Strichartz estimates for Schrödinger equation with singular and time dependent potentials and application to NLS equations

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Abstract. We establish inhomogeneous Strichartz estimates for the Schrödinger equation with singular and time-dependent potentials for some non-admissible pairs. Our work extends the results of Vilela (Trans Am Math Soc 359:2123–2136, 2007) and Foschi (J Hyperbolic Differ Equ 2:1–24, 2005), where they proved the results in the absence of potential. It also extends the works of Pierfelice (Asymptot Anal 47:1–18, 2006) and Burq et al. (J Funct Anal 203:519–549, 2003), who proved the estimates for admissible pairs. We also extend the recent work of Mizutani et al. (J Funct Anal 278:108350, 2020), and as an application of it, we improve the stability result of Kenig–Merle (Invent Math 166:645–675, 2006), which in turn establishes a proof (alternative to Yang in Commun Pure Appl Anal 20:77, 2020) of the existence of scattering solution for the energy-critical focusing NLS with inverse-square potentials.

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1. Introduction

Let us consider the following Cauchy problem for the Schrödinger equation

\[ i\partial_t u + \Delta u + Vu = F \text{ in } \mathbb{R} \times \mathbb{R}^d, \quad u(0, \cdot) = 0 \text{ on } \mathbb{R}^d \]  

(1.1)

where \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is the unknown, \( V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) is a real-valued potential and \( F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a given function. This equation plays an important role in quantum mechanics and has been studied extensively when \( V = 0 \), see [6,11,16,30,34]. In this case, since the operator \( \Delta \) is self-adjoint
in $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, by semi-group theory, the existence of the unique solution $e^{it\Delta}f$ of the corresponding \textit{homogeneous problem}

$$i\partial_t v + \Delta v = 0 \text{ in } \mathbb{R} \times \mathbb{R}^d, \quad v(0, \cdot) = f \text{ on } \mathbb{R}^d \quad (1.2)$$

is ensured for $f \in H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, in particular for $f \in L^2(\mathbb{R}^d)$. Note that, applying Fourier transform $\mathcal{F}$ to (1.2), and solving the resulting ODE for $\mathcal{F}v$ one finds $\mathcal{F}v(t) = e^{-4\pi^2 t|\xi|^2} \mathcal{F}f$. Then taking inverse Fourier transform, it follows that $e^{it\Delta}f = v(t)$ is given by $e^{it\Delta}f = M_t \mathcal{F} M_t f$ for $t \neq 0$, where $M_t w = e^{it|\cdot|^2/4t} w$, $D_t w = (4\pi t)^{-d/2} w \cdot / (4\pi t)$, see [5, Remark 2.2.5]. This formula suggests that the operators $e^{it\Delta}$, $t \neq 0$ has a lot of similarities with the Fourier transform operator $\mathcal{F}$. In fact it turns out that, $e^{it\Delta}f$ satisfies the $L^\infty-L^1$ estimates, called the \textit{dispersive estimate}

$$\|e^{\Delta} f\|_{L^\infty} \lesssim t^{-d/2}\|f\|_{L^1}, \quad t \neq 0, \quad (1.3)$$

which can be seen as a variant of the estimate $\|\mathcal{F}f\|_{L^\infty} \lesssim \|f\|_{L^1}$. Using the dispersive estimate (1.3), the \textit{inhomogeneous Strichartz estimate}

$$\|u\|_{L^q_tL^r_x} \lesssim \|F f\|_{L^\tilde{q}'_tL^{\tilde{r}'}_x} \quad (1.4)$$

is established for admissible pairs $(q,r), (\tilde{q},\tilde{r})$ with $q, \tilde{q} \neq 2$. From the special case $(q,r) = (\tilde{q},\tilde{r})$, using a duality argument the \textit{homogeneous Strichartz estimate}

$$\|e^{it\Delta} f\|_{L^q_tL^r_x} \lesssim \|f\|_{L^2} \quad (1.5)$$

is derived for admissible pair $(q,r)$ with $q \neq 2$, see [23, Chapter 4] for details. A standard scaling argument shows that

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad (1.6)$$

is necessary for the validity of the estimate (1.5). Recall that a pair of exponent $(q,r)$ is called an \textit{admissible pair} if $q, r \geq 2$, $(q,r,d) \neq (2,\infty,2)$ and the necessary condition (1.6) is satisfied.

The Strichartz estimates have many applications in solving non-linear Schrödinger (NLS) equations with various kinds of nonlinearities, e.g. power-type nonlinearity (i.e. $F = \pm |u|^{\alpha} u$), Hartree-type non-linearity (i.e. $F = \pm(|x|^{-\gamma} |u|^2) u$). Motivated by these non-linear problems (see also e.g. (NLSa)), in Sect. 4, our focus in this work is to establish Strichartz estimates for solutions to (1.1) involving some wide class of space-time spaces.

The inequalities (1.4) and (1.5) go back to 1977, when Strichartz [30] proved the special case $q = \tilde{q} = r = \tilde{r} = 2(d+2)/d$ as a Fourier restriction Theorem. Later Ginibre-Velo [11] in 1985, Yajima [34] in 1987 and Cazenave, Weissler [6] in 1988 proved (1.4), (1.5) assuming $(q,r), (\tilde{q},\tilde{r})$ are admissible pairs and $q \neq 2, \tilde{q} \neq 2$. The remaining case for admissible pairs $(q,r), (\tilde{q},\tilde{r})$ i.e. when at least one of $q, \tilde{q}$ is 2 (the endpoint case) is due to Keel and Tao [16] where they proved (1.4), (1.5) for more general settings in the year 1998.
Let us now concentrate on the inhomogeneous estimate i.e. (1.4). Again by a rescaling argument, \( q, \tilde{q}, r, \tilde{r} \) must satisfy

\[
\frac{2}{q} + \frac{2}{\tilde{q}} + d \left( \frac{1}{r} + \frac{1}{\tilde{r}} \right) = d
\]

whenever (1.4) holds. Note that the relation (1.7) is satisfied for many choices of \( q, \tilde{q}, r, \tilde{r} \) apart from those for which \((q, r), (\tilde{q}, \tilde{r})\) are admissible pairs. This indicates the possibility of (1.4) being true for non-admissible pairs. In fact, for non-admissible pairs, various authors including Cazenave, Weissler [7] in 1992, Kato [14] in 1994, Foschi [9] in 2005, Vilela [32] in 2007, Koh [22] in 2011, proved the inequality (1.4) for \( q, r, \tilde{q}, \tilde{r} \) satisfying (1.7) and other restrictions. But the problem of finding all possible exponents satisfying the estimate (1.4), is still open.

Now one question arises: what happens when \( V \) is non-zero? Assume that \((q, r), (\tilde{q}, \tilde{r})\) are admissible pairs. It follows from [16, Theorem 1.2] that, for any self-adjoint operator \( H \) in \( L^2(\mathbb{R}^d) \) satisfying the dispersive estimate

\[
\left\| e^{itH} f \right\|_{L^\infty} \lesssim t^{-d/2} \left\| f \right\|_{L^1}, \quad t \neq 0,
\]

the Strichartz estimates

\[
\left\| e^{itH} f \right\|_{L^q L^r} \lesssim \left\| f \right\|_{L^2}
\]

and (1.4) hold for \( u \) satisfying

\[
i \partial_t u + Hu = F \text{ in } \mathbb{R} \times \mathbb{R}^d, \quad u(0, \cdot) = 0 \text{ on } \mathbb{R}^d.
\]

Therefore, to have the Strichartz estimates (1.9), (1.4) for a solution to (1.10), it is sufficient to have inequality (1.8). Let us consider the case when \( H \) has the particular form \( H = \Delta + V \) where \( V : \mathbb{R}^d \to \mathbb{R} \) is a given function. This case is intensively studied, e.g. if the positive part of \( V \) is not too large, then it has been shown that \( H \) is self-adjoint, see Kato [15]. Schonbek [29] showed if \( \|V\|_{L^1 \cap L^\infty} \) is sufficiently small then the estimate (1.8) holds. It was also proved that if \( V \in C^\infty(\mathbb{R}^d) \) is non-positive and \( D^\alpha V \in L^\infty(\mathbb{R}^d) \) for all \( \alpha \geq 2 \) then (1.8) holds, see e.g. Fujiwara [10], Weinstein [33], Zelditch [37] and Oh [25].

In this work, we consider potentials \( V \) in \( L^{d/2, \infty}(\mathbb{R}^d) \) (and in \( L^\infty(\mathbb{R}, L^{d/2, \infty}(\mathbb{R}^d)) \)). Note that these \( V \)'s need not fall in the previous categories and hence the validity of the dispersive estimate (1.8) is not ensured. Therefore we possibly need a different kind of machinery to deal with such potentials. For time-independent \( V \), the operator \( \Delta + V \) is self-adjoint in \( L^2(\mathbb{R}^d) \) (via Friedrich’s extension) for the cases:

(i) \( V \) is of the form \( a/|x|^2 \) with \( a < (d-2)^2/4 \) for \( d \geq 3 \), by Hardy inequality, (or see [19, Introduction]),

(ii) \( V \) with sufficiently small \( \|V\|_{L^{d/2, \infty}} \), see [28, Section 2].

The case (i) above was studied by Burq–Planchon–Stalker–Tahvidar-Zadeh [3] in 2003. Using spherical harmonics and Hankel transforms the authors established the estimates (1.4) and (1.9) for admissible pairs \((q, r), (\tilde{q}, \tilde{r})\). On
the other hand, the case (ii) above was considered by Pierfelice [28] in 2006 to prove the inhomogeneous estimate
\[ \| u \|_{L^q L^r} \lesssim \| F \|_{L^{q'} L^{r'}} \]  \hspace{1cm} (1.11)
for admissible pairs \((q, r), (\tilde{q}, \tilde{r})\).

Note that by Calderón’s result i.e. Lemma 2.8 below, it follows that, (1.11) is stronger than (1.4) for \(2 \leq r, \tilde{r} < \infty\), see Corollary 2.9 below. The author in [28] also presented proof of the existence of a solution to (1.1) for time-dependent potentials via fixed point argument. A similar problem was studied by Bouclet and Mizutani [2] in 2018, where the authors provided estimates, for potentials in Morrey-Campanato spaces.

Here we would like to ask another question: what happens when the exponents \(q, r, \tilde{q}, \tilde{r}\) are such that \((q, r), (\tilde{q}, \tilde{r})\) are not admissible pairs? The only result according to our knowledge, answering the above two questions is the very recent (in 2020) work of Mizutani, Zhang, Zheng [24], where they proved the inhomogeneous Strichartz estimate (1.11), for some non-admissible pairs in the case (i) above, see Theorem 1.6 (ii) below. The case (ii) above is completely open for non-admissible pairs according to our knowledge. In this article, we establish the inhomogeneous estimate (1.11) for \(V\) satisfying the case (ii) above, with appropriate exponents \(1 \leq q, \tilde{q}, r, \tilde{r} \leq \infty\) for which \((q, r), (\tilde{q}, \tilde{r})\) need not be admissible, see Theorems 1.2, 1.5.

To achieve these estimates, first, we improve the result of Vilela [32, Theorem 2.4]. We would like to point out that the author in [32] proved the estimate (1.4) in the zero potential case, whereas we in the following result establish the stronger estimate (1.11):

**Theorem 1.1.** Let \(V = 0\) and \((q, r), (\tilde{q}, \tilde{r})\) satisfy (1.7), \(r, \tilde{r} > 2\) along with
\[ \begin{cases} \frac{d-2}{d} < \frac{r}{\tilde{r}} < \frac{d}{d-2}, & q, \tilde{q} \geq 2 \\ \frac{d}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right) < \frac{1}{\tilde{q}} & \text{if } r \leq \tilde{r} \\ \frac{d}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right) < \frac{1}{q} & \text{if } \tilde{r} \leq r. \end{cases} \]  \hspace{1cm} (1.12)

Then the inhomogeneous Strichartz estimate (1.11) holds for a solution \(u\) to (1.1).

