Sheets of Symmetric Lie Algebras and Slodowy Slices

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Abstract

Let \( \theta \) be an involution of the finite dimensional reductive Lie algebra \( \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the associated Cartan decomposition. Denote by \( K \) the adjoint group of \( \mathfrak{k} \). The \( K \)-module \( \mathfrak{p} \) is the union of the subsets \( \mathfrak{p}^{(m)} = \{ x : \text{dim} K.x = m \}, m \in \mathbb{N} \), and the \( K \)-sheets of \( (\mathfrak{g}, \theta) \) are the irreducible components of the \( \mathfrak{p}^{(m)} \). The sheets can be, in turn, written as a union of so-called Jordan \( K \)-classes.

We introduce conditions in order to describe the sheets and Jordan classes in terms of Slodowy slices. When \( \mathfrak{g} \) is of classical type, the \( K \)-sheets are shown to be smooth; if \( \mathfrak{g} = \mathfrak{gl}_N \) a complete description of sheets and Jordan classes is then obtained.

Contents

1 Generalities 4
   1.1 Notation ................................................................. 4
   1.2 Levi factors ............................................................ 5
   1.3 Jordan \( G \)-classes ................................................... 6
   1.4 Slodowy slices ........................................................ 6
   1.5 The regular \( G \)-sheet ................................................ 9
   1.6 The case \( \mathfrak{g} = \mathfrak{gl}_N \) .......................................... 9
      1.6.1 The setting ......................................................... 9
      1.6.2 The regular case and its consequences ....................... 11
   1.7 Reduction to simple Lie algebras .................................. 12

2 Symmetric Lie algebras 12
   2.1 Type 0 ................................................................. 13
   2.2 Root systems and semisimple elements ......................... 14
   2.3 Property (L) ......................................................... 17
   2.4 Jordan \( K \)-classes ................................................ 20
   2.5 \( K \)-sheets .......................................................... 22

3 Type A 27
   3.1 Involutions in type A ............................................... 28
      3.1.1 Case A0 ......................................................... 28
      3.1.2 Case A1 ......................................................... 28

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3.1.3 Case AII ................................................................. 29
3.1.4 Case AIII ............................................................... 29
3.1.5 Notation and remarks ............................................. 30
3.2 Properties of slices ................................................ 31
3.2.1 The slice property (1) ...................................... 31
3.2.2 The slice property (2) ...................................... 32
3.3 \( J_K \)-classes in type A ......................................... 35
3.3.1 Cases AI and AII ............................................. 35
3.3.2 Case AIII (1) .................................................. 36
3.3.3 Case AIII (2) .................................................. 39

4 Main theorem and remarks ........................................... 41
4.1 Main theorem ......................................................... 41
4.2 Remarks and comments .......................................... 42

Introduction

Let \( \mathfrak{g} \) be a finite dimensional reductive Lie algebra over an algebraically closed field \( k \) of characteristic zero. Fix an involutive automorphism \( \theta \) of \( \mathfrak{g} \); it yields an eigenspace decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) associated to respective eigenvalues +1 and -1. One then says that \( \langle \mathfrak{g}, \theta \rangle \), or \( \langle \mathfrak{g}, \mathfrak{t} \rangle \), is a symmetric Lie algebra, or a symmetric pair. Denote by \( G \) the adjoint group of \( \mathfrak{g} \) and by \( K < G \) the connected subgroup with Lie algebra \( \mathfrak{t} \cap \mathfrak{g} \). The adjoint action of \( g \in G \) on \( x \in \mathfrak{g} \) is denoted by \( g.x \). Recall that a \( G \)-sheet of \( \mathfrak{g} \) is an irreducible component of \( \mathfrak{g}^{(m)} := \{ x \in \mathfrak{g} \mid \dim_{G} x = m \} \) for some \( m \in \mathbb{N} \). This notion can be obviously generalized to \( \langle \mathfrak{g}, \theta \rangle \): the \( K \)-sheets of \( \mathfrak{p} \) are the irreducible components of the \( \mathfrak{p}^{(m)} := \{ x \in \mathfrak{p} \mid \dim_{K} x = m \} \), \( m \in \mathbb{N} \). The study of these varieties is related to various geometric problems occurring in Lie theory. For example, the study of the irreducibility of the commuting variety \( [\mathfrak{ri}] \) and of its symmetric analogue in \( [\mathfrak{pa}, \mathfrak{sy}, \mathfrak{py}] \) is based on some results about \( G \)-sheets and \( K \)-sheets.

Let us first recall some results about \( G \)-sheets. The \( G \)-sheets containing a semisimple element are called Dixmier sheets; they were introduced by Dixmier in \( [\mathfrak{dil}, \mathfrak{dil2}] \). Each \( G \)-sheet is Dixmier when \( \mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} \). In \( [\mathfrak{kr}] \), Kraft gave a parametrization of conjugacy classes of Dixmier sheets. This was generalized in \( [\mathfrak{bo}, \mathfrak{bk}] \) to all \( G \)-sheets. This parametrization relies on the induction of nilpotent orbits, cf. \( [\mathfrak{ls}] \), and the notion of Jordan \( G \)-classes. The Jordan \( G \)-class of an element \( x \in \mathfrak{g} \) can be defined by

\[
J_{G}(x) := \{ y \in \mathfrak{g} \mid \exists g \in G, \ g.x = y \}
\]

(where \( \mathfrak{g}^x \) is the centralizer of \( x \) in \( \mathfrak{g} \)). Clearly, Jordan \( G \)-classes are equivalence classes and one can show that \( \mathfrak{g} \) is a finite disjoint union of these classes. Then, it is easily seen that a \( G \)-sheet is the union of Jordan \( G \)-classes. A significant part of the work made in \( [\mathfrak{bo}, \mathfrak{bk}] \) consists in characterizing a \( G \)-sheet by the Jordan \( G \)-classes it contains. Basic results on Jordan classes (finiteness, smoothness, description of closures, . . . ) can be found in \( [\mathfrak{ty}, \text{Chapter 39}] \) and one can refer to \( [\mathfrak{br}] \) for more advanced properties (geometric quotients, normalization of closure, . . . ).

An important example of a \( G \)-sheet is the set of regular elements:

\[
\mathfrak{g}^r^g = \{ x \in \mathfrak{g} \mid \dim \mathfrak{g}^r(x) \leq \dim \mathfrak{g}^y(x) \text{ for all } y \in \mathfrak{g} \}.
\]

Kostant \( [\mathfrak{ko}] \) has shown that the geometric quotient \( \mathfrak{g}^r^g/G \) exists and is isomorphic to an affine space. This has been generalized to the so-called admissible \( G \)-sheets in \( [\mathfrak{ru}] \). Then, Katsylo proved in \( [\mathfrak{ka}] \) the
existence of a geometric quotient $S/G$ for any $G$-sheet $S$. More recently, [IH] showed that the $G$-sheets are smooth when $\mathfrak{g}$ is of classical type.

The parametrization of sheets used in [Ko, Ru, Ka, IH] differs from the one given in [Kr, Bo, BK] by the use of “Slodowy slices”. More precisely, let $S$ be a sheet containing the nilpotent element $e$ and embed $e$ into an $\mathfrak{sl}_2$-triple $(e, h, f)$. Following the work of Slodowy [Sl, §7.4], the associated Slodowy slice $e + X$ of $S$ is defined by

$$e + X := (e + \mathfrak{g}^f) \cap S.$$ 

Then, one has $S = G.(e + X)$ and $S/G$ is isomorphic to the quotient of $e + X$ by a finite group [Ka]. Furthermore, since the morphism $G \times (e + X) \to S$ is smooth [IH], the geometry of $S$ is closely related to that of $e + X$. We give a more detailed presentation of these results in the first section.

In the symmetric case, much less properties of sheets are known. The first important one was obtained in [KR] where the regular sheet $\mathfrak{p}^{reg}$ of $\mathfrak{p}$ is studied. In particular, similarly to [Ko], it is shown that $\mathfrak{p}^{reg} = K^\theta.(e^{reg} + \mathfrak{p}^f)$ where $K^\theta = \{g \in G \mid g \circ \theta = \theta \circ g\}$. Another interesting result is obtained in [Pa3, SY, PY] (where the symmetric commuting variety is studied): each even nilpotent element of $\mathfrak{p}$ belongs to some $K$-sheet containing a semisimple element. More advanced results can be found in [TY, §39]. The Jordan $K$-class of $x \in \mathfrak{p}$ is defined by

$$J_K(x) := \{y \in \mathfrak{p} \mid \exists k \in K, k.p^x = p^y\}.$$ 

One can find in [TY] some properties of Jordan $K$-classes (finiteness, dimension, . . . ) and it is shown that a $K$-sheet is a finite disjoint union of such classes.

Unfortunately, the key notion of “orbit induction” does not seem to be well adapted to the symmetric case. For instance, the definition introduced in [Ot3] does not leave invariant the orbit dimension anymore.

We now turn to the results of this paper. The inclusion $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$ is the starting point for studying the intersection of $G$-sheets, or Jordan classes, with $\mathfrak{p}$ in order to get some information about $K$-sheets.

We first consider the case of symmetric pairs of type 0 in section 2.1. A symmetric pair is said to be of type 0 if it is isomorphic to a pair $(\mathfrak{g}' \times \mathfrak{g}', \theta)$ with $\theta(x, y) = (y, x)$. This case, often called the “group case”, is the symmetric analogue of the Lie algebra $\mathfrak{g}'$. As expected, we show that the $K$-sheets of $\mathfrak{p}$ are in one to one correspondence with the $G'$-sheets of $\mathfrak{g}'$.

In the general case we study the intersection $J \cap \mathfrak{p}$ when $J$ is a Jordan $G$-class. Using the results obtained in sections 2.2 to 2.4, we show (see Theorem 2.4.4) that $J \cap \mathfrak{p}$ is smooth, equidimensional, and that its irreducible components are exactly the Jordan $K$-classes it contains.

We study the $K$-sheets, for a general symmetric pair, in section 2.5. After proving the smoothness of $K$-sheets in classical cases (Remark 2.5.4), we try to obtain a parametrization similar to the Lie algebra case by using generalized “Slodowy slices” of the form $e + X \cap \mathfrak{p}$, where $e \in \mathfrak{p}$ is a nilpotent element contained in the $G$-sheet $S$. To get this parametrization we need to introduce three conditions (labelled by (⊙), (◇) and (☆)) on the sheet $S$. Under these assumptions, we obtain the parametrization result in Theorem 2.5.11; it gives in particular the equidimensionality of $S \cap \mathfrak{p}$.

In the third section we show that the conditions (⊙), (◇), (☆) hold when $\mathfrak{g} = \mathfrak{gl}_N$ or $\mathfrak{sl}_N$ (type A).

In this case, up to conjugacy, three types of irreducible symmetric pairs exist (AI, AII, AIII in the notation of [He1]) and have to be analyzed in details. The most difficult one being type AIII, i.e. $(\mathfrak{g}, \mathfrak{t}) \cong (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_{N-p})$.

In Section 4 we prove the main result in type A (Theorem 4.1.2), which gives a complete description of the $K$-sheets and of the intersections of $G$-sheets with $\mathfrak{p}$. In particular, we give the dimension of a $K$-sheet in terms of the dimension of the nilpotent $K$-orbits contained in the sheet. One can also
determine the sheets which contain semisimple elements (i.e. the Dixmier K-sheets) and characterize nilpotent orbits which are K-sheets (i.e. the rigid nilpotent K-orbits).

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1 Generalities

1.1 Notation

We fix an algebraically closed field $k$ of characteristic zero and we set $k^\times = k \setminus \{0\}$. If $V,V'$ are $k$-vector spaces, $\text{Hom}(V,V')$ is the vector space of $k$-linear maps from $V$ to $V'$ and the dual of $V$ is $V^* = \text{Hom}(V,k)$. The space $gl(V) = \text{Hom}(V,V)$ inherits a natural Lie algebra structure by setting $[x,y] = x \circ y - y \circ x$ for $x,y \in gl(V)$. The action of $x \in gl(V)$ on $v \in V$ is written $x \cdot v = x(v)$ and $^t x$ is the transpose linear map of $x$. If $M$ is a subset of $\text{Hom}(V,V')$ we set $M = \bigcap_{\alpha \in M} \ker \alpha$.

If $v = (v_1, \ldots, v_N)$ is a basis of $V$, the algebra $gl(V)$ can be identified with $gl(v) = gl_N = M_N(k)$ (the algebra of $N \times N$ matrices). When $v^\prime = (v_{i_1}, \ldots, v_{i_k})$ is a sub-basis of $v$, we may identify $gl(v')$ with a subalgebra of $gl(V)$ by extending $x \in gl(v')$ as follows: $x.v_i = x.v_{i_j}$ if $i = i_j$ for some $j \in [1,k]$, $x.v_i = 0$ otherwise.

All the varieties considered will be algebraic over $k$ and we (mostly) adopt notations and conventions of [Ha] or [TY] for relevant algebraic and topological notions. In particular, $k[[X]]$ is the ring of globally defined algebraic functions on an algebraic variety $X$. Recall that when $V$ is a finite dimensional vector space one has $k[V] = S(V^*)$, the symmetric algebra of $V^*$.

As said in the introduction, $g$ denotes a finite dimensional reductive Lie $k$-algebra. We write $g = [g,g] \oplus j(g)$ where $j(g)$ is the centre of $g$ and we denote by $\text{ad}_g(x) : y \mapsto [x,y]$ the adjoint action of $x \in g$ on $y \in g$. Let $G$ be the connected algebraic subgroup of $\text{GL}(g)$ with Lie algebra $\text{Lie} G = \text{ad}_g(g) \cong [g,g]$. The group $G$ is called the adjoint group of $g$, see [TY, 24.8]. The adjoint action of $g \in G$ on $y \in g$ is denoted by $g.y = \text{Ad}(g).y$; thus, $G.y$ is the (adjoint) orbit of $y$.

We will generally denote Lie subalgebras of $g$ by small german letters (e.g. $I$) and the smallest algebraic subgroup of $G$ whose Lie algebra contains $\text{ad}_g(I)$ by the corresponding capital roman letter (e.g. $L$). When $I$ is an algebraic subalgebra of $g$ the subgroup $L$ acts on $I$ as its adjoint algebraic group, cf. [TY, 24.8.5]. We denote by $H^o$ the identity component of an algebraic group $H$.

Let $E \subset g$ be an arbitrary subset. If $I$, resp. $L$, is a subalgebra of $g$, resp. algebraic subgroup of $G$, we define the associated centralizers and normalizers by:

$$I^E = \{ x \in I \mid [x,E] = (0) \}, \quad L^E = Z_L(E) = C_G(E) = \{ g \in L \mid g.x = x \text{ for all } x \in E \}, \quad N_L(E) = \{ g \in L \mid g.E \subset E \}.$$  

When $E = \{x\}$ we simply write $I^x$, $L^x$, etc. Recall from [TY, 24.3.6] that $\text{Lie} L^E = I^E$. As in [TY], the set of “regular” elements in $E$ is denoted by:

$$E^* = \{ x \in E : \dim g^x = \min_{y \in \bar{E}} \dim g^y \} = \{ x \in E : \dim G.x = \max_{y \in \bar{E}} \dim G.y \}.$$  

Any $x \in g$ has a Jordan decomposition in $g$, that we will very often write $x = s + n$ (cf. [TY, 20.4.5, 20.5.9]). Thus: $s$ is semisimple, i.e. $\text{ad}_g(s) \in gl(g)$ is semisimple, $n$ is nilpotent, i.e. $\text{ad}_g(n)$ is nilpotent, and $[s,n] = 0$. The element $s$, resp. $n$, is called the semisimple, resp. nilpotent, part (or component) of $x$. An $sl_2$-triple is a triple $(e,h,f)$ of elements of $g$ satisfying the relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$
Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \); then, \( \mathfrak{h} = ([g, g] \cap \mathfrak{h}) \oplus \mathfrak{z}(g) \) and the rank of \( g \) is \( \text{rk} \mathfrak{g} = \text{dim} \mathfrak{h} \). We denote by \( R = R(\mathfrak{g}, \mathfrak{h}) = R([g, g], [g, g] \cap \mathfrak{h}) \subset \mathfrak{h}^* \) the associated root system. Recall that the Weyl group \( W = W(\mathfrak{g}, \mathfrak{h}) \) of \( R \) can be naturally identified with \( N_G(\mathfrak{h})/Z_G(\mathfrak{h}) \subset \text{GL}(\mathfrak{h}) \) (see, for example, [TY, 30.6.5]). The type of the root system \( R \), as well as the type of the reflection group \( W \), will be indicated by capital roman letters, frequently indexed by the rank of \([g, g]\), e.g. \( E_8 \). If \( \alpha \in R(\mathfrak{g}, \mathfrak{h}) \), \( g^\alpha = \{ x \in g \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \} \) is the root subspace associated to \( \alpha \). If \( M \) is a subset of \( R(\mathfrak{g}, \mathfrak{h}) \), we denote by \( \langle M \rangle \) the root subsystem \( (\sum_{\alpha \in M} \mathfrak{Q}(\alpha)) \cap R(\mathfrak{g}, \mathfrak{h}) \).

We use the notation \( \lfloor \cdot \rfloor \), resp. \( \lceil \cdot \rceil \), for the floor, resp. ceiling, function on \( \mathbb{Q} \); thus \( \lfloor \lambda \rfloor \), resp. \( \lceil \lambda \rceil \), is the largest, resp. smallest, integer \( \leq \lambda \), resp. \( \geq \lambda \).

### 1.2 Levi factors

We start by recalling the definition of Levi factors:

**Definition 1.2.1.** A Levi factor of \( \mathfrak{g} \) is a subalgebra of the form \( \mathfrak{l} = \mathfrak{g}^* \) where \( s \in \mathfrak{g} \) is semisimple. The subgroup \( L \subset G \) associated to the (algebraic) subalgebra \( \mathfrak{l} \) is called a Levi factor of \( G \).

Observe that the previous definition of a Levi factor of \( \mathfrak{g} \) is equivalent to the definition given in [TY, 29.5.6], see, for example, [Bou, Exercice 10, p. 223]. Recall that a Levi factor \( \mathfrak{l} = \mathfrak{g}^* \) is reductive [TY, 20.5.13] and \( L = G^s \), cf. [St, Corollary 3.11] and [TY, 24.3.6].

Let \( \mathfrak{h} \) be a Cartan subalgebra and \( \mathfrak{l} \) be a Levi factor containing \( \mathfrak{h} \). By [TY, 20.8.6] there exists a subset \( M = M_l \subset R(\mathfrak{g}, \mathfrak{h}) \) such that \( M = \langle M \rangle \) and

\[
\mathfrak{l} = I_M = \mathfrak{h} \oplus \bigoplus_{\alpha \in M} \mathfrak{g}^\alpha \tag{1.2}
\]

\[
\mathfrak{c}_g(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) = \{ t \in \mathfrak{h} \mid \alpha(t) = 0 \text{ for all } \alpha \in M \} \text{ and } \mathfrak{c}_g(\mathfrak{c}_g(\mathfrak{l})) = \mathfrak{l}. \tag{1.3}
\]

Conversely, if \( M \subset R(\mathfrak{g}, \mathfrak{h}) \) is a subset such that \( M = \langle M \rangle \), define \( \mathfrak{l} = I_M \) as in equation (1.2); then \( I_M \) is a Levi factor and:

\[
\mathfrak{h} \supset \{ s \in \mathfrak{g} \mid I_M = \mathfrak{g}^s \} = \ker M \setminus \left( \bigcup_{\alpha \notin M} \ker \alpha \right) \neq \emptyset. \tag{1.4}
\]

This construction gives a bijective correspondence \( \mathfrak{l} = I_M \leftrightarrow M = M_l \) between Levi factors and subsets of \( R(\mathfrak{g}, \mathfrak{h}) \) satisfying the above property. Remark that the Weyl group \( W = W(\mathfrak{g}, \mathfrak{h}) \) acts on the set of Levi factors by its action on \( R(\mathfrak{g}, \mathfrak{h}) \). Precisely, if \( g \in N_G(\mathfrak{h}) \) and \( I \supset \mathfrak{h} \) is a Levi factor, one has \( g \cdot I = w \cdot I \) where \( w = gZ_G(\mathfrak{h}) \in W \) is the class of \( g \). Let \( x, y \in \mathfrak{h} \); we will say that the Levi factors \( \mathfrak{g}^x, \mathfrak{g}^y \) are \( W \)-conjugate if there exists \( w \in W \) such that \( w.M_{g^x} = M_{g^y} \). From (1.4) one deduces that this definition is equivalent to \( w.\mathfrak{c}_g^{x}(\mathfrak{g}^y) = \mathfrak{c}_g^{y}(\mathfrak{g}^y) \) for some \( w \in W \).

Assume that \( \mathfrak{g} \) is semisimple and denote by \( \kappa \) the isomorphism \( \mathfrak{h} \cong \mathfrak{h}^* \) induced by the restriction of the Killing form of \( \mathfrak{g} \). Define a \( \mathbb{Q} \)-form of \( \mathfrak{h} \), or \( \mathfrak{h}^* \), by \( \mathfrak{h}_\mathbb{Q} \cong \mathfrak{h}^*_\mathbb{Q} = \mathbb{Q}R(\mathfrak{g}, \mathfrak{h}) \). Fix the Cartan subalgebra \( \mathfrak{h} \) and a fundamental system (i.e. a basis) \( B \) of \( R(\mathfrak{g}, \mathfrak{h}) \). We say that a Levi factor \( \mathfrak{l} \) is standard if \( \mathfrak{l} = \mathfrak{g}^s \) with \( s \in \mathfrak{h}_\mathbb{Q} \) in the positive Weyl Chamber of associated to \( B \). In this case, one can write \( M_l = (I_l) = Z I_l \cap R(\mathfrak{g}, \mathfrak{h}) \) where \( I_l \subset B \). The following proposition is consequence of the definition of a Levi factor and (1.4).

**Proposition 1.2.2.** Any Levi factor of \( \mathfrak{g} \) is \( G \)-conjugate to a standard Levi factor.

Let \( I \subset \mathfrak{g} \) be a Levi factor and \( L \) be the associated Levi factor of \( G \). There exists a unique decomposition \( I = \mathfrak{z}(I) \oplus \bigoplus_i I_i \), where \( \mathfrak{z}(I) \) is the centre and the \( I_i \) are simple subalgebras. Let \( L_i \subset G \) be the connected subgroup with Lie algebra \( l_i \) (cf. [TY, 24.7.2]). Under this notation we have:
Proposition 1.2.3. The subgroup \( L \subset G \) is generated by \( C_G(l) \) and the subgroups \( L_i \).

Proof. Recall that \( \text{Lie} \ L_i = l_i \) and \( \text{Lie} \ Z_G(l) = z(l) \). By [TY, 24.5.9] one gets that \( L \) is generated by the connected subgroups \( L_i \) and \( C_G(l)^o \). Writing \( l = g^* \) with \( s \) semisimple, we have already observed that \( L = G^s \), hence \( C_G(g^*) \subset G^s \) and the result follows. \( \square \)

1.3 Jordan \( G \)-classes

The description of \( G \)-sheets is closely related to the study of Jordan \( G \)-classes, also called decomposition classes. We now recall some facts about these classes (see, for example, [BK, Bo, Br, TY]).

Recall from §1.1 that any element \( x \in \mathfrak{g} \) has a unique Jordan decomposition \( x = s + n \). We then say that the pair \((g^*, n)\) is the datum of \( x \).

Definition 1.3.1. Let \( x = s + n \) be the Jordan decomposition of \( x \in \mathfrak{g} \). The Jordan \( G \)-class of \( x \), or \( J_G \)-class of \( x \), is the set \( J_G(x) := G.(c_\mathfrak{g}(g^*)^* + n) \). Two elements are Jordan \( G \)-equivalent if they have the same \( J_G \)-class.

Let \( L \) be a Levi factor of \( G \) with Lie algebra \( l \), and \( l.n \subset l \) be a nilpotent orbit. If \( J \) is a \( J_G \)-class, the pair \((l, L.n)\), or \((l,n)\), is called a datum of \( J \) if \((l,n)\) is the datum of an element \( x \in J \). Setting \( t := g^l \) it is then easy to see that \( J = G.(l^* + n) \). From this result one can deduce that Jordan \( G \)-classes are locally closed [TY, 39.1.7], and smooth [Br]. Furthermore, two elements of \( \mathfrak{g} \) are Jordan \( G \)-equivalent if and only if their data are conjugate under the diagonal action of \( G \) [TY, 39.1]. Then, \( \mathfrak{g} \) is the finite disjoint union of its Jordan \( G \)-classes (cf. [TY, 39.1.8]). Since each \( G \)-Jordan class is an irreducible subvariety of some \( \mathfrak{g}^{(i)} \), we get the following result:

Proposition 1.3.2. A \( G \)-sheet of \( \mathfrak{g} \) is a finite (disjoint) union of Jordan \( G \)-classes.

An immediate consequence of this proposition is that each \( G \)-sheet \( S \) contains a unique dense (open) Jordan \( G \)-class \( J \). It follows that we can define the datum of \( S \) to be any datum \((l, L.n)\), or \((l,n)\), of this dense class \( J \). For instance, if \( S \) is a \( G \)-sheet containing a semisimple element, i.e. \( S \) is a Dixmier sheet, then \( J \) is the class of semisimple elements of \( S \) and \((l,0)\) is a datum of \( S \), see [TY, 39.4.5].

1.4 Slodowy slices

We recall in this subsection some of the important results obtained by Katsylo [Ka]. One of the first fundamental properties of the sheets in \( \mathfrak{g} \), cf. [TY, 39.3.5], is:

Proposition 1.4.1. Each \( G \)-sheet contains a unique nilpotent orbit.

We fix a \( G \)-sheet \( S_G \), a datum \((l, L.n)\) of \( S_G \), cf. 1.3, and a Cartan subalgebra \( \mathfrak{h} \subset l \). Set \( t := g^l \) (thus \( t \subset \mathfrak{h} \)). Under this notation [Ka, Lemma 3.2] gives:

Proposition 1.4.2. Suppose that \( l \) is the Levi factor of a parabolic subalgebra of \( \mathfrak{g} \) with nilpotent radical \( n \). Let \( e \in n^* \); then, for any \( \mathfrak{sl}_2 \)-triple \((e,h,f)\) such that \( h \in \mathfrak{h} \), one has \( S_G = G.(e + t) \).

Adopt the notation of the previous proposition. From [Ka, Lemma 3.1] one knows that if \( e \in n^* \) there exists an \( \mathfrak{sl}_2 \)-triple \((e,h,f)\) such that \( h \in \mathfrak{h} \), hence \( e \in S_G \). We fix such a triple, \( \mathcal{S} = (e,h,f) \), for the rest of the subsection. The adjoint action of \( h \) on \( \mathfrak{g} \) yields a graduation

\[
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i,h), \quad \mathfrak{g}(i,h) = \{ v \in \mathfrak{g} : [h,v] = iv \}.
\]
One of the main constructions in [Ka] consists in deforming the “section” \( e + t \) into an other “section” having nice properties. The construction goes as follows. First, define a subset \( X(S_G, \mathcal{S}) \subset S_G \), depending only on the sheet and the choice of the \( s_2 \)-triple, by:

\[
e + X(S_G, \mathcal{S}) := S_G \cap (e + g^t).
\]

Then, the deformation is made by using a map \( \varepsilon_{S_G, \mathcal{S}}^g : e + t \to e + X(S_G, \mathcal{S}) \), whose definition is recalled below, see Remark 1.4.5. Before going into the details, note that when there is no ambiguity on the context, we write \( X \) instead of \( X(S_G, \mathcal{S}) \) and \( \varepsilon, \) or \( e, \) instead of \( \varepsilon_{S_G, \mathcal{S}}^g. \)

**Remark 1.4.3.** When \( g \) is of type A, there is a unique sheet containing a fixed nilpotent orbit (cf. [Kr, §2]). In this case we can therefore set \( X(\mathcal{S}) := X(S_G, \mathcal{S}) \) where \( S_G \) is the sheet containing the nilpotent element \( e \) of \( \mathcal{S}. \)

Define a one parameter subgroup \((F_t)_{t \in k} \subset \text{GL}(g)\) by setting \( F_t y = t(i-2)y \) for \( y \in g(i, h). \) One can show that \( F_t e = e, F_t(S_G = S_G, F_t X = X \text{ and } \lim_{t \to 0} F_t y = e \text{ for all } y \in e + X. \)

One can slightly modify [Ka, Lemma 5.1] to obtain the following result:

**Lemma 1.4.4.** There exists a polynomial map

\[
\epsilon : e + \bigoplus_{i \leq 0} g(2i, h) \longrightarrow e + g^t
\]

such that:

(i) \( e + z \in G.e(e + z) \text{ for all } z \in \bigoplus_{i \leq 0} g(2i, h); \)

(ii) let \( j \leq 0 \text{ and } P_j = (\pi_{2j} \circ \epsilon)|_{e + g(0, h)} \) where \( \pi_{2j} \) is the canonical projection from \( \bigoplus_{i \leq 0} g(2i, h) \) onto \( g(2j, h), \) then \( P_j \) is either 0 or a homogeneous polynomial of degree \( -j + 1. \)

**Proof.** We set \( g_i = g(i, h) \text{ for } i \leq 1. \) One can then define affine subspaces \( L_{2i} \) and \( M_{2i} \) by:

\[
L_{2i} := g^t \cap g_{2i}, \quad M_{2i} := e + L_{2} + L_0 + L_{-2} + \cdots + L_{2i} + g_{2i-2} + g_{2i-4} + \cdots
\]

It is clear that \( L_{2} = \{0\}, M_{2} = e + \bigoplus_{i \leq 0} g_{2i} \text{ and } M_{-2k} = e + g^t \) for \( k \) large enough. We fix such a \( k. \)

Now, define maps \( \epsilon_i : M_{2i} \to M_{2i-2} \) as follows. Denote the projections associated to the decomposition \( g_{2i-2} = [e, g_{2i-4}] \oplus L_{2i-2} \) by \( \text{pr}_1 : g_{2i-2} \to [e, g_{2i-4}] \) and \( \text{pr}_2 : g_{2i-2} \to L_{2i-2} \) (hence \( \text{pr}_1 + \text{pr}_2 = \text{Id}_{g_{2i-2}}.) \) Next, define \( \eta_{2i-2} : g_{2i-2} \to g_{2i-4} \) to be the linear map \((\text{ad} e)^{-1} \circ \text{pr}_1, \) which satisfies \( \eta_{2i-2}(x) e( x + e \in L_{2i-2} \text{ for all } x \in g_{2i-2}. \)

If \( e + z = e + \sum_{j=1}^{0} z_{2j} + \sum_{j+k}^{1} w_{2j} \) in \( M_{2i}, \) where \( z_{2j} \in L_{2j}, w_{2j} \in g_{2j}, \) set:

\[
\epsilon_i(e + z) = \exp(\text{ad} \eta_{2i-2}(w_{2i-2}))(e + z).
\]

Then, \( \epsilon_i \) is a polynomial map such that \( \epsilon_i(e + z) \in M_{2i-2}. \) Now, set:

\[
\epsilon'_i = \epsilon_i \circ \cdots \circ \epsilon_{i-1} \circ \epsilon_{0} \circ \epsilon_{1}, \quad \epsilon = \epsilon'_{-k}.
\]

Clearly, \( \epsilon \) is a polynomial map which satisfies (i).

