Size and power properties of some tests in the Birnbaum–Saunders regression model

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Abstract

The Birnbaum–Saunders distribution has been used quite effectively to model times to failure for materials subject to fatigue and for modeling lifetime data. In this paper we obtain asymptotic expansions, up to order $n^{-1/2}$ and under a sequence of Pitman alternatives, for the nonnull distribution functions of the likelihood ratio, Wald, score and gradient test statistics in the Birnbaum–Saunders regression model. The asymptotic distributions of all four statistics are obtained for testing a subset of regression parameters and for testing the shape parameter. Monte Carlo simulation is presented in order to compare the finite-sample performance of these tests. We also present an empirical application.

Keywords: Birnbaum–Saunders distribution, Fatigue life distribution, Gradient test, Lifetime data, Likelihood ratio test, Local power, Score test, Wald test.

1. Introduction

The Birnbaum–Saunders (BS) distribution has received considerable attention in the last few years. It was proposed by Birnbaum and Saunders (1969) and is also known as the fatigue life distribution. It describes the total time until the damage caused by the development and growth of a dominant crack reaches a threshold level and causes a failure. The random variable $T$ is said to have a BS distribution with parameters $\alpha$ and $\eta$, denoted by $T \sim BS(\alpha, \eta)$, if its probability density function is given by

$$f(t; \alpha, \eta) = \kappa(\alpha, \eta)t^{-3/2}(t + \eta) \exp\left\{ -\frac{\tau(t/\eta)}{2\alpha^2} \right\}, \quad t > 0,$$

where $\kappa(\alpha, \eta) = \exp(\alpha^{-2})/(2\alpha\sqrt{2\pi\eta})$, $\tau(z) = z + z^{-1}$, $\alpha > 0$ (shape parameter) and $\eta > 0$ (scale parameter). It is positively skewed, the skewness decreasing with $\alpha$. For any constant $k > 0$, it follows that $kT \sim BS(\alpha, k\eta)$. It is also noteworthy that the reciprocal property holds: $T^{-1} \sim BS(\alpha, \eta^{-1})$, which is in the same family of distributions (Saunders 1974). There are several recent articles considering the BS distribution; see

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for example, Wu and Wong (2004), Díaz–García and Leiva (2003), Lemonte et al. (2007, 2008), Balakrishnan et al. (2007), Kundu et al. (2008), Leiva et al. (2008), Gómez et al. (2009), Guiraud et al. (2009), Leiva et al. (2009), Xu and Tang (2010), Meintanis (2010), among others.

Rieck and Nedelman (1991) introduced a log-linear regression model based on the $BS$ distribution by showing that if $T \sim BS(\alpha, \eta)$, then $Y = \log(T)$ has a sinh-normal distribution with shape, location and scale parameters given by $\alpha$, $\mu = \log(\eta)$ and $\sigma = 2$, respectively, say $Y \sim SN(\alpha, \mu, 2)$. The regression model proposed by the authors is given by

$$ y_i = x_i^\top \beta + \varepsilon_i, \quad i = 1, \ldots, n, \quad (1) $$

where $y_i$ is the logarithm of the $i$th observed lifetime, $x_i = (x_{i1}, \ldots, x_{ip})$ contains the $i$th observation on $p$ covariates ($p < n$), $\beta = (\beta_1, \ldots, \beta_p)^\top$ is a vector of unknown regression parameters, and $\varepsilon_i \sim SN(\alpha, 0, 2)$. Diagnostic tools for the $BS$ regression model can be found in Galea et al. (2004), Xi and Wu (2007) and Leiva et al. (2007).

In the $BS$ regression model hypothesis testing inference is usually performed using the likelihood ratio, Rao score and Wald tests. A new criterion for testing hypothesis, referred to as the gradient test, has been proposed by Terrell (2002). Its statistic shares the same first order asymptotic properties with the likelihood ratio, Wald and score statistics and is very simple when compared with the other three classic tests. In fact, Rač (2005) wrote: “The suggestion by Terrell is attractive as it is simple to compute. It would be of interest to investigate the performance of the [gradient] statistic.” To the best of our knowledge, however, there is no mention in the statistical literature on the use of the gradient test in $BS$ regressions.

In this paper we compare the four rival tests from two different points of view. First, we invoke asymptotic arguments. We then move to a finite-sample comparison, which is accomplished by means of a simulation study. Our principal aim is to help practitioners to choose among the different criteria when performing inference in $BS$ regressions.

On asymptotic grounds, it is known that, to the first order of approximation, the likelihood ratio, Wald, score and gradient statistics have the same asymptotic distributional properties either under the null hypothesis or under a sequence of local alternatives, i.e. a sequence of Pitman alternatives converging to the null hypothesis at a convergence rate $n^{-1/2}$. On the other hand, up to an error of order $n^{-1}$ the corresponding criteria have the same size properties but their local powers differ in the $n^{-1/2}$ term. A meaningful comparison among the criteria can be performed by comparing the nonnull asymptotic expansions to order $n^{-1/2}$ of the distribution functions of these statistics under a sequence of Pitman alternatives. In this regard, we can benefit from the work by Hayakawa (1973), Harris and Peers (1980) and Lemonte and Ferrari (2010a). Hayakawa (1973) derived the nonnull asymptotic expansions up to order $n^{-1/2}$ for the likelihood ratio and Wald statistics, while an analogous result for the score statistic was obtained by Harris and Peers (1980). Recently, the asymptotic expansion up to order $n^{-1/2}$ for the density of the gradient statistic was derived by Lemonte and Ferrari (2010a). The expansions obtained by these authors are extremely general but it can be very difficult or even impossible to particularize their formulas for specific regression models. As we shall see below, we have been able to apply their results for the $BS$ regression model.

