Typical Relaxation of Isolated Many-Body Systems Which Do Not Thermalize

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We consider isolated many-body quantum systems which do not thermalize, i.e., expectation values approach an (approximately) steady longtime limit which disagrees with the microcanonical prediction of equilibrium statistical mechanics. A general analytical theory is worked out for the typical temporal relaxation behavior in such cases. The main prerequisites are initial conditions which appreciably populate many energy levels and do not give rise to significant spatial inhomogeneities on macroscopic scales. The theory explains very well the experimental and numerical findings in a trapped-ion quantum simulator exhibiting many-body localization, in ultracold atomic gases, and in integrable hard-core boson and XXZ models.

The long-standing task to explain macroscopic equilibrium phenomena in terms of the underlying microscopic quantum dynamics is presently regaining considerable attention \([1,2]\). Since open systems are beyond the realm of standard quantum mechanics, the common starting point is an isolated many-body system, possibly incorporating the environment of the subsystem of actual interest. The question whether and how such a system or subsystem approaches some thermal or non-thermal equilibrium state after a sufficiently long time has been the focus of numerous analytical \([3,4]\), numerical \([5,6]\), and experimental \([7,8]\) studies. Despite the reversible and everlasting motion of the microscopic degrees of freedom, it could be shown in Refs. \([9,10]\) under increasingly weak assumptions about the system Hamiltonian, the initial condition, and the considered observable that expectation values must remain extremely close to a constant value for the vast majority of all sufficiently late times (the exceptional times include initial transients and quantum revivals).

The natural next question is whether the system thermalizes, that is, whether the longtime behavior is well approximated by the pertinent microcanonical expectation value from equilibrium statistical mechanics. A first prominent criterion for thermalization is the so-called eigenstate thermalization hypothesis (ETH), postulating that every energy eigenstate yields expectation values close to the corresponding microcanonical values \([3,4]\). In other words, a violation of ETH is commonly considered an indicator of nonthermalization \([1,5,6]\). A related but different indicator is the existence of additional conserved quantities (besides the system Hamiltonian) which can be written as sums of local operators, and which play a particularly prominent role for so-called integrable systems \([1]\). Numerically, it has been found that such systems usually violate the ETH and do not thermalize \([5,6]\). Instead, the longtime behavior is well captured by a so-called generalized Gibbs ensemble (GGE), which is obtained by the standard working recipe to maximize the von Neumann entropy under the constraints that the expectation values of the conserved quantities must be correctly reproduced \([5]\). Yet another common distinction between integrable and nonintegrable systems is the statistics of the gaps between neighboring energy levels \(E_n\) \([1]\). Further prominent examples which do not thermalize are systems exhibiting many-body localization (MBL) \([2,11,12,13]\). Compared to integrable systems, they are structurally more robust against small changes of the model Hamiltonian, but they otherwise seem to be quite similar, e.g., regarding energy level statistics, conserved quantities, ETH violation, and the GGE \([1,2,21]\).

The objective of our Letter is a quantitative analytical description of the temporal relaxation in the absence of thermalization. Our approach is thus complementary to the numerical case studies, e.g., in Refs. \([2,12,21,22]\). Related analytical investigations are also quite numerous \([20,23,24]\). Yet, for each of them, a closer look at the considered systems and the obtained results reveals quite significant differences from our approach. For instance, some of them concern only thermalizing systems, others focus on special observables, or on deriving upper and lower bounds for the temporal relaxation, etc. Particularly little is known about equilibration time scales in isolated systems which do not thermalize. Likewise, pertinent experimental works are still rather scarce \([13,17]\).

A comparison of our theory with exemplary numerical and experimental results is provided later.

Going in medias res, let us consider a Hamiltonian \(H\) with eigenvalues \(E_n\) and eigenvectors \(|n\rangle\) and an arbitrary initial state \(\rho(0)\) (pure or mixed and, in general, far from equilibrium). According to textbook quantum mechanics, its temporal evolution is \(\rho(t) = \mathcal{U}_t \rho(0) \mathcal{U}_t^\dagger\) with \(\mathcal{U}_t := e^{-iHt/\hbar}\). Hence, the expectation value \(\langle A \rangle_\rho := \text{Tr}\{\rho A\}\) of an arbitrary observable \(A\) follows as

\[
\langle A \rangle_\rho(t) = \sum_{m,n} \rho_{mn}(0) A_{nm} e^{i(E_n - E_m)t/\hbar},
\]

where \(A_{nm} := \langle m | A | n \rangle\), \(\rho_{mn}(t) := \langle m | \rho(t) | n \rangle\) and where, depending on the specific model under consideration, \(m\) and \(n\) run from 1 to infinity or to some finite upper limit. Averaging Eq. (1) over all \(t \geq 0\) yields the result \(\langle A \rangle_\overline{\rho}\), where the diagonal ensemble \(\overline{\rho}\) is defined via \(\overline{\rho}_{mn} := \delta_{mn} \rho_{nn}(0)\) \([22]\). Hence, if the system equilibrates at all, \(\overline{\rho}\) must stay extremely close to \(\langle A \rangle_\overline{\rho}\) for practically all sufficiently large \(t\) (see above).

As usual, we focus on systems with a macroscopically well-defined energy; i.e., all energy levels \(E_n\) with non-
negligible populations $\rho_{nn}(0)$ must be contained in an interval $I_E := [E - \epsilon, E]$ of macroscopically small (but microscopically large) width $\epsilon$. Furthermore, we adopt the idealization that the probability $\rho_{nn}(0)$ to observe an energy $E_n$ outside $I_E$ can be approximated as strictly zero. The number of energies $E_n$ contained in $I_E$ is denoted by $D$ and, without loss of generality, we assume that $n \in \{1, \ldots, D\}$ for all those $E_n$. The Cauchy-Schwarz inequality $|\rho_{nn}|^2 \leq \rho_{nn} \rho_{nn}$ then implies that only $n$, $n \leq D$ actually matter in Eq. (1) and in all that follows. Specifically, the effectively relevant Hamiltonian is $H_1 := \sum_{n=1}^D E_n |n\rangle \langle n|$.

Denoting by $\pi$ any permutation of $\{1, \ldots, D\}$, we define

$$H_{\pi} := \sum_{n=1}^D E_n |\pi(n)\rangle \langle \pi(n)| = \sum_{n=1}^D E_{\pi^{-1}(n)} |n\rangle \langle n|. \quad (2)$$

Hence, $H_{\pi}$ is obtained by permuting either the eigenvalues or the eigenstates of the original Hamiltonian $H_1$.

In general, every $H_{\pi}$ entails a different evolution of $\rho(t)$. Accordingly, in Eq. (1) either the energies or the matrix elements must be permuted analogously as in (2).

On the other hand, one readily sees that the following important quantities and properties are invariant under arbitrary permutations $\pi$: (i) the energy spectrum, and hence the level statistics; (ii) the violation or nonviolation of the ETH; (iii) the conserved quantities (28); (iv) the initial expectation value $\langle A \rangle_{\rho(0)}$. (v) the large $t$, the expectation value $\langle A \rangle_{\rho(t)}$ stays extremely close to $\langle A \rangle_{\pi}$, with the same diagonal ensemble $\pi$ for all $H_{\pi}$, and likewise for the GGE.

