ASKEY-WILSON POLYNOMIALS
AND THE QUANTUM $SU(2)$ GROUP:
SURVEY AND APPLICATIONS

H.T. Koelink
Katholieke Universiteit Leuven
to appear in Acta Applicandae Mathematicae

Abstract. Generalised matrix elements of the irreducible representations of the quantum $SU(2)$ group are defined using certain orthonormal bases of the representation space. The generalised matrix elements are relatively infinitesimal invariant with respect to Lie algebra like elements of the quantised universal enveloping algebra of $sl(2)$. A full proof of the theorem announced by Noumi and Mimachi [Proc. Japan Acad. Sci. 66, Ser. A (1990), pp. 146–149] describing the generalised matrix elements in terms of the full four-parameter family of Askey-Wilson polynomials is given. Various known and new applications of this interpretation are presented.

Contents

1. Introduction
2. Representation theory of $SU(2)$
3. Preliminaries on basic hypergeometric orthogonal polynomials
4. Preliminaries on the quantum $SU(2)$ group
5. Koornwinder’s $(\sigma, \tau)$-spherical elements
6. Generalised matrix elements
7. Generalised matrix elements and Askey-Wilson polynomials
8. Some applications
9. Discrete orthogonality relations
10. Spherical and associated spherical elements
11. Characters
1. Introduction

The theory of special functions is an old but still lively branch of mathematics, which has its origins in the works of the great mathematicians of the 18th and 19th century trying to solve the differential equations of mathematical physics. One of the highlights in the theory is the introduction of the hypergeometric series

\[
1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \ldots
\]

by Gauss in 1812. It turned out that many special functions can be written in terms of Gauss’s hypergeometric series or in terms of one of the generalised hypergeometric series.

An example of such a hypergeometric special function is the Jacobi polynomial which forms a set of orthogonal polynomials, cf. §2. Jacobi polynomials, as many other special functions of hypergeometric type, satisfy various interesting properties. For the Jacobi polynomials it was understood in the 1950’s by the work of Gelfand and Šapiro, cf. the references in [56], that some of the properties of the Jacobi polynomials were reflections of underlying structures given by the complex group $SL(2, \mathbb{C})$, or its real form $SU(2)$, to which we get back in §2. This is just one of the many examples of a very fruitful relation between group theory, and in particular representation theory of Lie groups, and special functions of hypergeometric type, which is still a topic of current research, cf. the survey by Klimyk [23].

In 1846, 34 years after the introduction of the hypergeometric series by Gauss, Heine introduced the basic (or $q$-)hypergeometric series

\[
1 + \frac{(1 - q^a)(1 - q^b)}{(1 - q)(1 - q^c)} z + \frac{(1 - q^a)(1 - q^{a+1})(1 - q^b)(1 - q^{b+1})}{(1 - q)(1 - q^2)(1 - q^c)(1 - q^{c+1})} z^2 + \ldots,
\]

where the base $q$ is usually looked at as a fixed real number between 0 and 1. The limit $q \uparrow 1$ of the basic hypergeometric series yields the hypergeometric series.

Although, since its introduction, work was done on $q$-hypergeometric series and applications were known in several fields, such as the famous Rogers-Ramanujan identities in number theory, cf. Andrews [2], progress was much slower than for special functions of hypergeometric type. The collaboration of Andrews and Askey starting in the mid 1970’s was the beginning of an outburst of results on $q$-hypergeometric series and in particular on various $q$-hypergeometric orthogonal polynomials. This culminated in the introduction in their 1985 memoir by Askey and Wilson of a very general four-parameter set of orthogonal polynomials nowadays known as the Askey-Wilson polynomials.

Until recently there was not much knowledge on possible natural structures on which the $q$-hypergeometric series could ‘live’ in a similar fashion as the special functions of hypergeometric type ‘living’ on certain Lie groups. This problem seems to be solved with the introduction of quantum groups by Drinfeld [13], Jimbo [20] and Woronowicz [58], [59] around 1986. Quantum groups are no longer groups, but we consider them as deformations of the algebra of functions on a group so that the deformed algebra still carries properties that resemble group actions. More information can be found in the survey papers by
One of the oldest examples of a quantum group is the quantum $SU(2)$ group which is a quantum group analogue of the compact group $SU(2)$. A first indication of the relation between $q$-hypergeometric functions and quantum groups is the interpretation of the so-called little $q$-Jacobi polynomials as matrix elements of irreducible unitary representations of the quantum $SU(2)$ group by Vaksman and Soibelman [52], Masuda et al. [37], Koornwinder [30]. Since the Askey-Wilson polynomials can be viewed as $q$-analogues of the Jacobi polynomials, see the title of the Askey-Wilson memoir [7], it is natural to try to interpret the Askey-Wilson polynomials on the quantum $SU(2)$ group.

A decisive step in this direction is taken by Koornwinder [33]. In that paper he shows how to create sufficiently many 'quantum subgroups' of the quantum $SU(2)$ group to get a two-parameter family of Askey-Wilson polynomials as zonal spherical functions. We get back to this paper in more detail in §5. As follow-ups to Koornwinder’s paper there is the paper [24] by the author in which a quantum group theoretic derivation of the Rahman-Verma [46] addition formula for the continuous $q$-Legendre polynomials is given and the announcement by Noumi and Mimachi [40], [43] in which they claim the interpretation of the full four-parameter family of Askey-Wilson polynomials as matrix elements of irreducible unitary representations of the quantum $SU(2)$ group.

This paper grew out of an attempt to provide the announcements by Noumi and Mimachi [40], [43] with full proofs of the sort of proofs given in [24] in which the associated spherical elements on the quantum $SU(2)$ group are calculated explicitly. The purpose of this paper is to give a detailed proof of the relation between Askey-Wilson polynomials and matrix elements of irreducible unitary representations of the quantum $SU(2)$ group and to present a survey of some known as well as of some seemingly new applications of this interpretation of the Askey-Wilson polynomials. In this light the current paper may be viewed as a sequel to Koornwinder’s survey [31].

An alternative to the approach in this paper to $q$-special functions living on quantum groups is the approach in which representations of the so-called quantised universal enveloping algebra are $q$-exponentiated to obtain matrix coefficients expressible as $q$-special functions. This approach is the analogue of exponentiating a Lie algebra representation to find a representation of the corresponding Lie group. However, since the alternative approach uses a $q$-analogue of the exponential function the matrix coefficients neither end up on some group nor on a quantum group. Details of this ‘local’ approach can be found in papers by Floreanini and Vinet [15], Kalnins, Manocha and Miller [21], Zhedanov [60], see also the references in [15].

There are two other closely related non-compact quantum groups for which the relation with $q$-special functions is worked out in some detail. The quantum $SU(1,1)$ group is obtained from the quantum $SU(2)$ group by redefining a $*$-operation, which means that another real form of the quantum $SL(2,\mathbb{C})$ group is chosen. For more information on the representation theory of the quantum $SU(1,1)$ group and the corresponding special functions, which are $q$-analogues of Jacobi functions, the papers by Masuda et al. [35], [36] and by Burban and Klimyk [10] may be consulted. The quantum group of plane motions is obtained from the quantum $SU(2)$ group by a suitable contraction procedure similar
to the classical case. The representation theory of this quantum group and its connection with basic Bessel functions is worked out in papers by Vaksman and Korogodskii[51] and the author [27], [28]. The alternative ‘local’ approach for these quantum groups has also been developed, cf. Floreanini and Vinet [15] and references therein and Kalnins, Miller and Mukherjee [22].

The organisation of the paper is as follows. In §2 we consider briefly the representation theory of the compact Lie group $SU(2)$ and its relation with special functions. We also discuss the approach to the group case adapted for deformation. In order to make the paper more self-contained sections 3 and 4 contain some information on $q$-hypergeometric orthogonal polynomials and on the quantum $SU(2)$ group. These sections are also used to fix the notation. Sections 3 and 4 are by no means complete, but they present sufficient information concerning these subjects to be able to read the rest of the paper. References to the literature are given as well.

In §5 we elaborate on Koornwinder’s [30] approach to introduce sufficiently many ‘quantum subgroups’ and on the corresponding spherical elements. In §6 we introduce and investigate generalised matrix elements of the quantum $SU(2)$ group, which are linked to Askey-Wilson polynomials in §7. In §8 we discuss some known applications of the interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group, such as addition formulas for $q$-Legendre polynomials, quantum spheres and non-negative linearisation coefficients for $q$-Legendre polynomials. The last three sections are concerned with seemingly new applications. In §9 we consider some discrete orthogonality relations. The relation between spherical elements and associated spherical elements is studied in §10. Finally, in §11 we consider the characters of the irreducible unitary representations.

Acknowledgement. I thank George Gasper and especially Mizan Rahman for their help in checking a large formula in §9. Thanks are also due to Masatoshi Noumi for explaining the results and ideas of [44], which led to an improvement of the presentation.

2. Representation theory of $SU(2)$

The representation theory of the compact Lie group $SU(2)$ is the subject of this section. Its relation with special functions is shortly discussed. The point of view to this group and its representation theory, which is favourable for the quantum group approach, is given. This section is of an introductory nature. Complete results on the relation between $SU(2)$ and special functions can be found in Vilenkin [56, Ch. 4], Vilenkin and Klimyk [57, vol. 1, Ch. 6]. For more information on Lie groups and their representation theory the reader may consult Bröcker and tom Dieck [9] and Varadarajan [55].

The group

\begin{equation}
SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha \delta - \beta \gamma = 1, \ \bar{\alpha} = \delta, \ \bar{\beta} = -\gamma \right\}
\end{equation}

is a compact real three-dimensional Lie group of $2 \times 2$ unitary matrices with determinant 1. The group $SU(2)$ is a compact real form of the complex Lie group $SL(2, \mathbb{C})$, the group of $2 \times 2$ complex matrices with determinant 1. By Weyl’s unitary trick, cf. [55, § 4.11] the finite dimensional holomorphic representations of $SL(2, \mathbb{C})$ correspond to the finite dimensional representations of $SU(2)$. 

By $K$ we denote the diagonal subgroup

$$K = S(U(1) \times U(1)) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha \delta = 1, \ \alpha = \bar{\delta} \right\} \subset SU(2).$$

So $K$ is equal to the one-dimensional torus, i.e. the unit circle $\mathbb{T}$ in the complex plane viewed as a commutative one-dimensional group with multiplication given by $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$. The characters of this commutative group are given by

$$\psi_n \left( \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right) = \alpha^n, \quad n \in \mathbb{Z}.$$

For each $l \in \frac{1}{2}\mathbb{Z}_+$ there exists a $(2l+1)$-dimensional irreducible unitary representation $t^l$, i.e. a homomorphism $t^l: SU(2) \to GL(V^l)$ for a $(2l+1)$-dimensional vector space $V^l$ such that all operators $t^l(g)$ are unitary and that there is no invariant subspace of $V^l$ for all operators $t^l(g)$ other than $V^l$ and $\{0\}$. The restriction to $K$ of $t^l$ splits multiplicity-free;

$$t^l|_K = \bigoplus_{n=-l}^l \psi_{-2n}.$$ 

Let $\{e^l_n \mid n = -l, -l+1, \ldots, l\}$ be an orthonormal basis for $V^l$ such that $t^l(k)e^l_n = \psi_{-2n}(k)e^l_n$, $k \in K$. The matrix elements $SU(2) \ni g \mapsto t^l_{n,m}(g) = \langle t^l(g)e^l_m, e^l_n \rangle$ can be calculated in terms of Jacobi polynomials

$$P^{(\alpha,\beta)}_n(x) = \frac{(\alpha + 1)_n}{n!} 2F_1\left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} ; \frac{1-x}{2} \right)$$

for $\alpha, \beta \in \mathbb{Z}_+$. Here we use the standard notation

$$2F_1\left( \begin{array}{c} a, b \\ c \end{array} ; z \right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad (a)_k = \prod_{i=0}^{k-1} (a + i),$$

for Gauss’s hypergeometric series $2F_1$ and for Pochhammer’s symbol $(a)_k$.

The Schur orthogonality relations for the matrix elements are

$$\int_{SU(2)} t^l_{n,m}(g)t^k_{i,j}(g) dg = \delta_{l,k}\delta_{n,i}\delta_{m,j}(2l+1)^{-1},$$

where $dg$ denotes the normalised Haar measure on $SU(2)$. For $n = i, m = j$, these relations yield the orthogonality relations for the Jacobi polynomials

$$\int_{-1}^1 P^{(\alpha,\beta)}_l(x)P^{(\alpha,\beta)}_k(x)(1-x)^\alpha(1+x)^\beta dx = 0, \quad l \neq k,$$ 

for $\alpha, \beta \in \mathbb{Z}_+$.
Since $t^l(g)$ is a unitary matrix for $g \in SU(2)$ we have $\sum_k t^l_{m,k}(g) \overline{t^l_{n,k}(g)} = \delta_{n,m}$. This implies the finite discrete orthogonality relation

$$\sum_{x=0}^{N} K_m(x; p, N) K_n(x; p, N) \left( \begin{array}{c} N \\ x \end{array} \right) p^x (1-p)^{N-x} = \delta_{n,m} (1-p)^n p^{-n} \left( \begin{array}{c} N \\ n \end{array} \right)^{-1}$$

for the Krawtchouk polynomials

$$K_n(x; p, N) = \binom{-n}{x} \binom{-x}{N} p^x (1-p)^{N-x}, \tag{2.3}$$

where $N \in \mathbb{Z}_+$, $n \in \{0, 1, \ldots, N\}$, and $0 < p < 1$.

The commutative $\ast$-algebra of matrix elements of finite dimensional unitary representations of $SU(2)$ is a dense $\ast$-subalgebra of the $C^*$-algebra of continuous functions on $SU(2)$. Moreover, this algebra equals the $\ast$-algebra $P(SU(2))$ of polynomials in the coordinate functions $\alpha, \beta, \gamma, \delta$, cf. (2.1), subject to the relations $\alpha \delta - \beta \gamma = 1$, $\alpha^\ast = \delta$, $\beta^\ast = \gamma$, where the $\ast$-operator is complex conjugation. This algebra has much more structure, which stems from the group structure of $SU(2)$. So multiplication, unit and inverse give rise to the mappings

$$(\Delta p)(g, h) = p(gh), \quad \Delta: P(SU(2)) \to P(SU(2) \times SU(2)) \simeq P(SU(2)) \otimes P(SU(2)),$$

$$\varepsilon(p) = p\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \varepsilon: P(SU(2)) \to \mathbb{C},$$

$$(Sp)(g) = p(g^{-1}), \quad S: P(SU(2)) \to P(SU(2)).$$

and with these mappings $P(SU(2))$ becomes a Hopf $\ast$-algebra, cf. §4.1 for the definition. (The reader is invited to check the axioms of a Hopf $\ast$-algebra in this particular case.)

If we forget about the $\ast$-structure on the Hopf $\ast$-algebra $P(SU(2))$ we can consider the Hopf algebra $P(SU(2))$ as the algebra of polynomials in coordinate functions on the complex group $SL(2, \mathbb{C})$. The $\ast$-operator determines a real form of $SL(2, \mathbb{C})$ and in this case we can recover the real group $SU(2)$ from $P(SU(2))$ by

$$SU(2) = \left\{ g \in SL(2, \mathbb{C}) \mid p^\ast(g) = \overline{p(g)}, \forall p \in P(SU(2)) \right\}.$$

For the matrix elements $t^l_{n,m}$ of the irreducible unitary representation $t^l$ we get from $t^l(gh) = t^l(g) t^l(h)$, $t^l\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = id$, $t^l(g^{-1}) = t^l(g)^\ast$ the identities

$$\Delta(t^l_{n,m}) = \sum_{k=-l}^{l} t^l_{n,k} \otimes t^l_{k,m}, \quad \varepsilon(t^l_{n,m}) = \delta_{n,m}, \quad S(t^l_{n,m}) = (t^l_{m,n})^\ast,$$

cf. (4.18).
The Lie algebra $\mathfrak{su}(2) = \{X \in \mathfrak{gl}(2, \mathbb{C}) \mid trX = 0, X^* + X = 0\}$ is equal to $ihH + bB - \bar{b}C$ for $h \in \mathbb{R}$, $b \in \mathbb{C}$. Here
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
satisfying $[H, B] = 2B$, $[H, C] = -2C$, $[B, C] = H$, span the complexification $\mathfrak{sl}(2, \mathbb{C})$ of $\mathfrak{su}(2)$. The representations $t^l$ of $SU(2)$ can be differentiated yielding representations of the Lie algebra $\mathfrak{su}(2)$ which in turn can be extended to the whole universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. The universal enveloping algebra $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{C}))$ is another example of a Hopf algebra if we define $\Delta(1) = 1 \otimes 1$, $\varepsilon(1) = 1$, $S(1) = 1$, and
\[
\Delta(X) = 1 \otimes X + X \otimes 1, \quad \varepsilon(X) = 0, \quad S(X) = -X, \quad X \in \mathfrak{sl}(2, \mathbb{C}),
\]
and by extending $\Delta$, $\varepsilon$ as homomorphisms and $S$ as an antihomomorphism. We can introduce a $*$-operator making $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{C}))$ into a Hopf $*$-algebra and such that the $*$-operator can be used to recover the real Lie algebra $\mathfrak{su}(2)$ from it. Here the $*$-operator is defined by $H^* = H$, $B^* = C$, $C^* = B$, satisfying $[X^*, Y^*] = [Y, X]^*$. The $*$-operator is chosen in such a way that the Lie algebra $\mathfrak{su}(2)$ consists of the $-1$-eigenspace of the $*$-operator when restricted to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

For $X \in \mathfrak{sl}(2, \mathbb{C})$ and $p \in P(SU(2))$ we have the pairing
\[
\langle X, p \rangle = \frac{d}{dt} \bigg|_{t=0} p(e^{tX}),
\]
by using the exponential function from the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ to the Lie group $SL(2, \mathbb{C})$. This pairing extends to a pairing of $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{C}))$ and $P(SU(2))$ and this pairing makes the Hopf $*$-algebras $\mathfrak{U}(\mathfrak{sl}(2, \mathbb{C}))$ and $P(SU(2))$ in duality as Hopf $*$-algebras, cf. §4.4. The relation
\[
t^l_{n,m}(k_\theta g k_\psi) = (t^l(g) t^l(k_\psi)e_n^l, t^l(k_\theta) e_n^l) = e^{-2im\psi - 2in\theta} t^l_{n,m}(g)
\]
for $k_\theta = diag(e^{i\theta}, e^{-i\theta}) \in K$, $g \in SU(2)$, can be rephrased as
\[
\langle id \otimes iH, \Delta(t^l_{n,m}) \rangle = -2im t^l_{n,m}, \quad \langle iH \otimes id, \Delta(t^l_{n,m}) \rangle = -2im t^l_{n,m},
\]
where we use the duality (2.5) in the second, respectively first factor of the tensor product. If, instead of $K$, we start with two, possibly different, one-parameter subgroups of $SU(2)$, such as $\exp tX$ and $\exp tY$ for $X, Y \in \mathfrak{su}(2)$, then we may try to find matrix elements $s^l_{n,m}$ such that
\[
\langle id \otimes X, \Delta(s^l_{n,m}) \rangle = -2im s^l_{n,m}, \quad \langle Y \otimes id, \Delta(s^l_{n,m}) \rangle = -2im s^l_{n,m},
\]
i.e. matrix elements which are relatively right (left) invariant with respect to the one-parameter subgroup $\exp tX$ ($\exp tY$). However, this does not lead to a different interpretations of Jacobi polynomials, since it is equivalent to an affine transformation of the argument of the Jacobi polynomials. The underlying reason for this is that all one-parameter subgroups in $SU(2)$ are conjugated.

In the quantum $SU(2)$ group case this approach leads to interpretations of different $q$-analogues of the Jacobi polynomials.
3. Preliminaries on basic hypergeometric orthogonal polynomials

In this section we collect various results on basic hypergeometric orthogonal polynomials that appear in this paper. Per subsection references to the literature are given. The reader acquainted with $q$-hypergeometric series is advised to browse through this section. The notation follows the book [18] by Gasper and Rahman. See Andrews [2] for more information concerning applications of $q$-series.

3.1. Basic hypergeometric series. ([18, Ch. 1]) We consider $q \in (0, 1)$ fixed. The $q$-shifted factorials are defined by $(a; q)_k = \prod_{i=0}^{k-1} (1 - a q^i)$ for $a \in \mathbb{C}$, $k \in \mathbb{Z}_+$. The empty product corresponding to $k = 0$ equals 1 by definition. Since the absolute value of $q$ is less than 1, the limit $k \to \infty$ of any $q$-shifted factorial $(a; q)_k$ exists. This infinite product is denoted by $(a; q)_\infty$. The following abbreviation for $q$-shifted factorials

$$(a_1, a_2, \ldots, a_r; q)_k = \prod_{p=1}^{r} (a_p; q)_k, \quad k \in \mathbb{Z}_+ \cup \{\infty\}$$

is standard. Occasionally we use $\frac{[n]}{[k]}_q = (q^n; q^{-1})_k / (q; q)_k$, $n, k \in \mathbb{Z}_+$, $k \leq n$, as a $q$-analogue of the binomial coefficient. Note that $(x; q)_n$ is a polynomial of degree $n$ in $x$ and that $(a e^{i\theta}, a^{-1} e^{-i\theta}; q)_n = \prod_{i=0}^{n} (1 - 2 a q^i \cos \theta + a^2 q^{2i})$ is a polynomial of degree $n$ in $\cos \theta$ for $a \neq 0$.

Define basic hypergeometric series, or $q$-hypergeometric series, with upper parameters $a_1, \ldots, a_{r+1}$ and lower parameters $b_1, \ldots, b_r$ ($r \in \mathbb{Z}_+$) of argument $z$ and base $q$ by

$$(3.1) \quad r+1 \varphi_r \left( \begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(b_1, \ldots, b_r, q; q)_k} z^k.$$

For generic values of the parameters the radius of convergence of this series is 1. In this paper the $q$-hypergeometric series (3.1) are terminating series. This is the case if the upper parameter $a_1 = q^{-n}$, $n \in \mathbb{Z}_+$, since $(q^{-n}; q)_k = 0$ for $k > n$. If one of the lower parameters is of the form $q^{-N}$ for $N \in \mathbb{Z}_+$, then we require that there is an upper parameter of the form $q^{-n}$ for $n \in \{0, \ldots, N\}$ in order to have a well-defined $q$-hypergeometric series. Here we follow the convention that a $r+1 \varphi_r$-series with $q^{-N}$ as both an upper and lower parameter is considered as a terminating series of degree $N$.

