THE EQUIVARIANT BRAUER GROUPS OF COMMUTING FREE AND PROPER ACTIONS ARE ISOMORPHIC

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Abstract. If $X$ is a locally compact space which admits commuting free and proper actions of locally compact groups $G$ and $H$, then the Brauer groups $\text{Br}_H(G \setminus X)$ and $\text{Br}_G(X/H)$ are naturally isomorphic.

Rieffel’s formulation of Mackey’s Imprimitivity Theorem asserts that if $H$ is a closed subgroup of a locally compact group $G$, then the group $C^*$-algebra $C^*(H)$ is Morita equivalent to the crossed product $C_0(G/H) \rtimes G$. Subsequently, Rieffel found a symmetric version, involving two subgroups of $G$, and Green proved the following Symmetric Imprimitivity Theorem: if two locally compact groups act freely and properly on a locally compact space $X$, $G$ on the left and $H$ on the right, then the crossed products $C_0(G \setminus X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent. (For a discussion and proofs of these results, see [15].) Here we shall show that in this situation there is an isomorphism $\text{Br}_H(G \setminus X) \cong \text{Br}_G(X/H)$ of the equivariant Brauer groups introduced in [2].

Suppose $(G, X)$ is a second countable locally compact transformation group. The objects in the underlying set $\mathfrak{Br}_G(X)$ of the equivariant Brauer group $\text{Br}_G(X)$ are dynamical systems $(A, G, \alpha)$, in which $A$ is a separable continuous-trace $C^*$-algebra with spectrum $X$, and $\alpha : G \to \text{Aut}(A)$ is a strongly continuous action of $G$ on $A$ inducing the given action of $G$ on $X$. The equivalence relation on such systems is the equivariant Morita equivalence studied in [1], [3]. The group operation is given by $[A, \alpha] \cdot [B, \beta] = [A \otimes_{C(X)} B, \alpha \otimes \beta]$, the inverse of $[A, \alpha]$ is the conjugate system $[\overline{A}, \overline{\alpha}]$, and the identity is represented by $(C_0(X), \tau)$, where $\tau_s(f)(x) = f(s^{-1} \cdot x)$.

Notation. Suppose that $H$ is a locally compact group, that $X$ is a free and proper right $H$-space, and that $(B, H, \beta)$ a dynamical system. Then $\text{Ind}_H^X(B, \beta)$ will be the...
C*-algebra (denoted by $GC(X, B)^{\alpha}$ in [13] and by $\text{Ind}(B; X, H, \beta)$ in [11]) of bounded continuous functions $f : X \to B$ such that $\beta_h(f(x \cdot h)) = f(x)$, and $x \cdot H \mapsto \|f(x)\|$ belongs to $C_0(X/H)$.

We now state our main theorem.

**Theorem 1.** Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and a free and proper right $H$-action such that $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then there is an isomorphism $\Theta$ of $\text{Br}_H(G/X)$ onto $\text{Br}_G(X/H)$ satisfying:

1. if $(A, \alpha)$ represents $\Theta[B, \beta]$, then $A \rtimes_{\alpha} G$ is Morita equivalent to $B \rtimes_{\beta} H$;
2. $\Theta[B, \beta]$ is realised by the pair $(\text{Ind}^X_H(B, \beta)/J, \tau \otimes \text{id})$ in $\text{Br}_G(X/H)$, where $\tau \otimes \text{id}$ denotes left translation and, if $\pi_{G,x}$ is the element of $\tilde{B} = G\backslash X$ corresponding to $G \cdot x$,

$$J = \{ f \in \text{Ind}^X_H(B, \beta) : \pi_{G,x}(f(x)) = 0 \text{ for all } x \in X \}.$$

Item (1) is itself a generalization of Green’s symmetric imprimitivity theorem, and our proof of Theorem 1 follows the approach to Green’s theorem taken in [3]: prove that both $C_0(G\backslash X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent to $C_0(X) \rtimes_{\alpha} (G \times H)$, where $\alpha_{s,h}(f)(x) = f(s^{-1} \cdot x \cdot h)$, by noting that the Morita equivalences of $C_0(X) \rtimes G$ with $C_0(G\backslash X)$ and $C_0(X) \rtimes H$ with $C_0(X/H)$ ([7], [15, Situation 10]) are equivariant, and hence induce Morita equivalences

$$C_0(G\backslash X) \rtimes H \sim (C_0(X) \rtimes G) \rtimes H \cong C_0(X) \rtimes (G \times H) \cong (C_0(X) \rtimes H) \rtimes G.$$

The same symmetry considerations show that it will be enough to prove that $\text{Br}_H(G\backslash X) \cong \text{Br}_{G \times H}(X)$. Since we already know that $\text{Br}(G\backslash X) \cong \text{Br}_G(X)$ [2, §6.2], we just have to check that this isomorphism is compatible with the actions of $H$.

