Josephson junction on one edge of a two dimensional topological insulator affected by magnetic impurity

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Abstract

The current–phase relation in a Josephson junction formed by putting two s-wave superconductors on the same edge of a two dimensional topological insulator is investigated. We consider the case in which the junction length is finite and magnetic impurity exists. The similarities and differences with respect to a conventional Josephson junction are discussed. Both the \(2\pi\)- and \(4\pi\)-period current–phase relations \((I_{2\pi}(\phi), I_{4\pi}(\phi))\) are studied. There is a sharp jump at \(\phi = \pi\) and \(\phi = 2\pi\) for \(I_{2\pi}\) and \(I_{4\pi}\), respectively, in the clean junction. For \(I_{2\pi}\), the sharp jump is robust against the impurity strength and distribution. However, for \(I_{4\pi}\), an impurity makes the jump at \(\phi = 2\pi\) smooth. The critical (maximum) current \(I_{c,2\pi}\) of \(I_{2\pi}\) is given and we find it will be increased by an asymmetrical distribution of the impurity.

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently the topological insulator (TI) has excited great interest in the condensed-matter community [1, 2]. The unique feature of a TI is the existence of edge states (or surface states) which are protected by time reversal symmetry. The edge state of a two dimensional (2D) TI can be considered approximately as a 1D metal. But since the spin and momentum direction of carriers is locked together owing to strong spin–orbit coupling, it is only half that of the ordinary electron gas. This helical property is robust against nonmagnetic impurity due to its topological origin. If the edge state is in contact with a superconductor, a topological superconducting edge state will form in the interface because of the proximity effect [3, 4]. Also it can be viewed as a 1D topological superconductor (TS). Therefore it is possible to construct a Josephson junction on one edge of the 2D TI.

The conventional superconductor–normal metal–superconductor (SNS) junction has been investigated in detail in the past three decades [5–9]. Since the superconductor–TI–superconductor (STiS) junction is only half of the SNS junction, the corresponding Andreev bound state [5] and current–phase relation are similar for the clean junction if we suppose quasiparticles distribute thermodynamically \((2\pi\text{-period current case})^3\). However, for the STiS junction, a \(4\pi\)-period current–phase \((I_{4\pi}(\phi))\) relation (fractional Josephson effect) may arise if the thermodynamic distribution is partially destroyed while the superconducting phase difference is changed adiabatically [4, 10]. Both nonmagnetic and magnetic impurity play a significant role for the SNS junction. However, for the STiS junction only magnetic impurity can lead to a backscattering owing to time reversal symmetry. In dirty STiS junctions magnetic impurity contributes another significant difference, the extra \(\pi\) phase shift for hole reflection [11]. As a result even the \(2\pi\)-period current \((I_{2\pi}(\phi))\) and Andreev bound states of STiS junction would be quite different from those of the SNS junction.

There are two kinds of Josephson junctions, classified by the length \(L\) between two superconducting electrodes in

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\(^3\) For Andreev bound states there is a difference of degeneracy. For \(I_{2\pi}(\phi)\) there is a difference of factor 2.
comparison with the superconductor coherence length $\xi_0$ [7, 12]. One is the short junction with $L \ll \xi_0$. The other is the long junction with $L$ comparable with $\xi_0$. Both the short- and the long-junction regimes have been studied for SNS junctions [5–8] and junctions based on semiconductor nanowires with spin–orbit coupling [13–16]. Long junctions will present new characteristics. Some are different from those of short junctions and some are unique to long junctions [7].

Experimentally, the edge state, superconducting proximity effect and Andreev reflection have been detected in InAs/2GaSb quantum wells [17, 18]. As is analysed in [12] the length of the STIS junction in the experimental work, [18], is in the long-junction regime. Therefore a complete study of long STIS junctions is significant.

However, earlier works on STIS junctions mainly focus on short junctions [4, 10, 11, 19–27] and long junctions have received less attention [28]. We notice that, very recently, the work by Beenakker et al. [12] discusses the fractional Josephson effect in clean long junctions. As far as we know a detailed study of the long STIS junctions affected by magnetic impurity has never been reported. This is the gap which we want to fill here.

In this paper both the $2\pi$-period and $4\pi$-period current–phase relations are calculated. There is a sharp jump at $\phi = \pi$ and $\phi = 2\pi$ for $I_{2\pi}$ and $I_{4\pi}$, respectively, in the clean junction. For $I_{2\pi}$, the sharp jump at $\phi = \pi$ is robust against the magnetic impurity strength and distribution. However, for $I_{4\pi}$, the magnetic impurity makes the jump at $\phi = 2\pi$ smooth. Both the critical current and the shape of the current–phase curve are greatly influenced by the junction length and impurity distribution. We find that $I_{2\pi}$ will be increased by an asymmetrical distribution of impurity. The ratio $I_{2\pi}/I_{2\pi}$ increases with junction length and is affected by impurity, where $I_{2\pi}$ is the critical value of $I_{2\pi}$.

