On integral and differential representations of Jordan chains and the confluent supersymmetry algorithm

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Abstract
We construct a relationship between integral and differential representation of second-order Jordan chains. Conditions to obtain regular potentials through the confluent supersymmetry algorithm when working with the differential representation are obtained using this relationship. Furthermore, it is used to find normalization constants of wave functions of quantum systems that feature energy-dependent potentials. Additionally, this relationship is used to express certain integrals involving functions that are solution of Schrödinger equations through derivatives.

Keywords: Schrödinger equation, confluent supersymmetry algorithm, Jordan chain

1. Introduction
A Jordan chain for a linear operator is a sequence of vectors that satisfies a particular system of equations. In the most elementary case, the linear operator is a square matrix and the corresponding Jordan chain is a sequence of generalized eigenvectors that are needed for the construction of Jordan’s normal form [1]. If the linear operator associated with the Jordan chain is a Hamiltonian, then the corresponding system of equations appears in the confluent version of the supersymmetry formalism (SUSY). The general purpose of SUSY is the construction of solvable quantum models (SUSY partners) and the manipulation of their
spectra (spectral design). There is a large amount of literature on the topic, a selection of which can be found in the references of the reviews [2, 3] or the more recent work [4]. Within the formalism of SUSY, the standard algorithm and the confluent algorithm are to be distinguished. Being well understood, the vast majority of applications uses the standard algorithm, while its confluent counterpart is much less known. An introduction to the algorithm and discussion of mathematical properties can be found in [5–10] whereas some applications in [11–16] and references therein. In the confluent SUSY algorithm, partner models are essentially determined through Jordan chains. The system of equations associated with a Jordan chain is solved by transformation functions that are used to construct the SUSY partner to a given model. The solution of such a system of equations can be represented in integral form [10] or by means of derivatives [17]. While the process of obtaining the integral form is more straightforward [10, 18], a closed-form evaluation of the integrals is often not possible. Since the differential approach is much more feasible in calculations [19, 20], it is of particular interest to find the mathematical relationship between the two approaches. In the present work, we determined the latter relationship and presented several applications regarding the confluent SUSY algorithm and the resolution of integrals to find normalization constants for energy dependent quantum mechanical systems.

The paper is organized as follows. We introduce and discuss Jordan chains of second order in section 2. We present in section 3 two main application of the results obtained: the confluent SUSY algorithm and energy dependent potentials, showing that some integrals appearing both situations can be expressed through derivatives. Finally, our conclusions are presented in the last section.

2. Jordan chains of second order

Let \( D \subseteq \mathbb{R} \) be an open interval. The following system of differential equations is referred to as Jordan chain of second order

\[
\begin{align*}
\lambda & \frac{d}{dx} (u, \lambda) + [\lambda - V(x, \lambda)] u(x, \lambda) = 0, \\
\lambda & \frac{d}{dx} (v, \lambda) + [\lambda - V(x, \lambda)] v(x, \lambda) = \left[V_\lambda(x, \lambda) - 1 \right] u(x, \lambda), \quad (x, \lambda) \in (D, \mathbb{R}).
\end{align*}
\]

Here, the indices denote partial differentiation. We assume that the function \( V \) and its partial derivative \( V_\lambda \) are continuous. Furthermore, the solutions \( u \) and \( v \) are required to be three times continuously differentiable. Since equation (1) resembles the quantum-mechanical Schrödinger equation, the function \( V \) and the variable \( \lambda \) will be called potential and energy, respectively. In the particular case that the potential is independent of the energy, we have \( V_\lambda = 0 \), such that the Jordan chain simplifies to

\[
\begin{align*}
\lambda & \frac{d}{dx} (u, \lambda) + [\lambda - V(x)] u(x, \lambda) = 0, \\
\lambda & \frac{d}{dx} (v, \lambda) + [\lambda - V(x)] v(x, \lambda) = -u(x, \lambda), \quad (x, \lambda) \in (D, \mathbb{R}).
\end{align*}
\]

These equations play an important role in the confluent algorithm of the quantum-mechanical SUSY formalism, as will be discussed in detail below. Let us now return to our more general system (1), (2), where we are interested in the construction of solutions. Our starting point is the assumption that we know a solution \( u \) of the first equation (1). From here we will distinguish two options for determining a solution \( v \) to the second equation (2).
2.1. Integral representation

Our first option is the variation-of-constants formula, which requires two linearly independent solutions \( u_1 \) and \( u_2 \) of (1) in order to become applicable. For the sake of simplicity, let us make the following choices for these two functions

\[
\begin{align*}
    u_1(x, \lambda) & = u(x, \lambda), \\
    u_2(x, \lambda) & = u(x, \lambda) \int^t \frac{1}{u'^2(t, \lambda)} \, dt.
\end{align*}
\]

Note that the integral form of \( u_2 \) is obtained through reduction of order \([21]\). Observe further that due to the choice (5), the Wronskian \( W_{u_1 u_2} \) of \( u_1 \) and \( u_2 \) is equal to one. The variation-of-constants formula then gives the following particular solution \( v_{VC} \) to the second equation (2) of our Jordan chain.

