QUASILINEAR CHOQUARD EQUATIONS INVOLVING N-LAPLACIAN AND CRITICAL EXPONENTIAL NONLINEARITY

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Abstract

In the present paper, we study a class of quasilinear Choquard equations involving $N$-Laplacian and the nonlinearity with the critical exponential growth. We discuss the existence of positive solutions of such equations.

1. Introduction

This paper is concerned with the existence of the positive solutions for the following family of quasilinear equations with $N$-Laplacian and exponential Choquard type nonlinearity

\[
\begin{aligned}
-\Delta_N u - \Delta_N (u^2)u &= \left( \int_{\Omega} \frac{F(y,u)}{|x-y|^\mu} \, dy \right) f(x,u) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{aligned}
\]

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 < \mu < N$ and $\Delta_N := \text{div}(|\nabla u|^{N-2} \nabla u)$ is called the $N$-Laplacian. The function $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is given by $f(x,s) = g(x,s) \exp(|s|^\frac{N}{N-1})$, where $g \in C(\bar{\Omega} \times \mathbb{R})$ satisfies some appropriate assumptions described later. The function $F(y,s)$ is the primitive of $f(y,s)$ defined as $F(y,s) = \int_0^s f(y,t) \, dt$.

The problems involving the quasilinear operator $-\Delta_p u - \Delta_p (u^2)u$, $1 < p < \infty$, has been of interest to many researchers for long due to its significant applications in the modeling of the physical phenomenon such as in plasma physics and fluid mechanics \cite{4}, in dissipative quantum mechanics \cite{16}, etc. Solutions of such equation are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

\[
iu_t = -\Delta u + V(x)u - h_1(|u|^2)u - C\Delta h_2(|u|^2)h_2'(|u|^2)u, \quad x \in \mathbb{R}^N,
\]

where $V$ is a potential, $C$ is a real constant, $h_1$ and $h_2$ are real functions. Depending upon the different type of $h_2$, the quasilinear equations of the form (1.1) appear in the study of mathematical physics. For example, in the modeling of the superfluid film equation in plasma physics, $h_2(s) = s$ (see \cite{19}) and for studying the self-channeling of a high-power ultra short laser in matter $h_2(s) = \sqrt{1 + s^2}$ (see \cite{20}).

The main mathematical difficulty we face in studying the problem of type (P) occurs due to the quasilinear term $\Delta_N (u^2)u$, which doesn’t allow the natural energy functional corresponding to the problem (P) to be well defined for all $u \in W_0^{1,N}(\Omega)$ (defined in Section 3). Hence, we can not apply variational method directly for such problem. To overcome this inconvenience, several methods and arguments have been developed, such as the perturbation method (see for e.g., \cite{23, 26}) a constrained minimization technique (see for e.g., \cite{24, 25, 35, 37}), and a change of variables (see for e.g., \cite{6, 10, 11, 12, 17, 18}). The nonlinearity in the problem (P), which is nonlocal in nature, is driven by the Hardy-Littlewood-Sobolev inequality and the Trudinger-Moser inequality. Let us first recall the following well known Hardy-Littlewood-Sobolev inequality [Theorem 4.3, p.106] \cite{21}.

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Theorem 1.2. (Hardy-Littlewood-Sobolev inequality) Let \( t, r > 1 \) and \( 0 < \mu < N \) with \( 1/t + \mu/N + 1/r = 2 \), \( g_1 \in L^t(\mathbb{R}^N) \) and \( g_2 \in L^r(\mathbb{R}^N) \). Then there exists a sharp constant \( C(t,N,\mu,r) \), independent of \( g_1, g_2 \) such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^\mu} \, dx \, dy \leq C(t,N,\mu,r) \|g_1\|_{L^t(\mathbb{R}^N)} \|g_2\|_{L^r(\mathbb{R}^N)}.
\]

If \( t = r = \frac{2N}{2N-\mu} \) then

\[
C(t,N,\mu,r) = C(N,\mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-\mu}{2}\right)} \frac{\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\frac{N-\mu}{2}\right)}^{-1+\frac{\mu}{N}}.
\]

In this case there is equality in (1.2) if and only if \( g_1 \equiv (\text{constant})g_2 \) and

\[
g_1(x) = c_0(a^2 + |x-b|^2)^{-\frac{\mu}{2N-\mu}}
\]

for some \( c_0 \in \mathbb{C} \), \( 0 \neq a \in \mathbb{R} \) and \( b \in \mathbb{R}^N \).

Now a days, an ample amount of attention has been attributed to the study of Choquard type equations, which was started with the seminal work of S. Pekar [33], where the author considered the following nonlinear Schrödinger-Newton equation:

\[
-\Delta u + V(x)u = (K_{\mu} * u^2)u + \lambda f_1(x,u) \quad \text{in} \ \mathbb{R}^N,
\]

where \( \lambda > 0 \), \( K_{\mu} \) denotes the Riesz potential, \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous function and \( f_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function with some appropriate growth assumptions. When \( \lambda = 0 \), the nonlinearity in the right-hand side of (1.3) is termed as Choquard type nonlinearity. In the application point of view, such type of nonlinearity plays an important role in describing the Bose-Einstein condensation (see [7]) and also, appears in the modeling of the self gravitational collapse of a quantum mechanical wave function (see [34]). P. Choquard (see [20]) studied such elliptic equations of type (1.3) for construing the quantum theory of a polaron at rest and for modeling the phenomenon of an electron being trapped in its own hole in the Hartree-Fock theory. When \( V(x) = 1 \), \( \lambda = 0 \), the equations of type (1.3) were studied rigorously in [21] [22]. For more extensive study of Choquard equations, without attempting to provide a complete list, we refer to [23] [24] [50] [31] and the references therein.

The main feature of the problem (P) is that the nonlinear term \( f(x,t) \) has the maximal growth on \( t \), that is, critical exponential growth with respect to the following Trudinger–Moser inequality (see [32]):

Theorem 1.2. (Trudinger–Moser inequality) For \( N \geq 2 \), \( u \in W^{1,N}_0(\Omega) \)

\[
\sup_{\|u\| \leq 1} \int_{\Omega} \exp(\alpha|u|_N^N) \, dx < \infty
\]

if and only if \( \alpha \leq \alpha_N \), where \( \alpha_N = N \omega_{N-1}^{1/N} \) and \( \omega_{N-1} = (N-1) \cdot \text{dimensional surface area of} \ \mathbb{S}^{N-1} \).

Here the Sobolev space \( W^{1,N}_0(\Omega) \) and the corresponding norm \( \|\cdot\| \) are defined in Section 2. The critical growth non-compact problems associated to this inequality are initially studied by the work of Adimurthi [1] and de Figueiredo et al. [8]. More recently, authors in [13] [14] [15] studied the existence of multiple positive solutions for quasilinear equations involving exponential nonlinearities. In the case \( 1 < p < N \), the nonlinearity is of polynomial growth and the critical growth is \( p^* \), where \( p^* = Np/(N-p) \) (see [9] [11] [27]). When \( p = 2 \), the problem of type (P1) without the convolution term, that is the equation

\[
-\Delta u - \Delta(u^2)u + V(x)u = f(x,u) \quad \text{in} \ \mathbb{R}^2,
\]

where \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous potential and \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is a continuous function with some suitable assumption and is having critical exponential growth (exp(\( \alpha s^4 \))), was studied by Marcus do Ó et al. in [10]. Later, Wang et al. [38] extended these results to the case of \( p = N \geq 2 \).
Involving critical exponential Choquard type nonlinearity and Laplacian operator, Alves et al. \cite{2} studied the following problem

$$-\varepsilon^2 \Delta u + V(x)u = \left( \int_{\mathbb{R}^2} \frac{F(y, u)}{|x-y|^\mu} \, dy \right) f(x, u), \quad x \in \mathbb{R}^2,$$

where $\varepsilon > 0$, $0 < \mu < 2$, $V : \mathbb{R}^2 \to \mathbb{R}$ is the continuous potential function with some particular properties and the continuous function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ enjoys the critical exponential growth with some appropriate assumptions. Consequently, for the higher dimension, that is for $N \geq 2$, authors in \cite{3} discussed the Kirchhoff-Choquard problems involving the $N$-Laplacian with critical exponential growth.

Inspired from all the above mentioned works, in this article, we investigate the existence results for the problem (\(P_4\)), involving the Choquard type critical nonlinearity motivated by the above inequality (1.2). The main contribution in this work is to identify the first critical level for this problem and study the problem (\(P_4\)), due to the presence of the quasilinear term $\Delta_N(u^2)u$ in the principal operator, the critical growth in the critical dimension is exp$(|t|^{N/(N-1)})$. It turns out that the first critical level reduces by half (see Lemma 3.4). In this sense, the analysis presented here is new even for the equations without the presence of the nonlocal term $\left( \int_{\mathbb{R}^N} F(y, u)|x-y|^{-\mu} \, dy \right)$. Once we identify the first critical level, we prove some technically involved estimates to show the existence of non-trivial weak limit for Palais-Smale sequences at the mountain-pass min-max level. We would like to mention that to the best of our knowledge, there is no work on the existence of positive solutions to the elliptic equations involving quasilinear operator and critical exponential Choquard type nonlinearity. In this article, we study such equations for the first time. Our result is new even for the case $N = 2$.

