Existence of $C^{1,\alpha}$ Singular Solutions to Euler-Nernst-Planck-Poisson System on $\mathbb{R}^3$ with Free-Moving Charges

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Abstract

We construct a special $C^{1,\alpha}$ blow up solution to the three dimensional system modeling electro-hydrodynamics, which is a strongly coupled system of incompressible Euler equation and Nernst-Planck-Poisson equation. Our construction lies on the framework established in and relies on a special solution to variant spherical Laplacian.

1 Introduction

In this article, we consider Euler-Nernst-Planck-Poisson system with free-moving charges on $\mathbb{R}^3$

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nabla p &= \Delta \xi \nabla \xi, \\
\nabla \cdot u &= 0, \\
\partial_t v + (u \cdot \nabla) v &= 0, \\
\Delta \xi &= v, \\
(u, v, \xi)|_{t=0} &= (u_0, v_0, \xi_0),
\end{aligned}
\]

which is a special case of Navier-Stokes-Poisson-Nernst-Planck system

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \vartheta \Delta u + \nabla p &= \varepsilon \Delta \xi \nabla \xi, \\
\partial_t w_1 + (u \cdot \nabla) w_1 &= \nabla \cdot (D_1 \nabla w_1 - \iota_1 w_1 \nabla \xi), \\
\partial_t w_2 + (u \cdot \nabla) w_2 &= \nabla \cdot (D_2 \nabla w_2 + \iota_2 w_2 \nabla \xi), \\
\nabla \cdot u &= 0, \\
\varepsilon \Delta \xi &= w_1 - w_2, \\
(u, w_1, w_2, \xi)|_{t=0} &= (u_0, w_{1,0}, w_{2,0}, \xi_0).
\end{aligned}
\]

In the system (1.2), $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ is the velocity field; $p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ is the scalar pressure; $\xi : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$ is the electrostatic potential; $w_1$ and $w_2$ are the densities of binary diffuse negative and positive charges, respectively. $\vartheta$ is the fluid viscosity, $\varepsilon$ is the dielectric constant of the fluid. $D_1$, $D_2$, $\iota_1$, $\iota_2$ are the diffusion and mobility coefficients of the charges. In this article, we mainly consider the inviscid flow, namely, $\vartheta = 0$, and we let $\varepsilon = 1$ without loss of generality. Moreover, since diffusion in charges moving is a low order effect, which can be safely neglected, so we let $D_1 = D_2 = \iota_1 = \iota_2 = 0$. At last we let $v = w_1 - w_2$ and get the system (1.1).

First introduced by Rubinstein in [22], the system (1.2) is a well-established mathematical model for electro-hydrodynamics, coupling the Navier-Stokes of an incompressible fluid and the...
transported Poisson-Nernst-Planck equations. It describes the dynamical behavior of incompressible fluid with binary diffuse positive and negative charges (e.g., ions), which is relevant to understanding the behavior of different physical objects such as electrolyte solution, ion-exchangers, ion-selective membranes, and semiconductors. Over the years, (1.2) is well applied in many scientific areas such as electrical, fluid-mechanical and bio-chemical phenomena occurring in complex bio-hydril system, see [1], [12], [13], [25] for various applications and [19], [16] for computational results. For better understanding about the physical background and mathematical description, one would refer to [3], [2], [21], [6] and [24].

The mathematical analysis of (1.2) has been well studied by many experts for dozens of years, and their works can not be totally listed here. To our knowledge, the results date back to [15], where the author established the local wellposedness of the system by semigroup theory. Then, in [17] and [13] the authors proved the existence of global weak solutions with block boundary. The global existence for small data holds in [23] via linearization of a relative entropy functional. Besides, global existence for mild solutions with small data in various function spaces such as Triebel-Lizorkin space and negative-order Besov space were obtained in [9] and [27]. And global existence for large data in 2D was got by [24] via energy laws, mass conservation, non-negativity and pointwise bounds. Recently in [3], the authors used energy estimate and got exponential stability of steady states, and global existence for blocking and general selective boundary conditions was proved in [8] by exploiting Boltzmann states.

As for the blow up aspects, there has been several blowup criterions for (1.2) obtained in [20] and [18]. However, the existence of blowup solutions still remains open, let alone the blowup dynamics for some appropriately chosen initial data. Indeed, it is rather challenging to construct blow up solutions even for Euler and Navier-Stokes. But recently, the progress concerning formation of self-similar singular solutions in fluid dynamics is flourishing. For example, in the area of incompressible fluid, Elgindi constructed the $C^{1,\alpha}$ singular solution to Euler equation in [10]. At first, he established an approximation of Biot-Savart law naming fundamental model in a variant spherical coordinate. This model not only can be solved explicitly, but also wisely separated the solution into singular part and regular part. Then he performed linearization around the fundamental model and verified the coercivity of linearized operator in some weighted Sobolev space. At last with elliptic estimate and energy estimates, Elgindi finally established the formation of singular solutions to Euler in $\mathbb{R}^3$ with $C^{1,\alpha}$ data. While in the field of compressible fluid, by means of Riemann variable and 3D Burgers profile, the finite-time blowup with smooth data of 3D compressible Euler was obtained in [5] without symmetry assumptions. Besides, in [20], the authors constructed global self-similar profile of 3D compressible Euler by using classical phase portrait of nonlinear ODEs. In addition, self-similar profile of 2D Burgers and the spectrum of linearized operators were analyzed in [7] to get the singularity formation of 2D unsteady Prandtl’s system.

In this article, following the framework of [10] and [11] built by Elgindi, we prove the existence of finite-time blowup solutions to (1.1).

**Theorem 1.1.** There exists an $\alpha > 0$, a divergence-free $u_0 \in C^{1,\alpha}(\mathbb{R}^3)$ and a function $v_0 \in C^\infty(\mathbb{R}^3)$ with initial vorticity $|\omega_0(x)| \leq \frac{C}{|x|^{\alpha+1}}$ and initial charges density $|v_0(x)| \leq \frac{C}{|x|^{\frac{3}{2}\alpha+1}}$ for some constant $C > 0$ so that the unique local solution $(u, v) \in (C^{1,\alpha}_{x,t}([0,1] \times \mathbb{R}^3) \times C^\infty_{x,t}([0,1] \times \mathbb{R}^3))$ to (1.1) satisfies

$$\lim_{t \to 1} \int_0^t \|\omega\|_{L^\infty} ds = +\infty, \quad \lim_{t \to 1} \|v(t)\|_{L^\infty} = 0.$$

We shall explain our main ideas. First, the evolution of $v$ is a linear transport process with velocity of $u$, so $v$ can not develop singularity by itself because $\|u\|_{L^\infty}$ doesn’t blow up near the origin. So it is reasonable to construct $(u, v)$ where $v$ was kept small during the whole lifespan. In this way, the influence of $v$ on $u$ is under control once we make proper assumption on $v$. So in our proof, the electrostatic effect is regarded as a small perturbation to incompressible fluid dynamics. Next, $v$ has to be performed coordinate transformation following $u$, then we show the coercivity of linearized operator under the new coordinate. After that, we analyze the contribution of $v$ to
u, which is nonlocal because the effect is imposed by the electrostatic potential $\xi$ relating to $v$ by Poisson equation. To overcome the obstacle, with classical Strum-Liouville theory, we try to find a special explicit solution to the variant spherical Laplacian under new $(z, \theta)$ coordinate (see Section 4 for details), where there is only one parameter under determined. At last, we follow [11] to derive the modulation law and perform the energy estimate, finishing the proof of Theorem 1.1.

