Local Distributional Chaos

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Abstract
Distributional chaos was introduced in Schweizer and Smítal (Trans Am Math Soc 344:737–754, 1994) for continuous maps of the interval, as chaotic behavior based on development of distances between the orbits of points in the system. In Balibrea et al. (Chaos Solitons Fractals 23(5):1581–1583, 2005), this phenomenon was generalized to continuous maps of compact metric space and was distinguished into three different forms, chaos DC1, DC2 and DC3. In Loranty and Pawlak (Chaos 29:013104, 2019), the local idea of such behavior is studied, which leads to the definition of distributionally chaotic points (DC-points). It is also proved in Loranty and Pawlak (2019), that for interval maps, positive topological entropy implies existence of DC1-point. In this paper this result for interval maps is strengthened; it is proved that positive topological entropy implies existence of an uncountable set of DC1-points, and moreover this set can be chosen perfect. In greater dimensions than one, we deal with triangular systems on \( I^2 \). In this case the relationship between topological entropy and different types of distributional chaos is not clearly understood and several different results are possible. In the paper we use an example of map \( F \) given by Kolyada (Ergod Theory Dyn Syst 12:749–768, 1992) to prove that the corresponding two dimensional system \( (I^2, F) \) has positive topological entropy but without containing DC2-points, proving that there is no concentration of DC2-chaos.

Keywords Distributional chaos · Symbolic space · Chaos for interval maps · Chaos for triangular maps

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1 Introduction

Distributional chaos is a notion which studies behavior of orbits of a discrete dynamical system \((X, f)\) where \(X\) is the phase space and \(f \in C(X)\), using the properties of some density functions of probability generated by the values of the distances between the orbits (of pointwise form) of pairs of points belonging to certain subsets of the phase space. We wonder if it is possible that these types of chaos can concentrate around points of the phase space developing a local idea of this behavior. Answering this question, in 2019, the new notion of DC1-points was introduced by A. Loranty and R. Pawlak (see [9]). Their definition can be generalized to DC\(i\)-points, where \(i = 1, 2, 3\). Such special points are those in which distributional chaos of the different types concentrate and these behave like attracting points. In this paper we continue their research, strengthen one of their results and generalize the presumed spaces.

One of the main results of [9] states that if \(f\) is a continuous interval map with positive topological entropy \((h(f) > 0)\), then there is at least one DC1-point. In this paper we have tried to improve the result in the sense of better understanding of existence and generation of DC1-points. We wonder what is the situation in the symbolic spaces \((\Sigma, \sigma)\) where \(\sigma\) is the shift map. We obtain the following result: 

1) Every point of the symbolic system (on two symbols) \((\Sigma, \sigma)\) is a DC1-point. Such result is the key point to obtain the following result for \(f \in C(I)\): 

2) If the topological entropy of a system is positive, then the system has an uncountable set of DC1-points. Moreover such set can be chosen perfect, that is a Cantor type set. This result gives us an idea about the generation of DC1-points in \(I\).

Now we think that it will be important to try to understand the existence and production of DC\(i\)-points when \(X\) is not one dimensional. In previous literature there are a lot of papers devoted to systems \((I^2, F)\) where \(I^2\) is a square and \(F\) is a triangular map. A part of their studies has been devoted to cases with positive topological entropy. Our feeling is that the problem of getting DC\(i\)-points as a result is difficult and the unique progress we have made is the construction of one example. We have learned that in a general triangular systems we can have different results.

In the paper we use an example of map \(F\) given by Kolyada in 1992 obtaining the following result: 

3) There is a triangular system \((Q, F)\) with positive topological entropy, but containing no DC2-points.

Applying an important result by Downarowicz, the former system is DC2, that is, there is an uncountable set composed of DC2-pairs, but there is no concentration of this behavior. We have not been able to report anything concerning the existence of DC3-points. But the continuation of this research using the adequate tools would be interesting, it would make us know and understand more about the theory of chaos. Of course stating the above problems for systems of the form \((I^2, G)\) for a general continuous transformation \(G(x, y) = (f(x, y), h(x, y))\) is part of the future of the research.
2 Preliminaries

In this section we recall notions we are using. Throughout the paper we use standard notation. Let \((X, d)\) be compact metric space with metric \(d\) and \(f : X \to X\) a continuous map. Then \(\text{diam}(A) = \max\{d(x, y); x, y \in A\}\) for \(A \subseteq X\). We use \(f^n\) to denote \(n\)-th iteration of \(f\), which is \(f^n = f^{n-1} \circ f\), where \(n \in \mathbb{N}\) (and \(f^0\) is the identity map). By \(C(X)\) (or \(C(I)\)) we denote the set of all continuous maps from compact space \(X\) to itself (or of unit interval \(I\) to itself).