The rest of the paper is organized as follows. In section 2, we describe the model and give the analytical results. In section 3, the numerical results and analysis are given. In section 4, we give a brief conclusion. In appendix A, we give the reason for the similarity between the STIS junction and the conventional SNS junction. In appendices B and C, we derive the current operator and give the details of the calculation.

2. Model and analytical results

Two s-wave superconductors are in intimate contact with one edge of 2D TI. Because of the proximity effect, a 1D TS forms in the interface. Then we have a STIS Josephson junction on one edge of the 2D TI4. The effective Hamiltonian of the edge state is given as $H_0 = v_F \sigma_3 \psi \sigma_1 \psi$, in which $\psi = \psi_1$ and $\psi_1$ are Pauli matrices acting in the spin space and $v_F$ is the velocity of the edge states [2]. The proximity effect contributes a pairing term, such that the Hamiltonian of the 1D TS is given as [4],

$$H = \int dx \bar{\psi} (H_0 - \mu) \psi + \Delta \psi_1 \psi_1^\dagger + \Delta^* \psi_1^\dagger \psi_1$$

in which $\psi = (\psi_1, \psi_1^\dagger)^T$, $\psi_1^\dagger (\psi_1)$ annihilates a right (left)-moving electron. $\Delta = \Delta_0 \hat{e}^{\phi \hat{t}}$ is the pairing potential, $\Delta_0 = |\Delta|$, and $\phi \hat{t}$ is the phase of the superconductor. In the Nambu representation $\Psi = (\psi_1, \psi_1^\dagger, -\psi_1^\dagger, \psi_1)^T$, with $i\hbar \partial_t \Psi = H_{BdG} \Psi$, we derive the Bogoliubov–de Gennes (BdG) Hamiltonian [4, 29]

$$H_{BdG} = \gamma_{1r} \bar{\sigma}_3 \tau_3 - \mu \tau_3 + \Delta_0 \cos(\phi) \tau_1 - \sin(\phi) \beta \tau_2,$$

where $\mu$ is the chemical potential and $\tau_{1,2,3}$ are Pauli matrices mixing the $\psi$ and $\psi^\dagger$ blocks of $\Psi$. Particle hole symmetry is expressed as $[H_{BdG}, \Sigma] = 0$, in which $\Sigma = \sigma_2 \tau_2 K$ and $K$ is the complex conjugation operator. As a result these states are not independent. For an infinite TS, the dispersion relation is $\epsilon = \pm \sqrt{h^2 v_F^2 (k \pm \kappa)^2 + |\Delta|^2}$, in which $\kappa = h v_F k_F$. Also we neglect the self-consistency condition of $\Delta$ [29]. For the junction considered here, $\Delta = \Delta_0 e^{i\phi_0} \beta (-x) + \Delta_0 e^{i\phi_0} \beta (x-L)$, where $L$ is the length of the junction.

Magnetic impurity can change the direction of particles, and we describe the effect of a magnetic region located at $L_1 < x < L_2$ using the scattering matrices for electrons and holes [11]

$$S_e = \begin{pmatrix} r & t \\ -r^* & t^* \end{pmatrix}$$

and

$$S_h = \begin{pmatrix} -r & t^* \\ r^* & t \end{pmatrix}.$$ (3)

We denote the reflection coefficient $R = |r|^2$ and transmission coefficient $T = |t|^2$. For simplicity, we suppose that $R$ is a constant independent of energy and the length of the impurity region is considered to be short enough, i.e., $L_2 - L_1 \ll \xi_0$. The scattering matrices can be obtained from not only magnetic impurity but also the exchange field due to electron–electron interactions, an external magnetic field and the ferromagnetic proximity effect, which can be described by adding a scattering term $\propto \sigma_3 \tau_3 (x-L) \psi (x-L-x) \psi (x-L)$ in $H_{BdG}$ [30–32]5. Compared with the SNS junction6, there is an extra $\pi$ phase shift for hole reflection, which is the origin of the difference between the STIS junction and the SNS junction in Andreev bound states and $I_{2\pi}$ [9, 11].

