On arbitrarily slow convergence rates for strong numerical approximations of Cox–Ingersoll–Ross processes and squared Bessel processes

Mario Hefter1 · Arnulf Jentzen2

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Abstract Cox–Ingersoll–Ross (CIR) processes are extensively used in state-of-the-art models for the pricing of financial derivatives. The prices of financial derivatives are very often approximately computed by means of explicit or implicit Euler- or Milstein-type discretization methods based on equidistant evaluations of the driving noise processes. In this article, we study the strong convergence speeds of all such discretization methods. More specifically, the main result of this article reveals that each such discretization method achieves at most a strong convergence order of $\frac{\delta}{2}$, where $0 < \delta < 2$ is the dimension of the squared Bessel process associated to the considered CIR process.

Keywords Cox–Ingersoll–Ross process · Squared Bessel process · Stochastic differential equation · Strong (pathwise) approximation · Lower error bound · Optimal approximation

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✉ M. Hefter
hefter@mathematik.uni-kl.de

Arnulf Jentzen
arnulf.jentzen@sam.math.ethz.ch

1 Fachbereich Mathematik, Universität Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany

2 Seminar for Applied Mathematics, Eidgenössische Technische Hochschule Zürich, Rämistrasse 101, 8092 Zurich, Switzerland
1 Introduction

Stochastic differential equations (SDEs) are a key ingredient in a number of models from economics and the natural sciences. In particular, SDE based models are daily used in the financial engineering industry to approximately compute prices of financial derivatives. The SDEs appearing in such models are typically highly nonlinear and contain non-Lipschitz nonlinearities in the drift or diffusion coefficient. Such SDEs in almost all cases cannot be solved explicitly, and it has been and still is a very active topic of research to approximate SDEs with non-Lipschitz nonlinearities; see e.g. Gyöngy [14], Higham et al. [21], Hu [23], Hutzenthaler and Jentzen [24], Hutzenthaler et al. [25], Sabanis [36, 37] and the references therein. In particular, in the last five years, several results have been obtained demonstrating that approximation schemes may converge arbitrarily slowly; see Gerencsér et al. [12], Hairer et al. [16], Jentzen et al. [27], Yaroslavtseva [38], Yaroslavtseva and Müller-Gronbach [39]. For example, Theorem 1.2 in [27] demonstrates that there exists an SDE that has solutions with all moments bounded, but for which all approximation schemes that use only evaluation points of the driving Brownian motion converge in the strong sense with an arbitrarily slow rate; see also [12, Theorem 1.2], [16, Theorem 1.3], [38, Theorem 1] and [39, Theorem 3] for related results. All the SDEs in the above examples are purely academic with no connection to applications. The key contribution of this work is to reveal that such slow convergence phenomena also arise in concrete models from applications. To be more specific, in this work we reveal that Cox–Ingersoll–Ross (CIR) processes and squared Bessel processes cannot in general be solved approximately in the strong sense in a reasonable computational time by means of schemes using equidistant evaluations of the driving Brownian motion. The precise formulation of our result is the subject of the following theorem.

**Theorem 1.1** Let $T, a, \sigma \in (0, \infty)$, $b, x \in [0, \infty)$ satisfy $2a < \sigma^2$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions, let $W : [0,T] \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion, let $X : [0,T] \times \Omega \to [0, \infty)$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [0,T]$ $\mathbb{P}$-a.s. that

\[
X_t = x + \int_0^t (a - bX_s) \, ds + \int_0^t \sigma \sqrt{X_s} \, dW_s. \tag{1.1}
\]

Then there exists a real number $c \in (0, \infty)$ such that for all $N \in \mathbb{N}$, it holds that

\[
\inf_{\varphi : \mathbb{R}^N \to \mathbb{R} \text{ Borel-measurable}} \mathbb{E}\left[|X_T - \varphi(W_{T/N}, W_{2T/N}, \ldots, W_T)|\right] \geq c \, N^{-\frac{2a}{\sigma^2}}. \tag{1.2}
\]

Theorem 1.1 is proved in Sect. 5 below. Upper error bounds for strong approximations of CIR processes and squared Bessel processes, i.e., the opposite question of Theorem 1.1, have been intensively studied in the literature; see e.g. Alfonsi [1, 2], Berkaoui et al. [3], Bossy and Olivero [5], Chassagneux et al. [6], Cozma and Reisinger [9], Deelstra and Delbaen [10], Dereich et al. [11], Gyöngy and Rásonyi [15], Hefter and Herzwurm [17, 18], Higham and Mao [20], Hutzenthaler and Jentzen...
Corollary 1.2 Let $T, a, \sigma \in (0, \infty)$, $b, x \in [0, \infty)$ satisfy $4a < \sigma^2$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions, let $W : [0, T] \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion, let $X : [0, T] \times \Omega \to [0, \infty)$ be an $(\mathcal{F}_t)_{t \in [0, T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [0, T]$ $\mathbb{P}$-a.s. that

$$X_t = x + \int_0^t (a - bX_s) \, ds + \int_0^t \sigma \sqrt{X_s} \, dW_s.$$ 

Then there exist real numbers $c, C \in (0, \infty)$ such that for all $N \in \mathbb{N}$, it holds that

$$c N^{-2a/\sigma^2} \leq \inf_{\varphi : \mathbb{R}^N \to \mathbb{R}} \mathbb{E}[|X_T - \varphi(W_{T/N}, W_{2T/N}, \ldots, W_T)|] \leq C N^{-2a/\sigma^2}. \tag{1.3}$$

We conjecture that in the full parameter range $a, \sigma \in (0, \infty)$, the convergence order in (1.3) is equal to $\min(2a/\sigma^2, 1)$ since for scalar SDEs with coefficients satisfying standard assumptions, a convergence order of one is optimal; see e.g. Hofmann et al. [22], Müller-Gronbach [33]. Furthermore, this conjecture is in line with numerical experiments performed in the literature; see e.g. Alfonsi [1], Cozma and Reisinger [9], Hefter and Herzwurm [18].

An intuitive, i.e., without mathematical rigor, derivation of Theorem 1.1 is as follows. For every $x \in [0, \infty)$, denote by $X^x = (X^x_t)_{t \in [0, T]}$ the solution of (1.1) with initial value $x$. For simplicity, assume that the parameters in (1.1) satisfy $a < 2$, $b = 0$, $\sigma = 2$ and consider the final time $T = 1$. The comparison principle yields that for all $x, y \in [0, \infty)$ with $x \geq y$, it holds that $X^x \geq X^y \geq 0$. Furthermore, since stochastic integrals have vanishing expectations, it holds for all $x \in [0, \infty)$ that $\mathbb{E}[X^x_T] = x + a$. Combining these two facts, we get for all $N \in \mathbb{N}$ that

$$\frac{1}{N} = \mathbb{E}[X^{1/N}_1 - a] = \mathbb{E}[X^{1/N}_1 - X^0_1] = \mathbb{E}[|X^{1/N}_1 - X^0_1|]$$

$$= \mathbb{E}[|X^{1/N}_1 - X^0_1| \mathbbm{1}_{\{X^{1/N}_1 \neq 0, \forall \tau \in [0, 1]\}}]$$

$$+ \mathbb{E}[|X^{1/N}_1 - X^0_1| \mathbbm{1}_{\{\exists \tau \in [0, 1] : X^{1/N}_\tau = 0\}}]$$

$$= \mathbb{E}[|X^{1/N}_1 - X^0_1| \mathbbm{1}_{\{X^{1/N}_1 \neq 0, \forall \tau \in [0, 1]\}}]. \tag{1.4}$$

Furthermore, note that the hypothesis $a < 2$ yields that

$$\mathbb{P}[X^{1/N}_t \neq 0, \forall t \in [0, 1]] \approx N^{a/2 - 1}. \tag{1.5}$$
(see e.g. Borodin and Salminen [4, Sect. IV.6]). Combining (1.4) and (1.5) shows that
\[
\mathbb{E}[|X_{1/N}^1 - X_0^1| \mid \{X_{1/N}^1 \neq 0, \forall t \in [0, 1]\}] \approx N^{-a/2}. \tag{1.6}
\]
Next we observe that the random time \(\tau = \sup\{t \in [0, 1] : X_t = 0\}\) satisfies \(\tau \approx \frac{1}{2}\) with a positive probability. Moreover, note that it is known in the scientific literature that equidistant strong approximation schemes with \(N\) time steps (\(N\) evaluations of the driving Brownian motion) have at least a strong error of the order \(\frac{1}{N}\) (cf. e.g. Müller-Gronbach [33]). The strong approximation error between the exact solution of the SDE (1.1) and any strong approximation scheme of the form (1.2) should at time \(\frac{1}{2}\) be thus at least of order \(1/N\). Combining this with the intuition that the exact solution flow associated to (1.1) should be more accurate than any strong approximation scheme of the form (1.2) indicates that the strong approximation error between the exact solution of SDE (1.1) and any strong approximation scheme of the form (1.2) should be larger than or equal to the distance between \(X_{1/N}^1\) and \(X_0^1\). This and (1.6) suggest that the strong approximation error between the exact solution of the SDE (1.1) and any strong approximation scheme of the form (1.2) is at least of the order \(N^{-a/2}\), which finishes our intuition for the proof of the lower bound in (1.2).

The remainder of this article is organized as follows. In Sect. 2, we review a few elementary properties of CIR processes and squared Bessel processes. In Sect. 3, we present a piecewise construction of a Brownian motion. In Sect. 4, we prove the lower error bound for a specific parameter range, which is then generalized in Sect. 5.

2 Cox–Ingersoll–Ross (CIR) processes and squared Bessel processes

2.1 Setting

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(W : [0, \infty) \times \Omega \to \mathbb{R}\) a Brownian motion, and for every \(\delta \in (0, \infty), b, z \in [0, \infty)\) and every Brownian motion \(V : [0, \infty) \times \Omega \to \mathbb{R}\), let \(Z^{z,\delta,b,V} : [0, \infty) \times \Omega \to [0, \infty)\) be a stochastic process adapted to \((\sigma((\{V_s \leq a\} : a \in \mathbb{R}, s \in [0, t]) \cup \{A \in \mathcal{F} : \mathbb{P}[A] = 0\}))_{t \in [0, \infty)}\) with continuous sample paths which satisfies that for all \(t \in [0, \infty)\), it holds \(\mathbb{P}\)-a.s. that
\[
Z_t^{z,\delta,b,V} = z + \int_0^t (\delta - b Z_s^{z,\delta,b,V})ds + \int_0^t 2\sqrt{Z_s^{z,\delta,b,V}}dV_s. \tag{2.1}
\]
(In other words, \(Z\) is a mapping that produces a strong solution to the SDE (2.1).)

2.2 Basic results

**Lemma 2.1** In the setting of Sect. 2.1, suppose that the parameters \(\delta \in (0, \infty), b_1, b_2, z_1, z_2 \in [0, \infty)\) satisfy \(z_1 \leq z_2\) and \(b_1 \geq b_2\). Then
\[
\mathbb{P}[Z_t^{z_1,\delta,b_1,W} \leq Z_t^{z_2,\delta,b_2,W}, \forall t \in [0, \infty)] = 1. \tag{2.2}
\]
**Proof** This follows e.g. from Karatzas and Shreve [29, Proposition 5.2.18]. \(\square\)
Lemma 2.2 In the setting of Sect. 2.1, let $\delta, T \in (0, \infty), b, z \in [0, \infty), p \in [1, \infty)$. Then
\[ E \left[ \sup_{t \in [0,T]} |Z_t^{z,\delta,b,W}|^p \right] < \infty. \tag{2.3} \]

Proof This follows e.g. from Mao [31, Corollary 2.4.2].

Lemma 2.3 In the setting of Sect. 2.1, let $\delta \in (0, \infty), b, z, t \in [0, \infty)$. Then
\[ E[Z_t^{z,\delta,b,W}] = ze^{-bt} + \delta \int_0^t e^{-bs} \, ds = ze^{-bt} + \delta \left\{ \begin{array}{ll} \frac{1-e^{-bt}}{b}, & b \neq 0, \\ t, & b = 0. \end{array} \right. \]

Lemma 2.4 In the setting of Sect. 2.1, let $\delta \in (0, \infty), b, z_1, z_2, t \in [0, \infty)$. Then
\[ E[|Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}|] = e^{-bt}|z_1 - z_2|. \]

Proof Without loss of generality, assume $z_1 \geq z_2$. Lemmas 2.1–2.3 show that
\[ E[|Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}|] = E[Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}] = z_1 e^{-bt} - z_2 e^{-bt} = e^{-bt}|z_1 - z_2|. \]

Lemma 2.5 In the setting of Sect. 2.1, let $\delta, c \in (0, \infty), b, z \in [0, \infty)$. Then
\[ P[cZ_t^{c,\delta,cb,(c^{-1/2}W_{c\varepsilon})_{t \in [0,\infty)}} = Z_t^{z,\delta,b,W}, \forall t \in [0,\infty)] = 1. \tag{2.4} \]

Proof Equation (2.4) follows directly from the corresponding scaling property of Brownian motion and the stochastic integral (cf. e.g. Revuz and Yor [35, proof of Proposition XI.1.6]).

The next result is well known; see e.g. Göing-Jaeschke and Yor [13, Sect. 1] (cf. also Revuz and Yor [35, Sect. XI.1]).

