The effective hyper-Kähler potential
in the $N=2$ supersymmetric QCD

Sergei V. Ketov

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
ketov@itp.uni-hannover.de

Abstract

The effective low-energy hyper-Kähler potential for a massive $N=2$ matter in $N=2$ super-QCD is investigated. The $N=2$ extended supersymmetry severely restricts that $N=2$ matter self-couplings so that their exact form can be fixed by a few parameters, which is apparent in the $N=2$ harmonic superspace. In the $N=2$ QED with a single matter hypermultiplet, the one-loop perturbative calculations lead to the Taub-NUT hyper-Kähler metric in the massive case, and a free metric in the massless case. It is remarkable that the naive non-renormalization 'theorem' does not apply. There exists a manifestly $N=2$ supersymmetric duality transformation converting the low-energy effective action for the $N=2$ QED hypermultiplet into a sum of the quadratic and the improved (non-polynomial) actions for an $N=2$ tensor multiplet. The duality transformation also gives a simple connection between the low-energy effective action in the $N=2$ harmonic superspace and the component results.

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$^2$On leave of absence from: High Current Electronics Institute of the Russian Academy of Sciences,
Siberian Branch, Akademichesky 4, Tomsk 634055, Russia
1 Introduction

The effective action has proved to be very useful in quantum field theory. The full effective action is however non-local and highly complicated, and it is usually impossible to calculate it. Amongst the practical approximations capable to go beyond the standard (loop) perturbation theory, an expansion of the effective action in momenta (or in the number of spacetime derivatives) plays a prominent role. Its leading term — the so-called low-energy effective action (LEEA) — encodes important information about the spectrum and static couplings in the full quantum theory.

In the famous papers \[1, 2\] Seiberg and Witten described a construction of the exact (i.e. perturbative and non-perturbative) LEEA for N=2 supersymmetric Yang-Mills theories, and subsequently generalised it to the case of N=2 super-QCD, i.e. in a presence of N=2 matter. In N=2 superspace, the four-dimensional N=2 super-Yang-Mills (SYM) theory is described by the Lie algebra-valued (reduced) chiral N=2 superfield strength \(W\), which is sometimes called N=2 superpotential.\[3\] The most general Ansatz for the (Wilsonian) LEEA of spontaneously broken (abelian) N=2 SYM theory takes the form of a chiral N=2 superspace integral \[3\]

\[
S_{\mathcal{F}} = \frac{1}{4\pi \text{Im Tr}} \int d^4xd^4\theta \mathcal{F}(W)
\]

(1.1)

to be determined by a single holomorphic function \(\mathcal{F}(W)\). Demanding renormalizability of the initial (microscopic) N=2 SYM action requires the function \(\mathcal{F}\) to be quadratic in \(W\). It was shown by Seiberg and Witten \[1, 2\] how to determine the abelian LEEA function \(\mathcal{F}\) exactly, provided that N=2 supersymmetry is not dynamically broken. As regards the non-abelian LEEA, it also receives contributions from the non-holomorphic coupling

\[
S_{\mathcal{H}} = \int d^4xd^4\theta d^4\bar{\theta} \mathcal{H}(W,W)
\]

(1.2)

as was demonstrated in refs. \[4, 5\]. The real function \(\mathcal{H}\) was (partially) fixed in ref. \[4\].

The N=2 supersymmetric matter is described by hypermultiplets \[3\]. Each N=2 hypermultiplet contains a pair of complex scalars and a Dirac spinor, all transforming in the same gauge group representation, which can be different from the adjoint representation. By N=2 supersymmetry, the hypermultiplet scalars parametrise a hyper-Kähler manifold in the LEEA. It is therefore of interest to determine the associated hyper-Kähler metric or the corresponding effective hyper-Kähler potential

