Steep Points of Gaussian Free Fields in Any Dimension

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Received: 19 October 2019 / Revised: 3 May 2020 / Published online: 2 August 2020
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Abstract
This work aims to extend the existing results on the Hausdorff dimension of the classical thick point sets to a more general class of exceptional sets of a Gaussian free field (GFF). We adopt a circle or sphere averaging regularization to study a log-correlated or polynomial-correlated GFF in any dimension and introduce the notion of “f-steep points” of the GFF for a certain test function f. Roughly speaking, the f-steep points of the GFF are locations where, when weighted by the function f, the “rate of change” of the regularized field becomes unusually large. Different choices of f lead to the study of different exceptional behaviors of the GFF. We determine the Hausdorff dimension of the set consisting of f-steep points, from which not only can we recover the existing results on thick point sets for both log-correlated and polynomial-correlated GFFs, but we also obtain new results for exceptional sets that, to our best knowledge, have not been previously studied. Our method is inspired by the one used to study the thick point sets of the classical 2D log-correlated GFF.

Keywords Gaussian free field · Regularization processes · Steep point · Thick point · Exceptional set · Hausdorff dimension

Mathematics Subject Classification 2010 60G60 · 60G15

1 Introduction
Gaussian free fields (GFFs) have played essential roles in many recent achievements in quantum physics and statistical mechanics. Although originated in physics, the mathematical study of GFFs has been a fast developing field of probability theory, generating fruitful results on problems arising from discrete math, analysis, geometry...
and other subjects. Heuristically speaking, GFFs are analogs of the Brownian motion with multidimensional time parameters. Just as the Brownian motion are naturally viewed as a random univariate function, GFFs can be interpreted as random multivariate functions or generalized functions. Also, just as the path of the Brownian motion is taken to be a random curve, graphs of GFFs are considered as promising models of random surfaces or random manifolds, which ultimately lead to the study of random geometry. On one hand, GFFs have been applied to the construction of random geometric objects such as random measures; on the other hand, the study of geometric properties of GFFs themselves gives rise to meaningful problems in random geometry, most of which remain open to date.

The main reason that such random geometry problems are challenging, at least for a typical GFF concerned in our work, is that a generic element of the GFF, denoted by $\theta$, is only a tempered distribution which may not be point-wisely defined. The rigorous definition of $\theta$ will be given in Sect. 2.1. For now, we can think of $\theta$ as a random (generalized) function whose probability distribution is determined by the Green function of a certain differential operator or pseudo-differential operator, and the singularity of the Green function determines the singularity of the GFF. To tackle this matter of singularity, it is natural to consider the GFF in the discrete setting, such as on a discrete lattice, in which case the GFF will be defined on every vertex. A rich literature has been established on the geometry of discrete GFFs. For instance, the distribution of extrema and near-extrema of a discrete GFF has been extensively studied (e.g., [4,10,11]). Back to the continuum setting, to overcome the singularity of $\theta$, one needs to apply a procedure known as a regularization in physics literature to approximate point-wise values of $\theta$. Various regularization procedures have long been considered in the study of singular GFFs. Below we only allude to two commonly used regularization procedures.

The first one is based on the theory of Gaussian Multiplicative Chaos (GMC) introduced by Kahane in his seminal work [17]. The GMC theory enables one to define in any dimension a random Borel measure which formally takes the form $\text{e}^{\theta(x)}dx$, where $\theta$ is a generic element of a log-correlated GFF, i.e., the corresponding Green function has a logarithmic singularity, and $dx$ is the Lebesgue measure. Such a measure, known as the Liouville Quantum Gravity (LQG) measure, is an important object in quantum field theory. Kahane’s work has led to the multi-fractal analysis of the LQG measure by showing that such a measure is supported on a Borel set where the regularized $\theta$ achieves “unusually” large values. Over the past decade, further results on the support of the LQG measure and the geometry of log-correlated GFFs have been established under the framework of GMC (e.g., [2,3,13,19–21]). In addition, using the tool of GMC, the extreme values of the regularized $\theta$ are investigated in [18].

In addition to the GMC approach, one can also regularize $\theta$ by averaging it over some sufficiently “nice” Borel sets. Since taking a convolution or an integration is the natural way to overcome the singularity of a generalized function, it is reasonable to adopt an averaging procedure to study the “landscape” of $\theta$. For example, a more recent breakthrough on the topic of log-correlated GFFs is the work of Duplantier

\[1\] Similarly, the GFF is said to be polynomial-correlated when the corresponding Green function has a polynomial singularity. The rigorous definitions of these notions are given in Sect. 2.1.
and Sheffield [14]. Based on circular averages of θ, [14] gives a rigorous construction of the LQG measure in 2D, and proves the long celebrated Knizhnik–Polyakov–Zamolodchikov formula in the context of a quantitative relation between the scaling property of the LQG measure and that of the Lebesgue measure. Along the way, [14] also derives the same results for the support of the LQG measure as mentioned above, i.e., it is supported on a set where the regularized θ becomes unusually large. Meanwhile, also using circular averages of θ, Hu, Miller and Peres [16] study the points where the regularized θ is unusually large, introduce the notion of thick points and determine the Hausdorff dimension of the set consisting of thick points. Through a sphere averaging regularization, [6,7] generalize part of the results on the LQG measure to higher dimensional log-correlated GFFs, and [5,9] extend the study of thick points to log-correlated GFFs and, respectively, polynomial-correlated GFFs.

1.1 A Brief Review of Thick Point

In addition to being the support of the LQG measure, thick point sets also characterize a basic aspect of the landscape of a GFF, that is, where the “high peaks” occur, so thick points are important objects in understanding the geometry of a GFF. The purpose of this article is to consolidate the existing results on thick point sets and to extend our study to a more general class of exceptional sets for both log-correlated GFFs and polynomial-correlated GFFs. We start with a brief (and incomplete) review on the standard thick point sets.

1.1.1 Thick Point of Log-Correlated GFF

Following the same notations as above, let θ be a generic element of the GFF associated with the Laplacian operator Δ on a bounded domain $U \subseteq \mathbb{R}^2$, equipped with the Dirichlet boundary condition. Governed by the properties of the Green function of Δ in 2D, such a GFF is log-correlated, and for every $x \in U$ and $t > 0$, the average of θ over the circle centered at x with radius t, denoted by $\bar{\theta}(x,t)$, is a centered Gaussian random variable. To get an approximation of “θ(x),” we want to study the behaviors of $\bar{\theta}(x,t)$ as $t \downarrow 0$. As introduced in [16], for every $\gamma \geq 0$, the set of $\gamma$-thick points of θ is given by

$$T^{\gamma,\theta} := \left\{ x \in U : \lim_{t \downarrow 0} \frac{\bar{\theta}(x,t)}{-\ln t} = \sqrt{2\gamma/\pi} \right\}. \tag{1.1}$$

The definition of γ-thick point presented here adopts a different parametrization in γ from the original version in [16], but the spirit is the same. In fact, with x fixed, the circular average process $\{\bar{\theta}(x,t) : t \in (0,1]\}$ has the same distribution as a Brownian motion $\{B(\tau) : \tau \geq 0\}$ up to a deterministic time change $\tau = (-\ln t) / (2\pi)$, and hence, as $t \downarrow 0$, $\bar{\theta}(x,t)$ behaves just like $B(\tau)$ as $\tau \nearrow \infty$. Thus, for any given $x \in U$, the limit involved in (1.1) is equivalent to

$$\lim_{\tau \to \infty} \frac{B(\tau)}{\tau} = 2\sqrt{2\pi \gamma}.$$
which, for every $\gamma > 0$, occurs with probability zero. Therefore, we can view $\gamma$-thick points, so long as $\gamma > 0$, as locations where the value of $\theta$ is unusually large. It is proven in [16] that for almost every $\theta$, if $\gamma > 1$, then $T^{\gamma,\theta} = \emptyset$, and if $\gamma \in [0, 1]$, then $\dim_{\mathcal{H}} \left( T^{\gamma,\theta} \right) = 2 - 2\gamma$, where “$\dim_{\mathcal{H}}$” refers to the Hausdorff dimension; if $\gamma = 0$, then $x \in T^{\gamma,\theta}$ for Lebesgue-almost every $x \in U$.

1.1.2 Thick Point of Polynomial-Correlated GFF

If $\theta$ is a generic element of the GFF associated with the Bessel operator $I - \frac{\Delta}{\Delta_1}$ on $\mathbb{R}^\nu$ with $\nu \geq 3$, then $\theta$ is more singular compared with the 2D log-correlated GFF above because the GFF in this case is polynomial-correlated. Intuitively speaking, compared with the 2D case, the graph of $\theta$ is “rougher” when $\nu \geq 3$, and the higher $\nu$ is, the worse it becomes. But no matter what $\nu$ is, it is always possible to average $\theta$ over any codimensional-1 sphere. Again, we denote by $\bar{\theta}(x, t)$ a certain renormalization, whose exact definition is given in Lemma 2.2, of the spherical average of $\theta$ over the sphere centered at $x \in \mathbb{R}^\nu$ with radius $t > 0$, and study the properties of $\bar{\theta}(x, t)$ as $t \downarrow 0$. In this setting, for $\gamma \geq 0$, the set of $\gamma$-thick points of $\theta$ is defined in [5] as

$$T^{\gamma,\theta} := \left\{ x \in \mathbb{R}^\nu : \limsup_{t \downarrow 0} \frac{\bar{\theta}(x, t)}{\sqrt{-G(t) \ln t}} \geq \sqrt{2\nu \gamma} \right\} \quad (1.2)$$

where $G(t) := \mathbb{E} \left[ (\tilde{\theta}(x, t))^2 \right]$ for every $t > 0$. In a similar spirit to (1.1), when $\gamma > 0$, a $\gamma$-thick point is a location where $\theta$ is unusually large. It is established in [5] that almost surely (a.s.) if $\gamma > 1$, then $T^{\gamma,\theta} = \emptyset$, and if $\gamma \in [0, 1]$, then $\dim_{\mathcal{H}} \left( T^{\gamma,\theta} \right) = \nu (1 - \gamma)$.

One may have noticed that the definition (1.2) imposes a weaker constraint compared with (1.1), since an inequality involving “lim sup” is adopted instead of the equality of “lim.” It turns out that (1.2) is a more suitable thick point definition for polynomial-correlated GFFs, as it is shown in [5] that for every $\gamma > 0$, the perfect $\gamma$-thick point, i.e., $x$ such that the limit in (1.2) is achieved and is equal to $\sqrt{2\nu \gamma}$, does not exist a.s. [5] also investigates the set of sequential $\gamma$-thick points given by

$$ST^{\gamma,\theta} := \left\{ x \in \mathbb{R}^\nu : \lim_{n \to \infty} \frac{\bar{\theta}(x, r_n)}{\sqrt{-G(r_n) \ln r_n}} = \sqrt{2\nu \gamma} \right\} \quad (1.3)$$

where $\{r_n : n \geq 1\} \subseteq (0, 1]$ is a sequence such that $r_n \downarrow 0$ sufficiently fast as $n \to \infty$. Clearly, $ST^{\gamma,\theta} \subseteq T^{\gamma,\theta}$, but it is shown in [5] that $ST^{\gamma,\theta}$ has the same Hausdorff dimension as $T^{\gamma,\theta}$.

Compared with the log-correlated case, the higher-level singularity of $\theta$ in the polynomial-correlated setting makes the thick points more rare and hence harder to find. In fact, the most involved part of the work in [5] is to establish a lower bound for

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$^2$ The GFF associated with $\Delta$ is referred to as a massless GFF, and the one associated with $I - \Delta$ is a massive GFF. In $\mathbb{R}^\nu$ with $\nu \geq 3$, we consider the massive GFF because the Green function of $I - \Delta$ on $\mathbb{R}^\nu$ is technically more convenient to manipulate. See Remark 2.1 below.
dim \( H(\mathcal{W}^{T, \gamma, \theta}) \), which, by default, will be a lower bound for dim \( H(T^{T, \gamma, \theta}) \). For the majority of problems related to the geometry of GFFs, one expects that it is non-trivial to extend the study from log-correlated GFFs to polynomial-correlated GFFs.

1.2 An Outline of the Article

Our goal is to extend the framework of thick point sets, in order to study a more general class of exceptional sets for both log-correlated and polynomial-correlated GFFs. As a preparation, in Sect. 2 we give a brief review of the theory of Abstract Wiener Space, which provides a mathematical foundation for GFFs. We will also give a rigorous interpretation of the regularization \( \{ \bar{\theta}(x, t) : x \in \mathbb{R}^\nu, t \in (0, 1] \} \) based on the circular or spherical averages of \( \theta \).

In Sect. 3, we examine the behavior of \( \bar{\theta}(x, t) \) from a slightly different angle compared with the study of thick point. Instead of focusing on \( \bar{\theta}(x, t) \) itself, we consider the integral of some test function \( f(t) \) integrated against the increment of \( \bar{\theta}(x, t) \) in \( t \); instead of focusing on how large \( \bar{\theta}(x, t) \) becomes as \( t \downarrow 0 \), we study how large the involved integral becomes as \( t \downarrow 0 \), which reflects the “steepness” or the “rate of change” of \( \bar{\theta}(x, t) \) with respect to \( t \). To be specific, let \( f : (0, 1] \rightarrow \mathbb{R} \) be a properly chosen test function, where the requirements of \( f \) will be specified in Definition 3.1, and for every \( x \in \mathbb{R}^\nu \) and \( t \in (0, 1] \), set

\[
X^\theta_f(x, t) := \int_1^t f(s) \, d\bar{\theta}(x, s) \quad \text{and} \quad \Sigma_f(t) := \int_1^t f^2(s) \, dG(s).
\]

On the one hand, since the choice of \( f \) is general (upon satisfying Definition 3.1), \( X^\theta_f(x, t) \) captures well the dynamics of \( \bar{\theta}(x, t) \) as \( t \) changes. On the other hand, \( X^\theta_f(x, t) \) resembles the behavior of \( \bar{\theta}(x, t) \) in the sense that, for every fixed \( x \), \( \{ X^\theta_f(x, t) : t \in (0, 1] \} \) also has the distribution of the standard Brownian motion \( \{ B(\tau) : \tau \geq 0 \} \) up to the time change \( \tau = \Sigma_f(t) \). Inspired by the idea of thick point, we want to investigate the asymptotics of \( X^\theta_f(x, t) / \Sigma_f(t) \) and identify location \( x \) where this ratio becomes unusually large as \( t \downarrow 0 \).

We introduce in Definition 3.2 the notion of “\( f \)-steep point,” that is \( x \) where

\[
\lim_{t \downarrow 0} \frac{X^\theta_f(x, t)}{\Sigma_f(t)} = \sqrt{2\nu},
\]

and denote by \( D^\theta_f \) the collection of all the \( f \)-steep points. Heuristically speaking, if \( x \in D^\theta_f \), then \( X^\theta_f(x, t) \) achieves unusually large values as \( t \downarrow 0 \), which implies that \( \bar{\theta}(x, t) \) is unusually “steep,” or in other words, \( \bar{\theta}(x, t) \) changes unusually fast in \( t \), when weighted by \( f \). In addition to \( D^\theta_f \), we also consider two other exceptional sets that are closely related to \( D^\theta_f \):
Our main result is on the Hausdorff dimension of these exceptional sets. It turns out that $D_{f, \lim sup}^\theta$ and $D_{f, \lim inf}^\theta$ have non-trivial fractal structure only if $\Sigma_f (t)$ grows at a rate comparable to $(-\ln t)$ as $t \searrow 0$. In particular, we prove in Theorem 3.3 that if

$$0 < \lim inf_{t \searrow 0} \frac{\Sigma_f (t)}{-\ln t} \leq \lim sup_{t \searrow 0} \frac{\Sigma_f (t)}{-\ln t} \leq 1,$$

then

$$\dim_H \left( D_{f, \lim inf}^\theta \right) = \dim_H \left( D_{f, \lim sup}^\theta \right) = \nu \left( 1 - \lim sup_{t \searrow 0} \frac{\Sigma_f (t)}{-\ln t} \right) \text{ a.s.}$$

and

$$\nu \left( 1 - \lim sup_{t \searrow 0} \frac{\Sigma_f (t)}{-\ln t} \right) \leq \dim_H \left( D_{f, \lim sup}^\theta \right) \leq \nu \left( 1 - \lim inf_{t \searrow 0} \frac{\Sigma_f (t)}{-\ln t} \right) \text{ a.s.}$$

Although setting out to investigating a different perspective of the landscape of $\theta$, our work follows a similar line of strategy as that in [5,16].

We believe that analyzing the steep points can be a useful approach in the study of the geometry of GFFs. As mentioned earlier, the flexibility in choosing $f$ in $X_{f, \theta}^\theta (x, t)$ allows us to obtain information on different aspects of the behavior of $\theta$ near $x$. In Sect. 4, we investigate $f$-steep point sets with concrete examples of $f$. By choosing $f$ properly, we can use Theorem 3.3 to re-produce the existing results on Hausdorff dimensions of thick point sets for both log-correlated and polynomial-correlated GFFs. Moreover, certain choices of $f$ lead to natural generalizations of thick point, one of which we call “oscillatory thick point.” Heuristically speaking, $x$ is an oscillatory thick point if $\bar{\theta} (x, t)$ achieves unusually large values both in the positive direction and in the negative direction infinitely often as $t \searrow 0$, i.e., $\bar{\theta} (x, t)$ exhibits an oscillatory behavior with unusually large amplitude as $t \searrow 0$. We determine exact Hausdorff dimensions of oscillatory thick point sets for both log-correlated GFFs (Proposition 4.1) and polynomial-correlated GFFs (Proposition 4.4). Another variation of thick point we will consider is “lasting thick point,” and that is, intuitively speaking, a thick point $x$ where $\bar{\theta} (x, t)$ spends non-negligible portion of the total time maintaining an unusually large value. Again, using the result on steep point sets, we obtain the Hausdorff dimension of the set consisting of lasting thick points (Proposition 4.5), showing that, although being more rare than the standard thick points, lasting thick points can still be “detected.”

