Riemann Poisson manifolds and Kähler-Riemann foliations

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Abstract.
Riemann Poisson manifolds were introduced by the author in [1] and studied in more details in [2]. Kähler-Riemann foliations form an interesting subset of the Riemannian foliations with remarkable properties (see [3]).

In this paper we will show that to give a regular Riemann Poisson structure on a manifold $M$ is equivalent to to give a Kähler-Riemann foliation on $M$ such that the leafwise symplectic form is invariant with respect to all local foliate perpendicular vector fields. We show also that the sum of the vector space of leafwise cohomology and the vector space of the basic forms is a subspace of the space of Poisson cohomology.

Key words. Poisson manifold, Riemannian foliation.

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1 Regular Riemann Poisson manifolds

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman’s monograph [7].

Let $P$ be a Poisson manifold with Poisson tensor $\pi$. We have a bundle map $\pi : T^* P \longrightarrow TP$ defined by

$$\beta(\pi(\alpha)) = \pi(\alpha, \beta), \quad \alpha, \beta \in T^* P.$$ (1)

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On the space of differential 1-forms $\Omega^1(P)$, the Poisson tensor induces a Lie bracket

$$\left[ \alpha, \beta \right]_\pi = L_{\pi(\alpha)} \beta - L_{\pi(\beta)} \alpha - d(\pi(\alpha, \beta)) = i_{\pi(\alpha)} d\beta - i_{\pi(\beta)} d\alpha + d(\pi(\alpha, \beta)).$$

(2)

For this Lie bracket and the usual Lie bracket on vector fields, the bundle map $\pi$ induces a Lie algebra homomorphism $\pi : \Omega^1(P) \rightarrow \mathcal{X}(P)$:

$$\pi([\alpha, \beta]_\pi) = [\pi(\alpha), \pi(\beta)].$$

(3)

Let $\langle , \rangle$ be a Riemannian metric on the contangent bundle $T^*P$. The Levi-Civita contravariant connection $D$ associated with the couple $(\pi, \langle , \rangle)$ is defined by

$$2 \langle D_\alpha \beta, \gamma \rangle = \pi(\alpha). \langle \beta, \gamma \rangle + \pi(\beta). \langle \alpha, \gamma \rangle - \pi(\gamma). \langle \alpha, \beta \rangle + \langle [\alpha, \beta]_\pi, \gamma \rangle + \langle [\gamma, \alpha]_\pi, \beta \rangle + \langle [\gamma, \beta]_\pi, \alpha \rangle$$

(4)

where $\alpha, \beta, \gamma \in \Omega^1(P)$.

The notion of contravariant connection was introduced by Vaisman in [7] and studied in more details by Fernandes in [4].

The Levi-Civita contravariant connection $D$ associated with the couple $(\pi, \langle , \rangle)$ satisfies:

$$D_\alpha \beta - D_\beta \alpha = [\alpha, \beta]_\pi;$$

(5)

$$\pi(\alpha). \langle \beta, \gamma \rangle = \langle D_\alpha \beta, \gamma \rangle + \langle \beta, D_\alpha \gamma \rangle .$$

(6)

**Definition 1.1** With the notations above, the triple $(P, \pi, \langle , \rangle)$ is called a Riemann Poisson manifold if, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$D\pi(\alpha, \beta, \gamma) := \pi(\alpha). \pi(\beta, \gamma) - \pi(D_\alpha \beta, \gamma) - \pi(\beta, D_\alpha \gamma) = 0.$$

(7)

A regular Riemann Poisson manifold is a Riemann Poisson manifold with regular symplectic foliation.
Since $D$ has vanishing torsion and since the contravariant exterior differential $d_S$ associated with the bracket $[,]_\pi$ is given by $d_\pi = -[\pi, .]_S$, we can deduce obviously that, for any $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$-[\pi, \pi]_S(\alpha, \beta, \gamma) = D\pi(\alpha, \beta, \gamma) + D\pi(\beta, \gamma, \alpha) + D\pi(\gamma, \alpha, \beta).$$  \hspace{1cm} (8)

$[,]_S$ denote the Shouten bracket.

Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. We denote $TS$ the involutif distribution tangent to the symplectic foliation.

We have

$$T^*P = \text{Ker}\pi \oplus (\text{Ker}\pi)^\perp \hspace{1cm} (9)$$

where $(\text{Ker}\pi)^\perp$ is the orthogonal of $(\text{Ker}\pi)$ with respect to $<, >$.

The Riemannian metric gives an identification between the cotangent bundle $T^*P$ and the tangent bundle $TP$ which we denote $\#: T^*P \rightarrow TP$. We put $\mathcal{H} = \#(\text{Ker}\pi)$. We get

$$TP = TS \oplus \mathcal{H} \text{ and } \#((\text{Ker}\pi)^\perp) = TS. \hspace{1cm} (10)$$

The bundle map $\pi$ induces an isomorphism $\pi : (\text{Ker}\pi)^\perp \rightarrow TS$; we denote $\pi^{-1} : TS \rightarrow (\text{Ker}\pi)^\perp$ its inverse. The leafwise symplectic form $\omega$ belongs to $\Gamma(\wedge^2TS)$ and is given by

$$\omega(u, v) = \pi(\pi^{-1}(u), \pi^{-1}(v)), \hspace{1cm} u, v \in TS. \hspace{1cm} (11)$$

The following proposition gives some elementary properties of the Levi-Civita contravariant connection.

**Proposition 1.1** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. Let $D$ be the Levi-Civita contravariant connection associated with $(\pi, <, >)$. Then

1) $\pi(\beta) = 0 \Rightarrow \forall \alpha \in \Omega^1(P), \hspace{1cm} \pi(D_\alpha \beta) = 0$;

2) $\pi(\alpha) = 0 \Rightarrow D_\alpha = 0$;

3) If $\alpha, \beta \in \Gamma((\text{Ker}\pi)^\perp)$ then $D_\alpha \beta \in \Gamma((\text{Ker}\pi)^\perp)$ and $[\alpha, \beta]_\pi \in \Gamma((\text{Ker}\pi)^\perp)$.

**Preuve:**

1) Let $\alpha, \beta, \gamma \in \Omega^1(P)$ such that $\pi(\beta) = 0$. We have

$$\gamma[\pi(D_\alpha \beta)] = \pi(D_\alpha \beta, \gamma) = \pi(\alpha, \pi(\beta, \gamma) - D_\alpha \gamma[\pi(\beta)] = 0.$$

2) Let $\alpha, \beta \in \Omega^1(P)$ such that $\pi(\alpha) = 0$. We have $D_\alpha \beta = [\alpha, \beta]_\pi + D_\beta \alpha$. By 1) and (3) it follows that $\pi(D_\alpha \beta) = 0$. Now we show that $D_\alpha \beta \in \Gamma((\text{Ker}\pi)^\perp)$.  

3
Let $\gamma \in \Omega^1(P)$ such that $\pi(\gamma) = 0$. $\beta = \beta_1 + \beta_2$ with $\pi(\beta_1) = 0$ and $\beta_2 \in \Gamma((\text{Ker}\pi)^ot)$.

From (4) and the fact that if $\pi(\alpha) = 0$ and $\pi(\beta) = 0$ then $[\alpha, \beta]_\pi = 0$, we deduces that $<D_\alpha\beta_1, \gamma>$ = 0.

Finally, $<D_\alpha\beta_2, \gamma>$ = $- <\beta_2, D_\alpha\gamma>$ = 0 from (6) and 1).

3) Let $\alpha, \beta \in \Gamma((\text{Ker}\pi)^ot)$ and let $\gamma \in \Omega^1(P)$ such that $\pi(\gamma) = 0$. By (6) and 2) we have $<D_\alpha\beta, \gamma>$ = $- <\beta, D_\alpha\gamma>$ = 0. Hence $D_\alpha\beta \in \Gamma((\text{Ker}\pi)^ot)$ and since $[\alpha, \beta]_\pi = D_\alpha\beta - D_\beta\alpha$, we have also $[\alpha, \beta]_\pi \in \Gamma((\text{Ker}\pi)^ot)$. $\square$

**Proposition 1.2** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. We have

\[ L_X\pi(\alpha, \beta) = 0, \tag{12} \]

for any $\alpha, \beta \in \Gamma((\text{Ker}\pi)^ot)$ and for any $X$ tangent to $\mathcal{H}$.

