PSEUDO MAURER-CARTAN PERTURBATION ALGEBRA AND PSEUDO PERTURBATION LEMMA

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To Nodar Berikashvili

Abstract. We introduce the pseudo Maurer-Cartan perturbation algebra, establish a structural result and explore the structure of this algebra. That structural result entails, as a consequence, what we refer to as the pseudo perturbation lemma. This lemma, in turn, implies the ordinary perturbation lemma.

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1. Introduction

It is a pleasure to dedicate this paper to Nodar Berikashvili. In [11] I pointed out that there is an intimate relationship between Berikashvili’s functor \( \mathcal{D} \) and deformation theory. In particular, cf. [11, Section 5], there is a striking similarity between Berikashvili’s functor \( \mathcal{D} \) and a functor written in the deformation theory literature as \( \text{Def}_g \) for a differential graded Lie algebra \( g \). Here I develop a small aspect of that relationship. I introduce and explore the pseudo Maurer-Cartan perturbation algebra. This algebra relates to deformation theory in an obvious manner, and it so does as well with regard to Berikashvili’s functor \( \mathcal{D} \): One can view

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the members of the pseudo Maurer-Cartan perturbation algebra as operators on objects of
the kind that lead to Berikashvili’s functor \( \mathcal{D} \).

A recent result of Chuang and Lazarev \([5]\) shows that the ordinary perturbation lemma is a
consequence of a structural result for a differential graded bialgebra that arises by abstracting
from the operators acting on what these authors refer to as an abstract Hodge decomposition;
see Section 6 below for the latter notion. The underlying differential graded algebra results
from extending an observation in \([1, 2]\). I show here that a variant of the algebra in \([5]\), the
pseudo Maurer-Cartan perturbation algebra, leads to the same kind of conclusion. Indeed, a
similar structural result, Theorem 4.3 below, entails as well, as a consequence, the ordinary
perturbation lemma.

The notion of abstract Hodge decomposition is equivalent to that of contraction, a basic
concept in homological perturbation theory. A more general notion is this: A pseudocontrac-
tion consists of a chain complex \( N \), a chain endomorphism \( \tau: N \to N \), and a homogeneous
degree 1 operator \( h: N \to N \) such that \( dh + hd = \tau \) and \( h^2 = 0 \). Here \( \tau \) is not necessarily
an idempotent endomorphism nor are the data subject to any annihilation property (side
condition) beyond the vanishing of \( h^2 \). Abstracting from the formal properties of the algebra
of operators acting on a pseudocontraction together with a perturbation of the differential
leads to the pseudo Maurer-Cartan perturbation algebra. The pseudo Maurer-Cartan pertur-
bation algebra surjects non-trivially to the corresponding algebra in \([5]\) and hence recovers
all the members of this algebra. Thus the pseudo Maurer-Cartan perturbation algebra yields
all the relevant operators that act on any chain complex arising from an abstract Hodge de-
composition with a perturbation of the differential or, equivalently, from a contraction with
a perturbation of the differential. Theorem 4.3 below says that a structural result which
Chuang and Lazarev show to be valid for the algebra they consider still holds formally for
the pseudo Maurer-Cartan perturbation algebra. The structure of the pseudo Maurer-Cartan
perturbation algebra is somewhat simpler than that of the corresponding algebra in \([5]\): There
is no annihilation contraint beyond the vanishing of the square of \( h \), and \( \tau \) is not necessarily
an idempotent, which is equivalent to the axiom \( \pi \nabla = \text{Id} \) imposed on a contraction
\((M \xrightarrow{\nabla} N, h)\); see Section 6 below. The present terminology “pseudo Maurer-Cartan per-
turbation algebra” avoids confusion with the notions of Maurer-Cartan algebra \([27]\) and of
multi derivation Maurer–Cartan algebra \([16]\). A consequence of Theorem 4.3 is the pseudo
perturbation lemma. Corollary 5.1 and Corollary 5.4 below spell out two versions thereof.
The pseudo perturbation lemma implies the ordinary perturbation lemma, see Section 6 be-
low. The results of this paper admit extensions, not made precise here, relative to additional
algebraic structure like algebra or coalgebra structures, similar to such generalizations in \([18]\).

In \([17]\) I explained another small aspect of the relationship between Berikashvili’s functor \( \mathcal{D} \)
and the functor \( \text{Def}_g \) for a differential graded Lie algebra \( g \). Also, working out the connections
with \([25, 26]\) would be an exceedingly attractive project.

2. Preliminaries

The ground ring \( R \) is a commutative ring with unit. Henceforth “chain complex”, “algebra”
etc. means \( R \)-chain complex, \( R \)-algebra, etc. As in classical differential homological algebra,
cf., e.g., \([21]\), we denote the identity morphism on an object by the same symbol as the object.
3. Pseudo perturbation algebra

Let \( H \) be the differential graded algebra generated by \( s \) and \( \tau \) of degrees 1 and zero, respectively, with differential (lowering degree by \(-1\)) written as \( D \), subject to
\[
Ds = \tau, \quad s^2 = 0.
\] (3.1) (3.2)

We refer to \( H \) as the pseudocontraction algebra.

