MILNOR INVARIANTS FOR SPATIAL GRAPHS

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Abstract. Link homotopy has been an active area of research for knot theorists since its introduction by Milnor in the 1950s. We introduce a new equivalence relation on spatial graphs called component homotopy, which reduces to link homotopy in the classical case. Unlike previous attempts at generalizing link homotopy to spatial graphs, our new relation allows analogues of some standard link homotopy results and invariants.

In particular we can define a type of Milnor group for a spatial graph under component homotopy, and this group determines whether or not the spatial graph is splittable. More surprisingly, we will also show that whether the spatial graph is splittable up to component homotopy depends only on the link homotopy class of the links contained within it. Numerical invariants of the relation will also be produced.

1. Introduction

Where knot theory studies embeddings of circles into $S^3$, spatial graph theory studies embeddings of arbitrary graphs. Let $G$ be a finite abstract graph. Let $f$ be an embedding of $G$ into the 3-sphere. We will call $f(G)$ a spatial embedding of $G$ or simply a spatial graph. Since it is the map $f$ that matters in this construction, we will often refer to $f$ as a spatial embedding. We will often use the notation $\Phi := f(G)$ to emphasize that we are thinking of the spatial graph as a subcomplex of $S^3$.

Several analogues of link homotopy have been proposed for spatial graphs, such as the edge homotopy and vertex homotopy of Taniyama [9]. Two spatial embeddings of a graph $G$ are called edge-homotopic if they are transformed into each other by self crossing changes and ambient isotopies, where a self crossing change is a crossing change between two arcs of the same edge, and vertex-homotopic if they are transformed into each other by crossing changes between arcs of two adjacent edges and ambient isotopies. Each of these notions reduces to link homotopy in the case when the graph is a disjoint union of circles, and invariants of these relations have been produced [1, 8]. Classification of the embeddings of some graphs under these relations is possible. For example, spatial embeddings of $K_4$ up to edge-homotopy have been completely analyzed by Nikkuni [6].

Milnor invariants are sufficient to classify two and three component links up to link homotopy [5] and contributed to the eventual classification of all links [3]. So far, no analogue of these invariants exists for the spatial graph relations listed above. Since Milnor invariants are a useful tool for studying link homotopy, our goal is to introduce a generalization of link homotopy for spatial graphs that allows a reasonable generalization of Milnor’s link homotopy invariants.

Key words and phrases. spatial graph, Milnor numbers, Milnor group, link homotopy, edge homotopy, component homotopy.

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The equivalence relation we will use is called component homotopy. Let \( SE(G) \) denote the set of all spatial embeddings of an abstract graph \( G \). Two embeddings \( \Phi, \Phi' \in SE(G) \) are called component homotopic if \( \Phi \) can be transformed to \( \Phi' \) by ambient isotopy and crossing changes between edges that belong to the same component of \( G \). Note that if \( G \) is connected, then any two embeddings are component homotopic. This is the analogue of the fact that any knot is link homotopic to the unknot, though for most graphs there is no canonical choice of embedding to call the “unknot.” However, up to component homotopy, there is a clear choice for the analogue of the unlink, which we will call a completely split embedding. A spatial embedding \( \Phi \) of \( G \) is completely split if each component of \( \Phi \) can be separated from all others by an embedded \( S^2 \). Note that any two completely split spatial embeddings \( f, g \in SE(G) \) are component homotopic, and thus up to component homotopy, there is a canonical “unlink.”

If Milnor’s link homotopy invariants vanish for \( L \), then \( L \) is link homotopic to the unlink \([4]\). The vanishing of Milnor’s link homotopy invariants is equivalent to the Milnor group of the link, \( ML \), being isomorphic to the free Milnor group \( MF \). Choosing the correct analogue of these groups, which we will call \( CM(\Phi) \) and \( CMF(G) \), we are able to prove the following theorems, which are direct generalizations of Theorem 8 and Corollary 2 of Milnor’s original paper \([4]\).

**Theorem 2.3** Let \( \Phi \) be an embedding of a graph \( G \) into \( S^3 \). Then \( CMF(G) \cong CM(\Phi) \) where the isomorphism preserves generators (up to conjugacy) if and only if \( \Phi \) is component homotopic to a completely split embedding.

