$\mu - \tau$ symmetry breaking and CP violation in the neutrino mass matrix

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Abstract

The $\mu - \tau$ exchange symmetry in the neutrino mass matrix and its breaking as a perturbation are discussed. The exact $\mu - \tau$ symmetry restricts the 2-3 and 1-3 neutrino mixing angles as $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ at a zeroth order level. We claim that the $\mu - \tau$ symmetry breaking prefers a large CP violation to realize the observed value of $\theta_{13}$ and to keep $\theta_{23}$ nearly maximal, though an artificial choice of the $\mu - \tau$ breaking can tune $\theta_{23}$, irrespective of the CP phase. We exhibit several relations among the deviation of $\theta_{23}$ from $\pi/4$, $\theta_{13}$ and Dirac CP phase $\delta$, which are useful to test the $\mu - \tau$ breaking models in the near future experiments. We also propose a concrete model to break the $\mu - \tau$ exchange symmetry spontaneously and its breaking is mediated by the gauge interactions radiatively in the framework of the extended gauge model with $B - L$ and $L_{\mu} - L_{\tau}$ symmetries. As a result of the gauge mediated $\mu - \tau$ breaking in the neutrino mass matrix, the artificial choice is unlikely, and a large Dirac CP phase is preferable.
1 Introduction

The long baseline neutrino oscillation experiments are ongoing [1, 2], and it is expected that the 2-3 neutrino mixing and a CP phase will be measured more accurately [3, 4]. The 2-3 mixing angle $\theta_{23}$ for the atmospheric neutrino oscillations are nearly maximal $\sim 45^\circ$, and it has been questioned whether the angle is really $45^\circ$ or the angle deviates from it to the higher or lower octant. The current central value of $\theta_{23}$ for the global analysis [5, 6] is in the higher octant. The measurement of the Dirac CP phase $\delta$ in the neutrino oscillations is important since it may tell us something about the lepton number generation in the early universe. The current measurements imply a large CP violation, $\delta \sim -90^\circ$. The accurate measurements of them will be one of the most important issues in the next decade.

The 1-3 neutrino mixing angle $\theta_{13}$ has been measured accurately by reactor neutrino oscillations [7], and the angle $\theta_{13} \simeq 8^\circ - 9^\circ$ is much smaller than the other two, $\theta_{12} \sim 34^\circ$, $\theta_{23} \sim 45^\circ$. When the (nearly) maximal atmospheric neutrino mixing is revealed at Super-Kamiokande, the 1-3 neutrino mixing has been bounded from above by reactor neutrino at CHOOZ [8]. The $\mu$-$\tau$ exchange symmetry has been considered to realize such pattern of the neutrino mixings [9, 10]. Under the $\mu$-$\tau$ exchange symmetry, i.e., if the neutrino mass matrix has a symmetry under the $\nu_\mu$-$\nu_\tau$ exchange, the 2-3 mixing is maximal and the 1-3 mixing is zero (the 1-2 mixing is free). Surely, the observed 1-3 mixing is not zero, and the $\mu$-$\tau$ symmetry is an approximate symmetry. We claim that the separation of $\mu$-$\tau$ symmetric and $\mu$-$\tau$ breaking pieces is a good parametrization of the neutrino mass matrix to describe the deviation from the maximal angle of the 2-3 mixing ($\delta \theta_{23}$), 1-3 mixing $\theta_{13}$, and the CP phase $\delta$ (even if there is not an underlying $\mu$-$\tau$ symmetry in the Lagrangian).

If there is $\mu$-$\tau$ exchange symmetry and the symmetry is spontaneously broken in the neutrino sector, the deviation $\delta \theta_{23}$ is the similar size of $\theta_{13}$ very naively. In this sense, the observed 1-3 mixing angle is too large to explain the nearly maximal angle $\theta_{23}$, which may be the reason why people have fewer interests on the $\mu$-$\tau$ symmetry now. However, the small deviation $\delta \theta_{23}$ with the relatively large size of $\theta_{13}$ suggests that the CP violation in the neutrino sector is large. Due to this consciousness, we will work on the description of the $\mu$-$\tau$ exchange symmetry and its breaking in this paper.

We first describe the neutrino mixings using the parametrization to separate the $\mu$-$\tau$ symmetric and breaking pieces of the neutrino mass matrix. If the $\mu$-$\tau$ breaking parameters are free in general, any values of $\delta \theta_{23}$ and the CP phase $\delta$ is possible obviously due to the number of parameters. However, one can recognize that a large phase in the lepton sector is preferable to suppress $\delta \theta_{23}$. The separation of the $\mu$-$\tau$ symmetric and breaking pieces is useful to understand this feature. We exhibit several relations among $\delta \theta_{23}$, $\theta_{13}$ and the CP phase $\delta$ by special
conditions for the $\mu$-$\tau$ breaking parameters. It will be important to test the relations in the future when $\delta \theta_{23}$ and the CP phase $\delta$ are accurately measured, and to decode the underlying physics which determines the neutrino oscillation parameters.

We next construct a spontaneous $\mu$-$\tau$ symmetry breaking model, which can restrict the $\mu$-$\tau$ breaking parameters. In the model, the $\mu$-$\tau$ exchange symmetry is broken in a hidden sector, and extra gauge interactions mediate the symmetry breaking to the neutrino sector. Though one can easily build models in which the scalar fields to break the symmetry can couple with neutrino fields directly, the $\mu$-$\tau$ breaking parameters in such models can be anything by an artificial choice of the Yukawa-type interaction. In our model, on the other hands, the $\mu$-$\tau$ breaking parameters are related to the extra gauge boson masses and the standard model (SM) singlet neutrino mass spectrum, and thus, the physical meaning of the $\mu$-$\tau$ breaking parameters is clearer. We introduce so-called $B - L$ gauge boson for $\mu$-$\tau$ even gauge interaction, and $L_\mu - L_\tau$ gauge boson for $\mu$-$\tau$ odd gauge interaction. The mixings of those two gauge bosons are generated by the spontaneous breaking of the $\mu$-$\tau$ symmetry, and induce the $\mu$-$\tau$ breaking in the neutrino mass matrix by the gauge boson loops.

This paper is organized as follows: In Section 2, we describe the separation of the $\mu$-$\tau$ symmetric and breaking pieces in the neutrino mass matrix, and we discuss how the large CP phase is preferred exhibiting the relations among $\delta \theta_{23}$, $\theta_{13}$ and $\delta$. In Section 3, we build a model of the spontaneous $\mu$-$\tau$ exchange symmetry breaking. Section 4 is devoted to the conclusion of this paper. In Appendix A, we show the procedures of the diagonalization of the neutrino mass matrix in the basis where $\mu$-$\tau$ symmetric and breaking pieces are separated, in which the relations among the neutrino oscillation parameters are derived. In Appendix B, we discuss the relation of our description with the $\mu$-$\tau$ reflection symmetry.

2 Neutrino mixing angles and $\mu$-$\tau$ breaking

The $\mu$-$\tau$ symmetric neutrino mass matrix is given as

$$M^0_\nu = \begin{pmatrix} D & A & A \\ A & B & C \\ A & C & B \end{pmatrix},$$

(1)

By changing the sign of the third generation neutrino field, the (1,3) elements can be changed to be $-A$ (and $C$ to be $-C$ as well). The freedom of such field redefinition is surely unphysical. In this paper, the $\mu$-$\tau$ symmetric mass matrix is given as $(M_\nu)_{12} = (M_\nu)_{13}$, $(M_\nu)_{22} = (M_\nu)_{33}$ as a convention.
in the basis that the charged-lepton mass matrix is diagonal. We define a unitary matrix for 2-3 block rotation as

\[ U^{(23)}(\theta, \phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta e^{-i\phi} \\ 0 & -\sin \theta e^{i\phi} & \cos \theta \end{pmatrix}, \]  

and \( U^{(12)}, U^{(13)} \) for 1-2 and 1-3 rotation unitary matrices similarly. We obtain

\[ \bar{M}_\nu^0 \equiv U_0^T M_\nu^0 U_0 = \begin{pmatrix} D & \sqrt{2} A & 0 \\ \sqrt{2} A & E & 0 \\ 0 & 0 & F \end{pmatrix}, \quad U_0 = U^{(23)} \left(-\frac{\pi}{4}, 0\right), \]

where \( E = B + C \) and \( F = B - C \). The matrix \( \bar{M}_\nu^0 \) can be diagonalized by \( U^{(12)}(\theta_{12}, \phi) \) and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix can be written as

\[ U_{\text{PMNS}} = U^{(23)} \left(-\frac{\pi}{4}, 0\right) U^{(12)}(\theta_{12}, \phi). \]

One finds that the 2-3 neutrino mixing is maximal and the 1-3 mixing is zero under the \( \mu-\tau \) symmetry. Thus, the matrix is good to use a base to describe the observed neutrino mixings. The \( \mu-\tau \) breaking piece can be parametrized as

\[ M'_\nu = \begin{pmatrix} 0 & -A' & A' \\ -A' & -B' & 0 \\ A' & 0 & B' \end{pmatrix}, \]

and the neutrino mass matrix \( M_\nu = M_\nu^0 + M'_\nu \). The \( \mu-\tau \) breaking piece can be written by the same 2-3 rotation as

\[ \bar{M}'_\nu = U_0^T M'_\nu U_0 = \begin{pmatrix} 0 & 0 & \sqrt{2} A' \\ 0 & 0 & B' \\ \sqrt{2} A' & B' & 0 \end{pmatrix}. \]

The deviation from 45° of the 2-3 mixing and 1-3 mixing are generated by \( A' \) and \( B' \). Obviously, one can understand that the separation of the \( \mu-\tau \) symmetric and breaking pieces is just parametrization of the neutrino mass matrix elements and any values of mixings and Dirac CP phase are possible if one does not assume anything on \( A' \) and \( B' \).

