Diameter rigidity for Kähler manifolds with positive bisectional curvature

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Abstract Let $M^n$ be a compact Kähler manifold with bisectional curvature bounded from below by 1. If $\text{diam}(M) = \pi/\sqrt{2}$ and $\text{vol}(M) > \text{vol}(\mathbb{CP}^n)/2^n$, we prove that $M$ is biholomorphically isometric to $\mathbb{CP}^n$ with the standard Fubini-Study metric.

1 Introduction

In Riemannian geometry, the basic rigidity theorems under Ricci curvature lower bound are volume rigidity theorem [2], maximal diameter theorem [4] and Cheeger–Gromoll splitting theorem [3]. The counterpart for Kähler manifolds, in some sense, however, remains mysterious (cf. [5–7]). For instance, it is not clear to the authors whether or not the maximal volume is achieved by the Fubini-Study metric for any compact Kähler manifold with positive Ricci lower bound. On the other hand, Mok [8] proved some important metric rigidity theorems in Kähler geometry.

In this note, we are interested in the diameter rigidity in Kähler geometry when the bisectional curvature has a positive lower bound.

Definition [5,12] Let $(M, g, J)$ be a Kähler manifold. The bisectional curvature of $g$ is bounded below by a constant $K$ if

$$\frac{R(Z_1, Z_2, Z_2, Z_1)}{\|Z_1\|^2 \|Z_2\|^2 + |\langle Z_1, Z_2 \rangle|^2} \geq K$$

for any nonzero vectors $Z_1, Z_2 \in T^{(1,0)}M$, denoted by $B_K \geq K$.

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From now on, we assume that \((M, g, J)\) has holomorphic bisectional curvature bounded below by 1, i.e. \(BK \geq 1\). By the solution of the Frankel conjecture by Siu-Yau [11] and Mori [9], \(M\) is biholomorphic to the complex projective space \(\mathbb{CP}^n\). Moreover, by the volume comparison theorem proved by Li-Wang (Corollary 1.9. in [5]), the diameter \(d\) of \((M, g, J)\) is bounded above by \(\frac{\pi}{\sqrt{2}}\). Note that we use the normalization of metric as in [12] that is essentially the same as in [5] (up to a constant). In view of the Cheng’s maximal diameter theorem in the Riemannian case, it is natural to ask the following

**Question** If the diameter of \(M\) is \(\frac{\pi}{\sqrt{2}}\), is \(M\) isometric to \(\mathbb{CP}^n\)?

**Remark 1** Notice that we cannot replace the bisectional curvature lower bound by Ricci curvature bound. Indeed, the canonical Kähler-Einstein metric on \(\mathbb{CP}^1 \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1\) has diameter strictly greater than \(\mathbb{CP}^n\), if we normalize the metric so that the Ricci curvature are the same.

In [12], Tam and Yu solved the question affirmatively by assuming that there exist complex submanifolds \(P\) and \(Q\) of dimension \(k\) and \(n - k - 1\) so that \(d(P, Q) = \frac{\pi}{\sqrt{2}}\). In this note, we provide another partial answer to this question:

**Theorem 1** Let \((M^n, g, J)\) be a compact Kähler manifold with \(BK \geq 1\). If the diameter of \((M, g)\) is \(\frac{\pi}{\sqrt{2}}\), then there exists a totally geodesic, holomorphic isometric embedding \(\tau: \mathbb{CP}^1 \rightarrow (M, g, J)\), where the metric on \(\mathbb{CP}^1\) is the standard round metric with factor \(\frac{1}{2}\). As a consequence, \(\text{vol}(M) = \frac{\text{vol}(\mathbb{CP}^n)}{d^n}\) for some integer \(d \geq 1\). In particular, the volume of \(M\) can only take discrete values. If \(\text{vol}(M) > \frac{\text{vol}(\mathbb{CP}^n)}{2^n}\), then \(M\) is biholomorphically isometric to \(\mathbb{CP}^n\) with the standard Fubini-Study metric \(g_{FS}\).

**Remark 2** This theorem states that counterexample (if exists) to the question may not be found by small perturbation of the Fubini-Study metric.

Now we sketch the simple idea of the proof. First consider the Riemannian case. The key feature is the following: Given antipodal points \(p_1, p_2\) on the standard sphere, for any \(x\),

\[
d(p_1, x) + d(p_2, x) = \pi. \tag{1.1}
\]

Then we can apply the maximum principle for Laplacian or volume comparison to obtain the rigidity for diameter under Ricci lower bound. In standard \(\mathbb{CP}^n\) case, however, (1.1) is violated, unless \(p, q, x\) are collinear. Thus the traditional method in the Riemannian case cannot be directly extended to Kähler case. By a maximum principle and the Hessian comparison theorem, we manage to find a holomorphic curve with genus zero on which (1.1) holds. Combining the solution to Frankel conjecture and an elementary degree argument, we complete the proof of the theorem.