**Theorem 2.4** Let \( \Phi \) be an embedding of a graph \( G \) into \( S^3 \) with components \( \Phi_1 \ldots \Phi_n \). If the map \( \theta \) from \( CMF(G) \) to \( CM(\Phi) \) sends generators to conjugates of generators, and \( \theta^i \) is the induced map on the groups for \( \Phi^i := \Phi \setminus \Phi_i \), then \( \ker \theta \cong \ker \theta^i \) if and only if \( \Phi \) is component homotopic to an embedding where \( \Phi_i \) can be separated from the rest of \( \Phi \) by an embedded \( S^2 \).

A spatial subgraph of \( \Phi \) that is homeomorphic to \( \bigsqcup S^1 \) is called a constituent link.

The following theorem shows that the link homotopy classes of certain constituent links of \( \Phi \) determine whether \( \Phi \) is component homotopic to a completely split embedding. Note that this is in contrast to edge homotopy and vertex homotopy, where there are infinite families of examples of non-split embeddings, all of whose constituent links are link homotopic to trivial links \([1]\).

**Theorem 3.1** Let \( L \) be a constituent link of \( \Phi \), where at most one component of \( L \) is contained in each component of \( \Phi \). Every such constituent link of \( \Phi \) is link homotopic to the trivial link if and only if \( \Phi \) is component homotopic to a completely split embedding.

It is also possible to extract numerical invariants from the \( CM(\Phi) \) by looking at successive nilpotent quotients. Milnor’s numbers can be interpreted as arising from the elements in the kernel of the map \( MF \to ML \), and so by examining the kernel of the map \( CMF(G) \to CM(\Phi) \) we will be able to produce similar (though less subtle) numerical invariants. Naturally, any invariant of component homotopy is an invariant of edge homotopy and vertex homotopy, so these numerical invariants are invariants of those relations as well.

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2. The Milnor group for spatial graphs

In this chapter, we will study embeddings of graphs up to component homotopy. That is, up to colored edge homotopy, where every edge in a given component of a graph is labeled with the same color, and each component of the graph is given a distinct color. Notice that for the case \( G = \coprod S^3 \), this is simply link homotopy.

In an analogue to the colored Milnor group \( CMG \) of \([2]\), for an embedding \( f : G \to S^3 \) we can define a “free” group \( CM(G) \) associated to \( G \), and a group \( CM(\Phi) \) associated to the embedding.

The meridians of the edges of \( \Phi \) form a normally generating set for \( CM(\Phi) := \pi_1(S^3 \setminus \Phi)/\langle [m_{ij}^1, m_{ik}^2] \rangle \), where the generator \( m_{ij} \) is the meridian of the \( j \)th edge in the \( i \)th component, the \( g_i \) are elements of \( \pi_1(S^3 \setminus \Phi) \), and \( \langle S \rangle \) denotes the subgroup normally generated by the set \( S \). In fact, we need only one generator per loop in \( \Phi \), but by contracting a spanning tree, we may think of these generators as meridians for a subset of the edges.

Let \( CMF(G) \) denote the free colored Milnor group corresponding to \( G \), defined in the following way. Fix a spanning tree \( T \) of \( G \). Label the edges of component \( i \) in \( G/T \) by \( x_{ij} \). Define \( CMF(G) := F(x_{ij})/X \), where \( F(x_{ij}) \) is the free group with generators \( x_{ij} \), and \( X := \langle [x_{ij}^{g_1}, x_{ik}^{g_2}] \rangle \), with \( g_1, g_2 \) arbitrary words in the \( x_{ij} \). We will often refer to the relations induced by \( X \) as the Milnor relations. When \( CMF(G) \) is generated by \( x_{ij} \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq r_i \), then \( CMF(\Phi) \) is generated by meridians \( m_{ij} \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq r_i \), for the same values of \( n \) and \( r_i \).

For an edge \( x_{ij} \) in \( G \), there is a unique minimal path \( p_{ij} \) in \( T \) connecting the end points of \( x_{ij} \). Let \( l_{ij} := f(x_{ij} \cup p_{ij}) \in CM(\Phi) \).

