Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness

Masahiko Miyamoto *

Institute of Mathematics
University of Tsukuba
Tsukuba 305, Japan

Abstract

We investigate trace functions of modules for vertex operator algebras satisfying $C_2$-cofiniteness. For the modular invariance property, Zhu assumed two conditions in [Zh]: $A(V)$ is semisimple and $C_2$-cofiniteness. We show that $C_2$-cofiniteness is enough to prove a modular invariance property. For example, if a VOA $V = \bigoplus_{m=0}^{\infty} V_m$ is $C_2$-cofinite, then the space spanned by generalized characters (pseudo-trace functions of the vacuum element) of $V$-modules is a finite dimensional $SL_2(\mathbb{Z})$-invariant space and the central charge and conformal weights are all rational numbers. Namely we show that $C_2$-cofiniteness implies “rational conformal field theory” in a sense as expected in [GN]. Viewing a trace map as one of symmetric linear maps and using a result of symmetric algebras, we introduce “pseudo-traces” and pseudo-trace functions and then show that the space spanned by such pseudo-trace functions has a modular invariance property. We also show that $C_2$-cofiniteness is equivalent to the condition that every weak module is an $\mathbb{N}$-graded weak module which is a direct sum of generalized eigenspaces of $L(0)$.

1 Introduction

In this paper, we will consider a vertex operator algebra $V = \bigsqcup_{n=0}^{\infty} V_n$ with central charge $c$. One of the central concepts in conformal field theory (CFT) is “rationality,” a condition which is supposed to express a kind of finiteness of the theory. There exists various notions of finiteness. One of them is the complete reducibility of $\mathbb{N}$-graded weak modules, which is the condition called rationality by the most researchers on vertex operator algebras. There is another important finiteness condition called “$C_2$-cofiniteness”. Complete reducibility of $\mathbb{N}$-graded weak modules is a condition for modules. In this case, it was shown in [DLiM] that Zhu algebra $A(V) = V/O(V)$ is a finite dimensional semisimple algebra

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and \( V \) has only finitely many irreducible modules. On the other hand, \( C_2 \)-cofiniteness is a property of \( V \) itself, that is, if \( C_2(V) = \langle a(-2)b \mid a, b \in V \rangle \) is of finite codimension in \( V \), then \( V \) is called \( C_2 \)-cofinite. Here \( Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \) is the vertex operator of \( a \). Then it was shown in [DLiM3] that \( A(V) \) is of finite dimension and so \( V \) has only finitely many irreducible modules, say \( \{W^1, \ldots, W^k\} \). Under two hypotheses that \( A(V) \) is semisimple and \( V \) is \( C_2 \)-cofinite, (Condition \( C \) in his paper), Zhu proved in [Z] that the space spanned by trace functions

\[
S^W(v, \tau) = \text{tr}_{|W} o(v) q^L(0) - c/24 \quad (q = e^{2\pi i \tau} \text{ and } \tau \in \mathbb{H})
\]

has a modular-invariance (\( SL(2, \mathbb{Z}) \)-invariance) property, where \( c \) is the central charge of \( V \), \( \mathbb{H} \) is the upper half plane \( \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \} \) and \( o(v) \) is the grade-preserving operator of \( v \in V \). Namely, for \( \left( \begin{array}{c} a \\ d \end{array} \right) \in SL(2, \mathbb{Z}) \), there is a \( k \times k \)-matrix \( (\lambda_{ij}) \) with \( \lambda_{ij} \in \mathbb{C} \) such that

\[
\frac{1}{(c\tau + d)^n} \left( S^{W^1}(u, \frac{a\tau + b}{c\tau + d}) \cdots S^{W^k}(u, \frac{a\tau + b}{c\tau + d}) \right) = \left( S^{W^1}(u, \tau) \cdots S^{W^k}(u, \tau) \right) \left( \begin{array}{c} \lambda_{11} \cdots \lambda_{1k} \\ \vdots \\ \lambda_{k1} \cdots \lambda_{kk} \end{array} \right)
\]

for all \( u \in V[n] \) and \( n = 0, 1, \ldots \), where \( V = \oplus_{n=0}^\infty V[n] \) is the second grading on \( V \) introduced in [Zh], see Definition 2.3. Actually, Zhu assumed one more condition, but it is not necessary as mentioned in [DLiM3]. This is a fundamental result for modular invariance properties of vertex operator algebras and is extended by several authors, e.g. [DLiM3], [Miy1]-[Miy3] and [Y].

Since then, it becomes an important problem to study a relation between rationality and \( C_2 \)-cofiniteness. It was once conjectured the equivalency between \( C_2 \)-cofiniteness and rationality, but a few research don’t suggest that \( C_2 \)-cofiniteness implies the complete reducibility, see [GR] and [M1].

On the other hand, \( C_2 \)-cofiniteness was studied by Gaberdiel and Neitzke from a viewpoint of rationality in [GN] and they showed that if \( V \) is a \( C_2 \)-cofinite VOA of CFT type, then \( V \) satisfies the most conditions required for a rational conformal field theory except for a modular invariance property, where \( V \) is called CFT type if \( V = \oplus_{n=0}^\infty V_n \) and \( \text{dim } V_0 = 1 \), see also [Zh], [DLiM3], [L2].

Although \( C_2 \)-cofiniteness was introduced by Zhu as a technical assumption and it is a property of VOA itself, it is a natural condition to consider the characters of all (weak) modules. For, in order to define \( q^{L(0)} \) on a weak module \( W \), \( W \) has to be a direct sum of generalized eigenspaces of \( L(0) \), which is a condition equivalent to \( C_2 \)-cofiniteness.

**Theorem 2.7**  Let \( V \) be a vertex operator algebra. Then the following are equivalent.

1. \( V \) is \( C_2 \)-cofinite.
2. Every weak module is a direct sum of generalized eigenspaces of \( L(0) \).
3. Every weak module is an \( \mathbb{N} \)-graded weak module \( W = \oplus_{n=0}^\infty W(n) \) such that \( W(n) \) is a direct sum of generalized eigenspaces of \( L(0) \).
4. \( V \) is finitely generated and every weak module is an \( \mathbb{N} \)-graded weak module.
Therefore if $V$ is $C_2$-cofinite, then we can define several modules naturally. For example, for $\mathbb{N}$-graded weak $V$-modules $U$, the maximal weak $V$-submodule $D(U)$ of $\text{Hom}(U, \mathbb{C})$ introduced in $[\text{Li2}]$ is an $\mathbb{N}$-graded weak module. The most results about $C_2$-cofiniteness came from the existence of some spanning set, e.g. $[\text{LN}], [\text{Li2}], [\text{BG}], [\text{ABL}]$. We will prove the existence of the following spanning set for a general VOA without negative weights.

**Lemma 2.4** Let $A$ be a set of homogeneous elements of $V$ such that $V = C_2(V) + \langle A \rangle$. Assume that $V$ is $C_2$-cofinite and $W$ is a weak module generated from $w$ (by the action of vertex operator). Then $W$ is spanned by the following elements
\[
\{ v^1(i_1) \cdots v^k(i_k)w \mid v^i \in A, \quad i_1 < \cdots < i_k \}.
\]

In particular, if we set
\[
W(m) = \langle v^1(i_1) \cdots v^k(i_k)w \mid v^i \in V, \quad \deg(v^1(i_1) \cdots v^k(i_k)) = m \rangle,
\]
then $W(m) = 0$ for $m \ll 0$, where $\deg(v(i) \cdots u(j))$ denotes the degree of $v(i) \cdots u(j)$ as an operator.

The main purpose of this paper is to show that $C_2$-cofiniteness is enough for a modular invariance property without assuming that $A(V)$ is semisimple. As a result, the central charge $c$ and the conformal weights are all rational numbers. Namely, $C_2$-cofiniteness provides the conditions required for a rational conformal field in a sense.

In the proof of modular invariance property in $[\text{Zh}]$, Zhu introduced the space $C_1(V)$ of one point functions, see also $[\text{DLiM3}]$. He proved the modular invariance property by showing that $C_1(V)$ is spanned by trace functions $S^W(v, \tau) = \text{tr}_{W'}(v) q^{L(0) - c/24}$. However, if $A(V)$ is not semisimple, $C_1(V)$ may not be spanned by trace functions. One of our aims in this paper is to stuff suitable functions into a crevice. We introduce a new kind of trace map called a pseudo-trace map (different from pseudo trace in algebraic number field) on some kind of $\mathbb{N}$-graded weak V-modules with homogeneous spaces of finite dimension. One of the key steps in Zhu’s proof is that $C_1(V)$ is spanned by functions with the form
\[
\sum_{i=0}^N \left( \sum_{j=1}^d \left( \sum_{k=0}^\infty C_{ij,k}(v) q^k q^{r_{ij}} (\ln(q))^i \right) \right).
\]
such that the coefficient $C_{ij,0}$ of the lowest degree, satisfies the conditions:
(1) $C_{ij,0}(O(V)) = 0$,
(2) $C_{ij,0}(ab - ba) = 0$ for all $a, b \in A(V)$ and
(3) $C_{ij,0}(\omega - \omega^{-1} - 2r_{ij})^{N - i + 1} v = 0$.

In particular, $A(V)/\text{Rad}(C_{ij,0})$ is a symmetric algebra with a symmetric linear function $C_{ij,0}$, where $\text{Rad}(\phi) \overset{\text{def}}{=} \{ a \in A(V) \mid \phi(A(V)aA(V)) = 0 \}$. We are not interested in symmetric algebras, but symmetric linear functions. Originally, Nesbitt and Scott showed in $[\text{NS}]$ that $A$ is a symmetric algebra if and only if its basic algebra $P$ is symmetric. This is an idea to explain our strategy in this paper. We will show that $C_{ij,n}$ is a symmetric linear map of $n$-th Zhu algebra $A_n(V)$ (see §2.2) and so we have a symmetric algebra
$A = A_n(V)/\text{Rad}(C_{ij,a})$ and its symmetric basic algebra $P$ with a symmetric linear function $\phi$. We start from $(P, \phi)$ and construct a right $P$-module $W$ (a generalized Verma module) such that the basic algebra of $R = \text{End}_P(W)$ is $P$. We will call such a module $W$ “interlocked with $\phi$.” Nesbitt and Scott’s result tells that $R$ has a symmetric linear map $\text{tr}^\phi$ (we will call it pseudo-trace). Then we will define a pseudo-trace function

$$\text{tr}^\phi_W o(v) q^{L(0) - c/24}$$

and show that $C_1(V)$ is spanned by such pseudo-trace functions.

For example,

$$P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$$

is a basic symmetric algebra with a linear map $\phi(p) = b$. We note $J(P) = \text{soc}(P) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{C} \right\}$. Consider a right $P$-module $T = \mathbb{C}^m \oplus \mathbb{C}^m$. Then

$$\text{End}_P(T) = \left\{ \alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & A_\alpha \end{pmatrix} \mid A_\alpha, B_\alpha \in M_{m,m}(\mathbb{C}) \right\}$$

and the basic algebra of $\text{End}_P(T)$ is $P$. For any $\alpha \in \text{End}_P(T)$, if we define $\text{tr}_T^\phi(\alpha) = \text{tr} \alpha : T/TJ(P) \rightarrow T$, that is, $\text{tr}_T^\phi(\alpha) = \text{tr}(B_\alpha)$, then $\text{tr}_T^\phi$ is also a symmetric linear map.

We should note one more thing. Since we will treat the general cases, $L(0)$ may not act on $W^{(m)}(m)$ as a scalar. However, since $V$ is $C_2$-cofinite, every $n$-th Zhu algebra $A_n(V)$ is finite dimensional as we will see and so every generalized Verma $V$-module is a direct sum of modules $W = \oplus_{m=0}^\infty W(m)$ so that $L(0) - r - m$ acts on $W(m)$ as a nilpotent operator for some $r \in \mathbb{C}$, say $(L(0) - r - m)^s = 0$ on $W(m)$. Let $L^s(0)$ denote the semisimple part of $L(0)$, that is, $m + r$ on $W(m)$. Then trace function $\text{tr}_W^\phi o(v) q^{L(0)}$ on $W$ is defined by

$$\text{tr}_W^\phi o(v) q^{L(0) - c/24} = \text{tr}_W^\phi \left\{ o(v) \sum_{i=0}^{s-1} \frac{(2\pi i \tau)^i (L(0) - L^s(0))^i}{i!} \right\} q^{L^s(0) - c/24}.$$ 

These modules $W$ are called “logarithmic modules”, (see [3, 11]) and Flohr introduced in [1] a concept of generalized characters to interpret the modular invariance property of characters of logarithmic modules. As we will see, $S^{W(\tau)}_{V,1}(1, \tau)$ is a linear combination of (ordinary) characters with coefficients in $\mathbb{C}[\tau]$ and plays a role of a generalized character.

Our main theorem is:

**Theorem 5.5** Let $V$ be a $C_2$-cofinite VOA with central charge $c$. Then for $v \in V_{[m]}$, the set of pseudo-trace functions

$$\left\{ \text{tr}_W^\phi o(v) q^{L(0) - c/24} \mid W \text{ is interlocked with a symmetric linear map } \phi \text{ of } A_n(V) \right\}$$

is invariant under the action of $\text{SL}_2(\mathbb{Z})$ with weight $m$. In particular, $\dim C_1(V) = \dim A_n(V)/[A_n(V), A_n(V)] - \dim A_{n-1}(V)/[A_{n-1}(V), A_{n-1}(V)]$ for $n \gg 0$. 

