Characterization of the Pareto Social Choice Correspondence

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October 30, 2019

Abstract

Independent, necessary and sufficient conditions are derived for a social choice correspondence to be the one that selects exactly the Pareto optimal alternatives.

• **Keywords:** Pareto · tops-in · balancedness · monotonicity · stability
• **JEL:** D70 D71
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1 Introduction

We characterize the social choice correspondence that, at each profile of preferences, selects exactly the set of Pareto optimal alternatives. We use one condition, balancedness, introduced in (Kelly and Qi 2019) and a second, stability (related to a condition in Campbell, Kelly, and Qi 2018), as well as tops-in and the Pareto condition. Although the collection of Pareto optimal states at any situation has long been a central object of concern in welfare economics, there is no work we know of characterizing this mapping. Our main theorem shows independent, necessary, and sufficient conditions for the Pareto correspondence in the case of five or more alternatives. Three other results, using weaker conditions, for the cases of two, three, or four alternatives are also included.

2 Framework

Let $X$ with cardinality $|X| = m \geq 2$ be the finite set of alternatives and let $N = \{1, 2, \ldots, n\}$ with $n \geq 2$ be the set of individuals. A (strong) ordering on $X$ is a complete, asymmetric, transitive relation on $X$ (so we exclude non-trivial individual indifference). The top-ranked element of an ordering $r$ is denoted $r[1]$, the next highest is denoted $r[2]$, etc. The set of all orderings on $X$ is $L(X)$. A profile $u$ is an element $(u(1), u(2), \ldots, u(n))$ of the Cartesian product $L(X)^N$. If $x$ ranks above $y$ in $u(j)$, we sometimes write $x \succ^u y$.

A social choice correspondence $G$ is a map from the domain $L(X)^N$ to non-empty subsets of $X$. One example that will play a role here is the correspondence $T$ that maps profile $u$ to $T(u)$, the set of all top-ranked alternatives at $u$:

$$T(u) = \{ x \in X : \text{for some } i \in N, x = u(i)[1] \}$$

At profile $u$, alternative $x$ Pareto dominates $y$ if $x \succ^u y$ for all $i$. Social choice correspondence $G$ satisfies the Pareto condition if for all $x, y$, and $u$, whenever $x$ Pareto dominates $y$ at $u$, then $y \notin G(u)$. The Pareto correspondence, $G_P$, is defined by

$$G_P(u) = \{ x \in X : \text{there does not exist a } y \text{ that Pareto dominates } x \text{ at } u \}$$

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1. The author is indebted to Shaofang Qi, Somdeb Lahiri, and an anonymous reader for comments on earlier drafts.
2. Weymark (1984) characterizes the mapping to Pareto social preferences. Here we are concerned with social choice correspondences that map to social choice sets.
Thus, the Pareto condition is 

\[ G(u) \subseteq G_P(u) \text{ for all } u. \]

At profile \( u \), alternative \( x \) is **Pareto optimal** if there does not exist an alternative \( y \) that Pareto dominates \( x \). Thus \( G_P(u) \) is the set of all Pareto optimal alternatives at \( u \).

Clearly, the Pareto correspondence satisfies the Pareto condition, and this is one of the conditions we use to characterize the Pareto correspondence. This may seem odd at first, but the condition of excluding dominated alternatives is very weak and is used in characterizing many standard social choice correspondences including the following which will appear several times in this paper.

1. The **Borda correspondence**: Given profile \( u \) and alternative \( x \), the Borda score, \( N(x, u) \), is the sum of the ranks in which \( x \) appears in the orderings in \( u \). Alternative \( x \) is a Borda winner at \( u \) if \( N(x, u) \leq N(y, u) \) for all \( y \). The Borda correspondence selects at \( u \) the set of all Borda winners at \( u \).

2. The **plurality voting correspondence**: Given profile \( u \) and alternative \( x \), the plurality score, \( N(x, u) \), is the number of individuals who have \( x \) in the top rank. Alternative \( x \) is a plurality winner at \( u \) if \( N(x, u) \geq N(y, u) \) for all \( y \). The plurality correspondence selects at \( u \) the set of all plurality winners at \( u \).

3. The **Copeland correspondence**: Given profile \( u \) and alternative \( x \), the Copeland score, \( N(x, u) \), is the number of alternatives defeated or tied by \( x \) under simple majority voting. Alternative \( x \) is a Copeland winner at \( u \) if \( N(x, u) \geq N(y, u) \) for all \( y \). The Copeland correspondence selects at \( u \) the set of all Copeland winners at \( u \).

The issue of using the Pareto condition is addressed again in the conclusion.

The Pareto correspondence also satisfies other conditions commonly used in social choice theory:

- **Anonymity**: A social choice correspondence \( G \) satisfies anonymity if, for every permutation \( \rho \) on \( N \), and every profile \( u \),
  \[ G(u(1), u(2), \ldots, u(n)) = G(u(\rho(1)), u(\rho(2), \ldots, u(\rho(n))). \]

- **Neutrality**: Let \( \theta \) be a permutation of \( X \). If \( S = \{x, y, \ldots, z\} \) is a subset of \( X \), we set \( \theta(S) = \{\theta(x), \theta(y), \ldots, \theta(z)\} \). And if \( R \) is an ordering on \( X \), we define \( \theta(R) \) by \( (\theta(x), \theta(y)) \in \theta(R) \) if and only if \( (x, y) \in R \). Now we say a social choice correspondence \( G \) satisfies neutrality if, for every permutation \( \theta \) of \( X \), and every profile \( u \),
  \[ G(\theta(u(1)), \theta(u(2)), \ldots, \theta(u(n))) = \theta(G(u(1), u(2), \ldots, u(n))). \]

Those two additional conditions are not used in our characterization theorems but are referred to in examples.
3 Characterization with $m = 2$

We start with a property used in all the characterizations in this paper:

**Tops-in**: $T(u) \subseteq G(u)$ for all $u$

Of course the tops rule choosing $T(u)$, the constant rule $G(u) = X$, and the Pareto correspondence all satisfy tops-in. The plurality correspondence, the Borda correspondence, and the Copeland correspondence do not satisfy tops-in. Note that we are not restricting in this paper to "desirable" conditions. Since the Pareto correspondence itself is not desirable - if only because choice sets are often too large - any set of characterizing conditions must include some that are undesirable.

