Noether Gauge Symmetry of Modified Teleparallel Gravity Minimally Coupled with a Canonical Scalar Field

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Abstract: This paper is devoted to the study of Noether gauge symmetries of $f(T)$ gravity minimally coupled with a canonical scalar field. We explicitly determine the unknown functions of the theory $f(T), V(\phi), W(\phi)$. We have shown that there are two invariants for this model, one of which defines the Hamiltonian $H$ under time invariance (energy conservation) and the other is related to scaling invariance. We show that the equation of state parameter in the present model can cross the cosmological constant boundary. The behavior of Hubble parameter in our model closely matches to that of $\Lambda$CDM model, thus our model is an alternative to the later.

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I. INTRODUCTION

Observational data from Ia supernovae (SNIa) show that currently the observable Universe is in accelerated expansion phase [1]. This cosmic acceleration has also been confirmed by different observations of large scale structure (LSS) [2] and measurements of the cosmic microwave background (CMB) anisotropy [3]. The essence of this cosmic acceleration backs to “dark energy”, an exotic energy which generates a large negative pressure, whose energy density dominates the Universe (for a review see e.g. [4]). The astrophysical nature of dark energy confirms that it is not composed of baryonic matter. Now from cosmological observations we know that the Universe is spatially flat and consists of about 70% dark energy, 30% dust matter (cold dark matter plus baryons) and negligible radiation. But the nature of dark energy as well as its cosmological origin remains mysterious at present.

One of the methods for constructing a dark energy model is to modify the geometrical part of the Einstein equations. The general paradigm consists in adding into the effective action, physically motivated higher-order curvature invariants and non-minimally coupled scalar fields. But if we relax the Riemannian manifold, we can construct the models based on the torsion $T$ instead of the curvature. The representative models based on this strategy are termed ‘modified gravity’ and include $f(R)$ gravity [5], Horava-Lifshitz gravity [6–8], scalar-tensor gravity [9, 10] and the braneworld model [11, 12] and the newly $f(T)$ gravity [13].

The $f(T)$ theory of gravity is a meticulous class of modified theories of gravity. This theory can be obtained by replacing the torsion scalar $T$ in teleparallel gravity [14] with an arbitrary function $f(T)$. The dynamical equations of motion can be obtained by varying the Lagrangian with respect to the vierbein (tetrad) basis. Wu et al have proposed viable forms of $f(T)$ that can satisfy both cosmological and local gravity constraints [15]. Further Capozziello et al discussed the cosmographical method for reconstruction of $f(T)$ models [16]. But the $f(T)$ model has some theoretical problems. For example it’s not locally Lorentz invariance, possesses extra degrees of freedom, violate the first law of thermodynamics and inconsistent Hamiltonian formalism [17].

In the past, the use of scalar fields in certain physical theories, especially particle physics, has been explored. This led to study the role of scalar fields in cosmology as well. Recently some of the present authors investigated the behavior of scalar fields in $f(T)$ cosmology. In [18], we introduced a non-minimally conformally coupled scalar field and dark matter in $f(T)$ cosmology. We investigated the stability and phase space behavior of the parameters of the scalar field by choosing an exponential potential and cosmologically viable form of $f(T)$. We found that the dynamical
system of equations admit two unstable critical points; thus no attractor solution existed in that model. In another investigation \[19\], we studied the Noether symmetries (which are symmetries of the Lagrangian) of \(f(T)\) involving matter and dark energy. In that model, the dark energy was considered as a canonical scalar field with a potential. The analysis showed that \(f(T) \sim T^{3/4}\) and \(V(\phi) \sim \phi^2\). Therefore it becomes meaningful to reconstruct a scalar potential \(V(\phi)\) in the framework of \(f(T)\) gravity. It was demonstrated that dark energy driven by scalar field, decays to cold dark matter in the late accelerated Universe and this phenomenon yields a solution to the cosmic coincidence problem \[20\]. In this paper, we study the Noether gauge symmetries (NGS) of the model, which provide a more general notion of the Noether symmetry. This approach is useful in obtaining physically viable choices of \(f(T)\), and has been previously used for the \(f(R)\) gravity and generalized Saez-Ballester scalar field model as well \[21\].

