HIGH PRECISION NUMERICAL APPROACH FOR THE DAVEY-STEWARTSON II EQUATION FOR SCHWARTZ CLASS INITIAL DATA

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ABSTRACT. We present an efficient high-precision numerical approach for the Davey-Stewartson (DS) II equation, treating initial data from the Schwartz class of smooth, rapidly decreasing functions. As with previous approaches, the presented code uses discrete Fourier transforms for the spatial dependence and Driscoll’s composite Runge-Kutta method for the time dependence. Since the DS equation is a nonlocal, nonlinear Schrödinger equation with a singular symbol for the nonlocality, standard Fourier methods in practice only reach accuracies of the order of $10^{-6}$ or less for typical examples. This was previously demonstrated for the defocusing case by comparison with a numerical approach for DS via inverse scattering. By applying a regularization to the singular symbol, originally developed for D-bar problems, the presented code is shown to reach machine precision. The code can treat integrable and non-integrable DS II equations. Moreover, it has the same numerical complexity as existing codes for DS II. Several examples for the integrable defocusing DS II equation are discussed as test cases.

In an appendix by C. Kalla, a doubly periodic solution to the defocusing DS II equation is presented, providing a test for direct DS codes based on Fourier methods.

1. Introduction

The general Davey-Stewartson (DS) II system is

\[ i\Psi_t + \Psi_{xx} - \Psi_{yy} + 2\rho(\beta \Phi + |\Psi|^2)\Psi = 0, \]

\[ \Phi_{xx} + \Phi_{yy} + 2|\Psi|^2_{xx} = 0, \]

where $\beta$ is a positive constant, indices denote partial derivatives, $\rho$ takes values $\pm 1$, and where $\Phi$ denotes a mean field. The systems \(1\) are of considerable importance in applications since they are a simplification of the Benney-Roskes \cite{BR}, or Zakharov-Rubenchik \cite{ZR} systems being ‘universal’ models for the description of the interaction of short and long waves. They first appeared in the context of water waves \cite{DS,DR,AS,Lan} in the so-called modulational (Schrödinger) regime, i.e., in the study of the modulation of plane waves. In \cite{CL1,CL2} it is shown, via diffractive geometric optics, that DS systems provide good approximate solutions to general
quadratic hyperbolic systems. Furthermore, the DS systems appear in numerous physical contexts as ferromagnetism \[30\], plasma physics \[32\], and nonlinear optics \[33\]. The Davey-Stewartson systems can also be viewed as the two-dimensional version of the Zakharov-Schulman systems (see \[31, 32, 16\]. For more details on DS and its applications the reader is referred to \[26, 25\] where an abundance of references can be found.

Because of their importance in applications, many numerical approaches for nonlinear Schrödinger (NLS) equations have been developed. For Schwartz class or periodic functions, Fourier spectral methods generally are the most efficient, and there is also a multitude of adapted time integration schemes, see \[19, 18\] for a comparison and references. The DS system can be seen as a nonlocal NLS equation where the nonlocality corresponds to a singular Fourier symbol, see \((\text{DSIIbis})\) below.

Several numerical approaches have been presented for DS II systems along these lines, see \[38, 6, 23, 27\]. In \[20\] it was shown that the standard treatment of this singular symbol leads for typical examples to a numerical error of the order of at least \(10^{-6}\). The goal of this paper is to present an efficient numerical approach for the DS II systems so that machine precision can be reached, with the same computational complexity as existing codes.

If the second equation in \((\text{DSgen})\) is solved by formally inverting the Laplace operator \(\Delta = \partial_{xx} + \partial_{yy}\) with some boundary conditions at infinity, the DS II equation can be written as a nonlocal nonlinear Schrödinger equation (NLS),

\[
\begin{align*}
(2) \quad & i\epsilon \partial_t \psi + \epsilon^2 \partial_{xx} \psi - \epsilon^2 \partial_{yy} \psi + 2\rho \Delta^{-1}[(\partial_{yy} + (1 - 2\beta)\partial_{xx}) |\psi|^2] \psi = 0,
\end{align*}
\]

which involves the zeroth order nonlocal operator \(\Delta^{-1}[(\partial_{yy} + (1 - 2\beta)\partial_{xx})]\).

