Three dimensional Green function in cylindrical coordinates: application to the Aharonov–Bohm effect of double arc-slits

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Abstract

The object of this paper is the generalization of the theoretical calculation of the probability density of finding a charged particle crossing the double arc-slit in the presence of the vector potential. The explicit derivations is carried out in cylindrical coordinates for the Green functions for the underlying problem corresponding to a particle in free magnetic field, for a quantized as well as a non-quantized flux. The discussion in this paper will be limited to the uniform wavefunction of each particle on the surfaces of double arc-slits.

1. Introduction

In classical electromagnetism the vector potential is merely a mathematical entity, useful but of no physical significance. Nevertheless, in quantum physics, the Aharonov-Bohm effect can demonstrate clearly that the vector potential plays a fundamental role [1, 2]. To investigate the effect of the vector potential in the quantum world, we consider a double arc-slit experiment which demonstrates the wave nature of the electron beam. If one places a confined solenoidal magnetic field far away from the electron paths, then classically the solenoid should have no effect on the diffraction pattern. However, experiments were carried out and confirmed that the vector potential affect the electron’s motion [3–5].

In recent years, much progress of the AB effect has been made restricted to a model of the Aharonov–Bohm with ring structures [6–8] with applications in graphene [9–12] and in scattering of the AB effect [13, 14]. Although many applications of this effect have been carried out in various areas of physics, there has been much interest in further developments of the AB effect [15–18].

In this paper, we generalize the investigation carried out in [19] for a single-arc slit to a double one. In the latter reference, an expression of the two dimensional Green function in polar coordinates has been used to study the propagation of a charged particle emitted from a single arc-slit and then detected on the arc screen in the presence of the vector potential A and a nonquantized flux \( \Phi \), in particular, \( \lambda_0 = N + \delta_0 = q\Phi/2\pi hc \) with \( N = 0 \) and \( 0 < \delta_0 < 1 \) in a rather standard notation. We can see that there is room for further developments and improvements to the case of double-arc slit setup with some arbitrary \( N = 0, 1, 2, 3, \ldots \). This motivates us to carry out an analytical derivation for the three-dimensional Green function, in cylindrical coordinates, of a free particle and a charged particle of charge \( q \) for double arc-slits in the presence of the vector potential \( A \) originated from a very long solenoid of small diameter placed at the origin in such a way that the confined magnetic field is aligned along the z direction with some integer \( N = 0, 1, 2, 3, \ldots \). For additional clarity and further insight we also carry out a numerical analysis of the relevant expression in this work. For the convenience of the reader, we have added three appendices at the end of the paper spelling out in detail the technicalities involved in the derivations so the reader would have no difficulty in following the paper and this work should be, pedagogically, of interest for university students. A relativistic treatment of the problem, although of great interest, is beyond the scope of the present paper as it requires far more advanced mathematical techniques.

In this work, we use, in the process, the propagator approach which was particularly emphasised by Feynman [20–22] to calculate the pattern of probability density of a particle going through double arc-slit
apertures and strike the detection screen. We begin by introducing the famous expression for the one-dimensional free propagator [21]:

\[ G(x; x')_0 = \begin{cases} \left( \frac{m}{2\pi \hbar T} \right)^{1/2} \exp \left( -\frac{m(x-x')^2}{2\hbar T} \right), & t > t' \\ 0, & t < t' \end{cases} \]  

which satisfies a well-known partial differential equation:

\[ \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) G(x; x')_0 = i\hbar \delta(x - x') \delta(t - t') \]  

where \( T = t - t' \). With \( G(x; x')_0 \) interpreted as the amplitude that the system at \( x' \) at time \( t' \) to be found late at \( x \) at time \( t > t' \). Also by definition of \( G(x; x')_0 \), we may refer to this propagator as a Green function.

2. Derivation of the Green function

We first consider the three-dimensional retarded free Green function \( G^0_{r}(x; x') \) which satisfies the differential equation:

\[ \left( i\hbar \frac{\partial}{\partial t} - H^0 \right) G(x; x')_0 = i\hbar \delta(x - x') \delta(t - t') \]  

where \( H^0 = -\frac{\hbar^2}{2m} \nabla^2 \). The solution of above equation is given by

\[ G^0_{r}(x; x') = \frac{i\hbar}{(2\pi\hbar)^3} \int_0^\infty \frac{dp}{2\pi} \cdot \frac{e^{2i\phi p^0}}{p^0 + p^2 + i\varepsilon}, \quad \varepsilon \to +0 \]  

where

\[ dp = dp^0 dp \]  

\[ (x - x') \cdot p = (x - x') \cdot p - (t - t' p^0) \]  

with \( -\infty < p^0 < \infty \), \( -\infty < p^i < \infty \), \( i = 1, 2, 3 \). One may calculate the expression on the right-hand side of equation (4) by considering complex integration [19] which gives us the similar solution as one in equation (1):

\[ G^0_{r}(x; x') = \left( \frac{m}{2\pi \hbar T} \right)^{3/2} \exp \left( \frac{im|x - x'|^2}{2\hbar T} \right) \]  

where, in cylindrical coordinates,

\[ x = (r, \phi, z), \quad r = r(\cos \phi, \sin \phi), \quad T = t - t' > 0, \]  

and

\[ |x - x'|^2 = r^2 + r'^2 - 2rr' \cos(\phi - \phi') + (z - z')^2. \]  

