PARTIALLY REGULAR AND CSCK METRICS

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Abstract. A Kähler metric $g$ with integral Kähler form is said to be partially regular if the partial Bergman kernel associated to $mg$ is a positive constant for all integer $m$ sufficiently large. The aim of this paper is to prove that for all $n \geq 2$ there exists an $n$-complex dimensional manifold equipped with strictly partially regular and csck metric $g$. Further, for $n \geq 3$, the (constant) scalar curvature of $g$ can be chosen to be zero, positive or negative.

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1. Introduction

Let $M$ be an $n$-dimensional complex manifold endowed with a Kähler metric $g$. Assume that there exists a holomorphic line bundle $L$ over $M$ such that $c_1(L) = [\omega]_{dR}$, where $\omega$ is the Kähler form associated to $g$ and $c_1(L)$ denotes the first Chern class of $L$ (such an $L$ exists if and only if $\omega$ is an integral form). Let $h$ be an Hermitian metric on $L$ such that its Ricci curvature $\text{Ric}(h) = \omega$. Here $\text{Ric}(h)$ is the two–form on $M$ whose local expression is given by

$$\text{Ric}(h) = -\frac{i}{2\pi} \partial \overline{\partial} \log h(\sigma(x), \sigma(x)),$$

for a trivializing holomorphic section $\sigma : U \to L \setminus \{0\}$. Consider the separable complex Hilbert space $\mathcal{H}$ consisting of global holomorphic sections $s$ of $L$ such that

$$\langle s, s \rangle = \int_M h(s(x), s(x)) \frac{\omega^n}{n!} < \infty.$$

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Assume $\mathcal{H} \neq \{0\}$. Let $\mathcal{S} \subseteq \mathcal{H}$ be a complex subspace of $\mathcal{H}$ and let $s_j$, $j = 0, \ldots, N$ (dim $\mathcal{S} = N + 1 \leq \infty$) be an orthonormal basis of $\mathcal{S}$. In this paper we say that the metric $g$ is a partially balanced metric with respect to $\mathcal{S}$ if the smooth function, called partial Bergman kernel,

$$T_g^S(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x))$$

is a positive constant ($T_g^S$ really depends only on the metric $g$ and not on the orthonormal basis chosen). When $\mathcal{S} = \mathcal{H}$ then $T_g^\mathcal{H} = T_g$ is Rawnsley’s epsilon function (see [18], [7], [14] and references therein) and being $g$ a partially balanced metric with respect to $\mathcal{H}$ means that $g$ is balanced in Donaldson’s terminology (see [9] and [3] for the compact case and [1] for the noncompact case). Obviously, if $M$ is compact, $\mathcal{H} = H^0(L)$, where $H^0(L)$ is the (finite dimensional) space of global holomorphic sections of $L$. In the sequel we will say that a Kähler metric $g$ on a complex manifold $M$ is strictly partially balanced with respect to $\mathcal{S}$ if $T_g^S$ is a positive constant for $\mathcal{S}$ strictly contained in $\mathcal{H}$. Notice that given a (strictly) partially balanced metric $g$ on a complex manifold $M$ then, for all $x \in M$ there exists $s \in \mathcal{H}$ not vanishing at $x$ (the so called free based point condition in the compact case). Then the Kodaira’s map $\varphi : M \to \mathbb{C}P^N, x \to [s_0(x) : \cdots : s_N(x)]$ is well defined. Moreover, it is not hard to see that $\varphi^*\omega_{FS} = \omega + \frac{i}{2\pi} \partial \bar{\partial} \log T_g^S$ and hence in the partially balanced case $\varphi$ is indeed a Kähler immersion, i.e. $\varphi^*g_{FS} = g$ where $g_{FS}$ (resp. $\omega_{FS}$) is the Fubini-Study metric (resp. form) on $\mathbb{C}P^N$. Thus, by using the celebrated Calabi’s rigidity theorem ([5], [16]) one deduces that in the definition of partially balanced metric the space $\mathcal{S}$ is determined up to unitary transformations of $\mathcal{H}$ and one can then simply speak of strictly partially balanced metric without specifying the space $\mathcal{S}$. Partial Bergman kernels and their asymptotics have been recently considered, when $\mathcal{S}$ is the subspace of $\mathcal{H}$ consisting of those holomorphic sections of $L$ vanishing at a prescribed order on an analytic subvariety of $M$ (see [17], [19], [20], [21], [22], [23]). Notice that our definition is more general, since we are not fixing any analytic subvariety of $M$.

