THE CLOSEDNESS THEOREM
OVER HENSELIAN VALUED FIELDS

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Abstract. We prove the closedness theorem over Henselian valued fields, which was established over rank one valued fields in one of our recent papers. In the proof, as before, we use the local behaviour of definable functions of one variable and the so-called fiber shrinking, which is a relaxed version of curve selection. Now our approach applies also relative quantifier elimination for ordered abelian groups due to Cluckers–Halupczok. Afterwards the closedness theorem will allow us to achieve i.a. the Łojasiewicz inequality, curve selection and extending hereditarily rational functions as well as to develop the theory of regulous functions and sheaves.

1. Introduction

Throughout the paper, $K$ will be an arbitrary Henselian valued field of equicharacteristic zero with valuation $v$, value group $\Gamma$, valuation ring $R$ and residue field $k$. Examples of such fields are the quotient fields of the rings of formal power series and of Puiseux series with coefficients from a field $k$ of characteristic zero as well as the fields of Hahn series (maximally complete valued fields also called Malcev–Neumann fields; cf. [5]):

$$k((t^\Gamma)) := \left\{ f(t) = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma : a_\gamma \in k, \ \text{supp } f(t) \text{ is well ordered} \right\}.$$

We consider the ground field $K$ along with the three-sorted language $\mathcal{L}$ of Denef–Pas (cf. [1], [7]). Every valued field $K$ has a topology induced by its valuation $v$. Cartesian products $K^n$ are equipped with the product topology and subsets of Cartesian products $K^n$ inherit a topology, called the $K$-topology. The main purpose of this paper is to prove the following closedness theorem.

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Theorem 1.1. Let $D$ be an $\mathcal{L}$-definable subset of $K^n$. Then the canonical projection
\[ \pi : D \times R^m \to D \]
is definably closed in the $K$-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

In the case where the ground field $K$ is of rank one, it was established in our paper [7]. Of course, when $K$ is a locally compact field, the closedness theorem holds by a routine topological argument. In the proof given in Section 4, we use, as before, the local behaviour of definable functions of one variable and the so-called fiber shrinking, which is a relaxed version of curve selection. The former result over arbitrary Henselian valued fields was achieved in the paper [10], Proposition 5.1. The proof of the latter, in turn, is given in Section 2. Now our approach applies also relative quantifier elimination for ordered abelian groups due to Cluckers–Halupczok [2], which is recalled in Section 3.

Remark 1.2. Not all valued fields $K$ have an angular component map, but it exists if $K$ has a cross section, which happens whenever $K$ is $\aleph_1$-saturated (cf. [1, Chap. II]). Moreover, a valued field $K$ has an angular component map whenever its residue field $k$ is $\aleph_1$-saturated (cf. [12], Corollary 1.6). In general, unlike for $p$-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. The $K$-topology is, of course, definable in the language of valued fields, and therefore the closedness theorem is a first order property. Hence it is valid over arbitrary Henselian valued fields of equicharacteristic zero, because it can be proven using saturated elementary extensions and one may thus assume that an angular component map exists.

Afterwards Theorem 1.1 will allow us to establish i.a. the Łojasiewicz inequality, curve selection and extending hereditarily rational functions as well as to develop the theory of regulous functions and sheaves. Those results were achieved over rank one valued fields in our paper [1]. The theory of hereditarily rational functions on the real and $p$-adic varieties was provided in the paper [6]. The closedness theorem immediately yields five corollaries stated below. One of them, the descent property (Corollary 1.7), enables application of resolution of singularities and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field. Other its applications are provided in our recent papers [8, 9].

Corollary 1.3. Let $D$ be an $\mathcal{L}$-definable subset of $K^n$ and $\mathbb{P}^m(K)$ stand for the projective space of dimension $m$ over $K$. Then the canonical
projection
\[ \pi : D \times \mathbb{P}^m(K) \to D \]
is definably closed. \qed

**Corollary 1.4.** Let \( A \) be a closed \( \mathcal{L} \)-definable subset of \( \mathbb{P}^m(K) \) or \( \mathbb{R}^m \). Then every continuous \( \mathcal{L} \)-definable map \( f : A \to \mathbb{K}^n \) is definably closed in the \( \mathbb{K} \)-topology.

