SHARP MOMENT AND EXPONENTIAL TAIL ESTIMATES FOR U-STATISTICS

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Abstract.

We obtain in this paper a non-asymptotic non-improvable up to multiplicative constant moment and exponential tail estimates for distribution for $U$ -- statistics by means of martingale representation.

We show also the exactness of obtained estimations in one way or another by providing appropriate examples.

Key words: $U$ -- statistics, kernel, rank, random variables, Osekowski, Rosenthal, Jensen, Tchebychev, and triangle inequalities, martingales and martingale differences, martingale representation, Lebesgue-Riesz and Grand Lebesgue norm and spaces, symmetric function, lower and upper estimates, moments, examples, natural functions and norming, tails of distribution.

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1 Introduction. Notations. Statement of problem.

Let $(\Omega, F, P)$ be a probabilistic space, which will be presumed sufficiently rich when we construct examples (counterexamples). Let $\{\xi(i)\}$, $i = 1, 2, \ldots, n$, be independent identically distributed (i., i.d.) random variables (r.v.) with values in the certain measurable space $(X, S)$, $\Phi = \Phi(x(1), x(2), \ldots, x(d))$ be a symmetric measurable non-trivial numerical function (kernel) of $d$ variables: $\Phi : X^d \to R$, $U(n) = U_n = U(n, \Phi, d) =$

$$U(n, \Phi, d; \{\xi(i)\}) = \left(\frac{n}{d}\right)^{-1} \sum_{I \in I(d,n)} \Phi(\xi(i_1), \xi(i_2), \ldots, \xi(i(d))), \ n > d$$

be a so-called $U$ -- statistic. Denote $\deg \Phi = d$,
Φ = Φ(ξ(1), ξ(2), . . . , ξ(d)),  \( r = \text{rank } \Phi \in [1, 2, \ldots, d - 1] \),  

(1.1)

\[ \sigma(n) = \sigma_n = \sqrt{\text{Var}(U(n))}, \quad T(\Phi, x) := \sup_{n > d} \frac{T(U(n) - \mathbf{E}U(n))/\sigma(n), x),}{(1.2)} \]

i.e. the uniform tail function for our \( U \) - statistics under natural norming.

Let also \( I = I(n) = I(d; n) = \{i_1; i_2; \ldots; i_d\} \) be the set of indices of the form \( \{i\} = \{i_1, i_2, \ldots, i_d\} \) such that \( 1 \leq i_1 < i_2 < i_3 < i_{d-1} < i_d \leq n; \) \( J = J(n) = J(d; n) \) be the set of indices of the form (subset of \( I(d; n) \)) \( \{j\} = \{j_1; j_2; \ldots; j_{d-1}\} \) such that \( 1 \leq j_1 < j_2 < \ldots < j_{d-1} \leq n - 1. \)

Recall that

\[ \sigma^2(n) = \text{Var}(U(n)) \asymp n^{-r}, \quad n \to \infty. \]

The \textit{martingale representation} for the \( U \) - statistics as well as the exact value for its variance \( \sigma^2(n) = \text{Var}(U(n)) \) may be found, e.g. in [14], [19], chapter 1. Namely,

\[ U(n) - \mathbf{E}U(n) = \sum_{m=r}^{d} \binom{d}{m} U_{n,m}, \]

where

\[ U_{n,m} = \binom{n}{m}^{-1} \sum_{1 \leq i(1) < i(2) < \ldots < i(m) \leq n} g_i(\xi(i(1)), \xi(i(2)), \ldots, \xi(i(m))), \]

\[ g_i(x(1), x(2), \ldots, x(m)) = \]

\[ \int_X \int_X \cdots \int_X \Phi(y(1), y(2), \ldots, y(m)) \prod_{s=1}^{m} (\delta_{x(s)}(dy(s)) - P(dy(s))) \times \prod_{s=m+1}^{d} P(dy(s)). \]

The sequence

\[ n \to S_m(n) = \binom{n}{m} U_{n,m} \]

relative the natural filtration

\[ F_k = \sigma\{\xi(1), \xi(2), \ldots, \xi(k)\}, \quad F_0 = \{\emptyset, X\} \]

forme a martingale.

