Radial parts of Haar measures and probability distributions on the space of rational matrix-valued functions

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Abstract. We consider the space $\mathcal{C}$ of conjugacy classes of the unitary group $U(n+m)$ with respect to a smaller unitary group $U(m)$. It is known that to every element of $\mathcal{C}$ we can canonically assign a rational matrix-valued function (the Livshits characteristic function) on the Riemann sphere. We find an explicit expression for the natural measure on $\mathcal{C}$ obtained as the push-forward of the Haar measure of $U(n+m)$ in terms of characteristic functions.

Keywords: inner functions, characteristic functions, Haar measure, Cayley transform, random functions.

§ 1. Introduction. Statement of the results

1.1. Purpose of the paper. There is extensive literature (see, for example, [1]–[4] and references therein) on Gaussian random holomorphic functions, a topic known since at least the Paley–Wiener book [5], Ch. 10. Relatively recently Krishnapur [6] initiated the study of random matrix-valued holomorphic functions.

In the present paper we consider measures on the space of rational matrix-valued functions on the Riemann sphere. The origin of the question under consideration (which is somewhat removed from the issues in modern theory of random holomorphic functions) is the following. For example, consider the unitary group $U(n)$. The distribution of eigenvalues of unitary matrices is a measure on the set of $n$-point subsets of the circle with density of the form $C \cdot \prod_{k<l}|z_k - z_l|^2$ (Weyl’s formula, see, for example, [7], formula (3.2.2) or [8], Theorem 11.2.1). There is a multitude of similar formulae, which are usually referred to as ‘radial parts of the Haar measures on symmetric spaces’. Namely, consider a Riemannian symmetric space $G/K$, the space of double cosets $K \backslash G/K$ and the push-forward of the Haar measure under the map $G \to K \backslash G/K$ (in the example with the unitary group we have $G = U(n) \times U(n)$ and $K$ is the diagonal subgroup $U(n)$). General formulae can be found in [9], Propositions X.1.17, X.1.19.

This important example has numerous applications and ramifications. However, its extensions to other pairs $G \supseteq K$ of groups and subgroups are almost absent.\footnote{\textsuperscript{1}The case when $G = SU(2) \times \cdots \times SU(2)$ and $K = SU(2)$ is the diagonal subgroup is examined in [10].}

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One of the obstacles to such extensions is the difficulty in describing double cosets or conjugacy classes. There is one case for which such a description has been known for a long time. Namely, the solution for conjugacy classes of the unitary group $U(n + m)$ with respect to a smaller subgroup $U(m)$ is given in terms of the Livshits’ characteristic functions (see §1.2). A counterpart of these functions for double cosets of $U(n + m)$ by $O(m)$ was suggested in [11] (see also [12]). In [13], [14] characteristic functions were constructed for a large class of group-subgroup pairs. Since the characteristic functions in [13] originate from representation theory (see [15]), the question concerning radial parts of Haar measures arises naturally.

In the present paper we get an explicit formula for the radial part of the Haar measure in the case of conjugacy classes of $U(n + m)$ by $U(m)$.

1.2. The Livshits characteristic function. Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a block matrix of size $(n + m) \times (n + m)$. The characteristic function of $g$ (also referred to as the transfer function) is a function on $\mathbb{C}$ defined by

$$\chi(\lambda) := \alpha + \lambda \beta (1 - \lambda \delta)^{-1} \gamma.$$  (1.1)

It takes values in the space of matrices of size $n \times n$. We easily see that this function remains unchanged under conjugation of the matrix $g$ by matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. In other words, $\chi$ is an invariant of the conjugacy classes of $U(n + m)$ with respect to $U(m)$.

It is known that every rational matrix-valued function having no singularity at $\lambda = 0$ is the characteristic function of some matrix. The matrix $g$ (or, more precisely, its conjugacy class) is not uniquely determined by its characteristic function\(^2\) (see, for example, [16], Ch. 19).

Now let $g$ be a unitary matrix. Then the characteristic function possesses the following properties.

1) The values of $\chi(\lambda)$ for $|\lambda| = 1$ are unitary matrices.\(^3\)

2) When $|\lambda| < 1$, we have\(^4\) $\|\chi(\lambda)\| \leq 1$. It follows by the Riemann–Schwarz reflection principle that $\|\chi(\lambda)^{-1}\| \leq 1$ for $|\lambda| > 1$.

3) Write $\det \chi(\lambda)$ as a fraction $u(\lambda)/v(\lambda)$ that cannot be simplified. Then the degrees of the polynomials $u(\lambda), v(\lambda)$ do not exceed $m$.

