A One-dimensional Model for Thin-walled Box Girder Structures Considering the Deformable Cross-section in Dynamics

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Abstract. In this paper, a one-dimensional dynamic model is developed for thin-walled box girders considering both out-of-plane and in-plane cross-section deformations. The model considers the displacement field through the linear superposition of a set of deformation modes whose amplitudes vary along the beam axis. These deformation modes are defined based on the cross-section discretization, corresponding to contour deformations resulted by the imposing of unit displacements on discrete nodes. Then the deformation functions are approximated with nodal displacements through interpolation between adjacent nodes. Applying Hamilton's principle, the governing equation is derived and further interpolated for the finite element implementation. Numerical examples are also presented to show the accuracy in reproducing dynamic behaviors of thin-walled box girders.

Introduction

Box girder structures are widely used in civil and mechanical engineering for the high torsional stiffness and structural stability. However, they are generally susceptible to in-plane (distortion) and out-of-plane (warping) deformation due to “thin walls” comparing with the vast cross-sections. And the phenomenon has been verified to cast significant influences on the mechanical properties. Therefore, the cross-section deformation should be taken into consideration in the dynamic modeling of box girders [1].

Actually, researches on the distortion of box girders have been performed for several decades [2]. Recently, Ren et al. [3] investigate the distortion of cantilever box girders under in-plane shear strain. Park et al. [4] provide a method for distortional behavior of box girders. Recently, more attentions are focused on the shear lag and shear deformation effects due to the application in bridges [5]. Chen et al. [6] propose a shear lag warping function for a box girder with corrugated steel webs. Zhou [7] establishes an element considering the interaction of the bending and shear-lag deformation. However, these special models are not qualified in capturing a complete set of warping and distortion.

General thin-walled beam theories are useful in this aspect. The asymptotical method [8] and the expansion of displacement field [9] are able to describe deformable cross-sections provided a distinction between distortion and warping is not enormous. On the other end, generalized beam theory can account for both warping and distortion with shear deformation and transverse extension included [10]. Generalized eigenvectors [11] and the high-order model [12] also show similar performances in this index. The key point is the uncoupling of governing equations based on the solution of generalized eigenvalue problem which is involved and demanding. Actually, a simplified approach being not so full-featured but with high precision is more suitable in some cases.

In this paper, a one-dimensional model for thin-walled box girders is presented with cross-section deformation considered. First, the cross-section is discretized to approximate cross-section deformation modes with polynomial interpolation. Then with the linear superposition of these deformation modes, the three-dimensional displacement field is reduced to one dimension. With the application of Hamilton’s principle, the governing differential equation is obtained. Next, the
governing equation is interpolated to form the finite element scheme, which is further used in the numerical examples.

Model Formulation

Cross-section Analysis and Displacement Field

Figure 1 shows the box girder. The displacement field is defined with the axial \( u \), tangential \( v \) and normal \( w \) components, which are prescribed positive along the axes of local coordinate system \((n, s, z)\). Also shown is the global coordinate system \((x, y, z)\) with its origin located on one end. The cross-section is divided into a set of walls connected by 6 natural nodes which connect adjacent walls or locate at free ends (such as nodes 1 and 5, respectively). Since the breadth walls are much longer than the others, 2 intermediate nodes (nodes 7 and 8) are introduced to further divide the walls into segments which contribute to capture cross-section deformation.

The model considers the displacement field over the cross-section through linear superposition of a set of basis functions. Actually, each basis function is to describe one type of contour deformation mode. Here a total of 8 nodes lead to 32 deformation modes (Figure 2) since 4 degrees of freedom (shown in Figure 1c) are considered. The basis function is approximated with imposed unit nodal displacement through interpolation between adjacent nodes. Besides, two interpolation functions are adopted: linear Lagrange functions for the axial and tangential displacements, and cubic Hermite functions for the normal component.