Herewith
\[ \sigma^2(n) = \sum_{m=r}^{d} \binom{d}{m}^2 \cdot \binom{n}{m}^{-1} \cdot \text{Var} \Phi = r! \binom{d}{r}^2 n^{-r} \text{Var} \Phi + O(n^{-r-1}), \ n \to \infty. \]

Note in addition that it follows from Iensen inequality
\[ |g_i(\xi(i(1)), \xi(i(2)), \ldots, \xi(i(m))))|_p \leq |\Phi|_p. \]

Here and in the future for any r.v. \( \eta \) the function \( T_\eta(x) \) will be denote its tail function:
\[ T_\eta(x) \overset{\text{de}}{=} \max(\mathbf{P}(\eta > x), \mathbf{P}(\eta < -x)), \ x > 0. \]  
(1.3)

We denote as usually the \( L(p) \) norm of the r.v. \( \eta \) as follows:
\[ |\eta|_p = [\mathbf{E}|\eta|^p]^{1/p}, \ p \geq 1; \]

and correspondingly
\[ M(p) = M(p, \Phi) = M(p, \Phi, \{\xi(i)\}) \overset{\text{de}}{=} \sup_n |(U(n) - \mathbf{E}U(n))/\sigma(n)|_p. \]
(1.4)

We will derive the non-refined up to multiplicative constant moment and exponential tail estimations for distribution of normed \( U \) - statistics, indeed, to estimate the variables \( M(p, \Phi, \{\xi(i)\}) \) and \( T_{U(n)/\sigma(n)}(x) \).

Evidently, these estimates may be applied for building of a non-asymptotical confidence interval for unknown parameter by using the \( U \) - statistics in the statistical estimation.

There are many works about this problem; the next list is far from being complete: [3], [29], [12], [14], [19], [21], [27] etc.; see also reference therein.

Notice that in the classical book [19] there are many examples of applying of the theory of \( U \) - statistics. A new application, namely, in the modern adaptive estimation in the non-parametrical statistics may be found in the article [2] and in the book [23], chapter 5, section 5.13.

2 Main result: moments estimation for \( U \) - statistics.

It is reasonable to suppose that \( \mathbf{E}\Phi = 0, \text{Var}(\Phi) \in (0, \infty); \) moreover, we can and will assume without loss of generality \( \text{Var}(\Phi) = 1, \) as long as it is constant; and that
all the moments of r.v. $\Phi$ which are written below there exist; otherwise it nothing to prove.

**Theorem 2.1.** Let $E\Phi = 0$, $\text{Var}\Phi = 1$, $|\Phi|_p < \infty$ for some value $p \geq 2$. Then

\[
\left| \frac{U(n)}{\sigma(n)} \right|_p \leq C(d, r) \cdot \left[ \frac{p^d}{\log p} \right]^d \cdot |\Phi|_p, \ p \geq 2.
\] (2.1)

**Proof.**

1. **Previous result.** The most recent (foregoing) results in this direction was obtained in [24]:

\[
\left| \frac{U(n)}{\sigma(n)} \right|_p \leq C_0(d, r) \cdot \left[ \frac{p^d}{\log p} \right] \cdot |\Phi|_p, \ p \geq 2.
\] (2.2)

Thus, we update a dependency on the degree $p$. See also [13].

2. **Outline of the proof.** The inequality (2.2) was obtained in [24] by means of the so-called martingale representation for the $U$ statistics, see [14], [19], chapters 1,2; and using further the moment estimation for the centered homogeneous of the degree $d$ polynomial martingales $(\zeta_n, G_n)$, see [24], of the form

\[
\sup_n \left| \frac{\zeta_n}{\sqrt{\text{Var}(\zeta_n)}} \right|_p \leq C_2(d) \frac{p^d}{\log p},
\] (2.3)

if of course $|\zeta_n|_p < \infty$.

For the multiply series in rearrangement invariant spaces analogous result was obtained by S.V.Astashkin in [1]. More information about martingale inequalities may be found in the many works of D.L.Burkholder, see e.g. the articles [5]-[7]. A famous survey on the martingale inequalities belongs to G.Peshkir and A.N.Shirjaev [30].

But in the recent publication about martingales [27] relating in turn on the famous result belonging to A.Osekowski [22] the estimate(2.4) was improved:

\[
\sup_n \left| \frac{\zeta_n}{\sqrt{\text{Var}(\zeta_n)}} \right|_p \leq C_3(d) \left[ \frac{p}{\log p} \right]^d.
\] (2.4)

More exactly, the following important function was introduced by A.Osekowski (up to factor 2) in the article [22]:

\[
O_s(p) \overset{df}{=} 4 \sqrt{2} \cdot \left( \frac{p}{4} + 1 \right)^{1/p} \cdot \left( 1 + \frac{p}{\ln(p/2)} \right), \ p \geq 4;
\] (2.5)

the case $p \in [2,4)$ is simple and may be considered separately.