We now state all the hypotheses imposed on the continuous function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, given by $f(x, s) = g(x, s) \exp \left( \frac{s^{2N}}{\varepsilon^2} \right)$:

- (f1) $g \in C(\overline{\Omega} \times \mathbb{R})$ such that for each $x \in \overline{\Omega}$, $g(x, s) = 0$, for all $s \leq 0$ and $g(x, s) > 0$, for all $s > 0$.
- (f2) $\lim_{s \to 0} \frac{f(x, s)}{s^{2N-1}} = 0$, uniformly in $x \in \Omega$.
- (f3) For any $\varepsilon > 0$, $\lim_{s \to \infty} \sup_{x \in \overline{\Omega}} g(x, s) \exp \left( -\varepsilon s \right) = 0$ and $\lim_{s \to \infty} \inf_{x \in \Omega} g(x, s) \exp \left( \frac{s^{2N}}{\varepsilon^2} \right) = \infty$.
- (f4) There exist positive constants $s_0$ and $m_0$ such that $0 < s_0 m_0 F(x, s) \leq M_0 f(x, s)$ for all $(x, s) \in \mathbb{R}^2 \times [s_0, +\infty)$.
- (f5) There exists $\tau > N$ such that $0 < \tau F(x, s) \leq f(x, s)$, for all $(x, s) \in \Omega \times (0, \infty)$.
- (f6) We assume that

$$\lim_{s \to +\infty} \frac{sf(x, s)F(x, s)}{\exp \left( \frac{2s^{2N}}{\varepsilon^2} \right)} = \infty,$$

uniformly in $x \in \overline{\Omega}$. (1.4)

Example 1.3. Consider $f(x, s) = g(x, s) s^{\frac{2N}{\alpha N - 1}}$, where $g(x, s) = \begin{cases} \mu a_0 \exp(d_0 s^r), & \text{if } s > 0 \\ 0, & \text{if } s \leq 0 \end{cases}$ for some $a_0 > 0$, $0 < d_0 \leq \alpha N$ and $1 \leq r < \frac{2N}{N-1}$. Then $f$ satisfies all the conditions from (f1)-(f6).

Our main result reads as following:

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a smooth bounded domain and let the hypotheses (f1)-(f6) hold. Then the problem (\(P_4\)) has a non trivial positive weak solution.

**Notation.** Throughout this paper, we make use of the following notations:

- If $u$ is a measurable function, we denote the positive and negative parts by $u^+ = \max \{u, 0\}$ and $u^- = \max \{-u, 0\}$, respectively.
If $A$ is a measurable set in $\mathbb{R}^N$, we the Lebesgue measure of $A$ by $|A|$.

- The arrows $\rightharpoonup$, $\to$ denote weak convergence, strong convergence, respectively.
- $B_r(x)$ denotes the ball of radius $r > 0$ centered at $x \in \Omega$.
- $\overline{B_r}(x)$ denotes the closure of the set $B_r(x)$ with respect to $W^{1,N}_0(\Omega)$-norm topology.
- $\overline{S}$ denotes the closure of a set $S \subset \mathbb{R}^N$.
- $A \subset B$ implies $\overline{A}$ is compact in $B$.
- $c, C_1, C_2, \ldots, C_1, C_2, \ldots, C$ and $\tilde{C}$ denote (possibly different from line to line) positive constants.

### 2. Preliminaries and Variational Set-up

For $u : \Omega \to \mathbb{R}$, measurable function, and for $1 \leq p \leq \infty$, we define the Lebesgue space $L^p(\Omega)$ as

$$L^p(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} \mid \int_{\Omega} |u|^p dx < \infty \}$$

equipped with the usual norm denoted by $\|u\|_{L^p(\Omega)}$. Now the Sobolev space $W^{1,p}_0(\Omega)$ is defined as

$$W^{1,p}_0(\Omega) = \{ u \in L^p(\Omega) \mid \int_{\Omega} |\nabla u|^p dx < \infty \}$$

which is endowed with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u|^p dx \right)^{1/p}.$$

We have that the embedding $W^{1,N}_0(\Omega) \ni u \mapsto \exp(|u|^{\beta}) \in L^1(\Omega)$ is compact for all $\beta \in \left[1, \frac{N}{N-1}\right]$ and is continuous for $\beta = \frac{N}{N-1}$. Consequently, the map $\mathcal{M} : W^{1,N}_0(\Omega) \to L^q(\Omega)$, for $q \in [1, \infty]$, defined by $\mathcal{M}(u) := \exp \left( |u|^{\frac{N}{N-1}} \right)$ is continuous with respect to the norm topology.

The natural energy functional associated to the problem $[P_2]$ is the following:

$$I(u) = \frac{1}{N} \int_{\Omega} (1 + 2^{N-1} |u|^N) |\nabla u|^N dx - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{\mathcal{M}(y,u(y)) \mathcal{M}(x,u(x))}{|x-y|^p} dx dy.
$$

Observe that, the functional $I$ is not well defined in $W^{1,N}_0(\Omega)$ due to the fact that $\int_{\Omega} u^N |\nabla u|^N dx$ is not finite for all $u \in W^{1,N}_0(\Omega)$. So, it is difficult to apply variational methods directly in our problem $[P_2]$. In order to overcome this difficulty, we apply the following change of variables which was introduced in [6], namely, $w := h^{-1}(u)$, where $h$ is defined by

\[
\begin{align*}
    h'(s) &= \frac{1}{(1 + 2^{N-1} |h(s)|^N)^{\frac{1}{N}}} \text{ in } [0, \infty), \\
    h(s) &= -h(-s) \text{ in } (-\infty, 0].
\end{align*}
\]

Now we gather some properties of $h$, which we follow throughout in this article. For the detailed proofs of such results, one can see [6][10].

**Lemma 2.1.** The function $h$ satisfies the following properties:

- $(h_1)$ $h$ is uniquely defined, $C^\infty$ and invertible;
- $(h_2)$ $h(0) = 0$;
- $(h_3)$ $0 < h'(s) \leq 1$ for all $s \in \mathbb{R}$;
- $(h_4)$ $\frac{1}{2} h(s) \leq s h'(s) \leq h(s)$ for all $s > 0$;
- $(h_5)$ $|h(s)| \leq |s|$ for all $s \in \mathbb{R}$;
- $(h_6)$ $|h(s)| \leq 2^{1/(2N)} |s|^{1/2}$ for all $s \in \mathbb{R}$;
- $(h_7)$ $\lim_{s \to \pm \infty} h(s)/s^{\frac{3}{2}} = 2^{\frac{1}{2N}}$;
- $(h_8)$ $|h(s)| \geq h(1)|s|$ for $|s| \leq 1$ and $|h(s)| \geq h(1)|s|^{1/2}$ for $|s| \geq 1$;
- $(h_9)$ $h''(s) < 0$ when $s > 0$ and $h''(s) > 0$ when $s < 0$.

**Example 2.2.** One of the example of such functions is given in
After employing the change of variable $w = h^{-1}(u)$, we define the new functional $J : W^{1,N}_0(\Omega) \to \mathbb{R}$ as
\[
J(w) = \frac{1}{N} \int_\Omega |\nabla w|^N dx - \frac{1}{2} \int_\Omega \int_\Omega \frac{F(y, h(w))F(x, h(w))}{|x-y|^\mu} dx dy.
\] (2.2)

From the properties of $f, h$, it can be derived that the functional $J$ is well defined and $J \in C^1$. We observe that if $w \in W^{1,N}_0(\Omega)$ is a critical point of the functional $J$, then for every $v \in W^{1,N}_0(\Omega)$,
\[
\int_\Omega |\nabla w|^{N-2} \nabla w \nabla v dx - \int_\Omega \int_\Omega \frac{F(y, h(w))(y)f(x, h(w))(x)}{|x-y|^\mu} h'(w)v(x) dx dy = 0,
\]
then $w$ is a weak solution (solution, for short) to the following problem:
\[
\begin{aligned}
-\Delta_N w &= \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) f(x, h(w))h'(w) \text{ in } \Omega, \\
& w > 0 \text{ in } \Omega, \\
& w = 0 \text{ on } \partial \Omega.
\end{aligned}
\] (2.3)

It is easy to see that problem (2.3) is equivalent to our problem (2.2), which takes $u = h(w)$ as its solutions. Thus, our main objective is now reduced to proving the existence of the solution of (2.3).