The plan of this paper is as follows: In Section II, we present the formal framework of the \(f(T)\) action minimally coupled with a scalar field. In section III, we construct the governing differential equations from the Noether condition and solve them in an accompanying subsection. In section IV, we study the dynamics of the present model. Finally we conclude this work. In all later sections, we choose units \(c = 16\pi G = 1\).

## II. \(f(T)\) Gravity

If we limit ourselves to the validity equivalence principle, we must work with a gauge theory for gravity and such a gauge theory is possible only on curved manifold. Construction of a gauge theory on Riemannian manifolds is only one option and may be the simplest one. But it’s possible to write a gauge theory for gravity, with metric, non-metricity and torsion can be constructed easily \[22\]. Such theories are defined on a Weitzenböck spacetime, with globally non zero torsion but with vanishing local Riemannian tensor. In this theory, which is called teleparallel gravity, people are working on a non-Riemannian spacetime manifold. The dynamics of the metric is defined uniquely by the torsion \(T\). The basic quantities in teleparallel or the natural extension of it, namely \(f(T)\) gravity, is the vierbein (tetrad) basis \(e^i_\mu\) \[23\]. This basis of vectors is unique and orthonormal and is defined by the following equation

\[
g_{\mu\nu} = e^a_\mu e^b_\nu \eta^{ab}, \quad a, b = 0, 1, 2, 3
\]
This tetrad basis must be orthonormal and $\eta_{ab}$ is the flat Minkowski metric, $e^a\eta^\mu_b = \delta^a_b$. One suitable form of the action for $f(T)$ gravity in Weitzenböck spacetime is

$$S = \int d^4x e \left( (T + f(T)) + L_m \right),$$

(1)

where $f(T)$ is an arbitrary function of torsion $T$ and $e = \det(e^i_\mu)$. The dynamical quantity of the model is the scalar torsion $T$ and the matter Lagrangian $L_m$. The equation of motion derived from the action, by varying with respect to the $e^i_\mu$, is given by

$$e^{-1}\partial_\mu(eS^{\mu\nu}(1 + f_T) - e_i^\lambda T^\rho_{\mu\lambda} S^\nu_{\rho} f_T + S^i_{\mu\nu}\partial_\mu(T)f_{TT} - \frac{1}{4} e_i^\nu(1 + f(T)) = 4\pi e_i^\mu T^\nu_{\rho}.$$

As usual $T_{\mu\nu}$ is the energy-momentum tensor for matter sector of the Lagrangian $L_m$. It is a straightforward calculation to show that this equation of motion is reduced to Einstein gravity when $f(T) = 0$. Indeed, this is the equivalence between the teleparallel theory and the Einstein gravity.

III. OUR MODEL

We take a spatially flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)].$$

(2)

We add in the action (1) a scalar field with an unknown potential function $V = V(\phi)$ (sometimes also known as Saez-Ballester model [25]). However a slight redefinition of $(\Phi(\phi) = \int d\phi \sqrt{\pm V(\phi)},$ where $+/ -$ correspond to non-phantom/phantom phase, respectively [26]). The total action reads:

$$S = \int d^4x e \left( T + f(T) + \lambda(T + 6H^2) + V(\phi)\phi_i^\mu \phi^\mu - W(\phi) \right).$$

(3)

Here trace of the torsion tensor is $T = -6H^2$, $e = \det(e^i_\mu)$, $\lambda$ is the Lagrange multiplier and $H = \frac{\dot{a}}{a}$ is the Hubble parameter. For our convenience, we keep the original Saez-Ballester scalar field in our effective action. Varying (3) with respect to $T$, we obtain

$$\lambda = -(1 + f'(T)).$$

(4)

Here prime denotes the derivative with respect $T$. By substituting (4) in (3), and integrating over the spatial volume we get the following reduced Lagrangian:

$$L(a, \phi, T, \dot{a}, \dot{\phi}) = a^3 \left[ T + f(T) - [1 + f'(T)] \left( T + 6(\frac{\dot{a}}{a})^2 \right) + V(\phi)\dot{\phi}^2 - W(\phi) \right].$$