\[
v_{VC}(x, \lambda) = u_1(x, \lambda) \int^x u(t, \lambda) u_2(t, \lambda) \left[ 1 - V_2(t, \lambda) \right] dt \\
- u_2(x, \lambda) \int^x u(t, \lambda) u_1(t, \lambda) \left[ 1 - V_1(t, \lambda) \right] dr,
\]

Note that the function \( u \) appears in the nonhomogeneous term on the right side of (2). We can replace the latter function, as well as \( u_2 \), by their definitions shown in (5). We obtain the following expression

\[
v_{VC}(x, \lambda) = u(x, \lambda) \int^x \left[ \int^t \frac{1}{u'^2(t, \lambda)} \, dt \right] u^2(t, \lambda) \left[ 1 - V_2(t, \lambda) \right] dt \\
- u(x, \lambda) \left[ \int^x \frac{1}{u'^2(t, \lambda)} \, dt \right] \left[ \int^x u^2(t, \lambda) \left[ 1 - V_2(t, \lambda) \right] dt \right].
\]

After applying integration by parts to the first term on the right side, we arrive at the following integral representation of our solution \( v_{VC} \)

\[
v_{VC}(x, \lambda) = -u(x, \lambda) \int^x \left[ \int^s u^2(s, \lambda) \left[ 1 - V_2(s, \lambda) \right] ds \right] \frac{1}{u'^2(t, \lambda)} \, dr.
\] (6)

The principal disadvantage of this representation is its limited applicability. Inspection of (6) shows that the double integral can only be evaluated in closed form for relatively simple functions \( u \) and \( V \). For a solution \( u \) of the second-order equation (1) that is typically given through special functions, it is in general not possible to evaluate (6).

2.2. Differential representation

We will now derive a representation of the solution \( v \) to (2) that does not contain any integral. To this end, we substitute the derivative \( v = u_1 \) into the left side of (1). This gives

\[
\begin{align*}
    (u_1)_x(x, \lambda) + [\lambda - V(x, \lambda)] u_1(x, \lambda) \\
    = (u_2)_x(x, \lambda) + [\lambda - V(x, \lambda)] u_1(x, \lambda) \\
    = \left[ u_{xx}(x, \lambda) + [\lambda - V(x, \lambda)] u(x, \lambda) \right]_x - [\lambda - V(x, \lambda)] u(x, \lambda), \\
    = -[\lambda - V(x, \lambda)] u(x, \lambda).
\end{align*}
\] (7)

Note that in the last step we made use of (1). Next, we observe that (7) is the same as our initial equation (2). Consequently, we have shown that
\nu_{\text{DF}}(x, \lambda) = u_j(x, \lambda), \tag{8}

is a particular solution of (2), where the index is an abbreviation for differential formula. Since (8) is free of integrals, it is generally much easier to evaluate than (6), and as such more suitable for applications.

While both the variation-of-constants scheme and the differential formula provide methods for solving (2), the respective particular solutions \nu_{\text{VC}} and \nu_{\text{DF}} are in general different from each other. It is therefore desirable to know the precise relation between these functions, such that one function can be expressed through its counterpart. In order to find this relation, let \( u_1 \) and \( u_2 \) be linearly independent solutions of (1), the Wronskian of which is equal to one. Recall that such a choice can be made through the settings (5). Since both \nu_{\text{VC}} and \nu_{\text{DF}} are particular solutions of the nonhomogeneous equation (2), their difference must be a solution to the homogeneous equation (1), which can be given as a linear combination of \( u_1 \) and \( u_2 \). Equivalently, there are two parameters \( d_1 \) and \( d_2 \), such that the equation

\[ d_1(\lambda) u_1(x, \lambda) + d_2(\lambda) u_2(x, \lambda) = \nu_{\text{DF}}(x, \lambda) - \nu_{\text{VC}}(x, \lambda), \tag{9} \]

is fulfilled. The right side of this equation is completely determined by the choice of \( u_1 \) and \( u_2 \). Therefore, the remaining task is to calculate the functions \( d_1 \) and \( d_2 \). Since we need a second equation in order to find these two constants, we take the partial derivative of (9) with respect to \( x \), yielding

\[ d_1(\lambda) (u_1)_x(x, \lambda) + d_2(\lambda) (u_2)_x(x, \lambda) = (\nu_{\text{DF}})_x(x, \lambda) - (\nu_{\text{VC}})_x(x, \lambda). \tag{10} \]

Hence, we now have an algebraic system of equations (9) and (10) that is linear in \( d_1 \) and \( d_2 \). Application of Cramer’s rule and taking into account that \( W_{u_1 u_2} = 1 \), we arrive at the Wronskian representation of the solutions

\[ d_1(\lambda) = W_{\nu_{\text{DF}} - \nu_{\text{VC}}}(x_0, \lambda), \quad d_2(\lambda) = W_{\nu_{\text{DF}} - \nu_{\text{VC}}}(x_0, \lambda). \tag{11} \]

Note that the constant \( x_0 \in D \) can be chosen arbitrarily, since neither \( d_1 \) nor \( d_2 \) depend on the actual value of the variable \( x \). In summary, if the functions \( d_1, d_2 \) are chosen as in (11), then the particular solutions \nu_{\text{VC}} and \nu_{\text{DF}} of (2) are related to each other by means of the identity (9). Recall that the explicit form of the particular solutions can be found in (6) and (8).