Note that
\[ K = K_{O_s} \overset{\text{def}}{=} \sup_{p \geq 4} \left[ \frac{O_s(p)}{p/\ln p} \right] \approx 15.7858, \quad (2.6) \]

is the so-called Osekowski’s constant.

Let us define the following numerical sequence \( \gamma(d), \ d = 1, 2, \ldots : \gamma(1) := K_{O_s} = K, \) (initial condition) and by the following recursion

\[ \gamma(d + 1) = \gamma(d) \cdot K_{O_s} \cdot \left( 1 + \frac{1}{d} \right)^d. \quad (2.7) \]

Since

\[ \left( 1 + \frac{1}{d} \right)^d \leq e, \]

we conclude

\[ \gamma(d) \leq K_{O_s}^d \cdot e^{d-1}, \ d = 1, 2, \ldots. \quad (2.8) \]

It is proved in [27] in particular that the "constant" \( C_3(d) \) in (2.4) allows the following simple estimate: \( C_3(d) \leq \gamma(d). \)

The inequality (2.8) represents nothing more than \( d - \) dimensional and martingale generalization of a classical Rosenthal’s inequality for sums of independent random variables, [31], see also [18], the exact values of constants in the Rosenthal’s inequality see in [26].

We apply further the more modern estimate (2.5) instead (2.4) into the considerations of the report [24], we obtain what is desired.

**3. Some details.** Let as before \( E\Phi = 0, \ Var\Phi = 1, \ |\Phi|_p < \infty \) for some value \( p \geq 2. \) Let also the sequence \( \gamma(d) \) be defined in (2.7) (and in (2.8)). Then

\[ |U(n)|_p \leq \sum_{m=r}^{d} \gamma(m) \cdot \binom{d}{m} \cdot \binom{n}{m}^{-1/2} \cdot \left( \frac{p}{\ln p} \right)^m \cdot |\Phi|_p, \ p \geq 2. \quad (2.9) \]

and in turn after evident simplification

\[ |U(n)|_p \leq C(d, r) \cdot n^{-r/2} \cdot \left[ \frac{p}{\ln p} \right]^d \cdot |\Phi|_p, \ p \geq 2. \quad (2.10) \]

**Proof.** We can write using the martingale representation for \( U - \) statistics

\[ U(n) = \sum_{m=r}^{d} \binom{d}{m} \binom{n}{m}^{-1} \sum_{j \in J(m, n)} \mu_j, \quad (2.11) \]

where \((\mu_k, F_k)\) is certain centered martingale,

\[ \text{card} \ J(m, n) = \binom{n}{m}, \ Var \mu_j = Var \Phi = 1. \]
Denote for brevity
\[ N = N(m, n) = \binom{n}{m}, \]
then
\[ U(n) = \sum_{m=r}^{d} \binom{d}{m} N^{-1/2}(m, n) \zeta(m, n). \tag{2.12} \]

We apply the triangle inequality for the \( L(p) \) norm:
\[ | U(n) |_p \leq \sum_{m=r}^{d} \binom{d}{m} N^{-1/2}(m, n) | \zeta(m, n) |_p. \tag{2.13} \]

Each term \( \zeta(m, n) \) is the centered polynomial martingale of degree \( m \) generated by the function \( \Phi \). One can apply the Osekowski’s inequality (2.4):
\[ | \zeta(m, n) |_p \leq \gamma(m) \left[ \frac{p}{\ln p} \right]^m | \Phi |_p. \tag{2.14} \]

It remains to substitute into (2.13).

4. The assertion of theorem 2.1 follows immediately from the estimate (2.9), but this estimate gives us certain numerical estimate for the value \( |U_n|_p \).

**Remark 2.1.** As long as \( |U_n|_p \geq |U_n|_2 = \sigma_n \), we deduce that every time when \( | \Phi |_p < \infty \),
\[ | U(n) |_p \asymp n^{-r/2} \asymp \sigma(n), \ n \to \infty. \tag{2.15} \]

We generalize further this relation on more general than \( L_p(\Omega) \) spaces.

