METRIZED GRAPHS, ELECTRICAL NETWORKS, AND FOURIER ANALYSIS

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Abstract. A metrized graph is a finite weighted graph whose edges are thought of as line segments. In this expository paper, we study the Laplacian operator on a metrized graph and some important functions related to it, including the “j-function”, the effective resistance, and eigenfunctions of the Laplacian. We discuss the relationship between metrized graphs and electrical networks, which provides some physical intuition for the concepts being dealt with. We also discuss the relation between the Laplacian on a metrized graph and the combinatorial Laplacian matrix. We introduce the “canonical measure” on a metrized graph, which arises naturally when considering the Laplacian of the effective resistance function. Finally, we discuss a generalization of classical Fourier analysis which utilizes eigenfunctions of the Laplacian on a metrized graph. During the course of the paper, we obtain a proof of Foster’s network theorem and of the identity \( \min\{x, y\} = 8 \sum_{n \geq 1 \text{ odd}} \frac{\sin(n\pi x/2) \sin(n\pi y/2)}{n^2} \), for \( 0 \leq x, y \leq 1 \).

1. An informal discussion

Graphs are usually considered to be discrete objects, so issues of continuity and differentiability don’t typically appear in graph theory texts. Here is a picture of a graph:

![Graph Image](image)

Figure 1. A graph with four vertices and three edges.

The picture is misleading, in the sense that we might want to believe that the edge \( PQ \) is a line segment comprising a continuum of points. However, edges in graph theory are merely formal connections between vertices; they don’t “contain points.” But is there something wrong with thinking of \( PQ \) as a line segment? The notion...
of a metrized graph gives meaning to points between the vertices while retaining the salient combinatorial features of the graph. In a broader sense, metrized graphs will unite the discrete and the continuous for us.

The basic idea of a metrized graph is simple: identify each edge of a finite graph with a line segment and define the distance between two points of the graph to be the length of the shortest path connecting them. We will provide more details on this definition in §2.

Metrized graphs appear in the literature of several areas of science and mathematics. For example: in number theory, they are used to study arithmetic intersection theory on algebraic curves (see [CR, Zh]); in mathematical biology, they are used to study neuron transmission (see [Ni]); they are also used in physics, chemistry, and engineering (under the names metric graphs, quantum graphs, and $c^2$-networks) as wave-propagation models (see [Ku]).

In §3 we define a Laplacian operator on a metrized graph which is closely related to the Laplacian matrix (or Kirchhoff matrix) associated to a finite graph—see §5 for precise statements about this connection. Using a theory of eigenfunction expansions on a metrized graph that generalizes classical Fourier analysis on the circle, we will be able to prove intriguing series identities such as

$$\min\{x, y\} = 8 \sum_{n \geq 1 \text{ odd}} \frac{\sin\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)}{n^2}, \quad 0 \leq x, y \leq 1.$$ 

See §8 for more details.\footnote{Metrized graphs can be viewed as one-dimensional Riemannian manifolds with singularities, and from this point of view the Laplacian on a metrized graph is a nontrivial but computationally accessible variant of the Laplacian on a higher-dimensional Riemannian manifold.}

There is a well-known and useful interplay between the theories of finite graphs and resistive electrical networks (see e.g., [Br, Ch. II,IX]). This relationship extends beautifully to the setting of metrized graphs (cf. [DE]). For example, a theorem of Foster from 1949 (see [Fo]) asserts that

$$\sum_{\text{edges } e} \frac{r(e)}{L_e} = \#V - 1,$$

where $r(e)$ is the effective resistance in the electrical network between the endpoints of the edge $e$, $L_e$ is the resistance along the edge $e$, and $\#V$ is the number of nodes (vertices) in the network. In §7 we give a proof of Foster’s theorem using the “canonical measure” on a metrized graph.

The theory of electrical networks is itself closely related to the theory of random walks on graphs. We will not touch upon the connection with random walks in this paper, but we refer the interested reader to the delightful monograph [DS]. There is a nice proof of Foster’s theorem using random walks in [DE] (see also [Br, Theorem 25, Exercise 23, Chapter IX]).

This article is a follow-up to the 2003 summer REU on metrized graphs held at the University of Georgia and run by the first author and Robert Rumely. The participating students’ enthusiasm for the subject convinced us that a broader audience might appreciate an introduction to the ideas involved. Further information about the REU, its organizers and participants, and the research they performed can be found at \texttt{http://www.math.uga.edu/~mbaker/REU/REU.html}.
In keeping with the spirit of discovery that spawned this article, we have included a number of exercises to clarify the text or extend the ideas presented. We have also strived to keep the exposition as self-contained as possible with the hopes that it will inspire further students toward this subject.

2. Metrized graphs versus weighted graphs

There is a bijective correspondence between metrized graphs and equivalence classes of finite weighted graphs. In this section we give an overview of this correspondence, leaving many of the details to the reader. See [BR] for more detailed proofs of the assertions made in this section.

Definition 1. For the purposes of this paper, we define a weighted graph $G$ to be a finite, connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$, edge set $E(G) = \{e_1, \ldots, e_m\}$, and a collection of positive weights $\{w_{e_1}, \ldots, w_{e_m}\}$ associated to the edges of $G$. Further, we require that $G$ have no loop edges or multiple edges. The length of the edge $e$ is defined to be $L_e = 1/w_e$.

In classical graph theory, one associates weights to the edges of a graph. When studying metrized graphs, it makes more sense to work with lengths, since distance is the fundamental notion in a metric space. We will henceforth indicate lengths in our figures (e.g., Figure 3).

A weighted graph $G$ gives rise to a metric space $\Gamma$ in the following way. To each edge $e$, associate a line segment of length $L_e$, and identify the ends of distinct line segments if they correspond to the same vertex of $G$. The points of these line segments are the points of $\Gamma$. We call $G$ a model for $\Gamma$. The distance between two points $x$ and $y$ in $\Gamma$ is defined to be the length of the shortest path between them, where the length of a path is measured in the usual way along the line segments traversed. (A path between distinct points always exists because $G$ is connected.)

Exercise 1. Show that this notion of distance defines a metric on $\Gamma$ (which we call the path metric).

The space $\Gamma$, endowed with the path metric, is called a metrized graph. Here’s a more abstract definition, taken from [ZH]:

Definition 2. A metrized graph $\Gamma$ is a compact, connected metric space such that each $p \in \Gamma$ has a neighborhood $U_p$ isometric to a star-shaped set of valence $n_p \geq 1$, endowed with the path metric (see Figure 2). To be precise, a star-shaped set of valence $n_p$ is a set of the form

$$S(n_p, r_p) = \{ z \in \mathbb{C} : z = te^{k \cdot 2\pi i/n_p} \text{ for some } 0 \leq t < r_p \text{ and some } k \in \mathbb{Z} \}.$$
Exercise 2. Check that the metric space $\Gamma$ arising from a weighted graph $G$ satisfies the abstract definition (Definition 2) of a metrized graph.

The points $p \in \Gamma$ with valence different from 2 are precisely those where $\Gamma$ fails to look locally like an open interval, and the compactness of $\Gamma$ ensures that there are only finitely many such points. Let $V(\Gamma)$ be any finite, nonempty subset of $\Gamma$ such that:

- $V(\Gamma)$ contains all of the points with $n_p \neq 2$. (This implies that $\Gamma \setminus V(\Gamma)$ is a finite, disjoint union of subspaces $U_i$ isometric to open intervals.)
- For each $i$, the topological closure $\overline{U_i}$ of $U_i$ in $\Gamma$ is isometric to a line segment (as opposed to a circle). We call $e_i = \overline{U_i}$ a segment of $\Gamma$.
- For each $i \neq j$, $e_i \cap e_j = \emptyset$ or $\{p\}$, where $p$ is an endpoint of both $e_i$ and $e_j$.

Any finite set $V(\Gamma)$ satisfying these conditions will be called a vertex set for $\Gamma$, and the elements of $V(\Gamma)$ will be called vertices of $\Gamma$.

Exercise 3.

(a) Prove that a vertex set for $\Gamma$ always exists.

(b) Let $\Gamma$ be a circle. Show that any set consisting of only one or two points of $\Gamma$ cannot be a vertex set.

It should be remarked that $V(\Gamma)$ is not unique. For example, if $\Gamma$ is a circle, then any choice of three distinct points of $\Gamma$ is a vertex set. The choice of a vertex set $V(\Gamma)$ determines a finite set $\{e_i\}$ of segments of $\Gamma$. The endpoints of each segment $e_i$ are vertices of $\Gamma$. We emphasize that the segments of $\Gamma$ depend on our choice of a vertex set.

Given a metrized graph $\Gamma$, our next task will be to find a weighted graph $G$ that serves as a model for $\Gamma$ as above. Pick a vertex set $V(\Gamma)$ for $\Gamma$. Define a graph $G$ with vertices indexed by $V(\Gamma)$, and join two distinct vertices $p$ and $q$ of $G$ by an edge if and only if there exists a segment of $\Gamma$ with endpoints $p$ and $q$. (So edges of $G$ correspond to segments of $\Gamma$.) Define the length of the edge joining $p$ to $q$ to be the length of the segment $e$. Then $G$ is a weighted graph, with weights given by the reciprocals of the lengths; our definition of $V(\Gamma)$ guarantees that $G$ has no multiple edges or loop edges. Moreover, if we construct the metrized graph associated to $G$, it is easily seen to be isometric to $\Gamma$.

