Fix-Mahonian Calculus, II: further statistics

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Abstract. Using classical transformations on the symmetric group and two transformations constructed in Fix-Mahonian Calculus I, we show that several multivariable statistics are equidistributed either with the triplet \((\text{fix}, \text{des}, \text{maj})\), or the pair \((\text{fix}, \text{maj})\), where “fix,” “des” and “maj” denote the number of fixed points, the number of descents and the major index, respectively.

1. Introduction

First, recall the traditional notations for the \(q\)-ascending factorials

\[
(a; q)_n := \begin{cases} 1, & \text{if } n = 0; \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & \text{if } n \geq 1; 
\end{cases}
\]

\[
(a; q)_\infty := \prod_{n \geq 1} (1 - aq^{n-1});
\]

and the \(q\)-exponential (see [GaRa90, chap. 1])

\[
e_q(u) = \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_\infty}.
\]

Furthermore, let \((A_n(Y, t, q))\) and \((A_n(Y, q))\) \((n \geq 0)\) be the sequences of polynomials respectively defined by the factorial generating functions

\[
(1.1) \quad \sum_{n \geq 0} A_n(Y, t, q) \frac{u^n}{(t; q)_{n+1}} := \sum_{s \geq 0} t^s \left(1 - u \sum_{i=0}^s q^i\right)^{-1} \frac{(u; q)_{s+1}}{(uY; q)_{s+1}};
\]

\[
(1.2) \quad \sum_{n \geq 0} A_n(Y, q) \frac{u^n}{(q; q)_n} := \left(1 - \frac{u}{1 - q}\right)^{-1} \frac{(u; q)_\infty}{(uY; q)_\infty}.
\]

Of course, (1.2) can be derived from (1.1) by letting the variable \(t\) tend to 1, so that \(A_n(Y, q) = A_n(Y, 1, q)\). The classical combinatorial interpretation for those classes of polynomials has been found by Gessel and Reutenauer [GeRe93] (see Theorem 1.1 below). For each permutation \(\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)\) from the symmetric group \(S_n\) let \(i\sigma := \sigma^{-1}\) denote
the inverse of \( \sigma \); then let its set of fixed points, \( \text{FIX} \), descent set, \( \text{DES} \), idescent set, \( \text{IDES} \), be defined as the subsets:

\[
\text{FIX} \sigma := \{ i : 1 \leq i \leq n, \sigma(i) = i \}; \\
\text{DES} \sigma := \{ i : 1 \leq i \leq n - 1, \sigma(i) > \sigma(i + 1) \}; \\
\text{IDES} \sigma := \text{DES} \sigma^{-1}.
\]

Note that \( \text{IDES} \sigma \) is also the set of all \( i \) such that \( i + 1 \) is on the left of \( i \) in the linear representation \( \sigma(1) \sigma(2) \cdots \sigma(n) \) of \( \sigma \). Also let \( \text{fix} \sigma := \# \text{FIX} \sigma \) (the number of fixed points), \( \text{des} \sigma := \# \text{DES} \sigma \) (the number of descents), \( \text{maj} \sigma := \sum_i i \) (\( i \in \text{DES} \sigma \)) (the major index), \( \text{imaj} \sigma := \sum_i i \) (\( i \in \text{IDES} \sigma \)) (the inverse major index).

**Theorem 1.1** (Gessel, Reutenauer). For each \( n \geq 0 \) the generating polynomial for \( S_n \) by \( (\text{fix}, \text{des}, \text{maj}) \) (resp. by \( (\text{fix}, \text{maj}) \)) is equal to \( A_n(Y, t, q) \) (resp. to \( A_n(Y, q) \)). Accordingly,

\[
A_n(Y, t, q) = \sum_{\sigma \in S_n} Y^{\text{fix} \sigma} t^{\text{des} \sigma} q^{\text{maj} \sigma}; \\
A_n(Y, q) = \sum_{\sigma \in S_n} Y^{\text{fix} \sigma} q^{\text{maj} \sigma}.
\]

The purpose of this paper is to show that there are several other three-variable (resp. two-variable) statistics on \( S_n \), whose distribution is given by the generating polynomial \( A_n(Y, t, q) \) (resp. \( A_n(Y, q) \)). For proving that those statistics are equidistributed with \( (\text{fix}, \text{des}, \text{maj}) \) (resp. \( (\text{fix}, \text{maj}) \)) we make use the properties of the classical bijections \( F_2^{\text{loc}} \), \( F_2' \), \( \text{CHZ} \), \( \text{DW}^{\text{glo}} \), \( \text{DW}^{\text{loc}} \), plus two transformations \( F_3 \), \( \Phi \), constructed in our previous paper [FoHa07], finally a new transformation \( F_3' \) described in Section 3.

All those bijections appear as arrows in the diagram of Fig. 1. The nodes of the diagram are pairs or triplets of statistics, whose definitions have been, or will be, given in the paper. The integral-valued statistics are written in lower case, such as “fix” or “maj”, while the set-valued ones appear in capital letters, such as “FIX” or “DES”. We also introduce two mappings “Der” and “Desar” of \( S_n \) into \( S_m \) with \( m \leq n \).