The rest of the paper is organized as follows. Section 2 briefly describes the likelihood ratio, Wald, score and gradient tests. In Section 3 these tests are applied for testing...
hypotheses on the parameters of the BS regression model. In Section 4 we obtain and compare the local powers of the tests. Monte Carlo simulation results on the finite-sample performance of the tests are presented and discussed in Section 5. Section 6 contains an application to a real fatigue data set. Finally, Section 7 discusses our main findings and closes the paper with some conclusions.

2. Background

Consider a parametric model $f(\cdot; \theta)$ with corresponding log-likelihood function $\ell(\theta)$, where $\theta = (\theta_1^T, \theta_2^T)^T$ is a $k$-vector of unknown parameters. The dimensions of $\theta_1$ and $\theta_2$ are $k$ and $k - q$, respectively. Suppose the interest lies in testing the composite null hypothesis $H_0 : \theta_2 = \theta_2^{(0)}$ against $H_1 : \theta_2 \neq \theta_2^{(0)}$, where $\theta_2^{(0)}$ is a specified vector. Hence, $\theta_1$ is a vector of nuisance parameters. Let $\hat{U}_\theta$ and $K_\theta$ denote the score function and the Fisher information matrix for $\theta$, respectively. The partition for $\theta$ induces the corresponding partitions

\[
U_\theta = (U_{\theta_1}^T, U_{\theta_2}^T)^T, \quad K_\theta = \begin{pmatrix}
K_{\theta_{11}} & K_{\theta_{12}} \\
K_{\theta_{21}} & K_{\theta_{22}}
\end{pmatrix}, \quad K_\theta^{-1} = \begin{pmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{pmatrix},
\]

where $K_\theta^{-1}$ is the inverse of $K_\theta$.

Let $\hat{\theta} = (\hat{\theta}_1^T, \hat{\theta}_2^T)^T$ and $\hat{\theta}^{(0)} = (\hat{\theta}_{1}^{(0)T}, \hat{\theta}_{2}^{(0)T})^T$ denote the maximum likelihood estimators of $\theta = (\theta_1^T, \theta_2^T)^T$ under $H_1$ and $H_0$, respectively. The likelihood ratio ($S_1$), Wald ($S_2$), score ($S_3$) and gradient ($S_4$) statistics for testing $H_0$ versus $H_1$ are given by

\[
S_1 = 2\{\ell(\hat{\theta}) - \ell(\hat{\theta}^{(0)})\}, \quad S_2 = (\hat{\theta}_2 - \theta_2^{(0)})^T K_\theta^{-1} (\hat{\theta}_2 - \theta_2^{(0)}),
\]

\[
S_3 = \hat{U}_{\theta_1}^T K_{22} \hat{U}_{\theta_2}, \quad S_4 = \hat{U}_{\theta_2}^T (\hat{\theta}_2 - \theta_2^{(0)}),
\]

respectively, where $K_{22} = K^{22}(\hat{\theta})$, $K^{22} = K^{22}(\hat{\theta})$ and $U_{\theta_1} = U_{\theta_1}(\hat{\theta})$. The limiting distribution of $S_1, S_2, S_3$ and $S_4$ is $\chi^2_{k-q}$ under $H_0$ and $\chi^2_{k-q, \lambda}$, i.e., a noncentral chi-square distribution with $k - q$ degrees of freedom and an appropriate noncentrality parameter $\lambda$, under $H_1$. The null hypothesis is rejected for a given nominal level, $\gamma$, say, if the test statistic exceeds the upper $1 - \gamma$ quantile of the $\chi^2_{k-q}$ distribution. Clearly, $S_1$ has a very simple form and does not involve knowledge of the information matrix, neither expected nor observed, and any matrix, unlike $S_2$ and $S_3$.

3. Test statistics in the BS regression model

In what follows, we shall consider the tests which are based on the statistics $S_1, S_2, S_3$ and $S_4$ in the class of BS regression models for testing a composite null hypothesis. The log-likelihood function $\ell(\theta)$ for the vector parameter $\theta = (\beta^T, \alpha)^T$ from a random sample $y = (y_1, \ldots, y_n)^T$ obtained from model (1), except for constants, can be written as

\[
\ell(\theta) = \sum_{i=1}^{n} \log(\xi_{i1}) - \frac{1}{2} \sum_{i=1}^{n} \xi_{i2}^2,
\]

where $\xi_{i1} = \xi_{i1}(\theta) = 2\alpha^{-1} \cosh(y_i - \mu_i)/2$, $\xi_{i2} = \xi_{i2}(\theta) = 2\alpha^{-1} \sinh(y_i - \mu_i)/2$ and $\mu_i = x_i^T \beta$, for $i = 1, \ldots, n$. It is assumed that the model matrix $X = (x_1, \ldots, x_n)^T$
has full column rank, i.e., \( \text{rank}(X) = p \). The score function and the Fisher information matrix for \( \theta = (\beta^T, \alpha)^T \) are, respectively, given by

