The QCD phase diagram at nonzero baryon, isospin and strangeness chemical potentials: Results from a hadron resonance gas model

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We use a hadron resonance gas model to study the QCD phase diagram at nonzero temperature, baryon, isospin and strangeness chemical potentials. We determine the temperature of the transition from the hadronic phase to the quark gluon plasma phase using two different methods. We find that there are numerous types of color superconducting phases that emerge as the baryon chemical potential increases. Second, the quark-antiquark condensate for the u and d quark sectors decoupled. There are phases where the u and d quark sectors are decoupled. These results need to be confirmed by other methods, in particular on the lattice.

We study the QCD phase diagram at high temperature and nonzero baryon, isospin and strangeness chemical potentials using the hadron resonance gas model. It has been shown both experimentally and on the lattice that the hadronic phase is very well described by a weakly interacting hadron resonance gas. We use two different methods to determine the location of the transition. First, it has been found on the lattice that the phase transition that separates the hadronic phase from the quark gluon plasma phase corresponds to a surface of constant energy density: $\epsilon \sim 0.5 \text{–} 1.0 \text{ GeV/fm}^3$. Second, the quark-antiquark condensate for the u and d quarks should almost
vanish at the transition temperature. In this article, we determine the critical temperature, \( T_c \), at nonzero baryon, isospin and strangeness chemical potentials by using both approaches. We compute the surfaces of constant energy density as well as the quark-antiquark condensate in a hadron resonance gas model at nonzero temperature, baryon, isospin and strangeness chemical potentials. We show that both methods agree qualitatively as well as quantitatively. We find that the critical surface has small curvature, and that the critical temperature slowly decreases when either the baryon, the isospin, or the strangeness chemical potentials are increased.

II. THE MODEL

We assume that the pressure in the hadronic phase is given by the contributions of all the hadron resonances up to 2 GeV treated as a free gas, as in \[25\]. All the thermodynamic observables can be derived from the pressure since

\[
p = \lim_{V \to \infty} \frac{T}{V} \ln Z(T, \mu_B, \mu_I, \mu_S, V),
\]

where \( Z(T, \mu_B, \mu_I, \mu_S, V) \) is the grand canonical partition function in a finite volume \( V \), at nonzero temperature, \( T \), baryon chemical potential, \( \mu_B \), isospin chemical potential, \( \mu_I \), and strangeness chemical potential, \( \mu_S \). The energy density is given by

\[
\epsilon = T \frac{\partial p}{\partial T} - p + \mu_B \frac{\partial p}{\partial \mu_B} + \mu_I \frac{\partial p}{\partial \mu_I} + \mu_S \frac{\partial p}{\partial \mu_S}.
\]

For a quark \( q \) with mass \( m_q \), the quark-antiquark condensate is given by

\[
\langle \bar{q}q \rangle = \frac{\partial p}{\partial m_q}.
\]

At nonzero temperature, the contributions of massive states are exponentially suppressed \( \sim \exp(-m_H/T) \). Their interactions are also exponentially suppressed \( \sim \exp(-(m_H + m_d)/T) \). Therefore this approximation should be valid at low enough temperatures. However, since we are studying QCD at temperatures up to \( \sim 200 \) MeV, the lightness of the pions could be a problem, since \( m_\pi \sim 140 \) MeV. The hadron resonance gas model should be a good approximation for the other hadrons since they have a mass larger than \( \sim 500 \) MeV. The physics of pions at nonzero temperature has been extensively studied in chiral perturbation theory \[26\].

The pions’ contributions to the pressure have been calculated up to three loops in chiral perturbation theory \[26\]. It has been shown that the free gas approximation and chiral perturbation theory agree at the one loop level, and that, in chiral perturbation theory, the two-loop corrections to the pressure are below a few percents of the one-loop contributions for temperatures under 200 MeV \[26\]. Thus the hadron resonance gas model is a good approximation also for the pions. The hadron resonance gas model has already been used in the literature and has been shown to give a very good description of the hadronic phase and of the critical temperature \[26\].

