A Geometric Approach to the Standard Model

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A geometric approach to the standard model in terms of the Clifford algebra $\mathcal{C}_{7}$ is advanced. The gauge symmetries and charge assignments of the fundamental fermions are seen to arise from a simple geometric model involving extra space-like dimensions. The bare coupling constants are found to obey $g_s/g = 1$ and $g'/g = \sqrt{3}/5$, consistent with $SU(5)$ grand unification but without invoking the notion of master groups. In constructing the Lagrangian density terms, it is found that the Higgs isodoublet field emerges in a natural manner. A matrix representation of $\mathcal{C}_{7}$ is included as a computational aid.

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I. INTRODUCTION

In many physics equations of a fundamental nature, Clifford (geometric) algebras may be employed to recast conventional expressions into more holistic and aesthetically pleasing forms. This approach often reveals insights which were previously obscured by an inappropriate choice of architecture. The result may allow the consolidation of incongruous terms, suggest missing pieces, or reveal new restrictions imposed by identifying privileged subspaces within the chosen algebra.

The present paper explores the minimal standard model in terms of the Clifford algebra $\mathcal{C}_{7}$. The aim here is to demonstrate that the seemingly disparate gauge symmetries $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$ may be unified under a single simple geometric model. The implied formalism is then used to construct analogues of the conventional Lagrangian density terms.

In Sec. II the basic algebraic operations and conventions are laid out, as much of the notation used varies considerably within the Clifford algebra community. A brief recapitulation of the application of $\mathcal{C}_{3}$ to the description of flat space-time, as developed by Baylis, is presented as the proper foundation for the addition of higher space-like dimensions. A matrix representation of $\mathcal{C}_{7}$ is offered in order to better elucidate the structure of the higher-dimensional spinors used and to demonstrate the connection to the conventional Dirac algebra. Section III details the gauge symmetries of the standard model as plane-rotational invariances of the $\mathcal{C}_{3}$ (physical) part of an extended current-density expression involving all of the fermions of a single generation. The various group generators are worked out explicitly for the specific spinor representation used. In Sec. IV the gauge formalism developed is then applied to the construction of each of the various terms in the Lagrangian density. The bare coupling constants then follow immediately from the normalization of the double-sided set of generators. It is also shown that the minimal Higgs field has a natural origin, involving higher-dimensional components of the current.

II. ALGEBRAIC FOUNDATIONS

In the real Clifford algebra $\mathcal{C}_{3}$, the vector elements $\{e_1, e_2, e_3\}$ are chosen to represent the three physical space-like directions. The product of any number of vectors is completely determined by the anticommutator

$$\{e_j, e_k\} = 2\delta_{jk}. \quad (1)$$

All higher-order products of the vectors can be reduced to linear combinations of the 8 basis forms $\{1; e_1, e_2, e_3; e_1 e_2, e_2 e_3, e_3 e_1; e_1 e_2 e_3\}$. The three bivector forms and single trivector form are taken to represent planes and the volume element respectively.

Two basic real conjugations which are anti-automorphic involutions are frequently used in this paper. The reversion of $K \in \mathcal{C}_{3}$, denoted $K^*$, is obtained by reversing the order of appearance of all vector elements within $K$. For example,

$$(e_1 e_2 e_3)^* = e_3 e_2 e_1 = -e_1 e_2 e_3. \quad (2)$$
Both reversing the order and negating all vector elements of $K$ defines Clifford conjugation, denoted by $\bar{K}$. For example,

$$\overline{(e_1 e_2)} = (-e_2)(-e_1) = -e_1 e_2.$$  \hfill (3)

Within $\mathbb{C}l_3$, the latter operation serves to negate the vector and bivector portions while leaving the scalar and trivector invariant. For both of these operations we clearly have

$$\overline{(AB)} = \bar{B} \bar{A},$$

$$\overline{(AB)}^\dagger = B^\dagger A^\dagger. \hfill (4)$$

One type of structure that will be of central importance in this paper is that of projectors—pairs of real idempotent terms $P_{\pm}$ satisfying

$$P_{\pm}^2 = P_{\pm} = P_{\pm}^\dagger,$$

$$P_{\pm} P_{\mp} = 0,$$

$$P_{+} + P_{-} = 1. \hfill (5)$$

An example in $\mathbb{C}l_3$ are the elements

$$P_{\pm 3} = \frac{1}{2}(1 \pm e_3). \hfill (6)$$

In applying this formalism to relativistic physics, time may be allocated to the scalar part of a general real vector $V = v^0 + v^j e_j$. The Minkowski metric then arises naturally through the norm of $V$ given by

$$V \bar{V} = [v^\mu v_\mu]. \hfill (7)$$

The delimiters in Eq. (7) will be used throughout this paper to designate the prevailing non-algebraic notation, outside of which there is no metric implied in the summation convention for repeated indices.

One of the agreeable aspects of this formalism is that the Lorentz transformations have a particularly lucid form. Proper and orthochronous Lorentz transformations of vectors are represented by

$$V \rightarrow L V L^\dagger, \hfill (8)$$

where $L$ is any unimodular element: $LL^\dagger = 1$, which can be expressed as the product

$$L = \exp(\eta/2) \exp(-\theta/2), \hfill (9)$$

where $\eta = \eta^i e_i$ and $\theta = \theta^i e_i j e_j e_k$. Pure rotations $L_R = \exp(-\theta/2)$ are characterized by the unit plane of rotation $e_{ij} e_j e_k/(\theta^2)^{1/2}$. Similarly, pure boosts $L_B = \exp(\eta/2)$ are specified by the rapidity $(-\eta \bar{\eta})^{1/2}$ directed along a unit vector in the direction of the boost. The clear geometric interpretation of this method will be employed later when the gauge transformations are introduced.