(5)
For the Lagrangian (5), the equations of motion read as follows

\[ f_{TT}(T + 6H^2) = 0, \]  
\[ \ddot{a} = -\frac{1}{4(1 + f_T)} \left[ f - T f_T - \frac{T}{3}(1 + f_T) + V(\phi)\dot{\phi}^2 - W(\phi) + 4H\dot{T} f_{TT} \right], \]  
\[ \ddot{\phi} + 3H\dot{\phi} + \frac{1}{2V'}(V'\dot{\phi}^2 + W') = 0. \]  

Equation (6) indicates two possibilities: (1) \( f_{TT} = 0 \), which gives the teleparallel gravity and we are not interested in this case. (2) Another possibility is \( T = -6H^2 \) which is the standard definition of the torsion scalar in \( f(T) \) gravity. In the next section we will investigate the Noether gauge symmetries of the newly proposed model in (5).

**IV. NOETHER GAUGE SYMMETRY OF THE MODEL**

To calculate the Noether symmetries, we define it first. A vector field

\[ X = \mathcal{T}(t, a, T, \phi) \frac{\partial}{\partial t} + \alpha(t, a, T, \phi) \frac{\partial}{\partial a} + \beta(t, a, T, \phi) \frac{\partial}{\partial T} + \gamma(t, a, T, \phi) \frac{\partial}{\partial \phi}, \]  

is a Noether gauge symmetry corresponding to a Lagrangian \( \mathcal{L}(t, a, T, \phi, \dot{a}, \dot{T}, \dot{\phi}) \) if

\[ X^{[1]} \mathcal{L} + \mathcal{L} D_t(\mathcal{T}) = D_t B, \]  

holds, where \( X^{[1]} \) is the first prolongation of the generator \( X \), \( B(t, a, T, \phi) \) is a gauge function and \( D_t \) is the total derivative operator

\[ D_t = \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{T} \frac{\partial}{\partial T} + \dot{\phi} \frac{\partial}{\partial \phi}. \]  

The prolonged vector field is given by

\[ X^{[1]} = X + \alpha_t \frac{\partial}{\partial t} + \beta_t \frac{\partial}{\partial T} + \gamma_t \frac{\partial}{\partial \phi}, \]  

where

\[ \alpha_t = D_t \alpha - \dot{a} D_t \mathcal{T}, \quad \beta_t = D_t \beta - \dot{T} D_t \mathcal{T}, \quad \gamma_t = D_t \gamma - \dot{\phi} D_t \mathcal{T}. \]  

If \( X \) is the Noether symmetry corresponding to the Lagrangian \( \mathcal{L}(t, a, T, \phi, \dot{a}, \dot{T}, \dot{\phi}) \), then

\[ I = \mathcal{T} \mathcal{L} + (\alpha - T \dot{a}) \frac{\partial \mathcal{L}}{\partial \dot{a}} + (\beta - T \dot{T}) \frac{\partial \mathcal{L}}{\partial \dot{T}} + (\gamma - T \dot{\phi}) \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - B, \]
is a first integral or an invariant or a conserved quantity associated with $X$. The Noether condition \[(10)\] results in the over-determined system of equations

$$
\mathcal{T}_a = 0, \quad \mathcal{T}_\phi = 0, \quad \mathcal{T}_T = 0, \quad \alpha_T = 0,
$$
\[(15)\]

$$
\gamma_T = 0, \quad B_T = 0, \quad 2a^3V\gamma_T = B_\phi,
$$
\[(16)\]

$$
6(1 + f')\alpha_\phi - a^2V\gamma_T = 0,
$$
\[(17)\]

$$
12a(1 + f')\alpha_t + B_a = 0,
$$
\[(18)\]

$$
3V\alpha + aV'\gamma + 2aV\gamma_\phi - aV T_t = 0,
$$
\[(19)\]

$$
(1 + f')(\alpha + 2a\alpha_a - aT_t) + af''\beta = 0,
$$
\[(20)\]

$$
3a^2(f - T f' - W)\alpha - a^3T f''\beta - a^3W'\gamma + a^3(f - T f' - W)T_t = B_t.
$$
\[(21)\]