Because of the scaling condition (1.7), once we fix \(r, \tilde{r}, q\), the exponent \(\tilde{q}\) is determined. Theorem 1.1 tells that the estimate (1.11) with \(V = 0\), holds on the pentagon \(ACDFE\), for some \(q\)’s, see Fig. 1. To prove Theorem 1.1 we crucially use the Lemma 3.1 below due to Vilela [32, Lemma 2.2]. It is worth noticing here that the author in [32] also presented some negative result, namely, it was shown that if (1.7) is satisfied and \(1/r + 1/\tilde{r} < (d-2)/d\) or if \(1/r, 1/\tilde{r}\) is outside the pentagon \(ACD'F'EF\) in Fig. 1, then the estimate (1.4) (and hence (1.11)) does not hold for \(V = 0\).

As mentioned earlier, The above result is used to go from zero potential to non-zero potential case by using perturbation technique, incorporated from [28], followed by interpolations for mixed Lebesgue/Lorentz spaces. By \(2^*, p^*\) we mean the standard Sobolev conjugate \(2d/(d-2)\) of 2 and the number
For \( d = 1 \), symmetry with respect to the line 1

\[ L_{d/2,\infty} \]

estimates for the unperturbed equation). For the region DEFG i.e. the second

Our results Theorems 1.2 and 1.5 below extend the results of Pierfelice [28, Theorems 1, 3], Cazenave, Weissler [7], Kato [14], and Vilela [32, Theorem 2.4].

By symmetry of the problem we conclude that if the estimate (1.11) is true for \((q,r) = (q_0, r_0), (\hat{q}, \hat{r}) = (\tilde{q}_0, \tilde{r}_0)\) then the estimate (1.11) is also true for \((q,r) = (\tilde{q}_0, \tilde{r}_0), (\hat{q}, \hat{r}) = (q_0, r_0)\) provided \(\Delta + V\) is self-adjoint. In that case, the plot of \((1/r, 1/\hat{r})\), for which the estimate (1.11) holds for some \(q\), becomes symmetric with respect to the line \(1/r = 1/\hat{r}\), in \(1/r\) verses \(1/\hat{r}\) coordinate, see

\[ p(d - 1)/(d - 2) \] (for \(d \geq 3\)) respectively. Set \(2^*_s = (2^*)_s\) and note that \(2 < 2^*_s < 2^* < 2^*_s\). Here is our main result, answering the questions asked earlier:

**Theorem 1.2.** Let \(d \geq 3\) and \((q,r), (\hat{q}, \hat{r})\) satisfy (1.7), (1.12) and

\[
\begin{aligned}
&2^*_s < r < 2^*_s, q = 2 \text{ (the region BCDE)} \\
d (\frac{1}{r} - \frac{1}{2r^*}) < \frac{1}{q} &\quad \text{for } 2^*_s < r \leq 2^*, \text{ or } \\
&d (\frac{1}{r} - \frac{1}{2r^*}) < \frac{1}{q} \quad \text{for } 2^*_s < \hat{r} \leq 2^*, \tag{1.13}
\end{aligned}
\]

Let \(V\) be a real-valued potential with \(c_s\left(\frac{d}{2r + d}\right)'||V||_{L^{d/2,\infty}}\) (or \(c_s\left(\frac{d}{2\hat{r} + d}\right)'||V||_{L^{d/2,\infty}}\) respectively) \(< 1\) (here \(c_s\) is the constant appearing in the Strichartz estimates for the unperturbed equation). For the region DEFG i.e. the second set of conditions in (1.13), we further assume, \(||V||_{L^{d/2,\infty}}\) is so small that, \(\Delta + V\) is self-adjoint. Then the inhomogeneous Strichartz estimate (1.11) holds for a solution \(u\) to (1.1).

Moreover, a similar result holds for time-dependent potential in the region BCDE, if \(V\) satisfies the smallness condition \(c_s\left(\frac{d}{2r + d}\right)'||V||_{L^{\infty,L^{d/2,\infty}}} < 1\).

**Remark 1.3.** Our results Theorems 1.2 and 1.5 below extend the results of Pierfelice [28, Theorems 1, 3], Cazenave, Weissler [7], Kato [14], and Vilela [32, Theorem 2.4].
Fig. 1. Therefore it is enough to prove this result for the first set of conditions in \((1.13)\), as we impose further smallness of \(\|V\|_{L^{d/2,\infty}}\) for the region \(DEFG\) so that \(\Delta + V\) becomes self-adjoint.

**Remark 1.4.** Interpolating the results in the two regions mentioned in \((1.13)\), we would derive that the estimate \((1.11)\) also holds in the triangular region \(BGH\).

Now interpolating Theorem 1.2 (with \(r = 2^*\)) and the result of Pierfelice [28, Theorems 1, 3] we conclude the following:

**Theorem 1.5.** Let \(d \geq 3\) and \(V\) be a real-valued potential with \(c_d 2^* \|V\|_{L^{d/2,\infty}} < 1\). Then the inhomogeneous Strichartz estimate \((1.11)\) holds for a solution \(u\) to \((1.1)\) provided \((q, r), (\tilde{q}, \tilde{r})\) satisfy the scaling condition \((1.7)\) and one of the following

(i) \(\frac{d}{2} \left(\frac{1}{q} - \frac{1}{r}\right) < \frac{1}{2}, \quad \frac{d}{2} \left(\frac{1}{\tilde{q}} - \frac{1}{\tilde{r}}\right) < \frac{d-1}{d} \left(\frac{1}{2} - \frac{1}{r}\right), \quad 2 < \tilde{r} \leq r \leq 2^*\)

(ii) \(\frac{d}{2} \left(\frac{1}{q} - \frac{1}{r}\right) < \frac{1}{2}, \quad \frac{1}{q} + \frac{1}{\tilde{q}} > \frac{1}{2}, \quad 2 \leq r \leq 2^*, 2 < \tilde{r} \leq 2^*\).

Moreover, a similar result holds for time-dependent potential, if \(c_d 2^* \|V\|_{L^{\infty, L^{d/2,\infty}}} < 1\).

From Theorems 1.2, 1.5 and Remark 1.4, we conclude that if \(\|V\|_{L^{\infty, L^{d/2,\infty}}}\) is small enough, then the estimate \((1.11)\) holds for \(1/r, 1/\tilde{r}\) in the pentagonal region \(ACDEF\) (in Fig. 1, the line \(DE\) is excluded), with some \(q\)'s, for which the pairs \((q, r), (\tilde{q}, \tilde{r})\) need not be admissible.

Next, we state the inhomogeneous estimates for inverse-square potentials. Note that the first two results are from [3] and [24] and we derive the third case as a generalization of the first two cases.

**Theorem 1.6.** Let \(d \geq 3\), \(a \in (-\infty, \frac{(d-2)^2}{4})\), \(V = \frac{a}{|\cdot|^2}\) and \(0 < \gamma, \tilde{\gamma} \leq 1\). Then the Strichartz estimate

(i) \((1.4)\), \((1.9)\) holds for admissible pairs \((q, r), (\tilde{q}, \tilde{r})\),

(ii) \((1.11)\) holds for \(q = \tilde{q} = 2, r = \frac{2d}{d-2s}, \tilde{r} = \frac{2d}{d-2(2-s)}\) provided \(s \in A_{a,1}\),

(iii) \((1.4)\) holds for \(q = 2\gamma, \tilde{q} = 2\gamma, r = \frac{2d}{d-2(s+\gamma-1)}, \tilde{r} = \frac{2d}{d-2(\gamma-1-s)}\) provided \(s \in A_{a,\gamma,\tilde{\gamma}}\),

where \(A_{a,\lambda} = \left(1 - \frac{d-2}{2(d-1)}\lambda, 1 + \frac{d-2}{2(d-1)}\lambda\right) \cap R_{a,\lambda}, (0 < \lambda \leq 1)\) and \(R_{a,\lambda}\) is given by

\[
R_{a,\lambda} = \begin{cases} 
(1 - \frac{d}{2}, 1 + \frac{d}{2}), & \text{if } \sqrt{\frac{(d-2)^2}{4} - a} > \frac{1}{2} \\
\left(1 - \frac{(d-2)^2 - 4a}{2(2+4a-(d-2)^2)}, 1 + \frac{(d-2)^2 - 4a}{2(2+4a-(d-2)^2)}\lambda\right), & \text{if } 0 < \sqrt{\frac{(d-2)^2}{4} - a} \leq \frac{1}{2}.
\end{cases}
\]

Part (iii) of the above result can be rewritten in relatively less complicated way as follows: (1.4) holds provided \(q, \tilde{q}, r, \tilde{r}\) satisfy \((1.7)\), \(q, \tilde{q} \geq 2\) and \(|d\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{2}{q} - \frac{2}{\tilde{q}}| < \frac{2}{qq} \min\left\{\frac{2}{q}, \frac{(d-2)^2 - 4a}{2+4a-(d-2)^2}\right\}\). Now we compare the Theorems 1.2, 1.5 with Theorem 1.6: As inverse square potentials belong to \(L^{d/2,\infty}(\mathbb{R}^d)\), Theorems 1.2 and 1.5 are applicable to potentials of the form \(a/|\cdot|^2\), with \(|a|\) sufficiently small. Since \(\{a/|\cdot|^2 : a \in \mathbb{R}\} \subset L^{d/2,\infty}(\mathbb{R}^d)\),
Theorems 1.2 and 1.5 cover larger collection of potentials than Theorem 1.6\(^1\) when \(|a|\) is sufficiently small. Another advantage of Theorems 1.2, 1.5 is they accommodate time-dependent potentials which Theorem 1.6 does not cover. Note that we can take \((1/r, 1/\tilde{r})\) close to \((1/2^*, 1/2)\) in the shaded region in figure 1 so that estimate (1.11) holds (for certain choices of \(q\)) using Theorem 1.5, but such choice of \(r, \tilde{r}\) is not applicable to Theorem 1.6\(^2\). On the other hand, there are exponents which are in acceptable range of Theorem 1.6 but not covered by Theorems 1.2, 1.5 (e.g. one can choose \(r = \infty\) in Theorem 1.6 but it is not applicable to Theorems 1.2, 1.5). A big advantage of Theorem 1.6 is that, it covers all \(a/|·|^2\) with \(a < (d - 2)^2/4\) even if \(|a|\) is large. Thus Theorem 1.5 (together with Theorem 1.2) and Theorem 1.6 have edge over one another in the acceptable range of various parameters.

As an application of Theorem 1.6 (iii), we obtain a Long time perturbation result with inverse square potential (see Theorem 4.4), improving the result by Kenig–Merle [18, Theorem 2.14]. This, in turn, gives a proof (an alternative to [36]) of the scattering result for focusing energy-critical NLS with inverse-square potential, see Theorem 4.7.

We organize the material as follows: in Sect. 2 the notations and some known results are mentioned, in Sect. 3 we present the proofs of results. In the end, in Sect. 4, we provide the Long time perturbation result and its application to NLS.

2. Preliminaries

2.1. Notations

Throughout this article we denote by \(∥·∥\) and \(⟨·, ·⟩\) the \(L^2(\mathbb{R}^d)\) norm and inner product respectively unless otherwise specified.

By \(l^q_{β}\), we denote the weighted sequence space \(L^q(\mathbb{Z}, 2^β dj)\), where \(dj\) stands for counting measure.

The Lorentz space is the space of all complex-valued measurable functions \(f\) such that \(∥f∥_{L^{r,s}(\mathbb{R}^d)} < \infty\) where \(∥f∥_{L^{r,s}(\mathbb{R}^d)}\) is defined by
\[
∥f∥_{L^{r,s}(\mathbb{R}^d)} := r^{1/s}\left\| tμ\{|f| > t\}^{r/s}\right\|_{L^s((0,\infty), \frac{dt}{t})}
\]
with \(0 < r < \infty\), \(0 < s \leq \infty\) and \(μ\) denotes the Lebesgue measure on \(\mathbb{R}^d\).

\[
∥f∥_{L^{r,s}(\mathbb{R}^d)} = \begin{cases} r^{1/s} \left( \int_0^{\infty} t^{s-1}μ\{|f| > t\}^{s/r} dt \right)^{1/s} & \text{for } s < \infty \\
\sup_{t>0} tμ\{|f| > t\}^{1/s} & \text{for } s = \infty. \end{cases}
\]

\(^1\)The original version of Theorem 1.6 (ii) i.e. [24, Theorem 1.3] covers more general potential \(V\) of the form \(V(x) = v(θ)r^{-2}\) where \(r = |x|, θ = x/|x|, v ∈ C^1(\mathbb{S}^{d-1})\) but this is also a proper subclass of \(L^{d/2, ∞}(\mathbb{R}^d)\).