### 2 Proof of Theorem 1

Let \(p_1, p_2\) be two points on \(M\) realizing the diameter of \((M, g)\). Let \(l\) be a minimizing normal geodesic segment joint \(p_1\) and \(p_2\) with \(l(0) = p_1\) and \(l\left(\frac{\pi}{\sqrt{2}}\right) = p_2\). Fix any point \(q\) on \(l\) with \(q = l(t_0)\) for \(0 < t_0 < \frac{\pi}{\sqrt{2}}\). Then \(d(p_1, q) = t_0\) and \(d(p_2, q) = \frac{\pi}{\sqrt{2}} - t_0\). Let \(U\) be a small geodesic ball centered at \(q\) contained in a holomorphic coordinate chart with radius...
\[ \delta < \min \left\{ t_0, \frac{\pi}{\sqrt{2}} - t_0 \right\} . \] Moreover, we assume that \( U \) does not intersect the cut locus of \( p_1 \) and \( p_2 \).

We define \( r_1(x) = d(x, p_1), r_2(x) = d(x, p_2) \) and \( u(x) = r_1(x) + r_2(x) - \frac{\pi}{\sqrt{2}} \geq 0 \). Then \( r_1, r_2 \) and \( u \) are smooth functions on \( U \). For any \( x \in U \), as \( x \) is not in the cut locus of \( p_1 \), there exists a unique minimizing geodesic \( \gamma_1 \) connecting \( p_1 \) and \( x \) such that \( \gamma_1(0) = p_1 \) and \( \gamma_1(r_1(x)) = x \). Let \( X_1 \) be the unit tangent vector of \( \gamma_1 \) at \( x \). Similarly, \( \gamma_2, X_2 \) can be defined. Note that \( X_1(q) = -X_2(q) \).

**Claim 1** There exists a constant \( C > 0 \) (depending on \( U \)), such that
\[ \pi - Cu^2(x) \leq \theta(x) \leq \pi. \]

**Proof** Since \( M \) is compact, the sectional curvature has a lower bound. The lemma simply follows from the Toponogov comparison. \( \square \)

Let \( Z_1 = \frac{1}{\sqrt{2}}(X_1 - \sqrt{-1} J X_1) \in T_x^{(1,0)} M, Z_2 = \frac{1}{\sqrt{2}}(X_2 - \sqrt{-1} J X_2) \in T_x^{(1,0)} M \). Define an operator \( L \) by
\[ Lh(x) = (\nabla Z_1 \nabla Z_1 + \nabla Z_2 \nabla Z_2)h(x) \]
for smooth functions \( h(x) \) on \( U \).

**Proposition 1** There exists a constant \( C > 0 \) (depending on \( U \)), such that
\[ Lu(x) \leq Cu(x). \]

**Proof** Let \( e_1 = Z_1 \) and let \( \{ e_2, \ldots, e_n \} \) be parallel orthogonal along \( \gamma_1 \) such that \( \{ e_1, e_2, \ldots, e_n \} \) is an unitary frame. Write \( Z_2 = \sum_{\alpha=1}^{n} a_\alpha(x) e_\alpha \).

**Claim 1** There exists a constant \( C > 0 \) (depending on \( U \)) such that
\[ (1 - Cu(x))^{\frac{1}{2}} \leq |a_1(x)| \leq 1 \]
and thus
\[ \sum_{\alpha \geq 2} |a_\alpha(x)|^2 \leq Cu(x). \]

**Proof of Claim 1:** This just follows from the lemma above.

The complex Hessian comparison theorem derived by Tam-Y u (Theorem 2.1 in [12]) asserts
\[ (r_1^\alpha)_{\alpha \beta} \leq \sqrt{\frac{1}{2}} \cot \left( \sqrt{\frac{1}{2}} r_1 \right) g_{\alpha \beta} + \sqrt{2} \left( \cot \left( \sqrt{2} r_1 \right) - \cot \left( \sqrt{\frac{1}{2}} r_1 \right) \right) (r_1)_{\alpha} (r_1)_{\beta}, \]
Then we obtain
\[ L(r_1(x) + r_2(x)) \leq (I) + (II), \]
where
\[ (I) = \sqrt{\frac{1}{2}} \left( \sum_{\alpha=2}^{m} |a_\alpha|^2 \right) \left( \cot \left( \sqrt{\frac{1}{2}} r_1(x) \right) + \cot \left( \sqrt{\frac{1}{2}} r_2(x) \right) \right); \]
and

\[(II) = \frac{1}{\sqrt{2}} (1 + |a_1(x)|^2) \left( \cot \left(\sqrt{2}r_1(x)\right) + \cot \left(\sqrt{2}r_2(x)\right) \right).\]