\[3\]The term ‘superpotential’ is somewhat confusing here, since the \(W\) is a constrained superfield, unlike the unconstrained N=2 SYM pre-potential \(V^{++}\) to be introduced in sect. 2.
for that metric. In order to write down the N=2 superspace Ansatz, which would be analogous to eq. (1.1) but now for the N=2 matter, one needs a general manifestly N=2 supersymmetric off-shell formulation for the hypermultiplets. Such a formulation is known, and it is provided by the N=2 harmonic superspace (HSS) [7]. Both the conventional N=2 superspace methods and the N=1 superspace approach are not adequate for constructing the hypermultiplet LEEA, since either they have problems with a gauge-fixing and ghosts-for-ghosts (e.g., as in ref. [8]), or they suffer from the absence of manifest N=2 supersymmetry (e.g., as in refs. [5, 9]), which do not even allow one to write down the proper Ansatz for the N=2 matter LEEA.

In this Letter, I consider the N=2 super-QCD with the gauge group $SU(N_c)$ and the N=2 matter to be represented by $N_f$ charged massive hypermultiplets, each transforming in the fundamental representation $N_c$ of the gauge group. In sect. 2 the microscopic (renormalizable) N=2 supersymmetric gauge theory is formulated in the N=2 HSS. In sect. 3, I introduce the most general Ansatz for the hypermultiplet LEEA, which determines the form of the effective hyper-Kähler potential up to a few (finite) parameters whose actual appearance is connected to the (chiral) symmetry breaking. It is demonstrated in sect. 4 that the proposed terms in the N=2 matter LEEA do actually appear in the one-loop perturbation theory, by using the HSS Feynman rules of refs. [10, 11]. In sect. 5 a duality (Legendre) transformation is used to relate the simplest non-trivial N=2 matter LEEA to be written in the HSS for the N=2 QED, to the equivalent N=2 matter action in the projective N=2 superspace, which also provides a simple connection to the component results. The conclusions are summarized in sect. 6.

2 The setup

In the HSS formalism, the standard N=2 superspace $(x^m, x^5, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, $m = 0, 1, 2, 3; \alpha = 1, 2,$ and $i = 1, 2,$ is extended by adding the bosonic variables (or ‘zweibeins’) $u^{\pm i}$ parametrizing the sphere $S^2 \sim SU(2)/U(1)$:

$$\begin{pmatrix} u^{+i} \\ u^{-i} \end{pmatrix} \in SU(2), \text{ so that } u^{+i}u_i^- = 1, \quad u^{+i}u_i^+ = u^{-i}u_i^- = 0; \quad i = 1, 2. \quad (2.1)$$

\footnote{I use the notation and conventions as in ref. [7]. In particular, the $SU(2)$ indices are raised and lowered with the antisymmetric Levi-Civita symbols $\varepsilon_{ij}$ and $\varepsilon^{ij} = \varepsilon^{12} = -\varepsilon_{12} = 1$. The ordinary complex conjugation is detoned by bar. The extra bosonic coordinate $x^5$ is needed to take into account central charges [8].}
Instead of using an explicit parametrization for the sphere, it is more convenient to use the functions of zweibeins but consider only those of them which carry a definite $U(1)$ charge $q$ to be defined by $q(u_i^\pm) = \pm 1$. It leads to the simple integration rules \[7\]
\[
\int du = 1, \quad \int du\ u^{+i_1}\ldots u^{+i_m}u^{-j_1}\ldots u^{-j_n} = 0, \quad \text{when } m + n > 0. \tag{2.2}
\]
It follows that the integral over any $U(1)$-charged quantity vanishes.

In addition to the usual complex conjugation, there exists a star conjugation that only acts on the $U(1)$ indices, $(u_i^+)^* = u_i^-$ and $(u_i^-)^* = -u_i^+$. Accordingly, one has \[7\]
\[
\frac{\bar{u}_i^\pm}{u_i^\pm} = -u_i^\pm, \quad \frac{\bar{u}_i^\pm}{u_i^\pm} = u_i^\pm. \tag{2.3}
\]

One can also introduce the covariant derivatives (in the central basis) with respect to the zweibeins, which preserve the defining conditions (2.1), namely,
\[
D^{++}_c = u^{+i}\frac{\partial}{\partial u^{-i}}, \quad D^{-+}_c = u^{-i}\frac{\partial}{\partial u^{+i}}, \quad D^0_c = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}. \tag{2.4}
\]