**Proposition 3.1.** The algebra generators \( \tau \) and \( s \) of \( H \) commute. Hence the graded algebra that underlies \( H \) decomposes as \( \Lambda[s] \otimes R[\tau] \).

**Proof.**
\[
0 = Ds^2 = \tau s - s \tau. \tag*{□}
\]

Next, let \( P \) be the differential graded algebra having a single generator \( x \) of degree \(-1\), subject to
\[
Dx + x^2 = 0. \tag*{(3.3)}
\]

The canonical isomorphisms \( \varepsilon : H_0 \to R \) and \( \varepsilon : P_0 \to R \) turn \( H \) and \( P \) into augmented differential graded algebras. Let \( \mathcal{A} \) denote the augmented free product differential graded algebra \( P \ast H \), cf. [20]. We refer to \( \mathcal{A} = P \ast H \) as the pseudo perturbation algebra.

Here is an explicit description of that free product: For two chain complexes \( U \) and \( V \), let \( T_n(U, V) \) denote the chain complex which arises as an \( n \)-fold tensor product by alternatingly juxtaposing \( U \) and \( V \), starting with \( U \), that is,
\[
T_n(U, V) = U \otimes V \otimes ... (n \text{ factors}) \tag*{(3.4)}
\]

We use the notation \( I \) for the augmentation ideal functor. As a chain complex, the pseudo perturbation algebra \( \mathcal{A} = P \ast H \) decomposes as
\[
\mathcal{P} \ast \mathcal{H} = R \oplus \bigoplus_{n \geq 1} T_n(\mathcal{I}P, \mathcal{I}H) \oplus \bigoplus_{n \geq 1} T_n(\mathcal{I}H, \mathcal{I}P) \tag*{(3.5)}
\]
\[
= R \oplus IP \oplus IH \oplus \bigoplus_{n \geq 2} T_n(IP, IH) \oplus \bigoplus_{n \geq 2} T_n(IH, IP),
\]

**4. Pseudo Maurer-Cartan perturbation algebra**

Let \( u = sx \) and \( v = xs \). We also use the notation \( t = 1 - \tau \). The pseudo perturbation algebra \( \mathcal{A} = \mathcal{P} \ast \mathcal{H} \) has as well \( x \), \( s \), and \( t \) as algebra generators. Let \( \hat{\mathcal{A}} \) denote the graded \( R \)-algebra that arises by formally inverting the members \( 1 + u = 1 + sx \) and \( 1 + v = 1 + xs \) of \( \mathcal{A}_0 \). The differential \( D \) of \( \mathcal{A} \) extends to a differential on \( \hat{\mathcal{A}} \); we maintain the notation \( D \) for this differential. We refer to \( \hat{\mathcal{A}} \) as the pseudo Maurer-Cartan perturbation algebra.

Inspection shows that
\[
(1 + xs)^{-1} = 1 - x(1 + sx)^{-1}s \tag*{(4.1)}
\]
\[
(1 + sx)^{-1} = 1 - s(1 + xs)^{-1}x \tag*{(4.2)}
\]

**Remark 2.4**. Below we use the notation
\[
\alpha = (1 + u)^{-1} = (1 + sx)^{-1} \in \hat{\mathcal{A}}_0, \quad \beta = (1 + v)^{-1} = (1 + xs)^{-1} \in \hat{\mathcal{A}}_0. \tag*{(4.3)}
\]
In terms of this notation, (4.1) and (4.2) take the form

\[ \beta + x\alpha s = 1 \]  
\[ \alpha + s\beta x = 1. \]

**Proposition 4.1.** Setting

\[ \phi(x) = -x, \quad \phi(s) = \alpha s = s\beta, \quad \phi(t) = \alpha t\beta, \]

yields an involution \( \phi: \hat{A} \rightarrow \hat{A} \) of the graded \( R \)-algebra \( \hat{A} \) such that

\[ \phi(\alpha) = \alpha^{-1}, \quad \phi(\beta) = \beta^{-1}. \]

Under the involution \( \phi \) of \( \hat{A} \), the algebra differential \( D \) passes to the algebra differential \( D^\phi = \phi D \phi \) on \( \hat{A} \).

**Lemma 4.2.**

\[ D\alpha = -\alpha(\tau x + sx^2)\alpha \]  
\[ D\beta = \beta(x\tau + x^2s)\beta \]  
\[ x\alpha = \beta x \]  
\[ \alpha s = s\beta. \]

**Proof.** The identities \( 0 = D(\alpha\alpha^{-1}) \) and \( 0 = D(\beta\beta^{-1}) \) entail

\[ D\alpha = -\alpha(D\alpha^{-1})\alpha = -\alpha(D(1 + sx))\alpha = -\alpha(\tau x + sx^2)\alpha \]
\[ D\beta = -\beta(D\beta^{-1})\beta = -\beta(D(1 + xs))\beta = \beta(x\tau + x^2s)\beta. \]

Further,

\[ x\alpha - \beta x = x - xsx + xsxsx - \ldots - (x - xsx + xsxsx - \ldots) = 0. \]

On \( A \), the member \( x \) of \( A \) induces, in the standard manner, a twisted (or perturbed) differential \( D^x \). We recall that \( D^x(a) = Da + [x, a] \) \( (a \in A) \). This differential turns \( A \) into a differential graded algebra as well, and the twisted differential plainly extends to \( \hat{A} \). We denote the perturbed differential graded algebras by \( A^x \) and \( \hat{A}^x \).