In Sect. 5, we briefly discuss some generalizations and problems related to the notion of steep point, and possible directions in which we would like to further our study.
In addition to having a more general framework than that of the thick point, another motivation of studying \( X^\theta_f(x, t) \) and \( f \)-steep point is that they offer a possibility of extending the construction of the LQG measure to the setting of polynomial-correlated GFFs in higher dimensions. As mentioned above, the LQG measure is a random measure on a planar domain constructed with the classical 2D log-correlated GFF and supported on a thick point set of the GFF. For a polynomial-correlated GFF in higher dimensions, by choosing \( f \) properly, one can make the family of \( X^\theta_f(x, t) \) have similar covariance structure as the (regularized) 2D log-correlated GFF. So, we are hopeful that the same construction would work with \( X^\theta_f(x, t) \), and we could obtain an analog of the LQG measure that is supported on the \( f \)-steep point set. We are investigating this possibility in an ongoing work.

### 2 Gaussian Free Field and Circle/Sphere Averaging Regularization

#### 2.1 Abstract Wiener Space and GFF

A mathematically sound representation of GFFs is given by the theory of Abstract Wiener Space (AWS) [15], under whose framework not only can we define and construct GFFs rigorously, but we also have a natural interpretation of any regularization procedure such as those mentioned in Sect. 1. The connection between the AWS and GFFs has already been explained in §2 of [5]. We refer readers who are interested in the general theory of AWS to [8,15,23] and §8 of [24]. In this section, we will not repeat the entire theory but only review the key ideas for the sake of completeness.

Given \( \nu \in \mathbb{N}, \nu \geq 2 \) and \( p \in \mathbb{R} \), let \( H^p := H^p(\mathbb{R}^\nu) \) be the Sobolev space that is the closure of \( C_\infty^c(\mathbb{R}^\nu) \) under the inner product

\[
(\phi, \psi)_{H^p} := \left( (I - \Delta)^p \phi, \psi \right)_{L^2(\mathbb{R}^\nu)} = \frac{1}{(2\pi)^\nu} \int_{\mathbb{R}^\nu} \left( 1 + |\xi|^2 \right)^p \hat{\phi}(\xi) \hat{\psi}(\xi) d\xi
\]

for every \( \phi, \psi \in C_\infty^c(\mathbb{R}^\nu) \), where \((I - \Delta)^p\) is the Bessel operator of order \( p \) and \(" \cdot \)" refers to the Fourier transform. Clearly, \((H^p, \cdot, \cdot)_{H^p}\) forms a separable Hilbert space. One can identify \( H^{-p} \) as the dual space of \( H^p \), and the mapping

\[
\mu \in H^{-p} \mapsto h_\mu := (I - \Delta)^{-p} \mu \in H^p
\]

is an isometry. The theory of AWS guarantees that there exists a separable Banach space \( \Theta^p := \Theta^p(\mathbb{R}^\nu) \) with the norm \( \| \cdot \|_{\Theta^p} \) and a centered Gaussian measure \( \mathcal{N}^p := \mathcal{N}^p(\mathbb{R}^\nu) \) on \( \Theta^p \) such that

(i) \((H^p, \cdot, \cdot)_{H^p}\) is continuously embedded in \((\Theta^p, \| \cdot \|_{\Theta^p})\) as a dense subspace, so \( \Theta^p \) is also a space of functions or generalized functions;

(ii) if \( \lambda \in H^{-p} \) is a linear and bounded functional on \( \Theta^p \) with respect to the action \((\cdot, \cdot)_{L^2}\), or in other words, if \( \lambda \) is an element of \((\Theta^p)^*\) the dual space of \( \Theta^p \), then the mapping

\[
\theta \in \Theta^p \mapsto \mathcal{J}(h_\lambda)(\theta) := (\theta, \lambda)_{L^2} \in \mathbb{R},
\]
is a Gaussian random variable with $\mathbb{E}^\mathcal{W} [\mathcal{I} (h_\lambda)] = 0$ and $\text{Var} (\mathcal{I} (h_\lambda)) = \|h_\lambda\|^2_{H^p} = \|\lambda\|^2_{H^{-p}}$.

In this setting, we refer to the probability space $(\Theta^p, \mathcal{F} (\Theta^p), \mathcal{W}^p)$ as the dim-$v$ order-$p$ GFF; $\Theta \in \Theta^p$ sampled under $\mathcal{W}^p$ as a generic element of the GFF, and the Hilbert space $(H_p, (\cdot, \cdot)_{H_p})$ as the Cameron–Martin space associated with this GFF. Moreover, (i) and (ii) imply that the mapping $h_\lambda \in H^p \mapsto \mathcal{I} (h_\lambda) \in L^2 (\mathcal{W}^p)$ can be extended to the whole space $H^p$ as an isometry $\mathcal{I} : (H^p, (\cdot, \cdot)_{H^p}) \rightarrow L^2 (\mathcal{W}^p)$, and its image $\{\mathcal{I} (h) : h \in H^p\}$ forms a centered Gaussian family under $\mathcal{W}^p$ with covariance being

$$
\mathbb{E}^{\mathcal{W}^p} [\mathcal{I} (h) \mathcal{I} (g)] = (h, g)_{H^p} \text{ for every } h, g \in H^p.
$$

The isometry $\mathcal{I}$ is called the Paley–Wiener map, and its images $\{\mathcal{I} (h) : h \in H^p\}$ are known as the Paley–Wiener integrals. There are two facts about the Paley–Wiener integrals that we will use in later discussions.

1. If $\{h_n : n \geq 1\} \subseteq H^p$ is an orthonormal basis of $(H^p, (\cdot, \cdot)_{H^p})$, then $\{\mathcal{I} (h_n) : n \geq 1\}$ is a family of independent standard Gaussian random variables, and for $(\mathcal{W}^p)$-almost every $\theta \in \Theta^p$,

$$
\theta = \sum_{n \geq 1} \mathcal{I} (h_n) (\theta) h_n \text{ under } \|\cdot\|_{\Theta^p}. \tag{2.1}
$$

2. Under $\mathcal{W}^p$, $\{\mathcal{I} (h_\mu) : \mu \in H^{-p}\}$ is again a family of centered Gaussian random variables with covariance being

$$
\mathbb{E}^{\mathcal{W}^p} [\mathcal{I} (h_{\mu_1}) \mathcal{I} (h_{\mu_2})] = (h_{\mu_1}, h_{\mu_2})_{H^p} = \frac{1}{(2\pi)^v} \int_{\mathbb{R}^v} \frac{\mu_1 (\xi) \overline{\mu_2 (\xi)}}{1 + |\xi|^2} d\xi \text{ for every } \mu_1, \mu_2 \in H^{-p}. \tag{2.2}
$$

The formula (2.2) indicates that the covariance structure of the GFF is determined by the Green function of $(I - \Delta)^p$ on $\mathbb{R}^v$.

**Remark 2.1** Instead of $(I - \Delta)^p$, one can also use $\Delta^p$, equipped with proper boundary conditions, to construct GFFs on bounded domains in $\mathbb{R}^v$, and this is the case with the GFF treated in [14,16,22] and many other works. The field elements obtained in either way possess similar properties locally in space. We adopt $(I - \Delta)^p$ in our project for technical reasons. Specifically, $(I - \Delta)^p$ allows GFFs to be defined on the entire space $\mathbb{R}^v$, so we do not have to worry about any boundaries or boundary conditions. In addition, we can carry out computations with the Fourier transforms as in (2.2), which simplifies our tasks in many occasions.

With different choices of $p$ and $v$, $\theta$ possesses different levels of singularity or regularity (§8, [24]). In general, with $v$ fixed, the larger $p$ is, the more regular $\theta$ is,

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3. In physics literature, the term “GFF” only refers to the case when $p = 1$. Here we slightly extend the use of this terminology and continue to call it “GFF” when $p \neq 1$. 

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and the smaller \( p \) is, the more singular \( \theta \) becomes. Throughout the article, we assume that \( 2p \in \mathbb{N} \) and \( \nu \geq 2p \), in which case \( \theta \) is a generalized function and may not be pointwisely defined. In particular, (i) when \( p = \nu/2 \), the \( \dim - \nu \) order-(\( \nu/2 \)) GFF is a log-correlated GFF since the Green function of \((I - \Delta)^{\nu/2}\) on \( \mathbb{R}^{\infty} \) has a logarithmic singularity along the diagonal; (ii) when \( p < \nu/2 \), the field is a polynomial-correlated GFF because the corresponding Green function has a polynomial singularity of degree \( \nu - 2p \) along the diagonal. Since the level of singularity of the GFF is fully determined by the value of \( \nu - 2p \), we only focus on the case when \( p = 1 \) and \( \nu \geq 2 \). For general \( 2p \in \mathbb{N} \) and \( \nu \geq 2p \), one can adopt the regularization procedure introduced in §3 of [5] and in §3 of [7], and follow the method below without substantial changes to obtain the same results.

### 2.2 Circular or Spherical Average of GFF

For the rest of this article, we assume that \( p = 1 \), \( \nu \in \mathbb{N} \) and \( \nu \geq 2 \), and write \( H^1 \), \( \Theta^1 \), and \( \mathcal{W}^1 \) as, respectively, \( H \), \( \Theta \) and \( \mathcal{W} \) for simplicity. For every \( x \in \mathbb{R}^\nu \) and \( t \in (0, 1] \), let \( B(x, t) \) and \( \partial B(x, t) \) be the open disc/ball and, respectively, the circle/sphere centered at \( x \) with radius \( t \in (0, 1] \), \( \sigma(x, t) \) be the length/surface measure on \( \partial B(x, t) \), \( \alpha_\nu := 2\pi^{\nu/2}/\Gamma(\nu/2) \) be the dimensional constant, and \( \sigma^{ave}(x, t) := \sigma(x, t) / (\alpha_\nu t^{\nu-1}) \) be the circle/sphere averaging measure over \( \partial B(x, t) \). By straightforward computations, we see that for every \( x \in \mathbb{R}^\nu \) and \( t \in (0, 1] \), the Fourier transform of \( \sigma^{ave}(x, t) \) is

\[
\sigma^{ave}(x, t)(\xi) = \frac{(2\pi)^{\nu/2}}{\alpha_\nu} e^{i(x, \xi)_{\mathbb{R}^\nu}} (t |\xi|)^{\frac{2-\nu}{2}} J_{\frac{\nu-2}{2}}(t |\xi|) \quad \text{for every } \xi \in \mathbb{R}^\nu, \tag{2.3}
\]

where \( J_{\frac{\nu-2}{2}} \) is the standard Bessel function of the first kind with index \( \frac{\nu-2}{2} \). Using (2.3) and the asymptotics of \( J_{\frac{\nu-2}{2}} \) at infinity, it is easy to check that

\[
\sigma^{ave}(x, t) \in H^{-1}(\mathbb{R}^\nu) \quad \text{and} \quad h_{\sigma^{ave}(x, t)} := (I - \Delta)^{-1} \sigma^{ave}(x, t) \in H.
\]

Hence, \( \mathcal{I}(h_{\sigma^{ave}(x, t)}) \) is a centered Gaussian random variable under \( \mathcal{W} \). This is to say that, no matter what \( \nu \) is, no matter how singular \( \theta \) is, the average of \( \theta \) over a circle/sphere in \( \mathbb{R}^\nu \) is always well defined as a Gaussian random variable. Furthermore, \( \{ \mathcal{I}(h_{\sigma^{ave}(x, t)}) : x \in \mathbb{R}^\nu, t \in (0, 1] \} \) forms a centered Gaussian family with the following covariance: for \( x, y \in \mathbb{R}^\nu, t, s \in (0, 1] \), when \( x = y \),

\[
\mathbb{E}[\mathcal{I}(h_{\sigma^{ave}(x, t)}), \mathcal{I}(h_{\sigma^{ave}(x, s)})] = \frac{1}{\alpha_\nu (ts)^{\frac{\nu-2}{2}}} \int_0^\infty \frac{\tau J_{\frac{\nu-2}{2}}(\tau t) J_{\frac{\nu-2}{2}}(\tau s)}{1 + \tau^2} d\tau, \tag{2.4}
\]
and when \( x \neq y \),
\[
E^W \left[ \mathcal{J} \left( h_{\sigma^{ave}(x,t)} \right) \mathcal{J} \left( h_{\sigma^{ave}(y,s)} \right) \right] = \frac{(2\pi)^{\nu/2}}{\alpha_\nu^2 (ts |x - y|)^{\nu/2}} \int_0^\infty \frac{\tau^{\nu-2} J_{\nu-2} (\tau) J_{\nu-2} (s \tau) J_{\nu-2} (|x - y| \tau)}{1 + \tau^2} d\tau.
\]

(2.5)

The Gaussian family consisting of the circular/spherical averages has been carefully treated, and the integrals in (2.4) and (2.5) have been computed explicitly in §3 of [5]. Here, we only review the results from [5] that are relevant to our problem. Readers can turn to [5] for the detailed proofs of these results.

**Lemma 2.2** Let \( x \in \mathbb{R}^\nu \) be fixed. The distribution of the centered Gaussian family \( \{ \mathcal{J} \left( h_{\sigma^{ave}(x,t)} \right) : t \in (0, 1] \} \) does not depend on \( x \) and for every \( s, t \in (0, 1] \),
\[
E^W \left[ \mathcal{J} \left( h_{\sigma^{ave}(x,t)} \right) \mathcal{J} \left( h_{\sigma^{ave}(x,s)} \right) \right] = \frac{1}{\alpha_\nu (ts)^{\nu/2}} I_{\nu/2} (\tau) K_{\nu/2} (s \vee t),
\]
where \( s \land t := \min \{ s, t \}, s \lor t := \max \{ s, t \} \), and \( I_{\nu/2}, K_{\nu/2} \) are the modified Bessel functions with index \( \nu/2 \).

Further, if we carry out the renormalization by setting
\[
\tilde{\sigma} (x, t) := \frac{(t/2)^{\nu/2}}{\Gamma (\nu/2)} I_{\nu/2} (t) \sigma^{ave} (x, t) \quad \text{and} \quad \tilde{\theta} (x, t) := \mathcal{J} \left( h_{\tilde{\sigma}(x,t)} \right) (\theta)
\]
then \( \{ \tilde{\theta} (x, t) : t \in (0, 1] \} \) is a centered Gaussian process whose covariance is given by
\[
E \left[ \tilde{\theta} (x, t) \tilde{\theta} (x, s) \right] = \frac{\alpha_\nu}{(2\pi)^{\nu}} \frac{K_{\nu/2} (t)}{I_{\nu/2} (t)} =: G (t) \quad \text{for} \quad 0 < s \leq t \leq 1.
\]

(2.6)

In particular, \( \{ \tilde{\theta} (x, t) : t \in (0, 1] \} \) is a Brownian motion up to a time change in the sense that
\[
\left\{ B (\tau) := \tilde{\theta} \left( x, G^{-1} (\tau + G (1)) \right) - \tilde{\theta} (x, 1) : \tau \geq 0 \right\}
\]
has the distribution of a standard Brownian motion.

One can verify that
\[
\lim_{t \searrow 0} \frac{t^{\nu/2} 2^{2-\nu}}{\Gamma \left( \frac{\nu}{2} \right) I_{\nu/2} (t)} = 1,
\]
so \(\tilde{\theta} (x, t)\) still is a legitimate approximation of \(\theta (x)\).” Moreover, by the asymptotics of \(K_{\nu-2} (t)\) and \(I_{\nu-2} (t)\) near zero, the function \(G\) defined in (2.6) is positive, smooth and decreasing on \((0, \infty)\), and as \(t \searrow 0\),

\[
G (t) = \begin{cases} 
\frac{1}{2\pi} (- \ln t) + \mathcal{O} (1) & \text{if } \nu = 2, \\
\frac{1}{2\pi \alpha (\nu-2)} + \mathcal{O} (t^{3-\nu}) & \text{if } \nu \geq 3.
\end{cases}
\]

(2.7)

Throughout the rest of the article, we will adopt \(\{\tilde{\theta} (x, t) : x \in \mathbb{R}^\nu, t \in (0, 1]\) as the regularization to study the GFF. Not only does the concentric family reduce to a Brownian motion (up to a time change) for every \(x\), but the non-concentric family also possesses favorable properties under certain circumstances.

**Lemma 2.3** Assume that \(x, y \in \mathbb{R}^\nu\) and \(t, s \in (0, 1]\).

(i) If \(|x - y| \geq t + s\), i.e., \(B (x, t) \cap B (y, s) = \emptyset\), then

\[
\mathbb{E} \mathbb{W} \left[ \tilde{\theta} (x, t) \tilde{\theta} (y, s) \right] = (2\pi)^{-\nu/2} \frac{K_{\nu-2} ((x - y) | x - y|)}{|x - y|^{\nu/2}} =: C_{\text{disj}} (|x - y|) , \tag{2.8}
\]

and \(C_{\text{disj}} (|x - y|) = G (|x - y|) + \mathcal{O} (1)\) when \(|x - y|\) is small.

(ii) If \(t \geq |x - y| + s\), i.e., \(B (x, t) \supset B (y, s)\), then

\[
\mathbb{E} \mathbb{W} \left[ \tilde{\theta} (x, t) \tilde{\theta} (y, s) \right] = (2\pi)^{-\nu/2} \frac{I_{\nu-2} ((x - y) | x - y|) K_{\nu-2} (t)}{|x - y|^{\nu/2} I_{\nu-2} (t)} =: C_{\text{incl}} (t, |x - y|) , \tag{2.9}
\]

and \(C_{\text{incl}} (t, |x - y|) = G (t) + \mathcal{O} (1)\) when \(t\) is small.

We want to point out that (2.8) and (2.9) represent the advantage of this particular choice of regularization. Under the assumption (i) or (ii) of Lemma 2.3, small radius (radii) does not affect the covariance, which is a desirable property to have when studying the “convergence” in any reasonable sense as the radius (radii) tends to zero.