Indeed, this determinant can be written as\(^5\)

$$\det \chi(\lambda) = \frac{\det \begin{pmatrix} \alpha & -\lambda \beta \\ \gamma & 1 - \lambda \delta \end{pmatrix}}{\det(1 - \lambda \delta)}. \quad (1.2)$$

\(^2\)However, the matrix of minimum size with a given characteristic function is unique up to conjugation (see [16], Ch. 19). Moreover, an element of the categorical quotient of $GL(n + m, \mathbb{C})$ by $GL(m, \mathbb{C})$ is uniquely determined by its characteristic function (see [17]).

\(^3\)Proofs of this and the next statement are given in §2.1, Remark 2.3.

\(^4\)Thus, the characteristic functions are matrix-valued analogues of inner functions, which are well known in the classical theory of analytic functions (see, for example, [18]). The theory of matrix-valued inner functions was developed in [19]. For inner functions with a matrix-valued argument, see [13].

\(^5\)To do this, it suffices to apply the usual formula for the determinant of a block matrix (see (2.19) below).
Conversely, every rational function on \( \mathbb{C} \) with values in the space of \( n \times n \) matrices that has these properties is the characteristic function of some unitary matrix of size \( n + m \). We denote the set of such functions by \( R_n(m) \).

Consider block matrices of size \( n + m = n + (m_1 + m_2) \) and of the form

\[
g = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \kappa \end{pmatrix}.
\]

(1.3)

The characteristic function is easily seen to be independent of \( \kappa \). On the other hand, the eigenvalues of \( \kappa \) are invariants of the conjugacy class. It is easy to show that every matrix \( g \in U(n + m) \) can be reduced by conjugation by an element of \( U(m) \) to the form (1.3) with \( \|\delta\| < 1 \).

Finally, the spectral data determining a conjugacy class of \( U(n + m) \) by \( U(m) \) are as follows: the characteristic function and the set of eigenvalues of \( \delta \) that lie on the circle \( |\lambda| = 1 \).

Remark 1.1. I know no literature sources containing simple proofs of the statements about unitary matrices in the (very limited) general setting that is necessary for us. A short exposition for arbitrary matrices is contained in [16], Ch. 19 (see also [20]). The unitary case requires additional arguments. We are going to consider a particular case of Theorem 5.1 in [21]. Moreover, a unitary matrix can be restored from its characteristic function by Potapov’s method [19], using the expansion of a rational matrix-valued function into a Blaschke product.

In any case, for generic elements of \( U(n + m) \) the block \( \delta \) is purely contractive and the conjugacy class is determined by the characteristic function. Thus we have a map

\[
U(n + m) \to R_n(m)
\]

and we want to evaluate the image of the Haar measure under this map. The result will be presented in terms of the set of those points \( \lambda \) where \( \chi(\lambda) \) has eigenvalue \(-1\) and in terms of the corresponding eigenvectors.

1.3. The density of the measure. Take an element of \( R_n(m) \) in general position (that is, an element of a set of full measure, which will be chosen in due course). Consider the set of all points \( t_k \in \mathbb{C} \) such that the matrix \( \chi(t_k) \) has eigenvalue \(-1\). This set is contained in the circle \( |\lambda| = 1 \). In general position there are \( m \) such points. We order the points \( t_k \) by the rule

\[
2\pi > \arg t_1 > \cdots > \arg t_n > 0.
\]

In general position there is a unique corresponding eigenvector \( c_k \),

\[
\chi(t_k)c_k = -c_k, \quad c_k = (c^1_k, \ldots, c^n_k) \in \mathbb{C}^n.
\]

To fix coordinates on \( R_n(m) \), we normalize the vectors \( c_k \) by the condition

\[
\langle \chi'(t_k)c_k, c_k \rangle = -t_k^{-1}.
\]

Moreover, we fix the phase of each vector \( c_k \) putting \( c^1_k \geq 0 \).

\( ^6 \)For elements of \( U(n + m) \) in general position we have \( \|\alpha\| < 1 \). But \( \alpha = \chi(0) \), whence \( |\det \chi(0)| < 1 \). On the other hand, \( \|\chi(\lambda)\| = 1 \) on the unit circle. Applying the maximum principle for linear functionals on the space of matrices, we find that \( \|\chi(\lambda)\| < 1 \) inside the circle.
Lemma 1.2. The properties of characteristic functions listed in §1.2 are indeed satisfied in general position. The numbers $t_k$, the vectors $c_k$ and the unitary matrix $U := \chi(-1)$ uniquely determine the characteristic function in general position.

Let $C$ be the matrix of size $n \times m$ formed by the vector-columns $c_k$, and let $T$ be the diagonal matrix formed by the numbers $t_j$.

