Husimi distribution function and one-dimensional Ising model

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Abstract

Husimi distribution function for the one-dimensional Ising model is obtained. One-point and joint distribution functions are calculated and their thermal behaviour are discussed.

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1 Introduction

Husimi distribution function [1], is a positive definite distribution over the parameter space of a definite set of coherent states [2]. Positive definitiness of the Husimi distribution function, makes it as a good candidate for investigating quantum features using a classical like distribution over the phase space. In this case, to each coherent state |z⟩ of the system there corresponds a unique distribution function \( \mu(z) = \frac{1}{Z} |z e^{-\beta \hat{H}} z\rangle \), where \( Z \) is the partition function of the system. \( \hat{H} \) and \( \beta \) are Hamiltonian and inverse temperature parameter respectively. In this paper, using the single-fermion or spin-\( \frac{1}{2} \) coherent states, the coherent states of a one-dimensional spin chain are obtained. The Husimi distribution function is then obtained for the one-dimensional Ising model as the prototype of some important models in statistical mechanics. One-point and joint distribution functions or correlation functions are calculated and their thermal behaviour is discussed.

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2 One-dimensional Ising model

Hamiltonian of the one-dimensional Ising model in a homogeneous magnetic field \( B \) along the \( z \)-axis is

\[
\hat{H} = -J \sum_{k=1}^{N} \hat{S}_{k,z} \hat{S}_{k+1,z} + B \sum_{k=1}^{N} \hat{S}_{k,z},
\]

(1)

where periodic boundary condition is assumed, i.e., \( N + 1 \equiv 1 \). The Hilbert space \( \mathcal{H} \) of this model is a \( 2^N \)-dimensional vector space spanned by the following tensor products as basis vectors

\[
|\vec{i}\rangle = |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_N\rangle = \prod_{k=1}^{N} \otimes |i_k\rangle,
\]

(2)

where \( |i_k\rangle \in \{|+, -\rangle\} \), for \( k = 1...N \). The basis vectors \( |\vec{i}\rangle \), are Hamiltonian eigenvectors with respective eigenvalues \( E_{\vec{i}} \),

\[
\hat{H}|\vec{i}\rangle = (-J \sum_{k=1}^{N} i_k i_{k+1} + B \sum_{k=1}^{N} i_k)|\vec{i}\rangle,
\]

\[
E_{\vec{i}} = -J \sum_{k=1}^{N} i_k i_{k+1} + B \sum_{k=1}^{N} i_k,
\]

(3)

3 Ising model coherent states

Coherent states of the one-dimensional Ising model can be defined from the tensor product of the single-fermion coherent states \([2]\). A single-fermion coherent state for the spin half \( (s = \frac{1}{2}) \) representation can be written like this

\[
|z\rangle = \sin(\frac{\theta}{2}) \exp(-i\varphi)|+\rangle + \cos(\frac{\theta}{2})|\rangle - \rangle,
\]

(4)

where \( z = (\frac{\theta}{2})e^{\exp(-i\varphi)}, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \) and \( |+\rangle, |\rangle - \rangle \) are \( \hat{S}_z \) eigenvectors. The completeness of the fermion coherent states is

\[
\frac{1}{2\pi} \int d\Omega |z\rangle \langle z| = 1,
\]

(5)

where \( d\Omega = \sin \theta d\theta d\varphi \). Tensor product of these fermion coherent states makes Ising model coherent states

\[
|\vec{z}\rangle = \prod_{k=1}^{N} \otimes (\sin(\frac{\theta_k}{2}) \exp(-i\varphi_k)|+\rangle + \cos(\frac{\theta_k}{2})|\rangle - \rangle),
\]

(6)
the scalar product between a coherent state $|\vec{z}\rangle$ and an eigenvector $|\vec{i}\rangle$, of the Hamiltonian $\hat{H}$, is

$$
\langle \vec{z} | \vec{i} \rangle = \prod_{k=1}^{N} \left( \sin(\theta_k/2) \exp(-i \varphi_k) \delta_{i_k,1} + \cos(\theta_k/2) \delta_{i_k,-1} \right),
$$

so

$$
|\langle \vec{z} | \vec{i} \rangle|^2 = \prod_{k=1}^{N} \left( \sin^2(\theta_k/2) \delta_{i_k,1} + \cos^2(\theta_k/2) \delta_{i_k,-1} \right).
$$

Partition function of the Hamiltonian (1) is defined as

$$
Z = \sum_{\vec{i}} \langle \vec{i} | \exp(-\beta \hat{H}) | \vec{i} \rangle = tr(\hat{T}^N),
$$

where

$$
\hat{T} = \begin{pmatrix} \exp(\beta(J-B)) & \exp(-\beta J) \\ \exp(-\beta J) & \exp(\beta(J+B)) \end{pmatrix},
$$

is the transition matrix.

