CONTACT COHOMOLOGY OF THE PROJECTIVE PLANE

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ABSTRACT. We construct an associative ring which is a deformation of the quantum cohomology ring of the projective plane. Just as the quantum cohomology encodes the incidence characteristic numbers of rational plane curves, the contact cohomology encodes the tangency characteristic numbers.

1. INTRODUCTION

In this paper we construct an associative ring which we call the contact cohomology ring of the projective plane. We believe that an analogous construction should work for all homogeneous varieties, but in our proof of associativity we rely on certain technical results from our earlier paper on characteristic numbers of rational plane curves [EK98].

As we formulate it in section 3 of the present paper, the quantum product is actually a whole family of products parametrized by elements of the Chow ring $A^*(\mathbb{P}^2)$. Each product is defined on the formal power series ring $A^*(\mathbb{P}^2)[[T]]$ in one variable, and encodes the characteristic numbers

$$N_d = \text{the number of rational plane curves of degree } d \text{ through } 3d - 1 \text{ general points.}$$

The remarkable fact about the quantum product is that it is associative. This fact, together with the triviality $N_1 = 1$, suffices to determine all values of $N_d$.

The contact products are defined on the same formal power series ring. But now the parameter space is the Chow ring $A^1(\mathbb{P}^2)$ of the incidence variety of points and lines in $\mathbb{P}^2$, and these products encode (as we explain in section 6) a larger collection of characteristic numbers:

$$N_d(a, b, c) = \text{the number of rational plane curves of degree } d \text{ through } a \text{ general points, tangent to } b \text{ general lines, and tangent to } c \text{ general lines at a specified general point on each line (where } a + b + 2c = 3d - 1).$$

Section 5 is devoted to defining these products precisely and to showing that they are likewise associative. As we explain in section 6, the associativity implies a remarkable recursive relation among the characteristic numbers, but it is insufficient to determine all their values unless one already knows all values $N_d(a, b, c)$ for which $a < 3$. (Our previous paper [EK98] explains how to obtain this additional information.)

The associativity of the quantum product is a consequence of the recursive structure of the boundary divisors on the moduli space of stable maps to $\mathbb{P}^2$ (of a given degree and with a given number of markings). In brief, each boundary divisor is a fiber product of simpler instances of the same sort of moduli space. But in studying questions of tangency it is natural to look at a moduli space of stable lifts (defined in section 4), whose boundary divisors have a somewhat more complicated structure. To understand these divisors, consider a family of immersions $\mathbb{P}^1 \to \mathbb{P}^2$. Associated to each immersion is its lift, a map from $\mathbb{P}^1$ to the incidence correspondence $I$. Now suppose that the family of immersions

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degenerates to a map from a two-component curve onto the union of two plane curves of lower degree. Then one can show that the family of lifts degenerates to a map from a three-component curve, with the central component mapping two-to-one onto the fiber of \( I \) over a point of \( \mathbb{P}^2 \). Thus to create a moduli space with a recursive structure we form a fiber product of the space of stable lifts with a space of two-to-one covers of fibers of \( I \) over \( \mathbb{P}^2 \); we then exploit this recursive structure to define our associative product.

Our recursive relation specializes to that of Di Francesco and Itzykson \[\text{FI95}\], who also interpret their formula as the associativity of a certain product. We thank S. Colley for carefully reading earlier versions of this paper and making many helpful suggestions. We also want to indicate our indebtedness to B. Fantechi, W. Fulton, L. Göttsche and R. Pandharipande.

We should warn the reader of a potentially confusing clash in terminology: in symplectic geometry Y. Eliashberg and H. Hofer have introduced a new invariant of contact manifolds called “contact homology.” The Gromov-Witten invariants of a symplectic manifold with contact boundary take values in the contact homology of the boundary.

2. A “push-pull” formula

For an algebraic variety, scheme, or algebraic stack \( X \), we write \( A_*X \) for the rational equivalence group with coefficients in \( \mathbb{Q} \), and call it the homology; we write \( A^*_X \) for the operational cohomology ring. (See \[\text{Ful84}\] for the case of a variety or scheme, \[\text{Vis89}\] for intersection theory on stacks.) We will often use the fact that \( A_*X \) and \( A^*_X \) of a nonsingular variety \( X \) are naturally isomorphic.

We record here a formula which we will use repeatedly. It is the stack version of \[\text{Ful84, Proposition 1.7}\].

**Lemma 2.1.** Suppose that

\[
\begin{array}{ccc}
M_1 \times_X M_2 & \xleftarrow{q_1} & M_1 \\
\downarrow & & \downarrow \\
M_1 & \xleftarrow{g_1} & M_2
\end{array}
\]

\[
\begin{array}{ccc}
& \xleftarrow{q_2} & \\
\downarrow & & \downarrow \\
X & \xleftarrow{g_2} & \\
\uparrow & & \uparrow \\
& & \\
M_1 & \xleftarrow{g_1} & M_2
\end{array}
\]

is a fiber square of stacks over a nonsingular variety \( X \), with \( g_2 \) flat and \( g_1 \) proper. Then \( q_1 \) is flat, \( q_2 \) is proper, and for each class \( \alpha \in A^*_M \) we have

\[
q_2^*(q_1^*\alpha \cap [M_1 \times_X M_2]) = g_2^*g_1^*(\alpha \cap [M_1]) \cap [M_2].
\]

3. The quantum product

Here we recall the basic definitions of quantum cohomology and prove that the quantum product on \( \mathbb{P}^2 \) is associative. Our proof is basically the same as in section 8 of \[\text{FP96}\], except that we avoid the use of coordinates; we use this proof as a prototype for the longer proof in section 5. As general references for the material in this section we suggest \[\text{BM96}, \text{EK98}, \text{FP96}, \text{Kon93}, \text{KM94}, \text{LT97}\].

Let \( \overline{M}_{0,n+3}(\mathbb{P}^2,d) \) denote the stack of stable maps from curves of arithmetic genus 0, with \( n + 3 \) markings, to the projective plane. Let \( i, j, k : \overline{M}_{0,n+3}(\mathbb{P}^2,d) \rightarrow \mathbb{P}^2 \) be the evaluation maps associated to the first three markings; let \( e_1, \ldots, e_n \) be the others. Let \( A^*(\mathbb{P}^2) \) be the Chow cohomology ring of the plane with rational coefficients, and let \( A^*(\mathbb{P}^2)[[T]] \) be the ring of formal power series in one variable. Suppose that \( \alpha, \beta, \) and \( \delta \) are elements of \( A^*(\mathbb{P}^2) \). Then the **quantum product of \( \alpha \) and \( \beta \),**
deformed by \(\delta\), is the element of \(A^*(\mathbb{P}^2)[[T]]\) whose \(n\)th coefficient is

\[
\frac{1}{n!} \sum_{d \geq 0} k_*(i^*\alpha \cup j^*\beta \cup \bigcup_{t=1}^n \epsilon_t^*\delta \cap \overline{M}_{0,n+3}(\mathbb{P}^2,d)).
\]

Note that, since the dimension of \(\overline{M}_{0,n+3}(\mathbb{P}^2,d)\) is \(3d + 2 + n\), the sum is finite. We will denote the quantum product by \((\alpha \ast \beta)_\delta\) or simply \(\alpha \ast \beta\). Extending by \(Q[[T]]\)-linearity, we have a product on \(A^*(\mathbb{P}^2)[[T]]\). We call this ring the quantum cohomology of \(\mathbb{P}^2\), and denote it by \(QH^*(\mathbb{P}^2)\).