**Preuve:** For any $\alpha, \beta \in \Gamma((\text{Ker}\pi)^ot)$ and for any $X$ tangent to $\mathcal{H}$, we have $[\alpha, \beta]_\pi \in \Gamma((\text{Ker}\pi)^ot)$ and then $[\alpha, \beta]_\pi(X) = 0$. A straightforward calculation gives the relation

\[ [\alpha, \beta]_\pi(X) = L_X\pi(\alpha, \beta) \tag{13} \]

and the proposition follows. $\square$

We recall the definition of a foliate vector field. A vector field $X \in \mathcal{X}(P)$ is said to be foliate if, for all $Y$ tangent to $TS$, $[X, Y]$ is tangent to $TS$. This is equivalent to the fact that for any $\alpha \in \Omega^1(P)$, $[X, \pi(\alpha)]$ is tangent to $TS$.

### 2 The correspondence between regular Riemann Poisson manifolds and Kähler-Riemann foliations

We recall the definition of a Kähler-Riemann foliation (see [3]). A foliation $\mathcal{F}$ on a manifold is called a Kähler foliation if it is endowed with a complex structure $J$ and hermitien metric $h = S - 2i\omega$ on $T\mathcal{F}$ such that $d_J\omega = 0$. A Kähler foliation which is also a Riemannian foliation is called Kähler-Riemann foliation. Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold.
We define a Riemann metric on $TP$ by
\[
g(\#(\alpha), \#(\beta)) = \langle \alpha, \beta \rangle, \quad \alpha, \beta \in Ker\pi;
g(u, v) = \langle \pi^{-1}(u), \pi^{-1}(v) \rangle, \quad u, v \in TS;
g(u, \#(\alpha)) = 0, \quad \alpha \in Ker\pi, u \in TS.
\]

Let $S$ be a symplectic leaf. We denote $g_S$ and $\omega_S$ the restrictions of $g$ and $\omega$ to $S$. The Levi-Civita connection $\nabla^S$ of $g_S$ is given by
\[
\nabla^S_{\pi(\alpha)}\pi(\beta) = \pi(D(\alpha\beta)), \quad \alpha, \beta \in \Gamma((Ker\pi)^\perp)
\]
and we have
\[
\nabla^S\omega_S = 0.
\]
Furthermore, $S$ is a Kähler manifold (see [2]).

We denote $\Omega^0_b(P)$ the space of Casimir functions and $\Omega^1_b(P)$ the space of basic differential 1-forms. $\alpha \in \Omega^1_b(P)$ if and only if
\[
\pi(\alpha) = 0 \quad \text{and} \quad i_{\pi(\beta)}d\alpha = 0, \quad \forall \beta \in \Omega^1(P).
\]

We have from (2) and a careful verification that
\[
\alpha \in \Omega^1_b(P) \iff \forall \beta \in \Omega^1(P), \quad [\alpha, \beta]_\pi = 0.
\]

The following proposition gives an interesting characterization of the foliate vector fields which are tangent to $\mathcal{H}$.

**Proposition 2.1** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. If $\alpha \in \Gamma(Ker\pi)$, the following assertions are equivalent.

1) $\alpha$ is a basic 1-form.
2) $D\alpha = 0$.
3) $\#(\alpha)$ is a foliate vector field.
4) $L_{\#(\alpha)}\pi = 0$.

Furthermore, if $\alpha, \beta \in \Omega^1_b(P)$ then $\langle \alpha, \beta \rangle$ is a Casimir function.

**Preuve:** By (16) and Proposition 1.1 we have 1) $\iff$ 2).

$\#(\alpha)$ is a foliate vector field if and only if, for any $\beta \in \Gamma((ker\pi)^\perp)$, for any $x \in P$ and $\gamma_x \in Ker\pi_x$, $\gamma_x([\#(\alpha), \pi(\beta)]) = 0$. 

