Beyond No-Regret: Competitive Control via Online Optimization with Memory

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Abstract
This paper studies online control with adversarial disturbances using tools from online optimization with memory. Most work that bridges learning and control theory focuses on designing policies that are no-regret with respect to the best static linear controller in hindsight. However, the optimal offline controller can have orders-of-magnitude lower cost than the best linear controller. We instead focus on achieving constant competitive ratio compared to the offline optimal controller, which need not be linear or static. We provide a novel reduction from online control of a class of controllable systems to online convex optimization with memory. We then design a new algorithm for online convex optimization with memory, Optimistic Regularized Online Balanced Descent, that has a constant, dimension-free competitive ratio. This result, in turn, leads to a new constant-competitive approach for online control.

1. Introduction
The interface of learning and control is emerging as a vibrant research area. In recent years, considerable effort has been made to use ideas from learning to design optimal controllers for dynamical systems (Chowdhary et al., 2014; Berkenkamp et al., 2016; Chang et al., 2017; Dean et al., 2017; 2019; Fisac et al., 2018; Pan et al., 2018; Liu et al., 2019; Yin et al., 2019; Taylor et al., 2019). Of particular relevance to this paper are online approaches with convergence guarantees (Dean et al., 2018; Goel et al., 2019; Cohen et al., 2019; Agarwal et al., 2019a;b; Dean et al., 2017; 2018; Abbasi-Yadkori et al., 2018). For example, Agarwal et al. (2019b) achieves logarithmic regret under stochastic noise and strongly convex loss, and Agarwal et al. (2019a) achieves $O(\sqrt{T})$ regret under adversarial noise and convex loss.

However, the cost of the optimal linear controller may be far from the true offline optimal cost. In fact, we show a simple example (Example 1 in Section 5) where the gap is arbitrarily large. Thus, achieving small regret may still mean having a significantly larger cost than optimal.

Motivated by this drawback, we tackle a more challenging goal of finding a controller with low competitive ratio — the ratio of costs — with respect to the optimal offline controller (Borodin & El-Yaniv, 2005), which is not necessarily linear or static. We provide a novel reduction from a class of control problems to online optimization with memory. We then generalize recent advances in competitive online optimization (Lin et al., 2012; Chen et al., 2015; 2018; Goel & Wierman, 2019; Li et al., 2018; Goel et al., 2019; Li et al., 2019) to arrive at a new algorithm, Optimistic Regularized Online Balanced Descent (Optimistic ROBD), that achieves a constant competitive ratio for a class of online optimization problems with memory. To our knowledge, the only prior result for competitive online control is Goel & Wierman (2019), which considers a restricted form of (1) with invertible $B$ and known $w_t$ at step $t$.

Our contributions can be summarized as follows:

- We provide a reduction from a large class of online control problems, which we call Input-Disturbed Squared Regulators (IDSRs), to a class of online optimization problems with structured memory. The structure of the

\begin{equation}
x_{t+1} = Ax_t + Bu_t + w_t, \tag{1}
\end{equation}

where $x_t$ is the state, $u_t$ is the control or action taken by the agent or controller, and $w_t$ is the disturbance. At every time step $t$, the controller incurs a cost $c_t(x_t, u_t)$. The goal then is to design a controller that achieves low cost, typically quantified via comparing to a benchmark controller.

The predominant benchmark used in previous work is regret relative to the best linear controller in hindsight, i.e., $u_t = -K^*x_t$ (Cohen et al., 2018; Abeille & Lazaric, 2018; Agarwal et al., 2019a;b; Dean et al., 2017; 2018; Abbasi-Yadkori et al., 2018). For example, Agarwal et al. (2019b) achieves logarithmic regret under stochastic noise and strongly convex loss, and Agarwal et al. (2019a) achieves $O(\sqrt{T})$ regret under adversarial noise and convex loss.

A key feature of online learning for control is that an underlying dynamical system governs the state transitions from one time step to the next, which bears some affinity to online reinforcement learning (Auer & Ortner, 2007; Li & Li, 2019). One typically assumes a form for the dynamical system, such as a linear dynamical system with disturbances:
memory in the online optimization problem reflects the underlying dynamics of the system in consideration.

- We introduce a new algorithm, Optimistic ROBD, and prove that it has a constant, dimension-free competitive ratio for a class of online optimization problems with structured memory when the cost functions are $m$-strongly convex and $l$-smoothly even when only a noisy estimate of the cost function is known.
- Via the reduction, we show that Optimistic ROBD has a constant, dimension-free competitive ratio for the class of IDSR systems in controllable canonical form with adversarial disturbances.
- We demonstrate both analytically and via numerical experiments that the optimal linear controller can be arbitrarily worse than the optimal controller, even in simple systems; and that, as a result, the competitive bound on Optimistic ROBD ensures that it significantly outperforms the optimal linear controller in such cases.

2. Background & Model

In this section, we formally present the problem settings for online optimization and online control that we consider in this paper. We first survey prior work on OCO with memory and then introduce our new model of OCO with structured memory. Finally, we introduce a class of online control problems. We defer a formal reduction of online control to OCO with structured memory to Section 4. Throughout this paper, $M_{i,j}$ denotes either $\{M_i, M_{i+1}, \ldots, M_j\}$ if $i \leq j$, or $\{M_i, M_{i-1}, \ldots, M_j\}$ if $i > j$.

2.1. Online Convex Optimization with Memory

Online convex optimization (OCO) with memory is a variation of classical OCO that was first introduced in Anava et al. (2015). In contrast to classical OCO, in OCO with memory, the loss function depends on previous actions in addition to the current action. At time step $t$, the online agent picks $y_t \in \mathcal{K} \subseteq \mathbb{R}^d$ and then a loss function $g_t : \mathcal{K}^{p+1} \to \mathbb{R}$ is revealed. The agent incurs a loss of $g_t(y_{t-1}, \ldots, y_t)$. Thus, $p$ quantifies the length of the memory in the loss function.

Within this general model of OCO with memory, Anava et al. (2015) focuses on developing policies with small policy regret, which is defined as:

$$\text{PolicyRegret} = \sum_{t=p}^{T} g_t(y_{t-p}) - \min_{y \in \mathcal{K}} \sum_{t=0}^{T} g_t(y, \ldots, y).$$

The main result presents a memory-based online gradient descent algorithm can achieve $O(\sqrt{T})$ regret under some moderate assumptions on the diameter of $\mathcal{K}$ and the gradient of the loss functions.

Online Convex Optimization with Switching Costs.

While the general form of OCO with memory was introduced only recently, specific forms of OCO problems involving memory have been studied for decades. Perhaps the most prominent example is OCO with switching costs, often termed Smoothed Online Convex Optimization (SOCO) (Lin et al., 2012; Chen et al., 2015; 2018; Goel & Wierman, 2019; Li et al., 2018; Goel et al., 2019). In SOCO, the loss function is separated into two pieces: (i) a hitting cost $f_t$, which depends on only the current action $y_t$, and a switching cost $c(y_t, y_{t-1})$, which penalizes big changes in the action between rounds. Often the hitting cost is assumed to be of the form $\|y_t - v_t\|$ for some (squared) norm, motivated by tracking some unknown trajectory $v_t$, and the switching cost $c$ is a (squared) norm motivated by penalizing switching in proportion to the (squared) distance between the actions, e.g., a common choice is $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - y_{t-1}\|^2$ (Goel & Wierman, 2019; Li et al., 2018). The goal of the online learner is to minimize its total cost over $T$ rounds:

$$\text{cost}(ALG) = \sum_{t=1}^{T} f_t(y_t) + c(y_t, y_{t-1}).$$

Under SOCO, results characterizing the policy regret are straightforward, and the goal is instead to obtain stronger results that characterize the competitive ratio. The competitive ratio is the worst-case ratio of total cost incurred by the online learner and the offline optimal. The cost of the offline optimal is defined as the minimal cost of an algorithm if it has full knowledge of the sequence $\{f_t\}$, i.e.:

$$\text{cost}(OPT) = \min_{y_1, \ldots, y_T} \sum_{t=1}^{T} f_t(y_t) + c(y_t, y_{t-1}).$$

Using this, the competitive ratio is defined as:

$$\text{CompetitiveRatio}(ALG) = \sup_{f_1, \ldots, f_T} \frac{\text{cost}(ALG)}{\text{cost}(OPT)}.$$

Bounds for competitive ratio are stronger than for policy regret, since the dynamic offline optimal can change its decisions on different time steps (Anava et al., 2015).

In the context of SOCO, the first results bounding the competitive ratio focused on one-dimensional action sets (Lin et al., 2013; Bansal et al., 2015), but after a long series of papers there now exist algorithms that provide constant competitive ratios in high dimensional settings (Chen et al., 2018; Goel & Wierman, 2019; Goel et al., 2019). Among different choices of switching cost $c$, we are particularly interested in $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - y_{t-1}\|^2$ due to the form of our reduction from online control to OCO. The state-of-the-art algorithm for this switching cost is Regularized Online Balanced Descent (ROBD), introduced by Goel et al. (2019), which achieves the lowest possible competitive ratio of any online algorithm. Other recent results study the case where $c(y_t, y_{t-1}) = \|y_t - y_{t-1}\|$ (Bubeck et al., 2019; Argue et al., 2020; Sellke, 2020; Bubeck et al., 2020). Variations of the problem with predictions (Chen et al., 2015; 2016; Li et al., 2018), non-convex cost functions (Lin et al.,
2.2. OCO with Structured Memory

Our goal is to design competitive algorithms for online control, and working with the general model of OCO with memory is too ambitious for this goal. Instead, we introduce a model of OCO with structured memory that generalizes the form of 1-step memory used in SOCO, and provides a connection with online control (as shown in Section 4).

Specifically, we consider a loss function $g_t$ at time step $t$ that can be decomposed as the sum of a hitting cost function $f_t : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\}$ and a switching cost function $c : \mathbb{R}^{d \times (p+1)} \rightarrow \mathbb{R}^+ \cup \{0\}$. Additionally, we assume that the switching cost has the form:

$$c(y_{t:t-p}) = \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i (y_{t-i}) \right\|_2^2,$$

with known $C_i \in \mathbb{R}^{d \times d}$, $i = 1, \ldots, p$. As we show in Section 4, this form connects online optimization with online control. Intuitively, this connection results from the fact that the hitting cost penalizes the agent for deviating from a specific optimal point sequence, while the switching cost captures the cost of implementing a control action.

To summarize, we consider an online agent and an offline adversary interacting as follows in each time step $t$, and we assume $y_1$ is already fixed for $i = -p, -(p-1), \ldots, 0$.

1. The adversary reveals a function $h_t$ and a convex estimation set $\Omega_t \subseteq \mathbb{R}^d$. We assume $h_t$ is both $m$-strongly convex and $l$-strongly smooth, and that $\arg \min_{y_t} h_t(y_t) = 0$.
2. The agent picks $y_t \in \mathbb{R}^d$.
3. The adversary picks $v_t \in \Omega_t$.
4. The agent incurs hitting cost $f_t(y_t) = h_t(y_t - v_t)$ and switching cost $c(y_{t:t-p})$.

Notice that the hitting cost $f_t$ is revealed to the online agent in two separate steps. The geometry of $f_t$ (given by $h_t$ whose minimizer is at 0) is revealed before the agent picks $y_t$. After $y_t$ is picked, the minimizer $v_t$ of $f_t$ is revealed.

Because of the uncertainty about $v_t$, the online agent cannot determine the exact value of the hitting cost it incurs at time step $t$ when determining its action $y_t$. To keep the problem tractable, we assume an estimation set $\Omega_t$, which contains all possible $v_t$'s, is revealed to bound the uncertainty. The online agent can leverage this information when picking $y_t$. If $\Omega_t$ contains only one point, then the agent has a precise estimate of the minimizer $v_t$ when choosing its action.

The structured memory model we consider is a generalization of some of the most prominent models in the SOCO literature. For example, Goel & Wierman (2019); Goel et al. (2019) study a special case where the switching cost is the squared $\ell_2$ norm with 1-step memory and $\Omega_t = \{v_t\}$. Like SOCO, the offline optimal cost in the structured memory model is defined as $\text{cost}(\text{OPT}) = \min_{y_t} \sum_{t=1}^{T} f_t(y_t) + c(y_{t:t-p})$.

We introduce this structured memory model because of its connections to online control (see Section 4). Intuitively, $v_t$ is a function of the disturbances from step 0 to $t$ in a dynamical system, and $y_t$ is related to the control actions from step 0 to $t$.

2.3. Online Control

Our reduction in Section 4 enables studying a class of online control problems that we call Input-Disturbed Squared Regulators (IDSRs).