The notion of topological entropy we are using is that introduced by Adler, Konheim and McAndrew (see [1]) defined for continuous maps on topological space \((X, \tau)\), where \(\tau\) is its suitable topology.

We say that a system \((X, f)\) possesses a horseshoe (or \(n\)-horseshoe), if there are nonempty and non-degenerate sets \(J_1, \ldots, J_n\) with pairwise disjoint interiors such that \((J_1 \cup \cdots \cup J_n) \subset f(J_i)\) for all \(i = 1, \ldots, n\). If sets \(J_1, \ldots, J_n\) are disjoint, we call it a strict horseshoe.

Further we proceed to repeating the definition of distributional chaos, see e.g. [2]. Given a metric \(d\), for any positive integer \(n\), real number \(t\) and pair of points \(x, y\), we denote by \(\xi(x, y, t, n)\) the number of members of the set \(\{d(x, y), d(f(x), f(y)), d(f^2(x), f^2(y)), \ldots, d(f^{n-1}(x), f^{n-1}(y))\}\) which are less than \(t\). Clearly \(\xi(x, y, t, n) = 0\) for \(t \leq 0\), while for \(t > 0\), \(\xi(x, y, t, n) \in \{0, 1, \ldots, n\}\). Then we introduce the function \(\Psi_{xy}^n(f, t)\) associate to \(f\) as \(\Psi_{xy}^n(f, t) = \frac{1}{n} \xi(x, y, t, n)\) for map \(f\). Function \(\Psi_{xy}^n(f, t)\) is non-decreasing and its values are from interval \([0,1]\). Next, let

\[
\Psi_{xy}(f, t) = \liminf_{n \to \infty} \Psi_{xy}^n(f, t) \quad \text{and} \quad \Psi_{xy}^*(f, t) = \limsup_{n \to \infty} \Psi_{xy}^n(f, t)
\]

be lower and upper distributional function of \(f\) with respect to points \(x, y\). It is easy to see that \(\Psi_{xy}(f, t) \leq \Psi_{xy}^*(f, t)\) for all \(t \in \mathbb{R}\).

We say that pair of points \(x, y\) form a

- DC1-scrambled pair of \(f\) (or just DC1 pair), if \(\Psi_{xy}^*(f, t) \equiv 1\) and \(\Psi_{xy}(f, t) = 0\) for some \(t > 0,\)
- DC2-scrambled pair of \(f\) (or just DC2 pair), if \(\Psi_{xy}^*(f, t) \equiv 1\) and \(\Psi_{xy}(f, t) < \Psi_{xy}^*(f, t)\) for all \(t > 0,\)
- DC3-scrambled pair of \(f\) (or just DC3 pair), if \(\Psi_{xy}(f, t) < \Psi_{xy}^*(f, t)\) for all \(t > 0,\)

Map \(f\) is called DCi-scrambled (or simply DCi), if there exists an uncountable subset \(S \subseteq X\), where all pairs of points are DCi-scrambled pairs. Set \(S\) is called DCi-scrambled set.

Clearly, DC3 chaos is weaker than DC2 and DC2 chaos is weaker than DC1. It is known, for example, that positive topological entropy in general dynamical system implies DC2, but not DC1 (see [4, 13]). In the class of continuous maps of the interval, DC1, DC2 and DC3 are proven to be equivalent [2, 11]. There are also other versions of distributional chaos appearing in the theory of linear operators (see [3]).

Next definition is an extension of the one taken from [9] and introduces the notion of DCi-points.
Definition 1 [9] Let \((X, f)\) be a compact metric space. Point \(x_0\) is called a DCi-point of \(f\), for \(i = 1, 2, 3\), if for every \(\varepsilon > 0\) there are an uncountable set \(S_\varepsilon\), integer \(n(\varepsilon)\) and a closed set \(A_\varepsilon \supseteq S_\varepsilon\), such that \(S_\varepsilon\) forms a DCi-scrambled set for \(f\) and \(A_\varepsilon \subset f^{n(\varepsilon)}(A_\varepsilon) \subset B(x_0, \varepsilon)\).