**Theorem 4.3.** The algebra differential \( D^\phi \) on \( \hat{A} \) coincides with the twisted differential \( D^x \).
Proof. Using $Dt = 0$, $\beta \phi(\beta) = 1, \phi(\alpha)\alpha = 1, x\alpha = \beta x, \alpha = (1 + sx)^{-1}, \phi(\alpha) = \alpha^{-1} = 1 + sx, \beta = (1 + xs)^{-1}$, and $\phi(\beta) = \beta^{-1} = 1 + xs$, we find:

$$D^\phi(s) = \phi(D(as))$$
$$= \phi(D(\alpha)s + \alphaDs)$$
$$= \phi(D(\alpha))\phi(s) + \phi(\alpha\tau)$$
$$= \phi(D(\alpha))as + \phi(\alpha)\phi(1 - t)$$
$$= \phi(D(\alpha))as + \alpha^{-1}(1 - \alpha t \beta)$$
$$= \phi(D(\alpha))as + \alpha^{-1} - t\beta$$
$$= \phi(-\alpha(\tau x + sx^2)\alpha)as + \alpha^{-1} - t\beta$$
$$= -\phi(\alpha)((\phi(\tau)\phi(x) + \phi(s)\phi(x^2))\phi(\alpha)as + \alpha^{-1} - t\beta$$
$$= -\phi(\alpha)(\phi(1 - t)(-x) + ax^2)s + \phi(\alpha) - t\beta$$
$$= \phi(\alpha)(1 - \phi(t))xs - sx^2s + \phi(\alpha) - t\beta$$
$$= \phi(\alpha)(1 - \phi(t))xs - sx^2s + \phi(\alpha) - t\beta$$
$$= (1 + sx)xs - t\beta xs - sx^2s + 1 + sx - t\beta$$
$$= xs - t\beta xs + 1 + sx - t\beta$$
$$= 1 + [x, s] - t\beta xs - t\beta$$
$$= 1 + [x, s] - txs\beta - t\beta$$
$$= 1 + [x, s] - t(1 + xs)\beta$$
$$= 1 + [x, s] - t(1 + xs)(1 + xs)^{-1}$$
$$= 1 - t + [x, s]$$
$$= \tau + [x, s]$$
$$= D^\tau(s)$$

Likewise

$$D^\phi(x) = \phi(D(-x))$$
$$= \phi(x^2) = x^2$$
$$D^\tau(x) = Dx + [x, x]$$
$$= -x^2 + 2x^2 = x^2$$
Finally,
\[ D^\phi(t) = \phi(D(\phi(t))) = \phi(D(\alpha t \beta)) \]
\[ = \phi(D(\alpha t \beta) + \alpha D(t) \beta + \phi(\alpha D(\beta))) \]
\[ = \phi(D(\alpha)) \phi(t) \phi(\beta) + \phi(\alpha) \phi(D(t)) \phi(\beta) + \phi(\alpha) \phi(t) \phi(D(\beta)) \]
\[ = \phi(D(\alpha)) \phi(t) \phi(\beta) + \phi(\alpha) \phi(t) \phi(D(\beta)) \]
\[ = \phi(-\alpha(\tau x + sx^2) \alpha at \beta \phi(\beta) + \phi(\alpha) \alpha t \beta \phi(\beta(x^2 + s^2) \beta)) \]
\[ = \phi(-\alpha(\tau x + sx^2) \alpha at + \phi(\alpha) \alpha t \phi((x^2 + s^2) \beta) \]
\[ = \phi(-\alpha(\tau x + sx^2) t + t \phi((x^2 + s^2) \beta) \]
\[ = -\phi(\alpha t x) t - \phi(\alpha sx^2) t + t \phi((x^2) t) \phi((x^2 s) \beta) \]
\[ = \phi(\alpha) \phi(t) \phi(x) t - \phi(\alpha) \phi(s) \phi(x)^2 t + t \phi(x) \phi(\tau) \phi(\beta) + t \phi(x)^2 \phi(\phi(\beta)) \]
\[ = \phi(\alpha)(1 - \phi(t)) tx - \phi(\alpha) \alpha sx^2 t - tx(1 - \phi(t)) \phi(\beta) + tx^2 s \phi(\beta) \]
\[ = \phi(\alpha) t x - \phi(\alpha) \phi(t) tx - sx^2 t - tx \phi(\beta) + tx \phi(t) \phi(\beta) + tx^2 s \]
\[ = (1 + sx) tx - \phi(\alpha) \alpha t \beta x t - sx^2 t - tx(1 + xs) + tx \alpha t \phi(\beta) + tx^2 s \]
\[ = xt + sx^2 t - t \beta xt - sx^2 t - tx - tx^2 s + tx \alpha t + tx^2 s \]
\[ = xt - t \beta xt - tx + tx \alpha t \]
\[ = [x, t] + t(x - \beta x) t = [x, t] = D^\tau(t). \]

