SUSY in the Spacetime of Higher-Dimensional Rotating Black Holes

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(Dated: February 20, 2009)

Abstract

General higher-dimensional rotating black hole spacetimes of any dimensions admit the Killing and Killing-Yano tensors, which generate the hidden symmetries just as in four-dimensional Kerr spacetime. We study these properties of the black holes using the formalism of supersymmetric mechanics of pseudo-classical spinning point particles. We present two nontrivial supercharges, corresponding to the Killing-Yano and conformal Killing-Yano tensors of the second rank. We demonstrate that an unusual extended Poisson-Dirac algebra of these supercharges results in two independent Killing tensors in spacetime dimensions \( D \geq 6 \), giving explicit examples for the Myers-Perry black holes in \( D = 6 \) dimensions.
I. INTRODUCTION

Black holes are supposed to be one of the most enigmatic objects in nature and it is remarkable that general relativity provides an exact mathematical description of these objects. As is known, the exact solution of the Einstein field equations discovered by R. Kerr in 1963 describes a family of rotating black holes [1]. The Kerr metric is stationary and axisymmetric, that implies its invariance under two continuous spacetime symmetries: time-translational and rotational symmetries defined by two commuting Killing vector fields. Another important symmetry property of the Kerr metric was revealed when exploring its geodesics. In 1968, Carter [2] showed that the Hamilton-Jacobi equation for geodesics of the Kerr metric admits a complete separation of variables. The reason for this was the existence of an extra constant of motion not related to the global isometries of the spacetime. He was also able to show the separability of variables in the Klein-Gordon equation for charged particles by constructing explicitly the set of four, mutually commuting, differential operators [3]. In 1970, Walker and Penrose gave [4] an elegant mathematical interpretation of these results, pointing out that the Kerr metric admits hidden symmetries generated by a second rank Killing tensor. Namely, the Killing tensor plays a crucial role in the complete integrability of the geodesic and scalar field equations.

The existence of the Killing tensor has also motivated the study of electromagnetic and gravitational perturbations of the Kerr spacetime. In 1972, Teukolsky [5] showed that separation of variables occurs in equations for electromagnetic and gravitational perturbations and presented a master equation governing scalar, electromagnetic and gravitational perturbations of the Kerr spacetime. A further important step in the problem of separability in this background was made by achieving separation of variables in the Dirac equation [6, 7]. In order to explain this result, Carter and McIagan [8] managed to construct a new linear differential operator, commuting with the Dirac operator. For this purpose, the authors used the fact that the Kerr metric, in addition to the Killing tensor, also admits a second rank antisymmetric, the so-called Killing-Yano tensor [9]. In other words, it is the Killing-Yano tensor that provides the separability of variables in the Dirac equation and its very existence is the physical reason for many remarkable properties of the Kerr metric. These properties turned out to be so impressive that Chandrasekhar called them “miraculous” properties [10]. The miraculous properties played a profound role in astrophysical
implications of black holes, facilitating analytical studies of various classical and quantum processes around them.

A systematic exploration of hidden symmetries generated by the Killing-Yano tensor was undertaken in a remarkable paper by Gibbons et al [11]. In the strive to answer to mutually correlated questions of what is the classical analogue of the Carter and Mclenaghan result and what is the relation between the Killing-Yano tensor and the “fermionic constituent” of point particle dynamics, the authors explored the worldline supersymmetric mechanics of pseudo-classical spinning point particles in curved backgrounds [12, 13, 14, 15]. They found that the existence of the Killing-Yano tensor in the Kerr metric corresponds to the appearance of a new supersymmetry in the theory of spinning point particles in this background.

In recent developments, the efforts in the study of the hidden symmetries of black holes have been focused on higher dimensions. It turned that the hidden symmetries of the Kerr metric survive in higher dimensions as well. In 2007, Frolov and Kubizňák demonstrated that the higher-dimensional Myers-Perry metric [16], which is a generalization of the Kerr metric to all spacetime dimensions, admits the Killing and Killing-Yano tensors [17]. The authors [18] have also extended this result to the case of higher-dimensional Kerr-NUT-AdS spacetimes discovered by Chen, Lü and Pope [19], which has subsequently been the subject for many other studies (see a review paper [20]).

