Optimal Parameter-free Online Learning with Switching Cost

Zhiyu Zhang  Ashok Cutkosky  Ioannis Ch. Paschalidis
Boston University  Boston University  Boston University
zhiyuz@bu.edu  ashok@cutkosky.com  yannisp@bu.edu

Abstract
Parameter-freeness in online learning refers to the adaptivity of an algorithm with respect to the optimal decision in hindsight. In this paper, we design such algorithms in the presence of switching cost - the latter penalizes the optimistic updates required by parameter-freeness, leading to a delicate design trade-off. Based on a novel dual space scaling strategy, we propose a simple yet powerful algorithm for Online Linear Optimization (OLO) with switching cost, which improves the existing suboptimal regret bound [ZCP22a] to the optimal rate. The obtained benefit is extended to the expert setting, and the practicality of our algorithm is demonstrated through a sequential investment task.

1 Introduction
Online learning [CBL06, Haz16, Ora19] is a powerful setting to study sequential decision making tasks, such as neural network training, financial investment and robotic control. In each round, an agent picks a prediction $x_t$ in a convex domain $X$, receives a convex and Lipschitz loss function $l_t$ that depends on $x_1,\ldots,x_t$, and suffers the loss $l_t(x_t)$. The goal is to ensure that in any environment, the cumulative loss of the agent is never much worse than that of any fixed prediction $u \in X$. That is, one aims to upper-bound the regret

$$\sum_{t=1}^{T} [l_t(x_t) - l_t(u)],$$

for all time horizon $T \in \mathbb{N}_+$, comparator $u \in X$ and loss sequence $l_1,\ldots,l_T$.

Conventional solutions of this problem have a minimax nature. For example, if $X$ is bounded, then using gradient descent with a conservative learning rate schedule, one can guarantee the optimal $O(\sqrt{T})$ regret bound independent of $u$ [Zin03]. Despite its popularity, such an approach has a few limitations.

1. It requires setting the learning rate based on the diameter of the domain. Many practical problems are naturally unconstrained, making this approach inapplicable.

2. One may have prior information on the optimal fixed prediction (i.e., the comparator $u^*$ that maximizes the regret), possibly from domain knowledge or transfer learning. In that case, the minimax approach cannot utilize it to obtain a better guarantee.

Recent studies of parameter-free online learning [LS15, OP16, CO18] aim to address these issues. The domain does not need to be bounded, and the regret bound is an increasing function of $d(u^*,x_1)$, where $d(\cdot,\cdot)$ is some suitable distance measure. Intuitively, these algorithms are both optimistic and robust: When we have prior information on $u^*$, we can pick $x_1$ such that $d(u^*,x_1)$, and consequently the regret bound, are both low. Meanwhile, even when our initialization $x_1$ is wrong (i.e., $d(u^*,x_1)$ is large), the regret bound is still almost as good (up to logarithmic factors) as that of gradient descent with the best learning rate in hindsight. Such properties have shown benefits in many applications, e.g., [OT17, JO19, vdH19].

In this paper, we extend the design of parameter-free algorithms to a classical setting with switching costs. Here the agent is penalized not only by its loss, but also by how fast it changes its predictions. Practically, the switching cost is useful whenever the smooth operation of a system is favored, such as in network routing,
control of electrical grid, portfolio management with transaction cost, etc. Mathematically, with a given weight \( \lambda \geq 0 \) and a norm \( \| \cdot \| \)\(^1\), our goal is to show a parameter-free bound for the augmented regret

\[
\sum_{t=1}^{T} [l_t(x_t) - l_t(u)] + \lambda \sum_{t=1}^{T-1} \| x_t - x_{t+1} \| .
\]

While gradient descent can incorporate the switching cost by scaling its learning rate, extending parameter-free algorithms is a lot harder. Essentially, parameter-freeness is a form of adaptivity, and just like other adaptive algorithms, its key idea is to quickly respond to the incoming information and hedge aggressively. Switching cost, on the other hand, encourages the agent to stay still. Therefore, achieving our goal requires a delicate balance between the two opposite considerations.

Similar trade-offs between adaptivity and switching cost have led to interesting results in the past. For example, [Gof14] showed that the gradient variance adaptivity well-studied in the standard online learning setting is impossible with normed switching costs, thus establishing a clear separation between the two settings. [DM19] showed that a common analytical technique for switching costs is incompatible to the so-called “strong adaptivity” (i.e., a form of adaptivity w.r.t. nonstationary comparators). Regarding our goal, [ZCP22a] proposed the first parameter-free algorithm with switching cost, but the obtained regret bound is sub-optimal in multiple ways. The present work aims at closing this gap.

1.1 Main contribution

We develop parameter-free algorithms for two fundamental settings in online learning: (i) Online Linear Optimization (OLO) with switching cost; (ii) Learning with Expert Advice (LEA) with switching cost.

1. For one-dimensional unconstrained OLO with switching cost, assuming loss gradients \( |g_t| \leq 1 \) and initial prediction\(^2\) \( x_1 = 0 \), we propose an algorithm that guarantees

\[
\sum_{t=1}^{T} g_t(x_t - u) + \lambda \sum_{t=1}^{T-1} |x_t - x_{t+1}| = C \sqrt{\lambda T} + |u| O \left( \sqrt{\lambda T \log(C^{-1} \| u \|)} \right) ,
\]

where \( C > 0 \) is any hyperparameter chosen by the user. Our bound achieves multiple forms of optimality with respect to \( \lambda, |u| \) and \( T \). Notably, a doubling trick can convert it to \( C + |u| O \left( \sqrt{\lambda T \log(C^{-1} \| u \| T)} \right) \),

which is a substantial improvement over the existing suboptimal bound \( C + |u| O \left( \lambda \sqrt{T \log(C^{-1} \| u \| T)} \right) \) [ZCP22a]. Our improvement relies on a novel dual space scaling strategy for potential methods. Compared to [ZCP22a], both the algorithm and the analysis are arguably simpler. Extensions to bounded domains and high-dimensional domains are also presented.

2. Next, we consider the conversion from OLO to LEA. We demonstrate how classical techniques [LS15, OP16] are designed to have large switching costs, and then propose a fix which has a clear geometric interpretation. The final result is the first parameter-free algorithm for LEA with switching cost.

Concluding these theoretical results, our algorithm is applied to a portfolio management task with transaction costs. Numerical results support its superiority over the existing approach [ZCP22a].

1.2 Background

Online learning basics Throughout this paper we will only consider linear losses. The generality of our setting is preserved, since convex losses can be reduced to linear losses through the relation \( \sum_{t=1}^{T} [l_t(x_t) - l_t(u)] \leq \sum_{t=1}^{T} \langle \nabla l_t(x_t), x_t - u \rangle \) [Haz16, Ora19]. Online learning with linear losses is called Online Linear Optimization (OLO). As its important special case, Learning with Expert Advice (LEA) considers OLO on a probability simplex, but aims at a different form of regret bound due to its different geometry.

\(^1\)We specifically consider the \( L_1 \) norm for a unified view of OLO and LEA. Our strategy can be extended to other norms as well.

\(^2\)For general \( x_1 \), we can replace \( |u| \) in the regret bound by \( |u - x_1| \).
Classical minimax approaches in online learning include Online Mirror Descent (OMD) and Follow the Regularized Leader (FTRL), with Online Gradient Descent (OGD) being their most well-known special case. We write “gradient descent” as the minimax baseline for the ease of exposition. Moreover, both OMD and FTRL have elegant duality interpretations [Ora19, Section 6.4.1 and 7.3], involving simultaneous updates on the primal space (the domain $X$) and the dual space (the space of gradients). We will exploit this duality in our analysis.

**Parameter-free online learning** Also known as comparator-adaptivity, parameter-free online learning aims at matching the optimally-tuned gradient descent in hindsight, without knowing the correct tuning parameter (i.e., the optimal comparator $u^*$). The associated regret bound can appear in different forms, depending on the specific learning setting.

1. For LEA, a parameter-free bound has the form $O(\sqrt{T \cdot \text{KL}(u||\pi)})$, where $u$ and $\pi$ are distributions on the expert space representing the comparator and a user-chosen prior. Such an idea was initiated in [CFH09], and the analysis was improved and extended by a series of works [CV10, LS15, KVE15, CLW21, NBC+21]. Notably, a parameter-free LEA algorithm naturally induces a bound on the $\varepsilon$-quantile regret - the regret with respect to the $\varepsilon$-quantile best expert; this is particularly meaningful when the number of experts is large. Lower bounds were considered in [NBC+21].

   We will present a non-asymptotic improvement of the $\sqrt{T}$ divergence later in this paper. Frameworks that generalize root KL to $f$-divergences have been studied in [Alq21, NBC+21], but to our knowledge, no existing algorithm guarantees a better divergence term than root KL, even without switching costs.

2. For unconstrained OLO, typical parameter-free bounds are $C + \|u - x_1\|O(\sqrt{T \log(C^{-1} \|u - x_1\|_1 T)})$ or $C\sqrt{T} + \|u - x_1\|O(\sqrt{T \log(C^{-1} \|u - x_1\|_1 T)})$, where $C$ is any user-chosen constant. These two bounds are both Pareto-optimal [ZCP22b], as they represent different trade-offs on the loss (the regret at $u = x_1$) and the asymptotic regret (when $\|u - x_1\|$ is large). Existing works [MO14, CO18, FRS18, MK20, JC22] were mostly independent of the LEA setting, but unified views were presented in [FRS15, OP16]. Lower bounds were discussed in [SM12, Ora13, ZCP22b].

**Switching cost** Motivated by numerous applications, switching costs in online decision making have been studied from many different angles. For example, beside online learning, the online algorithm community has investigated settings like smoothed online optimization [CGW18, GLSW19, LQL20] and convex body chasing [BLLS19, Sel20], where the loss function $l_t$ is observed before the agent picks the prediction $x_t$. There, the switching cost is the key consideration that prevents the trivial strategy $x_t = \arg\min_x l_t(x)$. As for online learning, an additional complication is that $x_t$ (e.g., the investment portfolio) should be selected without knowing $l_t$ (e.g., tomorrow’s stock price).

   Even within online learning, there are several ways to model the switching cost. In cases like network routing, every switch means changing the packet route, which can be costly. Therefore, one needs a lazy agent whose amount of switches (or its expectation) $\|x_t - x_{t+1}\|$ is as low as possible - a good modeling candidate is $1[x_t \neq x_{t+1}]$. Alternatively, one could take a smooth view [ABL+13, BCKP21, WWYZ21, ZJLY21] where the aggregate cost as many switches as it wishes, as long as the cumulative distance of switching is low - in this view, switching cost can be a norm $\|x_t - x_{t+1}\|$ or its smoothed variant $\|x_t - x_{t+1}\|^2$. The present work considers the $L_1$ norm switching cost motivated by the transaction cost in some financial applications. Notably, for LEA, the $L_1$ norm unifies the lazy view and the smooth view [DM19, Section 5.2].

   Although switching costs have been extensively studied, existing works on the combination of adaptivity and switching cost are quite sparse. As one should carefully trade-off these two opposite requirements, there have been interesting impossibility results [Gof14, DM19], highlighted in our introduction. In this regard, one should not believe that every classical adaptivity can be naturally achieved with switching cost. Fortunately, we show that the optimal parameter-freeness can indeed be achieved, thus improving the suboptimal result in [ZCP22a].

**Relation to downstream problems** More generally, incorporating switching costs amounts to considering a history-dependent adversary: it can pick loss functions that depend not only on the instantaneous prediction $x_t$, but also on the previous prediction $x_{t-1}$. One could further generalize this setting to online learning with memory [CBD13, AHM15], where the loss depends on a fixed-length prediction history, and finally to dynamical systems [ABH+19, SSH20, Sim20], where the entire history matters. In fact, a common procedure in nonstochastic control
[ABH+19] is to bound the risk in the future by a properly scaled switching cost. Achieving parameter-freeness with switching cost may benefit these important downstream problems as well, by making algorithms easy to combine [Cut19, Cut20, ZCP22a].

1.3 Notation
Let $f^*$ be the Fenchel conjugate of a function $f$. $\Delta(d)$ represents the $d$-dimensional probability simplex; KL and TV denote the KL divergence and the total variation distance, respectively. For two integers $a \leq b$, $[a : b]$ is the set of all integers $c$ such that $a \leq c \leq b$. log represents the natural logarithm when the base is omitted. Throughout this paper, “increasing” and “positive” are not strict (i.e., include equality as well).

For a twice differentiable function $V(t, S)$ where $t$ represents time and $S$ represents a spatial variable, let $\nabla_t V$, $\nabla_{tt} V$, $\nabla_{SS} V$ and $\nabla_{SSS} V$ be the first, second and order partial derivatives. In addition, we define discrete derivatives as

$$\nabla_t V(t, S) = V(t, S) - V(t - 1, S),$$
$$\nabla_S V(t, S) = \frac{1}{2} [V(t, S + 1) - V(t, S - 1)],$$
$$\nabla_{SS} V(t, S) = V(t, S + 1) + V(t, S - 1) - 2V(t, S).$$

2 OLO with switching cost
This section presents our main result, a parameter-free OLO algorithm with switching cost. We will start with the one-dimensional unconstrained setting, followed by extensions to bounded domains and higher dimensions.

2.1 The 1D unconstrained setting
We consider the domain $\mathcal{X} = \mathbb{R}$, a Lipschitz constant $G > 0$ for the loss gradients, and a weight $\lambda \geq 0$ for switching costs. In the $t$-th round, the agent predicts $x_t \in \mathbb{R}$, receives a loss gradient $g_t \in [-G, G]$ that depends on past predictions $x_{1:t}$, and suffers an augmented loss $g_t x_t + \lambda |x_t - x_{t-1}|$ (w.l.o.g., let $x_0 = x_1 = 0$). We consider the augmented regret for all $u \in \mathbb{R}$ and $T \in \mathbb{N}_+$:

$$\text{Regret}_u^\lambda(T) := \sum_{t=1}^{T} g_t (x_t - u) + \lambda \sum_{t=1}^{T-1} |x_t - x_{t+1}|. \quad (1)$$

Ignoring the dependence on $G$ for now, our goal is to show a parameter-free bound $\tilde{O}(\sqrt{\lambda T} \log(C^{-1} |u| T))$ for any hyperparameter $C > 0$. These two cases are equivalent via the standard doubling trick [SS11].

For minmax algorithms like bounded domain gradient descent, one can use scaled learning rates $\eta_t \propto 1/\sqrt{\lambda t}$ to ensure that both sums in (1) are $O(\sqrt{\lambda T})$, thus obtaining a combined $O(\sqrt{\lambda T})$ regret bound. However, such a divide-and-conquer approach does not apply to parameter-free algorithms, as one cannot separately show the desirable bound on the two sums in (1). To see this, suppose one could guarantee the second sum alone is at most $1 + |u| O(\sqrt{T} \log(|u| T))$: here we only focus on the dependence on $|u|$ and $T$. Since this cumulative switching cost is an algorithmic quantity independent of the comparator, we can take infimum with respect to $u$ and obtain a “budget” of 1 for this sum. Following this argument, $|x_T| \leq |x_1| + \sum_{t=1}^{T-1} |x_t - x_{t+1}| = O(1)$. That is, the algorithm should only predict around the origin, which clearly leads to large regret with respect to far-away comparators, under certain loss sequences.

The challenge can be motivated in another way. As shown in [Ora19, Figure 9.1], the one-step switching cost $|x_t - x_{t+1}|$ of parameter-free algorithms can grow exponentially with respect to $t$, whereas such a quantity is uniformly bounded in gradient descent. In fact, the exponential growth is the key mechanism for standard parameter-free algorithms (i.e., without switching cost) to cover an unconstrained domain fast enough. This is however problematic when switching is penalized, as one can no longer control the switching cost by uniformly scaling $|x_t - x_{t+1}|$. 