**Input-Disturbed Systems.** We focus on systems in controllable canonical form defined by:

$$x_{t+1} = Ax_t + B(u_t + w_t),$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^d$ is the control, $w_t \in \mathbb{R}^d$ is a potentially adversarial disturbance to the system. We further assume that $(A, B)$ is in controllable canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix},$$

where each * represents a (possibly) non-zero entry, and the rows of $B$ with 1 are the same rows of $A$ with * (Luenberger, 1967). It is well-known that any controllable system can be linearly transformed to the canonical form. This system is more restrictive than the general form in (1). We call these Input-Disturbed systems, since the disturbance $w_t$ is in the control input/action space. There are many corresponding real-world applications that are well-described by Input-Disturbed systems, e.g., external/disturbance force in robotics (Shi et al., 2019; 2020; Chen et al., 2000).

**Squared Regulator Costs.** We consider the following cost model for the controller:

$$c_t(x_t, u_t) = \frac{q_t}{2} \left\| x_t \right\|^2_2 + \frac{1}{2} \left\| u_t \right\|^2_2,$$

where $q_t$ is a positive scalar. The sequence $q_0:T$ is picked by the adversary and revealed online. The objective of the controller is to minimize the total control cost $\sum_{t=0}^{T} c_t(x_t, u_t)$. 

We call this cost model the Squared Regulator model, which is a restriction of the classical quadratic cost model to a uniform diagonal cost matrix. This class of costs is general enough to address a fundamental trade-off in optimal control: the trade-off between the state cost and the control effort, i.e., bigger \( q_t \) implying relatively cheaper control effort (Kirk, 2004).

**Disturbances.** In the literature, a variety of assumptions have been made about the noise \( w_t \). In most works, the assumption is that the exact noise \( w_t \) is not known before \( u_t \) is taken. Many assume \( w_t \) is drawn from a certain known distribution, e.g., Agarwal et al. (2019b). Others assume \( w_t \) is chosen adversarially subject to \( \| w_t \|_2 \) being upper bounded by a constant \( W \), e.g., Agarwal et al. (2019a). In a closely related paper, Goel & Wierman (2019) connects SOCO with online control under the assumption that \( w_t \) can be observed before picking the control action \( u_t \). In contrast, in this paper we assume that the exact \( w_t \) is not observable before the agent picks \( u_t \). Instead, we assume a convex estimation set \( W_t \) (not necessarily bounded) that contains all possible \( w_t \) is revealed to the online agent to help the agent decide \( u_t \). Our assumption is a generalization of Goel & Wierman (2019), where \( W_t \) is a one-point set, and Agarwal et al. (2019a), where \( W_t \) is a ball of radius \( W \) centered at \( 0 \). Our setting can also naturally model time-Lipschitz noise, where \( w_t \) is chosen adversarially subject to \( \| w_t - w_{t-1} \|_2 \leq \epsilon \), by picking \( W_t \) as a sphere of radius \( \epsilon \) centered at \( w_{t-1} \), which has many real-applications such as smooth disturbances in robotics (Shi et al., 2019; 2020).

**Competitive Ratio.** Our goal is to develop policies with constant (small) competitive ratios. This is a departure from the bulk of the literature, which focuses on designing policies that have low regret compared to the optimal linear controller. We will show the optimal linear controller can have cost arbitrarily larger than the offline optimal in Section 5.1. We again denote the offline optimal cost, with full knowledge of the disturbance sequence \( w_{0:T} \), as \( \text{cost}(\text{OPT}) \):

\[
\text{cost}(\text{OPT}) = \min_{u_{0:T}} \sum_{t=0}^{T} c_t(x_t, u_t).
\]

For an online algorithm \( \text{ALG} \), let \( \text{cost}(\text{ALG}) \) be its cost on the same disturbance sequence \( w_{0:T} \). The competitive ratio is then the worst-case ratio of \( \text{cost}(\text{ALG}) \) and \( \text{cost}(\text{OPT}) \) over any disturbance sequence, i.e. \( \sup_{w_{0:T}} \frac{\text{cost}(\text{ALG})}{\text{cost}(\text{OPT})} \). We will show in Section 4 an exact correspondence between this \( \text{cost}(\text{OPT}) \) and the one defined in Section 2.2, so that the competitive ratio guarantees will directly translate.

To the best of our knowledge, the only prior work that studies competitive algorithms for online control is Goel & Wierman (2019), which considers a very restricted system with invertible \( B \) and known \( w_t \) at step \( t \). A related line of online optimization research studies dynamic regret, or competitive difference, defined as the difference between online algorithm cost and the offline optimal. For example, Li et al. (2019) bounds the dynamic regret of online control with time-varying convex costs with no noise. However, results for the dynamic regret depend on the path-length or variation budget and not just system properties. Bounding the competitive ratio is typically more challenging.

### 3. Algorithms for OCO with Memory

Before studying online control, we first focus on developing algorithms for OCO with structured memory. After, we show a reduction from online control to OCO with structured memory that highlights how to make use of the developed algorithms.

In our analysis of OCO with structured memory, there is a key differentiation depending on whether the online agent has knowledge of the hitting cost function (both \( h_t \) and \( v_t \)) when choosing its action or not, i.e., whether the estimation set \( \Omega_t \) is a single point, \( v_t \), or not. We deal with each of these cases in turn in the following.

#### 3.1. Case 1: Exact Prediction of \( v_t \) (\( \Omega_t = \{v_t\} \))

We first study the simplest case where \( \Omega_t = \{v_t\} \). Recall that \( \Omega_t \) is the convex set which contains all possible \( v_t \) and so, in this case, the online agent has exact knowledge of the hitting cost when determining its action. This assumption, while strict, is the standard assumption in the literature on SOCO, e.g., Goel & Wierman (2019); Goel et al. (2019). It is appropriate for situations where the hitting cost function can be observed before choosing an action.

Our main result in this setting is the following theorem, which shows that the ROBD algorithm (Algorithm 1), which is the state-of-the-art algorithm for SOCO, performs well in the more general case of structured memory. Note that, in this setting, the smoothness parameter \( l \) of hitting cost functions is not involved in the competitive ratio bound.

**Theorem 1.** Suppose the hitting cost functions are \( m \)-strongly convex and the switching cost is given by

\[
c(y_{t-1}, p) = \frac{1}{2} \| y_t - \sum_{i=1}^{p} C_i y_{t-i} \|_2^2,
\]

where \( C_i \in \mathbb{R}^{d \times d} \) and \( \sum_{i=1}^{l} \| C_i \|_2 = \alpha \). The competitive ratio of ROBD with
parameters \( \lambda_1 \) and \( \lambda_2 \) is upper bounded by:

\[
\max \left\{ \frac{m + \lambda_2}{m \lambda_1}, \frac{\lambda_1 + \lambda_2 + m}{(1 - \alpha^2) \lambda_1 + \lambda_2 + m} \right\},
\]

if \( \lambda_1 > 0 \) and \( (1 - \alpha^2) \lambda_1 + \lambda_2 + m > 0 \). If \( \lambda_1 \) and \( \lambda_2 \) satisfy \( m + \lambda_2 = \frac{m + \alpha^2 - 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2} \lambda_1 \), then the competitive ratio is:

\[
\frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( 1 + \frac{\alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right).
\]

A proof of Theorem 1 is given in Appendix B.

To get insight for Theorem 1, first consider the case when \( \alpha \) is a constant. In this case, the competitive ratio is of order \( O(1/m) \), which highlights that the challenging setting is when \( m \) is small. It is easy to see that this upper bound is in fact tight. To see this, note that the case of SOCO with \( \ell_2 \) squared switching cost considered in Goel & Wierman (2019); Goel et al. (2019) is a special case where \( p = 1 \), \( C_1 = I \), \( \alpha = 1 \). Substituting these parameters into Theorem 1 gives exactly the same upper bound (including constants) as Goel et al. (2019), which has been shown to match a lower bound on the achievable cost of any online algorithm, including constant factors. On the other hand, if we instead assume that \( m \) is a fixed positive constant. The competitive ratio can be expressed as \( 1 + O(\alpha^2) \). Therefore, the competitive ratio gets worse quickly as \( \alpha \) increases. This is also the best possible scaling, achievable via any online algorithm, as we show in Appendix C.

Surprisingly, the memory length \( p \) does not appear in the competitive ratio bound, which contradicts the intuition that the online optimization problem should get harder as the memory length increases. However, it is worth noting that \( \alpha \) becomes larger as \( p \) increases, so the memory length implicitly impacts the competitive ratio. For example, an interesting form of switching cost is

\[
c(y_{t:t-p}) = \frac{1}{2} \left\| \sum_{i=0}^{p} (-1)^i \binom{p}{i} y_{t-i} \right\|_2^2,
\]

which corresponds to the \( p \)th derivative of \( y \) and generalizes SOCO \( (p = 1) \). In this case, we have \( \alpha = 2^p - 1 \). Hence \( \alpha \) grows exponentially in \( p \).

3.2. Case 2: Inexact Prediction of \( v_t \) \( (v_t \in \Omega_t) \)

For general \( \Omega_t \), ROBD is no longer enough. It needs to be adapted to handle the uncertainty that results from the estimation set \( \Omega_t \). Note that this uncertainty set is crucial to the connection to online control with adversarial noise.

To handle this additional complexity, we propose Optimistic ROBD (Algorithm 2). Optimistic ROBD is based on two key ideas. The first is to ensure that the algorithm tracks the sequence of actions it would have made if given observations of the true cost functions before choosing an action. To formalize this, we define the accurate sequence \( \{y_1, \cdots, y_T\} \) to be the choices of ROBD (Algorithm 1) with \( \lambda_1 = \lambda, \lambda_2 = 0 \) when each hitting cost \( f_i \) is revealed before picking \( y_t \). The goal of Optimistic ROBD (Algorithm 2) is to approximate the accurate sequence. In order to track the accurate sequence, the first step is to recover it up to time step \( t-1 \) at time step \( t \). To do this, after we observe the previous minimizer \( v_{t-1} \), we can compute the accurate choice of ROBD as if both \( h_{t-1} \) and \( y_{t-1} \) are observed before picking \( y_t \). Therefore, Algorithm 2 can compute the accurate subsequence \( \{y_1, \cdots, y_{t-1}\} \) at time step \( t \). Picking \( y_t \) based on the accurate sequence \( \{y_1, \cdots, y_{t-1}\} \) instead of the noisy sequence \( \{y_1, \cdots, y_{t-1}\} \) ensures that the actions do not drift too far from the accurate sequence.

The second key idea is to be optimistic by assuming the adversary will give it the \( v \in \Omega_t \) that minimizes the cost it will experience. Specifically, before \( v_t \) is revealed, the algorithm assumes it is the point in \( \Omega_t \) which minimizes the weighted sum \( h_t(y - v) + \lambda c(y, y_{t-1:t-p}) \) if ROBD is implemented with parameter \( \lambda \) to pick \( y_t \). This ensures that additional cost is never taken unnecessarily, which could be exploited by the adversary. Note that \( \min_{v \in \Omega_t} h_t(y - v) + \lambda c(y) \) is strongly convex w.r.t. \( v \) (proof in Appendix D), so it is tractable even if \( \Omega_t \) is unbounded.

Our main result in this section is the following theorem, which bounds the competitive ratio of Optimistic ROBD.

**Theorem 2.** Suppose the hitting cost functions are both \( m \)-strongly convex and \( l \)-strongly smooth and the switching cost is given by \( c(y_{t:t-p}) = \frac{1}{2} \| y_t - \sum_{i=1}^{p} C_i y_{t-i} \|_2^2 \), where \( C_i \in \mathbb{R}^{d \times d} \) and \( \sum_{i=1}^{p} \| C_i \|_2 = \alpha \). For arbitrary \( \eta > 0 \), the cost of Optimistic ROBD with parameter \( \lambda > 0 \), is upper bounded by \( K_1 \text{cost}(OPT) + K_2 \), where:

\[
K_1 = (1 + \eta) \cdot \max \left\{ \frac{1}{\lambda}, \frac{\lambda + m}{(1 - \alpha^2) \lambda + m} \right\},
\]

\[
K_2 = \lambda \left( \frac{l}{1 + \eta - \lambda} + \frac{4 \alpha^2}{\eta} - \frac{m}{\lambda + m} \right) \sum_{i=1}^{T} \frac{1}{2} \| v_i - \tilde{v}_i \|_2^2.
\]
A proof of Theorem 2 is given in Appendix D. This proof is nontrivial and highly relies on the two key ideas we mentioned before. Although Theorem 2 does not apply to the case \( \lambda = 0 \), we discuss it separately in Appendix E.

We can choose \( \eta \) to balance \( K_1 \) and \( K_2 \) and obtain a competitive ratio, in particular the smallest \( \eta \) such that:

\[
\lambda \left( \frac{l}{1 + \eta} + \frac{4a^2}{\eta} - \frac{m}{\lambda + m} \right) \leq 0.
\]

Therefore, we have \( \eta = O(l + \alpha^2) \) and \( K_2 \leq 0 \). So the competitive ratio is upper bounded by:

\[
O \left( (l + \alpha^2) \max \left\{ \frac{1}{\lambda}, \frac{1}{(1 - \alpha^2)\lambda + m} \right\} \right).
\]

However, the reason we present Theorem 2 in terms of \( K_1 \) and \( K_2 \) is that, when the diameter of \( \Omega_t \) is small, we can pick a small \( \eta \) so that the ratio coefficient \( K_1 \) will be close to the competitive ratio of ROBD when \( v_2 \) is known before picking \( y_t \). This “beyond-the-worst-case” analysis is useful in many applications and we discuss it more in Section 4.3.