3.2. Askey-Wilson polynomials. ([7], [18, §7.5]) The Askey-Wilson polynomials are defined by

$$p_n(\cos \theta; a, b, c, d | q) = a^{-n} (ab, ac, ad; q)_n 4 \varphi_3 \left( q^{-n}, abcdq^{-n-1}, ae^{i\theta}, a^{-i\theta} \atop ab, ac, ad \right; q, q),$$

cf. Askey and Wilson [7, (1.15)]. The Askey-Wilson polynomial is symmetric in the parameters $a$, $b$, $c$ and $d$, cf. [7, p.6]. To stress the fact that Askey-Wilson polynomials generalise Jacobi polynomials (see the title of Askey-Wilson memoir [7]), we use the notation, cf. Noumi and Mimachi [40, (4.1)],

$$(3.2) \quad p_n^{(\alpha, \beta)}(x; s, t | q) = p_n(x; q^{1/2} t^s, q^{1/2+\alpha} s^t, q^{1/2}/(st), -stq^{1/2+\beta} | q),$$
which generalises Rahman’s notation \((s = t = 1)\) for continuous \(q\)-Jacobi polynomials, cf. [7, (4.17)]. With this notation we have, cf. (2.2),

\[
\lim_{q \uparrow 1} p_n^{(\alpha, \beta)}(x; s, t | q) = (s + s^{-1})^n (t + t^{-1})^n \frac{(\alpha + 1)_n}{n!} 
\times \binom{-n, n + \alpha + \beta + 1}{\alpha + 1} \binom{s/t - 2 \cos \theta + t/s}{(s + s^{-1})(t + t^{-1})}.
\]

### 3.3. Orthogonality relations for Askey-Wilson polynomials. ([7], [18, §7.5])

The orthogonality relations for the Askey-Wilson polynomials depend on the values of the parameters \(a, b, c\) and \(d\). First we introduce some notation;

\[
w\left(\frac{1}{2}(z + z^{-1})\right) = \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty},
\]

\[
h_n = \frac{(1 - q^{n-1}abcd)(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd)(abcd; q)_n} h_0,
\]

\[
h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},
\]

where we suppressed the dependence on \(a, b, c\) and \(d\) in the notation for \(w\) and \(h\). For \(z = e^{i\theta}\) we use \(w(\cos \theta)\).

**Proposition 3.1.** Let \(a, b, c\) and \(d\) be real and let all the pairwise products of \(a, b, c\) and \(d\) be less than 1. Then the Askey-Wilson polynomials \(p_n(x) = p_n(x; a, b, c, d | q)\) satisfy the orthogonality relations

\[
\frac{1}{2\pi} \int_0^\pi p_n(\cos \theta) p_m(\cos \theta) w(\cos \theta) d\theta + \sum_k p_n(x_k) p_m(x_k) w_k = \delta_{n,m} h_n.
\]

The points \(x_k\) are of the form \(\frac{1}{2}(eq^k + e^{-1}q^{-k})\) for \(e\) any of the parameters \(a, b, c\) or \(d\) with absolute value greater than 1; the sum is over \(k \in \mathbb{Z}_+\) such that \(|eq^k| > 1\) and \(w_k\) is the residue of \(z \mapsto w\left(\frac{1}{2}(z + z^{-1})\right)\) at \(z = eq^k\) minus the residue at \(z = e^{-1}q^{-k}\).

The orthogonality relations remain valid for complex parameters \(a, b, c\) and \(d\), if they occur in conjugate pairs. If all parameters have absolute value less than 1, the Askey-Wilson orthogonality measure is absolutely continuous.

We use the notation \(dm(x) = dm(x; a, b, c, d | q)\) for the normalised orthogonality measure. So for any polynomial \(p\)

\[
(3.3) \quad \int_R p(x) dm(x) = \frac{1}{h_0} \left(\frac{1}{2\pi} \int_{-1}^1 p(x) w(x) \frac{dx}{\sqrt{1 - x^2}} + \sum_k p(x_k) w_k\right).
\]
3.4. Continuous $q$-ultraspherical polynomials. ([5], [7, §4], [18, §7.4]) The continuous $q$-ultraspherical polynomials are special cases of the Askey-Wilson polynomials, but were already introduced by Rogers in 1894. They are defined by

$$C_n(\cos \theta; \beta \mid q) = \sum_{k=0}^{n} \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta}. \tag{3.4}$$

The orthogonality relation for the continuous $q$-ultraspherical polynomials is absolutely continuous for $-1 < \beta < 1$. Up to a normalisation factor the continuous $q$-ultraspherical polynomials correspond to Askey-Wilson polynomials with parameters $a = \sqrt{\beta}$, $b = \sqrt{q\beta}$, $c = -\sqrt{q\beta}$, $d = -\sqrt{\beta}$, cf. [7, §4]. There is also a relation between continuous $q$-ultraspherical polynomials of base $q^2$ and Askey-Wilson polynomials of base $q$ given by

$$C_n(\cos \theta; q^{1+2\alpha} \mid q^2) = \frac{(q^{1+2\alpha}; q)_n}{(q^{2+2\alpha}, q^{2}; q^2)_n} p_n^{(\alpha, \alpha)}(\cos \theta; 1, 1 \mid q), \tag{3.5}$$

cf. [7, (4.20), (4.2)], [18, (7.5.34)].

In case $\beta = 0$ we call $H_n(\cos \theta \mid q) = (q; q)_n C_n(\cos \theta; 0 \mid q)$ the continuous $q$-Hermite polynomials, cf. [5, §6]. The three-term recurrence relation for the continuous $q$-Hermite polynomials is

$$2xH_n(x \mid q) = H_{n+1}(x \mid q) + (1 - q^n)H_{n-1}(x \mid q). \tag{3.6}$$

The corresponding orthogonality relation can be obtained from proposition 3.1 and we get

$$\int_0^\pi H_n(\cos \theta \mid q)H_m(\cos \theta \mid q)(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta = \delta_{n,m} \frac{2\pi (q; q)_n}{(q; q)_\infty}. \tag{3.7}$$

3.5. Little $q$-Jacobi polynomials. ([3], [18, §7.3]) The little $q$-Jacobi polynomials

$$p_n(x; a, b; q) = \varphi_1\left(q^{-n}, abq^{n-1}; aq, qx \right) \tag{3.8}$$

are $q$-analogues of the Jacobi polynomials. The orthogonality relations are

$$\sum_{x=0}^{\infty} \frac{(bq; q)_x}{(q; q)_x} (aq)^x p_n(q^x; a, b; q)p_m(q^x; a, b; q) = 0, \quad n \neq m, \tag{3.9}$$

cf. Andrews and Askey [3, thm. 9]. The little $q$-Jacobi polynomials can be obtained as a limit case of the Askey-Wilson polynomials of §3.2, cf. Koornwinder [33, prop. 6.3] and the orthogonality relations, cf. proposition 3.1, go over into the orthogonality relations (3.9) for the little $q$-Jacobi polynomials.
3.6. \( q \)-Hahn polynomials. ([6], [18, §7.2]) The \( q \)-Hahn polynomials are defined by, cf. [18, (7.2.21)],

\[
Q_n(x) = Q_n(x; a, b, N; q) = 3\varphi_2 \left( \begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{array} ; q, q \right)
\]

for \( N \in \mathbb{Z}_+ \) and \( n \in \{0, 1, \ldots, N\} \). These polynomials are orthogonal with respect to a finite discrete measure;

\[
\sum_{x=0}^{N} Q_n(q^{-x})Q_m(q^{-x})(aq; q)_{x}(bq; q)_{N-x}(aq)^{-x} = 0, \quad n \neq m.
\]

The \( q \)-Hahn polynomials are special cases of the more general orthogonal polynomials, the so-called \( q \)-Racah polynomials introduced by Askey and Wilson [6].

3.7. (Dual) \( q \)-Krawtchouk polynomials. ([48], [6]) The \( q \)-Krawtchouk are defined by, cf. [48],

\[
K_n(x; q^\sigma, N; q) = 3\varphi_2 \left( \begin{array}{c} q^{-n}, x, -q^{-n-N-\sigma} \\ q^{-N}, 0 \end{array} ; q, q \right)
\]

and these polynomials are orthogonal with respect to a finite discrete measure;

\[
\sum_{x=0}^{N} (K_nK_m)(q^{-x}; q^\sigma, N; q)w_x(q^\sigma, N) = \delta_{n,m} (\alpha_n(q^\sigma, N))^{-1}
\]

with

\[
w_x(q^\sigma, N) = (-q^{N+\sigma})^{x} \left( \frac{q^{-N}; q}{q; q} \right),
\]

\[
\alpha_n(q^\sigma, N) = \frac{1 + q^{2n-N-\sigma}}{(-q^{n-2N-\sigma})} \left( \frac{q^{-N-\sigma}, q^{-N}; q}{q, -q^{1-\sigma}; q} \right)^n.
\]

The dual \( q \)-Krawtchouk polynomials are obtained from the \( q \)-Krawtchouk polynomials by interchanging the roles of the degree \( n \) and argument \( q^{-x} \),

\[
R_n(q^{-x} - q^{-x-N-\sigma}; q^\sigma, N; q) = 3\varphi_2 \left( \begin{array}{c} q^{-n}, q^{-x}, -q^{x-N-\sigma} \\ q^{-N}, 0 \end{array} ; q, q \right) = K_x(q^{-n}; q^\sigma, N; q).
\]

The orthogonality relations for the dual \( q \)-Krawtchouk polynomials follow from (3.12);

\[
\sum_{x=0}^{N} (R_nR_m)(q^{-x} - q^{-x-N-\sigma}; q^\sigma, N; q)h_x(q^\sigma, N) = \delta_{n,m} (\alpha_n(q^\sigma, N))^{-1}
\]

with \( w_x(q^\sigma, N) \) and \( h_x(q^\sigma, N) \) defined as above. Again both the \( q \)-Krawtchouk and dual \( q \)-Krawtchouk polynomials are special cases of the \( q \)-Racah polynomials [6].
4. Preliminaries on the quantum $SU(2)$ group

The necessary tools and definitions concerning the quantum $SU(2)$ group are given in this section. There is no notational standard on this subject. References to the literature are given per subsection.

4.1. Hopf $*$-algebras. ([1], [49], [54]) A complex associative algebra $A$ with unit 1 and multiplication $m: A \otimes A \to A$, $m: a \otimes b \mapsto ab$ and unit $e: \mathbb{C} \to A$, $e: z \mapsto z1$ is a bialgebra if there exist algebra homomorphisms $\Delta: A \to A \otimes A$, the comultiplication, and $\epsilon: A \to \mathbb{C}$, the counit, satisfying the coassociativity axiom $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ and the counit axiom $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$. A bialgebra $A$ is a Hopf algebra if there exists a linear map $S: A \to A$, the antipode, satisfying the antipode axiom $m \circ (S \otimes id) \circ \Delta = e \circ \epsilon = m \circ (id \otimes S) \circ \Delta$. The antipode $S$ in the Hopf algebra $A$ is unique. Moreover, it satisfies $S(1) = 1$, $\epsilon \circ S = \epsilon$, $S(ab) = S(b)S(a)$ and $\Delta \circ S = \omega \circ (S \otimes S) \circ \Delta$, where $\omega$ denotes the flip automorphism of $A \otimes A$, $\omega(a \otimes b) = b \otimes a$.

A $*$-operator on an algebra $A$ is an antilinear, antimultiplicative involution, i.e. $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(ab)^* = b^*a^*$, $(a^*)^* = a$ for $\lambda, \mu \in \mathbb{C}$, $a, b \in A$. A Hopf algebra $A$ with a $*$-operator is a Hopf $*$-algebra if $(\ast \otimes \ast) \circ \Delta = \Delta \circ \ast$, $\epsilon(a^*) = \bar{\epsilon}(a)$ for all $a \in A$, and then it is possible to prove that $S \circ \ast \circ S \circ \ast = id$.

4.2. Quantised algebra of polynomials on $SU(2)$. ([58], [59], [31]) Fix $q \in (0, 1)$. $A_q(SU(2))$ is the complex unital associative algebra generated by $\alpha, \beta, \gamma, \delta$ subject to the relations

$$
\begin{align*}
\alpha \beta &= q\beta \alpha, & \alpha \gamma &= q\gamma \alpha, & \beta \delta &= q\delta \beta, & \gamma \delta &= q\delta \gamma, \\
\beta \gamma &= \gamma \beta, & \alpha \delta - q\beta \gamma &= \delta \alpha - q^{-1}\beta \gamma = 1.
\end{align*}
$$

The algebra $A_q(SU(2))$ is an example of a Hopf $*$-algebra. The comultiplication $\Delta$, the counit $\epsilon$, the antipode $S$ and the $*$-operator are given on the generators by

$$
\begin{align*}
\Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\
\Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \gamma \otimes \beta + \delta \otimes \delta, \\
\epsilon \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & S \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} &= \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}, \\
\alpha^* &= \delta, & \beta^* &= -q\gamma, & \gamma^* &= -q^{-1}\beta, & \delta^* &= a.
\end{align*}
$$

It is possible to identify $A_1(SU(2))$ with the polynomial algebra $P(SU(2))$, cf. §2, on the Lie group $SU(2)$.

4.3. Quantised universal enveloping algebra. ([20], [31]) The quantised universal enveloping algebra $U_q(\mathfrak{su}(2))$ is the complex unital associative algebra generated by $A, B, C, D$ subject to the relations

$$
\begin{align*}
AD &= 1 = DA, & AB &= qBA, & AC &= q^{-1}CA, & BC - CB &= \frac{A^2 - D^2}{q - q^{-1}}.
\end{align*}
$$
The algebra $U_q(\mathfrak{su}(2))$ is also a Hopf $*$-algebra. The comultiplication, counit, antipode and $*$-operator are defined on the generators by

$$\Delta(A) = A \otimes A, \quad \Delta(B) = A \otimes B + B \otimes D,$$

$$\Delta(C) = A \otimes C + C \otimes D, \quad \Delta(D) = D \otimes D,$$

$$\varepsilon(A) = \varepsilon(D) = 1, \quad \varepsilon(C) = \varepsilon(B) = 0,$$

$$S(A) = D, \quad S(B) = -q^{-1}B, \quad S(C) = -qC, \quad S(D) = A,$$

$$A^* = A, \quad B^* = C, \quad C^* = B, \quad D^* = D.$$

The element

$$\Omega = \frac{q^{-1}A^2 + qD^2 - 2}{(q^{-1} - q)^2} + BC = \frac{qA^2 + q^{-1}D^2 - 2}{(q^{-1} - q)^2} + CB$$

is the Casimir element of the quantised universal enveloping algebra $U_q(\mathfrak{su}(2))$. The Casimir element is self-adjoint, $\Omega = \Omega^*$, and $\Omega$ belongs to the centre of $U_q(\mathfrak{su}(2))$.

### 4.4. Duality for Hopf $*$-algebras. ([54], [35])

The Hopf $*$-algebras are in duality as Hopf $*$-algebras if there exists a doubly non-degenerate pairing $\langle \cdot, \cdot \rangle: U_q(\mathfrak{su}(2)) \times A_q(SU(2)) \to \mathbb{C}$, i.e. $\langle X, \xi \rangle = 0$ for all $X \in U_q(\mathfrak{su}(2))$, respectively for all $\xi \in A_q(SU(2))$, implies $\xi = 0$, respectively $X = 0$, such that

$$\langle XY, \xi \rangle = \langle X \otimes Y, \Delta(\xi) \rangle, \quad \langle X, \xi \eta \rangle = \langle \Delta(X), \xi \otimes \eta \rangle,$$

$$\langle 1, \xi \rangle = \varepsilon(\xi), \quad \langle X, 1 \rangle = \varepsilon(X),$$

$$\langle S(X), \xi \rangle = \langle X, S(\xi) \rangle, \quad \langle X^*, \xi \rangle = \overline{\langle X, S(\xi)^* \rangle}.$$  

The duality can be used to define a left and right action of $U_q(\mathfrak{su}(2))$ on $A_q(SU(2))$. For $X \in U_q(\mathfrak{su}(2))$ and $\xi \in A_q(SU(2))$ we define elements $X.\xi$ and $\xi.X$ of $A_q(SU(2))$ by

$$X.\xi = (id \otimes X) \circ \Delta(\xi), \quad \xi.X = (X \otimes id) \circ \Delta(\xi),$$

where the pairing between $A_q(SU(2))$ and $U_q(\mathfrak{su}(2))$ is used in the second, respectively the first, part of the tensor product. Using the duality we can rewrite (4.13) as

$$\langle Y, X.\xi \rangle = \langle YX, \xi \rangle, \quad \langle Y, \xi.X \rangle = \langle XY, \xi \rangle.$$  

If $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$, then (4.10) implies

$$X.\xi \eta = \sum_{(X)} (X_{(1)}, \xi) (X_{(2)}, \eta), \quad (\xi \eta).X = \sum_{(X)} (\xi.X_{(1)}) (\eta.X_{(2)}).$$

Using (4.13) and (4.12) we prove

$$X.\xi^* = \sum_{(\xi)} \xi_{(1)}^* (X, \xi_{(2)}^*) = \left( \sum_{(\xi)} \xi_{(1)} \langle S(X)^*, \xi_{(2)} \rangle \right)^* = \langle S(X)^*, \xi \rangle^*.$$
The explicit duality between $\mathcal{A}_q(SU(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$ is given by
\[
\langle A, (\alpha \beta \gamma \delta) \rangle = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \langle B, (\alpha \beta \gamma \delta) \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
(4.17)
\[
\langle C, (\alpha \beta \gamma \delta) \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \langle D, (\alpha \beta \gamma \delta) \rangle = \begin{pmatrix} 0 & q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}.
\]

The duality between the Hopf $*$-algebras $\mathcal{A}_q(SU(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$ is completely determined by (4.17) and (4.10).

4.5. Representation theory. ([58], [52], [37], [30], [31]) A square matrix $t = (t_{n,m})$ of elements of the Hopf $*$-algebra $\mathcal{A}_q(SU(2))$ is called a unitary matrix corepresentation if
\[
\Delta(t_{n,m}) = \sum_k t_{n,k} \otimes t_{k,m}, \quad \varepsilon(t_{n,m}) = \delta_{n,m}, \quad S(t_{n,m}) = t^*_m,n.
\]
(4.18)

Dropping the prefix unitary means dropping the last requirement of (4.18). Using the duality a unitary matrix corepresentation of $\mathcal{A}_q(SU(2))$ gives a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$ by $(t(X))_{n,m} = \langle X, t_{n,m} \rangle$, for $X \in \mathcal{U}_q(\mathfrak{su}(2))$. There is precisely one irreducible unitary corepresentation of $\mathcal{A}_q(SU(2))$ in each finite dimension (up to equivalence). The corresponding $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$ can be realised on a $(2l+1)$-dimensional $(l \in \frac{1}{2}\mathbb{Z}_+)$ vector space with orthonormal basis $\{e^l_n\}$, $n = -l, -l+1, \ldots, l$ as the matrix representation $t^l = (t_{n,m}^l)_{n,m=-l,\ldots,l}$. The action of the generators is given by
\[
t^l(A)e^l_n = q^{-n}e^l_n, \quad t^l(D)e^l_n = q^ne^l_n,
\]
\[
t^l(B)e^l_n = \frac{\sqrt{(q^{-l+n-1} - q^{-l-1}) (q^{-l-n} - q^{l+n})}}{q^{-1} - q} e^l_{n-1}
\]
\[
t^l(C)e^l_n = \frac{\sqrt{(q^{-l+n} - q^{l-n}) (q^{-l+n-1} - q^{l+n+1})}}{q^{-1} - q} e^l_{n+1},
\]
(4.19)

where $e^l_{l+1} = 0 = e^l_{-l-1}$. The action of the Casimir follows from (4.9) and (4.19). Explicitly, $t^l(\Omega) = q^{1-2l}(1 - q^{2l+1} \frac{1}{2} (1 - q^2)^2}$.

The matrix elements $t^l_{n,m} \in \mathcal{A}_q(SU(2))$ are explicitly known in terms of the generators $\alpha$, $\beta$, $\gamma$ and $\delta$, Vaksman and Soibelman [52, prop. 6.6], Masuda et al. [37, thm. 2.8], Koornwinder [30, thm. 5.3]. These expressions involve little $q$-Jacobi polynomials;
\[
t^l_{n,m} = c^l_{n,m} x^{n+m} \gamma^{n-m} p_{l-k} (-q^{-1} \beta \gamma), \quad (n \geq m \geq -n),
\]
\[
t^l_{n,m} = c^l_{n,m} x^{n+m} \beta^{m-n} p_{l-k} (-q^{-1} \beta \gamma), \quad (m \geq n \geq -m),
\]
\[
t^l_{n,m} = c^l_{n,-m} x^{n-m} \alpha^{-m-n} p_{l-k} (-q^{2m+2n-1} \beta \gamma), \quad (-n \geq m \geq n),
\]
\[
t^l_{n,m} = c^l_{-n,m} x^{-m-n} \alpha^{m-n} p_{l-k} (-q^{2m+2n-1} \beta \gamma), \quad (-m \geq n \geq m),
\]
with $p_{l-k}(x) = p_{l-k}(x; q^{-2|n-m|}; q^{2|n+m|}; q^2)$, $k = \max(|n|, |m|)$, a little $q$-Jacobi polynomial, cf. (3.8), and the constant is given by
\[
c^l_{n,m} = \left[ \frac{l-m}{n-m} \right]^{1/2} \left[ \frac{l+n}{n-m} \right]^{1/2} q^{-(n-m)(l-n)}.
\]
4.6. Representations of the Hopf $*$-algebra $A_q(SU(2))$. ([52], [58], [59]) The irreducible $*$-representations of the Hopf $*$-algebra $A_q(SU(2))$ have been completely classified, cf. Vaksman and Soibelman [52, thm. 3.2]. The one-dimensional $*$-representations $\pi_\theta$ is defined by
\begin{equation}
\pi_\theta(\alpha) = e^{i\theta}, \quad \pi_\theta(\beta) = 0 = \pi_\theta(\gamma), \quad \pi_\theta(\delta) = e^{-i\theta}.
\end{equation}
Then $\pi_\theta(t^l_{n,m}) = \delta_{n,m}e^{-2ni\theta}$. For $\lambda \neq 0$ we also have one-dimensional representations of $A_q(SU(2))$ given by
\begin{equation}
\tau_\lambda(\alpha) = \lambda, \quad \tau_\lambda(\beta) = 0 = \tau_\lambda(\gamma), \quad \tau_\lambda(\delta) = \lambda^{-1}.
\end{equation}
Note that $\tau_\lambda$ is a $*$-representation of $A_q(SU(2))$ if and only if $\lambda = e^{i\theta}$, or $\tau_\lambda = \pi_\theta$, for some $\theta \in [0, 2\pi)$. Now $\tau_\lambda(t^l_{n,m}) = \delta_{n,m}\lambda^{-2n}$. The counit $\varepsilon$ coincides with the special case $\tau_1 = \pi_0$.