We put \(\xi = \xi_1 + i\xi_2\) where \(\xi_1\) and \(\xi_2\) are the dual Fourier variables to \(x\) and \(y\) respectively. The Fourier transform of a function \(\Phi\) as defined in \((\text{F})\) is denoted by \(\hat{\Phi} = \mathcal{F}\Phi\). In Fourier space, equation \((\text{DSfourier})\) takes the form

\[
\begin{align*}
(3) \quad & i\hat{\Psi}_t + \frac{1}{2}(\xi^2 + \tilde{\xi}^2)\hat{\Psi} - 2\rho \mathcal{F}^{-1} \left( \beta \left( \frac{\xi}{2\xi} + \frac{\tilde{\xi}}{2\xi} \right) \mathcal{F}|\Psi|^2 \right) \hat{\Psi} + (\beta - 1)|\Psi|^2 \hat{\Psi} = 0.
\end{align*}
\]

Note that DS II reduces to the standard cubic NLS equation in 1d for \(y\)-independent initial data. We remark that here, and throughout this paper, a function depending on a single complex variable is not necessarily holomorphic in this variable.

The problematic term for a Fourier spectral method is in the nonlocality in \((\text{DSfourier})\), i.e., with \(\xi = \chi \exp(i\psi), \chi, \psi \in \mathbb{R},\)

\[
\begin{align*}
(4) \quad & \frac{\xi}{2\xi} + \frac{\tilde{\xi}}{2\xi} = \cos(2\psi).
\end{align*}
\]

This means that the term does not have a well defined limit for \(\chi \to 0\). Previous codes, for example discussed in \[23\] simply fix the value there which leads to an integrand with a discontinuity for \(\chi = 0\). We will henceforth call these methods classical. For these classical methods, the discontinuity implies that the inverse discrete Fourier transform will not show spectral convergence, that is, an exponential decay of the numerical error with the resolution. It turns out that the precise value between \(-1\) and \(1\) chosen for \(\cos(2\psi)\) will not affect the numerical accuracy. Therefore we chose it to be \(0\) in previous codes.
A problem in developing numerical approaches for the DS II equation for Schwartz class or periodic data has been the absence of suitable exact test solutions. Although the equation is completely integrable for $\beta = 1$ and though there is consequently a wealth of exact solutions to choose from, none of these previously known explicit solutions appears suitable in this context: the solitons of the 1d NLS equation are solutions of the DS II equation called line solitons. But they either do not test the 2d aspect of DS II, or they hit the computational domain at an angle which leads to Gibbs’ phenomena if Fourier methods are applied. There is also a 2d soliton, the lump, which, however, shows algebraic decay towards infinity in all spatial directions. Such functions, in contrast to Schwartz class functions, in practice cannot be continued periodically as being smooth within the finite numerical precision. There is, however, a large class of quasi-periodic solutions to the integrable DS II equation (with $\beta = 1$), given in terms of multi-dimensional theta functions on general compact Riemann surfaces. They were constructed by Malanyuk and have been re-derived by Kalla via Fay’s trisecant identity and discussed in [17]. A numerical approach to evaluate these explicit solutions is presented in [18]. In general, these solutions are not exactly periodic, and certainly not in $x$ and $y$ (for example, they can be periodic on a non-rectangular lattice in the $(x,y)$ plane). This can lead (again) to problems with a Fourier approach.

In a previous work [23], which compares time integration schemes for DS II, the approach is to use a reference solution obtained from various such schemes with very high resolution. Whereas this works well to test the convergence of the time integration methods, this obviously does not test the potential mistakes, nor identify potential sources of errors, in the spatial discretization, since the latter is always the same and the accuracy is unknown. In the present paper, we address this problem in two ways: first in Section 1 of the Appendix, written by Kalla, we present a family of solutions both periodic in $x$ and in $y$. As an example we consider a genus 2 solution which is a travelling wave, see Fig. 1. Such a solution was first given for the Kadomtsev-Petviashvili equation, a generalization to 2d of the Korteweg-de Vries equation with solutions similar to DS II, in [13]. This solution provides a useful test as shown in [13, 22]. Secondly, we presented in [20, 21] a high-precision numerical approach for the scattering problem for DS II which allows one to generate reference solutions in the defocusing case with essentially machine precision. Such high precision will be useful to study the stability (or the lack thereof) of certain exact solutions as solitons, and a potential blow-up in this context.

Both types of tests are applied in this paper to show they are very useful if high precision approaches are looked for. Note that DS II equations are in general not integrable. However, for the tests of the codes we concentrate on the integrable DS II since we want to compare with a reference solution obtained via inverse scattering techniques as in [20]. The numerical techniques presented in this paper are directly applicable to the non-integrable case $\beta \neq 1$.

In [20, 21] we have shown how to regularize terms of the type arising in the context of D-bar equations with a hybrid approach, i.e., by regularizing the term via analytically reducing them to terms regular within a finite numerical precision and then doing a numerical computation. Note that the terms treated in this way in [20, 21] are less regular (actually unbounded) than the ones considered here. Therefore the regularization approach for DS II is less computationally expensive
than in the D-bar case. We show that this way we can reach machine precision even in cases for which this was not possible with previous codes.