### 4. Online Control via OCO with Memory

We now present a reduction from IDSR, introduced in Section 2.3, to OCO with structured memory. This reduction allows us to inherit the competitive ratio bounds on Optimistic ROBD for this class of online control problems.

#### 4.1. A Reduction to OCO with Structured Memory

Before presenting the reduction, we first introduce some important notations. The indices of non-zero rows in matrix \( B \) in (2) are denoted as \( \{k_1, \ldots, k_d\} := \mathcal{I} \). We define operator \( \psi : \mathbb{R}^n \rightarrow \mathbb{R}^d \) as:

\[
\psi(x) = (\psi(k_1), \ldots, \psi(k_d))^\top,
\]

which extracts the dimensions in \( \mathcal{I} \). Moreover, let \( t_i = k_i - k_{i-1} \) for \( 1 \leq i \leq n \), where \( k_0 = 0 \). The controllability index of the canonical-form \((A, B)\) is defined as \( p = \max\{t_1, \ldots, t_d\} \). We assume that the initial state is zero, i.e., \( x_0 = 0 \). In the reduction, we also need to use matrices \( C_i \in \mathbb{R}^{d \times d}, i = 1, \ldots, p \), which regroup the columns of \( A(\mathcal{I},:) \). We define \( C_i \) for \( i = 1, \ldots, p \) formally by constructing each of its columns. For \( j = 1, \ldots, d \), if \( i \leq p_j \), the \( j \) th column of \( C_i \) is the \( (k_j + 1 - i) \) th column of \( A(\mathcal{I},:) \); otherwise, the \( j \) th column of \( C_i \) is \( 0 \). Formally, for \( i \in \{1, \ldots, p\}, j \in \{1, \ldots, d\} \), we have:

\[
C_i(:,j) = \begin{cases} 
A(\mathcal{I}, k_j + 1 - i) & \text{if } i \leq p_j \\
0 & \text{otherwise}.
\end{cases}
\]

Based on coefficients \( q_{0:T} \), we define

\[
q_{\min} = \min_{0 \leq t \leq T - 1, 1 \leq i \leq d} \sum_{j=1}^{p_i} q_{t+j},
\]

where we assume \( q_t = 0 \) for all \( t > T \).

**Theorem 3.** Consider IDSR where the cost function and dynamics are specified by (3) and (2). We assume the coefficients \( q_{t:t+p-1} \) are observable at step \( t \). Any instance of IDSR in controllable canonical form can be reduced to an instance of OCO with Structured Memory by Algorithm 3.

A proof and an example of Theorem 3 are given in Appendix F. Notably, \( \text{cost}(\text{OPT}) \) and \( \text{cost}(\text{ALG}) \) remain unchanged in the reduction described by Algorithm 3. In fact, Algorithm 3, when instantiated with Optimistic ROBD, provides an efficient algorithm for online control. It only requires \( O(p) \) memory to compute the recursive sequences. As stated in Algorithm 3 the recursive computation of \( y_t \) and \( \zeta_t \) may have numerical issues. However this can be addressed in a straightforward manner when the algorithm is instantiated with Optimistic ROBD (see Appendix G).

#### 4.2. Case 1: Exact Prediction of \( w_t \) \((W_t = \{w_t\})\)

In the case where we know \( w_t \) exactly before picking \( u_t \) in the online control problem, we have \( W_t = \{w_t\} \) is a one-point set. So, after reducing it to OCO with structured memory, we have \( \Omega_t = \{v_t\} \) and can apply Optimistic ROBD directly. The pseudo code for this reduction is given in Algorithm 3, where we use Optimistic ROBD as the solver \text{ALG}. Combining Theorem 3 with Theorem 1, we obtain the performance bound in Corollary 1.

**Corollary 1.** In IDSR, assume that coefficients \( q_{t:t+p-1} \) are observable at time step \( t \). Let \( \alpha = \sum_{i=1}^{q} \|C_i\|_2 \), where \( C_i, i = 1, \ldots, p \) are defined as in Section 4.1. When \( W_t \) is a one-point set for all time steps \( t \), the competitive ratio of Algorithm 3, using Optimistic ROBD with parameter \( \lambda \), is...
upper bounded by:
\[
\max \left\{ \frac{1}{\lambda}, \frac{\lambda + q_{min}}{(1-\alpha^2)\lambda + q_{min}} \right\}.
\]

4.3. Case 2: Inexact Prediction of \( w_t \) (\( w_t \in W_t \))

We now move to the case with adversarial noise. As Theorem 2 suggests, we can tune \( \eta \) in Optimistic ROBD based on the quality of prediction. As a result, we present two forms of upper bounds for Algorithm 3 in Corollaries 2 and 3. Notably, Corollary 2 gives a tighter bound where good estimations are available, while Corollary 3 gives a bound that does not depend on the quality of the estimations.

In the first case, we assume that a good estimation of \( w_t \) is available before picking \( u_t \). Specifically, we assume the diameter of set \( W_t \) is upper bounded by \( \epsilon_t \) for all time steps \( t \), where \( \epsilon_t \) is a small positive constant. We derive Corollary 2 by setting \( \eta = 1 + \lambda \) in Theorem 2.

**Corollary 2.** In IDS, assume that coefficients \( q_{t:t+p-1} \) are observable at time step \( t \). Let \( \alpha = \sum_{i=1}^{q} \|C_i\|_2 \), where \( C_i, i = 1, \ldots, p \) are defined as in Section 4.1. When the diameter of \( W_t \) is upper bounded by \( \epsilon_t \) for all time steps \( t \), the total cost incurred by Algorithm 3 (using Optimistic ROBD with parameter \( \lambda \)) in the online control problem is upper bounded by:
\[
K_1 = (2 + \lambda) \cdot \max \left\{ \frac{1}{\lambda}, \frac{\lambda + q_{min}}{(1-\alpha^2)\lambda + q_{min}} \right\},
\]
\[
K_2 = \lambda \left( \frac{q_{max}}{2} + \frac{4\alpha^2}{1+\lambda} \right),
\]
\[
\sum_{t=0}^{T-1} 2 \epsilon_t^2.
\]

The residue term \( K_2 \) in Corollary 2 becomes negligible when the total estimation error \( \sum_{t=0}^{T-1} \epsilon_t^2 \) is small, leading to a pure competitive ratio guarantee. Further, if we ignore \( K_2 \), the coefficient \( K_1 \) is only constant factor worse than the ratio we obtain when exact prediction of \( w_t \) is available.

However, the bound in Corollary 2 can be significantly worse than the case where exact prediction is available when the diameter of \( W_t \) is large or unbounded. Hence we introduce a second corollary that does not use any information about \( w_t \) when picking \( u_t \). Specifically, we assume the diameter of set \( W_t \) cannot be bounded, and the upper bound given in Theorem 2 is meaningless. By picking the parameter \( \eta \) such that \( \lambda \left( \frac{1}{1+\eta^2} + \frac{q_{max}}{2} - \frac{1}{\lambda+m} \right) \leq 0 \) in Theorem 2, we obtain the following result.

**Corollary 3.** In IDS, assume that coefficients \( q_{t:t+p-1} \) are observable at time step \( t \). Let \( \alpha = \sum_{i=1}^{q} \|C_i\|_2 \), where \( C_i, i = 1, \ldots, p \) are defined as in Section 4.1. The competitive ratio of Algorithm 3, using Optimistic ROBD with \( \lambda \), is upper bounded by:
\[
O \left( \left( \frac{q_{max}}{2} + 4\alpha^2 \right) \max \left\{ \frac{1}{\lambda}, \frac{\lambda + q_{min}}{(1-\alpha^2)\lambda + q_{min}} \right\} \right).
\]

Compared with Corollary 2, Corollary 3 gives an upper bound that is independent of the size of \( W_t \). It is also a pure competitive ratio, without any additive term. However, the ratio is worse than the case where exact prediction of \( w_t \) is available, especially when \( q_{max} \) or \( \alpha \) is large.

5. Analytic and Numerical Examples

In this section we use simple examples to illustrate the contrast between the best linear controller in hindsight, which is the predominant benchmark, and the optimal offline controller, which is not necessarily linear or static. We highlight analytically that the optimal linear controller can be arbitrarily worse than the optimal offline controller, and then illustrate that both analytically and numerically that Optimistic ROBD can obtain near-optimal cost.

5.1. Example 1: A Scalar System

Consider the following scalar system:
\[
\min_{u_t} \sum_{t=0}^{T} q_x |x_t|^2 + |u_t|^2
\]
\[
s.t. \quad x_{t+1} = ax_t + u_t + w_t,
\]
where \( a > 1, x_0 = 0 \) and \( w_t \) is the disturbance. For this system, we have:
\[
\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a - 1)^2}{4}, \forall \{w_t\}_{t=0}^{T},
\]
where \( \text{cost}(LC) \) is the cost of the optimal linear controller in hindsight. Hence, \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \) is arbitrarily large as \( q \) and \( a \) increase. We emphasize that this lower bound holds for any disturbance sequence, and there exist many sequences making this lower bound even bigger. For example, if \( w_t = w, \forall t \):
\[
\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a - 1)^2}{4} \cdot \frac{q + (a - 1)^2}{q}.
\]
Alternatively, if \( w_t = (-1)^t \cdot w \):
\[
\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a - 1)^2}{4} \cdot \frac{q + (a + 1)^2}{q}.
\]
Proofs are given in Appendix H. This example highlights that the gap between \( \text{cost}(LC) \) and \( \text{cost}(OPT) \) can be arbitrarily large for strongly convex costs. Thus, even if an algorithm has no regret compared to the optimal linear controller, it has an unbounded competitive ratio.

Further, we can contrast the competitive ratio of the optimal linear controller derived above with that of Optimistic ROBD. For convenience, assume \( \text{cost}(OPT) = T \). First, notice that there exists \( \{w_t\}_{t=0}^{T} \) such that \( \text{cost}(LC) \geq \)
we consider two cases, in the first case

To further demonstrate the efficiency and performance of our

best linear controller, which uses perfect hindsight.

in the case when

linear controller even if it does not know

matches the true optimal when

ther, if

behavior of Optimistic ROBD is not sensitive to

given at time step

t is smooth, Optimistic ROBD is much better than the best

linear controller in hindsight, and almost

w is known/unknown at step

t is unpredictable and

w is a random walk, and also two different settings:

w is known/unknown at step

t.

Figure 1. Numerical results of Optimistic ROBD in 1-d and 2-d systems, with different \( \lambda \). LC means the best linear controller in hindsight and OPT means the global optimal controller in hindsight. LC is numerically searched in stable linear controller space. We consider two different types of \( w_t \): \( w_t \) is unpredictable and \( w_t \) is a random walk, and also two different settings: \( w_t \) is known/unknown at step \( t \).

\( O(\max\{a^2, q, a^4/q\} \cdot T) \) for big enough \( a \) and \( q \). From Corollary 1, in the case exact prediction of \( w_t \) is possible, Optimistic ROBD has \( \text{cost(ALG)} \leq O(\max\{1, a^2/q\} \cdot T) \), \( \forall \{w_t\}_t^T = 0 \), which is orders-of-magnitude lower than \( \text{cost(LC)} \).

In the case exact prediction is impossible and the estimation error is \( \epsilon_t = w_t - \hat{w}_t \), Optimistic ROBD guarantees \( \text{cost(ALG)} \leq O(\max\{1, a^2/q\} \cdot T + \max\{a^2, q\} \cdot \sum_{t=0}^{T-1} \epsilon_t^2) \) by Corollary 2. Moreover, Corollary 3 gives a constant competitive ratio, \( \text{cost(ALG)} \leq O(\max\{a^2, q, a^4/q\} \cdot T) \) for any \( \{w_t\}_t^T = 0 \), which is the same as the lower bound of \( \text{cost(LC)} \) we found. Thus, even without any estimate of the noise, our upper bound on the cost of Optimistic ROBD matches the lower bound on the cost of the optimal linear controller.

To further demonstrate the efficiency and performance of our algorithm, we implement Optimistic ROBD for this example with \( a = 2, q = 8 \) and \( T = 200 \). For the sequence \( \{w_t\}_t^T \), we consider two cases, in the first case \( \{w_t\}_t^T \) is generated by \( w_t \sim U(-1, 1) \) i.i.d., and in the second case the sequence is generated by \( w_{t+1} = w_t + \psi_t \) where \( \psi_t \sim U(-0.2, 0.2) \) i.i.d.. The first case corresponds to unpredictable disturbances, where the estimation set \( W_t = (-1, 1) \), and the second to smooth disturbances (i.e., a random walk), where \( W_t = w_{t-1} + (-0.2, 0.2) \). For both types of \( \{w_t\}_t^T = 0 \), we test Optimistic ROBD algorithms in two settings: \( w_t \) is known/unknown at step \( t \). In the first setting, \( w_t \) is directly given to the algorithm, and in the latter setting, only \( W_t \) is given at time step \( t \).