The main result of our Letter concerns the $\pi$ and $t$ dependent relaxation of $\langle A \rangle_{\rho(t)}$ and reads

$$\langle A \rangle_{\rho(t)} = \langle A \rangle_{\pi} + F(t) \left\{ \langle A \rangle_{\rho(0)} - \langle A \rangle_{\pi} \right\} + \xi_\pi(t), \quad (3)$$

$$F(t) := \left\{ \frac{D}{2} |\phi(t)|^2 - 1 \right\} \left( \frac{D}{2} - 1 \right), \quad (4)$$

$$\phi(t) := D^{-1} \sum_{n=1}^D e^{iE_n t}/\hbar. \quad (5)$$

The only $\pi$ dependent term on the right-hand side of Eq. (3) is $\xi_\pi(t)$ and satisfies, for $D \geq 6$, the following key properties:

$$[\xi_\pi(t)]_{\Pi} = 0, \quad [\xi_\pi^2(t)]_{\Pi} \leq (6\Delta_A)^2 \max_n \rho_{nn}(0), \quad (6)$$

where $\Pi$ denotes the set of all permutations of $\{1, \ldots, D\}$ and $\Pi_{\Pi}\Pi$ the average over all $\pi \in \Pi$. Furthermore, $\Delta_A$ is the measurement range of the observable $A$, i.e., the difference between its largest and smallest eigenvalues.

Equations (3)-(5) are exact analytic results when $D \geq 6$ and for arbitrary $H$, $A$, and $\rho(0)$ with $\rho_{nn}(0) = 0$ for $n > D$. Their detailed mathematical derivation is quite tedious and provides very little physical insight; hence, it has been postponed to the Supplemental Material.

Since a typical many-body system exhibits an extremely dense energy spectrum (exponential in the degrees of freedom), it is practically impossible (e.g., in an experiment) to notably populate only a few energy levels; hence, $\max_n \rho_{nn}(0)$ must be unimaginably small (18).

Observing that $1/D \leq \max_n \rho_{nn}(0)$ implies that $D \gg 1$ in Eqs. (4) and (5), that the number $D!$ of permutations $\pi \in \Pi$ is gigantic, and that $[\xi_\pi^2(t)]_{\Pi}$ in Eq. (6) is exceedingly small. As a consequence, $\xi_\pi(t)$ itself must be very small for the vast majority of all $\pi \in \Pi$; i.e., we can safely approximate Eq. (3) by

$$\langle A \rangle_{\rho(t)} = \langle A \rangle_{\pi} + F(t) \left\{ \langle A \rangle_{\rho(0)} - \langle A \rangle_{\pi} \right\}. \quad (7)$$

Specifically, this approximation also applies to the “true” system $H_1$, unless there are special reasons why its temporal relaxation should notably differ from that of practically all other $H_{\pi}$.

A first very strong argument why the true system may be expected to exhibit the typical relaxation behavior (1) is the abovementioned invariances (i)-(v) under arbitrary permutations $\pi$. In fact, when considering the corresponding Hamiltonians $H_{\pi}$ as a matrix ensemble, our situation is essentially just a particular instance of random matrix theory (25), whose predictions are well known to be surprisingly successful in many cases, provided that the ensemble preserves a few very basic properties of the true system of actual interest (e.g., symmetries, or the invariances (i)-(v) in our case).

On the other hand, usual model Hamiltonians $H_1$ only involve short-range interactions (or local operators) (1-2), while most other $H_{\pi}$ do not preserve this “local structure”. Spatial inhomogeneities of particle numbers, energy etc. are thus expected to be balanced out increasingly slowly over increasing distances when $H_1$ governs the dynamics, but not for most other $H_{\pi}$. Note that instead of permuting the energy eigenvectors $|n\rangle$ in Eq. (1) according to Eq. (2), one could replace $\rho(0)$ by $\rho_{\pi}(0) := U_\pi^\dagger \rho(0) U_{\pi}$, where the unitary $U_{\pi}$ is defined via $U_{\pi} |n\rangle = |\pi(n)\rangle$ (and likewise for $A$, while $H_1$ is now kept fixed). Once again, even when $\rho(0) = \rho_1(0)$ exhibits spatial inhomogeneities, one expects that most other $\rho_{\pi}(0)$ will appear (nearly) homogeneous; hence, the local structure of $H_1$ yields an unusually slow relaxation of $\rho(0)$ (compared to most other $\rho_{\pi}(0)$). In either case, it follows that our prediction (1) must be restricted to initial conditions without any significant spatial inhomogeneities on macroscopic scales.

A typicality result similar to Eqs. (3)–(6) was obtained by formally quite dissimilar methods in Ref. (24). Conceptually, the essential difference is that arbitrary unitary (Haar distributed) basis transformations rather than just eigenvector permutations in Eq. (2) were admitted in Ref. (24), resulting in the appearance of the microcanonical instead of the diagonal ensemble on the right-hand side of Eq. (3). In contrast to our work, the approach from Ref. (24) is thus restricted to systems which do thermalize. The main reason is that the permutations are a tiny subset (of measure zero) of all unitary basis transformations and thus may preserve additional key features of the true Hamiltonian $H_1$. For example, permutations preserve each of the abovementioned properties (i)-(v), but general unitaries preserve only (i) and (iv). In return, the smallness of $\max_n \rho_{nn}(0)$ on the right-hand side
of Eq. (6) is no longer required when admitting arbitrary unitaries [24]. In passing, we note that conditions similar to or even identical to \( \max_n \rho_{nn}(0) \ll 1 \) already arise in the general equilibration results from Refs. [19, 20].

Turning to the function \( F(t) \) in Eqs. (4) and (5), one readily sees that \( F(0) = 1 \) and \( 1 \geq F(t) > -1/D \) for all \( t \). Moreover, the following properties were derived previously in Ref. [23]: (i) \( F(t) \) remains negligibly small for the vast majority of all sufficiently large \( t \), provided the maximal energy degeneracy is much smaller than \( D \). (ii) Denoting by \( \Omega(E) \) the number of energies \( E_n \) below \( E \), by \( k_B \) and \( S(E) = k_B \ln(\Omega(E)) \) Boltzmann’s constant and entropy, respectively, and by \( T := 1/S'(E) \) the corresponding formal temperature, one can often approximate the sum in Eq. (5) by an integral over a suitably smoothened level density, yielding the approximation

\[
F(t) = 1/[1 + (t k_B T/h)\tau].
\]

Note that \( T \) and \( S(E) \) could be identified with the usual temperature and entropy for a thermalized system, but they have no immediate physical meaning for nonthermalizing systems.

Besides integrability and MBL, yet another (quite trivial) reason for nonthermalization may be that the non-negligible level populations \( \rho_{nn}(0) \) are not confined to a macroscopically small energy interval (see above Eq. (6)). Incidentally, this case can also be readily included in our present theory, namely, by choosing \( D \) and the labels \( n \) so that \( n \in \{1, ..., D\} \) if and only if \( \rho_{nn}(0) \) is non-negligible. As a consequence, Eq. (8) is, in general, no longer valid, while all other findings remain essentially unchanged.

