Projective Group Algebras

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ABSTRACT

In this paper we apply a recently proposed algebraic theory of integration to projective group algebras. These structures have received some attention in connection with the compactification of the $M$ theory on noncommutative tori. This turns out to be an interesting field of applications, since the space $\hat{G}$ of the equivalence classes of the vector unitary irreducible representations of the group under examination becomes, in the projective case, a prototype of noncommuting spaces. For vector representations the algebraic integration is equivalent to integrate over $\hat{G}$. However, its very definition is related only at the structural properties of the group algebra, therefore it is well defined also in the projective case, where the space $\hat{G}$ has no classical meaning. This allows a generalization of the usual group harmonic analysis. A particular attention is given to abelian groups, which are the relevant ones in the compactification problem, since it is possible, from the previous results, to establish a simple generalization of the ordinary calculus to the associated noncommutative spaces.

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1 Introduction

A very large class of algebras is obtained by considering the projective representations of an arbitrary group $G$. These algebras have been recently investigated in relation with the problem of compactification within the $M$-theory \cite{1,2}. In fact, to deal with this problem one needs to extend the matrix valued coordinates of the $D0$ branes, $X_{i_1i_2}^\mu$, $i_1, i_2 = 1, \ldots, N$, $\mu = 0, 1, \ldots, D$, where $D+1$ is the number of space-time dimensions, to matrices, $X_{(i_1,a_1)(i_2,a_2)}^\mu$, where $a_1$ and $a_2$ are elements of a group (in general discrete) $G$ of euclidean motions in the subspace $\mathbb{R}^n \in \mathbb{R}^D$, $n < D$, to be compactified \cite{4}. The compactification condition on the corresponding coordinates becomes a requirement of symmetry of the theory under the unitary transformation

$$U^{-1}(a)X^\mu U(a) = X^\mu + d^\mu_a$$

(1.1)

The quantity $d^\mu_a$ is the translation which defines the lattice structure of $\mathbb{R}^n$ induced by $G$. In this way, the theory is effectively defined on the coset space $\mathbb{R}^n/G$ (for instance, for $G = \mathbb{Z}^n$, the $n$-dimensional torus). The matrix $U(a)$ acts on the group indices of $X^\mu$, and must satisfy the group multiplication rule

$$U(a)U(b) = e^{ia(a,b)}U(ab)$$

(1.2)

Therefore, the matrices $U(a)$ are in the regular projective representation of the group $G$, that is they belong to the algebra of the right (or left) multiplications. Besides the interest of projective group algebras in the previous problem, they are an excellent ground for studying the nature of the algebraic theory of integration (ATI) that we have recently considered in ref \cite{4}. Strictly speaking a group algebra is defined by taking formal linear combinations of the group elements with coefficients in a given field $F$ (here we will take $F = \mathbb{C}$). A more convenient way to do this construction, is to consider in place of the abstract group $G$, a linear vector representation, $x_\lambda(a)$, $a \in G$, where $\lambda$ specifies which representation we are using. The linear combinations of $x_\lambda(a)$ span a vector space $\mathcal{A}(G)_\lambda$, which has also an algebra structure induced by the group product

$$x_\lambda(a)x_\lambda(b) = x_\lambda(ab)$$

(1.3)

In these cases it is convenient to introduce the set $\hat{G}$ of the equivalence classes of the unitary irreducible representations of $G$ (in the following, for simplicity,
we will consider only discrete and compact groups). From this point of view, an arbitrary function on the algebra

\[ \hat{f}(\lambda) = \sum_{a \in G} f(a) x_\lambda(a) \] (1.4)
can be regarded as follows: the quantities \( f(a) \) define a mapping from \( G \) to \( \mathbb{C} \) and eq. (1.4) can be thought to define the Fourier transform of \( f(g) \) with respect to the representation \( x_\lambda \). At each point \( \lambda \in \hat{G} \), \( \hat{f}(\lambda) \) takes values in \( \mathcal{A}(G)_\lambda \). Therefore, the concept of group algebra is strictly related to the harmonic analysis over a group. From an algebraic point of view, the elements \( x_\lambda(a) \) can be seen as the generators of the algebra, since any element of \( \mathcal{A}(G)_\lambda \) can be expressed as a linear combination of them. This scheme can be extended to the case of projective representations, where the only difference is the product rule (1.3) which is modified by a phase factor (since we are considering unitary representations)

\[ x_\lambda(a)x_\lambda(b) = e^{i\alpha(a,b)}x_\lambda(ab) \] (1.5)

\( \alpha(a,b) \) is called a cocycle. In this case we will speak of projective group algebras. The structure of these algebras is rather simple, but also very rich, and therefore it is possible to use them as a laboratory to study the properties of the ATI. In fact, we will show that this theory allows us to invert eq. (1.4). This means that, in the vector case, the integration defined by the ATI is equivalent to "sum", or "integrate" over all the unitary irreducible representations of \( G \). In other words it defines an integration measure in the space \( \hat{G} \). This is particularly simple in the case of the abelian groups, where \( \hat{G} \) is also an abelian group (the dual of \( G \) in the Pontryagin sense \[ \mathbb{R} \]). For instance, for \( G = \mathbb{R} \), we have \( \hat{G} = \mathbb{R} \), and the integral defined by the ATI becomes the Lebesgue integral over \( \hat{G} = \mathbb{R} \). However the situation is completely different in the case of projective representations. In fact, already for abelian groups, the space \( \hat{G} \) becomes a noncommutative space. By this we mean the following: a vector representation of \( G = \mathbb{R}^D \)

\[ x_q(\bar{a}) = e^{-i\bar{q} \cdot \bar{a}}, \quad \bar{a} \in \mathbb{R}^D, \quad \bar{q} \in \hat{G} = \mathbb{R}^D \] (1.6)
can be extended to a projective representation of \( \mathbb{R}^D \), at the price of considering \( \bar{q} \) not as numbers, but operators with non-vanishing commutation relations

