Solution of quantum Dirac constraints via path integral

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Abstract

The semiclassical solution of quantum Dirac constraints in generic constrained system is obtained by directly calculating in the one-loop approximation the gauge field path integral with relativistic gauge fixing procedure. The gauge independence property of this path integral is analyzed by the method of Ward identities with a special emphasis on boundary conditions for gauge fields. The calculations are based on the known reduction algorithms for functional determinants extended to gauge theories. The mechanism of transition from relativistic gauge conditions to unitary gauges, participating in the construction of this solution, is explicitly revealed. Implications of this result in problems with spacetime boundaries, quantum gravity and cosmology are briefly discussed.

1. Introduction

It is well known that the path integral is an efficient tool for solving the Schrodinger equation. Numerous results in modern physics in this or that way are related to the path integral method. The virtue of this method is that it is applicable to any evolutionary (parabolic) equation and for the Schrodinger equation gives its unitary evolution operator. At the same time there exists a class of problems in which the fundamental dynamical equations are not of a manifestly evolution type. A wide class of such problems is represented by Dirac quantization of constrained dynamical systems \cite{1,2}. In such systems the Schrodinger equation is supplemented by quantum Dirac constraints on quantum states, so that the problem amounts to solving the whole system of equations, not all of them being of the evolutionary type. Moreover, in the parametrized systems with a vanishing Hamiltonian there is no independent Schrodinger equation and their quantum dynamics is encoded in the Dirac constraints along with their gauge invariance properties. Applications of the path integral method in this context are much less known and generally look as follows.

Consider dynamical systems with the canonical action

\[ S \left[ q, p, N \right] = \int_{t_1}^{t_2} dt \left[ p_i \dot{q}^i - N^\mu T_\mu(q, p) \right] \] (1.1)
in configuration space of canonical coordinates and momenta \((q, p) = (q^i, p_i)\) and Lagrange multipliers \(N = N^\mu\). The variation of \(N^\mu\) leads to nondynamical equations – the constraints

\[
T_\mu(q, p) = 0.
\] (1.2)

The constraint functions on phase space \(T_\mu(q, p)\) belong to the first class when they satisfy the Poisson bracket algebra

\[
\{T_\mu, T_\nu\} = U^\alpha_{\mu\nu} T_\alpha
\] (1.3)

with some structure functions \(U^\alpha_{\mu\nu} = U^\alpha_{\mu\nu}(q, p)\). This algebra indicates that the theory possesses a local gauge invariance under the action of canonical transformations of \((q, p)\) generated by constraints themselves and by certain transformations of Lagrange multipliers \([3, 4]\).

Dirac quantization of the theory (1.1) consists in promoting initial phase-space variables and constraint functions to the operator level \((\hat{q}, \hat{p}, \hat{T}_\mu)\) and selecting the physical states \(|\Psi\rangle\) in the representation space of \((\hat{q}, \hat{p}, \hat{T}_\mu)\) by the equation

\[
\hat{T}_\mu |\Psi\rangle = 0 \quad [1, 2, 5, 6, 7].
\]

Operators \((\hat{q}, \hat{p})\) are subject to canonical commutation relations \([\hat{q}^k, \hat{p}_l] = i\hbar \delta^k_l\) and the quantum constraints \(\hat{T}_\mu\) as operator functions of \((\hat{q}, \hat{p})\) should satisfy the correspondence principle with classical \(c\)-number constraints and be subject to the commutator algebra

\[
[\hat{T}_\mu, \hat{T}_\nu] = i\hbar \hat{U}^\lambda_{\mu\nu} \hat{T}_\lambda.
\] (1.4)

with certain operator structure functions \(\hat{U}^\lambda_{\mu\nu}\) standing to the left of operator constraints. This algebra generalizes (1.3) to the quantum level and serves as integrability conditions for quantum constraints. In the coordinate representation, \(\langle q | \Psi\rangle = \Psi(q), \quad p_k = \hbar \partial / i \partial q^k\), the latter become the equations on the physical wave function

\[
\hat{T}_\mu(q, \hbar \partial / i \partial q) \Psi(q) = 0
\] (1.5)

with the differential operators of quantum Dirac constraints \(\hat{T}_\mu(q, \hbar \partial / i \partial q)\).

Note that without loss of generality we did not include in (1.1) the Hamiltonian \(H(q, p)\) nonvanishing on constraint equations (1.2). By extending the phase space of the theory with extra canonical pair \((q^0, p_0)\), \(q^0 \equiv t\), subject to the constraint \(p_0 + H(q, p) = 0\) one can always reduce the system to the case of action (1.1). At the quantum level this extra Dirac constraint plays the role of the Schrodinger equation for the wave function of the theory with parametrized time, \(\Psi(t, q) = \Psi(q^0, q)\),

\[
\left[ \frac{\hbar}{i} \frac{\partial}{\partial q^0} + \hat{H}(q, \hbar \partial / i \partial q) \right] \Psi(q^0, q) = 0.
\] (1.6)

In this sense the dynamical content of any theory can be encoded in the quantum Dirac constraints of the above type.
The path integral in this context arises as a special solution of quantum constraints (1.5) in the form of the two-point kernel $K(q, q')$ – the analogue of the two-point evolution operator for the Schrödinger equation [8, 9]

$$\hat{T}_\mu(q, h\partial/iq) K(q, q') = 0.$$ (1.7)

Similarly to the theory of the Schrödinger equation this is a path integral over configuration space variables $(q(t), N(t))$ in (spacetime) domain $t_- < t < t_+$ with the boundary conditions related to the arguments of this kernel $q(t_+) = q$, $q(t_-) = q'$. However, in view of the non-evolutionary nature of equations (1.7) the path integral construction here is much less straightforward than in the Schrödinger case.

Mainly the motivation for such applications comes from quantum gravity theory [8, 9, 10, 11, 12]. In this theory (1.1) is a canonical form of the Einstein action with $q^i = g_{ab}(x)$ – 3-metric coefficients and $N^\mu \sim \left((-g_{00})^{-1/2}(x), g_{0a}(x)\right)$ – lapse and shift functions in (3+1)-foliation of spacetime [13], (1.2) are the gravitational Hamiltonian and momentum constraints and (1.5) represents the system of Wheeler-DeWitt equations on the cosmological wavefunction $\Psi(q) = \Psi \left[ g_{ab}(x) \right]$ [2]. The construction of the path integral here is pursuing two main goals. The first goal is a two-point solution for the Wheeler-DeWitt equations (1.7) [8, 9, 11, 12], which in view of parametrized nature of time encodes the dynamical information. The second goal consists in specifying the distinguished cosmological quantum state – the model for initial conditions in quantum cosmology of the early universe [10]. The first formulation of the path integral for the two-point solution of Wheeler-DeWitt equations belongs to H.Leutwyler [8]. It was only qualitatively correct because at that time the structure of gauge fixing procedure and the role of ghost fields in the path integral have not yet been understood. The path integral with unitary gauge fixing procedure and exhaustive set of boundary conditions was later proposed in [9]. Then this canonical path integral was converted to the spacetime covariant form of the functional integral over Lagrangian variables in relativistic gauges [11, 12].

Of course, these results incorporate a well-known statement on equivalence of the canonical and covariant quantizations pioneered in [3]. In contrast to this, the works [8, 10, 11, 12] were focused on the nontrivial boundary conditions in spacetime. Correct treatment of these boundary conditions leads to the proof that this path integral solves the quantum Dirac constraints (1.7). However, this proof given in [3, 10, 11, 12] has a formal nonperturbative nature and does not even allow one to fix the operators of quantum constraints. One can only infer from this proof that these operators satisfy the correspondence principle with their classical counterparts and have quantum corrections which in a rather uncontrollable way depend on the calculational method for a path integral [14]. Thus, no check of the solution to quantum constraints was thus far given by direct calculations of the path integral. The goal of this paper is to perform such a check. This will be done for generic systems subject to first class constraints of the above type in the one-loop approximation of semiclassical expansion.
The nature of this check is certainly a comparison of two results: one obtained by solving the equations (1.7) and another by calculating the path integral. In the one-loop approximation for the Schrodinger equation such a comparison is based on the Pauli-Van Vleck-Morette formula [15] and the reduction method for functional determinants [16]. The preexponential factor in the subleading semiclassical order is given by the Van Vleck determinant – the solution of the corresponding continuity equation. On the other hand, it is given by the functional determinant of the inverse propagator of the theory – the contribution resulting from the one-loop (gaussian) approximation for the path integral. The equality of these two expressions follows from the reduction method for functional determinants considered in [16]. The semiclassical solution of quantum Dirac constraints in terms of the modified Van Vleck determinant was built in the author’s paper [17, 18]. Here we reproduce this solution from the Faddeev-Popov path integral of ref.[11].

The paper is organized as follows. In Sect.2 we recapitulate the semiclassical solution of quantum Dirac constraints of refs. [17, 18] with a particular emphasis on unitary gauges participating in its construction. Sect.3 serves as a link between the canonical formalism of the constrained system and its Lagrangian version with local gauge invariances presented in (space)time covariant form. The corresponding Faddeev-Popov path integral in relativistic gauges constructed as a solution of quantum Dirac constraints in [11] is presented in Sect.4 along with its semiclassical expansion up to the one-loop order. Sect.5 deals with the mechanism of its gauge independence based on Ward identities and, in particular, gauge independence of its local measure cancelling the strongest power divergences. In Sect.6 we calculate the contribution of the gauge field functional determinant by deriving the algorithm of its reduction to the Van Vleck elements of the semiclassical solution of Sect.2. Similar calculations are performed for the ghost field determinant in Sect.7. They accomplish the proof of the main result – path integral derivation of the solution to quantum Dirac constraints. In concluding section we discuss implications of this result regarding the aspects of gauge invariance in quantum cosmology and Euclidean quantum gravity.