Theorem 1.3. The image of the Haar probability measure under the map $U(n+m) \rightarrow \mathcal{R}_{n}(m)$ is of the form

$$\theta_{n,m} |\det(1 + T + C^*(1 + U)C)|^{-2n-2m} |\det(1 + U)|^{2m} \prod_{k=1}^{m} |1 + t_k|^{2m+2n}$$

$$\times \prod_{1 \leq k < l \leq n} |t_k - t_l|^2 \, d\sigma_n(U) \prod_{k=1}^{m} c_k^1 \, dc_k^1 \prod_{k=1}^{m} \prod_{j=2}^{n} d\text{Re} c_j^k \, d\text{Im} c_j^k \prod_{k=1}^{n} \frac{dt_k}{it_k}, \quad (1.4)$$

where $d\sigma_n(U)$ is the Haar measure on $U(n)$ and

$$\theta_{n,m} = 2^{-m-n} \pi^{-m-n} \frac{\prod_{j=1}^{m+n-1} j!}{\prod_{j=1}^{n-1} j! \prod_{j=1}^{m} j!}. \quad (1.5)$$

We mention two papers on random analytic functions in connection with Theorem 1.3.

First, measures on the space of scalar meromorphic functions in terms of distributions of their poles and residues were considered by Wigner [22]. His terminology is close to that of Theorem 1.3. Moreover, we literally encounter some distributions of poles and residues during a calculation in §2.4 (the main difference is that Wigner’s distributions of poles and residues are determined by independent random variables, which is not the case in our paper). The class of meromorphic functions considered in [22] can easily be transformed into the class of (scalar-valued) inner functions.

Second, Katsnelson [23] studied a measure on the space of inner functions. However, the object which appears in this work seems to be quite different from ours.

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§ 2. Proofs

We introduce the following notation:

- $\text{Mat}_{k,l}$ is the space of complex matrices of size $k \times l$;
- $\text{Herm}_k$ is the space of Hermitian matrices of size $k$;
- $\text{A}\text{Herm}_k$ is the space of anti-Hermitian ($X^* = -X$) matrices of size $k$;
- $T^k$ is a torus, that is, the product of $k$ circles with the corresponding group topology;
- $\mathbb{R}_+$ is the positive semi-axis.
2.1. A preparatory lemma. Let $T$ be a square matrix. We define its Cayley transform by

$$g \mapsto H = -1 + 2(1 + g)^{-1} = (1 + g)^{-1}(1 - g).$$

This transform is inverse to itself and sends unitary matrices into anti-Hermitian ones.

Lemma 2.1. Consider a block matrix $g$ of size $n + m$ and perform the following chain of manipulations:

1) apply the Cayley transform and denote the result by $H$;
2) take the characteristic function $\varphi(s)$ of the matrix $H$;
3) apply the Cayley transform to $\varphi(s)$ to get a function $\psi(s)$ with values in $\text{Mat}_{n,n}$;
4) consider the function $\psi((t + 1)/(t - 1))$.

Then the resulting function coincides with the characteristic function $\chi_g(t)$ of $g$.

First, we recall that the definitions of the Cayley transform and the characteristic function can be stated as follows.

Lemma 2.2. 1) Let $g$ be a square matrix of size $n$. Let vectors $u, v \in \mathbb{C}^n$ be related by the formula

$$(u - v) = g(u + v). \quad (2.1)$$

Then $v = Hu$, where $H$ is the Cayley transform of $g$.

2) Let $g$ be a block matrix of size $n + m$. Fix $\lambda \in \mathbb{C}$. Consider the set $L$ of all pairs $q, p \in \mathbb{C}^n$ for which there is an $x \in \mathbb{C}^m$ such that

$$\begin{pmatrix} q \\ x \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p \\ \lambda x \end{pmatrix}. \quad (2.2)$$

Then $L \subset \mathbb{C}^n \oplus \mathbb{C}^n$ is given by the equation

$$q = \chi_g(\lambda)p$$

for all $\lambda$ except the poles of the characteristic function.

Proof of Lemma 2.2. The first assertion is obvious: write (2.1) as

$$(1 - g)u = (1 + g)v.$$

To prove the second assertion, write (2.2) as

$$q = \alpha p + \lambda \beta x, \quad x = \gamma p + \lambda \delta x.$$

Eliminating $x$, we get the desired assertion of Lemma 2.2. □

Remark 2.3. We can now easily derive properties 1)–2) of the characteristic functions in §1.2. Indeed, if $g$ is a unitary matrix, then