Infinite dimensional $*$-representations of the Hopf $*$-algebra $A_q(SU(2))$ act in the Hilbert space $\ell^2(\mathbb{Z}_+)$. For an orthonormal basis $\{f_n\}_{n \in \mathbb{Z}_+}$ the action of the generators is given by
\begin{equation}
\pi^\infty_\theta(\alpha)f_n = \sqrt{1-q^{2n}}f_{n-1}, \quad \pi^\infty_\theta(\gamma)f_n = e^{i\theta}q^n f_n,
\end{equation}
with the convention $f_{-1} = 0$. The operators corresponding to $\beta$ and $\delta$ follow by (4.4)

4.7. Peter-Weyl theorem and Clebsch-Gordan series. ([58], [59], [29], [31]) The irreducible unitary matrix corepresentations $t^l$ of $A_q(SU(2))$ exhaust the set of irreducible unitary corepresentations of $A_q(SU(2))$ up to equivalence. The Peter-Weyl theorem for the quantum $SU(2)$ group states that the matrix elements $t^l_{n,m}$, $n,m=-l,-l+1,\ldots,l$, $l \in \frac{1}{2}\mathbb{Z}_+$, form a basis for the underlying linear space of $A_q(SU(2))$. If we denote by $A^l_q(SU(2))$ the span of the matrix elements $t^l_{n,m}$, $n,m=-l,-l+1,\ldots,l$, then we have the decomposition
\begin{equation}
A_q(SU(2)) = \bigoplus_{l \in \frac{1}{2}\mathbb{Z}_+} A^l_q(SU(2)).
\end{equation}
The tensor product of two irreducible matrix corepresentations $t^{l_1} \otimes t^{l_2}$ is the matrix corepresentation with matrix elements defined by $(t^{l_1} \otimes t^{l_2})_{i,n;j,m} = t^{l_1}_{i,j}t^{l_2}_{n,m}$ for $i,j = -l_1,\ldots,l_1$; $n,m = -l_2,\ldots,l_2$. Using the duality this can be defined equivalently as the $*$-representation of $U_q(\mathfrak{su}(2))$ given by $X \mapsto (t^{l_1} \otimes t^{l_2})\Delta(X)$. The Clebsch-Gordan series for the quantum $SU(2)$ group states that
\begin{equation}
t^{l_1} \otimes t^{l_2} = \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} t^l,
\end{equation}
or in terms of the decomposition (4.22),
\begin{equation}
A^{l_1}_q(SU(2)) \cdot A^{l_2}_q(SU(2)) = \bigoplus_{l=|l_1-l_2|}^{l_1+l_2} A^l_q(SU(2)).
\end{equation}
4.8. Haar functional and Schur orthogonality relations. ([58], [31]) On $A_q(SU(2))$ there exists a functional which is an analogue of the invariant integration on $SU(2)$. This functional, $h: A_q(SU(2)) \rightarrow \mathbb{C}$, called the Haar functional, is uniquely determined by the properties

(i) $h(1) = 1$,
(ii) $h(\xi^* \xi) \geq 0$ for all $\xi \in A_q(SU(2))$,
(iii) $(h \otimes id) (\Delta(a)) = h(a) 1 = (id \otimes h)(\Delta(a))$.

Condition (iii) is the left and right invariance of the Haar functional.

The Schur orthogonality for the matrix elements of the irreducible unitary matrix corepresentations of $A_q(SU(2))$ can be phrased as

$$h((t_{i,j}^l)^* t_{n,m}^k) = \delta_{l,k} \delta_{n,i} \delta_{m,j} q^{2(l-i)} \frac{1 - q^2}{1 - q^{d+2}}. \tag{4.24}$$

5. Koornwinder’s $(\sigma, \tau)$-spherical elements

In this section we treat the spherical functions on the quantum $SU(2)$ group introduced by Koornwinder [33] using the concept of infinitesimal invariance. These results are the basis for the further interpretation of the Askey-Wilson polynomials as generalised matrix elements.

To discover which elements in the quantised universal enveloping algebra $U_q(sl(2))$ play the role of Lie algebra like elements we recall that in the universal enveloping algebra $U(\mathfrak{sl}(2, \mathbb{C}))$ the Lie algebra elements satisfy, cf. (2.4), $\Delta(X) = 1 \otimes X + X \otimes S(1)$ for $X \in \mathfrak{sl}(2, \mathbb{C})$, where 1 satisfies $\Delta(1) = 1 \otimes 1$. The elements in $U_q(sl(2))$ satisfying $\Delta(X) = X \otimes X$, which are the so-called group-like elements, are precisely $A^n, n \in \mathbb{Z}$, cf. Masuda et al. [35, lemma 1(i)]. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is three-dimensional over $\mathbb{C}$ and the elements in $U_q(sl(2))$ satisfying $\Delta(X) = A^n \otimes X + X \otimes S(A^n)$ span a three-dimensional space if and only if $n = 1$, cf. [33, lemma 3.1], [43, §1]. Moreover, the elements satisfying $\Delta(X) = A \otimes X + X \otimes D$ are in the linear span of $B$, $C$ and $A - D$.

For $\sigma \in \mathbb{R}$ we define

$$X_\sigma = iq^{\frac{\sigma}{2}} B - iq^{-\frac{\sigma}{2}} C - \frac{q^\sigma - q^{-\sigma}}{q - q^{-1}} (A - D) \in U_q(sl(2)). \tag{5.1}$$

We also define

$$X_\infty = D - A = \lim_{\sigma \to \infty} (q^{-1} - q)^\sigma X_\sigma = \lim_{\sigma \to -\infty} (q - q^{-1}) q^{-\sigma} X_\sigma. \tag{5.2}$$

Using §4.3 it is easy to check that

$$\Delta(X_\sigma) = A \otimes X_\sigma + X_\sigma \otimes D, \quad S(X_\sigma) = -X_\sigma, \quad (X_\sigma A)^* = X_\sigma A. \tag{5.3}$$

An element $\xi \in A_q(SU(2))$ is a $(\sigma, \tau)$-spherical element if

$$X_\sigma \xi = 0, \quad \text{and} \quad \xi X_\tau = 0. \tag{5.4}$$
Theorem 5.1. ([33, prop. 4.7, thm. 5.3]) The space of \((\sigma, \tau)\)-spherical elements in the Hopf *-algebra \(A_q(SU(2))\) is a commutative algebra generated by the element

\[
(5.5) \quad \rho_{\tau, \sigma} = \frac{1}{2} \left( \alpha^2 + \delta^2 + q^{\gamma^2} + q^{-1} \beta^2 + i(q^{-\tau} - q^\tau)(q^\delta \gamma + \beta \alpha) - i(q^{-\tau} - q^\tau)(\delta \beta + q^\gamma \alpha) + (q^{-\sigma} - q^\sigma)(q^{-\tau} - q^\tau)\beta \gamma \right)
\]

satisfying \(\rho^{*}_{\tau, \sigma} = \rho_{\tau, \sigma}, \rho_{\tau, \sigma} \in A^1(SU(2))\). The Haar functional on the subalgebra of \((\sigma, \tau)\)-spherical elements is given by

\[
(5.6) \quad h(p(\rho_{\tau, \sigma})) = \int_{\mathbb{R}} p(x) \, dm(x; a, b, c, d \mid q^2)
\]

for any polynomial \(p\), where \(a = -q^{\sigma+\tau+1}, b = -q^{-\sigma-\tau+1}, c = q^{\sigma-\tau+1}, d = q^{-\sigma+\tau+1}\) and \(dm(x; a, b, c, d \mid q)\) denotes the normalised Askey-Wilson measure, cf. (3.3).

Since theorem 5.1 is an important tool in our proof of theorem 7.5, we give the idea of the proof of theorem 5.1. We start with the observation that the \((\sigma, \tau)\)-spherical elements form a subalgebra of \(A_q(SU(2))\), cf. proposition 6.4(i). Using the Peter-Weyl theorem it is possible to prove that the \((\sigma, \tau)\)-spherical elements in \(A^l_q(SU(2))\) are completely characterised by the kernels of the operators \(t^l(X_\sigma)\) and \(t^l(X_\tau^*)\), cf. the proof of proposition 6.4(ii). Koornwinder explicitly calculates the simplest non-trivial \((\sigma, \tau)\)-spherical element \(\rho_{\tau, \sigma}\) by considering the kernels of the operators \(t^1(X_\sigma)\) and \(t^1(X_\tau^*)\).

By an analysis of the spectrum of the operators \(t^l(X_\sigma)\) and \(t^l(X_\tau^*)\), which is essentially rephrased in proposition 5.2, it is shown that \(A^l_q(SU(2))\), \(l \in \frac{1}{2} + \mathbb{Z}_+\), doesn’t contain \((\sigma, \tau)\)-spherical elements and that \(A^l_q(SU(2))\), \(l \in \mathbb{Z}_+\), contains a unique, up to a scalar, \((\sigma, \tau)\)-spherical element, which has to be a polynomial in \(\rho_{\tau, \sigma}\). This polynomial is identified with an Askey-Wilson polynomial with parameters as in theorem 5.1 by identifying the action of the Casimir operator of \(\mathcal{U}_q(\mathfrak{su}(2))\) with the second order \(q\)-difference operator for the Askey-Wilson polynomials.

This is done as follows. For this polynomial \(p_l(\rho_{\tau, \sigma}) \in A^l_q(SU(2))\) we have for \(k \in \mathbb{Z}\) by §4.5 \(A^k \Omega, p_l(\rho_{\tau, \sigma})) = q^{1-2l}(1-q^{2l+1})^2(1-q^2)^{-2}p_l(\frac{1}{2}(q^k + q^{-k}))\), since testing against \(A^k\) is an algebra homomorphism. By rewriting \(A^k \Omega\) as a linear combination of \(A^{k+2}, A^k\) and \(A^{k-2}\) modulo \(\mathcal{U}_q(\mathfrak{su}(2))X_\sigma + X_\tau \mathcal{U}_q(\mathfrak{su}(2))\) we see that this expression also equals a linear combination of \(p_l(\frac{1}{2}(q^m + q^{-m}))\), \(m = k + 2, k, k - 2\). This gives precisely the second order \(q\)-difference equation [7, (5.7)] for the Askey-Wilson polynomials with the parameters as in theorem 5.1, which has a unique polynomial solution.

Equation (5.6) follows by identifying the Haar functional on the subalgebra of \((\sigma, \tau)\)-spherical elements with the Askey-Wilson orthogonality measure via the Schur orthogonality relations (4.24).

Proposition 5.2. ([33, thm. 4.3, lemma 4.4]) The self-adjoint operator \(t^l(X_\sigma A)\) has an orthonormal basis of eigenvectors \(v^{l,j}(\sigma) = \sum_{n=-l}^l v^{l,j}_n(\sigma) e^l_n\) corresponding to the eigenvalue

\[
\lambda_j(\sigma) = \frac{q^{-2j-\sigma} - q^{\sigma+2j} + q^{\sigma} - q^{-\sigma}}{q - q^{-1}}, \quad j = -l, -l + 1, \ldots, l.
\]
orthogonality relations for the dual $q$-Krawtchouk polynomials, and the second part of (5.7) is equivalent to the orthogonality relations

$$R_l n \quad \text{is a dual } q\text{-Krawtchouk polynomial, cf. (3.13), and the constant is given by}$$

$$C^{l,j}(\sigma) = q^{l+j} \left[ \frac{2l}{l-j} \right]^{1/2} \left( \frac{1 + q^{-4j-2\sigma}}{1 + q^{-2\sigma}} \right)^{1/2} \left( (-q^{-2\sigma}; q^2)_l (-q^{2+2\sigma}; q^2)_{l+j} \right)^{-1/2}.$$

To sketch the proof of this proposition we observe that the coefficients of an eigenvector $\sum c_n e_n^l$ of $t^l(X_{\sigma} A)$ satisfy a three-term recurrence relation by (5.1) and (4.19), which can be identified with the three-term recurrence relation for the dual $q$-Krawtchouk polynomials.

Since the vectors $v^{l,j}(\sigma)$ are orthonormal in a finite dimensional space, we have the orthogonality relations

$$(5.7) \quad \sum_{n=-l}^{l} v_{n}^{l,j}(\sigma) \overline{v_{n}^{l,j}(\sigma)} = \delta_{i,j}, \quad \sum_{i=-l}^{l} v_{m}^{l,i}(\sigma) \overline{v_{n}^{l,i}(\sigma)} = \delta_{n,m}.$$

The first part of (5.7) is equivalent to the orthogonality relations (3.12) for the $q$-Krawtchouk polynomials, and the second part of (5.7) is equivalent to the orthogonality relations (3.14) for the dual $q$-Krawtchouk polynomials.

The limit case $\sigma \to \infty$ of $t^l(X_{\sigma} A)v^{l,j}(\sigma) = \lambda_j(\sigma)v^{l,j}(\sigma)$ reduces to $t^l(1 - A^2)e_j^l = (1 - q^{-2j})e_j^l$, cf. (5.2), (4.19), because $q^\sigma(q^{-1} - q)\lambda_j(\sigma) \to (1 - q^{-2j})$ and $v_{n}^{l,j}(\sigma) \to \delta_{l,j}$ as $\sigma \to \infty$. The last limit can be established using transformation and summation formulas for basic hypergeometric series.

6. Generalised matrix elements

In this section we introduce the generalised matrix elements and we derive various properties of these elements by employing the Hopf $\ast$-algebra duality between the quantised polynomial algebra and the quantised universal enveloping algebra. These properties, especially proposition 6.4, will be crucial in introducing orthogonal polynomials in relation with these generalised matrix elements.

The vectors $v^{l,j}(\sigma), j = -l, -l + 1, \ldots, l$, of proposition 5.2 give an orthonormal basis for the representation space of the $\ast$-representation $t^l$ of $U_q(\mathfrak{su}(2))$. We define linear functionals on $U_q(\mathfrak{su}(2))$ by

$$(6.1) \quad a_{i,j}^l(\tau, \sigma)(X) = \langle t^l(X)v^{l,j}(\sigma), v^{l,i}(\tau) \rangle, \quad \sigma, \tau \in \mathbb{R}, \ i, j = -l, -l + 1, \ldots, l,$$

where the inner product on the right hand side of (6.1) is the inner product of the representation space of $t^l$. It follows from proposition 5.2 and §4.5 that

$$(6.2) \quad a_{i,j}^l(\tau, \sigma) = \sum_{n,m=-l}^{l} v_{m}^{l,i}(\sigma) \overline{v_{n}^{l,j}(\tau)} t_{n,m}^l \in \mathcal{A}_q(SU(2)),$$
so that we can write \( a^l_{i,j}(\tau, \sigma)(X) = \langle X, a^l_{i,j}(\tau, \sigma) \rangle \). Note that \( a^l_{n,m}(\infty, \infty) = i^{n-m}l^l_{n,m} \).

So the \( a^l_{i,j}(\tau, \sigma) \) are elements of the Hopf *-algebra \( A_q(SU(2)) \) and we can determine the action of the comultiplication, antipode and *-operator on such an element.

**Proposition 6.1.** The elements \( a^l_{i,j}(\tau, \sigma), \sigma, \tau \in \mathbb{R}, i, j = -l, -l + 1, \ldots, l, l \in \frac{1}{2}\mathbb{Z}_+ \), defined in (6.1), satisfy

\[
\Delta(a^l_{i,j}(\tau, \sigma)) = \sum_{p=-l}^l a^l_{i,p}(\tau, \mu) \otimes a^l_{p,j}(\mu, \sigma), \quad \forall \mu \in \mathbb{R},
\]

\[
(a^l_{i,j}(\tau, \sigma))^* = S(a^l_{j,i}(\sigma, \tau)), \quad \varepsilon(a^l_{i,j}(\tau, \sigma)) = \langle v^{l,j}(\sigma), v^{l,i}(\tau) \rangle.
\]

**Proof.** These statements are proved by testing against appropriate elements. Firstly,

\[
\langle X \otimes Y, \Delta(a^l_{i,j}(\tau, \sigma)) \rangle = \langle XY, a^l_{i,j}(\tau, \sigma) \rangle = \langle t^l(X)t^l(Y)v^{l,j}(\sigma), v^{l,i}(\tau) \rangle
\]

\[
= \sum_{p=-l}^l \langle t^l(X)v^{l,p}(\mu), v^{l,i}(\tau) \rangle \langle t^l(Y)v^{l,j}(\sigma), t^l(p)(\mu) \rangle
\]

\[
= \sum_{p=-l}^l \langle X, a^l_{i,p}(\tau, \mu) \rangle \langle Y, a^l_{p,j}(\mu, \sigma) \rangle, \quad \forall X, Y \in U_q(\mathfrak{su}(2)),
\]

by developing \( t^l(Y)v^{l,j}(\sigma) \) in the basis \( \{v^{l,p}(\mu)\}_{p=-l, \ldots, l} \), which proves (6.3). The first statement of (6.4) follows from

\[
\langle X, (a^l_{i,j}(\tau, \sigma))^* \rangle = \overline{\langle S(X)^*, (a^l_{i,j}(\tau, \sigma)) \rangle} = \langle t^l(S(X))^*v^{l,j}(\sigma), v^{l,i}(\tau) \rangle
\]

\[
= \langle t^l(S(X))v^{l,i}(\tau), v^{l,j}(\sigma) \rangle = \langle S(X), a^l_{j,i}(\sigma, \tau) \rangle = \langle X, S(a^l_{j,i}(\sigma, \tau)) \rangle
\]

for arbitrary \( X \in U_q(\mathfrak{su}(2)) \). Finally,

\[
\varepsilon(a^l_{i,j}(\tau, \sigma)) = \langle 1, a^l_{i,j}(\tau, \sigma) \rangle = \langle v^{l,j}(\sigma), v^{l,i}(\tau) \rangle
\]

proves the last statement. \( \square \)

Proposition 6.1 shows that by taking \( \sigma = \tau = \mu \) we obtain an irreducible unitary matrix corepresentation \( a^l(\sigma) = (a^l_{i,j}(\sigma, \sigma))_{i,j} \). This motivates to call \( a^l_{i,j}(\tau, \sigma) \) generalised matrix elements. Of course, \( a^l(\sigma) \) is equivalent to \( t^l \) by §4.5, so we can express the matrix elements in terms of each other, cf. (6.2). But a more general expression holds, which is given in part (ii) of the following corollary. Corollary 6.2 also gives unitarity properties of the generalised matrix elements.

**Corollary 6.2.** (i) The matrix \( a^l(\sigma) = (a^l_{i,j}(\sigma, \sigma))_{i,j=-l, \ldots, l}, l \in \frac{1}{2}\mathbb{Z}_+ \), is an irreducible \((2l + 1)\)-dimensional unitary matrix corepresentation of the Hopf *-algebra \( A_q(SU(2)) \).
(ii) The following connection between generalised matrix elements holds:

\[ a_{i,j}^l(\tau, \sigma) = \sum_{n,m=-l}^{l} \langle v^{l,n}(\rho), v^{l,i}(\tau) \rangle \langle v^{l,j}(\sigma), v^{l,m}(\mu) \rangle a_{n,m}^l(\rho, \mu). \]

(iii) The generalised matrix elements satisfy the ‘unitarity property’

\[ \sum_{p=-l}^{l} a_{i,p}^l(\tau, \mu) (a_{j,p}^l(\sigma, \mu))^* = \langle v^{l,j}(\sigma), v^{l,i}(\tau) \rangle = \sum_{p=-l}^{l} (a_{p,i}^l(\mu, \tau))^* a_{p,j}^l(\mu, \sigma). \]

Proof. Part (i) has been proved. To prove part (ii) we apply \((\varepsilon \otimes \text{id} \otimes \varepsilon) \circ (\Delta \otimes \text{id})\) to (6.3). Because of the Hopf algebra axiom \((\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta\), cf. §4.1, we see that the left hand side reduces to \(a_{i,j}^l(\tau, \sigma)\). An application of (6.3) shows that the right hand side after this mapping yields for arbitrary \(\rho \in \mathbb{R}\)

\[ \sum_{p=-l}^{l} \sum_{r=-l}^{l} \varepsilon(a_{i,r}^l(\tau, \rho)) a_{r,p}^l(\rho, \mu) \varepsilon(a_{p,j}^l(\mu, \sigma)) \]

and (ii) follows from (6.4).

To prove (iii) we apply \((\text{id} \otimes S)\) to (6.3), and we use the Hopf algebra axiom \(m \circ (\text{id} \otimes S) = e \circ \varepsilon\), cf. §4.1. The first equality follows from (6.4). The second equality is proved similariy using the map \(m \circ (S \otimes \text{id})\). \(\square\)

For the generalised matrix elements we have the relative bi-invariance property

\[ \langle (X_\tau A)^* Y X_\sigma A, a_{i,j}^l(\tau, \sigma) \rangle = \langle t^l(Y)t^l(X_\sigma A)v^{l,j}(\sigma), t^l(X_\tau A)v^{l,i}(\tau) \rangle = \lambda_j(\sigma) \lambda_i(\tau) \langle t^l(Y)v^{l,j}(\sigma), v^{l,i}(\tau) \rangle = \lambda_j(\sigma) \lambda_i(\tau) \langle Y, a_{i,j}^l(\tau, \sigma) \rangle, \quad \forall Y \in U_q(\text{su}(2)), \]

by proposition 5.2. Using (4.14) we can reformulate this into the following lemma, since \((X_\tau A)^* = X_\tau A\) by (5.3).

**Lemma 6.3.** The generalised matrix elements \(a_{i,j}^l(\tau, \sigma)\) satisfy the relative bi-invariance property

\[ (X_\sigma A) a_{i,j}^l(\tau, \sigma) = \lambda_j(\sigma) a_{i,j}^l(\tau, \sigma) \quad \text{and} \quad a_{i,j}^l(\tau, \sigma) (X_\tau A) = \lambda_i(\tau) a_{i,j}^l(\tau, \sigma). \]

Actually, lemma 6.3 is a key observation in determining the generalised matrix elements explicitly. But instead of determining \(a_{i,j}^l(\tau, \sigma)\) we determine

\[ b_{i,j}^l(\tau, \sigma) = A \cdot a_{i,j}^l(\tau, \sigma) \]

explicitly. Since \(A \cdot t_{n,m}^l = \sum_p t_{n,p}^l (t^l(A)e_m^l, e_p^l) = q^{-m}t_{n,m}^l\) we get from (6.2) the explicit expression

\[ b_{i,j}^l(\tau, \sigma) = \sum_{n,m=-l}^{l} v_{m}^{l,j}(\sigma) v_{n}^{l,i}(\tau) q^{-m}t_{n,m}^l \in A_q(SU(2)) \subset A_q(SU(2)). \]
Note that $A_{\theta}(SU(2)) \to A_{\theta}(SU(2))$, $\xi \mapsto A_{\theta}$ is an algebra homomorphism, because of (4.15) and (4.6). Moreover, this homomorphism is invertible, its inverse being $D_{\theta}: A_{\theta}(SU(2)) \to A_{\theta}(SU(2))$, $\xi \mapsto D_{\theta}$. Lemma 6.3 can be rewritten as

\[(6.8) \quad X_{\sigma}.b_{i,j}(\tau, \sigma) = \lambda_{j}(\sigma)D_{\theta}b_{i,j}(\tau, \sigma) \quad \text{and} \quad b_{i,j}(\tau, \sigma).X_{\tau} = \lambda_{i}(\tau)b_{i,j}(\tau, \sigma).D_{\theta}.\]

We have $\lambda_{0}(\sigma) = \lambda_{0}(\tau) = 0$, so that for $l \in \mathbb{Z}$ $b_{0,0}(\sigma, \tau)$ is a $(\sigma, \tau)$-spherical element in the sense of (5.4).

