Estimation of the Epidemic Branching Factor in Noisy Contact Networks

Wenrui Li  
*Boston University, Boston, USA.*

Daniel L. Sussman  
*Boston University, Boston, USA.*

Eric D. Kolaczyk†  
*Boston University, Boston, USA.*

**Summary.** Many fundamental concepts in network-based epidemic models depend on the branching factor, which captures a sense of dispersion in the network connectivity and quantifies the rate of spreading across the network. Moreover, contact network information generally is available only up to some level of error. We study the propagation of such errors to the estimation of the branching factor. Specifically, we characterize the impact of network noise on the bias and variance of the observed branching factor for arbitrary true networks, with examples in sparse, dense, homogeneous and inhomogeneous networks. In addition, we propose two estimators, by method-of-moments and bootstrap sampling. We illustrate the practical performance of our estimators through simulation studies and social contact networks in British secondary schools.

**Keywords:** Branching Factor; Noisy network; Method-of-moments; Bootstrap.

1. **Introduction**

The branching factor, \( \kappa \), is a measure of heterogeneity of a network. It captures a notion of the average degree of the vertex reached by following an edge from a vertex and, therefore, measures the rate of spreading across the network. Many key concepts in mathematical epidemiology depend on the branching factor, for example, the basic reproduction number \( R_0 \). The latter is generally defined as the number of secondary infections expected in the early stages of an epidemic by a single infective in a population of susceptibles (Anderson and May, 1991; Diekmann and Heesterbeek, 2000). In network-based susceptible-infected-removed (SIR) models, \( R_0 \) can be shown to equal \( \theta(\kappa - 1)/(\theta + \gamma) \), where \( \theta \) and \( \gamma \) are infection and recovery rates, respectively (Andersson (1997)). The importance of \( R_0 \) in the study of epidemics arises from its role in so-called threshold theorems, which state...
under which conditions the presence of an infective individual in a population will lead to an epidemic (Whittle, 1955). It is evident that knowing the value of $\kappa$ is vital for effective control responses in the early stages of an epidemic. In addition, various thresholds in epidemiological and percolation theory rely on the branching factor. In the discussion section, we provide details on how our estimation of the branching factor can be extended to those statistics.

Increasingly, contact networks are playing an important role in the study of epidemiology. Knowledge of the structure of the network allows models to take into account individual-level behavioral heterogeneities and shifts. Network-based approaches have been explored for investigating disease outbreaks in human (Eubank et al. (2004)), livestock (Kao et al. (2006)) and wildlife (Craft et al. (2009)) populations. Moreover, contact network information generally is available only up to some level of error – also known as, network noise. For example, there is often measurement error associated with network constructions, where, by ‘measurement error’ we will mean true edges being observed as non-edges, and vice versa. Such edge noise occurs in self-reported contact networks where participants may not perceive and recall all contacts correctly (Smieszek et al. (2012)). It can also be found in sensor-based contact networks where automated proximity loggers are used to report frequency and duration of contacts. (Drewe et al. (2012)). We investigate how network noise impacts on the observed value of $\kappa$ and, therefore, on our understanding of infectious diseases spreading.

Extensive work regarding uncertainty quantification has been done in the field of non-network epidemic modeling, where populations are assumed uniform and with homogeneous mixing. Given adequate data, estimates of the model parameters, such as $\theta$ and $\gamma$, can be produced with accompanying standard errors. Methods for this purpose are reviewed in Andersson and Britton (2012, Chapter 9–12) and Becker and Britton (1999). Many studies have explored the effects of uncertainty in parameter estimation on basic epidemic quantities. For instance, there have been efforts to quantify uncertainty in $R_0$ around recent high profile emergent events, including severe acute respiratory syndrome (SARS) (Chowell et al. (2004a)), the new influenza A (H1N1) (White et al. (2009)), and Ebola (Chowell et al. (2004b)). But, to our best knowledge, there has been little attention to date given towards uncertainty analysis of $\kappa$ and relevant quantities in network-based epidemic models. Exceptions include real-time estimation of $R_0$ at an early stage of an outbreak by considering the heterogeneity in contact networks (Davoudi et al. (2012)), and measurability of $R_0$ in highly detailed sociodemographic data with the clustered contact structure assumed of the population (Liu et al. (2018)).

As remarked above, there appears to be little in the way of a formal and general treatment of the error propagation problem in network-based epidemic models. However, there are several areas in which the probabilistic or statistical treatment of uncertainty enters prominently in network analysis. Model-based approaches include statistical methodology for predicting network topology or attributes with models that explicitly include a component for network noise (Jiang et al. (2011), Jiang and Kolaczyk (2012)), the ‘denoising’ of noisy networks (Chatterjee et al. (2015)), and the adaptation of methods for vertex classification using networks observed
with errors (Priebe et al. (2015)). The other common approach to network noise is based on a ‘signal plus noise’ perspective. For example, Balachandran et al. (2017) introduced a simple model for noisy networks that, conditional on some true underlying network, assumed we observed a version of that network corrupted by an independent random noise that effectively flips the status of (non)edges. Later, Chang et al. (2018) developed method-of-moments estimators for the underlying rates of error when replicates of the observed network are available. In a somewhat different direction, uncertainty in network construction due to sampling has also been studied in some depth. See, for example, Kolaczyk (2009, Chapter 5) or Ahmed et al. (2014) for surveys of this area. However, in this setting, the uncertainty arises only from sampling—the subset of vertices and edges obtained through sampling are typically assumed to be observed without error.

Our contribution in this paper is to quantify how such errors propagate to the estimation of the branching factor, and to provide estimators for $\kappa$ when replicates of the observed network are available. Adopting the noise model proposed by Balachandran et al. (2017), we characterize the impact of network noise on the bias and variance of the observed branching factor for arbitrary true networks, and we illustrate the asymptotic behaviors on networks for varying densities and degree distributions. Additionally, we propose two estimators of the branching factor, by parametric and nonparametric approaches. The parametric estimate is motivated by Chang et al. (2018), who recently developed method-of-moments estimators for network subgraph densities and the underlying rates of error when replicates of the observed network are available. The nonparametric approach is based on bootstrap sampling, inspired by Kucharski et al. (2018). Numerical simulation suggests that high accuracy is possible for networks of even modest size. We illustrate the practical use of our estimators in the context of social contact networks in British secondary schools, where a small number of replicates are available.

The organization of this paper is as follows. In Section 2 we provide background on the noise model and branching factor. In Section 3 we then present a general result for the bias of the observed branching factor, with examples provided for sparse, dense, homogeneous and inhomogeneous networks. Section 4 deals with the variance of the observed branching factor. Section 5 proposes our two estimators for the true branching factor. Numerical illustration is reported in Section 6. Proofs of our key results can be found in the appendix. All other proofs are relegated to supplementary materials.

2. Background

In this section, we provide essential notation and background.

2.1. Noise model

We assume the observed graph is a noisy version of a true graph. Let $G = (V, E)$ be an undirected graph and $G^{\text{obs}} = (V, E^{\text{obs}})$ be the observed graph, where we implicitly assume that the vertex set $V$ is known. Denote the adjacency matrix of $G$
by \( A = (A_{i,j})_{N_v \times N_v} \) and that of \( G^{\text{obs}} \) by \( \tilde{A} = (\tilde{A}_{i,j})_{N_v \times N_v} \). Hence \( A_{i,j} = 1 \) if there is a true edge between the \( i \)-th vertex and the \( j \)-th vertex, and 0 otherwise, while \( \tilde{A}_{i,j} = 1 \) if an edge is observed between the \( i \)-th vertex and the \( j \)-th vertex, and 0 otherwise. And denote the degree of the \( i \)-th vertex in \( G \) and \( G^{\text{obs}} \) by \( d_i \) and \( \tilde{d}_i \), respectively. We assume throughout that \( G \) and \( G^{\text{obs}} \) are simple.

We express the marginal distributions of the \( \tilde{A}_{i,j} \) in the form (Balachandran et al. (2017)):

\[
\tilde{A}_{i,j} \sim \begin{cases} 
\text{Bernoulli}(\alpha_{i,j}), & \text{if } \{i,j\} \in E^c \\
\text{Bernoulli}(1 - \beta_{i,j}), & \text{if } \{i,j\} \in E,
\end{cases}
\]

where \( E^c = \{\{i,j\} : i,j \in V; i < j\}\setminus E \). Drawing by analogy on the example of network construction based on hypothesis testing, \( \alpha_{i,j} \) can be interpreted as the probability of a Type-I error on the (non)edge status for vertex pair \( \{i,j\} \in E^c \), while \( \beta_{i,j} \) is interpreted as the probability of Type-II error, for vertex pair \( \{i,j\} \in E \).

Our interest is in characterizing the manner in which the uncertainty in the \( \tilde{A}_{i,j} \) propagates to the branching factor. Here we focus on a general formulation of the problem in which we make the following three assumptions.

**Assumption 1 (Constant marginal error probabilities).** Assume that \( \alpha_{i,j} = \alpha \) and \( \beta_{i,j} = \beta \) for all \( i < j \), so the marginal error probabilities are \( \mathbb{P}(\tilde{A}_{i,j} = 0|A_{i,j} = 1) = \beta \) and \( \mathbb{P}(\tilde{A}_{i,j} = 1|A_{i,j} = 0) = \alpha \).

**Assumption 2 (Independent noise).** The random variables \( \tilde{A}_{i,j} \), for all \( i < j \), are conditionally independent given \( A_{i,j} \).

**Assumption 3 (Large Graphs).** \( N_v \to \infty \).

In Assumption 1, we assume that both \( \alpha \) and \( \beta \) remain constant over different edges. Under Assumption 2, the distributions of \( \tilde{d}_i \) is

\[
\tilde{d}_i = \sum_{j=1}^{N_v} \tilde{A}_{j,i} \sim \text{Binomial}(N_v - d_i, \alpha) + \text{Binomial}(d_i, 1 - \beta).
\]

Assumption 2 is not strictly necessary. See Remark 5 in Section 5.1. Assumption 3 reflects both the fact that the study of large graphs is a hallmark of modern applied work in complex networks and, accordingly, our desire to understand the asymptotic behavior of the branching factor and provide concise descriptions in terms of the bias and variance for large graphs.

**Remark 1.** Note that \( \alpha \) and \( \beta \) can be constants or \( o(1) \) as \( N_v \to \infty \). For example, under Assumption 4, if \( \beta \) is constant and \( |E| = o(|E^c|) \), then \( \alpha = o(1) \). Thus, \( \alpha \) and \( \beta \) are actually \( \alpha(N_v) \) and \( \beta(N_v) \). For notational simplicity, we omit \( N_v \).

In addition to the core Assumptions 1 – 3, we add a fourth assumption, upon which we will call periodically throughout the paper when desiring to illustrate our results in the special case.
**Assumption 4 (Edge Unbiasedness).** \( \alpha |E^c| = \beta |E| \), so that the expected number of observed edges equals the actual number of edges.

Our use of Assumption 4 reflects the understanding that a ‘good’ observation \( G^{ob} \) of the graph \( G \) should at the very least have roughly the right number of edges.

**Remark 2.** Assumption 4 cannot guarantee the unbiasedness of higher-order subgraph counts. (Balachandran et al. (2017))

### 2.2. The branching factor in network-based epidemic models

Let \( G \) be a network graph describing the contact structure among \( N_v \) elements in a population. The branching factor is defined as follows.

**Definition 1.** For graph \( G \) with \( N_v \) nodes, the branching factor is

\[
\kappa = \begin{cases} 
\frac{\sum_{i=1}^{N_v} d_i^2 / N_v}{\sum_{i=1}^{N_v} d_i / N_v} & \text{if } \sum_{i=1}^{N_v} d_i > 0 \\
0 & \text{if } \sum_{i=1}^{N_v} d_i = 0,
\end{cases}
\]

where \( d_i \) is the degree of node \( i \).

Accordingly, we denote the branching factor in the noisy network by \( \tilde{\kappa} \). Besides the basic reproduction number, \( R_0 \), described in the introduction, there are other quantities depending on the observed branching factor. These include the percolation threshold \( 1/(\tilde{\kappa} - 1) \), the epidemic threshold \( 1/(\tilde{\kappa} - 1) \), and the immunization threshold \( 1 - 1/(\lambda \tilde{\kappa}) \), where \( \lambda \) is the spreading rate (Pastor-Satorras et al. (2015)).

### 3. Bias of the observed branching factor

In this section, we first quantify the asymptotic bias of the observed branching factor for arbitrary true networks. We then show specific results for four typical classes of networks: sparse and homogeneous, sparse and inhomogeneous, dense and homogeneous, and dense and inhomogeneous.

#### 3.1. Arbitrary network topology

**Theorem 1.** We define \( X = \sum_{i=1}^{N_v} d_i^2 \), \( Y = \sum_{i=1}^{N_v} d_i \) and we assume \( EY > 0 \), and \( EY = \Omega(N_v) \) \( (N_v \to \infty) \). Then, under Assumption 2, for any \( \eta > 0 \), we have

\[
\text{Bias}[\tilde{\kappa}] = \frac{EX}{EY} - \kappa + O\left(\frac{1}{(EY)^{1/(2+\eta)}} \frac{EX}{EY}\right) \quad \text{as } N_v \to \infty.
\]

**Remark 3.** Theorem 1 reflects the fact that, under certain assumptions, \( EX/EY \) is a good approximation of \( E(X/Y \cdot I_{\{Y>0\}}) \), i.e., \( E(\tilde{\kappa}) \).
Theorem 2. Under assumptions in Theorem 1 and Assumption 1 and 4, for any $\eta > 0$, we have
\[ \text{Bias}[\tilde{\kappa}] = (2 - \alpha - \beta)\left[\alpha(N_v - 1) + \beta - (\alpha + \beta)\kappa\right] + \mathcal{O}\left(\frac{1}{(\mathbb{E}Y)^{1/(2+\eta)}}\frac{\mathbb{E}X}{\mathbb{E}Y}\right) \]
as $N_v \to \infty$.

