Approximate Injectivity

J. Rosický · W. Tholen

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Abstract In a locally $\lambda$-presentable category, with $\lambda$ a regular cardinal, classes of objects that are injective with respect to a family of morphisms whose domains and codomains are $\lambda$-presentable, are known to be characterized by their closure under products, $\lambda$-directed colimits and $\lambda$-pure subobjects. Replacing the strict commutativity of diagrams by “commutativity up to $\epsilon$”, this paper provides an “approximate version” of this characterization for categories enriched over metric spaces. It entails a detailed discussion of the needed $\epsilon$-generalizations of the notion of $\lambda$-purity. The categorical theory is being applied to the locally $\aleph_1$-presentable category of Banach spaces and their linear operators of norm at most 1, culminating in a largely categorical proof for the existence of the so-called Gurarii Banach space.

Keywords Met-enriched category · Locally $\lambda$-presentable category · $\epsilon$-(co)limit · $\lambda$-$\epsilon$-pure morphism · $\epsilon$-injective object · Approximate $\lambda$-injectivity class · Urysohn space · Gurarii space

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J. Rosický
rosicky@math.muni.cz

W. Tholen
tholen@mathstat.yorku.ca

1 Department of Mathematics and Statistics, Faculty of Sciences, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

2 Department of Mathematics and Statistics, Faculty of Science, York University, 4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
1 Introduction

Recall that an object $K$ is injective to a morphism $f : A \to B$ in a category $\mathcal{K}$ if, for every morphism $g : A \to K$, there is a morphism $h : B \to K$ with $hf = g$. There is a well-developed theory of injectivity in locally presentable categories (see [2]), playing an important role in both algebra and topology. This theory applies to Banach spaces, too, because the category $\text{Ban}$ of (real or complex) Banach spaces and their linear operators of norm at most 1 is locally $\mathbb{N}_1$-presentable (see [2], 1.48). But in this category there is another – and probably more important – concept, of so-called approximate injectivity (see [16]), which is based on the fact that $\text{Ban}$ is enriched over metric spaces. The basic idea is to replace the commutativity of diagrams by their “commutativity up to $e$”. The aim of our paper is to develop a theory of approximate injectivity in metric enriched categories analogously to that of injectivity in ordinary locally presentable categories, and apply it to $\text{Ban}$. In particular, we present a largely categorical existence proof for the Gurarii space, which attracted renewed attention in several recent papers (see, for example, [3,7,13]).

Let us first clarify the categorical context of this paper in as concrete terms as possible. The category $\text{Met}$ of metric spaces and non-expansive maps is neither complete nor complete, and the tensor product $X \otimes Y$, which for $X = (X,d), Y = (Y,d')$ puts the $+$-metric $d \otimes d'(x,y), (x',y')) = d(x,x') + d'(y,y')$ on $X \times Y$, fails to make $\text{Met}$ monoidal closed. (Note that $X \otimes Y$ must not be confused with the Cartesian product $X \times Y$ in $\text{Met}$, which is given by the max-metric.) One therefore enlarges $\text{Met}$ to the category $\text{Met}_\infty$ of generalized metric spaces, by allowing distances to be $\infty$ while keeping all other requirements, as well as the type of morphisms. Then $\text{Met}_\infty$ is complete and cocomplete and monoidal closed, with the internal hom providing the hom-set $\text{Met}_\infty(X,Y)$ with the sup-metric $d''(f,g) = \sup\{d'(fx, gx) \mid x \in X\}$. In what follows, we will normally denote the (generalized) metric of a space by $d$, using annotations or variations only for the sake of clarity.

Throughout most of this paper, we will be considering a $\text{Met}_\infty$-enriched category $\mathcal{K}$. Hence, $\mathcal{K}$ has a class $\text{ob}\mathcal{K}$ of objects, and, for all $A, B, C \in \text{ob}\mathcal{K}$, there are hom-objects $\mathcal{K}(A, B)$ in $\text{Met}_\infty$, with the internal hom $\text{Met}_\infty(A,B)$ and units $1 \to \mathcal{K}(A, A)$ (where 1 is a one-point metric space), satisfying the expected associativity and unity conditions. Interpretation of a $\text{Met}_\infty$-arrow $1 \to \mathcal{K}(A, B)$ as a morphism $A \to B$ defines the underlying ordinary category $\mathcal{K}_0$ of $\mathcal{K}$, which must be carefully distinguished from $\mathcal{K}$ (see [9] for details). Often we will nevertheless call a morphism in $\mathcal{K}_0$ a morphism in $\mathcal{K}$. Should all hom-objects $\mathcal{K}(A, B)$ of the $\text{Met}_\infty$-enriched category $\mathcal{K}$ happen to be ordinary metric spaces, we will allow ourselves to call $\mathcal{K}$ briefly a $\text{Met}$-enriched category.

Our principal examples of $\text{Met}$-enriched categories arise from concrete categories $\mathcal{K}$ over $\text{Met}$, so that one has a faithful functor $U : \mathcal{K} \to \text{Met}$. Then the hom-set $\mathcal{K}(A, B)$ can be considered a subspace of $\text{Met}_\infty(UA, UB)$, and the composition maps of $\mathcal{K}$ remain non-expansive. The example of primary interest in this context is the category $\text{Ban}$, to be considered as a concrete category over $\text{Met}$ via the unit ball functor $U : \text{Ban} \to \text{Met}$, given by $\text{Ban}(I_1(1), -)$, where $I_1$ is the left adjoint to the unit ball functor $\text{Ban} \to \text{Set}$; hence, $I_1(1) = \mathbb{R}$ or $\mathbb{C}$.

We have to carefully distinguish between limits in $\mathcal{K}_0$ and (conical) limits in $\mathcal{K}$, the latter being limits of the former type (i.e., limits in $\mathcal{K}_0$) that are preserved by all representables $\mathcal{K}(\cdot, -) : \mathcal{K}_0 \to \text{Met}_\infty$ (see [9], Section 3.8). For concrete $\text{Met}$-categories, these are limits preserved by $U : \mathcal{K} \to \text{Met}$. Likewise, conical colimits in $\mathcal{K}$ are colimits in $\mathcal{K}_0$ preserved by the representables $\mathcal{K}(\cdot, K) : \mathcal{K}_0^{\text{op}} \to \text{Met}$, a property which, for concrete $\text{Met}$-categories, reduces to the preservation of colimits by $U$. 
Recall that, for a regular cardinal \( \lambda \), an object \( K \) of the ordinary category \( \mathcal{K}_0 \) is \( \lambda \)-
\textit{presentable} if its representable functor \( \mathcal{K}_0(K, -) : \mathcal{K}_0 \to \text{Set} \) preserves \( \lambda \)-directed colimits. \( \mathcal{K}_0 \) is \( \lambda \)-\textit{accessible} if it has all \( \lambda \)-directed colimits and a set \( \mathcal{A} \) of \( \lambda \)-presentable objects such that every object in \( \mathcal{K}_0 \) is a \( \lambda \)-directed colimit of objects in \( \mathcal{A} \); if all small colimits exist, \( \mathcal{K}_0 \) is \( \text{locally} \lambda \)-\textit{presentable}. Accessible (locally presentable) means \( \lambda \)-accessible (locally \( \lambda \)-presentable, respectively) for some \( \lambda \). The underlying ordinary category of \( \text{Met}_\infty \) is locally \( \aleph_1 \)-presentable (see [15], 4.5(3)), and for \( \lambda \) an uncountable regular cardinal, a generalized metric space \( X \) is \( \lambda \)-presentable if, and only if, \( |X| < \lambda \); moreover, the tensor product of two \( \lambda \)-presentable objects is again \( \lambda \)-presentable. By [10] (5.), (7.4), this latter property makes \( \text{Met}_\infty \) \( \text{locally} \lambda \)-\textit{presentable} as a symmetric monoidal closed category, for every uncountable regular \( \lambda \).

