INDUCED AND COMPLETE MULTINETS

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Abstract. Multinets are certain configurations of lines and points with multiplicities in the complex projective plane $\mathbb{P}^2$. They appear in the study of resonance and characteristic varieties of complex hyperplane arrangement complements and cohomology of Milnor fibers. In this paper, two properties of multinets, inducibility and completeness, and the relationship between them are explored with several examples presented. Specializations of multinets plays an integral role in our findings. The main result is the classification of complete 3-nets.

1. Introduction

Multinets are certain configurations of lines and points with multiplicities in the complex projective plane $\mathbb{P}^2$. More specifically they are multi-arrangements of projective lines partitioned into three blocks with some additional combinatorial properties (see section 2). They originally arose in the study of resonance and characteristic varieties of the complement of a complex hyperplane arrangement in [4, 7]. Multinets have also appeared in the studying the cohomology of Milnor fibers in [3].

Very few examples of multinets with non-trivial multiplicities were known initially. It was observed in [4] that several of the earliest known examples satisfied an extra property which implied the underlying arrangements were $K(\pi, 1)$-arrangements. These multinets are referred to as complete multinets. More recently, a systematic method of constructing multinets was introduced in [2] and produced a variety of new examples known as induced multinets. Not all multinets are induced. In fact, two line arrangements can support the same multinet structure while not being lattice equivalent. Such arrangements are referred to as specializations of a given multinet. In the paper, we recall some definitions and known results, give examples of induced and non-induced multinets, and investigate the completeness of the induced multinets presented in [2]. Specializations of multinets plays an integral role in our findings. The main result is the classification of complete 3-nets.

The paper is organized as follows. In section 2, we recall basic definitions and relevant properties of multinets. In section 3, we exhibit examples of induced and non-induced multinets while exploring the notion of specializations of a multinet. Section 4 discusses completeness of multinets and contains the main result of the paper, the classification of complete 3-nets. Finally some open problems are listed in section 5.

Key words and phrases. net, multinet, complete multinet, induced multinet, hyperplane arrangement, $K(\pi, 1)$ arrangement.
2. Preliminaries

2.1. Pencils of curves and multinets. There are several equivalent ways to define multinets. Here we present them using pencils of plane curves. A pencil of plane curves is a line in the projective space of homogeneous polynomials from $\mathbb{C}[x_1, x_2, x_3]$ of some fixed degree $d$. Any two distinct curves of the same degree generate a pencil, and conversely a pencil is determined by any two of its curves $C_1, C_2$. An arbitrary curve $C$ in the pencil (called a fiber) is $C = aC_1 + bC_2$ where $[a : b] \in \mathbb{P}^1$. Every two fibers in a pencil intersect in the same set of points $\mathcal{X} = C_1 \cap C_2$, called the base of the pencil. If fibers do not have a common component (called a fixed component), then the base is a finite set of points.

A curve of the form $\prod_{i=1}^{l} \alpha_i^{m_i}$, where $\alpha_i$ are distinct linear forms and $m_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq q$, is called completely reducible. Such a curve is called reduced if $m_i = 1$ for each $i$. We are interested in connected pencils of plane curves without fixed components and at least three completely reducible fibers. By connectivity we mean the nonexistence of a reduced fiber whose distinct components intersect only at $\mathcal{X}$. For conciseness we refer to such a pencil as a Ceva pencil.

**Definition 2.1.** The union of all completely reducible fibers (with a fixed partition into fibers, also called blocks) of a Ceva pencil of degree $d$ is called a $(k, d)$-multinet where $k$ is the number of the blocks. The base $\mathcal{X}$ of the pencil is determined by the multinet structure and called the base of the multinet.

If the intersection of each two fibers is transversal, i.e., $|\mathcal{X}| = d^2$ and hence all blocks are reduced then the multinet is called a net. If $|\mathcal{X}| < d^2$ then we call the multinet proper. If all blocks are reduced the multinet is said to be light. If there are non-reduced blocks we call the multinet heavy. A block of a multinet is said to be pencil if all of its lines intersect at a common point.

From a projective geometry perspective, a $(k, d)$-multinet is a multi-arrangement $\mathcal{A}$ of lines in $\mathbb{P}^2$ provided with multiplicities $m(\ell) \in \mathbb{Z}_{>0}$ ($\ell \in \mathcal{A}$) and partitioned into $k$ blocks $\mathcal{A}_1, \ldots, \mathcal{A}_k$ ($k \geq 3$) subject to the following two condition.

(i) Let $\mathcal{X}$ be the set of the intersections of lines from different blocks. For each point $p \in \mathcal{X}$, the number

$$n_p = \sum_{\ell \in \mathcal{A}_i, \ell \in \mathcal{X}} m(\ell)$$

is independent on $i$. This number is called the multiplicity of $p$.

(ii) For every two lines $\ell$ and $\ell'$ from the same block, there exists a sequence of lines from that block $\ell = \ell_0, \ell_1, \ldots, \ell_r = \ell'$ such that $\ell_{i-1} \cap \ell_i \notin \mathcal{X}$ for $1 \leq i \leq r$.

Multinets can be defined purely combinatorially using an incidence relation. Note that the multiplicity $m(\ell)$ for each $\ell \in \mathcal{A}$ equals the multiplicity of its corresponding linear factor in the completely reducible fibers of the Ceva pencil. From this viewpoint, a net is a multinet with $m(\ell) = n_p = 1$ for all $\ell \in \mathcal{A}$ and $p \in \mathcal{X}$.

From a combinatorial viewpoint, $(k, d)$-nets are the realization of $k - 2$ pairwise orthogonal Latin squares of size $d$ (after identifying all blocks). If $k = 3$, the Latin square gives a multiplication of a quasi-group $G$ and the associated net is said to realize $G$. Classification of groups which can be realized by nets has been completed in [5, 6, 11].

2.2. Properties of multinets and examples. Several important properties of multinets are listed below which have been collected from [4, 10, 12].
Proposition 2.2. Let $\mathcal{A}$ be a $(k, d)$-multinet. Then:

1. $\sum_{\ell \in \mathcal{A}} m(\ell) = d$, independent of $i$;
2. $\sum_{\ell \in \mathcal{A}} \ell m(\ell) = dk$;
3. $\sum_{p \in X} h_p^2 = d^2$ (Bézout’s theorem);
4. $\sum_{p \in X \cap A} n_p = d$ for every $\ell \in \mathcal{A}$;
5. There are no multinets with $k \geq 5$;
6. All multinets with $k = 4$ are nets.