The final remark, which is in order here, concerns our notations. Throughout the paper we use condensed DeWitt notations [2] which allow one in a manageable form to handle both the quantum mechanical and field theoretical problems within the language of the above type. We imply that the range of indices of $q^i$ and $N^\mu$

$$i = 1, \ldots, n, \quad \mu = 1, \ldots, m,$$  

(1.8)

in field models formally extends to infinite dimensionalities of phase space $n$ and space of gauge transformations $m$, and these canonical condensed indices together with discrete tensor labels carry also continuous labels of spatial coordinates $x$ (or certain discrete numbers when the fields are expanded in the basis of some countable set of spatial harmonics). Contraction of these indices will include spatial integration (or
the corresponding infinite summation\footnote{This formal approach does not certainly incorporate a rigorous handling of ultraviolet infinities and possible quantum anomalies which go beyond the scope of this paper. The justification of this approach is based on the fact that our considerations serve as a bridge between the manifestly noncovariant Dirac quantization and its Lagrangian path integral counterpart that can be cast into manifestly covariant form. It is the latter formalism that must be covariantly regulated to give a physically reasonable framework for renormalization and quantum anomalies.}. Starting from Sect. 3 we shall also need the covariant condensed indices including the time label and allowing one to represent the covariant (space)time operations of integration and differentiation in the form of a simple contraction.

2. Canonical solution of quantum constraints: unitary gauges

The solution of quantum constraints (1.7) found in \cite{17, 18, 6}

\[ K(q, q') = P(q, q') e^{i \bar{\hbar} S(q, q')} \]  

(2.1)
contains the Hamilton-Jacobi function \( S(q, q') \) which determines the tree-level approximation and the preexponential factor \( P(q, q') \) accumulating loop corrections. The system of Hamilton-Jacobi equations for \( S(q, q') \)

\[ T_\mu(q, \partial S/\partial q) = 0 \]  

(2.2)

enforces the quantum constraints in the leading order of \( \hbar \)-expansion. For their operator realization proposed in \cite{17, 18, 6} the subleading (one-loop) order yields the following quasi-continuity equations for the preexponential factor \( P(q, q') \) (see also \cite{19} in the gravitational context)

\[ \frac{\partial}{\partial q'} (\nabla^i T_\mu) = U^\lambda_\mu P^2, \]  

(2.3)

\[ \nabla^i_\mu \equiv \frac{\partial T_\mu}{\partial p_i} \bigg|_p = \partial S/\partial q' . \]  

(2.4)

A particular solution of eqs. (2.2) - (2.3) corresponds to the choice of the principal Hamilton function for \( S(q, q') \) – the classical action (1.1) calculated at the extremal of equations of motion \( g = g(t|q, q') \) joining the configuration space points \( q \) and \( q' \). Together with eq.(2.2) this function also satisfies the Hamilton-Jacobi equations with respect to its second argument

\[ T_\mu(q', -\partial S/\partial q') = 0. \]  

(2.5)

Similarly to the WKB theory of non-gauge quantum systems, this function gives rise to a special one-loop preexponential factor \cite{17, 18, 6} which is a generalization of...
the Pauli-Van Vleck-Morette formula [15] – calculating the determinant of the matrix of second order derivatives of the principal Hamilton function with respect to its two arguments. However, in contrast with non-gauge theories this factor cannot be directly constructed in terms of this determinant, because in view of eqs. (2.2) and (2.5) the matrix
\[ S_{ik'} = \frac{\partial^2 S(q, q')}{\partial q^i \partial q^{k'}} , \tag{2.6} \]
has left and right zero-eigenvalue eigenvectors [17, 18, 6] and, therefore, is degenerate
\[ \nabla_i S_{ik'} = 0 , \tag{2.7} \]
\[ S_{ik'} \nabla_{k'} = 0 , \quad \nabla_{k'} = \left. \frac{\partial T_{\nu}}{\partial p_{k'}} \right|_p = -\partial S/\partial q' . \tag{2.8} \]

The construction of the one-loop preexponential factor in terms of this degenerate matrix is equivalent to the Faddeev-Popov gauge-fixing procedure for gauge theories. It consists in adding the “gauge-breaking” term bilinear in “gauge conditions” \( X^\mu_i \) and \( X^\nu_{k'} \) – two sets of arbitrary covectors at the configuration space points \( q \) and \( q' \)
\[ D_{ik'} = S_{ik'} + X^\mu_i C_{\mu\nu} X^\nu_{k'} . \tag{2.9} \]
This allows one to replace the degenerate matrix \( S_{ik'} \) by the new invertible matrix \( D_{ik'} \), provided that the gauge-fixing matrix \( C_{\mu\nu} \) is also invertible and these gauge covectors produce invertible “Faddeev-Popov operators” [20]
\[ J^\mu_\nu = X^\mu_i \nabla_i^{\nu} , \quad J \equiv \det J^\mu_\nu \neq 0 , \tag{2.10} \]
\[ J'^\mu_\nu = X'^\mu_i \nabla'^i_\nu , \quad J' \equiv \det J'^\mu_\nu \neq 0 . \tag{2.11} \]

In terms of these objects the solution of the continuity equations (2.3) is given by the following expression [17, 18, 6]
\[ P = \left[ \frac{\det D_{ik'}}{J J' \det C_{\mu\nu}} \right]^{1/2} , \tag{2.12} \]
which can be regarded as an analogue of the one-loop expression for the effective action of gauge field theory – the contribution of gauge fields \( \det D_{ik'} \) partly compensated by the contribution of ghosts \( J \) and \( J' \). This compensation makes the prefactor independent of the introduced arbitrary elements of gauge-fixing procedure (\( X^\mu_i \), \( X'^\nu_{k'} \), \( C_{\mu\nu} \)) – the analogue of on-shell gauge independence in gauge field theory. The mechanism of this gauge independence is based on the “Ward identities” for the gauge field “propagator” (2.9)
\[ C_{\mu\nu} X'^\nu_{k'} D^{-1} k'^i = J^{-1}_{\mu\nu} \nabla^i_{\nu} , \tag{2.13} \]
easily obtained by contracting (2.9) with \( \nabla^i_\mu \) and using (2.7). The use of these identities shows that arbitrary variations of the quantities (\( X^\mu_i \), \( X'^\nu_{k'} \), \( C_{\mu\nu} \)) in the one-loop prefactor
vanish due to the cancellation of terms coming from the gauge field det \( D_{ik'} \) and ghost \( J, J' \) determinants.

The nature of covectors \((X_\mu^i, X_\nu^k')\) as matrices of gauge conditions does not only follow from the fact that they remove the degeneracy of the matrix \( S_{ik'} \) caused by gauge invariance of the theory. As shown in [17, 18] the quantum Hamiltonian reduction of the kernel \( K(q, q') \) with the one-loop factor (2.12) leads to the unitary evolution operator in the physical sector defined by the unitary gauge conditions

\[
X_\mu^i(q, t) = 0, \tag{2.14}
\]

such that

\[
X_\mu^i = \frac{\partial X_\mu}{\partial q^i}. \tag{2.15}
\]

The unitary gauge conditions are imposed only on phase space variables of the theory \((q, p)\) (in this case only on coordinates \(q\)). In contrast with relativistic gauges involving Lagrange multipliers, they manifestly incorporate unitarity and do not give rise to propagating ghosts. The price one usually pays for manifest unitarity is the absence of manifest covariance, that can be restored by going over to relativistic gauges. Thus, the solution (2.12) is obtained by directly solving the quantum Dirac constraints within the framework of unitary gauge conditions. In what follows we show that the same solution can be obtained by a direct calculation of the path integral in the relativistic gauge, and this derivation as a byproduct will establish the relation between unitary and relativistic gauge conditions.

3. **Lagrangian versus canonical formalisms**

As a first step towards the covariant path integral in the relativistic gauge let us consider the Lagrangian formalism of the theory with the canonical action (1.1). For this we introduce the collective notation for the full set of Lagrangian configuration space variables – canonical coordinates and Lagrange multipliers

\[
g^a = (q^i(t), N^{\mu}(t)). \tag{3.1}
\]

In what follows we shall need also (space)time condensed DeWitt notations in which the index \(a\) includes not only the spin labels and spatial coordinates \(x\) but also the time variable \(t\), and the contraction of these indices implies the time integration (as

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2 Explicit time dependence of unitary gauge conditions is necessary in the theories with parametrized time in order to have nontrivial time evolution with a nonvanishing physical Hamiltonian [4, 13].

3 In the context of Einstein gravity theory this collection of ten fields \(g^a \sim g_{\alpha\beta}(x, t)\) comprises the whole set of spacetime metric coefficients taken in a special parametrization adjusted to (3+1)-splitting: \(q^i = g_{ab}(x, t), N^{\alpha} \sim g_{0\alpha}(x, t)\).
mentioned in Introduction we shall call them covariant). In these notations the action has the form

\[ S[g] = \int_{t_-}^{t_+} dt \ L(q, \dot{q}, N) \]  

(3.2)

with the Lagrangian which does not involve time derivatives of the Lagrange multipliers \( N = N^\mu(t) \). This Lagrangian is related to the integrand of the canonical action (1.1)

\[ L(q, \dot{q}, N) = \left( p_i \dot{q}^i - N^\mu T_\mu(q, p) \right) \bigg|_{p=p^0(q, \dot{q}, N)} \]  

(3.3)

by the substitution of the expression for the canonical momentum \( p^0_i(q, \dot{q}, N) \) in terms of the velocities \( \dot{q} \)

\[ p^0_i(q, \dot{q}, N) = \frac{\partial L(q, \dot{q}, N)}{\partial \dot{q}^i} \]  

(3.4)

which is a solution of the canonical equation of motion

\[ \dot{q}^i = N^\mu \frac{\partial T_\mu(q, p)}{\partial p_i}. \]  

(3.5)

The classical action is invariant under gauge transformations with local (arbitrary time and space dependent) parameters \( f^\mu = f^\mu(t) \) vanishing on spacetime boundary \( f^\mu(t_\pm) = 0 \). For the canonical action (1.1) these transformations are canonical and, therefore, ultralocal in time for phase space variables, but involve the time derivative of the gauge parameter for Lagrange multipliers [3]

\[ \delta q^i = \{ q^i, T_\mu \} f^\mu, \quad \delta p_i = \{ p_i, T_\mu \} f^\mu, \]  

(3.6)

\[ \delta N^\mu = \dot{f}^\mu - U^\mu_{\alpha\nu} N^\alpha f^\nu. \]  

(3.7)

Thus the first class constraints \( T_\mu(q, p) \) serve as generators of infinitesimal gauge transformations on the phase space of \( (q, p) \). Note, in particular, that the transformations of phase space coordinates \( q \) when restricted to the Lagrangian surface of the Hamilton-Jacobi function, \( p = \partial S/\partial q \), read

\[ \delta q^i = \nabla^i f^\mu \]  

(3.8)

in terms of the vector field (2.4).