The results are shown in Figure 1 (a-b). We see that the behavior of Optimistic ROBD is not sensitive to \( \lambda \). Further, if \( w_t \) is known at step \( t \), Optimistic ROBD is much better than the best linear controller in hindsight, and almost matches the true optimal when \( w_t \) is smooth. In fact, when \( w_t \) is smooth, Optimistic ROBD is much better than the best linear controller even if it does not know \( w_t \) at step \( t \). Even in the case when \( w_t \sim U(-1, 1) \), and so is extremely unpredictable, Optimistic ROBD’s performance still matches the best linear controller, which uses perfect hindsight.

5.2. Example 2: Double Integrator

Our second example is slightly more complex than the first, but is still simple. We consider the following objective:

\[
\sum_{t=0}^{200} \|x_t\|^2 + \|u_t\|^2,
\]

with dynamics specified by:

\[
x_{t+1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t,
\]

where \( (A, B) \) is the canonical form of double integrator dynamics. For this 2-d system, similarly, we test the performance of Optimistic ROBD with two types of \( w_t \).

In this case, we present only a numerical comparison. The results are shown in Figure 1 (c-d) and reinforce the same observations we observed in Example 1. In particular, we see that the optimal linear controller can be significantly more costly than the offline optimal controller and that Optimistic ROBD can outperform the optimal linear controller, sometimes by a significant margin.

6. Concluding Remarks

In this paper, we provide the first constant competitive policy for a class of online control problems, called Input-Disturbed Squared Regulators (IDSRs), with adversarial disturbances. Our analysis is based on a novel reduction from online control to a class of online convex optimization problems with structured memory. We also highlight the benefits of our approach experimentally.

Following on our work, it will be interesting to understand the breadth of the class of online control problems that admit constant competitive algorithms. Our work shows that it is possible to be constant competitive for IDSRs in controllable canonical form, which comprise an interesting subclass of the more general systems. Obtaining results (positive or negative) about the existence of constant competitive algorithms for more general dynamics and more general classes of costs is an important and challenging future direction.
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A. Preliminaries

The appendices that follow provide the proofs of the results in the body of the paper. Throughout the proofs we use the following notation to denote the hitting and movement costs of the online learner: \( H_t := f_t(y_t) \) and \( M_t := c(y_{t:t-p}) \), where \( y_t \) is the point chosen by the online algorithm at time \( t \). Similarly, we denote the hitting and movement costs of the offline optimal as \( H^*_t := f_t(y^*_t) \) and \( M^*_t := c(y^*_{t:t-p}) \), where \( y^*_t \) is the point chosen by the offline optimal at time \( t \).

Before moving to the proofs, we summarize a few standard definitions that are used throughout the paper.

**Definition 1.** A function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is \( m \)-strongly convex with respect to a norm \( \| \cdot \| \) if for all \( x, y \) in the relative interior of the domain of \( f \) and \( \lambda \in (0, 1) \), we have

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{m}{2}\lambda(1 - \lambda)\|x - y\|^2.
\]

**Definition 2.** A function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is \( l \)-strongly smooth with respect to a norm \( \| \cdot \| \) if \( f \) is everywhere differentiable and if for all \( x, y \) we have

\[
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{l}{2}\|y - x\|^2.
\]

Finally, Lemma 13 in Goel et al. (2019) will be useful, and so we restate it here.

**Lemma 1.** If \( f : \mathcal{X} \rightarrow \mathbb{R} \) is a \( m \)-strongly convex function with respect to some norm \( \| \cdot \| \), and \( v \) is the minimizer of \( f \) (i.e. \( v = \arg \min_{x \in \mathcal{X}} f(x) \)), then we have for all \( y \in \mathcal{X} \),

\[
f(y) \geq f(v) + \frac{m}{2}\|y - v\|^2.
\]

B. Proof of Theorem 1

Our approach is to make use of strong convexity and properties of the hitting cost, the switching cost, and the regularization term to derive an inequality in the form of \( H_t + M_t + \Delta \phi_t \leq C(H^*_t + M^*_t) \) for some positive constant \( C \), where \( \Delta \phi_t \) is the change in potential, which satisfies \( \sum_{t=1}^T \Delta \phi_t \geq 0 \). We will give the formal definition of \( \Delta \phi_t \) later. The constant \( C \) is then an upper bound for the competitive ratio.

We use \( \| \cdot \| \) to denote \( \ell_2 \) norm or matrix norm induced by \( \ell_2 \) norm throughout the proof.

By assumption, we have \( y_t = y^*_t \) for \( i = 0, -1, \cdots, -(p - 1) \).

For convenience, we define

\[
\phi_t = \frac{\lambda_1 + \lambda_2 + m}{2}\|y_t - y^*_t\|^2.
\]

Recall that we define \( v_t = \arg \min_y f_t(y) \). Since the function

\[
g_t(y) = f_t(y) + \frac{\lambda_1}{2}\left\| y - \sum_{i=1}^p C_i y_{t-i} \right\|^2 + \frac{\lambda_2}{2}\left\| y - v_t \right\|^2
\]

is \((m + \lambda_1 + \lambda_2)-\)strongly convex and ROBD selects \( y_t = \arg \min_y g_t(y) \), we see that

\[
g_t(y_t) + \frac{m + \lambda_1 + \lambda_2}{2}\|y_t - y^*_t\|^2 \leq g_t(y^*_t),
\]

which implies

\[
H_t + \lambda_1 M_t + \left( \phi_t - \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \right)
\]

\[
\leq \left( H^*_t + \frac{\lambda_2}{2}\|y^*_t - v_t\|^2 \right) + \left( \frac{\lambda_1}{2}\|y^*_t - \sum_{i=1}^p C_i y_{t-i}\|^2 - \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \right). \tag{4}
\]
In the following steps, we bound the second term in the right hand side of (4) by the switching cost of the offline optimal.

\[
\sum_{i=1}^{p} \frac{||C_i||}{\alpha} \phi_{t-i} \\
= \frac{\lambda_1 + \lambda_2 + m}{2\alpha} \sum_{i=1}^{p} ||C_i|| \cdot ||y_{t-i} - y^*_{t-i}||^2 \\
\geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left( \sum_{i=1}^{p} ||C_i|| \cdot ||y_{t-i} - y^*_{t-i}|| \right)^2 (5a) \\
\geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left( \sum_{i=1}^{p} ||C_i y_{t-i} - C_i y^*_{t-i}|| \right)^2 (5b) \\
\geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left( \sum_{i=1}^{p} \sum_{i=1}^{p} C_i y_{t-i} - \sum_{i=1}^{p} C_i y^*_{t-i} \right)^2, (5c)
\]

where we use Jensen’s Inequality in (5a); the definition of the matrix norm in (5b); the triangle inequality in (5c).

For notation convenience, we define

\[
\delta_t = \sum_{i=1}^{p} C_i y_{t-i} - \sum_{i=1}^{p} C_i y^*_{t-i}.
\]

Therefore, we obtain that

\[
\frac{\lambda_1}{2} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y_{t-i} \right| \right|^2 - \sum_{i=1}^{p} \frac{||C_i||}{\alpha} \phi_{t-i} \\
\leq \frac{\lambda_1}{2} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y_{t-i} \right| \right|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 (6a) \\
= \frac{\lambda_1}{2} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right|^2 - \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 (6b) \\
\leq \frac{\lambda_1}{2} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right|^2 + \lambda_1 \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right| \cdot \|\delta_t\| \\
+ \frac{\lambda_1}{2} \cdot \|\delta_t\|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 (6c) \\
\leq \frac{\lambda_1}{2} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right|^2 + \frac{\alpha^2 \lambda_2^2}{2((1 - \alpha^2)\lambda_1 + \lambda_2 + m)} \left| \left| y^*_t - \sum_{i=1}^{p} C_i y^*_{t-i} \right| \right|^2 \\
+ \frac{(1 - \alpha^2)\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 - \frac{(1 - \alpha^2)\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 (6c) \\
= \frac{\lambda_1(\lambda_1 + \lambda_2 + m)}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M^*_t,
\]

where we use (5) in (6a); the triangle inequality in (6b); the AM-GM inequality in (6c).

We also notice that since \(f_t\) is \(m\)-strongly convex, the first term in the right hand side of (4) can be bounded by

\[
H^*_t + \frac{\lambda_2}{2} \left| \left| y^*_t - v_t \right| \right|^2 \leq \frac{m + \lambda_2}{m} H^*_t. (7)
\]
Substituting (6) and (7) into (4), we obtain that

\[ H_t + \lambda^2 M_t + \phi_t = \sum_{i=1}^q \|C_i\| \phi_{t-i} \leq \frac{m + \lambda^2}{m} H^*_t + \frac{\lambda^1 (\lambda^1 + \lambda^2 + m)}{(1 - \alpha^2)\lambda^1 + \lambda^2 + m} M^*_t. \]  

(8)

Define \( \Delta \phi_t = \phi_t - \sum_{i=1}^q \frac{\|C_i\|}{\alpha} \phi_{t-i} \). We see that

\[ \sum_{t=1}^T \Delta \phi_t = \frac{1}{\alpha} \sum_{t=1}^T \left( \sum_{i=1}^q \|C_i\| \right) \phi_{t-i} - \frac{1}{\alpha} \sum_{t=1}^T \left( \sum_{i=1}^q \|C_i\| \right) \phi_{t-i}. \]

Since \( \phi_t \geq 0, \forall t \) and \( \phi_0 = \phi_{-1} = \cdots = \phi_{-q+1} = 0 \), we have

\[ \sum_{t=1}^T \Delta \phi_t \geq 0. \]  

(9)

Summing (8) over timesteps \( t = 1, 2, \cdots , T \), we see that

\[ \sum_{t=1}^T (H_t + \lambda^1 M_t) + \sum_{t=1}^T \Delta \phi_t \leq \sum_{t=1}^T \left( \frac{m + \lambda^2}{m} H^*_t + \frac{\lambda^1 (\lambda^1 + \lambda^2 + m)}{(1 - \alpha^2)\lambda^1 + \lambda^2 + m} M^*_t \right). \]

Using (9), we obtain that

\[ \sum_{t=1}^T (H_t + \lambda^1 M_t) \leq \sum_{t=1}^T \left( \frac{m + \lambda^2}{m} H^*_t + \frac{\lambda^1 (\lambda^1 + \lambda^2 + m)}{(1 - \alpha^2)\lambda^1 + \lambda^2 + m} M^*_t \right), \]  

(10)

which implies

\[ \sum_{t=1}^T (H_t + M_t) \leq \sum_{t=1}^T \left( \frac{m^1 + \lambda^2}{m^1} H^*_t + \frac{\lambda^1 + \lambda^2 + m}{(1 - \alpha^2)\lambda^1 + \lambda^2 + m} M^*_t \right). \]

C. Lower Bound of Online Optimization with Structured Memory

Theorem 1 considers the problem setting where the hitting cost functions are \( m \)-strongly convex in feasible set \( \mathcal{X} \) and the switching cost is given by \( c(y_{t:p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2 \), where \( C_i \in \mathbb{R}^{d \times d} \) and \( \sum_{i=1}^p \|C_i\|_2 = \alpha \). We prove that the competitive ratio provided in Theorem 1 is optimal in parameters \( \alpha \) and \( m \) by showing a lower bound for a specific sequence of hitting costs and a specific form of switching cost, \( c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - \alpha y_{t-1}\|_2^2 \).

Notice that making improvements on the competitive ratio is still possible if we consider more specific matrix \( C_i \) or adding more assumptions on the hitting cost functions.