$$\|q\|^2 + \|x\|^2 = \|p\|^2 + \|\lambda x\|^2.$$

Assuming that $|\lambda| = 1$, we get $\|q\|^2 = \|p\|^2$. It follows that $\chi(\lambda)$ is a unitary matrix. If $|\lambda| < 1$, then $\|q\|^2 \geq \|p\|^2$, that is, $\|\chi(\lambda)\| \leq 1$. It is worth noting that a proof using (1.1) directly is not so easy.
Proof of Lemma 2.1. Let $g$ be a block square matrix of size $n + m$. Then $v = Hu$ is equivalent to the equality
$$u - v = g(u + v).$$
Next, $q = \varphi(s)p$ if there is a vector $x$ such that
$$p_{sx} - q_{sx} = g_{sx} + \left( p_{sx} + q_{sx} \right),$$
or
$$p_{sx} - q_{sx} = g_{sx}(s + 1)x.$$
We now apply the Cayley transform, that is, put $q = y - z, p = y + z$. It is more convenient to put $q = (y - z)/2, p = (y + z)/2$, which does not influence the result. Then we get
$$\left( \begin{array}{c} z \\ (s - 1)x \end{array} \right) = g \cdot \left( \begin{array}{c} y \\ (s + 1)x \end{array} \right),$$
recall that we consider pairs $z, y$ for which there is an $x$ such that this equality holds. Put $x' = (s - 1)x$. If we replace $x$ by $x'$, then the condition on $q, p$ remains unchanged. Thus,
$$\left( \begin{array}{c} z \\ x' \end{array} \right) = g \cdot \left( \begin{array}{c} y \\ \frac{s + 1}{s - 1}x' \end{array} \right).$$
We get the definition of the characteristic function of $g$ at the point $(s + 1)/(s - 1)$. □

2.2. Beginning of the proof of Theorem 1.3. The first Cayley transform.
We now intend to find what happens to the Haar measure under the transformations described in Lemma 2.1. It is important that all the operations over matrices commute with the conjugations by elements of $U(m)$.

The image of the Haar probability measure of $U(k)$ under the Cayley transform was evaluated by Hua Loo Keng. It has the form (see [7], § 3.1)
$$\tau_k \det(1 - X^2)^{-k} d\hat{X},$$
where $X$ is the Cayley transform of $g$,
$$d\hat{X} := \prod_{1 \leq k \leq l \leq n} d\text{Im} x_{kl} \prod_{1 \leq k < l \leq n} d\text{Re} x_{kl},$$
and $\tau_k$ is a normalization constant:
$$\tau_k = 2^{k^2 - k} \pi^{-(k + 1)/2} \prod_{j=1}^{k-1} j!.$$ Clearly, the density of the measure can be written as
$$\tau_n \det(1 + X)^{-k} \det(1 - X)^{-k} = \tau_n |\det(1 + X)|^{-2k}.$$
In our case $k = n + m$. By writing the anti-Hermitian matrix $H$ (the Cayley transform of $g$) in the block form $H = i \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$, we get a measure
$$\tau_{m+n} \det \left[ 1 + i \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \right]^{-2n - 2m} d\hat{A} d\hat{B} d\hat{D},$$
where $d\hat{A}$, $d\hat{B}$, $d\hat{D}$ are the natural Lebesgue measures on the space of matrices:

$$d\hat{B} := \prod_{1 \leq k \leq n} \prod_{1 \leq l \leq m} d\Re b_{kl} d\Im b_{kl},$$

and similarly for $d\hat{A}$, $d\hat{D}$ as in (2.4).

2.3. Quotients by the group action. We bring $D$ to a diagonal form. Let $\mu_1 > \cdots > \mu_m$ be the eigenvalues and let $M$ be the diagonal form, so that $M = V^{-1}DV$. Denote the set of all such tuples by $\Xi m \subset \mathbb{R}^m$. The distribution of eigenvalues of Hermitian matrices (see, for example, [7], §3.3 or [8], Theorem 10.1.4) is of the form

$$dw^m = \frac{\pi^{m(m-1)/2}}{\prod_{1 \leq j \leq m} j!} \prod_{1 \leq k < l \leq m} |\mu_k - \mu_l|^2 \prod_{k=1}^m d\mu_k. \quad (2.5)$$

Consider the action of $U(m)$ on the space $\frac{1}{i}H$ of Hermitian matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}^{-1} = \begin{pmatrix} A & BV \\ (BV)^* & V^{-1}DV \end{pmatrix}. \quad (2.6)$$

In fact, this group acts on the pairs $(D,B)$. The matrix $B$ can be regarded as an $n$-tuple of rows. Denoting them by $\beta_1, \ldots, \beta_n$, we get an action of the unitary group $U(m)$ on pairs consisting of a self-adjoint operator $D$ and an $n$-tuple of vectors $\beta_1, \ldots, \beta_n$.