\[
U_\theta = (U_\beta^T, U_\alpha)^T, \quad K_\theta = \text{diag}\{K_\beta, K_\alpha\},
\]

where \( U_\beta = (1/2)X^T s \), \( U_\alpha = -n/\alpha + (1/\alpha) \sum_{i=1}^n \xi_i^2 \), \( K_\beta = \psi(\alpha)(X^T X)/4 \), \( K_\alpha = 2n/\alpha^2 \), \( s = (s_1, \ldots, s_n)^T \) with \( s_i = \xi_1 \xi_2 - \xi_2/\xi_1 \) and \( \psi(\alpha) = 2 + 4/\alpha^2 - (\sqrt{2\pi}/\alpha) \{1 - \text{erf}(\sqrt{2}/\alpha)\} \exp(2/\alpha^2) \), \( \text{erf}(\cdot) \) denoting the error function: \( \text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt \) (see, for instance, Gradshteyn and Ryzhik, 2007). From the block-diagonal form of \( K_\theta \), we have that \( \beta \) and \( \alpha \) are globally orthogonal (Cox and Reid, 1987).

The hypothesis of interest is \( \mathcal{H}_0 : \beta_2 = \beta_2^{(0)} \), which will be tested against the alternative hypothesis \( \mathcal{H}_1 : \beta_2 \neq \beta_2^{(0)} \), where \( \beta \) is partitioned as \( \beta = (\beta_1^T, \beta_2^T)^T \), with \( \beta_1 = (\beta_1, \ldots, \beta_q)^T \) and \( \beta_2 = (\beta_{q+1}, \ldots, \beta_p)^T \). Here, \( \beta_2^{(0)} \) is a fixed column vector of dimension \( p - q \). The partition for \( \beta \) induces the corresponding partitions \( U_\beta = (U_{\beta_1}, U_{\beta_2})^T \), with \( U_{\beta_1} = (1/2)X_1 s \) and \( U_{\beta_2} = (1/2)X_2 s \),

\[
K_\beta = \begin{pmatrix}
K_{\beta 11} & K_{\beta 12} \\
K_{\beta 21} & K_{\beta 22}
\end{pmatrix} = \frac{\psi(\alpha)}{4} \begin{pmatrix}
X_1^T X_1 X_1^T X_2 \\
X_2^T X_1 X_1^T X_2
\end{pmatrix},
\]

with the matrix \( X \) partitioned as \( X = (X_1 \quad X_2) \). The likelihood ratio, Wald, score, and gradient statistics for testing \( \mathcal{H}_0 \) can be expressed, respectively, as

\[
S_1 = 2\{\ell(\hat{\theta}) - \ell(\hat{\theta}_0)\}, \quad S_2 = \frac{\psi(\hat{\alpha})}{4} (\hat{\beta}_2 - \beta_2^{(0)})^T (R^T R)(\hat{\beta}_2 - \beta_2^{(0)}),
\]

\[
S_3 = \frac{1}{\psi(\hat{\alpha})} \tilde{s}^T X_2 (R^T R)^{-1} X_2^T \tilde{s}, \quad S_4 = \frac{1}{2} \tilde{s}^T X_2 (\hat{\beta}_2 - \beta_2^{(0)}),
\]

where \( \hat{\theta} = (\hat{\beta}_1^T, \hat{\beta}_2^T, \hat{\alpha})^T \), \( \tilde{\theta} = (\tilde{\beta}_1^T, \tilde{\beta}_2^{(0)T}, \tilde{\alpha})^T \), \( R = X_2 - X_1 (X_1^T X_1)^{-1} X_1^T X_2 \) and \( \tilde{s} = s(\tilde{\theta}) \). The limiting distribution of all these statistics under \( \mathcal{H}_0 \) is \( \chi^2_{p-q} \). Notice that, unlike the Wald and score statistics, the gradient statistic does not involve the error function.

Now, the problem under consideration is that of testing a composite null hypothesis \( \mathcal{H}_0 : \alpha = \alpha^{(0)} \) against \( \mathcal{H}_1 : \alpha \neq \alpha^{(0)} \), where \( \alpha^{(0)} \) is a positive specified value for \( \alpha \), and \( \beta \) acts as a nuisance parameter. The four statistics are expressed as follows:

\[
S_1 = 2\{\ell(\hat{\beta}, \hat{\alpha}) - \ell(\hat{\beta}, \alpha^{(0)})\}, \quad S_2 = 2n \left( \frac{\hat{\alpha} - \alpha^{(0)}}{\alpha^{(0)}} \right)^2,
\]

\[
S_3 = \frac{n(\tilde{\xi}_2 - 1)^2}{2}, \quad S_4 = n(\tilde{\xi}_2 - 1) \left( \frac{\hat{\alpha} - \alpha^{(0)}}{\alpha^{(0)}} \right),
\]

where \( \tilde{\xi}_2 = \tilde{\xi}_2(\tilde{\theta}) = (1/n) \sum_{i=1}^n \xi_i^2(\tilde{\theta}) \), with \( \tilde{\theta} = (\tilde{\beta}^T, \tilde{\alpha})^T \) and \( \tilde{\theta} = (\tilde{\beta}^T, \alpha^{(0)})^T \) representing the unrestricted and restricted maximum likelihood estimators of \( \theta \) under \( \mathcal{H}_1 \) and \( \mathcal{H}_0 \), respectively.
4. Local power