\[ [q_i, q_j] = i\eta_{ij} \] (1.7)
where $\eta_{ij}$ is related to the cocycle which characterizes the projective representation. In this way, the space $\hat{G}$ looses its classical features and becomes an example of noncommutative space. At the same time the ordinary notion of integration over this space is meaningless. On the other hand, the ATI is based only on the algebraic properties of the projective group algebra $\mathcal{A}(G)$ and not on the representation $x_\lambda$ chosen to define it. In fact it depends only on the properties of the regular representation. In the classical case the integration through functions on $\hat{G}$ (the representations $x_\lambda$) allows the reconstruction of the integration over $\hat{G}$. In the noncommuting case the definition remains valid also if the "space" $\hat{G}$ has no meaning, since we are using the space $\mathcal{A}(G)$ for our construction.

The paper is organized as follows. In Section 2 we establish the integration properties for a projective group algebra, following the prescriptions of the ATI. To this end we need to determine a set of relations for the cocycles of the algebra that are necessary to obtain the integration rules. Section 3 is devoted to understand the meaning of the ATI, by examining the special case of vector representations. Here we prove that for discrete and compact groups, the ATI corresponds to integrate over the space $\hat{G}$, in such a way to provide the inversion formula for the Fourier transformation on the group, and to ensure the validity of the Plancherel formula. We discuss also how, in the case of projective representations, the Fourier analysis is generalized through the use of the ATI. The space $\hat{G}$ looses its meaning, but the ATI is still valid, and therefore it can be seen as a theory of integration over a noncommutative space $\hat{G}$. In Section 4 we consider the case of projective representations of abelian groups. In particular we compare our previous Fourier analysis with the more conventional one made in terms of the characters of the abelian groups. In this case, the Moyal product among the functions on the group $G$ arises in a very natural way. But this is avoided in our formulation, since we use the projective representations for the expansion rather than the characters. We show also that it is possible to introduce a derivation on the projective group algebra for abelian groups, such that its integral (in the sense of the ATI) is zero. This allows to extend the standard calculus to this noncommutative case. Explicit formulas are given for $G = \mathbb{R}^D$, $G = \mathbb{Z}^D$, and $G = \mathbb{Z}_n^D$. 

4
2 Projective group algebras

Let us start defining a projective group algebra. We consider an arbitrary projective linear representation, \( a \rightarrow x(a), a \in G, x(a) \in \mathcal{A}(G) \), of a given group \( G \). The representation \( \mathcal{A}(G) \) defines in a natural way an associative algebra with identity (it is closed under multiplication and it defines a generally complex vector space). This algebra will be denoted by \( \mathcal{A}(G) \). The elements of the algebra are given by the combinations

\[
\sum_{a \in G} f(a)x(a)
\] (2.1)

For a group with an infinite number of elements, there is no a unique definition of such an algebra. The one defined in eq. (2.1) corresponds to consider a formal linear combination of a finite number of elements of \( G \). This is very convenient because we will not be concerned here with topological problems. Other definitions correspond to take complex functions on \( G \) such that

\[
\sum_{a \in G} |f(g)| < \infty
\] (2.2)

Or, in the case of compact groups, the sum is defined in terms of the Haar invariant measure. When necessary we will be more precise about this point. The basic product rule of the algebra follows from the group property

\[
x(a)x(b) = e^{i\alpha(a,b)}x(ab)
\] (2.3)

where \( \alpha(a,b) \) is called a cocycle. This is constrained, by the requirement of associativity of the representation, to satisfy

\[
\alpha(a,b) + \alpha(ab,c) = \alpha(b,c) + \alpha(a,bc)
\] (2.4)

Changing the element \( x(a) \) of the algebra by a phase factor \( e^{i\phi(a)} \), that is, defining

\[
x'(a) = e^{-i\phi(a)}x(a)
\] (2.5)

we get

\[
x'(a)x'(b) = e^{i(\alpha(a,b)-\phi(ab)+\phi(a)+\phi(b))}x'(ab)
\] (2.6)

This is equivalent to change the cocycle to

\[
\alpha'(a,b) = \alpha(a,b) - [\phi(ab) - \phi(a) - \phi(b)]
\] (2.7)
In particular, if \(\alpha(a, b)\) is of the form \(\phi(ab) - \phi(a) - \phi(b)\), it can be transformed to zero, and therefore the corresponding projective representation is isomorphic to a vector one. For this reason the combination

\[
\alpha(a, b) = \phi(ab) - \phi(a) - \phi(b)
\]  

is called a trivial cocycle. Let us now discuss some properties of the cocycles. We start from the relation (\(e\) is the identity element of \(G\))

\[
x(e)x(e) = e^{i\alpha(e,e)}x(e)
\]  

By the transformation \(x'(e) = e^{-i\alpha(e,e)}x(e)\), we get

\[
x'(e)x'(e) = x'(e)
\]  

Therefore we can assume

\[
\alpha(e, e) = 0
\]  

Then, from

\[
x(e)x(a) = e^{i\alpha(e,a)}x(a)
\]  

multiplying by \(x(e)\) to the left, we get

\[
x(e)x(a) = e^{i\alpha(e,a)}x(e)x(a)
\]  

implying

\[
\alpha(e, a) = \alpha(a, e) = 0
\]  

where the second relation is obtained in analogous way. Now, taking \(c = b^{-1}\) in eq. (2.4), we get

\[
\alpha(a, b) + \alpha(ab, b^{-1}) = \alpha(b, b^{-1})
\]  

Again, putting \(a = b^{-1}\)

\[
\alpha(b^{-1}, b) = \alpha(b, b^{-1})
\]  