Lemma 2.6 In the setting of Sect. 2.1, let $\delta \in (0, \infty), b, z \in [0, \infty)$. Then
\[ P[Z_t^{z,\delta,b,W} > 0, \forall t \in (0, \infty)] = P[Z_t^{z,\delta,b,W} \neq 0, \forall t \in (0, \infty)] = \begin{cases} 1, & \delta \geq 2, \\ 0, & \delta < 2. \end{cases} \]

Lemma 2.7 In the setting of Sect. 2.1, assume $\delta \in (0, 2), b \in [0, \infty), T \in (0, \infty)$. Then there exists $c \in (0, \infty)$ such that for every $\varepsilon \in (0, T)$, it holds that
\[ P\left[ \inf_{t \in [\varepsilon, T]} Z_t^{0,\delta,b,W} > 0 \right] \leq c \varepsilon^{1-\delta/2}. \]
Proof Set \( \nu = 1 - \delta \frac{2}{\gamma} \in (0, 1) \) and let \( \Gamma : (0, \infty) \to (0, \infty) \) be the Gamma function, i.e.,

\[
\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} \, dx \quad \text{for all } r \in (0, \infty).
\]

Define \( P : \mathbb{R} \times (0, \infty) \to [0, \infty) \) by

\[
P(z, r) = \begin{cases} 
P(\inf_{t \in [0, r]} Z_t^{\delta,0,W} > 0), & z \geq 0, \\
0, & z < 0.
\end{cases}
\]

Observe that there exists a \( C \in (0, \infty) \) which satisfies for all \( z, r \in (0, \infty) \) that

\[
P(z, r) = C z^\nu \int_r^\infty t^{-(1+\nu)} \, dt.
\]

(2.5)

Next we observe that for every \( \varepsilon \in (0, \infty) \), the random variable \( \varepsilon^{-1} \sum_{0, \delta,0,W} \) is \( \chi^2 \)-distributed with \( \delta \) degrees of freedom (see e.g. Revuz and Yor \[35, Corollary XI.1.4\]). Hence, we obtain for all \( \varepsilon \in (0, \infty) \) that

\[
\mathbb{E} \left[ \left. \frac{Z_t^{0,\delta,0,W}}{\varepsilon} \right| \right]^\nu = \int_0^\infty x^\nu \frac{\left( \frac{1}{2} \right)^{\delta/2}}{\Gamma(\delta/2)} x^{-\delta/2-1} \exp \left( -\frac{x}{2} \right) \, dx
\]

\[
= \frac{\left( \frac{1}{2} \right)^{\delta/2}}{\Gamma(\delta/2)} \int_0^\infty \exp \left( -\frac{x}{2} \right) \, dx = \frac{2^{1-\delta/2}}{\Gamma(\delta/2)}.
\]

This and (2.5) imply that for all \( r \in (0, \infty) \), \( \varepsilon \in (0, r) \), it holds that

\[
\mathbb{E} \left[ \inf_{t \in [\varepsilon, r]} Z_t^{0,\delta,0,W} \right] = \mathbb{E}[P(Z_t^{0,\delta,0,W}, r - \varepsilon)]
\]

\[
\leq C \mathbb{E}[P(Z_t^{0,\delta,0,W}) \int_{r-\varepsilon}^\infty t^{-(1+\nu)} \, dt]
\]

\[
= C \int_{r-\varepsilon}^\infty t^{-(1+\nu)} \frac{2^{1-\delta/2} \varepsilon^\nu}{\Gamma(\delta/2)}
\]

\[
= C \frac{(r-\varepsilon)^{-\nu} 2^{1-\delta/2} \varepsilon^\nu}{\varepsilon \Gamma(\delta/2)}.
\]

Therefore, we obtain for all \( \varepsilon \in (0, T) \) that

\[
\mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_t^{0,\delta,0,W} > 0 \right] \leq \frac{C 2^{1-\delta/2} (T - \varepsilon)^{\delta/2-1} \varepsilon^{1-\delta/2}}{\Gamma(\delta/2) (1 - \delta/2)}.
\]
This and Lemma 2.1 show that for all \( \varepsilon \in (0, T/2] \), it holds that

\[
\mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_{t}^{0, \delta, b, W} > 0 \right] \leq \mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_{t}^{0, \delta, 0, W} > 0 \right] \leq \frac{C 2^{(2-\delta)} T^{\delta/2 - 1}}{\Gamma(\delta/2) (1 - \delta/2)} \varepsilon^{1-\delta/2}.
\]

Hence, we obtain that

\[
\sup_{\varepsilon \in (0, T/2]} \mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_{t}^{0, \delta, b, W} > 0 \right] \varepsilon^{1-\delta/2} < \infty.
\]

This assures that

\[
\sup_{\varepsilon \in (0, T/2]} \mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_{t}^{0, \delta, b, W} > 0 \right] \leq \sup_{\varepsilon \in (0, T/2]} \mathbb{P} \left[ \inf_{t \in [\varepsilon, T]} Z_{t}^{0, \delta, b, W} > 0 \right] \varepsilon^{1-\delta/2} + \sup_{\varepsilon \in (T/2, T]} \frac{1}{\varepsilon^{1-\delta/2}}\varepsilon^{1-\delta/2} < \infty.
\]

The proof of Lemma 2.7 is thus completed. \( \square \)

3 A piecewise construction of a Brownian motion

3.1 Setting

Let \( Z^{(1), (i)} = (Z^{(i)})_{z \in \mathbb{R}, t \in C([0, \infty); \mathbb{R})} : \mathbb{R} \times C([0, \infty); \mathbb{R}) \rightarrow C([0, \infty); \mathbb{R}) \) be a Borel-measurable and (see Kallenberg [28, Chap. 21]) universally adapted function and \( \alpha, \beta : \mathbb{R} \rightarrow \mathbb{R} \) continuous functions. Assume that for every complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), every filtration \( \mathbb{F} = (\mathcal{F}_{t})_{t \in [0, \infty)} \) on \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual conditions, every \( \mathbb{F} \)-Brownian motion \( W : [0, \infty) \times \Omega \rightarrow \mathbb{R} \), all continuous \( \mathbb{F} \)-adapted stochastic processes \( Z^{(1)}, Z^{(2)} : [0, \infty) \times \Omega \rightarrow \mathbb{R} \) with

\[
Z_{t}^{(i)} = Z_{0}^{(i)} + \int_{0}^{t} \alpha(Z_{s}^{(i)}) \, ds + \int_{0}^{t} \beta(Z_{s}^{(i)}) \, dW_{s} \quad \mathbb{P}\text{-a.s.} \tag{3.1}
\]

for \( i \in \{1, 2\}, t \in [0, \infty) \), and all \( t \in [0, \infty) \), it holds that

\[
\mathbb{P}[Z_{t}^{(1)} = Z_{t}^{(2)}] = 1.
\]

Assume also that for every complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), every filtration \( \mathbb{F} = (\mathcal{F}_{t})_{t \in [0, \infty)} \) on \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual conditions, every \( \mathbb{F} \)-Brownian motion \( W : [0, \infty) \times \Omega \rightarrow \mathbb{R} \), every \( \mathcal{F}_{0} \)-measurable function \( Z : \Omega \rightarrow \mathbb{R} \) and every \( t \in [0, \infty) \), it holds that

\[
\mathbb{P} \left[ Z_{t}^{Z, W} = Z + \int_{0}^{t} \alpha(Z_{s}^{Z, W}) \, ds + \int_{0}^{t} \beta(Z_{s}^{Z, W}) \, dW_{s} \right] = 1.
\]
3.2 Details on the construction

Lemma 3.1 In the setting of Sect. 3.1, let \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)} \) be a filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \) which satisfies the usual conditions, \( W: [0, \infty) \times \Omega \to \mathbb{R} \) an \( \mathbb{F} \)-Brownian motion, \( \tau: \Omega \to [0, \infty) \) an \( \mathbb{F} \)-stopping time and \( Z: \Omega \to \mathbb{R} \) an \( \mathcal{F}_0 \)-\( \mathbb{B}(\mathbb{R}) \)-measurable function. Define the stochastic process \( \tilde{W}: [0, \infty) \times \Omega \to \mathbb{R} \) by \( \tilde{W}_t = W_{t+\tau} - W_t \) and the random variable \( \tilde{Z}: \Omega \to \mathbb{R} \) by \( \tilde{Z} = Z_{\tau}^{\tilde{W}} \). Then:

(i) \( \tilde{W} \) is a Brownian motion.

(ii) \( \tilde{W} \) and \( \tilde{Z} \) are independent.

(iii) We have

\[
P[\mathcal{Z}_t^{\tilde{Z},\tilde{W}} = \mathcal{Z}_{t+\tau}^{Z,W}, \forall t \in [0, \infty)] = 1. \tag{3.2}
\]

Proof Define the filtration \( \tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0, \infty)} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) by \( \tilde{\mathcal{F}}_t = \mathcal{F}_{t+\tau} \). Observe that the fact that \( \mathcal{Z}_t^{Z,W} \) is \( \mathcal{F}_\tau \)-\( \mathbb{B}(\mathbb{R}) \)-measurable ensures that \( \tilde{Z} \) is \( \tilde{\mathcal{F}}_0 \)-\( \mathbb{B}(\mathbb{R}) \)-measurable. In addition, note that e.g. Kallenberg [28, Theorem 13.11] shows that \( \tilde{W} \) is an \( \tilde{\mathbb{F}} \)-Brownian motion. This and the fact that \( \tilde{Z} \) is \( \tilde{\mathcal{F}}_0 \)-\( \mathbb{B}(\mathbb{R}) \)-measurable show that \( \tilde{W} \) and \( \tilde{Z} \) are independent. Next observe that the stochastic process \( (\mathcal{Z}_t^{Z,W})_{t \in [0, \infty)} \) has continuous sample paths, is \( \tilde{\mathbb{F}} \)-adapted and satisfies for all \( t \in [0, \infty) \) that \( \mathbb{P} \)-a.s.,

\[
\mathcal{Z}_t^{Z,W} = Z + \int_0^t \alpha(Z_{s}^{Z,W}) \, ds + \int_0^t \beta(Z_{s}^{Z,W}) \, dW_s \\
= Z + \int_0^\tau \alpha(Z_{s}^{Z,W}) \, ds + \int_0^\tau \beta(Z_{s}^{Z,W}) \, dW_s \\
+ \int_\tau^{t+\tau} \alpha(Z_{s}^{Z,W}) \, ds + \int_\tau^{t+\tau} \beta(Z_{s}^{Z,W}) \, dW_s \\
= \tilde{Z} + \int_0^t \alpha(Z_{s+\tau}^{Z,W}) \, ds + \int_0^t \beta(Z_{s+\tau}^{Z,W}) \, d\tilde{W}_s.
\]

This establishes (3.2). \( \square \)

Lemma 3.2 In the setting of Sect. 3.1, let \( W, \tilde{W}: [0, \infty) \times \Omega \to \mathbb{R} \) be Brownian motions, \( \tau: \Omega \to [0, \infty] \) a random variable and assume for all \( t \in [0, \infty) \) that \( \mathbb{P}[W_{t \wedge \tau} = \tilde{W}_{t \wedge \tau}] = 1 \). Let \( Z: \Omega \to \mathbb{R} \) be a random variable and assume that \( W \) and \( Z \) as well as \( \tilde{W} \) and \( Z \) are independent. Then

\[
P[\mathcal{Z}^{Z,W}_t = \mathcal{Z}^{Z,W}_{t \wedge \tau} \mathbbm{1}_{\{t \leq \tau\}} = 0, \forall t \in [0, \infty)] = 1. \tag{3.3}
\]

Proof Observe that

\[
P[\forall t \in [0, \infty) \cap \mathbb{Q}: t \leq \tau \implies W_t = \tilde{W}_t] = 1.
\]
The fact that $W$ and $\tilde{W}$ have continuous sample paths hence shows that
\[ \mathbb{P}[\forall t \in [0, \infty) : t \leq \tau \implies W_t = \tilde{W}_t] = 1. \]

The assumption that $Z_{\cdot}^{(0)}$ is universally adapted therefore proves that
\[ \mathbb{P}\left[ (Z_t^Z)^W - Z_t^{\tilde{Z}, \tilde{W}} \mathbb{1}_{\{t \leq \tau\}} = 0, \forall t \in [0, \infty) \cap \mathbb{Q}\right] = 1. \]

This and the fact that the stochastic process $(Z_t^Z)^W - Z_t^{\tilde{Z}, \tilde{W}} \mathbb{1}_{\{t \leq \tau\}}, t \in [0, \infty)$ has left-continuous sample paths establish (3.3). \(\square\)

**Lemma 3.3** Assume the setting of Sect. 3.1. For $m \in \{0, 1\}$, let $\mathbb{P}^{\langle m \rangle} = (\mathcal{F}_t^{\langle m \rangle})_{t \in [0, \infty)}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and assume that we have $(\bigcup_{t \in [0, \infty)} \mathcal{F}_t^{\langle 0 \rangle}) \subseteq \mathcal{F}_0^{\langle 1 \rangle}$, let $W^{\langle m \rangle} : [0, \infty) \times \Omega \to \mathbb{R}$ be an $\mathbb{P}^{\langle m \rangle}$-Brownian motion and $\tau^{\langle m \rangle} : \Omega \to [0, \infty)$ an $\mathbb{P}^{\langle m \rangle}$-stopping time. Define the filtration $\tilde{\mathbb{P}} = (\mathcal{F}_t)_{t \in [0, \infty)}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by
\[ \tilde{\mathcal{F}}_t = \left\{ A \in \mathcal{F} : A \cap \{t < \tau^{(0)}\} \in \mathcal{F}^{(0)}_t \text{ and } A \cap \{t \geq \tau^{(0)}\} \in \mathcal{F}^{(1)}_{\max\{t-\tau^{(0)}, 0\}} \right\}, \]
define the random variable $\tilde{\tau} : \Omega \to [0, \infty) \text{ by } \tilde{\tau} = \tau^{(0)} + \tau^{(1)}$, consider the stochastic process $\tilde{W} : [0, \infty) \times \Omega \to \mathbb{R}$ given for $t \in [0, \infty) \text{ by }$
\[ \tilde{W}_t = W_t^{\langle 0 \rangle} \mathbb{1}_{\{t \leq \tau\}} + (W_t^{\langle 1 \rangle} \mathbb{1}_{\{t-\tau\}} + W_{\tau^{(0)}}^{\langle 0 \rangle}) \mathbb{1}_{\{t > \tau\}}, \]
let $Z : \Omega \to \mathbb{R}$ be a random variable which is $\mathcal{F}^{\langle 0 \rangle}\mathcal{B}(\mathbb{R})$-measurable, and define the random variable $\tilde{Z} : \Omega \to \mathbb{R}$ by $\tilde{Z} = Z_t^{\tilde{Z}, \tilde{W}}$. Then:

(i) $\mathcal{F}^{\langle 0 \rangle}_0 \subseteq \tilde{\mathcal{F}}_0$.

(ii) $(\bigcup_{t \in [0, \infty)} \tilde{\mathcal{F}}_t) \subseteq \sigma(\bigcup_{t \in [0, \infty)} \mathcal{F}^{\langle 1 \rangle}_t)$.

(iii) $\tilde{\tau}$ is an $\tilde{\mathbb{P}}$-stopping time.

(iv) $\tau^{(0)}$ is an $\mathbb{P}$-stopping time.

(v) $\tilde{W}$ is an $\tilde{\mathbb{P}}$-Brownian motion.

(vi) $\tilde{W}$ and $Z$ are independent.

(vii) $W^{\langle 1 \rangle}$ and $\tilde{Z}$ are independent.