**Proposition 3.2.** For each \(\delta\) in \(A^*(\mathbb{P}^2)\), the quantum product is commutative and associative. The identity element \(1 \in A^*(\mathbb{P}^2)\) for the ordinary cup product also serves as the identity element for the quantum product.

**Proof.** The commutativity is obvious. Unless \(n = d = 0\) the class

\[
i^*1 \cup j^*\beta \cup \bigcup_{t=1}^n \epsilon_t^*\delta
\]

is the pullback via the forgetful morphism \(\overline{M}_{0,n+3}(\mathbb{P}^2,d) \to \overline{M}_{0,n+2}(\mathbb{P}^2,d)\) of a class on the latter space. Since the fibers of this morphism have positive dimension, the projection formula tells us that the corresponding term of (3.1) vanishes. As for the remaining term \(n = d = 0\), note that in this case \(i, j, k\) are all the same isomorphism from \(\overline{M}_{0,3}(\mathbb{P}^2,0)\) to \(\mathbb{P}^2\). Hence 1 is the identity element for the quantum product.

To see that the quantum product is associative, consider the moduli space \(\overline{M}_{0,n+4}(\mathbb{P}^2,d)\). Denote the first four evaluation maps \(\overline{M}_{0,n+4}(\mathbb{P}^2,d) \to \mathbb{P}^2\) by \(i, j, k,\) and \(l\), and the remaining \(n\) maps by \(e_t\). Consider the “forgetful” morphism from \(\overline{M}_{0,n+4}(\mathbb{P}^2,d)\) to \(\overline{M}_{0,4}\), the space of stable 4-pointed curves, which associates to a stable map its source curve together with the first four markings, with any unstable components contracted to a point. The space \(\overline{M}_{0,4}\) is isomorphic to \(\mathbb{P}^1\). It has a distinguished point \(P(12\mid34)\) representing the two-component curve having the first two markings on one component and the latter two on the other; similarly there are two other distinguished points \(P(13\mid24)\) and \(P(14\mid23)\). Hence on \(\overline{M}_{0,n+4}(\mathbb{P}^2,d)\) there are three linearly equivalent divisors \(D(12\mid34), D(13\mid24),\) and \(D(14\mid23)\), which we call the special boundary divisors.

Kontsevich \cite{Kon95} (cf. \cite{FP96}) identifies the components of \(D(12\mid34)\). For a finite set \(A\), let \(\overline{M}_{0,A}(\mathbb{P}^2,d)\) denote the stack of stable maps with markings labeled by \(A\). Suppose that \(A_1 \cup A_2\) is a partition of \(\{1, \ldots, n + 4\}\) and that \(d_1 + d_2 = d\). Suppose that \(\ast\) is a single-element set. Then the fiber product

\[
D(A_1, A_2, d_1, d_2) = \overline{M}_{0,A_1 \cup \{\ast\}}(\mathbb{P}^2, d_1) \times_{\mathbb{P}^2} \overline{M}_{0,A_2 \cup \{\ast\}}(\mathbb{P}^2, d_2)
\]

is naturally a substack of \(\overline{M}_{0,n+4}(\mathbb{P}^2,d)\); the typical point represents a map from a curve with two components, with the point of attachment corresponding to the point labeled by \(\ast\), as indicated in Figure 3.1. The divisor \(D(12\mid34)\) is the sum

\[
D(12\mid34) = \sum D(A_1, A_2, d_1, d_2)
\]

over all partitions \(A_1 \cup A_2\) in which 1 and 2 belong to \(A_1\), and 3 and 4 belong to \(A_2\). There are corresponding statements for the other two special boundary divisors.
Since the divisors $D(12 \mid 34)$ and $D(14 \mid 23)$ are linearly equivalent, the equation

\[
\frac{1}{n!} \sum_{d \geq 0} l_d(\mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \mathbf{1}^* \gamma \cup \bigcup_{t=1}^{n} \mathbf{1}^* e_t \delta \cap [D(12 \mid 34)])
\]

is valid for each triple $\alpha, \beta, \gamma$ of classes in $A^*(\mathbb{P}^2)$ and for each $n$. According to (3.3), the divisor $D(12 \mid 34)$ is a sum of fiber products, each of which fits into a fiber diagram

\[
\begin{array}{ccc}
\overline{M}_{0, n+3}(\mathbb{P}^2, d_1) & \times_{\mathbb{P}^2} & \overline{M}_{0, n+3}(\mathbb{P}^2, d_2) \\
\downarrow q_1 & & \downarrow q_2 \\
\overline{M}_{0, n+3}(\mathbb{P}^2, d_1) & \times_{\mathbb{P}^2} & \overline{M}_{0, n+3}(\mathbb{P}^2, d_2) \\
\downarrow i_1 & & \downarrow i_2 \\
\mathbb{P}^2 & \cup & \mathbb{P}^2 \\
\downarrow e_1 & & \downarrow e_2 \\
\mathbb{P}^2 & \cup & \mathbb{P}^2 \\
\downarrow e_1 & & \downarrow e_2 \\
\end{array}
\]

in which $i = i_1 \circ q_1$, $j = j_1 \circ q_1$, $k = k_2 \circ q_2$, and $l = l_2 \circ q_2$. Furthermore each $e_t$ equals either $e_{1t} \circ q_1$ or $e_{2t} \circ q_2$: by relabeling we may assume that the former equation holds for $1 \leq t \leq n_1$. Note that, for specified partitions $n = n_1 + n_2$ and $d = d_1 + d_2$, the number of such fiber diagrams is $\binom{n}{n_1}$. Let $M_1 = \overline{M}_{0, n+3}(\mathbb{P}^2, d_1)$ and $M_2 = \overline{M}_{0, n+3}(\mathbb{P}^2, d_2)$. By Lemma 2.1 and the projection formula we have

\[
\begin{align*}
& l_* \left( \mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \mathbf{1}^* \gamma \cup \bigcup_{t=1}^{n} \mathbf{1}^* e_t \delta \cap [M_1 \times \mathbb{P}^2 M_2] \right) \\
& = l_{2*} q_{2*} \left( q_1 \left( \mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \bigcup_{t=1}^{n_1} \mathbf{1}^* e_{1t} \delta \cap [M_1 \times \mathbb{P}^2 M_2] \right) \right) \\
& = l_{2*} \left( k_{2}^* \mathbf{1}^* \gamma \cup \bigcup_{t=n_1+1}^{n} \mathbf{1}^* e_{2t} \delta \cup \left( q_1 \left( \mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \bigcup_{t=1}^{n_1} \mathbf{1}^* e_{1t} \delta \cap [M_1 \times \mathbb{P}^2 M_2] \right) \right) \right) \\
& = l_{2*} \left( k_{2}^* \mathbf{1}^* \gamma \cup \bigcup_{t=n_1+1}^{n} \mathbf{1}^* e_{2t} \delta \cup \left( g_1 \left( \mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \bigcup_{t=1}^{n_1} \mathbf{1}^* e_{1t} \delta \cap [M_1] \cap [M_2] \right) \right) \right) \\
& = l_{2*} \left( g_2^* \left( \mathbf{1}^* \alpha \cup \mathbf{1}^* \beta \cup \bigcup_{t=n_1+1}^{n} \mathbf{1}^* e_{1t} \delta \cap [M_1] \cap [M_2] \right) \right) \\
\end{align*}
\]