Remark 1. Here we only choose one special solution of spherical Laplacian, but there are still others can be used to construct the solutions for system, so the solution we find is not unique.

Remark 2. This result is NOT a stability one. In our construction the perturbation of electrostatic effect doesn’t occur in an open set. Indeed the profile of $v_0(x)$ is fixed, and what happens if we perturb the initial data is still unknown. To our sight, in order to do stability estimate, one must establish elliptic estimate to (5.3). However, we have to control $\| \sec \theta \Pi \|_{L^2}$ appearing in the $L^2$ estimate, which seems hard to handle.

Remark 3. In our construction, $u$ blows up in finite time, while $v$ goes to 0 near the blow up time. A natural question is whether there exists a solution that $u$ and $v$ blow up at the same time. However, in this case $v$ can no longer be regarded as the perturbation of $u$ so that the contribution of $v$ to $u$ seems hard to control near the blowup time, because the nontrivial profile of $v$ may change the profile of $u$.

This article is organized as follows: In Section 2 we recall the notations and some lemmas used in [10] and [11]. In Section 3 we follow the framework of [11] to perform the self-similar transformation and linearization. In Section 4 we discuss the coercivity of linearized operators. In Section 5 we state the elliptic estimate and construct the special solution of variant spherical Laplacian. In Section 6 we derive the modulation law. In Section 7 we perform the energy estimate so that main result follows.

2 Preliminary

We inherit the notations and some lemmas from [10] and [11].

2.1 Notations

$r$ will denote the two dimensional radial variable:

$$r = \sqrt{x_1^2 + x_2^2}.$$

$\rho$ will denote three dimensional variable:

$$\rho = \sqrt{r^2 + x_3^2}.$$

And $\rho^\alpha$ is denoted by $R$:

$$R = \rho^\alpha.$$

We write $\zeta$:

$$\zeta = \arctan \left( \frac{x_2}{x_1} \right)$$

and $\theta$:

$$\theta = \arctan \left( \frac{x_3}{r} \right).$$

$z$ will denote generally self-similar radial variable:

$$z = \frac{R}{(1 - (1 + \mu)t)^{(1+\lambda)}}.$$

where $\lambda$ and $\mu$ are small constants. Obviously, $\zeta, \theta \in [0, \frac{\pi}{2}], z \in [0, +\infty)$. And $\theta = 0$ corresponds to the plane $x_3 = 0$ while $\theta = \frac{\pi}{2}$ corresponds to the $x_3$ axis. The main parameters we will use are

$$0 < \alpha \ll 1, \quad \gamma = 1 + \frac{\alpha}{10}, \quad \eta = \frac{99}{100}. $$
Let $w$ be a weight function:

$$w = \frac{(1+z)^2}{z^2}, \quad z \in [0, +\infty).$$

Besides, we define

$$\Gamma(\theta) = (\sin \theta \cos^2 \theta)^{\frac{3}{2}}$$

and

$$K(\theta) = 3 \sin \theta \cos^2 \theta. \quad (2.1)$$

Define the differential operators:

$$D_\theta(f) = \sin 2\theta \partial_\theta f, \quad D_R(f) = R \partial_R f.$$ 

Define the $L^2$ inner product and norm:

$$(f,g)_{L^2} = \int_{\Omega} fg, \quad \|f\|_{L^2} = \sqrt{(f,f)_{L^2}}.$$ 

Define $L^\infty$ norm:

$$\|f\|_{L^\infty} = \sup_{x \in \Omega} |f(x)|.$$ 

Define norm of space $C^{0,\beta}$:

$$\|f\|_{C^{0,\beta}} = \sup_{x \in \Omega} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\beta}}.$$

Define weighted Sobolev norm $H^k([0, +\infty) \times [0, \pi/2])$:

$$\|f\|_{H^k}^2 = \sum_{i=0}^k \|D_R^i f \frac{w}{\sin^{\gamma/2} 2\theta}\|_{L^2}^2 + \sum_{0 \leq i+j \leq k, i \geq 1} \|D_R^j D_\theta^i f \frac{w}{\sin^{\gamma/2} 2\theta}\|_{L^2}^2. \quad (2.2)$$

For any $f : [0, +\infty) \times [0, \frac{\pi}{2}] \to \mathbb{R}$, we define non-local linear operator

$$L_K(f)(z) = \int_{\Omega}^{\pi/2} \int_{z}^{\infty} \frac{f(z') \Gamma(\theta)}{z'} dz' d\theta,$$

where $K$ is given in (2.1).

Define the linearized operators $L$, $L_P$, and $L_{P}^T$:

$$L(f) = f + z \partial_z f - 2 \frac{f}{1 + z},$$

$$L_P(f) = f + z \partial_z f - 2 \frac{f}{1 + z} - \frac{2z \Gamma(\theta)}{c(1+ z)^2} L_K(f),$$

and

$$L_{P}^T(f) = L_P(f) - \frac{3}{1 + z} \sin 2\theta \partial_\theta f,$$

where

$$P(f)(z, \theta) = f(z, \theta) - \frac{\Gamma(\theta)}{c} \frac{2z^2}{(1+ z)^2} L_K(f)(0), \quad c = \int_{0}^{\pi/2} \Gamma(\theta) K(\theta) d\theta.$$
2.2 Some Useful Facts

Let

\[ F_\ast(z, \theta) = \frac{\Gamma(\theta)}{c} \frac{4z}{(1+z)^2}, \]

we have

\[ L_K(F_\ast)(z) = \frac{4\alpha}{1+z}. \]

For any \( f \in \mathcal{H}^k \),

\[ L_K(z\partial_z f) = \int_0^\infty \int_0^{\pi/2} \partial_pf(p, \theta)K(\theta)d\theta dp = z\partial_z L_K(f)(z), \]

and in particular,

\[ L_K(z\partial_z f)(0) = -\int_0^{\pi/2} f(0, \theta)K(\theta)d\theta. \]  
(2.3)

Besides, there is the following boundness for \( L_K \):

**Proposition 2.1.** (10) There exists a universal constant \( C > 0 \) such that for all \( f \in \mathcal{H}^k \) with \( L_K(f)(0) = 0 \), we have

\[ \|L_K(f)\|_{\mathcal{H}^k} \leq \|f\|_{\mathcal{H}^k}. \]

And there are following properties holding for \( \mathcal{L}, \mathcal{L}_F, \) and \( \mathcal{L}^T_F \):

**Lemma 2.2.** (10) We have that

\[ L_K(\mathcal{L}_F(f)) = \mathcal{L}(L_K(f)), \]

and

\[ \mathcal{L}(g)w = gw + z\partial_z(gw), \]

hence

\[ (\mathcal{L}(g)w, gw)_{L^2} = \frac{1}{2}\|gw\|^2_{L^2}. \]

**Proposition 2.3.** (10) Fix \( \alpha < 10^{-14} \) and \( k \in \mathbb{N} \), there exists \( c_k > 0 \) such that for all \( f \in \mathcal{H}^k \) we have

\[ (\mathcal{L}^T_F(f), f)_{\mathcal{H}^k} \geq c_k\|f\|^2_{\mathcal{H}^k}. \]  
(2.8)

3 Coordinate Transformation, Dynamical Scaling and Linearization

We work in the framework of [11].