Different choices of a vertex set $V(\Gamma)$ yield distinct weighted graphs in the above construction. Write $G \sim G'$ if the two weighted graphs $G, G'$ admit a common refinement, where we refine a weighted graph by subdividing its edges in a manner that preserves total length (see Figure 3). This provides an equivalence relation on the collection of weighted graphs, and one can check that two weighted graphs are equivalent if and only if they give rise to isometric metrized graphs.

Having established this correspondence, we are now free to fix a particular model of a metrized graph, without worrying that we’ve lost some degree of generality in doing so. This will be especially convenient in the next section, when we work with functions on a metrized graph that are “nice” outside of some vertex set.

3. THE LAPLACIAN ON A METRIZED GRAPH

Our goal in this section is to motivate and define the Laplacian of a function on a metrized graph. The Laplacian on a metrized graph is a hybrid between the Laplacian on the real line (i.e., the negative of the second derivative) and the discrete Laplacian matrix studied in graph theory (cf. §4).
Choose a vertex set $V(\Gamma)$ for the metrized graph $\Gamma$. Let $p$ be a non-vertex point of $\Gamma$, and suppose $e$ is a segment of length $L$ containing $p$. Parametrize $e$ by an isometry $s_e : [0, L] \to e$ so that we have a real coordinate $t \in [0, L]$ to use for describing points of the segment. We say that $f$ is differentiable at $p$ if the quantity 

$$\frac{d}{dt} f(s_e(t)) \bigg|_{s_e(t)=p}$$

exists. There is precisely one other parametrization of this sort, namely $u_e(t) = s_e(L-t)$. The chain rule shows that 

$$\frac{d}{dt} f(u_e(t)) \bigg|_{u_e(t)=p} = -\frac{d}{dt} f(s_e(t)) \bigg|_{s_e(t)=p}.$$ 

Hence the value of the derivative of $f$ at $p$ depends on the parametrization, but only up to a sign. Picking one of the two parametrizations for a segment can be thought of as choosing an orientation for the segment, and we will use the two concepts interchangeably.

We can similarly determine if $f$ is $n$ times differentiable at $p$ by looking at the existence of the quantity $\frac{d^n}{dt^n} f(s_e(t)) \big|_{s_e(t)=p}$.

Exercise 4. Show that the second derivative $f''(p)$, when it exists, is well-defined independent of the choice of an orientation for $e$.

We also require a notion of differentiability that makes sense at the vertices. The abstract definition of a metrized graph tells us that each point $p \in \Gamma$ has a neighborhood isometric to a star-shaped set with $n_p \geq 1$ arms. Thus there are $n_p$ directions by which a path in $\Gamma$ can leave $p$. To each such direction, we associate a formal unit vector $\vec{v}$, and we write $\text{Vec}(p)$ for the collection of all $n_p$ directions at $p$. We make this convention so that we can write $p + \varepsilon \vec{v}$ for the point of $\Gamma$ at distance $\varepsilon$ from $p$ in the direction $\vec{v}$ for sufficiently small $\varepsilon > 0$.

Definition 3. Given a function $f : \Gamma \to \mathbb{R}$, a point $p \in \Gamma$, and a direction $\vec{v} \in \text{Vec}(p)$, the derivative of $f$ at $p$ in the direction $\vec{v}$, written $D_{\vec{v}} f(p)$, is given by 

$$D_{\vec{v}} f(p) = \lim_{\varepsilon \to 0^+} \frac{f(p + \varepsilon \vec{v}) - f(p)}{\varepsilon},$$

provided this limit exists. This will also be called a directional derivative.

Exercise 5. Given a function $f : \Gamma \to \mathbb{R}$ and a point $p \not\in V(\Gamma)$ at which $f$ is differentiable, show that the two directional derivatives of $f$ at $p$ exist and sum to zero. [Hint: Parametrize the segment containing $p$ and do the calculation explicitly.]

Here is the class of functions on which we intend to apply our Laplacian:
Definition 4. Define \( S(\Gamma) \) to be the class of all continuous functions \( f : \Gamma \to \mathbb{R} \) for which there exists a vertex set \( V_f(\Gamma) \) (with corresponding segments \( e_i \)) such that

(i) \( D_\vec{v} f(p) \) exists for each \( p \in \Gamma \) and each \( \vec{v} \in \text{vec}(p) \),

(ii) \( f \) is twice continuously differentiable on the interior of each segment \( e_i \), and

(iii) \( f'' \) is bounded on the interior of each segment \( e_i \).

We call \( S(\Gamma) \) the class of \textit{piecewise smooth functions} on \( \Gamma \). (This is, of course, a small abuse of terminology as these functions need not be infinitely differentiable away from the vertices.)

Exercise 6. Show that hypotheses (ii) and (iii) imply hypothesis (i), and that hypothesis (i) already implies that \( f \) is continuous.

We now define the Laplacian operator on a metrized graph. A conceptual obstacle to overcome is that the Laplacian of a function \( f \in S(\Gamma) \) is a \textit{bounded, signed measure}\(^2\) on \( \Gamma \), not a function. For readers unfamiliar with the notion of measure, we give a brief working definition in just a moment.

Definition 5. The \textit{Laplacian} of a function \( f \in S(\Gamma) \) is given by the measure

\[
\Delta f = -f''(x)dx - \sum_{p \in \Gamma} \sigma_p(f)\delta_p,
\]

where \( \sigma_p(f) = \sum_{\vec{v} \in \text{vec}(p)} D_\vec{v} f(p) \), \( dx \) denotes the Lebesgue measure on \( \Gamma \), and \( \delta_p \) is the Dirac measure (unit point mass) at \( p \).

By Exercise 6, the sum \( \sum_p \sigma_p(f) \) is actually finite as \( \sigma_p(f) = 0 \) for any \( p \) not in \( V_f(\Gamma) \). Also, \( f'' \) is well-defined away from the vertices in \( V_f(\Gamma) \), so \( \Delta f \) is independent of segment orientations. To perform computations with \( \Delta f \), however, we will need to choose parametrizations.

Let’s define some notation to make what lies ahead a little easier. Choose a model for \( \Gamma \), parametrize each segment \( e \) of \( \Gamma \) by \( s_e : [0, L_e] \to e \), and for \( f : \Gamma \to \mathbb{R} \) define \( f_e : [0, L_e] \to \mathbb{R} \) by \( f_e = f \circ s_e \).

Now for our working definition of measure. Intuitively, a measure is something we can integrate functions against. For our purposes, then, a measure on \( \Gamma \) will be an expression of the form

\[
\mu = \sum_{\text{segments } e} g_e(t)dt|_e + \sum_{i=1}^n c_i \delta_{p_i},
\]

where \( g_e : (0, L_e) \to \mathbb{R} \) is continuous and bounded, \( c_i \in \mathbb{R} \), and \( p_1, \ldots, p_n \) are points of \( \Gamma \). To integrate a continuous function \( f : \Gamma \to \mathbb{R} \) against the measure \( \mu \), define

\[
\int_{\Gamma} f(x)d\mu(x) = \sum_{\text{segments } e} \left\{ \int_0^{L_e} f_e(t)g_e(t)dt \right\} + \sum_{i} c_i f(p_i).
\]

A measure of the form \( \sum g_e(t)dt|_e \) is called a \textit{continuous measure}, and a measure of the form \( \sum_{i=1}^n c_i \delta_{p_i} \) is called a \textit{discrete measure}.

If \( g : \Gamma \to \mathbb{R} \) is a function such that \( g \circ s_e(t) = g_e(t) \) for all segments \( e \) of \( \Gamma \) and all \( t \in (0, L_e) \), we will usually write \( g(x)dx \) instead of \( \sum_e g_e(t)dt|_e \).

The \textit{total mass} of a measure \( \mu \) is defined to be \( \int_{\Gamma} 1(x)d\mu(x) \), where \( 1 \) denotes the constant function with value 1.

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\(^2\)We could also work with complex-valued functions, and then the Laplacian would be a complex measure.
Example. Consider the metrized graph $\Gamma$ modelled in Figure 4. Define a function on $\Gamma$ by

$$f_e(t) = \begin{cases} t + 1, & e = PQ \\ 3(t + \frac{1}{2}), & e = QS \\ t^2 + \frac{1}{2}, & e = RQ. \end{cases}$$

Then $\Delta f = -2dx_{RQ} - \delta_P + 3\delta_S$. Note that $\Delta f$ has total mass zero; we will see shortly that this is not an accident.

Our next goal is to put together some important facts about the Laplacian. It will be more convenient to work with an alternate formulation, though. For simplicity, we write $f'_e(0)$ for the right-hand derivative of $f_e$ at 0 (as the limit only makes sense from one side). Similarly, write $f'_e(L_e)$ for the left-hand derivative at $L_e$. If $p$ and $q$ are the endpoints of the segment $e$, with $s_e(0) = p$ and $s_e(L_e) = q$, we’ll say that $e$ begins at $p$ and ends at $q$. If $\vec{v} \in \text{Vec}(p)$ and $\vec{w} \in \text{Vec}(q)$ are the directions pointing inward along $e$, then

$$D_{\vec{v}} f(p) = f'_e(0), \quad D_{\vec{w}} f(q) = -f'_e(L_e).$$

Observe that $-\sigma_p(f)$ counts $-f'_e(0)$ for each segment $e$ beginning at $p$, and it counts $f'_e(L_e)$ for each segment $e$ ending at $p$. Thus $\Delta f$ can be written

$$(*) \quad \Delta f = \sum_{\text{segments } e} \left\{ -f''_e(x) \, dx|_e + f'_e(L_e)\delta_{s_e(L_e)} - f'_e(0)\delta_{s_e(0)} \right\}.$$ 

In particular, we now see that the contribution of the segment $e$ to the Laplacian is $-f''_e(x)dx|_e + f'_e(L_e)\delta_{s_e(L_e)} - f'_e(0)\delta_{s_e(0)}$. This measure is independent of the choice of parametrization of $e$, but it is necessary to choose a parametrization to write it down.