Summing—for fixed $n$—over all $d \geq 0$ and all components of $D(12 \mid 34)$, we find that the left side of (3.4) equals the $n$th coefficient of $(\mathbf{1}^* \beta) \circ \gamma$. A similar argument shows that the right side of (3.4) equals the $n$th coefficient of $\mathbf{1}^* (\beta \circ \gamma)$.

Invoking the $\mathbb{Q}[T]$-linearity, we conclude that the quantum product is associative. \qed
4. Stable maps to the incidence variety

Let $I$ be the incidence variety of points and lines in $\mathbb{P}^2$. Its cohomology ring $A^\ast(I)$ is generated by two classes $h$ and $h$ representing the pullbacks of, respectively, the class of a line in $\mathbb{P}^2$ and the class of a dual line in $\mathbb{P}^2$. The fundamental class of a curve is determined by its intersection numbers $d$ and $d$ with the two classes; we denote this class by $(d, d)$. Thus has dimension 3.

There is a substack $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$ of $\mathcal{M}_{0,n}(I, (d, 2d - 2))$ representing the lifts of immersions. We call its closure $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ the stack of stable lifts; it is birationally isomorphic to the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ and thus has dimension $3d - 1 + n$.

We will be working with two other special stacks of stable maps to the incidence variety. For $n \geq 2$, consider the stack $\mathcal{M}_{0,n}(I, (0, 1))$, whose typical member is an $n$-pointed isomorphism from $\mathbb{P}^1$ to a fiber of $I$ over $\mathbb{P}^2$. Let $e_1, \ldots, e_n$ be the evaluation maps.

**Lemma 4.2.** Let $\alpha$ be a class in $A^\ast(\mathbb{P}^2)$ and let $\delta_2, \ldots, \delta_n$ be classes in $A^\ast(I)$. Then

$$e_{2\ast}(e_1^\ast p^\ast \alpha \cup e_2^\ast \delta_2 \cup \cdots \cup e_n^\ast \delta_n \cap \overline{\mathcal{M}}_{0,n}(I, (0, 1))) = 0.$$

**Proof.** Since each element of $\overline{\mathcal{M}}_{0,n}(I, (0, 1))$ is a map to a fiber of $I$ over $\mathbb{P}^2$, we have that $p \circ e_1 = p \circ e_2$. Thus $e_1^\ast p^\ast = e_2^\ast p^\ast$, and by the projection formula

$$e_{2\ast}(e_1^\ast p^\ast \alpha \cup e_2^\ast \delta_2 \cup \cdots \cup e_n^\ast \delta_n \cap \overline{\mathcal{M}}_{0,n}(I, (0, 1))) = p^\ast \alpha \cap e_{2\ast}(e_2^\ast \delta_2 \cup \cdots \cup e_n^\ast \delta_n \cap \overline{\mathcal{M}}_{0,n}(I, (0, 1)))$$

The class in parentheses is the pullback of a class on $\overline{\mathcal{M}}_{0,n-1}(I, (0, 1))$ via the morphism

$$\overline{\mathcal{M}}_{0,n}(I, (0, 1)) \to \overline{\mathcal{M}}_{0,n-1}(I, (0, 1))$$

which forgets the first point. Furthermore $e_2$ factors through this forgetful morphism. Hence by the projection formula its pushforward via $e_2$ vanishes.

Similarly, let $\mathcal{M}_{0,n+2}(I, (0, 2))$ be the substack of $\overline{\mathcal{M}}_{0,n+2}(I, (0, 2))$ representing degree 2 maps from $\mathbb{P}^1$ to a fiber of $I$ over $\mathbb{P}^2$, with the two ramification points specially marked and with an additional $n$ markings. Let $\overline{\mathcal{M}}_{0,n+2}(I, (0, 2))$ be its closure, a stack of dimension $4 + n$. Denote by $e_1, \ldots, e_{n+2}$ the evaluation maps of $\overline{\mathcal{M}}_{0,n+2}(I, (0, 2))$. It does not matter which two are evaluation maps at the ramification; this flexibility is convenient in proving the next three lemmas. (Later we will use a different notation.)

**Lemma 4.3.** Let $\delta_2, \ldots, \delta_{n+2}$ be classes in $A^\ast(I)$. Then

$$e_{2\ast}(e_2^\ast \delta_2 \cup \cdots \cup e_{n+2}^\ast \delta_{n+2} \cap \overline{\mathcal{M}}_{0,n+2}(I, (0, 2))) = 0.$$

(Write that there is no condition on the marking corresponding to $e_1$.)
Proof. Each evaluation map $e_i$ ($i = 2, \ldots, n+2$) can be factored into the product map $f = e_2 \times \cdots \times e_{n+2}$ followed by projection $f_i$ onto the appropriate factor.

\[
\begin{array}{c}
\overline{M}_{0,n,2}(I, (0, 2)) \\
\downarrow f \\
I \times p_2 \cdots \times p_2 I \\
\downarrow f_i \\
\end{array}
\]

Therefore

\[
e_2^* (e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))) = f_2^* f_*(f_2^* \delta_2 \cup \cdots \cup f_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))) = f_2^* (\delta_2 \cup \cdots \cup f_{n+2}^* \delta_{n+2} \cap f_* \overline{M}_{0,n,2}(I, (0, 2))),
\]

which is zero because $\dim(\overline{M}_{0,n,2}(I, (0, 2))) = 4 + n$ and $\dim(I \times p_2 \cdots \times p_2 I) = 3 + n$.

\[\square\]

Lemma 4.4. Let $\alpha$ be a class in $A^*(P^2)$ and let $\delta_2, \ldots, \delta_{n+2}$ be classes in $A^*(I)$. Then $e_{2*} (e_1^* p^* \alpha \cup e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))) = 0$.

Proof. Since each element of $\overline{M}_{0,n,2}(I, (0, 2))$ is a map to a fiber of $I$ over $P^2$, we have that $p \circ e_1 = p \circ e_2$. Thus $e_1^* p^* = e_2^* p^*$, and by the projection formula

\[
e_{2*} (e_1^* p^* \alpha \cup e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))) = p^* \alpha \cap e_{2*} (e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))),
\]

which is zero by Lemma 4.3.

\[\square\]

Lemma 4.5. For any classes $\delta_2, \ldots, \delta_{n+2}$ in $A^*(I)$, the class $e_{1*} (e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2)))$ is in $A^*(P^2)$.