In this paper, we explore the hidden symmetries of the general higher-dimensional black hole spacetimes from the point of view of worldline supersymmetric mechanics of pseudo-classical spinning point particles. Following the work of [11], we show that the hidden symmetries of the black hole spacetimes enhance generic worldline supersymmetry for the spinning particles in these spacetimes. We begin with a brief review of the formalism of spinning point particles in curved backgrounds. Next, we consider two nongeneric supercharges, corresponding to the Killing-Yano and conformal (closed) Killing-Yano tensors of the second rank and underlying the extension of the usual worldline supersymmetry. We show that these supercharges and the standard supercharge of supersymmetric mechanics of the spinning particles are mutually commuting in the sense of Poisson-Dirac brackets.
However, the Poisson-Dirac bracket of each of the nongeneric supercharges with itself does not close on the Hamiltonian, forming an unusual algebra. We demonstrate that this gives rise to two independent Killing tensors in spacetime dimensions $D \geq 6$.

II. THE FORMALISM OF SPINNING PARTICLES

This formalism is based on the use of anticommuting Grassmann variables to introduce spin degrees of freedom in relativistic mechanics of point particles [12, 13, 14, 15]. In this sense, it is a pseudo-classical description of the relativistic Dirac particles. The history of a point particle in ordinary relativistic mechanics is described by its position vector $x^\mu(\tau)$ (a Grassmann-even variable) in a spacetime, where $\tau$ is a proper time parameter along the particle worldline. The extension of the configuration space of this particle by adding a Grassmann-odd variable $\psi^\mu(\tau)$, allows one to describe its spin degrees of freedom as well. Remarkably, this also results in the existence of supersymmetry relating these two variables $x^\mu(\tau)$ and $\psi^\mu(\tau)$. Below we present the action and general relations between the symmetries and constant of motions in supersymmetric mechanics of the pseudo-classical point particle.

A. The action

The action for a spinning particle in a $D$-dimensional curved spacetime with metric $g_{\mu\nu}(x)$ can be written in the form [12]

$$ S = \frac{1}{2} \int d\tau \left( e^{-1} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + i \psi_a \frac{D\psi^a}{D\tau} + i e^{-1} \chi \psi_a e_\mu^a \dot{x}^\mu \right), $$

(1)

where $e(\tau)$ is an ‘einbein’ field of the one-dimensional metric of the worldline, $e_\mu^a(x)$ a ‘vielbein’ field of the spacetime metric and $\chi$ is its fermionic counterpart. Thus, we have

$$ g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad \psi^a = e_\mu^a \psi^\mu, \quad \mu, a = 0, 1, \ldots, D, $$

(2)

where the indices $a$ and $b$ are the locally flat indices and $\eta_{ab}$ is a flat Minkowski metric. Furthermore, the overdot means the usual derivative $d/d\tau$ and the covariant derivative is given by

$$ \frac{D\psi^a}{D\tau} = \dot{\psi}^a - \dot{x}^\mu \omega^a_{\mu b} \psi^b, $$

(3)
where $\omega_{\mu b}^a$ is the spin connection. The action (1) in its present form is invariant with respect to both a reparametrization of $\tau \to \tau'(\tau)$ and local supersymmetry transformations which are given by

$$
\delta x^\mu = -i\epsilon e^\mu_a \psi^a, \quad \delta e = -2i\epsilon \chi, \quad \delta \chi = \dot{\epsilon},
$$

$$
\delta \psi^a = i\epsilon e^a_\mu \dot{x}^\mu + \dot{\delta x}^\mu \omega_{\mu b}^a \psi^b + i\epsilon e \chi \psi^a,
$$

(4)

where $\epsilon(\tau)$ is an infinitesimal Grassmann-odd parameter. It is important to note the use of these transformations along with those of the reparametrization invariance enables one to make a gauge choice $e = 1$ and $\chi = 0$. With this gauge, the action (1) reduces to the worldline supersymmetric one of the form

$$
S = \int d\tau \left( \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} \psi^a \frac{D\psi^a}{D\tau} \right).
$$

(5)