**Proposition 6.4.** (i) Let $\xi \in A_{\theta}(SU(2))$ be a $(\sigma, \tau)$-spherical element, cf. (5.4), and let $\eta \in A_{\theta}(SU(2))$ satisfy

\[(6.9) \quad X_{\sigma}.\eta = \lambda D_{\eta} \quad \text{and} \quad \eta.X_{\tau} = \mu \eta.D_{\theta}.\]

for $\lambda, \mu \in \mathbb{C}$. Then $\eta \xi$ satisfies (6.9) for the same $\lambda, \mu$. Moreover, if $\lambda, \mu \in \mathbb{R}$, then $\eta^{*}\eta$ is a $(\sigma, \tau)$-spherical element.

(ii) If $\eta \in A_{\theta}(SU(2))$ satisfies (6.9) for arbitrary $\lambda, \mu \in \mathbb{C}$ and $\eta$ is non-zero, then $\lambda = \lambda_{j}(\sigma), \mu = \lambda_{i}(\tau)$ for some $i, j \in \{-l, -l+1, \ldots, l\}$ and $\eta$ is a multiple of $b_{i,j}(\tau, \sigma)$.

**Proof.** To prove (i) we first consider $\eta \xi$, then, by (4.15), (5.3), (6.9), (5.4) and (4.6),

\[X_{\sigma}.(\eta \xi) = (A.\eta)(X_{\sigma}.\xi) + (X_{\sigma}.\eta)(D.\xi) = \lambda(D.\eta)(D.\xi) = \lambda D_{\theta}(\eta \xi).\]

Similarly we prove $(\eta \xi).X_{\tau} = \mu(\eta \xi).D_{\theta}$.

To prove the other statement of (i) we proceed as before to obtain

\[X_{\sigma}.(\eta^{*}\eta) = (A.\eta^{*})(X_{\sigma}.\eta) + (X_{\sigma}.\eta^{*})(D.\eta) = (\lambda - \bar{\lambda})(D.\eta)^{*}(D.\eta),\]

by (4.16), $S(A)^{*} = D$ and $S(X_{\sigma})^{*} = -X_{\sigma}$. This yields zero for $\lambda \in \mathbb{R}$. Similarly we prove $(\eta^{*}\eta).X_{\tau} = 0$ for real $\mu$.

To prove (ii) we take $\eta$ of the form $\sum_{n,m=-l}^{l} \gamma_{n,m}e_{n,m}$, then $\eta$ satisfies (6.9) if and only if

\[t^{l}(AX_{\sigma}) \sum_{m=-l}^{l} \gamma_{n,m}e_{m}^{l} = \lambda \sum_{m=-l}^{l} \gamma_{n,m}e_{m}^{l}, \quad \forall n \in \{-l, -l+1, \ldots, l\}\]

\[t^{l}(X_{\tau}A) \sum_{n=-l}^{l} \bar{\gamma}_{n,m}e_{n}^{l} = \bar{\mu} \sum_{n=-l}^{l} \bar{\gamma}_{n,m}e_{n}^{l}, \quad \forall m \in \{-l, -l+1, \ldots, l\}\]

by the Peter-Weyl theorem for $A_{\theta}(SU(2))$, cf. §4.7. The operator $t^{l}(X_{\tau}A)$ is considered in proposition 5.2 and the operator $t^{l}(AX_{\sigma}) = t^{l}(A)t^{l}(X_{\sigma}A)t^{l}(D)$ is conjugated to such an operator. Proposition 5.2 then gives $\gamma_{n,m} = cq^{-m}v_{m}^{i,j}(\sigma)v_{n}^{l,i}(\tau)$ for some non-zero constant $c$ independent of $n$ and $m$, and $\lambda = \lambda_{j}(\sigma), \mu = \lambda_{i}(\tau)$. \(\square\)

In the previous proposition we have shown that the product of two elements can behave nicely with respect to the identity (6.9). Let us now consider $\eta \in A_{\theta}(SU(2))$ satisfying
(6.9) and multiply from the left by an arbitrary \( \xi \in \mathcal{A}_q(SU(2)) \). We get similarly as in the proof of proposition 6.4

\[
X_\sigma.(\xi \eta) = (A.\xi)(X_\sigma.\eta) + (X_\sigma.\xi)(D.\eta) = [\lambda A.\xi + X_\sigma.\xi](D.\eta),
(\xi \eta).X_\tau = (\xi A)(\eta X_\tau) + (\xi X_\tau)(\eta D) = [\mu \xi A + \xi X_\tau](\eta D).
\]

So, if \( \xi \) satisfies

\[
(6.10) \quad [\lambda A.\xi + X_\sigma.\xi] = \lambda_1 D.\xi, \quad [\mu \xi A + \xi X_\tau] = \mu_1 \xi .D
\]

for some \( \lambda_1, \mu_1 \in \mathbb{C} \), then we get

\[
X_\sigma.(\xi \eta) = \lambda_1 D.(\xi \eta), \quad \text{and} \quad (\xi \eta).X_\tau = \mu_1 (\xi \eta).D.
\]

Next we restrict \( \xi \in \mathcal{A}_q^{1/2}(SU(2)) \), so that \( \xi = a\alpha + b\beta + c\gamma + d\delta \) for some complex constants \( a, b, c \) and \( d \). From §4.4 we compute

\[
A. \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} q^{1/2} \alpha \\ q^{1/2} \beta \\ q^{-1/2} \gamma \\ q^{-1/2} \delta \end{pmatrix}, \quad B. \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ 0 \\ \gamma \end{pmatrix}, \quad C. \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \\ \delta \\ 0 \end{pmatrix}.
\]

The first requirement of (6.10) is rewritten as

\[
(6.11) \quad [q^{1/2}\lambda_a - q^{\sigma - q^{-\sigma}}(q^{-1/2} - q^{-1/2})a + iq^{1/2}b - \lambda_1 q^{-1/2}a] \alpha + q^{-1/2}\lambda_b - q^{\sigma - q^{-\sigma}}(q^{-1/2} - q^{1/2})b - iq^{-1/2}a - \lambda_1 q^{-1/2}b] \beta + q^{1/2}\lambda_c - q^{\sigma - q^{-\sigma}}(q^{-1/2} - q^{-1/2})c + iq^{1/2}d - \lambda_1 q^{-1/2}c] \gamma + q^{-1/2}\lambda_d - q^{\sigma - q^{-\sigma}}(q^{-1/2} - q^{1/2})d - iq^{-1/2}c - \lambda_1 q^{-1/2}d] \delta = 0.
\]

Since the matrix elements are linearly independent in the algebra \( \mathcal{A}_q(SU(2)) \) we see that each of the terms in square brackets in (6.11) has to be zero, so we get two sets of two equations in the unknowns \( a, b, c \) and \( d \). Or

\[
A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

where \( A = A(\sigma, \lambda, \lambda_1) \) is the \( 2 \times 2 \)-matrix given by

\[
A = A(\sigma, \lambda, \lambda_1) = \begin{pmatrix} q^{1/2}\lambda - q^{\sigma - q^{-\sigma}}(q^{1/2} - q^{-1/2}) - \lambda_1 q^{-1/2}, & iq^{1/2} \\ -iq^{-1/2}, & q^{-1/2}\lambda - q^{\sigma - q^{-\sigma}}(q^{-1/2} - q^{1/2}) - \lambda_1 q^{1/2} \end{pmatrix}.
\]
There is only a non-trivial solution if the determinant of $A$ equals zero, and this depends on the choice of $\lambda$ and $\lambda_1$. From proposition 6.4(ii) we see that without loss of generality we may assume that $\lambda = \lambda_j(\sigma)$ and $\lambda_1 = \lambda_n(\sigma)$ for some $j, n \in \frac{1}{2}\mathbb{Z}_+$. Now $\det A = 0$ if and only if the product of the two diagonal elements equals 1, which means that the $\sigma$-dependence of this product must vanish. This gives $n = j \pm \frac{1}{2}$. We find

$$A(\sigma, \lambda_j(\sigma), \lambda_{j+1/2}(\sigma)) = \begin{pmatrix} q^{-2j-\sigma-1/2}, & iq^{1/2} \\ -iq^{-1/2}, & q^{\sigma+2j+1/2} \end{pmatrix},$$

$$A(\sigma, \lambda_j(\sigma), \lambda_{j-1/2}(\sigma)) = \begin{pmatrix} -q^{2j+\sigma-1/2}, & iq^{1/2} \\ -iq^{-1/2}, & -q^{1/2-\sigma-2j} \end{pmatrix}. \tag{6.13}$$

Note that the matrices in (6.13) only depend on $\sigma + 2j$, and that if $(x(\sigma), y(\sigma))^t$ is in the kernel of the first matrix with $j = 0$, then $(-x(-\sigma), -y(-\sigma))^t$ is in the kernel of the second matrix of (6.13) with $j = 0$.

The second requirement of (6.10) leads analogously to

$$B \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $B^t = A(\tau, \mu, \mu_1)$. So we only get non-trivial solutions for $\mu = \lambda_i(\tau), \mu_1 = \lambda_i(\tau)$. A similar remark on the relation between the kernels of the two possible choices for $B$ holds here as well. This proves the following proposition.

**Proposition 6.5.** Define elements in $A_q^{1/2}(SU(2))$ by

$$\alpha_{\tau,\sigma} = q^{1/2}\alpha - iq^{\sigma-1/2}\beta + iq^{\gamma+1/2}\gamma + q^{\tau-1/2}\delta,$$

$$\beta_{\tau,\sigma} = -q^{\sigma+1/2}\alpha - iq^{-1/2}\beta - iq^{\sigma+\gamma+1/2}\gamma + q^{\tau-1/2}\delta,$$

$$\gamma_{\tau,\sigma} = -q^{\gamma+1/2}\alpha + iq^{\tau+\sigma-1/2}\beta + iq^{1/2}\gamma + q^{\tau-1/2}\delta,$$

$$\delta_{\tau,\sigma} = q^{\tau+\gamma+1/2}\alpha + iq^{\tau-1/2}\beta - iq^{\sigma+1/2}\gamma + q^{-1/2}\delta,$$

and let $\eta$ satisfy (6.9) with $\lambda = \lambda_j(\sigma)$ and $\mu = \lambda_i(\tau)$, then

(i) $\alpha_{\tau+2i,\sigma+2j}\eta$ satisfies (6.9) with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i-1/2}(\tau)$,

(ii) $\beta_{\tau+2i,\sigma+2j}\eta$ satisfies (6.9) with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i+1/2}(\tau)$,

(iii) $\gamma_{\tau+2i,\sigma+2j}\eta$ satisfies (6.9) with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i+1/2}(\tau)$,

(iv) $\delta_{\tau+2i,\sigma+2j}\eta$ satisfies (6.9) with $\lambda = \lambda_{j+1/2}(\sigma)$ and $\mu = \lambda_{i+1/2}(\tau)$.

**Remark.** The normalisation for the elements $A(\tau, \sigma), B(\tau, \sigma), C(\tau, \sigma)$ and $D(\tau, \sigma)$ has been chosen such that

$$\frac{1}{\sqrt{(1 + q^{2\sigma})(1 + q^{2\tau})}} \begin{pmatrix} \alpha_{\tau,\sigma} & \beta_{\tau,\sigma} \\ \gamma_{\tau,\sigma} & \delta_{\tau,\sigma} \end{pmatrix} = b^{1/2}(\tau, \sigma) = \begin{pmatrix} b^{1/2}_{-1/2,-1/2} & b^{1/2}_{-1/2,1/2} \\ b^{1/2}_{1/2,-1/2} & b^{1/2}_{1/2,1/2} \end{pmatrix}.$$

In particular if we take $\eta = b^{1}_{i,j}(\tau, \sigma)$ we can write each of these products as a sum of two similar elements by the Clebsch-Gordan series, cf. §4.7. In the extremal cases, i.e.
\( l = \max\{|i|,|j|\} \), this sum reduces to one element and we get recursion relations. For instance, if \( l = i \), we obtain

\[
\begin{align*}
\delta_{\tau+2l,\sigma+2m} b_{l,m}^l(\tau,\sigma) &= c b_{l+1/2,m+1/2}^{l+1/2}(\tau,\sigma), \\
\gamma_{\tau+2l,\sigma+2m} b_{l,m}^l(\tau,\sigma) &= c b_{l+1/2,m-1/2}^{l+1/2}(\tau,\sigma)
\end{align*}
\]

(6.14)

for some non-zero constants. These relations can be iterated in several ways. Since the elements are non-commuting we have to be careful about the ordering in a product. We stick to the convention that \( \Pi_{j=0}^k \xi_k = \xi_0 \xi_1 \ldots \xi_k \).

Before giving the explicit expression and the corresponding proof we derive symmetries for the generalised matrix elements \( a_{i,j}^l(\tau,\sigma) \) enabling us to prove only one case. First we introduce the algebra map \( \Psi: A_q(SU(2)) \to A_q(SU(2)) \) defined by interchanging \( \beta \) and \( \gamma \). It follows directly from (4.1) that \( \Psi \) does preserve the commutation relations. Moreover, \( \Psi(t_{n,m}^l) = t_{m,n}^l \) as follows directly from (4.20). This has already been observed by Koornwinder [30] before knowing the explicit expressions for these matrix elements. We also introduce the antilinear mapping \( -: A_q(SU(2)) \to A_q(SU(2)) \) defined by taking complex conjugates of all coefficients.

**Lemma 6.6.** The generalised matrix elements \( a_{i,j}^l(\tau,\sigma) \) satisfy the symmetry relations

\[
a_{i,j}^l(\tau,\sigma) = \Psi\left((a_{j,i}^l(\sigma,\tau)^-\right) = (a_{-i,-j}^l(-\tau,-\sigma)^- = \Psi(a_{-j,-i}^l(-\sigma,-\tau)).
\]

**Proof.** The last relation follows from the first two. The first relation follows from \( \Psi(t_{n,m}^l) = t_{m,n}^l \) and (6.2). The second relation follows from (6.2) and

\[
v_{n,j}^{l,i}(\sigma) = v_{n,j}^{l,-i}(\sigma).
\]

To see that (6.15) is true we first note that \( C_{l,j}^l(\sigma) = C_{l,-j}^l(-\sigma) \). Next the transformation formula, cf. [18, (3.2.3)],

\[
3 \phi_2\left( q^{-n}, b, c : d, e ; q, q \right) = \frac{(de/bc;q)_n}{(e;q)_n} \left( \frac{bc}{d} \right)^n 3 \phi_2\left( q^{-n}, d/b, d/c : d, de/bc ; q, q \right)
\]

yields the identity

\[
R_{l-n}(q^{2j-2l}q^{-2j-2l-2\sigma} ; q^{2\sigma}, 2l) = (-1)^{l-n} q^{-2\sigma(l-n)} R_{l-n}(q^{-2j-2l} q^{2j-2l+2\sigma} ; q^{-2\sigma}, 2l; q^2),
\]

for the dual \( q \)-Krawtchouk polynomial \( R_{l-n} \) in proposition 5.2 and this proves (6.15). \qed

In the following corollary we only work out one of the many possibilities for one of the four cases. A similar expression for the other three cases can be be obtained using lemma 6.6 and the observation \( \Psi(A_i \xi) = \xi A \).
**Corollary 6.7.** With the notation of proposition 5.2 we have

\[
b^l_{i,m}(\tau, \sigma) = E^l_m(\tau, \sigma) \prod_{k=0}^{l+m-1} \delta_{\tau+2l-1-k, \sigma+2m-1-k} \\
\times \prod_{j=0}^{l-m-1} \gamma_{\tau+l-m-1-j, \sigma-l+m+1+j},
\]

with

\[
E^l_m(\tau, \sigma) = C^{l,m}(\sigma) C^{l,l}(\tau) q^{\sigma(m-l)} q^{\frac{1}{2}(l-m)(l-m-1)}.
\]

**Remark.** This corollary has been stated by Noumi and Mimachi in an unpublished announcement that extends the announcement [40].

**Proof.** Iteration of (6.14) proves the existence of the product as in the corollary. It remains to determine the constant. For this we apply the one-dimensional $*$-representation $\pi_{\theta/2}$ to it and we compare the coefficients of $e^{-il\theta}$ on both sides. The left hand side gives $C^{l,m}(\sigma) C^{l,l}(\tau) q^{\sigma(m-l)} q^{\frac{1}{2}(l-m)(l-m-1)}$, from which the corollary follows. □

**7. Generalised matrix elements and Askey-Wilson polynomials**

We have now developed all the necessary ingredients for the interpretation of Askey-Wilson polynomials on the quantum $SU(2)$ group. In this section we first show that orthogonal polynomials are of importance in describing generalised matrix elements. Next these polynomials are explicitly calculated in terms of Askey-Wilson polynomials, giving a full proof of the theorem announced by Noumi and Mimachi [40, thm. 3], see also [43, thm. 4].

**Theorem 7.1.** For fixed $i, j \in \frac{1}{2}\mathbb{Z}_+$ such that $i - j \in \mathbb{Z}$, there exists a system of orthogonal polynomials $(p_k)_{k \in \mathbb{Z}_+}$ of degree $k$ such that for $l \geq m = \max(|i|, |j|)$, $l - m \in \mathbb{Z}_+$,

\[
b^l_{i,j}(\tau, \sigma) = b^m_{i,j}(\tau, \sigma) p_{l-m}(\rho_{\tau,\sigma}),
\]

where $\rho_{\tau,\sigma} \in \mathcal{A}_q(SU(2))$ is the self-adjoint element given in (5.5).

**Proof.** We first prove that an expression of the form (7.1) exists. Consider for any polynomial $s_{l-m}$ of degree $l - m$ the expression $b^m_{i,j}(\tau, \sigma) s_{l-m}(\rho_{\tau,\sigma})$. If we decompose this product with respect to the decomposition of $\mathcal{A}_q(SU(2))$ in (4.22) we get

\[
b^m_{i,j}(\tau, \sigma) s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=\lfloor 2m-l \rfloor}^{l} b^k, \quad b^k \in \mathcal{A}_q^k(SU(2)),
\]

by the Clebsch-Gordan series (4.23). Now (4.18) implies that for any $X \in \mathcal{U}_q(\mathfrak{su}(2))$ the mappings $X.$ and $.X$ preserve $\mathcal{A}_q^k(SU(2))$. Proposition 6.4(i) shows that the left hand side
Remark 7.2. (7.2) satisfies (6.9) with \( \lambda = \lambda_j(\sigma) \) and \( \mu = \lambda_i(\tau) \). Consequently, each \( b^k \) has to satisfy (6.9) with \( \lambda = \lambda_j(\sigma) \) and \( \mu = \lambda_i(\tau) \). Proposition 6.4(ii) implies that \( b^k = 0 \) for \( k < m \) and \( b^k = c_kb^k_{i,j}(\tau,\sigma) \) for \( k \geq m \) and some constants \( c_k \). Hence, (7.2) reduces to

\[
b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) = \sum_{k=m}^{l} c^k_{i,j}(\tau,\sigma).
\]

Since both sides contain the same degree of freedom, (7.1) follows once we know that the mapping \( s_{l-m} \mapsto b^m_{i,j}(\tau,\sigma)s_{l-m}(\rho_{\tau,\sigma}) \) is injective. This can be seen by applying the one-dimensional \(*\)-representation \( \pi_\theta \) of \( \mathcal{A}_q(SU(2)) \), cf. (4.21), and use of the explicit expression of \( b^m_{i,j}(\tau,\sigma) \), cf. (6.7). This proves that an expression as in (7.1) exists.

For \( l, k \geq m, l-m, k-m \in \mathbb{Z}_+ \) we have \( b^l_{i,j}(\tau,\sigma) \in \mathcal{A}_q^l(SU(2)), b^k_{i,j}(\tau,\sigma) \in \mathcal{A}_q^k(SU(2)) \), so that the Schur orthogonality relations for the Haar functional \( h \), cf. (4.24), imply

\[
h \left( (b^l_{i,j}(\tau,\sigma))^*b^k_{i,j}(\tau,\sigma) \right) = \delta_{k,l}h_l, \quad h_l > 0.
\]

Now \( (b^m_{i,j}(\tau,\sigma))^*b^m_{i,j}(\tau,\sigma) = w_m(\rho_{\tau,\sigma}) \) for some polynomial \( w_m \) of degree \( 2m \), by proposition 6.4(i), theorem 5.1 and the Clebsch-Gordan series, cf. (4.23). Hence, (7.3) equals

\[
h \left( \bar{p}_{l-m}(\rho_{\tau,\sigma})p_{k-m}(\rho_{\tau,\sigma})w_m(\rho_{\tau,\sigma}) \right) = \delta_{l,k}h_l, \quad h_l > 0,
\]

since we have already established (7.1) for some polynomial and since \( \rho_{\tau,\sigma}^* = \rho_{\tau,\sigma} \). Consequently, by theorem 5.1 the polynomials \( p_{l-m}, l-m \in \mathbb{Z}_+ \), form a system of orthogonal polynomials with respect to the moment functional \( \mathcal{L} \) with moments given by

\[
\mathcal{L}[x^n] = \int_{\mathbb{R}} x^n w_m(x) \, dm(x; a, b, c, d \mid q^2) < \infty,
\]

where \( dm(x; a, b, c, d \mid q^2) \) is the measure described in theorem 5.1. □

Remark 7.2. (i) The constant \( h_l \) in (7.3) can be given explicitly using the Schur orthogonality relations (4.24) and (6.7). This yields

\[
h_l = \frac{(1 - q^2)q^{2l}}{1 - q^{4l+2}} \sum_{m=-l}^{l} |v^l_{i,j}(\sigma)|^2 q^{-2m} \sum_{n=-l}^{l} |v^l_{n,i}(\tau)|^2 q^{-2n}
\]

\[
= \frac{(1 - q^2)q^{2l}}{1 - q^{4l+2}} \tau \sqrt{q} (b^l_{j,i}(\sigma,\sigma)) \tau \sqrt{q} (b^l_{i,j}(\tau,\tau))
\]

\[
= \frac{(1 - q^2)q^{2l}}{1 - q^{4l+2}} \tau \sqrt{q} (b^l_{j,i}(\sigma,\sigma)) r_{l-|j|} ((q + q^{-1})/2) \tau \sqrt{q} (b^l_{i,j}(\tau,\tau)) s_{l-|i|} ((q + q^{-1})/2),
\]

for some polynomials \( r_{l-|j|} \) and \( s_{l-|i|} \) given by theorem 7.1. Here \( \tau \sqrt{q} \) is the one-dimensional representation of \( \mathcal{A}_q(SU(2)) \) defined in §4.6.
Proposition 7.3. \[\text{proposition by taking } \lambda \text{ in (7.6)}\]

Hence, it is sufficient to calculate \(\pi\) by the Clebsch-Gordan series, \(\S\), and (7.1). Using the injectivity of the map \(s_{l-m} \mapsto b_{i,j}^m(\tau, \sigma) s_{l-m}(\rho, \sigma)\) proves that the polynomials satisfy a three-term recurrence relation implying the orthogonality. See also remark 7.6(i).

