Combinatorial $p$-th Calabi flows on surfaces

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Abstract

For triangulated surfaces and any $p > 1$, we introduce the combinatorial $p$-th Calabi flow which precisely equals the combinatorial Calabi flows first introduced in H. Ge’s thesis [9] (or see H. Ge [13]) when $p = 2$. The difficulties for the generalizations come from the nonlinearity of the $p$-th flow equation when $p \neq 2$. Adopting different approaches, we show that the solution to the combinatorial $p$-th Calabi flow exists for all time and converges if and only if there exists a circle packing metric of constant (zero resp.) curvature in Euclidean (hyperbolic resp.) background geometry. Our results generalize the work of H. Ge [13], Ge-Xu [19] and Ge-Hua [14] on the combinatorial Calabi flow from $p = 2$ to any $p > 1$.

1 Introduction

Geometric flows are powerful tools to find canonical metrics on a given manifold which is an extremely important problem in differential geometry. R. Hamilton [8] introduced the Ricci flow, which has been used to prove the uniformization theorem and solve the Poincaré conjecture. Seeking constant curvature metrics, E. Calabi [1, 2] introduced the Calabi flow and studied the variational problem of minimizing the so-called Calabi energy in any fixed cohomology class of Kähler metrics. For dimension two, it is proved that both the Calabi flow and the normalized Ricci flow exist for all time and converge to a constant scalar curvature metric (cf. [3, 5] for further references).

Suppose $X$ is a closed surface with a triangulation $\mathcal{T} = (V, E, F)$, where $V, E, F$ denote the sets of vertices, edges, and faces respectively. Throughout this paper, a function defined on vertices is an $N$-dimensional column vector, where $N = |V|$ is the number of vertices. Moreover, all vertices, marked by $v_1, ..., v_N$, are ordered one by one.

**Definition 1.1.** A circle packing metric is defined to be a positive function

$$r : V \to (0, +\infty)$$

$$v_i \mapsto r_i, \ i = 1, ..., N$$
on the vertices.

Hence we may think of circle packing metrics as points in \( \mathbb{R}^N_{\geq 0} \), where \( \mathbb{R}^N_{\geq 0} \) means \( N \) times of Cartesian product of \((0, +\infty)\). A weight on the triangulation is defined to be a function \( \Phi : E \to [0, /2] \). A triangulated surface with a weight \( \Phi \) is denoted as \((X, T, \Phi)\). Let \( l : E \to (0, +\infty) \) be a positive function assigning each edge \( i, j \in E \) a length \( l_{ij} \). \( l \) is called a piecewise linear metric if for every triangle \( i, j, k \in F \), the three edge lengths \( l_{ij}, l_{jk}, l_{ik} \) satisfy triangle inequalities. For a fixed triangulated surface \((X, T, \Phi)\), every circle packing metric \( r \) determines a piecewise linear metric on \( X \) by setting the length of edge \( i, j \in E \) as

\[
l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j \cos(\Phi_{ij})},
\]

in Euclidean background geometry and

\[
l_{ij} = \cosh^{-1}(\cosh r_i \cosh r_j + \sinh r_i \sinh r_j \cos(\Phi_{ij})),
\]

in hyperbolic background geometry. As a consequence, each face in \( F \) is isometric to a Euclidean triangle (hyperbolic resp.). Specifically, each face \( i, j, k \in F \) is a Euclidean triangle (hyperbolic resp.) with edge lengths \( l_{ij}, l_{jk}, l_{ik} \) because \( l_{ij}, l_{jk}, l_{ik} \) satisfy triangle inequalities (Lemma 13.7.2 [21]). Furthermore, the triangulated surface \((X, T)\) is composed by gluing Euclidean triangles coherently (hyperbolic resp.).

Given a triangulated surface \((X, T, \Phi)\) with a circle packing metric \( r \), all inner angles of the triangles are determined by \( r_1, ..., r_N \). Denote \( \theta_{ik}^j \) as the inner angle at vertex \( i \) in the triangle \( i, j, k \in F \). The well-known combinatorial Gauss curvature \( K_i \) at vertex \( i \) is defined as

\[
K_i = 2\pi - \sum_{i,j,k \in F} \theta_{ik}^j,
\]

where the sum is taken over each triangle with \( i \) as one of its vertices. For every circle packing metric \( r \) on \((X, T, \Phi)\), we have the following combinatorial Gauss-Bonnet formula introduced by Thurston in [21].

\[
\sum_{i=1}^N K_i = 2\pi \chi(X) - \lambda \text{Area}(X),
\]

where \( \lambda = -1, 0, 1 \) correspond three geometries, i.e., hyperbolic geometry \( H^2 \), Euclidean geometry \( E^2 \) and spherical geometry \( S^2 \).

**Definition 1.2.** A constant curvature circle packing metric \( r_{av} \) is a circle packing metric that determines the constant combinatorial curvature \( K_{av} = K(r_{av}) = k_{av}(1, ..., 1)^T \), where \( k_{av} = 2\chi(X)/N \).
The constant (zero resp.) curvature circle packing metric is a good candidate for privileged metrics. Thurston [21] first studied this class of metrics, and found that there are combinatorial obstructions for the existence of constant (zero resp.) curvature metrics. Given a triangulated surface \((X, T, \Phi)\), Thurston introduced the circle packing metric and found that there are combinatorial obstructions for the existence of a circle packing metric with constant (zero resp.) combinatorial curvature (see section 13.7 in [21]).

Later, Chow-Luo [6] introduced the combinatorial Ricci flow in Euclidean background geometry

\[
\frac{dr_i}{dt} = -K_i r_i,
\]

and the normalized combinatorial Ricci flow

\[
\frac{dr_i}{dt} = (k_{av} - K_i) r_i,
\]

which is equivalent to (1.3). The combinatorial Ricci flow in hyperbolic background geometry is defined by Chow-Luo [6] as

\[
\frac{dr_i}{dt} = -K_i \sinh r_i.
\]

Then Chow-Luo proved that the combinatorial Ricci flows above exist for all time and converge exponentially fast to Thurston’s circle packing on surfaces if and only if there exists a circle packing metric of constant (zero resp.) curvature in Euclidean (hyperbolic resp.) background geometry. Since then, various discrete curvature flows were introduced and studied. We refer the reader to H. Ge [10–12], Ge-Hua-Jiang [15] Ge-Jiang [16–18], Zhang-Chang [22], Lin-Zhang [20] and Zhang-Lin [23,24]. Specially, motivated by the work of Chow-Luo, H. Ge in his thesis [9] (or see [13] [14] [19]) introduced the combinatorial Calabi flow in Euclidean background geometry

\[
\frac{dr_i}{dt} = \Delta K_i r_i = -(L^T K)_i r_i,
\]

and the combinatorial Calabi flow in hyperbolic background geometry

\[
\frac{dr_i}{dt} = \Delta K_i \sinh r_i = -(L^T K)_i \sinh r_i,
\]

where \(\Delta K_i = -\sum_{j \sim i} L_{ji} K_j, L_{ij} = \frac{\partial K_i}{\partial r_j} r_j\) in Euclidean background geometry, and \(L_{ij} = \frac{\partial K_i}{\partial r_j} \sinh r_j\) in hyperbolic background geometry.

Similar to the combinatorial Ricci flows, H. Ge [13], Ge-Xu [19] and Ge-Hua [14] proved the combinatorial Calabi flows have the long time existence and converge exponentially fast to Thurston’s circle packing on surfaces if and only if there exists a circle packing
metric of constant (zero resp.) curvature in Euclidean (hyperbolic resp.) background geometry.

In this paper, we generalize above results on the combinatorial Calabi flow for \( p = 2 \) to any \( p > 1 \). The difficulties for the generalizations mainly come from the fact that the \( p \)-th flow equations are nonlinear when \( p \neq 2 \). First for any \( p > 1 \) we introduce the combinatorial \( p \)-th Calabi flow.

