STOCHASTIC HAMILTONIAN FLOWS WITH SINGULAR COEFFICIENTS

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Dedicated to the 60th birthday of Professor Michael Röckner

ABSTRACT. In this paper we study the following stochastic Hamiltonian system in $\mathbb{R}^{2d}$ (a second order stochastic differential equation),
\[
\text{d} \dot{X}_t = b(X_t, \dot{X}_t) \text{d}t + \sigma(X_t, \dot{X}_t) \text{d}W_t, \quad (X_0, \dot{X}_0) = (x, v) \in \mathbb{R}^{2d},
\]
where $b(x, v) : \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\sigma(x, v) : \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions. We show that if $\sigma$ is bounded and uniformly non-degenerate, and $b \in H^{2/3,0}_p$ and $\nabla \sigma \in L^p$ for some $p > 2(2d + 1)$, where $H^{\alpha,\beta}_p$ is the Bessel potential space with differentiability indices $\alpha$ in $x$ and $\beta$ in $v$, then the above stochastic equation admits a unique strong solution so that $(x, v) \mapsto (X_t, \dot{X}_t)(x, v)$ forms a stochastic homeomorphism flow, and $(x, v) \mapsto Z_t(x, v)$ is weakly differentiable with $\text{ess.sup}_{x, v} \mathbb{E} \left( \sup_{t \in [0,T]} |\nabla Z_t(x, v)|^q \right) < \infty$ for all $q \geq 1$ and $T > 0$. Moreover, we also show the uniqueness of probability measure-valued solutions for kinetic Fokker-Planck equations with rough coefficients by showing the well-posedness of the associated martingale problem and using the superposition principle established by Figalli [14] and Trevisan [33].

1. Introduction

Consider the following second order time dependent stochastic differential equation (abbreviated as SDE):
\[
\text{d} \dot{X}_t = b_t(X_t, \dot{X}_t) \text{d}t + \sigma_t(X_t, \dot{X}_t) \text{d}W_t, \quad (X_0, \dot{X}_0) = (x, v) \in \mathbb{R}^{2d},
\]
where $b_t(x, v) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\sigma_t(x, v) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions, $\dot{X}_t$ denotes the first order derivative of $X_t$ with respect to $t$, and $W_t$ is a $d$-dimensional standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $\sigma = 0$, the above equation is the classical Newtonian mechanic equation, which describes the motion of a particle. When $\sigma \neq 0$, it means that the motion is perturbed by some random external force. More backgrounds about the above stochastic Hamiltonian system are referred to [29], [32], etc.

Key words and phrases. Stochastic Hamiltonian system, Weak differentiability, Krylov’s estimate, Zvonkin’s transformation, Kinetic Fokker-Planck operator.

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It is noticed that if we let \( Z_t := (X_t, \dot{X}_t) \), then \( Z_t \) solves the following one order (degenerate) SDE:
\[
\begin{align*}
\mathrm{d}Z_t &= (\dot{X}_t, b_t(Z_t))\mathrm{d}t + (0, \sigma_t(Z_t))\mathrm{d}W_t, \quad Z_0 = z = (x, v) \in \mathbb{R}^d, \quad (1.1)
\end{align*}
\]
and whose time-dependent infinitesimal generator is given by
\[
\mathcal{L}_t^{a,b} f(x, v) := \text{tr}(a_t \cdot \nabla_v^2 f)(x, v) + (v \cdot \nabla_x f)(x, v) + (b_t \cdot \nabla_v f)(x, v). \quad (1.2)
\]
Here \( a_t(x, v) := \frac{1}{2}(\sigma_t \sigma_t^*)(x, v) \), \( \nabla_v^2 f(x, v) \) stands for the Hessian matrix, the asterisk and \( \text{tr}(\cdot) \) denote the transpose and the trace of a matrix respectively. Moreover, let \( \mu_t \) be the probability distributional measure of \( Z_t \) in \( \mathbb{R}^{2d} \). By Itô’s formula, one knows that \( \mu_t \) solves the following Fokker-Planck equation in the distributional sense:
\[
\partial_t \mu_t = (\mathcal{L}_t^{a,b})^* \mu_t, \quad \mu_0 = \delta_z, \quad (1.3)
\]
where \( \delta_z \) is the Dirac measure at \( z \). More precisely, for any \( f \in C^2_\text{c}(\mathbb{R}^{2d}) \),
\[
\partial_t \mu_t(f) = \mu_t(\mathcal{L}_t^{a,b} f), \quad \mu_0(f) = f(z),
\]
where \( \mu_t(f) = \int f \mathrm{d}\mu_t = \mathbb{E} f(Z_t) \). In the literature \( \mathcal{L}_t^{a,b} \) is also called kinetic Fokker-Planck or Kolmogorov’s operator.

During the past decade, there is an increasing interest in the study of SDEs with singular or rough coefficients. In the non-degenerate case, Krylov and Röckner [21] showed the strong uniqueness to the following SDE in \( \mathbb{R}^d \):
\[
\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = x,
\]
where \( b \in L^q_\text{loc}(\mathbb{R}^d; L^p(\mathbb{R}^d)) \) with \( \frac{d}{p} + \frac{2}{q} < 1 \). The argument in [21] is based on Girsanov’s theorem and some estimates from the theory of PDE. In this framework, Federizzi and Flandoli [12, 11] studied the well-posedness of stochastic transport equations with rough coefficients. When \( b \) is bounded measurable, the Malliavin differentiability of \( X_t \) with respect to sample path \( \omega \) and the weak differentiability of \( X_t \) with respect to starting point \( x \) were recently studied in [23] and [25] respectively. We also mention that weak uniqueness was studied in [1] and [17] under rather weak assumptions on \( b \) (belonging to some Kato’s class). Moreover, the multiplicative noise case was studied in [38, 39, 41] by using Zvonkin’s transformation [42] and some careful estimates of second order parabolic equations.

In the degenerate case, Chaudru de Raynal [7] firstly showed the strong well-posedness for SDE (1.1) under the assumptions that \( \sigma \) is Lipschitz continuous and \( b \) is \( \alpha \)-Hölder continuous in \( x \) and \( \beta \)-Hölder continuous in \( v \) with \( \alpha \in (\frac{2}{3}, 1) \) and \( \beta \in (0, 1) \). The proofs in [7] strongly depend on some explicit estimates for Kolmogorov operator with constant coefficients and Zvonkin’s transformation. In a recent joint work [36] with F.Y. Wang, we also showed the strong uniqueness and homeomorphism property for (1.1) under weaker Hölder-Dini’s continuity assumption on \( b \). The proofs in [36] rely on a characterization of Hölder-Dini’s spaces and gradient estimates for
the semigroup associated with the kinetic operator. Notice that in \cite{7} and \cite{36}, more general degenerate SDEs are considered, while, the case with critical differentiability indices \( \alpha = \frac{2}{3} \) and \( \beta = 0 \) is left open.

The purpose of this work is to establish a similar theory for degenerate SDE (1.1) as in Krylov and Röckner’s paper \cite{21} (see also \cite{41}). In particular, the critical indices \( \alpha = \frac{2}{3} \) and \( \beta = 0 \) are covered. More precisely, we aim to prove that

**Theorem 1.1.** Suppose that for some \( K \geq 1 \) and all \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}\),

\[
K^{-1} |\xi| \leq |\sigma^*_t(x, v)\xi| \leq K |\xi|, \quad \forall \xi \in \mathbb{R}^d,
\]

where \( \sigma^* \) denotes the transpose of matrix \( \sigma \), and for some \( p > 2(2d + 1) \),

\[
\kappa_0 := \sup_{s > 0} \|\nabla \sigma_s\|_p^p + \int_0^\infty \left( \|(-\Delta)^{1/4} b_s\|_p^p \right) ds < \infty.
\]

Then for any \( z = (x, v) \in \mathbb{R}^{2d} \), SDE (1.1) admits a unique strong solution \( Z_t(z) = (X_t, \dot{X}_t) \) so that \((t, z) \mapsto Z_t(z)\) has a bi-continuous version. Moreover,

(A) There is a null set \( N \) such that for all \( \omega \notin N \) and for each \( t \geq 0 \), the map \( z \mapsto Z_t(z, \omega) \) is a homeomorphism on \( \mathbb{R}^{2d} \).

(B) For each \( t \geq 0 \), the map \( z \mapsto Z_t(z) \) is weakly differentiable a.s., and for any \( q \geq 1 \) and \( T > 0 \),

\[
\text{ess. sup} \mathbb{E} \left( \sup_{t \in [0, T]} |\nabla Z_t(z)|^q \right) < \infty, \quad (1.4)
\]

where \( \nabla \) denotes the generalized gradient.

(C) Let \( \sigma^n \) and \( b^n \) be the regularized approximations of \( \sigma \) and \( b \) (see (5.2) below for definitions). Let \( Z^n \) be the corresponding solution of SDE (1.1) associated with \( (\sigma^n, b^n) \). For any \( q \geq 1 \) and \( T > 0 \), there exits a constant \( C > 0 \) only depending on \( T, K, \kappa_0, d, p, q \) such that

\[
\mathbb{E} \left( \sup_{t \in [0, T]} |Z^n_t - Z_t|^q \right) \leq C \left( \|b^n - b\|_{L^p(T)}^q + n^{2(2d - 1)q} \right), \quad n \in \mathbb{N}.
\]

As a corollary, we have the following local well-posedness result by a standard localization argument.

**Corollary 1.2.** Suppose that for any \( T, R > 0 \), there exists a constant \( K_{T, R} \geq 1 \) such that for all \((t, x, v) \in [0, T] \times B_R \) and \( \xi \in \mathbb{R}^d \),

\[
K_{T, R}^{-1} |\xi| \leq |\sigma^*_t(x, v)\xi| \leq K_{T, R} |\xi|, \quad (1.5)
\]

where \( B_R := \{(x, v) : |(x, v)| \leq R \} \), and for some \( p > 2(2d + 1) \),

\[
\sup_{t \in [0, T]} \|\nabla (\sigma_t \chi_R)\|_p^p + \int_0^T \|(-\Delta)^{1/4} (b_s \chi_R)\|_p^p ds \leq K_{T, R},
\]
Proof. Let
\[ \sigma_t^R(z) := \sigma_t(z \chi_R(z)), \quad b_t^R(z) := b_t(z) \chi_R(z). \]

By the assumptions, one sees that \((\sigma^R, b^R)\) satisfies the conditions of Theorem 1.1. Hence, there exists a unique solution to the following SDE:
\[ dZ_t^R = (X_t^R, b_t^R(Z_t^R))dt + (0, \sigma_t^R(Z_t^R))dW_t, \quad Z_0^R = z = (x, v) \in \mathbb{R}^{2d}, \]
where \(Z_t^R = (X_t^R, X_t^v).\) Define
\[ \zeta_R := \inf \{ t \geq 0 : |Z_t^R| \geq R \}, \quad Z_t := Z_t^R, \quad t \in [0, \zeta_R]. \]

Since \(Z_t^R|_{[0, \zeta_R]} = Z_t^R|_{[0, \zeta_R]}\) for \(R' > R,\) one sees that \(R \mapsto \zeta_R\) is increasing and the above \(Z_t\) is well-defined. Clearly, \(\zeta = \lim_{R \to \infty} \zeta_R\) is the explosion time of \(Z_t\), and \(Z_t\) uniquely solves (1.1) before \(\zeta.\)

The strategy of proving Theorem 1.1 is still based on Zvonkin’s transformation. As in the non-degenerate case [41], we need to establish the \(L^p\)-maximal regularity estimate to the following degenerate parabolic equation (see Theorem 3.2 below):
\[ \partial_t u = \mathcal{L}_t^{a,b} u + f, \quad u_0 = 0. \]

Here we shall use the freezing coefficient argument and the \(L^p\)-estimate established in [6] and [5] for degenerate operators with constant coefficients (see also [3] for the case of nonlocal operators). Compared with [6] and [27], we not only consider the optimal regularity of \(u\) along the nondegenerate \(v\)-direction, but also the optimal regularity of \(u\) along the degenerate \(x\)-direction.

On the other hand, from the viewpoint of PDE, the well-posedness of Fokker-Planck equation (1.3) (especially uniqueness) with rough coefficients is a quite involved problem. Since \(a\) and \(b\) possess less regularities and \(\mathcal{L}_t^{a,b}\) is a degenerate operator, the direct analytical approach seems not work (cf. [3] [4]). Let \(\mathcal{P}(\mathbb{R}^{2d})\) be the set of all probability measures on \(\mathbb{R}^{2d}\). We shall use a probabilistic method to prove the following result.

**Theorem 1.3.** Suppose that \(\sigma\) satisfies (UE) and for any \(T > 0,\)
\[ \lim_{|z| \to 0} \sup_{t \in [0,T]} ||\sigma_t(z) - \sigma_t(z')|| = 0, \]
and \(b \in L^q_{\text{loc}}(\mathbb{R}^d; L^q(\mathbb{R}^{2d}))\) for some \(q \in (2d + 1), \infty].\) Then for any \(v \in \mathcal{P}(\mathbb{R}^{2d})\), there exists a unique probability measure-valued solution \(\mu_t \in \)
\( \mathcal{P}(\mathbb{R}^{2d}) \) to (1.3) in the distributional sense in the class that \( t \mapsto \mu_t \) is weakly continuous with \( \mu_0 = \nu \) and

\[
\int_0^t \int_{\mathbb{R}^{2d}} (|v| + |b_s(x, v)|) \mu_t(dx, dv)ds < \infty, \quad t > 0.
\]

The proof of this result is based on Figalli and Trevisan’s superposition characterization for the solutions of Fokker-Planck equation in terms of martingale problem associated with \( \sigma \) and \( b \). More precisely, Figalli [14] and Trevisan [33] showed that for any weakly continuous probability measure-valued solution \( \mu_t \) of (1.3) with initial value \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \), there exists a martingale solution for operator \( \mathcal{L}^{a,b}_t \) (a probability measure \( \mathbb{P}_\nu \) over the space of all continuous functions from \( \mathbb{R}_+ \) to \( \mathbb{R}^{2d} \) denoted by \( \Omega \)) such that for all \( t \in \mathbb{R}_+ \),

\[
\int_{\mathbb{R}^{2d}} \varphi(z) \mu_t(dz) = \int_{\Omega} \varphi(\omega_t) \mathbb{P}_\nu(d\omega),
\]

where \( t \mapsto \omega_t \) is the coordinate process over \( \Omega \). Hence, in order to prove Theorem 1.3, it suffices to show the well-posedness of martingale problem for \( \mathcal{L}^{a,b}_t \) in the sense of Stroock and Varadhan [31]. This will be achieved by proving some Krylov’s type estimate (see Theorem 4.3 below), which is also a key tool for proving Theorem 1.1. It is remarked that in [28], we have already used this technique to show the uniqueness of measure-valued solutions and \( L^p \)-solutions to possibly degenerate second order Fokker-Planck equations under some weak conditions on the coefficients (but not the case of Theorem 1.3).

This paper is organized as follows: in Section 2, we introduce some anisotropic fractional Bessel potential spaces, and prepare some useful estimates for later use. In Section 3, we show the \( L^p \)-maximal regularity estimate for kinetic Fokker-Planck equations. In Section 4, we study the martingale problem associated with \((\sigma, b)\) under the same assumptions as in Theorem 1.3 by showing the basic Krylov’s type estimate. In particular, we first prove Theorem 1.3. In Section 5, we then prove Theorem 1.1 by using Zvonkin’s transformation and Krylov’s estimate obtained in the previous section. In Appendix, a stochastic Gronwall’s type lemma used in Section 5 is given.

Convention: The letter \( C \) with or without subscripts will denote an unimportant constant, whose value may change in different places. Moreover, \( A \preceq B \) means that \( A \leq CB \) for some constant \( C > 0 \), and \( A \asymp B \) means that \( C^{-1}B \leq A \leq CB \) for some \( C > 1 \).

After this work was finished, I was informed by Professor Enrico Priola during “The 8th International Conference on Stochastic Analysis and its Applications” held at BIT that, very recently, Fedrizzi, Flandoli, Priola and Vovelle [13] also obtained the strong well-posedness together with their
Flow property of SDE (1.1) under the conditions $\sigma_i(z) = I$ and $b_i(z) = b(z)$ possessing the following regularity

$$\|(I - \Delta)^{s/2}b\|_p < \infty$$

for some $s > 2/3$ and $p > 6d$.