### 3. Applications

The Jordan chain and the differential representation of its solution has several interesting applications, two of which we will now introduce. The first application concerns the normalizability of solutions associated with Schrödinger equations for energy-dependent potentials. The second part of this section is devoted to the confluent SUSY formalism and several examples.

#### 3.1. The confluent SUSY algorithm

The second application of our results on second-order Jordan chains concerns the confluent SUSY algorithm. For sake of completeness we present here a brief introduction to that algorithm, for a detailed description see [4] and references therein.

We start out by considering the following stationary Schrödinger equation

\[ \psi_n(x) + [\epsilon - V(x)]\psi(x) = 0, \tag{12} \]

where \( \epsilon \) is a real constant and the potential \( V \) is a real function. Suppose further that we know two auxiliary functions \( u, v \) satisfying the Jordan chain
for a real constant $\lambda$. Observe that these equations are a special case of our general system (1), (2), where the potential does not depend on the variable $\lambda$. Then the function

$$
\phi(x, \lambda) = \frac{W_{u,v}(x, \lambda)}{W_{u,v}(x, \lambda)} = \frac{u^2(x, \lambda)}{W_{u,v}(x, \lambda)} \psi(x) + \left[ \lambda - e - \frac{u(x, \lambda) u_x(x, \lambda)}{W_{u,v}(x, \lambda)} \right] \psi(x),
$$

when $\epsilon \neq \lambda$, or in the opposite situation

$$
\phi(x, \lambda) = \frac{u(x, \lambda)}{W_{u,v}(x, \lambda)},
$$

fulfills the equation

$$
\phi_{xx}(x, \lambda) + \left[ \lambda - \tilde{V}(x, \lambda) \right] \phi(x, \lambda) = 0,
$$

where the potential $\tilde{V}$ is given by the expression

$$
\tilde{V}(x, \lambda) = V(x) - 2 \left\{ \log \left[ W_{u,v}(x, \lambda) \right] \right\}\lambda.
$$

This transformation is known as confluent SUSY transformation and the potentials $\tilde{V}$ and $V$ are called SUSY partners. Observe that the function $\phi$ in (15) depends on $\epsilon$. Since we are mainly interested in the dependence on $\lambda$, we do not include $\epsilon$ as an argument of $\phi$.

Now, as can be seen from (15), (16) and (18) the Wronskian $W_{u,v}$ plays a fundamental role to perform the transformation. Furthermore, to obtain regular potentials $\tilde{V}$ we should avoid zeros in this Wronskian. If the transformation function $v$ is obtained by the variation-of-constants method (see (6)), then $W_{u,v}$ can be expressed as

$$
W_{u,v}(x, \lambda) = \omega_0 - \int_{x_0}^{x} u^2(t, \lambda) dt,
$$

where $x_0$ is a point in the domain of $V$ and $\omega_0$ is an arbitrary constant. The conditions for obtaining a regular potential $\tilde{V}$ from the confluent SUSY transformation are known and can be presented as follows: Let the domain of $V$ be the interval $(x_f, x_r)$, then we have two non-excluding options:

- We use a solution of (13) as transformation function such that $u(x_f) = 0$. In this case the integral

$$
I_f \equiv \int_{x_f}^{x_0} u^2(y, \lambda) dy < \infty,
$$

and as a consequence if $\omega_0 \in (-\infty, -I_f]$, we avoid zeros in the Wronskian.

- The transformation function $u$ satisfies $u(x_r) = 0$. Thus, the integral

$$
I_r \equiv \int_{x_r}^{x_0} u^2(y, \lambda) dy < \infty,
$$

and if $\omega_0 \in [I_r, \infty)$, we guarantee that $W_{u,v}(x, \lambda)$ never vanishes.

When the solution $v$ of (14) is a superposition of the general solution of the corresponding homogeneous equation and the particular solution obtained by differentiating $u$ with respect to the parameter $\lambda$ (see (8)) the regularity conditions have not been fully explored. Let us express the Wronskian by
\[ W_{u,v}(x, \lambda) = K + W_{u,v}(x, \lambda), \quad (22) \]

where \( K \) is an arbitrary constant. A derivative with respect to the position, and using the Jordan chain (13), (14) to simplify second order derivatives, leads to

\[ \frac{\partial W_{u,v}(x, \lambda)}{\partial x} = -u^2(x, \lambda), \quad (23) \]

i.e., for a fixed \( \lambda \), \( W_{u,v} \) is a non increasing monotone function of \( x \). Thus, two conditions can lead to a regular potential:

- Non diverging transformation function \( u(x, \lambda) \) when \( x \rightarrow x_r \) and
  \[ K \leq -W_{u,v}(x_r, \lambda). \quad (24) \]

- Non diverging transformation function \( u(x, \lambda) \) when \( x \rightarrow x_r \) and
  \[ K \geq -W_{u,v}(x_r, \lambda). \quad (25) \]

Note that when the transformation function fulfills both boundary conditions, the domain of \( K \) is \((-\infty, -W_{u,v}(x_r, \lambda)] \cup [-W_{u,v}(x_r, \lambda), \infty)\).