Each arrow goes from one node to another node with the following meaning that we shall explain by means of an example: the vertical arrow \( (\text{fix}, \text{maz}, \text{Der}) \xrightarrow{F_3} (\text{fix}, \text{maf}, \text{Der}) \) (here rewritten horizontally for typographical reasons!) indicates that the bijection \( F_3 \) maps \( S_n \) onto itself and has the property: \( (\text{fix}, \text{maz}, \text{Der}) \sigma = (\text{fix}, \text{maf}, \text{Der}) F_3(\sigma) \) for all \( \sigma \). The remaining statistics are now introduced together with the main two decompositions of permutations: the fixed and pixed decompositions.
1.1. The fixed decomposition. Let $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ be a permutation and let $(i_1, i_2, \ldots, i_{n-m})$ (resp. $(j_1, j_2, \ldots, j_m)$) be the increasing sequence of the integers $k$ (resp. $k'$) such that $1 \leq k \leq n$ and $\sigma(k) = k$ (resp. $1 \leq k' \leq n$ and $\sigma(k') \neq k'$). Also let “red” denote the increasing bijection of $\{j_1, j_2, \ldots, j_m\}$ onto $[m]$. Let $Z\operatorname{Der}(\sigma) = x_1 x_2 \cdots x_n$ be the word derived from $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ by replacing each fixed point $\sigma(i_k)$ by 0 and each other letter $\sigma(j_{k'})$ by red $\sigma(j_{k'})$. As “DES,” “des” and “maj” can also be defined for arbitrary words with nonnegative letters, we further introduce:

\begin{align}
(1.5) \quad & \text{DEZ } \sigma := \text{DES } Z\operatorname{Der}(\sigma); \\
(1.6) \quad & \text{dez } \sigma := \text{des } Z\operatorname{Der}(\sigma), \quad \text{maz } \sigma := \text{maj } Z\operatorname{Der}(\sigma); \\
(1.7) \quad & \text{Der } \sigma := \text{red } \sigma(j_1) \text{ red } \sigma(j_2) \cdots \text{ red } \sigma(j_m); \\
(1.8) \quad & \text{maf } \sigma := \sum_{k=1}^{n-m} (i_k - k) + \text{maj } \circ \text{Der } \sigma.
\end{align}

The subword $\text{Der } \sigma$ of $Z\operatorname{Der}(\sigma)$ can be regarded as a permutation from $\mathfrak{S}_m$ (with the above notations). It is important to note that $\text{FIX } \text{Der } \sigma = \emptyset$, so that $\text{Der } \sigma$ is a derangement of order $m$. Also note that $\sigma$ is fully characterized by the pair $(\text{FIX } \sigma, \text{Der } \sigma)$, which is called the fixed decomposition.
of $\sigma$. Finally, we can rewrite $\text{maf } \sigma$ as

\begin{equation}
\text{maf } \sigma := \sum_{i \in \text{FIX } \sigma} i - \sum_{i=1}^{\text{fix } \sigma} i + \text{maj } \circ \text{Der } \sigma.
\end{equation}

**Example.** With $\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9)$ we have $\text{red } = (1 \ 3 \ 4 \ 7 \ 8 \ 9)$. 

$Z\text{Der}(\sigma) = 5 \ 0 \ 1 \ 2 \ 0 \ 0 \ 3 \ 6 \ 4$, $D\text{EZ } \sigma = \{1, 4, 8\}$, $\text{dez } \sigma = 3$, $\text{maz } \sigma = 13$, $D\text{er } \sigma = 5 \ 1 \ 2 \ 3 \ 6 \ 4$, $\text{FIX } \sigma = \{2, 5, 6\}$ and $\text{maf } \sigma = (2 - 1) + (5 - 2) + (6 - 3) + \text{maj}(5 \ 1 \ 2 \ 3 \ 6 \ 4) = 7 + 6 = 13$.

1.2. The *pixed* decomposition. Let $w = y_1 y_2 \cdots y_n$ be a word having no repetitions, without necessarily being a permutation of $1 \cdots n$. Say that $w$ is a *desarrangement* if $y_1 > y_2 > \cdots > y_{2k}$ and $y_{2k} < y_{2k+1}$ for some $k \geq 1$. By convention, $y_{n+1} = \infty$. We could also say that the *leftmost trough* of $w$ occurs at an *even* position. This notion was introduced by Désarménien [De84] and elegantly used in a subsequent paper [DW88].

A further refinement is due to Gessel [Ge91].

Let $\sigma = \sigma(1) \sigma(2) \cdots \sigma(n)$ be a permutation. Unless $\sigma$ is increasing, there is always a nonempty right factor of $\sigma$ which is a desarrangement. It then makes sense to define $\sigma^d$ as the *longest* such a right factor. Hence, $\sigma$ admits a unique factorization $\sigma = \sigma^p \sigma^d$, called the *pixed factorization*, where $\sigma^p$ is *increasing* and $\sigma^d$ is the longest right factor of $\sigma$ which is a desarrangement. The set (resp. number) of the letters in $\sigma^p$ is denoted by $\text{Pix } \sigma$ (resp. $\text{pix } \sigma$).