Let us discuss now the lower bounds for the theorem 2.1. To be more precise, we denote
\[ K_d(p) \overset{\text{def}}{=} \sup_{\{\xi(i)\}} \sup_{0 \neq \Phi \in L_p} \sup_n \left\{ \frac{M(p, \Phi, \{\xi(i)\})}{n^{-r/2} |\Phi|_p} \right\}. \tag{2.16} \]

**Theorem 2.2.** We conclude taking into account formulated above our definition and restrictions
\[ K_d(p) \asymp \left[ \frac{p}{\ln p} \right]^d, \ p \geq 2. \tag{2.17} \]

**Proof.** It remains to prove only the lower bound for the value \( K_d(p) \).
The relation (2.12) has been conjectured (hypothesis) in the article [24], page 19 for the polynomial martingales. It was proved (before!) for the polynomial
martingales from appropriate independent random variables in [13], [18]. The case of arbitrary polynomial martingales was grounded in authors report [27].

We represent here a very simple example in order to obtain the bottom border for $K_d(p)$ exactly for the $U$ statistics still for arbitrary value $d = 1, 2, \ldots$ Let $n = 1$ and let a r.v. $\eta$ has a standard Poisson distribution with unit parameter

$$P(\eta = k) = e^{-1}/k!, \; k = 0, 1, 2, \ldots$$

and define $\xi = \eta - 1$; then $\xi$ is centered, $\text{Var}(\xi) = 1$ and it is no hard to calculate

$$|\xi|_p \sim \frac{p}{e \cdot \ln p}, \; p \to \infty.$$

Let also $\xi(i)$ be independent copies of $\xi$. Then

$$| \prod_{i=1}^d \xi(i) |_p \sim e^{-d} \left( \frac{p}{\ln p} \right)^d, \; p \to \infty. \quad (2.18)$$

Therefore

$$\lim_{p \to \infty} \left\{ K_d(p) : \left[ \frac{p}{\ln p} \right]^d \right\} \geq e^{-d}. \quad (2.19)$$

Thus, obtained in this section estimate (2.1) of theorem (2.1) is essentially non-improvable relative the parameter $p$ for all the values of dimension $d$, of course, up to multiplicative constant.

3 Estimations of U-statistics in the Grand Lebesgue Spaces and in the exponential Orlicz space norms.

Let $\psi = \psi(p)$, $p \in [2, b)$, $b = \text{const}$, $1 < b \leq \infty$ (or $p \in [1, b]$) be certain bounded from below: $\inf \psi(p) > 1$ continuous inside the semi-open interval $[2, b)$ numerical function. We can and will suppose

$$b = \sup \{ p, \; \psi(p) < \infty \},$$

so that $\text{supp} \psi = [2, b)$ or $\text{supp} \psi = [2, b]$. The set of all such a functions will be denoted by $\Psi(b)$; $\Psi := \Psi(\infty)$.

For each such a function $\psi \in \Psi(b)$ we define

$$\psi_d(p) \overset{def}{=} \left[ \frac{p}{\ln p} \right]^d \cdot \psi(p). \quad (3.1)$$

Evidently, $\psi_d(\cdot) \in \Psi(b)$. 7
By definition, the (Banach) space $G_\psi = G_\psi(b)$ consists on all the numerical valued random variables $\{\zeta\}$ defined on our probability space $(\Omega, F, P)$ and having a finite norm

$$||\zeta||_{G_\psi} \overset{def}{=} \sup_{p \in (1, b)} \left[ \frac{|\zeta|_p}{\psi(p)} \right] < \infty.$$  \hspace{1cm} (3.2)

These spaces are suitable in particular for an investigation of the random variables and the random processes (fields) with exponential decreasing tails of distributions, the Central Limit Theorem in Banach spaces, study of Partial Differential Equations etc., see e.g. [17], [4], [20], [23], chapter 1, [9]-[11], [15]-[16] etc.

