \rho_0 \right\|_{L^1(\mathbb{R})} \sum_{j=1}^M \int_0^\tau \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt
\]
\[
+ 2 w_v(0) V''_{\text{max}} \sum_{j=1}^M \int_0^\tau \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt,
\]

where

\[
V''_{\text{max}} = \left\| v'' \right\|_{C^0([0,1];\mathbb{R}^M)} = \max_{j=1,\ldots,M} \left\| v''_j \right\|_{L^\infty((0,1];\mathbb{R})}.
\]

Collecting together (5.12), (5.13) and (5.14) we obtain

\[
\int_\mathbb{R} \left| \rho_j(\tau, x) - \pi_j(\tau, x) \right| \, dx \leq \int_\mathbb{R} \left| \rho_j(0, x) - \pi_j(0, x) \right| \, dx + C \sum_{j=1}^M \int_0^\tau \left\| \rho_j(t) - \pi_j(t) \right\|_{L^1(\mathbb{R})} \, dt,
\]

where

\[
C = 4 \mathcal{K} + 2 w_v(0) V''_{\text{max}} + w_v(0) V''_{\text{max}} \left( \sum_{j=1}^M \sup_{t \in [0,\tau]} \text{TV} (\rho_j(t)) \right)
\]
\[
+ \left( 2 (w_v(0))^2 V''_{\text{max}} + V''_{\text{max}} \left\| w'_v \right\|_{L^\infty((0,1))} \right) \left\| \rho_0 \right\|_{L^1(\mathbb{R})}.
\]

An application of Gronwall Lemma to (5.15) yields the desired result. □

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The following theorem, analogous to Theorem 2.2, collects the main result on problem (5.1), as well as some \textit{a priori} estimates on its solution.

**Theorem 5.7.** Let \( \rho_0 \in (L^1 \cap BV)(\mathbb{R}; [0,1]^M) \). Assume \( v \in C^2([0,1], \mathbb{R}) \). Then, for all \( T > 0 \), problem (5.1) has a unique solution \( \rho \in C^0([0,T]; L^1(\mathbb{R}; [0,1]^M)) \) in the sense of Definition 5.1. Moreover, the following estimates hold: for any \( t \in [0,T] \)

\[
\|\rho(t)\|_{L^1(\mathbb{R})} = \sum_{j=1}^{M} \|\rho_j(t)\|_{L^1(\mathbb{R})} = \|\rho_0\|_{L^1(\mathbb{R})},
\]

for \( j = 1, \ldots, M \) : \( 0 \leq \rho_j(t,x) \leq 1 \),

\[
\sum_{j=1}^{M} TV(\rho_j(t)) \leq e^{t(8K+w_\nu(0)V)} \sum_{j=1}^{M} TV(\rho_{j,o}).
\]

6 Numerical experiments

We present now some numerical examples. We divide this section in two parts: in the first part, we discuss an example with local flux and nonlocal source, as in (2.3), while in the second part we focus on nonlocal flux and source, as in (5.1). For simplicity, we restrict ourselves to only two lanes, i.e. \( M = 2 \), and scaling parameter \( K = 1 \).

6.1 Local flux and nonlocal source: results for model (2.3)

The first example is inspired by [19]. We consider the following velocity functions

\[
v_1(\rho) = 1.5(1 - \rho) \quad \text{and} \quad v_2(\rho) = 2.5(1 - \rho)
\]

and the initial data

\[
\rho_{1,o}(x) = \rho_{2,o}(x) = \sin(\pi x/2)^2.
\]

Figure 1 displays the density profiles with three different source terms at times \( T = 0.75 \) (left column) and \( T = 1.5 \) (right column). We use the nonlocal source term (2.1) with both (3.10) and (3.11), and constant kernel, namely \( w_\nu(x) = 1/\nu \) for (3.10) with \( \nu = 0.5 \), and \( w_\nu(x) = 1/(2\nu) \) for (3.11) with \( \nu = 0.25 \). Therefore, both nonlocal models have an interaction range equal to 0.5. To emphasize the influence of the nonlocality, we include also a local version of the source term (2.1) with

\[
R_j = \rho_j(t,x).
\]

Notice that such a local version differs from that used in [19]: Here the lane changing rate is also proportional to the density in the receiving lane. In the simulations, we consider \( \Delta x = 0.01 \) and \( \Delta t \) given by an adaptive version of the CFL condition (3.12), where \( V \) is computed at each time step using finite differences for the derivative of \( v_j \).

