A SUPERSPACE NORMAL COORDINATE DERIVATION OF
THE DENSITY FORMULA

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ABSTRACT

Using normal coordinate expansions we derive by purely superspace
methods the density formula giving the component action corresponding
to a superspace supergravity-matter action.

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1 Introduction

In recent years, normal coordinate expansions in superspace have been used for a variety of purposes. McArthur [1], extending to superspace Spivak’s ordinary space discussion [2], has given a general expansion of the vielbein and connection for four-dimensional supergravity (expanding with respect to both the bosonic and fermionic coordinates) and has used it for computing super-$b_4$ coefficients [3]. Atick and Dhar [4] have used normal coordinate methods to obtain an expansion with respect to the fermionic $\theta$ coordinates for the Green-Schwarz superstring action in the background of 10-dimensional supergravity. This expansion was subsequently used for studying properties of the corresponding $\sigma$-model action [5] and for computing $\beta$-functions [6]. In the present work we consider another application.

For many years, one small bone sticking in the collective throat of superspace practitioners has been the inability to derive by purely superspace methods the so-called density formula — the expression which starting from a superspace supergravity or matter-supergravity action of the form

$$S = \int d^4x d^N \theta E^{-1} \mathcal{L}$$

(1.1)
gives the corresponding component action. Here $E = \text{sdet} E^M_A$ is the vielbein determinant. To obtain the component action various techniques have been used on an ad hoc basis: a Noether-like procedure which starts with the knowledge of one term in the component action and obtains the rest by (local) supersymmetry transformations [7]; by explicit (covariant) $\theta$ expansions [8]; or some other ingenious acrobatics relying in part on explicit Wess-Zumino gauge information [9]. A special case, that of $(2,2)$ supergravity was recently treated by purely superspace methods [10], but again relying on some specific properties of the system. Recently, Gates [11] has proposed a method for obtaining locally supersymmetric component actions by considerations involving the topology of superspace. A variant of that method, as well as a brief account of the present work, can be found in [12].

The idea of using normal coordinate expansions is simple: one way to obtain a component action in terms of the superspace action is to just expand the latter in powers of the Grassmann coordinates $\theta^\alpha$, and simply do the $\theta$-integration. However, in this primitive form, one is led in general to noncovariant results, and field redefinitions are needed to recast them in terms of (component supergravity) covariant objects. A normal coordinate expansion with respect to suitable fermionic variables can obviate this problem because it leads in general to covariant expressions. As mentioned above, McArthur [1] has given a normal coordinate expansion of the vielbein. However, he expands $E^M_A(x, \theta)$ with respect to both bosonic and fermionic coordinates around the point $x = 0$, $\theta = 0$ and therefore his expansion is not suitable for our purpose. In their work on the expansion of the superstring...
action [4], Atick and Dhar considered $\theta$ expansions leading to covariant expressions which were sufficient for their, and subsequent, purposes, namely expansions for the quantity $V_i^A \equiv \partial_i Z^M E_M^A$ as well as various covariant field strengths. It was not necessary to separately expand the vielbein itself. However, their procedure can be easily adapted for the case at hand and leads to a suitable expansion of the vielbein and hence of the vielbein determinant, and a purely superspace derivation of the component density formula.

The outline of our paper is as follows: In section 2 we present the modified expansion of the inverse vielbein $E_M^A(x, \theta)$ with respect to normal coordinates around any point, although in the applications we will generally expand only with respect to fermionic normal coordinates. Because of the Grassmann nature of these coordinates, the expansion cuts off after a finite number of terms. The subsequent sections apply the general formalism to several specific examples: two-dimensional $(1, 0)$, $(1, 1)$ and $(2, 0)$ supergravity, and four-dimensional $N = 1$ minimal supergravity. Other cases can be treated straightforwardly. We observe that as the number of supersymmetries, and therefore the number of spinor coordinates, increases, so does the order of the normal coordinate expansion, leading to some algebraic complexity. Thus, for example, for $N = 1$ supergravity in four-dimensions one would a priori need to expand to fourth (and to some extent, fifth) order. We avoid this, in section 7, by expanding in two steps: first with respect to (dotted) spinor variables only (requiring only second order expansion), leading to the familiar reduction of the full superspace integral to a chiral integral, and then again with respect to the remaining spinor coordinates (requiring again only second order expansion). This procedure is first illustrated in section 6 for the simpler case of $(2, 0)$ supergravity. Section 8 contains our conclusions. The constraints for the various supergravity theories are given in the Appendix.

2 Normal Coordinate Expansion

Our procedure is straightforward and basically not different from that in ref. [4]: we parametrize superspace by coordinates $(x^m, y^\alpha)$ where the $y^\alpha$ describe a tangent vector in a fermionic direction at a point with coordinates $(x^m, 0)$. (Generically we use lower-case Greek letters for both dotted and undotted spinor indices, wherever no distinction needs to be made.) More precisely, we choose the origin of our normal coordinate system to be an arbitrary point with coordinates $z^M = (z^m, z^\mu)$. We parametrize a neighbourhood of the origin by normal coordinates $y^A = (y^a, y^\alpha)$. The point with coordinates $y^A$ is reached by parallel transport from the origin. Eventually we identify $z^m = x^m$ and $y^\alpha = \theta^\alpha$ and then set $z^\mu = y^\alpha = 0$. The normal coordinate expansion that we use gives the vielbein in Wess-Zumino gauge, as can be seen by examining its components and their spinor derivatives. WZ gauge for
the supergravity fields is completely equivalent to this normal coordinate expansion. The superspace action can then be written in the form

$$ S = \int d^d x \ d^N y \ E^{-1}(x, y) \mathcal{L}(x, y) $$

and after expansion in powers of $y^\alpha$, the fermionic integration is trivial, using the basic Grassmann integration rule $\int dy_\alpha y_\beta = \delta_\alpha^\beta$ or just picking out the highest-order terms in $E^{-1} \mathcal{L}$. The expansion of $E^{-1}$ automatically gives the usual factor of $e^{-1} = \det e_m^\alpha$.

