Characterizing multi-photon quantum interference with practical light sources and threshold single-photon detectors

Álvaro Navarrete, Wenyuan Wang, Feihu Xu and Marcos Curty

1 El Telecomunicación, Department of Signal Theory and Communications, University of Vigo, Vigo E-36310, Spain
2 Center for Quantum Information and Quantum Control, Department of Physics and Department of Electrical & Computer Engineering, University of Toronto, Toronto, Ontario, M5S 3G4, Canada
3 National Laboratory for Physical Sciences at Microscale, University of Science and Technology of China (Shanghai Branch), Hefei, Anhui 230026, People’s Republic of China

E-mail: anavarrete@com.uvigo.es

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Abstract

The experimental characterization of multi-photon quantum interference effects in optical networks is essential in many applications of photonic quantum technologies, which include quantum computing and quantum communication as two prominent examples. However, such characterization often requires technologies which are beyond our current experimental capabilities, and today’s methods suffer from errors due to the use of imperfect sources and photodetectors. In this paper, we introduce a simple experimental technique to characterize multi-photon quantum interference by means of practical laser sources and threshold single-photon detectors. Our technique is based on well-known methods in quantum cryptography which use decoy settings to tightly estimate the statistics provided by perfect devices. As an illustration of its practicality, we use this technique to obtain a tight estimation of both the generalized Hong–Ou–Mandel dip in a beamsplitter with six input photons and the three-photon coincidence probability at the output of a tritter.

1. Introduction

Multi-photon quantum interference is a key concept in quantum optics and quantum mechanics. It has been extensively studied by many authors over recent decades, from the seminal two-photon interference experiment performed by Hong, Ou and Mandel [1] to more recent experimental demonstrations which involve a higher number of indistinguishable photons in various scenarios [2–8]. Moreover, besides its indubitable inherent theoretical interest, multi-photon quantum interference also plays a pivotal role in several subfields and applications of quantum information science that use, for example, optical networks (ONs) to set up interference between photons. These applications include, among others, quantum computing [9], quantum cryptography [10, 11], boson sampling [12–16], quantum clock synchronization [17], and quantum metrology [18]. In any practical realization of these applications it is essential to confirm experimentally that the photons interfere as desired [19].

Unfortunately, however, to characterize multi-photon quantum interference in general ONs experimentally is usually a quite challenging task [20]. This is so because, for this, one would ideally need to use high-quality on-demand n-photon sources which are yet to be realized [21, 22], together with high-quality photon number resolving (PNR) detectors, which, besides being expensive in terms of experimental resources, can currently only distinguish up to a certain number of photons, and may also introduce noise [23–25]. As a result, we have that current experimental techniques to characterize the quantum interference behaviour of ONs at a few photons level typically suffer from inevitable errors due to the use of imperfect sources and detectors [20].

The main contribution of this paper is a novel technique to estimate the input–output photon number statistics of ONs experimentally when the input signals are tensor products of Fock states. To this end, we use simple laser sources to generate the input signals to the ON and practical threshold single-photon detectors to...
measure the output signals [26]. That is, our method is implementable with current technology, and allows the estimation of the conditional probability distribution \( P(x_1, \ldots, x_M | n_1, \ldots, n_N) \) that describes the behaviour of the ON on the input Fock states \( |n_1, \ldots, n_N) \), where \( n_i(x) \) with \( i = 1, \ldots, N \) (\( j = 1, \ldots, M \)) denotes the number of photons at the \( i \)th (\( j \)th) input (output) port of the ON. We emphasize, however, that, in practice, our method is specially suitable to the evaluation of mainly small-size ONs. This is so because, as we show later, it requires the experimental estimation of the probabilities of certain observable events whose estimation complexity may increase exponentially with the number of input/output ports of the ON [12, 27].

The key idea builds on two techniques that are extensively used in the field of quantum cryptography: the decoy-state method [28–30] and the so-called detector-decoy technique [31, 32]. We use the former at the input ports of the ON to estimate the statistics provided by ideal \( n \)-photon sources. Besides standard quantum key distribution [28–30, 33–35], the decoy-state method has also been used, for example, to estimate the yield of two single-photon pulses in measurement-device-independent quantum key distribution [10, 36, 37], to simulate single-photon sources with imperfect light sources [38], and to perform single-photon quantum state tomography with practical sources [39]. Indeed, our technique is closely related to the methodology introduced in [38]. However, while [38] and all other previous results which employ the decoy-state method to evaluate the behaviour of ONs, typically estimate only the statistics associated with those events where the ONs receive single-photon pulses at their input ports, here we extend these results to estimate the behaviour of ONs in the general case where they also receive multi-photon pulses. The second key difference between our analysis and that introduced in [38] is that here we employ the detector-decoy method [31, 32] at the output ports of the ON to estimate the statistics provided by ideal PNR detectors by means of threshold single-photon detectors.

To illustrate the practicality of our technique in the study of ONs, we evaluate two simple examples of interest. In the first, we estimate the generalized Hong–Ou–Mandel (HOM) dip [1] in a beamsplitter when the total number of input photons is six for two different conditional probabilities, \( P(3, 3|3, 3) \) and \( P(5, 1|5, 1) \). The first case has been experimentally studied in [5], where the authors used for this a spontaneous parametric down-conversion source in combination with a measurement setup with six threshold single-photon detectors. The second case, however, (to the best of our knowledge) has not yet been experimentally implemented, due to the difficulty of generating five-photon states to input to the beamsplitter. In both scenarios, we use our method to estimate the HOM dip by means of just two laser sources and two threshold single-photon detectors. In the second example, we estimate the three coincidence detection probability in a tritter [40] when there is just one single-photon pulse in each of its input ports—i.e. we estimate \( P(1, 1, 1|1, 1, 1) \). This example is also used to obtain a high precision estimation of the dependence of that probability on the triad phase, which arises when one considers more than two input photons [40]. While these two examples correspond to evaluating linear ONs, we remark that our method could also be used to study multi-photon quantum interference in non-linear ONs.

The paper is organized as follows. In section 2, we present our method in detail. Then, in section 3, we evaluate the two practical examples described above. Finally, we summarize the content of the paper in section 4. The paper also includes two appendixes with additional information.