This paper is organized as follows: in section 2 we give a brief summary of the numerical approaches for DS II via Fourier techniques and the regularization approaches of KM, KMS [20, 21] applied to this case. In section 3 we consider examples from the Schwartz class with and without radial symmetry which are compared to the numerical solution for the defocusing DS II for the same initial data constructed via the scattering approach. It is shown that machine precision can be essentially reached. In the first section of the appendix C. Kalla presents an exact solution to DS II periodic both in $x$ and $y$ on a genus 2 surface, and in the second section of the appendix we summarize the scattering and inverse scattering theory of the DS II equation.

2. HYBRID NUMERICAL APPROACH FOR DS II

In this section we briefly describe the numerical approach for the DS II equation. The Fourier transform of a function $\Phi$ is defined as

$$\hat{\Phi} = \mathcal{F} \Phi := \frac{1}{2\pi} \int_{\mathbb{R}^2} \Phi e^{-i(\xi \bar{z} + \bar{\xi} z)/2} dxdy,$$

$$\Phi = \mathcal{F}^{-1} \Phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi \bar{z} + \bar{\xi} z)/2} \hat{\Phi} d\xi d\xi.$$

The classical numerical approaches to DS II equations, see for instance [28, 10, 23], via Fourier spectral methods are based on an approximation of the Fourier transform (5) by a discrete Fourier transform, i.e., by essentially a truncated Fourier series. For the latter an efficient algorithm exists known as the fast Fourier transform (FFT). This leads for (5) to a finite dimensional system of ordinary differential equations (ODEs) for $\mathcal{F} \Psi$ (in an abuse of notation, we use the same symbol for the
continuous and the discrete Fourier transform). The ODE system has the form

$$\dot{\Psi} = L\dot{\Psi} + N[\dot{\Psi}],$$

where $L$ and $N$ denote linear and nonlinear operators, respectively:

$$L\dot{\Psi} = -\frac{1}{2}(\xi^2 + \bar{\xi}^2)\dot{\Psi}$$

$$N[\dot{\Psi}] = -2\rho F^{-1}\left(\beta \left(\frac{\xi}{2\xi} + \frac{\bar{\xi}}{2\xi}\right) F[|\Psi|^2] \Psi + (\beta - 1)|\Psi|^2 \Psi\right)$$

Since the operator $L$ is proportional to $|\xi|^2$ whereas $N$ is of order $|\xi|^0$, this is an example of a stiff system of equations, in particular if large values of $|\xi|$ are needed as in the case of rapid oscillations called dispersive shock waves as in [23] or in the case of a blow-up a diverging $L^\infty$ norm of the solution as studied in [22, 26, 27]. Loosely speaking the word stiff refers to ODEs for which explicit time integration schemes are not efficient for stability reasons which means that the condition of numerical stability requires considerably smaller time steps than imposed by the desired accuracy. For equations of the form (6) with stiffness in the linear term, many efficient integration schemes exist in the literature. Several such methods have been compared for DS II in [23] where the reader can find an abundance of references. The best method identified there for both the focusing and the defocusing case is Driscoll’s composite Runge-Kutta (RK) method [13]. It uses a stiffly stable third order RK method for the high wavenumbers in $L$ and the standard explicit fourth order RK method for the remaining terms. We apply this method in the following.

The task is thus to compute two singular integrals of the form $F^{-1}(S/\xi)$ and $F^{-1}(S/\xi)$. In [20, 21] we have shown that (we have $\tilde{\theta} = (\partial_x - i\partial_y)/2$

$$F^{-1}\left(\frac{\xi^n e^{-|\xi|^2}}{\xi}\right) = (-2i\tilde{\theta})^n \frac{i}{z} \left(1 - e^{-\frac{|\xi|^2}{4z}}\right), \quad n = 0, 1, \ldots,$$

and

$$F^{-1}\left(\frac{\xi^n e^{-|\xi|^2}}{\xi}\right) = (-2i\tilde{\theta})^n \frac{i}{z} \left(1 - e^{-\frac{|\xi|^2}{4z}}\right), \quad n = 0, 1, \ldots,$$

The idea to compute a singular integral of the form $F^{-1}(S(\xi)/\xi)$ is to subtract the first $M + 1$ terms of a Taylor series of $S$ in $\xi$ where $M$ is chosen such that the residual is smooth up to the finite numerical precision, i.e.,

$$F^{-1}\left(\frac{S(\xi)}{\xi}\right) = F^{-1}\left(\frac{S(\xi) - e^{-|\xi|^2} \sum_{n=0}^{M} \partial^n_x S(0) \bar{\xi}^n/n!}{\xi} + e^{-|\xi|^2} \sum_{n=0}^{M} \partial^n_x S(0)(-2i\tilde{\theta})^n \frac{i}{z} \left(1 - e^{-\frac{|\xi|^2}{4z}}\right).$$