3.1 Vorticity Formation under Cylindrical Coordinate

Conventionally, it is convenient to consider the blow up behavior of a fluid system in terms of vorticity. Let \( \omega = \nabla \times u \) and from [11] we get

\[ \partial_t\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nabla \times (v\nabla \xi). \]

By Biot-Savart law, if \( \tilde{\psi} \) is the stream field, we have

\[ -\Delta \tilde{\psi} = \omega, \quad \nabla \times \tilde{\psi} = u. \]

Furthermore, here we consider the axisymmetric case in cylindrical coordinate \( (r, \zeta, x^3) \) without swirl, i.e. the velocity field can be written to \( u(t, x) = u_r(t, r, x^3)e_r + u_\zeta(t, r, x^3)e_\zeta, v(t, x) = \)
\[ v(t, r, x^3), \text{ and } u_\zeta = 0, \text{ where } r = \sqrt{x_1^2 + x_2^2}, \zeta = \arctan(\frac{x_2}{x_1}). \] Meanwhile, \[ \omega(t, x) = \omega_\zeta(t, r, x^3)e_\zeta. \] Moreover, the cylindrical Biot-Savart law reads

\[ \begin{cases} 
\frac{1}{r} \frac{\partial_r (\frac{1}{r} \partial_r \psi)}{r} + \frac{1}{r} \frac{\partial_\zeta \psi}{r} = -\omega, \\
u_r = \frac{1}{r} \partial_\zeta \psi, \quad u_3 = -\frac{1}{r} \partial_r \psi.
\end{cases} \]

Now the vorticity form of (1.1) turns out to be (we just denote \( \omega = \omega_\zeta \) for simplicity)

\[ \begin{cases} 
\partial_\omega + (u_r \partial_r + u_3 \partial_3) \omega = \frac{\partial t}{\rho} + \partial_3 v \partial_r \xi - \partial_r v \partial_3 \xi, \\
(\partial_t + (u_r \partial_r + u_3 \partial_3)) v = 0, \\
-\partial_r \psi - \partial_3 \psi - \frac{1}{2} \partial_r \psi + \frac{1}{2} \omega = \omega, \\
(\partial_t + \frac{2}{\rho} \xi + \partial_3 \xi) = v,
\end{cases} \tag{3.1} \]

where \( r \psi = \tilde{\psi} \).

We further assume that the vorticity satisfies odd condition with respect to \( x_3 \), i.e. \( \omega(r, -x_3) = -\omega(r, x_3) \), so that \( \psi(r, 0) = \psi(0, x_3) = 0 \). Also, we assume charges density satisfies \( v(r, -x_3) = v(r, x_3) \) and electrical potential has zero tangential derivative across \( x_3 \) axis and \( x_3 = 0 \) plane, which means \( \partial_3 \xi |_{r=0} = 0 \) and \( \partial_3 \xi |_{x_3=0} = 0 \). In addition, we let \( \xi \to 0 \) as \( \rho \to \infty \).

### 3.2 Spherical Coordinate and Self-similar Transform

Now we pass to the spherical coordinate. We set \( \omega(r, x_3) = \Omega(R, \theta), \psi(r, x_3) = \rho^2 \Psi(R, \theta), v(r, x_3) = V(R, \theta) \) and \( \xi = \rho^2 \Xi(R, \theta) \), then we have

\[ \partial_r \to \cos \theta \rho \alpha R \partial_R - \frac{\sin \theta}{\rho} \partial_\theta, \quad \partial_3 \to \frac{\sin \theta}{\rho} \alpha R \partial_R + \cos \theta \rho \partial_\theta, \]

and

\[ u_r = \rho(2 \sin \theta \Psi + \alpha \sin \theta R \partial_R \Psi + \cos \theta \partial_\theta \Psi), \]
\[ u_3 = \rho(-\frac{1}{\cos \theta} \Psi - 2 \cos \theta \Psi - \alpha \cos \theta R \partial_R \Psi + \sin \theta \partial_\theta \Psi). \]

In this way, (3.1) turns out to be

\[ \begin{cases} 
\partial_t \Omega + \mathcal{U}(\Psi) \partial_\theta \Omega + V(\Psi) \alpha R \partial_R \Omega = \mathcal{R}(\Psi) \Omega + \alpha R \partial_R \Xi \partial_\theta V - \alpha R \partial_R V \partial_\theta \Xi + 2 \partial_\theta V \Xi, \\
\partial_t \psi + \mathcal{U}(\Psi) \partial_\theta V + V(\Psi) \alpha R \partial_R V = 0, \\
\alpha^2 R^2 \partial_R \Xi + \alpha(5 + \alpha) R \partial_R \Xi + 6 \Xi + \partial_\theta \Xi - \tan \theta \partial_\theta \Xi = V, \\
-\alpha^2 R^2 \partial_R \psi - \alpha(5 + \alpha) R \partial_R \psi - 6 \psi + \partial_\theta \psi + \partial_\theta(\tan \theta \Psi) = \Omega,
\end{cases} \tag{3.2} \]

where

\[ \mathcal{U}(\Psi) = -3 \Psi - \alpha R \partial_R \Psi, \quad \mathcal{V}(\Psi) = \partial_\theta \Psi - \tan \theta \Psi, \quad \mathcal{R}(\Psi) = \frac{1}{\cos \theta}(2 \sin \theta \Psi + \alpha \sin \theta R \partial_R \Psi + \cos \theta \partial_\theta \Psi). \]

Now we introduce dynamical scaling variables to explore the stability of profiles \( (F, \Phi_F) \):

\[ z = \frac{\mu R}{\lambda^{1+3}}, \quad ds = \frac{1}{\lambda}, \]

and

\[ \Omega(R, t, \theta) = \frac{1}{\lambda} W(z, s, \theta), \quad \Psi(R, t, \theta) = \frac{1}{\lambda} \Phi(z, s, \theta), \]
\[ \Xi(R, t, \theta) = \frac{1}{\lambda} \Pi(z, s, \theta), \quad V(R, t, \theta) = \frac{1}{\lambda} G(z, s, \theta). \tag{3.3} \]
So (3.2) becomes

\[
\begin{align*}
\partial_t W + \frac{\mu}{\mu} z \partial_z W - \frac{\lambda}{\lambda} S_\delta(W) + U(\Phi) \partial_\theta W + V(\Phi) \alpha z \partial_z W &= R(\Phi) W + N_2(\Pi, G), \\
\partial_t G + \frac{\mu}{\mu} z \partial_z G - \frac{\lambda}{\lambda} S_\delta(G) + U(\Phi) \partial_\theta G + V(\Phi) \alpha z \partial_z G &= 0,
\end{align*}
\]

(3.4)

where \( S_\delta(W) = W + (1 + \delta) z \partial_z W \) and \( N_2(\Pi, G) = \alpha z \partial_z \Pi \partial_\theta G - \alpha z \partial_z G \partial_\theta \Pi + 2 \partial_\theta G \Pi \).

### 3.3 Linearization

In [10] the author shows that there exists a solution to Euler equation of the form

\[
\Omega(R, t, \theta) = \frac{1}{T - t} F \left( \frac{R}{(T - t)^{1+\delta}}, \theta \right),
\]

where \( 0 < \delta \ll 1 \) depends on \( \alpha \).