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In particular, the space spanned by generalized characters \( \text{tr}_W q^{L(0) - c/24} \) is a finite-dimensional \( SL_2(\mathbb{Z}) \)-invariant space. As a corollary, we obtain:

**Corollary 5.10 and 5.11** If \( V \) is a \( C_2 \)-cofinite VOA, then the central charge and the conformal weights are all rational numbers. Moreover, we have

\[
\tilde{c} \leq \frac{\dim V/C_2(V) - 1}{2},
\]

where \( \tilde{c} \) is the effective central charge of \( V \) and \( h_{\min} \) is the smallest conformal weight.

They showed in [GN] that \( C_2 \)-cofiniteness implies \( C_m \)-cofiniteness for any \( m = 1, 2, \ldots \) if \( V \) is of CFT type. In this paper, we will consider only a VOA without negative weights. However, if we consider \( C_{2+s} \)-cofiniteness when \( V = \bigoplus_{n=-s}^{\infty} V_n \) has a negative weight, then it is not difficult to see that we will have a similar result as in Lemma 2.4 and the other results in this paper by replacing \( C_2(V) \) by \( C_{2+s}(V) \) and \( \text{wt}(v) \) by \( \text{wt}(v) - s \), respectively.

This paper is organized as follows. In Section 2, we will explain the notation and fundamental results. In Section 3, we will introduce a concept of modules interlocked with \( \phi \) and define a pseudo-trace map explicitly. In Section 4, we will define a pseudo-trace function for an \( \mathbb{N} \)-graded weak \( V \)-module interlocked with a symmetric function. In particular, we will explain that if we take a sufficiently large integer \( n \) and \( \phi \) is a symmetric linear map of \( A_n(V) \), then we can construct a generalized Verma VOA-modules \( W_T^{(n)} \), which is interlocked with \( \phi \). In Section 5, we will prove a modular invariance property.

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## 2 Fundamental results

### 2.1 Vertex operator algebras

**Definition 2.1** A weak module for VOA \((V, Y, 1, \omega)\) is a vector space \( M \); equipped with a formal power series

\[
Y^M(v, z) = \sum_{n \in \mathbb{Z}} v^M(n)z^{-n-1} \in (\text{End}(M))[z, z^{-1}]
\]

(called the module vertex operator of \( v \)) for \( v \in V \) satisfying:

1. \( v^M(n)w = 0 \) for \( n \gg 0 \) where \( v \in V \) and \( w \in M \);
2. \( Y^M(1, z) = 1_M \);
(3) \( Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L^M(n) z^{-n-2} \) satisfies:

(3.a) the Virasoro algebra relations:
\[
[L^M(n), L^M(m)] = (n-m)L^M(n+m) + \delta_{n+m,0} \frac{n^3 - n}{12} c,
\]

(3.b) the \( L(-1) \)-derivative property:
\[
Y^M(L(-1)v, z) = \frac{d}{dz} Y^M(v, z),
\]

(4) “Commutativity” holds;
\[
[v^M(n), u^M(m)] = \sum_{i=0}^{\infty} \binom{n}{i} (v(i)u)^M(n+m-i) \quad \text{and}
\]

(5) “Associativity” holds;
\[
(v(n)u)^M(m) = \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} \left\{ v^M(n-i)u^M(m+i) - (-1)^n u^M(n+m-i)v^M(i) \right\},
\]

where \( \binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!} \).

An \( \mathbb{N} \)-graded weak module is a weak \( V \)-module which carries an \( \mathbb{N} \)-grading, \( M = \bigoplus_{n=0}^{\infty} M(n) \), such that

\( (1') \) if \( v \in V_r \) then \( v^M(m)M(n) \subseteq M(n+r-m-1) \).

An ordinary module is an \( \mathbb{N} \)-graded weak \( V \)-module \( W = \bigoplus_{n=0}^{\infty} W(n) \) such that \( L(0) \) acts on \( W(n) \) semisimply and \( \dim W(n) < \infty \) for all \( n \). For a simple ordinary module \( W = \bigoplus_{i=0}^{\infty} W(i) \), \( L(0) \) acts on \( W(0) \) as a scalar, which is called a conformal weight of \( W \).

The main object in this paper is not an ordinary module, but an \( \mathbb{N} \)-graded weak module \( W = \bigoplus_{m=0}^{\infty} W(m) \) with \( \dim W(m) < \infty \). If \( v \in V_m \), then \( v^W(m-1) \) is a grade-preserving operator by \( (1') \) and we denote it by \( o(v) \) and extend it linearly.

**Definition 2.2** \( V \) is called \( C_2 \)-cofinite if the subspace \( \langle (v(-2))u : v, u \in V \rangle \) has a finite codimension in \( V \).

Zhu has introduced the second vertex operator algebra \( (V, Y[\cdot, \cdot], 1, \tilde{\omega}) \) associated to \( V \) in Theorem 4.2.1 of [Zh].

**Definition 2.3** The vertex operator \( Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} \) is defined for homogeneous \( v \) via the equality
\[
Y[v, z] = Y(v, e^{z-1}) e^{z|v|} \in \text{End}(V)[[z, z^{-1}]]
\]
and Virasoro element \( \tilde{\omega} \) is defined to be \( \omega - \frac{c}{24} 1 \), where \( |v| \) denotes the weight of \( v \).

Throughout this paper, we assume that \( V = \bigoplus_{n=0}^{\infty} V_n \) is a \( C_2 \)-cofinite VOA.
2.2 \( n \)-th Zhu algebras

Following [FZ], \( V \) has a product

\[
v * u = \text{Res}_x (1+x)^{|v|} Y(v, x) u
\]

for \( v \in V_{|v|} \) and \( u \in V \), where \( |v| \) denotes the weight of \( v \). Set

\[
O(V) = \left\{ \text{Res}_x (1+x)^{|v|} Y(v, x) u \mid v, u \in V \right\}
\]

and \( A(V) = V/O(V) \). Then it is known (Theorem 1.5.1 [FZ]) that \( A(V) \) is an associative algebra with a product \( * \). We call it Zhu algebra. Zhu has also shown that \( \omega + O(V) \) is in the center of \( A(V) \). From now on, abusing the notation, we use the same notation \( \omega \) for \( \omega + O(V) \). The essential property of Zhu algebra is that a top module \( W(0) \) of an \( \mathbb{N} \)-graded weak \( A(V) \)-module \( W = \bigoplus_{n=0}^{\infty} W(n) \) is an \( A(V) \)-module and every \( A(V) \)-module is a top module of some \( \mathbb{N} \)-graded weak \( V \)-module. This concept was naturally extended to \( n \)-th graded piece of \( \mathbb{N} \)-graded weak modules by Dong, Li and Mason in [DLiM2]. Set

\[
O_n(V) = \left\{ \text{Res}_x (1+x)^{|v|+n} Y(v, x) u \mid v, u \in V \right\}
\]

and \( A_n(V) = V/O_n(V) \). Like \( A(V) \), \( A_n(V) \) is an associative algebra with a product

\[
v \ast_n u = \sum_{m=0}^{n} \left( \frac{(-1)^m}{m!} \right) \text{Res}_x Y(v, x) u \frac{(1+x)^{|v|+n}}{x^{n+m+1}}
\]

and has a property that an \( n \)-th (and less ) graded piece \( W(n) \) of an \( \mathbb{N} \)-graded weak \( V \)-module \( W = \bigoplus_{n=0}^{\infty} W(n) \) is an \( A_n(V) \)-module and every \( A_n(V) \)-module is an \( n \)-th (or less) graded piece of an \( \mathbb{N} \)-graded weak \( V \)-module, (see Theorem 4.2 in [DLiM2]). \( A_n(V) \) is called an \( n \)-th Zhu algebra, in particular, \( 0 \)-th Zhu algebra is the original Zhu algebra. It is easy to see that there is a natural homomorphism from \( A_n(V) \) to \( A_{n-1}(V) \). The product \( \ast_n \) is characterized by the identity \( o(v \ast_n u) = o(v) o(u) \) on \( \bigoplus_{i=0}^{n} W(i) \) for every \( \mathbb{N} \)-graded weak \( V \)-module \( W \) so that \( A_n(V) \) is essentially the algebra of zero modes (grade-preserving operators) of fields. In particular, \( \omega \) is a central element of \( A_n(V) \) for any \( n \). Viewing \( A_n(V) \) as an algebra of zero modes, we will use the following notation:

\[
o(\alpha) = v(|v|-1+m) u(|u|-1-m) \quad \text{in} \quad A_n(V)
\]

which implies that \( o(\alpha) w = v(|v|-1+m) u(|u|-1-m) w \) for any \( \mathbb{N} \)-graded weak module \( W \) and \( w \in \bigoplus_{i=0}^{n} W(i) \).

We note that Garberdiel and Neitzke showed in [GN] that if \( V \) is a \( C_2 \)-cofinite VOA of CFT type, then \( A_n(V) \) is of finite dimension for any \( n \). They didn’t mention it directly. Later, Buhl proved it for irreducible modules in [Bul] by using a spanning set of module. We would like to explain their results and prove the finiteness of dimension of \( A_n(V) \) without
assuming condition of being of CFT type. First, they showed that if \( V = C_2(V) + \langle A \rangle \) for a set \( A \) of \( V \), then \( V \) is spanned by elements of the form
\[
v^1(-N_1) \cdots v^r(-N_r)1
\]
with \( N_1 > \cdots > N_r > 0 \) and \( v^i \in A \) by using a filtration, see Proposition 8 in [GN]. We will first show the existence of such a spanning set for general VOAs without negative weights.

**Lemma 2.4** Let \( V = \bigoplus_{m=0}^{\infty} V_m \) be a \( C_2 \)-cofinite VOA and \( A \) is a set of homogenous elements satisfying \( V = C_2(V) + \langle A \rangle \). Let \( W \) be a weak module generated from \( w \). Then \( W \) is spanned by the following elements
\[
\{v^1(i_1) \cdots v^k(i_k)w \mid v^i \in A, \ i_1 < \cdots < i_k \}.
\]

In particular, if we set
\[
W(m) = \langle v^1(i_1) \cdots v^k(i_k)w \mid v^i \in V, \ \deg(v^1(i_1) \cdots v^k(i_k)) = m \rangle,
\]
then \( W(m) = 0 \) for \( m \ll 0 \), where \( \deg(v(i) \cdots u(j)) \) denotes the degree of \( v(i) \cdots u(j) \) as an operator.

**Proof** Define a filtration on \( W \) by
\[
W(n, m, r) = \left\{ v^1(i_1) \cdots v^k(i_k)w \mid v^i \in V, \ \sum_{i=1}^{k} \wt(v^i) \leq n, \ \deg(v^1(i_1) \cdots v^k(i_k)) = m, \ k \leq r, \right\}.
\]

Clearly, \( W(n, m, r) \subseteq W(n+1, m, r) \) and
\[
W = \sum_{m \in \mathbb{Z}} \left( \bigcup_{n=0}^{\infty} \bigcup_{r=0}^{\infty} W(n, m, r) \right).
\]

We will prove that \( W(n, m, r) \) is spanned by the desired elements contained in \( W(n, m, r) \) for each \( m \). Suppose false and let \( (n, r) \) be a minimal counterexample with respect to lexicographical order. Let \( U(n, m, r) \) be the subspace spanned by the desired elements contained in \( W(n, m, r) \), then there is a nonzero element \( u \in W(n, m, r) - U(n, m, r) \) and \( W(n-1, m, r) + W(n, m, r-1) \subseteq U(n, m, r) \). We may assume
\[
u = v^1(i_1) \cdots v^r(i_r)w.
\]

Since \( v^i \in V = \langle A \rangle + C_2(V) \), there are \( u^i \in A \) and \( a^{ij}, b^{ij} \in V \) such that \( v^i = u^i + \sum a^{ij}(-1)b^{ij} \). Since \( C_2(V) \) and \( \langle A \rangle \) are direct sums of homogeneous spaces, we may assume \( \wt(v^i) = \wt(u^i) = \wt(a^{ij}(-2)b^{ij}) = \wt(a^{ij}) + \wt(b^{ij}) - 1 \). Using associativity
\[
(a(-2)b)(s) = \sum_{i=0}^{\infty} (-1)^i \binom{-2}{i} \left\{ a(-2-i)b(s+i) - b(-2+s-i)a(i) \right\},
\]
we may assume
\[ u = v^1(i_1) \cdots v^r(i_r)w \quad \text{with} \quad v^i \in A. \] (2.4)

We choose \( u \in \mathcal{W}(n, m, r) - U(n, m, r) \) so that \( u \) has a form (2.4) and \( \min\{i_1, \ldots, i_k\} \) is minimal. The existence of minimal one follows from the next arguments. Since \( v(i)u(j) = u(j)v(i) + [v(i), u(j)] = u(j)v(i) + \sum (\binom{j}{s})v(s)u(i + j - s) \) and \( \text{wt}(v(s)u) < \text{wt}(v) + \text{wt}(u) \) for \( s \geq 0 \), we may assume
\[ u = v^1(i_1) \cdots v^r(i_r)w \quad \text{with} \quad v^r \in A \quad \text{and} \quad i_1 \leq \cdots \leq i_r. \]

Since \( v^i \in A \) and \( A \) is a finite set, there is an integer \( N \) such that \( v(m)w = 0 \) for \( m > N \) and \( v \in A \). Therefore we have \( i_r > N \). In particular, the degree of \( v^p(i_p) \) as an operator is bounded below. Since the total degree of \( v^1(i_1) \cdots v^r(i_r) \) is \( m \), the degree of \( v^p(i_p) \) as an operator is bounded above and so \( i_1 \) is bounded below. By induction on \( r \), we may assume
\( i_2 < \cdots < i_r \). If \( i_1 < i_2 \) or \( i_1 > i_2 \), then \( u \in \mathcal{W}(n, m, r) \) by the minimality of \( i_1 \). The remaining case is \( i_1 = i_2 \). Set \( i = i_1 = i_2 \). The expansion of \( (v^1(-1)v^2)_{2i+1} \) by associativity if \( i < -1 \) and that of \( (v^2(-1)v^1)_{2i+1} \) by associativity if \( i \geq 0 \), contains a nonzero term \( v^1v^2 \) and the other terms are \( v^1(i-j)v^2(i+j) \) or \( v^2(i-j)v^1(i+j) \) with \( j \neq 0 \). For example, if \( i < -1 \), then there are constants \( \lambda_j, \mu_j \) such that
\[ v^1(i)v^2(i) = (v^1(-1)v^2)(2i+1) + \sum_{j \neq 0} (\lambda_j v^1(i-j)v^2(i+j) + \mu_j v^2(i-j)v^1(i+j)). \] (2.5)

We substitute the right-hand side for \( v^1(i_1)v^2(i_3) \) in \( u \). Since \( v^1(i-j)v^2(i+j)v^3(i_3) \cdots v^r(i_r)w \) has a form (2.4) and one of \( i-j \) and \( i+j \) is less than \( i \), we may assume \( u = (v^1(-1)v^2)(2i+1)v^2(i_3) \cdots w \), which is in \( \mathcal{W}(n, m, r-1) \). Therefore we have a contradiction. Since \( \dim(A) < \infty \), there is an integer \( N \) such that \( i_k < N \) and so \( \text{deg}(v^1(i_1) \cdots v^k(i_k)) \geq -kN + \frac{k(k-1)}{2} > -N^2 \) if \( i_1 < \cdots i_k < N \). Thus \( \mathcal{W}(m) = 0 \) if \( m < -N^2 \).