Objects such as Eq. (6) are clearly Lorentz invariant scalars, since by Eqs. (4) and (8) we have

$$V \bar{V} \rightarrow LV (L^\dagger L^\dagger) V \bar{L} = LV \bar{V} \bar{L} = (L \bar{L}) V \bar{V} = V \bar{V}. \hfill (10)$$

One must be careful here to distinguish between invariant scalars and the time component of a four-vector, which both occupy the same place in the algebra but transform differently through Eq. (8). The notation $e_0$ will often be used for the time component, with the understanding that it is algebraically just the identity element satisfying $\bar{e}_0 = e_0$. For example, a Lorentz invariant trivector representing the full space-time volume element can be written as

$$e_0 e_1 e_2 \bar{e}_3 \rightarrow L(e_0 e_1 e_2 \bar{e}_3) \bar{L} = e_0 e_1 e_2 \bar{e}_3. \hfill (11)$$

Again, we make no algebraic distinction between this and the trivector $e_1 e_2 e_3$, leaving it to be implicitly understood by the transformation rules which one is meant. In general, the same algebra will be used to construct a number of different objects, which may be further classified by how they transform.

Spinors may be defined as entities which transform according to the rule

$$\psi \rightarrow L \psi. \hfill (12)$$
Under this prescription, spinors span all of the basis forms, but contain some projector structure on the opposite side of the Lorentz transformation operator. To illustrate this point in $\mathbb{C}l_3$, consider the spinor

$$\psi = \sqrt{2}(\phi_{0r} + \phi_{1r}e_1 + \phi_{1i}e_2e_3 + \phi_{0i}e_1e_2e_3)P_{+3},$$

which has been contrived to transform under (12) in the same manner as the right-chiral portion of the conventional Weyl representation column spinor $\tilde{\psi}$. Once $\psi$ is explicitly defined in this manner, the projector structure is unaltered by any Lorentz transformation. Conversely, multiplying any fixed element onto $\psi$ from the right will not affect the Lorentz transformations. Writing a current vector as

$$\psi\psi^\dagger = \begin{bmatrix} \bar{\psi}\gamma^\mu \psi \end{bmatrix} e_\mu,$$

we can see that the choice of projector in this case is not unique, since $\psi$ may be multiplied on the right by a rotation operator $R$, having the same form as $L_R$. This leaves the current invariant by

$$\psi\psi^\dagger \rightarrow \psi R R^\dagger \psi^\dagger = \psi R \bar{R} \psi^\dagger = \psi\psi^\dagger,$$

and rotates the spatial vector in the projector through

$$\psi(1 + e_3)R_\theta = \psi R_\theta(1 + R_\theta e_3 R_\theta^\dagger).$$

One might be tempted to use this as the basis for some SU(2) current symmetry, but instead we shall fix the spatial projector into a specific direction by defining the current as

$$J = \psi P_{+3}\psi^\dagger.$$

Since $P_{+3}^\dagger = P_{+3}$, we need now only consider the $P_{+3}$ spinor forms. The current is then viewed as a sort of ‘pre-Lorentz’ transformation of the light-like vector $\frac{1}{2}(e_0 + e_3)$, with the spinors acting as the transformation operation. The Lorentz transformation of the current is then

$$J \rightarrow L\psi P_{+3}\psi^\dagger L^\dagger = LJL^\dagger,$$

where the internal structure reflects a choice in the form of the spinors and is treated as a Lorentz invariant. The form of Eq. (17) still admits a geometric U(1) gauge symmetry, since an internal rotation of the projector about the plane $e_1e_2$ yields

$$\psi_{+3}\exp(\theta^3 e_1e_2/2) = \exp(i\theta^3/2)\psi_{+3}.$$

This seemingly innocuous restriction that the physical projector should remain gauge invariant will now be extended to a higher-dimensional space, where a more elaborate projector structure on the spinors leads to an interesting framework in which the complete gauge symmetry of the standard model arises in a natural manner.

### A. Higher Dimensions

Each extra spatial dimension beyond the three physical vectors $\{e_1, e_2, e_3\}$ is accommodated simply by introducing a new $e_j$ in accordance with Eq. (1). The commutation of any higher-dimensional ($j > 3$) vector with any element $K \in \mathbb{C}l_3$ may be summarized by

$$e_j K = \bar{K}^\dagger e_j.$$

The definitions of reversal ($K^\dagger$) and Clifford conjugation ($\bar{K}$) exemplified in Eqs. (2) and (3) are extended to encompass higher dimensions with no additional modifications. As vector components, each of these extra dimensions is then manifestly Lorentz invariant. For example,

$$e_4 \rightarrow L e_4 L^\dagger = L\bar{L}e_4 = e_4.$$

\(^1\text{To avoid confusion with later double-sided transformations, the term “-chiral” is preferred in place of “-handed.”}\)
No further presumptions are made regarding the true nature of these additional dimensions, other than that they are, in a sense, inaccessible directly in flat space-time if Lorentz transformations cannot betray their presence.

Rotations in higher dimensions, which are key to the forthcoming discussion of gauge symmetries, are managed by directly extending the form of Eq. (1) to accommodate the exponentiation any element proportional to a unit bivector in \( \mathbb{C}_7 \) representing the plane of rotation. Since rotations in \( \mathbb{C}_3 \) leave all higher dimensions invariant, it is trivial to map the same framework to any plane in \( \mathbb{C}_7 \) by symmetry. Note that in euclidean spaces of dimensionality greater than three, there are an infinite number of planes orthogonal to a given direction. It is no longer sufficient outside of \( \mathbb{C}_3 \) to characterize a rotation by an axis.

The choice of adding four extra space-like dimensions \( \{e_4, e_5, e_6, e_7\} \) to form \( \mathbb{C}_7 \) is arrived at by simple counting arguments. Assuming four complex numbers (eight basis forms) for each fermion, one generation consisting of an electron, a neutrino (assuming both left- and right-chiral), three up quarks and three down quarks, requires 64 basis forms for the spinors. Each additional dimension doubles the number of basis forms, so \( \mathbb{C}_7 \), with 128 basis forms, is appropriate since only half of these will be used under the \( P_{+3} \) projection restriction.