These gauge invariance transformations in the Lagrangian formalism take the form of the infinitesimal transformations of configuration space variables

\[ \delta g^a = R^a_\mu f^\mu, \]  

(3.9)

\[ R^a_\mu \frac{\delta S[g]}{\delta g^a} = 0. \]  

(3.10)
with the Lagrangian generators $R^a_\mu = (R_i^\mu, R^\alpha_\mu)$. Here we use covariant condensed notations in which the index $a$ includes time, and its contraction implies the time integration. Thus, $R^a_\mu$ forms a delta-function type kernel with two entries $a \rightarrow (a, t), \mu \rightarrow (\mu, t')$

$$R^a_\mu = R^a_\mu (d/dt) \delta(t - t'),$$  

(3.11)

where $R^a_\mu (d/dt)$ denotes the differential (or ultralocal multiplication) operator acting on the first argument of the delta function. In view of the relation (3.3) between the canonical and Lagrangian formalisms different components of this kernel follow from the invariance transformations (3.6)-(3.7) \[4\]

$$R^i_\mu = \delta(t - t') \left. \frac{\partial T^\mu}{\partial p_i} \right|_{p = p^0(q, \dot{q}, N)},$$

(3.12)

$$R^\alpha_\mu = \left( \delta^\alpha_\mu \frac{d}{dt} - U^\alpha_\lambda N^\lambda \right) \delta(t - t').$$

(3.13)

The distinguished role of the Lagrange multipliers here manifests itself in the fact that only the $a = \alpha$ component of (3.11) forms the first order differential operator while the other components are ultralocal in time. In particular, the transformation (3.12) is a Lagrangian form of (3.8).

In what follows we shall have to use condensed notations of both canonical and covariant nature. For brevity we shall not introduce special labels to distinguish between them. As a rule, when the time argument is explicitly written we shall imply that the corresponding condensed index or indices are canonical, i.e. they contain only spin labels and spatial coordinates and their contraction does not involve implicit time integrals. For example, the left-hand side of gauge identities (3.10) can be written down in the form

$$R^a_\mu \frac{\delta S[g]}{\delta g^a} = R^a_\mu (d/dt) \frac{\delta S[g]}{\delta g^a(t)}, \mu \rightarrow (\mu, t),$$

(3.14)

where the time integral implicit in the contraction of the covariant condensed index $a$ removed the delta function contained in $R^a_\mu$ and, thus, the result boiled down to the action of the differential operator $R^a_\mu (d/dt)$ on $\delta S[g]/\delta g^a(t)$. This operator obviously differs from that of eq.(3.11) by the functional transposition – integration by parts, because in contrast with (3.11) it acts on test function with respect to upper condensed index $a$. This fact is indicated by the order of operator indices reversed relative to eq. (3.11).

\[4\] The equivalence of the full set of transformations (3.6)-(3.7) (including transformations of momenta) to (3.9)-(3.13) holds only up to terms proportional to equations of motion \[3\]. Only up to such terms holds the equality $\delta p|_{p^0(q^i, \dot{q}, N)} = \delta p^0(q, \dot{q}, N)$. For us, however, it is only important to know that Lagrangian gauge transformations can be obtained from the subset of transformations of coordinates $q^i$ and Lagrange multipliers $N^\mu$ in canonical theory.
Another important distinction between these two types of condensed notations concerns functional derivatives. We shall always reserve functional variational notations \( \frac{\delta}{\delta g^a} \equiv \frac{\delta}{\delta g^a(t)} \) for variational derivatives with respect to functions of time, while the variational derivatives with respect to functions of spatial coordinates will be denoted by partial derivatives. For example, in the gravitational context we have \( \frac{\delta}{\delta g^a} \equiv \frac{\delta}{\delta g_{\alpha\beta}(x,t)} \) vs \( \partial / \partial q^i \equiv \frac{\delta}{\delta g^{ab}(x)} \).

4. The path integral in relativistic gauge

The path integral solution of quantum Dirac constraints was constructed in [11]. This is a functional integral over the gauge fields \( g^a \) and grassman ghost fields \( C^\mu = C^\mu_{\nu} \) with a conventional Faddeev-Popov gauge fixing procedure

\[
K(q_+, q_-) = \int Dg \mu[g] DC D\bar{C} \exp \frac{i}{\hbar} \left\{ \left( S[g] - \frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu \right) + \bar{C}_\mu Q^\mu_{\nu} C^\nu \right\}. \tag{4.1}
\]

The total action in the exponential here contains the gauge fixed classical and ghost actions. The gauge fixed classical action

\[
S_{gf}[g] = S[g] - \frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu, \tag{4.2}
\]

includes the gauge breaking term

\[
\frac{1}{2} \chi^\mu c_{\mu\nu} \chi^\nu = \frac{1}{2} \int_{t_-}^{t_+} dt \chi^\mu(g, \dot{g}) c_{\mu\nu}(g, \dot{g}) \tag{4.3}
\]

quadratic in the relativistic gauge conditions

\[
\chi^\mu = \chi^\mu(g, \dot{g}), \tag{4.4}
\]
\[
a^\mu_\nu = - \frac{\partial \chi^\mu}{\partial N^\nu}, \quad \text{det } a^\mu_\nu \neq 0. \tag{4.5}
\]

An important distinction of these gauge conditions is that they involve the velocities of all Lagrange multipliers which results in propagating nature of all gauge and ghost fields. The corresponding Faddeev-Popov operator \( Q^\mu_{\nu} \), the kernel of the ghost action bilinear in ghost fields \( C \) and \( \bar{C} \)

\[
Q^\mu_{\nu} = \frac{\delta \chi^\mu}{\delta g^a} R^a_{\nu}, \tag{4.6}
\]

turns out to be a second order differential operator (cf. eqs. (3.13) and (4.3))

\[
Q^\mu_{\nu}(d/dt) \delta(t - t') = \left(-a^\mu_\nu \frac{d^2}{dt^2} + \ldots\right) \delta(t - t'). \tag{4.7}
\]

An important ingredient of the path integral (4.1) is the local integration measure \( \mu[g] \) which is built of the Hessian matrix of the classical Lagrangian \( G_{ik} \) and invertible
matrix \( c_{\mu \nu} \) fixing the gauge in (4.3)

\[
\mu[g] = \prod_t \left( \det G_{ik}(t) \det c_{\mu \nu}(t) \right)^{1/2} \equiv \left( \text{Det} G_{ik} \text{ Det} c_{\mu \nu} \right)^{1/2},
\]

(4.8)

\[
G_{ik} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}.
\]

(4.9)

Note that the symbol \( \det \) in (4.3) and in the middle part of eq.(4.8) denotes the determinants of matrices acting in the space of canonical condensed indices. The products over time points of local factors \( \det G_{ik}(t) \) and \( \det c_{\mu \nu}(t) \) can be regarded as determinants of higher functional dimensionality if we redefine these matrices as time-ultralocal operators acting in the space of condensed covariant indices \( G_{ik} \equiv G_{ik} \delta(t-t') \) and \( c_{\mu \nu} \equiv c_{\mu \nu} \delta(t-t') \). We shall denote such functional determinants for both ultralocal and differential operators in time by \( \text{Det} \). Thus the right-hand side of eq.(4.7) is formally written down in terms of such (purely divergent) functional determinant of the ultralocal operator. For the Hessian matrix it looks like

\[
\text{Det} G_{ik} \delta(t-t') = \exp \left\{ \delta(0) \int_{t_-}^{t_+} dt \ln \det G_{ik} \right\}.
\]

(4.10)

Similar definition holds for the ultralocal gauge-fixing matrix.

The final ingredient which accomplishes the definition of the path integral (4.1) is a specification of boundary conditions on integration variables. Integration in (4.1) runs over field histories with fixed values of canonical coordinates and ghost fields at \( t_\pm \)

\[
q^i(t_\pm) = q^i_\pm, \quad C^\mu(t_\pm) = 0, \quad \bar{C}_\nu(t_\pm) = 0,
\]

(4.11)

the coordinates \( q^i_\pm \) being the arguments of the two-point kernel \( K(q_+, q_-) \). On the contrary, the boundary values of Lagrange multipliers are integrated over in the infinite range

\[
-\infty < N^\mu(t_\pm) < +\infty.
\]

(4.12)

By using these boundary conditions, invariant with respect to BRST transformations of the total action in (4.1), one can show the gauge independence of the path integral and also give a formal proof that this integral solves quantum Dirac constraints in the coordinate representation of canonical commutation relations for \( q = q_+ \) and \( p_+ = \hbar \partial / i \partial q_+ \). This proof is based on an obvious consequence of the integration range for \( N^\mu(t_+) \)

\[
\int Dq DC D\bar{C} \frac{\delta}{\delta N^\mu(t_+)} \left( \ldots \right) = 0,
\]

(4.13)

\footnote{We don’t consider here the case when \( c_{\mu \nu} \) is a differential or nonlocal operator. The only difference in this case is that it is no longer a part of the local measure. It contributes to the path integral a nontrivial functional determinant usually represented as a gaussian integral over auxiliary Nielsen-Kallosh ghosts.}
where ellipses denote the full integrand of the path integral (4.1). The functional differentiation here boils down in the main to the deexponentiation of the constraint

\[
\frac{\delta S[g]}{\delta N^\mu(t_+)} = -T^\mu(t_+, p(t_+)) \bigg|_{p=p^0(q, \dot{q}, N)}
\]  

(4.14)

(the differentiation of gauge-breaking and ghost terms gives terms cancelling one another in virtue of Ward identities \[\text{[11]}\]). This deexponentiated constraint can be extracted from under the integral sign in (4.13) in the form of the differential operator acting on \(q\) and \(\dot{q}\), so that the equation (4.13) takes the form of the quantum Dirac constraint \[\text{[14]}\]. This nonperturbative derivation is, however, purely formal and in an uncontrollable way depends on the skeletonization of the path integral \[\text{[14]}\]. In contrast with these formal considerations in the following sections we prove that this path integral solves the quantum constraints at least in the one-loop order of semiclassical expansion.