**Theorem 1** (Self-adjointness of $\Delta$). Suppose $f, g \in S(\Gamma)$. Then

$$\int_{\Gamma} f \Delta g = \int_{\Gamma} g \Delta f = \int_{\Gamma} f'(x)g'(x) \, dx.$$ 

**Proof.** Choose a model for $\Gamma$ with vertex set $V(\Gamma) = V_f(\Gamma) \cup V_g(\Gamma)$. Then $f''$ is continuous on the interior of each segment of $\Gamma$, and the directional derivatives of $f$ and $g$ exist for all vertices in $V(\Gamma)$ (see Definition 4). Choose parametrizations for each segment of $\Gamma$ and define $f_e$ and $g_e$ as before. Using integration by parts,
we obtain:

\[
\int_{\Gamma} g \Delta f = \sum_{e} \left\{ f'_{e}(L_e)g_e(L_e) - f'_{e}(0)g_e(0) - \int_{0}^{L_e} g_e(t)f''_{e}(t)\, dt \right\}
\]

\[
= \sum_{e} \int_{0}^{L_e} f'_{e}(t)g'_e(t)\, dt = \int_{\Gamma} f'(x)g'(x)\, dx.
\]

The rest of the result follows by symmetry.

\[ \square \]

**Corollary 1.** If \( f \in S(\Gamma) \), then \( \Delta f \) has total mass 0.

**Proof.** Set \( g = 1 \) in the statement of Theorem 1. Then

\[
\int_{\Gamma} 1 \cdot \Delta f = \int_{\Gamma} f'(x)1'(x)\, dx = \int_{\Gamma} 0 \, dx = 0.
\]

\[ \square \]

Before stating the next result about the Laplacian, we need to define another useful class of functions:

**Definition 6.** Define \( A(\Gamma) \) to be the subclass of functions \( f \in S(\Gamma) \) such that for each oriented segment \( e \) of \( \Gamma \), there exist real constants \( A_e, B_e \) so that \( f_e(t) = A_e t + B_e \) for \( t \in [0, L_e] \). A function in \( A(\Gamma) \) is called **piecewise affine**.

**Exercise 7.** Show that a function \( f \in A(\Gamma) \) is completely determined by its values on a vertex set \( V_f(\Gamma) \) for \( f \). \ [{\text{Hint: Use linear interpolation.}} \]

**Exercise 8.** Show that if \( f \in S(\Gamma) \), then \( f \) is piecewise affine if and only if \( \Delta f \) is a discrete measure.

The next result is a graph-theoretic analogue of the second derivative test from calculus.

**Theorem 2** (The Maximum Principle). Suppose \( f \in A(\Gamma) \) is nonconstant. Then \( f \) achieves its maximum value on \( \Gamma \) at a vertex \( p \in V_f(\Gamma) \) for which \( \sigma_p(f) < 0 \).

**Proof.** It is easy to see that the function \( f \) must take on its maximum value at a vertex \( p \in V_f(\Gamma) \). Moreover, since \( f \) is nonconstant, we may select \( p \) so that \( f \) decreases along some segment \( e_0 \) having \( p \) as an endpoint. (This uses the fact that \( \Gamma \) is connected.) Re-parametrize each segment \( e \) having \( p \) as an endpoint, if necessary, so that \( s_e(0) = p \), where \( s_e : [0, L_e] \to e \). Then \( \sigma_p(f) = \sum f'_e(0) \), where the summation is over all segments \( e \) beginning at \( p \). Each of the slopes \( f'_e(0) \) must be non-positive; otherwise \( f \) would grow along \( e \), violating the fact that \( f \) is maximized at \( p \). We know \( f'_{e_0}(0) < 0 \) since \( f \) decreases along \( e_0 \). Hence \( \sigma_p(f) < 0 \), which completes the proof.

\[ \square \]

**Theorem 3.** Suppose \( f, g \in S(\Gamma) \). If \( \Delta f = \Delta g \) and \( f(p) = g(p) \) for some \( p \in \Gamma \), then \( f \equiv g \).

**Proof.** If \( h = f - g \), then \( \Delta h = 0 \). By Exercise 8 \( h \in A(\Gamma) \). As \( \Delta h = 0 \), it follows from the Maximum Principle that \( h \) is constant. The hypothesis that \( f(p) = g(p) \) for some \( p \) now implies that \( h \equiv 0 \), so that \( f \equiv g \) as desired.

\[ \square \]

Note in particular that if \( f \in S(\Gamma) \) is harmonic (i.e., \( \Delta f = 0 \)), then \( f \) must be constant.
4. Metrized graphs versus electrical networks

We now take a moment to give some physical intuition about the Laplacian coming from the theory of electrical networks. (For a more detailed account of the theory of electrical networks, see [Ba], [DS], and [CR].) For our purposes, a (resistive) electrical network is a physical model of a metrized graph $\Gamma$ obtained by viewing the vertices of $\Gamma$ as nodes of the network and the segments of $\Gamma$ as wires, each with a resistance given by its length.

Using an external device (such as a battery), one can force current to flow through the network; for simplicity, we consider only the case where a quantity $I > 0$ of current enters the circuit at some point $a$ and exits at some point $b$. At all other points of $\Gamma$, we have Kirchhoff’s current law: The total current flowing into any node equals the current flowing out of any node. Mathematically, current is a function which assigns to each oriented segment $e$ of $\Gamma$ a real number $i_e$, the current flow across $e$. Kirchhoff’s node law says that it is possible to define an electric potential function $\phi(x) \in A(\Gamma)$ such that for every oriented segment $e$, $\phi'(x) = -i_e$. (The minus sign is due to the convention that current flows from high potential to low potential.) In particular, if $p$ is the initial endpoint and $q$ the terminal endpoint of an oriented segment $e$, then Ohm’s law $\phi(p) - \phi(q) = i_e L_e$ holds. The potential function $\phi(x)$ is only determined up to an additive constant; one needs to pick a reference voltage at some point of $\Gamma$ in order to define the potential at other points.

In our language of directional derivatives, if $p$ is a point of $\Gamma$ and $\vec{v} \in \text{Vec}(p)$ is any direction at $p$, then the current flowing away from $p$ in the direction $\vec{v}$ is $-D_{\vec{v}}\phi(p)$. Mathematically, Kirchhoff’s current law states that for $p \notin \{a, b\}$, we have $-\sigma_p(\phi) = -\sum D_{\vec{v}}\phi(p) = 0$. We have $-\sigma_a(\phi) > 0$, which says that $a$ is a current source, and $-\sigma_b(\phi) < 0$, which says that $b$ is a current sink. The current entering the network at $a$ is $-\sigma_a(\phi)$; the current exiting the network at $b$ is $\sigma_b(\phi)$; and we have $-\sigma_a(\phi) = \sigma_b(\phi) = I$.

Taken together, Kirchhoff’s node and potential laws say that given $I > 0$, there is a function $\phi \in A(\Gamma)$ such that $\Delta \phi = I \cdot \delta_a - I \cdot \delta_b$. (This will be proved mathematically as a consequence of Corollary 3 in [4]) Note that $\phi$ is determined up to an additive constant by Theorem 3. Note also that we initially required $-\sigma_a(\phi) = \sigma_b(\phi) = I$ (conservation of current), which is demanded mathematically by Corollary 1.

In accordance with physical intuition, the Maximum Principle (Theorem 4) implies that the electric potential in the network is highest at $a$ (where current enters) and lowest at $b$ (where it exits). By convention, one often sets the potential at $b$ to be zero, in which case we say that the node $b$ is grounded.

5. The Laplacian on a weighted graph

In this section, we explain some connections between the classical Laplacian matrix on a weighted graph and the Laplacian on a metrized graph.

Suppose $G$ is a weighted graph with vertex set $V(G) = \{v_i\}$, edge set $E(G) = \{e_k\}$, and weights $\{w_{e_k}\}$. If the edge $e_k$ has endpoints $v_i$ and $v_j$, then we will use the notation $w_{ij} = w_{e_k} = w_{ji}$ to show the dependence of the weights on the vertices. For convenience, we set $w_{ij} = 0$ if $v_i$ and $v_j$ are not connected by an edge. In particular $w_{ii} = 0$ for all $i$. 


Definition 7. The Laplacian matrix associated to a weighted graph $G$ is the $n \times n$ matrix $Q$ with entries

$$Q_{ij} = \begin{cases} 
\sum_k w_{ik}, & \text{if } i = j \\
-w_{ij}, & \text{if } i \neq j.
\end{cases}$$

We should note that in the literature, our $Q$ is often called the combinatorial Laplacian or Kirchhoff matrix (see e.g., [Bo]).

