We characterize the full classes of M-estimators for semiparametric models of general functionals by formally connecting the theory of consistent loss functions from forecast evaluation with the theory of M-estimation. This novel characterization result opens up the possibility for theoretical research on efficient and equivariant M-estimation and, more generally, it allows to leverage existing results on loss functions known from the literature of forecast evaluation in estimation theory.

**Keywords:** M-estimation, loss function, strict consistency, characterization

1. Introduction

The task of regression is to model the effect of covariates $X$ on a response variable $Y$, or more precisely, the effect of $X$ on a functional $\Gamma$ of the conditional distribution of $Y$ given $X$, $F_{Y|X}$. The typical example for $\Gamma$ is the mean, resulting in mean regression. In many application, one is interested in other functionals such as quantiles (Value at Risk, VaR), expectiles, variance, or Expected Shortfall (ES) (Koenker and Bassett, 1978; Bollerslev, 1986; Patton et al., 2019).

A correctly specified parametric model $m(X, \theta)$ satisfies $\Gamma(F_{Y|X}) = m(X, \theta_0)$ for some unique parameter $\theta_0 \in \Theta$.

The statistician’s task is to estimate the parameter $\theta_0$ based on data $(Y_t, X_t)$, $t = 1, \ldots, N$. For the standard situation of linear mean regression, $\Gamma(F_{Y|X}) = E[Y|X]$, and $m(X, \theta) = X \theta$, one often employs the ordinary-least-squares (OLS) estimator of the form

$$\hat{\theta}_T = \arg \min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \rho(Y_t, m(X_t, \theta)),$$

based on a loss function $\rho$, which is the key ingredient of an M-estimator. Note that our results carry over to time-varying loss functions $\rho_t$. 
The core condition on $\rho$ for consistency of $\hat{\theta}_T$ is that $E[\rho(Y_t, m(X_t, \theta_0))] < E[\rho(Y_t, m(X_t, \theta))]$ for all $\theta \neq \theta_0$ and for all $t \in \mathbb{N}$, which we call strict model-consistency of $\rho$ for $m$. Apart from special cases, the classes of such loss functions $\rho$ are unfortunately not well understood in M-estimation yet. Our main result, Theorem 1, establishes that a loss function $\rho$ is (strictly) model-consistent for $m$ if and only if it is (strictly) consistent for the target functional $\Gamma$, meaning that $\int \rho(y, \Gamma(F))dF(y) \leq \int \rho(y, \xi)dF(y)$ for all $\xi \in \mathbb{R}^k$ and for all distributions $F$ in a sufficiently rich class, where strictness means that equality implies $\xi = \Gamma(F)$. Since there are well-understood characterization results for strictly consistent losses from the literature of forecast evaluation (Gneiting, 2011a; Fissler and Ziegel, 2016), Theorem 1 lifts these results to a novel characterization of the classes of consistent M-estimators for general (vector-valued) functionals.

Our result provides the first characterization of the full classes of consistent M-estimators for semiparametric models of general, possibly vector-valued functionals by formally connecting the two strands of literature on M-estimation and forecast evaluation. Understanding the full classes of M-estimators facilitates a deeper understanding of the (im)possibilities, e.g., in the following areas. It allows to derive M-estimation efficiency bounds, similar to efficient generalized method of moments estimation of Chamberlain (1987). Drawing on the literature of homogeneous and equivariant loss functions (Nolde and Ziegel, 2017; Fissler and Ziegel, 2019), our connection allows to classify such M-estimators having e.g. the advantage that they are invariant to linear rescaling in finite samples. Furthermore, estimators with the most beneficial integrability conditions in the sense that $E[|\rho(Y_t, m(X_t, \theta))|]$ is finite can be derived. Our result further allows to leverage recent discoveries in the forecast evaluation literature such as mixture representations (Ehm et al., 2016) or score decompositions (Dimitriadis et al., 2021b) for the purpose of M-estimation. Theorem 1 also provides the first formal argument why consistent M-estimation is basically impossible if there do not exist (strictly) consistent loss functions for the target functional $\Gamma$. This argument has already been used informally for the Expected Shortfall (Patton et al., 2019; Dimitriadis and Bayer, 2019), the Range Value at Risk (Barendse, 2022), and the mode (Kemp and Silva, 2012).

2. Notation and Definitions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a non-atomic, complete probability space where all random variables are defined. We introduce a class $\mathcal{Y}$ of $\mathbb{R}^d$-valued possible response variables, a corresponding class $\mathcal{X}$ or $\mathbb{R}^p$-valued regressors, and $\mathcal{Z}$ the class of possible response–regressor pairs $(Y, X)$. The class of marginal distributions $F_Y$ of $Y \in \mathcal{Y}$ is denoted by $\mathcal{F}_Y$, with a corresponding notation $\mathcal{F}_X$. $\mathcal{F}_Y|X$ is the class of regular versions of conditional distributions $F_{Y|X}$ for any $(Y, X) \in \mathcal{Z}$; see Appendix A for technical details. We will identify cumulative distribution functions with their corresponding measures where convenient. Let $\Gamma: \mathcal{F}_Y|X \rightarrow \Xi \subseteq \mathbb{R}^k$ be some $k$-dimensional, single-valued functional of the conditional distribution of $Y$ given $X$. Let $\Theta \subseteq \mathbb{R}^q$ be some parameter space with non-empty interior, int($\Theta$), and $m: \mathbb{R}^p \times \Theta \rightarrow \Xi$ a parametric model for the functional $\Gamma$. We shall work under the following assumption of a correctly specified model with a uniquely identified true model parameter.
Assumption 1. For all \( Z = (Y, X) \in \mathcal{Z} \) there is a unique parameter \( \theta_0 = \theta_0(F_Z) \in \text{int}(\Theta) \) such that almost surely
\[
m(X, \theta_0) = \Gamma(F_{Y|X}).
\] (2)

Assumption 1 is semiparametric in the sense that the finite-dimensional parameter \( \theta_0 \) in (2) does not fully describe the distribution of \((Y, X)\), but only \( \Gamma(F_{Y|X}) \), the component of the conditional distribution we are essentially interested in. In general, an estimator should be valid on a class of random variables \( \mathcal{Z} \) which is as large as possible, allowing to apply the estimator in many different situations and under uncertainty of the distributions of the underlying data. Hence, Assumption 1 is a minimal condition on the random variables that allows for semiparametric modeling of a functional of the conditional distribution.