Note that the free colored Milnor group \( CMF(G) \) is a colored Milnor group in the sense of Freedman and Teichner, and hence is nilpotent by Lemma 3.1 of \([2]\). As \( CM(\Phi) \) is a quotient of \( CMF(G) \), it is nilpotent as well. This is important, as it allows us to make use of the following theorem of Stallings \([7]\).

**Theorem 2.1 (Stallings).** Let \( G_1 \) and \( G_2 \) be nilpotent groups, and \( \psi : G_1 \to G_2 \) a homomorphism. If the induced map \( H_1(G_1) \to H_1(G_2) \) is an isomorphism, and the induced map \( H_2(G_1) \to H_2(G_2) \) is surjective, then \( \psi \) is an isomorphism.

With Stallings’ theorem at our disposal, it is possible to show that the group \( CM(\Phi) \) is invariant under component homotopy on the spatial graph \( \Phi \) in the same way as for the colored Milnor group in \([2]\). Roughly, the idea is this: given \( \Phi_0 \) a spatial embedding of \( G \), and \( h \) a component homotopy from \( \Phi_0 \) to \( \Phi_1 \), let \( H \) be the track of this homotopy in \( S^3 \times I \), and \( CM(W) := \pi_1(S^3 \times I \setminus H)/X \). Consider the maps \( \psi_i : CM(\Phi_i) \to CM(W) \) induced by inclusion. The inclusions carry meridians of \( \Phi_i \) to meridians of \( H \), so the \( \psi_i \) clearly induce isomorphisms \( H_1(CM(\Phi_i)) \to H_1(CM(W)) \). By using finger moves to introduce self intersections on \( H \), we can arrange \( CM(W) \cong \pi_1(S^3 \times I \setminus H) \). Then by Alexander duality, \( H_2(CM(W)) \) is generated by the handlebodies dual to \( \Phi_i \), and the linking tori at the self intersection points. These linking tori simply realize the relations in \( X \), so \( H_2(CM(\Phi_i)) \to H_2(CM(W)) \) is surjective. Thus, by Stallings’ Theorem, \( \psi_0 \) and \( \psi_1 \) are isomorphisms, and hence \( CM(\Phi_0) \cong CM(\Phi_1) \).

It will be necessary later to understand the structure of the groups \( CM(\Phi) \) in more detail. Given a spatial embedding \( \Phi \), a regular neighborhood of \( \Phi_1 \) is a handlebody, and the 2-cell of this handlebody induces the relation \( \prod_i [m_{ij}, l_{ij}] = 1 \).
in \( \pi_1(S^3 \setminus \Phi) \) and hence in \( CM(\Phi) \). We will call this relation the surface relation induced by \( \Phi_i \), and the element \( \prod [m_{ij}, l_{ij}] \) the surface element.

Just as is shown for the classical Milnor group in [4], it is possible to show that \( CM(\Phi) \) has the presentation below. The fact that the only relations in \( CM(\Phi) \) are the surface relations and the Milnor relations will be important in later arguments.

**Lemma 2.2.** The group \( CM(\Phi) \) has the presentation \( CM(\Phi) \cong \{m_{ij}\}_{i=1}^{r_i}, X = 1 \), where the \( m_{ij} \) are meridians to the edges \( x_{ij}, r_i \) is the surface relation given by \( \Phi_i \), and \( X = 1 \) denotes the relations arising from \( X \).

**Proof.** We begin with a Wirtinger presentation for \( \pi_1(S^3 \setminus \Phi) \), which is calculated from a projection of the spatial graph. Before projecting, we may first contract a spanning tree of \( \Phi \), as this does not affect \( \pi_1(S^3 \setminus \Phi) \). Given a projection, choose an arbitrary orientation for each edge, and label the generators corresponding to each arc of that edge in the diagram by \( m_{ij}^k \), \( 1 \leq k \leq r_{ij} \) in the order the arcs are met as the edge is traversed. At each vertex we have the relation that the product of the meridians to all the arcs meeting that vertex is trivial. In a fixed component, each edge begins and ends at the vertex, so we may choose a diagram such that the relation induced by that vertex is \( \prod m_{ij}^k (m_{ij}^r)^{-1} = 1 \).