Since we are only interested in the local behavior of \(\theta\), without loss of generality we will restrict \(\theta\) to \(S (\mathcal{O}, 1)\), the closed square/cube centered at the origin with side length 2. Similarly, for \(x \in S (\mathcal{O}, 1)\) and \(t \in (0, 1]\), \(S (x, t)\) and \(\tilde{S} (x, t)\) denote the open and, respectively, closed square/cube centered at \(x\) with side length \(2t\). An important factor of a regularization procedure is the continuity property possessed by the regularized GFF. The continuity of \(\{\tilde{\theta} (x, t) : (x, t) \in S (\mathcal{O}, 1) \times (0, 1]\) has been investigated in [5,16] via standard techniques such as Kolmogorov’s continuity criterion (e.g., §4 in [24]) and the classical entropy method (e.g., [1,12,25]). Following the same methods, we study the continuity modulus of \(\{\tilde{\theta} (x, t) / \sqrt{G (t)} : (x, t) \in S (\mathcal{O}, 1) \times (0, 1]\} in the lemma below, and the result will be useful in Sect. 3.1.

**Lemma 2.4** We denote by \(\tilde{d}\) the intrinsic metric associated with the Gaussian family

\[
\left\{ \frac{\tilde{\theta} (x, t)}{\sqrt{G (t)}} : (x, t) \in S (\mathcal{O}, 1) \times (0, 1]\right\}.
\]
that is,

\[
\tilde{d}(x, t; y, s) := \left( \mathbb{E}^{\tilde{\mathcal{W}}} \left[ \left( \frac{\tilde{\theta}(x, t)}{\sqrt{G(t)}} - \frac{\tilde{\theta}(y, s)}{\sqrt{G(s)}} \right)^2 \right] \right)^{1/2} \quad \text{for } x, y \in S(O, 1) \text{ and } t, s \in (0, 1],
\]

(2.10)

If, for every \( t \in (0, 1] \) and \( \delta > 0 \), \( \pi(\delta, t) \) is the random variable on \( \Theta \) given by

\[
\pi^\theta(\delta, t) := \sup \left\{ \frac{\tilde{\theta}(x, s)}{\sqrt{G(s)}} - \frac{\tilde{\theta}(y, s')}{\sqrt{G(s')}} : \tilde{d}(x, s; y, s') \leq \delta, x, y \in S(O, 1), s, s' \in [t, 1] \right\}
\]

for every \( \theta \in \Theta \),

then there exists a constant\(^4\) \( C > 0 \) such that for every \( t \in (0, 1] \) and \( 0 < \delta < \sqrt{G(t)} \),

\[
\mathbb{E}^{\tilde{\mathcal{W}}} [\pi(\delta, t)] \leq C\delta \sqrt{\ln(\delta^{-1}t^{-1/4})}.
\]

(2.11)

**Proof** We first recall from Lemma 6 in [5] that there exists a constant \( C > 0 \) such that for every \( x, y \in S(O, 1) \) and every \( 0 < s \leq t \leq 1 \),

\[
\mathbb{E}^{\tilde{\mathcal{W}}} \left[ \left( \frac{\tilde{\theta}(x, t)}{\sqrt{G(t)}} - \frac{\tilde{\theta}(y, s)}{\sqrt{G(s)}} \right)^2 \right] \leq C \left( t^{-3} - \sqrt{|x - y|} + G(s) - G(t) \right).
\]

(2.12)

By (2.6), (2.7) and (2.12), for every \( 0 < s < t \leq 1 \) and \( x, y \in S(O, 1) \),

\[
\tilde{d}^2(x, t; y, t) = \mathbb{E} \left[ \left( \frac{\tilde{\theta}(x, t)}{\sqrt{G(t)}} - \frac{\tilde{\theta}(y, t)}{\sqrt{G(t)}} \right)^2 \right] \leq C \sqrt{\frac{|x - y|}{t}} \quad \text{and} \quad \tilde{d}^2(y, t; y, s)
\]

\[
= 2 \left( 1 - \sqrt{\frac{G(t)}{G(s)}} \right) .
\]

(2.13)

We will apply the metric entropy method to prove (2.11). For every compact subset \( \mathcal{A} \subseteq S(O, 1) \times (0, 1] \) and every \( \epsilon > 0 \), let \( \text{diam}_{\tilde{d}}(\mathcal{A}) \) be the diameter of \( \mathcal{A} \) under \( \tilde{d} \) and \( N(\epsilon, \mathcal{A}) \) be the smallest number of \( \tilde{d} \)-discs/balls with radius \( \epsilon \) required to cover \( \mathcal{A} \). Then, \( N \) is the metric entropy function with respect to \( \tilde{d} \). For any fixed \( t \in (0, 1) \), set \( \mathcal{A}_t := S(O, 1) \times [t, 1] \), and let \( \pi(\cdot, t) \) be defined as above. Then, according to the standard metric entropy theory (e.g., Theorem 1.3.5 of [1]), there is a universal constant \( K > 0 \) such that

\[
\mathbb{E}^{\tilde{\mathcal{W}}} [\pi(\delta, t)] \leq K \int_0^\delta \sqrt{\ln N(\epsilon, \mathcal{A}_t)}d\epsilon.
\]

(2.14)

Below we will describe a specific finite covering of \( \mathcal{A}_t \) for every \( \epsilon > 0 \) sufficiently small.

\(^4\) Throughout the article, \( C \) refers to a constant that only depends on the dimension \( v \). \( C \)'s value may vary from line to line.
Set $s_\epsilon := \epsilon^4 t / (2 \cdot 3^4 C^2)$ ($C$ is taken to be larger than the constant in (2.12)). Let $\{B(y_l, s_\epsilon) : l = 1, \ldots, L_\epsilon\}$ be a finite covering of $\overline{S(O, 1)}$ where $\{y_l : l = 1, \ldots, L_\epsilon\} \subseteq S(O, 1)$ and $L_\epsilon$ be the smallest number of (Euclidean) balls $B(y_l, s_\epsilon)$ needed to cover $\overline{S(O, 1)}$, and hence,

$$L_\epsilon = O(s_\epsilon^{-\nu}) \leq C \left( \epsilon^{-1} t^{-1/4} \right)^{4\nu}.$$  

By (2.13), the choice of $s_\epsilon$ is such that for every $y, w \in B(y_l, s_\epsilon)$ and every $s \in [t, 1]$,

$$\tilde{d}^2 (y, s; w, s) \leq C \sqrt{2s_\epsilon/s} \leq C \sqrt{2s_\epsilon/s} \leq \epsilon^2 / 9.$$  

Next, take $\tau_0 := 2$ and define $\tau_m$ inductively as

$$\tau_m := G^{-1} \left( \left( 1 - \frac{1}{18} \epsilon^2 \right)^{-2} G(\tau_{m-1}) \right)$$

for $m = 1, \ldots, M_\epsilon + 1$,

where $M_\epsilon$ is the smallest integer such that $\tau_{M_\epsilon} \leq t$ and hence

$$M_\epsilon \leq \frac{C \ln G(t)}{-\ln \left( 1 - \frac{1}{18} \epsilon^2 \right)} \leq C \epsilon^{-2} \ln G(t).$$

Consider the covering of $\overline{S(O, 1)} \times [t, 1]$ that consists of the cylinders

$$\{B(y_l, s_\epsilon) \times (\tau_{m+1}, \tau_{m-1}) : l = 1, \ldots, L_\epsilon, m = 1, \ldots, M_\epsilon \}.$$  

Any pair $((y, t), (w, s))$ that lies in one of the cylinders above, e.g., $B(y_l, s_\epsilon) \times (\tau_{m+1}, \tau_{m-1})$, satisfies that

$$d(y, t; w, s) \leq d(y, \tau_m; y, \tau_m) + d(y, \tau_m; w, \tau_m) + d(w, \tau_m; w, s) \leq \epsilon.$$  

This implies that $N(\epsilon, \mathcal{A}_{\ell}) \leq L_\epsilon \cdot (M_\epsilon + 1)$ and hence by (2.14),

$$\mathbb{E}^{\mathbb{W}}[\pi(\delta, t)] \leq C \int_0^\delta \left( \sqrt{\ln L_\epsilon} + \sqrt{\ln M_\epsilon} \right) d\epsilon.$$  

Therefore, we only need to compute the two integrals in the right hand side above.

By a simple change of variable $u = \sqrt{\ln \left( \epsilon^{-1} t^{-1/4} \right)}$, we get that

$$\int_0^\delta \sqrt{\ln L_\epsilon} d\epsilon \leq C \int_0^\delta \sqrt{\ln \left( \epsilon^{-1} t^{-1/4} \right)} d\epsilon \leq C t^{-1/4} \int_0^\infty \frac{u^2}{\sqrt{\ln(u^{-1} t^{-1/4})}} e^{-u^2} du.$$  

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Since \( \int_{\alpha}^{\infty} e^{-u^2} u^2 \, du = \Theta \left( a e^{-a^2} \right) \) when \( a > 0 \) is sufficiently large, we arrive at

\[
\int_{0}^{\delta} \sqrt{\ln L_\epsilon} \, d\epsilon \leq C \delta \sqrt{\ln \left( \delta^{-1} t^{-1/4} \right)}.
\]

Similarly, one can derive that

\[
\int_{0}^{\delta} \sqrt{\ln M_\epsilon} \, d\epsilon \leq C \delta \sqrt{\ln \left( \delta^{-1} \sqrt{\ln G(t)} \right)}.
\]

Combining the inequalities above, we have proven (2.11).

\[\square\]

Remark 2.5  By Kolmogorov’s continuity theorem, (2.12) implies that \( \{ \tilde{\theta} (x, t) : (x, t) \in \overline{S(O, 1)} \times (0, 1] \} \) has a modification that is continuous in \((x, t)\). Without loss of generality, from now on we will assume that \( \{ \tilde{\theta} (x, t) : (x, t) \in \overline{S(O, 1)} \times (0, 1] \} \) is that modification, i.e., for every \( \theta \in \Theta \), the function

\[
(x, t) \in \overline{S(O, 1)} \times (0, 1] \mapsto \tilde{\theta} (x, t) \in \mathbb{R}
\]

is continuous.

3 Steep Point of Gaussian Free Field

As defined in (1.1) and (1.2) in Sect. 1.1, thick points of \( \theta \) are, intuitively speaking, locations of “high peaks” of the graph of \( \theta \); more rigorously, thick points are defined to be \( x \in \overline{S(O, 1)} \) such that the value of \( \tilde{\theta} (x, t) \) is unusually large for small \( t \). In this section, we will focus on another perspective of the behavior of \( \tilde{\theta} (x, t) \), that is, the rate of change of \( \tilde{\theta} (x, t) \) in \( t \) as \( t \searrow 0 \). If one could establish, in a proper sense, that for some \( x \in \overline{S(O, 1)} \), the rate of change of \( \tilde{\theta} (x, t) \) is unusually large when \( t \) is small, then one would expect that the landscape of \( \theta \) near \( x \) is unusually steep. Taking this consideration into account, we refer to \( x \in \overline{S(O, 1)} \) where \( \tilde{\theta} (x, t) \) changes unusually fast in \( t \) as a “steep point” of \( \theta \).

Although \( \left( \frac{d}{dt} \left( \tilde{\theta} (x, t) \right) \right)^{+} \) is the natural object to consider as the rate of change of \( \tilde{\theta} (x, t) \), it is clear from the last statement of Lemma 2.2 that at any given \( x, \tilde{\theta} (x, t) \) is a.s. nowhere differentiable in \( t \). To overcome the indifferentiability, we will study the rate of change of \( \tilde{\theta} (x, t) \) through some test function \( f : (0, 1] \rightarrow \mathbb{R} \). The choice of such test function \( f \) is rather general upon satisfying some basic requirements. On the one hand, since \( f \) has to overcome the singularity of \( \tilde{\theta} (x, t) \) when \( t \) is small, it is natural to require \(|f(t)|\) to decay to 0 sufficiently fast as \( t \searrow 0 \). On the other hand, \(|f(t)|\) should not decay too fast so that the unusual behaviors of \( \tilde{\theta} (x, t) \) for small \( t \) can still be captured. Furthermore, in order to pair \( f \) with \( \left( \frac{d}{dt} \left( \tilde{\theta} (x, t) \right) \right)^{+} \), \( f \) should have bounded variation at least locally on \((0, 1]\), i.e., \( f \in BV_{loc} ((0, 1]) \), and the total variation of \( f \) on \([t, 1]\) should not grow too fast as \( t \searrow 0 \). There is flexibility in choosing the class of \( f \) to which our methods and results apply. For the sake of convenience, we adopt the following class of \( f \).

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Definition 3.1 Define $\mathcal{C}$ to be the family of function $f \in BV_{\text{loc}} ((0, 1])$ satisfying that (a) there exist constants $C_f, \rho_f > 0$ such that for every $t \in (0, 1)$,

$$|f(t)| \sqrt{G(t)} \leq C_f (- \ln t)^{\rho_f} \text{ and } \int_0^t \sqrt{G(s)} dV_f (s) \leq C_f (- \ln t)^{\rho_f},$$

where $V_f (t)$ is the total variation of $f$ on $[t, 1]$ and the integral above is a Riemann–Stieltjes integral;

(b) if $\Sigma_f (t) := \int_1^t f^2 (s) \, dG (s)$ for every $t \in (0, 1]$, then $\lim_{t \searrow 0} \Sigma_f (t) = \infty$.

Given $f \in \mathcal{C}$, for every $(x, t) \in \overline{S (O, 1)} \times (0, 1]$ and $\theta \in \Theta$, since $f \in BV ([t, 1])$ and $s \in [t, 1] \mapsto \tilde{\theta} (x, s)$ is continuous, we know that

$$X^\theta_f (x, t) := \int_0^t f (s) \, d\tilde{\theta} (x, s) = f(t) \tilde{\theta} (x, t) - f(1) \tilde{\theta} (x, 1) - \int_1^t \tilde{\theta} (x, s) \, df (s)$$

is well defined as a Riemann–Stieltjes integral. Inspired by the idea of thick point, we want to identify point $x$ where $X^\theta_f (x, t)$ becomes unusually large.

Definition 3.2 Given $f \in \mathcal{C}$, for every $\theta \in \Theta$, $x \in \overline{S (O, 1)}$ is called an $f$-steep point of $\theta$ if

$$\lim_{t \searrow 0} \frac{X^\theta_f (x, t)}{\Sigma_f (t)} = \sqrt{2\nu}. \quad (3.1)$$

We denote by $D_\theta^f$ the set of all $f$-steep points of $\theta$.

We also introduce the set of super $f$-steep points of $\theta$:

$$D_{\theta, \lim sup}^f := \left\{ x \in \overline{S (O, 1)} : \limsup_{t \searrow 0} \frac{X^\theta_f (x, t)}{\Sigma_f (t)} \geq \sqrt{2\nu} \right\},$$

and the set of sub $f$-steep points of $\theta$:

$$D_{\theta, \lim inf}^f := \left\{ x \in \overline{S (O, 1)} : \liminf_{t \searrow 0} \frac{X^\theta_f (x, t)}{\Sigma_f (t)} \geq \sqrt{2\nu} \right\}.$$ 

Obviously,

$$D_f \subseteq D_{\theta, \lim sup}^f \subseteq D_{\theta, \lim inf}^f \subseteq D_{\theta, \lim sup}^f. \quad (3.2)$$

When there is no ambiguity, we will omit the superscript “$\theta$” and/or the subscript “$f$” in the notations above and write $X^\theta_f (x, t)$, $\Sigma_f (t)$, $D_{\theta, \lim sup}^f$, $D_{\theta, \lim inf}^f$ and $D_{\theta, \lim sup}^f$ as, respectively, $X (x, t)$, $\Sigma (t)$, $D$, $D_{\lim sup}$ and $D_{\lim inf}$ for simplicity. For every $(x, t) \in \overline{S (O, 1)} \times (0, 1]$, $X (x, t)$ is a random variable on $\Theta$. It is clear from
the last statement of Lemma 2.2 that, with \( x \) fixed, \( \{ X (x, t) : t \in (0, 1) \} \) is a continuous centered Gaussian process with independent increments (in the direction of \( t \) decreasing) and \( \mathbb{E} \left[ (X (x, t))^2 \right] = \Sigma (t) \) for every \( t \in (0, 1) \). In other words, \( \{ X (x, t) : t \in (0, 1) \} \) can be viewed as a Brownian motion running by the “clock” \( \Sigma (t) \), and according to (b) in Definition 3.1, the “clock” goes on forever. Hence, the law of the iterated logarithm says that for every \( x \in S (O, 1) \),

\[
\limsup_{t \searrow 0} \frac{X (x, t)}{\sqrt{2 \Sigma (t) \ln \ln \Sigma (t)}} = 1 \text{ a.s.,}
\]

which implies that

\[
\mathcal{W} (x \in D) = \mathcal{W} (x \in D_{\lim \inf}) = \mathcal{W} (x \in D_{\lim \sup}) = 0 \text{ for every } x \in S (O, 1).
\]

This is to say that (super/sub) \( f \)-steep point sets are exceptional sets of the GFF. Indeed, it is clear from (3.1) that \( f \)-steep point is the analog of thick point corresponding to the process \( \{ X (\cdot, t) : t \in (0, 1) \} \). As a consequence, one would expect that the results established on the Hausdorff dimension of thick point sets, as reviewed in Sect. 1, can be extended to steep point sets. This is the focus of this section. The following theorem contains our main result.

**Theorem 3.3** Given \( f \in \mathcal{C} \), set

\[
\tilde{c}_f := \limsup_{t \searrow 0} \frac{\Sigma (t)}{-\ln t} \text{ and } c_f := \liminf_{t \searrow 0} \frac{\Sigma (t)}{-\ln t}.
\]

(i) if \( \tilde{c}_f > 1 \), then \( D = D_{\lim \inf} = \emptyset \) a.s.; if \( 0 < c_f \leq \tilde{c}_f \leq 1 \), then

\[
\dim \mathcal{W} (D) = \dim \mathcal{W} (D_{\lim \inf}) = \nu \left( 1 - \tilde{c}_f \right) \text{ a.s.};
\]

(ii) if \( c_f > 1 \), then \( D_{\lim \sup} = \emptyset \) a.s.; if \( 0 < c_f \leq 1 \), then

\[
\nu \left( 1 - c_f \right) \leq \dim \mathcal{W} (D_{\lim \sup}) \leq \nu \left( 1 - c_f \right) \text{ a.s.}
\]

*In particular, when \( \tilde{c}_f = c_f =: c_f \), if \( c_f > 1 \), then

\[
D = D_{\lim \inf} = D_{\lim \sup} = \emptyset \text{ a.s.,}
\]

and if \( 0 < c_f \leq 1 \), then

\[
\dim \mathcal{W} (D) = \dim \mathcal{W} (D_{\lim \inf}) = \dim \mathcal{W} (D_{\lim \sup}) = \nu \left( 1 - c_f \right) \text{ a.s.}
\]

There are abundant examples of \( f \in \mathcal{C} \), which lead to the study of a variety of exceptional sets of the GFF. In Sect. 4, we will demonstrate a few concrete choices, including \( f \) being a constant function (in 2D), as well as \( f \) being a piecewise constant
function (in 2D and higher dimensions). We will see that, by choosing \( f \) properly and applying Theorem 3.3 to the corresponding \( f \)-steep point sets, not only can we recover the existing results on thick point sets, we also discover new exceptional sets of the GFF and are able to determine their Hausdorff dimensions.