The Feynman diagrammatic technique for the integral (4.1) was also built in \[\text{[11]}\]. Its main emphasis concerns, certainly, the boundary conditions at \(t_{\pm}\), because in other respects the \(\hbar\)-expansion produces a standard set of Feynman graphs. The first step in this expansion consists in finding the stationary point of the path integral subject to boundary conditions (4.11) - (4.12). As shown in \[\text{[11]}\], for gauge fixed action (4.2) this is a unique solution \(g = g(t \mid q_+, q_-)\) of the following boundary value problem for classical equations of motion in the chosen relativistic gauge

\[
\frac{\delta S[g]}{\delta g^a(t)} = 0, \quad \chi^\mu(g, \dot{g}, t) = 0, \quad t_- \leq t \leq t_+ \quad \text{(4.15)}
\]

\[
q(t_{\pm}) = q_{\pm}. \quad \text{(4.16)}
\]

Performing the gaussian integration over quantum fields in the vicinity of this stationary point one arrives at the semiclassical answer (2.1) for our path integral with the tree-level action

\[
S(q, q') = S[g(t \mid q_+, q_-)] \quad \text{(4.18)}
\]

and formal expression for the one-loop preexponential factor in terms of the functional determinants of the ghost and gauge field operators and the local measure

\[
P(q_+, q_-) = \mu[g] \frac{\text{Det} Q^\mu_\nu}{(\text{Det} F_{ab})^{1/2}} \bigg|_{g=g(t \mid q_+, q_-)}. \quad \text{(4.19)}
\]

\[\text{[12]}

The variation of the gauge fixed action gives for \(t_- \leq t \leq t_+\) classical equations amended by the gauge fixing term and two conditions at the boundaries \(t = t_{\pm}\) (requirement of vanishing coefficients of \(\delta N^\mu(t_{\pm})\)). In view of (4.13) these conditions reduce to gauge conditions \(\chi^\mu = 0\) enforced at \(t = t_{\pm}\). On the other hand, in view of the identity (4.16) the classical equations with a gauge fixing term result in homogeneous equations for these gauge conditions \(Q^\mu_\nu c_{\mu\alpha} \chi^\alpha = 0, t_- \leq t \leq t_+\). The second order differential operator \(Q^\mu_\nu\) is assumed to be invertible under the Dirichlet boundary conditions, so that gauge conditions are enforced for all \(t\).
The gauge field operator is given by the second order variational derivatives of the
gauge-fixed action (4.2). In the covariant condensed notations it equals

\[ F_{ab} = S_{ab} - \chi^\mu_a c_{\mu\nu} \chi^\nu_b, \quad (4.20) \]
\[ S_{ab} \equiv \frac{\delta^2 S[g]}{\delta g^a \delta g^b}, \quad (4.21) \]
\[ \chi^\mu_a \equiv \frac{\delta \chi^\mu}{\delta g^a}. \quad (4.22) \]

where the functional matrix \( \chi^\mu_a \) of linearized gauge conditions is a first-order differential
operator with the delta-function type kernel

\[ \chi^\mu_a = \chi^\mu_a (d/dt) \delta(t - t'). \quad (4.23) \]

A formal definition of functional determinants in (4.19) is incomplete unless one
specifies a functional space on which they are calculated. This is equivalent to specifying
the boundary conditions for Green’s functions of the gauge and ghost operators
\( G^{ba} = F^{-1}_{ba} \) and \( Q^{-1}_\mu^\nu \) which participate in the variational equations for their determinants [21]

\[ \delta \ln \text{Det} F_{ab} = G^{ba} \delta F_{ab}, \quad \delta \ln \text{Det} Q^\mu_\nu = Q^{-1}_\mu^\nu \delta Q^\mu_\nu \quad (4.24) \]

and multiloop graphs of semiclassical expansion.

Boundary conditions for the gauge propagator were derived in [11]. They actually
follow from the linearized version of the boundary value problem (1.15) - (1.17). The
propagator \( G^{ab}(t, t') \) is \((n + m) \times (n + m)\) matrix-valued function (cf. eq.(1.8)). Thus,
it requires \((n + m)\) boundary conditions at \( t_\pm \), \( n \) of them obviously being the Dirichlet
conditions for its \( a = i \)-components \((i = 1, \ldots n)\), while the rest of them coinciding with
\( m \) linearized gauge conditions at \( t = t_\pm \)

\[ F_{ca}(d/dt) G^{ab}(t, t') = \delta^b_c \delta(t - t'), \quad (4.25) \]
\[ G^{ib}(t_\pm, t') = 0, \quad (4.26) \]
\[ \chi^\mu_a (d/dt_\pm) G^{ab}(t_\pm, t') = 0. \quad (4.27) \]

The latter belong to Robin type because for relativistic gauges the operator (4.23)
contains derivatives transversal to the boundary. These boundary conditions have the
properties of the BRST-invariance, guarantee gauge independence of the path integral
(1.1) [11] and selfadjointness of the gauge field operator recently discussed in [22].

Finally, the propagator of ghost fields in (4.24) satisfies in view of (4.11) the Dirichlet
boundary value problem

\[ Q^\alpha_\mu (d/dt) Q^{-1}_\alpha^\beta(t, t') = \delta^\beta_\mu, \quad Q^{-1}_\alpha^\beta(t_\pm, t') = 0. \quad (4.28) \]
5. Local measure, Ward identities and gauge independence

The prescription of boundary conditions for Green’s functions does not, however, uniquely determine the variation of the determinants \((4.24)\). Problem is that kernels of propagators are not smooth functions of their arguments, and their irregularity enhances when they are acted upon by two derivatives contained in \(\delta F_{ab}(d/dt)\) and \(\delta Q_{\mu}^\alpha(d/dt)\). Therefore, one has to prescribe the way how these derivatives act on both arguments of Green’s functions and how to take the coincidence limit of the resulting singular kernels in the functional traces of \((4.24)\). It is well known that strongest divergences arising from these singular kernels are compensated by the local measure \(\mu [g] \) \([23]\). As far as it concerns the rest finite (or less divergent in field theories) part, its construction should maintain the gauge invariance properties encoded in the path integral at nonperturbative level. This might serve as a hint that can (at least partially) fix the ambiguities of the above type. For this reason, in this section we consider the role of local measure and Ward identities in the construction of the gauge-independent prefactor \((4.19)\).

It is well known that strongest power divergences of functional determinants are related to terms with second order derivatives in the ghost \((4.7)\) and gauge field operators \([23, 16]\)

\[
F_{ab}(d/dt) = -\frac{d}{dt} a_{ab} \frac{d}{dt} + ..., \quad (5.1)
\]

\[
a_{ab} = \frac{\partial^2 L_{gf}}{\partial \dot{g}^a \partial \dot{g}^b}. \quad (5.2)
\]

The Hessian matrix of the gauge-fixed Lagrangian is nondegenerate due to the relativistic nature of gauge conditions \((4.4)-(4.5)\). The gauge breaking term contributes the part quadratic in velocities of Lagrange multipliers \(\dot{N}^\mu\) that were initially absent in the original Lagrangian \(L(q, \dot{q}, N)\)

\[
L_{gf}(g, \dot{g}) = L(q, \dot{q}, N) - \frac{1}{2} \chi^\mu(g, \dot{g}) c_{\mu \nu} \chi^\nu(g, \dot{g}). \quad (5.3)
\]

Therefore, the Hessian matrix has the following components

\[
a_{ik} = G_{ik} - \frac{\partial \chi^\alpha}{\partial \dot{q}^i} c_{\alpha \beta} \frac{\partial \chi^\beta}{\partial \dot{q}^k}, \quad (5.4)
\]

\[
a_{i\mu} = \frac{\partial \chi^\alpha}{\partial \dot{q}^i} c_{\alpha \beta} a^\beta_{\mu}, \quad (5.5)
\]

\[
a_{\mu \nu} = -a^\alpha_{\mu} c_{\alpha \beta} a^\beta_{\nu}, \quad (5.6)
\]

where \(a^\alpha_{\mu}\) is the Hessian matrix \((4.3)\) of the ghost Lagrangian. In virtue of the relation

\[
det a_{ab} = det G_{ik} det c_{\alpha \beta} (det a^\mu_{\nu})^2 \quad (5.7)
\]
the expression for the local measure (4.8) can be rewritten as the following ratio of the determinants of the gauge field and ghost Hessian matrices
\[ \mu [g] = \frac{(\text{Det} a_{ab})^{1/2}}{\text{Det} a_{\xi}^\xi}. \quad (5.8) \]

Therefore the one-loop prefactor (4.19) takes the form
\[ P(q_+, q_-) = \left( \frac{\text{Det} F_{ab}}{\text{Det} a_{ab}} \right)^{-1/2} \frac{\text{Det} Q_\alpha^\alpha}{\text{Det} a_{\xi}^\xi} \bigg|_{g=g(t|q_+,q_-)}. \quad (5.9) \]

This form is especially adjusted to the manifest cancellation of strongest divergent parts of determinants: as shown in [16] the \( \delta(0) \)-type divergences for a second order differential operator are cancelled by the functional determinant of the corresponding ultralocal matrix coefficient of its second order derivatives.

The representation (5.8) for the local measure, which is similar to the ratio of gauge field and ghost functional determinants in (4.19), presents a simplest demonstration of gauge independence. The numerator and denominator separately depend on the choice of gauge conditions \( \chi^\mu (g, \dot{g}) \) (the corresponding matrix \( a^\mu_\alpha = -\partial \chi^\mu / \partial \dot{N}^\nu \)), but in the ratio this dependence cancels out and the total local measure (4.8) turns out to be gauge independent\(^7\). The mechanism of gauge independence for the rest part of the one-loop prefactor is more complicated and is based on Ward identities for gauge and ghost propagators.

These identities follow from the gauge invariance of the classical action (3.10). The functional differentiation of (3.10) shows that on shell, that is on the background satisfying classical equations of motion, the functional matrix \( S_{ab} \) is degenerate because it has zero-eigenvalue eigenvectors – the gauge generators
\[ R^\alpha_a S_{ab} = -S_{a\alpha} \frac{\delta R^\alpha_a}{\delta g^b} = 0 \quad (5.10) \]
(cf. the analogous property (2.7) in the canonical context). As a consequence the gauge operator \( F_{ab} \) satisfies the relation
\[ R^\alpha_a F_{ab} = -Q_\alpha^\alpha c_{\alpha\beta} \chi^\beta_b \quad (5.11) \]
which can be functionally contracted with matrices of the gauge \( G^{bc} \) and ghost \( Q_\alpha^{-1\mu} \) propagators introduced above. Integration by parts of the derivatives in the operators \( R^\alpha_a = R^\alpha_a (d/dt) \) and \( Q_\alpha^\alpha = Q_\alpha^\alpha (d/dt) \) does not produce additional surface terms in view of the boundary conditions for the propagators. Therefore, after using the equations (4.25) and (4.28) for gauge and ghost propagators one arrives at the needed Ward identity
\[ c_{\alpha\beta} \chi^\beta_b (d/dt) G^{bc}(t, t') = -Q_\alpha^{-1\beta}(t, t') R^\gamma_\beta (d/dt'). \quad (5.12) \]

\(^7\) Gauge independent expression for local measure (defined by the Hessian of the classical gauge action (4.9)) was first obtained in the context of Einstein gravity theory \([8]\) and then derived for generic gauge theory from the canonical path integral \([9]\).
Although this identity can be obtained from (5.11) by formal inversion of $F_{ab}$ and $Q_{\mu}^{\alpha}$ as finite-dimensional matrices, we emphasize the necessity of a cautious use of condensed notations with regard to possible surface terms following from integration by parts. The result is presented in the form clearly indicating the differential structure of operators acting on the arguments of Green’s function. In what follows we shall often label the differential operators by arrows to show the direction (right or left) in which they act.

