Percolation analysis of a disordered spinor Bose gas

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Abstract
We study the effects of an on-site disorder potential in a gas of spinor (spin-1) ultracold atoms loaded in an optical lattice corresponding to both ferromagnetic and antiferromagnetic spin-dependent interactions. Starting with a disordered spinor Bose–Hubbard model (SBHM) on a two-dimensional square lattice, we observe the appearance of a Bose glass phase using the fraction of the lattice sites having finite superfluid order parameter and non-integer local densities as an indicator. A precise distinction between different types of phases namely, superfluid, Mott insulator and Bose glass is done via a percolation analysis thereby demonstrating that a reliable enumeration of phases is possible at particular values of the parameters of the SBHM. Finally, we present the phase diagram based on the above information for both antiferromagnetic and ferromagnetic interactions.

Keywords: spinor bosons, Bose glass phase, percolation, scaling analysis

(Some figures may appear in colour only in the online journal)

1. Introduction

In a spinor Bose gas, the bosonic atoms have multiple internal degrees of freedom, namely the orientation of their spins. These additional features compared to those corresponding to the spinless case should create new possibilities for studying physical phenomena, such as quantum entanglement of spins [1], ground state properties [2], superfluidity of spin current [3], etc. However, experimental progress in spinor gases is hindered owing to the instability of the internal atomic or molecular states. The cause for the instability can be understood as follows: the excited states typically decay through spontaneous emission in time scales faster than that needed for the gas to equilibrate via collisions. However, due to huge technological advances, dilute atomic gases with internal degrees of freedom have been realized. Optically trapped spin-1 $^3$Na was among one of the first to be reported [4-8]. Progress has been rapid since then with the discovery of spin-1 $^9$Rb and spin-2 manifolds of $^8$Rb [8, 10-12]. A flurry of activity has taken place since. It is beyond the scope of the present work to elaborate on the large volume of literature that exists on the progress of the study of spinor Bose gases. There is an extensive review by Stamper-Kurn and Ueda on the subject [13].

Further, with the advent of high resolution techniques that detect single atoms [19, 20], the occupation densities of particles in different phases and their in situ tunneling dynamics can be investigated. This has led to the possibility of imaging the insulating phase of the interacting atomic species of bosons on an optical lattice with a single atom and single site resolution.

An interesting offshoot of the study of phase diagram of spinor Bose gases is to investigate the role of disorder present therein. Since the inclusion of disorder in the context of an atomic gas loaded in an optical lattice is experimentally possible via speckle or periodic potentials [14-18], investigation of its influence on the phase diagram is a pertinent task. The phase diagram of the spinor particles is in general richer owing to the different signs of the spin-dependent interactions that lead to a polar or a ferromagnetic nature of the superfluid phase [7, 21, 22].

The ‘theorem of inclusions’ [23] forbids a direct Mott insulator (MI) to superfluid (SF) transition in the presence of disorder. A Bose glass (BG) phase always intervenes in between. Since a BG phase originates from arbitrarily large, although rarely occurring, clusters of one type of phase present amidst another, and as statistically rare events are difficult to sample, elegant methods such as stochastic mean-field
theory [24–26], the quantum Monte Carlo (QMC) method [27–30], etc., were formulated and used to study the BG phase.

In the homogeneous case, the mean-field approach (MFA) [31], Gutzwiller [32] or the strong-coupling expansion [33] can accurately locate the MI-SF phase transition. While in the disordered case, neither the MFA nor the Gutzwiller approach can reduce the site-dependent Hamiltonian into a single-site Hamiltonian and thereby overestimate the location of BG-SF phase boundary. This is because, in the presence of disorder, the SF order parameter varies randomly and at the transition point there are some sites with vanishing SF order parameters, while the rest have finite ones.

To overcome the site inhomogeneities and to find the exact location of the phase boundaries, powerful numerical approaches, such as QMC [23, 27–30], SMFT [24–26] and the strong coupling expansion [33] have been applied on the disordered spin-0 (scalar) Bose–Hubbard model (BHM). In QMC, the location of the BG-SF phase transition is obtained from finite size scaling, while the MI-BG boundary is determined from the energy gap data owing to a gap in the particle–hole excitation spectra in the MI phase [23, 34]. The central idea of the SMFT approach is to compute the local SF order parameter self-consistently from a probability distribution function thereby showing a direct BG-SF phase transition [24–26].

Unlike the disordered scalar BHM, the disordered spinor BHM lacks such studies except the probabilistic mean-field perturbative approximation (PMFA), which is the simplest version of SMFT developed to overcome the site homogeneity in local-order parameters and obtains phase diagrams with the help of the Gutzwiller ansatz. These methods, however, provide no indication of the location of the transition points for the MI, BG and SF phases [35].

In this work we take a different approach where we focus on the site decoupled mean-field theory [31] and take into account the site inhomogeneities; we introduce a simple physical quantity, \( \chi \), defined as the fraction of the sites with a finite SF order parameter and a non-integer occupation density that will be used as indicators for the MI, BG or SF phases [36]. Although the location of MI-BG phase boundary can be determined depending on whether the indicator is zero or finite, for determining the extent of the BG phase, we turn our attention to the percolation analysis and finite size scaling [37].

A percolation of the insulating sites is seen to occur with patches of the superfluid order surviving, thereby maintaining a zero macroscopic superfluid order, although compressible. The percolation scenario is robustly tested and a percolation threshold, implying the onset of a SF phase, is obtained with the help of the Hoshen–Kopelman (HK) algorithm [38]. Equipped with all of the above, we map out a phase diagram for the (\( F = 1 \)) spinor bosons in a two-dimensional square (optical) lattice, where we specifically focus on the emergence and sustenance of the BG phase in both antiferromagnetic and ferromagnetic cases based only on the critical value of the indicator and compared our phase diagrams with [35].

We organize our paper as follows. In the next section, we briefly review the mean-field theory for the SBHM in the inhomogeneous case (in presence of disorder). In section 3, we present the results for the average SF order parameter and the compressibility in both the antiferromagnetic and ferromagnetic cases. Further, we study the variation of \( \chi \) in the presence of disorder and the finite size scaling properties. We obtain the phase diagram based on this information. Finally in section 4, we outline our concluding remarks.