3. Proof of the main theorem

3.1. Mountain pass geometry. We begin this section with the study of mountain pass structure and Palais-Smale sequences corresponding to the energy functional $J : W^{1,N}_0(\Omega) \to \mathbb{R}$ associated to (2.3). From the assumptions, $(f_2)$-$(f_3)$, we obtain that for any $\epsilon > 0$, $r \geq 1$, there exist $\tilde{C}(N, \epsilon)$ and $C(N, \epsilon) > 0$ such that
\[
|f(x, s)| \leq \epsilon|s|^{N-1} + \tilde{C}(N, \epsilon)|s|^r \exp \left( (1 + \epsilon)|s|^{\frac{2N}{2N-r}} \right), \quad \text{for all } (x, s) \in \Omega \times \mathbb{R},
\] (3.1)
\[
|F(x, s)| \leq \epsilon|s|^{N} + C(N, \epsilon)|s|^{r} \exp \left( (1 + \epsilon)|s|^{\frac{2N}{2N-r}} \right), \quad \text{for all } (x, s) \in \Omega \times \mathbb{R}.
\] (3.2)

Thus, for any $u \in W^{1,N}_0(\Omega)$, in light of Sobolev embedding, we have $u \in L^q(\Omega)$ for all $q \in [1, \infty)$, which implies that $F(x, u) \in L^q(\Omega)$ for any $q \geq 1$. (3.3)

Now by (2.1), we have if $w \in W^{1,N}_0(\Omega)$, then $h(w) \in W^{1,N}_0(\Omega)$, which together with Proposition 1.1 implies that
\[
\int_\Omega \left( \int_\Omega \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) dx \leq C(N, \mu) \|F(\cdot, h(w))\|^2_{L^{\frac{2N}{2N-r}}(\Omega)}. \] (3.4)
Lemma 3.1. Assume that the conditions \((f_1)-(f_5)\) hold and let \(h\) be defined as in (2.1). Then there exist \(\rho_* > 0\) and \(R_* > 0\) such that

\[
J(w) \geq R_* > 0 \quad \text{for any } w \in W^{1,N}_0(\Omega) \text{ with } \|w\| = \rho_*. 
\]

Proof. Let \(w \in W^{1,N}_0(\Omega)\). From \((3.4)-(3.2)\), Hölder inequality, Sobolev inequality and Lemma 2.1, we have

\[
\int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^{\mu}} \, dy \right) F(x, h(w)) \, dx 
\]

\[
\leq C(N, \mu) \|\|h(w)\|_{L^{2N/(N-\mu)}} \exp \left( (1+\epsilon) |h(w)|^{2N/N-\mu} \right) \|w\|^{2N/N-\mu} 
\]

\[
\leq C(\mu) \left[ 2^{2N/\mu} \left( \epsilon \int_{\Omega} |h(w)|^{2N/\mu} \, dx \right) + (\epsilon) \frac{2N}{2N-\mu} (1+\epsilon) |h(w)|^{2N/N-\mu} \right] \|w\|^{2N/N-\mu} \]

\[
\leq C(\mu) \frac{2N}{2N-\mu} \left( \epsilon \int_{\Omega} |h(w)|^{2N/\mu} \, dx \right) \|w\|^{2N/N-\mu} 
\]

\[
\leq C(\mu, N, \epsilon) \left[ \|w\|^{2N/\mu} \|w\|^{2N/N-\mu} \right] \left( \epsilon \int_{\Omega} \exp \left( \frac{4N(1+\epsilon)}{2N-\mu} \left( |w| \right)^{2N/\mu} \right) \, dx \right) \frac{2N}{2N-\mu} \]

\[
\leq C_1(N, \mu, \epsilon) \left[ \|w\|^{2N/\mu} \|w\|^{2N/N-\mu} \right] \left( \epsilon \int_{\Omega} \exp \left( \frac{4N(1+\epsilon)}{2N-\mu} \left( |w| \right)^{2N/\mu} \right) \, dx \right) \frac{2N}{2N-\mu} \]

So, for sufficiently small \(\epsilon > 0\) if we choose \(w\) such that \(\|w\|\) is small enough so that

\[
\frac{4N2^{2N/\mu}(1+\epsilon)\|w\|^{2N/N-\mu}}{2N-\mu} < \alpha_N, 
\]

then using Theorem 1.2 in (3.5), we obtain

\[
\int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^{\mu}} \, dy \right) F(x, h(w)) \, dx \leq C_1(N, \mu, \epsilon) \left( \|w\|^{2N/\mu} \right). 
\]

Using (2.2), (3.6), we have

\[
J(w) \geq \frac{1}{N} \|w\|^N - C_1(N, \mu, \epsilon) \left( \|w\|^{2N/\mu} \right). 
\]

Now by taking \(r > 0\) such that \(2r > N\), we can choose \(0 < \rho_* < 1\) sufficiently small so that, we finally obtain \(J(w) \geq R_* > 0\) for all \(w \in W^{1,N}_0(\Omega)\) with \(\|w\| = \rho_*\) and for some \(R_* > 0\) depending on \(\rho_*\). \(\square\)

Lemma 3.2. Assume that the conditions \((f_1)-(f_5)\) hold and let \(h\) be defined as in (2.1). Then there exists \(v_* \in W^{1,N}_0(\Omega)\) with \(\|v_*\| > \rho_*\) such that \(J(v_*) < 0\), where \(\rho_*\) is given as in Lemma 3.1.

Proof. The condition \((f_5)\) implies that there exist some positive constant \(C_1, C_2 > 0\) such that

\[
F(x, s) \geq C_1 s^r - C_2 \quad \text{for all } (x, s) \in \Omega \times [0, \infty). 
\]
Let \( \phi(\geq 0) \in C_c^\infty(\Omega) \) such that \( \| \phi \| = 1 \). Now by Lemma 2.1 \((h_0), (h_8) \) and (3.7), for large \( t \to 1 \), we obtain
\[
\int_\Omega \left( \int_\Omega \frac{F(y, h(\phi))}{|x-y|^\mu} dy \right) F(x, h(\phi)) \, dx \\
\geq \int_\Omega \int_\Omega (C_1(h(\phi(y))) - C_2)(C_1(h(\phi(x))) - C_2) \, dx \, dy \\
= \tau \int_\Omega \int_\Omega \frac{(h(\phi(y)))^\tau}{|x-y|^\mu} \, dx \, dy \\
- 2\tau C_1 C_2 \int_\Omega \int_\Omega \frac{(h(\phi(x)))^\tau}{|x-y|^\mu} \, dx \, dy + C_2^2 \int_\Omega \int_\Omega \frac{1}{|x-y|^\mu} \, dx \, dy \\
\geq \tau C_3^2 (h(1))^{2\tau} \int_\Omega \int_\Omega \frac{(\phi(y))^\tau}{|x-y|^\mu} \, dx \, dy \\
- 2\tau C_1 C_2 2\pi \tau \int_\Omega \int_\Omega \frac{(\phi(x))^\tau}{|x-y|^\mu} \, dx \, dy + C_2^2 \int_\Omega \int_\Omega \frac{1}{|x-y|^\mu} \, dx \, dy.
\]
From the last relation and (2.2), we obtain
\[
J(t\phi) \leq \| t\phi \|^N - \frac{1}{2} \int_\Omega \left( \int_\Omega \frac{F(y, h(\phi))}{|x-y|^\mu} dy \right) F(x, h(\phi)) \, dx \\
\leq C_3 t^N - C_4 t^\tau + C_5 \tau - C_6,
\] (3.8) where \( C_i's \) are positive constants for \( i = 3, 4, 5, 6 \). From (3.8), we infer that \( J(t\phi) \to -\infty \) as \( t \to \infty \), since \( \tau > N \). Thus, there exists \( t_0(>0) \in \mathbb{R} \) so that \( v_* := t_0\phi \in W_{0}^{1,N}(\Omega) \) with \( \| v_* \| > r_* \) such that \( J(v_*) < 0 \). \( \square \)

From the above two lemmas, we get that \( J \) satisfies the mountain pass geometry near 0. Let \( \Gamma = \{ \gamma \in C([0, 1], W_{0}^{1,N}(\Omega)) : \gamma(0) = 0, J(\gamma(1)) < 0 \} \) and define the mountain pass critical level
\[
\beta^* = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)).
\] (3.9)

Then by Lemma 3.1 Lemma 3.2 and the mountain pass theorem we know that there exists a Palais-Smale sequence \( \{w_k\} \subset W_{0}^{1,N}(\Omega) \) for \( J \) at level \( \beta^* \), that is, as \( k \to \infty \)
\[
J(w_k) \to \beta^*; \text{ and } J'(w_k) \to 0 \text{ in } (W_{0}^{1,N}(\Omega))^*,
\]
where \( (W_{0}^{1,N}(\Omega))^* \) denotes the dual of \( W_{0}^{1,N}(\Omega) \). Moreover, Lemma 3.1 guarantees that \( \beta^* > 0 \).

3.2. Analysis of Palais-Smale sequence.

Lemma 3.3. Let \((f_1)\) and \((f_5)\) hold. Let \( h \) be given as in 2.1. Then every Palais-Smale sequence of \( J \) is bounded in \( W_{0}^{1,N}(\Omega) \).

