Exact result for the polaron mass in a one-dimensional Bose gas

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We study the polaron quasiparticle in a one-dimensional Bose gas. In the integrable case described by the Yang-Gaudin model, we derive an exact result for the polaron mass in the thermodynamic limit. It is expressed in terms of the derivative with respect to the density of the ground-state energy per particle of the Bose gas without the polaron. This enables us to find high-order power series for the polaron mass in the regimes of weak and strong interaction.

Polarons were originally introduced to represent electrons that had changed their properties due to the influence of lattice distortions in ionic crystals [1]. The motion of electrons in such environments can be strongly modified as they move together with a surrounding cloud of phonons. This gives rise to a polaron mass that is larger than the electron one [2]. The concept of polarons is nowadays wider and applies well beyond solids, with particular importance for ultracold gases. Such environments offer numerous possibilities where polarons can be realized. Experiments provided both higher-dimensional [3–9] and one-dimensional [10, 11] systems which host polarons.

Many theoretical papers addressed polaron properties in one-dimensional bosonic crystals [12–31]. Several of them [12, 14, 15, 18, 23, 27] rely on models solvable by the Bethe ansatz [32–34]. Theoretical studies of such models are important for experiments, which are amenable to simulate them. Moreover, studies of integrable models can give rise to exact analytical results for physically relevant quantities, which are per se important.

A one-dimensional system of bosons with contact interaction is described by the Hamiltonian

\[ H = \frac{\hbar^2}{2m} \left[ -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + c \sum_{j \neq l} \delta(x_j - x_l) \right]. \tag{1} \]

Here \( m \) is the particle mass, \( c > 0 \) controls the interaction strength, and \( N \) is the total number of particles in the system. We consider the case with periodic boundary conditions. For a single-component Bose gas, the Hamiltonian (1) is exactly solved by the Bethe ansatz technique and is known as the Lieb-Liniger model [35, 36].

The Hamiltonian (1) is solvable by Bethe ansatz for more general symmetries of the wave function [32–34]. Let us consider a two-component (isospin-\( \frac{1}{2} \)) Bose gas [12, 32, 33]. In this case the Hamiltonian (1) is known as the Yang-Gaudin model for a Bose gas. Its eigenstates can be classified with respect to the value of the total isospin. In the sector where it has the maximal value \( N/2 \), one studies a single-component system described by the Lieb-Liniger model. It is characterized by \( N \) density quantum numbers \( I_1, I_2, \ldots, I_N \) that define \( N \) quasimomenta, \( k_1, k_2, \ldots, k_N \). In the case of the total isospin \( N/2 - 1 \), which will be the focus of this paper, the system acquires an additional spin quantum number that defines spin rapidity \( \eta \). The corresponding Bethe ansatz equations can be expressed as [18, 37]

\[ k_j L + \sum_{l=1}^{N} \theta(k_j - k_l) = 2\pi I_j + \pi + \theta(2k_j - 2\eta), \tag{2} \]

where \( L \) is the system size, \( I_j \) are integers for odd \( N \) and odd half-integers for even \( N \), and \( \theta(k) = 2 \arctan(k/c) \) is, up to the sign, the scattering phase shift. The momentum \( p \) and the energy \( E \) of the system are

\[ p = \frac{\hbar}{2} \sum_{j=1}^{N} k_j, \quad E = \frac{\hbar^2}{2m} \sum_{j=1}^{N} k_j^2. \tag{3} \]

Note that the spin rapidity \( \eta \) indirectly enters Eq. (3) through the quasimomenta \( k_j \) that depend on \( \eta \) via Eq. (2).