2. Preliminaries

For $\alpha \geq 0$ and $p \in (1, \infty)$, let $H^\alpha_p := H^\alpha_p(\mathbb{R}^d) := (I - \Delta)^{-\alpha}(L^p(\mathbb{R}^d))$ be the usual Bessel potential space with norm

$$\|f\|_{\alpha,p} := \|(I - \Delta)^{\alpha}f\|_p,$$

where $\| \cdot \|_p$ is the usual $L^p$-norm, and $\Delta$ is the Laplacian. For $\alpha \in (0, 2)$, let $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$ be the usual fractional Laplacian. Notice that up to a constant $C(\alpha, d) > 0$, an alternative definition of $\Delta^{\frac{\alpha}{2}}$ is given by

$$\Delta^{\frac{\alpha}{2}} f(x) := \lim_{\varepsilon \to 0} \int_{|y| < \varepsilon} \delta_x f(y) |y|^{d-\alpha} dy, \quad \delta_x f(x) := f(x + y) - f(x). \quad (2.1)$$

We will frequently use such a definition below. It is well-known that by the boundedness of Riesz’s transformation (cf. [30]),

$$\|\Delta^{\frac{\alpha}{2}} f\|_p \asymp \|\nabla f\|_p, \quad p > 1, \quad (2.2)$$

and an equivalent norm in $H^\alpha_p$ is given by

$$\|f\|_{\alpha,p} \asymp \|f\|_p + \|\Delta^{\frac{\alpha}{2}} \nabla |\alpha| f\|_p, \quad (2.3)$$

where $[\alpha]$ is the integer part of real number $\alpha$, and we have used the convention $\Delta^0 := I$. Notice that for $\alpha \in (0, 1]$ and $p > 1$,

$$\|f(\cdot + x) - f(\cdot)\|_p \leq \|\Delta^{\frac{\alpha}{2}} f\|_p |x|^{\alpha}, \quad (2.4)$$

and in particular,

$$\|f(\cdot + x) - f(\cdot)\|_p \leq \|f\|_{\alpha,p} (|x|^{\alpha} \wedge 1). \quad (2.5)$$

Moreover, we also have the following interpolation inequality: for any $0 \leq \alpha < \beta < \infty$,

$$\|f\|_{\alpha,p} \leq C(p, d, \alpha, \beta)\|f\|_p^{\frac{\beta - \alpha}{\beta}}\|f\|_\beta^{\frac{\alpha}{\beta}}, \quad (2.6)$$

and the following Sobolev embedding results hold: for any $\alpha \in (0, 1)$, if $p\alpha > d$, then

$$\|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} \leq C(p, d, \alpha, \gamma)\|f\|_{\alpha,p}, \quad \gamma \in (0, \alpha - \frac{d}{p}); \quad (2.7)$$

if $p\alpha < d$, then

$$\|f\|_q \leq C(p, d, \alpha, q)\|f\|_{\alpha,p}, \quad q \in [p, \frac{pd}{d-p\alpha}]. \quad (2.8)$$

All the above facts are standard and can be found in [2] and [30].
To treat the kinetic Fokker-Planck operator, we introduce the following anisotropic Bessel potential spaces. Let \( C_c^\infty(\mathbb{R}^2) \) be the space of all smooth functions on \( \mathbb{R}^2 \) with compact supports. For \( \alpha, \beta \geq 0 \), we define the Bessel potential space \( H^{\alpha,\beta}_p := H^{\alpha,\beta}_p(\mathbb{R}^2) \) as the completion of \( C_c^\infty(\mathbb{R}^2) \) with respect to norm:

\[
\|f\|_{\alpha,\beta,p} := \|(I - \Delta)^{\frac{\alpha}{2}} f\|_p + \|(I - \Delta)^{\frac{\beta}{2}} f\|_p.
\]

Notice that by the Mihlin multiplier theorem (cf. [2]),

\[
\|f\|_{\alpha,\beta,p} \asymp \|f\|_p + \|\Delta_x^\alpha f\|_p + \|\Delta_v^\beta f\|_p \asymp \|((I - \Delta_x)^{\frac{\alpha}{2}} + (I - \Delta_v)^{\frac{\beta}{2}}) f\|_p.
\]

In the following, we simply write

\[
H^{\infty,\infty}_p := \bigcap_{\alpha,\beta \geq 0} H^{\alpha,\beta}_p(\mathbb{R}^2).
\]

**Lemma 2.1.** (i) For any \( \alpha, \beta \geq 0, \theta \in [0, 1] \) and \( p > 1 \), there is a constant \( C = C(\alpha, \beta, \theta, p, d) > 0 \) such that

\[
\|\|I - \Delta\|^\theta f\|_p \|f\|_{\alpha,\beta,p} \leq C \|f\|_{\alpha,\beta,p},
\]

\[
\|\Delta_x^\alpha \Delta_v^{1-\theta} f\|_p \leq C \|\Delta_x + \Delta_v\|^{\frac{\theta}{2}} f\|_p.
\]

In particular, for any \( \alpha \geq 0, \beta \geq 1 \) and \( p > 1 \), we have

\[
\|\nabla_v f\|_{\alpha(\beta-1)/\beta-1,p} \leq C \|f\|_{\alpha,\beta,p}.
\]

(ii) Let \( \alpha, \beta \geq 0 \) and \( p > 1 \) with \( d \neq \frac{\alpha \beta}{\alpha + \beta} \). Set

\[
p^* := \begin{cases} 
  dp/(d - \frac{\alpha \beta}{\alpha + \beta}), & d > \frac{\alpha \beta}{\alpha + \beta}; \\
  \infty, & d < \frac{\alpha \beta}{\alpha + \beta}.
\end{cases}
\]

For any \( q \in [p, p^*] \), there is a constant \( C = C(\alpha, \beta, p, q, d) > 0 \) such that

\[
\|f\|_q \leq C \|f\|_{\alpha,\beta,p}.
\]

**Proof.** (i) It follows by the Mihlin multiplier theorem and (2.2).

(ii) For (2.12), by (2.7), (2.8) and (2.9) with \( \theta = \frac{\beta}{\alpha + \beta} \), we have

\[
\|f\|_q = \int_{\mathbb{R}^d} \|f(\cdot, v)\|_q dv \leq \int_{\mathbb{R}^d} \|(I - \Delta_x)^{\frac{\alpha}{2}} f(\cdot, v)\|_p dv
\]

\[
\leq \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \|\nabla_x (I - \Delta_x)^{\frac{\alpha}{2}} f(x, v)\|^q dv \right)^{p/q} dx \right)^{q/p}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \|(I - \Delta_x)^{\frac{\alpha}{2}} (I - \Delta_x)^{\frac{\beta}{2}} f(x, \cdot)\|_p dx \right)^{q/p} \leq \|f\|_{\alpha,\beta,p},
\]

where the second inequality is due to Minkovskii’s inequality.

Let \( a : \mathbb{R}^2 \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) be a measurable function. Write

\[
\mathcal{L}^a u := \text{tr}(a \cdot \nabla^2 u) + v \cdot \nabla u.
\]

We have
Lemma 2.2. Let \( \alpha \in (0, 1) \) and \( p > d/\alpha \). Suppose that
\[
\kappa_0 := \sup_v ||\Delta^{\frac{\alpha}{p}} a(\cdot, v)||_p + ||a||_\infty < \infty.
\]
For any \( \varepsilon \in (0, 1) \), there is a constant \( C_\varepsilon = C_\varepsilon(p, d, \alpha, \kappa_0) > 0 \) such that for all \( u \in H^{\varepsilon, 0} \),
\[
||[\Delta^{\frac{\alpha}{p}}, \mathcal{L}^a]u||_p \leq \varepsilon ||\Delta^{\frac{\alpha}{p}} \nabla^{2}_v u||_p + C_\varepsilon ||\nabla^{2}_v u||_p,
\]
where \([\Delta^{\frac{\alpha}{p}}, \mathcal{L}^a]u := \Delta^{\frac{\alpha}{p}}(\mathcal{L}^a u) - \mathcal{L}^a(\Delta^{\frac{\alpha}{p}} u)\).

Proof. Notice that by definition (2.1),
\[
\Delta^{\frac{\alpha}{p}}, \mathcal{L}^a]u = \text{tr}(\Delta^{\frac{\alpha}{p}} a \cdot \nabla^{2}_v u) + \int_{\mathbb{R}^d} \text{tr}(\delta_{(y,0)} a \cdot \nabla^{2}_v \delta_{(y,0)} u)|y|^{-d-\alpha} dy.
\]
Hence,
\[
||[\Delta^{\frac{\alpha}{p}}, \mathcal{L}^a]u||_p \leq ||\text{tr}(\Delta^{\frac{\alpha}{p}} a \cdot \nabla^{2}_v u)||_p + \int_{\mathbb{R}^d} ||\text{tr}(\delta_{(y,0)} a \cdot \nabla^{2}_v \delta_{(y,0)} u)||_p |y|^{-d-\alpha} dy.
\]
Let \( \beta \in (\frac{\alpha}{p}, \alpha) \). By (2.7), we have
\[
||\text{tr}(\Delta^{\frac{\alpha}{p}} a \cdot \nabla^{2}_v u)||_p \leq \int_{\mathbb{R}^d} ||\Delta^{\frac{\alpha}{p}} a(\cdot, v)||_p ||\nabla^{2}_v u(\cdot, v)||_p^p dv \\
\leq \sup_v ||\Delta^{\frac{\alpha}{p}} a(\cdot, v)||_p ||\nabla^{2}_v u||^p_{\beta,0,p},
\]
and for \( \gamma \in (0, \beta - \frac{\alpha}{p}) \),
\[
||\text{tr}(\delta_{(y,0)} a \cdot \nabla^{2}_v \delta_{(y,0)} u)||_p \leq \int_{\mathbb{R}^d} ||\delta_{v}(\cdot, v)||_p ||\nabla^{2}_v \delta_{(y,0)} u(\cdot, v)||_p^p dv \\
\leq |y|^{(\alpha+\gamma)p} \int_{\mathbb{R}^d} ||\Delta^{\frac{\alpha}{p}} a(\cdot, v)||_p ||\nabla^{2}_v u(\cdot, v)||_p^p dv \\
\leq |y|^{(\alpha+\gamma)p} \sup_v ||\Delta^{\frac{\alpha}{p}} a(\cdot, v)||_p ||\nabla^{2}_v u||^p_{\beta,0,p}.
\]
Moreover, it is easy to see that
\[
||\text{tr}(\delta_{(y,0)} a \cdot \nabla^{2}_v \delta_{(y,0)} u)||_p \leq ||a||_\infty ||\nabla^{2}_v u||_p.
\]
Therefore,
\[
||\text{tr}(\delta_{(y,0)} a \cdot \nabla^{2}_v \delta_{(y,0)} u)||_p \leq \kappa_0(|y|^{(\alpha+\gamma)p + 1}) ||\nabla^{2}_v u||_{\beta,0,p}.
\]
Combining the above calculations, we get for some \( C = C(p, d, \alpha, \beta) > 0 \),
\[
||[\Delta^{\frac{\alpha}{p}}, \mathcal{L}^a]u||_p \leq C \kappa_0 ||\nabla^{2}_v u||_{\beta,0,p},
\]
On the other hand, by the interpolation inequality (2.6) and Young’s inequality, we have for any \( \varepsilon \in (0, 1) \),
\[
||\nabla^{2}_v u||_{\beta,0,p} \leq ||\nabla^{2}_v u||^{\frac{\varepsilon}{1-\varepsilon}}_{\gamma,0,p} ||\nabla^{2}_v u||^{1-\varepsilon}_{\beta,0,p} \leq \varepsilon ||\nabla^{2}_v u||_{\alpha,0,p} + C_\varepsilon ||\nabla^{2}_v u||_p.
\]
Estimate (2.13) now follows by (2.14).

**Lemma 2.3.** For any \(\alpha, \beta \in (0, 1)\) and \(p > (\alpha + \beta)d/(\alpha \beta)\), there is a constant \(C = C(\alpha, \beta, p, d) > 0\) such that for all \(b \in H_p^{0,0}\) and \(u \in H_p^{\alpha,\beta}\),

\[
\|b \cdot \nabla u\|_{\alpha,0,p} \leq C\|b\|_{\alpha,0,p}\left(\|\Delta_x^\frac{\alpha}{\beta} \nabla u\|_{0,\beta,p} + \|\nabla u\|_{0,\beta,p}\right).
\]

**Proof.** Notice that by (2.3),

\[
\|b \cdot \nabla u\|_{\alpha,0,p} \leq \|b \cdot \nabla u\|_p + \|\Delta_x^\frac{\alpha}{\beta} (b \cdot \nabla u)\|_p.
\]

By definition (2.1), we have

\[
\|\Delta_x^\frac{\alpha}{\beta} (b \cdot \nabla u)\|_p \leq \|\Delta_x^\frac{\alpha}{\beta} b \cdot \nabla u\|_p + \|b \cdot \nabla \Delta_x^\frac{\alpha}{\beta} u\|_p
\]

\[
+ \int_{\mathbb{R}^d} \|\delta_{(0,0)} b \cdot \nabla \delta_{(0,0)} u\|_p |y|^{-d-\alpha} dy =: I_1 + I_2 + I_3.
\]

For \(I_1\), since \(p > (\alpha + \beta)d/(\alpha \beta)\), by (2.12) with \(q = \infty\), we have

\[
I_1 \leq \|\Delta_x^\frac{\alpha}{\beta} b\|_p \|\nabla u\|_\infty \leq \|b\|_{\alpha,0,p} \|\nabla u\|_{\alpha,\beta,p}.
\]

For \(I_2\), since \(p \alpha > d\), by (2.7) we have

\[
I_2^p = \int_{\mathbb{R}^d} \|b(\cdot, v) \cdot \nabla \Delta_x^\frac{\alpha}{\beta} u(\cdot, v)\|_p^p dv
\]

\[
\leq \int_{\mathbb{R}^d} \|b(\cdot, v)\|_\infty^p \|\nabla \Delta_x^\frac{\alpha}{\beta} u(\cdot, v)\|_p^p dv
\]

\[
\leq \int_{\mathbb{R}^d} \|b(\cdot, v)\|_{\alpha,p}^p \sup_v \|\nabla \Delta_x^\frac{\alpha}{\beta} u(\cdot, v)\|_p^p dv.
\]

For \(I_3\), by (2.5) and (2.7) again, we have for any \(\gamma \in (0, \alpha - \frac{d}{p})\),

\[
\int_{\mathbb{R}^d} \|\delta_x b(\cdot, v) \cdot \nabla \delta_x u(\cdot, v)\|_p^p dv \leq \int_{\mathbb{R}^d} \|\delta_x b(\cdot, v)\|_p^p \|\delta_x \nabla u(\cdot, v)\|_\infty^p dv
\]

\[
\leq \int_{\mathbb{R}^d} \|b(\cdot, v)\|_{\alpha,p}^p (|y|^{\alpha p} + 1) \|\nabla u(\cdot, v)\|_{\alpha,p}^p (|y|^{\alpha p} + 1) dv
\]

\[
\leq \left(\int_{\mathbb{R}^d} \|b(\cdot, v)\|_{\alpha,p}^p dv\right) \sup_v \|\nabla u(\cdot, v)\|_{\alpha,p}^p (|y|^{(\alpha + \gamma)p} + 1).
\]

On the other hand, notice that by \(p \beta > d\) and (2.7),

\[
\sup_v \|\nabla \Delta_x^\frac{\alpha}{\beta} u(\cdot, v)\|_p \leq \int_{\mathbb{R}^d} \sup_v \|\nabla \Delta_x^\frac{\alpha}{\beta} u(x, v)\|_p^p dx \leq \int_{\mathbb{R}^d} \|\nabla \Delta_x^\frac{\alpha}{\beta} u(x, \cdot)\|_{\beta,p}^p dx,
\]

and similarly,

\[
\|b \cdot \nabla u\|_p \leq \int_{\mathbb{R}^d} \|b(\cdot, v)\|_{\alpha,p}^p \|\nabla u(\cdot, v)\|_p^p dv \leq \|b\|_{\alpha,0,p}^p \|\nabla u\|_{\alpha,0,p}^p.
\]

Combining the above calculations, we obtain the desired estimate. □
Let \( \varrho : \mathbb{R}^2 \to [0, \infty) \) be a smooth function with support in the unit ball and \( \int \varrho = 1 \). Define
\[
\varrho_\varepsilon(z) := \varepsilon^{-2d} \varrho(\varepsilon^{-1} z), \quad \varepsilon \in (0, 1),
\] (2.15)
and for a locally integrable function \( u : \mathbb{R}^2 \to \mathbb{R} \),
\[
u_\varepsilon(z) := u * \varrho_\varepsilon(z) = \int_{\mathbb{R}^2} u(z') \varrho_\varepsilon(z - z') dz'.
\]

Let \( \mathcal{P} \) be an operator on the space of locally integrable functions. We define
\[
[\varrho_\varepsilon, \mathcal{P}] u := (\mathcal{P} u) * \varrho_\varepsilon - \mathcal{P}(u * \varrho_\varepsilon).
\] (2.16)

We need the following commutator estimate results.