**Theorem 4.** When the hitting cost functions are \( m \)-strongly convex in feasible set \( \mathcal{X} \) and the switching cost is given by

\[ c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - \alpha y_{t-1}\|_2^2 \]  

for a constant \( \alpha \geq 1 \), the competitive ratio of any online algorithm is lower bounded by

\[ \frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( \frac{1 + \alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right). \]

Theorem 4 is a generalization of (Goel et al., 2019)[Theorem 1], which only considers the case when \( \alpha = 1 \). Our proof uses a parallel approach but extends it to general \( \alpha \). Before giving the proof of Theorem 4, we first prove the generalization of (Goel et al., 2019)[Lemma 7]. To simplify presentation in the proofs, we use \( \mathcal{K}(n, y) \) to denote the set \( \{y \in \mathbb{R}^{n+2} \mid y_t \in \mathbb{R}, y_0 = 0, y_{n+1} = y \} \).
Then we have
\[ \lim_{n \to \infty} a_n = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}. \]

Proof of Lemma 2. Using a parallel approach to (Goel et al., 2019)[Lemma 7], we can show that sequence \( \{a_n\} \) satisfies the recursive relationship
\[ a_{n+1} = \frac{a_n + m}{a_n + m + \alpha^2}. \]

Solving the equation \( y = \frac{y + m}{y + m + \alpha^2} \), we find the two fixed points of the recursive relationship \( a_{n+1} = \frac{a_n + m}{a_n + m + \alpha^2} \) are
\[ y_1 = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}, \]
and
\[ y_2 = \frac{-m - \alpha^2 + 1 - \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}. \]

Notice that for \( i = 1, 2 \), we have
\[ m - (m + \alpha^2) y_i = -(1 - y_i) y_i. \]

Using this property, we obtain
\[ a_{n+1} - y_1 = \frac{a_n + m}{a_n + m + \alpha^2} y_1 - y_1 = \frac{(1 - y_1) a_n + m - (m + \alpha^2) y_1}{a_n + m + \alpha^2} = \frac{(1 - y_1) (a_n - y_1)}{a_n + m + \alpha^2}, \tag{11} \]
and
\[ a_{n+1} - y_2 = \frac{a_n + m}{a_n + m + \alpha^2} y_2 - y_2 = \frac{(1 - y_2) a_n + m - (m + \alpha^2) y_2}{a_n + m + \alpha^2} = \frac{(1 - y_2) (a_n - y_2)}{a_n + m + \alpha^2}. \tag{12} \]

Notice that \( a_{n+1} - y_2 > 0 \). By dividing equations (11) and (12), we obtain
\[ \left( \frac{a_{n+1} - y_1}{a_{n+1} - y_2} \right) = \frac{1 - y_1}{1 - y_2} \left( \frac{a_n - y_1}{a_n - y_2} \right), \forall n \geq 0. \]

Solving this in a parallel way to (Goel et al., 2019)[Lemma 7], we get
\[ a_n = \left( 1 - \frac{1 - y_1}{1 - y_2} \right)^{n+1} \left( y_1 - y_2 \cdot \frac{1 - y_1}{1 - y_2} \right)^{n+1}. \]

Since \( 0 < \left( \frac{1-y_1}{1-y_2} \right) < 1 \), we have
\[ \lim_{n \to \infty} a_n = y_1 = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}. \tag{13} \]

Now we come back to the proof of Theorem 4.

Proof of Theorem 4. We consider the counterexample where the starting point of the algorithm and the offline adversary is \( y_0 = y_0^* = 0 \), and the hitting cost functions are
\[ f_t(y) = \begin{cases} \frac{m}{2} y^2 & t \in \{1, 2, \cdots, n\} \\ \frac{m}{2} (y - 1)^2 & t = n + 1 \end{cases} \]
for some large parameter $m'$ that we choose later.

By a parallel approach to (Goel et al., 2019)[Theorem 1], we can show the cost incurred by any online algorithm has the lower bound

\[ \text{cost}(ALG) \geq \min_y \left( \frac{1}{2} y^2 + \frac{m'}{2} (y - 1)^2 \right) = \frac{1}{2} \left( 1 + \frac{1}{m'} \right). \] (14)

In contrast to the case when $\alpha = 1$, the offline adversary can leverage the factor $\alpha$ to approach $1$ quicker if $\alpha > 1$.

Let the sequence of points the adversary chooses be $y^* = (y_0^*, y_1^*, \cdots, y_{n+1}^*) \in \mathbb{R}^{n+2}$. We compute the cost incurred by the adversary as follows.

\[ a_n = 2 \min_{y^* \in K(n,1)} \sum_{i=1}^{n+1} (H_i^* + M_i^*) \]

\[ = 2 \min_{y^* \in K(n,1)} \left( \sum_{i=1}^{n} \frac{m}{2} (y_i^*)^2 + \sum_{i=1}^{n+1} \frac{1}{2} (y_i^* - \alpha y_{i-1}^*)^2 \right). \]

In words, $a_n$ is twice the minimal offline cost subject to the constraints $y_0^* = 0, y_{n+1}^* = 1$. Recall that we have derived the limiting behavior of the offline costs as $n \to \infty$ for general $\alpha$ in the Lemma 2. Given Lemma 2, the total cost of the offline adversary will be $\frac{a_n}{2}$. Finally, applying (14), we know $\forall n$ and $\forall m' > 0$,

\[ \frac{\text{cost}(ALG)}{\text{cost}(ADV)} \geq \frac{\frac{1}{2} \left( 1 + \frac{1}{m'} \right)}{\frac{a_n}{2}} = \frac{1}{\frac{1}{m'} a_n}. \]

By taking the limit $n \to \infty$ and $m' \to \infty$ and using Lemma 2, we obtain

\[ \frac{\text{cost}(ALG)}{\text{cost}(OPT)} = \lim_{n,m' \to \infty} \frac{\text{cost}(ALG)}{\text{cost}(ADV)} \geq \frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( 1 + \frac{\alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right). \]

D. Proof of Theorem 2

We use $\| \cdot \|$ to denote $\ell_2$ norm or matrix norm induced by $\ell_2$ norm throughout the proof. Before giving the proof of Theorem 2, we first prove three lemmas that we use later.

Recall that ROBD with parameters $\lambda_1 = \lambda, \lambda_2 = 0$ minimizes a weighted sum of the hitting cost $f_i$ and the switching cost $c$.

To pick the appropriate estimation of $v_t$ from the set $\Omega_t$, we want to study when the previous decision points $\hat{y}_{t-\eta x t-1}$ is fixed, how the position of $v_t$ will affect the minimum of this weighted sum. By a change of variable, we see this is equivalent to study when the hitting cost function is fixed, how the sum $\sum_{i=1}^{p} C_i \hat{y}_{t-i}$ will affect the weighted sum. We use $x$ to denote the sum $\sum_{i=1}^{p} C_i \hat{y}_{t-i}$ in Lemma 3.

**Lemma 3.** Suppose function $f : \mathbb{R}^d \to \mathbb{R}$ is $m$-strongly convex. Define function $g : \mathbb{R}^d \to \mathbb{R}$ as

\[ g(x) = \min_y f(y) + \frac{\lambda}{2} \| y - x \|^2. \]

Then $g$ is $\frac{\lambda m}{\lambda + m}$-strongly convex.

**Proof of Lemma 3.** Due to the definition of strongly convexity, we only need to show that for all $x_1, x_2 \in \mathbb{R}^d$ and $\eta \in (0, 1)$, we have

\[ g(\eta x_1 + (1 - \eta) x_2) \leq \eta g(x_1) + (1 - \eta) g(x_2) - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \| x_1 - x_2 \|^2. \]

For convenience, we define

\[ y_1 := \arg \min_y f(y) + \frac{\lambda}{2} \| y - x_1 \|^2, \]
and

\[ y_2 := \arg \min_y f(y) + \frac{\lambda}{2} \|y - x_2\|^2. \]

We have that

\[ \eta g(x_1) + (1 - \eta)g(x_2) - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \|x_1 - x_2\|^2 \]

\[ = \eta f(y_1) + (1 - \eta)f(y_2) + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta) \lambda}{2} \|y_2 - x_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \|x_1 - x_2\|^2 \] \hspace{1cm} (15a)

\[ \geq f(\eta y_1 + (1 - \eta)y_2) + \frac{m}{2} \eta (1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \|x_1 - x_2\|^2 \]

\[ + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta) \lambda}{2} \|y_2 - x_2\|^2 \] \hspace{1cm} (15b)

\[ \geq g(\eta x_1 + (1 - \eta)x_2) + \frac{m}{2} \eta (1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \|x_1 - x_2\|^2 \]

\[ + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta) \lambda}{2} \|y_2 - x_2\|^2 - \frac{\lambda}{2} \|\eta(y_1 - x_1) + (1 - \eta)(y_2 - x_2)\|^2 \] \hspace{1cm} (15c)

\[ \geq g(\eta x_1 + (1 - \eta)x_2) + \frac{m}{2} \eta (1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta (1 - \eta) \|x_1 - x_2\|^2 \]

\[ + \frac{\eta(1 - \eta) \lambda}{2} \|(y_1 - y_2) - (x_1 - x_2)\|^2, \]

where in (15a) and (15c) we use the definition of function \( g \); in (15b) we use the fact that \( f \) is \( m \)--strongly convex; in (15c) we use function \( \frac{1}{2} \|\cdot\|^2 \) is \( \lambda \)--strongly convex.

Notice that

\[ m \|y_1 - y_2\|^2 - \frac{\lambda m}{\lambda + m} \|x_1 - x_2\|^2 + \lambda \|(y_1 - y_2) - (x_1 - x_2)\|^2 \]

\[ \geq m \|y_1 - y_2\|^2 - \frac{\lambda m}{\lambda + m} \|x_1 - x_2\|^2 + \lambda \|y_1 - y_2\|^2 + \lambda \|x_1 - x_2\|^2 - 2\lambda \|y_1 - y_2\| \cdot \|x_1 - x_2\| \] \hspace{1cm} (16)

\[ = (m + \lambda) \|y_1 - y_2\|^2 + \frac{\lambda^2}{m + \lambda} \|x_1 - x_2\|^2 - 2\lambda \|y_1 - y_2\| \cdot \|x_1 - x_2\| \]

\[ \geq 0. \]

Substituting (16) into (15) finishes the proof.

In the second lemma, we show that if a function \( f \) is strongly smooth, the function value \( f(y) \) at point \( y \) can be upper bounded by a weighted sum of the function value \( f(x) \) at another point \( x \) and the squared distance between \( x \) and \( y \).

**Lemma 4.** If \( f : \mathbb{R}^d \to \mathbb{R}^+ \cup \{0\} \) is convex and \( l \)--strongly smooth, we have for all \( x, y \in \mathbb{R}^d \), the inequality

\[ f(y) \leq (1 + \eta) f(x) + \left( 1 + \frac{1}{\eta} \right) \cdot \frac{l}{2} \|x - y\|^2 \]

holds for all \( \eta > 0 \).

**Proof of Lemma 4.** Let \( v := \arg \min_z f(z) \).

Using the property of \( l \)--strongly smoothness, we see that

\[ f(x) \geq f(v) + \langle \nabla f(v), x - v \rangle + \frac{1}{2l} \|\nabla f(x) - \nabla f(v)\|^2 \] \hspace{1cm} (17a)

\[ \geq \frac{1}{2l} \|\nabla f(x)\|^2, \] \hspace{1cm} (17b)
where we use (Bubeck et al., 2015)[Lemma 3.5] in (17a); we use \( f(v) \geq 0, \nabla f(v) = 0 \) in (17b).

Therefore, we obtain that
\[
\begin{align*}
    f(y) &\leq f(x) + (\nabla f(x), y - x) + \frac{l}{2} ||y - x||^2 \\ &\leq f(x) + \|\nabla f(x)\| \cdot ||y - x|| + \frac{l}{2} ||y - x||^2 \\
    &\leq f(x) + \frac{\eta}{2l} \|\nabla f(x)\|^2 + \frac{l}{2\eta} ||y - x||^2 + \frac{l}{2} ||y - x||^2 \\
    &\leq f(x) + \eta f(x) + \left(1 + \frac{1}{\eta}\right) \cdot \frac{l}{2} ||y - x||^2 \\
    &= (1 + \eta) f(x) + \left(1 + \frac{1}{\eta}\right) \cdot \frac{l}{2} ||y - x||^2,
\end{align*}
\]

where we use that \( f \) is \( l \)-strongly smooth in (18a); Cauchy-Schwarz Inequality in (18b); AM-GM inequality in (18c); (17) in (18d).

Recall that \( \hat{y}_t \) is the decision point of ROBD which knows the exact \( v_t \) before picking \( \hat{y}_t \). \( y_t \) is the decision point of Optimistic ROBD which cannot observe the exact \( v_t \) before picking \( y_t \). In the third lemma, we show that \( y_t \) and \( \hat{y}_t \) will be close to each other once the estimated minimizer \( \hat{v}_t \) computed by Optimistic ROBD is close to the true minimizer \( v_t \).

**Lemma 5.** Under the same assumptions as Theorem 2, the distance between \( y_t \) and \( \hat{y}_t \) can be upper bounded by
\[
||y_t - \hat{y}_t|| \leq 2 ||\zeta_t||,
\]

where \( \zeta_t = v_t - \hat{v}_t \).

**Proof of Lemma 5.** Recall that by definition, the real hitting cost function which we used to pick \( \hat{y}_t \) is \( f_t(y) = h_t(y - v_t) \), and the estimated hitting cost function which we used to pick \( y_t \) is given by \( \hat{f}_t(y) = h_t(y - \hat{v}_t) \). Therefore, we have \( \hat{f}_t(y) = f_t(y + \zeta_t) \).