\(^2\)This can be proved by noting that \(A_{α,γ\tilde{γ}} = (1 - \frac{d-2}{2(d-1)} γ\tilde{γ}, 1 + \frac{d-2}{2(d-1)} γ\tilde{γ})\) for sufficiently small \(|a|\). For this choice of \(r, \tilde{r}\), one has \(γ\tilde{γ} \sim (s - 1)(2 - s)\). But \(s ∉ A_{α,γ\tilde{γ}}\).
For an interval $I \subset \mathbb{R}$ the norm of the space-time Lebesgue space $L^q(I, L^r(\mathbb{R}^d))$ is defined by $\|u\|_{L^q(I, L^r(\mathbb{R}^d))} := \left( \int_I \|u(t)\|_r^q dt \right)^{1/q}$. Similarly $L^q(I, L^{r,s}(\mathbb{R}^d))$ is defined. We write $\|u\|_{L^q(I, L^r(\mathbb{R}^d))}$ for $\|u\|_{L^q(I, L^r(\mathbb{R}^d))}$ and $\|u\|_{L^q(I, L^{r,s}(\mathbb{R}^d))}$ for $\|u\|_{L^q(I, L^{r,s}(\mathbb{R}^d))}$. By $\| \cdot \|_{S(I)}$ we denote

$$\|u\|_{S(I)} = \|u\|_{L^2(\mathbb{R}^d)} \left( I, L^2(\mathbb{R}^d) \right) , \quad \|u\|_{W(I)} = \|u\|_{L^2(\mathbb{R}^d)} \left( I, L^2(\mathbb{R}^d) \right).$$

The Fourier transform $\mathcal{F}f$ or $\hat{f}$ of a function $f \in L^1(\mathbb{R}^d)$ is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx$. Note that it has a unique extension as an operator in $S'(\mathbb{R}^d)$ with the property that for each $u \in S'(\mathbb{R}^d), \langle \mathcal{F}u, \varphi \rangle = \langle u, \mathcal{F}\varphi \rangle$ for all $\varphi \in S(\mathbb{R}^d)$.

The Sobolev conjugate $2d/(d-2)$ of 2 is denoted by $2^*$. We set $p_* = p(d-1)/(d-2)$ (for $d \geq 3$) and $2^*_* = (2^*)_\ast$. By $a \vee b$ we mean $\max\{a, b\}$ and $a \lesssim b$ stands for $a \leq cb$ for some (universal) constant $c$.

### 2.2. Sobolev spaces adapted with inverse-square potentials

We denote the Homogeneous Sobolev spaces by $\dot{W}^{s,p}(\mathbb{R}^d)$ which is defined as the completion of $C_c^\infty(\mathbb{R}^d)$ with the norm

$$\|u\|_{\dot{W}^{s,p}} = \|(-\Delta)^{s/2} u\|_{L^p} , \quad u \in C_c^\infty(\mathbb{R}^d).$$

where $(-\Delta)^{s/2} u(\xi) = (2\pi)^s |\xi|^s \hat{u}(\xi)$. We will denote $\dot{H}^s(\mathbb{R}^d) = \dot{W}^{s,2}(\mathbb{R}^d)$.

Consider the operator $\mathcal{L}_a := \Delta + a/|x|^2$ (defined on $L^2$ by standard Friedrichs extension) which is self-adjoint in $L^2(\mathbb{R}^d)$ for $a < (d-2)^2$. Let us denote the unitary group of operators $\{e^{it\mathcal{L}_a}\}_{t \in \mathbb{R}}$ in $L^2(\mathbb{R}^d)$ by $\{S_a(t)\}_{t \in \mathbb{R}}$.

We define as before: $\dot{W}^{s,p}_a(\mathbb{R}^d)$ is the completion of $C_c^\infty(\mathbb{R}^d)$ with $\| \cdot \|_{\dot{W}^{s,p}_a}$, where

$$\|u\|_{\dot{W}^{s,p}_a} = \|(-\mathcal{L}_a)^{s/2} u\|_{L^p} , \quad u \in C_c^\infty(\mathbb{R}^d)$$

and $\dot{H}^s_a(\mathbb{R}^d) = \dot{W}^{s,2}_a(\mathbb{R}^d)$. We often use $\| \cdot \|_{\dot{H}^1}, \| \cdot \|_{\dot{H}^1_a}$ for $\| \cdot \|_{\dot{H}^1(\mathbb{R}^d)}, \| \cdot \|_{\dot{H}^1_a(\mathbb{R}^d)}$ respectively.

The fractional derivative $D^\alpha u$ (or $|\nabla|^\alpha u$) of $u \in S'(\mathbb{R}^d)$ is defined as $(-\Delta)^{\alpha/2} u$ i.e. $\mathcal{F}D^\alpha u = (2\pi)^\alpha |\xi|^\alpha \hat{u} \hat{\xi}$. Note that $\|u\|_{H^\alpha}^2 = \langle (-\mathcal{L})^{1/2} u, (-\mathcal{L})^{1/2} u \rangle = \langle -\mathcal{L} u, u \rangle = \|\nabla u\|^2$ and $\|u\|_{\dot{H}^\alpha_a}^2 = \langle (-\mathcal{L}_a)^{1/2} u, (-\mathcal{L}_a)^{1/2} u \rangle = \langle -\mathcal{L}_a u, u \rangle = \|\nabla u\|^2 - a \|u/|x|^2\|_2^2$.

Therefore using the Hardy’s inequality

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \geq \left( \frac{d-2}{2} \right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx , \quad u \in C_c^\infty(\mathbb{R}^d)$$

we have the following result:

**Lemma 2.1.** The homogeneous spaces $\dot{H}^1(\mathbb{R}^d)$ and $\dot{H}^1_a(\mathbb{R}^d)$ are the same when $a < (d-2)^2/4$.

**Lemma 2.2.** (See Theorem 1.2 in [20]) Let $d \geq 3$ and $a < (d-2)^2/4$. Then the norms $\| \cdot \|_{\dot{W}^{s,p}}$ and $\| \cdot \|_{\dot{W}^{s,p}_a}$ are equivalent and hence $\dot{W}^{s,p}_a(\mathbb{R}^d) = \dot{W}^{s,p}(\mathbb{R}^d)$ under any of the following two conditions:
Here we recall some results on interpolation spaces. For details on this subject, see Lemma 2.6. (Section 3.13, exercise 5(b) in [1] and Theorems 1, 2 in [13]).

Corollary 2.3. Let $d \geq 3$ and $(q, r), (q, \tilde{r})$ are admissible pairs satisfying $q > \frac{2}{\sqrt{(\frac{d}{2})^2 - a}}$, then

(i) $\|\nabla S_{t}(t)\|_{L^{q}L^{r}} \lesssim \|f\|_{H^{1}}$,

(ii) $\left\| \nabla \int_{0}^{t} S_{t}(t-s)F(s,x)ds \right\|_{L^{q}L^{r}} \lesssim \|F\|_{L^{q'}L^{r'}}$.

Proof. Follows from Theorem 1.6 and Lemma 2.2. \qed

2.3. Interpolation spaces

Here we recall some results on interpolation spaces. For details on this subject, one can see the book [1] of Bergh and L"ofstr"om. We define the real interpolation space $(A_{0}, A_{1})_{\theta, \rho}$ (0 < $\theta < 1$, 1 $\leq \rho \leq \infty$) of two Banach spaces $A_{0}, A_{1}$ via the norm

$$
\|u\|_{(A_{0}, A_{1})_{\theta, \rho}} = \left( \int_{0}^{\infty} (t^{-\theta} K(t,u))^\rho dt \right)^{1/\rho}, \quad K(t,u) = \inf_{u = u_{0} + u_{1}} \|u_{0}\|_{A_{0}} + t\|u_{1}\|_{A_{1}}
$$

where the infimum is taken over $(u_{0}, u_{1}) \in A_{0} \times A_{1}$ such that $u = u_{0} + u_{1}$.

Lemma 2.4. (Theorem 3.4 in [27]) Let $\frac{1}{r} = \frac{1}{r_{0}} + \frac{1}{r_{1}} < 1$ and $s \geq 1$ is such that $\frac{1}{s} \leq \frac{1}{s_{0}} + \frac{1}{s_{1}}$. Then $f \in L^{r_{0},s_{0}}(\mathbb{R}^{d})$ and $g \in L^{r_{1},s_{1}}(\mathbb{R}^{d})$ imply $fg \in L^{r,s}(\mathbb{R}^{d})$ and $\|fg\|_{L^{r,s}} \leq \|f\|_{L^{r_{0},s_{0}}} \|g\|_{L^{r_{1},s_{1}}}$. 

Lemma 2.5. (See e.g. [1, 12]) Let $q_{j}, r_{j}, \tilde{q}_{j}, \tilde{r}_{j} \in [1, \infty]$, $j = 0, 1$. Let $q, r$ is such that $\frac{1}{q} = \frac{1}{q_{0}} + \frac{\theta}{q_{1}}$, $\frac{1}{r} = \frac{1}{r_{0}} + \frac{\theta}{r_{1}}$, $\frac{1}{\tilde{q}} = \frac{1}{q_{0}} + \frac{\theta}{q_{1}}$, $\frac{1}{\tilde{r}} = \frac{1}{r_{0}} + \frac{\theta}{r_{1}}$ for some $\theta \in [0, 1]$. Then for $T$ linear,

(i) $T : L^{q_{0},r_{0}} \rightarrow L^{\tilde{q}_{0},\tilde{r}_{0}}$ and $T : L^{q_{1},r_{1}} \rightarrow L^{\tilde{q}_{1},\tilde{r}_{1}}$ imply $T : L^{q,r} \rightarrow L^{\tilde{q},\tilde{r}}$.

(ii) $T : L^{q_{0},r_{0},2} \rightarrow L^{\tilde{q}_{0},\tilde{r}_{0},2}$ and $T : L^{q_{1},r_{1},2} \rightarrow L^{\tilde{q}_{1},\tilde{r}_{1},2}$ imply $T : L^{q,r,2} \rightarrow L^{\tilde{q},\tilde{r},2}$.

Lemma 2.6. (Section 3.13, exercise 5(b) in [1] and Theorems 1, 2 in [13]) Let $A_{0}, A_{1}, B_{0}, B_{1}, C_{0}, C_{1}$ are Banach spaces and $T$ be a bilinear operator such that

$$
T : \begin{cases} 
A_{0} \times B_{0} \longrightarrow C_{0}, \\
A_{0} \times B_{1} \longrightarrow C_{1}, \\
A_{1} \times B_{0} \longrightarrow C_{1},
\end{cases}
$$

then whenever $0 < \theta_{0}, \theta_{1} < \theta = \theta_{0} + \theta_{1} < 1$, $1 \leq p, q, r \leq \infty$ and $1 \leq \frac{1}{p} + \frac{1}{q}$, we have

$$
T : (A_{0}, A_{1})_{\theta_{0}, \theta_{1}} \times (B_{0}, B_{1})_{\theta_{1}, \theta_{1}} \longrightarrow (C_{0}, C_{1})_{\theta, \theta}.
$$

Lemma 2.7. (Theorems 5.2.1 and 5.6.1 in [1]) We have the following interpolation results:

(i) Let $r_{0} < r < r_{1}$ and $0 < \theta < 1$ be such that $\frac{1}{r} = \frac{1}{r_{0}} + \frac{\theta}{r_{1}}$, then for $r_{0} < p$
we have $(L^{r_{0},r_{1}})_{\theta,p} = L^{r,p}$.
(ii) Let $\beta_0 < \beta_1$ and $0 < \theta < 1$ be such that $(1 - \theta)\beta_0 + \theta\beta_1 = \beta$, then

$$\lambda_0^\beta, \lambda_1^\beta \in \mathcal{L}_1^\beta.$$

**Lemma 2.8.** (Calderón, see e.g. Lemma 2.5 in [27]) Let $1 < r < \infty$ and $s > \sigma$. Then $\|v\|_{L^r,s} \leq \left(\frac{2}{r}\right)^{1/\sigma - 1/s}\|v\|_{L^r,\sigma}$.