Incident particles with energy $\epsilon$ will be reflected at the superconductor–normal interface [33]. For the SNS junction, both Andreev and normal reflections can occur at the interface. But for the STIS junction, only the quantum Andreev reflection occurs at the interface [18, 34, 35]. If $|\epsilon| < \Delta_0$, incident particles will be reflected completely and, therefore, Andreev bound states will form [5]. Solving the BdG equation, we obtain the energy level equation of Andreev bound states. For a clean junction

$$-2\arccos \left( \frac{\epsilon}{\Delta_0} \right) + \frac{\epsilon}{\Delta_0} \frac{L}{\xi_0} = \pm \phi + 2\pi n$$ (4)

The term $\propto \sigma_3 \tau_3$ is an acceptable approximation for a magnetic impurity. The effect of a magnetic impurity can be given as $\sigma \cdot \mathbf{S}_\text{imp}$ in which $\mathbf{S}_\text{imp}$ is the potential due to the magnetic impurity. Replacing the operator $\mathbf{S}_\text{imp}$ with a classical number $S_t$ aligned along the $x$-direction then $\sigma \cdot \mathbf{S}_\text{imp} \rightarrow \sigma_1 S_t$, which is just $\propto \sigma_3 \tau_3$.

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6 For the SNS junction $S_h (-\epsilon) = S_h^* (\epsilon)$. 

4 The effect of the other side is neglected because of the large space difference.
where $\xi_0 = h v_F / (2 \Delta_0)$ is the superconducting coherence length, $\phi = \phi_2 - \phi_1$ is the phase difference and $n = 0, \pm 1, \pm 2, \ldots$. The second term on the left-hand side of equation (4) is equal to $(k_e - k_0) 2L$, where $k_0$ is the wavevector of the right-moving electron (left-moving hole) with energy $\epsilon$. Then we can interpret equation (4) in terms of Bohr–Sommerfeld quantization of the periodic electron–hole orbits in the TI region [36]. In the presence of impurity, the Andreev bound state is given as

$$-2\arccos \left( \frac{\epsilon}{\Delta_0} \right) + \frac{\epsilon}{\Delta_0 \xi_0} = \alpha$$

in which the phase difference is changed to $\alpha$,

$$\cos(\alpha) = T \cos(\phi) - R \cos \left( \frac{\epsilon L - 2L_1}{\Delta_0 \xi_0} \right),$$

which is different from that of the SNS junction [6].

The Josephson current $I(\phi)$ induced by the superconducting phase contains two parts, the discrete current $I_d(\phi)$ and the continuous current $I_c(\phi)$, carried by quasiparticles occupying Andreev bound states and continuous energy spectrum, respectively. To compute the current, we suppose that the system is nearly in thermodynamic equilibrium. Because the current is constant, we calculate the wavefunction and then obtain the average value of the current operator in the TI region. The current due to the scattering state (the eigenstate of junction Hamiltonian) $\varphi = (u(x), u'(x), v(x), v'(x))^T$ with eigenvalue $\epsilon$ is

$$J = e v_F [ (|u|^2 + |v|^2 - |u'|^2 - |v'|^2) f(\epsilon) - |v|^2 + |v'|^2 ]$$

where $e$ is the electron charge and $f(\epsilon)$ is the Fermi distribution function. The last two terms describe the current carried by the ‘vacuum’ (spin-down band and spin-up band filled by electrons) on which we can create quasiparticles occupying the ground state of $H_{BDG}$ to obtain the superconducting ground state [37]. There is an alternative statistical method by which the current is derived from free energy. In this paper we use the wavefunction method to calculate the continuous current and the quantum statistical method for the discrete current. In appendices B and C we give the calculation details and prove that the results according to both methods are equivalent for the discrete current.

The discrete current can be written as $I_d(\phi) = \sum_n I_n(\phi) f(\epsilon_n)$, where $I_n(\phi)$ is the current carried by the quasiparticle occupying the Andreev bound state with eigenvalue $\epsilon_n$. According to the quantum statistical method, the effective current due to the Andreev bound state with eigenvalue $\epsilon_n$ is $I_n(\phi) = \frac{e}{\hbar} \frac{d \epsilon_n}{d \phi}$ (derived in appendix C). For a dirty junction,

$$I_n(\phi) = \frac{e v_F}{2 L + 2\xi (\epsilon_n)} \sin(\epsilon) \gamma$$

$$\gamma = 1 + \frac{ev_F}{2\xi \Delta_0} \frac{R L - 2L_1}{\xi_0} \sin(\epsilon) \sin(\gamma)$$

For a clean junction,

$$I_n^c(\phi) = \pm \frac{e v_F}{2 L + 2\xi (\epsilon_n^c)}$$

where $\xi(\epsilon) = \xi_0 \frac{\Delta_0}{\sqrt{\Delta_0^2 - \epsilon^2}}$ is the energy dependent coherence.