Suppose that there is underlying \( \mu-\tau \) symmetry and the symmetry is broken, and \( A' \) and \( B' \) are the same order very naively. In this case, one expects\(^2\)

\[ \delta \theta_{23} \equiv \theta_{23} - 45^\circ \approx \pm \theta_{13}. \]

Namely, the non-zero 1-3 mixing angle implies that the 2-3 mixing angle is deviated from 45°. The observed 1-3 mixing angle is \( 8^\circ - 9^\circ \) and \( \theta_{23} \) is \( 41^\circ - 51^\circ \) for 3σ range under the current global

\(^2\)We surely use the Particle Data Group (PDG) convention [11] to describe the mixing angles in the mixing matrix; the mixing angles are put in the first quadrant by unphysical field redefinition if they are not there.
fit [5, 6]. The current global best fit for the 2-3 mixing by NuFIT4.1 [5] is \( \theta_{23} = (48.6_{-1.7}^{+1.0})^\circ \).

We expect more precise measurements of the 2-3 mixing angle to see if \( \delta \theta_{23} \) is non-zero in the up-coming experimental data to distinguish neutrino models.

It is worth to describe how \( \delta \theta_{23} \) and \( \theta_{13} \) (and the Dirac CP phase \( \delta \), as well) are generated from \( A' \) and \( B' \). The description depends on the neutrino mass hierarchy, so-called normal hierarchy (NH) and inverted hierarchy (IH). In NH, \( \theta_{13} \) is basically generated by \( A' \) and \( \delta \theta_{23} \) is generated by \( B' \). In IH, on the other hand, either \( A' \) or \( B' \) can generate both \( \theta_{13} \) and \( \delta \theta_{23} \). The detail descriptions are given in Appendix A. Since there are two complex parameters \( A' \) and \( B' \), there is no rigid relation among \( \delta \theta_{23} \), \( \theta_{13} \) and the Dirac CP phase \( \delta \). We here assume special conditions on \( A' \) and \( B' \) to express a relation among them. Derivation of the relations are given in Appendix A.

- **NH**
  
  We demand a condition that \( A' \sim B' \). We suppose that the (1,3) element of the \( \mu-\tau \) breaking matrix is zero after \( M_0^\nu \) is diagonalized by Eq.(4). This condition is satisfied if \( \mu-\tau \) symmetric matrix \( M_0^\nu \) is rank 2, and the neutrino mass matrix is rank 2 even after the \( \mu-\tau \) breaking term is added. Then, we obtain\(^3\)

\[
\theta_{23} - \frac{\pi}{4} \simeq \cot \theta_{12} \sin \theta_{13} \cos \delta. \tag{8}
\]

- **IH**
  
  In this case, either \( A' \) or \( B' \) can generate both \( \theta_{13} \) and \( \delta \theta_{23} \). In order to have a rigid relation, we just assume that one of \( A' \) and \( B' \) is zero to avoid their contributions to be cancelled simply. We obtain

\[
\begin{align*}
A' = 0 & \quad \Rightarrow \quad \theta_{23} - \frac{\pi}{4} \simeq \cot 2 \theta_{12} \sin \theta_{13} \cos \delta, \tag{9} \\
B' = 0 & \quad \Rightarrow \quad \theta_{23} - \frac{\pi}{4} \simeq -\tan 2 \theta_{12} \sin \theta_{13} \cos \delta. \tag{10}
\end{align*}
\]

It is interesting to remark that the Dirac CP phase \( \delta \) needs to be large (|\( \cos \delta \)| needs to be small) to satisfy the range of \( \theta_{23} \) for the current global fit in Eqs.(8) and (10) (for |\( \delta \theta_{23} \)| \( \lesssim 6^\circ \), one obtains |\( \cos \delta \)| \( \lesssim 0.45, 0.3 \) for Eqs.(8),(10), respectively). Small change for a large phase can be simply analogized to \(|1 + z|^2 = 1 + 2 \text{Re} \ z + |z|^2\), namely, \(|1 + z|^2 = 1 + O(|z|^2)\) for \(|z| \ll 1\) if \( \text{arg}(z) \) is nearly \( \pm \pi/2 \), while \(|1 + z|^2 = 1 \pm 2|z| + O(|z|^2)\) for \( \text{arg}(z) \approx 0 \) or \( \pi \). The description in Appendix A to derive the relations makes the meaning of this analogy clearer.

As a result, if the neutrino mass matrix has \( \mu-\tau \) symmetry and the symmetry is violated by \( A' \sim B' \), it is preferred to have a large phase to keep \(|\delta \theta_{23}| < \theta_{13}/2 \) roughly. Of course, in

\(^3\) We note that Eq.(8) holds if both \( M_0^\nu \) and \( M_0^\nu + M_0^\nu \) are rank 2, model-independently. Therefore, for example, it can hold in the minimal seesaw model (e.g., numerical calculations are found in [12]).
general, the large phase is not necessarily same as the Dirac CP phase which can be measured by the neutrino oscillations and/or $\delta \theta_{23}$ can be cancelled irrespective of the phase. If special cases are considered as above, the phase really corresponds to the Dirac CP phase, and those simple relations are satisfied.

It is also interesting to note that one needs $\cos \delta > 0$ ($< 0$) if $\theta_{23}$ is in the higher (lower) octant if the relations are given as in Eqs. (8) and (9). If $\delta \theta_{23} = 3^\circ$ as given by the current central value, one obtains $|\delta| = 76^\circ$, $32^\circ$, and $98^\circ$ for Eqs. (8), (9), and (10), respectively. It is interesting to test the relations and decode the $A'$, $B'$ parameters if $\delta \theta_{23}$ and $\delta$ are measured more accurately.

We emphasize that the CP phase is naively preferred to be large if the neutrino mass matrix has approximate $\mu$-$\tau$ symmetry, and $\theta_{13}$ is generated by spontaneous $\mu$-$\tau$ breaking, though the relation among $\theta_{13}$, $\delta \theta_{23}$ and $\delta$ cannot be rigid in general. It is worth to construct a $\mu$-$\tau$ symmetry breaking model.

3 A model for $\mu$-$\tau$ symmetry breaking

We have studied that the description of $\mu$-$\tau$ symmetric and $\mu$-$\tau$ breaking pieces is a good base to consider the size of $\delta \theta_{23}$ and the Dirac CP phase $\delta$, which will be measured more accurately at the near future long-baseline neutrino oscillation experiments. In this section, we will construct a model for the $\mu$-$\tau$ breaking.

3.1 Spontaneous breaking of $\mu$-$\tau$ exchange symmetry

In the beginning, let us consider a set of two complex scalars, named as $(\phi_\mu, \phi_\tau)$, and build a scalar potential, which has $\phi_\mu \leftrightarrow \phi_\tau$ exchange symmetry. Suppose that $\phi_\mu$ and $\phi_\tau$ have charges of a U(1) symmetry. Then, the potential can be written as

$$V = -m^2(|\phi_\mu|^2 + |\phi_\tau|^2) + \lambda_1(|\phi_\mu|^4 + |\phi_\tau|^4) + \lambda_2|\phi_\mu|^2|\phi_\tau|^2.$$  \hspace{1cm} (11)

One finds the minimum of the potential,

$$\phi_\mu^2 = \phi_\tau^2 = \frac{m^2}{2\lambda_1 + \lambda_2},$$  \hspace{1cm} (12)

for $2\lambda_1 + \lambda_2 > 0$, and

$$\phi_\mu^2 = \frac{m^2}{2\lambda_1}, \ \phi_\tau = 0, \ \text{or} \ \phi_\mu = 0, \ \phi_\tau^2 = \frac{m^2}{2\lambda_1},$$  \hspace{1cm} (13)

for $2\lambda_1 + \lambda_2 < 0$. The former keeps the exchange symmetry, and the latter violates it spontaneously. If the $\mu$-$\tau$ symmetric vacuum expectation value (vev) and $\mu$-$\tau$ breaking vev directly
couples to neutrino sector, one can easily construct a model with discrete symmetry such as $S_3$ [13, 14, 15, 16].