5. Pseudo perturbation lemma

From the introduction, we recall that a pseudocontraction consists of a chain complex \( N \), together with a chain endomorphism \( \tau: N \to N \) and a homogeneous degree 1 operator \( h: N \to N \), subject to, with \( h \) substituted for \( s \), (3.1) and (3.2). Pseudocontractions manifestly correspond bijectively to differential graded \( \mathcal{H} \)-modules. A pseudocontraction \((N, \tau, h)\) having \( \tau = N \) is an ordinary cone, together with a conical contraction, cf., e.g., [21, IV.1.5 p. 168] for this notion. This observation justifies, perhaps, our pseudocontraction terminology. In Proposition 6.4 we spell out the relationship between pseudocontractions and ordinary contractions.

Consider a pseudocontraction \((N, \tau, h)\). Recall that a perturbation \( \partial \) of the differential \( d \) on \( N \) is a homogeneous degree \(-1\) operator \( \partial \) on \( N \) such that the operator \( d + \partial \) on \( N \) has square zero, i.e., is itself a differential. The pseudocontraction structure \((h, \tau)\) on \( N \) being equivalent to an \( \mathcal{H} \)-module structure on \( N \) over the pseudocontraction algebra \( \mathcal{H} \), the perturbation \( \partial \) determines and is determined by a unique extension to an \( A \)-module structure on \( N \) over the pseudo perturbation algebra \( A = \mathcal{P} \ast \mathcal{H} \). Henceforth our convention is this: We distinguish in notation between \( s, x \in A \) and the operators they determine on \( N \) (provided that the degree zero endomorphisms \( N + h\partial \) and \( N + \partial h \) of \( N \) are invertible).

Let \( N_\partial \) denote the chain complex \((N, d + \partial)\), and write
\[ t\partial = \alpha t\beta: N \to N \quad (5.1) \]
\[ h\partial = \alpha h = h\beta: N \to N. \quad (5.2) \]

**Corollary 5.1** (Pseudo perturbation lemma). Suppose that the degree zero endomorphisms \( N + h\partial \) and \( N + \partial h \) of \( N \) are invertible, that is, that the \( A \)-module structure on \( N \) extends to
an $\hat{A}$-module structure on $N$ over the pseudo Maurer Cartan perturbation algebra $\hat{A}$. Then $(N_\partial, N - t_\partial, h_\partial)$ is a pseudocontraction as well.

**Proof.** The chain complex $N_\partial$ is a module over $\hat{A}x$. The composite $H \hookrightarrow \hat{A} \xrightarrow{\phi} \hat{A}x$ turns $N_\partial$ into an $H$-module in such a way that the members $\tau$ and $s$ act on $N$ as the operators $N - t_\partial$ and $h_\partial$. This establishes the assertion since $H$-module structures characterize pseudocontractions. □

**Remark 5.2.** Suppose that $N$ is a filtered chain complex, that the filtration is complete, see, e.g., [10, VIII.8 p. 292], and let $\partial$ be a perturbation of the differential $d$ of $N$ that lowers filtration. Then the series $\sum_{n \geq 0} (-h\partial)^n$ and $\sum_{n \geq 0} (-\partial h)^n$ converge, and hence the degree zero endomorphisms $N + h\partial$ and $N + \partial h$ of $N$ are invertible. In practice, for the degree filtration of a chain complex that is bounded below (e.g., concentrated in non-negative degrees), completeness is immediate. In fact, the convergence is then naive in the sense that, evaluated on a specific homogeneous element, $\sum_{n \geq 0} (-h\partial)^n$ and $\sum_{n \geq 0} (-\partial h)^n$ yield finite sums.

Define a weak contraction $(M \xleftarrow{\nabla} N, h)$ of chain complexes to consist of
- chain complexes $M$ and $N$,
- a surjective chain map $\pi: N \to M$ and an injective chain map $\nabla: M \to N$,
- a morphism $h: N \to N$ of the underlying graded modules of degree 1,
subject to the axioms

\begin{align*}
Dh &= N - \nabla \pi, \quad (5.3) \\
hh &= 0. \quad (5.4)
\end{align*}

Given a pseudocontraction $(N, \tau, h)$, let $M = tN \subseteq N$, let $\pi = t: N \to M$, and denote the injection $M \subseteq N$ by $\nabla: M \to N$. Since $t$ is a chain map, $M$ is a chain complex, $\pi$ and $\nabla$ are chain maps, and $(M \xleftarrow{\nabla} N, h)$ is a weak contraction. Further, $t = \nabla \pi$. Likewise, a weak contraction $(M \xleftarrow{\nabla} N, h)$ determines the pseudocontraction $(N, N - \nabla \pi, h)$. In this vein, the assignment to $(N, \tau, h)$ of $(M \xleftarrow{\nabla} N, h)$ yields an equivalence between pseudocontractions and weak contractions.