In this paper we address the study of those metrics $g$ such that $mg$ is strictly partially balanced (with respect to some complex subspace $\mathcal{S}_m \subset \mathcal{H}_m$) for $m$ sufficiently large, were $\mathcal{H}_m$ denotes the Hilbert space of global holomorphic sections of $L^m$ (the $m$-th tensor power of $L$) such that $\langle s, s \rangle_m = \int_M h_m(s(x), s(x)) < \infty$ and $h_m$ is the Hermitian metric on $L^m$ such that Ric($h_m$) = $mw$. Throughout the paper a metric satisfying the previous condition will be called a strictly partially regular metric. When $\mathcal{S}_m = \mathcal{H}_m$ a partially regular metric $g$ is regular as defined in [10] (see also [4] and references therein) and it follows that $g$ is a cscK (constant scalar curvature Kähler) metric.
Therefore it seems natural to address the following:

**Question:** Does there exist a complex manifold $M$ equipped with a cscK metric $g$ such that $g$ is strictly partially regular?

The aim of this paper is to provide a positive answer to the previous question in the noncompact case as expressed by the following theorem proved in the next section.

**Theorem 1.1.** For all positive integer $n \geq 2$ there exist an $n$-dimensional noncompact complex manifold $M$ equipped with a strictly partially regular cscK metric $g$. Furthermore for $n \geq 3$ the scalar curvature of $g$ can be chosen to be zero, positive or negative.

As we have already noticed above a partially balanced metric $g$ on a complex manifold $M$ is automatically projectively induced. On the other hand in the complex one-dimensional case a projectively induced cscK metric is regular, being homogeneous (actually a complex space form) and so it cannot be strictly partially regular. This is the reason why in Theorem 1.1 we assume $n \geq 2$.

Notice also that it is conjecturally true that a projectively induced cscK metric $g$ on a *compact* complex manifold $M$ is homogeneous and hence regular ([8], [13], see also [6] for an example of regular non homogeneous complete Kähler metric on the blow-up of $\mathbb{C}^2$ at the origin). Hence we believe that the previous question has a negative answer in the compact case.

Finally we still do not know if there exist complex 2-dimensional manifolds admitting strictly partially regular cscK metrics with non negative scalar curvature (see the proof of Theorem 1.1).

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### 2. Proof of Theorem 1.1

In order to prove Theorem 1.1 we first show that the punctured unit disk

$$D_* = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid 0 < |z|^2 < 1\}, \quad |z|^2 = |z_1|^2 + |z_2|^2,$$

can be equipped with a strictly partially regular cscK metric $g_*$ (see Proposition 2.1 below) whose associated Kähler form is given by:

$$\omega_* = \frac{i}{2\pi} \partial \bar{\partial} \Phi_* \quad \Phi_*(z) = \log |z|^2 - \log (1 - |z|^6).$$

A direct computation shows that its volume form is given by

$$\frac{\omega^2}{2!} = \frac{9r(1 + 2r^3)}{(1 - r^3)^3} (\frac{i}{2\pi})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2, \quad r = |z_1|^2 + |z_2|^2,$$

(2)
from which one easily sees (cfr. [11]) that \( g_* \) is a cscK (not Einstein) metric with constant scalar curvature
\[
s_{g_*} = -24\pi. \tag{3}
\]

**Remark 1.** Notice that the metric \( g_* \) is not complete at the origin while it is complete at \( \{ z \in \mathbb{C}^2 \mid |z|^2 = 1 \} \).

Let \( m \) be a positive integer and \( \mathcal{D}_* \times \mathbb{C} \) be the trivial holomorphic line bundle on \( \mathcal{D}_* \) endowed with the hermitian metric
\[
h^m_*(z, \xi) = e^{-m\Phi_*(z)}|\xi|^2 = \frac{(1 - |z|^6)^m|\xi|^2}{|z|^{2m}} = \frac{(1 - r^3)^m|\xi|^2}{r^m} \tag{4}
\]
satisfying \( \text{Ric}(h^m_*) = m\omega_* \) (cfr. [11]).