**Theorem 1.** For $m = 2$ and $n \geq 2$, let $G : L(X)^N \rightarrow 2^X \backslash \{\emptyset\}$ be a social choice correspondence satisfying both of:

1. The Pareto condition; and
2. Tops-in:

then $G = G_P$, the Pareto correspondence.

**Proof:** The necessity of these conditions is obvious. Let $X = \{x, y\}$ and let $G$ be a social choice correspondence satisfying the assumptions of the theorem.

**Case 1.** Only one of the two alternatives, say $x$, is in $T(u)$. Then by tops-in, $x \in G(u)$. By the Pareto condition, $y \notin G(u)$. Therefore, $G(u) = \{x\} = G_P(u)$.

**Case 2.** Otherwise both alternatives, $x$ and $y$, are in $T(u)$ and then, by tops-in, $G(u) = \{x, y\} = G_P(u)$. □

We now present examples showing that neither condition can be dropped in Theorem 2:

**Example 1.** A rule other than $G_P$ satisfying tops-in (as well as anonymity and neutrality) but not Pareto: $G(u) = X$ for all profiles $u$.

**Example 2.** A rule other than $G_P$ satisfying Pareto (as well as anonymity and neutrality) but not tops-in: Let $m = 2$ and $n \geq 3$ and set $G(u)$ equal to the set of plurality winners at $u$.

4 Characterization with $m = 3$

When there are more than two alternatives, the properties of Theorem 1, namely Pareto and tops-in, are *not* sufficient to characterize the Pareto correspondence.

**Example 3.** For $m \geq 3$, $G(u) = T(u)$ is distinct from $G_P(u)$ but satisfies Pareto and tops-in (as well as anonymity and neutrality).
Note that \( T(u) \) fails the following balancedness condition (Kelly and Qi 2019).

We say profile \( v \) is \textbf{constructed from profile} \( u \) \textbf{by transposition pair} \((x,y)\) \textbf{via individuals} \( i \) and \( j \) if at \( u \), \( x \) is immediately above \( y \) for \( i \) and \( y \) is immediately above \( x \) for \( j \), and profile \( v \) is just the same as \( u \) except that alternatives \( x \) and \( y \) are transposed for \( i \) and for \( j \). A social choice correspondence \( G \) will be called \textbf{balanced} if, for all \( x, y, u, v, i, \) and \( j \), whenever profile \( v \) is constructed from \( u \) by transposition pair \((x,y)\) via individuals \( i \) and \( j \), then \( G(v) = G(u) \).

The constant rule \( G(u) = X \), the Pareto correspondence, Borda’s rule, and the Copeland correspondence all satisfy balancedness, but \( T \) and the plurality correspondence do not. In (Kelly and Qi, 2019) it is observed:

\[ \text{[Balancedness] is a natural equity condition that simultaneously incorporates some equal treatment for individuals short of anonymity, some equal treatment of alternatives short of neutrality, and some equal treatment of position of alternatives in orderings (for example, raising \( x \) just above \( y \) in the bottom two ranks for individual \( j \) exactly offsets lowering \( x \) just below \( y \) in the top two ranks for \( i \)).} \]

We now show that, for three alternatives, incorporating balancedness with tops-in and the Pareto condition forces \( G = G_P \).

**Theorem 2.** For \( m = 3 \) and \( n \geq 2 \), let \( G : L(X)^N \to 2^X \setminus \{ \emptyset \} \) be a social choice correspondence satisfying all of:

1. The Pareto condition;
2. Tops-in;
3. Balancedness;

then \( G = G_P \), the Pareto correspondence.

**Proof:** The necessity of these conditions is obvious.

Now assume that \( G \) satisfies all three conditions. We need to show that if \( w \) is Pareto optimal at \( u \), then \( w \in G(u) \). Suppose that Pareto optimal \( w \notin G(u) \).

Alternative \( w \) can not be anyone’s top at \( u \) by tops-in. And \( w \) can not be everyone’s bottom since it is Pareto optimal. So someone, say \#1, has \( w \) in their second rank. Suppose that (1) \( w \) is Pareto optimal, (2) \( w \) is in \#1’s second rank at \( u \), (3) \( w \) is in no one’s top rank at \( u \), and (4) \( w \notin G(u) \). In particular, assume that \( x \neq z \) is at \#1’s top. Some individual, say \#2, has \( w \succ_2 x \) since \( w \) is Pareto optimal. For \#2, \( w \) must be adjacent to \( x \) with \( x \) bottom-ranked.

\[
\begin{array}{ccccc}
1 & 2 & \cdots \\
 x & \cdots & \\
w & w & \\
\vdots & x & \cdots \\
\end{array}
\]
By balancedness, $G(u') = G(u)$ where $u'$ is obtained from $u$ by transposition pair $(x, w)$ for #1 and #2. So $w \notin G(u')$. But that contradicts tops-in. □

To show the need for each condition in Theorem 2, we first observe that Example 3 exhibits a rule other than $G_P$ satisfying all conditions of Theorem 2 other than balancedness. Also, $G(u) = X$ for all profiles $u$ satisfies all conditions of Theorem 2 except Pareto.

**Example 4.** A rule other than $G_P$ satisfying all conditions of Theorem 2 except tops-in: Let $m = n = 3$ and let $u^*$ be a fixed voter’s paradox profile, say

\[
\begin{array}{ccc}
1 & 2 & 3 \\
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

Define $G(u^*) = \{x\}$ and $G(u) = G_P(u)$ for all $u \neq u^*$. Balancedness is satisfied by $G$ because there are no transposition pairs at a voter’s paradox profile. This rule is neither anonymous nor neutral.

### 5 Characterization with $m = 4$

When there are more than three alternatives, the properties of Theorem 2, namely Pareto, tops-in, and balancedness, are not sufficient to characterize the Pareto correspondence.