In the special case \( \eta \to +\infty \), the Bethe ansatz equations (2) describe the quasimomenta of the single-component (i.e., isospin-polarized) system described by the Lieb-Liniger model [35]. Its ground state is realized for a symmetric configuration of quasimomenta around zero, which occurs for

\[ I_j = j - \frac{N + 1}{2}, \quad j = 1, 2, \ldots, N. \tag{4} \]

The density of quasimomenta \( \rho(k_j) = [L(k_{j+1} - k_j)]^{-1} \) in the thermodynamic limit \( (N \to \infty \text{ and } L \to \infty) \) such that the density \( n = N/L \) is fixed satisfies the integral equation [35]

\[ \rho(k, Q) = \frac{1}{2\pi} \int_{-Q}^{Q} dk' \rho'(k - q) \rho(q, Q) = \frac{1}{2\pi}. \tag{5} \]

Here \( \rho'(k) = d\theta(k)/dk \) and \( Q \) is the Fermi rapidity, which denotes the largest occupied quasimomentum in the ground state. The density of quasimomenta enables us to express the particle density as

\[ n(Q) = \int_{-Q}^{Q} dk \rho(k, Q), \tag{6} \]

where we emphasized that \( n \) is a function of \( Q \). The ground-state momentum is zero and the ground-state energy per particle \( \epsilon_0 \) is given by [35]

\[ \epsilon_0 = \frac{\hbar^2}{2mn} \int_{-Q}^{Q} dk k^2 \rho(k, Q). \tag{7} \]
We note that the Fermi rapidity $Q$ naturally enters $\epsilon_0$ in Eq. (7). However we can eventually express $Q$ in terms of $\eta$ by inverting their connection (6).

At finite $\eta$, Eq. (2) describes the Bose gas with one isospin reversed. In the case of an unchanged set of quantum numbers (4), the system is in an excited state that is called spin-wave excitation [14] or a magnon [18]. In the context of this paper, such collective excitation of the system describes a polaron quasiparticle excitation.

The momentum of the system in the excited state coincides with the momentum of the polaron excitation. Substituting $k_j$ expressed from Eq. (2) into Eq. (3), in the thermodynamic limit we obtain [18]

$$p(Q, \eta) = \hbar \int_{-Q}^{Q} dk \rho(k, Q) \left[ \pi + \theta(2k - 2\eta) \right]. \quad (8)$$

We note that the momentum explicitly depends on the Fermi rapidity $Q$ and the spin rapidity $\eta$. As expected, at $\eta \to +\infty$ we obtain $p = 0$, i.e., the polaron is not excited.

The evaluation of the energy of the system in the excited state is more involved. At finite $\eta$, the quasimomentum become shifted by $\Delta k_j = k_j(\eta) - k_j(\eta \to +\infty) = O(1/L)$. From Eq. (2) we then find

$$\Delta k_j L = -\sum_{j=1}^{N} \theta'(k_j - k_i)(\Delta k_j - \Delta k_i) + \pi + \theta(2k_j - 2\eta) + O(1/N). \quad (9)$$

The formal expression $\rho(k, Q) = \frac{1}{L} \sum_{j=1}^{N} \delta(k - k_j)$ substituted into Eq. (5) gives

$$1 + \frac{1}{L} \sum_{j=1}^{N} \theta'(k_j - k) = 2\pi \rho(k, Q).$$

After introducing $g(k_j, Q) = L\rho(k_j, Q)\Delta k_j$, the latter equation enables us to express Eq. (9) as an integral equation

$$g(k, Q) = \frac{1}{2\pi} \int_{-Q}^{Q} dq \theta'(k - q)g(q, Q) = r(k, \eta), \quad (10a)$$

$$r(k, \eta) = \frac{1}{2} + \frac{\theta(2k - 2\eta)}{2\pi}. \quad (10b)$$

The energy of the system (3) in the thermodynamic limit now becomes $E = N\epsilon_0 + \mathcal{E}(Q, \eta)$, where $\epsilon_0$ is given by Eq. (7), while the energy of the polaron excitation corresponding to the momentum (8) is given by

$$\mathcal{E}(Q, \eta) = \frac{\hbar^2}{m} \int_{-Q}^{Q} dk k g(k, Q), \quad (11)$$

Here $g(k, Q)$ depends on $\eta$ and satisfies Eq. (10).