As can be seen in Figure 1 different source terms give rise to slight differences for small times, but as the time grows they become more significant. The nonlocal source terms transport mass faster from the slower lane (lane 1) to the faster one in comparison to the local model. Interestingly, there is only a slight difference between the two nonlocal models with
Figure 1: Density profiles on each lane at $T = 0.75$ (left column) and $T = 1.5$ (right column). The source term (2.1) is computed with (6.1) (top row), (3.10) with $w_\nu(x) = 1/\nu$ and $\nu = 0.5$ (middle row) and (3.11) with $w_\nu(x) = 1/(2\nu)$ and $\nu = 0.25$ (bottom row).

The source term (2.1) is computed with (6.1) (top row), (3.10) with $w_\nu(x) = 1/\nu$ and $\nu = 0.5$ (middle row) and (3.11) with $w_\nu(x) = 1/(2\nu)$ and $\nu = 0.25$ (bottom row). This is probably due to the fact that the nonlocal range and the form of the kernel are equal.

These observations are also supported by the evolution of the $L^1$-norm over time, see Figure 2. We can see that the nonlocal models transport the density faster from lane 1 to lane 2. In addition, the model with the forward looking kernel (3.10) is a bit faster than the one with (3.11).

Finally, we consider the nonlocal models with both (3.10) and (3.11) and let $\nu$ tend to zero. In particular, we choose the constant kernel, in the form $w_\nu(x) = 1/\nu$ for (3.10) and in the form $w_\nu(x) = 1/(2\nu)$ for (3.11), with $\nu \in \{0.64, 0.32, 0.16, 0.08, 0.04, 0.02\}$. Note that for simplicity, $\nu$ is the same for both models, even though one time the nonlocal range is in the interval $[x, x + \nu]$, one time in $[x - \nu, x + \nu]$. Figure 3 displays the lanes separately to better appreciate the convergence. The convergence against the source with (6.1) seems to hold for both models. Moreover, when focusing only on the second lane, also the two nonlocal models display some differences, e.g. compare (3.10) with $\nu = 0.64$ and (3.11) with $\nu = 0.32$ (parameters are chosen so that the interaction ranges have the same width in both models).

Table 1 presents the $L^1$-errors between the local source term (6.1) and both the nonlocal source terms: the data support the convergence against the local solution as $\nu \to 0$. Interestingly, the source term with back- and forward looking kernel (3.11) has smaller error terms on the first lane and only slightly larger errors on the second lane, even tough the nonlocal
Figure 2: Evolution of $L^1$-norm over time for the different source terms: Lane 1 (left) and Lane 2 (right). The dashed blue line represent the source term in (2.1) with (6.1), the continuous orange line refers to (3.10) and the dotted black line to (3.11).

range is twice that of the forward looking kernel (3.10).

| Source term | (3.10) | (3.11) |
|-------------|--------|--------|
| $\nu$       | Lane 1 | Lane 2 | Lane 1 | Lane 2 |
| 0.64        | 0.0311 | 0.0313 | 0.0330 | 0.0310 |
| 0.32        | 0.0239 | 0.0167 | 0.0208 | 0.0198 |
| 0.16        | 0.0159 | 0.0089 | 0.0131 | 0.0120 |
| 0.08        | 0.0095 | 0.0049 | 0.0078 | 0.0066 |
| 0.04        | 0.0054 | 0.0026 | 0.0045 | 0.0035 |
| 0.02        | 0.0030 | 0.0013 | 0.0023 | 0.0016 |

Table 1: $L^1$-errors computed between the solutions with local source (6.1) and nonlocal source terms (3.10) or (3.11).

6.2 Nonlocal flux and nonlocal source: results for model (5.1)

In the following we consider model (5.1) including a nonlocality in the source with parameter $\nu$ and a nonlocality in the flux with parameter $\iota$. We focus on the following two lanes example, inspired by [2]: The velocity function is the same on both lanes and given by

$$v_1(\rho) = v_2(\rho) = 1 - \rho^2,$$

(6.2)
Figure 3: Density profiles for the nonlocal source with (3.10) (left column) and (3.11) (right column) with different nonlocal ranges $\nu$, for lane 1 (top row) and lane 2 (bottom row). For lane 2 the zooms are into the spatial domain $x \in [0.3, 0.7]$ and $\rho \in [0.65, 0.69]$.

and the initial condition is given by

$$\rho_{1,0}(x) = q \left( 2x - \frac{1}{2} \right) \quad \text{and} \quad \rho_{2,0}(x) = q(x)$$

with

$$q(x) = 4x^2(1-x)^2 \chi_{(0,1)}(x),$$

$\chi_A$ being the characteristic function of the set $A$.