By using the Laplace operator in cylindrical coordinates, equation equation (3) may be rewritten as

\[ \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) \right] G^0_{r}(x; x') = i\hbar \frac{\delta(r - r')}{r} \delta(z - z') \delta(\phi - \phi') \delta(t - t'). \]  

Upon substituting equation (7) in equation (10) and using the generating function of the modified Bessel function \( I_\nu(\xi) \) [23]

\[ \exp \frac{\xi}{2} \left( s + \frac{1}{s} \right) = \sum_{k=-\infty}^{\infty} (s)^k I_\nu(\xi) \]  

where \( I_{-\nu}(\xi) = I_\nu(\xi) \) for integer \( k \) values. The modified Bessel functions \( I_{\nu}(\xi) \) satisfy the differential equation

\[ \left[ \xi^2 \frac{d^2}{d\xi^2} + \xi \frac{d}{d\xi} - (\xi^2 + \nu^2) \right] I_\nu(\xi) = 0 \]

with \( \nu \) not necessarily an integer. Upon choosing \( s = \exp(\phi - \phi') \), we may rewrite the Green function in equation (7) as

\[ G^0_r(x; x') = \sum_{k=-\infty}^{\infty} e^{i\phi (\phi - \phi')} g_0(zt; z't) F_{\nu k}(rt; r't) \]
where \( r = m \frac{r_t}{h^2} \) and

\[
F_{0k}(rt; r't') = \left( \frac{m}{2\pi i h^2} \right) \exp \left( \frac{im(r^2 + r'^2)}{2h^2} \right) \times I_{k\ell}(-i\rho) 
\]

\[
g_0(zt; z't') = \left( \frac{m}{2\pi i h^2} \right)^{1/2} \exp \left( \frac{im(z - z')^2}{2h^2} \right). 
\]

Since the angular part of the Dirac delta function is given by

\[
\int_{-\infty}^{\infty} \delta(\phi - \phi') = 2\pi \sum_{k=\infty}^{\infty} \exp[ik(\phi - \phi')]/2\pi \quad \text{and} \quad \text{the substitution of equation (13) in equation (10) leads us to}
\]

\[
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} + \frac{\partial^2}{\partial z^2} \right) \right] g_0(zt; z't') \times F_{0k}(rt; r't') = \frac{i\hbar}{2\pi} \frac{\delta(r - r')}{r} \delta(z - z') \delta(t - t')
\]

for integer \( k \) values.

### 3. Exact Green function in cylindrical coordinates: application to the Aharonov–Bohm effect

Let us consider a thought experiment. Suppose that an infinitely long, current carrying, solenoid of circular cross section of arbitrary small radius is situated along the \( z \)-axis (see figure 1 (left)). For the purpose of simple but concrete illustration, we will limit our study to the uniform wavefunction of each particle on the surfaces \( S_1, S_2 \) of arc-slit apertures (see figure 1 (right)) \(-\phi_0 - \Delta < \phi' < -\phi_0' + \Delta, \phi_0' - \Delta < \phi' < \phi_0' + \Delta \) and \(-a < z' < a\). This choice will simply our integrations.

The vector potential outside the solenoid is given by

\[
A = \frac{\Phi}{2\pi} \hat{\mathbf{e}}_\phi
\]

where the magnetic flux due to the magnetic field inside the solenoid is

\[
\Phi = \oint \mathbf{B} \cdot d\mathbf{S}.
\]

The Green function of a charged particle of charge \( q \) in the presence of the vector potential \( A \) satisfies the three dimensional differential equation

\[
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{1}{r^2}(k - \lambda_0)^2 + \frac{\partial^2}{\partial z^2} \right) \right] G_\lambda(x; x't') = i\hbar \frac{\delta(r - r')}{r} \delta(z - z') \delta(\phi - \phi') \delta(t - t')
\]

where \( c \) is the speed of light. The solution of above equation is the retarded Green function may be expressed in the form

\[
G_\lambda(x; x't') = \sum_{k=\infty}^{\infty} \delta^{(\phi - \phi')} g_0(zt; z't') F_{0k}(rt; r't')
\]

where the definition of \( g_0(zt; z't') \) is given in equation (15) and the latter \( F_{0k}(rt; r't') \) is going to be determined.