2.2 Switching-adjusted potential

To address these issues, one should bound the switching cost and the standard OLO regret in a unified framework, instead of treating them separately. The prior work [ZCP22a] used the coin-betting approach from [OP16, CO18]. In the $t$-th round, the algorithm maintains a sufficient statistic $\text{Wealth}_{t-1}$; by picking a betting fraction $\beta_t \in [0,1]$, the prediction is set to $x_t = \beta_t \text{Wealth}_{t-1}$. To ensure low switching cost, the betting fraction $\beta_t$ in [ZCP22a] is capped by a decreasing upper bound $O(1/\sqrt{t})$. Such a hard threshold is very conservative, which could be the reason of their suboptimal result.

In contrast, we follow the more general potential framework explored by a parallel line of works [MO14, FRS18, MK20, ZCP22b]. Coin-betting is essentially derived from certain types of potentials [OP16], and many theoretical results using coin-betting can be recovered by the latter. In general, a potential algorithm is defined with a potential function $V(t, S)$, where $t$ represents the time index, and $S$ represents a “sufficient statistic”. In each round, the algorithm computes $S_{t-1} = -\sum_{i=1}^{t-1} g_i / G$, and the prediction $x_t$ is the derivative $\nabla_S V$ evaluated at $(t, S_{t-1})$. We will specifically consider Algorithm 1, which is a variant based on the discrete derivative $\nabla_S V$ (see Section 1.3 for its definition).

Algorithm 1 One-dimensional unconstrained OLO with switching costs.

Require: A hyperparameter $C > 0$, the Lipschitz constant $G$, and a potential function $V(t, S)$ that implicitly depends on $\lambda$ and $G$.
1: Initialize $S_0 = 0$.
2: for $t = 1, 2, \ldots$ do
3:   Predict $x_t = \nabla_S V(t, S_{t-1})$, and receive the loss gradient $g_t$. Let $S_t = S_{t-1} - g_t / G$.
4: end for

One could think of the potential framework as the dual approach of FTRL - the potential function and the regularizer are naturally Fenchel conjugates. While the FTRL analysis relies on a one-step regret bound on the primal space (the domain $X$, cf. [Ora19, Lemma 7.1]), the potential framework constructs a similar one-step relation on the dual space (the space of $S_t$, cf. [ZCP22b, Lemma 3.1]). Along this interpretation, our key idea is to incorporate switching costs by scaling on the dual space, rather than only on the primal space. That is, given a potential function that works without switching costs, we scale the sufficient statistic sent to its second argument by a function of $\lambda$. Here is a clarification: we do not mean one shouldn’t scale on the primal space; on the contrary, uniformly scaling the magnitude of the predictions can be useful in some ways. However, it is the scaling on the dual space that allows a parameter-free bound, as we will see later.

To better demonstrate this idea, let us first consider a quadratic potential $V(t, S) = \frac{1}{2} C G S^2$. The potential method suggests the prediction $x_t = \nabla_S V(t, S_{t-1}) = C \sum_{i=1}^{t-1} g_i = x_{t-1} - C g_{t-1}$, which is simply gradient descent with learning rate $C$. Scaling on the primal space means scaling $V$ directly, while scaling on the dual space means scaling the sufficient statistic $S$. It is clear that both cases are equivalent to scaling the effective learning rate, which is the standard way to incorporate switching costs in bounded domain gradient descent. In other words, for this gradient descent potential, the two types of scaling are essentially the same.

Now, to achieve optimal parameter-freeness, we need a better potential where scaling on the dual space actually makes a difference. With a hyperparameter $\alpha$ that will eventually depend on $\lambda$, we consider Algorithm 1 induced by the potential

$$V_\alpha(t, S) = C \sqrt{\alpha t} \left[ 2 \int_0^{S/\sqrt{\alpha t}} \left( \int_0^u \exp(x^2) dx \right) du - 1 \right].$$

When the Lipschitz constant $G = 1$, it has been shown [ZCP22b] that $\alpha = 1/2$ leads to parameter-freeness without switching cost. Here we use $\alpha = 4 \lambda G^{-1} + 2$, which amounts to scaling both the primal space and the dual space: on the primal space, we scale up the overall prediction by $\Theta(\sqrt{\lambda G^{-1}})$, and on the dual space we scale down the sufficient statistic $S$ by $\Theta(1/\sqrt{\lambda G^{-1} + 1})$. The latter gives us the optimal parameter-free bound (i.e., Pareto-optimal rate in $|u|$ and $T$), while the former helps us obtain the optimal rate in $\lambda$. Due to incorporating $\lambda$ in $V_\alpha$, we call our approach the switching-adjusted potential method.

Finally, although the definition of $V_\alpha$ seems mysterious at first glance, it is actually derived from a clean continuous-time analysis presented in Appendix A.1. Such a limiting perspective provides an intuitive justification for our scaling strategy.
2.3 Optimal parameter-free bound

Despite its simplicity, our approach improves the existing result [ZCP22a] by a considerable margin. In the following, we present our 1D optimal parameter-free bound, discuss its significance, and sketch its proof.

**Theorem 1.** If $\alpha = 4\lambda G^{-1} + 2$, then Algorithm 1 induced by the potential $V_\alpha$ guarantees

$$\text{Regret}_T^\lambda(u) \leq \sqrt{(4\lambda G + 2G^2)T} \left[ C + |u| \left( \sqrt{4 \log \left( \frac{|u|}{C} \right) + 2} \right) \right],$$

for all $u \in \mathbb{R}$ and $T \in \mathbb{N}_+$.  

Theorem 1 simultaneously achieves several forms of optimality.

1. Pareto-optimal loss-regret trade-off: considering the dependence on $u$ and $T$, $\text{Regret}_T^\lambda(u) = O \left( |u| \sqrt{T \log |u|} \right)$, while the cumulative loss satisfies $\text{Regret}_T^\lambda(0) = O(\sqrt{T})$. An existing lower bound [ZCP22b, Theorem 10] shows that even without switching cost, all algorithms satisfying a $O(\sqrt{T})$ loss bound must suffer an $\Omega \left( |u| \sqrt{T \log |u|} \right)$ regret bound. In this sense, our algorithm attains a Pareto-optimal loss-regret trade-off, in a strictly generalized setting with switching costs.  

2. On $T$ alone: $\text{Regret}_T^\lambda(u) = O(\sqrt{T})$. Despite achieving parameter-freeness (i.e., adaptivity to $u$), the asymptotic rate on $T$ is still the optimal one, matching the well-known minimax lower bound.

3. On $\lambda$ alone: $\text{Regret}_T^\lambda(u) = O(\sqrt{\lambda})$. Our bound has the optimal dependence on the switching cost weight [GVW10, Theorem 5].

To compare Theorem 1 to [ZCP22a], we have to convert them to the same loss-regret trade-off, i.e., both guaranteeing $\text{Regret}_T^\lambda(0) = O(1)$ or $\text{Regret}_T^\lambda(0) = O(\sqrt{T})$. Here we take the first approach - details are presented in Appendix A.4. By a doubling trick, assuming $G = 1$ for clarity, our bound can be converted to $C + |u| O \left( \sqrt{\lambda T \log (C^{-1} \lambda |u| T)} \right)$, which improves the rate $C + |u| O \left( \sqrt{\lambda T \log (C^{-1} \lambda |u| T)} \right)$ from [ZCP22a, Theorem 1]. Specifically, our converted upper bound also attains Pareto-optimality in this regime (i.e., matching the lower bound in [Ora13]), whereas the existing approach does not.

We now sketch the proof of Theorem 1; the formal analysis is deferred to Appendix A.3. We mostly follows a standard potential argument, which is another benefit over the existing approach - the idea of this proof is easier to interpret and generalize.

**Proof sketch of Theorem 1**  

The first step is to show a one-step bound on the growth rate of the potential. If there is no switching cost, then the Discrete Ito formula [Kle13, HLPR20, ZCP22b] can serve this purpose. It applies to any convex potential $V$, hence does not use any special property of our potential $V_\alpha$.

**Lemma 2.1** (Lemma 3.1 of [ZCP22b]). *If the potential function $V(t, S)$ is convex in $S$, then against any adversary, Algorithm 1 guarantees for all $t \in \mathbb{N}_+$,

$$V(t, S_t) - V(t-1, S_{t-1}) \leq -G^{-1} g_t x_t + \nabla_t V(t, S_{t-1}) + \frac{1}{2} \nabla_{SS} V(t, S_{t-1}).$$

The core component is the following lemma, which incorporates switching costs into $V_\alpha$.

**Lemma 2.2.** For all $\alpha > 0$, consider Algorithm 1 induced by the potential function $V_\alpha$. For all $t \in \mathbb{N}_+$,

$$|x_t - x_{t+1}| \leq \nabla_S V_\alpha(t, S_{t-1} + 1) - \nabla_S V_\alpha(t, S_{t-1} - 1).$$

Combining the above, if we define

$$\Delta_t := \nabla_t V_\alpha(t, S_{t-1}) + \frac{1}{2} \nabla_{SS} V_\alpha(t, S_{t-1}) + G^{-1} \lambda \left[ \nabla_S V_\alpha(t, S_{t-1} + 1) - \nabla_S V_\alpha(t, S_{t-1} - 1) \right],$$

(3)
then a telescopic sum yields the following cumulative loss bound

$$\text{Regret}_T^A(0) \leq \sum_{t=1}^{T} (g_t x_t + \lambda |x_t - x_{t+1}|) \leq -G \cdot V_\alpha(T, S_T) + G \sum_{t=1}^{T} \Delta_t.$$  

To proceed, we need to control the residual term $\Delta_t$, which may seem problematic due to its complicated form. Fortunately, a careful analysis shows that $\Delta_t$ vanishes with a proper choice of $\alpha$!

**Lemma 2.3.** If $\alpha \geq 4\lambda G^{-1} + 2$, then for all $t$ and against any adversary, $\Delta_t \leq 0$.

Finally, with the updated loss bound $\text{Regret}_T^A(0) \leq -G \cdot V_\alpha(T, S_T)$, our regret bound follows from the classical loss-regret duality [MO14, Ora19].

### 2.4 Extension to bounded and higher-dimensional domains

To conclude our results on OLO, we now discuss the extension of Algorithm 1 to bounded domains and higher-dimensional domains. Due to limited space, details are presented in Appendix A.5.

First, for a constrained domain $\mathcal{X} \subset \mathbb{R}$, we use a well-known black-box reduction [CO18, Section 4] on top of Algorithm 1 such that the exact bound in Theorem 1 carries over (w.r.t. any $u \in \mathcal{X}$). Similar strategies apply to higher-dimensional problems, but here we emphasize the 1D special case due to an additional feature: if the domain $\mathcal{X}$ has a finite diameter $D$, then the switching cost alone of the combined algorithm has a $\tilde{O}(D \sqrt{T})$ bound on any time interval of length $T$. This could be useful when switching costs have high priority [SK21, WWYZ21] and should be independently bounded. Moreover, it allows the combination of parameter-free algorithms [ZCP22a] in settings with long term prediction effects (e.g., switching cost or memory).

**Theorem 2.** Consider the setting of Section 2.1, but on a (smaller) closed and convex domain $\mathcal{X} \subset \mathbb{R}$. Let $x^*$ be an arbitrary point in $\mathcal{X}$. For all $C > 0$, Algorithm 3 in Appendix A.5 guarantees

$$\text{Regret}_T^A(u) \leq \sqrt{(4\lambda G + 2G^2)T} \left[ C + |u - x^*| \left( \sqrt{4 \log \left( 1 + \frac{|u - x^*|}{C} \right)} + 2 \right) \right],$$

for all $u \in \mathcal{X}$ and $T \in \mathbb{N}_+$. Moreover, if $\mathcal{X}$ has a finite diameter $D$, then on any time interval $[T_1 : T_2] \subset \mathbb{N}_+$, the same algorithm guarantees

$$\sum_{t=T_1}^{T_2-1} |x_t - x_{t+1}| \leq 22 \sqrt{T_2 - T_1} \left[ 2D + C + 2D \sqrt{\log(1 + DC^{-1})} \right].$$

From a technical perspective, the second part of Theorem 2 is interesting due to its non-black-box use of the reduction approach: we characterize how this reduction (implicitly) controls the unconstrained base algorithm, resulting in the “concentration” of its sufficient statistic $S_t$ (i.e., $S_t = O(\sqrt{t})$) as if losses are stochastic. A similar bound was presented in [ZCP22a], but it critically relies on hard-thresholding a betting fraction, which, as we have shown, is suboptimal. In contrast, we use a different analysis on the improved base algorithm (Algorithm 1) to achieve this switching cost bound and an improved regret bound simultaneously.

As for higher dimensions, let us consider the setting where $\mathcal{X} = \mathbb{R}^d$, $\|g_t\|_{\infty} \leq G$, and the switching costs are measured by the $L_1$ norm. This serves as a nice bridge towards our LEA approach and financial applications. We run Algorithm 1 on each coordinate separately [SM12], and scale the hyperparameter $C$ by $1/d$.

**Theorem 3.** Consider OLO with switching costs on the domain $\mathcal{X} = \mathbb{R}^d$; assume loss gradients satisfy $\|g_t\|_{\infty} \leq G$. For all $C > 0$, Algorithm 4 in Appendix A.5 guarantees

$$\sum_{t=1}^{T} (g_t x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 \leq \sqrt{(4\lambda G + 2G^2)T} \left[ C + \|u\|_1 \left( \sqrt{4 \log \left( 1 + \frac{\|u\|_{\infty} d}{C} \right)} + 2 \right) \right],$$

for all $u \in \mathbb{R}^d$ and $T \in \mathbb{N}_+$. 

7
3 LEA with switching cost

Our Algorithm 1 can be applied to LEA with switching cost, resulting in the first parameter-free algorithm there. Conversion techniques without switching costs were studied in [LS15, OP16], and since then, they have become standard tools for the online learning community. Here we present a different view, based on their connection to the constrained domain reduction [CO18] adopted in our OLO analysis. In particular, it leads to a mechanism for incorporating switching costs, with a clear geometric interpretation.

The setting of LEA with switching cost is a special case of the high-dimensional OLO problem. Let $d$ be the number of experts. Then, compared to the setting of Theorem 3, we simply change $X$ to the probability simplex $\Delta(d)$. The main difference with OLO is the form of parameter-free bounds – here we aim at Regret$_T^x(u) = O(\sqrt{T} \cdot \KL(u||\pi))$, where $\pi \in \Delta(d)$ is a prior chosen at the beginning. Achieving such a root KL bound relies on special conversion techniques.

Existing approaches [LS15, OP16] have the following procedure. Given a 1D OLO algorithm that predicts on $\mathbb{R}_+$, independent copies are created for each coordinate and updated using certain surrogate losses. A meta-algorithm queries the coordinate-wise predictions $\{w_{t,i}; i \in [1 : d]\}$, collects them into a weight vector $w_t = [w_{t,1}, \ldots, w_{t,d}]$, and finally predicts the scaled weight $x_t = w_t/\|w_t\|_1$ on $\Delta(d)$. Despite its general success, such an approach has a discontinuity problem when switching costs are incorporated: if two consecutive weights $w_t$ and $w_{t+1}$ are both close to the origin, then simply scaling them to $\Delta(d)$ can lead to a large switching cost, even when $\|w_t - w_{t+1}\|_1$ is small. This problem is exacerbated by the typical setting$^3$ of $w_1 = 0$, due to the associated analysis. A graphical demonstration is provided in Figure 1 (Left).