2. Model

The behavior of ultracold atoms loaded in an optical lattice with hyperfine spin \( F = 1 \) can be well described by a SBHM [39, 40], which is written as

\[
\hat{H} = -t \sum_{\langle i,j \rangle} (\hat{a}_{i\sigma}^{\dagger} \hat{a}_{j\sigma} + \text{h.c.}) - \sum_{i} (\mu - \epsilon_i) \hat{n}_i
+ U_0 \sum_{i} \left( \hat{n}_i (\hat{n}_i - 1) - \frac{U_2}{2} \sum_{i} (\hat{s}_i^2 - 2 \hat{n}_i) \right) \tag{1}
\]

where \( t \) is the tunneling matrix, \((i, j)\) are the nearest neighbor sites, \( \mu \) is the chemical potential, \( U_0 \) describes the spin-independent on-site interaction and \( U_2 \) is the spin-dependent interaction which arises due to the difference in scattering lengths, \( a_0 \) and \( a_2 \), corresponding to \( S = 0 \) and \( S = 2 \) channels. \( U_0 \) and \( U_2 \) are related to the scattering lengths by, \( U_0 = (4 \pi \hbar^2 / M)(a_0 + 2a_2)/3 \) and \( U_2 = (4 \pi \hbar^2 / M)(a_2 - a_0)/3 \). The spin-dependent interaction, \( U_2 \), can have either positive or negative signs depending upon whether \( a_2 > a_0 \) or \( a_2 < a_0 \). \( U_2 \) is thus antiferromagnetic (where the SF phase is polar) for \( a_2 > a_0 \) and ferromagnetic (where the SF phase is ferromagnetic) for \( a_2 < a_0 \). The total spin at site \( i \) is given by, \( S_i = \hat{a}_{i\uparrow}^{\dagger} \hat{F}_{s\sigma} \hat{a}_{i\sigma}^{\dagger} \) where \( \hat{F}_{s\sigma} \) are the components of spin-1 matrices and \( \sigma = +1, 0, -1 \). The particle number operator, \( \hat{n}_i = \sum_{\sigma} \hat{n}_{i\sigma} \), \( \hat{n}_{i\sigma} = \hat{a}_{i\sigma}^{\dagger} \hat{a}_{i\sigma} \) where \( \hat{a}_{i\sigma}^{\dagger} (\hat{a}_{i\sigma}) \) is the boson creation (annihilation) operator at a site \( i \). \( \epsilon_i \) is the on-site disorder is introduced by the parameter \( i \), which is randomly chosen from a uniform probability distribution extended over \([-\Delta, \Delta]\) where \( \Delta \) is the strength of the disorder and an important parameter of our work. To find the ground state energy of the Hamiltonian in equation (1), we shall employ single site MFT (SSMFT) that decouples the system Hamiltonian into sum of the single site Hamiltonians. Now neglecting the second-order fluctuation term, the hopping part of the Hamiltonian can be decoupled as [21, 41],

\[
\hat{a}_{i\sigma}^{\dagger} \hat{a}_{i\sigma} = \langle \hat{a}_{i\sigma}^{\dagger} \rangle \hat{a}_{i\sigma} + \hat{a}_{i\sigma}^{\dagger} \hat{a}_{i\sigma} = - \langle \hat{a}_{i\sigma}^{\dagger} \rangle \langle \hat{a}_{i\sigma} \rangle \tag{2}
\]

where \( \langle \hat{\phantom{i}} \rangle \) denotes the equilibrium value of an operator. Now introducing the SF-order parameter at site \( i \) as,

\[
|\psi_{i\sigma}|^2 = \langle \hat{n}_{i\sigma} \rangle = \sqrt{\hat{n}_{i\sigma}} \xi \tag{3}
\]

where \( \psi_{i\sigma} = \sqrt{\psi_{i\sigma}^2} = \sqrt{\psi_{i\sigma}^2 + \psi_{i\sigma}^2 + \psi_{i\sigma}^2} \) and \( n_{i\sigma} \) is the SF density and \( \xi \) is a normalized spinor obeying \( \sum_{\sigma} \xi_{\sigma}^* \xi_{\sigma} = 1 \).

It is obvious that all spinors are related to each other by a gauge transformation, \( e^{i\theta} \), and spin rotations,
\( u(\alpha, \beta, \tau) = e^{-i\alpha}e^{-i\beta}e^{i\tau} \), where \((\alpha, \beta, \tau)\) are the Euler angles. The behavior of the SF-order parameter is different in the antiferromagnetic and ferromagnetic regimes and are enumerated in the following [42–44].

(i) Antiferromagnetic case: when \(U_2 > 0\), the spinor, \(\xi_\sigma\) is given by,

\[
\xi = e^{i\theta}u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{i\theta} \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-i\alpha \sin \beta} \\ \cos \beta \\ \frac{1}{\sqrt{2}} e^{i\alpha \sin \beta} \end{pmatrix}
\]

If \(\alpha = \beta = \frac{\pi}{2}\) then \(\psi_\sigma = \psi_\tau = 0, \psi_0 = 0\) or \(\alpha = 0, 2\pi\) and \(\beta = 0, \pi\) then \(\psi_\sigma = \psi_\tau = 0, \psi_0 \neq 0\).

(ii) Ferromagnetic case: when \(U_2 < 0\), \(\xi_\sigma\) is given by,

\[
\xi = e^{i\theta}u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{i\theta} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\alpha \cos \beta / 2 - i\alpha \sin \beta / 2} \\ \cos \beta / 2 \sin \beta / 2 \\ e^{i\alpha \sin \beta / 2} \end{pmatrix}
\]

If \(\beta = \pi / 2\) and \(\alpha = 0\) then \(\psi_\sigma = \psi_\tau, \psi_0 = i\psi_\sigma\).

Now, substituting the SF-order parameter in equation (1), the SBHM can be written as a sum of single-site Hamiltonians as,

\[
H = \sum_i H_{i}^{MF}
\]

where

\[
H_{i}^{MF} = -i \sum_\sigma (\phi^\dagger_\sigma \hat{a}_{i\sigma} + \phi_{i\sigma} \hat{a}^\dagger_\sigma) + i \sum_\sigma \phi^\dagger_\sigma \phi_{i\sigma} - (\mu - \epsilon_i) \hat{n}_i + \frac{U_0}{2} \hat{n}_i (\hat{n}_i - 1) + \frac{U_2}{2} (\hat{S}_i^2 - 2 \hat{n}_i)
\]

where \(\phi_{i\sigma} = \sum_j \psi_{j\sigma}\). The sum \(j\) includes all nearest neighbors at the site \(i\) of a square lattice with \(z = 4\), \(z\) being the coordination number.

