The Standard Model Prediction for Muon $g - 2$

Joaquim Prades

Centro Andaluz de Física de las Partículas Elementales (CAFPE) and Departamento de Física Teórica y del Cosmos, Universidad de Granada Campus de Fuente Nueva, E-18002 Granada, Spain

Abstract

The present Standard Model prediction for muon $g - 2$ is reviewed. Emphasis is put in discussing the main hadronic uncertainties.

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1 Introduction

Recently, the Muon $g - 2$ Collaboration from the E821 experiment at Brookhaven National Lab (BNL) \cite{1} reported a new result for the muon $g - 2$ with an uncertainty more than five times smaller than the last CERN experiment \cite{2}. The E821 result when combined with the previous experiments produce the present world average

$$a_\mu \equiv \frac{|g_\mu| - 2}{2} = (11 659 202.3 \pm 15.1) \cdot 10^{-10}. \quad (1)$$

The expected impressive final goal of E821 is to achieve an experimental uncertainty in $a_\mu$ of the order of $4 \cdot 10^{-10}$.

Accompanying this great experimental performance a lot of effort has been put in the theoretical side to get a Standard Model prediction for this quantity since the pioneering work of Schwinger \cite{3}. In the next Sections I will review the present Standard Model prediction putting emphasis in discussing the main hadronic uncertainties which at present are the dominant.

2 Quantum Electro-Weak-Dynamics Contributions

2.1 QED Contribution

The QED contribution to muon $g - 2$ is known to $O(\alpha^5)$

$$a_\mu^{\text{QED}} = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) + 0.765 \ 857 \ 376(27) \left( \frac{\alpha}{\pi} \right)^2 + 24.050 \ 508 \ 98(44) \left( \frac{\alpha}{\pi} \right)^3$$

$$+ 126.07(41) \left( \frac{\alpha}{\pi} \right)^4 + 930(170) \left( \frac{\alpha}{\pi} \right)^5 + \cdots \quad (2)$$

Higher orders are negligible compared with the experimental uncertainty. The original references leading to this result can be found in \cite{4,5,6}. The first, second, and third orders are known fully analytically. The fourth order contains contributions which are only known numerically \cite{2}. The fifth order is a numerical estimate of the dominant diagrams enough for the BNL expected uncertainty.

Using the present world average value $\alpha^{-1} = 137.035 \ 999 \ 76(50)$ \cite{6}, one gets

$$a_\mu^{\text{QED}} = (11 658 \ 470.6 \pm 0.3) \cdot 10^{-10}. \quad (3)$$

which gives the bulk of the experimental value of $a_\mu$.

\footnote{Recently, T. Kinoshita has found a numerical error in the fourth order contribution which final result is not public yet. However, the total Standard Model uncertainty is not upset by it.}
### 2.2 Electroweak Contribution

This contribution can be written as

\[
a_{\mu}^{\text{EW}} = \frac{5 G_F m_\mu^2}{3 8 \pi^2 \sqrt{2}} \left[ 1 + \frac{1}{5} \left( 1 - 4 \sin^2 \theta_W \right)^2 + (-97.0 \pm 8.8) \left( \frac{\alpha}{\pi} \right) + \cdots \right] \tag{4}
\]

where the first term is the one-loop contribution and the second term is the \(O(G_F m_\mu^2 \alpha)\) two-loop contribution. Using \[9\], one gets

\[
a_{\mu}^{\text{EW}} = (19.4808 \cdots + (-4.4 \pm 0.4)) \cdot 10^{-10} = (15.1 \pm 0.4) \cdot 10^{-10} \tag{5}
\]

for the one-loop plus two-loop EW contributions. The uncertainty in the two-loop result takes into account uncertainties in the Higgs mass, quark two-loop effects and \(\alpha^3\) and higher corrections. The leading \(\ln(M_Z/m_\mu)\) logs can be re-summed to all orders in \(\alpha\) in \[10\] and the result enlarge slightly the two-loop number but it is well within the quoted uncertainty, so that we keep the full two-loop result.

### 3 Hadronic Contributions

#### 3.1 Hadronic Light-by-Light Contribution

This is the contribution of a hadronic Green’s function with four legs coupled to electromagnetic two-quark currents. This four-point function is attached in all possible ways with three of its legs to the muon line. This contribution is of \(O(\alpha^3)\) and cannot be related to any measured quantity and we have to rely in our ability of treating the strong interactions at all energies.