Another important feature of the action (1) is that one can consistently derive the desired constraint equations, accompanying the usual equations of motion

$$
\frac{D^2 x^\mu}{D\tau^2} = -R^\mu_{\nu} \dot{x}^\nu, \quad \frac{D\psi^a}{D\tau} = 0,
$$

(6)

where

$$
R_{\mu\nu} = \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu}
$$

(7)

is the spin-valued curvature tensor. Varying the action (1) with respect to the auxiliary fields $e$ and $\chi$, we arrive at the following constraint equations

$$
H = -e^2 \frac{\delta S}{\delta e} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0,
$$

(8)

$$
Q = -ie \frac{\delta S}{\delta \chi} = e^a_\mu \dot{x}^\mu \psi^a = 0.
$$

(9)

From equation (8) it follows that the particle moves along a null curve, while equation (9) shows that its spin is spacelike.

**B. (Super)symmetries and Conserved Quantities**

The description of spacetime symmetries and the associated conserved charges in the motion of spinning point particles in its most complete form was given in [11, 15]. For our
purpose in the following, we recall some ingredients of this description in the Hamiltonian formalism. Introducing the basic phase-space variables \((x^\mu, \Pi_\mu, \psi^a)\), where the covariant momentum

\[
\Pi_\mu = p_\mu + \frac{i}{2} \omega_{\mu ab} \psi^a \psi^b = g_{\mu\nu} \dot{x}^\nu,
\]

we have the Hamiltonian in the form

\[
H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu.
\]

The proper time evolution of any phase-space function \(J(x, \Pi, \psi)\) is determined by the Poisson-Dirac bracket of this function with the Hamiltonian in (11). That is,

\[
\frac{dJ}{d\tau} = \{J, H\}.
\]

The Poisson-Dirac bracket of two arbitrary phase-space functions is defined as follows

\[
\{F, G\} = D_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} D_\mu G - R_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i (\epsilon(F) \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi_a}),
\]

where \(\epsilon(F)\) is either 0 or 1 depending on the Grassmann-even or odd parity of \(F\). The phase-space covariant derivative is given by

\[
D_\mu F = \partial_\mu F + \Gamma^\lambda_{\mu\nu} \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} + \omega^a_{\mu b} \psi^b \frac{\partial F}{\partial \psi^a}.
\]

Clearly, for the vanishing Poisson-Dirac bracket in (12)

\[
\{J, H\} = 0,
\]

the set of the phase-space functions \(J(x, \Pi, \psi)\) can be thought of as describing all conserved quantities and symmetries underlying the conservation. In this case, substituting the expansion

\[
J = \sum_{n=0}^{\infty} \frac{1}{n!} j^{(n)}_{\mu_1 \ldots \mu_n} (x, \psi) \Pi^{\mu_1} \ldots \Pi^{\mu_n}
\]

in equation (15), one obtains the chain of equations

\[
D_{(\mu_{n+1})} j^{(n)}_{\mu_1 \ldots \mu_n} + \frac{\partial j^{(n)}_{\mu_1 \ldots \mu_n}}{\partial \psi^a} \omega^a_{\mu_{n+1} b} \psi^b = R_{\nu(\mu_{n+1})} j^{(n+1)\nu}_{\mu_1 \ldots \mu_n},
\]

for different coefficients of the expansion \(j^{(n)}_{\mu_1 \ldots \mu_n} (x, \psi)\). We use round brackets to denote symmetrization over the indices enclosed. It is easy to verify that these coefficients can
be identified with Killing tensors of different ranks describing the symmetries of the action (5) (see for details [15]). The most simple solution of these equations for the second rank Killing tensor, $K_{\mu\nu} = g_{\mu\nu}$, gives rise to the Hamiltonian (11), which generates the symmetries with respect to proper time translations. Meanwhile, the simple solution for the Grassmann-odd Killing vector, $I^a = \psi^a$, gives us the supercharge $Q = \Pi_a \psi^a$ associated with the supersymmetry transformations of the form (4) with $e = 1$ and $\chi = 0$. Evaluating the Poisson-Dirac bracket of the supercharge $Q$ with the Hamiltonian (11) and with itself, we have the superalgebra

