Weight Distributions for Successive Cancellation Decoding of Polar Codes

Rina Polyanskaya, Mars Davletshin, and Nikita Polyanskii

Abstract

In this paper, we derive the exact weight distributions for the successive cancellation decoding of polar codes. The results allow to get an estimate of the decoding error probability and to show a link between the first nonzero components of the weight distribution and the partial order between the synthetic channels. Furthermore, we prove a statement on the minimal distance between cosets for the successive cancellation list decoding.

Keywords:  Polar codes, weight distribution, closest coset decoding, multistage decoding, partial order.

I. INTRODUCTION

Polar codes, introduced by Arikan [1], provably achieve the symmetric capacity of any binary-input memoryless symmetric channels (B-MSC) with encoding and decoding complexity $\Theta(N \log_2 N)$, where $N$ is the block length of the code. Within the 5G standardization process, polar codes have been adopted as channel coding for uplink and downlink control information for the enhanced mobile broadband communication service.

Multilevel codes are based on partitioning, thus multistage decoding is the most natural one to be performed [2]. Polar codes with the successive cancellation (SC) decoding can be represented in this way [1], [3]. A typical multilevel code construction employs small codes to get a larger one. Whereas polar codes are obtained by taking a Kronecker power of a square kernel matrix and expurgating some rows using a specific criterion. It is known that polar codes with good

Rina Polyanskaya is with the Institute for Information Transmission Problems, and with the Moscow Research Center, Huawei Technologies Co., Ltd (email: revenko.rina@huawei.com).

Mars Davletshin is with the Moscow Research Center, Huawei Technologies Co., Ltd (email: davletshin.mars1@huawei.com).

Nikita Polyanskii is with the Skolkovo Institute of Science and Technology. The research was conducted in part during May - October 2017 with the Moscow Research Center, Huawei Technologies Co., Ltd (email: nikitapolyansky@gmail.com).
distance properties turn out to have a poor performance under the SC decoding. To evaluate
the error rate provided by a multistage decoder, it is natural to calculate the weight distribution
(WD) between cosets (or spectrum of component codes) at all the stages [4]. However, only the
minimal distance for the SC decoding of polar codes is known at present. The aim of our paper
is to calculate WD at all the stages of the SC decoding.

A. Outline

The rest of the paper is organized as follows. In Section II, we give key definitions and
notations of polar codes and WDs associated with the SC decoding. We derive WD and focus
our attention on its first nonzero component in Section III. To obtain an algorithm calculating
them, we exploit a similar idea as in [5], where an \( |u|u+v| \) construction is investigated. Also, we
find a natural connection between the first nonzero component of WDs and the partial order [6],
[7]. The minimal distance between cosets for the SC list decoding is discussed in Section IV.
Finally, we conclude with some open problems in Section V.

II. NOTATIONS AND DEFINITIONS

For simplicity of presentation we shall use zero-based numbering. A vector of length \( n \) is
denoted by bold lowercase letters, such as \( \mathbf{x} \) or \( \mathbf{x}_{n-1} \), and the \( i \)th entry of the vector \( \mathbf{x} \) is referred
to as \( x_i \). Given a binary vector \( \mathbf{x} \), we define its support \( \text{supp}(\mathbf{x}) \) as the set of coordinates in
which the vector \( \mathbf{x} \) has nonzero entries. Let \( d(\mathbf{x}, \mathbf{y}) \) be the Hamming distance between \( \mathbf{x} \) and
\( \mathbf{y} \), and \( \text{wt}(\mathbf{x}) \) be the Hamming weight of \( \mathbf{x} \). The set of integers from \( i \) to \( j - 1 \), \( 0 \leq i < j \),
is abbreviated by \( [i,j) \) or simply \( [j - 1] \) if \( i = 0 \). Clearly, \( \text{wt}(\mathbf{x}) = d(\mathbf{x}, \mathbf{0}) \), where \( \mathbf{0} \) is the
all-zero vector. Given a \( (N \times N) \) binary matrix \( \mathbf{X} \) and \( A \subset [0, N) \), we write \( \mathbf{X}(A) \) to denote
the \( (|A| \times N) \) submatrix of \( \mathbf{X} \) formed by the rows of \( \mathbf{X} \) with indexes in \( A \).