Recall that \( F = F_* + \alpha^2 g \) with \( F_* = \frac{\alpha F}{\epsilon} \frac{4z}{(1+z)^2} \) and \( \|g\|_{H^k} \leq C \), where \( C \) independent of \( \alpha \). In particular,

\[
L_K(g)(0) = 0. \tag{3.5}
\]

Note that \( F \) and \( \Phi_F \) satisfy the equations

\[
F + (1 + \delta) z \partial_z F + U(\Phi_F) \partial_\theta F + \alpha V(\Phi_F) z \partial_z F = R(\Phi_F) F;
\]

and

\[
-\alpha^2 R^2 \partial_\theta R^2 \Phi_F - \alpha (\alpha + 5) R \partial_\theta \Phi_F - \partial_\theta \Phi_F + \partial_\theta (\tan \theta \Phi_F) - 6 \Phi_F = F,
\]

where \( U, V \) and \( R \) are as above. Now we perform linearization by

\[
W = F + \varepsilon, \quad \Phi = \Phi_F + \Phi_\varepsilon, \quad G = G + 0,
\]

and get

\[
\begin{align*}
\partial_\varepsilon \varepsilon + \frac{\mu}{\mu} z \partial_z \varepsilon - (1 + \frac{\lambda}{\lambda}) S_\delta(\varepsilon) + \mathcal{M}(\varepsilon) &= E + N_1(\varepsilon) + N_2(\Pi, G), \\
\partial_\varepsilon G + \frac{\mu}{\mu} z \partial_z G - (1 + \frac{\lambda}{\lambda}) S_\delta(G) + \mathcal{M}_G(G) &= N_3(\varepsilon, G),
\end{align*}
\]

(3.6a)

(3.6b)

\[
\begin{align*}
-\alpha^2 z^2 \partial_\varepsilon \Phi_\varepsilon - (\alpha + 5) \alpha z \partial_\varepsilon \Phi_\varepsilon - \partial_\theta \Phi_\varepsilon + \partial_\theta (\tan \theta \Phi_\varepsilon) - 6 \Phi_\varepsilon &= \varepsilon, \\
\alpha^2 z^2 \partial_\varepsilon \Pi + \alpha (\alpha + 5) \alpha z \partial_\varepsilon \Pi + 6 \Pi + \partial_\theta \Pi - \tan \theta \partial_\theta \Pi &= G,
\end{align*}
\]

(3.6c)

(3.6d)

where \( \mathcal{M} \) and \( \mathcal{M}_G \) are the linearized operators:

\[
\mathcal{M}(\varepsilon) = S_\delta(\varepsilon) + U(\Phi_F) \partial_\theta \varepsilon + V(\Phi_F) \alpha z \partial_z \varepsilon + U(\Phi_\varepsilon) \partial_\theta F + V(\Phi_\varepsilon) \alpha z \partial_z F - R(\Phi_\varepsilon) \varepsilon - R(\Phi_\varepsilon) F,
\]

\[
\mathcal{M}_G(G) = S_\delta(G) + U(\Phi_F) \partial_\theta G + V(\Phi_F) \alpha z \partial_z G.
\]

(3.7)

\[
E \text{ is the error term:}
\]

\[
E = -\frac{\mu}{\mu} z \partial_z F + (1 + \frac{\lambda}{\lambda}) S_\delta(F).
\]

(3.8)

And the nonlinear terms are

\[
\begin{align*}
N_1(\varepsilon) &= -U(\Phi_\varepsilon) \varepsilon - \alpha V(\Phi_\varepsilon) z \partial_z \varepsilon + R(\Phi_\varepsilon) \varepsilon, \\
N_2(\Pi, G) &= \alpha z \partial_\Pi \partial_\theta G - \alpha z \partial_z G \partial_\theta \Pi + 2 \partial_\theta G \Pi, \\
N_3(\varepsilon, G) &= -U(\Phi_\varepsilon) \partial_\theta G - V(\Phi_\varepsilon) \alpha z \partial_z G.
\end{align*}
\]

(3.9)
By elliptic estimate (5.2), \( U(\Phi_F), U(\Phi_\varepsilon) \) and other similar terms have simple asymptotic forms:

\[
U(\Phi_\varepsilon) = -\frac{3}{4\alpha} \sin 2\theta L_K(\varepsilon) + O(1),
\]
(3.10)

\[
V(\Phi_\varepsilon) = \frac{1}{4\alpha} (2\cos 2\theta - 2\sin^2 \theta)L_K(\varepsilon) + O(1),
\]
(3.11)

\[
R(\Phi_\varepsilon) = \frac{1}{2\alpha} L_K(\varepsilon) + O(1),
\]

where the \( O(1) \) terms are bounded by constant.

Recall \( F = F* + \alpha^2 g \) with

\[
F* = \Gamma \frac{4z}{c(1+z)^2},
\]

We also have

\[
U(\Phi_F) = -3 \sin 2\theta \frac{1}{1+z} + O(\alpha),
\]
(3.12)

\[
V(\Phi_F) = (2\cos 2\theta - 2\sin^2 \theta) \frac{1}{1+z} + O(\alpha),
\]
(3.13)

\[
R(\Phi_F) = 2 \frac{1}{1+z} + O(\alpha).
\]

Thus, we can rewrite \( M \) as

\[
M(\varepsilon) = L^T_F(\varepsilon) + \Gamma(\theta) \frac{2z^2}{c(1+z)^2} L_K(\frac{3}{1+z} \sin 2\theta \partial_\theta \varepsilon)(0) + \tilde{L}(\varepsilon),
\]
(3.14)

where

\[
\tilde{L}(\varepsilon) = -\frac{1}{\sqrt{\alpha}} \left[ \alpha V(F_*)z \partial_z \varepsilon + U(\Phi_\varepsilon) \partial_\theta F_* + \alpha V(\Phi_\varepsilon) z \partial_\theta F_* + l.o.t. \right].
\]

4 Coercivity

In [11], the author gives the coercivity of \( M \):

**Proposition 4.1.** ([11]) For all \( \varepsilon \in H^k \) with \( L_{12}(\varepsilon)(0) = 0 \), there exists \( C_M > 0 \) depending only on \( k \) such that if \( \alpha < 10^{-14} \), we have

\[
(M(\varepsilon),\varepsilon)_{H^k} \geq C_M \| \varepsilon \|_{H^k}^2.
\]

Then as for \( M_G, \) we have:

**Proposition 4.2.** For all \( G \in H^k \), there exists \( C_{M_G} > 0 \) depending only on \( k \) such that if \( \alpha \) is sufficiently small, we have

\[
(M_G(G),G)_{H^k} \geq C_{M_G} \| G \|_{H^k}^2.
\]

**Proof.** By (5.1), (3.12), (3.13), we get

\[
M_G(G) = S_3(G) + U(\Phi_F) \partial_\theta G + V(\Phi_F) \alpha z \partial_z G
\]

\[
= G + (1 + \delta)z \partial_z G - \frac{3\sin 2\theta}{1+z} \partial_\theta G + \frac{\alpha}{1+z} (2\cos 2\theta - 2\sin^2 \theta) \partial_\theta G + O(\alpha)
\]

\[
= G + z \partial_z G - \frac{3\sin 2\theta}{1+z} \partial_\theta G + l.o.t.
\]

\[
= L(G) - \frac{3\sin 2\theta}{1+z} \partial_\theta G + \frac{2G}{1+z} + l.o.t.,
\]

where \( l.o.t. \) stands for terms bounded by \( C\alpha \) in \( H^k \), which are much smaller than 1 if let \( \alpha \ll 1. \)

**Step 1:** \( k = 0. \)
By the definition of $\mathcal{M}_G$, (2.7) and integration by parts, we have

$$
\mathcal{M}_G(G, G \frac{w^2}{\sin 2\theta_n})_{L^2} = \int_0^\infty \int_0^{\pi/2} \mathcal{L} G - \frac{3}{1 + z} \sin 2\theta \partial \theta G + \frac{2G}{1 + z} G \frac{w^2}{\sin 2\theta_n} d\theta dz
$$

$$
= \frac{1}{2} \|G \frac{w^2}{\sin 2\theta_n}\|_{L^2}^2 + \int_0^\infty \int_0^{\pi/2} \frac{3(1 - \eta)}{1 + z} G^2 \cos 2\theta (\sin 2\theta)^{-\eta} w^2 d\theta dz
$$

$$
+ \int_0^\infty \int_0^{\pi/2} \frac{2G^2}{1 + z} \sin 2\theta_n d\theta dz.
$$

Because $3(1 - \eta) \cos 2\theta = \frac{3}{100} \cos 2\theta < 2$, the second term can be absorbed by the third term in the last formula.