This completes the proof.

Using this spanning set, we have the following theorem, (see Lemma 3 and Theorem 11 in [GN] if \( V \) is of CFT type.)

**Theorem 2.5** If \( V \) is \( C_2 \)-cofinite, then \( A_n(V) \) are all finite dimensional.

**Proof** We fix \( n \) and \( v \) and \( u \) denote homogeneous elements. Let \( O_{(\infty, 2n+2)} \) be the subspace of \( V \) spanned by elements of the form
\[ \langle v(-2 - N - 2n|v)|u \mid v, u \in V, \ |v| \geq 1, \ N \geq 0 \rangle. \]

We will show that \( O_{(\infty, 2n+2)} \) is of finite codimension. Since \( V \) is \( C_2 \)-cofinite, we can choose a finite set \( A \) of homogenous elements such that \( V \) is spanned by \( u_1(-N_1) \cdots u_r(-N_r)1 \) with \( N_1 > \cdots > N_r \) and \( w_i \in A \). Let \( t \) be the maximal weight of elements in \( A \). If \( N_1 \geq (2n)t + 2 \), then \( u_1(-N_1) \cdots u_r(-N_r)1 \in O_{(\infty, 2n+2)} \). This leaves us only finitely many choices for the \( N_i \), which gives a finite spanning set for \( V/O_{(\infty, 2n+2)} \).
Let $O_u$ be the subspace of $V$ spanned by elements of the form

$$v \circ_M u = \text{Res}_2 Y(v, z)u \frac{(z+1)^{(n+1)|v|}}{z^{2n|v|+2+M}}$$

with $|v| \geq 1$ and $M \geq 0$. Since $2n(|v|-1) \geq n(|v|-1)$ and

$$v \circ_M u = \text{Res}_2 Y(v, z)u \frac{(z+1)^{|v|+n+|v|-1}}{z^{2+M+2n(|v|-1)}}$$

$v \circ_M u \in O_n(V)$ by Lemma 2.1 in [DLiM2] and so it is sufficient to show

$$\dim V/O_u < \dim V/O_{(\infty 2n+2)}.$$ 

We note $v \circ_M u = \sum_{i=0}^{(n+1)|v|} \binom{n+1}{i}v(-2n|v|-2-M+i)u$,

that is, the weights of all terms are less than or equal to the weight of $v(-2n|v|-2-M)u$. Let $\{v_1, ..., v_N\}$ be a set of representatives for $V/O_{(\infty 2n+2)}$. Since $O_{(\infty 2n+2)}$ is a direct sum of homogenous spaces, we may assume that $v_i$ are all homogenous elements. We claim that $\langle v_1, ..., v_N \rangle + O_u = V$, which offers the desired conclusion. Suppose false and let $u \in V - \{\langle v_1, ..., v_N \rangle + O_u\}$ be a homogeneous element with minimal weight among them. By the choice of $\{v_i\}$, there are $a_r, b_j \in \mathbb{C}$ and homogeneous elements $v^r, u^r \in V$ and $N_r \in \mathbb{Z}_+$ such that

$$u = \sum_j b_jv_j + \sum_r a_r v^r(-N_r-2n|v|-2)u^r.$$

We may assume that the weights of all elements in the above equation are the same. But then

$$\hat{u} = u - \sum_j b_jv_j - \sum_r v^r \circ_{N_r} u^r$$

is a linear combination of vectors whose weights are strictly smaller than that of $u$. By the minimality of $u$, $\hat{u}$ is contained in $\langle v_1, ..., v_n \rangle + O_u$ and so is $u$.

This completes the proof of Theorem 2.5.

\[ \black\]

### 2.3 Generalized Verma module

In this paper, we will define a pseudo-trace function for a generalized Verma module $W_T^n$ constructed from an $A_n(V)$-module $T$. A generalized Verma module Verma$(X)$ for an $A(V)$-module $X$ is introduced in [3] as an extension of concept of a Verma module. It is a largest $\mathbb{N}$-graded weak $V$-module $W$ which has $X$ as a top module $W(0)$ and is generated from $X$. For an $A_n(V)$-module $T$, it is possible to consider a largest $\mathbb{N}$-graded weak $V$-module $W_T^{(n)} = \oplus_{i=0}^\infty W_T^{(n)}(i)$ which has $T$ as its $n$-th graded piece $W_T^{(n)}(n)$ and is generated from $T$. We should note that $W_T^{(n)}(0)$ might be zero if $T$ is also an $A_{n-1}(V)$-module. However, in this paper, we will only treat an $\mathbb{N}$-graded weak $V$-module $W_T^{(n)}$ constructed from an $A_n(V)$-module $T$ which satisfies the following condition:
(2.6) For any $A_n(V)$-submodules $T^1 \subseteq T^2$ of $T$ with an irreducible factor $T^2/T^1$, every $V$-module $W$ with $n$-th graded piece $T^2/T^1$ is irreducible.

More precisely, we will only consider an $\mathbb{N}$-graded weak $V$-module $W_T^{(n)}$ whose composition series has the same shape as does a composition series of $T = W_T^{(n)}(n)$. Therefore, a $V$-module $W$ with $T$ as an $n$-th graded piece is uniquely determined and so a generalized Verma module coincides with $L_n(T)$, which is defined in Theorem 4.2 of [DLiM2] and is the minimal one in a sense. So we don’t need a concept of generalized Verma module, but in order to emphasize that this module is naturally constructed from $A_n(V)$-module $T$, we will call it a generalized Verma module and denote it by $W_T^{(n)}$. As they showed in [GN], we obtain:

Lemma 2.6 If $V$ is a $C_2$-cofinite VOA and $T$ is a finite dimensional $A_n(V)$-module, then $W_T(m)$ has a finite dimension for any $m = 0, 1, ...$.

2.4 Elliptic functions

We adopt the same notation form [Zh]. The Eisenstein series $G_{2k}(\tau)$ ($k = 1, 2, ...$) are series

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}} \text{ for } k \geq 2 \quad \text{and}$$

$$G_{2}(\tau) = \frac{\pi^2}{3} + \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} \text{ for } k = 1.$$ 

They have the $q$-expansions

$$G_{2k}(\tau) = 2\xi(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n},$$

where $\xi(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ and $q = e^{2\pi i \tau}$. We make use of the following normalized Eisenstein series:

$$E_k(\tau) = \frac{1}{(2\pi i)^k} G_k(\tau) \text{ for } k \geq 2.$$ 

2.5 $C_2$-cofiniteness

Li showed in [Li2] that $C_2$-cofinite VOA is finitely generated and regularity implies rationality and $C_2$-cofiniteness. Conversely, Abe, Buhl and Dong proved in [ABD] that regularity comes from $C_2$-cofiniteness and rationality. The regularity means that every weak module is a direct sum of simple ordinary modules. This implies two conditions, that is, every weak module is an $\mathbb{N}$-graded weak module and $V$ is rational. In this section, we will show that they are equivalent separately. Namely, $C_2$-cofiniteness implies that every weak module is an $\mathbb{N}$-graded weak module. We will prove the following theorem.
Theorem 2.7 Let $V = \oplus_{m=0}^{\infty} V_m$ be a vertex operator algebra. Then the following are equivalent.

1. $V$ is $C_2$-cofinite.
2. Every weak module is a direct sum of generalized eigenspaces of $L(0)$.
3. Every weak module is an $\mathbb{N}$-graded weak module $W = \oplus_{n=0}^{\infty} W(n)$ such that $W(n)$ is a direct sum of generalized eigenspaces of $L(0)$.
4. $V$ is finitely generated and every weak module is an $\mathbb{N}$-graded weak module.

[Proof] Proof of (1) $\Rightarrow$ (3) (and so (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4))

The proof is essentially the same as in [ABD]. Let $W$ be a weak module. We note that $V$ has only finitely many irreducible modules. Let $\{r_1,\ldots, r_k\}$ be the set of conformal weights and set $R = \bigcup_{i=1}^{k} (r_i + \mathbb{Z}_{\geq 0})$. For $r \in \mathbb{C}$, $W_r = \{w \in W \mid (L(0) - r)^N w = 0 \text{ for } N \gg 0\}$ denotes a generalized eigenspace of $L(0)$ with eigenvalue $r$. Clearly, $v(|v|^{-1+h})W_r \subseteq W_{r-h}$ for $v \in V$. We will show $W = \bigoplus_{r \in R} W_r$, which implies (3). It is easy to see that there is a unique maximal $\mathbb{N}$-graded weak submodule $U = \oplus_{n=0}^{\infty} U(n)$ such that $U(n)$ is a direct sum of generalized eigenspaces of $L(0)$. Clearly, $U = \bigoplus_{r \in R} U \cap W_r$. Suppose $W/U \neq 0$. Since $W/U$ is also a weak module, we may assume $W/U$ is generated from $w \neq 0$. We use a similar argument as in [ABD]. Applying Lemma 2.4 to $W/U$, there is an integer $m$ such that $W(m) \neq 0$, but $W(t) = 0$ for $t < m$. Then $W(m)$ is an $A(V)$-module. Since $A(V)$ is a finite dimensional algebra, $W(m)$ is a direct sum of generalized eigenspaces of $L(0)$, which contradicts the choice of $U$.

Proof of (3) $\Rightarrow$ (2) is clear.

Proof of (2) $\Rightarrow$ (1).

As Li explained in [Li2], $V^* = \text{Hom}(V, \mathbb{C})$ contains a uniquely maximal weak submodule $W(V^*)$ and if $f(C_2(V)) = 0$, then $f \in W(V^*)$. By the assumption, $W(V^*)$ is a direct sum of generalized eigenspaces of $L(0)$. Suppose $\dim V/C_2(V) = \infty$. Let $S = \{i \mid V_i \neq (C_2(V))_i\}$. For $i \in S$, choose $v_i \in V_i - (C_2(V))_i$, and a hyperspace $T_i$ of $V_i$ containing $(C_2(V))_i$ such that $V_i = \mathbb{C}v_i + T_i$. Set $T_i = (C_2(V))_i$ for $i \notin S$. So $V$ is spanned by $T = \bigoplus T_i$ and $\{v_i \mid i \in S\}$. Define $f \in V^*$ so that $f(T) = 0$ and $f(v_i) = 1$. Since $(L(0)f)(v_i) = f(L(0)v_i) = f(|v_i|v_i) = |v_i|$, $\{L(0)^i f \mid i = 0,1,\ldots\}$ is a linearly independent. On the other hand, since $f \in W(V^*)$, $f$ is a sum of finite elements in generalized eigenspaces of $L(0)$ and so $\langle L(0)^i f \mid i = 0,1,\ldots\rangle$ is of finite dimension, which is a contradiction.

Proof of (4) $\Rightarrow$ (1).

Since $V$ is finitely generated, $V/C_2(V)$ is a finitely generated abelian group (product is given by $u(-1)v$). Therefore, there is a torsion-free element $u$, that is, $\{u(-1)^n1 \notin C_2(V)\}$ for any $m \in \mathbb{N}$. Define $f \in V^*$ such that $f(C_2(V)) = 0$ and $f(u(-1)^m1) = 1$ for any $m \in \mathbb{N}$. Then $v(2|v|^{-1})^m f \neq 0$ for any $m \in \mathbb{N}$, which implies that $f$ does not belong to an $\mathbb{N}$-graded weak module.

This completes the proof of Theorem 2.7.
3 Symmetric algebras and pseudo-trace maps

In this section, we always consider a finite dimensional algebra over \( \mathbb{C} \) with a unit 1. Let \( A \) be a ring and let \( L(a) \) and \( R(a) \) denote the left and right regular representations of \( a \in A \) given by a basis \( \{ e_i \} \) of \( A \). A is called a Frobenius algebra if \( L(a) \) and \( R(a) \) are similar: \( L(a) = Q^{-1}R(a)Q \) for some matrix \( Q \). In particular, \( A \) is called a symmetric algebra when the matrix \( Q \) can be chosen as a symmetric matrix. It is also well known (cf. \([CR]\)) that this is equivalent to that \( A \) has a symmetric linear map \( \phi : A \rightarrow \mathbb{C} \) such that \( \text{Rad}(\phi) = \{ a \in A \mid \phi(AaA) = 0 \} \) is zero and is also equivalent to that \( A \) has a symmetric associative nondegenerated bilinear form \( \langle \cdot , \cdot \rangle \), a relation is given by \( \phi(ab) = \langle a, b \rangle \), where a symmetric map implies \( \phi(ab) = \phi(ba) \) and an associative bilinear form means \( \langle ab, c \rangle = \langle a, bc \rangle \) for \( a, b, c \in A \), see \([CR]\). We denote a symmetric algebra \( A \) with a symmetric linear function \( \phi \) by \((A, \phi)\).