Furthermore, integration of (23) leads to the interesting result

\[ \int_{x_0}^{x_1} u^2(t, \lambda) \, dt = W_{u,v}(x_0, \lambda) - W_{u,v}(x, \lambda), \quad (26) \]

where \( x_0 \in (x_r, x_r) \). This last equation shows how to integrate the square of a function that is solution of a Schrödinger equation using a Wronskian and vice versa. This identity is useful when we are interested, for example, in finding normalization constants or probabilities in an interval.

In the following two examples we illustrate first with the infinite well quantum system how to obtain the normalization constant of its eigenfunctions using (26) and then how to obtain its SUSY partners using \( v_{DF} \) and the regularity conditions presented in (24) and (25).

Since SUSY partners of this potential have been obtained in [22] using the representation of the Wronskian as in (19), this example provides a good comparison of the two methods. In the second example we build new exactly solvable potentials departing from the radial oscillator system and study the regularity conditions for obtaining these potentials.

### 3.1.1. Particle in abox

Consider a particle included between two impenetrable potential walls at \( x = 0 \) and \( x = 1 \). The Schrödinger equation of the system is

\[ \psi_{\epsilon}(x) + \epsilon \psi(x) = 0, \quad x \in (0, 1), \quad (27) \]

and the boundary conditions are

\[ \lim_{x \to 0} \psi(x, \epsilon) = \lim_{x \to 1} \psi(x, \epsilon) = 0. \quad (28) \]

If \( \epsilon = k^2 \) the the general solution of (27) is given by

\[ \psi(x) = A \sin(kx) + B \cos(kx), \quad (29) \]

where \( A \) and \( B \) are arbitrary constants. To satisfy the boundary conditions (28) we need to set \( B = 0 \) and \( k = n\pi \), where \( n \) is a natural number, then the eigenfunctions and eigenvalues are
the normalization constant $A$ of which we can obtain by means of (26). In order to avoid confusion regarding the notation, we extend the solution (29) to a function of two variables $\psi = \psi(x, \epsilon)$, where it is understood that $|k| = \sqrt{\epsilon}$. The derivative with respect to $\epsilon$ can be obtained using the chain rule

$$
\psi_\epsilon(x, \epsilon) = \frac{\partial k}{\partial \epsilon} \frac{\partial \psi}{\partial k} = \frac{Ax}{2k} \sin(kx),
$$

(note that we did not include the variable of differentiation $\epsilon$ as an argument of $\psi$). We have

$$
W_{\psi, \psi_\epsilon} (x, \epsilon) = \frac{A^2}{2} \left[ \frac{\sin(2kx)}{2k} - x^2 \right].
$$

finally, the normalization condition leads us to

$$
1 = \int_0^1 \psi^2 dx = W_{\psi, \psi_\epsilon} (0, \epsilon) - W_{\psi, \psi_\epsilon} (1, \epsilon) = \frac{A^2}{2} \Rightarrow A = \sqrt{2}.
$$

To generate an exactly solvable potential through the confluent SUSY formalism using the differential representation of the Wronskian (22) set $\lambda = m^2 \pi^2$, where $m$ is a natural number, then the transformation functions $u$ and $v$ satisfying the Jordan chain (13), (14) for $V(x) = 0$. Let us consider

$$
u(x, \lambda) = \frac{1}{2 \sqrt{\lambda}} x \cos \left( \sqrt{\lambda} x \right) - \frac{K}{\sqrt{\lambda}} \cos \left( \sqrt{\lambda} x \right),
$$

where $K$ is an arbitrary constant. Note that the first term of $v$ is a particular solution of (14) obtained as $v_{DF} = u_j$, while the second term is solution of the corresponding homogeneous equation. Now, to obtain regular potentials we need to evaluate the Wronskian $W_{u, u_\epsilon}$ in $x = x_\ell$ and $x = x_r$:

$$
W_{u, u_\epsilon} (0, \lambda) = 0, \quad W_{u, u_\epsilon} (1, \lambda) = -\frac{1}{2}.
$$

According to the conditions (24) and(25), if the constant $K \in (-\infty, 0] \cup [1/2, \infty)$ then the produced potential is regular. The potential generated is given by (22) and can be expressed as
\[ \hat{V}(x, \lambda) = \frac{16\pi^2m^2[1 + m\pi(2K - x)\sin(2mx) - \cos(2mx)]}{[2m\pi(2K - x) + \sin(2mx)]^2}, \]  
(37)

recall that \( m = \sqrt{\lambda}/\pi \). Figure 1 shows a potential \( \hat{V} \) (see (37)) where transformation functions (34), (35) with a parameter \( \lambda = 4\pi^2 \) were used. On the right, it can be seen its first three eigenfunctions (continuous, dotted and dashed curves respectively), the first and third were generated with the rule (15) while the second with (16) since \( \lambda = 4\pi^2 \) is also an eigenvalue of the original system, see (30).

