MEET-DISTRIBUTIVE LATTICES HAVE THE INTERSECTION PROPERTY

HENRI MÜHLE

ABSTRACT. Meet-distributive lattices form an intriguing class of lattices, because they are precisely the lattices obtainable from a closure operator with the so-called anti-exchange property. Moreover, meet-distributive lattices are join semidistributive. Therefore, they admit two natural, secondary structures: the core label order is an alternative order on the lattice elements and the canonical join complex is the flag-simplicial complex on canonical join representations. In this article we present a characterization of finite meet-distributive lattices in terms of the core label order and the canonical join complex, and we show that the core label order of a finite meet-distributive lattice is always a meet-semilattice.

1. INTRODUCTION

A lattice $L$ is join semidistributive if every element admits a canonical expression as a join of join-irreducible elements [17, 18]. Consequently, the word problem can be solved efficiently in these lattices. The set of canonical join representations of a lattice forms a simplicial complex [15, Proposition 2.2]; the canonical join complex of $L$. If $L$ is join semidistributive, then the faces of the canonical join complex are naturally indexed by the elements of $L$.

Moreover, when $L$ is join semidistributive, canonical join representations can be computed easily with the help of a certain edge-labeling which is determined by a perspectivity relation [4]. This labeling is essentially unique and can be used to define an alternative partial order on $L$; the core label order.

This order first appeared N. Reading’s research on congruence-uniform lattices of regions of real hyperplane arrangements. We have investigated this order abstractly for congruence-uniform lattices in [11]. For some special cases the core label order was studied in [3, 5, 9, 10, 12–14].

An interesting subclass of join-semidistributive lattices are meet-distributive lattices, which have the property that every interval $[x, y]$—where $x$ is the meet of the elements covered by $y$—is isomorphic to a Boolean lattice [6, 7]. It turns out that we can use the core label order and the canonical join complex to characterize meet-distributive lattices.

Theorem 1.1. A finite join-semidistributive lattice $L$ is meet-distributive if and only if $\text{CLO}(L)$ is the face poset of the canonical join complex of $L$.

We want to point out that we can also use the core label order to characterize finite Boolean lattices. They are precisely the join-semidistributive lattices that are

2010 Mathematics Subject Classification. 06D75.

Key words and phrases. meet-distributive lattices, congruence-uniform lattices, canonical join complex, core label order, intersection property.

The author has received funding from the European Research Council (Grant Agreement no. 681988, CSP-Infinity).
isomorphic to their own core label order [11, Theorem 1.5]. Consequently, the canonical join complex of a finite Boolean lattice is a simplex.

In [16, Problem 9.5], N. Reading asked under what conditions the core label order is again a lattice. In [11, Section 4.2] we found one such property, which we call the intersection property. This property can be used to characterize the join-semidistributive lattices whose core label orders are meet-semilattices [11, Theorem 4.8]. We conclude this article with the observation that every meet-distributive lattice has the intersection property.

**Theorem 1.2.** Every finite meet-distributive lattice $L$ has the intersection property. Consequently, $\text{CLO}(L)$ is a meet-semilattice, and it is a lattice if and only if $L$ is isomorphic to a Boolean lattice.

We first recall the necessary basic notions in Section 2. After that we define the core label order of a lattice in Section 3.1, and we define the canonical join complex of a join-semidistributive lattice in Section 3.3, where we also prove Theorem 1.1. In Section 3.4 we define the intersection property and prove Theorem 1.2.

2. Preliminaries

2.1. Basic Notions. Let $P = (P, \leq)$ be a partially ordered set (poset for short). The dual poset of $P$ is $P^* \overset{\text{def}}{=} (P, \geq)$.

An element $x \in P$ is minimal in $P$ if $y \leq x$ implies $y = x$ for all $y \in P$. Dually, $x \in P$ is maximal in $P$ if it is minimal in $P^*$.

A cover relation of $P$ is a pair $(x, y)$ such that $x < y$ and there is no $z \in P$ such that $x < z < y$. We usually write $x \lessdot y$ for a cover relation, and we denote the set of all cover relations of $P$ by $E(P)$. Moreover, if $x \lessdot y$, then we call $x$ a lower cover of $y$, and $y$ an upper cover of $x$.