**Example 5.** Let $X = \{x, y, z, w\}$ with $n = 3$. Consider fixed profile $u^*$:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
x & y & z \\
y & w & w \\
z & x & x \\
w & z & y \\
\end{array}
\]

that has no transposition pairs. Observe that $G_P(u^*) = \{x, y, z, w\}$.

Now define social choice correspondence $G$ as follows:

1. $G(u^*) = \{x, y, z\} = T(u^*)$;
2. For all other $u$, set $G(u) = G_P(u)$.

This correspondence (which fails anonymity and neutrality) satisfies Pareto, tops-in, and balancedness, but differs from $G_P$ at $u^*$. So we need to add some new condition to those of Theorem 2 in order to characterize the Pareto correspondence for $m > 3$. What won’t work is using anonymity and neutrality. Example 5 could be modified by constructing subdomain $D$ consisting of all profiles obtained from the $u^*$ of that example by permuting either $X$ or $N$ or both. Then define
1. $G(u) = T(u)$ for all $u$ in $D$;
2. $G(u) = G_P(u)$ otherwise.

This $G$ satisfies Pareto, tops-in, balancedness, anonymity, and neutrality, but differs from $G_P$ at every profile in $D$. So we introduce a new property.

**Strong monotonicity**: A social choice correspondence $G$ satisfies **strong monotonicity** if, for every $x \in X$, $i \in N$, and every profile $u$, if $x \in G(u)$, and profile $u'$ is constructed from $u$ by raising $x$ in $i$’s ordering and leaving everything else unchanged, then

$$x \in G(u') \subseteq G(u)$$

Raising $x$ causes $x$ to be chosen again, but does not allow new alternatives to be chosen that weren’t chosen before. The constant rule $G(u) = X$, Pareto, Borda, the plurality correspondence, and $T$ all satisfy strong monotonicity.

Example 5, however, fails monotonicity in a significant way. If $z$, which is in $G(u^*)$, is raised one rank in profile $u^*$ for #2, then $G$ maps the resulting profile to $\{x, y, z, w\}$ and a new alternative, $w$, has been introduced to the choice set. We now show there is no way to incorporate monotonicity without forcing $G = G_P$.

In this and the next section, we will want to show that when a social choice correspondence $G$ satisfies certain properties it is $G_P$. If $G$ satisfies the Pareto condition, all alternatives that are not Pareto optimal at any profile $u$ are excluded from $G(u)$. What remains is to show that every alternative that is Pareto optimal at $u$ is contained in $G(u)$. That leads us to consider the possibility of alternatives $w$ in $G_P(u) \setminus G(u)$.

Given $G$, suppose that there exist profiles where some Pareto optimals are not chosen by $G$. Consider the non-empty collection $\mathcal{C} \subseteq L(X)^N$ of all profiles for which there exists at least one Pareto optimal alternative that is not chosen. For profile $v \in \mathcal{C}$, when there exists at least one individual $i$ and alternative $w$ such that $w = v(i)[t]$ and $w \in G_P(v) \setminus G(v)$ but there does not exist an $s < t$, an individual $j$ and alternative $y$ such that $y = v(j)[s]$ and $y \in G_P(v) \setminus G(v)$, set the **height** at $v$ as $h(v) = t$. While $h(u)$ is uniquely determined from $G$ and $u$, that is not true for the specification of individuals and alternatives. For example, a profile $u$ might have $h(u) = 2$, both because #1 has unchosen but Pareto optimal $x$ in second rank and #2 has unchosen but Pareto optimal $y$ in second rank. Of all these values of $h(v)$ for profiles $v$ in $\mathcal{C}$, let $H(G)$, the **height of $G$**, be the minimum $h(v)$ (corresponding to a highest ranked alternative from $G_P(v) \setminus G(v)$).

Suppose that for correspondence $G$, the height $H(G)$ is defined. Consider the possibility that $H(G) = m$. This requires in part that there is a profile $u$ and alternative $x$ such that:

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3This is the same as correspondence monotonicity in Moulin (1983, p. 35).
(1) $x$ is Pareto optimal at $u$;
(2) $x$ is not chosen;
(3) everyone has $x$ in rank $m$.
But (3) contradicts (1). Therefore,
$$H(G) \leq m - 1.$$ 

With $n > 2$, the plurality correspondence has height 1 as does the Copeland correspondence. For $m > 2$, height $H(G) > 1$ if and only if $G$ satisfies tops-in\[4\]. The correspondence $T$ has height 2. Height is not defined for $G_P$ or the constant rule $G(u) = X$, since those correspondences always choose all Pareto optimals.

Height is used in this section just as an organizational device in the proof of Theorem 3. But in the next section it is central to the proof of Theorem 4.

**Theorem 3.** For $m = 4$ and $n \geq 2$, let $G : L(X)^N \rightarrow 2^X \setminus \{\emptyset\}$ be a social choice correspondence satisfying all of:
1. The Pareto condition;
2. Tops-in;
3. Balancedness;
4. Strong monotonicity;
then $G = G_P$, the Pareto correspondence.

**Proof:** The necessity of these conditions is obvious.

Now assume that $G$ satisfies all four conditions. We need to show that if $w$ is Pareto optimal at $u$, then $w \in G(u)$. Assume otherwise; then $H(G)$ is defined. Height $H(G) = 1$ is excluded by tops-in and $H(G) = 4$ is excluded by the Pareto condition. Two possibilities then remain.

**Case 1.** $H(G) = 2$. Suppose that $w \in G_P(u) \setminus G(u)$ is in #1’s second rank at $u$ given in part by:

|   | 1 | 2 | ... |
|---|---|---|-----|
|   | x |   |     |
|   | w |   |     |
|   |   | w |     |
|   |   |   | x   |

where $x$ is at #1’s top (so $x \in G(u)$ by tops-in). Some individual, say #2, has $w \succ_2 x$ since $w$ is Pareto optimal. Construct $u'$ from $u$ by raising $x$ up to just below $w$ for #2; $w$ remains Pareto optimal. Now $x \in G(u') \subseteq G(u)$

\[4\]If $m = 2$, then $H(G) > 1$ is not possible.
by monotonicity. In particular, \( w \notin G(u') \). By balancedness, \( G(u'') = G(u') \) where \( u'' \) is obtained from \( u' \) by transposition pair \((x, w)\) via \#1 and \#2. So \( w \notin G(u'') \). But that contradicts tops-in since, at \( u'' \), \#1 now has \( w \) top-ranked.