Moreover, any integral of the form \( \int u^2dx \), where \( u \) is solution of (13), can be obtained with (26) even though the boundary condition (28) are not satisfied. Let us consider

\[ u_1(x, \lambda) = \sin \left( \sqrt{\lambda}x \right), \quad u_2(x, \lambda) = -\frac{1}{\sqrt{\lambda}} \cos \left( \sqrt{\lambda}x \right), \]  
(38)

then the following integrals can be obtained with (26):

\[ \int_0^x u_1^2(t, \lambda)dt = W_{u_1(u_1)}(0, \lambda) - W_{u_1(u_1)}(x, \lambda) = \frac{x}{2} - \frac{1}{4\sqrt{\lambda}} \sin \left( 2\sqrt{\lambda}x \right), \]  
(39)

\[ \int_0^x u_2^2(t, \lambda)dt = W_{u_2(u_2)}(0, \lambda) - W_{u_2(u_2)}(x, \lambda) = \frac{x}{2\lambda} + \frac{1}{4\lambda^{3/2}} \sin \left( 2\sqrt{\lambda}x \right). \]  
(40)

Furthermore, using the relations (9) and (11) doubles integrals of the form \( \int (\int u^2dx)dx \) can be found. The transformation function \( v \) using the variation of constant formula (6) in this example is given by

\[ v_{VC}(x, \lambda) = -u_1(x, \lambda) \int_{x_0}^x \left[ \int_{x_0}^t u_1^2(s, \lambda)ds \right] \frac{1}{u_1^2(t, \lambda)}dt, \]  
(41)

notice that the two integration constants associated with the integrals have been fixed by means of the integration limits, where \( x_0 \) is an arbitrary number in the domain \( D \) of our Jordan chain. Now, the function \( v_{VC} \) satisfies the conditions

\[ v_{VC}(x_0, \lambda) = 0. \]  
(42)

Keeping this in mind, the formulas in (11) reduce to

\[ d_1(\lambda) = W_{v_{VC}u_1}(x_0, \lambda), \quad d_2(\lambda) = W_{v_{VC}u_2}(x_0, \lambda). \]  
(43)

Let us point out that the functions in (43) do not depend on any integrals. Now, solving (9) for the double integral in it,

\[ \int_{x_0}^x \left[ \int_{x_0}^t u_1^2(s, \lambda)ds \right] \frac{1}{u_1^2(t, \lambda)}dt = \frac{1}{u_1(x, \lambda)} \left[ (u_1)_1(x, \lambda) - d_1(\lambda)u_1 - d_2(\lambda)u_2(x, \lambda) \right], \]  
(44)

recall that \( v_{VC} = (u_1)_1 \). Substitution of (38) in (44) leads to

\[ \int_{x_0}^x \left[ \int_{x_0}^t u_1^2(s, \lambda)ds \right] \frac{1}{u_1^2(t, \lambda)}dt = \frac{\cos^2 \left( \sqrt{\lambda}x_0 \right)}{2\lambda} - \frac{x - x_0}{2\sqrt{\lambda}} \sin \left( 2\sqrt{\lambda}x_0 \right) \cot \left( \sqrt{\lambda}x \right). \]  
(45)

In this simple case, integrals in (39), (40) and (45) can be verified by direct integration.
3.1.2. The radial oscillator system. The Schrödinger equation for the radial oscillator potential is given by [23]

\[
\psi_{\ell}(x) + \left[ e - x^2 - \frac{\ell(\ell + 1)}{x^2} \right] \psi(x) = 0, \quad x \in (0, \infty),
\]

where the potential \( V \) can be identified as

\[
V(x) = x^2 + \frac{\ell(\ell + 1)}{x^2},
\]

and we will consider in this example \( \ell \) as a natural number. The boundary conditions of this quantum problem are

\[
\lim_{x \to 0} \psi(x) = \lim_{x \to \infty} \psi(x) = 0.
\]

The general solution of (46) without considering the boundary condition is

\[
\psi(x) = x^{\ell+1} \exp\left(-\frac{x^2}{2}\right) \left\{ A \; \ confluent \; hypergeometric \left( \frac{1}{4}(2\ell + 3 - \epsilon); \ell + \frac{3}{2}; x^2 \right) + B \; x^{-(2\ell+1)} \ confluent \; hypergeometric \left( \frac{1}{4}(-2\ell + 1 - \epsilon); -\ell + \frac{1}{2}; x^2 \right) \right\},
\]

where \( _1F_1 \) is the confluent hypergeometric function [24], and \( A, B \) are arbitrary constants. Since the second term of \( \psi \) as given in (49) is divergent at the origin we need to set \( B = 0 \) in order to satisfy the first boundary condition. The second condition is fulfilled only when the first argument of the hypergeometric function is a negative integer. Thus, the eigenfunctions and eigenvalues of (46) are

\[
\psi(x) = A \; x^{\ell+1} \exp\left(-\frac{x^2}{2}\right) \ confluent \; hypergeometric \left( -n; \ell + \frac{3}{2}; x^2 \right), \quad \epsilon = 4n + 2\ell + 3, \quad n = 1, 2, 3, ...
\]

To obtain the confluent SUSY partners (18) of the potential (47) using the differential formula (22) we need a solution of the Jordan chain (13), (14). We will use the solution \( u \) of the first equation in the Jordan chain as given in (49) with \( A = 1, B = 0 \) and \( \lambda \) different from any of the eigenvalues:

\[
u(x, \lambda) = x^{\ell+1} \exp\left(-\frac{x^2}{2}\right) \ confluent \; hypergeometric \left( \frac{1}{4}(2\ell + 3 - \lambda); \ell + \frac{3}{2}; x^2 \right).
\]