To find the ground-state energy of the system, we first compute the matrix elements of the mean-field Hamiltonian \(H_{i}^{MF}\) in the site occupation number basis, \(|\hat{n}_{i\sigma}\rangle\) as,

\[
\langle \hat{n}_{i\sigma}, \hat{n}_{i\tau}|H_{i}^{MF}|\hat{n}_{i\sigma}', \hat{n}_{i\tau}'\rangle = h_{i}^{d} + h_{i}^{ad}
\]

where the \(h_{i}^{d}\) and \(h_{i}^{ad}\) part includes the matrix elements coming from the diagonal and off diagonal part of the mean-field Hamiltonian respectively. Hence we diagonalize the matrix in equation (8) to obtain the ground-state energy \(E_g(\psi_{i\sigma})\) and the eigenfunction \(\psi_g(\psi_{i\sigma})\) starting with some an estimated value for \(\psi_{i\sigma}\). Now from the updated wave function \(\psi_g(\psi_{i\sigma})\), we compute the SF-order parameter using,

\[
\psi_{i\sigma} = \langle \psi_g(\psi_{i\sigma}), \hat{a}_{i\sigma}, |\psi_g(\psi_{i\sigma})\rangle
\]

Using this new value of \(\psi_{i\sigma}\), we again reconstruct the matrix and repeat the diagonalization procedure until self-consistency is reached [31].

We confirm that the variation of the equilibrium ground-state energy, \(E_g\), with occupation number, \(n\), yields that it almost stabilizes corresponding to \(n = 7\), for which \(\langle \hat{n}_{i+}, \hat{n}_{i0}, \hat{n}_{i-}|H_{i}^{MF}|\hat{n}_{i+}', \hat{n}_{i0}', \hat{n}_{i-}'\rangle\) is a 120 × 120 matrix. The dimension is obtained by considering all possible combinations of \(\hat{n}_{i+}, \hat{n}_{i0}, \hat{n}_{i-}\) such that, \(\hat{n}_{i+} + \hat{n}_{i0} + \hat{n}_{i-} = \hat{n}_i\). The local density, \(\rho_i\) and compressibility, \(\kappa_i\) can also be computed using,

\[
\rho_i = \langle \Psi_g(\psi_{i\sigma}), \hat{a}_{i\sigma}, |\Psi_g(\psi_{i\sigma})\rangle; \quad \kappa_i = \langle \rho_i^2 \rangle - \langle \rho_i \rangle^2
\]

In the clean state, that is when \(\Delta / U_0 = 0\), the mean-field Hamiltonian is homogeneous and hence the SF-order parameter is uniform throughout the entire lattice. In the strong interaction limit \((U_0 \gg t)\), the SF-order parameter vanishes indicating that the system is in the MI regime where the atoms are forbidden from tunneling into neighboring sites and thereby are localized to individual sites with a fixed number of bosons per site. Thus the MI phase is characterized by a zero SF-order parameter and zero compressibility with a finite gap in the particle–hole excitation spectrum.

In the weak interaction limit \((t \gg U_0)\), the SF-order parameter shows a finite value and the atoms can easily tunnel between the nearest-neighbor sites. In the SF phase, atoms are delocalized over the entire lattice and the ground state is a coherent state, which in experiments is seen to form a constructive interference pattern in the momentum space. Thus the SF is characterized by a non-zero order parameter, non-integer boson density and finite compressibility with no particle–hole excitation gap in the spectrum.

In the presence of disorder, the mean-field Hamiltonian becomes inhomogeneous and hence the SF-order parameter will vary from one site to another. Further, in the pure case, the system can make a direct phase transition from MI to SF phase controlled by the system parameter \(t\) and \(U_0\). But as soon as disorder is present, the BG phase always intervenes between the SF and MI phases. The BG phase is defined as the zero SF-order parameter with no gap in the particle–hole excitation spectra. Thus the three types of phases can be characterized as

\begin{itemize}
  \item [(1)] SF phase: \(\psi_i = 0, \rho_i = \text{integer and } \kappa_i > 0\)
  \item [(2)] BG phase: \(\psi_i = 0, \rho_i = \text{integer and } \kappa_i > 0\)
  \item [(3)] MI phase: \(\psi_i = 0, \rho_i = \text{integer and } \kappa_i = 0\)
\end{itemize}

In order to see the effects of disorder, we first diagonalize the mean-field Hamiltonian matrix, equation (8) self-consistently on a two-dimensional (2D) square lattice of size \(L \times L\) where we have considered \(L = 30\) throughout our calculations (unless mentioned otherwise) to obtain the averaged value of \(\psi\) and \(\rho\) as,

\[
\bar{\psi} = \frac{1}{L^2} \sum_{i=1}^{L^2} \psi_i \quad \text{and} \quad \bar{\rho} = \frac{1}{L^2} \sum_{i=1}^{L^2} \kappa_i
\]

where sample in the subscript refer to the fact that results are averaged over different realizations of disorder. Here we have used a maximum of 40 different disorder realizations and confirmed that the fluctuations are satisfactorily equilibrated.
3. Results

In this section we shall present our results by numerically diagonalizing the mean-field Hamiltonian, in equation (8) by keeping $U_0 = 1$ and considered different values of spin-dependent interaction $U_2$. Since $U_2/U_0 = (a_2 - a_0)/(a_0 + 2a_2)$, so $U_2/U_0$ extends from -1 when $a_2 = 0$ to 0.5 when $a_0 = 0$. In our work, we choose $U_2/U_0 = 0.1$ (less than the maximum value 0.5) to study the odd and even MI lobes. It may be noted that at $U_2/U_0 \geq 0.5$, the odd MI lobes are completely consumed by the even MI lobes. For the ferromagnetic case, any value of $U_2$ in the interval [-1, 0] is seen to yield qualitatively similar results and we choose $U_2/U_0 = -0.2$.

The other important parameter is the disorder strength, $\Delta/U_0$. Since the MI phase is characterized by finite energy gap in their energy spectrum, in the atomic limit ($t = 0$) the width for the MI lobe is energy gap between the upper and lower values of the chemical potential, $\mu$ corresponding to the particle and hole excitations [45]. In the antiferromagnetic case, the width of the odd MI lobes is $E_x = \mu_+ - \mu_- = U_0 - 2U_2$, while for the even MI lobes, it is $E_x = U_0 + 2U_2$ [35]. In order to go from a gapped MI phase to a gapless BG phase, the disorder strength, $\Delta/U_0$, should be greater or equal to the width of the respective MI lobes. Since we are using a uniform probability distribution of disorder from $-\Delta$ to $\Delta$, so the critical disorder strength, $\Delta_c$, should extend as $-E_x/2$ to $+E_x/2$ that is $\Delta_c/U_0 = 0.4$ for the odd MI lobes and $\Delta_c/U_0 = 0.6$ for the even MI lobes corresponding to $U_2/U_0 = 0.1$ [35]. Similarly for the ferromagnetic case, the energy gap corresponds to $E_x = U_0 + U_2$ and hence the critical value is $\Delta_c/U_0 = 0.4$ corresponding to $U_2/U_0 = -0.2$ [35]. We have used same value of disorder strengths as used in [35] to see the closeness of our approach and present our results systematically in the following.