**Corollary 2.9.** The inequality (1.11) is stronger than the inequality (1.4) for $2 \leq r < \infty$ and $2 \leq \tilde{r} < \infty$.

**Proof.** By Lemma 2.8 above, for $1 < r, \tilde{r} < \infty$ we have $\|v\|_{L^r,L^r} = \|v\|_{L^2,L^r} \leq (2/r)^{1/2 - 1/r} \|v\|_{L^2,L^r,2}$ and $\|F\|_{L^s(L^r,L^r)} \leq (\tilde{r}'/\tilde{r})^{1/\tilde{r}' - 1/2} \|F\|_{L^s(L^r,L^r)} = \|F\|_{L^{\tilde{s}'}(L^{\tilde{r}'},L^{\tilde{r}'})}$. Since $r \geq 2$ and $2 \geq \tilde{r}'$, the result follows. \qed

### 2.4. Compactness in Hilbert space and dislocation

Here we discuss relation between the compactness in a Hilbert space $H$ and compactness in its quotient space $H/G$ associated with a Dislocation group $G$ in $H$. This will be useful in Sect. 4. Let $H$ be a Hilbert space and $G$ be a group of operators in $H$. Then we define an equivalence relation $\sim$ on $H$ by $x \sim y$ if $x = gy$ for some $g \in G$. Let $H/G$ denote the quotient space $H/\sim$. Before defining the dislocation group we recall the notions of strong and weak convergence.

**Definition 2.10.** (Strong convergence) We say a sequence $\{g_n\}$ in $G$ converges to $g \in G$ in the strong operator topology if $\|g_n x - gx\|_H \to 0$ for all $x \in H$.

**Definition 2.11.** (Weak convergence) We say a sequence $\{g_n\}$ in $G$ converges to $g \in G$ in the weak operator topology if $\langle g_n x, y \rangle_H \to \langle gx, y \rangle_H$ for all $x, y \in H$.

**Definition 2.12.** (Dislocation group) Let $H$ be a Hilbert space and $G$ be a group of unitary operators in $H$. Then $G$ is called a dislocation group if, for any $\{g_n\}$ in $G$ not converging to zero in the weak operator topology on $G$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and a non zero $g \in G$ such that $g_{n_k} \to g$ in the strong operator topology on $G$.

Now we state the main result in this subsection without proof relating compactness in $H$ and in $H/G$ which will be used in Sect. 4.2 to achieve the ‘almost’ compactness of the flow of a minimal blow-up solution.

**Theorem 2.13.** Let $G$ be a dislocation in a Hilbert space $H$. Then for any compact set $\tilde{K}$ in the quotient space $H/G$ with the quotient topology, there exists a compact set $K$ in $H$ such that $\tilde{K} = P(K)$, where $P : H \to H/G$ is the standard canonical projection.

We end this subsection with an example of dislocation to be used in Sect. 4. Let us consider the Hilbert space $\dot{H}^1(\mathbb{R}^d)$. For $\lambda > 0$ let $T_\lambda : \dot{H}^1(\mathbb{R}^d) \to \dot{H}^1(\mathbb{R}^d)$ be the unitary operator

$$T_\lambda f = \frac{1}{\lambda^{(d-2)/2}} f\left(\frac{\cdot}{\lambda}\right), \quad f \in \dot{H}^1(\mathbb{R}^d)$$

and set $G = \{T_\lambda : \lambda > 0\}$. Then $G$ is a group of unitary operators in $\dot{H}^1(\mathbb{R}^d)$ and we have the following:

**Lemma 2.14.** The group $G$ defined above is a dislocation in the Hilbert space $\dot{H}^1(\mathbb{R}^d)$.
3. Proof of the Theorems

This section is divided into three subsections. In Sect. 3.1 we prove the inhomogeneous estimates for solutions to (1.1) with $V = 0$ whereas in Sect. 3.2 we establish the same estimates to a solution to (1.1) with potentials in weak Lebesgue space $L^{d/2,\infty}$. In Sect. 3.3 we proved the estimates for solutions to (1.1) with inverse-square potentials.

3.1. Improvement in unperturbed case

The idea here is to use the $TT^*$ method as in [16] where the end-point Strichartz estimates are proved. Using the duality it essentially boils down to show that the operator $T : L^{\tilde{q}'}(\mathbb{R}, L^{r',2}(\mathbb{R}^d)) \times L^{q'}(\mathbb{R}, L^{r',2}(\mathbb{R}^d)) \to \mathbb{C}$ defined in (3.1) below, is bounded. This, in turn, further reduces to the show the operator

$$T := \{ T_j \}_{j \in \mathbb{Z}} : L^{\tilde{q}'}(\mathbb{R}, L^{r',2}(\mathbb{R}^d)) \times L^{q'}(\mathbb{R}, L^{r',2}(\mathbb{R}^d)) \to l^0_1,$$

where $T_j$ as in (3.2) below, is bounded. Now we choose $r_0, r_1$ near $r$ and $\tilde{r}_0, \tilde{r}_1$ near $\tilde{r}$ in a judicial way so that applying Lemma 3.1 below with $(q, \tilde{q}, r, \tilde{r}) = (q, \tilde{q}, r_0, \tilde{r}_0), (q, \tilde{q}, r_1, \tilde{r}_0), (q, \tilde{q}, r_0, \tilde{r}_1)$ we obtain three different bounds for $T$ (with appropriate domains and ranges). Then the result follows from interpolation results. Note that in [32], the author varied the exponents $q, \tilde{q}$ and established the weaker estimate (1.4) whereas here we vary the exponents $r, \tilde{r}$ and obtain the stronger estimate (1.11).

**Proof of Theorem 1.1.** Note that $u(t) = \int_0^t e^{i(t-\tau)\Delta}F(\tau, \cdot)d\tau$. Using $TT^*$ method we need to prove

$$|T(F, G)| \lesssim \|F\|_{L^{q'} L^{r',2}} \|G\|_{L^{q'} L^{r',2}}$$

where $T$ is given by

$$T(F, G) = \int_{\mathbb{R}} \int_{-\infty}^t \langle e^{-i\tau\Delta}F(\tau, \cdot), e^{-it\Delta}G(t, \cdot) \rangle d\tau dt. \quad (3.1)$$

Decomposing $T$ by $T = \sum T_j$ where

$$T_j(F, G) = \int_{\mathbb{R}} \int_{t-2^j}^{t-2^{j+1}} \langle e^{-i\tau\Delta}F(\tau, \cdot), e^{-it\Delta}G(t, \cdot) \rangle d\tau dt, \quad (3.2)$$

it is enough to prove

$$\sum_{j \in \mathbb{Z}} |T_j(F, G)| \lesssim \|F\|_{L^{q'} L^{r',2}} \|G\|_{L^{q'} L^{r',2}}. \quad (3.3)$$

Set $T_{F,G} = \{ T_j(F, G) \}$, then (3.3) is equivalent with

$$\|T_{F,G}\|_{l^0_1} \lesssim \|F\|_{L^{q'} L^{r',2}} \|G\|_{L^{q'} L^{r',2}}. \quad (3.4)$$

Now we quote a result due to Vilela, see [32, Lemma 2.2]. Using this Lemma for three different choices of $(r, \tilde{r})$ we would get three estimates. These estimates together with Lemmata 2.6 and 2.7 would finally imply (3.4).
Lemma 3.1. Let $d \geq 3$ and $r, \tilde{r}$ be such that $2 \leq r, \tilde{r} \leq \infty$ and
\[
\frac{d - 2}{d} \leq \frac{r}{\tilde{r}} \leq \frac{d}{d - 2}.
\] (3.5)

Then for all $q, \tilde{q}$ satisfying
\[
\begin{cases}
\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1 \\
\frac{d}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right) < \frac{1}{q} & \text{if } r \leq \tilde{r} \\
\frac{d}{2} \left( \frac{1}{\tilde{r}} - \frac{1}{r} \right) < \frac{1}{q} & \text{if } r \geq \tilde{r}
\end{cases}
\] (3.6)

the following estimates holds for all $j \in \mathbb{Z}$
\[
|T_j(F,G)| \leq c2^{-j\beta(\tilde{q}, q, r, \tilde{r})} \|F\|_{L^{q'}L^{r',2}} \|G\|_{L^{q'}L^{r',2}}
\] (3.7)

where $\beta(\tilde{q}, q, \tilde{r}, r) = \left( \frac{1}{q} - \frac{1}{\tilde{q}} \right) + \frac{d}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right) - 1$.

Let us fix $q, \tilde{q}$ as in (1.12) (this implies $1/q + 1/\tilde{q} \leq 1$ due to (1.7)) and assume we can choose $r_0, \tilde{r}_0, r_1, \tilde{r}_1 \geq 2$ satisfying
\[
\beta(\tilde{q}, q, \tilde{r}_0, r_1) = \beta(\tilde{q}, q, \tilde{r}_1, r_0) \iff \frac{1}{r_1} - \frac{1}{r_0} = \frac{1}{\tilde{r}_1} - \frac{1}{\tilde{r}_0}
\] (3.8)
\[
\frac{d - 2}{d} < \frac{r_j}{\tilde{r}_k} < \frac{d}{d - 2}, \quad \left\{ \frac{d}{2} \left( \frac{1}{r_j} - \frac{1}{\tilde{r}_k} \right) < \frac{1}{q} \quad \text{if } r_j \leq \tilde{r}_k \right\}
\] (3.9)

for $(j, k) = (0, 0), (1, 0), (0, 1)$, such that applying Lemma 3.1, we achieve
\[
T : \begin{cases}
L^{q'} L^{\tilde{r}_0'} \times L^{q'} L^{\tilde{r}_0'} \longrightarrow l^{\beta(\tilde{q}, q, \tilde{r}_0, r_0)}_{\infty} = l^{\beta_0}_{\infty}, \\
L^{q'} L^{\tilde{r}_0'} \times L^{q'} L^{r_1'} \longrightarrow l^{\beta(\tilde{q}, q, \tilde{r}_0, r_1)}_{\infty} = l^{\beta_1}_{\infty}, \\
L^{q'} L^{r_1'} \times L^{q'} L^{\tilde{r}_0'} \longrightarrow l^{\beta(\tilde{q}, q, \tilde{r}_1, r_0)}_{\infty} = l^{\beta_2}_{\infty}.
\end{cases}
\] (3.10)

Let us impose the conditions
\[
(1 - \theta) \beta_0 + \theta \beta_1 = 0 \quad \text{for some } 0 < \theta < 1,
\] (3.11)
\[
\frac{1}{r} = \frac{1 - \theta_0}{r_0} + \frac{\theta_0}{\tilde{r}_0} \quad \text{for some } 0 < \theta_0 < 1,
\] (3.12)
\[
\frac{1}{r} = \frac{1 - \theta_1}{r_0} + \frac{\theta_1}{r_1} \quad \text{for some } 0 < \theta_1 < 1,
\] (3.13)
\[
\theta_0 + \theta_1 = \theta
\] (3.14)

to apply Lemmas 2.6 and 2.7. Applying Lemma 2.6 we get
\[
T : (L^{q'} L^{\tilde{r}_0'}, L^{q'} L^{r_1'})_{\theta_0, 2} \times (L^{q'} L^{r_0'}, L^{q'} L^{r_1'})_{\theta_1, 2} \longrightarrow (l^{\beta_0}_{\infty}, l^{\beta_1}_{\infty})_{\theta, 1}
\] (3.15)
which implies $T : L^{q'} L^{r', 2} \times L^{q'} L^{r', 2} \longrightarrow l^{0}_{1}$ (by using $q, \tilde{q} \geq 2$, Lemma 2.7) this proves (3.4).