A particular solution of the second equation in the Jordan chain can be constructed as \( u_1 \). In order to obtain this derivative, the formula of the derivatives with respect the parameters of the confluent hypergeometric function can be found in [25]. Then the second transformation function can be expressed as:

\[
v(x, \lambda) = -\frac{1}{4} x^{\ell+1} \exp\left(-\frac{x^2}{2}\right) \left[ \sum_{m=0}^{\infty} \binom{\ell}{4} x^{2m} \sum_{p=0}^{m-1} \frac{1}{p!(2\ell + 3 - \lambda)} \right] - \frac{K}{2\ell + 1} x^{-\ell} \exp\left(-\frac{x^2}{2}\right) \ confluent \; hypergeometric \left( \frac{1}{4}(-2\ell + 1 - \lambda); -\ell + \frac{1}{2}; x^2 \right).
\]
where \( ( \cdot )_m \) is the Pochhammer symbol \(^{24}\) and \( K \) is an arbitrary constant. Note that the first term in (52) is the particular solution while the second is solution of the homogeneous equation. To simplify some expressions let us define an auxiliary function \( \lambda = h(x, \lambda) \) as:
\[
\lambda = \frac{(2\ell + 3 - \lambda)}{(2\ell + 3)} x^{\ell + 2} \exp\left(-\frac{x^2}{2}\right) _2F_1\left(\frac{1}{4}(2\ell + 7 - \lambda); \ell + \frac{5}{2}; x^2\right). \tag{53}
\]
Notice that the expression of the Wronskian \( W_{u_1u_2} \) is in (18) and for the radial oscillator can be expressed as
\[
\tilde{V}(x) = x^2 + \frac{\ell (\ell + 1)}{x^2} + \frac{4u(x, \lambda)u_x(x, \lambda)\left[K + u(x, \lambda)h_1(x, \lambda) - u_2(x, \lambda)h(x, \lambda)\right]}{\left[K + u(x, \lambda)h_1(x, \lambda) - u_2(x, \lambda)h(x, \lambda)\right]^2}. \tag{54}
\]
According to the condition (24), and since \( W_{u_1u_2}(0, \lambda) = 0 \), to obtain regular potentials the value of the constant \( K \) has to be a non positive real number. On the left of figure 2 it can be seen a potential \( \tilde{V} \) using a transformation function \( u \) with the parameters \( \ell = 1, \lambda = 8 \) and \( K = -0.01 \) (continuous curve) and as a reference the potential \( V \) (dotted curve). The parameters used are \( \lambda = 1, \ell = 8 \) and \( K = -0.01 \). On the right we plotted its first three eigenfunctions.

Before we conclude this section, let us comment on one more application that arises within the present example of the radial oscillator system. As we observe in (49), the general solution of equation (46) is a linear combination of two solutions that we will call \( u_1 \) and \( u_2 \). Using the abbreviations
\[
a = \frac{1}{4}(2\ell + 3 - \lambda), \quad b = \ell + \frac{3}{2}, \quad c = \frac{1}{4}(-2\ell + 1 - \lambda), \quad d = -\ell + \frac{1}{2}, \tag{55}
\]
the particular solutions in (49) can be written as
\[
u_1(x, \lambda) = x^{\ell+1} \exp\left(-\frac{x^2}{2}\right) _2F_1\left(\frac{1}{4}(2\ell + 7 - \lambda); \ell + \frac{5}{2}; x^2\right), \tag{56}
\]
\[
u_2(x, \lambda) = -\frac{1}{2\ell + 1} x^{-\ell} \exp\left(-\frac{x^2}{2}\right) _2F_1\left(\frac{1}{4}(2\ell + 7 - \lambda); \ell + \frac{5}{2}; x^2\right). \tag{57}
\]
Since \( u_1 \) and \( u_2 \) are solutions of equation (46) for \( \epsilon = \lambda \), the identity (26) can be used to find several integrals involving the latter functions. We obtain

\[
\int_0^x u_1^2(t, \lambda) \, dt = \frac{a}{2b} 2e^{2t} + 3 \exp(-x^2) \left\{ \sum_{n=0}^{\infty} \frac{(a + 1)_n x^{2n}}{(b + 1)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + a + 1} \right. \\
+ iF_1 \left( a + 1; b + 1; x^2 \right) \frac{iF_1 \left( a; b; x^2 \right)}{a} \\
- \sum_{n=0}^{\infty} \frac{(a)_n x^{2n}}{(b)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + a} \right\}
\]

and

\[
\int_x^\infty u_2^2(t, \lambda) \, dt = \frac{c}{2(2\ell + 1)^2} \exp(-x^2) \left\{ \sum_{n=0}^{\infty} \frac{(c + 1)_n x^{2n}}{(d + 1)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + c + 1} \right. \\
+ iF_1 \left( c + 1; d + 1; x^2 \right) \frac{iF_1 \left( c; d; x^2 \right)}{c} \\
- \sum_{n=0}^{\infty} \frac{(c)_n x^{2n}}{(d)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + c} \right\}.
\]