Proof. Since $p \circ e_1 = p \circ e_2$, we have

\[
p_* e_{1*} (e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))) = p_* e_{2*} (e_2^* \delta_2 \cup \cdots \cup e_{n+2}^* \delta_{n+2} \cap \overline{M}_{0,n,2}(I, (0, 2))),
\]

which is zero by Lemma 4.3. Now apply Lemma 4.1.

\[\square\]

5. The Contact Product

Denote by $\overline{M}(m, n, d)$ the fiber product

\[
\overline{M}_{0,m+3}^1(P^2, d) \times_I \overline{M}_{0,n,2}(I, (0, 2)).
\]

Denote the evaluation maps $\overline{M}(m, n, d) \to I$ coming from $\overline{M}_{0,m+3}^1(P^2, d)$ by $i, j, e_1, \ldots, e_m$ and $e_*$; denote those coming from $\overline{M}_{0,n,2}(I, (0, 2))$ by $f_1, \ldots, f_n, f_*$ and $k$, with the latter two being at the points of ramification. The fiber product over $I$ is defined using the maps $e_*$ and $f_*$. Thus the typical point of $\overline{M}(m, n, d)$ represents a map from a curve with two components, as depicted in Figure 5.1; the vertical component maps two-to-one to a fiber of $I$ and is ramified at the points marked $\times$. In addition
to the indicated markings, there are $m$ additional markings on the horizontal component and $n$ on the vertical component.

**Figure 5.1** Typical member of $\overline{M}(m, n, d)$.

Given cohomology classes $\alpha$ and $\beta$ in $A^*(\mathbb{P}^2)$ and a class $\delta$ in $A^*(I)$, let $(\alpha \bullet \beta)_\delta$ (or simply $\alpha \bullet \beta$) be the element of $A^*(I)[[T]]$ whose $q$th coefficient is

\[
\sum_{m+n=d, m+n=q} \frac{2}{m!n!} k_q \left( t^* \alpha \cup j^* \beta \cup \bigcup_{i=1}^m e_i^* \delta \cup \bigcup_{t=1}^n f^*_t \delta \cap [\overline{M}(m, n, d)] \right).
\]

Note that with our convention $A^*(\mathbb{P}^2)$ is a subring of $A^*(I)$, so it makes sense to pull back classes of $A^*(\mathbb{P}^2)$ via evaluation maps to $I$. The product formula (5.1) makes sense for any two classes $\alpha$ and $\beta$ in $A^*(I)$. However, to prove that the product is associative we will need to assume that the classes are in $A^*(\mathbb{P}^2)$.

We now prove that $\alpha \bullet \beta$ is in $A^*(\mathbb{P}^2)[[T]]$. (For this proof we don’t need to assume that $\alpha$ and $\beta$ are in $A^*(\mathbb{P}^2)$.) Let $i_1, j_1, e_{1*}$ and $e_{1t}$ indicate the evaluation maps from $\overline{M}^1_{0,m+3}(\mathbb{P}^2, d)$ to $I$; thus if $q_1$ is the projection of $\overline{M}(m, n, d)$ onto its first factor, we have $i = i_1 \circ q_1$, etc. Similarly let $q_2$ be the projection onto the second factor and $k_2, f_{2*}, f_{2t} \colon \overline{M}_{0,n,2}(I, (0, 2)) \to I$ be the evaluation maps, so that $k = k_2 \circ q_2$, etc.

Lemma 2.1 and the projection formula applied to the above diagram yield the following expression for the corresponding term of (5.1):

\[
\sum_{m+n=d, m+n=q} \frac{2}{m!n!} k_q \left( t^* \alpha \cup j^* \beta \cup \bigcup_{i=1}^m e_i^* \delta \cup \bigcup_{t=1}^n f^*_t \delta \cap [\overline{M}(m, n, d)] \right).
\]

It follows from Lemma 4.3 that $\alpha \bullet \beta$ is in $A^*(\mathbb{P}^2)[[T]]$.

Extending by $Q[[T]]$-linearity, we have a product $\bullet$ on $A^*(\mathbb{P}^2)[[T]]$. We now define the contact product of $\alpha$ and $\beta$, deformed by $\delta$, to be

$$
\alpha \ast \beta = \alpha \cup \beta + \alpha \bullet \beta.
$$

We call $A^*(\mathbb{P}^2)[[T]]$, together with this product, the contact cohomology ring of $\mathbb{P}^2$, and denote it by $Q^1H^*(\mathbb{P}^2)$.
Theorem 5.3. For each \(\delta\), the contact product is commutative and associative. The identity element \(1 \in \mathbb{A}^n(\mathbb{P}^2)\) for the ordinary cup product also serves as the identity element for the contact product.

Proof. The commutativity is obvious. The class

\[
i_1^* 1 \cup f_1^* \beta \cup \bigcup_{t=1}^{m} e_t^* \delta \cap [\overline{M}_{0,m+3}(\mathbb{P}^2,d)]
\]

is the pullback via the forgetful morphism

\[
\overline{M}_{0,m+3}(\mathbb{P}^2,d) \to \overline{M}_{0,m+2}(\mathbb{P}^2,d)
\]

of a class on the latter, and \(e_t^*\) factors through the forgetful morphism. Hence if \(\alpha = 1\) then (5.2) vanishes. Hence 1 is the identity for the contact product.

Now, let \(\alpha, \beta, \gamma\) be classes in \(\mathbb{A}^n(\mathbb{P}^2)\). The cup product is associative. Thus we must show that

\[
(\alpha \cup \beta) \cdot \gamma + (\alpha \cdot \beta) \cup \gamma = \alpha \cup (\beta \cdot \gamma) + \alpha \cdot (\beta \cup \gamma).
\]

(5.4)

As in the proof of associativity of the quantum product in section \(\text{[3]}\), we use a linear equivalence of divisors

\[
D(ij | kl) \simeq D(il | jk),
\]

this time on the stack of stable lifts \(\overline{M}_{0,m+4}(\mathbb{P}^2,d)\). This linear equivalence, which is derived from the forgetful map \(\overline{M}_{0,m+4}(\mathbb{P}^2,d) \to \overline{M}_{0,4}\), induces a linear equivalence on the fiber product \(\overline{M}_{0,m+4}(\mathbb{P}^2,d) \times_I \overline{M}_{0,n,2}(I, (0,2))\):

\[
D(ij | kl) \times_I \overline{M}_{0,n,2}(I, (0,2)) \simeq D(il | jk) \times_I \overline{M}_{0,n,2}(I, (0,2)).
\]

For these two fiber products, denote the evaluation maps to \(I\) coming from the first factor by \(i, j, k, l\) and \(e_1, \ldots, e_m\); denote the evaluation maps coming from the second factor by \(r\) and \(s\) (for evaluation at the ramification points) and by \(f_1, \ldots, f_n\). The fiber product is defined via the map \(l\) on the first factor and \(r\) on the second.