In this section we shall assume the following local alternative hypothesis $H_{1n} : \beta_2 = \beta_2^{(0)} + \epsilon$, where $\epsilon = (\epsilon_{r+1}, \ldots, \epsilon_q)^\top$ with $\epsilon_r = O(n^{-1/2})$ for $r = q + 1, \ldots, p$. We follow the notation in Ferrari et al. (1997). Let

$$A = \begin{pmatrix} A_\beta & 0 \\ 0 & \kappa_{\alpha,\alpha}^{-1} \end{pmatrix}, \quad M = \begin{pmatrix} M_\beta & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$A_\beta = \begin{pmatrix} K_{\beta 11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad M_\beta = K_{\beta 11}^{-1} - A_\beta.$$

It then follows that $m_{ra} = m_{ar} = m_{aa} = 0$, $a_{ra} = a_{ar} = 0$, for $r = 1, \ldots, p$, and $a_{aa} = \kappa_{\alpha,\alpha}^{-1} = \alpha^2 / (2n)$, where $m_{ra}$ and $a_{ra}$ are the $(r, p + 1)$ elements of the matrices $M$ and $A$, respectively, and $m_{aa}$ and $a_{aa}$ are the $(p + 1, p + 1)$ elements of the matrices $M$ and $A$, respectively. Additionally, let

$$\epsilon^* = \begin{pmatrix} K_{\beta 11}^{-1} K_{\beta 12} \\ -I_{p-q} \end{pmatrix} \epsilon,$$

where $I_{p-q}$ is a $(p-q) \times (p-q)$ identity matrix.

The nonnull distributions of the statistics $S_1, S_2, S_3$ and $S_4$ under Pitman alternatives for testing $H_0 : \beta_2 = \beta_2^{(0)}$ in the $BS$ regression model can be expressed as

$$\Pr(S_i \leq x) = G_{p-q, \lambda}(x) + \sum_{k=0}^{3} b_k G_{p-q+2k, \lambda}(x) + O(n^{-1}), \quad i = 1, 2, 3, 4,$$

where $G_{m, \lambda}(x)$ is the cumulative distribution function of a non-central chi-square variate with $m$ degrees of freedom and non-centrality parameter $\lambda$. Here, $\lambda = \epsilon^* \top K \epsilon^*$ and the coefficients $b_k$’s ($i = 1, 2, 3, 4$ and $k = 0, 1, 2, 3$) can be written as

$$b_{11} = -\frac{1}{6} \sum_{r,s,t=1}^{p} (\kappa_{rst} - 2\kappa_{r,s,t}) \epsilon_r^* \epsilon_s^* \epsilon_t^* - \frac{1}{2} \sum_{r,s,t=1}^{p} (\kappa_{rst} + 2\kappa_{r,s,t}) a_{r,s} \epsilon_r^* \epsilon_s^* \epsilon_t^* - \frac{1}{2} \sum_{r=q+1}^{p} \sum_{t=1}^{p} (\kappa_{r,\alpha, \alpha} + 2\kappa_{r,\alpha, \alpha}) \kappa_{\alpha,\alpha}^{-1} \epsilon_r^* \epsilon_t^*,$$

$$\quad - \frac{1}{2} \sum_{r,s,t=1}^{p} (\kappa_{rst} + \kappa_{r,s,t}) \epsilon_r^* \epsilon_s^* \epsilon_t^* - \frac{1}{2} \sum_{t=1}^{p} (\kappa_{\alpha,\alpha, t} + 2\kappa_{\alpha,\alpha, t}) \kappa_{\alpha,\alpha}^{-1} \epsilon_t^*,$$

$$b_{12} = -\frac{1}{6} \sum_{r,s,t=1}^{p} \kappa_{r,s,t} \epsilon_r^* \epsilon_s^* \epsilon_t^*, \quad b_{13} = 0,$$

$$b_{21} = -\frac{1}{2} \sum_{r,s,t=1}^{p} (\kappa_{rst} + 2\kappa_{r,s,t}) \epsilon_r^* \epsilon_s^* \epsilon_t^* + \sum_{r,s,t=1}^{p} \kappa_{rst} \epsilon_r^* \epsilon_t^* + \frac{1}{2} \sum_{r,s,t=1}^{p} (\kappa_{rst} + 2\kappa_{r,s,t}) \kappa_{r,s}^{-1} \epsilon_t^*$$