Suppose $G$ acts freely and properly on $X$, and $p : X \to G\backslash X$ is the orbit map. If $B$ is a a $C^*$-algebra with a nondegenerate action of $C_0(G\backslash X)$, then the pull-back $p^*B$ is the quotient of $C_0(X) \otimes B$ by the balancing ideal

$$I_{G\backslash X} = \overline{\text{span}}\{ f \cdot \phi \otimes b - \phi \otimes f \cdot b : \phi \in C_0(X), f \in C_0(G\backslash X), b \in B \}$$

in other words, $p^*B = C_0(X) \otimes_{C(G\backslash X)} B$. The nondegenerate action of $C_0(G\backslash X)$ on $B$ induces a continuous map $q$ of $\tilde{B}$ onto $G\backslash X$, characterized by $\pi(f \cdot b) = f(q(\pi))\pi(b)$. Then under the natural identification of $C_0(X) \otimes B$ with $C_0(\tilde{X}, B)$,

$$I_{G\backslash X} \cong \{ f \in C_0(X, B) : \pi(f(x)) = 0 \text{ for all } x \in q(\pi) \},$$

so that $p^*B$ has spectrum

$$\widehat{p^*B} = \{(x, \pi) \in X \times \tilde{B} : G \cdot x = q(\pi) \}.$$
If $B$ is a continuous-trace algebra with spectrum $G\setminus X$, then $p^*B$ is a continuous-trace algebra with spectrum $X$.

The isomorphism $\Theta : \text{Br}(G\setminus X) \cong \text{Br}_G(X)$ is given by $\Theta[A] = [p^*A, \tau \otimes \text{id}]$. To prove $\Theta$ is surjective in [2], we used [12, Theorem 1.1], which implies that if $(B, \beta) \in \mathcal{B}r_G(X)$, then $B \rtimes_{\beta} G$ is a continuous-trace algebra with spectrum $G\setminus X$ such that $(B, \beta)$ is Morita equivalent to $(p^*(B \rtimes_{\beta} G), \tau \otimes \text{id})$, and hence that $[B, \beta] = \Theta[B \rtimes_{\beta} G, \text{id}]$. In obtaining the required equivariant version of [12, Theorem 1.1], we have both simplified the proof and mildly strengthened the conclusion (see Corollary 4 below). However, with all these different group actions around, the notation could get messy, and we pause to establish some conventions.

**Notation.** We shall be dealing with several spaces carrying a left action of $G$ and/or a right action of $H$. We denote by $\tau$ the action of $G$ by left translation on $C_0(G)$, $C_0(X)$ or $C_0(G\setminus X)$, and by $\sigma$ any action of $H$ by right translation; we shall also use $\sigma^G$ to denote the action of $G$ by right translation on $C_0(G)$. Restricting an action $\beta$ of $G \times H$ on an algebra $A$ gives actions $\alpha : G \to \text{Aut}(A)$, $\gamma : H \to \text{Aut}(A)$ such that

$$\alpha_s(\gamma_h(a)) = \gamma_h(\alpha_s(a)) \quad \text{for all } h \in H, s \in G, a \in A.$$ 

Conversely, two actions $\alpha, \gamma$ satisfying (1) define an action of $G \times H$ on $A$, which we denote by $\alpha \gamma$; we write $\gamma$ for id $\gamma$ since it will be clear from context whether an action of $H$ or $G \times H$ is called for. If $\Phi : (A, G, \alpha) \to (B, G, \beta)$ is an equivariant isomorphism (i.e. $\Phi(\alpha_s(a)) = \beta_s(\Phi(a))$), then we denote by $\Phi \rtimes \text{id}$ the induced isomorphism in $A \rtimes_{\alpha} G$ onto $B \rtimes_{\beta} G$. Similarly, if $\alpha$ and $\gamma$ satisfy (1), we write $\alpha \rtimes \text{id}$ for the induced action of $G$ on $A \rtimes_{\alpha} H$.