Let \( W : \mathcal{X} \rightarrow \mathcal{Y} \) be a B-MSC channel with input alphabet \( \mathcal{X} = \{0, 1\} \), output alphabet \( \mathcal{Y} \),
and transition probabilities \( W(y|x) \) for \( x \in \mathcal{X} \) and \( y \in \mathcal{Y} \). By \( W^N \) we denote the vector channel
corresponding to \( N \) independent copies of \( W \), i.e., \( W^N : \mathcal{X}^N \rightarrow \mathcal{Y}^N \) with transition probabilities
\[
W^N(y_{0}^{N-1}|x_{0}^{N-1}) = \prod_{i=0}^{N-1} W(y_i|x_i).
\]
Arikan used a construction based on the following kernel matrix
\[
G_2 := \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
\]
Given $N = 2^n$, we consider an $(N \times N)$ binary matrix $G_N = G_2^\otimes n$ by performing the $n$th Kronecker power of $G_2$. We denote the $i$th row of $G_N$ by $g_i$. Usually a linear mapping $x = x(u) : \mathcal{X}^n \to \mathcal{X}^n$ is defined by

$$x = u B_N G_N,$$

where $B_N$ is an $(N \times N)$ bit-reversal permutation matrix defined in [1, Section VII-B], and the vectors $x, u$, and the vector space $\mathcal{X}^n$ are over $GF(2)$. However, since $B_N G_N = G_N B_N$, the latter being a simple permutation on $x$, we can dispense with $B_N$ in this paper and assume

$$x = u G_N. \quad (1)$$

Let us produce a vector channel $W_N : \mathcal{X}^N \to \mathcal{Y}^N$ as follows

$$W_N(y|u) := W^N(y|u G_N) = W^N(y|x).$$

Given $i \in [0, N)$, we define the synthetic channel $W^{(i)}_N : \mathcal{X} \to \mathcal{Y}^N \times \mathcal{X}^i$ as

$$W^{(i)}_N(y, u_{i+1}^{i-1}|u_i) := \sum_{u_i^{N-1} \in \mathcal{X}^{N-i-1}} \frac{1}{2^{N-1}} W_N(y|u).$$

A. Polar Coding

The generator matrix of a polar code is given by $G_N(A)$ for some set $A \subset [0, N)$, which is referred to as the information set. The indices $A^c := [0, N) \setminus A$ are usually called frozen ones and chosen carefully according to the reliabilities of the synthetic channels [1]. In other words, any message $u \in \{0, 1\}^N$ has $u_i = 0$ for all $i \in A^c$, and is mapped to the codeword $x$ by (1).

Let $x$ be sent over $W_N$, and let a channel output $y$ be received. Given $A$ and $y$, the decoder generates an estimate $\hat{u}$ of $u$. We shall briefly describe the SC decoding as the sequential use of the closest coset decoding [8].

For any binary vector $v \in \{0, 1\}^i$, let the set $C^{(n)}(v)$ induced by $v$ be defined as follows

$$C^{(n)}(v) := \sum_{j \in \text{supp}(v)} g_j + \langle g_{i+1}, \ldots, g_{N-1} \rangle,$$

where $\langle \cdot \rangle$ is a linear span of a set of vectors. By

$$C^{(n)}(v, 0) := \sum_{j \in \text{supp}(v)} g_j + \langle g_{i+1}, \ldots, g_{N-1} \rangle,$$

$$C^{(n)}(v, 1) := g_i + C^{(m)}(v, 0),$$
define the zero and the one cosets induced by \( v \), respectively. Obviously, the disjoint union of the zero and the one cosets coincides with \( C^{(n)}(v) \).

At the beginning of the \( i \)th stage of the SC decoding, we are given with a binary vector \( \hat{u}_0^{i-1} \in \{0, 1\}^i \), which can be treated as an estimate of \( u_0^{i-1} \). If \( i \in \mathcal{A}^c \), then the decoder makes a bit decision \( \hat{u}_i = 0 \). Otherwise, the decoder chooses which of the values \( \{w(\hat{u}_0^{i-1}, 0), w(\hat{u}_0^{i-1}, 1)\} \) is larger

\[
\begin{align*}
w(\hat{u}_0^{i-1}, 0) &:= \sum_{v \in C^{(n)}(\hat{u}_0^{i-1}, 0)} W^N(y|v), \\
w(\hat{u}_0^{i-1}, 1) &:= \sum_{v \in C^{(n)}(\hat{u}_0^{i-1}, 1)} W^N(y|v),
\end{align*}
\]

and makes a bit estimate \( \hat{u}_i \) of \( u_i \): \( \hat{u}_i = 0 \) if \( w(\hat{u}_0^{i-1}, 0) \geq w(\hat{u}_0^{i-1}, 1) \), and \( \hat{u}_i = 1 \) otherwise. This decision can be seen as choosing the “closest” (zero or one) coset to the received \( y \). If the wrong coset is selected at some decoding stage, then this decoding error is propagated to the next stages.