The obtained Ward identities lead to gauge independence of the one-loop prefactor (1.19) provided that we consistently fix the functional composition laws for $G^{ba}\delta F_{ab}$ and $Q_{\mu}^{-1\nu}\delta Q_{\nu}^{\mu}$ in (4.24). Let us assume that the time derivatives of varied differential operators $\delta F_{ab}(d/dt)$ and $\delta Q_{\nu}^{\mu}(d/dt)$ are understood as acting in two different ways

\begin{align*}
\delta \ln \text{Det} F_{ab} &= G^{ba} \delta F_{ab}, \\
\delta \ln \text{Det} Q_{\nu}^{\mu} &= Q_{\nu}^{-1\mu} \delta Q_{\nu}^{\mu}.
\end{align*}
(5.13)

(5.14)

In contrast with the ghost operator, for which both of its derivatives are acting on one argument of the Green’s function, eq.(5.13) here implies a symmetric action of $\delta F_{ab}(d/dt)$ on both arguments of $G^{ba}(t, t')$ in the sense that

\begin{equation}
L_{gt}^{(2)} = \frac{1}{2} \varphi^{a}(t) \delta F_{ab}(d/dt) \varphi^{b}(t)
\end{equation}

represents the quadratic part of the gauge-fixed Lagrangian in perturbations of field variables $\varphi^{a}$. It contains the squares of their velocities $\dot{\varphi}^{a}(t)$ rather than their second derivatives (and $\delta F_{ab}(d/dt)$ obviously implies the variation of this operator with respect to its background field dependence).

With these conventions the gauge variation of the gauge field and ghost determinants

\begin{align*}
\delta \chi \ln \text{Det} F_{ab} &= \delta \chi \delta F_{ab} G^{ba} \\
&= -2 \int_{t-}^{t+} dt \left[ c_{\alpha\beta} \delta \chi^{a}_{\beta} G^{ba}(d/dt) \delta \chi^{\alpha}_{a} \right]_{t=t'}, \quad (5.16) \\
\delta \chi \ln \text{Det} Q_{\nu}^{\mu} &= Q_{\mu}^{-1\nu} \delta \chi Q_{\nu}^{\mu} \equiv Q_{\alpha}^{-1\beta} \delta \chi^{a}_{\beta} R_{\beta}^{\nu} \delta \chi^{\alpha}_{a} \\
&= \int_{t-}^{t+} dt \left[ Q_{\alpha}^{-1\beta}(t, t') \delta \chi^{a}_{\beta} R_{\beta}^{\nu}(d/dt') \delta \chi^{\alpha}_{a}(d/dt') \right]_{t=t'} \quad (5.17)
\end{align*}

cancel out in virtue of the Ward identities (5.12) in the one-loop prefactor (4.19) which proves its gauge independence. Its independence from the choice of the gauge fixing matrix $c_{\mu\nu}$ is also based on these identities, though in this case it is cancelled by the Nielsen-Kallosh factor in the local measure.

For the one-loop effective action these Ward identities were in much detail analyzed in [24] in the framework of formal condensed notations. Here the main emphasis in the mechanism of Ward identities is focused on boundary conditions and accurate definition of the functional determinants in (4.24). Another choice of the functional composition law in these variational equations leads in general to extra surface terms breaking the gauge independence of the one-loop prefactor.
6. Gauge field functional determinant

Here we calculate the contribution of the gauge field determinant to the one-loop prefactor (5.9). For this purpose we introduce matrix notations for operators acting in the vector space of canonical indices. In these notations the gauge field operator

\[ F(d/dt) = F_{ab}(d/dt) = \left[ \begin{array}{cc} F_{ik}(d/dt) & F_{iv}(d/dt) \\ F_{mk}(d/dt) & F_{mv}(d/dt) \end{array} \right] \]  

(6.1)

acts in the space of columns

\[ \varphi(t) = \left[ \begin{array}{c} \varphi^i(t) \\ \varphi^\mu(t) \end{array} \right]. \]  

(6.2)

Generically it has the form of the second order differential operator

\[ F(d/dt) = -\frac{d}{dt} a \frac{d}{dt} - \frac{d}{dt} b + b^T \frac{d}{dt} + c, \]  

(6.3)

where the coefficients \( a = a_{ab}(t) \), \( b = b_{ab}(t) \) and \( c = c_{ab}(t) \) are the matrices acting in the vector space of \( \varphi(t) \), and the superscript \( T \) denotes their functional transposition \( (b^T)_{ab} \equiv b_{ba} \). These coefficients can be easily expressed as mixed second-order derivatives of the Lagrangian in the gauge-fixed action (4.2) with respect to \( g^a \) and \( \dot{g}^a \). In particular, the matrix of second order derivatives \( a_{ab} \) is given by the Hessian matrix (5.2).

In accordance with the notation (5.15) the "integrated by parts" version of the operator (6.1), \( \int F(d/dt) \), determines the quadratic part of the gauge fixed Lagrangian with two time derivatives of \( F \) acting symmetrically on two functions \( \varphi(t) \) and its transpose \( \varphi^T(t) \). For arbitrary two test functions \( \varphi_1 \) and \( \varphi_2 \) the operator of the form (5.3), in our notations, gives rise to the bilinear form

\[ \varphi_1^T \int F \varphi_2 = \varphi_1^T a \varphi_2 + \varphi_1^T b \varphi_2 + \varphi_1^T b^T \varphi_2 + \varphi_1^T c \varphi_2 \]  

(6.4)

and implies the following integration by parts

\[ \varphi_1^T \int F \varphi_2 = \varphi_1^T (F \varphi_2) + \frac{d}{dt} \left[ \varphi_1^T (W \varphi_2) \right], \]  

(6.5)

\[ W \equiv W(d/dt) = a \frac{d}{dt} + b. \]  

(6.6)

Here \( W \) is the Wronskian operator which enters the Wronskian relation for \( F \)

\[ \varphi_1^T (F \varphi_2) - (F \varphi_1)^T \varphi_2 = -\frac{d}{dt} \left[ \varphi_1^T (W \varphi_2) - (W \varphi_1)^T \varphi_2 \right] \]  

(6.7)

and also participates in the variational equation for the canonical momentum \( \partial L_{gf} / \partial \dot{g} \) valid for arbitrary field variations \( \delta g(t) \)

\[ \delta \frac{\partial L_{gf}}{\partial \dot{g}} = W(d/dt) \delta g(t). \]  

(6.8)
Since the velocities of Lagrange multipliers enter $L_{gf}(g, \dot{g})$ only through the gauge breaking term, the $\mu$-component of the Wronskian operator is given by

$$W_{\mu b}(d/dt) = -\frac{\partial \chi^\alpha}{\partial N^\mu} \vec{\chi}_b^\beta (d/dt),$$  \hspace{1cm} (6.9)

where, in particular, the differential operator of the linearized gauge conditions (4.23) coincides with that of the boundary value problem (4.27).

Now we introduce the notation for the matrix valued Green’s function of this boundary value problem

$$G^{ab}(t, t') = \left[ G^{ik}(t, t') \ G^{ic}(t, t') \ G^{j\kappa}(t, t') \ G^{j\nu}(t, t') \right] \equiv G(t, t'),$$ \hspace{1cm} (6.10)

and in accordance with equations (5.13) and (6.4) write down the variation of the gauge field determinant

$$\delta \ln \text{Det } F = \text{Tr } \delta \vec{F} G = \int_{t_-}^{t_+} dt \text{ tr } \left[ \delta \vec{F} G (t, t') \right]_{t'=t}$$

$$\equiv \int_{t_-}^{t_+} dt \text{ tr } \left[ (\delta a \frac{d^2}{dt^2} + \delta b \frac{d}{dt} + \delta b^T \frac{d}{dt} + \delta c) G(t, t') \right]_{t'=t}. \hspace{1cm} (6.11)$$

Here tr denotes the matrix trace operation with respect to condensed canonical indices of $\delta a = \delta a_{ab}(t), \delta b = \delta b_{ab}(t), \delta b^T = \delta b_{ba}(t), \delta c = \delta c_{ab}(t)$ and $G(t, t') = G^{ab}(t, t')$.

Our goal now will be to construct a special representation of the Green’s function $G(t, t')$ and integrate the variational equation (6.11) by the method of [16]. For this purpose, similarly to [16], we introduce two sets of basis functions $u_-$ and $u_+$ of the operator $F$

$$F u_{\pm} = 0, \quad u_{\pm} = u_{\pm}^a A(t)$$ \hspace{1cm} (6.12)

satisfying the full set of boundary conditions (1.20)-(1.27) respectively at $t_-$ and $t_+$. In view of (6.9) and invertibility of the matrix $\partial \chi^\alpha/\partial N^\mu$ the Robin type boundary conditions (1.27) can be rewritten in terms of the $\mu$-components of the Wronskian operator, so that the full set of these boundary conditions takes the form

$$u_{+}^i A(t_+) = 0, \quad u_{-}^i A(t_-) = 0, \hspace{1cm} (6.13)$$

$$u_{+}^i A(t_+) = 0, \quad u_{-}^i A(t_-) = 0. \hspace{1cm} (6.14)$$

In condensed notations we regard these basis functions, enumerated by the condensed index $A$ of arbitrary nature, as forming the square matrices with the first (contravariant) index $i$ and the second (covariant) index $A$.