A key feature of the N=2 HSS is an existence of the analytic subspace parametrised by the coordinates
\[
(\zeta, u) = \begin{cases}
  x^m_{\text{analytic}} = x^m - 2i\theta^i(\sigma^m\bar{\theta}^j)u^+_iu^-_j, & x^5_{\text{analytic}} = x^5 + i(\bar{\theta}^+\bar{\theta}^+ - \theta^+\theta^-), \\
  \theta^+_\alpha = \theta^i\bar{u}^+_i, & \bar{\theta}^+_\bar{\alpha} = \bar{\theta}^\alpha u^+_\alpha, & u^\pm_i
\end{cases} \tag{2.5}
\]
It is invariant under N=2 supersymmetry, and is closed under the combined conjugation of eq. (2.3) \[7\]. That allows one to define the analytic superfields of any $U(1)$ charge $q$, by the analyticity conditions
\[
D^{+}_i + \delta^{(q)} = \bar{D}^{i}_\alpha \phi^{(q)} = 0, \quad \text{where } D^+_i = D^i_\alpha u^+_\alpha \quad \text{and } \quad \bar{D}^+_i = \bar{D}^i_\bar{\alpha} \bar{u}^+_{\bar{\alpha}}, \tag{2.6}
\]
and introduce the analytic measure $d\zeta^{(-4)}du \equiv (d^4xdx^5)_{\text{analytic}}d^2\theta^+ d^2\bar{\theta}^+ du$ of charge $(-4)$ as well, so that the full measure in the N=2 HSS can be written down as
\[
d^4xdx^5d^4\theta d^4\bar{\theta} du = d\zeta^{(-4)}du(D^+)^4, \quad \text{where } (D^+)^4 = \frac{1}{16}(D^{+\alpha}D^+_\alpha)(\bar{D}^{+\bar{\alpha}}\bar{D}^{+\bar{\alpha}}). \tag{2.7}
\]
In the analytic subspace, the harmonic derivative $D^{++} = D^{++}_c - 2i\theta^+\sigma^m\bar{\theta}^+\partial_m$ obviously preserves analyticity, and it allows one to integrate by parts.

An action for the most general renormalizable N=2 supersymmetric gauge theory with matter in the N=2 HSS is given by
\[
S = \frac{1}{4\pi T(R)} \text{Im Tr} \int d^4xd^4\theta d^5x d\zeta^{(-4)}du \frac{1}{2}\tau W^2 + \int d\zeta^{(-4)}du \text{tr} \frac{\bar{\phi}^+(D^{++} + iV^{++})\phi^+}, \tag{2.8}
\]
where the coupling constant is \( \tau \equiv \frac{g}{2\pi} + \frac{4\pi i}{g} \) as usual, and the integration over \( x^5 \) is defined similarly to that in eq. (2.2), namely,

\[
\int dx^5 = 1, \quad \int dx^5 e^{imx^5} = 0, \quad \text{when } m \neq 0.
\] (2.9)

The chiral gauge N=2 superfield strength \( W \) is \( u \)-independent and \( x^5 \)-independent as in sect. 1, and it has to be considered as a (complicated) function of an unconstrained and analytic Lie algebra-valued gauge superfield \( V^{++} \). The \( V^{++} \) enters as an extension for the \( D^{++} \)-connection, satisfies a reality condition \( \overline{V^{++}} = V^{++} \), and has no dependence upon \( x^5 \) too. The explicit formula \( W(V^{++}) \) is given in refs. [7, 10], and it is not needed for our purposes. Each of the hypermultiplets representing N=2 matter in eq. (2.8) is described in the N=2 HSS by a complex analytic superfield \( \phi^+ \) transforming in a representation \( R \) of the gauge group with generators \( T^a \), \[ \text{tr}(T^a T^b) = T(R)\delta^{ab}. \] In the N=2 super-QCD, the gauge group is \( SU(N_c) \), and the representation \( R \) is a reducible combination of \( N_f \) fundamental representations. Unlike the \( V^{++} \), each superfield \( \phi^+ \) is assumed to be \( x^5 \)-dependent as \( \exp(imAx^5) \), thus describing a massive hypermultiplet of mass \( m_A \). As is well known, the mass of a hypermultiplet can only come from the central charges in the N=2 superalgebra since, otherwise, the number of the massive hypermultiplet components has to be increased.