**Definition 1.3.** The combinatorial \( p \)-th Calabi flow in Euclidean background geometry is defined as

\[
\frac{dr_i}{dt} = \Delta_p K_i r_i, \tag{1.8}
\]

where \( \Delta_p \) is the discrete Laplace operator defined as

\[
\Delta_p f_i = \sum_{j \sim i} B_{ij} |f_j - f_i|^{p-2} (f_j - f_i), \forall f \in \mathbb{R}^N \tag{1.9}
\]

with

\[
B_{ij} = \frac{\partial (\theta_{ijk} + \theta_{ikj})}{\partial r_j} r_j.
\]

Here \( \theta_{ijk} \) is the inner angle at the vertex \( v_i \) in a triangle \( \triangle ijk \in F \).

The combinatorial \( p \)-th Calabi flow in hyperbolic background geometry is defined as

\[
\frac{dr_i}{dt} = (\Delta_p K_i - A_i K_i) \sinh r_i, \tag{1.10}
\]

where \( A_i \) is defined as

\[
A_i = \sinh r_i \frac{\partial}{\partial r_i} \left( \sum_{\{ijk\} \in F} \text{Area}(\triangle v_iv_jv_k) \right).
\]

**Remark 1.** When \( p = 2 \), by Proposition 2.2 and (2.6) (see Section 2 for details), the combinatorial \( p \)-th Calabi flows (1.8) and (1.10) are exactly the combinatorial Calabi flows (1.6) and (1.7) introduced by H. Ge [9].

Now we state our main results in this paper as follows.

**Theorem 1.4.** Suppose \((T, \Phi)\) is a weighted generalized triangulation of a closed connected surface \( X \). Given any initial circle packing metric \( r(0) \in \mathbb{R}^N_+ \) satisfying \( \prod_{i=1}^N r_i(0) = 1 \), the solution to the combinatorial \( p \)-th Calabi flow (1.8) in Euclidean background geometry exists for \( t \in [0, +\infty) \). Additionally, \( r(t) \) converges if and only if there exists a constant curvature circle packing metric \( r_{av} \).
Theorem 1.5. Suppose \((T, \Phi)\) is a weighted generalized triangulation of a closed connected surface \(X\) of negative Euler characteristic. Given any initial circle packing metric \(r(0) \in \mathbb{R}_0^N\), the solution to the combinatorial \(p\)-th Calabi flow \(\text{(1.10)}\) in hyperbolic background geometry exists for \(t \in [0, +\infty)\). Additionally, \(r(t)\) converges if and only if there exists a zero curvature circle packing metric \(r_{ze}\).

Theorem 1.4 generalizes H. Ge’s results \([13]\) on the combinatorial Calabi flow in Euclidean background geometry. Theorem 1.5 generalizes the results of Ge-Xu \([19]\) and Ge-Hua \([14]\) on the combinatorial Calabi flow in hyperbolic background geometry. Especially, different from the the combinatorial Calabi flow \((p = 2)\), our results show that the combinatorial \(p\)-th Calabi flow does not have the exponential convergence for \(p > 1, p \neq 2\).

The key point of the proof of Theorem 1.4 is that we succeed showing the compactness of the solution \(\{r(t)\}\) to the combinatorial \(p\)-th Calabi flow \(\text{(1.8)}\). More precisely, in the case of Euclidean background geometry, similar to the combinatorial Ricci flow and the combinatorial Calabi flow, we show that the solution \(\{r(t)\}\) to the combinatorial \(p\)-th Calabi flow is always in a hypersurface. When restricted to the hypersurface, the combinatorial Ricci potential (see Section 2 for the definition) first introduced by de Verdiere \([7]\) satisfies the Palais-Smale condition and further is proper, which suffices to establish the the compactness of the solution \(\{r(t)\}\) to the combinatorial \(p\)-th Calabi flow. In the case of hyperbolic background geometry, we also show the long time existence and the uniform convergence. The difficulty of the proof of Theorem 1.5 comes from the long time existence in this case. By generalizing the work of Ge-Xu \([19]\) and Ge-Hua \([14]\), we can show that the boundness of the solution \(\{r(t)\}\) to the combinatorial \(p\)-th Calabi flow after getting rid of the additional assumption in \([19]\) on curvature.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish a uniform estimate which plays an important role in showing the long time existence of the solution to the combinatorial \(p\)-th Calabi flow in hyperbolic background geometry. Then in Section 4, we prove the long time existence parts of both Theorem 1.4 and Theorem 1.5. In Section 5, we prove the long time convergence parts of both Theorem 1.4 and Theorem 1.5. In section 6, we consider other combinatorial \(p\)-th curvature flows and discuss some interesting unsolved questions for future study.

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2 Preliminaries

In this section, we give necessary preliminaries.

2.1 Some basic propositions

Any circle packing metric $r$ determines an intrinsic metric structure on fixed $(X, T, \Phi)$ by Euclidean cosine law. The lengths $l_{ij}$, angles $\theta_{ik}^j$ and curvatures $K_i$ are elementary functions of $r = (r_1, ..., r_N)^T$.

Let the background geometry $K^2$ be one of the three geometries, $E^2$ (Euclidean), $H^2$ (hyperbolic), and $S^2$ (spherical), i.e., a simply connected surface with a complete Riemannian metric of curvature equal to 0, 1, and -1. For convenience, we choose coordinate transformations according to the background geometry. To be more precise, we choose $u_i = \ln r_i$ when $K^2 = E^2$, $u_i = \ln \tanh(r_i/2)$ when $K^2 = H^2$ and $u_i = \ln \tan(r_i/2)$ when $K^2 = S^2$.

Denote $j \sim i$ if the vertices $i$ and $j$ are adjacent. For any vertex $i$ and any edge $j \sim i$, in the case of $E^2$, set

$$B_{ij} = \frac{\partial(\theta_{ik}^j + \theta_{lj}^i)}{\partial r_j} r_j,$$

(2.1)

In the case of $H^2$, set

$$B_{ij} = \frac{\partial(\theta_{ik}^j + \theta_{lj}^i)}{\partial r_j} \sinh r_j,$$

(2.2)

$$A_i = \sinh r_i \frac{\partial}{\partial r_i} \left( \sum_{\{ijk\} \in F} \text{Area}(\triangle v_i v_j v_k) \right),$$

(2.3)

Note $B_{ij} = B_{ji}$ since $\frac{\partial \theta_{ik}^j}{\partial r_j} r_j = \frac{\partial \theta_{ik}^j}{\partial r_i} r_i$ (see Lemma 2.3 in [6]). We have the following estimate for $B_{ij}$ (see Lemma 2.2 in [6] or Proposition 3.1 in [13]).

**Proposition 2.1.** When $K^2 = H^2$, then for any $1 \leq i, j \leq N$ and $i \sim j$, we have

$$B_{ij} > 0.$$

(2.4)

When $K^2 = E^2$, then for any $1 \leq i, j \leq N$ and $i \sim j$, we have

$$0 < B_{ij} < 2\sqrt{3}.$$

(2.5)
Set $A = \text{diag}\{A_1, \cdots, A_N\}$ and $L_B = ((L_B)_{ij})_{N \times N}$, where

$$(L_B)_{ij} = \begin{cases} 
\sum_{k \sim i} B_{ik}, & j = i, \\
-B_{ij}, & j \sim i, \\
0, & \text{else}.
\end{cases}$$

(2.6)

Recall $L = (L_{ij}) = (\frac{\partial K}{\partial u_j})$, then we have the following proposition (see Theorem 3.1 in [19]).

**Proposition 2.2.** In the case of $E^2$, we have $L = L_B$. In the case of $H^2$, we have $L = A + L_B$. Furthermore, $L_B$ is semi-positive definite, and $A$ is positive definite.