There are always two topologies to a full four-point function; namely, first, two three-point form factors joined with full propagators, these we call three-point-like contributions, and second, the pure four-point form factor which we refer to as the four-point-like contribution. In all cases the three vector legs are joined to the muon line through full vector two-point functions. We calculate the leading \(O(N_c)\) contributions to the full four-point function as well as NLO in \(1/N_c\) corrections which are saturated by four-point-like charged pion and kaon loops. Being of different order in \(1/N_c\), they do not have to match the leading order in \(1/N_c\) contributions as the quark-loop contribution [11, 12].

Based in the \(1/N_c\) analysis of [11], there have been two full calculations of this contribution in [12] and [13]. Here, I pay more attention to [12] and a full comparison with [13] and references to previous work can be found in [12].

Recently, there have also calculations of the pseudo-scalar exchange in [14] and of the pseudo-scalar and scalar exchanges in [15]. In [16], the coefficient of the leading divergent logarithm has been obtained analytically for the point-like Wess-Zumino-Witten \(\pi^0\gamma\gamma\) form factor. This result has been confirmed in [17].
The results we get are\cite{12}

\[a_{\mu}^{L-b-L} = ((2.1 \pm 0.3) + (-0.68 \pm 0.2) + (8.5 \pm 1.3) + (0.25 \pm 0.1)) \cdot 10^{-10} + (-1.9 \pm 0.5) \cdot 10^{-10} = ((3.8 \pm 0.3) + (4.4 \pm 2.1)) \cdot 10^{-10} = (8.3 \pm 3.2) \cdot 10^{-10}, \quad (6)\]

where the first term is the four-point-like contribution, the second, third, and fourth terms are the three types of three-point-like contributions, namely, when the propagators are scalar, pseudo-scalar, or axial-vector. The first term in the second line is the NLO in $1/N_c$ contribution. The second split gives the contributions below 0.5 GeV and above 0.5 GeV, respectively.

With the same split as (6), the result obtained in \cite{13} is

\[a_{\mu}^{L-b-L} = ((0.97 \pm 1.1) + (0.0 \pm 0.0) + (8.27 \pm 0.60) + (0.17 \pm 0.1)) \cdot 10^{-10} + (-0.45 \pm 0.8)) \cdot 10^{-10} = (8.96 \pm 1.54) \cdot 10^{-10}. \quad (7)\]

Two main features appear from both results. The dominant contribution by far is the pseudo-scalar exchange and second, there is a very good agreement in the contribution from the pseudo-scalar exchange while for the rest of the $O(N_c)$ contributions the disagreement is pretty small. This disagreement is larger in the NLO in $1/N_c$ contribution. In fact both contributions cancel out very much and the full result is in quite good agreement. Further scrutiny of this cancellation in a as much as possible model independent way is needed.

For the low energy part (below 0.5 GeV) of the four-point-like and scalar exchange three-point-like contributions –they are related by Ward identities– we use the ENJL model \cite{18}. For the higher energy part of the four-point-like contribution we use a heavy quark which mass acts as an IR cut-off \cite{12}. The ENJL model has not free parameters, they are fixed to low-energy \(\pi\pi\) data. In the ENJL model, full vector two-point functions have a nonphysical behavior \cite{12, 19} at intermediate energies which we corrected in this work.

The pseudo-scalar three-point-like contribution is dominated by the $\pi^0$ exchange with non-negligible contributions of the \(\eta\) and \(\eta'\) exchanges \cite{12}. We used a variety of \(\pi^0\gamma^*\gamma^*\) form factors fulfilling all the known constraints. Fortunately, this form factor is very constrained by the $U_A(1)$ anomaly which gives its normalization at the origin and by $\rho^0 \to \pi^0\gamma$ which gives the slope at the origin. There are also data on \(\pi^0 \to \gamma^*\gamma\) between 0.5 GeV and 3.3 GeV \cite{20, 21}. All these constraints make the model dependence small and it is also the reason of the good agreement of (6) and (7). Nevertheless, the uncertainty in this contribution can be reduced using data on $e^+e^- \to \pi^0e^+e^-$ at intermediate energies \cite{22}. Data below 0.5 GeV on $\pi^0 \to \gamma^*\gamma$ can also help.