Example 2.3. A $(k, 1)$-net consists of $k$ lines intersecting all at one point with each block consisting of one line. This case corresponds to a so-called local resonance component. It is considered to be trivial and we will often tacitly assume that $d > 1$.

Example 2.4. For each $n \geq 2$, a Ceva pencil is generated by $x^n - y^n$ and $y^n - z^n$ with third completely reducible fiber given by $x^n - z^n$. They give the $(3, n)$-net which realizes $\mathbb{Z}/n\mathbb{Z}$. The lines of each block meet at a point outside of the base locus. For $n = 3$, this is one of the specializations of a Pappus arrangement (cf. subsection 3.2). Each block is pencil (i.e. all lines of the block meet at one point). Yuzvinsky showed in [11] that a $(3, n)$-net with all blocks pencil is projectively equivalent to the arrangement defined by $Q = [x^n - y^n][x^n - z^n][y^n - z^n]$.

Example 2.5. For each $n \geq 1$, a $(3, 2n)$-multinet is given by the pencil generated by polynomials $x^n(y^n - z^n)$ and $y^n(x^n - z^n)$ with the third completely reducible fiber being $z^n(x^n - y^n)$. These are the projectivizations of the reflection arrangements for the full monomial groups $G(n, 1, 3)$ (see [8]). For $n = 1$, it gives the only (up to projective isomorphism) $(3, 2)$-net of Coxeter type $A_3$; for $n = 2$, it is the $(3, 4)$-multinet of Coxeter type $B_3$. These multinets are heavy when $n > 1$.

Example 2.6. The cubics $xyz$ and $x^3 + y^3 + z^3$ generate a Ceva pencil with 4 completely reducible fibers. They give the $(4, 3)$-net known as the Hesse configuration. It is the only known multinet with 4 blocks. A long-standing conjecture is that the Hesse configuration is the unique 4-net.

2.3. Induced multinets. Few examples of multinets with non-trivial multiplicities were known initially. Then a systematic method of constructing multinets from $Q_n$ was introduced in [1, 2] and produced a variety of new examples known as induced multinets. We briefly describe the method of producing induced multinets and give a summary of their combinatorial properties.

The notion of multinets can be generalized to $\mathbb{P}^r$ ($r > 2$) by using pencils of homogeneous polynomials of $r + 1$ variables. Presently the only known multinets in $\mathbb{P}^r$ for $r > 2$ are the $(3, 2n)$-nets in $\mathbb{P}^3$ given for each $n \in \mathbb{Z}_{>0}$ by the defining polynomial

$$Q_n = [(x^n_0 - x^n_1)(x^n_2 - x^n_3)][(x^n_0 - x^n_2)(x^n_1 - x^n_3)][x^n_0 - x^n_2](x^n_1 - x^n_3)$$

where the brackets determine the blocks. This arrangement is the collection of all (projectivizations of) reflection hyperplanes of the finite complex reflection group known as the monomial group $G(n, n, 4)$ (see [8]). For $n = 2$ it is the Coxeter group of type $D_4$.

Each block of $Q_n$ is partitioned in two half-blocks (determined by parentheses) of degree $n$ each. Notice that all the planes of a half-block intersect at one line,
called the base of the half-block. For instance the base of the leftmost half-block is
given by the system $x_0 = 0, x_1 = 0$.

Multinets can be constructed as follows. Intersect $Q_n$ with a plane $H$ that does
not belong to $Q_n$. The resulting multi-arrangement in $H$ is denoted by $A^H$ and
referred to as the arrangement induced by $Q_n$. The pencil in $\mathbb{P}^3$ corresponding to $Q_n$
induces a pencil in $\mathbb{P}^2$ with 3 completely reducible fibers. It may happen that the
pencil has a fixed component. In this case, we cancel the fixed components obtaining
a smaller arrangement $A^H_0$ with a multinet structure. Abusing the notation slightly
we will call $A^H$ (if there is no fixed component) or $A^H_0$, provided with the partitions
into fibers of the induced pencil, the induced multinet.

A systematic study of the possible combinatorics of the induced multinets
obtained from $Q_n$ was performed in [1, 2]. Induced multinets from $Q_1$ are either
$(3,2)$-net realizing $\mathbb{Z}/2\mathbb{Z}$ or trivial (cf. Example 2.3). A summary of the findings
for $n > 1$ is given below. The first five cases are heavy multinets whereas the last
five cases are light multinets.

1. A heavy $(3,2n)$-multinet can have three lines of multiplicity $n$ and remaining
lines of multiplicity 1. This is projectively equivalent to the multinets realizing
$G(n,1,3)$ discussed in Example 2.5. For $n = 2$ this is the $(3,4)$-multinet of Coxeter
type $B_3$.

2. A heavy $(3,2n)$-multinet can have a unique line of multiplicity $n$ and all other
lines of multiplicity 1. The base $\mathcal{X}$ consists of two points of multiplicity $n$ and all
remaining points with multiplicity 1.

3. If $n > 1$ is even, a heavy $(3,2n)$-multinet can have three lines of multiplicity
2 and remaining lines of multiplicity 1. The base $\mathcal{X}$ consists of $3n - 3$ points of
multiplicity 2 and all remaining points of multiplicity 1. For $n = 2$ this is the
$(3,4)$-multinet of Coxeter type $B_3$.

4. If $n > 1$ is odd, a heavy $(3,2n)$-multinet can have two lines of multiplicity
2 and remaining lines of multiplicity 1. The base $\mathcal{X}$ consists of $2n - 1$ points of
multiplicity 2 and all remaining points of multiplicity 1.

5. A heavy $(3,2n)$-multinet can have an unique line of multiplicity 2 and all
other lines of multiplicity 1. The base $\mathcal{X}$ consists of $n$ points of multiplicity 2 and
all other points of multiplicity 1.

6. A light $(3,2n)$-multinet can have a unique point in $\mathcal{X}$ of multiplicity $n$. All
other points in $\mathcal{X}$ have multiplicity 1.

7. A light $(3,2n-1)$-multinet can have a unique point in $\mathcal{X}$ of multiplicity $n - 1$.
All other points in $\mathcal{X}$ have multiplicity 1. For $n = 2$ this gives a $(3,3)$-net realizing
$\mathbb{Z}/3\mathbb{Z}$ with each block in general position.