In order to be able to identify the the system of orthogonal polynomials described in theorem 7.1, we have to calculate the polynomial \(w_m\). Apply the one-dimensional \(*\)-representation \(\pi_{\theta/2}\) to the identity \(w_m(\rho, \sigma) = (b_{i,j}^m(\tau, \sigma))^* b_{i,j}^m(\tau, \sigma)\) to obtain

\[w_m(\cos \theta) = |\pi_{\theta/2}(b_{i,j}^m(\tau, \sigma))|^2.\]

Hence, it is sufficient to calculate \(\pi_{\theta/2}(b_{i,j}^m(\tau, \sigma))\). These values follow from the following proposition by taking \(\lambda = e^{i\theta/2}\), cf. \(\S\)\(4.6\).

**Proposition 7.3.** For \(i, j \in \frac{1}{2} \mathbb{Z}_+\), \(i - j \in \mathbb{Z}\) and \(m = \max(|i|, |j|)\) we have

(i) In case \(m = i\) or \(-i \leq j \leq i\)

\[\tau_\lambda(b_{i,j}^i(\tau, \sigma)) = C^{i,i}(\sigma) C^{i,i}(\tau) q^{-i} \lambda^{-2i}(\lambda^2 q^{1+\tau-\sigma}; q^2)_{i-j}(-\lambda^2 q^{1+\tau+\sigma}; q^2)_{i+j}.\]

(ii) In case \(m = j\) or \(-j \leq i \leq j\)

\[\tau_\lambda(b_{i,j}^j(\tau, \sigma)) = C^{j,j}(\tau) C^{j,j}(\sigma) q^{-j} \lambda^{-2j}(\lambda^2 q^{1+\sigma-\tau}; q^2)_{j-i}(-\lambda^2 q^{1+\tau+\sigma}; q^2)_{j+i}.\]

(iii) In case \(m = -i\) or \(i \leq j \leq -i\)

\[\tau_\lambda(b_{i,j}^{-i}(\tau, \sigma)) = C^{-i,-i}(-\sigma) C^{-i,-i}(-\tau) q^i \lambda^{2i}(\lambda^2 q^{1+\tau+\sigma}; q^2)_{j-i}(-\lambda^2 q^{1-\tau-\sigma}; q^2)_{-i-j}.\]

(iv) In case \(m = -j\) or \(j \leq i \leq -j\)

\[\tau_\lambda(b_{i,j}^{-j}(\tau, \sigma)) = C^{-j,-j}(-\tau) C^{-j,-j}(-\sigma) q^j \lambda^{2j}(\lambda^2 q^{1-\sigma+\tau}; q^2)_{i-j}(-\lambda^2 q^{1-\tau-\sigma}; q^2)_{-i-j}.\]

**Proof.** First we observe that the function \(\tau_\lambda(b_{i,j}^m(\tau, \sigma))\) of \(\lambda\) satisfies the symmetry relations

\[\tau_\lambda(b_{i,j}^l(\tau, \sigma)) = \tau_\lambda(b_{j,i}^l(\sigma, \tau)) = \tau_\lambda(b_{-i,-j}^l(-\sigma, -\tau)) = \tau_\lambda(b_{-j,i}^l(-\tau, -\sigma)).\]
This a straightforward consequence of lemma 6.6 and and $\tau_\lambda \circ \Psi = \tau_\lambda$.

By (7.7) it suffices to prove the first statement. Application of $\tau_\lambda$ to corollary 6.7 gives

$$
\tau_\lambda(b_{l,m}^i(\tau,\sigma)) = E_{m}^{l}(\tau,\sigma) \prod_{k=0}^{l+m-1} (q^{r+\sigma+2l+2m-2-2k+1/2}\lambda + q^{-1/2}\lambda^{-1})
\times \prod_{j=0}^{l-m-1} (-q^{r+l-m-1-j+1/2}\lambda + q^{\sigma-l+m+1+j-1/2}\lambda^{-1})
= C_{l,m}^{d}(\sigma)E_{m}^{l}(\tau)q^{-l}\lambda^{-2l}(\lambda^2 q^{1+r-\sigma}; q^2)_{l-m}(-\lambda^2 q^{1+r+\sigma}; q^2)_{l+m}
$$

and this proves (i). \(\Box\)

**Corollary 7.4.** A generating function for the dual $q$-Krawtchouk polynomials is

$$
\sum_{n=0}^{N} t^n q^{n(N+\sigma)/2} \frac{(q^{-N}; q)_n}{(q; q)_n} R_n(q^{-x} - q^{x-N-\sigma}; q^\sigma, N; q)
= (-tq^{-(N+\sigma)/2}; q)_x (tq(\sigma-N)/2; q)_{N-x}
$$

for $N \in \mathbb{N}$, $x \in \{0, \ldots, N\}$.

**Remark.** This generating function can be obtained from the generating function for the Askey-Wilson polynomials given by Ismail and Wilson [19, (1.9)], cf. [24, (3.3)] for this derivation. In [24] the generating function (7.8) is used to prove proposition 7.3 in a special case.

**Proof.** Apply the one-dimensional representation $\tau_\lambda$ to the expression for $b_{i,j}^i(\tau,\sigma)$ from (6.7). Since this also equals proposition 7.3(i) we obtain (7.8) with $q$, $N$, $n$, $x$, and $t$ replaced by $q^2$, $2i$, $i-n$, $i-j$ and $-\lambda^2 q^{r+2i+1}$. \(\Box\)

We can now prove the main result of this section in which we relate Askey-Wilson polynomials to generalised matrix elements. The Askey-Wilson polynomials involve four continuous and two discrete parameters. In the following theorem we establish an interpretation of the Askey-Wilson polynomials with two continuous and two discrete parameters.

**Theorem 7.5.** For $i, j \in \frac{1}{2}\mathbb{Z}_+$, $i - j \in \mathbb{Z}$ and $l - m \in \mathbb{Z}_+$, $m = \max(|i|, |j|)$ we have
(i) In case $m = i$ or $-i \leq j \leq i$:

$$b_{i,j}^i(\tau,\sigma) = d_{i,j}^l(\tau,\sigma) b_{i}^i(\tau,\sigma) p_{l-i}^{(i-j,i+j)}(\rho_{\tau,\sigma}; q^\tau, q^\sigma \mid q^2).
$$

(ii) In case $m = j$ or $-j \leq i \leq j$:

$$b_{i,j}^j(\tau,\sigma) = d_{j,i}^l(\tau,\sigma) b_{j}^j(\tau,\sigma) p_{l-j}^{(j-i,j+i)}(\rho_{\tau,\sigma}; q^\sigma, q^\tau \mid q^2).
$$

(iii) In case $m = -i$ or $i \leq j \leq -i$:

$$b_{i,j}^l(\tau,\sigma) = d_{-i,j}^l(-\tau,-\sigma) b_{-i}^j(\tau,\sigma) p_{l+i}^{(j-i,-i-j)}(\rho_{\tau,\sigma}; q^{-\tau}, q^{-\sigma} \mid q^2).$$
(iv) In case $m = -j$ or $j \leq i \leq -j$:

$$b_{i,j}^l(\tau, \sigma) = d_{i,-i}^l(-\sigma, -\tau) b_{i,-i}^l(\tau, \sigma) p_{l+i+j}^{(i-j,-i-j)}(\rho_{\tau, \sigma}; q^{-\sigma}, q^{-\tau} | q^2).$$

Here the constant is given by

$$d_{i,j}^l(\tau, \sigma) = \frac{C_{i,j}^l(\sigma)C_{i,i}^l(\tau)}{C_{i,j}^l(\sigma)C_{i,i}^l(\tau) (q^{4l}; q^{-2})_{l-i}}.$$

**Proof.** The explicit form (7.1) given in theorem 7.1, the symmetry relations of (7.7) and $\pi_{0/2}(\rho_{\tau, \sigma}) = \cos \theta$ being independent of $\sigma, \tau$ show that it suffices to prove the first statement.

From the explicit form of the Askey-Wilson weight measure, cf. proposition 3.1, we immediately get

$$(az, a/z; q)_r dm(x; a, b, c, d | q) = \frac{(ab, ac, ad; q)_r}{(abcd; q)_r} dm(aq^r, b, c, d | q)$$

for $r \in \mathbb{Z}_+, x = (z+z^{-1})/2$. A double application shows that for $r, s \in \mathbb{Z}_+, x = (z+z^{-1})/2$,

$$(az, a/z; q)_r (dz, d/z; q)_s dm(x; a, b, c, d | q) =$$

$$(ab, ac; q)_r (bd, cd; q)_s \frac{(ad; q)_{r+s}}{(abcd; q)_{r+s}} dm(aq^r, b, c, dq^s | q).$$

Use this in conjunction with (5.6), (7.6), proposition 7.3(i) and (7.4) to find that the looked-for polynomials are multiples of the Askey-Wilson polynomials

$$p_{l-i}(\rho_{\tau, \sigma}; -q^{\sigma+t+1+2i+2j}, -q^{-\tau}, q^{\sigma+t+1}, q^{\tau}, q^{-\sigma+1+2i+2j} | q^2),$$

which we rewrite using (3.2). □

It remains to calculate the constant. We apply the one-dimensional $\star$-representation $\pi_{0/2}$ to both sides of (i), and next we compare the coefficient of $e^{-i\theta}$ on both sides. The coefficient of $e^{-i\theta}$ on the left hand side is $v_{l,i}^i(\sigma) v_{l,j}^j(\tau) q^{-l} = C_{l,j}^l(\sigma) C_{l,i}^l(\tau) q^{-l}$. The coefficient of $e^{-i(l-i)\theta}$ of $p_{l-i}$ is $(q^{2l+2i+2}; q^2)_{l-i} = (q^{4i}; q^{-2})_{l-i}$, cf. [7, p.5], so that the coefficient of $e^{-i\theta}$ on the right hand side equals $C_{l,j}^l(\sigma) C_{l,i}^l(\tau) q^{-l} (q^{4i}; q^{-2})_{l-i}$, from which we obtain the value for $d_{i,j}^l(\tau, \sigma)$. □

**Remark 7.6.** (i) The orthogonal polynomials are determined in theorem 7.5 by explicitly determining the orthogonality measure for these polynomials as is done in [24] to determine associated spherical elements. Since there are other characteristics of orthogonal polynomials, theorem 7.5 can also be proved in other ways. We will sketch two other possible lines of proof.

The first alternative uses the three-term recurrence relation for orthogonal polynomials. Since $\rho_{\tau, \sigma} \in \mathcal{A}_q^1(SU(2))$ we obtain from the Clebsch-Gordan series, cf. §4.7, and proposition 6.4 the relation

$$b_{i,j}^l(\tau, \sigma) \rho_{\tau, \sigma} = A_{i} b_{i,j}^{l+1}(\tau, \sigma) + B_{i} b_{i,j}^l(\tau, \sigma) + C_{i} b_{i,j}^{l-1}(\tau, \sigma) \quad (7.9)$$
for certain constants $A_l, B_l, C_l$, cf. the proof of theorem 7.1 and remark 7.2(ii). Applying the one-dimensional $*$-representation $\pi_{\theta/2}$ to (7.9) and using (6.7) yields an identity for trigonometric polynomials from which $A_l, B_l, C_l \in \mathbb{C}$ can be determined explicitly by comparing the coefficients of $e^{i(l+1)\theta}, e^{i\theta \bar{e}^i(l-1)\theta}$. It follows that the polynomials of theorem 7.1 satisfy the recurrence relation

\begin{equation}
\cos \theta p_{l-m+1}(\cos \theta) = A_l p_{l-m+1}(\cos \theta) + B_l p_{l-m}(\cos \theta) + C_l p_{l-m-1}(\cos \theta)
\end{equation}

(7.10) with initial conditions $p_{-1}(\cos \theta) = 0, p_0(\cos \theta) = 1$. From this we can obtain the desired expression of $p_{l-m}$ in terms of the Askey-Wilson polynomials by identifying (7.10) with the three-term recurrence relation for the Askey-Wilson polynomials, cf. [7, (1.24)].

The second alternative proof uses the second order $q$-difference equation for the Askey-Wilson polynomials as the identifying characteristic. This approach is also used by Koornwinder [33] for the $(\sigma, \tau)$-spherical elements and the proof was sketched in §5. This time we have to use

$$
\langle A^k \Omega, b_{i,j}^m(\tau, \sigma) \rangle = q^{1-2l} \frac{(1-q^{2l+1})^2}{(1-q^2)^2} \langle A^k, b_{i,j}^m(\tau, \sigma) \rangle p_{l-m}( (q^k + q^{-k})/2 ),
$$

where $\Omega$ is the Casimir element of $U_q(\mathfrak{su}(2))$, cf. (4.9). But $\langle A^k, b_{i,j}^m(\tau, \sigma) \rangle$ is known by proposition 7.3, since $(A^k, \xi) = \tau_{q^{1/2}}(\xi)$ for all $\xi \in A_q(SU(2))$. By rewriting $A^k \Omega$ as a linear combination of $A^{k+2}, A^k, A^{k-2}$ modulo $U_q(\mathfrak{su}(2))(AX_s - \lambda_j(\sigma)) + (X_{\tau}A - \lambda_i(\tau))U_q(\mathfrak{su}(2))$ and using the relative bi-invariance property (6.8) we get a relation of the form

$$
q^{1-2l} \frac{(1-q^{2l+1})^2}{(1-q^2)^2} \langle A^k, b_{i,j}^m(\tau, \sigma) \rangle p_{l-m}( (q^k + q^{-k})/2 ) = \sum_{p=-1}^{1} C_p \langle A^{k+2p}, b_{i,j}^m(\tau, \sigma) \rangle p_{l-m}( (q^{k+2p} + q^{-k-2p})/2 ),
$$

where the value of $C_p$ follows from the rewriting of $A^k \Omega$. The resulting identity can be related to the second order $q$-difference equation for the Askey-Wilson polynomials, cf. [7, (5.7)], from which the identification of the $p_{l-m}$’s with the Askey-Wilson polynomials follows up to a scalar.

It must be noted that these two alternatives do not use the a priori knowledge of the explicit form of the Haar functional on $(\sigma, \tau)$-spherical elements as given in theorem 5.1. The second alternative is an extension of the method sketched in §5 to arbitrary matrix elements. It would be instructive to have a proof of the expression of the Haar functional as in theorem 5.1 without using the explicit expression of the zonal spherical elements in terms of the Askey-Wilson polynomials.

(ii) The polynomials in theorem 7.5 are identified using the orthogonality measure and the constant involved is calculated by comparing certain coefficients. So we have not used (7.5), but the expression in (7.5) is explicitly known by theorem 7.5 and proposition 7.3. On the other hand, now that we have identified the polynomials as Askey-Wilson polynomials,
the value of $h_l$ can also be read off from proposition 3.1. It is straightforward to check that these values agree.

Now that we have a full description of the matrix elements $b^l_{i,j}(\tau, \sigma)$ we can state the corresponding results on the generalised matrix elements $a^l_{i,j}(\tau, \sigma)$ defined in (6.1). The first part of each of the four statements of the following corollary is equivalent to the theorem announced by Noumi and Mimachi [40, thm. 3], [43, thm. 4].

**Corollary 7.7.** For $i, j \in \frac{1}{2} \mathbb{Z}_+$, $i - j \in \mathbb{Z}$ and $l - m \in \mathbb{Z}_+$, $m = \max(|i|, |j|)$ and the same constant $d^l_{i,j}(\tau, \sigma)$ as in theorem 7.5 we have

(i) In case $m = i$ or $-i \leq j \leq i$:

$$a^l_{i,j}(\tau, \sigma) = d^l_{i,j}(\tau, \sigma) a^i_{i,j}(\tau, \sigma) p^{(i-j,i+j)}_{l-i}(D,\rho_{\tau,\sigma}; q^\tau, q^\sigma | q^2),$$

$$\tau_\lambda(a^i_{i,j}(\tau, \sigma)) = C^{i,j}(\sigma) C^{i,i}(\tau) \lambda^{-2i}(\lambda^2 q^{\tau-\sigma}; q^2)_{i-j}(-\lambda^2 q^{\tau+\sigma}; q^2)_{i+j}.$$  

(ii) In case $m = j$ or $-j \leq i \leq j$:

$$a^l_{i,j}(\tau, \sigma) = d^l_{j,i}(\sigma, \tau) a^j_{i,j}(\tau, \sigma) p^{(j-i,j+i)}_{l-j}(D,\rho_{\tau,\sigma}; q^\sigma, q^\tau | q^2),$$

$$\tau_\lambda(a^j_{i,j}(\tau, \sigma)) = C^{i,j}(\tau) C^{j,j}(\sigma) \lambda^{-2j}(\lambda^2 q^{\sigma-\tau}; q^2)_{j-i}(-\lambda^2 q^{\tau+\sigma}; q^2)_{i-j}.$$ 

(iii) In case $m = -i$ or $i \leq j \leq -i$:

$$a^l_{i,j}(\tau, \sigma) = d^l_{-i,-j}(\tau, -\tau) a^{-i}_{i,j}(\tau, \sigma) p^{(j-i,j-i)}_{l+i}(D,\rho_{\tau,\sigma}; q^{-\tau}, q^{-\sigma} | q^2),$$

$$\tau_\lambda(a^{-i}_{i,j}(\tau, \sigma)) = C^{i,i}(\tau) C^{i,-i}(\sigma) \lambda^{2i}(\lambda^2 q^{-\tau+\sigma}; q^2)_{j-i}(-\lambda^2 q^{\tau-\sigma}; q^2)_{i-j}.$$  

(iv) In case $m = -j$ or $j \leq i \leq -j$:

$$a^l_{i,j}(\tau, \sigma) = d^l_{-j,-i}(\sigma, -\sigma) a^{-j}_{i,j}(\tau, \sigma) p^{(j-i,j-i)}_{l+j}(D,\rho_{\tau,\sigma}; q^{-\sigma}, q^{-\tau} | q^2),$$

$$\tau_\lambda(a^{-j}_{i,j}(\tau, \sigma)) = C^{j,j}(\tau) C^{j,-j}(\sigma) \lambda^{2j}(\lambda^2 q^{-\tau+\sigma}; q^2)_{i-j}(-\lambda^2 q^{\tau-\sigma}; q^2)_{-j-i}.$$ 

Proof. As already remarked, $D, b^l_{i,j}(\tau, \sigma) = a^l_{i,j}(\tau, \sigma)$, cf. (6.6). Moreover, $D$ is an algebra homomorphism, so the first statements in (i)–(iv) are nothing but theorem 7.5.

To prove the other statements we first observe that from the explicit duality (4.17) between $\mathcal{A}_q(SU(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$ we get

$$D,\alpha = q^{-1/2}\alpha, \quad D,\beta = q^{-1/2}\beta, \quad D,\gamma = q^{1/2}\gamma, \quad D,\delta = q^{1/2}\delta.$$ 

Hence, $\tau_\lambda(D,\xi) = \tau_{q^{1/2}}(\xi)$ for any $\xi \in \mathcal{A}_q(SU(2))$ since $D$ is an algebra homomorphism. This observation and proposition 7.3 prove the other statements. □

The special case $\sigma, \tau \to \infty$ of corollary 7.7 corresponds precisely to (4.20) if we use the transition of Askey-Wilson polynomials to the little $q$-Jacobi polynomials, cf. [33, §6]. Note that the only property of the matrix elements $t^l_{n,m} \in \mathcal{A}_q(SU(2))$ that we have used in the proofs until now is $\tau_\lambda(t^l_{n,m}) = \delta_{n,m}\lambda^{-2n}$, which can also be replaced by $\langle A^k, t^l_{n,m} \rangle = \delta_{n,m} q^{-nk}$, cf. (4.19). So we can obtain (4.20) as a special case of corollary 7.7 if we can determine the minimal elements $a^m_{i,j}(\infty, \infty)$, $m = \max(|i|, |j|)$, explicitly. This is possible using (4.19) and the explicit duality given for suitable bases of the underlying linear spaces of $\mathcal{A}_q(SU(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$, cf. Masuda et al. [35, lemma 5].
8. Some applications

Before giving some new applications of the interpretation of the Askey-Wilson polynomials on the quantum SU(2) group as established in the previous sections we give three recent applications. These applications concern addition formulas for q-analogues of the Legendre polynomial, the action of the quantum SU(2) group on quantum spheres and non-negative linearisation coefficients for products of q-Legendre polynomials. The proof in §8.3 is somewhat different from the proof given by Koornwinder [34, §7]. For detailed results and proofs the reader may consult the references cited in the following subsections.

8.1. Addition formulas for q-Legendre polynomials. The group theoretic proof of the addition formula for Legendre polynomials is a consequence of the classical group counterpart of (6.3) for $i = j = 0$, cf. Vilenkin [56, Ch. 3, §4.2], Vilenkin and Klimyk [57, Vol. 1, §6.6.2]. The addition formula for Legendre polynomials can also be found in Askey [4, Lecture 4] and Erdélyi et al. [14, Vol. 2, 10.11(47)]. However in the quantum group case (6.3) for $i = j = 0$ does not readily give addition formulas for q-analogues of the Legendre polynomials, since (6.3) is an identity involving non-commuting variables. So we have to use representations of $A_q(SU(2))$, cf. §4.6, to represent (6.3) for $i = j = 0$ as an identity involving $q$-special functions with ordinary arguments.

Application of the one-dimensional representation $\tau_{\sqrt{q}} \otimes \tau_{\sqrt{q}}$ to (6.3) for $i = j = 0$ yields the following addition formula for a two-parameter family of q-Legendre polynomials, cf. Nouni and Mimachi [40, (4.3)], see also Koelink [24, (3.15)];

\[
(q^2; q^2)q^{-l}p_i^{(0,0)}(\xi(\lambda); q^\tau, q^\sigma | q^2) = \frac{p_i^{(0,0)}(\xi(\lambda); q^\mu, q^\tau | q^2)p_i^{(0,0)}(\xi(\nu); q^\mu, q^\sigma | q^2)}{(-q^{2-2\mu}, -q^{2+2\mu}; q^2)_l}
\]

\[
+ \sum_{p=1}^{l} \frac{(1 + q^{4p-2\mu})(q^{2}; q^2)_{l+p}(\lambda\nu)^{-p}(\lambda q^{\mu-\tau}, -\lambda q^{\tau+\mu}, \nu q^{\mu-\sigma}, -\nu q^{\mu+\sigma}; q^2)_p}{(1 + q^{2\mu})(q^{2}; q^2)_{l-p}(-q^{2-2\mu}; q^2)_l(-q^{2+2\mu}; q^2)_{l+p}}
\]

\[
\times p_{l-p}^{(p,p)}(\xi(\lambda); q^\mu, q^\tau | q^2)p_{l-p}^{(p,p)}(\xi(\nu); q^\mu, q^\sigma | q^2)
\]

\[
+ \sum_{p=1}^{l} \frac{(1 + q^{4p-2\mu})(q^{2}; q^2)_{l+p}(\lambda\nu)^{-p}(\lambda q^{\tau+\mu}, -\lambda q^{\tau-\mu}, \nu q^{\sigma-\mu}, -\nu q^{\sigma+\mu}; q^2)_p}{(1 + q^{2\mu})(q^{2}; q^2)_{l-p}(-q^{2-2\mu}; q^2)_l(-q^{2+2\mu}; q^2)_{l+p}}
\]

\[
\times p_{l-p}^{(p,p)}(\xi(\lambda); q^{\mu-\sigma}, q^{-\tau} | q^2)p_{l-p}^{(p,p)}(\xi(\nu); q^{-\mu}, q^{-\sigma} | q^2),
\]

with $\xi(\lambda) = \frac{1}{2}(q^{-1}\lambda + q\lambda^{-1})$ by corollary 7.7 and straightforward calculation.