The linear equivalence implies the equality

\[
\sum_{m+n=q} \frac{2}{m! n!} s_\ast (i^* \alpha \cup j^* \beta \cup k^* \gamma \cup \bigcup_{t=1}^{m} e_t^* \delta \cup \bigcup_{t=1}^{n} f_t^* \delta \cap [D(ij | kl) \times_I \overline{M}_{0,n,2}(I, (0,2))])
\]

\[= \sum_{m+n=q} \frac{2}{m! n!} s_\ast (i^* \alpha \cup j^* \beta \cup k^* \gamma \cup \bigcup_{t=1}^{m} e_t^* \delta \cup \bigcup_{t=1}^{n} f_t^* \delta \cap [D(il | jk) \times_I \overline{M}_{0,n,2}(I, (0,2))]).
\]

We will show that the left side of (5.5) equals the \(q\)th coefficient of the left side of (5.4); an entirely similar argument shows that the right side of (5.5) equals the \(q\)th coefficient of the right side of (5.4).

In \(\text{[EK98, Section 5]}\) we have analyzed the components of \(D(ij | kl)\). We call such a component \(D\) numerically irrelevant if

\[
\int i_1^* \gamma_1 \cup j_1^* \gamma_2 \cup k_1^* \gamma_3 \cap \bigcup_{t=1}^{m} e_t^* \delta_t \cap [D] = 0.
\]

for all choices of cohomology classes \(\gamma_1, \gamma_2, \gamma_3, \) and \(\delta_1, \ldots, \delta_m\) in \(\mathbb{A}^n(I)\). Here \(i_1, j_1,\) etc. indicate the maps in the following diagram, for which \(i_1 \circ q_1 = i,\) etc.
To understand the contribution of a numerically irrelevant component to (5.5), consider the class

\[
s_*(i^*\alpha \cup j^*\beta \cup k^*\gamma \cup \bigcup_{t=1}^m e_t^*\delta \cup \bigcup_{t=1}^n f_t^*\delta \cap [D \times_I M_{0,n,2}(I,(0,2))]).
\]

Let \( \tau \) be a test class in \( A_*(I) \). Then the degree of its intersection with (5.6) equals, by repeated use of the projection formula, the degree of the class

\[
i_1^*\alpha \cup j_1^*\beta \cup k_1^*\gamma \cup \bigcup_{t=1}^m e_t^*\delta \cap q_1^* \left( q_2^* (s_2^*\tau \cup \bigcup_{t=1}^n f_t^*\delta) \cap [D \times_I M_{0,n,2}(I,(0,2))] \right).
\]

By Lemma 2.1, this class equals

\[
i_1^*\alpha \cup j_1^*\beta \cup k_1^*\gamma \cup \bigcup_{t=1}^m e_t^*\delta \cap l_t^*r_2^* \left( s_2^*\tau \cup \bigcup_{t=1}^n f_t^*\delta \cap [M_{0,n,2}(I,(0,2))] \right) \cap [D].
\]

Since \( D \) is numerically irrelevant, the degree is 0. And since this is true for every \( \tau \), the class (5.6) is zero.

Besides the numerically irrelevant components, the divisor \( D(ij \mid kl) \) on \( \overline{M}_{0,m+4}(\mathbb{P}^2,d) \) has three types of components [EK98, Proposition 5.9]. Thus there are three corresponding types of components of \( D(ij \mid kl) \times_I \overline{M}_{0,n,2}(I,(0,2)) \). A component of the first type is of the form

\[
\overline{M}_{0,m_1+3}(I,(0,0)) \times_I \overline{M}_{0,m_2+3}(\mathbb{P}^2,d) \times_I \overline{M}_{0,n,2}(I,(0,2))
\]

or

\[
\overline{M}_{0,m_1+3}(\mathbb{P}^2,d) \times_I \overline{M}_{0,m_2+3}(I,(0,0)) \times_I \overline{M}_{0,n,2}(I,(0,2)),
\]

where \( m_1 + m_2 = m \). A typical point of such a component represents a map from a curve with three components, as depicted in Figure 5.2. In each case the component on the right maps two-to-one to a fiber of \( I \) and is ramified at the points marked \( \times \); the horizontal component maps to \( I \) via the lift of an immersion; and the remaining component maps to a point of \( I \). In addition to the indicated markings, there are \( m_1 \) markings on the left component, \( m_2 \) on the middle component, and \( n \) on the right component.
Consider a component of type \( (5.7) \). Let us denote the three factors simply by \( M_1, M_2, M_3 \). We may relabel the markings so that the evaluation maps \( e_1, \ldots, e_{m_1} \), as well as \( i, j \), and the map \( g_1 \) used to create the fiber product, factor through projection onto \( M_1 \). We also note that all of these evaluation maps coincide. Hence we have the following fiber diagram.

By Lemma 2.1 and the projection formula, the contribution of our component to the left side of (5.5) is

\[
\frac{2}{m!n!} s_4 \left( t^* \alpha \cup j^* \beta \cup k^* \gamma \cup \bigcup_{t=1}^m e_t^* \delta \cup \bigcup_{t=1}^n f_t^* \delta \cap [M_1 \times_I M_2 \times_I M_3] \right)
\]

If \( m_1 > 0 \) then the fibers of the map from \( M_1 \) to \( I \) have positive dimension; hence

\[
g_1 \left( g_1^* \left( \bigcup_{t=1}^{m_1} \delta \cap [M_1] \right) \right) = \alpha \cup \beta \cup \bigcup_{t=1}^{m_1} \delta \cap g_1^*[M_1] = 0,
\]

and the component makes no contribution to (5.5). If \( m_1 = 0 \) then \( M_1 \) and \( I \) are isomorphic, and

\[
g_{23}^* g_1 \left( g_1^* \left( \bigcup_{t=1}^{m_1} \delta \cap [M_1] \right) \right) = g_{23}^* \left( \alpha \cup \beta \cap [I] \right).
\]

Thus if we sum the contributions from all such components with \( m + n = q \) we obtain the \( q \)th coefficient of \( (\alpha \cup \beta) \circ \gamma \).

Entirely similar arguments show that a component of type \( (5.8) \) makes no contribution to (5.5) unless \( m_2 = 0 \), and that the sum of contributions from all such components with \( m + n = q \) is the \( q \)th coefficient of \( (\alpha \circ \beta) \cup \gamma \).
The second type of component of $D(ij \mid kl) \times_I \overline{M}_{0,n,2}(I, (0, 2))$ is one of the form

\begin{equation}
\overline{M}_{0,m+3}(I, (0, 1)) \times_I \overline{C}_{0,m+2,1,*}^l(P^2, d) \times_I \overline{M}_{0,n,2}(I, (0, 2)),
\end{equation}

or

\begin{equation}
\overline{C}_{0,m+2,1,*}^l(P^2, d) \times_I \overline{M}_{0,m+3}(I, (0, 1)) \times_I \overline{M}_{0,n,2}(I, (0, 2))
\end{equation}

where $m_1 + m_2 = m$, and where the second (respectively first) factor is the stack of cuspidal stable lifts [EK98, Section 4]. A general point of such a component represents a map from a curve with the same configuration of components and markings as in Figure 5.2. The left and middle components map, in either order, to the lift of a degree $d$ rational curve with one cusp, and to the fiber of $I$ over the cusp, and the right component maps two-to-one to a fiber of $I$.