We can go farther by considering

\[
x(a)x(a^{-1}) = e^{i\alpha(a,a^{-1})}x(e)
\]  

and defining

\[
x'(a) = e^{-i\alpha(a,a^{-1})/2}x(a)
\]
from which
\[ x'(a)x'(a^{-1}) = e^{-i\alpha(a,a^{-1})}x(a)x(a^{-1}) = x(e) = x'(e) \quad (2.19) \]
Therefore we can transform \( \alpha(a, a^{-1}) \) to zero without changing the definition of \( x(e) \),
\[ \alpha(a, a^{-1}) = 0 \quad (2.20) \]
As a consequence, equation (2.15) becomes
\[ \alpha(a, b) + \alpha(ab, b^{-1}) = 0 \quad (2.21) \]
We can get another relation using \( x(a^{-1}) = x(a)^{-1} \)
\[ x(a^{-1})x(b^{-1}) = e^{i\alpha(a^{-1},b^{-1})}x(a^{-1}b^{-1}) = x(a)^{-1}x(b)^{-1} = (x(b)x(a))^{-1} = e^{-i\alpha(b,a)}x(a^{-1}b^{-1}) \quad (2.22) \]
from which
\[ \alpha(a^{-1}, b^{-1}) = -\alpha(b, a) \quad (2.23) \]
and together with eq. (2.21) we get
\[ \alpha(ab, b^{-1}) = \alpha(b^{-1}, a^{-1}) \quad (2.24) \]
This relation will be very useful in the following. From the product rule
\[ x(a)x(b) = e^{i\alpha(a,b)}x(ab) = \sum_{c\in G} f_{abc}x(c) \quad (2.25) \]
we get the structure constants of the algebra
\[ f_{abc} = \delta_{ab,c}e^{i\alpha(a,b)} \quad (2.26) \]
The delta function is defined according to the nature of the sum over the group elements.

In order to define an integration over \( A(G) \), according to the rules of the ATI, we start introducing a ket with elements given by \( x(a) \), that is \( |x\rangle_a = x(a) \), and the corresponding transposed bra \( \langle x| \). Then, we define left and right multiplications as
\[ R(a)|x\rangle = |x\rangle x(a), \quad \langle x|L(a) = x(a)\langle x| \quad (2.27) \]
where we have used the shorthand notation \( R(x(a)) = R(a) \), and similarly for \( L(a) \). From the algebra product, we get immediately

\[
(R(a))_{bc} = f_{bac} = \delta_{ba,c} e^{i\alpha(b,a)}, \quad (L(a))_{bc} = f_{acb} = \delta_{ac,b} e^{i\alpha(a,c)}
\] (2.28)

A self-conjugated algebra is defined as one equipped with a matrix \( C \) such that

\[
C^T = C, \quad CR(a)C^{-1} = L(a)
\] (2.29)

The second equation tells us that the bra \( \langle xC | \) is an eigenstate of \( R(a) \)

\[
\langle xC | R(a) = x(a) \langle xC |
\] (2.30)

with eigenvalue \( x(a) \), and

\[
(xC)_a = x(b) C_{ba}
\] (2.31)

Therefore, the matrix \( C \) can be determined by solving eq. (2.30). We get

\[
(xC)_b \delta_{ba,c} e^{i\alpha(b,a)} = (xC)_{ca^{-1}} e^{i\alpha(ca^{-1},a)} = x(a) (xC)_c
\] (2.32)

By putting

\[
(xC)_a = k_a x(a^{-1})
\] (2.33)

we obtain

\[
k_{ca^{-1}} x(ac^{-1}) e^{i\alpha(ca^{-1},a)} = k_c e^{i\alpha(a,c^{-1})} x(ac^{-1})
\] (2.34)

Then, from eq. (2.24)

\[
k_{ca^{-1}} = k_c
\] (2.35)

Therefore \( k_a = k_c \), and assuming \( k_c = 1 \), it follows

\[
(xC)_a = x(a^{-1}) = x(a)^{-1}
\] (2.36)

giving

\[
C_{a,b} = \delta_{ab,e}
\] (2.37)

This shows also that

\[
C^T = C
\] (2.38)

at least in the cases of discrete and compact groups. The mapping \( C : \mathcal{A} \to \mathcal{A} \) is an involution of the algebra. In fact, by defining

\[
x(a)^* = x(b) C_{b,a} = x(a^{-1}) = x(a)^{-1}
\] (2.39)
we have \((x(a))^* = x(a)\), and \(x(b)^* x(a)^* = (x(a)x(b))^*\). In refs. \[4, 6\] we have shown that for a self-conjugated algebra, it is possible to define an integration rule in terms of the matrix \(C\)

\[
\int_{(x)} x(a) = C^{-1}_{e,a} = \delta_{e,a} \quad (2.40)
\]

From this definition and the eq. \((2.29)\) it follows \[4, 6\]

\[
\int_{(x)} |x\rangle\langle xC| = \int_{(x)} |xC\rangle\langle x| = 1 \quad (2.41)
\]

Therefore we are allowed to expand a function on the group \(|f\rangle_a = f(a)\) as

\[
f(a) = \int_{(x)} x(a^{-1})\langle x|f\rangle \quad (2.42)
\]

with \(\langle x|f\rangle = \sum_{b\in G} x(b)f(b)\). It is also possible to define a scalar product among functions on the group. Defining, \(\langle f|a = \bar{f}(a)\), where \(\bar{f}(a)\) is the complex conjugated of \(f(a)\), we put

\[
\langle f|g \rangle = \int_{(x)} \langle f|xC\rangle\langle x|g \rangle = \sum_{a\in G} f^*(a)g(a) \quad (2.43)
\]

with

\[
f^*(a) = \langle f|xC \rangle = \bar{f}(a^{-1}) \quad (2.44)
\]

It is important to stress that this definition depends only on the algebraic properties of \(A(G)\) and not on the specific representation chosen for this construction.

