A lower bound of Ruzsa’s number related to the Erdős-Turán conjecture

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Abstract

For a set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_A(n)$ denote the number of ordered pairs $(a, a') \in A \times A$ such that $a + a' = n$. The celebrated Erdős-Turán conjecture says that, if $R_A(n) \geq 1$ for all sufficiently large integers $n$, then the representation function $R_A(n)$ cannot be bounded. For any positive integer $m$, Ruzsa’s number $R_m$ is defined to be the least positive integer $r$ such that there exists a set $A \subseteq \mathbb{Z}_m$ with $1 \leq R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. In 2008, Chen proved that $R_m \leq 288$ for all positive integers $m$. In this paper, we prove that $R_m \geq 6$ for all integers $m \geq 36$. We also determine all values of $R_m$ when $m \leq 35$.

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1 Introduction

Let $\mathbb{N}$ be all nonnegative integers. For any set $A, B \subseteq \mathbb{N}$, let

$$R_{A,B}(n) = \sharp \{ (a, b) : a \in A, b \in B, a + b = n \}.$$ 

Let $R_A(n) = R_{A,A}(n)$. If $R_A(n) \geq 1$ for all sufficiently large integers $n$, then we say that $A$ is a basis of $\mathbb{N}$. The celebrated Erdős-Turán conjecture [7] states that if $A$ is a basis of

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N, then \( R_A(n) \) cannot be bounded. Erdős [6] proved that there exists a basis \( A \) and two constants \( c_1, c_2 > 0 \) such that \( c_1 \log n \leq R_A(n) \leq c_2 \log n \) for all sufficiently large integers \( n \). Recently, Dubickas [5] gave the explicit values of \( c_1 \) and \( c_2 \). In 2003, Nathanson [14] proved that the Erdős-Turán conjecture does not hold on \( \mathbb{Z} \). In fact, he proved that there exists a set \( A \subseteq \mathbb{Z} \) such that \( 1 \leq R_A(n) \leq 2 \) for all integers \( n \). In the same year, Grekos et al. [8] proved that if \( R_A(n) \geq 1 \) for all \( n \), then \( \limsup_{n \to \infty} R_A(n) \geq 6 \). Later, Borwein et al. [2] improved 6 to 8. In 2013, Konstantoulas [11] proved that if the upper density \( d(\mathbb{N} \setminus (A + A)) \) of the set of numbers not represented as sums of two numbers of \( A \) is less than 1/10, then \( R_A(n) > 5 \) for infinitely many natural numbers \( n \). Chen [4] proved that there exists a basis \( A \) of \( \mathbb{N} \) such that the set of \( n \) with \( R_A(n) = 2 \) has density one. Later, the second author [17] and Tang [16] generalized Chen’s result. For the analogue of Erdős-Turán conjecture in groups, one can refer to [9], [10] and [12].

For a positive integer \( m \), let \( \mathbb{Z}_m \) be the set of residue classes mod \( m \). For \( A, B \subseteq \mathbb{Z}_m \), let \( R_{A,B}(n) \) be the number of solutions of equation \( a + b = n \), \( a \in A \), \( b \in B \). Let \( R_A(n) = R_{A,A}(n) \). If \( R_A(n) \geq 1 \) for all \( n \in \mathbb{Z}_m \), then \( A \) is called an additive basis of \( \mathbb{Z}_m \).

In 1990, Ruzsa [15] found a basis \( A \) of \( \mathbb{N} \) for which \( R_A(n) \) is bounded in the square mean. Ruzsa’s method implies that there exists a constant \( C \) such that for any positive integer \( m \), there exists an additive basis \( A \) of \( \mathbb{Z}_m \) with \( R_A(n) \leq C \) for all \( n \in \mathbb{Z}_m \). For each positive integer \( m \), Chen [3] defined Ruzsa’s number \( R_m \) to be the least positive integer \( r \) such that there exists an additive basis \( A \) of \( \mathbb{Z}_m \) with \( R_A(n) \leq r \) for all \( n \in \mathbb{Z}_m \). In this paper, Chen also proved that \( R_m \leq 288 \) for all positive integers \( m \) and \( R_{2p^2} \leq 48 \) for all prime numbers \( p \). Until now, this is the best upper bound about Ruzsa’s number and there is no nontrivial lower bound. In fact, in the same paper, Chen says “We have \( R_m \geq 3 \) for \( m \neq 1, 2, 3 \). Now we cannot improve this trivial lower bound”.