Now it is enough to find $r_0, \tilde{r}_0, r_1, \tilde{r}_1 > 2, \theta_0, \theta_1, \theta$ satisfying (3.8), (3.9), (3.11), (3.12), (3.13) and (3.14). Since the maps $(x, y) \mapsto \frac{x}{y}, (x, y) \mapsto \frac{d}{2} \left( \frac{1}{x} - \frac{1}{y} \right)$ are continuous on $(0, \infty) \times (0, \infty)$, because of (1.12), there exists $\delta > 0$ such that
\[
\frac{d - 2}{d} < \frac{1/r + a}{1/r + b} < \frac{d}{d - 2},
\]
\[
\begin{cases}
\frac{d}{2} \left( \frac{1}{r} + a - \frac{1}{\tilde{r}} - b \right) < \frac{1}{q} & \text{if } r \leq \tilde{r} \\
\frac{d}{2} \left( \frac{1}{r} + a - \frac{1}{\tilde{r}} - b \right) < \frac{1}{q} & \text{if } r \geq \tilde{r}
\end{cases}
\]

and \(1/r + a, 1/\tilde{r} + b > 2\) for all \(|a|, |b| \leq \delta\). Set

\[
\begin{align*}
\frac{1}{r_0} &= \frac{1}{r} - a, \quad \frac{1}{r_1} = \frac{1}{r} + b, \quad \frac{1}{\tilde{r}_0} = \frac{1}{\tilde{r}} - a, \quad \frac{1}{\tilde{r}_1} = \frac{1}{\tilde{r}} + b
\end{align*}
\]

with

\[
0 < a, b < \begin{cases}
\min \{ \delta, \frac{1}{r}, \frac{1}{\tilde{r}} - \frac{1}{r}, \frac{1}{\tilde{r}} - \frac{1}{\tilde{r}} - \frac{1}{2} \} & \text{if } r \neq \tilde{r} \\
\min \{ \delta, \frac{1}{r}, \frac{1}{\tilde{r}} - \frac{1}{r}, \frac{1}{\tilde{r}} - \frac{1}{\tilde{r}} - \frac{1}{2} \} & \text{if } r = \tilde{r}.
\end{cases}
\]

Then (3.8) and (3.9) are satisfied. Because of (3.12) and (3.13) we have 
\(a(1 - \theta_0) = b \theta_0\) and \(a(1 - \theta_1) = b \theta_1\). Adding them we have \(a(2 - \theta) = b \theta\) using (3.14). Therefore we have

\[
\theta = \frac{2a}{a + b}. \tag{3.16}
\]

Subtracting \(a(1 - \theta_0) = b \theta_0\) from \(a(1 - \theta_1) = b \theta_1\) we get \(\theta_0 = \theta_1\) and therefore \(\theta_0 = \theta_1 = \frac{a}{a + b}\). Since we require \(0 < \theta < 1\) we choose \(a < b\).

Note that (3.11) is equivalent to

\[
\left( \frac{1}{q} - \frac{1}{q} \right) + \frac{d}{2} \left( \frac{1}{r_0} - \frac{1}{r_0} \right) + \theta \frac{d}{2} \left[ \left( \frac{1}{r_0} - \frac{1}{r_0} \right) - \left( \frac{1}{r_0} - \frac{1}{r_0} \right) \right] = 1 \tag{3.17}
\]

Now using (1.12) we have \(\frac{1}{q} - \frac{1}{q} = 1 - \frac{1}{q} = 1 - \frac{d}{2} (1 - \frac{1}{r} - \frac{1}{\tilde{r}})\) and therefore (3.17) is equivalent with

\[
\frac{d}{2} \left( \frac{1}{r_0} - \frac{1}{r_0} \right) + \theta \frac{d}{2} \left( \frac{1}{r_0} - \frac{1}{r_1} \right) = \frac{d}{2} \left( 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right)
\]

\[
\iff \frac{1}{r_0} + \frac{1}{r_0} = \frac{1}{r} + \frac{1}{\tilde{r}} + \theta \left( \frac{1}{r_0} - \frac{1}{r_1} \right)
\]

and by our choice of \(r_0, \tilde{r}_0, r_1, \tilde{r}_1, \theta_0, \theta_1, \theta\) this is equivalent to \(2a = \theta(a + b)\) which is equivalent to (3.16). \(\square\)

3.2. Potential in \(L^{d/2, \infty}(\mathbb{R}^d)\)

**Proof of Theorem 1.2.** Let us split \(u = u_1 + u_2\) where \(u_1, u_2\) satisfy

\[
\begin{cases}
\begin{align*}
    i \partial_t u_1 + \Delta u_1 &= F \\
    u_1(0, \cdot) &= 0
\end{align*}
\end{cases}
\quad \begin{cases}
\begin{align*}
    i \partial_t u_2 + \Delta u_2 &= -Vu \\
    u_2(0, \cdot) &= 0
\end{align*}
\end{cases}
\]

Let \(r, \tilde{r}, q = 2, \tilde{q}\) satisfy (1.12). Using Theorem 1.1 for exponents \((q, r), (q, \tilde{r})\) we have that

\[
\|u_1\|_{L^q L^{r, 2}} \leq c_s \|F\|_{L^{q', r', 2}}
\]

and for exponent \((q, r), (q', (\frac{dr}{2r+d})')\) we have

\[
\|u_2\|_{L^q L^{r, 2}} \leq c_s \|Vu\|_{L^q L^{\frac{dr}{2r+d}, 2}}
\]
provided we farther assume
\[
\begin{align*}
&\left\{ \begin{array}{ll}
  d \left( \frac{1}{r} - \frac{1}{2r} \right) < \frac{1}{q'} & \text{for } 2(d-1) < 2r \leq 2^*, \\
  d \left( \frac{1}{r} - \frac{1}{2r} \right) < \frac{1}{q} & \text{for } 2^* \leq r < 2^{*} \frac{(d-1)}{d-2}.
\end{array} \right.
\end{align*}
\]
Now using H"older inequality for Lorentz spaces (see Lemma 2.4) we have
\[
\|Vu\|_{L^q L^{2r/(d-r)}} \leq \left( \frac{dr}{2r + d} \right) \|V\|_{L^{\infty} L^{d/(d+2) \infty}} \|u\|_{L^q L^{r,2}}
\]
and therefore
\[
\|u\|_{L^q L^{r,2}} \leq \|u_1\|_{L^q L^{r,2}} + \|u_2\|_{L^q L^{r,2}} \leq c_s \left( \|F\|_{L^{q'} L^{r',2}} + \|Vu\|_{L^{q'} L^{r',2}} \right)
\leq c_s \left( \|F\|_{L^{q'} L^{r',2}} + \left( \frac{dr}{2r + d} \right) \|V\|_{L^{\infty} L^{d/(d+2) \infty}} \|u\|_{L^{q'} L^{r',2}} \right).
\]
Then we have that
\[
\|u\|_{L^q L^{r,2}} \leq \frac{c_s}{1 - c_s \left( \frac{dr}{2r + d} \right) \|V\|_{L^{\infty} L^{d/(d+2) \infty}}} \|F\|_{L^{q'} L^{r',2}}
\]
provided \(c_s \left( \frac{dr}{2r + d} \right) \|V\|_{L^{\infty} L^{d/(d+2) \infty}} < 1\). \(\square\)

**Proof of Theorem 1.5. Case I:** Here we prove the result assuming the conditions in (i). Multiplying the equation (1.1) by \(\bar{u}\) and integrating by parts we get
\[
i \int_{\mathbb{R}^d} \partial_t \bar{u} \bar{u} - \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V|u|^2 = \int_{\mathbb{R}^d} F\bar{u}.
\]
Taking imaginary part of both side we get \(\text{Re} \left( \int_{\mathbb{R}^d} \partial_t \bar{u} \bar{u} \right) = \text{Im} \left( \int_{\mathbb{R}^d} F\bar{u} \right)\). Cauchy–Schwartz inequality now implies \(\partial_t \|u(t)\|^2 \leq 2\|u(t)\|\|F(t)\|\) which in turn gives (after cancelling one \(\|u(t)\|\) from both side and then integrating in time on \([0,t])\)
\[
\|u\|_{L^{\infty} L^{2,2}} \lesssim \|F\|_{L^1 L^{2,2}}
\]
(see proof of Proposition 3 in [28] for details). Now we would like to have the estimate
\[
\|u\|_{L^{q_1} L^{q_2,2}} \lesssim \|F\|_{L^{q'_1} L^{r',2}}
\]
(3.19)
using Theorem 1.2 for appropriate \(q_1, q'_1, r_1\). Choose \(0 \leq \theta \leq 1\) so that \(\frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{r'} \iff \theta = d \left( \frac{1}{2} - \frac{1}{r} \right)\), then take \(r_1 > 2\) so that \(\frac{1}{r_1} = \frac{1-\theta}{2} + \frac{\theta}{r'_1}\). Set \(q_1 = \theta q, q'_1 = \theta q'_1\). Let us now verify the conditions in Theorem 1.2 so that (3.19) holds. Note that by direct computation we have
\[
\begin{align*}
&\frac{1}{q_1} + \frac{1}{q'_1} + \frac{d}{2} \left( \frac{d-2}{2d} + \frac{1}{r_1} \right) = \frac{d}{2} \iff \frac{1}{q} + \frac{1}{q} + \frac{d}{2} \left( \frac{1}{q} + \frac{1}{q} \right) = \frac{d}{2} \iff (1.7), \\
&\frac{d}{d-2} < \frac{r_1}{2} < \frac{d}{d-2} \iff 0 < \frac{1}{2} - \frac{1}{r} < \frac{2(d-1)}{d-2} \left( \frac{1}{2} - \frac{1}{r} \right) \\
&\frac{1}{q_1} + \frac{1}{q'_1} = \frac{1}{\theta} \left( \frac{1}{q} + \frac{1}{q} \right) \leq 1 \iff \frac{1}{q} + \frac{1}{q} \leq d \left( \frac{1}{2} - \frac{1}{r} \right) \iff r \geq \tilde{r} \\
&\frac{d}{2} \left( \frac{1}{r_1} - \frac{1}{2r} \right) < \frac{1}{q_1} \iff \frac{d}{2} \left( \frac{1}{r} - \frac{1}{r} \right) < \frac{1}{q}.
\end{align*}
\]
Now for $c_2s^2\|V\|_{L^{\infty}L^{d/2}} < 1$, the above four conditions ensures (3.19). Interpolating (see Lemma 2.5) (3.18) and (3.19), we get the result.

**Case II:** Let us assume the conditions in (ii). As ($\infty, 2), (2, 2^*)$ are admissible pairs, by [28, Theorems 1, 3], we have

$$\|u\|_{L^{\infty}L^{2,2}} \lesssim \|F\|_{L^{2}L^{2^*,2}}$$  \hspace{1cm} (3.20)

for $c_2s^2\|V\|_{L^{d/2}} < 1$. Here again we would like to have the estimate of the form

$$\|u\|_{L^{q_1}L^{r_1,2}} \lesssim \|F\|_{L^{q_1^*}L^{r_1^*,2}}$$  \hspace{1cm} (3.21)

using Theorem 1.2 for appropriate $q_1, \tilde{q_1}, \tilde{r_1}$.

Choose $0 \leq \theta \leq 1$ so that $\frac{1}{\tilde{r}} = \frac{1}{2} - \frac{\theta}{2} + \frac{\theta}{q_1} \iff \theta = d\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right)$, then take $\tilde{r_1} > 2$ so that $\frac{1}{\tilde{r}} = \frac{1}{2} - \frac{\theta}{q_1}$. Set $q_1 = \theta q$ and $\tilde{q_1}$ so that $\frac{1}{q_1} = \frac{1}{2} - \frac{\theta}{q_1}$. Then again by direct computation we have

- $\frac{1}{q_1} + \frac{1}{\tilde{q_1}} + \frac{d}{2} \left(\frac{d-2}{2} + \frac{1}{r}\right) = d \left(\frac{1}{q_1} + \frac{1}{\tilde{q_1}} + \frac{d}{2}\right) = d \left(\frac{1}{\tilde{r}}\right) = (3.20).$

- $\frac{d}{d-2} < \frac{\tilde{r_1}}{d-2} < \frac{d}{d-2} \iff 2 < \tilde{r_1} < 2\left(\frac{d-2}{d}\right) \iff \frac{1}{q} > \frac{1}{\tilde{r}} > \frac{1}{2}$ and $\tilde{r} \leq 2^*$.