5
Let $\gamma \in \Gamma(\text{Ker}\pi)$. We have

\[
\gamma([\alpha, \pi(\beta)]) = -d\gamma(\alpha, \pi(\beta)) - \pi(\beta, \gamma(\alpha))
\]

\[
= -\pi(\beta, i_{\#(\alpha)}d\gamma) - \pi(\beta, d[\gamma(\alpha)]) - \pi(\beta, L_{\#(\alpha)}\gamma) = L_{\#(\alpha)}\pi(\beta, \gamma).
\]

Now, by (12) and since obviously $L_{\#(\alpha)}\pi(\beta, \gamma) = 0$ if $\gamma, \beta \in \Gamma(\text{Ker}\pi)$, we get $3) \iff 4$).

On the other hand, we have also

\[
\gamma([\alpha, \pi(\beta)]) = -d\gamma(\alpha, \pi(\beta)) - <D_{\beta}\alpha, \gamma> - <\alpha, D_{\beta}\gamma>.
\]

By Darboux theorem, for any $\gamma_x \in \text{Ker}\pi_x$ there is a local Casimir function $f$ such that $d_x f = \gamma_x$. This gives $2) \iff 3)$. If $\lambda, \beta$ are basic 1-forms, by (6) and 2) we have that $<\alpha, \beta>$ is a Casimir function. $\square$.

**Theorem 2.1** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. The symplectic foliation is a Kähler-Riemann foliation and $g$ is bundle-like metric.

**Preuve:** We have shown that the symplectic leaves are Kähler. We show now that $g$ is bundle-like. Following B. Reinhart [6], the metric $g$ is said to be bundle-like if it has the following property: for any open set $U$ in $P$ and for all vector fields $Y, Z$ that are foliate and perpendicular to the leaves, the function $g(Y, Z)$ is a basic function on $U$.

In our case, the perpendicular foliate vector fields are $Y = \#(\alpha)$ and $Z = \#(\beta)$ where $\alpha, \beta$ are basic 1-forms. Furthermore $g(Y, Z) = <\alpha, \beta>$ which is a Casimir function by Proposition 1.2 and the theorem follows. $\square$

**Remark.** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. If the Lie algebra of foliate vector fields is transitif on $P$ (which is the case if the symplectic foliation is transversally parallelizable) the symplectic leaves are symplectomorph.

Now we give the converse of Theorem 2.1.

**Theorem 2.2** Let $(P, F, g)$ be a differentiable manifold endowed with a Riemannian foliation $F$ and a bundle-like metric $g$. We suppose that there is $\omega \in \Gamma(\bigwedge^2 F)$ such that:
1) for any leaf $L$, the restriction of $\omega$ to $L$ is symplectic and parallel with respect to the Levi-Civita connection associated with the restriction of $g$ to $L$;  
2) for any local perpendicular foliate vector field $X$ and any couple $(U, V)$ of local vector fields tangent to $F$, 
\[ L_X \omega(U, V) = 0. \]

Then there is a Poisson tensor $\pi$ on $P$ and Riemann metric $<,>$ on $T^*P$ such that $(P, \pi, <,>)$ is a regular Riemann Poisson manifold whose symplectic foliation is $F$.

\textbf{Preuve:} We have
\[ TP = F \oplus F', \quad T^*P = F^o \oplus F'^o \]
where $F'$ is the distribution $g$-orthogonal to $F$ and $F^o = \{ \alpha \in T^*P/\alpha(F) = 0 \}$.

Denote $\# : T^*P \rightarrow TP$ the identification given by the Riemann metric $g$.

We have\[ \#(F^o) = F' \quad \text{and} \quad \#(F'^o) = F. \]

The leafwise symplectic form $\omega$ can be considered as a 2-form on $M$ by setting
\[ i_v \omega = 0 \]
for any $v \in F'$ and hence realizes an isomorphism $\omega : F \rightarrow F'^o, \quad v \mapsto \omega(v,.)$. Denote $\omega^{-1} : F'^o \rightarrow F$ its inverse.