An object \( K \) in a \( \text{Met}_\infty \)-enriched category \( \mathcal{K} \) is \( \lambda \)-\textit{presentable} (in the enriched sense) if \( \mathcal{K}(K, -) : \mathcal{K} \to \text{Met}_\infty \) preserves \( \lambda \)-directed colimits. Again, the enriched notion must be distinguished from the ordinary notion of \( K \) being \( \lambda \)-presentable in \( \mathcal{K}_0 \), which postulates only that the \text{Set}-valued functor \( \mathcal{K}(K, -) : \mathcal{K}_0 \to \text{Set} \) preserve \( \lambda \)-directed colimits. Since, for \( \lambda \) uncountable, the forgetful functor \( V : \text{Met}_\infty \to \text{Set} \) preserves \( \lambda \)-directed colimits, it is trivial that \( \lambda \)-presentability of an object in \( \mathcal{K} \) implies its \( \lambda \)-presentability in \( \mathcal{K}_0 \). But since \( V \) does not create \( \lambda \)-directed colimits, the converse statement generally fails. However, Kelly’s Proposition 7.5 of [10] in conjunction with his extended remarks 7.4 clarify that, for \( \lambda \) uncountable, the cocomplete \( \text{Met}_\infty \)-enriched category \( \mathcal{K} \) is locally \( \lambda \)-presentable if its underlying ordinary category \( \mathcal{K}_0 \) is locally \( \lambda \)-presentable, and if every locally \( \lambda \)-presentable object in \( \mathcal{K}_0 \) is locally \( \lambda \)-presentable in \( \mathcal{K} \). In the case \( \mathcal{K} = \text{Ban} \) we conclude:

**Lemma 1.1** For \( \lambda \) any uncountable regular cardinal, every Banach space \( \lambda \)-presentable in \( \text{Ban}_0 \) is \( \lambda \)-presentable in \( \text{Ban} \) and, consequently, the \( \text{Met}_\infty \)-enriched category \( \text{Ban} \) is locally \( \lambda \)-presentable.

**Proof** For an uncountable regular cardinal \( \lambda \), a Banach space \( A \) is \( \lambda \)-presentable in \( \text{Ban}_0 \) if and only if it has a dense subset of cardinality \( < \lambda \). Consider a \( \lambda \)-directed colimit \( (b_i : B_i \to B)_{i \in I} \) in \( \text{Ban} \) and a Banach space \( A \) which is \( \lambda \)-presentable in \( \text{Ban}_0 \). We have to show that \( \text{Ban}(A, b_i) : \text{Ban}(A, B_i) \to \text{Ban}(A, B) \) is a \( \lambda \)-directed colimit in \( \text{Met} \). Clearly, \( V \) sends this cocone to a \( \lambda \)-directed colimit in \( \text{Set} \). Thus, for any \( f : A \to B \), there is \( i \in I \) and \( f_i : A \to B_i \) such that \( f = b_i f_i \). Consider \( f, g : A \to B \). We have to show that \( d(f, g) = \inf_i d(f_i, g_i) \), where \( f = b_i f_i \) and \( g = b_i g_i \). Since \( \lambda \) is uncountable, for each \( a \in A \) there is \( i \in I \) such that \( d(f a, g a) = d(f_i a, g_i a) \). Since \( A \) has a dense subset of cardinality \( < \lambda \), there is \( j \in I \) such that \( d(f_j, g_j) \).

In the following section we briefly introduce the framework of \( \varepsilon \)-\textit{commutativity} (= “commutativity up to \( \varepsilon \)” in \( \text{Met}_\infty \)-enriched categories, as well as the ensuing concept of \( \varepsilon \)-(co)limit. Having presented \( \varepsilon \)-versions of the notion of pure subobject in Sect. 3, we proceed to give sufficient conditions for a class of objects in a locally presentable \( \text{Met}_\infty \)-enriched category to be an \( \varepsilon \)-injectivity class (Theorem 4.8), which leads us to a full characterization of approximate injectivity (= \( \varepsilon \)-injectivity, for all \( \varepsilon > 0 \)) classes, in terms of their closure under products, directed colimits and appropriately generalized pure subobjects (Theorem 5.5). The last section is devoted to presenting a categorical framework for constructing the Urysohn metric space and the Gurarii Banach space.

## 2 \( \varepsilon \)-Homotopy

For any \( \varepsilon \in [0, \infty) \) and morphisms \( f, g : A \to B \) in a \( \text{Met}_\infty \)-enriched category \( \mathcal{K} \), we say that \( f \) is \( \varepsilon \)-\textit{homotopic} (or \( \varepsilon \)-close [11]) to \( g \), if \( d(f, g) \leq \varepsilon \) in the generalized metric \( d \) of \( \mathcal{K}(A, B) \); we write

\( \varepsilon \) Springer
\[ f \sim_{\varepsilon} g \iff d(f, g) \leq \varepsilon. \]

\[ f : A \to B \text{ is an } \varepsilon\text{-homotopy equivalence} \text{ if there exists } f' : B \to A \text{ with } f'f \sim_{\varepsilon} \text{id}_A \text{ and } ff' \sim_{\varepsilon} \text{id}_B. \] (This concept is related to the Gromov–Hausdorff distance of \( A \) and \( B \); see [16], 2.4.)

The relation \( \sim_{\varepsilon} \) is preserved by composition from either side, and it is reflexive and symmetric, but generally not transitive; rather, one has the obvious transitivity rule for \( \varepsilon\)-homotopy, which just rephrases the triangle inequality:

\[ f \sim_{\varepsilon} g \text{ and } g \sim_{\delta} h \implies f \sim_{\varepsilon + \delta} h. \]

\( \varepsilon\)-commutativity of diagrams in \( K \) has the obvious meaning. For example, to say that

\[
\begin{array}{ccc}
B & \xrightarrow{g} & D \\
\downarrow{f} & & \uparrow{f'} \\
A & \xrightarrow{g} & C
\end{array}
\]

is an \( \varepsilon\)-commutative square simply means \( f'g \sim_{\varepsilon} gf \).

\textbf{Remark 2.1} Our motivation for using the homotopic terminology arises from the case \( K = \text{Met}_{\infty} \), as follows. For \( \varepsilon > 0 \), let \( 2_{\varepsilon} \) be the space with two points whose distance is \( \varepsilon \), and we put \( 2_0 = 1 \). Then \( 2_{\varepsilon} \) is not finitely presentable because it is a colimit of the chain formed by the spaces \( 2_{\varepsilon + \frac{1}{n}} \). Let

\[ i_{\varepsilon} : 1 + 1 \to 2_{\varepsilon} \]

be the inclusion map and, in the notation of [1], let \( i_{\varepsilon} \Box \) be the class of all morphisms \( h \) in \( \text{Met}_{\infty} \) with the right lifting property with respect to \( i_{\varepsilon} \), written as \( i_{\varepsilon} \Box h \), and let \( \text{cof}(i_{\varepsilon}) \) the smallest cofibrantly closed class containing \( i_{\varepsilon} \). Then, following [4] 1.3 and [1] Theorem III.6, we obtain the weak factorization system \( \text{(cof}(i_{\varepsilon}), i_{\varepsilon} \Box) \). Clearly, for any morphism \( h \) in \( \text{Met}_{\infty} \), one has \( i_{\varepsilon} \Box h \) if, and only if, \( d(hx, hy) \leq \varepsilon \) implies that \( d(x, y) \leq \varepsilon \) for all \( x, y \) in the domain of \( h \), and since \( h \) is non-expansive, we have the converse implication too. Consequently, \( 2_{\varepsilon} \) is the induced cylinder object, i.e., it is given by a weak factorization of the codiagonal

\[ \nabla : 1 + 1 \xrightarrow{c_{\varepsilon}} 2_{\varepsilon} \xrightarrow{s_{\varepsilon}} 1 \]

(see [14]). In general, for a space \( K \), the cylinder object \( C_K \) is given by a weak factorization

\[ \nabla : K + K \xrightarrow{c_K} C_K \xrightarrow{s_K} K. \]

Then, for morphisms \( f, g : K \to L \) to admit a morphism \( h : C_K \to L \) such that

\[
\begin{array}{ccc}
K + K & \xrightarrow{(f,g)} & L \\
\downarrow{c_K} & & \downarrow{h} \\
& C_K & \end{array}
\]

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commutes means precisely that $f$ and $g$ are $\varepsilon$-homotopic. Indeed, $K + K$ is obtained from $K$ by duplicating each $x \in K$ to $x'$ and $x''$ and putting $d(x', x'') = \infty$ while $d(x', x) = \varepsilon$ in $C_K$.