We will construct a model in which the $\mu$-$\tau$ exchange symmetry is broken in a hidden sector, namely the scalars $(\phi_\mu, \phi_\tau)$ does not couple to neutrinos directly, and the violation is mediated by a gauge interaction. We introduce gauge bosons which are even and odd under the exchange symmetry, and the mixing of the gauge bosons can generate the $\mu$-$\tau$ breaking in the neutrino mass matrix. Though the easy candidate of the $\mu$-$\tau$ even gauge boson is $Z$-boson, the sizable mixing of the $Z$-boson with the other gauge boson to generate $\theta_{13}$ contradicts with the precision measurements. Therefore, we consider $B-L$ gauge boson for the $\mu$-$\tau$ even gauge boson. More precisely speaking, the hypercharge is a linear combination of $U(1)_B-L$ and $U(1)_R$, and the extra $\mu$-$\tau$ even gauge boson we call as $Z'$ is the one for $U(1)'_B$ symmetry which is orthogonal to the hypercharge. The $\mu$-$\tau$ odd gauge boson, we call it as $Z''$, is a $U(1)$ gauge boson for $L_\mu - L_\tau$ charge. Namely, the gauge interactions to the lepton doublets are written as

$$
\mathcal{L} \supset ig'Z'(\bar{e}_e \gamma e + \bar{\ell}_\mu \gamma \ell_\mu + \bar{\ell}_\tau \gamma \ell_\tau) \quad (\mu - \tau \text{ even})
$$

$$
\mathcal{L} \supset ig''Z''(\bar{\ell}_\mu \gamma \ell_\mu - \bar{\ell}_\tau \gamma \ell_\tau) \quad (\mu - \tau \text{ odd})
$$

where $g'$ and $g''$ are gauge couplings ($g'$ is not a hypercharge gauge coupling $g_Y$ for our notation).

Suppose that the three scalars $\phi_\mu$, $\phi_\tau$ and $\phi$ has $U(1)'$ and $L_\mu - L_\tau$ charges as given in Table 1. The gauge boson mass term is obtained as

$$
\begin{pmatrix}
Z' & Z''
\end{pmatrix}
\begin{pmatrix}
g'^2(\phi_\mu^2 + \phi_\tau^2) & g'g''(\phi_\mu^2 - \phi_\tau^2) & g''^2(\phi_\mu^2 + \phi_\tau^2)
\end{pmatrix}
\begin{pmatrix}
Z'
Z''
\end{pmatrix}.
$$

Obviously, the $Z'$-$Z''$ mixing is absent in the $\mu$-$\tau$ symmetric vacua ($\phi_\mu = \phi_\tau$) and they are mixed in the $\mu$-$\tau$ breaking vacua. Suppose $\phi_\mu = 0$ and $\phi_\tau \neq 0$ for the $\mu$-$\tau$ breaking vacua, then we obtain the $Z'$-$Z''$ mixing angle $\alpha$ as

$$
\tan 2\alpha = \frac{2g'g''\phi_\tau^2}{g'^2\phi_\mu^2 + (g'^2 - g''^2)\phi_\tau^2}.
$$

One finds that they are maximally mixed if $g' = g''$ and $\phi \ll \phi_\tau$. Surely, there can be additional scalar fields to break the $U(1)$ symmetries. The request of the model will be that the $Z'$-$Z''$ mixing is not too small to generate the 1-3 neutrino mixing from the $\mu$-$\tau$ breaking.

|       | $\phi_\mu$ | $\phi_\tau$ | $\phi$ |
|-------|------------|-------------|--------|
| $U(1)'$ | 1          | 1           | 1      |
| $L_\mu - L_\tau$ | 1         | -1          | 0      |

Table 1: The U(1) charges of the scalar fields.
We comment that gauge kinetic mixing between $Z'$ and $Z''$ is possible in general. We can suppose that the kinetic mixing is absent by the $\mu$-$\tau$ exchange symmetry (since $Z'$ is even and $Z''$ is odd) and the $Z'$-$Z''$ mixing is generated by the spontaneous breaking of $\mu$-$\tau$ symmetry.

### 3.2 Loop-induced neutrino mass

We build a model where the $\mu$-$\tau$ breaking in the neutrino mass matrix is generated via the gauge boson loop. If there is a tree-level neutrino mass, however, the loop correction is tiny and the observed size of $\theta_{13}$ cannot be generated unless the gauge interaction is so strong. Therefore, we consider a situation that the $\mu$-$\tau$ symmetric neutrino mass is also generated by a loop effect. In order to realize such setup, we consider the following neutrino mass term (for one generation),

\[
(\nu \ N \ S) \left( \begin{array}{ccc} 0 & 0 & m \\ 0 & 0 & X \\ m & X & M \end{array} \right) \left( \begin{array}{c} \nu \\ N \\ S \end{array} \right),
\]

(18)

and study how the loop-induced neutrino mass is generated. We remark that a neutrino, which is mainly active neutrino $\nu$, is massless at the tree-level as long as $N$-$N$ and $\nu$-$N$ elements are zero. Surely, those elements are not necessarily exactly zero. The tree-level neutrino masses are supposed to be less than meV (if there are) as a setup. The active neutrino mass, to explain the observed neutrino oscillations, can be generated radiatively even if the tree-level mass is zero. Since we will gauge the $B-L$ symmetry, we assign $\nu \subset \ell$ has $B-L = -1$, and 0, +1 for $N$, $S$, respectively.

The induced neutrino mass via $Z$-boson and Higgs boson loops is [17]

\[
m_\nu = \frac{\alpha_2}{16\pi} \sum_{k=1,2} \frac{M_k}{M_W^2} C_{\nu N_k}^2 \left[ M_k^2 (f(M_k, M_Z) - f(M_k, M_H)) - 4M_Z^2 f(M_k, M_Z) \right],
\]

(19)

where $M_k$ is the mass of the mass eigenstate $N_k$ and $C_{\nu N_k}$ is the mixing (which is $Z$-$\nu$-$N_k$ coupling). The loop function $f$ is given as

\[
f(M_N, M) = \frac{M_N^2}{M_N^2 - M^2} \ln \frac{M_N^2}{M^2} + \ln \frac{M^2}{Q^2} - 1.
\]

(20)

The loop-induced mass does not depend on the renormalization scale $Q$ after all. The first term in the bracket in Eq.(19) corresponds to the contribution from the loop diagrams for Higgs and Nambu-Goldstone modes. These contributions are canceled if the singlet neutrino masses are much heavier than them, $M_i \gg M_{Z,H}$. The second term corresponds to the $Z$ boson loop, which we will calculate hereafter.
The diagonalization matrix can be written as

\[ U^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_R & -s_R \\ 0 & s_R & c_R \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ c_Rs_\theta & c_Rc_\theta & -s_R \\ s_Rs_\theta & s_Rc_\theta & c_R \end{pmatrix}, \] (21)

with an obvious notation, \( s_R = \sin \theta_R, \) \( s_\theta = \sin \theta, \) etc, and

\[
\begin{pmatrix} \nu \\ N \\ S \end{pmatrix} = \begin{pmatrix} U \end{pmatrix} \begin{pmatrix} \nu_1 \\ N_1 \\ N_2 \end{pmatrix}, \] (22)

where \( \nu_1, N_i \) are mass eigenstates. We obtain

\[
\tan \theta = \frac{m}{X}, \quad \tan 2\theta_R = \frac{2\sqrt{X^2 + m^2}}{M}. \] (23)

We have a relation

\[
c_R^2 M_1 + s_R^2 M_2 = 0, \] (24)

which corresponds to the condition that the tree-level seesaw neutrino mass is zero. The \( Z - \nu - N_k \) coupling are given as

\[
C_{\nu N_1} = c_\theta s_\theta c_R, \quad C_{\nu N_2} = c_\theta s_\theta s_R. \] (25)

The neutrino mass from \( Z \) boson loop is then

\[
m_\nu = -\frac{\alpha_Z}{4\pi} (s_\theta c_\theta)^2 (M_1 c_R^2 f(M_1, M_Z) + M_2 s_R^2 f(M_2, M_Z)) \] (26)

\[
= -\frac{\alpha_Z}{4\pi} (s_\theta c_\theta)^2 M_1 c_R^2 (f(M_1, M_Z) - f(M_2, M_Z)) \] (27)

\[
= -\frac{\alpha_Z}{4\pi} (s_\theta c_\theta)^2 M_1 c_R^2 \left( \frac{M_1^2}{M_1^2 - M_Z^2} \ln \frac{M_1^2}{M_Z^2} - \frac{M_2^2}{M_2^2 - M_Z^2} \ln \frac{M_2^2}{M_Z^2} \right), \] (28)

where \( \alpha_Z = (g_2^2 + g_Y^2)/(4\pi). \) We note that the loop-induced neutrino mass tends to be zero for a limit \( M/X \to 0 (M_1 \simeq -M_2). \) This can be understood as that a (global) lepton number symmetry remains for \( M = 0. \)