Furthermore, since \( u = u_1 + u_2 \) is also solution of the same differential equation and due to the linearity of the Wronskian we can obtain the integral of the the product \( u_1 u_2 \). This results in the comparatively long expression

\[
\int_{x_0}^x u_1(t, \lambda) u_2(t, \lambda) \, dt \\
= \frac{1}{8(2\ell + 1)} \left\{ (2\ell + 1) \exp(-x^2) \\
\times \left[ iF_1 \left( a; b; x^2 \right) \sum_{n=0}^{\infty} \frac{(c)_n x^{2n}}{(d)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + c} \right. \\
\left. - iF_1 \left( c; d; x^2 \right) \sum_{n=0}^{\infty} \frac{(a)_n x^{2n}}{(b)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + a} \right] \\
+ 2x^2 \exp(-x^2) \left[ - \frac{c}{d} iF_1 \left( a; b; x^2 \right) \\
\times \sum_{n=0}^{\infty} \frac{(c + 1)_n x^{2n}}{(d + 1)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + c + 1} \right. \\
\left. - \frac{a}{b} iF_1 \left( c; d; x^2 \right) \sum_{n=0}^{\infty} \frac{(a + 1)_n x^{2n}}{(b + 1)_n n!} \sum_{p=0}^{n-1} \frac{1}{p + a + 1} \right\}
\]

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While this expression is very difficult to manage by hand, it can easily be processed further through a symbolic calculator.

### 3.2. Energy-dependent potentials and normalization

It is well-known that quantum models involving energy-dependent interactions are subject to a modified theory [26]. In particular, the usual completeness relation and the corresponding $L^2$-norm of wavefunctions associated with energy-dependent potentials change their form. If we interpret (1) as a Schrödinger equation for an energy-dependent potential $V$ that admits a bound state solution $u$, then the norm $N(u)$ of such a solution must be calculated as follows [26]

$$
N (u) = \int_D \left[ 1 - V(t, \lambda) \right] |u(t, \lambda)|^2 \, dt. \quad (61)
$$

Here we use the symbol $N$ rather than the usual notation $\| \cdot \|$, because in the strict mathematical sense (61) is not a norm, as it can become negative. Keeping this in mind, we return to the Jordan chain (1), (2). According to (8), the system is solved by the two functions $u$ and $v = u_j$. Let us now calculate the Wronskian of these two functions. To this end, we first find the Wronskian’s derivative by taking into account both equations of the Jordan chain, obtaining

$$
\frac{\partial W_u}{\partial x}(x, \lambda) = - \frac{\partial W_v}{\partial x}(x, \lambda) = - u(x, \lambda) \left( u_j(x, \lambda) - u_j(x, \lambda) u_{xx}(x, \lambda) \right),
$$

$$
= u(x, \lambda) \left( -[\lambda - V(x, \lambda)] u_j(x, \lambda), - [\lambda - V(x, \lambda)] u_j(x, \lambda) + u_j(x, \lambda) \left( \lambda - V(x, \lambda) \right) u(x, \lambda),
\right.
$$

$$
= - \left[ 1 - V_j(x, \lambda) \right] u^2(x, \lambda).
$$

If we integrate this relation, we obtain the following representation of the Wronskian

$$
\int_{x_0}^x \left[ 1 - V_j(t, \lambda) \right] u^2(t, \lambda) \, dt = W_{u,u}(x_0, \lambda) - W_{u,u}(x, \lambda), \quad (62)
$$

for an arbitrary $x_0 \in D$. We observe that (62) is a generalization of (26) when $V$ depends on the parameter $\lambda$. In the next step we make the following observation: if the function $u$ in (61) is real-valued, then we can immediately match the latter relation with the normalization integral for systems that feature energy-dependent potentials. Setting $D = (x_0, x)$, we get
This identity says that the normalization integral (61) can be found entirely through calculating derivatives. As mentioned before, this is important in applications, because derivatives are generally much easier to find than integrals. Furthermore it is important to point out that the identities (62) and (63) neither require the function \( u \) to be a bound state solution, nor does the associated potential \( V \) have to admit such bound states. Both of the latter identities are valid independent of any physical meaning, that is, they hold for any solution of (1) and for any potential function, as long as the integrals exist. Before we present an example, let us comment on the case of a complex-valued function \( u \). Since (62) does not contain the absolute value, it is not directly applicable. Instead, we can split the integral by

\[
N(u) = \int_D [1 - V(t, \lambda)] \text{Re} \left[ u(t, \lambda)^2 \right] \, dt + \int_D [1 - V(t, \lambda)] \text{Im} \left[ u(t, \lambda)^2 \right] \, dt,
\]

note that \( \text{Re} \) and \( \text{Im} \) represent the real- and imaginary part of \( u \), respectively. Now, each of these two integrals can be identified with our formula (62), provided \( u \) is replaced by its real- and imaginary part.