**B. Weight Distribution**

Without loss of generality, we assume that the all-zero codeword is transmitted, i.e., \( u = x = 0 \). At the \( i \)th stage, the error occurs if the decoder selects \( C^{(n)}(\hat{u}_0^{i-1}, 1) \) instead of \( C^{(n)}(\hat{u}_0^{i-1}, 0) \) (in the very beginning \( C^{(n)}(1) \) instead of \( C^{(n)}(0) \), respectively). For \( i \in [0, 2^n) \) and \( w \in [2^n] \), let \( S_{i,w}^{(n)} \) be the number of words of weight \( w \) in \( C^{(n)}(\hat{u}_0^{i-1}, 1) \). For the BPSK transmission over the AWGN channel with variance \( \sigma^2 \) and \( i \in \mathcal{A} \), we can upper bound \([9]\) the error probability \( P_e(i) \) at the \( i \)th decoding stage as

\[
P_e(i) \leq P_{ub}(i) := \sum_{w=1}^{N} \frac{1}{2} S_{i,w}^{(n)} \text{erfc} \left( \sqrt{w/(2\sigma^2)} \right),
\]

where \( \text{erfc}(\cdot) \) is the complementary error function defined by

\[
\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.
\]

Taking into account all the words in \( C^{(n)}(\hat{u}_0^{i-1}, 0) \) (not only the all-zero codeword) may significantly tighten the bound \([4]\) especially for large \( \sigma \). However, we don’t see any practical framework which can be applied for this matter.

It is worth noting that there are several techniques allowing to calculate \( P_e(i) \) with inherent inaccuracy and to bound \( P_e(i) \). Among them are density evolution \([10]\), degrading and upgrading algorithms \([11]\) and Gaussian approximation \([3]\).
III. Weight Distribution for Successive Cancellation Decoding

In this section we give an algorithm for computing WDs for the SC decoding. Our analysis is similar to one in [5, Section 2], where WD for the closest coset decoding of \(|u|u+v|\) construction was established. Let us determine \(S_{i,w}^{(n)}\), the number of codewords of weight \(w\) in \(C^{(n)}(0_{0}^{i-1},1)\). If \(i \geq 2^{n-1}\), then any codeword in \(C^{(n)}(0_{0}^{i-1},1)\) represents a repetition of some codeword in \(C^{(n-1)}(0_{0}^{i-1-N/2},1)\). Thus \(S_{i,w}^{(n)} = 0\) for odd \(w\), and \(S_{i,w}^{(n)} = S_{i-N/2,w/2}^{(n-1)}\) for even \(w\). If \(i < 2^{n-1}\), then any codeword \(x \in C^{(n)}(0_{1}^{i-1},1)\) can be uniquely represented in the form

\[ x = g_i + \sum_{j \in I_1} g_j + \sum_{j \in I_2} g_j = (x_1,0) + (x_2,x_2), \]

where the index sets \(I_1 \subset [i+1,2^{n-1})\) and \(I_2 \subset [2^{n-1},2^{n})\), and the codewords \(x_1 \in C^{(n-1)}(0_{0}^{i-1},1)\) and \(x_2 \in \{0,1\}^{N/2}\). Moreover, any pair of \(x_1\) and \(x_2\) gives some \(x \in C^{(n)}(0_{0}^{i-1},1)\). Hence \(S_{i,w}^{(n)}\) can be determined using the following statement.

**Theorem 1.** For \(t \in \{0, \ldots, N/2 - w_1\}\), the contribution of \(x_1\) with \(\text{wt}(x_1) = w_1\) to \(S_{i,w_1+2t}^{(n)}\) is \(2^{w_1(N/2-w_1)}\).