We continue by recalling the use of loss functions in the closely related area of forecast evaluation, where the notion of a strictly consistent loss function is a crucial concept in the literature on forecast evaluation, since it incentivizes truthful reports (Murphy and Daan, 1985). Making use of a similar decision-theoretic terminology as in Gneiting (2011a) and Fissler and Ziegel (2016), let \( \mathcal{F} \) be some generic class of probability distributions on \( \mathbb{R}^d \), which is our observation domain, and \( \Xi \subseteq \mathbb{R}^k \), our action domain.

Definition 1 (Consistency and elicitability). A loss function \( \rho: \mathbb{R}^d \times \Xi \to \mathbb{R} \) is called \( \mathcal{F} \)-consistent for a functional \( \Gamma: \mathcal{F} \to \Xi \) if \( \rho(\cdot, \xi) \) is \( \mathcal{F} \)-integrable for all \( F \in \mathcal{F} \) and for all \( \xi \in \Xi \), and if
\[
\int \rho(y, \Gamma(F)) \, dF(y) \leq \int \rho(y, \xi) \, dF(y) \quad \text{for all } F \in \mathcal{F}, \text{ for all } \xi \in \Xi.
\] (3)
If equality in (3) implies \( \xi = \Gamma(F) \), then the loss function is called strictly \( \mathcal{F} \)-consistent for \( \Gamma \). A functional \( \Gamma: \mathcal{F} \to \Xi \) is elicitable if there is a strictly \( \mathcal{F} \)-consistent loss function for it.

On the class of distributions with a finite second moment, the squared loss \( \rho(y, \xi) = (y - \xi)^2 \) is strictly consistent for the mean functional. More generally, subject to regularity and integrability conditions on \( \rho \) and richness conditions on \( \mathcal{F} \), \( \rho \) is (strictly) \( \mathcal{F} \)-consistent for the mean if and only if it is a so-called Bregman loss
\[
\rho(y, \xi) = \phi(y) - \phi(\xi) + \phi'(\xi)(\xi - y) + \kappa(y),
\] (4)
where \( \phi \) is a (strictly) convex function on \( \mathbb{R} \) with subgradient \( \phi' \) and \( \kappa \) is any function of \( y \) (Savage, 1971; Gneiting, 2011a). Likewise, the well known pinball or asymmetric absolute loss \( \rho(y, \xi) = (\mathbb{1}\{y \leq \xi\} - \alpha)(\xi - y) \) is strictly consistent for the lower \( \alpha \)-quantile on the class of distributions with finite mean and where the lower \( \alpha \)-quantile coincides with the upper \( \alpha \)-quantile. Moreover, subject to regularity and integrability conditions on \( \rho \) and richness conditions on \( \mathcal{F} \), a loss \( \rho \) is (strictly) consistent for the lower \( \alpha \)-quantile, \( \alpha \in (0, 1) \), if and only if it is a generalized piecewise linear loss function
\[
\rho(y, \xi) = (\mathbb{1}\{y \leq \xi\} - \alpha)(g(\xi) - g(y)) + \kappa(y),
\] (5)
where \( g \) is (strictly) increasing (Gneiting, 2011b).

Similar characterization results exist for other functionals such as expectiles or the pairs consisting of the mean and variance or the quantile and Expected Shortfall (Gneiting, 2011a;
The characterization results in (4) and (5) rely on the fact that the related classes $\mathcal{F}$ are convex and rich enough. E.g., if we restrict attention to symmetric distributions only, the mean equals the median, and loss functions of the form given in (4) or (5) would elicit the mean and median.

The following definition develops similar notions of consistency for the setting of M-estimation with an underlying class $\mathcal{Z}$ implying that $\Gamma$ is defined on the class of conditional distributions $\mathcal{F}_{Y|X}$.

**Definition 2** (Model-consistency). Suppose Assumption 1 holds for the parametric model $m: \mathbb{R}^p \times \Theta \to \Xi$ and the functional $\Gamma: \mathcal{F}_{Y|X} \to \Xi$. Let $\rho: \mathbb{R} \times \Xi \to \mathbb{R}$ be a loss function such that $E[|\rho(Y, m(X, \theta))|] < \infty$ for all $(Y, X) \in \mathcal{Z}$ and for all $\theta \in \Theta$.

(i) The loss $\rho$ is **unconditionally $\mathcal{F}_Z$-model-consistent** for the model $m$ if

$$E[\rho(Y, m(X, \theta_0))] \leq E[\rho(Y, m(X, \theta))]$$

for all $(Y, X) \in \mathcal{Z}$, for all $\theta \in \Theta$. (6)

Moreover, $\rho$ is **strictly unconditionally $\mathcal{F}_Z$-model-consistent** for the model $m$ if equality in (6) implies that $\theta = \theta_0$.

(ii) The loss $\rho$ is **conditionally $\mathcal{F}_Z$-model-consistent** for the model $m$ if

$$E[\rho(Y, m(X, \theta_0)) | X] \leq E[\rho(Y, m(X, \theta)) | X] \text{ a.s.}$$

for all $(Y, X) \in \mathcal{Z}$, for all $\theta \in \Theta$. (7)

Moreover, $\rho$ is **strictly conditionally $\mathcal{F}_Z$-model-consistent** for the model $m$ if almost sure equality in (7) implies that $\theta = \theta_0$.

The concept of unconditional model-consistency is the central condition for consistent M-estimation: Gourieroux et al. (1987, Property 3.3 and 3.4) show equivalence of these two notions in terms of first order conditions and under some regularity assumptions. Also see condition (i) in Newey and McFadden (1994, Theorem 2.1), Huber (1967, Assumption (A-4)), where their remaining assumptions are merely regularity conditions. In contrast, the conditional notion is appealing since it bridges the gap between consistency for $\Gamma$ according to Definition 1 and unconditional model-consistency for $m$ in the proof of Theorem 1 below. It can still be practically useful when we resort to nonparametric kernel regressions or in the presence of repeated observations of $X$, e.g. if $X$ consists of categorical variables only.