\( \mathbb{C}_7 \) is also attractive in that the total volume element of the algebra

\[
e_t \equiv e_1 e_2 e_3 e_4 e_5 e_6 e_7
\]

commutes with all of the basis forms and squares to \(-1\), and can therefore be identified with the unit imaginary. This occurs for every \( \mathbb{C}_{4n+3} \). If one were to settle for \( \mathbb{C}_6 \), for example, the full volume element of that algebra would anticommute with all vectors, and one would be compelled to consider its inclusion as an additional ‘dimension’ regardless. This is also the case with the Dirac algebra, since \( \gamma^5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \) anticommutes with all \( \gamma^\mu \).

### B. \( \mathbb{C}_7 \) Matrix Representation

Building upon the Dirac algebra defined by

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},
\]

a faithful \( 8 \times 8 \) matrix representation of \( \mathbb{C}_7 \) may be constructed by

\[
1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_k \sim \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}, \quad e_4 \sim \begin{pmatrix} \gamma^1 & \gamma^2 \gamma^3 \\ -\gamma^1 & \gamma^2 \gamma^3 \end{pmatrix}, \\
\quad e_5 \sim \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}, \quad e_6 \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_7 \sim \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]

It is trivial to prove by contradiction that since the equivalent matrices satisfy the anticommutator \([1]\), the matrix representations of the 128 basis forms over the reals span all complex \( 8 \times 8 \) matrices. The total volume element \( e_t \) is equivalent to the diagonal matrix \([i]_{8 \times 8}\) and will be used interchangeably with the unit imaginary. For example, we may write algebraically

\[
e_1 e_2 e_3 = ie_4 e_5 e_6 e_7.
\]

This affords a complex space which will be useful when comparing algebraic terms to those formulated in conventional notation. Note that of the 128 basis forms, only 1 and \( i \) have traceless matrix representations, therefore the complex scalar portion of any algebraic term \( K \in \mathbb{C}_7 \) may be extracted by calculating the trace \( \frac{1}{2} \text{Tr}(M) \) of its equivalent matrix. Embedding the Dirac algebra in this manner supplants the need to introduce any explicit artificial metric as in Eq. (23). This is advanced as the proper means to add extra space-like dimensions to the Dirac algebra.

In developing the structure of spinors, the columns of an arbitrary complex \( 8 \times 8 \) matrix are factored out in terms of the \( \mathbb{C}_7 \) basis forms. This is directly analogous to representing spinors as column matrices in the conventional Dirac formalism, except now we may include several distinct fermion fields in the same term by using different columns for each. It is illustrative to adopt a particular \( \gamma \) matrix representation so that the columns may be factored explicitly. The Weyl representation is again chosen here because it yields a clear partition between the right- and left-chiral spinor components in both the column and algebraic forms.

Consider the top four complex elements of the first column, which are equivalent to

\[
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} \sim [(\phi_0 + \phi_1 e_1) - (\phi_2 - \phi_3 e_1)e_5]P_{+3}P_{-\alpha}P_{+\beta}.
\]
This retains a core spinor structure similar to that of spinors in $C\ell^3$, multiplied by two new projectors which have been abbreviated by

\[
P_{\pm \alpha} \equiv \frac{1}{2}(1 \pm ie_4e_5), \quad P_{\pm \beta} \equiv \frac{1}{2}(1 \pm ie_6e_7).
\]  

(27)

The bottom four components of the first column translate as

\[
\begin{pmatrix}
\zeta_2 \\
\zeta_3 \\
\zeta_0 \\
\zeta_1
\end{pmatrix}
\sim [\zeta_0 + \zeta_1e_1]e_5e_6 + \zeta_2 - \zeta_3e_1]e_6]P_{-3}P_{-\alpha}P_{+\beta},
\]

(28)

which has the same projector structure as the upper four components. The chiral designation has been interchanged in accordance with the conjugation of the lower $\gamma_0\gamma_k$ submatrix in $e_k$. Each of the eight possible products $P_{\pm 3}P_{\pm \alpha}P_{\pm \beta}$ corresponds to a real diagonal matrix which projects out the column associated with each projector set. For example,

\[
P_1 \equiv P_{-3}P_{-\alpha}P_{+\beta} \sim diag(1,0,...,0).
\]

(29)

Anticipating that an SU(2) doublet will arise from the degrees of freedom within a given column, and considering the submatrix structure in (24), the upper and lower four components are ascribed to distinct fermions. Transcribing the remaining columns into algebraic form using the Weyl representation, the four columns corresponding to $P_{+3}$ projections are summarized in table I, where

\[
\psi_R \equiv (\phi_0 + \phi_1e_1)P_{+3},
\]

\[
\psi_L \equiv (\phi_3 - \phi_2e_1)P_{+3},
\]

(30)

and the factor of $\sqrt{8}$ has been inserted for later normalizations. In general, other representations have a similar projector structure and are accessible through a different choice in Eq. (30).

Evidently, the chiral projector for all of the fermions in table I is given by

\[
P_{R/L} = \frac{1}{2}(1 \pm e_4e_5e_6e_7)
\]

(31)

operating from the left-hand side. This is easily seen by monitoring the sign changes in the element

\[
e_4e_5e_6e_7 = -(ie_4e_5)(ie_6e_7)
\]

(32)

as it both passes through the core spinor, where it commutes with any physical vector, anticommutes with any higher-dimensional vector, and is then absorbed into the column projectors. The parity operator is given by

\[
P : \Psi \rightarrow \Psi' = e_1e_2e_3e_4\Psi,
\]

(33)

where $\Psi$ denotes the entire spinor set of table I. Both the projector and operator also have clear matrix representations through (24).