2. Method

As mentioned above, we use the decoy-state (detector-decoy) method at the input (output) ports of the ON to estimate the statistics provided by ideal \( n \)-photon sources (PNR detectors). Of course, in contrast to the case where one really uses perfect \( n \)-photon sources and PNR detectors, the use of decoy settings does not provide single shot resolution about how many photons input and output each port of the ON each given time. However, it permits to estimate the full statistics that such perfect devices could give, which is enough for our purposes.

More precisely, we use as input signals to the ON Fock diagonal states each having different photon number statistics. This type of signal could be generated, for instance, using attenuated laser diodes emitting phase-randomised weak coherent pulses (WCPs), triggered spontaneous parametric down-conversion sources or practical single-photon sources, together with variable attenuators to vary the intensity of the individual light pulses. To implement the detector-decoy method, on the other hand, we also place variable attenuators on the output ports of the ON, together with threshold single-photon detectors. This general scenario is illustrated in figure 1. In so doing, as we show below, we have that the probability of each possible detection pattern observed on the threshold single-photon detectors can be written as a sum of linear terms where the only unknowns are the probabilities \( P(x_1, \ldots, x_M | n_1, \ldots, n_N) \equiv P(x|n) \) (where, for ease of notation, we use \( x \equiv x_1, \ldots, x_M \) and \( n \equiv n_1, \ldots, n_M \) in what follows). As a result, we obtain a set of linear equations which are functions of the probabilities \( P(x|n) \) and, in principle, one can estimate these quantities accurately. The more decoy-state/detector-decoy settings we use, the higher the number of linear equations that we obtain, and thus the better the
accuracy of the estimation. Indeed, in the asymptotic limit using an infinite number of decoy-state/detector-decoy settings, the probabilities \( P(x|n) \) could be estimated precisely. Importantly, however, as we show below, a small number of decoy-state/detector-decoy settings can typically provide quite a tight estimation of \( P(x|n) \) for small values of \( n \) and \( x \).

Our starting point is the input state to the ON. As shown in figure 1, this is the state of the \( N \) spatial modes after the input attenuators of transmittance \( \gamma_i \). This state can be written as

\[
\rho_{in}^\mu = \bigotimes_{i=1}^{N} \rho_i^\mu = \sum_n p_i^\mu(n) |n\rangle \langle n|,
\]

where \( \rho_i^\mu = \sum_{n_i=0}^{\infty} p_i(n_i) |n_i\rangle \langle n_i| \) is the Fock diagonal state at the \( i \)th input spatial mode of the ON, which in the case of phase-randomized WCPs satisfies \( p_i(n_i) = e^{-\mu_i} \mu_i^{n_i}/n_i! \). Here, the mean photon number \( \mu_i = \gamma_i \mu' \), with \( \mu' \) being the initial intensity of the laser sources. The quantity \( p_i^\mu = \prod_{i=1}^{N} p_i(n_i) \) on the other hand, represents the conditional probability of having the input state \( |n\rangle \equiv |n_1, \ldots, n_N\rangle \) given the set of input intensities \( \mu = \{ \mu_1, \ldots, \mu_N \} \).

Let us now consider the output state \( \rho_{out}^\mu = U \rho_{in}^\mu U^\dagger \) of the ON, where \( U \) denotes the evolution unitary operator applied by the network\(^4\). We can write this state in terms of the probabilities \( P(x|n) \). For this, for convenience, we first combine the effect of each output attenuator \( \omega_j \) with the detection efficiency of each threshold single-photon detector \( D_j \) (see figure 1). By doing so, we can conceptually consider that at the \( j \)th output port of the ON there is now a threshold single-photon detector with efficiency \( \kappa_j = \omega_j \eta_j \), \( j = 1, \ldots, M \), where \( \eta_j \) is the detection efficiency of the threshold single-photon detector \( D_j \) in the original scenario (note that here, for simplicity, we assume that all detectors \( D_j \) have the same detection efficiency \( \eta_j \)). This is so because when a detector has some finite detection efficiency \( \eta_j \), it can be mathematically described by a beamsplitter of transmittance \( \eta_j \) combined with a lossless detector [41]. Importantly, since the positive-operator valued measure (POVM) that characterizes the behaviour of a typical threshold single-photon detector is diagonal in the Fock bases, it follows that the resulting measurement statistics when measuring \( \rho_{out}^\mu \) remain unchanged if, before the actual measurements, we perform a quantum nondemolition (QND) measurement of the total number of photons at each output mode of the ON. This means, in particular, that for any \( \rho_{out}^\mu \), there is always a Fock diagonal state, which we shall denote by \( \rho_{out}^\mu_{FoS} \), of the form

\[
\rho_{out}^\mu_{FoS} = \sum_x |x\rangle \langle x| \rho_{out}^\mu_{FoS} \langle x| = \sum_x \langle x| U \rho_{in}^\mu U^\dagger \langle x| |x\rangle
\]

\[
= \sum_n \sum_{x \in N} P_n^\mu(x|n) \sum_{x \in N} U|n\rangle \langle n| \langle x| = \sum_{x \in N} P_n^\mu(x|n) \langle x|,
\]

\[\text{(2)}\]

\(^4\)Note that the output state \( \rho_{out}^\mu_{FoS} \) could also arise from tracing out some of the output modes of the network. That is, \( \rho_{out}^\mu = \text{Tr}_{B^c}[U \rho_{in}^\mu U^\dagger] \), where \( B \) denotes the set of output modes that is traced out.
that provides exactly the same measurement statistics as $\rho_{\text{out}}^{\mu}$. In equation (2), $|x\rangle = |x_0, \ldots, x_M\rangle$ is the Fock state after the QND measurements on $\rho_{\text{out}}^{\mu}$, and $P(x|n) = \langle x| U(n) |n\rangle^2$ denotes the conditional probability of having such a state $|x\rangle$ given that the input state to the ON is $|n\rangle$. Note that here, for simplicity, we consider passive networks that do not create photons, and therefore we assume that $\sum_j^M x_j \leq \sum_n^N n$. That is, the total number of output photons cannot be greater than the total number of input photons to the ON. This last condition is expressed in equation (2) with the symbol $x \leq n$. However, we remark that our method could as well be applied to evaluate active ONs.