For a short junction ($L \ll \xi_0$), it is enough to consider a discrete current only, because the continuous current is of the order of $L/\xi_0$. However, for a long junction the continuous current cannot be neglected. To calculate $I_c(\phi)$, we first construct the scattering state for an incident particle having energy $\epsilon$, and then apply the current formula given by equation (7). Also we take the semiconductor picture (both the positive and negative solutions of the BdG equation are used). The details of constructing the scattering states and computing the current are similar to [6], and some details are given in appendix B. Results are given below. For a clean junction,

$$I_c(\phi) = \frac{e}{\hbar} T \left[ \int_{-\infty}^{\infty} \int_{-\Delta_0}^{\Delta_0} d\epsilon f(\epsilon) |\tilde{v}_0| \sin(\phi) \right]$$

where

$$D(\epsilon, \alpha) = u_0^4 + v_0^4 - 2u_0^2 v_0^2 \cos \left( \frac{\epsilon L}{\Delta_0 \xi_0} + \alpha \right)$$

$$2u_0^2 = 1 + \sqrt{\epsilon^2 - \Delta_0^2}$$

$$2v_0^2 = 1 - \frac{\Delta_0}{\epsilon}, \quad u_0 v_0 = \frac{\Delta_0}{2\epsilon}.$$
Figure 1. Andreev bound states with several junction lengths and transition coefficients. Red line $T = 1$, blue line $T = 0.5$. Left: $L = 0$. Middle: $L = 8\xi_0$, $L_1 = L/2$. Right: $L = 8\xi_0$, $L_1 = 0.2L$.

Figure 2. Current–phase relation for several junction lengths with a certain transition coefficient. $T = 1 (0.5)$ shown as solid (dotted) lines. For the dotted lines the impurity distributes symmetrically. Current in units of $\frac{e\nu_F}{2}\sqrt{T}$. Length in units of $\xi_0$. Temperature is zero.

Figure 3. Current–phase relation for several $L_1$. Current in units of $\frac{e\nu_F}{2}\sqrt{T}$. $L_1$ in units of $\xi_0$. $T = 0.5$, $L = 8\xi_0$. Inset shows the dependence of the critical current on $L_1$. Temperature is zero.

and $\phi = (2n + 1)\pi$ (see the right panel of figure 1). But the crossing point at $\phi = \pi$, $\epsilon = 0$ remains for arbitrary length and cannot be broken by impurity scattering, which is different from the conventional SNS junction [6]. That specific crossing point is protected by the fermion parity conversion [4].

The zero-temperature current–phase characteristics for different junction length, impurity strength and distribution are shown in figures 2 and 3. With the length increasing, the curve changes from sinusoidal to sawtooth. Impurity reflection mainly decreases the critical current. There is a robust sharp jump at $\phi = \pi$. Since the continuous current is zero while $\phi = \pi$, the jump is rooted in the crossing point of the Andreev bound state at $\phi = \pi$, $\epsilon = 0$. It will not be destroyed by impurity reflection because the impurity cannot open a gap at $\phi = \pi$, $\epsilon = 0$, which is different from the case of a conventional SNS junction.

The critical (maximum) current $I_{c,2\pi}$ is reached when $\phi = \pi$, with $I_{c,2\pi} = I_d(\pi)$ due to $I_c(\pi) = 0$. For the dirty junction

$$I_{c,2\pi} = \frac{e\nu_F}{2}\sqrt{T} - \frac{\sqrt{T}}{(L + 2\xi_0)^2 - R(L - 2L_1)^2}$$  \hspace{1cm} (13)
current. An asymmetrical impurity distribution will enhance the current, as shown in the inset of figure 3. That is different from the conventional SNS case, where the critical current will decrease when the impurity leaves the centre. For a long enough junction with an extremely asymmetrical impurity distribution ($L \gg L_1, L \gg \xi_0$), we have $I_{c,2\pi} \approx \frac{1}{2} \frac{\epsilon_+}{\pi}$ for $T$ not too small, which almost reaches the result of the clean junction.

In the previous discussion, we suppose that there is some mechanism to make quasiparticles distribute nearly thermodynamically. Now we assume that the necessary mechanism is absent for the two eigenstates $\varphi_{\pm}(\phi)$ with energy $\epsilon_{\pm}(\phi)$ nearest to zero shown in figure 1. The two states are connected by electron–hole symmetry, $\varphi_+ = \Xi\varphi_-$ and $\epsilon_- = -\epsilon_+$. This assumption is preserved by fermion parity conservation for both short [4] and long [12] junctions at low temperature.