The set \(A_\varepsilon\) is called an envelope of the system \((X, f)\) where the neighbourhood \(B(x_0, \varepsilon)\) depends on the chosen metric in \(X\). In the case of interval map we will use common term DC-point for all three equivalent versions DC1, DC2, DC3.

A triangular map (sometimes also called skew-product maps usually when we are describing situations on spaces which are not of coordinates) on the square \([0, 1]^2 = I^2\) into itself is a continuous map \(F : I^2 \to I^2\) of the form \(F(x, y) = (f(x), g(x, y)) = (f(x), g_x(y))\), where \(I = [0, 1]\) and \(g_x : I \to I\) is a family of maps depending continuously on \(x\). The map \(f\) of the corresponding dynamical system is called the base for \(F\) and the maps \(g_x\) are called the fibre maps for all \(x \in I\).

3 DC-Points for the Interval Maps

The goal of this section is to strengthen the result from [9] that every positive entropy interval map possesses at least one DC-point. Here we show that in this case there is not only one DC-point, but uncountably many of them and moreover, their set can be chosen perfect (Cantor). We start with recalling that symbolic system is DC1 and is not only one DC-point, but uncountably many of them and moreover, their set can be the one-sided shift map defined as \(\sigma(a_1a_2a_3\ldots) = (a_2a_3\ldots)\) where \((a_1a_2\ldots) \in \Sigma\). It is known that system \((\Sigma, \sigma)\) contains (at least one) uncountable DC1-scrambled set. Its construction can be found for example in [11] (in the proof of Theorem 6.26). Let’s denote such set by \(S \subset \Sigma\).

Lemma 1 Every point of symbolic system \((\Sigma, \sigma)\) is a DC1-point.

Proof Equip the space \(\Sigma\) with the metric \(\rho\), where \(\rho(x, y) = \frac{1}{i}\) and \(i \in \mathbb{N}\) is the first position where symbols in \(x, y\) differ. Given \(B(x_0, \varepsilon)\), let \(m(\varepsilon)\) denote the lowest integer such that for any point \(x \in B(x_0, \varepsilon)\), sequences \(x\) and \(x_0\) coincide on (at least) first \(m(\varepsilon)\) positions.

We choose point \(x_0 \in \Sigma\) and fix it. If we want to show that \(x_0\) is a DC1-point, by definition we need to show that for every \(\varepsilon > 0\) there are: uncountable DC1-scrambled set \(S_{x_0, \varepsilon}\), integer \(n(x_0, \varepsilon)\) and a closed set \(A_{x_0, \varepsilon} \supseteq S_{x_0, \varepsilon}\) such that \(A_{x_0, \varepsilon} \subset \sigma^{n(x_0, \varepsilon)}(A_{x_0, \varepsilon}) \subset B(x_0, \varepsilon)\).

We use notation \(x_0|_{n(x_0, \varepsilon)}\) for finite block of first \(n(x_0, \varepsilon)\) symbols of sequence \(x_0\). For any \(\varepsilon > 0\) we set \(n(x_0, \varepsilon) := m(\varepsilon), S_{x_0, \varepsilon} := \{x_0|_{n(x_0, \varepsilon)}\} \in \Sigma, \forall y \in S\) and \(A_{x_0, \varepsilon} := \{x_0|_{n(x_0, \varepsilon)}\} \times \{x_0|_{n(x_0, \varepsilon)}\} \times \forall x \in \Sigma\) This makes \(\sigma^{n(x_0, \varepsilon)}(A_{x_0, \varepsilon}) = \{x_0|_{n(x_0, \varepsilon)}\} \times \forall x \in \Sigma\).