3.1. The behavior of the SF-order parameter and compressibility

In this section we present our results of the averaged-order parameter, $\tilde{\psi}$, and compressibility, $\tilde{\kappa}$, to characterize the MI, BG and SF phases as a function of tunneling strength $\gamma t/U_0$. In the pure case, the $\tilde{\psi}$ and $\tilde{\kappa}$ undergo a transition from zero to finite values indicating a direct MI-SF transition. That is, the system remains in the MI phase as long as the tunneling strength, $t$, is well below the critical tunneling strength $t_c$, where the latter is given by [46],

$$t_c = (1/3)(U_0 + 2U_2)[(2\rho + 3) - \sqrt{4\rho^2 + 12\rho}]$$

While in the disordered case, the BG phase appears and it tries to displace the MI phase making inroads for itself. The behaviors of $\tilde{\psi}$ and $\tilde{\kappa}$ in both antiferromagnetic and ferromagnetic regions are discussed below.

(i) Antiferromagnetic case: the spin-dependent interaction, $U_2$ for $^{23}$Na is antiferromagnetic since the atomic scattering length $a_2 > a_0$. For $U_2/U_0 > 0$, the nature of the MI-SF phase transition is either second or first order, corresponding to the MI lobes with odd or even local densities respectively, while for a scalar Bose gas, it is always a second-order phase transition.

In the MI phase, with an odd value for the local density per site, the SF-order parameter, $\psi$, and the compressibility, $\kappa$, changes continuously from zero, thereby indicating a second-order MI-SF phase transition. However, the MI phase with each site having even local density, the $\psi$ and $\kappa$ show finite jumps indicating the first-order MI-SF phase transition [21, 47].

We have also found that the three SF-order parameters assume values which are $\psi_{\uparrow\uparrow} = \psi_{\downarrow\downarrow} = 0$ and $\psi_{\uparrow\downarrow} = 0$ in the SF phase making the ground state only functions of $\bar{\psi}_{\uparrow\downarrow}$. Accordingly, the SF phase is a polar state, that was studied earlier in [21, 22, 42].

In order to visualize the different kinds of phase transition corresponding to the MI lobes with different occupation densities, we plot $\tilde{\psi}$ and $\tilde{\kappa}$ in the pure and disordered cases. We choose the values of $\mu$ in such a way so that we are at the MI lobes with local density $\rho = $ odd and even. At $\mu/U_0 = 0.4$, we are in the vicinity of the odd MI lobe with the local density $\rho = 1$ while for $\mu/U_0 = 1.4$, the local density comes out as $\rho = 2$ implying we are at the even MI lobe.

In the pure case, we find that, corresponding to $U_2/U_0 = 0.1$, the system remains in the MI phase with $\psi$ and $\kappa$ as zero till $\gamma t/U_0 = 0.16$ for the first odd MI lobe at $\mu/U_0 = 0.4$ and $\gamma t/U_0 = 0.26$ for the first even MI lobe at $\mu/U_0 = 1.4$, both of which lie below the critical tunneling strength $t_c/U_0$ (see equation (12)). Beyond this critical value, the system goes to a SF phase with non-integer local densities, $\rho$ and finite values of $\tilde{\psi}$ (see figure 1 (a)).

We shall now include the on-site disorder, $\epsilon_0$, and see how it affects the averaged SF-order parameter and compressibility. The inclusion of $\epsilon_0$ can equivalently be treated as a site-dependent chemical potential, $\mu_i (\mu_i = \mu - \epsilon_0)$. 

![Figure 1. The variation of $|\tilde{\psi}|$ and $|\tilde{\kappa}|$ at $U_2/U_0 = 0.1$ (antiferromagnetic) with disorder strengths $\Delta/U_0 = 0.3 (\times)$ and 0.5 (V), respectively, and $|\tilde{\psi}_{\uparrow\downarrow}| = 0$. The pure case ($\Delta/U_0 = 0$) is included for comparison (o). At $\mu/U_0 = 0.4$, $\tilde{\psi}$ and $\tilde{\kappa}$ show continuous variation indicating a second-order phase transition for MI lobes with $\rho = 1$ (a) and (b)). While at $\mu/U_0 = 1.4$, $\tilde{\psi}$ and $\tilde{\kappa}$ show continuous variation indicating a second-order transition for MI lobes with $\rho = 2$ at $\Delta/U_0 = 0.5$ (c)).](image-url)
Since the SF-order parameter and local densities become explicitly site-dependent, it is impossible to determine the sharp value of the tunneling strength, \( z/U_0 \), for which a MI-BG transition can be observed. The variation of averaged SF-order parameter \( \bar{\psi} \) and \( \bar{\kappa} \) for both the odd and even MI lobes for two different disorder strengths are shown in figure 1 at \( \Delta/U_0 = 0.3 \) (\( \times \)) and \( \Delta/U_0 = 0.5 \) (\( \nabla \)).

For the odd MI lobes, we found that the MI region starts to shrink due to appearance of the BG phase and the region spanned by the MI phase gradually decreases at the larger value of the disorder strength. When the disorder strength exceeds the critical value of disorder for the respective MI lobes, the BG phase completely destroys the MI phase, resulting in only the existence of the BG and SF phases as pointed out earlier. Thus at \( \Delta/U_0 = 0.5 \), the BG region extends till \( z/U_0 \approx 0.072 \) (see figures 1(a) and (b) (\( \nabla \))) as perceived from the vanishing of the average SF order, \( \bar{\psi} \), below 0.1 and finite average compressibility, \( \bar{\kappa} \).

We also study the behavior of even MI lobes in the antiferromagnetic case at different disorder strengths. In the pure case, we choose \( \mu/U_0 = 1.4 \) and we find that both \( \bar{\psi} \) and \( \bar{\kappa} \) show jumps from zero and a constant value respectively to distinct finite values signaling the first-order phase transition for the even MI lobes (see figures 1(c) and (d) (\( \circ \))). In the presence of disorder, the MI phase starts to diminish due to the appearance of the BG phase and the MI-SF phase transition continues to be first order for \( \Delta/U_0 = 0.3 \) (see figures 1(c) and (d) (\( \circ \))). While at \( \Delta/U_0 = 0.5 \) (see figures 1(c) and (d) (\( \nabla \))), the MI state still survives but the MI-SF transition becomes a second-order phase transition.