The first term on the right hand side of (10) can now be efficiently computed numerically via FFT, the second is computed analytically. The derivatives of $S$ are also computed via Fourier techniques,

$$\partial^n_x S(\xi) = F\left[(i\bar{z}/2)^n F^{-1} S\right]$$

Note that the derivatives are only needed for $\xi = 0$. To compute the first term on the right-hand-side of (10) at $\xi = 0$, one would also need $\partial_x S(0)$ (according to L'Hôpital’s rule).
Integrals of the form $F^{-1}(S(\xi)/\xi)$ are computed analogously. As already mentioned, for DS II the situation is less singular than the cases discussed in [20, 21]. Here we only need to compute the singular term in $\Phi$ or, equivalently, the singular part of the nonlinear term appearing in $\Phi$. This leads to

$$F^{-1}\left(\left[\frac{\tilde{\xi}}{2\xi} + \frac{\xi}{2\tilde{\xi}}\right] S(\xi)\right)$$

$$= F^{-1}\left(\left[\frac{\tilde{\xi}}{2\xi} + \frac{\xi}{2\tilde{\xi}}\right] S(\xi) - \frac{1}{2} e^{-|\xi|^2} \sum_{n=0}^{M} \frac{\partial^n S(0)(\xi^{n+1}/\xi) + \partial^n S(0)(\xi^{n+1}/\xi)}{n!}\right)$$

$$+ \frac{1}{2} e^{-|\xi|^2} \sum_{n=0}^{M} \frac{\partial^n S(0)(-2i\partial)^{n+1}}{n!} \left(1 - e^{-|z|^2}\right) + \partial^n S(0)(2i\tilde{\xi})^{n+1} \left(1 - e^{-|z|^2}\right).$$

A brief summary of the algorithm to solve the DS II equation is as follows.

1. Introduce a discrete Fourier grid on the spatial domain $x \in L_x[-\pi, \pi], y \in L_y[-\pi, \pi]$.
2. Compute and store the functions $W_n = F^{-1}\left(\left[\frac{\tilde{\xi}}{2\xi} + \frac{\xi}{2\tilde{\xi}}\right] S(\xi)\right)$.
3. In the implementation of Driscoll’s Runge-Kutta method, compute the nonlinear term $N(\hat{\Psi})$ as follows:
   (a) Compute the relevant Taylor coefficients of $S(\xi) = F(|\Psi|^2)$ directly from $|\Psi|^2$ via (13) (i.e. without computing any Fourier transforms):
   $$\partial^n F(|\Psi|^2) = \int_{\mathbb{R}^2} [(-iz/2)^n|\Psi|^2] \, dx \, dy,$$
   where the integral is simply approximated by a sum over all indices after the spatial discretization (note that the application of the trapezoidal rule in this context is a spectral method since it corresponds simply to the computation of the Fourier coefficient with indices 0).
   (b) Compute $S(\xi) = F(|\Psi|^2)$, then compute
   $$\beta F^{-1}\left(\left[\frac{\tilde{\xi}}{2\xi} + \frac{\xi}{2\tilde{\xi}}\right] S(\xi) - \frac{1}{2} e^{-|\xi|^2} \sum_{n=0}^{M} \frac{\partial^n S(0)(\xi^{n+1}/\xi) + \partial^n S(0)(\xi^{n+1}/\xi)}{n!}\right),$$
   using (a) above.
   (c) Now add to this quantity the exact term appearing in the last line of (12), and also the term $(\beta - 1)|\Psi|^2$.
   (d) Finally, the nonlinear term $N(\hat{\Psi})$ is obtained by multiplying the result by $\Psi$, and computing the Fourier transform of that product.

The reader will note that the functions $W_n$ in (2) need only be computed once, they are the same for all time steps and for all stages of the RK scheme. The computation of the nonlinear term $N(\hat{\Psi})$ using the regularization described in (a) through (d) above then requires the same number of Fourier transforms as the classical approach without regularization. Therefore, the numerical complexity of this algorithm is the same as the standard algorithm.
3. Examples

In this section we compare DS codes with and without the regularization of the nonlocal term of the previous section for various examples in the defocusing integrable case $\beta = 1$. We first consider the exact solution, and then more general initial data with a numerical scattering approach providing the reference solution.

3.1. Exact solution. As a first test we take as initial data with $t = 0$ the exact periodic solution of Fig. 1. We use $N_x = N_y = 2^{15}$ Fourier modes in $x$ and $y$ direction. In Fig. 2 we show the Fourier coefficients for this solution as computed with an FFT. It can be seen that the Fourier coefficients decrease in all directions to $10^{-15}$, the level of the rounding errors. This shows that the solution is numerically well resolved.