**Proof of Theorem 1.** It is easy to check that

\[
\text{wt}(x) = \text{wt}(x_2) + d(x_1,x_2) = \text{wt}(x_1)
+ (\text{wt}(x_2) + d(x_1,x_2) - \text{wt}(x_1)) \geq \text{wt}(x_1).
\]

We observe that the sum in the brackets is equal to the double number of coordinates \(i\) so that \(x_{1,i} = 0\) and \(x_{2,i} = 1\). Given \(x_1\) with \(\text{wt}(x_1) = w_1\), there are \(\binom{N/2-w_1}{t}\) different choices for placing \(t\) ones in \(x_2\) among \(N/2 - w_1\) coordinates corresponding to zeros in \(x_1\). Also \(x_2\) could have anything in the remaining \(w_1\) coordinates corresponding to ones in \(x_1\). Therefore the total number of choices for \(x_2\) is \(2^{w_1(N/2-w_1)}\).

Summarizing the arguments given above, WDs can be calculated in a recurrent manner with the help of Algorithm [7], where the notation \((k)^+\) stands for \(\max(0,k)\).

**Remark 1.** Algorithm [7] provides a practical way to determine WDs for SC. For instance, given \(p \geq 1\) the complexity of computing the \(p\) first nonzero components of \((S_{i,w}^{(n)})_{w=0}^{N}\) for \(i \in [0,N]\) is \(O(N)\) as \(N \to \infty\). Therefore, the complexity of getting upper bounds on the error probabilities for all the subchannels based on (4) (more precisely, its modification like the sphere bound) is also linear with code length.
Algorithm 1 Computing the weight distributions

Input: length \( N = 2^n \)

Output: weight distributions \( \left( S_{i,w}^{(n)} : w \in [N] \right)_{i=0}^{N-1} \)

Initialisation:
1: \( S^{(0)} := \left( S_{0,0}^{(0)}, S_{0,1}^{(0)} \right) \) with \( S_{0,0}^{(0)} := 0 \) and \( S_{0,1}^{(0)} := 1 \)
2: for Kroneker’s power \( j = 1 \) to \( n \) do
3: for row index \( i = 0 \) to \( 2^j - 1 \) do
4: if \( i < 2^j - 1 \) then
5: for weight \( w = 0 \) to \( 2^j \) do
6: \( S_{i,w}^{(j)} := \sum_{t=(w-2^j)}^{\lfloor w/2 \rfloor} S_{i,w-2t}^{(j-1)} \left( 2^{j-1} + 2t - w \right) 2^{w-2t} \)
7: end for
8: else
9: for weight \( w = 0 \) to \( 2^j-1 \) do
10: \( S_{i,2w}^{(j)} := S_{i-2^j,w}^{(j-1)} \); \( S_{i,2w+1}^{(j)} := 0 \)
11: end for
12: end if
13: end for
14: \( S^{(j)} := \left( \left( S_{i,w}^{(j)} : w \in \{0, \ldots, 2^j\} \right) \right)_{i=0}^{2^j-1} \)
15: end for
16: return \( S^{(n)} \)

Example 1. Let us illustrate the bound (4) by taking the code length \( N = 256 \) and the AWGN channel with variance \( \sigma^2 = 0.158 \). Using Algorithm 1 we compute the weight distributions and depict the pairs \( (P_{e}(i), P_{ub}(i)) \) in Figure 1, where the union bound \( P_{ub}(i) \) on the decoding error probability \( P_{e}(i) \) is computed with the help of (4).

Example 2. Now let us take the code length \( N = 128 \) and the synthetic channel with index \( i = 72 \). Using Algorithm 1 we compute the weight distributions and depict the union bound (4) on the decoding error probability \( P_{e}(72) \) along with this probability, calculated with the help of DE, in Figure 2.
Fig. 1. The union bound $P_{ub}(i)$ and the bit-channel probabilities of error $P_e(i)$ for a polar code of length $N = 256$ on AWGN with $\sigma^2 = 0.158$. Only those pairs $(P_e(i), P_{ub}(i))$ for which the error probability $P_e(i)$ is greater than $10^{-10}$ threshold are shown.