Consider a component of type (5.9). Denote the three factors by $M_1, M_2, M_3$ and consider the following fiber diagram.

By Lemma 2.1 and the projection formula, the contribution of our component to the left side of (5.5) is

\begin{equation}
\frac{2}{m!n!^{s_{23}}} \left( g_{23} g_1 \left( i_1^* \alpha \cup j_1^* \beta \cup \bigcup_{t=1}^{m_1} e_{1t}^* \delta \cap [M_1] \right) \cup k_{23}^* \cup \bigcup_{t=m_1+1}^{m} e_{23t}^* \delta \cup \bigcup_{t=1}^{n} f_{23t}^* \delta \cap [M_2 \times I M_3] \right).
\end{equation}

By Lemma 4.2, the class

\begin{equation}
g_1 \left( i_1^* \alpha \cup j_1^* \beta \cup \bigcup_{t=1}^{m_1} e_{1t}^* \delta \cap [M_1] \right) = 0.
\end{equation}

Hence the component makes no contribution. A similar argument applies to a component of type (5.10).

We come finally to the third type of component of $D(ij \mid kl) \times_I \overline{M}_{0,n,2}(I, (0, 2))$. Among them are components of the form

\begin{equation}
\overline{M}_{0,m+3}(P^2, d_1) \times_I \overline{M}_{0,m+2,2}(I, (0, 2)) \times_I \overline{M}_{0,m+3}(P^2, d_3) \times_I \overline{M}_{0,n,2}(I, (0, 2)),
\end{equation}

with $m_1 + m_2 + m_3 = m, d_1, d_3 > 0$ and $d_1 + d_3 = d$. A typical point of such a component represents a map from a curve with four components, as depicted at the top of Figure 5.3. The vertical components map two-to-one to fibers of $I$ and are ramified at the points marked $\times$; the horizontal components map to $I$ via the lifts of immersions. In addition to the indicated markings, there are $m + n$ others.
There are six other possibilities for a component of the third type, corresponding to the six different possible ways in which the four special markings $i, j, k, l$ can lie on the first three components of the typical curve. (The markings $i$ and $j$ may lie on either of the first two components; $k$ and $l$ may lie on either the second or third component; and the pair $i, j$ must be separated from the pair $k, l$ by a node.) We claim that in each of these six cases the component makes no contribution to (5.5). In five cases the argument is identical to that for a component of the second type: a configuration with $i, j$ or $k$ on a component mapping to a fiber (vertical component in Figure 5.3) does not contribute, as a consequence of Lemma 4.4.

In the sixth case we have a component of the form

\[
(5.12) \quad \left( \overline{M}_{0,m_1+3}(\mathbb{P}^2, d_1) \times_I \overline{M}_{0,m_2+1,2}(I, (0, 2)) \right) \times_I \overline{M}_{0,m_3+2}(\mathbb{P}^2, d_3) \times_I \overline{M}_{0,n,2}(I, (0, 2)),
\]

A typical point of such a component represents a map from the sort of curve shown at the bottom right of Figure 5.3. Denote the four factors by $M_1, M_2, M_3$ and $M_4$. Note that the fiber product of $M_1 \times_I M_2 \times_I M_3$ and $M_4$ is created by using an evaluation map from $M_1 \times_I M_2 \times_I M_3$ which comes from its second factor. Consider the following fiber diagram.

**Figure 5.3** The different configurations for components of the third type.
Using Lemma 2.1 one finds that the contribution to (5.5) is

$$\frac{2}{m!n!} s_4^* \left( \bigcup_{t=1}^n f_{4t}^* \delta \cup r_4^* i_{123}^* (i_{123}^* \alpha \cup j_{123}^* \beta \cup k_{123}^* \gamma \cup \bigcup_{t=1}^m e_{123t}^* \delta \cap [M_1 \times M_2 \times M_3]) \cap [M_4] \right).$$

We claim that the class

(5.13) $$l_{123}^* (i_{123}^* \alpha \cup j_{123}^* \beta \cup k_{123}^* \gamma \cup \bigcup_{t=1}^m e_{123t}^* \delta \cap [M_1 \times M_2 \times M_3])$$

is in $A^*(\mathbb{P}^2)$. It will then follow from Lemma 4.4 that the contribution is zero. To prove the claim we use the following diagram, in which there are three fiber squares.

By repeated use of Lemma 2.1 and the projection formula, the class (5.13) can be expressed as

$$l_{2*} \left( g_2^* g_1^* (i_{123}^* \alpha \cup j_{123}^* \beta \cup \bigcup_{t=1}^{m_1} e_{1t}^* \delta \cap [M_1]) \cup h_2^* h_3^* (k_3^* \gamma \cup \bigcup_{t=m_1+m_2}^{m_1+m_2+m_3} e_{3t}^* \delta \cap [M_3]) \cup \bigcup_{t=m_1+1}^{m_1+m_2} e_{3t}^* \delta \cap [M_2] \right).$$

Now $M_2 = \overline{M}_{0,n,2}(I, (0, 2))$, so by Lemma 4.5 this is a class in $A^*(\mathbb{P}^2)$ as we claimed.

It remains to consider the contribution of the component with the configuration on the top of Figure 5.3. Note that the product (5.11) can be written as the following fiber product:
Using Lemma 2.1 one finds that the contribution to the left side of (5.5) is

\[
\frac{2}{n!m!}k_2\gamma \cup \bigcup_{t=m_1+m_2+1} e_1^t \delta \cup \bigcup_{t=1}^{m_2} f_2^t \delta \cup g_2^{t_1} (i_1^t \alpha \cup j_1^t \beta \cup \bigcup_{t=m_1+1}^{m_2} e_1^{t_1} \delta \cap \bigcup_{t=m_1}^{m_2} e_1^{t_1} \delta \cap [\bar{M}(m_1, m_2, d_1)] \cap [\bar{M}(m_3, n, d_3)]).
\]

According to [EK98] each divisor component of this type appears in \(D(ij | kl)\) with a multiplicity of two. The reason is that \((d, 2d - 2)\) can be partitioned in two ways, either as \((d_1, 2d_1 - 2) + (d_2, 2d_2)\) or as \((d_1, 2d_1) + (d_2, 2d_2 - 2)\). If we keep \(q\) fixed, and sum the contributions for all \(d, d_1 + d_3 = d, m + n = q\) and all partitions of the \(m\) markings into three disjoint subsets with cardinalities \(m_1, m_2\) and \(m_3\), the result is the \(q\)th coefficient of \((\alpha \bullet \beta) \bullet \gamma\).