$$\{Q, H\} = 0, \quad \{Q, Q\} = -2iH. \tag{18}$$

In obtaining the second relation, we have used the cyclic identity for the curvature tensor in the expression

$$R_{\mu\nu} \frac{\partial Q}{\partial \Pi_\nu} = \frac{i}{2} R_{\mu[abc]} \psi^a \psi^b \psi^c = 0, \tag{19}$$

where square brackets denote antisymmetrization over the indices enclosed. We note that the above symmetries and the corresponding dual symmetries (dual supercharge and chiral charge) along with the Hamiltonian itself are of generic symmetries in the sense that they are built in the action (5) from the very beginning [11, 15]. A natural question that arises in this respect, is whether a classical system described by the action (5) admits additional (nongeneric) supersymmetries. Clearly, the existence of these supersymmetries will depend on the explicit form of the spacetime metric. In what follows, we discuss this question in the spacetime of general higher-dimensional rotating black holes.

**III. NONGENERIC SUPERSYMMETRIES**

The theory of spinning point particles in a curved spacetime possesses a new kind of supersymmetry if the spacetime metric admits a Killing-Yano tensor being a square root of the Killing tensor. In this respect, the new supersymmetry is an extension of the hidden symmetry generated by the Killing tensor. The authors of work [11] were the first to construct explicitly a new supercharge, which depends on the second rank Killing-Yano tensor and generates this supersymmetry. In particular, they found that a spinning particle in the Kerr-Newman spacetime acquires an additional worldline supersymmetry, corresponding to the Killing-Yano tensor discovered by Penrose and Floyd [9].
Recently, it was found that the most general rotating black hole spacetimes described by the Kerr-NUT-(anti)de Sitter metrics \[19\] admit a closed conformal Killing-Yano (CKY) two-form that, in essence, is of a “capsule” for all hidden symmetries of these metrics \[17, 18, 20\]. Building up the corresponding exterior products of this CKY two-form and taking their Hodge duals one can generate a “cascade” of Killing-Yano and Killing tensors, covering all spacetime dimensions. Here we use these results to construct new supercharges underlying the appearance of nongeneric supersymmetries for the spinning point particles in these spacetimes. In \[18\] it was shown that with a suitable analytical continuation the Kerr-NUT-(anti)de Sitter metrics can be written in terms of the orthonormal basis one-forms as follows

\[
d s^2 = \sum_{a=1}^{D} \sum_{b=1}^{D} \delta_{ab} e^a e^b = \sum_{\mu=1}^{n} (e^\mu e^\mu + E^\mu E^\mu) + \varepsilon \omega \omega ,
\]

where \( n = \lfloor D/2 \rfloor \) stands for the integer part of \( D/2 \), \( E^\mu = e^{n+\mu} \), \( \omega = e^{2n+1} \) and \( \varepsilon = D - 2n \), i.e. being 0 for even \( D \) and 1 for odd \( D \). For our purposes in the following, we do not need the explicit expressions for these basis one-forms. We recall that they can be found in \[18\]. In this orthonormal frame, the closed CKY two-form \( k \) existing in the Kerr-NUT-(anti)de Sitter metrics acquires a skew-diagonal form which is given by

\[
k = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge E^\mu ,
\]

where \( x_\mu \) are the eigenvalues corresponding to radial and latitude directions in the canonical form of the spacetime metrics. The wedge products of the \( j \)-th power of this closed CKY two-form

\[
k^j = k^\wedge j = k \wedge \ldots \wedge k
\]

is again a closed CKY form and taking its Hodge dual one can define the following nontrivial Killing-Yano \((D-2j)\)-form

\[
f^j = *k^j , \quad j = 1, 2, \ldots \lfloor D/2 - 1 \rfloor .
\]

The defining equations for the second rank closed CKY tensor have the form

\[
D_c k_{ab} = \eta_{ca} \xi_b - \eta_{cb} \xi_a ,
\]

\[
D_{[a} k_{bc]} = 0 ,
\]
where
\[ \xi_a = \frac{1}{D-1} D_c k_a^c, \tag{26} \]
while, the totally antisymmetric \((D - 2j)\) rank Killing-Yano tensor
\[ f^j_{a_1 a_2 \ldots a_{D-2j}} = f^j_{[a_1 a_2 \ldots a_{D-2j}]} \tag{27} \]
obeys the equation
\[ D(a_1 f^j_{a_2 \ldots a_{D-2j+1}}) = 0. \tag{28} \]