**Case 2.** \( H(G) = 3 \). Since \( w \) is Pareto optimal, it must be ranked higher than each of the other alternatives; without loss of generality, \( u \) is, in part:

|   | 1 | 2 | 3 | ... |
|---|---|---|---|-----|
| w | w | w | ... |
| x | y | z |

Take the alternative just above \( w \) for \#1 to be say, \( y \). Then construct \( u^* \) from \( u \) by transposition pair \((y, w)\) via \#1 and \#2. By balancedness, \( w \notin G(u^*) \) where \( w \) is Pareto optimal at \( u^* \) and ranked second by \#1 contrary to our assumption that \( H(G) = 3 \). \( \square \)

For examples showing the need for each condition in Theorem 3 (with \( n = 3 \) and \( m = 4 \)), we first observe that Example 5 exhibits a rule different from \( G_P \) satisfying all conditions of Theorem 3 other than monotonicity. And \( G(u) = X \) for all profiles \( u \) is a rule different from \( G_P \) satisfying all conditions of Theorem 3 except Pareto. (Another: set \( G(u) = G_P(u) \) except at profiles \( u \) where everyone has the same top and the same second-ranked alternative, at such profiles, \( G(u) \) is the set consisting of those two alternatives.) For the others:

**Example 6.** A rule different from \( G_P \) satisfying all conditions of Theorem 3 except tops-in: Fix one alternative \( t \), and then set \( G(u) = G_P(u) \setminus \{t\} \) unless \( G_P(u) = \{t\} \), in which case set \( G(u) = G_P(u) = \{t\} \).

**Example 7.** A rule different from \( G_P \) satisfying all conditions of Theorem 3 except balancedness: \( G(u) = T(u) \).

A variant of Theorem 3 will appear as Theorem 5 near the end of the next section.

### 6 Characterization with \( m \geq 5 \)

When there are more than four alternatives, the properties of Theorem 3, namely Pareto, tops-in, balancedness, and strong monotonicity, are not sufficient to characterize the Pareto correspondence.

**Example 8.** Let \( X = \{x, y, z, w, t\} \) and \( n = 2 \). Define social choice correspondence \( G \) as follows. First we identify a subdomain \( D \) of \( L(X)^N \) that consists of just the two fixed profiles
At these profiles in $D$, set $G(u) = G(u^*) = \{x, z\}$, the top-most alternatives (thus $w$ is not chosen even though it is Pareto optimal at these profiles). For all profiles $v$ in $L(X)^N \setminus D$, set $G(v) = G_P(v)$.

Clearly $G$ satisfies the Pareto condition and tops-in. For balancedness, observe that there does not exist a transposition pair at either of the profiles in $D$. Accordingly, if $v$ is obtained from $u$ by pairwise transposition, both $u$ and $v$ are in $L(X)^N \setminus D$ where $G(u) = G_P(u)$ and $G(v) = G_P(v)$. Since $G_P$ satisfies balancedness, so does $G$.

All that remains is monotonicity. If $v$ and $u$ are both in $L(X)^N \setminus D$, and $v$ arises from $u$ by raising a chosen alternative $x$, then, because $G = G_P$ there, and $G_P$ satisfies monotonicity, we cannot have a violation of monotonicity by $G$. Neither profile in $D$ can arise from the other by raising a chosen alternative. So, if $G$ fails monotonicity, it has to be because raising a chosen alternative takes you from $D$ into $L(X)^N \setminus D$ or from $L(X)^N \setminus D$ into $D$.

From $D$ to $L(X)^N \setminus D$: Suppose that $v$ is constructed by raising $x$ at $u$ where $G(u) = \{x, z\}$ (all other cases are dealt with by simple analogs of this argument). This must be for individual #2 and raising $x$ means that $x$ will now Pareto dominate $w$. Since $y$ and $t$ remain Pareto dominated, $G(v) = G_P(v)$ will be $\{x, z\}$ or $\{x\}$ (if $x$ is raised to #2’s top). In either case, we have $x \in G(v) \subseteq G(u)$.

From $L(X)^N \setminus D$ to $D$: Suppose that $u \in D$ is constructed by raising an alternative from a profile $q$ in $L(X)^N \setminus D$. So construct $q$ from $u$ by lowering an alternative for someone. If $y$ or $t$ is lowered, it remains Pareto dominated and so not in $G(q)$. If $w$ is lowered, it becomes Pareto dominated and so not in $G(q)$. If $x$ or $z$ is lowered and is chosen at $G(q)$, then we have e.g., $x \in G(u) \subseteq G(q)$.

In each case, strong monotonicity is confirmed. Note that $G$, while not neutral, is anonymous ($u^*$ is obtained from $u$ by a permutation on $N = \{1, 2\}$).

Our next example modifies Example 8 by allowing $n > 2$.

**Example 9.** Again we identify a subdomain $D$ of $L(X)^N$. We start from a list $C$ of eight orderings on $X = \{x, y, z, w, t\}$:

|   | 1 | 2 |
|---|---|---|
| $x$ | $y$ | $z$ |
| $w$ | $w$ | $y$ |
| $z$ | $x$ | $t$ |
| $t$ | $y$ | $w$ |

$u$: and $u^*$:

|   | 1 | 2 |
|---|---|---|
| $x$ | $z$ | $x$ |
| $y$ | $t$ | $y$ |
| $w$ | $w$ | $w$ |
| $z$ | $x$ | $y$ |
| $t$ | $w$ | $w$ |

Example 9. Again we identify a subdomain $D$ of $L(X)^N$. We start from a list $C$ of eight orderings on $X = \{x, y, z, w, t\}$:
The first two orderings in $C$ are the orderings in the profiles in the subdomain $D$ of Example 8. The next six orderings consist of the six possible ways of ordering \{x, y, z, t\} subject to $x \succ y$ and $z \succ t$ with $w$ then appended at the bottom. Subdomain $D$ consists of all profiles made up of just these orderings subject to the condition that each of the first two orderings occurs exactly once. For any profile $u \in D$, we set $G(u) = T(u) = \{x, z\}$. For all profiles $v$ in $L(X)^N \setminus D$, set $G(v) = G_P(v)$.