So for $k = 0$, we have

$$
\mathcal{M}_G(G, G \frac{w^2}{\sin 2\theta_n/2})_{L^2} \geq \frac{1}{2} \|G \frac{w^2}{\sin 2\theta_n/2}\|_{L^2}^2. \tag{4.2}
$$

**Step 2: k = 1.**

We need to estimate the terms involving $D_\theta$ and $D_z$.

1. Terms involving $D_\theta$: We have

$$
(D_\theta(\mathcal{M}_G(G)), D_\theta G \frac{w^2}{\sin 2\theta_n})_{L^2} + O(\alpha)
$$

$$
= \int_0^\infty \int_0^{\pi/2} (D_\theta \mathcal{L} G) - \frac{3}{1 + z} \sin 2\theta \partial \theta D_\theta G + \frac{2D_\theta G}{1 + z} D_\theta G \frac{w^2}{\sin 2\theta_n} d\theta dz,
$$

$$
= I_1 + I_2 + I_3.
$$

By (2.7),

$$
I_1 = \int_0^\infty \int_0^{\pi/2} (D_\theta \mathcal{L} G) G \frac{w^2}{\sin 2\theta_n} d\theta dz = \frac{1}{2} \|D_\theta G \frac{w}{\sin 2\theta_n/2}\|_{L^2}^2.
$$

For $I_2$ we use integration by parts and get

$$
I_2 = \int_0^\infty \int_0^{\pi/2} - \frac{3}{1 + z} \sin 2\theta \partial \theta D_\theta G D_\theta G \frac{w^2}{\sin 2\theta_n} d\theta dz
$$

$$
= \int_0^\infty \int_0^{\pi/2} \frac{3\alpha \cos 2\theta}{20(1 + z)} (D_\theta G)^2 \frac{w^2}{\sin 2\theta_n} d\theta dz,
$$

which can be absorbed by $I_3$ as long as $\alpha$ is small enough.

So we get

$$
(D_\theta(\mathcal{M}_G(G)), D_\theta G \frac{w^2}{\sin 2\theta_n})_{L^2} \geq \frac{1}{2} \|D_\theta G \frac{w}{\sin 2\theta_n/2}\|_{L^2}^2. \tag{4.3}
$$

2. Terms involving $D_z$: Direct computation gives

$$
D_z \mathcal{L} G = D_z(G + z \partial_z G - \frac{2G}{1 + z})
$$

$$
= D_z G + z \partial_z (D_z G) - \frac{2D_z G}{1 + z} + \frac{2zG}{(1 + z)^2}
$$

$$
= \mathcal{L}(D_z G) + \frac{2zG}{(1 + z)^2},
$$

and

$$
D_z \left( \frac{2G}{1 + z} \right) = \frac{2D_z G}{1 + z} - \frac{2zG}{(1 + z)^2}.
$$

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This, together with (4.1) and integration by parts, yields
\[
(D_z \mathcal{M}_G(G), D_z G \frac{w^2}{\sin 2\theta^2})_{L^2} + O(\alpha)
\]
\[
= \int_0^\infty \int_0^{\pi/2} \left[ D_z (\mathcal{L} G) - D_z \left( \frac{3}{1 + z} \sin 2\theta_0 G \right) + D_z \left( \frac{2G}{1 + z} \right) \right] D_z G \frac{w^2}{\sin 2\theta^2} d\theta dz
\]
\[
= \int_0^\infty \int_0^{\pi/2} \left[ L(D_z G) + \frac{3z}{(1 + z)^2} \sin 2\theta_0 G - \frac{3}{1 + z} \sin 2\theta_0 (D_z G) + \frac{2D_z G}{1 + z} \right] D_z G \frac{w^2}{\sin 2\theta^2} dz,
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]
Still by (2.7), \( I_1 = \frac{1}{\pi} \|D_z G \frac{w^2}{\sin 2\theta^2}\|_{L^2}^2 \).
And \( I_2 \) is bounded below by Cauchy-Schwarz
\[
I_2 = \int_0^\infty \int_0^{\pi/2} \frac{3z}{(1 + z)^2} D_\theta G D_z G \frac{w^2}{\sin 2\theta^2} d\theta dz 
\]
\[
\geq - \|D_\theta G \frac{w}{\sin 2\theta^2/2}\|_{L^2} \|D_z G \frac{w}{\sin 2\theta^2/2}\|_{L^2}
\]
\[
\geq - \|D_\theta G \frac{w}{\sin 2\theta^2/2}\|_{L^2} \|D_z G \frac{w}{\sin 2\theta^2/2}\|_{L^2}
\]
\[
\geq - \frac{1}{4} \|D_\theta G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2} - 4 \|D_z G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2},
\]
where the second inequality holds because \( \frac{90}{100} = \eta < \gamma = 1 + \frac{10}{9} \) and \( \sin 2\theta^2/2 > \sin 2\gamma/2, \theta \in [0, \pi/2] \), and the third inequality holds by Young’s inequality.
For \( I_3 \) we use integration by parts again and get
\[
I_3 = \int_0^\infty \int_0^{\pi/2} \frac{3}{1 + z} \sin 2\theta_0 (D_z G) \frac{w^2}{\sin 2\theta^2} d\theta dz
\]
\[
= \int_0^\infty \int_0^{\pi/2} \frac{3(1 - \eta) \cos 2\theta}{2(1 + z)} (D_\theta G)^2 \frac{w^2}{\sin 2\theta^2} d\theta dz,
\]
which can be absorbed by the fourth term for \( \frac{1}{2\theta^2} \cos 2\theta < 2 \).
So as for terms involving \( D_z \) we get
\[
(D_z \mathcal{M}_G(G), D_z G \frac{w^2}{\sin 2\theta^2})_{L^2} \geq \frac{1}{4} \|D_\theta G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2} - 4 \|D_z G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2}.
\]
Then gathering (4.2) (4.3) and (4.4), we have
\[
(\mathcal{M}_G(G), G \frac{w^2}{\sin 2\theta^2})_{L^2} + 9(D_\theta (\mathcal{M}_G(G)), D_\theta G \frac{w^2}{\sin 2\theta^2})_{L^2} + (D_z \mathcal{M}_G(G), D_z G \frac{w^2}{\sin 2\theta^2})_{L^2}
\]
\[
\geq \frac{1}{2} \|G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2} + \frac{1}{8} \|D_z G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2} + \frac{1}{2} \|D_\theta G \frac{w}{\sin 2\theta^2/2}\|^2_{L^2}.
\]
**Step 3:** \( k \geq 2 \).
Recall the definition of \( \mathcal{H}^k \) (2.2), we have the following equivalent relation
\[
(f, f)_{\mathcal{H}^k} \approx (f, f)_{\mathcal{H}^{k-1}} + \|D_\theta f\|^2_{\mathcal{H}^{k-1}} + \|D_z f\|^2_{\mathcal{H}^{k-1}} \approx \|f\|^2_{\mathcal{H}^k}.
\]
We prove by induction on \( k \). Assuming for \( k - 1 \)
\[
(\mathcal{M}_G(G), G)_{\mathcal{H}^{k-1}} \geq c_{k-1} \|G\|^2_{\mathcal{H}^{k-1}},
\]
we prove (4.1).
First we note that
\[
D_\theta (\mathcal{M}_G(G)) = D_\theta (\mathcal{L} G) - \frac{3}{1 + z} \sin 2\theta_0 D_\theta G + \frac{2D_z G}{1 + z}
\]
\[
= D_\theta (D_z G),
\]
so by (4.5) we have
\[ (D_\theta(M_G(G)), D_\theta(G))_{H^{k-1}} \geq c_{k-1} \| D_\theta G \|_{H^{k-1}}^2. \]  
(4.6)