**Lemma 2.4.**

(i) Let \( p \in [1, \infty) \) and \( q, r \in [p, \infty] \) with \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). For any \( b \in L^q(\mathbb{R}^2) \) and \( u \in H^1_p \), we have
\[
\lim_{\varepsilon \to 0} \| [\varrho_\varepsilon, b \cdot \nabla_y] u \|_p = 0.
\] (2.17)

(ii) Let \( a : \mathbb{R}^2 \to \mathbb{R}^d \otimes \mathbb{R}^2 \) be a bounded measurable function. For any \( p \in [1, \infty) \) and \( u \in H^1_p \), we have
\[
\lim_{\varepsilon \to 0} \| [\varrho_\varepsilon, \mathcal{L}^a] u \|_p = 0.
\] (2.18)

**Proof.** (i) It follows by [40, Lemma 4.2].

(ii) By definition, we can write for \( z = (x, v) \),
\[
[\varrho_\varepsilon, \mathcal{L}^a] u(z) = (\mathcal{L}^a u) * \varrho_\varepsilon(z) - \mathcal{L}^a(u * \varrho_\varepsilon)(z)
\]
\[
= \int_{\mathbb{R}^2} \text{tr}((a(z') - a(z)) \cdot \nabla_x u(z')) \varrho_\varepsilon(z - z') dz'
\]
\[
+ \int_{\mathbb{R}^2} (v' - v) \cdot \nabla_x u(z') \varrho_\varepsilon(z - z') dz'
\]
\[
=: I_1^\varepsilon(t, z) + I_2^\varepsilon(t, z).
\]

For \( I_1^\varepsilon(t, z) \), by Jensen’s inequality and the assumption, we have
\[
|I_1^\varepsilon(t, z)|^p \leq \int_{\mathbb{R}^2} |\text{tr}((a(z') - a(z)) \cdot \nabla_x u(z'))|^p \varrho_\varepsilon(z - z') dz'
\]
\[
\leq (2\|a\|_\infty)^p \int_{\mathbb{R}^2} |\nabla_x^2 u(z')|^p \varrho_\varepsilon(z - z') dz'.
\]

For \( I_2^\varepsilon(t, z) \), by the integration by parts and Hölder’s inequality, we have
\[
|I_2^\varepsilon(t, z)|^p = \left| \int_{\mathbb{R}^2} (v' - v) \cdot \nabla_x \varrho_\varepsilon(z - z') u(z') dz' \right|^p
\]
\[
\leq \varepsilon^p \left( \int_{\mathbb{R}^2} |\nabla_x \varrho_\varepsilon(z - z')| |u(z')| dz' \right)^p.
\]
Combining the above calculations, we get
\[
\| \mathcal{Q}_e, \mathcal{L}^\alpha u \|_p \leq C \| u \|_{0,2,p}.
\]
Hence, for any \( u \in H^{0,2}_p \), it is easy to see that
\[
\lim_{\epsilon' \to 0} \sup_{\epsilon \in (0,1)} \| \mathcal{Q}_e, \mathcal{L}^\alpha (u_{\epsilon'} - u) \|_p \leq C \lim_{\epsilon' \to 0} \| u_{\epsilon'} - u \|_{0,2,p} = 0. \tag{2.19}
\]
Moreover, for fixed \( \epsilon' \in (0, 1) \), since \( u_{\epsilon'} \in H^{s_0, \infty}_p \), by \([40]\) Lemma 4.2 we have
\[
\lim_{\epsilon \to 0} \| [\mathcal{Q}_e, \mathcal{L}^\alpha] u_{\epsilon'} \|_p = 0,
\]
which together with \((2.19)\) implies \((2.18)\).

Let \( \sigma_t(x, v) = \sigma_t \) be independent of \((x, v)\). Define for \( t < s \),
\[
P_{t,s} f(x, v) = \mathbb{E} f(x + (s-t)v + X_{t,s}, v + V_{t,s}), \tag{2.20}
\]
where
\[
(X_{t,s}, V_{t,s}) = \left( \int_t^s V_{t,s} \, dr, \int_t^s \sigma_t \, dW_r \right).
\]
We need the following basic \( L^p \)-regularity estimates related to \( P_{t,s} \), which plays a basic role in the next section.

**Theorem 2.5.** Let \( T > 0 \). Suppose that for some \( K > 0 \) and all \( t \in [0, T] \),
\[
K^{-1} |\xi| \leq |\sigma_t \xi| \leq K |\xi|, \quad \xi \in \mathbb{R}^d.
\]
(i) For any \( \alpha, \beta \geq 0 \) and \( p > 1 \), there exists a positive constant \( C = C(K, T, p, d, \alpha, \beta) \) such that for all \( f \in L^p(\mathbb{R}^{2d}) \) and \( 0 \leq t < s \leq T \),
\[
\| P_{t,s} f \|_{\alpha,0,p} \leq C(s-t)^{-\frac{3}{2}} \| f \|_p , \tag{2.21}
\]
\[
\| P_{t,s} f \|_{0,\beta,p} \leq C(s-t)^{-\frac{\beta}{2}} \| f \|_p.
\]
(ii) For any \( p > 1 \), there exists a positive constant \( C_p = C_p(K, d) \) such that for all \( \lambda \geq 0 \) and \( f \in L^p(T) = L^p([0, T] \times \mathbb{R}^{2d}) \),
\[
\| \nabla_x^2 u_1 \|_{L^p(T)} + \| \Delta_x^\frac{1}{2} u_1 \|_{L^p(T)} \leq C_p \| f \|_{L^p(T)}, \tag{2.22}
\]
where \( u_1(x, v) := \int_t^T e^{(t-s)} P_{t,s} f_s(x, v) \, ds \) satisfies
\[
\partial_t u_1 + \mathcal{L}_1^{\alpha,0} u_1 - \lambda u_1 + f = 0
\]
in the distributional sense.
Proof. (i) It follows by the following gradient estimate and the interpolation theorem (see [36, Theorem 2.1]),
\[ \|\nabla^k_x \nabla^m_y P_{t,x} f\|_p \leq C(s-t)^{\frac{k+m}{2}} \|f\|_p, \quad k, m \in \mathbb{N}_0. \]

(ii) It is a consequence of [6] and [5, Theorem 2.1] (see also [8, Theorem 3.3]). □

Remark 2.6. Notice that in the references [36] and [8], the positions of t and s are exchanged.

3. Maximal $L^p$-solutions of kinetic Fokker-Planck equations

Throughout this section, we fix $T > 0$. Let $p \in (1, \infty)$ and $\alpha, \beta \geq 0$. For $t \in [0, T]$, we introduce the following Banach spaces with natural norms:
\[ \mathbb{L}^p(t, T) := L^p([t, T]; L^p(\mathbb{R}^d)), \quad \mathbb{H}_p^{\alpha, \beta}(t, T) := L^p([t, T]; H_p^{\alpha, \beta}(\mathbb{R}^d)). \]

For simplicity of notation, we write
\[ \mathbb{L}^p(T) := \mathbb{L}^p(0, T), \quad \mathbb{H}_p^{\alpha, \beta}(T) := \mathbb{H}_p^{\alpha, \beta}(0, T). \]

We assume that $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is symmetric and satisfies that for some $K \geq 1$ and $\delta \in (0, 1),
\begin{gathered}
\left\{ \frac{K^{-1} \cdot 1}{K} \leq a_t(z) \leq K \cdot 1, \quad (t, z) \in [0, T] \times \mathbb{R}^d \right. \\
\left. \omega_\alpha(\delta) := \sup_{|z-z'| < \delta} \sup_{t \in [0, T]} ||a_t(z) - a_t(z')|| \leq \frac{1}{2(\omega_\alpha(1))} \right\},
\end{gathered}
(H^{\delta, p}_K)

where $C_p$ is the same as in (2.22). Here and in the remainder of this paper, $\| \cdot \|$ denotes the Hilbert-Schmidt norm. For $\lambda \geq 0$, consider the following backward kinetic Fokker-Planck equation
\[ \partial_t u + \mathcal{L}_{t,a}^{\alpha, \beta} u - \lambda u + f = 0, \quad u_T = 0, \quad (3.1) \]

where $f_j(x, v) : [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}$ is a Borel function. We first introduce the following notion of solutions to the above equation.

Definition 3.1. Let $p \in (1, \infty)$ and $f \in \mathbb{L}^p(T)$. A Borel function $u \in \mathbb{H}_p^{0, 2}(T)$ is called a solution of (3.1) if for any $\varphi \in C^\infty_c(\mathbb{R}^d)$ and all $t \in [0, T],
\begin{align}
\langle u_t, \varphi \rangle &= \int_t^T \langle \text{tr}(a_z \cdot \nabla^2 \varphi), \varphi \rangle ds - \int_t^T \langle v \cdot \nabla \varphi, u_s \rangle ds \\
&\quad + \int_t^T \langle b_z \cdot \nabla \varphi, u_s \rangle ds - \lambda \int_t^T \langle u_s, \varphi \rangle ds + \int_t^T \langle f_z, \varphi \rangle ds,
\end{align}
(3.2)

where $\langle u_t, \varphi \rangle := \int_{\mathbb{R}^d} u_t(z) \varphi(z) dz$.

The main aim of this section is to show that
Theorem 3.2. Let $\alpha \in [0, \frac{2}{3})$, $\beta \in (1, 2)$ and $p > \frac{2}{(2-3\alpha)(2-\beta)}$ be not equal to $\frac{d(\alpha+\beta)}{\alpha(\beta-1)}$. Suppose that $a$ satisfies $(H^5_{K})$, and for some $q \in [p \land \frac{d(\alpha+\beta)}{\alpha(\beta-1)}, \infty]$, $\kappa_0 := \|b\|_{L^p([0,T] \times \mathbb{R}^d)} < \infty$.

(i) For any $f \in L^p(T)$, there exists a unique solution $u = u^1$ to (3.1) in the sense of Definition 3.1 with

$$
\|u^1\|_{L^p(T)} \leq C \|f\|_{L^p(T)},
$$

(3.3)

and for all $t \in [0, T]$,

$$
\|u^1\|_{L^{p,\beta}(t)} \leq C((T - t) \land \lambda^{-1})^{\frac{1}{p} - \frac{(3\alpha + \beta)}{2}} \|f\|_{L^p(T)},
$$

(3.4)

where the constant $C$ only depends on $d, \delta, K, \alpha, \beta, p, q, T$ and $\kappa_0$.

(ii) If in addition, we also assume that $p > \frac{d(\delta - 1)}{2(2-\delta)}$ and

$$
\kappa_1 := \sup \|\Delta^\frac{1}{2} \sigma_t(\cdot, v)\|_{L^p} + \|b\|_{L^{p,\beta}(T)} < \infty,
$$

then for any $f \in H^{2,3,0}_p(T)$, the unique solution $u$ also satisfies

$$
\|\nabla_x \nabla_v u^1\|_{L^p(T)} + \|\Delta^\frac{1}{2} \nabla^2_v u^1\|_{L^p(T)} \leq C \|f\|_{L^p(T)},
$$

(3.5)

and for all $t \in [0, T]$,

$$
\|\Delta^\frac{1}{2} u^1\|_{L^{p,\beta}(t)} \leq C((T - t) \land \lambda^{-1})^{\frac{1}{p} - \frac{(3\alpha + \beta)}{2}} \|f\|_{L^p(T)},
$$

(3.6)

where the constant $C$ only depends on $d, \delta, K, \alpha, \beta, p, T$ and $\kappa_1$.

Remark 3.3. In order to emphasize the dependence of the unique solution $u$ on $a, b$ and $T$, $\lambda, f$, we sometimes denote $u = R^{a,b}_{\lambda,T}(f)$.

3.1 Case $b = 0$. In this subsection we first consider the case of $b = 0$ by using the freezing coefficient argument, and show the following basic existence and uniqueness result for equation (3.1).

Theorem 3.4. Let $p > 1$ and $\alpha \in [0, \frac{2}{3})$, $\beta \in [0, 2)$. Suppose $(H^5_{K})$ holds.

(i) For any $f \in L^p(T)$, there exists a unique solution $u = u^1$ to (3.1) in the sense of Definition 3.1 so that

$$
\|u^1\|_{L^p(T)} \leq C_1 \|f\|_{L^p(T)},
$$

(3.7)

(ii) If $p > \frac{2}{(2-3\alpha)(2-\beta)}$, then for all $t \in [0, T]$,

$$
\|u^1\|_{L^{p,\beta}(t)} \leq C_2((T - t) \land \lambda^{-1})^{\frac{1}{p} - \frac{(3\alpha + \beta)}{2}} \|f\|_{L^p(T)},
$$

(3.8)

Here $C_1 = C_1(d, \delta, K, p, T)$ and $C_2 = C_2(d, \delta, K, \alpha, \beta, p, T)$ are increasing with respect to $T$. 

13
Proof. We show the a priori estimates (3.7) and (3.8) by the freezing coefficient argument. The existence of a solution follows by the standard continuity argument. We divide the proof into five steps.

(a) First of all, we assume that \( u \in C([0, T]; H_0^{p, \infty}) \) satisfies (3.1) for Lebesgue almost all \( t \in [0, T] \). For given \( p \geq 1 \), let \( \phi \) be a nonnegative symmetric smooth function on \( \mathbb{R}^d \) with support in the unit ball and
\[
\int_{\mathbb{R}^d} |\phi(z)|^p \, dz = 1.
\]
Let \( \delta \in (0, 1) \) be as in (H\(_K^{\delta, p}\)) and set
\[
\phi_\delta(z) := \delta^{-2d/p} \phi(z/\delta),
\]
and for \( z^\delta = (x^\delta, v^\delta) \) and \( t \in [0, T] \), define
\[
z_t^\delta := (x^\delta - tv^\delta, v^\delta), \quad \phi_\delta^\delta(z) := \phi_\delta(z_t^\delta - z).
\]
By definition, it is easy to see that
\[
\int_{\mathbb{R}^d} |\phi_\delta^\delta(z)|^p \, dz = 1, \quad t \in [0, T], \quad z \in \mathbb{R}^d, \tag{3.9}
\]
and for \( j = 1, 2 \),
\[
\sup_{t \in [0, T]} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla^j \phi_\delta^\delta(z)|^p \, dz \leq C_\delta. \tag{3.10}
\]
Define the freezing functions at point \( z^\delta = (x^\delta, v^\delta) \) as follows:
\[
a_t^\delta := a_t(z_t^\delta), \quad u_t^\delta(z) := u_t(z)\phi_\delta^\delta(z).
\]
By (3.1) and easy calculations, one sees that
\[
\partial_t u_t^\delta + \text{tr}(a_t^\delta \cdot \nabla^2 u_t^\delta) + v \cdot \nabla u_t^\delta - \lambda u_t^\delta = g_t^\delta, \tag{3.11}
\]
where for \( z = (x, v) \),
\[
g_t^\delta(z) := \text{tr}(a_t^\delta \cdot \nabla^2 u_t^\delta(z)) - \text{tr}(a_t^\delta \cdot \nabla^2 u_t(z)\phi_\delta^\delta(z)) + (v - v^\delta) \cdot \nabla \phi_\delta^\delta(z)u_t(z) + f_t(z)\phi_\delta^\delta(z).
\]
We have the following claim:
\[
\left( \int_{\mathbb{R}^d} \|g_t^\delta\|_{L^p(t)}^p \, dz \right)^{1/p} \leq \omega_\delta(\|\nabla^2 u_t\|_{L_p(t)}) \tag{3.12}
\]
\[
+ C_\delta \left( \|u_t\|_{L_p(t)} + \|f_t\|_{L_p(t)} \right), \quad t \in [0, T].
\]
Proof of the claim: Observe that
\[
g_t^\delta(z) = \text{tr}(a_t^\delta - a_t) \cdot \nabla^2 u_t(z)\phi_\delta^\delta(z) + \text{tr}(a_t^\delta \cdot (\nabla u_t \otimes \nabla \phi_\delta^\delta))(z) + [\text{tr}(a_t^\delta \cdot \nabla^2 \phi_\delta^\delta)(z) + (v - v^\delta) \cdot \nabla \phi_\delta^\delta(z)u_t(z) + f_t(z)\phi_\delta^\delta(z) =: I_1^\delta(t, z, z^\delta) + I_2^\delta(t, z, z^\delta) + I_3^\delta(t, z, z^\delta) + I_4^\delta(t, z, z^\delta).
\]
For $I_1^i(t, z, z^o)$, since the support of $\phi_\delta$ is in $B_\delta := \{z \in \mathbb{R}^{2d} : |z| \leq \delta\}$, by the definition of $\omega_\delta(\delta)$ and (3.9), we have

$$\left(\int_{\mathbb{R}^{2d}} \|I_1^i(\cdot, \cdot, z^o)\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} \leq \omega_\delta(\delta) \left(\int_{\mathbb{R}^{2d}} \|\nabla_v^2 u \cdot \phi_\delta^\omega\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} = \omega_\delta(\delta) \|\nabla_v^2 u\|_{L^p(\Omega, T)}.$$

For $I_2^i(t, z, z^o)$, by (3.10) we have

$$\left(\int_{\mathbb{R}^{2d}} \|I_2^i(\cdot, \cdot, z^o)\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} \leq C_\delta \|\nabla_v u\|_{L^p(\Omega, T)}.$$

For $I_3^i(t, z, z^o)$, we similarly have

$$\left(\int_{\mathbb{R}^{2d}} \|I_3^i(\cdot, \cdot, z^o)\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} \leq C_\delta \|u\|_{L^p(\Omega, T)}.$$

For $I_4^i(t, z, z^o)$, by (3.9) we have

$$\left(\int_{\mathbb{R}^{2d}} \|I_4^i(\cdot, \cdot, z^o)\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} = \|f\|_{L^p(\Omega, T)}.$$

Combining the above calculations, and by the interpolation inequality (2.6) and Young’s inequality, we get the claim.