**Lemma 2.** Suppose a locally compact group $G$ acts freely and properly on a locally compact space $X$, and that $A$ is a $C^*$-algebra carrying a non-degenerate action of $C_0(X)$. If $\alpha : G \to \text{Aut}(A)$ is an action of $G$ on $A$ satisfying $\alpha_s(\phi \cdot a) = \tau_s(\phi) \cdot \alpha_s(a)$, then the map sending $f \otimes a$ in $C_0(X) \otimes A$ to the function $s \mapsto f \cdot \alpha^{-1}_s(a)$ induces an equivariant isomorphism $\Phi$ of $(C_0(X) \otimes C(G, X), A, G, \text{id} \otimes \alpha)$ onto $(C_0(G, A), G, \tau \otimes \text{id})$.

**Remark 3.** For motivation, consider the case where $A = C_0(X)$. Then the map $\Psi : C_b(X \times X) \to C_0(G \times X)$ defined by $\Psi(f)(s, x) = f(x, s \cdot x)$ maps $C_0$ to $C_0$ precisely when the action is proper, has range which separates the points of $G \times X$ precisely when the action is free, and has kernel consisting of the functions which vanish on the closed subset $\Delta = \{(x, y) : G \cdot x = G \cdot y\}$. Thus the free and proper actions are precisely those for which $\Psi$ induces an isomorphism of $C_0(X) \otimes_{C(G, X)} C_0(X)$ onto $C_0(G) \otimes C_0(X)$.

**Proof of Lemma 2.** If $\phi \in C_0(G \setminus X)$, then $f \cdot \phi \otimes a$ and $f \otimes \phi \cdot a$ have the same image in $C_0(G, A)$, and the map factors through the balanced tensor product as claimed. Further, $\Phi$ is related to the map $\Psi$ in Remark 3 by

$$\Phi(f \otimes g \cdot a) = (\Psi(f \otimes g)(s, \cdot)) \cdot \alpha^{-1}_s(a).$$
Thus it follows from the remark that (2) defines an element of $C_0(G, A)$ and that
the closure of the range of $\Phi$ contains all functions of the form $s \mapsto \xi(s)f \cdot \alpha^{-1}_s(a)$
for $\xi \in C_c(G)$, $f \in C_c(X)$, and $a \in A$. These elements span a dense subset of
$C_0(G, A)$, and hence $\Phi$ is surjective. The nondegenerate action of $C_0(X)$ on $A$ induces
a continuous equivariant map $q$ of $\hat{A}$ onto $X$ such that $\pi(f \cdot a) = f(q(\pi))\pi(a)$, and the
balanced tensor product $C_0(X) \otimes_{C(G)} A$ has spectrum $\Delta = \{(x, \pi) : G \cdot x = G \cdot q(\pi)\}$. 
Since each representation $(q(\pi), s \cdot \pi) = (q(\pi), \pi \circ \alpha^{-1}_s)$ in $\Delta$ factors through $\Phi$ and
the representation $b \mapsto \pi(b(s))$ of $C_0(G, A)$, the homomorphism $\Phi$ is also injective. Finally, to see the equivariance, we compute:

$$
\Phi(id \otimes \alpha_s(h \otimes a))(t) = h \cdot \alpha_t^{-1}(\alpha_s(a)) = \Phi(h \otimes a)(s^{-1}t) = \tau_s \otimes \text{id}(\Phi(h \otimes a))(t). \quad \square
$$

**Corollary 4.** (cf. [12, Theorem 1.1]) Let $(G, X)$ and $\alpha : G \to \text{Aut}(A)$ be as in
Lemma 2. Then there is an equivariant isomorphism of $(p^*(A \rtimes_\alpha G), G, p^* \text{id})$ onto
$(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \text{Ad} \rho)$.

**Proof.** A routine calculation shows that the equivariant isomorphism $\Phi$ of Lemma 2
gives an equivariant isomorphism

$$(3) \quad \Phi \rtimes \text{id} : \left((C_0(X) \otimes_{C(G)} A) \rtimes_{\text{id} \otimes \alpha} G, (\tau \otimes \text{id}) \times \text{id}\right)
\to \left(C_0(G, A) \rtimes_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha) \times \text{id}\right).$$