Recall \(U\) is a small open neighborhood of \(q\). If \(U\) is sufficiently small, then by Claim 1, \(I \leq Cu(x), II \leq Cu(x)\). This concludes the proof of the proposition. \(\Box\)

By the straightforward calculation we can write the complex Hessian operator \(L\) as the following real second order degenerate elliptic operator on \(U\).

**Lemma 2** Let \(\Box_d = X_1X_1 + (JX_1)(JX_1) + X_2X_2 + (JX_2)(JX_2), V = -\nabla_{JX_1}JX_1 - \nabla_{JX_2}JX_2\). Then

\[L = \Box_d - V \text{ on } U.\]

**Proof** The lemma follows from the straightforward calculation:

\[
L = \nabla_{X_1}\nabla_{X_1} + \nabla_{JX_1}\nabla_{JX_1} + \nabla_{X_2}\nabla_{X_2} + \nabla_{JX_2}\nabla_{JX_2} \\
= (X_1X_1 - \nabla_{X_1}X_1) \\
+ ((JX_1)(JX_1) - \nabla_{JX_1}JX_1) + (X_2X_2 - \nabla_{X_2}X_2) \\
+ ((JX_2)(JX_2) - \nabla_{JX_2}JX_2) \\
= \Box_d - \nabla_{JX_1}JX_1 - \nabla_{JX_2}JX_2.
\]

(2.1)

Let \(h(x) = -u(x)\) on \(U\). By Proposition 1 and Lemma 2, the nonpositive function \(h(x)\) satisfies the degenerate elliptic partial differential inequality

\[(L - C)h(x) \geq 0,
\]

where the positive constant \(C\) is from Proposition 1. Let \(S_U\) be the zero set of \(h(x)\) in \(U\). By Proposition 4 in [1] (cf. Theorem 2 in [10]), the maximum principle asserts that \(x \in S_U\) whenever \(x\) can be connected from \(q\) by a finite sequence of integral curves along \(X_1, JX_1, X_2, JX_2\). For such \(x\) with \(u(x) = -h(x) = 0\), the broken geodesic \(\gamma_1 \cup \gamma_2\) is a minimizing geodesic, implying \(X_1 = -X_2\).

Let \(B\) be a geodesic ball centered at \(p_1\) with radius \(\epsilon_0\) less than the injectivity radius of \(M\) such that \(B\) is contained in a coordinate chart at \(p_1\). Fix a point \(q_\lambda = \exp_{p_1}(\lambda l'(0)) \in B\) with \(d(q_\lambda, p_1) = \lambda \leq \epsilon_0\). Consider the integral curve \(c_\lambda(s)\) satisfying

\[
\frac{dc_\lambda(s)}{ds} = \lambda J\nabla r_1(c_\lambda(s)) \quad \text{and} \quad c_\lambda(0) = q_\lambda.
\]

(2.2)

As \(J\nabla r_1\) is perpendicular to \(\nabla r_1\), \(d(c_\lambda(s), p_1) = \lambda\) for all \(s\). Therefore \(c_\lambda(s) \in B\) and \(X_1 = \nabla r_1\) is always defined. Let \(s_0 = \sup|a|\) there exist a smooth family of minimal geodesics \(\tilde{l}_b(-a < b < a)\) containing \(p_1, c_\lambda(b), p_2\). As \(c_\lambda(s)\) is joint to \(q_\lambda\) by the integral curve along \(JX_1\), by applying Proposition 4 in [1], \(s_0 > 0\). If \(s_0\) is finite, by compactness, \(\tilde{l}_b\) is a smooth family of minimal geodesics for \(-s_0 < b < s_0\). By using the same argument, we can extend \(s_0\) a little bit more. This means \(s_0 = +\infty\).