This addition formula is probably not the most general addition formula which can be derived for the q-Legendre polynomial $p_i^{(0,0)}(x; q^\tau, q^\sigma | q^2)$ for two reasons. The first reason is that the Rahman-Verma addition formula for the continuous q-Legendre polynomial, cf. [46, (1.24)] with $a = q^{1/4}$ corresponding to the case $\sigma = \tau = 0$ is not contained in (8.1). The second reason is that the mapping $\tau_{\sqrt{q}} \otimes \tau_{\sqrt{q}}$ has a very large kernel, so that this map kills a lot of information contained in (6.3). However, it is possible to obtain the addition formula for Legendre polynomials in full generality from (8.1) by replacing $q^\tau$ by $t$ before letting $q \uparrow 1$.

In case we take $\sigma = \tau = \mu = 0$, $i = j = 0$ in (6.3) we obtain an identity for continuous q-ultraspherical polynomials from which it is possible to derive the Rahman-Verma addition
formula for the continuous $q$-Legendre polynomial, cf. Koelink [24, §4]. The method uses one-dimensional representations of the Hopf $*$-algebra $A_q(SU(2))$, but it also makes fundamental use of the relative bi-invariance properties, cf. lemma 6.3, to introduce an extra independent variable.  

The first example of an addition formula for $q$-Legendre polynomials derived using quantum group techniques is Koornwinder’s [32] addition formula for the little $q$-Legendre polynomials. Now we start from (6.3) with $i = j = 0$ and $\sigma = \tau = \mu = \infty$ with the explicit form of the matrix elements given by (4.20). Application of the infinite dimensional $*$-representation $\pi_0^{\infty} \otimes \pi_0^{\infty}$, cf. §4.6, yields an identity for bounded operators on the Hilbert space $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$. Letting act the operators on suitable vectors and taking inner products leads to the addition formula for the little $q$-Legendre polynomials.

Another way of deriving addition formulas for $q$-Legendre polynomials starts from corollary 6.2(ii). If we specialise $i = j = 0$, $\rho = \mu = \infty$ and we apply the $*$-representation $\pi_{\theta/2}$ to the resulting identity, then we get, after a suitable rescaling, the Fourier series

\begin{equation}
(8.2) \quad p_l^{(0,0)}(\cos \theta; q^\sigma, q^\tau \mid q^2) = C \sum_{n=-l}^{l} v_n^{l,0}(\sigma) v_n^{l,0} q^{-n} e^{-in\theta},
\end{equation}

for some explicitly known constant $C$, cf. Koornwinder [33, (5.9)]. Koornwinder [33, remark 5.4] also announces a generalisation of (8.2) to the corresponding $q$-ultraspherical polynomials. To see that (8.2) can be viewed as a $q$-analogue of the addition formula for the Legendre polynomials we recall that the $v_n^{l,0}(\sigma)$, cf. proposition 5.2, is given by a dual $q$-Krawtchouk polynomial which tends to the Krawtchouk polynomial (2.3) as $q \uparrow 1$. Now the Krawtchouk polynomial can be rewritten in terms of a Jacobi polynomial, cf. e.g. Nikiforov and Uvarov [38, §12, §22], and then we obtain the addition formula for the Legendre polynomials in the limit $q \uparrow 1$.

Another special case of corollary 6.2(ii), namely $\rho = \mu = \tau = \infty$, is turned into an addition formula for the big $q$-Legendre polynomials by use of an infinite dimensional $*$-representation of $A_q(SU(2))$, cf. Koelink [25].

8.2. Action on quantum spheres. The Lie group $SU(2)$ acts on the two-dimensional sphere $S^2$. This can be seen by noting that $SU(2)$ is a double cover of $SO(3)$, the group of $3 \times 3$ orthogonal matrices with determinant 1, i.e. $SU(2)/\{\pm I\} \cong SO(3)$, and by letting $SO(3)$ act on $S^2$ in the natural way. We can identify $S^2 \cong SU(2)/K$ with $K = S(U(1) \times U(1))$ as in §2. So $SU(2)$ also acts on the algebra of functions on $S^2$. Here we consider the analogue of these actions by letting act $A_q(SU(2))$ on certain deformed function algebras of quantum spheres originally introduced by Podleś [45]. As an alternative for this subsection there is an action of the quantised universal enveloping algebra $U_q(su(2))$ on the real two-dimensional sphere $S^2$, cf. Rideau and Winternitz [47] for details.

Define the subspace $A_{q,\sigma}(S^2) = \{ \xi \in A_q(SU(2)) \mid X_\sigma \cdot \xi = 0 \}$ of $A_q(SU(2))$ of right invariant elements with respect to $X_\sigma \in U_q(su(2))$, cf. (5.1). Applying similar techniques as in the proof of proposition 6.4(i) we see that $A_{q,\sigma}(S^2)$ is a $*$-subalgebra of $A_q(SU(2))$. From the coassociativity axiom, cf. §4.1, we get $(id \otimes X) \Delta(\xi) = \Delta(X_\xi)$ for any $X \in U_q(su(2))$, $\xi \in A_q(SU(2))$, cf. (4.13). This implies $\Delta : A_{q,\sigma}(S^2) \to A_q(SU(2)) \otimes A_{q,\sigma}(S^2)$. 

\textbf{ASKEY-WILSON POLYNOMIALS AND THE QUANTUM SU(2) GROUP} 33
A left corepresentation of the Hopf $*$-algebra $A_q(SU(2))$ in a linear space $V$ is a map $L : V \rightarrow A_q(SU(2)) \otimes V$ such that $(id \otimes L) \circ L = (\Delta \otimes id) \circ L$ and $(\varepsilon \otimes id) \circ L = id$. If we choose a basis in a left corepresentation space $V$, then the matrix elements in $A_q(SU(2))$ form a matrix corepresentation of $A_q(SU(2))$, cf. §4.1. The Hopf algebra axioms, cf. §4.1, imply that we have a left corepresentation in $A_{q,\sigma}(S^2)$ of the Hopf $*$-algebra $A_q(SU(2))$ with the left coaction $L$ given by the comultiplication $\Delta$. This is the analogue of the left regular representation of $SU(2)$ on $L^2(S^2) \cong L^2(SU(2)/K)$.

It is a consequence of proposition 6.4(ii) that $A_{q,\sigma}(S^2) = A_{q,\sigma}(S^2) \cap A_q(SU(2))$ equals $\{0\}$ for $l \leq \frac{1}{2} + \mathbb{Z}_+$ and that $A_{q,\sigma}(S^2)$ equals the $(2l + 1)$-dimensional subspace spanned by $b^l_{i,0}(\tau, \sigma)$, $i = -l, -l + 1, \ldots, l$, for $l \in \mathbb{Z}_+$ and for any $\tau \in \mathbb{R} \cup \{\infty\}$. So, by §4.5, the corepresentation of $A_q(SU(2))$ in $A_{q,\sigma}(S^2)$ splits multiplicity free as the direct sum of the finite dimensional matrix corepresentations $t^l$, $l \in \mathbb{Z}_+$.

Using $(id \otimes X.)\Delta(\xi) = \Delta(X.\xi)$ with $X = A$ and (6.3) we get

$$\Delta(b^l_{i,0}(\tau, \sigma)) = \sum_{p=-l}^{l} a^l_{i,p}(\tau, \mu) \otimes b^l_{p,0}(\mu, \sigma),$$

so that we recover the generalised matrix elements in this realisation of the irreducible representation of spin $l \in \mathbb{Z}_+$ where the associated spherical elements play the role of basis in $A_{q,\sigma}(S^2)$.

Two separate cases of (8.3) have been worked out by Noumi and Mimachi before this interpretation of the Askey-Wilson polynomials on the quantum $SU(2)$ group, namely for $\tau = \mu = \infty$, yielding the associated spherical elements as a basis of $A_{q,\sigma}(S^2)$ in terms of big $q$-Jacobi polynomials with $\alpha = \beta \in \mathbb{Z}_+$, cf. [41], and for $\tau = \sigma = \mu = 0$, yielding the associated spherical elements as a basis of $A_{q,0}(S^2)$ in terms of the continuous $q$-ultraspherical polynomials, cf. [42].

So we can view $A_{q,\sigma}(S^2)$ as the deformed algebra of polynomials on the sphere $S^2$. It is possible to identify $A_{q,\sigma}(S^2)$ with one of the quantum 2-spheres of Podleś [45], cf. Noumi and Mimachi [43, prop. 2] (without proof) and Dijkhuizen and Koornwinder [11, §2]. Moreover, each of Podleś’s quantum 2-spheres can be obtained in this way.

8.3. Non-negative linearisation coefficients for $q$-Legendre polynomials. In this subsection we consider the linearisation coefficients of products of $q$-Legendre polynomials corresponding to the matrix elements $a^l_{0,0}(\sigma, \sigma)$. The linearisation coefficients are shown to be non-negative as already proved by Koornwinder [34, §7] using the framework of compact quantum Gelfand pairs. See also Floris [16] and Vainerman [50]. Here we give a somewhat different proof of this fact using Clebsch-Gordan coefficients.

Let $V^I$ be the $(2l + 1)$-dimensional vector space in which the representation $t^l$ of $\mathcal{U}_q(\mathfrak{su}(2))$ acts, then we have for any $X \in \mathcal{U}_q(\mathfrak{su}(2))$, cf. §4.7,

$$(t^{l_1} \otimes t^{l_2}) \circ \Delta(X) = C^* \left( \sum_{l = |l_1 - l_2|}^{l_1 + l_2} t^l(X) \right) C$$

where $C : V^{l_1} \otimes V^{l_2} \rightarrow \bigoplus_{l = |l_1 - l_2|}^{l_1 + l_2} V^l$ is a unitary intertwining operator. Its matrix coefficients with respect to the standard bases are known in terms of $q$-Hahn polynomials, cf.
[29]. Here we consider $C$ with respect to the orthonormal basis $v^{l_1,j_1}(\sigma) \otimes v^{l_2,j_2}(\sigma)$, $j_1 = -l_1, \ldots, l_1$, $j_2 = -l_2, \ldots, l_2$ of $V^{l_1} \otimes V^{l_2}$ and the orthonormal basis $v^{l,j}(\sigma)$, $j = -l, \ldots, l$, $l = |l_1 - l_2|, \ldots, l_1 + l_2$ of $\bigoplus_{|l_1 - l_2|} V^l$. Then $C$ becomes a unitary matrix with matrix coefficients defined by

$$C: v^{l_1,j_1}(\sigma) \otimes v^{l_2,j_2}(\sigma) \mapsto \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{j=-l}^{l} C^{l_1,l_2,l}_{j_1,j_2,j}(\sigma) v^{l,j}(\sigma).$$

If we now take $l_1, l_2 \in \mathbb{Z}_+$ we see that $v^{l_1,0}(\sigma) \otimes v^{l_2,0}(\sigma)$ belongs to the kernel of $(t^{l_1} \otimes t^{l_2}) \Delta(X_\sigma A)$. Since $C$ is an intertwiner we obtain from proposition 5.2

$$0 = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{j=-l}^{l} C^{l_1,l_2,l}_{0,0,j}(\sigma) \lambda_j(\sigma) v^{l,j}(\sigma).$$

Since $v^{l,j}(\sigma)$ forms a basis and $\lambda_j(\sigma) \neq 0$ for $j \neq 0$ we get $C^{l_1,l_2,l}_{0,0,j}(\sigma) = 0$ for $j \neq 0$.

Using the intertwining property we obtain

$$\sum_{l=|l_1-l_2|}^{l_1+l_2} \left| C^{l_1,l_2,l}_{0,0,0}(\sigma) \right|^2 \langle t^l(X) v^{l,0}(\sigma), v^{l,0}(\sigma) \rangle =$$

$$\langle (t^{l_1} \otimes t^{l_2}) \circ \Delta(X) \rangle \langle v^{l_1,0}(\sigma) \otimes v^{l_2,0}(\sigma), v^{l_1,0}(\sigma) \rangle \otimes v^{l_2,0}(\sigma) \rangle =$$

$$\sum_{(X)} \langle t^{l_1} (X_{(1)}) v^{l_1,0}(\sigma), v^{l_1,0}(\sigma) \rangle \langle t^{l_2} (X_{(2)}) v^{l_2,0}(\sigma), v^{l_2,0}(\sigma) \rangle,$$

where $\Delta(X) = \sum_{(X)} X_{(1)} \otimes X_{(2)}$. Using the duality between $A_q(SU(2))$ and $U_q(su(2))$ and the notation of §6 we have

$$\sum_{l=|l_1-l_2|}^{l_1+l_2} \left| C^{l_1,l_2,l}_{0,0,0}(\sigma) \right|^2 \langle a^{l_1}_{0,0}(\sigma,\sigma), X \rangle = \langle a^{l_1}_{0,0}(\sigma,\sigma), a^{l_2}_{0,0}(\sigma,\sigma), X \rangle.$$

Apply the one-dimensional representation $\tau_{e^{i\theta/2}q^{1/2}}$ to the resulting identity in $A_q(SU(2))$ and use corollary 7.7 to find

$$(8.4) p_{l_1}^{(0,0)}(\cos \theta; q\sigma, q\sigma | q^2) p_{l_2}^{(0,0)}(\cos \theta; q\sigma, q\sigma | q^2) = \sum_{l=|l_1-l_2|}^{l_1+l_2} c_l(l_1,l_2) p_{l_1}^{(0,0)}(\cos \theta; q\sigma, q\sigma | q^2)$$

with non-negative coefficients $c_l(l_1,l_2) \geq 0$. This also remains valid for the little $q$-Legendre polynomials corresponding to $\sigma = \infty$, but then we also have an explicit formula for the linearisation coefficient in terms of the square of a certain $q$-Hahn polynomial, since the Clebsch-Gordan coefficient is explicitly known in that case, cf. [29].

The linearisation coefficients in (8.4) for $\sigma = 0$, corresponding to the continuous $q$-ultraspherical polynomials, were explicitly given by Rogers in 1894, cf. [5, (4.18)], [7, (4.8)], [18, §8.5], and non-negativity can be read off from that expression.

Finally, we note that $A_q(SU(2))$ and the $*$-invariant coideal $J$ of $U_q(su(2))$ generated by $X_{\sigma} A$ is a compact quantum Gelfand pair in the sense of Floris [16] and Koornwinder [34, §5].
9. Discrete orthogonality relations

In this section we translate some of the unitarity properties of the generalised matrix elements $a^l_{i,j} (\tau, \sigma)$, cf. corollary 6.2(iii), into identities for $q$-special functions. We obtain a formula for dual $q$-Krawtchouk polynomials which contains as special cases the generating function (7.8) and the Poisson kernel for the dual $q$-Krawtchouk polynomials and the orthogonality relations for $q$-Krawtchouk polynomials. Discrete orthogonality relations for certain functions extending the orthogonality relations for the dual $q$-Krawtchouk polynomials are also obtained.

Proposition 9.1. For $l \in \frac{1}{2} \mathbb{Z}_+$, $i, j = -l, -l + 1, \ldots, l$ we have

$$
\tau_\lambda(a^l_{i,j}(\tau, \sigma)) = (-1)^{l-i} C^{l,i}_j(\sigma) C^{l,i}_j(\tau) q^{(l-i)(l-i-1)}(\lambda-2i) \lambda^{-2i} (\lambda^{-2} q^{\tau-\sigma+2l+2i}; q^2)_{l-i} \times \left(\frac{-\lambda^2 q^{\tau+\sigma+2l+2i}; q^2}{-\lambda^2 q^{\tau+\tau+2l+2i}; q^2}\right)_{\infty} q^{2i l-2(l-i)} q^{2-2(l-j)} - q^{2\tau-2l+2i} - q^{2\tau-2l-2j} q^{4i l} q^{2-\sigma+\tau+2l+2i} \lambda^{-2} q^{\tau-\sigma+2l+2i} \lambda^2 ; q^2.$$ 

Proof. We have to show that each of the four expressions in corollary 7.7 can be rewritten in this form. In the cases (i) and (ii) of corollary 7.7 we rewrite the terminating $4\varphi_3$-series of the Askey-Wilson polynomial by reversing the series summation. So we use, cf. [18, ex. 1.4.(ii)]

$$
(9.1) \quad 4\varphi_3\left(q^{-n}, a b c d q^{n-1}, a z, a / z ; q, q \right) = \frac{(a b c d q^{n-1}, a z, a / z ; q)_n (-1)^n q^{-\frac{1}{2} n(n-1)} 4\varphi_3\left(q^{-n}, q^{1-n} / (a b), q^{1-n} / (a c), q^{1-n} / (a d) ; q^2 q^{2n} / (a b c d), q^{1-n} / (a z), q^{1-n} / z / a ; q, q \right)}{(a b, a c, a d ; q)_n},
$$

with $q$ replaced by $q^2$. In case (i) we use (9.1) with $a = q^{1+\sigma-\tau}$ and in case (ii) we use (9.1) with $a = q^{1+\sigma+\tau+2j-2i}$. The values of the other parameters in the Askey-Wilson polynomials follow from corollary 7.7 and (3.2). Note that we use the symmetry of the Askey-Wilson polynomials in its four parameters. The complete expression of the proposition follows by a straightforward calculation.

In case (iii) and (iv) we use (9.1) as well. In case (iii) we take $a = -q^{1+\sigma-\tau}$ and in case (iv) we take $a = -q^{1-\sigma-\tau-2i-2j}$. The resulting $4\varphi_3$-series are not of the form as in the proposition, but they are transformed to it by use of Sears’s transformation, cf. [18, (2.10.4)],

$$
(9.2) \quad 4\varphi_3\left(q^{-n}, a, b, c ; d, e, f ; q, q \right) = a^n \frac{(e / a, f / a ; q)_n}{(e, f ; q)_n} 4\varphi_3\left(q^{-n}, a, d / b, d / c ; a q^{1-n} / e, a q^{1-n} / f ; q, q \right),
$$

where $abc = def q^{n-1}$. The observation $C^{d, -j}(-\sigma) = C^{d, j}(\sigma)$ and a straightforward calculation prove the correctness of the factor in front of the $4\varphi_3$-series in the proposition. □

Remark. Proposition 9.1 can also be proved from rewriting case (i) of corollary 7.7 as in the proof given above and then using the symmetry relations of lemma 6.6, or (7.7), for
the other cases. However, we still have to use transformations for \( \varphi_3 \)-series to obtain proposition 9.1 in full generality.

Two applications of proposition 9.1 are given in the rest of this section. First we derive a formula for dual \( q \)-Krawtchouk polynomials, from which we can obtain some special cases.

**Proposition 9.2.** For \( N \in \mathbb{Z}_+ \), \( i, j \in \{0, 1, \ldots, N\} \), \( \sigma, \tau \in \mathbb{R} \) we have

\[
\sum_{n=0}^{N} q^{\frac{1}{2}n(n+1)} q^{\frac{1}{2}n(n-1)} (q^N; q^{-1})_n t^n R_n(q^{-j} - q^{i-N-s}; q^{3}; N; q) R_n(q^{-i} - q^{i-N-\tau}; q^{3}, N; q) \\
= (-1)^{i} q^{\frac{1}{2}i(i-1)} q^{\frac{1}{2}i(\sigma-\tau)} t^i (t^{-1} q^{\frac{1}{2}(\sigma-\tau)+1-i}; q)_i (t q^{\frac{1}{2}(\tau-\sigma)-i}; q)_j \\
\times 4\varphi_3 \left( \begin{array}{c} q^{-i}, q^{-j}, -q^{-i}, -q^{j-N-\sigma} \\ q^{-N}, t^{1-i} q^{\frac{1}{2}(\sigma-\tau)+1-i}, t q^{\frac{1}{2}(\tau-\sigma)-i} \\ q, q \end{array} \right).
\]

**Proof.** Apply the one-dimensional representation \( \tau_\lambda \) to (6.2) and use proposition 9.1 to rewrite the left hand side as a \( 4\varphi_3 \)-series. Then use the explicit value for \( v_{n}^{l_{i,j}}(\sigma) \) of proposition 5.2 and replace \( q^2 \), \( l-j \), \( l-i \), \( l-n \), \( 2l \), \( \lambda^2 \) by \( q, j, i, n, N, t \) to obtain the proposition. \( \square \)

**Remark.** Proposition 9.2 can also be proved analytically as follows. Use for the dual \( q \)-Krawtchouk polynomials the \( 2\varphi_1 \)-series representation which can be obtained from [18, (III.7)]. Interchange the summations and note that the summation over \( n \) from 0 to \( N \) can be summed by the \( q \)-binomial formula [18, (II.4)]. Write the resulting double sum as a sum with a terminating balanced \( 3\varphi_2 \)-series in the summand. Using the \( q \)-Saalschütz summation formula [18, (II.12)] we obtain a \( 4\varphi_3 \)-series, which can be transformed using Sears’s transformation formula (9.2) to get a \( 4\varphi_3 \)-series as in the right hand side of proposition 9.2. I thank Mizan Rahman for providing me with this analytic proof of proposition 9.2.

Let us note some special cases of proposition 9.2. Firstly, take \( i = 0 \), so that one of the dual \( q \)-Krawtchouk polynomials and the \( 4\varphi_3 \)-series reduce to 1. We obtain the generating function (7.8) for the dual \( q \)-Krawtchouk polynomials as a special case of proposition 9.2. Secondly, take \( \sigma = \tau, t = 1 \) and note that

\[
(q^{1-i}; q)_i (q^{-i}; q)_j 4\varphi_3 \left( \begin{array}{c} q^{-i}, q^{-j}, -q^{3i}, -q^{j-N-\sigma} \\ q^{-N}, q^{1-i}, q^{-i} \\ q, q \end{array} \right) \\
= \sum_{p=0}^{\min(i,j)} \frac{(q^{-i}, q^{-j}, -q^{3i}, -q^{j-N-\sigma}; q)_p}{(q^{-N}, q; q)_p} q^p (q^{1-i+p}; q)_{i-p} (q^{p-i}; q)_{j-p} \\
= \frac{\delta_{i,j} (q^{-i}, q^{-i}, -q^{3i}, -q^{j-N-\sigma}; q)_i (q^{-N}, q; q)_i}{(q^{-N}, q; q)_i},
\]

since \( (q^{1-i+p}; q)_{i-p} = \delta_{i,p} \) and \( (q^{p-i}; q)_{j-p} = 0 \) for \( j > i \). We obtain the orthogonality relations for the \( q \)-Krawtchouk polynomials, cf. (3.12), as a special case of proposition
9.2. Finally, note that if we only specialise $\sigma = \tau$ in proposition 9.2, we obtain a Poisson kernel for the dual $q$-Krawtchouk polynomials. The Poisson kernel is a special case of the Gasper-Rahman formula for the Poisson kernel for the $q$-Racah polynomials, cf. [17, (3.11)], [18, (8.7.13)], but introduce a factor $(q; q)_s$ in the denominator. This calculation is far from trivial and I thank George Gasper and especially Mizan Rahman for making this identification precise. Note that for $\sigma \neq \tau$ proposition 9.2 does not fit into the Gasper-Rahman Poisson kernel.