The projectors \(\{P_{\pm 3}, P_{\pm \alpha}, P_{\pm \beta}\}\) all satisfy $P^\dagger = P$ and are mutually commuting \(\Box\). Consequently, in adopting Eq. (17) as the form for the particle current, the only surviving terms are those arising from within the same column. The remaining columns \(\{2,3,5,8\}\) have algebraic forms akin to those of table I, but are $P_{-3}$ projections and therefore do not contribute. The total current obtained from simply adding the algebraic equivalents of all eight column-spinor doublets into a single element $\Psi$ is then

\[
J = \Psi P_{-3}\Psi^\dagger
\]

\[
= \sum_{P_{+3}} \sum_{u,l} \left[ J_{(u)}^\mu \right] e_\mu + \text{(higher-dim. terms)}.
\]

(34)

The sum here runs over the spinors assigned to the upper and lower portions of the $P_{+3}$ columns. This is most easily verified by computing the trace of the matrix representations of the algebraic products $J_\mu$. The eight currents of a single generation of fermions are now incorporated into one expression. The residual part of the current involves cross-current terms and mass-like terms of the form $[\bar{\psi}\psi]$ between the upper and lower fermions of the same column, all
projected onto higher-dimensional elements. For the moment, we will be concerned with only the physical components of the current, deferring the higher-dimensional results to a later discussion of the gauge coupling terms and Higgs field in the Lagrangian.

The above construction of the spinors can also be performed without any matrix representation scaffolding. It is sufficient to specify that the spinor doublets should be minimal left-ideals of $C\ell_7$. However, the matrix representation will prove a useful instructive device when making introductory comparisons with the conventional gauge transformations.

**III. GAUGE SYMMETRIES**

The various gauge symmetries of the standard model will now be constructed by considering rotational transformations on the total spinor which leave the physical parts of the current in Eq. (34) invariant. **Internal** rotations are transformations

$$\Psi \to \Psi R \quad (35)$$

from the right-hand side of the spinor that leave $e_3$ invariant, since then

$$J \to \Psi R P_{+3} R^\dagger \Psi^\dagger = J. \quad (36)$$

The space of available internal planes of rotation in $C\ell_7$ is spanned by the set of fifteen bivector generators

$$e_j e_k : (j, k) \in \{1, 2, 4, 5, 6, 7\}, \ j < k. \quad (37)$$

**External** rotations are comprised of the more limited set of transformations

$$\Psi \to R' \Psi \quad (38)$$

from the left of the spinor that leave the physical vectors of the current invariant. The external rotation generators are spanned by the six planes

$$e_j e_k : (j, k) \in \{4, 5, 6, 7\}, \ j < k. \quad (39)$$

An additional restriction is made in that one of the spinors, namely the right-chiral neutrino, is to be exempt from any such transformations. This will prove sufficient to completely specify the conventional gauge structure of the standard model.

We begin by arbitrarily assigning the lepton doublet to the first column, with the neutrino and electron occupying the upper and lower spinors respectively. The remaining three $P_{+3}$ columns are likewise assigned to up and down quark doublets. In excluding the right-chiral neutrino, the six external generators are reduced to three, since the set may be regrouped into the three listed in table II, which implicitly contain the left-chiral projector, while the right-chiral partners are discarded. Each generator characterizes a simultaneous independent rotation about two commuting planes. These generators satisfy

$$[T'_a, T'_b] = f_{abc} T'_c, \quad (40)$$

with the antisymmetric structure constants $f_{123} = 1$. The conventional presence of the unit imaginary in front of $T'_c$ in Eq. (40) has been absorbed into the properties of bivectors.

In the matrix representation (24), the generators $T'_a$ are nonzero only for the central $4 \times 4$ submatrix. These act on the left-chiral portions of the spinors yielding

$$T'_a \sim [-i\sigma_a/2 \otimes 1_{2 \times 2}]_L, \quad (41)$$

where $\sigma_a$ are the standard Pauli matrices. Therefore, the effect of the transformation

$$\Psi \to \exp(\theta_a T'_a) \Psi \quad (42)$$

on the total spinor is identical to that of the prevailing $SU(2)$ prescriptions

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \to \exp(-i\theta_a \sigma_a/2) \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad (43)$$

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \to \exp(-i\theta_a \sigma_a/2) \begin{pmatrix} u \\ d \end{pmatrix}_L.$$
In considering the broader set of internal transformations, it is useful to think in terms of shuffling entire columns about in the matrix representation, which is the only structural change a rotation solely from the right-hand side can accomplish, aside from introducing phase factors. To remove the right-chiral neutrino from this process we insulate the first column by discarding any candidate generators \( T \) for which \( P,T \neq 0 \). This determines the surviving terms \( T_1 \) through \( T_7 \) arranged in Table I. For example, \( P,T = 0 \) while \( P_1(e_4e_6 + e_5e_7) \neq 0 \). This reduces the degrees of freedom from fifteen to eight, and the remaining plane \( e_1e_2 \) can then be fitted into \( T_8 \) under the same restriction to complete an \( SU(3) \) symmetry. These generators satisfy

\[
[T_a, T_b] = -f_{abc} T_c,
\]

where the antisymmetric structure constants are given by

\[
\begin{align*}
  f_{123} &= 1, \\
  f_{147} &= f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \\
  f_{458} &= f_{678} = \frac{\sqrt{2}}{2}.
\end{align*}
\]

Note that a similar symmetry would have existed from the left-hand side, if it were not that the vectors \( e_1 \) and \( e_2 \) were required to remain invariant as elements of the spatial current. The negative sign in Eq. (44) has been introduced simply to maintain consistent required to remain invariant as elements of the spatial current. The negative sign in Eq. (44) has been introduced simply to maintain consistent required to remain invariant as elements of the spatial current. The negative sign in Eq. (44) has been introduced simply to maintain consistent required to remain invariant as elements of the spatial current. The negative sign in Eq. (44) has been introduced simply to maintain consistent.

To verify that these generators induce the same effect as in the conventional arrangement, label columns 4, 6, and 7 as red, green and blue respectively, then transcribe the generators \( T_a \) back into matrix form using (24). Extracting the 4 \( \times \) 4 submatrix formed by the rows and columns 1, 4, 6 and 7 yields only a lower-right 3 \( \times \) 3 submatrix \(-i\lambda^*_a\), where \( \lambda_a \) are the well-known Gell-Mann matrices. Therefore, the transformation

\[
\Psi \rightarrow \Psi \exp(\theta_a T_a)
\]

leaves the first column invariant and is identical in its effect on the remaining \( P_{4\text{-}3} \) spinor components to

\[
(\psi_R, \psi_G, \psi_B) \rightarrow (\psi_R, \psi_G, \psi_B) \exp(-i\theta_a \lambda^*_a/2),
\]

which is equivalent to the more familiar

\[
\begin{pmatrix}
\psi_R \\
\psi_G \\
\psi_B
\end{pmatrix}
\rightarrow
\exp(-i\theta_a \lambda_a/2)
\begin{pmatrix}
\psi_R \\
\psi_G \\
\psi_B
\end{pmatrix}.
\]

The difference here is that the symmetry arises naturally from the geometric architecture and is not formed by arbitrarily imposing an \( SU(3) \) symmetry acting in some abstract space. Of course, one may deduce the effects of both of these sets of generators by dealing strictly with the algebraic elements and reach the same conclusions. It is a useful exercise for some of the manipulations that will follow.