Thus we have shown that \((\alpha \ast \beta) \ast \gamma = \alpha \ast (\beta \ast \gamma)\) when \(\alpha, \beta, \gamma\) are elements of \(A^*(\mathbb{P}^2)\). Invoking the \(\mathbb{Q}[[T]]\)-linearity, we conclude that the contact product is associative.

6. The recursive relation among characteristic numbers

The associativity of the quantum product implies Kontsevich’s recursive formula for the characteristic numbers

\[N_d = \text{the number of rational plane curves of degree } d \text{ through } 3d - 1 \text{ general points.}\]

For details of this story, see [FI95], [FP96], [KM94]. Here we show that, in a similar fashion, the associativity of the contact product implies a recursive formula for the numbers

\[N_d(a, b, c) = \text{the number of rational plane curves of degree } d \text{ through } a \text{ general points, tangent to } b \text{ general lines, and tangent to } c \text{ general lines at a specified general point on each line (where } a + b + 2c = 3d - 1).\]

Our formula will specialize both to Kontsevich’s formula and the more general formula of Di Francesco and Itzykson [FI95, Equation 2.95].

We will use the following ordered basis for the cohomology of the incidence correspondence:

\[\{T_0, T_1, T_2, T_3, T_4, T_5\} = \{1, h, h^2, \bar{h}, \bar{h}^2, h^2\bar{h}\}.\]
With respect to this basis the fundamental class of the diagonal $\Delta$ in $I \times I$ has the simple decomposition

$$[\Delta] = \sum_{s=0}^{5} [T_s] \times [T_{5-s}].$$

Suppose that $d$ is a positive integer, and that $\gamma_1, \ldots, \gamma_n$ are elements of $A^*I$. Then the first-order Gromov-Witten invariant is

$$N_d(\gamma_1 \cdots \gamma_n) = \int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [\overline{M}_{0,n}^{\gamma}(\mathbb{P}^2, d)].$$

According to \cite[Section 4]{EK98}, we have the following interpretations:

1. Suppose that $a$ of the $\gamma_i$’s equal the class $h^2$, that $b$ of them equal the class $\hat{h}^2$, and that the remaining $c$ of them equal $h^2 \hat{h}$, where $a + b + 2c = 3d - 1$. Then the Gromov-Witten invariant is the characteristic number $N_d(a, b, c)$.
2. For all $d$ and all $\gamma_1, \ldots, \gamma_{n-1}$,

$$N_d(\gamma_1 \cdots \gamma_{n-1} \cdot 1) = 0.$$

3. If $\gamma_n$ is the class of a divisor, then for all $d$ and all $\gamma_1, \ldots, \gamma_{n-1}$,

$$N_d(\gamma_1 \cdots \gamma_n) = N_d(\gamma_1 \cdots \gamma_{n-1}) \int \gamma_n \cap [C],$$

where $[C] = d \hat{h}^2 + (2d - 2)h^2$.

Using a general element

$$\gamma = y_0T_0 + \cdots + y_5T_5$$

of $A^*I$, we define the quantum potential to be the following formal power series in $y_0, \ldots, y_5$:

$$\mathcal{N} = \sum_{d \geq 1} \frac{1}{d!} \int e_1^*(\gamma) \cup \cdots \cup e_m^*(\gamma) \cap [\overline{M}_{0,m}^{\gamma}(\mathbb{P}^2, d)],$$

where $e_1, \ldots, e_m$ are the evaluation maps. By the previous remarks

$$\mathcal{N} = \sum_{\substack{d \geq 1 \\ a+b+2c=3d-1 \\ a,b,c \geq 0}} \frac{N_d(a, b, c) y_0^a y_1^b y_2^c \exp(dy_1 + (2d - 2)y_3)}{a!b!c!}.$$

In a similar way, we define a potential associated to the stacks $\overline{M}_{0,n,2}(I, (0, 2))$. This will be a formal power series in two sets of indeterminates. Let $\delta = z_0T_0 + \cdots + z_5T_5$ be a second general element of $A^*I$. Then

$$\mathcal{R} = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_{n+1}^*(\delta) \cup e_{n+2}^*(\delta) \cap [\overline{M}_{0,n,2}(I, (0, 2))],$$

where $e_{n+1}$ and $e_{n+2}$ are evaluation at the points of ramification. As we show in \cite[Section 6]{EK98},

$$\mathcal{R} = \left\{ \frac{z_1^2}{2}(y_4^2 + y_5) + z_3 z_4 y_4 + \frac{z_3 z_5}{2} + \frac{z_2^2}{4} \right\} \exp(2y_3).$$

Similarly, we define a potential associated to the stacks $\overline{M}_{0,m+1}(\mathbb{P}^2, d) \times \overline{M}_{0,n,2}(I, (0, 2)).$
Let \( e_1, \ldots, e_m \) and \( e \) be the evaluation maps coming from the first factor; let \( f \) be evaluation at the points of ramification; let \( f_1, \ldots, f_n \) be the other evaluations coming from the second factor; let \( e_\star \) and \( f_\star \) be the maps used to create the fiber product. We define \( K \) to be

\[
\sum_{m,n \geq 0} \frac{1}{m!n!} \int e_\star^1(\gamma) \cup \cdots \cup e_\star^m(\gamma) \cup f_\star^1(\gamma) \cup \cdots \cup f_\star^n(\gamma) \cup k^\star(\delta) \cap [\overline{M}_{0,m+1}(\mathbb{P}^2, d) \times_I \overline{M}_{0,n,2}(I, (0, 2))].
\]

Then by Propositions 6.2 and 6.3 of [EK98], the three potentials are related by the differential equation

\[
(6.2) \quad K = \frac{2}{5} \sum_{s=0}^{5} \frac{\partial N}{\partial y_i} \frac{\partial R}{\partial z_{5-s}};
\]

from the explicit form (6.1) of \( R \), however, we see that the sum needs to run only from \( s = 0 \) to 2.

Proposition 6.3 also tells us how the product \( \cdot \) is related to the potential \( K \): for \( 0 \leq i, j \leq 2 \) we have

\[
T_i \cdot T_j = \sum_{s=0}^{5} \frac{\partial^3 K}{\partial y_i \partial y_j \partial z_{5-s}} T_s.
\]

Again we note that the sum needs to run only from \( s = 0 \) to 2, since the product of two elements of \( A^\star(\mathbb{P}^2) \) is a formal power series whose coefficients are likewise in \( A^\star(\mathbb{P}^2) \). Thus

\[
T_i \cdot T_j = \frac{\partial^3 K}{\partial y_i \partial y_j \partial z_5} T_0 + \frac{\partial^3 K}{\partial y_i \partial y_j \partial z_4} T_1 + \frac{\partial^3 K}{\partial y_i \partial y_j \partial z_3} T_2.
\]