We substitute equation (20) in equation (19) this gives the differential equation

\[
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \frac{1}{r^2}(k - \lambda_0)^2 + \frac{\partial^2}{\partial z^2} \right) \right] g_0(zt; z't')
\]

\[
\times F_{0k}(rt; r't') = i\hbar \frac{\delta(r - r')}{2\pi r} \delta(z - z') \delta(t - t')
\]

\[115007\]
where
\[ \lambda_0 = \frac{q\Phi}{2\pi\hbar c} \] (22)

The comparison of equation (21) and equation (16) suggests us that the solution of equation (21) should be expressed in a form similar to the one equation (14) as given below
\[ F_i^+(rt; rt') = \left( \frac{m}{2\pi i\hbar T} \right) \exp\left( \frac{im(r^2 + r'^2)}{2\hbar T} \right) F^+(k, \xi) \] (23)
where
\[ F^+(k, \xi) = I_{k-\lambda_0}(-i\rho) \] (24)

with \( \xi = -i\rho = -imr' / \hbar T \). Here \( F^+(k, \xi) \) satisfies the modified Bessel differential equation. To this end, upon substituting equation (23) in equation (21), a careful analysis gives
\[
i\hbar\delta(T)g_0(zt; z't')F_i^+(rt; rt') = \left( \frac{m}{2\pi i\hbar T} \right) \exp\left( \frac{im(r^2 + r'^2)}{2\hbar T} \right) \left( \frac{mr'^2}{2T^2\xi^2} \right) \times \left\{ \xi^2 \frac{\partial^2}{\partial\xi^2} F^+(k, \xi) + \xi \frac{\partial}{\partial \xi} F^+(k, \xi) - \left[ \xi^2 + (k - \lambda_0)^2 \right] \right\} \]
\[ \times F^+(k, \xi) = \frac{i\hbar}{2\pi} \frac{\delta(r - r')}{r} \delta(z - z') \delta(t - t'). \] (25)

Consider the term inside the square bracket on the left-hand side of above equation, with equation (24), this suggests us that this term has to vanish satisfying the modified Bessel differential equation, i.e.,
\[
\left[ \xi^2 \frac{\partial^2}{\partial\xi^2} + \xi \frac{\partial}{\partial \xi} - (\xi^2 + (k - \lambda_0)^2) \right] F^+(k, \xi) = 0.
\] (26)

Therefore, the leading term should satisfy
\[
i\hbar\delta(T)g_0(zt; z't')F_i^+(rt; rt') = \frac{i\hbar}{2\pi} \frac{\delta(r - r')}{r} \delta(z - z') \delta(t - t'). \] (27)

We note that the solution of equation (26) decomposes into \( I_{\nu}(\xi), I_{-\nu}(\xi) \) and \( K_{\nu}(\xi) \), where \( \nu = k - \lambda_0 \) not necessarily an integer value and \( \xi = -i\rho \). \( K_{\nu}(\xi) \) diverges for \( \xi \to 0 \) and \( I_{-\nu}(\xi) \), for \( \nu \) not an integer, also diverges at \( \xi \to 0 \), therefore we are left with \( I_{\nu}(\xi) \) and the asymptotic behaviour of \( F^+(k, \xi) = I_{\nu}(\xi) \) in equation (27) for \( |\xi| \to \infty \) or \( T \to 0 \) will be verified now.

We use the asymptotic behavior of \( I_{\nu}(\xi) \) and \( K_{\nu}(\xi) \) [18, 23, 24]
\[
I_{\nu}(\xi) = \frac{i}{2\pi\rho} \left[ e^{-i\rho} + \frac{1}{i} e^{i|\nu|-i\pi} \right], \quad \rho \to \infty
\] (28)

for \( \xi = -i\rho = -imr' / \hbar T \) and \( r, r' > 0 \), i.e., \( \rho \) is real and positive. This means that when \( \rho \to \infty \) then \( T \to 0 \) and by using the identity \( \delta(x) = \lim_{a \to 0} \frac{1}{\pi|a|} e^{-x^2/\sigma^2} \) the left-hand side of equation (27) may be rewritten as
\[
i\hbar\delta(T)\left( \frac{m}{2\pi i\hbar T} \right)^{3/2} \exp\left( \frac{im(r^2 + r'^2 + (z - z')^2)}{2\hbar T} \right) \times I_{\nu}(\xi) = \frac{i\hbar}{2\pi\rho} \delta(r - r') - \delta(r + r') e^{-i|\nu|i\pi} \times \delta(z - z') \delta(T)
\] (29)

Since \( r, r' > 0 \) then the second term on the right-hand side has to be equal to zero. We put this result in equation (28) to confirm that \( F^+(k, \xi) = I_{\nu}(\xi) \) satisfies equation (26). Finally, we obtain the exact Green function in cylindrical coordinates for a charged particle moving under the effect of the vector potential \( A \) generated by a long solenoid situated along the \( z \) axis (see figure 1):
\[ G_i(xt; x't') = \left( \frac{m}{2\pi i\hbar T} \right)^{3/2} \exp\left( \frac{im(r^2 + r'^2 + (z - z')^2)}{2\hbar T} \right) \times \sum_{k=-\infty}^{\infty} e^{ik(\varphi - \phi')} h_{k-\lambda_0}(-i\rho). \] (30)