Now by (3.9), we have

$$\|\nabla_v^2 u\|_{L^p(\Omega, T)} = \left(\int_{\mathbb{R}^{2d}} \|\nabla_v^2 u \cdot \phi_\delta^\omega\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} \leq \left(\int_{\mathbb{R}^{2d}} \|\nabla_v^2 (u \phi_\delta^\omega)\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} + \left(\int_{\mathbb{R}^{2d}} \|\nabla_v^2 u \cdot \phi_\delta^\omega\|_{L^p(\Omega, T)}^p \, dz^o\right)^{1/p} = I_1 + I_2. \tag{3.13}$$

For $I_1$, by (3.10) and the interpolation inequality, we have

$$I_1 \leq C_\delta \left(\|\nabla_v u\|_{L^p(\Omega, T)} + \|u\|_{L^p(\Omega, T)}\right) \leq \omega_\delta(\delta) \|\nabla_v^2 u\|_{L^p(\Omega, T)} + C_\delta \|u\|_{L^p(\Omega, T)}.$$

For $I_2$, noticing that by (3.11) and Duhamel’s formula,

$$u_{\delta, a}^\omega(z) = \int_t^T e^{(t-s)} P_{\lambda,s, \delta, a}^\omega(z) \, ds, \tag{3.14}$$
where \( P^{\alpha}_{t,s} \) is defined by (2.20) in terms of \( \sigma^\alpha_{t,i} := (a^\alpha_{t,i})^{1/2} \), by (2.22) and (3.12), we have

\[
I_2 \leq C_0 \left( \int_{\mathbb{R}^{2d}} \left\| \tilde{g}^\omega_{\delta, s} \right\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} \leq C_0 \omega_\delta(\delta) \left\| \nabla^2 u \right\|_{L^p(\mathbb{R}^d)} + C_\delta \left( \left\| u \right\|_{L^p(\mathbb{R}^d)} + \left\| f \right\|_{L^p(\mathbb{R}^d)} \right).
\]

Substituting these two estimates into (3.13) and by \( \omega_\delta(\delta) \leq \frac{1}{c_0+1} \), we get

\[
\left\| \nabla^2 u \right\|_{L^p(\mathbb{R}^d)} \leq C \left( \left\| u \right\|_{L^p(\mathbb{R}^d)} + \left\| f \right\|_{L^p(\mathbb{R}^d)} \right).
\]

Similarly, one can show that (see also step (c) below)

\[
\left\| \Delta^1 u \right\|_{L^p(\mathbb{R}^d)} \leq C \left( \left\| u \right\|_{L^p(\mathbb{R}^d)} + \left\| f \right\|_{L^p(\mathbb{R}^d)} \right).
\]

(b) By (3.14) and the contraction of operator \( P^{\alpha}_{t,s} \) in \( L^p(\mathbb{R}^{2d}) \), we have

\[
\left\| u \right\|_p = \int_{\mathbb{R}^{2d}} \left\| u \right\|_p^p dx^p \leq \int_{\mathbb{R}^{2d}} \left( \int_t^T \left\| g_{\delta, s}^\omega \right\|_p ds \right)^p dx^p \leq \left( \int_t^T e^{p\lambda(t-s)/(p-1)} ds \right)^{p-1} \int_{\mathbb{R}^{2d}} \left\| g_{\delta, s}^\omega \right\|_p^p ds dx^p \leq (T - t) / \lambda - 1 \cdot \left\| \nabla^2 u \right\|_{L^p(\mathbb{R}^d)} + \left\| u \right\|_{L^p(\mathbb{R}^d)} + \left\| f \right\|_{L^p(\mathbb{R}^d)} \right) \leq \left( T - t \right) \cdot \lambda - 1 \cdot \left( \int_t^T \left\| u \right\|_p^p ds + \left\| f \right\|_{L^p(\mathbb{R}^d)} \right),
\]

which yields by Gronwall’s inequality that

\[
\left\| u \right\|_p \leq C \left( \left( T - t \right) \cdot \lambda - 1 \right) \cdot \left\| f \right\|_{L^p(\mathbb{R}^d)}, \quad t \in [0, T].
\]

Substituting it into (3.15) and (3.16), we obtain (3.7), and also by (3.12),

\[
\int_{\mathbb{R}^{2d}} \left\| g_{\delta, s}^\omega \right\|_{L^p(\mathbb{R}^d)}^p dx^p \leq C \left\| f \right\|_{L^p(\mathbb{R}^d)}^p, \quad t \in [0, T].
\]

(c) Let \( \alpha \in (0, \frac{3}{2}) \). By (3.14), (2.21) and Hölder’s inequality, we have

\[
\left\| u \right\|_{a_0, p} \leq \int_t^T e^{4(t-s)} \left\| P^{\alpha}_{t,s} \right\|_{L^p(\mathbb{R}^d)} \left\| a_0 \right\|_p ds \leq \int_t^T e^{4(t-s)} (s - t) \cdot \frac{2\gamma}{s^{p-1}} \left\| g_{\delta, s}^\omega \right\|_p ds \leq \left( \int_t^T e^{4(t-s)} (s - t) \cdot \frac{2\gamma}{s^{p-1}} ds \right)^{1/p} \left\| g_{\delta, s}^\omega \right\|_{L^p(\mathbb{R}^d)} \leq \left( (T - t) / \lambda - 1 \right)^{1/p} \left\| g_{\delta, s}^\omega \right\|_{L^p(\mathbb{R}^d)}.
\]
By (3.9) again, we have
\[
\|\Delta_\chi^\phi u_t\|_p = \int_{\mathbb{R}^d} \|\Delta_\chi^\phi u_t\phi_\delta^\alpha\|_p^p dz^\alpha \leq \int_{\mathbb{R}^d} \|\Delta_\chi^\phi u_t\|_p^p dz^\alpha + \int_{\mathbb{R}^d} \|\Delta_\chi^\phi (u_t \phi_\delta^\alpha) - (\Delta_\chi^\phi u_t) \phi_\delta^\alpha\|_p^p dz^\alpha.
\] (3.20)

By (3.19) and (3.18), we have
\[
\int_{\mathbb{R}^d} \|\Delta_\chi^\phi u_t\phi_\delta^\alpha\|_p^p dz^\alpha \leq ((T - t) \wedge \lambda^{-1})^{p-1} \frac{1}{\lambda^p} \int_{\mathbb{R}^d} \|g_\delta^\alpha\|_{L^p(t,T)} dz^\alpha 
\leq ((T - t) \wedge \lambda^{-1})^{p-1} \|f\|_{L^p(t,T)}^p.
\] (3.21)

On the other hand, noticing that by definition (2.1),
\[
\Delta_\chi^\phi (u_t \phi_\delta^\alpha) - (\Delta_\chi^\phi u_t) \phi_\delta^\alpha = u_t \cdot \Delta_\chi^\phi \phi_\delta^\alpha + \int_{\mathbb{R}^d} \delta_{(0,0)} u_t \cdot \delta_{(0,0)} \phi_\delta^\alpha |y|^{-d-\alpha} dy,
\]
and
\[
\sup_z \int_{\mathbb{R}^d} |\Delta_\chi^\phi \phi_\delta^\alpha(z) - z|^p dz^\alpha \leq C_\delta,
\]
by Minkovskii’s inequality, we have
\[
\int_{\mathbb{R}^d} \|\Delta_\chi^\phi (u_t \phi_\delta^\alpha) - (\Delta_\chi^\phi u_t) \phi_\delta^\alpha\|_p^p dz^\alpha \leq \int_{\mathbb{R}^d} \|u_t \cdot \Delta_\chi^\phi \phi_\delta^\alpha\|_p^p dz^\alpha + \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta_{(0,0)} u_t(z) \cdot (|y| \wedge 1)| |y|^{-d-\alpha} dy \right)^p dz
\leq \|u_t\|_p^p + \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta_{(0,0)} u_t(z) \cdot (|y| \wedge 1)| |y|^{-d-\alpha} dy \right)^p dz
\leq \|u_t\|_p^p + \left( \int_{\mathbb{R}^d} \|\delta_{(0,0)} u_t\|_p \cdot (|y| \wedge 1)|y|^{-d-\alpha} dy \right)^p \leq \|u_t\|_p^p.
\] (3.22)

Combining (3.20), (3.21) and (3.22) with (3.17), we arrive at
\[
\|u_t\|_{p,0,p}^p \leq ((T - t) \wedge \lambda^{-1})^{p-1} \frac{1}{\lambda^p} \|f\|_{L^p(t,T)}^p
\]
Similarly, for any $\beta \in (0, 2)$, one can show that
\[
\|u_t\|_{0,p,\beta,p}^p \leq ((T - t) \wedge \lambda^{-1})^{p-1-\beta} \|f\|_{L^p(t,T)}^p
\]
Combining the above two estimates, we obtain (3.8).

**d** Below we assume that $u \in H_{p,0}^0(T)$ is a solution of (3.1) in the sense of Definition 3.1 Let $q_\varepsilon$ be defined by (2.15) and set
\[
u_\varepsilon := u * q_\varepsilon, \quad f_\varepsilon := f * q_\varepsilon.
\]
By the apriori estimate (3.7), we have
\[ \beta_1 u_e + \mathcal{L}_t^a u_e - \lambda u_e + [\mathcal{Q}_e, \mathcal{L}_t^a] u_e + f_e = 0, \quad u_e, \beta_1 = 0, \]
where \( \mathcal{L}_t^a := \mathcal{L}_t^{a,0} \) and \([\mathcal{Q}_e, \mathcal{L}_t^a]\) is defined by (2.16). By what we have proved, it holds that
\[ \|u_e\|_{L_2^{3/2}(T)} \leq C_1(\|f_e\|_{L_p(T)} + \|[\mathcal{Q}_e, \mathcal{L}_t^a]\|_{L_p(T)}). \]

Since \( \nabla_2^2 u \in L_p(T) \), by the property of convolutions and (ii) of Lemma 2.4, we get (3.7) by taking limits. Similarly, we also have (3.8).

(e) Finally, we use the standard continuity argument to show the existence of a solution (see [20]). Consider the following parametrized equation:
\[ \partial_t u + \mathcal{L}_t^a u - \lambda u + f = 0, \quad u_t = 0, \quad (3.23) \]
where \( \tau \in [0, 1] \) and \( a_\tau := K(1 - \tau) \mathbb{I} + \tau a \). Since \( K^{-1} \cdot \mathbb{I} \leq a \leq K \cdot \mathbb{I}, \) we obviously have
\[ K^{-1} \cdot \mathbb{I} \leq a_\tau \leq K \cdot \mathbb{I}, \quad \omega_{a_\tau}(\delta) = \omega_{a}(\delta). \]

Hence the apriori estimate (3.7) holds for (3.23) with constant \( C_1 \) independent of \( \tau \in [0, 1] \). Suppose that (3.23) is solvable for some \( \tau_0 \in [0, 1] \). We want to show that (3.23) is also solvable for any \( \tau \in [\tau_0, \tau_0 + \frac{1}{4KC_1}] \), where \( C_1 \) is the constant in (3.7). Let \( u^0 = 0 \) and for \( n \in \mathbb{N} \), define \( u^n \) recursively by
\[ \partial_t u^n + \mathcal{L}_t^a u^n - \lambda u^n + \text{tr}((a_\tau - a_{\tau_0}) \cdot \nabla_2^2 u^{n-1}) + f = 0, \quad u_t^n = 0. \]

By the apriori estimate (3.7), we have
\[ \|u^n\|_{H^{2/3}_p(T)} \leq C_1\|\text{tr}((a_\tau - a_{\tau_0}) \cdot \nabla_2^2 u^{n-1}) + f\|_{L_p(T)} \leq 2KC_1(\tau - \tau_0)\|u^{n-1}\|_{H^{2/3}_p(T)} + C_1\|f\|_{L_p(T)} \leq \frac{1}{2}\|u^{n-1}\|_{H^{2/3}_p(T)} + C_1\|f\|_{L_p(T)}, \]
and similarly,
\[ \|u^n - u^m\|_{H^{2/3}_p(T)} \leq \frac{1}{2}\|u^{n-1} - u^{m-1}\|_{H^{2/3}_p(T)}, \]
which imply that \( u^n \) is a Cauchy sequence in \( H^{2/3}_p(T) \). It is easy to see that the limit \( u \) of \( u^n \) satisfies (3.23). Since (3.23) is solvable for \( \tau = 0 \) by (ii) of Theorem 2.5, by repeatedly using what we have proved finitely many times, we get the solvability of (3.23) for \( \tau = 1 \).

Next we show further regularity of the solution under extra assumption.

**Theorem 3.5.** Suppose that for some \( p > 3d/2 \), \( (H^{\delta,p}_K) \) holds and
\[ \kappa_1 := \sup_{(t, v) \in [0, T] \times \mathbb{R}^d} \|\Delta_1^{\frac{1}{p}} \sigma_1(\cdot, v)\|_p < \infty. \]
(i) For any \( f \in H_p^{2/3,0}(T) \), the unique solution \( u \) of PDE (3.1) also satisfies
\[
\| \nabla_x \nabla_t u \|_{L^p(T)} + \| \Delta_x \nabla^2_t u \|_{L^p(T)} \leq C_3 \| f \|_{H_p^{2/3,0}(T)}.
\] (3.24)

(ii) Let \( \alpha \in [0, \frac{2}{3}) \) and \( \beta \in [0, 2) \). If \( p > \frac{2}{(2-3\alpha)(2-\beta)} \sqrt{\frac{3d}{2}} \), then
\[
\| \Delta_x u_t \|_{\alpha, \beta, p} \leq C_4((T-t) \wedge \lambda^{-1})^{1-\frac{1}{p}} \left( \frac{3d}{2} \right)^{\frac{1}{p}} \| f \|_{H_p^{2/3,0}(T)}.
\] (3.25)

Here \( C_3 = C(d, \delta, K, \kappa_1, p, T) \) and \( C_4 = C(d, \delta, K, \kappa_1, \alpha, \beta, p, T) \) are increasing with respect to \( T \).