Clearly $G$ satisfies the Pareto condition and tops-in. It is straightforward to check that balancedness and strong monotonicity hold. Note that $G$ is anonymous (as any permutation of $N$ takes a profile in $D$ to another profile in $D$).

We next illustrate the need for the restriction that each of the first two orderings occurs exactly once. First, we need both to occur so that $w$ is Pareto optimal. To illustrate the need for the orderings to not occur more than once, suppose that $n = 3$ and consider the domain $D^*$ of profiles made up of the eight orderings above (without the restriction that each of the first two orderings occurs exactly once). Define the correspondence $G^*$ by $G^*(u^*) = T(u^*)$ if $u^* \in D^*$ and $G^*(u) = G_P(u)$ if $u \notin D^*$. Then look at the profiles

$$
\begin{array}{ccc}
1 & 2 & 3 \\
x & z & z \\
y & t & t \\
w & w & w \\
z & x & x \\
t & y & y \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
x & z & z \\
y & t & t \\
w & x & w \\
z & w & x \\
t & y & y \\
\end{array}
$$

$G(u) = \{x, z\}$ since $u \in D^*$; then $v$ is obtained by raising $x$ above $w$ for #2. But $G(v) = G_P(v) = \{x, z, w\}$, a violation of strong monotonicity.

Thus, to achieve a characterization of the Pareto correspondence for $m \geq 5$, we need another new condition. We introduce a correspondence analog of a property of social choice functions that are resolute, i.e., single alternatives are chosen at every profile (Campbell, Kelly, and Qi, 2018). A social choice function $g$ satisfies stability if for every pair of profiles, $u$ and $u^*$, and for every pair $x, y$ of alternatives, and every individual $i$, if $x = g(u)$ and, in ordering $u(i)$, alternative $y$ is adjacent to and just below $x$, then if $u^*$ is obtained from $u$ by only lowering $x$ to just below $y$ for $i$, we must have either $g(u^*) = x$ or $g(u^*) = y$. A small change in the profile results in a restricted set of possible outcomes. Here, where we are dealing with correspondences, we consider the same alteration of profiles, but allow somewhat different consequences, but still with a focus on "small changes cause small effects." $^5$

A social choice correspondence $G$ satisfies strong stability if for every pair of profiles, $u$ and $u^*$, and for every pair $x, y$ of alternatives, and every individual $i$, if $x \in G(u)$ and, in ordering $u(i)$,

\[\text{In (Campbell, Kelly, and Qi, 2018), it is observed that in Richard Bellman’s autobiography [1984, p. 181], he writes: "Change one small feature, and the structure of the solution was strongly altered. There was no stability."} \]
alternative $y$ is adjacent to and just below $x$, then if $u^*$ is obtained from $u$ by only lowering $x$ to just below $y$ for $i$, we must have exactly one of the following outcomes hold at $u^*$:

a. $G(u^*) = G(u)$;  
b. $G(u^*) = G(u) \setminus \{x\}$ where $x \in G(u)$;  
c. $G(u^*) = G(u) \cup \{y\}$ where $y \notin G(u)$.

If $G(u^*)$ differs from $G(u)$, it must either drop $x$ or add $y$, but not both (stability here is "strong" because of the "not both" requirement).

The Pareto correspondence $G_P$ satisfies the strong stability property. Consider profile $u$ with $x \in G_P(u)$, where $y$ is just below $x$ in $u(i)$, and $u^*$ differs from $u$ only in that $x$ is brought down just below $y$ in $i$’s ordering. If $z$ is any element of $X \setminus \{x, y\}$, then $z$ is dominated by an element at $u^*$ if and only if it is dominated by that same element at $u$. Hence $z \in G_P(u^*)$ if and only if $z \in G_P(u)$. So $G_P(u^*)$ can differ from $G_P(u)$ only by losing $x$ or gaining $y$. But if $y$ is gained, then $y \notin G_P(u)$. It must be that $y$ was Pareto dominated by $x$ at $u$. But then at $u^*$, all individuals other than $i$ still prefer $x$ to $y$ and so $x$ is still Pareto optimal at $u^*$, i.e., $G_P(u)$ can not both lose $x$ and gain $y$.

Clearly $G$ given by $G(u) = X$ for all $u$ satisfies strong stability. So does the following rule: Set $G(u) = X$ unless, at $u$, there is a common alternative $t$ at everyone’s bottom rank; then set $G(u) = X \setminus \{t\}$.

Many rules fail strong stability. Dictatorship fails strong stability. $G(u) = T(u)$, the Borda correspondence, and the plurality correspondence all fail strong stability. Also, the correspondences of Examples 8 and 9 do not satisfy strong stability. For example, in Example 8, consider profile $u \in D$:

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
x & z & \\
y & t & \\
w & w & \\
z & x & \\
t & y & \\
\hline
\end{array}
$$

where $G(u) = \{x, z\}$. Construct $u^* \notin D$ from $u$ by bringing $x$ down just below $y$ for individual 1. $G(u^*) = G_P(u) = \{x, y, z, w\}$, a violation of strong stability.

That strong stability fails for plurality and $T$ is seen at profile $u$:

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
x & y & z \\
a & : & : \\
w & : & : \\
\hline
\end{array}
$$
Bringing $x$ down just below $a$ for #1 both loses $x$ and gains $a$. Borda and Copeland also fail strong stability; consider profile $u$:

|   | 1  | 2  | 3  |
|---|----|----|----|
| x | x  | y  | x  |
| y | y  | x  |
|   |   | y  |

Bringing $x$ down just below $y$ for #1 both loses $x$ and gains $y$.

Note that strong stability does not imply strong monotonicity. Consider the correspondence that always selects individual #1’s bottom-ranked alternative. Strong monotonicity treats raising a chosen $y$ above anything. Strong stability only treats raising (any) $y$ above an adjacent chosen $x$.

We will employ strong stability to complete a characterization of the Pareto correspondence with five or more alternatives.