In order to further transform Eq. (11), we introduce the Green’s function for the linear operator in Eq. (10) by [38, 39]

$$G(k, k') = -\frac{1}{2\pi} \int_{-Q}^{Q} dq \theta'(k - q)G(k', q) = \delta(k - k'). \quad (12)$$

It is symmetric, $G(k, k') = G(k', k)$, as we can show by iterations. Multiplying Eq. (12) by $r(k', \eta)$, after the integration over $k'$ we obtain Eq. (10) provided

$$g(k, Q) = \int_{-Q}^{Q} dk' G(k, k')r(k', \eta).$$

Equation (11) then becomes $\mathcal{E}(Q, \eta) = \int_{-Q}^{Q} dq \sigma(k, Q) r(k, \eta)$ where we have defined $\sigma(k, Q) = (\hbar^2/m) \int_{-Q}^{Q} dk k G(k, k')$. Multiplying Eq. (12) by $k'$ and performing the integration over it, we obtain that $\sigma(k, Q)$ satisfies the integral equation

$$\sigma(k, Q) - \frac{1}{2\pi} \int_{-Q}^{Q} dq \theta'(k - q)\sigma(q, Q) = \frac{\hbar^2}{m} k. \quad (13)$$

Since $\sigma(k, Q)$ is an odd function of $k$ we eventually obtain

$$\mathcal{E}(Q, \eta) = \frac{1}{2\pi} \int_{-Q}^{Q} dq \sigma(k, Q)\theta(2k - 2\eta). \quad (14)$$

Equation (14) gives $\mathcal{E} = 0$ at $\eta \to +\infty$, as expected.

Equations (8) and (14) are exact and determine the dispersion of the polaron excitation with the momentum $0 \leq p \leq 2\pi \hbar \ln$ in the parametric form. It is difficult to find analytically the explicit form of the dispersion $E(p)$ for arbitrary values of the interaction. Instead, here we study the dispersion at the smallest momenta. This is achieved if we expand $\theta(2k - 2\eta)$ at $\eta/c \gg 1$ in Eqs. (8) and (14). Accounting for the leading-order term, we obtain the low-momentum spectrum [14, 18, 40]

$$\mathcal{E}(p) = \frac{p^2}{2m^*}, \quad (15)$$

where the mass of the polaron excitation $m^*$ is given by

$$\frac{1}{m^*} = \frac{S(Q)}{\pi \hbar^2 c n^2}, \quad S(Q) = \int_{-Q}^{Q} dk k \sigma(k, Q). \quad (16)$$

Equation (16) is the exact result.

The function $S(Q)$ [Eq. (16)] and $\epsilon_0$ [Eq. (7)] are not independent. We found that they satisfy a remarkable relation

$$S(Q) = 2\pi n^2 \frac{\partial \epsilon_0(n)}{\partial n}. \quad (17)$$

Equation (17) can be shown by differentiating Eq. (7) with respect to $n$ [which is a function of $Q$, see Eq. (6)], and then using the relation [41]

$$\frac{\rho(Q, Q)}{\rho(Q, Q)} + \frac{\sigma_k(Q, Q)}{\sigma_k(Q, Q)} = \frac{2\pi \hbar^2}{m \sigma(Q, Q)} \rho(Q, Q) \quad (18)$$

between the functions $\rho(k, Q)$ and $\sigma(k, Q)$ satisfying Eqs. (5) and (13), respectively. Equation (18) can be shown by differentiating Eq. (5) with respect to $Q$ and Eq. (13) with respect to $k$, and then comparing the obtained expressions. From Eq. (17) we finally obtain

$$\frac{1}{m^*} = \frac{2\hbar^2 c}{\partial \epsilon_0(n) \partial n}, \quad (19)$$

which is our main result.

Equation (19) is exact and thus valid at arbitrary interaction $c$. It shows that the polaron mass in the Yang-Gaudin model
depends only on the derivative with respect to the density of the ground-state energy per particle of the single-component Lieb-Liniger Bose gas (7). The latter can be expressed in terms of the dimensionless function $e(\gamma)$ defined by [35]

$$e_0 = \frac{\hbar^2 n^2}{2m} e(\gamma).$$  \hspace{1cm} (20)

Here $\gamma = c/n$ is the dimensionless parameter that accounts for the interaction strength. The particular form (20) enables us to express the polaron mass as

$$\frac{m}{m^*} = -\gamma^2 \frac{\partial}{\partial \gamma} \left( e(\gamma) \gamma^{-2} \right),$$ \hspace{1cm} (21)

which is an alternative form of Eq. (19). Equations (19) and (21) apply in the thermodynamic limit.

