The Seven Messengers and the 
“Buzzati sequence”

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Abstract

A young Prince decides to explore the Father’s Kingdom and aims
to reach its furthest boundaries. He starts from the City with a
Caravan and seven fast and strong Messengers. They have the task
to maintain the communications between the Caravan and the City
during the exploration, going back and forth between the Caravan and
the City while the Caravan inexorably goes away.

Drawing inspiration from a fantasy tale by the Italian writer Dino
Buzzati, I derive the geometric progression (named “Buzzati sequence”
in his honor) which governs the duration of the Messenger’s trips, re-
newing further the fascination of the tale. I also note with wonder how
all this apparently hidden mathematical structure was already known
to the author. An extension of the “Buzzati sequence” to relativistic
velocities of the Caravan and the Messengers is finally presented as
exercise.

1 Introduction

Some years ago a dear friend of mine gave me a collection of fantasy tales
by Dino Buzzati, named ‘La Boutique del Mistero’ [1]. Dino Buzzati (San

* I am very grateful to my longlasting friend Eng. Giorgio Borghini.
Pellegrino (1906– Milano 1972) was an Italian journalist, a writer, a poet and a painter. He had been an extraordinary explorer of the human mind, of his anguish and fears, and his writings perfectly reflect his deep vision of the human condition. I already had the pleasure to read something by Buzzati and to go into his troubled and fascinating Weltanschauung, but the first tale of this collection, titled ‘I sette messaggeri’ (The seven messengers) strongly impressed my ‘mathematical imagination’: just more than thirty years old, a Prince decides to explore the Father’s Kingdom and aims to reach its furthermost boundaries. He starts from the City with a Caravan and seven fast and strong Knights. These Knights have the task to maintain the communications between the Caravan and the City during the exploration, in other words they are the Messengers. The first Messenger starts two days after the departure of the Caravan, the second three days after and so on. Many years pass and the Kingdom seems to be endless. Even though the Messengers ride night and day, every day, their encounters with the Caravan become ever more rare and the contacts between the Caravan and the City absurdly far away in time.

Soon after having read the tale I tried to derive the time elapsed, for each Messenger, between each tour using the informations given by the author and, to my amazement, I obtained really the same values described in the tale. My wonder was mainly due to the fact that, by my experience, this particular attention to mathematical accuracy is a bit unusual in this kind of literature.

In the following Section I derive the mathematical expression of the duration of the tours. Let me refer to this relation as to the “Buzzati sequence”, named after the great writer. Suggested by the fascinating atmosphere of the tale, where men seem to be lost in time, in the third Section I make the exercise of extending the previous formula to cases in which the velocities of the Caravan and Messengers are relativistic (namely, closer to the velocity of light).

2 Buzzati sequence

Before deriving the “Buzzati sequence” it is appropriate to clarify all the approximations introduced with the aim of simplifying the derivation. I assume that all the velocities are constant in modulus ($V_c$ is the velocity of
the Caravan, while $V_m$ is the velocity of all Messengers. Obviously it must be $V_m > V_c$, otherwise the Caravan misses the Messengers) and that Messengers change direction \textit{instantaneously} soon after having reached the City or the Caravan (in other words they never stop). With $T_{n,i}$ I represent the time when the $i$–th Messenger leaves the Caravan for the $n$–th tour (for sake of simplicity time counting starts when the Caravan leaves the City) and $\Delta T_{n,i}$ is the duration of the $n$–th tour (i.e. $\Delta T_{n,i} = T_{n+1,i} - T_{n,i}$).

With this assumptions it is not difficult to see that for first tours the following relation holds

$$V_m \Delta T_{1,i} = 2V_c T_{1,i} + V_c \Delta T_{1,i}, \quad (1)$$

which simply states that the distance covered by the $i$–th Messenger in his first tour must be equal to two times the distance covered by the Caravan just before this departure plus the distance covered by the Caravan during Messenger’s travel. So from equation (1) we have for $\Delta T_{1,i}$

$$\Delta T_{1,i} = \frac{2V_c T_{1,i}}{V_m - V_c} = 2q T_{1,i}, \text{ if we define } q \equiv \frac{V_c}{V_m - V_c}. \quad (2)$$