While the classical notion of consistency in (3) and the unconditional model version in (6) are closely related, they are crucially different in that in the latter, the expectation is also taken with respect to the covariates. Establishing and finding reasonable conditions for their equivalence is indeed not trivial as shown in Theorem 1 below.

### 3. Main Result and Discussion

To present our main Theorem 1, we introduce and discuss the following two assumptions.

**Assumption 2.** For all $X \in \mathcal{X}$, the map $m(X, \cdot): \Theta \to \Xi$ is surjective almost surely. For all $(Y, X) \in \mathcal{Z}$ the conditional expectation $E[\rho(Y, m(X, \theta)) | X]$ is continuous in $\theta$ almost surely.
The surjectivity in Assumption 2 can usually be fulfilled by a sensible choice of Ξ, and the smoothness condition on the expected loss is standard in the literature, see e.g., Newey and McFadden (1994, Section 2.3).

**Assumption 3.** For any $Z = (Y, X) \in Z$ and any event $A \in \sigma(X)$ with positive probability $\mathbb{P}(A) > 0$, there is some $\tilde{Z} \in Z$ such that $\mathbb{P}(\tilde{Z} \in B) = \mathbb{P}(Z \in B | A)$ for all Borel sets $B \subseteq \mathbb{R}^{d+p}$.

Assumption 3 is a richness condition on the class of possible data generating processes (DGP) in $Z$ as for any process $Z = (Y, X) \in Z$, and any set $A$ with positive probability, it stipulates that $Z$ is rich enough to contain a process $\tilde{Z} = (\tilde{Y}, \tilde{X})$ as specified in Assumption 3. Crucially, it yields that $\mathbb{P}(\tilde{Y} \in C | \tilde{X}) = \mathbb{P}(Y \in C | X, A)$ for all Borel sets $C \subseteq \mathbb{R}$. This, together with Assumption 1, implies that the correctly specified parameter and hence the semiparametric model, is the same under the distributions $\tilde{F}_Z$ and $F_Z$; in formulae, $\theta_0(\tilde{F}_Z) = \theta_0(F_Z)$. Recall that in estimation, $Z$ captures the flexibility about the underlying and in practice unknown DGP, such that a large $Z$ is desirable in order to obtain an estimation method which is applicable to a wide range of distributions of $Z \in Z$. Thus, Assumption 3 intuitively means that given a certain plausible and correctly specified DGP, and given a measurable set $B \subset \mathbb{R}^p$ of possible values for the covariates $X$ which is attained with positive probability, i.e. $\mathbb{P}(X \in B) > 0$, restricting the DGP to these values of covariates must be feasible. E.g., if income $Y$ is studied in dependence of years after graduation, $X_1$, and further covariates $X_2, \ldots, X_p$, one might as well study income of persons at most 5 years after their graduation, $X_1 \leq 5$. Then, in a correctly specified model, the true but unknown parameter $\theta_0$ remains the same, no matter whether considering the whole population or only persons within 5 years after their graduation. Further recall that as discussed after (5), the characterization results for strictly consistent loss functions already rely on richness conditions on the classes of distributions.

**Theorem 1.** Under Assumption 1 the following holds for a loss $\rho: \mathbb{R} \times \Xi \rightarrow \mathbb{R}$.

(i) If $\rho$ is (strictly) $\mathcal{F}_{Y|X}$-consistent for $\Gamma$ then it is (strictly) conditionally $\mathcal{F}_Z$-model-consistent for the model $m$.

(ii) Under Assumption 2 if $\rho$ is conditionally $\mathcal{F}_Z$-model-consistent for $m$, there is a modification $\tilde{\mathcal{F}}_{Y|X}$ of $\mathcal{F}_{Y|X}$ such that $\rho$ is $\tilde{\mathcal{F}}_{Y|X}$-consistent for $\Gamma$.

(iii) If $\rho$ is (strictly) conditionally $\mathcal{F}_Z$-model consistent for $m$ then it is (strictly) unconditionally $\mathcal{F}_Z$-model-consistent for $m$.

(iv) Under Assumption 3 if $\rho$ is (strictly) unconditionally $\mathcal{F}_Z$-model-consistent for $m$ then it is also (strictly) conditionally $\mathcal{F}_Z$-model-consistent for $m$.

Theorem 1, whose proof can be found in Appendix B, provides two main implications; see Figure 1 for a visualisation: First, a combination of (i) and (iii) justifies the use of strictly $\mathcal{F}_{Y|X}$-consistent losses for $\Gamma$ in the context of M-estimation. This is well known in the literature, e.g. Gneiting and Raftery (2007, Section 9) describe this under the term optimum score estimation. The proofs of (i) and (iii) are straight-forward, and for special cases they can be found, e.g. in the proof of Patton et al. (2019, Theorem 1).
Second, and more important for our purposes is the reverse implication, combining (ii) and (iv). It asserts that, under appropriate assumptions, an unconditionally $F_Z$-model-consistent loss for $m$ is necessarily $F_{Y|X}$-consistent for $\Gamma$. Thus, exploiting known characterization results for $F_{Y|X}$-consistent losses for many relevant functionals $\Gamma$, it constitutes an effective and original bound on the class of consistent M-estimators. Notice that strictness of the $\tilde{F}_{Y|X}$-consistent losses cannot be established in part (ii) of Theorem 1. While a stronger version of this result including the strictness would be desirable, its lack hardly diminishes the applicability of the results since characterization results for non-strict $F_{Y|X}$-consistent losses are available in the literature and these are almost as strong as the ones for strictly consistent losses (Gneiting, 2011a). E.g., we obtain all consistent losses for the mean functional when possibly non-strictly convex functions $\phi$ are used in (4), and for quantiles when possibly non-strictly increasing functions $g$ are used in (5). Further note that for practical or intuitive purposes, the technical distinction between $F_{Y|X}$ and a modification $\tilde{F}_{Y|X}$ thereof is inessential, see Appendix A.

E.g., Theorem 1 implies that the class of M-estimators for conditional mean models are characterized by loss functions of the form (4) while for conditional quantiles, a loss of the form (5) must be used. This enables to find beneficial choices of $\phi$, $g$ and $\kappa$ in terms of efficiency and equivariance (Dimitriadis et al., 2021a) or integrability of the loss. Similarly, one can leverage the flexibility of the class of consistent losses for VaR and ES provided in Fissler and Ziegel (2016).