3 What is the meaning of the algebraic integration?

As we have said in the previous Section, the integration formula we have obtained is independent on the group representation we started with. In fact, it is based only on the structure of right and left multiplications, that is on the abstract algebraic product. This independence on the representation suggests that in some way we are "summing" over all the representations. To
understand this point, we will study in this Section vector representations. To do that, let us introduce a label $\lambda$ for the vector representation we are actually using to define $\mathcal{A}(G)$. Then a generic function on $\mathcal{A}(G)_\lambda$

$$\hat{f}(\lambda) = \sum_{a \in G} f(a)x_\lambda(a) \quad (3.1)$$

can be thought as the Fourier transform of the function $f : G \rightarrow \mathbb{C}$. Using the algebraic integration we can invert this expression (see eq. (2.42))

$$f(a) = \int_{(x_\lambda)} \hat{f}(\lambda)x_\lambda(a^{-1}) \quad (3.2)$$

But it is a well known result of the harmonic analysis over the groups, that in many cases it is possible to invert the Fourier transform, by an appropriate sum over the representations. This is true in particular for finite and compact groups. Therefore the algebraic integration should be the same thing as summing or integrating over the labels $\lambda$ specifying the representation. In order to show that this is the case, let us recall a few facts about the Fourier transform over the groups [5]. First of all, given the group $G$, one defines the set $\hat{G}$ of the equivalence classes of the irreducible representations of $G$. Then, at each point $\lambda$ in $\hat{G}$ we choose a unitary representation $x_\lambda$ belonging to the class $\lambda$, and define the Fourier transform of the function $f : G \rightarrow \mathbb{C}$, by the eq. (3.1). In the case of compact groups, instead of the sum over the group element one has to integrate over the group by means of the invariant Haar measure. For finite groups, the inversion formula is given by

$$f(a) = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\hat{f}(\lambda)x_\lambda(a^{-1})] \quad (3.3)$$

where $n_G$ is the order of the group and $d_\lambda$ the dimension of the representation $\lambda$. Therefore, we get the identification

$$\int_{(x)} \{\cdots\} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_\lambda tr[\{\cdots\}] \quad (3.4)$$

A more interesting way of deriving this relation, is to take in (3.1), $f(a) = \delta_{e,a}$, obtaining for its Fourier transform, $\hat{\delta} = x_\lambda(e) = 1_\lambda$, where the last symbol
means the identity in the representation $\lambda$. By inserting this result into (3.3) we get the identity

$$\delta_{e,a} = \frac{1}{n_G} \sum_{\lambda \in \hat{G}} d_{\lambda} tr[\hat{x}_\lambda(a^{-1})]$$

(3.5)

which, compared with eq. (2.40), gives (3.4). This shows explicitly that the algebraic integration for vector representations of $G$ is nothing but the sum over the representations of $G$.

An analogous relation is obtained in the case of compact groups. This can also be obtained by a limiting procedure from finite groups, if we insert $1/n_G$, the volume of the group, in the definition of the Fourier transform. That is one defines

$$\hat{f}(\lambda) = \frac{1}{n_G} \sum_{a \in G} f(a) x_\lambda(a)$$

(3.6)

from which

$$f(a) = \sum_{\lambda \in G} d_{\lambda} tr[\hat{f}(\lambda)x_\lambda(a^{-1})]$$

(3.7)

Then one can go to the limit by substituting the sum over the group elements with the Haar measure

$$\hat{f}(\lambda) = \int_G d\mu(a) f(a) x_\lambda(a)$$

(3.8)

The inversion formula (3.7) remains unchanged. We see that in these cases the algebraic integration sums over the elements of the space $\hat{G}$, and therefore it can be thought as the dual of the sum over the group elements (or the Haar integration for compact groups). By using the Fourier transform (3.1) and its inversion (3.2), one can easily establish the Plancherel formula. In fact by multiplying together two Fourier transforms, one gets

$$\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a \in G} \left( \sum_{b \in G} f_1(b)f_2(b^{-1}a) \right) x_\lambda(a)$$

(3.9)

from which

$$\int_G \hat{f}_1(\lambda)\hat{f}_2(\lambda)x_\lambda(a^{-1}) = \sum_{b \in G} f_1(b)f_2(b^{-1}a)$$

(3.10)

and taking $a = e$ we obtain

$$\int_G \hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{b \in G} f_1(b)f_2(b^{-1})$$

(3.11)
This formula can be further specialized, by taking \( f_2 \equiv f \) and for \( f_1 \) the involuted of \( f \). That is

\[
\hat{f}^*(\lambda) = \sum_{a \in G} \bar{f}(a)x_\lambda(a^{-1})
\]  

(3.12)

where use has been made of eq. (2.39). Then, from eq. (3.11) we get the Plancherel formula

\[
\int (x) \hat{f}^*(\lambda) \hat{f}(\lambda) = \sum_{a \in G} \bar{f}(a)f(a)
\]  

(3.13)

Let us also notice that eq. (3.9) says that the Fourier transform of the convolution of two functions on the group is the product of the Fourier transforms.

We will consider now projective representations. In this case, the product of two Fourier transforms is given by

\[
\hat{f}_1(\lambda)\hat{f}_2(\lambda) = \sum_{a \in G} h(a)x_\lambda(a)
\]  

(3.14)

with

\[
h(a) = \sum_{b \in G} f_1(b)f_2(b^{-1}a)e^{i\alpha(b,b^{-1}a)}
\]  

(3.15)

Therefore, for projective representations, the convolution product is deformed due to the presence of the phase factor. However, the Plancherel formula still holds. In fact, since in

\[
h(e) = \sum_{b \in G} f_1(b)f_2(b^{-1})
\]  

(3.16)

using eq. (2.20), the phase factor disappears, the previous derivation from eq. (3.11) to eq. (3.13) is still valid. Notice that eq. (3.14) tells us that the Fourier transform of the deformed convolution product of two functions on the group, is equal to the product of the Fourier transforms.