Suppose $G$ is a correspondence satisfying balancedness. Let $x$ be an alternative, $u$ a profile with $x$ Pareto optimal but $x \not\in G(u)$, and $i$ an individual with $x$ in rank $k$ where $x \not\in G(u)$. The following two propositions are inconsistent:

A. $H(G) = k$;

B. There is an alternative $\alpha$ adjacent to and just above $x$ for individual $i$ and adjacent to and just below $x$ for some other individual $j$.

If both were true at $u$, construct profile $u^*$ by transposing $x$ and $\alpha$ for individuals $i$ and $j$. Then $x$ is also Pareto optimal at $u^*$ and, by balancedness, $x \not\in G(u^*)$. But then $H(G) < k$.

We will use this fact in two different ways:

(i) If we have assumed $H(G) = k$, then we must exclude the existence of a profile with an alternative $\alpha$ adjacent to and just above $x$ (with $x$ in rank $k$) for individual $i$ and adjacent to and just below $x$ for some other individual $j$.

(ii) If we have a profile for which there exists an alternative $\alpha$ adjacent to and just above Pareto optimal but not chosen $x$ (in rank $k$) for individual $i$ and adjacent to and just below $x$ for some other individual $j$, we must reject $H(G) = k$.

Either of these two uses will be called a **height argument**.

Note that in the following we do not assume monotonicity.

**Theorem 4.** For $m \geq 5$ and $n \geq 2$, let $G : L(X)^N \rightarrow 2^X \setminus \{\emptyset\}$ be a social choice correspondence satisfying all of:

1. The Pareto condition;
2. Tops-in;
3. Balancedness;
4. Strong stability; then \( G = G_P \), the Pareto correspondence.

**Proof:** The necessity of these conditions is obvious.

Now assume that \( G \) satisfies all four conditions but \( G \neq G_P \). Then \( H(G) \) is defined; say \( H(g) = k \) for \( 1 \leq k \leq m \). So there exists a profile \( u \) and alternative \( w \) such that \( w \) is Pareto optimal, \( w \) is not chosen, and \( w \) is in the \( k \)th rank for, say, individual #1. We will show that each possible value of \( k \) leads to a contradiction. Some values of \( H(G) \) are easily dealt with. \( H(G) = 1 \) violates tops-in while \( H(G) = m \) violates the Pareto condition. But we seek a more general, systematic approach to deal with all possible values of \( k \).

Without loss of generality, suppose profile \( u \) is given by

\[
\begin{array}{cccc}
1 & 2 & \cdots \\
| & | & | \\
a & b & \cdots \\
| & | & | \\
\vdots & c & w & \cdots \\
| & | & | \\
d & w & \cdots \\
| & | & | \\
\vdots & e & \cdots \\
\end{array}
\]

with \( w \) in the \( k \)th rank for #1. We will methodically work down the top set of alternatives in #1’s ranking in order from \( a \) to \( c \).

We know \( a \in G(u) \) by tops-in.

**Case 1.** Some \( i > 1 \) has \( a \succ_i^u w \).

Without loss of generality, we take this to be #2

\[
\begin{array}{cccc}
1 & 2 & \cdots \\
| & | & | \\
a & b & \cdots \\
| & | & | \\
\vdots & c & a & \cdots \\
| & | & | \\
d & w & \cdots \\
| & | & | \\
\vdots & e & \cdots \\
\end{array}
\]

**Step 1.** If \( a \) is at #2’s top, go to Step 2. Otherwise there is a \( p \) just above \( a \) for #2; construct \( u' \) from \( u \) by raising \( a \) just above \( p \). By tops-in, \( a \) is still chosen. Also \( w \) is still Pareto optimal and not chosen (otherwise going from \( u' \) to \( u \) would violate strong stability). Do this until \( a \) is at the top of #2’s order.
with \( w \) still Pareto optimal but not chosen.

**Step 2.** If, for \#1, \( a \) is not adjacent to \( w \), bring \( a \) down one rank. Then \( a \) is still chosen by \#2 having \( a \) at the top with \( w \) still Pareto optimal but not chosen (otherwise going back would violate strong stability). Do this until \( a \) is adjacent and just above \( w \) for \#1. Call the resulting profile \( u^* \).

**Step 3.** Since \( w \) is Pareto optimal, someone, say \#3, must have \( w \succ u^*_3 a \). If \( a \) is not adjacent to \( w \) for \#3, raise \( a \) one rank to get profile \( u^*_1 \). By tops-in, \( a \) is still chosen. Continue in this way until \( a \) is adjacent to and just below \( w \) for \#3 with \( w \) still Pareto optimal but not chosen (otherwise going back would violate strong stability). Then a height argument for \( a \) and \( w \) for \#1 and \#3 shows \( H(G) = k \) is violated.

**Case 2.** For all \( i > 1 \), \( w \succ u^*_i a \).

For this case, we shift attention to \( b \), the next lower alternative in \#1’s ranking. If \( k = 2 \), go to the argument right after Subcase 2-2B. Assuming then \( k \geq 3 \), so \( b \) is above \( w \), there are several possibilities:

**Subcase 2-1.** \( b \) is not Pareto optimal. Only \( a \) can Pareto dominate \( b \) and, since for all \( i > 1 \), \( w \succ u^*_i a \), we also have all \( i > 1 \), \( w \succ u^*_i b \).

**Subcase 2-2.** If \( b \) is Pareto optimal then because \#1 ranks \( b \) above position \( H(G) \), alternative \( b \) must be chosen. Then we divide our analysis just as we did for \( a \).

**Subcase 2-2A.** Suppose some \( i > 1 \) has \( b \succ u^*_i w \). Without loss of generality, we take this to be \#2:

|   | 1 | 2 | ... |
|---|---|---|-----|
| a |   |   |     |
| b |   |   |     |
| . | . |   |     |
| c |   | b |     |
| w |   | w |     |
| d | w |   |     |
| . |   | . | a   |
| e | a |   |     |

\( w \):

If \( b \) is not \#2’s top, raise \( b \) one rank. Then \( b \) is still Pareto optimal and \( b \) is chosen because \#1 ranks \( b \) above \( H(G) \). Alternative \( w \) is still Pareto optimal. More, \( w \) is not chosen; for if it were then bringing chosen \( b \) back down one rank would violate strong stability. Continue in this way, raising \( b \) to \#2’s top.