In this paper, we give a nontrivial lower bound of Ruzsa’s number.

**Theorem 1.** \( R_m = 2 \) if and only if \( m = 2, 3 \); \( R_m = 3 \) if and only if \( m = 4, 5, 7 \).

**Remark 1.** If \( m > 1 \) and \( A \subseteq \mathbb{Z}_m \) is an additive basis, then \( |A| \geq 2 \). It follows that there exist two distinct elements \( a, a' \in A \), and so \( R_A(a + a') \geq 2 \). Hence \( R_m \geq 2 \) if and only if \( m = 1 \).

**Theorem 2.** \( R_m = 4 \) if and only if \( m = 6, 8, 9, 10, 11, 12, 13, 14, 15, 19 \); \( R_m = 5 \) if and only if \( m = 16, 17, 18, 20, 21, 22, 23, 24, 25, 27, 28, 35 \).

By Theorems 1 and 2, we have the following Corollary.

**Corollary 1.** If \( m \geq 36 \), then \( R_m \geq 6 \).

**Remark 2.** Furthermore, if \( m \leq 35 \), then \( R_m \leq 6 \). We list all the values of \( R_m \) (\( 2 \leq m \leq 35 \)) and a set \( A \subseteq \mathbb{Z}_m \) such that \( 1 \leq R_A(n) \leq R_m \) for all \( n \in \mathbb{Z}_m \) in the Appendix.
2 Proofs

In order to prove Theorems 1 and 2, we need some lemmas in the following. The first lemma due to Lev and Sárközy [13] is the main tool of our proofs.

**Lemma 1.** (Lev and Sárközy’s lower bound) If $A$ is a subset of a finite non-trivial abelian group $G$, then for any real number $c$ we have

$$\sum_{g \in G} (R_A(g) - c)^2 \geq \frac{1}{|G| - 1} \left( \frac{|A|^4}{|G|^3} - 2|A|^3 + |A|^2 |G| \right).$$

**Lemma 2.** Let $A \subseteq \mathbb{Z}_m$. If $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, then $|A| > \sqrt{2m} - 1/2$.

**Proof.** Since $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, we have

$$|A|^2 = \sum_{n=0}^{m-1} R_A(n) \geq |\{ n : n \in \mathbb{Z}_m, R_A(n) = 1 \}| + 2|\{ n : n \in \mathbb{Z}_m, R_A(n) \geq 2 \}|$$

$$= 2|\{ n : n \in \mathbb{Z}_m \}| - |\{ n : n \in \mathbb{Z}_m, R_A(n) = 1 \}|$$

$$= 2m - |\{ n : n \in \mathbb{Z}_m, R_A(n) = 1 \}| \geq 2m - |A|.$$ 

Hence $(|A| + 1/2)^2 > 2m$, that is, $|A| > \sqrt{2m} - 1/2$. \( \square \)

**Lemma 3.** Let $A \subseteq \mathbb{Z}_m$ and $c$ be a positive integer. If $R_A(n) \leq c$ for all $n \in \mathbb{Z}_m$, then $|A| \leq \sqrt{cm}$.

This lemma follows from $|A|^2 = \sum_{n=0}^{m-1} R_A(n) \leq cm$ immediately.

**Lemma 4.** (See [1, P. 827, Test C].) Suppose that $v, \lambda, k$ ($v \geq k \geq \lambda$) are positive integers. Let $p$ be a prime divisor of $k - \lambda$ and let $w \geq 1$, $(w, p) = 1$, be a divisor of $v$ for which there exists an integer $f > 0$ such that $p^f \equiv -1 \pmod{w}$. If $p^e$ exactly divides $k - \lambda$ and $p^f$ ($l \geq 0$) exactly divides $v$, then there exists a set $A \subseteq \mathbb{Z}_v$ with $|A| = k$ such that the congruence $a - a' \equiv b \pmod{v}$, $a, a' \in A$ has exactly $\lambda$ distinct solutions for all $b \neq 0 \pmod{v}$ if and only if

$$p^\lfloor e/2 \rfloor < (v/w)p^{-l},$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$.