The Laplacian matrix encodes interesting information about the graph $G$ (see e.g., [Mo], [GR, §13]). For example, zero appears as an eigenvalue of $Q$ with multiplicity equal to the number of connected components of $G$ (so exactly once in our case). Kirchhoff’s famous Matrix-Tree Theorem (see [Bo, Corollary 13, Chapter II]) equates the weighted number of spanning trees of the graph with the absolute value of the determinant of the matrix obtained by deleting any row and column from $Q$.

Returning to metrized graphs, we’ve already noted in Exercise 7 that a function $f \in A(\Gamma)$ is completely determined by its values on the finite set $V_f(\Gamma)$. Thus, a piecewise affine function on $\Gamma$ yields a function on the vertices of a certain model for $\Gamma$, and conversely, given a model $G$ and a function on $V(G)$, we can linearly interpolate to obtain a piecewise affine function on $\Gamma$. Our two notions of Laplacian honor this correspondence:

**Theorem 4.** Suppose $\Gamma$ is a metrized graph, $f \in A(\Gamma)$, and $G$ is a model of $\Gamma$ with vertex set $V_f(\Gamma) = \{v_1, \ldots, v_n\}$. Let $\tilde{f}$ be the $n \times 1$ vector with $\tilde{f}_i = f(v_i)$. Then

$$\Delta f = \sum_i \left[Q \tilde{f}\right]_i \delta_{v_i}.$$ 

**Proof.** We already know that $\Delta f$ is discrete if $f$ is piecewise affine. So it suffices to show that $\left[Q \tilde{f}\right]_i = -\sigma_{v_i}(f)$ for any vertex $v_i$. To that end, we parametrize each segment $e$ having $v_i$ as an endpoint so that $s_e(0) = v_i$. As $f$ is piecewise affine, the directional derivatives of $f$ at $v_i$ are given by $f'_e(0) = \frac{f_e(L_e) - f_e(0)}{L_e}$. Recall that the weight of an edge is the reciprocal of its length. We conclude that

$$\sigma_{v_i}(f) = \sum_{\text{segments } e \text{ adjacent to } v_i} \frac{f_e(L_e) - f_e(0)}{L_e} = \sum_j w_{ij} \left\{f(v_j) - f(v_i)\right\}$$

$$= -\left\{\left(\sum_k w_{ik}\right) f(v_i) - \sum_{j \neq i} w_{ij} f(v_j)\right\} = - \left[Q \tilde{f}\right]_i.$$ 

□

As a bonus, we now deduce a few useful facts about the Laplacian matrix:

**Corollary 2.** If $G$ is a weighted graph with $n \times n$ Laplacian matrix $Q$, then

(i) The kernel of $Q$ is 1-dimensional with basis $[1, \ldots, 1]^t$.

(ii) If $x \in \mathbb{R}^n$ is a vector, then $\sum_i |Qx|_i = 0$.

**Proof.** Identify $\mathbb{R}^n$ with the $n$-dimensional vector space spanned by the vertices of $G$. A vector $x \in \mathbb{R}^n$ can be interpreted as a function on the vertices of $G$, and this function can be linearly interpolated to yield a piecewise affine function $f$ on the associated metrized graph $\Gamma$. If $Qx = 0$, then Theorem 4 implies that $\Delta f = 0$. 

The Maximum Principle shows \( f \) must be constant, so \( x = [c, \ldots, c]^t \) for some real number \( c \). This proves (i). For (ii), use Corollary 11 and Theorem 14 to get
\[
\sum_i [Qx]_i = \int_\Gamma \Delta f = 0.
\]
\[\square\]

Now we know the relationship between the Laplacian operator acting on \( A(\Gamma) \) and the Laplacian matrix. In fact, one can prove that the Laplacian of a piecewise smooth function \( f \) is a limit of Laplacians of piecewise affine approximations of \( f \).

To state the result, we introduce the following notation. If \( f \in S(\Gamma) \) and \( G_N \) is a model of \( \Gamma \) whose vertices contain \( V_f(\Gamma) \), define \( f_N \) to be the unique piecewise affine function with \( f_N(p) = f(p) \) for each vertex \( p \) of \( G_N \) (restrict \( f \) to the vertices of \( G_N \) and linearly interpolate).

**Theorem 5.** Suppose \( f \in S(\Gamma) \). There exists a sequence of models \( \{G_N\} \) for \( \Gamma \) such that for all continuous functions \( g \) on \( \Gamma \), we have
\[
\int_\Gamma g \Delta f_N \to \int_\Gamma g \Delta f \quad \text{as } N \to \infty.
\]
That is, the sequence of measures \( \{\Delta f_N\} \) converges weakly to \( \Delta f \) on \( \Gamma \). By Theorem 14, the discrete measures \( \Delta f_N \) can be computed using the Laplacian matrix.

Theorem 5 is not hard to prove, but we will not give the proof here. (A complete proof can be found in [Fa].) We mention the theorem in order to display the very close connection between the Laplacian matrix on a weighted graph and the Laplacian operator on a metrized graph.

6. **The \( j \)-function**

In this section, we introduce a three-variable function \( j_z(x, y) \) on the metrized graph \( \Gamma \) which allows us, in a sense to be made precise, to invert the Laplacian operator.

Let \( \text{Meas}_0(\Gamma) \) denote the space of measures of total mass zero on \( \Gamma \). We know from Corollary 11 that if \( f \in S(\Gamma) \) then \( \Delta f \in \text{Meas}_0(\Gamma) \). The following result is a partial converse to this fact.

**Theorem 6.** Let \( \nu = \sum c_i \delta_{p_i} \in \text{Meas}_0(\Gamma) \) be a discrete measure. Then there exists a piecewise affine function \( f \) on \( \Gamma \) such that \( \Delta f = \nu \).

**Proof.** Let \( S = \{p_1, \ldots, p_k\} \), and fix a model \( G \) for \( \Gamma \) with vertex set \( V(G) \) containing \( S \). Let \( n = \#V(G) \), and let \( W \) be the \( n \)-dimensional real vector space spanned by the vertices of \( G \), which we identify with \( \mathbb{R}^n \). If \( Q \) is the Laplacian matrix associated to \( G \), then we know \( \text{Ker}(Q) \) is 1-dimensional by Corollary 14(i). The rank-nullity theorem implies that \( \text{Im}(Q) \) is \((n - 1)\)-dimensional.

By Theorem 14, solving \( \Delta f = \nu \) is equivalent to finding a vector \( x \in W \) with \( Qx = [c_1, \ldots, c_n]^t \). Let \( W_0 \) be the \((n - 1)\)-dimensional subspace of \( W \) consisting of vectors \([a_1, \ldots, a_n]^t \) such that \( \sum a_i = 0 \). Corollary 2(ii) shows \( \text{Im}(Q) \) is contained in \( W_0 \). As these two spaces have the same dimension, they must be equal. The condition \( \nu \in \text{Meas}_0(\Gamma) \) says \( \sum c_i = 0 \), so \([c_1, \ldots, c_n]^t \) lies in the image of \( Q \). \[\square\]

\[\text{In comparison with the Riemannian manifold setting, integrating } j_z(x, y) \text{ will yield the associated Green’s function for the metrized graph Laplacian.}\]
We now single out a special case of this result which is of particular interest. In what follows, we write $\Delta_x$ instead of $\Delta$ if we wish to emphasize that we are taking the Laplacian with respect to the variable $x$.

**Corollary 3.** For fixed $y, z \in \Gamma$, there exists a unique piecewise affine function $j(x) = j_z(x, y)$ satisfying

$$\Delta_x j_z(x, y) = \delta_y(x) - \delta_z(x), \quad j_z(z, y) = 0.$$  

**Proof.** The existence of $j(x)$ follows from Theorem 6, and uniqueness follows from Theorem 3. □

We now justify our assertion that the $j$-function allows us to “invert the Laplacian” on the space $\text{Meas}_0(\Gamma)$. Recall from Theorem 6 that given a discrete measure $\nu \in \text{Meas}_0(\Gamma)$, there exists a function $f \in A(\Gamma)$ (unique up to an additive constant) that satisfies the differential equation $\Delta f = \nu$. The next result shows that we can explicitly describe such a function $f$ using the $j$-function:

**Theorem 7.** Let $\nu = \sum c_i \delta_{p_i} \in \text{Meas}_0(\Gamma)$ be a discrete measure. Then for any fixed $z \in \Gamma$, the function

$$f(x) = \int_{\Gamma} j_z(x, y) d\nu(y) = \sum_i c_i j_z(x, p_i)$$

is piecewise affine and satisfies the equation $\Delta f = \nu$.

**Proof.** The condition $\nu \in \text{Meas}_0(\Gamma)$ means that $\sum c_i = 0$. Therefore

$$\Delta f = \sum_i c_i (\delta_{p_i} - \delta_z) = \nu.$$  

Since the $j$-function is piecewise affine, $f$ is as well. □

We mention (see [BR] for a proof) that Theorem 7 admits the following generalization to arbitrary (not necessarily discrete) measures $\nu \in \text{Meas}_0(\Gamma)$: For fixed $z \in \Gamma$, the function $f(x) = \int_{\Gamma} j_z(x, y) d\nu(y)$ is in $S(\Gamma)$ and satisfies the equation $\Delta f = \nu$. In particular, if $\nu$ is a measure on $\Gamma$, then we can solve the differential equation $\Delta f = \nu$ if and only if $\nu \in \text{Meas}_0(\Gamma)$.