We have also investigated the behavior of the individual SF-order parameter and local density components at different disorder strengths at both values of chemical potential, namely \( \mu/U_0 = 0.4 \) and 1.4. At \( \Delta/U_0 = 0 \), it is found that the occupation densities becomes \( \rho_+ = \rho_- \approx 0.5 \) and \( \rho_0 \approx 0 \) at \( \mu/U_0 = 0.4 \) and \( \rho_+ = \rho_- = \rho_0 = 2/3 \) at \( \mu/U_0 = 1.4 \) in the MI phase while \( \rho_+ , \rho_- , \rho_0 \) behave similar to that of \( \rho \) in the SF phase satisfying the condition \( \rho = \sum \rho_i \), and correspondingly \( \bar{\psi}_s \) and \( \bar{\kappa}_s \) at \( \Delta/U_0 = 0.3 \) are shown in figure 1. The compressibility, \( \bar{\kappa}_s \), is constant in the MI phase and gradually increases in the SF phase for even MI lobes.

The above scenario is equivalent to the MI phase with fractional occupation densities for an individual spinor components, although the summed over spinor components yields an integer.

(ii) Ferromagnetic case: for \(^{87}\text{Rb}\) atoms, the experimental data yield \( a_2 < a_0 \) for the scattering lengths in the respective channels for which the spin-dependent interaction \( U_2 \leq 0 \) and the spin-1 ultracold atoms show similar behavior as that of scalar or spinless Bose gas where the MI-SF phase transition is now a second-order one.

In the ferromagnetic regime, all the three components of the SF-order parameter are non-zero in the SF phase and the ground state is now functions of \( \psi_{s1-} \) and \( \psi_0 \). In the pure case, we found that \( \psi_{s1-} = \psi_{s1+} \) and \( \psi_0 = \sqrt{2} \psi_{s1+} \) and correspondingly the SF phase is ferromagnetic [21, 22, 42].

The variation of \( \bar{\psi} \) and \( \bar{\kappa} \) are shown in figures 2(a) and (b) for both pure and disordered cases at \( U_2/U_0 = -0.2 \) and \( \mu/U_0 = 0.4 \). When \( \Delta/U_0 = 0 \), the local density, \( \rho = 1 \), \( \bar{\psi} \), and \( \bar{\kappa} \) are zero due to \( \rho = 0.13 \), \( \bar{\psi}(U_0) \) indicating the system is in the MI regime. At \( z/U_0 \), the system smoothly goes toward the SF regime with finite \( \bar{\psi} \) and \( \bar{\kappa} \) and non-integer local densities (see figures 2(a) and (b) (\( \circ \)).

As disorder is introduced, the BG phase intervenes in between the MI and SF phases. Thus the width of the MI phase at a disorder strength, \( \Delta/U_0 = 0.5 \) (see figures 2(a) and (b) (\( \nabla \))) becomes much smaller as compared to that of \( \Delta/U_0 = 0.3 \) (see figures 2(a) and (b) (\( \times \))). From figure 2, we see that at \( \Delta/U_0 = 0.5 \), the region spanned by the MI phase is completely occupied by the BG phase, resulting in only the survival of the BG and the SF phases owing to similar reasons as that of the antiferromagnetic case discussed earlier.

We also study the behavior of the individual SF-order parameter and local density components at different disorder strengths at \( \mu/U_0 = 0.4 \). At \( \Delta/U_0 = 0 \), \( \psi_+ = \psi_- = \psi_0 = 0 \) and \( \rho_+ = \rho_- \approx 0.25 \) and \( \rho_0 \approx 0.5 \) in MI phase. The compressibility, \( \bar{\kappa}_s \), is constant in the MI phase and corresponding \( \psi_{s1-} \) and \( \bar{\kappa}_s \) at \( \Delta/U_0 = 0.3 \), 0.5 are shown in figures 2(a) and (b).

3.2. Indicators of MI, BG and SF phases

In the previous section we have studied the effects of on-site disorder on the averaged SF-order parameter and the compressibility. Via these averaged quantities, it becomes very difficult to locate the precise parameter values for a MI-BG and BG-SF transition, owing to the fact that the \( \psi_\ell \) and \( \rho_\ell \) are explicitly site-dependent. Thus at the transition point there are some sites with zero SF order parameter and integer occupation densities, while other sites may have zero-SF-order parameters and non-integer occupation densities.

In order to circumvent this difficulty, we define a measurable quantity, \( \chi \), which is defined as,

\[
\chi = \frac{\text{Sites with } \psi_\ell = 0 \text{ and } \rho_\ell \neq \text{integer}}{\text{Total numbers of sites}}
\]  

The main reason to define \( \chi \) is to distinguish between the MI, BG and SF phases where depending upon the value of \( \chi \), we will be able to easily characterize the three different types of phases.
In a similar manner, we can also define the $\chi_\sigma$ involving $\psi_\sigma$ and $\rho_\sigma$ which will aid in the study of the individual spinor components.

At this point, one can ask, what will be the value of $\psi_i$ and $\rho_i$ which can define the MI or the SF phase. Since we see from the behavior of $\psi$ and $\delta$ in the pure case that the system will remain in the MI phase till $zt < zt_c$ having vanishingly small $\psi_\sigma$ values below $O(10^{-3})$ and the occupation density $\rho_i = N \pm \delta$ where $N = 0, 1, 2, 3...$ and $\delta$ is of $O(10^{-3})$. These values are set for numerical convergence of the parameters corresponding to the MI phase. Again as earlier, we include discussions on the antiferromagnetic and ferromagnetic regimes one by one.

(i) Antiferromagnetic case: for antiferromagnetic interaction with $U_2/U_0 = 0.1$, we study the the variation of $\chi$ with tunneling strength, $zt/U_0$ in both pure and disordered cases at two different values of $\mu/U_0$ corresponding to the odd and even MI lobes respectively as shown in figures 3 and 4.

In the disorder-free case for $\mu/U_0 = 0.4$, we see that $\chi$ makes direct transition from 0 to 1 emphasizing an unhindered MI-SF transition. When $zt < zt_c$, the system is in MI phase, that is all sites have $\psi_i = 0$ and $\rho_i = \text{integer}$ and thus $\chi$ takes the value 0. While for $zt > zt_c$, the system becomes a SF phase with $\psi_i = 0$ and $\rho_i = \text{integer}$ and thereby $\chi$ assumes a value 1 (see figure 3(dotted $\circ$)). As soon as disorder is included, the direct transition of $\chi$ from 0 to 1 is prohibited, as indicated by a gradual increase due to the presence of the BG phase.