We can bring $D$ to a diagonal form. Then the remaining freedom in (2.6) is to apply conjugation by elements of the diagonal subgroup $\mathbb{T}^m$. Such conjugations enable us to make all coordinates of the vector

$$\beta_1 = (b_{11}, \ldots, b_{1m})$$

real and positive. Then the coordinates of the remaining vectors $\beta_2, \ldots, \beta_n$ become fixed.

Lemma 2.4. Define a map

$$\Pi: \text{Herm}_m \times \text{Mat}_{n,m} \to \Xi m \times (\mathbb{R}_+)^m \times \text{Mat}_{n-1,m} \quad (2.7)$$

sending each pair $(D,B)$ into its canonical form $(M,B')$. Then the image of the Lebesgue measure under $\Pi$ is

$$dw^m(2\pi)^m \prod_{k=1}^m b'_{1k} db'_{1k} \prod_{1 \leq l \leq n, 2 \leq k \leq n} d\Re b'_{kl} d\Im b'_{kl}.$$ 

Proof. We write $\text{Fl}_m$ for the space whose points are ordered $m$-tuples $\ell = (\ell_1, \ldots, \ell_m)$ of pairwise orthogonal (complex) lines in $\mathbb{C}^m$. Clearly, $\text{Fl}_m$ is a flag space. The group $U(m)$ acts on $\text{Fl}_m$ transitively and the stabilizer of a point is isomorphic to $\mathbb{T}^m$, that is, $\text{Fl}_n \simeq U(m)/\mathbb{T}^m$. Since $U(m)$ is compact, there is a unique $U(m)$-invariant probability measure $d\ell$ on $\text{Fl}_n$. To every Hermitian matrix $D$ we
assign the \(m\)-tuple \(\mu_1 > \cdots > \mu_m\) of its eigenvalues and the \(m\)-tuple \(\ell_1, \ldots, \ell_m\) of its eigenlines. This yields a map 

\[
    \text{Herm}_m \to \Xi_m \times \text{Fl}_m,
\]

which is defined a.e. and bijective a.e. The image of the Haar measure under this map is \(dw^m d\ell\). This is equivalent\(^7\) to (2.5).

Thus the action of \(U(m)\) on 

\[
    \text{Herm}_m \times \text{Mat}_{n,m} \simeq \text{Herm}_m \times (\mathbb{C}^m)^n
\]

can be regarded as an action on the space 

\[
    \Xi_m \times \text{Fl}_m \times \mathbb{C}^m \times (\mathbb{C}^m)^{n-1}
\]

which is trivial on the first factor.

Consider the quotient of \(\text{Fl}_m \times \mathbb{C}^m\) by \(U(m)\). Fix a point \(\ell \in \text{Fl}_m\) and denote its stabilizer by \(T^m_{\ell} \subset U(m)\). The distribution of \(T^m_{\ell}\)-invariants in the fibre \(\mathbb{C}^m\) is the same for all \(\ell\). Therefore, on the space 

\[
    (\text{Fl}_m \times \mathbb{C}^m)/ U(m) \simeq (\mathbb{R}_+)^m
\]

we get the same distribution of invariants 

\[
    \prod_{k=1}^{m} b'_{1k} db'_{1k}.
\]

We can now identify the following spaces by means of a \(U(m)\)-equivariant measure-preserving transformation: 

\[
    \text{Fl}_m \times \mathbb{C}^m \simeq (\mathbb{R}_+)^m \times U(m).
\]

Thus we arrive at an action of \(U(m)\) on 

\[
    \Xi_m \times (\mathbb{R}_+)^m \times U(m) \times (\mathbb{C}^m)^{n-1}.
\]

This action is trivial on the first two factors. On \(U(m)\) the group acts by right shifts, and on vector-rows it acts in a natural way. The description of the quotient now becomes trivial. \(\square\)

We shall use a slightly modified version of Lemma 2.4. Define a map 

\[
    \pi: \Xi_m \times \text{Mat}_{n,m} \to \Xi_m \times (\mathbb{R}_+)^m \times \text{Mat}_{n-1,m}
\]

as a reduction of the first row to the canonical form using the action of the torus \(T^m\).

**Corollary 2.5.** The image of the measure on \(\Xi_m \times \text{Mat}_{n,m}\) under \(\pi\) coincides with that of the measure on \(\text{Herm}_m \times \text{Mat}_{n,m}\) under \(\Pi\). In other words, the natural identification of the spaces 

\[
    (\text{Herm}_m \times \text{Mat}_{n,m})/ U(m) \leftrightarrow (\Xi_m \times \text{Mat}_{n,m})/T^m
\]

is measure preserving.