Theorem 1 shows the asymptotic bias of the observed branching factor in terms of the expectations of the first and second moments of the observed under Assumption 2. Theorem 2 relies on Assumptions 1 – 4 and provides a more explicit expression for the leading term of the asymptotic bias in this special case.

3.2. Specific network topology

By making assumptions on the network density and degree distribution, we can obtain a more nuanced understanding of the limiting behavior of the observed branching factor in terms of bias when the number of nodes tends towards infinity. Specifically, we consider the combinations of sparse versus dense and homogeneous versus inhomogeneous networks. By the term sparse we will mean a graph for which the average degree $\bar{d} = \Theta(\log N_v)$, and by dense, $\bar{d} = \Theta(N_v^c)$, where $0 < c < 1$. By the term homogeneous we mean the degrees follow a Poisson distribution, and by inhomogeneous, the degrees follow a truncated Pareto distribution.

Corollary 1 (Sparse and homogeneous). In the sparse homogeneous graph, where the average degree $\bar{d} = \Theta(\log N_v)$ and the asymptotic degree distribution is the Poisson distribution with mean $\bar{d}$, under the assumptions in Theorem 2 and $\beta = \mathcal{O}(1)$ ($N_v \to \infty$), for any $\eta > 0$, we have
\[ \text{Bias}[\tilde{\kappa}] = \mathcal{O}\left(\frac{\log N_v}{(N_v \log N_v)^{1/(2+\eta)}}\right) \text{ as } N_v \to \infty, \]
where $\kappa = \Theta(\log N_v)$.

Corollary 2 (Sparse and inhomogeneous). In the sparse inhomogeneous graph where the average degree $\bar{d} = \Theta(\log N_v)$ and the asymptotic degree distribution is truncated Pareto distribution with shape $\zeta$, lower bound $d_L$ and upper bound $N_v - 1$, under the assumptions in Theorem 2 and $\beta = \mathcal{O}(1)$ ($N_v \to \infty$), for any $\eta > 0$, we have
\[ \text{Bias}[\tilde{\kappa}] = \begin{cases} -\beta(2 - \alpha - \beta)\kappa + \mathcal{O}\left(\max\left\{\log N_v, \frac{\kappa}{(N_v \log N_v)^{1/(2+\eta)}}\right\}\right) \text{ if } 0 < \zeta \leq 2 \\ -\beta(2 - \alpha - \beta)\frac{\kappa}{(\zeta - 1)^2} + \mathcal{O}(1) \text{ if } \zeta > 2 \end{cases} \]
as $N_v \to \infty$, where

$$\kappa = \begin{cases} 
\Theta(N_v), & \text{if } 0 < \zeta < 1 \\
\Theta(N_v / \log N_v), & \text{if } \zeta = 1 \\
\Theta(N_v^{2-\zeta} \cdot \log^{\zeta-1} N_v), & \text{if } 1 < \zeta < 2 \\
\Theta(\log^2 N_v), & \text{if } \zeta = 2 \\
\Theta(\log N_v), & \text{if } \zeta > 2.
\end{cases}$$

**Remark 4.** In Corollary 2, by the definition of expectation, $\zeta$, $d_L$, $\bar{d}$ and $N_v$ satisfy the equation

$$\bar{d} = \int_{d_L}^{N_v - 1} x \cdot \zeta \frac{d_L^\zeta}{1 - \left(\frac{d_L}{N_v - 1}\right)^\zeta} x^{-(\zeta+1)} dx.$$

Under the condition $\bar{d} = \Theta(\log N_v)$, the relationship among them can be simplified. See the supplementary materials for details. Similar relationships also hold in Corollary 4, 6 and 8.

Note that the $O$ term in Corollary 2 is dominated by the corresponding $\kappa$ asymptotically, so $\text{Bias}(\hat{\kappa}) = \Theta(\kappa)$, reflecting the challenges of estimating $\kappa$ in under heterogeneous degree distributions. In contrast, $\text{Bias}(\hat{\kappa}) = o(\kappa)$ in Corollary 1.

**Corollary 3 (Dense and homogeneous).** In the dense homogeneous graph where the average degree $\bar{d} = \Theta(N_v^c)$, $0 < c < 1$, and the asymptotic degree distribution is the Poisson distribution with mean $\bar{d}$, under the assumptions in Theorem 2 and $\beta = \mathcal{O}(1)$ ($N_v \to \infty$), for any $\eta > 0$, we have

$$\text{Bias}[\hat{\kappa}] = \Theta\left(N_v^{c - \frac{\zeta + 1}{2 + \eta}}\right) \quad \text{as } N_v \to \infty,$$

where $\kappa = \Theta(N_v^c)$.

**Corollary 4 (Dense and inhomogeneous).** In the dense inhomogeneous graph where the average degree $\bar{d} = \Theta(N_v^c)$, $0 < c < 1$, and the asymptotic degree distribution is truncated Pareto distribution with shape $\zeta$, lower bound $d_L$ and upper bound $N_v - 1$, under the assumptions in Theorem 2 and $\beta = \mathcal{O}(1)$ ($N_v \to \infty$), for any $\eta > 0$, we have

$$\text{Bias}[\hat{\kappa}] = \begin{cases} 
-\beta(2 - \alpha - \beta)\kappa + \mathcal{O}\left(\max\left\{N_v^c, \frac{\kappa}{N_v^{(c+1)/(2+\eta)}}\right\}\right) & \text{if } 0 < \zeta \leq 2 \\
-\beta(2 - \alpha - \beta)\kappa \frac{1}{(\zeta - 1)^2} + \mathcal{O}(\max\{N_v^{2c-1}, 1\}) & \text{if } \zeta > 2
\end{cases} \quad \text{as } N_v \to \infty,$$

where $\kappa = \Theta(N_v^c)$, $\Theta(N_v / \log N_v)$, $\Theta(N_v^{2-\zeta+\zeta-1})$, $\Theta(N_v^{\zeta} \cdot \log N_v)$, and $\Theta(N_v^{\zeta})$ for $0 < \zeta < 1$, $\zeta = 1$, $1 < \zeta < 2$, $\zeta = 2$, and $\zeta > 2$, respectively.
Note that the $O$ term in Corollary 4 is dominated by the corresponding $\kappa$ asymptotically, so $\text{Bias}(\tilde{\kappa}) = \Theta(\kappa)$. In contrast, $\text{Bias}(\tilde{\kappa}) = o(\kappa)$ in Corollary 3.

In summary, the observed branching factor is asymptotically unbiased in the homogeneous network setting, but asymptotically biased in the inhomogeneous network setting. The bias of the observed branching factor is negative which reflects the fact that the observed graph is typically more homogeneous then the true graph in the inhomogeneous setting. The bias depends on $\alpha$, $\beta$, and $\zeta$, and when the shape $\zeta > 2$, the bias decreases as $\zeta$ increases. The different results in the homogeneous and inhomogeneous network setting also reflect Remark 2 since the branching factor is related to the second-order moment.

4. Variance of the observed branching factor

In this section, we first compute the asymptotic variance of the observed branching factor for arbitrary true networks. We then show specific results for the same four types of networks as in Section 3.2.

4.1. Arbitrary network topology

Under certain assumptions, we provide upper bounds for asymptotic variances of the observed branching factors and derive good approximations of asymptotic variances for arbitrary true networks. Considering that variances are important and involved, we briefly show a main outcome here and give details in Appendix 9.2.

We assume $EY > 0$, $EY = \Omega(N_v)$, and $1 - \beta = \Omega(N_v)$. Then, under Assumption 1, 2, and 4, we have

$$\text{Var}[\tilde{\kappa}] = O \left( \frac{E[(X \bar{E} Y - Y \bar{E} X)^2]}{(\bar{E} Y)^4} \right) \text{ as } N_v \to \infty.$$  

This provide upper bounds for asymptotic variances of the observed branching factors. By making additional assumptions on the network density and degree distribution, we can obtain the order of $O$ term and therefore the order of the variance.

4.2. Specific network topology

Again, by making assumptions on the network density and degree distribution, we can describe the limiting behavior of the observed branching factor in term of variance when the number of nodes tends towards infinity.

Corollary 5 (Sparse and homogeneous). In the sparse homogeneous graph where the average degree $\bar{d} = \Theta(\log N_v)$ and the asymptotic degree distribution is the Poisson distribution with mean $\bar{d}$, under the assumptions in Theorem 2 and $\beta = O(1)$ ($N_v \to \infty$), we have

$$\text{Var}[\tilde{\kappa}] = O \left( \left( \frac{\log N_v}{N_v} \right)^{1/2} \right) \text{ as } N_v \to \infty.$$
Corollary 6 (Sparse and inhomogeneous). In the sparse inhomogeneous graph where the average degree $\bar{d} = \Theta(\log N_v)$ and the asymptotic degree distribution is truncated Pareto distribution with shape $\zeta$, lower bound $d_L$ and upper bound $N_v - 1$, under the assumptions in Theorem 2 and $\beta = O(1)$ ($N_v \to \infty$), we have

$$\text{Var}[\bar{\kappa}] = \begin{cases} O(N_v/\log N_v), & 0 < \zeta < 1 \\ O(N_v/\log^2 N_v), & \zeta = 1 \\ O((N_v/\log N_v)^{2-\zeta}), & 1 < \zeta < 5/2 \\ O((\log N_v/N_v)^{1/2}), & \zeta \geq 5/2 \end{cases}$$

as $N_v \to \infty$.

Corollary 7 (Dense and homogeneous). In the dense homogeneous graph where the average degree $\bar{d} = \Theta(N_v^c)$, $0 < c < 1$, and the asymptotic degree distribution is the Poisson distribution with mean $\bar{d}$, under the assumptions in Theorem 2 and $\beta = O(1)$ ($N_v \to \infty$), we have

$$\text{Var}[\bar{\kappa}] = O(N_v^{(c-1)/2}) \text{ as } N_v \to \infty.$$

Corollary 8 (Dense and inhomogeneous). In the dense inhomogeneous graph where the average degree $\bar{d} = \Theta(N_v^c)$, $0 < c < 1$, and the asymptotic degree distribution is truncated Pareto distribution with shape $\zeta$, lower bound $d_L$ and upper bound $N_v - 1$, under the assumptions in Theorem 2 and $\beta = O(1)$ ($N_v \to \infty$), we have

$$\text{Var}[\bar{\kappa}] = \begin{cases} O(N_v^{1-c}), & 0 < \zeta < 1 \\ O(N_v^{1-c}/\log N_v), & \zeta = 1 \\ O(N_v^{2-\zeta}(1-c)), & 1 < \zeta < 5/2 \\ O(N_v^{(c-1)/2}), & \zeta \geq 5/2 \end{cases}$$

as $N_v \to \infty$.

Note that the orders of the variances are asymptotically dominated by the corresponding biases for all four cases. Therefore, in noisy contact networks, bias would appear to be the primary concern for the observed branching factor. The $O$ notations for variances in the homogeneous networks are bounded above by those in the inhomogeneous networks of the same network density.

5. Estimators for the true branching factor

As we saw in Section 3, the observed branching factor is biased in the inhomogeneous network setting. Due to the presence of heterogeneity in most real-world network data, it is important to have new estimators for bias reduction. We present a method-of-moments estimator in Section 5.1 and a bootstrap sampling estimator in Section 5.2. Both estimators require network replicates. The method-of-moments estimator needs a minimum of three replicates, and the bootstrap sampling estimator requires a minimum of two replicates.
5.1. Method-of-moments estimator

We adapt the method-of-moments estimators (MME) of subgraph density in Chang et al. (2018), which require at least three replicates of the observed network. Let \( C_{V_1} \) and \( C_{V_2} \) denote the edge density and the two-stars density, respectively. Then

\[
C_{V_1} = \frac{1}{|V_1|} \sum_{v=(i_1,i'_1) \in V_1} A_{i_1,i'_1}
\]

and

\[
C_{V_2} = \frac{1}{|V_2|} \sum_{v=(i_1,i'_1,i_2,i'_2) \in V_2} A_{i_1,i'_1} A_{i_2,i'_2},
\]

where \( V_1 = \{(i_1,i'_1) : i_1 < i'_1\} \) and \( V_2 = \{(i_1,i'_1,i_2,i'_2) : i'_1 = i_2, i_1 \neq i_2 \neq i'_2\} \).

Next we define

\[
\hat{d} = (N_v - 1) \hat{C}_{V_1},
\]

\[
\hat{d}^2 = (N_v - 1)(N_v - 2) \hat{C}_{V_2} + \hat{d},
\]

where \( \hat{C}_{V_1} \) and \( \hat{C}_{V_2} \) are method-of-moments estimators of \( C_{V_1} \) and \( C_{V_2} \), which we will define later. Thus, our estimator of \( \kappa \) is given by:

\[
\hat{\kappa} = \frac{\hat{d}^2}{\hat{d}} = (N_v - 2) \frac{\hat{C}_{V_2}}{\hat{C}_{V_1}} + 1. \tag{5.1}
\]

Note that \( \hat{\kappa} \) is an asymptotically unbiased estimator for \( \kappa \) and its asymptotic distribution is normal. Details can be obtained from Chang et al. (2018) Section 4.3. Specifically, Chang et al. (2018) provide joint inference of higher-order subgraph densities with unknown error rates. Mimicking their proofs, we can easily obtain the asymptotic joint normal distribution of \( \hat{C}_{V_1} \) and \( \hat{C}_{V_2} \). Then, by the delta method, we can derive the asymptotic normal distribution of \( \hat{\kappa} \).