The derived modes are numbered and arranged in matrix form with each mode related to the associated node (row) and unit displacement (column). These modes are described with basis functions, varying along the coordinate \( s \) for the tangential, normal and axial components, separately. The displacement field is reduced to the cross-section, \( \mathbf{u} = [u(s, z), v(s, z), w(s, z)]^T \), reading:

\[
u(s, z) = \psi_1 x, \ v(s, z) = \psi_2 x, \ w(s, z) = \psi_3 x
\]

where \( \psi_1, \psi_2 \) and \( \psi_3 \) correspond to the set of basis functions, reading \( \psi_1 = [\phi_1(s), \phi_2(s), \ldots, \phi_{32}(s)] \), \( \psi_2 = [\psi_1(s), \psi_2(s), \ldots, \psi_{32}(s)] \) and \( \psi_3 = [\omega_1(s), \omega_2(s), \ldots, \omega_{32}(s)] \). Then the three-dimensional displacement \( \mathbf{U}=[U_x(n, s, z), U_y(n, s, z), U_n(n, s, z)]^T \) is obtained as

![Figure 1](image1.png)

![Figure 2](image2.png)
\[ U_z = \psi_3 x - n \psi_3 (\partial \psi_3 / \partial z), \quad U_y = \psi_2 x - n (\partial \psi_3 / \partial s) x, \quad U_n = \psi_3 x. \]  

The deformation \( \varepsilon = [\varepsilon_{zz}(n, s, z), \varepsilon_{ss}(n, s, z), \gamma_{zz}(n, s, z)]^T \) and stress \( \sigma = [\sigma_{zz}(n, s, z), \sigma_{ss}(n, s, z), \tau_{zz}(n, s, z)]^T \) fields are obtained under the small displacements hypothesis as follows:

\[
\varepsilon_{zz} = \frac{\partial \psi_z}{\partial z} - n \frac{\partial \psi_3}{\partial z}, \quad \varepsilon_{ss} = \frac{\partial \psi_s}{\partial s} - n \frac{\partial \psi_3}{\partial s}, \quad \gamma_{zz} = \frac{\partial^2 \psi_z}{\partial z \partial s} + \frac{\partial^2 \psi_s}{\partial z \partial s} - 2n \frac{\partial \psi_3}{\partial z} \frac{\partial \psi_3}{\partial s}.
\]

\[
\sigma_{zz} = E' \varepsilon_{zz} + E' \nu \varepsilon_{ss}, \quad \sigma_{ss} = E' \varepsilon_{ss} + E' \varepsilon_{zz}, \quad \tau_{zz} = G \gamma_{zz}, \quad E' = \frac{E}{1 - \nu^2}
\]

where \( E, \nu \) and \( G \) are the material Young’s modulus, Poisson’s ratio and shear modulus, respectively.

**Governing Equation and Finite Element Scheme**

The strain energy, potential energy and kinetic energy are calculated by definition:

\[
U = \frac{1}{2} \int_V \varepsilon^T \sigma \text{d}V, \quad U_p = -\int_V U^T \text{pd}A \text{d}z, \quad T = \frac{1}{2} \int_V \rho \partial \dot{U}^T / \partial t \partial U / \partial t \text{d}V
\]

where \( V, A \) and \( L \) are the volume, cross-section area and length of the girder; \( \rho \) is material density; \( \text{p} \) is the loading vector. In the deduction of the governing equation, Hamilton principle is employed as

\[
\delta \int_{t_{1}}^{t_{2}} L_{a} \text{d}t = 0, \quad \delta \dot{x}|_{t_{1}} \equiv \delta \dot{x}|_{t_{2}} = 0
\]

where \( L_{a} \) is the Lagrangian as \( L_{a} = T - U - U_{p} \), and \( t_{1} \) and \( t_{2} \) are start and end times. Substituting Eq. 3, 4 and 5 into 6 leads to the governing equation of the box girder, reading

\[
\int_{V} \delta x^{T} H^{T} H^{T} \eta \text{d}A + \int_{V} \delta x^{T} (c^{T} Ec) \text{d}A_{z} - \int_{V} \dot{x}^{T} H^{T} \text{p} \text{d}A_{z} = 0
\]

where \( H \) meets \( U = Hx, \ c \) is defined by \( \varepsilon = cU \), and \( E \) is derived from \( \sigma = Ec \).

Then quadratic functions are involved to approximate the displacement within an element, reading

\[
x = N X = [ N_1 , N_2 , N_3 ] X, \quad N_1 = \left(2z^2/l^2 - 3z/l + 1\right) I_{12}, \quad N_2 = \left(4z/l - 4z^2/l^2\right) I_{32}, \quad N_3 = \left(2z^2/l^2 - z/l\right) I_{32}
\]

where \( N \) and \( X \) are the shape function matrix and the nodal displacement vector, respectively; \( l \) is the element length; the coordinate \( z \) varies from 0 to \( l \) within an element.