If $\sigma^d = (n - m + 1) \sigma(n - m + 2) \cdots \sigma(n)$ and if “red” is the increasing bijection mapping the set $\{\sigma(n - m + 1), \sigma(n - m + 2), \ldots, \sigma(n)\}$ onto $\{1, 2, \ldots, m\}$, define

\begin{equation}
\text{Desar } \sigma := \text{red } \sigma(n - m + 1) \text{ red } \sigma(n - m + 2) \cdots \text{ red } \sigma(n);
\end{equation}

\begin{equation}
\text{mag } \sigma := \sum_{i \in \text{PIX } \sigma} i - \sum_{i=1}^{\text{pix } \sigma} i + \text{imaj } \circ \text{Desar } \sigma.
\end{equation}

Note that $\text{Desar } \sigma$ is a desarrangement and belongs to $\mathfrak{S}_m$, for short, a desarrangement of order $m$. Also note that $\sigma$ is fully characterized by the pair $(\text{Pix } \sigma, \text{Desar } \sigma)$, which will called the *pixed decomposition* of $\sigma$. Form the inverse $(\text{Desar } \sigma)^{-1} = y_1 y_2 \cdots y_m$ of $\text{Desar } \sigma$ and define $Z\text{Desar}(\sigma)$ to be the unique shuffle $x_1 x_2 \cdots x_n$ of $0^{n-m}$ and $y_1 y_2 \cdots y_m$, where $x_i = 0$ if and only if $i \in \text{PIX } \sigma$.

**Example.** With $\sigma = 3 \ 5 \ 7 \ 4 \ 2 \ 8 \ 1 \ 9 \ 6$, then $\sigma^p = 3 \ 5 \ 7$, $\sigma^d = 4 \ 2 \ 8 \ 1 \ 9 \ 6$, $\text{PIX } \sigma = \{3, 5, 7\}$, $\text{pix } \sigma = 3$. Also, $\text{Desar } \sigma = 3 \ 2 \ 5 \ 1 \ 6 \ 4$, $\text{imaj } \circ \text{Desar } \sigma = 1 + 2 + 4 = 7$, $(\text{Desar } \sigma)^{-1} = 4 \ 2 \ 1 \ 6 \ 3 \ 5$, $Z\text{Desar } \sigma = 4 \ 2 \ 0 \ 1 \ 6 \ 0 \ 3 \ 5$ and $\text{mag } \sigma = (3 + 5 + 7) - (1 + 2 + 3) + 7 = 16$. 


FURTHER STATISTICS

Referring to the diagram in Fig. 1 the purpose of this paper is to prove the next two theorems.

**Theorem 1.2.** In each of the following four groups the pairs of statistics are equidistributed on $\mathcal{S}_n$:

1. $(\text{fix, maj}), (\text{fix, maf}), (\text{fix, maz}), (\text{pix, mag}), (\text{pix, inv}), (\text{pix, imaj})$;
2. $(\text{FIX, maf}), (\text{PIX, mag}), (\text{PIX, inv})$;
3. $(\text{fix, DEZ}), (\text{fix, DES}), (\text{pix, IDES})$;
4. $(\text{FIX, DEZ}), (\text{PIX, IDES})$.

**Theorem 1.3.** In each of the following two groups the triplets of statistics are equidistributed on $\mathcal{S}_n$:

1. $(\text{fix, maf, Der})$ and $(\text{fix, maz, Der})$;
2. $(\text{pix, mag, Desar})$ and $(\text{pix, imaj, Desar})$.

Furthermore, the following diagram, involving the bijections $\text{DW}^{\text{loc}}, F_3$ and $F_3'$ is commutative.

![Diagram](Fig. 2)

2. The bijections

2.1. *The transformations $\Phi$ and “CHZ”.* In our preceding paper [FoHa07] we have given the constructions of two bijections $\Phi, F_3$ of $\mathcal{S}_n$ onto itself. The latter one will be re-studied and used in Section 3. As was shown in our previous paper [FoHa07], the first one has the following property:

\[(2.1) \quad (\text{fix, DEZ, Der}) \sigma = (\text{fix, DES, Der}) \Phi(\sigma) \quad (\sigma \in \mathcal{S}_n).\]

This shows that over $\mathcal{S}_n$ the pairs $(\text{fix, maz})$ and $(\text{fix, maj})$ are equidistributed over $\mathcal{S}_n$, their generating polynomial being given by the polynomial $A_n(Y, q)$ introduced in (1.2). Also the triplets $(\text{fix, dez, maz})$ and $(\text{fix, des, maj})$ are equidistributed, with generating polynomial $A_n(Y, t, q)$ introduced in (1.1).
In [CHZ97] the authors have constructed a bijection, here called “CHZ”, satisfying

\[(\text{fix, maf, Der}) \sigma = (\text{fix, maj, Der}) \text{CHZ}(\sigma) \quad (\sigma \in \mathfrak{S}_n).\]