Originally, Nesbitt and Scott showed in \([NS]\) that \( A \) is a symmetric algebra if and only if its basic algebra \( P \) is symmetric. Later, Oshima gave a simpler proof of this equivalence, see \([O]\). This is an idea to explain our strategy in this paper. Given an symmetric linear map \( \pi \) of \( P/J \) we have started from \((P, \pi)\). Since there is a symmetric linear map \( \pi \) of \( P/J \) such that \( \pi(e_i) = \phi(e_i) \) for all \( i \). Since such a map is a linear sum of ordinary trace maps, we may assume that we have started from \( \phi - \pi \) and consider only the following case.

\[
(3.4) \quad \phi(e_i) = 0 \quad \text{for all} \quad i.
\]

If \( P \) is decomposable as a ring, say \( P = P_1 \oplus P_2 \), then \((P, \phi)\) is a sum of \((P_1, \phi|_{P_1})\) and

\[
(3.4) \quad \phi(e_i) = 0 \quad \text{for all} \quad i.
\]
\((P_2, \phi|P_2)\). We hence assume that \(P\) is indecomposable. In particular, there is \(r \in \mathbb{C}\) and \(\mu(r) \in \mathbb{Z}_+\) such that
\[
(3.5) \quad (\omega - r)^{\mu(r)}P = 0.
\]

First of all, we have:

**Lemma 3.1**  \(\text{soc}(P) \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k\) as \(P \times P\)-modules and \(\text{soc}(P) \subseteq J(P) = \text{soc}(P)\perp\), where \(\text{soc}(P)\) denotes the socle of \(P\), that is, the sum of all minimal left ideals.

**Proof** Let \(M\) be a minimal ideal. If \(e_i Me_j = M\) for \(i \neq j\), then \(\phi(m) = \phi(e_i m - me_i) = 0\) for all \(m \in M\), which contradicts \(\text{Rad}(\phi) = 0\). Hence \(M\) satisfies \(e_i Me_i = M\) for some \(i\). For \(i = 1, \ldots, k\), \(M_i = (\mathbb{C}e_1 + \cdots + \mathbb{C}e_{i-1} + \mathbb{C}e_{i+1} + \mathbb{C}e_k + J(P))\perp\) is a minimal ideal with \(e_i M_i = M_i\). If \(P\) has two minimal ideals \(M\) and \(N\) which are isomorphic together, then \(L = \{(a, b) \in M \oplus N \mid \phi(a) + \phi(b) = 0\}\) is a nonzero ideal of \(P\) with \(\phi(L) = 0\), which is a contradiction. Hence \(\text{soc}(P) \equiv \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k = J(P)\perp\). Since \(P\) is indecomposable, \(\text{soc}(P) \subseteq J(P)\).

Let \(\{f_i \in \text{soc}(P) \mid i = 1, \ldots, k\}\) be the dual basis of \(\{e_i \mid i = 1, \ldots, k\}\). We note \(\phi(f_i) = \langle 1, f_i \rangle = \langle e_i, f_i \rangle = 1\). Set \(d_{ij} = \dim_{\mathbb{C}} e_i J(P)e_j / e_i \text{soc}(P)e_j\). Then we can find the following basis of \(P\).

**Lemma 3.2** \(P\) has a basis
\[
\Omega = \{\rho^{ij}_0, \rho^{ij}_{d_{ij}+1}, \rho^{ij}_{s_{ij}} \mid i, j = 1, \ldots, k, s_{ij} = 1, \ldots, d_{ij}\}
\]
satisfying
\begin{enumerate}
\item \(\rho^{ij}_0 = e_i, \rho^{ij}_{d_{ij}+1} = f_i\),
\item \(e_i \rho^{ij}_s e_j = \rho^{ij}_s\) for all \(i, j, s\),
\item \(\langle \rho^{ij}_s, \rho^{ab}_{d_{ab}-t} \rangle = \delta_{ib}\delta_{ja}\delta_{st}\) and
\item \(\rho^{ij}_s \delta_{d_{ij}-s} = f_i\).
\end{enumerate}

**Proof** We first choose \(\rho^{ij}_0 = e_i\) and \(\rho^{ij}_{d_{ij}+1} = f_i\), which satisfy (2), (3) and (4). As we showed, \(J(P)\perp = \text{soc}(P) \subseteq J(P)\) and so \(J(P) / \text{soc}(P)\) has a nondegenerated symmetric bilinear form and so do \(e_i J(P)e_i / e_i \text{soc}(P)e_i\) and \(e_i J(P)e_j / e_i \text{soc}(P)e_j + e_j J(P)e_i / e_j \text{soc}(P)e_i\). Since \(J(P)\) is nilpotent, there is a sequence of ideals \(\mathfrak{A}_t\) of \(e_i Pe_i\):
\[
e_i Pe_i = \mathfrak{A}_0 \supsetneq J(P) \cap e_i Pe_i = \mathfrak{A}_1 \supsetneq \cdots \supsetneq \mathfrak{A}_{d_{ij}+1} = \text{soc}(P) \cap e_i Pe_i \supsetneq 0
\]
such that \(\mathfrak{A}_s / \mathfrak{A}_{s+1}\) is a simple \(P\)-module and \(\mathfrak{A}_s \mathfrak{A}_{d-s+1} \subseteq e_i \text{soc}(P)e_i\). In particular, we may choose \(\mathfrak{A}_s\) so that \(\mathfrak{A}_s^\perp = \mathfrak{A}_{d_{ij}+2-s}\) for all \(s\). Choosing a base \(\rho^{ii}_s\) in \(\mathfrak{A}_s - \mathfrak{A}_{s+1}\) and its dual base \(\rho^{ii}_{d_{ij}+s-1} = (\rho^{ii}_s)^*\) in \(\mathfrak{A}_{s+1}^\perp\) for \(s \leq \frac{1}{2}(d_{ij})\) so that \(\langle \rho^{ii}_s, \rho^{ii}_t \rangle = \delta_{s,t,d_{ij}+1}\) inductively, we have a desired set for \(e_i Pe_i\). For \(i \neq j\), we first note \(d_{ij} = d_{ji}\). Then there are sequences of ideals \(\mathfrak{B}_s\) and \(\mathfrak{C}_s\) of \(e_j Pe_j\) and \(e_j Pe_j:\n\[
e_i Pe_j = \mathfrak{B}_1 \supsetneq \mathfrak{B}_2 \supsetneq \cdots \supsetneq \mathfrak{B}_{d_{ij}} \supsetneq 0
\]
\[
e_j Pe_i = \mathfrak{C}_1 \supsetneq \mathfrak{C}_2 \supsetneq \cdots \supsetneq \mathfrak{C}_{d_{ij}} \supsetneq 0
\]
such that \( \mathcal{B}_s \mathcal{C}_{d_j-s+1} = \mathbb{C} f_i \) and \( \mathcal{C}_{d_j-s+1} \mathcal{B}_s = \mathbb{C} f_j \). So we can take \( \rho^{ij}_s \in \mathcal{B}_s - \mathcal{B}_{s+1} \) and choose suitable \( \rho^{ij}_{d_j-s+1} \in \mathcal{C}_{d_j-s+1} - \mathcal{C}_{d_j-s} \) so that they satisfy (3) and (4). It is easy to see that the set of all \( \rho^{ij}_s \) satisfies the desired conditions.

For \( \rho \in \Omega \), \( \rho^* \) denotes its dual, that is, a unique element \( \rho^* \in \Omega \) such that \( \langle \rho, \rho^* \rangle = 1 \).

**Lemma 3.3** Set \( P^0 = (\mathbb{C} e_1 + \cdots + \mathbb{C} e_k)^\perp \). Then for \( \rho, \mu \in \Omega \), if \( \mu \neq \rho^* \), then \( \rho \mu \in P^0 \).

**Proof** Assume \( e_i \rho = \rho \). Then we have \( \langle e_i, \rho \mu \rangle = \langle e_i \rho, \mu \rangle = \langle \delta_{ij} \rho, \mu \rangle = \delta_{ij} \langle \rho, \mu \rangle = 0 \) for all \( i \).

We assume that \( P \) has a basis satisfying the conditions in Lemma 3.2. As we explain, we would like to call a right \( P \)-module \( W \) “interlocked with \( \phi \)” if a basic algebra of \( \text{End}_P(W) \) is \( P \). However, we will need a symmetric linear map \( \text{tr}^\phi \) explicitly and so we will just introduce a sufficient condition for that.

**Definition 3.4** Let \( (P, \phi) \) be a basic symmetric algebra with a symmetric linear map \( \phi \) satisfying (3.4) and \( W \) is a finite dimensional right \( P \)-module. We will call that \( W \) is interlocked with \( \phi \) if \( \text{Ker}(f_i) \overset{\text{def}}{=} \{ w \in W \mid w f_i = 0 \} \) is equal to \( \sum_{\rho \in \Omega - \{e_i\}} W \rho \) for each \( i = 1, \ldots, k \), where \( f_i = \rho^{ii}_{d_i+1} \) is a dual base of \( e_i \).

Set \( R = \text{End}_P(W) \). Then \( W \alpha \) is an \( R \)-module for each \( \alpha \in P \) and the condition above implies that \( W f_i \cong W/\{ w \in W \mid w f_i = 0 \} = W/\sum_{\alpha \in \Omega - \{e_i\}} W \alpha \cong W/(\sum_{i \neq j} W e_j + W J(P)) \) as \( R \)-modules. Namely, the definition says that the socle part \( \bigoplus_{i=1}^k W f_i \) and the semisimple part \( W/W J(P) \) are isomorphic together. Not only the top and the bottom, but we also have:

**Lemma 3.5** \( W e_i/W J(P) e_i \rightarrow W e_i \rho e_j/W J(P) e_i \rho e_j \) is also an \( R \)-isomorphism for any \( e_i \rho e_j \in \Omega \).

**Proof** Since the composition map

\[
(e_i \rho e_j)^* (e_i \rho e_j) : W e_i/ \sum_{\mu \leq e_i} W \mu e_i \rho e_j/ \sum_{\mu \leq e_i} W \mu e_i \rho e_j \rightarrow W e_i \rho e_j e_j \rho^* e_i = W f_i
\]

is an \( R \)-isomorphism, so is

\[
e_i \rho e_j : W e_i/ \sum_{\mu \leq e_i} W \mu \rightarrow W e_i \rho e_j/ \sum_{\mu \leq e_i} W \mu e_i \rho e_j
\]

for any \( e_i \rho e_j \in \Omega \), where \( \mu < e_i \) implies \( W \mu \not\subset W e_i \).

Therefore, it is easy to see that \( W \) is interlocked with \( \phi \) if and only if there are vector spaces \( T_p \) (\( \cong W f_p \)) for each \( p \) such that

\[
W \cong \bigoplus_{e_i \rho e_j \in \Omega} T_p \otimes e_p \rho e_s
\]
as right $P$-modules. Now we will define a symmetric linear map $\text{tr}^\phi_W$ of $R = \text{End}_P(W)$. Let $\{v_i^{e_p} \mid i = 1, \ldots, \dim T_p\}$ be a basis of $T_p$.

**Definition 3.6** Let $(P, \phi)$ be a basic symmetric algebra and $W$ a right $P$-module inter- locked with $\phi$. For $\alpha \in R = \text{End}_P(W)$, we have a $(\sum_{p=1}^k \dim T_p) \times (\sum_{e_p \in \Omega} \dim T_p)$-matrix $(\alpha_{ji}^{e_p e_s})$ such that

$$\alpha(v_j^{e_s} \otimes e_s) = \sum_{e_p \in \Omega} \left( \sum_{i=1}^{\dim T_p} \alpha_{ji}^{e_p e_s} v_i^{e_p} \otimes e_p e_s \right)$$

for $v_j^{e_s} \otimes e_s$. Then we define a pseudo-trace map $\text{tr}^\phi_W(\alpha)$ by the sum of traces of matrices $(\alpha_{ji}^{e_p})_{ji}$ for $s = 1, \ldots, k$. Namely, we define

$$\text{tr}^\phi_W(\alpha) = \sum_{s=1}^k \text{tr}(\alpha_{ji}^{e_s}) = \sum_{s=1}^k \sum_{i=1}^{\dim T_p} \alpha_{ji}^{e_s}. $$

Namely, we take the trace of $\text{Hom}(W/WJ(P) \to W\text{soc}(P))$ as an example in the introduction.