![Fourier coefficients for exact solution](fourierexact)

Figure 2. Modulus of the Fourier coefficients of the doubly periodic solution to the defocusing DS II equation discussed in the appendix for $t = 0$.

As similar exact solutions to KdV and KP, this solution is numerically resolved with a comparatively small number of Fourier modes. As discussed for instance in [23], this implies that the system (6) is not stiff, and that this example is thus not challenging for the codes. We use $N_t = 1000$ time steps for $t \leq 10$. The $L^2$ norm of the DS II solution, which is a conserved quantity for DS II, is not automatically conserved by the code and thus a test of the time resolution. In this example, it is conserved to the order of $10^{-16}$ during the entire computation. The difference to the exact solution at the final time is of the order of $10^{-14}$, the estimated accuracy for the exact solution.

This shows that the time evolution code reproduces the solution with the accuracy with which it is known. Note that this is both the case for the codes with and without the regularization of the previous section. Thus this exact solution does not test this aspect of the codes. In the next subsection we thus present examples that require higher resolution in space.
3.2. Scattering approach. The defocussing DSII equation \( \text{DS}_{\text{gen}}^1 \) with \( \beta = 1 \) was shown to be integrable in [2]. Scattering and inverse scattering theory were developed in a number of works, including [3], and [37]. Asymptotic behavior of solutions was considered in [37], and more recently in [35] and separately [34], asymptotic results were obtained with initial data in natural function spaces. In Section 2 of the Appendix, we provide a brief summary of the scattering and inverse scattering theory that provides a solution procedure for the integrable case.

In [20] a spectral method is developed to solve the D-bar equations that appear in the scattering and inverse scattering theory associated to the integrable defocusing DSII equation. The method is shown to have spectral convergence in the number of Fourier modes, for Schwartz class functions. The approach is based on the integral equation in the Fourier domain associated to the D-bar system \( \text{eq:specprob2} \), which contains a singular integral operator. The singular part appearing in the singular integral operator is subtracted off using polynomials and exponentials, and then the offending terms are handled via exact calculations, just as we are implementing in \( \text{DSreg}^1 \) above.

We consider the computational method in [20] as a source of a completely independent collection of reference solutions to the DSII equation. Here we use two examples to study the accuracy of the new method we present in Section 2.

Example 1: The first example is the initial data \( \Psi_0 = e^{-|z|^2} \). We use the method from [20] to produce numerical approximations \( \Psi(z,t) \) at time \( t = 0 \). The solution is shown in Fig. 3. It can be seen that the real and the imaginary part of the solution show oscillations.

We study the convergence of the solution in dependence of the number \( N \) of Fourier modes for \( N = 2^5, 2^6, 2^7, \) and \( 2^8 \), on spatial domains \( [-\pi L(N), \pi L(N)] \) that grow with \( N \) (essentially \( L(N) \) scales as \( \sqrt{N} \)). For each reference solution for the same parameters, we compare to the solution to the DSII equation obtained using the direct solver as described in Section 2. The table below contains the relevant parameters, and maximum error between the two methods.

The error of Table 4 is plotted in Fig. 5. It can be seen that the error decays exponentially with \( N \) as expected which means that spectral convergence is achieved. On the other hand the error for the classical codes for DS decreases only very slowly and essentially saturates at the order of \( 10^{-6} \).

Example 2: The second example is the initial data \( \Psi_0 = e^{-(x^2+xy+2y^2)} \), i.e., initial data in the Schwartz class without radial symmetry. We again use the method from [20] to produce numerical approximations \( \Psi(z,t) \), this time to time \( t = 0.2 \). The solution is shown in Fig. 6.

The convergence of the numerical approach is again tested for \( N = 2^5, 2^6, 2^7, \) and \( 2^8 \), on spatial domains \( [-\pi L(N), \pi L(N)] \) that scale with \( \sqrt{N} \). For each reference solution with the same parameters, we compare to the solution to the DSII equation obtained using the direct solver described in Section 2. The table below contains the relevant parameters, and maximum error between the two methods.
Figure 3. Gaussian initial data (top) $\Psi(z,0) = e^{-|z|^2}$, and DS II evolution to time $t = 0.4$ (bottom - $\text{Imag}(\Psi(z,t))$ on right, $\text{Re}(\Psi(z,t))$ on left) with $N = 256$ Fourier modes, $L = 4.56$, using the direct method described in Section Sec:HybNum.

| Fourier Modes | L   | Error with classical method | Error with new method |
|---------------|-----|-----------------------------|-----------------------|
| $2^3$         | 1.65| 9.5e-05                     | 1.2e-05               |
| $2^4$         | 2.6 | 1.5e-05                     | 1.1e-07               |
| $2^5$         | 4.0 | 2.7e-06                     | 1.6e-10               |
| $2^6$         | 6.15| 4.9e-07                     | 1.2e-14               |

Figure 4. Comparison of maximum error for the simulations with Gaussian initial data.