Next, following the works of [11, 28], we consider the set of supercharges
\[ \Omega^j = \Pi^a f^j_{a_1 a_2 \ldots a_{D-2j}} \psi^{a_2} \ldots \psi^{a_{D-2j}} - \frac{i}{D - 2j + 1} D_a f^j_{a_1 a_2 \ldots a_{D-2j+1}} \psi^a \psi^{a_2} \ldots \psi^{a_{D-2j+1}}, \tag{29} \]
corresponding to the Killing-Yano tensors in various spacetime dimensions. It is straightforward to show that the Poisson-Dirac bracket of these supercharges both with the Hamiltonian \((11)\) and the standard supercharge \(Q\) vanishes. Evaluating first the Poisson-Dirac bracket of \(\Omega^j\) with \(Q\), we find that
\[ \{Q, \Omega^j\} = -\left( \psi^a D_a \Omega^j + i \Pi^a \frac{\partial \Omega^j}{\partial \psi^a} \right) = -\left( D_a f^j_{a_1 a_2 \ldots a_{D-2j}} \Pi^a \psi^{a_2} \ldots \psi^{a_{D-2j}} \right) \]
\[ - \frac{i}{D - 2j + 1} D_a D_a f^j_{a_1 a_2 \ldots a_{D-2j+1}} \psi^a \psi^{a_2} \ldots \psi^{a_{D-2j+1}} \]
\[ + D_a f^j_{a_2 \ldots a_{D-2j+1}} \Pi^a \psi^{a_2} \ldots \psi^{a_{D-2j+1}}. \tag{30} \]
In obtaining this expression we have used the relation in \([19]\). Substituting in this expression the integrability condition
\[ D_a D_a f^j_{a_2 \ldots a_{D-2j+1}} = \frac{(-1)^{D-2j+1}}{2} (D - 2j + 1) R^b_{a[a_1 a_2 f^j_{a_3 \ldots a_{D-2j+1}]b]} \tag{31} \]
and taking into account equations \([19]\) and \([28]\), we see that it vanishes. Thus, we have
\[ \{Q, \Omega^j\} = 0. \tag{32} \]
The Jacobi identity for two supercharges \(Q\), and \(\Omega^j\) implies that
\[ \{\Omega^j, H\} = 0. \tag{33} \]

Let us now consider a supercharge
\[ \Upsilon = \Pi^a k_{ab} \psi^b, \tag{34} \]
which corresponds to the second rank closed conformal Killing-Yano tensor. Evaluating the Poisson-Dirac bracket of $Q$ with this supercharge, we find that

$$\{Q, \Upsilon\} = -Q \psi_a \xi^a. \quad (35)$$

Clearly, this expression vanishes due to the constraint [9]. On the other hand, from the Jacobi identity, as in the case of (33), it also follows that

$$\{\Upsilon, H\} = 0. \quad (36)$$

Thus, both $\Upsilon$ and $\Omega^i$ commute with the standard supercharge $Q$ in the sense of Poisson-Dirac brackets and therefore, they are superinvariants.

Next, we are interested in vanishing Poisson-Dirac bracket of $\Upsilon$ and $\Omega^i$. It turns out that only for the supercharge $\Omega \equiv \Omega^{[D/2-1]}$ we have the vanishing Poisson-Dirac bracket. The novel feature of this supercharge is that it corresponds to the second rank Killing-Yano tensor in all even spacetime dimensions, whereas in all odd dimensions it depends on the third rank Killing-Yano tensor. We wish now to show that

$$\{\Upsilon, \Omega\} = 0. \quad (37)$$

For certainty, we begin with even spacetime dimensions. In this case, we have the second rank Killing-Yano tensor

$$f_{ab} \equiv f_{ab}^{(D-2)/2} = \frac{1}{(D-2)!} \varepsilon_{ab}^{d_1 d_2 \ldots d_{D-3} d_{D-2}} k_{d_1 d_2} k_{d_3 d_4} \cdots k_{d_{D-3} d_{D-2}}, \quad (38)$$

where $\varepsilon_{a \ldots b \ldots}$ is the usual totally antisymmetric Levi-Civita symbol, and the associated supercharge

$$\Omega = \Pi^a f_{ab} \psi^b - \frac{i}{3} D_a f_{bc} \psi^a \psi^b \psi^c. \quad (39)$$