\footnote{Notice that we correct for an errata in the sign of the pseudo-scalar and axial-vector exchange contributions.}
The authors of [14] were able to obtain analytical formulas for the pseudo-scalar exchange for a general class of $\pi^0\gamma^*\gamma^*$ form factors that fulfill the OPE and large $N_c$ QCD constraints, and which are compatible with the data. Their result for this contribution $(8.3 \pm 1.2) \cdot 10^{-10}$ agrees very well with both [12] and [13] after correcting for the sign mistake.

For the NLO in $1/N_c$ contribution we cannot use the ENJL model. We saturate them with charged pion and kaon loops coupled to photons and need $P^+ P^- \gamma^* \gamma^*$ and $P^+ P^- \gamma^*$ vertices for which we take complete VMD which works very well for one-photon couplings at all energies. For two-photons there is no data beyond $\pi^+ \pi^- \rightarrow \gamma \gamma$. In [13], a HGS symmetry model was used and the discrepancy with [12] already at low-energy is large but being of CHPT order $\rho^6$ there is no data to disentangle it. It is possible that information on $e^+ e^- \rightarrow \pi^+ \pi^- e^+ e^-$ could help to eliminate this large model dependence.

The uncertainty in (6) is obtained by adding linearly the individual errors plus $0.8 \cdot 10^{-10}$ added linearly to take into account the discrepancy between our result and the one in [4]. The uncertainty in (7) is obtained adding quadratically their individual errors.

I take the average of (6) and (7)

$$a_{\mu}^{L-b-L} = (8.6 \pm 3.2) \cdot 10^{-10} \quad (8)$$

as the present value for this hadronic light-by-light contribution. The uncertainty is the one from (6) since already takes into account the discrepancy between both results as explained above and is more realistic. Improving both in the $\pi^0 \gamma^* \gamma^*$ and $\pi^+ \pi^- \gamma^* \gamma^*$ vertices could decrease this uncertainty to around $2 \cdot 10^{-10}$.

### 3.2 Hadronic Vacuum Polarization Contribution

This contribution starts at order $\alpha^2$ and it is the one with the largest uncertainty at present. It is a hadronic Green’s function with two legs coupled to electromagnetic two-quark currents and attached to the muon line.

The $O(\alpha^2)$ contribution can be written as [23],

$$a_{\mu}^{(2)\text{hvp}} = \int_{4m_e^2}^{\infty} dt \ K(t) \ \frac{\sigma^{(0)}(e^+ e^- \rightarrow \text{hadrons})(t)}{\sigma^{(0)}(e^+ e^- \rightarrow \mu^+ \mu^-)} \quad (9)$$

where $K(t)$ is a known function of $t$. There have been many calculations with increasing accuracy due to better data and theoretical input, see [24] for recent calculations and [25] for a critical review. References to previous work can be found in [4, 11, 13, 25]. Since its uncertainty can be reduced systematically with accurate data one should consider this contribution mainly of experimental origin.

The most recent $e^+ e^-$ data [26] from BES-II at Beijing, and SND and CMD-2 at Novosibirsk has been used in [27]

$$a_{\mu}^{e^+ e^-\text{hvp}} = (697.4 \pm 10.5) \cdot 10^{-10} \quad (10)$$
Also just using $e^+e^-$ data [28] gets

$$a_{\mu}^{e^+e^-\text{hvp}} = (698.2 \pm 9.7) \cdot 10^{-10}. \quad (11)$$

These results give the present average for this contribution

$$a_{\mu}^{e^+e^-\text{hvp}} = (697.8 \pm 10.5) \cdot 10^{-10}. \quad (12)$$

Further reduction of the uncertainty in the $e^+e^-$ data below the tau mass to the order or below 1% is expected from VEPP-2M (CMD-2, SND) at Novosibirsk, DAΦNE (KLOE) at Frascati, BEPC (BES) at Beijing and other low energy facilities. In the theory side, the possibility of measuring the pion form factor at DAΦNE and discussion of the relevant QED corrections was presented in [29]. The complete $O(\alpha)$ QED initial state, final state, and initial-final state radiation corrections to $e^+e^- \to \pi^+\pi^-$ has been recently presented [30].