We note that the sup-metric $d(f, g) = \sup \{d(fx, gx) \mid x \in K\}$ of $\text{Met}(K, L)$ may be recovered from the $\varepsilon$-homotopy relation, as

$$d(f, g) = \inf \{\varepsilon \geq 0 \mid f \sim_{\varepsilon} g\}.$$ 

**Definition 2.2** An $\varepsilon$-pushout of morphisms $f : A \to B$, $g : A \to C$ in a $\text{Met}_{\infty}$-enriched category $K$ is given by an $\varepsilon$-commutative square

$$
\begin{array}{ccc}
B & \xrightarrow{\bar{g}} & D \\
\downarrow{f} & & \downarrow{f'} \\
A & \xrightarrow{g} & C
\end{array}
$$

such that, for any $\varepsilon$-commutative square

$$
\begin{array}{ccc}
B & \xrightarrow{g'} & D' \\
\downarrow{f} & & \downarrow{f'} \\
A & \xrightarrow{g} & C
\end{array},
$$

there is a unique morphism $t : D \to D'$ such that $t\bar{f} = f'$ and $t\bar{g} = g'$. An $\varepsilon$-coequalizer of a pair of parallel morphisms is defined likewise. In the presence of coproducts we define the $\varepsilon$-colimit of a diagram $D$ in $K$ as the $\varepsilon$-coequalizer of the standard pair of morphisms between coproducts of the objects of the diagram that one uses to construct the (ordinary) colimit of $D$ from coproducts and coequalizers.

Up to isomorphism, $\varepsilon$-colimits are uniquely determined; we denote the $\varepsilon$-colimit of $D$ by colim$_{\varepsilon} D$. 0-colimits are simply colimits. In case of a discrete diagram, the $\varepsilon$-notion of colimit coincides with the ordinary one, for every $\varepsilon \in [0, \infty]$: $\varepsilon$-coproducts are precisely coproducts.

**Lemma 2.3** $\text{Met}_{\infty}$ has $\varepsilon$-pushouts.

**Proof** Since $\text{Met}_{\infty}$ has pushouts, there is nothing to be shown in case $\varepsilon = 0$. For $\varepsilon > 0$, consider $f : A \to B$ and $g : A \to C$. In the coproduct $B + C$ we have $d(fx, gx) = \infty$ for all $x \in A$. Changing all distances $d(fx, gx)$ to $\varepsilon$ gives a distance function that satisfies all axioms of a generalized metric but the triangle inequality. Such structures are called semimetrics and, following [15] 4.5(3), $\text{Met}_{\infty}$ is reflective in the corresponding category $\text{SMet}_{\infty}$: the reflector provides a semimetric space $(X, d)$ with the metric $\overline{d}$ given by

$$
\overline{d}(x, z) = \inf \left\{ \sum_{i=0}^{n-1} d(y_i, y_{i+1}) \mid n \geq 1, y_i \in X, y_0 = x, y_n = z \right\}.
$$
It now suffices to take the reflection $D$ of the resulting semimetric space to $\text{Met}_\infty$ to obtain an $\varepsilon$-pushout in $\text{Met}_\infty$. \hfill $\square$

**Corollary 2.4** $\text{Met}_\infty$ has $\varepsilon$-colimits.

**Proof** An $\varepsilon$-coequalizer

$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\quad & \searrow{g} & \downarrow{h} \\
& \downarrow{(f, \text{id}_B)} & \downarrow{h} \\
A + B & \xrightarrow{(g, \text{id}_B)} & B,
\end{array}$

may be given by an $\varepsilon$-pushout

$\begin{array}{ccc}
B & \xrightarrow{h} & D \\
\downarrow{(f, \text{id}_B)} & & \downarrow{h} \\
A + B & \xrightarrow{(g, \text{id}_B)} & B,
\end{array}$

and $\varepsilon$-colimits are constructed with the help of coproducts and $\varepsilon$-coequalizers. \hfill $\square$

**Proposition 2.5** $\text{Ban}$ has $\varepsilon$-colimits.

**Proof** Let $S$ be a representative full subcategory of separable Banach spaces. $S$ is dense in the ordinary category $\text{Ban}_0$, and by Lemma 1.1 even in the $\text{Met}_\infty$-enriched category $\text{Ban}$ (see [10](7.3)). Consequently, the functor

$E : \text{Ban} \to \text{Met}^{\text{Sop}}$

given by $E(A)(K) = \text{Ban}(K, A)$, is a full embedding, and it makes $\text{Ban}$ a reflective full subcategory of $\text{Met}^{\text{Sop}}$. Following Corollary 2.4, $\text{Met}^{\text{Sop}}$ has $\varepsilon$-colimits, and they are calculated pointwise. Given an $\varepsilon$-diagram $D : \mathcal{D} \to \text{Ban}$, its $\varepsilon$-colimit is given by a reflection of the $\varepsilon$-colimit of $E D$. \hfill $\square$

**Remark 2.6** For a diagram $D$ in a $\text{Met}_\infty$-enriched category $\mathcal{K}$ with the needed ($\varepsilon$-)colimits one has canonical morphisms

$\bigcup_{i \in \mathcal{D}} D_i \simeq \text{colim}_\infty D \to \text{colim}_\varepsilon D \to \text{colim}_0 D \simeq \text{colim} D,$

with the morphisms

$q_\varepsilon : \text{colim}_\varepsilon D \to \text{colim} D \quad (\varepsilon > 0)$

presenting colim $D$ as a colimit of the chain $(\text{colim}_\varepsilon D \to \text{colim}_\delta D)_{\varepsilon \geq \delta > 0}$.

### 3 $\varepsilon$-Purity

The notion of $\lambda$-pure morphism in a locally $\lambda$-presentable category as given in [2] allows for an obvious generalization in the case of a $\text{Met}_\infty$-enriched category, as follows. The latter notion entails the former when one puts $\varepsilon = 0$. \hfill $\spadesuit$ Springer
**Definition 3.1** Let $\mathcal{K}$ be a $\text{Met}_\infty$-enriched category and $\lambda$ a regular cardinal. We say that a morphism $f : K \to L$ is $\lambda,\varepsilon$-pure if for any $\varepsilon$-commutative square

$$
\begin{array}{c}
K \\
v \\
A \\
g \\
B
\end{array}
\xymatrix{
A \ar[r]^g & B \\
A \ar[u]^u \ar[r]^{tg} & A \ar[u]^v
}
$$

with $A$ and $B$ $\lambda$-presentable in $\mathcal{K}$ there exists $t : B \to K$ such that $tg \sim_\varepsilon u$.

**Remark 3.2** (1) A composite of $\lambda,\varepsilon$-pure morphisms is $\lambda,\varepsilon$-pure.

(2) If $f_2 f_1$ is $\lambda,\varepsilon$-pure, then $f_1$ is $\lambda,\varepsilon$-pure.

(3) Every split monomorphism is $\lambda,\varepsilon$-pure, for any $\lambda$. (Indeed, when $pf = id_A$, consider the $\varepsilon$-commutative square of 3.1. Then $fu \sim_\varepsilon v g$ implies $u = pf u \sim_\varepsilon pv g$.)

(4) Every $\lambda,\varepsilon$-pure morphism is $\lambda',\varepsilon$-pure, for all $\lambda' \leq \lambda$.

(5) The $\lambda,0$-pure morphisms are precisely the $\lambda$-pure morphisms (as defined in [2]).

Before discussing $\lambda,\varepsilon$-purity further, let us also consider some variations of the notion.

**Definition 3.3** Let $f : K \to L$ be a morphism in the $\text{Met}_\infty$-enriched category $\mathcal{K}$. We say that a morphism $f : K \to L$ is weakly (barely) $\lambda,\varepsilon$-pure if for every $\varepsilon$-commutative (commutative) square as in 3.1, with $A$ and $B$ $\lambda$-presentable in $\mathcal{K}$, there exists $t : B \to K$ such that $tg \sim_\varepsilon u$ ($tg \sim_\varepsilon u$, respectively). We also say that $f$ is $\varepsilon$-split if there is $p : L \to K$ in $\mathcal{K}$ with $pf \sim_\varepsilon id_l$. Finally, we say that $f$ is an $\varepsilon$-monomorphism if $fg = fh$ implies $g \sim_\varepsilon h$.

The following easily verified statements all rely on the transitivity rule for $\varepsilon$-homotopy:

**Lemma 3.4** (1) Every $\lambda,\varepsilon$-pure morphism is weakly $\lambda,\varepsilon$-pure and barely $\lambda,\varepsilon$-pure.

(2) Every weakly $\lambda,\varepsilon$-pure morphism is barely $\lambda,2\varepsilon$-pure.