As of yet, implications in the direction of the points (ii) and (iv) of Theorem 1 have only been provided for special cases or under much stronger conditions: First, Gourieroux et al. (1987) consider M-estimators for general semiparametric models by restricting attention to parameters identified by a set of conditional moment restrictions, as given by their equation (4.5). As a consequence, their main result, Property 4.7, characterizes the functional form of the losses’ first-order conditions and is hence closest to our Proposition S3 in the Supplementary Material. In contrast, our Theorem 1 allows us to conveniently connect M-estimation to known classes of strictly consistent loss functions from the literature on forecast evaluation. Gourieroux et al. (1987) further operate under a stronger and less interpretable richness condition on the class of distributions; compare their Assumption A.7 i) to our Assumption 3. Second, Komunjer (2005, Theorem 2) shows a necessary condition for M-estimation if $\Gamma$ is some quantile. However, a richness condition, corresponding to our Assumption 3, is only assumed implicitly in the proof when quantifying over all model classes before their equation (19). In contrast, our Theorem 1 rigorously shows this relation for semiparametric models for any elicitable functional.
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Supplementary Material

The Supplementary Material derives results corresponding to Theorem 1 for zero (Z) estimators and identification functions.

Appendix

A. Technical Details on the Class of Conditional Distributions

\( \mathcal{F}_{Y\mid X} \) is a collection of distributions such that for any \((Y, X) \in Z\) a regular version of the conditional distribution \( F_{Y\mid X} \) is an element of \( \mathcal{F}_{Y\mid X} \) almost surely. Recall that \( F_{Y\mid X} \) is unique almost surely. Therefore, \( \mathcal{F}_{Y\mid X} \) is induced by a map \( Z \ni Z \mapsto \Omega_Z \in \{ \Omega' \in A \mid \mathbb{P}(\Omega') = 1 \} \) such that \( \mathcal{F}_{Y\mid X} \) is the union of all collections \( \{ F_{Y\mid X}(\cdot, \omega) \mid \omega \in \Omega_Z \} \) where \( Z \in Z \). Since there are several such maps \( Z \mapsto \{ \Omega' \in A \mid \mathbb{P}(\Omega') = 1 \} \), \( \mathcal{F}_{Y\mid X} \) is not unique, but has several modifications corresponding to the choices of this map \( Z \mapsto \Omega_Z \in \{ \Omega' \in A \mid \mathbb{P}(\Omega') = 1 \} \).

B. Proof of Theorem 1

**Proof of Theorem 1.** (i) Let \( Z = (Y, X) \in Z \) and \( \theta_0 = \theta_0(F_Z) \). Suppose that \( \theta \neq \theta_0 \). Then

\[
\mathbb{E}[\rho(Y, m(X, \theta_0)) | X] = \mathbb{E}[\rho(Y, \Gamma(F_{Y\mid X})) | X] \leq \mathbb{E}[\rho(Y, m(X, \theta)) | X] \quad \text{due to the } \mathcal{F}_{Y\mid X}-\text{consistency of } \rho. \]

Invoking Assumption 1, we get \( \mathbb{P}(m(X, \theta_0) \neq m(X, \theta)) > 0 \). Therefore, if \( \rho \) is (strictly) \( \mathcal{F}_{Y\mid X} \)-consistent for \( \Gamma \), it is also (strictly) \( \mathcal{F}_Z \)-model-consistent for \( m \).

(ii) Suppose that \( \rho \) is conditionally \( \mathcal{F}_Z \)-model consistent and let \( (Y, X) \in Z \). Then for all \( \theta \in \Theta \) it holds that

\[
\mathbb{P}\left( \mathbb{E}[\rho(Y, m(X, \theta_0)) | X] \leq \mathbb{E}[\rho(Y, m(X, \theta)) | X] \right) = 1.
\]

Since the countable union of null sets is again a null set, this implies that

\[
\mathbb{P}\left( \mathbb{E}[\rho(Y, m(X, \theta_0)) | X] \leq \mathbb{E}[\rho(Y, m(X, \theta)) | X] \quad \forall \theta \in \Theta \cap \mathbb{Q}' \right) = 1,
\]

where \( \mathbb{Q} \) is the set of all rationals. Using the fact that \( \Theta \cap \mathbb{Q}' \) is dense in \( \Theta \) and due to the stipulated continuity of the conditional expectations of the losses in Assumption 2, we obtain that \( \mathbb{P}(A) = 1 \) where

\[
A = \left\{ \omega \in \Omega : \mathbb{E}[\rho(Y, m(X, \theta_0)) | X](\omega) = \mathbb{E}[\rho(Y, \Gamma(F_{Y\mid X})) | X](\omega) \leq \mathbb{E}[\rho(Y, m(X, \theta)) | X](\omega) \quad \forall \theta \in \Theta \right\} \in \mathcal{A},
\]

where we also used Assumption 1. Let \( A' \in \mathcal{A} \) be the set with probability one such that
\(m(X(\omega), \cdot)\) is surjective for all \(\omega \in A'\). Then for all \(\omega \in A \cap A'\) we get that
\[
\int \rho(y, \Gamma(F_{Y|X}(\cdot, \omega)))F_{Y|X}(dy, \omega) \leq \int \rho(y, m(X(\omega), \theta))F_{Y|X}(dy, \omega)
\]
for all \(\theta \in \Theta\). Finally, exploiting the surjectivity of the model, we arrive at the claim. Clearly, it is only possible to establish this assertion on a modification of \(F_{Y|X}\); see the first paragraph of Section 2 and Appendix A.

(iii) This is a standard application of the tower property together with the positivity of the expectation.