A chain of $P$ is a totally ordered subset of $P$, and it is saturated if it can be written as a sequence of cover relations. A saturated chain is maximal if it contains a minimal and a maximal element of $P$.

We say that $P$ is a lattice if for every two elements $x, y \in P$ there exists a greatest lower bound $x \land y$ (the meet) and a least upper bound $x \lor y$ (the join). Observe that every finite lattice has a unique minimal element (denoted by $\emptyset$) and a unique maximal element (denoted by $\emptyset'$).

A lattice is Boolean if it is isomorphic to the family of subsets of some set $M$ ordered by inclusion. If $|M| = n$, then we write $\text{Bool}(n)$ for the Boolean lattice with $2^n$ elements.

2.2. Join-Semidistributive Lattices. Let $L = (L, \leq)$ be a lattice. A join representation of $x \in L$ is a set $X \subseteq L$ with $x = \bigvee X$. A join representation $X$ of $x$ join-refines a join representation $X'$ of $x$ if for every $y \in X$ there exists some $y' \in X'$ such that $y \leq y'$. A join representation $X$ of $x$ is irredundant if no proper subset of $X$ joins to $x$, and it is canonical if it join-refines every other join representation of $x$. We denote the canonical join representation of $x \in L$ by $\Gamma(x)$ (if it exists).

It turns out that the finite lattices in which every element admits a canonical join representation can be characterized algebraically. A lattice $L = (L, \leq)$ is join semidistributive if for all $x, y, z \in L$ the following implication holds:

\[(\text{JSD}) \quad x \lor y = x \lor z \implies x \lor y = x \lor (y \land z).\]
2.3. Meet-Distributive Lattices. We now move to a subfamily of the join-semidistributive lattices. Let \( L = (L, \leq) \) be a lattice. For \( x \in L \), we define its \textit{nucleus} to be

\[
x_{\downarrow} \overset{\text{def}}{=} x \land \bigwedge_{y \in L : y \leq x} y.
\]

We call the interval \([x_{\downarrow}, x]\) the \textit{core} of \( x \). Then, \( L \) is \textit{meet distributive} if for every \( x \in L \), the core \([x_{\downarrow}, x]\) is isomorphic to a Boolean lattice. Figure 1a shows a join-semidistributive lattice that is not meet distributive, and Figure 2a shows a meet-distributive lattice.

The following result characterizes meet-distributive lattices. Recall that \( L \) is \textit{lower semimodular} if for all \( x, y \in L \) whenever \( x, y \leq x \lor y \), then \( x \land y \leq x, y \).

\textbf{Theorem 2.2} ([1, Theorem 1.9]). A finite lattice is meet distributive if and only if it is join semidistributive and lower semimodular.

Meet-distributive lattices are precisely the lattices that arise from a closure operator satisfying the so-called anti-exchange property, see [1,2,7].

3. THE CORE LABEL ORDER OF A JOIN-SEMDISTRIBUTIVE LATTICE

3.1. The Core Label Order. Motivated by the study of the poset of regions of real hyperplane arrangements, N. Reading introduced an alternate way to order
the elements of a congruence-uniform lattice [16, Section 9-7.4]. In fact, we may
generalize this construction to arbitrary, finite lattices.

Let \( L = (L, \leq) \) be a finite lattice, let \( M \) be a set and let \( \lambda : E(L) \to M \) be an edge
labeling of \( L \). The core label set of \( x \in L \) (with respect to \( \lambda \)) is
\[
\Psi_{\lambda}(x) \overset{\text{def}}{=} \{ \lambda(u,v) \mid x \downarrow u \ll v \leq x \}.
\]
We may now define \( x \leq_{\text{clo}} y \) if and only if \( \Psi_{\lambda}(x) \subseteq \Psi_{\lambda}(y) \). In general, this results
in a quasi-ordered set \( \text{CLO}_{\lambda}(L) = (L, \leq_{\text{clo}}) \).