More generally, we expand with respect to some subset of the coordinates: We first divide the coordinates into sets $(z^i, z^s)$ and $(y^i, y^s)$. After using the algorithm we set $z^s = y^i = 0$ (in fact the $z^s$ can be set equal to some arbitrary constants) and use $(z^i, y^s)$ as a new coordinate system. The surviving $y^i$’s are the normal-gauge-fixed coordinates, while the remaining $z^s$’s are still arbitrary coordinates. Such gauges are useful, e.g., for “compactification”, where expansions are made in some of the coordinates, while coordinate invariance is still desired in the remaining coordinates. The most familiar case is that of Riemann normal coordinates, where we set all the $z$’s to vanish, and keep all the $y$’s as our new coordinates. A more relevant example is Gaussian normal coordinates, where we choose $y$ to be a single timelike coordinate, and $z$ the spacelike coordinates, by setting $z^0 = y^i = 0$. This construction then gives the timelike gauge $g_{m0} = \eta_{m0}$, fixing the time coordinate while leaving the space coordinates arbitrary. For covariant component expansions in supersymmetry, the idea is to fix the fermionic coordinates, while maintaining coordinate invariance in the bosonic coordinates.

Superspace integrations are often performed over “subsuperspaces” parametrized by the usual spacetime coordinates plus a subset of fermionic coordinates, for example (anti)chiral superspace for $D = 4$, $N = 1$. In such cases normal coordinate expansions can be used (1) to reduce a full superspace action, e.g. $\int d^4 x \ d^4 \theta \ E^{-1} \mathcal{L}$ to a subsuperspace action, e.g. $\int d^4 x \ d^\theta \ E^{-1} \mathcal{L}_{ch}$, and (2) to derive the component expansion of the subsuperspace action e.g. $\int d^4 x e^{-1} \mathcal{L}$. In fact, these are the two steps we use in practice to evaluate the component expansion of a full superspace action. We can interpret the chiral integral as being obtained from the full integral by expanding (and integrating over) only with respect to the $\bar{\theta}^\dot{a}$. (Of course, in any situation coordinates can be integrated out one at a time, but the result will not always be simple. The reduction produces a manifestly covariant result only when scalars, e.g. chiral scalars in curved $D = 4$, $N = 1$ superspace, can be defined on such subsuperspaces.) The procedure in both steps is the same; the only difference is the choices of the various sets of coordinates ($z$’s and $y$’s). For our chiral superspace example, which we will discuss in more detail below, the first step expands with respect to $\bar{\theta}^\dot{a}$, dividing up the superspace coordinates as $(z^m, z^\mu; y^\dot{a})$, while the second step expands with respect to $\theta^a$, dividing up the coordinates as $(z^m, z^\mu; y^a)$.
A familiar analogue is Euler angles, coordinates for SU(2) group space defined by a succession of rotations through each angular coordinate. Of course, performing the steps in succession, rather than all at once, produces a different coordinate choice since the covariant derivatives that enter in the parallel transport do not commute (even in flat superspace), so the expression for the component lagrangian may differ by a total derivative, although the component action is the same.

The point with coordinates \((y^a, y^\alpha)\) is obtained by an active coordinate transformation — parallel transport — from the point with coordinate \((z^m, z^\mu)\). In particular, under such a transformation, the superspace covariant derivative

\[
\nabla_A = E_A^M D_M + \omega_A M
\]

(2.2)

with \(D_M = (\partial_m, D_\mu)\) and \(\omega_A M = \omega_{AB}^C M^B\), \(M\) a Lorentz generator, transforms in standard fashion as

\[
\nabla_A \rightarrow e^{y^B \nabla_B} \nabla_A e^{-y^B \nabla}
\]

(2.3)

with \(y \cdot \nabla = y^B \nabla_B\) and \(y^B = E^B_M y^M\). Infinitesimally\(^1\) this gives

\[
\delta \nabla_A = [y^B \nabla_B, \nabla_A] = y^B (T_{BA}^C \nabla_C + \omega_{BA} M) - (\nabla_A y^B) \nabla_B
\]

(2.4)

Here \(T_{BA}^C\) and \(R_{BA}\) are the torsion and curvature respectively, defined by \([\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + \omega_{AB} M\) and \(R_{AB} = R_{ABC}^D M^C\). Under the coordinate transformation tensors undergo the variation

\[
\delta T = y \cdot \nabla T
\]

(2.5)

Since by definition of normal coordinates the tangent vector \(y^A\) is parallel transported, we have

\[
\delta y^A = \delta (y \cdot \nabla) = 0
\]

(2.6)

From the infinitesimal variation of the covariant derivative we deduce the corresponding variations of the vielbein and connection:

\[
E^M_A \delta E^B_M = \nabla_A y^B - y^C T_{CA}^B
\]

\[
E^M_A \delta \omega^A_M = y^B R_{BA}
\]

(2.7)

The finite transformation corresponding to (2.3) is obtained by iteration. For this purpose it is most convenient to consider the variation of the differential forms rather than the covariant derivatives. Thus, with the definition

\[
E^A = d\omega^M E^A_M \quad , \quad \omega^A_B = d\omega^M \omega^A_{MB}
\]

(2.8)

\(^1\) One normally introduces an affine parameter \(t\), and considers expansions in powers of \(t\), with the affine parameter being set to unity, \(t = 1\), at the end of the calculation. We dispense with introducing it explicitly.
we deduce from (2.7),
\[ \delta E^A = D y^A + y^C E^B T_{BC}^A \] (2.9)
where the covariant differential is
\[ D y^A = E^B \nabla_B y^A = dy^A - y^B \omega_B^A = dy^A - y^B E^C \omega_{CB}^A \] (2.10)
From the variation of the connection we also obtain
\[ \delta D y^A = -y^B E^C y^D R_{DCB}^A \] (2.11)
These equations are, of course, essentially identical to those of ref. [1].