Proof. Let \( \{w_k\} \subset W_{0}^{1,N}(\Omega) \) be a Palais-Smale sequence of \( J \) at level \( c \in \mathbb{R} \), that is, as \( k \to \infty \)
\[
J(w_k) \to c, \text{ } J'(w_k) \to 0 \text{ in } (W_{0}^{1,N}(\Omega))^*.
\]
Then we have
\[
\frac{1}{N} \| w_k \|^N - \frac{1}{2} \int_\Omega \left( \int_\Omega \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx \to c \text{ as } k \to \infty,
\]
\[
\int_\Omega |\nabla w_k|^{N-2} \nabla w_k \nabla \phi dx = \int_\Omega \left( \int_\Omega \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \phi \, dx \leq \epsilon_k \| \phi \|,
\] (3.10)
where \( \epsilon_k \to 0 \) as \( k \to \infty \). In the last relation, taking \( \phi = w_k \) we get
\[
\| w_k \|^N - \int_\Omega \left( \int_\Omega \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \leq \epsilon_k \| w_k \|.
\] (3.11)
Now set
\[
v_k := \frac{h(w_k)}{h'(w_k)}.
\]
Since by Lemma 2.1 \((h_4)\), we have \(|v_k| \leq 2|w_k|\) and by using (2.1) and Lemma 2.1 \((h_5)\), we get
\[
|\nabla v_k| = \left(1 + \frac{2^{N-1}|h(w_k)|^N}{1 + 2^{N-1}|h(w_k)|^N}\right) |\nabla w_k|^N \leq 2|\nabla w_k|,
\]
combining these, we obtain
\[
\|v_k\| \leq 2\|w_k\|.
\]
Therefore, \(v_k \in W_0^{1,N}(\Omega)\). Then by choosing \(\phi = v_k\) and inserting this into (3.10), we deduce
\[
|\langle J'(w_k), v_k \rangle| = \left| \int_\Omega \left(1 + \frac{2^{N-1}|h(w_k)|^N}{1 + 2^{N-1}|h(w_k)|^N}\right) |\nabla w_k|^N dx - \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k))h(w_k)dx \right| \\
\leq \epsilon_k\|v_k\| \leq 2\epsilon_k\|w_k\|,
\]
where \(\langle \cdot, \cdot \rangle\) denotes the dual pairing of between \(W_0^{1,N}(\Omega)\) and its dual \(\left(W_0^{1,N}(\Omega)\right)^*\). Using (3.12) and \(f_5\), we get
\[
C + \epsilon_k\|w_k\| \geq J(w_k) - \frac{1}{2\tau}\langle J'(w_k), v_k \rangle \\
\geq \int_\Omega \left[ \frac{1}{N} - \frac{1}{2\tau} \left(1 + \frac{2^{N-1}|h(w_k)|^N}{1 + 2^{N-1}|h(w_k)|^N}\right) \right] |\nabla w_k|^N dx \\
- \frac{1}{2} \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) \left[F(x,h(w_k)) - \frac{1}{\tau} f(x,h(w_k))h(w_k)\right] dx \\
\geq \int_\Omega \left(\frac{1}{N} - \frac{1}{\tau}\right) |\nabla w_k|^N dx = \left(\frac{1}{N} - \frac{1}{\tau}\right) \|w_k\|^N.
\]
Hence, \(\{w_k\}\) must be bounded in \(W_0^{1,N}(\Omega)\).

Next, we have the following lemma:

**Lemma 3.4.** If \(f\) satisfies \((f_1)-(f_6)\) and \(h\) is defined as in (2.1), then
\[
0 < \beta^* < \frac{1}{2N} \left(\frac{2N-\mu}{2N} \alpha_N\right)^{N-1},
\]
where \(\beta^*\) is given as in (3.9).

**Proof.** By Lemma 3.2 for \(w \in W_0^{1,N}(\Omega)\) with \(\|w\| = 1\), \(J(tw) \to -\infty\) as \(t \to +\infty\). Also, from (3.9), we get
\[
\beta^* \leq \max_{\substack{t \in [0,1], w \in W_0^{1,N}(\Omega)\setminus\{0\}}} J(tw)
\]
with \(J(w) < 0\). So, it is sufficient to prove that there exists some \(v \in W_0^{1,N}(\Omega)\) such that \(\|v\| = 1\) and
\[
\max_{t \in [0,\infty)} J(tv) < \frac{1}{2N} \left(\frac{2N-\mu}{2N} \alpha_N\right)^{N-1}.
\]
To show this, let us consider the sequence of Moser functions \(\{M_k\}\) defined as
\[
M_k(x) = \begin{cases} \frac{1}{\omega_{N-1}} \left(\frac{\log k^{\frac{N-1}{\delta}}}{kn}\right)^{\frac{\delta}{k}} & \text{if } 0 \leq |x| \leq \frac{\delta}{k}, \\
\frac{1}{\omega_{N-1}} \left(\frac{\log k^{\frac{N-1}{\delta}}}{\log k}\right)^{\frac{\delta}{k}} & \text{if } \frac{\delta}{k} \leq |x| \leq \delta \\
0 & \text{if } |x| \geq \delta. \end{cases}
\]
Then it is easy to see that \(\text{supp}(M_k) \subset B_\delta(0)\) and \(\|M_k\| = 1\) for all \(k \in \mathbb{N}\). Now we assert that there exists some \(k \in \mathbb{N}\) such that
\[
\max_{t \in [0,\infty)} J(tM_k) < \frac{1}{2N} \left(\frac{2N-\mu}{2N} \alpha_N\right)^{N-1}. \tag{3.13}
\]
Indeed, if this doesn’t hold then for all $k \in \mathbb{N}$, there exists $t_k > 0$ such that
\[
\max_{t \in [0, \infty)} J(t_\delta M_k) = J(t_k M_k) \geq \frac{1}{2N} \left( \frac{2N - \mu}{2N} \right)^{N-1}. \tag{3.14}
\]
Since $(f_1)$ implies that $F(x, h(t_k M_k)) \geq 0$ for all $k \in \mathbb{N}$, by the definition of $J(t_k w_k)$ together with (3.14), we obtain
\[
t_k^N \geq \frac{1}{2} \left( \frac{2N - \mu}{2N} \right)^{N-1}. \tag{3.15}
\]
Also, in view of (3.14), we have $\frac{d}{dt}(J(t_k M_k)) |_{t=t_k} = 0$. This combining with Lemma 2.1-$(h_4)$ yields that
\[
t_k^N = \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(t_k M_k))}{|x-y|^\mu} dy \right) \left( f(x, h(t_k M_k)) h'(t_k M_k) t_k M_k \right) dx
\geq \frac{1}{2} \int_{B_{\delta/k}(0)} f(x, h(t_k M_k)) h(t_k M_k) \left( \int_{B_{\delta/k}(0)} \frac{F(y, h(t_k M_k))}{|x-y|^\mu} dy \right) dx. \tag{3.16}
\]
Now from (3.14), we know that for each $b > 0$ there exists a constant $R_b$ such that
\[
s f(x,s) F(x,s) \geq b \exp \left( 2 |s|^{\frac{2N}{\alpha N}} \right), \text{ whenever } s \geq R_b.
\]
From (3.15), we infer that $t_k M_k \to \infty$ as $k \to \infty$ in $B_{\delta/k}(0)$. Now by Lemma 2.1-$(h_b)$, we get $h(t_k M_k) \to \infty$ as $k \to \infty$, uniformly in $B_{\delta/k}(0)$. So, we can choose $s_b \in \mathbb{N}$ such that in $B_{\delta/k}(0)$,
\[
h(t_k M_k) \geq s_b, \text{ for all } k \geq s_b.
\]
Furthermore, by Lemma 2.1-$(h_\gamma)$, for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,
\[
|h(t_k M_k)| \geq \left( t_k M_k \right)^{\frac{2N}{\alpha N} - \epsilon} \tag{3.17}
\]
In addition, using the same idea as in $\cite{2}$ (see equation (2.11)), we can have
\[
\int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \frac{dx dy}{|x-y|^\mu} \geq C_{\mu,N} \left( \frac{\delta}{k} \right)^{2N-\mu}, \tag{3.18}
\]
where $C_{\mu,N}$ is a positive constant depending on $\mu$ and $N$. Using these estimates in (3.16), (3.17) and (3.18) for sufficiently large $b$ and for $k \geq k_0$, we get
\[
t_k^N \geq \frac{b}{2} \int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \frac{dx dy}{|x-y|^\mu} \exp \left( 2(h(t_k M_k))^{\frac{2N}{\alpha N}} \right)
\geq \frac{b}{2} \exp \left( \log \left( \frac{2(\frac{2N}{\alpha N} - \epsilon) t_k^{\frac{2N}{\alpha N}}}{\omega_{N-1}^{\frac{2N}{\alpha N}}} \right) \right) \int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \frac{dx dy}{|x-y|^\mu}
\geq \frac{b}{2} \exp \left( \log \left( \frac{2(\frac{2N}{\alpha N} - \epsilon) t_k^{\frac{2N}{\alpha N}}}{\omega_{N-1}^{\frac{2N}{\alpha N}}} \right) \right) C_{\mu,N} \left( \frac{\delta}{k} \right)^{2N-\mu}
\geq \frac{b}{2} \exp \left( \log \left[ \frac{2(\frac{2N}{\alpha N} - \epsilon) t_k^{\frac{2N}{\alpha N}}}{\omega_{N-1}^{\frac{2N}{\alpha N}}} \right] \right) - (2N - \mu) \right) C_{\mu,N} \delta^{2N-\mu}.
\]
Now taking $\epsilon \to 0$ in the above expression and using (3.15), we obtain
\[
t_k^N \geq \frac{b}{2} C_{\mu,N} \delta^{2N-\mu}, \tag{3.19}
\]
uniformly in $k \in \mathbb{N}$. On the other hand, from the proof of Lemma 3.2 it follows that $J(t_\delta M_k) \to -\infty$ as $t \to \infty$ uniformly in $k \in \mathbb{N}$. Using this with (3.14), we get that the sequence $\{t_k\}$ is bounded, which contradicts (3.19), since $b$ is arbitrary. This establishes our claim (3.13) and hence, we conclude the proof. \(\square\)
3.3. Convergence results. Let \( \{w_k\} \subset W^{1,N}_0(\Omega) \) be a Palais-Smale sequence for \( J \). Now Lemma 3.3 yields that \( \{w_k\} \) is bounded in \( W^{1,N}_0(\Omega) \). Thus, there exists \( w \in W^{1,N}_0(\Omega) \) such that up to a subsequence, still denoted by \( \{w_k\} \), \( w_k \rightharpoonup w \) weakly in \( W^{1,N}_0(\Omega) \), \( w_k \to w \) strongly in \( L^q(\Omega) \), \( q \in [1, \infty) \), \( w_k(x) \to w(x) \) point-wise a.e. in \( \Omega \), as \( k \to \infty \). Also, from (3.10) and (3.11), we obtain that for some positive constants \( C', C'' \),