8. If $n > 2$, a light $(3,2n-2)$-multinet can have a unique point in $\mathcal{X}$ of multiplicity
$n - 2$. All other points in the base locus have multiplicity 1. For $n = 3$ this gives
a $(3,4)$-net realizing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with each block having exactly three concurrent
lines and fourth line in general position (cf. Example 2.7).
The base locus consists of the following sixteen points of multiplicity 1.

Using this choice of labels, we can see that this net realizes \( \mathbb{Z} \)-nonisomorphic intersection lattices which support the same multinet structure.}

10. A light \((3,2n)\)-multinet can be a net which realizes the dihedral group of order \(2n\).

**Example 2.7.** Intersect \( Q_4 \) by the hyperplane \( H \) defined by \( \ell_{x_0} = (\xi + 1)x_1 - \xi x_2 \) where \( \xi \) is a primitive 3rd root of unity. Then \( A^H \) has two common factors \( x_1 - x_2 \) and \( x_1 - \xi x_2 \). Canceling results in a \((3,4)\)-multinet realizing \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Using a convenient choice of labels for the lines, the three blocks are \( A_1 = \{ \ell_{11}, \ell_{12}, \ell_{13}, \ell_{14} \} \), \( A_2 = \{ \ell_{21}, \ell_{22}, \ell_{23}, \ell_{24} \} \), and \( A_3 = \{ \ell_{31}, \ell_{32}, \ell_{33}, \ell_{34} \} \) where the equations for the lines in \( (\mathbb{P}^2)^* \) are:

\[
\begin{align*}
\ell_{11} & = [0 : 1 : -1] & \ell_{21} & = [1 : 0 : -1] & \ell_{31} & = [1 : \xi^2 : \xi] \\
\ell_{12} & = [2 \xi : 1 : 0] & \ell_{22} & = [1 : 0 : -\xi] & \ell_{32} & = [1 : \xi^2 : 1] \\
\ell_{13} & = [0 : 1 : -\xi] & \ell_{23} & = [1 : 0 : -\xi^2] & \ell_{33} & = [1 : -\xi^2 : 0] \\
\ell_{14} & = [0 : 1 : -\xi^2] & \ell_{24} & = [\xi : 2 : 0] & \ell_{34} & = [\xi : 1 : 1].
\end{align*}
\]

The base locus consists of the following sixteen points of multiplicity 1.

\[
\begin{align*}
\ell_{11} \cap \ell_{21} \cap \ell_{31} & = [1 : 1 : 1] & \ell_{13} \cap \ell_{21} \cap \ell_{33} & = [1 : \xi : 1] \\
\ell_{11} \cap \ell_{22} \cap \ell_{32} & = [\xi : 1 : 1] & \ell_{13} \cap \ell_{22} \cap \ell_{34} & = [\xi : \xi : 1] \\
\ell_{11} \cap \ell_{23} \cap \ell_{33} & = [\xi^2 : 1 : 1] & \ell_{13} \cap \ell_{23} \cap \ell_{31} & = [\xi^2 : \xi : 1] \\
\ell_{11} \cap \ell_{24} \cap \ell_{34} & = [-2 \xi^2 : 1 : 1] & \ell_{13} \cap \ell_{24} \cap \ell_{32} & = [-2 : \xi : 1] \\
\ell_{12} \cap \ell_{21} \cap \ell_{32} & = [1 : -2 \xi : 1] & \ell_{14} \cap \ell_{21} \cap \ell_{34} & = [1 : \xi^2 : 1] \\
\ell_{12} \cap \ell_{22} \cap \ell_{31} & = [\xi : -2 \xi^2 : 1] & \ell_{14} \cap \ell_{22} \cap \ell_{33} & = [\xi : \xi^2 : 1] \\
\ell_{12} \cap \ell_{23} \cap \ell_{34} & = [\xi^2 : -2 : 1] & \ell_{14} \cap \ell_{23} \cap \ell_{32} & = [\xi^2 : \xi^2 : 1] \\
\ell_{12} \cap \ell_{24} \cap \ell_{33} & = [0 : 0 : 1] & \ell_{14} \cap \ell_{24} \cap \ell_{31} & = [-2 \xi^2 : \xi^2 : 1].
\end{align*}
\]

Using this choice of labels, we can see that this net realizes \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) by observing that the point \( \ell_{1i} \cap \ell_{2j} \cap \ell_{3c} \in \mathcal{X} \) appears in the associated Latin square as \( c \) in the \((i,j)\)-th position.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{bmatrix}
\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

Note that each block has exactly three concurrent lines and a fourth line in general position.

3. Specializations of Multinets and Inducibility from \( Q_n \)

Although induced multinets provides a wealth of examples of multinets, it is known that not all multinets can be obtained in this manner. For instance, every \((3,2n+1)\)-net for \( n \in \mathbb{Z}_{>0} \) is not an induced multinet. Another example of a proper multinet which is not induced is given in Problem 4 of [2]. Unlike the examples for nets, there is a specialization of this latter example which is an induced multinet from \( Q_n \). To be precise, a specialization of a multinet is any line arrangement in \( \mathbb{P}^2 \) multinet which supports the given multinet structure. A given multinet may have more than one specializations. That is, there may be line arrangements with nonisomorphic intersection lattices which support the same multinet structure.
In [9] Stipins presents a method of constructing \((k, d)\)-nets from \(k - 2\) mutually orthogonal Latin squares of order \(d\). These Latin squares contain the combinatorial data regarding the incidence relations of the associated nets and are used to generate a parameterized family of explicit defining equations. In this section, we explore the possible specializations of several multinets using results and adaptations of the techniques introduced by Stipins.

3.1. \((3, 2)\)-nets. Any \((3, 2)\)-net in \(\mathbb{P}^2\) is projectively equivalent to arrangement with defining polynomial \([x(y - z)][y(x - z)][z(x - y)]\). In particular, it consists of classes which are all in general position (which coincides with pencil for \(d = 2\)) and is associated with the Latin square

\[
\begin{bmatrix}
1 & 2 & 2 \\
2 & 1 & 1
\end{bmatrix} \cong \mathbb{Z}/2\mathbb{Z}.
\]

These nets can be induced from \(Q_1\).