In general, we evaluate the transformed vielbein
\[ E'^A(z; y) = E^A + \delta E^A + \frac{1}{2!} \delta^2 E^A + \frac{1}{3!} \delta^3 E^A + \ldots \]
\[ = E^B(z) F_B^A + (D y^B) G_B^A \] (2.12)
where \( F \) and \( G \) depend explicitly on \( T \) and \( R \) and their covariant derivatives, and on \( y \). The procedure is now straightforward; higher order terms in the expansion are obtained by iterating the variations in (2.7, 2.9, 2.11). Thus, the second variation is
\[ \delta^2 E^A = -y^B E^C y^D R_{DCB}^A + y^C E^B y^D \nabla_D T_{BC}^A + y^C (D y^B + y^E E^D T_{DE}^B) T_{BC}^A \] (2.13)
and the third is
\[ \delta^3 E^A = \]
\[ = y^D (D y^B + y^E E^C T_{GF}^B) y^C R_{CB}^A - y^D E^B y^C y^F \nabla_F R_{CB}^A + y^C (D y^B + y^E E^C T_{GF}^B) y^D \nabla_D T_{BC}^A + y^C E^B y^D \nabla_D T_{BC}^A - y^C y^D y^F y^E \nabla_{E} R_{CB}^A + y^D \nabla_D T_{BC}^A + y^C y^F E^D y^E \nabla_D T_{BC}^A + y^C y^F E^D y^E \nabla_D T_{BC}^A \] (2.14)
The fourth variation is:
\[ \delta^4 E^A = \]
\[ = y^D y^E E^F y^G R_{GF}^B y^C R_{CB}^A - y^D D y^B y^C y^F \nabla_F R_{CB}^A - y^D y^F (D y^B + y^E T_{GH}^E) T_{EF}^B y^C R_{CB}^A - y^D y^F E^C y^E \nabla_E R_{CB}^A - y^D (D y^B + y^E E^C T_{GF}^B) y^C y^E \nabla_E R_{CB}^A - y^D E^B y^C y^F \nabla_F R_{CB}^A - y^C y^E E^F y^G R_{GF}^B y^D \nabla_D T_{BC}^A + y^C D y^B y^D y^E \nabla_E \nabla_D T_{BC}^A + y^C (D y^B + y^E E^C T_{GF}^B) y^C y^E \nabla_D T_{BC}^A + y^C y^E (D y^E + y^G T_{HG}^E) T_{EF}^B y^D \nabla_D T_{BC}^A \]
\[ + y^C y^E (D y^E + y^G E^H T_{HG}^E) T_{EF}^B y^D \nabla_D T_{BC}^A \]
by modifications can be made for other cases. Thus, we proceed as follows:

\[ y^C y^F E^H y^G (\nabla_G T_{HF}^B) y^D \nabla_D T_{BC}^A \]
\[ + y^C y^F E^H T_{HF}^B y^G \nabla_G \nabla_D T_{BC}^A \]
\[ + y^C (D y^B + y^F E G T_{GF}^B) y^D y^F \nabla_D \nabla_D T_{BC}^A \]
\[ + y^C E^B y^D y^F \nabla_F \nabla_D T_{BC}^A \]
\[ - y^C y^D (D y^F + y^G E^H T_{HF}^B) y^E R_{EFDB} T_{BC}^A \]
\[ - y^C y^D E^F y^G (\nabla_G R_{EFDB}) T_{BC}^A \]
\[ - y^C y^D E^F y^G R_{EFDB} y^G \nabla_G T_{BC}^A \]
\[ - y^C y^F E^G y^G R_{EFDB} y^D \nabla_D T_{BC}^A \]
\[ + y^C D y^B y^D y^F \nabla_F \nabla_D T_{BC}^A \]
\[ - y^C y^F E^G y^H y^R_{HGF} T_{DE}^{FB} T_{BC}^A \]
\[ + y^C y^F D y^D [y^F (\nabla_D T_{DF}^B) T_{BC}^A + T_{DF}^B y^E \nabla_E T_{BC}^A] \]
\[ + y^C y^F y^G E (D y^F + y^I E^H T_{HI}^F) T_{FE}^{DB} T_{BC}^A \]
\[ + y^C y^G y^E [y^H (\nabla_H T_{FE}^{D}) T_{DG}^{FB} T_{BC}^A + T_{FE}^{DB} y^H (\nabla_H T_{DG}^{B}) T_{BC}^A] \]
\[ + T_{FE}^{DB} y^H (\nabla_H T_{BC}^A)] \]
\[ + y^C y^G (D y^D + y^G E^H T_{HG}^D) y^E \nabla_D T_{DF}^B T_{BC}^A \]
\[ + y^C y^F E^D y^G (\nabla_G \nabla_D T_{DF}^B) T_{BC}^A \]
\[ + y^C y^F E^D y^G (\nabla_D T_{DF}^B) y^G \nabla_G T_{BC}^A \]
\[ + y^C y^F (D y^D + y^G E^H T_{HG}^D) T_{DF}^B y^E \nabla_E T_{BC}^A \]
\[ + y^C y^F E^D y^G (\nabla_G T_{DF}^B) y^F \nabla_E T_{BC}^A \]
\[ + y^C y^F E^D T_{DF}^B y^E y^G \nabla_G \nabla_E T_{BC}^A \]

(2.15)

(In the third and fourth variations, we have not collected like terms, for ease of comparison.)

For an action integrated with \( d^N y \) one needs in fact the variation up to \( \delta^{N+1} \) because it also gives rise to terms proportional to \( y^N \) (times \( D y \)). However, to obtain the relevant terms in \( \delta^{N+1} E^A \) for example, one needs only to make two changes on the right hand side of \( \delta^N E^A \): first, one replaces the \( E^C \) factors by \( D y^C \), and second, for the terms proportional to \( D y \), one replaces the tensor \( T \) appearing in such terms, by \( \delta T = y \cdot \nabla T \). There is then no actual need to work out the variation beyond \( \delta^N E^A \).