The gauge-invariant HSS action (2.8) has to be supplemented by a gauge-fixing term and the FP ghost term. For HSS perturbative calculations in the background-field method, the N=2 supersymmetric Feynman gauge is convenient since the gauge-fixed kinetic term (without a \( \theta \)-term) for the \( V^{++} \) superfield is particularly simple,

\[
S_{V-\text{kin.}} = \frac{1}{4T(R)g^2} \text{Tr} \int d\zeta^{(-4)} du V^{++} \Box V^{++}.
\] (2.10)

The full list of the HSS Feynman rules can be found in refs. [10, 11] (see also sect. 4).

### 3 The effective hyper-Kähler potential and symmetry breaking in N=2 QCD

Let’s first summarize the classical symmetries of the microscopic N=2 super-QCD action (2.8). The theory has the local \( SU(N_c) \) gauge symmetry, the global N=2 supersymmetry, a global \( SU(2)_A \) symmetry which is an automorphism of the N=2 superalgebra (it rotates its two supercharges), and an R-symmetry \( U(1)_R \). If all the hypermultiplets have the same mass \( m \), the theory has an \( SU(N_f) \) flavour symmetry.

\[ ^5 \text{The superfield } V^{++} \text{ in the second term of eq. (2.8) is also valued in the R-representation.} \]
which is broken down to a smaller subgroup when the masses are not equal. If all the masses are different, the $SU(N_f)$ is broken down to $U(1)^{N_f}$.

In quantum theory some of the above symmetries may be broken or become anomalous. Based on calculations of the Witten index [12], one can argue that $N=2$ supersymmetry should remain unbroken in the $N=2$ QCD. Accordingly, the global $SU(2)_A$ symmetry should also be a symmetry of the quantum theory [1, 2]. The R-symmetry is well-known to be anomalous due to the non-perturbative instanton effects, so that it is actually broken to a discrete subgroup. The (spontaneous) gauge symmetry breaking is dependent upon the vacuum which, in its turn, can be found by minimizing the component (tree) scalar potential. The vacuum solutions with unbroken supersymmetry are known [13]. In particular, if the hypermultiplet masses do not vanish, one has $\langle q^i \rangle = 0$, where $q^i \equiv \phi^i$ is the leading scalar component of the on-shell hypermultiplet, $\phi^+ = \phi^i(x, \theta, \bar{\theta})u_i^+$, so that only the leading scalar component $A \equiv W$ of the vector $N=2$ multiplet can have a non-vanishing vacuum expectation value (a Coulomb phase). If all the hypermultiplet masses are zero, there exist solutions with $\langle q^i \rangle \neq 0$ (the scalar potential has flat directions) but $\langle A \rangle = 0$ (see ref. [13] for details) – it is a Higgs phase, according to the classification of refs. [1, 2]. In the Higgs phase, there are no monopoles or dyons, and there should be, therefore, no non-perturbative corrections to the hyper-Kähler potential. In the Coulomb phase, the abelian gauge symmetry remains unbroken, while all the hypermultiplets are massive, and one expects the one-loop hyper-Kähler potential to be exact, since there are no instantons [2].