It particular, when \( \lambda_0 = 0 \), the above result reduces to the generalized free Green function \( G_i^0(xt; x't') \) as one given in equation (13). In the presence of the vector potential \( A \), and the flux \( \Phi \) is quantized, we obtain \( \lambda_0 = N = q\Phi / 2\pi\hbar c \) is some integer then the Green function in equation (30) should be expressed as
\[ G_r(x_t; x't') = \left( \frac{m}{2\pi i\hbar T} \right)^{3/2} \exp \left( \frac{im(r^2 + r'^2 + (z - z')^2)}{2\hbar T} \right) \times e^{iN(\phi - \phi')} \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')} \tilde{I}_k(-i\rho) = e^{iN(\phi - \phi')} G_0^r(x_t; x't') \]  

which coincides with the free Green function. When the flux not quantized, \( \lambda_0 \) may be written as

\[ \lambda_0 = N + \delta_0 \]  

is not an integer, where \( N \) is some integer such that \( N = 0, 1, 2, 3, \ldots \) and \( 0 < \delta_0 < 1 \). The Green function in equation (31) may be rewritten as

\[ G_r(x_t; x't') = \left( \frac{m}{2\pi i\hbar T} \right)^{3/2} \exp \left( \frac{im(r^2 + r'^2 + (z - z')^2)}{2\hbar T} \right) \times e^{iN(\phi - \phi')} \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')} \tilde{I}_k(-i\rho) \]  

We may see that if \( \lambda = 0 \) that is an analog of setting \( N = 0 \) and \( \delta = 0 \), above equation will be collapse into the free Green function as shown in equation (13). In practice, one is interested in the region of \( r, r' \to \infty \) with \( T \approx 0 \) and \( r^2 + r'^2 \gg z^2 + z'^2 \). Thus, above expression may be rewritten as

\[ G_r(x_t; x't') \sim \left( \frac{m}{2\pi i\hbar T} \right)^{3/2} \times \exp \left( \frac{-imzz'}{\hbar T} \right) \exp \left( \frac{im(r^2 + r'^2)}{2\hbar T} \right) \times e^{iN(\phi - \phi')} \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')} \tilde{I}_k(-i\rho) \]  

In this work, by using the Green propagators, we will calculate and compare the amplitude and the probability density for a particle in the absence of the vector potential and then a charged particle in the presence of vector potential of quantized and non-quantized magnetic flux.

### 4. Free Green function

In the first situation, a particle crosses double arc-slits in the absence of the vector potential \( \mathbf{A} \). The corresponding Green function to a particle is the free Green function. The amplitude of finding a particle on detection screen at an angle \( \phi \) for \( r, r' \to \infty, T > 0 \) is obtained by considering the free Green function with asymptotic region. For \( r, r' \to \infty \), with equation (28), the factor \( (m/(2\pi i\hbar T))^{3/2} \exp(i\rho)/\sqrt{2\pi i\rho} \) would not be relevant in this case, we obtain

\[ G_r(x_t; x't') = e^{\frac{-imzz'}{\hbar T}} \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')} [\text{e}^{-2i\rho} + \text{e}^{-i|k||\tau|}] \]

\[ = e^{\frac{-imzz'}{\hbar T}} \left[ 2\pi i(\phi - \phi') e^{-2i\rho} + \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')} e^{-|k|\tau} \right], \rho \to \infty. \]  

The amplitude of a particle to be found at an angle \( \phi \) is obtained by integrating over all surface areas of double arc-slits. Since \( \pi/2 < \phi < 3\pi/2 \) with \( \phi \) not in the range \(-\phi_0^0 - \Delta, \phi_0^0 + \Delta\), the second term on the right-hand side of equation (35) gives rise a zero contribution. Because we limit our study for a uniform wavefunction which sharp cut-offs outside the slits, then the amplitude of interest is given by

\[ A = 2a \left[ \sin \left( \frac{mzz}{\hbar T} \right) \sum_{k=-\infty}^{\infty} e^{-ik\sin(k\Delta)} \cos(k\phi'_0) \times e^{-ik\phi} + \sum_{k=-\infty}^{\infty} e^{-ik\sin(k\Delta)} \cos(k\phi'_0) e^{ik\phi} \right]. \]  

The leading term is just a Sinc function in \( z \) direction. In particular, we are interested in the amplitude at \( z = 0 \) that makes this term converges to \( 2a \) which is the height of the slits measured in \( z \) direction. Now the calculation of \( A \) in above equation at \( z = 0 \) becomes

\[ A(z = 0) = 2a \left[ 4\Delta + 2 \sum_{k=1}^{\infty} \cos(k\pi) \frac{4\sin(k\Delta)}{k} \times \cos(k\phi'_0) \cos(k\phi) \right]. \]  