The Wronskian relation (5.7) can be used to form the $t$-independent matrix of the Wronskian inner products $\Delta_{12} = \varphi_1^T (W \varphi_2) - (W \varphi_1)^T \varphi_2$ of basis functions $\varphi_{1,2} =$
\( u_\pm(t) \). In view of the boundary conditions of the above type this matrix has only two nonvanishing blocks given by the two mutually transposed matrices

\[
\Delta_{+-} = u_+^T (W u_-) - (W u_+)^T u_-, \quad \Delta_{+-} \equiv (\Delta_{+-})_{AB}, 
\]

(6.15)

\[
\Delta_{-+} = u_-^T (W u_-) - (W u_-)^T u_+, \quad \Delta_{-+} \equiv (\Delta_{-+})_{AB}, 
\]

(6.16)

\[
\Delta_{++} = -\Delta_{--}. 
\]

(6.17)

This fact can also be represented in the form of the following matrix relation

\[
\begin{bmatrix}
    u_-^T - (W u_-)^T & Wu_+ & Wu_-
\end{bmatrix}
\begin{bmatrix}
    Wu_+ & Wu_-
    u_+ & u_-
\end{bmatrix}
= \begin{bmatrix}
    \Delta_{--} & 0 \\
    0 & \Delta_{++}
\end{bmatrix}. 
\]

(6.18)

Under the assumption of absence of zero modes of \( F \) subject to boundary conditions (4.26)-(4.27) (absence of linear dependence of \( u_+ \) and \( u_- \)) this relation implies the invertibility of \( \Delta_{+-} \). It also allows one to establish the following important relations for equal-time bilinear combinations of basis functions [16]

\[
\begin{align*}
    u_+(t) (\Delta_{--})^{-1} u_+(t) + u_-(t) (\Delta_{++})^{-1} u_+^T (t) &= 0, \\
    a \left[ \dot{u}_+(t) (\Delta_{--})^{-1} u_+^T (t) + \dot{u}_-(t) (\Delta_{++})^{-1} u_+^T (t) \right] &= I.
\end{align*}
\]

(6.19)

(6.20)

Here \( I = \delta^a_b \) denotes the unity matrix in the space of canonical indices \( a \). To clarify their use in these equations and in what follows, we note that in the transposed matrix \( u_+^T = u_+^a A \) the covariant index \( A \) is considered to be the first one (in contrast with \( u_\pm \)), so that the matrix composition law with \( (\Delta_{+-})^{-1} = [(\Delta_{+-})^{-1}]^{AB} \) gives rise in eqns. (6.20) to the matrices with two indices \( a \) and \( b \).

By the method of ref. [16] (applied there in the case of Dirichlet boundary conditions) one can show that the Green’s function of the mixed Dirichlet-Robin boundary value problem (4.25)-(4.27) has the following representation

\[
G(t, t') = -\theta(t - t') u_+(t) (\Delta_{--})^{-1} u_+^T (t) \\
+ \theta(t' - t) u_-(t) (\Delta_{++})^{-1} u_+^T (t'),
\]

(6.21)

where \( \theta(x) \) is the step function: \( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x < 0 \). This expression is the analogue of positive-negative frequency decomposition for the Feynman propagator in scattering theory.

Substituting (6.21) to (6.11) and repeating the calculations of [16] one can see that in view of relations (6.19)-(6.20) the \( \delta(0) \)-type terms with derivatives of step functions reduce to the variation of the logarithm of the local measure for \( F \). Therefore, the variational equation (6.11) acquires the form taking into account the cancellation of \( \delta(0) \)-type divergences

\[
\delta \ln \frac{\text{Det } F}{\text{Det } a} = -\text{tr} (\Delta_{+-})^{-1} \int_{t_-}^{t_+} dt \ u_+^T \delta \dot{F} u_+.
\]

(6.22)
Here we reserve the same notation \( tr \) for the trace operation in the space of indices \( A \) enumerating the basis functions of \( F \), so that the matrix multiplication here should read \( [(\Delta\_+)\_1]^{AB} u_T^B \delta F^A u_A \).

The further transformation of the time integral in (6.22) is based on integration by parts which uses the identities (6.5), (6.7) and their corollaries obtained by replacing the operators \( F \) and \( W \) with their variations \( \delta F \) and \( \delta W \). This allows one in a systematic way to reduce the integrand to a total derivative modulo the terms vanishing in virtue of equations (6.12) for basis functions or their varied versions \( \delta F u^\pm = -F \delta u^\pm \). The result boils down to the contribution of surface terms at \( t = t^\pm \). In view of boundary conditions for \( u^\pm \) they takes the following form

\[
\int_{t^-}^{t^+} dt \; u_T^T \delta F^T u^+_\pm = \left[ u_\pm^i \delta(W u^\pm)_i - (W u^-_\mu)_\mu \delta u^\mu_\pm \right]_{t^+} + \left[ (W u^-_i)\delta u^i_\pm - u^\mu_- \delta(W u^+_\mu) \right]_{t^-}, \tag{6.23}
\]

where for brevity we introduced notations

\[
(W u^\pm)_i \equiv \overrightarrow{W}_{ia} u^a_\pm, \quad (W u^\pm)_\mu \equiv \overrightarrow{W}_{\mu a} u^a_\pm. \tag{6.24}
\]

It is useful to rewrite this result in the matrix form

\[
\int_{t^-}^{t^+} dt \; u_T^T \delta F^T u^+_\pm = \left[ u_\pm^i - (W u^-)_\mu \right] \left[ \frac{\delta(W u^+_\mu)}{\delta u^\mu_+} \right]_{t^+} \left[ \frac{\delta u^i_-}{\delta u^\mu_+} \right]_{t^-} - \left[ -(W u^-_i) u^\mu_- \right] \left[ \frac{\delta u^i_-}{\delta(W u^+_\mu)} \right]_{t^-}, \tag{6.25}
\]

For brevity here the indices \( A, B, \ldots \) enumerating the basis functions are omitted: they can be regarded as included into subscripts \( \pm \) – the rule which we shall imply in what follows. Thus all the matrices in this relation are square (but certainly not symmetric). One index of these matrices, with respect to which the contraction takes place, is a combination of \( i \) and \( \mu \) (being in superscript and subscript positions respectively or vicy versa), while another index is \( A \) encoded as we agreed in \( \pm \).

The expression for the time-independent Wronskian matrix \( \Delta_{\_+} \) can also be rewritten in terms of similar matrices. When calculated at \( t^+ \) and \( t^- \) it looks respectively as

\[
\Delta_{\_+} = \left[ u^-_\mu \right] \left[ (W u^+_\mu)_i \right]_{t^+}, \tag{6.26}
\]

\[
\Delta_{\_-} = \left[ -(W u^-_i) u^\mu_- \right] \left[ (W u^+_\mu)_i \right]_{t^-}. \tag{6.27}
\]

The nondegeneracy of \( \Delta_{\_+} \) guarantees the invertibility of matrix factors in these two expressions.
Combining the obtained expressions (6.25), (6.26), (6.27) with equation (6.22) one can convert its right-hand side into the total variation
\[
\delta \ln \frac{\text{Det} F}{\text{Det} a} = -\delta \text{ tr ln} \left[ \begin{array}{c} (W u_+)_i \\ u_+^\mu \end{array} \right]_{t_+} + \delta \text{ tr ln} \left[ \begin{array}{c} u_+^i \\ (W u_+)_\mu \\ \nu \end{array} \right]_{t_-} = \delta \ln \text{det} D \tag{6.28}
\]
of the logarithm of the determinant of the matrix \( D = D_{ab} \) with indices \( a = (i, \mu) \) and \( b = (k', \nu) \)
\[
D = \left[ \begin{array}{c} (W u_+)_i \\ u_+^\mu \end{array} \right]_{t_+} (\Delta_{-}^{-1}) \left[ -(W u_-)_{k'} u_-^{\nu} \right]_{t_-}. \tag{6.29}
\]
The block form of this matrix
\[
D = \left[ \begin{array}{c} S_{ik'} \ C_{\mu\nu} \\ -X_{k'}^\mu \ \ X_i^\nu \end{array} \right],
\]
\[
S_{ik'} = -(W u_+)_{i(t_+)} \Delta_{-}^{-1} (W u_-)_{k(t_-)} \tag{6.30}
\]
\[
X_i^\nu = (W u_+)_{i(t_+)} \Delta_{-}^{-1} u_-^{\nu} (t_-), \tag{6.31}
\]
\[
X_{k'}^\mu = u_+^\mu (t_+) \Delta_{-}^{-1} (W u_-)_{k(t_-)}, \tag{6.32}
\]
\[
C_{\mu\nu} = u_+^\mu (t_+) \Delta_{-}^{-1} u_-^{\nu} (t_-), \tag{6.33}
\]
allows one to rewrite its determinant in the form
\[
\text{det} D = \text{det} \left( S_{ik'} + X_i^\mu C_{\mu\nu} X_{k'}^{\nu} \right) \text{det} C_{\mu\nu}, \quad C_{\mu\nu} = (C_{\mu\nu})^{-1}. \tag{6.34}
\]
The notation \( S_{ik'} \) chosen for the block (6.30) of the matrix \( D \) is not accidental. This expression really yields the matrix of second-order derivatives of the Hamilton-Jacobi function (2.6) with respect to its arguments \( q^i \) and \( q^{k'} \). Indeed, this matrix coincides with the derivative of the canonical momentum \(-p_{k'} = \partial S/\partial q^{k'} \) with respect to \( q^i \). The momentum is taken at the initial moment \( t_- \) on the classical extremal joining the points \( q \) and \( q' \). In view of the relation (5.8)
\[
S_{ik'} = -W_{k'\nu} (d/dt_-) \frac{\partial g^{\nu}(t_-|q_+, q_-)}{\partial q_+^i}, \tag{6.35}
\]
where the derivative \( \partial g^{\nu}(t|q_+, q_-)/\partial q_+^i \) is given in terms of the Green’s function \( G^{ab}(t, t') \) (see ref.[25])
\[
\frac{\partial g^{\nu}(t|q_+, q_-)}{\partial q_+^i} = -W_{ia} (d/dt_+) G^{ab}(t_+, t), \tag{6.36}
\]
whence one gets the equation (6.30) after using the basis functions representation for \( G^{ab}(t_+, t) \).

Thus, up to field independent normalization constant, the one-loop contribution of gauge fields equals
\[
\left( \frac{\text{Det} F}{\text{Det} a} \right)^{-1/2} = \text{Const} \left( \frac{\text{det} \left( S_{ik'} + X_i^\mu C_{\mu\nu} X_{k'}^{\nu} \right)}{\text{det} C_{\mu\nu}} \right)^{1/2}. \tag{6.37}
\]
This is a part of expression (2.12) for the one-loop prefactor provided we identify arbitrary elements \((X_\mu^i, X_\nu^k, C_{\mu\nu})\) of its canonical gauge fixing procedure with the quantities \((6.31), (6.32)\) and \((6.33)\) above. Let us now show that the contribution of ghost fields gives the remaining part of (2.12).