In this section, we choose and fix \( f \in \mathcal{C} \). The proof of Theorem 3.3 follows a similar strategy as that adopted in \([5,16]\), combined with an analysis of the continuity modulus of \( \{X(x, t) : x \in S(O, 1), t \in (0, 1)\} \), which will be carried out in Lemma 3.4. We study the Hausdorff dimension of each of \( D, D_{\text{lim inf}} \) and \( D_{\text{lim sup}} \) by establishing the “upper bound” and the “lower bound” separately. We want to point out that the condition “\( c_f > 0 \)” (and “\( c_f > 0 \)” imposed in Theorem 3.3 is not necessary in proving the upper bounds in Sect. 3.1 and can be relaxed under certain circumstances in deriving the lower bounds in Sect. 3.2 (see Remark 3.10).

### 3.1 Upper Bound

This subsection is devoted to establishing the upper bounds for the Hausdorff dimension of the concerned exceptional sets. To get started, we need to develop estimates related to the modulus of continuity for \( \{\bar{\theta}(x, t) / \sqrt{G(t)} : (x, t) \in S(O, 1) \times (0, 1)\} \). Instead of studying this Gaussian family directly, we will make use of the existing results on the modulus of continuity of \( \{\bar{\theta}(x, t) / \sqrt{G(t)} : (x, t) \in S(O, 1) \times (0, 1)\} \).

**Lemma 3.4** For every \( n \geq 1 \), let \( B_n \) be the subset of \( S(O, 1)^2 \times (0, 1) \) that

\[
B_n := \{(x, y, t) : x, y \in S(O, 1), |x - y| < 2^{-(n+1)^2} 2 \sqrt{\nu}, t \in [2^{-n^2}, 2^{-(n-1)^2}]\}.
\]

When \( n \) is sufficiently large, we have that

\[
\mathbb{E} \sup_{(x, y, t) \in B_n} \left| \frac{X(x, t)}{\Sigma(t)} - \frac{X(y, t)}{\Sigma(t)} \right| \leq 2^{-n/4},
\]

and hence

\[
\mathbb{P} \left( \sup_{(x, y, t) \in B_n} \left| \frac{X(x, t)}{\Sigma(t)} - \frac{X(y, t)}{\Sigma(t)} \right| > 2^{-n/8} \ i.o. \right) = 0. \quad (3.3)
\]

**Proof** We only need to prove the first statement, since the second statement is an immediate consequence of the first one by the Borel–Cantelli lemma. To facilitate the proof, for every \( n \geq 1 \) and \( t \in [2^{-n^2}, 1] \), we consider the random variable \( \tilde{m}_n(t) \) where

\[
\tilde{m}_n(t) := \sup \left\{ \frac{\tilde{\theta}(x, s)}{\sqrt{G(s)}} - \frac{\tilde{\theta}(y, s)}{\sqrt{G(s)}} : x, y \in S(O, 1), |x - y| \leq 2^{-(n+1)^2} 2 \sqrt{\nu}, s \in [t, 1] \right\}
\]

for every \( \theta \in \Theta \).
We observe that if $\tilde{d}$ is the metric defined in (2.10), then by (2.12), we have that

$$\tilde{d}(x, s; y, s) \leq C_s^{-1/4}|x - y|^{1/4} \leq Ct^{-1/4}2^{-(n+1)^2/4} \leq C2^{-n/2},$$

whenever $|x - y| \leq 2^{-(n+1)^2}2\sqrt{v}$ and $s \in [t, 1]$, and hence according to (2.11),

$$\mathbb{E}^\mathcal{W}[\bar{m}_n(t)] \leq \mathbb{E}^\mathcal{W}[\bar{\pi}(C2^{-n/2}, t)] \leq C2^{-n/2}n. \quad (3.4)$$

In addition, if we set

$$mn(t): \theta \mapsto m^\theta_n(t)$$

$$:= \sup \left\{ |\tilde{\theta}(x, s) - \tilde{\theta}(y, s)| : x, y \in S(O, 1), |x - y| \leq 2^{-(n+1)^2}2\sqrt{v}, s \in [t, 1] \right\}.$$

then

$$\mathbb{E}^\mathcal{W}[m_n(t)] \leq \mathbb{E}^\mathcal{W}[\bar{m}_n(t)] \sqrt{G(t)} \leq C\sqrt{G(t)}2^{-n/2}n.$$

Now we turn our attention to the desired statement. For every $n$ sufficiently large and $(x, y, t) \in B_n$, the assumption (b) in Definition 3.1 guarantees that $\Sigma(t) \geq \Sigma\left(2^{-(n+1)^2}\right) > 1$, and hence

$$\left| \frac{X(x, t)}{\Sigma(t)} - \frac{X(y, t)}{\Sigma(t)} \right| \leq |X(x, t) - X(y, t)|$$

$$\leq |f(t)(\tilde{\theta}(x, t) - \tilde{\theta}(y, t))| + |f(1)(\tilde{\theta}(x, 1) - \tilde{\theta}(y, 1))|$$

$$+ \int_1^t |\tilde{\theta}(x, s) - \tilde{\theta}(y, s)| dV_f(s)$$

$$\leq |f(t)| \sqrt{G(t)} |\bar{m}_n(t)| + |f(1)| \sqrt{G(1)} |\bar{m}_n(1)|$$

$$+ \int_1^t m_n(s) dV_f(s)$$

Therefore, by (3.4) and (a) in Definition 3.1, there exists $C > 0$ such that for all sufficiently large $n$,

$$\mathbb{E}^\mathcal{W}\left[ \sup_{(x, y, t) \in B_n} \left| \frac{X(x, t)}{\Sigma(t)} - \frac{X(y, t)}{\Sigma(t)} \right| \right]$$

$$\leq Cn^{2\rho_f} \mathbb{E}^\mathcal{W}[\bar{m}_n(2^{-n^2})] + \int_1^{2^{-n^2}} \mathbb{E}^\mathcal{W}[m_n(s)] dV_f(s)$$

$$\leq Cn^{2\rho_f + 1}2^{-n/2} + Cn2^{-n/2} \int_1^{2^{-n^2}} \sqrt{G(s)} dV_f(s)$$

$$\leq Cn^{2\rho_f + 1}2^{-n/2} + Cn^{1 + 2\rho_f}2^{-n/2} \leq 2^{-n/4}.$$
We are now ready to prove the upper bound for \( \dim_H(D) \), \( \dim_H(D_{\lim \sup}) \) and \( \dim_H(D_{\lim \inf}) \).

**Proposition 3.5**  Under the setting of Theorem 3.3, we have that

(i) if \( \zeta_f > 1 \), then

\[
D = D_{\lim \inf} = D_{\lim \sup} = \emptyset \text{ a.s.};
\]

(ii) if \( \zeta_f \leq 1 < \bar{c}_f \), then

\[
D = D_{\lim \inf} = \emptyset \text{ and } \dim_H(D_{\lim \sup}) \leq \nu \left( 1 - \zeta_f \right) \text{ a.s.};
\]

(iii) if \( \bar{c}_f \leq 1 \), then

\[
\dim_H(D) \leq \dim_H(D_{\lim \inf}) \leq \nu \left( 1 - \bar{c}_f \right) \text{ a.s.}
\]

**Proof** We will first prove the result concerning \( D_{\lim \inf} \), from which the claim about \( D \) follows due to (3.2), and the rest of the statement can be proven by similar arguments with minor changes. Assume \( \bar{c}_f > 0 \). Otherwise, the inequality on \( \dim_H(D_{\lim \inf}) \) is satisfied trivially.

For each \( n \geq 0 \), consider a finite lattice partition of \( S(O, 1) \) with cell size \( 2 \cdot 2^{-n} \) (i.e., the length of each side of the cell is \( 2 \cdot 2^{-n} \)). Let \( \left\{ x_j^{(n)} : j = 1, \ldots, J_n \right\} \) be the collection of the lattice cell centers where \( J_n = 2^{\nu n^2} \) is the total number of the cells. Fix \( c_f' \in (0, \bar{c}_f) \) and let \( c_f' \) be arbitrarily close to \( \bar{c}_f \). If \( \bar{c}_f = \infty \), then we take \( c_f' \) to be arbitrarily large. There exists a sequence \( \{ s_k : k \geq 1 \} \subseteq (0, 1) \) such that \( s_k \searrow 0 \) as \( k \to \infty \), and \( \Sigma(s_k) > c_f' (-\ln s_k) \) for all sufficiently large \( k \geq 1 \). For each \( k \geq 1 \), set \( n_k \) to be the unique positive integer such that \( 2^{-n_k^2} < s_k \leq 2^{-(n_k+1)^2} \). According to (3.3), with probability one, for all but finitely many \( n \) and every \( j = 1, \ldots, J_{n+1} \),

\[
\sup_{(y,t) \in S(x_j^{(n+1)}, 2^{-(n+1)^2}) \times [2^{-n^2}, 2^{-(n-1)^2}]} \left| \frac{X(y,t)}{\Sigma(t)} - \frac{X(x_j^{(n+1)}, t)}{\Sigma(t)} \right| \leq 2^{-n/8}. \tag{3.5}
\]

If \( y_0 \in D_{\lim \inf} \), then (3.5) guarantees that for arbitrarily small \( \epsilon > 0 \) and sufficiently large \( k \), we have that

\[
\frac{X(x_j^{(n_k+1)}, s_k)}{\Sigma(s_k)} > (1 - \epsilon) \sqrt{2\nu}, \tag{\dagger}
\]

where \( x_j^{(n_k+1)} \) is the center of the lattice cell (at the \( (n_k + 1) \)st level) where \( y_0 \) lies, i.e., \( y_0 \in S(x_j^{(n_k+1)}, 2^{-(n_k+1)^2}) \). Equivalently, if we denote by \( J_k \) the set of \( x_j^{(n_k+1)} \),
\[ j = 1, \ldots, J_{n_k+1}, \text{ such that \( (\dagger) \) holds, then} \]

\[
D_{\lim \inf} \subseteq \bigcup_{K \geq 1} \bigcap_{k \geq K} \bigcup_{j=1}^{J_{n_k+1}} \left\{ S \left( x_j^{(n_k+1)}, 2^{-(n_k+1)^2} \right) : x_j^{(n_k+1)} \in \mathcal{J}_k \right\} \text{ a.s. (3.6)}
\]

Meanwhile, since \( X \left( x_j^{(n_k+1)}, s_k \right) \) has the centered Gaussian distribution with variance \( \Sigma (s_k) \), we have that for all sufficiently large \( k \),

\[
\mathcal{W} \left( x_j^{(n_k+1)} \in \mathcal{J}_k \right) \leq C \exp \left[ -v \Sigma (s_k) (1 - \epsilon)^2 \right] \leq C S_k^{v c_f (1-\epsilon)^2}. \tag{3.7}
\]

Assume that \( c_f' < 1, c_f' \) sufficiently close to \( c_f \) and \( \epsilon \) sufficiently small, one can always make \( c_f' (1-\epsilon)^2 > 1 + \epsilon \). Therefore, (3.7) implies that for sufficiently large \( k \).

\[
\mathcal{W}' (D_{\lim \inf} \neq \emptyset) \leq \mathcal{W} \left( \bigcup_{K \geq 1} \bigcap_{k \geq K} \bigcup_{j=1}^{J_{n_k+1}} \left\{ x_j^{(n_k+1)} \in \mathcal{J}_k \right\} \right)
\]

\[
\leq \sum_{K \geq 1} \limsup_{K \leq k, k \to \infty} J_{n_k+1} \cdot \mathcal{W} \left( x_j^{(n_k+1)} \in \mathcal{J}_k \right)
\]

\[
\leq C \sum_{K \geq 1} \limsup_{K \leq k, k \to \infty} 2^{v(n_k+1)^2} S_k^{v(1+\epsilon)} = 0.
\]

That is, \( D_{\lim \inf} = \emptyset \) a.s. when \( c_f' > 1 \).

Next, assume that \( c_f' \in (0, 1] \). Note that, with probability one, the right hand side of (3.6) forms a covering of \( D_{\lim \inf} \), and the diameter of \( S \left( x_j^{(n_k+1)}, 2^{-(n_k+1)^2} \right) \) is \( 2 \sqrt{v} 2^{-(n_k+1)^2} \). Thus, if \( \mathcal{H}^\eta \) is the Hausdorff–\( \eta \) measure for \( \eta > 0 \), then

\[
\mathcal{H}^\eta (D_{\lim \inf}) \leq \liminf_{k \to \infty} \sum_{j=1, \ldots, J_{n_k+1}} 2^{-\eta(n_k+1)^2} \left( 2 \sqrt{v} 2^{-(n_k+1)^2} \right)^\eta
\]

\[
= (2 \sqrt{v})^\eta \liminf_{k \to \infty} 2^{-\eta(n_k+1)^2} \text{card} \left( \mathcal{J}_k \right).
\]

Again, it follows from (3.7) and Fatou’s lemma that

\[
\mathbb{E}^\mathcal{W} \left[ \mathcal{H}^\eta (D_{\lim \inf}) \right] \leq (2 \sqrt{v})^\eta \liminf_{k \to \infty} 2^{-\eta(n_k+1)^2} \mathbb{E}^\mathcal{W} \left[ \text{card} \left( \mathcal{J}_k \right) \right]
\]

\[
\leq (2 \sqrt{v})^\eta \liminf_{k \to \infty} 2^{(v-\eta)(n_k+1)^2} \mathcal{W} \left( x_j^{(n_k+1)} \in \mathcal{J}_k \right)
\]

\[
\leq (2 \sqrt{v})^\eta \liminf_{k \to \infty} 2^{(v-\eta)(n_k+1)^2 - v c_f' (1-\epsilon)^2(n_k-1)^2}.
\]
Given any \( \eta > \nu \left( 1 - \bar{c}_f \right) \), we can choose \( c'_f \) and \( \epsilon \) such that \( \eta > \nu - \nu c'_f \left( 1 - \epsilon \right)^2 \). In this case, we see from the inequality above that \( \mathcal{H}^\eta \left( D_{\lim \inf} \right) = 0 \) a.s., which implies that \( \dim \mathcal{H} \left( D_{\lim \inf} \right) \leq \eta \) a.s.. Since \( \eta \) can be arbitrarily close to \( \nu \left( 1 - \bar{c}_f \right) \), we conclude that

\[
\dim \mathcal{H} \left( D_{\lim \inf} \right) \leq \nu \left( 1 - \bar{c}_f \right) \text{ a.s.}
\]

Now we turn our attention to \( D_{\lim \sup} \). Again, without loss of generality, we will assume \( c_f > 0 \). Let \( c''_f = \left( 0, c_f \right) \) be arbitrarily close to \( c_f \), and \( \epsilon > 0 \) be arbitrarily small. Obviously, \( \Sigma \left( t \right) > c'_f \left( - \ln t \right) \) for all sufficiently small \( t \). Meanwhile, for every \( y_0 \in D_{\lim \sup} \), one can find a sequence \( \{ u_k : k \geq 1 \} \subseteq (0, 1] \) such that \( u_k \searrow 0 \) as \( k \rightarrow \infty \), and

\[
\frac{X \left( y_0, u_k \right)}{\Sigma \left( u_k \right)} > \left( 1 - \frac{\epsilon}{4} \right) \sqrt{2 \nu}
\]

for sufficiently large \( k \). Similarly, we choose \( n_k \) to be the unique integer such that \( 2^{-n_k^2} < u_k \leq 2^{-\left( n_k - 1 \right)^2} \). Note that this time the choice of \( \{ u_k : k \geq 1 \} \) and \( \{ n_k : k \geq 1 \} \) will depend on the specific instance \( \theta \), but we can still make the arguments above work. Since the estimate (3.5) still applies, when \( k \) is sufficiently large,

\[
\frac{X \left( x_j^{(n_k + 1)}, u_k \right)}{\Sigma \left( u_k \right)} > \left( 1 - \frac{\epsilon}{2} \right) \sqrt{2 \nu}
\]

where \( x_j^{(n_k + 1)} \) is again the center of the cell (at the \( (n_k + 1) \)st level) that contains \( y_0 \), and hence,

\[
\sup_{t \in \left[ 2^{-n_k^2}, 2^{-\left( n_k - 1 \right)^2} \right]} \frac{X \left( x_j^{(n_k + 1)}, t \right)}{\Sigma \left( t \right)} > \left( 1 - \frac{\epsilon}{2} \right) \sqrt{2 \nu}. \quad (\dagger\dagger)
\]

For each \( n \geq 1 \), Denote by \( \mathcal{H}_n \) the set of \( x_j^{(n + 1)} \), \( j = 1, \ldots, J_{n+1} \), such that (\( \dagger\dagger \)) holds (with \( n_k \) replaced by \( n \)). Then,

\[
D_{\lim \sup} \subseteq \bigcap_{k \geq 1} \bigcup_{n \geq k} \bigcup_{j = 1}^{J_{n + 1}} \left\{ S \left( x_j^{(n + 1)}, 2^{-\left( n + 1 \right)^2} \right) : x_j^{(n + 1)} \in \mathcal{H}_n \right\}.
\]

Following the same method as above, to proceed, we need to study the probability of the event (\( \dagger\dagger \)). Different from (\( \dagger \)), (\( \dagger\dagger \)) concerns the maximum of a family of Gaussian random variables, for which we will use the Borell–TIS inequality (e.g., §2.3 of [1]).
The Borell–TIS inequality tells us that, if
\[ M := \sup_{t \in \left[ 2^{-n^2}, 2^{-(n-1)^2} \right]} \frac{X \left( x_j^{(n+1)}, t \right)}{\Sigma (t)} \quad \text{and} \quad \varsigma^2 := \sup_{t \in \left[ 2^{-n^2}, 2^{-(n-1)^2} \right]} \text{Var} \left( \frac{X \left( x_j^{(n+1)}, t \right)}{\Sigma (t)} \right) \]
then \( M \) satisfies the exponential tail probability estimate, i.e.,
\[ \mathbb{W} \left( M - \mathbb{E}[M] > c \right) \leq 2 \exp \left( -\frac{c^2}{2\varsigma^2} \right) \quad \text{for every } c > 0. \]
Obviously, \( \varsigma^2 = \left( \Sigma \left( 2^{-(n-1)^2} \right) \right)^{-1} \). Since \( t \mapsto X \left( x_j^{(n+1)}, t \right) / \Sigma (t) \) can be identified with \( \tau \mapsto B(\tau)/\tau \) where \( B(\tau) \) is the standard Brownian motion, it is easy to check that there exists a constant \( C > 0 \) such that
\[ \mathbb{E}[M] \leq C \left( \Sigma \left( 2^{-(n-1)^2} \right) \right)^{-1/2} \]
which tends to zero as \( n \to \infty \) according to (b) in Definition 3.1. Thus, when \( n \) is sufficiently large,
\[ \mathbb{W} \left( x_j^{(n+1)} \in \mathcal{K}_n \right) = \mathbb{W} \left( M > \left( 1 - \frac{\epsilon}{2} \right) \sqrt{2\nu} \right) \]
\[ \leq \mathbb{W} \left( M - \mathbb{E}[M] > \left( 1 - \epsilon \right) \sqrt{2\nu} \right) \]
\[ \leq 2 \exp \left[ -\nu \left( 1 - \epsilon \right)^2 \Sigma \left( 2^{-(n-1)^2} \right) \right] \]
\[ \leq C \left( 2^{-(n-1)^2} \right)^{\nu/2} (1-\epsilon)^2 \]
The rest follows in exactly the same way as in the proof of the upper bound of \( \dim_{\mathcal{H}} (D_{\lim\inf}) \). \( \square \)

### 3.2 Lower Bound

We now move on to the lower bound of the Hausdorff dimension of the concerned exceptional sets. We summarize in the proposition below the result we want to prove.