We say that \( \lambda \) is a core labeling if the assignment \( x \mapsto \Psi_{\lambda}(x) \) is injective. If \( \lambda \) is a
core labeling, then it is quickly checked that \( \text{CLO}(L) \) is in fact a partial order; the
core label order. Figures 1 and 2 illustrate this construction.

### 3.2. A Perspectivity Labeling

Two cover relations \((x_1, y_1), (x_2, y_2) \in E(L)\) are 
perspective if either \( y_1 \lor x_2 = y_2 \) and \( y_1 \land x_2 = x_1 \) or \( y_2 \lor x_1 = y_1 \) and \( y_2 \land x_1 = x_2 \).
We write \( (x_1, y_1) \nleq (x_2, y_2) \) in this case. This definition is illustrated in Figure 3.

Recall another useful fact about join-semidistributive lattices. An element \( j \in L \)
is join irreducible if whenever \( j = x \lor y \), then \( j \in \{x, y\} \). The set of join-irreducible
elements of \( L \) is denoted by \( \text{JoinIrr}(L) \). In particular, if \( L \) is finite and \( j \in \text{JoinIrr}(L) \),
then there exists a unique element \( j_* \in L \) such that \((j_*, j) \in E(L)\).

**Lemma 3.1** ([1, Lemma 1.8]). Let \( L \) be a finite join-semidistributive lattice. For \((x, y) \in E(L)\), the set 
\( \{ z \in L \mid z \leq y \text{ and } z \nleq x \} \) has a unique minimal element \( j \), and \( j \) is join
irreducible.

This gives rise to the following edge-labeling of a finite, join-semidistributive
lattice \( L \):

\[
\lambda_{\text{jad}} : E(L) \to \text{JoinIrr}(L), \quad (x, y) \mapsto \bigwedge \{ z \in L \mid z \leq y \text{ and } z \nleq x \}.
\]

This labeling is illustrated in Figures 1a and 2a.

**Lemma 3.2.** Let \((x, y) \in E(L)\) and \( j \in \text{JoinIrr}(L) \). If \((x, y) \nleq (j_*, j) \), then \( j \leq y \).

**Proof.** If \((x, y) \nleq (j_*, j) \), then either \( j \leq y \) or \( y \leq j \). The latter case, however, forces
the existence of two lower covers of \( j \), contradicting that \( j \) is join irreducible. \( \square \)

We now show that the labeling \( \lambda_{\text{jad}} \) is a canonical labeling of a finite, join-semi-
distributive lattice, because it is determined by the perspectivity relation.

**Lemma 3.3.** Let \((x, y) \in E(L)\). Then \( \lambda_{\text{jad}}(x, y) = j \) if and only if \((j_*, j) \nleq (x, y) \).

![Figure 3. The green edges represent perspective cover relations.](image-url)
Proof. Suppose that $\lambda_{\text{jsd}}(x, y) = j$. By definition, $x \lor j = y$ and thus $x \land j < j$. Since $j$ is minimal with the property that $j \leq y$ and $j \not\leq x$, we see that $j_1 \leq x$. This implies $x \land j = j_1$, and it follows that $(j_1, j) \nleq (x, y)$.

Conversely, suppose that $(j_1, j) \nleq (x, y)$. By Lemma 3.2, we get $j \lor x = y$ and $j \land x = j_1$. Thus, $\lambda_{\text{jsd}}(x, y) \leq j$. But, $j_1 \leq x$, which means that $\lambda_{\text{jsd}}(x, y) \not\leq j$. Since $j \in \text{Joinirr}(L)$, we must have $\lambda_{\text{jsd}}(x, y) = j$. □

The labeling $\lambda_{\text{jsd}}$ also allows for a simple computation of canonical join representations.

**Proposition 3.4 ([4, Lemma 19]).** If $L = (L, \leq)$ is a finite, join-semidistributive lattice, then for every $x \in L$:

$$\Gamma(x) = \{\lambda_{\text{jsd}}(x', x) \mid x' < x\}.$$  

**Proposition 3.5.** The edge-labeling $\lambda_{\text{jsd}}$ of a finite, join-semidistributive lattice is a core labeling.