The original state remains while the phase difference is changed adiabatically. Starting from the ground state while $\phi = 0$, for $\phi < 2\pi$ the state $\epsilon_-$ is occupied. The current due to a pair of Andreev bound states is

$$I_c = \frac{e}{\hbar} \frac{\partial \epsilon_+}{\delta \phi} f(\epsilon_-) - \frac{e}{\hbar} \frac{\partial \epsilon_-}{\delta \phi} (1 - f(\epsilon_-))$$

(14)

and the distribution is $f(\epsilon_-) = 1$, independent of energy, then we have $I_c = \frac{e}{\hbar} \frac{\partial \epsilon_-}{\delta \phi}$ for $0 < \phi < 2\pi$. While $\phi = 2\pi$, the system is in an excited state. Also, it cannot decay to the ground state because of fermion parity conversion [4, 12, 38]. For $2\pi < \phi < 4\pi$, the state $\epsilon_+$ is occupied. While $\phi = 4\pi$, the system reaches the original state we started with [4, 12, 38]. Therefore $I_c$ is $4\pi$ periodic. The net current will be $4\pi$ periodic since $I_c$ contributes significantly to the current. The current at $\phi = 2\pi$ can be given as $I_{c,2\pi} (2\pi - 0^+) = -I_{4\pi} (2\pi + 0^+) = \frac{2e}{\hbar} \frac{\epsilon_{\pi}}{\pi} |\phi = 2\pi - 0^+|$

The current–phase curve is shown in figure 4. There is a sharp jump at $\phi = 2\pi$ for $I_{4\pi}$ in the long clean junction. For $I_{2\pi}$, the jump at $\phi = \pi$ is robust against impurity reflection. However, impurity reflection will make the jump located at $\phi = 2\pi$ smoother for $I_{4\pi}$. The reason is that for a clean junction the energy crossing of the Andreev bound state at $\phi = 2\pi$ has a non-zero slope, i.e., $\frac{\partial \epsilon_{\pi}}{\delta \phi} \neq 0$, which results in the jump since $I_{4\pi} (2\pi - 0^+) = -I_{4\pi} (2\pi + 0^+) \neq 0$. However, magnetic impurity can open the degeneracy at $\phi = 2\pi$ making the slope and, therefore, the current, zero. Thus the jump is smoothed. The maximum current $I_{c,4\pi}$ of $I_{4\pi}$ is different from $I_{c,2\pi}$, with a ratio factor $g$ denoted as $g = \frac{I_{c,4\pi}}{I_{c,2\pi}}$. The ratio increases with increasing length. For a long enough clean junction, i.e., $L \gg \xi_0$, we have $g = 2 [12]$. This is apparent if we notice that the energy level located deep in the pairing potential well is nearly linear for the long clean junction and the system is in the first excited state for $\pi < \phi < 3\pi$. For a short junction, $g$ is independent of reflection and we have $g = 1$. For the sufficiently long junction case, impurity reflection will make the ratio decrease; on varying the reflection coefficient from 0 to 1, $g$ changes from 2 to 1.

### Figure 4. Current–phase relation to show the $4\pi$ period. Dotted (solid) line for $I_{4\pi}$ ($I_{2\pi}$). Left: $T = 1$. Right: $L = 8\xi_0$, $L_1 = 0.5L$. Current in units of $\frac{e}{\pi \xi_0}$. Length in units of $\xi_0$. Temperature is zero.

### 4. Conclusion

In summary, the current–phase relation of a long STiS junction with magnetic impurity is investigated. We consider both the $2\pi$- and $4\pi$-period case. With the length increasing, the current–phase curve evolves from a sinusoidal shape into a sawtooth shape. There is a sharp jump at $\phi = \pi$ and $\phi = 2\pi$ for $I_{2\pi}$ and $I_{4\pi}$, respectively, in the clean junction. For $I_{2\pi}$, the sharp jump at $\phi = \pi$ is robust against the impurity strength and distribution. However, for $I_{4\pi}$, the impurity makes the jump at $\phi = 2\pi$ smooth. The critical current is greatly influenced by the junction length and impurity. Also we find $I_{c,2\pi}$ will be increased by an asymmetrical distribution of impurity.

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Appendix A

This appendix explains the origin of the similarity between the STIS junction and conventional SNS junction.

For the SNS junction, the Nambu basis can be selected as \( \psi = (\psi_1, \psi_\uparrow) \). With \( \text{i} \hbar \partial_t \psi = H_{\text{BdG}} \psi \), we can derive \( H_{\text{BdG}} = (p^2/2m + \mu) \mathbf{1} + \Delta_0 [\cos(\delta) \mathbf{1} - \sin(\delta) \gamma_x] \), where \( \gamma_x = -\text{i} \hbar \partial_x \) and \( m \) is the effective mass of an electron. Taking the Andreev approximation \cite{33} and denoting the eigenvector as \( \varphi = \chi e^{\text{i}(\phi_x + \delta \theta)} \), \( \sigma = \pm \) for incident particles with wave vector near \( \pm k_F \), \( \chi \) is a vector independent of \( x \). Then we arrive at the Andreev equation \cite{33}

\[
\left( \begin{array}{cc} \sigma \sqrt{p_x} - \hbar k_F \Delta & \Delta^* \\ -\sigma \sqrt{p_x} + \hbar k_F \Delta^* \\ \end{array} \right) \varphi = \epsilon \varphi. \tag{15}
\]