---

\(^7\)Actually, proofs of (2.5) are usually based on this fact.
We arrive at the following problem. Consider the space \( \text{Herm}_n \times \text{Mat}_{n,m} \times \Xi_m \) equipped with the measure

\[
\tau_{m+n} \left| \det \left( 1 + i \begin{pmatrix} A & B \\ B^* & M \end{pmatrix} \right) \right|^{-2n-2m} dw_m(M) d\hat{A} d\hat{B}.
\]

We must describe the behaviour of this measure under the remaining transformations listed in Lemma 2.1.

2.4. Taking the characteristic function. Write the characteristic function of a matrix \( i \begin{pmatrix} A & B \\ B^* & M \end{pmatrix} \) as

\[
\varphi(s) = iA - sB(1 - isM)^{-1}B^* = iA - \sum_{k=1}^{m} \frac{sb_k b_k^*}{1 - is\mu_k},
\]

where \( b_k \) are the columns of the matrix \( B \).

We can see that the poles of \( \varphi(s) \) are located at the points \( s_k = 1/(i\mu_k) \), and the residues at these poles are matrices of rank 1 given by

\[
\text{Res}_{s=1/i\mu_k} \varphi(s) = -\frac{b_k b_k^*}{\mu_k^2}.
\]

Notice that \( \varphi(0) = iA \), and the vector \( b_k \) can be restored up to a phase from the matrix \( b_k b_k^* \). Hence there is no need to introduce new coordinates: the matrices \( A, B, M \) can be reconstructed from \( \varphi(s) \).

Notice that all the vectors \( b_k \) are non-zero in general position.

2.5. The second Cayley transform. Consider the function

\[
\psi(s) = -1 + 2(1 + \varphi(s))^{-1}.
\]

Notice that \( \psi(s) \) is a unitary matrix for purely imaginary \( s \) since \( \varphi(s) \) is anti-Hermitian. Put \( U := \psi(0) \). Clearly,

\[
iA = -1 + 2(1 + U)^{-1}.
\]

Next, the matrix \( (1 + \varphi(s))^{-1} \) is degenerate at the points \( s = 1/(i\mu_k) \), whence \( \psi(s) \) has eigenvalue \(-1\). Let us write the Taylor expansions of the functions \( 1 + \varphi(s) \) and \( (1 + \varphi(s))^{-1} \) at \( s = 1/(i\mu_k) \):

\[
1 + \varphi(s) = -\frac{b_k b_k^*}{\mu_k^2(s - 1/(i\mu_k))} + V + \cdots, \quad (1 + \varphi(s))^{-1} = W + \left(s - \frac{1}{i\mu_k}\right)Y + \cdots,
\]

where \( V, W, Y \) are certain matrices. Then

\[
(1 + \varphi(s))^{-1}(1 + \varphi(s)) = 1 = -\frac{Wb_k b_k^*}{\mu_k^2(s - 1/(i\mu_k))} + (WV - \mu_k^{-2}Yb_k b_k^*) + \cdots.
\]

Therefore,

\[
Wb_k b_k^* = 0, \quad WV - \mu_k^{-2}Yb_k b_k^* = 1.
\]

(2.10)
We have $Wb_k = 0$. Otherwise we would get a non-zero matrix by multiplying the vector-column $Wb_k$ by the non-zero row $b_k$. Next, we write down the equality
\[
\langle (WV - \mu_k^{-2}Yb_k^*b_k) b_k, b_k \rangle = \langle b_k, b_k \rangle
\]  
(2.11)
for the entries $\langle (\ldots) b_k, b_k \rangle$ on both sides of (2.10). Notice that
\[
\langle WVb_k, b_k \rangle = \langle Vb_k, W^*b_k \rangle = \langle Vb_k, 0 \rangle = 0.
\]
Indeed, $W = (\psi(s) + 1)/2$ is a normal matrix, therefore $\ker W = \ker W^*$. Next,
\[
(Yb_k b_k^*)b_k = Yb_k (b_k^* b_k) = \langle b_k, b_k \rangle Yb_k.
\]
Hence (2.11) transforms into
\[
\langle Yb_k, b_k \rangle = -2\mu_k^2
\]
or, equivalently, into
\[
\langle \psi'(1/i\mu_k), b_k \rangle = -\mu_k^2.
\]
In particular, we can see that the vectors $b_k$ can be reconstructed up to a phase from $\psi(s)$.

2.6. Linear-fractional change of the argument. It remains to perform the last step: passing to the function
\[
\chi(t) = \psi\left(\frac{t + 1}{t - 1}\right).
\]
We define new parameters of the conjugacy classes:
1) points $t_k$ of the unit circle determined by the condition
\[
t_k := \frac{1 + i\mu_k}{1 - i\mu_k} \quad \text{or} \quad \frac{t_k - 1}{t_k + 1} = i\mu_k;
\]
2) a matrix $U \in U(n)$,
\[
U := -1 + 2(1 + iA^{-1})^{-1};
\]
3) vectors $c_k \in \mathbb{C}^n$,
\[
c_k := \frac{1}{2} e^{i\theta_k} (1 + t_k) b_k.
\]
(2.13)
Here the phase factor $e^{i\theta_k}$ is fixed by the condition $c_k^1 > 0$, where $c_k^1$ is the first coordinate.