To get (ii), we now show, by decreasing induction on \( i \leq 2, \) that \( (\pi_{2j} \circ \epsilon'_i)|_{e + g_i} \) is either 0 or a homogeneous polynomial of degree \( -j + 1. \) Set \( \epsilon'_2 = \text{Id}, \) for which the claim is obviously true. Assume that the assertion is true for a given integer \( i_0 = i + 1 \leq 2. \) Remark the construction of \( \epsilon_i, \epsilon'_i \) gives

\[
\epsilon'_i(e + t) = \epsilon_i \circ \epsilon'_{i_0}(e + t) = \exp(\text{ad} \eta_{2i-2}(\pi_{2i-2} \circ \epsilon'_{i_0}(e + t)))) \epsilon'_{i_0}(e + t)
\]
for all $e + t \in e + g_0$. By induction, $u_i := \eta_{2i-2}(\pi_{2i-2} \circ e'_i) : e + g_0 \to g_{2i-4}$ is 0 or homogeneous of degree $-i + 2$; thus

$$\pi_{2j} \circ e'_i(e + t) = \sum_{l \geq 0} \frac{(\operatorname{ad} u_l(e + t))^{l}}{l!} \circ \pi_{2j+l(-2i+4)} \circ e'_{i+1}(e + t)$$

is either 0 or homogeneous of degree $l(-i + 2) + (-j - l(-i + 2) + 1) = -j + 1$, as desired. \[\square\]

**Remark 1.4.5.** The polynomial map $\varepsilon$ constructed in the proof of Lemma 1.4.4 will be denoted by $\varepsilon^g = \varepsilon^g_{\mathcal{Y}}$. By restriction, it induces a map $\varepsilon = \varepsilon^g = \varepsilon^g_{\mathcal{Y}}$ from $e + h$ to $e + g^f$ and Lemma 1.4.4(ii) implies that $\varepsilon$ maps $e + t$ into $e + X$. One can therefore define $\varepsilon^g_{S_G, \mathcal{Y}}$ to be the polynomial map $(\varepsilon^g_{\mathcal{Y}})_{|e+1}$.

Furthermore, one may observe that the construction of $\varepsilon^g$ made in the proof of the previous proposition yields that $\varepsilon^g$ does not depend on $g$ in the following sense: if $g'$ is a reductive Lie subalgebra of $g$ containing $\mathcal{Y}$, then $\varepsilon^{g'} = \varepsilon^g_{\mathcal{Y} \cap g'}$. In the sequel, we will often write $\varepsilon$ when the subscript is obvious from the context.

The next lemma is due to Katsylo [Ka], see [IH] for a purely algebraic proof.

**Lemma 1.4.6.** Under the previous notation:

(i) $S_G = G(e + X)$;

(ii) the action of $G$ on $g$ induces an action of $A = G^e/(G^e)^o$ on $e + X$;

(iii) for all $x \in e + X$, one has $A.x = G.x \cap (e + X)$.

These results enable us to define a quotient map (of sets) by:

$$\psi = \psi_{S_G, \mathcal{Y}} : S_G \longrightarrow (e + X)/A, \quad \psi(x) = A.y \text{ if } G.y = G.x.$$

Since $e + X$ is an affine algebraic variety on which the finite group $A$ acts rationally, it follows from [TY, 25.5.2] that $(e + X)/A$ can be endowed (in a canonical way) with a structure of algebraic variety and that the quotient map

$$\gamma : e + X \longrightarrow (e + X)/A$$

is the geometric quotient of $e + X$ under the action of $A$. Using Lemma 1.4.4(i) and Lemma 1.4.6 one obtains:

$$\psi = \gamma \circ \varepsilon \text{ on } e + t.$$

The following theorem is the main result in [Ka]:

**Theorem 1.4.7.** The map $\psi : S_G \rightarrow (e + X)/A$ is a morphism of algebraic varieties and gives a geometric quotient $S_G/G$ of the sheet $S_G$.

**Remark 1.4.8.** One has $\dim S_G/G = \dim X = \dim t$, see [Bo, §5]. It is shown in [IH, Corollary 4.6] that, when $g$ is classical, the map $\varepsilon : e + t \rightarrow e + X$ is quasi-finite (it is actually finite by [IH, Chaps. 5 & 6]).

The variety $e + X$ will be called a Slodowy slice of $S_G$. One of the main results of [IH] is that $e + X$ is smooth when $g$ is of classical type, cf. Theorem 1.4.10. This result relies on some properties of $e + g^f$ that we now recall (see [Si, 7.4]).

**Proposition 1.4.9.** (i) The intersection of $G.x$ with $e + g^f$ is transverse for any $x \in e + X$ (i.e. $T_x(G.x) \cap T_x(e + g^f) = \{x\}$.)

(ii) The morphism $\delta : G \times (e + g^f) \rightarrow g, (g, x) \mapsto g.x$, is smooth.

(iii) Let $Y$ be a $G$-stable subvariety of $g$ and set $Z = Y \cap (e + g^f)$. Then the restricted morphism $\delta' : G \times Z \rightarrow Y$ is smooth. In particular, when $Y = G.Z$, $Y$ is smooth if and only if $Z$ is smooth.
Proof. Claims (i) and (ii) are essentially contained in [SI, 7.4, Corollary 1].

(iii) We merely repeat the argument given in [IH]. Let \( \hat{Z} = Y \cap_{\text{sch}} (e + g^f) = Y \times \delta (e + g^f) \) be the schematic intersection of \( Y \) and \((e + g^f)\) (cf. [Ha, p. 87]). Writing \((G \times (e + g^f)) \times_Y Y \cong G \times ((e + g^f) \times_Y Y) = G \times \hat{Z} \), the base extension \( Y \to g \) gives the following diagram:

\[
\begin{array}{ccc}
G \times (e + g^f) & \overset{\delta}{\to} & g \\
\uparrow & & \cup \\
G \times \hat{Z} & \overset{\delta''}{\to} & Y.
\end{array}
\]

By [Ha, III, Theorem 10.1] \( \delta'' \) is smooth. Thus, as \( Y \) is reduced, [AK, VII, Theorem 4.9] implies that \( \hat{Z} \) is reduced. Since \( Y \) is \( G \)-stable, it is easy to see that \( \delta' \) factorizes through \( \delta'' \), hence \( \delta' = \delta'' \). When \( Y = G \cdot Z \), the morphism \( \delta' \) is surjective and [AK, VII, Theorem 4.9] then implies that \( Z \) is smooth if, and only if, \( Y \) is smooth.

Applying Proposition 1.4.9(iii) to a sheet \( Y = S_G \), one deduces that \( S_G \) is smooth if and only if the Slodowy slice \( e + X \) is smooth. Using this method, the following general result was obtained by Im Hof:

**Theorem 1.4.10 ([IH]).** The sheets of a classical Lie algebra are smooth.

Recall that the smoothness of sheets was proved by Kraft [Kr] in the \( \mathfrak{sl}_N \) case, and that this fact is not always true in the exceptional cases, e.g. when \( g \) is of type \( G_2 \) (see [IH, Introduction, p. 1]).

1.5 The regular \( G \)-sheet

It is known [Ko, TY] that the set \( g^{reg} \) of regular elements in \( g \) is a sheet, called the regular \( G \)-sheet, that we will denote by \( S_G^{reg} \). We will use the notation and results of the previous subsection with \( S_G = S_G^{reg} \).

One has \( t = \mathfrak{h} \) and \( G . (e + h) = S_G^{reg} \) for any principal \( \mathfrak{sl}_2 \)-triple \((e, h, f)\) such that \( e \) is regular and \( h \in \mathfrak{h} \).

Moreover, \( e + g^f \subset S_G^{reg} \) and therefore \( S_G^{reg} = G . (e + g^f) \).

**Lemma 1.5.1.** Adopt the previous notation.

(i) The semisimple part of an element \( e + x \in e + \mathfrak{h} \) is conjugate to \( x \).

(ii) Two regular elements are conjugate if and only if their semisimple parts are in the same \( G \)-orbit.

(iii) Two elements \( e + x, e + y \in e + \mathfrak{h} \) lie in the same \( G \)-orbit if and only if \( W . x = W . y \).

**Proof.** The assertions (i) and (ii) follow from [Ko, Lemma 11, Theorem 3], whence (iii) is a direct consequence of (i) and (ii).

We will need the following important result [Ko, Theorem 8]:

**Lemma 1.5.2.** The group \( A \) acts trivially on \( e + g^f \), thus \( \psi : S_G^{reg} \to e + g^f = \varepsilon (e + \mathfrak{h}) \) is a geometric quotient of \( S_G^{reg} \).

1.6 The case \( g = \mathfrak{gl}_N \)

1.6.1 The setting

In this section we assume that \( g = \mathfrak{gl}(V) \), where \( V \) is a \( k \)-vector space of dimension \( N \). By [Kr, §2], we know that there exist two natural bijections from \( G \)-sheets to partitions of \( N \):

- the first one, associates to a \( G \)-sheet \( S \) the partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\delta_c}) \) of the unique nilpotent orbit \( O \) contained in \( S \) (cf. Proposition 1.4.1);
then the subspace $\mathfrak{b}$ that we will need later is the following isomorphism:

$$
\mathfrak{e} \cong \mathfrak{e} \oplus \mathfrak{f}
$$

where $\mathfrak{f}$ be a well known similar formula gives $f$. Precisely, we can define $e_i \in \mathfrak{e}$, where $e_i \in \mathfrak{g}$ is defined by:

$$
e_i v_j^{(k)} = \begin{cases} v_{j-1}^{(i)} & \text{if } k = i \text{ and } j = 2, \ldots, \lambda_i; \\ 0 & \text{otherwise.} \end{cases}
$$

Set $\mathfrak{q}_i = \mathfrak{gl}(v_j^{(i)} \mid j \in [1, \lambda_i])$, which is a reductive Lie algebra isomorphic to $\mathfrak{gl}_{\lambda_i}$, and define

$$
\mathfrak{q} := \bigoplus_i \mathfrak{q}_i.
$$

Let $\mathfrak{pr}_i : \mathfrak{q} \to \mathfrak{q}_i$ be canonical projection. For $x \in \mathfrak{q}$ we set $x_i = \mathfrak{pr}_i(x)$; conversely, for any family $(y_i)_i$ of elements $y_i \in \mathfrak{q}_i$, we can define $y = \sum_i y_i \in \mathfrak{q}$. We apply this construction to get an $\mathfrak{sl}_2$-triple $(e, h, f) \subset \mathfrak{q}$ as follows. Fixing the basis $(v_1^{(i)}, \ldots, v_{\lambda_i}^{(i)})$, one can identify $\mathfrak{q}_i$ with $\lambda_i \times \lambda_i$-matrices. Using this identification, embed $e_i$ in the standard $\mathfrak{sl}_2$-triple $(e_i, h_i, f_i)$ of $\mathfrak{q}_i$ afforded by the irreducible representation of $\mathfrak{sl}_2$ of dimension $\lambda_i$, i.e.:

$$
e_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad h_i = \begin{pmatrix} \lambda_i - 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i - 3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i - 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & - \lambda_i + 1 \end{pmatrix}
$$

(a well known similar formula gives $f_i$). Then, $h = \sum_i h_i$ and $f = \sum_i f_i$.

Clearly, the subspace

$$
\mathfrak{l} = \bigoplus_j \mathfrak{gl}(v_j^{(i)} \mid i \in [1, \lambda_j])
$$

is a Levi factor of $\mathfrak{g}$. Denote by $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$ the Cartan subalgebra of diagonal matrices with respect to the chosen basis $\mathfrak{v}$. If $\mathfrak{t}$ is the center of $\mathfrak{l}$ we then have $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{l} \cap \mathfrak{q}$.

Let $E_{j,j}^{i,i}$ be the element of $\mathfrak{h}$ defined by $E_{j,j}^{i,i} v_k^{(l)} = v_j^{(i)}$ if $(i, j) = (l, k)$, and $E_{j,j}^{i,i} v_k^{(l)} = 0$ otherwise. Each $t \in \mathfrak{t}$ can then be written $t = \sum_{i,j} t_{i,j} E_{j,j}^{i,i}$ and one has the following easy characterization of $t$:

$$
t = \{ t \in \mathfrak{h} \mid t_{i,j} = t_{\ell,j} \text{ for all } i \leq i' \text{; } j \in [1, \lambda_{i'}] \}.
$$

We will need later the following isomorphism:

$$
\alpha : \begin{cases} \mathbb{R}^{\lambda_1} & \rightarrow \mathfrak{t} \\ \langle x_j \rangle_{j \in [1, \lambda_1]} & \mapsto (t_{i,j})_{i,j} \end{cases}
$$

where $t_{i,j} = x_j$ for all $i \in [1, \lambda_{i'}, \lambda_i]$, $1 \leq j \leq \lambda_i$.

Order, lexicographically, the elements of $\mathfrak{v}$ by: $v_j^{(i)} < v_k^{(i)}$ if $j < \ell$ or $j = \ell$ and $i < k$. Denote by $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ consisting of upper triangular matrices with respect to this ordering of $\mathfrak{v}$. Then, the subspace $\mathfrak{b} + \mathfrak{t}$ is a parabolic subalgebra having $\mathfrak{t}$ as Levi factor. Observe that $\mathfrak{h} \in \mathfrak{h} \subset \mathfrak{l}$ and that $\mathfrak{e}$ is regular in the nilradical of $\mathfrak{b} + \mathfrak{t}$. Therefore, by Proposition 1.4, the results of §1.4.2 can be applied: we have $S_G = G.(e + \mathfrak{t})$ and we can construct the map $\varepsilon : e + \mathfrak{h} \to e + \mathfrak{g}^{\mathfrak{t}}$ as in Lemma 1.4.4.
Lemma 1.6.1. (i) The group $G^e$ is connected.
(ii) The map $\psi$ induces a bijection between $G$-orbits in $S_G$ and points in $X$.

Proof. Part (i) is a classical result, see for example [CM, 6.1.6]. Since the group $A = G^e/(G^e)^0$ is then trivial, part (ii) follows from Lemma 1.4.6.

By Remark 1.4.5 we may assume that $\varepsilon = \varepsilon^q = \sum_i \varepsilon_i$ where
\[
\varepsilon_i := \varepsilon^q_i : e_i + h_i \to e_i + q_i^t_i.
\]
As $e_i \in q_i$ is regular, the study of $\varepsilon$ is therefore reduced to the regular case.

1.6.2 The regular case and its consequences

We need to study in more details the maps $\varepsilon_i : e_i + h_i \to e_i + q_i^t_i$ introduced at the end of the previous subsection, where, as already said, $e_i$ is regular in $q_i \cong g_{t_i}$. To simplify the notation we (temporarily) replace $g_{t_i}$ by $g_{t_i}^N$ and $e_i$ by $e^\text{reg}$, the regular element of $g = g_{t_i}^N$. Hence,
\[
e^\text{reg}, v_j = \begin{cases} v_{j-1} & \text{if } j = 2, \ldots, N; \\
0 & \text{if } j = 1.
\end{cases}
\]
Recall that $h \subset g_{t_i}^N$ is the set of diagonal matrices in the basis $v^\text{reg} = (v_j)_j$. We can then define the canonical principal triple $(e^\text{reg}, h^\text{reg}, f^\text{reg})$ with respect to this basis (see the definition of the triple $(e_i, h_i, f_i)$ in 1.6.1). In this case, $e^\text{reg} : e^\text{reg} + h \to e^\text{reg} + g^\text{reg}$ can be considered as the restriction of the geometric quotient map of $g^\text{reg}$ (cf. Lemma 1.5.2).

Lemma 1.6.2. Denote by $f^{(k)}$ the subspace formed by the elements on the $k$-th subdiagonal of $g = g_{t_i}^N$. The map $e^\text{reg}$ is given by
\[
e^\text{reg}(e^\text{reg} + t) = e_i + \sum_{j \leq 0} P_j(t) \quad \text{for all } t \in h,
\]
where each $P_j : h \to f^{(-j)}$ is a homogeneous polynomial map of degree $-j+1$, symmetric in the eigenvalues of the elements of $h$.

Proof. Recall that $g(2j, h^\text{reg})$ is the $2j$-th eigenspace of $\text{ad}_g$ $h^\text{reg}$. It is easily seen that $g(2j, h^\text{reg}) = f^{(-j)}$ when $j \leq 0$. Using Lemma 1.4.4(ii), the only fact remaining to be proved is that the polynomial map $P_j$ is symmetric. Observe that the Weyl group $W = W(g, h)$ acts as the permutation group of $[1, N]$ on the eigenvalues of $h$ and recall that, by Lemma 1.5.2, $e^\text{reg}$ is a quotient map with respect to $W$. Consequently, for all $t \in h$ and $w \in W$ one has $e^\text{reg}(e^\text{reg} + w.t) = e^\text{reg}(e^\text{reg} + t)$. Thus $P_j$ is symmetric.

If $t$ is a semisimple element of $g$ we denote by $sp(t)$ the set of eigenvalues of $t$ and by $m(t, c)$ the multiplicity of $c \in k$ as an eigenvalue of $t$, with the convention that $m(t, c) = 0$ if $c \notin sp(t)$. The next lemma is a direct consequence of Lemma 1.5.1.

Lemma 1.6.3. Let $t \in h$ and $c \in sp(t)$. In a Jordan normal form of $e^\text{reg} + t$, there exists exactly one Jordan block associated to $c$, and its size is $m(t, c)$.

Recall that we want to apply Lemma 1.6.3 to the regular elements $e_i$ in $q_i \cong g_{t_i}^N$; we therefore generalize the previous notation as follows. For $t = \sum_i t_i \in h \subset \bigoplus_i q_i$ and $c \in k$, let $m_i(t, c)$ be the multiplicity of $c$ as an eigenvalue of $t_i$. Then, $\sum_i m_i(t, c) = m(t, c)$ and we have the following easy consequence of lemmas 1.5.1 and 1.6.3.

Corollary 1.6.4. Let $t \in h$. The semisimple part of $e + t$ is conjugate to $t$. Its nilpotent part is associated to the partition of $N$ given by the integers $m_i(t, c)$, $c \in sp(t)$ and $i \in [1, \delta(t)]$. 

11
1.7 Reduction to simple Lie algebras

Let \( \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \) be a decomposition of \( \mathfrak{g} \) as a direct sum of reductive Lie (sub)algebras. Let \( G_i \) be the adjoint group of \( \mathfrak{g}_i \), thus \( G = \prod_i G_i \).

Lemma 1.7.1. The \( G \)-sheets of \( \mathfrak{g} \) are of the form \( \prod S_i \) where each \( S_i \) is a \( G_i \)-sheet of \( \mathfrak{g}_i \).

Proof. Clearly, an obvious induction reduces the proof to the case where \( \mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \). Denote by \( (S^m_{i,j})_j \) the sheets contained in \( \mathfrak{g}_1^{(m)} = \{ x \in \mathfrak{g}_1 | \dim G_1 x = m \} \), \( i = 1,2 \). One can decompose \( \mathfrak{g}^{(m)} \) in a finite union of irreducible subsets as follows:

\[
\mathfrak{g}^{(m)} = \bigcup_{p+q=m} S^p_{1,j} \times S^q_{2,k}.
\]

(1.9)

Pick \( x \in S^p_{1,j} \times S^q_{2,k} \) such that \( x \notin S^p_{1,j} \times S^q_{2,k} \). Then (by symmetry) we may assume that \( x = (x_1, x_2) \) with \( x_i \in S^p_{1,j}, i = 1, 2 \), and \( x_1 \notin S^p_{1,j} \). It follows that \( x_1 \in \mathfrak{g}_1^{(p')} \), \( x_2 \in \mathfrak{g}_1^{(q')} \) where \( p' < p \) and \( q' \leq q \), which implies \( p' + q' \neq m \) and \( x \notin \mathfrak{g}^{(m)} \). Therefore, (1.9) gives a decomposition of \( \mathfrak{g}^{(m)} \) into irreducible closed subsets of \( \mathfrak{g}^{(m)} \). We want to show that (1.9) is the decomposition of \( \mathfrak{g}^{(m)} \) into irreducible components.

Suppose that \( S^p_{1,j} \times S^q_{2,j} \subset S^p_{1,j'} \times S^q_{2,k'} \). Then, \( p \leq p' \), \( q \leq q' \), and, since \( p+q = m = p' + q' \), we get that \( p = p' \) and \( q = q' \). Hence, \( S^p_{1,j} \subset S^p_{1,j'} \) and \( S^q_{2,k} \subset S^q_{2,k'} \), forcing \( j = j' \) and \( k = k' \). This proves that \( S^p_{1,j} \times S^q_{2,k} \) is an irreducible component of \( \mathfrak{g}^{(m)} \).

Recall that, since \( \mathfrak{g} \) is reductive, there exists a decomposition \( \mathfrak{g} = \mathfrak{z} \times \prod \mathfrak{g}_i \) where \( \mathfrak{z} \) is the centre of \( \mathfrak{g} \) and \( \mathfrak{g}_i \) is a simple Lie algebra for all \( i \).

Corollary 1.7.2. The \( G \)-sheets of \( \mathfrak{g} \) are the sets of the form \( \mathfrak{z} \times \prod S_i \) where each \( S_i \) is a \( G_i \)-sheet of \( \mathfrak{g}_i \).

Proof. Since \( \mathfrak{z} \) is the unique sheet contained in \( \mathfrak{z} \), the claim follows from Lemma 1.7.1.

The previous corollary allows us to restrict to the case when \( \mathfrak{g} \) is simple. Furthermore, it shows that the study of sheets of \( \mathfrak{g} \) and of \([\mathfrak{g}, \mathfrak{g}]\) are obviously related by adding the centre. Therefore, we may for instance work with \( \mathfrak{g} = \mathfrak{sl}_n \) to study of the \( \mathfrak{sl}_n \)-case.

2 Symmetric Lie algebras

We now turn to the symmetric case. We will denote a symmetric Lie algebra either by \((\mathfrak{g}, \theta), (\mathfrak{g}, \mathfrak{k})\) or \((\mathfrak{g}, \mathfrak{t}, \mathfrak{p})\), where: \( \theta \) is an involution of \( \mathfrak{g} \), \( \mathfrak{t} \) (resp. \( \mathfrak{p} \)) is the +1(resp. −1)-eigenspace of \( \theta \) in \( \mathfrak{g} \). Then, \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \), \( \mathfrak{t} \) is a Lie subalgebra and \( \mathfrak{p} \) is a \( \mathfrak{t} \)-module under the adjoint action. Recall from §1.1 that \( K \) is the connected subgroup of \( G \) such that \( \text{Lie}(K) = \text{ad}_\mathfrak{g}(\mathfrak{t}) \) and that \( K \) is the connected component of

\[
K^\theta = \{ g \in G | g \circ \theta = \theta \circ g \} = N_G(\mathfrak{t}).
\]

(2.1)

Sheets and Jordan classes can naturally be defined in this setting, see [TY, 39.5 & 39.6]. One has, cf. [KR],

\[
\dim K.x = \frac{1}{2} \dim G.x \quad \text{for all } x \in \mathfrak{p}
\]

and we set:

\[
\mathfrak{p}^{(m)} := \{ x \in \mathfrak{p} | \dim K.x = m \} \subset \mathfrak{g}^{(2m)}.
\]

Definition 2.0.3. The \( K \)-sheets of \((\mathfrak{g}, \theta)\) are the irreducible components of the \( \mathfrak{p}^{(m)} \), \( m \in \mathbb{N} \).

Let \( x = s + n \) (where \( s, n \in \mathfrak{p} \)) be the Jordan decomposition of an element \( x \in \mathfrak{p} \). The Jordan \( K \)-class of \( x \), or \( J_K \)-class of \( x \), is the set

\[
J_K(x) := K.(\mathfrak{c}_\mathfrak{p}(\mathfrak{p}^s)^* + n) \subset \mathfrak{p}.
\]
It is easily seen that $\mathfrak{p}$ is the finite disjoint union of its $J_K$-classes and that a $K$-sheet is the union of the $J_K$-classes it contains [TY, 39.5.2].

There exists a symmetric analogue to the notion of $\mathfrak{sl}_2$-triple. An $\mathfrak{sl}_2$-triple $(e, h, f)$ is called normal if $e, f \in \mathfrak{p}$ and $h \in \mathfrak{k}$. Similarly to the Lie algebra case, there is a bijection between $K$-orbits of nilpotent elements and $K$-orbits of normal $\mathfrak{sl}_2$-triples, see [TY, 38.8.5].

Any semisimple symmetric Lie algebra can be decomposed as $(\mathfrak{g}, \theta) = \prod_i (\mathfrak{g}_i, \theta_{(i)})$ where $(\mathfrak{g}_i, \theta_{(i)})$ is a symmetric Lie subalgebra of one of the following two types:

(a) $\mathfrak{g}_i$ simple;

(b) $\mathfrak{g}_i = \mathfrak{g}_i^1 \oplus \mathfrak{g}_i^2$, with $\mathfrak{g}_i^j$ simple, $\theta_{|\mathfrak{g}_i^j}$ isomorphism from $\mathfrak{g}_i^j$ onto $\mathfrak{g}_i^{3-j}$, $j = 1, 2$.

Each $(\mathfrak{g}_i, \theta_{(i)})$ is called an irreducible factor of $(\mathfrak{g}, \theta)$; this decomposition is unique (up to permutation of the factors).

### 2.1 Type 0

When $(\mathfrak{g}, \theta)$ is the sum of two simple factors as in the above case (b), then $\mathfrak{g}$ is said to be of “type 0”. We slightly enlarge this definition by saying that a pair $(\mathfrak{g}, \theta)$ is a symmetric pair of type 0 if $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}'$, $\theta(x, y) = (y, x)$, $\mathfrak{k} = \{ (x, x) \mid x \in \mathfrak{g}' \}$, $\mathfrak{p} = \{ (x, -x) \mid x \in \mathfrak{g}' \}$, where $\mathfrak{g}'$ is only assumed to be reductive. Recall the following easy observations. Let $\text{pr}_1$ be the projection on the first coordinate. Via $\text{pr}_1$, the Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{g}'$, thus $K$ is isomorphic to the adjoint group $G'$ of $\mathfrak{g}'$. Moreover, the $K$-module $\mathfrak{p}$ is isomorphic to the $G'$-module $\mathfrak{g}'$. If $Y$ is a subset of $\mathfrak{p}$, we set $\phi(Y) = \text{pr}_1(Y) \times \text{pr}_1(-Y) \subset \mathfrak{g}$.

**Lemma 2.1.1.** (i) The $G$-sheets of $\mathfrak{g} = \mathfrak{g}' \times \mathfrak{g}'$ are the $S' \times S''$ where $S'$ and $S''$ are $G'$-sheets of $\mathfrak{g}'$.

(ii) The sets $\{(x, -x) \mid x \in S' \}$, where $S'$ is a $G'$-sheet of $\mathfrak{g}'$, are the $K$-sheets of $\mathfrak{p}$.

**Proof.** (i) This is a particular case of Lemma 1.7.1.

(ii) Note that an element $(x, -x)$ belongs to $\mathfrak{p}^{(m)}$ if, and only if, $x$ is in $(\mathfrak{g}')^{(m)}$. This shows that a $K$-sheet of $\mathfrak{p}^{(m)}$ is contained in a set of the form $(S' \times S'') \cap \mathfrak{p}$ where $S'$ and $S''$ are sheets of $(\mathfrak{g}')^{(m)}$.

Observe now that Jordan classes, and consequently sheets, are stable under the transformation $x \mapsto -x$. This implies that $(S' \times S'') \cap \mathfrak{p} = \{ (x, -x) \mid x \in S' \cap S'' \}$.

In particular, we have $(S' \times S'') \cap \mathfrak{p} \subset (S' \times S') \cap \mathfrak{p}$, where $(S' \times S') \cap \mathfrak{p}$ is an irreducible subset of $\mathfrak{p}^{(m)}$. The result then follows from: $\{ (x, -x) \mid x \in S' \} \subset \{ (x, -x) \mid x \in S'' \}$ if, and only if, $S' \subset \overline{S''}$.

**Proposition 2.1.2.** (i) If $Y$ is a $K$-orbit (resp. a $J_K$-class or a $K$-sheet) of $\mathfrak{p}$, then $\phi(Y)$ is a $G$-orbit (resp. a $J_G$-class or a $G$-sheet) of $\mathfrak{g}$.

(ii) If $Z$ is a $G$-orbit (resp. a $J_G$-class) of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then $Z \cap \mathfrak{p}$ is a $K$-orbit (resp. a $J_K$-class) of $\mathfrak{p}$

(iii) Distinct sheets of $\mathfrak{g}'$ have an empty intersection if, and only if, the intersection of each $G$-sheet of $\mathfrak{g}$ with $\mathfrak{p}$ is either empty or a single $K$-sheet.

**Proof.** Let $x = s + n \in \mathfrak{g}'$ and set $y = (x, -x) \in \mathfrak{p}$, $Y = K.y$; then $\text{pr}_1(Y) = G'.x$ and $G.y = (G' \times G').(x, -x) = (G', G'.(-x)) = \phi(Y)$. 


Conversely, if \( Z = G.y \) then \( Z \cap p = \{(z, -z) \mid z \in G'.x\} = K.y \).
Now, let \( Y = J_K(y) \). We have \( pr_1(c_p(p^*)) = c_{p^*}(g'(g^*)^*)\) and the equivariance of the isomorphism \( pr_1 : p \to g' \) shows that the regularity condition is preserved. It follows that \( pr_1(Y) = G'(c_{g'}(g'^*)^* + n) \) and
\[
J_G(y) = (G' \times G').(c_{g'}(g'^*)^* + n, -c_{g'}(g'^*)^* - n) = \phi(Y).
\]
Conversely, if \( Z = J_G(y) \), then \( Z \cap p = \phi(Y) \cap p = \{(z, -z) \mid z \in K.(c_{g'}(g'^*)^* + n)\} = J_K(y) \).
If \( Y \) is an irreducible component of \( p^{(m)} \), one has \( pr_1(Y) = pr_1(k^*Y) = pr_1(-Y) \) and \( pr_1(Y) \) is a \( G' \)-sheet of \( g' \). As \( \phi(Y) \) is an irreducible subset of \( g'^{(2m)} \), there exists a \( G' \)-sheet \( S \) containing \( \phi(Y) \). Then, \( S \) decomposes as the product of two \( G' \)-sheets of \( g' \) and therefore \( S = \phi(Y) \). This ends the proof of (i) and (ii)
(iii) Let \( Z \) be a \( G' \)-sheet of \( g' \) and write \( Z \) as the product of two \( G' \)-sheets of \( g' \), say \( Z = Z_1 \times Z_2 \). If \((x, -x) \in Z\), it follows that \( x \in Z_1 \cap Z_2 \) and, in particular, \( Z_1 \cap Z_2 \neq \emptyset \). If \( Z_1 = Z_2 \), then Lemma 2.1.1 shows that \( Z \cap p \) is a \( K \)-sheet. Otherwise, one has \( Z \cap p \subseteq (Z_1 \times Z_1) \cap p \) and \( Z \cap p \) is not a \( K \)-sheet of \( p \).