As a corollary, we have the following proposition (see Proposition 3.3 in [13]).

**Proposition 2.3.** If $K^2 = H^2$, then $L$ is positive definite. If $K^2 = E^2$, then $L$ is a semi-positive definite $N \times N$ matrix, whose rank is $N - 1$. Moreover, the null space of $L$ is $\text{Ker}(L) = \{t(1, \ldots, 1)^T | t \in \mathbb{R}\}$.

### 2.2 Combinatorial Ricci Potential

For $K^2 = E^2$ or $H^2$, we have $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ by $\frac{\partial \theta^i_k}{\partial r_j} r_j = \frac{\partial \theta^j_k}{\partial r_i} r_i$ (see Lemma 2.3 in [6]). Thus the smooth differential 1-form $\sum_{i=1}^N (K_i - k_{av}) du_i$ is closed, so we can consider the combinatorial Ricci potential.

**Definition 2.4.** When $K^2 = E^2$, the following function

$$F(u) \triangleq \int_a^u \sum_{i=1}^N (K_i - k_{av}) du_i, \; u \in \mathbb{R}^N.$$  

(2.7)

is called *combinatorial Ricci potential*, where $a$ is any point in $\mathbb{R}^N$. In the case of $H^2$ the *combinatorial Ricci potential* is defined as

$$F(u) \triangleq \int_a^u \sum_{i=1}^N K_i du_i, \; u \in \mathbb{R}^N_{<0}.$$  

(2.8)

where $a$ is any point in $\mathbb{R}^N_{<0}$.

These integrals were first introduced by de Verdiere [7], who also first proved the convexity of $F$ in the zero weight case. Chow-Luo proved the following convex theorem (see Proposition 3.9 in [6]).
Proposition 2.5. When $K^2 = H^2$, then the combinatorial Ricci potential \((2.8)\) $F(u) : \mathbb{R}^N_{>0} \to \mathbb{R}$ is strictly convex. When $K^2 = E^2$, then \((2.7)\) $F(u) : \mathbb{R}^N \to \mathbb{R}$ is convex. Furthermore, it satisfies $F(u + c(1, 1, \cdots, 1)^T) = F(u)$ for any $a \in \mathbb{R}^N$ and $F$ is strictly convex when restricted in the hyperplane $U_0 = \{(u_1, \cdots, u_N)^T \mid \sum_{i=1}^{N} u_i = 0\}$.

As a consequence of the above proposition, we have the following rigidity theorem (see Corollary 3.11 in \[6\]).

Proposition 2.6. Let $\Pi : \mathbb{R}^N_{>0} \to \mathbb{R}^N$ be the map sending a metric $r_1, \cdots, r_N)^T$ to the corresponding curvature $(K_1, \cdots, K_N)^T$ where $K^2 = H^2$ or $E^2$. Then (a) In the case of $H^2$, $\Pi$ is injective, i.e., the metric is determined by its curvature. (b) In the case of $E^2$, $\Pi$ restricted to the hypersurface $P_1 = \{(r_1, \cdots, r_N)^T \in \mathbb{R}^N_{>0} \mid \prod_{i=1}^{N} r_i = 1\}$ is injective, i.e., the metric is determined by its curvature up to a scalar multiplication.

To prove our main results, it is key to show that $F$ satisfies the Palais-Smale condition. In fact, Chow-Luo \[6\] has proved the following theorem

Proposition 2.7. If $K^2 = H^2$, then the combinatorial Ricci potential $F$ \((2.8)\) satisfies the Palais-Smale condition if the following Thurston’s combinatorial conditions hold:

(c1) for any three edges $e_1, e_2, e_3$ forming a null homotopic loop in $M$, if $\sum_{i=1}^{3} \Phi(e_i) \geq \pi$, then $e_1, e_2, e_3$ form the boundary of a triangle in $F$;

(c2) for any four edges $e_1, e_2, e_3, e_4$ forming a null homotopic loop in $M$, if $\sum_{i=1}^{4} \Phi(e_i) \geq 2\pi$, then $e_1, e_2, e_3, e_4$ form the boundary of the union of two adjacent triangles.

If $K^2 = E^2$, then the combinatorial Ricci potential $F$ \((2.7)\) restricted to the hyperplane $U_0 = \{(u_1, \cdots, u_N)^T \mid \sum_{i=1}^{N} u_i = 0\}$, i.e., $F|_{U_0}$ satisfies the Palais-Smale condition if for any proper subset $I \subset V$,

$$2\pi |I| \chi(X)/N > - \sum_{(e, v) \in \text{Lk}(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I),$$

where $F_I$ is the set of all cells in $T$ whose vertices are in $I$, and $\text{Lk}(I)$ is the link of $I$ which is the set of pairs $(e, v)$ of an edge $e$ and a vertex $v$ so that (1) the end points of $e$ are not in $I$ and (2) the vertex $v$ is in $I$ and (3) $e, v$ form a triangle.

Remark 2. By the work of Thurston \[21\] on circle packing, conditions (c1) and (c2) are equivalent to the existence of the zero curvature circle packing metric when $K^2 = H^2$, and condition \((2.9)\) is equivalent to the existence of the constant curvature circle packing metric $K^2 = E^2$.

Furthermore, we can show the properness of $F$ under conditions above. In fact, H. Ge and Ge-Xu have proved the following theorem (see Theorem B.2 in \[13\] and Lemma B.1 in \[19\]).
Proposition 2.8. Making the same assumptions as in Proposition [2.7] then for the combinatorial Ricci potential \( F \) (2.7) and (2.8)

\[
\lim_{\|u\| \to +\infty} F(u) = +\infty.
\] (2.10)

Moreover, the combinatorial Ricci potential \( F \) (2.8) is proper in the case of \( H^2 \). The combinatorial Ricci potential \( F \) (2.7) is proper when restricted to the hyperplane \( U_0 \) in the case of \( E^2 \).

2.3 Combinatorial \( p \)-th Calabi flows

Recall the combinatorial \( p \)-th Calabi flow in the case of \( E^2 \) is

\[
\frac{dr_i}{dt} = \Delta_p K_i r_i,
\]

where

\[
\Delta_p K_i = \sum_{j \sim i} B_{ij} |K_j - K_i|^{p-2}(K_j - K_i),
\]

and the combinatorial \( p \)-th Calabi flow in the case of \( H^2 \) is

\[
\frac{dr_i}{dt} = (\Delta_p K_i - A_i K_i) \sinh r_i,
\]

where

\[
A_i = \sinh r_i \frac{\partial}{\partial r_i} \left( \sum_{\{ijk\} \in F} \text{Area}(\Delta v_i v_j v_k) \right)
\]

By coordinate transformations, more precisely, \( u_i = \ln r_i \) when \( K^2 = E^2 \), and \( u_i = \ln \tanh(r_i/2) \) when \( K^2 = H^2 \), then we can rewrite the combinatorial \( p \)-th Calabi flow as

\[
u'_i(t) = \Delta_p K_i,
\] (2.11)

in the case of \( E^2 \) and the combinatorial \( p \)-th Calabi flow

\[
u'_i(t) = L_p K_i = \Delta_p K_i - A_i K_i,
\] (2.12)

in the case of \( H^2 \).

We can also write the combinatorial \( p \)-th Calabi flow equations in a matrix form as

\[
u'(t) = \Delta_p K
\] (2.13)

in the case of \( E^2 \) and

\[
u'(t) = L_p K = (\Delta_p - A) K
\] (2.14)

in the case of \( H^2 \), where \( A = \text{diag}\{A_1, A_2, \ldots, A_N\} \). The matrix \( A \) is positive definite by Proposition [2.2].
3 A uniform estimate

In order to prove Theorem 1.5 we need the following uniform estimate which plays an important role in our arguments. First recall a lemma (see Lemma 13.7.3 in [21]) proved by Thurston as follows:

Lemma 3.1. Let \( \triangle v_i v_j v_k \) be a hyperbolic triangle which is patterned by three circles with fixed weights \( \Phi_{ij}, \Phi_{jk}, \Phi_{ik} \in [0, \pi/2] \) as intersection angles. Let \( \theta_{ij}^k \) be the inner angle at \( v_i \). Then \( \partial \theta_{ij}^k / \partial r_j > 0 \), \( \partial \theta_{ij}^k / \partial r_i < 0 \) and \( \partial (\theta_{ij}^k + \theta_{ik}^j + \theta_{jk}^i) / \partial r_i < 0 \).