We obtain the solution of the above system of linear partial differential equations for $f(T)$, $V(\phi)$, $W(\phi)$, $T$, $\alpha$, $\beta$ and $\gamma$. We have:

$$
f(T) = \frac{1}{2}t_0T^2 - T + c_2,
$$
\[(22)\]

$$
V(\phi) = V_0\phi^{-4},
$$
\[(23)\]

$$
W(\phi) = W_0\phi^{-4} + c_2,
$$
\[(24)\]

$$
T = t + c_1,
$$
\[(25)\]

$$
\alpha = a,
$$
\[(26)\]

$$
\beta = -2T,
$$
\[(27)\]

$$
\gamma = \phi,
$$
\[(28)\]

where $t_0, V_0, W_0, c_2$ and $c_1$ are constants. It is interesting to note that quadratic $f(T) = T^2$ has been used to model static wormholes in $f(T)$ gravity \cite{27}. Also the scalar potential is proportional to $\phi^{-4}$ which has been previously reported in \cite{21} for $f(R)$-tachyon model. Recently Iorio & Saridakis \cite{28} studied solar system constraints on the model (22) and found the bound $|t_0| \leq 1.8 \times 10^4 m^2$. Further a $T^2$ term can cure all the four types of the finite-time future singularities in $f(T)$ gravity, similar to that in $F(R)$ gravity \cite{29}. The quadratic correction to teleparallel model is quite vital as a next approximation in astrophysical context. Hence the Noether gauge symmetry approach generates a cosmologically viable model of $f(T)$ gravity.

It is clear from (22)-(28) that the Lagrangian (5) admits two Noether symmetry generators

$$
X_1 = \frac{\partial}{\partial T},
$$
\[(29)\]

$$
X_2 = t \frac{\partial}{\partial t} + a \frac{\partial}{\partial a} - 2T \frac{\partial}{\partial T} + \phi \frac{\partial}{\partial \phi}.
$$
\[(30)\]
The first symmetry $X_1$ (invariance under time translation) gives the energy conservation of the dynamical system in the form of (31) below, while the second symmetry $X_2$ (scaling symmetry) and a corresponding conserved quantity of the form (32) below. The two first integrals (conserved quantities) which are
\[
I_1 = -\frac{1}{2} t_0 T^2 a^3 + 6t_0 T a\dot{a}^2 - V_0 a^3 \phi^{-4} \dot{\phi}^2 - W_0 a^3 \phi^{-4},
\]
\[
I_2 = -\frac{1}{2} t_0 T^2 a^3 + 6t_0 T a\dot{a}^2 - V_0 a^3 \phi^{-4} \dot{\phi}^2 - W_0 a^3 \phi^{-4}
- 12t_0 Ta\ddot{a} + 2V_0 a^3 \phi^{-3} \dot{\phi}.
\]
Also the commutator of generators satisfies $[X_1, X_2] = X_1$ which shows that the algebra of generators is closed.

V. COSMOLOGICAL IMPLICATIONS

We rewrite Eq. (3) using the solutions for $f(T)$, $V(\phi)$, $W(\phi)$ as obtained in the previous section in the following form
\[
\mathcal{L} = \frac{\dot{a}^4}{a} + a^4 \frac{(V_0 \dot{\phi}^2 - W_0 - c_2 \phi^4)}{\phi^4},
\]
where $t_0$ is an arbitrary constant. We choose $t_0 = \frac{1}{18}$ for simplification and further we take $c_2 \neq 0$.

The Euler-Lagrange equations read
\[
\ddot{a} = \frac{H}{2} a + \frac{1}{4a\phi^4 H^2} \left( V_0 \dot{\phi}^2 - W_0 - c_2 \phi^4 \right),
\]
\[
\ddot{\phi} + 3H \dot{\phi} - \frac{2\dot{\phi}^2}{\phi} = \frac{2W_0}{V_0 \phi}.
\]