Since a \( G' \)-sheet of \( g' \) contains exactly one nilpotent orbit of \( g' \), two \( G' \)-sheets of \( g' \) have a non-empty intersection if and only if they contain the same nilpotent orbit (cf. [TY, 39.3.2]). A necessary and sufficient condition for \( g' \) to have intersecting sheets is therefore to have more sheets than nilpotent orbits. Using [Bo] one can show that there are only two cases where sheets are in bijection with nilpotent orbits: when \( g' \) is of type \( A \) or \( D_4 \). Therefore we have:

**Corollary 2.1.3.** Any \( G' \)-sheet of \( g' \) intersects \( p \) along one \( K \)-sheet if and only if the simple factors of \( g' \) are of type \( A \) or \( D_4 \).

The next (easy) result is true in type 0, but false in general.

**Proposition 2.1.4.** Let \( S_G \) be a \( G' \)-sheet of \( g' \) intersecting \( p \). Let \( \mathcal{S} = (e, h, f) \) be a normal \( \mathfrak{sl}_2 \)-triple containing a nilpotent element \( e \in S_G \cap p \). Then, if \( e + X(S_G, \mathcal{S}) = (e + g'^*G') \cap S_G \), one has
\[
S_G \cap p = K.(e + X(S_G, \mathcal{S}) \cap p).
\]

**Proof.** Write \( S_G = S_1 \times S_2 \) with \( S_1, S_2 \) sheets of \( g' \)(cf. Lemma 2.1.1) and set \( e = (e', -e'), f = (f', -f') \), \( e', f' \in g' \). Recall that \( pr_1 \) yields an isomorphism between \( p \) and \( g' \) and that \( pr_1(S_G \cap p) = S_1 \cap S_2 \). If \( X_i \subseteq g' \) is defined by \( (e' + X_i) = (e' + g'^*) \cap S_i \), one has \( pr_1(e + X \cap p) = e' + X_1 \cap X_2 \). Moreover, \( pr_1(K.(e + X \cap p)) = G'.(e' + X_1 \cap X_2) = S_1 \cap S_2 = pr_1(S_G \cap p) \). Since \( pr_1|_p \) is an isomorphism, we get the desired result.

### 2.2 Root systems and semisimple elements

Let \((g, \mathfrak{t}, p)\) be a semisimple symmetric Lie algebra associated to the involution \( \theta \). Fix a Cartan subspace \( a \) of \( p \); recall that the rank of the symmetric pair \((g, \mathfrak{t}) = (g, \theta) \) is \( \text{rk}(g, \theta) = \text{dim} \ a \). Let \( \mathfrak{d} \) be a Cartan subalgebra of \( c_1(a) \). Then, \( \mathfrak{b} = a \oplus \mathfrak{d} \) is a \( \theta \)-stable Cartan subalgebra of \( g \) ([TY, 37.5.2]). If \( V = \mathfrak{b}^* \) and \( \sigma \) denotes the transpose of \( \theta \), one can consider the \( \sigma \)-stable root system \( R = R(g, \mathfrak{b}) \subset V \) and we set (see [TY, 36.1]):
\[
V' = \{ x \in \mathfrak{b}^* \mid \sigma(x) = x \} = \{ x \mid x|_a = 0 \},
V'' = \{ x \in \mathfrak{b}^* \mid \sigma(x) = -x \} = \{ x \mid x|_\theta = 0 \},
R^0 = R \cap V' = \{ \alpha \in R \mid \sigma(\alpha) = \alpha \}, \quad R^1 = \{ \alpha \in R \mid \sigma(\alpha) \neq \alpha \}.
\]
Recall that $R^0$ is a root system. One has $V = V' + V''$; more precisely, $x \in V$ decomposes as $x = x' + x''$, where $x' = \frac{1}{2}(x + \sigma(x)) \in V'$, $x'' = \frac{1}{2}(x - \sigma(x)) = x_1 a \in V''$. When $x \in R$ is a root, $x''$ is called its restricted root. Set:

$$S = \{\alpha'' | \alpha \in R^1\}.$$

Then, $S \subset \mathfrak{a}^*$ is a (not necessarily reduced) root system, see [TY, 36.2.1], which is called the restricted root system of $(\mathfrak{g}, \theta)$. We denote by $W$, resp. $W_S$, the Weyl group of the root system $R$, resp. $S$, and we set

$$W_\sigma = \{w \in W | w \circ \sigma = \sigma \circ w\}.$$

If $B \subset R$ is a fundamental system (i.e. a basis of $R$), denote by $R_+$ (resp. $R_-$) the set of positive (resp. negative) roots associated to $B$. In order to define the Satake diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{t})$ one needs to work with some special fundamental systems for $R$. Setting

$$R^1_\pm = R^1 \cap R_\pm,$$

one can give the following definition:

**Definition 2.2.1.** ([TY, 36.1.4], [Ar, 2.8]) A $\sigma$-fundamental system $B \subset R$ is a fundamental system satisfying the following conditions:

(i) $\sigma(R^1_+) = R^1_-$;

(ii) If $\alpha \in R^1_+$, $\beta \in R$ and $\alpha - \beta \in V''$, then $\beta \in R^1_+$;

(iii) $(R^1_+ + R^1_-) \cap R \subset R^1_+;

Let $V_Q$ be the Q-vector space spanned by $R$; then $V_Q = V_Q' + V_Q''$ where $V_Q' = V_Q \cap V'$, resp. $V_Q'' = V_Q \cap V''$, are Q-forms of $V'$, resp. $V''$ (cf. [TY, proof of 36.1.4]). Denote by $a_Q$ be the Q-form of $a$ given by the dual of $V_Q''$. The choice of a Q-basis $C = (e_1, \ldots, e_l)$ of $V_Q$ gives rise to a lexicographic ordering $\prec$ on $V_Q$ and, therefore, to a set of positive roots $R_{+;C} = \{\alpha \in R | \alpha > 0\}$. Recall [TY, 18.7] that for each choice of such a basis $C$, there exists a unique fundamental system $B_C$ such that $R_{+;C}$ is the set of positive roots with respect to $B$. The existence of a $\sigma$-fundamental system is ensured by the next lemma, which provides all the $\sigma$-fundamental systems, see Proposition 2.2.3(iv).

**Lemma 2.2.2.** Let $(e_1, \ldots, e_p)$, resp. $(e_{p+1}, \ldots, e_l)$, be a basis of $V_Q''$, resp. $V_Q'$, and set $C = (e_1, \ldots, e_l)$. Then $B_C$ is a $\sigma$-fundamental system such that $B_0 = B_C \cap V'$ is a fundamental system of $R^0$.

**Proof.** By [TY, 36.1.4] $B_C$ is a $\sigma$-fundamental system. The second statement follows from the fact that $B_C \cap V'$ is the set of simple roots associated to the lexicographic ordering associated to the basis $(e_{p+1}, \ldots, e_l)$. \qed

**Proposition 2.2.3.** (i) The map $w \mapsto w|_{V''}$ induces a surjective homomorphism $W_\sigma \to W_S$ whose kernel is $W^0$, the Weyl group of $R^0$.

(ii) For $x \in V_Q''$, one has $W_S.x = W.x \cap V_Q''$. Dually, $W_S.a = W.a \cap a_Q$ for all $a \in a_Q$.

(iii) Let $B$ be a $\sigma$-fundamental system. Then, the restricted fundamental system $B'' = \{\alpha'' | \alpha \in B\}$ is a fundamental system of the restricted root system $S$.

(iv) $W_\sigma$ acts transitively on the set of $\sigma$-fundamental systems.

**Proof.** Claims (i) and (ii) are proved in [TY, 36.2.5, 36.2.6], while (iii) and (iv) can be found in [Ar, 2.8 and 2.9]. \qed
**Remarks 2.2.4.** (1) The restriction to \( a \) yields an isomorphism \( N_K(a)/Z_K(a) \rightarrow W_S \), cf. [TY, 38.7.2].

(2) Let \( w \in W_S \), then there exists \( k \in K \) such that \( k_\hbar = w \). This can be shown as follows. Recall that \( \hbar = a \oplus \mathfrak{d} \), where \( \mathfrak{d} \) is a Cartan subalgebra of \( u = c_1(a) \). Note that \( w.\mathfrak{a} = a \) and \( w.\mathfrak{d} = \mathfrak{d} \). Pick \( k_1 \in K \) such that \( k_1|_a = w|_a \in W_S \). Let \( U \subset C_K(a) \) be the connected subgroup of \( K \) with Lie algebra \( u \). The Weyl group of the root system \( R^0 = R(u, \mathfrak{d}) \) is \( W^0 \cong N_U(\mathfrak{d})/Z_U(\mathfrak{d}) \), see [TY, 38.2.1]. By composing \( k_1 \) with an element of \( U \) we may assume that \( k_1.\hbar = \hbar \) and \( k_1|_a = w|_a \). Set \( w_0 = (w \circ k_1^{-1})|_\hbar \in W \); one has \( w_0|_a = \text{Id}_a \), therefore \( w_0 \in W^0 \) and we can find \( k_0 \in N_U(\mathfrak{d}) \) such that \( k_0|_\hbar = w_0|_\hbar = w_0 \circ k_1^{-1}|_\hbar \). Setting \( k = k_0k_1 \in K \) we obtain \( k|_\hbar = k_1|_a = w|_a \) and \( k_\hbar = k_0 \circ k_1 = w_\hbar \), thus \( k_\hbar = w \).

Fix a \( \sigma \)-fundamental system \( B \); from the Dynkin diagram \( D \) associated to \( B \) one can construct the Satake diagram \( \tilde{D} \) of \( (\mathfrak{g}, \theta) \) as follows. The nodes \( \alpha \) of \( D \) such that \( \alpha'' = 0 \) are colored in black, the other nodes being white; two white nodes \( \alpha \neq \beta \) of \( D \) such that \( \alpha'' = \beta'' \) are related by a two-sided arrow. This defines the new diagram \( \tilde{D} \). Recall that the Satake diagram of \( (\mathfrak{g}, \theta) \) does not depend on the choice of the \( \sigma \)-fundamental system \( B \), and that two semisimple symmetric Lie algebras are isomorphic if and only if they have the same Satake diagram (cf. [Ar, Theorem 2.14]). A classification of symmetric Lie algebras together with their Satake diagrams and restricted root systems is given in [He1, Ch. X].

We now recall the (well-known) links between \( G \)-conjugacy and \( W \)-conjugacy, and their analogues for a symmetric Lie algebra.

**Lemma 2.2.5.** (i) Two elements of \( \hbar \) (resp. \( a \)) are \( G \)(resp. \( K \))-conjugate if and only if they are \( W \)(resp. \( W_S \) or, equivalently, \( W_\sigma \))-conjugate.

(ii) Let \( x, y \in \hbar \) (resp. \( x, y \in a \)), then the Levi factors \( \mathfrak{g}^x \) and \( \mathfrak{g}^y \) are \( G \)(resp. \( K \))-conjugate if, and only if, they are \( W \)(resp. \( W_S \) or, equivalently, \( W_\sigma \))-conjugate.

**Proof.** (i) We write the proof for \( x, y \in a \). Thanks to [TY, 29.2.3 & 37.4.10] applied to \( (\mathfrak{g}^y, \mathfrak{p}^y) \), the elements \( x, y \) are \( K \)-conjugate if, and only if, there exists an element \( g \in K \) such that \( g.x = y, g.\hbar = \hbar \) and \( g \circ \sigma = \sigma \circ g \). It follows from [TY, 30.6.5] that \( g \) induces an element of \( W \), and therefore of \( W_\sigma \). Observe finally that Proposition 2.2.3(i) implies the equivalence of \( W_\sigma \) and \( W_S \)-conjugacy. Conversely, [TY, 38.7.2] shows that the conjugation under \( W_S \) implies the \( K \)-conjugation.

(ii) The proof is analogue to (i). Indeed, one can show that \( G.\mathfrak{g}^x = G.\mathfrak{g}^y \) (resp. \( K.\mathfrak{g}^x = K.\mathfrak{g}^y \)) if, and only if, there exists \( g \in G \) (resp. \( K \)) such that \( g.\mathfrak{g}^x = \mathfrak{g}^y \) and \( g.\hbar = \hbar \).

In general, if \( x \in \mathfrak{p} \), the intersection of \( G.x \) with \( \mathfrak{p} \) contains more than one orbit (cf. [TY, 38.6.1(i)]). But, when \( x \) is semisimple one can prove the following result, for which we provide a proof since we did not find a reference in the literature.

**Proposition 2.2.6.** Let \( s \in \mathfrak{p} \) be semisimple. Then, \( G.s \cap \mathfrak{p} = K.s \).

**Proof.** Recall that any semisimple element of \( \mathfrak{p} \) is \( K \)-conjugate to an element of \( a \), cf. [TY, 37.4.10]. Therefore, by Lemma 2.2.5(i), it suffices to show that the property (ii) of Proposition 2.2.3 holds for all \( a \in a \), i.e. \( W_S.a = W.a \cap a \). Denote by \( \mathbb{L} \) one of the fields \( \mathbb{Q} \) or \( \mathbb{k} \). For \( (w, w') \in W \times W_S \), define linear subspaces of \( a_\mathbb{L} = a_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{L} \) by:

\[
E^w_w = \ker_{a_\mathbb{L}}(w - w') = \{ a \in a_\mathbb{L} \mid w.a = w'.a \}, \quad E^w_w = w^{-1}(a_\mathbb{L}) \cap a_\mathbb{L}.
\]

From Proposition 2.2.3(ii) one gets that \( E^w_w = \bigcup_{w' \in W_S} E^w_{w,w'} \); thus, there exists \( w' \in W_S \) such that \( E^w_w = E^w_{w,w'} \). The flatness of \( - \otimes_{\mathbb{Q}} k \) yields:

\[
E^w_{w,w'} = E^w_{w,w'} \otimes_{\mathbb{Q}} k, \quad E^w_k = E^w_k \otimes_{\mathbb{Q}} k.
\]

Therefore, for any \( w \in W \), there exists \( w' \in W_S \) such that \( w'\mid_{E^w_k} = w \mid_{E^w_k} \). It follows that Proposition 2.2.3(ii) is satisfied for all \( a \in a = a_k \).
Consequence. Proposition 2.2.6 yields a bijection between $K$-orbits of semisimple elements of $p$ and $G$-orbits of semisimple elements intersecting $p$.

Recall [Ko, KR] that the set of semisimple $G$(resp. $K$)-orbits is parameterized by the categorical quotient $g//G$ (resp. $p//K$), and that $k[g//G] \cong k[h//W] = S(h^*)^W$, $k[p//K] \cong k[a//W_S] = S(a^*)^{W_S}$. The previous consequence can then be interpreted as follows.

Let $\gamma$ be the map which associates to the $W_S$-orbit of $a \in a$, the orbit $W.a \subset h$; hence, $\gamma : a//W_S \to h//W$. Define $Z = \gamma(a//W_S) \subset h//W$ to be the image of $\gamma$ and let $\phi : a//W_S \to Z$ be the induced surjective map. Write $\gamma = \iota \circ \phi$, where $\iota : Z \to h//W$ is the natural inclusion. Since $\gamma$ is a finite morphism, $Z$ is closed and $\phi$ is a finite surjective morphism. We then have the two following commutative (dual) diagrams:

\[
\begin{array}{ccc}
    a//W_S & \xrightarrow{\phi} & Z \\
    \gamma \downarrow & & \downarrow \iota \\
    h//W & & S(h^*)^W
\end{array}
\]

\[
\begin{array}{ccc}
    S(a^*)^{W_S} & \xleftarrow{\phi^*} & k[Z] \\
    \gamma^* \uparrow & & \uparrow \\
    S(h^*)^W & &
\end{array}
\]

The significance of Proposition 2.2.6 is that $\gamma$ is injective; equivalently, $\phi$ is bijective. Since $a//W_S$ is a normal variety ($S(a^*)^{W_S}$ is a polynomial ring), [TY, 17.4.4] shows that $\phi$ is birational and that $S(a^*)^{W_S}$ is an integral extension of $k[Z]$. Hence:

Corollary 2.2.7. The morphism $\phi : a//W_S \to Z$ is a bijective birational map, and $a//W_S$ is the normalization of $Z$.

One must observe that the injective map $\phi^*$ is not surjective, i.e. $Z$ is not normal, in general. This question has been studied in [He2, He3, Ri2, Pa3]. The notation being as in [He1, Ch. X], the results obtained in the previous references show that $\phi$ is an isomorphism when $g$ is of classical type, and in the exceptional cases of type EI, EII, EV, EVI, EVIII, FI, FII, G. In cases EIII, EIV, EVII, EIX, it is known that $\phi^*$ (or, equivalently, $\gamma^*$) is not surjective, cf. [He2, Ri2].

Remark 2.2.8. By standard arguments one can see that the results obtained in 2.2.4, 2.2.5 and 2.2.6 remain true when $(g, \theta)$ is a reductive symmetric Lie algebra.

2.3 Property (L)

Let $(g, \theta) = (g, t, p)$, $a, h, R, R^0, R^1, S$ be as in 2.2, and fix a $\sigma$-fundamental system $B$ of $R$ (cf. Definition 2.2.1). The next definition introduces an important property in order to study the $K$-conjugacy classes of Levi factors of the form $g^s$, $s \in p$ semisimple.

Definition 2.3.1. The pair $(g, t)$ satisfies the property (L) if, for all semisimple elements $s, u \in p$:

$$\{ \exists g \in G, g.g^s = g^u \} \iff \{ \exists k \in K, k.g^s = g^u \}. \quad \text{(L)}$$

Remark 2.3.2. More generally, when $(g, \theta)$ is a reductive symmetric Lie algebra, the condition (L) holds if and only if it holds for $([g, g], \theta)$.

We are going to show that it is sufficient to check (L) for some Levi factors $g^s$ of a particular type, cf. Remark 2.3.6.

Definition 2.3.3. Let $s \in h_Q$ be in the positive Weyl chamber defined by $B$. One says the standard Levi factor $g^s$ arises from $p$ if one can choose $s$ in $a_Q$. 

17
Recall from Section 1 that there is a natural one to one correspondence between standard Levi factors and subsets of $B$. In this correspondence, to a Levi factor $I$ one associates the subset

$$I_l := \{ \alpha \in B \mid \alpha(s) = 0 \}$$

where $s$ is any element in $(\mathfrak{g})^\ast$. Conversely, from any subset $I \subset B$ one gets a Levi subalgebra by setting:

$$I_l = \mathfrak{h} \oplus (\oplus_{\alpha \in \langle I \rangle} \mathfrak{g}^\alpha)$$

where $\langle I \rangle = Z_I \cap R$. Remark that $\mathfrak{g}^{I_l} = \{ h \in \mathfrak{h} : \alpha(h) = 0 \text{ for all } \alpha \in I \}$.

Let $\bar{D}$ be the Dynkin diagram defined by $B$ and denote by $\bar{D}$ the associated Satake diagram. Let $B^0 \subset B$ be the set of black nodes of $\bar{D}$; recall that $B^0$ is a fundamental system of $R^0$ (cf. Lemmas 2.2.2 and 2.2.3). Set

$$B^2 = \{ (\alpha_1, \alpha_2) \in B \times B : \alpha_1 \neq \alpha_2, \alpha_1'' = \alpha_2'' \}, \quad B^3 = \{ \alpha_1 - \alpha_2 \mid (\alpha_1, \alpha_2) \in B^2 \} \subset \mathfrak{h}_Q^\ast.$$

Thus, $B^2$ is the set of pairs of white nodes $(\alpha_1 \neq \alpha_2)$ of $\bar{D}$ connected by a two-sided arrow (note that $(\alpha_1, \alpha_2) \in B^2 \iff (\alpha_2, \alpha_1) \in B^2$). Denote by $B^2 \subset B$ the set of all nodes pointed by such an arrow, i.e. $B^2 = \{ \alpha \in B : \exists \beta \in B, (\alpha, \beta) \in B^2 \}$. A subset $I \subset B$ is said to be stable under arrows if $(\alpha_1, \alpha_2) \in B^2$ with $\alpha_1 \in I$ implies $\alpha_2 \in I$.

**Remark 2.3.4.** The subspace $a_Q \subset \mathfrak{h}_Q$ is the intersection of the kernels of elements of $B^0 \cup B^3$. A standard Levi factor $I$ arises from $\mathfrak{p}$ if, and only if, $I_l$ is stable under arrows and contains $B^0$.

We now want to describe the subalgebra $\mathfrak{g}^s$ when $s \in a$ semisimple. Observe that $(\mathfrak{g}^s, \mathfrak{t}^s)$ is a reductive symmetric pair and set

$$E_s = \{ \varphi \in \mathfrak{h}_Q^\ast = V_Q : \varphi(s) = 0 \}, \quad R_s = E_s \cap R.$$

Then, $R_s$ is a root subsystem of $R$ (cf. [TY, 18.2.5]) and, with obvious notation, the $\mathbb{Q}$-vector space $F_s$ spanned by $R_s$ decomposes as $F'_s \oplus F''_s$. The restriction to $\mathfrak{h}_s \subset \mathfrak{h}$ identifies $F_s$ with $\mathfrak{h}_s \cap [\mathfrak{g}^s, \mathfrak{g}^s]$. One can therefore apply to $R_s$ the results of section 2.2. Let $S_s$ be the restricted root system of $R_s$. As $s \in a$, one has:

$$S_s = \{ x'' \mid x \in R^1, x(s) = 0 \} = \{ x'' \mid x \in R^1, x''(s) = 0 \} = S \cap F''_s. \quad (2.2)$$

Let $B_s$ be a $\sigma$-fundamental system of $R_s$. One can write $B_s = B^0_s \cup B^1_s$ with $B^0_s \subset R^0$, $B^1_s \subset R^1$ and we denote by $B^0_s$ the restricted fundamental system of $S_s$ associated to $B_s$.

We can now prove the following result:

**Proposition 2.3.5.** Each Levi factor $\mathfrak{g}^s$, $s \in \mathfrak{p}$, is $K$-conjugate to a standard Levi factor that arises from $\mathfrak{p}$.

**Proof.** Since the element $s \in \mathfrak{p}$ is semisimple, it is $K$-conjugate to an element of $a$ and we may as well suppose that $s \in a$. We will use the previous notation relative to $R_s, S_s$ and a fixed $\sigma$-fundamental system $B_s \subset R_s$.

We first show that there exists $w \in W_\sigma$ such that $B_s \subset w.B$. Since $V_Q' \subset E_s$ one has $R^0 \subset R_s$, and $B^0_s$ being a fundamental system of the root system $R^0$, it can be conjugated to $B^0$ by an element of $W^0$. As $B''_s$ is a fundamental system of $S_s = S \cap F''_s$ (see (2.2)), [TY, 18.7.9(ii)] implies that $B''_s$ is a $W_s$-conjugate of a subset of $B''$. Combining these two facts and Lemma 2.2.3(i), one gets the existence of $w \in W_\sigma$ such that $B^0_s = w.B^0$ and $B''_s \subset w.B''$. We claim that $B_s \subset w.B$, i.e. $B^1_s \subset w.B$. Let $\alpha \in B^1_s$. Since $w.B$ is a $\sigma$-fundamental system of $R$, there exist integers $(n_\gamma)_{\gamma \in w.B}$, of the same sign, such that $\alpha = \sum_{\gamma \in w.B} n_\gamma \gamma$ and $\alpha'' = \sum_{\gamma \in w.B} n_\gamma \gamma''$. 18
As \( \alpha'' \in w.B'' \), the \( n_\gamma \)'s must be positive and there exists a unique \( \beta \in w.B^1 \) such that: \( \alpha'' = \beta'' \), \( n_\beta = 1, n_\gamma = 0 \) for \( \gamma \in w.B^1 \setminus \{ \beta \} \). One then gets \( \beta = \alpha - \sum_{\gamma \in w.B^0 = B_0} n_\gamma \gamma \), hence \( \beta \in R_s \). But \( B_s \) is a fundamental system of \( R_s \), thus the previous decomposition of \( \beta \) as a sum of positive and negative elements of \( B_s \) forces \( n_\gamma = 0 \) for \( \gamma \in B^0_0 \). Therefore \( \alpha = \beta \in w.B \), as desired.

Pick \( \dot{w} \in K \) such that \( \dot{w}.s = w.s \), see Remark 2.2.4(2); replacing \( \mathfrak{g}^a \) by \( \mathfrak{g}^{\dot{w}.s} \) we may assume that \( w = \text{Id} \) and \( B_s \subset B \). Define \( t \in B_0 \) by the conditions: \( \alpha(t) = 0 \) for \( \alpha \in B_s \) and \( \beta(t) = 1 \) for \( \beta \in B \setminus B_s \). Then, \( t \in \cap_{\varphi \in B^0_0 \cup B^1} \ker \varphi = a_Q \) (cf. Remark 2.3.4). Finally, since \( B_s \) is a fundamental system of \( R_s \), it is easily seen that \( \mathfrak{g}' = \mathfrak{g}^t \).

From the previous proposition one deduces the announced result:

**Corollary 2.3.6.** The property (L) is equivalent to: “Two standard Levi factors arising from \( \mathfrak{p} \) are G-conjugate if and only if, they are \( K \)-conjugate”.

We can now show that (L) is satisfied by any symmetric Lie algebra \( (\mathfrak{g}, \mathfrak{t}) \).

**Theorem 2.3.7.** (i) If \( l_1 \) and \( l_2 \) are two standard G-conjugate Levi factors arising from \( \mathfrak{p} \) such that \( B^0 \cup \overline{B^2} \subset I_{l_1} \), then \( \mathfrak{g}^{l_1} \subset \mathfrak{a} \subset \mathfrak{p} \) and \( l_1 \) and \( l_2 \) are \( K \)-conjugate.

(ii) Assume that there is no arrow in the Satake diagram of \( (\mathfrak{g}, \mathfrak{t}) \). Then, for any semisimple element \( s \in \mathfrak{p} \) one has \( c_s(\mathfrak{g}^s) = c_s(\mathfrak{p}^s) \subset \mathfrak{p} \), and the property (L) is satisfied by \( (\mathfrak{g}, \mathfrak{t}) \).

(iii) If \( (\mathfrak{g}, \mathfrak{t}) \) is irreducible of type AIII, DI, DIII, EII, EIII, then it satisfies (L).

(iv) Every reductive symmetric Lie algebra satisfies the property (L).

**Proof.** (i) The first assertion follows from the characterization of \( \mathfrak{a} \) given in Remark 2.3.4. Let \( s \in (\mathfrak{g}^{l_2})^* \), hence \( \mathfrak{g}^s = \mathfrak{g}^{l_2} \); by hypothesis, there exists \( g \in G \) such that \( g.s \in (\mathfrak{g}^{l_1})^* \subset \mathfrak{p} \). Proposition 2.2.6 then implies the existence of \( k \in K \) such that \( g.s = k.s \), thus: \( l_1 = \mathfrak{g}^{k.s} = k.l_2 \).

(ii) Observe that, here, \( \overline{B^2} = \emptyset \). By Proposition 2.3.5 we may assume that \( \mathfrak{g}^s = \mathfrak{a} \) with \( s \in \mathfrak{a}_Q \). Then, obviously, \( B^0 \subset I_l \) and from (i) one deduces \( \mathfrak{g}^s \subset \mathfrak{p} \). Therefore, \( c_s(\mathfrak{p}^s) = \mathfrak{g}^s \cap \mathfrak{p} = \mathfrak{g}^s \subset \mathfrak{p} \) (cf. [TY, 38.8.3]). The proof of (L) follows as in (i).

(iii) Let \( \mathfrak{g}^{s_i}, s_i \in \mathfrak{a}_Q, i = 1, 2 \), be two standard Levi factors arising from \( \mathfrak{p} \). Observe first that Proposition 2.2.5(ii) yields: \( \mathfrak{g}^{s_1}, \mathfrak{g}^{s_2} \) are \( G \)-conjugate \( \iff \mathfrak{g}^{s_1}, \mathfrak{g}^{s_2} \) are \( W \)-conjugate, and \( \mathfrak{g}^{s_1}, \mathfrak{g}^{s_2} \) are \( K \)-conjugate \( \iff \mathfrak{g}^{s_1}, \mathfrak{g}^{s_2} \) are \( W_\sigma \)-conjugate. Let \( B \) be a \( \sigma \)-fundamental system; denote by \( \Phi \) the set of all subsets of \( B \) which contain all black nodes and which are sable under arrows. Observe that \( E \in \Phi \) is equivalent to \( E = I_l \) for some standard Levi factor \( l \) arising from \( \mathfrak{p} \). Therefore, by the previous remark, we need to show that two elements of \( \Phi \) are \( W \)-conjugate if and only if they are \( W_\sigma \)-conjugate.

For \( E \in \Phi \) we define a subset \( \phi(E) \) of \( B'' \), the fundamental system of the restricted root system \( S \), by setting \( \phi(E) = \{ \alpha'' : \alpha \in E \} \setminus \{ \emptyset \} \). It is easy to see that \( \phi \) defines a bijection from \( \Phi \) onto \( \Phi'' \), the set of all subsets of \( B'' \), and that two elements of \( \Phi \) are \( W_\sigma \)-conjugate if and only if their images by \( \phi \) are \( W_\sigma \)-conjugate. By abuse of notation, we denote by \( \Phi/W \) and \( \Phi/W_\sigma \) resp. \( \Phi''/W_\sigma \), the set of orbits under \( W \) and \( W_\sigma \), resp. \( W_\sigma \), of elements of \( \Phi \), resp. \( \Phi'' \). Since \( W_\sigma \subset W \), there exists a natural surjection \( \pi \) from \( \Phi/W_\sigma \) onto \( \Phi/W \), hence \( \# \Phi/W \leq \# \Phi''/W_\sigma = \# \Phi/W_\sigma \), and we need to show that \( \pi \) is bijective. We have remarked above that \( \phi^{-1} \) yields a bijection between \( \Phi''/W_\sigma \) and \( \Phi/W_\sigma \). Let \( \delta : \Phi''/W_\sigma \rightarrow \Phi/W \) be the surjection induced by \( \pi \circ \phi^{-1} \). It remains to show that \( \delta \) is injective, or, equivalently, that \( \# \Phi/W \geq \# \Phi''/W_\sigma \).