$$\quad - \frac{1}{2} \sum_{r=q+1}^{p} \sum_{s,t=1}^{p} (\kappa_{r,\alpha, s} + \kappa_{r,\alpha, t}) \epsilon_r^* \epsilon_s^* \epsilon_t^* - \frac{1}{2} \sum_{t=1}^{p} (\kappa_{\alpha,\alpha, t} + 2\kappa_{\alpha,\alpha, t}) \kappa_{\alpha,\alpha}^{-1} \epsilon_t^*. $$
\[
\begin{align*}
\beta_{22} &= \frac{1}{2} \sum_{r,s,t=1}^{p} \kappa_{r,st} \epsilon'^*_s \epsilon'^*_t + \frac{1}{2} \sum_{r,s,t=1}^{p} \kappa_{r,st} m_{r,st} \epsilon'^*_t, \\
\beta_{23} &= \frac{1}{6} \sum_{r,s,t=1}^{p} \kappa_{r,st} \epsilon'^*_s \epsilon'^*_t, \\
\beta_{31} &= -\frac{1}{6} \sum_{r,s,t=1}^{p} (\kappa_{r,st} - 2\kappa_{r,t}) \epsilon'^*_s \epsilon'^*_t + \frac{1}{2} \sum_{r,s,t=1}^{p} \kappa_{r,st} m_{r,st} \epsilon'^*_t - \frac{1}{2} \sum_{r,s,t=1}^{p} (\kappa_{r,t} + 2\kappa_{r,s}) \varrho_{r,st} \epsilon'^*_t \\
&- \frac{1}{2} \sum_{r=s+1}^{p} \sum_{s,t=1}^{p} (\kappa_{r,s} + \kappa_{r,t}) \epsilon'^*_s \epsilon'^*_t - \frac{1}{2} \sum_{t=1}^{p} (\kappa_{r,st} + 2\kappa_{r,at}) \kappa^{-1}_{r,at} \epsilon'^*_t, \\
\beta_{32} &= -\frac{1}{2} \sum_{r,s,t=1}^{p} \kappa_{r,s,t} m_{r,s,t} \epsilon'^*_t, \\
\beta_{33} &= \frac{1}{6} \sum_{r,s,t=1}^{p} \kappa_{r,s,t} \epsilon'^*_s \epsilon'^*_t,
\end{align*}
\]
where the \( \kappa \)'s are defined as \( \kappa_{r,s} = \mathbb{E}(\partial^2 \ell(\theta)/\partial \beta_r \partial \beta_s) \), \( \kappa_{r,t} = \mathbb{E}(\partial^2 \ell(\theta)/\partial \beta_r \partial \beta_t) \), \( \kappa_{r,s} = \mathbb{E}(\{\partial \ell(\theta)/\partial \beta_r\}(\partial^2 \ell(\theta)/\partial \beta_s \partial \beta_t)) \), \( \kappa_{a,at} = \mathbb{E}(\partial^3 \ell(\theta)/\partial \beta_a \partial \beta_t) \), etc., and \( \kappa^{-1} \) is the \((r,s)\)th element of inverse of \( \mathbf{K}_\beta \). The coefficients \( b_{10} \) are obtained from \( b_{10} = -(b_{11} + b_{12} + b_{13}) \), for \( i = 1, 2, 3, 4 \).

After some algebra, it is possible to show that, in the \( BS \) regression model,
\[
\kappa_{r,s} = \kappa_{r,st} = \kappa_{r,a} = \kappa_{a,at} = 0, \quad r, s, t = 1, \ldots, p.
\]
Therefore, \( b_{ik} = 0 \) for \( i = 1, 2, 3, 4 \) and \( k = 0, 1, 2, 3 \), and we can write
\[
\Pr(S_i \leq x) = G_{p-q,\lambda}(x) + O(n^{-1}), \quad i = 1, 2, 3, 4.
\]
This is a very interesting result, which implies that the likelihood ratio, score, Wald and gradient tests for testing the composite null hypothesis \( H_0 : \beta_2 = \beta_2^{(0)} \) have exactly the same local power up to an error of order \( n^{-1} \).

We now turn to the problem of testing hypotheses on \( \alpha \), the shape parameter. The nonnull asymptotic distributions of the statistics \( S_1, S_2, S_3 \) and \( S_4 \) for testing \( H_0 : \alpha = \alpha^{(0)} \) under the local alternative \( H_{1n} : \alpha = \alpha^{(0)} + \epsilon \), where \( \epsilon = \alpha - \alpha^{(0)} \) is assumed to be \( O(n^{-1/2}) \), is
\[
\Pr(S_i \leq x) = G_{1,\lambda}(x) + \sum_{k=0}^{3} b_{ik} G_{1+2k,\lambda}(x) + O(n^{-1}), \quad i = 1, 2, 3, 4,
\]
with \( \lambda = 2n\epsilon^2/\alpha^2 \). The \( b_{ik} \)'s for the test of \( H_0 : \alpha = \alpha^{(0)} \) are easy to obtain and are given by \( b_{11} = (\kappa_{aa} - 2\kappa_{a,a})\epsilon^3/6 + \sum_{r,s=1}^{p} (\kappa_{r,s} + 2\kappa_{r,a})\kappa^{r,s} \epsilon^2/2 - (\kappa_{aa} + \epsilon) \).
Hence, we arrive at the following inequalities: \( \Pi_3 = \Pi_1 - \Pi_2 = 5 \frac{\epsilon}{\alpha} g_5, \lambda(x) + \frac{10n \epsilon^3}{3 \alpha^3} g_7, \lambda(x), \) 
\( \Pi_1 - \Pi_3 = - \left\{ 4 \frac{\epsilon}{\alpha} g_5, \lambda(x) + \frac{8n \epsilon^3}{3 \alpha^3} g_7, \lambda(x) \right\}, \)
\( \Pi_1 - \Pi_4 = - \left\{ 5 \frac{\epsilon}{2 \alpha} g_5, \lambda(x) + \frac{5n \epsilon^3}{3 \alpha^3} g_7, \lambda(x) \right\}, \)
\( \Pi_3 - \Pi_4 = 3 \frac{\epsilon}{2 \alpha} g_5, \lambda(x) + \frac{n \epsilon^3}{\alpha^3} g_7, \lambda(x). \)