Their evolutionary behavior is obtained by numerically solving the Euler-Lagrange equations (34)-(35) for an appropriate set of the parameters and the initial conditions. To obtain the equation of state parameter numerically, first we note that for $f(T)$-Saez-Ballester theory, the energy-momentum tensor reads
\[
T_{\mu\nu} = V(\phi) \left[ g_{\mu\nu} \phi,\alpha \phi^\alpha - 2\phi,\mu \phi,\nu \right] - g_{\mu\nu} W(\phi).
\]
It is easy to calculate the total energy density and averaged pressure
\[
\rho = V(\phi) \dot{\phi}^2 - W(\phi),
\]
\[
p = -V(\phi) \dot{\phi}^2 - W(\phi).
\]
FIG. 1: (Left) Cosmological evolution of $H(t), \phi(t)$ vs time $t$. The model parameters chosen as $V_0 = 1$, $W_0 = 2V_0$. Various curves correspond to (solid, $H(t)$), (dots, $\phi(t)$). (Right) Variation of $w$ vs time $t$. The model parameters are chosen as $\alpha_0 = 2$, $\beta = \rho m_0 \frac{V}{V_0}$, $\omega = 1$. In both figures, we chose the initial conditions $\dot{a}(0) = H_0$, $\phi(0) = 0.1$, $a(0) = 1$, $\dot{\phi}(0) = 1$.

The EoS parameter is constructed via

$$w \equiv \frac{p}{\rho} = \frac{V(\phi)\dot{\phi}^2 + W(\phi)}{-V(\phi)\dot{\phi}^2 + W(\phi)}. \quad (39)$$

Putting the potential functions (23) and (24) in (39), we get

$$w = \frac{\dot{\phi}^2 + \beta \phi^4 + \alpha_0}{-\dot{\phi}^2 + \beta \phi^4 + \alpha_0}, \quad \beta = \frac{c_2}{V_0}, \quad \alpha_0 = \frac{W_0}{V_0}. \quad (40)$$

The numerical simulation of $w, \dot{w}$ is drawn in the figure which shows that $w$ behaves like the phantom energy for a brief period of time. This conclusion is exciting since there exists convincing astrophysical evidence that the observable Universe is currently in the phantom phase [30].

VI. CONCLUSION

In this paper, motivated by some earlier works on Noether symmetry in $f(T)$ gravity, we introduced a new model containing a canonical scalar field with a potential. Firstly we showed that this model obeys a quadratic term of torsion, with potential proportional to $\phi^{-4}$ which also appears for tachyonic field in $f(R)$ model. It is also interesting to note that quadratic $f(T) = T^2$ has been used to model static wormholes in $f(T)$ gravity in literature.

We mention our key results and comments below:
• Our numerical simulations show that there happens a phantom crossing scenario for a brief period in this toy model, after which the state parameter evolves to cosmological constant asymptotically.

• The behavior of Hubble parameter in our model closely mimics to that of ΛCDM model, thus our model is an alternative to the later.

• Since at the same time ΛCDM model cannot explain the phantom crossing as is observed from the empirical astrophysical results, one should prefer alternatives to ΛCDM model such as the present one. It is curious to note that such a result is obtained from a Lorentz invariance violating \( f(T) \) theory, however, the same theory is consistent with solar system tests, contains attractor solutions, and is free of massive gravitons.

• The results reported here are significantly different from \([18,19]\) since there we calculated the ‘Noether symmetries’ while here only ‘Noether gauge symmetries’ are obtained. As one can see, the results obtained here are different from the ones previously obtained in the literature.

• The most important feature of \( f(T) \) gravity which differs it from any other “curvature invariant” model like \( f(R) \) theory, is its irreducibility to a scalar model in the Jordan frame unlike \( f(R) \). As we know, \( f(R) \) gravity can be reduced to a scalar field by a simple identification between the scalar field and the gravity sector of the action in the Jordan frame. But here, since the Lorentz symmetry is broken locally, and also for leakage of finding such formal transformation between the scalar field and the Torsion action, these two models are not equivalent. So unless the \( f(R) \), here proposition of the scalar field is not artificial and do not add any additional degree of freedom to the model. From another point of the view, the \( f(T) \) gravity is not conformal invariant, so reduction of the gravity sector to the scalar matter is not possible. So indeed by introducing the scalar field in the action, we avoided from a similar formal extensions like two scalar components models, like Quintessence. Note that proposition of the scalar field to \( f(T) \) is completely new irreducible action but to \( f(R) \) is just the quintom model or in it’s extreme form reduces to the multi-scalar models with less symmetry than the original action.
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