![Figure 1: Switching costs in LEA-OLO reductions. Left: existing approaches. Right: ours, where the projection of $w_t$ is separated into two cases. (i) $\|w_t\|_1 \geq 1$, shown in green; (ii) $\|w_t\|_1 < 1$, shown in black.](image-url)

Our solution is based on a unified view of the LEA-OLO reduction and the constrained domain reduction [CO18]. Starting from the setting without switching costs, we observe that the general Banach version of the latter can also convert OLO to LEA, therefore specialized techniques are not required for this task. Algorithmically, we set $x_t = \arg\min_{x \in \Delta(d)} \|x - w_t\|_1$ as opposed to $x_t = w_t/\|w_t\|_1$. The surrogate losses for the base algorithms are also different, which we elaborate in Appendix B.3.

A major benefit of this unified view is the non-uniqueness of the $L_1$ norm projection: if $\|w_t\| < 1$, then any $x_t \in \Delta(d)$ satisfying $\{x_{t,i} \geq w_{t,i}; \forall i\}$ minimizes $\|x - w_t\|_1$ on $\Delta(d)$. This brings more flexibility to the algorithm design: for the setting with switching costs, we adopt (i) the orthogonal projection $x_t = w_t + d^{-1}(1 - \|w_t\|_1)$ when $\|w_t\|_1 \leq 1$, and (ii) the scaling $x_t = w_t/\|w_t\|_1$ when $\|w_t\|_1 > 1$. The orthogonal projection is better for controlling switching costs, as shown in Figure 1 (Right). Concretely, this leads to the first parameter-free algorithm for LEA with switching cost.

**Theorem 4.** For LEA with switching cost, given any prior $\pi$ in the relative interior of $\Delta(d)$, Algorithm 5 from Appendix B.2 guarantees

$$\sum_{t=1}^{T} \langle g_t, x_t - u \rangle + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 = \left[ \sqrt{\TV(u||\pi)} \cdot \KL(u||\pi) + 1 \right] \cdot O \left( \sqrt{(\lambda G + G^2)T} \right),$$

$^3$When $w_1 = 0$, $x_t$ can be arbitrary on $\Delta(d)$ by definition. However, as $w_t$ changes continuously w.r.t. the observed information, it could hover around 0 at some point, thus experiencing the sketched problem.
for all $u \in \Delta(d)$ and $T \in \mathbb{N}_+$.

We make two comments on this bound.

1. Since our algorithm is parameter-free, this bound only implicitly depends on $d$ through the divergence term $\sqrt{TV \cdot KL}$. In favorable cases we may have a good prior $\pi$ such that $TV(u||\pi) \cdot KL(u||\pi) = O(1)$; this will save us a $\sqrt{\log d}$ factor compared to minimax algorithms (with switching costs), such as Follow the Lazy Leader [KV05] and Shrinking Dartboard [GVW10].

2. Even without switching costs, we improve the $\sqrt{KL}$ divergence term in existing parameter-free bounds [CFH09, LS15, OP16] to $\sqrt{TV \cdot KL}$. The latter is better since (i) TV is always less than 1, and (ii) there exist $p, q \in \Delta(d)$ such that $TV(p||q) \cdot KL(p||q) \leq 1$ but $KL(p||q) \geq \sqrt{\log d} - o(1)$ (cf. Appendix B.3). Generalizations of root $KL$ to $f$-divergences have been considered in [Alq21, NBC+21], but to our knowledge, no existing algorithm guarantees a better divergence term than root $KL$.

4 Unconstrained investment with transaction cost

Finally, we present applications to a portfolio selection problem with transaction costs. Online portfolio selection has been studied by multiple communities, resulting in a large amount of literature (see [LH14, Doc16] for general expositions). Here we consider an unconstrained setting, allowing both short selling (i.e., holding unconstrained wealth), and margin trading (i.e., borrowing money to buy assets). Its connections and differences to the classical rebalancing setting [Cov91, CO96, HSSW98, KV02, LWZ18] are detailed in Appendix C.1.

We consider a market with $d$ assets and discrete trading period $t \in \mathbb{N}_+$. In the $t$-th round, an algorithm chooses a portfolio vector $x_t = [x_{t,1}, \ldots, x_{t,d}] \in \mathbb{R}^d$, where $x_{t,i}$ is the number of shares of the $i$-th asset that the algorithm suggests to hold. Compared to the previous round, we need to buy $x_{t,i} - x_{t-1,i}$ shares (or sell, if negative), which requires paying a $\lambda |x_{t,i} - x_{t-1,i}|$ transaction cost. Then, the market reveals a number $g_{t,i} \in [-G, G]$, which represents the price change per share (of the $i$-th asset) in this round. This effectively increases the value of our portfolio by $\langle g_{t,i}, x_{t,i} \rangle$.

The considered performance metric is the increased amount of wealth on any time horizon $[1 : T] \subset \mathbb{N}_+$, and such wealth includes the total value of our portfolio plus cash. Our goal is to show that the performance of our algorithm is never much worse than that of any unconstrained Buy-and-Hold (BAH) strategy, which picks a portfolio vector $u \in \mathbb{R}^d$ at the beginning and holds that amount throughout the considered time horizon. That is, we aim to upper bound $\sum_{t=1}^{T} \langle -g_{t}, x_{t} - u \rangle + \lambda \sum_{t=1}^{T-1} \|x_{t} - x_{t+1}\|_1$ for all $u \in \mathbb{R}^d$ and $T \in \mathbb{N}_+$. This is exactly the setting of Theorem 3 with flipped gradients, therefore the same theoretical result carries over.

To complement the theory, we present some numerical results on a synthetic market. Let $G = 1$, $\lambda = 0.1$, and the market contains five assets with different return characteristics. Each $g_{t,i}$ is the summation of a i.i.d. noise, a periodic fluctuation and a constant trend, e.g.,

$$g_{t,i} = 0.6 \cdot \text{Uniform}[-1, 1] + 0.2 \sin \left(\frac{(t/500 + 1) \pi}{\pi} \right) + 0.2.$$ 

Two algorithms are tested, our Algorithm 4 (i.e., “ours”), and the baseline [ZCP22a, Algorithm 1, adapted]. Both algorithms require a confidence parameter (our $C$, and the initial wealth for the baseline, also denoted by $C$). They are set to 1 following the practice of parameter-free algorithms [OP16, CLO20, ZCP22a]. Each algorithm is tested in 50 random trials, and the increased wealth $\sum_{t=1}^{T} \langle g_{t}, x_{t} \rangle - \lambda \sum_{t=1}^{T-1} \|x_{t} - x_{t+1}\|_1$ (mean ± std) is plotted in Figure 2, higher is better. In this setting, our algorithm beats the baseline by a considerable margin, due to being a lot less conservative.

---

\[W.l.o.g., \text{assume } x_0 = x_1.\]

\[\text{The coefficient } \lambda \text{ can depend on } i, \text{ the sign of } x_{t,i} - x_{t+1,i} \text{ and the sign of } x_{t,i}, \text{ but for simplicity we use the same } \lambda \text{ for all cases.}\]
Finally, detailed settings and further experiments, including preliminary results on historical US stock data, are deferred to Appendix C.2 and C.3. Specifically, we also test different $\lambda$ to show that our algorithm scales to transaction costs better.

5 Conclusion

The present work investigates the design of parameter-free algorithms in the presence of switching cost. By carefully trading off the two opposite considerations, we propose a simple algorithm for OLO with switching cost, which improves the suboptimal regret bound [ZCP22a] to the optimal rate. Extensions of this algorithm lead to new results for bounded domain OLO, parameter-free LEA, and unconstrained portfolio selection.

More broadly, our contribution lies in the intersection of adaptive online learning and online learning with long-term effects. Future works may continue this direction to make learning-based control and decision algorithms more capable in complex environments.

References

[ABH+19] Naman Agarwal, Brian Bullins, Elad Hazan, Sham Kakade, and Karan Singh. Online control with adversarial disturbances. In International Conference on Machine Learning, pages 111–119. PMLR, 2019.

[ABL+13] Lachlan Andrew, Siddharth Barman, Katrina Ligett, Minghong Lin, Adam Meyerson, Alan Roytman, and Adam Wierman. A tale of two metrics: Simultaneous bounds on competitiveness and regret. In Conference on Learning Theory, pages 741–763. PMLR, 2013.

[AHM15] Oren Anava, Elad Hazan, and Shie Mannor. Online learning for adversaries with memory: price of past mistakes. Advances in Neural Information Processing Systems, 28, 2015.

[Alo21] Pierre Alquier. Non-exponentially weighted aggregation: Regret bounds for unbounded loss functions. In International Conference on Machine Learning, pages 207–218. PMLR, 2021.

[AT18] Jason Altschuler and Kunal Talwar. Online learning over a finite action set with limited switching. In Conference On Learning Theory, pages 1569–1573. PMLR, 2018.

[BCKP21] Aditya Bhaskara, Ashok Cutkosky, Ravi Kumar, and Manish Purohit. Power of hints for online learning with movement costs. In International Conference on Artificial Intelligence and Statistics, pages 2818–2826. PMLR, 2021.

[BK99] Avrim Blum and Adam Kalai. Universal portfolios with and without transaction costs. Machine Learning, 35(3):193–205, 1999.

[BLLS19] Sébastien Bubeck, Yin Tat Lee, Yuanzhi Li, and Mark Sellke. Competitively chasing convex bodies. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, pages 861–868, 2019.

[CBDS13] Nicolo Cesa-Bianchi, Ofer Dekel, and Ohad Shamir. Online learning with switching costs and other adaptive adversaries. Advances in Neural Information Processing Systems, 26, 2013.

[CBL06] Nicolo Cesa-Bianchi and Gábor Lugosi. Prediction, learning, and games. Cambridge university press, 2006.

[CFH09] Kamalika Chaudhuri, Yoav Freund, and Daniel J Hsu. A parameter-free hedging algorithm. Advances in neural information processing systems, 22, 2009.

[CGW18] Niangjun Chen, Gautam Goel, and Adam Wierman. Smoothed online convex optimization in high dimensions via online balanced descent. In Conference On Learning Theory, pages 1574–1594. PMLR, 2018.
[CLO20] Keyi Chen, John Langford, and Francesco Orabona. Better parameter-free stochastic optimization with ode updates for coin-betting. arXiv preprint arXiv:2006.07507, 2020.

[CLW21] Liyu Chen, Haipeng Luo, and Chen-Yu Wei. Impossible tuning made possible: A new expert algorithm and its applications. In Conference on Learning Theory, pages 1216–1259. PMLR, 2021.

[CO96] Thomas M Cover and Erik Ordentlich. Universal portfolios with side information. IEEE Transactions on Information Theory, 42(2):348–363, 1996.

[CO18] Ashok Cutkosky and Francesco Orabona. Black-box reductions for parameter-free online learning in banach spaces. In Conference On Learning Theory, pages 1493–1529. PMLR, 2018.

[Cov91] Thomas M Cover. Universal portfolios. Mathematical Finance, 1(1):1–29, 1991.

[Cut19] Ashok Cutkosky. Combining online learning guarantees. In Conference on Learning Theory, pages 895–913. PMLR, 2019.

[Cut20] Ashok Cutkosky. Parameter-free, dynamic, and strongly-adaptive online learning. In International Conference on Machine Learning, pages 2250–2259. PMLR, 2020.

[CV10] Alexey Chernov and Vladimir Vovk. Prediction with advice of unknown number of experts. In Proceedings of the Twenty-Sixth Conference on Uncertainty in Artificial Intelligence, pages 117–125, 2010.

[CYLK20] Lin Chen, Qian Yu, Hannah Lawrence, and Amin Karbasi. Minimax regret of switching-constrained online convex optimization: No phase transition. Advances in Neural Information Processing Systems, 33:3477–3486, 2020.

[DM19] Amit Daniely and Yishay Mansour. Competitive ratio vs regret minimization: achieving the best of both worlds. In Algorithmic Learning Theory, pages 333–368. PMLR, 2019.

[Doc16] Robert Dochow. Online algorithms for the portfolio selection problem. Springer, 2016.

[FRS15] Dylan J Foster, Alexander Rakhlin, and Karthik Sridharan. Adaptive online learning. Advances in Neural Information Processing Systems, 28:3375–3383, 2015.

[FRS18] Dylan J Foster, Alexander Rakhlin, and Karthik Sridharan. Online learning: Sufficient statistics and the burkholder method. In Conference On Learning Theory, pages 3028–3064. PMLR, 2018.

[GLSW19] Gautam Goel, Yiheng Lin, Haoyuan Sun, and Adam Wierman. Beyond online balanced descent: An optimal algorithm for smoothed online optimization. Advances in Neural Information Processing Systems, 32, 2019.

[Gof14] Eyal Gofer. Higher-order regret bounds with switching costs. In Conference on Learning Theory, pages 210–243. PMLR, 2014.

[GVW10] Sascha Geulen, Berthold Vöcking, and Melanie Winkler. Regret minimization for online buffering problems using the weighted majority algorithm. In Conference on Learning Theory, pages 132–143, 2010.

[Haz16] Elad Hazan. Introduction to online convex optimization. Foundations and Trends® in Optimization, 2(3-4):157–325, 2016.

[HLPR20] Nicholas JA Harvey, Christopher Liaw, Edwin A Perkins, and Sikander Randhawa. Optimal anytime regret for two experts. In 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), pages 1404–1415. IEEE, 2020.

[HSSW98] David P Helmbold, Robert E Schapire, Yoram Singer, and Manfred K Warmuth. On-line portfolio selection using multiplicative updates. Mathematical Finance, 8(4):325–347, 1998.
[JC22] Andrew Jacobsen and Ashok Cutkosky. Parameter-free mirror descent. *arXiv preprint arXiv:2203.00444*, 2022.

[JO19] Kwang-Sung Jun and Francesco Orabona. Parameter-free locally differentially private stochastic subgradient descent. *arXiv preprint arXiv:1911.09564*, 2019.

[Kle13] Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

[KV02] Adam Tauman Kalai and Santosh Vempala. Efficient algorithms for universal portfolios. *Journal of Machine Learning Research*, pages 423–440, 2002.

[KV05] Adam Kalai and Santosh Vempala. Efficient algorithms for online decision problems. *Journal of Computer and System Sciences*, 71(3):291–307, 2005.

[KVE15] Wouter M Koolen and Tim Van Erven. Second-order quantile methods for experts and combinatorial games. In *Conference on Learning Theory*, pages 1155–1175. PMLR, 2015.

[LH14] Bin Li and Steven CH Hoi. Online portfolio selection: A survey. *ACM Computing Surveys (CSUR)*, 46(3):1–36, 2014.

[LQL20] Yingying Li, Guannan Qu, and Na Li. Online optimization with predictions and switching costs: Fast algorithms and the fundamental limit. *IEEE Transactions on Automatic Control*, 66(10):4761–4768, 2020.

[LS15] Haipeng Luo and Robert E Schapire. Achieving all with no parameters: Adanormalhedge. In *Conference on Learning Theory*, pages 1286–1304. PMLR, 2015.

[LWZ18] Haipeng Luo, Chen-Yu Wei, and Kai Zheng. Efficient online portfolio with logarithmic regret. *Advances in Neural Information Processing Systems*, 31, 2018.

[MK20] Zakaria Mhammedi and Wouter M Koolen. Lipschitz and comparator-norm adaptivity in online learning. In *Conference on Learning Theory*, pages 2858–2887. PMLR, 2020.

[MO14] H Brendan McMahan and Francesco Orabona. Unconstrained online linear learning in hilbert spaces: Minimax algorithms and normal approximations. In *Conference on Learning Theory*, pages 1020–1039. PMLR, 2014.

[MR22] Zakaria Mhammedi and Alexander Rakhlin. Damped online newton step for portfolio selection. *arXiv preprint arXiv:2202.07574*, 2022.

[NBC+21] Jeffrey Negrea, Blair Bilodeau, Nicoló Campolongo, Francesco Orabona, and Dan Roy. Minimax optimal quantile and semi-adversarial regret via root-logarithmic regularizers. *Advances in Neural Information Processing Systems*, 34, 2021.