To compute \( \hat{\kappa} \), we first estimate \( \hat{C}_{V_1} \) and \( \hat{C}_{V_2} \) by methods used in Chang et al. (2018). Define relevant quantities as follows:

\[
u_1 = (1 - \delta)\alpha + \delta(1 - \beta),
\]

\[
u_2 = (1 - \delta)\alpha(1 - \alpha) + \delta\beta(1 - \beta),
\]

\[
u_3 = (1 - \delta)\alpha(1 - \alpha)^2 + \delta\beta^2(1 - \beta),
\]

where \( \delta \) is the edge density in the true network, \( \nu_1 \) is the expected edge density in one observed network, \( \nu_2 \) is the expected density of edge differences in two observed networks, and \( \nu_3 \) is the average probability of having an edge between two arbitrary nodes in one observed network but no edge between same nodes in the other two
observed networks. The method-of-moments estimators for $u_1$, $u_2$ and $u_3$ are

$$\hat{u}_1 = \frac{2}{N_v(N_v - 1)} \sum_{i<j} \hat{A}_{i,j},$$

$$\hat{u}_2 = \frac{1}{N_v(N_v - 1)} \sum_{i<j} |\hat{A}_{i,j} - \bar{A}_{i,j}|,$$

$$\hat{u}_3 = \frac{2}{3N_v(N_v - 1)} \sum_{i<j} I(\text{Exactly one of } \tilde{A}_{i,j,*}, \tilde{A}_{j,i,*}, \tilde{A}_{i,j} \text{ equals } 1).$$

(5.2)

where $\hat{A}_* = (\hat{A}_{i,j})_{N_v \times N_v}$, $\tilde{A}_* = (\tilde{A}_{i,j,*})_{N_v \times N_v}$ are independent and identically distributed replicates of $A$. Calculation of the estimator $\hat{\kappa}$ in (5.1) and the estimation of its asymptotic variance can be accomplished as detailed in Algorithm 1 below and Algorithm 3 in the Appendix, respectively. The variance estimation is based on a nonstandard bootstrap.

**Algorithm 1** Method-of-moments estimator $\hat{\kappa}$

**Input:** $\hat{A} = (\hat{A}_{i,j})_{N_v \times N_v}$, $\bar{A} = (\tilde{A}_{i,j,*})_{N_v \times N_v}$. $\bar{A} = (\tilde{A}_{i,j,**})_{N_v \times N_v}$, $\alpha_0$, $\varepsilon$

**Output:** $\hat{\alpha}$, $\hat{\beta}$, $\hat{\kappa}$

Compute $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ defined in (5.2);

Initialize $\hat{\alpha} = \alpha_0$, $\alpha_0 = \hat{\alpha} + 10\varepsilon$;

while $|\hat{\alpha} - \alpha_0| > \varepsilon$ do

$$\alpha_0 \leftarrow \hat{\alpha}, \quad \hat{\beta} \leftarrow \frac{\hat{u}_2 - \alpha_0 + \hat{u}_1 \alpha_0}{\hat{u}_1 - \alpha_0}, \quad \hat{\delta} \leftarrow \frac{(\hat{u}_1 - \alpha_0)^2}{\hat{u}_1 - 2\hat{u}_2 - 2\hat{u}_1 \alpha_0 + \alpha_0^2}, \quad \hat{\alpha} \leftarrow \hat{u}_3 - \hat{\delta} \hat{\beta}^2 (1 - \hat{\beta}) \frac{1}{(1 - \hat{\delta}) (1 - \alpha_0)^2}.$$

Compute $k_3 = 1 - \hat{\alpha} - \hat{\beta}$, $\hat{C}_{v_1} = \frac{2}{k_3 N_v(N_v - 1)} \sum_{i<j} (\tilde{A}_{i,j} - \hat{\alpha})$,

$$\hat{C}_{v_2} = \frac{1}{k_3^2 N_v(N_v - 1)(N_v - 2)} \sum_{i \neq j \neq l} (\tilde{A}_{i,j} - \hat{\alpha})(\tilde{A}_{j,l} - \hat{\alpha}), \quad \hat{\kappa} = (N_v - 2) \frac{\hat{C}_{v_2}}{\hat{C}_{v_1}} + 1.$$

**Remark 5.** Since our estimation of the unknown parameters is based on moment estimation, the independent noise dictated by Assumption 2 is not strictly necessary. As is shown in the proof of Chang et al. (2018), the convergence rate for the moment estimation of the unknown parameters is determined by the convergence rates of $\hat{u}_1 - u_1$, $\hat{u}_2 - u_2$ and $\hat{u}_3 - u_3$. When some limited dependency among observed edges is present, the convergence rates of $\hat{u}_1 - u_1$, $\hat{u}_2 - u_2$ and $\hat{u}_3 - u_3$ still are $O(1/N_v)$.

**5.2. Bootstrap sampling estimator**

A generic bootstrap method has been proposed in the context of contact networks by Kucharski et al. (2018), for the purpose of assessing various summaries of network structure. We adapt and formalize this method to undirected networks, and obtain a bootstrap sampling estimator for $\kappa$ when a minimum of two replicates of the observed network are available.
Suppose we have \( m \) replicates \( \tilde{A}^{(1)} = (\tilde{A}^{(1)}_{i,j})_{N_v \times N_v}, \ldots, \tilde{A}^{(m)} = (\tilde{A}^{(m)}_{i,j})_{N_v \times N_v}, \) where \( m \geq 2 \). In each iteration, we construct a bootstrap resample matrix \( \tilde{A}^b \) as follows: for entries \( \tilde{A}^b_{i,j} \), \( 1 \leq i < j \leq N_v \), we randomly select one of \( m \) observed adjacency matrices, and use the \((i,j)\) entry of the selected matrix as the value of \( \tilde{A}^b_{i,j} \). Then, we let the lower triangular elements equal to the corresponding upper triangular elements and make the diagonal elements to be 0s. This generates a bootstrap network from which we can calculate the branching factor. We then perform iterations of bootstrap sampling to estimate the mean and confidence interval. See Algorithm 2 for details.

In general, the bootstrap sampling estimator does not work well since bootstrap samples are drawn from observed degree distributions, which introduce errors into the branching factor. When error rates are small and satisfy Assumption 4, we expect performance to be better. Recall Remark 2 that higher-order subgraph counts may not be unbiased under Assumption 4. Thus, Assumption 4 can’t guarantee the good performance of the bootstrap sampling estimator. The performance also depends on other network characteristics, such as the degree distribution. For the sake of comparison, we show results of the bootstrap sampling estimator using Algorithm 2 in Section 6. Moreover, extensive work has been done on bootstrapping networks without replicates. For example, Bhattacharyya et al. (2015) proposed bootstrap subsampling methods for finding empirical distribution of count features.

**Algorithm 2** Bootstrap sampling estimator

| Input: \( \tilde{A}^{(1)} = (\tilde{A}^{(1)}_{i,j})_{N_v \times N_v}, \ldots, \tilde{A}^{(m)} = (\tilde{A}^{(m)}_{i,j})_{N_v \times N_v}, N_b \) |
| Output: \( \kappa^1, \ldots, \kappa^{N_b} \) |
| Initialize \( \tilde{A}^{b_1} = \ldots = \tilde{A}^{b_{N_b}} = (0)_{N_v \times N_v} \); |
| for \( n_b = 1 : N_b \) do |
| for \( i = 1 : N_v \) do |
| for \( j = i + 1 : N_v \) do |
| Randomly select one element from \( \{1, \ldots, m\} \), denoted by \( l \); |
| \( \tilde{A}^{b_{n_b}}_{i,j} \leftarrow \tilde{A}^{(l)}_{i,j} \); |
| \( \tilde{A}^{b_{n_b}}_{j,i} \leftarrow \tilde{A}^{(l)}_{i,j} \); |
| Compute the branching factors for \( \tilde{A}^{b_{n_b}} \), denoted by \( \kappa^{n_b} \). |

### 6. Numerical illustration: British secondary school contact networks

We conduct some simulations and experiments to illustrate the finite sample properties of the proposed estimation methods. We consider the data and network construction described in Kucharski et al. (2018). These data were collected from 460 unique participants across four rounds of data collection conducted between January and June 2015 in year 7 groups in four UK secondary schools, with 7,315 identifiable contacts reported in total. They used a process of peer nomination as a method for data collection: students were asked, via the research questionnaire, to
list the six other students in year 7 at their school that they spend the most time with. For each pair of participants in a specific round of data collection, a single link was defined if either one of the participants reported a contact between the pair (i.e. there was at least one unidirectional link, in either direction). Our analysis focuses on the single link contact network. We consider two settings, a simulation setting where noise is added to a ‘true’ network and an application setting where the four replicates are each treated as noisy versions of an unknown true network.

6.1. Simulations
For each school, we construct a ‘true’ adjacency matrix $A$: if an edge occurs between a pair of vertices more than once in four rounds, we view that pair to have a true edge. The noisy, observed adjacency matrices $\tilde{A}$, $\tilde{A}_\ast$, $\tilde{A}_{\ast\ast}$ are generated according to (2.1). We set $\alpha = 0.005$ or 0.010, and $\beta = 0.01$, 0.15, or 0.20. We assume that both $\alpha$ and $\beta$ are unknown.

We evaluate the two types of point estimates for $\kappa$ and 95% confidence intervals. For the method-of-moments estimator, we follow Algorithms 1 and 3. For the bootstrap sampling estimator, we perform 10,000 iterations of bootstrap sampling to estimate the mean and 95% confidence interval. Figure 6.1 shows the simulation results, in which we replicate 500 times for each setting. The mean absolute errors (MAE) for the point estimates for the branching factor $\kappa$ and the relative frequency (RF) of coverage for the estimated 95% confidence interval for $\kappa$ are shown in Figure 6.1. Note that, for the method-of-moments estimator, $\text{MAE}(\tilde{\kappa}) = \frac{1}{500} \sum_{i=1}^{500} |\tilde{\kappa}_i - \kappa|$, where $\tilde{\kappa}_1, \cdots, \tilde{\kappa}_{500}$ denote the estimated values in 500 replications of simulation, and $\kappa$ denotes the true value. For the bootstrap sampling estimators, we define $\text{MAE}(\bar{\kappa}) = \frac{1}{500} \sum_{i=1}^{500} |\bar{\kappa}^b_i - \kappa|$, where $\bar{\kappa}^b_1, \cdots, \bar{\kappa}^b_{500}$ denote the mean of estimates across 10,000 bootstrap samples in 500 replications of simulation.

For the method-of-moments estimator, the estimation errors for $\kappa$ increase when $\alpha$ and $\beta$ increase. And the estimated coverage probabilities are indeed around 95%. The average interval lengths are slightly larger than those of bootstrap sampling estimators. For the bootstrap sampling estimators, the estimation errors and estimated coverage probabilities depend on the relationship of $\alpha$ and $\beta$ and the degree distribution. For example, when $\beta/\alpha = 20$ in school 2, the estimation of $\kappa$ is quite accurate and the estimated coverage probability is high. This may due to the fact that $\alpha$ and $\beta$ satisfy Assumption 4. When $\alpha = 0.005$ and $\beta = 0.2$, the estimated coverage probabilities are low for all four schools.

6.2. Application
Again, we use the British secondary school contact networks. Considering the nodes in four rounds are not same, we choose the common nodes in four rounds and their edges to obtain four replicates of the noisy networks. Since our estimation methods only need three replicates, we select rounds 1, 2, and 3.

We evaluate the two types of point estimates for $\kappa$, 95% confidence intervals, and the observed branching factor $\bar{\kappa}$. Point estimates and 95% confidence intervals for $\alpha$ and $\beta$ are reported in Table 6.2. Figure 6.2 show the point estimates for the
Fig. 6.1. Mean absolute errors (MAE) of $\hat{\kappa}$, and 95% confidence intervals for $\kappa$ in the simulation with 500 replications for noisy networks in four schools. Reported in the plots are the relative frequencies (RF) of the event that a confidence interval covers the corresponding true value, and also the average Length of the intervals.
Estimation of the Epidemic Branching Factor in Noisy Contact Networks

Table 6.1. Point estimates and 95% confidence intervals for $\alpha$ and $\beta$ in four schools.

| School  | Estimates  | CI          | Estimates  | CI          |
|---------|------------|-------------|------------|-------------|
| School 1| 0.005      | (0.004, 0.007) | 0.207      | (0.140, 0.275) |
| School 2| 0.013      | (0.012, 0.015) | 0.141      | (0.092, 0.191) |
| School 3| 0.013      | (0.012, 0.015) | 0.000      | (-0.057, 0.057) |
| School 4| 0.020      | (0.014, 0.025) | 0.123      | (0.025, 0.222) |

branching factor $\kappa$ and the observed branching factor $\bar{\kappa}$ in each round. The error bars are the estimated 95% confidence interval for $\kappa$.