7. Functional determinant for ghost fields

The ghost operator \((4.6)\) is not even formally symmetric because its right and left action is defined correspondingly on contravariant and covariant vector fields in the space of gauge indices

\[
Q_\nu^\mu (d/dt) f^\nu(t) = \chi_\mu^a (d/dt) R_\nu^a (d/dt) f^\nu(t),
\]

\[
Q_\nu^\mu (d/dt) f_\mu(t) = R_\nu^\mu (d/dt) \chi_\mu^a (d/dt) f_\mu(t),
\]

where the form of the first-order differential operators \(R_\mu^a (d/dt)\) and \(\chi_\mu^a (d/dt)\) varies depending they are acting on test functions \((\varphi, f)\) with respect to their condensed indices \(a\) or \(\mu\)

\[
\chi_\mu^a (d/dt) \varphi^a(t) = \left( \frac{\partial \chi_\mu^a}{\partial g^a} \frac{dt}{dX^a} + \frac{\partial \chi_\mu^a}{\partial g^a} \right) \varphi^a(t),
\]

\[
\chi_\mu^a (d/dt) f_\mu(t) = \left( -\frac{d}{dt} \frac{\partial \chi_\mu^a}{\partial g^a} + \frac{\partial \chi_\mu^a}{\partial g^a} \right) f_\mu(t)
\]

and

\[
R_\mu^a (d/dt) f^\mu(t) = \left( \delta_\mu^a \frac{dt}{dX^a} + \ldots \right) f^\mu(t),
\]

\[
R_\nu^\mu (d/dt) \varphi_\nu(t) = \left( -\delta_\nu^\mu \frac{dt}{dX^a} + \ldots \right) \varphi_\nu(t).
\]

Pairs of relations \((7.1)-(7.2), (7.3)-(7.4)\) and \((7.5)-(7.6)\) obviously differ from one another by integration by parts in bilinear integral forms with \(Q_\mu^\nu\), \(\chi_\mu^a\) and \(R_\nu^\mu\) as functional two-point kernels.

Because of the absence of symmetry the Wronskian relation for \(Q_\nu^\mu\)

\[
f_{1\mu} (\hat{Q}_\nu^\mu f_2^\nu) - (f_{1\mu} \hat{Q}_\nu^\mu) f_2^\nu = -\frac{d}{dt} \left[ f_{1\mu} (\hat{W}_\nu^\mu f_2^\nu) - (f_{1\mu} \hat{W}_\nu^\mu) f_2^\nu \right]
\]

involves two different Wronskian operators \(\hat{W}_\nu^\mu\) and \(\hat{W}_\nu^\mu\) with the arrows indicating the direction in which the differential operators are acting. Their actions on test functions

\[
\hat{W}_\nu^\mu f^\nu = -\frac{\partial \chi_\mu^a}{\partial g^a} R_\nu^a (d/dt) f^\nu(t)
\]

\[
f_\mu \hat{W}_\nu^\mu \equiv \hat{W}_\nu^\mu f_\mu = \left( -\frac{d}{dt} \frac{\partial \chi_\mu^a}{\partial N^\nu} + \frac{\partial \chi_\mu^a}{\partial N^\nu} \right) f_\mu(t) = \chi_\mu^a (d/dt) f_\mu(t)
\]
are essentially different from one another and do not differ by a simple functional transposition of one and the same operator\footnote{We shall not introduce for these two Wronskian operators separate notations and will only distinguish them by the (upper or lower) position of their indices. The order of indices depends on the direction in which the operator is acting: when acting to the right its second index (irrespective of its covariant or contravariant nature) gets contracted with the index of the test function. For operators acting to the left the order of indices reverses, so that with these conventions $\tilde{W}^\nu_{\mu} = \tilde{W}_{\nu}^\mu$.}. Note, in particular, that the operator (7.9) coincides with with the $\nu$-component of the linearized gauge operator (7.4).

With these notations one can again construct the Green’s function of $Q^\nu_{\mu}$ with Dirichlet boundary conditions in terms of \textit{doubled} set of the right and left basis functions $r^{\mu}_{\pm}(t)$ and $v^{\nu}_{\pm}(t)$

\begin{equation}
\begin{aligned}
\tilde{Q}^\mu_{\nu} r^{\nu}_{\pm}(t) &= 0, \quad r^{\nu}_{\pm}(t) = 0, \\
\tilde{Q}^\mu_{\nu} v^{\nu}_{\pm}(t) &= 0, \quad v^{\nu}_{\pm}(t) = 0.
\end{aligned}
\end{equation}

(7.10)

(7.11)

The matrix of Wronskian inner products of these basis functions has only the following nonvanishing block components

\begin{equation}
\begin{aligned}
\theta^{++} &= r^{\mu}_{\nu} (\tilde{W}^\mu_{\nu} v^{\nu}_{+}) - (r^{\mu}_{\nu} \tilde{W}^\mu_{\nu}) v^{\nu}_{+}, \\
\theta^{+-} &= r^{\mu}_{\nu} (\tilde{W}^\mu_{\nu} v^{\nu}_{-}) - (r^{\mu}_{\nu} \tilde{W}^\mu_{\nu}) v^{\nu}_{-}
\end{aligned}
\end{equation}

(7.12)

(7.13)

participating in the analogue of matrix relations (6.18)

\begin{equation}
\begin{bmatrix}
r^{\mu}_{-} & -(W r_{-})^\nu \\
r^{\mu}_{+} & -(W r_{+})^\nu
\end{bmatrix}
\begin{bmatrix}
(Wv^{+})_{\mu} & (Wv^{-})_{\mu} \\
v^{\nu}_{+} & v^{\nu}_{-}
\end{bmatrix}
= \begin{bmatrix}
\theta^{++} & 0 \\
0 & \theta^{+-}
\end{bmatrix},
\end{equation}

(7.14)

where for brevity we introduced the notations $(W r_{\pm})^\nu \equiv r^{\mu}_{\nu} \tilde{W}^\mu_{\nu} = \tilde{W}^\nu_{\mu} r^{\mu}_{\nu}$ and $(Wv^{\pm})_{\mu} = \tilde{W}^\nu_{\mu} v^{\nu}_{\pm}$. Similarly to (6.19) these equations imply the equal-time bilinear relations for ghost basis functions

\begin{equation}
\begin{aligned}
(Wv^{+})_{\mu} (\theta^{++})^{-1} r^{\nu}_{-} + (Wv^{-})_{\mu} (\theta^{+-})^{-1} r^{\nu}_{+} &= \delta^{\nu}_{\mu}, \\
v^{\nu}_{+} (\theta^{++})^{-1} (W r_{-})^\nu + v^{\nu}_{-} (\theta^{+-})^{-1} (W r_{+})^\nu &= -\delta^{\nu}_{\mu}, \\
v^{\nu}_{+} (\theta^{++})^{-1} r^{\nu}_{-} + v^{\nu}_{-} (\theta^{+-})^{-1} r^{\nu}_{+} &= 0.
\end{aligned}
\end{equation}

(7.15)

(7.16)

(7.17)

In terms of these basis functions the ghost Green’s function equals

\begin{equation}
Q^{\mu}_{\nu}(t, t') = -\theta(t - t') \frac{v^{\nu}_{\mu}(t) (\theta^{++})^{-1} r^{\nu}_{-}(t')}{\theta(t' - t)} + \theta(t' - t) \frac{v^{\nu}_{\mu}(t) (\theta^{+-})^{-1} r^{\nu}_{+}(t')}{\theta(t' - t)}.
\end{equation}

(7.18)

Substituting this expression to (5.14) and again repeating the calculations of [16] one gets the $\delta(0)$-type terms absorbed by the local measure and arrives at the expression

\begin{equation}
\begin{aligned}
\delta \ln \frac{\text{Det} Q^{\mu}_{\nu}}{\text{Det} a^{\mu}_{\nu}} &= - \frac{1}{2} \text{tr} (\theta^{++})^{-1} \int_{t_{-}}^{t_{+}} dt r^{\nu}_{-}(t) \delta Q^{\mu}_{\nu} v^{\nu}_{\mu}(t) \\
& \quad + \frac{1}{2} \text{tr} (\theta^{+-})^{-1} \int_{t_{-}}^{t_{+}} dt r^{\nu}_{+}(t) \delta Q^{\nu}_{\mu} v^{\nu}_{\mu}(t),
\end{aligned}
\end{equation}

(7.19)
where the 1/2 coefficients originate from the symmetric prescription for the theta function $\theta(0) = 1/2$. It differs from the analogous expression (6.22) by the presence of two terms built of two different sets of right and left basis functions – the consequence of asymmetry for the ghost operator as compared to the symmetric gauge field one. These terms are different here in contrast with the case of $F_{ab}$ for which they coincide and, thus, lead to 1/2 coefficients adding up to unity (or, irrespective of the rule $\theta(0) = 1/2$, lead to using the identity $\theta(t-t') + \theta(t'-t) = 1$). Their further transformation repeats the calculations of [16] briefly explained in the previous section and leads to the result

$$
-\mathrm{tr} \left( \theta_+^{-1} \right) \int_{t_-}^{t_+} dt \ r_\mu^\nu (t) \ \delta Q_{\mu}^\nu \ v^\mu_\mu (t) \\
= \delta \ln \det \left[ v^\mu_\mu (t_-) (\theta_+^{-1} r^\mu_\mu (t_+)) \right] + \delta \ln \det a_\mu^\mu (t_-),
$$

(7.20)

$$
\mathrm{tr} \left( \theta_+^{-1} \right) \int_{t_-}^{t_+} dt \ r^\nu_\mu (t) \ \delta Q_{\mu}^\nu \ v_\mu^\mu (t) \\
= \delta \ln \det \left[ v_\mu^\mu (t_+) (\theta_+^{-1} r^\mu_\mu (t_-)) \right] + \delta \ln \det a_\mu^\mu (t_+).
$$

(7.21)