It is clear from the above that \(\tilde{l}_b\) depends on \(\lambda\). Now let \(\lambda \to 0^+\). Then we obtain a family of minimal geodesics \(\tau_\epsilon\) connecting \(p_1\) and \(p_2\). Moreover, we show that the unit tangent vector of \(\tau_\epsilon\) at \(p_1\) is \(l'(0) \cos \epsilon + Jl'(0) \sin \epsilon\). The proof is simple as the Kähler metric \(g\) is locally Euclidean. Nevertheless we include the proof here for the sake of completeness.
Consider the variation \( \gamma(s, \lambda) := c_{\lambda}(s) \) for \( \lambda \) sufficiently small, \( s \in (-\infty, \infty) \) of the base curve \( \gamma(0, \lambda) = l(\lambda) \). By the regularity of the ordinary differential equation (2.2), \( \gamma(s, \lambda) \) is a smooth variation. Let \( x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \) be the real coordinate of \( B \) with \( x(p_1) = 0 \) such that

- \( J \frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial x_{\alpha+n}} \) for \( 1 \leq \alpha \leq n \);
- \( l' = \frac{\partial}{\partial s} \sin t \);
- \( g(x) = \sum_{1 \leq i, j \leq 2n} (\delta_{ij} + o(|x|^2)) dx_i \otimes dx_j \).

Then the Eq. (2.2) can be written in terms of local coordinates \( x(s, \lambda) := x(\gamma(s, \lambda)) \):

\[
\frac{\partial x(s, \lambda)}{\partial s} = \lambda J \nabla r_1(x(s, \lambda)) \quad \text{and} \quad x(0, \lambda) = (\lambda, 0, \ldots, 0).
\]  

(2.3)

Since the Kähler metric \( g \) is locally Euclidean, \( \nabla r_1(x(s, \lambda)) = \frac{1}{\lambda} \left( \sum_{1 \leq j \leq n} x_j \frac{\partial}{\partial x_j} + o(\lambda) \right) \) and \( J \nabla r_1(x(s, \lambda)) = \frac{1}{\lambda} \left( \sum_{1 \leq j \leq n} (x_j \frac{\partial}{\partial x_{j+n}} - x_{n+j} \frac{\partial}{\partial x_j}) + o(\lambda) \right) \). Therefore, the solution of the Eq. (2.3) is given by

\[
x(s, \lambda) = (\lambda \cos s, 0, \ldots, 0, \lambda \sin s, 0, \ldots, 0) + o(\lambda).
\]

Hence, for any fixed \( s, x(\tau_s(t)) = (t \cos s, 0, \ldots, 0, t \sin s, 0, \ldots, 0) \). Therefore, this family of geodesics closes up with period \( 2\pi \).

**Proposition 2** \( S = \bigcup_{0 \leq s < 2\pi} \tau_s \) is an embedded holomorphic sphere in \( M \). Moreover, \( S \) is totally geodesic and isometric to the standard 2-sphere up to a factor \( \frac{1}{2} \).

**Proof** It is clear that the length of \( \tau_s \) is constant. Let \( X = \frac{\partial}{\partial t} \tau_s(t), Y = \frac{\partial}{\partial s} \tau_s(t) \) for \( 0 \leq t \leq \frac{\pi}{\sqrt{2}} \). Then \( Y \) is a Jacobi field with initial condition

\[
Y(0) = 0, \quad Y'(0) = JX.
\]

(2.4)

By the second variation of arc length, for any vector field \( Z \) orthogonal to \( X \) along \( \tau_s \) and vanishing at \( p_1 \) and \( p_2 \),

\[
0 \leq \int_0^{\frac{\pi}{\sqrt{2}}} |\nabla_X Z|_t^2 - R(Z, X, X, Z)dt =: I(Z).
\]

(2.5)

If we take \( Z = \sin(\sqrt{2}t)JX \), then by \( BK \geq 1 \),

\[
R(X, JX, JX, X) = 2
\]

(2.6)

along \( \tau_s \). Thus

\[
I(\sin(\sqrt{2}t)JX) = 0.
\]

(2.7)

**Claim 2** \( R(JX, X, X, Z) = 0 \) for any \( Z \) orthogonal to \( JX \) and \( X \). Equivalently, \( R(Z, X)X \in \text{span}(X, JX) \perp \) and \( R(JX, X)X \in \text{span}(JX) \).