The relations in the Wirtinger presentation for this diagram are those coming from the vertices and those of the form \( m_{ij}^{k+1} = w_{ij}^k m_{ij}^k (w_{ij}^k)^{-1} \) induced by the crossings, where the \( w_{ij}^k \) are arbitrary words in the \( m_{ij}^k \). Using the Milnor relations, the relations from the crossings, and a double induction as in [4], we can eliminate the relations from the crossings and reduce the set of generators to only the \( m_{ij} := m_{ij}^1 \). The only remaining relations are those induced by the vertices, which are of the form \( \prod m_{ij} (m_{ij}^{w_{ij}})^{-1} = \prod [m_{ij}, w_{ij}] = 1 \). These are precisely the surface relations discussed above.

\[ \square \]

Recall that a spatial embedding \( \Phi \) of \( G \) is completely split if each component of \( \Phi \) can be separated from all others by an embedded \( S^2 \). We may now formulate an analogue of Theorem 8 of [4].

**Theorem 2.3.** Let \( \Phi \) be an embedding of a graph \( G \) into \( S^3 \). Then \( CMF(G) \cong \Phi \) if and only if \( \Phi \) is component homotopic to a completely split embedding.

**Proof.** Suppose that \( \Phi \) is component homotopic to a completely split embedding. Since \( CM(\Phi) \) is invariant under component homotopy, we may assume that \( \Phi \) is completely split.

The group \( CM(\Phi) \) is a free group modulo the set \( X \) and the surface relations generated by the handlebodies forming the boundaries of neighborhoods of each component of \( \Phi \). The surface relations in \( CM(\Phi) \) are of the form \( \prod [m_{ij}, l_{ij}] = 1 \) for fixed \( i \). Since each component is separated from all others, for a fixed \( i \), each \( l_{ij} \) is a word in the \( m_{ij} \). Since \( m_{ij} \) commutes with all other generators with index \( i \), \( [m_{ij}, l_{ij}] = 1 \) for each pair \( m_{ij}, l_{ij} \). Thus, in \( CM(\Phi) \) the surface relations are trivial, so we have only the relations introduced by \( X \), and the map \( x_{ij} \to m_{ij} \) is an isomorphism \( CMF(G) \to CM(\Phi) \).

Suppose that an isomorphism \( \theta : CMF(G) \to CM(\Phi) \) exists. We will induct on the number of edges in the final component. Clearly if \( \Phi \) has one component, it is component homotopic to a split embedding, so the base case is trivial.
Suppose \( \Phi = \bigsqcup \Phi_i \) is a spatial graph of \( n \) components, and the first Betti number \( b_1(\Phi_n) = r \). Then there are \( r \) generators of \( CM(\Phi) \) arising from \( \Phi_n \), specifically \( m_{nj}, 1 \leq j \leq r \).

If deleting an edge of \( \Phi_n \) does not change \( b_1(\Phi_n) \), that edge is contractible and so the argument is trivial. So suppose that deleting an edge \( e \) does change \( b_1(\Phi_n) \). We may choose the spanning tree \( T \) to avoid \( e \), so let \( m_{nr} \) be the generator of \( CM(\Phi) \) represented by a meridian of \( e \).

Let \( G' := G \setminus e \) and \( \Phi' \) be the restriction of \( f \) to \( G' \). There is an obvious surjection \( CMF(G) \to CMF(G') \) given by \( x_{nr} = 1 \), and the kernel of this map is clearly \( \langle x_{nr} \rangle \). Let \( \theta_e \) be the restriction of \( \theta \) to this kernel and \( \theta' : CMF(G') \to CM(\Phi) \) be the map induced by \( \theta \).

We want to show that \( \theta' \) is an isomorphism. By the 5 Lemma, this is equivalent to showing that \( \theta_e \) is an isomorphism.

Since \( \theta \) sends generators to conjugates of generators, \( \theta(x_{nr}) = m_{nr}^g \), and so \( \theta_e(\langle x_{nr} \rangle) \subset \langle m_{nr} \rangle \). Clearly, as \( \theta_e \) is the restriction of an isomorphism it is injective, so it suffices to show that it is surjective. Given \( m_{nr}^l \in \langle m_{nr} \rangle \), we have \( \theta^{-1}(m_{nr}^l) = x_{nr}^{-1}(g^{-1}g) \in \langle x_{nr} \rangle \). Thus \( \theta_e \) is an isomorphism, and so is \( \theta' \).