The general expression for $\Delta T_{n,i}$ can be easily obtained from equation (2) by substituting $T_{1,i}$ with the time when the $n$–th departure takes place. Proceeding by steps

$$\Delta T_{2,i} = 2q (T_{1,i} + \Delta T_{1,i}) = \Delta T_{1,i} + 2q \Delta T_{1,i} = \Delta T_{1,i}(1 + 2q),$$

since the $i$–th Messenger starts his second tour at $T_{2,i} = T_{1,i} + \Delta T_{1,i},$

$$\Delta T_{3,i} = 2q (T_{1,i} + \Delta T_{1,i} + \Delta T_{2,i}) = \Delta T_{2,i} + 2q \Delta T_{2,i} = \Delta T_{2,i}(1 + 2q) = \Delta T_{1,i}(1 + 2q)^2,$$

$$\Delta T_{4,i} = 2q (T_{1,i} + \Delta T_{1,i} + \Delta T_{2,i} + \Delta T_{3,i}) = \Delta T_{3,i} + 2q \Delta T_{3,i} = \Delta T_{3,i}(1 + 2q) =$$

$$\Delta T_{1,i}(1 + 2q)^3,$$

$$\ldots,$$

and so on. So we can write by induction

$$\Delta T_{n,i} = \Delta T_{1,i}(1 + 2q)^{n-1} = 2q T_{1,i}(1 + 2q)^{n-1}. \quad (3)$$
Moreover comparing equation (2) with equation (3) it is easy to see that the time \( T_{n,i} \) when the \( n \)-th departure takes place is simply

\[
T_{n,i} = T_{1,i}(1 + 2q)^{n-1},
\]  

which is a straightforward geometric progression.

If we adopt the relation \( V_m = 3/2V_c \) (so \( q = 2 \)) and we put \( T_{1,i} = (i + 1) \text{days}, i = 1, \ldots, 7 \), like in the tale by Buzzati, we can verify how fast (4) grows with \( n \). In the following table some values of \( T_{n,i} \) are shown.

| \( q = 2 \) | Mess. 1 | Mess. 2 | Mess. 3 | Mess. 4 | Mess. 5 | Mess. 6 | Mess. 7 |
|---|---|---|---|---|---|---|---|
| \( T_{1,i} \) | 2 days | 3 days | 4 days | 5 days | 6 days | 7 days | 8 days |
| \( T_{2,i} \) | 10 " | 15 " | 20 " | 25 " | 30 " | 35 " | 40 " |
| \( T_{3,i} \) | 50 " | 75 " | 100 " | 125 " | 150 " | 175 " | 200 " |
| \( T_{4,i} \) | 250 " | 375 " | ~ 1.4 yrs | ~ 1.7 yrs | ~ 2.05 yrs | ~ 2.4 yrs | ~ 2.7 yrs |
| \( T_{5,i} \) | ~ 3.4 yrs | ~ 5.1 yrs | ~ 6.8 yrs | ~ 8.6 yrs | ~ 10.3 yrs | ~ 12.0 yrs | ~ 13.7 yrs |
| \( T_{6,i} \) | ~ 17.1 " | ~ 25.7 " | ~ 34.2 " | ~ 42.8 " | ~ 51.4 " | ~ 60.0 " | ~ 68.5 " |
| \( T_{7,i} \) | ~ 85.6 " | ~ 128.4 " | ~ 171.2 " | ~ 214.0 " | ~ 256.8 " | ~ 299.6 " | ~ 342.5 " |

### 3 Relativistic treatment

It is appropriate to note that the way in which relations (3) and (4) have been derived in the previous section doesn’t take differences between ‘reference systems’ into account, i.e. the time \( T_{n,i} \) is the same for the City, the Caravan and the \( i \)-th Messenger; moreover (3) and (4) seem to hold for every absolute value of \( V_m \) and \( V_c \).

However, form the Theory of Special Relativity (TSR, A. Einstein, 1905) it is well known that nothing can overcome the speed of light and that the measure of space and time intervals depends on the particular reference system.

Here we are dealing with special relativity so time and space intervals will be measured with respect to inertial reference systems, that is, systems in which free motion of bodies (motion not subject to any kind of external force) is a constant speed motion.