(iv) Assume that \(\rho\) is not strictly conditionally \(F_Z\)-model-consistent for \(m\). That means there exists \(Z = (Y, X) \in \mathcal{Z}\) with true parameter \(\theta_0 = \theta_0(F_Z)\) such that for some \(\theta \neq \theta_0\) the event \(A = \{\omega \mid \mathbb{E}[\rho(Y, m(X, \theta)) - \rho(Y, m(X, \theta_0))|X](\omega) \leq 0\}\) has positive probability. Let \(\tilde{Z} = (\tilde{Y}, \tilde{X}) \in \mathcal{Z}\) be the pair given by Assumption 3 with \(A\) specified above. Then clearly
\[
\mathbb{E}[\rho(\tilde{Y}, m(\tilde{X}, \theta)) - \rho(\tilde{Y}, m(\tilde{X}, \theta_0))] = \mathbb{E}\left[\mathbb{E}[\rho(\tilde{Y}, m(\tilde{X}, \theta)) - \rho(\tilde{Y}, m(\tilde{X}, \theta_0))|\tilde{X}]\right] \leq 0. \tag{8}
\]
This means that \(\rho\) is not strictly unconditionally \(F_Z\)-model-consistent for \(m\). The argument when we assume that \(\rho\) is merely conditionally \(F_Z\)-model-consistent works analogously, where we replace the inequalities in the definition of \(A\) and in (8) with strict inequalities.

\[\square\]

References

Barendse, S. (2022). Efficiently weighted estimation of tail and interquartile expectations. Preprint. https://dx.doi.org/10.2139/ssrn.2937665.

Bierens, H. J. (1990). A consistent conditional moment test of functional form. Econometrica, 58(6):1443–1458.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3):307–327.

Brown, B. W. (1983). The identification problem in systems nonlinear in the variables. Econometrica, 51(1):175–196.

Chamberlain, G. (1987). Asymptotic efficiency in estimation with conditional moment restrictions. Journal of Econometrics, 34(3):305–334.

Dimitriadis, T. and Bayer, S. (2019). A joint quantile and expected shortfall regression framework. Electronic Journal of Statistics, 13(1):1823–1871.

Dimitriadis, T., Fissler, T., and Ziegel, J. (2022). Osband’s principle for identification functions. Preprint. https://arxiv.org/abs/2208.07685.

Dimitriadis, T., Fissler, T., and Ziegel, J. F. (2021a). The efficiency gap. Preprint, (version v2). https://arxiv.org/abs/2010.14146v2.

Dimitriadis, T., Gneiting, T., and Jordan, A. I. (2021b). Stable reliability diagrams for probabilistic classifiers. Proceedings of the National Academy of Sciences, 118(8):e2016191118.

Dimitriadis, T., Patton, A. J., and Schmidt, P. W. (2021c). Testing forecast rationality for measures of central tendency. Preprint. https://arxiv.org/abs/1910.12545.

Ehm, W., Gneiting, T., Jordan, A., and Krüger, F. (2016). Of quantiles and expectiles: consistent scoring functions, Choquet representations and forecast rankings. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 78(3):505–562.

Fissler, T. and Ziegel, J. F. (2016). Higher order elicitability and Osband’s principle. Annals of Statistics, 44(4):1680–1707.

Fissler, T. and Ziegel, J. F. (2019). Order-sensitivity and equivariance of scoring functions. Electronic Journal of Statistics, 13(1):1166–1211.
Gneiting, T. (2011a). Making and evaluating point forecasts. *Journal of the American Statistical Association*, 106(494):746–762.

Gneiting, T. (2011b). Quantiles as optimal point forecasts. *International Journal of Forecasting*, 27(2):197–207.

Gneiting, T. and Raftery, A. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477):359–378.

Gourieroux, C., Monfort, A., and Renault, E. (1987). Consistent M-estimators in a semi-parametric model. *CEPREMAP Working Paper 8720*. http://www.cepremap.fr/depot/couv_orange/co8720.pdf.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, 50(4):1029–54.

Huber, P. J. (1967). The behavior of maximum likelihood estimates under nonstandard conditions. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, pages 221–233. Berkeley: University of California Press.

Kemp, G. C. and Silva, J. S. (2012). Regression towards the mode. *Journal of Econometrics*, 170(1):92–101.

Koenker, R. and Bassett, G. (1978). Regression quantiles. *Econometrica*, 46(1):33–50.

Komunjer, I. (2005). Quasi-maximum likelihood estimation for conditional quantiles. *Journal of Econometrics*, 128(1):137–164.

Komunjer, I. (2012). Global identification in nonlinear models with moment restrictions. *Econometric Theory*, 28(4):719–729.

Murphy, A. H. and Daan, H. (1985). Forecast evaluation. In Murphy, A. H. and Katz, R. W., editors, *Probability, Statistics and Decision Making in the Atmospheric Sciences*, pages 379–437. Westview Press, Boulder, Colorado.

Newey, W. K. (1985). Maximum likelihood specification testing and conditional moment tests. *Econometrica*, 53(5):1047–1070.

Newey, W. K. (1990). Semiparametric efficiency bounds. *Journal of Applied Econometrics*, 5(2):99–135.

Newey, W. K. (1993). Efficient estimation of models with conditional moment restrictions. In Maddala, G., Rao, C., and Vinod, H., editors, *Handbook of Statistics, Volume 11: Econometrics*.

Newey, W. K. and McFadden, D. (1994). Large sample estimation and hypothesis testing. In Engle, R. F. and McFadden, D., editors, *Handbook of Econometrics*, volume 4, chapter 36, pages 2111–2245. Elsevier.

Nolde, N. and Ziegel, J. F. (2017). Elicitability and backtesting: Perspectives for banking regulation. *Annals of Applied Statistics*, 11(4):1833–1874.

Patton, A. J., Ziegel, J. F., and Chen, R. (2019). Dynamic semiparametric models for expected shortfall (and value-at-risk). *Journal of Econometrics*, 211(2):388 – 413.

Roehrig, C. S. (1988). Conditions for identification in nonparametric and parametric models. *Econometrica*, 56(2):433–447.

Rothenberg, T. J. (1971). Identification in parametric models. *Econometrica*, 39(3):577–591.

Savage, L. J. (1971). Elicitation of personal probabilities and expectations. *Journal of the American Statistical Association*, 66(336):783–801.
This supplement provides results on linking zero (Z) estimation to identification functions from forecast evaluation. It further illustrates why a full analogon of Theorem 1 in the main text is not achievable by providing corresponding counterexamples. Section S1 introduces the setup and notation and Section S2 provides results and a discussion thereof. All proofs are given in Section S3.