**Proposition 3.7** $\text{tr}^\phi_W$ is a symmetric linear map.

**Proof** It is easy to see that $\text{tr}^\phi_W$ does not depend on the choice of bases of $T_p$’s. Let $\alpha, \beta \in R = \text{End}_P(W)$. Then there are $\alpha_{ji}^{e_p e_s}, \beta_{ji}^{e_p e_s} \in \mathbb{C}$ such that

$$\alpha(v_j^{e_s} \otimes e_s) = \sum_{e_p \in \Omega} \left( \sum_{i} \alpha_{ji}^{e_p e_s} v_i^{e_p} \otimes e_p e_s \right)$$

and

$$\beta(v_i^{e_p} \otimes e_p) = \sum_{e_p \in \Omega} \left( \sum_{h} \beta_{ih}^{e_p e_s} v_h^{e_p} \otimes e_t e_p \right)$$

By direct calculation, we obtain

$$\beta \alpha(v_j^{e_s} \otimes e_s) = \beta \left( \sum_{e_p \in \Omega} \left( \sum_{i} \alpha_{ji}^{e_p e_s} v_i^{e_p} \otimes e_p e_s \right) \right)$$

$$= \sum_{e_p \in \Omega} \left( \sum_{i} \alpha_{ji}^{e_p e_s} \beta(v_i^{e_p} \otimes e_p) e_p e_s \right) \text{ since } \beta \in \text{End}_P(W),$$

$$= \sum_{e_p \in \Omega} \left( \sum_{i} \alpha_{ji}^{e_p e_s} \left( \sum_{t,h} \beta_{ih}^{e_p e_s} v_h^{e_t} \otimes e_t e_p \right) e_p e_s \right)$$

$$= \sum_{e_p \in \Omega} \sum_{i} \sum_{t,h} \alpha_{ji}^{e_p e_s} \beta_{ih}^{e_p e_s} v_h^{e_t} \otimes e_t e_p e_s.$$
Hence we obtain
\[
\text{tr}^\phi_W(\beta \alpha) = \sum_{s=1}^{k} \text{tr}(\alpha_{j_i}^{e_p \rho e_s})(\beta_{i_h}^{e_p \rho e_s}) = \sum_{s=1}^{k} \text{tr}(\beta_{i_h}^{e_p \rho e_s})(\alpha_{j_i}^{e_p \rho e_s}) = \text{tr}^\phi_W(\alpha \beta),
\]
as desired. \[\square\]

We will call \(\text{tr}^\phi_W\) a pseudo-trace map. We should note that we will treat an ordinary trace map as one of pseudo-trace map although \(\phi\) does not satisfy (3.4). For a basic symmetric algebra \((P, \phi) = (\mathbb{C}, 1)\), then \(\text{tr}^\phi_W\) coincides with the ordinary trace map. We next investigate the properties of pseudo-trace maps. Let \(\omega\) be a central element of \(P\) such that \((\omega - r)^s P = 0\) and \((\omega - r)^{s-1} P \neq 0\) for \(r \in \mathbb{C}\). Set \(\mathfrak{N} = \{a \in P \mid (\omega - r)a = 0\}\).
It is easy to see that if we define \(\phi'\) by
\[
\phi'(a) = \phi((\omega - r)a),
\]
then \(\phi'\) is also a symmetric linear function of \(P/\mathfrak{N}\). We denote it by \((\omega - r)\phi\) and the right action of \(\omega - r\) by \((\omega - r)_P\).

**Proposition 3.8** If \(W\) is a right \(P\)-module interlocked with \(\phi\), then \(W/W\mathfrak{N}\) is a right \(P/\mathfrak{N}\)-module interlocked with \((\omega - r)\phi\) and
\[
\text{tr}^\phi_W(g(\omega - r)_P) = \text{tr}^\phi_{W/W\mathfrak{N}}g\tag{3.6}
\]
for \(g \in \text{End}_P(W)\). (We also view \(g \in \text{End}_{P/\mathfrak{N}}(W/W\mathfrak{N})\).)

**Proof** Set \(R = \text{End}_P(W)\). The first assertion is clear. Since \(\omega - r \in Z(P)\), we obtain \(\omega - r \in Z(R)\), where \(Z(A)\) denotes the center of \(A\). Let \(\mathfrak{D}\) be an ideal of \(R\) such that \(\mathfrak{D}/\mathfrak{N} = \text{soc}(R/\mathfrak{N})\). Then \((\omega - r)\mathfrak{D} \subseteq \text{soc}(R) \cap \text{Im}(\omega - r)\) and \((\omega - r)\mathfrak{D} = \mathfrak{D}/\mathfrak{N}\). On the other hand, \(\mathfrak{C} = \{a \in R \mid (\omega - r)a \in \text{soc}(R)\}\) satisfies \(\mathfrak{C}/\mathfrak{N} = \text{soc}(R/\mathfrak{N})\). Hence \((\omega - r)\) is an isomorphism from \(\text{soc}(R/\mathfrak{N})\) to \(\text{soc}(R) \cap \text{Im}(\omega - r)\). From the definition of pseudo-trace maps, \(\text{tr}^\phi_W\) is given by the traces of \(\text{Hom}_P(R/J(R), \text{soc}(R))\) and so \(\text{tr}^\phi_W(\omega - r)_P\) is given by the traces of \(\text{Hom}_P(R/J(R), (\omega - r)^{-1}\text{soc}(R)/\mathfrak{N})\), which equals the traces of \(\text{Hom}_P(R/J(R), \text{soc}(R/\mathfrak{N}))\). We hence have \(\text{tr}^\phi_W(g(\omega - r)_P) = \text{tr}^\phi_{W/W\mathfrak{N}}g\), as desired. \[\square\]

Let \(A\) be a finite dimensional symmetric algebra with a symmetric linear map \(\phi\). Let \(A/J(A) = A_1 \oplus \cdots \oplus A_k\) be the decomposition into the direct sum of simple components. Let \(\{e_i \mid i = 1, \ldots, k\}\) be a set of orthogonal idempotents such that \(e_i = e_i + J(A)\) is a primitive idempotent of \(A_i\) for each \(i\). Set \(e = \sum_{i=1}^k e_i\). Then \(eAe\) is a basic algebra and \(eAe\) has a symmetric map \(\phi\) (we use the same notation \(\phi\)) with zero radical. Viewing \(Ae\) as a right \(eAe\)-module, it is easy to check that \(Ae\) is interlocked with \(\phi\) and so we can define a pseudo-trace map \(\text{tr}^\phi_{Ae}\) of \(A \subseteq \text{End}_{eAe}(Ae)\) on \(Ae\). From the definition of \(\text{tr}^\phi_{Ae}\) and the previous arguments, it is easy to see the following.

**Lemma 3.9** \(\text{tr}^\phi_{Ae}(a) = \phi(a)\) for all \(a \in \text{soc}(A)\).
So we have the following theorem, which we need later.

**Theorem 3.10** Let $A$ be a finite dimensional associative algebra over $\mathbb{C}$ and $\phi$ a linear function of $A$ satisfying $\phi(ab) = \phi(ba)$ for every $a, b \in A$. Let $\omega$ be in the center of $A$ and assume that $\phi((\omega - r)^{\mu(r)}a) = 0$ for every $a \in A$. Then there are linear symmetric functions $\phi_i$ of $A$ ($i = 1, \ldots, s$) and basic symmetric subalgebras $P_i$ of factor rings $A/\mathfrak{N}_i$ with symmetric linear functions $\phi_i$ and $A \times P_i$-modules $M^i = (A/\mathfrak{N}_i)e$ satisfying $(\omega - r)^{\mu(r)}M^i = 0$ such that

$$\phi(b) = \sum_{i=1}^{s} \text{tr}^\phi_i M_i(b)$$

for every $b \in A$, where $\mathfrak{N}_i = \text{Rad}(\phi_i)$.

**Proof** We will use induction on $\dim A$. If $\phi$ has a nonzero radical $M$, then $\phi : A/M \rightarrow \mathbb{C}$ satisfies the same condition, but $\dim A/M < \dim M$. So we may assume that $\text{Rad}(\phi)$ is zero and $A$ is a symmetric algebra with a symmetric linear map $\phi$. We may also assume that $A$ is indecomposable. By the previous lemma, there is an $A \times eAe$-module $Ae$ and a symmetric linear map $\psi$ such that $\text{tr}^\psi_{Ae}(a) = \phi(a)$ for all $a \in \text{soc}(A)$. Set $\phi' = \phi - \text{tr}^\psi_{Ae}$. Then $\phi'$ satisfies the same condition and $\text{Rad}(\phi') \supseteq \text{soc}(A)$. By induction on $\dim A$, $\phi'$ is a sum of pseudo-trace maps and so is $\phi$.

---

**4 Pseudo-trace functions**

Let $\{W^1, \ldots, W^k\}$ be the set of all irreducible $V$-modules and set $W^i = \bigoplus_{m=0}^{\infty} W^i(m)$ with $W^i(0) \neq 0$. It may happen that $W^i(h) = 0$ for some $h > 0$ and $W^i(h-1) \neq 0$. Then, since $\text{sl}_2(\mathbb{C}) \cong \langle L(-1), 2L(0), L(1) \rangle$ acts on $\bigoplus_{m=0}^{h-1} W^i(m)$, the conformal weight is $1-h$. Therefore, there is an integer $l$ such that $W^i(m) \neq 0$ for any $m > l$ and $i$.

In this section, we assume:

(4.1) $A_n(V)$ has a symmetric linear map $\phi$ satisfying $\phi((\omega - r)^s \ast v) = 0$ for some $r \in \mathbb{C}$ and $s \in \mathbb{N}$ and the real part $\text{Re}(r)$ of $r$ is greater than the real parts of conformal weights of all irreducible modules by $l$.

Set $A = A_n(V)/\text{Rad}(\phi)$, which is a symmetric algebra with a symmetric linear map $\phi$. Let $A/J(A) = A_1 \oplus \cdots \oplus A_k$ be the decomposition into the direct sum of simple components $A_i$. Let $\{e_i \mid i = 1, \ldots, k\}$ be a set of mutually orthogonal primitive idempotents of $A$ such that $\bar{e}_i = e_i + J(A)$ is a primitive idempotent of $A_i$. Set $e = e_1 + \cdots + e_k$. Then $P = eAe$ is a basic symmetric algebra with a symmetric linear map $\phi$. As we showed in the previous section, $T = Ae$ is interlocked with $\phi$.

Let $W^{(n)}_T$ denote a generalized Verma $V$-module generated from $A_n(V)$-module $T$. We assert that

$W^{(n)}_T$ is interlocked with $\phi$. 

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We first note that $W_T^{(n)}(m)$ has a finite dimension for every $m$ and $L(0) - r$ acts on $W_T^{(n)}(n) \cong T$ as a nilpotent operator. By the definition of generalized Verma module $W_T^{(n)}$, $W_T^{(n)}$ is a right $P$-module by the action
\[
(\sum v^1(i_1) \cdots v^s(i_s)x)g = \sum v^1(i_1) \cdots v^s(i_s)(xg)
\]
for $v^i \in V, \ x \in T$ and $g \in P$. Since the real part $\text{Re}(r)$ of $r$ is greater than the real parts of conformal weights of any modules by $l$, every nonzero submodule of $W_T^{(n)}$ has a nonzero intersection with $W_T^{(n)}(n)$ and so all irreducible factors $W/U$ of composition series of $W_T^{(n)}$ with isomorphic $n$-th graded pieces as $A_n(V)$-modules are isomorphic together as $V$-modules. In particular, the semisimple part $W_T^{(n)}/W_T^{(n)}J(P)$ and the socle part $W_T^{(n)}\text{soc}(P)$ are isomorphic together as $V \times P$-modules. Thus, $W_T^{(n)}$ is interlocked with $(P, \phi)$.

We note that $W_T^{(n)}(0)$ may be zero if $T$ is also an $A_{n-1}(V)$-module. We should also note that we have defined a pseudo-trace map for a finite dimensional vector space $W$ and so we have to say that for each $N$, $\oplus_{h=0}^N W_T^{(n)}(h)$ is interlocked with $\phi$. However, by the definition of pseudo-trace map, $\text{tr}_W^\phi$ on $W = \oplus_{h=0}^N W_T^{(n)}(h)$ does not depend on the choice of $N$ and so it is uniquely defined on any $W_T^{(n)}(h)$ as does an ordinary trace map. We also note that
\[
\text{tr}_{W_T^{(n)}}^\phi(a) = \text{tr}_T^\phi(a) = \phi(a)
\]
for $a \in \text{soc}(A)$ by definition.

### 4.1 Logarithmic modules

In this subsection, $\omega$ denotes Virasoro element and we fix $r \in \mathbb{C}$ and $s \in \mathbb{N}$. Assume $(\omega - n - r)^sT = 0$ and $(\omega - n - r)T \neq 0$, thus $L(0) = o(\omega)$ does not act on the graded piece $W_T^{(n)}(m)$ semisimply. However, since $(L(0) - m - r)^sW_T^{(n)}(m) = 0$, we are able to understand
\[
e^{2\pi i L(0)\tau} = e^{2\pi i(m+r)\tau} \left( \sum_{j=0}^{s-1} \frac{1}{j!} (2\pi i (L(0) - m - r))^j \right) \text{ on } W_T^{(n)}(m)
\]
and define $q^{L(0)}$ on $W_T^{(n)}(m)$ by
\[
q^{m+r} \left( \sum_{j=0}^{s-1} \frac{1}{j!} (2\pi i (L(0) - m - r))^j \right).
\]

Such a module is called a logarithmic module. We note that the left action $L(0) - m - r$ on $W_T^{(n)}(m)$ is equal to the right action of $\omega - n - r \in P$ on $W(m)$ and we denote it by $(\omega - n - r)_p$. Set $\mathfrak{N}_i = \{a \in P \mid (\omega - n - r)^i a = 0\}$ and let $L^*(0)$ be a degree operator which acts on $W(m)$ as $m+r$, that is, the semisimple part of $L(0)$ and $L(0) - L^*(0)$ is nilpotent. By Proposition 3.8, we have:
Lemma 4.1

\[ \text{tr}^\phi_W(L(0) - L^*(0))^i g = \text{tr}^\phi_W g(\omega - n - r)^i = \text{tr}_{W/W_{2l}}^\omega \phi \]  

Proposition 4.2 For any \( v, u \in V \) and a generalized Verma module \( W = W_T^{(n)} \) interlocked with \( \phi \). Although we are studying a general (nonsemisimple) operator \( L(0) \) and a pseudo-trace function, they satisfy the following properties as do the action of \( L(0) \) on modules and trace map:

1. \( \text{tr}^\phi_W \) is a symmetric function,
2. \( [L(0), v(m)] = (|v| - m - 1)v(m) \) and
3. \( \text{tr}^\phi_W o(\omega)q^{L(0)} = \frac{1}{2\pi i} \frac{d}{d\tau} \text{tr}^\phi_W o(v)q^{L(0)}, \)

which are the properties that Zhu used in the proof for ordinary trace functions. Therefore we have the following results by exactly the same arguments as in [Zh].