2.2. The Désarménien-Wachs bijection. For each \( n \geq 0 \) let \( D_n \) denote the set of permutations \( \sigma \) from \( \mathfrak{S}_n \) such that \( \text{FIX} \sigma = \emptyset \). The elements of \( D_n \) are referred to as the derangements of order \( n \). Let \( K_n \) be the set of permutations \( \sigma \) from \( \mathfrak{S}_n \) such that \( \text{PIX} \sigma = \emptyset \). The class \( K_n \) was introduced by Désarménien [De84], who called its elements desarrangements of order \( n \). He also set up a one-to-one correspondence between \( D_n \) and \( K_n \). Later, by means of a symmetric function argument Désarménien and Wachs [DW88] proved that for every subset \( J \subset [n-1] \) the following equality

\[(2.3) \quad \#\{\sigma \in D_n : \text{DES} \sigma = J\} = \#\{\sigma \in K_n : \text{IDES} \sigma = J\}
\]

holds. In a subsequent paper [DW93] they constructed a bijection \( \text{DW} : D_n \to K_n \) having the expected property, that is,

\[(2.4) \quad \text{IDES} \circ \text{DW}(\sigma) = \text{DES} \sigma.
\]

Although their bijection is based on an inclusion-exclusion argument, leaving the door open to the discovery of an explicit correspondence, we use it as such in the sequel. For the very definition of “DW” we refer the reader to their original paper [DW93].

We now make a full use of the fixed and pixed decompositions introduced in §§1.1 and 1.2. Let \( \tau \in \mathfrak{S}_n \) and consider the chain

\[(2.5) \quad \tau \mapsto (\text{FIX} \tau, \text{Der} \tau) \mapsto (\text{FIX} \tau, \text{DW} \circ \text{Der} \tau) \mapsto \sigma,
\]

where \( \sigma \) is the permutation defined by

\[(2.6) \quad (\text{PIX} \sigma, \text{Desar} \sigma) := (\text{FIX} \tau, \text{DW} \circ \text{Der} \tau).
\]

Then, the mapping \( \text{DW}^{\text{loc}} \) defined by

\[(2.7) \quad \text{DW}^{\text{loc}}(\tau) := \sigma,
\]

is a bijection of \( \mathfrak{S}_n \) onto itself satisfying \( \text{FIX} \tau = \text{PIX} \sigma \) and \( \text{DES} \circ \text{Der} \tau = \text{IDES} \circ \text{Desar} \sigma \). In particular, \( \text{fix} \tau = \text{pix} \sigma \). Taking the definitions of “maf” and “mag” given in (1.8) and (1.9) into account we have:

\[
\text{maf} \tau = \sum_{i \in \text{FIX} \tau} i - \sum_{i=1}^{\text{fix} \tau} i + \text{maj} \circ \text{Der} \tau
= \sum_{i \in \text{PIX} \sigma} i - \sum_{i=1}^{\text{pix} \sigma} i + \text{imaj} \circ \text{Desar} \sigma = \text{mag} \sigma.
\]

We have then proved the following proposition.
FURTHER STATISTICS

Proposition 2.1. Let $\sigma := \text{DW}^{\text{loc}}(\tau)$. Then

\begin{equation}
\text{FIX} \tau = \text{PIX} \sigma, \quad \text{DES} \circ \text{Der} \tau = \text{IDES} \circ \text{Desar} \sigma, \quad \text{maf} \tau = \text{mag} \sigma.
\end{equation}

Corollary 2.2. The pairs (\text{FIX}, \text{maf}) and (\text{PIX}, \text{mag}) are equidistributed over $\mathfrak{S}_n$.

Example. Assume that the bijection “\text{DW}” maps the derangement 512364 onto the desarrangement 623145. On the other hand, the fixed decomposition of $\tau = 182453697$ is equal to $\{1,4,5\}, 512364$ and $\{1,4,5\}, 623145$ is the pixed decomposition of the permutation $\sigma = 145936278$. Hence $\text{DW}^{\text{loc}}(182453697) = 145936278$.

We verify that $\text{DES} \circ \text{Der} \tau = \text{DES}(512364) = \{1,5\} = \text{IDES}(623145) = \text{IDES} \circ \text{Desar} \sigma$. Also $\text{maf} \tau = (1+4+5) - (1+2+3) + (1+5) = 10 = \text{mag} \sigma$.