\[
\int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx \leq C',
\]

(3.20)

\[
\int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \leq C''.
\]

(3.21)

Now we have the next two results, where we consider the Palais-Smale sequence \( \{w_k\} \) to be satisfying all the above facts.

Lemma 3.5. Assume that the assumptions \((f_1)-(f_5)\) hold and let \( h \) be defined as in (2.1). Let \( \{w_k\} \subset W^{1,N}_0(\Omega) \) be a Palais-Smale sequence for \( J \). Then we have

\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} \frac{|F(y, h(w_k)) - F(y, h(w))|}{|x-y|^\mu} dy \right) |F(x, h(w_k)) - F(x, h(w))| \, dx = 0.
\]

(3.22)

Proof. We know that if a function \( F \in L^1(\Omega) \) then for any \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that 

\[
\left| \int_{\Omega} F(x) \, dx \right| < \epsilon
\]

for any measurable set \( \Omega' \subset \Omega \) with \( |\Omega'| \leq \delta(\epsilon) \). Also, if \( F \in L^1(\Omega) \) then for any fixed \( \delta_* > 0 \) there exists \( \alpha > 0 \) such that

\[
|[x \in \Omega : |F(x)| \geq \alpha]| \leq \delta_*.
\]

Now (3.20) gives that

\[
\left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(\cdot, h(w_k)) \in L^1(\Omega)
\]

and similarly, (3.4) gives that

\[
\left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(\cdot, h(w)) \in L^1(\Omega).
\]

Thus we fix \( \delta_* > 0 \) and choose \( \alpha > \max \left\{ 1, \frac{2C''M_0}{\delta_*} \right\} \), where \( C'' \) is given in (3.21), such that

\[
\int_{\Omega \cap \{|h(w)| \geq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx \leq \delta_*. \quad (3.23)
\]

Now by using \((f_4), \text{Lemma 2.1}, (h_4)\) and (3.20), we deduce

\[
\int_{\Omega \cap \{|h(w_k)| \geq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]

\[
\leq M_0 \int_{\Omega \cap \{|h(w_k)| \geq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) \, dx
\]

\[
\leq M_0 \int_{\Omega \cap \{|h(w_k)| \geq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) (h(w_k))^{m_0} \, dx
\]

\[
\leq \frac{2M_0}{M_0+1} \alpha^{m_0+1} \int_{\Omega \cap \{|h(w_k)| \geq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx < \delta_*. \quad (3.24)
\]

Combining (3.24) and (3.23), we have

\[
\left| \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx \right| - \int_{\Omega \cap \{|h(w)| \leq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx
\]

\[
\leq 2\delta_* + \int_{\Omega \cap \{|h(w_k)| \leq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]

\[
- \int_{\Omega \cap \{|h(w)| \leq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx.
\]
Next, we show that as \( k \to \infty \)
\[
\int_{\Omega \cap \{h(w_k) \leq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx \to \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx.
\]

Since \( \left( \int_{\Omega} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(\cdot, h(w)) \in L^1(\Omega) \), Fubini’s theorem yields that
\[
\lim_{\Lambda \to \infty} \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx = 0.
\]

Therefore, we can fix \( \Lambda > \max \left\{ \left\{ \frac{2C'' M_0}{\delta_0} \right\}^{\frac{1}{\mu-1}}, s_0 \right\} \), where \( C'' \) is given in (3.21), such that
\[
\int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx \leq \delta_*.
\]

Next, using (3.20), (f4) and Lemma 2.1 (h4), we deduce
\[
\int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]
\[
\leq M_0 \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{f(y, h(w_k))}{(h(w_k))^m |x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]
\[
\leq M_0 \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{f(y, h(w))h(w_k)}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]
\[
\leq \frac{2M_0}{\Lambda^{m_0+1}} \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cup \{h(w) \geq \Lambda\}} \frac{f(y, h(w_k))h(w_k)w_k(y)}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]
\[
\leq \frac{2M_0}{\Lambda^{m_0+1}} \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) f(x, h(w_k))w_k \, dx \leq \delta_*.
\]

Thus, we obtain
\[
\left| \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx
\]
\[
- \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx \right| \leq 2\delta_.*
\]

Now we claim that as \( k \to \infty \), for fixed positive real numbers \( \alpha \) and \( \Lambda \), the following holds:
\[
\left| \int_{\Omega \cap \{h(w_k) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \leq \alpha\}} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k)) \, dx
\]
\[
- \int_{\Omega \cap \{h(w) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w) \leq \alpha\}} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w)) \, dx \right| \to 0. \tag{3.25}
\]

It is easy to compute that
\[
\left( \int_{\Omega \cap \{h(w_k) \leq \Lambda\}} \frac{F(y, h(w_k))}{|x-y|^\mu} dy \right) F(x, h(w_k))\chi_{\Omega \cap \{h(w_k) \leq \alpha\}}
\]
\[
\to \left( \int_{\Omega \cap \{h(w) \leq \Lambda\}} \frac{F(y, h(w))}{|x-y|^\mu} dy \right) F(x, h(w))\chi_{\Omega \cap \{h(w) \leq \alpha\}} \tag{3.26}
\]
point-wise a.e. as \( k \to \infty \). Now taking \( N = r \) in (3.2) and using Lemma 2.1 \((h_5)\) and (1.2), we get a constant \( C_{\alpha, \Lambda} > 0 \) depending on \( \alpha \) and \( \Lambda \) such that

\[
\int_{\Omega \cap \{h(w_k) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w_k) \leq \Lambda\}} \frac{F(y, h(w_k))}{|x - y|^\mu} \, dy \right) F(x, h(w_k)) \, dx \\
\leq C_{\alpha, \Lambda} \int_{\Omega \cap \{h(w_k) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w_k) \leq \Lambda\}} \frac{|h(w_k(y))|^N}{|x - y|^\mu} \, dy \right) |h(w_k(x))|^N \, dx \\
\leq C_{\alpha, \Lambda} \int_{\Omega \cap \{h(w_k) \leq \alpha\}} \left( \int_{\Omega \cap \{h(w_k) \leq \Lambda\}} \frac{|w_k(y)|^N}{|x - y|^\mu} \, dy \right) |w_k(x)|^N \, dx \\
\leq C_{\alpha, \Lambda} C(N, \mu) \|w_k\|_{L^\infty(\Omega)}^{2N} \to C_{\alpha, \Lambda} C(N, \mu) \|w\|_{L^\infty(\Omega)}^{2N} \quad \text{as } k \to \infty,
\]

since \( w_k \to w \) strongly in \( L^q(\Omega) \) for each \( q \in [1, \infty) \). This combining with Theorem 4.9 in \([3]\) implies that there exists \( G \in L^1(\Omega) \) such that up to a subsequence, for each \( k \in \mathbb{N} \), we have

\[
\left| \left( \int_{\Omega \cap \{h(w_k) \leq \Lambda\}} \frac{F(y, h(w_k))}{|x - y|^\mu} \, dy \right) F(x, h(w_k)) \chi_{\Omega \cap \{h(w_k) \leq \alpha\}} \right| \leq |G(x)|.
\]

Therefore, using (3.26) and applying the Lebesgue dominated convergence theorem, we obtain (3.25) and hence, as \( k \to \infty \), we get

\[
\int_\Omega \left( \int_\Omega \frac{F(y, h(w_k))}{|x - y|^\mu} \, dy \right) F(x, h(w_k)) \, dx \to \int_\Omega \left( \int_\Omega \frac{F(y, h(w))}{|x - y|^\mu} \, dy \right) F(x, h(w)) \, dx. \tag{3.27}
\]