**Corollary 1.5.** Let \( \phi_i, i = 0, \ldots, m \), be regular functions on \( \mathbb{K}^n \), \( D \) be an \( \mathcal{L} \)-definable subset of \( \mathbb{K}^n \) and \( \sigma : Y \to \mathbb{K}^n \) the blow-up of the affine space \( \mathbb{K}^n \) with respect to the ideal \((\phi_0, \ldots, \phi_m)\). Then the restriction
\[ \sigma : Y(\mathbb{K}) \cap \sigma^{-1}(D) \to D \]
is a definably closed quotient map.

*Proof.* Indeed, \( Y(\mathbb{K}) \) can be regarded as a closed algebraic subvariety of \( \mathbb{K}^n \times \mathbb{P}^m(K) \) and \( \sigma \) as the canonical projection. \( \Box \)

**Corollary 1.6.** Let \( X \) be a smooth \( \mathbb{K} \)-variety, \( \phi_i, i = 0, \ldots, m \), regular functions on \( X \), \( D \) be an \( \mathcal{L} \)-definable subset of \( X(\mathbb{K}) \) and \( \sigma : Y \to X \) the blow-up of the ideal \((\phi_0, \ldots, \phi_m)\). Then the restriction
\[ \sigma : Y(\mathbb{K}) \cap \sigma^{-1}(D) \to D \]
is a definably closed quotient map. \( \Box \)

**Corollary 1.7.** *(Descent property)* Under the assumptions of the above corollary, every continuous \( \mathcal{L} \)-definable function
\[ g : Y(\mathbb{K}) \cap \sigma^{-1}(D) \to \mathbb{K} \]
that is constant on the fibers of the blow-up \( \sigma \) descends to a (unique) continuous \( \mathcal{L} \)-definable function \( f : D \to \mathbb{K} \). \( \Box \)

2. Fiber shrinking

Consider a Henselian valued field \( \mathbb{K} \) of equicharacteristic zero along with the three-sorted language \( \mathcal{L} \) of Denef–Pas. In this section, we remind the reader the concept of fiber shrinking introduced in our paper [7, Section 6].

Let \( A \) be an \( \mathcal{L} \)-definable subset of \( \mathbb{K}^n \) with accumulation point
\[ a = (a_1, \ldots, a_n) \in \mathbb{K}^n \]
and \( E \) an \( \mathcal{L} \)-definable subset of \( \mathbb{K} \) with accumulation point \( a_1 \). We call an \( \mathcal{L} \)-definable family of sets
\[ \Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A \]
an $L$-definable $x_1$-fiber shrinking for the set $A$ at $a$ if
\[
\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),
\]
i.e. for any neighbourhood $U$ of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood $V$ of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, $A$ is itself a fiber shrinking for the subset $A$ of $K$ at an accumulation point $a \in K$.

**Proposition 2.1.** (Fiber shrinking) Every $L$-definable subset $A$ of $K^n$ with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $L$-definable $x_1$-fiber shrinking at $a$.

In the case where the ground field $K$ is of rank one, the proof of Proposition 2.1 was given in [7, Section 6]. In the general case, it can be repeated verbatim once we demonstrate the following result on definable subsets in the value group sort $\Gamma$.

**Lemma 2.2.** Let $\Gamma$ be an ordered abelian group and $P$ be a definable subset of $\Gamma^n$. Suppose that $(\infty, \ldots, \infty)$ is an accumulation point of $P$, i.e. for any $\delta \in \Gamma$ the set
\[
\{x \in P : x_1 > \delta, \ldots, x_n > \delta\} \neq \emptyset
\]
is non-empty. Then there is an affine semiline $L$ passing through a point $\gamma = (\gamma_1, \ldots, \gamma_n) \in P$:
\[
L = \{(r_1 t + \gamma_1, \ldots, r_n t + \gamma_n) : t \in \Gamma, t \geq 0\},
\]
where $r_1, \ldots, r_n \in \mathbb{N}$ are positive integers, such that $(\infty, \ldots, \infty)$ is an accumulation point of the intersection $P \cap L$ too.

In [7, Section 6], Lemma 2.2 was shown for archimedean groups by means of quantifier elimination in the Presburger language. But in the general case, it follows in a similar fashion via relative quantifier elimination for ordered abelian groups (apply Theorem 3.1 along with Remarks 3.2 and 3.3), recalled in the next section.