We are now in a position to consider the LEEA Ansatz for the effective hypermultiplet self-couplings in the $N=2$ HSS. On dimensional grounds, it has to be an integral of a local quantity over the analytic subspace. Since the local quantity has to (i) be constructed out of $\phi^+, \bar{\phi}^+$ and $D^{++}$, (ii) have the $U(1)$ charge $q = +4$, and (iii) preserve the $SU(2)_A$ global invariance, the most general Ansatz appears to be fixed up to a finite number of parameters ($\lambda, \beta, \gamma$) as follows (cf. ref. [10]):

$$S_{\text{hyper-K.}} = \frac{1}{2} \int d\zeta^{(-4)} du \left( \frac{\bar{\phi}^+}{\phi^+} D^{++} \phi^+ + L_{\text{int.}}^{(4)} \right), \quad (3.1)$$

where

$$L_{\text{int.}}^{(4)} = \frac{\lambda}{2} (\bar{\phi}^+)^2 (\phi^+)^2 + \left[ \beta \frac{\bar{\phi}^+}{\phi^+} (\phi^+)^3 + \gamma (\phi^+)^4 + \text{h.c.} \right]. \quad (3.2)$$

In the case of $N=2$ QCD with the charged massive hypermultiplets in the fundamental (complex) representation of the gauge group (in the microscopic action), the unbroken (abelian) symmetries of the quantum theory still require $\beta = \gamma = 0$. Hence, we are
left with the quartic self-interactions

\[ \mathcal{L}_{\text{int.}}^{\text{QCD(+4)}} = \sum_{\text{flavour}}^{N_f} \sum_{\text{colour}}^{N_c} \lambda_{ABCD} \frac{\phi^*_A}{\phi_B^*} \phi_C \phi_D^* , \quad (3.3) \]

whose particular structure (i.e. the non-vanishing \( \lambda \)'s) is dependent upon the chiral and flavour symmetry breaking under consideration.

In the case of the abelian N=2 super-QED with a single complex hypermultiplet, eq. (3.3) has only one term,

\[ \mathcal{L}_{\text{int.}}^{\text{QED(+4)}} = \frac{\lambda}{2} (\phi^+) (\bar{\phi}^+) . \quad (3.4) \]

It is worth mentioning that the constant \( \lambda \) has dimension \([m]^{-2}\) in units of mass. It is also clear that the whole Ansatz (3.1) is in conflict with the naive non-renormalization ‘theorem’ in the ordinary superspace [14] and in the HSS [10, 11], that formally forbids quantum contributions of the form of an integral over a chiral or an analytic subspace of superspace. Hence, it has to be explained how the quantum (finite) analytic corrections to the effective hyper-Kähler potential are nevertheless possible (see the next sect. 4).

4 The one-loop effective hyper-Kähler potential in the N=2 super-QED

The one-loop local contribution to the effective hyper-Kähler potential in the N=2 super-QED can be easily calculated by using the HSS Feynman rules for the theory (2.8) in the N=2 super-Feynman gauge [10, 11]. One expands the action to the second order around a hypermultiplet background \( \Phi^+, \phi^+ = \Phi^+ + \varphi^+ \). The propagators for the quantum HSS fields \( \varphi^+ \) and \( V^{++} \) are defined by the kinetic terms, as in refs. [10, 11]:

\[ \langle \varphi^+(p_1, \theta_1, u_1) \varphi^+(p_2, \theta_2, u_2) \rangle = \frac{i}{(p_2 - p_1)^2 + m^2} \frac{(D^+_1)^4(D^+_2)^4}{(u_1^{+*}u_2^+)^3} \delta^8(\theta_1 - \theta_2) , \quad (4.1) \]

and (after rescaling \( V \) by the factor of \( g \))

\[ \langle V^{++}(p_1, \theta_1, u_1)V^{++}(p_2, \theta_2, u_2) \rangle = \frac{i}{(p_2 - p_1)^2} (D^+_1)^4 \delta^8(\theta_1 - \theta_2) \delta^{(-2,2)}(u_1, u_2) , \quad (4.2) \]

where \( \delta^{(-2,2)}(u_1, u_2) \) is one of the harmonic delta-functions on \( S^2 \), defined by the equations

\[ \int dv \delta^{(q,-q)}(u, v) f^{(q)}(v) = \delta^{pq} f^{(q)}(u) \quad (4.3) \]
for any regular function $f^{(\nu)}(u)$ of the $U(1)$ charge ($p$). The explicit form of $\delta^{(q,-q)}(u,v)$ can be found in ref. [10], but it is not really needed for perturbative calculations.