It is noticeable that the amplitude \( A \) in equation (36) may be summed exactly by using the identities

\[ \sum_{k=1}^{\infty} e^{ikx} = -\ln(1 - e^{ix}) \text{ for } e^{imx} \geq 1, \text{ and } e^{ix} = 1 \]  

\[ \sum_{k=1}^{\infty} e^{-ikx} = -\ln(1 - e^{-ix}) \text{ for } e^{imx} \leq 1, \text{ and } e^{ix} = 1. \]  

At first, we can note that (see figure 2) the segment between \( (\pi/2, 3\pi/2) \) is divided into five segments \( (\pi/2, \pi - \phi_0^0 - \Delta), (\pi - \phi_0^0 - \Delta, \pi - \phi_0^0 + \Delta), (\pi - \phi_0^0 + \Delta, \pi + \phi_0^0 - \Delta), \)
In the second situation, we consider a charged particle in the presence of the vector potential \( \mathbf{A} \), and for the intervals (I.1), (I.2), (I.3), (I.4) and (I.5), respectively.

The identities in equation (38) and equation (39) applied to equation (36) give the identities which recorded here for future reference:

\[
\sum_{k=1}^{\infty} e^{-ikx} \frac{4 \sin(k\Delta)}{k} \cos(k\phi_0) e^{-ik\phi} = -2\Delta + \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi + \phi_0 + \Delta)}{1 + \cos(\phi - \phi_0 - \Delta)} \right] + \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi - \phi_0 + \Delta)}{1 + \cos(\phi + \phi_0 - \Delta)} \right]
\]

for \( \phi \) in the regions (I.1), (I.3), (I.5) and

\[
\sum_{k=1}^{\infty} e^{-ikx} \frac{4 \sin(k\Delta)}{k} \cos(k\phi_0) e^{-ik\phi} = -2\Delta + \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi + \phi_0 + \Delta)}{1 + \cos(\phi - \phi_0 - \Delta)} \right] - \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi - \phi_0 + \Delta)}{1 + \cos(\phi + \phi_0 - \Delta)} \right]
\]

for \( \phi \) in the regions (I.2) and (I.4). Also, we have

\[
\sum_{k=1}^{\infty} e^{-ikx} \frac{4 \sin(k\Delta)}{k} \cos(k\phi_0) e^{-ik\phi} = -2\Delta + \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi + \phi_0 + \Delta)}{1 + \cos(\phi - \phi_0 - \Delta)} \right] - \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi - \phi_0 + \Delta)}{1 + \cos(\phi + \phi_0 - \Delta)} \right]
\]

for \( \phi \) in the regions (I.1), (I.3), (I.5) and

\[
\sum_{k=1}^{\infty} e^{-ikx} \frac{4 \sin(k\Delta)}{k} \cos(k\phi_0) e^{-ik\phi} = -2\Delta + \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi + \phi_0 + \Delta)}{1 + \cos(\phi - \phi_0 - \Delta)} \right] - \frac{1}{2i} \ln \left[ \frac{1 + \cos(\phi - \phi_0 + \Delta)}{1 + \cos(\phi + \phi_0 - \Delta)} \right]
\]

for \( \phi \) in the regions (I.2) and (I.4). These identities give us the amplitude of finding a charged particle at angle \( \phi \) which lies in the intervals (I.1)–(I.5) are given by

\[
|A(\epsilon = 0)|^2 = \begin{cases} 0, & \text{if } \phi \text{ in (I.1), (I.3), (I.5)} \\ 16\pi^2a^2, & \text{if } \phi \text{ in (I.2), (I.4)}. \end{cases}
\]

Figure 3 shows the comparison of probability density calculated numerically from equation (37) with \( k = 1, 2, \ldots, 500 \) and the probability density calculated theoretically from equation (44). In practical, we set \( \phi_0 = \pi/15, \Delta = \pi/100 \) and \( a = 0.50 \) mm.

5. Quantized flux: \( \lambda_0 = N \)

In the second situation, we consider a charged particle in the presence of the vector potential \( \mathbf{A} \) when the magnetic flux \( \Phi \) is quantized, then \( \lambda_0 = N \) where \( N = 0, 1, 2, 3, \ldots \). In this case, we set \( \epsilon_0 \) in equation (34) to zero, we obtain

\[
G_r(\mathbf{r}; \mathbf{r}') = \left( \frac{m}{2\pi i\hbar} \right)^{3/2} \exp \left( \frac{im(r^2 + r'^2 + (z - z')^2)}{2\hbar T} \right) \times e^{iN(\phi - \phi')} \sum_{k=\infty}^{\infty} e^{ik(\phi - \phi')} I_{|k|} (-i\rho)
\]
Thus, the amplitude of a particle to be found at an angle $\phi \leq \phi'$, we obtain

$$G_r(x, t; x', t') = \exp\left(\frac{-imz'}{\hbar T}\right)e^{iN(\phi - \phi')}\sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')}$$

$$\times \left[ie^{-2i\rho} + e^{-ik|\rho|}\right] = \exp\left(\frac{-imz'}{\hbar T}\right)e^{iN(\phi - \phi')}$$

$$\times \left[2\pi i\delta(\phi - \phi')e^{-2i\rho} + \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')}e^{-i|\rho|}\right].$$