Since \( \hat{y}_t = ROBD(f_t, \hat{y}_{t-1:t-q}) = \arg \min_{y} f_t(y) + \lambda c(y, \hat{y}_{t-1:t-p}) \), by strongly convexity, we have that
\[
\begin{align*}
    &f_t(\hat{y}_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{m + \lambda}{2} \|\hat{y}_t - y_t - \zeta_t\|^2 \\
    &\leq f_t(y_t + \zeta_t) + \frac{\lambda}{2} \left\| y_t + \zeta_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2.
\end{align*}
\]

Similarly, using \( y_t = ROBD(\hat{f}_t, \hat{y}_{t-1:t-q}) = \arg \min_{y} \hat{f}_t(y + \zeta_t) + \lambda c(y, \hat{y}_{t-1:t-p}) \), we obtain that
\[
\begin{align*}
    &\hat{f}_t(y_t + \zeta_t) + \frac{\lambda}{2} \left\| y_t + \zeta_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{m + \lambda}{2} \|\hat{y}_t - y_t - \zeta_t\|^2 \\
    &\leq \hat{f}_t(\hat{y}_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \zeta_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2.
\end{align*}
\]

Adding (19) and (20) together, we obtain that
\[
\begin{align*}
    &\left( m + \lambda \right) \|\hat{y}_t - y_t - \zeta_t\|^2 \\
    &\leq \frac{\lambda}{2} \left( \left\| y_t + \zeta_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 - \left\| y_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \left\| \hat{y}_t - \zeta_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 - \left\| \hat{y}_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 \right) \\
    &= \lambda \zeta_t^T (y_t + \zeta_t - \hat{y}_t) \\
    &\leq \lambda \|\zeta_t\| \cdot ||\hat{y}_t - y_t - \zeta_t||.
\end{align*}
\]
Therefore, we see that
\[ \| \hat{y}_t - y_t - \zeta_t \| \leq \| \zeta_t \|, \]
which implies
\[ \| y_t - \hat{y}_t \| \leq 2 \| \zeta_t \|. \]

Now we come back to the proof of Theorem 2.

Define function \( \psi : \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{0\} \) as
\[ \psi(v) = \min_y h_t(y - v) + \lambda c(y, \hat{y}_{t-1:t-q}). \]

By a change of variable \( y \leftarrow z + v \), we can rewrite function \( \psi \) as
\[ \psi(v) = \min_z h_t(z) + \frac{\lambda}{2} \left\| z - \left( -v + \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right) \right\|^2. \quad (22) \]

By Lemma 3, we see that function \( \psi \) is \( \frac{\lambda m}{\lambda + m} \)-strongly convex.

Recall that
\[ y_t = ROBD(\tilde{f}_t, \hat{y}_{t-1:t-q}) = \arg \min_y h_t(y - \bar{v}_t) + \lambda c(y, \hat{y}_{t-1:t-q}), \quad (23) \]

and
\[ \hat{y}_t = ROBD(f_t, \hat{y}_{t-1:t-q}) = \arg \min_y h_t(y - v_t) + \lambda c(y, \hat{y}_{t-1:t-q}). \quad (24) \]

Since \( \bar{v}_t \) minimizes \( \psi \) and \( \psi \) is \( \frac{\lambda m}{\lambda + m} \)-strongly convex, using (23) and (24), we obtain that
\[
\begin{align*}
& h_t(y_t - \bar{v}_t) + \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{1}{2} \cdot \frac{m\lambda}{\lambda + m} \| v_t - \bar{v}_t \|^2 \\
& \leq h_t(\hat{y}_t - v_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2.
\end{align*} \quad (25)
\]

Using Lemma 4, we see that for any \( \eta_1 > 0 \),
\[ \frac{1}{1 + \eta_1} h_t(y_t - v_t) \leq h_t(y_t - \bar{v}_t) + \frac{1}{2\eta_1} \| v_t - \bar{v}_t \|^2. \quad (26) \]

Since function \( \frac{\lambda}{2} \| y_t - y_t \|^2 \) is \( \lambda \)-strongly smooth in \( y \), by Lemma 4, we see that for any \( \eta_2 > 0 \),
\[ \frac{1}{1 + \eta_2} \cdot \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \leq \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{\lambda}{2\eta_2} \left\| \sum_{i=1}^{p} C_i (y_{t-i} - \hat{y}_{t-i}) \right\|^2. \quad (27) \]

Notice that
\[
\begin{align*}
\frac{1}{2} \left\| \sum_{i=1}^{p} C_i (y_{t-i} - \hat{y}_{t-i}) \right\|^2 & \leq \frac{1}{2} \left( \sum_{i=1}^{p} \| C_i \| \cdot \| y_{t-i} - \hat{y}_{t-i} \| \right)^2 \quad (28a) \\
& \leq \alpha \left( \sum_{i=1}^{p} \| C_i \| \cdot \| y_{t-i} - \hat{y}_{t-i} \| \right)^2 \quad (28b)
\end{align*}
\]
We pick $\eta$. As in Section 3, we first consider the case when $\Omega$. Although Theorem 2 does cover the case when $\Omega$, the agent may choose any point in $\Omega$, and has access to the exact $v_t$. Therefore, the same upper bound of $\sum_{t=1}^{T} (H_t + \lambda M_t)$ given in (10) in the proof of Theorem 1 also applies here. It shows that

$$\sum_{t=1}^{T} (H_t + \lambda M_t) \leq \frac{1 + \eta}{\lambda} \sum_{t=1}^{T} (\hat{H}_t + \lambda \hat{M}_t) + \lambda \left( \frac{l}{1 + \eta - \lambda} + \frac{4\alpha^2 \lambda - m\lambda}{\eta (\lambda + m)} \right) \sum_{t=1}^{T} \frac{1}{2} \|v_t - \bar{v}_t\|^2.$$  \hfill (32)

Recall that the point sequence $\{\hat{y}_t\}_{1 \leq t \leq T}$ is identical with the one picked by ROBD, which has parameters $\lambda_1 = \lambda, \lambda_2 = 0$ and has access to the exact $v_t$ before picking $\hat{y}_t$. Therefore, the same upper bound of $\sum_{t=1}^{T} (H_t + \lambda M_t)$ given in (10) in the proof of Theorem 1 also applies here. It shows that

$$\sum_{t=1}^{T} (H_t + \lambda M_t) \leq \sum_{t=1}^{T} \left( H_t^* + \frac{\lambda (\lambda + m)}{(1 - \alpha^2) \lambda + m} M_t^* \right).$$  \hfill (33)

Substituting (33) into (32) finishes the proof.

### E. Optimistic ROBD with $\lambda = 0$

Although Theorem 2 does cover the case when $\lambda = 0$, it is possible to extend the analysis to cover this setting. Notice that the agent may choose any point in $\Omega_t$ in Algorithm 2 when $\lambda = 0$. Thus, a tiebreaking rule is needed to cover the case of $\lambda = 0$. We break the tie by choosing the projection of $\sum_{i=1}^{p} C_i v_{t-i}$ on $\Omega_t$, which is natural if we consider $\lambda \to 0^\dagger$. We give the pseudo for this specific case in Algorithm 4.

As in Section 3, we first consider the case when $\Omega_t$ is a one-point set, i.e. $\Omega_t = \{v_t\}$.

**Theorem 5.** Suppose the hitting cost functions are $m$–strongly convex and the switching cost is given by $c(y_{t:t-p}) = \frac{1}{2} \|y_{t} - \sum_{i=1}^{p} C_i y_{t-i}\|^2$, where $C_i \in \mathbb{R}^{d \times d}$ and $\sum_{i=1}^{p} \|C_i\|_2 = \alpha$. When $\Omega_t = \{v_t\}$, the competitive ratio of Algorithm 4 is upper bounded by $1 + \frac{(1 + \alpha)^2}{m}$. 

\[
\leq 2\alpha \left( \sum_{i=1}^{p} \|C_i\| \cdot \|v_{t-i} - v_{t-i}\|^2 \right), \tag{28c}
\]

where we use the triangle inequality and the definition of matrix norm in (28a); Jensen’s Inequality in (28b); Lemma 5 in (28c). Substituting (28) into (27) gives

$$\frac{1}{1 + \eta_2} \cdot \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \leq \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{2\alpha \lambda}{\eta_2} \left( \sum_{i=1}^{p} \|C_i\| \cdot \|\hat{v}_{t-i} - v_{t-i}\|^2 \right). \tag{29}$$

Substituting (26) and (29) into (25), we obtain that

$$\frac{1}{1 + \eta_1} h_t(y_t - v_t) + \frac{\lambda}{2(1 + \eta_2)} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \leq h_t(\hat{y}_t - v_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^{p} C_i \hat{y}_{t-i} \right\|^2 + \frac{1}{2} \|v_t - \bar{v}_t\|^2 + \frac{2\alpha \lambda}{\eta_2} \left( \sum_{i=1}^{p} \|C_i\| \cdot \|\hat{v}_{t-i} - v_{t-i}\|^2 \right). \tag{30}$$

Summing up (30) over all time steps, we see that

$$\min\{\frac{1}{1 + \eta_1}, \frac{\lambda}{1 + \eta_2}\} \sum_{t=1}^{T} (H_t + M_t) \leq \sum_{t=1}^{T} (\hat{H}_t + \lambda \hat{M}_t) + \left( \frac{l}{\eta_1} - \frac{4\alpha^2 \lambda - m\lambda}{\eta_2} \right) \sum_{t=1}^{T} \frac{1}{2} \|v_t - \bar{v}_t\|^2. \tag{31}$$

We pick $\eta_2 = \eta$ and $\eta_1 = \frac{1 + \eta - \lambda}{\lambda}$ so that $\frac{1}{1 + \eta_1} = \frac{\lambda}{1 + \eta_2}$. Substituting into (31) gives

$$\sum_{t=1}^{T} (H_t + M_t) \leq \frac{1 + \eta}{\lambda} \sum_{t=1}^{T} (\hat{H}_t + \lambda \hat{M}_t) + \lambda \left( \frac{1}{1 + \eta - \lambda} + \frac{4\alpha^2 \lambda - m\lambda}{\eta (\lambda + m)} \right) \sum_{t=1}^{T} \frac{1}{2} \|v_t - \bar{v}_t\|^2. \tag{32}$$
Algorithm 4 Optimistic ROBD with $\lambda = 0$

for $t = 1, 2, \cdots, T$ do

\begin{itemize}
  \item Observe: $v_{t-1}, h_t, \Omega_t$
  \item $s_t \leftarrow \sum_{i=1}^{p} C_i v_{t-i}$
  \item Let $y_t$ be the projection of $s_t$ on $\Omega_t$
  \item Output: $y_t$ (the decision at time step $t$)
\end{itemize}

end for

**Proof of Theorem 5.** Notice that when $\Omega_t = \{v_t\}$, Algorithm 4 will pick $y_t = v_t$ for all time step $t$.

Since $v_t = \arg \min_{y} f_t(y)$ and $f_t$ is $m$-strongly convex, we have that

$$f_t(v_t) + \frac{m}{2} \|y_t^* - v_t\|^2 \leq f_t(y_t^*). \tag{34}$$

On the other hand, we can bound the switching cost of Algorithm 4 by

$$\frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2$$

$$= \frac{1}{2} \left\| y_t^* - \sum_{i=1}^{p} C_i y_{t-i}^* \right\|^2 + \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\| \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\| + \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}^*) \right\|^2$$

$$\leq \frac{1}{2} \left\| y_t^* - \sum_{i=1}^{p} C_i y_{t-i}^* \right\|^2 + \left\| y_t^* - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 + \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}) \right\|^2$$

$$\leq \left(1 + \frac{(1 + \alpha)^2}{m}\right) \cdot \frac{1}{2} \left\| y_t^* - \sum_{i=1}^{p} C_i y_{t-i}^* \right\|^2 + \left(1 + \frac{m}{(1 + \alpha)^2}\right) \cdot \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}) \right\|^2,$$ \tag{35a}

where we use Cauchy-Schwarz inequality in (35a); we use AM-GM inequality in (35b).

Notice that

$$\left\| (v_t - y_t^*) - \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}^*) \right\|^2 \leq \left\| v_t - y_t^* \right\|^2 + \sum_{i=1}^{p} \left\| C_i \right\| \left\| v_{t-i} - y_{t-i}^* \right\|^2$$

$$\leq (1 + \alpha) \cdot \left( \left\| v_t - y_t^* \right\|^2 + \sum_{i=1}^{p} \left\| C_i \right\| \left\| v_{t-i} - y_{t-i}^* \right\|^2 \right),$$ \tag{36a}

where we use the triangle inequality in (36a) and the Cauchy-Schwarz inequality in (36b).