As a first example, we consider the experiment by Smith et al. [16] with \( N = 10 \) ions in a linear Paul trap, emulating the disordered Ising Hamiltonian

\[
H = \sum_{i<j} J_{ij} \sigma^x_i \sigma^x_j + \frac{B}{2} \sum_i \sigma^z_i + \sum_i \frac{D}{2} \sigma^z_i
\]

with \( i, j = 1, ..., N \), the Pauli matrices \( \sigma^{x,z}_i \), the couplings \( J_{ij} = J_{\max}/|i - j|^{1+\delta} \), the homogeneous field \( B = 4 J_{\max} \), the uniformly distributed random fields \( D_i \in [-W, W] \), and \( h = 1 \). Initializing the spins in the Néel state \( |\uparrow\uparrow\cdots\uparrow\rangle \), the system exhibits MBL for disorder strengths beyond about \( W = J_{\max} \). As was noted in Ref. [22], the experimentally measured Hamming distance \( \tilde{D}(t) \) from Ref. [16] can be recovered as the expectation value of the observable \( A := (1 - M)/2 \) with \( M := \sum_i (-1)^i \sigma^z_i \) (staggered magnetization).

In Fig. 1, the experimental results are compared with our theoretical approximation [24, 25] by introducing the numerically determined energies \( E_n \) of the Hamiltonian into Eq. (5). Furthermore, as in the experiment, we averaged the so obtained results for \( \tilde{D}(t) \) over 30 realizations of the disorder in Eq. (4). Since there are only \( N = 10 \) spins, \( \max_n \rho_{nn}(0) \) is typically not yet very small and increases with \( W \). We therefore focused on a moderate disorder of \( W = 4 J_{\max} \), and we considered labels \( n \) with \( \rho_{nn}(0) < 0.01 \) as negligible (see above), resulting in typical values \( \max_n \rho_{nn}(0) \approx 0.1 \) and \( D \approx 20 \). The concomitant approximations for \( \langle A \rangle_{\rho(0)} \) turned out to exhibit particularly strong finite-\( N \) effects; hence, we used the \textit{a priori} known actual value \( \langle A \rangle_{\rho(0)} = 0 \) in Eq. (7).

Besides those disorder averaged results, individual realizations of Eq. (4) would also seem interesting. Since experimental data are not available, we replicated the numerical solutions of the Schrödinger equation with Hamiltonian (5) from Refs. [16, 22]. The results for two realizations are shown in the insets of Fig. 1. The theoretical curves have been obtained as described above, employing the same realization of Eq. (4) as in the numerics in each inset. In view of those quite notable finite size fluctuations, the theory explains the “real” temporal relaxation remarkably well.

Next, we consider the equilibration of a coherently split Bose gas, as observed experimentally by Kuhnert.
et al. in Ref. 14 via the mean integrated squared contrast \( \langle C^2(L, t) \rangle \) of the matter-wave interference pattern for various integration lengths \( L \). This experiment (approximately) realizes an integrable system, exhibiting prethermalization rather than thermalization. The data from Ref. 14 are compared in Fig. 2 with our theory equation (7). Since modeling the quite intricate observable of the actual experiment goes beyond our present scope, we treated \( \langle A \rangle_{\rho(0)} \) and \( \langle A \rangle_{\rho} \) in Eq. (7) as fit parameters for any given \( L \). Similarly, estimating the experimentally relevant “effective temperature” \( T \) in Eq. (8) from first principles is beyond our present scope; hence, it was treated as fit parameter (common to all Eq. (8) from first principles is beyond our present scope; therefore, we treated \( T \) as fit parameter. The experimental estimate of \( T \) in Fig. 2 (at \( t_c = 0 \) ms) in Ref. 13 may also be a reasonable approximation in our case. Here, \( T \) is yet another effective temperature, which would agree with \( T \) at thermal equilibrium, but may well be different from \( T \) in our present case. Furthermore, the experimental estimate of \( T_{\text{eff}} \) is based on a quite involved procedure 13, whose implicit premises may only be approximately satisfied. In conclusion, \( T = 3 \) nK seems still compatible with the experimental findings, and the resulting theoretical curves in Fig. 2 explain the main features of the data quite well.

As a third example, we turn to the numerical results for an integrable model by Rigol 9, consisting of eight hard-core bosons on a periodic one-dimensional lattice with 24 sites, and exhibiting nonthermal longtime temperature values. The detailed definition of the considered observable \( \delta n_k(t) \) from Ref. 9 is not repeated here since only the initial and longtime values are actually needed in Eq. (8), whose quantitative values cannot be estimated theoretically anyway, and hence are treated as fit parameters. Furthermore, we adopted the approximation 8 with the estimate \( T = 2 \) from Ref. 9 (in units with \( k_B = h = 1 \)). The resulting agreement with the numerical data in Fig. 3 is remarkably good, considering that the system consists of just eight bosons.

Our last example is the integrable XXZ model of Torres-Herrera et al. from Ref. 26. Similarly as before, the initial value \( \langle A \rangle_{\rho(0)} = 0.25 \) in Eq. (10) is known \textit{a priori} for the specific observable under consideration, while \( \langle A \rangle_{\rho} \) is treated as a fit parameter. On the other hand, \( F(t) \) is now evaluated via Eq. (4) by approximating the discrete levels on the right-hand side of Eq. (5) by a continuous level density. In view of Table 1 and Fig 3(b) in Ref. 26, we roughly approximated this density as constant within the energy interval \( I_E = [-1.8, 1.8] \) and as zero otherwise. The resulting agreement with the numerics in Fig. 4 speaks for itself.

In conclusion, we devise in this Letter a general analytical theory for the temporal relaxation behavior of isolated many-body systems which do not thermalize. The main prerequisites are initial conditions which appreciably populate many energy levels and do not give rise to significant spatial inhomogeneities on macroscopic scales. Specifically, the relaxation must not entail any significant transport currents, caused by some unbalanced local densities (of particles, energy, etc.). On the other hand, the particular reason for the absence of thermalization (MBL, integrability, broad energy distribution) seems largely irrelevant. In fact, our theory also applies to systems which do thermalize, provided that the diagonal and microcanonical ensembles yield identical expectation values due to, e.g., the validity of the ETH. Compared to previous related studies, our main new concept consists in admitting only permutations of basis vectors in Eq. (2), rather than arbitrary (Haar distributed) basis transformations, thus preserving all local constants of motion, the diagonal ensemble which governs the longtime behavior, and the violation (or not) of the ETH. The adequate treatment of inhomogeneous initial conditions remains an important challenge for future research.

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**SUPPLEMENTAL MATERIAL**

I. AVERAGES OVER PERMUTATIONS

As in the main text, we denote by $\Pi$ the set of all permutations of $\{1, \ldots, D\}$ for an arbitrary but fixed $D \in \mathbb{N}$. Hence, there are $D!$ elements $\pi \in \Pi$.

Next, we consider an arbitrary but fixed $n \in \{1, \ldots, D\}$ and define

$$
\Pi_n := \{ \pi \in \Pi \mid \pi(n) = \nu \}
$$  \hspace{1cm} (10)

for any $\nu \in \{1, \ldots, D\}$. It follows that every $\pi \in \Pi$ belongs to one and only one of those subsets $\Pi_n \subset \Pi$. For symmetry reasons, each of the $D$ subsets $\Pi_n$ contains the same number of elements, and hence this number must be $(D-1)!$.