Generalizing Lemma 3.2 in [14], we have the following uniform estimate which plays an important role in our arguments.

Lemma 3.2. For any \( \lambda > 1 \), there exists a constant \( C \), depending on \( \lambda \), such that if \( r_i \geq C \), then

\[
\frac{\partial}{\partial r_i} (\lambda \text{Area}(\triangle v_i v_j v_k) + \theta_{ij}^k) \geq 0. \tag{3.1}
\]

Proof. The proof is similar to the proof of Lemma 3.2 in [14]. Adopt the argument from the hyperbolic geometry. Suppose that the triangle \( \triangle v_i v_j v_k \) is embedded in the half hyperbolic plane \( \mathbb{H}^2 \), with \( v_j, v_k \) and the corresponding radii \( r_j, r_k \) fixed. Let \( \hat{v}_i \) be the new vertex with a larger radius \( \hat{r}_i > r_i \). Then \( \hat{l}_{v_i v_j} > l_{v_i v_j} \) and \( \hat{l}_{v_i v_k} > l_{v_i v_k} \) due to \( \hat{r}_i > r_i \).

Draw two triangles, \( \hat{\triangle} v_i v_j v_k \) and \( \hat{\triangle} v_i v_j v_k \) whose common edge \( v_j v_k \) is in the same half hyperbolic plane separated by the (extended) geodesic \( v_j v_k \). By Lemma 3.1 for fixed \( r_j \) and \( r_k \) the angles \( \theta_{ij}^k \) and \( \theta_{ik}^j \) are increasing which implies that the vertex \( v_i \) lies in the interior of the triangle \( \hat{\triangle} v_i v_j v_k \) (please refer to Figure 2. in [14]). Denote by \( \hat{\theta}_{ij}^k \), \( \hat{\theta}_{ik}^j \) and \( \hat{\theta}_{jk}^i \) three inner angles of the new triangle \( \hat{\triangle} v_i v_j v_k \) respectively. To prove (3.1), it is sufficient to show that for any \( \hat{r}_i > r_i \) which is sufficiently close to \( r_i \),

\[
\lambda \text{Area}(\hat{\triangle} v_i v_j v_k) - \lambda \text{Area}(\triangle v_i v_j v_k) + \hat{\theta}_{ij}^k - \theta_{ij}^k \geq 0. \tag{3.2}
\]

Set \( x = \hat{\theta}_{ij}^k - \theta_{ij}^k \) and \( \hat{\theta}_{jk}^i - \theta_{jk}^i \), then we have

\[
\lambda \text{Area}(\hat{\triangle} v_i v_j v_k) - \lambda \text{Area}(\triangle v_i v_j v_k) + \hat{\theta}_{ij}^k - \theta_{ij}^k
\]

\[
= (\lambda - 1) \text{Area}(\hat{\triangle} v_i v_j v_k) - (\lambda - 1) \text{Area}(\triangle v_i v_j v_k) - x - y
\]

\[
= [(\lambda - 1) \text{Area}(\hat{\triangle} v_i v_j v_k) - x] + [(\lambda - 1) \text{Area}(\triangle v_i v_j v_k) - y] \tag{3.3}
\]

Then it suffices to prove the following inequalities

\( (\lambda - 1) \text{Area}(\hat{\triangle} v_i v_j v_k) \geq x \), and \( (\lambda - 1) \text{Area}(\triangle v_i v_j v_k) \geq y \).

Without loss of generality, by the symmetry, we show that

\( (\lambda - 1) \text{Area}(\hat{\triangle} v_i v_j v_k) \geq x. \tag{3.4} \)
Let $s$ be the point which attains the minimum distance from the vertex $v_i$ to a point on the geodesic $\hat{v}_i v_j$, then $s$ is in the interior of the geodesic $\hat{v}_i v_j$ (please refer to Figure 2. in [14]) since $l_{\hat{v}_i v_j} > l_{v_i v_j}$. Suppose that $\hat{r}_i$ is sufficiently close to $r_i$ such that $l_{\hat{v}_i v_j} \leq 1$. By the hyperbolic cosine law,

$$\cos x = \frac{\cosh l_{v_i v_j} - \cosh l_{\hat{v}_i v_j}}{\sinh l_{v_i v_j} \sinh l_{\hat{v}_i v_j}} \to 1,$$

uniformly as $r_i \to \infty$. Hence there is a universal constant $C_1$ such that if $r_i \geq C_1$, then

$$x \leq \frac{\pi(\lambda - 1)}{4\lambda}. \quad (3.5)$$

Set $\beta = \angle sv_i v_j$, then there are two possibilities.

If $\beta < \frac{\pi}{4}$, then $\beta + x + \frac{x}{\lambda - 1} < \frac{\pi}{2}$ by (3.5). Considering the Gauss-Bonnet theorem in hyperbolic background geometry, i.e., $\beta + x + \frac{\pi}{2} = \pi - \text{Area}(\triangle sv_i v_j)$, we have $\text{Area}(\triangle \hat{v}_i v_i v_j) \geq \text{Area}(\triangle sv_i v_j) > \frac{\pi}{\lambda - 1}$, which leads to (3.4). If $\beta \geq \frac{\pi}{4}$, then

$$\frac{\sinh l_{sv_j}}{\sinh l_{v_i v_j}} = \sin \beta \geq \frac{\sqrt{2}}{2},$$

For the hyperbolic right triangle $\triangle sv_i v_j$, using the hyperbolic cosine law, we have

$$\cos \beta = \sin x \cosh l_{sv_j}, \quad \cos x = \tanh l_{sv_j} / \tanh l_{v_i v_j}, \quad \sin \beta = \sinh l_{sv_j} / \sinh l_{v_i v_j}$$

This yields

$$\sinh(\text{Area}(\triangle sv_i v_j)) = \cos(x + \beta) = \cos x \cos \beta - \sin x \sin \beta$$

$$= \frac{\tanh l_{sv_j} \sin x \cosh l_{sv_j} - \sin x \sinh l_{sv_j}}{\tanh l_{v_i v_j} \sinh l_{sv_j}}$$

$$= \sin x (\cosh l_{v_i v_j} - 1) \frac{\sinh l_{sv_j}}{\sinh l_{v_i v_j}}$$

$$\geq \sin x \frac{\cosh r_i - 1}{\sqrt{2}}. \quad (3.6)$$

When $\lambda \geq 2$, by (3.6) there exists a universal constant $C_2$ such that if $r_i \geq C_2$,

$$\sin(\text{Area}(\triangle sv_i v_j)) \geq \sin x.$$

Noting that both $x$ and $\text{Area}(\triangle sv_i v_j)$ are in $(0, \frac{\pi}{2})$, we have $\text{Area}(\triangle sv_i v_j) \geq x$, which yields $(\lambda - 1)\text{Area}(\triangle sv_i v_j) \geq \text{Area}(\triangle sv_i v_j) \geq x$.