The function $e(\gamma)$ defined by Eq. (20) can be routinely evaluated numerically [35, 42]. On the other hand, to date there is no simple closed expression for $e(\gamma)$. It was calculated perturbatively at $\gamma \ll 1$ and $\gamma \gg 1$ at low orders [35] and recently to very high orders [42–44]. From $e(\gamma)$ at weak interaction calculated in Refs. [42, 44] we obtain the power series

$$\frac{m}{m^*} = 1 - \frac{2}{3\pi} \sqrt{\gamma} + \frac{4 - 3\zeta(3)}{16\pi^3} \gamma^{3/2} + \frac{4 - 3\zeta(3)}{24\pi^4} \gamma^2 + \frac{3[32 - 60\zeta(3) + 45\zeta(5)]}{2048\pi^5} \gamma^{5/2} + O(\gamma^3)$$ \hspace{1cm} (22)

in the regime $\gamma \ll 1$. Equation (22) shows that decreasing the interaction strength, the polaron mass approaches the bare impurity mass. This is expected, since the impurity in this limit becomes decoupled from the system. We notice the absence of the term proportional to $\gamma$ in Eq. (22), which is obvious from the form of the derivative in Eq. (21). The first two terms of the expression for $m/m^*$ were calculated in Ref. [15] using the discrete Bethe ansatz and in Ref. [14] using the hydrodynamic description [45]. Our expression (22) that relies on the Bethe ansatz is in agreement with Ref. [14]. However, we calculated several further terms of the series, which would be very hard to obtain using alternative methods.

The function $e(\gamma)$ at strong interaction was calculated in Ref. [43]. It enables us to find

$$\frac{m}{m^*} = \frac{2\pi^2}{3\gamma} - \frac{4\pi^2}{\gamma^2} + \frac{16\pi^2}{\gamma^3} - \frac{32\pi^2(15 - \pi^2)}{9\gamma^4} + O(\gamma^{-5})$$ \hspace{1cm} (23)

in the regime $\gamma \gg 1$. At strong interaction, the impurity motion implies the motion of many surrounding bosons. The resulting polaron quasiparticle is therefore heavy due to the surrounding cloud; its effective mass is determined by Eq. (23). The leading-order term in Eq. (23) was known before [14, 15]. We note that the first subleading term was calculated in Ref. [15], but it disagrees within a numerical coefficient from our result (23). Plotted in Fig. 1 is the exact result for the polaron mass obtained by numerically solving the integral equations (5) and (13) and its comparison with the two analytical expressions (22) and (23).

In conclusion, analyzing the Bethe ansatz equations of the Yang-Gaudin model for a Bose gas, we have found the exact result for the polaron mass $m^*$. It is given by Eq. (19) or its equivalent form (21). The latter expressions show that $m^*$ is fully determined by the dependence of the ground-state energy per particle of the Lieb-Liniger Bose gas on density or equivalently on $\gamma$. The latter function, and thus $m^*$, can be calculated analytically at very high orders at weak or strong interaction [43, 44]. We note that the specific form of the scattering phase shift is not used in Eqs. (5)–(14) and (18). Therefore, the described procedure might be applicable in studies of other two-component models [47] that can also describe polarons.

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[1] S. I. Pekar, Research in electron theory of crystals (U.S. Atomic Energy Commission, 1963).
[2] L. D. Landau and S. I. Pekar, “Effective mass of a polaron,” Zh. Eksp. Teor. Fiz. 18, 419 (1948).
[3] A. Schirotzek, C.-H. Wu, A. Sommer, and M. W. Zwierlein, “Observation of Fermi Polarons in a Tunable Fermi Liquid of Ultracold Atoms,” Phys. Rev. Lett. 102, 230402 (2009).
[4] M. Koschorreck, D. Pertot, E. Vogt, B. Fröhlich, M. Feld, and M. Köhl, “Attractive and repulsive Fermi polarons in two dimensions,” Nature (London) 485, 619 (2012).
[5] C. Kohstall, M. Zaccanti, M. Jag, A. Trenkwalder, P. Massignan, G. M. Bruun, F. Schreck, and R. Grimm, “Metastability and coherence of repulsive polarons in a strongly interacting Fermi mixture,” *Nature (London)* **485**, 615 (2012).