Then, just as in Step 2 above, bring \( b \) down for \#1 until it is just above \( w \). At the resulting profile, \( b \) is chosen by tops-in (via \#2) while \( w \) is Pareto optimal but not chosen. Since \( w \) is Pareto optimal, someone, say \#3, has \( b \) below \( w \). If \( b \) is not adjacent to \( w \) for \#3, raise \( b \) one rank to get profile \( u^{**} \).
stops in only a very few possible ways. Since \( b \) is Pareto optimal and \#1 ranks \( b \) above position \( H(G) \), we see \( b \) is still chosen. Continue in this way until \( b \) is adjacent to and just below \( w \) for \#3. Then a height argument shows \( H(G) = k \) is violated.

**Subcase 2-2B.** Otherwise for all \( i > 1, w \succ_i b \).

Continue in this fashion, for each alternative \( t \) that \#1 ranks above \( w \). For each alternative we find a contradiction from a height argument or we learn that for everyone other than \#1 that alternative is ranked below \( w \). This procedure stops in only a very few possible ways.

1. If some alternative \( t \) ranked above \( w \) by \#1 is chosen and someone else has \( t \) ranked above \( w \), then we move \( t \) around in ways that lead to a height argument contradicting \( H(G) = k \). In particular this has to happen if \( H(G) - 1 > m - H(G) \).

2. If \( H(G) - 1 = m - H(G) \) and we haven’t run into a contradiction from a height argument, then every \( i > 1 \) has all of \( \{a, b, \ldots, c\} \) ranked below \( w \) and everything in \( \{d, \ldots, e\} \) ranked above \( w \) with everyone ranking \( w \) in position \( k \). Then \#2’s top, say \( d \), will be ranked above \( w \) by someone else (when \( n \geq 3 \); the case \( n = 2 \) can be dealt with straightforwardly) and then an argument like that for \( a \) will lead to a contradiction of \( H(G) = k \).

3. If \( H(G) - 1 < m - H(G) \) and no inconsistency had been found, raise \( a \) up to just below \( w \) for some \( i > 1 \). Next lower \( a \) to just above \( w \) for \#1 as before; at each step in this process, alternative \( a \) is ranked above all of \( \{b, \ldots, c\} \) by individual \( i \) and ranked above all of \( \{w, d, \ldots, e\} \) for \#1, so \( a \) always remains Pareto optimal. Since, for \#1, \( a \) appears above the \( k \)th rank, it can not be chosen or \( H(G) = k \) would be violated. Finally apply a height argument. \( \square \)

Regarding the need for each condition, Example 9 shows a rule different from \( G_P \) satisfying all conditions of Theorem 4 except strong stability. A rule different from \( G_P \) satisfying all conditions of Theorem 4 except Pareto is: \( G(u) = X \) for all profiles \( u \). For tops-in consider the next example.

**Example 10.** \( m \geq 5, n > m \). \( G(u) = G_P(u) \setminus \{x : \text{for some } y, \text{ exactly } n - 1 \text{ of the } n \text{ individuals prefer } y \text{ to } x\} \). (Since \( n > m \), at profile \( u \) there is at least one alternative at the top for at least two individuals. This alternative is Pareto optimal and can’t lose \((n - 1)\)-to-1 to any alternative. Thus \( G(u) \) is non-empty.) This clearly satisfies Pareto (and anonymity and neutrality). It fails tops-in as \( x \) could be \#1’s top and everyone else’s bottom and thus lose in some \((n - 1)\)-to-1 votes. For balancedness, construction of \( u^* \) by an interchange of transposition pairs from \( u \) isn’t possible if one Pareto dominates the other and doesn’t change any \((n - 1)\)-to-1 vote; \( G(u^*) = G(u) \). For stability, suppose that \( u^* \) is constructed from \( u \) by bringing \( x \) down just below adjacent \( y \). Can we have \( G(u^*) = G(u) \setminus \{x\} \) where \( x \in G(u) \)? It can’t be because now \( y \) Pareto dominates \( x \) for then previously \( y \) beat \( x \) by \((n - 1)\)-to-1 and \( x \in G(u) \) would not be true. But it could be that at \( u \), alternative \( x \) lost to \( y \) by \((n - 1)\)-to-1
but now only $n-2$-to-$2$. But then for $y$ to move into $G(u)$ it would have to have been excluded at $u$. This requires a $z \neq x$ such that at least $n-1$ individuals prefer $z$ to $y$ at $u$; but then this must also be true at $u^*$ so $y$ can’t move into $G(u^*)$.

For balancedness, consider the next correspondence which is also anonymous and neutral:

**Example 11.** Assume that $n = 4$ and $m = 5$ (or more generally, $n = m-1$).

Partition $L(X)^N$ as $D \cup D^*$ where $D^*$ is the set of profiles satisfying the following three properties:

1. $u \in D^*$ implies there is a (unique) alternative $\phi(u)$ that in everyone’s ordering at $u$ has $\phi(u)$ in the next-to-last rank;
2. For every alternative in $u \setminus \{\phi(u)\}$, one person has that alternative ranked at the bottom (just below $\phi(u)$);
3. Every alternative is Pareto optimal at $u$ (note $\phi(u)$ is already guaranteed to be Pareto optimal by property 2);

and $D = L(X)^N \setminus D^*$.

Define collective choice correspondence $G$ as follows:

If $u \in D$, then $G(u) = G_P(u)$, the Pareto optimals at $u$;
If $u \in D^*$, then $G(u) = G_P(u) \setminus \{\phi(u)\} = X \setminus \{\phi(u)\}$.

Note that $G \neq G_P$, as at every $u \in D^*$, $\phi(u) \in G_P(u) \setminus G(u)$.

Observe that $G$ fails balancedness. Let $u$ be a profile in $D^*$ with $\phi(u) = x$, say. Let $y$ be the alternative ranked immediately above $x$ in #1’s ordering (nothing special about #1 here). Some other individual, say $i$, has $y$ ranked immediately below $x$. Construct $u'$ from $u$ by transposition pair $(x,y)$ via 1 and $i$. Profile $u' \in D$, so $G(u) = G_P(u) = x \neq G(u') = X \setminus \{x\}$.