When $1 < \lambda < 2$, by (3.6), we have

$$\sin((\lambda - 1)\text{Area}(\triangle sv_i v_j)) \geq (\lambda - 1) \sin(\text{Area}(\triangle sv_i v_j)) \geq \sin x \frac{\cosh r_i - 1}{\sqrt{2}} (\lambda - 1).$$
Note that both \( x \) and \( (\lambda - 1)\text{Area}(\triangle sv_i v_j) \) are in \((0, \frac{\pi}{2})\), thus there exists a a universal constant \( C_3 \), depending on \( \lambda \), such that if \( r_i \geq C_3 \),

\[
(\lambda - 1)\text{Area}(\triangle sv_i v_j) \geq x.
\]

\( \square \)

**Proposition 3.3.** For any \( \epsilon > 0 \), there exists a universal number \( C > 0 \), such that if \( r_i > C \),

\[
\epsilon A_i \geq \sum_{j \sim i} B_{ij}.
\]

**Proof.** By (3.1), we have

\[
\epsilon A_i \geq \sum_{j \sim i} B_{ij}.
\]

\[
= \epsilon \sum_{\{ijk\} \in F} \frac{\partial \text{Area}(\triangle v_i v_j v_k)}{\partial r_i} \sinh r_i - \sum_{j \sim i} \left( \frac{\partial \theta^i_{jk}}{\partial r_j} \sinh r_j + \frac{\partial \theta^i_{jl}}{\partial r_j} \sinh r_j \right)
\]

\[
= \epsilon \sum_{\{ijk\} \in F} \frac{\partial \text{Area}(\triangle v_i v_j v_k)}{\partial r_i} \sinh r_i - \sum_{\{ijk\} \in F} \left( \frac{\partial \theta^i_{jk}}{\partial r_j} \sinh r_j + \frac{\partial \theta^i_{jl}}{\partial r_j} \sinh r_j \right)
\]

\[
= \sum_{\{ijk\} \in F} \frac{\partial \left( \epsilon \text{Area}(\triangle v_i v_j v_k) - \theta^i_{jk} - \theta^i_{jl} \right)}{\partial r_i} \sinh r_i
\]

\[
= \sum_{\{ijk\} \in F} \frac{\partial \left( (\epsilon + 1)\text{Area}(\triangle v_i v_j v_k) + \theta^i_{jk} \right)}{\partial r_i} \sinh r_i \geq 0.
\]

\( \square \)

### 4 The long time existence

In this section we prove the long time existence parts of both Theorem 1.4 and Theorem 1.5. In fact, we get the following theorem.

**Theorem 4.1.** For any initial circle packing metric \( r(0) \in \mathbb{R}^N_{>0} \), both the solutions to the combinatorial \( p \)-th Calabi flows (1.8) in \( E^2 \) and (1.10) in \( H^2 \) exist for all time \( t \in [0, +\infty) \).

Before showing the long time existence of combinatorial \( p \)-th Calabi flow, we generalize Ge-Hua’s a theorem ( Theorem 3.4 in [14] ) and give the following result, which show that the solution \( r(t) \) to the combinatorial \( p \)-th Calabi flow (1.10) in \( H^2 \) remains bounded away from 0 and bounded from above as well, as long as time remains bounded.
**Theorem 4.2.** Let \((M, T)\) be a triangulated compact surface with an edge weight \(\Phi : E \to [0, \pi/2]\). Let \(r(t)\) be the unique solution to the combinatorial \(p\)-th Calabi flow (1.10) in hyperbolic background geometry on the maximal time interval \([0, T)\). Then all \(r_i(t)\) are uniformly bounded above on \([0, T)\).

**Proof.** The result is arrived at by contradiction. The proof is similar to the proof of Theorem 3.4 in [14]. Assume that it is not true, then there exists at least one vertex \(i \in V\), such that

\[
\lim_{t \to T} r_i(t) = +\infty. \tag{4.1}
\]

For this vertex \(i\), using Lemma 3.5 in [6], we can choose a sufficient large positive number \(l\) such that \(r_i > l\) and the inner angle \(\theta_i\) is smaller than \(\frac{\pi}{2d_i}\), where \(d_i\) is the degree of the vertex \(i\). Then we have \(K_i > \pi\).

Set \(L = \max\{l, C, r_i(0) + 1\}\), where \(C\) is given in Proposition 3.3. We claim that if \(r_i(t) > L\), then

\[
\frac{dr_i}{dt} < 0 \tag{4.2}
\]

for any \(t \in (0, T)\). To see this, first by the definition of the combinatorial curvature, i.e., \(K_i = 2\pi - \sum_{j \in F} \theta_{ij}\), we have \(K_i < 2\pi\).

Combining above inequalities on curvature, we have \(K_j - K_i < \pi\) for \(j \sim i\). Note that if \(a \leq b\), then \(|a|^{p-2}a \leq |b|^{p-2}b\). Consequently, by the \(p\)-th combinatorial Calabi flow (1.10) and Proposition 3.3 for \(\epsilon = \frac{1}{\pi^{p-2}}\), we have

\[
\frac{1}{\sinh r_i} \frac{dr_i}{dt} = \sum_{j \sim i} B_{ij} |K_j - K_i|^{p-2}(K_j - K_i) - A_i K_i \\
\leq \sum_{j \sim i} B_{ij} \pi^{p-1} - A_i \pi \\
= \pi^{p-1} \left( \sum_{j \sim i} B_{ij} - \frac{1}{\pi^{p-2}} A_i \right) \\
\leq 0
\]

This proves the claim.

We may choose \(t_0 \in (0, T)\) such that \(r_i(t_0) > c\) by (4.2). Suppose that \(t_0 \in [0, t_0]\) attains the maximum of \(r_i(t)\) in \([0, t_0]\). Then by the definition of \(L\), \(t_1 > 0\) Thus

\[
\frac{dr_i}{dt}(t_1) \geq 0,
\]

which contradicts to (4.2). This completes the proof. \(\square\)
Now we can prove Theorem 4.1 as follows.

**Proof.** In the case of $E^2$, the proof is similar to the proof of Theorem 3.5 in [13]. Let $d_i$ denote the degree at vertex $v_i$, which is the number of edges adjacent to $v_i$. Set $d = \max\{d_1, \ldots, d_N\}$, then $(2 - d)\pi < K_i < 2\pi$, hence

$$|K_j - K_i| < d\pi, \forall j \sim i.$$

By the estimation of $B_{ij}$ in Proposition 2.1, all $|\Delta_pK_i|$ are uniformly bounded by a positive constant $c = 2\sqrt{3} \cdot N \cdot (d\pi)^{p-1}$, which depends only on the triangulation. Then we have

$$c_0e^{-ct} \leq r_i(t) \leq c_0e^{ct}$$

where $c_0 = c(r(0))$, which implies that the combinatorial $p$-th Calabi flow (1.8) has a solution for all time $t \in [0, +\infty)$ for any $r(0) \in \mathbb{R}_{>0}^N$. In the case of $H^2$, Theorem 4.2 show that as long as time remains bounded, then the solution $r_i(t)$ to the $p$-th combinatorial Calabi flow (1.10) is bounded away from 0 and bounded from above. This implies that the solution $r_i(t)$ to the $p$-th combinatorial Calabi flow (1.10) exists for all time $t \geq 0$. $\square$

5 The long time convergence

In this section we prove the convergence parts of both Theorem 1.4 and Theorem 1.5. More precisely, we show the long time convergence of both the solutions to the combinatorial $p$-th Calabi flows (1.8) in $E^2$ and (1.10) in $H^2$. In fact, we have

**Theorem 5.1.** Suppose $\{r(t)|t \in [0, +\infty)\}$ is a long time solution to the combinatorial $p$-th Calabi flow (1.10) (1.8 resp.) in hyperbolic (Euclidean resp.) background geometry, then $\{r(t)|t \in [0, +\infty)\}$ converges if and only if there exists a zero curvature circle packing metric $r_{ze}$. Suppose $\{r(t)|t \in [0, +\infty)\}$ is a long time solution to the combinatorial $p$-th Calabi flow (1.8) in Euclidean background geometry with any initial value $r(0)$ satisfying $\prod_{i=1}^N r_i(0) = 1$, then $\{r(t)|t \in [0, +\infty)\}$ converges if and only if there exists a constant curvature circle packing metric $r_{av}$.