Before calculating the Poisson-Dirac bracket (13) for $\Upsilon$ and $\Omega$, we first consider the last term in it, for which we have

$$-i \frac{\partial \Upsilon}{\partial \psi^a} \frac{\partial \Omega}{\partial \psi_a} = -\Pi_a k^{ab} \left( i \Pi^c f_{cb} + D_b f_{cd} \psi^c \psi^d \right). \quad (40)$$

Comparing equations (21) and (38) in even dimensions, we obtain the identity

$$k^{ab} f_{cb} = \frac{1}{D!!} \delta_c^a \varepsilon_{d_1 d_2 \ldots d_{D-1} d_D} k_{d_1 d_2} \cdots k_{d_{D-1} d_D}. \quad (41)$$
Using this identity along with equations (8), (9) and (28) we transform equation (40) into the form

\[-i \frac{\partial \Upsilon}{\partial \psi^a} \frac{\partial \Omega}{\partial \psi^a} = -\Pi^a D_k^b k^c_{ab} f^b_d \psi^c \psi^d.\]  

(42)

Performing now the similar calculations with the first two terms in (13) for \( \Upsilon \) and \( \Omega \), and using the integrability condition (31), we find that

\[\{ \Upsilon, \Omega \} = -3\Pi^a D_k^b k^c_{ab} f^b_d \psi^c \psi^d + i R_{hebc} f^b_d \psi^a \psi^b \psi^c \psi^d.\]  

(43)

The first term in this expression vanishes due to equation (25). On the other hand, using this equation one can construct the identity

\[D_a D_k^b k^c_{bc} f^b_d \psi^a \psi^c \psi^d = 0,\]  

(44)

which, in turn, results in

\[R_{a[eb} k^c_{cd]} f^e_d \psi^a \psi^b \psi^c \psi^d = 0.\]  

(45)

This expression along with (19) shows that the second term in (43) vanishes as well, thereby leaving us with (37).

Turning now to odd spacetime dimensions, we note that in this case the calculations for (37) are entirely similar to those described above. We have the third rank Killing-Yano tensor

\[f_{abc} \equiv f_{abc}^{(D-3)/2} = \frac{1}{(D-3)!} \varepsilon_{abc} d_1 d_2 \ldots d_{D-4} d_{D-3} k_{d_1 d_2} \ldots k_{d_{D-4} d_{D-3}},\]  

(46)

and the supercharge

\[\Omega = \Pi^a f_{abc} \psi^b \psi^c - i D_a f_{bcd} \psi^a \psi^b \psi^c \psi^d.\]  

(47)

With equations (21) and (46) in odd dimensions, we find that

\[k^{ad} f_{cbd} = \frac{1}{(D-1)!!} \delta^{a}_{[c} \varepsilon_{d]d_1 d_2 \ldots d_{D-2} d_{D-1}} k_{d_1 d_2} \ldots k_{d_{D-2} d_{D-1}}.\]  

(48)

Using this identity and repeating all steps made above for the case of even dimensions, we again confirm equation (37). Thus, we have the set of three mutually commuting, in the sense of Poisson-Dirac brackets, supercharges \( Q, \Omega \) and \( \Upsilon \).

A. Extended superalgebra

Though all the three supercharges commute with each other however, unlike the case of two standard supercharges, as given in equation (18), the Poisson-Dirac bracket of two
supercharges Ω as well as that of two supercharges Υ, both do not close on the Hamiltonian. Instead, they give rise to an extended, in some sense, unusual superalgebra. Following [11], we find that in all even dimensions

$$\{\Omega, \Omega\} = -2i \left( \frac{1}{2} K_{(1)}^{\mu\nu} \Pi_\mu \Pi_\nu + I^\mu \Pi_\mu + G \right), \quad \text{even } D,$$

where $K_{(1)}^{\mu\nu}$ is a symmetric second rank Killing tensor given by

$$K_{(1)}^{\mu\nu} = f_\lambda^{\mu} f^{\nu\lambda},$$

$I^\mu$ is a spin-valued Killing vector,

$$I^\mu = i (f_a^\mu D_\nu f_b^\mu + f^{\mu\nu} D_\nu f_{ab}) \psi^a \psi^b,$$

and the Killing scalar

$$G = \frac{1}{4} (R_{\mu\nu ab} f_c^\mu f_d^\nu + 2 D_a f_b^\mu D_\mu f_{cd}) \psi^a \psi^b \psi^c \psi^d.$$