Now we denote \( \psi_{\uparrow} = \psi_{\uparrow+} + \psi_{\downarrow-} \), \( \psi_{\downarrow} = (\psi_{\downarrow+}, \psi_{\downarrow-}) \), and \( \psi_{\pm} = (\psi_{\uparrow \pm}, \psi_{\downarrow \pm}) \), where \( \pm \) corresponds to the right-/left-moving component. If we reset Nambu basis as \( \Psi = (\psi_{\uparrow+}, \psi_{\downarrow-}) \) and take \( \sigma = 1 \) (\( \sigma = -1 \) for \( \psi_{\uparrow} \), \( \psi_{\downarrow} \)), we will find the corresponding BdG Hamiltonian is identical to the BdG Hamiltonian of the STS junction. This explains the similarity between the STIS junction and the conventional SNS junction. For the Nambu bases there is a difference of a diagonal matrix \( P = \text{diag}(1, 1, 1, -1) \) with spin neglected in the SNS junction due to its degeneracy. \( P \) matrix indicates that for the dirty STIS junction there will be an extra \( \pi \) phase shift for hole reflection, as is shown in equation \( 3 \).

Appendix B

This appendix derives the current formula equation \( 7 \) and gives some details of the calculation of the current.

The system is given as

\[
H = \int \text{d}x \psi^\dagger (H_0 - \mu) \psi + \Delta \psi^\dagger \psi^\dagger + \Delta^* \psi \psi_{\uparrow}. \tag{16}
\]

With \( \text{i} \hbar \partial_t \Psi = H_{\text{BdG}} \Psi \), the BdG Hamiltonian is yielded as

\[
H_{\text{BdG}} = \left( \begin{array}{cc} H_0 - \mu & \Delta \\ \Delta^* & -\hat{T}(H_0 - \mu)\hat{T}^{-1} \end{array} \right) \tag{17}
\]

with the time reversal operator \( \hat{T} = -\text{i} \sigma_2 K \). In fact, equation \( 17 \) is appropriate for arbitrary \( H_0 \) but with the corresponding time reversal operator for different systems. The BdG equation can be written as

\[
H_{\text{BdG}} \psi_{i,v}(x) = \epsilon_{i,v} \psi_{i,v}(x) \tag{18}
\]

where \( \psi_{i,v} = (u_{i,v}(x), u'_{i,v}(x), v_{i,v}(x), v'_{i,v}(x)) \) is the eigenvector and \( \epsilon_{i,v} \) is the eigenvalue. Because of electron–hole symmetry \( [H_{\text{BdG}}, \mathbf{z}] = 0 \), \( \mathbb{Z} \psi_{i,v}(x) \) is also an eigenvector with eigenvalue \( -\epsilon_{i,v} \). Here, \( v \) and \( i \) denote energy and the extra degeneracy, respectively. For a continuous spectrum \( i = 1, 2, 3, 4, \psi_{3,v}, \psi_{4,v} = -\mathbb{Z} \psi_{2,v}, \psi_{4,v} = -\mathbb{Z} \psi_{1,v} \), where \( \psi_{i,v} \) is the scattering state constructed from the incident state \( \langle \psi_{i,v} \rangle \) shown in figure B.1. However, for Andreev bound states we only have \( i = \pm, \psi_{\pm,v} = \mathbb{Z} \psi_{\mp,v} \). For simplicity we denote \( \psi_{-,v} = \psi_{1,v}, \psi_{+,v} = \psi_{3,v}, \psi_{2,v} = \psi_{2,v} = 0 \).

To diagonalize the Hamiltonian we first rewrite it as \( H = \frac{1}{2} \int \text{d}x \psi^\dagger H_{\text{BdG}} \psi + \text{constant} \). The Bogoliubov transformation is given as \( \Psi = \sum_v \mathcal{S}_v \psi_v \), in which \( \mathcal{S}_v = (\psi_{1,v}, \psi_{2,v}, \psi_{3,v}, \psi_{4,v}) \), \( \psi_v = (\gamma_{1,v}, \gamma_{2,v}, \gamma_{3,v}, \gamma_{4,v})^\dagger \), \( \gamma_{4,v} = -\gamma_{1,v}, \gamma_{3,v} = \gamma_{2,v} \), where \( \gamma_{2,v} \) annihilates a quasiparticle in eigenstate \( \psi_{v} \). Then we have \( H = \frac{1}{2} \sum_{i,v} \epsilon_{i,v} \gamma_{i,v}^\dagger \gamma_{i,v} + \text{constant} \).