Proof. Let $L = (L, \leq)$ be a finite, join-semidistributive lattice, and let $x \in L$.

If $j \in \Psi_{\lambda_{\text{jsd}}}(x)$, then there exist $x_1, x_2 \in L$ such that $x_1 \leq x_1 \land x_2 \leq x$ such that $\lambda_{\text{jsd}}(x_1, x_2) = j$. By Lemma 3.3, this means that $(j_1, j) \nleq (x_1, x_2)$ and by Lemma 3.2 it follows that $j \leq x_2 \leq x$. As a consequence, $\bigvee \Psi_{\lambda_{\text{jsd}}}(x) \leq x$. Moreover, by Proposition 3.4, we have $\Gamma(x) \subseteq \Psi_{\lambda_{\text{jsd}}}(x)$, and therefore $x = \bigvee \Gamma(x) \subseteq \bigvee \Psi_{\lambda_{\text{jsd}}}(x)$.

It follows that $\bigvee \Psi_{\lambda_{\text{jsd}}}(x) = x$.

Now, if there exist $x, y \in L$ such that $\Psi_{\lambda_{\text{jsd}}}(x) = \Psi_{\lambda_{\text{jsd}}}(y)$, then

$$x = \bigvee \Psi_{\lambda_{\text{jsd}}}(x) = \bigvee \Psi_{\lambda_{\text{jsd}}}(y) = y.$$ 

Hence, the assignment $x \mapsto \Psi_{\lambda_{\text{jsd}}}(x)$ is injective, and $\lambda_{\text{jsd}}$ is a core labeling. □

**Theorem 3.6.** Let $L = (L, \leq)$ be a finite, join-semidistributive lattice. Then we have $\Gamma(x) = \Psi_{\lambda_{\text{jsd}}}(x)$ for all $x \in L$ if and only if $L$ is meet distributive.

Proof. If $L$ is meet distributive, then every core $[x_1, x]$ is isomorphic to a Boolean lattice. If Bool($k$) is the Boolean lattice with the ground set $M = \{1, 2, \ldots, k\}$, then it is easy to verify that $\Gamma(M) = M = \Psi_{\lambda_{\text{jsd}}}(M)$. This proves that $\Gamma(x) = \Psi_{\lambda_{\text{jsd}}}(x)$ for all $x \in L$.

Conversely, suppose that $L$ is not meet distributive. By Theorem 2.2, $L$ is not lower semimodular, which means that there exist two elements $x, y \in L$ such that $x, y \lneq x \lor y$ and—without loss of generality—$(x \land y, x) \notin \mathcal{E}(L)$. This means that there exists $z \in L$ with $x \land y < z \lneq x$. Suppose that $\lambda_{\text{jsd}}(z, x) = j$. By construction, $j \in \Psi_{\lambda_{\text{jsd}}}(x \lor y)$. By perspectivity, $j \neq \lambda_{\text{jsd}}(x, x \lor y)$.

Since $j \lneq x$ and $j \lneq z$, the assumption $x \land y < z$ implies that $j \lneq y$. Moreover, $z \lneq y$ because otherwise $z = x \land y$. This implies that $j \lor y = x \lor y = z \lor y$ and by (JSD) we get $y \neq x \lor y = y \lor (z \land j) = y \lor j_1$. Thus, $j_1 \neq y$ and since $j \in \text{Joinirr}(L)$, we find $y \land j \neq j$. It follows that $\lambda_{\text{jsd}}(y, x \lor y) \neq j$.

If $x, y$ are the only lower covers of $x \lor y$, then we have just shown that $j \notin \Gamma(x \lor y)$, which yields $\Gamma(x \lor y) \subseteq \Psi_{\lambda_{\text{jsd}}}(x \lor y)$.