The description of \( \phi, \Phi \) and \( \Phi'' \) can be deduced from [He1, p. 532]. In [BC, p. 5] are given the \( W \)-conjugacy classes of subsets of \( B \). Using these results we are now going to make a case by case comparison of \( \Phi/W \) and \( \Phi''/W_\sigma \). We mostly adopt the notation of [BC]; in particular, an element of \( \Phi/W \) will be identified with a Dynkin subdiagram of the Dynkin diagram defined by \( B \). We make a
distinction between the diagrams of type $A_1, B_1$, resp. $2A_1, D_2$, resp. $A_3, B_3$, as in [Os, §10]. A similar notation is used for the elements of $\Phi''/W_S$.

Cases EIII & EII: In case EIII, one finds that $\Phi/W = \{E_6, A_5, D_4, A_3\}$ and $\Phi''/W_S = \{B_2, B_1, A_1, 0\}$. In case EII, one easily sees that $\#\Phi/W = \#\Phi''/W_S = 12$.

Cases DI & DII: By (i), one can reduce the comparison to the elements of $\Phi$ which do not contain elements of $W^2$. Denote by $\Phi_1 \subset \Phi$ these subsets and set $\Phi''_1 = \phi(\Phi_1) \subset \Phi''$. The conjugacy classes of $\Phi_1, \Phi''_1$ are, respectively, denoted by $\Phi_1/W$ and $\Phi''_1/W_S$.

In type DI we are then reduced to consider the case $(g, t) = (so_{2n}, so_{n-1} \times so_{n+1})$, where $R$ is of type $D_n$ and $S$ of type $B_{n-1}$. Under the above notation, any element $E$ in $\Phi_1/W$ has type $\sum_k A_{i_k} + D_j$ where: $i_k \in \mathbb{N}$, $\sum_k (i_k + 1) + j = n$ (the Bala-Carter conditions) and $j = 0$, $i_{k_0} = 0$ for at least one $k_0$ (because $E \in \Phi_1/W$). Since $i_{k_0} = 0$ is even, it follows from [BC] that there exists a unique conjugacy class in $\Phi_1$ satisfying these properties. Then, $\delta^{-1}(E)$ contains an unique element, of type $\sum_{k \neq k_0} A_{i_k} + B_j$, determined by the previous conditions.

The remaining case, $(g, t) = (so_{4n+2}, gl_{2n+1})$, in type DII is similar.

Case AIII: Here, $(g, t) = (sl_{p+q}, sl_p \times sl_q \times k)$ with $p \geq q$. The restricted root system is of type $B_q$ (when $p > q$) or $C_q$ (when $p = q$).

The $W$-conjugacy class $N$ of an element of $\Phi$ is given by a diagram of type $\sum_i n_i A_i$, where $\sum_i n_i (i+1) = p + q$ and at most one $n_i$ is odd. One case $p - q = 1$ if $i_0$ does not exist, and $i_0 \geq p - q - 1$ otherwise.

The $W_S$-conjugacy class of $\phi(N)$ is given by a diagram $\sum_i m_i A_i + BC_j$ defined by the following rule: $m_i = n_i/2$ if $i \neq i_0$, $m_{i_0} = (n_{i_0} - 1)/2$, and

$$BC_j = \begin{cases} B_{(i_0-(p-q-1))/2} & \text{if } i_0 \text{ exists and } p \neq q; \\ B_0 & \text{if } p - q = 1 \text{ and } i_0 \text{ does not exist}; \\ C_{(i_0+1)/2} & \text{if } p = q \text{ and } i_0 \neq 0; \\ C_0 & \text{if } p = q \text{ and } i_0 = 0. \end{cases}$$

It follows from [BC] that this class depends only on the class $N$. Therefore $\delta$ is injective, proving (iii) in type AIII.

(iv) When $(g, \theta)$ is of type $0$ there is an obvious bijection between $W$-conjugacy classes of elements $\Phi$ and $W_S$-conjugacy classes in $\Phi''$. Since $g$ is a direct product of irreducible symmetric Lie algebras, the result then follows from (ii) and (iii).

\[ \Box \]

### 2.4 Jordan $K$-classes

Let $(g, t)$ be a reductive symmetric Lie algebra. We adopt the notation of §1.3 and Definition 2.0.3.

Observe the following easy result:

**Lemma 2.4.1.** The intersection of a $J_G$-class with $p$ is either empty or the union of $J_K$-classes it contains.

**Proof.** Let $J$ be a Jordan $G$-class intersecting $p$ and $x = s + n \in J \cap p$. Then $J_K(x) = K.(c_p(p^s)^s + n) \subset G.(c_p(g^s)^s + n) = J_G(x)$.

In Lemma 2.4.2 we fix a $J_G$-class $J$ such that $J \cap p \neq \emptyset$, and an element $x = s + n \in J \cap p$. Let $l = g^s$ and $L = G^s \subset G$ be the associated Levi factors. Observe that:

$$L = Z_G(c_p(g^s)^s). \tag{2.3}$$

Then, $(g^s, t^s)$ is a symmetric pair and $K_L = (K \cap L)^g \subset K^g$ acts naturally on $p^s$. Denote by $O_1$ the orbit $L.n \in l$, so that $(l, O_1)$ is a datum of $J$. Let $O_1 \subset l (i > 1)$ be the $L$-orbits (if they exist) different
Lemma 2.4.2. (i) One has $J \cap p = \bigcup_{i,j} K.(c_p(p^*)^i + n_i^j)$.
(ii) Any $J_K$-class contained in $J \cap p$ has dimension $\dim c_p(p^*) + \dim K.x$.

Proof. Let $y = s' + n' \in J \cap p$. Since $x$ and $y$ belong to the same $J_K$-class, $g_{s'}$ is $G$-conjugate to $g$ [TY, 39.1.3]. By Property (L), see Theorem 2.3.7, the subalgebra $g_{s'}$ is then $K$-conjugate to $g$. We can therefore assume that $s' \in c_p(p^*)$. It follows that $n'$ belongs to one of the orbits $K.n_i^j$, hence $J_K(y) = K.(c_p(p^*)^i + n_i^j) \subset J \cap p$.

By [TY, 39.5.8] one knows that $\dim J_K(y) = \dim K.y + \dim c_p(p^*) = \dim K.x + \dim c_p(p^*) = \dim J_K(x)$. This proves (i) and (ii).

Note that the union in Lemma 2.4.2(i) is not necessarily a disjoint union.

Lemma 2.4.3. (i) Let $g \in G$ and $s \in p$ semisimple be such that $g.s \in p$; then $g.c_p(p^*) \subset p$.
(ii) For $x, y \in p$ such that $G.x = G.y$, one has $G.J_K(x) = G.J_K(y)$.

Proof. (i) By Lemma 2.2.6 there exists $k \in K$ such that $k.(g.s) = s$, hence $k.g \in L = Z_G(c_p(g_{s'}^*)^i)$ (see (2.3)) and $k.g.c_p(p^*) = c_p(p^*)$. This gives $g.c_p(p^*) = k^{-1}.c_p(p^*) \subset p$.

(ii) By Lemma 2.2.6, again, we may assume that $x = s + n$ and $y = s + n'$. Then, $J_K(x) = K.(c_p(p^*)^i + n)$ and $J_K(y) = K.(c_p(p^*)^i + n')$. Write $y = g.x$, $g \in G$; from (2.3) it follows that $g.(s + n) = s' + n'$ for all $s' \in c_p(p^*)^i$.

We can now describe the intersection of a $J_G$-class with $p$.

Theorem 2.4.4. Let $J$ be a Jordan $G$-class. The variety $J \cap p$ is smooth. The $J_K$-classes contained in $J \cap p$ are its (pairwise disjoint and smooth) irreducible components.

Proof. We may obviously assume that $J \cap p \neq \emptyset$; pick $x \in J \cap p$. Recall [Br] that $J$ is smooth and that the tangent space $T_xJ$ is equal to $[x, g] \subset c_p(g_{s'})$, see [TY, 39.2.8, 39.2.9]. By [TY, 39.5.5] there exists a dominant morphism $\mu : K \times c_p(p^*)^i \rightarrow J_K(x)$, $(k, u) \mapsto k.u$. Therefore $d_{(\text{Id}, x)} \mu(\xi \times c_p(p^*)) = [x, \xi] \subset c_p(p^*)$ (cf. [TY, 39.5.7]) is a subspace of the tangent space $T_xJ_K(x)$, and we then obtain:

$$T_x(J \cap p) \subset T_xJ \cap p = ([x, g] \subset c_p(g_{s'}))^i \cap p = [x, \xi] \subset c_p(p^*) \subset T_xJ_K(x) \subset T_x(J \cap p).$$

Thus $T_x(J \cap p) = T_xJ_K(x)$ has dimension $\dim J_K(x) = \dim c_p(p^*) + \dim K.x$. By Lemma 2.4.2(ii), this dimension does not depend on the element $x$ chosen in $J \cap p$. Therefore $J_K(x), J \cap p$ are smooth and each element of $J \cap p$ belongs to a unique irreducible component (see, for example, [TY, 17.1.3]). Then, Lemma 2.4.1 yields the desired result.

The smoothness of $J \cap p$ can be deduced from a general result that we now recall, see, for example, [IV, Proposition 1.3] or [PV, 6.5, Corollary].

Theorem 2.4.5. Let $\Gamma$ be a reductive group acting on a smooth variety $X$. Then the subvariety of fixed points $X^{\Gamma} = \{x \in X \mid \Gamma.x = x\}$ is smooth, and $T_xX^{\Gamma} = (T_xX)^{\Gamma}$ for all $x \in X^{\Gamma}$.

This theorem can be applied to a $J_G$-class $J$ as follows. Let

$$\Gamma = \{\text{Id}, \tilde{\theta}\} \subset \text{GL}(g)$$

be the group, of order two, generated by $\tilde{\theta} = -\theta$ (thus $\tilde{\theta}$ is an anti-automorphism of $g$). Recall [TY, 39.1.7] that $J = J_G(x) = G.c_p(g_{s'}^*)^i$. From this description it is easily seen that, when $J \cap p \neq \emptyset$, $J$ is stable under the $\kappa^x$-action $y \mapsto \lambda y$, $\lambda \in \kappa^x$, and that $\tilde{\theta}(J) = \theta(J) = J$. Therefore, the group $\Gamma$ acts on the smooth variety $J$ and we get from Theorem 2.4.5 that $J^{\Gamma} = J \cap p$ is smooth.
2.5 \( K \)-sheets

We continue with the same notation. Fix a \( G \)-sheet \( S = S_G \subset \mathfrak{g}^{(2m)} \), \( m \in \mathbb{N} \). We aim to describe the irreducible components of \( S_G \cap \mathfrak{p} \). One important remark is that if \( S_G \cap \mathfrak{p} \neq \emptyset \), then the unique nilpotent orbit \( \mathcal{O} \) contained in \( S_G \) intersects \( \mathfrak{p} \) (cf. [TY, 39.6.2]). The description of the irreducible components of \( S_G \cap \mathfrak{p} \) will be given in terms of the \( K \)-orbits contained in \( \mathcal{O} \), see Theorem 2.5.11.

We first want to prove that when \( S \) is smooth, and \( (\mathfrak{g}, \theta) \) has no irreducible factor of type 0, the intersection \( S \cap \mathfrak{p} \) (which can be empty) is also smooth. To obtain this result we will apply Theorem 2.4.5, as in the case of a Jordan \( G \)-class. We adopt the notation of the end of the previous subsection, in particular we set \( \Gamma = \{ \text{Id}, \hat{\theta} = -\theta \} \). Observe that \( S \) is stable under the \( \mathbb{R}^* \)-action, thus \( \hat{\theta}(S) = \theta(S) \); but, contrary to the case of a Jordan class, the stability of \( S \) under \( \Gamma \) requires some hypothesis, even in the case where \( S \cap \mathfrak{p} \neq \emptyset \).

We begin with the following, probably known, technical result. Recall [CM, 7.1] that a nilpotent orbit \( \mathcal{O} \) is called rigid if it can not be obtained by induction of a proper parabolic subalgebra of \( \mathfrak{g} \); equivalently, when \( \mathfrak{g} \) is semisimple, \( \mathcal{O} \) is rigid if \( \mathcal{O} \) is a \( G \)-sheet, cf. [Bo, §4]. Recall also that the only rigid orbit in type \( A \) is \( \{0\} \), see [Kr, 2.4] or [CM].

**Lemma 2.5.1.** Let \( \mathfrak{l} \) be a Levi factor of a simple Lie algebra \( \mathfrak{g} \) and \( \mathcal{O} \) be a rigid nilpotent orbit of \( \mathfrak{l} \). Then, \( \tau(\mathcal{O}) = \mathcal{O} \) for all \( \tau \in \text{Aut}(\mathfrak{l}) \).

**Proof.** Observe first that \( \tau(\mathcal{O}) \) is a rigid nilpotent orbit. Decompose \( \mathfrak{l} \) as the direct sum of its center and simple factors, \( \mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \bigoplus_j \mathfrak{l}_j \). Write \( \mathcal{O} = \prod_j \mathcal{O}_j \) where \( \mathcal{O}_j \subset \mathfrak{l}_j \) is a nilpotent orbit. It is easy to see that the orbits \( \mathcal{O}_j \) are rigid; observe also that if each \( \mathfrak{l}_j \) is of type \( A \), \( \mathcal{O} \) is zero and, obviously, \( \tau(\mathcal{O}) = \mathcal{O} \).

From the classification of Dynkin diagrams one deduces that for each subdiagram of the Dynkin diagram of \( \mathfrak{g} \), there exists at most one connected component of type different from \( A \). Therefore, we are reduced to the case when there exists an index \( j \) such that \( \mathfrak{l}_j \) is not of type \( A \). By uniqueness of \( j \) one has \( \tau(\mathfrak{l}_j) = \mathfrak{l}_j \). Set \( \mathfrak{m} = \mathfrak{l}_j \) and \( \Omega = \mathcal{O}_j \); by the previous remarks it remains to show that \( \Omega = \tau(\Omega) \subset \mathfrak{m} \).

Recall that \( \text{Aut}(\mathfrak{m}) \) is the semidirect product of the adjoint group \( M \) of \( \mathfrak{m} \) and of the group \( \text{Out}(\mathfrak{m}) \), isomorphic to the group of automorphisms of its Dynkin diagram, cf. [Bou, Ch. VIII, §5.3].

- If \( \mathfrak{m} \) is of type \( B_n, C_n, E_7, E_8, F_4 \) or \( G_2 \), then \( \text{Aut}(\mathfrak{m}) = M \) and \( \tau(\Omega) = \Omega \) is clear.
- Suppose that \( \mathfrak{m} \) is of type \( E_6 \); then, \( \text{Out}(\mathfrak{m}) = \{\text{Id}, \omega\} \) has order two. Recall that weighted Dynkin diagrams are in one to one correspondence with nilpotent orbits, see [CM]. Then, going through the list of weighted Dynkin diagrams in type \( E_6 \) (cf. [CM, 8.4]), one sees that each of these diagrams is fixed by the automorphism induced by \( \omega \). Thus, \( \omega(\Omega) = \Omega \) and it follows that \( \tau(\Omega) = \Omega \) for all \( \tau \in \text{Aut}(\mathfrak{m}) \).

- Assume that \( \mathfrak{m} \cong \mathfrak{so}_{2n} \) is of type \( D_n \) with \( n \geq 5 \); here, as in the previous case, \# \text{Out}(\mathfrak{m}) = 2. Let \( \lambda \) be the partition of \( 2n \) associated to \( \Omega \). From the classification of rigid orbits one deduces that \( \lambda \) is not very even and it follows that \( \Omega \) is stable under \( \text{Aut}(\mathfrak{m}) \) (see [CM, 7.3] for these assertions).

- The last remaining case is when \( \mathfrak{m} \) is of type \( D_4 \). By [CM, 7.3] (for example), one gets that there exist two nonzero rigid orbits in \( \mathfrak{m} \), which have different dimensions. Thus \( \Omega \) is stable under \( \text{Aut}(\mathfrak{m}) \). \( \square \)

The next lemma ensures that when \( \mathfrak{g} \) is simple, the smoothness of \( S \) is inherited by \( S \cup \theta(S) \).

**Lemma 2.5.2.** Let \( \mathfrak{g} \) be a simple Lie algebra. If \( \mathfrak{g} \) is not of type \( D \), then \( \theta(S) = S \).

If \( \mathfrak{g} \) is of type \( D \), one has either \( \theta(S) = S \) or \( S \cap \theta(S) = \emptyset \).

**Proof.** Let \( \mathfrak{l}_1 \) be the dense Jordan class contained in \( S \) and let \( (\mathfrak{l}, \mathcal{O}) \) be a datum of \( \mathfrak{l}_1 \). Then, the dense Jordan class \( \mathfrak{l}_2 \) in the sheet \( \theta(S) \) has datum \( (\theta(\mathfrak{l}_1), \theta(\mathcal{O})) \).

If \( \mathfrak{g} \) is of type different from \( D \) or \( E_7 \), it follows from the classification of Levi factors in [BC, Proposition 6.3] that \( \theta(l) \) is \( G \)-conjugate to \( l \). In these cases we can therefore assume that \( \theta(l) = l \), and Lemma 2.5.1 yields \( \theta(\mathcal{O}) = \mathcal{O} \). Thus, \( \mathfrak{l}_1 = \mathfrak{l}_2 \) and \( \theta(S) = S \).
If \( g \) is of type \( E_7 \), there exists no outer automorphism of \( g \) so \( \theta(S) = G.S = S \).
Suppose that \( g \) is of type \( D \). If \( I \) and \( \theta(I) \) are \( G \)-conjugate, the previous argument applies and one gets \( \theta(S) = S \). Otherwise, [IH, Corollary 3.15] implies that \( S \cap \theta(S) = J_1 \cap J_2 = \emptyset \).

We can now prove the desired result:

**Proposition 2.5.3.** (i) Let \( (g, \theta) \) be a reductive symmetric Lie algebra which has no irreducible factor of type \( 0 \). If \( S \) is a smooth \( G \)-sheet, the intersection \( S \cap p \) is smooth.
(ii) Let \( (g, \theta) \) be a symmetric Lie algebra and \( S' \) be a \( K \)-sheet contained in a smooth \( G \)-sheet \( S \). Then \( S' \) is smooth.

**Proof.** Decompose the symmetric algebra \( (g, \theta) \) as \( (g, \theta) = \bigoplus_j (g_j, \theta_j) \) where each \( (g_j, \theta_j) \) is an irreducible factor (see the beginning of this section).
(i) We want to apply Theorem 2.4.5 with \( \Gamma = \{ \text{Id}, \hat{\theta} = -\theta \} \) and \( X = S \cup \theta(S) \subset g \). Note that \( X' = (S \cap p) \cup (\theta(S) \cap p) \) and that \( \theta(S) \) is smooth.

If \( g \) is simple, Lemma 2.5.2 yields that \( X = S \) or \( S \cup \theta(S) \) (in type \( D \)) is smooth; therefore \( X' \), and consequently \( S \cap p \), is smooth. Suppose that \( g \) is not simple. By hypothesis, each \( g_i \) is simple and the result then follows from Corollary 1.7.2.
(ii) By a similar reduction to the irreducible factors \( (g_i, \theta_i) \) the only remaining case to consider is that of type \( 0 \), i.e., \( g = g^1 \oplus g^2 \) with \( \theta : g^1 \overset{\sim}{\rightarrow} g^2 \). From the results of §2.1 it follows that there exists a \( G^1 \)-sheet \( S^1 \subset g^1 \) such that \( S' \equiv \{ x - \theta(x) \mid x \in S^1 \} \). Then \( S = S^1 \times \theta(S^1) \), which is smooth if and only if \( S^1 \) is smooth. As \( S^1 \) is isomorphic to \( S' \), one gets the desired result.

**Remarks 2.5.4.** (1) The sheets in a classical Lie algebra are smooth, see Theorem 1.4.10. Therefore if \( g \) is simple of classical type, Proposition 2.5.3 implies that \( S \cap p \) is smooth for each sheet \( S \) of \( g \).
(2) When \( g = gl_N \), case which will be studied in details in Section 3, the smoothness of \( S_G \cap p \) can be explained in different (equivalent) terms. Indeed, recall first that, if \( g = gl_N \), a nilpotent orbit is contained in a unique \( G \)-sheet, cf. Remark 1.4.3. Assume that the sheet \( S = S_G \) intersects \( p \) and let \( O = G.e \) be the nilpotent orbit contained in \( S \). Then, since we may assume that \( e \in p \), it follows from \( G.\theta(e) = G.\theta(-e) = G.e \subset \theta(S) \cap S \) that \( \theta(S) = S \). Therefore, the group \( \Gamma \) acts on \( S \) and \( S' = S \cap p \) is smooth.

Assume that the sheet \( S_G \) intersects \( p \), pick \( e \in O \cap p \) and set
\[
O_e = K.e \subset O \cap p.
\]
Denote by \( \mathcal{S} = (e, h, f) \) a normal \( sl_2 \)-triple containing \( e \). We are going to apply the results recalled in §1.4 to various triples deduced from \( \mathcal{S} \).
Let \( Z \subset G \) be a subset such that \( \{ g.e \}_{g \in Z} \) is a set of representatives of the \( K \)-orbits contained in \( O \cap p \); we assume that \( \text{Id} \in Z \). Observe that, since the \( sl_2 \)-triples containing \( g.e \) are conjugate, we may also assume that \( g.\mathcal{S} := (g.e, g.h, g.f) \) is a normal \( sl_2 \)-triple for all \( g \in Z \). Recall that \( X(S_G, g.\mathcal{S}) \) is defined by
\[
g.e + X(S_G, g.\mathcal{S}) = S_G \cap (g.e + g^{sl_2}) = g.(S_G \cap (e + g^{sl_2})) = g.(e + X(S_G, \mathcal{S})).
\]
(Hence \( X(S_G, g.\mathcal{S}) = g.X(S_G, \mathcal{S}) \).) Set
\[
X_p(S_G, g.\mathcal{S}) = X(S_G, g.\mathcal{S}) \cap p.
\]  

**Remark 2.5.5.** Recall that \( S \subset g^{(2m)} \). Let \( \emptyset \neq Y \subset g.e + X_p(S_G, g.\mathcal{S}) \); then, each \( G \)-orbit (resp. \( K \)-orbit) of an element of \( Y \) has dimension \( \dim G.e = 2m \) (resp. \( \dim K.e = m \)). Lemma 1.4.6 implies that the fibers of the morphisms \( G \times Y \to G.Y \) and \( K \times Y \to K.Y \) are of respective dimension \( \dim G.e \) and \( \dim K.e \). Then, by [TY, 15.5.5], we get that \( \dim G.Y = \dim Y + 2m \) and \( \dim K.Y = \dim Y + m \).
We now introduce some conditions which will be sufficient to give a description of the irreducible components of \( S_G \cap \mathfrak{p} \) in terms of the \( X_p(S_G, \mathcal{I}) \), see Theorem 2.5.11.

Recall that \( S_G = G.(e + X(S_G, \mathcal{I})) \). The first condition ensures that \( e + X_p \) is large enough:

\[
G.(g.e + X_p(S_G, g.\mathcal{I})) = G.(S_G \cap \mathfrak{p}) \quad \text{for all } g \in \mathbb{Z}.
\]

The condition \((\bigtriangledown)\) was established for pairs of type 0 in Proposition 2.1.4, and we will see that it also holds for all symmetric pairs when \( g = \mathfrak{gl}_N \) (cf. Theorem 3.2.1). Set:

\[
A(g.e) = G^{g.e}/(G^{g.e})^\circ.
\]

By Theorem 1.4.7 the Slodowy slice \( g.e + X(S_G, g.\mathcal{I}) \) provides the geometric quotient

\[
\psi_{S_G, g.\mathcal{I}} : S_G \rightarrow (g.e + X(S_G, g.\mathcal{I}))/A(g.e)
\]

and we will be interested in some cases where the following property is satisfied:

\[
G^e \text{ is connected.}
\]

Recall that \((*)\) is true when \( g = \mathfrak{gl}_N \) (see Lemma 1.6.1). Clearly, \((*)\) implies that \( g.e + X(S_G, g.\mathcal{I}) \) is the geometric quotient of \( S_G \). In this case, the restriction of \( \psi_{S_G, g.\mathcal{I}} \) to the subset \((g.e + \bigoplus_{i \geq 0} \mathfrak{g}(2i, g.h)) \cap S_G\)

\[
\text{is given by the map } \varepsilon_{S_G, g.\mathcal{I}} \text{ constructed in Lemma 1.4.4, and if hypothesis } (\bigtriangledown) \text{ is also satisfied, one has:}
\]

\[
\psi_{S_G, g.\mathcal{I}}(S_G \cap \mathfrak{p}) = g.e + X_p(S_G, g.\mathcal{I}).
\]

Let \( J_1 \) be a \( J_K \)-class contained in \( S_G \cap \mathfrak{p} \). As \( J_1 \) is \( K \)-stable, the dimension of \( J_1 \cap (g.e + \mathfrak{p}^{g.f}) \) does not depend on the representative element \( g.e \) of the orbit \( K.g.\mathcal{I} \). Since \( K \)-orbits of normal \( \mathfrak{sl}_2 \)-triples are in one to one correspondence with \( K \)-orbits of their nilpositive part, we may introduce the following definition.

**Definition 2.5.6.** Let \( g \in \mathbb{Z} \). A \( J_K \)-class \( J_1 \) contained in \( S_G \) is said to be well-behaved with respect to \( O_{g.e} = K.g.e \), if:

\[
\dim J_1 \cap (g.e + \mathfrak{p}^{g.f}) = \dim J_1 - m.
\]

**Remark 2.5.7.** It follows from Lemma 2.4.2(ii) that a \( J_K \)-class \( J_1 = K.(c_p(p^*)^\bullet + n) \) is well-behaved w.r.t. \( O_{g.e} \) if and only if \( Y = J_1 \cap (g.e + \mathfrak{p}^{g.f}) \) satisfies \( \dim Y = \dim c_p(p^*) (= \dim J_1 - m) \). By Remark 2.5.5 this is also equivalent to \( \dim K.Y = \dim J_1 \), which is in turn equivalent to \( J_1 \subset K.Y \). In this case one has \( J_1 \cap \overline{K.(g.e + X_p(S_G, g.\mathcal{I}))} \), property which will be of importance for the description of \( S_G \cap \mathfrak{p} \).

The following lemma shows that, assuming \((\bigtriangledown)\), well-behaved \( J_K \)-classes exist.

**Lemma 2.5.8.** Let \( J \) be a \( J_G \)-class contained in \( S_G \) such that \( J \cap \mathfrak{p} \neq \emptyset \). Fix \( g \in \mathbb{Z} \) and set \( \psi = \psi_{S_G, g.\mathcal{I}} \).

Assume that the property \((\bigtriangledown)\) is satisfied.

(i) Let \( J_1 \subset J \cap \mathfrak{p} \) be a \( J_K \)-class. There exists a subvariety \( Y \subset g.e + X_p(S_G, g.\mathcal{I}) \) such that: \( Y \) is irreducible and \( \psi(Y) \) is dense in \( \psi(J_1) \). Moreover, if \( Y \subset g.e + X_p(S_G, g.\mathcal{I}) \) is maximal for these two properties, then \( \psi(Y) = \psi(J_1) \) and \( J_2 = \overline{K.Y} \cap J \) is a \( J_K \)-class (contained in \( J \)) which is well-behaved w.r.t. \( O_{g.e} \).

(ii) The class \( J_1 \) is well-behaved w.r.t. \( O_{g.e} \) if and only if one can find \( Y \), as in (i), such that \( J_1 = \overline{K.Y} \cap J \).

(iii) If \((*)\) holds, there exists a unique maximal \( Y \) as in (i), namely \( Y = \psi_{S_G, g.\mathcal{I}}(J_1) \), thus \( J_2 = \overline{K.\psi_{S_G, g.\mathcal{I}}(J_1)} \cap J \).

**Proof.** In order to simplify the notation, we suppose that \( g = \text{Id} \) and we set \( X = X(S_G, \mathcal{I}), X_p = X \cap \mathfrak{p}, \psi = \psi_{S_G, \mathcal{I}}, S = S_G \), etc.
(i) Consider the following commutative diagram

\[
\begin{array}{ccc}
e + X_p & \xrightarrow{i} & e + X \\
\gamma_p & \downarrow & \downarrow \gamma \\
(e + X)/A = \psi(S) & & & \\
\end{array}
\]

where \( i \) is the natural closed embedding and \( \gamma \) is the quotient morphism, see (1.5). Observe that, the group \( A \) being finite, the morphisms \( \gamma \) and \( \gamma_p \) are finite, hence closed. Moreover, (\( \forall \)) implies that \( \text{im}(\gamma_p) = \psi(S \cap p) \). Let \( Y' \) be any irreducible component of \( \gamma_p^{-1}(\psi(J_1)) \) dominating \( \overline{\psi(J_1)} \subset (e + X)/A \) and set:

\[
Y = \gamma_p^{-1}(\psi(J_1)) \cap Y'.
\]

Then, \( Y \subset J \) is a dense irreducible subset of \( Y' \) such that \( \psi(Y) = \gamma_p(Y) = \psi(J_1) \). Since the fibers of \( \psi \) are of dimension \( m \) and \( \gamma_p \) is finite, one has \( \dim Y = \dim J_1 - m \). Set

\[
J_2 = \overline{K.Y} \cap J.
\]

As \( K.Y \subset J_2 \subset J \cap p \), \( J_2 \) is a closed irreducible subset of \( J \cap p \) of dimension \( \dim K.Y = \dim Y + m = \dim J \cap p \) (cf. Remark 2.5.5). One obtains from Theorem 2.4.4 that \( J_2 \) is a \( J_K \)-class, which is well-behaved w.r.t. \( O_e \) (recall that \( J_2 \subset \overline{K.Y} \)).

Suppose now that \( Y_1 \subset e + X_p \) is maximal for the properties: \( Y_1 \) irreducible and \( \psi(Y_1) \) dense in \( \psi(J_1) \). Observe that the closure \( Y_1' \) of \( Y_1 \) inside \( e + X_p \) is irreducible, and \( \gamma_p(Y_1') = \overline{\psi(Y_1')} = \psi(J_1) \). The argument of the previous paragraph, together with the maximality of \( Y_1 \), implies that \( Y_1 = \gamma_p^{-1}(\psi(J_1)) \cap Y_1' \). As above, we then get that \( \overline{K.Y} \cap J \) is a well-behaved \( J_K \)-class contained in \( J \cap p \).