First we give the following compactness theorem which is key to prove Theorem 5.1.

**Proposition 5.2.** Suppose $u(t)$ for $t \in [0, +\infty)$ is a solution to the combinatorial $p$-th Calabi flow (1.10) (1.8 resp.) in the hyperbolic (Euclidean resp.) background geometry so that the set $\{u(t)|t \in [0, +\infty)\}$ lies in a compact region in $\mathbb{R}^N$. Then $u(t)$ converges a zero (constant resp.) curvature circle packing metric $u_{ze}$ ($u_{av}$ resp.).
5.1 Some properties on the combinatorial $p$-Laplace operator

In this subsection we show some properties with respect to the combinatorial $p$-Laplace operator $\Delta_p$ (1.9).

For any smooth closed manifold $(M, g)$ with Riemannian metric $g$, considering the smooth $p$-Laplace operator $\Delta_p f = \text{div}(|\nabla f|^{p-2} \nabla f)$, where $f$ is an arbitrary smooth function on $M$, and div denotes the divergence. By the divergence theorem, we have $\int_M \Delta_p fd\mu = 0$. For a combinatorial surface $(X, T, \Phi)$ with a circle packing metric $r$, similarly we have

**Lemma 5.3.** For any $f \in C(V)$, we have

$$\sum_{i=1}^{N} \Delta_p f_i = 0.$$

Specially, for the curvature function $K_i$ we have

$$\sum_{i=1}^{N} \Delta_p K_i = 0.$$

By above lemma and the combinatorial $p$-th Calabi flow equation 2.11 we generalize Ge’s Proposition 3.4 in [13] and obtain the following important invariance property.

**Proposition 5.4.** As long as the solution $u(t)$ (or equivalently $r(t)$) to the combinatorial $p$-th Calabi flow 2.11 (or equivalently 1.8) in $E^2$ exists, both $\prod_{i=1}^{N} r_i(t) \equiv \prod_{i=1}^{N} r_i(0)$ and $\sum_{i=1}^{N} u_i(t) \equiv \sum_{i=1}^{N} u_i(0)$ are constants.

**Lemma 5.5.** For the $p$-Laplace operator $\Delta_p$ and For any $f \in C(V)$, we have the following identity

$$\sum_{i=1}^{N} f_i \Delta_p f_i = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j \sim i} B_{ij} |f_j - f_i|^p.$$

Specially, for the curvature function $K_i$ we have

$$\sum_{i=1}^{N} K_i \Delta_p K_i = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j \sim i} B_{ij} |K_j - K_i|^p.$$

**Proof.** By the definition (1.9) of $\Delta_p$ and the symmetry of $B_{ij}$, we have

$$\sum_{i=1}^{N} f_i \Delta_p f_i = \sum_{i=1}^{N} \sum_{j \sim i} B_{ij} |f_j - f_i|^{p-2} (f_j - f_i) f_i.$$
This completes the proof. \(\blacksquare\)

For any \(c > 0\), denote \(\mathcal{P}_c = \{r = (r_1, ..., r_N)^T \in \mathbb{R}^N_+ | \prod_{i=1}^N r_i = c\}\) and denote \(\mathcal{U}_a = \{u = (u_1, ..., u_N)^T | \sum_{i=1}^N u_i = a\}\). Then \(\mathcal{P}_c = \text{Exp}(\mathcal{U}_a)\), where \(a = \text{ln} c\). By Proposition 5.4 \(\{r(t)\} \subset \mathcal{P}_c\) and \(\{u(t)\} \subset \mathcal{U}_a\) along the combinatorial \(p\)-th Calabi flow in \(E^2\), where \(c = \prod_{i=1}^N r_i(0)\) and \(a = \sum_{i=1}^N u_i(0)\).

### 5.2 Proof of Proposition 5.2

Now we give the proof of Proposition 5.2.

**Proof.** In the case of \(H^2\), suppose \(u(t)\) for \(t \in [0, +\infty)\) is a solution to the combinatorial \(p\)-th Calabi flow (2.12). By the assumption, we know \(\{u(t) | t \in [0, +\infty)\} \subset M\), where \(M\) is a compact subset in \(\mathbb{R}^N\). Consider the combinatorial Ricci potential

\[
F(u) = \int_{u_{ze}}^u \sum_{i=1}^N K_i du_i, \quad u \in \mathbb{R}^N_{<0},
\]

where \(u_{ze} = \ln r_{ze}\).

Set \(\varphi(t) = F(u(t))\), then we carry out the proof in two steps.

**Step 1.** The solution \(u(t)\) to the combinatorial \(p\)-th Calabi flow (2.12) sequently converges to a zero curvature circle packing metric \(u_{ze}\).

By Proposition 2.6 the gradient map of the combinatorial Ricci potential

\[
\nabla F : \mathbb{R}^N_{<0} \rightarrow \mathbb{R}^N
\]

\[u \mapsto K\]

is injective, and \(u = u_{ze}\) is the only critical point. Furthermore, the Hessian of \(F\) is exactly \(L = (L_{ij})\), which is positive definite by Proposition 2.3. Therefore, \(F(u) \geq F(u_{ze}) = 0\).

Note that \(B_{ij} = B_{ji}\), and by Lemma 5.5 we have

\[
K^T \Delta_p K = -\frac{1}{2} \sum_{i=1}^N \sum_{j \sim i} B_{ij} |K_j - K_i|^p \leq 0.
\]
Moreover, $A = \text{diag}\{A_1, A_2, \ldots, A_N\}$ is positive definite. Using the $p$-th combinatorial Calabi flow equation (2.12), we can compute the derivative of $\varphi(t)$ as follows

$$\varphi'(t) = \sum_i K_i L_p K_i = K^T L_p K = K^T (\Delta_p - A) K \leq 0,$$

which implies that $\varphi(t)$ is decreasing and bounded, i.e., $0 \leq F(u(t)) = \varphi(t) \leq \varphi(0) = F(u(0))$. Hence $\varphi(t)$ converges, i.e., when $t \to +\infty$, $\varphi(t) \to \varphi(+\infty)$. By the mean value theorem, there exists $t_n \in (n, n+1)$ satisfying $\varphi(n+1) - \varphi(n) = \varphi'(t_n)$, thus $\varphi'(t_n) \to 0$ when $n \to +\infty$.

By the assumption, we have $u(t_n) \subset M$, which means that there exists a convergent subsequence $t_{n_k} \to +\infty$ s.t. $\lim_{k \to +\infty} u(t_{n_k}) = u^*$. Consider $K_i, B_{ij}$ as smooth functions of $u$, which means there exist $K_1^*, B_{1j}^*$ s.t. $\lim_{k \to +\infty} K_i(t_{n_k}) = K_i^*$ and $\lim_{k \to +\infty} B_{ij}(t_{n_k}) = B_{ij}^*$. By equalities above $\varphi'(t) = K^T (\Delta_p - A) K$ and

$$K^T \Delta_p K = -\frac{1}{2} \sum_{i=1}^N \sum_{j \sim i} B_{ij} |K_j - K_i|^p \leq 0,$$

we have

$$0 \geq -\frac{1}{2} \sum_{i=1}^N \sum_{j \sim i} B_{ij}^* |K_j^* - K_i^*|^p = (K^*)^T A^* K^* \geq 0,$$

which implies $(K^*)^T A^* K^* = 0$. Note that $A = \text{diag}\{A_1, A_2, \ldots, A_N\}$ is always positive definite on the compact subset $M$, hence $A^*$ is positive definite, which implies that $K_i^* = 0$, for $i = 1, \ldots, N$, hence $u^*$ is exactly a zero curvature circle packing metric $u_{ze}$, i.e.,