Following the similar arguments as discussed above in this proof, we can find as \( \Lambda \to \infty \)

\[
\int_\Omega \int_{\Omega \cap \{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|x - y|^\mu} F(x, h(w)) \, dy \, dx = o(\Lambda), \tag{3.28}
\]

\[
\int_\Omega \int_{\Omega \cap \{h(w_k) \geq \Lambda\}} \frac{F(y, h(w_k))}{|x - y|^\mu} F(x, h(w_k)) \, dy \, dx = o(\Lambda), \tag{3.29}
\]

\[
\int_\Omega \int_{\Omega \cap \{h(w_k) \geq \Lambda\}} \frac{F(y, h(w_k))}{|x - y|^\mu} F(x, h(w)) \, dy \, dx = o(\Lambda), \tag{3.30}
\]

and

\[
\int_\Omega \int_{\Omega \cap \{h(w_k) \geq \Lambda\}} \frac{F(y, h(w_k))}{|x - y|^\mu} F(x, h(w)) \, dy \, dx = o(\Lambda). \tag{3.31}
\]

So,

\[
\int_\Omega \left( \int_\Omega \frac{|F(y, h(w_k)) - F(y, h(w))|}{|x - y|^\mu} \, dy \right) |F(x, h(w_k)) - F(x, h(w))| \, dx \\
\leq 2 \int_\Omega \left( \int_\Omega \frac{\chi_{\{h(w_k) \geq \Lambda\}}(y)F(y, h(w))}{|x - y|^\mu} \, dy \right) F(x, h(w_k)) \, dx \\
+ 4 \int_\Omega \left( \int_\Omega \frac{\chi_{\{h(w_k) > \Lambda\}}(x)F(x, h(w))}{|x - y|^\mu} \, dy \right) \frac{DF(y, h(w_k))}{|x - y|^\mu} \, dy \\
+ 4 \int_\Omega \left( \int_\Omega \frac{\chi_{\{h(w_k) \geq \Lambda\}}(y)F(y, h(w_k))F(x, h(w))}{|x - y|^\mu} \, dy \right) \frac{DF(y, h(w_k))}{|x - y|^\mu} \, dy \\
+ 2 \int_\Omega \left( \int_\Omega \frac{\chi_{\{h(w) \geq \Lambda\}}(y)F(y, h(w))}{|x - y|^\mu} \, dy \right) F(x, h(w)) \, dx \\
+ \int_\Omega \left[ \left( \int_\Omega \frac{F(y, h(w_k))\chi_{\{h(w_k) \leq \Lambda\}}(y) - F(y, h(w))\chi_{\{h(w) \leq \Lambda\}}(y)}{|x - y|^\mu} \, dy \right) \right] \left( \int_\Omega \frac{|F(x, h(w_k))\chi_{\{h(w_k) \leq \Lambda\}}(x) - F(x, h(w))\chi_{\{h(w) \leq \Lambda\}}(x)|}{|x - y|^\mu} \, dy \right) \, dx.
\]

Then employing the Lebesgue dominated convergence theorem we infer that the last integrand tends to 0 as \( k \to \infty \). Hence, making use of (3.28)–(3.31), we finally can conclude (3.22). \( \square \)
Lemma 3.6. Assume that \((f_1)-(f_5)\) hold and \(h\) is defined as in (2.1). Let \(\{w_k\} \subset W^{1,N}_0(\Omega)\) be a Palais-Smale sequence for \(J\). Then for all \(\varphi \in W^{1,N}_0(\Omega)\), we have

\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \varphi \, dx = \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w))}{|x - y|^\mu} dy \right) f(x, h(w)) h'(w) \varphi \, dx.
\]

Proof. Let \(\Omega' \subset \subset \Omega\) and \(\psi \in C_c^\infty(\Omega)\) such that \(0 \leq \psi \leq 1\) and \(\psi \equiv 1\) in \(\Omega'\). One can easily compute that

\[
\left\| \psi \right\|_{1+w_k}^N = \int_{\Omega} \frac{\nabla \psi}{1+w_k} - \psi \nabla w_k (1+w_k)^2 \right\|_{1+w_k}^N dx \leq 2N-1(\|\psi\|^N_2 + \|w_k\|^N), \tag{3.32}
\]

which yields that \(\psi \in W^{1,N}_0(\Omega)\). Now taking \(\varphi = \frac{\psi}{1+w_k}\) as a test function and using Lemma \ref{2.1}(h3) and \ref{3.32}, we obtain

\[
\int_{\Omega'} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \psi \, dx \leq \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \psi \, dx \leq \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) \psi \, dx
\]

\[
= \epsilon_k \left\| \psi \right\|_{1+w_k} + \int_{\Omega} \frac{\nabla w_k}{1+w_k} \nabla \left( \frac{\psi}{1+w_k} \right) dx \leq \epsilon_k 2^{\frac{N-1}{2}}(\|\psi\|_N + \|w_k\|_N) + \int_{\Omega} \frac{\nabla w_k}{1+w_k} \nabla \left( \frac{\psi}{1+w_k} \right) dx \leq \epsilon_k 2^{\frac{N-1}{2}}(\|\psi\|_N + \|w_k\|_N) + \int_{\Omega} \frac{\nabla w_k}{1+w_k} \nabla \left( \frac{\psi}{1+w_k} \right) dx
\]

\[
\leq \epsilon_k 2^{\frac{N-1}{2}}(\|\psi\|_N + \|w_k\|_N) + \|\psi\|_N \|w_k\|_N \|w_k\|_N \leq C_1,
\]

where \(C_1\) is a positive constant and in the last line we used the fact that \(\{w_k\}\) is bounded in \(W^{1,N}_0(\Omega)\). Again, using the boundedness of the sequence \(\{w_k\}\), from \ref{3.11}, we get

\[
\int_{\Omega'} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \leq \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \leq \epsilon_k \|w_k\|_N \|w_k\|_N \leq C_2 \tag{3.34}
\]

for some constant \(C_2 > 0\). Combining \ref{3.33} and \ref{3.34}, we deduce

\[
\int_{\Omega'} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \, dx \leq 2 \int_{\Omega' \cap \{w_k < 1\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \, dx + \int_{\Omega' \cap \{w_k \geq 1\}} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) w_k f(x, h(w_k)) h'(w_k) \, dx \leq 2 \int_{\Omega'} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \, dx + \int_{\Omega'} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) w_k f(x, h(w_k)) h'(w_k) \, dx \leq 2C_1 + C_2 \leq C_3.
\]

Thus, the sequence \(\{v_k\} := \left\{ \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \right\}\) is bounded in \(L_{loc}^1(\Omega)\). Therefore, there exists a Radon measure \(\zeta\) such that, up to a subsequence, \(v_k \to \zeta\) in the weak*-topology as \(k \to \infty\). Hence, we have

\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} dy \right) f(x, h(w_k)) h'(w_k) \eta \, dx = \int_{\Omega} \eta d\zeta, \forall \eta \in C_c^\infty(\Omega).
\]
Since \( w_k \) satisfies (3.10), we achieve
\[
\int_A \eta d\zeta = \lim_{k \to \infty} \int_A |\nabla w_k|^{N-2}\nabla w_k \nabla \eta \, dx, \quad \forall A \subset \Omega,
\]
which together with Lemma 4.1 yields that the Radon measure \( \zeta \) is absolutely continuous with respect to the Lebesgue measure. So, there exists a function \( \varrho \in L^1_{\text{loc}}(\Omega) \) such that for any \( \eta \in C^\infty_c(\Omega) \), it holds that \( \int_A \eta d\zeta = \int_A \eta \varrho \, dx \), thanks to Radon-Nikodym theorem. Therefore, we obtain
\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{|x-y|^\mu} F(y, h(w_k)) \, dy \right) f(x, h(w_k)) h'(w_k) \eta(x) \, dx
\]
\[
= \int_{\Omega} \eta \varrho \, dx = \int_{\Omega} \left( \int_{|x-y|^\mu} \frac{F(y, h(w))}{|x-y|^\mu} \, dy \right) f(x, h(w)) h'(w) \eta(x) \, dx, \quad \forall \eta \in C^\infty_c(\Omega),
\]
since \( C^\infty_c(\Omega) \) is dense in \( W^{1,N}_0(\Omega) \), this completes the proof. \( \square \)

**Proof of Theorem 1.4** Let \( \{w_k\} \) be a Palais-Smale sequence at the level \( \beta^* \). Then \( \{w_k\} \) can be obtained as a minimizing sequence associated to the variational problem (3.9). Then by Lemma 3.3 there exists \( w \in W^{1,N}_0(\Omega) \) such that, up to a subsequence, \( w_k \rightharpoonup w \) weakly in \( W^{1,N}_0(\Omega) \) as \( k \to \infty \).