Each of the first terms on the right hand sides of these two relations is equivalent to the equations (3.19)-(3.20) of the reference [16] for a symmetric operator subject to Dirichlet boundary conditions. On the contrary, the terms with det $a_\mu^\mu (t_-)$ are new and originate from asymmetric action of $\delta Q_{\mu}^\nu$ in the variational definition of the ghost determinant (5.14). Using $\theta_+$ and $\theta_-$ calculated from (7.12)-(7.13) respectively at $t_-$ and $t_+$ and taking into account that $(\mathcal{W} r_\pm)^\mu_\mu (t_\pm) = a_\mu^\mu (t_\pm) \hat{r}^\mu_\mu (t_\pm)$ one finds the following expressions for matrices

$$
v^\mu_\mu (t_-) (\theta_+^{-1} r^\mu_\mu (t_+)) = -[r_\mu^\mu (t_+) (\hat{r}^\nu_\nu (t_-))^{-1}]_\mu^\alpha [a^{-1} (t_-)]^\alpha_\mu,
$$

(7.22)

$$
v_\mu^\mu (t_+) (\theta_+^{-1} r^\mu_\mu (t_-)) = -[r_\mu^\mu (t_-) (\hat{r}^\nu_\nu (t_+))^{-1}]_\mu^\alpha [a^{-1} (t_+)]^\alpha_\mu.
$$

(7.23)

Here we imply that $r_\pm (t) = r^\mu_\mu (t)$ and $\hat{r}_\pm (t) = \hat{r}^\mu_\mu (t)$ are square matrices with the first index $\mu$ and the second index encoded in their subscripts $\pm$. Substituting these expressions to (7.20)-(7.21) and then to (7.19) one finds that the variations of det $a_\mu^\mu (t_-)$ cancel out and the result can be functionally integrated

$$
\frac{\mathrm{Det} \ Q_{\mu}^\nu}{\mathrm{Det} a_\mu^\nu} = \text{const} \ (\det J_\mu^\nu \det J^\mu_\nu)^{-1/2},
$$

(7.24)

$$
J_\mu^\nu = [\hat{r}_- (t_-) (r_-)^{-1} (t_-)]_\nu^\mu,
$$

(7.25)

$$
J^\mu_\nu = -[\hat{r}_+ (t_+) (r_+)^{-1} (t_-)]_\nu^\mu.
$$

(7.26)

up to numerical normalization constant. The coincidence of notations for the last two matrices with those of Faddeev-Popov operators in unitary gauges (2.10)-(2.11) is again not accidental. The proof of these equalities is as follows.

---

9 This prescription is apparently related to symmetric ordering of equal-time operator products which makes them Hermitian, but we shall not trace back to the roots of this rule in operator quantization.
As it follows from the previous section, to compare our one-loop preexponential factor with that of the solution (2.12) we have to identify the quantities (6.31)-(6.32) with matrices of canonical gauges. The Faddev-Popov matrix corresponding to the gauge matrix (6.31) is therefore

\[ J_\mu^i = X_\mu^i \nabla^i_\nu = \nabla^i_\nu (Wu_+)_{i}(t_+ \rightarrow t_-). \]  

(7.27)

We show now that this expression equals (7.25). For this purpose, note that the combination \( \nabla^i_\nu (Wu_+)_{i}(t_+) \) can be identified with the linearized constraint under the variation of canonical coordinates and momenta induced by variations of Lagrangian variables \( \delta q^a(t) = u^a_+(t) \):

\[ \delta q_i^i(t_+) \equiv u^i_+(t_+) = 0, \]

(7.28)

\[ \delta p_i(t_+) \equiv W^S_{ia}(d/dt_+) u^a_+(t) = W_{ia}(d/dt_+) u^a_+(t). \]

(7.29)

Here \( W^S_{ia}(d/dt) \) is a Wronskian operator built similarly to (6.8) but with respect to the original Lagrangian \( L(q, \dot{q}, N) \) instead of the gauge-fixed one. The second equality in (7.29) follows from the Robin-boundary conditions on basis functions (6.13). Thus, since \( \nabla^i_\mu \) is the momentum derivative of the constraint (2.4), the combination of the above type coincides with the linearized constraint which in view of (4.14) equals the variation of the \( \mu \)-component of the classical equations of motion

\[ \nabla^i_\nu (Wu_+)_{i}(t_+) = \delta T_\mu (q(t_+), p(t_+)) = -S_{\mu a}(d/dt_+) u^a_+(t_+). \]

(7.30)

Then, in virtue of equations (6.12) for basis functions and relation (7.9) for the Wronskian operator of \( Q^\nu_{\mu}(d/dt) \)

\[ \nabla^i_\nu (Wu_+)_{i}(t_+) = -W^\nu_\alpha d/dt_+ \left( c_{\alpha \beta} \dot{\chi}^\beta \right) u^a_+(t_+). \]

(7.31)

Substitution of this relation to (7.27) leads to the expression bilinear in basis functions \( u^a_+(t) \) with one of them acted upon by the operator of linearized gauge conditions. Such an expression can be transformed by using the Ward identity (5.12). Substituting the basis function representations (6.21) and (7.18) into it at \( t > t' \) one gets

\[ c_{\alpha \beta} \dot{\chi}^\beta (d/dt) u^a_+(t) \Delta^{-1}_+ u^b_+(t') = -v^a_+(t) (\theta_+^{-1})^1 r^\beta(t') \dot{R}^b_{\beta}(d/dt'). \]

(7.32)

Using this relation with \( t = t_+ \) and \( t' = t_- \) together with (7.31) in the right hand side of (7.27) and using the Dirichlet boundary conditions for ghost basis functions one finally arrives at the equation (7.25). Similar proof holds for (7.26).

This accomplishes the derivation of the equation (7.24). It represents a reduction algorithm that expresses the ghost functional determinant in terms of determinants of the canonical Faddeev-Popov matrices with qualitatively lower functional dimensionality. Simultaneously this algorithm relates the gauge fixing procedure in relativistic gauges to the special unitary gauges of the form (2.14) with the matrix (2.13) given by (6.31). This reduction algorithm together with the the result for the gauge field determinant (6.37) finalizes the the proof that the path integral with the one-loop prefactor (4.14) indeed represents the semiclassical solution (2.1), (2.12) of quantum Dirac constraints.
8. Conclusions

The equality of expressions (2.12) and (4.19) comprises the equality of two gauge independent objects – the function (2.12) independent of the unitary gauge conditions \((X^\mu_i, X^\nu_k)\) and the functional (4.19) independent of the relativistic ones \(\chi^\mu_a\). The meaning of this seemingly vacuous equation is that it establishes the mechanism of identical transition from unitary to relativistic gauge conditions. Such a transition is very important because it proves intrinsic unitarity of a manifestly covariant quantization in terms of Lagrangian path integral. This transition is generally a very nontrivial procedure, because in contrast with unitary gauges in relativistic gauges the theory possesses qualitatively different number of propagating degrees of freedom: gauge modes of \(g^a\), Lagrange multipliers \(N^\mu\) as well as ghost fields \((C^\mu, \check{C}^\nu)\) are dynamical. It is a subtle mechanism of BRST invariance that guarantees the cancellation of contributions of all these modes in physical quantities. This mechanism makes the result equivalent to quantization in the physical sector, that is in unitary gauge. However, this transition is usually performed at the level of formal identical transformation in the path integral, which is reached only in a singular limit \(\epsilon \to 0\) of the so-called \(\epsilon\)-procedure of ref. [3].

On the contrary, the equivalence of (2.12) and (4.19) is actually achieved without any singular limiting operations – by independently calculating these quantities in unitary and relativistic gauges and observing their coincidence when these two sets of gauge conditions are related by equations (6.31)-(6.33). These relations are nonlocal in time – the matrices of unitary gauges and gauge fixing matrix \(C_{\mu\nu}\) express in terms of basis functions of the gauge field operator, nonlocally depending on its relativistic gauge fixing elements \(\chi^\mu_a\) and \(c_{\mu\nu}\).

In physical applications, loop expansion of the path integral as a means of solving quantum Dirac constraints has a very important advantage. This technique implies that solutions of canonical constraints (attached to time foliation) admit the spacetime covariant description in terms of Feynman diagrams [25]. This property is of crucial importance for correct regularization of inevitable ultraviolet divergences. This regularization should maintain the covariance in the form not splitted by time foliation. Spacetime covariance is not manifest due to the canonical origin of Dirac constraints, but it gets restored in the proposed calculation technique, because the arising one-loop functional determinants can be cast into spacetime covariant form by a suitable choice of relativistic gauge conditions. Important implication of this technique is the theory of loop effects in quantum cosmology including, in particular, effective equations for expectation values in the early inflationary universe [26].

The aspects of gauge independence considered above are important in gauge theory applications in spacetimes with boundaries or nontrivial time foliations. There exists a long list of examples when the calculations of a formally gauge independent quantity – one-loop effective action – gives different results in different gauges [27, 28]. No exhaustive explanation for these discrepancies has thus far been given, and there is a
hope that a careful analysis of Ward identities with regard to boundary conditions can resolve this problem.

The last but not the least problem that belongs to the scope of our result is the theory of quantum gravitational tunneling and the physics of wormholes [29]. There is a widespread opinion that the predictions of Euclidean quantum gravity modelling these phenomena have a questionable status due to the indefiniteness of the Euclidean gravitational action [30]. The conclusion drawn in [31] about the mechanism of transitions with changing spacetime topology is based on the existence of a negative mode on the wormhole instanton – a formal extrapolation of the mechanism which is directly applicable only to non-gravitational systems [32]. As it was pointed out in [31] (see also [33]), this issue should be revised from the viewpoint of the Wheeler-DeWitt equation. The necessity of such a revision is obvious because this negative mode belongs to the conformal sector which is not dynamically independent in the Lorentzian quantum gravity. Its contribution is, therefore, cancelled by ghost fields in relativistic gauges. One should expect that similar cancellation should take place also in Euclidean theory (despite the hyperbolic vs elliptic nature of field equations in Lorentzian and Euclidean theories, they demand, after all, the same total number of boundary conditions)\[^{10}\]. Other difficulties in Euclidean gravity theory related to this issue also find place in current literature: the lack of strong ellipticity of the Dirichlet-Robin boundary value problem (1.25)-(1.27) observed in [34] seems to be explained by the indefiniteness of the Euclidean gravitational action. The technique proposed here can be regarded as a direct avenue towards the resolution of these issues – the problem which is currently under study.

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\[^{10}\] The difference between the Euclidean and Lorentzian settings mainly concerns the place where the boundary conditions are consistently imposed: in the hyperbolic case two conditions are fixed at one boundary – initial Cauchy surface, for the elliptic case of Euclidean theory they should be imposed separately on two different boundaries.
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