(viii) We have
\[ \mathbb{P}\left[ Z_t^Z, \tilde{W} = Z_t^Z, W^{\langle 0 \rangle} \mathbb{1}_{\{t \leq \tau\}} + Z_t^Z, W^{\langle 1 \rangle} \mathbb{1}_{\{t > \tau\}}, \forall t \in [0, \infty) \right] = 1. \]

**Proof** Denote by $F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ the distribution function of the standard normal distribution and for every $t \in [0, \infty)$, set $\rho_t = \max\{t - \tau^{(0)}, 0\}$. Observe that $\rho_t$ is an $\mathbb{P}^{\langle 1 \rangle}$-stopping time for every $t \in [0, \infty)$. The fact that $\{0 < \tau^{(0)}\} \in \mathcal{F}_0^{\langle 0 \rangle}$ ensures that for every $A \in \mathcal{F}_0^{\langle 0 \rangle}$, we have $A \cap \{0 < \tau^{(0)}\} \in \mathcal{F}_0^{\langle 0 \rangle}$ and
\[ A \cap \{0 \geq \tau^{(0)}\} \in \mathcal{F}_0^{\langle 0 \rangle} \subseteq \mathcal{F}_0^{\langle 1 \rangle} = \mathcal{F}_{\max\{0-\tau^{(0)}, 0\}}^{\langle 1 \rangle}. \]
This proves (i). Next observe that for every \( t \in [0, \infty) \), \( A \in \mathcal{F}_t \), it holds that

\[
A \cap \{ t < \tau^{(0)} \} \in \mathcal{F}_t^{(0)} \subseteq \mathcal{F}_t^{(0)} \subseteq \sigma \left( \bigcup_{s \in [0, \infty)} \mathcal{F}_s^{(1)} \right),
\]

\[
A \cap \{ t \geq \tau^{(0)} \} \in \mathcal{F}_{\max\{t-\tau^{(0)}, 0\}}^{(1)} \subseteq \sigma \left( \bigcup_{s \in [0, \infty)} \mathcal{F}_s^{(1)} \right).
\]

Hence, we obtain for every \( t \in [0, \infty) \), \( A \in \mathcal{F}_t \) that

\[
A = (A \cap \{ t < \tau^{(0)} \}) \cup (A \cap \{ t \geq \tau^{(0)} \}) \in \sigma \left( \bigcup_{s \in [0, \infty)} \mathcal{F}_s^{(1)} \right).
\]

This proves (ii). For every \( t \in [0, \infty) \), we have \( \{ \bar{t} \leq t \} \cap \{ t < \tau^{(0)} \} = \emptyset \in \mathcal{F}_t^{(0)} \) and

\[
\{ \bar{t} \leq t \} \cap \{ t \geq \tau^{(0)} \} = \{ \bar{t} \leq t \} = \{ \tau^{(1)} \leq t - \tau^{(0)} \}
\]

\[
= \{ \tau^{(1)} \leq \max\{ t - \tau^{(0)}, 0 \} \} \cap \{ t - \tau^{(0)} \geq 0 \} \in \mathcal{F}_{\max\{t-\tau^{(0)}, 0\}}^{(1)}.
\]

This proves (iii). Because \( \{ \tau^{(0)} \leq t \} \cap \{ t < \tau^{(0)} \} = \emptyset \in \mathcal{F}_t^{(0)} \) for every \( t \in [0, \infty) \) and

\[
\{ \tau^{(0)} \leq t \} \cap \{ t \geq \tau^{(0)} \} = \{ \tau^{(0)} \leq t \} \in \mathcal{F}_t^{(0)} \subseteq \mathcal{F}_0^{(0)} \subseteq \mathcal{F}_{\max\{t-\tau^{(0)}, 0\}}^{(1)},
\]

we get (iv). Due to the strong Markov property of Brownian motion, each process \( (\mathcal{W}_{\rho_s}^{(1)} - \mathcal{W}_{\rho_s}^{(1)})_{u \in [0, \infty)} \) is a Brownian motion independent of \( \mathcal{F}_{\rho_s}^{(1)} \) for \( s \in [0, \infty) \).

This and the fact that \( A \cap \{ s \geq \tau^{(0)} \} \in \mathcal{F}_{\rho_s}^{(1)} \) for every \( s \in [0, \infty) \), \( A \in \mathcal{F}_s \) demonstrate that for every \( s \in [0, \infty) \), \( t \in (s, \infty) \), \( A \in \mathcal{F}_s \), \( \alpha \in \mathbb{R} \), it holds that

\[
\mathbb{P}[\{ \bar{W}_t - \bar{W}_s \leq \alpha \} \cap A \cap \{ s \geq \tau^{(0)} \}] = \mathbb{P}[\{ (\mathcal{W}_{\rho_s}^{(1)} - \rho_s)_{u \in [0, \infty)} - \mathcal{W}_{\rho_s}^{(1)} \leq \alpha \} \cap A \cap \{ s \geq \tau^{(0)} \}]
\]

\[
= F \left( \frac{a}{\sqrt{t-s}} \right) \mathbb{P}[A \cap \{ s \geq \tau^{(0)} \}]. \tag{3.4}
\]

Moreover, observe that for every \( s \in [0, \infty) \), \( t \in (s, \infty) \), \( A \in \mathcal{F}_s \), we have that

(a) \( W_{t \wedge \tau^{(0)} \vee s}^{(0)} - W_s^{(0)} \) is \( \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \)-\( \mathcal{B}(\mathbb{R}) \)-measurable;

(b) \( A \cap \{ s < \tau^{(0)} \} \in \mathcal{F}_s^{(0)} \subseteq \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \);

(c) \( t - (t \wedge \tau^{(0)} \vee s) \) is \( \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \)-\( \mathcal{B}(\mathbb{R}) \)-measurable.

Next, \( \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \subseteq \sigma(\bigcup_{u \in [0, \infty)} \mathcal{F}_u^{(0)}) \subseteq \mathcal{F}_0^{(1)} \) for every \( s \in [0, \infty) \), \( t \in (s, \infty) \) ensures that for every \( s \in [0, \infty) \), \( t \in (s, \infty) \), the Brownian motion \( W^{(1)} \) is independent of \( \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \). Moreover, the strong Markov property of Brownian motion implies that for every \( s \in [0, \infty) \), \( t \in (s, \infty) \), the process \( (\mathcal{W}_{\rho_s}^{(0)} - \mathcal{W}_{\rho_s}^{(0)})_{u \in [0, \infty)} \) is a Brownian motion independent of \( \mathcal{F}_{t \wedge \tau^{(0)} \vee s}^{(0)} \). Combining (a)–(c) with the fact that the
Brownian motion $W^{(1)}$ is independent of $\mathcal{F}_{t \wedge \tau^{(0)} \vee s}$ for every $s \in [0, \infty)$, $t \in (s, \infty)$ therefore shows that for every $s \in [0, \infty)$, $t \in (s, \infty)$, $A \in \mathcal{F}_s$, $a \in \mathbb{R}$, it holds that
\[
\mathbb{P}\left[ (\tilde{W}_t - \tilde{W}_s \leq a) \cap A \cap \{ s < \tau^{(0)} \} \right] = \mathbb{P}\left[ (W^{(1)}_{t- (t \wedge \tau^{(0)} \vee s)} + W^{(0)}_{t \wedge \tau^{(0)} \vee s} - W^{(0)}_s \leq a) \cap A \cap \{ s < \tau^{(0)} \} \right] = \mathbb{P}\left[ (W^{(0)}_t - W^{(0)}_s \leq a) \cap A \cap \{ s < \tau^{(0)} \} \right].
\]
The fact that $A \cap \{ s < \tau^{(0)} \} \in \mathcal{F}_s$ for every $s \in [0, \infty)$, $A \in \mathcal{F}_s$, hence shows that for all $s \in [0, \infty)$, $t \in (s, \infty)$, $A \in \mathcal{F}_s$, $a \in \mathbb{R}$, it holds that
\[
\mathbb{P}\left[ (\tilde{W}_t - \tilde{W}_s \leq a) \cap A \cap \{ s < \tau^{(0)} \} \right] = \mathbb{P}\left[ A \cap \{ s < \tau^{(0)} \} \right].
\]
Together with (3.4), this implies for all $s \in [0, \infty)$, $t \in (s, \infty)$, $A \in \mathcal{F}_s$, $a \in \mathbb{R}$ that
\[
\mathbb{P}\left[ (\tilde{W}_t - \tilde{W}_s \leq a) \cap A \right] = \mathbb{P}\left[ A \right].
\]
This proves (v). Item (i) implies that $Z$ is $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R})$-measurable. This proves (vi). Because $\tilde{Z}$ is $\sigma(\bigcup_{u \in [0, \infty)} \mathcal{F}^{(0)}_u)$-$\mathcal{B}(\mathbb{R})$-measurable and $\sigma(\bigcup_{u \in [0, \infty)} \mathcal{F}^{(0)}_u) \subseteq \mathcal{F}^{(0)}_0$, we obtain that $W^{(1)}$ and $\tilde{Z}$ are independent. This proves (vii). Lemma 3.2 implies that
\[
\mathbb{P}\left[ \forall t \in [0, \infty) : t \leq \tau^{(0)} \implies \tilde{Z}_t \tilde{W} = Z_t W^{(0)} \right] = 1.
\]
Therefore, we obtain that
\[
\mathbb{P}\left[ \tilde{Z} = Z_t \tilde{W} \right] = 1.
\]
As $\tilde{W}_{t,T^{(0)}} - \tilde{W}_{t^{(0)}} = W_t^{(1)}$ for all $t \in [0, \infty)$, Lemma 3.1 and (3.6) imply that
\[
\mathbb{P}\left[ Z_{t,T^{(0)}} = Z_t W^{(1)}, \forall t \in [0, \infty) \right] = 1.
\]
Combining (3.5) and (3.7) establishes (viii). □

**Lemma 3.4** Assume that we have the setting of Sect. 3.1. For every $m \in \mathbb{N}_0$, let $\mathbb{F}^{(m)}(\mathcal{F}^{(m)}_t)_{t \in [0, \infty)}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and assume that $(\bigcup_{u \in [0, \infty)} \mathcal{F}^{(m)}_u) \subseteq \mathcal{F}^{(m+1)}_0$, let $W^{(m)} : [0, \infty) \times \Omega \to \mathbb{R}$ be an $\mathbb{F}^{(m)}$-Brownian motion and $\tau^{(m)} : \Omega \to [0, \infty)$ an $\mathbb{F}^{(m)}$-stopping time. Assume that we have $\sum_{m=0}^{\infty} \tau^{(m)} = \infty$ and for every $m \in \mathbb{N}_0$, define $T^{(m)} = \sum_{i=0}^{m-1} \tau^{(i)}$. Further, let $W : [0, \infty) \times \Omega \to \mathbb{R}$ be the stochastic process given for all $m \in \mathbb{N}_0$, $t \in [0, \infty)$ by $W_0 = 0$ and
\[
(W_t - W_{T^{(m)}} - W^{(m)}_{[t-T^{(m)}]} \mathbb{1}_{[T^{(m)} \leq t \leq T^{(m+1)}]} = 0,
\]
let $\tilde{Z} : \Omega \to \mathbb{R}$ be an $\mathcal{F}_0^{(0)}$-$\mathcal{B}(\mathbb{R})$-measurable function, let $Z^{(m)} : \Omega \to \mathbb{R}$, $m \in \mathbb{N}_0$, be random variables, assume for every $m \in \mathbb{N}_0$ that $Z^{(m)}$ is $\mathcal{F}_0^{(m)}$-$\mathcal{B}(\mathbb{R})$-measurable, assume for all $m \in \mathbb{N}_0$ that $Z^{(0)} = \tilde{Z}$ and

$$Z^{(m+1)} = \tilde{Z}^{Z^{(m)}, W^{(m)}},$$

and let $\tilde{Z} : [0, \infty) \times \Omega \to \mathbb{R}$ be a stochastic process with continuous sample paths which satisfies that

$$\mathbb{P}[\left(\tilde{Z}_t - \tilde{Z}_r^{Z^{(m)}, W^{(m)}}\right) \mathbb{1}_{\{t \leq r \leq T^{(m+1)}\}} = 0, \forall m \in \mathbb{N}_0, t \in [0, \infty)] = 1.$$

Then:

(i) $W$ is a Brownian motion.

(ii) $W$ and $\tilde{Z}$ are independent.

(iii) We have

$$\mathbb{P}[Z\tilde{Z}_t, W_t = \tilde{Z}_t, \forall t \in [0, \infty)] = 1. \quad (3.8)$$

**Proof** Define the set $\mathcal{D}$ by

$$\mathcal{D} = \{(\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}):$$

for $i \in \{1, 2\}$, $\mathcal{G}^{(i)} = (\mathcal{G}^{(i)}_t)_{t \in [0, \infty)}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$

satisfying the usual conditions and $(\bigcup_{u \in [0, \infty)} \mathcal{G}^{(0)}_u) \subseteq \mathcal{G}^{(1)}_0$,

$V^{(i)} : [0, \infty) \times \Omega \to \mathbb{R}$ is a $\mathcal{G}^{(i)}$-Brownian motion,

$\upsilon^{(i)} : \Omega \to [0, \infty)$ is a $\mathcal{G}^{(i)}$-stopping time\}.

Define

$$\tilde{\mathcal{G}} : \mathcal{D} \to \{\mathcal{G} = (\mathcal{G}_t)_{t \in [0, \infty)} \text{ filtration on } (\Omega, \mathcal{F}, \mathbb{P}) \text{ satisfying the usual conditions}\},$$

$$\tilde{\upsilon} : \mathcal{D} \to \{\upsilon : \Omega \to [0, \infty) \text{ is a random variable}\},$$

$$\tilde{W} : \mathcal{D} \to \{V : [0, \infty) \times \Omega \to \mathbb{R} \text{ is a stochastic process}\}$$

by

$$\begin{align*}
(\tilde{\mathcal{G}}(\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}),
= \{A \in \mathcal{F} : A \cap \{t < \upsilon^{(0)}\} \in \mathcal{G}^{(0)}_t \text{ and } A \cap \{t \geq \upsilon^{(0)}\} \in \mathcal{G}^{(1)}_{\max\{t - \upsilon^{(0)}, 0\}}\},
\tilde{\upsilon}(\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}) = \upsilon^{(0)} + \upsilon^{(1)},
(\tilde{W}(\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)})_t = \begin{cases}
V^{(0)}_t, & t \leq \upsilon^{(0)},
V^{(1)}_{t - \upsilon^{(0)}} + V^{(0)}_{\upsilon^{(0)}}, & t \geq \upsilon^{(0)},
\end{cases}
\end{align*}$$

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respectively. For every $m \in \mathbb{N}_0$, let $\bar{F}^{(m)} = (\bar{F}^{(m)}_t)_{t \in [0, \infty)}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions, $\bar{W}^{(m)} : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ an $\bar{F}^{(m)}$-Brownian motion and $\bar{\tau}^{(m)} : \Omega \rightarrow [0, \infty)$ an $\bar{F}^{(m)}$-stopping time such that

$$(\bar{F}^{(0)}, \bar{W}^{(0)}, \bar{\tau}^{(0)}) = (\mathbb{F}^{(0)}, W^{(0)}, \tau^{(0)})$$

and

$$\bar{F}^{(m)} = \bar{F}(\bar{F}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)}),$$

$$\bar{W}^{(m)} = \bar{W}(\bar{F}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)}),$$

$$\bar{\tau}^{(m)} = \bar{\tau}(\bar{F}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)})$$

(the unique existence of $(\bar{F}^{(m)}, \bar{W}^{(m)}, \bar{\tau}^{(m)}), m \in \mathbb{N}_0,$ follows from Lemma 3.3). Observe that for every $m \in \mathbb{N}_0, t \in [0, \infty)$, it holds that $\bar{\tau}^{(m)} = T^{(m+1)}$ and

$$(W_t - \bar{W}_t^{(m)}) 1_{\{t \leq \bar{\tau}^{(m)}\}} = 0. \quad (3.9)$$

Hence, we obtain for every $t \in [0, \infty)$ that

$$\lim_{m \rightarrow \infty} \bar{W}_t^{(m)} = W_t. \quad (3.10)$$

This shows that $W$ is a Brownian motion. For every $m \in \mathbb{N}_0$, Lemma 3.3 implies that $\bar{Z}$ is $\mathbb{F}^{(m)}_0$-$\mathcal{B}(\mathbb{R})$-measurable and therefore independent of $\bar{W}^{(m)}$. Combining this with (3.10) implies that $\bar{Z}$ and $W$ are independent. Next we prove by induction on $m$ that for all $m \in \mathbb{N}_0$, it holds that

$$\mathbb{P}\left[(\bar{Z}_t, \bar{W}^{(m)}_t - \bar{Z}_t) 1_{\{t \leq \bar{\tau}^{(m)}\}} = 0, \forall t \in [0, \infty)\right] = 1. \quad (3.11)$$