Denoting, as in the main text, by $[\ldots]_\Pi$ the average over all $\pi \in \Pi$, we can conclude for an arbitrary but fixed $n \in \{1, \ldots, D\}$ and any function $f : \{1, \ldots, D\} \to \mathbb{C}$ that

$$
[f(\pi(n))]_\Pi := \frac{1}{D!} \sum_{\pi \in \Pi} f(\pi(n)) = \frac{1}{D!} \sum_{\nu=1}^{D} \sum_{\pi \in \Pi_n} f(\pi(n))(11)
$$

where the last identity is due to the above observation that $\Pi$ is the disjoint union of all the $\Pi_n$’s. Considering that $\pi(n) = \nu$ for all $\pi \in \Pi_n$, according to (10), we can conclude that

$$
[f(\pi(n))]_\Pi = \frac{1}{D!} \sum_{\nu=1}^{D} (D-1)! f(\nu) = \frac{1}{D} \sum_{\nu=1}^{D} f(\nu) \quad (12)
$$

independent of $n$.

Analogously, we consider an arbitrary but fixed pair of integers $m, n \in \{1, \ldots, D\}$ with $m \neq n$ and define

$$
\Pi_{\mu \nu} := \{ \pi \in \Pi \mid \pi(m) = \mu, \pi(n) = \nu \}
$$  \hspace{1cm} (13)

for any $\mu, \nu \in \{1, \ldots, D\}$ with $\mu \neq \nu$. As before, it follows that every $\pi \in \Pi$ is contained to one and only one $\Pi_{\mu \nu}$ and that every $\Pi_{\mu \nu}$ consists of $(D-2)!$ elements. For an arbitrary complex valued function $f$ of two arguments $m, n \in \{1, \ldots, D\}$ we thus obtain

$$
[f(\pi(m), \pi(n))]_\Pi := \frac{1}{D!} \sum_{\pi \in \Pi} f(\pi(m), \pi(n)) = \frac{1}{D!} \sum_{\mu \nu} \sum_{\pi \in \Pi_{\mu \nu}} f(\pi(m), \pi(n)) . \quad (14)
$$

Here and in the following, the prime symbol in $\sum'$ means that all summation indices run from 1 to $D$ and must be pairwise distinct. As before, one now can infer that

$$
[f(\pi(m), \pi(n))]_\Pi = \frac{1}{D!} \sum_{\mu \nu} (D-2)! f(\mu, \nu) = \frac{1}{D(D-1)} \sum_{\mu \nu} f(\mu, \nu) \quad (15)
$$

for any $m \neq n$.

A straightforward generalization of the above line of reasoning yields

$$
[f(\pi(n_1), \ldots, \pi(n_K))]_\Pi = \frac{(D-K)!}{D!} \sum_{\nu_1 \ldots \nu_K} f(\nu_1, \ldots, \nu_K) \quad (16)
$$

for any $K$-tuple $(n_1, \ldots, n_K)$ of pairwise distinct integers $n_k \in \{1, \ldots, D\}$ and any complex valued function $f(n_1, \ldots, n_K)$.

II. SUBSETS OF QUADRUPLES

For any given $D \in \mathbb{N}$ we define the set of quadruples

$$
I := \{ (k, l, m, n) \mid k, l, m, n \in \{1, \ldots, D\} \} \ . \quad (17)
$$

Next, we introduce 15 subsets $I_a$ of $I$, $a = 1, \ldots, 15$, defined via the following properties of its elements $(k, l, m, n)$:

$$
I_1 : k = l = m = n \\
I_2 : l = m = n, k \neq l \\
I_3 : k = m = n, k \neq l \\
I_4 : k = l = m, n \neq m \\
I_5 : k = l = m, k \neq l \\
I_6 : m = n \text{ and } k, l \text{ pairwise distinct} \\
I_7 : k = l \text{ and } k, m \text{ pairwise distinct} \\
I_8 : k = m \text{ and } k, l \text{ pairwise distinct} \\
I_9 : k \neq m \text{ and } k \neq l \\
I_{10} : k, l = m, n \neq m \\
I_{11} : k = m \text{ and } k, l \text{ pairwise distinct} \\
I_{12} : l = n \text{ and } k, m \text{ pairwise distinct} \\
I_{13} : l = m \text{ and } k, l \text{ pairwise distinct} \\
I_{14} : k = n \text{ and } k, l \text{ pairwise distinct} \\
I_{15} : k, l = m, n \text{ pairwise distinct}
$$  \hspace{1cm} (18)

The sequence of labels of $I_6, \ldots, I_{10}$ may appear peculiar but will turn out to be convenient later on. It is quite obvious that every given quadruple $(k, l, m, n) \in I$ is contained in one and only one of the 15 subsets $I_a$: Either all indices $k, l, m, n$ are equal ($I_1$), or three of them are equal and the fourth distinct ($I_2, \ldots, I_5$), or they can be grouped into two distinct pairs ($I_6, I_9, I_{10}$), or three are pairwise distinct and one among them is equal to the fourth ($I_6, I_7, I_{11}, \ldots, I_{14}$), or all of them are pairwise distinct ($I_{15}$). In other words, $I$ is the disjoint union of the 15 subset $I_a$.

Finally, we consider the set of quadruples

$$
J := \{ (k, l, m, n) \in I \mid k \neq l, m \neq n \} \quad (19)
$$

Since $J \subset I$ and since $I$ is the disjoint union of the 15 subsets $I_a$, it follows that $J$ is the disjoint union of the 15 subsets $J_a := J \cap I_a$. The definitions (15) and (19)
readily imply that the subsets $J_1, \ldots, J_8$ are empty and that $J_a = I_a$ for $a = 9, \ldots, 15$. In conclusion $J$ is the disjoint union of the subsets $I_a$ with $a = 9, \ldots, 15$.

III. DERIVATION OF THE MAIN RESULT

A. Preliminaries

As in the main text, we consider an arbitrary but fixed Hamiltonian $H_1 = \sum_{n=1}^{D} E_n |n\rangle \langle n|$. Furthermore, we recall the definition from Eq. (2) in the main text, namely

$$H_\pi := \sum_{n=1}^{D} E_n |\pi(n)\rangle \langle \pi(n)| .$$

(20)

Similarly as around Eq. (1) of the main text, the time evolution induced by $H_\pi$ is $\rho(t) = U_t \rho(0) U_t^\dagger$ with $U_t := e^{-iH_\pi t/\hbar}$ and the expectation value $\langle A \rangle_{\rho(t)} := \text{Tr} \{ \rho(t) A \}$ takes the form

$$\langle A \rangle_{\rho(t)} = \sum_{m,n=1}^{D} g(\pi(m), \pi(n)) e_m^n,$$

(21)

$$g(m,n) := \rho_{mn}(0) A_{nm},$$

(22)

$$e_m^n := e^{i(E_n - E_m)t/\hbar},$$

(23)

where, for notational convenience, the $t$ dependence of $e_m^n$ is omitted, and where, as in the main text, $A_{mn} := \langle m|A|n\rangle$, $\rho_{mn}(t) := \langle m|\rho(t)|n\rangle$.

Denoting, as usual, by $[\ldots]_\Pi$ the average over all $\pi \in \Pi$, the first objective of the following calculations is to explicitly evaluate the average $\langle [ A \rangle_{\rho(t)} \rangle_\Pi$ of the expectation value from (21). Denoting the deviation from the average by

$$\xi_\pi(t) := \langle A \rangle_{\rho(t)} - [ \langle A \rangle_{\rho(t)} ]_\Pi,$$

(24)

our second objective is to show that this definition implies (3) of the main text and that the variance of $\xi_\pi(t)$ satisfies the second relation in (6) from the main text. Note that the first relation in (6) follows immediately from the definition (24).