According to the theory what is really invariant is the infinitesimal ‘length’ element in space–time

\[
ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,
\]  

where \( c \) is the speed of light.
namely, the ‘distance’ between two events infinitesimally close in space and
time. Indeed, in any inertial reference system we can assign a time and three
spatial coordinates \((t, x, y, z)\) to any event and (5) can be written for two
events infinitesimally close; the main point is that the value of \(ds^2\) is the
same for all those systems. So, for two inertial reference systems, say \(O\) and
\(\overline{O}\), the following holds
\[
c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,
\]
(6)
o no matter what is the constant relative velocity between \(O\) and \(\overline{O}\).

From (6) it is not difficult to obtain the expression for the ‘proper time’
of an observer, i.e. the time measured by a clock at rest in the observer
reference system (for example, like we will see, the proper time of the Caravan
or Messengers). In our inertial system \((O)\) the infinitesimal displacement of
the observer in space–time is of the form of (5), while in the reference system
of the observer \((\overline{O})\), by definition, there is no spatial displacement and (5)
becomes
\[
\overline{ds}^2 = c^2 dt^2.
\]
But now we know that \(ds^2 = \overline{ds}^2\), so
\[
c^2 dt^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2.
\]
(7)

Moreover if \(\overrightarrow{v}(t)\) is the observer velocity (not necessarily uniform) that
we measure in our inertial system, we can write, for infinitesimal spatial
displacements,
\[
dx = v_x(t) dt, \quad dy = v_y(t) dt, \quad dz = v_z(t) dt,
\]
so, for proper time \(\overline{t}\), we have (from (7))
\[
\overline{t} = \int_0^t d\overline{t} = \int_0^t \sqrt{1 - \frac{\overrightarrow{v}(\tau) \cdot \overrightarrow{v}(\tau)}{c^2}} \, d\tau.
\]
(8)

For a more detailed description of these basic principles and a more rig-
orous derivation of the formulas you can see, for example, the book by L. D.
Landau and E. M. Lifšits, *The Classical Theory of Fields* [2].
Now we are able to deal with the extension of ‘Buzzati sequence’ to cases in which $V_c/c \to 1$, $V_m/c \to 1$, where $c$ is the speed of light; I will use the same notations and approximations of Section 2.

Suppose you live in the City (our inertial reference system) and you want to find out the proper time of the Caravan and Messengers, then all you need is (3), (4) and (8). It is natural to consider $T_{1,i}$ like a time measured in the reference frame of the Caravan (in fact before their first tours the Messengers belong to the Caravan), while in the City we have, by (8)

$$T_{1,i} = \sqrt{1 - \frac{V^2}{c^2}} T_{1,i,\text{City}} \quad \to \quad T_{1,i,\text{City}} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} T_{1,i}. \quad (9)$$

Obviously I suppose that all the clocks have been synchronized at the origin. So for $T_{n,i,\text{City}}$, which gives the time of the $n$–th departure in the reference system of the City, we have

$$T_{n,i,\text{City}} = T_{1,i,\text{City}} (1 + 2q)^{n-1} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} T_{1,i}(1 + 2q)^{n-1}. \quad (10)$$

Multiplying (10) by $\sqrt{1 - V_c^2/c^2}$ we will obtain the proper time $T_{n,i}$ of the Caravan which is obviously $T_{1,i}(1 + 2q)^{n-1}$ again. Concerning the proper time of Messengers the situation is quite different. The proper time $T_{n,i,\text{Mes}}$ when Messengers start their $n$–th tour, is the sum of the following two terms (see (8) again)

$$T_{n,i,\text{Mes}} = \int_0^{T_{1,i,\text{City}}} \sqrt{1 - \frac{V_c^2}{c^2}} \, d\tau + \int_{T_{1,i,\text{City}}} \sqrt{1 - \frac{V_m^2}{c^2}} \, d\tau, \quad (11)$$

since before $T_{1,i,\text{City}}$ the Caravan and the Messengers travel together at the same speed $V_c$. The integration (11) is trivial and using equations (9) and (10) we obtain

$$T_{n,i,\text{Mes}} = \frac{\sqrt{1 - V_c^2/c^2}}{\sqrt{1 - V_c^2/c^2}} T_{1,i}(1 + 2q)^{n-1} + \left(1 - \sqrt{1 - \frac{V_m^2}{c^2}} \right) T_{1,i}. \quad (12)$$

Now it is easy to verify that both (10) and (12) reduce themselves to (4) when $V_c/c \to 0$ and $V_m/c \to 0$. Moreover the time $T_{n,i,\text{Mes}}$ is asymptotically
lower than both proper time of the Caravan and $T_{n,i,\text{City}}$ (this phenomenon is of the same kind of the so–called ‘Twin Paradox’).