We also have equivariant isomorphisms

$$(4) \quad (C_0(G, A) \rtimes_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha) \times \text{id}) \cong \left(A \otimes (C_0(G) \rtimes \gamma G), \alpha \otimes (\sigma^G \times \text{id})\right),
\cong \left(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \text{Ad} \rho\right)$$

and

$$(5) \quad (C_0(X) \otimes_{C(G \setminus X)} (A \rtimes_\alpha G), \tau \otimes \text{id}) \cong \left((C_0(X) \otimes_{C(G \setminus X)} A) \rtimes_{\text{id} \otimes \alpha} G, (\tau \otimes \text{id}) \times \text{id}\right);$$
combining (3), (4), and (5) gives the result. \quad \square

**Lemma 5.** In addition to the hypotheses of Lemma 2, suppose that $H$ is a locally
compact group acting on the right of $X$, and that $(A, H, \gamma)$ is a dynamical system such
that $\alpha$ and $\gamma$ commute and $\gamma_h(f \cdot a) = \sigma_h(f) \cdot \gamma_h(a)$ for $h \in H$, $f \in C_0(X)$, $a \in A$. Then the action $\tau \sigma \otimes \gamma$ of $G \times H$ on $C_0(X) \otimes A$ preserves the balancing ideal $I_{G \setminus X}$,
and hence induces an action of $G \times H$ on $C_0(X) \otimes_{C(G \setminus X)} A$, also denoted $\tau \sigma \otimes \gamma$. The equivariant isomorphism of Lemma 2 induces an equivariant isomorphism

$$
((C_0(X) \otimes_{C(G \setminus X)} A) \rtimes_{\text{id} \otimes \alpha} G, (\tau \sigma \otimes \gamma) \times \text{id}) \cong (C_0(G, A) \rtimes_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha \gamma) \times \text{id}).
$$
Proof. The first assertion is straightforward. For the second, we can consider the actions of $H$ and $G$ separately. We have already observed in (3) that $\Phi \times \text{id}$ intertwines the $G$-actions. On the other hand, if $h \in H$ and $t \in G$, then

$$\Phi(\sigma_h \otimes \gamma_h(f \otimes a))(t) = \sigma_h(f) \cdot \alpha_t^{-1}(\gamma_h(a)) = \sigma_h(f) \cdot \gamma_h(\alpha_t^{-1}(a))$$

$$= \gamma_h(\Phi(f \otimes a)(t)). \square$$

**Corollary 6.** Let $G \times_H$ and $\alpha : G \to \text{Aut}(A)$, $\gamma : H \to \text{Aut}(A)$ be as in the lemma. Denote by $p$ the orbit map of $X$ onto $G \backslash X$. Then there is an equivariant isomorphism

$$(p^*(A \rtimes_{\alpha} G), G \times H, \tau \sigma \otimes (\gamma \rtimes \text{id})) \cong (A \otimes \mathcal{K}(L^2(G)), G \times H, \alpha \gamma \rtimes \text{Ad} \rho).$$

**Proof.** Compose the isomorphism of Lemma 5 with (4) and (5). $\square$

We are now ready to define our map of $\text{Br}_G(G \backslash X)$ into $\text{Br}_{G \times H}(X)$. Suppose $(B, \beta) \in \mathfrak{B}_H(X)$. Then the action $\tau \sigma \otimes \beta$ of $G \times H$ preserves the balancing ideal $I_{G \backslash X}$: if $\phi \in C_0(G \backslash X)$ then

$$(\tau \sigma \otimes \beta)_\phi(f \otimes b) = \sigma_h(\tau_s(f) \otimes \beta_h(b)) \otimes \beta_h(\phi \cdot b) - \sigma_h(\tau_s(f) \otimes \beta_h(b)) \otimes \beta_h(\phi \cdot b).$$

Since $p^*(B)$ is a continuous-trace $C^*$-algebra with spectrum $X$ [12, Lemma 1.2], and $\tau \sigma \otimes \beta$ covers the canonical $G \times H$-action on $X$, we can define $\theta : \mathfrak{B}_H(G \backslash X) \to \mathfrak{B}_H(G \times H)$ by $\theta(B, \beta) = (p^*(B), \tau \sigma \otimes \beta)$.

Similarly if $(A, \alpha) \in \mathfrak{B}_H(G \times H)$, then $A \rtimes_{\alpha} G$ is a continuous-trace $C^*$-algebra with spectrum $G \backslash X$ by [12, Theorem 1.1]. Since $\gamma$ is compatible with $\sigma$, we have $\gamma_h(\phi \cdot z(s)) = \gamma_h(\phi) \cdot \gamma_h(z(s))$ for $z \in C_c(G, A)$, and hence $\gamma \rtimes \text{id}$ covers the given action of $H$ on $X$. Thus we can define $\lambda : \mathfrak{B}_H(G \times H) \to \mathfrak{B}_H(G \backslash X)$ by $\lambda(A, \alpha \gamma) = (A \rtimes_{\alpha} G, \gamma \rtimes \text{id})$.