We emphasize that the loop-induced neutrino mass does not depend on \( Z \) boson mass, if the singlet neutrinos are heavy. In fact, one finds

\[
m_\nu = -\frac{\alpha_Z}{4\pi} (s_\theta c_\theta)^2 M_1 c_R^2 \ln \frac{M_1^2}{M_Z^2}, \] (29)

for \( M_1, M_2 \gg M_Z. \) Therefore, the contributions from \( Z' \) and \( Z'' \) gauge bosons are the same order as the one from \( Z \) boson if they are lighter than the heavy neutrino mass and gauge couplings are the same size of the electroweak coupling. The neutrino mass \( M \) violates the
$B - L$ symmetry and the $B - L$ gauge boson and the neutrino mass may be similar size. Using the freedom of the mass spectrum of gauge bosons and singlet neutrinos, we will control the induced $\mu$-$\tau$ symmetric and violating terms.

Now we understand the loop-induced neutrino mass for one-generation, and it can be extended to the three-generation version. The neutrino mass terms are given as

$$
(\nu \ N \ S) \begin{pmatrix} 0 & 0 & m \\ 0 & 0 & X \\ m^T & X^T & M \end{pmatrix} \begin{pmatrix} \nu \\ N \\ S \end{pmatrix},
$$

where each element is $3 \times 3$ matrix. We assume that the active and singlet neutrino mixings are small ($mX^{-1}$ is small). Then, by extending the same process to obtain the loop-induced neutrino mass for the 1-generation case, the neutrino mass matrix for 3-generation case from $Z$-boson loop is approximately given as

$$M_\nu = -\frac{\alpha_Z}{4\pi} \sum_{a=1,2} mX^{-1} U_a [M^{(a)} f(M^{(a)}, M_Z)] U_a^T X^{-1} T M^T. \quad (31)$$

Here, $[M^{(a)} f(M^{(a)}, M_Z)]$ denotes a diagonal matrix: $[A] \equiv \text{diag}(A_1, A_2, A_3)$, and

$$\begin{pmatrix} U_1^T & U_3^T \\ U_2^T & U_4^T \end{pmatrix} \begin{pmatrix} 0 & X \\ X^T & M \end{pmatrix} \begin{pmatrix} U_1^* & U_2^* \\ U_3^* & U_4^* \end{pmatrix} = \begin{pmatrix} M^{(1)} & 0 \\ 0 & M^{(2)} \end{pmatrix}, \quad (32)$$

and $M^{(1,2)}$ are $3 \times 3$ diagonal matrices. By definition,

$$U_1 M^{(1)} U_1^T + U_2 M^{(2)} U_2^T = 0. \quad (33)$$

In the following, we denote the $Z$-loop-induced neutrino mass factoring out the gauge coupling as

$$M_{\nu} = g_Z^2 \hat{M}(M_Z). \quad (34)$$

The loop-induced neutrino mass depends on the heavy singlet neutrino masses explicitly, but we omit the dependence to write the equations shortly. We assume that tree-level term has $\mu$-$\tau$ exchange symmetry, and then, the matrix $\hat{M}_\nu(M_Z)$ has the symmetry.

### 3.3 Loop-induced $\mu$-$\tau$ breaking in the neutrino mass matrix

The gauge interaction of the extra gauge bosons to the leptons can be written as

$$i \bar{\ell}_i (g' Z' \mu I_{ij} + g'' Z'' \mu T_{ij}) \gamma^\mu \ell_j, \quad (35)$$

where $I$ is an identity matrix and

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (36)$$
The gauge bosons are mixed, and the mass eigenstates $Z_1$ and $Z_2$ are written as
\[ Z_1 = c_\alpha Z' + s_\alpha Z'', \quad Z_2 = -s_\alpha Z' + c_\alpha Z''. \] (37)

Then, the gauge interaction is written as
\[ i\bar{\nu}_i ((c_\alpha g' I_{ij} + s_\alpha g'' T_{ij}) Z_1^\mu + (-s_\alpha g' I_{ij} + c_\alpha g'' T_{ij}) Z_2^\mu) \gamma_\mu \ell_j. \] (38)

We obtain the loop-induced neutrino mass as
\[
M_\nu = g^2_Z \hat{M}(M_Z) + (c_\alpha g'I + s_\alpha g''T)\hat{M}(M_{Z_1})(c_\alpha g'I + s_\alpha g''T)
+(-s_\alpha g'I + c_\alpha g''T)\hat{M}(M_{Z_2})(-s_\alpha g'I + c_\alpha g''T)
\]
\[= g^2_Z \hat{M}(M_Z) + g^2 c^2_\alpha \hat{M}(M_{Z_1}) + g^2 s^2_\alpha \hat{M}(M_{Z_2})
+g''s^2_\alpha T\hat{M}(M_{Z_1})T + g''c^2_\alpha T\hat{M}(M_{Z_2})T
+c_\alpha s_\alpha g'g''(T(\hat{M}(M_{Z_1}) - \hat{M}(M_{Z_2})) + (\hat{M}(M_{Z_1}) - \hat{M}(M_{Z_2}))T). \] (40)

The last term is the $\mu$-$\tau$ breaking term. As we have remarked, $\hat{M}$ does not depend on the gauge boson mass if the singlet neutrinos are much heavier than the gauge boson. Therefore, in that case, the $\mu$-$\tau$ breaking term vanishes.

In order to calculate the loop-induced neutrino mass matrix expressed in Eq.(40), one needs to diagonalize the $6 \times 6$ heavy neutrino mass matrix as given in Eq.(32). The diagonalization can be done numerically in any setup. We here assume a setup to express the loop-induced mass matrix in a simple form and to illustrate the essence of the structure. We assume that the singlet neutrino mass matrices $X$ and $\hat{M}$ are also $\mu$-$\tau$ symmetric. We note that the $L_\mu - L_\tau$ symmetry breaking vev can be directly applied to the SM singlet masses, but the $L_\mu - L_\tau$ breaking in the Dirac mass matrix $m$ will be higher order\(^4\). Due to the $\mu$-$\tau$ symmetry, one can suppose that the matrices are given after 2-3 generation are rotated by $\pi/4$ as
\[
\bar{m} = \begin{pmatrix} d & a & 0 \\ a' & e & 0 \\ 0 & 0 & f \end{pmatrix}, \quad \bar{X} = \begin{pmatrix} X_1 & X_4 & 0 \\ X_5 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}. \] (41)

The matrix $\hat{M}$ has been diagonalized by rotating 1-2 generation of the $S$ field, and $a, a'$ in $\bar{m}$ are generated by the rotation. One can parameterize so that $X_4$ or $X_5$ are zero by rotating the $N$ field, but both $\bar{X}$ and $\hat{M}$ cannot be diagonalized simultaneously in general. In order to illustrate the induced mass matrix simply, we here assume $X_4 = X_5 = 0$ so that the diagonalization of

\(^4\) We comment that the $L_\mu - L_\tau$ charges for the lepton doublets are assigned as 0, 1, $-1$, and those of the right-handed leptons are 0, $-1, 1$. The charged lepton mass matrix is then diagonal up to the higher order terms.
the singlet neutrino mass matrix in Eq.(32) are split in each generation. Then, we find that
the induced neutrino mass matrix can be written as

$$\tilde{M}_\nu = M_1 F_1(X_1, M_1) + M_2 F_1(X_2, M_2) + M_3 F_1(X_3, M_3) + M_4 F_2(X_1, M_1) + M_5 F_2(X_2, M_2) + M_6 F_2(X_3, M_3) + M_6 F_3(X_1, M_1) + M_7 F_3(X_2, M_2) + M_9 F_3(X_3, M_3),$$

where the matrices $M_i$ are\(^5\)

$$M_1 = \begin{pmatrix} a^2 & a' & d & 0 \\ a' & a' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a^2 & ae & 0 & 0 \\ ae & e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 0 & a' & 0 \\ 0 & 0 & 0 & a' \\ 0 & 0 & a'^2 & 0 \\ a'^2 & a'^2 & 0 & 0 \end{pmatrix}, \quad M_8 = \begin{pmatrix} 0 & 0 & ae & 0 \\ 0 & 0 & e^2 & 0 \\ ae & e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ (44)