In the free gas approximation, the contribution to the pressure due to a particle of mass \( m_H \), baryon charge \( B \), isospin \( I \), strangeness \( S \), and degeneracy \( g \) is given by

\[
\Delta p = g m_H^2 T^2 \sum_{n=1}^{\infty} \frac{(-\eta)^{n+1}}{2n^2 \pi^2} \exp \left( \frac{n(B \mu_B - I_3 \mu_I - S \mu_S)}{T} \right) K_2 \left( \frac{nm_H}{T} \right),
\]

where \( \eta = +1 \) for fermions and \( \eta = -1 \) for bosons, and \( K_n(x) \) is the modified Bessel function. This particle’s contribution to the energy density is given by

\[
\Delta \epsilon = g m_H^2 T \sum_{n=1}^{\infty} \frac{(-\eta)^{n+1}}{2n^2 \pi^2} \exp \left( \frac{n(B \mu_B - I_3 \mu_I - S \mu_S)}{T} \right) \left[ 3TK_2 \left( \frac{nm_H}{T} \right) + nm_H K_1 \left( \frac{nm_H}{T} \right) \right],
\]

and its contribution to the quark-antiquark condensate is given by

\[
\Delta \langle \bar{q}q \rangle = -g m_H^2 \frac{\partial m_H}{\partial m_q} \sum_{n=1}^{\infty} \frac{(-\eta)^{n+1}}{2n^2 \pi^2} \exp \left( \frac{n(B \mu_B - I_3 \mu_I - S \mu_S)}{T} \right) K_1 \left( \frac{nm_H}{T} \right).
\]

In order to compute \( \Delta \langle \bar{q}q \rangle \), we need to know \( m_H \) as a function of \( m_u \) or \( m_d \). We make two assumptions in order to compute \( \Delta \langle \bar{q}q \rangle \). First, we assume that the Gell-Mann–Oakes–Renner relation is valid

\[
F_\pi^2 m_\pi^2 = (m_u + m_d) \langle \bar{q}q \rangle_0,
\]

where \( F_\pi \) is the pion decay constant.
where $F_\pi = 93$ MeV is the pion decay constant, and $\langle \bar{q}q \rangle_0 = \langle \bar{u}u \rangle_0 = \langle \bar{d}d \rangle_0$ is the quark-antiquark condensate at zero temperature and chemical potentials. Second, based on lattice results, we assume that the pion mass dependence of the hadron masses is given by

$$\frac{\partial m_H}{\partial (m_\pi^2)} \approx \frac{A}{m_H},$$

where $0.9 \lesssim A \lesssim 1.2$. Therefore, combining (4) and (5), we assume that

$$\frac{\partial m_H}{\partial m_q} \approx \frac{A\langle \bar{q}q \rangle_0}{F_\pi^2 m_H}.$$ 

Notice that since the hadron spectrum is isospin symmetric, we have that $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle$ in the hadron resonance model at any temperature, baryon, isospin and strangeness chemical potentials. Therefore, the rich structure of the phase diagram found in [21, 22, 23] cannot be seen in this model. Finally at fixed $T$, it can be readily seen from (4) and (6) that an increase in either $\mu_B$, or $\mu_I$, or $\mu_S$ will increase $\epsilon$ and decrease $\langle \bar{q}q \rangle$. Thus using either $\epsilon$ or $\langle \bar{q}q \rangle$ as a criterion to determine the critical temperature, we find that an increase in either $\mu_B$, or $\mu_I$, or $\mu_S$ decreases $T_c$, and that at fixed $\mu_B$ an increase in $\mu_I$ or $\mu_S$ results in a decrease of $T_c$ as well.