## 4 Husimi distribution function

Husimi distribution function is a normalized positive definite distribution in the parameter space of coherent states and is defined as follows

$$
\mu(\vec{z}) = \frac{1}{Z} \langle \vec{z} | \exp(-\beta \hat{H}) | \vec{z} \rangle,
$$

$$
= \frac{1}{Z} \sum_{\vec{i}} \langle \vec{z} | \exp(-\beta \hat{H}) | \vec{i} \rangle |i\rangle \langle \vec{i} | \vec{z} \rangle,
$$

$$
= \frac{1}{Z} \sum_{\vec{i}} e^{\beta J \sum_{k=1}^{N} i_k i_{k+1} - \beta B \sum_{k=1}^{N} i_k} |\langle \vec{z} | \vec{i} \rangle|^2,
$$

substituting (8) in (11) and doing some routine calculations, we get the following relation for Husimi distribution

$$
\mu(\vec{z}) = \frac{1}{2^N} \{ 1 - \sum_{m=1}^{N} \langle \hat{S}_{m,z} \rangle u_m + \sum_{m<n} \langle \hat{S}_{m,z} \hat{S}_{n,z} \rangle u_m u_n + \cdots \\
+ (-1)^N \langle \hat{S}_{1,z} \hat{S}_{2,z} \cdots \hat{S}_{N,z} \rangle u_1 u_2 \cdots u_n \},
$$

(12)
where for simplicity $u_k = \cos \theta_k$ is assumed and a general $n$-point correlation function $\langle \hat{S}_{1,z} \hat{S}_{2,z} \cdots \hat{S}_{r,z} \rangle$ is defined as

$$\langle \hat{S}_{1,z} \hat{S}_{2,z} \cdots \hat{S}_{r,z} \rangle = \frac{1}{Z} \sum_i \prod_{k=1}^r i_k e^{\beta J} \sum_{r_k} i_k i_{k+1} - \beta B \sum_{r_k} i_k,$$

where $\hat{S}_{k,z} |i_k \rangle = i_k |i_k \rangle$, from (12) it is clear that the Husimi distribution function is independent of parameters $\varphi_k$ and contains the information about all the $n$-point correlation functions, i.e., it is the generator of the $n$-point functions of the Ising model.

$$\langle \hat{S}_{1,z} \hat{S}_{2,z} \cdots \hat{S}_{r,z} \rangle = (-3)^m \int \cdots \int \mu(\vec{z}) u_{r_1} u_{r_2} \cdots u_{r_m} \prod_{k=1}^N du_k,$$

for calculating $\text{tr}[\hat{T} \sigma_z]$ and $\text{tr}[\hat{T}^{N-n+m} \sigma_z \hat{T}^{m-n} \sigma_z]$ we can diagonalize the matrix $\hat{T}$, and write the matrix $\sigma_z$ in the basis of eigenvectors of $\hat{T}$, the eigenvalues of $\hat{T}$ are

$$\lambda_{\pm} = \exp(\beta J) \cosh(\beta B) \pm \sqrt{\exp(2\beta J) \cosh^2(\beta B) - 2 \sinh(\beta J)},$$

and the transformation matrix between the standard basis $\{|+,|-\}$ and the eigenvectors of $\hat{T}$ is

$$U = \begin{pmatrix} \sin(\omega) & -\cos(\omega) \\ \cos(\omega) & \sin(\omega) \end{pmatrix},$$
where \( |\lambda_+\rangle = \begin{pmatrix} \sin(\omega) \\ \cos(\omega) \end{pmatrix} \) and \( |\lambda_-\rangle = \begin{pmatrix} -\cos(\omega) \\ \sin(\omega) \end{pmatrix} \) are normalized eigenvectors of \( \hat{T} \) belonging to eigenvalues \( \lambda_+ \) and \( \lambda_- \) respectively and

\[
\tan(\omega) = \frac{\exp(-\beta J)}{\exp(\beta J) \sinh(\beta B) + \sqrt{\exp(2\beta J) \sinh^2(\beta B) + \exp(-2\beta J)}}. \tag{20}
\]