Defining

$$F(X, M, M_z) = \frac{e_R^2}{X^2} M^{(1)}(f(M^{(1)}, M_z) - f(M^{(2)}, M_z)),$$ (47)

where $e_R = \cos \theta_R$, $\tan 2\theta_R = 2X/M$ and $M^{(1,2)} = (M \pm \sqrt{M^2 + 4X^2})/2$, we give the functions as

$$F_1(X, M) = \frac{1}{16\pi^2} [g_2^2 F(X, M, M_z) + g''_R c_\alpha^2 F(X, M, M_{Z_1}) + s_\alpha^2 F(X, M, M_{Z_2})],$$

$$F_2(X, M) = \frac{1}{16\pi^2} g''_R (s_\alpha^2 F(X, M, M_{Z_1}) + c_\alpha^2 F(X, M, M_{Z_2})),$$

$$F_3(X, M) = \frac{1}{16\pi^2} g'' g''_R s_\alpha c_\alpha (F(X, M, M_{Z_1}) - F(X, M, M_{Z_2})).$$

One finds that the matrices $M_7, 8, 9$ correspond to the $\mu - \tau$ breaking terms. We list the property of the function $F(X, M, M_z)$, which is useful to control the neutrino mass hierarchy:

1. For $X, M \gg M_z$, the function $F$ is less dependent on $M_z$.

\(^5\) $M_1 = \tilde{m} \text{diag}(1, 0, 0) \tilde{m}^T$, $M_2 = \tilde{m} \text{diag}(0, 1, 0) \tilde{m}^T$, $M_3 = \tilde{m} \text{diag}(0, 0, 1) \tilde{m}^T$, $M_4 = \tilde{T} M_1 \tilde{T}$, $M_5 = \tilde{T} M_2 \tilde{T}$, $M_6 = \tilde{T} M_3 \tilde{T}$, $M_7 = \tilde{T} M_1 + \tilde{M}_1 \tilde{T}$, $M_8 = \tilde{T} M_2 + \tilde{M}_2 \tilde{T}$, $M_9 = \tilde{T} M_3 + \tilde{M}_3 \tilde{T}$, where $\tilde{T}$ is the matrix for $\mu - \tau$ odd gauge interaction, Eq.(36), in the basis that the 2-3 generation is rotated by $\pi/4$.

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ (43)
2. For $X, M \ll M_z$, the function $F$ is suppressed by $\sim X^2/M_2^2, M^2/M_2^2$.

3. For $M \ll X$, the function $F$ is suppressed as a result of a global lepton number symmetry.

Now let us consider how the neutrino mixings are reproduced from $M_i$. Remind that they are written in the basis that the 2-3 generation is rotated by $\pi/4$. The neutrino mass matrices are given as Eqs.(3) and (6). In the NH case, the $(3,3)$ element is the largest. The simplest realization is that $f$ is the largest in $M_3$. The other elements are smaller than the $(3,3)$ element by the original Dirac Yukawa coupling for $m$ or by the spectrum of the singlet neutrinos. For example, if $f > a, a', d, e$ in the matrix $\bar{m}$, the choice of $X_3, M_3 < M_{Z_1}, M_{Z_2}$ can suppress $F(X_3, M_3, M_{Z_{1,2}})$ not to disturb the hierarchical structure by the $M_6, 9$ terms. Surely, for $X_3, M_3 > M_Z$, $M_3$ term is not suppressed. Because the neutrino mass ratio $m_2/m_3 \approx \sqrt{\Delta m^2_{\text{sol}}/\Delta m^2_{\text{atm}}}$ is the same size as the 1-3 mixing in NH, $g_Z^2 \sim g'g''s_a c_a$ is needed naively. For another choice, we can also reproduce the mass hierarchy by the size of the gauge couplings: the contribution from $M_4$ or $M_5$ is the largest. This can be done if $L_\mu - L_\tau$ gauge coupling $g''$ is a bit (2-3 times) stronger than $g_Z$ and $g'$.

Naively speaking, the large solar mixing is generated by $a'd$ and $ae$ in $M_1$ and $M_2$. Then, if the 1-3 mixing is generated by the $\mu-\tau$ breaking terms, $M_7$ and $M_8$, one can understand that the 2-3 mixing is also deviated from the maximal mixing due to $A' \sim B'$. Therefore, as explained, a large CP phase in the components is preferable to keep the 2-3 mixing nearly maximal, though there can be freedom for cancellation between (2,3) components in $M_{7,8}$ in general. If the neutrino mass matrix is rank 2, such cancellation can be avoided. The rank-2 mass matrix can be easily constructed: for example, the $M_{1,4,7}$ contributions are tiny if $M_1 \ll X_1$. As we have remarked, $F(X_1, M_1, M)$ is small if $M_1 \ll X_1$. Then, we have a relation given in Eq.(8). In this case, a large CP phase $\delta$ is definitely favored. We comment that the rank 2 condition is a sufficient one to realize Eq.(8). One can find that Eq.(8) holds if $(d, a')$ and $(a, e)$ are “orthogonal” (i.e. $da^* + a'e^* = 0$), supposing that the neutrino mass matrix is given as a linear combination of $M_{1,2,4,7}$.

In the case of IH, one can build the neutrino mass matrix so that the $M_{1,2}$ contributions are dominant, and the others are subdominant, namely, the $Z'$ and $Z''$ loop contributions are $\sim 10\%$ compared to the $Z$ boson contribution. Such a situation can be easily achieved if the gauge coupling is small, and/or $Z'-Z''$ mixing is small, and/or $Z', Z''$ masses are a bit larger than the singlet neutrino masses. If one chooses $M_{7,8}$ to be suppressed by $M_{1,2}, X_{1,2} \ll M_{Z_{1,2}}$ and $M_9$ to be a dominant contribution for $\mu-\tau$ breaking, a situation for $A' = 0$ can be obtained, and then, Eq.(9) is satisfied approximately.

Finally, we comment about the scales of the singlet neutrino masses and the extra gauge boson masses. If the singlet neutrinos are much heavier than $Z$-boson, the induced neutrino
mass is roughly given as

\[ m_\nu \approx \frac{\alpha Z}{4\pi} m^2 \frac{M}{X^2}, \tag{51} \]

and therefore, the neutrino mass scale is similar to the usual seesaw (or inverse seesaw) up to the loop factor. For the Dirac mass \( m \sim 100 \, \text{GeV} \), one finds \( M \sim X \sim 10^{13} \, \text{GeV} \) to obtain \( m_\nu \sim 0.05 \, \text{eV} \). If \( M \sim X \sim 1 \, \text{TeV} \), one finds \( m \sim 1 \, \text{MeV} \). If the SM singlet neutrinos are lighter than \( Z \) boson and \( Z', Z'' \) bosons are heavier than \( Z \) boson, the \( \mu-\tau \) breaking in the neutrino mass matrix will be small. If \( Z_1 \) and/or \( Z_2 \) are around \( O(100) \, \text{GeV} \) and the neutrinos are as light as them, it may be possible to build the \( \mu-\tau \) breaking neutrino mass matrix for light SM singlet neutrinos and to make the extra gauge boson contribute to generate lepton flavor non-universality [18, 19, 20], as long as they are allowed by experiments though we do not survey the possibility in this paper. The very light (MeV-scale) extra gauge boson may also have phenomenological interests [21]. As described, the extra \( U(1) \) symmetries are broken by the field given in Table 1. The field \( \phi_\tau \neq 0 \) (with \( \phi_\mu = 0 \)) breaks \( \mu-\tau \) exchange symmetry and \( U(1)' \) and \( L_\mu - L_\tau \) symmetry is broken to a linear combination. The remained symmetry is broken by the vev of \( \phi \), which has a \( B - L \) charge. The \( \mu-\tau \) breaking can occur at a high scale, and the vev of \( \phi \) can be much lower than it. Then, the \( Z'-Z'' \) mixing angle is determined by the gauge couplings: \( \tan \alpha \simeq g''/g' \), and \( M_{Z_1} \ll M_{Z_2} \). The SM singlet neutrinos can lie around the mass of \( Z_1 \). In this situation, the proper size of the \( \mu-\tau \) breaking in the neutrino mass matrix can be easily generated. The neutrino mass hierarchy and the size of \( \theta_{13} \) can be adjusted by the singlet neutrino mass spectrum and the choice of the gauge couplings \( g' \) and \( g'' \), as we have described. The mass scale of the remained \( U(1) \) breaking, namely \( \sim M_{Z_1} \), can be around TeV scale, and there may have an opportunity to be observed experimentally.