**Proof** Assume the claim is not true. Say at some \( x = \tau_{s_0}(t_0) \), for some tangent vector \( Z \in T_x M \),

\[
R(JX, X, X, Z) > 0, \quad Z \perp JX, \quad Z \perp X.
\]

(2.8)

It is clear that we can find \( Z \) satisfying (2.8) in a neighborhood of \( x \). Say for \( 0 < t_1 < t < t_2 < \frac{\pi}{\sqrt{2}}, s = s_0 \). Thus without loss of generality, we may assume that \( 0 < t \leq \frac{\pi}{\sqrt{2}} \). Let \( \square \)
us consider a cut-off function $\xi$ satisfying $\xi \geq 0$ on $[0, \frac{\pi}{\sqrt{2}})$, and $\xi$ has compact support in $(t_1, t_2)$. Moreover, $\xi = 1$ at $t_0$. For any $\lambda \geq 0$, consider the vector field $Z_\lambda(t) = \xi(t)\lambda Z + \sin(\sqrt{2}t)JX$. Let us plug $Z_\lambda$ in (2.5). According to (2.5) and (2.7), $I(Z_\lambda) \geq 0$ and $I(Z_0) = 0$. Thus

\[ \frac{d}{d\lambda} |_{\lambda=0} I(Z_\lambda) \geq 0. \]  

(2.9)

However, by direct calculation,

\[ \frac{d}{d\lambda} |_{\lambda=0} I(Z_\lambda) = \int_0^{\frac{\pi}{\sqrt{2}}} -2\xi(t) \sin(\sqrt{2}t) R(JX, X, X, Z) dt < 0. \]

(2.10)

This is a contradiction. \hfill \Box

**Lemma 3** $Y = \frac{1}{\sqrt{2}} \sin(\sqrt{2}t)JX$ on $\tau_s$. Therefore $S$ is smooth at $p_2$. $S$ is an immersed holomorphic sphere in $M$.

**Proof** Set $Y = Y_1 + Y_2$, where $Y_1$ is parallel to $JX$ and $Y_2$ is orthogonal to $JX$ and $X$. $Y$ satisfies the Jacobi field equation

\[ \nabla_X \nabla_X Y = -R(Y, X)X. \]

(2.11)

Let us rewrite it as

\[ \nabla_X \nabla_X Y_1 + \nabla_X \nabla_X Y_2 = -R(Y_1, X)X - R(Y_2, X)X. \]

(2.12)

Observe that $\nabla_X \nabla_X Y_1 \in \text{span}\{JX\}$ and $\nabla_X \nabla_X Y_2 \in \text{span}\{X, JX\}^\perp$. With the help of claim 2, we find

\[ \nabla_X \nabla_X Y_1 = -R(Y_1, X)X, \]

(2.13)

\[ \nabla_X \nabla_X Y_2 = -R(Y_2, X)X. \]

(2.14)

Notice that $Y_1(0) = 0, Y_1'(0) = JX$. With the help of (2.6) and (2.13), we find $Y_1 = \frac{1}{\sqrt{2}} \sin(\sqrt{2}t)JX$. Also note $Y_2(0) = 0, Y_2'(0) = 0$. Then from (2.14) and the uniqueness of ode, we find $Y_2 \equiv 0$. The proof of Lemma 3 is complete. \hfill \Box

Lemma 3 indicates a holomorphic isometry from the rescaled standard sphere to $S$. Next we prove $S$ is embedded. Suppose $\tau_{s_1}t_1 = \tau_{s_2}t_2$. As $d(p_1, \tau_2(t)) = t, t_1 = t_2$. We may assume $0 < t_1 < \frac{\pi}{\sqrt{2}}$. If $X_{s_1}t_1 \neq X_{s_2}t_2$, by standard triangle inequality, we see that $\tau_{s_1}$ cannot be a minimizing geodesic connecting $p_1$ and $p_2$. Therefore, by the uniqueness of geodesic, $\tau_{s_1}$ is the same as $\tau_{s_2}$. By checking the initial tangent vector at $p_1$, we find $s_1 = s_2$ modulo $2\pi$. Now we prove that $S$ is totally geodesic. It is clear that $\nabla_X X, \nabla_X Y \in \text{span}\{X, Y\}, \nabla_Y X = \nabla_X Y + [Y, X] \in \text{span}\{X, Y\}$ and $\nabla_Y Y = J\nabla_Y (\frac{1}{\sqrt{2}} \sin(\sqrt{2}t)X) \in \text{span}\{X, Y\}$. This completes the proof of proposition 2. \hfill \Box

According to Mori [9] and Siu-Yau [11] solution to the Frankel conjecture, $M$ is biholomorphic to $\mathbb{C}P^n$. Proposition 2 says $S$ is an embedded holomorphic sphere. Let us assume the degree of $S$ is $d$ for some integer $d \geq 1$. Then $\text{Vol}(M) = \frac{\text{Vol}(\mathbb{C}P^n)}{d^n}$. If $d = 1$, from the volume rigidity result in [5], $M$ is isometric to $\mathbb{C}P^n$.

**Remark 3** To prove $d = 1$, one may estimate the integration of the Ricci form on $S$. However, there are some difficulties when the points are near $p_1$ or $p_2$. 

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