Obviously \(S_{x_0, \varepsilon} \subset A_{x_0, \varepsilon} \subset \sigma^{n(x_0, \varepsilon)}(A_{x_0, \varepsilon}) = B(x_0, \varepsilon)\). Moreover \(S_{x_0, \varepsilon}\) forms an uncountable DC1-scrambled set of \(\Sigma\), because \(S\) is an uncountable DC1-scrambled set of \(\Sigma\) and \(A_{x_0, \varepsilon}\) is closed set, because limit point of sequence of points of type \(x_0|_{n(x_0, \varepsilon)}\) \(x_0|_{n(x_0, \varepsilon)}\) \(x\) is again point of type \(x_0|_{n(x_0, \varepsilon)}\) \(x_0|_{n(x_0, \varepsilon)}\) \(x\). \(\square\)
Theorem 1 Let \( f : [0, 1] \to [0, 1] \) be a continuous map such that \( h(f) > 0 \). Then the system \( ([0,1],f) \) has an uncountable set of DC1-points. Moreover this set can be chosen perfect.

Proof We start with the Misiurewicz’s result that the positive topological entropy of \( f \) implies existence of a horseshoe for some iteration of \( f \) (the result can be found for example as Theorem 4.7 in [11]). Let us denote such iteration by \( f^k \) and for simplicity \( g := f^k \). From now on we work with map \( g \) and we will get back to \( f \) in the end of the proof.

Map \( g \) possesses a horseshoe, so we get sets \( J_0, J_1 \subset I \) such that \( (g(J_0) \cap g(J_1)) \supset (J_0 \cup J_1) \). By analysis similar to that from Proposition 5.15 from [11], we choose a Cantor set in \( I \). The construction is following: we know that \( g(J_0) \supset (J_0) \) and \( g(J_0) \supset (J_1) \). By Lemma 1.13(i) from [11] there are sets \( J_{00} \subset J_0 \) and \( J_{01} \subset J_0 \) such that \( g(J_{00}) = J_0 \) and \( g(J_{01}) = J_1 \). Similarly we can find subsets \( J_{10}, J_{11} \) of \( J_1 \), which map by \( g \) to \( J_0 \) or \( J_1 \), respectively. By Lemma 1.13(i) it is also true that sets \( J_{00}, J_{01}, J_{10}, J_{11} \) are disjoint, their boundaries map on boundaries of \( J_0 \) or \( J_1 \) and their interiors on interiors of \( J_0 \) or \( J_1 \).

Now by \( J_{00}, J_{01} \subset J_0 \) and \( g(J_{00}) = J_0 \) we get \( g(J_{00}) \supset J_0 \) and \( g(J_{01}) \supset J_0 \). By application of Lemma 1.13 on these chains, we get subsets \( J_{00}, J_{01} \subset J_0 \) such that \( g(J_{000}) = J_0 \) and \( g(J_{001}) = J_0 \). Similarly we create sets \( J_{0i1} \subset J_0, i = 0, 1 \) where \( g(J_{0i1}) = J_{1i}, i = 0, 1 \) and after application of the same process to \( J_{10}, J_{11} \) we get subsets \( J_{10i} \) and \( J_{11i} \) for \( i = 0, 1 \). Note that interiors still map to interiors and boundaries to boundaries.

By additional applications of Lemma 1.13(i) from [11], we get system of sets

\[
J_{\bar{\alpha}} = \bigcap_{n=1}^{\infty} J_{\alpha_1 \ldots \alpha_n}, \quad \bar{\alpha} = \alpha_1 \alpha_2 \ldots \in \Sigma.
\]

As in Proposition 5.15 from [11], we created an invariant Cantor set \( X \subset I \) and also there is a continuous map \( \varphi : X \to \Sigma \) such that \( \varphi \) is a semi-conjugacy between \( (X, g|_X) \) and \( (\Sigma, \sigma) \) (in particular \( (\varphi \circ g)(x) = (\sigma \circ \varphi)(x) \) for all \( x \in X \)), the system \( (X, g|_X) \) is transitive and there exists countable set \( E \subset X \) such that \( \varphi \) is one-to-one on \( X \setminus E \) and two-to-one on \( E \).