And note that
\[ D_z(M_G(G)) = D_z(LG) - D_z\left(\frac{3}{1+z} \sin 2\theta \partial_\theta G\right) + D_z \frac{2G}{1+z} \]
\[ = L(D_z G) + \frac{3z}{(1+z)^2} \sin 2\theta \partial_\theta G - \frac{3}{1+z} \sin 2\theta \partial_\theta (D_z G) + \frac{2D_z G}{1+z} \]
\[ = M_G(D_z G) + \frac{3z}{(1+z)^2} \sin 2\theta \partial_\theta G, \]
by (4.5) and Young’s inequality we get
\[ (D_z(M_G(G)), D_z(G))_{H^{k-1}} \]
\[ = (M_G(D_z G), D_z(G))_{H^{k-1}} + \frac{3z}{(1+z)^2} \sin 2\theta \partial_\theta G, D_z(G))_{H^{k-1}} \]
\[ \geq c_{k-1} \| D_z G \|_{H^{k-1}}^2 - \frac{c_{k-1}}{2} \| D_z G \|_{H^{k-1}}^2 - C_{k-1} \| D_\theta G \|_{H^{k-1}}^2, \]
that is
\[ (D_z(M_G(G)), D_z(G))_{H^{k-1}} \geq \frac{c_{k-1}}{2} \| D_z G \|_{H^{k-1}}^2 - C_{k-1} \| D_\theta G \|_{H^{k-1}}^2. \]  
(4.7)

Gathering the (4.5), (4.6) and (4.7), there exist \( c_k > 0 \) such that
\[ (M_G(G), G)_{H^k} \geq c_k \| G \|_{H^k}^2. \]

\[ \square \]

5 Analysis of Variant Spherical Laplacian

Now we consider the elliptic equations with boundary conditions assumed in Section 2. First we consider
\[ \begin{cases} -\alpha^2 z^2 \partial_{zz} \Psi - \alpha(5 + \alpha) z \partial_\theta \Psi - \partial_\theta \Psi + \partial_\theta (\tan \theta \Psi) - 6 \Psi = F, \\ \Psi(z, 0) = 0, \quad \Psi(z, \frac{\pi}{2}) = 0, \quad \Psi(\infty, \theta) = 0, \end{cases} \]
(5.1)
which arises from Biot-Savart law. In [10] the author gives the following estimate on (5.1):

**Proposition 5.1.** Let \( \alpha > 0 \) and assume \( F \in H^k \). Let \( \Psi \) be the unique solution to (5.1) which vanishes at \( \theta = 0, \frac{\pi}{2} \) as \( z \to \infty \). Then
\[ \alpha^2 \| D_z^2 \Psi \|_{H^k} + \alpha \| D_z \Psi \|_{H^k} + \| \partial_\theta \Psi - \frac{1}{4\alpha} \sin 2\theta L_K(F) \|_{H^k} \leq C_k \| F \|_{H^k}, \]
(5.2)

where \( L_K(F) = \int_{\pi}^{\pi/2} \int_0^\infty \frac{F(\rho, \theta) e^{iK(\rho)}}{\rho} d\rho d\theta. \)

Next, we consider
\[ \begin{cases} \alpha^2 z^2 \partial_{zz} \Pi + \alpha(5 + \alpha) z \partial_\theta \Pi + \partial_\theta \Pi - \tan \theta \partial_\theta \Pi + 6 \Pi = G, \\ \partial_\theta \Pi(z, 0) = 0, \quad \partial_\theta \Pi(z, \frac{\pi}{2}) = 0, \quad \Pi(\infty, \theta) = 0, \end{cases} \]
(5.3)
which is derived from variant spherical Laplacian corresponding to electrical potential. We want to find a special solution of (5.3).

First we consider the part relevant to \( z \):
\[ \begin{cases} \alpha^2 z^2 \partial_{zz} \Pi_1 + \alpha(5 + \alpha) z \partial_\theta \Pi_1 + \Pi_1 = 0, \\ \Pi_1(\infty) = 0, \end{cases} \]
(5.4)
which is an ODE of Euler type, and \( \Pi_1(z) = c_1 z^{\frac{5\alpha - 5}{2 \alpha}} \) is a special solution, where \( c_1 \) is under determined. Secondly, by Sturm-Liouville theory, direct computation shows that \( \Pi_2(\theta) = \cos^2 \theta - \frac{2}{3} \) is a solution of
\[
\begin{cases}
\partial_\theta \Pi_2 - \tan \theta \partial_\theta \Pi_2 + 6 \Pi_2 = 0, \\
\partial_\theta \Pi_2(0) = \partial_\theta \Pi_2(\frac{\pi}{2}) = 0.
\end{cases}
\]

Then, we construct \( \Pi_0 \) as
\[\Pi_0 = \tilde{\Pi}(s) \Pi_1(z) \Pi_2(\theta),\]
where \( \tilde{\Pi}(s) \) is under determined. Obviously, \( \Pi_0 \) satisfies
\[\alpha^2 z^2 \partial_{zz} \Pi_0 + \alpha(5 + \alpha) z \partial_z \Pi_0 + \partial_\theta \Pi_0 - \tan \theta \partial_\theta \Pi_0 + 6 \Pi_0 = -\Pi_0.\]

However, \( \Pi_0 \not\in \mathcal{H}^k \) because \( z^{\frac{5\alpha - 5}{2 \alpha}} \) grows too fast near origin, so we let
\[\Pi = \chi \Pi_0, \quad (5.5)\]
where
\[\chi(z) = \begin{cases} 0 & z < \frac{1}{2}, \\
1 & z > 1. \end{cases}\]

We also have the following properties hold for \( \Pi \),

**Proposition 5.2.** Let \( P = \alpha^2 z^2 \partial_{zz} + \alpha(5 + \alpha) z \partial_z + \partial_\theta - \tan \theta \partial_\theta + 6 \text{Id}, \) and \( G = \Pi \), then the following hold:

- \( \Pi \in \mathcal{H}^k. \) In particular, \( \|\Pi\|_{\mathcal{H}^k} = C \tilde{\Pi}(s) \) for some \( C; \)
- \( |G| = |\Pi| \leq C|\Pi|, \) where \( C \) is independent of \( z, \) so \( (G,G)_{\mathcal{H}^k} = \tilde{\Pi}^2(s) \) for suitable \( c_1; \)
- \( \partial_\theta \Pi(0) = \partial_\theta \Pi(\frac{\pi}{2}) = \partial_z G(0,\theta) = 0. \)

**Proof.** We have
\[\|\Pi\|_{\mathcal{H}^k} = \tilde{\Pi}(s) \|\chi(z)z^{\frac{5\alpha - 5}{2 \alpha}} (\cos^2 \theta - \frac{2}{3})\|_{\mathcal{H}^k} = C \tilde{\Pi}(s),\]
so the first assertion holds. Meanwhile,
\[\Pi \Pi = P(\chi \Pi_0) = (\alpha^2 z^2 \partial_{zz} + \alpha(5 + \alpha) z \partial_z)(\chi \Pi_0) = \alpha^2 z^2 (\partial_{zz} \chi \Pi_0 + 2 \partial_z \chi \partial_z \Pi_0) + \alpha(5 + \alpha) z \partial_z \chi \Pi_0 + \chi \Pi_0.\]

It is easy to check \( |\alpha^2 z^2 \partial_{zz} \chi| + |\alpha(5 + \alpha) z \partial_z \chi| \leq C|\chi| \) and \( |\alpha^2 z^2 \partial_z \chi \partial_z \Pi_0| \leq C|\chi \Pi_0|, \) so the second assertion holds. The third assertion is obvious.