**Proposition 3.6** Under the setting of Theorem 3.3, if \( \tilde{c}_f > 0 \), then
\[ \dim_{\mathcal{H}} (D_{\lim\sup}) \geq \dim_{\mathcal{H}} (D_{\lim\inf}) \geq \dim_{\mathcal{H}} (D) \geq \nu \left( 1 - \tilde{c}_f \right) \quad \text{a.s.} \]

Without loss of generality, we will assume that \( \tilde{c}_f < 1 \). Otherwise the lower bound in (i) is trivially satisfied. Let \( \tilde{\epsilon} \in \left( \tilde{c}_f, 1 \right) \) and \( \epsilon \in \left( 0, \tilde{c}_f \right) \) be arbitrarily close to \( \tilde{c}_f \) and, respectively, \( \tilde{c}_f \). Then, for \( t > 0 \) sufficiently small,
\[ \epsilon (-\ln t) \leq \Sigma (t) \leq \tilde{\epsilon} (-\ln t). \quad (3.8) \]
The main strategy to prove the desired lower bound is to create a setting in which we can apply Frostman’s lemma. We need to prepare two “ingredients” to achieve this purpose. Let us start with “discretizing” the problem, similarly as what we did in Sect. 3.1, by considering $X(\cdot, t)$ and $\Sigma(t)$ along a sequence in $t$. To be specific, we set $t_0 := 1$ and $t_n := \exp(-2n^2)$ for every $n \geq 1$. For simplicity, we write, for every $x \in S(O, 1)$ and $n \geq 1$,

$$
\Delta X_n(x) := X(x, t_n) - X(x, t_{n-1}) \quad \text{and} \quad \Delta \Sigma_n := \Sigma(t_n) - \Sigma(t_{n-1}).
$$

For every $x \in S(O, 1)$ and $n \geq 1$, set

$$
P_{x,n} := \left\{ \theta \in \Theta : \sup_{t \in [t_0, t_n]} \left| X(x, t) - X(x, t_{n-1}) - \sqrt{2\nu} (\Sigma(t) - \Sigma(t_{n-1})) \right| \leq \sqrt{\Delta \Sigma_n} \right\},
$$

(3.9)

and $\Phi_{x,n} := \left( \bigcap_{i=1}^n P_{x,i} \right)$. The first ingredient we need is the probability estimate for $P_{x,n}$ and $\Phi_{x,n}$.

**Lemma 3.7** For every $n \geq 1$, $P_{x,i}, i = 1, \ldots, n$, are mutually independent. Moreover, there is a constant $p \in (0, 1)$ such that for every $n \geq 1$,

$$
\exp\left( \ln p - \nu \Delta \Sigma_n - \sqrt{2\nu \Delta \Sigma_n} \right) \leq \mathcal{W}(P_{x,n}) \leq \exp\left( \ln p - \nu \Delta \Sigma_n + \sqrt{2\nu \Delta \Sigma_n} \right)
$$

(3.10)

and hence

$$
\exp\left[ n \ln p - \nu \Sigma(t_n) - \sqrt{2\nu n \Sigma(t_n)} \right] \leq \mathcal{W}(\Phi_{x,n}) \leq \exp\left[ n \ln p - \nu \Sigma(t_n) + \sqrt{2\nu n \Sigma(t_n)} \right].
$$

(3.11)

**Proof** The independence of $\{P_{x,j} : j = 1, \ldots, n\}$ is obvious from the fact that $X(x, t)$ has independent increments as $t$ decreases. We only need to show (3.10), since (3.11) follows from (3.10), the independence and the Cauchy inequality. For every $n \geq 1$ and $x \in S(O, 1)$, assume that $h$ is the unique element in $H$ such that the corresponding Paley–Wiener integral is given by $\mathcal{I}(h) = -\Delta X_n(x)$. Then, for $t \in [t_n, t_{n-1}]$,

$$
(h, h_{\tilde{\sigma}(x,t)})_H = \mathbb{E}^{\mathcal{W}} \left[ \mathcal{I}(h) \cdot \tilde{\theta}(x, t) \right] = -\int_{t_n}^{t} f(s) \, dG(s).
$$
which implies that for every $\theta \in \Theta$,

$$X^{\theta + \sqrt{2v}h} (x, t) - X^{\theta + \sqrt{2v}h} (x, t_{n-1})$$

$$= \int_{t_{n-1}}^t f (s) \, d \left( \tilde{\theta} (x, s) + \sqrt{2v} \left( h, h \tilde{\sigma} (x, s) \right)_H \right)$$

$$= X^{\theta} (x, t) - X^{\theta} (x, t_{n-1}) - \sqrt{2v} \int_{t_{n-1}}^t f^2 (s) \, dG (s)$$

$$= X^{\theta} (x, t) - X^{\theta} (x, t_{n-1}) - \sqrt{2v} \left[ \Sigma (t) - \Sigma (t_{n-1}) \right].$$

Moreover,

$$\|h\|_H^2 = \mathbb{E}^H \left[ (\mathcal{J} (h))^2 \right] = \mathbb{E}^H \left[ (\Delta X_n (x))^2 \right] = \Delta \Sigma_n.$$ 

Therefore, by the Cameron–Martin formula (e.g., §8 in [24]), we get that

$$\mathcal{W} (P_{x,n}) = \mathcal{W} \left( \sup_{t \in [t_n, t_{n-1}]} \left| X^{\theta + \sqrt{2v}h} (x, t) - X^{\theta + \sqrt{2v}h} (x, t_{n-1}) \right| \leq \sqrt{\Delta \Sigma_n} \right)$$

$$= \mathbb{E}^\mathcal{W} \left[ e^{\sqrt{2v} \mathcal{J} (h) - \mathcal{J} (h)_H^2} \sup_{t \in [t_n, t_{n-1}]} \left| X (x, t) - X (x, t_{n-1}) \right| \leq \sqrt{\Delta \Sigma_n} \right]$$

$$= e^{-\mathcal{J} (h)} \mathbb{E}^\mathcal{W} \left[ e^{\sqrt{2v} \mathcal{J} (h)} \sup_{t \in [t_n, t_{n-1}]} \left| X (x, t) - X (x, t_{n-1}) \right| \leq \sqrt{\Delta \Sigma_n} \right],$$

where, for an integrable random variable $F$ on $\Theta$ and a measurable set $A$, the notation “$\mathbb{E}^\mathcal{W} [F; A]$” refer to $\int_A F \, d\mathcal{W}$. If the constraint in the right hand side above is satisfied, then $- \sqrt{\Delta \Sigma_n} \leq \mathcal{J} (h) \leq \sqrt{\Delta \Sigma_n}$. Meanwhile, we recall that

$$\{ X (x, t) - X (x, t_{n-1}) : t \in [t_n, t_{n-1}] \}$$

has the distribution of the standard Brownian motion $\{ B (\tau) : 0 \leq \tau \leq T \}$ with $\tau := \Sigma (t) - \Sigma (t_{n-1})$ and $T := \Delta \Sigma_n$. Thus,

$$\mathcal{W} \left( \sup_{t \in [t_n, t_{n-1}]} \left| X (x, t) - X (x, t_{n-1}) \right| \leq \sqrt{\Delta \Sigma_n} \right)$$

$$= \mathbb{P} \left( \sup_{\tau \in [0, T]} \left| B (\tau) \right| \leq \sqrt{T} \right)$$

$$= \mathbb{P} \left( \sup_{\tau \in [0, 1]} \left| B (\tau) \right| \leq 1 \right) := p \in (0, 1).$$

Combining all of the above leads to (3.10). 

The events $P_{x,n}$ and $\Phi_{x,n}$, which concern the discrete family $\{ \Delta X_n (x) : n \geq 0 \}$, will be used to design a specific collection of $f$-steep points, i.e., a subset of $D$, and whose Hausdorff measure or Hausdorff dimension is convenient to study. This is the second
ingredient needed for the proof of the main result. For every \( n \geq 0 \), we consider the lattice partition of \( S(\mathcal{O}, 1) \) with cell size \( t_n \). Let \( \{x_j^{(n)} : j = 1, \ldots, J_n := t_n^{-\nu} \} \) be the collection of all the cell centers and set

\[
\mathcal{L}_n := \left\{ x_j^{(n)} : j = 1, \ldots, J_n, \Phi_{x_j^{(n)}, n} \text{ occurs} \right\}.
\]

**Lemma 3.8** If \( \Upsilon := \bigcap_{k \geq 1} \bigcup_{n \geq k} \bigcup_{x \in \mathcal{L}_n} S(x, t_n) \), then \( \Upsilon \subseteq D \) a.s..

**Proof** Take \( y \in \Upsilon \). It is easy to see that one can always find a subsequence \( \{n_j : j \geq 1\} \subseteq \mathbb{N} \) with \( n_j \not\to \infty \) as \( j \to \infty \) and a sequence of cell centers \( \{x^{(n_j)} \} \subseteq \mathcal{L}_{n_j} \) such that \( \lim_{j \to \infty} |y - x^{(n_j)}| = 0 \). For any \( t \in (0, 1] \), assume that \( \ell \in \mathbb{N} \) is the unique integer such that \( t \ell \leq t < t \ell - 1 \). When \( \ell \) is sufficiently large, by (3.8), we have that for every \( n_j \geq \ell \) and \( x^{(n_j)} \in \mathcal{L}_{n_j} \),

\[
\left| \frac{X\left(x^{(n_j)}, t\right)}{\Sigma(t)} - \sqrt{2v} \right| \leq \frac{\ell\Sigma(t)}{\Sigma(t) - 1} \leq \frac{\ell\Sigma(t)}{\Sigma(t) - 1} = \frac{\sqrt{\ell\Sigma(t)}}{\ell\Sigma(t)} \leq \frac{\sqrt{\ell}}{\ell^{\ell-1}},
\]

(3.12)

which tends to zero as \( \ell \) goes to infinity, or equivalently, as \( t \) goes to zero.

Moreover, the function \( y \mapsto X(y, t) / \Sigma(t) \) is continuous and hence uniformly continuous on \( S(\mathcal{O}, 1) \), so one can also make \( \left| X\left(x^{(n_j)}, t\right) - X(y, t) / \Sigma(t) \right| \) arbitrarily small by choosing sufficiently large \( n_j \). This is sufficient for us to conclude that \( y \in D \).

Now we are ready to start the proof of Proposition 3.6. Our goal is to apply Frostman’s lemma to bound \( \dim \mathcal{H}(\Upsilon) \) from below, which requires us to find a non-trivial Borel measure \( \mu \) supported on \( \Upsilon \) and to study the \( \alpha \)-energy of \( \mu \) for certain values of \( \alpha \). We will achieve our goal in two steps: first we consider a naturally chosen family of Borel measures supported on \( S(\mathcal{O}, 1) \), denoted by \( \{\mu_n : n \geq 1\} \), and verify that the family is “nice” in the sense that \( \mu_n \)’s have uniformly bounded first and second moments in their total mass, as well as uniformly bounded expectation of \( \alpha \)-energy for certain \( \alpha > 0 \); next we combine a compactness argument and the Hewitt–Savage 0–1 law to extract, with probability one, a limit measure \( \mu \) from \( \{\mu_n : n \geq 1\} \) and confirm that \( \mu \) is a non-trivial measure supported on \( \Upsilon \) with finite \( \alpha \)-energy for \( \alpha \) in a proper range.

**Proof of Proposition 3.6** We choose and fix \( \epsilon \) to be an arbitrarily small positive number. For every \( n \geq 1 \), we consider the same lattice with cell size \( t_n \) as above. In this proof, since it will not cause any confusion if we drop the superscript “\(^{(n)}\)” in “\( x_j^{(n)} \),” we will write the cell centers as \( \{x_j : j = 1, \ldots, J_n\} \) for simplicity. We define the (random) finite measure \( \mu_n \) on \( S(\mathcal{O}, 1) \) as

\[
\mu_n(B) := \frac{1}{J_n} \sum_{j=1}^{J_n} \mathbb{I}_{\mathcal{W}_n}(x_j) \mathbb{I}_{\mathcal{W}_n}(x_{\Phi_{x_j, n}}) \frac{\text{vol}(B \cap S(x_j, t_n))}{\text{vol}(S(x_j, t_n))},
\]

(3.13)
where “vol” refers to the volume under the Lebesgue measure on $\mathbb{R}^\nu$. It is clear that
\[
\mathbb{E}^W \left[ \mu_n \left( S(O, 1) \right) \right] = 1 \text{ for every } n \geq 1. \tag{3.14}
\]

In addition to the uniformity in its first moment, we want to show that the second moments of $\mu_n \left( S(O, 1) \right)$ are also bounded in $n$, i.e.,
\[
\sup_{n \geq 1} \mathbb{E}^W \left[ \left( \mu_n \left( S(O, 1) \right) \right)^2 \right] < \infty. \tag{3.15}
\]

To this end, we write
\[
\mathbb{E}^W \left[ \left( \mu_n \left( S(O, 1) \right) \right)^2 \right] = \frac{1}{J_n^2} \sum_{j,k=1}^{J_n} \frac{\mathbb{W} \left( \Phi_{x_j,n} \cap \Phi_{x_k,n} \right)}{\mathbb{W} \left( \Phi_{x_j,n} \right) \mathbb{W} \left( \Phi_{x_k,n} \right)}. \tag{3.16}
\]

It is easy to see that, by (3.8), (3.10) and (3.11),
\[
\frac{1}{J_n^2} \sum_{j,k=1}^{J_n} \frac{1}{\mathbb{W} \left( \Phi_{x_j,n} \right) \mathbb{W} \left( \Phi_{x_k,n} \right)} \leq t_n^\nu \cdot \exp \left[ -n \ln p + \nu \Sigma (t_n) + \sqrt{2vn \Sigma (t_n)} \right]
\]
\[
\leq \exp \left[ -n \ln p + \nu (1 - \tilde{c}) 2^{-n^2} + \sqrt{2vcn^2} / \nu \right] \to 0 \text{ as } n \to \infty. \tag{3.17}
\]

This means that the diagonal component of the sum in (3.16) is negligible when $n$ is large. To treat the off-diagonal terms (i.e., $(j, k)$ such that $j \neq k$) in (3.16), we rely on the following inequality.

Claim 3.9 When $n$ is sufficiently large, for every $x_j$ and $x_k$ where $x_j \neq x_k$,
\[
\frac{\mathbb{W} \left( \Phi_{x_j,n} \cap \Phi_{x_k,n} \right)}{\mathbb{W} \left( \Phi_{x_j,n} \right) \mathbb{W} \left( \Phi_{x_k,n} \right)} \leq \exp \left[ \nu \Sigma \left( |x_j - x_k|^{1+\epsilon} \right) + o \left( -\ln |x_j - x_k| \right) \right]. \tag{3.18}
\]

Clearly (3.18) is an uncorrelation inequality for the events $\Phi_{x_j,n}$ and $\Phi_{x_k,n}$. The proof of this inequality is lengthy and involved, and not closely related to the main content, so we will leave it to the Appendix. Assuming (3.18), we have that
\[
\frac{1}{J_n^2} \sum_{j,k=1 \atop j \neq k}^{J_n} \frac{\mathbb{W} \left( \Phi_{x_j,n} \cap \Phi_{x_k,n} \right)}{\mathbb{W} \left( \Phi_{x_j,n} \right) \mathbb{W} \left( \Phi_{x_k,n} \right)}
\]
\[
\leq \frac{1}{J_n^2} \sum_{j,k=1}^{J_n} \exp \left[ \nu \Sigma \left( |x_j - x_k|^{1+\epsilon} \right) + o \left( -\ln |x_j - x_k| \right) \right],
\]

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and as \( n \to \infty \), the right hand side is asymptotically bounded by

\[
\frac{1}{4^n} \int \int_{(S(O,1))^2} \exp \left[ n \Sigma \left( |x - y|^{1+\nu} \right) + o \left( -\ln |x - y| \right) \right] \, dx \, dy.
\]

When \( x \) and \( y \) are sufficiently close,

\[
\exp \left[ n \Sigma \left( |x - y|^{1+\nu} \right) + o \left( -\ln |x - y| \right) \right] \leq |x - y|^{-\frac{1+2\nu}{1-\nu}} \nu^2
\]

which is integrable over \( (S(O,1))^2 \) provided that \( \tilde{c} < 1 \) and \( \epsilon > 0 \) is sufficiently small so that \( \tilde{c} (1 + 2\epsilon) < 1 - \epsilon \). Therefore, we have proven (3.15).