By Lemma 1, every point of \( (\Sigma, \sigma) \) is a DC-point. As in its proof, we denote the maximal uncountable DC1-scrambled set in \( (\Sigma, \sigma) \) by \( S_{\Sigma, \sigma} \). Given \( \beta \in \Sigma \) and \( \varepsilon > 0 \) define the transformation \( \psi_{\alpha, \varepsilon} : \Sigma \to \Sigma \) as

\[
\psi_{\alpha, \varepsilon}(x) = \bar{\alpha}|_{n(\bar{\alpha}, \varepsilon)} \bar{\alpha}|_{n(\bar{\alpha}, \varepsilon)} x \quad \text{for all} \quad x \in \Sigma.
\]

The transformation maps all space \( \Sigma \) to \( A_{\bar{\alpha}, \varepsilon} \) and DC1-scrambled set \( S \) into \( S_{\bar{\alpha}, \varepsilon} \). Moreover, map \( \psi_{\alpha, \varepsilon} \) is continuous for every \( \beta \in \Sigma \) and \( \varepsilon > 0 \), because for arbitrary points \( x \neq y \in \Sigma \), the images \( \psi_{\alpha, \varepsilon}(x), \psi_{\alpha, \varepsilon}(y) \) have \( 2n(\bar{\alpha}, \varepsilon) \) more common first spaces than \( x, y \). As \( \varphi \) maps one point from \( X \) to one point from \( \Sigma \), \( \psi_{\alpha, \varepsilon} \) maps one point in \( \Sigma \) to one point in \( \psi_{\alpha, \varepsilon}(\Sigma) \).

Let us take the point \( x_0 = \varphi^{-1}(\bar{\alpha}) \subset X \setminus E \). We considered \( S_{\bar{\alpha}, \varepsilon} \) to be a DC1-scrambled set of \( \sigma \). Moreover, as stated earlier, set \( E \subset X \) where \( \varphi \) acts as a two-to-one, is countable. Consequently, the set \( S_{\bar{\alpha}, \varepsilon} \setminus \psi(E) \) is still uncountable and map \( \varphi|_{(X \setminus E)} \) is a conjugacy between \( X \setminus E \) and \( \varphi(X \setminus E) \). It is known that conjugacy preserves the distributional chaos (see [13]), hence (together with continuity of \( \varphi \)) for each \( \varepsilon > 0 \).
there is $\varepsilon' > 0$ such that $S_{x_0, \varepsilon'} := \varphi^{-1}(S_{\alpha, \varepsilon}) \setminus E = \varphi^{-1}(\psi_{\alpha, \varepsilon}(S)) \setminus E$ forms a DC1-scrambled set of $g$. It is easy to see that $S_{x_0, \varepsilon'} \subset B(x_0, \varepsilon')$ and as a suitable envelope we can take $A_{x_0, \varepsilon'} := \varphi^{-1}(\psi_{\alpha, \varepsilon}(\Sigma))$. This makes $x_0$ a DC1-point of map $g$. Consequently the set $X \setminus E$ is an uncountable set of DC1-points of $g$ and by the above construction it is a Cantor set.

As a last step we want to show that $X \setminus E$ is also uncountable set of DC1-points for $f$ (remember that $g = f^k$). We fix an arbitrary pair $x, y \in X \setminus E$ and $t' > 0$. Naturally, $\Psi_{xy}(g, t') < 1$ implies $\Psi_{xy}(f, t') < 1$. For showing $\Psi_{xy}^*(f, t) = 1$ for every $t \in (0, 1)$ we use continuity of $f$. For any $t_1 > 0$ there is $t_2 > 0$ such that $d(x, y) < t_2$ implies $d(f^i(x), f^i(y)) < t_1$ for $i = 0, 1, \ldots, k-1$. Since $x, y \in X \setminus E$ is distributionally scrambled pair for $g$, we have $\Psi_{xy}^*(g, t_2) = 1$ for every $t_2 \in (0, 1)$. Then, by continuity we also get $\Psi_{xy}^*(f, t_1) = 1$ for all $t_1 \in (0, 1)$, as desired. Finally, since $x, y$ and $t'$ was arbitrary, we showed that $X \setminus E$ is uncountable set of DC1-points of map $f$, which finishes the proof.

\[\square\]

4 DC-Points in More General Spaces

In this section we study existence of DC$i$-points in higher classes of maps, general compact metric spaces with continuous maps. There is no longer equivalence between various types of distributional chaos.