The function $j_z(x, y)$ has an interpretation in terms of electrical networks. Recalling our description of the electrical network associated to a metrized graph given in §4, the function $j_z(x, y)$ is the electric potential at $x$ if one unit of current enters the network at $y$ and exits at $z$, and the node $z$ is grounded. So one could build a real-life model of the metrized graph $\Gamma$ with wires, hook up a battery, and empirically determine the values of the $j$-function!

**Exercise 9.** Physical intuition suggests that the $j$-function should be nonnegative; prove more precisely that

$$0 \leq j_z(x, y) \leq j_z(y, y)$$

for all $x, y, z \in \Gamma$. [Hint: For fixed $y$ and $z$, apply the Maximum Principle to $j_z(x, y)$ and its negative.]

The three-variable function $j_z(x, y)$ satisfies a magical four-term identity, which will be used in various guises throughout this section and the next. The proof of this identity is an excellent illustration of the theory developed in §3.
Theorem 8 (Magical Identity). For all \( x, y, z, w \in \Gamma \), we have the identity
\[
j_z(x, y) - j_z(w, y) = j_w(y, x) - j_w(z, x).
\]

Proof. Fix \( x, y, z, w \in \Gamma \). On one hand, we have
\[
\int_{\Gamma} j_z(u, y) \Delta_u (j_w(u, x)) = \int_{\Gamma} \{ \delta_x(u) - \delta_w(u) \} = j_z(x, y) - j_z(w, y).
\]
By Theorem 1 this is equal to
\[
\int_{\Gamma} j_w(u, x) \Delta_u (j_z(u, y)) = \int_{\Gamma} \{ \delta_y(u) - \delta_z(u) \} = j_w(y, x) - j_w(z, x).
\]

□

The Magical Identity allows us to prove two useful symmetries for the \( j \)-function.

Corollary 4. For \( x, y, z \in \Gamma \), the \( j \)-function satisfies

(i) \( j_z(x, y) = j_z(y, x) \)

(ii) \( j_z(x, x) = j_z(z, z) \)

Proof. For (i), if we set \( w = z \) in the Magical Identity, we obtain
\[
j_z(x, y) - j_z(z, y) = j_z(y, x) - j_z(z, x).
\]
Since \( j_z(z, x) = j_z(z, y) = 0 \), the result follows.

For (ii), substitute \( x = z, y = w \) into the Magical Identity to get
\[
j_z(z, w) - j_z(w, w) = j_w(w, z) - j_w(z, z).
\]
Since \( j_z(z, w) = j_w(w, z) = 0 \), the result follows by swapping \( w \) for \( x \). □

In passing, we mention that \( j_z(x, y) \) has a very strong continuity property: it is jointly continuous in \( x, y, \) and \( z \). That is, the value of the \( j \)-function varies continuously if we make small variations to \( x, y, \) and \( z \) simultaneously. Our electrical network interpretation makes this statement quite plausible: the value on our voltmeter should vary continuously when we move the battery terminals and the point at which we’re reading the voltage. A mathematical proof is outlined in the next exercise (see [CR] for a different approach).

Exercise 10.

(a) Let \( I, I' \) be closed intervals in \( \mathbb{R} \). Suppose \( f : I \times I' \to \mathbb{R} \) has the property that \( f(x, y) \) is affine in \( x \) and \( y \) separately. Then \( f(x, y) = c_1 + c_2 x + c_3 y + c_4 xy \) for some \( c_1, \ldots, c_4 \in \mathbb{R} \).

(b) Use (a) to show that for fixed \( z \in \Gamma \), \( j_z(x, y) \) is jointly continuous as a function of \( x \) and \( y \).

(c) Use Theorem 8 to prove the five-term identity
\[
j_z(x, y) = j_w(x, y) - j_w(x, z) - j_w(z, y) + j_w(z, z).
\]

(d) Deduce from (b) and (c) that \( j_z(x, y) \) is jointly continuous in \( x, y, \) and \( z \).

We now define another useful function motivated by the theory of electrical networks:

Definition 8. The effective resistance between two points \( x, y \) of a metrized graph is given by
\[
r(x, y) = j_y(x, x) = j_z(y, y).
\]
The fact that \( j_y(x, x) = j_x(y, y) \) is just a restatement of the second symmetry of the \( j \)-function in Corollary 4. In terms of electrical networks, the effective resistance between two nodes \( x \) and \( y \) is the absolute value of the potential difference between \( x \) and \( y \) when a unit current enters the network at \( x \) and exits at \( y \).

We now introduce some useful techniques for calculating the \( j \)-function and the effective resistance function. Rules (ii) and (iii) in Theorem 9 below are essentially the familiar series and parallel transforms from circuit theory. The proofs of Theorems 9 and 10 below are adapted from [Zh].

A subgraph of the metrized graph \( \Gamma \) is a subspace of \( \Gamma \) which is a metrized graph in its own right. In the statement of Proposition 9, \( \Gamma_1 \) and \( \Gamma_2 \) will always denote subgraphs of \( \Gamma \). We let \( j_x(z, y) \) (resp. \( j_{x, 1}(w, z), j_{x, 2}(w, y) \)) denote the \( j \)-function on \( \Gamma \) (resp. on \( \Gamma_1, \Gamma_2 \)), and similarly we let \( r(x, y) \) (resp. \( r_{1}(x, y), r_{2}(x, y) \)) denote the effective resistance function on \( \Gamma \) (resp. on \( \Gamma_1, \Gamma_2 \)).

**Theorem 9.** Let \( \Gamma \) be a metrized graph, and let \( \Gamma_1 \) and \( \Gamma_2 \) be subgraphs.

(i) Suppose \( e \) is a segment in \( \Gamma \) of length \( L \) with endpoints \( x, y \), and assume that \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup e \) with \( \Gamma_1 \cap e = \{x\} \), \( \Gamma_2 \cap e = \{y\} \), and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). (Compare Figure 5(i).) Then \( r(x, y) = L \).

(ii) Suppose \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 = \{z\} \). (Compare Figure 5(ii).) Then for all \( x \in \Gamma_1 \) and \( y \in \Gamma_2 \), we have \( r(x, y) = r_{1}(x, z) + r_{2}(z, y) \).

(iii) Suppose \( \Gamma = \Gamma_1 \cup \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 = \{x, y\} \). (Compare Figure 5(iii).) Then

\[
\frac{1}{r(x, y)} = \frac{1}{r_{1}(x, y)} + \frac{1}{r_{2}(x, y)}
\]

**Figure 5.** These three figures illustrate the three parts of Proposition 9. The solid lines (resp. dashed lines) indicate the segments of the diagram belonging to \( \Gamma_1 \) (resp. to \( \Gamma_2 \)).

**Proof.** For (i), we pick a parametrization \( s_e : [0, L] \to e \) such that \( s_e(0) = x \) and \( s_e(L) = y \). Let \( t : e \to [0, L] \) be the inverse of \( s_e \). We claim that

\[
j_x(z, y) = \begin{cases} 
0, & \text{if } z \in \Gamma_1 \\
t(z), & \text{if } z \in e \\
L, & \text{if } z \in \Gamma_2.
\end{cases}
\]

Indeed, it is easily verified that the Laplacian of the right-hand side with respect to \( z \) is \( \delta_y - \delta_x \), and that the two sides agree when \( z = x \). The claim therefore follows from Theorem 3 and the desired result follows by setting \( z = y \).

For (ii), we claim that

\[
j_x(w, y) = \begin{cases} 
j_{x, 1}(w, z), & \text{if } w \in \Gamma_1 \\
r_{1}(x, z) + j_{x, 2}(w, y), & \text{if } w \in \Gamma_2.
\end{cases}
\]
The point is that the right-hand side is continuous at \( w = z \), has Laplacian equal to \((\delta_z - \delta_x) + (\delta_y - \delta_z) = \delta_y - \delta_x\), and is zero when \( w = x \). The result then follows by setting \( w = y \).

We leave the proof of (iii) as an exercise for the reader. \(\square\)

**Exercise 11.** Verify part (iii) of Theorem 9 by first showing that

\[
  j_x(z, y) = \begin{cases} \frac{r_2(x,y)}{r_1(x,y)+r_2(x,y)} j_x,1(z, y), & \text{if } z \in \Gamma_1 \\ \frac{r_1(x,y)}{r_1(x,y)+r_2(x,y)} j_x,2(z, y), & \text{if } z \in \Gamma_2. \end{cases}
\]

**Exercise 12.** Show that the function \( r(x, y) \) is jointly continuous in \( x \) and \( y \), and that for fixed \( y \in \Gamma \), \( r(x, y) \) is continuous and piecewise quadratic in \( x \). \[\text{[Hint: Use Exercise 10]}\]

Using Theorem 9 we can derive an explicit description of the function \( r(x, y) \) when \( x \) varies along a single segment of \( \Gamma \) having \( y \) as an endpoint. To state the result, we define a quantity \( R_e \) associated to a segment \( e \) of \( \Gamma \) as follows. Let \( e^o \) denote the interior of the segment \( e \) and let \( \Gamma_e \) be the complement of \( e^o \) in \( \Gamma \). If \( \Gamma_e \) is connected, then \( \Gamma_e \) is a subgraph of \( \Gamma \), and we define \( R_e \) to be the effective resistance \( r(y, z) \) between the endpoints \( y \) and \( z \) of \( e \) computed on \( \Gamma_e \). If \( \Gamma_e \) is not connected (i.e., if \( e \) is not part of a cycle), we define \( R_e \) to be \( \infty \). Loosely speaking, \( R_e \) is the effective resistance between the endpoints of \( e \) in the subgraph obtained by deleting \( e \).