4 Conclusion

The separation of the \( \mu-\tau \) symmetric and breaking terms is useful to discuss the deviation from \( \pi/4 \) of the 2-3 neutrino mixing angle and the Dirac CP phase \( \delta \) as a parametrization of the neutrino mass matrix. The description can be applied when one constructs any neutrino models. Since the observed 1-3 neutrino mixing is sizable to keep the 2-3 mixing angle nearly maximal naively, one may think that the \( \mu-\tau \) exchange symmetry is not a good frame to describe the observed pattern of the mixing angles. However, we have shown that it leads a large CP violation in the neutrino sector. In general, the CP phase is not necessarily equal to the Dirac CP phase \( \delta \), which can be measured by the neutrino oscillation experiments. If the \( \mu-\tau \) breaking parameters are fixed, one can write a relation among the 2-3 and 1-3 mixings and the Dirac CP phase such as given in Eqs.(8), (9), (10). It will be interesting to test such relations and to
see if there is underlying symmetry in the neutrino mass matrix.

We have built a model in which the mixing of the gauge bosons which are even and odd under the $\mu$-$\tau$ exchange symmetry generates the $\mu$-$\tau$ breaking terms in the neutrino mass matrix by loop effects. The $\mu$-$\tau$ exchange symmetry is spontaneously broken in a hidden sector, and the extra gauge bosons are messengers to generate the $\mu$-$\tau$ breaking terms in the neutrino mass matrix. One can surely construct a model in which the scalar fields, which breaks the $\mu$-$\tau$ exchange symmetry, can couple to the neutrino sector via Yukawa-type interactions, but the pattern of the $\mu$-$\tau$ breaking is left to the model-builder’s discretion in such models. In our model, on the other hand, the pattern is determined by the gauge boson masses and mixing and the SM singlet heavy neutrino spectrum. The pattern of the $\mu$-$\tau$ breaking in the neutrino mass matrix can lead a large CP phase preferably. The extra gauge boson and SM singlet neutrinos can lie around TeV for Dirac neutrino mass to be around MeV. The $B - L$ and $L_\mu - L_\tau$ gauge bosons can play the $\mu$-$\tau$ even and odd interactions, and then, their gauge couplings are constrained.

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**Appendix A: Diagonalization of the neutrino matrix**

In this Appendix, we show the (approximate) diagonalization matrices of the neutrino mass matrix given in Section 2.

Before the description, we write a diagonalization matrix of $2 \times 2$ matrix. A $2 \times 2$ symmetric matrix can be diagonalized as

$$
\begin{pmatrix}
\cos \theta & -\sin \theta e^{i\phi} \\
\sin \theta e^{-i\phi} & \cos \theta
\end{pmatrix}
\begin{pmatrix}
a & b \\
b & d
\end{pmatrix}
\begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & \cos \theta
\end{pmatrix} =
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix},
$$

where

$$
\tan 2\theta = \frac{2b}{e^{i\phi}d - e^{-i\phi}a}, \quad \phi = \arg(ab^* + bd^*). \quad (53)
$$

The eigenvalues $\lambda_{1,2}$ (they are not eigenvalues unless the elements are real, though) are

$$
\lambda_1 = a \cos^2 \theta - e^{i\phi}b \sin 2\theta + e^{2i\phi}d \sin^2 \theta = e^{i\phi} \left( \frac{ae^{-i\phi} + de^{i\phi}}{2} - b \csc 2\theta \right), \quad (54)
$$

$$
\lambda_2 = d \cos^2 \theta + e^{-i\phi}b \sin 2\theta + e^{-2i\phi}a \sin^2 \theta = e^{-i\phi} \left( \frac{ae^{-i\phi} + de^{i\phi}}{2} + b \csc 2\theta \right). \quad (55)
$$
We note that, for the solar neutrino mixing in the case of IH, one needs,

$$|ae^{-i\phi} + de^{i\phi}| \ll 2|b| \csc \theta_{12},$$  \hspace{1cm} (56)

to obtain $|\lambda_1| \simeq |\lambda_2|$. If both $a$ and $d$ are much smaller than $b$, the solar mixing angle becomes nearly $\pi/4$, and thus one needs a cancellation to obtain the mass degeneracy.

Now we move to the diagonalization of $3 \times 3$ neutrino matrix, rotated by $U^{(23)}(-\pi/4, 0)$,

$$\bar{M}_\nu = \begin{pmatrix} D & \sqrt{2}A & \sqrt{2}A' \\ \sqrt{2}A & E & B' \\ \sqrt{2}A' & B' & F \end{pmatrix},$$  \hspace{1cm} (57)

which is $\bar{M}_\nu^0 + \bar{M}_\nu'$ given in Eqs.(3) and (6). The description of the diagonalization depends on the neutrino mass hierarchy. We explain the diagonalization processes below for 3 cases: general NH, IH cases, and NH for rank-2 matrix.

• NH (general)

We assume $D, \sqrt{2}A, \sqrt{2}A', E, B' \sim \lambda F$, where $\lambda \sim 0.2$. ($D$ can be much smaller than the others). The approximate PMNS matrix up to $O(\lambda^2)$ can be written as

$$U_{\text{PMNS}} \simeq U^{(23)}(-\pi/4, 0)U^{(23)}(\theta_{23}', \phi_1)U^{(13)}(\theta_{13}', \phi_2)U^{(12)}(\theta_{12}', \phi_3).$$  \hspace{1cm} (58)

The $\theta_{23}'$ angle rotation makes the $(2,3)$ element of $\bar{M}_\nu$ to be zero, and $\theta_{13}'$ angle rotation makes the $(1,3)$ element. Then, a quantity $O(\lambda^2)F$ appears in $(2,3)$ element, but we ignore it. Finally, 1-2 block is diagonalized by $\theta_{12}'$ rotation. The diagonalization process of $\bar{M}_\nu$ can be illustrated by

$$\begin{pmatrix} \lambda X & \lambda X & \lambda X \\ \lambda X & \lambda X & \lambda X \\ \lambda X & \lambda X & 1 \end{pmatrix} \xrightarrow{U^{(23)}(\theta_{23}', \phi_1)} \begin{pmatrix} \lambda X & \lambda X & \lambda X \\ \lambda X & \lambda X & 0 \\ \lambda X & 0 & 1 \end{pmatrix} \xrightarrow{U^{(13)}(\theta_{13}', \phi_2)} \begin{pmatrix} \lambda X & \lambda X & 0 \\ \lambda X & \lambda X & \lambda^2 X \\ 0 & \lambda^2 X & 1 \end{pmatrix} \xrightarrow{U^{(12)}(\theta_{12}', \phi_3)} \begin{pmatrix} \lambda X & 0 & \lambda^2 X \\ 0 & \lambda X & \lambda^2 X \\ \lambda^2 X & \lambda^2 X & 1 \end{pmatrix},$$  \hspace{1cm} (59)

where $X$’s denote arbitrary numbers.

One finds\(^6\)

$$\tan^2 \theta_{23} = \frac{|e_{23}' - e^{-i\phi_1}s_{23}'|^2}{|e_{23}' + e^{-i\phi_1}s_{23}'|^2} = \frac{1 - \sin 2\theta_{23}' \cos \phi_1}{1 + \sin 2\theta_{23}' \cos \phi_1},$$  \hspace{1cm} (60)

\(^6\) We remark that $\theta_{23}$ can be kept to be $\pi/4$ for $\phi_1 = \pm \pi/2$ even if $\theta_{23}' \sim 1$ ($B' \sim E \sim F$). Since $\phi_1 = \arg(EB'^* + B'F^*) = \arg(\text{Re}(BB'^*) + i \text{Im}(CB'^*))$, one finds $B'/B$ is pure imaginary, and $|B + B'| = |B - B'|$ for $\phi_1 = \pm \pi/2$. Therefore, for $\phi_1 = \pm \pi/2$, it is more convenient to reparameterize $B'$ to be zero by rephasing the fields before the 2-3 rotation by $U_0$, though the physical Dirac CP phase surely does not depend on the diagonalization processes.
and
\[ \theta_{23} - \frac{\pi}{4} \simeq -\frac{1}{2} \sin 2\theta'_{23} \cos \phi_1. \] (61)

Remarking that one can obtain
\[ U^{(23)}(-\frac{\pi}{4}, 0)U^{(23)}(\theta'_{23}, \phi_1) = PU^{(23)}(\theta_{23}, \phi'_1), \] (62)
where \( P \) is an unphysical diagonal phase matrix, and \( \phi'_1 = \pi + \arg(c'_{23} - e^{-i\phi_1}s'_{23})(c'_{13} + e^{-i\phi_1}s'_{23}) = \pi + \arctan(s'_{23}^2 \sin 2\phi_1/(1 - 2s'_{23}^2 \cos^2 \phi_1)), \) one finds that the Dirac CP phase is
\[ \delta \simeq \phi_2 - \phi_3 - \pi + O(s'_{23}). \] (64)

The 1-3 and 1-2 mixing angles corresponds to \( \theta^0_{13} \) and \( \theta^0_{12} \). One can easily find that \( A' \) leads to the 1-3 mixing and \( B' \) provides \( \delta \theta_{23} \). If we suppose \( A' = B' \), then \( \theta_{13} = \theta'_{23} \) and \( \phi_1 = \phi_2 \). Thus, to make \( \delta \theta_{23} \) in the allowed region, we need a large phase \( \phi_1 \). However, the Dirac CP phase mixes with \( \phi_3 \) which is a phase to diagonalize the 1-2 block, and the phase \( \phi_1 \) is not necessarily same as the Dirac CP phase, which is measured by the neutrino oscillations.