3. **Quantifier elimination for ordered abelian groups**

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated are quantifier elimination results for non-archimedean groups (especially those with infinite rank), going back as far as Gurevich [3]. He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas. This was done in his doctoral dissertation "The decision problem for some algebraic
theories” (Sverdlovsk, 1968), and next also by Schmitt in his habilitation thesis ”Model theory of ordered abelian groups” (Heidelberg, 1982); see also the paper [13]. Such a transfer is a kind of relative quantifier elimination, which allows Gurevich–Schmitt [14] in their study of the NIP property to lift model theoretic properties from ordered sets to ordered abelian groups or, in other words, to transform statements on ordered abelian groups into those on coloured chains.

Instead Cluckers–Halupczok [2] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort $\Gamma$ and auxiliary imaginary sorts which carry the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

Fortunately, sometimes it is possible to directly deduce information about ordered abelian groups without any knowledge of the auxiliary sorts. For instance, this may be illustrated by their theorem on piece-wise linearity of definable functions [2, Corollary 1.10] as well as by Proposition 2.2 and application of quantifier elimination in the proof of the closedness theorem in Section 4.

Now we briefly recall the language $\mathcal{L}_{qe}$ taking care of points essential for our applications.

The main group sort $\Gamma$ is with the constant 0, the binary function $+$ and the unary function $−$. The collection $\mathcal{A}$ of auxiliary sorts consists of certain imaginary sorts:

$$\mathcal{A} := \{ S_p, T_p, T_p^+: p \in \mathbb{P}\};$$

here $\mathbb{P}$ stands for the set of prime numbers. By abuse of notation, $\mathcal{A}$ will also denote the union of the auxiliary sorts. In this section, we denote $\Gamma$-sort variables by $x, y, z, \ldots$ and auxiliary sorts variables by $\eta, \theta, \zeta, \ldots$.

Further, the language $\mathcal{L}_{qe}$ consists of some unary predicates on $S_p$, $p \in \mathbb{P}$, some binary order relations on $\mathcal{A}$, a ternary relation

$$x \equiv_{m,\alpha}^{m'} y \text{ on } \Gamma \times \Gamma \times S_p \text{ for each } p \in \mathbb{P}, \ m, m' \in \mathbb{N},$$

and finally predicates for the ternary relations

$$x \circ_{\alpha} y + k_{\alpha} \text{ on } \Gamma \times \Gamma \times \mathcal{A},$$

where $\circ \in \{=, <, \equiv_m\}$, $m \in \mathbb{N}, \ k \in \mathbb{Z}$ and $\alpha$ is the third operand running any of the auxiliary sorts $\mathcal{A}$. 
We now explain the meaning of the above ternary relations, which are defined by means of certain definable subgroup $\Gamma_\alpha$ and $\Gamma_{\alpha m'}$ of $\Gamma$ with $\alpha \in A$ and $m' \in \mathbb{N}$. Namely we write
\[
x \equiv_{m,\alpha} y \iff x - y \in \Gamma_{\alpha m'} + m\Gamma.
\]
Further, let $1_\alpha$ denote the minimal positive element of $\Gamma / \Gamma_\alpha$ if $\Gamma / \Gamma_\alpha$ is discrete and $1_\alpha := 0$ otherwise, and set $k_\alpha := k \cdot 1_\alpha$ for all $k \in \mathbb{Z}$. By definition we write
\[
x \diamond_\alpha y + k_\alpha \iff x \pmod{\Gamma_\alpha} \diamond y \pmod{\Gamma_\alpha} + k_\alpha.
\]
(Thus the language $\mathcal{L}_{qe}$ incorporates the Presburger language on all quotients $\Gamma / \Gamma_\alpha$.) Note also that the ordinary predicates $<$ and $\equiv_m$ on $\Gamma$ are $\Gamma$-quantifier-free definable in the language $\mathcal{L}_{qe}$.

Now we can readily formulate quantifier elimination relative to the auxiliary sorts ([4, Theorem 1.8]).