Finally, to estimate the unknown probabilities $P(x|n)$, we need to relate them to some observable quantities. For this, we use the fact that the probability $r_{\text{out}}^{\mu}$ of observing the detection pattern $\theta \equiv (\theta_1, \ldots, \theta_M)$, where $\theta_j$ is equal to zero (one) for a no-click (click) event in the threshold single-photon detector $D_j$, given the state $\rho_{\text{out}}^{\mu}$, and the detectors’ efficiencies $\kappa = \kappa_1, \ldots, \kappa_M$, is given by

$$P_{\theta}^{\mu,\kappa} = \text{Tr} \left[ \rho_{\text{out}}^{\mu} \otimes \prod_{j=1}^M \Pi_j^{\kappa} \right],$$

with the POVM elements $\Pi_j^{\kappa}$ given by

$$\Pi_j^{\kappa} = (1 - p_{\text{dark}}) \sum_{n=0}^{\infty} (1 - \kappa_j)^n |n\rangle \langle n|,$$

$$\Pi_j^{\kappa} = 1 - \Pi_0^{\kappa},$$

where $p_{\text{dark}}$ denotes the dark count probability of the detector $D_j$, which we assume for simplicity to be equal for all $j = 1, \ldots, M$. That is, the operator $\Pi_j^{\kappa}$ is associated with a no-click (click) event at the detector $D_j$. After substituting equations (2) and (4) into equation (3), we finally obtain

$$P_{\theta}^{\mu,\kappa} = \sum_{n \leq n} P_n (x|n) P^{\kappa}(\theta|x).$$

Here, $P^{\kappa}(\theta|x) = \langle x| \otimes_{j=1}^M \Pi_j^{\kappa} |x\rangle$ denotes the probability of observing the detection pattern $\theta$ given the output state $|x\rangle$, the detection efficiencies $\kappa$ and the dark count probability $p_{\text{dark}}$. If the detectors $D_j$ are well-characterized, this quantity is known. Importantly, equation (5) relates the observed probabilities $P_{\theta}^{\mu,\kappa}$, which can be directly measured in the actual experiment, to the unknown probabilities $P(x|n)$ via the statistics $P_n^{\kappa}$ and $P^{\kappa}(\theta|x)$, which are both known a priori given the experimental parameters $\mu$, $\eta_0$ and $p_{\text{dark}}$ together with the attenuator settings $\gamma = \{\gamma_1, \ldots, \gamma_N\}$ and $\omega = \{\omega_1, \ldots, \omega_M\}$. Indeed, as mentioned previously, each decoy-state/detector-decoy setting provides a new linear equation which has the same unknowns $P(x|n)$ but different coefficients $P_n^{\kappa}$ and $P^{\kappa}(\theta|x)$ and constant terms $P_{\theta}^{\mu,\kappa}$. Thus, by solving the set of linear equations given by equation (5) one can, in principle, estimate any conditional probability $P(x|n)$.

3. Evaluation

In what follows, we illustrate this method with two simple examples of practical interest.

For simulation purposes, in both examples we set the detection efficiency of the threshold single-photon detectors to $\eta_0 = 80\%$, and their dark count probability to $p_{\text{dark}} = 10^{-6}$, which are values which can be achieved with current technology [42]. We remark, however, that our method can also provide tight estimations when it employs detectors with lower detection efficiency and higher dark count rate. That is, the method is in principle quite robust to typical imperfections of the detectors, given that they are well-characterized. Indeed, when the detection efficiency is low, in principle one can mitigate its effect by increasing the number of decoy-state/detector-decoy settings. This is illustrated in figure C1. Also, if the value of the dark count rate is stable, then it is basically a scaling factor in the equations of the linear program and thus its value does not greatly affect the estimation. For simplicity, in our simulations we disregard other imperfections of the detectors such as after-pulsing. Note that in a practical implementation of the method, one could strongly reduce the number of after-pulses by just imposing—for example—a dead-time to all detectors after observing a detection click [42]. During this dead-time period no detector is able to produce further detection clicks, and thus no after-pulse can occur.

Also, for simplicity, we consider the asymptotic scenario where the number of signals transmitted is infinite, and disregard imperfections (like, for instance, intensity fluctuations or imperfect phase randomisation) of the light sources. The realistic scenario where the number of signals transmitted is actually finite, could be analysed by means of standard techniques in quantum key distribution (see e.g. [43]), which use concentration inequalities—like, for instance, Chernoff’s [44] and Hoeffding’s [45] bounds—to relate the observed experimental data and its expected value except for a minuscule error probability. As a result, the accuracy of the estimation obviously depends on the number of trials of the experiment. Similarly, the case of imperfect light sources could be evaluated using techniques from quantum key distribution. For example, to consider the effect
that intensity fluctuations have on the estimation results provided by the decoy-state method, one could use—for instance—the techniques introduced in [46]. There, the authors study the case where the intensity fluctuations are bounded within a certain interval except for a minuscule probability. Finally, a simple method to generate phase-randomized WCPs is to directly modulate the lasers using gain-switching conditions—i.e. far above and below threshold. For this, please note that it is essential that the time where the laser is off is sufficiently long in comparison with its reset time. Thus, it is simultaneously guaranteed that the cavity field is sufficiently attenuated that any prior coherence vanishes, and inputs amplified spontaneous emission due to vacuum fluctuations which results in a field with a truly random phase. As a result, the phases of the optical pulses generated are truly random and not intercorrelated. In so doing, there is no need to use quantum random number generators, together with phase modulators, to select the phase of each outgoing pulse randomly.

Indeed, this technique is commonly used in decoy-state quantum cryptography, and also to generate random numbers using phase diffusion in semiconductor lasers [47, 48]. The analysis of the finite regime case with imperfect light sources is, however, beyond the scope of this paper.