Table 6.2 indicates there exist nontrivial noise in all four schools. Figure 6.2 shows that, in schools 2 and 3, the resulting method-of-moments estimates for $\kappa$ are lower than all of their observed values, indicating a nontrivial downward adjustment for network noise. And the observed branching factors are not in the estimated 95% confidence intervals, which further reinforces the evidence that the true branching factor is less than those observed empirically. In schools 1 and 4, the resulting method-of-moments estimates for $\kappa$ are close to their observed values. For all four schools, the bootstrap sampling estimators tend to be closer to the observed branching factors than the method-of-moments estimators. Additionally, the interval lengths are relatively small, which is consistent with the simulation results. The simulation results would suggest that the method-of-moments estimates are preferable here, and hence that the bootstrap approach is less able to adjust for the bias induced by noise in our observed networks.

Ultimately, we see that the ability to account for network noise appropriately in reporting the branching factor can lead to substantially different conclusions than use of the original, empirically observed branching factor.

7. Discussion

Here we have quantified the bias and variance of the observed branching factor in noisy networks and developed a general framework for estimation of the true branching factor in contexts wherein one has observations of noisy networks. One of our approaches requires as few as three replicates of network observations, and employs method-of-moments techniques to derive estimators and establish their asymptotic consistency and normality. The other approach relies on bootstrapping of the observed networks to construct many sample networks, from which in turn we obtain estimators and confidence intervals. Simulations demonstrate that substantial inferential accuracy by method-of-moments estimators is possible in networks of even modest size when nontrivial noise is present. And our application to social contact networks in British secondary schools shows that the gains offered by our approach over presenting the observed branching factor can be pronounced.

We have pursued a frequentist approach to the problem of uncertainty quantification for the branching factor. If the replicates necessary for our approach are unavailable in a given setting, a Bayesian approach is a natural alternative. For
example, posterior-predictive checks for goodness-of-fit based on examination of a handful of network summary measures is common practice (e.g., Bloem-Reddy and Orbanz (2018)). Note, however, that the Bayesian approach requires careful modeling of the generative process underlying $G$ and typically does not distinguish between signal and noise components. Our analysis is conditional on $G$, and hence does not require that $G$ be modeled. It is effectively a ‘signal plus noise’ model, with the signal taken to be fixed but unknown. Related work has been done in the context of graphon modeling, with the goal of estimating network motif frequencies (e.g., Latouche and Robin (2016)). However, again, one typically does not distinguish between signal and noise components in this setting. Additionally, we note that the problem of practical graphon estimation itself is still a developing area of research.

Our work here sets the stage for extensions to various thresholds and statistics which depend on the branching factor. Recall that these include the percolation threshold $1/(\kappa - 1)$, the epidemic threshold $1/(\kappa - 1)$, and the immunization threshold $1 - 1/(\lambda \kappa)$, where $\lambda$ is the spreading rate (Pastor-Satorras et al. (2015)). Replacing $\kappa$ with $\hat{\kappa}$, we obtain asymptotically unbiased estimators for the corresponding thresholds. The asymptotic distributions can be derived from the delta method. In addition, the total branching factor of the network is important for epidemic spreading and immunization strategy in multiplex networks (e.g., Buono et al. (2014)).

Our choice to work with independent network noise is both natural and motivated
by convenience. And our results of method-of-moments estimators still hold when there is some dependency across (non)edges. A precise characterization of the dependency is typically problem-specific and hence a topic for further investigation.

8. Acknowledgement

This work was supported in part by ARO award W911NF1810237. This work was also supported by the Air Force Research Laboratory and DARPA under agreement number FA8750-18-2-0066 and by a grant from MIT Lincoln Labs.

9. Appendix

In the appendix, you will find proofs of theorems for bias of the observed branching number, theorems for variance of the observed branching number and the proofs, and the algorithm for estimation of asymptotic variance of method-of-moments estimator $\hat{\kappa}$. Proofs of corollaries can be found in supplementary materials.

9.1. Proofs of theorems for bias of the observed branching number

**Proof (Theorem 1).** Recall $X = \sum_i \tilde{d}_i^2$ and $Y = \sum_i \tilde{d}_i$. Note that $\text{Bias}[\hat{\kappa}] = \frac{EX}{EY} - \kappa + \mathcal{O}\left(\frac{1}{(EY)^{1/(2+\eta)}} \frac{EX}{EY}\right)$ is equivalent to

$$\mathbb{E}[\hat{\kappa}] - \frac{EX}{EY} = \mathcal{O}\left(\frac{1}{(EY)^{1/(2+\eta)}} \frac{EX}{EY}\right).$$

(9.1)

By Jensen’s inequality, we have

$$\left|\mathbb{E}[\hat{\kappa}] - \frac{EX}{EY}\right| = \frac{1}{EY} \mathbb{E}\left[\frac{X(EY - Y)}{Y} \cdot I_{\{Y > 0\}}\right] \leq \frac{1}{EY} \cdot \mathbb{E}\left[\frac{X|EY - Y|}{Y} \cdot I_{\{Y > 0\}}\right].$$

(9.2)

Then by additivity of expectation, for $0 < \delta < 1$, $\mathbb{E}\left[\frac{X|EY - Y|}{Y} \cdot I_{\{Y > 0\}}\right]$ in (9.2) equals

$$\mathbb{E}\left[\frac{X|EY - Y|}{Y} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| \geq \delta EY\}}\right] + \mathbb{E}\left[\frac{X|EY - Y|}{Y} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| < \delta EY\}}\right].$$

(9.3)

Next, we find the upper bounds of two terms in (9.3). For the first term, by definitions of $X$ and $Y$, $X/Y \cdot I_{\{Y > 0\}} < N_v$ and $|EY - Y| < N_v^2$. Thus, we have

$$\mathbb{E}\left[\frac{X|EY - Y|}{Y} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| \geq \delta EY\}}\right] < N_v^3 \cdot \text{Pr}(|Y - EY| \geq \delta EY).$$
Then, by Chernoff Bound, we obtain
\[
\mathbb{E}\left[\frac{X|\mathbb{E}Y - Y|}{Y} \cdot I\{Y > 0\} \cdot I\{|Y - \mathbb{E}Y| \geq \delta \mathbb{E}Y\}\right] < 2N_v^3 \cdot \exp\left(-\frac{\delta^2 \cdot \mathbb{E}Y}{6}\right).
\]
For the second term, when \(|Y - \mathbb{E}Y| < \delta \mathbb{E}Y\), \(Y > (1 - \delta) \mathbb{E}Y\). So, we obtain
\[
\mathbb{E}\left[\frac{X|\mathbb{E}Y - Y|}{Y} \cdot I\{Y > 0\} \cdot I\{|Y - \mathbb{E}Y| < \delta \mathbb{E}Y\}\right] < \delta \cdot \mathbb{E}X.
\]
By (9.2), we show
\[
\left|\mathbb{E}[\tilde{\kappa}] - \frac{\mathbb{E}X}{\mathbb{E}Y}\right| \leq 2N_v^3 \mathbb{E}Y \cdot \exp\left(-\frac{\delta^2 \mathbb{E}Y}{6}\right) + \frac{\delta}{(1 - \delta)} \cdot \frac{\mathbb{E}X}{\mathbb{E}Y}.
\]
(9.4)

Let \(L_1(N_v)\) and \(L_2(N_v)\) denote two terms on the right sides in (9.4). We choose \(\delta = \frac{\mathbb{E}Y}{(2 + \eta)}\), \(\eta > 0\), such that
\[
L_1(N_v) = o\left(\frac{\mathbb{E}X}{\mathbb{E}Y}\right) \text{ and } L_2(N_v) = o\left(\frac{\mathbb{E}X}{\mathbb{E}Y}\right) \text{ as } N_v \to \infty,
\]
under the assumption \(\mathbb{E}Y = \Omega(N_v)\) as \(N_v \to \infty\), \(1 - \delta = O(1)\). By L'Hopital's rule, we have
\[
L_1(N_v) = o(L_2(N_v)) \text{ as } N_v \to \infty.
\]
These imply (9.1).

**Proof (Theorem 2).** We compute \(\mathbb{E}Y\) and \(\mathbb{E}X\) under Assumption 1 and 2,
\[
\mathbb{E}Y = \sum_{i=1}^{N_v} \mathbb{E}[\tilde{d}_i] = \sum_{i=1}^{N_v} \alpha(N_v - 1 - d_i) + (1 - \beta)d_i
\]
\[
= \alpha N_v(N_v - 1) + (1 - \alpha - \beta) \sum_{i=1}^{N_v} d_i, \text{ and}
\]
\[
\mathbb{E}X = \sum_{i=1}^{N_v} \mathbb{E}[\tilde{d}_i^2] = \sum_{i=1}^{N_v} (\text{var}[\tilde{d}_i] + (\mathbb{E}[\tilde{d}_i])^2)
\]
\[
= \sum_{i=1}^{N_v} \left[\alpha(1 - \alpha)(N_v - 1 - d_i)
\right.
\]
\[+ \beta(1 - \beta)d_i + \left[\alpha(N_v - 1 - d_i) + (1 - \beta)d_i\right]^2\]\n\[= (1 - \alpha - \beta)^2 \sum_{i=1}^{N_v} d_i^2 + [\beta(1 - \beta) - \alpha(1 - \alpha)
\]
\[+ 2\alpha(N_v - 1)(1 - \alpha - \beta)] \sum_{i=1}^{N_v} d_i + \alpha N_v(N_v - 1)[1 - \alpha + \alpha(N_v - 1)]. \quad (9.5)
\]
Then, under Assumption 4, (9.5) leads to
\[ E_Y = \sum_{i=1}^{N_v} d_i, \]
\[ E_X = (1 - \alpha - \beta)^2 \sum_{i=1}^{N_v} d_i^2 + (2 - \alpha - \beta) \left[ \alpha(N_v - 1) + \beta \right] \sum_{i=1}^{N_v} d_i. \]

Plugging the value of \( E_X/E_Y \) into the bias expression in Theorem 1 completes the proof.

9.2. Theorems for variance of the observed branching number in arbitrary network topology

**Theorem 3.** We assume \( E_Y > 0 \), and \( E_Y = \Omega(N_v) \) \((N_v \to \infty)\). Then, under Assumption 2,

(i) \[ \text{Var}[\hat{\kappa}] = \mathcal{O}\left( \max \left\{ E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right], P(Y = 0) \cdot \left[ \frac{E X}{E_Y} \right]^2 \right\} \right) \]
as \( N_v \to \infty \).

(ii) For any \( \eta, \lambda > 0 \),

\[ \text{Var}[\hat{\kappa}] = E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right] + \mathcal{O}\left( \max \left\{ (E_Y)^{-1/(2 + \eta)} \cdot E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right], (E_Y)^{-2/(2 + \lambda)} \cdot \left[ \frac{E X}{E_Y} \right]^2 \right\} \right) \]
as \( N_v \to \infty \).

**Theorem 4.** Under the assumptions in Theorem 3, Assumption 1 and 4, and \( 1 - \beta = \Omega(N_v) \) \((N_v \to \infty)\),

(i) \[ \text{Var}[\hat{\kappa}] = \mathcal{O}\left( E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right] \right) \text{ as } N_v \to \infty. \]

(ii) For any \( \eta, \kappa > 0 \),

\[ \text{Var}[\hat{\kappa}] = E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right] \]
\[ + \mathcal{O}\left( \max \left\{ (E_Y)^{-1/(2 + \eta)} \cdot E\left[ \left( \frac{X E_Y - Y E_X}{E_Y} \right)^2 \right], (E_Y)^{-2/(2 + \lambda)} \cdot \left[ \frac{E X}{E_Y} \right]^2 \right\} \right) \]
as \( N_v \to \infty \).
Theorem 3 (i) and Theorem 4 (i) provide upper bounds for variances of the observed branching factors. And Theorem 3 (ii) and Theorem 4 (ii) derive good approximations of variances if the $O$ terms are dominated by the corresponding first terms asymptotically.

9.3. Proofs of theorems for variance of the observed branching number

To show Theorem 3 and Theorem 4, we first introduce a useful lemma.

**Lemma 1.** Under assumptions in Theorem 3, for any $\eta > 0$, we have

$$
E \left[ \left( \hat{\kappa} - \frac{EX}{EY} \right)^2 \right] = E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] + O \left( \max \left\{ (EY)^{-1/(2+\eta)}, E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right], \Pr(Y = 0) \cdot \left[ \frac{EX}{EY} \right]^{2} \right\} \right)
$$

as $N_\nu \to \infty$.