S1. Setup, Notation and Definitions

Recall the estimation problem for semiparametric models of a general functional \( \Gamma \) as outlined in the main text, where we shall use the same notation. A correctly specified parametric model \( m(X, \theta) \) satisfies \( \Gamma(F_Y | X) = m(X, \theta_0) \in \Xi \subseteq \mathbb{R}^k \) for some unique parameter \( \theta_0 \in \Theta \subseteq \mathbb{R}^q \). The task is to estimate the parameter \( \theta_0 \) based on data \( (Y_t, X_t), t = 1, \ldots, N \). A standard alternative to M-estimation is (zero) Z-estimation

\[
\hat{\theta}_{Z,T} = \arg \min_{\theta \in \Theta} \left\| \frac{1}{T} \sum_{t=1}^{T} \psi(Y_t, X_t, \theta) \right\|^2,
\]

(S1)

based on some \( s \)-dimensional functions \( \psi(Y_t, X_t, \theta) \), that could also be time-varying. The latter are often called moment conditions or identification functions for \( \theta_0 \), satisfying the strict unconditional identification condition

\[
\left( \mathbb{E}[\psi(Y_t, X_t, \theta)] = 0 \iff \theta = \theta_0 \right) \quad \text{for all } \theta \in \Theta \quad \text{for all } t \in \mathbb{N}.
\]

(S2)

Motivated by the zero-condition in (S2), the estimator in (S1) is called Z-estimator. We restrict attention to the exactly identified case of \( s = q \), implying as many moment conditions as model parameters. Extensions to the case \( s > q \) are possible in the framework of generalized method of moments (GMM) estimation (Hansen, 1982; Newey and McFadden, 1994).

Similar to the strict model-consistency of loss functions, the strict unconditional identification (S2) is the core condition for the consistency of the Z-estimator \( \hat{\theta}_{Z,T} \) and the choice of \( \psi \) is the key ingredient of Z-estimation, see e.g., Gourieroux et al. (1987), Newey (1990), Newey and McFadden (1994). We continue to formally introduce the notions of identification functions from the literature on forecast evaluation, where these functions are deployed to check (conditional) calibration of forecasts (Nolde and Ziegel, 2017; Dimitriadis et al., 2021c), akin to a goodness-of-fit test.

**Definition S1** (Identification function and identifiability). A map \( \varphi: \mathbb{R}^d \times \Xi \to \mathbb{R}^k \) is called an \( \mathcal{F} \)-identification function for a functional \( \Gamma: \mathcal{F} \to \Xi \subseteq \mathbb{R}^k \) if \( \varphi(\cdot, \xi) \) is \( F \)-integrable for all \( F \in \mathcal{F} \).
and for all $\xi \in \Xi$, and if
\[
\int \varphi(y, \Gamma(F)) \, dF(y) = 0 \quad \text{for all } F \in \mathcal{F}.
\]
If additionally
\[
\left( \int \varphi(y, \xi) \, dF(y) = 0 \implies \xi = \Gamma(F) \right) \quad \text{for all } F \in \mathcal{F}, \text{ for all } \xi \in \Xi,
\]
it is a strict $\mathcal{F}$-identification function for $\Gamma$. A functional $\Gamma: \mathcal{F} \to \Xi$ is called identifiable if there is a strict $\mathcal{F}$-identification function for it.

In this definition, we restrict attention to $k$-dimensional identification functions. This is motivated by the characterization result of Dimitriadis et al. (2022), who show that, given a strict identification function $\varphi(y, \xi)$, the whole class of strict identification functions is given by the set
\[
\left\{ h(\xi)\varphi(y, \xi) \mid h: \Xi \to \mathbb{R}^{k \times k}, \det(h(\xi)) \neq 0 \text{ for all } \xi \in \Xi \right\}. \quad (S3)
\]
This implies that identification functions of lower dimension than $k$ cannot be strict, while ones with dimension greater $k$ contain redundancies; see Dimitriadis et al. (2022, Remark 6) for a detailed discussion.

For estimation of semiparametric models, and similar to Definition 1 we introduce an unconditional and a conditional notion of identification functions for models, where the latter coincides with the notion of conditional moment conditions (or restrictions), as given, e.g. in Newey (1993) and the references therein.

For this, we make use of the observation that by Assumption 1, the parameter $\theta_0 = \theta_0(F_Z)$ can be interpreted as a functional $\theta_0: \mathcal{F}_Z \to \Theta \subseteq \mathbb{R}^q$ by mapping a distribution $F_Z \in \mathcal{F}_Z$ to the unique $\theta_0(F_Z) \in \text{int}(\Theta)$ such that (2) is satisfied.

**Definition S2.** Under Assumption 1, let $\theta_0: \mathcal{F}_Z \to \Theta \subseteq \mathbb{R}^q$ be the functional given by (2). Let $\psi: \mathbb{R}^d \times \mathbb{R}^p \times \Theta \to \mathbb{R}^q$ be a function such that $\mathbb{E}\|\psi(Y, X, \theta)\|_1 < \infty$ for all $(Y, X) \in Z$ and for all $\theta \in \Theta$.

(i) The function $\psi$ is an unconditional $\mathcal{F}_Z$-identification function for $\theta_0: \mathcal{F}_Z \to \Theta$ if
\[
\mathbb{E}[\psi(Y, X, \theta_0(F_Z))] = 0 \quad \text{for all } Z = (Y, X) \in Z.
\]

It is a strict unconditional $\mathcal{F}_Z$-identification function for $\theta_0$ if additionally
\[
\left( \mathbb{E}[\psi(Y, X, \theta)] = 0 \implies \theta = \theta_0(F_Z) \right) \quad \text{for all } Z = (Y, X) \in Z, \text{ for all } \theta \in \Theta.
\]

(ii) The function $\psi$ is a conditional $\mathcal{F}_Z$-identification function for $\theta_0: \mathcal{F}_Z \to \Theta$ if
\[
\mathbb{E}[\psi(Y, X, \theta_0(F_Z)) | X] = 0 \quad \text{a.s.} \quad \text{for all } Z = (Y, X) \in Z.
\]
It is a strict conditional $F_Z$-identification function for $\theta_0$ if additionally
\[
\left( \mathbb{E}[\psi(Y,X,\theta)|X] = 0 \quad \text{a.s.} \implies \theta = \theta_0(F_Z) \right)
\] for all $Z = (Y,X) \in Z$, for all $\theta \in \Theta$.