Substituting (36) into (35) and summing up through time steps, we obtain that

$$\sum_{t=1}^{T} \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 \leq \sum_{t=1}^{T} \left(1 + \frac{(1 + \alpha)^2}{m}\right) M_t^* + \left(1 + \frac{m}{(1 + \alpha)^2}\right) \cdot \frac{1}{2} \left\| v_t - y_t^* \right\|^2.$$ \tag{37}

Substituting (34) gives that

$$\sum_{t=1}^{T} \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 \leq \sum_{t=1}^{T} \left(1 + \frac{(1 + \alpha)^2}{m}\right) M_t^* + \left(1 + \frac{(1 + \alpha)^2}{m}\right) \cdot (H_t^* - f_t(v_t)),$$

which implies

$$\sum_{t=1}^{T} \left( f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 \right) \leq \left(1 + \frac{(1 + \alpha)^2}{m}\right) \sum_{t=1}^{T} (H_t^* + M_t^*).$$ \tag{38}
Now we consider the case when $\Omega_t$ is a general convex set.

**Theorem 6.** Suppose the hitting cost functions are both $m-$strongly convex and $l-$strongly smooth and the switching cost is given by $c(y_{t-i} - p) = \frac{1}{2} \| y_t - \sum_{i=1}^{p} C_i y_{t-i} \|^2$, where $C_i \in \mathbb{R}^{d \times d}$ and $\sum_{i=1}^{p} \| C_i \|_2 = \alpha$. For arbitrary $\eta > 0$, the cost of Algorithm 4 is upper bounded by $K_1 COSE(\text{OPT}) + K_2$, where:

\[
K_1 = (1 + \eta) \left( 1 + \frac{(1 + \alpha)^2}{m} \right),
\]

\[
K_2 = \left( l + \frac{1}{\eta} \alpha^2 - (1 + \eta) \right) \cdot \sum_{i=1}^{T} \frac{1}{2} \| y_t - v_t \|^2.
\]

Like Theorem 2, we can choose $\eta$ to balance $K_1$ and $K_2$ and obtain a competitive ratio, in particular the smallest $\eta$ such that:

\[
l + \left( 1 + \frac{1}{\eta} \right) \alpha^2 - (1 + \eta) \leq 0.
\]

Therefore, we have $\eta = O(l + \alpha^2)$ and $K_2 \leq 0$. So the competitive ratio is upper bounded by:

\[
O \left( (l + \alpha^2) \cdot \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \right).
\]

**Proof of Theorem 6.** Since $y_t$ is the projection of $\sum_{i=1}^{p} C_i v_{t-i}$ on $\Omega_t$, and $\Omega_t$ is a convex set, we have that

\[
\frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 \leq \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 - \frac{1}{2} \| v_t - y_t \|^2.
\]

Because the hitting cost function $f_t$ is $l$-strongly smooth, and $v_t$ is the minimizer of $f_t$, we see that

\[
\frac{1}{\eta_1} f_t(y_t) \leq \frac{l}{2\eta_1} \| v_t - y_t \|^2 + \frac{1}{\eta_1} f_t(v_t)
\]

holds for any $\eta_1 \geq 1$.

Since function $\frac{1}{2} \| y_t - y \|^2$ is 1-strongly smooth in $y$, by Lemma 4, we see that for any $\eta_2 > 0$,

\[
\frac{1}{1 + \eta_2} \cdot \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \leq \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 + \frac{1}{2\eta_2} \left\| \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}) \right\|^2.
\]

Notice that

\[
\frac{1}{2} \left\| \sum_{i=1}^{p} C_i (v_{t-i} - y_{t-i}) \right\|^2 \leq \frac{1}{2} \left( \sum_{i=1}^{p} \| C_i \| \cdot \| y_{t-i} - v_{t-i} \| \right)^2 \leq \frac{\alpha}{2} \left( \sum_{i=1}^{p} \| C_i \| \cdot \| y_{t-i} - v_{t-i} \|^2 \right),
\]

where we use the triangle inequality and the definition of matrix norm in \ref{eq:2}; Jensen’s Inequality in \ref{eq:3}.

Substituting \eqref{eq:2} into \eqref{eq:1} gives

\[
\frac{1}{1 + \eta_2} \cdot \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \leq \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 + \frac{\alpha}{2\eta_2} \left( \sum_{i=1}^{p} \| C_i \| \cdot \| y_{t-i} - v_{t-i} \|^2 \right).
\]
Substituting (40) and (43) into (39) gives
\[
\frac{1}{\eta_1} f_t(y_t) + \frac{1}{1 + \eta_2} \left( \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \right) 
\leq \frac{1}{\eta_1} f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 + \left( \frac{l}{\eta_1} - 1 \right) \cdot \frac{1}{2} \left\| y_t - y_{t-i} \right\|^2 + \frac{\alpha}{2\eta_2} \left( \sum_{i=1}^{p} \left\| C_i \right\| \cdot \left\| y_{t-i} - v_{t-i} \right\|^2 \right). \tag{44}
\]

Summing up (44) through time steps, we obtain that
\[
\min\left\{ \frac{1}{\eta_1}, \frac{1}{1 + \eta_2} \right\} \sum_{t=1}^{T} \left( f_t(y_t) + \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \right) 
\leq \sum_{t=1}^{T} \left( f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^{p} C_i v_{t-i} \right\|^2 \right) + \left( \frac{l}{\eta_1} + \frac{\alpha^2}{\eta_2} - 1 \right) \cdot \frac{1}{2} \left\| y_t - v_t \right\|^2. \tag{45}
\]

Let \( \eta_2 = \eta \) and \( \eta_1 = 1 + \eta \). Combining with (38), we obtain that
\[
\sum_{t=1}^{T} \left( f_t(y_t) + \frac{1}{2} \left\| y_t - \sum_{i=1}^{p} C_i y_{t-i} \right\|^2 \right) 
\leq (1 + \eta) \cdot \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \sum_{t=1}^{T} (H_t^* + M_t^*) + \left( l + \left( \frac{1}{\eta} \right) \alpha^2 - (1 + \eta) \right) \cdot \frac{1}{2} \| y_t - v_t \|^2. \tag{46}
\]

\( \square \)

**F. Proof and Example of Theorem 3**

The proof will proceed as follows. First, we extract the controllable dimensions in \( x_t, \{x_t^{(k_1)}, \cdots, x_t^{(k_d)}\} \), to construct a new vector \( z_t \). Then we can represent \( x_t \) by \( z_t, z_{t-1}, \cdots, z_{t-p} \). Therefore, we can rewrite the dynamics in sequence \( \{z_t\}_{0 \leq t \leq T} \), control action \( u_t \), and noise \( w_t \). By this approach, we can remove the control matrix \( B \) before \( (u_t + w_t) \) in the dynamics. Finally, we can convert the resulting dynamics to an OCO problem with structured memory.

We use \( \| \cdot \| \) to denote \( \ell_2 \) norm throughout the proof.

Recall that the objective is given as
\[
\frac{1}{2} \sum_{t=0}^{T} \left( q_t \| x_t \|^2 + \| u_t \|^2 \right), \tag{47}
\]
where \( q_t > 0 \) for all \( 0 \leq t \leq T \). Without loss of generality, we assume \( q_t = 0 \) for all \( t > T \).

Recall that we define operator \( \psi : \mathbb{R}^n \to \mathbb{R}^m \) as
\[
\psi(x) = \left( x^{(k_1)}, \cdots, x^{(k_d)} \right)^\top.
\]

Using this notation, we define vector \( z_t \) as
\[
z_t := \psi(x_t), t \geq 0.
\]

Notice that \( z_t^j = x_t^{(k_j)} \) for \( j = 1, \cdots, d \). Since we have \( x_t^{(i)} = x_{t-1}^{(i+1)} \) for \( i \notin I \), \( x_t \) can be represented by
\[
x_t = \left( z_{t-p+1}^{(1)}, \cdots, z_t^{(1)}, \cdots, z_{t-p+d}^{(1)}, \cdots, z_t^{(d)} \right)^\top. \tag{48}
\]

Since \( x_0 = 0 \), we have \( z_t = 0 \) for \( t \leq 0 \).
Using (48), we can rewrite the objective function as a function of sequence \( \{z_t\} \) and \( \{u_t\} \). Notice that

\[
\sum_{t=0}^{T} q_t \|x_t\|^2 = \sum_{t=0}^{T} q_t \sum_{i=1}^{d} \sum_{j=1}^{p_i} (z_{t+1-j}^{(i)})^2 \\
= \sum_{t=0}^{T-1} \sum_{i=1}^{d} \sum_{j=1}^{p_i} q_{t+j} (z_{t+1}^{(i)})^2, \tag{49a}
\]

where in (49a) we use \( z_t = 0 \) for all \( t \leq 0 \) and \( q_t = 0 \) for all \( t > T \).

Therefore, we define function \( h_t : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\} \) as

\[
h_t(y) = \frac{1}{2} \sum_{i=1}^{d} \left( \sum_{j=1}^{p_i} q_{t+j} \right) \left( y^{(i)} \right)^2.
\]

Using this definition, the objective (47) can be rewritten as

\[
\frac{1}{2} \sum_{t=0}^{T} \left( q_t \|x_t\|^2 + \|u_t\|^2 \right) = \sum_{t=0}^{T-1} h_t(z_{t+1}) + \frac{1}{2} \|u_t\|^2, \tag{50}
\]

where we notice that the optimal choice of control action \( u_T \) is always zero because it will not affect any state.

We also see that \( u_t \) can be determined by \( z_{t-p+1:t+1} \) because

\[
u_t = z_{t+1} - w_t - A(I,)x_t, \tag{51}
\]

where \( A(I,) \) consists of \( k_1, \cdots, k_n \) rows of \( A \) and \( t \geq 0 \).

Notice that \( A(I,)x_t \) can be written as \( \sum_{i=1}^{p} C_i z_{t-i+1} \) by the definition of \( C_i, i = 1, \cdots, p \). Therefore, we can rewrite (51) as

\[
u_t = z_{t+1} - w_t - \sum_{i=1}^{p} C_i z_{t-i+1}, \tag{52}
\]

which is equivalent to

\[
z_{t+1} = u_t + u_t + \sum_{i=1}^{p} C_i z_{t-i+1}.
\]

We recursively define sequence \( \{y_t\}_{t \geq -p} \) as the accumulation of control actions, i.e.

\[
y_t = u_t + \sum_{i=1}^{p} C_i y_{t-i}, \forall t \geq 0,
\]

where \( y_t = 0 \) for all \( t < 0 \). We also define sequence \( \{\zeta_t\}_{t \geq -p} \) as the accumulation of control noises, i.e.

\[
\zeta_t = u_t + \sum_{i=1}^{p} C_i \zeta_{t-i}, \forall t \geq 0,
\]

where \( \zeta_t = 0 \) for all \( t < 0 \).

Recall that we have \( x_0 = 0 \) by assumption. Therefore,

\[
z_{t+1} = y_t + \zeta_t \tag{53}
\]

holds for all \( t \geq -1 \).
Using (50) and (53), we can formalize the problem as *online optimization with memory*, where the hitting cost function is given by

\[ f_t(y) = h_t(y + \zeta_t), \]

and the switching cost is \( \frac{1}{2} \| y_t - \sum_{i=1}^{p} C_i y_{t-i} \|^2. \)

Although \( h_t \) is revealed before the agent picks \( y_t \), we need the knowledge of \( v_t = -\zeta_t \) to construct the hitting cost function \( f_t \), which depends on previous noises \( w_{0:t} \). At time step \( t \), we know the exact \( w_{t} \) for all \( t \leq t - 1 \), thus we can compute the exact \( \zeta_t \) for all \( t \leq t - 1 \). Since the set \( W_t \) contains all possible noise \( w_t \), we can construct the set \( \Omega_t = \{ -w - \sum_{i=1}^{p} C_i \zeta_{t-i} \mid w \in W_t \} \) which contains all possible \( v_t \).