Now, define a bivector $\pi$ by
\[ \pi(\alpha, \beta) = \begin{cases} \omega(\omega^{-1}(\alpha), \omega^{-1}(\beta)) & \text{if} \ \alpha, \beta \in F'^o, \\ 0 & \text{if} \ \alpha \text{ or } \beta \text{ belongs to } F^o, \end{cases} \]

and a Riemann metric on $T^*P$ by
\[ < \alpha, \beta > = \begin{cases} g(\omega^{-1}(\alpha), \omega^{-1}(\beta)) & \text{if} \ \alpha, \beta \in F'^o, \\ g(\#, \#) & \text{if} \ \alpha, \beta \in F^o, \\ 0 & \text{if} \ \alpha \in F'^o \text{ and } \beta \in F^o. \end{cases} \]

Now, we will compute the Levi-Civita contravariant connection $D$ associated with $(\pi, <,>)$. Before, we do some remarks on the Poisson bracket $[\ , \]_\pi$. For $\alpha, \beta, \gamma \in F'^o$ and $v$ a tangent vector to $F'$ in some $x$, a straightforward calculation gives
\[ [\alpha, \beta]_\pi(\omega^{-1}(\gamma)) = -d\omega[\omega^{-1}(\alpha), \omega^{-1}(\beta), \omega^{-1}(\gamma)] - \omega([\omega^{-1}(\alpha), \omega^{-1}(\beta)], \omega^{-1}(\gamma)). \]

\[ [\alpha, \beta]_\pi(v) = L_X \pi(\alpha, \beta) \]
where $X$ is any vector field such that $X_x = v$. If we choose $X$ a foliate perpendicular vector field, it’s easy to see that

$$L_X \pi(\alpha, \beta) = L_X \omega(\omega^{-1}(\alpha), \omega^{-1}(\beta)) = 0.$$ 

$d\omega$ also vanishes and we get

$$[\alpha, \beta]_{\pi}(\omega^{-1}(\gamma)) = -\omega([\omega^{-1}(\lambda), \omega^{-1}(\beta)], \omega^{-1}(\gamma)).$$

$$[\alpha, \beta]_{\pi}(v) = 0.$$

From this relation we can deduce that

$$D_{\alpha\beta} = \pi^{-1}(\nabla^L_{\pi(\alpha)}\pi(\beta))$$

where $\nabla^L$ is the Levi-Civita connection of the restriction of $g$ to a leaf $L$. If $\alpha, \beta \in F^o$ and $\gamma \in F^o$, we have

$$[\alpha, \gamma]_{\pi}(\pi(\beta)) = L_{\pi(\alpha)}\gamma(\pi(\beta)) = d\gamma(\pi(\alpha), \pi(\beta)) = -\gamma([\pi(\alpha), \pi(\beta)]) = 0$$

since $F$ is involutif.

A straightforward calculation gives

$$2 < D_{\alpha\beta}, \gamma > = \begin{cases} 0 & \text{if } \alpha \in F^o, \beta \in F^o, \gamma \in F^o. \\
L_{\pi(\beta)}g(\#(\alpha), \#(\gamma)) & \text{if } \alpha \in F^o, \beta \in F^o, \gamma \in F^o. \\
0 & \text{if } \alpha \in F^o, \beta \in F^o, \gamma \in F^o. \\
L_{\pi(\gamma)}g(\#(\alpha), \#(\beta)) & \text{if } \alpha \in F^o, \beta \in F^o, \gamma \in F^o. \end{cases}$$

As a characterization of a bundle-like metric, we have

$$L_{\pi(\gamma)}g(\#(\alpha), \#(\beta)) = 0 \quad \text{for} \quad \alpha, \beta \in F^o, \gamma \in F^o. \quad (17)$$

Now, it is easy to verify that $D\pi = 0$ and the theorem follows. $\Box$

### 3 The Poisson cohomology of a regular Riemann Poisson manifold

The Poisson cohomology of a Poisson manifold $(P, \pi)$ is the cohomology of the chain complex $(\mathcal{X}^p(P), d\pi)$ where, for $0 \leq p \leq dim P$, $\mathcal{X}^p(P)$ is the
\( C^\infty(P, \mathbb{R}) \)-module of \( p \)-multivector fields and \( d_\pi \) is given by

\[
d_\pi Q(\alpha_0, \ldots, \alpha_p) = \sum_{j=0}^{p} (-1)^j \pi(\alpha_j) Q(\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_p) + \sum_{i<j} (-1)^{i+j} Q([\alpha_i, \alpha_j]_\pi, \alpha_0, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_p).
\]

We denote \( H^*_\pi(P) \) the spaces of cohomology.