(ii) Set \( Y_1 = J_1 \cap (e + X_p) \) and suppose that \( J_1 \) is well-behaved w.r.t. \( O_e \), thus \( \dim J_1 = \dim Y_1 + m \). Let \( Y_2 \subset Y_1 \) be an irreducible component of maximal dimension; since \( \gamma_p \) is finite, one has \( \dim \gamma_p(Y_2) = \dim Y_1 = \dim \psi(J_1) \), hence \( \psi(Y_2) \) is dense in \( \psi(J_1) \). We then deduce from (i) that \( J_2 = \overline{K.Y} \cap J \) is a \( J_K \)-class; since \( Y_2 \subset J_2 \cap J_1 \), it follows that \( J_1 = J_2 \) is well-behaved w.r.t. \( O_e \). The converse is clear.

(iii) Here, \( \gamma_p : e + X_p \xrightarrow{\sim} \psi(S \cap p) \) is the identity; thus \( Y' = \overline{\psi(J_1)} \) and \( Y = \psi(J_1) \).

**Remarks 2.5.9.** (1) In part (i) of the previous lemma, the \( J_K \)-class \( J_2 (\subset J \subset S_G) \) is contained in the following variety:

\[
S_K(S_G,g.\mathcal{F}) := \overline{K.(g.e + X_p(S_G,g.\mathcal{F}))}.
\]  

(2.6)

Since \( K \)-orbits of \( \mathfrak{sl}_2 \)-triples are in bijection with nilpotent \( K \)-orbits, \( S_K(S_G,g.\mathcal{F}) \) depends only on the sheet \( S_G \) and the orbit \( O_{g.e} = K.g.e \). Therefore we can write

\[
S_K(S_G,g.\mathcal{F}) = S_K(S_G,O_{g.e}).
\]

Furthermore when \( g \) is of type \( A \), thanks to Remark 1.4.3, we may also write \( S_K(S_G,g.\mathcal{F}) = S_K(g.\mathcal{F}) = S_K(O_{g.e}) \).

(2) Under assumption (\( \ast \)), Lemma 2.5.8(iii) yields a well defined application

\[
J_1 \mapsto J_2 = J \cap \overline{K.\psi(S_G,g.\mathcal{F})(J_1)}
\]

from the set of \( J_K \)-classes contained in \( S_G \cap p \) to the set of \( J_K \)-classes contained in \( S_K(S_G,g.\mathcal{O}) \).

In case \( A \), we will show in Lemma 3.3.5 and Lemma 3.3.11 that each \( J_K \)-class contained in \( S_G \cap p \) is in the image of such an application, for an appropriate choice of \( g \in \mathbb{Z} \).

We now introduce a condition ensuring that the varieties \( S_K(S_G,O_e) \) are irreducible:

\[
X_p(S_G,g.\mathcal{F}) \text{ is irreducible for all } g \in \mathbb{Z}.
\]  

(\( \diamond \))
Corollary 2.5.10. Assume that conditions ( Disorder) and ( Disorder) hold. Then, \( S_K(S_G, O_{g,e}) \) is an irreducible component of \( S_G \cap p \) of maximal dimension.

Proof. Let \( J_1 \) be a \( J_K \)-class of maximal dimension contained in \( S_G \cap p \) and \( J \subset S_G \) be the \( J_G \)-class containing \( J_1 \). Since ( Disorder) is satisfied, one can find \( Y \) as in Lemma 2.5.8(ii) such that \( J_2 = KY \cap J \) is a \( J_K \)-class contained in \( J \). Then, \( J_2 \subset S_K(S_G, O_{g,e}) \subset S_G \cap p \) and Theorem 2.4.4 implies that dim \( J_2 = \dim J_1 = \dim S_G \cap p \). Therefore \( S_K(S_G, O_{g,e}) = \overline{J_2} \) is an irreducible component of \( S_G \cap p \) of maximal dimension.

In view of the previous corollary, it is then natural to ask: Are all the irreducible components of \( S_G \cap p \) of the form \( S_K(S_G, O_{g,e}) \)? We introduce the next additional condition to answer that question:

For each \( J_K \)-class \( J_1 \) in \( S_G \cap p \), there exists \( g \in Z \) such that \( J_1 \) is well-behaved w.r.t. \( O_{g,e} \). ( Disorder)

Theorem 2.5.11. Under conditions ( Disorder), ( Disorder) and ( Disorder), the irreducible components of \( S_G \cap p \) are the \( S_K(S_G, O_{g,e}) \) with \( g \in Z \). Consequently, \( S_G \cap p \) is equidimensional. Moreover, there exists a unique \( J_G \)-class \( J \) such that \( S_G \cap p = \overline{J} \cap p \) and, for each \( g \in Z \), \( S_K(S_G, O_{g,e}) = \overline{J_g} \) for a unique \( J_K \)-class \( J_g \). In particular, the map \( S_K(S_G, O_{g,e}) \to J_g \) gives a bijection between irreducible components of \( S_G \cap p \) and the set of \( J_K \)-classes contained in \( J \cap p \).

Proof. Write \( S_G \cap p = \bigcup_{J \subset S_G} J \cap p \), where the union is taken over the \( J_G \)-classes \( J \) intersecting \( p \). For any such \( J, J \cap p \) is the union of the \( J_K \)-classes it contains (cf. Lemma 2.4.1), thus ( Disorder) and Lemma 2.5.8(ii) imply that \( S_G \cap p = \bigcup_{g \in Z} S_K(S_G, O_{g,e}) \). Since ( Disorder) and ( Disorder) are satisfied, one may apply Corollary 2.5.10 to get the two first claims.

Now, let \( J_1 \) be a \( J_K \)-class of maximal dimension contained in \( S_G \cap p \) and denote by \( J \subset S_G \) the \( J_G \)-class containing \( J_1 \). Let \( g \in Z \); as in the proof of Corollary 2.5.10 one can find a \( J_K \)-class \( J_g \subset J \cap p \) such that \( S_K(S_G, O_{g,e}) = \overline{J_g} \). It then follows from the previous paragraph that \( S_G \cap p = \overline{J} \cap p \). Furthermore, as \( J_K \)-classes are locally closed, \( J_g \) is the unique dense \( J_K \)-class in \( S_K(S_G, O_{g,e}) \). This implies the unicity of the class \( J \). Finally, the bijection follows from the first claim and Theorem 2.4.4.

We have observed in Remark 2.5.4 that when \( g \) is a simple classical Lie algebra, the variety \( S_G \cap p \) is smooth. We are going to introduce two new conditions in order to obtain the smoothness of \( S_G \cap p \) in the general case.

Recall from Lemma 1.4.6(ii) that the \( G \)-action induces an action of the group \( A(g,e) \cong G^\circ / (G^\circ)^c \) on \( g.e + X(S_G, g, \mathcal{F}) \). We consider the following condition:

\[
A(g,e).(g.e + X_p(S_G, g, \mathcal{F})) \text{ is a smooth variety for all } g \in Z. \tag{2.7}
\]

Remarks 2.5.12. (1) Set \( X_p = X_p(S_G, g, \mathcal{F}), A = A(g,e) \) and suppose that (2.7) is satisfied. Using the \((F_t)_{t \in k}\)-action (see §1.4), one can see that \( g.e \) belongs to each irreducible component of \( g.e + X_p \). Since \( A \) is finite, this implies that, for all \( a \in A, g.e + X_p = a.(g.e + X_p) = A.(g.e + X_p) \) is a smooth irreducible variety. In particular, \( X_p \) is smooth and irreducible, and the condition (2.7) is stronger than ( Disorder).

(2) Using an analogue of Theorem 1.4.9 it is possible to show that the smoothness \( S_G \cap p \) implies the smoothness of \( X_p(S_G, g, \mathcal{F}) \), and that \( K.(g.e + X_p(S_G, g, \mathcal{F})) \) is then also smooth. But the later variety is in general not an irreducible component of \( S_G \cap p \), even for some cases in types AI or AIII.

The last condition we want to consider is:

For each \( J_K \)-class \( J_1 \subset S_G \cap p \) and all \( x, y \in J_1 \), one has: \( G.x = G.y \iff K.x = K.y \). \tag{2.8}

Even if conditions (2.7) and (2.8) will not be used in the sequel of the paper, it is worth noticing that they have interesting consequences on the smoothness of the intersection \( S_G \cap p \), see Proposition 2.5.14.
It can be shown that these conditions hold when $g = \mathfrak{gl}_N$.

We first prove a slight improvement of Lemma 2.5.8 under the conditions (♣) and (2.8).

**Lemma 2.5.13.** Let $J$ be a $J_G$-class contained in $S_G$ such that $J \cap p \neq \emptyset$ and adopt the notation of Lemma 2.5.8(i). Assume that the properties (♣) and (2.8) hold. Then the class $J_2 \subset J$ satisfies $J_2 = K.Y$.

**Proof.** Under the notation of the proof of Lemma 2.5.8(i), choose $x \in Y$ such that $x \in G.J_1$. By Lemma 2.4.3(ii) one gets $G.J_1 = G.J_2$ and $\psi(J_1) = \psi(J_2)$. It follows from $\gamma_p(Y) = \psi(J_2)$ and (2.8) that $J_2 = K.Y$.

We can now summarize in the following result the consequence of the four conditions previously introduced:

**Proposition 2.5.14.** Under the four conditions (♣), (♠), (2.7), (2.8), the variety $S_G \cap p$ is smooth and its irreducible components are of the form $S_K(S_G, O_{g,e})$ with $g \in Z$.

**Proof.** Set $S = S_G$ and recall that $S' = G.(S \cap p) = G.(g.e + X_p(S_G, g.J))$ for all $g \in Z$, cf. (♣). Let $J_1 \subset S \cap p$ be a $J_K$-class. By (♠), $J_1$ is well-behaved w.r.t. $O_{g,e}$ for some $g \in Z$. Then, Lemma 2.5.8(ii) and Lemma 2.5.13 imply $J_1 = K.Y$ with $Y \subset g.e + X_p(S_G, g.J) \subset S \cap p$. It follows that:

$$S \cap p = S' \cap p = \bigcup_{g \in Z} K.(g.e + X_p(S_G, g.J)).$$

(2.9)

Fix $g \in Z$ and set $X_p = X_p(S_G, g.J)$. From (♣) and Lemma 1.4.6(iii) we deduce that $S' \cap (g.e + g^2.f) = A(g.e).(g.e + X_p)$. Since $S' = G.(g.e + X_p)$, Proposition 1.4.9(iii) gives that $S'$ is smooth if and only if $S' \cap (g.e + g^2.f) = A(g.e).(g.e + X_p)$ is smooth. Hence, by (2.7), $S'$ is smooth.

We now compute the tangent space to $S' \cap p$ at a point $g.e + x \in g.e + X_p$. Using Remark 2.5.12(1) one gets that $S' \cap (g.e + g^2.f) = g.e + X_p$ and Proposition 1.4.9, again, yields:

$$T_{g.e + x}S' = [g, x] \oplus T_x X_p = [t, x] \oplus [p, x] \oplus T_x X_p.$$

It follows that $T_{g.e + x}(S' \cap p) \subseteq T_{g.e + x}S' \cap p = [t, x] \oplus T_x X_p$. On the other hand, by (2.9), $T_{g.e + x}(S' \cap p) \supseteq T_{g.e + x}K.(g.e + X_p) \supseteq [t, x] \oplus T_x X_p$. Thus:

$$T_{g.e + x}(S' \cap p) = [t, x] \oplus T_x X_p.$$

Since $X_p$ is smooth, we obtain: $\dim T_{g.e + x}(S' \cap p) = \dim K.x + \dim T_x X_p = m + \dim X_p$. Therefore, each element of $g.e + X_p$ is a smooth point of $S' \cap p = S \cap p$, and (2.9) then implies that $S \cap p$ is smooth.

The last assertion is given by Theorem 2.5.11.

### 3 Type A

We show in this section that the conditions (♣), (♠) and (♠), introduced in Section 2.5 in order to describe the $K$-sheets of a reductive (or semisimple, see Corollary 1.7.2) symmetric Lie algebra $(\mathfrak{g}, \theta)$, are satisfied in type A, i.e. when $\mathfrak{g} = \mathfrak{gl}_N$ (or $\mathfrak{sl}_N$).

Thereafter, unless otherwise specified, e.g. in 3.1.1, we set $\mathfrak{g} = \mathfrak{gl}_N$, $N \in \mathbb{N}^*$, and if $\theta$ is an involution on $\mathfrak{g}$ we adopt the notation of Section 2 relative to the symmetric pair $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{t})$. The natural action of $\tilde{G} = \text{GL}_N$ on $\mathfrak{g}$ factorizes through the adjoint action to give the surjective morphism:

$$\rho : \tilde{G} \longrightarrow G \cong \tilde{G}/k^x \text{Id} = \text{PGL}_N = \text{PSL}_N$$
Recall that $K^\theta = \{ g \in G \mid g \circ \theta = \theta \circ g \}$ and $K = (K^\theta)^o$. If $H$ is an algebraic subgroup of $G$ we set:

$$\hat{H} = \rho^{-1}(H).$$

Thus, $H.x = \hat{H}.x$ for all $x \in \mathfrak{g}$. After recalling the three different possible types of involutions, we will establish the three aforementioned conditions:

- $(\bigtriangledown)$ in Theorem 3.2.1 (types AI, AII) and Proposition 3.2.6 (type AIII);
- $(\diamondsuit)$ in Remark 3.2.3 (types AI, AII) and Remark 3.2.8 (type AIII);
- $(\clubsuit)$ in Corollary 3.3.5 (types AI, AII) and Proposition 3.3.11 (type AIII).

### 3.1 Involutions in type A

We recall below a construction of the involutions on $\mathfrak{gl}_N = \mathfrak{gl}(V)$. We will also have to consider the involution by permutation of factors on $\mathfrak{gl}_N \times \mathfrak{gl}_N$, cf. 2.1; this case will be called “type A0”.

Recall that the nilpotent orbits in $\mathfrak{g}$ are in bijection with the partitions of $N$ and that, to each partition $\mu = (\mu_1 \geq \cdots \geq \mu_k)$, one associates a Young diagram having $\mu_i$ boxes on the $i$-th row. We fix a $G$-sheet $S_G \subset \mathfrak{g}$ and an element $e$ in the nilpotent orbit $O \subset S_G$. The partition associated to $e$ is denoted by

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\delta_O}).$$

We adopt the notation introduced in 1.6.1; in particular, the basis $\mathbf{v}$ (in which $e = \sum e_i$ has a Jordan normal form, see (1.6)) and the subalgebras $\mathfrak{q}_i \cong \mathfrak{gl}_{\lambda_i}$, $\mathfrak{q} = \oplus \mathfrak{q}_i$, $l, t$ are fixed.

We want to construct symmetric pairs $(\mathfrak{g}, \theta) \equiv (\mathfrak{g}, \mathfrak{t}, \mathfrak{p}) \equiv (\mathfrak{g}, \mathfrak{q})$ such that $e \in \mathfrak{p}$. These constructions are inspired from [Ot1, Ot2]. The notation being as in [He1, GW], one obtains three types of non-isomorphic symmetric pairs: AI, AII and AIII. Recall that the involution $\theta$ is outer in types AI, AII and inner in type AIII.

The most complicated case is type AIII, where it is possible to embed $e$ in several non-isomorphic ways in different $\mathfrak{p}$’s. These possibilities will be parameterized by functions $\Phi : [1, \delta_O] \to \{a, b\}$, where $a, b$ are different symbols.

#### 3.1.1 Case A0

Let $\theta$ be the involution on $\mathfrak{g} = \mathfrak{gl}_N \times \mathfrak{gl}_N$ sending $(x, y)$ to $(y, x)$. Recall that $\mathfrak{t} = \{(x, x) \mid x \in \mathfrak{gl}_N\} \cong \mathfrak{gl}_N$, $\mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{gl}_N\}$. The $\mathfrak{t}$-module $\mathfrak{p}$ is isomorphic to the $\text{ad} \mathfrak{gl}_N$-module $\mathfrak{gl}_N$; thus, $G.y \cap \mathfrak{p} = K.y$ for $y = (x, -x) \in \mathfrak{p}$. Suppose that $y = (x, -x)$ is nilpotent, i.e. $x \in \mathfrak{gl}_N$ is nilpotent. The elements $x$ and $-x$ share the same Young diagram $\mu = (\mu_1 \geq \cdots \geq \mu_k)$ and the orbit $K.y$ is uniquely determined by $\mu$.

#### 3.1.2 Case AI

Let $\chi$ be the nondegenerate symmetric bilinear form on $V$ defined, in the basis $\mathbf{v}$, by:

$$\chi(v^{(i)}_j, v^{(l)}_k) = \begin{cases} 1 & \text{if } l = i \text{ and } j + k = \lambda_i + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathfrak{t} = \{x \in \mathfrak{g} \mid \forall u, v \in V, \chi(k.u, v) = -\chi(u, k.v)\} \cong \mathfrak{so}_N, \quad \mathfrak{p} = \{p \in \mathfrak{g} \mid \forall u, v \in V, \chi(p.u, v) = \chi(u, p.v)\}.$$  

The symmetric Lie algebra $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ is of type AI with associated involution $\theta$ on $\mathfrak{g}$ having $\mathfrak{t}$ (resp. $\mathfrak{p}$) as $+1$ (resp. $-1$) eigenspace. In particular $\mathfrak{z}(\mathfrak{g}) = k \text{Id} \subset \mathfrak{p}$.

In this case, each $(\mathfrak{q}_i, \mathfrak{t} \cap \mathfrak{q}_i)$ is a simple symmetric pair of type AI isomorphic to $(\mathfrak{gl}_{\lambda_i}, \mathfrak{so}_{\lambda_i})$. Denote
by $s_k$ the $(k \times k)$-matrix with entries equal to 1 on its antidiagonal and 0 elsewhere, as in [GW, 3.2].

The involution $\theta$ associated to $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ acts on each element $x \in \mathfrak{q}_1$ by $\theta(x) = -s_{\lambda_1}^{-1} xs_{\lambda_1}$ (which is the opposite of the symmetric matrix of $x$ with respect to the antidiagonal).

The group $K^\theta = \rho^{-1}(K^\theta)$, cf. (3.1), is a nonconnected group isomorphic to the orthogonal group $O_N$ and $K^\theta \cong O_N/\{\pm \text{Id}\}$. Fix $\tilde{\omega} \in K^\theta \setminus (K^\theta)^\circ$, then: $K^\theta = (K^\theta)^\circ \cup \tilde{\omega}(K^\theta)^\circ$, $K^\theta = K \cup \omega K$, where $\omega = \rho(\tilde{\omega})$. When $N$ is odd, $\omega \in \mathfrak{K} = K^{\theta} \cong SO_N$ and $K^\theta$ is connected. If $N$ is even, one has $K^\theta = K \cup \omega K$ and $K \cong SO_N/\{\pm \text{Id}\}$.

Let $(\mathfrak{g}, \mathfrak{t}', \mathfrak{p}')$ be another symmetric Lie algebra of type AI, then $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$ and, moreover, for any element $e' \in \mathcal{O} \cap \mathfrak{p}'$ there exists an isomorphism of symmetric Lie algebras $\tau : (\mathfrak{g}, \mathfrak{t}', \mathfrak{p}') \rightarrow (\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ such that $\tau(e') = e$ (see [GW]).

### 3.1.3 Case AII

Assume that $\theta'$ is an involution of type AII on $\mathfrak{g}$ such that $\theta'(e) = -e$; the following condition is then necessarily satisfied:

$$\lambda_{2i+1} = \lambda_{2i+2} \text{ for all } i.$$  

We therefore assume, in this subsection, that the previous condition holds. In particular, $N$ is even and we write $N = 2N'$.

Define a symplectic form $\chi$ on $V$ by setting

$$\chi(v_j^{(i)}, v_k^{(l)}) = \begin{cases} 
1 & \text{if } i + 1 = l \equiv 0 \pmod{2} \text{ and } j + k = \lambda_i + 1; \\
-1 & \text{if } l + 1 = i \equiv 0 \pmod{2} \text{ and } j + k = \lambda_i + 1; \\
0 & \text{otherwise}. 
\end{cases}$$

The subspaces $\mathfrak{t}$ and $\mathfrak{p}$ are then defined, through $\chi$, as in the AI case and, $\theta$ being the associated involution, one has:

$$\mathfrak{t} \cong \mathfrak{sp}_{2N'}, \quad K = K^\theta \cong \tilde{K}^\theta \cong \mathfrak{sp}_{2N'}, \quad \mathfrak{3}(\mathfrak{g}) \subset \mathfrak{p}.$$  

Set $\mathfrak{q}_{2i+1} = \mathfrak{gl}(v_j^{(2i+1)}, v_j^{(2i+2)}) \mid j = 1, \ldots, \lambda_{2i+1}$; then, $(\mathfrak{q}_{2i+1}, \mathfrak{t} \cap \mathfrak{q}_{2i+1})$ is a simple symmetric pair of type AII isomorphic to $(\mathfrak{gl}_{2\lambda_{2i+1}}, \mathfrak{sp}_{2\lambda_{2i+1}})$. We can identify $\mathfrak{q}_{2i+1}$ with $\mathfrak{q}_{2i+2}$ via the isomorphism $u_i : \mathfrak{q}_{2i+1} \cong \mathfrak{q}_{2i+2}$ defined as follows:

$$u_i(x)v_j^{(2i+2)} = x v_j^{(2i+1)} \text{ for all } j \in [1, \lambda_{2i+1}] \text{ and } x \in \mathfrak{q}_{2i+1}.$$  

The involution $\theta$ associated to $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ acts on each element $x \in \mathfrak{q}_{2i+2}$, resp. $x \in \mathfrak{q}_{2i+1}$, by $\theta(x) = -u_i^{-1}(s_{\lambda_{2i+2}}^{-1} xs_{\lambda_{2i+2}})$, resp. $\theta(x) = -u_i(s_{\lambda_{2i+1}}^{-1} xs_{\lambda_{2i+1}})$.

As in case AI, if $(\mathfrak{g}, \mathfrak{t}', \mathfrak{p}')$ is an other symmetric pair of type AII then $\mathcal{O} \cap \mathfrak{p}' \neq \emptyset$ and, for any element $e' \in \mathcal{O} \cap \mathfrak{p}'$, there exists an isomorphism of symmetric pair $\tau : (\mathfrak{g}, \mathfrak{t}', \mathfrak{p}') \rightarrow (\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ with $\tau(e') = e$.

### 3.1.4 Case AIII

Following [Ot1, Ot2] we will use the notion of $ab$-diagram to classify nilpotent orbits in classical reductive symmetric pairs of type AIII, i.e. $(\mathfrak{g}, \mathfrak{t}) = (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$.

**Definition 3.1.1.** An $ab$-diagram is a Young diagram in which each box is labeled by an $a$ or a $b$, in such a way that these two symbols alternate along rows. Two $ab$-diagrams are considered to be equivalent if they differ by permutations of lines of the same length.

Recall that $\mathcal{O} \subset \mathfrak{g}$ is a nilpotent orbit with associated partition $\lambda$. To any function

$$\Phi : [1, \delta_{\mathcal{O}}] \rightarrow \{a, b\}$$

29
one can associate an ab-diagram $\Delta(\Phi)$ of shape $\lambda$ as follows: label the first box of the $i$-th row (of size $\lambda_i$) of $\Delta(\Phi)$ by $\Phi(i)$, and continue the labeling to get an ab-diagram as defined above. Observe that we may have $\Phi \neq \Psi$ and $\Delta(\Phi) = \Delta(\Psi)$.

Fix a such a function $\Phi$ and decompose $V$ in a direct sum $V = V^\Phi_a \oplus V^\Phi_b$ by defining (cf. [Ot2])

\[
V^\Phi_a = \langle \psi^{(i)}_j \mid (\Phi(i) = a \text{ and } \lambda_i - j \equiv 0 \mod 2) \text{ or } (\Phi(i) = b \text{ and } \lambda_i - j \equiv 1 \mod 2) \rangle \\
V^\Phi_b = \langle \psi^{(i)}_j \mid (\Phi(i) = b \text{ and } \lambda_i - j \equiv 0 \mod 2) \text{ or } (\Phi(i) = a \text{ and } \lambda_i - j \equiv 1 \mod 2) \rangle.
\]

Set $N_a = \dim V^\Phi_a$ and $N_b = \dim V^\Phi_b$, hence $N = N_a + N_b$. Now, if

\[
t^\Phi = gl(V^\Phi_a) \oplus gl(V^\Phi_b) \subset g, \quad p^\Phi = Hom(V^\Phi_a, V^\Phi_b) \oplus Hom(V^\Phi_b, V^\Phi_a) \subset g
\]

we obtain a symmetric Lie algebra

\[
(g, \mathfrak{t}, p) = (g, t^\Phi, p^\Phi),
\]

such that $([g, g], \mathfrak{t} \cap [g, g])$ is irreducible of type AIII and $\mathfrak{g}(g) \subset \mathfrak{t}$. One has: $K = \rho(GL(V^\Phi_a) \times GL(V^\Phi_b))$ and, $\theta$ being the associated involution, $K = K^\theta$ if and only if $N_a = N_b$. It is easily seen that $(q_i, t^\Phi \cap q_i)$ is a reductive symmetric pair (of type AIII) isomorphic to $(gl_{\lambda_i}, gl_{\lambda_i} \oplus gl_{\lambda_i})$.

The ab-diagram associated to a nilpotent element $e' \in p^\Phi$ is defined in the following way (see, for example, [Ot2, (1.4)]). Let $\mu = (\mu_1 \geq \cdots \geq \mu_k)$ be the partition associated to $e'$. Fix a normal $sl_2$-triple $(e', h', f')$ and a basis of $V$

\[
\{ e_j^{(i)} \mid i \in [1, a_i], \ j \in [1, \mu_i] \}
\]

such that: $e_j^{(i)}$ belongs either to $V^\Phi_a$ or $V^\Phi_b$, $(\zeta_j^{(i)})_i$ is a basis of ker $f'$ and $e'(\zeta_j^{(i)}) = 0$. Then, label the $j$-th box in the $i$-th row of the Young diagram associated to $\mu$ by $a$, resp. $b$, if $\zeta_j^{(i)} \in V^\Phi_a$, resp. $\zeta_j^{(i)} \in V^\Phi_b$.

This ab-diagram is uniquely determined by $e'$ and will be denoted by $\Gamma^\Phi(e')$. The map $K.x \mapsto \Gamma^\Phi(x)$ gives a parameterization of the nilpotent $K$-orbits in $p^\Phi$, see [Ot2, Proposition 1(2)].

Remark that the element $e_i$, defined in 1.6.1, belongs to $p^\Phi \cap q_i$ in the symmetric Lie algebra $(q_i, t^\Phi \cap q_i)$; its ab-diagram has only one row, with first box labeled with $\Phi(i)$. An ab-diagram of the form $\Gamma^\Phi(x)$ is said to be admissible for $\Phi$. For example, $\Gamma^\Phi(e) = \Delta(\Phi)$ is admissible. It is easy to see that a necessary and sufficient condition for an ab-diagram to be admissible is to have exactly $N_a$ labels equal to $a$ and $N_b$ labels equal to $b$.

The number $N_a - N_b$ is called the parameter of the symmetric pair $(g, t^\Phi)$. Its absolute value $|N_a - N_b|$ can be read from the Satake diagram of the symmetric pair $(g, t^\Phi)$. The parameter is different from 0 when all the white nodes are connected by arrows; then, its absolute value is the number of black nodes plus one, cf. [Hel1, p. 532]. Two symmetric pairs $(g, t)$ of type AIII are isomorphic if and only if their parameters have the same absolute value.

Assume that $(g, t^\Phi, p^\Phi)$ is a symmetric Lie algebra of type AIII such that $O \cap p^\Phi \neq \emptyset$. Then, for every element $e' \in O \cap p$ with ab-diagram $\Gamma'$, there exists a function $\Psi : [1, \delta_O] \rightarrow \{a, b\}$ such that $\Gamma'$ is admissible for $\Psi$. Furthermore, it is not difficult to show that, in this case, there exists an isomorphism of symmetric Lie algebras $\tau : (g, t^\Phi, p^\Phi) \rightarrow (g, t^\Phi, p^\Phi)$ such that $\tau(e') = e$.

3.1.5 Notation and remarks

Let $(g, t, p)$ be a symmetric Lie algebra with $g = gl_N = gl(V)$ and $S_G$ be a G-sheet intersecting $p$. We follow the notation introduced in sections 1.6.1 and 2.5.

Recall that the nilpotent orbit $O \subset S_G$ intersects $p$ and fix $e \in O \cap p$. Then, the symmetric pair $(g, t)$ can be described as in 3.1.2, 3.1.3 or 3.1.4. The notation for $v, q = \bigoplus q_i$, $l, t \subset h \subset l \cap q$, being as in 1.6.1, set:

\[
\xi_i = q_i \cap t, \quad p_i = q_i \cap p, \quad p_i = q_i \cap p, \quad \theta_i = \theta_i|q_i.
\]
The normal $\mathfrak{s}\mathfrak{t}_2$-triple $\mathcal{S} = (e, f, h)$ is then given by $e = \sum_i e_i$, $h = \sum_i h_i$, $f = \sum_i f_i$. The map
\[ \varepsilon = \varepsilon^g : e + h \to e + g^f \]
is defined as in Remark 1.4.5; it is the restriction of the polynomial map $\varepsilon$ from Lemma 1.4.4.

Recall also that the subset $Z \subset G$ is chosen such that: $\text{Id} \in Z$, $\{g.e\}_{g \in Z}$ is a set of representatives of the $K$-orbits contained in $G.e \cap p$ and $g.\mathcal{S} = (g.e, g.h, g.f)$ is a normal $\mathfrak{s}\mathfrak{t}_2$-triple. The “Slodowy slices” are defined by:
\[ g.e + X(S_G, g.\mathcal{S}) = S_G \cap (g.e + g^f), \quad X_p(S_G, g.\mathcal{S}) = X(S_G, g.\mathcal{S}) \cap p. \]

As observed in Remark 1.4.3, we may simplify the notation by setting:
\[ X = X(\mathcal{S}) = X(S_G, \mathcal{S}), \quad X_p = X_p(\mathcal{S}) = X(S_G, \mathcal{S}) \cap p. \]

It follows from the results of Section 1.4 that: $X$ is smooth, $e + X = \varepsilon(e + t)$ is irreducible, $S_G = G.(e + X)$ and $\psi : S_G \to e + X$ is a geometric quotient of the sheet $S_G$, cf. Theorem 1.4.7 (recall that the group $G^e$ is connected).

Since $e, g.e \in p$, the remarks at the end of the previous subsections show that there exists an isomorphism $\tau$ (depending on $g$) of symmetric Lie algebras sending $(g, t, p)$ to a symmetric pair of the same type and $e$ to $g.e$. It is not hard to see that we can further assume that $\tau(\mathcal{S}) = g.\mathcal{S}$. The main consequence of this observation is that, applying $\tau$, any property obtained for $e + X_p(\mathcal{S})$ also holds for $g.e + X_p(g.\mathcal{S})$. In particular, we will mainly work with $e + X_p(\mathcal{S})$.