Consider a weak contraction $(M \xleftarrow{\nabla} N, h)$. Let $\partial$ be a perturbation of the differential on $N$, and suppose that the degree zero endomorphisms $N + h\partial$ and $N + \partial h$ of $N$ are invertible. Let

\begin{align*}
\mathcal{D} &= \pi \partial \alpha \nabla = \pi \beta \partial \nabla: M \to M \quad (5.5) \\
\nabla \partial &= \alpha \nabla: M \to N \quad (5.6) \\
\pi \partial &= \pi \beta: N \to M, \quad (5.7)
\end{align*}

and let $M_\partial$ denote the graded object $M$, endowed with the operator $d + \mathcal{D}$. Plainly,

\begin{equation}
t_\partial (= \alpha t \beta) = \nabla \partial \pi \partial. \quad (5.8)
\end{equation}
Lemma 5.3. The operator $D$ on $M$ satisfies the identities

\[ \pi_\partial (d + \partial) = (d + D) \pi_\partial \]  
(5.9)

\[ (d + \partial) \nabla_\partial = \nabla_\partial (d + D). \]  
(5.10)

Hence $D$ is a perturbation of the differential on $M$, and $\pi_\partial : N_\partial \to M_D$ and $\nabla_\partial : M_D \to N_\partial$ are chain maps. Furthermore,

\[ \pi_\partial \nabla_\partial (d + D) = (d + \partial) \nabla_\partial = (d + D) \pi_\partial \nabla_\partial. \]  
(5.11)

Proof. Identity (4.9) entails $D\beta = \beta(\partial \tau + \partial^2 h)\beta$. Hence

\[ \pi_\partial \circ (d + \partial) = \pi \beta \circ (d + \partial) \]
\[ = \pi (d\beta - \beta(\partial \tau + \partial^2 h)\beta) + \pi \beta \partial \]
\[ = \pi d\beta - \pi \beta \partial(\tau \beta + \partial h \beta) + \pi \beta \partial \]
\[ = d \pi \beta - \pi \beta \partial((1 - t) \beta + \partial h \beta) + \pi \beta \partial \]
\[ = d \pi \beta - \pi \beta \partial(1 + \partial h) \beta + \pi \beta \partial t \beta + \pi \beta \partial \]
\[ = d \pi \beta + \pi \beta \partial t \beta \]
\[ = d \pi \beta + \pi \beta \partial \nabla \pi \beta \]
\[ = (d + \pi \beta \partial \nabla) \circ \pi \beta \]
\[ = (d + \partial) \circ \pi_\partial. \]

Likewise, identity (4.8) entails $D\alpha = -\alpha(\tau \partial + h \partial^2)\alpha$. Hence

\[ (d + \partial) \nabla_\partial = (d + \partial) \alpha \nabla \]
\[ = d \alpha \nabla + \partial \alpha \nabla \]
\[ = (ad - \alpha(\tau \partial + h \partial^2)\alpha) \nabla + \partial \alpha \nabla \]
\[ = ad \nabla - \alpha((1 - t) + \partial h) \partial \alpha \nabla + \partial \alpha \nabla \]
\[ = \alpha \nabla d - \alpha(1 + \partial h) \partial \alpha \nabla + \alpha \partial t \alpha \nabla + \partial \alpha \nabla \]
\[ = \alpha \nabla d + \partial \alpha \nabla \]
\[ = \alpha \nabla d + \alpha \nabla \pi \partial \alpha \nabla \]
\[ = \alpha \nabla (d + \pi \partial \alpha \nabla) \]
\[ = \nabla_\partial (d + D). \quad \square \]

Corollary 5.4 (Pseudo perturbation lemma; second version). Let $(M \xleftarrow{\nabla} \pi N, h)$ be a weak contraction of chain complexes, let $\partial$ be a perturbation of the differential on $N$, and suppose that the degree zero endomorphisms $N + h \partial$ and $N + \partial h$ of $N$ are invertible. Then

\[ \left( M_D \xleftarrow{\nabla_\partial} \pi_\partial N_\partial, h_\partial \right) \]  
(5.12)

is a weak a contraction.
Proof. The weak contraction \( (M \overset{\nabla}{\rightarrow} \pi N, h) \) determines the pseudocontraction
\[
(N, \tau, h) = (N, N - \nabla \pi, h),
\]
and the pseudocontraction structure and the perturbation \( \partial \) determine an \( \hat{A} \)-module structure on \( N \) over the pseudo Maurer-Cartan perturbation algebra \( A = P \ast H \). By Corollary 5.1, \((N_\partial, N - t_\partial, h_\partial)\) is a pseudocontraction, that is,
\[
\begin{align*}
h_\partial^2 &= 0 \\
(d + \partial) \circ t_\partial &= t_\partial \circ (d + \partial) \\
(d + \partial) \circ h_\partial + h_\partial \circ (d + \partial) &= N - t_\partial \\
&= N - \nabla \partial \pi \partial,
\end{align*}
\]
cf. (5.8) above. In view of Lemma 5.3, we conclude that (5.12) is a weak contraction. \( \square \)