**Lemma 5.** Let $A$ be an additive basis of $\mathbb{Z}_m$ and $k, l$ be positive integers with $(l, m) = 1$. Then $A + k$, $lA$ is also an additive basis and

$$\max_{n \in \mathbb{Z}_m} R_A(n) = \max_{n \in \mathbb{Z}_m} R_{A+k}(n) = \max_{n \in \mathbb{Z}_m} R_{lA}(n).$$

This lemma follows from $R_A(n) = R_{A+k}(n + 2k) = R_{lA}(ln)$ for all $n \in \mathbb{Z}_m$ immediately.
Proof of Theorem 1. If \( m \leq 11 \), by the computer-based calculation, then we obtain that \( R_m = 2 \) if and only if \( m = 2, 3 \) and \( R_m = 3 \) if and only if \( m = 4, 5, 7 \). Now it suffices to prove that \( R_m \leq 3 \) implies \( m \leq 11 \). Suppose that \( m \geq 12 \) and there exists a subset \( A \subseteq \mathbb{Z}_m \) such that \( 1 \leq R_A(n) \leq 3 \) for all \( n \in \mathbb{Z}_m \).

Putting \( G = \mathbb{Z}_m \) and \( c = 2 \), by Lemma 1, we obtain that for any subset \( A \subseteq \mathbb{Z}_m \),

\[
(1) \quad \frac{1}{m-1} \sum_{n=0}^{m-1} (R_A(n) - 2)^2 \geq \frac{|A|^2 (m - |A|)^2}{m(m-1)}.
\]

Since \( 1 \leq R_A(n) \leq 3 \), it follows that

\[
(R_A(n) - 2)^2 = \begin{cases} 
1, & \text{if } R_A(n) \text{ is odd;} \\
0, & \text{if } R_A(n) \text{ is even.}
\end{cases}
\]

Furthermore, if \( R_A(n) \) is odd, then there exists \( a \in A \) such that \( n = 2a \), and so

\[
(2) \quad \frac{1}{m-1} \sum_{n=0}^{m-1} (R_A(n) - 2)^2 = \frac{1}{2|R_A(n)|} \leq \sum_{a \in A} 1 = |A|.
\]

By (1) and (2), we have

\[
|A|(m - |A|)^2 \leq m(m - 1) < m^2.
\]

On the other hand, by Lemmas 2 and 3, we have \( \sqrt{2m - 1/2} < |A| \leq \sqrt{3m} \). Hence

\[
|A|(m - |A|)^2 > (\sqrt{2m - 1/2})(m - \sqrt{3m})^2 > m^2,
\]

because \( \sqrt{2m - 1/2} > 4 \) and \( \sqrt{3m} \leq m/2 \) for \( m \geq 12 \). This is a contradiction.

Proof of Theorem 2. We first prove that \( R_m \leq 5 \) implies that \( m \leq 500 \). Suppose that \( m > 500 \) and there exists \( A \subseteq \mathbb{Z}_m \) such that \( 1 \leq R_A(n) \leq 5 \) for all \( n \in \mathbb{Z}_m \). By Lemma 1, taking \( G = \mathbb{Z}_m \) and \( c = 3 \), we get

\[
(3) \quad \frac{1}{m-1} \sum_{n=0}^{m-1} (R_A(n) - 3)^2 \geq \frac{|A|^2 (m - |A|)^2}{m(m-1)}.
\]

If \( R_A(n) \) is odd, then \( (R_A(n) - 3)^2 \leq 4 \). If \( R_A(n) \) is even, then \( (R_A(n) - 3)^2 = 1 \). Hence

\[
(4) \quad \frac{1}{m-1} \sum_{n=0}^{m-1} (R_A(n) - 3)^2 \leq 4|\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is odd}\}| + |\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is even}\}| = m + |\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is odd}\}| \leq m + 3|A|.
\]

By (3) and (4), we have

\[
(5) \quad |A|^2 (m - |A|)^2 \leq (m + 3|A|)m(m - 1).
\]
On the other hand, by Lemmas 2 and 3, we have $\sqrt{2m} - 1/2 < |A| \leq \sqrt{5m}$. Hence

$$|A|^2(m - |A|)^2 > (\sqrt{2m} - 1/2)^2(m - \sqrt{5m})^2 > (1.9 \cdot 0.9^2)m^3$$
$$> 1.3m^3 > (m + 3\sqrt{5m})m^2 > (m + 3|A|)(m - 1),$$

because $\sqrt{2m} - 1/2 > \sqrt{1.9m}, m - \sqrt{5m} > 0.9m$ and $m + 3\sqrt{5m} < 1.3m$ for $m > 500$. This contradicts with the inequality (5). Thus, if $m > 500$, then $R_m \geq 6$.