We have shown that the contact product is associative. In particular \( (T_1 \cdot T_1) \cdot T_2 = T_1 \cdot (T_1 \cdot T_2) \). Equating the coefficients of \( T_0 \) on the two sides of this equation, we find that

\[
\frac{\partial^3 K}{\partial y_1 \partial y_1 \partial z_4} \frac{\partial^3 K}{\partial y_1 \partial y_2 \partial z_5} + \frac{\partial^3 K}{\partial y_2 \partial y_2 \partial z_5} + \frac{\partial^3 K}{\partial y_1 \partial y_1 \partial z_3} \frac{\partial^3 K}{\partial y_2 \partial y_2 \partial z_5} = \frac{\partial^3 K}{\partial y_1 \partial y_1 \partial z_5} \frac{\partial^3 K}{\partial y_1 \partial y_2 \partial z_4} + \frac{\partial^3 K}{\partial y_1 \partial y_2 \partial z_5} \frac{\partial^3 K}{\partial y_1 \partial y_2 \partial z_3}.
\]

Applying (6.2) throughout and simplifying, we obtain the following partial differential equation for the potential \( N \):

\[
(6.3) \quad N_{222} = \exp(2y_3) \left( N_{112}^2 - N_{111} N_{122} + 2y_4 (N_{112} N_{122} - N_{111} N_{222}) + (2y_4^2 + 2y_5) (N_{122}^2 - N_{112} N_{222}) \right).
\]

Note that if we set \( y_3 = y_5 = 0 \) we recover equation (2.95) of [FI95], and that if furthermore we set \( y_4 = 0 \) we recover equation (5.16) of [KM94].

Our equation (6.3) can be rewritten as a formula for the characteristic number \( N_d(a, b, c) \). Note we must assume that \( a \geq 3 \). In this formula \( d_1 \) and \( d_2 \) are greater than zero; thus it determines our
characteristic number if we assume we already know those for curves of lower degree.

\[(6.4)\]

\[N_d(a, b, c) = \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) N_{d_2}(a_2, b_2, c_2) \left[ d_1^2 d_2 \left( \frac{a - 3}{a_1 - 1} \right) - d_1 d_2 \left( \frac{a - 3}{a_1} \right) \right] \left( \frac{b}{b_1} \right) \left( \frac{c}{c_1} \right) \]

\[+ 2 \cdot \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) N_{d_2}(a_2, b_2, c_2) \left[ d_1^2 d_2 \left( \frac{a - 3}{a_1 - 1} \right) - d_1^2 \left( \frac{a - 3}{a_1} \right) \right] \left( \frac{b}{b_1 b_2} \right) \left( \frac{c}{c_1} \right) \]

\[+ 4 \cdot \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) N_{d_2}(a_2, b_2, c_2) \left[ d_1 d_2 \left( \frac{a - 3}{a_1 - 2} \right) - d_1 d_2 \left( \frac{a - 3}{a_1 - 1} \right) \right] \left( \frac{b}{b_1 b_2} \right) \left( \frac{c}{c_1 c_2} \right) \cdot \]

If we set \(c = 0\) we recover the recursive formula of Di Francesco and Itzykson \[\text{[FI95]}\]. They never state the formula explicitly, but it is an immediate consequence of their equation (2.95), and they have clearly used it in calculating the table of values (2.97). If we also set \(a = 3d - 1\) and \(b = 0\) we recover the formula (5.17) of \[\text{[KM94, Claim 5.2.1]}\].

Formula (5.2) by itself is not enough to determine all characteristic numbers, since we need to supply it with all those values for which \(a \leq 2\). Now it may appear that we could perhaps derive additional information by writing down the associativity equations for three other basis elements (rather than \(T_1, T_1, T_2\)) or by extracting the coefficients with respect to some other basis element (rather than \(T_0\)). But in fact all such derivations lead—if not to a triviality—to the same differential equation (6.3). We will omit the proof of this fact. To determine all characteristic numbers it is necessary to use the other relations of \[\text{[EK98, Section 7]}\].

Using the formalism of Fulton and Pandharipande \[\text{[FP96, 9 Prop.10]}\], it is possible to give a presentation of the contact cohomology ring, based on the standard presentation

\[0 \to (z^3) \to Q[z] \to A^*(\mathbb{P}^2) \to 0\]

of \(A^*(\mathbb{P}^2)\) as an algebra over \(Q\), in which \(z^i\) is sent to the class \(h^i = h \cup \cdots \cup h\). Using their arguments, we see that \(1, h, h^2, h^3\) form a \(Q[[y_0, \ldots, y_5]]\)-basis for \(Q^1 H^*(\mathbb{P}^2)\), and that we have a similar presentation for the \(Q[[y_0, \ldots, y_5]]\)-algebra \(Q^1 H^*(\mathbb{P}^2)\):

\[0 \to (z^3 - \xi_2 z^2 - \xi_1 z - \xi_0) \to Q[[y_0, \ldots, y_5]][z] \to Q^1 H^*(\mathbb{P}^2) \to 0.\]

Here \(z^i\) is sent to \(h^{*i} = h \ast \cdots \ast h\), and thus the elements \(\xi_i\) are the coefficients of \(h \ast h \ast h\) with respect to this basis. Explicitly,

\[(6.5) \quad Q^1 H^*(\mathbb{P}^2) = Q[[y_0, \ldots, y_5]][z]/\left( z^3 - \exp(2y_3) \left( N_{111} + 4y_4 N_{112} + (2y_4^2 + 2y_5) N_{122} \right) z^2 \right.\]

\[+ \left. (2N_{112} + 2y_4 N_{122}) z + N_{122} \right) - \exp(4y_3) (N_{111} N_{122} - N_{112}^2) ((2y_4^2 + 2y_5) z^2 + 2y_4) \].

If we set \(y_4 = y_5 = 0\) we recover their presentation of \(QH^*(\mathbb{P}^2)\) \[\text{[FP96, 9 Eqn.64]}\].
References

[BM96] K. Behrend and Y. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85 (1996), no. 1, 1–60.

[EK98] L. Ernström and G. Kennedy, Recursive formulas for the characteristic numbers of rational plane curves, J. Algebraic Geom. 7 (1998), 141–181.

[FI95] P. Di Francesco and C. Itzykson, Quantum intersection rings, The Moduli Space of Curves (Texel Island, 1994), Birkhäuser, Boston, MA, 1995, pp. 81–148.

[FP96] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96.

[Ful84] W. Fulton, Intersection Theory, Springer-Verlag, Berlin, 1984.

[KM94] M. Kontsevich and Y. Manin, Gromov-Witten classes, quantum cohomology and enumerative geometry, Comm. Math. Phys. 164 (1994), no. 3, 525–562.

[Kor95] M. Kontsevich, Enumeration of rational curves via torus action, The Moduli Space of Curves (Texel Island, 1994), Birkhäuser, Boston, MA, 1995, pp. 335–368.

[LT97] J. Li and G. Tian, The quantum cohomology of homogeneous varieties, J. Algebraic Geom. 6 (1997), no. 2, 269–305.

[Vis89] A. Vistoli, Intersection theory on algebraic stacks and their moduli, Inv. Math. 97 (1989), 613–670.