**Theorem 3.1.** In the theory $T$ of ordered abelian groups, each $\mathcal{L}_{qe}$-formula $\phi(\vec{x}, \vec{\eta})$ is equivalent to an $\mathcal{L}_{qe}$-formula $\psi(\vec{x}, \vec{\eta})$ in family union form, i.e.
\[
\psi(\vec{x}, \vec{\eta}) = \bigvee_{i=1}^{k} \exists \vec{\theta} \left[ \chi_i(\vec{\eta}, \vec{\theta}) \land \omega_i(\vec{x}, \vec{\theta}) \right],
\]
where $\vec{\theta}$ are $A$-variables, the formulas $\chi_i(\vec{\eta}, \vec{\theta})$ live purely in the auxiliary sorts $A$, each $\omega_i(\vec{x}, \vec{\theta})$ is a conjunction of literals (i.e. atomic or negated atomic formulas) and $T$ implies that the $\mathcal{L}_{qe}(A)$-formulas
\[
\{ \chi_i(\vec{\eta}, \vec{\alpha}) \land \omega_i(\vec{x}, \vec{\alpha}) : i = 1, \ldots, k, \vec{\alpha} \in A \}
\]
are pairwise inconsistent.

**Remark 3.2.** The sets definable (or, definable with parameters) in the main group sort $\Gamma$ resemble to some extent the sets which are definable in the Presburger language. Indeed, the atomic formulas involved in the formulas $\omega_i(\vec{x}, \vec{\theta})$ are of the form
\[
t(\vec{x}) \diamond_{\theta_j} k_{\theta_j},
\]
where $t(\vec{x})$ is a $\mathbb{Z}$-linear combination (respectively, a $\mathbb{Z}$-linear combination plus an element of $\Gamma$), the predicates
\[
\diamond \in \{ =, <, \equiv_m, \equiv_{m'} \} \text{ with some } m, m' \in \mathbb{N},
\]
$\theta_j$ is one of the entries of $\vec{\theta}$ and $k \in \mathbb{Z}$; here $k = 0$ if $\diamond$ is $\equiv_{m'}$. Clearly, equality and inequalities define polyhedra and congruence conditions define sets which consist of entire cosets of $m\Gamma$ for finitely many $m \in \mathbb{N}$. 

Remark 3.3. Note also that the sets given by atomic formulas \( t(\bar{x}) \diamond \theta, k_{\theta_j} \) consist of entire cosets of the subgroups \( \Gamma_{\theta_j} \). Therefore, the union of those subgroups \( \Gamma_{\theta_j} \) which essentially occur in a formula in family union form, describing a proper subset of \( \Gamma^m \), is not cofinal with \( \Gamma \). This observation is often useful as, for instance, in the proofs of fiber shrinking and Theorem 1.1.

4. Proof of the closedness theorem

Generally, we shall follow the idea of the proof from [7, Section 7]. Again, the proof reduces easily to the case \( m = 1 \) and next, by means of fiber shrinking (Proposition 2.1), to the case \( n = 1 \) and \( a = 0 \in K \).

Whereas in the paper [7] preparation cell decomposition (due to Pas; see [11, Theorem 3.2] and [7, Theorem 2.4]) was combined with quantifier elimination in the \( \Gamma \) sort in the Presburger language, here it is combined with relative quantifier elimination due to Cluckers–Halupczok. In the same manner as before, we can now assume that \( B \) is a subset \( F \) of a cell \( C \), as explained below. Let

\[
a(x, \xi), b(x, \xi), c(x, \xi) : D \to K
\]

be three \( L \)-definable functions on an \( L \)-definable subset \( D \) of \( \mathbb{K}^2 \times \mathbb{K}^m \) and let \( \nu \in \mathbb{N} \) is a positive integer. For each \( \xi \in \mathbb{K}^m \) set

\[
C(\xi) := \left\{ (x, y) \in K^n_x \times K_y : (x, \xi) \in D, v(a(x, \xi)) \prec_1 v((y - c(x, \xi))^\nu) \prec_2 v(b(x, \xi)), \overline{\nu c}(y - c(x, \xi)) = \xi_1 \right\},
\]

where \( \prec_1, \prec_2 \) stand for \( <, \leq \) or no condition in any occurrence. A cell \( C \) is by definition a disjoint union of the fibres \( C(\xi) \). The subset \( F \) of \( C \) is a union of fibers \( F(\xi) \) of the form

