ON THE ORTHOGONAL SUB-GRID SCALE PRESSURE STABILIZATION IN FINITE DEFORMATION J2 PLASTICITY
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Key words: Stabilization, Sub-grid scales, Multiscale, J2 plasticity, Finite elements.

Abstract. Use of stabilization methods is becoming an increasingly well-accepted technique due to their success in dealing with numerous numerical pathologies that arise in a variety of applications in computational mechanics. In this paper, a multi-scale finite element method technique to deal with pressure stabilization of incompressibility and nearly incompressibility problems in nonlinear solid mechanics, using low order finite elements, is presented. An Orthogonal Sub-Grid Scales (OSGS) method for both incompressible elasticity and J2-plasticity at finite deformations is proposed. Standard mixed finite element formulations, particularly those using low order interpolations, perform poorly or totally fail to perform for nearly incompressibility or incompressibility problems, producing results thoroughly polluted by spurious oscillations of the pressure. The goal here is to avoid this undesirable effect while retaining the use of low order elements. To achieve this goal we consistently derive, within the framework of the OSGS method, a modified variational mixed formulation of the original problem with enhanced stability properties. An assessment of the behavior of the formulation is presented. Results are compared with standard Galerkin and Q1P0 mixed large strain formulations.
1 INTRODUCTION

Use of stabilization methods is becoming an increasingly well-accepted technique due to their success in dealing with numerous numerical pathologies that arise in a variety of applications in computational mechanics. In this paper, a multiscale finite element method technique to deal with pressure stabilization of incompressible and nearly incompressible problems in nonlinear solid mechanics, using low order finite elements, is presented. An Orthogonal Sub-Grid Scales (OSGS) method for both incompressible elasticity and J2-plasticity at finite deformations is proposed. The goal is to consistently derive, within the framework of the OSGS method, a modified variational mixed formulation of the original problem with enhanced stability properties.

It is well known that the standard irreductible Galerkin finite element method with low-order piecewise polynomials perform miserably in nearly incompressible problems, exhibiting spurious wild oscillations of the mean pressure and leading to a response which is almost completely locked due to the incompressibility constraint. In the computational literature these devastating numerical difficulties are referred to as locking phenomena. Actually, the exact incompressibility problem does not admit an irreductible formulation and, consequently, a mixed displacement/pressure framework is necessary in that case. Even though, many standard mixed finite element formulations, particularly those using low order interpolations, also perform poorly or totally fail to perform for nearly incompressibility or incompressibility problems, producing results thoroughly polluted by spurious oscillations of the pressure.

To overcome these difficulties, over the years different strategies were suggested to reduce or avoid volumetric locking and pressure oscillations in finite element solutions. For an engineering oriented presentation see the well known books of Zienkiewicz & Taylor [1], Hughes [2] and Simo & Hughes [3]. For a more mathematically oriented presentation see the book of Brezzi & Fortin [4]. Different mixed and enhanced finite element formulations were proposed and degrees of success were obtained. See, e.g., Simo, Taylor & Pister [5], Simo [6],[7], Simo & Miehe [8], Miehe [9], Simo & Rifai [10], Simo & Armero [11]. Unfortunately, few approaches were successfully applied to low order finite elements, as shown for instance in Reddy & Simo [12] for the enhanced assumed strain method. This was due to the strictness of the inf-sup or Ladyzhenskaya-Babuska-Brezzi (LBB) condition when the standard Galerkin finite element projection was straightforwardly applied to mixed low order finite elements, as it imposes severe restrictions on the compatibility of the interpolations used for the displacement and pressure fields [1-4]. One significant effort in that direction was the so called mini element [13], an attractive linear displacement/pressure triangle enhanced with a cubic displacement bubble function. The mini element satisfies the LBB condition, but it is only marginally stable and it does not perform very well in many practical situations. Despite these not very good satisfactory results, there still exists a great practical interest in the use of stable low order elements, mainly motivated by the fact that, nowadays, tetrahedral finite element meshes are relatively easy to generate for real life complex geometries. Therefore, stabilization techniques for low order finite elements is a very
active research area in solid mechanics. Some recent formulations have been proposed by Zienkiewicz et al. [14], Klaas, Maniatty & Shephard [15], Oñate et al. [16] and Maniatty et al. [17]-[19].