$$\lim_{k \to +\infty} u(t_{n_k}) = u_{ze}.$$

**Step 2.** $u(t) \to u_{ze}, t \to +\infty$.

We claim

$$\lim_{t \to +\infty} \varphi(t) = 0. \quad (5.1)$$

Recall $\varphi(t)$ converges, i.e., when $t \to +\infty$, $\varphi(t) \to \varphi(+\infty)$. By $\lim_{k \to +\infty} u(t_{n_k}) = u_{ze}$, we have

$$\varphi(t_{n_k}) = F(u(t_{n_k})) \to F(u_{ze}) = 0, k \to +\infty,$$

therefore $\varphi(+\infty) = 0$, i.e.,

$$\lim_{t \to +\infty} \varphi(t) = 0.$$
Now we prove
\[ \lim_{t \to +\infty} u(t) = u_{ze} \]
in fact by contradiction. If not, then there exists \( \delta > 0 \) and \( \xi_n \uparrow +\infty \) such that
\[ |u(\xi_n) - u_{av}| > \delta, \]
which means \( \{u(\xi_n)\} \subset \mathbb{R}^N \setminus B(u_{av}, \delta) \). Recall \( \{u(t) \mid t \in [0, +\infty)\} \subset M \) where \( M \) is a compact subset of \( \mathbb{R}^N \). Hence \( \{u(\xi_n)\} \subset M \cap (\mathbb{R}^N \setminus B(u_{av}, \delta)) \). It is obvious that continuous differentiable function \( F(u) \) has a positive lower bound in the compact set \( M \cap (\mathbb{R}^N \setminus B(u_{av}, \delta)) \), i.e.,
\[ F(u) \geq c > 0, \forall u \in M \cap (\mathbb{R}^N \setminus B(u_{av}, \delta)). \]
Hence we have
\[ \varphi(\xi_n) = F(u(\xi_n)) \geq c > 0, \]
which causes a contradiction considering \( \varphi(\xi_n) \to 0, n \to +\infty \) by (5.1). Therefore
\[ \lim_{t \to +\infty} u(t) = u_{ze}. \]

In the case of \( E^2 \), consider the combinatorial Ricci potential
\[ F(u) = \int_{u_{av}}^{u} \sum_{i=1}^{N} (K_i - k_{av}) du_i, \quad u \in \mathbb{R}^N, \]
where \( u_{av} = \ln r_{av} \).

By Proposition 2.6, the gradient map
\[ \nabla F \mid_{u_{av}} : U_a \to \mathbb{R}^N \]
\[ u \mapsto K - K_{av} \]
is injective, where \( a = \sum_{i=1}^{N} u_i(0) \), and \( u = u_{av} \) is the only critical point. Furthermore, the Hessian of \( F \) is exactly \( L = (L_{ij}) \), which is semi-positive definite by Proposition 2.3. Therefore, \( F(u) \geq F(u_{av}) = 0 \).

Set \( \varphi(t) = F(u(t)) \), then we compute the derivative of \( \varphi(t) = F(u(t)) \) as follows
\[ \varphi'(t) = \sum_{i=1}^{N} (K_i - k_{av}) \Delta_p K_i \]
\[ = \sum_{i=1}^{N} K_i \Delta_p K_i \]

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\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \sim i} B_{ij} |K_j - K_i|^p \]
\[ \leq 0, \]

where the second equality is due to Lemma 5.3, the third equality is due to Lemma 5.5 and the last inequality is due to Proposition 2.1.

This implies that \( \varphi(t) \) is decreasing and bounded, i.e., \( 0 \leq \varphi(t) \leq \varphi(0) = F(u(0)) \). Then by the same arguments above in the case of \( H^2 \), we can show that \( u(t) \) converges a constant curvature circle packing metric \( u_{av} \). \( \square \)

### 5.3 Proof of Theorem 5.1

Finally, we give the proof of Theorem 5.1.

**Proof.** First we show that there exists a zero curvature circle packing metric \( r_{ze} \) if the long time solution \( \{r(t) | t \in [0, +\infty)\} \) or equivalently \( \{u(t) | t \in [0, +\infty)\} \) to the combinatorial \( p \)-th Calabi flow (2.12) in \( H^2 \) converges, i.e.,

\[ u^* = \lim_{t \to +\infty} u(t), u_i^* = \lim_{t \to +\infty} u_i(t) \]

exists. Hence by the mean value theorem, there exists \( t_n \in (n, n + 1) \) satisfying

\[ u_i(n + 1) - u_i(n) = u'_i(t_n). \]

It is obvious that the limit of the left of above equation is 0 when \( n \to +\infty \), so by the combinatorial \( p \)-th Calabi flow equation (2.12) we have

\[ L_p K_i |_{u=u^*} = (\Delta_p - A_i) K_i |_{u=u^*} = \lim_{n \to +\infty} (\Delta_p - A_i) K_i |_{u=u(t_n)} = \lim_{n \to +\infty} u'_i(t_n) = 0. \]

Therefore \( \sum_i K_i L_p K_i |_{u=u^*} = K^T (\Delta_p - A) K |_{u=u^*} = 0 \), by the same arguments in the proof above, we have

\[ \lim_{n \to +\infty} K_i(t_n) = K_i(u^*) = 0, i = 1, \cdots, N. \]

This means \( u^* \) is actually a zero curvature circle packing metric \( u_{ze} \). Using the same argument we can show that there exists a constant curvature circle packing metric \( r_{av} \) if the long time solution \( \{r(t) | t \in [0, +\infty)\} \) or equivalently \( \{u(t) | t \in [0, +\infty)\} \) to the combinatorial \( p \)-th Calabi flow (2.11) in \( E^2 \) converges.

As for the “if” part of the proof, we divide it into cases:

In the case of \( H^2 \), suppose \( \{r(t) | t \in [0, +\infty)\} \) or equivalently \( \{u(t) | t \in [0, +\infty)\} \) is a solution to the combinatorial \( p \)-th Calabi flow (2.11) in \( H^2 \). If there exists a zero curvature circle packing metric \( r_{ze} \) or equivalently \( u_{ze} \), consider the the combinatorial
Ricci potential $F(2.8)$, and set $\varphi(t) = F(u(t))$. We know that $\varphi(t)$ is decreasing and bounded, i.e., $0 \leq \varphi(t) \leq \varphi(0) = F(u(0))$ by the proof above. By Theorem 2.8 the combinatorial Ricci potential $F(2.8)$ is proper, thus $F^{-1}[0,F(u(0))]$ is a compact subset, which implies $\{u(t) \mid t \in [0, +\infty)\}$ lies in a compact region $F^{-1}[0,F(u(0))]$ in $\mathbb{R}^N$. By Proposition 5.2, $\{u(t) \mid t \in [0, +\infty)\}$ converges the zero curvature circle packing metric $u_{ze}$.

In the case of $E^2$, if there exists a constant curvature circle packing metric $r_{av}$ or equivalently $u_{av}$, and suppose $u(t)$ for $t \in [0, +\infty)$ is a solution to the combinatorial $p$-th Calabi flow (2.11) in $E^2$ with any initial value $\sum_{i=1}^{N} u_i(0) = 0$ (equivalently $\prod_{i=1}^{N} r_i(0) = 1$). Note by Proposition 5.3, $\{u(t) \mid t \in [0, +\infty)\} \subset U_a$ along the combinatorial $p$-th Calabi flow (2.11), where $a = \sum_{i=1}^{N} u_i(0) = 0$ in this case. Then consider the combinatorial Ricci potential

$$F(u) = \int_{u_{ze}}^{u} \sum_{i=1}^{N} (K_i - k_{av}) du_i, \quad u \in \mathbb{R}^N,$$

and set $\varphi(t) = F(u(t))$. By the proof above we know that $\varphi(t)$ is decreasing and bounded, i.e., $0 \leq \varphi(t) \leq \varphi(0) = F(u(0))$. Hence $\{u(t) \mid t \in [0, +\infty)\} \subset F^{-1}[0,\varphi(0)]$. By Proposition 2.8 $F|_{U_0}$ is proper, thus $F|_{U_0}^{-1}[0,\varphi(0)]$ is a compact subset of $U_0$, which implies $\{u(t) \mid t \in [0, +\infty)\}$ lies in a compact region $F^{-1}[0,F(u(0))]$ in $\mathbb{R}^N$. By Proposition 5.2, $\{u(t) \mid t \in [0, +\infty)\}$ converges the constant curvature circle packing metric $u_{av}$. □

6 Further discussions

Theorem 1.4 and Theorem 1.5 generalize the work of H. Ge, Ge-Xu and Ge-Hua from the combinatorial Calabi flow to the combinatorial $p$-th Calabi flow. Moreover, our results show that the combinatorial $p$-th Calabi flow does not have the exponential convergence in general $p > 1$. There are still several interesting questions related to our results.