- $\frac{1}{q_1} + \frac{1}{\tilde{q_1}} = \frac{1}{q_1} + \frac{1}{\tilde{q_1}} \iff 1 \times \tilde{r} \leq 2^*$

- $\frac{d}{2} \left(\frac{1}{\tilde{r}} - \frac{1}{q_1}\right) < \frac{1}{q_1} \iff \frac{d}{2} \left(\frac{1}{\tilde{r}} - \frac{1}{2}\right) < \frac{1}{q_1}$

The above set of assumption together with $c_2s^2\|V\|_{L^{\infty}L^{d/2}} < 1$ imply (3.21). Now we interpolate (3.20) and (3.21) to get the result. \hfill \Box

### 3.3. Inverse square potential

**Proof of Theorem 1.6.** Note we have to prove the case (iii) only. We prove the result using interpolation twice in two steps.

**Step I:** Set $\frac{1}{q_0} = \frac{1}{q_0} = \frac{1}{2}, \frac{1}{r_0} = \frac{d-2s}{2d}, \frac{1}{r_0} = \frac{d-2(2-\sigma)}{2d}, \frac{1}{q_1} = \frac{1}{2}$ and $\frac{1}{r_1} = \frac{d-2}{2d}$. From Theorem 1.6 (ii) we have

$$\left\| \int_0^t e^{i(t-\tau)L_0} F(\tau) d\tau \right\|_{L^{2}\tilde{L}^{d/2\sigma,\sigma}} \lesssim \|F\|_{L^{2}\tilde{L}^{d/2\sigma,\sigma}}$$  \hspace{1cm} (3.22)

for appropriate $\sigma$. Let us choose $\theta$ so that

$$\frac{1}{r_0} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}. \hspace{1cm} (3.23)$$

Note that $\frac{1}{r_0} - \frac{1}{r_1} = \frac{d-2(2-\sigma)}{2d} - \frac{d-2(2-s)}{2d} = \sigma - \frac{s}{d}$ and $\frac{1}{r_0} - \frac{1}{r_1} = \frac{d-2(2-\sigma)}{2d} - \frac{d-2(2-s)}{2d} = \frac{2(\sigma - 2)}{2d} - \frac{2(2-s)}{2d} = \frac{1-(2-s)}{d} = \frac{\sigma - 1}{d}$. These together with (3.23) imply $\theta = \frac{1}{r_0} - \frac{1}{r_1} = \frac{\sigma - 1}{d}$.

In order to make $\theta \in (0, 1)$ we must have

$$1 < \sigma < \sigma \quad \text{or} \quad \sigma < s < 1.$$  \hspace{1cm} (3.24)

Set $q_1$ so that $\frac{1}{q_1} = \frac{1}{q_0} + \frac{\theta}{q_1}$. To make $q_1 > 0$ we need

$$\frac{1}{q_1} = 1 - \frac{1}{q_0} - (1-\theta)\frac{1}{2} > 0 \iff \gamma > \frac{s-1}{\sigma-1} \iff \gamma > \frac{s-1}{\sigma-1}.$$  \hspace{1cm} (3.25)
Then $\frac{1}{2} - \frac{1}{q_1} = \frac{1}{\bar{q}} \left( \frac{1}{2} - \frac{1}{q} \right) \geq 0 \iff q_1 \geq 2$ as $q \geq 2$. Now choose $r_1$ so that $(q_1, r_1)$ is an admissible pair. Then by Theorem 1.6 (i) we have

$$\left\| \int_0^t e^{i(t-s)L_a} F(s) ds \right\|_{L^{q_1}_{2;1}} \lesssim \| F \|_{L^{2;2'}}. \quad (3.26)$$

Interpolating (see Lemma 2.5) of (3.22) and (3.26) we have

$$\left\| \int_0^t e^{i(t-\tau)L_a} F(\tau) d\tau \right\|_{L^{2/\gamma}_{2;1} L^{\frac{2d}{d-2(\gamma+\gamma'-\gamma)}_\gamma}} \lesssim \| F \|_{L^{2;2'}(\frac{2d}{d-2(\gamma+\gamma'-\gamma)})}, \quad (3.27)$$

where $s \in A_{a, \gamma} = \left( 1 - \frac{d-2}{2(d-1)} \gamma, 1 + \frac{d-2}{2(d-1)} \gamma \right) \cap R_{a, \gamma}$. Note that (3.25) is equivalent to

$$\begin{cases} s < 1 + (\sigma-1)\gamma & \text{if } \sigma > 1 \\ s > 1 - (1-\sigma)\gamma & \text{if } \sigma < 1. \end{cases}$$

This ensures (3.24) and sets the conditions $s \in A_{a, \gamma}$.

Step II: Set $\frac{1}{q_0} = 2, \frac{1}{r_0} = \frac{d-2(\sigma+\gamma-\gamma'-\gamma)}{2d}$, $\frac{1}{q_1} = \frac{1}{2}, \frac{1}{r_1} = \frac{d-2(2-\sigma)}{2d}, \frac{1}{q_1} = \frac{1}{\gamma}$ and $\frac{1}{r_1} = \frac{d-2}{2\gamma}$. From Step I we have

$$\left\| \int_0^t e^{i(t-\tau)L_a} F(\tau) d\tau \right\|_{L^{2/\gamma}_{2;1} L^{\frac{2d}{d-2(\gamma+\gamma'-\gamma)}_\gamma}} \lesssim \| F \|_{L^{2;2'}(\frac{2d}{d-2(\gamma+\gamma'-\gamma)})}, \quad (3.28)$$

for appropriate $\sigma$. Set $\theta$ so that $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Then $\frac{1}{r} - \frac{1}{r_0} = \frac{\sigma-\gamma}{d}$, $\frac{1}{r_1} - \frac{1}{r_0} = \frac{\sigma-1}{d}$ and hence $\theta = \frac{\sigma-\gamma}{d}$. Take $\bar{q}_1$ so that $\frac{1}{q_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. To make $\bar{q}_1 > 0$ as before we need $\bar{q}_1 > \frac{s+1}{\gamma-1}$. This ensures $\theta \in (0, 1)$ and sets the condition $s \in A_{a, \gamma}$. At last choose $\bar{r}_1$ so that $(\bar{q}_1, \bar{r}_1)$ is an admissible pair. Then by Theorem 1.6 (i) we have

$$\left\| \int_0^t e^{i(t-s)L_a} F(s) ds \right\|_{L^{\bar{q}_1}_{2;1} L^{\bar{r}_1}} \lesssim \| F \|_{L^{\bar{q}_1}_{2;1} L^{\bar{r}_1}}. \quad (3.29)$$

Now the theorem follows from interpolation of (3.28) and (3.29).

4. Application

In this section, we study the scattering solutions of the Cauchy problem

$$i \frac{\partial}{\partial t} u(t, x) + L_a u(t, x) + |u(t, x)|^{4\gamma} u(t, x) = 0, \quad u(t_0, x) = u_0(x). \quad \text{(NLS}_a)$$

We show that as an application of Theorem 1.6 (iii), we can establish a stability result for this problem with $a \neq 0$, similar to that of [18, Theorem 2.14] for the case $a = 0$. This stability result in turn will establish the existence of scattering solution for radial data in dimensions 3, 4 and 5 by proceeding exactly as in Kenig and Merle [18]. In fact, when this project was in its final stage, we came across the very recent work of Yang [36] where the same result has been established using a different argument. Therefore our work serves as an alternative proof of Theorem 4.7.
In Sect. 4.1, we present the stability of solutions to \((\text{NLS}_a)\) in detail, and in Sect. 4.2 we outline the proof of the scattering result without details as the proof deviates very little from that of \([18]\).

### 4.1. Stability of solution

Let \(I\) be an open interval in \(\mathbb{R}\), \(t_0 \in I\) and \(u_0 \in \dot{H}^1(\mathbb{R}^d)\). We say that \(u \in C(I, \dot{H}^1(\mathbb{R}^d))\) is a solution of \((\text{NLS}_a)\) if \(\|\nabla u\|_{W(I)} < \infty\) for all \(I \subset \subset I\) and satisfy the integral equation

\[
u(t) = e^{i(t-t_0)\mathcal{L}_a}u_0 + i \int_{t_0}^{t} S_a(t-\tau) f(u(\tau))d\tau,
\]

with \(f(u) = |u|^4 u\). Then proceeding exactly as in the proof of Theorem 2.5 in \([18]\) by using Strichartz estimates with inverse square potential i.e. Theorem 1.6 (and Corollary 2.3) we can establish the following local existence theorem.

**Proposition 4.1.** (Local existence) Let \(d \in \{3, 4, 5\}\) and \(a < \left(\frac{d-2}{2}\right)^2 - \left(\frac{d-2}{d+2}\right)^2\).

Then for every \(A > 0\) there exists \(\delta = \delta(A) > 0\) such that for any interval \(I \subset \mathbb{R}\) containing \(t_0\) and \(u_0 \in \dot{H}^1(\mathbb{R}^d)\) satisfying \(\|u_0\|_{\dot{H}^1} < A\) and \(\|S_a(t-t_0)u_0\|_{S(I)} < \delta\), the Cauchy problem \((\text{NLS}_a)\) has a unique solution in \(I\) with \(\|\nabla u\|_{W(I)} < \infty\), \(\|u\|_{S(I)} \leq 2\delta\). Moreover, if \(u_{0,k} \to u_0\) in \(\dot{H}^1(\mathbb{R}^d)\), the corresponding solutions \(u_k \to u\) in \(C(I, \dot{H}^1(\mathbb{R}^d))\).

**Remark 4.2.** Note that in the above result we have further restriction on \(a\). This restriction comes to achieve equivalence of the norms \(\|\cdot\|_{W_a^{1, r}}\) and \(\|\cdot\|_{W^{1, r}}\) with \(r = \frac{2(4d+2)}{d^2+4}\) (which would be used to prove Proposition 4.1, in a way similar to the proof of \([18, \text{Theorem 2.5}]\)), see Corollary 2.3 and Lemma 2.2.

Using the above Proposition, we can define the maximal interval of existence. It is easy to see from Proposition 4.1 by using the Sobolev inequality that \((\text{NLS}_a)\) has a global solution when the initial data is small enough. Also following the very same arguments as \([18, \text{Lemma 2.11}]\) we have,

**Lemma 4.3.** (Finite time blow-up criterion) Let \(I = (-T_-(u_0), T_+(u_0))\) be the maximal interval of existence of solution to \((\text{NLS}_a)\). If \(T_+(u_0) < \infty\), then \(\|u\|_{S([t_0, T_+(u_0)])} = \infty\). A similar result holds for \(T_-(u_0)\).

Now we can state the main theorem of this subsection:

**Theorem 4.4.** (Long time perturbation) Let \(d \in \{3, 4, 5\}\), \(a < \left(\frac{d-2}{2}\right)^2 - \left(\frac{d-2}{d+2}\right)^2\)

and \(I\) be an open interval in \(\mathbb{R}\) containing \(t_0\). Let \(\tilde{u}\) be defined on \(I \times \mathbb{R}^d\) and satisfy \(\sup_{t \in I} \|\tilde{u}(t)\|_{\dot{H}^1} \leq A\), \(\|\tilde{u}\|_{S(I)} \leq M\) for some constants \(M, A > 0\). Assume that \(\tilde{u}\) satisfies \(i\partial_t \tilde{u} + \mathcal{L}_a \tilde{u} + f(\tilde{u}) = g\), i.e.,

\[
\tilde{u}(t) = S_a(t-t_0)\tilde{u}(t_0) + i \int_{t_0}^{t} S_a(t-\tau)(f(\tilde{u}(\tau)) - g(\tau))d\tau = 0.
\]

Then for every \(A' > 0\), there exists \(\epsilon_0 = \epsilon_0(M, A, A', d) > 0\) such that whenever

\[
\|u_0 - \tilde{u}(t_0)\|_{\dot{H}^1} \leq A', \quad \|\nabla g\|_{L^2(I, L^{\frac{2d}{d+2}})} \leq \epsilon, \quad \|S_a(t-t_0)[u_0 - \tilde{u}(t_0)]\|_{S(I)} \leq \epsilon
\]
for some $0 < \epsilon < \epsilon_0$, then the Cauchy Problem \((\text{NLS}_a)\) has a solution \(u\) defined on \(I\) satisfying the estimate
\[
\|u\|_{S(I)} \leq C(M, A, A', d) \quad \text{and} \quad \|u(t) - \tilde{u}(t)\|_{H^1} \leq C(A, M, d)(A' + \epsilon) \quad \text{for all} \; t \in I.
\]

**Proof.** First note that for any \(u_0\) as in the statement of the theorem, the Cauchy problem \((\text{NLS}_a)\) has a solution in a maximal interval of existence by Proposition 4.1. We prove that this solution satisfies the required a priori estimates. By blow-up alternative i.e. Lemma 4.3, this will immediately imply that solution has to exist in all of \(I\) as \(\|u\|_{S(I)} \leq C(M, A, A', d) < \infty\).