Generalization of the result from Sect. 3 for DC1-points is not possible here without supposing some additional properties, because there is generally no relation between positive topological entropy and chaos DC1, see [8, 10]. On the other hand, Downarowicz proved in [4] that positive topological entropy implies chaos DC2 (hence also DC3). Despite this result, positive topological entropy does not imply existence of a single DC2-point, even in the class of triangular maps of the square. In the paper by Kolyada [6], there appears a construction of a whole class of continuous triangular maps of type $2^\infty$ with positive topological entropy. Later, it was showed by Smítal and Štefánková [13] that these maps have no DC1-scrambled pair, but there is uncountable DC2-scrambled set. We use this construction to show that there is no DC2-point in any of the maps.

By Definition 1, the envelope $A_\varepsilon$ of a DCi-point $x_0$ must satisfy the inclusions $A_\varepsilon \subset f^{n(\varepsilon)}(A_\varepsilon) \subset B(x_0, \varepsilon)$. We prove, that in the construction by Kolyada, no possible envelope can satisfy the first inclusion, while it is possible to satisfy the second one. It could be discussed whether the first inclusion is necessary in the definition, or how it can be replaced. But in this paper we omit such discussion and we take the definition as presented in [9].

First we recall the Kolyada’s construction and then, with help of Lemma 2 we prove Theorem 2, the main result of this section.

Example 1 We recall the construction of class of triangular maps $F(x, y) = (f(x), g_x(y))$ from Theorem 10 in [6], for more details see the cited paper.

Let base map $f$ be the logistic function $f(x) = \lambda x(1 - x)$, where $\lambda$ is irrational number $\lambda = 3.569 \ldots$, such that $f$ is of type $2^\infty$. Such map is known to contain a
system of intervals $J^n_k$, where $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, 2^n - 1$ such that:

the $J^n_k$ with fixed $n$ are mutually disjoint, \( f(J^n_k) = J^n_{k+1} \) for $k = 0, \ldots, 2^n - 2$ and $f(J^n_{2^n-1}) \subseteq J^n_0$, \( \quad (3) \)

every interval $J^{n-1}_k$ contains exactly two intervals $J^n_k, J^n_{k+2^{n-1}}$ and
the set $J^{n-1}_k \setminus (J^n_k \cup J^n_{k+2^{n-1}})$ contains periodic point of type $2^n - 1$, \( \quad (4) \)

\[
\lim_{n \to \infty} \max_{0 \leq k \leq 2^n - 1} |J^n_k| = 0 \quad \text{and} \quad Q = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} J^n_k \text{ is a minimal Cantor set.} \quad (5)
\]

Let $y \in I$. Fibre maps $g_x(y) \equiv 0$ for $x = 0, \frac{1}{2}, 1$ and in all periodic points of $f$. On set $Q$, $g_x$ are tent maps divided by different powers of $2$. In particular if $\tau(y) = 1 - |1 - 2y|$ is a tent map, we set

\[
g_x(y) = \frac{\tau(y)}{2^{i-1}} \quad \text{for} \quad x \in Q \cap J^n_{2^{n-1}}, \quad (7)
\]

where $n \in [n_i, n_i + 1 - 1]$ and sequence $\{n_i\}_{i=1}^{\infty}$ is to be defined in the next paragraph. The remaining fibre maps are filled also in by tent maps divided by suitable real numbers chosen in such a way that heights of $g_x(I)$ increase or decrease linearly with respect to $x$, which secures the continuity of $F$.

Sequence $\{n_i\}_{i=1}^{\infty}$ must satisfy the conditions $n_1 = 1$ and

\[
\sum_{i=1}^{\infty} \frac{1}{2n_i} < 1, \quad (8)
\]

so it grows faster than sequence of integers. With various choice of sequences $\{n_i\}$ we get whole class of triangular maps, which is denoted by $\mathcal{F}$. With given conditions for $\{n_i\}$ satisfied, every map $F \in \mathcal{F}$ has positive topological entropy (see [6, 13]) and there is no DC1-scrambled pair [13]. On the other hand there exists an uncountable DC2-scrambled set in each fibre above point of $Q$. Originally, in [13] is shown the existence of one DC2-scrambled pair, but by Downarowicz’s result [4], there can be found a whole uncountable DC2-scrambled set, which can be chosen as a Cantor set.

Denote by $I_x$ the “fibre” in point $x$, i.e. the set $\{x\} \times I$. For any $x \in I$ and any $F \in \mathcal{F}$, let $R(F, x, j)$ be the range of $F^j$ restricted to the fibre $I_x$. Also let denote by $K_0 = \{x \in I; g_x = 0\}$ and by $K$ the union of all both-sided $f$–orbits of points from $K_0$. Set $K$ is a countable subset of $I$.