Now we only need to consider cases $m \leq 500$.

If $m \leq 20$, then the computer-based calculation can run over all the sets $A \subseteq \mathbb{Z}_m$ with $\sqrt{2m} - 1/2 \leq |A| \leq \sqrt{5m}$ and we can determine these values of $R_m$. We obtain that $R_{16} = 4$ for $m \in \{6, 8, 9, 10, 11, 12, 13, 14, 15, 19\}$ and $R_{16} = 4$ for $m > 500$.

Next we assume that $21 \leq m \leq 500$. A routine computer-based calculation gives that the maximal pair of $(m, k)$ satisfying that

$$(6) \quad 21 \leq m \leq 500, \quad \sqrt{2m} - 1/2 \leq |A| = k \leq \sqrt{5m}$$

and the inequality (5) holds is $(m, k) = (91, 13)$. The value for such $(m, k)$ is too large for the computer-based calculation to run over all the sets $A \subseteq \mathbb{Z}_m$ with $|A| = 13$.

In the following, we need three steps to reduce these values.

Our task is to find all exact pairs of $(m, k)$ with the following property: There exists $A \subseteq \mathbb{Z}_m$ with $|A| = k$ such that $1 \leq R_A(n) \leq 5$ for all $n \in \mathbb{Z}_m$. In the first step, for $i \in \{1, 2, 3, 4, 5\}$, let

$$k_i = |\{n : n \in \mathbb{Z}_m, R_A(n) = i\}|.$$

Then

$$k_1 + k_2 + k_3 + k_4 + k_5 = k, \quad k_i \in \mathbb{N} \ (1 \leq i \leq 5),$$

$$(8) \quad k^2 = |A|^2 = \sum_{n=0}^{m-1} R_A(n) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5,$$

and

$$k_1 + k_3 + k_5 \leq |A| = k, \quad \text{and the equality holds when } m \text{ is odd.}$$

By Lemma 1, taking $c = k^2/m$, we have

$$(10) \quad \sum_{n=0}^{m-1} \left( R_A(n) - \frac{k^2}{m} \right)^2 = \sum_{i=1}^{5} \left( i - \frac{k^2}{m} \right)^2 k_i \geq \frac{|A|^2(m - |A|)^2}{m(m-1)} = \frac{k^2(m - k)^2}{m(m-1)}.$$

By the computer-based calculation, the maximal values of $(m, k)$ such that there exists nonnegative integers $k_1, k_2, k_3, k_4, k_5$ satisfying (6)-(10) is $(50, 12)$. This value is
also too large for the computer-based calculation to run over all subsets $A \subseteq \mathbb{Z}_{50}$ with $|A| = 12$.

In the second reduction step, we shall delete all pairs $(m, k)$ for which $42 \leq m \leq 50$. Here we need to improve the Lev-Sárközy’s bound. Clearly,

$$\sum_{n=0}^{m-1} \left( R_A(n) - \frac{k^2}{m} \right)^2 = \sum_{n=0}^{m-1} R_A^2(n) - \frac{2k^2}{m} \sum_{n=0}^{m-1} R_A(n) + \frac{k^4}{m}$$

$$= \sum_{n=0}^{m-1} R_A^2(n) - \frac{2k^2}{m} \cdot k^2 + \frac{k^4}{m} = \sum_{n=0}^{m-1} R_A^2(n) - \frac{k^4}{m}.$$

Next we use Lev-Sárközy’s arguments to obtain a better lower bound for $\sum_{n=0}^{m-1} \left( R_A(n) - \frac{k^2}{m} \right)^2$. Clearly, the sum $\sum_{n=0}^{m-1} R_A^2(n)$ counts the number of solutions of the equation

$$a_1 + a_2 = a_3 + a_4, \quad a_1, a_2, a_3, a_4 \in A.$$  

Rearranging these terms, one can rewrite this equation as $a_1 - a_3 = a_4 - a_2$. Hence

$$\sum_{n=0}^{m-1} R_A^2(n) = \sum_{n=0}^{m-1} R_{A,-A}(n) = k^2 + \sum_{n=1}^{m-1} R_{A,-A}(n).$$