Remark 5.5. Under the circumstances of Corollary 5.4, the perturbed pseudocontraction \((N_\partial, N - t_\partial, h_\partial)\) determines the weak contraction \( \left( (t_\partial N, (d + \partial)|_{t_\partial N}) \overset{j}{\rightarrow} N_\partial, h_\partial \right) \). Inspection of the diagram
\[
\begin{array}{ccc}
N & \overset{\pi}{\rightarrow} & M \\
\beta \downarrow & & \nabla \downarrow & \alpha \\
N & \overset{t_\partial}{\rightarrow} & t_\partial N & \overset{j}{\rightarrow} N
\end{array}
\]
shows that the values of \( \nabla \partial = \alpha \nabla \) lie in \( t_\partial N \) in such a way that \( \nabla \partial \) is chain isomorphism
\[
\nabla \partial : M_\partial = (M, d + D) \rightarrow (t_\partial N, (d + \partial)|_{t_\partial N}).
\]
The morphism \( \nabla \partial \) being a chain map of the kind (5.13) is the content of identity (5.10).

6. RELATIONSHIP WITH ORDINARY HOMOLOGICAL PERTURBATION THEORY

The reader can find details about H(omological) P(erturbation) T(hory) in [12, 13, 14, 15, 18, 19]. Among the classical references are [4, 6, 7, 8, 9].

A contraction of chain complexes is a weak contraction \( (M \overset{\nabla}{\rightarrow} \pi N, h) \) subject to, furthermore, the axioms
\[
\begin{align*}
\pi \nabla &= M, \\
\pi h &= 0, \ h \nabla &= 0 \quad \text{(annihilation properties or side conditions).}
\end{align*}
\]

Remark 6.1. In the definition of a contraction, as opposed to that of a weak contraction, there is no need to require \( \pi \) to be surjective and \( \nabla \) to be injective since these properties are consequences of (6.1).

For a contraction of chain complexes of the particular kind \( (H(N) \overset{\nabla}{\rightarrow} \pi N, h) \), letting \( H = \ker(h) \cap \ker(d) = \nabla H(N) \), we see that the homogeneous degree \( j \) constituent \( N_j \) \( (j \in \mathbb{Z}) \) of \( N \) decomposes as
\[
N_j = dN_{j+1} \oplus H_j \oplus h(dN_j).
\]
In the situation of Example 6.2 below, (6.3) plays the role of a Hodge decomposition. On p. 19 of [24], Nijenhuis and Richardson indeed refer to a decomposition of the kind (6.3) (not using the language of homological perturbation theory) as a “Hodge decomposition”.

**Example 6.2** (Kodaira-Spencer Lie algebra). See [22, 23]. Take the ground ring to be the field $\mathbb{C}$ of complex numbers, consider a complex manifold $M$, let $\tau_M$ denote the holomorphic tangent bundle of $M$, let $\overline{\partial}$ be the corresponding Dolbeault operator, and let $g = (\mathcal{A}^{(0,*)}(M, \tau_M), \overline{\partial})$ be the Kodaira-Spencer algebra of $M$, endowed with the homological grading

$$
g_0 = \mathcal{A}^{(0,0)}(M, \tau_M), \quad g_{-1} = \mathcal{A}^{(0,1)}(M, \tau_M), \quad g_{-2} = \mathcal{A}^{(0,2)}(M, \tau_M), \text{ etc.} \quad (6.4)
$$

Thus, with our convention on degrees, $H^*(g) = H^{-*}(M, \tau_M)$, the cohomology of $M$ with values in the sheaf of germs of holomorphic vector fields. A Hodge decomposition of $g$ now yields a special kind of contraction.

Following [5], define an abstract Hodge decomposition of a chain complex $X$ to consist of operators $t$ and $h$ on $X$ of degree 0 and 1, respectively, such that

$$
h^2 = 0 \quad (6.5)
$$

$$
Dh = 1 - t \quad (6.6)
$$

$$
Dt = 0 \quad (6.7)
$$

$$
t^2 = t \quad (6.8)
$$

$$
th = ht = 0. \quad (6.9)
$$

**Remark 6.3.** The conditions characterizing an abstract Hodge decomposition are not independent. For example, $ht = 0$ implies $t^2 = t$: $0 = D(ht) = (Dh)t = (1 - t)t$.