Next, we turn our attention to the \( \alpha \)-energy, \( \alpha > 0 \), of the measure \( \mu_n \), i.e.,

\[
I_\alpha (\mu_n) := \int \int_{(S(O,1))^2} |y - w|^{-\alpha} \, \mu_n (dy) \, \mu_n (dw).
\]

Our goal is to verify that, whenever \( 0 < \alpha < \nu (1 - \tilde{c}) \),

\[
\sup_{n \geq 1} \mathbb{E}^{\mu_n} [I_\alpha (\mu_n)] < \infty.
\]

To this end, we fix \( \alpha \in (0, \nu (1 - \tilde{c})) \) and use (3.13) to write \( \mathbb{E}^{\mu_n} [I_\alpha (\mu_n)] \) as

\[
\frac{1}{J_n^2} \sum_{j,k=1}^{J_n} \mathbb{E} \left( \Phi_{x_j,n} \cap \Phi_{x_k,n} \right) \frac{\int \int_{S(x_j,t_n) \times S(x_k,t_n)} |y - w|^{-\alpha} \, dy \, dw}{\text{vol}^2 (S(x_j,t_n)) \text{vol} (S(x_k,t_n))}.
\]

Obviously, for the diagonal terms in the summation in (3.20), we have that

\[
\frac{\int \int_{S(x_j,t_n)} ^2 |y - w|^{-\alpha} \, dy \, dw}{\text{vol}^2 (S(x_j,t_n))} = Ct_n^{-\alpha} \text{ for some universal constant } C > 0,
\]

so it follows from (3.8) and (3.11) that

\[
\frac{1}{J_n^2} \sum_{j=1}^{J_n} \frac{1}{\mathbb{E} \left( \Phi_{x_j,n} \right)} \frac{\int \int_{S(x_j,t_n)} ^2 |y - w|^{-\alpha} \, dy \, dw}{\text{vol}^2 (S(x_j,t_n))} \leq C \exp \left[ -n \ln p + (\nu (1 - \tilde{c}) - \alpha) 2^{-n^2} + \sqrt{2 \nu \epsilon n^2 / 2} \right] \cdot 0 \text{ as } n \to \infty.
\]

Now we assume that \( x_j \neq x_k \). When \( |x_j - x_k| \leq 4 \sqrt{vt_n} \), by possibly enlarging the constant \( C \) above, we can make

\[
\frac{\int \int_{S(x_j,t_n) \times S(x_k,t_n)} |y - w|^{-\alpha} \, dy \, dw}{\text{vol} (S(x_j,t_n)) \text{vol} (S(x_k,t_n))} \leq Ct_n^{-\alpha} \leq C |x_j - x_k|^{-\alpha};
\]
when $|x_j - x_k| > 4\sqrt{v}t_n$, we notice that for every $y', w' \in S(O, t_n)$,

$$|y' - w'| \leq \frac{1}{2} |x_j - x_k|$$

and hence $|x_j - x_k - (y' - w')| \geq \frac{1}{2} |x_j - x_k|$.

Therefore,

$$\frac{\iint_{S(x_j, t_n) \times S(x_k, t_n)} |y - w|^{-\alpha} dydw}{\text{vol} (S (x_j, t_n)) \text{vol} (S (x_k, t_n))} = \frac{\iint_{S(O, t_n)} |x_j - x_k - (y' - w')|^{-\alpha} dy' dw'}{\text{vol} (S (x_j, t_n)) \text{vol} (S (x_k, t_n))} \leq C |x_j - x_k|^{-\alpha}.$$

By (3.18), we have that

$$\frac{1}{J_n^2} \sum_{j, k=1, j \neq k} J_n \mathcal{W}(\Phi_{x_j, n} \cap \Phi_{x_k, n}) \frac{\iint_{S(x_j, t_n) \times S(x_k, t_n)} |y - w|^{-\alpha} dydw}{\mathcal{W}(\Phi_{x_j, n}) \mathcal{W}(\Phi_{x_k, n}) \text{vol} (S (x_j, t_n)) \text{vol} (S (x_k, t_n))} \leq \frac{C}{J_n^2} \sum_{j, k=1, j \neq k} \exp \left[ v \Sigma \left( |x_j - x_k|^{\frac{1 + \epsilon}{1 - \epsilon}} \right) + o \left( -\ln |x_j - x_k| \right) \right] |x_j - x_k|^{-\alpha}.$$

Similarly as above, we observe that when $|x - y|$ is sufficiently small,

$$\exp \left[ v \Sigma \left( |x - y|^{\frac{1 + \epsilon}{1 - \epsilon}} \right) + o \left( -\ln |x - y| \right) \right] |x - y|^{-\alpha} \leq |x - y|^{-\frac{1 + 2\epsilon}{1 - \epsilon} \nu - \alpha},$$

and the right hand side is integrable over $(S(O, 1))^2$ provided that $0 < \alpha < \nu \left( 1 - \frac{1 + 2\epsilon}{1 - \epsilon} \right)$, which can be made possible by taking $\epsilon$ sufficiently small. Therefore, we have proven (3.19) for every $\alpha \in (0, \nu (1 - \tilde{c}))$.

To proceed, fix any $\alpha \in (0, \nu (1 - \tilde{c}))$. After showing (3.15) and (3.19), we have two positive real numbers

$$A_1 := \sup_{n \geq 1} \mathbb{E}^\mathcal{W} \left[ (\mu_n (S(O, 1)))^2 \right] \text{ and } A_2 := \sup_{n \geq 1} \mathbb{E}^\mathcal{W} [I_\alpha (\mu_n)].$$

For constants $c_1 > 1$, $c_2 > 0$, we define $\Lambda_n^\alpha$ to be the event where

$$c_1^{-1} \leq \mu_n (S(O, 1)) \leq c_1 \text{ and } I_\alpha (\mu_n) \leq c_2.$$  \hspace{1cm} (3.21)

Set $\Lambda^\alpha := \limsup_{n \to \infty} \Lambda_n^\alpha$. Clearly,

$$\sup_{n \geq 1} \mathcal{W}(I_\alpha (\mu_n) > c_2) \leq A_2 c_2^{-1} \text{ and } \sup_{n \geq 1} \mathcal{W}(\mu_n (S(O, 1)) > c_1) \leq c_1^{-1}.$$
Moreover, by (3.14) and the Paley–Zygmund inequality,
\[
\sup_{n \geq 1} \mathcal{W}(\mu_n(S(O, 1)) < c_1^{-1}) \leq 1 - \left(1 - c_1^{-1}\right)^2 \frac{A}{A_1}.
\]
As a consequence, by choosing \(c_1\) and \(c_2\) sufficiently large, we can make
\[
\mathcal{W}(\Lambda_n^\alpha) > \frac{(1 - c_1^{-1})^2}{A_1} - c_1^{-1} - A_2c_2^{-1} > \frac{1}{2A_1}(1 - c_1^{-1})^2 - A_2c_2^{-1},
\]
for every \(n \geq 1\), and hence, \(\mathcal{W}(\Lambda^\alpha) \geq \frac{1}{2A_1}\).

Finally, we are ready to extract a limit measure \(\mu\) from the family \(\{\mu_n : n \geq 0\}\) given \(\Lambda^\alpha\). Assume that \(\Lambda^\alpha\) occurs. Then, (3.21) is satisfied for infinitely many \(n\)’s. Since \(I\), as a mapping from the space of finite measures on \(S(O, 1)\) to \([0, \infty]\), is lower semi-continuous with respect to the weak topology,
\[
\mathcal{M} := \left\{ \mu \text{ Borel measure on } S(O, 1) : c_1^{-1} \leq \mu(S(O, 1)) \leq c_1, \ I_\alpha(\mu) \leq c_2 \right\}
\]
is compact, and hence, there exists a Borel measure \(\mu\) on \(S(O, 1)\) such that \(\mu_n\) weakly converges to \(\mu\) along a subsequence, say, \(\{\mu_{n_k} : k \geq 0\}\). In addition, we know that \(\mu\) also satisfies the inequalities in (3.21). The weak convergence relation between \(\{\mu_{n_k} : k \geq 0\}\) and \(\mu\), combined with the fact that \(\mu_{n_k}\) is supported on \(\bigcup_{x \in \mathcal{L}_{n_k}} S(x, t_{n_k})\) for every \(k \geq 1\), implies that
\[
\mu(\Upsilon) \geq \limsup_{k \to \infty} \mu_{n_k}\left(\bigcup_{x \in \mathcal{L}_{n_k}} S(x, t_{n_k})\right) \geq c_1^{-1}.
\]
This means that \(\Upsilon\) has strictly positive \(\alpha\)-capacity, i.e.,
\[
\sup \left\{ \left(\int_{\Upsilon \times \Upsilon} \frac{\mu \times \mu (dydw)}{|y - w|^\alpha}\right)^{-1} : \mu \text{ is a probability measure on } \Upsilon \right\} > 0.
\]
By Frostman’s lemma, \(\dim_{\mathcal{M}}(\Upsilon) \geq \alpha\) and hence \(\dim_{\mathcal{M}}(D) \geq \alpha\). Therefore, we have established that if \(\Lambda^\alpha\) occurs, then \(\dim_{\mathcal{M}}(D) \geq \alpha\), and hence,
\[
\mathcal{W}(\dim_{\mathcal{M}}(D) \geq \alpha) \geq \mathcal{W}(\Lambda^\alpha) \geq \frac{1}{2A_1}.
\]
At last, we recall from (2.1) that for almost every \(\theta \in \Theta\), \(\theta = \sum_{n \geq 1} \mathcal{I}(h_n)(\theta)h_n\) where \(\{h_n : n \geq 1\}\) is an orthonormal basis of the Cameron–Martin space \(H\) and \(\{\mathcal{I}(h_n) : n \geq 1\}\) under \(\mathcal{W}\) forms a sequence of independent standard Gaussian random variables. By a simple application of the Hewitt–Savage 0–1 law, we have that
\[ W(\dim H(D) \geq \alpha) = 1. \] Since \( \alpha \) is arbitrary in \((0, \nu (1 - \tilde{c}))\) with \( \tilde{c} \) being arbitrarily close to \( \tilde{c}_f \), we get that
\[ W(\dim H(D) \geq \nu (1 - \tilde{c}_f)) = 1. \]

This completes the proof of Proposition 3.6. \(\square\)

**Remark 3.10** We will finish this section by two remarks on the choice of \(\{t_n : n \geq 0\}\) and the condition \(\tilde{c}_f > 0\)" that are involved in the proof of Proposition 3.6 (as well as in Theorem 3.3). First, it is clear from above that the proof does not rely on the specific choice of \(\{t_n : n \geq 0\}\) and can apply it to any sequence that has the proper decay rate. Second, the condition \(\tilde{c}_f > 0\)" is used in deriving (3.12) which would remain true if we had only required \(\Sigma (t_n) / (-\ln t_n)\) to stay bounded from below away from zero for large \(n\)'s. Following these observations, we can obtain the results in Proposition 3.6 under slightly different conditions. Combined with the upper bound we found in Proposition 3.5, we get the following version of Theorem 3.3 that will be used in the discussions of examples in Sect. 4.

**Proposition 3.11** Under the setting of Theorem 3.3, if \(0 \leq \xi f \leq \tilde{c}_f \leq 1\) and there exists a sequence \(\{r_n : n \geq 0\} \subseteq (0, 1]\) with \(\ln r_n \rightarrow 0\) as \(n \rightarrow \infty\) such that
\[ \liminf_{n \rightarrow \infty} \frac{\sum (r_n)}{-\ln r_n} > 0, \quad \lim_{n \rightarrow \infty} \frac{\ln r_{n-1}}{\ln r_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sqrt{-n \ln r_n} = 0, \] then
\[ \nu (1 - \xi f) \geq \dim \mathcal{H} (D_{\limsup}) \geq \dim \mathcal{H} (D_{\liminf}) = \dim \mathcal{H} (D) = \nu (1 - \tilde{c}_f) \ a.s. \]

**4 Examples of Steep Point of Gaussian Free Field**

By varying the choices of \(f \in \mathcal{C}\), the definition (3.1) of \(f\)-steep point leads to different exceptional sets of GFFs, including the classical thick point set for the log-correlated GFF as defined in (1.1), as well as the counterpart for the polynomial-correlated GFF as defined in (1.2) and (1.3). Theorem 3.3 and Proposition 3.11 provide information on the Hausdorff dimension of these sets. Since in this section we will discuss multiple choices of \(f\), we put back the subscript “\(f\)” to the notations of variables and sets (e.g., \(X_f(x, t), \Sigma_f(t), D_f, \text{etc.}\)) to avoid confusion.

**4.1 Log-Correlated GFF**

When \(\nu = 2\), the GFF we have studied in the previous sections is log-correlated. Since the framework developed in Sect. 3 applies well when \(\Sigma_f(t)\) is “comparable” with \((-\ln t)\) as \(t \searrow 0\), a natural choice of \(f\) is a constant function. As we will see in this subsection, modification of this simple case will already lead us to interesting new exceptional sets of the log-correlated GFF.
4.1.1 Thick Point, Revisited

By setting \( f \equiv \sqrt{2\pi\gamma} \) for \( \gamma > 0 \), it is clear that for every \( t \in (0, 1] \) and \( x \in S(O, 1) \), the random integral studied in Sect. 3 is

\[
X^\theta_f (x, t) = \sqrt{2\pi\gamma} \left[ \tilde{\theta} (x, t) - \tilde{\theta} (x, 1) \right] \quad \text{and} \quad \Sigma_f (t) = 2\pi\gamma [G(t) - G(1)].
\]

By (2.7),

\[
c_f = \lim_{t \downarrow 0} \frac{\Sigma_f (t)}{-\ln t} = \gamma,
\]

so \( x \) is an \( f \)-steep point of \( \theta \in \Theta_1 \) if and only if

\[
\lim_{t \downarrow 0} \frac{\tilde{\theta} (x, t)}{-\ln t} = \sqrt{2\gamma / \pi}.
\]

In other words, when \( \gamma > 0 \), according to the definition (1.1), \( D^\theta_f \) coincides with \( T^{\gamma, \theta} \) the set of \( \gamma \)-thick points.

Theorem 3.3 implies that if \( \gamma > 1 \), then

\[
D^\theta_f = D^\theta_{f, \lim \inf} = D^\theta_{f, \lim \sup} = \emptyset \ \text{a.s.}; \quad (4.1)
\]

if \( 0 < \gamma \leq 1 \), then

\[
\dim_H \left( D^\theta_f \right) = \dim_H \left( D^\theta_{f, \lim \inf} \right) = \dim_H \left( D^\theta_{f, \lim \sup} \right) = 2 - 2\gamma \ \text{a.s.}, \quad (4.2)
\]

which agrees with the results on \( T^{\gamma, \theta} \) obtained in [16].

4.1.2 Oscillatory Thick Point

Apart from the standard thick point set, the framework of steep point also allows us to study certain variations of thick point. Again, we fix a constant \( \gamma \in (0, 1] \). (4.2) states that

\[
\dim_H \left( \left\{ x \in S(O, 1) : \limsup_{t \downarrow 0} \frac{\tilde{\theta} (x, t)}{-\ln t} \geq \sqrt{2\gamma / \pi} \right\} \right) = 2 - 2\gamma \ \text{a.s.}
\]

Since \( H \) is invariant under the transformation \( \theta \mapsto -\theta \), we also have that

\[
\dim_H \left( \left\{ x \in S(O, 1) : \liminf_{t \downarrow 0} \frac{\tilde{\theta} (x, t)}{-\ln t} \leq -\sqrt{2\gamma / \pi} \right\} \right) = 2 - 2\gamma \ \text{a.s.}
\]
However, if we require the two conditions above to be met at the same time and set

\[ T_{\gamma, \theta}^{oscil.} := \left\{ x \in S(O, 1) : \limsup_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{-\ln t} \geq \sqrt{2\gamma/\pi} \text{ and } \liminf_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{-\ln t} \leq -\sqrt{2\gamma/\pi} \right\}, \]

then intuitively \( T_{\gamma, \theta}^{oscil.} \) consists of oscillatory thick points where \( \tilde{\theta}(\cdot, t) \) oscillates and achieves an unusually large magnitude in both the positive and the negative directions. One would expect that \( T_{\gamma, \theta}^{oscil.} \) is smaller than either of the two sets mentioned above when only one condition is imposed. But we will show that, at least in terms of the Hausdorff dimension, \( T_{\gamma, \theta}^{oscil.} \) is as big as either of the two sets.

**Proposition 4.1** If \( \gamma \in [0, 1] \), then \( \dim_H \left( T_{\gamma, \theta}^{oscil.} \right) = 2 - 2\gamma \) a.s..