The exponential decay of the error is shown in Fig. Sec:AsymmErrorPlot, whereas the error of the classical approach is once more essentially stagnant at the order of $10^{-6}$.

4. Outlook

In this paper we have presented a high precision numerical approach for DS II systems for Schwartz class initial data. Existing approaches based on a Fourier spectral method for the spatial dependence have been amended by regularizing the singular Fourier symbol of the nonlocal nonlinear term of the DS II systems via a hybrid method: the symbol is regularized via explicitly computable functions. Except for the numerical computation of these functions, which needs to be done only once for the whole time evolution, the same number of Fourier transforms (which are responsible for the main computational cost) are needed as in classical approaches. The accuracy of the method was tested via an independent numerical
Figure 5. Convergence comparison of the direct DS solver and the solution via inverse scattering, for \( N = 2^5, 2^6, 2^7, \) and \( 2^8 \), for defocusing DS II evolution of Gaussian initial data to time \( t = 0.4 \) (see Fig. 4).

Figure 6. Asymmetric initial data (top) \( \Psi_0 = e^{-(x^2 + xy + 2y^2)} \), and DS II evolution to time \( t = 0.2 \) (bottom - Imag(\( \Psi(z,t) \)) on right, Re(\( \Psi(z,t) \)) on left) with \( N = 256 \) Fourier modes, \( L = 4.56 \), using the direct method described in Section 2.
solution of the defocussing integrable DS II system by a numerical inverse scattering approach. It was shown that essentially machine precision can be reached with both methods.

The Davey-Stewarson systems derived in the modulational regime of surface waves occur only in two-spatial dimensions. However the Zakharov-Rubenchik systems make physical sense also in three spatial dimensions. They may then degenerate to three-dimensonal Davey-Stewartson systems

$$i\psi_t + \delta\psi_{xx} + \Delta\psi = (\chi|\psi|^2 + \beta\phi)\psi,$$
$$\Delta\phi = \partial_{xx}|\psi|^2,$$

where $\Delta_L = \partial^2_x + \partial^2_y$ and $\Delta = \partial^2_x + \partial^2_y + \partial^2_z$, and where $\delta$, $\chi$, and $\beta$ are real constants. This can be once more written as a nonlocal NLS equation, this time in 3D, where the nonlocality corresponds to a singular Fourier symbol. The numerical study of these 3D DS II systems with a regularized symbol of the nonlocality along the lines of the present paper will be the subject of further research.
Appendix A. Doubly periodic solution to DS II

by C. Kalla

In this section we derive a family of travelling wave solutions to the defocusing DS II equation which are both periodic in \( x \) and \( y \). We only briefly touch the theory of algebro-geometric solutions to integrable equations. The reader is referred to [BK] for an overview and a comprehensive list of references.

It was shown in [Kal] that the DS solutions in terms of multi-dimensional theta functions by Malanyuk [Mal] can be written in the form

\[
\Psi(x,y,t) = \sqrt{|q_2(a,b)|} \frac{\Theta(z + r)}{\Theta(z)} e^{i(-N_1 \Xi - N_2 \eta + N_3 t / 2)},
\]

where

\[
\Xi = x + iy, \quad \eta = x - iy,
\]

and

\[
z = iV_a \Xi - iV_b \eta + i(W_a - W_b) \frac{t}{2};
\]

here the multi-dimensional theta function is defined as the series

\[
\Theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\langle n | B n \rangle / 2 + \langle n | z \rangle),
\]

which is uniformly convergent since \( B \) is a Riemann matrix, i.e., satisfies \( B^t = B \) and has a negative definite real part; \( \langle \cdot | \cdot \rangle \) defines the euclidean scalar product, \( a, b \) are points on a Riemann surface of genus \( g \), and \( q_2, N_1, N_2, N_3, r, V, W \) are certain integrals respectively their periods on this Riemann surface (see [Kal]). We do not specify them here since they will be just parameters to be chosen in a way to get periodic solutions.