6.1 Normalized combinatorial $p$-th Calabi flow

The normalized combinatorial $p$-th Calabi flow in Euclidean background geometry is defined as follows.

$$u_i'(t) = -\nabla_{u_i} E_p = p \sum_{j=1}^{N} L_{ij} |k_{av} - K_j|^{p-2}(k_{av} - K_j), \quad (6.1)$$

where $E_p(u)$ is the $p$-th Calabi energy, i.e., $E_p(u) = \sum_{i=1}^{N} |k_{av} - K_i|^p$.

It is obvious that (6.1) is precisely the negative gradient flow of the $p$-th Calabi energy $E_p$. We can similarly show the long time existence of (6.1). If there exists the compactness
of the closure of the solution subset \( \{ u(t) \mid t \in [0, +\infty) \} \), then we can show the long time convergence for the curvature flow \( \{ K(u(t)) \mid t \in [0, +\infty) \} \) when \( t \to +\infty \). To see this, by the compactness of the closure of the solution subset \( \{ u(t) \mid t \in [0, +\infty) \} \) and the similar argument of the proof of Proposition 5.2 we can show that the solution \( u(t) \to (6.1) \) sequently converges a constant curvature circle packing metric \( u_{av} \). That is to say, there exists a sequence \( \xi_n \to +\infty \) such that

\[
\lim_{n \to +\infty} u(\xi_n) = u_{av}.
\]

By (6.1) we have

\[
\frac{\partial E_p(u(t))}{\partial t} = \sum_{i=1}^{N} \nabla u_i E_p u'_i = -\sum_{i=1}^{N} (u'_i)^2 \leq 0,
\]

which implies that \( E_p(u(t)) \) is decreasing and bounded, i.e., \( 0 \leq E_p(u(t)) \leq E(u(0)) \). Hence

\[
\lim_{t \to +\infty} E_p(u(t)) = A
\]

exists. In fact \( A = 0 \). Note that

\[
\lim_{n \to +\infty} u(\xi_n) = u_{av},
\]

thus

\[
A = \lim_{t \to +\infty} E_p(u(\xi_n)) = E_p(u_{av}) = 0.
\]

Hence we have

\[
\lim_{t \to +\infty} E_p(u(t)) = 0.
\]

Further we have

\[
0 \leq |k_{av} - K_i(u(t))|^p \leq E_p(u(t)), \forall i = 1, \cdots, N
\]

thus

\[
\lim_{t \to +\infty} K_i(u(t)) = k_{av}, \forall i = 1, \cdots, N.
\]

By the above argument, the key question is to show the compactness of the closure of the solution subset \( \{ u(t) \mid t \in [0, +\infty) \} \) to (6.1). Or can one directly show the long time convergence of the solution \( \{ u(t) \mid t \in [0, +\infty) \} \) to (6.1) when \( t \to +\infty \) using different approaches?
6.2 Another combinatorial $p$-th Calabi type flow

Let $G = (V, E)$ be a finite graph, where $V$ denotes the vertex set and $E$ denotes the edge set. Fix a vertex measure $\mu : V \to (0, +\infty)$ s.t., $\mu_i = \mu(i) \equiv 1$, $i = 1, \cdots, N$ and an edge measure $\omega : E \to (0, +\infty)$ on $G$. The edge measure is assumed to be symmetric, that is, $\omega_{ij} = \omega_{ji}$ for each edge $i \sim j$. Then we study another form of the combinatorial $p$-th curvature flow. For any given combinatorial $p$-th Laplace operator, namely fix $\Delta^p f_i = \frac{1}{\mu_i} \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2} (f_j - f_i) = \sum_{j \sim i} \omega_{ij} |f_j - f_i|^{p-2} (f_j - f_i),$

where $\mu_i = 1$ and $\omega_{ij} \geq 0$ are independent on time $t$.

Consider the combinatorial $p$-th Calabi type flow

$$u'_i = \frac{du_i}{dt} = \Delta^p K_i. \quad (6.2)$$

For these equations, we can similarly prove the long time existence. Then consider the $p$-th energy

$$E(u) = \int_E |\nabla K|^p d\omega = \sum_{j \sim i} \omega_{ij} |K_j - K_i|^p,$$

and we have

$$\frac{\partial E}{\partial K_j} = -p \Delta^p K_j.$$

Hence

$$\frac{dE(u(t))}{dt} = \sum_{i=1}^N \frac{\partial E}{\partial u_i} u'_i = \sum_{i=1}^N \sum_{j=1}^N \frac{\partial E}{\partial K_j} \frac{\partial K_j}{\partial u_i} u'_i = -p \sum_{i=1}^N \sum_{j=1}^N u'_i L_{ij} u'_j \leq 0,$$

where the last inequality holds because the matrix $L = (L_{ij}) \geq 0$.

Therefore $E(u(t))$ is decreasing and bounded. If there exists the compactness of the closure of the solution subset $\{ u(t) \mid t \in [0, +\infty) \}$ to (6.2), then we can show the long time convergence for the curvature flow $\{ K(u(t)) \mid t \in [0, +\infty) \}$ when $t \to +\infty$ using the same argument in above subsection. To see this, we first similarly show

$$\lim_{t \to +\infty} E(u(t)) = 0.$$

Next we we prove

$$\lim_{t \to +\infty} K_i(t) = K_{av}.$$

For this we need a lemma due to H. Ge [19].
Lemma 6.1. Let $G = (V, E)$ be a finite graph. For all functions $\varphi \in C(V)$ with $\bar{\varphi} = 0$, there exists some positive constant $C_{p,G}$ depending on $G$ and $p$ such that

$$
\int_V |\varphi|^p d\mu \leq C_{p,G} \int_E |\nabla \varphi|^p d\omega.
$$

(6.3)

Let $\varphi = K_{av} - K$, then $\bar{\varphi} = 0$ because

$$
\bar{\varphi} = \sum_{i=1}^N (k_{av} - K_i) \mu_i / Vol(G) = \sum_{i=1}^N (k_{av} - K_i) / Vol(G) = 2\pi \chi(X) - \sum_{i=1}^N K_i = 0,
$$

where we use the combinatorial Gauss-Bonnet formula in the case of $E^2$.

By the lemma above, for $i = 1, \ldots, N$ we have

$$
|k_{av} - K_i|^p \leq \sum_{i=1}^N |k_{av} - K_i|^p = \int_V |\varphi|^p d\mu \leq C_{p,G} \int_E |\nabla \varphi|^p d\omega = E(u(t)),
$$

which implies

$$
\lim_{t \to +\infty} K_i(t) = k_{av}, \forall i, \ldots, N.
$$

Therefore the key question is also to show the compactness of the closure of the solution subset $\{u(t) \mid t \in [0, +\infty)\}$ to (6.2). Or can one directly show the long time convergence of the solution $\{(u(t) \mid t \in [0, +\infty)\}$ to (6.2) when $t \to +\infty$ using different approaches?

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