### 3.2 Properties of slices

We continue with the notation of 3.1.5. Hence $S_G \subset g$ is a $G$-sheet, $e \in S_G \cap p$ is a fixed nilpotent element and $\mathcal{S} = (e, f, h)$, $v, q$, etc., are as defined in 1.6.1.

#### 3.2.1 The slice property (1)

In this subsection we give a proof of Theorem 3.2.1 for types AI and AII. It asserts that the condition $\langle \triangledown \rangle$ holds; we also refer to it as the slice property.

**Theorem 3.2.1.** Assume that $(g, \theta)$ is of type A. Then, one has:
\[ G.(e + X_p) = G.(S_G \cap p). \tag{\triangledown} \]
Moreover, in types AI and AII a stronger version holds, namely: $e + X \subset p$.

**Proof.** Since $S_G = G.(e + X)$ and $e + X_p = (e + X) \cap p \subset S_G \cap p$, the inclusion $G.(e + X_p) \subset G.(S_G \cap p)$ is obvious. Clearly, $e + X \subset p$ yields $G.(e + X_p) = G.(e + X) = S_G \supset G.(S_G \cap p)$. We prove below that the inclusion $e + X \subset p$ is true when $(g, t)$ is of type AI or AII. The proof of the theorem in type AIII is postponed to subsection 3.2.2, see Proposition 3.2.6.

Type AI: As said in subsection 3.1.2, each $(q_i, t_i)$ is a symmetric pair of type AI. Since this pair has maximal rank and $e_i \in p_i$ is a regular element, one has $q_i^{t_i} = p \cap q_i^{f_i}$. Therefore the image of each map $\varepsilon_i : e_i + h_i \to e_i + q_i^{f_i}$, as defined in 1.6.1, is contained in $q_i^{f_i} \subset p_i$. From $\varepsilon = \sum_i \varepsilon_i$ one gets that $e + X = \varepsilon(e + t) \subset S_G \cap p$.

Type AII: Recall that $\lambda_{2i+1} = \lambda_{2i+2}$ if $2i + 2 \leq \delta_G$. Let $x = \sum_i x_i \in q$; then $x \in p \cap q$ if and only if, for all $i$, $x_{2i+1} = -\theta_{2i+2}(x_{2i+2})$, which is the symmetric of $u^{-1}_i(x_{2i+2})$ with respect to the antidiagonal (cf. §3.1.3). Fix $t \in t$, hence $e + t \subset S_G$; from the description of $t$ given in (1.7), one
deduces that \( u_i(e_{2i+1} + t_{2i+1}) = e_{2i+2} + t_{2i+2} \). Set \( x = \varepsilon(e + t) \). It follows from \( u_i \circ \varepsilon_{2i+1} = \varepsilon_{2i+2} \circ u_i \) that \( u_i(x_{2i+1}) = x_{2i+2} \). Since \( e_{2i+1} + q_{2i+1} \) is fixed under the conjugation by \( s_{\lambda_{2i+1}} \), one obtains \(-\theta_{2i+2}(x_{2i+2}) = s_{\lambda_{2i+1}}^{-1} x_{2i+1} s_{\lambda_{2i+1}} = x_{2i+1} \). Hence \( \varepsilon(e + t) \in p \) and, therefore, \( \varepsilon(e + t) = e + X \subset p \). □

**Corollary 3.2.2.** Every \( G \)-orbit contained in \( S_G \) and intersecting \( p \), also intersects \( (q \cap p)^* \).

**Proof.** It suffices to observe that \( e + X \subset q^* \) and \( (q \cap p)^* \subset q^* \). □

**Remark 3.2.3.** (1) One can deduce Theorem 3.2.1 from Corollary 3.2.2. Indeed, let \( x \in S_G \) and suppose that \( y \in G.x \cap (q \cap p)^* \). Since \( e \) is regular in \( q \), it follows from [KR] that \( y \) is \((Q \cap K)^c\)-conjugate to an element of \( e + X_p \).

(2) Assume that \((g, t)\) is of type AI or AII. Then, since \( e + X_p \) is irreducible and smooth in type A (see §3.1.5), Theorem 3.2.1 yields that the conditions \((\heartsuit)\) and \((\clubsuit)\), cf. §2.5, hold.

### 3.2.2 The slice property (2)

We assume in this section that \((g, t, p) = (g, t^\Phi; p^\Phi)\) is of type AIII. Let \( a \subset p \) be Cartan subspace and \( b' \subset g \) be a Cartan subalgebra containing \( a \). Denote by \( B \) a \( \sigma \)-fundamental system of the root system \( R(g, b') = R([g, g], b' \cap [g, g]) \), see §2.2 with \( h \) replaced by \( h' \cap [g, g] \). Let \( D \) be the Satake diagram of type AIII associated to \( B \) (cf. [He1, p. 532]). Since \( a \subset [g, g] \), see 3.1.4, one can define a \( \mathbb{Q} \)-form of \( a \) by

\[
a_Q = \{ a \in a \mid \alpha(a) \in \mathbb{Q} \text{ for all } \alpha \in R(g, b') \}.
\]

The nodes of \( \hat{D} \) can be labeled by the elements \( \alpha_1, \ldots, \alpha_{N-1} \) of \( B \). Set \( \alpha'_i = \alpha_{N-i}, 1 \leq i \leq N-1 \), hence \( \alpha'_i = \alpha'_i \); there exists an arrow between \( \alpha_i \) and \( \alpha'_i \) when these nodes are colored in white and \( i \neq N/2 \).

Let \( s \in g \) be semisimple and let \( c \in \text{sp}(s) \) be an eigenvalue of \( s \) on \( V \). Denote by \( V_{s,c} \) the eigenspace associated to \( c \); thus, \( m(s, c) = \dim V_{s,c} \) is the multiplicity of \( c \). More generally, see §1.6.2, we set \( V_{s,d} = \ker(s - d \text{Id}_V) \) and \( m(s, d) = \dim V_{s,d} \) for every \( d \in k \). One can identify \( \text{gl}(V_{s,c}) \) with a Lie subalgebra of \( \text{gl}(V) \) by extending an element \( x \in \text{gl}(V_{s,c}) \) by \( 0 \) on \( \bigoplus_{c' \neq c} V_{s,c'} \). Under this identification, \( \text{sl}(V_{s,c}) \) is a simple factor of \( l \) if and only if \( m(s, c) \geq 2 \). Setting

\[
w_{s,c}' = \text{sl}(V_{s,c}), \quad w_{s,c} = \text{gl}(V_{s,c}),
\]

one has:

\[
g^s = \bigoplus_{c \in \text{sp}(s)} w_{s,c} = c_\Phi(g^s) \oplus \bigoplus_{m(s, c) \geq 2} w_{s,c}'.
\]

Denote by \( M_{s,c} \) the connected algebraic subgroup of \( G \) with Lie algebra \( w_{s,c}' \). Then, \( M_{s,c} \) acts on \( w_{s,c} \) via the adjoint action and the group \( G^s \) is generated by \( C_G(g^s) \) and the \( M_{s,c}, c \in \text{sp}(s) \) (see §1.2 and Proposition 1.2.3).

The group \( \{ \pm 1 \} \) acts by multiplication on \( \text{sp}'(s) = \{ c \in \text{sp}(s) \mid -c \in \text{sp}(s) \} \); let \( \text{sp}'_\pm(s) = \text{sp}'(s)/\{ \pm 1 \} \) be the orbit space. The class of \( c \in \text{sp}'(s) \) in \( \text{sp}'_\pm(s) \) is denoted by \( \pm c \). When \( 0 \in \text{sp}(s) \) we simply write \( \pm 0 = 0 \). We then set

\[
g_{s,\pm} = w_{s,c} \oplus w_{s,-c}, \quad g_{s,0} = w_{s,0}.
\]

If \( 0 \neq c \in \text{sp}'(s) \), the connected subgroup of \( G \) generated by \( M_{s,c} \) and \( M_{s,-c} \) is denoted by \( G_{s,\pm c} \) and we set \( G_{s,0} = M_{s,0} \). One has \( \text{Lie}(G_{s,\pm c}) = [g_{s,\pm c}, g_{s,\pm c}] \).

Recall that we have written \( V = V_\alpha^\phi \oplus V_b^\phi \); we set \( V_a = V_\alpha^\phi, V_b = V_b^\phi \). The parameter of \((g, t)\) is \( N_a - N_b \) where \( N_a = \dim V_a, N_b = \dim V_b \), see 3.1.4.
Lemma 3.2.4. Let \( s \in p \) be a semisimple element. Then:

(1) \( m(s, c) = m(s, -c) \) for all \( c \in k; \)

(2) the symmetric Lie algebra \((g^s, \mathfrak{t}^s)\) decomposes as \( \bigoplus_{s \in \mathfrak{sp}^+(s)} (g_{s,\pm c}, \mathfrak{t}_{s,\pm c}) \), where \( \mathfrak{t}_{s,\pm c} = \mathfrak{t} \cap g_{s,\pm c}; \)

(3) if \( c \neq 0 \), \((g_{s,\pm c}, \mathfrak{t}_{s,\pm c})\) is a reductive symmetric pair whose semisimple part is irreducible of type \( A0; \)

(4) \( V_{s,0} = (V_{s,0} \cap V_a) \oplus (V_{s,0} \cap V_b) \) and the symmetric Lie algebra \((g_{s,0}, \mathfrak{t}_{s,0})\) is a reductive symmetric pair whose semisimple part is irreducible of type \( AIII \), with the same parameter as \((g, \mathfrak{t})\). In particular, the parameter of \((g, \mathfrak{t})\) is 0 when \( 0 \notin \mathfrak{sp}(s) \).

Proof. (1) Since the involution \( \theta \) is inner, the claim follows from the following elementary observation. Suppose that \( A \in GL_N, x \in gl_N \), and set \( x' = AxA^{-1} \). Then, \( m(x, c) = \dim \ker(x - c \text{Id}) = \dim \ker(x' - c \text{Id}); \) in particular, \( m(x, c) = m(x', c) = m(x, -c) \) when \( x' = -x \).

(2) The assertion is an easy consequence of (3.2) and \( \theta(\mathfrak{m}_{s,c}) = \mathfrak{w}_{s,-c}. \)

(3) & (4). We may assume that \( N_a \geq N_b \) and, by Proposition 2.3.5, \( s \in a_Q \). Then, the claims can be read on the Satake diagram of type \( AIII \), except for the equality of the parameters when \( c = 0 \) (one only sees in this way that the absolute values are equal). A complete proof can be given in this way. Let \( \{v_a, v_b\} \) be a basis of \( V_a \) and \( V_b \). For each \( i \in [1, N_b], \) define \( u_i \in p \) by

\[
u_i(v_{d,i}) = \begin{cases} v_{d,i} & \text{if } i = j, \\ 0 & \text{otherwise,}
\end{cases}
\]

where \( d \) is the element of \( \{a, b\} \setminus \{d\} \). The subspace generated by the \( u_i, \) \( i \in [1, N_b], \) is a Cartan subspace of \( p. \) If \( s = \sum_i c_i u_i, \) the eigenvalues of \( s \) are given by square roots of the \( c_i's \) and one has \( V_{s,0} = \langle \{v_{a,i}, v_{b,i} \mid c_i = 0\} \cup \{v_{a,i} \mid i > N_b\} \rangle. \) It is then not difficult to get the desired assertions. \( \Box \)

Recall from §1.6.2 that if \( t = \sum_i t_i \in q = \bigoplus_i q_i \) is semisimple, \( m_i(t, c) \) denotes the multiplicity of the eigenvalue \( c \) for \( t_i \in q_i; \) recall also that \( h \subset q. \)

Lemma 3.2.5. Let \( t \in h \) be such that \( G.(e + t) \cap p \neq \emptyset. \) Then:

\[ m_i(t, c) = m_i(t, -c) \quad \text{for all } c \in k. \quad (3.3) \]

Proof. Let \( s_1 + n_1 \) be the Jordan decomposition of \( e + t \) and pick \( g \in G \) such that \( g.(e + t) \in p. \) Therefore, \( s = g.s_1 \in p \) and \( n = g.n_1 \in p \cap g^s. \) By Corollary 1.6.4 we know that \( t, s_1 \) and \( s \) are in the same \( G \)-orbit. Then, Lemma 3.2.4(1) gives \( m(t, c) = m(s, c) = m(s, -c) = m(t, -c). \) On the other hand, \( n \in p \cap g^s \) is a nilpotent element of the subsymmetric pair \((g^s, g^s \cap \mathfrak{t}) = \bigoplus_{s \in \mathfrak{sp}^+(s)} (g_{s,\pm c}, \mathfrak{t}_{s,\pm c}), \) cf. Lemma 3.2.4(3,4).

With obvious notation, one can decompose the orbit \( K.n \) of this direct product as follows:

\[ K.n = \prod_{s \in \mathfrak{sp}^+(s)} O_{s,c} \]

The result in the case \( c = 0 \) is clear. Recall that when \( c \neq 0 \) one has \( g_{s,\pm c} = w_{s,c} \oplus w_{s,-c}, \) and we can further decompose each orbit \( G_{s,\pm c}: O_{s,c} \) as \( O_c \times O_{-c} \subset w_{s,c} \times w_{s,-c} \). Then, \( G_{s,\pm c}: O_{s,c} \), is characterized by the Young diagrams of the nilpotent orbits \( O_c, O_{-c}. \) Since \((g_{s,\pm c}, \mathfrak{t}_{s,\pm c})\) is of type \( A0, \) these two Young diagrams are equal (cf. §3.1.1). The results of §1.6.4 then yield that the partition of \( O_{s,c}, \delta \in \{-1,1\}, \) is given by the sequence \( (m_i(t, \delta c))_i. \) As these two sequences are decreasing on \( i, \) cf. (1.7), one obtains \( m_i(t, c) = m_i(t, -c) \) for all \( i. \) \( \Box \)

The following proposition completes the proof of Theorem 3.2.1 and Corollary 3.2.2 in case AIII.

Proposition 3.2.6. Let \( e + t \in e + t. \)

(i) If \( t \) satisfies (3.3), then \( e.(e + t) \in e + X_p. \)

(ii) One has \( G.(e + t) \cap p \neq \emptyset \) if and only if \( t \) satisfies (3.3).

(iii) The condition \((\bigwedge)\) holds, i.e. \( G.(S_G \cap p) = G.(e + X_p). \)
Proof. (i) Recall that \( t = \sum_i t_i, \quad e = \sum_i e_i \) with \( t_i \in \mathfrak{q}_i \) and \( e_i \in \mathfrak{p} \cap \mathfrak{q}_i \) regular in \( \mathfrak{q}_i \cong \mathfrak{g}(\lambda_i) \). The map \( \varepsilon \) writes \( \sum_i \varepsilon_i \), where \( \varepsilon_i \) is given by Lemma 1.6.2 applied in the algebra \( \mathfrak{q}_i \). Thus \( \varepsilon_i(e_i + t_i) = e_i + \sum_{j \leq 0} P_j(t_i) \). From (3.3) and the symmetry of the polynomials \( P_j \) one obtains \( P_j(t_i) = 0 \) if \( j \) is even. One can deduce from the construction made in 3.1.4 that the subspaces \( \mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{q}_i \) are the sum of the \( j \)-subdiagonals and \( j \)-supdiagonals of \( \mathfrak{q}_i \) for \( j \) odd. It follows that \( \varepsilon_i(e_i + t_i) \in e_i + \mathfrak{p}_i^L \), hence \( \varepsilon(e+t) \in e + X \cap \mathfrak{p} \).

(ii) By Lemma 1.4.4 one has \( G.(e+t) = G.e(e+t) \), thus part (i) shows that the condition is sufficient. Lemma 3.2.5 gives the converse.

(iii) The inclusion \( G.(e+X_p) \subset G.(S_G \cap \mathfrak{p}) \) is always true. By Proposition 1.4.2, every \( x \in S_G \cap \mathfrak{p} \) is \( G \)-conjugate to an element \( e + t \in e + t \); parts (i) and (ii) give \( \varepsilon(e+t) \in G.x \cap (e + X_p) \) and the result follows. \( \square \)

We now find a convenient subspace \( \mathfrak{c} \subset \mathfrak{t} \) such that \( \varepsilon(e+\mathfrak{c}) = e + X_p \). For \( i \in [1, \lambda_{\delta_0}] \) and \( j \in [0, [(\lambda_i - \lambda_{i+1})/2] - 1] \), define elements \( c(i,j) = (c(i,j)_k)_{k} \in k^\lambda_i \) by:

\[
c(i,j)_k = \begin{cases} 
1 & \text{if } k = \lambda_i + 2j + 1; \\
-1 & \text{if } k = \lambda_i + 2j + 2; \\
0 & \text{otherwise.}
\end{cases}
\] (3.4)

Let \( \mathfrak{c}' \) be the subspace of \( k^{\lambda_i} \) generated by the elements \( c(i,j) \). Recall from (1.8) the isomorphism \( \alpha : k^{\lambda_i} \cong \mathfrak{t} \) and set:

\[
\mathfrak{c} = \alpha(\mathfrak{c}') \subset \mathfrak{t}.
\] (3.5)

The main property of the subspace \( \mathfrak{c} \) is the following. By construction every element of \( \mathfrak{c} \) satisfies (3.3); conversely, Lemma 1.5.1 applied in each \( \mathfrak{q}_i \) implies that any element \( e + t \) (with \( e = \sum_i e_i, \quad t = \sum_i t_i \)) satisfying (3.3) is conjugate to an element of \( \mathfrak{c} \).

**Proposition 3.2.7.** Under the previous notation one has: \( \varepsilon(e+\mathfrak{c}) = e + X_p \) and \( G.(e+\mathfrak{c}) = G.(S_G \cap \mathfrak{p}) \). Moreover,

\[
\dim \mathfrak{c} = \sum_{i=1}^{\delta_{\mathfrak{g}}} \left\lceil \frac{\lambda_i - \lambda_{i+1}}{2} \right\rceil.
\] (3.6)

which only depends \( \lambda \).

**Proof.** The formula (3.6) follows without difficulty from the definition of \( \mathfrak{c}' \). Since the elements of \( e + \mathfrak{c} \) satisfy (3.3), Proposition 3.2.6(ii) gives \( \varepsilon(e+\mathfrak{c}) \subset e + X_p \). Conversely, let \( e + x \in e + X_p \). As \( e + X = \varepsilon(e+t) \), the element \( e + x = \varepsilon(e+t) \), \( t \in \mathfrak{t} \), is the unique point of \( e + X \) intersecting the orbit \( G.(e+x) = G.\varepsilon(e+t) = G.(e+t) \) (see Lemma 1.4.4(i)). By Proposition 3.2.6(ii), \( e + t \) satisfies (3.3) and, as noticed above, \( e + t \) is conjugate to an element \( e + c \in e + \mathfrak{c} \subset e + \mathfrak{t} \). It follows that \( \{ e + x \} = G.(e+x) \cap (e + X) = G.\varepsilon(e+c) \cap (e + X) = \{ \varepsilon(e+c) \} \). Hence, \( e + x = \varepsilon(e+c) \in \varepsilon(e+\mathfrak{c}) \).

Finally, \( G.(S_G \cap \mathfrak{p}) = G.(e+X_p) = G.\varepsilon(e+\mathfrak{c}) = G.(e+\mathfrak{c}) \). \( \square \)

**Remark 3.2.8.** Proposition 3.2.7 implies that condition (\( \diamond \)) holds in case AIII, i.e., \( e + X_p \) is irreducible. Actually, using similar arguments to [IH, Chapter 5] it is possible to show that \( e + X_p \) is isomorphic to the quotient of \( \mathfrak{c} \) by a reflection group of type B, hence is smooth. In particular, since \( G^c \) is connected, the stronger condition (2.7) holds.

Corollary 3.2.2 says that in each \( G \)-orbit contained in \( S_G \) and intersecting \( \mathfrak{p} \) one can find an element \( x = s + n \in (\mathfrak{q} \cap \mathfrak{p})^* \). The next corollary summarizes various results which can be deduced from Lemma 3.2.4. Recall that \( \mathfrak{q} = \bigoplus \mathfrak{q}_i \) and that \( (\mathfrak{q}_i, \mathfrak{t} \cap \mathfrak{q}_i) \) is a symmetric Lie algebra of type AIII. Applying Lemma 3.2.4 in each symmetric pair \( (\mathfrak{q}_i, \mathfrak{t} \cap \mathfrak{q}_i) \) yields:
Corollary 3.2.9. Let \( x = s + n \in (q \cap p)^* \) and write \( s = \sum_i s_i, n = \sum_i n_i \) with \( s_i, n_i \in p \cap q_i \), as in 1.6.1.

1. The Levi factor \( q_i^n \) of \( q_i \) has the following decomposition:

\[
q_i^n = \bigoplus_{c \in k} w_{i,s,c}
\]

where \( w_{i,s,c} \) is identified with \( \mathfrak{g}((\ker(s_i - c \text{Id})) \subset q_i \).

2. The symmetric pair \( (q_i^n, q_i^n \cap \mathfrak{t}) \) decomposes as

\[
(q_i^n, q_i^n \cap \mathfrak{t}) = \bigoplus_{\pm \in \mathfrak{sp}_r(s_i)} (q_{i,s,\pm c}, \mathfrak{t}_{i,s,\pm c})
\]

where \( (q_{i,s,0}, \mathfrak{t}_{i,s,0}) = (w_{i,s,0}, w_{i,s,0} \cap \mathfrak{t}) \) is of type AIII and, when \( c \neq 0 \), \( (q_{i,s,\pm c}, \mathfrak{t}_{i,s,\pm c}) = (w_{i,s,c} + w_{i,s,-c}, (w_{i,s,c} + w_{i,s,-c}) \cap \mathfrak{t}) \) is of type A0.

3. The factor \( (q_{i,s,0}, \mathfrak{t}_{i,s,0}) \) has the same parameter as \( (q_i, q_i \cap \mathfrak{t}) \). In particular, the ranks of \( q_i \) and \( q_{i,s,0} \) have the same parity.

4. The nilpotent element \( n_i \) is regular in \( q_i^n \); thus, the orbit \( (Q \cap K)^o.n_i \) is uniquely determined by its one row ab-diagram (see 3.1.4).

### 3.3 \( J_K \)-classes in type A

Knowing that \( (\triangledown) \) holds, we want to prove below that condition \( (\clubsuit) \), introduced in §2.5, is satisfied. As above, \( S_G \subset \mathfrak{g}^{(2m)} \) is a \( G \)-sheet and \( c \in S_G \) is a nilpotent element. We fix a Jordan \( G \)-class \( J \subset S_G \) such that \( J \cap p = \emptyset \). Recall from Theorem 2.4.4 that \( J \cap p \) is a (disjoint) union of \( J_K \)-classes.

#### 3.3.1 Cases AI and AII

In this subsection we assume that \( (\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{t}, p) \) is a symmetric Lie algebra of type AI or AII, as described in 3.1.2 and 3.1.3.

We will need the following result, which is a formulation of [Ot1, Proposition 4] in a slightly more general setting. (Its proof is exactly the same.)

**Proposition 3.3.1** (Otha). Let \( \kappa \) be a linear involution of the associative algebra \( \mathfrak{g} = \mathfrak{gl}_N \) and \( x \mapsto x^* \) be a linear anti-involution of the associative algebra \( \mathfrak{g} \) which commutes with \( \kappa \). Define:

\[
G' = \mathfrak{g}^\kappa \cap \text{GL}_N, \quad G'' = \{ g \in G' : g^* = g^{-1} \}.
\]

Set \( \sigma(x) = -x^* \) and let \( \eta, \eta' \) be elements of \( \{ \pm 1 \} \). Then, via the adjoint action, \( G' \) acts on \( \mathfrak{g}^{\eta'} \) and \( G'' \) acts on \( \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta'}. \) The elements \( x, y \in \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\eta'}. \) are conjugate under \( G'' \) if and only if they are conjugate under \( G' \).

We may apply this proposition in the two following situations. Fixing \( \eta = -1, \eta' = 1 \), we take:

\( (\kappa = \text{Id}, x^* = t \, x) \) in type AI, \( (\kappa = \text{Id}, x^* = -J^t \, x \, J) \) in type AII, where \( J = \begin{bmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{bmatrix} \in \mathfrak{gl}_2N. \) Observe that \( \mathfrak{g}' = \mathfrak{g} = \mathfrak{gl}_N, \quad G'' = \text{O}_N, \text{ resp. } G'' = \text{Sp}_N, \) and that the action of \( G' = \text{GL}_N = \tilde{G} \) factorizes through \( G \cong \tilde{G}/(\text{K^x \text{Id})}. \) Then, \( \sigma \) is an involution of the Lie algebra \( \mathfrak{g} \) of type AI, resp. AII (cf. [GW, Theorem 3.4]). Using an isomorphism \( \tau \) as explained in 3.1, we may assume that \( \mathfrak{t} = \tau(\mathfrak{g}^\sigma) \) and \( p = \tau(\mathfrak{g}^{-\sigma}) \). Moreover, in each case \( \rho(\tau(G'')) = K^\theta \) (cf. 3.1.2 and 3.1.3).

We therefore have obtained the (well known) result:

**Proposition 3.3.2.** Let \( (\mathfrak{g}, \theta) \) be of type AI or AII. If \( x, y \in p \) one has the equivalence:

\[
K^\theta. x = K^\theta. y \iff G.x = G.y
\]
Corollary 3.3.3. If \((g, \theta)\) is of type AI or AII, the \(J_K\)-classes contained in \(J \cap p\) are conjugate under \(K^\theta\).

Proof. Let \(J_1 = K.(\epsilon_p(p^*)^\bullet + n)\) be the Jordan \(K\)-class containing \(x = s + n \in J \cap p\) and denote by \(J_2 = K.(\epsilon_p(p^*)^\bullet + n')\) another Jordan \(K\)-class contained in \(J \cap p\) (cf. 2.4.2(ii)). Since \(J = G.\epsilon_p(g^*)^\bullet + n'\), there exists \(g \in G\) such that \(g.x \in \epsilon_p(g^*)^\bullet + n'\) and Lemma 2.3.7(ii) implies that \(g.x \in J_2\). Now, by Proposition 3.3.2, we may assume that \(g \in K^\theta\). Then, \(g.J_1\) is an irreducible subvariety of \(J \cap p\) of dimension \(\dim J \cap p\) (see Lemma 2.4.2(ii)) which intersects \(J_2\). It follows from Theorem 2.4.4 that \(g.J_1 = J_2\).

\(\square\)

Remark 3.3.4. As \(K^\theta = K \cup \omega K\) in type AI (cf. 3.1.2), there are at most two Jordan \(K\)-classes in \(J \cap p\). In type AII one has \(K^\theta = K\) and \(J \cap p\) is a Jordan \(K\)-class.

Corollary 3.3.5. The condition (\(\clubsuit\)) of section 2.5 is satisfied.

Proof. Let \(J_1 \subset J \cap p\) be a \(J_K\)-class. By Lemma 2.5.8 there exists a \(J_K\)-class \(J_2 \subset J \cap p\) such that \(J_2\) is well-behaved w.r.t. \(K.e\), and Corollary 3.3.3 gives \(k \in K^\theta\) such that \(J_1 = k.J_2\). Since \(k\) defines an automorphism of the symmetric Lie algebra \((g, t, p)\), the class \(J_1 = k.J_2\) is well-behaved w.r.t. \(K(k.e) = k.(K.e)\).

\(\square\)

3.3.2 Case AIII (1)

We fix \((g, \theta) = (g, t, p) = (g, t^\theta, p^b)\) of type AIII as in section 3.1.4 and we use the notation introduced in 3.2.2. For simplicity we assume that the numbers \(N_a, N_b\) are such that \(N_b \leq N_a\).

Let \(a \subset p\) be a Cartan subspace. Since the involutions of type AIII are conjugate, and the Cartan subspaces are \(K\)-conjugate, one can find a Cartan subalgebra \(h'\) containing \(a\) and satisfying the following conditions (see, for example, [GW, Polarizations-Type AIII, p. 20]). There exists a basis \((w_1, \ldots, w_N)\) of \(h'^*\) such that: \(w_j(t), 1 \leq j \leq N,\) are the eigenvalues of \(t \in h'\) and \(B = \{\omega_j = w_j - w_{j+1} | 1 \leq j \leq N - 1\}\) is a \(\sigma\)-fundamental system of the root system \(R = R(g, h')\). Recall that the Weyl group \(W(g, h') = N_G(h')/Z_G(h')\) can be naturally identified with the group \(\mathfrak{S}(\{w_1, \ldots, w_N\}) \cong \mathfrak{S}_N = \mathfrak{S}([1, N])\), where we denote by \(\mathfrak{S}(E)\) the permutation group of a set \(E\). Moreover, the action of \(\theta\) on \(h'\) is defined by:

\[
\varpi_i(\theta(t)) = \begin{cases} 
\varpi_{N+1-i}(t) & \text{if } \min(i, N+1-i) \leq N_b; \\
\varpi_i(t) & \text{otherwise.}
\end{cases}
\] (3.7)

Fix the semisimple part \(s\) of an element belonging to \(J \cap p\). By Lemma 2.4.2, \(J \cap p\) is the union of \(J_K\)-classes of the form \(K.(\epsilon_p(p^*)^\bullet + n)\) where \(n \in p^*\) is nilpotent. Thanks to Proposition 2.3.5 we may assume that \(s \in a_Q\) is in the positive Weyl chamber defined by \(B\). Recall from (3.2) that we write

\[
g^s = \bigoplus_{c \in \sp(s)} w_{s,c}, \quad w_{s,c} = \gl(V_{s,c}),
\]

where \(\gl(V_{s,c})\) is naturally embedded in \(g = \gl(V)\). Note that \(\epsilon_p(g^*) = \bigoplus_{c \in \sp(s)} \kappa \Id_{V_{s,c}}\). Let \(g \in N_G(g^s)\); then \(s' = g.s \in \epsilon_p(g^s)\), hence \(s'_{V_{s,c}} = c' \Id_{V_{s,c}}\) for some \(c' \in \sp(s)\), that is to say \(V_{s,c} \subset V'_{s',c'}\). It is then easily seen that the map \(\eta : c \mapsto c'\) defines a permutation of \(\sp(s)\) such that \(V_{s,c} = V'_{s',c'}\). If \(\tau(g) := \eta^{-1} \in \mathfrak{S}(\sp(s))\) one has \(V'_{s',c} = g.V_{s,c} = V_{s,r(g)(c)}\) and it follows that:

\[
g.w_{s,c} = w_{s,r(g)(c)} \quad \text{for all } c \in \sp(s).
\]

From this observation one deduces a group homomorphism

\[
r : N_G(\epsilon_p(g^s)) = N_G(g^s) \longrightarrow \mathfrak{S}(\sp(s)), \quad g \mapsto \tau(g).
\]
that we denote by $\gamma$. (3.8)\[

This condition characterizes the elements of the image of $r$:

**Lemma 3.3.6.** An element $\gamma \in \mathfrak{S}(\mathfrak{sp}(s))$ is in the image of the morphism $r$ if and only if it satisfies (3.8).