Clearly,
1) $U = \chi(-1)$,
2) $t_k$ are the points where $\chi(t)$ has eigenvalue $-1$,
3) $c_k$ are normalized solutions of the equation $\chi(t_k)v_k = -v_k$.

We form a diagonal matrix $T$ with entries $t_j$ and a matrix $C$ of size $n \times m$ with columns $c_k$. 
Condition (2.12) can be written as
\[
\langle \chi'(t_k)b_k, b_k \rangle = -\frac{4}{(t_k + 1)^2} = -\frac{4t_k^{-1}}{|t_k + 1|^2}
\]
or
\[
\langle \chi'(t_k)c_k, c_k \rangle = -t_k^{-1}.
\]

We can now see what happens with the measure (2.8) under this change. We shall simultaneously pass to the quotient by the action of \(T^m\), which was deferred in §2.3.

The Lebesgue measure \(d\dot{B}\) takes the following form after the change (2.13) and passing to the quotient by the torus action:
\[
2^{2m(n-1)}(2\pi)^m \prod_{k=1}^m |1 + t_k|^{-2n} \prod_{k=1}^m c_{1k}d_{c_{1k}} \prod_{1 \leq l \leq n, 2 \leq k \leq n}^m d\text{Re}c_{kl}d\text{Im}c_{kl}.
\]
(2.14)

The measure \(dw^m\) (see (2.5)) transforms into
\[
2^{m(n-1)/2}(2\pi)^m 2^{m(m+1)} \prod_{k=1}^m |1 + t_k|^{-2m} \prod_{1 \leq k < l \leq n}^m |t_k - t_l|^2 \prod_{k=1}^m \frac{dt_k}{it_k}.
\]
(2.15)

It remains to examine the factor
\[
\tau_{m+n}|\det\left(1 + i\binom{A}{B^*}\binom{B}{M}\right)|^{-2n-2m}d\dot{A},
\]

Using the Hua formula (2.3) for the Haar measure, we transform this factor into the form
\[
\tau_{m+n}\left|\det\left(1 + i\binom{A}{B^*}\binom{B}{M}\right)\right|^{-2n-2m}d\dot{A}
= \frac{\tau_{m+n}}{\tau_n}\left|\det\left(1 + i\binom{A}{B^*}\binom{B}{M}\right)\right|^{-2n-2m}|\det(1 + iA)^{2n}\tau_n|\det(1 + iA)^{-2n}d\dot{A}
= \frac{\tau_{m+n}}{\tau_n}\left|\det\left(1 + i\binom{A}{B^*}\binom{B}{M}\right)\right|^{-2n-2m}|\det(1 + iA)^{2n}d\sigma_n(U).
\]
(2.16)

Next we transform the two determinants in (2.16):
\[
\det(1 + iA) = \det(1 - 1 + 2(1 + U)^{-1}) = 2^n \det(1 + U)^{-1},
\]
(2.17)
\[
\det\left(1 + i\binom{A}{B^*}\binom{B}{M}\right) = \det\left(1 + iA \begin{pmatrix} iB \\ iB^* \end{pmatrix} \begin{pmatrix} 1 + iM \\ 1 + iM \end{pmatrix}\right)
= \det\begin{pmatrix} 2(1 + U)^{-1} & 2iC(1 + T)^{-1} \\ 2i(1 + T^{-1})C^* & 1 + (T + 1)^{-1}(T - 1) \end{pmatrix}.
\]
(2.18)

The lower right block of the last matrix is equal to \(2T(1 + T)^{-1} = 2(1 + T^{-1})^{-1}\). Using the formula
\[
\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det a \det(d - ca^{-1}b)
\]
(2.19)
for the determinant of a block matrix, we thus obtain

\[
2^{m+n} \det \begin{pmatrix} 1 & 0 \\ 0 & (1+T^{-1})^{-1} \end{pmatrix} \det \begin{pmatrix} (1+U)^{-1} & iC \\ iC^* & 1+T \end{pmatrix} \det \begin{pmatrix} 1 & 0 \\ 0 & (1+T)^{-1} \end{pmatrix}
= 2^{m+n} \prod_{k=1}^m (1+t_k)^{-2} \det(1+U)^{-1} \det(1+T+C^*(1+U)C).
\]  

(2.20)

Theorem 1.3 now follows from (2.14)–(2.20).