We now try to identify some of the simplest periodic solutions amongst (14). Let \( L_x \) and \( L_y \) be the \( x \)-period and \( y \)-period respectively and let

\[
\tilde{\Psi}(x,y,t) = \Psi(x + k_1 L_x, y + k_2 L_y, t), \quad k_1, k_2 \in \mathbb{Z}.
\]

We want solutions such that

\[
\tilde{\Psi}(x,y,t) = \Psi(x,y,t) \quad \forall x,y,t,k_1,k_2.
\]

We can see that

\[
\tilde{\Psi}(x,y,t) = \sqrt{|q_2(a,b)|} \frac{\Theta(z + r + L_1)}{\Theta(z + L_1)} e^{i(-N_1 \Xi - N_2 \eta + N_3 t / 2)} e^{L_2},
\]

where

\[
L_1 = k_1 L_x (V_a - V_b)i - k_2 L_y (V_a + V_b),
\]

\[
L_2 = -k_1 L_x (N_1 + N_2)i + k_2 L_y (N_1 - N_2).
\]

In order to obtain periodic solutions, we need that

\[
L_1 = 2i\pi n + \mathbb{B} m, \quad n, m \in \mathbb{Z}^2,
\]

which implies

\[
\tilde{\Psi}(x,y,t) = \sqrt{|q_2(a,b)|} \frac{\Theta(z + r)}{\Theta(z)} e^{i(-N_1 \Xi - N_2 \eta + N_3 t / 2)} e^{L_2 - \langle r, m \rangle}.
\]

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Hence, if (16) is satisfied, we need that
\[ \Re(L_2) = \langle \Re(r), m \rangle. \]

Now, recall that in order to get a regular solution to the defocusing DS II equation, one must have
\[ V_b = -\overline{V_a}, \quad N_2 = \overline{N_1}. \]
This implies
\[ L_1 = 2i (k_1 L_x \Re(V_a) - k_2 L_y \Im(V_a)), \]
and
\[ L_2 = 2i (-k_1 L_x \Re(N_1) + k_2 L_y \Im(N_1)). \]
Now, if we assume that (for defocusing DS II)
\[ B = B, \]
we deduce from (16) and (19) that
\[ m = 0, \]
and (17) is verified. Hence, always by (16) and (19), we search \( L_x, L_y \in \mathbb{R} \) such that for a given \( n \in \mathbb{Z}^2 \)
\[ k_1 L_x \Re(V_a) - k_2 L_y \Im(V_a) = \pi n. \]
Since this equation has to hold for all \( k_1, k_2 \in \mathbb{Z} \), we get
\[ m_1 \Re(V_{a,1}) = m_2 \Re(V_{a,2}), \]
\[ m_1 \Im(V_{a,1}) = m_2 \Im(V_{a,2}) \]
for some \( m_1, m_2, n_1, n_2 \in \mathbb{Z} \). A possible choice is \( \Re(V_{a,1}) = \Im(V_{a,2}) = 0 \). If \( \Re(V_{a,1}), \Im(V_{a,1}) \) do not vanish, one has
\[ L_x = -\frac{\pi n_1}{\Re(V_{a,1})}, \quad L_y = -\frac{\pi m_1}{\Im(V_{a,1})}. \]

The remaining condition to be satisfied is (20) which reads
\[ k_2 L_y \Im(N_1) - k_1 L_x \Re(N_1) = k \pi, \quad k \in \mathbb{Z}. \]
Since this has to hold for all \( k_1, k_2 \), this is again equivalent to two equations,
\[ \frac{\Re(N_1)}{\Re(V_{a,1})} = l_1, \quad \frac{\Im(N_1)}{\Im(V_{a,1})} = l_2, \quad l_1, l_2 \in \mathbb{Z}. \]

The task is thus to find a point \( a \) on an M-curve of genus 2 (which means a hyperelliptic curve with only real branch points) such that (22) and (24) are satisfied and such that the solution is non trivial. Thus we consider the hyperelliptic curve
\[ \mu^2 = \prod_{n=1}^{6} (\lambda - \lambda_n), \quad \lambda_n \in \mathbb{R}. \]
The vector \( V \) on this curve has the form \( A(1, a)^t / \mu(a) \), where \( A \) is the inverse of the matrix of \( a \)-periods of the holomorphic one-forms times \( 2\pi i \). One possibility is to choose the curve and the point \( a \) such that \( V_{a,1} = \overline{V_{a,2}} \). In this case equation (22) is obviously satisfied. This can be achieved on a curves with a symmetry with respect to \( \lambda \mapsto -\lambda \) and a point \( a \) projecting to the imaginary axis. But this
leaves two conditions to be satisfied with a single real parameter which implies that though a whole family of such solutions exist, there must be a special relation between some of its parameters.