Hence, we arrive at the following inequalities: \( \Pi_3 > \Pi_4 > \Pi_1 > \Pi_2 \) if \( \alpha > \alpha^{(0)} \), and \( \Pi_3 < \Pi_4 < \Pi_1 < \Pi_2 \) if \( \alpha < \alpha^{(0)} \).

The most important finding obtained so far is that the likelihood ratio, score, and gradient tests for testing the null hypothesis \( \mathcal{H}_0 : \beta_2 = \beta_2^{(0)} \) share the same null
size and local power up to an error of order $n^{-1}$. To this order of approximation the
null distribution of the four statistics is $\chi^2_{p-q}$. Therefore, if the sample size is large, type
I error probabilities of all the tests do not significantly deviate from the true nominal
level, and their powers are approximately equal for alternatives that are close to the null
hypothesis.

The natural question now is how these tests perform when the sample size is small
or of moderate size, and which one is the most reliable. In the next section, we shall use
Monte Carlo simulations to put some light on this issue.

5. Finite-sample performance

In this section we shall present the results of a Monte Carlo simulation in which we
evaluate the finite sample performance of the likelihood ratio, Wald, score and gradient
tests. The simulations were based on the model

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \varepsilon_i,$$

where $x_{i1} = 1$ and $\varepsilon_i \sim SN(\alpha, 0, 2)$, $i = 1, \ldots, n$. The covariate values were selected
as random draws from the uniform $U(0, 1)$ distribution and for fixed $n$ those values
were kept constant throughout the experiment. The number of Monte Carlo replications
was 15,000, the nominal levels of the tests were $\gamma = 10\%$, $5\%$ and $1\%$, and all simula-
tions were performed using the Ox matrix programming language (Doornik, 2007). Ox
is freely distributed for academic purposes and available at http://www.doornik.com
All log-likelihood maximizations with respect to $\beta$ and $\alpha$ were carried out using the BFGS
quasi-Newton method with analytic first derivatives through the MaxBFGS subroutine.
This method is generally regarded as the best-performing nonlinear optimization method
(Mittelhammer et al., 2000, p. 199). The initial values in the iterative BFGS scheme were
$\tilde{\beta} = (X^\top X)^{-1}X^\top y$ for $\beta$ and $\sqrt{\tilde{\alpha}^2}$ for $\alpha$, where

$$\tilde{\alpha}^2 = \frac{4}{n} \sum_{i=1}^{n} \sinh^2 \left( \frac{y_i - x_i^\top \tilde{\beta}}{2} \right).$$

First, the null hypothesis is $H_0 : \beta_{p-1} = \beta_p = 0$, which is tested against a two-sided
alternative. The sample size is $n = 25$, $\alpha = 0.5$, $1.0$ and $p = 3, 4, \ldots, 7$. The values of the
response were generated using $\beta_1 = \cdots = \beta_{p-2} = 1$. The null rejection rates of the four
tests are presented in Table 1. It is evident that the likelihood ratio ($S_1$) and Wald ($S_2$)
tests are markedly liberal, more so as the number of regressors increases. The score ($S_3$)
and gradient ($S_4$) tests are also liberal in most of the cases, but much less size distor-
ted than the likelihood ratio and Wald tests in all cases. For instance, when $\alpha = 0.5$, $p = 6$
and $\gamma = 5\%$, the rejection rates are $10.12\%$ ($S_1$), $12.77\%$ ($S_2$), $7.12\%$ ($S_3$) and $7.32\%$
($S_4$). It is noticeable that the score test is much less liberal than the likelihood ratio
and Wald tests and slightly less liberal than the gradient test. The score and gradient
tests are slightly conservative in some cases. Additionally, the Wald test is much more
liberal than the other tests. Similar results hold for $\alpha = 1.0$. Table 2 reports results
for $\alpha = 0.5$ and $p = 5$ and sample sizes ranging from 15 to 200. As expected, the null
rejection rates of all the tests approach the corresponding nominal levels as the sample
size grows. Again, the score and gradient tests present the best performances.
Table 1: Null rejection rates (%); $\alpha = 0.5$ and 1.0, with $n = 25$.