Quite generally, the algorithm can be used for any division of \( z^M \) and \( y^A \) into complementary sets. The procedure applies for expansion about any subset of the coordinates; only the range of the indices changes. However, to simplify notation we will use indices corresponding to the special case of expansion of the full superspace over all the fermionic coordinates, with the understanding that appropriate modifications can be made for other cases. Thus, we proceed as follows:

- On the right hand side all quantities are evaluated at the origin of the normal
coordinates, chosen as $z^M = (z^m, 0)$ (and with $dz^m = 0$). In particular

$$E^A(z^m, 0) = dz^n E_n^A(z^m, 0)$$

(2.16)

$$= \begin{cases} 
  dz^n e_n^a(z^m) \\
  -dz^n \psi_n^a(z^m)
\end{cases}$$

Here $e_n^a$ is the component vielbein and $\psi_n^a$ the gravitino field. The minus sign arises because we usually define the gravitino from the inverse, $E_{c\nu} = \psi_{c\nu}$, c.f. ref. [7], eq. (5.6.6).

- We specialize $y^A$ to be purely spinorial, $(0, y^a)$ (and also $dy^a = 0$). We observe that splitting up the covariant differential $Dy^a$ into an ordinary differential $dy^a$ and a connection term, c.f. [2.11], introduces the latter explicitly as a noncovariant dependence in the normal coordinate expansion. However, in any Lorentz invariant quantity, such as the vielbein determinant, the dependence drops out. Alternatively, one can remove it by an additional Lorentz rotation accompanying the transport of the vielbein from the origin of the normal coordinates. Keeping it at intermediate stages provides a check on the calculation, but to simplify the algebra we will simply drop it.

- We use the torsion and curvature constraints appropriate to the supergravity theory under consideration, in particular the fact that in $R_{ABC}^D$, $C$ and $D$ must be both either spinorial or vectorial.

- We write $E^A(z^m, 0; 0, y^a) = E^A(z^m, y^a) = dz^m E_m^A(z, y) + dy^\beta E_\beta^A(z, y)$ and read off the normal coordinate expansion of the components from the coefficients of $dz^m$ and $dy^\beta$. On the right-hand-side of the normal coordinate expansion we have in each term (aside from those proportional to $Dy^C$) one factor

$$E^C(z^m, 0) = dz^m E_m^C = dz^m e_m^b \delta_b^C - dz^m \psi_m^\beta \delta_\beta^C = e^b [\delta_b^C - \psi_\gamma^b \delta_\gamma^C]$$

$$\equiv E^b(z^m, 0) \hat{E}_b^A(z^m, 0)$$

(2.17)

with $e^b = dz^m e_m^b$, the component (gravitational) vielbein. We thus have

$$E^A(z, y) = \hat{E}^b(z, 0) \hat{E}_b^A(z, y) + (Dy^\beta) \hat{E}_\beta^A(z, y)$$

(2.18)

where

$$\hat{E}_b^A = \hat{E}^b(z, 0) F_c^A(z, y) = F_b^A - \psi_\gamma^b F_\gamma^A$$

$$\hat{E}_\beta^A = G_\beta^A(z, y)$$

(2.19)

Therefore we obtain

$$E^{-1} = \text{sdet} \begin{pmatrix} 
  e_m^b \hat{E}_b^a & e_m^b \hat{E}_b^a \\
  \hat{E}_\beta^a & \hat{E}_\beta^a
\end{pmatrix} = e^{-1} \cdot \text{sdet} \hat{E}_B^A$$

(2.20)
where $\hat{E}_\beta^A$ is obtained directly as the coefficient of $Dy^\beta$ while $\hat{E}_b^A$ is obtained by substituting on the right-hand-side of the normal coordinate expansion $\delta^a_b$ and $-\psi_b^\gamma$, respectively, wherever $e_m^c$ and $-\psi_m^\gamma$ would appear. The expansion involves now only objects with (component) tangent space indices. We will generally drop the hat on $\hat{E}_B^A$. The superdeterminant itself is computed, in standard fashion:

$$\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D} \quad (2.21)$$

although sometimes it is simpler to evaluate it from $\text{sdet}E = \exp(\text{str} \ln E)$.

In the next section we consider, as a simple illustration of the procedure, the density formula for the two-dimensional $(1,0)$ system.

### 3 Two-dimensional $(1,0)$ Supergravity

The $(1,0)$ supergravity case is the simplest to consider. The superspace is parametrized by two bosonic coordinates $x^m$ and one fermionic coordinate $y^+$. We consider an action of the form

$$S = \int d^2x dy^+ E^{-1}(x,y)\mathcal{L}^+(x,y) \quad (3.1)$$

where $\int dy^+ y^+ = 1$. We need the expansion of the vielbein and the lagrangian to first order in $y^+$. For the latter we simply use

$$\mathcal{L}^+(x,y) = \mathcal{L}^+| + y^+(\nabla_+ \mathcal{L}^+|) \quad (3.2)$$

where the vertical bar $|$ indicates as usual evaluation at $y^+=0$. In the expansion of the vielbein determinant, for terms linear in $y^+$ we need to go up to the second variation of $E^A$ (but keeping only one term, the one proportional to $Dy^+$, from that variation). We have

$$E^A(x,y) = E^A(x,0) + Dy^A + y^+ E^C T_{C+}^A + \frac{1}{2} y^+ Dy^+ T_{++}^A \quad (3.3)$$

The supergravity constraints for the torsion that enter here are $T_{++}^A = 2i\delta_+^A$, $T_{C+}^A = 2i\delta_C^+ \delta_+^A$. (The complete set of constraints can be found in the Appendix.) Using $E_m^+| = -\psi_m^+$ we find then

$$E_m^a = (e_m^a - 2iy^+ \psi_m^+)\delta^a_+ = e_m^b(\delta_b^a - 2iy^+ \psi^+_b \delta^a_+)$$

$$E_+^m = -\psi_m^+ = -e_m^b \psi_b^+$$

$$E_{\beta}^a = -iy^+ \delta_+^\beta \delta^a_+$$

$$E^+_\beta = \delta^+_\beta \quad (3.4)$$
We obtain

$$E^{-1} = e^{-1[1 - iy^+\psi^+_*]}$$  \hspace{1cm} (3.5)$$

Therefore, the action in (3.1) is

$$S = \int d^2x dy_+ e^{-1[1 - iy^+\psi^+_*]} \cdot (\mathcal{L}^+| + y^+\nabla_+ \mathcal{L}^+|)$$

$$= \int d^2x \ e^{-1[\nabla_+ - iy^+\psi^+\mathcal{L}^+|}$$ \hspace{1cm} (3.6)$$

which is the familiar density formula for this case \cite{13}.