Now by using Lemma 4.1 and Lemma 3.6 we infer that \( w \) forms a weak solution of (2.3). We claim that \( w \neq 0 \). Indeed, if not, that is, if \( w \equiv 0 \) then using (3.27), we have
\[
\int_{\Omega} \left( \int_{|x-y|^\mu} F(y, h(w_k)) \, dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \to 0 \quad \text{as} \quad k \to \infty,
\]
which yields that \( \lim_{k \to \infty} J(w_k) = \frac{1}{N} \lim_{k \to \infty} \|w_k\|^N = \beta^* N \). That is,
\[
\lim_{k \to \infty} \|w_k\|^N = \beta^* N.
\]

Therefore, recalling Lemma 3.3 there exists a real number \( l > 0 \), very close to 1, and there exists \( k_0 \in \mathbb{N} \) depending on \( l \) such that
\[
\|w_k\|^N < \frac{1}{2 \pi^N} \frac{2N - \mu}{2N} \alpha_N (1 - l), \quad \text{for all} \quad k \geq k_0.
\]

Now we show that
\[
\int_{\Omega} \left( \int_{|x-y|^\mu} F(y, h(w_k)) \, dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \to 0 \quad \text{as} \quad k \to \infty.
\]

Now using \((f_3)\), Proposition 1.1 (3.1), Lemma 2.1 (h4)-(h6), Hölder’s inequality, we deduce
\[
\int_{\Omega} \left( \int_{|x-y|^\mu} F(y, h(w_k)) \, dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx
\]
\[
\leq \frac{1}{r} \int_{\Omega} \left( \int_{|x-y|^\mu} \frac{f(y, h(w_k)) h'(w_k) w_k}{|x-y|^\mu} \, dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx
\]
\[
\leq \frac{1}{r} \mathcal{C}(N, \mu) \left( \int_{\Omega} \left[ f(x, h(w_k)) h(w_k) \right]^{\frac{2N}{2N - \mu}} \, dx \right)^{- \frac{2N - \mu}{2N}}
\]
\[
\leq \mathcal{C}_1(N, \mu, \epsilon) \left( \int_{\Omega} \left[ h(w_k)^N + |h(w_k)|^r \exp((1 + \epsilon) |h(w_k)|^{\frac{2N}{2r - 1}}) \right] \frac{2N}{2N - \mu} \, dx \right)^{- \frac{2N - \mu}{2N}}
\]
\[
\leq \mathcal{C}_2(N, \mu, \epsilon) \left[ \left\| w_k \right\|_{L^2(\Omega)}^{2N \frac{2N - \mu}{2N - 2r + 2}} + \left\| w_k \right\|_{L^{2N/(2N - 2r + 2)}(\Omega)}^{2r} \left( \int_{\Omega} \exp \left( q(1 + \epsilon) \frac{2N}{2N - \mu} |h(w_k)|^{\frac{2N}{2r - 1}} \, dx \right) \right)^{\frac{2N - \mu}{2N}} \right]
\]
\[
\leq \mathcal{C}_3(N, \mu, \epsilon) \left[ \left\| w_k \right\|_{L^2(\Omega)}^{2N \frac{2N - \mu}{2N - 2r + 2}} \left( \int_{\Omega} \exp \left( q(1 + \epsilon) \frac{2N}{2N - \mu} \left\| w_k \right\|_{L^\infty(\Omega)}^{\frac{2N}{2r - 1}} \, dx \right) \right)^{\frac{2N - \mu}{2N}} \right],
\]
\[(3.37)\]
where \( q' = \frac{q}{q-1} \) is the conjugate of the exponent \( q > 1 \) and \( C_i' \)'s, \( i = 1, 2, 3 \) are some positive constants. Now we can choose \( q > 1 \) very close to \( 1 \) and \( \epsilon > 0 \) sufficiently small such that, for sufficiently large \( k \in \mathbb{N} \), from (3.35), it follows that
\[
q(1 + \epsilon) \frac{2N}{2N - \mu} \frac{2N}{N - \mu} \| w_k \| \frac{2N}{N - \mu} \frac{q}{q-1} < q(1 + \epsilon)(1 - \alpha_k) < \alpha_N.
\]
Therefore, in the light of Theorem 1.2 for sufficiently large \( k \in \mathbb{N} \), it follows that
\[
\int_{\Omega} \exp \left( q(1 + \epsilon) \frac{2N}{2N - \mu} \frac{2N}{N - \mu} \| w_k \| \frac{2N}{N - \mu} \frac{q}{q-1} \right) < C.
\]
Using this combining with the fact that \( w_k \to 0 \) strongly in \( L^q(\Omega) \) for \( 1 \leq q < \infty \), as \( k \to \infty \), from (3.37), we finally obtain (3.36). Since, \( \{ w_k \} \) is a Palais-Smale sequence for \( J \), we have \( \lim_{k \to \infty} (J'(w_k), w_k) = 0 \) which together with (3.36) gives that \( \lim_{k \to \infty} \| w_k \| \to = 0 \). Thus, from Lemma 3.5 it follows that \( \lim_{k \to \infty} J(w_k) = 0 = \beta^* \) which is a contradiction to the fact that \( \beta^* > 0 \). Hence, \( w \neq 0 \).

Next, we prove that \( w > 0 \) in \( \Omega \). Since, \( w \) is a weak solution to (2.3),
\[
\int_{\Omega} | \nabla w |^{N-2} \nabla w \nabla \varphi \, dx = \int_{\Omega} \left( \int_{\Omega} F(y, h(w)) \frac{dy}{| x - y |^{\mu}} \right) f(x, h(w)) h'(w) \varphi \, dx,
\]
for all \( \varphi \in W_0^{1,N}(\Omega) \). In particular, taking \( \varphi = w^- \) in the above equation, we obtain \( \| w^- \| \to = 0 \) which implies that \( w^- \to = 0 \) a.e. in \( \Omega \). Therefore, \( w \geq 0 \) a.e. in \( \Omega \).

Now from Theorem 1.2 we have \( f(\cdot, h(w)) \in L^q(\Omega) \), for \( 1 \leq q < \infty \). Also by (3.3), \( F(\cdot, h(w)) \in L^q(\Omega) \), for any \( q \in [1, \infty) \). Since \( \mu \in (0, N) \) and \( \Omega \) is bounded, it follows that \( y \to | x - y |^{-\mu} \in L^q(\Omega) \) for all \( q \in (1, \frac{N}{\mu}) \) uniformly in \( x \in \Omega \). Thus, from Hölder’s inequality, we can deduce
\[
\int_{\Omega} F(y, h(w)) \frac{dy}{| x - y |^{\mu}} \in L^\omega(\Omega).
\]
Therefore, recalling \( h'(s) \leq 1 \), we get
\[
\left( \int_{\Omega} F(y, h(w)) \frac{dy}{| x - y |^{\mu}} \right) f(x, h(w)) h'(w) \in L^q(\Omega),
\]
for \( 1 \leq q < \infty \). Now by employing the regularity results for the elliptic equations, we infer that \( w \in L^\omega(\Omega) \) and \( w \in C^1(\overline{\Omega}) \) for some \( \gamma \in (0, 1) \). Finally, from the strong maximum principle, we draw the conclusion that \( w > 0 \) in \( \Omega \). This completes the proof of Theorem 1.4.

4. Appendix A

In this section, we give the proof of the following almost everywhere convergence of gradients of Palais-Smale sequences using the concentration compactness arguments.

**Lemma 4.1.** Suppose the assumptions \((f_1)-(f_6)\) hold and \( h \) is defined as in (2.1). Let \( \{ w_k \} \subset W_0^{1,N}(\Omega) \) be a Palais-Smale sequence for \( J \). Then \( \nabla w_k \to \nabla w \) a.e. in \( \Omega \). Moreover, we have, as \( k \to \infty \)
\[
| \nabla w_k |^{N-2} \nabla w_k \to | \nabla w |^{N-2} \nabla w \text{ weakly in } (L^{\frac{N}{N-\gamma}}(\Omega))^N.
\]

**Proof.** Since Lemma 3.3 yields that \( \{ w_k \} \) is bounded in \( W_0^{1,N}(\Omega) \), there exists \( w \in W_0^{1,N}(\Omega) \) such that, in the sense of subsequence, we have \( w_k \to w \) weakly in \( W_0^{1,N}(\Omega) \), \( w_k \to w \) strongly in \( L^q(\Omega) \), \( q \in [1, \infty) \), \( w_k(x) \to w(x) \) point-wise a.e. in \( \Omega \), as \( k \to \infty \). From the properties of the sequence \( \{ w_k \} \), it is evident that the sequence \( \{ | \nabla w_k |^{N-2} \nabla w_k \} \) must be bounded in \( (L^{\frac{N}{N-\gamma}}(\Omega))^N \), which implies that there exists \( u \in (L^{\frac{N}{N-\gamma}}(\Omega))^N \) such that,
\[
| \nabla w_k |^{N-2} \nabla w_k \to u \text{ weakly in } (L^{\frac{N}{N-\gamma}}(\Omega))^N \text{ as } k \to \infty.
\]

Also we have, \( \{ | \nabla w_k | \} \) is bounded in \( L^1(\Omega) \), which yields that there exists a non-negative radon measure \( \sigma \) such that, up to a subsequence, we have
\[
| \nabla w_k | \to \sigma \text{ in } (C(\overline{\Omega}))^* \text{ as } k \to \infty.
\]

Our aim is to show \( u = | \nabla w |^{N-2} \nabla w \). For that, first we take \( \nu > 0 \) and set \( X_\nu := \{ x \in \Omega : \sigma(B_l(x) \cap \Omega) \geq \nu \text{ for all } l > 0 \} \).
Claim 1: \( X_\nu \) is a finite set.