**Proposition 7.** Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and an $H$-action such that $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then $\theta$ and $\lambda$ above preserve Morita equivalence classes, and define homomorphisms $\Theta : \text{Br}_G(G \backslash X) \to \text{Br}_{G \times H}(X)$ and $\Lambda : \text{Br}_{G \times H}(X) \to \text{Br}_H(G \backslash X)$. In fact, $\Theta$ is an isomorphism with inverse $\Lambda$, and if $\Theta[B, \beta] = [A, \alpha]$ then $B \rtimes_{\alpha} H$ is Morita equivalent to $A \rtimes_{\alpha} (G \times H)$.

**Proof.** If $(\mathfrak{Y}, \nu)$ implements an equivalence between $(B, \beta)$ and $(B', \beta')$ in $\mathfrak{B}_H(G \backslash X)$, then, the external tensor product $Z = C_0(X) \hat{\otimes} \mathfrak{Y}$, as defined in [9, §1.2] or [2, §2], is a $C_0(X) \otimes B - C_0(X) \otimes B'$-imprimitivity bimodule. A routine argument, similar to that in [2, Lemma 2.1], shows that the Rieffel correspondence [14, Theorem 3.1] between the lattices of ideals in $C_0(X) \otimes B$ and in $C_0(X) \otimes B'$ maps the balancing ideal $I = I_{C(G \times X)}$ in $C_0(X) \otimes B$ to the balancing ideal $J = J_{C(G \times X)}$ in $C_0(X) \otimes B'$. Thus [14, Corollary 3.2] implies that $X = \mathfrak{Z}/\mathfrak{Z}$, $J$ is a $p^*(B) - p^*(B')$-imprimitivity
bimodule. Since $f \cdot x = x \cdot f$ for all $x \in X$ and $f \in C_0(X)$, it follows from [10, Proposition 1.11] that $X$ implements a Morita equivalence over $X$. More tedious but routine calculations show that the map defined on elementary tensors in $Z_0 = C_0(X) \otimes Y$ by $u \circ (f \otimes y) = \sigma_h(\tau_s(f)) \otimes v_h(y)$ extends to the completion $Z$, and defines a strongly continuous map $u : G \times H \rightarrow \text{Iso}(X)$ such that $(X, u)$ implements an equivalence between $(p^*(B), \tau \sigma \otimes \beta)$ and $(p^*(B'), \tau \sigma \otimes \beta')$. Thus $\Theta$ is well-defined.

Observe that

$$\Theta([B, \beta][B', \beta']) = \Theta([B \otimes_{C(G \setminus X)} B', \beta \otimes \beta'])$$

(6)

$$= [p^*(B \otimes_{C(G \setminus X)} B'), \tau \sigma \otimes (\beta \otimes \beta')].$$

But (6) is the class of

$$(C_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} B', \tau \sigma \otimes \beta \otimes \beta')$$

$$\sim (C_0(X) \otimes_{C(G \setminus X)} C_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} B', \tau \sigma \otimes \tau \sigma \otimes \beta \otimes \beta')$$

$$\sim (C_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} C_0(X) \otimes_{C(G \setminus X)} B', \tau \sigma \otimes \beta \otimes \tau \sigma \otimes \beta'),$$

which represents the product of $\Theta[B, \beta]$ and $\Theta[B', \beta']$. Thus $\Theta$ is a homomorphism.

Now suppose that $(A, \alpha \gamma) \sim (A', \alpha' \gamma')$ in $\mathfrak{B}t_{G \times H}(X)$ via $(Z, w)$. Then $u_s = w_{(s,e)}$ and $v_h = w_{(e,h)}$ define actions of $G$ and $H$, respectively, on $Z$. In particular, $(Z, u)$ implements an equivalence between $(A, \alpha)$ and $(A', \alpha')$ in $\mathfrak{B}t_G(X)$. It follows from [1, §6] that $Y_0 = C_c(G, Z)$ can be completed to a $A \rtimes_\alpha G - A' \rtimes_{\alpha'} G$-imprimitivity bimodule $Y$. One can verify that the induced $C_0(G \setminus X)$-actions on $Y_0$ are given by $(\phi \cdot x)(t) = \phi \cdot (x(t))$ and $(x \cdot \phi)(t) = (x(t)) \cdot \phi$, and [10, Proposition 1.11] implies that $Y$ is an imprimitivity bimodule over $G \setminus X$. Now define $\tilde{\nu}_h^\varnothing$ on $Y_0$ by $\tilde{\nu}_h^\varnothing(x)(t) = v_h(x(t))$.