**Proof.** As in the proof of Theorem 3.4, it suffices to show the apriori estimates (3.24) and (3.25). Notice that using (2.11) with \( \alpha = \frac{2}{3}, \beta = 2 \) and by (3.7),
\[
\| \Delta_x^\frac{1}{2} \nabla_t u \|_{L^p(T)} \leq \| u \|_{H_p^{2/3,2}(T)} \leq \| f \|_{L^p(T)}.
\] (3.26)

Assume \( u \in C([0, T]; H_p^{\infty, \infty}) \) and let \( w_\ell(x, v) := \Delta_x^\frac{1}{2} u_t(x, v) \). By (3.1) we have
\[
\partial_t w + \mathcal{L}_v^a w + [\Delta_x^\frac{1}{2}, \mathcal{L}_v^a] u + \Delta_x^\frac{1}{2} f = 0, w_T = 0.
\]

By definition (2.1) and the assumptions, it is easy to see that
\[
\sup_{(t,v) \in [0,T] \times \mathbb{R}^d} \| \Delta_x^\frac{1}{2} a_t(\cdot, v) \|_p < \infty.
\]

Hence, by (3.26), (3.7) and (2.13), we have for any \( \varepsilon > 0 \),
\[
\| \nabla_x \nabla_t u \|_{L^p(T)} + \| \Delta_x^\frac{1}{2} u \|_{L^p(T)} + \| \Delta_x^\frac{1}{2} \nabla^2_t u \|_{L^p(T)}
\leq \| \Delta_x^\frac{1}{2} \nabla_v w \|_{L^p(T)} + \| \Delta_x^\frac{1}{2} w \|_{L^p(T)} + \| \nabla^2_v w \|_{L^p(T)}
\leq \| [\Delta_x^\frac{1}{2}, \mathcal{L}_v^a] u + \Delta_x^\frac{1}{2} f \|_{L^p(T)}
\leq \| [\Delta_x^\frac{1}{2}, \mathcal{L}_v^a] u \|_{L^p(T)} + \| f \|_{H_p^{2/3,0}(T)}
\leq \varepsilon \| \Delta_x^\frac{1}{2} \nabla^2_v u \|_{L^p(T)} + C_\varepsilon (\| \nabla^2_v u \|_{L^p(T)} + \| f \|_{H_p^{2/3,0}(T)})
\leq \varepsilon \| \Delta_x^\frac{1}{2} \nabla^2_v u \|_{L^p(T)} + C_\varepsilon f \|_{H_p^{2/3,0}(T)},
\]
which implies the desired estimate (3.24) by letting \( \varepsilon \) be small enough. As for (3.25), it follows by applying (3.8) to \( w \) and using the above estimate.

\[ \Box \]

3.2. **Proof of Theorem 3.2.** By a standard fixed point argument or Picard’s iteration, it suffices to prove the apriori estimate (3.3). Let \( \alpha \in (0, \frac{2}{3}), \beta \in (1, 2) \) and \( p > \frac{2}{(2-3\alpha)(2-\beta)} \) be not equal to \( \frac{d(\alpha+\beta)}{\alpha(\beta-1)} \).
Thus, by (3.8) we have
\[ \|b \cdot \nabla u\|_{L^p(I,T)} \leq \int_I \|b_s\|_{L^p} \|\nabla u_s\|_{L^p} ds \leq \int_I \|b_s\|_{L^p} \|u_s\|_{L^p,\beta,p} ds. \]

Thus, by (3.8) we have
\[ \|u_t\|_{L^p,\alpha,\beta,p} \leq ((T - t) \land \lambda^{-1})^{p-1-\frac{\mu(3\gamma_0)}{2}} \|b \cdot \nabla u + f\|_{L^p(I,T)} \]
\[ \leq ((T - t) \land \lambda^{-1})^{p-1-\frac{\mu(3\gamma_0)}{2}} \|f\|_{L^p(I,T)} + \int_I \|b_s\|_{L^p} \|u_s\|_{L^p,\alpha,\beta,p} ds, \]
which implies (3.4) by Gronwall’s inequality.

On the other hand, by (3.7) we have
\[ \|u_t\|_{L^{2,3/0}(T)} \leq \|b \cdot \nabla u + f\|_{L^p(I,T)} \leq \int_I \|b_s\|_{L^p} \|u_s\|_{L^p,\alpha,\beta,p} ds + \|f\|_{L^p(I,T)}, \]
which in turn implies (3.3) by (3.4).

(ii) Let \( p > \frac{d(\beta - 1)}{2\beta - 1}. \) By Lemma 2.3 with \( \alpha = \frac{2}{3}, \) we have
\[ \|b \cdot \nabla u\|_{L^{2,3/0}(I,T)} \leq \int_I \|b_s\|_{L^{2,3/0}} \left( \|\Delta_x^{1/2} \nabla u_s\|_{L^{2,3}} + \|\nabla u_s\|_{L^{2,3/0}} \right) ds \]
\[ \leq \int_I \|b_s\|_{L^{2,3/0}} \left( \|\Delta_x^{1/2} u_s\|_{L^{2,3}} + \|u_s\|_{L^{2,3/0}} \right) ds. \]

Thus, by (3.25) and (3.4), we have
\[ \|\Delta^{1/2} u_t\|_{L^{2,3/0}(I,T)} \leq ((T - t) \land \lambda^{-1})^{p-1-\frac{\mu(3\gamma_0)}{2}} \|b \cdot \nabla u + f\|_{L^{2,3/0}(I,T)} \]
\[ \leq ((T - t) \land \lambda^{-1})^{p-1-\frac{\mu(3\gamma_0)}{2}} \|f\|_{L^{2,3/0}(I,T)} \]
\[ + \int_I \|b_s\|_{L^{2,3/0}} \|\Delta^{1/2} u_s\|_{L^{2,3/0}} ds, \]
which yields (3.6) by Gronwall’s inequality.

Moreover, by (3.24) we have
\[ \|\nabla \nabla u\|_{L^p(I,T)} + \|\Delta^{1/2} \nabla^2 u\|_{L^p(I,T)} \leq C \|b \cdot \nabla u + f\|_{L^{2,3/0}(I,T)} \leq C \|f\|_{L^{2,3/0}(I,T)}, \]
which gives (3.5). The proof is complete.

4. Well-posedness of martingale problem

Let \( \Omega = C(\mathbb{R}_+; \mathbb{R}^{2d}) \) be the space of all continuous functions from \( \mathbb{R}_+ \) to \( \mathbb{R}^{2d}, \) which is endowed with the locally uniform convergence topology. Let
\[ Z_t(\omega) := \omega_t \]
be the coordinate process on $\Omega$. For $t \geq 0$, let
\[ F_t := \sigma \{ Z_s : s \in [0, t] \}, \quad \mathcal{F} := \vee_{t \geq 0} \mathcal{F}_t. \]
We first recall the following notions of martingale solution and weak solution (see [31]).

**Definition 4.1.** Let $\sigma : \mathbb{R}_+ \times \mathbb{R}^{2d} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ be Borel measurable functions and $\mathcal{L}^{a,b}_t$ be defined by (1.2) with $a = \frac{1}{2} \sigma \sigma^*$. 

(i) (Martingale solution) For given $\sigma$, $b$, and $P$, let
\[ \mathbb{P}(Z_t = z, t \in [0, r]) \text{ and } \int_r (|a_s(Z_s)| + |b_s(Z_s)|)ds < \infty, \quad t \geq r \]
for all $\varphi \in C^\infty_c(\mathbb{R}^d)$,
\[ [r, \infty) \ni t \mapsto \varphi(Z_t) - \int_r^t \mathcal{L}^{a,b}_s \varphi(Z_s)ds =: M_t^{r,\varphi} \quad (4.1) \]
is an $\mathcal{F}_t$-martingale with respect to $\mathbb{P}$ after time $r$. We denote by $\mathcal{P}^{\sigma,b}_{r,z}$ the set of all martingale solutions associated with $(\sigma, b)$ and starting from $(r, z)$.

(ii) (Well-posedness) One says that the martingale problem for $\mathcal{L}^{a,b}_t$ is well-posed if for each $(r, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$, there is exactly one solution $\mathbb{P}_{r,z}$ to the martingale problem for $\mathcal{L}^{a,b}_t$ starting from $(r, z)$.

(iii) (Weak solution) A triple $(\hat{Z}, \hat{W}; (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}); (\hat{\mathcal{F}}_t)_{t \geq 0})$ is called a weak solution of SDE (1.1) with starting point $(r, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$ if $(\hat{\mathcal{F}}_t)_{t \geq 0}$ satisfies the usual conditions, $\hat{W}$ is a $d$-dimensional $\hat{\mathcal{F}}_t$-Brownian motion, and $\hat{Z} = (\hat{X}, \hat{\mathcal{F}})$ is an $\hat{\mathcal{F}}$-adapted $\mathbb{R}^{2d}$-valued process satisfying
\[ \int_r^t (|a_s(\hat{Z}_s)| + |b_s(\hat{Z}_s)|)ds < \infty, \quad t \geq r, \quad \hat{P} - a.s., \]
and
\[ \hat{Z}_t = z + \int_r^t (\hat{X}_s, b_s(\hat{Z}_s))ds + \int_r^t (0, \sigma_s(\hat{Z}_s)d\hat{W}_s). \]

**Remark 4.2.** It is well known that the martingale solutions and weak solutions are equivalent, for example, see [18] p.318, Proposition 4.11.

4.1. **Krylov’s type estimate.** In this subsection we first show the following important estimate of Krylov’s type for weak solutions.

**Theorem 4.3.** Suppose that $\sigma$ satisfies (UE) and
\[ \lim_{|z-z'| \rightarrow 0} \sup_{t \in [0,T]} \| \sigma_t(z) - \sigma_t(z') \| = 0, \quad T > 0, \quad (4.2) \]
and \( b \in \cap_T L^q(T) \) for some \( q > (2(2d + 1), \infty) \). Then for any \( p > 2d + 1 \) and \( T > 0 \), there is a constant \( C > 0 \) depending on \( T, \sigma, p, q, d, \|b\|_{L^q(T)} \) such that for any \( (r, z) \in [0, T) \times \mathbb{R}^d \), any weak solution \((\hat{Z}_t, \hat{W}_t; (\hat{\Omega}_t, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}); (\hat{\mathcal{F}}_t)_{t \geq 0}\) of SDE (1.1) with starting point \((r, z)\), and \( r \leq t_0 < t_1 \leq T \) and \( f \in L^p(T) \),

\[
\mathbb{E}\left( \int_{t_0}^{t_1} f_s(\hat{Z}_s) \, ds \right) \leq C(t_1 - t_0)^{\frac{1}{2d+1}} \|f\|_{L^p(T)},
\]

where \( C > 0 \) is increasing in \( T, \omega_\sigma(\delta) \) and \( \|b\|_{L^q(T)} \).

**Proof.** Without loss of generality, we assume \((r, z) = (0, 0)\) and drop the tilde in the definition of weak solutions for simplicity. We divide the proof into four steps.

(a) Let \( a_t(z) := \frac{1}{2}(\sigma_t^x \sigma_t^y)(z) \) and \( p \in (2(2d + 1), q] \). For any \( 0 \leq t_0 < t_1 \leq T \), \( \lambda \geq 1 \) and \( f \in L^p(t_0, t_1) \), let \( u = R_{t_0}^{a,b}(f) \) be the solution of PDE (3.3) with terminal time \( T = t_1 \). By (2.12) and (3.4) with \( \alpha = \frac{4d}{3(2d + 1)} \) and \( \beta = \frac{4d}{2d + 1} \),

\[
\|R_{t_0}^{a,b}(f)\|_{L^p(t_0, t_1)} \leq \|R_{t_0}^{a,b}(f)\|_{L^p(t_0, t_1)} \leq ((t_1 - t_0) \wedge \lambda^{-1})^\frac{1}{2d+1} \|f\|_{L^p(t_0, t_1)},
\]

and by (2.11), (2.12), (3.4) with \( \alpha = \frac{4d+1}{3(2d+1)} \) and \( \beta = \frac{4d+1}{2d+1} \),

\[
\|R_{t_0}^{a,0}(f)\|_{L^p(t_0, t_1)} \leq \|R_{t_0}^{a,0}(f)\|_{L^p(t_0, t_1)} \leq ((t_1 - t_0) \wedge \lambda^{-1})^\frac{1}{2d+1} \|f\|_{L^p(t_0, t_1)},
\]

where the \( C \) in the above \( \leq \) only depend on \( d, p, q, K, T, \omega_\sigma(\delta) \) and \( \|b\|_{L^q(T)} \).

Now, for any \( R > 0 \), define a stopping time

\[
\tau_R := \left\{ t \geq 0 : \int_0^t |b_s(Z_s)| \, ds > R \right\}.
\]

Let \( \varrho_\sigma \) be as in (2.15). We introduce a \( d \times d \) matrix-valued function, which is crucial for us below. For \( t \geq 0 \) and \( z \in \mathbb{R}^d \), let

\[
a_t^{e,R}(z) := \begin{cases} \frac{\mathbb{E}(\varrho_\sigma(Z_0) \mathbb{1}_{t < \tau_R})}{\mathbb{E}(\varrho_\sigma(Z_0) \mathbb{1}_{t < \tau_R})}, & \text{if } \mathbb{E}(\varrho_\sigma(Z_t - z) \mathbb{1}_{t < \tau_R}) \neq 0, \\ a_t(z), & \text{if } \mathbb{E}(\varrho_\sigma(Z_t - z) \mathbb{1}_{t < \tau_R}) = 0. \end{cases}
\]

By (UE) and the definitions, we have

\[
K^{-2} \cdot I \leq a_t^{e,R}(z) \leq K^2 \cdot I,
\]

and for all \( |z - z'| \leq \varepsilon \),

\[
\|a_t^{e,R}(z) - a_t^{e,R}(z')\| \leq \|a_t(z) - a_t(z')\| + \frac{\mathbb{E}(\varrho_\sigma(Z_t - z) \mathbb{1}_{t < \tau_R})}{\mathbb{E}(\varrho_\sigma(Z_0) \mathbb{1}_{t < \tau_R})} \|a_t(z) - a_t(z')\| + \frac{\mathbb{E}(\varrho_\sigma(Z_t - z) \mathbb{1}_{t < \tau_R})}{\mathbb{E}(\varrho_\sigma(Z_0) \mathbb{1}_{t < \tau_R})} \|a_t(z) - a_t(z')\| \leq 3\omega_\sigma(\varepsilon).
\]
Let $C_0$ be the same as in (2.22). By (4.2), one may choose $\delta_0$ small enough such that for all $\varepsilon \in (0, \delta_0)$ and $R > 0$,

$$\omega_{\varepsilon, R}(\varepsilon) \leq 3\omega_0(\delta_0) \leq \frac{1}{2(C_0 + 1)}, \quad (4.7)$$

(b) In this step we show that for any $p > 2(2d + 1)$ and $f \in L^p(T)$,

$$\mathbb{E} \left( \int_0^T f_s(Z_s)ds \right) \leq C\|f\|_{L^p(T)}. \quad (4.8)$$

By a standard density argument, we may assume $f \in C_c([0, T] \times \mathbb{R}^d)$. Let

$$u^{e, R} := \mathcal{P}_{\lambda, T}^{\varepsilon, R}(f), \quad u^{e, R}_e := u^{e, R} * \varrho_e, \quad f_e := f * \varrho_e.$$ 

By Itô’s formula, we have

$$\mathbb{E}(u^{e, R}_{s,T \wedge T_e}(Z_{T \wedge T_e}) - u_{s,0}(Z_0)) = \mathbb{E} \left( \int_0^{T \wedge T_e} (\partial_s u^{e, R}_{s,\varepsilon} + \mathcal{L}_s a^{e, R}) u^{e, R}_{s,\varepsilon} + f_e)ds \right). \quad (4.9)$$