The next result is motivated by the following intuition: to calculate \( r(x, y) \) on the segment \( e \), we can think of \( e \) and its complement \( \Gamma_e \) as being connected in parallel, and \( x \) splits \( e \) into two segments connected in series. We can then use the parallel and series transforms to calculate \( r(x, y) \).

**Theorem 10.** Let \( e \) be a closed segment of \( \Gamma \) of length \( L_e \), let \( y, z \) be the endpoints of \( e \), and parametrize \( e \) by \( s_e : [0, L_e] \to e \) with \( s_e(0) = y \) and \( s_e(L_e) = z \). Suppose \( t : e \to [0, L_e] \) is the inverse of \( s_e \). Then for \( x \in e \), we have

\[
  r(x, y) = t(x) - \frac{1}{L_e + R_e} t(x)^2,
\]

where \( \frac{1}{L_e + R_e} = 0 \) if \( R_e = \infty \).

**Proof.** If \( \Gamma_e \) is not connected, then \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup e \), with \( \Gamma_1 \cap e = \{y\} \), \( \Gamma_2 \cap e = \{z\} \), and \( \Gamma_1 \cap \Gamma_2 = \emptyset \). Then \( r(x, y) = t(x) \) by part (i) of Proposition 9.

Now suppose that \( \Gamma_e \) is connected. Then \( x \) breaks \( e \) into two closed segments \( \Gamma_1 = [t(y), t(x)] \) and \( \Gamma_2 = [t(x), t(z)] \), and \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_e \). Letting \( \Gamma_3 = \Gamma_2 \cup \Gamma_e \), we have (with the obvious notation):

\[
  \frac{1}{r(x, y)} = \frac{1}{r_1(x, y)} + \frac{1}{r_3(x, y)}, \quad \text{by Prop. 9(iii)}
\]

\[
  = \frac{1}{r_1(x, y)} + \frac{1}{r_2(x, z) + R_e}, \quad \text{by Prop. 9(ii)}
\]

\[
  = \frac{1}{t(x)} + \frac{1}{L_e - t(x) + R_e}, \quad \text{by Prop. 9(i)}.
\]
The desired formula now follows because of the simplification
\[
\left( \frac{1}{t(x)} + \frac{1}{L_e - t(x) + R_e} \right)^{-1} = t(x) - \frac{1}{L_e + R_e}t(x)^2.
\]
\[\square\]

Exercise 13. Let \( \Gamma \) be a metrized graph of total length \( L \). Fix \( a, b \in \Gamma \), and choose a vertex set \( V(\Gamma) \) containing \( a \) and \( b \). Let \( e \) be an oriented segment of \( \Gamma \) beginning at \( p \) and ending at \( q \). For \( x \in \Gamma \), let \( \phi(x) = j_b(x, a) \), and define \( i_e = \frac{\phi(p) - \phi(q)}{L_e} = -\phi'_e. \) (In terms of electrical networks, \( i_e \) is the current flowing across \( e \) when a unit current enters the network at \( a \) and exits at \( b \).) Also, define \( r(e) = r(p, q) \).

(a) Show that \( r(e) \leq L_e \). [Hint: Use Prop. 9]
(b) Show that \( r(x, y) \) is a metric on \( \Gamma \). [Hint: For the triangle inequality, use Exercise 11]
(c) Deduce that \( r(a, b) \) is bounded above by the length of any path from \( a \) to \( b \), and conclude that \( 0 \leq r(x, y) \leq L \) for all \( x, y \in \Gamma \).

7. The canonical measure and Foster's Theorem

Calculating the Laplacian of the effective resistance function \( r(x, y) \) for fixed \( y \) is not so easy just from the definitions, but our explicit description in Theorems 10 and 11 below will allow us to do it in a slick way. The first half of this section will be devoted to figuring out \( \Delta_x r(x, y) \), and in the second half we reap the benefits of this calculation by proving some interesting results from graph theory, including Foster’s theorem. The method presented here is a simplified version of §2 of [CR].

Example. If \( \Gamma = [0, 1] \), then Theorem 9(i) shows that \( r(x, y) = |x - y| \), and a simple calculation shows that \( \Delta_x r(x, y) = \delta_0(x) + \delta_1(x) - 2\delta_y(x) \). Interestingly, we see that \( \Delta_x r(x, y) + 2\delta_y(x) \) is independent of \( y \). This simple example actually illustrates a general phenomenon.

Theorem 11. For any metrized graph, \( \Delta_x r(x, y) + 2\delta_y(x) \) is a measure which is independent of \( y \).

Proof. Let \( z, w \in \Gamma \) be arbitrary. Set \( x = y \) in the Magical Identity of §9 to get
\[
j_z(y, y) - j_z(w, y) = j_w(y, y) - j_w(z, y).
\]
Applying Corollary 4 we obtain
\[
r(y, z) - j_z(y, w) = r(y, w) - j_w(y, z).
\]
Taking the Laplacian of both sides with respect to \( y \) and recalling that \( \Delta_y j_z(y, x) = \delta_x - \delta_z \), we get
\[
\Delta_y r(y, z) - \delta_w + \delta_z = \Delta_y r(y, w) - \delta_z + \delta_w.
\]
Rearranging, we see that \( \Delta_y r(y, z) + 2\delta_z = \Delta_y r(y, w) + 2\delta_w \). As \( w \) and \( z \) were arbitrary, the result follows. \( \square \)

Definition 9. The canonical measure on a metrized graph \( \Gamma \) is given by
\[
\mu_{\text{can}} = \frac{1}{2} \Delta_x r(x, y) + \delta_y(x),
\]
where \( y \in \Gamma \) is arbitrary. Theorem 11 shows that \( \mu_{\text{can}} \) is independent of the choice of \( y \).
Recall from Corollary 1 that $\Delta_x r(x, y)$ is a measure of total mass zero, so we see from Definition 9 that $\mu_{\text{can}}$ has total mass 1.

**Exercise 14.** Show that the quantity

$$\tau(\Gamma) = \frac{1}{2} \int_{\Gamma} r(x, y) d\mu_{\text{can}}(x)$$

is independent of the choice of $y \in \Gamma$. [**Hint:** Use Theorem 11.]

We now give an explicit description of the measure $\mu_{\text{can}}$.

**Theorem 12.** Let $n_p$ denote the valence of a vertex $p \in V(\Gamma)$. Then

$$\mu_{\text{can}} = \sum_{\text{vertices } p} \left(1 - \frac{1}{2} n_p\right) \delta_p + \sum_{\text{segments } e} \frac{1}{R_e + L_e} dx|_e.$$ 

*Proof.* We compute the discrete and continuous parts of $\mu_{\text{can}}$ separately.

**Continuous part:** Let $e$ be an oriented segment of $\Gamma$ which begins at $y$ and ends at $z$. If $x$ lies on $e$, then we’re in the situation of Theorem 10, and we calculate that

$$(\dagger) \quad \Delta_x \{r(x, y)|_e\} = \frac{2}{L_e + R_e} dx|_e + \delta_z - \delta_y.$$ 

Since $\mu_{\text{can}} = \frac{1}{2} \Delta_x r(x, y) + \delta_y$ is independent of our choice of $y$, $(\dagger)$ shows that the continuous part of $\mu_{\text{can}}$ along $e$ must be $\frac{2}{L_e + R_e} dx|_e$.

**Discrete part:** If $y$ is an endpoint of a segment $e$, then $r(x, y)$ is quadratic along the interior of $e$ by Theorem 10. It follows that the discrete part of $\mu_{\text{can}}$ is supported on $V(\Gamma)$. Let $p \in V(\Gamma)$ be a vertex. Using $(\dagger)$ in §3, we calculate from Equation $(\dagger)$ that $\frac{1}{2} \Delta_x r(x, p)$ contributes $-\frac{1}{2} \delta_p$ to the discrete part of $\mu_{\text{can}}$ at $p$ for each segment $e$ beginning at $p$. Recalling that $\mu_{\text{can}} = \frac{1}{2} \Delta_x r(x, p) + \delta_p$, the coefficient of $\delta_p$ in $\mu_{\text{can}}$ must therefore be $1 - \frac{1}{2} n_p$.

$\square$

**Example.** If $\Gamma$ is a circle of length 1, then every vertex has valence 2, so $\mu_{\text{can}}$ has no discrete part. For the continuous part, divide the circle into three segments $e_1, e_2, e_3$, each of length 1/3. Then we get

$$\mu_{\text{can}} = dx|_{e_1} + dx|_{e_2} + dx|_{e_3} = dx.$$ 

**Example.** Let $\Gamma$ be the star of Figure 4. Then $\mu_{\text{can}}$ has no continuous part because $R_e$ is infinite for all edges. Therefore

$$\mu_{\text{can}} = \frac{1}{2} \delta_p - \frac{1}{2} \delta_Q + \frac{1}{2} \delta_R + \frac{1}{2} \delta_S.$$ 

Theorem 12 has some interesting consequences for weighted graphs. For example, we have the following result from [CR]:

**Corollary 5.** Let $G$ be a weighted graph with vertex set $V(G)$ and edge set $E(G)$. Then

$$\sum_{\text{edges } e} \frac{L_e}{R_e + L_e} = 1 + \#E(G) - \#V(G).$$
Proof. Integrating both sides of the formula in Theorem 12 over $\Gamma$, we obtain:

$$1 = \sum_{\text{vertices } p} \left(1 - \frac{1}{2} n_p\right) + \sum_{\text{edges } e} \frac{L_e}{R_e + L_e}.$$  

(Here we have summed over edges of $G$ instead of segments of $\Gamma$, but the two sets are in bijective correspondence.) As each edge in $G$ connects exactly 2 vertices, we have

$$\sum_{\text{vertices } p} n_p = 2 \{\#E(G)\}.$$

Therefore

$$1 = \#V(G) - \#E(G) + \sum_{\text{edges } e} \frac{L_e}{R_e + L_e},$$

which is equivalent to the desired formula. \qed

It is a well-known fact from graph theory that $1 + \#E(G) - \#V(G)$ is the number of linearly independent cycles on $G$ (see [Bo, Theorem 9, Chapter II]). This is a topological invariant which only depends on the associated metrized graph $\Gamma$.