Using the inner products defined in [1, §6],

$$\langle \tilde{\nu}_h^\varnothing(x), \tilde{\nu}_h^\varnothing(y) \rangle = \int_G \langle \tilde{\nu}_h^\varnothing(x)(s) , \Delta(t^{-1}s)u_t(\tilde{\nu}_h^\varnothing(y)(t^{-1}s)) \rangle \, ds$$

$$= \int_G \langle v_h(x(s)) , \Delta(t^{-1}s)u_t(v_h(y)(t^{-1}s)) \rangle \, ds$$

$$= \gamma_h(\langle x, y \rangle(t)),$$

where, in the last equality, we use $u_s \circ v_h = v_h \circ u_s$. A similar computation shows that $\langle \tilde{\nu}_h^\varnothing(x), \tilde{\nu}_h^\varnothing(y) \rangle_{A' \rtimes_{\gamma' \times \id} G} = \gamma_{\lambda}'(\langle x, y \rangle_{A' \rtimes_{\gamma' \times \id} G}(t))$. Thus $\tilde{\nu}_h^\varnothing$ extends to all of $Y$ and defines a map $\tilde{\nu} : H \rightarrow \text{Iso}(Y)$, and it is not hard to verify that $\tilde{\nu}$ is strongly continuous. Therefore $(A \rtimes_\alpha G, \gamma \times \id) \sim (A' \rtimes_{\alpha'} G, \gamma' \times \id)$ in $\mathfrak{B}t_{G \times H}(X)$, and $\Lambda$ is well-defined.

Now it will suffice to show that, for $a \in \mathfrak{B}t_H(G \setminus X)$ and $b \in \mathfrak{B}t_{G \times H}(X)$, $\theta(\lambda(b)) \sim b$ and $\lambda(\theta(a)) \sim a$. For the first of these, suppose that $(A, \alpha \gamma) \in \mathfrak{B}t_{G \times H}(X)$. Then $\theta(\lambda(A, \alpha \gamma) = (p^*(A \rtimes_\alpha G), (\tau \sigma \otimes \gamma) \times \id)$, which by Corollary 6 is equivalent to
Remark 8. We showed that $\Lambda$ is a well-defined map of $\text{Br}_{G \times H}(X)$ into $\text{Br}_H(G \backslash X)$, and that it is a set-theoretic inverse for $\Theta$; since $\Theta$ is a group homomorphism, it follows that $\Lambda$ is also a homomorphism. This seems to be non-trivial: it implies that if $(A, \alpha)$, $(B, \beta)$ are in $\mathcal{B}_G(X)$, then $(A \otimes_C(X) B) \rtimes_{\alpha \otimes \beta} G$ is Morita equivalent to $A \rtimes \alpha G \otimes_C(G \backslash X) (B \rtimes \beta G)$. We do not know what general mechanism is at work here. Certainly, it is a Morita equivalence rather than an isomorphism: if $G$ is finite and the algebra commutative, one algebra is $|G|$-homogeneous and the other $|G|^2$-homogeneous. The only direct way we have found uses [8, Theorem 17], which seems an excessively heavy sledgehammer.

Proof of Theorem 1. It follows from Proposition 7 that there are isomorphisms $\Theta_H : \text{Br}_H(G \backslash X) \to \text{Br}_{G \times H}(X)$, and $\Lambda_G : \text{Br}_{G \times H}(X) \to \text{Br}_G(X/H)$. Therefore $\Lambda_G \circ \Theta_H$ is an isomorphism of $\text{Br}_H(G \backslash X)$ onto $\text{Br}_G(X/H)$. Assertion (1) also follows from Proposition 7. The isomorphism $\Lambda_G \circ \Theta_H$ maps the class of $(B, \beta)$ in $\mathcal{B}_H(G \backslash X)$ to the class of $(p^*(B) \rtimes_{\sigma \otimes \beta} H, (\tau \otimes \text{id}) \rtimes \text{id})$, so it remains to show that the latter is equivalent to $(A/J, \tau)$.