Proposition 2.3. Let $\sigma := \text{DW}^{\text{loc}}(\tau)$. Then

\begin{equation}(\text{FIX}, \text{DEZ}) \tau = (\text{PIX}, \text{IDES}) \sigma; \end{equation}

\begin{equation}(\text{fix}, \text{maz}) \tau = (\text{pix}, \text{imaj}) \sigma.
\end{equation}

Proof. It suffices to prove (2.9) and in fact only $\text{DEZ} \tau = \text{IDES} \sigma$. Let $\sigma^{\text{p}}\sigma^{\text{d}}$ be the pixed factorization of $\sigma$. Then $\sigma^{\text{p}}$ is the increasing sequence of the elements of $\text{PIX} \sigma = \text{FIX} \tau$. We have $i \in \text{DEZ} \tau$ if and only if $\tau(i) \neq i$ and one of the following conditions holds:

\begin{enumerate}
\item $\tau(i) > \tau(i + 1)$ and $\tau(i + 1) \neq i + 1$;
\item $\tau(i + 1) = i + 1$.
\end{enumerate}

In case (1) the letters red $\tau(i)$ and red $\tau(i + 1)$ are adjacent letters in Der $\tau$ and red $\tau(i) >$ red $\tau(i + 1)$. As $\text{DES} \circ \text{Der} \tau = \text{IDES} \circ \text{Desar} \sigma$, the letter red$(i + 1)$ is to the right of the letter red$(i)$ in Desar $\sigma$ and then $(i + 1)$ is to the right of $i$ is $\sigma$, so that $i \in \text{IDES} \sigma$.

In case (2) we have $(i+1) \in \text{FIX} \tau = \text{PIX} \sigma$ and red $i$ is a letter of Desar $\sigma$. Again $i \in \text{IDES} \sigma$. 

Corollary 2.4. The pairs (\text{FIX}, \text{DEZ}) and (\text{PIX}, \text{IDES}) are equidistributed over $\mathfrak{S}_n$.

Using the same example as above we have: $\tau = 182453697$, $\text{FIX} \tau = \{1,4,5\}$, $\tau^0 = 082003697$, so that $\text{DEZ} \tau = \{2,3,8\}$. Moreover, $\sigma = \text{DW}^{\text{loc}}(182453697) = 145 \mid 936278$. Hence $2,8 \in \text{IDES} \sigma$ (case (1)) and $3 \in \text{IDES} \sigma$ (case (2)).

2.3. The second fundamental transformation. As described in [Lo83, p. 201, Algorithm 10.6.1] by means of an algorithm, the second fundamental transformation, further denoted by $F_2$, can be defined on permutations
as well as on words. Here we need only consider the case of permutations. As usual, the number of inversions of a permutation $\sigma = \sigma(1) \sigma(2) \cdots \sigma(n)$ is defined by $\text{inv} \, \sigma := \# \{(i, j) \mid 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}$. Its construction was given in [Fo68]. Further properties have been proved in [FS78], [BjW88]. Here we need the following result.

**Theorem 2.5** [FS78]. The transformation $F_2$ defined on the symmetric group $S_n$ is bijective and the following identities hold for every permutation $\sigma \in S_n$: $\text{inv} \, F_2(\sigma) = \text{maj} \, \sigma$; $\text{IDES} \, F_2(\sigma) = \text{IDES} \, \sigma$.

Using the composition product $F'_2 := i \circ F_2 \circ i$ we therefore have:

$$\text{inv} \, F'_2(\sigma) = \text{imaj} \, \sigma; \quad \text{IDES} \, F'_2(\sigma) = \text{DES} \, \sigma$$

for every $\sigma \in S_n$.

As the descent set “DES” is preserved under the transformation $F'_2$, each desarrangement is mapped onto another desarrangement. It then makes sense to consider the chain:

$$\sigma \mapsto (\text{PIX} \, \sigma, \text{Desar} \, \sigma) \mapsto (\text{PIX} \, \sigma, F'_2 \circ \text{Desar} \, \sigma) \mapsto \rho,$$

where $(\text{PIX} \, \rho, \text{Desar} \, \rho) := (\text{PIX} \, \sigma, F'_2 \circ \text{Desar} \, \sigma)$. The mapping $F'^{\text{loc}}_2 : \sigma \mapsto \rho$ is a bijection of $S_n$ onto itself. Moreover, $\text{PIX} \, \sigma = \text{PIX} \, \rho$, $\text{imaj} \circ \text{Desar} \, \sigma = \text{inv} \circ \text{Desar} \, \rho$ and $\text{DES} \circ \text{Desar} \, \sigma = \text{DES} \circ \text{Desar} \, \rho$. Hence,

$$\text{mag} \, \sigma = \sum_{i \in \text{PIX} \, \sigma} i - \sum_{i=1}^{\text{pix} \, \sigma} i + \text{imaj} \circ \text{Desar} \, \sigma$$

$$= \sum_{i \in \text{PIX} \, \rho} i - \sum_{i=1}^{\text{pix} \, \rho} i + \text{inv} \circ \text{Desar} \, \rho$$

$$= \# \{ (i, j) : 1 \leq i < \text{pix} \, \rho < j \leq n, \rho(i) > \rho(j) \}$$

$$+ \# \{ (i, j) : \text{pix} \, \rho < i < j \leq n, \rho(i) > \rho(j) \}$$

$$= \text{inv} \, \rho.$$

We have then proved the following proposition.