**Proof.** Note that

$$
E \left[ \left( \frac{EX}{EY} \right)^2 \right] - E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] = E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \cdot I_{\{Y > 0\}} \right] + \left( \frac{EX}{EY} \right)^2 \cdot \Pr(Y = 0)
$$

By triangle inequality and Jensen’s inequality, we obtain

$$
\left| E \left[ \left( \frac{X}{Y} \cdot I_{\{Y > 0\}} - \frac{EX}{EY} \right)^2 \right] - E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] \right| \leq \frac{1}{(EY)^4} E \left[ \frac{(XEY - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \right] + \left( \frac{EX}{EY} \right)^2 \cdot \Pr(Y = 0) \cdot \Pr(Y = 0).
$$

Next, we find an upper bound of

$$
E \left[ \frac{(XEY - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \right].
$$

By additivity of expectation, for any $\delta \in (0, 1)$, (9.7) equals

$$
E \left[ \frac{(XEY - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| \geq \delta EY\}} \right] + E \left[ \frac{(XEY - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| < \delta EY\}} \right].
$$

For the first term in (9.8), by definitions of $X$ and $Y$, we obtain $\left| \frac{XEY - YEX}{Y^2} \right| \cdot I_{\{Y > 0\}} < N_\nu$ and $|EY|^2 - Y^2| < N_\nu$. Thus, we have

$$
E \left[ \frac{(XEY - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| \geq \delta EY\}} \right] < N_\nu \cdot \Pr(|Y - EY| \geq \delta EY).
$$
Then, by Chernoff Bound, we obtain
\[
\mathbb{E}
\left[
\frac{(X_{EY} - YEX)^2}{Y^2} \cdot I_{\{Y > 0\}} \cdot I_{\{|Y - EY| \geq \delta Y\}}\right] < 2N_v^{10} \cdot \exp\left(-\frac{\delta^2 \cdot Y}{6}\right).
\]

For the second term in (9.8), when \(|EY - Y| < \delta EY\), \(Y > (1 - \delta)EY\) and \(Y < (1 + \delta)EY\). And notice \(|(EY)^2 - Y^2| = (EY + Y)|EY - Y|\), we have
\[
\mathbb{E}
\left[
\frac{(X_{EY} - YEX)^2}{(EY)^2} \cdot Y^2 \cdot I_{\{EY > 0\}} \cdot I_{\{|Y - EY| < \delta EY\}}\right] \leq \frac{\delta(2 + \delta)}{(1 - \delta)^2} \cdot \mathbb{E} \left[ (X_{EY} - YEX)^2 \right].
\]

By (9.6), we show
\[
\left| \mathbb{E} \left[ \left( \tilde{\kappa} - \frac{EX}{EY} \right)^2 \right] - \mathbb{E} \left[ \frac{(X_{EY} - YEX)^2}{(EY)^4} \right] \right| \leq \frac{2N_v^{10}}{(EY)^4} \cdot \exp\left(-\frac{\delta^2 \cdot Y}{6}\right) + \delta(2 + \delta) \cdot \mathbb{E} \left[ \frac{(X_{EY} - YEX)^2}{(EY)^4} \right] + \left( \frac{EX}{EY} \right)^2 \cdot \Pr(Y = 0).
\]

(9.10)

L1(Nv) and L2(Nv) denote the first two terms on the right sides in (9.10). We choose \(\delta = (EY)^{-1/(2+\eta)}\), \(\eta > 0\), such that
\[L1(N_v) = o\left(\mathbb{E} \left[ \frac{(X_{EY} - YEX)^2}{(EY)^4} \right] \right) \quad \text{and} \quad L2(N_v) = o\left(\mathbb{E} \left[ \frac{(X_{EY} - YEX)^2}{(EY)^4} \right] \right)\]
as \(N_v \to \infty\). Under the assumption \(EY = \Omega(N_v)\) as \(N_v \to \infty\), \(1 - \delta = \mathcal{O}(1)\). By L’Hopital’s rule, we have
\[L1(N_v) = o(L2(N_v)) \quad \text{as} \quad N_v \to \infty.\]

These complete the proof.

**Proof (Theorem 3).** By the definition of variance, we have
\[
\Var[\tilde{\kappa}] = \Var \left[ \tilde{\kappa} - \frac{EX}{EY} \right] = \mathbb{E} \left[ \left( \tilde{\kappa} - \frac{EX}{EY} \right)^2 \right] - \mathbb{E} \left[ \frac{EX}{EY} \right]^2.
\]

(i) By Lemma 1, for any \(\eta > 0\), we obtain
\[
\Var[\tilde{\kappa}] = \mathcal{O}\left( \mathbb{E} \left[ \left( \tilde{\kappa} - \frac{EX}{EY} \right)^2 \right] \right) = \mathcal{O}\left( \max \left\{ \mathbb{E} \left[ \frac{(X_{EY} - YEX)^2}{(EY)^4} \right], \mathbb{P}(Y = 0) \cdot \left( \frac{EX}{EY} \right)^2 \right\} \right).
\]
(ii) By triangle inequality, we have
\[ |\text{Var}[\tilde{\kappa}] - E\left[\frac{(X E Y - Y E X)^2}{(E Y)^4}\right]| \leq E\left[\left(\tilde{\kappa} - \frac{E X}{E Y}\right)^2\right] - E\left[\frac{(X E Y - Y E X)^2}{(E Y)^4}\right] + E(\tilde{\kappa}) - \frac{E X}{E Y}^2.\]

Apply Lemma 1 and Theorem 1 and the rest follows.

**Proof (Theorem 4).** Under assumptions in Theorem 4, we have
\[ P(Y = 0)\left[\frac{E X}{E Y}\right]^2 = o\left( E\left[\frac{(X E Y - Y E X)^2}{(E Y)^4}\right]\right),\]
and there exist \( \eta_0, \lambda_0 > 0 \), such that
\[ P(Y = 0)\left[\frac{E X}{E Y}\right]^2 = o\left( \max \left\{ \frac{1}{(E Y)^{1/(2+\eta_0)}} E\left[\frac{(X E Y - Y E X)^2}{(E Y)^4}\right], \frac{1}{(E Y)^{2/(2+\lambda_0)}} \left[\frac{E X}{E Y}\right]^2 \right\} \right).\]

Apply Theorem 3 and the rest follows.

### 9.4. Algorithm for estimation of asymptotic variance of method-of-moments estimator \( \tilde{\kappa} \)

To evaluate the asymptotic variance of method-of-moments estimator \( \tilde{\kappa} \), we first estimate the asymptotic variance of \( (\tilde{C}_\nu, \tilde{C}_\nu) \) by the method in Section 4 of Chang et al. (2018). Then, we use the delta method to obtain the estimation of the asymptotic variance of \( \tilde{\kappa} \). The detail is shown in Algorithm 3.

\[
\hat{\Sigma} = (\hat{\sigma}_{ij})_{3x3},
\]
\[
\hat{\Delta} = \hat{k}_3^{-1} \cdot \left( \begin{array}{ccc} \tilde{C}_\nu - 1 & \tilde{C}_\nu & \tilde{C}_\nu \\ 2\tilde{C}_\nu - 2\tilde{C}_\nu & \tilde{C}_\nu & \tilde{C}_\nu \\ \end{array} \right),
\]
\[
\hat{G} = \hat{k}_3^{-2} \cdot \left( \begin{array}{ccc} (1 - \delta)^{-1} \{ (1 - 2\hat{\beta})\hat{\alpha} + \hat{\beta}^2 \} & (1 - \delta)^{-1} \{ (1 - 2\hat{\beta})\hat{\alpha}^2 + (1 - \delta)^{-1} \{ (1 - 2\hat{\beta})\hat{\alpha} + (1 - \delta)^{-1} \} \\
-\hat{\delta}^{-1} \{ (1 - 2\hat{\beta})\hat{\alpha} + \hat{\beta}^2 \} & \hat{\delta}^{-1} \{ (1 - 2\hat{\alpha})\hat{\beta} + 1 \} & \hat{\delta}^{-1} \{ (1 - 2\hat{\alpha})\hat{\beta} + 1 \}
\end{array} \right).
\]
\[
\hat{H} = \frac{1}{3} \cdot \left( \begin{array}{ccc} \tilde{C}_\nu & \tilde{C}_\nu \tilde{C}_\nu \\ 2\tilde{C}_\nu - 2\tilde{C}_\nu & \tilde{C}_\nu \\ \end{array} \right).
\]

where \( \hat{k}_1 = \hat{\alpha}(1 - \hat{\beta}), \hat{k}_2 = \hat{\beta}(1 - \hat{\beta}), \hat{k}_3 = 1 - \hat{\alpha} - \hat{\beta}, \hat{k}_4 = \hat{\beta} - \hat{\alpha}, \hat{\sigma}_{11} = \delta \hat{k}_2 + (1 - \hat{\delta})\hat{k}_1, \hat{\sigma}_{22} = \delta \hat{k}_2(1/2 - \hat{k}_2) + (1 - \hat{\delta})\hat{k}_1(1/2 - \hat{k}_2), \hat{\sigma}_{33} = \delta \hat{k}_2(1/3 - \hat{k}_2) + (1 - \hat{\delta})\hat{k}_1(1/3 - \hat{k}_2), \hat{\sigma}_{12} = \delta \hat{k}_2(\hat{\beta} - 1/2) + (1 - \hat{\delta})\hat{k}_1(1/2 - \hat{\alpha}), \hat{\sigma}_{13} = \hat{\sigma}_{31} = \delta \hat{k}_2(\hat{\beta}^2/3 - 2\hat{k}_2/3) + (1 - \hat{\delta})\hat{k}_1(1 - \alpha)^2/3 - 2\hat{k}_1/3); \hat{\sigma}_{23} = \hat{\sigma}_{32} = \delta \hat{k}_2(1 - \hat{k}_2) + (1 - \hat{\delta})(1 - \hat{\alpha})\hat{k}_1(1/3 - \hat{k}_1). \]
Algorithm 3 Estimation of asymptotic variance of method-of-moments estimator $\hat{\kappa}$

Input: $\hat{A} = (\hat{A}_{i,j})_{N_v \times N_v}$, $\varepsilon$, $N_b$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{k}_3$, $\hat{C}_{V_1}$, $\hat{C}_{V_2}$, $\delta$

Output: $\text{Var}(\hat{\kappa})$

if $|\hat{\alpha} - \hat{\beta}| < \varepsilon$ then
  $\xi_2 = \hat{\alpha}$, $\xi_1 = 1 - 2\xi_2$;
if $\hat{\beta} - \hat{\alpha} > \varepsilon$ then
  $t_1 = \sqrt{1 - 4\hat{\alpha}(1 - \hat{\beta})}$, $t_2 = \sqrt{1 - 4\hat{\beta}(1 - \hat{\alpha})}$, $\xi_2 = (1 - t_1)/2$;
  if $t_1 + t_2 < 0.5$ then
    $\xi_1 = (t_1 + t_2)/2$;
else
  $\xi_1 = (t_1 - t_2)/2$;
if $\hat{\alpha} - \hat{\beta} > \varepsilon$ then
  $t_1 = \sqrt{1 - 4\hat{\alpha}(1 - \hat{\beta})}$, $t_2 = \sqrt{1 - 4\hat{\beta}(1 - \hat{\alpha})}$, $\xi_2 = (1 + t_1)/2$, $\xi_1 = (t_2 - t_1)/2$;

for $n_b = 1 : N_b$ do
  for $i = 1 : N_v$ do
    for $j = i + 1 : N_v$ do
      Draw $\eta_{i,j}$ from distribution $\mathbb{P}(\eta_{i,j} = 0) = \xi_1$, $\mathbb{P}(\eta_{i,j} = 1) = \xi_2$ and
      $\mathbb{P}(\eta_{i,j} = -1) = 1 - \xi_1 - \xi_2$;
      Compute $\hat{A}_{i,j}^\dagger = \hat{A}_{i,j}I(\eta_{i,j} = 0) + I(\eta_{i,j} = 1)$;
      Compute $\hat{A}_{i,j}^\dagger = \hat{A}_{j,i}^\dagger = \hat{A}_{j,i} - \hat{A}_{i,j} - \xi_1 - \xi_2$;
      Compute $\hat{V}_{1,n_b} = \hat{V}_{2,n_b} + \hat{V}_{3,n_b}$, where
      $\hat{V}_{1,n_b} = \left[\begin{array}{c} \text{Var}(\hat{S}_{V_{1,i}}) \\ \text{Cov}(\hat{S}_{V_{1,i}}, \hat{S}_{V_{2,i}}) \\ \text{Cov}(\hat{S}_{V_{1,i}}, \hat{S}_{V_{2,i}}) \\ \text{Var}(\hat{S}_{V_{2,i}}) \end{array}\right]$, $\hat{V}_{2,n_b} = \hat{\Delta} \hat{G} \hat{\Sigma} \hat{G}^\top \Delta^\top$, $\hat{V}_{3,n_b} = (\hat{H} \hat{G}^\top \Delta^\top + \hat{\Delta} \hat{G} \hat{H}^\top)/2$, $\hat{\Delta}$, $\hat{G}$, $\hat{\Sigma}$, and $\hat{H}$ defined in (9.11);
      Compute $\text{Var}(\hat{\kappa}) = (N_v - 2)^2\left[-\hat{C}_{V_2}/\hat{C}_{V_1}^2, 1/\hat{C}_{V_1}\right]\hat{V}_{N_v}\left[-\hat{C}_{V_2}/\hat{C}_{V_1}^2, 1/\hat{C}_{V_1}\right]^\top$. 

References

Ahmed, N. K., Neville, J. and Kompella, R. (2014) Network sampling: From static to streaming graphs. ACM Transactions on Knowledge Discovery from Data (TKDD), 8, 7.

Anderson, R. M. and May, R. (1991) Infectious diseases of humans. 1991. New York: Oxford Science Publication Google Scholar.

Andersson, H. (1997) Epidemics in a population with social structures. Mathematical biosciences, 140, 79–84.
Andersson, H. and Britton, T. (2012) *Stochastic epidemic models and their statistical analysis*, vol. 151. Springer Science & Business Media.

Balachandran, P., Kolaczyk, E. D. and Viles, W. D. (2017) On the propagation of low-rate measurement error to subgraph counts in large networks. *The Journal of Machine Learning Research*, 18, 2025–2057.

Becker, N. G. and Britton, T. (1999) Statistical studies of infectious disease incidence. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 61, 287–307.

Bhattacharyya, S., Bickel, P. J. et al. (2015) Subsampling bootstrap of count features of networks. *The Annals of Statistics*, 43, 2384–2411.

Bloem-Reddy, B. and Orbanz, P. (2018) Random-walk models of network formation and sequential monte carlo methods for graphs. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80, 871–898.