### 3.1. First example: beamsplitter

In this case, we have that the creation operators, \( \hat{a}_i^\dagger \) and \( \hat{a}_j^\dagger \), for the input modes of a beamsplitter and those, \( \hat{b}_i^\dagger \) and \( \hat{b}_j^\dagger \), for its output modes satisfy the relations \( \hat{b}_i^\dagger = \hat{a}_i^\dagger + \hat{a}_j^\dagger \) and \( \hat{b}_j^\dagger = \hat{r} \hat{a}_i^\dagger + \hat{r}^* \hat{a}_j^\dagger \), where the parameters \( r, t, r' \) and \( t' \) fulfill \( |r|^2 + |r'|^2 = 1, |t|^2 = |t'|, |r| = |r'| \) and \( t + r' = 0 \) [49]. That is, if the state at the input spatial modes \( a_1 \) and \( a_2 \) is say \( |n_1, n_2\rangle_{a_1,a_2} \) (i.e. it consists of \( n_1 \) and \( n_2 \) indistinguishable photons respectively), the state at the output modes \( b_1 \) and \( b_2 \) is given by the following coherent superposition of Fock states

\[
|\psi_{\text{out}}\rangle_{b_1,b_2} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} \binom{(n_2-j+i)(n_1-i+j)!}{n_2!n_1!} \eta^{n_2+n_j-i} (1-\eta)^{j+i} (-1)^i |n_1-i+j, n_2-j+i\rangle_{b_1,b_2},
\]

where, for simplicity, we have considered the particular case in which \( r = -\sqrt{1-\eta}, r' = -r \) and \( t' = t = \sqrt{\eta} \), with \( \eta \) being the transmittance of the beamsplitter. From equation (6) one could directly calculate the theoretical probability distribution \( P(x_1, x_2|n_1, n_2) = |\langle x_1|x_2|n_1, n_2\rangle|^2 \) of finding respectively \( x_1 \) and \( x_2 \) photons at the output ports \( b_1 \) and \( b_2 \) of the beamsplitter, given that there are \( n_1 \) and \( n_2 \) photons at its input ports \( a_1 \) and \( a_2 \). Importantly, according to quantum mechanics, the value of this probability strongly differs from that of a classical scenario, where the photons are considered distinguishable particles which do not interfere. The HOM dip [1] is a well-known example of this fact. Indeed, when two photons input a 50:50 beamsplitter through a different input port, classical mechanics predicts a probability equal to 1/2 of finding the two photons at different output ports of the beamsplitter, while quantum mechanics predicts (for indistinguishable photons) that this probability is equal to zero. In general, this difference between the predictions of quantum and classical mechanics can be quantified by means of the visibility, which is defined as

\[
V_{x_1,x_2|n_1,n_2} = \frac{P(x_1, x_2|n_1, n_2) - P(x_1, x_2|n_1, n_2)}{P(x_1, x_2|n_1, n_2)},
\]

where the subindex \( c \) denotes the classical case, i.e. when the photons are perfectly distinguishable.

Equation (7) has been experimentally evaluated in many different experiments over recent years. For instance, in [44] and [50], the authors obtain visibilities \( V_{x_1,x_2|n_1,n_2} \) equal to 88% for a four-photon interference scheme within an asymmetric beamsplitter and \( V_{x_1,x_2|n_1,n_2} \) equal to 92% for a six-photon interference scheme, respectively. For this, they use type-II parametric down-conversion sources to generate pairs of entangled photons and a measurement setup with four and six threshold single-photon detectors, respectively, in combination with beamsplitters. Also, in the experiment reported in [50], the authors interfere two bosonic atoms (instead of photons) and observe a visibility equal to about 65%.

We now apply our method based on two sources of phase-randomized WCPs and two threshold single-photon detectors to evaluate the visibility \( V_{x_1,x_2|n_1,n_2} \). As in the general case considered in the previous section, it is straightforward to show that by varying the intensity \( \mu_i \) of the input signals at the \( i \)th input port of the beamsplitter, as well as the attenuator’s transmittance \( \omega_j \) (and thus the effective detector’s efficiency \( \kappa_j \)) at its \( j \)th output port, with \( j = 1, 2 \), one can generate an arbitrary number of inequalities that involve the unknown probabilities \( P(x_1, x_2|n_1, n_2) \). The final system of linear equations, particularized from equation (5), is given by

\[
P_{\theta}^{n_1} = \sum_{n_1,n_2} \sum_{x_1,x_2} \sum_{x_1+x_2=n_1+n_2} P^{n_1}(x_1, x_2|n_1, n_2) P^{n_2}(\theta|x_1, x_2),
\]

for each one of the four possible detection patterns \( \theta \equiv (\theta_1, \theta_2) \in \{00, 01, 10, 11\} \). Again, in a real experiment, the probabilities \( P_{n_1}^{n_2}(\theta|x_1, x_2) \) are given by equation (4), are known given the experimental sets \( \mu \) and \( \kappa \) together with the value of the dark count
probability of the detectors, while the probabilities $P_{\theta}^{\nu,\kappa}$ can be directly observed in the experiment, once performed. For our simulations, we use as observed values $P_{\theta}^{\nu,\kappa}$ those predicted by quantum mechanics (see appendix A for more details).

To solve the set of linear equations given by equation (8), one can use analytical or numerical tools. For simplicity, here we solve equation (8) numerically. For this, we first transform the set of equalities given by equation (8), which contains an infinite number of unknowns $P(x_0, x_1|n_1, n_2)$, into a set of inequalities with a finite number of unknowns, as shown in appendix B. Also, we use the linear programming solver Gurobi [50] and the Matlab interface Yalmip [51].

Just as an example, figure 2 shows our results for the conditional probabilities $P(3, 3|3, 3)$ and $P(5, 1|5, 1)$ in beamsplitters with transmittance $\eta = 1/2$ and $\eta = 5/6$ respectively, as a function of the relative delay $dT/\Delta T$ between the arrival times of the phase-randomized WCPs at the two input ports of the beamsplitter. Here $dT$ denotes the absolute delay between the arrival times of the optical pulses at each input port of the beamsplitter and $\Delta T$ is the full-width-half-maximum (FWHM) of the pulses, which for simplicity we assume is equal for all of them. We have chosen these particular examples because quantum mechanics predicts that these probabilities are equal to zero (i.e. complete destructive interference) when $dT/\Delta T = 0$. As we can see from figure 2, our estimations approximate the theoretical value very well; the simulated lower bounds for the visibilities $V_{5,3|3,3}$ and $V_{5,1|5,1}$ are very close to one. To be precise, we obtain $V_{5,3|3,3} \geq 0.99994$ and $V_{5,1|5,1} \geq 0.99996$. The main reasons for the slightly noisy behaviour of the estimated values, as well as for the small discrepancy between these and the theoretical values predicted by quantum mechanics (especially when $dT/\Delta T \approx 0$), are twofold. First, as we have already mentioned above, in our simulations we use a relatively small number of decoy-state/detector-decoy settings. In particular, for each value of $dT/\Delta T$, we choose an optimized set of six possible values for the input parameters $\mu_1$ and $\mu_2$ and five possible values for the output parameters $\kappa_1$ and $\kappa_2$. By using a larger number of settings, one could in principle approximate the theoretical value as closely as desired. The second reason is the limited numerical precision of the linear solver together with the fact that, as explained in appendix B, to solve equation (8) numerically we reduce the number of unknowns $P(x_0, x_1|n_1, n_2)$ to a final set. Also, we emphasise that the upper and lower bounds illustrated in figure 2 depend on the absolute value of $dT/\Delta T$. This is because the experimental data $P_{\theta}^{\nu,\kappa}$ that we use in our simulations depend on $|dT/\Delta T|$ (see appendix A for further details).