Fig. 2. The DE computations and the union bound on $P_e(72)$ under SC

A. First Nonzero Component

It was known that the first nonzero component of $(S^{(n)}_{i,w})_{w=0}^{N}$ corresponds to weight $w^{(n)}_i := \text{wt}(g_i)$. We extend this line of research and find an explicit formula for $S^{(n)}_{i,w^{(n)}}$. Given $j \in [0, n)$ and $i \in [0, N)$, let $b(j, i)$ be the $j$th bit in the binary representation of integer $i$, and $p(j, k)$ be a partial sum of the first $j + 1$ bits, i.e.,

$$i =: \sum_{j=0}^{n-1} b(j, i)2^j, \quad p(j, k) := \sum_{s=0}^{j} b(s, k).$$
Theorem 2. Given $N = 2^n$ and $i \in [0, N)$, the first nonzero component of $(S_{i,w}^{(n)})^N_{w=0}$ is $S_{i,w_i}^{(n)}$, where $w_i^{(n)}$ satisfies $\log_2(w_i^{(n)}) = p(n-1, i)$. Moreover,

$$\log_2 S_{i,w_i}^{(n)} = \sum_{j=0}^{n-1} 2^{p(j,i)} (1 - b(j,i)).$$

(7)

Proof of Theorem 2. We shall prove the statement of this theorem by induction on $n$. The base case is evident as $S_{0,0}^{(0)} = 0$. We assume for a moment that the equality (7) holds for $S_{i,w_i}^{(n)}$ and $i \in [0, N)$. Following line 6 of Algorithm 1, for $i < N = 2^n$, we get

$$S_{i,2w_i^{(n)}}^{(n+1)} = S_{i,w_i}^{(n)} 2^{w_i^{(n)}}$$

and $S_{i,w}^{(n+1)} = 0$ for $w < w_i^{(n)}$. Similarly for $i \geq N$,

$$S_{i,2w_i^{(n)}}^{(n+1)} = S_{i-N,w_i}^{(n)}$$

and $S_{i,w}^{(n+1)} = 0$ for $w < 2w_i^{(n)}$.

In other words, if the $(j + 1)$th bit in the binary expansion of $i$ equals 0, i.e., $b(j+1, i) = 0$, then the first nonzero component of WD parametrized by $i$ and $j + 1$ has $w_i^{(j+1)} = w_i^{(j)}$ and is $2^{w_i^{(j)}} = 2^{2^{p(j,i)}} = 2^{2^{p(j+1,i)}}$ times as much as one of the previous WD. In other case, that is $b(j + 1, i) = 1$, the value of the first nonzero component parametrized by $i, i \geq N$, and $j + 1$ remains the same as one parametrized by $i - N$ and $j$. Combining these arguments completes the proof of the inductive step.

It was observed [6], [7] that there is a partial order between the synthetic channels, which holds for any B-MSC channel. Let us rephrase this result using our notation.

Theorem 3 (The partial order [6, Definition 8]). $W_{(i)}^{(i)}$ is stochastically degraded by $W_{(j)}^{(j)}$ if there exists a finite sequence $a_0, a_1, \ldots, a_k \in [0, N)$, $k \geq 0$, such that $a_0 = i$, $a_k = j$ and for all $m \in [0, k)$, one of the following two properties holds:

1) There exists $s_m', s_m'' \in [n-1]$ so that $s_m' < s_m''$ and

$$b(s, a_m) = b(s, a_{m+1}) \text{ for all } s \in [n-1] \setminus \{s_m', s_m''\},$$

$$b(s_m', a_m) = b(s_m', a_{m+1}) = 1,$$

$$b(s_m'', a_m) = b(s_m', a_{m+1}) = 0;$$

2) $b(s, a_m) \leq b(s, a_{m+1})$ for all $s \in [n-1]$.
Note that for any such sequence \( a_0, a_1, \ldots, a_k \), we have
\[
\sum_{k=n-1-s}^{n-1} b(k, a_m) \leq \sum_{k=n-1-s}^{n-1} b(k, a_{m+1}) \text{ for all } s \in [n - 1].
\]
In particular, this means
\[
\sum_{k=n-1-s}^{n-1} b(k, i) \leq \sum_{k=n-1-s}^{n-1} b(k, a_j) \text{ for all } s \in [n - 1]. \tag{8}
\]

Theorem 2 shows us a natural one-way connection between the partial order given in Theorem 3 and the first nonzero components of WDs. Namely, if the \( i \)th synthetic channel is worse than the \( j \)th one by the partial order, then the property (8) holds, and according to (7) the first nonzero component of WD at the \( i \)th decoding stage of SC is worse than that at the \( j \)th decoding stage, i.e., either \( w_i^{(n)} < w_j^{(n)} \) or \( w_i^{(n)} = w_j^{(n)} = w \) and \( S_{i,w}^{(n)} > S_{j,w}^{(n)} \).