We note that in odd spacetime dimensions Ω is Grassmann-even and therefore, its Poisson-Dirac bracket with itself identically vanishes

$$\{\Omega, \Omega\} = 0, \quad \text{odd } D.$$

Similarly, for the Poisson-Dirac bracket of two supercharges Υ, we have

$$\{\Upsilon, \Upsilon\} = -i K_{(2)}^{\mu\nu} \Pi_\mu \Pi_\nu,$$

where the Killing tensor $K_{(2)}^{\mu\nu}$ is given by

$$K_{(2)}^{\mu\nu} = k_\lambda^\mu k^{\nu\lambda}.$$

We recall that in our constructions we use the constraint (8), which corresponds to the null trajectories.

IV. THE MYERS-PERRY METRICS

As we have emphasized above, our construction of the nongeneric supercharges refers to the most general rotating black hole spacetimes described by the Kerr-NUT-(anti)de Sitter metrics [19], though it does not require the explicit form of these metrics. Below, for
some illustrations of the results obtained in the previous section, we consider the general higher-dimensional Myers-Perry metrics given by

\[ ds^2 = -dt^2 + \frac{U dr^2}{V - 2M} + \frac{2M}{U} \left( dt - \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i \right)^2 + \sum_{i=1}^{n} \left( r^2 + a_i^2 \right) \left( \mu_i^2 d\phi_i^2 + d\mu_i^2 + \varepsilon r^2 d\mu_{n+\varepsilon}^2 \right), \quad (56) \]

where the metric functions

\[ V = r^{n-2} \prod_{i=1}^{n} \left( r^2 + a_i^2 \right), \quad \frac{U}{V} = 1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \quad (57) \]

and the latitude coordinates obey the relation

\[ \sum_{i=1}^{n} \mu_i^2 + \varepsilon r^2 d\mu_{n+\varepsilon}^2 = 1, \quad n = \left( (D - 1)/2 \right). \quad (58) \]

We note that here \( n = \left( (D - 1)/2 \right) \), and \( \varepsilon = 1 \) for even and \( \varepsilon = 0 \) for odd spacetime dimensions. In [17] it was shown that these metrics admit the second rank closed conformal Killing-Yano tensor, which has the following form

\[ k = \sum_{i=1}^{n} a_i \mu_i d\mu_i \wedge \left[ a_i dt - \left( r^2 + a_i^2 \right) d\phi_i \right] + r d\rho \wedge \left( dt - \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i \right), \quad (59) \]

while, the associated Killing tensor is given by

\[ K^{\mu\nu} = \sum_{i=1}^{n} \left[ a_i^2 \left( \mu_i^2 - 1 \right) g^{\mu\nu} + a_i^2 \mu_i^2 \delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{\mu_i^2} \delta_\mu^\phi_i \delta_\nu^\phi_i \right] \\
+ \sum_{i=1}^{n-1+\varepsilon} \delta_\mu^\rho \delta_\nu^\mu - 2Z^{(\mu} Z^{\nu)} - 2\xi^{(\mu} \zeta^{\nu)}, \quad (60) \]

where

\[ Z = \sum_{i=1}^{n-1+\varepsilon} \mu_i \delta_\mu_i, \quad \xi = \delta_t, \quad \zeta = \sum_{i=1}^{n} a_i \delta_\phi_i. \quad (61) \]

It is interesting to compare these results with the Killing tensors in (50) and (55) obtained from the Poisson-Dirac brackets of the corresponding nongeneric supercharges. We begin with the \( D = 4 \) case. Then, the metrics (56) reduce to the usual Kerr metric and the second rank Killing-Yano tensor in (38) takes the form

\[ f_{\mu\nu} = \frac{1}{2} \xi_{\mu} \alpha^\alpha \kappa_{\alpha\beta}. \quad (62) \]