As \( \Phi' \) has fewer edges, and \( \theta' \) is an isomorphism carrying generators to conjugates of generators, we may apply the induction hypothesis and use component homotopy to convert \( \Phi' \) to a completely split embedding. If we perform this component homotopy on \( \Phi \) simply carrying the extra edge \( e \) along, we arrive at the embedding shown in Figure 1.

![Figure 1](image-url)

**Figure 1.** The embedding of \( \Phi \) after the inductive hypothesis.

The edge labeled \( e \) winds through some or all of the components.

Since the embedding \( \Phi' \) is completely split, in \( CM(\Phi') \) the loop \( l_{ni} \) is a word in the \( m_{nj} \). Thus, in \( CM(\Phi) \), \( l_{ni} \) is a word in \( m_{nj} \), \( j < r \) and \( m_{nr}^g \) for any \( g \in \pi_1(S^3 \setminus \Phi) \). Now, since \( [m_{ni}, m_{nj}^g] = 1 \) in \( CM(\Phi) \), we know that \( [m_{ni}, l_{ni}] = 1 \in CM(\Phi) \) for all \( i < r \).
We have the relation $\prod_{i \leq r} [m_{ni}, l_{ni}] = 1$ in $CM(\Phi)$ from the 2-cell of the neighborhood of $\Phi_n$, but as we just saw that $[m_{ni}, l_{ni}] = 1$, $i < r$, we may conclude that $[m_{nr}, l_{nr}] = 1$ in $CM(\Phi)$.

Using the isomorphism $\theta$, we have that $[x_{nr}, l_{nr}] = 1$ in $CMF(G)$. In $CMF(G)$, the only elements that commute with a generator $x_{ij}$ are conjugates of generators of the same color [2]. Thus, we know $l_{nr} = \Pi(x_{nj})^{g_j}$ for some $g_j \in CMF(G)$, and hence has the same form in $CM(\Phi)$.

This means that $l_{nr} = \prod m_{rj}^{g_{nj}} [m_{ijr}^{g_{ij}}, m_{ijk}^{g_{ijr}}]$ in $\pi_1(S^3 \setminus \Phi')$. We may eliminate the commutators through component homotopy as shown in Figure 2. To eliminate the terms of $m_{nj}^{g_{nj}}$, we use the component homotopy move as in Figure 3.

We may reduce to $l_{nr} = 1$ in $\pi_1(S^3 \setminus \Phi')$. Thus the loop $l_{nr}$ is contractible and may be isotoped into the sphere containing $\Phi_n$, so $\Phi$ is component homotopic to a completely split embedding.

\[\square\]

**Figure 2.** Using component homotopy to cancel elements of the form $[m_{ij}, m_{ik}]$.

**Figure 3.** Using component homotopy to cancel elements of the form $m_{nj}^{g_{nj}}$. 
Let $CM(\Phi^i)$ denote the colored Milnor group of $\Phi^i := \Phi \setminus \Phi_i$, the spatial graph obtained from $\Phi$ by removing the $i$th component $\Phi_i$. That is $CM(\Phi^i) := CM(\Phi)/(m_{ij} = 1)$ for a fixed $i$. Given a map $\theta : CMF(G) \to CM(\Phi)$, we have the sequence

$$\text{ker } \theta \longrightarrow CMF(G) \xrightarrow{\theta} CM(\Phi)$$

Let $G^i := G \setminus G_i$, and let $\theta^i$ denote the restriction of the map $\theta$ to $G^i$. This induces the sequence

$$\text{ker } \theta^i \longrightarrow CMF(G^i) \xrightarrow{\theta^i} CM(\Phi^i)$$

We are now able to produce a theorem analogous to Milnor’s Corollary 2 of [4].

**Theorem 2.4.** Let $\Phi$ be an embedding of a graph $G$ into $S^3$. Suppose the map $\theta : CMF(G) \to CM\Phi$ sends generators to conjugates of generators. Then $\text{ker } \theta^i \cong \text{ker } \theta$ if and only if $\Phi$ is component homotopic to an embedding where $\Phi_i$ can be separated from the rest of $\Phi$ by an embedded $S^2$.