4 The case of abelian groups

In this Section we consider the case of abelian groups, and we compare the Fourier analysis made in the framework of the ATI with the more conventional one made in terms of the characters. A fundamental property of the
abelian groups is that the set \( \hat{G} \) of their vector unitary irreducible representations (VUIR), is itself an abelian group, the dual of \( G \) (in the sense of Pontryagin [5]). Since the VUIR’s are one-dimensional, they are given by the characters of the group. We will denote the characters of \( G \) by \( \chi_\lambda(a) \), where \( a \in G \), and \( \lambda \) denotes the representation of \( G \). For what we said before, the parameters \( \lambda \) can be thought as the elements of the dual group. The parameterization of the group element \( a \) and of the representation label \( \lambda \) are given in Table 1, for the most important abelian groups and for their dual groups, where we have used the notation \( a = \vec{a} \) and \( \lambda = \vec{q} \).

The characters are given by
\[
\chi_\lambda(a) \equiv \chi_\vec{q}(\vec{a}) = e^{-i\vec{q} \cdot \vec{a}}
\] (4.1)
and satisfy the relation (here we use the additive notation for the group operation)
\[
\chi_\lambda(a + b) = \chi_\lambda(a)\chi_\lambda(b)
\] (4.2)
and the dual

\[ \chi_{\lambda_1 + \lambda_2}(a) = \chi_{\lambda_1}(a)\chi_{\lambda_2}(a) \]  

That is they define vector representations of the abelian group \( G \) and of its dual, \( \hat{G} \). Also we can easily check that the operators

\[ D_\vec{q}\chi_\vec{q}(\vec{a}) = -i\vec{a}\chi_\vec{q}(\vec{a}) \]  

are derivations on the algebra \( \{4.2\} \) of the characters for any \( G \) in Table 1.

We can use the characters to define the Fourier transform of the function \( f(g) : G \to \mathbb{C} \)

\[ \tilde{f}(\lambda) = \sum_{a \in G} f(a)\chi_\lambda(a) \]  

If we evaluate the Fourier transform of the deformed convolution of eq. \( \{3.13\} \), we get

\[ \tilde{h}(\lambda) = \sum_{a \in G} h(a)\chi_\lambda(a) = \sum_{a,b \in G} f(a)\chi_\lambda(a)e^{i\alpha(a,b)}g(b)\chi_\lambda(b) \]  

In the case of vector representations the Fourier transform of the convolution is the product of the Fourier transforms. In the case of projective representations, the result, using the derivation introduced before, can be written in terms of the Moyal product (we omit here the vector signs)

\[ \tilde{h}(\lambda) = \tilde{f}(\lambda) \star \tilde{g}(\lambda) = e^{-i\alpha(D_{\lambda'},D_{\lambda''})}\tilde{f}(\lambda')\tilde{g}(\lambda'') \bigg|_{\lambda' = \lambda'' = \lambda} \]  

Therefore, the Moyal product arises in a very natural way from the projective group algebra. On the other hand, we have shown in the previous Section, that the use of the Fourier analysis in terms of the projective representations avoids the Moyal product. The projective representations of abelian groups allow a derivation on the algebra, analogous to the one in eq. \( \{4.4\} \), with very special features. In fact we check easily that

\[ \vec{D}_\lambda(\vec{a}) = -i\vec{a}\chi_\lambda(\vec{a}) \]  

is a derivation, and furthermore

\[ \int_{\chi_\lambda} \vec{D}_\lambda(\vec{a}) = 0 \]
From this it follows, by linearity, that the integral of $\vec{D}$ applied to any function on the algebra is zero

$$\int_{(x,\lambda)} \vec{D} \left( \sum_{\vec{a} \in \mathcal{G}} f(\vec{a}) x_{\lambda}(\vec{a}) \right) = 0 \quad (4.10)$$

This relation is very important because, as we have shown in [6], the automorphisms generated by $\vec{D}$, that is $\exp(\vec{a} \cdot \vec{D})$, leave invariant the integration measure of the ATI (see also later on). Notice that this derivation generalizes the derivative with respect to the parameter $\vec{q}$, although this has no meaning in the present case. In the case of nonabelian groups, a derivation sharing the previous properties can be defined only if there exists a mapping $\sigma : \mathcal{G} \rightarrow \mathbb{C}$, such that

$$\sigma(ab) = \sigma(a) + \sigma(b), \quad a, b \in \mathcal{G} \quad (4.11)$$

since in this case, defining

$$D x(a) = \sigma(a) x(a) \quad (4.12)$$

we get

$$D(x(a)x(b)) = \sigma(ab)x(a)x(b) = (\sigma(a) + \sigma(b))x(a)x(b) = (Dx(a))x(b) + x(a)(Dx(b)) \quad (4.13)$$

Having defined derivations and integrals one has all the elements for the harmonic analysis on the projective representations of an abelian group.