[OLL17] Laurent Orseau, Tor Lattimore, and Shane Legg. Soft-bayes: Prod for mixtures of experts with log-loss. In *International Conference on Algorithmic Learning Theory*, pages 372–399. PMLR, 2017.

[OP16] Francesco Orabona and Dávid Pál. Coin betting and parameter-free online learning. *Advances in Neural Information Processing Systems*, 29, 2016.

[Ora13] Francesco Orabona. Dimension-free exponentiated gradient. *Advances in Neural Information Processing Systems*, 26, 2013.

[Ora19] Francesco Orabona. A modern introduction to online learning. *arXiv preprint arXiv:1912.13213*, 2019.

[OT17] Francesco Orabona and Tatiana Tommasi. Training deep networks without learning rates through coin betting. *Advances in Neural Information Processing Systems*, 30:2160–2170, 2017.
[Sel20] Mark Sellke. Chasing convex bodies optimally. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1509–1518. SIAM, 2020.

[Sim20] Max Simchowitz. Making non-stochastic control (almost) as easy as stochastic. Advances in Neural Information Processing Systems, 33:18318–18329, 2020.

[SK21] Uri Sherman and Tomer Koren. Lazy oco: Online convex optimization on a switching budget. In Conference on Learning Theory, pages 3972–3988. PMLR, 2021.

[SM12] Matthew Streeter and Brendan McMahan. No-regret algorithms for unconstrained online convex optimization. Advances in neural information processing systems, 25, 2012.

[SS11] Shai Shalev-Shwartz. Online learning and online convex optimization. Foundations and trends in Machine Learning, 4(2):107–194, 2011.

[SSH20] Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. In Conference on Learning Theory, pages 3320–3436. PMLR, 2020.

[vdH19] Dirk van der Hoeven. User-specified local differential privacy in unconstrained adaptive online learning. Advances in Neural Information Processing Systems, 32, 2019.

[WWYZ21] Guanghui Wang, Yuanyu Wan, Tianbao Yang, and Lijun Zhang. Online convex optimization with continuous switching constraint. Advances in Neural Information Processing Systems, 34, 2021.

[ZAK22] Julian Zimmert, Naman Agarwal, and Satyen Kale. Pushing the efficiency-regret pareto frontier for online learning of portfolios and quantum states. arXiv preprint arXiv:2202.02765, 2022.

[ZCP22a] Zhiyu Zhang, Ashok Cutkosky, and Ioannis Paschalidis. Adversarial tracking control via strongly adaptive online learning with memory. In International Conference on Artificial Intelligence and Statistics, pages 8458–8492. PMLR, 2022.

[ZCP22b] Zhiyu Zhang, Ashok Cutkosky, and Ioannis Paschalidis. PDE-based optimal strategy for unconstrained online learning. arXiv preprint arXiv:2201.07877, 2022.

[Zin03] Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In Proceedings of the 20th International Conference on Machine Learning, pages 928–936, 2003.

[ZJLY21] Lijun Zhang, Wei Jiang, Shiyin Lu, and Tianbao Yang. Revisiting smoothed online learning. Advances in Neural Information Processing Systems, 34, 2021.
Appendix

**Organization** Appendix A, B and C respectively present details on OLO with switching cost, LEA with switching cost and financial applications.

A Details on OLO

This section presents detailed discussions and omitted proofs for our parameter-free OLO strategies. Let us start by justifying our use of the potential $V_\alpha$, through a continuous-time analysis.

### A.1 Continuous-time derivation of $V_\alpha$

[ZCP22b] proposed a framework that generates unconstrained OLO potentials by solving a PDE. Here we present an analogous argument that takes switching costs into account. We will see that it naturally leads to the parametric form of $V_\alpha$ (2) and our key idea of scaling on the dual space. Before starting, we need a generalized definition of the discrete derivative, with a tunable gap increment $\delta$.

$$\nabla^\delta_S V(t, S) = \frac{1}{2\delta} [V(t, S + \delta) - V(t, S - \delta)].$$

The choice of $\delta = 1$ recovers $\nabla_S V(t, S)$ in the main paper. The Lipschitz constant $G$ will be set to 1 for simplicity.

First, let us consider the following inequality that characterizes “admissible” potentials for Algorithm 1. For all $t$ and $S$,

$$V(t - 1, S) \geq \max_{g \in [-1, 1]} \{V(t, S - g) + g\nabla^1_S V(t, S) + \lambda [\nabla^1_S V(t, S) - \nabla^1_S V(t + 1, S - g)]\}. \tag{4}$$

Indeed, if this holds, then we can plug $S = S_{t - 1}$ into this inequality and guarantee that for all $g_t \in [-1, 1]$,

$$g_t x_t + \lambda |x_t - x_{t+1}| \leq V(t - 1, S_{t-1}) - V(t, S_t),$$

which further leads to a cumulative loss bound $\text{Regret}^\alpha_T(0) \leq V(0, 0) - V(T, S_T)$ via telescoping. Bounding $\text{Regret}^\alpha_T(u)$ for general $u$ then follows from the standard loss-regret duality [MO14].

Now, since we ideally need *optimal potential functions* that satisfy the inequality (4) without any slack, let us turn (4) into an equality and try to (approximately) solve it. A continuous-time argument comes in handy: given any $\varepsilon > 0$ that will later approach 0, we scale (i) the unit time by $\varepsilon^2$; and (ii) the loss gradient $g$, the switching cost weight $\lambda$ and the gap increment by $\varepsilon$. Both scaling factors are justified in [ZCP22b, Appendix A.2]; notably, since $g$ and $\lambda$ have the same unit, it is natural that they are scaled by the same rate. With that, we obtain

$$V(t - \varepsilon^2, S) = \max_{g \in [-1, 1]} \{V(t, S - \varepsilon g) + \varepsilon g \nabla^\varepsilon_S V(t, S) + \varepsilon \lambda [\nabla^\varepsilon_S V(t, S) - \nabla^\varepsilon_S V(t + \varepsilon^2, S - \varepsilon g)]\}. \tag{5}$$

Taking second-order Taylor approximations,

$$V(t - \varepsilon^2, S) = V(t, S) - \varepsilon^2 \nabla_t V(t, S) + o(\varepsilon^2),$$

$$V(t, S - \varepsilon g) = V(t, S) - \varepsilon g \nabla_S V(t, S) + \frac{1}{2} \varepsilon^2 g^2 \nabla_{SS} V(t, S) + o(\varepsilon^2),$$

$$\nabla^\varepsilon_S V(t, S) = \frac{1}{2\varepsilon} [V(t, S + \varepsilon) - V(t, S - \varepsilon)] = \nabla_S V(t, S) + o(\varepsilon),$$

$$\nabla^\varepsilon_S V(t + \varepsilon^2, S - \varepsilon g) = \frac{1}{2\varepsilon} [V(t + \varepsilon^2, S - \varepsilon g + \varepsilon) - V(t + \varepsilon^2, S - \varepsilon g - \varepsilon)],$$

where

$$V(t + \varepsilon^2, S - \varepsilon g + \varepsilon) = V(t, S) + \varepsilon^2 \nabla_t V(t, S) + (1 - g)\varepsilon \nabla_S V(t, S) + \frac{1}{2} (1 - g)^2 \varepsilon^2 \nabla_{SS} V(t, S) + o(\varepsilon^2),$$

$$V(t + \varepsilon^2, S - \varepsilon g - \varepsilon) = V(t, S) + \varepsilon^2 \nabla_t V(t, S) + (-1 - g)\varepsilon \nabla_S V(t, S) + \frac{1}{2} (1 + g)^2 \varepsilon^2 \nabla_{SS} V(t, S) + o(\varepsilon^2).$$
Combining everything, (5) becomes
\[ \nabla_t V(t,S) + \max_{g\in[-1,1]} \left\{ \frac{1}{2} g^2 \nabla_{SS} V(t,S) + \lambda |g\nabla_{SS} V(t,S)| + o(1) \right\} = 0. \]

Within the above, the \( o(1) \) term vanishes as \( \varepsilon \to 0 \). Moreover, as typical potential functions are convex in the sufficient statistic, we restrict our search to \( \nabla_{SS} V(t,S) \geq 0 \). The result is the 1D backward heat equation
\[ \nabla_t V + \alpha \nabla_{SS} V = 0, \]
where \( \alpha = \lambda + 1/2 \). Compared to the case without switching cost [ZCP22b, Eq. 5], we effectively change the negative thermal diffusivity \( \alpha \) from 1/2 to 1/2 + \( \lambda \).

Despite the relatively simple form of the backward heat equation, solving it on \( t \in \mathbb{R}_+ \) is a challenging task due to a continuity issue (often called the “ill-posedness” in literature). Here we follow the argument of [ZCP22b] based on a particular ansatz: with a hyperparameter \( c \), if we consider solutions of the form
\[ V_\alpha(t,S) = t^c g \left( \frac{S}{\sqrt{4\alpha t}} \right), \]
then the backward heat equation reduces to the Hermite ODE
\[ g''(z) - 2zg'(z) + 4cg(z) = 0, \]
which is independent of \( \alpha \). In other words, (i) if we do not consider switching costs, then we can take any solution \( g(z) \) of the Hermite ODE, plug in the argument \( z = S/\sqrt{2t} \) and obtain a potential function \( V_\alpha \); (ii) when switching costs are considered, the continuous-time argument suggests us to take the same function \( g(z) \) as before, and plug in a scaled argument \( z = S/\sqrt{4\alpha t} \). Such an argument justifies our dual space scaling in the continuous-time limit. For our concrete analysis later, we will consider \( c = 1/2 \), which recovers the special solution \( V_\alpha \) presented in (2).

Finally, we emphasize the gap between the above continuous-time analysis and our main goal, which is proving a regret bound for the original discrete-time problem. The continuous-time analysis is useful for providing intuition, but translating it into a regret bound relies on an obscure argument that has not been made concrete yet: “\( V_\alpha \) derived in the continuous time also serves as a good potential in the discrete time.” Indeed, this step is technically challenging, and doing so requires a more conservative choice of \( \alpha \) (i.e., \( 4\lambda + 2 \)) than what is suggested above. We will present the details next.

To proceed, we first provide a few basic lemmas on Algorithm 1 and \( V_\alpha \), which will be useful later on.

### A.2 A few basic lemmas

The first lemma shows the monotonicity of the discrete derivative strategy, which is quite intuitive.

**Lemma A.1.** If the potential \( V(t,S) \) is even and convex in \( S \), then \( \nabla_S V(t,S) \) is odd and monotonically increasing in \( S \).

**Proof of Lemma A.1.** \( \nabla_S V(t,S) \) is odd due to the simple relation
\[ \nabla_S V(t,-S) = \frac{1}{2} [V(t,-S+1) - V(t,-S-1)] = \frac{1}{2} [V(t,S+1) - V(t,S-1)] = -\nabla_S V(t,S). \]

As for the monotonicity, it is equivalent to showing for all \( \delta \geq 0 \),
\[ V(t,S+1+\delta) - V(S+1+\delta) \geq V(t,S+1) - V(S-1). \]
This follows from the convexity of \( V(t,\cdot) \), as
\[ V(t,S+1) \leq \frac{2}{2+\delta} V(t,S+1+\delta) + \frac{\delta}{2+\delta} V(t,S-1), \]
\[ V(t,S-1+\delta) \leq \frac{\delta}{2+\delta} V(t,S+1+\delta) + \frac{2}{2+\delta} V(t,S-1). \]

As for \( V_\alpha \), we compute its continuous partial derivatives. The proof is straightforward calculation, therefore omitted.

**Lemma A.2.** For any \( \alpha > 0 \), \( V_\alpha \) defined in (2) is even and convex. Moreover,
\[ \nabla S V_\alpha(t, S) = C \int_0^{S/\sqrt{4at}} \exp(x^2) \, dx, \]
\[ \nabla_{SS} V_\alpha(t, S) = \frac{C}{2\sqrt{at}} \exp \left( \frac{S^2}{4at} \right), \]
\[ \nabla_{SSS} V_\alpha(t, S) = \frac{CS}{4(a(t))^{3/2}} \exp \left( \frac{S^2}{4at} \right), \]
\[ \nabla_t V_\alpha(t, S) = \frac{C\sqrt{t}}{2\sqrt{at}} \exp \left( \frac{S^2}{4at} \right). \]

Based on the above, the discrete derivative \( \nabla S V_\alpha \) has the following properties.

**Lemma A.3.** For all \( \alpha > 0 \), \( t \geq 0 \) and \( S \geq 0 \),

1. \( \nabla S V_\alpha(t, S) \) as a function of \( t \) is decreasing and convex;
2. \( \nabla S V_\alpha(t, S) \) as a function of \( S \) is convex.

**Proof of Lemma A.3.** Considering the first part of the lemma,

\[ \nabla_t [\nabla S V_\alpha(t, S)] = \frac{1}{2} [\nabla_t V_\alpha(t, S + 1) - V_\alpha(t, S + 1)] = -\frac{C\sqrt{t}}{4\sqrt{at}} \exp \left( \frac{S^2 + 1}{4at} \right) \sinh \left( \frac{S}{2\alpha t} \right), \]

which, when \( S \geq 0 \), is negative and increasing in \( t \). Therefore, \( \nabla S V_\alpha(t, S) \) as a function of \( t \) is decreasing and convex. Similarly,

\[ \nabla_S [\nabla S V_\alpha(t, S)] = \frac{1}{2} [\nabla S V_\alpha(t, S + 1) - \nabla S V_\alpha(t, S - 1)] = \frac{C}{2} \int_{(S-1)/\sqrt{4at}}^{(S+1)/\sqrt{4at}} \exp(x^2) \, dx, \]

which is increasing in \( S \). Therefore, \( \nabla S V_\alpha(t, S) \) as a function of \( S \) is convex. \( \square \)

### A.3 Details on 1D parameter-free OLO

In this subsection, we prove Theorem 1, the regret bound of our 1D OLO algorithm with switching cost. As sketched in Section 2.3, our proof relies on two important lemmas, Lemma 2.2 and 2.3. We prove them first.

**Lemma 2.2.** For all \( \alpha > 0 \), consider Algorithm 1 induced by the potential function \( V_\alpha \). For all \( t \in \mathbb{N}_+ \),

\[ |x_t - x_{t+1}| \leq \nabla S V_\alpha(t, S_{t-1} + 1) - \nabla S V_\alpha(t, S_{t-1} - 1). \]

**Proof of Lemma 2.2.** First, since \( \nabla S V_\alpha(t, S) \) is monotonic in \( S \) due to Lemma A.1, we have

\[ |x_t - x_{t+1}| = |\nabla S V_\alpha(t, S_{t-1}) - \nabla S V_\alpha(t + 1, S_{t-1})| \leq \max_{c=\pm 1} |\nabla S V_\alpha(t, S_{t-1}) - \nabla S V_\alpha(t + 1, S_{t-1} + c)|. \]

For clarity, from the RHS we define

\[ f(t, S) := \max_{c=\pm 1} |\nabla S V_\alpha(t, S) - \nabla S V_\alpha(t + 1, S + c)|. \]

It is even in \( S \), as

\[ f(t, -S) = \max_{c=\pm 1} |\nabla S V_\alpha(t, -S) - \nabla S V_\alpha(t + 1, -S + c)| \]
\[ = \max_{c=\pm 1} |\nabla S V_\alpha(t, S) + \nabla S V_\alpha(t + 1, S - c)| \quad \text{(Lemma A.1)} \]
\[ = \max_{c=\pm 1} |\nabla S V_\alpha(t, S) - \nabla S V_\alpha(t + 1, S - c)| = f(t, S). \]

Therefore, we can restrict the rest of the proof to \( S \geq 0 \), and the remaining task is to upper bound \( f(t, S) \) for all \( 0 \leq S \leq t - 1 \).