III. RESULTS

A. Energy density criterion

Lattice simulations have shown that the transition from the hadronic phase to the quark gluon plasma phase takes place at a constant energy density $\epsilon \simeq 0.5 - 1.0$ GeV/fm$^3$ [25]. We use this criterion to determine the critical temperature as a function of baryon, isospin and strangeness chemical potentials. Our results for the critical temperature as a function of $\mu_B$ at fixed $\mu_I$ and $\mu_S$ are shown in Figure 1. We find that this criterion constrains the critical temperature in a band of $\sim 15$ MeV. At zero chemical potentials, we find that $T_c = 176 \pm 8$ MeV, which is in good agreement with lattice simulations [8, 9, 10, 11]. As expected, the critical temperature decreases as $\mu_B$ increases, as expected, but the decrease is slow. At the accuracy we can achieve using this method, an increase in $\mu_I$ does indeed decrease the critical temperature at fixed $\mu_B$, but this effect is small at best. The decrease of the critical temperature is more important when $\mu_S$ is increased. In Figure 2, we compare the critical temperature as a function of $\mu_B$ at $\mu_I = \mu_S = 0$ with the critical temperature as a function of $\mu_I$ at $\mu_B = \mu_S = 0$. Notice that we limit ourselves to $\mu_I \lesssim m_\pi$ and $\mu_S \lesssim m_K$ in order to avoid the pion and kaon superfluid phases [18, 24, 28]. We find that the critical temperature curves are almost identical in both cases. This is in agreement with results from the lattice [12, 13, 14, 15, 16].

We can fit our result for the critical temperature as a function of $\mu_B$, $\mu_I$ and $\mu_S$. By construction, since the pressure is an even function of $\mu_I$ in the hadron resonance model, the critical temperature is also even in $\mu_I$. We find

$$\frac{T_c}{T_0} = 1 - 0.021(3) \left( \frac{\mu_B}{T_0} \right)^2 - 0.039(1) \left( \frac{\mu_I}{T_0} \right)^2 - 0.037(2) \left( \frac{\mu_S}{T_0} \right)^2 - 0.031(3) \left( \frac{\mu_B \mu_S}{T_0^2} \right) + \cdots,$$

where $T_0$ is the critical temperature at zero chemical potentials. The fit is excellent, with a linear regression coefficient $R^2 = 0.994$.

B. Quark-antiquark condensate criterion

The critical temperature can also be computed from the quark-antiquark condensate. Indeed $\langle \bar{q}q \rangle$ is of the order of the light quark masses at the phase transition and therefore almost vanishes. We determine the critical temperature by finding the point where $\langle \bar{q}q \rangle = 0$ in the hadron resonance gas model. We obtain a range of critical temperatures, since in the relation [9] the constant $A \simeq 0.9 - 1.2$ is not precisely known [25]. Our results for the critical temperature as a function of $\mu_B$ at fixed $\mu_I$ and $\mu_S$ using $\langle \bar{q}q \rangle$ are shown in Figure 3. We find that this criterion constrains the critical temperature to a band of $\sim 15$ MeV. At zero chemical potentials, we find that $T_c = 185 \pm 6$ MeV, which is in good agreement both with the result obtained above using the $\epsilon$-criterion, as well as with lattice simulations [8, 9, 10, 11]. As expected, the critical temperature decreases as $\mu_B$ increases. As in the previous method, we find that an increase in the isospin chemical potential might reduce the critical temperature, but not in a significant way. The strangeness chemical potential has a stronger effect on the critical temperature. In Figure 4, we compare the critical
FIG. 1: Critical temperature as a function of $\mu_B$ determined by lines of constant energy density: $\epsilon \simeq 0.5 - 1.0$ GeV/fm$^3$. In the upper two plots $\mu_S = 0$, the dark shading with full curves corresponds to $\mu_I = 0$, and the light shading with dashed curves corresponds to $\mu_I = 100$ MeV. In the lower two plots $\mu_I = 0$, the dark shading with full curves corresponds to $\mu_S = 0$, and the light shading with dashed curves corresponds to $\mu_S = 200$ MeV. $T_0$ is the critical temperature at zero chemical potentials.

FIG. 2: Critical temperature as a function of $\mu_B$ at $\mu_I = \mu_S = 0$ (dark shading with full curves), and as a function of $\mu_I$ at $\mu_B = \mu_S = 0$ (light shading with dashed curves), determined by lines of constant energy density: $\epsilon \simeq 0.5 - 1.0$ GeV/fm$^3$. $T_0$ is the critical temperature at zero chemical potentials.

temperature as a function of $\mu_B$ at $\mu_I = \mu_S = 0$ with the critical temperature as a function of $\mu_I$ at $\mu_B = \mu_S = 0$. We find that the critical temperature curves are almost identical in both cases.