Suppose that there exists another lower cover $u$ of $x \lor y$ (different from $x$ and $y$). If $z \leq u$, then we get $x \lor y = x \lor u = y \lor u$ and therefore by (JSD) $x \lor y = u \lor (x \land y) \leq u \lor z = u$; which is a contradiction. If $j \leq u$, then $\lambda_{\text{jsd}}(u, x \lor y) \neq j$ by Lemma 3.3. Otherwise, we get $j \lor u = x \lor y = z \lor u$ and therefore $u \neq x \lor y = u \lor x \lor y = x \lor y = x \lor y$. This proves that $\Gamma(x \lor y) = \Psi_{\lambda_{\text{jsd}}}(x \lor y)$.
Boolean defect of a join-semidistributive lattice \( L = (L, \leq) \) by
\[
\text{bdef}(L) \overset{\text{def}}{=} \sum_{x \in L} |\Psi_{\lambda_{\text{jsd}}}(x) \setminus \Gamma(x)|.
\]

Theorem 3.6 has the following consequence, which strengthens [11, Proposition 5.2].

**Corollary 3.7.** A finite join-semidistributive lattice \( L \) has \( \text{bdef}(L) = 0 \) if and only if \( L \) is meet distributive.

### 3.3. The Canonical Join Complex of a Join-Semidistributive Lattice

Given a finite set \( M \), a *simplicial complex* on \( M \) is a family \( \Delta(M) \) of subsets of \( M \), such that for every \( F \in \Delta(M) \) and every \( F' \subseteq F \) we have \( F' \in \Delta(M) \). The members of \( \Delta(M) \) are faces. The face poset of \( \Delta(M) \) is the poset \((\Delta(M), \subseteq)\).

N. Reading has observed in [15, Proposition 2.2] that the set of canonical join representations of a lattice is closed under taking subsets. In other words, it forms a simplicial complex; the *canonical join complex* of \( L \), denoted by \( \text{Can}(L) \).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( L = (L, \leq) \) be a finite, join-semidistributive lattice. By definition, the face poset of \( \text{Can}(L) \) is precisely \( \left\{ \Gamma(x) \mid x \in L \right\}, \subseteq \), and \( \text{CLO}_{\lambda_{\text{jsd}}}(L) \) is isomorphic to \( \left\{ \Psi(x) \mid x \in L \right\}, \subseteq \).

If \( L \) is meet distributive, then Theorem 3.6 states that these two posets are isomorphic.

If \( L \) is not meet distributive, then by Theorem 3.6, there exists some \( x \in L \) such that \( \Gamma(x) \subseteq \Psi_{\lambda_{\text{jsd}}}(x) \). In particular, there exists \( j \in \Psi_{\lambda_{\text{jsd}}}(x) \setminus \Gamma(x) \). It follows that \( \{j\} \subseteq \Psi_{\lambda_{\text{jsd}}}(x) \), but \( \{j\} \not\subseteq \Gamma(x) \), so that the core label order of \( L \) is not isomorphic to the face poset of \( \text{Can}(L) \).

Figure 4 illustrates Theorem 1.1 on a bigger example.

### 3.4. The Intersection Property

N. Reading asked in [16, Problem 9.5] for conditions on a congruence-uniform lattice \( L \) which would imply that \( \text{CLO}(L) \) is a lattice, too. We gave one such property in [11, Section 4.2], which extends to arbitrary lattices as follows. A finite \( L = (L, \leq) \) with edge labeling \( \lambda \) has the intersection property if for all \( x, y \in L \) there exists \( z \in L \) such that \( \Psi_{\lambda}(x) \cap \Psi_{\lambda}(y) = \Psi_{\lambda}(z) \).

Provided that \( \lambda \) is a core labeling, the proof of [11, Theorems 1.3 and 4.7] carries over essentially verbatim to the more general case.

**Theorem 3.8** ([11, Theorems 1.3 and 4.7]). Let \( L \) be a finite lattice with core labeling \( \lambda \). The core label order \( \text{CLO}_{\lambda}(L) \) is a meet-semilattice if and only if \( L \) has the intersection property. It is a lattice if and only if \( \hat{1}_L = \hat{0} \).