Finally, let us remark that when we try to estimate the conditional probabilities $P(x_0, x_1|n_1, n_2)$ for higher total input photon numbers, the accuracy of the estimation decreases. This is so because the value of the coefficients $P^\kappa_{\theta} P^\nu(\theta|x)$ decreases very rapidly when $n$ increases, which renders the estimation problem difficult to solve numerically even with strong scaling methods. Moreover, increasing the value of the intensity setting $\mu$ is not of much help here, since it entails an increase of the leftover term (see appendix B). Possible solutions might be to try to solve the set of linear equations analytically by means of—say—Gaussian elimination, or to develop more efficient numerical estimation methods with higher numerical precision. Also, one could replace the detector-decoy method with practical photon number resolving detectors like, for example, those based on time-multiplexing [52], or those introduced in [23]. A potential advantage of this latter approach would be that

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Figure 2. Hong–Ou–Mandel dip for the conditional probabilities $P(3, 3|3, 3)$ and $P(5, 1|5, 1)$ at a beamsplitter of transmittance $\eta = 1/2$ and $\eta = 5/6$ respectively, as a function of the relative delay $dT/\Delta T$. Here, $dT$ denotes the absolute delay between the arrival times of the optical pulses at each input port of the beamsplitter and $\Delta T$ is the FWHM of the pulses. The theoretical values predicted by quantum mechanics are illustrated with solid and dashed black lines, respectively. The blue (red) dots and crosses show the upper (lower) bound for these probabilities obtained with our method based on the use of two laser sources emitting phase-randomized WCPs and two threshold single-photon detectors. In our simulations, we consider that the dark count probability is $P_{\text{dark}} = 10^{-6}$, and the transmittances $\gamma_i(n_i)$ take six (five) different values.
now the linear program that estimates the input-output statistics of the ON would be simpler and involve fewer unknowns, and thus it could handle a larger number of photons. The main drawback of this approach is, however, that it requires the use of PNR detectors, which are more expensive resources than threshold single-photon detectors. In any case, it would be definitively interesting to investigate these three options further.

3.2. Second example: tritter

We now estimate the three coincidence detection probability $P(1, 1, 1|1, 1, 1)$ for a tritter for two different scenarios. Both scenarios have been experimentally analysed very recently in [40], where the authors used heralded single-photon sources (based on spontaneous four-wave mixing in silica-on-silicon waveguides together with three threshold single-photon detectors for heralding) in combination with a measurement setup with five threshold single-photon detectors. If we denote by $(\psi_j|\psi_k) = r_{jk} e^{i\phi_{jk}}$ the inner product between the states of the single-photon signals at the $j$th and $k$th input ports of the tritter, quantum mechanics predicts that the probability $P(1, 1, 1|1, 1, 1)$ is given by [40, 53]

$$P(1, 1, 1|1, 1, 1) = \frac{1}{9} (2 + 4n_2r_{23}r_{32} \cos(\phi) - r_{12}^2 - r_{23}^2 - r_{31}^2),$$

where $\phi = \phi_{12} + \phi_{23} + \phi_{31}$ is the so-called collective triad phase.

The first scenario that we consider is shown in figure 3(a). In this case, the input pulses to the tritter have the same polarization state, but their arrival times at the various input ports of the tritter vary. The result predicted by quantum theory for $P(1, 1, 1|1, 1, 1)$ in this situation is shown with a solid line in figure 3(a), while our estimations are shown with dots. Again, we can see that the estimated upper and lower bounds for $P(1, 1, 1|1, 1, 1)$ fit the theoretical probability tightly. In the second scenario, the polarization states of the input light pulses are now chosen to compensate the temporal distinguishability between the arriving photons, and might be different for the signals at each input port. The motivation for this scenario is to observe the dependence that the three-photon coincidence probability $P(1, 1, 1|1, 1, 1)$ has on the triad phase $\phi$ by keeping constant all those terms $r_{jk}$ that affect such probability but arise from two-photon distinguishability [40]. The results are shown in figure 3(b), where once again we can see that our method provides a tight estimation of the theoretical values, thus showing its practicality. Also, we remark that, as in the case of figure 2, the upper and lower bounds illustrated in figure 3 depend on $|dT/\Delta T|$ because in our simulations the experimental data $F_\theta^{\mu\nu}$ depend on $|dT/\Delta T|$. Here we use the expected values predicted by quantum mechanics for $F_\theta^{\mu\nu}$. Furthermore, for each value of $dT/\Delta T$ in figure 3(a), and for each value of $\phi$ in figure 3(b), we choose three different values for the intensities of the phase-randomized WCPs, as well as two possible values for the output attenuators.

4. Conclusion

In this paper, we have proposed a simple method to experimentally characterize the behaviour of small-size optical networks (ONs) for input signals that are tensor products of Fock states. More precisely, our method
could be used to obtain a tight estimation of the input—output photon number statistics of an ON. Importantly, our technique could easily be implemented with current technology, such as—for instance—phase-randomized weak coherent pulses together with threshold single-photon detectors. The main idea of the method is rather simple: it estimates the statistics provided by ideal n-photon sources at the input ports of an ON by means of decoy-state techniques and estimates the statistics provided by ideal photon number resolving detectors at its output ports by means of detector—decoy techniques.