The induction base case $m = 0$ is clear. If (3.11) holds for some $m \in \mathbb{N}_0$, note that the induction hypothesis implies that

$$\mathbb{P}\left[\bar{Z}_{\bar{\tau}^{(m)}} = \bar{Z}^{(m)} = \bar{Z}_{\bar{\tau}^{(m)}}^{(m+1)} = W^{(m)}_t \right] = 1.$$ 

Lemma 3.3 hence implies that it holds $\mathbb{P}$-a.s. for all $t \in [0, \infty)$ that

$$\bar{Z}_t^{(m+1)} = \begin{cases} 
\bar{Z}_t^{(m+1)}, & t \leq \bar{\tau}^{(m)}, \\
\bar{Z}_{t-\bar{\tau}^{(m)}}^{(m+1)}, & t \geq \bar{\tau}^{(m)}. 
\end{cases}$$

This proves (3.11) for $m + 1$. Combining (3.9) and (3.11) with Lemma 3.2 demonstrates (3.8).
4 Lower error bounds for CIR processes and squared Bessel processes in the case of a special choice of the parameters

4.1 Setting

For every $\delta \in (0, 2)$, $b \in [0, \infty)$, let

$$Z^{(\cdot),\delta,b,\cdot} = (Z^{\cdot},\delta,b,\cdot)_{\cdot \in \mathbb{R}, \cdot \in C([0,\infty];\mathbb{R})} : \mathbb{R} \times C([0,\infty); \mathbb{R}) \to C([0,\infty); \mathbb{R})$$

be a Borel-measurable and universally adapted function such that for every complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every filtration $(\mathcal{F}_t)_{t \in [0,\infty)}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions, every $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R})$-measurable function $Z : \Omega \to \mathbb{R}$, every $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion $W : [0,\infty) \times \Omega \to \mathbb{R}$ and every $t \in [0,\infty)$, it holds $\mathbb{P}$-a.s. that

$$Z_t^{Z,\delta,b,W} = Z + \int_0^t (\delta - b Z_s^{Z,\delta,b,W}) \, \mathrm{d}s + \int_0^t 2\sqrt{|Z_s^{Z,\delta,b,W}|} \, \mathrm{d}W_s. \quad (4.1)$$

(In other words, $Z$ is a mapping giving a strong solution to the SDE (4.1), and we also note that strong uniqueness holds for (4.1).) Let $\delta \in (0, 2)$, $b \in [0, \infty)$, define $C_0 = \{ f \in C([0, \infty); \mathbb{R}) : f(0) = 0 \}$, $C_0 = \{ f \in C([0, 1]; \mathbb{R}) : f(0) = f(1) = 0 \}$, and define $\nu : \{ \triangle, \square \} \to \mathbb{N}$ by $\nu(\triangle) = 3$ and $\nu(\square) = 4$. For every $n \in \mathbb{N}$, $* \in \{ \triangle, \square \}$, define $G_n^* : C_0 \times C_0 \to C_0$ by

$$\left( G_n^*(w, f) \right)_t = \begin{cases} (n w_1/n t + \frac{1}{\sqrt{n}} f_{nt}) (\nu(*) - 3) + w_t (4 - \nu(*)), & 0 \leq t \leq \frac{1}{n}, \\ w_t, & \frac{1}{n} \leq t < \infty, \end{cases}$$

and $F_n^* : [0, \infty) \times (C_0)^3 \times C_0 \to C_0$ by

$$\left( F_n^*(r, w^{(1)}, w^{\triangle}, w^{(2)}, f) \right)_t = \begin{cases} w_t^{(1)}, & t \leq r, \\ (G_n^*(w^{\triangle}, f))_{t-r} + w_t^{(1)}, & r \leq t \leq r + \frac{1}{n}, \\ w_{t-(r+1)/n}^{(2)} + w_1/n + w_{r}^{(1)}, & r + \frac{1}{n} \leq t. \end{cases}$$

For every $n \in \mathbb{N}$, $k \in \{ 1, 2 \}$, define $\Upsilon_n^k : [0, \infty) \to [(k-1)/n, k/n)$ by

$$\Upsilon_n^k(t) = \min \left( \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \right\} \cap [t, \infty) \right) - t + \frac{(k-1)}{n},$$

and for every $n \in \mathbb{N}$, define

$$S_n : [0, \infty) \times (C_0)^3 \times C_0 \to [1/n, \infty],$$

$$T_n : [0, \infty) \times (C_0)^3 \times C_0 \to [1/n, \infty],$$

$$\Phi_n = (\Phi_{n,1}, \ldots, \Phi_{n,6}) : [0, \infty) \times (C_0)^3 \times C_0$$

$$\to [0, \infty)^2 \times (C([0, \infty); \mathbb{R}))^2$$

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by
\[ S_n(t, y) = \max_{s \in [\Delta, \Box]} \inf \{ s \in [\Sigma_n^2(t), \infty) : Z_s^{0, \delta, b, F^*_n(\Sigma_n^1(t), y)} = 0 \} \cup \{ \infty \}, \]
\[ T_n(t, y) = \begin{cases} S_n(t, y), & S_n(t, y) \neq \infty, \\ \Sigma_n^2(t), & S_n(t, y) = \infty, \end{cases} \]
\[ \Phi_n(t, y) = \left( \Phi_{n, 1}(t, y), \ldots, \Phi_{n, 6}(t, y) \right) \]
\[ = \left( t, t + T_n(t, y), F_n^\Box(\Sigma_n^1(t), y), F_n^\triangle(\Sigma_n^1(t), y), \right. \\
\left. Z^{0, \delta, b, F^*_n(\Sigma_n^1(t), y)}, Z^{0, \delta, b, F^\Box(\Sigma_n^1(t), y)} \right). \]

(4.2)

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, \(\tilde{W}, \tilde{W}^{(1)}, \tilde{W}^\triangle, \tilde{W}^{(2)} : \Omega \to C_0\) Brownian motions, \(B : \Omega \to C_0\) a Brownian bridge, \(Z : \Omega \to [0, \infty)\) a random variable, \(Y^{[n]} : \Omega \to (C_0)^3 \times C_0\), \(n \in \mathbb{N}_0\), i.i.d. random variables such that we have
\[ Y^{[0]} = (\tilde{W}^{(1)}, \tilde{W}^\triangle, \tilde{W}^{(2)}, B), \]
and let
\[ X^{n, m} = (X_1^{n, m}, \ldots, X_6^{n, m}) : \Omega \to [0, \infty)^2 \times (C_0)^2 \times (C(\mathbb{R}, \mathbb{R}))^2, \]
for \(n \in \mathbb{N}, m \in \mathbb{N}_0\), be given by \(X^{n, m} = \Phi_n(X_2^{n, m-1}, Y^{[m]})\) and
\[ X^{n, 0} = \begin{cases} (0, 0, \tilde{W}, \tilde{W}, Z^{\triangle, \delta, b, \tilde{W}}, Z^{\triangle, \delta, b, \tilde{W}}) \\ \text{if } Z_t^{\triangle, \delta, b, \tilde{W}} \neq 0, \forall t \in [0, \infty), \\ (0, \inf \{ t \in [0, \infty) : Z_t^{\triangle, \delta, b, \tilde{W}} = 0 \}, \tilde{W}, \tilde{W}, Z^{\triangle, \delta, b, \tilde{W}}, Z^{\triangle, \delta, b, \tilde{W}}) \\ \text{if } \exists t \in [0, \infty) \text{ with } Z_t^{\triangle, \delta, b, \tilde{W}} = 0. \end{cases} \]

(4.3)

For every \(n \in \mathbb{N}, * \in \{\triangle, \Box\}\), let \(W^{(n), *} : \Omega \to C_0\) be a stochastic process which satisfies for all \(m \in \mathbb{N}_0, t \in [0, \infty)\) that \(W^{(n), *}_0 = 0\) and
\[ (W^{(n), *}_t - W^{(n), *}_{X_{1, n}^{m}} - (X_0^{n, m})_{|t - X_{1, n}^{m}|}) \mathbb{1}_{\{X_1^{n, m} \leq t \leq X_2^{n, m}\}} = 0, \]
let \(Z^{(n), *} : [0, \infty) \times \Omega \to \mathbb{R}\) be a stochastic process with continuous sample paths which satisfies for all \(m \in \mathbb{N}_0\) that
\[ \mathbb{P} \left[ \left( Z^{(n), *}_t - (X_0^{n, m} + X_{1, n}^{m})_{|t - X_{1, n}^{m}|} \mathbb{1}_{\{X_1^{n, m} \leq t \leq X_2^{n, m}\}} = 0, \forall t \in [0, \infty) \right) \right] = 1, \]
and for every \(n \in \mathbb{N}\), let \(\mathcal{M}_n : \Omega \to \mathbb{N}_0\) and \(\gamma_n : \Omega \to [0, 1] \cup \{ \infty \}\) be given by \(\mathcal{M}_n = \sup(\{0\} \cup \{m \in \{0, 1, \ldots, n + 1\} : X_1^{n, m} \leq 1\})\) and
\[ \gamma_n = \begin{cases} X_1^{n, \mathcal{M}_n}, & \mathcal{M}_n \neq 0, \\
\infty, & \mathcal{M}_n = 0. \end{cases} \]

Finally, assume that \(\tilde{W}, \tilde{W}^{(1)}, \tilde{W}^\triangle, \tilde{W}^{(2)}, B, Z, Y^{[1]}, Y^{[2]}, Y^{[3]} \ldots\) are independent.
4.2 Properties of the constructed random objects

4.2.1 The Feller boundary condition revisited

Lemma 4.1 In the setting of Sect. 4.1, we have
\[ P\left[ \exists t \in [0, \infty) \text{ with } Z_t^{Z,\delta,b,\tilde{W}} = 0 \right] = 1. \]

Proof Note that \( \delta \in (0, 2) \) and Lemma 2.6 ensure that for all \( z \in [0, \infty) \), we have
\[ P\left[ Z_t^{Z,\delta,b,\tilde{W}} \neq 0, \forall t \in [0, \infty) \right] = 0. \quad (4.4) \]

Next observe that the integral transformation theorem, the fact that \( Z \) and \( \tilde{W} \) are independent and Fubini’s theorem ensure that
\[ P\left[ Z_t^{Z,\delta,b,\tilde{W}} \neq 0, \forall t \in [0, \infty) \right] = \mathbb{E}\left[ \mathbb{1}_{\{v \in C([0,\infty); \mathbb{R}) : v(t) \neq 0, \forall t \in [0,\infty)\}} \left( Z_t^{Z,\delta,b,\tilde{W}} \right) \right] \]
\[ = \int_{[0,\infty)} \int_{C([0,\infty); \mathbb{R})} \mathbb{1}_{\{v \in C([0,\infty); \mathbb{R}) : v(t) \neq 0, \forall t \in [0,\infty)\}} \left( Z_t^{Z,\delta,b,\tilde{W}} \right) \tilde{W}(\mathbb{P}) B(C([0,\infty); \mathbb{R})) (dw) Z(\mathbb{P}) B([0,\infty)) (dz). \]

Combining this and (4.4) assures that
\[ P\left[ Z_t^{Z,\delta,b,\tilde{W}} \neq 0, \forall t \in [0, \infty) \right] = \int_{0}^{\infty} P\left[ Z_t^{Z,\delta,b,\tilde{W}} \neq 0, \forall t \in [0, \infty) \right] Z(\mathbb{P}) B([0,\infty)) (dz) = 0. \]

Hence the result follows. \( \square \)

4.2.2 One step in the construction of the Brownian motions

The following result is well known.

Lemma 4.2 Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \( T \in (0, 1) \), \( W : [0, 1] \times \Omega \to \mathbb{R} \) a Brownian motion and \( B : [0, 1] \times \Omega \to \mathbb{R} \) a Brownian bridge. Assume that \( W \) and \( B \) are independent, and define \( \mathcal{W} : [0, T] \times \Omega \to \mathbb{R} \) by
\[ \mathcal{W}_t = \frac{t}{T} W_T + \sqrt{T} B_t. \]

Then \( \mathcal{W} \) is a Brownian motion.

Lemma 4.3 In the setting of Sect. 4.1, let \( n \in \mathbb{N} \), let \( \tau : \Omega \to [0, \infty) \) be a random variable, assume \( Y^{[0]} \) and \( \tau \) are independent, and define \( W^{\square}, W^{\Delta}, W^{\Box} : \Omega \to C_0 \) by
\[ W^{\square} = G_n^{\square}(\tilde{W}^{\Delta}, B), \quad W^{\Delta} = F_n^{\Delta}(\tau, Y^{[0]}), \quad W^{\Box} = F_n^{\Box}(\tau, Y^{[0]}). \]

Then the stochastic processes \( W^{\square}, W^{\Delta}, \) and \( W^{\Box} \) are Brownian motions.
Proof For a constant \( \tau \), this follows from Lemma 4.2. The case of a general \( \tau \) follows from the result with a constant \( \tau \) by using the independence of \( Y[0] \) and \( \tau \).  

Lemma 4.4 In the setting of Sect. 4.1, let \( n \in \mathbb{N} \), let \( \tau : \Omega \rightarrow [0, \infty) \) be a random variable, assume \( Y[0] \) and \( \tau \) are independent, define \( \rho : \Omega \rightarrow [1/n, 2/n] \) by \( \rho = \mathbb{E}^2_n(\tau) \), and for \( * \in \{ \triangle, \square \} \), define \( W^* : \Omega \rightarrow C_0 \) and \( Z^* : \Omega \rightarrow C([0, \infty); \mathbb{R}) \) by

\[
W^* = F^*_n(\mathbb{E}^1_n(\tau), Y[0]) \quad \text{and} \quad Z^* = \mathbb{E}^{0, \delta, b, W^*}_n.
\]

Then:

(i) \( \tilde{W}^{(2)} \) and \((Z^\triangle, Z^\square)\) are independent.

(ii) For \( * \in \{ \triangle, \square \} \), we have

\[
\mathbb{P}[Z_{t+\rho}^* = Z_{t+\rho}^*, \forall t \in [0, \infty)] = 1.
\]

(iii) We have

\[
\mathbb{P}[Z^\triangle_{\rho} \geq Z^\square_{\rho} \iff (Z^\triangle_{t+\rho} \geq Z^\square_{t+\rho}, \forall t \in [0, \infty))] = 1.
\]

(iv) We have

\[
\mathbb{P}[Z^\square_{\rho} \geq Z^\triangle_{\rho} \iff (Z^\square_{t+\rho} \geq Z^\triangle_{t+\rho}, \forall t \in [0, \infty))] = 1.
\]

(v) We have

\[
\mathbb{P}[S_n(\tau, Y[0]) = T_n(\tau, Y[0])]
= \inf(\{ \infty \cup \{ t \in [0, \infty) : t \geq \rho \text{ and } \max_{* \in \{ \triangle, \square \}} Z^*_t = 0 \})]\]

= 1.