There remains one symmetry which has not been exploited as yet. We may consider a synchronized double-sided rotation which has the effect of various phase transformations on the spinors and which conspires to cancel out in the case of the right-chiral neutrino. The sole candidate for such an operation is

\[
\Psi \rightarrow \exp(\theta_0 T'_0) \Psi \exp(\theta_0 T_0),
\]

with the \( U(1) \) double-sided generators given by

\[
\begin{align*}
  T'_0 &= \beta_1 e_4 e_5 + \beta_2 e_6 e_7, \\
  T_0 &= \beta_3 e_1 e_2 + \beta_4 e_4 e_5 + \beta_5 e_6 e_7,
\end{align*}
\]

where the \( \beta_k \) are real coefficients satisfying

\[
\beta_1 - \beta_2 + \beta_3 + \beta_4 - \beta_5 = 0.
\]

The latter restriction follows directly from the projector structure of the right-chiral neutrino spinor. As this is to represent a distinct symmetry, the left- and right-side generators must commute with all \( SU(2) \) and \( SU(3) \) generators respectively. This imposes the additional constraints

\[
\begin{align*}
  \beta_1 \psi_R &= \beta_2 \psi_G = \beta_3 \psi_B, \\
  \beta_4 \psi_R &= \beta_5 \psi_G = \beta_3 \psi_B.
\end{align*}
\]
\[ \beta_1 = -\beta_2, \]
\[ \beta_3 = \beta_4 = -\beta_5. \]  
(52)

The trivial solution may be normalized to
\[ T'_0 = \frac{1}{2}(e_5e_4 + e_6e_7), \]
\[ T_0 = \frac{1}{3}(e_1e_2 + e_4e_5 + e_7e_6). \]  
(53)

Applying this operation to each spinor in turn proves to be identical to the conventional \( U(1)_Y \) transformation
\[ [\psi_{(j)} \rightarrow \exp(-i\theta_0 Y_{(j)})\psi_{(j)}], \]  
(54)

with the \( Y_j \) weak hypercharge assignments listed in Table IV. This is most easily verified by examining the matrix representation of the infinitesimal transformation
\[ \Psi \rightarrow \Psi + \theta_0 T'_0 \Psi + \Psi \theta_0 T_0 \]  
(55)

for each of the \( P_{+3} \) columns. All of these weak hypercharge values now arise naturally from a single algebraic operator which was derived from symmetry constraints, and there is no need to artificially insert the charges for each particle.

In order to derive the electromagnetic charges we must look for linear combinations of the generators \( \{T'_0 \pm T_0, T'_3\} \) yielding assignments that are invariant under the parity transformation of Eq. (53). Note that the generators \( T'_3 \) and \( T'_8 \) have been excluded here since no linear combination has a uniform effect on the three quarks. Since the generators from the left must commute with the parity operator \( e_1e_2e_3e_4 \), the only possibility is to isolate the generator \( e_6e_7 = (T'_3 + \frac{1}{2}T'_0) \) on the left. The \( U(1)_{em} \) symmetry is then specified by the generators
\[ T'_em = \frac{1}{2}e_6e_7, \]
\[ T_{em} = \frac{1}{6}(e_1e_2 + e_4e_5 + e_7e_6) \]  
(56)

operating simultaneously from the left and right respectively. It is readily verified that these furnish the correct electromagnetic charge assignments for each of the fermions with
\[ \Psi \rightarrow \exp(\theta_{em} T'_em) \Psi \exp(\theta_{em} T_{em}) \]  
(57)

being identical to
\[ [\psi_{(j)} \rightarrow \exp(-i\theta_{em} Q_{(j)})\psi_{(j)}]. \]  
(58)

This is essentially the basis of the Gell-Mann-Nishijima formula
\[ Q = T_3 + \frac{1}{2}Y. \]  
(59)

One may be tempted here to think that the restrictions in Eq. (52) are simply a circuitous way of establishing the same situation, since Eq. (53) is contrived to satisfy the symmetry constraints
\[ Y(\nu_L) = Y(e_L), \quad Y(u_L) = Y(d_L), \]
\[ Q(\nu_L) = Q(\nu_R), \quad Q(u_L) = Q(u_R), \]
\[ Q(e_L) = Q(e_R), \quad Q(d_L) = Q(d_R), \]  
(60)

which completely determines the relative lepton assignments if \( Y(\nu_R) = 0 \) is imposed. However, no analogy of the latter restriction exists in the quark sector, as there are no such assumptions concerning \( Y(u_R) \). This leaves the quark system indeterminate with an infinite choice of possible charges under Eq. (59). The solution in Eq. (53) contains additional structural information which is lost in the standard prescription and effectively derives the observed charge assignments for the quarks. Ultimately, this may be traced to the identification of the privileged subspace of bivectors within the larger algebra, a choice which is unavailable in the conventional notation.

All of the above transformations may now be combined into a single expression
\[ \Psi \rightarrow \exp(\theta_0 T'_0 + \theta'_a T'_a) \Psi \exp(\theta_0 T_0 + \theta_b T_b), \]  
(61)

which exhausts the plane-rotational symmetries under the condition that the right-chiral neutrino is to be disengaged from all gauge transformations.
IV. LAGRANGIAN TERMS

In this section, the preceding gauge formalism and overall strategy of consolidating terms will be applied to the Lagrangian density. The previous disregard for higher-dimensional products in the current will now be justified in that the form chosen for the fermion terms extracts only select components of the current.