In order to construct a particular such solution, we choose the branch points in \( \alpha, 2, 1, -1, -2, -\alpha \) with \( \alpha > 2 \) and \( V_a = \beta i [1, -1] \) iteratively in order to satisfy conditions for given \( \frac{n_2}{n_1}, \frac{m_1}{m_2} \). This is done with some initial guess with the optimization algorithm \( \text{lagarias} \) implemented in Matlab as the command \( \text{fminsearch} \). The quantities on the hyperelliptic curve are computed with the code \( \text{FK} \). To give a concrete example, we choose \( \frac{n_2}{n_1} = 1 \) and \( \frac{m_2}{m_1} = -4 \) and get \( \alpha = 2.870255599870804, \beta = 2.119032837086884 \) and thus the solution shown in Fig. 1. Internal tests of the code \( \text{FK} \) indicate that the solution satisfies DS II to the order of \( 10^{-14} \).

Appendix B. Integrability of the DS II equation

In this section of the Appendix, we provide a brief summary of the scattering and inverse scattering theory that provides a solution procedure for the integrable case of the defocusing DS II equation.

Here we follow the exposition in \( \text{KM} \). From the initial data \( \Psi_0(x) = \Psi(x, 0) \), we determine the reflection coefficient (the scattering transform of the initial data, sometimes also called a nonlinear Fourier transform) by solving the following system of linear elliptic partial differential equations

$$
\begin{pmatrix}
\partial & 0 \\
0 & \partial
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
0 & \Psi_0(z) \\
\Psi_0(z) & 0
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}.
$$

The operators \( \partial \) and \( \overline{\partial} \) are defined via

$$
\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \overline{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
$$

We seek a column vector \( \psi = \psi(z, k) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \) solving the system, with the following asymptotic behavior as \( |z| \to \infty \):

$$
\lim_{|z| \to \infty} \psi_1 e^{-kz/\epsilon} = 1,
\lim_{|z| \to \infty} \psi_2 e^{-\overline{k}z/\epsilon} = 0.
$$

Here the quantity \( k \) is a complex parameter, \( k = k_1 + ik_2 \) with \( (k_1, k_2) \in \mathbb{R}^2 \), which plays the role of a spectral variable. In keeping with the literature on inverse problems, we refer to the quantity \( \psi \) as a complex geometric optics (CGO) solution.

Given initial data \( \Psi_0(z) \) for the DSII equation, the CGO solution \( \psi \) is uniquely determined, and the reflection coefficient \( r = r^e(k) \), is obtained from the sub-leading term in the asymptotic expansion of \( \psi \) as \( z \to \infty \), via

$$
\psi_2 e^{-\overline{k}z} = \frac{\overline{r^e(k)}}{2|z|} + \mathcal{O} \left( \frac{1}{|z|^2} \right).
$$

The mapping from \( \Psi_0 \) to \( r \) is a transformation from the potential \( \Psi_0(x, y) \) (again a function of two real variables) to a function \( r(k_1, k_2) \), which extends to a Lipschitz continuous and invertible mapping on the function space \( L^2 (\mathbb{C}) \) (see \( \text{KM} \), \( \text{Perry} \), and the references contained therein).
Now, if $\Psi = \Psi(x, y, t)$ evolves according to the DSII equation, then the reflection coefficient evolves according to

$$r = r(k, t) = r(k, 0)e^{-it(k^2 + k^2)}.$$  

It is well-known that the inverse problem, reconstructing the potential $q(x, y, t, \epsilon)$ from the reflection coefficient $r(k, t)$, is also a D-bar problem, this time with respect to the complex variable $k$. Indeed, setting

$$\phi_1 := e^{-kz}\psi_1 \quad \text{and} \quad \phi_2 := e^{-kz}\psi_2,$$

one may verify that for each $z \in \mathbb{C}$,

$$\partial_k \phi_1 = \frac{1}{2}e^{-(k^2 - k^2)z}r(k, t)\phi_2, \quad \partial_k \phi_2 = \frac{1}{2}e^{-(k^2 - k^2)z}r(k, t)\phi_1,$$

where,

$$\partial_k \phi_1 = \frac{1}{2} \left( \frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2} \right), \quad \partial_k \phi_2 = \frac{1}{2} \left( \frac{\partial}{\partial k_1} - i \frac{\partial}{\partial k_2} \right),$$

and the quantities $\phi_1$ and $\phi_2$ possess the following asymptotic behavior:

$$\lim_{|k| \to \infty} \phi_1(k; z, t) = 1 \quad \text{and} \quad \lim_{|k| \to \infty} \phi_2(k; z, t) = 0.$$ 

The functions $\phi_1$ and $\phi_2$ are uniquely determined by this elliptic PDE system and boundary conditions. The potential $q(x, y, t, \epsilon)$ is similarly determined through the asymptotic behavior as $|k| \to \infty$:

$$\phi_2 = \frac{q(x, y, t, \epsilon)}{2k} + O\left(|k|^{-2}\right).$$

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