**Proposition 2.1.** The Kähler metric \( g_* \) on \( \mathcal{D}_* \) is strictly partially regular.

**Proof.** Consider the Hilbert space
\[
\mathcal{H}_m = \{ f \in \text{Hol}(\mathcal{D}_*) \mid \| f \|_m^2 = \int_{\mathcal{D}_*} e^{-m\Phi_*(z)}|f(z)|^2\omega_*^2 < \infty \}.
\]
We show that \( \mathcal{H}_m \neq \{0\} \) for \( m \geq 3 \) and an orthonormal basis of \( \mathcal{H}_m \) is given by the monomials \( \left\{ \frac{z^j\bar{z}^k}{\|z_j\bar{z}_k\|_m} \right\}_{j + k > m - 3} \), where
\[
\|z^j\bar{z}^k\|_m^2 = \frac{3j!k!}{4(j + k + 1)!} \frac{\Gamma\left(\frac{1}{3}j + \frac{1}{3}k + 1\right)(m - 3)!}{\Gamma\left(\frac{1}{3}j + \frac{1}{3}k + m - 1\right)} \left[ 1 + \frac{2\left(\frac{1}{3}j + \frac{1}{3}k + 1\right)}{\frac{1}{3}j + \frac{1}{3}k} + \frac{1}{4} \right]. \tag{5}
\]

Let us first see for which values of \( j \) and \( k \) the monomial \( z_j^k \) belongs to \( \mathcal{H}_m \), namely when its norm \( \|z^j\bar{z}^k\|_m^2 \) is finite. By passing to polar coordinates \( z_1 = \rho_1 e^{i\theta_1}, z_2 = \rho_2 e^{i\theta_2}, r = \rho_1^2 + \rho_2^2 \) one gets:
\[
\|z^j\bar{z}^k\|_m^2 = 9 \int_{\mathcal{D}_*} \left( \frac{1 + 2r^3}{r^{m-1}} \right)^{m-3} |z_1|^{2j} |z_2|^{2k} \frac{i}{2\pi} d\rho_1 d\rho_2
\]
\[
= 9 \int_{r=0}^{\rho} \rho_1^{2j+1} \rho_2^{2k+1} \frac{(1 + 2r^3)^{m-3}}{r^{m-1}} d\rho_1 d\rho_2
\]
Now, by setting \( r^* = \rho = \sqrt{\rho_1^2 + \rho_2^2} \) we can make the substitution \( \rho_1 = \rho \cos \theta, \rho_2 = \rho \sin \theta \), \( 0 < \rho < 1, 0 < \theta < \frac{\pi}{2} \), so that \( d\rho_1 d\rho_2 = \rho d\rho d\theta \) and the integral becomes
\[
9 \int_{\theta=0}^{\frac{\pi}{2}} \frac{(\cos \theta)^{2j+1}}{2(j + k + 1)!} \int_{\rho=0}^{1} \rho^{2j+2k-2m+5} (1 + 2\rho^6)(1 - \rho^6)^{m-3} d\rho
\]
\[
= \frac{9j!k!}{2(j + k + 1)!} \int_{\rho=0}^{1} \rho^{2j+2k-2m+5} (1 + 2\rho^6)(1 - \rho^6)^{m-3} d\rho \tag{6}
\]
Let us make the change of variable
and \(\mathcal{H}_m\) rewrites

\[
\|z_1^j z_2^k\|_m^2 = \frac{3j!k!}{4(j + k + 1)!} \int_0^1 x^2 \frac{1}{(1 + 2x)(1 - x)^{m-3}} dx. \]

This integral converges if and only if \(m \geq 3\) and \(j + k > m - 3\). Moreover, formula (5), easily follows by using the well-known fact that, for any \(\alpha, \beta \in \mathbb{C}\) with \(Re(\alpha) > 0, Re(\beta) > 0\), we have

\[
\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \]

By radiality it follows the monomials \(z_1^j z_2^k\) form a complete orthogonal system and hence \(\{z_1^j z_2^k\}_{j+k=m-3}\) turns out to be an orthonormal basis for \(\mathcal{H}_m\).