The leafwise cohomology of a manifold \( P \) endowed with a foliation defined by an involutive distribution \( F \) is the cohomology of the chain complex \((\Omega^p_F(P), d_F)\) where, for \( 0 \leq p \leq \text{rang} F \), \( \Omega^p_F(P) \) is the \( C^\infty(P, \mathbb{R}) \)-module of \( p \)-forms on the vector bundle \( F \) and \( d_F \) is given by

\[
d_F \omega(X_0, \ldots, X_p) = \sum_{j=0}^{p} (-1)^j X_j \omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).
\]

We denote \( H^*_F(P) \) the spaces of cohomology.

Let \((P, \pi, \langle, \rangle)\) be a Poisson manifold.

For \( 1 \leq p \leq \text{dim} P \), we consider the subspace \( X^p_0(P) \subset X^p(P) \) of \( p \)-multivector fields \( Q \) such that

\[ i_\alpha Q = 0 \quad \text{for any} \quad \alpha \in \text{Ker} \pi. \]

It’s easy to verify that \( d_\pi(X^p_0) \subset X^{p+1}_0 \). The natural injection \( X^p_0 \hookrightarrow X^p \) induces a linear map \( H^*(X^p_0) \rightarrow H^*_\pi(P) \) which is not injective in general.

Let \( \pi : \Omega^p_F(P) \rightarrow X^p_0(P) \) be the map given by

\[ \pi(\omega)(\alpha_1, \ldots, \alpha_p) = \omega(\pi(\alpha_1), \ldots, \pi(\alpha_p)). \]

It is easy to verify that \( \pi \) is an isomorphism and \( \pi(d_F \omega) = d_\pi \pi(\omega) \) and hence \( \pi \) induces an isomorphism

\[ \pi^* : H^p_F(P) \rightarrow H^p(X^p_0(P)) \]
Let $\mathcal{X}^p_P(\mathcal{P})$ be the space of $p$-multivector fields $Q$ such that

$$Q(\alpha_1, \ldots, \alpha_p) = 0 \quad \text{for any } \alpha_1, \ldots, \alpha_p \in (\ker\pi)^\perp.$$ 

We have

$$\mathcal{X}^p_P = \mathcal{X}^p_0(P) \oplus \mathcal{X}^p_1(P), \quad d_\pi(\mathcal{X}^p_i) \subset \mathcal{X}^{p+1}_i, \ i = 0, 1$$

and hence, for $1 \leq p \leq \dim P$,

$$H^p_\pi(P) = H^p(\mathcal{X}^0_\pi(P)) \oplus H^p(\mathcal{X}^*_1(P)).$$

The isomorphism (20) gives an injective linear map

$$\pi^* : H^p_\pi(P) \hookrightarrow H^p(P).$$

We consider now the space $\Omega^p_b(\mathcal{P})$ of basic $p$-forms. Let $\omega \in \Omega^p_b(\mathcal{P})$, we denote $\#(\omega)$ the $p$-multivector field in $\mathcal{X}^p_1(\mathcal{P})$ given by

$$\#(\omega)(\alpha_1, \ldots, \alpha_p) = \omega(\#(\alpha_1), \ldots, \#(\alpha_p)).$$

**Lemma 3.1** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. For any basic $p$-form $\omega$,

$$d_\pi \#(\omega) = 0.$$  

**Proof:** It’s obvious that $d_\pi \#(\omega)(\alpha_0, \ldots, \alpha_p) = 0$ if all the $\alpha_i$ belong to $\Gamma(\ker\pi)$. On other hand since $\omega$ is basic and since $\#$ maps $\Gamma(\ker\pi)^\perp$ on $\Gamma(TS)$, $d_\pi \#(\omega)(\alpha_0, \ldots, \alpha_p) = 0$ if there is $i \neq j$ such that $\alpha_i$ and $\alpha_j$ belong to $\Gamma((\ker\pi)^\perp)$.