S2. Main result and discussion

The following Proposition gives counterparts of Theorem 1 (i) and (ii), with similar attenuations with respect to the strictness as in Theorem 1 (ii).

Proposition S1. Under Assumption 1, the following implications hold for $\varphi : \mathbb{R}^d \times \Xi \to \mathbb{R}^k$:

(i) If $\varphi$ is a (strict) $F_{Y\mid X}$-identification function for $\Gamma$ then $\mathbb{R}^d \times \mathbb{R}^p \times \Theta \ni (y,x,\theta) \mapsto \varphi(y,m(x,\theta))$ is a (strict) conditional $F_Z$-identification function for $\theta_0$.

(ii) If $(y,x,\theta) \mapsto \varphi(y,m(x,\theta))$ is a conditional $F_Z$-identification function for $\theta_0$ then $\varphi$ is an $F_{Y\mid X}$-identification function for $\Gamma$.

While counterparts to the conclusions of Theorem 1 (iii) and (iv), connecting the conditional with the unconditional notion, would be desirable for identification functions, they seem out of reach in the general case. To arrive at a counterpart of Theorem 1 (iii) note that, by the tower property, any (strict) conditional $F_Z$-identification for $\theta_0$ is an unconditional $F_Z$-identification function for $\theta_0$. However, it generally fails to be strict; see Example S1 below and the results of Newey and McFadden (1994), Rothenberg (1971), Brown (1983), Roehrig (1988), and Komunjer (2012) among many others.

Henceforth, and similar to Gourieroux et al. (1987, Equation (4.5)), we restrict attention to conditional moment conditions of the form
\[
\psi_A(Y,X,\theta) = A(X,\theta)\varphi(Y,m(X,\theta)),
\] (S4)
where $\varphi$ is a strict $F_{Y\mid X}$-identification function for $\Gamma$, and $A(X,\theta)$ is a $(q \times k)$ instrument matrix. This construction is motivated by the well-known fact that $\mathbb{E}[(\varphi(Y,m(X,\theta))|X] = 0$ a.s. is equivalent to
\[
\mathbb{E}[a(X)^\intercal \varphi(Y,m(X,\theta))] = 0 \quad \text{for all measurable } a: \mathbb{R}^p \to \mathbb{R}^k,
\] (S5)
such that $\mathbb{E}[(a(X)^\intercal \varphi(Y,m(X,\theta))|X] < \infty$, where the functions $a(X) = a(X,\theta)$ can also depend on $\theta$. To render the equivalence in (S5) statistically feasible, we reduce the number of functions $a(X,\theta)^\intercal$ to be finite, resulting in the instrument matrix $A(X,\theta) \in \mathbb{R}^{s \times k}$ in (S4). Also see Newey (1985), Newey (1990), Bierens (1990), Nolde and Ziegel (2017) for similar constructions in estimation and testing based on conditional moment restrictions.

Remark S1. In light of the characterization result of Dimitriadis et al. (2022), this has the advantage that the choice of the identification function $\varphi$ in (S4) is actually irrelevant. Indeed, suppose one considers $\varphi' : \mathbb{R}^d \times \Xi \to \mathbb{R}^k$ rather than $\varphi$ in (S4). This means there is a matrix-valued function $h: \text{int}(\Xi) \to \mathbb{R}^{k \times k}$ of full rank such that $\varphi'(y,m(x,\theta)) = h(m(x,\theta))\varphi(y,m(x,\theta))$.
Then we can use the matrix $A'(x,\theta) = A(x,\theta)(h(m(x,\theta)))^{-1}$ such that $A'(x,\theta)\varphi'(y,m(x,\theta)) =
\(A(x, \theta) \varphi(y, m(x, \theta))\). Consequently, it is no loss of generality to fix a certain strict \(F_{Y \mid X}\)-identification function \(\varphi\) for \(\Gamma\) since the remaining flexibility can always be captured through the choice of the instrument matrix \(A\).

The following Proposition S2 provides a sufficient condition on the instrument matrix \(A(X, \theta)\) such that \(A(X, \theta) \varphi(Y, m(X, \theta))\) becomes a strict identification function for \(\theta_0\).

**Proposition S2.** Under Assumption 1, let \(\varphi: \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k\) be a strict \(F_{Y \mid X}\)-identification function. Let \(A: \mathbb{R}^p \times \Theta \to \mathbb{R}^{p \times k}\) be an instrument matrix such that the matrix \(E[A(X, \theta)D(X, \theta')]\) has full rank, where \(D(X, \theta') = \nabla_{\theta} E[\varphi(Y, m(X, \theta)) | X] |_{\theta = \theta'}\) for all \((Y, X) \in \mathcal{Z}\) and for all \(\theta, \theta' \in \Theta\) such that there is a \(\lambda \in [0, 1]\) with \(\theta' = (1 - \lambda)\theta_0 + \lambda\theta_0\). Then \(A(x, \theta) \varphi(y, m(x, \theta))\) is a strict unconditional \(F_Z\)-identification function for \(\theta_0\).

The following proposition takes the angle of ‘reverse engineering’ establishing a counterpart to Theorem 1 (iv): When an unconditional strict identification function is of the form (S4) it establishes a sufficient condition on the instrument matrix \(A\) to ensure that \(\varphi\) is a conditional strict identification function.

**Proposition S3.** Suppose that \(q \geq k\) and that Assumptions 1 and 3 hold. Moreover, assume that for all \(Z = (Y, X) \in \mathcal{Z}\) with correctly specified parameter \(\theta_0\) the map \(A: \mathbb{R}^p \times \Theta \to \mathbb{R}^{q \times k}\) satisfies

\[
\mathbb{P}\left( \text{rank}(A(X, \theta_0(F_Z))) = k \right) = 1. \tag{S6}
\]

If \(\psi_A: \mathbb{R}^d \times \mathbb{R}^p \times \Theta \to \mathbb{R}^q, \psi_A(y, x, \theta) = A(x, \theta)\psi(y, x, \theta)\) is a strict unconditional \(F_Z\)-identification function for \(\theta_0\): \(F_Z \to \Theta\), then \(\psi: \mathbb{R}^d \times \mathbb{R}^p \times \Theta \to \mathbb{R}^k\) is a strict conditional \(F_Z\)-identification function for \(\theta_0\).