### 6 Modulation

In this section, for any \( s > 0, \) we choose proper \( \lambda \) and \( \mu \) such that
\[L_K(\varepsilon)(0) = 0, \quad (6.1)\]
to impose (5.4) to \( \varepsilon(z,s,\theta). \) And note that if \( \partial_z \varepsilon(0,s,\theta) = 0 \) then
\[L_K(z \partial_z \varepsilon)(s) = 0, \quad s > 0. \quad (6.2)\]
Proposition 6.1. For any $\theta \in [0, \pi/2]$, $s > 0$, in order to keep $L_K(\varepsilon)(0) = 0$ and $\partial_z \varepsilon(0, s, \theta) = 0$, it suffices to impose $\lambda$ and $\mu$ to satisfy
\[
\alpha \left( \frac{\lambda_s}{\lambda} + 1 \right) - 3L_K \left( \frac{\sin 2\theta}{1 + z} \partial_\theta \varepsilon \right)(0) + \sqrt{\alpha}L_K \left( \bar{L}(\varepsilon) \right)(0) + L_K(N_1(\varepsilon))(0) + L_K(N_2(\Pi, G))(0) = 0, \tag{6.3}
\]
and
\[
\frac{\mu_s}{\mu} = (2 + \delta) \left( \frac{\lambda_s}{\lambda} + 1 \right). \tag{6.4}
\]
In particular, we have the following estimates:
\[
|\alpha \left( \frac{\lambda_s}{\lambda} + 1 \right) - 3L_K \left( \frac{\sin 2\theta}{1 + z} \partial_\theta \varepsilon \right)(0)| \lesssim \sqrt{\alpha} \|\varepsilon\|_{H^k} + \alpha \|\varepsilon\|_{H^k}^2 + \tilde{\Pi}^2(s), \tag{6.5}
\]
and
\[
\left| \frac{\lambda_s}{\lambda} + 1 \right| \lesssim \frac{1}{\alpha} \|\varepsilon\|_{H^k}. \tag{6.6}
\]

Proof. At first, since $\partial_z N_2(\Pi, G)(0) = 0$, the proof of (6.4) is just a repetition of that in [11], and $\partial_z \varepsilon(0, s, \theta) = 0$ follows.

It remains to keep $L_K(\varepsilon)(0) = 0$. We impose $L_K$ on the first equation of (3.6a) and let $z = 0$ to get
\[
\partial_s L_K(\varepsilon)(0) + \frac{\mu_s}{\mu} L_K(z \partial_z \varepsilon)(0) - \left( \frac{\lambda_s}{\lambda} + 1 \right) L_K(S_3(\varepsilon))(0) = L_K(E)(0) - L_K(\mathcal{M}\varepsilon)(0)
+ L_K(N_1(\varepsilon))(0) + L_K(N_2(\Pi, G))(0).
\]
By (2.3), (3.8) and (6.2), as well as $F = F_* + \alpha^2 g$, we have
\[
L_K(E)(0) = L_K \left( -\frac{\mu_s}{\mu} z \partial_z F + (1 + \frac{\lambda_s}{\lambda}) S_3(F) \right)(0)
= L_K \left( (1 + \frac{\lambda_s}{\lambda}) F \right)(0)
= L_K \left( (1 + \frac{\lambda_s}{\lambda})(F_* + \alpha^2 g) \right)(0)
= 4\alpha \left( 1 + \frac{\lambda_s}{\lambda} \right).
\]
Meanwhile, (3.13) and (2.3) yield
\[
L_K(\mathcal{M}\varepsilon)(0) = L_K \left( L_{F, \varepsilon}^T \left( \varepsilon \right) + \Gamma(\theta) \frac{2z^2}{c(1 + z)^3} L_K \left( \frac{3}{1 + z} \sin 2\theta \partial_\theta \varepsilon \right)(0) + \sqrt{\alpha} \bar{L}(\varepsilon) \right)(0)
= L_K \left( L_{F, \varepsilon} \right)(0) - 3L_K \left( \frac{\sin 2\theta}{1 + z} \partial_\theta \varepsilon \right)(0) + \sqrt{\alpha} L_K(\bar{L}(\varepsilon))(0)
= \mathcal{L}(L_K(\varepsilon))(0) - 3L_K \left( \frac{\sin 2\theta}{1 + z} \partial_\theta \varepsilon \right)(0) + \sqrt{\alpha} L_K(\bar{L}(\varepsilon))(0).
\]
Then,
\[
\partial_s L_K(\varepsilon)(0) - \left( \frac{\lambda_s}{\lambda} + 1 \right) L_K(\varepsilon)(0) = 4\alpha \left( 1 + \frac{\lambda_s}{\lambda} \right) + \mathcal{L}(L_K(\varepsilon))(0) - 3L_K \left( \frac{\sin 2\theta}{1 + z} \partial_\theta \varepsilon \right)(0)
+ \sqrt{\alpha} L_K(\bar{L}(\varepsilon))(0) + L_K(N_1(\varepsilon))(0) + L_K(N_2(\Pi, G))(0).
\]
So (6.3) implies
\[
\partial_s L_K(\varepsilon)(0) = (\frac{\lambda_s}{\lambda} + 1)L_K(\varepsilon)(0) + L(\varepsilon)(0)
\]
\[
= (\frac{\lambda_s}{\lambda} + 1)L_K(\varepsilon)(0) + (z\partial_z L_K(\varepsilon)(0) - (\frac{2}{1 + z} L_K(\varepsilon)(0))
\]
\[
= \frac{\lambda_s}{\lambda} L_K(\varepsilon)(0) - \int_0^{\pi/2} \varepsilon(0, s, \theta) K(\theta) d\theta
\]
where we used (2.3) in the third equality and \(\varepsilon(0, s, \theta) = 0\) in the last equality. Then we have
\[
L_K(\varepsilon)(0, s) = \frac{L_K(\varepsilon)(0, 0)}{\lambda(0)} \lambda(s),
\]
so \(L_K(\varepsilon)(0)\) will vanish identically once \(L_K(\varepsilon)(0)\)|\(_{s=0}\) = 0 and \(\lambda(0) = 1\).

By (6.3) and \([11]\), it is easy to get
\[
|4\alpha (\frac{\lambda_s}{\lambda} + 1) - 3L_K(\sin \frac{2\theta}{1 + z} \partial_y \varepsilon)(0)| \lesssim \sqrt{\alpha} \|\varepsilon\|_{\mathcal{H}^k} + \alpha \|\varepsilon\|_{\mathcal{H}^k}^2 + \tilde{\Pi}^2(s)
\]
and
\[
|\frac{\lambda_s}{\lambda} + 1| \lesssim \frac{1}{\alpha} |L_K(\sin \frac{2\theta}{1 + z} \partial_y \varepsilon)(0)| \lesssim \frac{1}{\alpha} \|\varepsilon\|_{\mathcal{H}^k}.
\]

7 Energy Estimate

In this section we derive energy estimate for (3.6a) and (3.6b).