To illustrate the practicality of the method, we have evaluated two simple examples. In the first, we have estimated the generalized Hong—Ou—Mandell dip in a beamsplitter for a total number of six input photons, while in the second we have estimated the three coincidence detection probability in a tritter when it receives one single-photon pulse at each of its input ports. In both cases, we have obtained tight estimates that approximate the theoretical values very well.

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Appendix A. Toy model for the experimental data \( P_{\theta}^{\mu_{k}, \kappa} \)

In order to evaluate the performance of our technique, we need to generate the experimental data \( P_{\theta}^{\mu_{k}, \kappa} \) which is required to run the simulations. For this, and in the absence of a real experiment, we use a simple mathematical model that we detail below. In particular, let \( A' \) (\( B' \)) be the creation operators for the input (output) spatial modes of the ON. That is, \( A' = [\hat{a}^+_1, \ldots, \hat{a}^+_N]^{T} \) and \( B' \) is defined similarly. These vectors satisfy

\[
A' = UB',
\]

where \( U \) is the unitary transformation that describes the behaviour of the ON.

In the case of WCPs, the input state to the ON can be written as \( |\psi_{\text{in}}\rangle = \bigotimes_{k=1}^N |\psi_{\text{in},k}\rangle \), where

\[
|\psi_{\text{in},k}\rangle = \exp\left(\int (\alpha_k(\omega)\hat{a}^+_k(\omega) - \alpha_k^+(\omega)\hat{a}_k(\omega))d\omega\right) |0_k\rangle,
\]

is the coherent state at the \( k \)th input mode [54]. Here, the parameters \( \alpha_k(\omega) \) are defined as

\[
\alpha_k(\omega) = \sqrt{\mu_k} \exp\left(-\frac{\omega^2}{4\sigma^2}\right) e^{i\phi_k-k\omega},
\]

That is, for simplicity, we assume that each \( |\alpha_k(\omega)|^2 \) follows a Gaussian distribution of mean zero and standard deviation \( \sigma \) which is multiplied by the intensity \( \mu_k \) to guarantee that the condition \( \int |\alpha_k(\omega)|^2d\omega = \mu_k \) holds.

The temporal parameter \( t_k \) represents the arrival time of the optical pulse that enters the ON through its \( k \)th input port. We remark that in the definition of the states \( |\psi_{\text{in},k}\rangle \), we have not yet included the fact that their phases \( \phi_k \) are randomized. We will return to this point later.

Let \{ \( u_{jk} \) \} be the elements of the unitary matrix \( U \). By applying equation (A.1), and due to the linearity of the integral, we have that the state at the output ports of the ON can be written as \( |\psi_{\text{out}}\rangle = \bigotimes_{k=1}^M |\psi_{\text{out},k}\rangle \), where

\[
|\psi_{\text{out},k}\rangle = \exp\left(\int (\beta_k(\omega)\hat{b}^+_k(\omega) - \beta_k^+(\omega)\hat{b}_k(\omega))d\omega\right) |0_k\rangle,
\]

and \( \beta_k(\omega) = \sum_{j=1}^N \alpha_j(\omega)u_{jk} \). This means that the state \( |\psi_{\text{out}}\rangle \) at the output ports of the attenuators of efficiency \( \kappa \) is given by

\[
|\psi_{\text{out}}\rangle = \exp\left(\sum_{k=1}^M \sqrt{\kappa_k} \int (\beta_k(\omega)\hat{b}^+_k(\omega) - \beta_k^+(\omega)\hat{b}_k(\omega))d\omega\right) |0\rangle.
\]

For convenience, note that here—as in the main text—we have included the effect of the efficiencies \( \eta_D \) of the threshold single-photon detectors into the efficiency of the attenuators.

The probability of having vacuum in a specific output mode \( k \) is related to the mean photon number \( \bar{n}_k = \int |\beta_k(\omega)|^2d\omega \) of the coherent state in that mode by \( P_0 = e^{-\bar{n}_k} \). In order to calculate \( \bar{n}_k \), let \( \varphi_{jk} \) be the phase of the element \( u_{jk} \) of \( U \), i.e. \( u_{jk} = |u_{jk}|e^{i\varphi_{jk}} \). It is then straightforward to show that
\[
|\beta_k(\omega)|^2 = \sum_{i=1}^{N} \left|\alpha_i(\omega)\right|^2 |u_{ik}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} 2|\alpha_i(\omega)||\alpha_j(\omega)||u_{ik}||u_{kj}| \times \cos(\phi_j - \phi_i + \varphi_{kj} - \varphi_{ki} + \omega(t_j - t_i)).
\] (A.6)

This means, in particular, that
\[
\bar{n}_k \equiv \int |\beta_k(\omega)|^2 d\omega = \sum_{j=1}^{N} |\mu_j u_{jk}|^2 + \sum_{i=1}^{N-1} \sum_{j=1}^{N-i} 2\sqrt{\mu_j \mu_k} |u_{ik}| |u_{kj}| \exp \left(-\frac{\tau_{ij}^2 \ln 2}{\Delta T^2}\right) \times \cos(\phi_j - \phi_i + \varphi_{kj} - \varphi_{ki}),
\] (A.7)

where \(\tau_{ij} = t_j - t_i\) represents the delay between the arrival times of the pulses that enter the ON through its \(i\)th and \(j\)th input ports, and \(\Delta T\) is their FWHM. Finally, we have that the joint probability of detecting a certain pattern \(\theta\) on the threshold single-photon detectors is given by
\[
P^\mu_{\theta} = \prod_{k=1}^{M} \left[ \frac{1 - (1 - \theta_k)P_{\text{dark}}}{2} + (-1)^{\theta_k}(1 - P_{\text{dark}})e^{-\gamma_{nk}} \right],
\] (A.8)

where \(\theta = \{\theta_1, \ldots, \theta_N\}\) represents the dependence of that probability on the phase of each input coherent pulse. This is so because the probability of having no click at output port \(k\) (that is, \(\theta_k = 0\)) is given by \(P^\mu_{\theta} = (1 - P_{\text{dark}})e^{-\gamma_{nk}}\), and thus the probability of having a click \((\theta_k = 1)\) has the form \(P^\mu_{\theta} = 1 - (1 - P_{\text{dark}})e^{-\gamma_{nk}}\).