4.2 One-point functions

By the investigation of trace functions in [Zh], Zhu showed

\[ S^M(v[-2]u + \sum_{k=1}^\infty (2k - 1)E_{2k}(\tau)v[2k - 2]u, \tau) = 0 \]

\[ S^M(L[-2]u + \sum_{k=1}^\infty E_{2k}(\tau)L[2k - 2]u, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S^W(u, \tau) \]
for a $V$-module $M$. Since $[o(v), o(u)] = o(v)[0]u$, (4.2) implies that $S_W(\ast, \tau)$ is a symmetric linear function on $\langle o(v) \mid v \in V \rangle$ in a sense.

Consider $V[E_4(q), E_6(q)] \subseteq V[[q]]$. $O_q(V)$ is the submodule of $V[E_4(q), E_6(q)]$ generated by elements of the type $v[0]u$

and

$$v[-2]u + \sum_{k=2}^{\infty} (2k-1)E_{2k}(\tau) \otimes v[2k-2]u \quad v, u \in V.$$

We first prove the following lemma.

**Lemma 4.3** For $\alpha \in V$ and a fixed integer $n > 1$, there are $v^i$ and $u^i \in V$ ($i = 1, ..., p$) such that

$$o(\alpha) = \sum_{i=1}^{p} v^i(|v^i|^{-1}n)u^i(|u^i|^{-1}n) \text{ in } A(V),$$

where the statement “$o(\beta) = v^i(|v^i|^{-1}n)u^i(|u^i|^{-1}n) \text{ in } A(V)$” implies $o(\beta)u = v^i(|v^i|^{-1}n)u^i(|u^i|^{-1}n)u$ for any $\mathbb{N}$-graded weak $V$-module $W$ and $u \in W(0)$.

**Proof** We first note the following fact, which is a natural consequence of associativity. For any $v, u \in V$ and $r, s \in \mathbb{Z}$ and $m \in \mathbb{N}$, there is an element $\beta$ of $V$ which is a linear combination $\beta = \sum \lambda_i v(i)u + \sum \mu_i u(i)v$ of $v(i)u$ and $u(i)v$ such that

$$\beta(|\beta|-1+r-s) = v(|v|-1-s)u(|u|-1+r) + \sum_{i \geq m} a_i v(|v|-1-s+r-i)u(|u|-1+i)$$

$$+ \sum_{i \geq m} b_i u(|u|-1-s+r-i)v(|v|-1+i)$$

with $a_i, b_i \in \mathbb{C}$. In particular, when $s = r = 0$, then $\beta = v \ast_m u$ is the product in $A_m(V)$. We will use this argument in several places.

Set $D_n = \langle \alpha \in V \mid \alpha = v(|v|-1+n)u(|u|-1-n) \text{ in } A(V) \text{ for some } v, u \in V \rangle$.

Clearly, $D_n$ contains $O(V)$ and $D_n/O(V)$ is an ideal of $A(V)$. If $W$ is an irreducible $V$-module and $0 \neq w \in W(0)$, then there is an element $v \in V$ such that $0 \neq v(|v|-1-n)w \in W(n)$ since $W(n) \neq 0$. There is also an element $u \in V$ such that $u(|u|-1+n)v(|v|-1-n)w \neq 0$, (see [DM]). Hence $D_n$ covers $\text{End}(W(0))$ for all simple modules $W$ and so $A(V) = D_n + J(A(V))$, which means $D_n = V$.

The purpose in this subsection is to prove the following three propositions, which we will use in the next section.

**Proposition 4.4** If $S(v, \tau) = \sum_{i=0}^{\infty} S_i(v)q^i \in \mathbb{C}[[q]]$ satisfies that $S(\alpha, \tau) = 0$ for all $\alpha \in O_q(V)$, then $S(v, \tau) \in \mathbb{C}[[q]]q^{n+1}$ for all $v \in O_n(V)$.
In particular, $S_i$ is a symmetric linear function of $A_i(V)$.

**Proposition 4.5** Assume that $S(v, \tau) = \sum_{j=0}^{N} \left( \sum_{i=0}^{\infty} S_{ji}(v) q^{i+r} \right) (2\pi i \tau)^j \in \mathbb{C}[q]q^r[\tau]$ satisfies $S(\alpha, \tau) = 0$ for all $\alpha \in O_q(V)$ and

$$S(L[-2]v - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]v, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S(v, \tau)$$

for all $v \in V$. Then $S_{jm}((\omega - r - \frac{c}{24} - m)^{N-j+1} *_m v) = 0$ for any $m$ and $j$.

**Proposition 4.6** Assume that $S(v, \tau) = \sum_{i=0}^{\infty} S_i(v) q^i \in \mathbb{C}[q]$ satisfies $S(\alpha, \tau) = 0$ for all $\alpha \in O_q(V)$ and $S_m((\omega - r - \frac{c}{24} - m)^{s} *_m v) = 0$ for all $v \in V$ and $m$. If $S_n = 0$ for some $n > l+r$, then $S_0 = 0$, where $l$ is given in the beginning of this section.

**Proof of Proposition 4.4, 4.5 and 4.6**

In order to prove the three propositions above at once, we will review the proof of Proposition 4.3.3 in [Zh]. Zhu first obtained

$$\text{tr}_M v(|v| - 1 - k) u(|u| - 1 + k) q^{L(0)} = \text{tr}_M \frac{-q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 - k}{s} \right) o(v(s)u) q^{L(0)}$$

$$\text{tr}_M u(|u| - 1 - k) v(|v| - 1 + k) q^{L(0)} = \text{tr}_M \frac{q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 + k}{s} \right) o(v(s)u) q^{L(0)},$$

for $V$-modules $M$ and $k \neq 0$, see the proof of Proposition 4.3.2 in [Zh].

Using the equation above, Zhu denoted

$$\text{tr}_M w^{[|v|] z^{[|v|]} Y(v, w) Y(u, z) q^{L(0)}}$$

as an infinite linear combinations of $o(v)o(u), o(v[0]u) = o(v)o(u) - o(u)o(v)$,

$$\text{tr}_M \frac{-q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 - k}{s} \right) o(v(s)u) q^{L(0)} \left( \frac{w}{z} \right)^k$$

and

$$\text{tr}_M \frac{q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 + k}{s} \right) o(v(s)u) q^{L(0)} \left( \frac{z}{w} \right)^k.$$

Then substituting them into the normal product

$$Y(v(i)u, z) = \text{Res}_w \{ (w - z)^i Y(v, w) Y(u, z) - (-z + w)^i Y(u, z) Y(v, w) \}$$

and using an expansion $v[-1]u = \sum_{i \geq -1} c_i v(i)u$ with $c_i \in \mathbb{C}$, he expressed the term $o(v[-1]u)$ between $\text{tr}_M$ and $q^{L(0)}$ as a linear combinations of $o(v)o(u), o(v[0]u), \frac{-q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 - k}{s} \right) o(v(s)u)$ and $\frac{q^k}{1 - q^k} \sum_{s=0}^{\infty} \left( \frac{|v| - 1 + k}{s} \right) o(v(s)u).$
The next step is the crucial part of his paper. He changed the shape of the above expression of \( o(v[-1]u) \) into

\[
\frac{o(v)o(u) - o(v[0]u)}{o(u)o(v)} + \sum_{k=1}^{\infty} E_{2k}(q)o(v[2k-1]u)
\]

using the equations (c.f. (4.3.8)-(4.3.11) and Proposition 4.3.2 in [Zh]). Namely, he proved

\[
\text{tr}_{|M} \left\{ o(v[-1]u) - o(v)o(u) + o(v[0]u) - \sum_{k=1}^{\infty} E_{2k}(q)o(v[2k-1]u) \right\} q^{L(0)} = 0.
\]

He obtained two important equations from it. The first is, by substituting \( \tilde{\omega} \) for \( v \),

\[
\text{tr}_{|M} \left\{ o(L[-2]u) - \sum_{k=1}^{\infty} E_{2k}(\tau)o(L[2k-2]u) \right\} q^{L(0)-c/24} = \frac{1}{2\pi i} \frac{d}{d\tau} \text{tr}_{|M} o(u)q^{L(0)-c/24}.
\]

The second is, by substituting \( L[-1]v \) for \( v \), he had

\[
\text{tr}_{|M} \left\{ v[-2]u + \sum_{k=2}^{\infty} (2k-1)v[2k-2]u E_{2k}(q) \right\} q^{L(0)} = 0.
\]

These equations are so beautiful that we can see the modular invariance property of \( O_q(V) \) clearly. Once we know the modular invariance property of \( O_q(V) \), we don’t need these forms. We are interested in the structure of \( O_q(V) \) from a view point of ordinary vertex operators \( Y(v,z) = \sum_{i \in \mathbb{Z}} v(i)z^{-i-1} \). We go back to the former form of

\[
o(u)o(v) + \sum_{k=1}^{\infty} E_{2k}(\tau)o(v[2k-1]u).
\]

Namely, we change an expansion of \( o(v[-1]u) \):

\[
o(v[-1]u) = \sum_{i \geq -1} c_i o(v(i)u)
\]

\[
= \sum_{i \geq -1} c_i \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \left\{ v(i-j)u(|v|+|u|-i-2-j) - (-1)^i u(|v|+|u|-2-j)v(j) \right\}
\]

by replacing \( v(|v|-1-k)u(|u|-1+k) \) and \( u(|u|-1-k)v(|v|-1+k) \) by \( \frac{-q^k}{1-q^k} \sum_{s=0}^{\infty} \left( \frac{|v|-1-k}{s} \right) v(s)u \)

and \( \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \left( \frac{|v|-1+k}{s} \right) v(s)u \), respectively, for \( k \neq 0 \). We note \( c_{-1} = 1 \).

To simplify the arguments, we express this process by notation \( \theta \), that is, for \( \alpha = \sum_i \lambda_i v(i)u \), we first develop \( \sum_i \lambda_i o(v(i)u) \) as a linear sum

\[
ao(v)o(u) + bo(u)o(v) + \sum_{k \neq 0} \lambda_k v(|v|-1-k)u(|u|-1+k) + \sum_{k \neq 0} \mu_k u(|u|-1-k)v(|v|-1+k)
\]
with \( a, b, \lambda, \mu \in \mathbb{C} \) by using associativity, then define

\[
\theta \left( \sum_i \lambda_i v(i)u - ao(v)o(u) - bo(u)o(v) \right) = \sum_{k \neq 0} \lambda_k \frac{-q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1-k}{s} v(s)u + \sum_{k \neq 0} \mu_k \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} v(s)u.
\]

What Zhu has obtained are

\[
v[-1]u - \theta(v[-1]u-o(u)o(v)) = v[-1]u - \sum_{k=1}^{\infty} E_{2k}(\tau) v[2k-1]u \quad (4.3)
\]

and

\[
v[-2]u - \theta(v[-2]u) = v[-2]u + \sum_{k=1}^{\infty} (2k-1) E_{2k}(\tau) v[2k-2]u \in O_q(V) \quad (4.4)
\]

On the other hand, if \( i \geq 0 \), then we have

\[
o(v(i)u) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} \{ v(i-j)u(|v|+|u|-i-2+j) - (-1)^i u(|v|+|u|-2-j)v(j) \}
\]

\[
= \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \{ v(j)u(|v|+|u|-2+j) - u(|v|+|u|-2-j)v(j) \}.
\]

If we replace \( v(|v|-1+k)u(|u|-1-k) - u(|u|-1-k)v(|v|-1+k) \) by

\[
\frac{1}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} v(s)u - \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} v(s)u = \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} v(s)u
\]

for each \( k \neq 0 \), then

\[
o(v(i)u) = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \{ v(j)u(|v|+|u|-2+j) - u(|v|+|u|-2-j)v(j) \}
\]

is replaced by

\[
\sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \sum_{s=0}^{\infty} \binom{j}{s} v(s)u = v(i)u.
\]

Namely, \( \theta(v(i)u) = v(i)u \) for \( i \geq 0 \). By cancelling \( v(i)u (i \geq 0) \) from both sides of \( v[-1]u \) and \( \theta(v[-1]u) \) in (4.3) and also both sides of \( v[-2]u \) and \( \theta(v[-2]u) \) in (4.4), we obtain

\[
v(-1)u - \theta(v(-1)u-o(u)o(v)) = v[-1]u - \sum_{k=1}^{\infty} E_{2k}(\tau) v[2k-1]u \quad (4.5)
\]
\[
\begin{align*}
\omega(-1)u - \frac{c}{24}u - \theta(\omega(-1)u - o(u)o(\omega)) &= L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u, \quad \text{and} \quad (4.6) \\
v(-2)u + |v|v(-1)u - \theta(v(-2)u) + |v|v(-1)u \\
&= v[-2]u + \sum_{k=1}^{\infty} (2k-1)E_{2k}(\tau)v[2k-2]u \in O_q(V), \quad (4.7)
\end{align*}
\]

since \( \theta(1(-1)u - o(u)) = 0 \). Substituting \( L(-1)^m v \) into \( v \) of (4.7), we have

\[
v(-2-m)u + \frac{|v|+m}{m+1}v(-1-m)u - \theta(v(-2-m)u + \frac{|v|+m}{m+1}v(-1-m)u) \in O_q(V).
\]

If \( a \in O_n(V) \), then \( a \) is a linear combination of

\[
v \circ_n u = \text{Res}_z Y(v, z)u \frac{(1+z)^{|v|+n}}{z^{2+2n}}
\]

for some \( v, u \in V \) and the expansion of \( v \circ_n u \) by associativity is an infinite linear sums of \( v(|v|-1-k)u(|u|-1+k) \) and \( u(|u|-1-k)v(|v|-1+k) \) with \( k \geq n+1 \). Since for \( k \geq n+1 \),

\[
\theta(v(|v|-1-k)u(|u|-1+k)) = \frac{-q^k}{1-q^k} \sum_{s=0}^{\infty} \left( \frac{|v|-1-k}{s} \right) v(s)u
\]

and

\[
\theta(u(|u|-1-k)v(|v|-1+k)) = \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \left( \frac{|v|-1+k}{s} \right) v(s)u
\]

are contained in \( V[q]q^{n+1} \), we obtain \( S(a, q) \in V[[q]]q^{n+1} \), which proves Proposition 4.4.