We are now ready to prove that \(g_*\) is strictly partially regular for \(m \geq 3\) with respect to the subspace \(\mathcal{S}_m \subset \mathcal{H}_m\) spanned by \(\{z_1^j z_2^k\}_{j+k=m=3i}\), for \(i = 0, 1, \ldots\).

One needs to verify that there exists a positive constant \(C_m\) (depending on \(m\)) such that:

\[
T_{\mathcal{S}_m g_*}(z) = \frac{(1 - |z|^2)^m}{|z|^{2m}} \sum_{j+k=m+3i, i=0,1,\ldots} \frac{|z_1|^{2j}|z_2|^{2k}}{\|z_1^j z_2^k\|_m^2} = C_m, \quad (7)
\]

Equation (5) for \(j + k = m + 3i\) becomes

\[
\|z_1^j z_2^k\|_m^2 = \frac{3}{4(m-1)(m-2)} \binom{m+3i}{j}^{-1} \binom{m+i-1}{i}^{-1}. \quad (8)
\]

By using

\[
\frac{1}{(1-x)^m} = \sum_{i=0}^{\infty} \frac{m(m+1) \cdots (m+i-1)}{i!} x^i = \sum_{i=0}^{\infty} \binom{m+i-1}{i} x^i
\]

(and \(|z|^2 = |z_1|^2 + |z_2|^2\) one has

\[
\frac{|z_1|^{2m}}{(1 - |z|^6)^m} = \sum_{j+k=m+3i, i=0,1,\ldots} \binom{m+i-1}{j} \binom{m+3i}{j} |z_1|^{2j}|z_2|^{2k}. \quad (9)
\]

By combining (8) and (9) one sees that (7) is satisfied with \(C_m = \frac{4}{3}(m-1)(m-2)\), and we are done.

**Remark 2.** Notice that the metric \(g_*\) is radial, namely it admits a Kähler potential depending only on \(|z_1|^2 + |z_2|^2\). Moreover, a simple computation shows that \(|R_{g_*}|^2 - 4|Ric_{g_*}|^2\) is a constant (given by \(-960n^2\)), where \(Ric_{g_*}\) and \(R_{g_*}\), are, respectively, the Ricci tensor and the Riemann curvature tensor of the metric \(g_*\). By using the classification results on radial cscK metrics given in [11] (see also [12]) one can prove the following: if \(g\) is a radial, strictly partially regular cscK metric on an \(n\)-dimensional complex manifold \(M\) such that \(|R_{g}|^2 - 4|Ric_{g}|^2\) is constant, then \(n = 2\).
and there exist three positive constants $\mu, \lambda$ and $\xi$ such that $\mu$ and $\mu^\frac{1}{\lambda}$ are positive integers, $M = \left\{ r = |z_1|^2 + |z_2|^2 \mid r < \xi^{-\frac{1}{\lambda+1}} \right\}$ and $\omega = \frac{i}{\pi} \partial \bar{\partial} \Phi(m, \lambda, \xi)$ where

$$\Phi(m, \lambda, \xi) = m \log \frac{(|z_1|^2 + |z_2|^2)\frac{\lambda}{\lambda+1}}{1 - \xi(|z_1|^2 + |z_2|^2)(\lambda+1)}.$$ \hspace{1cm} (10)

Moreover the scalar curvature $s_g$ of $g$ is given by

$$s_g = -\frac{24\pi}{m}.$$ \hspace{1cm} (11)

Notice that when $m = 1$, $\lambda = 2$ and $\xi = 1$ one regains $M = D_*$ and $g = g_*$. 

In the proof of Theorem 1.1 we need the following:

**Lemma 2.2.** Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Kähler manifolds such that $g_1$ is strictly partially balanced and $g_2$ is (strictly) partially balanced. Then the metric $g_1 \oplus g_2$ on $M_1 \times M_2$ is strictly partially balanced. In particular, if $g_1$ is strictly partially regular and $g_2$ is (strictly) partially regular then $g_1 \oplus g_2$ is strictly partially regular.