Corollary 6 (Foster’s Theorem). For an edge $e$ in a weighted graph $G$, let $r(e)$ denote the effective resistance $r(x, y)$ between the endpoints $x$ and $y$ of $e$ on the associated metrized graph $\Gamma$. Let $w_e = 1/L_e$ be the weight of the edge $e$. Then

$$\sum_{\text{edges } e} w_e r(e) = \sum_{\text{edges } e} \frac{r(e)}{L_e} = \#V(G) - 1.$$  

Proof. If $R_e = \infty$, then $r(e) = L_e$ by Proposition 9(i). Otherwise, by Theorem 10 we have

$$r(e) = L_e - \frac{L_e^2}{L_e + R_e} = \frac{L_e R_e}{L_e + R_e}.$$  

Combining these observations, we see that

$$\sum_{\text{edges } e} \frac{r(e)}{L_e} = \sum_{\text{edges } e \text{ with } R_e = \infty} 1 + \sum_{\text{edges } e \text{ with } R_e \neq \infty} \left\{ \frac{R_e}{L_e + R_e} - 1 \right\}$$

$$= \#E(G) + \sum_{\text{edges } e \text{ with } R_e \neq \infty} \left\{ \frac{R_e}{L_e + R_e} - 1 \right\}$$

$$= \#E(G) - \sum_{\text{edges } e \text{ with } R_e \neq \infty} \frac{L_e}{L_e + R_e}.$$  

The result follows immediately from Corollary 5. \qed

Example. If $G$ is a tree, then we have $r(e) = L_e$ for all $e$ by Proposition 9(i), and $\#E(G) = \#V(G) - 1$. Therefore $\sum_e \frac{r(e)}{L_e} = \#E(G) = \#V(G) - 1$ as predicted by Foster’s theorem.

More generally, for arbitrary $G$ it follows from Exercise 13(a) that $0 \leq \frac{r(e)}{L_e} \leq 1$ for each edge $e$, so that a priori we have $\sum_e \frac{r(e)}{L_e} \leq \#E(G)$. Foster’s theorem is equivalent to the assertion that the difference $\#E(G) - \sum_e \frac{r(e)}{L_e}$ is equal to the number of independent cycles in $G$. 
Example. Foster’s theorem can be a useful tool for calculating effective resistances, especially in the presence of symmetry. For example, let $G = K_n$ be the complete graph on $n \geq 2$ vertices, with all edge weights equal to 1. By symmetry, the effective resistance $r(x, y)$ between distinct points $x, y \in V(G)$ is independent of $x$ and $y$; let $r$ denote the common value. Foster’s theorem gives

$$\sum_{\text{edges } e} w_e \cdot r(e) = \left(\frac{n}{2}\right) \cdot r = n - 1,$$

so that $r = 2/n$.

8. Eigenfunctions of the Laplacian

Suppose $\Gamma$ is a circle of length 1. Then for $f \in S(\Gamma)$, we have

$$\Delta f = -f''(x)dx + (\text{discrete measure}).$$

A standard computation shows that the nonzero piecewise smooth functions $\phi$ that satisfy the equation

$$\Delta \phi = \lambda \phi(x)dx$$

for some $\lambda \in \mathbb{R}$ are precisely the constant multiples of $\sin(2\pi nx)$ and $\cos(2\pi nx)$ for $n \in \mathbb{Z}$. These functions will be called the eigenfunctions of the Laplacian on $\Gamma$. The corresponding eigenvalues are $\lambda_n = 4\pi^2 n^2$.

It is convenient to normalize each eigenfunction $\phi$ of the Laplacian so that $\int_{\Gamma} \phi(x)^2 dx = 1$; i.e., $\phi$ has $L^2$-norm equal to 1. By standard calculus facts, for $m \neq n$ and $n \neq 0$ we have

$$\int_{\Gamma} \sin^2(2\pi nx) dx = \int_{\Gamma} \cos^2(2\pi nx) dx = \frac{1}{2},$$

$$\int_{\Gamma} \sin(2\pi nx) \sin(2\pi mx) dx = \int_{\Gamma} \cos(2\pi nx) \cos(2\pi mx) dx = 0,$$

$$\int_{\Gamma} \sin(2\pi nx) \cos(2\pi mx) dx = 0.$$

Therefore

$$\Lambda = \{1\} \cup \{\sqrt{2} \cos(2\pi nx)\}_{n \geq 1} \cup \{\sqrt{2} \sin(2\pi nx)\}_{n \geq 1}$$

is an $L^2$-orthonormal set of eigenfunctions for the Laplacian.

We make the following observations about the set $\Lambda$:

- Each $L^2$-normalized eigenfunction of $\Delta$ occurs exactly once on this list.
- Each nonzero eigenvalue $4\pi^2 n^2$ occurs twice, and the eigenvalue 0 occurs with multiplicity 1.

A standard result in Fourier analysis is the following:

**Theorem 13.** Let $\Gamma$ be a circle of length 1. Then any $f \in S(\Gamma)$ can be expanded as a uniformly convergent series

$$f(x) = a_0 + \sum_{n \geq 1} a_n \sqrt{2} \cos(2\pi nx) + \sum_{n \geq 1} b_n \sqrt{2} \sin(2\pi nx),$$
where the Fourier coefficients $a_n, b_n$ are determined by
\[
\begin{align*}
a_0 &= \int_\Gamma f(x) \, dx \\
a_n &= \int_\Gamma f(x) \sqrt{2} \cos(2\pi nx) \, dx \quad n \geq 1 \\
b_n &= \int_\Gamma f(x) \sqrt{2} \sin(2\pi nx) \, dx \quad n \geq 1.
\end{align*}
\]

Viewed in this light, Fourier analysis on the circle is the theory of eigenfunctions of the Laplacian on the underlying metrized graph.\(^4\)

A nice fact is that one can generalize Fourier analysis to an arbitrary metrized graph. One way to do this is as follows. Fix a measure $\mu$ on $\Gamma$; for simplicity, we will assume that $\mu$ has total mass 1. Let
\[
S_\mu(\Gamma) = \{ f \in S(\Gamma) : \int_\Gamma f(x) \, d\mu(x) = 0 \}.
\]

We now make the following somewhat non-intuitive definition:

**Definition 10.** A nonzero function $\phi \in S(\Gamma)$ is an eigenfunction of the Laplacian with respect to $\mu$ if $\phi \in S_\mu(\Gamma)$ and satisfies the equation
\[
\Delta \phi = \lambda \phi(x) \, dx - C \mu
\]
for some $\lambda, C \in \mathbb{R}$.

Note that the value of the constant $C$ is completely determined by $\lambda$ and $\phi$ in the above equation. Indeed, integrating both sides and recalling that $\Delta \phi$ has total mass zero and $\mu$ has total mass one shows that
\[
C = \lambda \int_\Gamma \phi(x) \, dx.
\]

A sequence $\{\phi_n\}$ of distinct eigenfunctions is orthonormal if $\int_\Gamma \phi_i(x) \phi_j(x) \, dx = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. For our purposes, the sequence is called complete if every eigenfunction of the Laplacian with respect to $\mu$ is a scalar multiple of some $\phi_n$.\(^5\)

It may not be clear a priori what role the $C \mu$ term is playing in the definition of an eigenfunction. However, this definition turns out to be quite flexible and useful, as illustrated by the following result (see [BR] for a proof):

**Theorem 14.** Suppose $\Gamma$ is a metrized graph. Let $\mu$ be a measure of total mass 1 on $\Gamma$, and consider a complete orthonormal sequence $\{\phi_n\}_{n \geq 1}$ of eigenfunctions of the Laplacian with respect to $\mu$. The corresponding eigenvalues $\{\lambda_n\}$ are all positive and each occurs with finite multiplicity. Furthermore, every $f \in S_\mu(\Gamma)$ can be expanded as a uniformly convergent series
\[
f(x) = \sum_{n \geq 1} a_n \phi_n(x),
\]

---

\(^4\)Of course, there are many variants and generalizations of Theorem 13, and much sophisticated mathematics has been developed to address what happens if $f$ satisfies hypotheses weaker than piecewise smoothness (for example, if $f$ is merely continuous). But this article is not the place to discuss such matters!