(3) Every split monomorphism is $\varepsilon$-split, and every $\varepsilon$-split morphism is both, weakly and barely $\lambda,\varepsilon$-pure, for any $\lambda$, and it is a $2\varepsilon$-monomorphism.

**Remark 3.5** (1) Note that an $\varepsilon$-split morphism does not need to be a monomorphism, not even an $\varepsilon$-monomorphism: in $\text{Met}_\infty$, for $0 < \varepsilon < \infty$, consider $\{a, b, c\} \to 1$ with $d(a, b) = d(b, c) = \varepsilon$ and $d(a, c) = 2\varepsilon$.

(2) For $\lambda$ uncountable, every barely $\lambda,\varepsilon$-pure morphism in $\text{Met}_\infty$ is $2\varepsilon$-monomorphic. Indeed, for $f : K \to L$ barely $\lambda,\varepsilon$-pure, consider $a, b \in K$ with $fa = fb$. With $\delta = d(a, b)$ we exploit the commutative square

$$
\begin{array}{c}
K \\
2\delta \ar[u]u \ar[r]^f & L \\
2\delta \ar[u]^v \ar[r] & 1
\end{array}
\xymatrix{
A \ar[r]^{2\delta} & B \\
A \ar[u]^u \ar[r]^{tg} & A \ar[u]^v
}
$$

with $u$ mapping $2\delta$ onto $\{a, b\}$. Since $2\delta$ and $1$ are $\lambda$-presentable, there is $t : 1 \to K$ such that $tg \sim_\varepsilon u$. With $c$ the image of $t$, this forces $d(a, b) \leq d(a, c) + d(b, c) \leq 2\varepsilon$. 

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Let us also record to which extent $\lambda$-$\varepsilon$-purity gets transported along $\varepsilon$-homotopy:

**Lemma 3.6** Let $f \sim_{\varepsilon} f'$.

1. If $f$ is $\lambda$-$2\varepsilon$-pure, then $f'$ is weakly $\lambda$-$\varepsilon$-pure.
2. If $f$ is $\lambda$-$\varepsilon$-pure, then $f'$ is barely $\lambda$-$\varepsilon$-pure.

In conjunction with Remark 3.2 (2), Lemma 3.6 gives:

**Corollary 3.7** Let $gf \sim_{\varepsilon} h$.

1. If $h$ is $\lambda$-$2\varepsilon$-pure, then $f$ is weakly $\lambda$-$\varepsilon$-pure.
2. If $h$ is $\lambda$-$\varepsilon$-pure, then $f$ is barely $\lambda$-$\varepsilon$-pure.

We are now ready to prove an important stability property of $\lambda$-$\varepsilon$-pure morphisms:

**Proposition 3.8** Let $\mathcal{K}$ be a $\text{Met}_{\omega}$-enriched category and $\lambda$ be an uncountable regular cardinal. Then $\lambda$-$\varepsilon$-pure morphisms in $\mathcal{K}$ are closed under $\lambda$-directed colimits in $\mathcal{K}$.

**Proof** Let $E : \mathcal{E} \rightarrow \mathcal{K}^\omega$ be a $\lambda$-directed diagram in the morphism category of $\mathcal{K}$ with $Ee : K_e \rightarrow L_e$ $\lambda$-$\varepsilon$-pure for all $e$ in $\mathcal{E}$. For $f = \text{colim} E : K \rightarrow L$ in $\mathcal{K}$ with a colimit cocone $(k_e, l_e) : Ee \rightarrow f$ we have that $k_e : K_e \rightarrow K$ and $l_e : L_e \rightarrow L$ are colimits in $\mathcal{K}$. Consider an $\varepsilon$-commutative square

$$
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow u & & \downarrow v \\
A & \xrightarrow{g} & B
\end{array}
$$

with $A$ and $B$ $\lambda$-presentable in $\mathcal{K}$.

We will show that there are $e_0$ in $\mathcal{E}$ and $u_{e_0} : A \rightarrow K_{e_0}, v_{e_0} : B \rightarrow L_{e_0}$ in $\mathcal{K}$, such that $u = k_{e_0} u_{e_0}, v = l_{e_0} v_{e_0}$ and $(Ee_0) u_{e_0} \sim_{\varepsilon} v_{e_0} g$. As $Ee_0$ is $\lambda$-$\varepsilon$-pure, there is then $t : B \rightarrow K_{e_0}$ in $\mathcal{K}$ with $tg \sim_{\varepsilon} u_{e_0}$. Hence, $u = k_{e_0} u_{e_0} \sim_{\varepsilon} k_{e_0} tg$, and the proof for $f$ to be $\lambda$-$\varepsilon$-pure will be complete.

Indeed, since $A$ and $B$ are $\lambda$-presentable in $\mathcal{K}_0$, first one finds $e$ in $\mathcal{E}$ and $u_{e} : A \rightarrow K_{e}$ and $v_{e} : B \rightarrow L_{e}$ with $u = k_{e} u_{e}$ and $v = l_{e} v_{e}$. Since $l_{e}(Ee) u_{e} = f k_{e} u_{e} = fu$ and $l_{e} v_{e} g = vg$, $d(l_{e}(Ee) u_{e}, l_{e} v_{e} g) \leq \varepsilon$, follows. Since $A$ is $\lambda$-presentable in $\mathcal{K}$, $\mathcal{K}(A, l_{e}) : \mathcal{K}(A, L_{e}) \rightarrow \mathcal{K}(A, L)$ is a colimit in $\text{Met}_{\infty}$. By the construction of directed colimits in $\text{Met}_{\infty}$, whereby

$$
d(l_{e}(Ee) u_{e}, l_{e} v_{e} g) = \inf_{e' \geq e} d(l_{e, e'}(Ee) u_{e}, l_{e, e'} v_{e} g),
$$

with $(k_{e, e'}, l_{e, e'}) : Ee \rightarrow Ee'$ given by the diagram $E$, for all $n = 1, 2, \ldots$, there are then $e_n \geq e$ in $\mathcal{E}$ with $d(l_{e, e_n}(Ee) u_{e}, l_{e, e_n} v_{e} g) \leq \varepsilon + \frac{1}{n}$. Finally, since $\lambda$ is uncountable, we can find $e_0 \geq e_n$ for all $n$ and obtain $u = k_{e_0} u_{e_0}, v = l_{e_0} v_{e_0}$ and $(Ee_0) u_{e_0} \sim_{\varepsilon} v_{e_0} g$. \qed

**Remark 3.9** As in Proposition 3.8 one proves that the classes of weakly and barely $\lambda$-$\varepsilon$-pure morphisms are both closed under $\lambda$-directed colimits in $\mathcal{K}$, for $\lambda$ uncountable.
**Corollary 3.10** Let \( \lambda \) be a regular uncountable cardinal and \( \mathcal{K} \) be a \( \text{Met}_\infty \)-enriched category with \( \lambda \)-directed colimits such that \( \mathcal{K}_0 \) is locally \( \lambda \)-presentable. Then every \( \lambda \)-pure morphism is \( \lambda \)-\( \varepsilon \)-pure, for all \( \varepsilon \geq 0 \).

**Proof** Since \( \lambda \)-pure morphisms are \( \lambda \)-directed colimits of split morphisms (see [2] 2.30), the result follows from Remark 3.2(3) and Proposition 3.8. \( \square \)

We can finally give the following characterization of barely \( \lambda \)-\( \varepsilon \)-pure morphisms in a large class of categories.

**Proposition 3.11** Let \( \lambda \) be an uncountable regular cardinal and \( \mathcal{K} \) be a \( \text{Met}_\infty \)-enriched category with \( \lambda \)-directed colimits and \( \varepsilon \)-pushouts such that \( \mathcal{K}_0 \) is locally \( \lambda \)-presentable. Then the following assertions are equivalent for a morphism \( f \) in \( \mathcal{K} \):

(i) \( f \) is barely \( \lambda \)-\( \varepsilon \)-pure;

(ii) there are \( g, h \) such that \( gf \sim_\varepsilon h \), with \( h \) being \( \lambda \)-pure;

(iii) there are \( g, h \) such that \( gf \sim_\varepsilon h \), with \( h \) being \( \lambda \)-\( \varepsilon \)-pure.