**Proof.** This proof is similar to that of Theorem 2.3. Recall that the only relations in $CM(\Phi)$ are those induced by $X$ and the surface relations. Under $\theta$, the set $X$ in $CM(\Phi)$ is hit only by elements of $X$ in $CMF(G)$, so $\text{ker } \theta$ is generated by the preimages of the surface relations. Suppose that $\Phi$ is component homotopic to an embedding where $\Phi_1$ can be separated from the other components by an embedded $S^2$. Splitting along this $S^2$, Van Kampen’s theorem implies that $\pi_1(S^3 \setminus \Phi) = \pi_1(S^3 \setminus \Phi') \ast \pi_1(S^3 \setminus \Phi_1)$, where $\Phi' := \Phi \setminus \Phi_1$. As the surface element $\Pi[m_{ij}, l_{ij}]$ lies in $\pi_1(S^3 \setminus \Phi_1)$ and the surface elements $\Pi[m_{ij}, l_{ij}]$ lie in $\pi_1(S^3 \setminus \Phi')$ for $i \geq 2$, the surface relation given by $\Phi_1$ is trivial in $CM(\Phi)$, and the surface elements given by the $\Phi_i, i > 1$ can be written without the generators $m_{ij}$. Thus, $\text{ker } \theta \cong \text{ker } \theta^i$ under the map taking $x_{ij}$ to 1 and $x_{ij}$ to $x_{ij}$ for $i > 1$.

Now suppose that the component of interest is $\Phi_n$ and $\text{ker } \theta \cong \text{ker } \theta^n$. We now induct on the number of edges in $\Phi_n$ as before. Since $\text{ker } \theta \cong \text{ker } \theta^n$, we have that the surface relation from $\Phi_n$ is trivial, that is $\prod_{i}[x_{nij}, l_{nj}] = 1$ in $CMF(G)$. However, by induction we know that $\prod_{i}[x_{nij}, l_{nj}] = [x_{nr}, l_{nr}] = 1$ in $CMF(G)$. We may now use the same techniques as in the proof of Theorem 2.3 to alter $\Phi$ by component homotopy so that $\Phi_n$ can be separated from the other components.

3. Numerical invariants and relations to links

Recall that a constituent link is a subgraph of $\Phi$ that is homeomorphic to a disjoint union of circles.

**Theorem 3.1.** Let $L$ be a constituent link of $\Phi$, where at most one component of $L$ is contained in each component of $\Phi$. Every such constituent link of $\Phi$ is link homotopic to the trivial link if and only if $\Phi$ is component homotopic to a completely split embedding.

**Proof.** Let $\Phi$ be a spatial graph component homotopic to a completely split embedding $\Phi'$, and let $L$ be a constituent sublink of $\Phi$, with at most one component of $L$ contained in each component of $\Phi$. The component homotopy from $\Phi$ to $\Phi'$ induces a link homotopy carrying $L$ to $L'$. Each component of $\Phi'$ can be separated
from the others by an embedded $S^2$. Each separating 2-sphere contains a single component of $L'$, which by additional component homotopy moves (contained in that sphere) can be unknotted. Thus, $L$ is link homotopic to the unlink.

Assume that all the relevant constituent links of $\Phi$ are link homotopically trivial. We will work by induction on the number of components in $\Phi$. For a one component graph, all spatial embeddings trivially satisfy the condition to be completely split.

Suppose $G = \bigwedge_1^n G_i$ is an $n$ component graph and $\Phi := f(G)$ is not split up to component homotopy. Then by Theorem 2.3, no map from $CMF(G)$ to $CM(\Phi)$ preserving generators (up to conjugacy) is an isomorphism. So given the map $\theta : x_{ij} = m_{ij}$, there must exist a nontrivial element in the kernel. Relabeling the components of $\Phi$ if necessary, it must be of the form $\prod_i [x_{ij}, l_{ij}]$. So some $[x_{ij}, l_{ij}] \neq 1$ in $CMF(G)$. Delete all edges in $G_1$ except for the edges forming $l_{ij}$, to produce $G^1 := S^1 \bigwedge_j G_i$, and form the embedding $\Phi^1$ by restricting $f$ to $G^1$. We have reduced $\Phi_1$ to a circle, so we may now think of $l_{ij}$, as the zero framed pushoff.