Since the FP ghosts do not couple to the hypermultiplet, they can be ignored at one loop. Then the only relevant vertices are $\bar{\Phi} (p_1)V(k)\varphi(p_2)$ and $\bar{\varphi}(p_1)V(k)\Phi(p_2)$, each contributing $-g(2\pi)^4\delta(p_1-p_2-k)$ to the Feynman rules, like that in the ordinary QED (the momentum integration is implied). It leaves us with only one HSS graph $\Gamma_4$ that has to be calculated (Fig. 1).

First, one does all the $\theta$-integrations but one, by taking the factor $(D_4^+)^4(D_2^+)^4$ off a hypermultiplet propagator and using the identity [10]

$$\delta^8(\theta_1-\theta_2)(D_4^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2) = (u_1^+u_2^+)^4\delta^8(\theta_1-\theta_2).$$

As a result, the full N=2 HSS Grassmann measure $d^8\theta$ is restored, just in agreement with the non-renormalization theorem (cf. refs. [10, 11]),

$$\Gamma_4 = g^4 \int \frac{d^4p_1d^4p_2d^4p_3d^4p_4}{(2\pi)^4} \int \frac{du_1du_2}{(u_1^+u_2^+)^2} \int d^8\theta \bar{\Phi}^+(1)\Phi^+(2)\Phi^+(3)\bar{\Phi}^+(4) \times$$

$$\times \int \frac{d^4k}{k^2(p_1-p_2+p_3-k)} \frac{d^4k}{k^2(k-p_1-p_2)^2([k-p_2]^2+m^2)(k-p_3)^2+m^2}.$$  

It is worth mentioning that the loop momentum integral is UV-convergent.

Since the external legs are on-shell, $D^{++}\Phi^+ = D^{++}\bar{\Phi}^- = 0$, the identities [10]

$$\Phi^+(u) = D^{++}\Phi^+(u) \quad \text{and} \quad D_4^{++} \frac{1}{(u_1^+u_2^+)^2} = D_1^{--}\delta^{(2,-2)}(u_1,u_2)$$

(4.6)

can be used to eliminate one of the $u$-integrations in the low-energy approximation. The remaining harmonic superspace integral has just two $D^{--}$-insertions, and it leads to the appearance of an analytic term because of eq. (2.7) and another identity [10]

$$-\frac{1}{2}(D_4^+)^4(D^{--})^2\Phi^+(u) = \Box \Phi^+(u) = m^2\Phi^+(u).$$

(4.7)

The low-energy analytic contribution takes the form

$$\Gamma_4 = g^4 \int \frac{d^4k}{(2\pi)^4} \frac{m^2}{k^4(k^2+m^2)^2} \int d(4^{-4})du(\Phi^+)^2(\Phi^+)^2,$$

(4.8)

where the loop integral has to be regularised by restricting $k^2 \geq \mu^2$, with $\mu$ being an IR (Wilsonian) cutoff. One easily finds that

$$\lambda = \frac{g^4}{8\pi^2\mu^2}f(y), \quad \text{where} \quad f(y) = \frac{\ln(1+y)}{y} - \frac{1}{1+y}, \quad \text{with} \quad y \equiv \frac{m^2}{\mu^2} \quad \text{and} \quad f(0) = 0.$$

(4.9)

The loophole in the non-renormalization ‘theorem’, which is apparent above, seems to be quite similar to the other counter-examples found earlier in ref. [13] for some supersymmetric theories with massless fields.
The effective hyper-Kähler metric and a duality transformation for N=2 QED