IV. TOWARD WEIGHT DISTRIBUTION FOR SUCCESSIVE CANCELLATION LIST DECODING

Let us briefly recall the high level description of the successive cancellation list (SCL) decoder [12] with the list size \( L \). At the \( i \)th decoding stage for \( i \in A \), we split each path \( \hat{u}_0^{i-1} \) from the list of candidates, abbreviated by \( L \), into two paths by taking \( \hat{u}_i = 0 \) and \( \hat{u}_i = 1 \) and calculate two values \( w(\hat{u}_0^{i-1}, 0) \) and \( w(\hat{u}_0^{i-1}, 1) \) by (5). Since the number of paths is doubled, we keep in \( L \) only the \( L \) most likely paths at each stage. The pruning criterion is based on the values \( \{w(\hat{u}_0^{i-1}, 0), w(\hat{u}_0^{i-1}, 1)\}_{\hat{u}_0^{i-1} \in L} \). If index \( i \in A^c \), then for any path \( \hat{u}_0^{i-1} \) from \( L \), the decoder makes a bit decision \( \hat{u}_i = 0 \) and keep this \( \hat{u}_0^i \) in \( L \).

Assume that after the \( j \)-th decoding stage, the SCL decoder keeps (at least) the following two paths: true path \( 0_0^j \) and path \( u_0^j \) mistaken in only two positions \( i \) and \( j \), \( i < j \), i.e., \( \text{supp}(u_0^j) = \{i, j\} \). Our goal is to estimate the minimal distance between sets induced by these two pathes. For simplicity of notation we abbreviate \( C^{(n)}(u_0^j) \) by \( C^{(n)}(i, j) \).

**Theorem 4.** The minimal weight of any word in \( C^{(n)}(i, j) \) is equal to the weight \( wt(g_i + g_j) \).

**Proof of Theorem 4.** We shall prove the statement of this theorem by induction on \( n \), \( N = 2^n \).

The base case \( n = 1 \) is obviously true. Now assume that the theorem statement holds for every \( C^{(n)}(i, j) \) and all \( n \leq \bar{n} \). We prove that it holds for \( n = \bar{n} + 1 \). Let us consider one of the three cases.
Case 1: \(2^{n-1} \leq i < j < 2^n\). Let \(i'\) and \(j'\) be the residues of \(i\) and \(j\) modulo \(2^{n-1}\), respectively. Given \(\alpha_r \in \{0, 1\}\) for \(r \in [j+1, 2^n)\), the weight of any binary vector \(x \in C(n)(i, j)\) represented by

\[
x = g_i + g_j + \sum_{r \in [j+1,2^n)} \alpha_r g_r
\]

is exactly two times larger as the weight of the binary vector

\[
x' = g_{i'} + g_{j'} + \sum_{r \in [j'+1,2^{n-1})} \alpha_{r+2^{n-1}} g_r,
\]

Therefore we deduce from the induction that

\[
\min_{x \in C(n)(i, j)} \text{wt}(x) = 2 \min_{x' \in C(n-1)(i', j')} \text{wt}(x') = 2 \text{wt}(g_{i'} + g_{j'}) = \text{wt}(g_i + g_j).
\]

Case 2: \(i < j < 2^{n-1}\). Any binary vector in \(x \in C(n)(i, j)\) can be written in the form \(x = x_1 + x_2\), where

\[
x_1 = g_i + g_j + \sum_{r \in [j+1,2^{n-1})} \alpha_r g_r, \quad x_2 = \sum_{r \in [2^{n-1},2^{n-1}-1)} \alpha_r g_r.
\]

Applying (5)-(6) and the inductive assumption, we establish \(\text{wt}(x) \geq \text{wt}(x_1) \geq \text{wt}(g_i + g_j)\).

Case 3: \(i < 2^{n-1} \leq j < 2^n\). Let \(\ell\) be the minimal integer such that \(g_{i,2^\ell} = 0\). By \(I\) denote the collection of indices \(k\) such that \(k = k_m2^\ell + k_r\), \(0 \leq k_r < 2^\ell\), and \(k_m\) is odd. It is easy to see that \(g_{i,k} = 0\) for any \(k \in I\) and \(|I| = 2^{n-1}\). For any \(k\) from \(I_c := [0,2^n) \setminus I\), the value of \(g_{i,k}\) can be either 1, or 0. If \(\ell = n - 1\), then \(g_i = (1,0)\), and the weight of any vector \(x \in C(n)(i, j)\) represented by (9) is exactly \(\text{wt}(g_i + g_j) = \text{wt}(g_i) = 2^{n-1}\). If \(\ell < n - 1\), then we consider two subcases.