Buono, C., Alvarez-Zuzek, L. G., Macri, P. A. and Braunstein, L. A. (2014) Epidemics in partially overlapped multiplex networks. *PloS one*, 9, e92200.

Chang, J., Kolaczyk, E. D. and Yao, Q. (2018) Estimation of subgraph density in noisy networks. *arXiv preprint arXiv:1803.02488*.

Chatterjee, S. et al. (2015) Matrix estimation by universal singular value thresholding. *The Annals of Statistics*, 43, 177–214.

Chowell, G., Castillo-Chavez, C., Fenimore, P. W., Kribs-Zaleta, C. M., Arriola, L. and Hyman, J. M. (2004a) Model parameters and outbreak control for sars. *Emerging Infectious Diseases*, 10, 1258.

Chowell, G., Hengartner, N. W., Castillo-Chavez, C., Fenimore, P. W. and Hyman, J. M. (2004b) The basic reproductive number of ebola and the effects of public health measures: the cases of congo and uganda. *Journal of theoretical biology*, 229, 119–126.

Craft, M. E., Volz, E., Packer, C. and Meyers, L. A. (2009) Distinguishing epidemic waves from disease spillover in a wildlife population. *Proceedings of the Royal Society B: Biological Sciences*, 276, 1777–1785.

Davoudi, B., Miller, J. C., Meza, R., Meyers, L. A., Earn, D. J. and Pourbohloul, B. (2012) Early real-time estimation of the basic reproduction number of emerging infectious diseases. *Physical Review X*, 2, 031005.

Diekmann, O. and Heesterbeek, J. A. P. (2000) *Mathematical epidemiology of infectious diseases: model building, analysis and interpretation*, vol. 5. John Wiley & Sons.
Drewe, J. A., Weber, N., Carter, S. P., Bearhop, S., Harrison, X. A., Dall, S. R., McDonald, R. A. and Delahay, R. J. (2012) Performance of proximity loggers in recording intra-and inter-species interactions: a laboratory and field-based validation study. *PLoS One*, 7, e39068.

Eubank, S., Guclu, H., Kumar, V. A., Marathe, M. V., Srinivasan, A., Toroczkai, Z. and Wang, N. (2004) Modelling disease outbreaks in realistic urban social networks. *Nature*, 429, 180.

Jiang, X., Gold, D. and Kolaczyk, E. D. (2011) Network-based auto-probit modeling for protein function prediction. *Biometrics*, 67, 958–966.

Jiang, X. and Kolaczyk, E. D. (2012) A latent eigenprobit model with link uncertainty for prediction of protein–protein interactions. *Statistics in Biosciences*, 4, 84–104.

Kao, R. R., Danon, L., Green, D. M. and Kiss, I. Z. (2006) Demographic structure and pathogen dynamics on the network of livestock movements in great britain. *Proceedings of the Royal Society B: Biological Sciences*, 273, 1999–2007.

Kolaczyk, E. D. (2009) *Statistical Analysis of Network Data*. Springer.

Kucharski, A. J., Wenham, C., Brownlee, P., Racon, L., Widmer, N., Eames, K. T. and Conlan, A. J. (2018) Structure and consistency of self-reported social contact networks in british secondary schools. *PloS one*, 13, e0200090.

Latouche, P. and Robin, S. (2016) Variational bayes model averaging for graphon functions and motif frequencies inference in w-graph models. *Statistics and Computing*, 26, 1173–1185.

Liu, Q.-H., Ajelli, M., Aleta, A., Merler, S., Moreno, Y. and Vespignani, A. (2018) Measurability of the epidemic reproduction number in data-driven contact networks. *Proceedings of the National Academy of Sciences*, 115, 12680–12685.

Pastor-Satorras, R., Castellano, C., Van Mieghem, P. and Vespignani, A. (2015) Epidemic processes in complex networks. *Reviews of modern physics*, 87, 925.

Priebe, C. E., Sussman, D. L., Tang, M. and Vogelstein, J. T. (2015) Statistical inference on errorfully observed graphs. *Journal of Computational and Graphical Statistics*, 24, 930–953.

Smieszek, T., Burri, E. U., Scherzinger, R. and Scholz, R. W. (2012) Collecting close-contact social mixing data with contact diaries: reporting errors and biases. *Epidemiology & infection*, 140, 744–752.

White, L. F., Wallinga, J., Finelli, L., Reed, C., Riley, S., Lipsitch, M. and Pagano, M. (2009) Estimation of the reproductive number and the serial interval in early phase of the 2009 influenza a/h1n1 pandemic in the usa. *Influenza and other respiratory viruses*, 3, 267–276.

Whittle, P. (1955) The outcome of a stochastic epidemic–a note on bailey’s paper. *Biometrika*, 42, 116–122.
Supplementary Materials for “Estimation of the Branching Factor in Noisy Networks”

Wenrui Li
Boston University, Boston, USA.
Daniel L. Sussman
Boston University, Boston, USA.
Eric D. Kolaczyk†
Boston University, Boston, USA.

Summary. In this Supplementary Materials document, we provide proofs of all corollaries presented in the main paper.

A. Proofs of corollaries for bias of the observed branching number

A.1. Proof of Corollary 1

By homogeneity, we obtain

\[ \sum_{i=1}^{N_v} d_i^2 = (\bar{d} + 1)\bar{d}N_v. \]

By edge unbiasedness, we have

\[ \alpha = \frac{\beta \bar{d}}{N_v - 1 - \bar{d}}. \] (A.1)

Thus,

\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \sum_{i=1}^{N_v} d_i^2 = -\alpha, \] (A.2)

and

\[ \frac{EX}{EY} = \bar{d} + 1 - \alpha(2 - \alpha - \beta). \]

By Theorem 2, for any \( \eta > 0 \), we have

\[ \text{Bias}[\kappa] = O\left( \frac{\log N_v}{(N_v \log N_v)^{1/(2+\eta)}} \right) \text{ as } N_v \to \infty. \]
A.2. Proof of Corollary 2

First, we compute the first and second moments of truncated Pareto distribution.

\[
\int_{d_L}^{N_v - 1} x \cdot \frac{\zeta d_L^\zeta}{1 - \left(\frac{d_L}{N_v - 1}\right)^\zeta} x^{-(\zeta + 1)} dx = \begin{cases} 
\zeta d_L \cdot \frac{\log \left(\frac{N_v - 1}{d_L} \right)}{1 - \left(\frac{d_L}{N_v - 1}\right)^\zeta}, & \text{if } \zeta = 1 \\
\zeta d_L \cdot \frac{\log \left(\frac{N_v - 1}{d_L} \right)}{\zeta - 1}, & \text{otherwise},
\end{cases}
\]

(A.3)

\[
\int_{d_L}^{N_v - 1} x^2 \cdot \frac{\zeta d_L^\zeta}{1 - \left(\frac{d_L}{N_v - 1}\right)^\zeta} x^{-(\zeta + 1)} dx = \begin{cases} 
\zeta d_L^2 \cdot \frac{\log \left(\frac{N_v - 1}{d_L} \right)}{1 - \left(\frac{d_L}{N_v - 1}\right)^\zeta}, & \text{if } \zeta = 2 \\
\zeta d_L^2 \cdot \frac{\log \left(\frac{N_v - 1}{d_L} \right)}{\zeta - 2}, & \text{otherwise}.
\end{cases}
\]

Note that \( \bar{d} = \sum_{i=1}^{N_v} d_i / N_v = \Theta(\log N_v) \). So, as \( N_v \to \infty \), we obtain

\[
d_L \sim \begin{cases} 
\left(1 - \frac{\zeta}{\zeta} \cdot \frac{\bar{d}}{N_v^{1-\zeta}}\right)^{1/\zeta}, & \text{if } 0 < \zeta < 1 \\
\bar{d} / \log N_v, & \text{if } \zeta = 1 \\
\zeta - 1 \cdot d, & \text{if } \zeta > 1.
\end{cases}
\]

(A.4)

Thus, as \( N_v \to \infty \), we have

\[
\sum_{i=1}^{N_v} d_i^2 = \begin{cases} 
\frac{N_v}{2 - \zeta}, & \text{if } 0 < \zeta < 1 \\
\frac{N_v}{\log N_v}, & \text{if } \zeta = 1 \\
\frac{N_v^{2-\zeta}}{2 - \zeta} \cdot (\bar{d})^{\zeta-1}, & \text{if } 1 < \zeta < 2 \\
d \cdot \log N_v / 2, & \text{if } \zeta = 2 \\
\frac{(\zeta - 1)^2 d}{\zeta(\zeta - 2)}, & \text{if } \zeta > 2.
\end{cases}
\]

(A.5)

By edge unbiasedness, we have

\[
\alpha = \frac{\beta \bar{d}}{N_v - 1 - \bar{d}} = \Theta \left(\frac{\log N_v}{N_v}\right).
\]

(i) \( 0 < \zeta \leq 2 \)
Note that
\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} = -\beta \kappa + \alpha(N_v - \kappa - 1) + \beta, \]
and
\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i}\right). \]

By Theorem 2, for any \( \eta > 0 \), we have
\[ \text{Bias}[\tilde{\kappa}] = -\beta(2 - \alpha - \beta)\kappa + (2 - \alpha - \beta) \left[ \alpha(N_v - \kappa - 1) + \beta \right] + O\left( \frac{1}{(\frac{\mathbb{E}X}{\mathbb{E}Y})^{1/(2+\eta)}} \right) \frac{\mathbb{E}X}{\mathbb{E}Y} \]
\[ = -\beta(2 - \alpha - \beta)\kappa + O\left( \max \left\{ \log N_v, \frac{\kappa}{(N_v \log N_v)^{1/(2+\eta)}} \right\} \right) \]
as \( N_v \to \infty. \)

(ii) \( \zeta > 2 \)

Note that
\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} \sim -\frac{\beta \bar{d}}{\zeta(\zeta - 2)} - \frac{\beta(d)^2}{\zeta(\zeta - 2)(N_v - 1 - d)} + \beta, \]
and
\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = O(\bar{d}). \]

By Theorem 2, for any \( \eta > 0 \), we have
\[ \text{Bias}[\tilde{\kappa}] = -(2 - \alpha - \beta) \frac{\beta \bar{d}}{\zeta(\zeta - 2)} + O(1) \]
\[ = -\beta(2 - \alpha - \beta) \frac{\kappa}{(\zeta - 1)^2} + O(1) \]
as \( N_v \to \infty. \)

A.3. Proof of Corollary 3

By homogeneity, we obtain
\[ \sum_{i=1}^{N_v} d_i^2 = (\bar{d} + 1)d_N v. \]

By edge unbiasedness, we have
\[ \alpha = \frac{\beta \bar{d}}{N_v - 1 - \bar{d}}. \] (A.6)
Thus,

\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} = -\alpha, \quad (A.7) \]

and

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \bar{d} + 1 - \alpha(2 - \alpha - \beta). \]

By Theorem 2, for any \( \eta > 0 \), we have

\[ \text{Bias}[\kappa] = \mathcal{O}\left( N_v^{c-\frac{c+1}{2+\eta}} \right) \text{ as } N_v \to \infty. \]

### A.4. Proof of Corollary 4

Note that the asymptotic notations for \( d_L \) and \( \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} \) are same as equation (A.4) and equation (A.5).

By edge unbiasedness, we have

\[ \alpha = \frac{\beta \bar{d}}{N_v - 1 - \bar{d}} = \Theta(N_v^{-1}). \]

(i) \( 0 < \zeta \leq 2 \)

Note that

\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} \sim -\beta \bar{d} \left( \frac{\zeta}{\zeta - 2} \right) - \beta (\bar{d})^2 \left( \frac{\zeta}{\zeta - 2}(N_v - 1 - \bar{d}) \right) + \beta, \]

and

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left( \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} \right). \]

By Theorem 2, for any \( \eta > 0 \), we have

\[ \text{Bias}[\kappa] = -\beta(2 - \alpha - \beta)\kappa + (2 - \alpha - \beta) \left[ \alpha(N_v - \kappa - 1) + \beta \right] + \mathcal{O}\left( \frac{1}{(\mathbb{E}Y)^{1/(2+\eta)}} \frac{\mathbb{E}X}{\mathbb{E}Y} \right) \]

as \( N_v \to \infty. \)

(ii) \( \zeta > 2 \)

Note that

\[ \alpha(N_v - 1) + \beta - (\alpha + \beta) \frac{\sum_{i=1}^{N_v} d_i^2}{\sum_{i=1}^{N_v} d_i} \sim -\frac{\beta \bar{d}}{\zeta(\zeta - 2)} - \frac{\beta (\bar{d})^2}{\zeta(\zeta - 2)(N_v - 1 - \bar{d})} + \beta, \]
and
\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = O(d). \]

By Theorem 2, for any \( \eta > 0 \), we have
\[ \text{Bias}[\tilde{\kappa}] = -(2 - \alpha - \beta) \frac{\beta \bar{d}}{\zeta(\xi - 2)} + O(\max\{N_v^{2c-1}, 1\}) \]
\[ = -\beta(2 - \alpha - \beta) \frac{\kappa}{(\zeta - 1)^2} + O(\max\{N_v^{2c-1}, 1\}) \]
as \( N_v \to \infty \).