(46)

to the limit $\rho \rightarrow \infty$, involving large distance $r \rightarrow \infty$ and/or $r' \rightarrow \infty$ with $T \neq 0$. The amplitude of finding a charged particle on the detection screen is obtained by integrating along the surfaces of double arc-slit. The integration of the first term on the right-hand side of equation (46) is vanished, then we obtain

$$A = 2a\sin\left(\frac{mz}{\max|\rho|}\right)\sum_{k=-\infty}^{\infty} e^{-ik|\rho|}\left(e^{i(k+N)\rho} - e^{i(k+N)\rho + (\phi' - \phi)}\right)$$

$$\times \left[e^{i(k+N)(\phi' - \phi)} - e^{i(k+N)(\phi' - \phi + \Delta)}\right].$$

(47)

Note that $e^{iN\rho} = e^{-iN\rho}$ and $e^{ik\pi} = e^{-ik\pi}$. We also remove the irrelevant multiplicative factor $e^{iN\rho}$, we obtain the amplitude

$$A = 2a\sin\left(\frac{mz}{\max|\rho|}\right)\frac{4\Delta + 2\sin(k\Delta)}{k}\times \cos(k\phi')\cos(k\phi).$$

(48)

This result coincides with one in equation (37) which is obtained from free Green propagator. Therefore, the probability density at $z = 0$ is $16\pi^2a^2$ where $\phi$ lies in the interval (I.2) and (I.4) and zero in (I.1), (I.3) and (I.5).

6. Non-quantized flux: $\lambda_0 = N + \delta_0$

In the last situation, when the magnetic flux is not quantized with $\lambda_0 = N + \delta_0$ for $N = 0, 1, 2, 3, \ldots$ and $0 < \delta_0 < 1$. The retarded Green function for a particle in the presence of the vector potential is given by equation (34). For practical purpose, when $r, r' \rightarrow \infty$ and $T > 0$ we obtain

$$G_r(x, t; x', t') = \exp\left(\frac{-imz'}{\hbar T}\right)e^{iN(\phi - \phi')}\left[2\pi i\delta(\phi - \phi')e^{-2i\rho} + \sum_{k=-\infty}^{\infty} e^{ik(\phi - \phi')}e^{-i|\rho|}\right], \rho \rightarrow \infty.$$  

(49)

Thus, the amplitude of a particle to be found at an angle $\phi$ with $\pi/2 < \phi < 3\pi/2$ is given by

$$\int_{\phi' - \Delta}^{\phi' + \Delta} d\phi' \sum_{k=-\infty}^{\infty} e^{i(k+N)(\phi - \phi')}e^{-i|\rho|k\phi'} + \int_{\phi' - \Delta}^{\phi' + \Delta} d\phi' \sum_{k=-\infty}^{\infty} e^{i(k+N)(\phi - \phi')}e^{-i|\rho|k\phi'}$$

(50)

where $\phi$ not in the range $(-\phi' - \Delta, \phi' + \Delta)$ that gives us zero for Dirac delta function integrations and the $z$-dependent leading term is left temporarily here. After calculating some simple integrations, we obtain the
amplitude
\[ \sum_{k=-\infty}^{\infty} \frac{e^{-i[k-N]\pi}}{-i(k+N)} \left[ (e^{i(k+N)(\phi+\phi_0')-\Delta}) - (e^{i(k+N)(\phi-\phi_0')-\Delta}) + (e^{i(k+N)(\phi+\phi_0')-\Delta}) - (e^{i(k+N)(\phi-\phi_0')-\Delta}) \right]. \] (51)