Before starting with those calculations, we introduce some useful definitions and relations. As in the main text, the diagonal ensemble $\bar{\pi}$ is defined via its matrix elements as

$$\bar{\pi}_{mn} := \delta_{mn} \rho_{mn}(0),$$

(25)

where $\delta_{mn}$ is the Kronecker delta. One readily sees that $\bar{\pi}$ is hermitian, non-negative and of unit trace, i.e. a well-defined density operator. Likewise, we introduce the auxiliary hermitian operators $B$ and $C$ via

$$B_{mn} := \delta_{mn} A_{nn}, \quad C_{mn} := A_{mn} - B_{mn}.$$  

(26)

Denoting by $\| A \|$ the operator norm of $A$, i.e. the largest eigenvalue in modulus, one readily concludes that

$$\| B \| \leq \| A \|, \quad \| C \| \leq 2 \| A \| .$$

(27)

As in the main text, we denote by $\Delta_A$ the difference between the largest and smallest eigenvalues of $A$. It is intuitively obvious, and can also be readily verified rigorously, that $\xi_\pi(t)$ from (21) remains unchanged upon adding an arbitrary constant to $A$. Without loss of generality, we thus can assume that the largest and smallest eigenvalues of $A$ are of equal modulus and opposite sign. Hence, we can take for granted that

$$\| A \| = \Delta_A/2$$

(28)

as far as the properties of $\xi_\pi(t)$ are concerned.

Denoting the eigenvalues and eigenvectors of $\rho$ as $p_n$ and $|\chi_n\rangle$, respectively, and observing that $p_n \geq 0$ for all $n = 1, \ldots, D$, it follows that $\sqrt{\rho} := \sum_{n=1}^{D} \sqrt{p_n} |\chi_n\rangle \langle \chi_n|$ is a well-defined Hermitian operator with $(\sqrt{\rho})^2 = \rho$. Given an arbitrary but fixed pair of vectors $|\psi\rangle$ and $|\phi\rangle$ we define $|\psi'\rangle := \sqrt{\rho} |\psi\rangle$ and $|\phi'\rangle := \sqrt{\rho} |\phi\rangle$. Rewriting $\langle \psi|\rho|\phi\rangle$ as $\langle \psi'|\phi'\rangle$ and invoking the Cauchy–Schwarz inequality, we can conclude that

$$|\langle \psi|\rho|\phi\rangle|^2 \leq |\langle \psi'|\phi'\rangle|^2 \leq \langle \psi'|\rho|\phi'\rangle \langle \psi'|\rho|\phi'\rangle.$$  

(29)

Finally, we recall the definitions (4) and (5) from the main text, reading

$$F(t) := (D |\phi(t)|^2 - 1)/(D - 1),$$

(30)

$$\phi(t) := D^{-1} \sum_{n=1}^{D} e^{i(E_n - E_m)t/\hbar}.$$  

(31)

By exploiting (23) one readily concludes that

$$F(t) = \frac{1}{D(D - 1)} \left[ \sum_{m,n=1}^{D} e^{i(E_n - E_m)t/\hbar} - D \right]$$

$$= \frac{1}{D(D - 1)} \sum_{m,n} e_n^m.$$  

(32)

Here and in the following, and in accordance with the notation in (14)-(16), the prime symbol in $\sum'$ indicates that all summation indices run from 1 to $D$ and must be pairwise distinct.

B. Evaluation of the average

In view of (21), we can infer that

$$[\langle A \rangle_{\rho(t)} ]_\Pi = \sum_{m,n=1}^{D} [g(\pi(m), \pi(n)) ]_\Pi e_m^n = Q + R.$$  

(33)

$$Q := \sum_{n=1}^{D} [g(\pi(n), \pi(n)) ]_\Pi$$

(34)

$$R := \sum_{m,n} [g(\pi(m), \pi(n)) ]_\Pi e_m^n.$$  

(35)

In (34) we exploited that $e_m^n = 1$ according to (23), and the primed sum in (35) is defined below (32). Choosing
With (32) we thus can rewrite (42) as

\[ Q = \sum_{\nu=1}^{D} g(\nu, \nu) \]  

(36)

and choosing \( f(m, n) := g(m, n) \) in (15) that

\[ R = \sum_{mn}^{'} e_{mn}^n \frac{1}{D(D-1)} \sum_{\mu, \nu} e_{\mu \nu}^m g(\mu, \nu) . \]  

(37)

Taking into account (32) and (36), the last relation can be rewritten as

\[ R = F(t) \left[ \sum_{\mu, \nu=1}^{D} g(\mu, \nu) - Q \right]. \]  

(38)

The definition (22) implies that

\[ \sum_{\mu, \nu=1}^{D} g(\mu, \nu) = \sum_{\mu, \nu=1}^{D} \langle \mu | \rho(0) | \nu \rangle \langle \nu | A | \mu \rangle \]

\[ = \sum_{\mu=1}^{D} \langle \mu | \rho(0) | A | \mu \rangle \]

\[ = \text{Tr} \{ \rho(0) A \} . \]  

(39)

Likewise, one readily sees that

\[ \sum_{\nu=1}^{D} g(\nu, \nu) = \text{Tr} \{ \pi A \} . \]  

(40)

By introducing (39) and (40) into (36) and (38), one finds that

\[ Q = \text{Tr} \{ \pi A \} \]

\[ R = F(t) \left[ \text{Tr} \{ \rho(0) A \} - \text{Tr} \{ \pi A \} \right] . \]  

(41)

(42)

Adopting the definition \( \langle A \rangle_{\rho(t)} := \text{Tr} \{ \rho A \} \) from the main text, we finally can rewrite (39) as

\[ [(A)_{\rho(t)}]_{H} = \langle A \rangle_{\pi} + F(t) \left\{ \langle A \rangle_{\rho(0)} - \langle A \rangle_{\pi} \right\} \]  

(43)

and with (22) we recover Eq. (3) from the main text.

For later use, we employ (26) and (28) to conclude

\[ \text{Tr} \{ \pi A \} = \sum_{n=1}^{D} \rho_{nn}(0) A_{nn} = \text{Tr} \{ \rho(0) B \} \]  

(44)

and hence

\[ \text{Tr} \{ \rho(0) A \} - \text{Tr} \{ \pi A \} = \text{Tr} \{ \rho(0) (A-B) \} \]

\[ = \text{Tr} \{ \rho(0) C \} . \]  

(45)

With (32) we thus can rewrite (42) as

\[ R = \text{Tr} \{ \rho(0) C \} \frac{1}{D(D-1)} \sum_{mn}^{'} e_{mn}^n . \]  

(46)

---

C. Evaluation of the Variance

From (21), (23) we can infer that

\[ ((A)_{\rho(t)})^2 = \sum_{klnm \in I} h(\pi(k), \pi(l), \pi(m), \pi(n)) e_{km}^{ln} \]  

(47)

where

\[ h(k, l, m, n) := g(k, l) g(m, n) \]  

(48)

\[ e_{km}^{ln} := e_{k}^{l} e_{m}^{n} . \]  

(49)