4 \hspace{1cm} (1, 1) \hspace{1cm} Supergravity

We turn now to the case of (1, 1) supergravity where superspace is parametrized by two spinorial coordinates, $y^+$ and $y^-$, in addition to the bosonic ones. We consider an action of the form

$$S = \int d^2x dy_+ dy_- E^{-1}(x,y) \mathcal{L}(x,y)$$ \hspace{1cm} (4.1)$$

We need the normal coordinate expansion to second order in $y$. The Lagrangian expanded to that order is

$$\mathcal{L}(x,y) = (1 + y^+\nabla_+ + y^-\nabla_- + 1/2 y^+ y^-[\nabla_+, \nabla_-])\mathcal{L}$$ \hspace{1cm} (4.2)$$

The relevant constraints are \cite{13, 16, 17}

$$T^+ = 2i , \hspace{1cm} T^- = 2i$$

$$T^+ = \frac{i}{2} R , \hspace{1cm} T^- = -\frac{i}{2} R$$ \hspace{1cm} (4.3)$$

We find for the expansion of the vielbein (actually, as discussed in section 2, $\hat{E}_b^A$)

$$E_b^+ = \delta_b^+ - 2iy^+\psi_b^+ - \frac{i}{2} y^+ y^- \delta_b^+ R$$

$$E_b^- = \delta_b^- - 2iy^-\psi_b^- - \frac{i}{2} y^+ y^- \delta_b^- R$$

$$E_b^+ = -\psi_b^+ + \frac{i}{2} y^+ \delta_b^+ R - \frac{i}{2} \delta_b^+ y^+ y^- \nabla_+ R - \frac{i}{2} y^+ y^- \psi_b^+ R$$

$$E_b^- = -\psi_b^- - \frac{i}{2} y^- \delta_b^- R - \frac{i}{2} \delta_b^- y^+ y^- \nabla_- R - \frac{i}{2} y^+ y^- \psi_b^- R$$

$$E_\beta^+ = -iy^+\delta_\beta^+$$

$$E_\beta^- = -iy^-\delta_\beta^-$$

$$E_\alpha^+ = \frac{i}{2} y^+ y^- \delta_\alpha^+ R$$

$$E_\alpha^- = \frac{i}{2} y^+ y^- \delta_\alpha^- R$$ \hspace{1cm} (4.4)$$

with $R$ and $\nabla_\alpha R$ evaluated at $(x,0)$.  

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From this we compute the superdeterminant and find

\[ E^{-1} = e^{-1} [1 + iy^+ \psi_+^* + iy^- \psi_- - \frac{1}{2} y^+ y^- R + y^+ y^- (\psi_+^* \psi_- - \psi_-^* \psi_+) ] \]  \hspace{1cm} (4.5)

The action (4.1) is therefore

\[ S = \int d^2 x dy_+ dy_- e^{-1} [1 + iy^+ \psi_+^* + iy^- \psi_- - \frac{1}{2} y^+ y^- R + y^+ y^- (\psi_+^* \psi_- - \psi_-^* \psi_+)] \cdot (1 + y^+ \nabla_+ + y^- \nabla_- - y^+ y^- \nabla_+ \nabla_-) L | \]  \hspace{1cm} (4.6)

which agrees with the result in [16].

5 \hspace{1cm} (2,0) Supergravity

We move on to the case of (2,0) supergravity, with superspace parametrized by \( x^m \), as well as \( y^+, y^- \). The action for this case is

\[ S = \int d^2 x dy_+ dy_- E^{-1} (x, y) \mathcal{L}^{++}(x, y) \]  \hspace{1cm} (5.1)

Again, the Lagrangian is expanded to second order as

\[ \mathcal{L}^{++}(x, y) = (1 + y^+ \nabla_+ + y^- \nabla_- + \frac{1}{2} y^+ y^- [\nabla_+, \nabla_-]) \mathcal{L}^{++} \]  \hspace{1cm} (5.2)

The required torsions, from the appendix, are

\[ T^{++}_+ = 2i \]

\[ T^{++}_- = -\Sigma^+ , \quad T^{+-}_+ = -\Sigma^- \]

\[ T^{+-}_- = i G_- , \quad T^{--}_- = -i G_- \]

The vielbein components (for \( \hat{E} \)) are:

\[ E_+^+ = 1 - 2i (y^+ \psi_+^* + y^- \psi_-^*) \]
\[ E_+^- = -2i (y^+ \psi_+^* + y^- \psi_-^*) - 2y^+ y^- G_- \]
\[ E_+^0 = 0 \]
\[ E_+^= = 1 \]
\[ E_-^+ = -iy^+ \]
\[ E_-^- = -iy^- \]
\[ E_-^0 = 0 \]
\[ E_-^= = 0 \]
\[
E_A^+ = -\psi_A^+ \\
E_A^\pm = -\psi_A^\pm \\
E_A^\mp = -\psi_A^\mp - iy^+ G_A^\mp + iy^+ y^\mp \Sigma^\mp \\
E_A^\pm = -\psi_A^\pm + iy^+ G_A^\pm - iy^+ y^\mp \Sigma^\mp \\
E_B^\alpha = \delta_B^\alpha (5.3)
\]