**Proof.** For $\alpha = 1, 2$ let $(L_\alpha, h_\alpha)$ be the Hermitian line bundle over $M_\alpha$ such that $\text{Ric}(h_\alpha) = \omega_\alpha$ (cfr. [11] in the introduction), where $\omega_\alpha$ is the Kähler form associated to $g_\alpha$. Let $(L_{1,2} = L_1 \otimes L_2, h_{1,2} = h_1 \otimes h_2)$ be the holomorphic hermitian line bundle over $M_1 \times M_2$ such that $\text{Ric}(h_{1,2}) = \omega_1 \otimes \omega_2$ and

$$\mathcal{H}_\alpha = \left\{ s \in H^0(L_\alpha) \mid \int_{M_\alpha} h_\alpha(s, s)\frac{\omega_\alpha^{n_\alpha}}{n_\alpha!} < \infty \right\},$$

where $n_\alpha$ is the complex dimension of $M_\alpha$. Let $\mathcal{S}_\alpha \subset \mathcal{H}_\alpha$, $\alpha = 1, 2$, be the subspace of $\mathcal{H}_\alpha$ with respect to which $g_\alpha$ is (strictly) partially balanced. Notice that $\mathcal{S}_1 \subseteq \mathcal{H}_1$ since $g_1$ is strictly partially balanced.

Let $\{s^1_1\}$ (resp. $\{s^2_2\}$) be an orthonormal basis for $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) with respect to the $L^2$-product induced by $h_1$ (resp. $h_2$).

It is not hard to see (cfr. [15] Lemma 7]) that $\{s^1_j \otimes s^2_k\}$ is an orthonormal basis for the subspace $\mathcal{S}_{1,2} = \mathcal{S}_1 \otimes \mathcal{S}_2$ of the Hilbert space

$$\mathcal{H}_{1,2} = \left\{ s \in H^0(L_{1,2}) \mid \int_{M_1 \times M_2} h_{1,2}(s, s)\frac{(\omega_1 + \omega_2)^{n_1+n_2}}{(n_1+n_2)!} < \infty \right\}.$$ 

Thus

$$T^{S_{1,2}}_{g_1 \oplus g_2}(x, y) = \sum_{j,k} h_{1,2}(s^1_j(x) \otimes s^2_k(y), s^1_j(x) \otimes s^2_k(y)) = \sum_j h_1(s^1_j(x), s^1_j(x)) \sum_k h_2(s^2_k(y), s^2_k(y)) = T^{S_1}_{g_1}(x)T^{S_2}_{g_2}(y) = C_1C_2.$$
for two positive constant $C_1$ and $C_2$. Then $g_1 \oplus g_2$ is strictly partially balanced with respect to the subspace $S_{1,2} \subseteq H_{1,2}$. Assume now that $g_\alpha$ is (strictly) partially regular and let $m_\alpha$ be such that $mg_\alpha$ is (strictly) partially balanced for $m \geq m_\alpha$. Then, by the first part, $m(g_1 \oplus g_2)$ is strictly partially balanced for $m \geq \max\{m_1, m_2\}$, i.e. $g_1 \oplus g_2$ is strictly partially regular.

**Proof of Theorem 7.1** It is well-known (see, e.g. [1]) that the Fubini-Study metric $g_{FS}$ on the complex sphere $\mathbb{C}P^1$, and the flat metric $g_0$ on the complex Euclidean space $\mathbb{C}^k$, $k \geq 1$, are regular cscK metrics of constant scalar curvatures $s_{g_{FS}} = 8\pi$ and $s_{g_0} = 0$ respectively. By Proposition 2.1 and Lemma 2.2 the metric $g_\ast \oplus g_0$ is a strictly partially regular cscK metric with negative scalar curvature on the complex $n$-dimensional manifold $\mathcal{D}_* \times \mathbb{C}^{n-2}$, for all $n \geq 2$.

Further, by combining Proposition 2.1, Lemma 2.2, (3) and (11) it follows that the Kähler metrics $3g_\ast \oplus g_{FS} \oplus g_0$ and $4g_\ast \oplus g_{FS} \oplus g_0$ on the complex $n$-dimensional manifold $\mathcal{D}_* \times \mathbb{C}P^1 \times \mathbb{C}^{n-3}$, $n \geq 3$, are strictly partially regular metrics with vanishing scalar curvature and positive scalar curvature respectively.

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