For convenience, write $I$ for the balancing ideal $I_{C(G \backslash X)}$ in $C_0(X) \otimes B$. Then

$$p^*(B) \rtimes_{\sigma \otimes \beta} H = ((C_0(X) \otimes B)/I) \rtimes_{\sigma \otimes \beta} H = (C_0(X, B) \rtimes_{\sigma \otimes \beta} H)/(I \rtimes_{\sigma \otimes \beta} H)$$

by, for example, [8, Proposition 12]. By [13, Theorem 2.2], $X_0 = C_c(X, B)$ can be completed to a $C_0(X, B) \rtimes_{\sigma \otimes \beta} H - A$-imprimitivity bimodule $X$. The irreducible representations of $A$ are given by $M_{(x, \pi_{G, y})}(f)(x) = \pi_{G, y}(f(x))$ [13, Lemma 2.6]. In the proof of [13, Theorem 2.5], it was shown that the representation $X^{M_{(x, \pi_{G, y})}}$ of $C_0(X, B) \rtimes_{\sigma \otimes \beta} H$ induced from $M_{(x, \pi_{G, y})}$ via $X$ is equivalent to $\text{Ind}^G_{\{e\}} N_{(x, G, y)}$, where $N_{(x, G, y)}$ is the analogous irreducible representation of $C_0(X, B)$. Since the orbit space for a proper action is Hausdorff, [5] implies that $(C_0(X, B), H, \sigma \otimes \beta)$ is regular. Since $R = \bigoplus_{x \in X} N_{(x, G, x)}$ is a faithful representation of $p^*(B)$, it follows from [8, Theorem 24] that $\text{Ind}^G_{\{e\}}(R)$ is a faithful representation of $p^*(B) \rtimes_{\sigma \otimes \beta} H$, and so has
Corollary 10. Suppose that $\ker I \rtimes_{\sigma \otimes \beta} H$. On the other hand, $\text{Ind}^G_{H}(R)$ is equivalent to $\bigoplus_{x \in X} X^{M_{r,G,x}}$. It follows from [14, §3] that $I^X = X/I \cdot X$ is an $p^*(B) \rtimes_{\sigma \otimes \beta} H_{/X/H} A/J$-imprimitivity bimodule. Then the map $u^*_x : X_0 \to X_0$ defined by $u^*_x(x) = x(s^{-1} \cdot x)$ induces a map $u : G \to \text{Iso}(I^X)$ such that $(I^X, u)$ implements the desired equivalence. 

We close with two interesting special cases where the isomorphism takes a particularly elegant form. Recall that if $B$ is a continuous-trace $C^*$-algebra with spectrum $X$, then we may view $B$ as the sections $\Gamma_0(\xi)$ of a $C^*$-bundle $\xi$ vanishing at infinity.

Corollary 9. Suppose that $H$ is a closed subgroup of a second countable locally compact group $G$, and that $X$ is a second countable locally compact right $H$-space. Then $G \times X$ is a free and proper $H$-space via the diagonal action $(s, x) \cdot h = (sh, x \cdot h)$. Thus $(G \times X)/H$ is a locally compact $G$-space via $s \cdot [r, x] = [sr, x]$, and the map $(B, \beta) \mapsto (\text{Ind}^G_H(B, \beta), \tau)$ induces an isomorphism of $\text{Br}_H(X)$ onto $\text{Br}_G((X \times G)/H)$.

Proof. We apply Theorem 1 to $G(G \times X)_H$, where $G$ acts on the left of the first factor, obtaining an isomorphism of $\text{Br}_G(X) \cong \text{Br}_H(G \setminus (G \times X))$ onto $\text{Br}_G((G \times X)/H)$ sending the class of $(B, \beta)$ to the class of $\text{Ind}^G_H(B, \beta)/J$ where $J = \{ f : f(s, x)(x) = 0 \}$.