Proof We prove Lemma 4.4 in two steps. In the first step, we assume that there exists a \( t \in [0, \infty) \) such that \( \tau(\omega) = t \) for all \( \omega \in \Omega \). Observe that

\[
\tilde{W}^{(2)} \quad \text{and} \quad (W^\triangle|_{[0, \mathbb{E}^2_n(\tau)] \times \Omega}, W^\square|_{[0, \mathbb{E}^2_n(\tau)] \times \Omega}) \text{ are independent.} \quad \text{(4.5)}
\]

Moreover, note that for every \( * \in \{ \triangle, \square \}, t \in [0, \infty) \), it holds that

\[
W^*_t + \mathbb{E}^1_n(t) = \tilde{W}^{(2)}.
\]

Combining this and (4.5) proves (i) and (ii). Next note that Lemma 2.1, (i) and (ii), establish (iii) and (iv). Moreover, observe that Lemma 2.6 implies that

\[
\mathbb{P}[S_n(\tau, Y[0]) = T_n(\tau, Y[0])] = 1.
\]

This together with (iii) and (iv) establishes (v). The case of a general \( \tau \) follows immediately from the case of a constant \( \tau \) by using that \( Y[0] \) and \( \tau \) are independent.  

\[ \square \]
4.2.3 Properties of the constructed random times

**Lemma 4.5** In the setting of Sect. 4.1, let \( n \in \mathbb{N} \). Then:

(i) For all \( m \in \mathbb{N}_0 \), we have

\[
0 \leq X_1^{n,m} \leq X_2^{n,m} = X_1^{n,m+1} \leq X_2^{n,m+1}.
\]  

(ii) We have

\[
\sup_{m \in \mathbb{N}_0} X_1^{n,m} = \sup_{m \in \mathbb{N}_0} X_2^{n,m} = \infty.
\]  

(iii) For all \( m \in \mathbb{N}, i \in \{5, 6\} \), we have

\[
P[(X_i^{n,m})_0 = 0] = 1.
\]

(iv) For all \( m \in \mathbb{N}_0, i \in \{5, 6\} \), we have

\[
P[(X_i^{n,m})_{X_2^{n,m} - X_1^{n,m}} = 0] = 1.
\]

**Proof** First, observe that (4.6) is a direct consequence from (4.3). Next note that for all \( t \in [0, \infty), y \in (C_0)^3 \times C_0, \) we have \( \tilde{T}_n(t, y) \geq \frac{1}{n} \). Combining this with (4.2) establishes (4.7). Next, we observe that for every \( m \in \mathbb{N}, i \in \{3, 4\} \), the stochastic process \( X_i^{n,m} \) is a Brownian motion (see Lemma 4.3). This establishes (4.8). It thus remains to prove (4.9). For this, note that Lemma 4.1 assures that for \( i \in \{5, 6\} \),

\[
P[(X_i^{n,0})_{X_2^{n,0} - X_1^{n,0}} = 0] = 1.
\]

In addition, observe that Lemma 4.4 (v) ensures that for all \( m \in \mathbb{N}, i \in \{5, 6\} \),

\[
P[(X_i^{n,m})_{X_2^{n,m} - X_1^{n,m}} = 0] = 1.
\]

Combining (4.10) and (4.11) establishes (4.9).

\( \Box \)

4.2.4 Properties of the constructed Brownian motions and squared Bessel processes

**Lemma 4.6** In the setting of Sect. 4.1, let \( n \in \mathbb{N} \). Then for every \( t \in \{0, 1/n, 2/n, \ldots\} \), we have

\[
W_t^{(n)} \triangleq W_t^{(n)} = W_t^{(n)}.
\]

**Proof** First, observe that \( (X_3^{n,0})_t = (X_4^{n,0})_t \) for all \( t \in [0, \infty) \). Hence, we obtain for all \( k \in \mathbb{N}_0 \) that

\[
(X_3^{n,0})_{k/n} = (X_4^{n,0})_{k/n},
\]

\[
(X_3^{n,0})_{X_2^{n,0}} = (X_4^{n,0})_{X_2^{n,0}}.
\]

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Next note that for all \( r \in [0, \infty), \, y \in (C_0)^3 \times C_{00}, \, t \in [0, \, r] \cup [r + 1/n, \infty) \), we have
\[
\left( F_n^\Delta (r, y) \right)_t = \left( F_n^\square (r, y) \right)_t.
\]

Moreover, observe that Lemma 4.5 (i) ensures that for all \( m \in \mathbb{N} \), we have
\[
X^{n,m}_2 - X^{n,m}_1 = T_n(X^{n,m}_2, \, Y^{[m]}_1) \geq \Sigma_n^2(X^{n,m}_2, -1) = \Sigma_n^1(X^{n,m}_1) + 1/n.
\]

This and (4.15) yield that for all \( m \in \mathbb{N}, \, k \in \mathbb{N}_0 \), one has
\[
(X^{n,m}_3)_{\Sigma_n^1(X^{n,m}_1) + k/n} = \left( F_n^\Delta (\Sigma_n^1(X^{n,m}_1), \, Y^{[m]}_1) \right)_{\Sigma_n^1(X^{n,m}_1) + k/n} = \left( F_n^\square (\Sigma_n^1(X^{n,m}_1), \, Y^{[m]}_1) \right)_{\Sigma_n^1(X^{n,m}_1) + k/n} = (X^{n,m}_4)_{\Sigma_n^1(X^{n,m}_1) + k/n},
\]
Combining this with (4.13), (4.14) and (4.16) establishes (4.12).

\[\Box\]

**Lemma 4.7** In the setting of Sect. 4.1, let \( n \in \mathbb{N}, \, * \in \{ \Delta, \square \} \). Then:

(i) \( W^{(n),*} \) is a Brownian motion.

(ii) \( W^{(n),*} \) and \( Z \) are independent.

(iii) We have
\[
\mathbb{P}[Z^{(n),*}_t = Z^Z_{\delta,b,W^{(n),*}}, \forall t \in [0, \infty)] = 1.
\]

**Proof** We present the proof in the case \(* = \Delta\). The case \(* = \square\) is handled similarly. For every \( m \in \mathbb{N}_0 \), let \( W^{(m)} : \Omega \rightarrow C_0 \) be the Brownian motion given by \( W^{(m)} = X^{n,m}_3 \), define \( \tau^{(m)} : \Omega \rightarrow [0, \infty) \) by \( \tau^{(m)} = X^{n,m}_2 - X^{n,m}_1 \) and the filtration \( \mathbb{P}^{(m)} = (\mathcal{F}_t^{(m)})_{t \in [0, \infty)} \) on \((\Omega, \mathcal{F}, \mathbb{P})\) by
\[
\mathcal{F}_t^{(0)} = \sigma(\sigma(W^{(0)}_s : s \in [0, t]) \cup \sigma(Z) \cup \{ A \in \mathcal{F} : \mathbb{P}[A] = 0 \}),
\]
\[
\mathcal{F}_t^{(m)} = \sigma(\sigma(W^{(m)}_s : s \in [0, t]) \cup \sigma(Z, \tilde{W}, Y^{[m]}_4) \cup (1 \cup \{ A \in \mathcal{F} : \mathbb{P}[A] = 0 \}).
\]

For every \( m \in \mathbb{N}_0 \), define the mappings \( \tilde{t}_1^{(m)} : \Omega \rightarrow [0, 1/n] \) by \( \tilde{t}_1^{(m)} = \Sigma_n^1(X^{n,m}_2, -1) \) and \( \tilde{t}_2^{(m)} : \Omega \rightarrow [1/n, 2/n) \) by \( \tilde{t}_2^{(m)} = \Sigma_n^2(X^{n,m}_2, -1) \). Note that \( W^{(m)} \) is an \( \mathbb{P}^{(m)} \)-Brownian motion for every \( m \in \mathbb{N}_0 \). Next note that for every \( m \in \mathbb{N}_0 \), it holds that
\[
\bigcup_{n \in [0, \infty)} \mathcal{F}_n^{(m)} \subseteq \mathcal{F}_{n+1}^{(m)}.
\]

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By Lemma 4.1, \( \tau^{(0)} \) is an \( \mathbb{P}^{(0)} \)-stopping time. Note that \( \tau_{1}^{(m)} \) is \( \mathcal{F}_{0}^{(m)} \)-B(\( [0, 1/n] \))-measurable for every \( m \in \mathbb{N} \). Moreover, observe that \( \tau_{2}^{(m)} \) is \( \mathcal{F}_{0}^{(m)} \)-B(\( [1/n, 2/n] \))-measurable for every \( m \in \mathbb{N} \). Lemma 4.4 (v) implies that for all \( m \in \mathbb{N} \), it holds \( \mathbb{P} \)-a.s. that

\[
\tau^{(m)} = X_{2}^{n,m} - X_{1}^{n,m} = T_{n}(X_{2}^{n,m-1}, Y^{[m]}) = S_{n}(X_{2}^{n,m-1}, Y^{[m]}) = \max_{s \in \{\Delta, \Box\}} \left( \inf \left\{ t \in [0, \infty) : t \geq \Xi_{n}(X_{2}^{n,m-1}) \right\} \right)
\]

\[
= \max_{s \in \{\Delta, \Box\}} \left( \inf \left\{ t \in [0, \infty) : t \geq \Xi_{n}(X_{2}^{n,m-1}) \right\} \right)
\]

Observe that for every \( m \in \mathbb{N} \), \( t \in [0, \infty) \), it holds that

\[
(X_{4}^{n,m})_{t} = \begin{cases} 
(X_{3}^{n,m})_{t}^{m}, & 0 \leq t \leq \tau_{1}^{(m)}, \\
\left( n \left( (X_{3}^{n,m})_{t}^{m} - (X_{3}^{n,m})_{t}^{m} \right) (t - \tau_{1}^{(m)}) \right) + \frac{1}{\sqrt{n}} (Y_{4}^{[m]})_{t \wedge \tau_{1}^{(m)}}, & \tau_{1}^{(m)} \leq t \leq \tau_{2}^{(m)}, \\
(X_{3}^{n,m})_{t}, & \tau_{2}^{(m)} \leq t < \infty.
\end{cases}
\]

Hence, we obtain for every \( m \in \mathbb{N} \), \( t \in [0, \infty) \), \( s \in [0, t] \) that

\[
(X_{4}^{n,m})_{s} \cdot 1_{\{t \geq \tau_{2}^{(m)}\}} = W_{s}^{(m)} \cdot 1_{\{s \leq \tau_{1}^{(m)}\} \cup \{s \geq \tau_{2}^{(m)}\}} \cdot 1_{\{t \geq \tau_{2}^{(m)}\}} + \left( n \left( W_{s}^{(m)} - W_{t \wedge \tau_{2}^{(m)}}^{(m)} \right) (s - \tau_{1}^{(m)}) + \frac{1}{\sqrt{n}} (Y_{4}^{[m]})_{t \wedge \tau_{1}^{(m)}} \right) \cdot 1_{\{t \geq \tau_{2}^{(m)}\}}.
\]

This demonstrates for every \( m \in \mathbb{N} \), \( t \in [0, \infty) \), \( s \in [0, t] \) that the function

\[
\Omega \ni \omega \mapsto (X_{4}^{n,m})_{s}(\omega) \cdot 1_{\{t \geq \tau_{2}^{(m)}\}}(\omega) \in \mathbb{R}
\]

is \( \mathcal{F}_{t}^{(m)} \)-\( \mathbb{B}(\mathbb{R}) \)-measurable. Hence, we obtain that \( (X_{i}^{n,m})_{s} \cdot 1_{\{s \geq \tau_{2}^{(m)}\}} \), \( t \in [0, \infty) \), is an \( \mathbb{P}^{(m)} \)-adapted stochastic process for every \( i \in \{5, 6\}, m \in \mathbb{N} \). This implies for every \( i \in \{5, 6\}, m \in \mathbb{N}, t \in [0, \infty) \) that

\[
\sup \left\{ (X_{i}^{n,m})_{s} : s \in [0, t] \text{ and } s \geq \tau_{2}^{(m)} \right\} = \sup \left\{ (X_{i}^{n,m})_{s} \cdot 1_{\{s \geq \tau_{2}^{(m)}\}} : s \in [0, t] \text{ and } s \geq \tau_{2}^{(m)} \right\},
\]

\[
\inf \left\{ (X_{i}^{n,m})_{s} : s \in [0, t] \text{ and } s \geq \tau_{2}^{(m)} \right\} = \inf \left\{ (X_{i}^{n,m})_{s} \cdot 1_{\{s \geq \tau_{2}^{(m)}\}} : s \in [0, t] \text{ and } s \geq \tau_{2}^{(m)} \right\}.
\]
are \( F_t^{(m)} \)-\( \mathcal{B}([0, \infty)) \)-measurable. So \( \inf \{ t \in [0, \infty) : t \geq \tau_t^{(m)} \} \) and \( (X_t^{n,m})_t = 0 \) is an \( F^{(m)} \)-stopping time for all \( i \in \{5, 6\} \), \( m \in \mathbb{N} \). This implies that for all \( m \in \mathbb{N} \),

\[
\max_{i \in \{5, 6\}} \inf \{ t \in [0, \infty) : t \geq \tau_t^{(m)} \} \text{ and } (X_t^{n,m})_t = 0
\]

is an \( F^{(m)} \)-stopping time. Combining this and (4.18) assures that \( \tau_t^{(m)} \) is an \( F^{(m)} \)-stopping time for all \( m \in \mathbb{N} \). Next observe that for all \( m \in \mathbb{N} \), \( \tau_t^{(m)} \geq 1/n \) ensures that

\[
\sum_{m \in \mathbb{N}_0} \tau_t^{(m)} = \infty.
\]  

(4.19)

Furthermore, note that Lemma 4.5 (iv) implies that for all \( m \in \mathbb{N}_0 \), it holds that

\[
P[(X_5^{n,m})_{\tau_t^{(m)}} = 0] = 1.
\]  

(4.20)

Combining (4.17), the fact that \( W_t^{(m)} \) is an \( F^{(m)} \)-Brownian motion and \( \tau_t^{(m)} \) is an \( F^{(m)} \)-stopping time for every \( m \in \mathbb{N}_0 \), (4.19), the fact that \( Z \) is \( F_t^{(0)} \)-\( \mathcal{B}([0, \infty)) \)-measurable, (4.20) and Lemma 3.4 completes the proof. \( \Box \)

4.2.5 On conditional distributions of the considered random objects

**Lemma 4.8** In the setting of Sect. 4.1, let \( n \in \mathbb{N} \) and for every \( r \in [0, \infty) \), define the probability measure \( \mathbb{P}_r : \mathcal{B}([0, \infty)^2 \times (C_0)^2 \times (C([0, \infty); \mathbb{R}))^2) \to [0, 1] \) by

\[
\mathbb{P}_r(B) = \mathbb{P}\left[ \Phi_n(r, Y^{[0]}_t) \in B \mid \Phi_n,2(r, Y^{[0]}_t) > 1 \right].
\]