An abstract Hodge decomposition is a special kind of pseudocontraction, and contractions and abstract Hodge decompositions are equivalent notions: Let $(M \xrightarrow{\nabla} N, h)$ be a contraction of chain complexes, and let $t = \nabla \pi$. Then $t$ and $h$ yield an abstract Hodge decomposition of $N$. Likewise, let $(N, \tau, h)$ be a pseudocontraction, let $t = \text{Id} - \tau: N \to N$, let $M = tN$, and let $j: M \to N$ denote the inclusion.

**Proposition 6.4.** Let $(N, \tau, h)$ be a pseudocontraction. The following are equivalent.

(i) The operators $h$ and $t = 1 - \tau$ yield an abstract Hodge decomposition of $N$.

(ii) The operators $h$ and $t = 1 - \tau$ satisfy (6.8) and (6.9).

(iii) Beyond the side condition $h^2 = 0$, the operators $h$ and $t = 1 - \tau$ satisfy the side conditions $th = 0$ and $hj = 0$, cf. (6.2), that is, $(M \xrightarrow{\nabla} N, h)$ is an ordinary contraction.

**Proof.** This is straightforward. We only note that (6.8) is equivalent to (6.1). \qed

**Corollary 6.5** (Ordinary perturbation lemma). Let $(M \xrightarrow{\nabla} N, h)$ be a contraction of chain complexes, let $\partial$ be a perturbation of the differential on $N$, and suppose that the degree zero endomorphisms $N + h\partial$ and $N + \partial h$ of $N$ are invertible. Then

$$
(M_D \xrightarrow{\nabla_0} N_\partial, h_\partial)
$$

constitutes a contraction.
Remark 6.6. Writing out (5.2) and (5.5) – (5.7) explicitly yields the standard expressions in the perturbation lemma, see, e.g., [12, Lemma 9.1]:

\[ D = \pi \partial (1 + h \partial)^{-1} \nabla = \sum_{n \geq 0} \pi \partial (-h \partial)^n \nabla \]
\[ = \pi (1 + \partial h)^{-1} \partial \nabla = \sum_{n \geq 0} \pi (-\partial h)^n \partial \nabla \]
\[ \nabla_\partial = (1 + h \partial)^{-1} \nabla = \sum_{n \geq 0} (-h \partial)^n \nabla \]
\[ \pi_\partial = \pi (1 + \partial h)^{-1} = \sum_{n \geq 0} \pi (-\partial h)^n \]
\[ h_\partial = (1 + h \partial)^{-1} h = \sum_{n \geq 0} (-h \partial)^n h \]
\[ = h (1 + \partial h)^{-1} = \sum_{n \geq 0} h (-\partial h)^n \]

Proof. In view of Corollary 5.4, it remains to confirm (6.1) and (6.2) for the perturbed data, that is, we must show that \( \pi_\partial \nabla_\partial = M \) and \( \pi_\partial h_\partial = 0 = h_\partial \nabla_\partial \). Using (6.1) and (6.2) for the unperturbed data, we find

\[ \pi_\partial \nabla_\partial = \pi \beta \alpha \nabla \]
\[ = \pi (1 + xs)^{-1} (1 + sx)^{-1} \nabla \]
\[ = \pi \sum_{n \geq 0} (-\partial h)^n \sum_{n \geq 0} (-h \partial)^n \nabla \]
\[ = \pi (-\partial h - h \partial + (\partial h)^2 + \partial hh \partial + h \partial h + (h \partial)^2 + \ldots) \nabla \]
\[ = \pi \nabla \]
\[ = M. \]

The same kind of reasoning shows that \( \pi_\partial h_\partial = 0 = h_\partial \nabla_\partial \).

Remark 6.7. Chuang-Lazarev refer to [5, Theorem 3.5] as the “abstract version of the HPL” (homological perturbation lemma) and claim that the “ordinary HPL is a consequence of the abstract one”. They spell out this consequence as [5, Corollary 3.7]. [5, Theorem 3.5] is similar to Theorem 4.3 above, except that it incorporates the side conditions (6.2) and (6.1) (or an equivalent condition), and [5, Corollary 3.7] yields a result similar to Corollary 5.1 above, but again with the side conditions (6.2) and a condition of the kind (6.1) incorporated. From the resulting perturbed abstract Hodge decomposition of the kind \((N_\partial, t_\partial, h_\partial)\), we can at once deduce the contraction

\[ \left( (t_\partial N, (d + \partial)|t_\partial N) \xrightarrow{j} N_\partial, h_\partial \right). \tag{6.11} \]

However, cf. Remark 5.5 above, when we start with a contraction \( M \xrightarrow{\nabla} N, h \) and a perturbation \( \partial \) of the differential on \( N \), we cannot deduce, from (6.11), the perturbation of the kind \( D \) of the differential on \( M \), cf. (5.5), without further thought. Lemma 5.3 provides the requisite further thought.
7. Insight into the structure of the pseudo Maurer-Cartan perturbation algebra

As before, let \( u = sx \) and \( v = xs \). We use the notation \( p(u, \tau), p_1(u, \tau), p_2(u, \tau) \), etc. for non-commutative monomials in \( u \) and \( \tau \) that involve \( u \) non-trivially (but do not necessarily involve \( \tau \)) and the notation \( q(v, \tau), q_1(v, \tau), q_2(v, \tau) \), etc. for non-commutative monomials in \( v \) and \( \tau \) that involve \( v \) non-trivially (but do not necessarily involve \( \tau \)). Further, we occasionally write the multiplication map (product operation) of \( \mathcal{A} \) as \( \cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \).