To generalize the amplitude, we carefully consider both cases when \( N = 0 \) and \( N = 1, 2, 3, \ldots \), this gives us the amplitude of finding a charged particle on detection screen for non-quantized magnetic flux as shown below

\[
4\Delta e^{-i\delta_0\pi} - i2 \sin(\delta_0\pi)(1 - \delta_{0N}) \sum_{k=1}^{N} e^{-2\pi k} \frac{4\sin(k\Delta)}{k} \times \cos(k\phi')e^{ik\phi} + e^{-i\delta_0\pi} \sum_{k=1}^{\infty} e^{-2\pi k} \frac{4\sin(k\Delta)}{k} \cos(k\phi')e^{-ik\phi}
\]

where \( \delta_{0N} \) is a Kronecker delta. Upon using the identities in equations (40)–(43) to calculate the infinite sum in the above equation we obtain the generalized exact amplitude for both cases of quantized and non-quantized magnetic flux as shown below

\[
A(z = 0) = 2a \left\{ 2\pi \cos(\delta_0\pi) + 8(1 - \delta_{0N}) \times \sin(\delta_0\pi) \sum_{k=1}^{N} \cos(k\pi) \frac{\sin(k\Delta)}{k} \cos(k\phi') \sin(\delta_0\pi) - \sin(\delta_0\pi) \left[ \ln \left( \frac{1 + \cos(\phi + \phi_0' + \Delta)}{1 + \cos(\phi - \phi_0' - \Delta)} \right) \right]
\]

\[
+ \ln \left[ \frac{1 + \cos(\phi + \phi_0' + \Delta)}{1 + \cos(\phi - \phi_0' - \Delta)} \right] \right\} \right\} - i2\pi \sin(\delta_0\pi) \left\{ 4\Delta + 8(1 - \delta_{0N}) \sum_{k=1}^{N} \frac{\sin(k\Delta)}{k} \right\} \times \cos(k\pi) \cos(\phi) \cos(k\phi') \right\}.
\] (53)

Note that the term \( 2\pi \cos(\delta_0\pi) \) will only be appeared when \( \phi \) lies in the intervals (1.2) and (1.4) \( \pi - \phi_0' - \Delta < \phi < \pi - \phi_0' + \Delta \) and \( \pi + \phi_0' - \Delta < \phi < \pi + \phi_0' + \Delta \), respectively. One may see that even \( 0 < \delta_0 < 1 \), but when the parameter \( \delta_0 \) is forced to be zero then the above amplitude exactly coincides with the cases of the free Green propagator in equation (44) and the quantized magnetic flux in equation (48). Therefore, the amplitude in equation (53) may be represented as a generalized amplitude in all cases. Note that with the uniform wave function with sharp cut-offs outside the regions \( S_1 \) and \( S_2 \), these densities become arbitrarily large in the limits \( \phi = \pi - \phi_0' \pm \Delta \) and \( \phi = \pi + \phi_0' \pm \Delta \) then these points are avoided.

For the comparison with the case of quantized flux, we choose \( N = 0 \) and \( \delta_0 = 0.25 \). The probability density of finding a charged particle on the detection screen at angle \( \phi \) may be simplified as:

\[ |A(z = 0, N = 0)|^2 = 4\pi^2 \left\{ 2\pi \cos(\delta_0\pi) \times \left[ \Theta(\phi - (\pi - \phi_0' - \Delta)) - \Theta(\phi - (\pi - \phi_0' + \Delta)) \right] \right\}^2
\]

\[
+ \left[ \frac{1 + \cos(\phi + \phi_0' + \Delta)}{1 + \cos(\phi - \phi_0' - \Delta)} \right] \right\} \right\} \right\}^2 \]

\[ + 6\pi^2\Delta^2 \sin^2(\delta_0\pi), \] (54)

and its plot is shown in figure 4. Here, \( \Theta(x) \) is a unit step function with \( \Theta(x) = 1 \) for \( x > 0 \) and \( \Theta(x) = 0 \) for \( x < 0 \). We observe that the numerical calculation agrees with the theoretical result. Furthermore, the comparison of this result with the result of free Green propagator in figure 3 shows that the presence of the vector potential does effect the amplitude of the system even the magnetic field is zero outside the solenoid. This result is verified the Aharonov–Bohm effect of the double arc-slit thought experiment in cylindrical coordinates.

Figure 5 shows the evolution of the probability density for some cases of \( \delta_0 \) with \( N \) fixed, e.g., \( N = 20 \). Note at the top part of this figure when \( \delta_0 = 0.0 \), this is the case of quantized flux. Thus we can not observe the
Aharonov–Bohm effect and therefore the plot of the probability density does analog with one obtained from free Green propagator (see figure 3).

Another interesting example related to the Aharonov–Bohm effect is the case of $\delta_0 = 0.50$. It is sufficient to look at the point $\phi = \pi$. In the presence of the solenoid with $\delta_0 = 0.50$, non vanishing of relative intensity is a statement of the existence of this effect.