Noticing that by Definition 3.1

$$\partial_s u^{e, R}_{s,\varepsilon} + (\mathcal{L}_s a^{e, R}) u^{e, R}_{s,\varepsilon} + f_e = 0,$$

and by the definitions of $u^{e, R}_e$ and $a^{e, R}$,

$$\mathbb{E} \left( \text{tr}(a_s \cdot \nabla_s u^{e, R}_e)(z) \right) = \mathbb{E} \left( \text{tr}(a_s \cdot \nabla_s u^{e, R}_e)(z) \right) \mathbb{E}(\varrho_e(Z_s - z)1_{s < T_e})dz$$

by easy calculations, one sees that

$$\mathbb{E}(\partial_s u^{e, R}_{s,\varepsilon} + \mathcal{L}_s a^{e, R}) u^{e, R}_{s,\varepsilon} + f_e)(z)dz$$

$$\leq \|u^{e, R}_{s,\varepsilon}\|_\infty \left( \lambda + \int_{\mathbb{R}^d} \|v\| \|\nabla \varrho_e(x, v)\|_\infty dv \right)$$

Substituting this into (4.9), we obtain

$$\mathbb{E} \left( \int_0^{T \wedge T_e} f_e(Z_s)ds \right) \leq (\|\nabla \varrho\|_1 + 2 + \lambda)\|u^{e, R}_{s,\varepsilon}\|_{L^p(T)}$$

$$+ \|\nabla \varrho u^{e, R}_{s,\varepsilon}\|_{L^p(T)} \mathbb{E} \left( \int_0^{T \wedge T_e} |b_s(Z_s)|ds \right).$$
By (4.7), (4.4) and (4.5) with $b \equiv 0$, there is a $C = C(d, p, K, T, \omega_\epsilon(\delta_0)) > 0$ such that for all $\epsilon \in (0, \delta_0)$ and $\lambda > 1$,

$$
\begin{align*}
\mathbb{E}\left(\int_0^{T_\wedge \tau_\epsilon} f_{e, t}(Z_s)ds\right) \leq C(1 + \lambda)T^\frac{1}{2d+1} \|f\|_{L^p(T)} + C\lambda \frac{1}{\frac{1}{2d+1}} \|f\|_{L^p(T)}^\frac{1}{2d+1},
\end{align*}
$$

which implies, by letting $\epsilon \to 0$ and $\lambda$ large enough, that for any $\delta > 0$, there is a $C_\delta > 0$ such that for all $f \in L^p(T)$,

$$
\mathbb{E}\left(\int_0^{T_\wedge \tau_\epsilon} f_s(Z_s)ds\right) \leq C_\delta + \delta \mathbb{E}\left(\int_0^{T_\wedge \tau_\epsilon} |b_s(Z_s)|ds\right) \|f\|_{L^p(T)}. 
$$

In particular, choosing $f_s = |b_s|$ and $\delta \leq 1/(2\|b\|_{L^p(T)})$, we get

$$
\mathbb{E}\left(\int_0^{T_\wedge \tau_\epsilon} |b_s(Z_s)|ds\right) \leq C\|b\|_{L^p(T)}. 
$$

Substituting this into (4.10) and letting $R \to \infty$, we get (4.8).

(c) In this step we show that (4.3) holds for $p = q > 2(2d + 1)$. Let $0 \leq t_0 < t_1 \leq T$ and $f \in L^q(t_0, t_1)$, and write

$$
u := -\mathcal{R}_{0,t_1}^{a,b}(f) \in H^{0,2}_q(t_1), \quad u_\epsilon := u * \mathcal{Q}_\epsilon. $$

Noticing that by definitions,

$$
\partial_t u_{e, t} + \mathcal{L}_t^{a,b} u_{e, t} = f_{e, t} + [\mathcal{Q}_t, \mathcal{L}_t^{a,b}] u_t, \quad u_{e, t_0} = 0,
$$

by Itô’s formula, we have

$$
\mathbb{E}\mathcal{F}_0\left(u_{e, t_1}(Z_{t_1}) - u_{e, t_0}(Z_{t_0})\right) = \mathbb{E}\mathcal{F}_0\left(\int_{t_0}^{t_1} f_{e, s} + [\mathcal{Q}_s, \mathcal{L}_s^{a,b}] u_s(Z_s)ds\right),
$$

where $\mathbb{E}\mathcal{F}_0(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_0)$. Since by (4.8), (4.5) and Lemma 2.4,

$$
\lim_{\epsilon \to 0} \mathbb{E}\left(\int_{t_0}^{t_1} [\mathcal{Q}_s, \mathcal{L}_s^{a,b}] u_s(Z_s)ds\right) = \lim_{\epsilon \to 0} ||[\mathcal{Q}_s, \mathcal{L}_s^{a,b}] u||_{L^q(t_0, t_1)} = 0,
$$

and

$$
\lim_{\epsilon \to 0} \mathbb{E}\left(\int_{t_0}^{t_1} |f_{e, s} - f_s(Z_s)|ds\right) \leq \lim_{\epsilon \to 0} ||f_{e} - f||_{L^q(t_0, t_1)} = 0,
$$

taking limits $\epsilon \to 0$ for both sides of (4.11) and by (4.4), we obtain

$$
\mathbb{E}\mathcal{F}_0\left(\int_{t_0}^{t_1} f_s(Z_s)ds\right) \leq 2\|u\|_{L^q(t_0, t_1)} \leq C(t_1 - t_0)^\frac{1}{2d+1} \|f\|_{L^q(t_0, t_1)}. 
$$

(d) Let $0 \leq t_0 < t_1 \leq T$. By (4.12) with $f_s = |b_s|$ and Corollary 4.4 below, we have for any $\lambda > 0$,

$$
\mathbb{E}\mathcal{F}_0 \exp\left(\lambda \int_{t_0}^{t_1} |b_s(Z_s)|ds\right) \leq C(\lambda, \|b\|_{L^p(T)}) < \infty. 
$$

(4.13)
Define
\[ E_{t_0,t_1} := \exp \left\{ \int_{t_0}^{t_1} (\sigma_s^{-1} b_s) (Z_s) dW_s - \frac{1}{2} \int_{t_0}^{t_1} (\sigma_s^{-1} b_s)^2 (Z_s)^2 ds \right\}. \]

By Novikov’s criterion, \( E(E_{t_0,t_1}) = 1 \), and for any \( \gamma \in \mathbb{R} \), by (4.13) and Hölder’s inequality,
\[ \mathbb{E}(E_{t_0,t_1}^\gamma | \mathcal{F}_{t_0}) < \infty. \tag{4.14} \]

Define a new probability \( Q_{t_0,t_1} := E_{t_0,t_1} \mathbb{P} \). By Girsanov’s theorem, \( \tilde{W} := W + \int_{t_0}^{t} b_s(Z_s) ds \) is still a Brownian motion under \( Q_{t_0,t_1} \). Moreover, \( Z_t \) satisfies
\[ Z_t = Z_{t_0} + \int_{t_0}^{t} (\dot{X}_s, 0) ds + \int_{t_0}^{t} (0, \sigma_s(Z_s)) d\tilde{W}_s. \]

By the same argument as used in (c) with \( b \equiv 0 \), since in this case, we only need to control \( ||u||_{L^\infty(t_0,t_1)} \) and (4.4) holds for any \( p > 2d + 1 \), we obtain that there is a constant \( C > 0 \) such that for all \( p > 2d + 1 \) and \( f \in L^p(T) \),
\[ \mathbb{E}^{Q_{t_0,t_1}} \left( \int_{t_0}^{t} f_s(Z_s) ds \middle| \mathcal{F}_{t_0} \right) \leq C(t_1 - t_0)^{\frac{1}{2d+1}} ||f||_{L^p(T)}. \]

Finally, by (4.14) and Hölder’s inequality, we obtain (4.3). \( \square \)

We have the following useful corollary.

**Corollary 4.4.** (Khasminskii’s type estimate) In the same framework of Theorem 4.3, letting \( \beta := \frac{1}{2d+1} - \frac{1}{p} \) and \( C \) be the same as in (4.3), we have
(i) For each \( m \in \mathbb{N} \) and \( 0 \leq t_0 < t_1 \leq T \), it holds that
\[ \mathbb{E}^{\mathbb{F}_0} \left( \int_{t_0}^{t_1} f_s(\tilde{Z}_s) ds \right)^m \leq m! (C ||f||_{L^p(T)} (t_1 - t_0)^\beta)^m, \]
where \( \mathbb{E}^{\mathbb{F}_0} (\cdot) = \mathbb{E}(\cdot | \mathbb{F}_0) \).

(ii) For any \( \lambda > 0 \) and \( 0 \leq t_0 < t_1 \leq T \), it holds that
\[ \mathbb{E}^{\mathbb{F}_0} \exp \left( \lambda \int_{t_0}^{t_1} f_s(\tilde{Z}_s) ds \right) \leq 2^{T(2C ||f||_{L^p(T)})^1/\beta}. \]

**Proof.** (i) Still we drop the tilde below. For \( m \in \mathbb{N} \), noticing that
\[ \left( \int_{t_0}^{t_1} g(s) ds \right)^m = m! \int_{\Delta^m} g(s_1) \cdots g(s_m) ds_1 \cdots ds_m, \]
where \( \Delta^m := \{(s_1, \cdots, s_m) : t_0 \leq s_1 \leq s_2 \leq \cdots \leq s_m \leq t_1\}, \)
by (4.3), we have
\[ \mathbb{E}^{\mathbb{F}_0} \left( \int_{t_0}^{t_1} f_s(Z_s) ds \right)^m = m! \mathbb{E}^{\mathbb{F}_0} \left( \int_{\Delta^m} f_{s_1} (Z_{s_1}) \cdots f_{s_m} (Z_{s_m}) ds_1 \cdots ds_m \right). \]
Proof into three steps.

Proof.

Then by (i) we have

\[ \text{□} \]

The proof is complete.

4.2. Well-posedness of martingale problem. In this subsection we show the well-posedness of martingale problem for \( \mathcal{L}^{a,b}_t \). More precisely,

**Theorem 4.5.** Suppose that (UE) holds, and for any \( T > 0 \),

\[ \lim_{|z-z'| \to 0} \sup_{t \in [0,T]} ||\sigma_t(z) - \sigma_t(z')|| = 0, \quad (4.15) \]

and \( b \in \mathbb{L}^q(T) \) for some \( q \in (2(2d+1), \infty] \). For each \( (r, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d} \), the set \( \mathcal{R}^{r,b}_z \) has one and only one point. In particular, the martingale problem for \( \mathcal{L}^{a,b}_t \) is well-posed.

**Proof.** Below, we shall fix starting point \( (r, z) \in \mathbb{R}_+ \times \mathbb{R}^{2d} \) and divide the proof into three steps.
(a) We first show the uniqueness. For \( \varphi \in C^\infty_c(\mathbb{R}^{2d}) \) and \( t_1 > r \), let \( u = \mathcal{P}_{0,v}^{a,b}(\varphi) \in \mathbb{H}^{2/3,2}(t_1) \), which satisfies
\[
\partial_t u + \mathcal{L}^{a,b} u + \varphi = 0, \quad u_t = 0.
\] (4.16)
Let \( u^\varepsilon(t) := u_t \ast \varrho_\varepsilon(z) \) and \( \mathbb{P} \in \mathcal{P}_{r,\varepsilon}^{r,b} \). By (4.1), (4.3) and a standard approximation for the time variable, one sees that
\[
t \mapsto u^\varepsilon(t, Z_t) - \int_0^t (\partial_s + \mathcal{L}^{a,b}) u^\varepsilon(s, Z_s) ds
\]
is an \( \mathcal{F}_t \)-martingale under \( \mathbb{P} \) after time \( r \). Thus, by (4.16), we have
\[
u^\varepsilon(t) = -\mathbb{E} \left( \int_0^t (\partial_s u + \mathcal{L}^{a,b}) u^\varepsilon(s, Z_s) ds \right) = \mathbb{E} \left( \int_0^t (\varrho_\varepsilon, \mathcal{L}^{a,b}) u_s + \varphi^\varepsilon(s, Z_s) ds \right).
\]
By Theorem 4.3 and Lemma 2.4, taking limits \( \varepsilon \to 0 \) for both sides yields
\[
u_r(t) = \mathbb{E} \left( \int_0^t \varphi(Z_s) ds \right), \quad t \geq r.
\]
In particular, we have for any \( \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}_{r,\varepsilon}^{r,b} \) and \( t \geq r \),
\[
\mathbb{E}_{\mathbb{P}_1} \varphi(Z_t) = \mathbb{E}_{\mathbb{P}_2} \varphi(Z_t).
\]
By [31 Theorem 6.2.3], we get the uniqueness.

(b) For \( n \in \mathbb{N} \), let \( \varrho_{1/n} \) be defined by (2.15) with \( \varepsilon = 1/n \), and define
\[
b^n := b_t \ast \varrho_{1/n}, \quad \sigma^n := \sigma_t \ast \varrho_{1/n}.
\]
Clearly,
\[
b^n \in L^p([0, T]; C^\infty_b(\mathbb{R}^{2d})), \quad \sigma^n \in L^\infty([0, T]; C^\infty_b(\mathbb{R}^{2d}))
\]
and
\[
\|b^n\|_{L^p(T)} \leq \|b\|_{L^p(T)}, \quad \omega_p(\delta) \leq \omega_p(\delta).
\]
By the classical theory of SDEs, the following SDE admits a unique strong solution \( Z^n_t = (X^n_t, X^n_t) \)
\[
dZ^n_t = (X^n_t, b^n(Z^n_t)) dt + (0, \sigma^n(Z^n_t)) dW_t, \quad Z^n_{t_0} = z.
\]
By Corollary 4.4, for any \( m \in \mathbb{N} \) and \( T > 0 \), there is a constant \( C = C(m, T) > 0 \) such that for all \( r \leq t_0 < t_1 \leq T \) and \( n \in \mathbb{N} \),
\[
\mathbb{E} \left| \int_{t_0}^{t_1} b^n(Z^n_s) ds \right|^m \leq C\|b\|_{L^m(T)}^{m/(m+1 - \frac{1}{2})} (t_1 - t_0)^{m(\frac{1}{m+1} - \frac{1}{2})}.
\]
Let \( \mathbb{P}_n \) be the probability distribution of \( Z^n \) in \( \Omega \). By the above moment estimate, it is by now standard to show that \( (\mathbb{P}_n)_{n \in \mathbb{N}} \) is tight.

(c) By extracting a subsequence, without loss of generality, we may assume that \( \mathbb{P}_n \) weakly converges to some probability measure \( \mathbb{P} \). To see that
\( P \in \mathcal{P}^{n,b}_r \) it suffices to show that for any \( \varphi \in C_c^\infty(\mathbb{R}^{2d}) \), \( M_{t_1}^{\varphi} \) defined by (4.1) is an \( \mathcal{F}_r \)-martingale under \( P \). Equivalently, for any \( r < t_0 < t_1 \) and any bounded continuous \( \mathcal{F}_{t_0} \)-measurable \( G \),

\[
\mathbb{E}_P(M_{t_1}^{\varphi} G) = \mathbb{E}_P(M_{t_0}^{\varphi} G).
\] (4.17)

Notice that

\[
\mathbb{E}_P(M_{n,t_1}^{\varphi} G) = \mathbb{E}_P_n(M_{n,t_0}^{\varphi} G),
\] (4.18)

where

\[
M_{n,t_i}^{\varphi} := \varphi(Z_{t_i}) - \int_{r}^{t_i} \mathcal{L}_s^{\alpha,b} \varphi(Z_s)ds, \ i = 0, 1.
\]

Let us prove the following limit: for \( i = 0, 1 \),

\[
\lim_{n \to \infty} \mathbb{E}_P_n \left( G \int_{r}^{t_i} (b^m_s \cdot \nabla \varphi)(Z_s)ds \right) = \mathbb{E}_P \left( G \int_{r}^{t_i} (b_s \cdot \nabla \varphi)(Z_s)ds \right). \tag{4.19}
\]

For any \( p \in (2d+1, q) \) and \( T > 0 \), by Theorem 4.3, there is a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \) and \( f \in \mathbb{L}^p(T) \),

\[
\mathbb{E}_P_n \left( \int_{r}^{T} f_i(Z_s)ds \right) \leq C\|f\|_{\mathbb{L}^p(T)}.
\]

Let \( f \in C_c([0, T] \times \mathbb{R}^{2d}) \). By taking weak limits, we have

\[
\mathbb{E}_P \left( \int_{r}^{T} f_i(Z_s)ds \right) = \lim_{n \to \infty} \mathbb{E}_P_n \left( \int_{r}^{T} f_i(Z_s)ds \right) \leq C\|f\|_{\mathbb{L}^p(T)}.
\]

By a monotone class argument, the above estimate still holds for all \( f \in \mathbb{L}^p(T) \). Let the support of \( \varphi \) be contained in \( B_R \). Thus, if we let \( P_\infty = P \), then

\[
\lim_{m \to \infty} \sup_{n \in \mathbb{N}} \mathbb{E}_P \left( G \int_{r}^{t_i} |(b^m_s - b_s) \cdot \nabla \varphi(Z_s)ds \right) \leq C\|G\|_{\infty}\|\nabla \varphi\|_{\infty} \lim_{m \to \infty} \|(b^m - b)1_{B_R}\|_{\mathbb{L}^p(r,T)} = 0. \tag{4.20}
\]

On the other hand, for each \( m \in \mathbb{N} \), since \( \omega \mapsto G(\omega) \int_{r}^{t_i} (b^m_s \cdot \nabla \varphi)(Z_s(\omega))ds \) is a continuous and bounded functional, we have

\[
\lim_{n \to \infty} \mathbb{E}_P_n \left( G \int_{r}^{t_i} (b^m_s \cdot \nabla \varphi)(Z_s)ds \right) = \mathbb{E}_P \left( G \int_{r}^{t_i} (b_s^m \cdot \nabla \varphi)(Z_s)ds \right).
\]

Combining this with (4.20), we get (4.19). Similarly, one can show

\[
\lim_{n \to \infty} \mathbb{E}_P_n \left( G \int_{r}^{t_i} \mathcal{L}_s^{\alpha,0} \varphi(Z_s)ds \right) = \mathbb{E}_P \left( G \int_{r}^{t_i} \mathcal{L}_s^{\alpha,0} \varphi(Z_s)ds \right). \tag{4.21}
\]

Finally, by taking weak limits for both sides of (4.18) and using (4.19) and (4.21), we get (4.17). The proof is complete. \( \square \)
Remark 4.6. When $b$ is bounded measurable ($q = \infty$), this result has been proven in [27] and [23]. Therein, more general equations were considered. However, by comparing with the original proof of Stroock and Varadhan [31, Chapter 7], our proof is quite different from [27, 23] as our starting point is a global apriori Krylov’s estimate (see Theorem 4.3). In principle, it is reasonable to believe that our argument is applicable for more general equations as studied in [27, 23].