We find for the superdeterminant
\[
E^{-1} = e^{-1}[1 - iy^+ \psi_A^+ - iy^+ \psi_A^+] (5.4)
\]
and therefore the action (5.1) becomes
\[
S = \int d^2x dy_+ dy_+ e^{-1}[1 - iy^+ \psi_A^+ - iy^+ \psi_A^+] \\
\cdot \left(1 + y^+ \nabla_+ + y^+ \nabla_+ + \frac{i}{2} y^+ y^+ [\nabla_+, \nabla_+] \right) \mathcal{L}^{++} \\
= \int d^2x e^{-1} \{ \frac{i}{2} [\nabla_+, \nabla_+] + iv_A^+ \nabla_+ - iv_A^+ \nabla_+ \} \mathcal{L}^{++} (5.5)
\]
which is consistent with the density formula in [13].

We note for future reference, using \( \frac{i}{2} [\nabla_+, \nabla_+] = \nabla_+ \nabla_+ - i \nabla_+ \) and evaluating at the origin of normal coordinates with \( \nabla_+ = D_+ + \psi_+^+ \nabla_+ + \psi_+^+ \nabla_+ \) (the \( D_+ \) term gives a total derivative that we drop), that the action can be rewritten as
\[
S = \int d^2x dy_+ dy_+ e^{-1} [\nabla_+ - 2iv_A^+] \nabla_+ \mathcal{L}^{++} (5.6)
\]
suggesting that the density formula could be obtained in a two-step process going through the intermediate stage of a “chiral” integral. We discuss this in the next section.

6 A Chiral Decomposition in (2, 0) Supergravity

In this section, we recover the last density formula in the previous section by going through an intermediate stage of rewriting the (2, 0) superspace integral as a \( dy_+ \) integral of a “chiral” integrand.

As discussed in section 2, instead of considering a normal coordinate expansion around the origin \( (z^m, 0, 0) \) in a direction parametrized by \( y^+ \) and \( y^\perp \), we can consider an expansion in a single fermionic coordinate, with the origin \( (z^m, z^\perp, 0) \), in the direction parametrized by \( y^\perp \). The expansion of the (2, 0) vielbein is similar to that in the (1, 0) theory:
\[
E^A(z, y^\perp) = E^A + Dy^\perp \delta^A + y^\perp E^C T^A_C + \frac{1}{2} y^\perp Dy^\perp T^A_{++} (6.1)
\]
with the quantities on the right hand side evaluated at $y^+ = 0$. Note that in $(2,0)$ theory $T_{++}^A = 0$.

The analogue of (2.17) is $E^C = dz^m[E_m^b \delta^C_b + E_m^+[\delta^C_+]]$ where we have introduced roman letters to indicate collectively space-time and + spinor indices: $m = (m, +)$, $b = (b, +)$. We use a modification of the procedure explained in (2.17) and (2.18) by extracting from the full vielbein matrix a factor

$$E^m_a = \left( \begin{array}{c} E^m_a \\ E^m_\mu \\ E^\mu_a \\ E^\mu_\alpha \end{array} \right)$$

(6.2)

The entries in $E^m_a$ are the vielbein components evaluated at $z^+ = 0$. We have then (ellipses indicate irrelevant terms)

$$E^{-1} = \text{sdet} \left( E^m_a + y^+ E^m c T_{c^+}^a \right)$$

(6.3)

which equals

$$\text{sdet} E^c \times \text{sdet} [\delta^c_a + y^+ T_{c^+}^a]$$

(6.4)

But the second superdeterminant is unity, c.f. (A.3), and we simply obtain $\tilde{E}^{-1} = \text{sdet} E^c$. Consequently we have the reduction to a chiral density formula

$$\int d^2x dy_+ dy \, E^{-1} \mathcal{L}^{++} = \int d^2x dy_+ \tilde{E}^{-1} \nabla_+ \mathcal{L}^{++} \bigg|_{y^+ = 0}$$

(6.5)

We proceed now to the next step in the reduction, using an expansion with respect to $y^+$ similar to the one above

$$E^a(x, y^+) = E^a + Dy^+ \delta^a_+ + y^+ E^c T_{c^+}^a + \frac{1}{2} y^+ D y^+ T_{++}^a$$

(6.6)

We find

$$\tilde{E}^{-1} = e^{-1} \text{sdet} \left( \begin{array}{cc} 1 - 2i y^+ \psi^+ \star & -2i y^+ \psi^+ \star \\ 0 & 1 \end{array} \right)$$

$$= e^{-1} (1 - 2i y^+ \psi^+ \star)$$

(6.7)

When this is substituted in the chiral integral, we recover the final expression in the previous section.

### 7 Four-dimensional $N = 1$ Supergravity

As a final example we present the derivation of the density formula for this case. The same derivation will work for the density formula of $(2, 2)$ supergravity in two dimensions \[10\]. We will consider the case of minimal, $n = -1/3$ supergravity, but the
same method should work for any value of \( n\). \( N = 1 \) superspace is parametrized by \((x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})\) and corresponding covariant derivatives. The complete set of constraints defining the torsions and curvatures, is given in the Appendix.

We will use the same approach as in section 6 for \((2, 0)\) supergravity, namely reduce first the \( d^4\theta \) integral to a \( d^2\theta \) chiral integral, and then reduce the latter. In this manner, it is not necessary to use the normal coordinate expansion to fourth order, as a direct reduction would require, but only to second order. We consider therefore an expression of the form

\[
S = \int d^4x d^2\theta d^2\tilde{\theta} E^{-1} \mathcal{L} = \int d^4x d^2\theta \int d^2\tilde{\theta} E^{-1} \mathcal{L}
\]

and expand the integrand with respect to normal coordinates \( y^{\dot{\alpha}} \) around an origin with coordinates \((z^m, z^\mu, 0)\). We split up the vielbein components \( E^A_M \) into components \( E^{\alpha}_a, E_{\dot{\alpha}}^\dot{\mu}, \) and \( E^\mu_{\dot{\alpha}} \) where we use roman letters to indicate collectively space-time indices as well as undotted spinor indices, e.g. \( a = (\alpha, \alpha) \).