In (17), the symbol \( \sum_{klnm \in I} \) indicates a summation over all quadruples of indices contained in \( I \) from (17). As pointed out below (18), this set \( I \) is the disjoint union of the 15 subsets \( I_a \). We thus can conclude from (17) that

\[ [(\langle A \rangle_{\rho(t)})^2]_{H} = \sum_{a=1}^{15} S_a \]  

(50)

\[ S_a := \sum_{klnm \in I_a} [h(\pi(k), \pi(l), \pi(m), \pi(n))]_{H} e_{km}^{ln} . \]  

(51)

Next, we turn to the evaluation of the 15 terms \( S_a \) from (51). To begin with, one readily sees that \( I_2 \) goes over into \( I_4 \) in (18) upon interchanging the labels \( k \) and \( m \) and simultaneously interchanging the labels \( l \) and \( n \). Similarly, \( I_3 \) goes over into \( I_5 \) and \( I_6 \) into \( I_7 \). On the other hand, the terms (48) and (49) and hence the summands in (51) are invariant under such an interchange of labels. It follows that

\[ S_4 = S_2 \]

(52)

\[ S_5 = S_3 \]

(53)

\[ S_7 = S_6 . \]  

(54)

Likewise, upon interchanging the labels \( k \) and \( l \) simultaneously, we obtain \( S_1, S_8, S_9, S_{12}, S_{13}, S_{14} \) and \( S_{15} \). Further, one can conclude that the terms (48) and (49) and hence the summands in (51) go over into their complex conjugate under such an interchange of labels. It follows that

\[ S_1 = S_{11} \]

(55)

\[ S_{12} = S_{12} \]

(56)

D. Evaluation of \( S_1, ..., S_8 \)

For \( a = 1 \), all four indices in (18) are equal. With (26) it follows that the term (49) is unity and (51) takes the form

\[ S_1 = \sum_{n=1}^{D} [h(\pi(n), \pi(n), \pi(n), \pi(n))]_{H} . \]  

(57)

Choosing \( f(n) := h(n, n, n, n) \) in (12), we thus obtain

\[ S_1 = \sum_{\nu=1}^{D} h(\nu, \nu, \nu, \nu) . \]  

(58)
Combining (58) and (59) yields
\[ S_8 = \sum_{kn} [h(\pi(k), \pi(l), \pi(n))]_{H} e_{kn}^{\prime}. \]
In the last step, we exploited (16) with \( K \) that
\[ f \]
Combining (58) and (59) yields
\[ S_1 + S_8 = \sum_{D} h(\lambda, \lambda, \nu, \nu). \] 
Observing (48) and (50) we thus can rewrite (60) as
\[ S_1 + S_8 = Q^2. \]
Along the same line of reasoning one finds for \( a = 6 \) that
\[ S_6 = \sum_{kl} [h(\pi(k), \pi(l), \pi(n))]_{H} e_{kn}^{\prime}. \]
In the last step, we exploited (11) with \( K = 3 \) and \( f(k, l, n) := g(k, l, n, n) \). Due to (49) and (49) it follows that \( e_{kn}^{\prime} = e_{k}^{\prime} \), hence the summands in (62) are independent of \( n \). Since the prime indicates that all the summands \( k, l, n \) must be pairwise distinct, it follows that for every given pair \( (k, l) \), the index \( n \) can take on \( D - 2 \) different values. Performing the summation over \( n \) thus yields a factor \( D - 2 \) and we obtain
\[ S_6 = \sum_{kl} e_{k}^{\prime} \frac{1}{D(D-1)} \sum_{\lambda \nu} h(\kappa, \lambda, \kappa, \kappa). \] 
The evaluation of \( S_2 \) and \( S_3 \) is similar to (but simpler than) that of \( S_6 \), yielding
\[ S_2 = \sum_{kl} e_{k}^{\prime} \frac{1}{D(D-1)} \sum_{\lambda \nu} h(\kappa, \lambda, \lambda, \lambda). \]
\[ S_3 = \sum_{kl} e_{k}^{\prime} \frac{1}{D(D-1)} \sum_{\lambda \nu} h(\kappa, \kappa, \kappa). \]
Altogether (63)–(65) sum up to
\[ S_2 + S_3 + S_6 = \sum_{kl} e_{k}^{\prime} \frac{1}{D(D-1)} \sum_{\lambda \nu} h(\kappa, \lambda, \nu, \nu). \] 
Considering (48) and exploiting (40), (47) we finally can conclude that
\[ S_2 + S_3 + S_6 = Q R \] 
and with (42)–(44), (61) that
\[ \sum_{a=1}^{8} S_a = Q^2 + 2 Q R. \] 
E. Upper bounds for \( S_9, \ldots, S_{14} \)
In view of \( I_9 \) in (18), the sum \( S_9 \) from (51) takes the form
\[ S_9 = \sum_{mn} [h(\pi(m), \pi(n))]_{H} e_{mn}^{\prime}. \]
(69)
where we exploited (15) with \( f(m, n) := g(m, n, n, n) \) in the last step. With (49) and (23) it follows that \(|e_{mn}^{\prime}| = 1 \). Since the primed sum over \( m \) and \( n \) consists of \( D(D-1) \) summands, we can conclude that
\[ |S_9| \leq \sum_{mn} |e_{mn}^{\prime}| \frac{1}{D(D-1)} \sum_{\mu \nu} |h(\mu, \nu, \mu, \nu)| \]
\[ = \sum_{\mu \nu} \rho_{\mu \nu} |A_{\mu \nu}|^2. \] 
In the last relation, we utilized (22) and (48), and we adopted the abbreviation
\[ \rho_{mn} := \rho_{mn}(0). \] 
The Cauchy-Schwarz inequality (29) implies \(|\rho_{mn}|^2 \leq \rho_{mm} \rho_{nn} \). Upon extending the sum in (70) over all \( \mu \) and \( \nu \), we thus obtain
\[ |S_9| \leq \sum_{\mu \nu=1}^{D} \rho_{\mu \nu} |A_{\mu \nu}|^2. \] 
Denoting by \( a \) and \( b \) two arbitrary \( D \times D \) matrices with matrix elements \( a_{\mu \nu} \) and \( b_{\mu \nu} \), the definition \( (a, b) := \sum_{\mu \nu=1}^{D} a_{\mu \nu} b_{\mu \nu} \) amounts to a well-defined scalar product for any \( \epsilon > 0 \). Invoking the Cauchy-Schwarz inequality and then letting \( \epsilon \rightarrow 0 \) we can conclude that
\[ \left| \sum_{\mu \nu=1}^{D} a_{\mu \nu} b_{\mu \nu} |A_{\mu \nu}|^2 \right|^2 \leq \sum_{\mu \nu=1}^{D} |a_{\mu \nu}|^2 |A_{\mu \nu}|^2 \sum_{\mu \nu=1}^{D} |b_{\mu \nu}|^2 |A_{\mu \nu}|^2 \]
For the particular choice \( a_{\mu \nu} := \rho_{\mu \mu} \) (independent of \( \nu \)) and \( b_{\mu \nu} := \rho_{\nu \nu} \) (independent of \( \mu \)), the right-hand side of (72) can thus be further bounded to yield
\[ |S_9| \leq \sqrt{T U} \]
\[ T := \sum_{\mu, \nu=1}^{D} \rho_{\mu \nu}^2 |A_{\mu \nu}|^2 \]
\[ U := \sum_{\mu, \nu=1}^{D} \rho_{\mu \nu}^2 |A_{\mu \nu}|^2 \]
Since \(|A_{\mu \nu}|^2 = |\mu| A_\nu^2 |A_{\mu}|^2 \), the sum over \( \nu \) in (74) can be readily performed, yielding
\[ T = \sum_{\mu=1}^{D} \rho_{\mu \mu}^2 (\mu A_\mu^2 |A_{\mu}|) \] 
(76)
The last factor can be estimated from above by \(\|A^2\|\), which in turn is equal to \(\|A\|^2\). The remaining sum over \(\mu\) can be identified with \(\text{Tr}p^2\) by exploiting (73) and (74). Altogether, we thus obtain

\[ T \leq \|A\|^2\text{Tr}p^2. \] (77)