3.2. \((3, 3)\)-nets. Up to isotopy, there is a unique Latin square of order 3, namely

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix} \cong \mathbb{Z}/3\mathbb{Z}.
\]

Assuming that the block \(A_1 = \{\ell_{11}, \ell_{12}, \ell_{13}\}\) is in general position, Stipins derives a family of line arrangements in \(\mathbb{P}^2\) indexed by \([s_0 : s_1] \times [t_0 : t_1] \in \mathbb{P} \times \mathbb{P}\) given by

\[
\ell_{11} = [1 : 0 : 0] \quad \ell_{21} = [1 : 1 : 1] \quad \ell_{31} = [s_0 : s_1 : s_1] \\
\ell_{12} = [0 : 1 : 0] \quad \ell_{22} = [s_0 t_1 : s_1 t_1 : s_1 t_0] \quad \ell_{32} = [t_0 : t_1 : t_0] \\
\ell_{13} = [0 : 0 : 1] \quad \ell_{23} = [s_0 t_0 : s_0 t_1 : s_1 t_0] \quad \ell_{33} = [s_0 t_1 : s_0 t_1 : s_1 t_0].
\]

This line arrangement is a \((3, 3)\)-net realizing \(\mathbb{Z}_3\) for generic index with the other blocks given by \(A_2 = \{\ell_{21}, \ell_{22}, \ell_{23}\}\) and \(A_3 = \{\ell_{31}, \ell_{32}, \ell_{33}\}\). Note that the nine lines are distinct if and only if \(s_0, s_1, t_0, t_1 \neq 0\), \(s_0 \neq s_1\), \(t_0 \neq t_1\), and \(s_0/s_1 \neq t_0/t_1\).

This family can be reindexed by the parameters \(\lambda\) and \(\mu\) by normalizing the original indices. That is, \([s_0 : s_1] = [1 : s_1/s_0] = [1 : \lambda]\) and \([t_0 : t_1] = [1 : t_1/t_0] = [1 : \mu]\) where \(\lambda, \mu \neq 0, 1\) and \(\lambda \neq \mu\). In terms of this reparameterization, the family of line arrangements can be written in \((\mathbb{P}^2)^*\) as

\[
\ell_{11} = [1 : 0 : 0] \quad \ell_{21} = [1 : 1 : 1] \quad \ell_{31} = [1 : \lambda : \lambda] \\
\ell_{12} = [0 : 1 : 0] \quad \ell_{22} = [\mu : \lambda \mu : \lambda] \quad \ell_{32} = [1 : \mu : 1] \\
\ell_{13} = [0 : 0 : 1] \quad \ell_{23} = [1 : \mu : \lambda] \quad \ell_{33} = [\mu : \mu : \lambda].
\]

Denoting points \(\ell_{11} \cap \ell_{2j} \cap \ell_{3c} \in \mathcal{X}\) as the triple \((i, j, c)\), the nine points of \(\mathcal{X}\) are

\[
(1, 1, 1) = [0 : 1 : -1] \quad (1, 2, 2) = [0 : 1 : -\mu] \quad (1, 3, 3) = [0 : \lambda : -\mu] \\
(2, 1, 2) = [1 : 0 : -1] \quad (2, 2, 3) = [\lambda : 0 : -\mu] \quad (2, 3, 1) = [\lambda : 0 : -1] \\
(3, 1, 3) = [1 : -1 : 0] \quad (3, 2, 1) = [\lambda : -1 : 0] \quad (3, 3, 2) = [\mu : -1 : 0].
\]
The intersections points within the three blocks of this family are

\[
\begin{align*}
\ell_{11} \cap \ell_{12} &= [0 : 0 : 1] \\
\ell_{11} \cap \ell_{13} &= [0 : 1 : 0] \\
\ell_{12} \cap \ell_{13} &= [1 : 0 : 0] \\
\ell_{21} \cap \ell_{22} &= [\lambda(1 - \mu) : \mu - \lambda : \mu(\lambda - 1)] \\
\ell_{21} \cap \ell_{23} &= [\lambda - \mu : 1 - \lambda : \mu - 1] \\
\ell_{22} \cap \ell_{23} &= [\lambda(\lambda - 1) : \lambda(1 - \mu) : \mu(\mu - \lambda)] \\
\ell_{31} \cap \ell_{32} &= [\lambda(1 - \mu) : \lambda - 1 : \mu - \lambda] \\
\ell_{31} \cap \ell_{33} &= [\lambda(\lambda - \mu) : \lambda(\mu - 1) : \mu(1 - \lambda)] \\
\ell_{32} \cap \ell_{33} &= [\mu(\lambda - 1) : \mu - \lambda : \mu(1 - \mu)]
\end{align*}
\]

Each block consists of three lines which can either be pencil or in general position. Recall it was assumed that the block $A_1$ is in general position. Direct computations show that the block $A_2$ is pencil if and only if $\mu\lambda^2 - 3\mu\lambda + \lambda + \mu^2 = 0$. Similarly, $A_3$ is pencil if and only if $\lambda\mu^2 - 3\lambda\mu + \mu + \lambda^2 = 0$. If $A_2$ and $A_3$ are both pencil, solving the pair of equations for $\mu$ gives $\mu = \lambda$ or $\mu = 1$. However, these values for $\mu$ do not define a (3,3)-net as noted above. On the other hand, given a generic value for $\lambda$, the equation associated with $A_2$ can be solved for $\mu$ to find parameters so that the $A_2$ is pencil and $A_3$ is in general position (see Figure 1a). For generic $\lambda$ and $\mu$, both equations will not be satisfied and gives a (3,3)-net with all blocks in general position (see Figure 1b).

The remaining possible specialization of the (3,3)-net realizing $\mathbb{Z}/3\mathbb{Z}$ consists of all blocks being pencil. This configuration is possible and was discussed in Example 2.4. We summarize our findings in the following result.

**Theorem 3.1.** Any $(3, n)$-net realizes $\mathbb{Z}/3\mathbb{Z}$ and has exactly one of the following blocks structures:

1. all blocks are in general position;
2. one blocks is pencil, two classes are in general position;
3. all blocks are pencil.
In particular, there does not exist a $(3, 3)$-net with one class generic and two classes pencil in $\mathbb{P}^2$. The specialization with all blocks in general position can be induced from $Q_2$ using cancellation. The other cases are not inducible from $Q_n$, however note that the specialization with all classes pencil appears as a subarrangement of the $(3, 6)$-net realizing the dihedral group of order 6 induced from $Q_3$.