**Lemma 2** Given $\{n_i\}$ and $F \in \mathcal{F}$ as in the previous example, for any point $x \in (Q \setminus K)$:

(i) $\limsup_{j \to \infty} R(F, x, j) = 1$,
(ii) $\liminf_{j \to \infty} R(F, x, j) = 0$.

**Proof** As was mentioned, the sequence $\{n_i\}_{i=1}^{\infty}$ is bounded from below by sequence of integers. Let’s now suppose the case of triangular map $F$ ($\not\in \mathcal{F}$) where $n_i = i$
and point \((x_0, 0)\) such that \(x_0 \in Q \setminus K\) and \(g_{x_0}\) is full tent map (such \(x_0\) exists, because \(n_1 = 1\)). By (7), starting at \(x_0\), we apply fibre maps in order \(\tau, \frac{\tau}{2}, \tau, \frac{\tau}{4}, \tau, \frac{\tau}{8}, \tau, \frac{\tau}{2}, \tau, \frac{\tau}{4}, \tau, \frac{\tau}{8}, \tau, \frac{\tau}{16}, \tau, \ldots\). Taking small set \(\{x_0\} \times [0, \frac{1}{2^j}]\) for some \(j\), every full tent map spreads its length 2 times, while every application of \(\frac{\tau}{2^j}\) shrinks the vertical length \(2^{j-1}\) times. It is easy to count that before we get to application of map \(\frac{\tau}{2^m}\) for chosen \(m\), our set spreads its vertical length \(2^m\)-times (or it reaches the full length first).

Now we take sequence \(\{n_i\}\) satisfying (8), as desired. In that case the vertical length of set \(\{x_0\} \times [0, \frac{1}{2^j}]\) will grow even faster. The same estimate can be done for any point \(x_0 \in Q \setminus K\) (not necessarily starting from fibre with full tent map), which proves (i).

To prove (2) it is enough to realize that the vertical length of every \(\{x_0\} \times [0, \frac{1}{2^j}]\) can maximally spread to full length 1 and periodically, maps \(\frac{\tau}{2^j}\) for all \(j = 1, 2, \ldots\) are applied. Every such map shrinks the length of the set to \(\frac{1}{2^k}\), which for \(k \to \infty\) gives (ii).

**Theorem 2** There is a triangular map \(F : I^2 \to I^2\) with positive topological entropy, hence is DC2, but containing no DC2-point.

**Proof** Suppose the class \(\mathcal{F}\) from Example 1. We will try to find sequence \(\{n_i\}\) with corresponding map \(F\) and a DC2-point \(x_{DC} = (x_0, y_0)\) for \(F\). By the Definition 1, \(x_{DC}\) is DC2-point, if for every \(\varepsilon > 0\) there are distributionally scrambled set \(S_{x_{DC}, \varepsilon}\) (of type DC2), integer \(n(x_{DC}, \varepsilon)\) and envelope \(A_{x_{DC}, \varepsilon} \supset S_{x_{DC}, \varepsilon}\) such that

\[
A_{x_{DC}, \varepsilon} \subseteq F^n(x_{DC}, \varepsilon)(A_{x_{DC}, \varepsilon}) \supseteq B(x_{DC}, \varepsilon).
\]  

Let us gather some facts about our example:

(1) Coordinate \(x_0 \in Q\), because \(Q \times I\) is the support of topological entropy of \(F\) (see [6], or [13]).

(2) If point \(x \in Q\) lies in the both-sided \(f\)-orbit of a point where the fibre map equals zero, then \(R(F, x, j) = 0\) eventually. Therefore \(x_0 \in Q \setminus K\).

(3) Possible DC2-scrambled sets are always contained in the single fibers \(I_x\), because different points of \(Q\) are contained in different \(I_k\)'s, which keep their distance from each other when iterated.

(4) On the other hand, the envelopes cannot be subsets of the single fibers, because every fiber gets arbitrarily close to itself by \(f\)-iterating, but never exactly on itself.