Then it holds for all \( B \in \mathcal{B}([0, \infty)^2 \times (C_0)^2 \times (C([0, \infty); \mathbb{R}))^2) \) that

\[
\mathbb{P}\left[ 1_{\{0 \leq \gamma_n \leq 1\}} \mathbb{P}[X_n,M_n \in B \mid \sigma(\gamma_n)] = 1_{\{0 \leq \gamma_n \leq 1\}} \mathbb{P}_{\min(\gamma_n, 1)}[B] \right] = 1.
\]

**Proof** Set \( \mathbb{B} = [0, \infty)^2 \times (C_0)^2 \times (C([0, \infty); \mathbb{R}))^2 \), define \( \mathbb{P}_\infty : \mathcal{B}(\mathbb{B}) \to [0, 1] \) by \( \mathbb{P}_\infty[B] = 0 \) and \( \mathbb{Q}_r : \mathcal{B}(\mathbb{B}) \to [0, 1] \), \( r \in [0, \infty) \), by

\[
\mathbb{Q}_r[B] = \mathbb{P}\left[ \{\Phi_n(r, Y^{[0]}_t) \in B\} \cap \{\Phi_n,2(r, Y^{[0]}_t) > 1\} \right]
\]

and \( \mathbb{Q}_\infty[B] = 0 \), and let \( A \in \mathcal{B}([0, 1]) \), \( B \in \mathcal{B}(\mathbb{B}) \). We need to prove that

\[
\mathbb{P}\left[ \{X_n,M_n \in B\} \cap \{\gamma_n \in A\} \right] = \mathbb{E}\left[ \mathbb{P}_{\gamma_n}(B) 1_A(\gamma_n) \right].
\]  

(4.21)

For this, we observe that

\[
\mathbb{P}\left[ \{X_n,M_n \in B\} \cap \{\gamma_n \in A\} \right] = \sum_{m=0}^\infty \mathbb{E}\left[ 1_B(X_n,M_n) 1_A(\gamma_n) 1_{[m]}(M_n) \right] = \sum_{m=1}^\infty \mathbb{E}\left[ 1_B(X_n,M_n) 1_A(\gamma_n) 1_{[m]}(M_n) \right] = \sum_{m=1}^\infty \mathbb{E}\left[ 1_B(X_n,m) 1_A(X_1^{n,m}) 1_{[0,1]}(X_1^{n,m}) 1_{(1,\infty)}(X_1^{n,m+1}) \right].
\]  

(4.22)
Next we recall that for all $m \in \mathbb{N}$, it holds that

(a) $X^{n,m} = \Phi_n(X^{n,m-1}_n, Y^{[m]}_n)$;
(b) $Y^{[m]}_n$ and $Y^{[0]}_n$ have the same distribution;
(c) $X^{n,m-1}_n$ and $Y^{[m]}_n$ are independent.

Note that (a) and Lemma 4.5 (i) ensure that for all $m \in \mathbb{N}$, it holds that

\[
\mathbb{E}[\mathbb{1}_B(X^{n,m}) \mathbb{1}_A(X^{n,m}_1) \mathbb{1}_{[0,1]}(X^{n,m}_1) \mathbb{1}_{(1,\infty)}(X^{n,m+1}_1)]
= \mathbb{E}[\mathbb{1}_B(X^{n,m}) \mathbb{1}_A(X^{n,m-1}_2) \mathbb{1}_{[0,1]}(X^{n,m-1}_2) \mathbb{1}_{(1,\infty)}(X^{n,m}_2)]
= \mathbb{E}[\mathbb{1}_A(X^{n,m-1}_2) \mathbb{1}_{[0,1]}(X^{n,m-1}_2) \times \mathbb{1}_B(Y^{[m]}_n) \mathbb{1}_{(1,\infty)}(\Phi_{n,2}(X^{n,m-1}_2, Y^{[m]}_n))].
\]

Items (b) and (c) hence show that for all $m \in \mathbb{N}$, one has

\[
\mathbb{E}[\mathbb{1}_B(X^{n,m}) \mathbb{1}_A(X^{n,m}_1) \mathbb{1}_{[0,1]}(X^{n,m}_1) \mathbb{1}_{(1,\infty)}(X^{n,m+1}_1)]
= \mathbb{E}[\mathbb{1}_A(X^{n,m-1}_2) \mathbb{1}_{[0,1]}(X^{n,m-1}_2) \mathbb{1}_{(1,\infty)}(X^{n,m}_2)]
= \mathbb{E}[\mathbb{1}_A(X^{n,m-1}_2) \mathbb{1}_{[0,1]}(X^{n,m-1}_2) \mathbb{1}_{(1,\infty)}(\Phi_{n,2}(X^{n,m-1}, Y^{[m]}_n))].
\]

Lemma 4.5 (i) therefore proves that for all $m \in \mathbb{N}$, it holds that

\[
\mathbb{E}[\mathbb{1}_B(X^{n,m}) \mathbb{1}_A(X^{n,m}_1) \mathbb{1}_{[0,1]}(X^{n,m}_1) \mathbb{1}_{(1,\infty)}(X^{n,m+1}_1)]
= \mathbb{E}[\mathbb{1}_A(X^{n,m-1}_2) \mathbb{1}_{[0,1]}(X^{n,m-1}_2) \mathbb{1}_{(1,\infty)}(X^{n,m}_2)]
= \mathbb{E}[\mathbb{1}_A(X^{n,m}_1) \mathbb{1}_{[0,1]}(X^{n,m}_1) \mathbb{1}_{(1,\infty)}(\Phi_{n,2}(X^{n,m-1}, Y^{[m]}_n))].
\]

Combining (4.22) with (4.23) yields

\[
\mathbb{E}[\mathbb{1}_B(X^{n,M_n}) \mathbb{1}_A(\gamma_n)] = \sum_{m=1}^{\infty} \mathbb{E}[\mathbb{1}_A(\gamma_n) \mathbb{1}_{[0,1]}(M_n) \mathbb{1}_{[m]}(M_n)] = \mathbb{E}[\mathbb{1}_A(\gamma_n) \mathbb{1}_{[0,1]}(B) \mathbb{1}_{[m]}(M_n)].
\]

This establishes (4.21).

4.3 Lower bounds for strong $L^1$-distances between the constructed squared Bessel processes

4.3.1 A first very rough lower bound for strong $L^1$-distances between the constructed squared Bessel processes

Lemma 4.9 In the setting of Sect. 4.1, let $z \in [0, \infty)$ and define the Brownian motion $\tilde{W}^\triangle: \Omega \rightarrow C_0$ by $\tilde{W}^\triangle = G_1^\triangle(\tilde{W}^\triangle, B)$. Then the following are equivalent:

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(i) It holds that
\[ \mathbb{E}[|Z_{1}^{z,\delta,b},\tilde{W}^{\triangle} - Z_{1}^{z,\delta,b},\tilde{W}^{\square}|] = 0. \]

(ii) There exists a \(\mathcal{B}(\mathbb{R})-\mathcal{B}(\mathbb{R})\)-measurable function \(f : \mathbb{R} \to \mathbb{R}\) which satisfies
\[ \mathbb{P}[Z_{1}^{z,\delta,b},\tilde{W}^{\triangle} = f(\tilde{W}_{1}^{\triangle})] = 1. \]

**Proof** To prove (ii) \(\Rightarrow\) (i), observe that (ii) ensures that \(\mathbb{P}[Z_{1}^{z,\delta,b},\tilde{W}^{\square} = f(\tilde{W}_{1}^{\square})] = 1\).

Combining this with the fact that \(\tilde{W}^{\triangle} = \tilde{W}^{\square}\) hence demonstrates that
\[ \mathbb{E}[|Z_{1}^{z,\delta,b},\tilde{W}^{\triangle} - Z_{1}^{z,\delta,b},\tilde{W}^{\square}|] = \mathbb{E}[|f(\tilde{W}^{\triangle}) - f(\tilde{W}^{\square})|] = 0. \]

To prove (i) \(\Rightarrow\) (ii), note that independence of \(\tilde{W}^{\triangle}\) and \(B\) and (i) assure that \(\mathbb{P}\)-a.s.,
\[ \mathbb{E}[Z_{1}^{z,\delta,b},\tilde{W}^{\triangle} | \sigma(\tilde{W}_{1}^{\triangle})] = \mathbb{E}[Z_{1}^{z,\delta,b},\tilde{W}^{\triangle} | \sigma(\tilde{W}_{1}^{\triangle}, B)] = \mathbb{E}[Z_{1}^{z,\delta,b},\tilde{W}^{\square} | \sigma(\tilde{W}_{1}^{\triangle}, B)] = Z_{1}^{z,\delta,b},\tilde{W}^{\triangle}. \]

The factorization lemma for conditional expectations hence establishes (ii). \(\Box\)

**Lemma 4.10** In the setting of Sect. 4.1, let \(z \in [0, \infty)\) and define the Brownian motion \(\tilde{W}^{\square} : \Omega \to C_0\) by \(\tilde{W}^{\square} = G_1^{\square}(\tilde{W}^{\triangle}, B)\). Then
\[ \inf_{r \in [0,1]} \inf_{\beta \in [0,b]} \mathbb{E}[|Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{\triangle}} - Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{\square}}|] > 0. \quad (4.24) \]

**Proof** In the case \((\delta, b) = (1, 0)\), (4.24) follows from Lemma 4.9 and Hefter and Herzzwurm [17, Eq. (2.4)]. For \((\delta, b) \in ((0, 2) \times [0, \infty)) \setminus \{(1, 0)\}\), inequality (4.24) follows from Lemma 4.9 and Hefter et al. [19, Theorem 6]. \(\Box\)

**Lemma 4.11** In the setting of Sect. 4.1, define for every \(r \in [0,1]\), \(* \in \{\triangle, \square\}\) the Brownian motion \(\mathcal{W}^{r,*} : \Omega \to C_0\) by \(\mathcal{W}^{r,*} = F_1^{r}(r, Y^{[0]}(\cdot))\). Then
\[ \inf_{r \in [0,1]} \inf_{\beta \in [0,b]} \mathbb{E}[|Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,\triangle}} - Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,\square}}|] > 0. \quad (4.25) \]

**Proof** Define the random fields \(U^{\triangle}, U^{\square}, V : [0, 1] \times [0, \infty) \times \Omega \to \mathbb{R}\) by
\[ U^*(r, \beta) = Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,*}} \quad \text{for } * \in \{\triangle, \square\}, \]
\[ V(r, \beta) = Z_r^{0,\delta,\beta,\tilde{W}^{(1)}}, \]
the function \(g : [0, 1] \times [0, \infty) \to \mathbb{R}\) by
\[ g(r, \beta) = \mathbb{E}[|U^{\triangle}(r, \beta) - U^{\square}(r, \beta)|], \]
\(\Box\)
and the Brownian motion $\tilde{W}^\square: \Omega \to C_0$ by $\tilde{W}^\square = G_1^{\square}(\tilde{W}^\triangle, B)$. Observe that for every $* \in \{\triangle, \square\}$, $r, t \in [0, 1]$, it holds that $(\mathcal{W}^{r,*})_{t+r} - (\mathcal{W}^{r,*})_t = \tilde{W}_t^*$. Moreover, note that for every $* \in \{\triangle, \square\}$, $r \in [0, 1]$, $t \in [0, r]$, it holds that $(\mathcal{W}^{r,*})_t = \tilde{W}_t^{(1)}$. Hence, we obtain for every $* \in \{\triangle, \square\}$, $r, t \in [0, 1]$, $\beta \in [0, \infty)$ almost surely that

$$
U^*(r, \beta) = Z_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,*}} = Z_r^{0,\delta,\mathcal{W}^{r,*}} = Z_1^{V(r,\beta),\delta,\beta,\tilde{W}^*} (4.26)
$$

(see Lemma 3.1). Combining Lemma 4.10, (4.26) and the independence of $V(r, \beta)$ and $(\tilde{W}^\triangle, \tilde{W}^\square)$ for $r \in [0, 1]$, $\beta \in [0, \infty)$ yields for all $r \in [0, 1]$, $\beta \in [0, \infty)$ that

$$
g(r, \beta) = \mathbb{E}\left[|U_\triangle(r, \beta) - U_\square(r, \beta)|\right] = \mathbb{E}\left[|Z_1^{V(r,\beta),\delta,\beta,\tilde{W}^\triangle} - Z_1^{V(r,\beta),\delta,\beta,\tilde{W}^\square}|\right] > 0. (4.27)
$$

In the next step, we combine Lemma 2.4, (4.26) and the independence of $\tilde{W}^*$ and $(V(r, \beta), V(t, \beta))$ for $* \in \{\triangle, \square\}$, $r, t \in [0, 1]$, $\beta \in [0, \infty)$ to obtain for every $* \in \{\triangle, \square\}$, $r, t \in [0, 1]$, $\beta \in [0, \infty)$ that

$$
\mathbb{E}\left[|U^*(r, \beta) - U^*(t, \beta)|\right] = \mathbb{E}\left[|Z_1^{V(r,\beta),\delta,\beta,\tilde{W}^*} - Z_1^{V(t,\beta),\delta,\beta,\tilde{W}^*}|\right] = e^{-\beta} \mathbb{E}\left[|V(r, \beta) - V(t, \beta)|\right] = e^{-\beta} \mathbb{E}\left[|Z_r^{0,\delta,\beta,\tilde{W}^*(\beta)} - Z_r^{0,\delta,\beta,\tilde{W}^*(\beta)}|\right]. (4.28)
$$

In addition, we note that Lemma 2.1 ensures that for all $* \in \{\triangle, \square\}$, $t \in [0, 1]$, $\beta_1, \beta_2 \in [0, \infty)$, one has

$$
\mathbb{E}\left[|U^*(t, \beta_1) - U^*(t, \beta_2)|\right] = |\mathbb{E}[U^*(t, \beta_1)] - \mathbb{E}[U^*(t, \beta_2)]|.
$$

The triangle inequality and (4.28) therefore imply that for all $* \in \{\triangle, \square\}$, $r, t \in [0, 1]$, $\beta_1, \beta_2 \in [0, \infty)$, it holds that

$$
\mathbb{E}\left[|U^*(r, \beta_1) - U^*(t, \beta_2)|\right] \leq \mathbb{E}\left[|U^*(r, \beta_1) - U^*(t, \beta_1)|\right] + \mathbb{E}\left[|U^*(t, \beta_1) - U^*(t, \beta_2)|\right] = e^{-\beta_1} \mathbb{E}\left[|Z_r^{0,\delta,\beta_1,\tilde{W}^*(\beta)} - Z_r^{0,\delta,\beta_1,\tilde{W}^*(\beta)}|\right] + |\mathbb{E}[U^*(t, \beta_1)] - \mathbb{E}[U^*(t, \beta_2)]| \leq \mathbb{E}\left[|Z_r^{0,\delta,\beta_1,\tilde{W}^*(\beta)} - Z_r^{0,\delta,\beta_1,\tilde{W}^*(\beta)}|\right] + |\mathbb{E}[U^*(t, \beta_1)] - \mathbb{E}[U^*(t, \beta_2)]|. (4.29)
$$

Next observe that for all $r \in [0, 1]$, $\beta \in [0, \infty)$, we have

$$
\limsup_{t \to 0, \beta \to [0, 1]} \mathbb{E}\left[|Z_r^{0,\delta,\beta,\tilde{W}^*(\beta)} - Z_r^{0,\delta,\beta,\tilde{W}^*(\beta)}|\right] = 0 (4.30)
$$

(cf. e.g. Mao [31, Theorem 2.4.3]). Moreover, we note that Lemma 2.3 ensures that for all $* \in \{\triangle, \square\}$, $r \in [0, 1]$, $\beta_1 \in [0, \infty)$, one has

$$
\lim_{\beta_2 \to \infty} \sup_{(r, \beta_1)} |\mathbb{E}[U^*(t, \beta_2)] - \mathbb{E}[U^*(t, \beta_1)]| = 0. (4.31)
$$
Combining (4.29)–(4.31) yields for all \( * \in \{\triangle, \square\}, \ r \in [0, 1], \ \beta_1 \in [0, \infty) \) that

\[
\limsup_{(t, \beta_2) \to (r, \beta_1), \ (t, \beta_2) \in [0, 1] \times [0, \infty)} \mathbb{E}[|U^*(t, \beta_2) - U^*(r, \beta_1)|] = 0.
\]

Therefore \( g \) is a continuous function. Combining this with (4.27) establishes (4.25).

\[\square\]

4.3.2 On conditional \( L^1 \)-distances between the constructed squared Bessel processes

**Lemma 4.12** In the setting of Sect. 4.1, let \( n \in \mathbb{N} \cap [5, \infty) \), take \( t_0 \in [0, \frac{1}{2}] \), set \( t_1 = \Xi_n(t_0) \in [0, 1/n) \), \( t_2 = \Xi_n^2(t_0) \in [1/n, 2/n) \), \( t_3 = 1 - t_0 \in [\frac{1}{2}, 1] \), and for \( * \in \{\triangle, \square\} \), define the Brownian motion \( W^* : \Omega \to \mathbb{C}_0 \) by \( W^* = F_n^*(t_1, Y^{[0]}) \). Then:

(i) \( t_1 < t_2 < t_3 \).