Let us start considering $\mathcal{G} = \mathbb{R}^D$. In the case of vector representations we have

$$x_{\vec{q}}(\vec{a}) = e^{-i\vec{q} \cdot \vec{a}} \quad (4.14)$$

with $\vec{a} \in \mathcal{G}$, and $\vec{q} \in \hat{\mathcal{G}} = \mathbb{R}^D$ labels the representation. The Fourier transform is

$$\hat{f}(\vec{q}) = \int d^D \vec{a} f(\vec{a}) e^{-i\vec{q} \cdot \vec{a}} \quad (4.15)$$

Here the Haar measure for $\mathcal{G}$ coincides with the ordinary Lebesgue measure. Also, since $\hat{\mathcal{G}} = \mathbb{R}^D$, we can invert the Fourier transform by using the Haar measure on the dual group, that is, again the Lebesgue measure. In the projective case, eq. (4.14) still holds true, if we assume $\vec{q}$ as a vector operator satisfying the commutation relations

$$[q_i, q_j] = i\eta_{ij} \quad (4.16)$$
with \( n_{ij} \) numbers which can be related to the cocycle, by using the Baker-Campbell-Hausdorff formula

\[
e^{-i\vec{q} \cdot \vec{a}} e^{-i\vec{q} \cdot \vec{b}} = e^{-in_{ij}a_ib_j/2} e^{-i\vec{q} \cdot (\vec{a} + \vec{b})}
\]

(4.17)
giving

\[
\alpha(\vec{a}, \vec{b}) = -\frac{1}{2} n_{ij}a_ib_j
\]

(4.18)
The inversion of the Fourier transform can now be obtained by the ATI in the form

\[
f(\vec{a}) = \int_{\hat{G}} \hat{f}(\vec{q}) x_{\vec{q}} (-\vec{a})
\]

(4.19)
where the dependence on the representation is expressed in terms of \( \vec{q} \), thought now they are not coordinates on \( \hat{G} \). We recall that in this case, eq. (2.40) gives

\[
\int_{\hat{G}} x_{\vec{q}} (\vec{a}) = \delta^D (\vec{a})
\]

(4.20)
Therefore, the relation between the integral in ATI and the Lebesgue integral in \( \hat{G} \), in the vector case is

\[
\int_{\hat{G}} = \int \frac{d^D \vec{q}}{(2\pi)^D}
\]

(4.21)
In the projective case the right hand side of this relation has no meaning, whereas the left hand side is still well defined. Also, we cannot maintain the interpretation of the \( q_i \)'s as coordinates on the dual space \( \hat{G} \). However, we can define elements of \( \mathcal{A}(G) \) having the properties of the \( q_i \)'s (in particular satisfying eq. (4.16)), by using the Fourier analysis. That is we define

\[
q_i = \int d^D \vec{a} \left( -i \frac{\partial}{\partial a_i} \delta^D (\vec{a}) \right) x_{\vec{q}} (\vec{a})
\]

(4.22)
which is an element of \( \mathcal{A}(G) \) obtained by Fourier transforming a distribution over \( G \), which is a honestly defined space. From this definition we can easily evaluate the product

\[
q_i x_{\vec{q}} (\vec{a}) = \int d^D \vec{b} \left( -i \frac{\partial}{\partial b_i} \delta^D (\vec{b}) \right) x_{\vec{q}} (\vec{b}) x_{\vec{q}} (\vec{a})
\]

(4.23)
Using the algebra and integrating by parts, one gets the result

$$q_i x_{\vec{q}}(\vec{a}) = i \nabla_i x_{\vec{q}}(\vec{a})$$  \hspace{1cm} (4.24)

where

$$\nabla_i = \frac{\partial}{\partial a_i} + i \alpha_{ij} a_j$$  \hspace{1cm} (4.25)

where $\alpha_{ij} = \alpha(\vec{e}_{(i)}, \vec{e}_{(j)})$, with $\vec{e}_{(i)}$ an orthonormal basis in $\mathbb{R}^D$. In a completely analogous way one finds

$$x_{\vec{q}}(\vec{a}) q_i = i \nabla_i x_{\vec{q}}(\vec{a})$$  \hspace{1cm} (4.26)

where

$$\nabla_i = \frac{\partial}{\partial a_i} - i \alpha_{ij} a_j$$  \hspace{1cm} (4.27)

Then, we evaluate the commutator

$$[q_i, \hat{f}(\vec{q})] = \int d^D \vec{a} \left[ -i \left( \nabla_i - \nabla_i \right) f(\vec{a}) \right] x_{\vec{q}}(\vec{a})$$  \hspace{1cm} (4.28)

where we have done an integration by parts. We get

$$[q_i, \hat{f}(\vec{q})] = -2i \alpha_{ij} D_{q_j} \hat{f}(\vec{q})$$  \hspace{1cm} (4.29)

where $D_{q_j}$ is the derivation (4.18), with $q_j$ a reminder for the direction along which the derivation acts upon. In particular, from

$$D_{q_j} q_i = \int d^D \vec{a} \left( -i \frac{\partial}{\partial a_i} \delta^D(\vec{a}) \right) (-ia_j) e^{-i \vec{q} \cdot \vec{a}} = \delta_{ij}$$  \hspace{1cm} (4.30)

we get

$$[q_i, q_j] = -2i \alpha_{ij}$$  \hspace{1cm} (4.31)

in agreement with eq. (4.10), after the identification $\alpha_{ij} = -\eta_{ij}/2$.