**Proposition 7.1.** The degree zero algebra \( \mathcal{A}_0 \) of the graded algebra \( \mathcal{A} \) has the following structural properties.

(i) As an \( R \)-module, \( \mathcal{A}_0 \) is free, having as basis the monomials in the union of the four families of the following kind:

- the monomials in \( \tau \),
- the monomials of the kind \( p(u, \tau) \),
- the monomials of the kind \( p(v, \tau) \),
- the monomials of the kind \( p(u, \tau)p(v, \tau) \).

(ii) Iuxtaposition realizes products in \( \mathcal{A}_0 \) of the kind

\[
\begin{align*}
p(u, \tau) \cdot \tau^i \cdot p(u, \tau), & \quad q(v, \tau) \cdot \tau^j \cdot q(v, \tau), \\
p(u, \tau) \cdot q(v, \tau), & \quad p_2(u, \tau) \cdot p_1(u, \tau)q(v, \tau), \\
p(u, \tau)q_1(v, \tau) \cdot q_2(v, \tau).
\end{align*}
\]

(iii) Products of the kind

\[
q(v, \tau) \cdot p(u, \tau), \quad p_1(u, \tau)q(v, \tau) \cdot p_2(u, \tau), \quad q_2(v, \tau) \cdot p(u, \tau)q_1(v, \tau)
\]

are zero.

(iv) Hence, for a monomial of the kind \( p(u, \tau)p(v, \tau) \),

\[
(p(u, \tau)p(v, \tau))^2 = 0.
\]

(v) As an \( R \)-algebra, \( \mathcal{A}_0 \) has the multiplicative generators \( u, v, \) and \( \tau \), subject to the relations

\[
v\tau^ju = 0, \quad j \geq 0.
\]

**Proof.** Consider a non-commutative monomial of the kind

\[
u^{k_1}v^{\ell_1}\tau^{m_1}u^{k_2}v^{\ell_2}\tau^{m_2} \ldots u^{k_a}v^{\ell_a}\tau^{m_a}, \quad k_j, \ell_j, m_j \geq 0, \quad 0 \leq j \leq a.
\]

Suppose that (7.5) is non-zero in \( \mathcal{A}_0 \). If \( \ell_1 = \ldots = \ell_a = 0 = k_1 = \ldots = k_a \), (7.5) is a monomial in \( \tau \). Now suppose that (7.5) is not merely a monomial in \( \tau \). If \( \ell_1 = \ldots = \ell_a = 0 \), (7.5) is of the kind \( p(u, \tau) \). If \( k_1 = \ldots = k_a = 0 \), (7.5) is of the kind \( q(v, \tau) \). Suppose that some \( k_i \) and some \( \ell_j \) are non-zero, and let \( \ell_a \) be the smallest member among the non-zero \( \ell_j \)s. Then \( \ell_1 = \ldots = \ell_{a-1} = 0 \) and, since \( v\tau^ju = xs\tau^jsx = 0 \in \mathcal{A}_0 \) and since (7.5) is non-zero, we conclude \( k_{u+1} = \ldots = k_a = 0 \), that is, (7.5) is of the kind \( q(u, \tau)q(v, \tau) \).

The homology algebras of the differential graded algebras \( \mathcal{H}, \mathcal{P}, \) and \( \mathcal{A} \) plainly reduce to isomorphisms \( \varepsilon : H(\mathcal{H}) \to R, \varepsilon : H(\mathcal{P}) \to R, \varepsilon : H(\mathcal{A}) \to R \). More precisely:
Proposition 7.2. The differential graded algebras $\mathcal{H}$ and $\mathcal{P}$ admit obvious algebra contractions
\[ (R \xrightarrow{\varepsilon} \mathcal{H}, h_{\mathcal{H}}) \quad (7.6) \]
\[ (R \xrightarrow{\varepsilon} \mathcal{P}, h_{\mathcal{P}}) \quad (7.7) \]
and these contractions induce an algebra contraction
\[ (R \xrightarrow{\varepsilon} \mathcal{A}, h_{\mathcal{A}}) \quad (7.8) \]
Furthermore, application of the perturbation lemma yields an algebra contraction
\[ (R \xrightarrow{\varepsilon} \mathcal{A}^\ast, h_{\mathcal{A}^\ast}) \quad (7.9) \]

Proof. This is straightforward. We leave the details to the reader. \qed

Remark 7.3. An obvious question is whether the contracting homotopy $h_{\mathcal{A}}$ in (7.8) extends to a contracting homotopy for the pseudo Maurer-Cartan perturbation algebra $\hat{\mathcal{A}}$.

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