Consider the algebraic form
\[ \Psi \bar{\psi}_\mu \Psi. \] (62)

Since the matrix representations for the vectors of \( \mathcal{C}_7 \) are Hermitian, the matrix \( M \) associated with any general element \( K \in \mathcal{C}_7 \) satisfies the isomorphism
\[ (M^*)^T \sim K^\dagger. \] (63)

The term \( \Psi \bar{\psi}_\mu \) may then be viewed as a stack of complex conjugate row spinors. As multiplication by \( \bar{\psi}_\mu \) from the left cannot mix the columns of \( \Psi \), the diagonal of the matrix representation of (62) contains the contraction of each row spinor with elements from its associated column spinor. From the representation of \( \bar{\psi}_\mu \) in (24), it is then clear that
\[ \langle \Psi \bar{\psi}_\mu \Psi \rangle_s = \sum_f [J_{(f)}^\mu], \] (64)

where the s-bracket denotes the real scalar part. Note that the sum includes the lower chiral-inverted spinor, since the sign of \( \gamma^0 \gamma^k \) is reversed in the lower right \( 4 \times 4 \) submatrix of \( e_k \). Taking the real scalar part here, and with later terms, is simply convenient shorthand notation for what could also be achieved by adding various symmetric counterparts to the algebraic expressions.

It is sometimes convenient to invoke the algebraic equivalent of the square-matrix trace theorem
\[ tr(AB) = tr(BA) \] (65)

when dealing with the real scalar part. For example, we have
\[ \langle \Psi \bar{\psi}_\mu \Psi \rangle_s = \langle \bar{\psi}_\mu \Psi \Psi \bar{\psi}_\mu \rangle_s, \] (66)

which is then seen to be manifestly gauge invariant for physical vectors \( \bar{\psi}_\mu \) since the physical part of the current \( \Psi \Psi \bar{\psi}_\mu \) remains invariant.

The algebraic spatial derivative is defined as
\[ \bar{\partial} \equiv \partial_0 + \partial_1 \bar{e}_1 + \partial_2 \bar{e}_2 + \partial_3 \bar{e}_3 \] (67)

operating to the right. A unidirectional derivative is sufficient since only the real scalar part is to be extracted, symmetrizing the operation regardless. In analogy to the usual fermion derivative term in the Lagrangian, we then have
\[ \mathcal{L}_\partial = \langle \Psi \bar{\psi}_\mu \bar{\partial} \Psi \rangle_s = \sum_f \bar{\psi}_{(f)} i \gamma^\mu \partial_\mu \psi_{(f)}. \] (68)

It must be emphasized that although this expression appears similar to the conventional version, it contains all fermion fields of one generation in a single algebraic term.

Explicitly writing \( i = -\bar{e}_0 \bar{e}_1 \bar{e}_2 \bar{e}_3 \bar{e}_4 \bar{e}_5 \bar{e}_6 \bar{e}_7 \), the Lagrangian component (68), for example, is then manifestly Lorentz invariant via
\[ \mathcal{L}_\partial \rightarrow \langle \Psi \bar{\psi}_\mu L^\dagger \bar{L}^\dagger \bar{\partial} LL \Psi \rangle_s = \mathcal{L}_\partial. \] (69)

To ensure local gauge invariance, we introduce gauge fields \( \{ B, W_a, G_a \} \in \mathcal{C}_3 \) which transform according to
\[ B \rightarrow B + \frac{2}{g} \bar{\partial} \theta_0, \]
\[ W_a \rightarrow W_a + \frac{1}{g} \bar{\partial} \theta^a_0 + f_{abc} \bar{\theta}^b_0 \bar{W}_c, \]
\[ G_a \rightarrow G_a + \frac{1}{g_s} \bar{\partial} \theta_a + f_{abc} \bar{\theta} b \bar{G}_c \] (70)
into the additional Lagrangian terms

\[
\begin{align*}
\mathcal{L}_1 &= -\frac{g'}{2} \langle \Psi^\dagger i\hat{B} (T'_a \Psi + \Psi T_0) \rangle_s, \\
\mathcal{L}_2 &= -g \langle \Psi^\dagger i\hat{\bar{W}}_a T'_a \Psi \rangle_s, \\
\mathcal{L}_3 &= -g_s \langle \Psi^\dagger i\hat{G}_a \Psi T_a \rangle_s.
\end{align*}
\]

(71)

It is sufficient to verify invariance to first order in \( \theta \) using the infinitesimal transformation

\[
\Psi \rightarrow \Psi + (\theta'_a T'_a + \theta T_0) \Psi + \Psi (\theta T_0 + \theta_a T_a).
\]

(72)

Note that \( T'^\dagger = -T \) for all generators, and all external \( T' \) commute with the physical gauge fields.

Rather than belabor the comparison of these terms with conventional expressions, it will simply be mentioned that the usual charge currents are contained within Eq. (71). For example,

\[
\mathcal{L}_1 = -\frac{g'}{2} \left\{ i Y^{(f)} B^{\mu} J^{(f)}_{\mu} \right\}_s.
\]

(73)

Note that \( T_0 \) by itself would be the singlet generator associated with the often ruminated “ninth gluon”, which has now been absorbed into the definition of the weak hypercharge.

In constructing the free-field expressions, the design here is to consolidate the internal and external transformations into two separate terms. The physical part of the tensor associated with each generator \( T_c \) occupies the six vectors and bivectors of \( \mathcal{G}_3 \) and may written in the form

\[
F_c = (\partial W_c - W_c \partial) - g W_a W_b f_{abc}.
\]

(74)

The full generator portion is handled through the contraction

\[
\mathcal{L}_F = -\frac{1}{2} \left\{ F'_a F'_b \{ T'_a, T'_b \} + F_a F_b \{ T_a, T_b \} \right\}_s.
\]

(75)

where \( a = 0 \) and \( b = 0 \) are now included in the sum, since \( T'_a \) and \( T_0 \) commute with all other generators on their respective sides and also do not contract any scalar elements with them. The anticommutator in Eq. (75) is evidently necessary in the case of internal rotations since products such as \( T_4 T_5 = \frac{1}{2} (e_7 e_0 + e_2 e_1) \) would otherwise introduce spurious terms through the presence of a physical plane. This also ensures that under the gauge transformations of Eq. (76), which induce the transformation

\[
F_a \rightarrow F_a + f_{abc} \theta_b F_c,
\]

(76)

the scalar part in (74) remains invariant since the only purely physical element contracted by the generators is then the identity element.