The automorphisms induced by the derivations (4.18) are easily evaluated

$$S(\vec{a}) x_{\vec{q}}(\vec{a}) = e^{\vec{a} \cdot D_{q} \vec{q}} x_{\vec{q}}(\vec{a}) = e^{-i \vec{a} \cdot \vec{q}} x_{\vec{q}}(\vec{a}) = x_{\vec{q} - \vec{a}}(\vec{a})$$  \hspace{1cm} (4.32)

where the last equality follows from

$$\int d^D \vec{a} \left( -i \frac{\partial}{\partial a_i} \delta^D(\vec{a}) \right) e^{\vec{a} \cdot D_{q} \vec{q}} (\vec{a}) = q_i + \alpha_i$$  \hspace{1cm} (4.33)
Meaning that in the vector case, $S(\vec{\alpha})$ induces translations in $\hat{G}$. Since $D_{\vec{q}}$ satisfies the eq. (4.10), it follows from [3] that the automorphism $S(\vec{\alpha})$ leaves invariant the algebraic integration measure

$$\int_{(\vec{q})} = \int_{(\vec{q} + \vec{\alpha})}$$

(4.34)

This shows that it is possible to construct a calculus completely analogous to the one that we have on $\hat{G}$ in the vector case, just using the Fourier analysis following by the algebraic definition of the integral. We can push this analysis a little bit further by looking at the following expression

$$\int_{(\vec{q})} \hat{f}(\vec{q}) \ q_i \ x_{\vec{q}} (-\vec{a}) = -i \left( \frac{\partial}{\partial a_i} + i\alpha_{ij}a_j \right) f(\vec{a})$$

(4.35)

where we have used eq. (4.24). In the case $D = 2$ this equation has a physical interpretation in terms of a particle of charge $e$, in a constant magnetic field $B$. In fact, the commutators among canonical momenta are

$$[\pi_i, \pi_j] = ieB\epsilon_{ij}$$

(4.36)

where $\epsilon_{ij}$ is the 2-dimensional Ricci tensor. Therefore, identifying $\pi_i$ with $q_i$, we get $\alpha_{ij} = -eB\epsilon_{ij}/2$. The corresponding vector potential is given by

$$A_i(\vec{a}) = -\frac{1}{2}\epsilon_{ij}Ba_j = \frac{1}{e}\alpha_{ij}a_j$$

(4.37)

Then, eq. (4.35) tells us that the operation $\hat{f}(\vec{q}) \rightarrow \hat{f}(\vec{q})q_i$, corresponds to take the covariant derivative

$$-i \frac{\partial}{\partial a_i} + eA_i(\vec{a})$$

(4.38)

of the inverse Fourier transform of $\hat{f}(\vec{q})$. An interesting remark is that a translation in $\vec{q}$ generated by $\exp(\vec{\alpha} \cdot \vec{D})$, gives rise to a phase transformation on $f(\vec{a})$. First of all, by using the invariance of the integration measure we can check that

$$\hat{f}(\vec{q} + \vec{\alpha}) = e^{\vec{\alpha} \cdot \vec{D}} \hat{f}(\vec{q})$$

(4.39)

In fact

$$\int_{(\vec{q})} \hat{f}(\vec{q} + \vec{\alpha}) x_{\vec{q}} (-\vec{a}) = \int_{(\vec{q} - \vec{\alpha})} \hat{f}(\vec{q}) x_{\vec{q} - \vec{a}} (-\vec{a}) = e^{-i\vec{\alpha} \cdot \vec{a}} f(\vec{a})$$

(4.40)
Then, we have

$$\int_{(\vec{q})} \left( e^{\vec{\alpha} \cdot \vec{D}} \hat{f}(\vec{q}) \right) x_{\vec{q}}(-\vec{a}) = \int_{(\vec{q})} \hat{f}(\vec{q}) \left( e^{-\vec{\alpha} \cdot \vec{D}} x_{\vec{q}}(-\vec{a}) \right) = e^{-i\vec{\alpha} \cdot \vec{a}} f(\vec{a}) \quad (4.41)$$

where we have made use of eq. (4.32). This shows eq. (4.39), and at the same time our assertion. From eq. (4.33), this is equivalent to a gauge transformation on the gauge potential $A_i = \alpha_{ij} a_j, A_i \rightarrow A_i - \partial_i \Lambda$, with $\Lambda = \vec{\alpha} \cdot \vec{a}$. Therefore, we see here explicitly the content of a projective representation in the basis of the functions on the group. One starts assigning the two-form $\alpha_{ij}$. Given that, one makes a choice for the vector potential. For instance in the previous analysis we have chosen $\alpha_{ij} a_j$. Any possible projective representation corresponds to a different choice of the gauge. In the dual Fourier basis this corresponds to assign a fixed set of operators $q_i$, with commutation relations determined by the two-form. All the possible projective representations are obtained by translating the operators $q_i$’s. Of course, this is equivalent to say that the projective representations are the central extension of the vector ones, and that they are determined by the cocycles. But the previous analysis shows that the projective representations generate noncommutative spaces, and that the algebraic integration, allowing us to define a Fourier analysis, gives the possibility of establishing the calculus rules.

Consider now the case $G = \mathbb{Z}^D$. Let us introduce an orthonormal basis on the square lattice defined by $\mathbb{Z}^D$, $\vec{e}_{(i)}, i = 1, \cdots, D$. Then, any element of the algebra can be reconstructed in terms of a product of the elements

$$U_i = x(\vec{e}_{(i)}) \quad (4.42)$$

corresponding to a translation along the direction $i$ by one lattice site. In general we will have

$$x(\vec{m}) = e^{i\theta(\vec{m})} U_1^{m_1} \cdots U_D^{m_D} \quad (4.43)$$

with $\theta$ a convenient phase factor such to reproduce correctly the phase in the algebra product. The quantities $U_i$ play the same role of $\vec{q}$ of the previous example. The Fourier transform is defined by

$$\hat{f}(\vec{U}) = \sum_{\vec{m} \in \mathbb{Z}^D} f(\vec{m}) x_{\vec{U}}(\vec{m}) \quad (4.44)$$
where the dependence on the representation is expressed in terms of \( \vec{U} \), denoting the collections of the \( U_i \)'s. The inverse Fourier transform is defined by

\[
f(\vec{m}) = \int_{\vec{U}} \hat{f}(\vec{U}) x_{\vec{U}}(-\vec{m})
\]

where the integration rule is

\[
\int_{(\vec{U})} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}}
\]