Given the hypermultiplet LEEA in the form (3.1) with the interaction (3.4), it is still non-trivial to extract the explicit hyper-Kähler metric of the associated non-linear sigma-model (NLSM) which arises after eliminating the infinite tower of auxiliary fields contained in the harmonic expansion of the HSS superfield $\phi(\zeta, u)$ via equations of motion. Fortunately, just in the case under consideration, it was already done in ref. [16] by solving constraints appearing in the full HSS equation of motion

$$D^{++}\phi^+ + \lambda(\phi^+\phi^+)^{1/2}(\phi^+\phi^+)^{1/2} = 0 .$$

(5.1)

The remaining ‘true’ equation in components has the form of a NLSM equation of motion, whose metric can be written down in proper (‘spherical’) coordinates as the Taub-NUT metric [17]

$$ds^2 = \frac{1}{2}\left[\frac{r+M}{r-M}dr^2 + \frac{1}{2}(r^2-M^2)(d\theta^2 + \sin^2 \theta d\phi^2) + 2M^2 \frac{r-M}{r+M}(d\psi + \cos \theta d\phi)^2\right] .$$

(5.2)

where $M \equiv \frac{1}{2}\lambda^{-1/2} \sim g^{-2}$ is the mass of the Taub-NUT gravitational instanton.

It is worth mentioning that the solitonic mass $M$ is proportional to the inverse gauge coupling constant squared, so that its origin is non-perturbative with respect to the microscopic action (2.8). That is because its derivation involves a kind of duality transformation relating weak and strong couplings, similarly to that considered in refs. [1, 2] (see e.g., ref. [18] for a review).

There exists a manifestly N=2 supersymmetric duality (Legendre) transformation in the N=2 HSS [19] that transforms the hypermultiplet action

$$S_{\text{hyper}} = \frac{1}{2} \int d\zeta^{(-4)}du \left[ \frac{z}{\phi} + \frac{z}{D^{++}\phi^+} + \frac{\lambda}{2}(\phi^+)^2(\phi^+)^2 \right]$$

(5.3)

into the dual one to be written in terms of dimensionless real analytic HSS superfields $L^{++}$ and $\omega$,

$$S_{\text{dual}} = \lambda S_{\text{free}} + S_{\text{improved}} ,$$

(5.4)

where

$$S_{\text{free}} = \frac{\mu}{2} \int d\zeta^{(-4)}du \left[ (L^{++})^2 + \omega D^{++}L^{++} \right]$$

(5.5)
describes a (non-conformal) free N=2 tensor multiplet in the N=2 HSS, whereas
\[
S_{\text{improved}} = \frac{\mu^2}{2} \int d\zeta (-4) d\nu \left[ (g^{++})^2 + \omega D^{++} L^{++} \right],
\]
with \[19\]
\[
g^{++}(L, u) \equiv \frac{2(L^{++} - 2iu_1^+u_2^+)}{1 + \sqrt{1 - 4u_1^+u_2^+ - 2iL^{++}u_1^-u_2^-}},
\]
describes the improved (i.e. N=2 superconformally invariant) action \[20\] for the same N=2 tensor multiplet. The duality transformation reads as follows \[19 \]:
\[
\phi^+ = -i(2u_1^+ + ig^{++}u_1^-)e^{-i\omega/2},
\]
\[
\bar{\phi}^+ = +i(2u_2^+ - ig^{++}u_2^-)e^{+i\omega/2},
\]
and implies \[\bar{\phi}^+\phi^+ = 2iL^{++}\] in particular. Therefore, the N=2 HSS action (5.3) is dual to a sum of the naive (quadratic) and improved (non-polynomial) actions for an N=2 tensor multiplet. Both actions are known both in components \[20\] and in terms of the ordinary N=1 superfields \[21\]. The most elegant formulation with a finite number of auxiliary fields exists in the projective N=2 superspace where a single complex CP(1) coordinate \(\xi\) plays the role of the HSS zweibeins, \(u_i \rightarrow \xi_i = (1, \xi)\). The defining N=2 tensor multiplet constraints in the standard N=2 superspace (see the footnote \# 6) imply
\[
\nabla_\alpha G \equiv (D^1_\alpha + \xi D^2_\alpha)G = 0 \quad \text{and} \quad \Delta^* G \equiv (\bar{D}^1_\alpha + \xi \bar{D}^2_\alpha)G = 0
\]
for any function \(G(Q(\xi), \xi)\) with \(Q(\xi) \equiv \xi_i\xi_jL^{ij}\). Hence, after integrating the function \(G\) over the rest of the standard N=2 superspace coordinates, one gets an N=2 superinvariant \[22, 23\]
\[
S_{\text{dual}} = \mu^4 \int d^4x \frac{1}{2\pi i} \oint_C d\xi \tilde{\nabla}^2 \Delta^2 G(Q, \xi),
\]
where \(\tilde{\nabla}_\alpha \equiv \xi D^1_\alpha - D^2_\alpha\) and \(\tilde{\Delta}^* \equiv \xi \bar{D}^1_\alpha - \bar{D}^2_\alpha\). The sum of the naive and improved N=2 tensor multiplet actions in the projective N=2 superspace is given by \[22\]
\[
S_{\text{dual}} = \mu^4 \int d^4x \tilde{\nabla}^2 \tilde{\Delta}^2 \frac{1}{2\pi i} \left[ \lambda \oint_{C_1} d\xi \frac{Q^2}{2\xi} + \mu^{-2} \oint_{C_2} d\xi Q \ln Q \right],
\]
where the contour \(C_1\) goes around the origin while the contour \(C_2\) encircles two roots of the quadratic equation \(Q(\xi) = 0\) in the complex \(\xi\)-plane.