More detail, suppose $0 < ||\zeta|| := ||\zeta||_{G_\psi} < \infty$. Define the function

$$\nu(p) = \nu_\psi(p) = p \ln \psi(p), \hspace{0.5cm} 2 \leq p < b$$

and put formally $\nu(p) := \infty$, $p < 2$ or $p > b$. Recall that the Young-Fenchel, or Legendre transform $f^*(y)$ for arbitrary function $f : R \to R$ is defined (in the one-dimensional case) as follows

$$f^*(y) \overset{def}{=} \sup_x (xy - f(x)).$$

It is known that

$$T_\zeta(y) \leq \exp \left( -\nu_\psi^*(\ln(y/||\zeta||)) \right), \hspace{0.5cm} y > e \cdot ||\zeta||.$$ \hspace{1cm} (3.3)

Conversely, if (3.3) there holds in the following version:

$$T_\zeta(y) \leq \exp \left( -\nu_\psi^*(\ln(y/K)) \right), \hspace{0.5cm} y > e \cdot K, \hspace{0.5cm} K = \text{const} > 0,$$

and the function $\nu_\zeta(p), \hspace{0.5cm} 2 \leq p < \infty$ is positive, continuous, convex and such that

$$\lim_{p \to \infty} \psi(p) = \infty,$$

then $\zeta \in G_\psi$ and besides

$$||\zeta||_{G_\psi} \leq C(\psi) \cdot K.$$ \hspace{1cm} (3.5)

Moreover, let us introduce the exponential Orlicz space $L^{(M)}$ over the source probability space $(\Omega, F, P)$ with proper Young-Orlicz function

$$M(u) := \exp \left( \nu_\psi^*(\ln |u|) \right), \hspace{0.5cm} |u| > e$$

or correspondingly

$$M_d(u) := \exp \left( \nu_\psi^d*(\ln |u|) \right), \hspace{0.5cm} |u| > e$$

and as ordinary $M(u) = M_d(u) = \exp(C u^2) - 1, \hspace{0.5cm} |u| \leq e$. It is known [28] that the $G_\psi$ norm of arbitrary r.v. $\zeta$ is complete equivalent to the its norm in Orlicz space $L^{(M)}$. 

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\[ ||\zeta||_{G^{\psi}} \leq C_1 ||\zeta||_{L^{(M)}} \leq C_2 ||\zeta||_{G^{\psi}}, \quad 1 \leq C_1 \leq C_2 < \infty; \]
\[ ||\zeta||_{G^{\psi_d}} \leq C_3 ||\zeta||_{L^{(M_d)}} \leq C_4 ||\zeta||_{G^{\psi_d}}, \quad 1 \leq C_3 \leq C_4 < \infty. \]

**Example 3.1.** The estimate for the r.v. \( \xi \) of a form
\[ ||\xi||_p \leq C_1 p^{1/m} \ln^r p, \quad p \geq 2, \]
where \( C_1 = \text{const} > 0, \ m = \text{const} > 0, \ r = \text{const}, \) is quite equivalent to the following tail estimate
\[ T_{\xi}(x) \leq \exp \left\{ -C_2(C_1, m, r) x^m \log^{-mr} x \right\}, \ x > e. \]

It is important to note that the inequality (3.4) may be applied still when the r.v. \( \xi \) does not have the exponential moment, i.e. does not satisfy the famous Kramer’s condition. Namely, let us consider next example.

**Example 3.2.** Define the following \( \Psi - \) function.
\[ \psi_{[\beta]}(p) := \exp \left( C_3 p^{\beta} \right), \ p \in [2, \infty), \ \beta = \text{const} > 0. \]

The r.v. \( \xi \) belongs to the space \( G\psi_{[\beta]} \) if and only if
\[ T_{\xi}(x) \leq \exp \left\{ -C_4(C_3, \beta) \left[ \ln(1 + x) \right]^{1+1/\beta} \right\}, \ x \geq 0. \]

See also [24].

Let us return to the source problem. Assume that there exists certain function \( \psi(\cdot) \in \Psi(b), \ b = \text{const} \in (2, \infty) \) such that \( \Phi \in G\psi(b) \). For instance, this function may be picked by the following natural way:
\[ \psi_{\Phi}(p) := ||\Phi||_p, \quad (3.6) \]
if of course there exists and is finite at last for some value \( p \) greatest than 2, obviously, with the correspondent value \( b \).

**Theorem 3.1.** We propose under formulated above conditions
\[ \sup_n ||U(n)/\sigma(n)||_{G\psi_d} \leq C(\psi) \ ||\Phi||_{G\psi}. \quad (3.7) \]