### 3.2.1. Energy-dependent harmonic oscillator system

Let us now look at a model that features an energy-dependent harmonic oscillator potential. Our problem is governed by the following boundary-value problem

\[
\begin{align*}
\lambda \frac{d^2 u}{dx^2}(x, \lambda) + (\lambda - \lambda x^2)u(x, \lambda) &= 0, \quad (x, \lambda) \in \mathbb{R}^2, \\
\lim_{|x| \to \infty} u(x, \lambda) &= 0, \quad \lambda \in \mathbb{R}.
\end{align*}
\]

This problem is exactly solvable and admits an infinite discrete spectrum \( (\lambda_n) \) for an associated solution set \( (u_n) \), \( n \) a nonnegative integer [26], of the form

\[
\lambda_n = (2n + 1)^2, \quad u_n(x, \lambda_n) = \exp \left[ -\frac{1}{2} (2n + 1) x^2 \right] H_n \left( \sqrt{2n + 1} x \right),
\]

where \( H_n \) stands for the Hermite polynomial of order \( n \) [24]. For the sake of simplicity, we will restrict ourselves to the simplest case \( n = 0 \). We extract from (66)

\[
\lambda_0 = 1, \quad u_0(x, \lambda_0) = \exp \left( -\frac{1}{2} x^2 \right).
\]

We will now determine the norm of this function \( u_0 \) by means of the integral (61). Afterwards, the result is verified by using the differential approach (63). The normalization integral (61) of this function can be evaluated in a straightforward manner

\[
N(u_0) = \int_{\mathbb{R}} \left[ 1 - (\lambda x^2)_{\lambda=\lambda_0} \right] \exp \left( -x^2 \right) \, dx = \int_{\mathbb{R}} \left( 1 - x^2 \right) \exp \left( -x^2 \right) \, dx = \frac{\sqrt{\pi}}{2}.
\]

Let us now demonstrate that the same result can be found using our formula (63). First, we must determine the partial derivative \( (u_0)_\lambda \). This involves a little trick, because we are not given a solution of (64) in terms of \( \lambda \). But since the relation between \( \lambda \) (or \( \lambda_n \)) and \( n \) in (66) is differentiable with differentiable inverse due to nonnegativity of both \( \lambda_n \) and \( n \), we can interpret \( n \) as a continuous variable and use the chain rule to obtain
\[(u_0)_\lambda(x, \lambda) = \left[ \frac{\partial u_0}{\partial n}(x, \lambda_n) \frac{\partial n}{\partial \lambda_n}(\lambda_n) \right]_{n=0},\]

\[= \left[ \frac{\partial u_2}{\partial n}(x, \lambda_n) \frac{1}{\partial \lambda_n(n)} \right]_{n=0},\]

\[= -\frac{1}{4} x^2 \exp\left(-\frac{1}{2} x^2\right) + \frac{1}{4} \exp\left(-\frac{1}{2} x^2\right) \left[ \frac{\partial H_n}{\partial n}(\sqrt{2} n + 1, x) \right]_{n=0}.\]

We incorporate this into the Wronskian in (63) and arrive after some simplification at the following result

\[W_{u_n(u_0)} = -\frac{1}{2} \exp\left(-x^2\right)[x - H_{-1}(x)].\]

In the final step we plug this Wronskian into formula (63) for our norm. Since the numbers \(x_0\) and \(x\) correspond to negative and positive infinity, respectively, we must apply limits. This yields

\[N(u_0) = \lim_{x_0 \to -\infty} W_{u_n(u_0)}(x_0, \lambda) - \lim_{x \to \infty} W_{u_n(u_0)}(x, \lambda),\]

\[= -\lim_{x_0 \to -\infty} \frac{1}{2} \exp\left(-x_0^2\right)[x_0 - H_{-1}(x_0)] + \lim_{x \to \infty} \frac{1}{2} \exp\left(-x^2\right)[x - H_{-1}(x)],\]

\[= -\lim_{x_0 \to -\infty} \frac{1}{2} \exp\left(-x_0^2\right)[x_0 - H_{-1}(x_0)],\]

\[= \frac{\sqrt{\pi}}{2}.\]

This result coincides with (67), as expected. At first sight it seems that the calculations using formula (63) are cumbersome when compared to the simple integration in (67). This impression is merely due to the simplicity of the present example. If we choose a more general solution of (64), resolving the integral in (67) can become very difficult. At the same time, the use of our formula (63) will also result in complicated calculations, but it involves only differentiation.

4. Conclusions

In this work we constructed a relationship between second-order Jordan chains represented in integral and differential forms. This relationship, in a general scenario, allows to find certain integrals involving functions that are solutions of Schrödinger equations. We applied this relationship to obtain conditions to generate regular potentials through the confluent SUSY algorithm using the differential representation, to illustrate we used the particle in a box and the radial oscillator systems and in both cases it was shown how to find integrals related to the involved special functions of each system entirely through derivatives. It is worth mention that the exactly solvable SUSY partner potentials of the radial oscillator presented in this work as example had not been reported. Also, since the limits of the found integrals are not fixed, they can be used to calculate probabilities in any arbitrary interval inside the domain of definition of the potential. Moreover, the relationship was used to find normalization constants of wave functions of energy dependent potentials in quantum mechanics, it
was exemplified with the ground state of an energy dependent harmonic oscillator. The examples explored in this article are far from being exhaustive but have rather exemplary character since for every quantum system integrals with different special functions can be analyzed.

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