We need the expansion to second order in \( y^{\dot{\alpha}} \), and the following nonzero torsions and curvatures (see Appendix A)

\[
T_{\dot{\alpha} b}^{\dot{\alpha}} = i\delta_{\dot{\alpha}}^b \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad b \equiv (\beta \dot{\beta}) \tag{7.2}
\]

\[
T_{\dot{\alpha} b}^{\gamma} = iC_{\alpha \beta} G_{\gamma \dot{\beta}}, \quad T_{\dot{\alpha} b}^{\gamma} = -iC_{\alpha \beta} \delta_{\dot{\gamma}}^{\dot{\beta}} R \tag{7.3}
\]

where the Lorentz generator is

\[
(M^\alpha)^{\beta} = \delta^\alpha_{\dot{\beta}} \delta_{\gamma}^{\dot{\alpha}} - \frac{1}{2} \delta^\alpha_{\dot{\beta}} \delta_{\gamma}^{\dot{\beta}} \tag{7.4}
\]

(Torsions with an odd number of spinor indices vanish, as do, here, the curvatures with a vector or dotted spinor last index.)

Once more we modify the discussion in section 2, extracting a factor \( E^b_m (z^m, z^\mu, 0) \) with \( m = (m, \mu) \) and \( b = (b, \beta) \). (One has to pay some attention to minus signs when bringing ahead of spinors some spinorial components of \( E^b_m \)). We compute

\[
E^{-1}(z^m, z^\mu, 0, 0, y^{\dot{\alpha}}) = \tilde{E}^{-1} \hat{E}^{-1} \tag{7.5}
\]

where

\[
\tilde{E}^{-1} = \text{sdet} \left( \begin{array}{cc} E^a_m & E^\alpha_m \\ E^\mu_a & E^\alpha_\mu \end{array} \right) \tag{7.6}
\]

and \( \hat{E}^{-1} = \text{sdet} \hat{E}_B^A \), with

\[
\hat{E}_B^A = \begin{pmatrix}
  \delta_{\dot{\alpha}}^\alpha - i\bar{\psi}^{\dot{\alpha}} \psi_{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^\alpha \bar{\psi}^{\dot{\beta}} y^2 R & -\psi_{\dot{\alpha}}^\alpha + i\bar{\psi}^{\dot{\alpha}} C_{\dot{\gamma} \dot{\beta}} \delta_{\dot{\alpha}}^\gamma R + \cdots & -\psi_{\dot{\alpha}}^\alpha + \cdots \\
  -i\bar{\psi}^{\dot{\alpha}} \delta_{\dot{\beta}}^\gamma + \cdots & \delta_{\dot{\beta}}^\alpha + \bar{\psi}_{\dot{\gamma}}^\beta \delta_{\dot{\alpha}}^\gamma R & \cdots \\
  0 & 0 & \delta_{\dot{\beta}}^\alpha + \frac{1}{2} \delta_{\dot{\beta}}^\alpha \bar{\psi}^{\dot{\alpha}} y^2 R
\end{pmatrix} \tag{7.7}
\]
where the ellipses indicate irrelevant terms.

We find, evaluating the superdeterminant

$$E^{-1} = \tilde{E}^{-1}[1 - \tilde{y}^2 R]$$

(7.8)

where $\tilde{y}^2 = \frac{1}{2}\bar{y}^\alpha \bar{y}_\alpha$.

At the same time, the expansion of the lagrangian $L$ is

$$L = \left[1 + \bar{y}_\gamma \nabla_\gamma + \frac{1}{2}\bar{y}_\gamma \bar{y}_\delta \nabla_\gamma \nabla_\delta \right] L|_{\theta = 0}$$

(7.9)

and the action becomes

$$S = \int d^4x d^4\theta E^{-1} L = \int d^4x d^2\theta \tilde{E}^{-1}(\nabla^2 + R) L|.$$ 

(7.10)

$\tilde{E}^{-1}$ is the measure of the chiral subspace. (One can verify in the $n = -1/3$ theory that $\tilde{E}^{-1} = \phi^3$ where $\phi$ is the superconformal compensator.)

The same procedure can be applied to the next step. The coordinate restriction is now the antichiral one

$$z^\mu = 0, \ (y^a, \bar{y}^\dot{\alpha}) = 0$$

(7.11)

We first need to evaluate $E_{m^a}$, now treated as a function of the new $z^M$ and $y^A$. We can use the result of the previous calculation by: (1) replacing (7.7) with the hermitian conjugate (effectively just switching dotted and undotted indices), and (2) deleting the second row and column before taking the superdeterminant, so we get the contribution to the smaller superdeterminant of (7.6). The result is

$$\tilde{E}^{-1} = e^{-1}[1 - iy^a \bar{\psi}_{\alpha\dot{\alpha}} - y^2(3\tilde{R} + \frac{1}{2}C^{\alpha\beta}\bar{\psi}_{\alpha\dot{\alpha}}\bar{\psi}_{\beta\dot{\beta}})]$$

(7.12)

Comparing to (7.5) we identify the new $\tilde{E}^{-1}$ as $e^{-1} = det e_{m^a}$ and the rest as the new $\tilde{E}^{-1}$. The expansion of the Lagrangian is similar to the previous case (just switching $\bar{y} \rightarrow y$), giving the final result

$$S = \int d^4x d^4\theta E^{-1} L = \int d^4x e^{-1}D^2(\nabla^2 + R) L| = \int d^4x e^{-1}D^4 L|$$