**Proof** is very simple. Assume \( ||\Phi||_{G\psi} \in (0, \infty) \). It follows immediately from the direct definition of the norm in the Grand Lebesgue Spaces
\[ ||\Phi||_p \leq ||\Phi||_{G\psi \cdot \psi(p)}. \]
We apply the inequality (2.1) of theorem 2.1 for the values \( p \) from the set \( p \in [2, b) : \)
\[
\sup_n \left| \frac{U(n)}{\sigma(n)} \right|_p \leq C(d, r) \cdot \left( \frac{p}{\log p} \right)^d \cdot |\Phi|_p \leq \\
C(d, r) \cdot \left( \frac{p}{\log p} \right)^d \cdot \|\Phi\| G\psi \cdot \psi(p) = \\
C(d, r) \cdot \|\Phi\| G\psi \cdot \psi_d(p),
\]

or equally

\[
\sup_n \left| \frac{U(n)}{\sigma(n)} \right| G\psi_d \leq C(d, r) \cdot \|\Phi\| G\psi.
\]

**Example 3.3.** Suppose

\[
T_{\Phi}(x) \leq \exp \left\{ -C_5 x^m \log^{-m} x \right\}, \quad x > e,
\]

where \( C_5 = \text{const} > 0, \ m = \text{const} > 0, \ r = \text{const} \). As we know,

\[
|\Phi|_p \leq C_6(C_5, m, r) p^{1/m} \ln^r p, \quad p \geq 2.
\]

We use theorem 3.1

\[
\sup_n \left| \frac{U(n)}{\sigma(n)} \right|_p \leq C_7 p^{d+1/m} (\ln p)^{r-d}, \quad p \geq 2,
\]

and we conclude returning to the tail of distribution

\[
\sup_n T_{U(n)/\sigma(n)}(x) \leq \exp \left\{ -C_8 \frac{x^{m/(1+dm)}}{\log^{(1+dm)/m} x} \right\}, \quad x \geq e. \quad (3.9)
\]

Thus, we obtained in this way the exponential bounds for distribution for the normed \( U \) - statistics, expressed only through the very simple source data (3.8).

But the authors are not convinced of the finality of these estimations in the considered case; cf., e.g. [3], [12], [19], chapter 2, [29].

**Example 3.4.** Assume now

\[
T_{\Phi}(x) \leq \exp \left( -C_9 [\ln(1 + x)]^{1+1/\beta} \right), \quad x \geq 0. \quad (3.10)
\]

or more strictly

\[
\exp \left( -\tilde{C}_9 [\ln(1 + x)]^{1+1/\beta} \right) \leq T_{\Phi}(x) \leq \\
\exp \left( -C_9 [\ln(1 + x)]^{1+1/\beta} \right), \quad x \geq 0, \ 0 < C_9 \leq \tilde{C}_9 < \infty. \quad (3.10a)
\]

Then

\[
|\Phi|_p \leq \exp \left( C_{10} p^{\beta} \right), \quad p \geq 2,
\]
therefore by virtue of theorems 2.1 and 3.1

\[
\sup_n |U(n)/\sigma(n)|_p \leq C_{11} \ p^d (\ln p)^{-d} \ \exp \left( C_{10} \ p^\beta \right) \leq \\
\exp \left( C_{12} \ p^\beta \right), \ p \geq 2,
\]

and we have returning again to the tails of distribution

\[
\sup_n T_{U(n)/\sigma(n)}(x) \leq \exp \left( -C_{13} \ [\ln(1 + x)]^{1+1/\beta} \right), \ x \geq 0. \quad (3.11)
\]

As long as

\[
\sup_n T_{U(n)/\sigma(n)}(x) \geq T_{U(1)/\sigma(1)}(x) = T_{\Phi}(x),
\]

cf. (3.10) and (3.10a), we conclude that the assertion of theorem 3.1, i.e. exponential bound for tail distribution, is also in general case non improvable.

**Remark 3.1.** As we promised to prove, the proposition of Remark 2.1 it remains valid still for the Grand Lebesgue Spaces \( G\psi \) and for exponential Orlicz spaces of the form \( L^{(M)} \) : every time when \( \| \Phi \|_{G\psi} < \infty \) or equally \( \| \Phi \|_{L^{(M)}} < \infty \)

\[
\| U(n) \|_{G\psi} \sim n^{-r/2} \sim \sigma(n) \sim \| U(n) \|_{L^{(M)}}, \ n \to \infty. \quad (3.12)
\]

### 4 Concluding remarks.

**A.** It is interest, by our opinion, to obtain analogous estimates for dependent source random variables, for instance, for martingales or mixingales. Some preliminary results in this directions may be found in [3], [12].

**B.** We do not aim to derive the best possible values of appeared in this report constants. It remains to be done.

**C.** Perhaps, the case of the so-called \( V \) – statistics may be investigated analogously.

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