**Proposition 2.6.** Let $\rho := F'^{\text{loc}}_2(\sigma)$. Then

$$\text{(PIX, mag)} \, \sigma = (\text{PIX, inv}) \, \rho.$$  

**Corollary 2.7.** The pairs $(\text{PIX, mag})$ and $(\text{PIX, inv})$ are equidistributed over $S_n$.

Finally, go back to Properties (2.11) and let $\xi := F'_2(\rho)$. Also let $\xi^p \xi^d$ and $\rho^p \rho^d$ be the pixed factorizations of $\xi$ and $\rho$, respectively. We do not have $\xi^p = \rho^p$ necessarily, but as $\rho$ and $\xi$ have the same descent set, the factors $\xi^p$ and $\rho^p$ have the same length, i.e., pix $\xi = \text{pix} \, \rho$. Let us state this result in the next proposition.
FURTHER STATISTICS

Proposition 2.8. Let \( \xi := F'_2(\rho) \). Then

\[
(\text{pix, inv}) \xi = (\text{pix, imaj}) \rho.
\]

Corollary 2.9. The pairs \((\text{pix, inv})\) and \((\text{pix, imaj})\) are equidistributed over \( \mathcal{S}_n \).

Finally, \( \text{DWglo} \) attached to the unique oblique arrow in Fig. 1 refers to the global bijection constructed by Désarménien and Wachs ([DW93], §5). It has the property: \((\text{fix, DES}) \text{DWglo}(\sigma) = (\text{pix, IDES}) \sigma\) for all \( \sigma \) in \( \mathcal{S}_n \). The big challenge is to find two explicit bijections \( f \) and \( g \), replacing \( \text{DWglo} \) and \( \text{DWloc} \), such that \((\text{fix, DES}) g(\sigma) = (\text{pix, IDES}) \sigma\) and \((\text{PIX, IDES}) f(\sigma) = (\text{FIX, DEZ}) \sigma\), which would make the bottom triangle commutative, that is, \( \Phi = g \circ f \).

3. The bijections \( F_3 \) and \( F'_3 \)

Let \( 0 \leq m \leq n \) and let \( v \) be a nonempty word of length \( m \), whose letters are positive integers (with possible repetitions). Designate by \( \text{Sh}(0^{n-m}v) \) the set of all shuffles of the words \( 0^{n-m} \) and \( v \), that is, the set of all rearrangements of the juxtaposition product \( 0^{n-m}v \), whose longest subword of positive letters is \( v \). Let \( w = x_1x_2\cdots x_n \) be a word from \( \text{Sh}(0^{n-m}v) \). It is convenient to write: \( \text{Pos} w := v, \text{Zero} w := \{i : 1 \leq i \leq n, x_i = 0\}, \text{zero} w := \# \text{Zero} w = n - m \), so that \( w \) is completely characterized by the pair \((\text{Zero} w, \text{Pos} w)\). Besides the statistic “maj” we will need the statistic “majz” that associates the number

\[
(3.1) \quad \text{mafz} w := \sum_{i \in \text{Zero} w} i - \sum_{i=1}^{\text{zero} w} i + \text{maj} \text{Pos} w.
\]

with each word from \( \text{Sh}(0^{n-m}v) \). In ([FoHa07], §4) we gave the construction of a bijection \( F_3 \) of \( \text{Sh}(0^{n-m}v) \) onto itself having the following property:

\[
(3.2) \quad \text{maj} w = \text{mafz} F_3(w) \quad (w \in \text{Sh}(0^{n-m}v)).
\]

The bijection \( F_3 \) is now applied to each shuffle class \( \text{Sh}(0^{n-m}v) \), when \( v \) is a derangement, or the inverse of a desarrangement. Let

\[
\mathcal{S}^\text{Der}_n := \bigcup_{m,v} \text{Sh}(0^{n-m}v) \quad (0 \leq m \leq n, v \in D_n);
\]

\[
\mathcal{S}^\text{Desar}_n := \bigcup_{m,v} \text{Sh}(0^{n-m}v) \quad (0 \leq m \leq n, v^{-1} \in K_n).
\]
As already seen in § 1.1, the mapping $Z\text{Der}$ is a bijection of $\mathcal{S}_n$ onto $\mathcal{S}_n^{\text{Der}}$ satisfying

\begin{align*}
(3.3) \quad \text{FIX } \sigma &= \text{Zero } Z\text{Der}(\sigma); \quad \text{Der } \sigma = \text{Pos } Z\text{Der}(\sigma); \\
\text{maf } \sigma &= \text{mafz } Z\text{Der}(\sigma); \quad \text{DEZ } \sigma = \text{DES } Z\text{Der}(\sigma).
\end{align*}

**Example 3.2.** Let $\sigma = 1 7 3 5 2 6 4$. Then $w := Z\text{Der}(\sigma) = 0 4 0 3 1 0 2$.

We have $\text{FIX } \sigma = \text{Zero } w = \{1, 3, 6\}$; $\text{Der } \sigma = \text{Pos } w = 4312$; $\text{maf } \sigma = \text{mafz } w = (1+3+6)-(1+2+3)+(1+2+3) = 10$, $\text{DEZ } \sigma = \text{DES } w = \{2, 4, 5\}$.