- IH

We assume \( \sqrt{2}A', B', F \sim \lambda(\sqrt{2}A, B, D) \), where \( \lambda \sim 0.2 \). \((F \text{ can be much smaller than the others)}\). The approximate PMNS matrix up to \( O(\lambda^2) \) can be written as
\[ U_{\text{PMNS}} = U^{(23)}(-\frac{\pi}{4}, 0)U^{(12)}(\theta^0_{12}, \phi_1)U^{(13)}(\theta^0_{13}, \phi_2)U^{(23)}(\theta'_{23}, \phi_3). \] (65)

By \( U^{(23)}(-\frac{\pi}{4}, 0)U^{(12)}(\theta^0_{12}, \phi_1) \), the \( \mu-\tau \) symmetric matrix is diagonalized as \( \check{M}'_{\nu} = \text{diag}(m_1, m_2, F) \).

The \( \mu-\tau \) breaking matrix is, then, in the basis,
\[ \check{M}'_{\nu} = \begin{pmatrix}
0 & 0 & \sqrt{2}A' c^0_{12} - s^0_{12} e^{i\phi_1} B' \\
0 & 0 & \sqrt{2}A' s^0_{12} e^{-i\phi_1} + c^0_{12} B' \\
\sqrt{2}A' c^0_{12} - s^0_{12} e^{i\phi_1} B' & \sqrt{2}A' s^0_{12} e^{-i\phi_1} + c^0_{12} B' & 0
\end{pmatrix}. \] (66)

We made 1-3 and 2-3 rotation further to eliminate the (1,3) and (2,3) elements. Those two angles \( \theta'_{23} \) and \( \theta'_{13} \) are small, and we neglect the quantities in the off-diagonal elements after the rotations. The diagonalization process of \( \check{M}'_{\nu} \) can be illustrated by
\[ \begin{pmatrix}
X & X & \lambda X \\
X & X & \lambda X \\
\lambda X & \lambda X & \lambda X
\end{pmatrix}
\rightarrow U^{(12)}(\theta^0_{12}, \phi_1)
\rightarrow
\begin{pmatrix}
m_1 & 0 & \lambda X \\
0 & m_2 & \lambda X \\
\lambda X & \lambda X & \lambda X
\end{pmatrix}
\rightarrow U^{(13)}(\theta^0_{13}, \phi_2)
\rightarrow
\begin{pmatrix}
m'_1 & \lambda^2 X & \lambda^3 X \\
\lambda^2 X & m'_2 & \lambda X \\
0 & \lambda X & \lambda X
\end{pmatrix}
\rightarrow U^{(23)}(\theta'_{23}, \phi_3)
\rightarrow
\begin{pmatrix}
m'_1 & \lambda^2 X & \lambda^3 X \\
\lambda^2 X & m'_2 & 0 \\
\lambda^3 X & 0 & \lambda X
\end{pmatrix}. \] (67)

\[ ^7 \text{We note that one can obtain an identity equation,}
\]
\[ U^{(23)}(\theta_{23}, \phi_1)U^{(13)}(\theta_{13}, \phi_2)U^{(12)}(\theta_{12}, \phi_3)
= \text{diag}(e^{-i\phi_3}, 1, e^{i\phi_1})U^{(23)}(\theta_{23}, 0)U^{(13)}(\theta_{13}, \phi_2 - \phi_1 - \phi_3)U^{(12)}(\theta_{12}, 0)\text{diag}(e^{i\phi_3}, 1, e^{-i\phi_1}). \] (63)
Because the diagonal elements are modified by the rotation, primes (′) are attached to them.

We remark that the mixing angles \( \theta_{ij} \) of a unitary matrix \( U \) defined by the PDG convention are obtained as

\[
\tan \theta_{23} = \left| \frac{U_{23}}{U_{33}} \right|, \quad \tan \theta_{12} = \left| \frac{U_{12}}{U_{11}} \right|, \quad \sin \theta_{13} = |U_{13}|, \quad (68)
\]

and the CP phase \( \delta \) can be extracted by using Jarlskog invariant

\[
J = U_{12} U_{23} U_{13}^* U_{22}^*.
\]

Expanding Eq.(58), we obtain

\[
\tan \theta_{23} = \frac{c_{13} + e^{i(\phi_1 - \phi_2)} s_{12} s_{13} - e^{-i\phi_3} c_{12} s'_{23}}{c_{13} - e^{i(\phi_1 - \phi_2)} s_{12} s_{13} + e^{-i\phi_3} c_{12} s'_{23}}, \quad (70)
\]

\[
\tan \theta_{12} = \frac{c_{12} - e^{-i(\phi_1 - \phi_2 + \phi_3)} s_{23} l_{12}}{c_{13} - e^{i(\phi_1 - \phi_2)} s_{12} s_{13} + e^{-i\phi_3} c_{12} s'_{23}} = | \tan \theta_{12}^0 + O(\lambda^2) |, \quad (71)
\]

\[
\sin \theta_{13} = | s_{12} s_{23} e^{-i\phi_3} + c_{12} c_{23} s'_{13} e^{i(\phi_1 - \phi_2)} |, \quad (72)
\]

\[
J = -\frac{1}{4} \sin 2\theta_{12} s_{13}' c_{23}' (e^{i\phi_3} s_{12} s_{23}' + e^{i(\phi_2 - \phi_1)} c_{12} c_{23}' s_{13}') + O(\lambda^2). \quad (73)
\]

The approximate expressions are rewritten as

\[
\tan \theta_{23} \simeq \left| \frac{1 + \Delta}{1 - \Delta} \right|, \quad \sin \theta_{13} \simeq | \Theta_{13} |, \quad \delta \approx \pi - \arg \Theta_{13}, \quad (75)
\]

\[
\Delta \equiv (-c_{12} s_{23}' e^{-i\phi_1} + s_{12} s_{13}' e^{i\phi_1}), \quad \Theta_{13} \equiv s_{12} s_{23}' e^{-i\phi_3} + c_{12} c_{23}' e^{i\phi_1}, \quad (76)
\]

where \( \phi_{12} = \phi_1 - \phi_2 \). We note that one obtains by Taylor expansion as

\[
\arctan \left| \frac{1 + \Delta}{1 - \Delta} \right| - \frac{\pi}{4} = \Re \Delta + O(\Delta^3), \quad (77)
\]

and \( O(\Delta^2) \) is absent, and thus, it gives a good approximation up to \( O(\lambda^2) \). Defining

\[
\frac{s'_{13}}{s'_{23}} = \tan \theta'. \quad (78)
\]

we express

\[
\theta_{23} - \frac{\pi}{4} \simeq \Re \left( -c_{12} c' e^{-i\phi_3} + s_{12} s' e^{i\phi_12} \Theta_{13} \right). \quad (79)
\]

\[8\] Under the PDG convention, one obtains

\[
\delta = \arg(J + s_{12}^2 s_{23}^2 s_{13}'^2 c_{13}). \quad (74)
\]

The \( \sim \lambda^2 \) terms of the expansion in Eq.(73) corresponds to the \( s_{12}^2 s_{23}^2 s_{13}'^2 c_{13}^2 \) term, and thus, one finds \( \delta \simeq \arg(-\Theta_{13}) \).
Calculating the real part of the expression, one obtains
\[
\theta_{23} - \frac{\pi}{4} \simeq \frac{s_{12}c_{12}(c^2 - s^2)}{s_{12}^2c^2 + c_{12}^2s^2 + 2s_{12}c_{12}c's'\cos \phi_{123}} \theta_{13} \tag{80}
\]
where \( \phi_{123} = \phi_{12} + \phi_{3} \). If one tune as
\[
\sin 2\theta_{12} \cot 2\theta' \cos \delta + \cos 2\theta_{12} \cos \phi_{123} \cos \delta = \sin \delta \sin \phi_{123}, \tag{81}
\]
\( \theta_{23} \) can be kept to be \( \pi/4 \) irrespective of the CP phases. As a consequence, the CP phase cannot be predicted in general. However, if one wants to avoid such a cancellation between \( A' \) and \( B' \) contributions, the choice of \( \cos \delta \simeq 0 \) and \( \sin \phi_{123} \simeq 0 \) can be a simple solution to avoid a large deviation from \( \pi/4 \). If one of \( A' \) and \( B' \) is zero as a special point, such cancellation is avoided obviously, and a relation among \( \theta_{13}, \delta \theta_{23} \) and \( \delta \) can be obtained. Let us calculate the relation in such cases. For \( A' = 0 \), one obtains \( \tan \theta' = \tan \theta_{12} \). For \( B' = 0 \), one obtains \( \tan \theta' = - \cot \theta_{12} \). Remarking \( m_1 : m_2 \simeq e^{i\phi_1} : -e^{-i\phi_1} \) in the convention we use, one finds \( \phi_{123} = 0 \). As a result, we obtain Eqs.(9) and (10), which can be verified by numerical calculations.