In presence of disorder, that is at $\Delta/U_0 = 0.3$ (see figure 3(dashed $\circ$)), $\chi$ remains zero till $zt/U_0 = 0.133$, beyond which $\chi$ starts to increase indicating that some sites with $\rho_i = \text{integer}$ in the MI regime start evolving and the system moves towards the BG regime with $\rho_i = \text{integer}$. Further, the gradual increase of the slope of $\chi$ refers to the intrusion of the BG phase into a territory which used to be the MI phase without disorder. As disorder strength increases, that is at $\Delta/U_0 = 0.5$ (see figure 3(solid $\circ$)), we see that $\chi$ only takes finite values and ultimately go unity as a function of $zt/U_0$, indicating that the region spanned by the BG phase continues to increase and results in vanishing of the MI phase. Thus at $\Delta/U_0 = 0.5$, the system only consists of the BG and SF phases as expected, since the disorder strength $\Delta/U_0$ exceeds the energy gap for the first odd MI lobe ($\rho = 1$).

It is now interesting to study the behavior of $\chi_\sigma$ in presence of disorder. In order to study $\chi_\sigma$, we use $\psi_\sigma$ and $\rho_\sigma$ and remember that they satisfy the same condition as the total SF-order parameter, $\psi_t$ and the local density, $\rho_i$ for rendering the MI and the SF phases, respectively.

At $\mu/U_0 = 0.4$, following the definition of $\chi$, we compute $\chi_\pm$ by suitably modifying the numerator of the equation (13) with $\psi_\pm = 0$ and $\rho_\pm = 0.5$, respectively. In figure 3($\times$), we have plotted $\chi_\pm$ with $zt/U_0$. We find that $\chi_+ = \chi_-$ where they show same behavior as that of $\chi$ in the clean and disordered cases. In the antiferromagnetic case, $\psi_0$ is always zero and $\rho_0$ behaves as $\rho_i$ corresponding to the MI and SF regimes, respectively. So for determining $\chi_0$, we set the condition $\psi_{\pm} = 0$ and $\rho_{\pm} > 0.001$ in the numerator of equation (13). Its variation with respect to $zt/U_0$ is also shown in the figure 3($\times$).

Similar to the discussion above, at $\mu/U_0 = 1.4$, for $\chi_\pm$, we replace the numerator of the equation (13) with $\psi_\pm = 0$ and $\rho_\pm = 0.667 \pm 0.5$. For computing $\chi_0$, we set the condition $\psi_{\pm} = 0$ and $\rho_{\pm} = 0.667 \pm 0.5$ and the variation of $\chi$ and $\chi_\sigma$ with different disorder strengths are shown in figure 4.

From the behavior of $\chi$, we see that at $\Delta/U_0 = 0.3$ (see figure 4(dashed $\circ$)), although there is a very few values in between 0 and 1, but the region occupied by the MI phase shrinks as compared with the pure case, owing to the emergence of the BG phase. For $\Delta/U_0 = 0.5$ (see figure 4(solid $\circ$)), the BG phase appears as $\chi$ takes finite values in between 0 and 1, while the MI phase still exists since the critical disorder strength, $\Delta_i/U_0$ for the $\rho = 2$ MI lobe is a much higher as discussed in the previous section. Components of $\chi_\sigma$ show similar kinds of behavior as that of $\chi$ in the presence of disorder.

(ii) Ferromagnetic case: for ferromagnetic interactions with $U_2/U_0 = -0.2$ at $\mu/U_0 = 0.4$, the variation of $\chi_\sigma$ with tunneling strength $zt/U_0$ is shown in figure 5. In the ferromagnetic case, since the spin-1 ultracold atoms shows similar phase diagram as that of a scalar Bose gas (spinless), so there is no distinction between the odd and the even MI lobes, and thus showing results involving either will suffice.

At $\mu/U_0 = 0.4$, the MI lobes have local density $\rho_i = 1$ and $\chi$ is zero till $zt/U_0 = 0.133$. Beyond this value, the system goes to the SF phase with $\chi = 1$ in the pure case (see figure 5 (dotted $\circ$)). In the presence of disorder, that is at $\Delta/U_0 = 0.3$ (see figure 5 (dashed $\circ$)), $\chi$ remains zero till
and non-integer $\varrho_j$ for lattice size $L \times L = 50 \times 50$ for a single realization of the disorder. The parameter corresponding to the plots are $\Delta/\varrho_0 = 0.5$ and $\mu/\varrho_0 = 0.4$ in the antiferromagnetic case ($U_2/\varrho_0 = +0.1$). The light green circles are the indicators for the integer densities, while the deep red circles denote the non integer densities. The $\varrho_j$ for the MI phase at $\varrho/\varrho_0 = 0.001$ and for the BG phase at $\varrho/\varrho_0 = 0.06$ and for the SF phase at $\varrho/\varrho_0 = 0.3$.

In the clean state, the system splits into two, the MI patches with all sites having zero $\varphi_i$ (not shown here) and integer densities $\varrho_i$ as shown by light dots in figure 6(a) till $\varrho/\varrho_0 < \varrho_c/\varrho_0$ and the SF islands with finite $\varphi_i$ and non-integer densities $\varrho_i$ at $\varrho/\varrho_0 > \varrho_c/\varrho_0$ separated by a sharp boundary. As soon as disorder is introduced, the BG phase tries to mix two phases by removing the boundary between them. So the sites with zero $\varphi_i$ (not shown here) and integer densities attempt to evolve towards zero $\varphi_i$ and non-integer densities as shown in figure 6(b) (BG phase), which finally percolates for the first time towards a state with non zero $\varphi_i$ (not shown here) and non-integer densities figure 6(c), the latter being the SF phase.

To extract different phases of the system, we shall concentrate on the appearance of the SF clusters which percolate throughout the entire lattice for the first time. In the MI phase, all sites have integer particle densities and hence there will be no SF cluster, while in the BG phase, some of the sites have non-integer densities and thus the SF clusters are trapped by other clusters with sites having integer densities. Finally with the increase in the tunneling strength, the SF clusters start to percolate throughout the lattice resulting in an SF phase for the system. Thus the BG region can be identified in between the MI and the SF phases where SF clusters exist, however they do not percolate across the lattice [18].

When an SF cluster percolates through an infinite lattice, it is known as an infinite cluster, while when it percolates through a finite lattice it is known as a percolating or spanning cluster. Since we are dealing with a finite size of the system

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The real space plots of the occupation densities, $\varrho_j$ for lattice size $L \times L = 50 \times 50$ at $\varrho/\varrho_0 = 0.5$ with $U_2/\varrho_0 = +0.1$ and $\mu/\varrho_0 = 0.4$ corresponding to three different phases that are the MI, BG and SF phases of the system, respectively. The light green circles (light shades) represent integer densities and deep red circles (deep shades) represent the non integer densities. The parameter value is at $\varrho/\varrho_0 = 0.001$ in the MI phase (a) and $\varrho/\varrho_0 = 0.06$ in the BG phase (b) and $\varrho/\varrho_0 = 0.3$ for the SF phase (c), respectively.}
\end{figure}
for our numerical work, we shall aim to find a spanning cluster by using the HK algorithm [38].