\[
F(\xi) := \left\{ (x, y) \in C(\xi) : \exists \bar{\theta} \chi(\bar{\theta}) \wedge \right. \left. \bigwedge_{i \in I_a} v(a_i(x, \xi)) \prec_{1, \theta_i} v((y - c(x, \xi))^\nu_i), \bigwedge_{i \in I_b} v((y - c(x, \xi))^\nu_i) \prec_{2, \theta_i} v(b_i(x, \xi)) \right. \left. \wedge \bigwedge_{i \in I_f} v((y - c(x, \xi))^\nu_i) \prec_{\theta_i} v(f_i(x, \xi)) \right\},
\]

where \( I_a, I_b, I_f \) are finite (possibly empty) sets of indices, \( a_i, b_i, f_i \) are \( L \)-definable functions, \( \nu_i, M \in \mathbb{N} \) are positive integers, \( \prec_1, \prec_2 \) stand for
< or \leq\), the predicates
\[ \diamond \in \{ \equiv_M, \neg \equiv_M, \equiv_M^\prime, \neg \equiv_M^\prime \} \]
with some \( m' \in \mathbb{N} \),
and \( \theta_{ji} \) is one of the entries of \( \bar{\theta} \).

As before, since every \( \mathcal{L} \)-definable subset in the Cartesian product \( \Gamma^n \times \mathbb{K}^m \) of auxiliary sorts is a finite union of the Cartesian products of definable subsets in \( \Gamma^n \) and in \( \mathbb{K}^m \), we can assume that \( B \) is one fiber \( F(\xi') \) for a parameter \( \xi' \in \mathbb{K}^m \). For simplicity, we abbreviate
\[ c(x,\xi'), a(x,\xi'), b(x,\xi'), a_i(x,\xi'), b_i(x,\xi'), f_i(x,\xi') \]
to
\[ c(x), a(x), b(x), a_i(x), b_i(x), f_i(x) \]
with \( i \in I_a, i \in I_b \) and \( i \in I_f \). Denote by \( E \subset K \) the common domain of these functions; then \( 0 \) is an accumulation point of \( E \).

By the theorem on existence of the limit, established over arbitrary Henselian valued fields in our paper \[10\], Proposition 5.1], we can assume that the limits
\[ c(0), a(0), b(0), a_i(0), b_i(0), f_i(0) \]
of the functions
\[ c(x), a(x), b(x), a_i(x), b_i(x), f_i(x) \]
when \( x \to 0 \) exist in \( R \). Moreover, there is a neighbourhood \( U \) of \( 0 \) such that, each definable set
\[ \{(v(x), v(f_i(x))): x \in (E \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}), \quad i \in I_f, \]
is contained in an affine line with rational slope
\[ l = \frac{p_i}{q} \cdot k + \beta_i, \quad i \in I_f, \]
with \( p_i, q \in \mathbb{Z}, q > 0, \beta_i \in \Gamma \), or in \( \Gamma \times \{\infty\} \).

The role of the center \( c(x) \) is, of course, immaterial. We may assume, without loss of generality, that it vanishes, \( c(x) \equiv 0 \), for if a point \( b = (0, w) \in K^2 \) lies in the closure of the cell with zero center, the point \( (0, w + c(0)) \) lies in the closure of the cell with center \( c(x) \).

Observe now that if \( \triangleleft_1 \) occurs and \( a(0) = 0 \), the set \( F(\xi') \) is itself an \( x \)-fiber shrinking at \( (0, 0) \) and the point \( b = (0, 0) \) is an accumulation point of \( B \) lying over \( a = 0 \), as desired. And so is the point \( b = (0, 0) \) if \( \triangleleft_{1, \theta_{ji}} \) occurs and \( a_i(0) = 0 \) for some \( i \in I_a \), because then the set \( F(\xi') \) contains the \( x \)-fiber shrinking
\[ F(\xi') \cap \{(x, y) \in E \times K: v(a_i(x)) \triangleleft_1 v(y^m)\}. \]
So suppose that either only \( \prec_2 \) occur or \( \prec_1 \) occur and, moreover, 
\( a(0) \neq 0 \) and \( a_i(0) \neq 0 \) for all \( i \in I_a \). By elimination of \( K \)-quantifiers, 
the set \( v(E) \) is a definable subset of \( \Gamma \). Further, it is easy to check, 
applying Theorem \[3.1\] ff. likewise as it was in Lemma \[2.2\], that the set 
\( v(E) \) is given near infinity only by finitely many congruence conditions 
of the form