Now define the bijection $F_3$ of $\mathcal{S}_n$ onto itself by the chain

\begin{align*}
(3.4) \quad F_3 : \sigma \mapsto w \mapsto w' \mapsto Z\text{Der}^{-1} \mapsto \sigma'.
\end{align*}

Then, by (3.2),

\begin{align*}
(\text{fix, maz, Der}) \sigma &= (\text{zero, maj, Pos}) w \\
&= (\text{zero, mafz, Pos}) w' \\
&= (\text{fix, maz, Der}) \sigma',
\end{align*}

\begin{align*}
(3.5) \quad (\text{fix, maz, Der}) \sigma &= (\text{fix, maz, Der}) F_3(\sigma).
\end{align*}

The map “Desar” has been defined in (1.10) and it was noticed that each permutation $\sigma$ was fully characterized by the pair $(\text{PIX } \sigma, \text{Desar } \sigma)$. Another way of deriving $Z\text{Desar}(\sigma)$ introduced in § 1.2 is to form the inverse $\sigma^{-1} = \sigma^{-1}(1)\sigma^{-1}(2) \cdots \sigma^{-1}(n)$ of $\sigma$. As $\sigma^{-1}(i) \geq \text{pix } \sigma + 1$ if and only if $i \in [n] \setminus \text{PIX } \sigma$, we see that $Z\text{Desar}(\sigma)$ is also the word $w = x_1 x_2 \cdots x_n$, where

\[ x_i := \begin{cases} 
0, & \text{if } i \in \text{PIX } \sigma; \\
\sigma^{-1}(i) - \text{pix } \sigma, & \text{if } i \in [n] \setminus \text{PIX } \sigma.
\end{cases} \]

The word $\sigma^{-1}$ contains the subword $1 2 \cdots \text{pix } \sigma$. We then have: $i \in \text{IDES } \sigma \iff i \in \text{DES } \sigma^{-1} \iff \sigma^{-1}(i) \geq \text{pix } \sigma + 1$ and $\sigma^{-1}(i) > \sigma^{-1}(i+1) \iff x_i \geq 1$ and $x_i > x_{i+1} \iff i \in \text{DES } w$, so that $\text{IDES } \sigma = \text{DES } w$.

On the other hand, as $\text{PIX } \sigma = \text{Zero } w$ and $(\text{Desar } \sigma)^{-1} = \text{Pos } w$, we also have, by (1.11)

\[ \text{mag } \sigma = \sum_{i \in \text{PIX } \sigma} i - \sum_{i=1}^{\text{pix } \sigma} i + \text{imaj } \circ \text{Desar } \sigma \]

\[ = \sum_{i \in \text{Zero } w} i - \sum_{i=1}^{\text{zero } w} i + \text{maj } \circ \text{Pos } w = \text{mafz } w. \]

As a summary,

\begin{align*}
(3.6) \quad \text{PIX } \sigma &= \text{Zero } Z\text{Desar}(\sigma); \quad \text{Desar } \sigma = \text{Pos } Z\text{Desar}(\sigma); \\
\text{mag } \sigma &= \text{mafz } Z\text{Desar}(\sigma); \quad \text{IDES } \sigma = \text{DES } Z\text{Desar}(\sigma).
\end{align*}
FURTHER STATISTICS

Example 3.3. Let $\sigma = 1 3 6 5 4 7 2$. Then $\sigma^{-1} = 1 7 2 5 4 3 6$; $w := ZDesar(\sigma) = 0 4 0 2 1 0 3$, $\text{PIX} \sigma = \text{Zero} w = \{1, 3, 6\}$; $\text{Desar} \sigma = 3 2 4 1$, $(\text{Desar} \sigma)^{-1} = \text{Pos} w = 4 2 1 3$; $\text{mag} \sigma = \text{mafz} w = (1 + 3 + 6) - (1 + 2 + 3) + (1 + 2) = 7$, $\text{IDES} \sigma = \text{DES} w = \{2, 4, 5\}$.

Next define the bijection $F_3'$ of $S_n$ onto itself by the chain

$$(3.7) \quad F_3' : \sigma \mapsto Z\text{Desar} \mapsto w \mapsto Z\text{Desar}^{-1} \mapsto \sigma'.$$

Then, by (3.2),

$$
(\text{pix, imaj, Desar}) \sigma = (\text{zero, maj, Pos}) w
= (\text{zero, mafz, Pos}) w'
= (\text{pix, mag, Desar}) \sigma',
$$

With (3.5) and (3.8) the first part of Theorem 1.3 is proved.