Remark 4.7. By suitable localization techniques as developed in [27], it is possible to weaken the global assumptions in Theorem 4.5 as local ones together with some non-explosion conditions.

4.3. Proof of Theorem 1.3 Let $\nu \in \mathcal{P}(\mathbb{R}^{2d})$. By Theorem 4.5, the probability measure $\mathbb{P}_\nu(A) := \int_{\mathbb{R}^{2d}} \mathbb{P}_0(z) \nu(dz)$ is the unique martingale solution for $\mathcal{L}_t^{a,b}$ starting from $\nu$ at time $0$. The conclusion of Theorem 1.3 now follows by [33, Theorem 2.5].

5. Proof of Theorem 1.1

In this section we assume that $\sigma$ satisfies (UE) and for some $p > 2(d+1)$,

$$k_0 := |b|_{L^p([-1,1];H^2_{-1})} + \|\nabla \sigma\|_{L^\infty([-1,1];L^p)} < \infty. \tag{5.1}$$

For $n \in \mathbb{N}$, let $a_{1/n}$ be defined by (2.15) with $\epsilon = 1/n$, as in the previous, define

$$b^n_t := b_t * a_{1/n}, \quad \sigma^n_t := \sigma_t * a_{1/n}, \quad b^\infty_t := b_t, \quad \sigma^\infty_t := \sigma_t. \tag{5.2}$$

Lemma 5.1. Assume (UE) and (5.1). Then $(H^K_{d,p})$ holds for $a^n := \frac{t}{2} \partial^n(\sigma^n)^*$ uniformly with respect to $n$, and there exists a constant $C = C(d, p) > 0$ such that for all $s$,

$$\|\sigma^n_s - \sigma^s\|_\infty \leq C\|\nabla \sigma_s\|_p n^{2d/p - 1}, \quad \sup_v \|\Delta_{\sigma_s}^\frac{1}{2} \sigma^n \|_p \leq C\|\nabla \sigma_s\|_p.$$

Proof. Since $p > 2d$, by Morrey’s inequality, there is a constant $C = C(d, p) > 0$ such that

$$|\sigma_s(z) - \sigma_s(z')| \leq C|z - z'|^{1 - \frac{2d}{p}}\|\nabla \sigma_s\|_p, \quad z, z' \in \mathbb{R}^{2d}.$$

From this, it is easy to see that $(H^K_{d,p})$ holds for $a^n := \frac{t}{2} \partial^n(\sigma^n)^*$ uniformly with respect to $n$, and

$$\|\sigma^n_s - \sigma^s\|_\infty \leq \int_{\mathbb{R}^{2d}} \|\sigma_s(\cdot + z) - \sigma_s(\cdot)\|_\infty a_{1/n}(z)dz \leq C\|\nabla \sigma_s\|_p \int_{\mathbb{R}^{2d}} |z|^{1 - \frac{2d}{p}} a_{1/n}(z)dz \leq C\|\nabla \sigma_s\|_p n^{2d/p - 1}.$$
Moreover, since $p > 4d$, by (2.7) and (2.10), we have
\[
\sup_v \|\Delta_\lambda^\frac{1}{2} \sigma^p \|_p^p \leq \int \sup_v |\Delta_\lambda^\frac{1}{2} \sigma^p(x, v)|_p^p \, dx \leq \|\Delta_\lambda^\frac{1}{2} \sigma^p\|_p^p + \|\Delta_\lambda^\frac{1}{2} \Delta_\lambda^\frac{1}{2} \sigma^p\|_p^p \\
\leq \|\Delta_\lambda^\frac{1}{2} \sigma^p\|_p^p \leq \|\Delta_\lambda^\frac{1}{2} \sigma^p\|_p^p \leq \|\nabla \sigma^p\|_p^p.
\]
Here $\Delta = \Delta_\lambda + \Delta_\nu$. The proof is complete.

Let $T > 0$ and $\lambda \geq 1$. For $n \in \mathbb{N} \cup \{\infty\}$, let $u^n_\lambda \in H^{0,2}(T)$ uniquely solve the following PDE:
\[
\partial_t u^n_\lambda + \mathcal{L}_t a^n b^n u^n_\lambda - \lambda u^n_\lambda + b^n = 0, \quad u^n_\lambda(T) = 0,
\]
where $a^n : = \frac{1}{2} a^\beta(\sigma^n)^\beta$. By Lemma 7.1 and Theorem 3.2, there is a constant $C = C(d, p, \kappa_0, K) > 0$ such that for all $n \in \mathbb{N} \cup \{\infty\}$,
\[
\|\nabla \nabla u^n_\lambda\|_{L^p(T)} \leq C\|b\|_{H^{2,1,0}(T)}, \tag{5.3}
\]
and by (2.11), (2.12), (3.4) with $\alpha = \frac{4d+1}{2d+1}$ and $\beta = \frac{4d+1}{2d+1}$,
\[
\|\nabla \nabla u^n_\lambda\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \leq C(T \wedge \lambda^{-1})^{\frac{d+1}{2d+1}} \|b\|_{L^p(T)}. \tag{5.4}
\]

Lemma 5.2. Under (UE) and (5.1), there is a constant $C > 0$ depending only on $d, p, \kappa_0, K, T, \lambda$ such that for all $n, m \in \mathbb{N}$,
\[
\|u^n_\lambda - u^m_\lambda\|_{L^\infty(\mathbb{R}^d)} + \|\nabla \nabla u^n_\lambda - \nabla \nabla u^m_\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C\left(\|b^n - b^m\|_{L^p(\mathbb{R}^d)} + (n \wedge m)^{\frac{d-1}{p}}\right).
\]

Proof. Let $w^{m,n} := u^n_\lambda - u^m_\lambda$. By equation (3.1), we have
\[
\partial_t w^{m,n} + (\mathcal{L}_t a^n b^n - \lambda) w^{m,n} + (\mathcal{L}_t a^m b^m - \mathcal{L}_t a^n b^n) u^n_\lambda + b^n - b^m = 0.
\]
Noticing that
\[
g^{m,n}_{\lambda, \lambda} := (\mathcal{L}_t a^n b^n - \mathcal{L}_t a^m b^m) u^n_\lambda = \text{tr}(a^n - a^m) \cdot \nabla \nabla u^n_{\lambda, \lambda} + (b^n - b^m) \cdot \nabla \nabla u^n_{\lambda, \lambda},
\]
by (3.3), (3.4) and Lemma 5.1, we have
\[
\|g^{m,n}_{\lambda, \lambda}\|_{L^p(T)} \leq \|a^n - a^m\|_{L^\infty(\mathbb{R}^d)} \|\nabla \nabla u^n_{\lambda, \lambda}\|_{L^p(T)} + \|b^n - b^m\|_{L^p(T)} \|\nabla \nabla u^n_{\lambda, \lambda}\|_{L^\infty(\mathbb{R}^d)} \\
\leq (n \wedge m)^{\frac{d-1}{p}} + \|b^n - b^m\|_{L^p(T)}.
\]
By (2.12) and (3.4) with $\alpha = \frac{4d+1}{2d+1}$ and $\beta = \frac{4d+1}{2d+1}$, we have
\[
\|w^{m,n}_{\lambda}\|_{L^\infty} \leq \|w^{m,n}_{\lambda}\|_{L^p(T)} \|g^{m,n}_{\lambda, \lambda}\|_{L^p(T)} + \|w^{m,n}_{\lambda}\|_{L^\infty(\mathbb{R}^d)} \\
\leq (n \wedge m)^{\frac{d-1}{p}} + \|b^n - b^m\|_{L^p(T)},
\]
and by (2.11), (2.12), (3.4) with $\alpha = \frac{4d+1}{2d+1}$ and $\beta = \frac{4d+1}{2d+1}$,
\[
\|\nabla \nabla w^{m,n}_{\lambda}\|_{L^\infty} \leq \|\nabla \nabla w^{m,n}_{\lambda}\|_{L^p(T)} \|g^{m,n}_{\lambda, \lambda}\|_{L^p(T)} + \|w^{m,n}_{\lambda}\|_{L^\infty(\mathbb{R}^d)} \\
\leq (n \wedge m)^{\frac{d-1}{p}} + \|b^n - b^m\|_{L^p(T)}.
\]
The proof is complete. \(\square\)
For $n \in \mathbb{N} \cup \{\infty\}$, let

$$H^n_t(x, v) := v + u^n_{\lambda, t}(x, v).$$

(5.5)

By (5.4), one can choose $\lambda$ large enough (being independent of $n$ and fixed below) so that

$$\|\nabla_v u^n_t\|_{L^\infty(T)} \leq \frac{1}{2},$$

(5.6)

and thus,

$$\frac{1}{2}|v - v'| \leq |H^n_t(x, v) - H^n_t(x, v')| \leq \frac{3}{2}|v - v'|.$$

(5.7)

Observing that

$$\partial_t H^n + \mathcal{L}_t \lambda H^n - \lambda u^n = 0,$$

by Itô’s formula, we have

$$H^n_t(Z^n_t) = H^n_0(Z^n_0) + \lambda \int_0^t u^n_{\lambda, s}(Z^n_s)ds + \int_0^t \Theta^n_s(Z^n_s)dW_s,$$

(5.8)

where $\Theta^n_s(z) := (\nabla_v H^n_s \cdot \sigma^n_s)(z)$ satisfies by (5.6) and (5.3) that

$$\|\Theta^n\|_{\infty} \leq 2\|\sigma\|_{\infty},$$

and for the given $p > 2(2d + 1)$,

$$\|\nabla \Theta^n\|_{L^p(\Omega)} \leq 2\|\nabla \sigma^n\|_{L^p(T_0)} + \|\sigma^n\|_{\infty}\|\nabla \nabla H^n\|_{L^p(\Omega)} + \|\sigma^n\|_{\infty}\|b\|_{L^p(T_0)}.$$

(5.9)

For $n \in \mathbb{N}$, since $b^n \in L^p([0, T]; C^0_b(\mathbb{R}^{2d}))$ and $\sigma^n \in L^\infty([0, T]; C^\infty(\mathbb{R}^{2d}))$, the following SDE admits a unique solution $Z^n_t = (X^n_t, \dot{X}^n_t)$

$$dZ^n_t = (X^n_t, \dot{X}^n_t)dt + (0, \sigma^n_t(Z^n_s))dW_t, \quad Z^n_0 = z = (x, v) \in \mathbb{R}^{2d}.$$

(5.10)

We have

**Lemma 5.3.** Under (UE) and (5.1), for any $q \geq 2$, there is a constant $C > 0$ such that for all $n, m \in \mathbb{N}$,

$$\left\| \sup_{t \in [0, T_0]} |Z^n_t - Z^m_t| \right\|_{L^q(\Omega)} \leq C \left( \|b^n - b^m\|_{L^p(T_0)} + (n \wedge m)^{\frac{2d}{p} - 1} \right).$$

(5.11)

**Proof.** By (5.8) and Itô’s formula, we have

$$|H^n_t(Z^n_t) - H^m_t(Z^m_t)|^2 = |H^n_0(Z^n_0) - H^m_0(Z^m_0)|^2 + \int_0^t |\Theta^n_s(Z^n_s) - \Theta^m_s(Z^m_s)|^2 ds$$

$$+ 2\lambda \int_0^t \langle H^n_s(Z^n_s) - H^m_s(Z^m_s), u^n_{\lambda, s}(Z^n_s) - u^m_{\lambda, s}(Z^m_s) \rangle ds$$

$$+ 2 \int_0^t \langle H^n_s(Z^n_s) - H^m_s(Z^m_s), (\Theta^n_s(Z^n_s) - \Theta^m_s(Z^m_s))dW_s \rangle,$$

and also,

$$|X^n_t - X^m_t|^2 = 2 \int_0^t \langle X^n_s - X^m_s, X^n_s - X^m_s \rangle ds.$$
If we set
\[ \xi_t := |H_s^n(Z^s_t) - H_t^n(Z^m_t)|^2 + |X^n_t - X^m_t|^2, \]
then
\[ \xi_t \leq 2\|H_t^n - H^m_t\|^2 + 2\|H_t^n(Z^s_t) - H^m_t(Z^m_t)\|^2 + |X^n_t - X^m_t|^2 \]
\[ \leq \xi_0 + \int_0^t \xi_s^{(1)} \, ds + \int_0^t \xi_s^{(2)} \, dW_s + \int_0^t \xi_s \beta_s \, ds + \int_0^t \xi_s \alpha_s \, dW_s, \]
where \( \xi_0 := 3\|u^n - u^m\|^2_{L^\infty(T_0)} \) and
\[ \xi_s^{(1)} := 4\|\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)\|^2 + 6\lambda H^m_t(Z^n_t) - H^m_t(Z^m_t), \]
\[ \xi_s^{(2)} := 4(\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t))' (H_t^n(Z^s_t) - H_t^m(Z^m_t)) - 4\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)')(H_t^n(Z^s_t) - H_t^m(Z^m_t)), \]
\[ \beta_s := 4\|\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)\|^2/\xi_s + 6\lambda |\dot{X}^n_s - \dot{X}^m_s|^2/\xi_s + 2\langle X^n_s - X^m_s, \dot{X}^n_s - \dot{X}^m_s \rangle/\xi_s, \]
\[ \alpha_s := 4(\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t))' (H_t^n(Z^s_t) - H_t^m(Z^m_t))/\xi_s. \]

By (5.7), one has
\[ \xi_t \leq |Z^n_t - Z^m_t|^2 = |X^n_t - X^m_t|^2 + |\dot{X}^n_t - \dot{X}^m_t|^2, \tag{5.12} \]
which implies by (6.6) that
\[ |\beta_s| + |\alpha_s|^2 \leq 1 + (M|\nabla \Theta_s^n(Z^s_t)|^2 + (M|\nabla \Theta_s^m(Z^m_t)|^2 =: G_{s,m}. \tag{5.13} \]
In view of \( p > 2(2d + 1) \) and \( \sup_n ||M|\nabla \Theta^n||_{L^p(T)} < \infty, \) by (ii) of Corollary 4.4 we have for any \( \gamma > 0, \)
\[ \sup_{n,m} \mathbb{E} \exp \left\{ \gamma \int_0^T G_{s,m} \, ds \right\} < \infty. \]
Therefore, by Lemma 6.1 below with \( q_0 = q, q_1 = q_2 = q_3 = 3q/2, \) we have
\[ ||\xi_T^u||_q \leq ||u^n - u^m||^2_{L^\infty(T)} + \left\| \int_0^T \xi_s^{(1)} \, ds \right\|_{3q/2} + \left\| \int_0^T \xi_s^{(2)} \, ds \right\|_{3q/4}, \tag{5.14} \]
where \( \xi_T^u := \sup_{n>0,T} \xi_T. \)

Now, letting
\[ \ell_{n,m} := ||\sigma^n - \sigma^m||_{L^\infty(T)} + ||u^n - u^m||_{L^\infty(T)} + ||\nabla u^n - \nabla u^m||_{L^\infty(T)}, \]
by definitions, we have
\[ ||\xi_s^{(1)}|| \leq \ell_{n,m}, \]
and by (6.6) and (5.12),
\[ ||\xi_s^{(2)}|| \leq ||\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)||^2 (H_t^n(Z^s_t) - H_t^m(Z^m_t))^2 \]
\[ + ||\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)||^2 (H_t^n(Z^s_t) - H_t^m(Z^m_t))^2 \]
\[ \leq \left( ||\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)||^2 + ||\Theta_s^n(Z^s_t) - \Theta_s^m(Z^n_t)||^2 \right) \]
Letting $Z^m$, Proof.