\[
(4.2) \quad v(E) = \left\{ k \in \Gamma : k > \beta \land \exists \bar{\theta} \omega(\bar{\theta}) \land \bigwedge_{i=1}^{s} m_i k \odot_{N,\theta_{j_i}} \gamma_i \right\},
\]

where \( \beta, \gamma_i \in \Gamma \), \( m_i, N \in \mathbb{N} \) for \( i = 1, \ldots, s \), the predicates
\[
\odot \in \{ \equiv_{N}, \neg \equiv_{N}, \equiv_{N}^{m'}, \neg \equiv_{N}^{m'} \}
\]
with some \( m' \in \mathbb{N} \),
and \( \theta_{j_i} \) is one of the entries of \( \bar{\theta} \). Obviously, after perhaps shrinking 
the neighbourhood of zero, we may assume that 
\[
v(a(x)) = v(a(0)) \quad \text{and} \quad v(a_i(x)) = v(a_i(0))
\]
for all \( i \in I_a \) and \( x \in E \setminus \{0\} \), \( v(x) > \beta \).

Now, take an element \((u, w) \in F(\xi')\) with \( u \in E \setminus \{0\} \), \( v(u) > \beta \).
In order to complete the proof, it suffices to show that \((0, w)\) is an 
accumulation point of \( F(\xi') \). To this end, observe that, by equality \[1.2\], 
there is a point \( x \in E \) arbitrarily close to 0 such that 
\[
v(x) \in v(u) + qMN \cdot \Gamma.
\]
By equality \[1.1\] we get
\[
v(f_i(x)) \in v(f_i(u)) + p_i MN \cdot \Gamma, \quad i \in I_f,
\]
and hence
\[
(4.3) \quad v(f_i(x)) \equiv_M v(f_i(u)), \quad i \in I_f.
\]
Clearly, in the vicinity of zero we have
\[
v(y') \prec_2 v(b(x, \xi))
\]
and
\[
\bigwedge_{i \in I_b} v(y^{'i}) \prec_2 v(b_i(x, \xi)).
\]
Therefore equality \[1.3\] along with the definition of the fibre \( F(\xi') \) yield 
\((x, w) \in F(\xi')\), concluding the proof. \( \square \)
References

[1] G. Cherlin, *Model Theoretic Algebra, Selected Topics*, Lect. Notes Math. 521, Springer-Verlag, Berlin, 1976.

[2] R. Cluckers, E. Halupczok, *Quantifier elimination in ordered abelian groups*, Confluentes Math. 3 (2011), 587–615.

[3] Y. Gurevich, *Elementary properties of ordered abelian groups*, Algebra i Logika Seminar, 3 (1964), 5–39 (in Russian); Amer. Math. Soc. Transl., II Ser. 46 (1965), 165–192 (in English).

[4] Y. Gurevich, P.H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Trans. Amer. Math. Soc. 284 (1984), 171–182.

[5] I. Kaplansky, *Maximal fields with valuations I and II*, Duke Math. J. 9 (1942), 303–321 and 12 (1945), 243–248.

[6] J. Kollár, K. Nowak, *Continuous rational functions on real and p-adic varieties*, Math. Zeitschrift 279 (2015), 85–97.

[7] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. 23 (2017), 455–495.

[8] K.J. Nowak, *Hölder and Lipschitz continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.03463 [math.AG].

[9] K.J. Nowak, *Piecewise continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.07849 [math.AG].

[10] K.J. Nowak, *On functions given by algebraic power series over Henselian valued fields*, arXiv:1703.08203 [math.AG].

[11] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. 399 (1989), 137–172.

[12] J. Pas, *On the angular component map modulo p*, J. Symbolic Logic 55 (1990), 1125–1129.

[13] P.H. Schmitt, *Model and substructure complete theories of ordered abelian groups*; In: *Models and Sets* (Proceedings of Logic Colloquium ’83), Lect. Notes Math. 1103, Springer-Verlag, Berlin, 1984, 389–418.

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