The main reason for displaying the free-field terms in this manner is to emphasize a key point concerning the field normalizations. The \( W \) and \( G \) fields may be entered into Eq. (72) directly, since they share a common factor of

\[
\langle T^2_W \rangle_s = \langle T^2_G \rangle_s = \frac{1}{8}.
\]

(77)

In the case of \( B \) we have

\[
\langle T^2_0 \rangle_s = -\frac{1}{7}, \quad \langle T^2_0 \rangle_s = -\frac{1}{3},
\]

(78)

and are obliged to insert \( W_0 = G_0 = \sqrt{3/20} B \) in order to recover the conventional expression

\[
\mathcal{L}_F = -\frac{1}{4} \left\{ B_{\mu\nu} B^{\mu\nu} + W_{\mu\nu} \cdot W^{\mu\nu} + G_{\mu\nu} \cdot G^{\mu\nu} \right\}.
\]

(79)

At unification energies, where one would expect pure geometry to dominate, the gauge transformations of \( W_a \) and \( G_a \) via Eq. (70) should share a conjoint coupling constant relating the geometric rotation angle to the field strengths. It then follows immediately that the bare coupling constants obey

\[
g = g_s = 1, \quad \tan \theta_w = \frac{g'}{g} = \sqrt{\frac{3}{5}}.
\]

(80)
which results in a Weinberg angle \([10]\) of \(\sin^2 \theta_w = \frac{3}{8}\). Radiative corrections are assumed to lower the weak mixing angle to the observed value of \(\sin^2 \theta_w \approx 0.23\) at accelerator energies. It has not been examined as yet what role the extra space-like dimensions might play in such a renormalization procedure carried out within this framework.

Conspicuously absent in the previous Lagrangian terms are those projecting out the higher-dimensional mass-like components of the current in Eq. (34). It should be mentioned that these reside on the 8-dimensional Lorentz-invariant space \(\{e_4, e_5\} \otimes P_{+\beta} \oplus \{e_6, e_7\} \otimes P_{-\alpha}\), and provide a natural inclusion of the Higgs complex scalar isodoublet \(H\) and anti-Higgs \(\tilde{H}\) through the identification

\[
H = - (\phi_1 e_6 + \phi_2 e_7) P_{+\alpha} - (\phi_3 e_5 + \phi_4 e_4) P_{-\beta}
\]

\[
\tilde{H} = - (\xi_1 e_6 + \xi_2 e_7) P_{-\alpha} - (\xi_3 e_5 + \xi_4 e_4) P_{+\beta}
\]

\[
H \sim \left[ \begin{array}{c} \phi_1 + i \phi_2 \\ \phi_3 + i \phi_4 \end{array} \right],
\]

\[
\tilde{H} \sim \left[ \begin{array}{c} -\xi_3 + i \xi_4 \\ \xi_1 - i \xi_2 \end{array} \right],
\]

(81)

in the expression

\[
\mathcal{L}_M = \frac{G_f}{\sqrt{2}} (\Psi^\dagger (H + \tilde{H}) \Psi)_{\alpha\beta}.
\]

These algebraic elements form a carrier space for the set of external gauge generators where, for example, the transformation required for gauge invariance,

\[
H \to \exp(\theta_0 T_0' + \theta_a T_a') H \exp(-\theta_0 T_0' - \theta_a T_a'),
\]

(83)

is equivalent to the conventional notation

\[
\left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right) \to \exp(-i Y \theta_0 - i \theta_a \sigma_a / 2) \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right).
\]

(84)

The weak hypercharge assignment of \(Y = 1 (Y = -1)\) for the Higgs particle (antiparticle) is recovered naturally from the double-sided algebraic transformation. Here again, the Higgs field is no longer an artificial appendage cast in some abstract space, but emerges readily from the geometry and is associated with the higher-dimensional vector components of the current.

The form of Eq. (82) is an illustrative example for equal fermion masses. Distinct masses may be introduced through weighted projectors on both sides of the spinor set \(\Psi\). This particular area is not understood at present, but it is encouraging that there seems to be additional structure to work with here.

For completeness, the remaining parts of the minimal Higgs sector may also be written algebraically. The gauge-invariant free-field and potential terms respectively are

\[
\mathcal{L}_H = \langle (\bar{\partial}H - \frac{g'}{2} \bar{B}[T_0', H] - g \bar{W}_j [T_j', H])^2 \rangle_s,
\]

\[
\mathcal{L}_V = - (\mu H^2 + \lambda H^4 + \cdots)_s,
\]

(85)

where Eq. (20) provides for the Minkowski contraction of the physical components. The gauge symmetry may broken by choosing a vacuum expectation value

\[
H_0 = -ve_5 P_{-\beta}, \ v \equiv -\mu^2 / \lambda,
\]

(86)

analogous to the conventional choice. This leads directly to the vector-boson mass relations of the Weinberg-Salam model \([10,11]\), the relevant initial term being

\[
\mathcal{L}_{H_0} = \frac{\nu^2}{8} [g^2 \bar{W}_j W_j - gg' (\bar{W}_3 B + \bar{B} W_3) + g^2 \bar{B} B] + \cdots.
\]

(87)
V. CONCLUSION

A common criticism on the application of Clifford algebras is that, despite any alluring framework that may be constructed, geometric approaches are merely reformulations of conventional expressions and are consequently devoid of additional insights. In the present work, such an amelioration prompted the addition of extra space-like dimensions and was essential in identifying the otherwise obscured geometric subspace in which the gauge symmetries abide. These symmetries are no longer relegated to abstract spaces, but are seen to arise naturally from the architecture of higher-dimensional spinors. As a consequence, the proposed unification of the elementary-particle forces follows from purely geometric concerns.