- **NH (rank-2 matrix)**

The reason why the PMNS matrix for NH in Eq.(58) is different from the one in IH given in Eq.(65) is that \( O(\lambda) \) mixing remains in 1-2 block in general and one needs additional 1-2 rotation if the diagonalization is executed by Eq.(65):
\[
\begin{pmatrix}
\lambda X & \lambda X & \lambda X \\
\lambda X & \lambda X & \lambda X \\
\lambda X & \lambda X & 1
\end{pmatrix}
\xrightarrow{U^{(12)}(\theta_{12}^{\prime}, \phi_1)}
\begin{pmatrix}
\lambda X & 0 & \lambda X \\
0 & \lambda X & \lambda X \\
\lambda X & \lambda X & 1
\end{pmatrix}
\xrightarrow{U^{(13)}(\theta_{13}^{\prime}, \phi_2)}
\begin{pmatrix}
\lambda X & \lambda^2 X & 0 \\
\lambda^2 X & \lambda X & \lambda X \\
0 & \lambda X & 1
\end{pmatrix}
\xrightarrow{U^{(23)}(\theta_{23}^{\prime}, \phi_3)}
\begin{pmatrix}
\lambda X & \lambda^2 X & \lambda^3 X \\
\lambda^2 X & \lambda X & 0 \\
\lambda^3 X & 0 & 1
\end{pmatrix}. \tag{82}
\]

If the (1,3) element of \( \tilde{M}_\nu' \), \( \sqrt{2}A'c_{12}^0 - s_{12}^0 e^{i\phi_1 B'} \), given in Eq.(66) is small, such additional 1-2 rotation is not needed, and thus, the PMNS matrix for NH can be also given as in Eq.(65)\(^9\). One can find that the condition \( \sqrt{2}A'c_{12}^0 - s_{12}^0 e^{i\phi_1 B'} = 0 \) is satisfied if the \( \mu-\tau \) symmetric matrix and the total neutrino mass matrix are rank 2. Then, the relation is given by Eq.(80) with \( s_{13}' = 0 \) (\( \tan \theta' = 0 \)). The diagonalization process of \( \tilde{M}_\nu \) for the rank-2 case can be illustrated by
\[
\begin{pmatrix}
\lambda^2 a & \lambda a b & \lambda a c \\
\lambda b a & \lambda b^2 & \lambda b c \\
\lambda a c & \lambda b c & d
\end{pmatrix}
\xrightarrow{U^{(12)}(\theta_{12}^{\prime}, \phi_1)}
\begin{pmatrix}
0 & 0 & 0 \\
0 & \lambda X^2 & \lambda X c \\
0 & \lambda X c & d
\end{pmatrix}
\xrightarrow{U^{(23)}(\theta_{23}^{\prime}, \phi_3)}
\begin{pmatrix}
0 & 0 & 0 \\
0 & \lambda X'^2 & 0 \\
0 & 0 & d'
\end{pmatrix}. \tag{83}
\]

\(^9\) The PMNS matrix in NH case can be also given by Eq.(65) if the (2,3) element of \( \tilde{M}_\nu \) is small. In this case, one finds \( s_{23}' = 0 \) (\( c' = 0, s' = 1 \)) in Eq.(80), and the relation is given as \( \theta_{23} - \pi/4 \simeq -\tan \theta_{12} \sin \theta_{13} \cos \delta \).
where $X = \sqrt{a^2 + b^2}$, and primes attached to the diagonal elements are because of the modification by the last 2-3 rotation. As a result, we find that the relation for rank-2 matrix in NH is given as in Eq.(8). One can verify the relation by numerical diagonalization of the neutrino mass matrix.

We comment on the case of general rank-2 neutrino matrix. The rank-2 matrix can be given by two row vectors $x_1$ and $x_2$ as $M_\nu = x_1^T x_1 + x_2^T x_2$. Without loss of generality (up to unphysical redefinitions), one can parametrize as $x_1 = (e, -d, d)$ and $x_2 = (x, y, z)$, for $e, x, y, z \sim \lambda d$. This can be understood as that the rank-2 symmetric matrix has 5 complex degrees of freedom. One can find that $e \neq 0$ and $y - z \neq 0$ break the $\mu$-$\tau$ exchange symmetry. The PMNS matrix for the rank-2 matrix can be written as

$$U_{\text{PMNS}} = U^{(23)}(-\frac{\pi}{4}, 0)U^{(13)}(\theta'_{13}, \phi_1)U^{(12)}(\theta_0^0, \phi_2)U^{(23)}(\theta'_{23}, \phi_3).$$  \hfill (84)  

By the $U^{(23)}(-\frac{\pi}{4}, 0)U^{(13)}(\theta'_{13}, \phi_1)$ rotation, $x_1 \rightarrow x'_1 \propto (0, 0, 1)$. The $U^{(12)}(\theta_0^0, \phi_2)$ rotation makes the 1st element of $x_2$ to be zero. Finally, $U^{(23)}(\theta'_{23}, \phi_3)$ diagonalize the remained $2 \times 2$ matrix. Expanding the expression similarly to the previous case, one finds

$$\theta_{23} - \frac{\pi}{4} \simeq \theta_{13} \frac{c_{12} s'_{23} (s_{12} s'_{23} \cos \delta + s'_{13} \cos(\phi_{123} - \delta))}{s_{13}^2 + s_{12}^2 s_{23}^2 + s_{13}^2 s_{12}^2 s_{23}^2 \cos \phi_{123}}.$$  \hfill (85)  

One can find that $\theta'_{13} = 0$ if $e = 0$, and then, Eq.(8) holds.

**Appendix B: Comparison with the $\mu$-$\tau$ reflection symmetry**

We comment on the $\mu$-$\tau$ reflection symmetry considered in [22]. Lagrangian is assumed to be invariant under $\nu_e \leftrightarrow \nu_e^*$, $\nu_\mu \leftrightarrow \nu_\mu^*$, and thus, the neutrino mass matrix is given as

$$M_\nu = \begin{pmatrix} W & X & X^* \\ X & Y & Z \\ X^* & Z & Y^* \end{pmatrix},$$  \hfill (86)  

where $W$ and $Z$ are real. By the field redefinitions $\nu_\mu \rightarrow \nu_\mu \eta$, $\nu_\tau \rightarrow \nu_\tau \eta^*$, $\eta^2 = e^{-i \arg Y}$, one obtains $Y \rightarrow |Y|$, $X \rightarrow X \eta$. Therefore, without loss of generality, $Y$ can be also considered to be real and only $X$ is complex. Comparing with Eqs.(3) and (6), one finds

$$U_0^T P_\eta M_\nu P_\eta^* U_0 = \begin{pmatrix} W & \sqrt{2} \text{Re}(X \eta) & -i \sqrt{2} \text{Im}(X \eta) \\ \sqrt{2} \text{Re}(X \eta) & |Y| + Z & 0 \\ -i \sqrt{2} \text{Im}(X \eta) & 0 & |Y| - Z \end{pmatrix}, \quad P_\eta = \text{diag}(1, \eta, \eta^*),$$  \hfill (87)  

and $A = \text{Re}(X \eta)$, $A' = -i \text{Im}(X \eta)$, $B' = 0$. Because $A'$ is pure imaginary and the other elements are real, one easily finds $\delta = \pm \pi/2$ and $\theta_{23} = \pi/4$. The $\mu$-$\tau$ reflection symmetry is
a kind of “CP” transformation property to make $\delta = \pm \pi/2$. On the other hand, we do not assume any CP property under the $\mu$-$\tau$ exchange symmetry. Even without the CP property, we find that a large CP violation is preferable in the neutrino mass matrix with the $\mu$-$\tau$ exchange symmetry breaking to keep the 2-3 mixing angle to be nearly maximal, as explained in the text. In fact, the configuration of $\mu$-$\tau$ reflection symmetry is a special choice of the $\mu$-$\tau$ exchange symmetry breaking. One can easily construct a model with $\mu$-$\tau$ reflection symmetry, if a scalar field, which breaks the $\mu$-$\tau$ exchange symmetry, also breaks CP symmetry spontaneously.

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