Model (5.1) fits into the model framework proposed in [2], if we consider $\nu = \iota$, the same kernel functions for the source and the flux and a forward looking nonlocal term as in (3.10). Therefore we consider, if not stated otherwise, the parameters $\iota = \nu = 0.5$ and the kernels

$$w_\iota(x) = 2 \frac{\iota - x}{\iota^2}, \quad w_\nu(x) = 2 \frac{\nu - x}{\nu^2}.$$

Figure 4 compares models (2.3) and (5.1) and clearly shows the impact of the nonlocal flux. For both models the same nonlocal term (3.10) is used. Because of the nonlocal transport, the solutions display completely different dynamics, mainly due to the high nonlocal range. Indeed, the density does not decrease at the front part of its support on each lane since the vehicles just behind the leading ones anticipate the free space ahead, so that the average density is lower than in the local case.
As already mentioned, the model introduced in [2] has the same nonlocal term, i.e. the same kernel and nonlocal range, both in the source and in the flux. On the other hand, the model (5.1) presented in this paper has more flexibility since it is able to deal with different types of nonlocality in the flux and in the source, the latter being independent of the forward nonlocal term. Therefore, we now focus on varying the nonlocality in the source term.

First of all, we observe that the nonlocal range in the flux and in the source term do not necessarily have to be equal: If a driver wants to overtake a car and thus starts to accelerate, getting ready to change lane, he/she might look further ahead when performing a lane change than if he/she keeps on driving in the same lane. In Figure 5 we display the solutions to (5.1) with initial datum (6.3), velocities (6.2), kernels (6.4), $i = 0.5$ and varying the nonlocal range in the source, thus varying the parameter $\nu$. Due to the initial condition, the main influence of the different parameters $\nu$ in the source term can be seen at the back of the support of the density in lane 1: the smaller the range $\nu$, the smaller the average density (and thus the larger the velocity on lane 1), the more vehicles move from lane 2 to lane 1. An analogous situation happens at the front of the support of the density in lane 2. To sum up, the greater the nonlocal range $\nu$, the less the effect of the source term: When $\nu$ is large, cars get a better awareness of the actual free space ahead so that lane changing may be evaluated as not necessary.

The second reasonable aspect to keep in mind when performing a lane change is to take into account also the backward traffic, both in the present lane and in the target lane. This can be done by considering model (5.1) with back- and forward looking kernel (3.11) in the

Figure 4: Density profiles at $T = 0.5$ (left) and $T = 1$ (right) for the local flux model (2.3) (top row) and the nonlocal flux model (5.1) (bottom row), both with the nonlocal source term using (3.10) with (6.4). Velocity functions as in (6.2) and initial datum as in (6.3).
source term. For this example, we consider a linear symmetric kernel, i.e.

\[ w_\nu(x) = \frac{\nu - |x|}{\nu^2}. \]  

(6.5)

Figure 6 considers the solutions to model (5.1) with velocities (6.2), initial datum (6.3), \( \nu = 0.5, w_i \) as in (6.4) and the following choices of \( w_\nu \) and \( \nu \):

(a) the back- and forward looking kernel (3.11)-(6.5) with \( \nu = 0.25 \), to have the same nonlocal influence as in the flux;

(b) the back- and forward looking kernel (3.11)-(6.5) with \( \nu = 0.5 \), to have the same look ahead parameter as in the flux;

(c) the forward looking kernel (3.10)-(6.3) with \( \nu = 0.5 \), exactly as in the flux.

In cases (a) and (b) with the nonlocal term of type (3.11), more mass is transported from lane 1 to lane 2, especially in the front part of the support of the density of lane 2, even though the leading part of lane 1 is aware of the density on lane 2. In addition, more mass is transported with smaller nonlocal range due to similar effects as already described above. In contrast, more mass seems to be transported from the rear part of lane 2 to lane 1 when the nonlocal term with forward looking kernel (3.10) is used. This may be due to the fact that
for the back- and forward looking kernel (3.11) the nonlocal velocities on both roads depend on free space and density, but for the forward looking kernel (3.11) the velocity of the second lane does not include some free space and lane changing becomes favourable.

Figure 6: Density profiles for the model (5.1) at $T = 0.5$ (left column) and $T = 1$ (right column), velocities (6.2), initial datum (6.3), $v = 0.5$, $w_i$ as in (6.4). Concerning $w_\nu$ and the parameter $\nu$: first row represents case (a), second row case (b), third row case (c).