Lemma 1.2 requires no separate proof since our step-by-step calculations were based on passing from one parametrization of the double coset space to another.

Bibliography

[1] Y. Peres and B. Virág, “Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process”, Acta Math. 194:1 (2005), 1–35.

[2] M. Sodin, “Zeros of Gaussian analytic functions”, European congress of mathematics, Eur. Math. Soc., Zürich 2005, pp. 445–458.

[3] M. Sodin and B. Tsirelson, “Random complex zeroes. I. Asymptotic normality”, Israel J. Math. 144:1 (2004), 125–149.

[4] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág, Zeros of Gaussian analytic functions and determinantal point processes, Univ. Lecture Ser., vol. 51, Amer. Math. Soc., Providence, RI 2009.

[5] R. E. A. C. Paley and N. Wiener, Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., New York 1934; Russian transl., Nauka, Moscow 1964.

[6] M. Krishnapur, “From random matrices to random analytic functions”, Ann. Probab. 37:1 (2009), 314–346.

[7] L. K. Hua, Harmonic analysis of functions of several complex variables in the classical regions, Inostr. Lit., Moscow 1959; English transl., Transl. Math. Monogr., vol. 6, Amer. Math. Soc., Providence, RI 1963.

[8] J. Faraut, Analysis on Lie groups. An introduction, Cambridge Stud. Adv. Math., vol. 110, Cambridge Univ. Press, Cambridge 2008.

[9] S. Helgason, Differential geometry and symmetric spaces, Pure Appl. Math., vol. XII, Academic Press, New York–London 1962; Russian transl., Mir, Moscow 1964.

[10] Yu. A. Neretin, “Double cosets for SU(2) × · · · × SU(2) and outer automorphisms of free groups”, Int. Math. Res. Not. IMRN, 2011, no. 9, 2047–2067.

[11] Yu. A. Neretin, Categories of symmetries and infinite-dimensional groups, London Math. Soc. Monogr. (N.S.), vol. 16, The Clarendon Press, Oxford Univ. Press, New York 1996; Russian transl., URSS, Moscow 1998.

[12] G. I. Olshanskij, “Unitary representations of infinite dimensional pairs (G, K) and the formalism of R. Howe”, Representation of Lie groups and related topics, Adv. Stud. Contemp. Math., vol. 7, Gordon and Breach, New York 1990, pp. 269–463.

[13] Yu. A. Neretin, “Multi-operator colligations and multivariate characteristic functions”, Anal. Math. Phys. 1:2-3 (2011), 121–138.

[14] Yu. A. Neretin, Multiplication of conjugacy classes, colligations, and characteristic functions of matrix argument, arXiv:1211.7091.
Yu. A. Neretin, “Sphericity and multiplication of double cosets for infinite-dimensional classical groups”, Funktsional. Anal. i Prilozhen. 45:3 (2011), 79–96; English transl., Funct. Anal. Appl. 45:3 (2011), 225–239.

H. Dym, Linear algebra in action, Grad. Stud. Math., vol. 78, Amer. Math. Soc., Providence, RI 2007.

Yu. A. Neretin, “On p-adic colligations and ‘rational maps’ of Bruhat–Tits trees”, Geometric methods in physics (Bialowieża, Poland, June 28–July 4, 2015), Trends Math., Birkhäuser/Springer, Basel 2016, pp. 139–158.

J. B. Garnett, Bounded analytic functions, Pure Appl. Math., vol. 96, Academic Press, Inc., New York–London 1981; Russian transl, Mir, Moscow 1984.

V. P. Potapov, “The multiplicative structure of J-contractive matrix functions”, Trudy Moskov. Mat. Obshch., vol. 4, GITTL, Moscow 1955, pp. 125–236; English transl., Amer. Math. Soc. Transl. (2), vol. 15, Amer. Math. Soc., Providence, RI 1960, pp. 131–243.

H. Bart, I. Gohberg, and M. A. Kaashoek, Minimal factorization of matrix and operator functions, Oper. Theory Adv. Appl., vol. 1, Birkhäuser Verlag, Basel–Boston, Mass. 1979.

M. S. Brodskii, “Unitary operator colligations and their characteristic functions”, Uspekhi Mat. Nauk 33:4(202) (1978), 141–168; English transl., Russian Math. Surveys 33:4 (1978), 159–191.

E. P. Wigner, “On the connection between the distribution of poles and residues for an R function and its invariant derivative”, Ann. of Math. (2) 55:1 (1952), 7–18.

V. Katsnelson, “A generic Schur function is an inner one”, Interpolation theory, systems theory and related topics (Tel Aviv/Rehovot 1999), Oper. Theory Adv. Appl., vol. 134, Birkhäuser, Basel 2002, pp. 243–286.

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