\(^5\)This is a non-standard definition of complete. In [BR] it is proved that $L^2(\Gamma)$ admits a complete orthonormal basis of eigenfunctions of the Laplacian in the standard sense of complete; i.e., the only $L^2$-function orthogonal to all of the eigenfunctions is the zero function.
where the generalized Fourier coefficients \( a_n \) are determined by the formula

\[
a_n = \int_{\Gamma} f(x) \phi_n(x) \, dx.
\]

If we add in the constant function 1, then it follows from Theorem 14 that every \( f \in S(\Gamma) \) can be uniquely expressed as

\[
f(x) = a_0 + \sum_{n \geq 1} a_n \phi_n(x),
\]

where \( a_0 = \int_{\Gamma} f(x) d\mu(x) \) and the \( a_n \)'s are as before.

Though the main interest of this result is the fact that it applies to arbitrary metrized graphs, we illustrate what’s happening in Theorem 14 by considering the special case where \( \Gamma = [0, 1] \) and \( \mu = \delta_0 \) is a point mass at 0.

What are the eigenfunctions of the Laplacian in this case? By Definition 10 and the fact that an eigenfunction is required to be in \( S_\mu(\Gamma) \), we demand that

\[
-\phi''(x) dx + \phi'(1) \delta_1 - \phi'(0) \delta_0 = \lambda \phi(x) dx - C \delta_0, \quad \phi(0) = 0,
\]

for some \( \lambda, C \in \mathbb{R} \).

Thus \( \phi''(x) = -\lambda \phi(x) \), \( \phi'(1) = 0 \), and \( \phi(0) = 0 \). A computation now shows that \( \phi(x) \) must be a constant multiple of \( \sin(\pi n x/2) \) for some odd positive integer \( n \). It is then easy to verify that

\[
\Lambda = \left\{ \sqrt{2} \sin \left( \frac{\pi n x}{2} \right) \right\}_{n \geq 1 \text{ odd}}
\]

forms a complete orthonormal set of eigenfunctions for the Laplacian with respect to \( \delta_0 \). The corresponding eigenvalues are \( \pi^2 n^2/4 \), each of which occurs with multiplicity one.

The next result follows immediately from Theorem 14, but in order to show the connection to classical Fourier analysis, we will deduce it directly from Theorem 13.

**Theorem 15.** Every \( f \in S([0, 1]) \) can be written as a uniformly convergent generalized Fourier series of the form

\[
f(x) = f(0) + \sum_{n \geq 1 \text{ odd}} a_n \sqrt{2} \sin \left( \frac{\pi n x}{2} \right),
\]

where

\[
a_n = \int_{\Gamma} f(x) \sqrt{2} \sin \left( \frac{\pi n x}{2} \right) \, dx.
\]

**Proof.** For \( f \in S([0, 1]) \), subtract \( f(0) \) if necessary so that we may assume \( f(0) = 0 \). Define

\[
\tilde{f}(x) = \begin{cases} 
  f(4x), & 0 \leq x \leq \frac{1}{4} \\
  f(2 - 4x), & \frac{1}{4} \leq x \leq \frac{1}{2} \\
  -f(4x - 2), & \frac{1}{2} \leq x \leq \frac{3}{4} \\
  -f(4 - 4x), & \frac{3}{4} \leq x \leq 1.
\end{cases}
\]

Then \( \tilde{f} \) is piecewise smooth and periodic with period 1, so we may consider it as a function on the circle. Theorem 13 now applies, and the Fourier coefficients are
easily calculated to be:
\[ a_0 = \int_0^1 \tilde{f}(x) \, dx = 0, \]
\[ \tilde{a}_n = \int_0^1 \tilde{f}(x) \sqrt{2} \cos(2\pi nx) \, dx = 0, \]
\[ \tilde{b}_n = \int_0^1 \tilde{f}(x) \sqrt{2} \sin(2\pi nx) \, dx = \begin{cases} 
\int_0^1 f(x) \sqrt{2} \sin \left( \frac{\pi nx}{2} \right) \, dx, & \text{if } n \text{ is odd} \\
0, & \text{if } n \text{ is even.}
\end{cases} \]

We can now represent \( \tilde{f} \) by a uniformly convergent series, which in turn gives a representation of \( f(4x) \) for \( x \in [0, \frac{1}{4}] \):
\[ f(4x) = \sum_{n \geq 1 \text{ odd}} \tilde{b}_n \sqrt{2} \sin(2\pi nx). \]
Replacing \( 4x \) with \( x \) gives precisely the result we want on \([0, 1]\).

As an application, we prove the following irresistible identity, which was mentioned in §1.

**Theorem 16.** For all real numbers \( 0 \leq x, y \leq 1 \), we have
\[ \min\{x, y\} = 8 \sum_{n \geq 1 \text{ odd}} \frac{\sin \left( \frac{\pi nx}{2} \right) \sin \left( \frac{\pi ny}{2} \right)}{\pi^2 n^2}. \]

**Proof.** We provide two proofs of this result; the first one is quicker, but the second proof generalizes better and uses more explicitly the theory of metrized graphs.

**First proof:** Fix \( y \in [0, 1] \) and set \( f(x) = \min\{x, y\} \). Using Theorem 15, we compute that
\[ a_n = \sqrt{2} \int_0^1 f(x) \sin \left( \frac{\pi nx}{2} \right) \, dx \]
\[ = \sqrt{2} \left\{ \int_0^y x \sin \left( \frac{\pi nx}{2} \right) \, dx + \int_y^1 y \sin \left( \frac{\pi nx}{2} \right) \, dx \right\} = \frac{4\sqrt{2}}{\pi^2 n^2} \sin \left( \frac{\pi ny}{2} \right). \]
Noting that \( f(0) = 0 \) and inserting the coefficients \( a_n \) into Theorem 15 yields the result.

**Second proof:** Thinking of \([0, 1]\) as a metrized graph, we see that \( \min\{x, y\} \) coincides with the function \( j_0(x, y) \) (they have the same Laplacian and agree at 0). For \( n \geq 1 \) odd, let \( \phi_n(x) = \sqrt{2} \sin(\pi nx/2) \) and set \( \lambda_n = \pi^2 n^2/4 \).

To prove the result, fix \( y \in [0, 1] \) and use Theorem 15 to write
\[ j_0(x, y) = \sum_{n \geq 1 \text{ odd}} a_n \phi_n(x). \]

Each \( \phi_n \), being an eigenfunction of the Laplacian, satisfies
\[ \Delta \phi_n = \frac{\phi_n(x) \, dx - C_n \delta_0}{\lambda_n} \]
for some \( C_n \in \mathbb{R} \) (cf. Definition 11). It follows that
\[ \phi_n(x) \, dx = \frac{\Delta \phi_n}{\lambda_n} + C_n \delta_0. \]
Applying this to calculate the Fourier coefficients of \( j_0(x, y) \), we have
\[
a_n = \int_{\Gamma} j_0(x, y) \phi_n(x) dx = \int_{\Gamma} j_0(x, y) \left( \frac{\Delta \phi_n(x)}{\lambda_n} + C_n \delta_0(x) \right)
\]
\[
= \left( \int_{\Gamma} \frac{\phi_n(x)}{\lambda_n} \Delta_x j_0(x, y) \right) + C_n j_0(0, y) \quad \text{by Theorem 11}
\]
\[
= \int_{\Gamma} \frac{\phi_n(x)}{\lambda_n} \{ \delta_y(x) - \delta_0(x) \} = \frac{\phi_n(y)}{\lambda_n}.
\]
Substituting our formula for \( a_n \) into (13), we obtain
\[
j_0(x, y) = \sum_{n \geq 1 \text{ odd}} \frac{\phi_n(x) \phi_n(y)}{\lambda_n},
\]
which is equivalent to the desired result. \( \square \)

Theorem 14 together with an argument similar to the second proof of Theorem 16 yields the following more general fact:

**Theorem 17.** Let \( \Gamma \) be a metrized graph, and let \( z \in \Gamma \). Suppose \( \{ \phi_n(z) \}_{n \geq 1} \) is a complete orthonormal set of eigenfunctions of the Laplacian relative to the measure \( \delta_z \), with corresponding eigenvalues \( \lambda_n \). Then for all \( x, y \in \Gamma \), we have
\[
j_z(x, y) = \sum_{n \geq 1} \frac{\phi_n(x) \phi_n(y)}{\lambda_n}.
\]

A proof of this theorem and many more results concerning Fourier analysis on metrized graphs can be found in [BR].

**Exercise 15.** Find other nice identities like the one in Theorem 16 by taking a metrized graph \( \Gamma \), working out the eigenfunctions of the Laplacian with respect to \( \delta_z \) for some point \( z \in \Gamma \), and applying Theorem 17.

9. Epilogue

The material in this expository paper was adapted from a series of lectures given by the first author and Robert Rumely for the “Analysis on Metrized Graphs” REU in summer 2003, and represents the jumping-off point for several research questions explored during the REU. These questions included:

- Is there a good discrete analogue of the canonical measure? (Yes.)
- Do the eigenvalues of the Laplacian matrix on a sequence of models for a metrized graph \( \Gamma \) converge (under suitable hypotheses) to the eigenvalues of the Laplacian operator on \( \Gamma \)? (Yes.)
- If \( \Gamma \) is normalized to have total length 1, can the quantity \( \tau(\Gamma) \) from Exercise 14 be arbitrarily small? (No.)

A more detailed discussion of these questions, and of the results obtained, can be found at [http://www.math.uga.edu/~mbaker/REU/REU.html](http://www.math.uga.edu/~mbaker/REU/REU.html)

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