(ii) \( \mathbb{P}[\inf_{t \in [t_2, t_3]} \max_{* \in \{\triangle, \square\}} \mathbb{Z}^{0, \delta, b, W^*}_s > 0] > 0 \).

(iii) We have

\[
\mathbb{E}[|\mathbb{Z}^{0, \delta, b, W^\square}_{t_3} - \mathbb{Z}^{0, \delta, b, W^\square}_{t_3}|] \geq \frac{\mathbb{E}[|\mathbb{Z}^{0, \delta, b/n, F_n^\square(t_1, n, Y^{[0]})}_{t_1 n + 1} - \mathbb{Z}^{0, \delta, b/n, F_n^\square(t_1, n, Y^{[0]})}_{t_1 n + 1}|]}{2ne^{b(t_3 - t_2)}} \mathbb{P}[\mathbb{Z}^{0, \delta, b, W^\square}_s > 0, \forall s \in [2/n, 1/2)]](4.32)
\]

**Proof** For \( * \in \{\triangle, \square\} \), define the mappings \( U^* : \Omega \to \mathbb{R} \) by \( U^* = \mathbb{Z}^{0, \delta, b, W^*}_{t_2} \) and \( \tilde{Y} : \Omega \to (\mathcal{C}_0)^3 \times \mathcal{C}_{00} \) by

\[
\tilde{Y} = ((\sqrt{n} \tilde{W}^{(1)}_{t/n})_{t \in [0, \infty)}, (\sqrt{n} \tilde{W}^{\square}_{t/n})_{t \in [0, \infty)}, (\sqrt{n} \tilde{W}^{(2)}_{t/n})_{t \in [0, \infty)}, B).
\]

Observe that \( n \geq 5 \) ensures that

\[
t_2 \leq \frac{2}{n} \leq \frac{2}{5} < \frac{1}{2} \leq t_3.
\]

This and the fact that \( 0 \leq t_1 < 1/n \leq t_2 \) prove (i). Using that \( W^\triangle \) is a Brownian motion, \( \delta > 0 \) and \( t_3 < \infty \) demonstrates that

\[
\mathbb{P}\left[\inf_{s \in [t_2, t_3]} \max_{* \in \{\triangle, \square\}} \mathbb{Z}^{0, \delta, b, W^*}_s > 0\right] \geq \mathbb{P}\left[\inf_{s \in [t_2, t_3]} \mathbb{Z}^{0, \delta, b, W^{\triangle}}_s > 0\right] = \mathbb{P}[\mathbb{Z}^{0, \delta, b, W^{\triangle}} > 0, \forall s \in [t_2, t_3]] > 0.
\]

Moreover, observe that Lemma 4.4, (i) and (ii), and the fact that pathwise uniqueness holds for the SDE (4.1) imply

\[
\mathbb{E}[|\mathbb{Z}^{0, \delta, b, W^{\square}}_{t_3} - \mathbb{Z}^{0, \delta, b, W^{\square}}_{t_3}|] = \mathbb{E}[|\mathbb{Z}^{U^{\triangle}, \delta, b, W^{(2)}}_{t_3 - t_2} - \mathbb{Z}^{U^{\square}, \delta, b, W^{(2)}}_{t_3 - t_2}|] = 0.
\]

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Hence, we obtain

\[
\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}| \mid \inf_{s \in [t_2,t_3]} \max_{s \in \{\Delta,\square\}} Z_{s}^{0,\delta,b,W^*} > 0] \\
= \frac{\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}| \mid \inf_{s \in [t_2,t_3]} \max_{s \in \{\Delta,\square\}} Z_{s}^{0,\delta,b,W^*} > 0]}{\mathbb{P}[\max_{s \in \{\Delta,\square\}} Z_{s}^{0,\delta,b,W^*} > 0, \forall s \in [t_2,t_3]]}
\]

(4.34)

In the next step, we note that Lemma 4.4, (iii) and (iv), and (4.33) imply

\[
\mathbb{P}\left[\max_{s \in \{\Delta,\square\}} Z_{s}^{0,\delta,b,W^*} > 0, \forall s \in [t_2,t_3]\right] \leq 2 \mathbb{P}\left[Z_{s}^{0,\delta,b,W^\Delta} > 0, \forall s \in [t_2,t_3]\right] \\
\leq 2 \mathbb{P}\left[Z_{s}^{0,\delta,b,W^\Delta} > 0, \forall s \in \left[\frac{2}{n}, \frac{1}{2}\right]\right].
\]

(4.35)

In addition, we observe that Lemma 4.4 (ii) ensures

\[
\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}|] = \mathbb{E}[|Z_{t_3-t_2}^{U^\Delta,\delta,b,\tilde{W}^{(2)}} - Z_{t_3-t_2}^{U^\square,\delta,b,\tilde{W}^{(2)}}|].
\]

(4.36)

Furthermore, we note that Lemma 4.4 (i) implies that \((U^\Delta, U^{\square})\) and \(\tilde{W}^{(2)}\) are independent. Combining this with (4.36) and Lemma 2.4 assures

\[
\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}|] = e^{-b(t_3-t_2)} \mathbb{E}[|U^\Delta - U^{\square}|] \\
= e^{-b(t_3-t_2)} \mathbb{E}[|Z_{t_3-t_2}^{0,\delta,b,W^\Delta} - Z_{t_2}^{0,\delta,b,W^{\square}}|].
\]

Combining this with Lemma 2.5 assures

\[
\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}|] \\
= e^{-b(t_3-t_2)} \frac{1}{n} \mathbb{E}[|Z_{n(t_3-t_2)}^{0,\delta,b/n,(\sqrt{n}W_{t_3}^{\square})_{t \in [0,\infty)}} - Z_{n(t_3-t_2)}^{0,\delta,b/n,(\sqrt{n}W_{t_2}^{\square})_{t \in [0,\infty)}}|].
\]

Using \(t_2n = t_1n + 1\) and the fact that \(\sqrt{n}W_{t_3}^{\ast} = \sqrt{n}(F_n^{\ast}(t_1,Y^{0}))_{t/n} = (F_1^{\ast}(t_1n,\tilde{Y}))_{t}\) for every \(* \in \{\Delta,\square\}, t \in [0,\infty)\) therefore demonstrates that

\[
\mathbb{E}[|Z_{t_3}^{0,\delta,b,W^\Delta} - Z_{t_3}^{0,\delta,b,W^{\square}}|] \\
= e^{-b(t_3-t_2)} \frac{1}{n} \mathbb{E}[|Z_{t_1n+1}^{0,\delta,b/n,F_1^{\Delta}(t_1n,\tilde{Y})} - Z_{t_1n+1}^{0,\delta,b/n,F_1^{\square}(t_1n,\tilde{Y})}|] \\
= e^{-b(t_3-t_2)} \frac{1}{n} \mathbb{E}[|Z_{t_1n+1}^{0,\delta,b/n,F_1^{\Delta}(t_1n,Y^{0})} - Z_{t_1n+1}^{0,\delta,b/n,F_1^{\square}(t_1n,Y^{0})}|].
\]

(4.37)

Combining (4.34), (4.35) and (4.37) yields (4.32).
Lemma 4.13 In the setting of Sect. 4.1, we have

\[
\inf_{n \in \mathbb{N}} \inf_{r \in [0,1/2]} \mathbb{E}\left[ \left| \mathcal{Z}_{1-r}^{0,\delta,b,F_{n}^{\Delta}(\mathcal{I}_{n}^{1}(r),Y^{[0]}_k)} - \mathcal{Z}_{1-r}^{0,\delta,b,F_{n}^{\Box}(\mathcal{I}_{n}^{1}(r),Y^{[0]}_k)} \right| \right]
\]

\[
\max_{* \in \{\Delta, \Box\}} \mathcal{Z}_{s}^{0,\delta,b,F_{n}^{*}(\mathcal{I}_{n}^{1}(r),Y^{[0]}_k)} > 0, \forall s \in [\mathcal{I}_{n}^{2}(r), 1 - r] \right) > 0. \tag{4.38}
\]

Proof This follows from Lemmas 2.7, 4.11 and 4.12. \hfill \Box

Lemma 4.14 In the setting of Sect. 4.1, fix \( n \in \mathbb{N} \). For every \( * \in \{\Delta, \Box\}, r \in [0,1] \), define the Brownian motion \( \mathcal{W}^{*,r} : \Omega \to C_{0} \) by \( \mathcal{W}^{*,r} = F_{n}^{*}(\mathcal{I}_{n}^{1}(r),Y^{[0]}_k) \), and for every \( r \in [0,1] \), define

\[
E_{r} = \mathbb{E}\left[ \left| \mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{\Delta,r}} - \mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{\Box,r}} \right| \right] \max_{* \in \{\Delta, \Box\}} \mathcal{Z}_{s}^{0,\delta,b,\mathcal{W}^{*,r}} > 0, \forall s \in [\mathcal{I}_{n}^{2}(r), 1 - r] \right].
\]

Then it holds \( \mathbb{P} \)-a.s. that

\[
\mathbb{E}\left[ \left| \mathcal{Z}_{1}^{0,\delta,b,W^{(n),\Delta}} - \mathcal{Z}_{1}^{0,\delta,b,W^{(n),\Box}} \right| \right] \mathbb{E}_{\{0 \leq \gamma_{n} \leq 1\}} = E_{\min\{\gamma_{n}, 1\}}. \tag{4.39}
\]

Proof Set \( E_{\infty} = 0 \) and \( \mathbb{B} = [0, \infty) \times (C_{0})^{2} \times (C([0, \infty); \mathbb{R})). \) For \( r \in [0, \infty) \), define the probability measure \( \mathbb{P}_{r} : \mathcal{B}(\mathbb{B}) \to [0,1] \) by

\[
\mathbb{P}_{r}[B] = \begin{cases} 
\mathbb{P} [ \Phi_{n}(r, Y^{[0]}_k) \in B | \Phi_{n,2}(r, Y^{[0]}_k) > 1], & \text{if } r < \infty, \\
\mathbb{P} [ \Phi_{n}(0, Y^{[0]}_k) \in B | \Phi_{n,2}(0, Y^{[0]}_k) > 1], & \text{if } r = \infty,
\end{cases}
\]

and define the function \( G : \mathbb{B} \to [0, \infty) \) by

\[
G(x) = |x_{5}(|1 - x_{1}|) - x_{6}(|1 - x_{1}|)|.
\]

Observe that for \( * \in \{\Delta, \Box\} \), it holds that

\[
\mathbb{P}\left[ \mathcal{Z}_{1}^{(n),*} = (X^{n,\mathcal{M}_{n}}_{b^{(n)}+2} - X^{n,\mathcal{M}_{n}}_{-1}) \right] = 1.
\]

Lemma 4.7 hence implies that for \( * \in \{\Delta, \Box\} \), one has

\[
\mathbb{P}\left[ \mathcal{Z}_{1}^{0,\delta,b,W^{(n)*}} = (X^{n,\mathcal{M}_{n}}_{b^{(n)}+2} - X^{n,\mathcal{M}_{n}}_{-1}) \right] = 1. \tag{4.40}
\]

Next observe for \( r \in [0,1] \) that \( \Phi_{n,2}(r, Y^{[0]}_k) > 1 \) if and only if \( \mathcal{T}_{n}(r, Y^{[0]}_k) > 1 - r \). Lemma 4.4 (v) therefore assures that for all \( r \in [0,1] \), it holds that

\[
\mathbb{P}\left[ \Phi_{n,2}(r, Y^{[0]}_k) > 1 \iff \left( \max_{* \in \{\Delta, \Box\}} \mathcal{Z}_{s}^{0,\delta,b,\mathcal{W}^{*,r}} > 0, \forall s \in [\mathcal{I}_{n}^{2}(r), 1 - r] \right) \right] = 1.
\]

Hence, we obtain that for all \( r \in [0,1] \),
\[ \int G(x) \mathbb{P}_r(dx) \]
\[ = \mathbb{E}\left[ G(\Phi_n(r, Y^{[0]})) \mid \Phi_{n,2}(r, Y^{[0]}) > 1 \right] \]
\[ = \mathbb{E}\left[ Z_{1-r}^{0,\delta,b,W^r} - Z_{1-r}^{0,\delta,b,W^\square} \bigg| \bigcap_{s \in [s_2^2(r), 1-r]} \left\{ \max_{s \in \{\triangle, \square\}} Z_{s}^{0,\delta,b,W^s} > 0 \right\} \right] \]
\[ = E_r. \tag{4.41} \]

Next observe that (4.40) yields
\[ \mathbb{P}\left[ Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square} \mid \sigma(\gamma_n) \right] = G(X^{n,M_n}) = 1. \]

Hence, we obtain that \( \mathbb{P} \)-a.s. one has
\[ \mathbbm{1}_{[0,1]}(\gamma_n) \mathbb{E}\left[ Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square} \mid \sigma(\gamma_n) \right] \]
\[ = \mathbbm{1}_{[0,1]}(\gamma_n) \mathbb{E}[G(X^{n,M_n}) \mid \sigma(\gamma_n)] \]
\[ = \mathbbm{1}_{[0,1]}(\gamma_n) \int G(x) \mathbb{P}[X^{n,M_n} \in dx \mid \sigma(\gamma_n)]. \]