3.3. $(3, 2n)$-multinets of type $G(n, 1, 3)$. A specialization of this multinet was presented in Example 2.5. It consists of three lines of multiplicity $n$ and $3n$ lines of multiplicity 1. The base $X$ has three points of multiplicity $n$ and $n^2$ points have multiplicity 1. This multinet contains the $(3, n)$-multinet realizing $\mathbb{Z}/n\mathbb{Z}$ with all blocks pencil as a subarrangement (see Example 2.4). Applying Yuzvinsky’s result to this subarrangement, the lines of multiplicity 1 are projectively equivalent to the arrangement defined by $[x^n - y^n][x^n - z^n][y^n - z^n]$. It remains to add the three lines of multiplicity $n$ to this subarrangement.

Observe that the three points of multiplicity $n$ are the intersection of two lines of multiplicity $n$ and $n$ lines of multiplicity 1. Thus these three points are the common intersection point of each block, namely $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$. The lines of multiplicity $n$ are the three lines which pass through exactly two of these points, namely $x$, $y$, and $z$. Taking multiplicity into account, we can make the following conclusion.

**Theorem 3.2.** Any $(3, 2n)$-multinet of type $G(n, 1, 3)$ is projectively equivalent to the arrangement with defining polynomial $[x^n(y^n - z^n)][y^n(x^n - z^n)][z^n(x^n - y^n)]$.

These multinets are inducible from $Q_n$ (see subsection 2.3).

3.4. Light $(3, 4)$-multinet with unique double point. Next we turn our attention to specializations of the light $(3, 4)$-multinets with a unique point of multiplicity 2 and all other points of multiplicity 1 in the base $X$. Two such examples are seen in Figure 2 with different block structures. The specialization with two blocks in general position depicted in Figure 2b first appeared in [4].

Here each block consists of four lines. There are three possibilities for each block: (1) the four lines are pencil; (2) the lines are in general position; or (3) exactly three of the four lines meet at a common point.

We can eliminate the first possibility with the following lemma.

**Lemma 3.3.** If a light and proper multinet has a block which is pencil, then it is the trivial multinet with $|X| = 1$.

**Proof.** Let $p$ be the common intersection point from the block which is pencil. Since the multinet is proper, $p \in X$. It follows that $p$ lies on every line from the other two blocks, hence $X = \{p\}$. This is the trivial multinet. \hfill $\Box$

It follows that each block is either (1) in general position with six double points or (2) consists of unique triple point and 3 double points. To determine the specializations possible for this multinet, we adapt Stipins methods to construct a family of line arrangements satisfying the defining intersection relationships of the multinet.

We may assume the unique point $p \in X$ with multiplicity 2 has coordinates $[0 : 0 : 1]$ in $\mathbb{P}^2$. Since no block is pencil, we may choose coordinates on $(\mathbb{P}^2)^*$
and labels for the lines in each block, namely $A_1 = \{\ell_{11}, \ell_{12}, \ell_{13}, \ell_{14}\}$, $A_2 = \{\ell_{21}, \ell_{22}, \ell_{23}, \ell_{24}\}$, and $A_3 = \{\ell_{31}, \ell_{32}, \ell_{33}, \ell_{34}\}$, so that:

$$\begin{align*}
\ell_{11} &= [1 : 1 : 1] \\
\ell_{12} &= [0 : 1 : \lambda] \\
\ell_{13} &= [0 : 1 : \mu] \\
\ell_{14} &= [x_0 : x_1 : x_2]
\end{align*}$$

$$\begin{align*}
\ell_{21} &= [1 : 0 : 0] \\
\ell_{22} &= [0 : 1 : 0] \\
\ell_{23} &= [0 : 0 : 1] \\
\ell_{24} &= [y_0 : y_1 : y_2]
\end{align*}$$

$$\begin{align*}
\ell_{31} &= [s_0 : s_1 : s_2] \\
\ell_{32} &= [0 : 1 : t] \\
\ell_{33} &= [0 : 1 : u] \\
\ell_{34} &= [v_0 : v_1 : v_2].
\end{align*}$$

Here we assumed that $p = \ell_{12} \cap \ell_{13} \cap \ell_{22} \cap \ell_{23} \cap \ell_{32} \cap \ell_{33}$. Using the incidence relations imposed by the multinet structure, we can express this family of line arrangements in terms of the two parameters $\lambda$ and $\mu$. The blocks $A_1$ and $A_2$ can be used to compute the coordinates for the twelve points of $\mathcal{X}$ with multiplicity 1 as

$$\begin{align*}
\ell_{11} \cap \ell_{21} &= [0 : 1 : -1] \\
\ell_{11} \cap \ell_{22} &= [1 : 0 : -1] \\
\ell_{11} \cap \ell_{23} &= [1 : -1 : 0] \\
\ell_{11} \cap \ell_{24} &= [y_1 - y_2 : y_2 - y_0 : y_0 - y_1] \\
\ell_{12} \cap \ell_{21} &= [0 : \lambda : -1] \\
\ell_{12} \cap \ell_{22} &= [0 : 1 : -1] \\
\ell_{12} \cap \ell_{23} &= [\lambda y_1 - y_2 : -\lambda y_0 : y_0]
\end{align*}$$

where we have

$$\begin{align*}
z_0 &= x_2 y_1 - x_1 y_2, \\
z_1 &= x_0 y_2 - x_2 y_0, \\
z_2 &= x_1 y_0 - x_0 y_1.
\end{align*}$$

Consider the line $\ell_{32}$ and its intersections with lines from $A_1$ and $A_2$. Each intersection point lies in $\mathcal{X}$. The double point $p$ is the intersection with $\ell_{12}, \ell_{13}, \ell_{22}$, and $\ell_{23}$. It follows that $\ell_{32}$ passes through either $\ell_{11} \cap \ell_{21}$ and $\ell_{14} \cap \ell_{24}$, or $\ell_{11} \cap \ell_{24}$ and $\ell_{14} \cap \ell_{21}$. We may choose our labels so that $\ell_{11} \cap \ell_{21}$ and $\ell_{14} \cap \ell_{24}$ lie on $\ell_{32}$ which implies $t = 1$. It follows that $\ell_{11} \cap \ell_{24}$ and $\ell_{14} \cap \ell_{21}$ lie on $\ell_{33}$.