**B. Proofs of corollaries for variances of the observed branching number**

To prove corollaries for variances of the observed branching number, we first compute
\[ \mathbb{E}\left[ \frac{(X \mathbb{E}Y - Y \mathbb{E}X)^2}{(\mathbb{E}Y)^4} \right]. \]
Note that
\[ \mathbb{E}\left[ \frac{(X \mathbb{E}Y - Y \mathbb{E}X)^2}{(\mathbb{E}Y)^4} \right] = \left( \frac{\mathbb{E}X}{\mathbb{E}Y} \right)^2 \cdot \left[ \frac{\text{Var}X}{(\mathbb{E}X)^2} - 2 \frac{\text{Cov}(X, Y)}{\mathbb{E}X \mathbb{E}Y} + \frac{\text{Var}Y}{(\mathbb{E}Y)^2} \right]. \]

Under Assumption 1, 2 and 4, we have
\[ \mathbb{E}Y = \sum_{i=1}^{N_v} d_i, \]
\[ \mathbb{E}X = (1 - \alpha - \beta)^2 \sum_{i=1}^{N_v} d_i^2 + (2 - \alpha - \beta) \left[ \alpha(N_v - 1) + \beta \sum_{i=1}^{N_v} d_i \right], \]
\[ \text{Var}Y = 2 \beta(2 - \alpha - \beta) \sum_{i=1}^{N_v} d_i, \]
\[ \text{Cov}(X, Y) = 4(\beta - \alpha)(1 - \alpha - \beta)^2 \sum_{i=1}^{N_v} d_i^2 + 2 \left\{ \beta \left[ (1 - \alpha)(1 - 2\alpha) - (1 - 2\beta)(1 - 2\beta) \right] \right. \]
\[ + 2\alpha(N_v - 1) \left[ \beta(1 - \beta) + (1 - \alpha)^2 \right] \} \sum_{i=1}^{N_v} d_i, \]
\[ \text{Var}X = 4(\beta - \alpha)(1 - \alpha - \beta)^3 \sum_{i=1}^{N_v} d_i^3 \]
\[ + 2(1 - \alpha - \beta)^2 \left[ 19\alpha^2 + 9\beta^2 - 6\alpha(1 + 3\beta) - 4\beta + 2\alpha(1 + 4\beta - 5\alpha)N_v \right] \sum_{i=1}^{N_v} d_i^2 \]
\[ + \left\{ (1 - \alpha - \beta) \left[ -38\alpha^3 + 2\alpha^2(17 + 11\beta) + \beta(1 - 6\beta + 6\beta^2) - \alpha(5 + 14\beta^2) \right] \right\}. \]
+ 4\alpha(1 + 11\alpha^2 + 2\beta^2 - \alpha(9 + 5\beta))N_v + 4\alpha^2(2 - 3\alpha + \beta)N_v^2
+ (1 - \alpha)(\alpha + \beta)
+ 2(1 - \alpha - \beta)[3\alpha^2 - \beta(1 - \beta) - \alpha(1 + 2\beta) + \alpha(1 - 2\alpha + \beta)N_v]
+ 4(1 - \alpha - \beta)[-8\alpha^3 + 3\alpha(1 - \beta)\beta + (1 - \beta)^2\beta + 4\alpha^2(1 + \beta)]
- 8\alpha[5\alpha^3 + (1 - \beta)^2\beta - \alpha^2(8 - 3\beta) + \alpha(3 - 2\beta - \beta^2)]N_v
+ 4\alpha^2[2 + 3\alpha^2 - \beta - \beta^2 - \alpha(5 - 2\beta)]N_v^2
+ 2\alpha(1 - \alpha)(\alpha + \beta)(N_v - 2)[1 + 2\alpha(N_v - 2)]
+ \beta[\alpha(\alpha - 3) - \beta(\beta - 3)] + 2\alpha(N_v - 1)[\beta(1 - \beta) + (1 - \alpha)^2]\sum_{i=1}^{N_v} d_i
+ 4(1 - \alpha - \beta)^2[\alpha(1 - \alpha)\sum_{i=1}^{N_v} \sum_{j \neq i} d_id_j I_{\{A_{ij}=0\}}]
+ \beta(1 - \beta)\sum_{i=1}^{N_v} \sum_{j \neq i} d_id_j I_{\{A_{ij}=1\}}.

B.1. Proof of Corollary 5

By edge unbiasedness, we have

$$\alpha = \frac{\beta\bar{d}}{N_v - 1 - \bar{d}} = \Theta\left(\frac{\log N_v}{N_v}\right).$$ (B.1)

By homogeneity, we obtain

$$\sum_{i=1}^{N_v} d_i^2 = \Theta(N_v \log^2(N_v)),
\sum_{i=1}^{N_v} d_i^3 = \Theta(N_v \log^3(N_v)).$$

Besides, by Young’s inequaility, we have

$$\sum_{i=1}^{N_v} \sum_{j \neq i} d_id_j I_{\{A_{ij}=1\}} \leq \left(\sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=1\}}\right)^{1/2} = O(N_v^{3/2} \log^{5/2} N_v),
\sum_{i=1}^{N_v} \sum_{j \neq i} d_id_j I_{\{A_{ij}=0\}} \leq \left(\sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=0\}}\right)^{1/2} = O(N_v^2 \log^2 N_v).$$
Thus, we obtain
\[
\frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta(\log N_v),
\]
\[
\text{Var}(X) \left( \frac{\mathbb{E}X}{\mathbb{E}Y} \right)^2 = \mathcal{O}\left( \frac{1}{N_v^{1/2} \log^{3/2} N_v} \right),
\]
\[
\text{Cov}(X, Y) \left( \frac{\mathbb{E}X}{\mathbb{E}Y} \right)^2 = \Theta\left( \frac{1}{N_v \log N_v} \right),
\]
\[
\text{Var}(Y) \left( \frac{\mathbb{E}Y}{\mathbb{E}Y} \right)^2 = \Theta\left( \frac{1}{N_v \log N_v} \right).
\]

Then, we have
\[
\mathbb{E}\left[ \left( \frac{X \mathbb{E}Y - Y \mathbb{E}X}{\mathbb{E}Y} \right)^2 \left( \frac{\mathbb{E}Y}{\mathbb{E}Y} \right)^4 \right] = \mathcal{O}\left( \left( \frac{\log N_v}{N_v} \right)^{1/2} \right).
\]

By Theorem 4,
\[
\text{Var}[\tilde{\kappa}] = \mathcal{O}\left( \left( \frac{\log N_v}{N_v} \right)^{1/2} \right) \text{ as } N_v \to \infty.
\]

### B.2. Proof of Corollary 6

By edge unbiasedness, we have
\[
\alpha = \frac{\beta \bar{d}}{N_v - 1 - d} = \Theta\left( \frac{\log N_v}{N_v} \right). \tag{B.2}
\]

Next, we compute \(\sum_{i=1}^{N_v} d_i^3\) for different values of \(\zeta\).
\[
\int_{d_L}^{N_v - 1} x^3 \cdot \frac{\zeta d_L}{1 - \left( \frac{d_L}{N_v - 1} \right)^{\zeta}} x^{-(\zeta+1)} dx = \begin{cases} 
\frac{\log \left( \frac{N_v - 1}{d_L} \right)}{1 - \left( \frac{d_L}{N_v - 1} \right)^{\zeta}}, & \text{if } \zeta = 3 \\
\frac{\zeta d_L^3}{1 - \left( \frac{d_L}{N_v - 1} \right)^{\zeta-3}} \cdot \frac{1 - \left( \frac{d_L}{N_v - 1} \right)^{\zeta-3}}{\zeta - 3}, & \text{if } \zeta > 3.
\end{cases}
\]
\[
\tag{B.3}
\]

Thus, we obtain
\[
\sum_{i=1}^{N_v} d_i^3 = \begin{cases} 
\Theta(N_v^3 \log N_v), & \text{if } 0 < \zeta < 1 \\
\Theta(N_v^3), & \text{if } \zeta = 1 \\
\Theta(N_v^{4-\zeta} \log^\zeta N_v), & \text{if } 1 < \zeta < 3 \\
\Theta(N_v \log^4 N_v), & \text{if } \zeta = 3 \\
\Theta(N_v \log^3 N_v), & \text{if } \zeta > 3.
\end{cases}
\]
Equation (A.3) leads to

\[
\sum_{i=1}^{N_v} d_i^2 = \begin{cases} 
\Theta(N_v^2 \log N_v), & \text{if } 0 < \zeta < 1 \\
\Theta(N_v^2), & \text{if } \zeta = 1 \\
\Theta(N_v^{3-\zeta} \log^\zeta N_v), & \text{if } 1 < \zeta < 2 \\
\Theta(N_v \log^3 N_v), & \text{if } \zeta = 2 \\
\Theta(N_v \log^2 N_v), & \text{if } \zeta > 2.
\end{cases}
\]

In addition, by Young’s inequality, we have

\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=1\}} \leq \left( \sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=1\}} \right)^{1/2},
\]

\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=0\}} \leq \left( \sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=0\}} \right)^{1/2}. \tag{B.4}
\]

Thus, we obtain

\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=1\}} = \begin{cases} 
\mathcal{O}(N_v^{5/2} \log^{3/2} N_v), & \text{if } 0 < \zeta < 1 \\
\mathcal{O}(N_v^{5/2} \log^{1/2} N_v), & \text{if } \zeta = 1 \\
\mathcal{O}(N_v^{7/2-\zeta} \log^{\zeta+1/2} N_v), & \text{if } 1 < \zeta < 2 \\
\mathcal{O}(N_v^{3/2} \log^{7/2} N_v), & \text{if } \zeta = 2 \\
\mathcal{O}(N_v^{3/2} \log^{5/2} N_v), & \text{if } \zeta > 2,
\end{cases}
\]

and

\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=0\}} = \begin{cases} 
\mathcal{O}(N_v^3 \log N_v), & \text{if } 0 < \zeta < 1 \\
\mathcal{O}(N_v^3), & \text{if } \zeta = 1 \\
\mathcal{O}(N_v^{4-\zeta} \log^\zeta N_v), & \text{if } 1 < \zeta < 2 \\
\mathcal{O}(N_v^2 \log^3 N_v), & \text{if } \zeta = 2 \\
\mathcal{O}(N_v^2 \log^2 N_v), & \text{if } \zeta > 2.
\end{cases}
\]

(i) \(0 < \zeta < 1\)

Note that

\[
\frac{\mathbb{E} X}{\mathbb{E} Y} = \Theta\left(\frac{N_v}{N_v}\right), \\
\frac{\text{Var} X}{(\mathbb{E} X)^2} = \Theta\left(\frac{1}{N_v \log N_v}\right), \\
\frac{\text{Cov}(X,Y)}{\mathbb{E} X \mathbb{E} Y} = \Theta\left(\frac{1}{N_v \log N_v}\right), \\
\frac{\text{Var} Y}{(\mathbb{E} Y)^2} = \Theta\left(\frac{1}{N_v \log N_v}\right).
\]
Then, we have

$$E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] = O \left( \frac{N_v}{\log N_v} \right).$$

By Theorem 4,

$$\text{Var}[\tilde{\kappa}] = O \left( \frac{N_v}{\log N_v} \right) \text{ as } N_v \to \infty.$$

(ii) $\zeta = 1$

Note that

$$\frac{EX}{EY} = \Theta \left( \frac{N_v}{\log N_v} \right),$$

$$\text{Var}X \left( \frac{EX}{EY} \right)^2 = \Theta \left( \frac{1}{N_v} \right),$$

$$\frac{\text{Cov}(X,Y)}{EXEY} = \Theta \left( \frac{1}{N_v \log N_v} \right),$$

$$\frac{\text{Var}Y}{(EY)^2} = \Theta \left( \frac{1}{N_v \log N_v} \right).$$

Then, we have

$$E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] = O \left( \frac{N_v}{\log^2 N_v} \right).$$

By Theorem 4,

$$\text{Var}[\tilde{\kappa}] = O \left( \frac{N_v}{\log^2 N_v} \right) \text{ as } N_v \to \infty.$$

(iii) $1 < \zeta < 2$

Note that

$$\frac{EX}{EY} = \Theta \left( N_v^{2-\zeta} \cdot \log^{-\zeta+1} N_v \right),$$

$$\text{Var}X \left( \frac{EX}{EY} \right)^2 = \Theta \left( \frac{1}{N_v^{2-\zeta} \cdot \log^{\zeta} N_v} \right),$$

$$\frac{\text{Cov}(X,Y)}{EXEY} = \Theta \left( \frac{1}{N_v \log N_v} \right),$$

$$\frac{\text{Var}Y}{(EY)^2} = \Theta \left( \frac{1}{N_v \log N_v} \right).$$

Then, we have

$$E \left[ \frac{(XEY - YEX)^2}{(EY)^4} \right] = O \left( \left( \frac{N_v}{\log N_v} \right)^{2-\zeta} \right).$$
By Theorem 4,
\[ \text{Var}[\tilde{\kappa}] = O\left(\left(\frac{N_v}{\log N_v}\right)^{2-\zeta}\right) \text{ as } N_v \to \infty. \]

(iv) $$\zeta = 2$$

Note that

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\log^2 N_v\right), \]
\[ \text{Var}X = \Theta\left(\frac{1}{\log^4 N_v}\right), \]
\[ \frac{\text{Cov}(X, Y)}{\mathbb{E}X \mathbb{E}Y} = \Theta\left(\frac{1}{N_v \log N_v}\right), \]
\[ \text{Var}Y = \Theta\left(\frac{1}{N_v \log N_v}\right). \]

Then, we have
\[ \mathbb{E}\left[\left(\frac{X \mathbb{E}Y - Y \mathbb{E}X}{\mathbb{E}Y}\right)^2\right] = O\left(1\right). \]

By Theorem 4,
\[ \text{Var}[\tilde{\kappa}] = O\left(1\right) \text{ as } N_v \to \infty. \]