Now, it is easy to see that

\[ K^{\mu\nu}_{(1)} = \frac{1}{2} g^{\mu\nu} - K_{(2)}^{\mu\nu}. \quad (63) \]
With the constraint (5), these two Killing tensors are just proportional to each other, resulting in only one constant of motion. This is in agreement with (60) in $D = 4$. In the five-dimensional case $D = 5$ we have the only Killing tensor $K^{\mu\nu}_{(2)}$, which given by the Poisson-Dirac brackets of two supercharges $\Upsilon$. This also agrees with (60).

However, the situation is completely different in dimensions $D \geq 6$, where the supercharges $\Upsilon$ and $\Omega$ define two independent Killing tensors. To demonstrate this explicitly, let us consider $D = 6$. Using equation (59), it is straightforward to show that the Killing tensor $K^{\mu\nu}_{(2)}$ in (55) agrees with the $D = 6$ limit of the expression (60), while the Killing tensor $K^{\mu\nu}_{(1)}$ defined by the Killing-Yano tensor

$$f_{\mu\nu} = \frac{1}{4!} \varepsilon_{\mu\nu\alpha\beta\lambda\tau} k^{\alpha\beta} k^{\lambda\tau},$$

has the following compact form

$$K^{\mu\nu}_{(1)} = \frac{1}{36} \left( 3 K^{\mu\nu}_{(2)} K^{\rho\lambda}_{(2)} - 2 k^2 K^{\mu\nu}_{(2)} + k_{\mu} k_{\nu} K^{\rho\tau}_{(2)} k^{\rho\tau} \right),$$

where $k^2 = k_{\mu\nu} k^{\mu\nu}$. Thus, we have explicitly shown that the extended algebra of the non-generic supercharges $\Omega$ and $\Upsilon$, discussed in the previous section, results in two independent Killing tensors in the spacetime of Myers-Perry black holes with $D = 6$. Clearly, this is also true in all $D > 6$ dimensions.

V. CONCLUSION

The existence of hidden symmetries generated by the Killing-Yano tensors in rotating black holes spacetimes has fundamental significance for the study of properties of the black holes in various dimensions. It is the Killing tensor that lies at the root of the appearance of a “hidden supersymmetry” in the motion of pseudo-classical spinning point particles in the Kerr-Newman spacetime [11]. The hidden supersymmetry appears as an extension of the usual worldline supersymmetry of the spinning particles and unlike, supersymmetry of black holes in supergravity, does not require any special relation between the physical parameters of the black holes.

In this paper, we have studied the hidden supersymmetries for the model of the spinning point particles in the spacetime of higher-dimensional rotating black holes. Using the fact that the Kerr-NUT-(anti)de Sitter metrics, describing the most general rotating black holes
in all higher dimensions possess the hidden symmetries, we have presented two nontrivial supercharges based on the corresponding Killing-Yano and conformal Killing-Yano tensors of these metrics. Evaluating the associated Poisson-Dirac brackets, we have shown that these supercharges along with the generic supercharge of the spinning particle model constitute a set of three mutually commuting supercharges. On the other hand, the hidden supersymmetries generated by the nongeneric supercharges do not obey the standard algebra of the generic supercharge, but form an unusual extended algebra. The latter, in turn, results in two independent Killing tensors in spacetime dimensions $D \geq 6$. These Killing tensors along with the three commuting Killing vectors and the spacetime metric itself guarantee a complete separation of variables for the Hamilton-Jacobi equation in six dimensions. For these dimensions, our construction may also shed some light on the physical reason of separation of variables in the Dirac equation [21]. On the other hand, it is known that in both cases the separability occurs in all higher dimensions [21, 29]. Therefore, our construction raises a natural question: What are the new extra supercharges underlying the separability in all $D \geq 8$ spacetime dimensions. This is a challenging task for future work.

VI. ACKNOWLEDGMENTS

The authors thank Nihat Berker and Teoman Turgut for their stimulating encouragements and support.

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