| $\alpha$ = 0.5 | $\gamma$ = 10% | $\gamma$ = 5% | $\gamma$ = 1% |
|----------------|----------------|----------------|----------------|
| $p$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 3  | 13.10 | 15.73 | 10.20 | 10.46 | 7.25 | 9.41 | 4.78 | 4.97 | 1.71 | 3.32 | 0.48 | 0.57 |
| 4  | 14.37 | 17.05 | 11.51 | 11.66 | 8.04 | 10.61 | 5.35 | 5.51 | 2.01 | 3.70 | 0.73 | 0.77 |
| 5  | 15.69 | 18.64 | 12.68 | 12.99 | 8.87 | 11.86 | 5.97 | 6.21 | 2.56 | 4.37 | 1.02 | 1.09 |
| 6  | 17.13 | 20.04 | 13.79 | 14.13 | 10.12 | 12.77 | 7.12 | 7.32 | 2.83 | 4.99 | 1.01 | 1.05 |
| 7  | 19.04 | 22.07 | 15.36 | 15.73 | 11.30 | 14.57 | 7.86 | 8.15 | 3.51 | 5.90 | 1.47 | 1.57 |

| $\alpha$ = 1.0 | $\gamma$ = 10% | $\gamma$ = 5% | $\gamma$ = 1% |
|----------------|----------------|----------------|----------------|
| $p$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 3  | 12.69 | 16.27 | 10.18 | 9.20 | 6.82 | 9.91 | 4.00 | 4.71 | 1.66 | 3.46 | 0.41 | 0.55 |
| 4  | 13.94 | 17.60 | 9.84 | 11.07 | 7.64 | 10.87 | 4.23 | 5.07 | 1.89 | 3.81 | 0.55 | 0.75 |
| 5  | 15.91 | 20.41 | 11.15 | 12.69 | 8.76 | 12.53 | 4.92 | 6.05 | 2.35 | 4.59 | 0.64 | 0.93 |
| 6  | 16.74 | 20.65 | 12.28 | 13.86 | 9.77 | 13.61 | 5.77 | 6.89 | 2.71 | 5.39 | 0.84 | 1.09 |
| 7  | 18.67 | 22.90 | 13.49 | 15.51 | 11.27 | 15.59 | 6.77 | 8.03 | 3.55 | 6.59 | 1.00 | 1.47 |

Table 2: Null rejection rates (%); $\alpha = 0.5$, $p = 5$ and different sample sizes.

| $n$ | $\gamma$ = 10% | $\gamma$ = 5% | $\gamma$ = 1% |
|-----|----------------|----------------|----------------|
| $n$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ |
| 15  | 21.37 | 26.38 | 15.41 | 15.85 | 13.41 | 19.02 | 7.47 | 7.89 | 4.81 | 9.47 | 0.88 | 0.97 |
| 20  | 17.65 | 21.24 | 13.73 | 14.11 | 10.81 | 14.41 | 6.83 | 7.10 | 3.41 | 6.20 | 0.99 | 1.07 |
| 30  | 14.71 | 16.95 | 12.39 | 12.50 | 8.24 | 10.45 | 5.84 | 6.07 | 2.17 | 3.60 | 0.88 | 0.93 |
| 40  | 13.49 | 15.38 | 11.73 | 11.91 | 7.25 | 8.93 | 5.47 | 5.66 | 1.65 | 2.57 | 0.88 | 0.90 |
| 50  | 12.93 | 14.27 | 11.54 | 11.80 | 6.87 | 8.21 | 5.69 | 5.73 | 1.54 | 2.14 | 0.96 | 1.04 |
| 100 | 11.77 | 12.30 | 10.95 | 11.01 | 5.79 | 6.28 | 5.25 | 5.31 | 1.23 | 1.51 | 0.93 | 0.97 |
| 200 | 10.89 | 11.11 | 10.49 | 10.57 | 5.61 | 5.90 | 5.33 | 5.38 | 1.10 | 1.25 | 1.00 | 0.99 |
Figure 1: Power of four tests: $n = 25$, $p = 4$, $\alpha = 0.5$ and $\gamma = 5\%$.

We now turn to the finite-sample power properties of the four tests. The simulation results above show that the tests have different sizes when one uses their asymptotic $\chi^2$ distribution in small and moderate-sized samples. In evaluating the power of these tests, it is important to ensure that they all have the correct size under the null hypothesis. To overcome this difficulty, we used 500,000 Monte Carlo simulated samples, drawn under the null hypothesis, to estimate the exact critical value of each test for the chosen nominal level. We set $n = 25$, $p = 4$, $\alpha = 0.5$ and $\gamma = 5\%$. For the power simulations we computed the rejection rates under the alternative hypothesis $\beta_3 = \beta_4 = \delta$, for $\delta$ ranging from $-2.0$ to $2.0$. Figure 1 shows that the power curves of the four tests are indistinguishable from each other. As expected, the powers of the tests approach 1 as $|\delta|$ grows. Power simulations carried out for other values of $n$, $p$ and $\gamma$ showed a similar pattern.

Overall, in small to moderate-sized samples the best performing tests are the score and the gradient tests. They are less size distorted than the other two and are as powerful as the others.

We also performed Monte Carlo simulations considering hypothesis testing on $\alpha$. To save space, the results are not shown. The score and gradient tests exhibited superior behavior than the likelihood ratio and Wald tests. For example, when $n = 35$, $p = 4$, $\gamma = 5\%$ and $H_0 : \alpha = 0.5$, we obtained the following null rejection rates: 9.99% ($S_1$), 15.89% ($S_2$), 5.29% ($S_3$) and 6.99% ($S_4$). Again, the best performing tests are the score and gradient tests.

6. Application

This application focuses on modeling the die lifetime ($T$) in the process of metal extrusion. The data were taken from Lepadatu et al. (2005). According to the authors,
“the estimation of tool life (fatigue life) in the extrusion operation is important for scheduling tool changing times, for adaptive process control and for tool cost evaluation.”