Theorem 5.4. (Existence of strong solutions) Under we obtain the desired estimate (5.11) from (5.15).

In fact, for any $q$, there is a continuous $X_t$ solving SDE (1.1) by taking limits for (5.10). For any $q$, there is a continuous $X_t$ solving SDE (1.1) by taking limits for (5.10).

First of all, by (5.11), there exists a continuous $X_t$ such that $X_t$ solving SDE (1.1) by taking limits for (5.10). For any $q$, there is a continuous $X_t$ solving SDE (1.1) by taking limits for (5.10).

Below we show that $X_t$ solves SDE (1.1) by taking limits for (5.10). For this, we only need to show the following two limits:

\[ \lim_{n \to \infty} \mathbb{E} \left| \int_0^T b^n_s(Z^n_s)ds - \int_0^T b_s(Z_s)ds \right| = 0, \quad (5.19) \]

\[ \lim_{n \to \infty} \mathbb{E} \left| \int_0^T \sigma^n_s(Z^n_s)dW_s - \int_0^T \sigma_s(Z_s)dW_s \right| = 0. \quad (5.20) \]
For (5.19), by (5.17) and (5.18) we have
\[
\lim_{n \to \infty} \sup_{m \in \mathbb{N}} \mathbb{E} \left( \int_0^t |b^m(Z^m_s) - b_s(Z_s^m)|^2 \, ds \right) \leq C \lim_{n \to \infty} \|b^n - b\|_{L^p} = 0,
\]
and for each \( n \in \mathbb{N} \), by the dominated convergence theorem,
\[
\lim_{m \to \infty} \mathbb{E} \left( \int_0^t |b^m(Z^m_s) - b^m_s(Z_s^m)|^2 \, ds \right) = 0.
\]
Limit (5.20) is similar. The proof is complete. \( \square \)

Now, by a result of Cherny [9], the existence of strong solutions together with the weak uniqueness (see Theorem 4.5) implies the pathwise uniqueness. However, to show the homeomorphism property of \( z \mapsto Z_t(z) \), one needs the following \( q \)-order moment estimate for all \( q \in \mathbb{R} \), which clearly implies the pathwise uniqueness.

**Lemma 5.5.** For any \( q \in \mathbb{R} \), there is a constant \( C > 0 \) such that for any two solutions \( Z_t \) and \( Z_t' \) of SDE (1.1) with starting points \( z = (x, \nu) \) and \( z' = (x', \nu') \) respectively,
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |Z_t - Z_t'|^2q \right) \leq C |z - z'|^{2q}.
\]
Moreover, we also have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} (1 + |Z_t|^2)^q \right) \leq C (1 + |z|^2)^q.
\]

**Proof.** Let \( H = H^\infty \) be defined by (5.5). By (5.10) and Itô’s formula, we have
\[
|H_t(Z_t) - H_t(Z_t')|^2 = |H_0(Z_0) - H_0(Z_0')|^2 + \int_0^t \left\| \Theta_s(Z_s) - \Theta_s(Z_s') \right\|^2 \, ds
+ 2 \lambda \int_0^t \langle H_s(Z_s) - H_s(Z_s'), u_{\lambda,s}(Z_s) - u_{\lambda,s}(Z_s') \rangle \, ds
+ 2 \int_0^t \langle H_s(Z_s) - H_s(Z_s'), \Theta_s(Z_s) - \Theta_s(Z_s') \rangle \, dW_s.
\]
As in the proof of Lemma 5.3, if we set
\[
\xi_t := |H_t(Z_t) - H_t(Z_t')|^2 + |X_t - X_t'|^2,
\]
then
\[
\xi_t = \xi_0 + \int_0^t \xi_s \beta_s \, ds + \int_0^t \xi_s \alpha_s \, dW_s,
\]
where
\[
\beta_s := [||\Theta_s(Z_s) - \Theta_s(Z_s')||^2 + 2 \langle X_s - X_s', \dot{X}_s - \dot{X}_s' \rangle / \xi_s
+ 2 \lambda \langle H_s(Z_s) - H_s(Z_s'), u_{\lambda,s}(Z_s) - u_{\lambda,s}(Z_s') \rangle / \xi_s
\]
\[
\alpha_s := \frac{1}{\xi_s} \left( 2 \lambda Z_s - H_s(Z_s') \right).
\]
\[ \alpha_s := 2(\Theta_s(Z_s) - \Theta_s(Z'_s))(H_s(Z_s) - H_s(Z'_s))/\xi_s. \]

By Doléans-Dade’s exponential formula, we have
\[ \xi_t = \xi_0 \exp \left\{ \int_0^t \alpha_s dW_s + \int_0^t \left[ \beta_s - \frac{1}{2} |\alpha_s|^2 \right] ds \right\} \quad (5.23) \]

By (5.7), one has
\[ \xi_t \asymp |Z_t - Z'_t|^2 = |X_t - X'_t|^2 + |\dot{X}_t - \dot{X'}_t|^2. \]

Hence, by (5.5) and (6.6) below,
\[ |\beta(s) + |\alpha(s)|^2 \leq 1 + (M|\nabla \Theta_s(Z_s)|)^2 + (M|\nabla \Theta_s(Z'_s)|)^2. \]

Since \( \|M|\nabla \Theta\|_{L_p(T)} < \infty \) and \( p > 2(2d + 1) \), by Corollary 4.4 we have for any \( \gamma > 0 \),
\[ \mathbb{E} \exp \left\{ \gamma \int_0^T (|\beta_s| + |\alpha_s|^2) ds \right\} < \infty. \quad (5.24) \]

For \( q \in \mathbb{R} \), let
\[ \xi_t^q := \exp \left\{ q \int_0^t \alpha_s dW_s - \frac{q^2}{2} \int_0^t |\alpha_s|^2 ds \right\}. \]

By (5.24) and Novikov’s criterion, \( t \mapsto \xi_t^q \) is a continuous martingale. Therefore, by (5.23), Hölder’s inequality and Doob’s maximal inequality,
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |Z_t - Z'_t|^{2q} \right)
\leq |z - z'|^{2q} \mathbb{E} \left( \sup_{t \in [0,T]} \exp \left\{ q \int_0^t \alpha_s dW_s + q \int_0^t \left[ \beta_s - \frac{1}{2} |\alpha_s|^2 \right] ds \right\} \right)
\leq |z - z'|^{2q} \left( \mathbb{E} (\xi_t^q)^{2} \right)^{\frac{1}{2}} \left( \mathbb{E} \exp \left\{ \int_0^T ((q^2 + q)|\alpha_s|^2 + q\beta_s) ds \right\} \right)^{\frac{1}{2}}
\leq |z - z'|^{2q} \left( \mathbb{E} (\xi_t^q)^{2} \right)^{\frac{1}{2}} \leq |z - z'|^{2q},
\]
which gives (5.21).

On the other hand, if we let
\[ \xi_t := 1 + |H_t(Z_t)|^2 + |X_t|^2, \]
then for any \( q \in \mathbb{R} \), by Itô’s formula, we have
\[
\xi_t^q = \xi_0^q + q \int_0^t \xi_s^{q-1} (2\langle X_s, \dot{X}_s \rangle + ||\Theta_s(Z_s)||^2 - \lambda \langle H_s(Z_s), u_{1,s}(Z_s) \rangle) ds
+ 2q \int_0^t \xi_s^{q-1} \langle H_s(Z_s), \Theta_s(Z_s) dW_s \rangle + 2q(q - 1) \int_0^t \xi_s^{q-2} ||\Theta'_s(Z_s)H_s(Z_s)||^2 ds.
\]
Noticing that
\[ |H_t(Z_t)| = 1 + |X_t|, \quad \xi_t = 1 + |Z_t|^2 = 1 + |X_t|^2 + |\dot{X}_t|^2, \]
by Burkholder’s inequality, we have
\[ \mathbb{E}\left( \sup_{s \in [0,t]} \xi_s^{2q} \right) \leq \xi_0^{2q} + \mathbb{E} \int_0^t \xi_s^{2q} \, ds, \]
which in turn gives (5.22) by Gronwall’s inequality. \(\square\)

Now we can give

Proof of Theorem 1.1. The existence and uniqueness of a strong solution in time interval \([0, T]\) follows by Theorem 5.4 and Lemma 5.5. The bi-continuous version of \((t, z) \mapsto Z_t(z)\) follows by (5.21) and Kolmogorov’s continuity criterion.

(A) As for the homeomorphism property, it follows by Lemma 5.5 and Kunita’s argument (see [22, 39]).

(B) The weak differentiability of \(z \mapsto Z_t(z)\) and estimate (1.4) follow by (5.21) and [37, Theorem 1.1].

(C) It follows by (5.16). \(\square\)

6. APPENDIX

The following stochastic Gronwall’s type lemma is probably well-known. Since we can not find it in the literature, a proof is provided here for the reader’s convenience.

Lemma 6.1. For given \(T > 0\), let \((\xi_t)_{t \in [0,T]}\) and \((\beta_t)_{t \in [0,T]}\) (resp. \((\alpha_t)_{t \in [0,T]}\)) be two real-valued (resp. \(\mathbb{R}^d\)-valued) measurable \(\mathcal{F}_t\)-adapted processes. Let \(\xi_t\) be an Itô process with the form:
\[ \xi_t = \xi_0 + \int_0^t \xi^{(1)}_s \, ds + \int_0^t \xi^{(2)}_s \, dW_s. \]
Suppose that for any \(\gamma > 0\),
\[ \kappa_\gamma := \mathbb{E} \exp \left\{ \gamma \int_0^T (|\beta_s| + |\alpha_s|^2) \, ds \right\} < \infty, \quad (6.1) \]
and
\[ 0 \leq \xi_t \leq \xi_0 + \int_0^T \xi_s \beta_s \, ds + \int_0^T \xi_s \alpha_s \, dW_s. \quad (6.2) \]
Then for any \(q_0 \in [1, \infty)\) and \(q_1, q_2, q_3 > q_0\), there is a constant \(C > 0\) only depending on \(q_i, \kappa_\gamma\), \(i = 0, 1, 2, 3\) such that
\[ \|\xi_T^*\|_{q_0} \leq C \left( \|\xi_0\|_{q_0} + \left\| \int_0^T |\xi^{(1)}_s| \, ds \right\|_{q_1} + \left\| \int_0^T |\xi^{(2)}_s|^2 \, ds \right\|_{q_3/2}^{1/2} \right), \quad (6.3) \]
where \(\xi_T^* := \sup_{t \in [0,T]} \xi_t\) and \(\|\cdot\|_{q_i}\) denotes the norm in \(L^{q_i}(\Omega)\).
Proof. Write
\[
\eta_t := \zeta_t + \int_0^t \xi_s \beta_s \, ds + \int_0^t \xi_s \alpha_s \, dW_s
\]
and
\[
\eta_t := \zeta_t + \int_0^t \eta_s \tilde{\beta}_s \, ds + \int_0^t \eta_s \tilde{\alpha}_s \, dW_s,
\]
where \( \tilde{\beta}_s := \xi_s \beta_s / \eta_s \) and \( \tilde{\alpha}_s := \xi_s \alpha_s / \eta_s \). Here we use the convention \( \frac{0}{0} := 0 \).

Define
\[
M_t := \exp \left\{ \int_0^t \tilde{\alpha}_s \, dW_s + \int_0^t (\tilde{\beta}_s - \frac{1}{2} |\tilde{\alpha}_s|^2) \, ds \right\}.
\]
By Itô’s formula, we have
\[
M_t = 1 + \int_0^t M_s \tilde{\beta}_s \, ds + \int_0^t M_s \tilde{\alpha}_s \, dW_s
\]
and
\[
\eta_t = M_t \left[ \zeta_0 + \int_0^t M_s^{-1} (\zeta_s^{(1)} - \langle \tilde{\alpha}_s, \zeta_s^{(2)} \rangle) \, ds + \int_0^t M_s^{-1} \zeta_s^{(2)} \, dW_s \right].
\]
Hence,
\[
\|\eta_t^+\|_{q_0} \leq \|M_T^+ \zeta_0\|_{q_0} + \left\| M_T^+ (M^{-1})^+_T \int_0^T |\zeta_s^{(1)}| \, ds \right\|_{q_0}
\]
\[
+ \left\| M_T^+ (M^{-1})^+_T \int_0^T |\tilde{\alpha}_s| \cdot |\zeta_s^{(2)}| \, ds \right\|_{q_0}
\]
\[
+ \left\| M_T^+ \sup_{t \in [0, T]} \left| \int_0^T M_s^{-1} \zeta_s^{(2)} \, dW_s \right| \right\|_{q_0}
\]
\[
=: I_1 + I_2 + I_3 + I_4.
\]
Noticing that by (6.2),
\[
|\tilde{\beta}_s| \leq |\beta_s|, \quad |\tilde{\alpha}_s| \leq |\alpha_s|,
\]
for any \( p \in \mathbb{R} \), by (6.1), Hölder’s inequality and Doob’s maximal inequality, we have
\[
\mathbb{E} \left( \sup_{t \in [0, T]} M_t^p \right) < \infty.
\]
Thus, by Hölder’s inequality and (6.5), we have
\[
I_1 \leq C \|\zeta_0\|_{q_1}, \quad I_2 \leq C \left\| \int_0^T |\zeta_s^{(1)}| \, ds \right\|_{q_2},
\]
and by (6.4),
\[
I_3 \leq \left\| M_T^+ (M^{-1})^+_T \left( \int_0^T |\tilde{\alpha}_s|^2 \, ds \right)^{1/2} \left( \int_0^T |\zeta_s^{(2)}|^2 \, ds \right)^{1/2} \right\|_{q_0}
\]
\[
=: I_4.
\]
\[ \leq C \left( \int_0^T |\zeta_x(t)|^2 \, dt \right)^{1/2} = C \left( \int_0^T |\zeta_x(t)|^2 \, dt \right)^{1/2}. \]

Similarly, by Hölder and Burkholder’s inequalities, we also have

\[ I_3 \leq C \left( \int_0^T |\zeta_x(t)|^2 \, dt \right)^{1/2}. \]

Combining the above estimates, we obtain (6.3).

Let \( f \) be a locally integrable function on \( \mathbb{R}^d \). The Hardy-Littlewood maximal function is defined by

\[ Mf(x) := \sup_{0<r<\infty} \frac{1}{|B_r|} \int_{B_r} f(x+y) \, dy, \]

where \( B_r := \{ x \in \mathbb{R}^d : |x| < r \} \). The following result can be found in [10, Appendix A].

**Lemma 6.2.** (i) There exists a constant \( C_d > 0 \) such that for all \( f \in C^\infty(\mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \),

\[ |f(x) - f(y)| \leq C_d |x - y| (M|\nabla f|(x) + M|\nabla f|(y)). \tag{6.6} \]

(ii) For any \( p > 1 \), there exists a constant \( C_{d,p} \) such that for all \( f \in L^p(\mathbb{R}^d) \),

\[ \|Mf\|_p \leq C_{d,p}\|f\|_p. \tag{6.7} \]

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