(5) If \(y_0 \neq 0\), \(x_{DC}\) is not a DC2-point. Regular applications of maps \(\frac{\tau}{2^i}\) for high \(i\) shrink possible envelopes arbitrarily close to zero, which makes them not invariant. Therefore \(y_0 = 0\). [(1)]

Consequently \(x_{DC} = (x_0, 0)\), \(x_0 \in Q \setminus K\), \(S_{x_{DC}, \varepsilon} \subseteq I_{x_0}\) and \(A_{x_{DC}, \varepsilon} \subseteq (J_{k}^m \cap Q) \times [0, \varepsilon]\) for some \(k\) and \(m\). For every \(\varepsilon > 0\) there are unique numbers \(m \in \mathbb{N}\) such that \(\frac{1}{2^m} \leq \varepsilon < \frac{1}{2^{m-1}}\) and \(k_0\) such that \(J_{k_0}^m\) contains \(x_0\). Let us fix them. Clearly \(A_{x_{DC}, \varepsilon}\) can be chosen closed and such that \(S_{x_{DC}, \varepsilon} \subseteq A_{x_{DC}, \varepsilon}\). Envelope \(A_{x_{DC}, \varepsilon}\) will be "jagged" from above. Indeed, fibers in \((J_{k_0}^m \cap Q \cap K) \times [0, \varepsilon]\) can only consists of points \((J_{k_0}^m \cap Q \cap K) \times [0, \varepsilon]\), otherwise the envelope would not be invariant.

As \(J_{k_0}^m\) “travels” through different \(J_{k_0}^m\)'s, as described in (7), different fiber maps \(\frac{\tau}{2^i}\) are applied. Since there are \(2^m\) portions \(J_k^m\) \((k = 0, \ldots, 2^m - 1)\) and infinitely
many maps \( \frac{r}{2^n} \) acting on them (sequence \( \{n_i\} \) is increasing), there is one set \( J^m_k \) and number \( P \in \mathbb{N} \), such that all maps \( \frac{r}{2^n}, j > P \) are acting in points of \( J^m_p \). Obviously the higher \( j \), the smaller subportion of \( J^m_k \) where map \( \frac{r}{2^n} \) is acting on.

The portion \( J^m_k \) is \( f \)-periodic with period \( 2^m \), and by (6), \( J^m_k \cap Q \) is also \( f \)-periodic with period \( 2^m \). Also any subportion \( J^{m+i}_k \subset J^m_k \) for arbitrary \( i \in \mathbb{N} \) is periodic with period \( 2^{m+i} \).

To make the \( y \)-axis of \( A_{x_{DC},\varepsilon} \subset (J^m_k \cap Q) \times \{0, \frac{1}{2^m}\} \) also shrink back to \( B(x_{DC}, \varepsilon) \) with period of suitable power of 2 it is enough to choose \( x_0 \) from such subportion \( J^{m+i}_k \subset f(J^m_k) \) that in points of \( f^{-1}(J^{m+i}_k) \) only maps \( \frac{r}{2^n} \) or lower are applied (see (ii) from Lemma 2). In such case the vertical length of fibres of \( A_{x_{DC},\varepsilon} \) will sufficiently shrink at iteration \( 2^m \) and the chosen subportion \( J^{m+i}_k \) is periodic with period \( 2^{m+i} \).

Consequently \( F^{2^m+i}(A_{x_{DC},\varepsilon}) \subset B(x_{DC}, \varepsilon) \).

On the other hand it is impossible to satisfy the first inclusion in (9). Exactly once in every \( 2^m \) iterations, \( J^m_k \) maps to \( J^m_p \), where all fiber maps \( \frac{r}{2^n} \), \( j > P \) are applied. Therefore for any \( n \in \mathbb{N} \) always there is a part of \( A_{x_{DC},\varepsilon} \) such that the fibers in \( F^n(A_{x_{DC},\varepsilon}) \) are shorter than the original ones in \( A_{x_{DC},\varepsilon} \). \( \square \)

It is worth mentioning that triangular map from [6] can be linearly extended to the triangular map of the square, see [6]. So in this class of maps, positivity of topological entropy itself does not assure existence of a DC2-point. But it might be possible assuming some additional properties on the map, which prompts us to state the following open problem:

**Open problem:** Under which additional (and reasonable) properties on triangular map of the square would positivity of topological entropy imply existence of a DC2-point?

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