**STEP I:** Let us show that \(\|\nabla \tilde{u}(t)\|_{W(I)} \leq M' = M'(A, M, d) < \infty\).

For \(\eta > 0\) split \(I\) into \(\gamma = \gamma(M, \eta)\) intervals \(I_1, I_2, \ldots, I_\gamma\) so that \(\|\tilde{u}(t)\|_{S(I_j)} \leq \eta\) for \(j = 1, 2, \ldots, \gamma\). Then
\[
\tilde{u}(t) = S_a(t - t_j)\tilde{u}(t_j) + i \int_{t_j}^t S_a(t - \tau)f \circ \tilde{u}(\tau)d\tau + i \int_{t_j}^t S_a(t - \tau)g(\tau)d\tau
\]
for some \(t_j \in I_j\) fixed. Then
\[
\|\nabla \tilde{u}\|_{W(I_j)} \leq cA + c\|\tilde{u}\|_{S(I_j)}^{\frac{2}{4}} \|\nabla \tilde{u}\|_{W(I_j)} + c\|\nabla g\|_{L^2(I_j, L^{\frac{2d}{d+2}}(\mathbb{R}^d))}
\]
\[
\leq cA + c\eta^{\frac{4}{d-2}} \|\nabla \tilde{u}\|_{W(I_j)} + c\|\nabla g\|_{L^2(I_j, L^{\frac{2d}{d+2}}(\mathbb{R}^d))}
\]
\[
\leq c(A + \epsilon) + \frac{1}{2}\|\nabla \tilde{u}\|_{W(I_j)}
\]
choosing \(\eta = \eta(d) > 0\) small enough. Hence we have \(\|\nabla \tilde{u}\|_{W(I_j)} \leq 2c(A + \epsilon)\) consequently by taking \(\epsilon_0 \leq 1\), we have \(\|\nabla \tilde{u}\|_{W(I)} \leq 2\gamma(\eta(d), M)c(A + 1) =: M'(A, M, d)\).

**STEP II:** A priori estimate.

Here we follow [17], where the case \(\alpha = 0\) is dealt. Let us set \(q, r, \tilde{q}, \tilde{r}\) by \(q = \frac{2(d+2)}{d-2}, \frac{1}{r} = \frac{d-2}{2(d+2)} + \frac{\alpha}{2}, \tilde{q} = 2\) and \(\frac{1}{\tilde{r}} = \frac{d^2+2(1-\alpha)d-4\alpha}{2d(d+2)}\). If we write \(\frac{1}{q} = \frac{\gamma}{2}, \frac{1}{r} = \frac{\gamma}{2}, \frac{1}{\tilde{r}} = \frac{d^2-2(1-s)}{2d}\) then we have \(\gamma = \frac{d-2}{4d} < 1, \tilde{\gamma} = 1\) and \(s-1 = 1-\alpha\). Since Theorem 1.6 (iii) is valid for \(s\) in a neighbourhood of 1, we conclude
\[
\left\| \int_0^t S_a(t - \tau)h(\tau)d\tau \right\|_{L^{\tilde{q}}L^{\tilde{r}}} \lesssim \|h\|_{L^qL^r}
\]
(4.1)
is valid for \(0 < \alpha < 1\) close enough to 1. Then we have by fractional Hardy inequalities
\[
\|f\|_{S(I)} \lesssim \|D^\alpha f\|_{L^qL^r} \lesssim \|\nabla f\|_{W(I)}
\]
(4.2)
by interpolation
\[
\|D^\alpha f\|_{L^q(I, L^{\tilde{r}})} \lesssim \|f\|_{S(I)}^{1-\alpha}\|\nabla f\|_{W(I)}^\alpha
\]
(4.3)
by Holder
\[
\|u|^{4/(d-2)} D^\alpha u\|_{L^{\tilde{q}}L^{\tilde{r}}} \leq \|u\|_{S(I)}^{4/(d-2)}\|D^\alpha u\|_{L^qL^r}.
\]
(4.4)
Let $\eta > 0$. Again split $I$ into $l = l(M, M', \eta)$ intervals $I_0, I_1, \ldots, I_{l-1}$ with $I_j = [t_j, t_{j+1}]$ so that $\|u\|_{S(I_j)} \leq \eta$ and $\|D^\alpha \tilde{u}\|_{L^q(I_j, L^r)} \leq \eta$ for $j = 0, 1, \ldots, l - 1$. Let us write $u = \tilde{u} + w$. Then $w$ solves

$$i\partial_t w + \mathcal{L}_a w + f(\tilde{u} + w) - f(\tilde{u}) = -g$$

with $w(t_0) = u_0 - \tilde{u}(t_0)$ if $u$ solves (NLS$_a$). Now in order to solve for $w$, we need to solve, in $I_j$, the integral equation

$$w(t) = S_a(t - t_j)e^{i(t - t_j)\mathcal{L}_a}w(t_j) + i \int_{t_j}^t S_a(t - \tau)[f(\tilde{u} + w) - f(\tilde{u})](\tau)\,d\tau + i \int_{t_j}^t S_a(t - \tau)g(\tau)\,d\tau.$$  \hspace{1cm} (4.5)

Put $B_j = \|D^\alpha w\|_{L^q(I_j, L^r)}$, $\gamma_j = \|D^\alpha e^{i(t - t_j)\mathcal{L}_a}w(t_j)\|_{L^q(I_j, L^r)} + \varepsilon$ and $N_j(w, \tilde{u}) = \|D^\alpha [(f \circ (\tilde{u} + w)) - (f \circ \tilde{u})]\|_{L^q(I_j, L^r)}$. Then by (4.1) (see also Remark 4.5 below also)

$$B_j \leq \gamma_j + cN_j(w, \tilde{u}).$$

Now by fractional Leibnitz and chain rule

$$N_j(w, \tilde{u}) \lesssim \left(\|\tilde{u}\|^4_{S(I_j)} + \|w\|^4_{S(I_j)}\right) \|D^\alpha w\|_{L^q(I_j, L^r)}$$

$$+ \|w\|_{S(I_j)} \left(\|\tilde{u}\|^6_{S(I_j)} + \|w\|^6_{S(I_j)}\right) \left(\|D^\alpha \tilde{u}\|_{L^q(I_j, L^r)} + \|D^\alpha w\|_{L^q(I_j, L^r)}\right).$$

Therefore $B_j \leq \gamma_j + c\eta^{\frac{4}{d-2}}B_j + cB_j^{\frac{\frac{4}{d-2}+1}{\frac{4}{d-2}}}$. and choosing $\eta > 0$ small

$$B_j \leq 2\gamma_j + cB_j^{\frac{\frac{4}{d-2}+1}{\frac{4}{d-2}}} = 2\gamma_j + cB_j^{\frac{4}{d-2}}B_j.$$  

This implies if $B_j \leq \left(\frac{1}{2c}\right)^{\frac{4}{d-2}} =: c_0$ (so that $cB_j^{\frac{4}{d-2}} \leq \frac{1}{2}$) then $B_j \leq 4\gamma_j$. Hence we have

$$\|\nabla w\|_{W(I_j)} \leq 4 \left(\|e^{-i(t - t_j)\mathcal{L}_a}w(t_j)\|_{W(I)} + \varepsilon\right)$$

provided $B_j \leq c_0$.

Now put $t = t_{j+1}$ in the integral formula (4.5), and apply $S_a(t - t_{j+1})$ to we obtain

$$S_a(t - t_{j+1})w(t_{j+1}) = S_a(t - t_{j})w(t_{j}) + i \int_{t_{j}}^{t_{j+1}} S_a(t - \tau)[f(\tilde{u} + w) - f(\tilde{u})](\tau)\,d\tau$$

$$+ i \int_{t_{j}}^{t_{j+1}} S_a(t - \tau)g(\tau)\,d\tau.$$  

Therefore as before provided $B_j \leq c_0$ we have

$$\|D^\alpha S_a(t - t_{j+1})w(t_{j+1})\|_{L^q(I_j, L^r)}$$

$$\leq \|D^\alpha S_a(t - t_{j})w(t_{j})\|_{L^q(I_j, L^r)} + \varepsilon + c\eta^{\frac{4}{d-2}}B_j + cB_j^{\frac{\frac{4}{d-2}+1}{\frac{4}{d-2}}}$$

$$\leq \gamma_j + c\eta^{\frac{4}{d-2}}B_j + 2\gamma_j \leq 3\gamma_j + c\eta^{\frac{4}{d-2}}4\gamma_j$$
and choosing \( \eta > 0 \) small we get \( \gamma_j + 1 \leq 5\gamma_j \). Note that by (4.3)
\[
\|D^\alpha e^{-i(t-t_j)\mathcal{L}_a}w(t_j)\|_{L^q(I,L^r)} \lesssim \|e^{-i(t-t_j)\mathcal{L}_a}w(t_j)\|^\alpha_{L^q(I)} \lesssim \|e^{-i(t-t_j)\mathcal{L}_a}w(t_j)\|^\alpha_{L^q(I)} \|w(t_j)\|^\alpha_{H^1(I)}.
\]
Therefore by the hypothesis that \( \gamma_0 \leq \varepsilon^{1-\beta}A' + c\varepsilon \). Iterating, we have \( \gamma_j \leq 5^j(\varepsilon^{1-\beta}A' + c\varepsilon) \) if \( B_j \leq c_0 \). Thus \( B_j \leq 4\gamma_j \leq 5^4(\varepsilon^{1-\beta}A' + c\varepsilon) \) if \( B_j \leq c_0 \). Choose \( \varepsilon_0 = \varepsilon_0(c,l) = \varepsilon_0(c,M,M',\eta) = \varepsilon_0(c,M,A,d) > 0 \) so that \( 5^4(\varepsilon_0^{1-\beta}A' + c\varepsilon_0) = c_0 \).

Therefore for \( 0 < \varepsilon < \varepsilon_0 \) we have \( \|D^\alpha w\|_{L^q(I,L^r)} \leq 5^4 l^4(\varepsilon^{1-\beta}A' + c\varepsilon) \) and hence by (4.2) \( \|w\|_{S(I)} \leq c 5^4 l^4(\varepsilon^{1-\beta}A' + c\varepsilon) \). Using Strichartz again we get \( \|w(t)\|_{H^1} \leq C(\varepsilon^{1-\beta}A' + c\varepsilon) \) for all \( t \in I \). This proves the required estimates and hence the theorem.

\[\Box\]

**Remark 4.5.** Since we are applying the Strichartz estimates on the \( \alpha \)-fractional derivative, the equivalences of the homogeneous Sobolev norms \( \|\cdot\|_{W^\alpha,\varepsilon} \), \( \|\cdot\|_{W^\alpha,\varepsilon'} \), \( \|\cdot\|_{\tilde{W}^\alpha,\varepsilon} \), and \( \|\cdot\|_{\tilde{W}^\alpha,\varepsilon'} \) play roles here. Note that the first equivalence is a consequence of the restriction on \( a \) and the second one is true for all \( a \). In the case when \( a = 0 \) such issue does not arise.

### 4.2. Scattering of solutions

In this subsection, we outline a proof of the scattering result, see Theorem 4.7 below for the exact statement. First, we define the ground state solution \( W_a \) and energy of a solution of (NLS\(_a\)):

**Definition 4.6.** (i) Given \( a < \left(\frac{d-2}{2}\right)^2 \), we define \( \beta > 0 \) via \( a = \left(\frac{d-2}{2}\right)^2[1-\beta^2] \).

Then define the function (ground state solution) by \( W_a(x) := [d(d-2)]^{1/2} \left[ \frac{|x|^{\beta-1}}{1+|x|^\beta} \right]^{(d-2)/2} \).