The same estimate readily carries over to \(U\) from (75), implying for (76) that

\[ |S_9| \leq \|A\|^2\text{Tr}p^2. \] (78)

Along the same line of reasoning, one finds for \(a = 10\) that

\[ S_{10} = \sum_{m,n} e_{mn}^1 (1) = \sum_{m,n} e_{mn}^1 (1) \cdot \sum_{\nu \mu} \sum_{\nu \mu} h(\mu, \nu, \nu, \mu). \] (79)

With (19) and (23) it follows that \(e_{nm}^1 = 1\). Since the primed sum over \(m\) and \(n\) consists of \(D(D-1)\) summands, we can conclude with (22) and (15) that

\[ S_{10} = \sum_{\nu \mu} |\rho_{\nu \mu}|^2 |A_{\nu \mu}|^2. \] (80)

The last sum is identical to that in (70). Hence, one finds exactly as in (71) the estimate

\[ |S_{10}| \leq \|A\|^2\text{Tr}p^2. \] (81)

Likewise, one finds for \(a = 11\) that

\[ S_{11} = \sum_{k,l} e_{kk}^1 (1) = \sum_{k,l} e_{kk}^1 (1) \cdot \sum_{\nu \mu} \sum_{\nu \mu} h(\kappa, \lambda, \nu, \nu). \] (82)

Since \(|e_{kk}^1| = 1\) and the primed sum over \(k, l\) consists of \(D(D-1)\) summands, we can conclude that

\[ |S_{11}| \leq \sum_{\nu \mu} |h(\kappa, \lambda, \nu, \nu)| \leq \sum_{\nu \mu} |\rho_{\nu \mu}|^2 |A_{\nu \mu}|^2. \] (83)

In the last step, we utilized (22), (15), and (71). The primed sum extends over all \(\kappa, \lambda, \nu\) under the constraint that they must be pairwise distinct. It can be rewritten as a sum over all \(\kappa, \lambda, \nu\) if we multiply each summand by an extra factor \((1 - \delta_{\nu \lambda})(1 - \delta_{\nu \kappa})(1 - \delta_{\nu \lambda})\). Indeed, the latter factor is unity if \(\kappa, \lambda, \nu\) are pairwise distinct, and zero otherwise. Observing that \((1 - \delta_{\nu \lambda})A_{\lambda \kappa} = C_{\lambda \kappa}\) and \((1 - \delta_{\nu \kappa})A_{\nu \kappa} = C_{\nu \kappa}\) according to (26), we can conclude that

\[ |S_{11}| \leq \sum_{\nu \mu} |\rho_{\nu \mu}|^2 |A_{\nu \mu}|^2 \leq V + W \] (84)

Choosing \(|\psi\rangle = |\kappa\rangle\) and \(|\phi\rangle = C|\kappa\rangle\) in (29), it follows that

\[ V \leq \sum_{\kappa = 1} \rho_{\kappa \kappa} |C|\rho C|\kappa\rangle \langle \kappa| \] (85)

Choosing \(|\psi\rangle = |\kappa\rangle\) and \(|\phi\rangle = C|\kappa\rangle\) in (29), it follows that

\[ V \leq \sum_{\kappa = 1} \rho_{\kappa \kappa} |C|\rho C|\kappa\rangle \langle \kappa|. \] (86)

Evaluating the trace by means of the eigenbasis of \(C\), one sees that \(\text{Tr}\{C\rho C\} \leq \|C\|^2\). With (27) we thus can conclude that

\[ V \leq 4 \|A\|^2 \max_n \rho_{nn}(0). \] (89)

To evaluate \(S_{11}\), one proceeds similarly as in (70)-(78), yielding

\[ W \leq \|C\|^2\text{Tr}p^2 \leq 4 \|A\|^2\text{Tr}p^2 \] (90)

In combination with (84) and (85), we thus can conclude that

\[ |S_{11}| \leq 4 \|A\|^2 \text{Tr}p^2 + \max_n \rho_{nn}(0). \] (91)

Along the same line of reasoning, one finds for \(a = 13\) that

\[ |S_{13}| \leq 4 \|A\|^2 \text{Tr}p^2 + \max_n \rho_{nn}(0). \] (92)

Taking into account (73), (71), (91), (92) we finally can infer that

\[ \sum_{a=9}^{14} |S_a| \leq \|A\|^2 \left(18 \text{Tr}p^2 + 16 \max_n \rho_{nn}(0) \right). \] (93)
F. Evaluation of $S_{15}$

We start with the trivial identities

$$S_{15} = X + Y + Z + R^2$$  \hspace{1cm} (94)
$$X := S_{15} - \alpha$$  \hspace{1cm} (95)
$$Y := \alpha - \beta$$  \hspace{1cm} (96)
$$Z := \beta - R^2$$  \hspace{1cm} (97)
$$\alpha := (\text{Tr}(\rho(0)C))^2 \frac{(D-4)!}{D!} \sum_{klmn} \epsilon_{km}^{ln}$$  \hspace{1cm} (98)
$$\beta := (\text{Tr}(\rho(0)C))^2 \frac{(D-4)!}{D!} \sum_{klmn} \epsilon_{km}^{ln}$$  \hspace{1cm} (99)

For $a = 15$, all four indices in (98) are pairwise distinct, and with (11) it follows that

$$S_{15} = \sum_{klmn} \epsilon_{km}^{ln} \frac{(D-4)!}{D!} \sum_{\kappa \lambda \mu \nu} h(\kappa, \lambda, \mu, \nu) .$$  \hspace{1cm} (100)

In the last step we exploited (10) with $K = 4$.

Upon comparison of (37) and (46) one sees that

$$\text{Tr}(\rho(0)C) = \sum_{\mu, \nu} g(\mu, \nu)$$  \hspace{1cm} and (98) can be rewritten with (18) as

$$\alpha = \sum_{klmn} \epsilon_{km}^{ln} \frac{(D-4)!}{D!} \sum_{\kappa \lambda \mu \nu} h(\kappa, \lambda, \mu, \nu) .$$  \hspace{1cm} (101)

Introducing (100) and (101) into (95) yields

$$X = X_1 X_2$$  \hspace{1cm} (102)
$$X_1 := \sum_{klmn} \epsilon_{km}^{ln} \frac{(D-4)!}{D!} \sum_{\kappa \lambda \mu \nu} h(\kappa, \lambda, \mu, \nu) .$$  \hspace{1cm} (103)
$$X_2 := \sum_{klmn \in I_{15}} h(k, l, m, n) - \sum_{klmn \in J} h(k, l, m, n) .$$  \hspace{1cm} (104)

For later convenience, we employed $k, l, m, n$ instead of $\kappa, \lambda, \mu, \nu$ as summation indices in (104). Furthermore, the sets $I_{15}$ and $J$ are defined in (18) and (19), respectively.