We can fit our result for the critical temperature as a function of $\mu_B$, $\mu_I$ and $\mu_S$. We find

$$
\frac{T_c}{T_0} = 1 - 0.017(1) \left( \frac{\mu_B}{T_0} \right)^2 - 0.109(4) \left( \frac{\mu_I}{T_0} \right)^2 - 0.032(2) \left( \frac{\mu_S}{T_0} \right)^2 - 0.024(2) \frac{\mu_B \mu_S}{T_0^2} + \cdots,
$$

(11)

where $T_0$ is the critical temperature at zero chemical potentials. The fit is excellent, with a linear regression coefficient.
FIG. 3: Critical temperature as a function of $\mu_q$ determined by $\langle \bar{q}q \rangle = 0$. In the upper two plots $\mu_S = 0$, the dark shading with full curves corresponds to $\mu_I = 0$, and the light shading with dashed curves corresponds to $\mu_I = 100$ MeV. In the lower two plots $\mu_I = 0$, the dark shading with full curves corresponds to $\mu_S = 0$, and the light shading with dashed curves corresponds to $\mu_S = 200$ MeV. $T_0$ is the critical temperature at zero chemical potentials.

FIG. 4: Critical temperature as a function of $\mu_B$ at $\mu_I = \mu_S = 0$ (dark shading with full curves), and as a function of $\mu_I$ at $\mu_B = \mu_S = 0$ (light shading with dashed curves), determined by $\langle \bar{q}q \rangle = 0$. $T_0$ is the critical temperature at zero chemical potentials.

$R^2 = 0.991$.

Finally we can compare the critical temperatures obtained using these two different approaches: constant energy density and disappearance of the quark-antiquark condensate. We present our results in Figure 5 at nonzero $\mu_B$ with $\mu_I = \mu_S = 0$, nonzero $\mu_I$ with $\mu_B = \mu_S = 0$, and nonzero $\mu_B$ with $\mu_I = \mu_S = 0$, respectively. We find that the critical temperature at zero chemical potentials, $T_0$, is lower when we use the $\epsilon$-criterion than when we use the $\langle \bar{q}q \rangle$-criterion. If the critical curve is normalized with respect to $T_0$, we find that the two methods are in very good agreement. If we compare the fits (10) and (11), we find that the $\mu_B$ and $\mu_S$ coefficients are very close in both cases,
FIG. 5: Comparison of the critical temperatures as a function of $\mu_B$ at $\mu_I = 0$ and $\mu_S = 0$ (upper panel), as a function of $\mu_I$ at $\mu_B = 0$ and $\mu_S = 0$ (lower left panel), and as a function of $\mu_S$ at $\mu_B = 0$ and $\mu_I = 0$ (lower right panel) obtained using the energy density method (dark shading with full curves) and the quark-antiquark condensate method (light shading with dashed curves). $T_0$ is the critical temperature at zero chemical potentials.

whereas the $\mu_I$ coefficients almost differ by a factor of three. However, this large difference in the coefficients leads to critical temperatures that only differ by a few percents in the region of interest. We therefore conclude that both methods yield critical temperatures that are in very good agreement.

IV. CONCLUSION

We have used the hadron resonance gas model to determine the temperature of the transition from the hadronic phase to the quark gluon plasma phase as a function of baryon, isospin and strangeness chemical potentials. This is of interest for heavy ion collision experiments, since baryon, isospin and strangeness chemical potentials are nonzero in this case. We have used two different methods to determine the critical temperature. The first one relies on the observation on the lattice that the quark gluon plasma phase emerges at a constant energy density [25]. The second method is based on the fact that the quark-antiquark condensate for the light quarks should almost disappear at the transition between the hadronic phase and the quark gluon plasma phase. We find that the critical temperatures found in both methods are in very good agreement.

In the hadron resonance gas model, the critical temperature decreases as the baryon, isospin, or strangeness chemical potentials increase, albeit slowly. In agreement with recent lattice simulations [12, 13, 14, 15, 18] and several models [21, 22, 23], we find that the critical temperature as a function of the quark chemical potential at zero isospin chemical potential is almost identical to the critical temperature as a function of isospin chemical potential at zero quark chemical potential. We also find that the critical temperature decreases slightly when the isospin chemical potential is increased at fixed baryon chemical potential. This might be important for heavy ion collision experiments: A choice of different isotopes should reduce the critical temperature that separates the hadronic phase from the quark gluon plasma phase.
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