\[\text{Footnote 6}\] An equivalence to the standard N=2 superspace formulation of the N=2 tensor multiplet, given by the constraints \(D^1_\alpha L^{jk} = \bar{D}^1_\alpha L^{jk} = 0\), can be easily established one shell: a variation of the action (5.5) with respect to the Lagrange multiplier \(\omega\) yields a HSS constraint \(D^{++} L^{++} = 0\) which implies \(L^{++}(\zeta, u) = u_1^+u_2^+ L^{ij}\) as well as the defining constraints for the \(L^{ij}(x, \theta, \bar{\theta})\).
In the massless limit $m \to 0$ one has $\lambda \to 0$ too, so that the action $S_{\text{dual}}$ is reduced to the improved $N=2$ tensor multiplet action alone. The latter is $N=2$ superconformally invariant, and is equivalent to a free action [2], as it should have been expected for the massless hypermultiplet LEEA. It agrees with the statements made in ref. [2] about the absence of quantum corrections to the hyper-Kähler effective potential for the massless hypermultiplets, and the $N=1$ superspace calculations of ref. [3] as well.

6 Conclusion

The $N=2$ extended supersymmetry severely restricts the effective hyper-Kähler potential in the LEEA for $N=2$ matter fields. That becomes apparent in the $N=2$ HSS where the $N=2$ matter is described by off-shell hypermultiplets, and the automorphism invariance $SU(2)_A$ of the $N=2$ supersymmetry algebra restricts the effective hypermultiplet self-couplings even further. As a result, the exact form of the effective hyper-Kähler potential is determined by a few parameters to be related to a symmetry breaking in quantum theory. In the simplest case of $N=2$ QED with a single massive hypermultiplet, one finds a unique solution given by the Taub-NUT metric. Any non-trivial solution thus represents a counter-example to the naive non-renormalization ‘theorem’, that can be tested in one-loop perturbation theory. There exists a manifestly $N=2$ supersymmetric duality transformation which converts the hypermultiplet LEEA in the $N=2$ HSS into the equivalent action in terms of an $N=2$ tensor multiplet. The duality transformation in that case also gives a simple root from the $N=2$ HSS to the components – it is highly non-trivial in a generic situation.

I assumed that non-analytic effective hypermultiplet self-couplings (they are similar to that in eq. (1.2), but in the $N=2$ HSS) do not contribute to the low-energy effective hyper-Kähler potential. Though being quite natural by dimensional reasons, that assumption has yet to be justified in the non-abelian case.

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Fig. 1. A one-loop HSS graph contributing to the effective hyper-Kähler potential in the N=2 super-QED.