Clearly, $\sum_{n=0}^{m-1} R_{A,-A}(n) = k^2 - k$. Let $k^2 - k = q(m-1) + r$, where $q, r$ are nonnegative integers and $0 \leq r < m - 1$. Then

$$q = \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor \quad \text{and} \quad r = k^2 - k - \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor (m - 1).$$

Hence

$$\sum_{n=0}^{m-1} R_A^2(n) = k^2 + \sum_{n=1}^{m-1} R_{A,-A}(n)$$

$$\geq k^2 + (q + 1)^2 r + q^2 (m - 1 - r)$$

$$= k^2 + (2q + 1) r + q^2 (m - 1)$$

$$= k^2 + \left(2 \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor + 1 \right) \left( k^2 - k - \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor (m - 1) \right) + \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor^2 (m - 1).$$

By (10), (11) and (12), we get the following better lower bound instead of (10).

$$\sum_{i=1}^{5} \left( i - \frac{k^2}{m} \right)^2 k_i \geq k^2 + \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor^2 (m - 1) - \frac{k^4}{m}$$

$$+ \left(2 \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor + 1 \right) \left( k^2 - k - \left\lfloor \frac{k^2 - k}{m - 1} \right\rfloor (m - 1) \right).$$

By the computer-based calculation, we list all pairs of $(m, k)$ such that there exist nonnegative integers $k_1, k_2, k_3, k_4, k_5$ satisfying (6)-(9) and (13) in the following.

$$(m, k) \in \{(21, 7),(21, 8),(21, 9),(22, 7),(22, 8),(22, 9),(23, 7),(23, 8),(23, 9),(24, 8),(24, 9),(25, 8),(25, 9),(26, 8),(26, 9),(27, 8),(27, 9),(28, 8),(28, 9),(28, 10),(29, 8),(29, 9),(29, 10),\}$$
In the last step, we deal with cases $(m, k) = (40, 11), (41, 11), (45, 12)$, since such values are also too large for the computer-based calculation.

Now we first deal with the largest case $(m, k) = (45, 12)$. Take $v = 45$, $\lambda = 3$, $k = 12$, $p = 3$, $w = 5$, $f = 2$, $l = 2$. By Lemma 4, it follows that there is no subset $A \subseteq \mathbb{Z}_{45}$ with $|A| = 12$ such that $R_{A, -A}(n) = 3$ for all $n \not\equiv 0 \pmod{45}$. In other words, for any set $A \subseteq \mathbb{Z}_{45}$, there exists $n \not\equiv 0 \pmod{45}$ such that $R_{A, -A}(n) \neq 3$. Noting that $\sum_{n=1}^{44} R_{A, -A}(n) = k^2 - k = 132$, we have

$$\sum_{n=1}^{44} R_{A, -A}^2(n) \geq 3 \times 42 + 2^2 + 4^2 = 398.$$ 

Hence, by (11) and (12), we have

$$\sum_{n=0}^{44} \left( R_A(n) - \frac{12^2}{45} \right)^2 = 12^2 + \sum_{n=1}^{44} R_{A, -A}^2(n) - \frac{12^4}{45} \geq 81.2.$$ 

On the other hand, we list all values of $(k_1, k_2, k_3, k_4, k_5)$ when $(m, k) = (45, 12)$ in the following.

| $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ |
|-------|-------|-------|-------|-------|
| 0     | 24    | 0     | 9     | 12    |
| 1     | 22    | 0     | 11    | 11    |
| 2     | 20    | 0     | 13    | 10    |
| 4     | 16    | 0     | 17    | 8     |
| $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ |
| 5     | 14    | 0     | 19    | 7     |
| 6     | 12    | 0     | 21    | 6     |
| 7     | 10    | 0     | 23    | 5     |
| 8     | 8     | 0     | 25    | 4     |
| 9     | 6     | 0     | 27    | 3     |
| 10    | 4     | 0     | 29    | 2     |
| 11    | 2     | 0     | 31    | 1     |
| 12    | 0     | 0     | 33    | 0     |

For all the values list above, we have

$$\sum_{n=0}^{44} \left( R_A(n) - \frac{12^2}{45} \right)^2 = \sum_{i=1}^{5} \left( i - \frac{12^2}{45} \right)^2 k_i = 79.2.$$ 

This is a contradiction.