We conclude this article with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let \( L = (L, \leq) \) be a finite meet-distributive lattice. For \( x, y \in L \) we conclude from Theorem 3.6 that \( \Psi_{\lambda_{\text{jsd}}}(x) = \Gamma(x) \) and \( \Psi_{\lambda_{\text{jsd}}}(y) = \Gamma(y) \). It follows that \( Z = \Gamma(x) \cap \Gamma(y) \) is a face of \( \text{Can}(L) \), which means that there exists \( z \in L \) with \( Z = \Gamma(z) = \Psi_{\lambda_{\text{jsd}}}(z) \). We have thus established that \( L \) has the intersection property.
Lemma 3.9 of [11] states that $\text{CLO}(L)$ has a greatest element if and only if $\hat{1} \downarrow = \hat{0}$. Now, if $L$ is meet-distributive, then the interval $[\hat{1} \downarrow, \hat{1}]$ is isomorphic to a Boolean lattice. Thus, $\hat{1} \downarrow = \hat{0}$ if and only if $L$ is Boolean. The claims then follows from Theorem 3.8. □

REFERENCES

[1] Kira V. Adaricheva, Viktor A. Gorbunov, and V. I. Tumanov, Join-Semidistributive Lattices and Convex Geometries, Advances in Mathematics 173 (2003), 1–49.
[2] Drew Armstrong, The Sorting Order on a Coxeter Group, Journal of Combinatorial Theory, Series A 116 (2009), 1285–1305.
[3] Erin Bancroft, The Shard Intersection Order on Permutations (2011), available at arXiv:1103.1910.
[4] Emily Barnard, The Canonical Join Complex, The Electronic Journal of Combinatorics 26 (2019), Research paper P1.24, 25 pages.
[5] Alexander Clifton, Peter Dillery, and Alexander Garver, The Canonical Join Complex for Biclosed Sets, Algebra Universalis 79 (2018).
[6] Robert P. Dilworth, Lattices with Unique Irreducible Decompositions, Annals of Mathematics 41 (1940), 771–777.
[7] Paul H. Edelman, Meet-Distributive Lattices and the Anti-Exchange Closure, Algebra Universalis 10 (1980), 290–299.
[8] Ralph Freese, Jaroslav Ježek, and James B. Nation, Free Lattices, American Mathematical Society, Providence, 1995.
[9] Alexander Garver and Thomas McConville, Enumerative Properties of Grid-Associahedra (2017), available at arXiv:1705.04901.
[10] Alexander Garver and Thomas McConville, Oriented Flip Graphs of Polygonal Subdivisions and Noncrossing Tree Partitions, Journal of Combinatorial Theory (Series A) 158 (2018), 126–175.
[11] Henri Mühle, The Core Label Order of a Congruence-Uniform Lattice, Algebra Universalis 80 (2019), Research paper 10, 22 pages.
[12] Henri Mühle, Noncrossing Arc Diagrams, Tamari Lattices, and Parabolic Quotients of the Symmetric Group, Annals of Combinatorics (2021). To appear.
[13] T. Kyle Petersen, On the Shard Intersection Order of a Coxeter Group, SIAM Journal on Discrete Mathematics 27 (2013), 1880–1912.
[14] Nathan Reading, Noncrossing Partitions and the Shard Intersection Order, Journal of Algebraic Combinatorics 33 (2011), 483–530.
[15] Nathan Reading, Noncrossing Arc Diagrams and Canonical Join Representations, SIAM Journal on Discrete Mathematics 29 (2015), 736–750.
[16] Nathan Reading, Lattice Theory of the Poset of Regions (George Grätzer and Friedrich Wehrung, eds.), Vol. 2, Birkhäuser, Cham, 2016.
[17] Philip M. Whitman, Free Lattices, Annals of Mathematics 42 (1941), 325–330.
[18] Philip M. Whitman, Free Lattices II, Annals of Mathematics 43 (1942), 104–115.

TECHNISCHE UNIVERSITÄT DRESDEN, INSTITUT FÜR ALGEBRA, ZELLESCHER WEG 12–14, 01069 DRESDEN, GERMANY.

Email address: henri.muehle@tu-dresden.de