**Proof.** Let $c_1, \ldots, c_\ell$ be the distinct eigenvalues of $s$. By construction, $\gamma$ can be identified with the element $\gamma \in \mathfrak{S}_\ell$ such that $\gamma(c_i) = c_{\gamma(i)}$, $1 \leq i \leq \ell$. Write $[1, N]$ as a disjoint union $\bigcup_{j=1}^{\ell} J_j$, where $J_j = \{k : \omega_k(s) = c_j\}$. By (3.8) one has $\#J_j = m(s, c_j) = m(s, \gamma(c_j))$. One can therefore find $w \in \mathfrak{S}_N \cong W(\mathfrak{g}, \mathfrak{h}')$ such that $w(J_j) = J_{\gamma(j)}$ for $j = 1, \ldots, \ell$. Let $g \in N_G(\mathfrak{h}')$ be a representative of $w$. One then gets $g \in N_G(\mathfrak{g}^*) = N_G(\mathfrak{h}^*)$ and $\gamma = r(g)$.

Recall from §3.2.2 that we denote by $\mathfrak{sp}_\pm(s)$ the set of classes $\{\pm c : c \in \mathfrak{sp}(s)\}$. For every $k \in N_K(\mathfrak{g}^*)$ we have

$\mathfrak{w}_{s,k}(c) = k \mathfrak{w}_{s,-c} = k \theta(\mathfrak{w}_{s,c}) = \theta(k \mathfrak{w}_{s,c}) = \mathfrak{w}_{s,-r(k)(c)}$.

Thus $r(k)(-c) = -r(k)(c)$ and, since $\mathfrak{g}_{s,\pm c} = \mathfrak{w}_{s,c} \oplus \mathfrak{w}_{s,-c}$, one gets $k \mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm r(k)(c)}$. Therefore, any element of $r(N_K(\mathfrak{g}^*))$ induces a permutation of $\mathfrak{sp}_\pm(s)$. By Lemma 3.2.4, if $0 \in \mathfrak{sp}(s)$, the factor $(\mathfrak{g}_{s,0}, \mathfrak{t}_{s,0})$ is the unique factor of type AIII in the decomposition of the symmetric Lie algebra $(\mathfrak{g}^*, \mathfrak{t}^*)$ and, as $k \in N_K(\mathfrak{g}^*)$ defines an automorphism of this symmetric pair, one necessarily has $r(k)(0) = 0$. It follows that $r$ induces a homomorphism:

$r' : N_K(\mathfrak{r}(\mathfrak{p}^*)) \rightarrow \mathfrak{S}(\mathfrak{sp}_\pm(s) \setminus \{0\}), \ k \mapsto r'(k)$,

with the convention that $\mathfrak{sp}_\pm(s) \setminus \{0\} = \mathfrak{sp}_\mp(s)$ when $0 \notin \mathfrak{sp}(s)$.

**Lemma 3.3.7.** (1) Let $c_0, c_1 \in \mathfrak{sp}(s) \setminus \{0\}$ be such that $m(s, c_0) = m(s, c_1)$. There exists $k \in N_K(\mathfrak{r}(\mathfrak{p}^*))$ such that: $r'(k)(\pm c_i) = r'(k)(\pm c_{1-i})$, for $i = 0, 1$, and $r'(k)(\pm c) = \pm c$ for all $\pm c \in \mathfrak{sp}_\pm(s) \setminus \{\pm c_0, \pm c_1\}$.

(2) A permutation $\gamma$ of $\mathfrak{sp}_\pm(s) \setminus \{0\}$ belongs to $r'(N_K(\mathfrak{g}^*))$ if and only if

$m(s, \pm c) = m(s, \gamma(\pm c))$ for all $\pm c \in \mathfrak{sp}_\pm(s) \setminus \{0\}$.

In particular, for such a permutation $\gamma$ there exists $k \in N_K(\mathfrak{g}^*)$ such that

$k \mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\gamma(\pm c)}$

where $\gamma$ is, if necessary, extended to $\mathfrak{sp}_\mp(s)$ by $\gamma(0) = 0$.

**Proof.** (1) Recall that $s \in \mathfrak{a}_Q$ is in the positive Weyl chamber defined by $B$. Therefore, for $i = 0, 1$, $I_i = \{j \mid c_i = \varpi_j(s) \in [1, N]\}$ is an interval; set $I_i = [d_i, d_i^2]$. In the case $c_0 = c_1$ the element $k = Id$ obviously works. Otherwise, we may replace $c_i$ by $-c_i$ to ensure that $d_i^2 \leq N_b \leq N/2$ and we define a permutation $\gamma \in \mathfrak{S}_N$ by:

$\gamma(j) = \begin{cases} j - d_i^2 + d_{1-i} & \text{if } j \in I_i; \\ j & \text{if } j \leq (N+1)/2 \text{ and } j \notin I_1 \cup I_2; \\ N + 1 - \gamma(N + 1 - j) & \text{if } j > (N+1)/2. \end{cases}$

One has: $\varpi_j(s) = \pm c_{1-i}$ if $\varpi_j(s) = \pm c_i$, $i = 0, 1$ and $\pm \varpi_{\gamma(j)}(s) = \pm \varpi_j(s)$ otherwise. Denote by $w \in W = W(\mathfrak{g}, \mathfrak{h}') \cong \mathfrak{S}_N$ the element corresponding to the permutation $\gamma$, hence $w \varpi_j = \varpi_{\gamma(j)}$. From (3.7) one deduces that:

$\theta(w \varpi_j) = \begin{cases} \varpi_{N+1-\gamma(j)} = \varpi_{\gamma(N+1-j)} = w \theta(\varpi_j) & \text{if } \min(j, N + 1 - j) \leq N_b; \\ w \varpi_j = \gamma(\varpi_{\gamma(j)}) & \text{otherwise.} \end{cases}$
This implies $\theta \circ w(\alpha) = w \circ \theta(\alpha)$ for all $\alpha \in R(\mathfrak{g}, \mathfrak{h}')$; thus $\theta$ commutes with $w$, i.e. $w \in W_\sigma$ in the notation of §2.2. By Remark 2.2.4(2) there exists $k \in K$ acting like $w$ on $\mathfrak{h}'$. Therefore $k \in N_\mathfrak{k}(\mathfrak{c}_\mathfrak{g}(\mathfrak{g}^k))$, $r(k) = \gamma$ and $k$ has the desired properties.

(2) It suffices to write an element of $\mathcal{S}(\mathfrak{sp}_\pm(s) \setminus \{0\})$ as a product of transpositions and to apply part (1). \hfill $\square$

If $x = t + n \in \mathfrak{g}^s$ we write $x = \sum_c x_{s,c} = \sum_c (t_{s,c} + n_{s,c})$ where $t_{s,c} + n_{s,c}$ is the Jordan decomposition of $x_{s,c} \in \mathfrak{w}_{s,c}$ (thus $n_{s,c}$ is the nilpotent part of $x_{s,c}$).

We first state consequences of Lemma 3.2.4 for a nilpotent element $x = n \in \mathfrak{p}^s$. As $\theta$ sends $n_{s,c}$ onto $-n_{s,-c}$, the Young diagram of $n_{s,c} \in \mathfrak{w}_{s,c}$ is the same as the Young diagram of $n_{s,-c} \in \mathfrak{w}_{s,-c}$. Moreover, the $(K^*)^n$-orbit of $n$ in $\mathfrak{p}^s$ is characterized by the Young diagrams of the $n_{s,c}$ for $c \neq 0$ and the $ab$-diagram of $n_{s,0}$.

**Lemma 3.3.8.** Let $x = t + n$ and $x' = t' + n'$ be $G$-conjugate elements of $\mathfrak{p}$ with $t, t' \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^*)$. Then $n_{s,0}$ and $n'_{s,0}$ have the same Young diagram. Furthermore, if $x$ and $x'$ are $K$-conjugate, $n_{s,0}$ and $n'_{s,0}$ have the same ab-diagram.

**Proof.** If $m(s,0) \leq 1$ one has $n_{s,0} = n'_{s,0} = 0$; we will therefore assume that $0 \in \mathfrak{sp}(s)$ and $\mathfrak{w}'_{s,0} = \mathfrak{sl}(V_s,0) \neq \{0\}$. One can define equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on $\mathfrak{sp}(s)$ as follows. Say that $cRd$ if the two following conditions are satisfied: $\mathfrak{w}_{s,c}$ is isomorphic to $\mathfrak{w}_{s,d}$, i.e. $m(s,c) = m(s,d)$, and $n_{s,c}, n_{s,d}$ have the same Young diagram. The relation $\mathcal{R}'$ is defined similarly with $n'$ instead of $n$. As observed above, the elements $c$ and $-c$ are in the same equivalence class. Consequently, the class containing $0$ is the only class, for $\mathcal{R}$ or $\mathcal{R}'$, having odd cardinality.

Since $t, t' \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^*)$, there exists $g \in N_G(\mathfrak{g}^s)$ such that $g.x' = x$ and we can set $\gamma = r(g)$. One then has $n_{s,\gamma(c)} = g.n_{s,c}$, therefore $\gamma$ sends each $\mathcal{R}'$-equivalence class to an $\mathcal{R}$-equivalence class. Thus, as the cardinality of the equivalence class of $\gamma(0)$ is odd, $\gamma(0)R0, g.n_{s,0} = n_{s,\gamma(0)}$ and $n_{s,0}, n'_{s,0}$ have the same Young diagram. This proves the first statement.

Now assume that $g \in K$, hence $g \in N_\mathfrak{k}(\mathfrak{g}^s)$. We have already shown before lemma 3.3.7 that, in this situation, $\gamma(0) = 0$. Thus $g.n_{s,0} = n'_{s,0}$ with $g \in K$, as desired. \hfill $\square$

Let $y = t + n \in J \cap \mathfrak{p}$. Then $(\mathfrak{g}_t,0, \mathfrak{t},0)$ is either $(0)$ or a reductive factor of type AIII. By Lemma 3.2.4 the parameter of this factor is the same as the parameter of $(\mathfrak{g}, \mathfrak{t})$, thus it does not depend on the choice of $y \in J \cap \mathfrak{p}$. Recall that $n_{t,0}$ is the component of $n$ lying in $\mathfrak{g}_t,0 = \mathfrak{w}_{t,0}$ and define $\Gamma^\Phi(y)$ to be the $ab$-diagram of $n_{t,0}$ in $(\mathfrak{g}_t,0, \mathfrak{t},0)$. Remark that one can recover the $ab$-diagram of $n_{t,0}$ in $(\mathfrak{g}, \mathfrak{t})$ by adding to $\Gamma^\Phi(y)$ some pairs of rows of length 1, one row beginning by $a$ and the other by $b$.

**Proposition 3.3.9.** (1) Let $x^1, x^2 \in J \cap \mathfrak{p}$. The following conditions are equivalent

(i) $\Gamma^\Phi(x^1) = \Gamma^\Phi(x^2)$ ; (ii) $J_K(x^1) = J_K(x^2)$.

Set $\Gamma^\Phi(J_K(x)) = \Gamma^\Phi(x)$ for $x \in J \cap \mathfrak{p}$.

(2) The map $J_1 \mapsto \Gamma^\Phi(J_1)$ gives an injection from the set of $J_K$-classes contained in $J \cap \mathfrak{p}$ to the set of admissible $ab$-diagrams for the symmetric pair $(\mathfrak{g}_s,0, \mathfrak{t},0)$.

**Proof.** (1) Write $x^i = t^i + n^i$ for $i = 1, 2$. By Lemma 2.4.2 there exists $k^i \in K$ such that $k^i.t^i \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^*)$. Observe that $\Gamma^\Phi(k^i.x^i) = \Gamma^\Phi(x^i)$ and $J_K(k^i.x^i) = J_K(x^i)$, therefore we may assume that $x^i \in \mathfrak{g}^s$ and $t^i \in \mathfrak{c}_\mathfrak{p}(\mathfrak{p}^*)$ for $i = 1, 2$. We may also assume that $m(t^1,0) = m(t^2,0) \geq 1$, otherwise each $n^i_{t^i,0}$ is zero and the equivalence is clear.

As $n^i_{t^i,0}$ belongs to the unique simple factor of type AIII of $(\mathfrak{g}^{t^i}, \mathfrak{t}^{t^i})$, one has $n^i_{t^i,0} \in \mathfrak{w}_{s,0}$, thus $n^i_{t^i,0} = n^i_{s,0}$ and we can set $n^i_{t^i,0} = n^i_{s,0} = n^i_{s,c}$. For $0 \neq \gamma \in \mathfrak{sp}(s)$, set $n^i_{\gamma} = n^i_{s,c}$. Recall that the $J_K$-class of $x^i$ is

38
\[J_K(x^1) = K.(\mathfrak{c}_p(\mathfrak{p}^s)^* + n^1).\]

(ii) ⇒ (i): By hypothesis there exists an element of \(K.(\mathfrak{c}_p(\mathfrak{p}^s)^* + n^1)\) which is \(K\)-conjugate to \(x^2\). Lemma 3.3.8 then shows that \(n^1\) has the same \(ab\)-diagram as \(n^2\) for the pair \((\mathfrak{g}, \mathfrak{t})\), which implies that \(\Gamma^\Phi(x^1) = \Gamma^\Phi(x^2)\) (cf. remark above).

(i) ⇒ (ii): Suppose that \(n^1\) and \(n^2\) have the same \(ab\)-diagram in \((\mathfrak{g}_{s,0}, \mathfrak{t}_{s,0})\). We want to show that \(n^1\) is \(N_K(\mathfrak{g}^s)\)-conjugate to \(n^2\). Observe that \(n^1\) and \(n^2\) have the same orbit under the group \(K_{s,0}\), where we set \(K_{s,\pm c} = (G_{s,\pm c} \cap K)^o\). As \(n^1\) is \(G\)-conjugate to \(n^2\) there exists \(g \in N_G(\mathfrak{g}^s)\) such that \(g.n^1 = n^2\), which defines \(\gamma = r(g) \in \mathcal{G}(\mathfrak{sp}(s))\). Since \(n^1, n^2\) have the same diagrams for all \(c\), there exists \(\gamma' \in \mathcal{G}(\mathfrak{sp}(s))\) such that:

\[w_{s,c} \cong w_{s,\gamma(c)}, \quad n^1_c \text{ has the same diagram as } n^2_{\gamma(c)}, \quad \gamma'(-c) = -\gamma(c),\]

for all \(c \in \mathfrak{sp}(s)\). The permutation \(\gamma'\) fixes 0 and induces \(\gamma'' \in \mathcal{G}(\mathfrak{sp}_{\pm}(s))\). Lemma 3.3.7(2) gives an element \(k \in N_K(\mathfrak{g}^s)\) such that \(k.\mathfrak{g}_{s,\pm c} = \mathfrak{g}_{s,\pm \gamma'(c)}\) for \(c \in \mathfrak{sp}_\pm(s)\). Set \(n^3 = k.n^1\); then \(n^3\) has the same diagram as \(n^2\) for all \(c \neq 0\), and the same \(ab\)-diagram when \(c = 0\). By the results on type A0, \(n^3\) and \(n^2\) are \(K_{s,\pm c}\)-conjugate for \(c \neq 0\). This proves the existence of \(k' \in Z_K.(\mathfrak{c}_p(\mathfrak{p}^s)^*) \subseteq N_K(\mathfrak{g}^s)\) such that \(k'.n^3 = n^2\) and \(k'.k.n^1 = n^2\). In particular, \(K.(\mathfrak{c}_p(\mathfrak{p}^s)^* + n^1) = K.(\mathfrak{c}_p(\mathfrak{p}^s)^* + n^2)\) and the result follows.

(2) is an obvious consequence of (1).

Remark. On can show, by a similar proof to that of Proposition 3.3.9, that condition (2.8) of section 2.5 holds in case AIII.

3.3.3 Case AIII (2)

We continue with the same notation. Thus: \(e \in \mathfrak{g} = \mathfrak{gl}_N\) is a nilpotent element, the partition of \(N\) associated to \(\mathcal{O} = G.e\) is denoted by \(\lambda = (\lambda_1 \geq \cdots \geq \lambda_{k_0})\), \(\Phi : [1, \delta_\mathcal{O}] \rightarrow \{a, b\}\) is an arbitrary function and \((\mathfrak{g}, \mathfrak{θ}) = (\mathfrak{g}, \mathfrak{t}, \mathfrak{p}) = (\mathfrak{g}, \mathfrak{t}^\Phi, \mathfrak{p}^\Phi)\) is the symmetric Lie algebra defined in §3.1.4, hence \(e \in \mathfrak{p} = \mathfrak{p}^\Phi\). As above, \(S = S_G\) is the \(G\)-sheet containing \(e\) and \(J\) is a \(J_G\)-class of \(S\) intersecting \(\mathfrak{p}\). Recall from section 2.5 that the set \(\{g.e\}_{g \in \mathbb{Z}}\) parameterizes the \(K\)-orbits \(\mathcal{O}_{g,e} = K.(g.e)\) contained in \(\mathcal{O} \cap \mathfrak{p}\). We aim to show that the condition (3) introduced in 2.5 holds, see Proposition 3.3.11.

Let \(\Gamma_1 := \Delta(\Phi)\) be the admissible \(ab\)-diagram associated to \(e \in \mathfrak{p}^\Phi\) and let \(J_1 \subset J \cap \mathfrak{p}\) be a \(J_K\)-class. By Theorem 3.2.1 and Lemma 1.6.1 the conditions (\(\bigvee\)) and (\(\bigwedge\)) are satisfied; therefore, Lemma 2.5.8(iii) can be applied in this situation. Let \(J_2\) be given by this lemma (for \(g = 1d\)), thus \(J_2 \subset J\) is a \(J_K\)-class which is well behaved w.r.t \(\mathcal{O}_e\). Set \(Y := J_2 \cap (e + \mathfrak{X}_p) \subset J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet\); as observed in Remark 2.5.7, we have:

\[\dim Y = \dim J \cap \mathfrak{p} - m.\]

(3.9)

Let \(s\) be the semisimple part of an element of \(J \cap \mathfrak{p}\) and recall that \(\Gamma^\Phi(J_1), \text{ resp. } \Gamma^\Phi(J_2),\) is the admissible \(ab\)-diagram, for \((\mathfrak{g}_{s,0}, \mathfrak{t}_{s,0})\), associated to \(J_1\), resp. \(J_2\), by Proposition 3.3.9(2). We are going to compare these diagrams with \(\Gamma_1\) in order to obtain an element \(g.e\) \((g \in \mathbb{Z})\) such that \(J_1\) is well behaved w.r.t \(\mathcal{O}_e\).

Let \(\mathfrak{q} = \bigoplus_i \mathfrak{q}_i\), as in 1.6.1 and \(x = s + n\) be an element of \(J \cap (\mathfrak{q} \cap \mathfrak{p})^\bullet\), cf. Corollary 3.2.2. Recall that we write \(n = \sum n_{i,0}\), with \(n_i \in \mathfrak{q}_i\). Let \(O' \subset \mathfrak{g}_{s,0}\) be the nilpotent orbit \(G_{s,0}.n_{s,0}\) and let \(\mu = (\mu_1 \geq \cdots \geq \mu_{k_{\mathfrak{O}'}})\) be the associated partition of \(m(s,0)\). Remark that the shape of the Young diagram underlying \(\Gamma^\Phi(J_1)\) or, equivalently, \(\Gamma^\Phi(J_2)\), is given by \(\mu\).

On the other hand \(n = \sum_{c \in \mathfrak{sp}(s)} n_{s,c}\) with \(n_{s,c} \in \mathfrak{w}_{s,c}\) and, by Corollary 3.2.9, one can write \(n_{s,0} = \sum n_{i,s,0}\) where each \(n_{i,s,0} \in \mathfrak{q}_{i,s,0} \cap \mathfrak{p}^\Phi\) is regular. This yields in particular that \(\delta_{\mathfrak{O}' \setminus \mathfrak{O}} \leq \delta_{\mathfrak{O}}\). We can therefore define a map

\[\hat{\mathfrak{e}}^\Phi(x) : [1, \delta_{\mathfrak{O}' \setminus \mathfrak{O}}] \rightarrow \{a, b\}\]

(3.10)
where \( b_\Phi(x)(i) \) is the first symbol of the one row ab-diagram of \( n_{i,s,0} \in q_{i,s,0} \cap p_\Phi \). Observe that when \( \lambda_i \) is odd, Corollary 3.2.9(3-4) yields

\[
\mu_i \equiv 1 \mod 2 \text{ and } b_\Phi(x) = \Phi(i) \text{ for all } x \in J \cap (q \cap p)^*. \tag{3.11}
\]

It is not difficult to see that the ab-diagram \( \Delta(b_\Phi(x)) \) associated to the function \( b_\Phi(x) \), see §3.1.4, coincides with the ab-diagram \( \Gamma_\Phi(x) \) defined before Proposition 3.3.9. Thus, according to the previous notation:

\[
\Delta(b_\Phi(y)) = \Gamma_\Phi(y) = \Gamma_\Phi(J_2) \text{ for all } y \in Y \subset J_2 \cap (q \cap p)^*.
\]

**Remark.** One may have \( b_\Phi(x) \neq b_\Phi(x') \) with \( K.x' = K.x \). Such examples can be easily obtained by permuting blocks \( q_i \) and \( q_j \) such that \( \lambda_i = \lambda_j \).

Now, let \( \Psi' : [1, \delta_\bigtriangleup] \rightarrow \{a, b\} \) be a map such that its associated ab-diagram, \( \Delta(\Psi') \), is equal to \( \Gamma_\Phi(J_1) \). Under this notation, we want to construct \( \Psi : [1, \delta_\bigtriangleup] \rightarrow \{a, b\} \) such that \( \Psi' = b_\Psi(y) \) and

\[
\Delta(b_\Psi(y)) = \Gamma_\Psi(y) = \Gamma_\Psi(J_1) \text{ for all } y \in Y.
\]

Fix \( y \in Y \) and define \( \Psi \) as follows:

\[
\Psi(i) = \begin{cases} 
\Phi(i) & \text{if } \Psi'(i) = b_\Psi(y)(i) \text{ and } i \leq \delta_\bigtriangleup; \\
\Phi(i) & \text{if } \Psi'(i) = b_\Psi(y)(i) \text{ and } i \leq \delta_\bigtriangleup; \\
\Phi(i) & \text{for } i \in [\delta_\bigtriangleup + 1, \delta_\bigtriangleup].
\end{cases}
\]

By (3.11), for each \( i \in [1, \delta_\bigtriangleup] \) such that \( \lambda_i \) is odd one has \( \Psi'(i) = \Psi(i) \).

**Lemma 3.3.10.** The ab-diagram \( \Gamma_2 := \Delta(\Psi) \) is admissible for the symmetric pair \((g, t^\Phi)\).

**Proof.** The only thing to prove is that \( N_a' \) (resp. \( N_b' \)), the number of \( a \) (resp. \( b \)) in \( \Gamma_2 \) is equal to \( N_a \) (resp. \( N_b \)). This is equivalent to showing that \( N_a - N_b' = N_a - N_b \). From (3.11) and the definition of \( \Phi \) one deduces:

\[
N_a - N_b - (N_a' - N_b') = \#\{i \mid \Phi(i) = a \text{ and } \lambda_i \equiv 1 \mod 2\} - \#\{i \mid \Phi(i) = b \text{ and } \lambda_i \equiv 1 \mod 2\}
- \#\{i \mid \Psi(i) = a \text{ and } \lambda_i \equiv 1 \mod 2\} + \#\{i \mid \Psi(i) = b \text{ and } \lambda_i \equiv 1 \mod 2\}
= \#\{i \mid b_\Psi(y)(i) = a \text{ and } \lambda_i \equiv 1 \mod 2\} - \#\{i \mid b_\Psi(y)(i) = b \text{ and } \lambda_i \equiv 1 \mod 2\}
- \#\{i \mid \Psi'(i) = a \text{ and } \lambda_i \equiv 1 \mod 2\} + \#\{i \mid \Psi'(i) = b \text{ and } \lambda_i \equiv 1 \mod 2\}.
\]

Since the diagrams \( \Delta(\Psi') = \Gamma_\Phi(J_1) \) and \( \Delta(b_\Psi(y)) = \Gamma_\Phi(J_2) \) are admissible in the same symmetric pair \((g_{s,0}, t_{s,0})\), the previous equation implies that \( N_a - N_b - (N_a' - N_b') = 0 \).

From the function \( \Psi \) one constructs, as in §3.1.4, the symmetric Lie algebra \((g, t', p') = (g, t^\Phi, p^\Phi)\) with \( V = V_a^\Phi \oplus V_b^\Phi \). Since \( q_i \cap t \) and \( q_i \cap t' \) are both spanned by even sup- and sub-diagonals, we obtain the same symmetric Lie subalgebras \((q_i, q_i \cap t, q_i \cap p) = (q_i, q_i \cap t', q_i \cap p') \). It follows that the function \( b_\Psi(z) : [1, \delta_\bigtriangleup] \rightarrow \{a, b\} \) is well defined for all \( z \in J \cap (q \cap p)^* = J \cap (q \cap p')^* \).

Recall that \( y \in (q \cap p)^* = (q \cap p')^* \), thus \( b_\Psi(y) \) is defined; we claim that \( b_\Psi(y) = \Psi' \). Set \( V_a^\Phi(i) = \langle i_j(1) : 1 \leq j \leq \lambda_i \rangle \cap V_a^\Phi, V_b^\Phi(i) = \langle i_j(1) : 1 \leq j \leq \lambda_i \rangle \cap V_b^\Phi \), and define \( V_a^\Psi(i), V_b^\Psi(i) \) accordingly. Observe that: \( V_a^\Phi(i) = V_a^\Psi(i), V_b^\Phi(i) = V_b^\Psi(i) \) when \( \Phi(i) = \Psi(i) \), and \( V_a^\Phi(i) = V_b^\Psi(i), V_b^\Phi(i) = V_a^\Psi(i) \) otherwise. Suppose that \( \Phi(i) \neq \Psi(i) \); by definition of \( b_\Phi, b_\Psi \) one has \( b_\Phi(y)(i) \neq b_\Psi(y)(i) \), therefore \( b_\Psi(y)(i) = \Psi'(i) \) by definition of \( \Psi \). The equality \( b_\Psi(y)(i) = \Psi'(i) \) is obtained in the same way when \( \Phi(i) = \Psi(i) \). The equality \( b_\Psi(y) = \Psi' \) implies in particular \( \Gamma_\Psi(y) = \Gamma_\Phi(J_1) \).

We can now show that the condition \((\bullet)\) is satisfied in type AIII:
Proposition 3.3.11. For each $J_K$-class $J_1 \subset J \cap \mathfrak{p}$, there exists $g \in \mathbb{Z}$ such that $J_1$ is well-behaved w.r.t. $\mathcal{O}_{g,e}$.

Proof. By Lemma 3.3.10 one can find $g' \in G_L$ such that $g'.V_{\mathfrak{a}}^{\psi} = V_{\mathfrak{a}}^{\Phi}$ and $g'.V_{\mathfrak{b}}^{\psi} = V_{\mathfrak{b}}^{\Phi}$. Then, $g = \rho(g') \in G$ induces an isomorphism of symmetric Lie algebras between $(\mathfrak{g}, \mathfrak{t}', \mathfrak{p}')$ and $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ (cf. end of 3.1.4). Since $e \in q \cap \mathfrak{p}$, one has $e \in \mathfrak{p}'$ and $g.e \in \mathfrak{p}$; therefore, up to conjugation by an element of $K^\Phi$ (the algebraic group associated to $\mathfrak{t} = \mathfrak{t}^\Phi$), we may assume that $g \in \mathbb{Z}$ (see §3.1.5). These remarks imply that $\Gamma^\Phi(g,y) = \Gamma^\Phi(y) = \Gamma^\Phi(J_1)$ is the $ab$-diagram associated to $J_1$ with respect to $\Phi$, cf. Proposition 3.3.9.

From $Y \subset q \cap \mathfrak{p} = q \cap \mathfrak{p}'$ one gets $g.y \in Y \subset J \cap \mathfrak{p}$ and, since $g.Y$ is irreducible, one has $g.Y \subset J_1$. In particular, $g.Y \subset g.(e + X_\mathfrak{p}(g.J)) \cap \mathfrak{p} \subset g.e + X_\mathfrak{p}(g.J)$ is contained in $J_1$ with $\dim g.Y = \dim J_1 - m$.

The result then follows from Remarks 2.5.5 and 2.5.7.

4 Main theorem and remarks

4.1 Main theorem

In this subsection we give the description of the $K$-sheets when $(\mathfrak{g}, \theta)$ is of type A. Thus, $\mathfrak{g} \cong \mathfrak{gl}_N$ and $(\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ is a symmetric Lie algebra. Suppose that $S_G \subset \mathfrak{g}$ is a $G$-sheet intersecting $\mathfrak{p}$. In (2.6), cf. Remark 2.5.9, we have defined, for any nilpotent element $e \in S_G \cap \mathfrak{p}$ and any normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$, the following subvariety of $S_G \cap \mathfrak{p}$:

$$S_K(S_G, \mathscr{S}) = S_K(\mathscr{S}) = S_K(K.e) = \overline{K.(e + X_\mathfrak{p}(\mathscr{S}))}.$$  

We aim to describe the $K$-sheets and the varieties $S_G \cap \mathfrak{p}$ in terms of the $S_K(K.e)$.

Recall from Remark 2.5.4(2) that $S_G \cap \mathfrak{p}$ is smooth; in particular, its irreducible components are disjoint. The next lemma reduces the study of $K$-sheets to the study of irreducible components of $S_G \cap \mathfrak{p}$; this result may be false in some cases of type 0, see Corollary 2.1.3.

Lemma 4.1.1. Let $S_G$ be a $G$-sheet of $\mathfrak{g}$ intersecting $\mathfrak{p}$, then each irreducible component of $S_G \cap \mathfrak{p}$ is a $K$-sheet.

Proof. Let $S_K$ be an irreducible component of $S_G \cap \mathfrak{p}$. As $S_G \cap \mathfrak{p}$ is a union of $K$-orbits of same dimension, there exists a $K$-sheet $S_K'$ containing $S_K$. Recall that, as $\mathfrak{g} \cong \mathfrak{gl}_N$, two distinct $G$-sheets are disjoint (see the discussion previous to Corollary 2.1.3). It follows that $S_K'$ must be contained in $S_G$ and, therefore, in $S_G \cap \mathfrak{p}$. This proves that $S_K' = S_K$, hence the result.