From Lemma A.1 and A.3,

\[ \nabla S V_\alpha(t + 1, S - 1) \leq \nabla S V_\alpha(t + 1, S) \leq \nabla S V_\alpha(t, S), \]
\( \nabla_S V_\alpha(t+1, S - 1) \leq \nabla_S V_\alpha(t+1, S + 1). \)

Therefore, if \( \nabla_S V_\alpha(t+1, S - 1) \leq \nabla_S V_\alpha(t, S) \leq \nabla_S V_\alpha(t+1, S + 1), \) then

\[
f(t, S) = \max \left\{ \left| \nabla_S V_\alpha(t, S) - \nabla_S V_\alpha(t+1, S - 1) \right|, \left| \nabla_S V_\alpha(t, S) - \nabla_S V_\alpha(t+1, S + 1) \right| \right\}
\leq \nabla_S V_\alpha(t+1, S + 1) - \nabla_S V_\alpha(t+1, S - 1).
\]

On the other hand, if \( \nabla_S V_\alpha(t+1, S + 1) \leq \nabla_S V_\alpha(t, S), \) then

\[
f(t, S) = \nabla_S V_\alpha(t, S) - \nabla_S V_\alpha(t+1, S - 1).
\]

Combining the above,

\[
f(t, S) \leq \max \left\{ \nabla_S V_\alpha(t, S) - \nabla_S V_\alpha(t+1, S - 1), \nabla_S V_\alpha(t+1, S + 1) - \nabla_S V_\alpha(t+1, S - 1) \right\}.
\]

Our goal next is to upper bound \( f(t, S) \) by \( \nabla_S V_\alpha(t+1, S + 1) - \nabla_S V_\alpha(t, S - 1), \) which can be divided into two cases.

**Case 1** We aim to show that

\[
\nabla_S V_\alpha(t, S) - \nabla_S V_\alpha(t+1, S - 1) \leq \nabla_S V_\alpha(t+1) - \nabla_S V_\alpha(t, S - 1),
\]

which is equivalent to

\[
\nabla_S V_\alpha(t, S - 1) - \nabla_S V_\alpha(t + 1, S - 1) \leq \nabla_S V_\alpha(t+1) - \nabla_S V_\alpha(t, S - 1).
\]

Note that this trivially holds when \( 0 \leq S < 1: \) due to Lemma A.3, the RHS is always positive; however, the LHS is negative due to \( \nabla_S V_\alpha(t, S - 1) \) being increasing in \( t \) (Lemma A.1 and A.3 Part 1). Therefore, we only need to show (6) for all \( S \geq 1. \)

To this end, with \( S \geq 1, \) we apply the convexity of \( \nabla_S V_\alpha \) from Lemma A.3:

\[
\nabla_S V_\alpha(t, S + 1) - \nabla_S V_\alpha(t, S) \geq \nabla_S \left[ \nabla_S V_\alpha(t, S) \right],
\]

\[
\nabla_S V_\alpha(t, S - 1) - \nabla_S V_\alpha(t + 1, S - 1) \leq -\nabla_t \left[ \nabla_S V_\alpha(t, S - 1) \right].
\]

Consequently, it suffices to show that

\[
-\nabla_t \left[ \nabla_S V_\alpha(t, S - 1) \right] \leq \nabla_S \left[ \nabla_S V_\alpha(t, S) \right].
\]

Now it is time to invoke the specific form of \( V_\alpha. \) We may reuse \( \nabla_S \left[ \nabla_S V_\alpha(t, S) \right] \) and \( \nabla_t \left[ \nabla_S V_\alpha(t, S) \right] \) calculated from the proof of Lemma A.3.

\[
\nabla_S \left[ \nabla_S V_\alpha(t, S) \right] = \frac{C}{2} \int_{(S-1)/\sqrt{4at}}^{(S+1)/\sqrt{4at}} \exp \left( x^2 \right) dx \geq \frac{C}{2\sqrt{\alpha t}} \exp \left( \frac{S^2}{4at} \right),
\]

and for all \( 1 \leq S \leq t - 1, \)

\[
-\nabla_t \left[ \nabla_S V_\alpha(t, S - 1) \right] = \frac{C\sqrt{\alpha}}{4\sqrt{t}} \exp \left( \frac{(S-1)^2 + 1}{4at} \right) \sinh \left( \frac{S-1}{2at} \right)
= \frac{C\sqrt{\alpha}}{8\sqrt{t}} \exp \left( \frac{S^2}{4at} \right) \left[ 1 - \exp \left( \frac{-S+1}{\alpha t} \right) \right]
\leq \frac{C\sqrt{\alpha}}{8\sqrt{t}} \exp \left( \frac{S^2}{4at} \right) \left[ 1 - \exp \left( \frac{-1}{\alpha} \right) \right]
\leq \frac{C}{8\sqrt{\alpha t}} \exp \left( \frac{S^2}{4at} \right). \quad (S - 1 \leq t)
\]

Therefore, \( -\nabla_t \left[ \nabla_S V_\alpha(t, S - 1) \right] \leq \nabla_S \left[ \nabla_S V_\alpha(t, S) \right], \) which proves (6) and concludes Case 1.
Case 2 We aim to show that
\[ \nabla_S V_\alpha(t + 1, S + 1) - \nabla_S V_\alpha(t + 1, S - 1) \leq \nabla_S V_\alpha(t, S + 1) - \nabla_S V_\alpha(t, S - 1). \]

This is straightforward, as
\[
\nabla_t \left[ \nabla_S V_\alpha(t, S + 1) - \nabla_S V_\alpha(t, S - 1) \right] = \frac{1}{2} \left[ \nabla_t V_\alpha(t, S + 2) + \nabla_t V_\alpha(t, S - 2) - 2 \nabla_t V_\alpha(t, S) \right]
\]
\[ = -C \sqrt{\alpha} \frac{4}{4 \sqrt{t}} \left[ \exp \left( \frac{(S + 2)^2}{4 \alpha t} \right) + \exp \left( \frac{(S - 2)^2}{4 \alpha t} \right) - 2 \exp \left( \frac{S^2}{4 \alpha t} \right) \right] \leq 0. \]

Combining the two cases, we can upper bound \( f(t, S) \) by \( \nabla_S V_\alpha(t, S + 1) - \nabla_S V_\alpha(t, S - 1) \), which completes the proof.

Next, we present the proof of Lemma 2.3, which bounds the residual term \( \Delta_t \) defined in (3).

**Lemma 2.3.** If \( \alpha \geq 4\lambda G^{-1} + 2 \), then for all \( t \) and against any adversary, \( \Delta_t \leq 0 \).

**Proof of Lemma 2.3.** We restate the definition of \( \Delta_t \) for easier reference.
\[
\Delta_t = \nabla_t V_\alpha(t, S_{t-1}) + \frac{1}{2} \nabla_{SS} V_\alpha(t, S_{t-1}) + G^{-1} \lambda \left[ \nabla_S V_\alpha(t, S_{t-1} + 1) - \nabla_S V_\alpha(t, S_{t-1} - 1) \right].
\]

Let us define a function \( g(t, S) \) as
\[
g(t, S) := \frac{1}{2} V_\alpha(t, S + 1) + \frac{1}{2} V_\alpha(t, S - 1) - V_\alpha(t, S) + \frac{\lambda}{2G} \left[ V_\alpha(t, S + 2) + V_\alpha(t, S - 2) - 2V_\alpha(t, S) \right],
\]
then from the definition of discrete derivatives, \( \Delta_t = g(t, S_{t-1}) \). Also note that \( g(t, S) \) is even in \( S \), so we can only focus on positive values of \( S \). The rest of the proof will show \( g(t, S) \leq 0 \) for all \( t \in \mathbb{N}_+ \) and \( S \geq 0 \).

Let us start from the special case, \( t = 1 \). \( S \) can only take the value 0, therefore \( g(1, S) = g(1, 0) \). We now present a general result that upper bounds \( g(t, 0) \) for all \( t \in \mathbb{N}_+ \):
\[
g(t, 0) = V_\alpha(t, 1) - V_\alpha(t, 1) + G^{-1} \lambda V_\alpha(t, 2) - G^{-1} \lambda V_\alpha(t, 0)
\]
\[ = C \sqrt{\alpha t} \left[ 2 \int_0^{1/\sqrt{4 \alpha t}} \left( \int_0^u \exp(x^2)dx \right) du + 2G^{-1} \lambda \int_0^{1/\sqrt{4 \alpha t}} \left( \int_0^u \exp(x^2)dx \right) du + \sqrt{\frac{t-1}{t}} - 1 \right]
\]
\[ \leq C \sqrt{\alpha t} \left[ 2 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{4 \alpha t}} \int_0^{1/\sqrt{4 \alpha t}} \exp(x^2)dx + 2G^{-1} \lambda \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{4 \alpha t}} \int_0^{1/\sqrt{4 \alpha t}} \exp(x^2)dx + \sqrt{\frac{t-1}{t}} - 1 \right]
\]
\[ \leq C \sqrt{\alpha t} \left[ \frac{1}{4 \alpha t} \exp \left( \frac{1}{4 \alpha t} \right) + \frac{\lambda}{G \alpha t} \exp \left( \frac{1}{\alpha t} \right) + \sqrt{\frac{t-1}{t}} - 1 \right].
\]

Since \( \sqrt{1+x} \leq 1 + x/2 \) for all \( x \geq -1 \), we have \( \sqrt{(t-1)/t} - 1 \leq -1/(2t) \). Therefore, if \( \alpha \geq 4\lambda G^{-1} + 2 \), then
\[
g(t, 0) \leq C \sqrt{\alpha t} \left[ \frac{\lambda G^{-1} + (1/4)}{\alpha t} \exp \left( \frac{1}{\alpha t} \right) - \frac{1}{2} \right] \leq C \sqrt{\alpha t} \left[ \frac{\lambda G^{-1} + (1/4)}{\alpha t} \exp \left( \frac{1}{2} \right) - \frac{1}{2} \right] \leq 0.
\]

As its special case, we have \( g(1, 0) \leq 0 \), thus concluding the proof of the special case (\( t = 1 \)).

Next, we prove \( g(t, S) \leq 0 \) for general \( t \), i.e., \( t \geq 2 \). Our overall strategy is to show that for all \( 0 \leq S \leq t - 1 \), \( g(t, S) \leq g(t, 0) \), and then using the argument above we have \( g(t, 0) \leq 0 \). Concretely, let us calculate the first and second order derivatives of \( g(t, S) \), using Lemma A.2.
\[
\nabla_S g(t, S) = C \frac{1}{2} \left[ \int_0^{(S+1)/\sqrt{4 \alpha t}} \exp(x^2)dx + \int_0^{(S-1)/\sqrt{4 \alpha t}} \exp(x^2)dx - 2 \int_0^{S/\sqrt{4 \alpha (t-1)}} \exp(x^2)dx \right]
\]
\[ + \frac{\lambda C}{2G} \left[ \int_0^{(S+2)/\sqrt{4 \alpha t}} \exp(x^2)dx + \int_0^{(S-2)/\sqrt{4 \alpha t}} \exp(x^2)dx - 2 \int_0^{S/\sqrt{4 \alpha t}} \exp(x^2)dx \right],
\]
\[ \nabla_{ssg}(t, S) = \frac{C}{4\sqrt{\alpha t}} \left[ \frac{\lambda}{G} \exp \left( \frac{(S+2)^2}{4\alpha t} \right) + \exp \left( \frac{(S+1)^2}{4\alpha t} \right) - \frac{2\lambda}{G} \exp \left( \frac{S^2}{4\alpha t} \right) + \exp \left( \frac{(S-1)^2}{4\alpha t} \right) + \frac{\lambda}{G} \exp \left( \frac{(S-2)^2}{4\alpha t} \right) - \frac{C}{2\sqrt{\alpha(t-1)}} \exp \left( \frac{S^2}{4\alpha(t-1)} \right) \right] \]

Notice that \( \nabla_{ssg}(t, 0) = 0 \). To proceed, we aim to prove \( \nabla_{ssg}(t, S) \leq 0 \) for all \( S \geq 0 \), which then shows \( g(t, S) \leq g(t, 0) \). To this end, we will show the sum inside the bracket in (8) is negative. Denote it as \( h(t, S) \), and more specifically,

\[ h(t, S) := \frac{\lambda}{G} \exp \left( \frac{1}{\alpha t} \right) \cosh \left( \frac{S}{\alpha t} \right) + \exp \left( \frac{1}{4\alpha t} \right) \cosh \left( \frac{S}{2\alpha t} \right) - \frac{\lambda}{G} - \sqrt{\frac{t}{t-1}} \exp \left( \frac{S^2}{4\alpha(t-1)} \right). \]

The rest of the proof is divided into two steps: we first prove (i) \( h(t, 0) \leq 0 \); and then prove (ii) \( \nabla_{sh}(t, S) \leq 0 \) for all \( S \geq 0 \).

**Step 1:** prove \( h(t, 0) \leq 0 \). From the definition of \( h(t, S) \),

\[ h(t, 0) = \frac{\lambda}{G} \exp \left( \frac{1}{\alpha t} \right) + \exp \left( \frac{1}{4\alpha t} \right) - \frac{\lambda}{G} - \sqrt{\frac{t}{t-1}}. \]

Letting \( x = 1/t \), then to prove \( h(t, 0) \leq 0 \) for all \( t \geq 2 \), it suffices to prove

\[ \psi(x) := \frac{\lambda}{G} \exp \left( \frac{x}{\alpha} \right) + \exp \left( \frac{x}{4\alpha} \right) - \frac{\lambda}{G} - \sqrt{\frac{1}{1-x}} \leq 0, \]

on the range \( x \in (0, 1/2] \). \( \psi(0) = 0 \), and

\[ \nabla_x \psi(x) = -\frac{\lambda}{\alpha G} \exp \left( \frac{x}{\alpha} \right) + \frac{1}{4\alpha} \exp \left( \frac{x}{4\alpha} \right) - \frac{1}{2} (1-x)^{-3/2} \leq \frac{4\lambda G^{-1} + 1}{4\alpha} \exp \left( \frac{1}{2\alpha} \right) - \frac{1}{2}, \]

which is negative when \( \alpha \geq 4\lambda G^{-1} + 2 \). Therefore, \( h(t, 0) \leq 0 \) for all \( t \geq 2 \).

**Step 2:** prove \( \nabla_{sh}(t, S) \leq 0 \). Taking the derivative of \( h(t, S) \),

\[ \nabla_{sh}(t, S) = \frac{\lambda}{\alpha G} \exp \left( \frac{1}{\alpha t} \right) \sinh \left( \frac{S}{\alpha t} \right) + \frac{1}{2\alpha t} \exp \left( \frac{1}{4\alpha t} \right) \sinh \left( \frac{S}{2\alpha t} \right) - \sqrt{\frac{t}{t-1}} \cdot \frac{S}{2\alpha t(t-1)} \exp \left( \frac{S^2}{4\alpha(t-1)} \right) \leq \frac{\lambda}{\alpha G} + \frac{1}{2\alpha t} \exp \left( \frac{1}{\alpha t} \right) \sinh \left( \frac{S}{\alpha t} \right) - \frac{S}{2\alpha t^2} \sqrt{\frac{t}{t-1}}. \]

Note that for all \( x, \exp(-x) \geq 1 - x \), therefore for all \( 0 \leq x < 1 \), \( \exp(x)/2 \leq \sqrt{1/(1-x)} \). Assigning \( x \) to \( 1/t \) which is less than 1, we have for all \( \alpha \geq 2 \),

\[ \exp \left( \frac{1}{\alpha t} \right) \leq \exp \left( \frac{1}{2t} \right) \leq \sqrt{\frac{t}{t-1}}. \]

Moreover, for all \( 0 \leq x \leq 1 \), \( \sinh(x) \leq 2x \). Therefore,

\[ \nabla_{sh}(t, S) \leq \sqrt{\frac{t}{t-1}} \left[ \frac{\lambda G^{-1} + (1/2)}{\alpha t} \sinh \left( \frac{S}{\alpha t} \right) - \frac{S}{2\alpha t^2} \right] \leq \frac{S}{\alpha^2 t^2} \sqrt{\frac{t}{t-1}} \left[ 2\lambda G^{-1} + 1 - \frac{\alpha}{2} \right]. \]
When $\alpha \geq 4\lambda G^{-1} + 2$, $\nabla_S h(t, S) \leq 0$ for all $t \geq 2$ and $S \geq 0$.