### 7. Conclusions

We have carried out a theoretical calculation of the Green function of the double arc-slits thought experiment and applied the result to demonstrate the Aharonov–Bohm effect when a charged particle moving in the presence of the vector potential. The probability density of finding the particle on detection screen is carefully verified to get the generalized form that can be used to calculate the intensity of detecting the particle in three cases: the absence of the vector potential, the presence of the vector potential with quantized magnetic flux and the presence of the vector potential with non-quantized magnetic flux. We also show that the probability density calculated from our theoretical expression agrees well with the numerical calculation and we hope an experiment of this sort may be actually carried out in the future.
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Appendix A. Derivation of the derivatives on the left-hand side of equation (25)

Upon substituting equations (15) and (23) in equation (21) and by using the identity ∂Θ(τ)/∂τ = δ(τ), we then obtain the first term for the time derivative on the left-hand side of equation (21)

\[ i\hbar \frac{\partial}{\partial r} g_0(z; z') F_k^+(r; r') + i\hbar g_0(z; z') \frac{\partial F_k^-(r; r')}{\partial t} = \frac{-i}{2} \left( 1 + \frac{im(z - z')^2}{2\hbar T^2} \right) + i\hbar \left( \frac{m}{2\pi i/nT} \right) \exp \left( \frac{im(r^2 + r'^2)}{2\hbar T} \right) \]

The second term for the radial part on the left-hand side of equation (21) may be written as

\[ \frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right\} g_0(z; z') F_k^+(r; r') = g_0(z; z') \left( \frac{m}{2\pi i/nT} \right) \exp \left( \frac{im(r^2 + r'^2)}{2\hbar T} \right) \times F^+(k, \xi) - \frac{m(r^2)}{2\hbar T^2} F^+(k, \xi) \times \frac{i\hbar}{T} \frac{\partial}{\partial r} F^+(k, \xi) \frac{m(r^2)}{2\hbar T^2} \xi \frac{\partial^2}{\partial \xi^2} F^+(k, \xi) \]

and the last term for the derivative on the left-hand side of equation (21) may be written as

\[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} g_0(z; z') F_k^+(r; r') = \frac{i\hbar}{2T} \left( \frac{im(z - z')^2}{2\hbar T} \right) + \frac{m}{2\pi i/nT} \exp \left( \frac{im(r^2 + r'^2)}{2\hbar T} \right), \]

Appendix B. Derivation of equation (29)

We may rewrite the left-hand side of equation (29) as

\[ \left( \frac{m}{2\pi i/nT} \right)^{3/2} \sqrt{\frac{2m}{2\pi i/nT}} e^{i(m(r^2 + r'^2 + (z - z')^2)/2\hbar T)} I_0(\xi) \]

Upon introducing the variable a where

\[ \frac{2}{a^2} = \frac{im}{2\hbar T}, \quad \frac{1}{|a|/\sqrt{\pi}} = \sqrt{\frac{m}{2\pi i/nT}} \]

we may rewrite equation (B.1) as

\[ \left( \frac{m}{2\pi i/nT} \right)^{3/2} \sqrt{\frac{2m}{2\pi i/nT}} e^{i(m(r^2 + r'^2 + (z - z')^2)/2\hbar T)} I_0(\xi) \]

By using the identity δ(x) = lim_{a→0} \frac{1}{|a|/\sqrt{\pi}} e^{-x^2/|a|^2}, the right-hand side of equation (B.3) may be rewritten as

\[ \frac{1}{2\pi \sqrt{T}} \delta(r - r') + \frac{1}{i} \delta(r + r') e^{-i|\xi|\tau} \]

\[ \delta(z - z'). \]
We may rewrite $\frac{\delta(r-r')}{\sqrt{r'^2}} = \frac{\delta(r-r')}{r'}$ and $\frac{\delta(r+r')}{\sqrt{r^2}} = \delta(r+r')$. Upon substituting equation (B.4) to the right-hand side of equation (B.1), we then multiply equation (B.1) by $i\hbar(T)$, we obtain the expression given in equation (29).

**Appendix C. Simplification of identities in equation (38) and equation (39)**

To evaluate the summations in equations (36), (37), (48) and (52), we use the identities in equations (38) and (39) which can be simplified as

$$\sum_{k=1}^{\infty} \frac{e^{ikx}}{k} = -\frac{i}{2} - \frac{i\pi}{2} - \ln 2 - \ln \left[ \sin \frac{x}{2} \right],$$  \hspace{1cm} (C.1)

for $0 < x < 2\pi$,

$$\sum_{k=1}^{\infty} \frac{e^{ikx}}{k} = -\frac{i}{2} - \frac{i\pi}{2} - \ln 2 - \ln \left[ \sin \frac{-x}{2} \right],$$  \hspace{1cm} (C.2)

for $-2\pi < x < 0$,

$$\sum_{k=1}^{\infty} \frac{e^{-ikx}}{k} = \frac{i}{2} - \frac{i\pi}{2} - \ln 2 - \ln \left[ \sin \frac{x}{2} \right],$$  \hspace{1cm} (C.3)

for $0 < x < 2\pi$ and

$$\sum_{k=1}^{\infty} \frac{e^{-ikx}}{k} = \frac{i}{2} + \frac{i\pi}{2} - \ln 2 - \ln \left[ \sin \frac{-x}{2} \right],$$  \hspace{1cm} (C.4)

for $-2\pi < x < 0$.

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