Case 3.a: \(g_{j,k} = 0\) for all \(k \in I\). For any given vector \(x \in C(n)(i, j)\), we split the terms in (9) into two groups: in the first one, \(g_{r,k} = 0\) for all \(k \in I\); the remaining terms go to the second group. Let \(x_1\) and \(x_2\) be the sums of the vectors in the first and the second groups, respectively. For \(x_1\), the vector projection of all the terms onto coordinates indexed by \(I_c\) maintains the assumption in the induction for \(n' = n - 1\). Hence \(\text{wt}(x_1) \geq \text{wt}(g_i + g_j)\). Since the terms \(g_r\) included in \(x_2\) satisfy the equality \(g_{r,k} = g_{r,k-2^\ell}\) for all \(k \in I\), we apply the arguments as in (5)-(6) and conclude that

\[
\text{wt}(x) = \text{wt}(x_1 + x_2) \geq \text{wt}(x_1) \geq \text{wt}(g_i + g_j).
\]
Case 3.b: \( g_{j,k} = g_{j,k-2^e} \) for all \( k \in I \). For any given vector \( x \in C(n)(i,j) \), we consider its projections \( x|_I \) and \( x|_{I^c} \) onto coordinates indexed by \( I \) and \( I^c \), respectively. For the vector \( x|_{I^c} \) and all the terms in (9) restricted onto \( I^c \), we apply the inductive assumption. Therefore
\[
\text{wt}(x|_{I^c}) \geq \text{wt}(g_i|_{I^c} + g_j|_{I^c}).
\]
For the vector \( x|_I \), we deduce from the induction that
\[
\text{wt}(x|_I) \geq \text{wt}(g_j|_I).
\]
Finally, we have
\[
\text{wt}(x) = \text{wt}(x|_I) + \text{wt}(x|_{I^c}) \geq \text{wt}(g_i|_{I^c} + g_j|_{I^c}) + \text{wt}(g_j|_I) = \text{wt}(g_i + g_j).
\]

**Example 3.** If we impose some linear constraints [13] on information vector \( u \), then we can check, based on Theorem 4, how the minimal distance between sets induced by the true path and a path mistaken in one position is changed. For instance, let us take \( n = 5 \) and include into the information set \( A \) all the indexes \( i \) so that \( \text{wt}(g_i) \geq 8 \). In particular, index 14 belongs to \( A \) and index 17 is in \( A^c \). Suppose \( u_{0}^{17} \) has \( \text{supp}(u_{0}^{17}) = \{14\} \). The minimal weight of any word in \( C(5)(u_{0}^{17}) \) is then 8. Now let us impose constraint \( u_{17} = u_{14} \). At the 17th decoding step for path \( u_{0}^{16} \), the decoder makes the bit decision \( \hat{u}_{17} = 1 \) to satisfy the constraint, and continues with path \( \hat{u}_{0}^{17} \), where \( \text{supp}(\hat{u}_{0}^{17}) = \{14, 17\} \). The minimal weight of any word in \( C(5)(\hat{u}_{0}^{17}) \) is then 10.

**V. Conclusion**

In this paper, we discuss the exact weight distribution of the coset associated with each synthetic channel \( W_{N}^{(i)} \). Also, we prove some statement on the minimal distance between cosets associated with pathes different in two positions for the successive cancellation list decoding. This study represents the initial steps towards understanding the performance of the polar codes under the successive cancellation list decoding.

The upper bound (4) take into account only weight distributions of the one coset \( C(0_{0}^{1}, 1) \). The drawback of this approach is evident: for low and medium signal-to-noise ratio, the bounds could not be tight. Based on Algorithm 1, weight distributions of any zero coset \( C(0_{0}^{0}, 0) \) can be calculated. However, we do not know how to use it in order to get a more accurate bound of the error probability. It is still unknown how to calculate efficiently the minimal weight (and
the first nonzero component) of a set $C(u_0^j)$ with an arbitrary $u_0^j$. We believe that such an analysis can be helpful for constructing polar codes under the SC list decoding and estimating the spectrum of polar codes. In addition, it may be reasonable to use polar codes with dynamic frozen symbols [13].

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