With (19) and (23) it follows that $|\epsilon_{km}^{ln}| = 1$. Since the primed sum over $k, l, m, n$ in (103) consists of $D!/(D-4)!$ summands, we can conclude that

$$|X_1| \leq 1 .$$  \hspace{1cm} (105)

As observed below (19), the set $J$ is the disjoint union of $I_9, \ldots, I_{15}$, implying that (104) can be re-written as

$$X_2 = - \sum_{a=9}^{14} S_a'$$  \hspace{1cm} (106)
$$S_a' := \sum_{klmn \in I_a} h(k, l, m, n) .$$  \hspace{1cm} (107)

With (19) and (23) it follows that $\epsilon_{km}^{ln} = 1$ for $t = 0$. Hence, $S_a'$ from (107) can be identified with $S_a$ from (11) by taking into account (16) and setting $t = 0$. Since the estimate from (98) is valid independently of $t$, we can conclude that

$$\sum_{a=9}^{14} |S_a'| \leq \| A \|^2 \left[ 18 \text{Tr}$^{2}$ + 16 \max_{n} \rho_{nn}(0) \right] .$$  \hspace{1cm} (108)

 Altogether, (102) with (105)–(108) implies

$$X \leq \| A \|^2 \left[ 18 \text{Tr}$^{2}$ + 16 \max_{n} \rho_{nn}(0) \right] .$$  \hspace{1cm} (109)

In the same vein, we rewrite (90) with (89) and (90) as

$$Y = Y_1 Y_2$$  \hspace{1cm} (110)
$$Y_1 := (\text{Tr}(\rho(0)C))^2 \frac{(D-4)!}{D!}$$  \hspace{1cm} (111)
$$Y_2 := \sum_{klmn \in I_{15}} \epsilon_{km}^{ln} - \sum_{klmn \in J} \epsilon_{km}^{ln} .$$  \hspace{1cm} (112)

where $I_{15}$ and $J$ are defined in (18) and (19), respectively.

Evaluating the trace by means of the eigenbasis of $C$, one sees that $|\text{Tr}(\rho(0)C)| \leq \| C \|$. With (27) we thus can conclude that

$$|Y_1| \leq 4 \| A \|^2 \frac{(D-4)!}{D!} .$$  \hspace{1cm} (113)

Since $|\epsilon_{km}^{ln}| = 1$ (see (19) and (23)) and since $J$ is the disjoint union of $I_9, \ldots, I_{15}$ (see below (19)), we can conclude from (112) that

$$|Y_2| \leq \sum_{a=9}^{14} |I_a| ,$$  \hspace{1cm} (114)

where $|I_a|$ denotes the size (number of elements) of the set $I_a$. From (18) one readily concludes that $|I_a| = D(D-1)$ for $a = 9, 10$ and $|I_a| = D(D-1)(D-2)$ for $a = 11, \ldots, 14$, yielding

$$\sum_{a=9}^{14} |I_a| = 4D(D-1)(D-3/2) .$$  \hspace{1cm} (115)

Introducing (113)–(115) into (110) implies

$$|Y| = \| A \|^2 q(D)/D = q(D)$$  \hspace{1cm} (116)
$$q(D) := \frac{16 \ D - 3/2}{D - 3} .$$  \hspace{1cm} (117)

Observing that the last two factors on the right hand side of (117) are both decreasing functions of $D$, we obtain $q(D) \leq q(6) = 36$ for $D \geq 6$. Altogether, we thus arrive at

$$|Y| \leq 36 \| A \|^2/D$$  \hspace{1cm} for $D \geq 6 .$$  \hspace{1cm} (118)
With the help of (46) we can conclude from (97) and (99) that
\[ Z = Z_1 Z_2 \]  
(119)

\[ Z_1 := (\text{Tr}\{\rho(0)C\})^2 \sum_{kl} \sum_{mn} e_{km}^{ln} \]  
(120)

\[ Z_2 := \frac{(D - 4)!}{D!} - \frac{1}{D^2(D - 1)^2} \]  
(121)

As before, we exploit that \(|\text{Tr}\{\rho(0)C\}| \leq 2\|A\|\), that the two double sums in (120) give rise to \(D^2(D - 1)^2\) sum-mands, and that \(|e_{km}^{ln}| = 1\) to infer
\[ |Z_1| \leq 4 \|A\|^2 D^2(D - 1)^2 . \]  
(122)

Introducing (121) and (122) into (119) readily yields the same upper bound for \(|Z|\) as for \(|Y|\) in (110). Like in (113), we thus obtain
\[ |Z| \leq 36 \|A\|^2/D \text{ for } D \geq 6 . \]  
(123)

**G. Final result**

From the definition (24) of \(\xi_\pi(t)\) we can infer that
\[ [\xi_\pi^2(t)]_{\Pi} = [(\langle A \rangle_{\rho(t)}^2)_{\Pi} - (\langle A \rangle_{\rho(t)}\Pi)]^2 \]  
(124)

and from (33) that
\[ (\langle A \rangle_{\rho(t)}\Pi)^2 = Q^2 + 2QR + R^2 . \]  
(125)

Introducing (33) and (24) into (10) yields
\[ [(\langle A \rangle_{\rho(t)}^2)_{\Pi}] = Q^2 + 2QR + \sum_{a=9}^{14} S_a + X + Y + Z + R^2 . \]  
(126)

From (124)-(126) we can conclude that
\[ [\xi_\pi^2(t)]_{\Pi} = \sum_{a=9}^{14} S_a + X + Y + Z \]  
(127)

and with (33), (109), (118), (123) that
\[ [\xi_\pi^2(t)]_{\Pi} \leq 4 \|A\|^2 [9\text{Tr}\rho^2 + 8 \max_n \rho_{nn}(0) + 18/D] \]  
(128)

for \(D \geq 6\). Observing that \(D^{-1} \leq \text{Tr}\rho^2\) and (28), we obtain
\[ [\xi_\pi^2(t)]_{\Pi} \leq \Delta_A^2 [27 \text{Tr}\rho^2 + 8 \max_n \rho_{nn}(0)] \]  
(129)

for \(D \geq 6\). Finally, we exploit the inequality
\[ \text{Tr}\rho^2 = \sum_n \rho_{nn}^2(0) \leq \max_n \rho_{nn}(0) \]  
(130)

and 35 < 6² to rewrite (129) as
\[ [\xi_\pi^2(t)]_{\Pi} \leq (6\Delta_A)^2 \max_n \rho_{nn}(0) \]  
(131)

for \(D \geq 6\). This is identical to the second relation in (6) of the main text.

We also tried to derive a slightly stronger version of (131) with \(\text{Tr}\rho^2\) instead of \(\max_n \rho_{nn}(0)\) appearing on the right hand side. The main problem is the last summand in (129), whose origin is the estimate in (87). Unfortunately, we did not succeed to find an analogous estimate but in terms of \(\text{Tr}\rho^2\) instead of \(\max_n \rho_{nn}(0)\).