Whether a SF cluster will percolate can be well understood from the following quantity, \( \chi_\infty \) which is defined as [37],

\[
\chi_\infty = \frac{\text{Sites in a spanning cluster}}{\text{Total number of occupied sites}} \quad \text{(14)}
\]

Here we briefly outline the idea used in the HK algorithm for the sake of completeness. It is based on the special application of the unified algorithm and aims to assign a label to each cluster. For that we represent our 2D lattice as a matrix and label all occupied sites initially with -1 and unoccupied sites with 0. Our cluster index starts with 1. Corresponding to each occupied site, every time we check the neighbors at the top and left corner of the current site. If both sites are empty, we label a new cluster number that has not used so far. Else, if the site has one occupied neighbor, then we assign same cluster number to the current site. If both neighboring sites are occupied, then we set the smallest number cluster label of the occupied neighbors to use as the label for the current site. To link two clusters, we create a union between both labels and set the site as the lowest of the two labels. When we burn the lattice for a second time, we collect the unions and update the lattice. In figure 7(a) we study \( \chi_\infty \) as a function of \( \chi \) at a disorder strength \( \Delta/U_0 = 0.5 \) and \( \mu/U_0 = 0.4 \) in the antiferromagnetic case. It is found that \( \chi_\infty \) is zero till \( \chi = \chi_c = 0.581 \) for a lattice size \( L \times L = 128 \times 128 \) which is close to the critical threshold value for the occupation probability \( p_c = 0.592 \) for random-site percolation problem in an infinite 2D square lattice [37] and shows finite value above \( \chi > \chi_c \). This similarity is only coincidental as the random site percolation model is a classical problem where sites are randomly occupied, while we have a system of quantum particles and the driving parameters are systematically varied at a given value of random disorder.

Another useful quantity in this regards, is the mean cluster size, \( M_{cs} \) defined as [37]-

\[
M_{cs}(\chi) = \frac{\sum_{p} p^2 s_p(\chi)}{\sum_{p} p s_p(\chi)} \quad \text{(15)}
\]

where \( p s_p \) is the number of occupied sites belonging to a \( p \)th cluster and the spanning clusters are excluded from the sum.

Since the spanning clusters are excluded from the sum, the mean cluster size continues to increase with \( \chi \) till the appearance of a spanning cluster for the first time and starts to decay immediately after that. The variation of \( M_{cs} \) with \( \chi \) at a disorder strength \( \Delta/U_0 = 0.5 \) for \( \mu/U_0 = 0.4 \) (see figure 7(b)) shows that it reaches its peak value just below \( \chi_c \) and falls off after \( \chi_c \), which confirms that the system is in the BG phase till \( \chi_c \), and beyond that it goes toward the SF phase.

To deal with the finite size effects and also to determine the percolation transition, we resort to the finite size scaling. For a given system size, \( L \), we shall now study the percolation probability \( P_{\text{perc}} \) which is the probability of having a percolating cluster with the system parameter \( \eta = \chi U_0 \).

Figure 8 (a) shows the variation of \( P_{\text{perc}} \) with different lattice sizes \( L \) at a particular disorder strength, \( \Delta/U_0 = 0.5 \) and \( \mu/U_0 = 0.4 \) in the antiferromagnetic case. We averaged our results over 1000 and 100 different realizations of disorder for \( L = 10 \) (●) and \( L = 30 \) (□), 50 (◦) system sizes respectively. \( P_{\text{perc}} \) is assumed to follow a scaling law near the critical tunneling strength, \( \eta_c \) [37] and is described by,

\[
P_{\text{perc}}(L, \eta) = \tilde{p}(L^2 (\eta - \eta_c)) \quad \text{(16)}
\]

where \( P_{\text{perc}} \) for different \( L \) values intersect approximately at one critical point given by \( \eta_c = \chi U_0 \) and \( \tilde{p} \) is the scaling function which approaches zero in \( (\eta - \eta_c) \) < 0 and unity for \( (\eta - \eta_c) \) > 0, \( \nu \) is the critical exponent which is equal to 1.33 for conventional random site percolation problem in two dimension. The finite size scaling plot, where all data corresponding to different system sizes collapse on a single curve, is shown in figure 8(b), where the corresponding critical tunneling strength is given by \( \eta_c = \chi U_0 = 0.073 \). For the sake of brevity we have not included the discussion corresponding to the ferromagnetic case.

We are now in a position to distinguish the three different types of phases and able to ascertain the critical tunneling strength, \( \eta_c = \chi U_0 \) corresponding to the MI-BG and the BG-SF phase transitions. As we have already pointed out earlier that the MI regime corresponds to \( \chi = 0 \), while the BG regime corresponds to \( 0 < \chi < \chi_c \) till \( \chi_c = 0 \) and the SF regime is given by, \( \chi_c \leq \chi \leq 1 \) when \( \chi_c \neq 0 \). For that purpose we shall tune \( \chi U_0 \) in a controlled manner to closely monitor the value of \( \chi \) and \( \chi_\infty \) for a given lattice size namely,
$L = 30$. We find that in the antiferromagnetic case (at $\mu/U_0 = 0.4$), $\chi$ is zero till $zt/U_0 = 0.122$ at $\Delta/U_0 = 0.3$, while it is non-zero at a higher disorder concentration $\Delta/U_0 = 0.5$. Thus at $\mu/U_0 = 0.4$, the onset for the MI-BG transition occurs at $zt/U_0 = 0.122$ and the BG phase extends up to $zt/U_0 = 0.133$, where $\chi_c = 0.524$ with $\chi_\infty = 0$. Hence the BG-SF transition takes place at $zt/U_0 = 0.133$ corresponding to $\Delta/U_0 = 0.3$. For $\Delta/U_0 = 0.5$, since there is no MI phase, so the BG-SF phase transition occur at $zt/U_0 = 0.073$ with $\chi_c = 0.549$.

3.4. Phase diagram

In this section we will now present the phase diagram of spinor ultracold atoms in $t - \mu$ plane for both the pure and disorder cases based upon the information obtained from $\psi$, $\chi$ and $\chi_\infty$. In the pure case ($\Delta/U_0 = 0.0$), the odd MI lobes start to shrink while the even MI lobes expanded and the color variation corresponds to the magnitude of the SF order parameter $\psi$ shown in the legend of (a). Phase diagram based on the information obtain from $\chi$ for the disordered cases, namely $\Delta/U_0 = 0.3$ is shown in (b) and $\Delta/U_0 = 0.5$ in (c).