By the same arguments, if

\[
S(v, \tau) = \sum_{j=0}^{N} S_j(v, \tau)q^j(2\pi i \tau)^j = \sum_{j=0}^{N} \left( \sum_{i=0}^{\infty} S_{ji}(v)q^i \right) q^j(2\pi i \tau)^j
\]

satisfies \( S(\alpha, \tau) = 0 \) for all \( \alpha \in O_q(V) \) and

\[
S(L[-2]v - \sum E_{2k}(\tau)L[2k-2]v, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S(v, \tau)
\]

for all \( v \in V \), then

\[
S(\sum_{i>-M} \lambda_i \omega(i)u, \tau) = s \left( \frac{1}{2\pi i} \frac{d}{d\tau} S(u, \tau) + \frac{c}{24} S(u, \tau) \right) + S(\theta(\sum_{i>-M} \lambda_i \omega(i)u - so(\omega)o(u)), \tau)
\]

for \( \sum_{i>-M} \lambda_i \omega(i)u \), where \( s \) is given by

\[
o(\sum_{i>-M} \lambda_i \omega(i)u) = s \ o(\omega)o(u) + \sum_{j>0} (a_j \omega(1-j)u(|u|-1+j) + b_j u(|u|-1-j)\omega(1+j))
\]
Therefore we obtain

\[
S(\omega * u, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} S(u, \tau) + \frac{c}{24} S(u, \tau) + S(\theta(\omega * u - o(\omega) o(u)), \tau).
\]

Therefore we obtain

\[
\sum_{j=0}^{N} \sum_{i=0}^{\infty} S_{ji}(\omega * u)q^{jr}(2\pi i \tau)^j = \sum_{j=0}^{N} S_{j}(\omega * u, \tau)q^{jr}(2\pi i \tau)^j = S(\omega * u, \tau)
\]

\[
= \frac{1}{2\pi i} \frac{d}{d\tau} S(u, \tau) + \frac{c}{24} S(u, \tau) + S(\theta(\omega * u - o(\omega) o(u)), \tau)
\]

\[
= \frac{1}{2\pi i} \frac{d}{d\tau} \left( \sum_{j=0}^{N} S_{ij}(u)q^{jr}(2\pi i \tau)^j \right) + \frac{c}{24} \sum_{j=0}^{N} S_{ij}(u)q^{jr}(2\pi i \tau)^j
\]

\[
+ \sum_{j=0}^{\infty} S_j(\theta(\omega * u - o(\omega) o(u)), \tau)(2\pi i \tau)^j
\]

and so

\[
\sum_{i=0}^{\infty} S_{ji}(\omega * u)q^{jr}(2\pi i \tau)^j
\]

\[
= \sum_{i=0}^{\infty} S_{j,i}((i+r - \frac{c}{24}) u)q^{jr}(2\pi i \tau)^j + \sum_{j=0}^{\infty} S_{j+1,i}(u)q^{jr}(2\pi i \tau)^j
\]

\[
+ S_j(\theta(\omega * u - o(\omega) o(u)), \tau)(2\pi i \tau)^j
\]

for each \( j \). Thus we have

\[
S_{j,i}((\omega - i - r - \frac{c}{24}) * u) = S_{j+1,i}(u)
\]

for \( i \leq n \) since \( \theta(\omega * u - o(\omega) o(u)) \in V[[q]]q^{n+1} \). It follows from \( S_{N+1,n}(u) = 0 \) that \( S_{j,n}((\omega - n - r - \frac{c}{24})^{N-j+1} * u) = 0 \), which proves Proposition 4.5.

Suppose that Proposition 4.6 is false. Namely, there is an integer \( n > l + r \) such that \( S_n = 0 \) and \( S_0(\alpha) \neq 0 \) for some \( \alpha \in V \). By Proposition 4.4, \( S_0 \) is a (symmetric) linear map of \( A(V) = V/O(V) \). Set \( A = A(V)/\text{Rad}(S_0) \). Then \( (\omega - r - \frac{c}{24})^* A = 0 \). By Lemma
4.3, there are $v^i, u^i \in V$ such that $\sum_{i=1}^{n} v^i(|v^i|-1+n)u^i(|u^i|-1-n) = o(\alpha)$ in $A(V)$. By the choice of $\alpha$, we may assume $v(|v|-1+n)u(|u|-1-n) = o(\alpha)$ in $A(V)$ for some $v, u \in V$.

As we mentioned in the proof of Lemma 4.3, there is an element $\beta \in V$ such that

$$o(\beta) = u(|u|-1-n)v(|v|-1+n) + \sum_{i>n} a_i v(|v|-1-i)u(|u|-1+i)$$

$$+ \sum_{i>n} b_i u(|u|-1-i)v(|v|-1+i)$$

for some $a_i, b_i \in \mathbb{C}$. Then we obtain

$$o(\alpha) = v(|v|-1+n)u(|u|-1-n) = [v(|v|-1+n), u(|u|-1-n)]$$

$$= \sum_{i=0}^{\infty} \left(\frac{|v|-1+n}{i}\right) o(v(i)u)$$

on $W(0)$ for any $\mathbb{N}$-graded weak $V$-modules $W$ and so $\alpha = \sum_{i=0}^{\infty} \left(\frac{|v|-1+n}{i}\right) v(i)u$ in $A(V)$.

On the other hand, since $n > 1$, we have $\beta \in O(V)$ and

$$S(\beta, \tau) = S(\theta(\beta), \tau)$$

$$\in \frac{q^n}{1-q^n} S(\sum_{i=0}^{\infty} \left(\frac{|v|-1+n}{i}\right) v(i)u, \tau) q^n + q^{n+1} \mathbb{C}[[q]]$$

$$= S(\sum_{i=0}^{\infty} \left(\frac{|v|-1+n}{i}\right) v(i)u, \tau) q^n + q^{n+1} \mathbb{C}[[q]] = S(\alpha, \tau) q^n + q^{n+1} \mathbb{C}[[q]].$$

Since coefficients of $S(\beta, \tau)$ at $q^n$ are always zero and the constant term of $S(\alpha, \tau)$ is nonzero, we have a contradiction.

This completes the proofs of three propositions.

5 The space of one point functions on the torus

In this section, we will just follow the proofs in [ZH] and [DLiM3] with suitable modification since we use pseudo-trace functions which satisfy the same properties as do the ordinary trace functions and so we will skip the most part of the proof. The differences between our case and Zhu’s case (and also the case in [DLiM3]) are that we will treat $A_n(V)$ and our $A_n(V)$ is not a semisimple algebra, for example, $\omega$ might not act on $A_n(V)$ semisimply. However, since $A_n(V)$ is a finite dimensional algebra, there are $r_i \in \mathbb{C}$ and $\mu(r_i) \in \mathbb{Z}$ such that $\prod_{i=1}^{s} (\omega - \frac{r_i}{24} - r_i - n)^{\mu(r_i)} A_n(V) = 0$.

Let’s recall the following notation from [ZH] and [DLiM3]: Consider $V[E_4(q), E_6(q)] \subseteq V[[q]]$. $O_q(V)$ is the submodule of $V[E_4(q), E_6(q)]$ generated by elements of the type

$$v[-2]u + \sum_{k=2}^{\infty} (2k-1)E_{2k}(\tau) \otimes v[2k-2]u \quad \text{with} \quad v, u \in V$$

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Theorem 5.2 (Modular-Invariance). For \( C \) and \( W \) for a generalized Verma module \( u \)
In particular, \( S \) is holomorphic on the upper half plane. In order to prove this fact, we will show that
\[
\sum_{n=1}^{\infty} |n|^2 \frac{d^2}{d\tau^2} S(n,\tau) = \frac{1}{2\pi i} d\tau S(u,\tau)
\]
By exactly the same proof, we have the following modular invariance property of \( C(V) \), (see Theorem 5.1.1 in [Zh] and Theorem 5.4 in [DLIM3].)

Theorem 5.2 (Modular-Invariance). For \( S \in C_1(V) \) and \( \gamma \in \left( \frac{a}{c}, \frac{b}{d} \right) SL(2, \mathbb{Z}) \), define \( S|\gamma(v,\tau) = \frac{1}{(cr+dz)^m} S(v, \gamma(\tau)) \) for \( v \in V[n] \) and extend linearly. Then \( S|\gamma \in C_1(V) \).

Set
\[
S^W(u,\tau) = tr_W \phi o(u) q^{L(0)-c/24}
\]
for a generalized Verma module \( W \) interlocked with a symmetric linear function \( \phi \) of \( A_n(V) \) and an element \( u \in V \), where \( c \) is the central charge of \( V \). Then extend it linearly for \( V[E_4(q), E_6(q)] \).

We next prove that \( S^W(u,\tau) \in C_1(V) \). What we have to do is to show that \( S^W(u,\tau) \) is holomorphic on the upper half plane. In order to prove this fact, we will show that \( S^W \) and all \( S \in C_1(V) \) satisfy differential equations. The proof is essentially the same as the arguments (4.4.11) in [Zh] and Lemma 6.1 in [DLIM3].

Theorem 5.3 Assume that \( V \) is \( C_2 \)-cofinite. If \( S(v,\tau) = 0 \) for \( v \in O_q(V) \) and
\[
S(L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u,\tau) = \frac{1}{2\pi i} d\tau S(u,\tau)
\]
for \( u \in V \), then there are \( m \in \mathbb{N} \) and \( r_i(\tau) \in \mathbb{C}[E_4(q), E_6(q)] \), \( 0 \leq i \leq m-1 \), such that
\[
S(L[-2]^m v,\tau) + \sum_{i=0}^{m-1} r_i(\tau) S(L[-2]^i v,\tau) = 0.
\]
In particular, \( S(u,\tau) \) converges absolutely and uniformly in every closed subset of the domain \( \{ q \mid |q| < 1 \} \) for every \( u \in V \) and the limit function can be written as a linear sum of \( q^h f(q) \), where \( f(q) \) is some analytic function in \( \{ q \mid |q| < 1 \} \). In particular, \( S \in C_1(V) \).
As a corollary, we obtain:

**Corollary 5.4** $S^W(u, \tau)$ is holomorphic on the upper half-plane for $u \in V$ and $S^W(\ast, \tau) \in C_1(V)$.

### 5.1 Spanning set of $C_1(V)$

The remaining is to show that $C_1(V)$ is spanned by pseudo-trace functions $S^W(\ast, \tau)$. So we will prove the following main theorem, which covers a nonsemisimple version of Theorem 5.3.1 in [Zh].

**Theorem 5.5** Suppose that $V = \bigoplus_{m=0}^{\infty} V_m$ is a $C_2$-cofinite VOA with central charge $c$. Take an integer $n$ sufficiently large and let $\{W^1, \ldots, W^m\}$ be the set of $n$-th generalized Verma $V$-modules $W^i$ interlocked with some symmetric linear function $\phi^i$ of $A_n(V)$. Then $C_1(V)$ is spanned by

$$\{S^{W^1}(\cdot, \tau), \ldots, S^{W^m}(\cdot, \tau)\}.$$ In particular, $\dim C_1(V) = \dim A_n(V)/[A_n(V), A_n(V)] - \dim A_{n-1}(V)/[A_{n-1}(V), A_{n-1}(V)]$.

We note that the dimension of $A_n(V)$ is finite and so $C_1(V)$ is also of finite dimension.