Theorem 4.1.2. (i) The $K$-sheets of $\mathfrak{p}$ are disjoint, they are exactly the smooth irreducible varieties $S_K(\mathcal{O}_K)$ where $\mathcal{O}_K \subset \mathfrak{p}$ is a nilpotent $K$-orbit.

(ii) Let $S_G$ be a $G$-sheet intersecting $\mathfrak{p}$. Then, $S_G \cap \mathfrak{p}$ is a smooth equidimensional variety and each of its irreducible component is some $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K \subset S_G \cap \mathfrak{p}$ is a nilpotent $K$-orbit.

(iii) Let $S_K \subset \mathfrak{p}$ be a $K$-sheet and $e$ be a nilpotent element of $S_K$ embedded in a normal $\mathfrak{sl}_2$-triple $\mathscr{S} = (e, h, f)$. Define $Y$ by $e + Y = S_K \cap (e + \mathfrak{p}^f)$. Then $S_K = \overline{K.(e + Y)}$.

Proof. We need to summarize the conditions introduced in §2.5 and proved in cases AI, AII and AIII: ($\heartsuit$) has been proved in Theorem 3.2.1 (with proof in Proposition 3.2.6 for type AIII); ($\clubsuit$) was established in Remark 3.2.3 (types AI, AII) and Remark 3.2.8 (type AIII); ($\spadesuit$) has been obtained in Corollary 3.3.5 (types AI, AII) and Proposition 3.3.11 (type AIII).

Claim (ii) is therefore consequence of Remark 2.5.4(2) (or equivalently Proposition 2.5.3) and Theorem 2.5.11.

Recall that $G$-sheets are disjoint. Then, from $\mathfrak{p}^{(m)} \subset \mathfrak{g}^{(2m)}$, it follows that each $K$-sheet is contained in
a unique $G$-sheet. So, (i) is consequence of (ii) and Lemma 4.1.1.
Under the hypothesis in (iii), $e$ belongs to $S_K$, hence $S_K = S_K(\mathcal{S})$ is the unique $K$-sheet containing $e$. Therefore,
\[ e + Y \subset e + X(\mathcal{S}) \cap p \subset S_K(\mathcal{S}) \cap (e + p^f) = e + Y. \]
The assertion in (iii) then follows from the definition of $S_K(\mathcal{S})$.

**Remark.** One can be more precise about the number of irreducible components of $S_G \cap p$, see §4.2(4).

Fix a sheet $S_G$ intersecting $p$. One can compute the dimension of $S_G \cap p$ in terms of the partitions associated to the nilpotent orbit $O \subset S_G$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\delta_O})$ and $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_{\tilde{\delta}_O})$ be the partitions of $N$ defined in 1.6.1. Pick $e \in O \cap p$ and recall that if $S = (e, h, f)$ is a normal $\mathfrak{sl}_2$-triple we set $S_K(K.e) = K.(e + X_p(\mathcal{S})).$

**Proposition 4.1.3.** Under the previous notation one has
\[
\dim S_G \cap p = \dim S_K(K.e) = \lambda_1 + \frac{1}{2} \left( N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2 \right)
\]
in types AI and AII, and
\[
\dim S_G \cap p = \dim S_K(K.e) = \sum_{i=1}^{\delta_O} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2} \left( N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2 \right).
\]
in type AIII.

**Proof.** Recall that $\dim G.e = N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2$, see [CM], and $\dim K.e = \frac{1}{2} \dim G.e$. By Theorem 4.1.2 and Remark 2.5.5 one has
\[
\dim S_G \cap p = \dim S_K(K.e) = \dim K.e + \dim X_p(\mathcal{S}).
\]
We know that $X_p(\mathcal{S}) = X(\mathcal{S})$ in types AI and AII, cf. Theorem 3.2.1. Therefore, Remark 1.4.8 and equation (1.8) yield $\dim S_G \cap p = \dim K.e + \dim X(\mathcal{S}) = \dim K.e + \dim t = \dim K.e + \lambda_1$. Hence:
\[
\dim S_G \cap p = \lambda_1 + \frac{1}{2} \left( N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2 \right).
\]
Since the morphism $\varepsilon$ is quasi-finite, see Remark 1.4.8, one has $\dim X_p(\mathcal{S}) = \dim \varepsilon$ in type AIII by Proposition 3.2.7. It then follows from (3.6) that
\[
\dim S_G \cap p = \dim K.e + \dim \varepsilon = \dim K.e + \sum_{i=1}^{\delta_O} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor.
\]
Thus
\[
\dim S_G \cap p = \sum_{i=1}^{\delta_O} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{2} \right\rfloor + \frac{1}{2} \left( N^2 - \sum_{i=1}^{\lambda_1} \tilde{\lambda}_i^2 \right)
\]
as desired.

**4.2 Remarks and comments**

We collect here various remarks and comments about the results obtained in the previous sections. To keep the length of the exposition reasonable we will not give full details of the proofs, leaving them to the interested reader.
If not otherwise specified, we assume that \((\mathfrak{g}, \theta) \cong (\mathfrak{g}(N), \theta)\) is of type AI-II-III; we then retain the notation of Section 3 and §4.1. In particular, \(S_G \subset \mathfrak{g}\) is a \(G\)-sheet which intersects \(\mathfrak{p}, \mathcal{O} = G.e, e \in S_G \cap \mathfrak{p}\), is the nilpotent orbit contained in \(S_G, \lambda = (\lambda_1, \ldots, \lambda_{\delta(N)})\) is the associated partition of \(N\), \(\mathbf{v}\) is the basis of \(V\) introduced in §1.6.1, \(e + X = e + X(\mathcal{J})\), with \(\mathcal{J} = (e, h, f)\), is a Slodowy slice of \(S_G, X_{\mathfrak{p}} = X_{\mathfrak{p}}(\mathcal{J}) = X \cap \mathfrak{p}\), \(c \subset \mathfrak{t}\) is such that \(\varepsilon(e + c) = e + X_{\mathfrak{p}}\) in case III (cf. (3.5)), etc.

For simplicity, we will sometimes assume that \(\mathfrak{g} = \mathfrak{sl}_N\). When this is the case, the above notation refers to their intersection with \(\mathfrak{sl}_N\).

(1) Theorems 3.2.1 and 4.1.2 show that \(e + X_{\mathfrak{p}}\) is “almost” a slice for \(S_G \cap \mathfrak{p}\), or for a \(K\)-sheet contained in \(S_G\) and containing \(e\), meaning that the \(G\)-orbit of an element of \(S_G \cap \mathfrak{p}\) intersects \(e + X_{\mathfrak{p}}\). But, contrary to the Lie algebra case, \(e + X_{\mathfrak{p}}\) does not necessarily intersect each \(K\)-orbit contained in the given \(K\)-sheet, even when (2.8) is satisfied. This phenomenon already occurs for the regular sheet in \((\mathfrak{g}, \mathfrak{t}) = (\mathfrak{sl}_2, \mathfrak{so}_2)\). Indeed, using the one parameter subgroup \((F_t)_{t \in \mathbb{R}}\) introduced in §1.4, it is possible to show that \(K(e + X_{\mathfrak{p}})\) contains only one nilpotent orbit, namely \(K.e\), while \(S_{G_{reg}}^\mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}^\mathfrak{reg}\) is the regular \(K\)-sheet of \(\mathfrak{p}\) and contains two nilpotent \(K\)-orbits.

Nevertheless, by [KR, Theorem 11] \(e + X_{\mathfrak{p}} = e + \mathfrak{p}^J\) is a set of representative elements for \(K^\mathfrak{g}\)-orbits in \(\mathfrak{p}^\mathfrak{reg}\). Using Proposition 3.3.2 and Theorem 3.2.1, one can show that \(K^\mathfrak{g}.(e + X_{\mathfrak{p}}) = S_G \cap \mathfrak{p}\) in type AI and AII. In type AIII this result is far from being true, as shown by the following example.

**Example.** Consider a symmetric Lie algebra \((\mathfrak{g}, \mathfrak{t})\) isomorphic to \((\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{k})\). Let \(S_G\) be the \(G\)-sheet containing the nilpotent orbit associated to the partition \(\lambda = (2, 2)\). Choose the nilpotent element \(e \in S_G\) in Jordan canonical form and fix a function \(\Phi\) determining the symmetric pair \((\mathfrak{g}, \mathfrak{t})\). If we set \(\Phi(1) = \Phi(2) = a\), the \(ab\)-diagram of \(e\) given by these choices is \(\Gamma^\Phi(e) = \Delta(\Phi) = ab_{ba}\), see 3.1.4. Here, \(\mathfrak{t} = \mathfrak{c}\) is the set of diagonal elements of the form \(\text{diag}(c, -c, c, -c)\), \(c \in \mathfrak{k}\) (cf. (1.7) and (3.5)). Then, \(S_G = G.(e + t)\) contains exactly two Jordan \(G\)-classes: \(J_{G_{reg}}^\mathfrak{p} = G.e\) (obtained for \(c = 0\)) and \(J_{G_{reg}}^\mathfrak{t} = G.t\), Moreover, one has \(e + X = \varepsilon(e + t) \subset \mathfrak{p}\), see Proposition 3.2.7.

If \(x \in J_{G_{reg}}^\mathfrak{t} \cap \mathfrak{p}\), the slice \(e + X\) contains a unique element \(y\) such that \(G.x = G.y\) (cf. Lemma 1.6.1) and, by Proposition 2.2.6, \(y\) is \(K\)-conjugate to \(x\). Again, it is possible to show that \(e\) is the unique nilpotent element in \(e + X\). Therefore, \(J_{K_{reg}}^\mathfrak{g} = \overline{J_{G_{reg}}^\mathfrak{g} \cap \mathfrak{p}} = K.((e + X) \setminus \{e\})\) is the dense \(J_{K_{reg}}\)-class contained in \(S_K(K.e)\). The same results hold for each \(g, e\) such that \(g \in \mathbb{Z}\), see 3.1.5, and one gets that \(S_G \cap \mathfrak{p} = \overline{J_{K_{reg}}^\mathfrak{g}} = S_K(K.e)\).

A study of the centralizers of nilpotent elements then shows that \(\text{Aut}(\mathfrak{g}, \mathfrak{t}).(e + X) \not\subset S_K(K.e)\). This proves that \(e + X \cap \mathfrak{p}\) is not a slice of \(S_K(K.e) = S_G \cap \mathfrak{p}\) for the action of \(\text{Aut}(\mathfrak{g}, \mathfrak{t})\).

(2) Suppose that \((\mathfrak{g}, \theta) = (\mathfrak{g}, \mathfrak{t})\) is an arbitrary reductive symmetric Lie algebra. Recall [TY, 39.4] that a \(G\)-sheet containing a semisimple element is called a *Dixmier sheet*. Similarly, we will say that a \(K\)-sheet which contains a semisimple element is a *Dixmier \(K\)-sheet*.

If \(\mathfrak{g}\) is semisimple of type \(\Lambda\), all \(G\)-sheets are Dixmier sheets [TY, 33.4.6, 39.4.8]. This implies that, for each sheet \(S_G\) and \(\mathfrak{sl}_2\)-triple \(\mathcal{J} = (e, h, f)\) as in §1.4, the set \(e + X(S_G, \mathcal{J}) = e + X(\mathcal{J})\) contains a semisimple element. For symmetric pairs of type AI or AII, the \(K\)-sheets are all of the form \(S_K(K.e) = S_K(\mathcal{J}) = K.(e + X(\mathcal{J}))\) (cf. Theorems 3.2.1 and 4.1.2); thus, in these cases, any \(K\)-sheet is a Dixmier \(K\)-sheet.

In type AIII there exist \(K\)-sheets containing no semisimple element and one can characterize them in terms of the partition \(\lambda\) associated to the nilpotent element \(e \in S_G \cap \mathfrak{p}\). Recall first that we have shown (see Propositions 3.2.7 and 3.2.6) that

\[
G.(S_G \cap \mathfrak{p}) = G.(e + X_{\mathfrak{p}}) = G.(e + c).
\]

Therefore, if \(e + c\) contains a semisimple element, the same is true for \(e + X_{\mathfrak{p}}(\mathcal{J})\) and \(S_K(K.e) = K.(e + X_{\mathfrak{p}})\). Conversely, if a \(K\)-sheet in \(S_G \cap \mathfrak{p}\) is Dixmier, then \(e + c\) contains a semisimple element.
Thus, a $K$-sheet contained in $S_G \cap \mathfrak{p}$ is Dixmier if and only if $e + c$ contains a semisimple element. By Corollary 1.6.4, an element $e + c$, $c = \sum c_i \in \mathfrak{c} \subset \mathfrak{h} = \bigoplus \mathfrak{h}_i$, is semisimple if and only if the eigenvalues of each $c_i$ are distinct. It follows from the construction of $\mathfrak{c} = \alpha(c')$, see (3.4), that this condition reduces to: 0 is an eigenvalue of $c_i$ of multiplicity at most one for all $i$. One then deduces from the definition of $c'$ the following result:

**Claim 4.2.1.** In type AIII, a $K$-sheet is Dixmier if and only if the partition $\lambda$ satisfies: $\lambda_i - \lambda_{i+1}$ is odd for at most one $i \in \{1, \delta_c\}$ (where we set $\delta_{c+1} = 0$). 

Observe that the condition for a $K$-sheet to be Dixmier depends only on the nilpotent orbit $G.e$ and that $S_K(K.e)$ is Dixmier if and only if $S_K(K.g.e)$, $g \in \mathbb{Z}$, is Dixmier.

(3) Recall from Section 2.5 that a nilpotent orbit of $\mathfrak{g}$ is rigid when it is a sheet of $[\mathfrak{g}, \mathfrak{g}]$. When $\mathfrak{g}$ is of type A the only rigid nilpotent orbit is $\{0\}$. In other cases it may happen that a rigid orbit $\mathcal{O}_1$ contains a non-rigid orbit $\mathcal{O}_2$ in its closure (see the classification of rigid nilpotent orbits in [CM]). Observe that, since the nilpotent cone is closed, a sheet containing $\mathcal{O}_2$ cannot be contained in the closure of $\mathcal{O}_1$. One gets in this way some sheets whose closure is not a union of sheets. One can ask if similar facts occur for symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, in particular when $\mathfrak{g}$ is of type A.

Let $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ be a symmetric Lie algebra; a nilpotent $K$-orbit in $\mathfrak{p}$ which is a $K$-sheet in $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$ will be called rigid. We remarked in (2) that, in types AI and AII, each $K$-sheet contains a semisimple element; thus, $\{0\}$ is the only rigid nilpotent $K$-orbit in these cases. Assume that $(\mathfrak{g}, \mathfrak{t}, \mathfrak{p})$ is of type AIII, $\mathfrak{g}(\mathfrak{g}) \subset \mathfrak{t}$, and recall from the proof of Proposition 4.1.3 (using Remark 1.4.8) that $S_K(K.e) = K.e$ if and only if $\dim e = 0$. The arguments given in (2) about $K$-sheets can be adapted to prove:

**Claim 4.2.2.** The orbit $K.e$ is rigid if and only if the partition $\lambda$ satisfies: $\lambda_i - \lambda_{i+1} \leq 1$ for all $i \in \{1, \delta_c\}$. 

Note that the previous result depends only on the partition $\lambda$ and not on the $ab$-diagram of $e$. In particular, $K.e$ is rigid if and only if each $K$-orbit contained in $G.e \cap \mathfrak{p}$ is rigid.

*Example.* Consider the symmetric pair $(\mathfrak{gl}_6, \mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$ and a rigid $K$-orbit $\mathcal{O}_1$ associated to the partition $\lambda = (3,2,1)$. This orbit contains in its closure a nilpotent $K$-orbit $\mathcal{O}_2$ with partition $(3,1,1,1)$, cf. [Ot2], but $\mathcal{O}_2$ is not rigid.

In type AIII, we can construct in this way $K$-sheets whose closures are not a union of sheets. 

(4) We have shown in Theorem 4.1.2 that the irreducible components of $S_G \cap \mathfrak{p}$ are $K$-sheets and are of the form $S_K(\mathcal{O}_K)$, where $\mathcal{O}_K$ is a (nilpotent) $K$-orbit contained in $\mathcal{O} = G.e$. The number of these irreducible components thus depends on the analysis of the equality $S_K(\mathcal{O}_K) = S_K(\mathcal{O}_K')$ where $\mathcal{O}_K, \mathcal{O}_K'$ are nilpotent $K$-orbits. An obvious necessary condition is $\mathcal{O} = G.\mathcal{O}_K = G.\mathcal{O}_K'$. We begin by showing that $S_G \cap \mathfrak{p}$ is irreducible in types AI and AII.

Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AI. Then, Proposition 3.3.2 and the equality $K = K^\theta$ imply that $\mathcal{O} \cap \mathfrak{p} = \mathcal{O}_K^1$. Therefore, the map $\mathcal{O}_K \rightarrow S_K(\mathcal{O}_K)$ is in this case bijective and $S_G \cap \mathfrak{p}$ is a single $K$-sheet.

Assume that $(\mathfrak{g}, \mathfrak{k})$ is of type AI and choose $\omega \in K^\theta \setminus K$ (if it exists). We need to compare $S_K(\mathcal{O}_K)$ and $S_K(\mathcal{O}_K')$ when $\mathcal{O}_K^1 = \omega.\mathcal{O}_K^2$. Since $\omega \in N_G(\mathfrak{p})$, one has $S_K(\mathcal{O}_K^1) = \omega.S_K(\mathcal{O}_K^2)$. As the $K$-sheets are Dixmier, cf. (2), $S_K(\mathcal{O}_K)$ contains a $J_k$-class $J_k^\mathcal{O}_K$ of semisimple elements. But Lemma 2.2.6 then implies that $J_k^\mathcal{O}_K$ is stable under $\omega$; thus, $S_K(\mathcal{O}_K) = S_K(\mathcal{O}_K')$ since distinct $K$-sheets are disjoint. Hence, in type AI, $S_G \cap \mathfrak{p}$ is a $K$-sheet.

The situation in type AIII is more complicated and one can find $G$-sheets having a nonirreducible intersection with $\mathfrak{p}$. The characterization of the equality $S_K(\mathcal{O}_K^1) = S_K(\mathcal{O}_K^2)$ is given in Claim 4.2.3.

We will simply sketch the proof of this result, which can decomposed in two steps. First, one has to characterize the unique $J_G$-class $J$ such that $S_G \cap \mathfrak{p} = J \cap \mathfrak{p}$, see Theorem 2.5.11. By the same theorem
we know that \( S_K(O^i_K) = J_i \) for a unique \( J_K \)-class \( J_i \). Since Proposition 3.3.9 says that \( J_i \) is determined by its \( ab \)-diagram, it remains to relate this \( ab \)-diagram to that of \( O^i_K \).

In order to state Claim 4.2.3 we first have to define the notion of “rigidified \( ab \)-diagram”. Let \( \Gamma \) be an \( ab \)-diagram corresponding to a nilpotent \( K \)-orbit \( O_K \subset p \); remove from \( \Gamma \) the maximum number of pairs of consecutive columns of the same length. The new \( ab \)-diagram obtained in this way is uniquely determined and is called the the rigidified \( ab \)-diagram deduced from \( \Gamma \), or associated to \( O_K \). The terminology can be justified by the following remark: a rigidified \( ab \)-diagram corresponds to a rigid nilpotent \( K \)-orbit in some other symmetric pair of type \( \text{AIII} \).

Claim 4.2.3. The two orbits \( O^1_K \) and \( O^2_K \) are contained in the same \( K \)-sheet, i.e. \( S_K(O^1_K) = S_K(O^2_K) \), if and only if their associated rigidified \( ab \)-diagrams are equal.

Example. Let \( (g, t) = (\text{gl}_6, \text{gl}_4 \oplus \text{gl}_4) \) and \( O \) be the nilpotent \( G \)-orbit with associated partition \( \lambda = (4, 3, 1) \). The set \( O \cap p \) splits into four \( K \)-orbits \( O^j_K \), \( 1 \leq j \leq 4 \), whose respective \( ab \)-diagrams are

\[
\begin{align*}
\Gamma(O^1_K) &= \text{abab} \quad \Gamma(O^2_K) = \text{abab} \quad \Gamma(O^3_K) = \text{baba} \quad \Gamma(O^4_K) = \text{baba}.
\end{align*}
\]

The associated rigidified \( ab \)-diagrams are, respectively:

\[
\begin{align*}
&ab \quad ab \quad ba \quad ba \\
&b \quad a \quad a \quad a \\
&a ; \quad a ; \quad a ; \quad a \\
&b \quad b \quad b \quad b
\end{align*}
\]

The previous result implies that \( S_G \cap p \) is the disjoint union of \( S_K(O^1_K) = S_K(O^2_K) \) and \( S_K(O^3_K) = S_K(O^4_K) \).

The proof of Claim 4.2.3 being rather technical, we will only try to give below an idea of the main ingredients. Define first a family \((\ell_i)_{i \in [\lambda, \lambda O]}\) of integers by:

\[
\ell_i := \left[ \frac{\lambda_i \lambda_{i+1}}{2} \right], \quad \ell_{i+1} := \ell_i + \left[ \frac{\lambda_i - \lambda_{i+1}}{2} \right].
\]

Note that \( \ell := \sum_i \ell_i = \dim \mathfrak{c} \) (see the construction of \( c' \) in (3.4)). Then, for any \( \ell \)-tuple \((t_1, \ldots, t_\ell) \in \mathbb{k}^\ell \), define \( y(t_1, \ldots, t_\ell) \in q \subset g \) by its action on \( \mathfrak{v} \):

\[
y(t_1, \ldots, t_\ell).v_j^{(i)} = \begin{cases} 
 t_1v_2^{(i)} & \text{if } j = 1; \\
 (1 + t_j/2)v_j^{(i)} & \text{if } j \in 2[1, \ell]; \\
 t_{(j+1)/2}v_{j+1}^{(i)} + v_{j-1}^{(i)} & \text{if } j \in 2[1, \ell - 1] + 1; \\
 v_j^{(i)} & \text{otherwise.}
\end{cases}
\]

Set \( Y = \{ y(t_1, \ldots, t_\ell) \mid (t_1, \ldots, t_\ell) \in (\mathbb{k} \setminus \{-1\})^\ell \} \). Is is easily seen that \( c \in Y \) and one can check that

\[
G.Y = G.(e + c) = G.(S_G \cap p), \quad Y \subset S_G \cap (q \cap p)^*,
\]

see Proposition 3.2.7 for the second equality. Recall that \( J_K \)-classes are locally closed and observe that \( Y \) is irreducible; it follows that there exists a unique \( J_K \)-class \( J_i \) such that \( \overline{Y} \cap J_1 = \overline{Y} \). Set \( Y_1 = Y \cap J_1 \) and let \( J \subset S_G \) be the \( J_G \)-class containing \( J_1 \). From (4.1) we deduce:

\[
S_G \cap p \subseteq (G.Y_1) \cap p \subseteq (G.J_1) \cap p \subseteq J \cap p \subseteq S_G \cap p.
\]

Hence, we can get in this way the Jordan \( G \)-class \( J \) such that \( S_G \cap p = J \cap p \). The datum \((g, O')\) of \( J \) can be computed from any element of \( Y_1 \). One finds in particular that the Young diagram of \( O' \) can be
obtained from the Young diagram of \( \mathcal{O} \) by removing pairs of consecutive columns of the same length. Now, as previously said, we need to compare the \( ab \)-diagram of \( \mathcal{O}^i_K \) with the \( ab \)-diagram of the dense \( J_K \)-class contained in \( S_K(O^i_K) \). We have observed after Claim 4.2.2 that the rigidity property depends only on \( \mathcal{O} \), thus, we may assume that \( O^i_K = \mathcal{O}^i \). Recall that the \( K \)-sheets are disjoint, cf. Theorem 4.1.2, as \( e \in T \) this forces \( T \) to be the dense \( J_K \)-class in \( S_K(O^i_K) \). Since \( Y_1 \subset J \cap (q \cap p)^* \), the \( ab \)-diagram \( \Gamma^p(J_1) \) associated to \( J_1 \), cf. Proposition 3.3.9, can be computed with the help of the function \( \Phi^p \) defined in (3.10). One finds that \( \Phi^p(x) \) is a \( K \)-semisimple element in \( C \subset C \). Using the fact that \( \Phi \) runs over \( C \x C \) and \( \mathcal{O} \) is irreducible; consequently, \( x \in C \) is an irreducible subvariety of \( \mathcal{O} \). To be more explicit, it is shown in [TY, 38.10.1]. Under the previous notation, this forces the \( ab \)-diagram associated to \( K.e \) to be zero, which says that \( \mathcal{O} \) is irreducible subvariety of \( \mathcal{O} \).

Claim 4.2.4. Let \( x = s + n \) be such that \( \mathcal{O}(J_K(x)) \) is an irreducible component of \( \mathcal{O}(p) \). Then \( n \) is \( p \)-distinguished in \( [\mathfrak{g}^*, \mathfrak{g}] \cap \mathfrak{p} \).

Proof. Suppose on the contrary that there exists a semisimple element \( 0 \neq s' \in [\mathfrak{g}^*, \mathfrak{g}^*] \cap \mathfrak{p} \) such that \( [s', n] = 0 \) ans set \( x = s + s' \). It follows easily from the description of the closures of \( J_K \)-classes given in [TY, 39.2.2] that the semisimple part of an element of \( J_K(x) \) is conjugate to an element of \( J_K(s) \). Since \( x' \) is semisimple and \( \dim \mathfrak{g}^* < \dim \mathfrak{g}^* \) we have \( x' \notin J_K(s) \), hence \( x' \notin J_K(x) \). Therefore, \( (x', x) \in \mathcal{O}(J_K(x)) \). This gives a contradiction. 

Recall that the \( K \)-orbits of \( p \)-distinguished elements are classified, see [PT] or [Bu]. Using Claim 4.2.4, in order to upper bound the number of irreducible components of \( \mathcal{O}(p) \) we can look for \( K \)-sheets containing a dense Jordan class of the form \( J_K((\mathfrak{c}_p(\mathfrak{p}^*) + n) \), where the element \( n \) is \( p \)-distinguished in \( [\mathfrak{g}^*, \mathfrak{g}^*] \).

In type A1 or AII, each sheet is Dixmier, cf. (2), and therefore contains a dense \( J_K \)-class consisting of semisimple elements [TY, 39.6.7]. Under the previous notation, this forces the \( p \)-distinguished element \( n \) to be zero, which says that \( \mathfrak{a} = \mathfrak{c}_p(\mathfrak{p}^*) \) is a Cartan subspace. Hence, \( \mathcal{O}(p) = \mathcal{O}(\mathfrak{a} \times \mathfrak{a}) \) is irreducible and we recover a result proved in [Pa1, Pa2].

In type AIII we are going to illustrate the method in symmetric rank 1 and 2. Recall that the symmetric rank is the dimension of any Cartan subspace, so in this case it is equal to \( \min(N_a, N_b) \). In the symmetric rank one case with \( N_a > N_b = 1 \) (see §3.1.4), following [Pa1] one can obtain in this way that \( \mathcal{O}(p) \) has three irreducible components. In the symmetric rank two case we will see what happens when \( N_b = 2 \) and \( N_a \geq 4 \), the other cases being quite similar. For simplicity, we remove the center and assume \( (\mathfrak{g}, \mathfrak{k}) = (\mathfrak{s}_4(k) \oplus \mathfrak{s}_4 \oplus \mathfrak{s}_2 \oplus \mathfrak{k}) \). We find that \( \mathcal{O}(p) \) has at most seven irreducible components, while five is given as a lower bound in [PY]. To be more explicit, it is shown in [PY] that \( \mathcal{O}(p) = \bigcup_{q=0}^4 P_q \) where the \( P_q \) are distinct closed subsets such that \( P_q \not\subseteq \bigcup_{q' \neq q} P_{q'} \) for each \( q \). Applying our method, one gets

\[
P_2 = \mathcal{O}(\mathfrak{a} \times \mathfrak{a}) = P_1 = \mathcal{O}(J_1) = \mathcal{O}(J_1), \quad P_0 = \mathcal{O}(J_1),
\]

46
where $a$ is a Cartan subspace of $p$, $J_1$ is a non-nilpotent $J_K$-class and $O_K^1, O_K^0$ are nilpotent $K$-orbits. Furthermore, neither $C(J_1)$ nor $C(O_K^1)$ is contained in $\bigcup_{q \neq 1} P_q$. The description of the variety $P_3$, resp. $P_4$, is analogous to that of $P_1$, resp. $P_0$. Since all the elements of $C(O_K^1)$ are nilpotent, the determination of the number of irreducible components of $C(p)$ reduces in this case to the question: do we have $C(O_K^1) \subset C(J_1)$?

Unfortunately, the upper bound we obtain increases rapidly with the symmetric rank of $P$. Bala and R. W. Carter, Classes of unipotent elements in simple algebraic groups. II, Math. Proc. Camb. Phil. Soc, 80 (1976), 1-18.

(6) A natural problem is, using section 2.5, to generalize the results obtained in type A to other types. The action of $\varepsilon$ is well described in [HI] for classical Lie algebras and one may ask if conditions $(\bigtriangledown)$, $(\bigtriangledown')$ or $(\downarrow)$ hold in this case. Concerning $(\bigtriangledown)$, the author made some calculations when $(\mathfrak{g}, \mathfrak{k})$ is of type CI. Im-Hof, cf. [HI], splits this type in three cases that we label CI-I, CI-II and CI-III. It is likely that $(\bigtriangledown)$ remains true for the first two cases. In case CI-III one finds the following counterexample. Consider $(\mathfrak{g}, \mathfrak{k}) = \mathfrak{sp}_6, \mathfrak{g}l_3$ and the sheet $S_G$ with datum $(1, 0)$ where $I$ is isomorphic to $\mathfrak{gl}_2 \oplus \mathfrak{sp}_2$. Let $e$ and $e'$ be nilpotent elements in $S_G \cap p$ with respective $ab$-diagrams $\Gamma(e) = abab$ and $\Gamma(e') = babb$. Embed $e$, resp. $e'$, in an $\mathfrak{sl}_2$-triple $\mathcal{J}$, resp. $\mathcal{J}'$. One can show that $\dim \mathcal{X}_p(S_G, \mathcal{J}) = 1$, $\dim \mathcal{X}_p(S_G, \mathcal{J}') = 2$ and we then get $G.(e + \mathcal{X}_p(S_G, \mathcal{J})) \subseteq G.(e' + \mathcal{X}_p(S_G, \mathcal{J}'))$, showing that $(\bigtriangledown)$ is not satisfied. Moreover, we see that the similarity observed in the case $\mathfrak{g} = \mathfrak{gl}_N$ between properties of $\mathcal{X}_p(S_G, \mathcal{J})$ and $\mathcal{X}_p(S_G, \mathcal{J}')$, when $g \in \mathbb{Z}$, is no longer valid.

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