Indeed, if not, then there exists a sequence of distinct points \( \{ z_k \} \) in \( X_\nu \) such that \( \sigma(B_l(z_k) \cap \overline{\Omega}) \geq \nu \) for all \( l > 0 \) and for all \( k \in \mathbb{N} \). This gives that \( \sigma(\{ z_k \}) \geq \nu \) for all \( k \). Therefore, \( \sigma(X_\nu) = +\infty \). But this is a contradiction to the fact that

\[
\sigma(X_\nu) = \lim_{k \to \infty} \int_{X_\nu} |\nabla w_k|^N \, dx \leq C.
\]

Hence, the claim holds. Thus, we can take \( X_\nu = \{ z_1, z_2, \ldots, z_n \} \).

Claim 2: We can choose \( \nu > 0 \), such that \( \nu \frac{1}{2N} < \frac{1}{2N - \mu} \alpha_N \) and we have

\[
\lim_{k \to \infty} \int_{S} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} \, dy \right) f(x, h(w_k))h'(w_k)w_k \, dx
\]

\[
= \int_{S} \left( \int_{\Omega} \frac{F(y, h(w))}{|x - y|^\mu} \, dy \right) f(x, h(w))h'(w)w \, dx,
\]

(4.2)

where \( S \) is any compact subset of \( \overline{\Omega} \setminus X_\nu \).

Let \( z_0 \in S \) and \( l_0 > 0 \) be such that \( \sigma(B_{l_0}(z_0) \cap \overline{\Omega}) < \nu \) that is \( z_0 \notin X_\nu \). Also, we consider \( \phi \in C_c^\infty(\Omega) \) satisfying \( 0 \leq \phi(x) \leq 1 \) for \( x \in \Omega \), \( \phi \equiv 1 \) in \( B_{l_0}(z) \cap \overline{\Omega} \) and \( \phi \equiv 0 \) in \( \Omega \setminus (B_{l_0}(z) \cap \overline{\Omega}) \). Then

\[
\lim_{k \to \infty} \int_{B_{l_0}(z_0) \cap \Omega} |\nabla w_k|^N \, dx \leq \lim_{k \to \infty} \int_{B_{l_0}(z_0) \cap \Omega} |\nabla w_k|^N \phi dx \leq \sigma(B_{l_0}(z_0) \cap \overline{\Omega}) < \nu.
\]

Hence, for sufficiently large \( k \in \mathbb{N} \) and sufficiently \( \epsilon > 0 \) small, it follows that

\[
\int_{B_{l_0}(z_0) \cap \Omega} |\nabla w_k|^N \leq \nu(1 - \epsilon).
\]

(4.3)

Now we estimate the following using \((f_3), (4.3)\) and Lemma 2.1-(h6):

\[
\int_{B_{l_0}(z_0) \cap \Omega} |f(x, h(w_k))|^q \, dx = \int_{B_{l_0}(z_0) \cap \Omega} |g(x, h(w_k))|^q \exp \left( qh(w_k) \frac{2^N}{2N} \right) \, dx
\]

\[
\leq C \int_{B_{l_0}(z_0) \cap \Omega} \exp \left( 1 + \epsilon \right) 2^{\frac{N}{2}} q |h(w_k)|^{\frac{2^N}{2N}} \, dx
\]

\[
\leq C \int_{B_{l_0}(z_0) \cap \Omega} \exp \left( 1 + \epsilon \right) 2^{\frac{N}{2}} q |w_k|^N \, dx
\]

\[
\leq C \int_{B_{l_0}(z_0) \cap \Omega} \exp \left( 1 + \epsilon \right) q \nu \frac{1}{\nu} \left( 1 - \epsilon \right) \frac{1}{\nu} \left( \frac{|w_k|^N}{\int_{B_{l_0}(z_0) \cap \Omega} |\nabla w_k|^N \, dx} \right) \, dx.
\]

Thus, we can choose \( q > 1 \) and \( \epsilon > 0 \) such that \( (1 + \epsilon) 2^{\frac{N}{2}} q \nu \frac{1}{\nu} < \alpha_N \) and then using Theorem 1.2 in the last relation, we obtain

\[
\int_{B_{l_0}(z_0) \cap \Omega} |f(x, h(w_k))|^q \, dx \leq C
\]

(4.4)

for some constant \( C > 0 \), independent of \( k \). Next, we consider

\[
\int_{B_{l_0}(z_0) \cap \Omega} \left( \int_{\Omega} \frac{F(y, h(w_k))}{|x - y|^\mu} \, dy \right) f(x, h(w_k))h'(w_k)w_k - \left( \int_{\Omega} \frac{F(y, h(w))}{|x - y|^\mu} \, dy \right) f(x, h(w))h'(w)w \, dx
\]

\[
\leq \int_{B_{l_0}(z_0) \cap \Omega} \left( \int_{\Omega} \frac{F(y, h(w))}{|x - y|^\mu} \, dy \right) (f(x, h(w_k))h'(w_k)w_k - f(x, h(w_k))h'(w)w) \, dx
\]

\[
+ \int_{B_{l_0}(z_0) \cap \Omega} \left( \int_{\Omega} \frac{F(y, h(w)) - F(y, h(w_k))}{|x - y|^\mu} \, dy \right) f(x, h(w_k))h'(w_k)w_k \, dx
\]

\[
:= J_1 + J_2.
\]

(4.5)
From the asymptotic growth assumptions on $f$, we obtain
\[ \lim_{s \to \infty} \frac{f(x,s)}{(f(x,s))^r} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } r > 1. \] (4.6)

Using (3.38), we get
\[ J_1 \leq C \int_{B_{\frac{ho}{2}}(z_0) \cap \Omega} |f(x,h(w_k))h'(w_k)w_k - f(x,h(w))h'(w)w| \, dx, \]
where $C > 0$ is a constant, independent on $k$. Moreover, (4.4) and (4.6) together with Lemma 2.1 (i) imply that \{ $f(x,h(w_k))h'(w_k)w_k$ \} is an equi-integrable family of functions over $B_{\frac{ho}{2}}(z_0) \cap \Omega$. Also, by the continuity of the functions $f$ and $h$, we have $f(x,h(w_k))h'(w_k)w_k \to f(x,h(w))h'(w)w$ a.e. in $\Omega$ as $k \to \infty$. Hence, by applying Vitali’s convergence theorem, we obtain
\[ J_1 \to 0 \quad \text{as} \quad k \to \infty. \]

Next, we show that $J_2 \to 0$ as $k \to \infty$.

For that, first we recall the semigroup property of the Riesz potential and get the following estimate:
\[ J_2 = \int_{\Omega} \left( \int_{\Omega} \frac{F(y,h(w_k)) - F(y,h(w))}{|x-y|^\mu} \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(x)f(x,h(w_k))h'(w_k)w_k \, dy \right) \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(x) f(x,h(w_k))h'(w_k)w_k \, dx \]
\[ \leq C \left( \int_{\Omega} \left( \int_{\Omega} \frac{|F(y,h(w_k)) - F(y,h(w))|}{|x-y|^\mu} \right) \ |F(x,h(w_k)) - F(x,h(w))| \, dx \right)^{1/2} \]
\[ \times \left( \int_{\Omega} \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(x) f(x,h(w_k))h'(w_k)w_k \, dx \right)^{1/2}, \]
(4.7)
where $C > 0$ is some constant, independent of $k$. Combining (4.4), (4.6), Lemma 2.1 (i), Proposition 1.1 and $\nu \frac{1}{|x-y|^\mu} < \frac{1}{2N - \mu} \omega_N$, it follows that
\[ \left( \int_{\Omega} \left( \int_{\Omega} \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(y) \frac{f(y,h(w_k))h'(w_k)w_k}{|x-y|^\mu} \, dy \right) \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(x) f(x,h(w_k))h'(w_k)w_k \, dx \right)^{1/2} \]
\[ \leq \left( \int_{\Omega} \left( \int_{\Omega} \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(y) \frac{|f(y,h(w_k))h(w_k)|}{|x-y|^\mu} \, dy \right) \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega}(x) f(x,h(w_k))h(w_k) \, dx \right)^{1/2} \]
\[ \leq \| \chi_{B_{\frac{ho}{2}}(z_0) \cap \Omega} f(\cdot,h(w_k))h(w_k) \|_{L^{\frac{2N}{N-\mu}}(\Omega)} \leq C. \]
This together with (4.7) and Lemma 3.5 gives that
\[ J_2 \to 0 \quad \text{as} \quad k \to \infty. \]

Hence, from (4.5), we can infer that
\[ \lim_{k \to \infty} \int_{B_{\frac{ho}{2}}(z_0) \cap \Omega} \left| \left( \int_{\Omega} \frac{F(y,h(w_k))}{|x-y|^\mu} \, dy \right) f(x,h(w_k))h'(w_k)w_k \right| \, dx = 0. \]
Since $S$ is compact, we can repeat this procedure over a finite covering of balls and hence, (4.2) is achieved. Finally, the proof of (4.1) can be concluded by the similar standard arguments as in the proof of Lemma 4 in [12].

\[ \square \]

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