The current density operator can be derived using the current density conversion equation, \( \partial_t \hat{\rho}(x) + \partial_x \hat{J}(x) = 0 \), in which the electron density operator \( \hat{\rho}(x) = \langle \psi_\uparrow \psi_\uparrow(x) \rangle \), \( \hat{\rho}(x) = \langle \psi_\downarrow \psi_\downarrow(x) \rangle \). In the TI region, it can be derived as \( \hat{J}(x) = \epsilon_v \psi_\uparrow \psi_\uparrow(x) - \psi_\downarrow \psi_\downarrow(x) \). In the TS region, the pairing potential will contribute an additional term \( -\partial_x \hat{J}_s = 2 \epsilon(\Delta \psi_\downarrow \psi_\uparrow - \text{H.C.})/\hbar \), which describes exchanging Cooper pairs between quasiparticles and condensate. However, this term vanishes for energies larger than the pairing potential, thus it makes no contribution to the continuous current. But it will make the discrete current gradually transform into a supercurrent carried by the condensate in the superconducting region \cite{39}.

Take the ensemble average \( J(x) = \langle \hat{J}(x) \rangle \), with Bogoliubov transformation and \( \langle \gamma_{i,v}^\dagger \gamma_{i,v} \rangle = f(\epsilon_{i,v}) \). In the TI region we find \( J = \sum_i J_{i,1,v} + J_{i,2,v} \),

\[
J_{i,v} = \epsilon_v [\langle u_{i,v}^2 \rangle f(\epsilon_{i,v}) - \langle v_{i,v}^2 \rangle (1 - f(\epsilon_{i,v}))] - [\langle u_{i,v}^2 \rangle f(\epsilon_{i,v}) - \langle v_{i,v}^2 \rangle (1 - f(\epsilon_{i,v}))] \tag{19}
\]

which is just equation \( 7 \) that we wanted to derive. The extra current due to the pairing potential is \( -\partial_x \hat{J} = \frac{\hbar v}{2e} \text{Im} \left[ \sum_{i=1,2,v} [u_{i,v}^2 f(\epsilon_{1,v}) - u_{i,v}^2 f(\epsilon_{1,v})] \right] \).

Now we prove that the contributions from electron-like and hole-like injected states are equal. \( J_{e,v}(h_{e,v}) \) is the current due to the electron-like (hole-like) state \( \psi_{e,v} \) (\( \psi_{h,v} \)), which is the eigenvalue \( \epsilon_{e,v} \) (\( \epsilon_{h,v} \), where \( e = \{1, 2\}, h = \{3, 4\} \) \( J_{e,h,v} \) is

**Figure B.1.** Continuous energy spectrum for an infinite 1D TS. Red and green lines correspond to electron-like and hole-like eigenstates of the 1D TS, respectively. \( \psi_{i,v} \) is the incident state, from which one can construct the scattering state \( \psi_{i,v} \).

The dispersion curve is approximated to be linear near \( \pm k_F \), which is valid for \( \mu \gg |\Delta| \).
given by equation (19). Since \( \phi_{n,v} = \mp \phi_{e,v} \), \( \epsilon_{n,v} = -\epsilon_{e,v} \) and \( 1 - f(\epsilon) = f(-\epsilon) \), we can obtain \( J_{e,v} = J_{h,v} \).

For a continuous spectrum, the eigenstate with a certain energy is 4-fold degenerate. The continuous current can be written as, \( I_e = \oint d\epsilon \frac{N(\epsilon)J(\epsilon)}{2}, J(\epsilon) = \sum_{\nu} J_n(\epsilon_n) = e\nu \sum \left[ (|u_{\nu}^r|^2 + |v_{\nu}^r|^2 - |u_{\nu}^l|^2 - |v_{\nu}^l|^2) f(\epsilon_n) - \sum |\psi_{\nu}^r|^2 \right], \) where \( N(\epsilon) \) is the density of states of the TS. With the eigenvectors solved we find the last two terms cancel with each other. Then we have \( I_e = -\frac{1}{2} \oint d\epsilon N(\epsilon)|J_n(\epsilon)| + J_n(\epsilon)\tanh(\frac{\epsilon - \epsilon_n}{T}) \), where \( J_n(\epsilon) = e\nu \left( |u_{\nu}^r|^2 + |v_{\nu}^r|^2 - |u_{\nu}^l|^2 - |v_{\nu}^l|^2 \right), \) which is the equation we used to derive equations (10)–(12).