Concluding the above two steps, we have shown $h(t, S) \leq 0$. Plugging it back into (8), we have $\nabla_{SS} g(t, S) \leq 0$, which shows that for all $t \geq 2$ and $S \geq 0$, $g(t, S) \leq g(t, 0)$. Finally, $g(t, 0) \leq 0$ following (7).

Now, given the two important lemmas above (Lemma 2.2 and 2.3), our Theorem 1 follows from a standard loss-regret duality. Details are presented below.

**Theorem 1.** If $\alpha = 4\lambda G^{-1} + 2$, then Algorithm 1 induced by the potential $V_\alpha$ guarantees

$$\text{Regret}_T^\lambda(u) \leq \sqrt{4\lambda G + 2G^2}T \left[ C + |u| \left( \sqrt{4\log \left( 1 + \frac{|u|}{C} \right)} + 2 \right) \right],$$

for all $u \in \mathbb{R}$ and $T \in \mathbb{N}_+$.

**Proof of Theorem 1.** Combining Lemma 2.1, 2.2 and 2.3, we have

$$\sum_{t=1}^{T} (g_t x_t + \lambda|x_t - x_{t+1}|) \leq -G \cdot V_\alpha(T, S_T).$$

Consider $V_\alpha(T, S_T)$ as a function of $S_T$; let us write $V_\alpha^*(\cdot)$ as its Fenchel conjugate. Then, the augmented regret can be bounded as

$$\text{Regret}_T^\lambda(u) = \sum_{t=1}^{T} g_t (x_t - u) + \lambda \sum_{t=1}^{T-1} |x_t - x_{t+1}|$$

$$\leq G \cdot u S_T + \sum_{t=1}^{T} (g_t x_t + \lambda|x_t - x_{t+1}|)$$

$$\leq G |u S_T - V_\alpha(T, S_T)| \leq G \cdot V_\alpha^*(u).$$

The last step is to bound $V_\alpha^*(u)$, which also follows from a standard proof strategy.

$$V_\alpha^*(u) = \sup_{S \in \mathbb{R}} u S - V_\alpha(T, S).$$

It is clear that the supremum is uniquely achieved; let $S^*$ be the maximizing argument. Then,

$$u = \nabla_S V_\alpha(T, S^*) = C \int_{S^*/\sqrt{4\alpha T}}^{S/\sqrt{4\alpha T}} \exp(x^2) \, dx.$$ 

If we define $\text{erfi}(z) = \int_{-z}^{z} \exp(x^2) \, dx$ (note that it a scaled version of the conventional imaginary error function), then $S^* = \sqrt{4\alpha T} \cdot \text{erfi}^{-1}(u C^{-1})$.

$$V_\alpha^*(u) = u S^* - V_\alpha(T, S^*) \leq u S^* - V_\alpha(T, 0) = C\sqrt{\alpha T} + |u| \sqrt{4\alpha T} \cdot \text{erfi}^{-1}(u C^{-1}).$$

Finally, as shown in [ZCP22b, Theorem 4], $\text{erfi}^{-1}(x) \leq 1 + \sqrt{\log(1 + x)}$. Combining the above completes the proof.

**A.4 Conversion of loss-regret trade-offs**

In this subsection we discuss the conversion of loss-regret trade-offs in unconstrained OLO. Our Theorem 1 guarantees a loss bound $\text{Regret}_T^\lambda(0) = O(\sqrt{T})$ and an accompanying regret bound $\text{Regret}_T^\lambda(u) = O \left( |u| \sqrt{T \log |u|} \right)$. By a doubling trick (effectively, a meta-algorithm), we can turn such guarantees into $\text{Regret}_T^\lambda(0) = O(1)$ and $\text{Regret}_T^\lambda(u) = O \left( |u| \sqrt{T \log |u|} \right)$. These can be directly compared to [ZCP22a]. Concretely, we present the doubling trick as Algorithm 2.
Algorithm 2 Conversion of loss-regret trade-offs.

**Require:** A hyperparameter \( C > 0 \), and a base unconstrained OLO algorithm \( \mathcal{A} \). Here we define \( \mathcal{A} \) as the algorithm considered in Theorem 1, with \( \alpha = 8\lambda G^{-1} + 2 \).

1. for \( m = 0, 1, 2, \ldots \) do
2. Initialize a copy of \( \mathcal{A} \) as \( \mathcal{A}_m \), whose hyperparameter is set to \( C/(2^m \sqrt{\alpha} G) \).
3. Run \( \mathcal{A}_m \) for \( 2^m \) rounds: \( t = 2^m, 2^m + 1, \ldots, 2^{m+1} - 1 \).
4. end for

**Theorem 5.** With any hyperparameter \( C > 0 \), Algorithm 2 guarantees for all \( u \in \mathbb{R} \) and \( T \in \mathbb{N}_+ \),

\[
\text{Regret}_T^\lambda(u) \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left[ C + |u| \sqrt{(8\lambda G + 2G^2)T} \left( \frac{\sqrt{8 \log \left( 1 + \frac{|u| \sqrt{8\lambda G + 2G^2}}{C} \right)} + 2\sqrt{2}}{\sqrt{2} - 1} \right) \right].
\]

**Proof of Theorem 5.** Algorithm 2 divides the time axis into intervals of doubling lengths. On the \( m \)-th interval, following Theorem 1, Algorithm 2 guarantees

\[
\sum_{t=2^m}^{2^{m+1} - 1} [g_t(x_t - u) + \lambda |x_t - x_{t+1}|] \leq \sum_{t=2^m}^{2^{m+1} - 1} g_t(x_t - u) + 2\lambda \sum_{t=2^m}^{2^{m+1} - 1} |x_t - x_{t+1}| \quad (x_{2^{m+1}} = 0; \Delta\text{-inequality})
\]

\[
\leq \frac{C}{\sqrt{2^m}} + |u| G\sqrt{\alpha} \cdot 2^m \left( \sqrt{4 \log \left( 1 + \frac{|u| \cdot 2^m \sqrt{\alpha} G}{C} \right)} + 2 \right).
\]

Now consider any time horizon \( T \).

\[
\text{Regret}_T^\lambda(u) \leq \sum_{m=0}^{\left\lfloor \log_2 T \right\rfloor} \left[ \frac{C}{\sqrt{2^m}} + |u| G\sqrt{\alpha} \cdot 2^m \left( \sqrt{4 \log \left( 1 + \frac{|u| \cdot 2^m \sqrt{\alpha} G}{C} \right)} + 2 \right) \right]
\]

\[
\leq C \cdot \sum_{m=0}^{\left\lfloor \log_2 T \right\rfloor} \left( \frac{1}{\sqrt{2}} \right)^m + |u| G\sqrt{\alpha} \left( \sqrt{4 \log \left( 1 + \frac{|u| \sqrt{\alpha} GT}{C} \right)} + 2 \right) \cdot \sum_{m=0}^{\left\lfloor \log_2 T \right\rfloor} \sqrt{2^m}
\]

\[
\leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left[ C + |u| G\sqrt{\alpha T} \left( \sqrt{8 \log \left( 1 + \frac{|u| \sqrt{\alpha} GT}{C} \right)} + 2\sqrt{2} \right) \right]. \quad \Box
\]

Let us compare it to [ZCP22a, Theorem 1], which guarantees

\[
\text{Regret}_T^\lambda(u) \leq C + |u| (\lambda + G)\sqrt{2T} \left( \frac{3}{2} + \log \frac{\sqrt{2} |u| (\lambda + G) T^{3/2}}{C} \right). \quad (9)
\]

If we only care about the dependence on \( \lambda, |u| \) and \( T \), then with the same loss bound \( \text{Regret}_T^\lambda(0) = O(1) \), our algorithm improves the regret bound \( \text{Regret}_T^\lambda(u) \) from \( O(|u| \lambda \sqrt{T} \log(\lambda |u| T)) \) to \( O(|u| \sqrt{\lambda T} \log(\lambda |u| T)) \). The latter matches a lower bound [Ora13] (in \( |u| \) and \( T \)), thus achieving Pareto-optimality in this regime.

### A.5 Details on bounded and higher dimensional domains

We now present extensions of our 1D unconstrained algorithm (Algorithm 1) to bounded domains and higher-dimensional domains.

First, we review a classical meta-algorithm [CO18, Cut20] that adds constraints to our Algorithm 1. The pseudo-code is shown as Algorithm 3, where \( \Pi \) denotes the projection function in 1D. It immediately leads to a clean result: the regret bound of Algorithm 1 (i.e., Theorem 1) can be translated in a black-box manner to the constrained setting. Moreover, the same algorithm has guaranteed switching cost bound on any time interval.
Algorithm 3 1D constrained OLO with switching costs.

**Require:** A hyperparameter $C > 0$, a closed and convex domain $\mathcal{X} \subset \mathbb{R}$, and an unconstrained algorithm $\mathcal{A}$ (Algorithm 1 induced by $V_{\lambda G^{-1} + 2}$ and the hyperparameter $C$). Let $x^*$ be an arbitrary point in $\mathcal{X}$.

1: for $t = 1, 2, \ldots$ do
2: Query $\mathcal{A}$ for its prediction $\hat{x}_t$.
3: Predict $x_t = \Pi_{\mathcal{X}}(\hat{x}_t + x^*)$ and receive a loss gradient $g_t$.
4: Define a surrogate loss gradient $\tilde{g}_t$ as
   \[
   \tilde{g}_t = \begin{cases} 
   g_t, & \text{if } g_t(\hat{x}_t + x^*) \geq g_t x_t, \\
   0, & \text{otherwise},
   \end{cases}
   \]
   and send $\tilde{g}_t$ to $\mathcal{A}$.
5: end for

**Theorem 2.** Consider the setting of Section 2.1, but on a (smaller) closed and convex domain $\mathcal{X} \subset \mathbb{R}$. Let $x^*$ be an arbitrary point in $\mathcal{X}$. For all $C > 0$, Algorithm 3 in Appendix A.5 guarantees

\[
\text{Regret}^C_T(u) \leq \sqrt{(4\lambda G + 2G^2)T \left[ C + |u - x^*| \left( \sqrt{4\log \left( 1 + \frac{|u - x^*|}{C} \right)} + 2 \right) \right]},
\]

for all $u \in \mathcal{X}$ and $T \in \mathbb{N}_+$. Moreover, if $\mathcal{X}$ has a finite diameter $D$, then on any time interval $[T_1 : T_2] \subset \mathbb{N}_+$, the same algorithm guarantees

\[
\sum_{t=T_1}^{T_2-1} |x_t - x_{t+1}| \leq 22\sqrt{T_2-T_1} \left[ 2D + C + 2D\sqrt{\log(1 + DC^{-1})} \right].
\]

**Proof of Theorem 2.** The first part of the theorem directly follows from [Cut20, Theorem 2] and the contraction property of 1D projections. As for the second part, the key idea is that the sufficient statistic $S_t = -\sum_{i=1}^{t} \tilde{g}_i/G$ “concentrates” due to the use of the surrogate loss $\tilde{g}_t$ instead of $g_t$. Let us make this argument clear. Without loss of generality, we assume $S_{t-1} \geq 0$. Considering the prediction $\hat{x}_t = \nabla S V_{\alpha}(t, S_{t-1})$ of the unconstrained base algorithm, there are two cases.

**Case 1:** $\hat{x}_t \leq D$. Due to convexity,

\[
\hat{x}_t = \nabla S V_{\alpha}(t, S_{t-1}) = C\sqrt{\alpha t} \int_{(S_{t-1}-1)/\sqrt{4\alpha t}}^{(S_{t-1}+1)/\sqrt{4\alpha t}} \left( \int_0^u \exp(x^2)dx \right) du \geq C \int_0^{S_{t-1}/\sqrt{4\alpha t}} \exp(x^2)dx.
\]

Similar to the proof of Theorem 1, if we define erfi$(z) = \int_0^z \exp(x^2)dx$, then $S_{t-1} \leq \sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1})$. As for the next round, $|S_t| \leq S_{t-1} + |\tilde{g}_t|/G \leq \sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) + 1$.

**Case 2:** $\hat{x}_t > D$. In this case, since $x^* \in \mathcal{X}$, we have $\hat{x}_t + x^*$ larger than the maximum element of $\mathcal{X}$, hence $\hat{x}_t + x^* > x_t$. Due to the definition of the surrogate loss, $\tilde{g}_t \geq 0$. Therefore, $|S_t| \leq \max\{|S_{t-1}|, |\tilde{g}_t/G|\} \leq \max\{|S_{t-1}|, 1\}$.

Combining the two cases and their analogous arguments for $S_{t-1} \leq 0$, we can see that for all $t$, $|S_t| \leq \max \left\{ \sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) + 1, |S_{t-1}|, 1 \right\}$. By induction, we obtain for all $t$,

\[
|S_t| \leq \sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) + 1.
\]

The concentration of $S$ is connected to upper bounds on the switching cost. Still, assume $S_{t-1} \geq 0$ without
loss of generality. From Lemma 2.2,

$$|x_t - x_{t+1}| \leq \nabla_S V_\alpha(t, S_{t-1} + 1) - \nabla_S V_\alpha(t, S_{t-1} - 1)$$

$$= C \sqrt{\alpha t} \left[ \int_{S_{t-1}/\sqrt{4\alpha t}}^{(S_{t-1}+2)/\sqrt{4\alpha t}} \left( \int_0^u \exp(x^2)dx \right) du - \int_{(S_{t-1}-2)/\sqrt{4\alpha t}}^{S_{t-1}/\sqrt{4\alpha t}} \left( \int_0^u \exp(x^2)dx \right) du \right]$$

$$\leq C \sqrt{\alpha t} \left[ \frac{2}{\sqrt{4\alpha t}} \int_0^{(S_{t-1}+2)/\sqrt{4\alpha t}} \exp(x^2)dx - \frac{2}{\sqrt{4\alpha t}} \int_0^{(S_{t-1}-2)/\sqrt{4\alpha t}} \exp(x^2)dx \right]$$

$$= C \int_{(S_{t-1}-2)/\sqrt{4\alpha t}}^{(S_{t-1}+2)/\sqrt{4\alpha t}} \exp(x^2)dx \leq \frac{2C}{\sqrt{\alpha t}} \exp \left( \frac{(S_{t-1}+2)^2}{4\alpha t} \right).$$

In the next step, we use the upper bound on $S_{t-1}$ to show that $|x_t - x_{t+1}| = O(Ct^{-1/2} \exp(\text{erfi}^{-1}(DC^{-1}))^2)$.

To this end, we discuss two cases regarding how the “concentration” bound (i.e., $O(\sqrt{t})$) compares to the trivial bound (i.e., $S_t \leq t$).