Conclusion

Inspired by the models presented in [2] and [19], we have introduced a multilane traffic model that allows for nonlocality in the source and in the flux term. For both approaches we have shown existence and uniqueness of solutions. Based on a Godunov type discretization, we also present a numerical study comparing the influence of the nonlocality and different kernels. Future works include the consideration of the continuum limit for infinitely many lanes and comparisons to real data.

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References

[1] P. Amorim, R. M. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM Math. Model. Numer. Anal.*, 49(1):19–37, 2015.

[2] A. Bayen, A. Keimer, L. Pfug, and T. Veeravalli. Modeling multi-lane traffic with moving obstacles by nonlocal balance laws. Preprint, 2020.

[3] S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, 132(2):217–241, 2016.

[4] C. Chalons, P. Goatin, and L. M. Villada. High-order numerical schemes for one-dimensional nonlocal conservation laws. *SIAM J. Sci. Comput.*, 40(1):A288–A305, 2018.

[5] F. A. Chiarello, J. Friedrich, P. Goatin, and S. Göttlich. Micro-macro limit of a nonlocal generalized Aw-Rascle type model. *SIAM J. Appl. Math.*, 80(4):1841–1861, 2020.

[6] F. A. Chiarello, J. Friedrich, P. Goatin, S. Göttlich, and O. Kolb. A non-local traffic flow model for 1-to-1 junctions. *European J. Appl. Math.*, 31(6):1029–1049, 2020.

[7] F. A. Chiarello and P. Goatin. Global entropy weak solutions for general non-local traffic flow models with anisotropic kernel. *ESAIM Math. Model. Numer. Anal.*, 52(1):163–180, 2018.

[8] F. A. Chiarello and P. Goatin. Non-local multi-class traffic flow models. *Netw. Heterog. Media*, 14(2):371–387, 2019.

[9] F. A. Chiarello, P. Goatin, and E. Rossi. Stability estimates for non-local scalar conservation laws. *Nonlinear Anal. Real World Appl.*, 45:608–687, 2019.

[10] J. Chien and W. Shen. Stationary wave profiles for nonlocal particle models of traffic flow on rough roads. *NoDEA Nonlinear Differential Equations Appl.*, 26(6):Paper No. 53, 2019.

[11] R. M. Colombo, A. Corli, and M. D. Rosini. Non local balance laws in traffic models and crystal growth. *ZAMM Z. Angew. Math. Mech.*, 87(6):449–461, 2007.

[12] M. G. Crandall and A. Majda. Monotone difference approximations for scalar conservation laws. *Math. Comp.*, 34(149):1–21, 1980.

[13] J. Friedrich and O. Kolb. Maximum principle satisfying CWENO schemes for nonlocal conservation laws. *SIAM J. Sci. Comput.*, 41(2):A973–A988, 2019.

[14] J. Friedrich, O. Kolb, and S. Göttlich. A Godunov type scheme for a class of LWR traffic flow models with non-local flux. *Netw. Heterog. Media*, 13(4):531–547, 2018.

[15] P. Goatin and E. Rossi. A multilane macroscopic traffic flow model for simple networks. *SIAM J. Appl. Math.*, 79(5):1967–1989, 2019.

[16] P. Goatin and F. Rossi. A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit. *Commun. Math. Sci.*, 15(1):261–287, 2017.

[17] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. *Netw. Heterog. Media*, 11(1):107–121, 2016.

[18] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied Mathematical Sciences*. Springer, Heidelberg, second edition, 2015.

[19] H. Holden and N. H. Risebro. Models for dense multilane vehicular traffic. *SIAM J. Math. Anal.*, 51(5):3694–3713, 2019.
[20] A. Keimer and L. Pflug. Existence, uniqueness and regularity results on nonlocal balance laws. J. Differential Equations, 263(7):4023–4069, 2017.

[21] A. Keimer and L. Pflug. Nonlocal conservation laws with time delay. NoDEA Nonlinear Differential Equations Appl., 26(6):Paper No. 54, 34, 2019.

[22] A. Keimer and L. Pflug. On approximation of local conservation laws by nonlocal conservation laws. J. Math. Anal. Appl., 475(2):1927–1955, 2019.

[23] S. N. Kružkov. First order quasilinear equations with several independent variables. Mat. Sb. (N.S.), 81 (123):228–255, 1970.

[24] R. J. LeVeque. Numerical methods for conservation laws. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 1992.

[25] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. Proc. Roy. Soc. London. Ser. A., 229:317–345, 1955.

[26] P. I. Richards. Shock waves on the highway. Oper. Res., 4:42–51, 1956.

[27] J. Ridder and W. Shen. Traveling waves for nonlocal models of traffic flow. Discrete Contin. Dyn. Syst., 39(7):4001–4040, 2019.