On the other hand, research on stabilization methods for incompressibility, as well as other phenomena, in Computational Fluid Dynamics (CFD) has been always in the front line of research because of the innumerable practical applications of the field [20]-[24]. In Hughes [25] and Hughes et al. [26] the variational multiscale method was introduced as a new computational mechanics paradigm to address stabilization problems in CFD. Within the multiscale method it is assumed that there is a component of the continuous (exact) solution which can not be captured by the finite element solution. This component which is not captured by the finite element solution is called the subgrid scale or the subscale. The consideration of this subgrid scale leads to a modified variational formulation with enhanced stability properties and allows the use of a convenient mixed velocity/pressure equal linear interpolation. Since then, multiscale methods have been extensively and successfully used in CFD. In Codina [27], [28] the Orthogonal Subgrid Scales (OSGS) method was introduced, leading to a better sustained and better performing stabilization procedures.

In Computational Solid Mechanics (CSM), variational multiscale techniques have been used by Garikipati & Hughes [29], [30] in strain localization problems. Recently, a variational multiscale stabilization method based on the OSGS has been applied to both incompressibility and nearly incompressibility problems in small deformations elasticity by Chiumenti et al. [31] and Cervera et al. [32], [33], J2 plasticity by Chiumenti et al. [34] and Cervera et al. [32], [33], softening and localization in J2 plasticity by Cervera, Chiumenti and Agelet de Saracibar [35] and shear band localization using a J2 continuum damage model by Cervera, Chiumenti and Agelet de Saracibar [36], [37]. Effectiveness and robustness of the method have encouraged the authors to extend the approach to finite deformation problems and particularly to finite deformation J2 plasticity. An assessment of the behaviour of the formulation is presented. Results are compared with standard Galerkin and Q1P0 mixed large strain formulations in either elastic or elasto-plastic incompressible problems.

2 ORTHOGONAL SUB-GRID SCALE PRESSURE STABILIZATION IN FINITE DEFORMATION J2 PLASTICITY

Let us consider a multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ into elastic $\mathbf{F}^e$ and (volume preserving) plastic $\mathbf{F}^p$ parts. Let $\mathbf{J} = |\mathbf{F}| = |\mathbf{F}^e|$ be the determinant of the deformation gradient and consider the uncoupled volumetric/isochoric stored energy function

$$W(J, \mathbf{b}^e) = U(J) + \overline{W}(\mathbf{b}^e), \quad U(J) = \frac{1}{2} \kappa J^2, \quad \overline{W}(\mathbf{b}^e) = \frac{1}{2} \mu \left[ r(\mathbf{b}^e) - 3 \right]$$

where $\mathbf{b}^e = J^{-2/3} \mathbf{b}^e$, $\mathbf{b}^e = \mathbf{F}^e \mathbf{F}^{eT}$ and $\kappa$ and $\mu$ are the bulk and shear modulus, respectively.
The mixed formulation of the problem at hand can be expressed as: find a displacement field $u$ and a Kirchhoff pressure field $\pi$ such that:

$$\begin{align*}
J \nabla (J^{-1} \pi) &+ J \nabla \cdot (J^{-1} s) + f = 0 \\
\pi - J U'(J) & = 0
\end{align*}$$

(2)

where $s(u) = \mu \text{dev} \left[ \overline{\mathbf{B}} \right]$ is the deviatoric part of the Kirchhoff stress tensor. Using a compact notation, (2) can be written as $\mathbb{L}(u) = F$ where $U = [u, \pi]^T$. The Galerkin finite element projection of the variational form associated to (2) can be expressed as: find $U_h = [u_h, \pi_h]^T \in W_h$ such that $B(U_h, V_h) = L(V_h)$ $\forall V_h \in W_h$. Explicitly:

$$\begin{align*}
\left( \nabla^T \mathbf{v}_h, s_h \right) + (\nabla \cdot \mathbf{v}_h, \pi_h) &= (\mathbf{v}_h, f) + (\mathbf{v}_h, \overline{t}) \\
(q_h, -\pi_h + JU'(J)) &= 0
\end{align*}$$

(3)