Lemma 4.8 therefore assures that \( \mathbb{P} \)-a.s. we have
\[ \mathbbm{1}_{[0,1]}(\gamma_n) \mathbb{E}\left[ Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square} \mid \sigma(\gamma_n) \right] = \int G \mathbb{d}\mathbb{P}\min[\gamma_n,1] \mathbbm{1}_{[0,1]}(\gamma_n). \]

Combining this with (4.41) establishes (4.39). \( \square \)

### 4.3.3 A lower bound for hitting time probabilities

**Lemma 4.15** In the setting of Sect. 4.1, we have
\[ \inf_{n \in \mathbb{N}} \mathbb{P}\left[ 0 \leq \gamma_n \leq \frac{1}{2} \right] > 0. \]

**Proof** First, observe that Lemma 4.5 (i) ensures that for all \( n \in \mathbb{N}, \)
\[ \gamma_n \in [0, 1] \iff \mathcal{M}_n \neq 0 \]
\[ \iff \exists m \in \{1, 2, \ldots, n + 1\} \text{ with } X_i^{n,m} \leq 1 \]
\[ \iff X_i^{n,1} \leq 1 \]
\[ \iff X_i^{n,0} \leq 1. \]

Hence, we obtain that for all \( n \in \mathbb{N}, \)
\[ \mathbb{P}\left[ \gamma_n \in [0, 1] \iff \exists t \in [0, 1] \text{ with } Z_t^{Z,\delta,b,W} = 0 \right] = 1. \]
This ensures that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$,

$$\mathbb{P}\left[ \gamma_n \in [0, 1] \iff \exists t \in [0, 1] \text{ with } Z^{(n),*}_t = 0 \right] = 1.$$  

This and Lemma 4.7 demonstrate that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$,

$$\mathbb{P}\left[ \gamma_n \in [0, 1] \iff \exists t \in [0, 1] \text{ with } Z^{Z,\delta,b,W^{(n)},*}_t = 0 \right] = 1. \tag{4.42}$$

Next note that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$, one has

$$\mathbb{P}\left[ Z^{(n),*}_{X_{n,m}} = 0, \forall m \in \mathbb{N} \right] = 1.$$

This shows that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$,

$$\mathbb{P}\left[ \gamma_n \in [0, 1] \implies Z^{(n),*}_{\min(\gamma_n, 1)} = 0 \right] = 1. \tag{4.43}$$

Combining (4.42) and (4.43) demonstrates that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$,

$$\mathbb{P}\left[ 0 \leq \gamma_n \leq \frac{1}{2} \right] \geq \mathbb{P}\left[ \exists t \in \left[ 0, \frac{1}{2} \right] \text{ with } Z^{Z,\delta,b,W^{(n)},*}_t = 0 \right] \cap \mathbb{P}\left[ \gamma_n \in [0, 1] \implies Z^{Z,\delta,b,W^{(n)},*}_{\min(\gamma_n, 1)} = 0 \right].$$

Lemma 4.7 hence shows that for all $n \in \mathbb{N}$,

$$\mathbb{P}\left[ 0 \leq \gamma_n \leq \frac{1}{2} \right] \geq \mathbb{P}\left[ \exists t \in \left[ 0, \frac{1}{4} \right] \text{ with } Z^{Z,\delta,b,\tilde{W}}_t = 0 \right] \cap \mathbb{P}\left[ \exists t \in \left[ 0, \frac{1}{2} \right] \text{ with } Z^{Z,\delta,b,\tilde{W}}_t > 0, \forall t \in \left[ \frac{1}{2}, 1 \right] \right].$$
Combining this, the independence of $Z$ and $\tilde{W}$, the fact that $\tilde{W}$ is a Brownian motion and $0 < \delta < 2$ with Lemma 2.6 ensures that the right-hand side is strictly positive. This completes the proof. □

4.3.4 A refined lower bound for strong $L^1$-distances between the constructed squared Bessel processes

Lemma 4.16 In the setting of Sect. 4.1, we have

$$\inf_{n \in \mathbb{N} \cap [5, \infty)} (n^{\delta/2} \mathbb{E}[|Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square}|]) > 0. \quad (4.44)$$

Proof For every $n \in \mathbb{N} \cap [5, \infty)$, $* \in \{\triangle, \square\}$, $r \in [0, 1]$, define the Brownian motion $Y_{n,*} \colon \Omega \to C_0$ by $Y_{n,*} = F_n^* (\mathbb{X}_2^1 (r), Y^{[0]})$ and for every $n \in \mathbb{N} \cap [5, \infty)$, $r \in [0, 1]$, set

$$E_{n,r} = \mathbb{E} \left[ |Z_{1-r}^{0,\delta,b,Y_{n,*},\triangle} - Z_{1-r}^{0,\delta,b,Y_{n,*},\square}| \cap \max_{s \in [\mathbb{X}_2^1 (r), 1-r]} \{ \sup_{* \in \{\triangle, \square\}} Z_s^{0,\delta,b,Y_{n,*},r} > 0 \} \right].$$

Combining the tower property for conditional expectations and Lemma 4.14 implies that for all $n \in \mathbb{N} \cap [5, \infty)$,

$$\mathbb{E}[|Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square}|] \geq \mathbb{E} \left[ \mathbb{E}[|Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square}| | \sigma(\gamma_n)] \mathbb{1}_{\{0 \leq \gamma_n \leq 1\}} \right] \geq \mathbb{E} \left[ E_{n,\gamma_n} \mathbb{1}_{\{0 \leq \gamma_n \leq \frac{1}{2}\}} \right].$$

Hence, we obtain for all $n \in \mathbb{N} \cap [5, \infty)$ that

$$\mathbb{E}[|Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square}|] \geq \mathbb{P} \left[ \gamma_n \in \left[0, \frac{1}{2}\right] \right] \inf_{r \in [0,\frac{1}{2}]} E_{n,r}.$$ 

This assures that

$$\inf_{n \in \mathbb{N} \cap [5, \infty)} (n^{\delta/2} \mathbb{E}[|Z_1^{Z,\delta,b,W^{(n)},\triangle} - Z_1^{Z,\delta,b,W^{(n)},\square}|]) \geq \inf_{n \in \mathbb{N} \cap [5, \infty)} \left( \mathbb{P} \left[ \gamma_n \in \left[0, \frac{1}{2}\right] \right] n^{\delta/2} \inf_{r \in [0,\frac{1}{2}]} E_{n,r} \right) \geq \left( \inf_{n \in \mathbb{N} \cap [5, \infty)} \mathbb{P} \left[ \gamma_n \in \left[0, \frac{1}{2}\right] \right] \right) \left( \inf_{n \in \mathbb{N} \cap [5, \infty)} (n^{\delta/2} \inf_{r \in [0,\frac{1}{2}]} E_{n,r}) \right).$$

Combining this with Lemmas 4.15 and 4.13 establishes (4.44). □
4.4 Proofs for the lower error bounds

Lemma 4.17 In the setting of Sect. 4.1, there exists a $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$, it holds that

$$\inf_{\varphi: \mathbb{R}^n \to \mathbb{R}} \mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{2/n}, \ldots, \tilde{W}_1)|] \geq c n^{-\delta/2}.$$  

**Proof** Define the function $e = (e_n)_{n \in \mathbb{N}}: \mathbb{N} \to [0, \infty]$ by

$$e_n = \inf_{\varphi: \mathbb{R}^n \to \mathbb{R}} \mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{2/n}, \ldots, \tilde{W}_1)|]$$

and the constants $c, C \in [0, \infty]$ by

$$C = \inf_{n \in \mathbb{N} \cap [5, \infty)} (n^{\delta/2} \mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \triangle} - Z_1^{Z, \delta, b, W^{(n)}, \square}|])$$

and $c = \frac{C}{24}$. Note that Lemma 4.7, (i) and (ii), ensure that for all $n \in \mathbb{N}$ and all Borel-measurable functions $\varphi: \mathbb{R}^n \to \mathbb{R}$, one has

$$\mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)|] = \frac{1}{2} (2 \mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)|])$$

$$= \frac{1}{2} (\mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \triangle} - \varphi(W_{1/n}^{(n), \triangle}, W_{2/n}^{(n), \triangle}, \ldots, W_1^{(n), \triangle})|]$$

$$+ \mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \square} - \varphi(W_{1/n}^{(n), \square}, W_{2/n}^{(n), \square}, \ldots, W_1^{(n), \square})|]). \quad (4.45)$$

Next observe that Lemma 4.6 ensures

$$\varphi(W_{1/n}^{(n), \triangle}, W_{2/n}^{(n), \triangle}, \ldots, W_1^{(n), \triangle}) = \varphi(W_{1/n}^{(n), \square}, W_{2/n}^{(n), \square}, \ldots, W_1^{(n), \square}).$$

Combining this with (4.45) proves that

$$\mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)|]$$

$$= \frac{1}{2} (\mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \triangle} - \varphi(W_{1/n}^{(n), \triangle}, W_{2/n}^{(n), \triangle}, \ldots, W_1^{(n), \triangle})|]$$

$$+ \mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \square} - \varphi(W_{1/n}^{(n), \square}, W_{2/n}^{(n), \square}, \ldots, W_1^{(n), \square})|]).$$

The triangle inequality hence implies

$$\mathbb{E}[|Z_1^{Z, \delta, b, \tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)|] \geq \frac{1}{2} \mathbb{E}[|Z_1^{Z, \delta, b, W^{(n)}, \triangle} - Z_1^{Z, \delta, b, W^{(n)}, \square}|].$$
This establishes for all \( n \in \mathbb{N} \cap [5, \infty) \), that
\[
e_n \geq \frac{1}{2} \mathbb{P} \left[ |Z_1^{\delta,b,W^{(n)},\square} - Z_1^{\delta,b,W^{(n),\triangle}}| \right] \geq C n^{-\delta/2}.
\]
Combining this with \( e_n \geq e_{12} \) for \( n \in \{1, 2, 3, 4\} \) hence implies for all \( n \in \mathbb{N} \) that
\[
e_n \geq \min \left\{ e_1, e_2, e_3, e_4, \frac{C}{2} n^{-\delta/2} \right\} \geq \min \left\{ \frac{C}{2} \frac{12^{-\delta/2}}{n^{-\delta/2}}, \frac{C}{2} n^{-\delta/2} \right\}
\]
\[
\geq \frac{C}{2} 12^{-\delta/2} n^{-\delta/2} \geq \frac{C}{24} n^{-\delta/2} = c n^{-\delta/2}.
\]
(4.46)
Because Lemma 4.16 proves that \( C > 0 \), we obtain \( c \in (0, \infty) \). This and (4.46) complete the proof of Lemma 4.17.

\[5 \text{ Lower error bounds for CIR processes and squared Bessel processes in the general case}\]

**Corollary 5.1** Let \( \delta \in (0, 2) \), \( b, x \in [0, \infty) \), let \((\Omega, \mathfrak{F}, \mathbb{P})\) be a probability space with a filtration \((\mathcal{F}_t)_{t \in [0,1]}\) satisfying the usual conditions, let \( W : [0, 1] \times \Omega \rightarrow \mathbb{R} \) be an \((\mathcal{F}_t)_{t \in [0,1]}\)-Brownian motion, let \( X : [0, 1] \times \Omega \rightarrow [0, \infty) \) be an \((\mathcal{F}_t)_{t \in [0,1]}\)-adapted stochastic process with continuous sample paths satisfying for all \( t \in [0, 1] \) \( \mathbb{P}\)-a.s.

\[
X_t = x + \int_0^t (\delta - b X_s) \, ds + \int_0^t 2 \sqrt{X_s} \, dW_s.
\]

Then there exists a \( c \in (0, \infty) \) such that for all \( N \in \mathbb{N} \), it holds that
\[
\inf_{\varphi : \mathbb{R}^N \rightarrow \mathbb{R} \text{ Borel-measurable}} \mathbb{E} \left[ |X_1 - \varphi(W_{1/N}, W_{2/N}, \ldots, W_1)| \right] \geq c N^{-\delta/2}.
\]

**Proof** This is a consequence of Lemma 4.17. We only need to observe that all objects in Sect. 4.1 exist. See Kallenberg [28, Theorem 21.14] for the existence of \( Z^{(\cdot),\delta,b,(\cdot)} \) and Lemma 4.5 for the existence of \( W^{(n),*} \) and \( Z^{(n),*} \).

We are now in a position to present the proof of Theorem 1.1.

**Proof of Theorem 1.1** Define the filtration \((\mathcal{F}_t)_{t \in [0,1]}\) on \((\Omega, \mathfrak{F}, \mathbb{P})\) by \( \mathcal{F}_t = \mathcal{F}_{tT} \), the \((\mathcal{F}_t)_{t \in [0,1]}\)-Brownian motion \( W : [0, 1] \times \Omega \rightarrow \mathbb{R} \) by \( W_t = \frac{1}{\sqrt{T}} W_{tT} \), let \( \delta = \frac{4a}{\sigma^2} \), \( b = Tb \), \( \rho = \frac{4}{(T \sigma^2)} \in (0, \infty) \), \( x = \rho x \), and define the \((\mathcal{F}_t)_{t \in [0,1]}\)-adapted stochastic process \( X : [0, 1] \times \Omega \rightarrow [0, \infty) \) with continuous sample paths by \( X_t = \rho X_{tT} \). Observe that
\[
\delta \in (0, 2), \quad b \in [0, \infty), \quad x \in [0, \infty).
\]
Moreover, note that for all \( t \in [0, 1] \), it holds \( \mathbb{P}\)-a.s. that
\[ X_t = \rho X_{tT} \]
\[ = \rho x + \rho \int_0^{tT} (a - b X_s) \, ds + \rho \int_0^{tT} \sigma \sqrt{X_s} \, dW_s \]
\[ = \rho x + \rho T \int_0^{t} (a - b X_s T) \, ds + \rho \sqrt{T} \int_0^{t} \sigma \sqrt{X_s T} \, dW_s \]
\[ = \rho x + \rho T \int_0^{t} (a - b X_s / \rho) \, ds + \rho \sqrt{T} \int_0^{t} \sigma \sqrt{X_s / \rho} \, dW_s \]
\[ = x + \int_0^{t} (\delta - b X_s) \, ds + 2 \int_0^{t} \sqrt{X_s} \, dW_s. \] (5.2)

Next observe that for all \( N \in \mathbb{N} \), one has
\[ \inf_{\varphi : \mathbb{R}^N \to \mathbb{R} \text{ Borel-measurable}} \mathbb{E} \left[ |X_T - \varphi(W_{T/N}, W_{2T/N}, \ldots, W_T)| \right] \]
\[ = \frac{1}{\rho} \inf_{\varphi : \mathbb{R}^N \to \mathbb{R} \text{ Borel-measurable}} \mathbb{E} \left[ |X_1 - \varphi(W_{1/N}, W_{2/N}, \ldots, W_1)| \right]. \] (5.3)

Combining (5.1)–(5.3) with Corollary 5.1 establishes (1.2).

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