If we consider now the fact that the input coherent states are phase-randomized, we find that the probability of detecting the pattern \(\theta\) on the threshold single-photon detectors \(D_k\) is given by
\[
P^\mu_{\theta,k} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\mu_{\theta} d\phi_1 d\phi_2 \cdots d\phi_N,
\] (A.9)

which can be calculated numerically, or even analytically for the simplest cases.

### Appendix B. Numerical estimation with linear programming

For small values of the intensities \(\mu = \{\mu_1, \ldots, \mu_N\}\), we have that the coefficients \(P^\mu_{\theta} P^\nu(\theta|x)\) of the set of linear equations given by equation (5) drop quickly to zero as the number of photons \(n \equiv n_1, \ldots, n_M\) increases. Therefore, one can neglect some of the terms in equation (5) to decrease the number of unknowns \(P(x|n)\) to a finite set. For instance, one can discard all the summation terms that satisfy \(\sum_{i=1}^{N} n_i > M_{\text{cut}}\), for a certain prefixed parameter \(M_{\text{cut}}\). In this way, we obtain that
\[
P^\mu_{\theta,k} \geq \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x),
\] (B.1)

where \(S_{\text{cut}}\) is the subset that contains all possible \(n\) such that \(\sum_{i=1}^{N} n_i \leq M_{\text{cut}}\). Similarly, one could also obtain an upper bound on \(P^\mu_{\theta,k}\) that depends on the same finite number of unknowns \(P(x|n)\). For this, note that
\[
P^\mu_{\theta} = \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x) + \sum_{n \notin S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x)
\leq \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x) + \sum_{n \notin S_{\text{cut}}} P^\mu_{n} \sum_{x \in n} P(x|n)
= \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x) + \left[ 1 - \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} \right]
= \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n} P(x|n) P^\nu(\theta|x) + \Lambda_{S_{\text{cut}}},
\] (B.2)

where the first inequality is due to the fact that \(P^\nu(\theta|x) \leq 1\) and the second equality comes from \(\sum_{x \in n} P(x|n) = 1\) and \(\sum_{n} P^\mu_{n} = 1, \forall n\). Obviously, the leftover term \(\Lambda_{S_{\text{cut}}} = 1 - \sum_{n \in S_{\text{cut}}} \sum_{x \in n} P^\mu_{n}\) should be as small as possible.

By using this result, one can numerically obtain an upper bound for the probability \(P(x|n)\) by solving the following linear program:
The method proposed in this work allows the estimation of the interference probabilities \(P(\mathbf{x}|\mathbf{n})\) for general ONs even when the threshold detectors have a relatively low detection efficiency. Indeed, the main limitation of low detection efficiencies arises because the values \(\kappa_i \in [0, \eta_D]\) of the set of output effective attenuators are upper bounded by \(\eta_D\), with \(j = 1, 2, \ldots, M\). When \(\eta_D \ll 1\), this significantly reduces the range of \(\kappa_i\), and thus the accuracy of the estimation for a fixed number of decoy settings. Importantly, however, it is possible to mitigate this effect by increasing the number of decoy-state/detector-decoy settings, as this generates more constraints for the linear program. This is illustrated in figure C1, where—for simplicity—we consider upper bounds on the input—output statistics \(P(1, 1|1, 1)\) and \(P(2, 2|2, 2)\) of a beamsplitter of transmittance \(\eta = 1/2\). The roughness of the upper bounds is mainly due to the finite size of the optimization grid. Note that, in theory, for a continuous optimization grid we have that \(U_b [P(\mathbf{x}|\mathbf{n})] \leq U_b [P(\mathbf{x}|\mathbf{n})]\) for \(\eta_D > \eta_D^I\), being \(U_b [P(\mathbf{x}|\mathbf{n})]\) the upper bound on \(P(\mathbf{x}|\mathbf{n})\) calculated with detectors of efficiency \(\eta_D^I\). This is because the optimal input and output settings when the efficiency is \(\eta_D^I\) are always also accessible when the efficiency is \(\eta_D\).

### Appendix C. Effect of the detection efficiency in the estimation

The method proposed in this work allows the estimation of the interference probabilities \(P(\mathbf{x}|\mathbf{n})\) for general ONs even when the threshold detectors have a relatively low detection efficiency. Indeed, the main limitation of low detection efficiencies arises because the values \(\kappa_i \in [0, \eta_D]\) of the set of output effective attenuators are upper bounded by \(\eta_D\), with \(j = 1, 2, \ldots, M\). When \(\eta_D \ll 1\), this significantly reduces the range of \(\kappa_i\), and thus the accuracy of the estimation for a fixed number of decoy settings. Importantly, however, it is possible to mitigate this effect by increasing the number of decoy-state/detector-decoy settings, as this generates more constraints for the linear program. This is illustrated in figure C1, where—for simplicity—we consider upper bounds on the input—output statistics \(P(1, 1|1, 1)\) and \(P(2, 2|2, 2)\) of a beamsplitter of transmittance \(\eta = 1/2\). The roughness of the upper bounds is mainly due to the finite size of the optimization grid. Note that, in theory, for a continuous optimization grid we have that \(U_b [P(\mathbf{x}|\mathbf{n})] \leq U_b [P(\mathbf{x}|\mathbf{n})]\) for \(\eta_D > \eta_D^I\), being \(U_b [P(\mathbf{x}|\mathbf{n})]\) the upper bound on \(P(\mathbf{x}|\mathbf{n})\) calculated with detectors of efficiency \(\eta_D^I\). This is because the optimal input and output settings when the efficiency is \(\eta_D^I\) are always also accessible when the efficiency is \(\eta_D\).