Next consider the line $\ell_{22}$. A similar argument shows $\ell_{22}$ passes through either $\ell_{11} \cap \ell_{31}$ and $\ell_{14} \cap \ell_{34}$, or $\ell_{11} \cap \ell_{34}$ and $\ell_{14} \cap \ell_{31}$. We may choose our labels so that $\ell_{11} \cap \ell_{31}$ and $\ell_{14} \cap \ell_{34}$ lie on $\ell_{22}$. It follows that $\ell_{11} \cap \ell_{34}$ and $\ell_{14} \cap \ell_{31}$ lie on $\ell_{23}$. The
incidence relations for twelve points in $\mathcal{X}$ of multiplicity 1 based on these choice of labels are:

\[
\begin{align*}
\ell_{11} \cap \ell_{21} \cap \ell_{32} & \quad \ell_{12} \cap \ell_{21} \cap \ell_{31} \quad \ell_{14} \cap \ell_{21} \cap \ell_{33} \\
\ell_{11} \cap \ell_{22} \cap \ell_{31} & \quad \ell_{12} \cap \ell_{24} \cap \ell_{34} \quad \ell_{14} \cap \ell_{22} \cap \ell_{34} \\
\ell_{11} \cap \ell_{23} \cap \ell_{34} & \quad \ell_{13} \cap \ell_{21} \cap \ell_{34} \quad \ell_{14} \cap \ell_{23} \cap \ell_{31} \\
\ell_{11} \cap \ell_{24} \cap \ell_{33} & \quad \ell_{13} \cap \ell_{24} \cap \ell_{31} \quad \ell_{14} \cap \ell_{24} \cap \ell_{32}.
\end{align*}
\]

It is straightforward to use these incidence relations to express this family in terms of the parameters $\lambda$ and $\mu$. We summarize these computations below.

\[
\begin{align*}
\ell_{11} &= [1 : 1 : 1] & \ell_{21} &= [1 : 0 : 0] & \ell_{31} &= [\lambda : 1 : \lambda] \\
\ell_{12} &= [0 : 1 : \lambda] & \ell_{22} &= [0 : 1 : 0] & \ell_{32} &= [0 : 1 : 1] \\
\ell_{13} &= [0 : 1 : \mu] & \ell_{23} &= [0 : 0 : 1] & \ell_{33} &= [1 : 1 : \lambda\mu] \\
\ell_{14} &= [\lambda : 1 : \lambda\mu] & \ell_{24} &= [\lambda : 1 + \lambda : \lambda(1 + \mu)] & \ell_{34} &= [1 : 1 : \mu].
\end{align*}
\]

The base $\mathcal{X}$ consists of thirteen points. The twelve points of multiplicity 1 are

\[
\begin{align*}
\ell_{11} \cap \ell_{21} \cap \ell_{32} &= [0 : 1 : -1] & \ell_{13} \cap \ell_{21} \cap \ell_{34} &= [0 : \mu : -1] \\
\ell_{11} \cap \ell_{22} \cap \ell_{31} &= [1 : 0 : -1] & \ell_{13} \cap \ell_{24} \cap \ell_{31} &= [\lambda - \mu : \lambda\mu : -\lambda] \\
\ell_{11} \cap \ell_{23} \cap \ell_{34} &= [1 : -1 : 0] & \ell_{14} \cap \ell_{21} \cap \ell_{33} &= [0 : \lambda : -1] \\
\ell_{11} \cap \ell_{24} \cap \ell_{33} &= [1 - \lambda\mu : \lambda\mu : -1] & \ell_{14} \cap \ell_{22} \cap \ell_{34} &= [\mu : 0 : -1] \\
\ell_{12} \cap \ell_{21} \cap \ell_{31} &= [0 : \lambda : -1] & \ell_{14} \cap \ell_{23} \cap \ell_{31} &= [1 : -\lambda : 0] \\
\ell_{12} \cap \ell_{24} \cap \ell_{34} &= [\lambda - \mu : -\lambda : 1] & \ell_{14} \cap \ell_{24} \cap \ell_{32} &= [\lambda\mu - 1 : \lambda : -\mu] \\
\end{align*}
\]

and the point of multiplicity 2 is

\[
\ell_{12} \cap \ell_{13} \cap \ell_{22} \cap \ell_{23} \cap \ell_{32} \cap \ell_{33} = [1 : 0 : 0].
\]

For the twelve lines and thirteen points of $\mathcal{X}$ to all be distinct, the parameters must satisfy the following conditions: $\lambda, \mu \not= 0, 1$, $\lambda \not= \mu$, and $\lambda\mu \not= 1$. Observe that the intersection points within each block are

\[
\begin{align*}
\ell_{11} \cap \ell_{12} &= [1 - \lambda : \lambda : -1] & \ell_{22} \cap \ell_{23} &= [1 : 0 : 0] \\
\ell_{11} \cap \ell_{13} &= [1 - \mu : \mu : -1] & \ell_{22} \cap \ell_{24} &= [\lambda + \lambda\mu : 0 : -\lambda] \\
\ell_{11} \cap \ell_{14} &= [\lambda\mu - 1 : \lambda - \lambda\mu : 1 - \lambda] & \ell_{23} \cap \ell_{24} &= [1 + \lambda : -\lambda : 0] \\
\ell_{12} \cap \ell_{13} &= [1 : 0 : 0] & \ell_{31} \cap \ell_{32} &= [\lambda - 1 : \lambda : -\lambda] \\
\ell_{12} \cap \ell_{14} &= [1 - \mu : -\lambda : 1] & \ell_{31} \cap \ell_{33} &= [\mu - 1 : -\lambda\mu : 1] \\
\ell_{13} \cap \ell_{14} &= [\mu - \lambda\mu : -\lambda\mu : \lambda] & \ell_{31} \cap \ell_{34} &= [\lambda - \mu - \lambda\mu : -\lambda - 1] \\
\ell_{21} \cap \ell_{22} &= [0 : 0 : 1] & \ell_{32} \cap \ell_{33} &= [1 : 0 : 0] \\
\ell_{21} \cap \ell_{23} &= [0 : 1 : 0] & \ell_{32} \cap \ell_{34} &= [1 - \mu : -1 : 1] \\
\ell_{21} \cap \ell_{24} &= [0 : \lambda + \lambda\mu : -\lambda - 1] & \ell_{33} \cap \ell_{34} &= [\lambda\mu - \mu - \lambda\mu : 1].
\end{align*}
\]

Recall that each block is either (1) in general position with six double points or (2) consists of unique triple point and 3 double points. $A_1$ is in general position except if $\mu = 2 - \lambda$ or $\mu = \lambda/(2\lambda - 1)$; $A_2$ is in general position unless $\lambda = -1$ or $\mu = -1$; and $A_3$ is in general position unless $\mu = 1/(2 - \lambda)$ or $\mu = (2\lambda - 1)/\lambda$. All three blocks are in general position for generic choice of $\lambda$ and $\mu$. Exactly two blocks are in general position by choosing $\lambda = -1$ and a generic value of $\mu$ (see Figure 2b).