Suppose $[x_{ij}, l_{ij}] = 1$ in $CMF(G^1)$, then $l_{ij} = x_{ij}^g$ in $CMF(G^1)$ by 2. The map $CMF(G) \to CMF(G^1)$ is given by $x_{ij} = 1$ for $j \neq i$, so the kernel is normally generated by those $x_{ij}$. So, in $CMF(G)$, $l_{ij} = \Pi x_{ij}^g$, and thus $[x_{ij}, l_{ij}] = [x_{ij}, \Pi x_{ij}^g] = 1$ in $CMF(G)$. This is a contradiction, so $[x_{ij}, l_{ij}] \neq 1$ in $CMF(G^1)$, and thus $\theta^1 : CMF(G^1) \to CMF(\Phi^1)$ given by the restriction of $\theta$ is not an isomorphism. So by Theorem 2.3, $\Phi^1$ is not completely split.

Suppose $[x_{ij}, l_{ij}] = 1$ in $CMF(G^1)$ for all $j$. Then as discussed in the proof of Theorem 2.3, $l_{ij} = \Pi x_{ij}^g \in CMF(G^1)$, and so $l_{ij} = \prod i \Pi m_{ij}^{g_{ij}}$ in $\pi_1(S^3 \setminus \Phi^1)$. We may now do component homotopy as in Figure 2 and Figure 3 to separate $\Phi_2$ from the rest of $\Phi^1$.

Let $G^{12} := G^1 \setminus G_2$, and $\Phi^{12} := \phi(G^{12})$. Then, $\Phi^1$ is component homotopic to $\Phi^{12} \bigwedge_2 \Phi_2$. The component homotopy induces a link homotopy on the relevant constituent links, so by assumption, these constituent links of $\Phi^{12}$ are link homotopic to trivial links. The spatial graph $\Phi^{12}$ has $n - 1$ components, so by induction, it is completely split up to component homotopy, and hence so is $\Phi^1$. This is a contradiction.

Thus, there exists some $x_{2j}$ with $[x_{2j}, l_{2j}] \neq 1$ in $CMF(G^1)$. Now, as before, delete edges of $G_2$ to reduce it to the circle defined by $l_{2j}$, forming $G^{12} := S^1 \bigwedge_i S^1 \bigwedge_2 G_i$. Again, restricting $f$, we obtain an embedding $\Phi^{12}$, and $[m_{2j}, l_{2j}] \neq 1$ in $CM(\Phi^{12})$.

Repeat the argument as above until each component of $G$ has been reduced to a circle, forming $G' := G^{12 \cdots n}$. Form $\Phi'$ by restricting $f$ to $G'$. Notice that $[x_{nj}, l_{nj}] \neq 1$ in $CMF(G')$. Now, $\Phi'$ is an $n$ component link $L$ where each component of $L$ came from a distinct component of $G$. Thus $CM(\Phi')$ is $ML$, and the map $\theta : CMF(G) \to CM(\Phi)$ induces a map $\theta' : MF \to ML$, where $MF$ is the free Milnor group on $n$ generators.

The commutator $[x_{nj}, l_{nj}] \neq 1$ in $MF$, but is an element of the kernel of $\theta'$. Since the map $\theta' : MF \to ML$ takes generators to conjugates of generators but is not an isomorphism, by Theorem 2.3, $L$ is not link homotopically trivial.

To determine when a spatial graph $\Phi$ is completely split up to component homotopy, Theorem 6.1 implies that it is sufficient to check that the constituent links (of
a certain type) are all split instead of checking the conditions of Theorem 2.3. Note that a similar statement is not true for Theorem 2.4. As the following example shows, it is possible for a every constituent link of $\Phi$ to be $i$-trivial, but $\Phi$ to not be component homotopic to $\Phi_i \sqcup \Phi_i$.

**Example 3.2.** Let $\Phi$ be the spatial graph shown in Figure 4. Then clearly every constituent link is 3-trivial. However, we have the relation $[m_{31}, [m_{11}, m_{21}]] = 1 \in CM(\Phi)$, but $[x_{31}, [x_{11}, x_{21}]] \neq 1 \in CM(G)$. Thus, for the map $\theta(x_{ij}) = m_{ij}$, we have $ker \, \theta \nsubseteq ker \, \theta^3$, and so by Theorem 2.4 the embedding $\Phi$ is not component homotopic to $\Phi^3 \sqcup \Phi_3$.