### References

[1] Hong C.K, Ou Z.Y and Mandel L. 1987 Phys. Rev. Lett. 59 2044–6
[2] Ou Z.Y, Rhee J.K and Wang I.I 1998 Phys. Rev. Lett. 83 959–62
[3] Liu B.H, Sun F.W, Gong Y.X, Huang Y.F, Ou Z.Y and Guo G.C 2007 Europhys. Lett. 77 24003
[4] Liu B.H, Sun F.W, Gong Y.X, Huang Y.F, Guo G.C and Ou Z.Y 2007 Opt. Lett. 32 1320–2
[5] Xiang G.Y, Huang Y.F, Sun F.W, Zhang P., Ou Z.Y and Guo G.C 2006 Phys. Rev. Lett. 97 023604
[6] Niu X.L, Gong Y.X, Liu B.H, Huang Y.F, Guo G.C and Ou Z.2009 Opt. Lett. 34 1297–9
[7] Kobayashi T, Ikuta R, Yasu S, Miki S, Yamashita T, Terai H, Yamamoto T, Koashi M and Imoto N 2016 Nat. Photonics 10 441–4
[8] Lopes R, Imanaliev A, Aspect A, Cheneau M, Boiron D and Westbrook C 2015 Nature 520 66–8
[9] Knill E, Laﬂamme R and Milburn G J 2001 Nature 409 46–52
[10] Lo H.K, Curty M and Qi B 2012 Phys. Rev. Lett. 108 130503
[11] Yin H.L et al 2016 Phys. Rev. Lett. 117 190501
[12] Aaronson S and Arkhipov A 2011 Proc. of the forty-third Annual ACM Symposium on Theory of Computing (New York: ACM) pp 335–42
[13] Broome M.A, Fedrizzi A, Rahimi-Keshari S, Dove J, Aaronson S, Ralph T and White A G 2013 Science 339 794–8
[14] Spring J B et al 2013 Science 339 798–801
[15] Tillmann M, Dakić B, Heilmann R, Nolte S, Szameit A and Walther P 2013 Nat. Photonics 7 540–4
[16] Crespi A, Osellame R, Ramponi R, Brod D J, Galvão E F, Spagnolo N, Vitelli C, Maiorino E, Mataloni P and Sciarrino F 2013 Nat. Photonics 7 545–9
[17] Quan R, Zhai Y, Wang M, Hou F, Wang S, Xiang X, Liu T, Zhang S and Dong R 2016 Sci. Rep. 6 30453
[18] Bell B, Kannan S, McMillan A, Clark A S, Wadsworth W J and Rarity J G 2013 Phys. Rev. Lett. 111 093603
[19] O’Brien J L, Furusawa A and Vulečkovic J 2009 Nat. Photonics 3 687–95
[20] Lobino M, Korystov D, Kupchak C, Figueroa E, Sanders B C and Lvovsky A 2008 Science 322 563–6
[21] Wang X. et al 2016 Phys. Rev. Lett. 117 210502
[22] Ding X et al 2016 Phys. Rev. Lett. 116 020401
[23] Kardynal B E, Yuan Z L and Shields A J 2008 Nat. Photonics 2 425–8
[24] Lita A E, Miller A J and Nam S W 2008 Opt. Express 16 3032–40
[25] Harder G, Silberhorn C, Rehacek J, Hradil Z, Motka L, Stoklasa B and Sánchez-Soto L I 2016 Phys. Rev. Lett. 116 133601
[26] Navarrete A 2015 Interferencia cuántica en redes ópticas pasivas Bachelors Thesis University of Vigo
[27] Troyansky L and Tishby N 1996 Proceedings of the Fourth Workshop on Physics and Computation (Proc. PhysComp’96) (Boston, MA)
[28] Hwang W Y 2003 Phys. Rev. Lett. 91 057901
[29] Lo H K, Ma X and Chen K 2005 Phys. Rev. Lett. 94 230504
[30] Wang X B 2005 Phys. Rev. Lett. 94 230503
[31] Moroder T, Curty M and Lütkenhaus N 2009 New J. Phys. 11 045008
[32] Lim C C W, Walenta N, Légré M, Gisin N and Zbinden H 2015 IEEE J. Sel. Topics Quantum Electron. 21 192–6
[33] Peng C Z, Zhang J, Yang D, Gao W B, Ma H X, Yin H, Zeng H P, Yang T, Wang X B and Pan J W 2007 Phys. Rev. Lett. 98 010505
[34] Rosenberg D, Harrington J W, Rice P R, Hiskett P A, Peterson C G, Hughes R J, Lita A E, Lita A E, Nam S W and Nordholt J E 2007 Phys. Rev. Lett. 98 010503
[35] Yuan Z L, Kardynal B E, Sharpe A W and Shields A J 2007 Appl. Phys. Lett. 91 041114
[36] Curty M, Xu F, Cui W, Lim C C W, Tamaki K and Lo H K 2014 Nat. Commun. 5 3732
[37] Xu F, Curty M, Qi B and Lo H K 2013 New J. Phys. 15 113007
[38] Yuan X, Zhang Z, Lütkenhaus N and Ma X 2016 Phys. Rev. A 94 062305
[39] Valente P and Lezama A 2017 J. Opt. Soc. Am. B 34 924–9
[40] Menssen A I, Jones A E, Metcalf B J, Tichy M C, Barz S, Kolthammer W S and Walmsley I A 2017 Phys. Rev. Lett. 118 153603
[41] Yurke B 1985 Phys. Rev. A 32 311
[42] Hadfield R H 2009 Nat. Photonics 3 696–705
[43] Lim C C W, Curty M, Walenta N, Xu F and Zbinden H 2014 Phys. Rev. A 89 022307
[44] Chernoff H 1952 Ann. Math. Stat. 435–97
[45] Hoeffding W 1963 J. Amer. Statist. Assoc. 58 13–30
[46] Mizutani A, Curty M, Lim C C W, Imoto N and Tamaki K 2015 New J. Phys. 17 093011
[47] Jofre M, Curty M, Steinlechner F, Anzolin G, Torres J P, Mitchell M W and Pruneri V 2011 Opt. Express 19 20665–72
[48] Abellan C, Amaya W, Jofre M, Curty M, Acín A, Capmany J, Pruneri V and Mitchell M W 2014 Opt. Express 22 1645–54
[49] Ou Z Y 2007 Multi-Photon Quantum Interference 1st edn (Berlin: Springer)
[50] Gurobi Optimization, Inc. 2016 Gurobi Optimizer reference manual http://gurobi.com
[51] Löfberg J 2004 Proc. of the CACSD Conf. (Taipei, Taiwan) pp 284–9
[52] Achilles D, Silberhorn C, Siwa C, Banaszek K, Walmsley I A, Fitch M J, Jacobs B C, Pittman T B and Franson J D 2004 J. Modern Optics 51 1499
[53] Tichy M C 2015 Phys. Rev. A 91 022316
[54] Blow K J, Loudon R, Phoenix S J and Shepherd T J 1990 Phys. Rev. A 42 4102–14