Now, we suppose that $\alpha_0 \in \Gamma((\ker\pi)^\perp)$ and, for $1 \leq j \leq p$, $\alpha_j \in \Gamma(\ker\pi)$. We have

$$d_\pi \#(\omega)(\alpha_0, \ldots, \alpha_p) = \pi(\alpha_0).\omega(\#(\alpha_1), \ldots, \#(\alpha_p))$$

$$+ \sum_{j=1}^{p} (-1)^j \omega(\#:([\alpha_0, \alpha_j]_\pi), \#(\alpha_1), \ldots, \hat{\alpha}_j, \ldots, \#(\alpha_p)).$$
Now we will compute the perpendicular component of $\#([\alpha_0, \alpha_j])$. Let $\gamma \in \Gamma((Ker\pi)$.

\[
\gamma[\#([\alpha_0, \alpha_j])] = \gamma[\#(L_{\pi(\alpha_0)}\alpha_j)] = < L_{\pi(\alpha_0)}\alpha_j, \gamma > \\
= - < \alpha_j, L_{\pi(\alpha_0)}\gamma > + \pi(\alpha_0). < \alpha_j, \gamma > \quad \text{by (17)} \\
= - < \alpha_j, i_{\pi(\alpha_0)}d\gamma > + \pi(\alpha_0). < \alpha_j, \gamma > \\
= -d\gamma(\pi(\alpha_0), \#(\alpha_j)) + \pi(\alpha_0). < \alpha_j, \gamma > \\
= \gamma([\pi(\alpha_0), \#(\alpha_j)]).
\]

So $[\pi(\alpha_0), \#(\alpha_j)]$ is the perpendicular component of $\#([\alpha_0, \alpha_j])$ and

\[
d_{\pi}(\omega)(\alpha_0, \ldots, \alpha_p) = L_{\pi(\alpha_0)}\omega(\#(\alpha_1), \ldots, \#(\alpha_p)) = 0
\]
since $\omega$ is basic. $\square$

We get a linear map $\# : \Omega^p_b(P) \rightarrow H^p(\mathcal{A}^p_1(P))$ which is obviously injective.

We have shown the following theorem.

**Theorem 3.1** Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. For $1 \leq p \leq \dim P$, there is an injective linear map

\[
\Omega^p_b(P) \oplus H^p_1(P) \hookrightarrow H^p_\pi(P).
\]

In particular, we have

\[
\Omega^1_b(P) \oplus H^1_1(P) \simeq H^1_\pi(P).
\]

**Remarks.**

Let $(P, \pi, <, >)$ be a regular Riemann Poisson manifold. On can verify that the injection $H^1_\pi(P) \hookrightarrow H^1_\pi(P)$ maps the modular class of the symplectic foliation which is an obstruction lying in $H^1_\pi(P)$ to the existence of an invariant transverse volume form to $2\mod(P)$ where $\mod(P)$ is the modular class of the Poisson manifold which is an obstruction lying $H^1_\pi(P)$ to the existence of a volume form invariant with respect to Hamiltonian flows. Since the modular class vanishes for a Riemannian foliation, we get that $(P, \pi)$ is unimodular and we have another proof of the result given in [2].

As a final we give some examples:

1) Any Poisson manifold such that the Poisson tensor is parallel with respect to the Levi-Civita connection for a Riemannian metric is a regular Riemann Poisson manifold ( see [1]).
2) Let \((M, \omega, g)\) be a Kähler manifold endowed with a symplectic, isometric and free action of the circle \(S^1\). Let \(Z\) denote de fundamental vector field of this action and suppose that there is a vector field \(Y\) such that \(Z\) and \(Y\) commute and are orthogonal. The 1-form \(i_Z \omega\) project on the space of orbits \(M/S^1\) and define a Riemannian foliation which verify the hypothesis of Theorem 2.2.
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