Let us consider an enhanced space $W = W_h \oplus \tilde{W}$ and let $U = U_h + \tilde{U}$ be the exact solution, where $\tilde{U}$ is the subgrid scale variable. The problem at hand can be expressed as: find $U_h \in W_h$ and $\tilde{U} \in \tilde{W}$ such that $B(U_h + \tilde{U}, V_h) = L(V_h)$ $\forall V_h \in W_h$ and $B(U_h + \tilde{U}, \tilde{V}) = L(\tilde{V})$ $\forall \tilde{V} \in \tilde{W}$. Within the OSGS method we consider as complementary space $\tilde{W}$ the space orthogonal to the finite element solution space and therefore $\tilde{U} \in \tilde{W} \approx W_{h}^\perp$. Denoting by $P_{h}^\perp$ the orthogonal projection onto $W_{h}^\perp$, the subgrid scales $\tilde{U}$ are (locally) approximated as $\tilde{U} \approx \alpha \left[ \nabla \pi_h - P_{h}^\perp (\nabla \pi_h) \right] \in W_{h}^\perp$, where $\alpha = c h^2/2\bar{\mu}$ is a stabilization parameter which depends of the average element length $h$. The subgrid scale shear modulus is defined as $\bar{\mu} = \mu J^{-2/3} \left\| \text{dev} \left[ \overline{\mathbf{B}} \right] \right\| / \left\| \text{dev} \left[ \overline{\mathbf{B}} \right] \right\|$. Introducing the linearization of the deviatoric Kirchhoff stress and the Kirchhoff pressure and the approximation found for the subscales, the variational multiscale (OSGS) problem can be expressed as: find $U_h \in W_h$ such that

$$B_{\text{stab}}(U_h, V_h) = L(V_h) \quad \forall V_h \in W_h$$

(5)

where $B_{\text{stab}}(U_h, V_h) = B(U_h, V_h) + \sum_{e=1}^{\text{n elem}} (a \nabla q_h, (\nabla \pi_h - \Pi_h))_{e \Omega}$, where $\Pi_h = P_h (\nabla \pi_h)$ is the projection of the pressure gradient onto the finite element space and it has been introduced as a new variable field.

The resulting stabilized system of equations takes the form:
\[
\begin{aligned}
\left[ \nabla^2 \mathbf{v}_h , \mathbf{s}_h \right] + \left( \nabla \cdot \mathbf{v}_h , \tau_h \right) &= \left( \mathbf{v}_h , \mathbf{f} \right) \\
\left( q_h , \log(\mathbf{J}_h) - \frac{\tau_h}{\kappa} \right) - \sum_{e=1}^{n_{elem}} (\alpha \nabla q_h , (\nabla \tau_h - \mathbf{\Pi}_h))_{\mathbf{\Omega}_e} &= 0 \\
\left( \mathbf{w}_h , \mathbf{\Pi}_h \right) - \left( \mathbf{w}_h , \nabla \tau_h \right) &= 0
\end{aligned}
\]  

(6)

The monolithic solution of the system seems to be rather expensive. However, a staggered solution, in which the pressure gradient projection is solved independently and explicitly, can be considered as a reliable and suitable option to overcome the undesirable increase in CPU time [31]-[37].

3 COMPUTATIONAL ASSESSMENT

An assessment of the stabilization method proposed, through numerical simulations, has been performed. An upsetting of a 3D block is considered by prescribing the vertical displacement on the top surface up to a 15% of its initial eight. Material is assumed to follow J2 flow plasticity theory at finite deformation. Fig. 1 shows the comparison of pressure distribution for tri-linear tetrahedral elements using standard Galerkin, standard linear mixed displacement/pressure and OSGS stabilized formulations, and hexahedral elements using the Q1/P0 element. Numerical results show locking behaviour typical of standard Galerkin and mixed linear elements, while locking is avoided using the OSGS stabilized formulation proposed by the authors.

Figure 1: Pressure distributions using standard Galerkin, mixed, OSGS and Q1/P0 formulations
4 CONCLUDING REMARKS

In this paper a stabilization technique for incompressible J2-flow theory plasticity, within the framework of finite deformation theory, has been presented. The stabilization technique, which falls within the variational multiscale technique, is based on the Orthogonal Subgrid Scale (OSGS) method. Within the paradigmatic framework of the multiscale techniques the Subgrid Scale (SGS) method seeks to approximate the effect of the component of the continuous solution which can not be captured by the finite element mesh used to obtain the discrete finite element solution. The unresolved component is referred as the subgrid scale or subscale. Within the OSGS method we take the orthogonal space to the finite element solution space as the natural space of the subgrid scales. An approximate solution for the subgrid scales is considered and a suitable simple nonlinear expression for the stabilization parameter is proposed. Computational aspects and details of implementation have been shown. Computational simulations show that the OSGS stabilized formulation proposed, allows to obtain pressure stable results within the framework of an elastoplastic J2-flow theory model at finite deformations.

5 ACKNOWLEDGEMENTS

The authors are thankful for the financial support given by the Spanish “Ministerio de Ciencia y Tecnología” through the project 2FD1997-0512-CO2-02 and by the European Commission through the Growth project GRD1-2000-25243. Useful discussions with Prof. R. Codina and Prof. E. Oñate are gratefully acknowledged.

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