To resolve the remaining cases, choose $\lambda = -1$. Then $A_1$ is in general position unless $\mu = 3$ or $\mu = 1/3$. Either choice of values for the parameters also satisfies the relation for $A_4$, hence no blocks are in general position (see Figure 2a). It is straightforward to verify that if any two blocks are not in general position, then the third block is also not in general position. This gives us the following result.
Theorem 3.4. Any light \((3,4)\)-multinet with base locus \(\mathcal{X}\) consisting of a unique point and all other points of multiplicity 1 has exactly one of the following blocks structures:

1. all blocks are in general position;
2. two blocks are in general position, one block has exactly three lines which coincide;
3. no blocks are in general position, all block has exactly three lines which coincide.

The specialization with all blocks in general position can be induced from \(Q_2\). In addition, the specialization with no blocks in general position can be induced from \(Q_3\) with double cancellation. However, it is not possible to induce the specialization with exactly two blocks in general position.

4. Complete Multinets

A Riemann-Hurwitz type formula was obtained for multinets in [4] by calculating the Euler characteristic of the blowup of \(\mathbb{P}^2\) at the points of \(\mathcal{X}\) using the Ceva pencil. This formula can be used to determine whether all singular fibers of a Ceva pencil associated to a multinet are completely reducible. In this case, the complement of the arrangement is aspherical and the multinet is referred to as an \(K(\pi,1)\)-arrangement.

4.1. Previously known results and the classification of complete 3-nets.

We briefly summarize the previously known results. Then we present and establish our main result, namely the classification of complete 3-nets.

We begin by introducing some additional notation. Let \(\bar{\mathcal{X}}\) be the set of intersection points of \(\mathcal{A}\) not contained in \(\mathcal{X}\). For \(p \in \bar{\mathcal{X}}\), let \(m_p\) be the multiplicity of \(p\) in \(\mathcal{A}\). The next two results and subsequent definition were introduced in [4].

Theorem 4.1. Let \(\mathcal{A}\) be a \((k,d)\)-multinet, and let \(\pi : \mathbb{P}^2 \to \mathbb{P}^1\) be the associated Ceva pencil. Then

\[
3 + |\mathcal{X}| \geq (2-k)(3d-d^2 + \sum_{p \in \mathcal{X}} (n_p^2 - n_p)) + 2|\mathcal{A}| - \sum_{p \in \bar{\mathcal{X}}}(m_p - 1)
\]

with equality if and only if the blocks of \(\mathcal{A}\) form the only singular fibers of \(\pi\).

Corollary 4.2. Equality holds in (1) if and only if the restriction of \(\pi\) to the complement \(M = \mathbb{P}^2 - (\cup \mathcal{A})\) of \(\mathcal{A}\) is a smooth bundle projection with base \(B = \mathbb{P}^1 - (k \text{ points})\) and fiber a smooth surface with some points removed. In particular, \(\mathcal{A}\) is a \(K(\pi,1)\)-arrangement.

Definition 4.3. A \((k,d)\)-multinet or its associated Ceva pencil is called complete if the equality holds in (1). When \(k = 3\) this condition reduces to

\[
\sum_{p \in \mathcal{X}} (m_p - 1) \geq 2|\mathcal{A}| - |\mathcal{X}| - 3(d + 1) + \sum_{p \in \bar{\mathcal{X}}} n_p.
\]

Thus the underlying arrangement of a complete multinet is a \(K(\pi,1)\)-arrangement.

Falk and Yuzvinsky present several examples of complete multinets in [4], specifically the arrangements presented in Example 2.4, Example 2.5, and Example 2.6.

It follows that any arrangement which is lattice equivalent to the one of the arrangements defined by \([x^n-y^n][x^n-z^n][y^n-z^n], [x^n(y^n-z^n)][y^n(x^n-z^n)][z^n(x^n-y^n)]\), or the Hesse configuration is complete. Currently, these examples are the only known
examples of complete multinets. In fact, we show that the family of examples given by the Fermat pencil are the only complete 3-nets.

**Theorem 4.4.** A complete \((3, d)\)-net is projectively equivalent to the arrangement with defining polynomial \([x^d - y^d][x^d - z^d][y^d - z^d]\).

**Proof.** It suffices to show each block is pencil. Using Definition 4.3 and Proposition 2.2, the hypotheses imply the Riemann-Hurwitz type formula (2) becomes

\[
\sum_{p \in \bar{X}} (m_p - 1) = 2|A| - |X| - 3(d + 1) + \sum_{p \in \bar{X}} n_p
\]

\[
= 2(3d) - |X| - 3(d + 1) + |\mathcal{X}|
\]

\[
= 3(d - 1).
\]

If each class is pencil, then \(|\mathcal{X}| = 3\), \(m_p = d\) for each \(p \in \mathcal{X}\), and equality holds. On the other hand, consider the block \(A_i\) and select a line \(\ell_0 \in A_i\). Since \(\ell \cap \ell_0 \in \bar{X} \cap \mathcal{X}\) for each \(\ell \in A_i \setminus \{\ell_0\}\), it follows that

\[
\sum_{p \in \bar{X} \cap \ell_0} (m_p - 1) = \left( \sum_{p \in \bar{X} \cap \ell_0} m_p \right) - |\mathcal{X} \cap \ell_0|
\]

\[
= [(d - 1) + |\mathcal{X} \cap \ell_0|] - |\mathcal{X} \cap \ell_0|
\]

\[
= d - 1.
\]