![Figure 4](image.png)

**Figure 4.** A spatial graph that is not split up to component homotopy.

Teichner and Freedman introduced the notion of colored link homotopy in [2]. In this notion, each component of link is assigned a color, and crossing changes are allowed between components of the same color. Thereom 3.1 can be easily extended to the statement that $\Phi$ is component homotopic to a completely split embedding if and only if every constituent link of $\Phi$ is colored link homotopic to the unlink, where two components of the link have the same color if and only if they are contained in the same component of $\Phi$. This shows that in some sense, the group $CM(\Phi)$ contains information about colored link homotopy classes of all the constituent links.

It is possible to extract numerical invariants from the $CM(\Phi)$ which are the analogue of the length of the shortest non-vanishing Milnor invariant for a link.

It is well known that the Milnor group of a link is nilpotent, and the group $CM(\Phi)$ is also nilpotent. The kernel of the map $CMF(G) \to CM(\Phi)$ is generated by surface elements, which are products of commutators of elements. This suggests that examining successive quotients by all commutators of a fixed length may be productive.

Let $H$ be a subgraph of $G$, and $\Phi := f(G)$ a spatial embedding. Let $\Phi^H$ be the spatial embedding of $G^H := G \setminus H$ that is induced by $f$. Let $CM(\Phi^H)$ denote the colored Milnor group of $\Phi^H$. Given a map $\theta : CMF(G) \to CM(\Phi)$, we have the sequence below:

$$
ker \, \theta \xrightarrow{\theta} CMF(G) \xrightarrow{\theta} CM(\Phi)
$$

Let $\theta^H$ be the map induced by the map $\theta$. 


Then we may define $\lambda_\Phi(H)$, a component homotopy invariant of $\Phi$ as follows:

$$
\lambda_\Phi(H) + 1 := \min \{ n \mid \ker \theta^H/[n] \not\cong \ker \theta/[n] \}
$$

Since the groups $CM(\Phi)$ and $CM(\Phi^H)$ are invariant under component homotopy, so is $\lambda_\Phi(H)$. Here we consider $ker \theta$ as a subgroup of $CMF(G)$, and $[n]$ denotes all commutators of length $n$ whose entries are elements of $CMF(G)$. Notice that if $G = \coprod S^1$, then $\lambda_{L_i}(L_i)$ is the length of the index of the shortest nonvanishing Milnor homotopy invariant whose index contains $i$.

**Example 3.3.** The first nonvanishing Milnor invariant of the Borromean rings is $\mu(123)$. If $L$ is the Borromean rings, then the kernel of the map $\theta : CMF(L) \to CML$ is generated by the element $[x_1, [x_2, x_3]]$. When the first component is removed, however, the resulting link is the unlink, so $\ker \theta^{L_1}$ is trivial. Thus, $\lambda_{L}(L_1) = 3$.

\[ \text{Figure 5. A spatial graph.} \]

**Example 3.4.** Let $\Phi$ be the spatial embedding of $\Theta_3 \coprod S^1 \coprod H_2 \coprod K_4$ shown in Figure 3. For this graph, $CM(\Phi)$ is isomorphic to $CMF(G)$ modulo the following additional relations:

$$
[m_{13}, [m_{21}, m_{41}]] = [m_{21}, [m_{31}, m_{13}]] = [m_{31}, [m_{13}, m_{21}]] [m_{32}, m_{41}] = [m_{41}, m_{32}] = 1
$$

So, for this spatial graph, $\lambda_\Phi(G_1) = \lambda_\Phi(G_2) = 3$ and $\lambda_\Phi(G_3) = \lambda_\Phi(G_4) = 2$.

Notice that while the surface relation given by $\Phi_3$ contains multiple commutators, only the shortest matters for the computation of $\lambda_\Phi(G_3)$. Also, the presence of the length two commutator involving $\Phi_3$ and $\Phi_4$ does not affect the computation of $\lambda_\Phi(G_1)$.

In fact, by the proof of Theorem 3.1 when $H = G_i$ the value of $\lambda_\Phi(H)$ will be the length of the shortest non-vanishing (colored) Milnor invariant of any link that contains a cycle from $G_i$. 
MILNOR INVARIANTS FOR SPATIAL GRAPHS

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