### Appendix C

This appendix proves that the discrete current values obtained by the wavefunction method and the quantum statistical method are identical if the states are occupied thermodynamically [40].

In this appendix we take the Nambu basis given as \( \Psi = (\psi_1^\dagger, \psi_1^\dagger, \psi_2^\dagger, -\psi_2^\dagger)^T \) for simplicity. The corresponding BdG Hamiltonian is

\[
H_{BDG}^\prime = \begin{pmatrix}
(vFp_\xi - \mu)\tau_3 + \hat{\Lambda} & M(x)\tau_0 \\
-M(x)\tau_0 & (vFp_\xi - \mu)\tau_3 - \hat{\Lambda}
\end{pmatrix}
\]

where \( M(x) = M0(x-L_1)\theta(L_2 - x) \), \( \tau_0 = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \) is a 2 × 2 unit matrix and \( \hat{\Lambda} = \left( \frac{\Delta \sigma_0 - i\phi_0}{\Delta \sigma_0 + i\phi_0} \right) \), with \( \sigma_0 = x|\psi|^2 \).

A pair of Andreev bound states connected by electron–hole transformation is given as \( \psi_{\pm}^\dagger \), with energy \( \epsilon_{\pm}, \psi_{\mp}^\dagger = \mp \psi_{\pm}^\dagger \) where \( \psi_{\pm} = (u_{\pm}(x), v_{\pm}(x), u_{\mp}(x), v_{\mp}(x))^T \) and \( \epsilon_{\pm} = -\epsilon_{\mp} \). The corresponding current is \( J = evF(|u_{\pm}^r|^2 + |v_{\pm}^r|^2 - |u_{\mp}^r|^2 - |v_{\mp}^r|^2)f(\epsilon_{\mp} - f(\epsilon_{\pm})) \)/2 + evF(|u_{\pm}^r|^2 - |v_{\pm}^r|^2 - |u_{\mp}^r|^2 - |v_{\mp}^r|^2)\)/2. With the solved eigenvectors, we find that the second term on the right-hand side vanishes. Then the current is derived as

\[
J = I(\epsilon_{\mp})f(\epsilon_{\mp}) + I(\epsilon_{\pm})f(\epsilon_{\pm})
\]

in which \( I(\epsilon_{\pm}) = evF(|u_{\pm}^r|^2 + |v_{\mp}^r|^2 - |u_{\mp}^r|^2 - |v_{\mp}^r|^2)\)/2 and \( I(\epsilon_{\mp}) = -I(\epsilon_{\pm}) \). \( I(\epsilon_{\pm}) \) can be seen as the effective current carried by the eigenstate \( \psi_{\pm}^\dagger \).

Rewrite the current as \( J = I(\epsilon_{\pm})f(\epsilon_{\mp}) - f(\epsilon_{\pm}) \), and then act with the operator \( p_x = -\partial_\xi \) on both sides. With a straightforward calculation, we have \( 2p_xJ/e = \psi_{\mp}^\dagger \left( \begin{pmatrix} \Delta_0 \\ \tau_3 \end{pmatrix} \right) \psi_{\pm} \cdot (f(\epsilon_{\mp}) - f(\epsilon_{\pm})) \). We derive \( 2p_xJ/e = \psi_{\mp}^\dagger \partial_\phi \psi_{\pm} \cdot (f(\epsilon_{\mp}) - f(\epsilon_{\pm})) \). We integrate over the whole region. Since the current in the TI region is constant and decays to zero gradually in the superconductor, the left-hand side gives \( h\xi = 0 \) with the help of the Feynman–Hellmann theorem the right-hand side gives \( \frac{\partial}{\partial \phi} f(\epsilon_{\pm}) \). Then we obtain

\[
J = \frac{e}{h} \frac{\partial}{\partial \phi} f(\epsilon_{\mp}) + \frac{e}{h} \frac{\partial}{\partial \phi} f(\epsilon_{\pm}).
\]

Comparing this with equation (21) we have \( I(\epsilon_{\pm}) = \frac{e}{h} \frac{\partial}{\partial \phi} f(\epsilon_{\pm}) \). So the result reduces to (21).

It is important to point out that the quantum statistical method has taken both particle energy levels and hole energy levels into consideration. For a clean STJ junction with length \( L = 0 \), one can take the Nambu basis to be \( \psi = (\psi_1, \psi_2)^T \) and the corresponding BdG Hamiltonian is \( 2 \times 2 \). In this case there is only one energy level with energy \( \epsilon_{\pm}(\phi) \) contributing to the current. However as we have discussed, the wrong result, \( J = \frac{e}{h} \frac{\partial}{\partial \phi} f(\epsilon_{\mp}) \), will be derived if we use the quantum statistical method.

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