Moreover, \(\mathcal{X} \cap \ell_0 \subseteq \mathcal{X} \cap \mathcal{A}_i\) for each \(i\) and \(m_p \geq 2\) for each \(p \in \mathcal{X}\), so

\[
\sum_{p \in \mathcal{X} \cap \mathcal{A}_i} (m_p - 1) \geq \sum_{p \in \mathcal{X} \cap \ell_0} (m_p - 1) = d - 1
\]

for each \(i\). It follow that

\[
\sum_{p \in \mathcal{X}} (m_p - 1) \geq 3 \cdot \min_i \left( \sum_{p \in \mathcal{X} \cap \mathcal{A}_i} (m_p - 1) \right)
\]

\[
\geq 3(d - 1).
\]

Equality holds if and only if \(\mathcal{X} \cap \ell = \mathcal{X} \cap \mathcal{A}_i\) for each \(i\), or equivalently \(\mathcal{A}_i\) is pencil for each \(i\). The result now follows directly from Proposition 3.3 in [11]. \(\square\)

### 4.2. Completeness of Induced Multinets from \(Q_n\).

The known infinite families of complete 3-multinets are both related to induced multinets from \(Q_n\). The \((3, 2n)\)-multinets of type \(G(n, 1, 3)\) are induced multinets from \(Q_n\) by choosing \(H\) to be the plane \(x_0 = 0\). Furthermore the \((3, n)\)-nets from the Fermat pencil appear as their subarrangements. On the other hand, note that the latter family of nets are not inducible directly from \(Q_n\) with one exception, namely any \((3, 2)\)-net is complete and can be induced from \(Q_1\).

With induced multinets from \(Q_n\) providing numerous examples of multinets, we investigate this class of multinets for completeness. A useful tool is the following local test for completeness presented in [4].

**Proposition 4.5.** Suppose \(A\) is a complete multinet. Then, for each \(p \in X\),

\[
2n_p - 2 = \sum_{\ell \in A_p} (m(\ell) - 1).
\]
In particular, if the multinet is light, then the multinet is complete only if it is a net.

Note that (3) holds for each \( p \in \mathcal{X} \) of a net. Also since the only proper \( k \)-multinets occur when \( k = 3 \) (see Proposition 2.2), we focus our attention on this situation. Using combinatorial properties of multinets, Proposition 4.5 can be freshly reformulated to give a local test for completeness which is convenient to implement. Let \( \mathcal{A}_p = \{ \ell \in \mathcal{A} : p \in \ell \} \) denote the lines of \( \mathcal{A} \) passing through the point \( p \).

**Corollary 4.6.** Suppose \( \mathcal{A} \) is a complete \((3,d)\)-multinet. Then, for each \( p \in \mathcal{X} \),

\[
|\mathcal{A}_p| = n_p + 2. 
\]

**Proof.** For 3-multinets, observe

\[
\sum_{\ell \in \mathcal{A}_p} (m(\ell) - 1) = 3n_p - |\mathcal{A}_p|.
\]

Substituting into (3) and simplifying gives the statement. \( \square \)

**Theorem 4.7.** The only complete multinets induced from \( Q_n \) are the \((3,2n)\)-multinets of type \( G(n,1,3) \).

**Proof.** A complete description of multinets induced from \( Q_n \) was given in section 2.3 and classified into ten types. We refer to a specific type based on the numbering conventions used there.

It follows from Theorem 4.4 and Proposition 4.5 that any light induced multinet from \( Q_n \) is not complete. It remains to investigate the completeness of the heavy induced multinets.

Induced multinets of type 1 which realize \( G(n,1,3) \) are complete. Also the induced multinet of type 3 from \( Q_2 \) realizes \( G(2,1,3) \), hence is also complete. We use Corollary 4.6 to show the remaining types are not complete by exhibiting a point \( p \in \mathcal{A} \) where (4) does not hold. For type 2, choose \( p \in \mathcal{X} \) with multiplicity \( n \) and observe \( |\mathcal{A}_p| = 2n + 1 \). Next consider an induced multinet of type 3 (with \( n > 2 \), 4, or 5). Choose \( p \in \mathcal{X} \) to be a double point which lies on exactly one of the lines of multiplicity 2. Then \( |\mathcal{A}_p| = 5 \). This completes the proof. \( \square \)

## 5. Open Problems

**Problem 1.** Are there examples of complete multinets other than the ones exhibited in Examples 2.4, 2.5, and 2.6?

**Problem 2.** How many specializations are possible for the nets realizing Latin squares of small order such as \( \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \) and Latin square of order 5 which is not isotopic to the multiplication table of a group (cf. [9])? These are nets constructed by Stipins.

**Problem 3.** Are there any general properties regarding the number of specializations of a given multinet?
References

[1] J. Bartz, *Multinets in $\mathbb{P}^2$ and $\mathbb{P}^3$*, Ph. D. thesis, University of Oregon, 2013. 2.3

[2] J. Bartz, S. Yuzvinsky, *Multinets in $\mathbb{P}^2$*, Bridging Algebra, Geometry, and Topology, Springer Proceedings in Mathematics and Statistics, vol. 96, 2014. 1, 2.3, 3

[3] G. Denham, A. Suciu, *Multinets, parallel connections, and Milnor fibrations of arrangements*, Proceedings of the London Mathematical Society 108 (2014) no. 6, 1435-1470. 1

[4] M. Falk, S. Yuzvinsky, *Multinets, Resonance Varieties, and pencils of plane curves*, Compositio Math. 143 (2007), 1069-1088. 1, 2.2, 3.4, 4, 4.1, 4.1, 4.2

[5] G. Korchmaros, G.P. Nagy, N. Pace, *k-nets embedded in a projective plane over a field*, arXiv:1306.5779. 2.1

[6] G. Korchmaros, G. Nagy, N.Pace, *3-nets realizing a group in a projective plane*, arXiv:1104.4439v3. 2.1

[7] A. Libgober, S. Yuzvinsky, *Cohomology of the Orlik-Solomon algebras and local systems*, Compositio Math. 121 (2000), 337-361. 1

[8] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, 1992. 2.5, 2.3

[9] J. Stipins, *Old and new examples of k-nets in $\mathbb{P}^2$*, math.AG/0701046. 3, 5

[10] J. Stipins, *On finite k-nets in the complex projective plane*, Ph. D. thesis, The University of Michigan, 2007. 2.2

[11] S. Yuzvinsky, *Realization of finite Abelian groups by nets in $\mathbb{P}^2$*, Compositio Math. 140 (2004), 1614–1624. 2.1, 2.4, 4.1

[12] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of curves*, Proc. AMS 137 (2009), 1641-1648. 2.2

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