**Proof** Given \( \theta \in \Theta \), obviously \( T_{\gamma, \theta}^{oscil.} \subseteq D_{\theta, \limsup}^{\theta} \) with \( f \equiv \sqrt{2\pi\gamma} \), so

\[ \dim_H \left( T_{\gamma, \theta}^{oscil.} \right) \leq \dim_H \left( D_{\theta, \limsup}^{\theta} \right) \leq 2 - 2\gamma. \]

To show the other direction, we consider a sequence \( \{r_n : n \geq 1\} \subseteq (0, 1) \) such that \( r_n \searrow 0 \) as \( n \to \infty \) and

\[ \lim_{n \to \infty} n \frac{(-\ln r_{n-1})}{-\ln r_n} = 0. \quad (4.3) \]

Set \( r_0 = 1 \), and define the piecewise constant function

\[ f_{oscil.} : t \in (0, 1) \mapsto f_{oscil.}(t) := \sum_{n \geq 1} I_{(r_n, r_{n-1}]}(t) (-1)^n \sqrt{2\pi\gamma}. \]

It is still the case that \( \Sigma f_{oscil.}(t) = 2\pi\gamma [G(t) - G(1)] \), but this time

\[ X_{f_{oscil.}}^\theta(x, t) = \sqrt{2\pi\gamma} \sum_{n \geq 0} (-1)^n \left[ \tilde{\theta}(x, t \lor r_n) - \tilde{\theta}(x, t \lor r_{n-1}) \right]. \]

Again by (4.1), for almost every \( \theta \in \Theta \),

\[ \limsup_{t \searrow 0} \frac{1}{-\ln t} |\tilde{\theta}(x, t)| \leq \sqrt{\gamma/\pi} \text{ for every } x \in S(O, 1). \]

Combining this with (4.3), it is clear that

\[ \lim_{n \to \infty} \frac{1}{G(r_n)} \sum_{j=1}^{n-1} |\tilde{\theta}_{r_j}(x)| = 0 \text{ for every } x \in S(O, 1). \]
Therefore, if \( x \in D_{oscil}^{\theta} \), then
\[
\lim_{k \to \infty} \frac{X_{oscil}^{\theta}(x, r_{2k})}{\Sigma_{oscil}(r_{2k})} = 2 = \lim_{k \to \infty} \frac{X_{oscil}^{\theta}(x, r_{2k-1})}{\Sigma_{oscil}(r_{2k-1})},
\]
which is equivalent to
\[
\lim_{k \to \infty} \frac{\tilde{\theta}(x, r_{2k})}{- \ln r_{2k}} = \sqrt{\frac{2\gamma}{\pi}} = - \lim_{k \to \infty} \frac{\tilde{\theta}(x, r_{2k-1})}{- \ln r_{2k-1}}.
\]
This implies that \( D_{oscil}^{\theta} \subseteq T_{oscil}^{\gamma, \theta} \) a.s., and hence by Theorem 3.3,
\[
\dim \mathcal{H}(T_{oscil}^{\gamma, \theta}) \geq \dim \mathcal{H}(D_{oscil}^{\theta}) = 2 - 2\gamma \text{ a.s.}
\]
\[\square\]

The results above also apply to the case of asymmetric oscillations. Namely, if, for \( \gamma_1, \gamma_2 > 0 \),
\[
T_{oscil}^{(\gamma_1, -\gamma_2), \theta} := \left\{ x \in S(O, 1) : \limsup_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{- \ln t} \geq \sqrt{\frac{2\gamma_1}{\pi}} \text{ and } \liminf_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{- \ln t} \leq -\sqrt{2\gamma_2/\pi} \right\},
\]
then we have the following fact.

**Corollary 4.2** If \( \gamma_1, \gamma_2 \in (0, 1] \), then \( \dim \mathcal{H}(T_{oscil}^{(\gamma_1, -\gamma_2), \theta}) = 2 - 2(\gamma_1 \lor \gamma_2) \) a.s.

**Proof** For almost every \( \theta \in \Theta \), since \( T_{oscil}^{(\gamma_1, -\gamma_2), \theta} \subseteq D_{oscil}^{\theta} \) with \( f = \sqrt{2\pi \gamma_1} \), we know from Theorem 3.3 that \( \dim \mathcal{H}(T_{oscil}^{(\gamma_1, -\gamma_2), \theta}) \leq 2 - 2\gamma_1 \). By the invariance of \( \mathcal{H} \) under \( \theta \mapsto -\theta \), we also have that
\[
\dim \mathcal{H}(T_{oscil}^{(\gamma_1, -\gamma_2), \theta}) = \dim \mathcal{H}(T_{oscil}^{(\gamma_2, -\gamma_1), \theta}) \leq 2 - 2\gamma_2 \text{ a.s.}
\]
In other words,
\[
\dim \mathcal{H}(T_{oscil}^{(\gamma_1, -\gamma_2), \theta}) \leq 2 - 2(\gamma_1 \lor \gamma_2) \text{ a.s.}
\]
On the other hand, if \( \gamma := \gamma_1 \lor \gamma_2 \), then \( T_{oscil}^{\gamma, \theta} \subseteq T_{oscil}^{(\gamma_1, -\gamma_2), \theta} \) and by Proposition 4.1,
\[
\dim \mathcal{H}(T_{oscil}^{(\gamma_1, -\gamma_2), \theta}) \geq \dim \mathcal{H}(T_{oscil}^{\gamma, \theta}) = 2 - 2(\gamma_1 \lor \gamma_2) \text{ a.s.}
\]
\[\square\]
4.2 Polynomial-Correlated GFF

Let \( \nu \geq 3 \) and \( \theta \) be the polynomial-correlated GFF on \( \mathbb{R}^\nu \). In this subsection, we want to revisit the thick point sets considered in [5] and introduce other related exceptional sets, and this will be done again by taking \( f \) to be piece-wise constant.

4.2.1 Thick Point, Revisited

Recall from (1.2) that for \( \gamma \geq 0 \), the \( \gamma \)-thick point set of \( \theta \) is

\[
T^{\gamma, \theta} := \left\{ x \in S(O, 1) : \limsup_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{\sqrt{-G(t) \ln t}} \geq \sqrt{2 \nu \gamma} \right\},
\]

and it is proven in [5] that for almost every \( \theta \), \( T^{\gamma, \theta} = \emptyset \) when \( \gamma > 1 \), and \( \dim (T^{\gamma, \theta}) = \nu (1 - \gamma) \) when \( \gamma \in [0, 1] \). As mentioned in Sect. 1.1.2, due to the higher-order singularity of the covariance function in this case, the proof of the lower bound of \( \dim (T^{\gamma, \theta}) \) in [5] is considerably more technical and involved than that in the log-correlated case; in fact, the lower bound on \( \dim (T^{\gamma, \theta}) \) was established indirectly through the sequential \( \gamma \)-thick point sets, which, we recall from (1.3), is

\[
ST^{\gamma, \theta} := \left\{ x \in S(O, 1) : \lim_{n \to \infty} \frac{\tilde{\theta}(x, r_n)}{\sqrt{-G(r_n) \ln r_n}} = \sqrt{2 \nu \gamma} \right\},
\]

where \( \{r_n : n \geq 1\} \subseteq (0, 1] \) is a sequence satisfying (4.3). Since \( ST^{\gamma, \theta} \subseteq T^{\gamma, \theta} \), a lower bound of \( \dim (ST^{\gamma, \theta}) \) leads to a lower bound of \( \dim (T^{\gamma, \theta}) \).

Below we will reproduce the lower bound on \( \dim (ST^{\gamma, \theta}) \) under the framework of steep point with a much shorter and easier proof than the one given in [5]. There is nothing to be done when \( \gamma = 0 \), so we will focus on the case when \( \gamma > 0 \). It is shown in [5], as part of the upper bound result, that with probability one,

\[
\liminf_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{\sqrt{-G(t) \ln t}} \geq -\sqrt{2 \nu} \quad \text{and} \quad \limsup_{t \searrow 0} \frac{\tilde{\theta}(x, t)}{\sqrt{-G(t) \ln t}} \leq \sqrt{2 \nu} \quad \text{for every} \quad x \in S(O, 1).
\]

Let \( \{r_n : n \geq 1\} \subseteq (0, 1] \) be any sequence satisfying (4.3). Since we can always insert more terms into \( \{r_n : n \geq 1\} \) which will make \( ST^{\gamma, \theta} \) even smaller, without loss of generality we will assume that

\[
\lim_{n \to \infty} \frac{\sqrt{-n \ln r_n}}{\ln r_n} = 0,
\]

(4.5)
i.e., \( r_n \) does not decay to zero too fast as \( n \to \infty \). Given a constant \( \gamma \in (0, 1] \), we set \( r_0 = 1 \) and define the function

\[
g : t \in (0, 1] \mapsto g(t) := \sum_{n \geq 1} \sqrt{-\gamma \ln r_n} \frac{G(r_n)}{G(r_{n-1})} (t) . \tag{4.6}
\]

For every \( n \geq 1 \) and every \( t \in (r_n, r_{n-1}] \), it is clear that \( |g(t)| \sqrt{G(t)} \leq \sqrt{-\gamma \ln r_n} \) and

\[
\int_1^t \sqrt{G(s)} dV_g(s) = \sum_{k=1}^{n-1} \sqrt{-\gamma \ln r_k} \left[ \sqrt{G(r_k)} - \sqrt{G(r_{k-1})} \right] 
\leq \sqrt{-\gamma (n - 1) \ln r_{n-1}} \leq C (-\ln t) ,
\]

where the last inequality follows from (4.3), and \( C > 0 \) is a constant that can be chosen uniformly in \( n \) and \( t \). In addition, when \( n \) is large and \( t \in (r_n, r_{n-1}] \),

\[
\sum_{j=1}^n \sqrt{-\gamma \ln r_j} \frac{G(r_j)}{G(r_{j-1})} \left[ \bar{\theta} (x, r_j) - \bar{\theta} (x, r_{j-1}) \right] . \tag{4.7}
\]

By (4.4), we know that

\[
\sup_{x \in S(O, 1)} \left| \bar{\theta} (x, r_n) \right| \leq \left( \sqrt{2\nu + 1} \right) \sqrt{-G(r_n) \ln r_n} \text{ for all but finitely many } n\text{'s}. 
\]

Combining this with (4.3) and (4.7) leads to

\[
\lim_{n \to \infty} \frac{1}{\Sigma_g(r_n)} \left[ \sqrt{-\ln r_n} \frac{G(r_n)}{G(r_{n-1})} \left| \bar{\theta} (x, r_n) \right| + \sum_{j=1}^{n-1} \sqrt{-\ln r_j} \frac{G(r_j)}{G(r_{j-1})} \left| \bar{\theta} (x, r_j) - \bar{\theta} (x, r_{j-1}) \right| \right] = 0 . \tag{4.8}
\]
Thus, if \( x \in D_\theta^g \), then
\[
\sqrt{2 \nu} = \lim_{n \to \infty} \frac{X_\theta^g (x, r_n)}{\Sigma_g (r_n)} = \lim_{n \to \infty} \frac{\sqrt{-\gamma \ln r_n \tilde{\theta} (x, r_n)}}{\sqrt{G (r_n) \Sigma_g (r_n)}},
\]
which implies that
\[
\lim_{n \to \infty} \frac{\tilde{\theta} (x, r_n)}{\sqrt{G (r_n) \ln r_n}} = \sqrt{2 \nu \gamma}
\]
Therefore, we can conclude that \( D_\theta^g \subseteq ST^{\gamma, \theta} \) a.s., and by Proposition 3.11,
\[
\dim H \left( T^{\gamma, \theta} \right) \geq \dim H \left( ST^{\gamma, \theta} \right) \geq \dim H \left( D_\theta^g \right) = \nu (1 - \gamma) \text{ a.s.}
\]

**Remark 4.3** Here, we only re-prove the lower bound of \( \dim H \left( T^{\gamma, \theta} \right) \) because our new proof via the framework of steep point is a considerable improvement compared with the proof given in [5]. We will not revisit the upper bound of \( \dim H \left( T^{\gamma, \theta} \right) \), since our method in establishing the upper bound, as presented in Sect. 3.1, does not differ substantially from that used to obtain the upper bound of \( \dim H \left( T^{\gamma, \theta} \right) \) in [5].

### 4.2.2 Oscillatory Thick Point

Similarly as in the log-correlated case, we can consider the oscillatory thick point for the polynomial-correlated GFF. Namely, for \( \gamma_1, \gamma_2 > 0 \), we define
\[
T^{(\gamma_1, -\gamma_2), \theta}_{\text{oscil.}} := \left\{ x \in S (O, 1) : \limsup_{t \searrow 0} \frac{\tilde{\theta} (x, t)}{\sqrt{-G (t) \ln t}} \geq \sqrt{2 \nu \gamma_1} \text{ and } \liminf_{t \searrow 0} \frac{\tilde{\theta} (x, t)}{\sqrt{-G (t) \ln t}} \leq -\sqrt{2 \nu \gamma_2} \right\}.
\]

**Proposition 4.4** If \( \gamma_1, \gamma_2 \in (0, 1] \), then \( \dim H \left( T^{(\gamma_1, -\gamma_2), \theta}_{\text{oscil.}} \right) = \nu - \nu (\gamma_1 \vee \gamma_2) \text{ a.s.} \)

**Proof** Set \( \gamma := \gamma_1 \vee \gamma_2 \). Again, it follows directly from the upper bound result in [5] and the invariance of \( H \) under \( \theta \mapsto -\theta \) that
\[
\dim H \left( T^{(\gamma_1, -\gamma_2), \theta}_{\text{oscil.}} \right) \leq \min \left\{ \dim H \left( T^{\gamma_1, \theta} \right), \dim H \left( T^{\gamma_2, \theta} \right) \right\} \leq \nu (1 - \gamma) \text{ a.s.}
\]
Choose the same \( \{ r_n : n \geq 0 \} \) as in Sect. 4.2.1 and define the following function
\[
g_{\text{oscil.}} : t \in (0, 1] \mapsto g_{\text{oscil.}} (t) := \sum_{n=1}^{\infty} (-1)^n \sqrt{-\gamma \ln r_n \frac{\ln r_n}{G (r_n)}} I_{(r_n, r_n-1]} (t).
\]
Following exactly the same arguments as above, one can show that for every \( \theta \), if \( x \in D_{g_{\text{oscil}}}^{\theta} \), then

\[
\lim_{k \to \infty} \frac{\tilde{\theta} (x, r_{2k})}{\sqrt{-G (r_{2k}) \ln r_{2k}}} = \sqrt{2\nu \gamma} = \lim_{k \to \infty} \frac{\tilde{\theta} (x, r_{2k-1})}{\sqrt{-G (r_{2k-1}) \ln r_{2k-1}}},
\]

which means that \( D_{g_{\text{oscil}}}^{\theta} \subseteq T_{\text{oscil}}^{(\gamma_1, -\gamma_2), \theta} \). Therefore,

\[
\dim_{\mathcal{H}} \left( T_{\text{oscil}}^{(\gamma_1, -\gamma_2), \theta} \right) \geq \dim \left( D_{g_{\text{oscil}}}^{\theta} \right) = \nu (1 - \gamma) \text{ a.s.}
\]

4.2.3 Lasting Thick Point

The choice of function \( g \) in Sect. 4.2.1 also leads to another exceptional set. For \( \gamma > 0 \), we call \( x \in S(O, 1) \) a lasting \( \gamma \)-thick points of \( \theta \in \Theta \), i.e., if

\[
\limsup_{t \downarrow 0} \frac{1}{G (t)} \int_{t}^{\infty} \left( \frac{\tilde{\theta} (x, s)}{\sqrt{-G (s) \ln s}} \right) dG (s) > 0.
\]

Denote by \( LT_{\gamma, \theta} \) the collection of the lasting \( \gamma \)-thick points of \( \theta \). Heuristically speaking, a lasting \( \gamma \)-thick point is a location where not only \( \tilde{\theta} (x, t) \) exceeds the unusually large value \( \sqrt{-2\nu \gamma G (t) \ln t} \) for arbitrarily small \( t \), it also remains above this level for a non-negligible fraction of the total duration of the process. Since it is obvious that \( LT_{\gamma, \theta} \subseteq T_{\gamma, \theta} \), when \( \gamma > 1 \), \( LT_{\gamma, \theta} = \emptyset \) a.s., and when \( \gamma \in (0, 1] \),

\[
\dim_{\mathcal{H}} \left( LT_{\gamma, \theta} \right) \leq \dim_{\mathcal{H}} \left( T_{\gamma, \theta} \right) \leq \nu (1 - \gamma) \text{ a.s.}
\]

Below we will determine \( \dim_{\mathcal{H}} \left( LT_{\gamma, \theta} \right) \) by drawing the connection between \( LT_{\gamma, \theta} \) and the steep point set \( D_{g}^{\theta} \) considered in Sect. 4.2.1.

Proposition 4.5 If \( \gamma \in (0, 1] \), then \( \dim_{\mathcal{H}} \left( LT_{\gamma, \theta} \right) = \nu (1 - \gamma) \text{ a.s.} \)

Proof Only the lower bound requires proof. Taking the same sequence \( \{r_{n} : n \geq 0\} \) as in Sect. 4.2.1, we define \( g \) as in (4.6) with \( \gamma \) replaced by \( \gamma' \), where \( \gamma' > \gamma \) and \( \gamma' \) is arbitrarily close to \( \gamma \). Again, when \( n \) is sufficiently large,

\[
\Sigma_{g} (r_{n}) = \left( \gamma' + o (1) \right) (-\ln r_{n}),
\]
and for every $t \in (r_n, r_{n-1}]$, $x \in S(O, 1)$ and $\theta \in \Theta$, 

$$\Sigma_g(t) = -\gamma' \ln r_n \cdot \frac{G(t)}{G(r_n)} - \gamma' \ln r_{n-1} + o\left(-\ln t\right), \quad X^\theta_g(x, t)$$

$$= \sqrt{-\gamma' \ln r_n} \left[\bar{\theta}(x, t) - \bar{\theta}(x, r_{n-1})\right] + \sum_{j=1}^{n-1} \sqrt{-\gamma' \ln r_j} \left[\bar{\theta}_j(x, r_j) - \bar{\theta}_{j-1}(x, r_{j-1})\right].$$

We set $\gamma'' := \frac{1}{2} \left(\gamma + \gamma'\right)$ and $A := (\gamma'/\gamma'')^{\frac{1}{\gamma'}}$. It is clear from above that for every $t \in (r_n, Ar_n]$,

$$\Sigma_g(t) \geq \Sigma_g(Ar_n) \geq \left(\gamma' A^{\gamma'} + o(1)\right) (-\ln r_n) = \left(\gamma'' + o(1)\right) (-\ln r_n).$$

Following the same arguments as earlier, we have that when $n$ is sufficiently large,

$$-\sqrt{-\gamma' \ln r_n} \bar{\theta}(x, r_{n-1}) + \sum_{j=1}^{n-1} \sqrt{-\gamma' \ln r_j} \left[\bar{\theta}(x, r_j) - \bar{\theta}(x, r_{j-1})\right]$$

$$= o\left(-\ln r_n\right) = o\left(\Sigma_g(t)\right).$$

Therefore, if $x \in D^\theta_g$, then for sufficiently large $n$ and $t \in (r_n, Ar_n]$,

$$\sqrt{2\nu} + o(1) = \frac{X^\theta_g(x, t)}{\Sigma_g(t)} = \frac{-\gamma' \ln r_n \bar{\theta}(x, t)}{G(r_n) \Sigma_g(t)} + o(1),$$

which implies that

$$\frac{\bar{\theta}(x, t)}{\sqrt{-G(t) \ln t}} \geq \sqrt{2\nu} \frac{\gamma''}{\gamma'} + o(1) > \sqrt{2\nu} \gamma'.$$

This is to say that

$$\limsup_{n \to \infty} \int_{(\sqrt{2\nu} \gamma', \infty)} \left(\frac{\bar{\theta}(x, s)}{\sqrt{-G(s) \ln s}}\right) dG(s) \geq \lim_{n \to \infty} \frac{G(r_n) - G(Ar_n)}{G(r_n)} = 1 - \frac{\gamma''}{\gamma'} > 0.$$

We can conclude that $D^\theta_g \subseteq LT^{\gamma', \theta}$ a.s. and hence, by Proposition 3.11,

$$\dim_{\mathcal{H}}(LT^{\gamma', \theta}) \geq \dim_{\mathcal{H}}(D^\theta_g) \geq \nu \left(1 - \gamma'\right) \text{ a.s.}$$
Finally, since $\gamma' > \gamma$ is arbitrarily close to $\gamma$, we have that $\dim_{\mathcal{H}} \left( LT^{\gamma', \theta} \right) \geq \nu (1 - \gamma)$ a.s.. \hfill $\square$

5 Generalizations and Further Questions

At the end of the article, we briefly mention two related problems and directions in which we would like to further our study.