(v) $$2 < \zeta < 5/2$$

Note that

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\log N_v\right), \]
\[ \text{Var}X = \Theta\left(\frac{1}{N_v^{5-2} \log^{4-\zeta} N_v}\right), \]
\[ \frac{\text{Cov}(X, Y)}{\mathbb{E}X \mathbb{E}Y} = \Theta\left(\frac{1}{N_v \log N_v}\right), \]
\[ \text{Var}Y = \Theta\left(\frac{1}{N_v \log N_v}\right). \]

Then, we have
\[ \mathbb{E}\left[\left(\frac{X \mathbb{E}Y - Y \mathbb{E}X}{\mathbb{E}Y}\right)^2\right] = O\left(\left(\frac{\log N_v}{N_v}\right)^{\zeta-2}\right). \]

By Theorem 4,
\[ \text{Var}[\tilde{\kappa}] = O\left(\left(\frac{\log N_v}{N_v}\right)^{\zeta-2}\right) \text{ as } N_v \to \infty. \]

(vi) $$\zeta \geq 5/2$$
Note that
\[
\frac{EX}{EY} = \Theta(\log N_v),
\]
\[
\frac{\text{Var}X}{(EX)^2} = O\left(\frac{1}{N_v^{1/2} \log^{3/2} N_v}\right),
\]
\[
\frac{\text{Cov}(X,Y)}{EXEY} = \Theta\left(\frac{1}{N_v \log N_v}\right),
\]
\[
\frac{\text{Var}Y}{(EY)^2} = \Theta\left(\frac{1}{N_v \log N_v}\right).
\]

Then, we have
\[
\mathbb{E}\left[\left(\frac{X \cdot EY - Y \cdot EX}{(EY)^4}\right)^2\right] = O\left(\left(\frac{\log N_v}{N_v}\right)^{1/2}\right).
\]
By Theorem 4,
\[
\text{Var}[\tilde{\kappa}] = O\left(\left(\frac{\log N_v}{N_v}\right)^{1/2}\right) \text{ as } N_v \to \infty.
\]

### B.3. Proof of Corollary 7

By edge unbiasedness, we have
\[
\alpha = \frac{\beta \bar{d}}{N_v - 1 - d} = \Theta(N_v^{-1}). \tag{B.5}
\]
By homogeneity, we obtain
\[
\sum_{i=1}^{N_v} d_i^2 = \Theta(N_v^{2c+1}),
\]
\[
\sum_{i=1}^{N_v} d_i^3 = \Theta(N_v^{3c+1}).
\]
Besides, by Young’s inequality, we have
\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=1\}} \leq \left(\sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=1\}}\right)^{1/2} = O(N_v^{(5c+3)/2}),
\]
\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{\{A_{ij}=0\}} \leq \left(\sum_{i=1}^{N_v} \sum_{j \neq i} d_i^2 d_j^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N_v} \sum_{j \neq i} I_{\{A_{ij}=0\}}\right)^{1/2} = O(N_v^{2c+2}).
\]
Thus, we obtain
\[
\frac{EX}{EY} = \Theta(N_v^c),
\]
\[
\frac{\text{Var}X}{(EX)^2} = \mathcal{O}\left(\frac{1}{N_v^{(3c+1)/2}}\right),
\]
\[
\frac{\text{Cov}(X, Y)}{EXEY} = \Theta\left(\frac{1}{N_v^{c+1}}\right),
\]
\[
\frac{\text{Var}Y}{(EY)^2} = \Theta\left(\frac{1}{N_v^{c+1}}\right).
\]

Then, we have
\[
\mathbb{E}\left[\frac{(XEX - YEX)^2}{(EY)^4}\right] = \mathcal{O}(N_v^{(c-1)/2}).
\]

By Theorem 4,
\[
\text{Var}[\tilde{\kappa}] = \mathcal{O}(N_v^{(c-1)/2}) \text{ as } N_v \to \infty.
\]

**B.4. Proof of Corollary 8**

By edge unbiasedness, we have
\[
\alpha = \frac{\beta d}{N_v - 1 - d} = \Theta(N_v^{c-1}). \quad \text{(B.6)}
\]

By equation (B.3), we have
\[
\sum_{i=1}^{N_v} d_i^3 = \begin{cases} 
\Theta(N_v^{c+3}), & \text{if } 0 < \zeta < 1 \\
\Theta(N_v^{c+3}/\log N_v), & \text{if } \zeta = 1 \\
\Theta(N_v^{3-c+c\zeta}), & \text{if } 1 < \zeta < 3 \\
\Theta(N_v^{3c+1} \cdot \log N_v), & \text{if } \zeta = 3 \\
\Theta(N_v^{3c+1}), & \text{if } \zeta > 3.
\end{cases}
\]

Equation (A.3) leads to
\[
\sum_{i=1}^{N_v} d_i^2 = \begin{cases} 
\Theta(N_v^{c+2}), & \text{if } 0 < \zeta < 1 \\
\Theta(N_v^{c+2}/\log N_v), & \text{if } \zeta = 1 \\
\Theta(N_v^{3-c+c\zeta}), & \text{if } 1 < \zeta < 2 \\
\Theta(N_v^{2c+1} \cdot \log N_v), & \text{if } \zeta = 2 \\
\Theta(N_v^{2c+1}), & \text{if } \zeta > 2.
\end{cases}
\]

In addition, by equation (B.4), we obtain
\[
\sum_{i=1}^{N_v} \sum_{j \neq i} d_i d_j I_{(A_{ij} = 1)} = \begin{cases} 
\mathcal{O}(N_v^{(3c+5)/2}), & \text{if } 0 < \zeta < 1 \\
\mathcal{O}(N_v^{(3c+5)/2}/\log N_v), & \text{if } \zeta = 1 \\
\mathcal{O}(N_v^{7/2-c+c(1/2)}), & \text{if } 1 < \zeta < 2 \\
\mathcal{O}(N_v^{(5c+3)/2} \cdot \log N_v), & \text{if } \zeta = 2 \\
\mathcal{O}(N_v^{(5c+3)/2}), & \text{if } \zeta > 2.
\end{cases}
\]
and

$$\sum_{i=1}^{N_v} \sum_{j \neq i} d_id_j I_{\{A_{ij}=0\}} = \begin{cases} \mathcal{O}(N_v^{c+3}), & \text{if } 0 < \zeta < 1 \\ \mathcal{O}(N_v^{c+3}/\log N_v), & \text{if } \zeta = 1 \\ \mathcal{O}(N_v^{-\zeta+c\zeta}), & \text{if } 1 < \zeta < 2 \\ \mathcal{O}(N_v^{2c+2} \cdot \log N_v), & \text{if } \zeta = 2 \\ \mathcal{O}(N_v^{2c+2}), & \text{if } \zeta > 2. \end{cases}$$

(i) $0 < \zeta < 1$

Note that

$$\frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left( N_v \right), \quad \frac{\text{Var}X}{\left(\mathbb{E}X\right)^2} = \Theta\left( \frac{1}{N_v^{c+1}} \right), \quad \frac{\text{Cov}(X,Y)}{\mathbb{E}X\mathbb{E}Y} = \Theta\left( \frac{1}{N_v^{c+1}} \right), \quad \frac{\text{Var}Y}{\left(\mathbb{E}Y\right)^2} = \Theta\left( \frac{1}{N_v^{c+1}} \right).$$

Then, we have

$$\mathbb{E}\left[ \frac{(X\mathbb{E}Y - Y\mathbb{E}X)^2}{\left(\mathbb{E}Y\right)^4} \right] = \mathcal{O}\left( N_v^{1-c} \right).$$

By Theorem 4,

$$\text{Var}[\hat{\kappa}] = \mathcal{O}\left( N_v^{1-c} \right) \quad \text{as} \quad N_v \to \infty.$$  

(ii) $\zeta = 1$

Note that

$$\frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left( \frac{N_v}{\log N_v} \right), \quad \frac{\text{Var}X}{\left(\mathbb{E}X\right)^2} = \Theta\left( \frac{\log N_v}{N_v^{c+1}} \right), \quad \frac{\text{Cov}(X,Y)}{\mathbb{E}X\mathbb{E}Y} = \Theta\left( \frac{1}{N_v^{c+1}} \right), \quad \frac{\text{Var}Y}{\left(\mathbb{E}Y\right)^2} = \Theta\left( \frac{1}{N_v^{c+1}} \right).$$

Then, we have

$$\mathbb{E}\left[ \frac{(X\mathbb{E}Y - Y\mathbb{E}X)^2}{\left(\mathbb{E}Y\right)^4} \right] = \mathcal{O}\left( \frac{N_v^{1-c}}{\log N_v} \right).$$
By Theorem 4,

\[ \text{Var}[\tilde{\kappa}] = O\left(\frac{N_v^{1-c}}{\log N_v} \right) \text{ as } N_v \to \infty. \]

(iii) \( 1 < \zeta < 2 \)

Note that

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\frac{N_v^{2-\zeta+c(\zeta-1)}}{N_v} \right), \]

\[ \text{Var} \frac{X}{(\mathbb{E}X)^2} = \Theta\left(\frac{1}{N_v^{2-\zeta+c(\zeta-1)}} \right), \]

\[ \frac{\text{Cov}(X, Y)}{\mathbb{E}X \mathbb{E}Y} = \Theta\left(\frac{1}{N_v^{c+1}} \right), \]

\[ \frac{\text{Var} Y}{(\mathbb{E}Y)^2} = \Theta\left(\frac{1}{N_v^{c+1}} \right). \]

Then, we have

\[ \mathbb{E} \left[ \frac{(X \mathbb{E} Y - Y \mathbb{E} X)^2}{(\mathbb{E} Y)^4} \right] = O\left(\frac{1}{N_v^{2-\zeta+c(\zeta-1)}} \right). \]

By Theorem 4,

\[ \text{Var}[\tilde{\kappa}] = O\left(\frac{1}{N_v^{2-\zeta+c(\zeta-1)}} \right) \text{ as } N_v \to \infty. \]

(iv) \( \zeta = 2 \)

Note that

\[ \frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\frac{N_v^c \cdot \log N_v}{N_v} \right), \]

\[ \text{Var} \frac{X}{(\mathbb{E}X)^2} = \Theta\left(\frac{1}{N_v^{2c} \cdot \log^2 N_v} \right), \]

\[ \frac{\text{Cov}(X, Y)}{\mathbb{E}X \mathbb{E}Y} = \Theta\left(\frac{1}{N_v^{c+1}} \right), \]

\[ \frac{\text{Var} Y}{(\mathbb{E}Y)^2} = \Theta\left(\frac{1}{N_v^{c+1}} \right). \]

Then, we have

\[ \mathbb{E} \left[ \frac{(X \mathbb{E} Y - Y \mathbb{E} X)^2}{(\mathbb{E} Y)^4} \right] = O\left(\frac{1}{N_v^{c+1}} \right). \]

By Theorem 4,

\[ \text{Var}[\tilde{\kappa}] = O\left(\frac{1}{N_v^{c+1}} \right) \text{ as } N_v \to \infty. \]

(v) \( 2 < \zeta < 5/2 \)
Note that

\[
\frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\frac{N_v^{c}}{N_v}\right),
\]

\[
\text{Var}_X\left(\frac{\mathbb{E}X}{(\mathbb{E}X)^2} = \Theta\left(\frac{1}{N_v^{c-2+\epsilon(4-\zeta)}}\right),
\]

\[
\frac{\text{Cov}(X, Y)}{\mathbb{E}X\mathbb{E}Y} = \Theta\left(\frac{1}{N_v^{c+1}}\right),
\]

\[
\text{Var}_Y\left(\frac{\mathbb{E}Y}{(\mathbb{E}Y)^2} = \Theta\left(\frac{1}{N_v^{c+1}}\right).
\]

Then, we have

\[
\mathbb{E}\left[\frac{(X\mathbb{E}Y - Y\mathbb{E}X)^2}{(\mathbb{E}Y)^4}\right] = \mathcal{O}\left(\frac{N_v^{(2-\zeta)(1-c)}}{N_v}\right).
\]

By Theorem 4,

\[
\text{Var}[\hat{\kappa}] = \mathcal{O}\left(\frac{N_v^{(2-\zeta)(1-c)}}{N_v}\right) \text{ as } N_v \to \infty.
\]

(vi) \(\zeta \geq 5/2\)

Note that

\[
\frac{\mathbb{E}X}{\mathbb{E}Y} = \Theta\left(\frac{N_v^{c}}{N_v}\right),
\]

\[
\text{Var}_X\left(\frac{\mathbb{E}X}{(\mathbb{E}X)^2} = \mathcal{O}\left(\frac{1}{N_v^{(3c+1)/2}}\right),
\]

\[
\frac{\text{Cov}(X, Y)}{\mathbb{E}X\mathbb{E}Y} = \Theta\left(\frac{1}{N_v^{c+1}}\right),
\]

\[
\text{Var}_Y\left(\frac{\mathbb{E}Y}{(\mathbb{E}Y)^2} = \Theta\left(\frac{1}{N_v^{c+1}}\right).
\]

Then, we have

\[
\mathbb{E}\left[\frac{(X\mathbb{E}Y - Y\mathbb{E}X)^2}{(\mathbb{E}Y)^4}\right] = \mathcal{O}\left(\frac{N_v^{(c-1)/2}}{N_v}\right).
\]

By Theorem 4,

\[
\text{Var}[\hat{\kappa}] = \mathcal{O}\left(\frac{N_v^{(c-1)/2}}{N_v}\right) \text{ as } N_v \to \infty.
\]