Table 1. The transition points for the MI-BG and BG-SF phases obtained from $\chi$ and $\chi_\infty$ for both antiferromagnetic (AF) and ferromagnetic (F) interactions are presented in this table. The corresponding parameter values are mentioned in the table.

| Interactions | MI lobes | $\Delta/U_0 = 0.0$ | $\Delta/U_0 = 0.3$ | $\Delta/U_0 = 0.5$ |
|-------------|----------|-------------------|-------------------|-------------------|
| AF ($U_2/U_0 = 0.1$) | first odd | $\mu/U_0$ | $zt/U_0$ | $zt/U_0$ | $zt/U_0$ | $zt/U_0$ |
|                | 0.4      | 0.16              | 0.122             | 0.133             | No            | 0.073        |
| F ($U_2/U_0 = -0.2$) | first even | 1.4              | 0.26              | 0.210             | 0.225         | 0.123        | 0.174        |
|                | 0.4      | 0.133             | 0.085             | 0.103             | No            | 0.047        |
(i) Antiferromagnetic case: in the disorder-free case, the phase diagram of spinor ultracold atoms in the presence of antiferromagnetic interaction with $U_2/U_0 = +0.1$ is shown in figure 9(a). It shows that the MI lobes with even occupation densities are somewhat enhanced as compared to the corresponding lobes with odd occupation densities [21, 46]. The MI phase with even number of bosons per site where each pair of bosons favor the formation of singlet pairs and are likely to be more stable and keep themselves isolated from nearest neighbor tunneling. Whereas the MI phase, with odd number of bosons per site, although each pair of bosons can make singlet states, the remaining one is free to tunnel to nearest neighbor sites rendering the odd MI lobes unstable and smaller in width as compared to the even MI lobes.

In the presence of disorder, we now try to obtain our phase diagram entirely based on the information obtained from $\chi$. In the previous section, we distinguished between the three different kind of phases depending upon $\chi$ and $\chi_0$. Thus, for obtaining the phase diagram, we set $\chi = 0$ as the indicator for the MI phase, $0 < \chi < \chi_c (= 0.524)$ for the BG phase and $0.524 \leq \chi \leq 1$ for the SF phase at $\Delta/U_0 = 0.3$. Similarly for $\Delta'/U_0 = 0.5$, where $\chi_c$ is equal to 0.549. The phase diagrams for both the disorder strengths (above and below the critical values, $\Delta_c/U_0$) are shown in figures 9(b) and 9(c). At $\Delta/U_0 = 0.3$, the intervening BG region for the odd MI lobes is more noticeable as compared to the even MI lobes (see figure 9(b)) [35]. At $\Delta/U_0 = 0.5$, the BG phase completely destroys the first odd MI lobe, but the even MI lobe still exists as expected from the earlier discussion (see figure 9(c)).

(ii) Ferromagnetic case: in figure 10(a), we plot our numerically obtained phase diagram corresponding to the pure case with a ferromagnetic interaction namely, $U_2/U_0 = -0.2$. The phases bear similar characteristic features as that of scalar or spinless Bose gas. In the disordered case, as earlier, we determine the phase diagram by invoking the critical value of $\chi_c = 0.5034$ at $\Delta/U_0 = 0.3$ and $\chi_c = 0.557$ at $\Delta/U_0 = 0.5$ corresponding to the onset of the SF phase. The phase diagram corresponding to both the disorder values are shown in figures 10(b) and 10(c). For $\Delta/U_0 = 0.5$, the MI lobes disappear since the disorder strength is higher than the critical disorder strength and the system is left only with the BG and the SF phases.

4. Conclusion

We have studied the effect of on-site disorder in a two-dimensional SBHM for both the antiferromagnetic and ferromagnetic interactions. The appearance of the BG phase is observed via the average SF-order parameter $\psi$ and compressibility $\kappa$. The $\psi$ are zero while $\kappa$ are zero or constant in MI phase and gradually increase with disorder yielding signature of a BG phase. In the antiferromagnetic case, the MI-SF phase transition for odd MI lobes is always second order while for even MI lobes, it shows second-order phase transition with increasing disorder strength. We also find that the MI phase exists till the disorder strength is below the critical value, while it vanishes above the critical value.

Furthermore, the three different types of phases, namely the MI, BG and SF phases, are identified based on the concept of the SF percolating clusters. We define the MI phase where no SF cluster exists and the SF phase with at least one SF percolating cluster exists. The BG region corresponds in between the MI and SF phase where the SF cluster exists but does not percolate. We locate the transition point for the MI-BG and the BG-SF phase by calculating the probability of having a percolating cluster $\chi_c (P_{perc})$. Percolation analysis of an SF percolating cluster is studied by using the HK algorithm with different system sizes $L$ as a function of tunneling strength, $\eta = zt/\mu_0$ which obey finite size scaling law.

Finally, we obtain our phase diagram based on $\chi$ for the antiferromagnetic and ferromagnetic cases. For that purpose, we determine the critical value of $\chi$ where SF clusters percolate for the first time throughout the lattice and set the respective ranges for $\chi$ for three different phases. Thus a reliable enumeration of different phases can be obtained at desired parameter values of the SBHM Hamiltonian. Since these parameters can be precisely controlled in experiments using the properties of the laser fields which superimpose to form optical lattices, our study may have significant impact on the experimental scenario to determine these phases [48].

The transition points for the MI-BG and BG-SF phases obtained from $\chi$ and $\chi_c$ are presented in the table 1 for both antiferromagnetic (AF) and ferromagnetic (F) interactions in pure and disordered cases.

In conclusion, the phase diagrams obtained by us via the indicator $\chi$ which determine the existence of the BG phase and the extent and location of the MI, BG and SF phases as a function of $\eta = zt/\mu_0$ using percolation analysis and finite size scaling agree well with the phase diagrams obtained in [35]. Significantly, we have been able to compute the transition points between the various phases (MI, BG and SF) even at the mean-field level with reasonable accuracy, a feature that is absent in [35]. We have also studied the effect of disorder in the spin-dependent interaction terms and the corresponding phase diagrams are in good agreement with results obtained via PMFA in [35].

Now we shall include some comments concerning the drawbacks in the mean-field theory. Since our work is mainly a focus on the effects of disorder in SBHM, thus the single-site mean-field theory always is not able to handle the site inhomogeneity of the system properly and it works well in higher dimensions $d \geq 2$. Although we take into account the effect the site inhomogeneity by defining $\chi$, ignoring quadratic fluctuations introduces a tiny error in the calculations. For example, the MI phase will vanish above the critical disorder strength, but our results are in good agreement against this claim within a 3% error.

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