[6] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinson, R. S. Christensen, G. M. Bruun, and J. J. Arlt, “Observation of Attractive and Repulsive Polarons in a Bose-Einstein Condensate,” *Phys. Rev. Lett.* **117**, 055302 (2016).

[7] M.-G. Hu, M. J. Van de Graaff, D. Kedar, J. P. Corson, E. A. Cornelli, and D. S. Jin, “Bose Polarons in the Strongly Interacting Regime,” *Phys. Rev. Lett.* **117**, 055301 (2016).

[8] Z. Z. Yan, Y. Ni, C. Robens, and M. W. Zwierlein, “Bose polarons near quantum criticality,” *Science* **368**, 190 (2020).

[9] M. G. Skou, T. G. Skov, N. B. Jørgensen, K. K. Nielsen, A. Camacho-Guardian, T. Pohl, G. M. Bruun, and J. J. Arlt, “Non-equilibrium quantum dynamics and formation of the Bose polaron,” *Nat. Phys.* **17**, 731 (2021).

[10] J. Catani, G. Lamporesi, D. Naik, M. Gring, M. Inguscio, F. Minardi, A. Kiantian, and T. Giamarchi, “Quantum dynamics of impurities in a one-dimensional Bose gas,” *Phys. Rev. A* **85**, 023623 (2012).

[11] F. Meinert, M. Knap, E. Kirilov, K. Jag-Lauber, M. B. Zvonarev, E. Demler, and H.-C. Nägerl, “Bloch oscillations in the absence of a lattice,” *Science* **356**, 945 (2017).

[12] Y.-Q. Li, S.-J. Gu, Z.-J. Ying, and U. Eckern, “Exact results of their quantum fluctuations,” *Phys. Rev. A* **80**, 033610 (2004).

[13] J. N. Fuchs, D. M. Gangardt, T. Keilmann, and G. V. Shlyapnikov, “Spin Waves in a One-Dimensional Spinor Bose Gas,” *Phys. Rev. Lett.* **95**, 150402 (2005).

[14] M. T. Batchelor, M. Bortz, X. W. Guan, and N. Oelkers, “Collective dispersion relations for the one-dimensional interacting two-component Bose and Fermi gases,” *J. Stat. Mech.* **2006**, P03016 (2006).

[15] K. Sacha and E. Timmermans, “Self-localized impurities embedded in a one-dimensional Bose-Einstein condensate and their quantum fluctuations,” *Phys. Rev. A* **73**, 063604 (2009).

[16] A. Kamenev and L. I. Glazman, “Dynamics of a one-dimensional spinor Bose liquid: A phenomenological approach,” *Phys. Rev. A* **80**, 011603(R) (2009).

[17] M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, “Edge exponent in the dynamic spin structure factor of the Yang-Gaudin model,” *Phys. Rev. B* **80**, 201102(R) (2009).

[18] M. Schecter, D. M. Gangardt, and A. Kamenev, “Dynamics and Bloch oscillations of mobile impurities in one-dimensional quantum liquids,” *Ann. Phys. (Amsterdam)* **327**, 639 (2012).

[19] C. J. M. Mathy, M. B. Zvonarev, and E. Demler, “Quantum flutter of supersonic particles in one-dimensional quantum liquids,” *Nat. Phys.* **8**, 881 (2012).

[20] E. Burovski, V. Cheianov, O. Gamayun, and O. Lychkovskiy, “Momentum relaxation of a mobile impurity in a one-dimensional quantum gas,” *Phys. Rev. A* **89**, 041601 (2014).

[21] A. Petković and Z. Ristivojević, “Dynamics of a Mobile Impurity in a One-Dimensional Bose Liquid,” *Phys. Rev. Lett.* **117**, 105301 (2016).

[22] N. J. Robinson, J.-S. Caux, and R. M. Konik, “Motion of a Distinguishable Impurity in the Bose Gas: Arrested Expansion Without a Lattice and Impurity Snaking,” *Phys. Rev. Lett.* **116**, 145302 (2016).