**Proof of Theorem 5.5**

Let $S \in C_1(V)$. We will prove that $S$ is a sum of pseudo-trace functions. By the same arguments as in [Zh], it follows from Theorem 5.3 that there are integers $d, N_1, \ldots, N_d$ which does not depend on $v$ such that

$$S(v, \tau) = \sum_{s=0}^{d} S_s(v, \tau)q^{r_s}$$

and each $S_s(v, \tau)$ can be further decomposed as

$$S_s(v, \tau) = \sum_{j=0}^{N_s} S_{sj}(v, \tau)(2\pi i \tau)^j,$$

where $r_1, \ldots, r_d$ are complex numbers independent of $v$, $r_{s_1} - r_{s_2} \notin \mathbb{Z}$ for $s_1 \neq s_2$ and $S_{sj}(v, \tau)$ has a $q$-expansion $S_{sj}(v, \tau) = \sum_{i=0}^{\infty} C_{sji}(v)q^k$ with $C_{sji}(v) \in \mathbb{C}$ and for each $j$ there is $s$ such that $C_{sj0} \neq 0$. Since $r_{s_1} - r_{s_2} \notin \mathbb{Z}$ for $s_1 \neq s_2$, each $S_s(v, \tau)q^{r_s}$ satisfies (C2)~(C4) and also (C1) by Theorem 5.3. Hence we may assume $d = 1$ and

$$S(v, \tau) = \sum_{j=0}^{N} S_j(v, \tau)(2\pi i \tau)^j = \sum_{j=0}^{N} \left( \sum_{i=0}^{\infty} C_{ji}(v)q^{i+r} \right)(2\pi i \tau)^j, \quad (5.1)$$

with $r \in \mathbb{C}$. In the case where $A(V)$ is semisimple, they proved in [Zh] and [DL1] that $N = 0$ and $\sum_{i=0}^{\infty} S_i(v)q^{i+r}$ is a sum of trace functions. However, if $A_n(V)$ is not semisimple, we may have nonzero $N$ since we consider logarithmic modules, too. In order to continue the proof, we need the following two lemmas.
Lemma 5.6 Let $W = W_T^{(n)}$ be an $n$-th generalized Verma module interlocked with $(P, \phi)$ satisfying $(\omega - r - n - \frac{e}{24})^{N+1} W_T^{(n)}(n) = 0$ for some $r \in \mathbb{C}$ and $N \in \mathbb{N}$. Then there are constants $b_1, \ldots, b_{N-1}$ such that

$$\text{tr}^\phi_W o(v)q^{L^*(0)-c/24} = S^W(v, \tau) - \sum_{i=1}^N b_i S^{W/W^\Phi}(v, \tau)(2\pi \iota)^i,$$

where $\Phi = \{ a \in P | (\omega - r - n - \frac{e}{24}) a = 0 \}$ and $L^*(0)$ is a semisimple part of $L(0)$, which acts on $W(m)$ as $m + r + \frac{e}{24}$ and $(L(0) - L^*(0))^{N+1} = 0$ on $W$.

**Proof** We first note that there are constants $b_0 = 1, b_1, \ldots$ such that $e^{2\pi i \alpha} = 1 + b_1 \alpha e^{2\pi i \alpha} + b_2 \alpha^2 e^{2\pi i \alpha} + \ldots + b_{N-1} \alpha^{N-1} e^{2\pi i \alpha} + \ldots$. Hence we obtain

$$S^W(v, \tau) = \text{tr}^\phi_W o(v)q^{L^*(0)-\frac{e}{24}} = \text{tr}^\phi_W o(v)q^{(L^*(0)-\frac{e}{24})(L(0) - L^*(0))}$$

$$= \text{tr}^\phi_W o(v)q^{L^*(0)-\frac{e}{24}} + \sum_{i=1}^N \text{tr}^\phi_W o(v)b_i (L(0) - L^*(0))^{i}(2\pi \iota)^i q^{(L(0) - L^*(0))^{i}(2\pi \iota)^i}$$

$$= \text{tr}^\phi_W o(v)q^{L^*(0)-\frac{e}{24}} + \sum_{N=1}^N b_i \left( \text{tr}^\phi_W o(v)q^{L^*(0)-\frac{e}{24}}\right)^i$$

by (3.6)

$$= S^W(v, \tau) + \sum_{i=1}^N b_i S^{W/W^\Phi}(v, \tau)(2\pi \iota)^i. \quad \blacksquare$$

We will prove that $S_0(v, \tau) = \sum_{i=0}^\infty C_{0i}(v) q^{i+r}$ is a linear sum of pseudo-trace functions with a semisimple grading operator $L^*(0)$, say $S_0(v, \tau) = \sum_P a_P \text{tr}^\phi_W o(v)q^{L^*(0)-c/24}$. Then

$$\tilde{S}(v, \tau) = S(v, \tau) - \sum_P a_P \left( S^{W_P}(v, \tau) - \sum_{i=1}^{s-1} b_i S^{W/W^\Phi}(v, \tau)(2\pi \iota)^i \right) \in \mathcal{C}_1(V),$$

but if we express it by

$$\tilde{S}(v, \tau) = \sum_{s=0}^N \sum_{j=0}^\infty \tilde{S}_{sj}(v) q^{j+r}(2\pi \iota)^s,$$

then $\tilde{S}_{0j}(v) = 0$ for all $j$ and $v \in V$. However, since $\tilde{S}(L[-2]u - \sum_{k=1}^\infty E_{2k}(\tau)L[2k-2]u, \tau) = \frac{1}{2\pi \iota \tau} \tilde{S}(u, \tau)$, $S_{0j} = 0$ implies $S_{1j} = 0$ and so on. We hence have $\tilde{S}(v, \tau) = 0$ for all $v \in V$ as desired.

So it is sufficient to prove that $S_0(v, \tau) = \sum_{i=0}^\infty C_{0i}(v) q^{i+r}$ is the coefficient of $(2\pi \iota)^0$-term of a linear sum of pseudo-trace functions $S^{W_P}(v, \tau)$.

Since $S(L[-2]u - \sum_{k=1}^\infty E_{2k}(\tau)L[2k-2]u, \tau) = \frac{1}{2\pi \iota \tau} S(v, \tau)$, we obtain the following lemma from Proposition 4.4 and 4.5.
Lemma 5.7 We have

\[ C_{0,n}(v *_n u) = C_{0,n}(u *_n v) \]
\[ C_{0,n}(v) = 0 \text{ for } v \in O_{n+1}(V) \]
\[ C_{0,n}(\omega - \frac{c}{24} - r_s - n)^{N+1} *_n v) = 0 \]

In particular, \( A_n(V) \) has a symmetric linear map \( C_{0,n} \). Let \( A_1 \oplus \cdots \oplus A_k \) be the decomposition of \( A_n(V)/J(A_n(V)) \) into the direct sum of simple algebras \( A_i \) and \( \{e_i \mid i = 1, \ldots, k\} \) a set of mutually orthogonal primitive idempotents of \( A_n(V) \) such that \( e_i + J(A_n(V)) \in A_i \). Set \( e = \sum e_i \). By Theorem 3.10, there are symmetric linear functions \( \phi_p \) of \( A_n(V) \) and \( A_n(V) \times e A_n(V) e \)-modules \( (A_n(V)/\mathfrak{H}_p)e \) such that \( (\omega - \frac{c}{24} - r_n)^{N+1} \in \mathfrak{H}_p \) and \( C_{0,n} \) is a sum of pseudo-trace maps, that is,

\[ C_{0,n} = \sum_p a_p \text{tr} \phi_p(A_n(V)/\mathfrak{H}_p) e, \]

where \( \mathfrak{H}_p = \text{Rad}(\phi_p) \) and \( (A_n(V)/I_p) e \) are all indecomposable \( A_n(V) \)-modules.

We have assumed that \( n \) is large enough so that there are no conformal weights greater than \( r + n - l \). Construct \( n \)-th generalized Verma modules

\[ W^p = W^{(n)}_p \]

from \( A_n(V) \)-module \( T_p = (A_n(V)/\mathfrak{H}_p) \tilde{e} \). As we showed, \( W^p \) is interlocked with \( \phi_p \) and \( L(0) - \frac{c}{24} - r - m \) acts on \( W^p(m) \) as a nilpotent operator. Define a pseudo-trace function

\[ S^{W^p}(v, \tau) = \text{tr} \phi_p o(v) g^{L(0) - c/24}. \]

Then \( \tilde{S}(v, \tau) = S(v, \tau) - \sum_p a_p S^{W^p}(v, \tau) = \sum_{s=0}^N (\sum_{i=0}^\infty \tilde{C}_{si}(v) q^{i+r})(2\pi i \tau)^s \) satisfies the same properties, but

\[ \tilde{C}_{00}(v, \tau) = 0 \text{ for all } v \in V. \]

Then by Proposition 4.5, we have \( \tilde{C}_{00} = 0 \). Since

\[ \tilde{S}(L[-2]u - \sum_{k=1}^\infty E_{2k}(\tau) L[2k-2]u, \tau) = \frac{1}{2\pi i} \frac{d}{d\tau} \tilde{S}(u, \tau), \]

\( \tilde{C}_{s0}(u) = 0 \) for all \( u \in V \) and \( s = 0, 1, \ldots, N \). Namely, the lowest weight \( \tilde{r} \) of \( \tilde{S}(\ast, \tau) \) is greater than that of \( S(\ast, \tau) \). Repeating these steps, we obtain the desired result, since \( V \) has only finitely many lowest weights of pseudo-trace functions. This completes the proof of Theorem 5.5.

We next consider the case \( v = 1 \). Then by Theorem 5.5,

\[ \left\langle \text{tr}^\phi_W q^{L(0) - c/24} \mid W \text{ is interlocked with } \phi \text{ of } A_n(V) \right\rangle \]

is \( SL_2(\mathbb{Z}) \)-invariant. \( \text{tr}^\phi_W q^{L(0) - c/24} \) plays a role of a generalized character introduced in \( \text{[F]} \) and so we will call it a generalized character of \( W \). Let’s study generalized characters for a while. If \( \phi(1) = 0 \) and \( (L(0) - L^s(0)) W = 0 \), then \( \text{tr}^\phi_W q^{L(0)} = 0 \) and so we have:
Theorem 5.8 Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a $C_2$-cofinite VOA. Then the space spanned by generalized characters is $SL_2(\mathbb{Z})$-invariant. In particular, if there is no logarithmic module, then the space spanned by the set of all (ordinary) characters is $SL_2(\mathbb{Z})$-invariant.

We may assume $\phi(1) = 0$ and $(L(0) - L^s(0))^m W = 0$. Let $r$ be a conformal weight of $W$. Then

$$\text{tr}_W^\phi q^{L(0) - c/24} = \text{tr}_W^\phi \sum_{j=0}^{m} \frac{1}{j!} (L(0) - L^s(0))^j q^{L^s(0) - c/24} (2\pi i \tau)^j \in \mathbb{C}[[q]] q^{-c/24} [\tau].$$

By Lemma 4.1, $\text{tr}_W^\phi (L(0) - L^s(0))^j q^{L^s(0) - c/24} = \text{tr}_{W/W_N}^\phi r^j q^{L^s(0) - c/24}$ is a linear combination of characters. Therefore, we obtain:

Proposition 5.9 A generalized character is a linear combination of characters with coefficients in $\mathbb{C}[\tau]$.

As an application of Theorem 5.5, $\langle \text{ch}_W(\tau)^{SL_2(\mathbb{Z})} \rangle$ is of finite dimension for an irreducible $V$-module $W$ and $\text{ch}_W(\tau) \in \sum_{i=1}^{k} \mathbb{C}[[q]] q^{-c/24}$. Therefore, we can apply the same arguments as in the proofs of Proposition 3 in [AM] and Theorem 11.3 in [DLiM3] with suitable modifications ($q$-powers should be replaced by elements in $\mathbb{C}[[q]][\tau]$) and so we obtain the following corollary.

Corollary 5.10 If $V = \bigoplus_{m=0}^{\infty} V_m$ is a $C_2$-cofinite VOA, then the central charge and the conformal weights are all rational numbers.

We will next prove a bound of the effective central charge $\tilde{c} = c - 24h_{\text{min}}$, where $h_{\text{min}}$ is the smallest conformal weight.

Corollary 5.11 Let $V$ be a $C_2$-cofinite VOA. Then $\tilde{c} \leq \frac{\dim(V/C_2(V)) - 1}{2}$.

[Proof] The proof is essentially the same as in [AM] with slight modifications. Set $k = \dim V/C_2(V) - 1$ and define $f_2(q) = \sqrt{2} q^{1/24} \prod_{n=1}^{\infty} (1 + q^n)$. By using a spanning set of irreducible module $W$ with a conformal weight $r$ given in Lemma 2.4, there is a polynomial $g(q) = \sum_{i=0}^{s} g_i q^i \in \mathbb{C}[q]$ such that $g_i \geq 0$ and

$$\text{ch}_W(\tau) \leq 2^{-k/2} q^{-k/24} f_2(q)^k g(q) q^r.$$

Here and in the following we shall always assume that $0 < q < 1$. As we showed,

$$\text{ch}_{W^{(-1/\tau)}} = \sum_{X} a^W_X(1/\tau) \text{ch}_X(\tau),$$

where $\text{ch}_X(\tau)$ runs over the set of distinct characters and $a^W_X(\tau) \in \mathbb{C}[\tau]$. Hence

$$| \sum_{X} a_X(\tau) \text{ch}_X(\tau) | \leq \tilde{q}^{-(k+c)/24} 2^{-k/2} f_2(\tilde{q})^k g(\tilde{q}),$$

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where $\tilde{q} = e^{-2\pi i/\tau}$ and $f_4(q) = q^{-1/48} \prod_{n=1}^{\infty} (1-q^n)^{1/2}$. In the limit $\tau \to i\infty$ ($q \to 0, \tilde{q} \to 1$),

$$\text{ch}_W(\tilde{q}) = |\tau|^m q^{h-c/24} (a + o(1)) g(1)$$

for some integer $m$ and constants $a$, where $h$ is a minimal one among conformal weights which appear in $\sum_X a^W_X(\tau) \text{ch}_X(\tau)$. Since $\tau \to -1/\tau$ is an involution, there is an irreducible $V$-module $W$ such that a character with a minimal conformal weight $h_{\text{min}}$ appears in $\text{ch}_W(\tilde{q})$. Hence there is a constant $C$ such that

$$|\tau|^m q^{h_{\text{min}}-c/24} \leq q^{-k/48} (C + O(q))$$

and so we have $h_{\text{min}}-c/24 \geq -k/48$ as desired. 

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