That this correspondence $G$ satisfies tops-in and Pareto is easy to see. Strong stability can be established by a detailed case-by-case analysis.

Now that we have defined strong stability we can alter Theorem 3 by substituting stability for monotonicity.

**Theorem 5.** For $m = 4$ and $n \geq 2$, let $G : L(X)^N \to 2^X \setminus \{\emptyset\}$ be a social choice correspondence satisfying all of:

1. The Pareto condition;
2. Tops-in;
3. Balancedness;
4. Strong stability;

then $G = G_P$, the Pareto correspondence.

Proof: Suppose $w = \{a, b, c, w\}$ so that $m = 4$ and that $G$ is a correspondence satisfying the Pareto condition, tops-in, balancedness and the strong stability
condition. We need to show that if alternative \( w \) is Pareto-optimal at \( u \), then it is in \( G(u) \). Suppose otherwise, i.e., there exists a profile \( u \in L(w)^N \) for which \( w \) is Pareto optimal at \( u \) but \( w \notin G(u) \). Then \( H(G) \) is defined. We show that each possible value of \( H(G) \) leads to a contradiction.

1. \( H(G) = 1 \) is ruled out by tops-in.

2. \( H(G) = 4 \) is ruled out by the Pareto condition.

3. Suppose \( H(G) = 3 \). Then with a possible renumbering of individuals and relabeling of alternatives, we have \( n \geq 3 \) and

\[
\begin{array}{ccc|c}
1 & 2 & 3 & \cdots \\
\hline
\vdots & \vdots & \vdots & \\
\hline
w & w & w & \cdots \\
\hline
a & b & c & \end{array}
\]

The alternative adjacent to but just above \( w \) for \#1 is either \( b \) or \( c \). But then a height argument shows a violation of \( H(G) = 3 \).

4. Finally, suppose \( H(G) = 2 \). There exists a profile \( u \) with \( w \) in say \#1’s second rank (with \( a \neq w \) top-ranked), \( w \) Pareto optimal at \( u \) but \( w \notin G(u) \).

\[
\begin{array}{cc|c}
1 & 2 & \cdots \\
\hline
a & \vdots & \\
\hline
w & \vdots & \cdots \\
\end{array}
\]

Because \( w \) is Pareto optimal, someone, say \#2, must prefer \( w \) to \( a \). And \( a \) can’t be just below \( w \) for \#2, for if it were, we could construct \( u^* \) by transposing \( a \) and \( w \) for \#1 and \#2, resulting in \( w \) at \#1’s top but, by balancedness, \( w \notin G(u^*) \), a violation of tops-in. Without loss of generality, \#2 has \( b \) between \( w \) and \( a \):

\[
\begin{array}{ccc|c}
1 & 2 & \cdots \\
\hline
a & c & \vdots & \\
\hline
w & w & b & \cdots \\
\hline
\end{array}
\]

Now \( c \) can not be adjacent to but just below \( w \) for \#1, or a transposition of \( w \) and \( c \) for \#1 and \#2 would generate a profile with \( w \) at \#2’s top but \( w \) not chosen. So we must have:

\[
\begin{array}{ccc|c}
1 & 2 & \cdots \\
\hline
a & c & \vdots & \\
\hline
w & w & b & \cdots \\
\hline
b & a & \end{array}
\]
But then look at profile

|   | 1 | 2 | ··· |
|---|---|---|-----|
|   | a | c |     |
| w | w |   |     |
| c | b |   |     |
| b | a |   |     |

Alternative $c \in G(u')$ by tops-in. Profile $u$ can be obtained from $u'$ by lowering chosen $c$ just below $b$ for #1. By strong stability, this can not cause $w$ be kicked out of $G(u')$. But $w \notin G(u)$, so we must have had $w \notin G(u')$. But then a height argument shows a contradiction with $H(G) = 2$. □

7 Final remarks

A perhaps unusual feature of the results in this paper is the use of the Pareto condition in characterizing the Pareto social choice correspondence. We make four observations regarding this:

1. What might at first appear to be an excessively strong condition is not sufficient for five or more alternatives even when supplemented with balancedness, tops-in, anonymity, and neutrality.

2. As noted earlier, the condition of excluding dominated alternatives is actually extremely weak and has been used in characterizing a wide variety of standard social choice correspondences, e.g., Borda, plurality voting, and the Copeland correspondence.

3. It helps to compare with an analogous use of a plurality condition. Suppose that we wanted to characterize plurality voting by using a plurality condition that excludes all alternatives that are not plurality winners and then added enough additional conditions to ensure that all plurality winners are included. This plurality condition would seem quite artificially constructed, solely for the purpose of the one characterization theorem. That’s quite different from the Pareto condition. Almost all standard social choice correspondences fail the plurality condition.

4. Something like the Pareto condition is required. Without Pareto, we can get rules far from $G_P$: As seen by $G(u) = X$, all our other conditions combined, balancedness, monotonicity, and tops-in, plus anonymity, neutrality, and strong stability are insufficient to rule out correspondences that differ from $G_P$. And we can get rules very far from $G(u) = X$ as well, even very close.

---

6One reader has suggested that we can simply characterize the Pareto correspondence as the coarsest rule satisfying the Pareto condition. Of course, in the same way, the plurality correspondence is the coarsest rule satisfying the plurality condition; and the Borda correspondence is the coarsest rule satisfying a Borda condition. None of these "characterizations" really involve social choice properties (like monotonicity, tops-in, stability, etc.) that relate social outcomes to individual preferences. "Coarseness" is not a social choice property.
Example 12. Set $G(u) = G_P(u)$ except for those profiles with complete agreement: $u(i) = u(j)$ (not just same top alternatives) for all $i$ and $j$. At profiles of complete agreement, set $G(u)$ to be the set consisting of everyone’s top two alternatives (although the common second is Pareto-dominated). This social choice correspondence satisfies balancedness, monotonicity, and tops-in, plus anonymity, neutrality, and strong stability.

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