**Example.** To illustrate the reduction, consider the following example:

\[
\begin{pmatrix}
  x_t^{(1)} \\
  x_t^{(2)} \\
  x_t^{(3)} \\
  x_t^{(4)} \\
  x_t^{(5)}
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 \\
  a_1 & a_2 & a_3 & a_4 & a_5 \\
  0 & 0 & 0 & 0 & 0 \\
  b_1 & b_2 & b_3 & b_4 & b_5 \\
  C_1 & C_2 & C_3 & C_4 & C_5
\end{pmatrix}
\begin{pmatrix}
  y_t^{(1)} \\
  y_t^{(2)} \\
  y_t^{(3)} \\
  y_t^{(4)} \\
  y_t^{(5)}
\end{pmatrix}
+ \begin{pmatrix}
  1 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  w_t^{(1)} \\
  w_t^{(2)} \\
  w_t^{(3)} \\
  w_t^{(4)} \\
  w_t^{(5)}
\end{pmatrix}.
\]

(54)

Notice that since \( x_t^{(1)} = x_t^{(2)} = x_t^{(3)} = x_t^{(4)} \), we can rewrite (54) in a more compact form:

\[
\begin{pmatrix}
  y_t^{(2)} \\
  y_t^{(3)} \\
  y_t^{(4)} \\
  y_t^{(5)}
\end{pmatrix}
= \begin{pmatrix}
  a_2 & a_3 & a_4 & a_5 \\
  b_2 & b_3 & b_4 & b_5 \\
  C_2 & C_3 & C_4 & C_5
\end{pmatrix}
\begin{pmatrix}
  y_t^{(1)} \\
  y_t^{(2)} \\
  y_t^{(3)} \\
  y_t^{(4)} \\
  y_t^{(5)}
\end{pmatrix}
+ \begin{pmatrix}
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  w_t^{(1)} \\
  w_t^{(2)} \\
  w_t^{(3)} \\
  w_t^{(4)} \\
  w_t^{(5)}
\end{pmatrix}.
\]

(55)

In this example \( p_1 = 2, p_2 = 3, \mathcal{I} = \{ k_1, k_2 \} = \{ 2, 5 \} \) and thus \( p = 3 \) and \( n = 2 \). From (55) we have

\[ z_{t+1} = C_1 z_t + C_2 z_{t-1} + C_3 z_{t-2} + u_t + w_t. \]

(56)

Recall the definition of \( y_t \) and \( \zeta_t \):

\[ y_t = u_t + \sum_{i=1}^{3} C_i y_{t-i}, \forall t \geq 0, \quad \zeta_t = w_t + \sum_{i=1}^{3} C_i \zeta_{t-i}, \forall t \geq 0. \]

(57)

Then the original system could be translated to the compact form:

\[ z_{t+1} = y_t + \zeta_t. \]

(58)

If the objective is given as (47), we have that

\[ h_t(z) = \frac{1}{2} (q_{t+1} + q_{t+2}) \left( z^{(1)} \right)^2 + \frac{1}{2} (q_{t+1} + q_{t+2} + q_{t+3}) \left( z^{(2)} \right)^2. \]

Lastly, we want to point out that our reduction can work for more general forms of objectives than (47). Specifically, we only require that the objective can be transformed to

\[ \sum_{t=0}^{\tau-1} h_t(z_{t+1}) + \frac{1}{2} \| u_t \|^2, \]

where \( h_t \) is a strongly convex and strongly smooth function that is observable before the agent picks \( u_t \). Therefore, our reduction is more general than the reduction given in (Goel et al., 2019)[Corollary 2], which considered the case when \( B = I \). Notice that when \( B = I \), we have \( p = 1 \) and \( z_t = x_t \).
We have presented Algorithm 3 in as simple and intuitive a manner as possible but, as a result, there is a potential numerical
issue that may arise for large horizon $T$. Although the sequence $\{z_t\}$ is naturally bounded and we always have $z_{t+1} = y_t + \zeta_t$, the magnitudes of $y_t$ and $\zeta_t$ may grow exponentially since they accumulate the actions and the noises separately. However, this is not a fundamental problem, and there is a straightforward solution when the $Solver$ in Algorithm 3 is Optimistic ROBD (Algorithm 2). The key insight is to solve optimization in $\{u_t, w_t, z_t\}$ space, instead of $\{y_t, \zeta_t, z_t\}$ space.

More specifically, when instantiated with Optimistic ROBD, we can rewrite the pseudo code of Algorithm 3 as Algorithm 5 so that variables $y_t$ and $\zeta_t$ are not involved. While equivalent to Algorithm 3 with Optimistic ROBD as the $Solver$, Algorithm 5 is numerically stable because we avoid the potentially unstable recursive calculation of $\zeta_t$ and the sequence $\{w_t\}$ is bounded.

### H.1. Lower bound of cost($L_C$) for any noise sequence $\{w_t\}_{t=0}^P$

For any stable linear controller $u_t = -k x_t$, we have the following closed-loop dynamics

\[ x_{t+1} = (a-k)x_t + w_t. \]

Our technique is to consider the sum of squares of two consecutive states $x_{t+1}$ and $x_t$. Due to the constraints given by the dynamics and the linear controller itself, $x_{t+1}$ and $x_t$ cannot reach zero simultaneously. Specifically, we define $\beta = a - k$. Since the controller is stable, we have $-1 < \beta < 1$. Consider $|x_{t+1}|^2 + |x_t|^2$, $\forall t \geq 0$, we have:

\[
|\begin{array}{l}
|x_{t+1}|^2 + |x_t|^2 \\
= (\beta x_t + w_t)^2 + x_t^2 \\
= (\beta^2 + 1)x_t^2 + 2\beta x_t w_t + w_t^2 \\
= (\beta^2 + 1)(x_t + \frac{\beta}{\beta^2 + 1} w_t)^2 + \frac{1}{\beta^2 + 1} w_t^2
\end{array}\]

#### G. A Numerical Issue in Algorithm 3 and Its Solution

We have presented Algorithm 3 in as simple and intuitive a manner as possible but, as a result, there is a potential numerical issue that may arise for large horizon $T$. Although the sequence $\{z_t\}$ is naturally bounded and we always have $z_{t+1} = y_t + \zeta_t$, the magnitudes of $y_t$ and $\zeta_t$ may grow exponentially since they accumulate the actions and the noises separately. However, this is not a fundamental problem, and there is a straightforward solution when the $Solver$ in Algorithm 3 is Optimistic ROBD (Algorithm 2). The key insight is to solve optimization in $\{u_t, w_t, z_t\}$ space, instead of $\{y_t, \zeta_t, z_t\}$ space.

More specifically, when instantiated with Optimistic ROBD, we can rewrite the pseudo code of Algorithm 3 as Algorithm 5 so that variables $y_t$ and $\zeta_t$ are not involved. While equivalent to Algorithm 3 with Optimistic ROBD as the $Solver$, Algorithm 5 is numerically stable because we avoid the potentially unstable recursive calculation of $\zeta_t$ and the sequence $\{w_t\}$ is bounded.

### Example 5.1

In this section, we establish the lower bound of the cost incurred by any linear controller and the upper bound of the offline optimal cost for different noise sequences. Specifically, we show a lower bound of the linear controller’s cost on any noise sequence in Section H.1. We also give an upper bound of the offline optimal cost on any noise sequence in Section H.2. We further show that the upper bound of the offline optimal cost can be improved on two specific noise sequences in Section H.3 and H.4. Based on these results, we derive the lower bound of the competitive ratio for any linear control with respect to the these noise sequences in Section H.5, H.6, and H.7.

### Algorithm 5: Adaptive Control via Optimistic ROBD

**Parameters:** $\lambda > 0$

**Input:** Transition matrix $A$ and control matrix $B$

**for** $t = 0, 1, \cdots, T - 1$ **do**

**Observe:** $x_t$, $W_t$, and $q_{t:t+p-1}$

**if** $t > 0$ **then**

$w_{t-1} \leftarrow \psi(x_t - Ax_{t-1} - Bu_{t-1})$

$\tilde{z}_t \leftarrow \psi(x_t)$

**end if**

Define function $h_t(z) = \frac{1}{2} \sum_{i=1}^{d} (\sum_{j=1}^{p} q_{t+j})(z(i))^2$

$\tilde{w}_t \leftarrow \arg \min_{w \in W_t} \min_{z} h_t(z) + \frac{\lambda}{2} \|z - w - \sum_{i=1}^{p} C_i \tilde{z}_{t+1-i}\|^2$

$z_t \leftarrow \arg \min_{z} h_t(z) + \frac{\lambda}{2} \|z - \tilde{w}_t - \sum_{i=1}^{p} C_i \tilde{z}_{t+1-i}\|^2$

$u_t \leftarrow z_t - \tilde{w}_t - \sum_{i=1}^{p} C_i \tilde{z}_{t-i}$

**Output:** $u_t$

**end for**

**Output:** $u_T = 0$

### H. Proofs for Example 5.1

In this section, we establish the lower bound of the cost incurred by any linear controller and the upper bound of the offline optimal cost for different noise sequences. Specifically, we show a lower bound of the linear controller’s cost on any noise sequence in Section H.1. We also give an upper bound of the offline optimal cost on any noise sequence in Section H.2. We further show that the upper bound of the offline optimal cost can be improved on two specific noise sequences in Section H.3 and H.4. Based on these results, we derive the lower bound of the competitive ratio for any linear control with respect to the these noise sequences in Section H.5, H.6, and H.7.
When the controller has the full knowledge of the future noise sequence, the simplest strategy is to correct the noise greedily
\[c\]
where
\[u\]
where the first part is a quadratic function w.r.t.
\[-\]
where the last step comes from the fact
\[-1 < a - k < 1\] and \(a > 1\).

**H.2. Upper bound of \(\text{cost}(OPT)\) for any \(\{w_t\}_{t=0}^T\)**

When the controller has the full knowledge of the future noise sequence, the simplest strategy is to correct the noise greedily at the start of each time step so that the agent always stays at state 0.

Formally, for \(\text{cost}(OPT)\), consider controller \(u_t = -w_t, \forall t \neq T\) and \(u_t = 0, t = T\). Then we will have \(x_t = 0, \forall t \leq T\) so the cost would be \(\sum_{t=0}^{T-1} w_t^2\). Therefore we have
\[
\text{cost}(OPT) \leq \sum_{t=0}^{T-1} w_t^2.
\]

**H.3. Upper bound of \(\text{cost}(OPT)\) for \(w_t = w\)**

Compared with Section H.2, since \(w_t\) is a constant case, we can balance the hitting cost and the switching cost by keeping the agent at non-zero stationary state that is close to the zero state.

Formally, we consider the following control strategy:
\[
u_t = \begin{cases} 
\frac{u+w}{1-a} - w, & t = 0 \\
u, & t \geq 1,
\end{cases}
\]
where \(u\) is another constant. This controller yields \(x_t = \frac{u+w}{1-a}, t \geq 1\). Then, we have
\[
\text{cost}(u) = T(q(\frac{u+w}{1-a})^2 + u^2) + (\frac{u+w}{1-a} - w)^2,
\]
where the first part is a quadratic function w.r.t. \(u\) and the minimum is \(\frac{q}{q+(a-1)^2} \cdot Tw^2\) with minimizer \(u^* = \frac{-qw}{q+(a-1)^2}\). Therefore we get
\[
\text{cost}(OPT) \leq \frac{q}{q+(a-1)^2} Tw^2 + c_1,
\]
where \(c_1 = \left(\frac{u^*+w}{1-a} - w\right)^2\) is a constant.

**H.4. Upper bound of \(\text{cost}(OPT)\) for \(w_t = (-1)^t \cdot w\)**

Instead of keeping the noise \(w_t\) at a fixed value, we let it oscillate between two values \(w\) and \(-w\). The resulting offline optimal controller will also oscillate between a positive state and a negative state. We show that in this case, the offline optimal cost can be even smaller than the one when \(w_t\) is fixed at \(w\) (Section H.3).

In this case the dynamics follows
\[
\begin{align*}
x_{2k+1} &= ax_{2k} + u_{2k} + w, & k \geq 0 \\
x_{2k+2} &= ax_{2k+1} + u_{2k+1} - w, & k \geq 0.
\end{align*}
\]
Consider controller class
\[
u_t = \begin{cases} 
\frac{u-w}{a+1} - w, & t = 0 \\
u, & t = 2k + 1, k \geq 0 \\
-w, & t = 2k + 2, k \geq 0.
\end{cases}
\]
Following this controller class, we have
\[
    x_t = \begin{cases} 
        \frac{-u-w}{a+1}, & t = 2k + 1, k \geq 0 \\
        \frac{u-w}{a+1}, & t = 2k + 2, k \geq 0.
    \end{cases}
\]

For simplicity, assume \( T \) is an even number. Then, we have
\[
    \text{cost}(u) = T\left(q\left(\frac{u-w}{a+1}\right)^2 + u^2\right) + \left(\frac{u-w}{a+1} + w\right)^2.
\]

Similarly, the first part of \( \text{cost}(u) \) is a quadratic function and the minimum is \( \frac{q}{q+(a+1)^2} \cdot Tw^2 \). Therefore, we have
\[
    \text{cost}(OPT) \leq \frac{q}{q+(a+1)^2} T w^2 + c_2,
\]
where \( c_2 \) is also a constant.

**H.5. Lower bound of \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \) for any \( \{w_t\}_{t=0}^T \)**

Combining H.1 and H.2 we will have, for any \( \{w_t\}_{t=0}^T \):
\[
    \frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{q + (a-1)^2}{4} \sum_{t=0}^{T-1} w_t^2 = q + (a-1)^2.
\]

**H.6. Lower bound of \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \) for \( w_t = w \)**

Combining H.1 and H.3, we will have, if \( w_t = w \):
\[
    \frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{q + (a-1)^2}{4} Tw^2.
\]

Therefore as \( T \to \infty \), \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a-1)^2}{4} Tw^2 + c_1 \).

**H.7. Lower bound of \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \) for \( w_t = (-1)^t \cdot w \)**

Combining H.1 and H.4, we will have, if \( w_t = (-1)^t \cdot w \):
\[
    \frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{q + (a-1)^2}{4} Tw^2.
\]

Therefore as \( T \to \infty \), \( \frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a-1)^2}{4} Tw^2 + c_2 \).