**Case 1:** $\sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) \geq t$. In this case, note that $S_{t-1} + 1 \leq t$ and $\alpha \geq 2$,

$$|x_t - x_{t+1}| \leq \frac{2C}{\sqrt{\alpha t}} \exp \left( \frac{(S_{t-1}+2)^2}{4\alpha t} \right) = \frac{2C}{\sqrt{\alpha t}} \exp \left( \frac{S_{t-1}^2}{4\alpha t} \right) \exp \left( \frac{4S_{t-1}+4}{4\alpha t} \right) \leq \frac{2\sqrt{7}C}{\sqrt{\alpha t}} \exp \left( \frac{S_{t-1}^2}{4\alpha t} \right).$$

Since $\sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) \geq t$, we have $t \leq 4\alpha(\text{erfi}^{-1}(DC^{-1}))^2$. Therefore,

$$\exp \left( \frac{S_{t-1}^2}{4\alpha t} \right) \leq \exp \left( \frac{t}{4\alpha} \right) \leq \exp \left[ (\text{erfi}^{-1}(DC^{-1}))^2 \right],$$

$$|x_t - x_{t+1}| \leq \frac{2\sqrt{7}C}{\sqrt{\alpha t}} \exp \left[ (\text{erfi}^{-1}(DC^{-1}))^2 \right].$$

**Case 2:** $\sqrt{4\alpha t} \cdot \text{erfi}^{-1}(DC^{-1}) < t$. Plugging the $O(\sqrt{t})$ bound for $S_{t-1}$ into $|x_t - x_{t+1}|$,

$$|x_t - x_{t+1}| \leq \frac{2C}{\sqrt{\alpha t}} \exp \left( \frac{\sqrt{4\alpha(t-1) \cdot \text{erfi}^{-1}(DC^{-1})} + 3}{4\alpha t} \right)$$

$$\leq \frac{2C}{\sqrt{\alpha t}} \exp \left[ (\text{erfi}^{-1}(DC^{-1}))^2 \right] \exp \left( \frac{6t + 9}{4\alpha t} \right)$$

$$\leq \frac{2C^2}{\sqrt{\alpha t}} \exp \left[ (\text{erfi}^{-1}(DC^{-1}))^2 \right].$$

Combining the above, we have

$$|x_t - x_{t+1}| \leq \frac{2C^2}{\sqrt{\alpha t}} \exp \left[ (\text{erfi}^{-1}(DC^{-1}))^2 \right].$$

The remaining task is to upper bound the function $\exp [(\text{erfi}^{-1}(x))^2]$. Let us consider a related function $\int_0^x \text{erfi}(u)du$. Using integration by parts,

$$\int_0^x \text{erfi}(u)du = u \cdot \text{erfi}(u) \bigg|_{u=0}^x - \int_0^x u \exp(u^2)du$$

$$= x \cdot \text{erfi}(x) - \frac{1}{2} \exp(x^2) + \frac{1}{2}.$$
Then, as we did in Theorem 1, we plug in \( \text{erf}^{-1}(x) \leq 1 + \sqrt{\log(1 + x)} \) and obtain
\[
|x_t - x_{t+1}| \leq \frac{11}{\sqrt{t}} \left\{ 2D \left[ 1 + \sqrt{\log(1 + DC^{-1})} \right] + C \right\}.
\]

Finally, note that
\[
\sum_{t=T_i}^{T_2-1} \frac{1}{\sqrt{t}} \leq \int_{T_i-1}^{T_2-1} \frac{1}{\sqrt{x}} dx \leq 2\sqrt{T_2 - 1} - 2\sqrt{T_1 - 1} \leq 2\sqrt{T_2 - T_1}.
\]
Combining it with our bound on \(|x_t - x_{t+1}|\) completes the proof.

As for the extension to higher dimensions, we consider \( \mathcal{X} = \mathbb{R}^d \), \( \|g_t\| \leq G \) and \( L_1 \) norm switching costs. Our solution is shown as Algorithm 4.

**Algorithm 4** \( d \)-dimensional OLO with \( L_1 \) norm switching costs.

**Require:** A hyperparameter \( C > 0 \) and Algorithm 1.

1. For each dimension \( i \in [1 : d] \), initialize a copy of Algorithm 1 as \( \mathcal{A}_i \). It uses the hyperparameter \( C/d \) and our potential \( V_\alpha \), with \( \alpha = 4\lambda G^{-1} + 2 \).
2. for \( t = 1, 2, \ldots \) do
3. \( \mathbf{for} \) all \( i \), query \( \mathcal{A}_i \) and assign its prediction to \( x_{t,i} \). Define a vector as \( x_t = [x_{t,1}, \ldots, x_{t,d}] \in \mathbb{R}^d \).
4. Predict \( x_t \) and receive a loss gradient \( g_t = [g_{t,1}, \ldots, g_{t,d}] \).
5. For all \( i \), send \( g_{t,i} \) to \( \mathcal{A}_i \) as a new surrogate loss gradient.
6. \( \mathbf{end for} \)

**Theorem 3.** Consider OLO with switching costs on the domain \( \mathcal{X} = \mathbb{R}^d \); assume loss gradients satisfy \( \|g_t\|_\infty \leq G \). For all \( C > 0 \), Algorithm 4 in Appendix A.5 guarantees
\[
\sum_{t=1}^{T} (g_t, x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 \leq \sqrt{(4\lambda G + 2G^2)T} \left[ C + \|u\|_1 \left( \sqrt{4 \log \left( 1 + \frac{\|u\|_\infty d}{C} \right)} + 2 \right) \right],
\]
for all \( u \in \mathbb{R}^d \) and \( T \in \mathbb{N}_+ \).

**Proof of Theorem 3.** We simply combine the regret on each coordinate:
\[
\sum_{t=1}^{T} (g_t, x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 = \sum_{i=1}^{d} \left[ \sum_{t=1}^{T} g_{t,i}(x_{t,i} - u_i) + \lambda \sum_{t=1}^{T-1} |x_{t,i} - x_{t+1,i}| \right]
\leq \sqrt{(4\lambda G + 2G^2)T} \sum_{i=1}^{d} \frac{C}{d} + |u_i| \left( \sqrt{4 \log \left( 1 + \frac{|u_i| d}{C} \right)} + 2 \right)
\leq \sqrt{(4\lambda G + 2G^2)T} \left[ C + \|u\|_1 \left( \sqrt{4 \log \left( 1 + \frac{\|u\|_\infty d}{C} \right)} + 2 \right) \right].
\]

**B Details on LEA**

In this section we present techniques that extend our 1D OLO algorithm to LEA with switching cost. We show that with a streamlined analysis, the general Banach version of the constrained domain reduction [CO18] can already convert 1D OLO algorithms to LEA, therefore appears to be more general than the specialized techniques [LS15, OP16]. Our approach is presented as Algorithm 5.
Algorithm 5 Converting OLO to LEA via the constrained domain reduction.

**Require:** A prior \( \pi = [\pi_1, \ldots, \pi_d] \) in the relative interior of \( \Delta(d) \), and Algorithm 3.

1. For each dimension \( i \in [1 : d] \), initialize a copy of Algorithm 3 as \( A_i \). We set \( \lambda = 4\lambda \) and \( \tilde{G} = 2G \) as the switching cost weight and the Lipschitz constant considered by \( A_i \). Moreover, \( A_i \) uses the domain \( X = \mathbb{R}_+ \), the offset \( \tilde{x}^* = \pi_i \), the hyperparameter \( \pi_i \), and our potential \( V_\alpha \), where \( \alpha = 4\tilde{\lambda}\tilde{G}^{-1} + 2 \).

2. \textbf{for} \( t = 1, 2, \ldots \) \textbf{do}
   
   3. For all \( i \), query \( A_i \) and assign its prediction to \( w_{t,i} \). Define the weight vector as \( w_t = [w_{t,1}, \ldots, w_{t,d}] \in \mathbb{R}_+^d \).

   4. Compute the LEA prediction \( x_t = [x_{t,1}, \ldots, x_{t,d}] \) from
      \[
      x_{t,i} = \frac{w_{t,i} + \frac{1}{2} \max\{0, 1 - \|w_t\|_1\}}{\max\{\|w_t\|_1, 1\}}.
      \]

   5. Predict \( x_t \) and receive a loss vector \( g_t \in [-G, G]^d \).

   6. For all \( i \), compute
      \[
      z_{t,i} = \begin{cases} 
      g_{t,i} - \|g_t\|_\infty, & \text{if } \|w_t\|_1 < 1, \\
      g_{t,i}, & \text{if } \|w_t\|_1 = 1, \\
      g_{t,i} + \|g_t\|_\infty, & \text{if } \|w_t\|_1 > 1,
      \end{cases}
      \]
      and return \( z_{t,i} \) to \( A_i \) as a new surrogate loss.

7. \textbf{end for}

B.1 An auxiliary lemma

Before presenting the performance guarantee of Algorithm 5, we first prove an auxiliary lemma.

**Lemma B.1.** For all \( x \geq 0 \),
\[
|x - 1| \log(1 + |x - 1|) \leq 2(1 - x + x \log x).
\]

**Proof of Lemma B.1.** Define \( LHS - RHS = h(x) \). Clearly, \( h(1) = 0 \). When \( x > 1 \),
\[
h'(x) = 1 - \log x - x^{-1}.
\]
It equals 0 when \( x = 1 \), and \( h''(x) = (1 - x)/x^2 \) which is negative for all \( x > 1 \). Therefore, \( h(x) \leq 0 \) for all \( x \geq 1 \).

As for the case of \( x < 1 \),
\[
h'(x) = -\log(2 - x) - \frac{1 - x}{2 - x} - 2 \log x,
\]
\[
h''(x) = -\frac{x^2 - x + 2}{(x - 2)^2 x} < 0,
\]
therefore \( h(x) \leq 0 \) for all \( 0 \leq x \leq 1 \). \( \square \)

B.2 Analysis of Algorithm 5

Next, we present our result on LEA with switching cost.

**Theorem 4.** For LEA with switching cost, given any prior \( \pi \) in the relative interior of \( \Delta(d) \), Algorithm 5 from Appendix B.2 guarantees
\[
\sum_{t=1}^{T} (g_t, x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 = \left[\sqrt{\text{TV}(u||\pi)} \cdot \text{KL}(u||\pi) + 1\right] \cdot O\left(\sqrt{(\lambda G + G^2)T}\right),
\]
for all \( u \in \Delta(d) \) and \( T \in \mathbb{N}_+ \).

**Proof of Theorem 4.** We divide the proof into three steps.
Step 1  The first step is to show that (i) for all \( u \in \Delta(d) \), \( \langle g_t, x_t - u \rangle \leq \langle z_t, w_t - u \rangle \); and (ii) \(|x_t - x_{t+1}|_1 \leq O(||w_t - w_{t+1}||_1)\). In this way, we can translate the LEA problem to a \( d \)-dimensional OLO problem with the loss vector \( z_t \), despite not achieving the root KL bound yet.

To prove the goal (i), we consider two cases, \(|w_t|_1 < 1 \) and \(|w_t|_1 > 1 \) (the case of \(|w_t|_1 = 1 \) trivially holds). If \(|w_t|_1 < 1 \), we have \( x_t = w_t + d^{-1}(1 - |w_t|_1) \) and \( z_t = g_t - \parallel g_t \rVert_{\infty} \).

\[
\langle g_t, x_t - u \rangle = \langle g_t, w_t - u \rangle + (1 - \parallel w_t \parallel_1) \left( \sum_i g_{t,i} / d \right),
\]

\[
\langle z_t, w_t - u \rangle = \langle g_t, w_t - u \rangle + (1 - \parallel w_t \parallel_1) \parallel g_t \parallel_{\infty},
\]

therefore \( \langle g_t, x_t - u \rangle \leq \langle z_t, w_t - u \rangle \). As for the case of \(|w_t|_1 > 1 \), similarly, \( x_t = w_t / |w_t|_1 \), \( z_t = g_t + \parallel g_t \parallel_{\infty} \), \( \langle g_t, x_t - u \rangle = \langle g_t, w_t / |w_t|_1 - u \rangle \), and \( \langle z_t, w_t - u \rangle = \langle g_t, w_t - u \rangle + \parallel g_t \parallel_{\infty} (\parallel w_t \parallel_1 - 1) \).

\[
\langle g_t, x_t - u \rangle - \langle z_t, w_t - u \rangle = - (\langle g_t, x_t \rangle + \parallel g_t \parallel_{\infty})(\parallel w_t \parallel_1 - 1) \leq 0.
\]

Now consider the goal (ii). To avoid cluttered notations, define \( a_t = w_t + d^{-1}\max\{0, 1 - \parallel w_t \parallel_1\} \) and \( A_t = \max\{\parallel w_t \parallel_1, 1\} \). Note that \( A_t = \parallel a_t \parallel_1 \).

\[
\parallel x_t - x_{t+1} \parallel_1 = \left\| \frac{a_t - a_{t+1}}{A_t} \right\|_1
= \left\| \frac{(a_t - a_{t+1})A_{t+1} + a_{t+1}(A_t - A_{t+1})}{A_t A_{t+1}} \right\|_1
\leq \frac{1}{A_t} \parallel a_t - a_{t+1} \parallel_1 + \frac{1}{A_t} (A_{t+1} - A_t) \leq 2 \parallel a_t - a_{t+1} \parallel_1.
\]

\[
\parallel a_t - a_{t+1} \parallel_1 = \parallel w_t + d^{-1}\max\{0, 1 - \parallel w_t \parallel_1\} - w_{t+1} - d^{-1}\max\{0, 1 - \parallel w_{t+1} \parallel_1\} \parallel_1
\leq \parallel w_t - w_{t+1} \parallel_1 + \max\{0, 1 - \parallel w_t \parallel_1\} - \max\{0, 1 - \parallel w_{t+1} \parallel_1\}
\leq \parallel w_t - w_{t+1} \parallel_1 + \parallel w_t \parallel_1 - \parallel w_{t+1} \parallel_1 \leq 2 \parallel w_t - w_{t+1} \parallel_1.
\]

Therefore, \( \parallel x_t - x_{t+1} \parallel_1 \leq 4 \parallel w_t - w_{t+1} \parallel_1 \).