**Proof** (i)\( \Rightarrow \) (ii): Assume that \( f : A \to B \) is barely \( \lambda \)-\( \varepsilon \)-pure. We proceed as in the proof of [2] 2.30(ii) and express \( f \) as a \( \lambda \)-directed colimit of morphisms \( f_i : A_i \to B_i \) (\( i \in I \)), with \( A_i \) and \( B_i \) \( \lambda \)-presentable. Since \( f \) is barely \( \lambda \)-\( \varepsilon \)-pure, for every \( i \) there is \( t_i : B_i \to A \) such that \( t_i f_i \sim_\varepsilon u_i \). Therefore, in the \( \varepsilon \)-pushouts

\[
\begin{array}{ccc}
A & \xrightarrow{f_i} & B_i \\
\downarrow{u_i} & & \downarrow{u_i} \\
A_i & \xrightarrow{f_i} & B_i \\
\end{array}
\]

every \( f_i \) is a split monomorphism. We get a \( \lambda \)-directed diagram \( (\text{id}_A, \bar{t}_{ij}) : \bar{f}_i \to \bar{f}_j \). Its colimit \( \bar{f} : A \to \bar{B} \) is \( \lambda \)-pure (as a \( \lambda \)-directed colimit of of split monomorphisms), and it may be realized as the the \( \varepsilon \)-pushout

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\text{id}_A} & & \downarrow{g} \\
A & \xrightarrow{f} & B \\
\end{array}
\]

so that we have \( gf \sim_\varepsilon \bar{f} \).

(ii)\( \Rightarrow \) (iii): Corollary 3.10.

(iii)\( \Rightarrow \) (i): Corollary 3.7(2). \( \square \)
4 $\varepsilon$-Injectivity

**Definition 4.1** Let $\mathcal{K}$ be a $\text{Met}_\infty$-enriched category. Given a morphism $f : A \to B$, we say that an object $K$ is $\varepsilon$-injective to $f$ if, for every $g : A \to K$, there exists $h : B \to K$ in $\mathcal{K}$ with $hf \sim_\varepsilon g$.

**Remark 4.2**
1. 0-injectivity coincides with the ordinary injectivity notion.
2. If $K$ is $\varepsilon$-injective to $f$, then $K$ is also $\varepsilon'$-injective to $f$, for all $\varepsilon' \geq \varepsilon$.
3. $K$ is $\infty$-injective to $f : A \to B$ precisely when $\mathcal{K}(B, K) = \emptyset$ only if $\mathcal{K}(A, K) = \emptyset$.

For a class $\mathcal{F}$ of morphisms in $\mathcal{K}$, we denote by

$$\text{Inj}_\varepsilon \mathcal{F}$$

the class of objects $\varepsilon$-injective to every $f \in \mathcal{F}$. Trivially, following 4.2(2), $\text{Inj}_\varepsilon \mathcal{F} \subseteq \text{Inj}_{\varepsilon'} \mathcal{F}$ whenever $\varepsilon \leq \varepsilon'$. A class of objects in $\mathcal{K}$ is an $\varepsilon$-injectivity class if, for some $\mathcal{F}$, it is of the form $\text{Inj}_\varepsilon \mathcal{F}$, and if $\mathcal{F}$ is a set, then $\text{Inj}_\varepsilon \mathcal{F}$ is called a small $\varepsilon$-injectivity class. If the domains and codomains of morphisms in $\mathcal{F}$ are all $\lambda$-presentable in $\mathcal{K}$, $\text{Inj}_\varepsilon \mathcal{F}$ is called a $\lambda$-$\varepsilon$-injectivity class. If $\mathcal{K}_0$ is locally $\lambda$-presentable then every $\lambda$-$\varepsilon$-injectivity class is a small $\varepsilon$-injectivity class.

Compatibility of $\sim_\varepsilon$ with the category composition immediately gives the expected closure properties of $\varepsilon$-injectivity classes, as follows.

**Lemma 4.3** Let $\mathcal{L}$ be an $\varepsilon$-injectivity class in the $\text{Met}_\infty$-enriched category $\mathcal{K}$ with products. Then $\mathcal{L}$ is closed under retracts, and $\mathcal{L}$ is also closed under products in $\mathcal{K}$.

**Proof** Closure under retracts is obvious. For the product of a family of $\varepsilon$-injective objects $K_i$, since the canonical

$$\mathcal{K}\left(A, \prod_{i \in I} K_i\right) \to \prod_{i \in I} \mathcal{K}(A, K_i), \quad g \mapsto (gp_i)_{i \in I},$$

(with product projections $p_i$) is an isomorphism, one has

$$\forall i \in I \ (p_i g \sim_\varepsilon p_i g') \implies g \sim_\varepsilon g'$$

whenever $g, g' : A \to \prod_{i \in I} K_i$ in $\mathcal{K}$, a property which is immediately seen to guarantee product stability of the $\varepsilon$-injectivity class. \qed

**Lemma 4.4** Let $\mathcal{K}$ be a $\text{Met}_\infty$-enriched category such that $\mathcal{K}_0$ is locally presentable. Then every small $\varepsilon$-injectivity class in $\mathcal{K}$ is closed under $\lambda$-directed colimits, for some regular cardinal $\lambda$.

**Proof** For any given set $\mathcal{F}$ of morphisms in $\mathcal{K}$ we can find $\lambda$ such that the domains of morphisms from $\mathcal{F}$ are all $\lambda$-presentable. The proof that $\text{Inj}_\varepsilon \mathcal{F}$ is closed under $\lambda$-directed colimits is then straightforward. \qed

We say that a class $\mathcal{L}$ of objects is closed under (weakly) $\lambda$-$\varepsilon$-pure morphisms in $\mathcal{K}$ if, for every (weakly) $\lambda$-$\varepsilon$-pure morphism $K \to L$, with $L$ in $\mathcal{L}$ one has also $K$ in $\mathcal{L}$.

**Lemma 4.5** Let $\mathcal{K}$ be a $\text{Met}_\infty$-enriched category having all objects presentable. Then every small $\varepsilon$-injectivity class in $\mathcal{K}$ is closed under $\lambda$-$\varepsilon$-pure morphisms, for some regular cardinal $\lambda$.\qed
Proof Take $\lambda$ such that the domains and the codomains of morphisms in $\mathcal{F}$ are all $\lambda$-presentable in $\mathcal{K}$. Let $p : K \to L$ be $\lambda$-$\varepsilon$-pure with $L$ in Inj$_{\varepsilon} \mathcal{F}$, and consider $f : A \to B$ in $\mathcal{F}$ and any $g : A \to K$. $\varepsilon$-injectivity of $L$ gives $h : B \to L$ such that $gf \sim_{\varepsilon} pg$, and then $\lambda$-$\varepsilon$-purity of $p$ gives a morphism $t : B \to K$ with $tf \sim_{\varepsilon} g$. \qed

We now have the tools enabling us to state:

**Proposition 4.6** Let $\mathcal{K}$ be a Met$_{\infty}$-enriched category with products, such that all objects in $\mathcal{K}$ are presentable and the ordinary category $\mathcal{K}_0$ is locally presentable. Then every small $\varepsilon$-injectivity class in $\mathcal{K}$ is a small injectivity class.

Proof According to Theorem 4.8 in [2], it suffices to show that a small $\varepsilon$-injectivity class is accessible and accessibly embedded into the ambient locally presentable category, as well as closed under products. While the latter condition is satisfied by Lemma 4.3, the former two conditions are guaranteed by Lemmas 4.4 and 4.5, in conjunction with Corollary 2.36 in [2]. \qed

**Remark 4.7** By Theorem 2.2 in [17], in a locally $\lambda$-presentable category, $\lambda$-injectivity (= $\lambda$-$0$-injectivity) classes are characterized by closure under products, $\lambda$-directed colimits and $\lambda$-pure subobjects. Consequently, every $\lambda$-$\varepsilon$-injectivity class in a category satisfying the hypotheses of Proposition 4.6 is a $\lambda$-injectivity class.

**Theorem 4.8** Let $\lambda$ be an uncountable regular cardinal and $\mathcal{K}$ a Met$_{\infty}$-enriched category with $\lambda$-directed colimits, such that $\mathcal{K}_0$ is locally $\lambda$-presentable and any $\lambda$-presentable object in $\mathcal{K}_0$ is $\lambda$-presentable in $\mathcal{K}$. Then every class $\mathcal{L}$ of objects in $\mathcal{K}$ closed under products, $\lambda$-directed colimits and weakly $\lambda$-$\varepsilon$-pure morphisms is a $\lambda$-$\varepsilon$-injectivity class and, in particular, a small injectivity class.