5.1 Dependence or Independence on the Choice of $f$

As we have mentioned in Sect. 1, to overcome the singularity of GFFs in general, various regularization procedures have been introduced and adopted in the study of GFFs. Although different regularization procedures may work equally well in the study of certain properties of GFFs, it is unclear, in most cases, whether a result obtained is dependent on the specific regularization, or it is intrinsic about the GFF itself and independent of the choice of regularization. For example, it remains open, in the general setting, whether two thick point sets obtained through two different regularizations have any connection, as well as whether there is an intrinsic way to define thick points without the use of any regularization.

In our project it is clear that, if $f$ are $g$ are two different functions in $C$ with $c_f = c_g := c \in (0, 1]$, then

$$\dim_{\mathcal{H}} \left( D^\theta_f \right) = \dim_{\mathcal{H}} \left( D^\theta_g \right) = \nu (1 - c)$$ a.s..

Thus, when the two choices of test functions have the same key parameter, at least the Hausdorff dimensions of the corresponding steep point sets are identical. We are interested in further studying the relation between $D^\theta_f$ and $D^\theta_g$. In particular, we hope to use the framework developed in this article to determine the conditions on $f$ and $g$ under which the difference set between $D^\theta_f$ and $D^\theta_g$ is small, as well as to design examples of $f$ and $g$ such that the difference set between $D^\theta_f$ and $D^\theta_g$ is big.

5.2 Liouville quantum gravity measure in $\mathbb{R}^\nu$ for $\nu \geq 3$

In Sect. 1, we briefly reviewed the Liouville quantum gravity (LQG) measure on a planar domain, which is a random measure that formally takes the form of “$e^{\theta(x)} dx$” where $\theta$ is a generic element of the 2D log-correlated GFF and $dx$ is the Lebesgue measure on the domain. Since the formal density with respect to the Lebesgue measure is always positive, the LQG measure can be viewed as the induced measure of the Lebesgue measure under a random conformal transformation, providing a model of 2D random geometry. The fact that the covariance function of the GFF has a singularity that is no worse than logarithmic plays an essential role in the mathematical construction of the LQG measure. Therefore, the straightforward analog of the LQG measure in $\mathbb{R}^\nu$ for $\nu \geq 3$ is not accessible in the same way.
On the other hand, if one is interested in modeling random geometry in \( \mathbb{R}^v \) for \( v \geq 3 \) using the GFF, then a possible approach is to construct the analog of the LQG measure with \( X_f^\theta(x, t) : x \in \mathbb{R}^v, t \in (0, 1] \) for some \( f \in \mathcal{C} \). Upon choosing \( f \) properly, the family of \( X_f^\theta(x, t) \) has the desired logarithmic singularity. In addition, since the LQG measure has a thick point set as its support, one can expect that an analogous random measure will be supported on the corresponding \( f \)-steep point set. We believe it is also possible to extend further results on the LQG measure to the proposed random measure, such as the Knizhnik–Polyakov–Zamolodchikov formula which gives the correspondence between the scaling dimension of a set under the random measure and the counterpart under the Lebesgue measure.

Appendix

The Appendix is dedicated to proving Claim 3.9, i.e., the uncorrelation inequality (3.18) between \( \Phi_{i,j,n} \) and \( \Phi_{i,k,n} \), which concern random integrals \( X(x_j, t_n) \) and \( X(x_k, t_n) \). Let \( n \) be large, and take \( x_j \) and \( x_k, j, k = 1, \ldots, J_n \), where \( x_j \neq x_k \). For convenience, we write \( \delta := |x_j - x_k| \). Assume \( i \in \{0, 1, \ldots, n - 1\} \) is the unique integer such that \( 2t_{i+1} \leq \delta < 2t_i \) (without loss of generality, we assume that \( i \) is large). Let \( \epsilon \) be an arbitrarily small positive constant.

For \( \delta \) concerns the annulus \( R(x_j, t \to s) \), and similarly, for \( 0 < t' < s' \leq 1 \), \( X(x_k, t') - X(x_k, s') \) concerns \( R(x_k, t' \to s') \). Due to the properties of the covariance function of the family \( \{X(x, t) : (x, t) \in S(O, 1) \times (0, 1]\} \), particularly (2.8) and (2.9), we know that \( X(x_j, t) - X(x_j, s) \) and \( X(x_k, t') - X(x_k, s') \) are independent if the two corresponding annuli do not intersect, which will happen if either \( B(x_j, s) \cap B(x_k, s') = \emptyset \) or \( B(x_j, s) \subseteq B(x_k, t') \) (or \( B(x_k, s') \subseteq B(x_j, t) \)). This implies that when \( i \) is large,

\[
\Delta X_l(x_j) \text{ with } l \in \{i + 2, \ldots, n\}, \quad \Delta X_{l'}(x_k) \text{ with } l' \in \{1, \ldots, i - 1, i + 2, \ldots, n\}
\]

(5.1)

are mutually independent.

To further “extract” independent relationships from the rest of the random variables, we need to look at annuli that intersect and carry out a more careful analysis. Our strategy is to divide the intersected annuli into refined sub-annuli. First, depending on the range of \( \delta \), we will disregard certain sections of the annuli so that the remaining part of the annuli become non-intersecting, which leads to independence of the corresponding integrals. Second, we will verify that removing those sections will only cause a negligible discrepancy in computing the probability of the concerned event, thanks to (a) in Definition 3.1.

We will treat the following two cases separately according to the range of \( \delta \).
Case 1. Suppose that $t_{i+1}^{1-\epsilon} \leq \delta < 2t_i$. In this case,

$$\delta \gg \delta^{1/\epsilon/2} > t_{i+1}^{1-\epsilon/2} \gg t_{i+1}.$$  

$\Delta X_{i+1}(x_j)$ corresponds to the annulus $R\left(x_j, t_{i+1} \rightarrow t_i\right)$, from which we remove the outer section $R\left(x_j, \delta^{1-\epsilon/2} \rightarrow t_i\right)$, and only consider the inner annulus $R\left(x_j, t_{i+1} \rightarrow \delta^{1-\epsilon/2}\right)$ and set the corresponding integral to be

$$Z := X\left(x_j, t_{i+1}\right) - X\left(x_j, \delta^{1/\epsilon/2}\right).$$

As for $\Delta X_l(x_k)$ and $\Delta X_{i+1}(x_k)$, we remove from the corresponding annulus $R\left(x_k, t_{i+1} \rightarrow t_{i-1}\right)$ the middle section $R\left(x_j, \delta - \delta^{1-\epsilon/2} \rightarrow \delta + \delta^{1-\epsilon/2}\right)$, and set the integral over the remaining two sections as

$$W_1 := X\left(x_k, t_{i+1}\right) - X\left(x_k, \delta - \delta^{1/\epsilon/2}\right), \quad W_2 := X\left(x_k, \delta + \delta^{1/\epsilon/2}\right) - X\left(x_k, t_{i-1}\right).$$

$Z$, $W_1$ and $W_2$, as well as all the relevant annuli are indicated in Fig. 1. Now we have that

$$Z, W_1, W_2, \Delta X_l(x_j) \text{ with } l \in \{i + 2, \ldots, n\}, \Delta X_{l'}(x_k) \text{ with } l' \in \{1, \ldots, i - 1, i + 2, \ldots, n\}$$

are all mutually independent centered Gaussian variables.

Next, we turn our attention to the probabilities of the events involving $Z$, $W_1$ and $W_2$. We know that

$$\text{Var}(Z) = \Sigma(t_{i+1}) - \Sigma\left(\delta^{1/\epsilon/2}\right)$$

and

$$\text{Var}(W_1 + W_2) = \Sigma(t_{i+1}) - \Sigma\left(\delta - \delta^{1/\epsilon/2}\right) + \Sigma\left(\delta + \delta^{1/\epsilon/2}\right) - \Sigma(t_{i-1}).$$

The event $P_{x_j, i+1}$ implies the event, denoted by $E_1$, that

$$\left|Z - \sqrt{2\nu} \text{Var}(Z)\right| \leq 2\sqrt{\Delta \Sigma_{i+1}},$$

and $P_{x_k, i+1} \cap P_{x_k, i}$ implying $E_2$, which is the event that

$$\left|W_1 + W_2 - \sqrt{2\nu} \text{Var}(W_1 + W_2)\right| \leq 4\sqrt{\Delta \Sigma_{i+1}}.$$
By basic computations, we get that

\[ W(E_1) \leq \exp\left[ -\nu \text{Var}(Z) + C \sqrt{\Delta \Sigma_{i+1}} \right] \text{ and } W(E_2) \leq \exp\left[ -\nu \text{Var}(W_1 + W_2) + C \sqrt{\Delta \Sigma_{i+1}} \right]. \]

Combining the estimates above with (3.10) and (3.11), we have that

\[
\begin{align*}
\mathcal{W}(\Phi_{x_j,n} \cap \Phi_{x_k,n}) &\leq \frac{\mathcal{W}(E_1 \cap (\bigcap_{l=i+2}^n P_{x_j,l}) \cap (\bigcap_{l=1}^{i-1} P_{x_j,l}) \cap E_2 \cap (\bigcap_{l=i+2}^n P_{x_j,l}))}{\mathcal{W}(\Phi_{x_j,n}) \mathcal{W}(\Phi_{x_k,n})} \\
&\leq \frac{\mathcal{W}(E_1) \mathcal{W}(E_2)}{\mathcal{W}(\Phi_{x_j,i+1}) \mathcal{W}(P_{x_k,l+1})} \\
&\leq \exp\left[ \nu \Sigma \left( \delta - \delta\frac{1}{t_{i+1}} \right) - \nu \Sigma \left( \delta + \delta\frac{1}{t_{i+1}} \right) + C \sqrt{i \Sigma (t_{i+1})} \right].
\end{align*}
\]
Assuming $i$ is large, the assumption (a) in Definition 3.1 implies that

$$
\Sigma \left( \delta - \delta^{1-\epsilon/2} \right) - \Sigma \left( \delta + \delta^{1-\epsilon/2} \right) = \int_{\delta-\delta^{1-\epsilon/2}}^{\delta+\delta^{1-\epsilon/2}} f^2 (s) \, dG (s)
$$

\begin{equation}
\leq C \int_{\delta-\delta^{1-\epsilon/2}}^{\delta+\delta^{1-\epsilon/2}} \frac{(- \ln s)^2 \rho_f^2}{G (s)} \, dG (s) \leq C \delta^{\frac{\epsilon}{2-\epsilon}}
\end{equation}

which is negligible; while (3.8) and the range of $\delta$ guarantee that

$$
\int_{\delta-\delta^{1-\epsilon/2}}^{\delta+\delta^{1-\epsilon/2}} (- \ln s)^2 \rho_f^2 \, dG (s) \leq C \delta^{\epsilon/2} = o \left( -\ln \delta \right).
$$

Therefore, in Case 1, when $i$ is sufficiently large,

$$
\mathcal{W} \left( \Phi_{x_j, n} \cap \Phi_{x_k, n} \right) \leq \exp \left[ \nu \Sigma \left( \delta^{1-\epsilon/2} \right) + o \left( -\ln \delta \right) \right].
$$

(5.3)

Case 2. Suppose that $2 t_{i+1} \leq \delta < t_{i+1}^{1-\epsilon}$. In this case, $t_i \gg \delta + t_{i+1}$ and $\Delta X_i (x_k)$ is independent of the family (5.1). We will take the annulus $R (x_j, t_{i+2} \to t_{i+1})$ and remove from it the outer section $R \left( x_j, t_{i+1}^{1+\epsilon} \to t_{i+1} \right)$, and set the integral over the remaining inner section $R \left( x_j, t_{i+2} \to t_{i+1}^{1+\epsilon} \right)$ to be

$$
\mathcal{Z}' = X (x_j, t_{i+2}) - X \left( x_j, t_{i+1}^{1+\epsilon} \right).
$$

We also remove from $R (x_k, t_{i+1} \to t_i)$ the middle section $R \left( x_k, \delta - t_{i+1}^{1+\epsilon} \to \delta + t_{i+1}^{1+\epsilon} \right)$, and set the integral over the remaining two sections as

$$
\mathcal{W}' = X (x_k, t_{i+1}) - X \left( x_k, \delta - t_{i+1}^{1+\epsilon} \right) + X \left( x_k, \delta - t_{i+1}^{1+\epsilon} \right) - X (x_k, t_i).
$$

See Fig. 2 for an illustration of the division of the annuli. By doing the above, we have that

$$
\mathcal{Z}' \quad \text{and} \quad \Delta X_i (x_j) \quad \text{with} \quad l \in \{ i + 3, \ldots, n \}, \quad \Delta X_{i'} (x_k) \quad \text{with} \quad l' \in \{ 1, \ldots, i, i + 2, \ldots, n \}
$$

are mutually independent. Furthermore,

$$
\text{Var} (\mathcal{Z}') = \Sigma (t_{i+2}) - \Sigma \left( t_{i+1}^{1+\epsilon} \right)
$$
and

$$\Var(W') = \sum (t_{i+1}) - \sum \left( \delta - t_{i+1}^{1+\epsilon} \right) + \sum \left( \delta + t_{i+1}^{1+\epsilon} \right) - \sum (t_i).$$

Similarly as in the previous case, we have that

$$P_{x,j,i+2} \subseteq E'_1 := \left\{ \left| Z' - \sqrt{2\nu \Var(Z')} \right| \leq 2\sqrt{\Delta \Sigma_{i+2}} \right\}, \quad \mathcal{W}(E'_1) \leq \exp \left( -\nu \Var(Z') + C\sqrt{\Delta \Sigma_{i+2}} \right).$$

and

$$P_{x,k,i+1} \subseteq E'_2 := \left\{ \left| W' - \sqrt{2\nu \Var(W')} \right| \leq 3\sqrt{\Delta \Sigma_{i+1}} \right\}, \quad \mathcal{W}(E'_2) \leq \exp \left( -\nu \Var(W') + C\sqrt{\Delta \Sigma_{i+1}} \right).$$

Following the same arguments as in (5.2), we get that

$$\sum \left( \delta - t_{i+1}^{1+\epsilon} \right) - \sum \left( \delta + t_{i+1}^{1+\epsilon} \right) = \int_{\delta+t_{i+1}^{1+\epsilon}}^{\delta-t_{i+1}^{1+\epsilon}} f^2(s) \, dG(s) \leq C \int_{\delta+t_{i+1}^{1+\epsilon}}^{\delta-t_{i+1}^{1+\epsilon}} \left( -\ln s \right)^{2\nu / \rho} \frac{dG(s)}{G(s)} \leq Ct_{i+1}^{\epsilon/4}.$$
Combining the arguments above with (3.10) and (3.11) leads to

\[
\frac{\mathcal{W}(\Phi_{x_j,n} \cap \Phi_{x_k,n})}{\mathcal{W}(\Phi_{x_j,n}) \mathcal{W}(\Phi_{x_k,n})} \leq \frac{\mathcal{W}(E_1' \cap (\bigcap_{l'=i+3} P_{x_j,l'}) \cap E_2' \cap (\bigcap_{l'=i+2} P_{x_k,l'}))}{\mathcal{W}(\Phi_{x_j,n}) \mathcal{W}(\Phi_{x_k,n})} \leq \frac{\mathcal{W}(E_1') \mathcal{W}(E_2')}{\mathcal{W}(\Phi_{x_j,i+2}) \mathcal{W}(P_{x_k,i+1})} \leq \exp \left[ \nu \Sigma \left( \frac{1}{i+1} \right) + C \sqrt{t \Sigma (t_{i+2})} \right].
\]

When \( i \) is large, again by (3.8) and the range of \( \delta \), we have that \( \Sigma \left( \frac{1}{i+1} \right) \leq \Sigma \left( \frac{1+\epsilon}{i+1} \right) \) and \( \sqrt{t \Sigma (t_{i+2})} = o (- \ln \delta) \). In other words, in Case 2 we have that

\[
\frac{\mathcal{W}(\Phi_{x_j,n} \cap \Phi_{x_k,n})}{\mathcal{W}(\Phi_{x_j,n}) \mathcal{W}(\Phi_{x_k,n})} \leq \exp \left[ \nu \Sigma \left( \frac{1+\epsilon}{i+1} \right) + o (- \ln \delta) \right].
\]

Combining (5.3) and (5.4) together, since \( \frac{1+\epsilon}{1-\epsilon} > \frac{1}{1-\epsilon/2} \), we have arrived at (3.18).

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