Step 2  The second step is to add up the regret bound for each coordinates. Consider the \( i \)-th coordinate. Note that \( |z_{t,i}| \leq 2G \). Using Theorem 2, for all \( u_{1d} \in \mathbb{R}_+ \),

\[
\sum_{t=1}^{T} z_{t,i} (w_{t,i} - u_{1d}) + \tilde{\lambda} \sum_{t=1}^{T-1} |w_{t,i} - w_{t+1,i}| \leq \sqrt{(4\lambda G + 2G^2)T} \left[ \pi_i + |u_{1d} - \pi_i| \left( \frac{4 \log \left( 1 + \frac{|u_{1d} - \pi_i|}{\pi_i} \right) + 2}{\pi_i} \right) \right]
= \sqrt{(32\lambda G + 8G^2)T} \left[ \pi_i + |u_{1d} - \pi_i| \left( \frac{4 \log \left( 1 + \frac{|u_{1d} - \pi_i|}{\pi_i} \right) + 2}{\pi_i} \right) \right].
\]
Then, by summing up all the coordinates, for all \( u \in \Delta(d) \),

\[
\sum_{t=1}^{T} (g_t, x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 \\
\leq \sum_{t=1}^{T} (z_t, w_t - u) + 4\lambda \sum_{t=1}^{T-1} \|w_t - w_{t+1}\|_1 \\
= \sum_{i=1}^{d} \left[ \sum_{t=1}^{T} z_{t,i}(w_{t,i} - u_i) + \lambda \sum_{t=1}^{T-1} |w_{t,i} - w_{t+1,i}| \right] \\
\leq \sqrt{(32\lambda G + 8G^2)T} \left[ 1 + 2\|u - \pi\|_1 + 2\sum_{i=1}^{d} |u_i - \pi_i| \sqrt{\log \left( 1 + \frac{|u_i - \pi_i|}{\pi_i} \right)} \right] \\
\leq \sqrt{(32\lambda G + 8G^2)T} \left[ 1 + 2\|u - \pi\|_1 + 2\|u - \pi\|_1 \right] \left[ \sum_{i=1}^{d} |u_i - \pi_i| \log \left( 1 + \frac{|u_i - \pi_i|}{\pi_i} \right) \right].
\]

(Cauchy-Schwarz)

Observe that since \( u \) and \( \pi \) both belong to \( \Delta(d) \), \( \|u - \pi\|_1 \leq 2 \). If we define a function \( f \) as

\[
f := |x - 1| \log(1 + |x - 1|),
\]

then using the standard definition of \( f \)-divergence

\[
D_f(u||\pi) := \sum_{i=1}^{d} \pi_i f \left( \frac{u_i}{\pi_i} \right),
\]

we have

\[
\sum_{t=1}^{T} (g_t, x_t - u) + \lambda \sum_{t=1}^{T-1} \|x_t - x_{t+1}\|_1 = \left[ \sqrt{\text{TV}(u||\pi) \cdot D_f(u||\pi)} + 1 \right] \cdot O \left( \sqrt{\lambda G + G^2} T \right).
\]

**Step 3** The last step is to upper bound \( D_f(u||\pi) \) by \( \text{KL}(u||\pi) \). To this end, notice that \( \text{KL}(u||\pi) = D_g(u||\pi) \), where

\[
g(x) := 1 - x + x \log x.
\]

By Lemma B.1, \( f(x) \leq 2g(x) \) for all \( x \geq 0 \), therefore \( D_f(u||\pi) \leq 2D_g(u||\pi) = 2\text{KL}(u||\pi) \).

\[\Box\]

**B.3 Discussion on Algorithm 5**

Here are some discussions to conclude our LEA analysis. First, the surrogate loss \( z_t \) defined in Line 6 follows exactly the definition in [CO18, Algorithm 3]. We adopt this choice just to show the power of this general reduction technique. However, one could use other choices of \( z_t \) and obtain the same guarantee, although the empirical performance could be different. For example, one can use

\[
z_{t,i} = \begin{cases} 
\max_i g_{t,i}, & \text{if } \|w_t\|_1 < 1, \\
\min_i g_{t,i}, & \text{if } \|w_t\|_1 > 1
\end{cases}
\]

and clearly, the exact same proof still holds. Another possible choice is

\[
z_{t,i} = \begin{cases} 
\sum_i g_{t,i}, & \text{if } \|w_t\|_1 < 1, \\
g_{t,i}, & \text{if } \|w_t\|_1 = 1, \\
g_{t,i} - \langle g_t, x_t \rangle, & \text{if } \|w_t\|_1 > 1
\end{cases}
\]

This is more analogous to the surrogate losses in existing specialized approaches [LS15, OP16].
Also, to justify the improvement of $\sqrt{TV \cdot KL}$ over $\sqrt{KL}$, here is an example. For all $d \geq 3$, define $p, q \in \Delta(d)$ from

$$
p_1 = \frac{1}{\sqrt{\log d}}, \quad q_1 = \frac{1}{d \sqrt{\log d}}
$$

and

$$
p_i = \frac{1 - p_1}{d - 1}, \quad q_i = \frac{1 - q_1}{d - 1}, \quad \forall i \in [2 : d].
$$

Then,

$$
TV(p || q) = \frac{1}{2} \left[ |p_1 - q_1| + (d - 1) \left| \frac{1 - p_1}{d - 1} - \frac{1 - q_1}{d - 1} \right| \right] = |p_1 - q_1| = \frac{d - 1}{d \sqrt{\log d}}.
$$

$$
KL(p || q) = p_1 \log \frac{p_1}{q_1} + (d - 1) \cdot \frac{1 - p_1}{d - 1} \log \frac{1 - p_1}{1 - q_1}
$$

$$
= \sqrt{\log d} + \left( 1 - \frac{1}{\sqrt{\log d}} \right) \log \left( 1 - \frac{d - 1}{d \sqrt{\log d} - 1} \right)
$$

$$
\geq \sqrt{\log d} + \log \left( 1 - \frac{d}{d \sqrt{\log d} - 1} \right) = \sqrt{\log d} - o(1).
$$

Since we also have

$$
KL(p || q) = \sqrt{\log d} + (1 - p_1) \log \frac{1 - p_1}{1 - q_1} \leq \sqrt{d},
$$

we can combine the above and obtain $TV(p || q) \cdot KL(p || q) \leq 1$ and $KL(p || q) \geq \sqrt{\log d} - o(1)$. If our comparator $u$ and prior $\pi$ take the value of $p$ and $q$ respectively, then even without switching costs, Theorem 4 saves a $(\log d)^{1/4}$ factor from the existing parameter-free bounds.

C  Details on the financial application

In this final section of the appendix, we present details on the application of our algorithm to portfolio selection. We will introduce the connection of our setting to prior works, discuss our synthetic market results, and finally present additional experiments on historical US stock data.

C.1  Comparison to the rebalancing setting

Online portfolio selection has been studied by different communities with diverse perspectives. Here we mainly compare our setting to existing works with adversarial guarantees [Cov91, CO96, HSSW98, KV02, OLL17, LWZ18, MR22, ZAK22]. Differences are the following:

1. Existing works typically forbid short selling (i.e., $x_{t,i} < 0$) and margin trading (i.e., borrowing cash to buy an asset), therefore the decision is modeled as a rebalancing distribution $p_t \in \Delta(d)$. In contrast, our setting allows both\(^6\), so we call it “unconstrained”. Similar to the loss-regret trade-off in OLO, allowing margin trading introduces a risk-return trade-off in some sense: based on its own risk tolerance, one can trade off the best-case return with the worst-case loss on a Pareto-optimal frontier.

2. Related to the above, existing works consider Constant Rebalanced Portfolios (CRP, i.e., $p_t = p^* \in \Delta(d)$) as the benchmark class, and the goal is to lower bound the ratio of the growth rate of the considered algorithm to the growth rate of the benchmark. Here we consider unconstrained Buy-and-Hold (BAH) strategies as benchmarks, and we aim at an additive bound on the wealth. There have been discussions on the correct choice of benchmarks, but as suggested by a series of works [Cov91, HSSW98, BK99], a major weakness of CRPs is the incorporation of transaction costs: such benchmark strategies lose money due to constant rebalancing in every round, which makes the performance guarantee vacuous in certain cases. In contrast, BAH benchmarks do not suffer from this issue.

\(^6\)Although we only consider the ideal case with zero interest rate on loans.
3. Finally, transaction costs can take many forms. Here we consider the special case that charges a fixed price per share. This is different from the proportional transaction cost in some prior works [BK99, Gof14], which is proportional to the total value of the transaction.

We also note that our Algorithm 5 for LEA with switching cost is essentially a parameter-free improvement of the Exponentiated Gradient (EG) algorithm adopted in [HSSW98]. Therefore, it can be applied to the rebalancing setting, following the same argument there.

C.2 Synthetic market

We first introduce the detailed setting of our experiment. The market vector \( g \) is generated by

\[
\begin{align*}
    g_{t,1} &= 0.4 \cdot \text{Uniform}[-1, 1] + 0.4 \sin[(t/500) \cdot \pi] + 0.2, \\
    g_{t,2} &= 0.5 \cdot \text{Uniform}[-1, 1] + 0.3 \sin[(t/500 + 1/2) \cdot \pi] + 0.2, \\
    g_{t,3} &= 0.6 \cdot \text{Uniform}[-1, 1] + 0.2 \sin[(t/500 + 1) \cdot \pi] + 0.2, \\
    g_{t,4} &= 0.7 \cdot \text{Uniform}[-1, 1] + 0.1 \sin[(t/500 + 3/2) \cdot \pi] + 0.2, \\
    g_{t,5} &= 0.8 \cdot \text{Uniform}[-1, 1] + 0.2.
\end{align*}
\]

Specifically, the stochastic noise term on each coordinate is independent.

As for the baseline, we adapt it from [ZCP22a, Algorithm 1]. The latter is originally designed for a bounded domain \([0, R]\), but here we run it on the domain \( \mathbb{R} \), which effectively removes its internal unconstrained-to-constrained reduction. Its one-dimensional regret bound (9) still holds, with at most changes on the constants. After that, this one-dimensional algorithm is extended to higher dimensions using the coordinate-wise decomposition; this is the same procedure as our Algorithm 4, just with a different base algorithm.

Discussion on \( C \) We remark that setting \( C = 1 \) in Section 4 may complicate our direct comparison. Let us clarify it here. At first glance, one may question the existence of hyperparameters in such “parameter-free” algorithms. The classical rationale is the following: As shown in Theorem 3, parameter-free regret bounds depend on \( C \) logarithmically, whereas minimax regret bounds depend on the learning rate (and its inverse) linearly. In this regard, parameter-free algorithms are provably less sensitive to the correct setup, therefore as a rule of thumb, most practices [OP16, CLO20, ZCP22a] simply use \( C = 1 \) without requiring any domain knowledge. Such a default setup removes hyperparameter tuning, which is the most attractive feature of such algorithms. Our result in Section 4 (Figure 2) clearly shows the advantage of our algorithm when both algorithms are in this default, parameter-free implementation.

Nonetheless, for specific tasks like portfolio selection, tuning \( C \) can affect the actual performance one cares about (although violating the purpose of parameter-freeness). Intuitively, fixing the market, an aggressive trader with a worse strategy could make more profit than a conservative trader with a better strategy. Reflected in our experiment, since the market model does not depend on the invested amount, the baseline with a 10 times larger default, parameter-free implementation.

Our result in Section 4 (Figure 2) clearly shows the advantage of our algorithm when both algorithms are in this default setup removes hyperparameter tuning, which is the most attractive feature of such algorithms. Our result in Section 4 (Figure 2) clearly shows the advantage of our algorithm when both algorithms are in this default, parameter-free implementation.

Therefore, if tuning \( C \) is allowed, then comparing our algorithm to the baseline amounts to comparing two algorithm classes both parameterized by \( C \). A skeptical reader may wonder if the superior performance of our algorithm in the parameter-free setting is due to the confidence encoding rather than a better algorithm design. That is, is it possible that a baseline with a larger \( C \) can consistently outperform our algorithm with \( C = 1 \)? We provide evidence against this hypothesis, by increasing \( \lambda \) while keeping \( C = 1 \) for our algorithm and \( C = 10 \) for the baseline; results are plotted in Figure 4. It shows that even when the baseline is given an advantage \( (C = 10) \), our algorithm is still better at handling transaction costs due to an improved design. This is aligned with the superiority of our theoretical results.

Finally, we present some results on different synthetic markets. The market vectors are defined by

\[
\begin{align*}
    g_{t,1} &= 0.2 \cdot \text{Uniform}[-1, 1] + 0.4 \sin[(t/500) \cdot \pi] + 0.4, \\
    g_{t,2} &= 0.3 \cdot \text{Uniform}[-1, 1] + 0.3 \sin[(t/500 + 1/2) \cdot \pi] + 0.4,
\end{align*}
\]
Figure 3: Synthetic market experiment with tuned $C$; not a parameter-free implementation. Left: only tuning the baseline. Right: tuning both our algorithm and the baseline.

Figure 4: Synthetic market experiment with increasing $\lambda$. Left: $\lambda = 0.1$. Middle: $\lambda = 0.5$. Right: $\lambda = 1$. The baseline is given an advantage ($C = 10$), while our algorithm is in its default parameter-free implementation ($C = 1$). It shows our algorithm indeed handles transaction costs better.

$$g_{t,3} = 0.4 \cdot \text{Uniform}[-1, 1] + 0.2 \sin((t/500 + 1) \cdot \pi) + 0.4,$$

$$g_{t,4} = 0.5 \cdot \text{Uniform}[-1, 1] + 0.1 \sin((t/500 + 3/2) \cdot \pi) + 0.4,$$

$$g_{t,5} = 0.55 \cdot \text{Uniform}[-1, 1] + 0.45,$$

(Figure 5 (Left))

$$g_{t,5} = 0.5 \cdot \text{Uniform}[-1, 1] + 0.5.$$  

(Figure 5 (Right))

We implement both algorithms in the parameter-free way ($C = 1$) and plot the results in Figure 5. These markets have less uncertainty, therefore are easier to gain profit from. In both cases, our algorithm outperforms the baseline by a large margin.

Figure 5: Synthetic market experiment with different market models.
C.3 Historical stock data

Finally, we present some preliminary results on historical US stock data\(^7\). Eight stocks (Table 1) are considered on a time period of 5 years (1/1/2013 to 1/1/2018). Our algorithm trades once per day after the market closes, based on the closing price. We take the difference between the closing price on the \((t+1)\)-th day and the closing price on the \(t\)-th day, and define it as the market vector \(g_t\). The largest single day price change for any stock is less than $15, therefore \(G\) is set in a posterior manner to 15. We consider a hypothetical broker that charges $0.1 per share, therefore define \(\lambda = 0.1\).

| Company                          | Symbol |
|----------------------------------|--------|
| Apple Inc.                       | AAPL   |
| Berkshire Hathaway Inc. Class B  | BRK.B  |
| Meta Platforms Inc.              | FB     |
| Johnson & Johnson                | JNJ    |
| JPMorgan Chase & Co.             | JPM    |
| Microsoft Corporation            | MSFT   |
| Pfizer Inc.                      | PFE    |
| Exxon Mobil Corporation          | XOM    |

Table 1: List of considered stocks

Same as the synthetic market experiment, we test our algorithm against the baseline from [ZCP22a]. Our algorithm is in its default parameter-free implementation \((C = 1)\). However, setting \(C = 1\) also for the baseline is too conservative, which means the baseline hardly makes any investment, making the comparison less interesting. Therefore we set \(C = 150\) for the baseline, thus giving it an advantage at the beginning. In this way, the increased wealth of the two algorithms is roughly comparable.

We plot the results in Figure 6. Specifically, Figure 6 (Left) shows the increased wealth (in USD) over the considered time period. Figure 6 (Right) shows the cumulative amount of investment (in USD), which is the total net amount of cash the investor uses to buy stocks (i.e., increases when buying, and decreases when selling), plus the transaction costs paid to the broker. Before analyzing this result, we note that such a “cumulative investment” only makes sense in our setting, due to a fundamentally different mechanism compared to the rebalancing approach [Cov91]: in the latter, the investor is self-financed, i.e., it is given a certain budget at the beginning and never adds more money from external sources after that. In contrast, the investor in our setting can add more money at any time it wishes.

![plot](image)

Figure 6: Experiment on historical US stock data. Left: the increased wealth of the two algorithms. Right: total amount of investment since the start of the experiment (1/1/2013), including the transaction costs paid to the broker.

\(^7\)US stock price data is publicly available. We retrieved the data from Yahoo Finance website. [https://finance.yahoo.com/](https://finance.yahoo.com/)
From the plot we can see that the baseline is more aggressive at the beginning, due to a much larger $C$. Therefore, it slightly makes more profit during 2013-2014. When the market oscillates and declines in 2015 and 2016, the two algorithms perform roughly the same, while the baseline has a lower risk due to holding a smaller portfolio at the time. However, the major difference starts after mid-2016, when the market grows rapidly. Our algorithm is able to identify this trend and quickly increase the amount of investment. This brings a lot more profit than the baseline, which hardly recovers its confidence from the declining market in the previous year. Such an advantage of our algorithm is partly due to the better control of switching costs, and partly due to a better risk-return trade-off discussed in Appendix A.4 and C.1.

Our experiment also shows a limitation of our unconstrained investment setting. Throughout this five year period, our algorithm invests a total amount of $\sim$6.5 (including the transaction costs), and makes a total profit of $\sim$3. However, in practice, one typically invests a lot more than this (let’s say, $10,000), and expect a similar rate of return. Our setting does not model such a budget explicitly; instead, it relies on the parameter-freeness of the trading algorithms to increase the invested amount. Such a process can be slow, especially since we only consider trading once per day. Therefore, to use our algorithm in real trading situations, one has to tune the confidence parameter $C$ to implicitly take his budget and tolerable risk into account. For example, using our algorithm with $C = 1000$ would result in investing $6,500 throughout the five year period, and make a total profit of $3,000. The connection of this approach to rebalancing could be an interesting direction for future works.