Proof Let $\mathcal{L}$ be closed under products, $\lambda$-directed colimits and weakly $\lambda$-$\varepsilon$-pure morphisms. We will follow the proof of [17] 2.2. According to 3.10 and [2] 2.36 and 4.8, $\mathcal{L}$ is weakly reflective. This means that every $K$ in $\mathcal{K}$ comes with a morphism $r_K : K \to K^*$, $K^* \in \mathcal{L}$, such that every object of $\mathcal{L}$ is injective to $r_K$. Let $\mathcal{F}$ consist of all morphisms $f : A \to B$ such that $A$ and $B$ are $\lambda$-presentable and every object of $\mathcal{L}$ is $\varepsilon$-injective to $f$. By the definition of $\mathcal{F}$ we have $\mathcal{L} \subseteq \text{Inj}_{\varepsilon} \mathcal{F}$, and the converse inclusion $\text{Inj}_{\varepsilon} \mathcal{F} \subseteq \mathcal{L}$ will follow from the closure of $\mathcal{L}$ under weakly $\lambda$-$\varepsilon$-pure morphisms once we have shown that, for $K \in \text{Inj}_{\varepsilon} \mathcal{F}$, any weak reflection of $K$ into $\mathcal{L}$ is weakly $\lambda$-$\varepsilon$-pure.

Thus, given $K \in \text{Inj}_{\varepsilon} \mathcal{F}$ and a weak reflection $r : K \to K^*$ in $\mathcal{L}$, we are to prove that in any $\varepsilon$-commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{u} & & \downarrow{v} \\
K & \xrightarrow{r} & K^*
\end{array}
\]

with $A$ and $B$ $\lambda$-presentable the morphism $u$ $2\varepsilon$-factors through $h$. We will say that $(u, v) : h \to r$ is an $\varepsilon$-morphism in this situation.

**Claim:** There is a factorization $u = u_2 \cdot u_1$ and an $\varepsilon$-morphism $(u_1, v_1) : h \to \tilde{r}$ where $\tilde{r} : \tilde{K} \to \tilde{K}^*$ is a weak reflection into $\mathcal{L}$ of a $\lambda$-presentable $\tilde{K}$. \[\square\]
Proof of Claim. Consider all \( \varepsilon \)-morphisms \((u_1, v_1) : h \to \tilde{r}\) where \( \tilde{r} : \tilde{K} \to \tilde{K}^* \) is a weak reflection of \( \tilde{K} \) in \( \mathcal{L} \) and \( u = u_2 \cdot u_1 \) for some \( u_2 \). Since \((u, v) : h \to r\) is such an \( \varepsilon \)-morphism, we can take the smallest \( \alpha \) such that \( \tilde{K} \) is \( \alpha \)-presentable. We are to prove \( \alpha \leq \lambda \); indeed, assuming \( \alpha > \lambda \) we will obtain a contradiction. As in [17], we express \( \tilde{K} \) as a colimit of a smooth chain \( k_{ij} : K_i \to K_j \) \((i \leq j < \alpha)\) of objects \( K_i \) of presentability less than \( \alpha \). This provides weak reflections \( r_i : K_i \to K_i^\alpha \) into \( \mathcal{L} \) such that their colimit \( r_\alpha : \tilde{K} \to K_\alpha^* \) factorizes through \( \tilde{r} \), i.e., \( r_\alpha = s\tilde{r} \) for some \( s : \tilde{K}^* \to K_\alpha^* \). Since \( \tilde{r}u_1 \sim_\varepsilon v_1h \), we have \( r_\alpha u_1 = s\tilde{r}u_1 \sim_\varepsilon sv_1h \), so that \((u_1, sv_1) : h \to r_\alpha \) is an \( \varepsilon \)-morphism. In the same way as in the proof of 3.8, this \( \varepsilon \)-morphism \( \varepsilon \)-factors through some \( r_i, i < \alpha \). This means that there is an \( \varepsilon \)-morphism \( h \to r_i \), which contradicts the minimality of \( \alpha \) and proves the claim.

We are ready to prove that \( u \) \( 2\varepsilon \)-factors through \( h \). Let us consider a factorization \( u = u_2 \cdot u_1 \) and a morphism \((u_1, v_1) : h \to \tilde{r}\) as in the above claim. Let us express \( \tilde{K}^* \) as a \( \lambda \)-directed colimit of \( \lambda \)-presentable objects \( Q_t, t \in T \), with a colimit cocone \( q_t : Q_t \to \tilde{K}^* \). Since both \( \tilde{K} \) and \( B \) are \( \lambda \)-presentable, the morphisms \( \tilde{r} \) and \( v_1 \) both factor through \( q_{t_0} \) for some \( t_0 \in T \). Since \( A \) is \( \lambda \)-presentable, there then exists \( t_1 \geq t_0 \) in \( T \) with an \( \varepsilon \)-commutative diagram, as follows:

\[
\begin{array}{c}
A \\
\downarrow
\end{array}
\begin{array}{ccc}
\overrightarrow{h} & \rightarrow & B \\
\downarrow & & \downarrow \\
\tilde{K} & \rightarrow & \tilde{K}^*
\end{array}
\]

Since all objects of \( \mathcal{L} \) are injective to \( \tilde{r} \), they are also injective to \( \tilde{r} \); moreover, \( \tilde{K} \) and \( Q_{t_1} \) are both \( \lambda \)-presentable. Thus \( \tilde{r} \in \mathcal{F} \). This implies that \( K \) is \( \varepsilon \)-injective to \( \tilde{r} \). Choosing \( d : Q_{t_1} \to K \) with \( u_2 \sim_\varepsilon d\tilde{r} \) we obtain

\[
u = u_2u_1 \sim_\varepsilon d\tilde{r} u_1 \sim_\varepsilon d\tilde{v}_1 h.
\]

Hence, \( r \) is weakly \( \lambda \)-\( \varepsilon \)-pure, and thus \( K \) lies in \( \mathcal{L} \). \( \square \)

Remark 4.9 (1) Let

\[
\begin{array}{ccc}
B & \rightarrow & D \\
\downarrow & \rightleftharpoons & \downarrow \\
A & \rightarrow & C
\end{array}
\]

be an \( \varepsilon \)-pushout and \( K \) be \( \varepsilon \)-injective to \( f \). Then \( K \) is injective to \( \overline{f} \). Indeed, considering \( u : C \to K \) we obtain \( v : B \to K \) such that \( vf \sim_\varepsilon u \). Thus there is \( w : D \to K \) such that \( w\overline{f} = u \).

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(2) Let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow \id_A & & \uparrow p \\
A & \xrightarrow{f} & B
\end{array}
\]

be an \(\varepsilon\)-pushout as in the proof of 3.11 (which corresponds to the mapping cylinder in homotopy theory). Then an object \(K\) is \(\varepsilon\)-injective to \(f\) if and only if it injective to \(\overline{f}\). Indeed, the “if”-part of this statement follows from (1), and the converse is evident.

(3) If \(\lambda\) is an uncountable regular cardinal and \(A, B\) are \(\lambda\)-presentable, then \(\overline{B}\) in (2) is \(\lambda\)-presentable. The verification is analogous to that in the proof of 3.8.

This yields, under the presence of \(\varepsilon\)-pushouts, a direct proof of 4.8.

**Problem 4.10** Let \(\lambda\) be an uncountable regular cardinal and \(K\) a \(\Met_\infty\)-enriched category with \(\lambda\)-directed colimits, such that \(K_0\) is locally \(\lambda\)-presentable and any \(\lambda\)-presentable object in \(K_0\) is \(\lambda\)-presentable in \(K\). Are \(\lambda\)-\(\varepsilon\)-injectivity classes in \(K\) precisely classes closed under products, \(\lambda\)-directed colimits and \(\lambda\)-\(\varepsilon\)-pure morphisms?

## 5 Approximate Injectivity

The following definition is motivated by [16] 3.2.

**Definition 5.1** Let \(\mathcal{K}\) be a \(\Met_\infty\)-enriched category. We say that an object \(K\) is approximately injective to \(f : A \to B\) in \(\mathcal{K}\) if it is \(\varepsilon\)-injective to \(f\) for every \(\varepsilon > 0\).