The following Example S1 illustrates possible choices of instrument matrices in a linear mean regression context. In particular, it shows that—in certain situations—we could relax the assumption that \(A(X, \theta_0)\) needs to have full rank almost surely. However, relaxing the assumptions of Proposition S3 does not seem to be a fruitful direction from our point of view, because one would need to tailor the relaxed assumptions almost on a case by case basis.

**Example S1.** Let \(d = k = 1\), \(q = p \geq 1\) and \(\mathcal{X}\) be such that \(E[XX^\top]\) has full rank for all \(X \in \mathcal{X}\). Then define

\[
\mathcal{Z} = \{(m(X, \theta_0) + \varepsilon, X) \mid X \in \mathcal{X}, \ \theta_0 \in \Theta = \mathbb{R}^q, \ E[\varepsilon | X] = 0\},
\]

where \(m(X, \theta) = X^\top \theta\). Clearly, \(\Gamma(F_{Y \mid X}) := \int_y dF_{Y \mid X}(y) = X^\top \theta_0\). The condition that \(E[XX^\top]\) has full rank implies that the model \(m\) uniquely identifies the parameter \(\theta_0\) such that Assumption 1 holds. Indeed, for any \(\theta' \neq \theta\) it holds that \(0 < (\theta - \theta')^\top E[XX^\top] (\theta - \theta') = E[||m(X, \theta) - m(X, \theta')||^2]\). Employing the canonical identification function for the mean, we obtain a strict conditional \(F_Z\)-identification function \(\varphi(Y, m(X, \theta)) = m(X, \theta) - Y = X^\top \theta - Y\). Indeed \(E[\varphi(Y, m(X, \theta)) | X] = X^\top (\theta - \theta_0)\). If \(\theta \neq \theta_0\) and \(X^\top (\theta - \theta_0)\) were zero a.s., then \(E[XX^\top](\theta - \theta_0) = 0\), violating the full rank property of \(E[XX^\top]\). However, \(\varphi(Y, m(X, \theta))\) is in general not a strict unconditional \(F_Z\)-identification function for \(\theta_0\). It could be the case,
for example, that there is an \( X \in \mathcal{X} \) with \( \mathbb{E}[X] = 0 \) such that we obtain \( \mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta))] = \mathbb{E}[X^\top(\theta - \theta_0)] = 0 \) for all \( \theta \).

Now, choosing an instrument matrix \( A(X,\theta) \) such that \( \mathbb{E}[A(X,\theta)X^\top] \) has full rank for all \( \theta \) is a sufficient and necessary condition to ensure that \( \psi_A(Y,X,\theta) = A(X,\theta)\varphi(Y,\mathcal{A}(X,\theta)) \) is a strict unconditional \( \mathcal{F}_Z \)-identification function for \( \theta_0 \). Indeed, we obtain \( \mathbb{E}[\psi_A(Y,X,\theta)] = \mathbb{E}[A(X,\theta)X^\top](\theta - \theta_0) \). In particular, the choice \( A(X,\theta) = X \) yields a strict unconditional \( \mathcal{F}_Z \)-identification function for \( \theta_0 \). \( \square \)

S3. Proofs

**Proof of Proposition S1.** Part (i) is a direct application of the definitions, using similar arguments to the ones in the proof of Theorem 1 (i). For part (ii) we have under Assumption 1 that
\[ 0 = \mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta_0))|X] = \mathbb{E}[\varphi(Y,\Gamma(F_{Y|X}))|X]. \]

**Proof of Proposition S2.** The tower property implies \( \mathbb{E}[A(X,\theta_0)\varphi(Y,\mathcal{A}(X,\theta_0))] = 0 \). For \( \theta \neq \theta_0 \) the mean value theorem yields
\[
\mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta))|X] = \mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta_0))|X] - \mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta_0))|X]
= \nabla_\theta \mathbb{E}[\varphi(Y,\mathcal{A}(X,\theta_0))|X]|_{\theta=\theta'}(\theta - \theta_0) = D(X,\theta')(\theta - \theta_0)
\]
Therefore \( \mathbb{E}[A(X,\theta)\varphi(Y,\mathcal{A}(X,\theta_0))] = \mathbb{E}[A(X,\theta)D(X,\theta')(\theta - \theta_0)] \neq 0. \) \( \square \)

**Proof of Proposition S3.** Assume that \( \psi \) is not a strict conditional \( \mathcal{F}_Z \)-model-identification function for \( \theta_0 \). That means there is \( Z = (Y,X) \in \mathcal{Z} \) with true model parameter \( \theta_0 = \theta_0(F_Z) \) such that
\[ \mathbb{P}(\mathbb{E}[\psi(Y,X,\theta_0)|X] \neq 0) > 0 \] (S7)

or
\[ \exists \theta' \neq \theta_0: \mathbb{E}[\psi(Y,X,\theta')|X] = 0 \quad \text{a.s.} \quad (S8) \]

If (S8) holds, then we can directly apply the tower property to obtain that for some \( \theta' \neq \theta_0 \) we have \( \mathbb{E}[\psi_A(Y,X,\theta')] = \mathbb{E}[A(X,\theta')\mathbb{E}[\psi(Y,X,\theta')|X]] = 0 \), which means that \( \psi_A \) is not a strict unconditional \( \mathcal{F}_Z \)-identification function for \( \theta_0 \). Now, we assume that (S7) holds. Then using (S6) we can conclude that \( \mathbb{P}(\mathbb{E}[\psi_A(Y,X,\theta_0)|X] \neq 0) = \mathbb{P}(A(X,\theta_0)\mathbb{E}[\psi(Y,X,\theta_0)|X] \neq 0) \geq \mathbb{P}(\{\text{rank}(A(X,\theta_0)) = k\} \cap \{\mathbb{E}[\psi(Y,X,\theta_0)|X] \neq 0\}) > 0 \). Then, we can again argue that there exists a component \( j \in \{1,\ldots,q\} \) such that \( \mathbb{P}(\mathbb{E}[\psi_{A,j}(Y,X,\theta_0)|X] < 0) > 0 \) or \( \mathbb{P}(\mathbb{E}[\psi_{A,j}(Y,X,\theta_0)|X] > 0) > 0 \), and we continue as in the proof of Theorem 1 (iv). \( \square \)