Substituting Eq. 8 into 7 leads to the finite element formulation

\[
m \ddot{x} + k x = f
\]

where the mass matrix \( m \), the stiffness matrix \( k \) and the force matrix \( f \) are separately calculated as

\[
m = \int_{A} N^{T} H^{T} \eta H \text{d}A_{z}, \quad k = \int_{A} \int_{A} N^{T} H^{T} c^{T} E c H \text{d}A_{z}, \quad f = \int_{A} \int_{A} N^{T} H^{T} \text{p} \text{d}A_{z}
\]

**Numerical Examples**

To validate the versatility, numerical examples are carried out. Related parameters are set as \( L = 24 \) m, \( h = 2.4 \) m, \( b_1 = 4.8 \) m, \( b_2 = 2.4 \) m, \( t = 0.3 \) m, \( E = 200 \) GPa, \( \nu = 0.3, \rho = 7830 \) Kg/m$^3$.

**Regarding a Cantilevered Box Girder**

On the basis of convergence check, the present element is used to model a cantilevered box girder for free vibration analysis. Table 1 shows the first 8 natural frequencies obtained with the present model and ANSYS shell theory and the relative errors. The results of present model are calculated with 40 elements, while the ANSYS model is discretized into 1280 Shell 181 4-node elements, distributed as 40 elements along the length and 32 over the cross-section.
Table 1. Comparison of the first 10 natural frequencies of the cantilevered box girder.

| Mode | Present model [Hz] | ANSYS shell [Hz] | Relative errors [%] |
|------|--------------------|------------------|---------------------|
| 1st  | 4.9191             | 4.8764           | 0.88                |
| 2nd  | 10.685             | 10.599           | 0.81                |
| 3rd  | 16.138             | 15.748           | 2.48                |
| 4th  | 22.125             | 22.272           | -0.66               |
| 5th  | 29.103             | 28.864           | -0.83               |
| 6th  | 30.451             | 30.669           | -0.71               |
| 7th  | 33.231             | 33.984           | -2.22               |
| 8th  | 35.019             | 34.776           | 0.70                |

The results in Table 1 show that natural frequencies derived from the present model are very close to those from ANSYS shell theory with relative errors less than 2.5%. To further check the dynamic behaviors, Figure 3 provides the comparison concerning the 1st ~ 8th modal shapes. The comparison proves the capability of reproducing three-dimensional behaviors of thin-walled box girders.

![Figure 3. Comparison of modal shapes of the cantilevered box girder with ANSYS shell model (the right ones).](image)

Regarding a Fixed-fixed Box Girder

To check if the model is equally applicable to different boundaries, another example is carried out on the same box girder with two ends restrained. Table 2 presents the comparison between present model and ANSYS shell theory about the first 6 modes.

Table 2. Comparison of the first 6 natural frequencies of the fixed-fixed box girder.

| Mode | Present model [Hz] | ANSYS shell [Hz] | Relative errors [%] |
|------|--------------------|------------------|---------------------|
| 1st  | 20.616             | 20.723           | -0.52               |
| 2nd  | 25.240             | 25.728           | -1.90               |
| 3rd  | 31.666             | 32.706           | -2.26               |
| 4th  | 32.833             | 33.664           | -2.47               |
| 5th  | 35.653             | 36.538           | -2.42               |
| 6th  | 38.057             | 39.650           | -2.50               |
The results in Table 2 validate that the present model is qualified for box girders with fixed-fixed constraints. The comparison of modal shapes in Figure 4 also proves its capability in reproducing three-dimensional behaviors of thin-walled box girders.

Figure 4. Comparison of modal shapes of the fixed-fixed box girder with ANSYS shell model (the right ones).

Summary

A one-dimensional model for thin-walled box girders has been presented considering cross-section deformations. In the process, cross-section discretization is considered to approximate sectional deformation modes with interpolations of nodal displacements. Then the box girder displacement field is approximated as the linear superposition of deformation modes. Applying Hamilton’s principle, the governing differential equation is obtained, and subsequently interpolated to establish the corresponding finite element. Numerical examples prove that the present model agrees well with three-dimensional shell models with much less computation and is of easy accessibility.

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