The class of objects in \(\mathcal{K}\) approximately injective to a class \(\mathcal{F}\) of morphisms in \(\mathcal{K}\) will be denoted \(\text{Inj}_{\text{ap}} \mathcal{F}\). If \(\mathcal{F}\) is a set, then \(\text{Inj}_{\text{ap}} \mathcal{F}\) is called an approximate small injectivity class. If the domains and the codomains of all morphisms in \(\mathcal{F}\) are \(\lambda\)-presentable in \(\mathcal{K}\), \(\text{Inj}_{\text{ap}} \mathcal{F}\) is called an approximate \(\lambda\)-injectivity class. If \(\mathcal{K}_0\) is locally presentable, then any approximate \(\lambda\)-injectivity class is an approximate small injectivity class.

**Definition 5.2** A morphism in \(\mathcal{K}\) is (weakly, barely) \(\lambda\)-ap-pure if it is (weakly, barely) \(\lambda\)-\(\varepsilon\)-pure for every \(\varepsilon > 0\).

**Remark 5.3**

(1) A composite of \(\lambda\)-ap-pure morphisms is \(\lambda\)-ap-pure.

(2) If the composite morphism \(f_2 f_1\) is \(\lambda\)-ap-pure, then \(f_1\) is also \(\lambda\)-ap-pure.

(3) Let \(\lambda\) be an uncountable regular cardinal and \(\mathcal{K}\) be a \(\Met_\infty\)-enriched category with \(\lambda\)-directed colimits and \(\varepsilon\)-pushouts, such that \(\mathcal{K}_0\) is locally \(\lambda\)-presentable. Then every barely \(\lambda\)-ap-pure morphism is a monomorphism. Indeed, considering the \(\varepsilon\)-pushouts
and applying the characterization 3.11(ii), since $\lambda$-pure morphisms in an accessible category are monomorphisms, we see that every $f_\varepsilon$ is monic. Consequently, as a directed colimit of these morphisms, also $f$ is a monomorphism.

(4) It follows easily from Lemma 3.4(2) that every weakly $\lambda$-ap-pure morphism is barely $\lambda$-ap-pure and, hence, a monomorphism, by (3). Consequently, rather than referring to its closure under weakly $\lambda$-ap-pure morphisms we may say that a class $\mathcal{L}$ of objects be closed under weakly $\lambda$-ap-pure subobjects.

**Lemma 5.4** Let $\lambda$ be an uncountable regular cardinal and $\mathcal{K}$ a $\text{Met}_\infty$-enriched category with $\lambda$-directed colimits, such that $\mathcal{K}_0$ is locally $\lambda$-presentable. Then every $\lambda$-pure morphism is $\lambda$-ap-pure.

**Proof** The statement follows from Corollary 3.10. □

**Theorem 5.5** Let $\lambda$ be an uncountable regular cardinal and $\mathcal{K}$ a $\text{Met}_\infty$-enriched category with $\lambda$-directed colimits, such that $\mathcal{K}_0$ is locally $\lambda$-presentable and every $\lambda$-presentable object in $\mathcal{K}_0$ is $\lambda$-presentable in $\mathcal{K}$. Then the approximate $\lambda$-injectivity classes in $\mathcal{K}$ are precisely the full subcategories closed under products, $\lambda$-directed colimits and weakly $\lambda$-ap-pure morphisms.

**Proof** Since

$$\text{Inj}_{\text{ap}} \mathcal{F} = \bigcap_{\varepsilon > 0} \text{Inj}_\varepsilon \mathcal{F},$$

every approximate $\lambda$-injectivity class is closed under products and $\lambda$-directed colimits (see 4.3 and 4.4). We will show that $\text{Inj}_{\text{ap}} \mathcal{F}$ is closed under weakly $\lambda$-ap-pure subobjects. Let $p : K \rightarrow L$ be weakly $\lambda$-ap-pure and $L$ belong to $\text{Inj}_{\text{ap}} \mathcal{F}$. Consider $f : A \rightarrow B$ in $\mathcal{F}, \varepsilon > 0$ and $u : A \rightarrow K$. There is $v : B \rightarrow L$ such that $pu \sim_\varepsilon vf$. Since $p$ is weakly $\lambda$-$\varepsilon$-pure, there exists $t : B \rightarrow K$ with $tf \sim_\varepsilon u$. Thus $K$ is $\varepsilon$-injective to $f$.

Let $\mathcal{L}$ be closed under products, $\lambda$-directed colimits and weakly $\lambda$-ap-pure subobjects. We will proceed in the same way as in 4.8. Let $\mathcal{F}$ consist of all morphisms $f : A \rightarrow B$ such that $A$ and $B$ are $\lambda$-presentable and every object of $\mathcal{L}$ is approximately injective to $f$. We have $\mathcal{L} \subseteq \text{Inj}_{\text{ap}} \mathcal{F}$, and the converse inclusion will follow from the fact that every weak reflection of $K \in \text{Inj}_{\text{ap}} \mathcal{F}$ into $\mathcal{L}$ is weakly $\lambda$-ap-pure.

Since $K \in \text{Inj}_\varepsilon \mathcal{F}$, a weak reflection $r : K \rightarrow K^*$ is weakly $\lambda$-$\varepsilon$-pure. Hence $r$ is weakly $\lambda$-ap-pure. □

In the presence of $\varepsilon$-pushouts, we can speak about the closure under weakly $\lambda$-ap-pure subobjects (see 5.3(4)).

**Corollary 5.6** Under the hypotheses of Theorem 5.5, every approximate $\lambda$-injectivity class in $\mathcal{K}$ is a $\lambda$-injectivity class.

**Proof** The statement follows from 5.4, 5.5 and [17] 2.2. □

**Remark 5.7** Continuing to work under the hypotheses of Theorem 5.5, we let $\mathcal{L}$ be an approximate $\lambda$-injectivity class in $\mathcal{K}$. Then, following 5.6 and [2] 4.8, $\mathcal{L}$ is weakly reflective. We claim that every object $K$ that is approximately injective to its weak reflection $r : K \rightarrow K^*$ must lie in $\mathcal{L}$. Indeed, $r \varepsilon$-splits for every $\varepsilon > 0$ and then, by 3.4, must be weakly $\lambda$-ap-pure, for any $\lambda$. Hence, $K \in \mathcal{L}$ follows.
Recall that Vopěnka’s Principle is a large-cardinal principle which guarantees that injectivity classes in a locally presentable category are characterized by their closure under products and split subobjects (see [2] 6.26). We are now ready to conclude the validity of an “ap-version” of this theorem. To state it, we say that \( f \) is ap-split if it is \( \varepsilon \)-split for every \( \varepsilon > 0 \).

**Theorem 5.8** Under Vopěnka’s Principle, the following conditions are equivalent for a full subcategory \( \mathcal{L} \) of a category \( K \) satisfying the hypotheses of Theorem 5.5:

1. \( \mathcal{L} \) is closed under products and ap-split subobjects,
2. \( \mathcal{L} \) is an ap-injectivity class,
3. \( \mathcal{L} \) is weakly reflective and closed under ap-split subobjects.

**Proof** 
(3) \( \Rightarrow \) (2): By Remark 5.7, \( \mathcal{L} \) is an ap-injectivity class w.r.t. weak reflections of \( K \)-objects to \( \mathcal{L} \).
(2) \( \Rightarrow \) (1) follows from 3.4 and 5.5.
(1) \( \Rightarrow \) (3) follows from [2] 6.26, since closure under ap-split subobjects trivially entails closure under split subobjects. \( \square \)

6 The Countable Case

Regular monomorphisms in \( \text{Met}_\infty \) are isometries, and these are stable under pushout. Every finite generalized metric space \( A \) is \( \aleph_0 \)-generated, in the sense that \( \text{Met}_\infty (A, -) : \text{Met}_\infty \to \text{Set} \) preserves directed colimits of isometries.

A generalized metric space \( K \) is \( \aleph_0 \)-saturated if, for any isometry \( f : A \to B \) between finite generalized metric spaces and any isometry \( g : A \to K \), there is an isometry \( h : B \to K \) with \( hf = g \). This means that \( K \) is injective to morphisms between finite (i.e., \( \aleph_0 \)-generated) objects in the category of generalized metric spaces and isometries.

A generalized metric space is called rational if all of its distances are either rational or \( \infty \). By an \( \aleph_0 \)-saturated generalized rational metric space we mean a space which is injective to morphisms between finite spaces in the category of generalized rational metric spaces and isometries.