(ii) By \( E_a(u(t)) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} |\nabla u(t,x)|^2 - \frac{a}{2|x|^2} |u(t,x)|^2 - \frac{c}{2^*} |u(t,x)|^{2^*} \right\} dx \), we define the energy \( E_a(u) \) of a solution \( u \) corresponding to our problem.

For details of ground state solutions, one can see [4,8,31]. Note that the energy \( E(u) \) is conserved for a solution \( u \) to (NLS\(_a\)) throughout the maximal interval of existence, see [26, Lemma 3.6]. Now we are in a position to state the scattering result:

**Theorem 4.7.** (Scattering of solution) Let \( d \in \{3,4,5\} \) and \( a < \left(\frac{d-2}{2}\right)^2 - \left(\frac{d-2}{d+2}\right)^2 \). Assume that \( E_a(u_0) < E_{a\varepsilon_0}(W_{a\varepsilon_0}) \) and \( \|u_0\|_{H^1_a} < \|W_{a\varepsilon_0}\|_{H^1_{a\varepsilon_0}} \) and \( u_0 \) is radial. Then the solution \( u \) to (NLS\(_a\)) with data at \( t = 0 \) is defined for all time with \( \|u\|_{S(\mathbb{R})} < \infty \) and there exists \( u_{0,+},u_{0,-} \) in \( H^1 \) such that

\[
\lim_{t \to +\infty} \|u(t) - e^{it\mathcal{L}_a}u_{0,+}\|_{H^1} = 0, \quad \lim_{t \to -\infty} \|u(t) - e^{it\mathcal{L}_a}u_{0,-}\|_{H^1} = 0.
\]

Before giving the proof of Theorem 4.7, we state a few preliminaries from early works:
Theorem 4.8. (Coercivity, see Corollary 7.6 in [19]) Let $d \geq 3$ and $a < \frac{(d-2)^2}{2}$. Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to (NLS$_a$) with initial data $u(t_0) = u_0 \in H^1(\mathbb{R}^d)$ for some $t_0 \in I$. Assume $E_0(u_0) \leq (1 - \delta_0)E_{a\nu_0}(W_{a\nu_0})$ for some $\delta_0 > 0$. Then there exist positive constants $\delta_1$ and $c$ depending on $d, a, \delta_0$, such that if $\|u_0\|_{\dot{H}^1_a} \leq \|W_{a\nu_0}\|_{\dot{H}^1_{a\nu_0}}$, then for all $t \in I$

(i) $\|u(t)\|_{\dot{H}^1_a} \leq (1 - \delta_1)\|W_{a\nu_0}\|_{\dot{H}^1_{a\nu_0}}$.

(ii) $\int_{\mathbb{R}^d} |\nabla u(t, x)|^2 + \frac{a}{|x|^2}|u(t, x)|^2 - |u(t, x)|^\frac{2d}{d+2} \ dx \geq c\|u(t)\|^2_{\dot{H}^1_a}$.

(iii) $c\|u(t)\|^2_{\dot{H}^1_a} \leq 2E_a(u) \leq \|u(t)\|^2_{\dot{H}^1_a}$.

Theorem 4.9. (Concentration compactness, see Lemma 4.3 in [18], Theorem 3.1 in [19,35]) Assume $a < \frac{(d-2)^2}{2} - \frac{(d-2)^2}{d+2}$. Let $\{v_{0,n}\} \in \dot{H}^1(\mathbb{R}^d)$, $\|v_{0,n}\|_{\dot{H}^1} < A$, $v_{0,n}$ is radial for all $n \in \mathbb{N}$. Assume that $\|e^{itL_a}v_{0,n}\|_{S(\mathbb{R})} \geq \delta > 0$, where $\delta = \delta(A)$ is as in Proposition 4.1. Then there exist a sequence $\{V_{0,j}\}_{j=1}^\infty$ in $\dot{H}^1(\mathbb{R}^d)$, a subsequence of $\{v_{0,n}\}$ (which we still call $\{v_{0,n}\}$) and a couple $(\lambda_{j,n}, t_{j,n}) \in (0, \infty) \times \mathbb{R}$, with

$$\lambda_{j,n} + \lambda_{j',n} + |t_{j,n} - t_{j',n}| \lambda_{j,n} \to \infty$$

as $n \to \infty$ for $j \neq j'$ such that $\|V_{0,1}\|_{\dot{H}^1} \geq \alpha_0(A) > 0$. If $V_j(x,t) := e^{itL_a}V_{0,j}(x)$, then, given $\epsilon_0 > 0$, there exists $J = J(\epsilon_0)$ and $\{w_n\}_{n=1}^\infty \in \dot{H}^1(\mathbb{R}^d)$, so that

(i) $v_{0,n} = \sum_{j=1}^J \frac{1}{\lambda_{j,n}^2}V_j(\frac{x}{\lambda_{j,n}}) + w_n$

(ii) $\|e^{itL_a}w_n\|_{S(\mathbb{R})} \leq \epsilon_0$

(iii) $\|v_{0,n}\|^2_{\dot{H}^1_a} = \sum_{j=1}^J \|V_{0,j}\|^2_{\dot{H}^1_a} + \|w_n\|^2_{\dot{H}^1_a} + o(1)$ as $n \to \infty$

(iv) $E_a(v_{0,n}) = \sum_{j=1}^J E_a(V_j(\frac{x}{\lambda_{j,n}})) + E_a(w_n) + o(1)$ as $n \to \infty$.

In addition we may assume that for each $j$ either $\frac{t_{j,n}}{\lambda_{j,n}} \equiv 0$ or $\frac{t_{j,n}}{\lambda_{j,n}} \to \infty$ as $n \to \infty$.

Remark 4.10. The original result [19, Theorem 3.1] says we would get a sequence $\{x_{j,n}\}$ along with $\{\lambda_{j,n}\}, \{t_{j,n}\}$. But due to the radial situation we can take $x_{j,n} = 0$ for all $j, n$.

Proposition 4.11. (Localized virial identity) Let $\phi \in C_0^\infty(\mathbb{R}^d), t \in [0, T_+(u_0))$. Then for $u$ satisfying $i\partial_t u + \Delta u - Vu + |u|^{4/(d-2)}u = 0$ we have

(i) $\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 \phi = 2\text{Im} \int_{\mathbb{R}^d} \bar{u} \nabla u \cdot \nabla \phi \ dx$

(ii) $\frac{d^2}{dt^2}\int_{\mathbb{R}^d} |u|^2 \phi = 4 \sum_{i,j} \text{Re} \int_{\mathbb{R}^d} \partial_{x_i,x_j} \phi \partial_{x_i} u \partial_{x_j} \bar{u} - \int_{\mathbb{R}^d} [\Delta^2 \phi + 2\nabla \phi \cdot \nabla V] |u|^2 - \frac{4}{d} \int_{\mathbb{R}^d} \Delta \phi |u|^2$.

Proof. See [21, Lemma 7.2] by Killip and Visan. □

Now let us give a shorthand notation to an $u_0 \in \dot{H}^1(\mathbb{R}^d)$ for which scattering happens:
Definition 4.12. Let \( u_0 \in \dot{H}^1_a(\mathbb{R}^d) \) with \( \| u_0 \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a} \) and \( E_a(u_0) < E_{a\nu 0}(W_{a\nu 0}) \). We say that \((SC)(u_0)\) holds, if the maximal interval \( I \) of existence of the solution \( u \) to \((\text{NLS}_a)\) with initial data \( u_0 \) at \( t_0 \), is \( \mathbb{R} \) and \( \| u \|_{S(\mathbb{R})} < \infty \).

Note that, because of Proposition 4.1, Strichartz and Sobolev inequality, if \( \| u_0 \|_{\dot{H}^1_a} \leq \delta \), \((SC)(u_0)\) holds. Thus, in light of Theorem 4.8, there exists \( \eta_0 > 0 \) such that \( \| u_0 \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a} \), \( E_a(u_0) < \eta_0 \), then \((SC)(u_0)\) holds. Thus, there exists a number \( E_C \), with \( 0 < \eta_0 \leq E_C \leq E_{a\nu 0}(W_{a\nu 0}) \), such that, if \( \| u_0 \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a} \) and \( E_a(u_0) < E_C \), then \((SC)(u_0)\) holds and \( E_C \) is optimal with this property. Note that

\[
E_C = \sup \left\{ E \in (0, E_{a\nu 0}(W_{a\nu 0})) : \| u_0 \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a}, E_a(u_0) < E \Rightarrow (SC)(u_0) \text{ holds} \right\}
\]

and \( E_C \leq E_{a\nu 0}(W_{a\nu 0}) \). Assuming \( E_C < E_{a\nu 0}(W_{a\nu 0}) \), we have existence of a critical solution with some compactness property, namely we have the following result:

Proposition 4.13. Let \( E_C < E_{a\nu 0}(W_{a\nu 0}) \). Then there exists \( u_{0,C} \in \dot{H}^1(\mathbb{R}^d) \) with

\[
E_a(u_{0,C}) = E_C < E_{a\nu 0}(W_{a\nu 0}), \quad \| u_{0,C} \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a}
\]

such that, if \( u_C \) is the solution of \((\text{NLS}_a)\) with initial data \( u_{0,C} \) at \( t = 0 \) and maximal interval of existence \( I \), then \( \| u_C \|_{S(I)} = \infty \). In addition \( u_C \) has the following property: If \( \| u_C \|_{S(I_+)} = \infty \) then there exists a function \( \lambda : I_+ \rightarrow (0, \infty) \) such that the set

\[
K = \left\{ v(t,x) : v(t,x) = \frac{1}{\lambda(t)^{(d-2)/2}} u_C \left( t, \frac{x}{\lambda(t)} \right) \right\}
\]

has compact closure in \( \dot{H}^1(\mathbb{R}^d) \). A corresponding conclusion is reached if \( \| u_C \|_{S(I_-)} = \infty \), where \( I_+ = (0, \infty) \cap I, I_- = (-\infty, 0) \cap I \).

Proof. The existence of \( u_C \) follows exactly in the same way as in \cite[Proposition 4.1]{18} once we have Theorems 4.4 and 4.9. For the existence of \( \lambda \) we go in the way of proof of \cite[Proposition 4.2]{18} along with Theorem 2.13 (with \( G \) defined as Lemma 2.14 and \( H = \dot{H}^1(\mathbb{R}^d) \)).

Now we have the following rigidity result:

Proposition 4.14. Let \( u_0 \in \dot{H}^1(\mathbb{R}^d) \) such that \( E_a(u_0) < E_{a\nu 0}(W_{a\nu 0}) \), \( \| u_0 \|_{\dot{H}^1_a} < \| W_{a\nu 0} \|_{\dot{H}^1_a} \) and \( u \) be the solution to \((\text{NLS}_a)\) with \( u(0, \cdot) = u_0 \). Assume there exists a function \( \lambda : I_+ \rightarrow (0, \infty) \) such that the set

\[
K = \left\{ v(t,x) : v(t,x) = \frac{1}{\lambda(t)^{(d-2)/2}} u \left( t, \frac{x}{\lambda(t)} \right) \right\}
\]

has compact closure in \( \dot{H}^1(\mathbb{R}^d) \). Then \( u = 0 \).

Proof. The proof is similar to that of \cite[Proposition 5.3]{18} once we have Theorem 4.8 and Proposition 4.11.
Proof of Theorem 4.7. Note that Theorem 4.7 is the assertion $E_C = E_{a\vee 0}$ ($W_{a\vee 0}$). If not assume $E_C < E_{a\vee 0}(W_{a\vee 0})$. By Proposition 4.13 we have existence of a minimal solution $u_C$ satisfying the assumption of Proposition 4.14. Applying Proposition 4.14 to $u_C$ we conclude that $u_C = 0$ which is a contradiction as we had $\|u_C\|_{S(I)} = \infty$ from Proposition 4.13. □

Remark 4.15. The non-radial data can also be dealt in this technique provided one can bound the sequence $\{x_{1,n}\}$ in concentration compactness result, see Remark 4.10 and Theorem 4.9. In fact this is proved in [35] for dimension $d = 4, 5$. However, the non-radial case in dimension $d = 3$ is still open.

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