As another application of proposition 9.1 we rewrite the orthogonality relations of corollary 6.2(iii) for $\sigma = \tau$ in terms of the $4\varphi_3$-series derived in proposition 9.1. Unfortunately, the orthogonality relations we find are not the orthogonality relations for the quantum discrete orthogonality relations for the so-called quantum $q$-Racah polynomials. It is known that in the special case $q$-Hahn polynomials, cf. (3.14). Moreover, (9.4) reduces to the $q$-Krawtchouk polynomials if $\tau \to \infty$.

As already stated these orthogonality relations do not correspond to the orthogonality relations for the $q$-Racah polynomials. It is known that in the special case $\sigma = \tau = \infty$ the discrete orthogonality relations for the so-called quantum $q$-Krawtchouk polynomials can be obtained in this way, cf. Koornwinder [30, §6]. This result is extended to the special case $\sigma = \infty$ by Noumi and Mimachi, cf. [44, §4], from which the discrete orthogonality relations of the (dual) $q$-Hahn polynomials, cf. §3.6, can be obtained. These derivations take place on the non-commutative level, in the algebra $A_q(SU(2))$ in case $\sigma = \tau = \infty$ and in an extension of the algebra $A_q(SU(2))$ in case $\sigma = \infty$. It might be possible that the orthogonality relations for the $q$-Racah polynomials can be obtained in a still larger extension of $A_q(SU(2))$ in a way similar as in Noumi and Mimachi [44]. I thank Masatoshi Noumi for explaining the ideas and backgrounds of [44] to me.

**Proposition 9.3.** The function $r_n(q^p) = r_n(q^p; \sigma, \tau, e^{i\theta}, N; q)$ satisfies the following discrete orthogonality relations:

\[
\sum_{p=0}^{N} r_n(q^p) r_m(q^p) \frac{(1 + q^{2p-N-\sigma})(q^{N-\sigma}; q)_p}{(-q^{2p-2N})^p} = \delta_{n,m} h_n^{-1}
\]

where

\[
h_n = (-q^{2N-\sigma+1})^n \frac{q^{2N+\sigma+\tau}(1 + q^{2n-N-\tau})(q^{N-\sigma}; q)_n}{(-q^{\tau}, q^{-\sigma}; q)_{N+1} (q, q^{-1-\tau}; q)_n} \times \frac{(q^{1/2}(\sigma-\tau), \xi^\theta, q^{1/2}(\sigma-\tau) e^{-i\theta}; q)_n}{(-q^{-1/2}(\tau+\sigma), -q^{1/2}(\tau+\sigma) e^{-i\theta}; q)_n}.
\]

**Remark.** Note that the weights in (9.4) are equal to weights in the orthogonality relations for the dual $q$-Krawtchouk polynomials, cf. (3.14). Moreover, (9.4) reduces to the orthogonality relations for the dual $q$-Krawtchouk polynomials if $\tau \to \infty$. 

orthonormal basis for all \(i, j\), cf. corollary 6.2(i). Then we have

\[
\tau \quad \text{since the algebra homomorphisms} \quad \text{operator. This section is motivated by the paper [8] by Badertscher and Koornwinder.}
\]

\[
\begin{aligned}
\pi_{\theta/2}(a_{l, p}^l(\tau, \sigma)) &= (-1)^{l-i} C^{l,i}(\tau) q^{l-i}(l-i-1) q^{l-i}(\sigma-\tau) e^{-ii\theta} \frac{(e^{-i\theta} q^{\tau - \sigma + 2 - 2l + 2i}; q^2)_{l-i}}{(-e^{i\theta} q^{\tau + \sigma - 2l + 2i}; q^2)_{l-i}} \\
&\times C^{l,p}(\sigma) r_{l-i}(q^{2(l-p)}; \sigma, \tau, e^{i\theta}, 2l; q^2),
\end{aligned}
\]

(9.5)

where we used \(\pi_{\theta/2} = \pi_{e_{\theta/2}}\), cf. §4.6. From corollary 6.2(iii) we obtain the orthogonality relations

\[
\sum_{p=-l}^{l} \pi_{\theta/2}(a_{l, p}^l(\tau, \sigma)) \pi_{\theta/2}(a_{l, p}^l(\tau, \sigma)) = \delta_{i,j},
\]

since \(\pi_{\theta/2}\) is a *-representation of \(A_q(SU(2))\). In this expression we use (9.5) and

\[
|C^{l,p}(\sigma)|^2 = \frac{q^{4l+2\sigma}(1 + q^{-4p - 2\sigma})}{(-q^{2(l-p)-8l-2\sigma})^{l-p}(-q^{2\sigma}; q^2)_{2l+1}},
\]

cf. proposition 5.2. Now replace \(l - i, l - j, l - p 2l, q^2\) by \(n, m, p, N, q\) to obtain the statement of the proposition. □

The orthogonality relations dual to (9.4), which are obtainable from

\[
\sum_{p=-l}^{l} \pi_{\theta/2}(a_{l, p}^l(\tau, \sigma)) \pi_{\theta/2}(a_{l, p}^l(\tau, \sigma)) = \delta_{i,j},
\]

cf. corollary 6.2(iii), and proposition 9.1, can be rewritten as the orthogonality relations (9.4) with \(\sigma\) and \(\tau\) interchanged by lemma 6.6, cf. also (7.7).

### 10. Spherical and associated spherical elements

In this section we show how the associated spherical elements can be obtained from the spherical elements. In case of the continuous \(q\)-ultraspherical polynomials we explicitly give an operator mapping the continuous \(q\)-Legendre polynomial to the continuous \(q\)-ultraspherical polynomials. This operator is a \(q\)-Hahn polynomial in a very simple shift operator. This section is motivated by the paper [8] by Badertscher and Koornwinder.

Consider the irreducible unitary matrix corepresentation \(a^l(\sigma) = (a_{i,j}^l(\sigma, \sigma))_{i,j=-l,\ldots,l}\), cf. corollary 6.2(i). Then we have

\[
\langle t^l(D^2) v^{l,j}(\sigma), v^{l,i}(\sigma) \rangle = \langle D^2, a_{i,j}^l(\sigma, \sigma) \rangle = \tau_{q^{-1}}(a_{i,j}^l(\sigma, \sigma)),
\]

since the algebra homomorphisms \(\tau_{q^{k/2}}: A_q(SU(2)) \to \mathbb{C}\) given in §4.6, and \(\xi \mapsto \langle A^k, \xi \rangle\), \(k \in \mathbb{Z}\), coincide on the generators of \(A_q(SU(2))\). From corollary 7.7 we get

\[
\langle t^l(D^2) v^{l,j}(\sigma), v^{l,i}(\sigma) \rangle = C(q^{-2}; q^2)_{|i-j|}
\]

for all \(i, j = -l, -l + 1, \ldots, l\). So \(t^l(D^2)\) is a tridiagonal matrix with respect to the orthonormal basis \(v^{l,i}(\sigma), i = -l, -l + 1, \ldots, l\). The subdiagonal and superdiagonal are non-zero for \(\sigma \neq \pm \infty\).

In the remainder of this section we assume \(l \in \mathbb{Z}_+\) fixed. Let \(A(\sigma) = (A_{i,j}(\sigma))\) be the matrix of \(t^l(D^2)\) with respect to the basis \(\{v^{l,i}(\sigma) \mid i = -l, -l + 1, \ldots, l\}\).
Proposition 10.1. $A(\sigma)$ is a tridiagonal, real, symmetric matrix with positive diagonal entries. Moreover, $A_{-i,-j}(-\sigma) = A_{i,j}(\sigma)$.

Proof. The operator $t^l(D^2)$ is self-adjoint, and

$$A_{i,j}(\sigma) = \langle t^l(D^2)v^{l,j}(\sigma), v^{l,i}(\sigma) \rangle = \sum_{m=-l}^l v^{l,j}_m(\sigma)\overline{v^{l,i}_m(\sigma)}q^{2m}.$$ 

The explicit value of the coefficients $v^{l,j}_m(\sigma)$ given in proposition 5.2 shows that this yields a real value, which is obviously positive for $i = j$. The last statement follows from (6.15). □

If we denote the $k$-th power of $A(\sigma)$ by $A^k(\sigma)$, then we get $A^k_{i,j}(\sigma) = A^k_{-i,-j}(-\sigma)$ by induction with respect to $k$. Indeed, the cases $k = 0, 1$ being known, we get

$$A^{k+1}_{i,j}(\sigma) = \sum_{p=-l}^l A_{i,p}(\sigma)A^k_{p,j}(\sigma) = \sum_{p=-l}^l A_{-i,-p}(\sigma)A^k_{-p,-j}(\sigma) = A^{k+1}_{-i,-j}(\sigma).$$

Consequently, for $k = 0, 1, \ldots, l$ and with $A_k(\sigma) = A^k_{k,0}(\sigma)$

(10.1) $$t^l(D^{2k})v^{l,0}(\sigma) = A_k(\sigma)v^{l,k}(\sigma) + A_k(-\sigma)v^{l,-k}(\sigma) + \sum_{p=-k+1}^{k-1} c_p(\sigma)v^{l,p}(\sigma)$$

for certain constants $c_p(\sigma) = A^k_{p,0}(\sigma)$ satisfying $c_p(\sigma) = c_{-p}(-\sigma)$.

So the (ordered) set of vectors

$$v^{l,0}(\sigma), \quad t^l(D^2)v^{l,0}(\sigma), \quad t^l(D^4)v^{l,0}(\sigma), \quad \ldots, \quad t^l(D^{2l})v^{l,0}(\sigma)$$

is linearly independent. Applying the Gram-Schmidt orthogonalisation process to these vectors yields

$$v^{l,0}(\sigma), \quad A_1(\sigma)v^{l,1}(\sigma) + A_1(-\sigma)v^{l,-1}(\sigma), \quad \ldots, \quad A_l(\sigma)v^{l,l}(\sigma) + A_l(-\sigma)v^{l,-l}(\sigma),$$

as a set of orthogonal vectors because of the specific form of (10.1).

Proposition 10.2. There exists a set of monic orthogonal polynomials $\{p_k\}_{k=0}^l$ of degree $k$ such that

(10.2) $$t^l(p_k(D^2))v^{l,0}(\sigma) = A_k(\sigma)v^{l,k}(\sigma) + A_k(-\sigma)v^{l,-k}(\sigma)$$

for $k = 0, 1, \ldots, l$.

Proof. The Gram-Schmidt orthogonalisation process yields the $k$-th orthogonal vector as a linear combination of the zeroth until the $k$-th vector of the linearly independent set to be orthogonalised. This shows the existence of the polynomial $p_k$ in (10.2), which has to have degree $k$ and leading coefficient 1 because of (10.1).
Consider the moment functional $\mathcal{L}_\sigma^l$ defined by

\begin{equation}
\mathcal{L}_\sigma^l(x^k) = \langle t^l(D^{2k})v^{l,0}(\sigma), v^{l,0}(\sigma) \rangle,
\end{equation}

then, for $0 \leq n, m \leq l$,

\begin{equation}
\mathcal{L}_\sigma^l(\tilde{p}_n(x)p_m(x)) = \langle t^l(p_n(D^2))v^{l,0}(\sigma), t^l(p_n(D^2))v^{l,0}(\sigma) \rangle = \delta_{n,m} \begin{cases}
A_n^2(\sigma) + A_n^2(-\sigma), & n > 0,
1, & n = 0,
\end{cases}
\end{equation}

since $D^* = D$. \(\Box\)

More explicit formulas for the moments of the moment functional $\mathcal{L}_\sigma^l$ are given by

\begin{equation}
\mathcal{L}_\sigma^l(x^k) = \sum_{n=-l}^l |v_n^{l,0}(\sigma)|^2 q^{2kn} = 4\varphi_3 \left( q^{-2l}, q^{2+2l}, q^{-2k}, q^{2+2k} ; q^{2}, q^{2} \right).
\end{equation}

The first equality follows from proposition 5.2, (4.19) and (10.3), and the second equality follows from the observation that (10.3) equals $\tau_{q^{-k}}(d_{0,0}^l(\sigma, \sigma))$ and corollary 7.7. So the orthogonality measure corresponding to the moment functional $\mathcal{L}_\sigma^l$ is the finite discrete measure with positive weights $|v_n^{l,0}(\sigma)|^2$ at the points $x_n = q^{2n}$, $n = -l, -l + 1, \ldots, l$.

Unfortunately, I have not (yet) been able to obtain an explicit formula for the orthogonal polynomials $\{p_k\}_{k=0}^l$ unless $\sigma = 0$. In this case we can obtain special q-Hahn polynomials by identifying the orthogonality measures. We start with the lemma in which we do the necessary computation, cf. [33, p. 805].

**Lemma 10.3.** In case $\sigma = 0$ the weights for $l - n$ even are given by

\begin{equation}
|v_n^{l,0}(0)|^2 = q^{l+n} \frac{(q^2; q^4)_{(l-n)/2}(q^2; q^4)_{(l+n)/2}}{(q^4; q^4)_{(l-n)/2}(q^4; q^4)_{(l+n)/2}}.
\end{equation}

and $|v_n^{l,0}(0)|^2 = 0$ for $l - n$ odd.

**Proof.** The proof is based on Andrews’s summation formula, cf. [18, (II.17)],

\begin{equation}
4\varphi_3 \left( q^{-n}, aq^n, c, -c^{2}, q, q^2 \right) = \begin{cases}
0, & \text{if } n \text{ is odd},
c^n(q, aq/c^2; q^2)_{n/2}, & \text{if } n \text{ is even}.
\end{cases}
\end{equation}

An application of this formula with $a$, $c$, $n$ and $q$ replaced by $0$, $q^{-2l}$, $l - n$ and $q^2$ shows that, cf. (3.13),

\begin{equation}
R_{l-n}(q^{-2l} - q^{-2l}; 1, 2l; q^2) = \begin{cases}
0, & \text{if } l - n \text{ is odd},
q^{-2(l-n)}(q^2; q^4)_{(l-n)/2} \frac{(q^2-4l; q^4)_{(l-n)/2}}{(q^2; q^4)_{(l-n)/2}}, & \text{if } l - n \text{ is even}.
\end{cases}
\end{equation}
This proves that \(|v_n^{l,0}(0)|^2 = 0\) for \(l - n\) odd, so it remains to calculate the value for \(l - n\) even.

It follows from proposition 5.2 that we can rewrite \(|C_n^{l,0}(0)|^2 = q^{2l}(q^2; q^4)_l/(q^4; q^4)_l\). Hence, proposition 5.2 and the evaluation of the dual \(q\)-Krawtchouk polynomial in this case imply that

\[
|v_n^{l,0}(0)|^2 = q^{2l}(q^2; q^4)_l/(q^4; q^4)_l \left(\frac{(q^2; q^4)_l}{(q^4; q^4)_l}\right)^{l-n} \left(\frac{(q^2; q^4)_{l-n}}{(q^4; q^4)_{l-n}}\right)^2
\]

by inverting one of the \(q\)-shifted factorials. Now we use

\[
\frac{(q^2; q^4)_{l-n}/2}{(q^4; q^4)_{l-n}} = \frac{1}{(q^4; q^4)_{l-n}/2}, \quad \frac{(q^4; q^4)_{l-n}}{(q^4; q^4)_{l-n}/2} = (q^4; q^4)_{l-n}/2,
\]

\[
\frac{(q^4; q^4)_{l-n}/2}{(q^4; q^4)_{l-n}} = \frac{1}{(q^4; q^4)_{l-n}/2}, \quad \frac{(q^4; q^4)_{l-n}}{(q^4; q^4)_{l-n}/2} = (q^2; q^4)_{l-n}/2.
\]

Collecting the powers of \(q\) proves the lemma. \(\square\)

Using the moment functional \(\mathcal{L}_0^l\) introduced in the proof of lemma 10.3 we see that in case \(\sigma = 0\) we have to find orthogonal polynomials \(\{p_n\}_{n=0}^l\) satisfying

\[
\sum_{n=-l, \ l-n \in 2\mathbb{Z}} q^{l+n}(q^2; q^4)_{(l-n)/2}(q^4; q^4)_{(l+n)/2} \tilde{p}_k(q^{2n})p_m(q^{2n}) = 0, \quad k \neq m,
\]

\[
\sum_{n=0}^l \frac{q^{2l}n(q^2; q^4)_n(q^4; q^4)_l-n}{(q^4; q^4)_n(q^4; q^4)_l-n} \tilde{p}_k(q^{2l-4n})p_m(q^{2l-4n}) = 0, \quad k \neq m.
\]

(10.4)

The weights in (10.4) almost coincide with the Fourier coefficients of the continuous \(q\)-Legendre polynomial, cf. §3.4, [5, (3.1)], [18, (7.4.2)] with \(\beta = q^{1/2}\), which is analogous to the general classical case described in [8, prop. 5.2]. Comparing (10.4) with the orthogonality relations for the \(q\)-Hahn polynomials, cf. (3.11), leads to

\[
p_m(x) = q^{2lm}(q^2; q^4)_m(q^4; q^4)_m Q_m(q^{-2l}x; q^{-2}, q^{-2}, l; q^4), \quad m = 0, 1, \ldots, l,
\]

(10.5)

where the constant follows from the condition that \(p_m\) is monic. So we have for the polynomial \(p_m\) defined in (10.5)

\[
t^l(p_m(D^2))v^{l,0}(0) = A_m(0)(v^{l,m}(0) + v^{l,-m}(0))
\]

and consequently, using the notation of §§4, 6

\[
\langle X, p_m(D^2), a_{i,0}(\tau, 0) \rangle = \langle t^l(X)t^l(p_m(D^2))v^{l,0}(0), v^{l,i}(\tau) \rangle
\]

\[
= A_m(0)(\langle t^l(X)v^{l,m}(0), v^{l,i}(\tau) \rangle + \langle t^l(X)v^{l,-m}(0), v^{l,i}(\tau) \rangle)
\]

\[
= A_m(0)(\langle X, a_{i,m}(\tau, 0) \rangle + \langle X, a_{i,-m}(\tau, 0) \rangle),
\]
so that as an identity in $\mathcal{A}_q(SU(2))$ we have

$$p_m(D^2).a_{i,0}^l(\tau,0) = A_m(0)(a_{i,m}^l(\tau,0) + a_{i,-m}^l(\tau,0)),$$

with $p_m$ defined in terms of a $q$-Hahn polynomial as in (10.5).

**Proposition 10.4.** For $m, l \in \mathbb{Z}_+$, $m \leq l$ we have

$$Q_m(q^{-l}E_;q^{-1}, q^{-1}, l; q^2)p_{l,m}^{(0,0)}(\cos \theta; 1, q^{\frac{1}{2}} q^2 | q) =$$

$$\frac{1}{2}(q^{l+1}; q)_m e^{-i\theta}\left\{ (e^{i\theta} q^{\frac{1}{2}(1-\tau)}, -e^{i\theta} q^{\frac{1}{2}(1+\tau)}; q)_m p_{l-m}^{(m,m)}(\cos \theta; 1, q^{\frac{1}{2}} q^2 | q) + (e^{i\theta} q^{\frac{1}{2}(1+\tau)}, -e^{i\theta} q^{\frac{1}{2}(1-\tau)}; q)_m p_{l-m}^{(m,m)}(\cos \theta; 1, q^{\frac{1}{2}} q^2 | q) \right\},$$

where $Q_m$ denotes a $q$-Hahn polynomial (3.10), $p_{l-m}^{(m,m)}$ an Askey-Wilson polynomial (3.2) and $E_-$ is the operator acting on trigonometric polynomials $f(e^{i\theta})$ by $(E_- f)(e^{i\theta}) = f(q^{-1}e^{i\theta})$.

In particular, for $\tau = 0$ we have

$$Q_m(q^{-l}E_-; q^{-1}, q^{-1}, l; q^2)C_l(\cos \theta; q | q^2) =$$

$$\frac{(q; q^2)_m}{(q^{2l-2m+2}; q^2)_m} e^{-i\theta}(qe^{2i\theta}; q^2)_m C_{l-m}(\cos \theta; q^{1+2m} | q^2),$$

where $C_l(\cos \theta; \beta | q)$ denotes a continuous $q$-ultraspherical polynomial (3.4).

**Remark.** A generalisation of (10.8) to arbitrary continuous $q$-ultraspherical polynomials using a generating function for the dual $q$-Hahn polynomials is given in [26, prop. 1.1].

**Proof.** The proof consists of applying the one-dimensional representation $\tau_\lambda$ to (10.6) for $i = 0$. In the proof of corollary 7.7 we already noted that $\tau_\lambda(D, \xi) = \tau_{\lambda q^{-1/2}}(\xi), \xi \in \mathcal{A}_q(SU(2)).$ More generally we have $\tau_\lambda(D^{2j}, \xi) = \tau_{\lambda q^{-j}}(\xi), \xi \in \mathcal{A}_q(SU(2))$. Introduce the operator $F_-$ acting on Laurent polynomials in $\lambda$ by $(F_- f)(\lambda) = f(q^{-1}\lambda)$, then we get $\tau_\lambda(p(D^2), \xi) = p(F_-)\tau_\lambda(\xi)$ for arbitrary polynomials $p$.

From this observation, corollary 7.7 and (10.5) we get for some constant $C$ the expression

$$Q_m(q^{-2l}F_-; q^{-2}, q^{-2}, l; q^4)p_{l,m}^{(0,0)}((q^{-1}\lambda^2 + q\lambda^{-2})/2; 1, q^\tau | q^2) =$$

$$C\lambda^{-2m}\left\{ (\lambda^2 q^{-\tau}, -\lambda^2 q^{\tau}; q^2)_m p_{l-m}^{(m,m)}((q^{-1}\lambda^2 + q\lambda^{-2})/2; 1, q^\tau | q^2) + (\lambda^2 q^{\tau}, -\lambda^2 q^{-\tau}; q^2)_m p_{l-m}^{(m,m)}((q^{-1}\lambda^2 + q\lambda^{-2})/2; 1, q^{-\tau} | q^2) \right\}$$

after applying $\tau_\lambda$ to (10.6) for $i = 0$. In this expression we replace $q^{-1}\lambda^2, q^2$ by $e^{i\theta}, q$, so that the operator $F_-$ goes over into $E_-$. We obtain the expression of the proposition apart from the explicit expression for the constant involved. The constant is calculated by comparing the coefficients of $e^{-il\theta}$ on both sides of (10.7). Since the coefficient of $e^{-i(l-m)\theta}$ in $p_{l-m}^{(m,m)}(\cos \theta; 1, q^{\frac{1}{2}} q^2 | q)$ is $(q^{2l}; q^{-1})_{l-m}$, cf. [7, p.5], and the $q$-Hahn polynomial of argument 1 equals 1 we have to solve $(q^{2l}; q^{-1})_{l-m} = 2C(q^{2l}; q^{-1})_{l-m}$. This proves (10.7).