The authors noted that “die fatigue cracks are caused by the repeat application of loads which individually would be too small to cause failure.” The BS regression model is then appealing in this context since the main motivation for the BS distribution is the fatigue failure time due to propagation of an initial crack.

We consider the following regression model:

\[ y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \beta_5 x_{2i} x_{3i} + \beta_6 x_{2i} x_{4i} + \beta_7 x_{3i} x_{4i} + \varepsilon_i, \]

where \( y_i = \log(T_i) \), \( \varepsilon_i \sim SN(\alpha, 0, 2) \), \( x_{1i} = 1 \) and the covariates are \( x_{2i} \) (friction coefficient), \( x_{3i} \) (angle of the die) and \( x_{4i} \) (work temperature), for \( i = 1, \ldots, 15 \). We wish to test the significance of the interaction effects, i.e., the interest lies in testing \( H_0 : \beta_5 = \beta_6 = \beta_7 = 0 \). The likelihood ratio, Wald, score and gradient test statistics equal 6.387 (\( p \)-value: 0.094), 8.039 (\( p \)-value: 0.045), 5.144 (\( p \)-value: 0.162) and 5.206 (\( p \)-value: 0.157), respectively. Hence, the null hypothesis is rejected at the 10% nominal level when inference is based on the likelihood ratio or the Wald test, but the opposite decision is reached when either the score or the gradient test is used. Recall that our simulation results indicated that the likelihood ratio and Wald tests are markedly liberal in small samples (here, \( n = 15 \)), which leads us to mistrust the inference delivered by the likelihood ratio and Wald tests. Therefore, we removed the interaction effects from the model as indicated by the score and gradient tests.

The model containing only main effects is \( y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \varepsilon_i \), for \( i = 1, \ldots, 15 \). The null hypothesis \( H_0 : \beta_3 = 0 \) is strongly rejected by the four tests at the usual significance levels. All tests also suggest the individual and joint exclusions of the friction coefficient and angle of the die from the model. Hence, we end up with the regression model \( y_i = \beta_1 + \beta_4 x_{4i} + \varepsilon_i \), for \( i = 1, \ldots, 15 \). The maximum likelihood estimates of the parameters are (standard errors in parentheses): \( \hat{\beta}_1 = 6.2453 (0.326) \), \( \hat{\beta}_4 = 0.0052 (0.001) \) and \( \hat{\alpha} = 0.2039 (0.037) \).

7. Discussion

The BS regression model is becoming increasingly popular in lifetime analyses and reliability studies. In this paper, we dealt with the issue of performing hypothesis testing concerning the parameters of this model. We considered the three classic tests, likelihood ratio, Wald and score tests, and a recently proposed test, the gradient test. For the discussion that follows, let us concentrate on tests regarding the regression parameters, which are, in general, of primary interest.

The four tests have the same distribution, under either the null hypothesis or a sequence of local alternatives, up to an error of order \( n^{-1} \), as we showed. Our Monte Carlo simulation study added some important information. It revealed that the likelihood ratio and the Wald tests can be remarkably oversized if the sample is small. The score and the gradient tests are clearly much less size distorted than the other two tests. Our power simulations suggested that all the four tests have similar power properties when estimated correct critical values are used. Overall, this is an indication that the score and the gradient tests should be preferred.
At this point, a discussion on small-sample corrections for the classic tests is in order. A Bartlett correction for the likelihood ratio statistic and a Bartlett-type correction for the score statistic were derived in Lemonte et al. (2010) and Lemonte and Ferrari (2010b), respectively; see also Cordeiro and Ferrari (1991). The corrected statistics have the following interesting properties: (i) the uncorrected and corrected statistics have the same asymptotic distribution under the null hypothesis; (ii) the order of the error of the approximation for the distribution of the test statistics by $\chi^2$ is smaller for the corrected statistics than for the uncorrected statistics; (iii) the corrections have no effect on the $n^{-1/2}$ term of the local power of the corresponding tests; (iv) simulation results in Lemonte et al. (2010) and Lemonte and Ferrari (2010b) show that these corrections reduce the size distortion of the tests and that the best performing test in small and moderate-sized samples is the test which uses the corrected score statistic. Therefore, the Bartlett-type-corrected score test is an excellent alternative to the tests under consideration in the present paper. The slight disadvantage of such an alternative is the extra computational burden involved in computing the Bartlett-type correction.

We computed the corrected versions of likelihood ratio and score statistics for the hypotheses tested in the real data application presented in Section 6. Recall that the hypothesis of no interaction effects is rejected by the likelihood ratio and Wald tests, but not rejected by the score and gradient tests. It is noteworthy that the decision reached by either the corrected likelihood ratio test or the corrected score test is in agreement with that obtained by the later two tests and in disagreement with the likelihood ratio and Wald tests, which tend to reject the null hypothesis much more often than indicated by the significance level.

Finally, our overall recommendations for practitioners when performing testing inference in BS regressions are as follows. The score test or the gradient test should be preferred as both perform better than the likelihood ratio and Wald tests in small and moderate-sized samples. While the gradient test is a little more liberal than the score test, it is easier to calculate. The Bartlett-type corrected score test is a further better option although it requires a small extra computational effort.

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