Using CVC, one can relate the $\tau^- \to \pi^-\pi^0\nu_\tau$ with $I = 1$ vector channel data to the $e^+e^- \to \pi^+\pi^-$ data. The very precise tau hadronic decay data [31] from ALEPH and OPAL at CERN and CLEO at CESR when supplemented with the SU(2) breaking corrections can help in increasing the accuracy of the $e^+e^- \to \pi^+\pi^-$ data which gives 70% of the contribution to $a_{\mu}^{\text{hvp}}$ and 80% of its uncertainty when integrated in (1) from $4m_{\tau}^2$ to $t = 0.8$ GeV. This program was started in [32] and continued in [33] where pQCD was also pushed down up to 1.8 GeV helping to reduce the final uncertainty sizeably. This use has been recently confirmed by the BES-II data which give compatible results [28, 33].

Adding the rest of the hadronic vacuum polarization contributions and the known isospin breaking corrections, [33] gets

$$a_{\mu}^{\tau\text{hvp}} = (692.4 \pm 6.2) \cdot 10^{-10}. \quad (13)$$

The leading isospin breaking corrections studied in [34] agree quite well within errors with the same corrections applied in [28]. Numerically, the isospin corrections applied in [28] to the tau data are very similar to the ones in [33]. The result from tau data in [28] is

$$a_{\mu}^{\tau\text{hvp}} = (695.2 \pm 6.4) \cdot 10^{-10}. \quad (14)$$

From these results the present average for the $a_{\mu}^{\text{hvp}}$ from tau data reads

$$a_{\mu}^{\tau\text{hvp}} = (693.8 \pm 6.4) \cdot 10^{-10}. \quad (15)$$

The recent calculation [35] is also compatible within errors though no isospin breaking effects are included. For the combined final result of (12) and (15) I take the weighted average

$$a_{\mu}^{(\alpha^2)\text{hvp}} = (694.9 \pm 6.4) \cdot 10^{-10}. \quad (16)$$
There are some $O(\alpha^3)$ contributions already included in (16), namely, the intermediate $\pi^0\gamma^*$ and $\eta\gamma^*$ states in $\sigma(e^+e^- \rightarrow \text{hadrons})$. The rest of the $O(\alpha^3)$ corrections have been calculated in [32, 37] and are well under control

$$a_{\mu}^{(\alpha^3)\text{hvp}} = (-10.0 \pm 0.6) \cdot 10^{-10}. \quad (17)$$

It was pointed out in [36] that final state radiative corrections are eliminated up to 80% in the Novosibirsk data analysis of $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$. They were estimated in [28] and added back to their final number (11). These $O(\alpha^3)$ corrections depend obviously on the experimental set-up and on the particular analysis of the data. In fact, they were already taken into account in the ALEPH tau data used by [32] and [28] and therefore should not be added back to the results (13) and (14).

Summing (16) and (17), one gets for the total hadronic vacuum polarization contribution

$$a_{\mu}^{\text{hvp}} = (684.9 \pm 6.4) \cdot 10^{-10}. \quad (18)$$

The expected $e^+e^-$ data accuracy below 1% supplemented with theoretical efforts like [29, 30] will reduce the uncertainty in $a_{\mu}^{\text{hvp}}$ from $e^+e^-$ up to the order of $6 \cdot 10^{-10}$. Joint works of tau and $e^+e^-$ groups [33, 38] are also announced and will reduce this uncertainty further. Isospin breaking studies like [34] will help to take these corrections under better control. Chiral symmetry can also help to reinforce the accuracy of the $\pi\pi$ dominant contribution [39].

### 4 Results and Summary

Summing all the Standard Model contributions, (3), (5), (8), and (18) to muon $g - 2$, one gets

$$a_{\mu}^{\text{SM}} = (11659.179.2 \pm 9.4) \cdot 10^{-10}. \quad (19)$$

where uncertainties of the QED, EW, and hadronic light-by-light contributions have been added linearly and afterwards added quadratically to the hadronic vacuum polarization uncertainty.

As a final result, we get

$$a_{\mu} - a_{\mu}^{\text{SM}} = (23.1 \pm 16.9) \cdot 10^{-10} \quad (20)$$

i.e. there is at present a bit more than one sigma of discrepancy. The significance of this discrepancy could be largely enhanced by the aimed experimental uncertainty BNL goal of $4 \cdot 10^{-10}$. The announced improvements on the hadronic contributions are also very interesting and can reduce the uncertainty of the muon $g - 2$ Standard Model prediction to the order of $6 \cdot 10^{-10}$ to $7 \cdot 10^{-10}$. The near future of muon $g - 2$ reveals thus very exciting.

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