Now in the basis \( |\lambda_+\rangle, |\lambda_-\rangle \), we have

\[
\text{tr}[\hat{T}\sigma_z] = \text{tr}\left( \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} \begin{pmatrix} -\cos(2\omega) & -\sin(2\omega) \\ -\sin(2\omega) & \cos(2\omega) \end{pmatrix} \right),
= \text{tr}\left[ \begin{pmatrix} -\lambda_-^N \cos(2\omega) & -\lambda_-^N \sin(2\omega) \\ -\lambda_-^N \sin(2\omega) & \lambda_-^N \cos(2\omega) \end{pmatrix} \right],
= (\lambda_-^N - \lambda_-^N) \cos(2\omega), \tag{21}
\]

and similarly

\[
\text{tr}[\hat{T}^{N-j+i}\sigma_z\hat{T}^{j-i}\sigma_z] = \lambda_+^N \cos^2(2\omega) + \lambda_+^{N-j+i}\lambda_-^{j-i} \sin^2(2\omega) + \lambda_-^{N-j+i}\lambda_-^{j-i} \sin^2(2\omega) + \lambda_-^N \cos^2(2\omega), \tag{22}
\]

substituting (21) and (22) in (15) and (17) respectively, we obtain

\[
\mu(u_k) = \frac{1}{2}(1 - \cos(2\omega) |\lambda_-^N - \lambda_+^N| u_k), \tag{23}
\]

\[
\mu(u_i, u_j) = \frac{1}{4}(1 - (u_i + u_j) \cos(2\omega) |\lambda_-^N - \lambda_+^N| u_k + u_i u_j \cos^2(2\omega) + u_i u_j \sin^2(2\omega) + \frac{\lambda_-^{N-j+i}\lambda_-^{j-i} + \lambda_-^{N-j+i}\lambda_-^{j-i}}{\lambda_-^N + \lambda_+^N}), \tag{24}
\]

in thermodynamic limit, i.e., \( N \rightarrow \infty \), we have

\[
\mu(u_k) = \frac{1}{2}(1 + \cos(2\omega) u_k),
\]

\[
\mu(u_i, u_j) = \frac{1}{4}(1 + (u_i + u_j) \cos(2\omega) + u_i u_j \cos^2(2\omega) + u_i u_j (\lambda_-^{N-j+i} \sin^2(2\omega))), \tag{25}
\]
such that

\[
\cos(2\omega) &= \frac{2 \exp(\beta J) \sinh(\beta B)}{\sqrt{4 \exp(2\beta J) \sinh^2(\beta B) + 4 \exp(-2\beta J)}}, \\
\tan(2\omega) &= \frac{\exp(-2\beta J)}{\sinh(\beta B)},
\]

where \( Sgn(B) \) is the sign function. The behaviour of one-point distribution function \( \mu(u_k) \), in high and low temperatures, is

\[
\mu(u_k) = \begin{cases} 
\frac{1}{2} & \beta \to 0, \\
\frac{1}{2}(1 + u_k) & \beta \to \infty,
\end{cases}
\]

in zero temperature, \( \mu(u_k) \) attains it’s maximum for \( \theta_k = 0 \), which from (4) corresponds to the spin down state, \(|-\rangle\), as expected.

Similarly we can find the following behaviour for joint distribution \( \mu(u_i, u_j) \),

\[
\mu(u_i, u_j) = \begin{cases} 
\frac{1}{4} & \beta \to 0, \\
\frac{1}{4}(1 + u_i + u_j + u_i u_j) & \beta \to \infty,
\end{cases}
\]

in this case the disjoint distribution attains it’s maximum value for \( \theta_i = \theta_j = 0 \), which means that most probably the two spins are in the spin down state, i.e., \(|-\rangle\), in derivation of these behaviours it is assumed that \( Sign(B) = 1 \), by reversing the magnetic field \( (Sign(B) = -1) \), the spins flip to the spin up state.

**References**

[1] K. Husimi, Proc. Phys. Math. Soc. Jpn. 22, 264 (1940)

[2] W. M. Zhang, D. H. Feng and R. Gilmore, Rev. Mod. Phys. Vol.62, No.4, 867 (1990)