First we define the energy
\[
\mathcal{E}(s) = \langle \varepsilon, \varepsilon \rangle_{\mathcal{H}^k} + \tilde{\Pi}^2(s).
\]

Proposition 7.1. If \(\alpha \ll 1\), we have:
\[
\frac{d}{ds} \mathcal{E} \leq -c \mathcal{E} + \frac{C}{\alpha^{3/2}} \mathcal{E}^{3/2}.
\]

for some \(c, C > 0\).

Proof. First, we check the boundness of \(\frac{d}{ds}(\varepsilon, \varepsilon)_{\mathcal{H}^k}\). By (3.6a), we have
\[
\frac{1}{2} \frac{d}{ds}(\varepsilon, \varepsilon)_{\mathcal{H}^k} \leq -\langle \mathcal{M}\varepsilon, \varepsilon \rangle_{\mathcal{H}^k} + \langle E, \varepsilon \rangle_{\mathcal{H}^k} + \frac{\mu_s}{\mu} \langle (z\partial_z \varepsilon, \varepsilon)_{\mathcal{H}^k} | (1 + \frac{\lambda_s}{\lambda}) \rangle (S_\delta \varepsilon, \varepsilon)_{\mathcal{H}^k}
\]
\[
+ \langle N_1(\varepsilon), \varepsilon \rangle_{\mathcal{H}^k} + \langle N_2(\Pi, G), \varepsilon \rangle_{\mathcal{H}^k}.
\]
As in [11], we have
\[
-\langle \mathcal{M}\varepsilon, \varepsilon \rangle_{\mathcal{H}^k} + \langle E, \varepsilon \rangle_{\mathcal{H}^k} \leq -\alpha \|\varepsilon\|_{\mathcal{H}^k}^2 + \alpha \|\varepsilon\|_{\mathcal{H}^k}^2 + \alpha \|\varepsilon\|_{\mathcal{H}^k}^2 + \alpha \|g\|_{\mathcal{H}^k} \|\varepsilon\|_{\mathcal{H}^k}^2,
\]
\[
\frac{\mu_s}{\mu} \langle (z\partial_z \varepsilon, \varepsilon)_{\mathcal{H}^k} \leq \frac{C}{\alpha^{3/2}} \|\varepsilon\|_{\mathcal{H}^k}^3,
\]
\[
\frac{\lambda_s}{\lambda} + 1| \langle S_\delta \varepsilon, \varepsilon \rangle_{\mathcal{H}^k} | \leq \frac{C}{\alpha^{3/2}} \|\varepsilon\|_{\mathcal{H}^k}^3,
\]
and
\[
\langle N_1(\varepsilon), \varepsilon \rangle_{\mathcal{H}^k} \leq \frac{C}{\alpha^{3/2}} \|\varepsilon\|_{\mathcal{H}^k}^3.
\]
As for \((N_2(\Pi, G, \varepsilon))\), since \(\Pi = \tilde{\Pi}(s)\chi(z)\frac{\tilde{\mu}}{\tilde{\nu}}(\cos^2 \theta - \frac{3}{4})\), \(|\Pi| \leq C|\Pi|\), and \(|\Pi|) = C\tilde{\Pi}(s), we get

\[\|\partial_\theta G\| H^k \leq C\|\partial_\theta \Pi\| H^k \leq C|\Pi|) H^k \leq C\tilde{\Pi}(s).\]

Then, by the product lemmas in the Appendix of [11], we have

\[(\alpha z_2 G_\theta, G) H^k \leq C\alpha^{1/2}\|\Pi\| H^k|G| H^k \leq C\alpha^{1/2}\tilde{\Pi}^2(s)|\varepsilon| H^k,\]

\[(\alpha z_2 G_\theta, G) H^k \leq C\alpha^{1/2}\|\Pi\| H^k|G| H^k \leq C\alpha^{1/2}\tilde{\Pi}^2(s)|\varepsilon| H^k,\]

\[(\partial_\theta G_\theta, G) H^k \leq \frac{C}{\alpha^{1/2}\|\Pi\| H^k|G| H^k \leq \frac{C}{\alpha^{1/2}\tilde{\Pi}^2(s)|\varepsilon| H^k.}\]

So by (3.10),

\[(N_2(\Pi, G, \varepsilon)) H^k \leq \frac{C}{\alpha^{1/2}} + C\alpha^{1/2}\tilde{\Pi}^2(s)|\varepsilon| H^k.\]

Now we check the boundness of \(\frac{d}{ds}(G, G) H^k \leq -(M_G(G, G)) H^k + \|\mu_\alpha\|((z_2 G_\theta, G) H^k + |C| + \lambda_s)|)(S_\delta G, G) H^k + (N_3(\varepsilon, G)) H^k.\]

By (3.6A), we have

\[-(M_G(G, G)) H^k \leq -c|G| H^k = -c\tilde{\Pi}^2(s),\]

and meanwhile,

\[|\mu_\alpha|((z_2 G_\theta, G) H^k \leq \frac{C}{\alpha^{1/2}}|G| H^k \leq |\varepsilon| H^k,\]

\[|(1 + \lambda_s)|)(S_\delta G, G) H^k \leq \frac{C}{\alpha^{1/2}}|G| H^k \leq |\varepsilon| H^k.\]

And for the nonlinear term \((N_3(\varepsilon, G)) H^k, because\)

\[\|z_2 G_\theta\| H^k \leq C|z_2 \Pi\| H^k \leq C|\Pi| H^k \leq C\tilde{\Pi}(s),\]

as well as (3.10), (2.4), (5.1) and product lemmas in [11], we have

\[(\mathcal{U}(\Phi_\varepsilon)z_2 G_\theta) H^k \leq \frac{C}{\alpha}|G| H^k \leq L_K(\varepsilon)|\Pi| H^k \leq \frac{C}{\alpha}\tilde{\Pi}^2(s)|\varepsilon| H^k,\]

and

\[(\mathcal{V}(\Phi_\varepsilon)z_2 G_\theta) H^{k+1} \leq C\tilde{\Pi}^2(s)|\varepsilon| H^k.\]

Gathering preceding estimates together, \(\tilde{\Pi}(s)\) follows.

Now we get another version of Theorem 1.1:

**Theorem 7.2.** For \(k \geq 4\), there exists \(0 < \alpha_0 \ll 1\) and \(\nu, \kappa_0 > 0\), such that if \(\Pi\) is defined as (3.5), and for every initial \(\varepsilon(\Pi(s_0)) < \nu_0\alpha^{1/2}\) and \(L_K(\varepsilon) = 0\), there exists a unique solution to (3.6A) so that:

\[|\frac{\mu_\alpha}{\mu}| + \frac{\lambda_s}{\lambda} + 1 + \mathcal{E}(\varepsilon, \Pi) \leq C\mathcal{E}(\varepsilon_0, \Pi(s_0)) + \kappa_0.\]

**Proof.** By (3.1), (2.4) and (6.6), modulation and Gronwall’s inequality, the rapid decay of \(|\mu_\alpha|\) and \(|\frac{\lambda_s}{\lambda} + 1|\), as well as energy estimate. And the existence and uniqueness can be obtained by weak compactness and the decay of energy similarly in [10].

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