Therefore, the Fourier transform of \( U_i \) is simply \( \delta_{\vec{m},\vec{e}_i} \). The algebraic integration for the vector case is

\[
\int_{(\vec{U})} \rightarrow \int_0^L \frac{d^D \vec{q}}{L^D}
\]

Since the set \( \vec{U} \) is within the generators of the algebra, to establish the rules of the calculus is a very simple matter. Eq. (4.42) is the definition of the set \( \vec{U} \), analogous to eq. (4.22). In place of eq. (4.29) we get

\[
U_i \hat{f}(\vec{U}) U_i^{-1} = e^{-2\alpha_{ij} D_j} \hat{f}(\vec{U})
\]

Here \( D_j \) is the \( j \)-th component of the derivation \( \vec{D} \) which acts upon \( U_i \) as

\[
D_i U_j = -i\delta_{ij} U_j
\]

By choosing \( \hat{f}(\vec{U}) = U_k \) we have

\[
U_i U_k U_i^{-1} U_k^{-1} = e^{2\alpha_{ik}}
\]

which is the analogue of the commutator among the \( q_i \)'s. The automorphisms generated by \( \vec{D} \) are

\[
S(\vec{\phi}) x_{\vec{U}}(\vec{m}) = e^{\vec{\phi} \cdot \vec{D}} x_{\vec{U}}(\vec{m}) = e^{-i\vec{\phi} \cdot \vec{m}} x_{\vec{U}}(\vec{m})
\]

From which we see that

\[
U_i \rightarrow S(\vec{\phi}) U_i = e^{-i\phi_i} U_i
\]
This transformation corresponds to a trivial cocycle. As in the case $G = R^D$ it gives rise to a phase transformation on the group functions

$$
\int_{\{\vec{U}\}} \left( e^{\vec{\alpha} \cdot \vec{D}} \hat{f}(\vec{U}) \right) x_{\vec{U}}(-\vec{m}) = \int_{\{\vec{U}\}} \hat{f}(\vec{U}) \left( e^{-\vec{\alpha} \cdot \vec{D}} x_{\vec{U}}(-\vec{m}) \right) = e^{i\vec{\phi} \cdot \vec{m}} f(\vec{m}) \quad (4.53)
$$

Of course, all these relations could be obtained by putting $U_i = \exp(-iq_i)$, with $q_i$ defined as in the case $G = R^D$.

Finally, in the case $G = Z^D$, the situation is very much alike $Z^D$, that is the algebra can be reconstructed in terms of a product of elements

$$U_i = x(\vec{e}_{(i)}) \quad (4.54)$$

satisfying

$$U_i^n = 1 \quad (4.55)$$

Therefore we will not repeat the previous analysis but we will consider only the case $D = 2$, where $U_1$ and $U_2$ can be expressed as [7]

$$(U_1)_{a,b} = \delta_{a,b-1} + \delta_{a,n} \delta_{b,1}, \quad (U_2)_{a,b} = e^{2\pi i n (a-1)} \delta_{a,b}, \quad a, b = 1, \ldots, n \quad (4.56)$$

The elements of the algebra are reconstructed as

$$x_{\vec{U}}(\vec{m}) = e^{i\pi m_1 m_2} U_1^{m_1} U_2^{m_2} \quad (4.57)$$

The cocycle is now

$$\alpha(\vec{m}_1, \vec{m}_2) = -\frac{2\pi}{n} \epsilon_{ij} m_1 m_2 j \quad (4.58)$$

In this case we can compare the algebraic integration rule

$$\int_{\{\vec{U}\}} x_{\vec{U}}(\vec{m}) = \delta_{\vec{m},\vec{0}} \quad (4.59)$$

with

$$Tr[x_{\vec{U}}(\vec{m})] = n \delta_{\vec{m},\vec{0}} \quad (4.60)$$

A generic element of the algebra is a $n \times n$ matrix

$$A = \sum_{m_1, m_2=0}^{n-1} c_{m_1 m_2} x_{\vec{U}}(\vec{m}) \quad (4.61)$$
and therefore
\[ \int \vec{U} A = \frac{1}{n} Tr[A] \] (4.62)

In ref. [6] we have shown that the algebraic integration over the algebra of the $n \times n$ matrices $\mathcal{A}_n$ is given by
\[ \int_{\mathcal{A}_n} A = Tr[A] \] (4.63)

implying
\[ \int_{\vec{U}} A = \frac{1}{n} \int_{\mathcal{A}_n} A \] (4.64)

5 Conclusions

During the last few years many theoretical hints came about the possible relevance of noncommutative spaces in physics. Among them we recall that $D0$ branes in $M$ theory are described by noncommuting $N \times N$ hermitian matrices. Within the same framework, noncommutative spaces arise from compactification in various dimensions. The noncommutative compactification has to do with the use of projective representations of abelian groups of motion acting on the subspace to be compactified. To study the related geometry it would be important to dispose of tools allowing to mimic the usual calculus in classical spaces. In this paper we have done an attempt along this direction by using, as a principal instrument, an algebraic theory of integration that we have developed in some recent paper. Here we have shown that the ATI allows to generalize the usual harmonic analysis over vector group representations to the projective case. This is done in a way which is very much alike the spirit of noncommutative geometry [8]. That is, the ATI deals with the algebra of functions, in this case with the algebra of the projective representations, rather than with the base space, which is the noncommutative analogue of the space of the equivalence classes of the representations of the group under consideration. For abelian groups we have shown that it is possible, through the harmonic analysis based on the ATI, to extend the usual tools of the calculus over classical spaces, as derivatives, integrals, and so on. For instance, it is possible to give a meaning to the analogue of concepts as space coordinates, in terms of distributions over the
group. Many of the concepts used in this study need a more refined mathematical analysis, but we feel that these ideas may have some interest for physical applications of noncommutative geometry.

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