To prove the special case $\tau = 0$ we note that both terms on the right hand side of (10.7) are equal and that we can combine the $q$-shifted factorials involving $e^{i\theta}$. Because of (3.5) the last statement of the proposition follows after a short manipulation of $q$-shifted factorials. □
11. Characters

The characters of the irreducible unitary representations \( t^l \) of the quantum \( SU(2) \) group are investigated in this section. We obtain expansions for the Chebyshev polynomial of the second kind in terms of Askey-Wilson polynomials.

For any finite dimensional representation \( t \) of the algebra \( U_q(sl(2)) \) we define the character \( \chi \) of the representation as a linear functional on \( U_q(sl(2)) \) by \( \langle X, \chi \rangle = \text{tr}(t(X)) \). Here \( \text{tr} \) denotes the trace operator. In particular for the irreducible \( * \)-representations \( t^l \) of \( U_q(sl(2)) \), cf. §4.5, of spin \( l \in \frac{1}{2}\mathbb{Z}_+ \), we define the character \( \chi_l \) by \( \langle X, \chi_l \rangle = \text{tr}(t^l(X)) \). Since the matrix elements of the irreducible \( * \)-representation \( t^l \) are elements of \( A_q(sl(2)) \) we see that \( \chi_l \in A_q(sl(2)) \) and

\[
\chi_l = \sum_{n=-l}^l t^l_{n,n} = \sum_{n=-l}^l a^l_{n,n}(\sigma, \sigma),
\]

where the last equality follows from corollary 6.2(i) and §4.5, or from the fact that the trace operator is independent of the basis.

Since \( t^l \) is a representation we find for \( X, Y \in U_q(sl(2)) \)

\[
\langle XY, \chi_l \rangle = \text{tr}(t^l(X)t^l(Y)) = \text{tr}(t^l(Y)t^l(X)) = \langle YX, \chi_l \rangle
\]

and from (4.10) we then find that the character \( \chi_l \), and, for that matter, any character of a finite dimensional representation, is cocalgebraic. This means \( \Delta(\chi_l) = \omega \circ \Delta(\chi_l) \) where \( \omega: \eta \otimes \xi \mapsto \xi \otimes \eta \) denotes the flip automorphism of \( A_q(sl(2)) \otimes A_q(sl(2)) \) and \( \Delta \) is the comultiplication of \( A_q(sl(2)) \), cf. §4.2.

The characters of the irreducible unitary representations and the algebra of cocalgebraic elements have been investigated by Woronowicz [58, §5, App. A.1]. We collect some of Woronowicz's results in the following proposition.

**Proposition 11.1.** (i) The algebra of cocalgebraic elements is generated by \( (\alpha + \delta)/2 \).

(ii) The Haar functional on the algebra of cocalgebraic elements is given by

\[
h\left(p\left(\frac{1}{2}(\alpha + \delta)\right)\right) = \frac{2}{\pi} \int_{-1}^1 p(x) \sqrt{1 - x^2} \, dx
\]

for any polynomial \( p \).

(iii) For \( l \in \frac{1}{2}\mathbb{Z}_+ \) we have \( \chi_l = U_{2l}((\alpha + \delta)/2) \), where \( U_n(\cos \theta) = \sin(n+1)\theta / \sin \theta \) is the Chebyshev polynomial of the second kind.

By (11.1) and proposition 11.1(iii) we get

\[
U_{2l}((\alpha + \delta)/2) = \sum_{n=-l}^l a^l_{n,n}(\sigma, \sigma)
\]
for arbitrary $\sigma \in \mathbb{R}$. Next we apply the one-dimensional representation $\tau_{q^{1/2}e^{i\theta/2}}$, cf. §4.6, to this identity. We obtain from corollary 7.7 the identity

$$(11.2)$$

$$
\sum_{m=0}^{[l-1]} \left| C_{l-m}(\sigma) \right|^2 q^{-l} \frac{(q^{4l}; q^{-2})_m}{(q^{4l}; q^{-2})_l} e^{-i(l-m)\theta} (-e^{i\theta} q^{1+2\sigma}; q^2)^{2l-2m} p_m^{(0,2l+2m)}(\cos \theta; q^\sigma, q^\sigma | q^2) 
$$

$$
+ \sum_{m=0}^{[l-1]} \left| C_{l-m}(-\sigma) \right|^2 q^{-l} \frac{(q^{-4l}; q^{-2})_m}{(q^{4l}; q^{-2})_l} e^{-i(l-m)\theta} (-e^{i\theta} q^{1-2\sigma}; q^2)^{2l-2m} p_m^{(0,2l-2m)}(\cos \theta; q^{-\sigma}, q^{-\sigma} | q^2) 
$$

$$
+ \frac{1}{(q^{4l}; q^{-2})_l} \frac{|C_l(\sigma)|^2 q^{-l}}{p_l^{(0,0)}(\cos \theta; q^\sigma, q^\sigma | q^2)} U_{2l}((q^{1/2}e^{i\theta/2} + q^{-1/2}e^{-i\theta/2})/2),
$$

where we have replaced $l - n$ by $m$. The $+$ indicates that the last term in (11.2) is only occurring for $l \in \mathbb{Z}_+$, $[a]$ denotes the smallest integer greater than or equal to $a \in \mathbb{R}$ and $C_{l-m}(\sigma)$ is given in proposition 5.2. Note that the right hand side of (11.2) is independent of $\sigma$. Askey and Wilson [7, (2.18)] already noted that the Chebyshev polynomials of the second kind can be obtained as a special case of the Askey-Wilson polynomials. Equation (11.2) is a $q$-analogue of Vilenkin [56, Ch. 3, §7.1(10)], Vilenkin and Klimyk [57, Vol. 1, §6.9.2(6)].

Now that we have represented (11.1) for $\sigma \in \mathbb{R}$ as (11.2) we can proceed to find an identity for special functions corresponding to the case $\sigma \to \infty$ of (11.1), i.e. the first equality of (11.1). Applying the one-dimensional $*$-representation $\pi_\theta$ gives the known equality $U_{2l}((q^{1/2}e^{i\theta/2} + q^{-1/2}e^{-i\theta/2})/2)$.

To obtain another identity we apply the infinite dimensional $*$-representation $\pi_\infty = \pi_0^\infty$, cf. §4.6, to the first equality in (11.1). So we obtain an identity for bounded linear operators acting on $l^2(\mathbb{Z}_+)$. To rewrite this as an identity for functions we try to find eigenvectors for $\pi_\infty(\frac{1}{2}(\alpha + \delta))$. Note that the matrix elements $t_{n,m}^l$ already act nicely on the standard orthonormal basis $\{f_n\}_{n \in \mathbb{Z}_+}$ of $l^2(\mathbb{Z}_+)$ by (4.20), cf. §4.6.

Now $\sum_{n=0}^{\infty} c_n \lambda^n$ is an eigenvector for the eigenvalue $\lambda$ of the self-adjoint operator $\pi_\infty(\frac{1}{2}(\alpha + \delta))$ if and only if

$$
c_{n+1} \sqrt{1 - q^{2n+2} + c_{n-1} \sqrt{1 - q^{2n}}} = 2\lambda c_n, \quad n \in \mathbb{Z}_+
$$

with the convention $c_{-1} = 0$, $c_0 = 1$. So we can consider $c_n$ as a polynomial of degree $n$ in $\lambda$. A suitable normalisation, $c_n = (q^2; q^2)_n^{-1/2} p_n(\lambda)$, leads to the three-term recurrence relation

$$
p_{n+1}(\lambda) + (1 - q^{2n})p_{n-1}(\lambda) = 2\lambda p_n(\lambda), \quad n \in \mathbb{Z}_+,
$$

with initial conditions $p_{-1}(\lambda) = 0$, $p_0(\lambda) = 1$. The solution $c_n = (q^2; q^2)_n^{-1/2} H_n(\lambda | q^2)$ follows by comparing this with the three-term recurrence relation for the continuous $q$-Hermite polynomials, cf. (3.6).

However, the corresponding eigenvector $\sum_{n=0}^{\infty} (q^2; q^2)_n^{-1/2} H_n(\lambda | q^2) f_n$ is not an element of the Hilbert space $l^2(\mathbb{Z}_+)$, as can be seen from the asymptotic expansion for the Askey-Wilson polynomials, cf. Ismail and Wilson [19, (1.11)-(1.13)], or from the $q$-analogue of
Mehler’s formula for the continuous $q$-Hermite polynomials, cf. Askey and Ismail [5, (6.5)]. To circumvent this problem we introduce the vector $v_\lambda(N) = \sum_{n=0}^{N}(q^2; q^2)_n^{-1/2}H_n(\lambda \mid q^2)f_n$ for which we have

$$
\pi^\infty \left( \frac{1}{2}(\alpha + \delta) \right) v_\lambda(N) = \lambda v_\lambda(N) + \frac{H_N(\lambda \mid q^2)}{2\sqrt{(q^2; q^2)_N}} \sqrt{1 - q^{2N+2}} f_{N+1} - \frac{H_{N+1}(\lambda \mid q^2)}{2\sqrt{(q^2; q^2)_N}} f_N
$$

and

$$
(11.3) \quad \pi^\infty \left( U_{2l} \left( \frac{1}{2}(\alpha + \delta) \right) \right) v_\lambda(N) = U_{2l}(\lambda)v_\lambda(N) + \sum_{p=N-2l+1}^{N+2l} c_p f_p
$$

for certain constants $c_p$, $N > 2l$ and $\lambda \in \frac{1}{2}\mathbb{Z}_+$.

**Proposition 11.2.** For $l \in \frac{1}{2}\mathbb{Z}_+$, $p \in \mathbb{Z}_+$ we have

$$
U_{2l}(\cos \theta) H_p(\cos \theta \mid q^2) = \sum_{m=0}^{[l-1]} p_m(q^{2p}; 1, q^{4l-4m}; q^2) H_{p+2l-2m}(\cos \theta \mid q^2)
$$

$$
+ \sum_{m=0}^{[l-1]} (q^{2p}; q^{-2})_{2l-2m} p_m(q^{2p-4l+4m}; 1, q^{4l-4m}; q^2) H_{p-2l+2m}(\cos \theta \mid q^2)
$$

$$
+ \frac{1}{2} p_l(q^{2p}; 1, q^2) H_p(\cos \theta \mid q^2)
$$

where $\frac{1}{2}$ means that the last term is absent for $l \in \frac{1}{2} + \mathbb{Z}_+$ and $[a]$ denotes the smallest integer greater than or equal to $a \in \mathbb{R}$. Here $U_{2l}$ denotes a Chebyshev polynomial of the second kind, $H_p$ a continuous $q$-Hermite polynomial, cf. §3.4, and $p_m$ a little $q$-Jacobi polynomial (3.8).

**Proof.** From (4.20) and §4.6 we get for $l \in \frac{1}{2}\mathbb{Z}_+$, $l - n \in \mathbb{Z}_+$, $n \geq 0$, the operators, which act on a basis vector $f_p$ by

$$
\pi^\infty (t_{n,n}^l) f_p = p_{l-n}(q^{2p}; 1, q^{4n}; q^2) \sqrt{(q^{2p+2}; q^2)_{2n}} f_{p+2n}
$$

$$
\pi^\infty (t_{-n,-n}^l) f_p = p_{l-n}(q^{2p-4n}; 1, q^{4n}; q^2) \sqrt{(q^{2p}; q^2)_{2n}} f_{p-2n},
$$

with the convention that $f_{-n} = 0$ for $n \in \mathbb{N}$. Now we have the identity

$$
(11.5) \quad \langle \sum_{n=-l}^{l} \pi^\infty (t_{n,n}^l) f_p, v_\lambda(N) \rangle = \langle U_{2l}((\alpha+\delta)/2) f_p, v_\lambda(N) \rangle = \langle f_p, U_{2l}((\alpha+\delta)/2) v_\lambda(N) \rangle,
$$

since $\alpha + \delta$ is a self-adjoint element of $A_q(SU(2))$. Now we use (11.4) and (11.3) in (11.5). For $N \geq p + 2l$ the additional terms on the right hand side of (11.3) are of no importance and we can use the explicit value $\langle f_{p \pm 2n}, v_\lambda(N) \rangle = (q^2; q^2)^{-1/2} H_{p \pm 2n}(\lambda \mid q^2)$. We obtain the proposition after substitution $m = l - n$. □
Instead of a chopped-off eigenvector \( v_\lambda(N) \) of \( \pi^\infty(\frac{1}{2}(\alpha+\delta)) \) we can also use the spectral theorem for the self-adjoint operator \( \pi^\infty(\frac{1}{2}(\alpha+\delta)) \), which is a Jacobi matrix with respect to the basis \( f_n \) of \( \ell^2(\mathbb{Z}_+) \), cf. e.g. Dombrowski [12].

Since \( U_{2l}(\cos \theta) \) and \( H_n(\cos \theta | q) \) are continuous \( q \)-ultraspherical polynomials for \( \beta = q \) and \( \beta = 0 \), the result in proposition 11.2 can also be derived from Rogers’s linearisation and connection formulas for the continuous \( q \)-ultraspherical polynomials, cf. [5, (4.15), (4.18)], [7, (4.7), (4.8)].

The limit \( q \uparrow 1 \) of proposition 11.2 to the relation can be handled using the devices developed by Van Assche and Koornwinder [53, thm. 1] to show that Koornwinder’s addition formula for the \( q \)-Legendre polynomials [32] tends to the addition formula for the Legendre polynomials as \( q \uparrow 1 \). See also Koelink [25, §5] for another application of that theorem.

A straightforward limit sending \( q \uparrow 1 \) of proposition 11.2 can also be given. Since

\[
\lim_{q \uparrow 1} H_p(\cos \theta | q^2) = 2^p \cos^p \theta \]

by either letting \( q \uparrow 1 \) in the three-term recurrence relation (3.6) for the continuous \( q \)-Hermite polynomials or by letting \( q \uparrow 1 \) in the explicit expression, cf. (3.4), for the continuous \( q \)-Hermite polynomials and using the binomial theorem, we obtain, after replacing \( 2l \) by \( l \),

\[
U_l(\cos \theta) 2^p \cos^p \theta = \sum_{m=0}^{\lfloor l/2 \rfloor} \binom{-m, l - m + 1}{1} 2^{p+l-2m} \cos^{p+l-2m} \theta,
\]

where \( \lfloor a \rfloor \) is the largest integer smaller than or equal to \( a \in \mathbb{R} \). Applying Vandermonde’s formula \( \binom{-n, b; c}{1} = (c-b)_n/(c)_n \), cf. [14, Vol. 1, §2.8(18)], yields the classical formula for the Chebyshev polynomial of the second kind, cf. [14, Vol. 2, §10.11(23)],

\[
U_l(\cos \theta) = \sum_{m=0}^{\lfloor l/2 \rfloor} \frac{(m - l)m! 2^{l-2m} \cos^{l-2m} \theta}{m!}
\]

The orthogonality relations for the continuous \( q \)-Hermite polynomials, cf. (3.7), lead to the following integral representation of the little \( q \)-Legendre polynomials, i.e. little \( q \)-Jacobi polynomials (3.8) with \( a = b = 1 \). For \( l \in \mathbb{Z}_+ \) multiply the result of proposition 11.2 by \( H_p(\cos \theta | q^2) \) and integrate over \([0, \pi]\) with respect to the orthogonality measure \((e^{2i\theta}, e^{-2i\theta}; q)_\infty\).

**Corollary 11.3.** The little \( q \)-Legendre polynomial has the integral representation

\[
p_l(q^p; 1, 1; q) = \frac{(q^{p+1}; q)_\infty}{2\pi} \int_0^\pi U_{2l}(\cos \theta) (H_p(\cos \theta | q))^2 (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta.
\]

This integral representation is an alternative for the \( q \)-integral representation for the little \( q \)-Legendre polynomial derived by Koornwinder [32, thm. 5.1].

**References**

1. E. Abe, *Hopf Algebras*, Cambridge University Press, 1980.
2. G.E. Andrews, *q*-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, CBMS-NSF Regional Conference Series in Math. 66, Amer. Math. Soc., Providence RI, 1986.

3. G.E. Andrews and R. Askey, *Enumeration of partitions: the role of Eulerian series and *q*-orthogonal polynomials*, “Higher Combinatorics” (M. Aigner, ed.), Reidel, Dordrecht, 1977, pp. 3–26.

4. R. Askey, *Orthogonal Polynomials and Special Functions*, CBMS-NSF Regional Conference Series in Applied Math. 21, SIAM, Philadelphia PA, 1975.

5. R. Askey and M.E.H. Ismail, *A generalization of ultraspherical polynomials*, “Studies in Pure Mathematics” (P. Erdős, ed.), Birkhäuser, Basel, 1983, pp. 55–78.

6. R. Askey and J. Wilson, *A set of orthogonal polynomials that generalize the Racah coefficients or the 6-j symbols*, SIAM J. Math. Anal. 10 (1979), 1008–1016.

7. , *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. vol. 54, no. 319, Amer. Math. Soc., Providence RI, 1985.

8. E. Badertscher and T.H. Koornwinder, *Continuous Hahn polynomials of differential operator argument and analysis on Riemannian symmetric spaces of constant curvature*, Can. J. Math. 44 (1992), 750–773.

9. T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, GTM 98, Springer-Verlag, New York, 1985.

10. I.M. Burban and A.U. Klimyk, *Representations of the quantum algebra* $U_q(su_{1,1})$, J. Phys. A: Math. Gen. 26 (1993), 2139–2151.

11. M.S. Dijkhuizen and T.H. Koornwinder, *Quantum homogeneous spaces, duality and quantum 2-spheres*, Geom. Dedicata (to appear).

12. J. Dombrowski, *Orthogonal polynomials and functional analysis*, “Orthogonal Polynomials: Theory and Practice” (P. Nevai, ed.), NATO ASI series C, vol. 294, Kluwer, Dordrecht, 1990, pp. 147–161.

13. V.G. Drinfeld, *Quantum groups*, “Proc. Intern. Congress Math. 1986” (A. Gleason, ed.), Amer. Math. Soc., Providence RI, 1987, pp. 798–820.

14. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, 3 volumes, McGraw-Hill, 1953, 1955, 1965.

15. R. Floreanini and L. Vinet, *On the quantum group and quantum algebra approach to q-special functions*, Lett. Math. Phys. 27 (1993), 179–190.

16. P.G.A. Floris, *Gelfand pair criteria for compact matrix quantum groups*, Indag. Math. (to appear).

17. G. Gasper and M. Rahman, *Product formulas of Watson, Bailey and Bateman types and positivity of the Poisson kernel for q-Racah polynomials*, SIAM J. Math. Anal. 15 (1984), 768–789.

18. , *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications 35, Cambridge University Press, 1990.

19. M.E.H. Ismail and J. Wilson, *Asymptotic and generating relations for the q-Jacobi and 4φ3 polynomials*, J. Approx. Theory 36 (1982), 43–54.

20. M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. 10 (1985), 63–69.

21. E.G. Kalnins, H.L. Manocha and W. Miller, *Models of q-algebra representations: tensor products of special unitary and oscillator algebras*, J. Math. Phys. 33 (1992), 2365–2383.

22. E.G. Kalnins, W. Miller and S. Mukherjee, *Models of q-algebra representations: the group of plane motions*, SIAM J. Math. Anal. 25 (1994), 513–527.

23. A.U. Klimyk, *Classical Lie groups, quantum groups and special functions*, CWI Quarterly 5 (1992), 271–291.

24. H.T. Koelink, *The addition formula for continuous q-Legendre polynomials and associated spherical elements on the SU(2) quantum group related to Askey-Wilson polynomials*, SIAM J. Math. Anal. 25 (1994), 197–217.

25. , *Addition formula for big q-Legendre polynomials from the quantum SU(2) group*, Can. J. Math. (to appear).

26. , *Identities for q-ultraspherical polynomials and Jacobi functions*, Proc. Amer. Math. Soc. (to appear).
45. P. Podleś, Quantum spheres, J. Math. Anal. 174 (1993), 392–407.
46. M. Rahman and A. Verma, Product and addition formulas for the continuous $q$-ultraspherical polynomials, SIAM J. Math. Anal. 17 (1986), 1461–1474.
47. G. Rideau and P. Winternitz, Representations of the quantum algebra $su_q(2)$ on a real two-dimensional sphere, J. Math. Phys. 34 (1993), 6030–6044.
48. D. Stanton, Orthogonal polynomials and Chevalley groups, “Special Functions: Group Theoretical Aspects and Applications” (R.A. Askey, T.H. Koornwinder and W. Schempp, eds.), Reidel, Dordrecht, 1984, pp. 87–128.
49. M.E. Sweedler, Hopf Algebras, Benjamin, 1969.
50. L. Vainerman, Gelfand pairs of quantum groups, hypergroups and $q$-special functions, “Applications of hypergroups and related measure algebras”, Contemporary Math. (W. Connett, O. Gebuhrer, A. Schwartz, eds.) (to appear).
51. L.L. Vaksman and L.I. Korogodskii, An algebra of bounded functions on the quantum group of motions of the plane, and $q$-analogues of the Bessel function, Soviet Math. Dokl. 39 (1989), 173–177.
52. L.L. Vaksman and Ya.S. Soibelman, Algebra of functions on the quantum group SU(2), Funct. Anal. Appl. 22 (1988), 170–181.
53. W. Van Assche and T.H. Koornwinder, *Asymptotic behaviour for Wall polynomials and the addition formula for little q-Legendre polynomials*, SIAM J. Math. Anal. **22** (1991), 302–311.

54. A. Van Daele, *Dual pairs of Hopf *∗*-algebras*, Bull. London Math. Soc. **25** (1993), 209-230.

55. V.S. Varadarajan, *Lie Groups, Lie Algebras, and their Representation Theory*, Prentice Hall, Englewood Cliffs NJ, 1974.

56. N.J. Vilenkin, *Special Functions and the Theory of Group Representations*, Transl. Math. Monographs **22**, Amer. Math. Soc., Providence RI, 1968.

57. N.J. Vilenkin and A.U. Klimyk, *Representation of Lie Groups and Special Functions*, 3 volumes, Kluwer, Dordrecht, 1991, 1993.

58. S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

59. ———, *Twisted SU(2) group. An example of non-commutative differential calculus*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **23** (1987), 117–181.

60. A.S. Zhedanov, *Q rotations and other Q transformations as unitary nonlinear automorphisms of quantum algebras*, J. Math. Phys. **34** (1993), 2631–2647.

**H.T. KOELINK**

**Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Leuven (Heverlee), Belgium**

*E-mail address: erik%twi%wis@cc3.KULeuven.ac.be*