[23] F. Grusdt, G. E. Astrakharchik, and E. Demler, “Bose polarons in ultracold atoms in one dimension: beyond the Fröhlich paradigm,” *New J. Phys.* **19**, 103035 (2017).

[24] L. Parisi and S. Giorgini, “Quantum Monte Carlo study of the Bose-polaron problem in a one-dimensional gas with contact interactions,” *Phys. Rev. A* **95**, 023619 (2017).

[25] A. G. Volosniev and H.-W. Hammer, “Analytical approach to the Bose-polaron problem in one dimension,” *Phys. Rev. A* **96**, 031601 (2017).

[26] N. J. Robinson and R. M. Konik, “Excitations in the Yang–Gaudin Bose gas,” *J. Stat. Mech.* **2017**, 063101 (2017).

[27] B. Kain and H. Y. Ling, “Analytical study of static beyond-Fröhlich Bose polarons in one dimension,” *Phys. Rev. A* **98**, 033610 (2018).

[28] G. Panochko and V. Pastukhov, “Mean-field construction for spectrum of one-dimensional Bose polaron,” *Ann. Phys. (Amsterdam)* **409**, 167933 (2019).

[29] A. Petković, “Microscopic theory of the friction force exerted on a quantum impurity in one-dimensional quantum liquids,” *Phys. Rev. B* **101**, 104503 (2020).

[30] J. Jager, R. Barnett, M. Will, and M. Fleischhauer, “Strong-coupling Bose polarons in one dimension: Condensate deformation and modified Bogoliubov phonons,” *Phys. Rev. Research* **2**, 033142 (2020).

[31] M. Gaudin, “Un systeme a une dimension de fermions en interaction,” *Phys. Lett. A* **24**, 55 (1967).

[32] C. N. Yang, “Some Exact Results for the Many-Body Problem in One Dimension with Repulsive Delta-Function Interaction,” *Phys. Rev. Lett.* **19**, 1312 (1967).

[33] B. Sutherland, “Further Results for the Many-Body Problem in One Dimension,” *Phys. Rev. Lett.* **20**, 98 (1968).

[34] E. H. Lieb and W. Liniger, “Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State,” *Phys. Rev.* **130**, 1605 (1963).

[35] E. H. Lieb, “Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum,” *Phys. Rev. 130*, 1616 (1963).

[36] Note that Eq. (2) conveniently contains an extra additive term $\pi$ and thus it has different quantum numbers $J_y$ from those in Refs. [12, 14, 27].

[37] B. Reichert, G. E. Astrakharchik, A. Petković, and Z. Ristivojević, “Exact Results for the Boundary Energy of One-Dimensional Bosons,” *Phys. Rev. Lett.* **123**, 250602 (2019).

[38] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, 1999).

[39] Note that in the language of magnetic systems, the quadratic spectrum (15) is expected for a long-wavelength spin-wave in ferromagnets.

[40] K. A. Matveev and M. Pustilnik, “Effective mass of elementary excitations in Galilean-invariant integrable models,” *Phys. Rev. B* **94**, 115436 (2016).

[41] Z. Ristivojević, “Conjectures about the ground-state energy of the Lieb-Liniger model at weak repulsion,” *Phys. Rev. B* **100**, 081110(R) (2019).

[42] Z. Ristivojević, “Excitation Spectrum of the Lieb-Liniger Model,” *Phys. Rev. Lett.* **113**, 015301 (2014).

[43] M. Mariño and T. Reis, “Exact Perturbative Results for the Lieb–Liniger and Gaudin–Yang Models,” *J. Stat. Phys.* **177**, 1148 (2019).

[44] Note that the first two term in Eq. (22) follow from Eq. (D15) of Ref. [46] taken at $M = m$, $G = g$, and $k_F = 0$.

[45] B. Reichert, A. Petković, and Z. Ristivojević, “Quasiparticle decay in a one-dimensional Bose-Fermi mixture,” *Phys. Rev. B* **95**, 045426 (2017).

[46] B. Sutherland, *Beautiful models* (World Scientific, Singapore, 2004).