Note that the value of $\sin^2 \theta_w = \frac{3}{8}$ at unification energies is identical to the often touted result from minimal $SU(5)$ grand unification [12]. This is not surprising, as both originate from the normalization of the weak hypercharge operator. However, the contention here is distinctly different. In the geometric approach, the second ‘symmetry-breaking’ of the strong and electroweak forces arises naturally by invoking double-sided transformations. The introduction of a master group such as $SU(5)$ to accomplish this adds extra gauge bosons and the hierarchy problem associated with their experimental absence. Furthermore, there is seldom any physical justification for a particular abstract master group, other than its suiting a selection process under various constraints. This has been done here to some extent in that the choice of $\mathcal{O}_7$ was to accommodate the number of observed fermions. However, the notion that higher dimensions are directly responsible for the presence of gauge groups is more fundamental, as the physical basis for those symmetries is apparent.

An obvious deficiency of this model in its present form is the need to suppress the right-chiral neutrino. Also lacking is a geometric rationale for the inclusion of three generations of fermions and for some indication as to the origin of their disjointed masses. Work is continuing on these and other aspects of the standard model within this framework.

ACKNOWLEDGMENTS

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[1] See, for example, P. Lounesto, Clifford Algebras and Spinors (Cambridge University Press, Cambridge, England, 1997).
[2] W. E. Baylis, Phys. Rev. A 45, 4293 (1992).
[3] P. W. Higgs, Phys. Rev. 145, 1156 (1966).
[4] Within $\mathcal{O}_3$, the trivector element both commutes with all of the basis forms and squares to $-1$, and can therefore be identified with the unit imaginary: $j \equiv e_1e_2e_3$. This reduces the eight-dimensional real space of the basis forms to a complex four-dimensional space over the basis elements $\{1, e_1, e_2, e_3\}$. For example, we may write Eq. (13) as $\psi = \sqrt{2}(\phi_0 + \phi_1 e_1)P_{+3}$. This has been tactfully avoided here since the unit imaginary will be identified with the full volume element of $\mathcal{O}_7$. The results of this work can also be derived using the $\mathcal{O}_3$ definition, but there is the added algebraic hazard that $j$ does not commute with higher-dimensional vectors. See, for example, W. E. Baylis, J Huschilt, and Jiansu Wei, Am. J. Phys. 60 (9), 788 (1992).
[5] W. E. Baylis and G. Jones, J. Phys. A 22, 1 (1989).
[6] There are various conventions for the Weyl representation. To avoid confusion, the choice used throughout this work is explicitly stated as

$$
\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},
$$

$$
\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad g^{\mu\nu} = (1, -1, -1, -1).
$$

This is consistent with, for example, M. Kaku, Quantum Field Theory (Oxford University Press, New York, 1993).
[7] A similar primitive idempotent structure for particle doublets has been proposed for the algebra $\mathcal{O}_{1,6}$, but there are crucial differences both in the algebras themselves and in allotting time to the scalar component here. See, for example, J. S. R. Chisholm, and R. S. Farwell, Properties of Clifford Algebras for Fundamental Particles, in Clifford (Geometric) Algebras, ed. W. E. Baylis (Birkhäuser, 1996), pp 365-388.
TABLE I. The algebraic $P_{+3}$ spinors.

| column # | lower spinor | upper spinor |
|----------|--------------|--------------|
| 1        | $\sqrt{8}(\psi_{R)e_6e_1})P_{+3}P_{+\beta}$ | $\sqrt{8}(\psi_{R}e_1e_5)P_{-\alpha}P_{+\beta}$ |
| 4        | $\sqrt{8}(\psi_{R)e_1e_6}P_{+3}P_{+\beta}$ | $\sqrt{8}(\psi_{R}e_1e_6)P_{+\alpha}P_{+\beta}$ |
| 6        | $\sqrt{8}(\psi_{R}e_1e_6 + \psi_{R}e_5e_1)P_{-\alpha}P_{-\beta}$ | $\sqrt{8}(\psi_{R}e_1e_6 + \psi_{L}e_6e_5)P_{-\alpha}P_{-\beta}$ |
| 7        | $\sqrt{8}(\psi_{R} + \psi_{L}e_5e_1)P_{+\alpha}P_{-\beta}$ | $\sqrt{8}(\psi_{R}e_6e_5 + \psi_{L}e_6e_1)P_{+\alpha}P_{-\beta}$ |

TABLE II. The algebraic SU(2) generators.

$T'_1 = \frac{1}{4}(e_6e_4 + e_5e_7)$
$T'_2 = \frac{1}{4}(e_4e_7 + e_5e_6)$
$T'_3 = \frac{1}{4}(e_6 + e_7)$

TABLE III. The algebraic SU(3) generators.

$T_1 = \frac{1}{4}(e_6e_4 + e_5e_7)$
$T_2 = \frac{1}{4}(e_4e_6 + e_5e_7)$
$T_3 = \frac{1}{4}(e_4e_5 + e_6e_7)$
$T_4 = \frac{1}{4}(e_1e_7 + e_2e_6)$
$T_5 = \frac{1}{4}(e_6e_1 + e_2e_7)$
$T_6 = \frac{1}{4}(e_1e_4 + e_2e_5)$
$T_7 = \frac{1}{4}(e_1e_4 + e_2e_5)$
$T_8 = \frac{1}{4\sqrt{3}}(2e_1e_2 + e_5e_4 + e_6e_7)$

TABLE IV. Charge assignments for fermions.

|   | $T_3$ | $Y$ | $Q$ |
|---|-------|-----|-----|
| $\nu_L$ | 1/2  | -1  | 0   |
| $\nu_R$ | 0    | 0   | 0   |
| $e_L$    | -1/2 | -1  | -1  |
| $e_R$    | 0    | -2  | -1  |
| $u_L$    | 1/2  | 1/3 | 2/3 |
| $u_R$    | 0    | 4/3 | 2/3 |
| $d_L$    | -1/2 | 1/3 | -1/3|
| $d_R$    | 0    | -2/3| -1/3|