(7.13)

where we have defined a superdifferential operator, the “chiral density projector” $D^2$, which (for the present case of old minimal supergravity) takes the form

$$D^2 \equiv \nabla^2 + i\bar{\psi}_{\alpha\dot{\alpha}} \nabla_\alpha + 3\tilde{R} + \frac{1}{2}C^{\alpha\beta}\bar{\psi}_{\alpha\dot{\alpha}}\bar{\psi}_{\beta\dot{\beta}}$$

(7.14)

and the general density projector $D^4 \equiv D^2(\nabla^2 + R)$. We could use instead the complex conjugate $D^4 = \tilde{D}^2(\nabla^2 + \tilde{R})$, which differs only by a total space-time derivative. The final result is the standard density formula for minimal four-dimensional $N = 1$ supergravity.
8 Conclusions

The normal coordinate procedure we have used in this paper for deriving the density formula is straightforward and can be applied to any supergravity theory. Although straightforward, it is not necessarily the most economical way for obtaining the component supergravity-matter action; one has to go through the two stages of first computing the vielbein, and then the superdeterminant, and algebraically this can be quite complicated. Obviously, in any specific case, or even in general, some short-cut or particular trick may be preferable. However, we feel that in principle one should have an unambiguous, and extra-assumption-free procedure for deriving the result, relying on nothing more than the general properties of superspace. We believe the normal coordinate expansion provides the appropriate technique.

We conclude with the following remark. In ref. [7], section 5.6.b, a closely related technique for obtaining $\theta = 0$ components of covariant derivative was developed. It also allows one, in principle, to extract from these components an expansion for the vielbein. However, we believe that the normal coordinate expansion we have used in this paper is more systematic and simpler.

A Appendix: Supergravity Constraints

We list here our conventions and the constraints for the supergravity theories discussed in the main text. In general we follow the conventions used in [7]. The constraints for the various cases are listed below.

(1,0) Supergravity:

$$\{\nabla_+ , \nabla_+ \} = 2i\nabla_\# \ , \ [\nabla_+ , \nabla_\#] = -2i\Sigma^+ M$$
$$\{\nabla_+ , \nabla_\#\} = 0 \ , \ [\nabla_\# , \nabla_\#] = -\Sigma^+ \nabla_+ + RM$$

where $R = 2\nabla_+ \Sigma^+$.

(1,1) Supergravity:

$$\{\nabla_+ , \nabla_+ \} = 2i\nabla_\# \ , \ \{\nabla_- , \nabla_- \} = 2i\nabla_\#$$
$$\{\nabla_+ , \nabla_- \} = RM$$

$$[\nabla_+ , \nabla_\#] = i(\nabla_- R) M + \frac{1}{2} R \nabla_- \ , \ \ [\nabla_- , \nabla_\#] = i(\nabla_+ R) M - \frac{1}{2} R \nabla_+$$

(2,0) Supergravity:

$$\{\nabla_+ , \nabla_+ \} = 0 \ , \ \{\nabla_+ , \nabla_\#\} = 0$$
\{\nabla_+, \nabla_+\} = 2i\nabla_+ \quad (A.3) \\
[\nabla_+, \nabla_-] = iG_\pm \nabla_+ - 2i(\nabla_+ G_\pm)M \\
[\nabla_-, \nabla_+] = -iG_\pm \nabla_- + 2i(\nabla_- G_\pm)M \\
[\nabla_\pm, \nabla_\mp] = (\nabla_+ G_\pm) \nabla_+ - (\nabla_+ G_\mp) \nabla_- + ([\nabla_+, \nabla_+] G_\mp)M \\

(2.2) Supergravity:

\{\nabla_+, \nabla_+\} = 2i\nabla_+ \quad , \quad \{\nabla_-, \nabla_-\} = 2i\nabla_- \\
\{\nabla_+, \nabla_-\} = -\bar{R}M \quad , \quad \{\nabla_+, \nabla_-\} = R\bar{M} \\
[\nabla_+, \nabla_\mp] = -\frac{i}{2}\bar{R}\nabla_- - i(\nabla_- \bar{R})\bar{M} \\
[\nabla_-, \nabla_+] = \frac{i}{2}\bar{R}\nabla_- + i(\nabla_- R)M \\
[\nabla_-, \nabla_\pm] = -\frac{i}{2}R\nabla_+ - i(\nabla_+ \bar{R})\bar{M} \\
[\nabla_\pm, \nabla_\mp] = -\frac{i}{2}R\nabla_+ + i(\nabla_+ R)M \quad (A.4)

N=1 Four-Dimensional Supergravity:

\{\nabla_\alpha, \nabla_\beta\} = -2\bar{R}M_{\alpha\beta} \quad , \quad \{\nabla_\alpha, \nabla_\bar{\alpha}\} = i\nabla_\alpha \bar{\alpha} \quad (A.5) \\
[\nabla_\alpha, \nabla_\beta] = -iC_{\alpha\beta}[\bar{R}\nabla_\beta - G^\gamma_\beta \nabla_\gamma] \\
\quad + iC_{\alpha\beta}[\nabla_\beta M_{\gamma\delta}^\gamma - (\nabla_\gamma G^\delta_{\gamma\delta})M_{\delta\gamma}] - i(\nabla_\beta \bar{R})M_{\alpha\beta} \\
[\nabla_\alpha\bar{\alpha}, \nabla_\beta\bar{\beta}] = [C_{\alpha\gamma}W_{\beta\gamma}^\gamma + C_{\alpha\beta}(\nabla_\alpha G^\gamma_\beta) - C_{\alpha\beta}(\nabla_\alpha R)\delta_\beta^\gamma] \nabla_\gamma + C_{\alpha\beta}G^\gamma_\beta \nabla_{\gamma\bar{\alpha}} \\
\quad + [C_{\bar{\alpha}\beta}(\nabla_\alpha W_{\beta\gamma}^\gamma + (\nabla^2 \bar{R} + 2R\bar{R})C_{\gamma\beta\delta}^\delta\gamma) - C_{\bar{\alpha}\beta}(\nabla_\alpha \nabla^\delta G^\gamma_{\gamma\delta})]M_{\delta\gamma} + h.c.
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