**Theorem 6.1** There is a countable \( \aleph_0 \)-saturated generalized rational metric space.

**Proof** Isometries in \( \text{Met}_\infty \) are stable under pushout; moreover, if the given spaces are rational, so is the pushout. Up to isomorphism, there are only countably many finite generalized metric spaces and isometries. Let \( S \) be the set of all isometries between them; \( S \) is countable again.

We express \( S \) as a union of a countable chain of finite subsets \( S_n \) and will construct a countable chain \( (k_{ij} : K_i \to K_j)_{i < j < \omega} \) of finite generalized rational metric spaces and isometries, as follows. Let \( K_0 = \emptyset \). Having \( (k_{ij} : K_i \to K_j)_{i < j \leq n} \), we consider all spans \( (u, h) \) where \( h : X \to Y \) is in \( S_n \) and \( u : X \to K_n \) is an isometry. Since \( K_n \) is finite, the number of these spans is finite, and we can enumerate them as \( (u_0, h_0), \ldots, (u_{m-1}, h_{m-1}) \) where \( h_i : X_i \to Y_i \) for \( i = 0, \ldots, m - 1 \). We will construct a finite chain \( (p_{ij} : P_i \to P_j)_{0 \leq i < j \leq m} \) of finite generalized metric spaces and isometries as follows. Let \( P_0 = K_n \). Having \( (p_{ij} : P_i \to P_j)_{0 \leq i < j \leq l} \), let \( p_{l,l+1} \) be given by the pushout.
Then \( P_{l,l+1} = P_{l,l+1}P_{l,l} \). We put \( K_{n+1} = P_m \) and \( k_{n,n+1} = P_{0m} \).

As the colimit of a chain of isometrically embedded rational generalized metric spaces, also \( K = \text{colim} K_n \) is rational. We claim that \( K \) is \( \aleph_0 \)-saturated. Indeed, consider \( h : X \to Y \) in \( S \) and an isometry \( u : X \to K \). Since \( X \) is \( \aleph_0 \)-small, there is \( u' : X \to K_n \) such that \( k_nu' = u \); here \( k_n : K_n \to K \) is a colimit injection. Without any loss of generality, we may assume \( h \in S_n \). Thus \( (u', h) = (u_l, h_l) \) for some \( 0 \leq l \leq m - 1 \). We get the isometry \( k_{n+1,p_{l+1,m}h} : Y \to K \) and
\[
k_{n+1,p_{l+1,m}h} = \left( k_{n+1,p_{l+1,m}p_{l,l+1}P_{l,l}}u' \right) = \left( k_{n+1,p_{l,m}u'} \right) = \left( k_{n+1,k_{n,n+1}u'} \right) = k_{n+1}u' = u.
\]

\( \square \)

A countable \( \aleph_0 \)-saturated generalized rational metric space \( U_0 \) is, in fact, uniquely determined, up to isomorphism: see, e.g., [18] Theorem 2; it is the Fraïssé limit of finite generalized rational metric spaces (see [12]). Its completion \( \hat{U} \) is an \( \aleph_0 \)-saturated complete separable metric space, called Urysohn space in the literature: see [8] or [12] for a proof of its so-called universality and homogeneity, from which one easily concludes its \( \aleph_0 \)-saturatedness in \( \text{Met} \).

In order for us to establish a corresponding result in \( \text{Ban} \), we introduce the needed definitions more generally at the level of \( \text{Met}_{\infty} \)-enriched categories.

**Definition 6.2** Let \( K \) be a \( \text{Met}_{\infty} \)-enriched category. A morphism \( f : A \to B \) is called an isometry if, for every \( \varepsilon \geq 0 \) and all \( u, v : C \to A \), one has
\[
fu \sim_\varepsilon fv \Rightarrow u \sim_\varepsilon v.
\]

**Example 6.3** In both, \( \text{Met} \) and \( \text{Ban} \), isometries have their usual meaning. In \( \text{Met} \), it suffices to test them on \( u, v : 1 \to A \), and in \( \text{Ban} \) on \( u, v : I_1(1) \to A \).

**Definition 6.4** Let \( K \) be a \( \text{Met}_{\infty} \)-enriched category. An object \( A \) in \( K \) is \( \lambda \)-\( \varepsilon \)-generated if, for any \( \lambda \)-directed diagram of isometries \( (k_{ij} : K_i \to K_j)_{i \leq j \in I} \) with colimit cocone \( k_i : K_i \to K \) and every morphism \( f : A \to K \), there is \( i \in I \) such that

1. \( f \varepsilon \)-factorizes through \( k_i \), i.e., \( f \sim_\varepsilon k_ig \) for some \( g : A \to K_i \),
2. the \( \varepsilon \)-factorization is \( \varepsilon \)-essentially unique, in the sense that, if \( f \sim_\varepsilon k_i g \) and \( f \sim_\varepsilon k_ig' \), then \( k_ig \sim_\varepsilon k_ig' \) for some \( j \geq i \).

We say that \( A \) is \( \lambda \)-ap-generated if it is \( \lambda \)-\( \varepsilon \)- generated for every \( \varepsilon > 0 \).

**Example 6.5** (1) In \( \text{Ban} \), every finite-dimensional space \( A \) is \( \aleph_0 \)-ap-generated.

(2) Since every Banach space is a directed colimit of finite-dimensional Banach spaces and isometries, every \( \aleph_0 \)-\( \varepsilon \)-generated Banach space admits an \( \varepsilon \)-split morphism to a finite-dimensional Banach space, for any \( \varepsilon > 0 \).
(3) More generally, for every isometry \( f : X \to Y \) between \( \aleph_0 \)-ap-generated Banach spaces and every \( \varepsilon > 0 \), there is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}
\]

in \( \text{Ban} \) with an isometry \( g \) between finite-dimensional Banach spaces \( A, B \), as well as morphisms \( s : X \to A, t : Y \to B \), such that \( us \sim_{\varepsilon} \text{id}_X, vt \sim_{\varepsilon} \text{id}_Y \) and \( gs \sim_{\varepsilon} tf \).

**Definition 6.6** Let \( K \) be a \( \text{Met}_{\infty} \)-enriched category. We say that an object \( K \) is \( \aleph_0 \)-ap-saturated if it is approximately injective to morphisms between ap-\( \aleph_0 \)-generated objects in the category of \( K \)-objects and isometries.

**Theorem 6.7** There is a separable \( \aleph_0 \)-ap-saturated Banach space.

**Proof** By 6.5(2), a Banach space is \( \aleph_0 \)-ap-saturated if, and only if, it is approximately injective to isometries between finite-dimensional Banach spaces. Now we proceed in the same way as in 6.1. Isometries are pushout stable in \( \text{Ban} \) (see [3], 2.1) and, following [7] 2.7 and [13], a Banach space is approximately injective to finite-dimensional Banach spaces if, and only if, it is approximately injective to rational isometries between finite-dimensional rational Banach spaces. (For the meaning of “rational” in the Banach space context, see [7].) We will show that \( K \), constructed analogously to the construction in 6.1 from the countable set \( S \) of relevant isometries as a colimit of finitely-dimensional spaces \( K_n \) (with \( K_0 \) the null space), is \( \aleph_0 \)-ap-saturated. Given a finite-dimensional Banach space \( X \), there are uncountable many isometries \( v : X \to K_n \). But there is a countable set \( \mathcal{U} \) of isometries \( u : X \to K_n \) such that for any \( v \) and any \( \varepsilon > 0 \) there is \( u \in \mathcal{U} \) such that \( v \sim_{\varepsilon} u \). We use only \( u \in \mathcal{U} \) in the construction.

For that, consider \( h : X \to Y \) in \( S \) and \( v : X \to K_n \), and let \( \varepsilon > 0 \). There is \( u : X \to K \) in \( \mathcal{U} \) such that \( v \sim_{\varepsilon} u \). Since \( X \) is \( \aleph_0 \)-ap-generated, there is \( u' : X \to K_n \) such that \( k_n u' \sim_{\varepsilon} u \). Without loss of generality we may assume that \( h \in S_n \). Consequently, \( k_{n,n+1} u' = wh \) for some \( w : Y \to K_{n,n+1} \) and, hence,

\[
u \sim_{\varepsilon} k_n u' = k_{n+1} k_{n,n+1} u' = k_{n+1} wh.
\]

Thus \( v \sim_{\varepsilon} k_{n+1} wh \) as desired. ☐

A separable \( \aleph_0 \)-ap-saturated Banach space is in fact uniquely determined, up to isomorphism; it coincides with the **Gurarii space** (see [13]).

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