A little more difficult exercise is to derive the expression of proper time of Messengers when they reach the City during their $n$–th tour ($T_{n,i,\text{Mes}}^{\text{City}}$). In the reference system of the City the $i$–th Messenger starts his $n$–th tour at $T_{n,i,\text{City}}$, when the Caravan is at distance $T_{n,i,\text{City}} \times V_c$, so he will reach the City after $T_{n,i,\text{City}} \times V_c/V_m$ or $T_{n,i,\text{City}} \times q/(1 + q)$ (see (2)). Thus the proper time will be, using (8) again,

$$
T_{n,i,\text{Mes}}^{\text{City}} = T_{n,i,\text{Mes}} + \sqrt{1 - \frac{V_m^2}{c^2}} \frac{q}{1 + q} T_{n,i,\text{City}},
$$

which becomes, substituting $T_{n,i,\text{City}}$ and $T_{n,i,\text{Mes}}$ with (10) and (12),

$$
T_{n,i,\text{Mes}}^{\text{City}} = \sqrt{1 - \frac{V_m^2}{c^2}} \left(1 + \frac{q}{1 + q}\right) T_{1,i} (1 + 2q)^n - 1 + \left(1 - \frac{1 - \frac{V_c^2}{c^2}}{\sqrt{1 - \frac{V_m^2}{c^2}}}\right) T_{1,i}.
$$

In the following table some values of $T_{n,4,\text{City}}$, $T_{n,4,\text{Mes}}$, $T_{n,4}$ and $T_{n,4,\text{Mes}}^{\text{City}}$ for Messenger 4 are shown; $q$ is still equal to 2 and for $V_c$ I have chosen a test value of $c/2$ (so $V_m = 3/4c$)

| $n$ | $T_{n,4,\text{City}}$ | $T_{n,4,\text{Mes}}$ | $T_{n,4}$ (non relativistic) | $T_{n,4,\text{Mes}}^{\text{City}}$ |
|-----|------------------|-----------------|-----------------------------|-----------------------------|
| 1   | 5.8 days         | 5 days          | 5 days                      | 7.54 days                   |
| 2   | 28.9 "           | 20.3 "          | 25 "                        | 33.0 "                      |
| 3   | 144.3 "          | 96.6 "          | 125 "                       | 160.3 "                     |
| 4   | ~ 1.9 yrs        | ~ 1.3 yrs       | ~ 1.7 yrs                    | ~ 2.2 yrs                   |
| 5   | ~ 9.9 "          | ~ 6.5 "         | ~ 8.6 "                     | ~ 10.9 "                    |
| 6   | ~ 49.4 "         | ~ 32.7 "        | ~ 42.8 "                    | ~ 54.5 "                    |
| 7   | ~ 247.1 "        | ~ 163.5 "       | ~ 214.0 "                   | ~ 272.5 "                   |

Lastly suppose that the Caravan uses electromagnetic waves to communicate, namely $V_m = c$. Hence, it is easy to see that, at first order in $V_c/c$, equation (10) becomes $T_{n,i,\text{City}} = T_{n,i} = T_{1,i} (1 + 2(n - 1)V_c/c)$ (remember that the exchange of informations between Caravan and Messengers –electromagnetic Messengers too– was supposed instantaneous).

And the last thought, why not to see Buzzati’s marvelous tale as a metaphor for the future space exploration of the mankind?
References

[1] Dino Buzzati, *La Boutique del Mistero*, Arnoldo Mondadori Editore, Milano, 1968 (in Italian).

[2] L. D. Landau and E. M. Lifšits, *The Classical Theory of Fields*, Revised Second Edition, Pergamon Press, 1962.