The second part of Theorem 1.3 is proved as follows. Remember that, if zero $w = n - m$, the pair $(\text{Zero} w, \text{Pos} w)$ uniquely determines the shuffle of $(n - m)$ letters equal to 0 into the letters of $\text{Pos} w$. The bijection “dw” defined by $dw := Z\text{Desar} \circ \text{DW} \circ Z\text{Desar}^{-1}$ can also be derived by the chain

$$
(3.9) \quad dw : w \mapsto (\text{Zero} w, \text{Pos} w)
\mapsto (\text{Zero} w, (\text{DW} \circ \text{Pos} w)^{-1}) = (\text{Zero} w', \text{Pos} w') \mapsto w',
$$

where $\text{DW}$ denotes the Désarménien-Wachs bijection. It maps $S_n^{\text{Der}}$ onto $S_n^{\text{Desar}}$. In particular,

$$
\text{Pos} \circ dw w = (\text{DW} \circ \text{Pos} w)^{-1}; \quad \text{Zero} \circ dw (w) = \text{Zero} w.
$$

Because of (2.4) we also have $\text{DES} \circ \text{Pos} w = \text{DES} \circ \text{Pos} w'$. As shown in our previous paper ([FoHa07], Proposition 4.1), the latter property implies that

$$
\text{Zero} \circ F_3 (w) = \text{Zero} \circ F_3 (w').
$$

Furthermore,

$$
\text{Pos} \circ F_3 (w) = \text{Pos} w, \quad \text{Pos} \circ F_3 (w') = \text{Pos} w',
$$

since $F_3$ maps each shuffle class onto itself. Hence,

$$
\text{Zero} \circ dw \circ F_3 (w) = \text{Zero} \circ F_3 (w);\\
\text{Pos} \circ dw \circ F_3 (w) = (\text{DW} \circ \text{Pos} F_3 (w))^{-1} = (\text{DW} \circ \text{Pos} w)^{-1};\\
\text{Zero} \circ F_3 \circ dw (w) = \text{Zero} \circ F_3 (w);\\
\text{Pos} \circ F_3 \circ dw (w) = \text{Pos} \circ dw (w) = (\text{DW} \circ \text{Pos} w)^{-1}.
$$
The word \( \text{dw} \circ \text{F}_3(w) \) is characterized by the pair
\[
(\text{Zero} \circ \text{dw} \circ \text{F}_3(w), \text{Pos} \circ \text{dw} \circ \text{F}_3(w)),
\]
which is equal to the pair
\[
(\text{Zero} \circ \text{F}_3 \circ \text{dw}(w), \text{Pos} \circ \text{F}_3 \circ \text{dw}(w)),
\]
which corresponds itself to the word \( \text{F}_3(w) \circ \text{dw}(w) \). Hence,
\[
(3.10) \quad \text{dw} \circ \text{F}_3 = \text{F}_3 \circ \text{dw}.
\]
This shows that the top square in Fig. 2 is a commutative diagram, so is the bottom one. 

Example 3.4. Noting that \( \text{DW}(4312) = 3241 \) we have the commutative diagram
\[
\begin{array}{ccc}
7431562 & \xrightarrow{\text{DW}^{\text{loc}}} & 3564271 \\
1735264 & \xrightarrow{\text{DW}^{\text{loc}}} & 1365472
\end{array}
\]
\[
\begin{array}{ccc}
\text{F}_3 & \uparrow & \text{F}'_3 \\
\end{array}
\]
References

[BjW88] Anders Björner, Michelle L. Wachs. Permutation Statistics and Linear Extensions of Posets, *J. Combin. Theory, Ser. A*, 58 (1991), pp. 85–114.

[CHZ97] Robert J. Clarke, Guo-Niu Han, Jiang Zeng. A combinatorial interpretation of the Seidel generation of $q$-derangement numbers, *Annals of Combinatorics*, 4 (1997), pp. 313–327.

[De84] Jacques Désarménien. Une autre interprétation du nombre de dérangements, *Séminaire Lotharingien de Combinatoire*, [B08b], 1984, 6 pages.

[DW88] Jacques Désarménien, Michelle L. Wachs. Descentes des dérangements et mots circulaires, *Séminaire Lotharingien de Combinatoire*, [B19a], 1988, 9 pages.

[DW93] Jacques Désarménien, Michelle L. Wachs. Descent Classes of Permutations with a Given Number of Fixed Points, *J. Combin. Theory, Ser. A*, 64 (1993), pp. 311–328.

[Fo68] Dominique Foata. On the Netto inversion number of a sequence, *Proc. Amer. Math. Soc.*, 19 (1968), pp. 236–240.

[FoHa07] Dominique Foata, Guo-Niu Han. Fix-Mahonian Calculus, I: two transformations, preprint, 2007, 14 pages.

[FS78] Dominique Foata, M.-P. Schützenberger. Major Index and Inversion number of Permutations, *Math. Nachr.*, 83 (1978), pp. 143–159.

[GaRa90] George Gasper, Mizan Rahman. *Basic Hypergeometric Series*, London, Cambridge Univ. Press, 1990 (Encyclopedia of Math. and Its Appl., 35).

[Ge91] Ira Gessel. A coloring problem, *Amer. Math. Monthly*, 98 (1991), pp. 530–533.

[GeRe93] Ira Gessel, Christophe Reutenauer. Counting Permutations with Given Cycle Structure and Descent Set, *J. Combin. Theory Ser. A*, 64 (1993), pp. 189–215.

[Lo83] M. Lothaire. *Combinatorics on Words*, Addison-Wesley, London 1983 (Encyclopedia of Math. and its Appl., 17).

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