Finally, we deal with the cases $(m, k) = (41, 11)$ and $(40, 11)$, since the number of sets $A$ for which the computer-based calculation can run over is about $\binom{39}{9}$. If $m = 41$, by Lemma 5, then we can assume that $0, 40 \in A$. Hence the number of such $A$ is $\binom{39}{8}$, and the computer-based calculation can run over all such sets $A$. Now we consider the case $m = 40$. If there is an element in $A$ coprime with 40, by Lemma 5, then we can assume that $0, 39 \in A$, and so the computer-based calculation can also deal with the case. If there is no element in $A$ coprime with 40, then we can assume that $0 \in A$ and

$$A \subseteq \{0, 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 25, 26, 28, 30, 32, 34, 35, 36, 38\}.$$
In this case, there are only $\binom{23}{10}$ sets $A$ and we can deal with it with the computer-based calculation.

Using these idea, by the computer-based calculation, we obtain

\[ R_m = 4 \quad \text{if and only if} \quad m = 6, 8, 9, 10, 11, 12, 13, 14, 15, 19; \]

\[ R_m = 5 \quad \text{if and only if} \quad m = 16, 17, 18, 20, 21, 22, 23, 24, 25, 27, 28, 35. \]

\[ \square \]

3 Appendix

| $m$ | $R_m$ | the set $A$ | $m$ | $R_m$ | the set $A$ |
|-----|-------|-------------|-----|-------|-------------|
| 2   | 2     | $\{0, 1\}$  | 19  | 4     | $\{0, 1, 5, 7, 8, 15, 18\}$ |
| 3   | 2     | $\{0, 1\}$  | 20  | 5     | $\{0, 1, 2, 5, 6, 13, 16\}$ |
| 4   | 3     | $\{0, 1, 2\}$| 21  | 5     | $\{0, 1, 2, 3, 4, 6, 13, 16\}$ |
| 5   | 3     | $\{0, 1, 2\}$| 22  | 5     | $\{0, 1, 2, 4, 5, 9, 15, 17\}$ |
| 6   | 4     | $\{0, 3, 4, 5\}$| 23  | 5     | $\{0, 1, 2, 3, 5, 11, 14, 18\}$ |
| 7   | 3     | $\{0, 1, 2, 4\}$| 24  | 5     | $\{0, 1, 2, 6, 9, 10, 12, 17\}$ |
| 8   | 4     | $\{0, 3, 5, 6, 7\}$| 25  | 5     | $\{0, 1, 2, 4, 9, 12, 20, 22\}$ |
| 9   | 4     | $\{0, 4, 6, 7, 8\}$| 26  | 6     | $\{0, 1, 2, 5, 15, 19, 20, 22\}$ |
| 10  | 4     | $\{0, 1, 2, 3, 6\}$| 27  | 5     | $\{0, 1, 2, 3, 5, 11, 15, 18, 23\}$ |
| 11  | 4     | $\{0, 4, 6, 8, 9\}$| 28  | 5     | $\{0, 1, 2, 4, 5, 8, 10, 17, 22\}$ |
| 12  | 4     | $\{0, 1, 6, 8, 9, 11\}$| 29  | 6     | $\{0, 1, 2, 3, 4, 6, 10, 17, 22\}$ |
| 13  | 4     | $\{0, 5, 7, 8, 11, 12\}$| 30  | 6     | $\{0, 1, 2, 3, 4, 5, 7, 11, 17, 22\}$ |
| 14  | 4     | $\{0, 4, 8, 9, 11, 12\}$| 31  | 6     | $\{0, 1, 2, 3, 4, 5, 9, 13, 20, 25\}$ |
| 15  | 4     | $\{0, 6, 8, 11, 12, 14\}$| 32  | 6     | $\{0, 1, 2, 3, 4, 5, 8, 15, 20, 26\}$ |
| 16  | 5     | $\{0, 1, 2, 3, 4, 7, 11\}$| 33  | 6     | $\{0, 1, 2, 3, 4, 6, 10, 14, 21, 26\}$ |
| 17  | 5     | $\{0, 1, 2, 3, 4, 7, 12\}$| 34  | 6     | $\{0, 1, 2, 3, 4, 6, 13, 19, 26, 29\}$ |
| 18  | 5     | $\{0, 1, 2, 3, 5, 8, 12\}$| 35  | 5     | $\{0, 1, 4, 5, 10, 12, 16, 19, 26, 34\}$ |
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