Given $f \in \text{Ind}^{G \times X}_H(B, \beta)$ and $s \in G$, let $\Phi(f)(s)$ be the function from $X$ to $\xi$ defined by $\Phi(f)(s)(x) = f(s, x)(x)$. We claim $\Phi(f)(s) \in \Gamma_0(\xi)$. If $x_0 \in X$, then $x \mapsto f(s, x_0)(x)$ is in $\Gamma_0(\xi)$, and $\|\Phi(f)(s) - f(s, x_0)(x)\|$ tends to zero as $x \to x_0$. It follows from [6, Proposition 1.6 (Corollary 1)] that $\Phi(f)(s)$ is continuous. To see that $\Phi(f)(s)$ vanishes at infinity, suppose that $\{ x_n \} \subset X$ satisfies

$$\|\Phi(f)(s)(x_n)\| \geq \epsilon > 0$$

for all $n$. Then $\|f(s, x_n)\| \geq \epsilon$ for all $n$, and passing to a subsequence and relabeling if necessary, there must be $h_n \in H$ such that $(s \cdot h_n, x_n \cdot h_n) \to (r, x)$. Then $h_n \to s^{-1}r \in H$, and $x_n \to x \cdot (r^{-1}s)$. In sum, $\Phi(f)(s) \in \Gamma_0(\xi) = B$. Now the continuity of $f$ easily implies that $s \mapsto \Phi(f)(s)$ is continuous from $G$ to $B$. Furthermore, since $\beta$ covers $\sigma$ (i.e., $\beta_h(\phi \cdot b)(x) = \phi(x \cdot h)\beta_h(b)(x)$),

$$f(\phi, x)(x) = \beta^{-1}_h(\Phi(f)(\phi))(x),$$

and $\Phi$ is a $*$-homomorphism of $\text{Ind}^{G \times X}_H(B, \beta)$ into $\text{Ind}_G^G(B, \beta)$, which clearly has kernel $J$.

Finally, it is not difficult (cf., e.g., [13, Lemma 2.6]) to see that $\Phi(\text{Ind}^{G \times X}_H(B, \beta))$ is a rich subalgebra of $\text{Ind}^G_H(B, \beta)$ as defined in [4, Definition 11.1.1]. Thus $\Phi$ is surjective by [4, Lemma 11.1.4]. 

Corollary 10. Suppose that $X$ is a locally compact left $G$-space, and that $H$ is a closed normal subgroup of $G$ which acts freely and properly on $X$. Then there is an isomorphism of $\text{Br}_{G/H}(H \setminus X)$ onto $\text{Br}_G(X)$ taking $[B, \beta]$ to $[p^*(B), p^*(\beta)] = [p^*(B), \tau \otimes \beta]$. 
Proof. View $Y = X \times G/H$ as a left $G$-space via the diagonal action, and a right $G/H$-space via right translation on the second factor. Both actions are free, and the second action is proper. To see that the first action is proper, suppose that $(x_n, t_nH) \to (x, tH)$ while $(s_n \cdot x_n, s_n t_n H) \to (y, rH)$. Then $s_n H \to s H$ for some $s \in G$. Passing to a subsequence and relabeling, we can assume that there are $h_n \in H$ such that $h_n s_n \to s$ in $G$. But then $s_n \cdot x_n \to y$ while $h_n \cdot (s_n \cdot x_n) \to s \cdot x$. Since the $H$-action is proper, we can assume that $h_n \to h$ in $H$. Thus $s_n \to h^{-1} s$, and this proves the claim.

The map $G \cdot (x, tH) \mapsto H t^{-1} \cdot x$ is a bijection $\phi$ of $G \setminus Y$ onto $H \setminus X$. Further, $G \setminus Y$ is a right $G/H$-space and $H \setminus X$ is a left $G/H$-space with

$$\phi(v \cdot (s^{-1} H)) = s H \cdot \phi(v).$$

(That is, $\phi$ is equivariant when the $G/H$-action on $G \setminus Y$ is viewed as a left-action.) Therefore,

$$(7) \quad \text{Br}_{G/H}(G \setminus Y) \cong \text{Br}_{G/H}(H \setminus X)$$

Similarly, $Y/(G/H)$ and $X$ are isomorphic as left $G$-spaces so that

$$(8) \quad \text{Br}_G(Y/(G/H)) \cong \text{Br}_G(X).$$

Finally, Theorem 1 implies that

$$(9) \quad \text{Br}_G(Y/(G/H)) \cong \text{Br}_{G/H}(G \setminus Y).$$

Thus, Equations (7)–(9) imply that there is an isomorphism of $\text{Br}_{G/H}(H \setminus X)$ onto $\text{Br}_G(X)$ sending $(B, \beta)$ to $(\text{Ind}^{X \times G/H}_{G/H}(B, \beta)/J, \tau \otimes \text{id})$ with

$$J = \{ f \in \text{Ind}^{X \times G/H}_{G/H}(B, \beta) : f(x, rH)(H r^{-1} \cdot x) = 0 \text{ for all } x \in X \}.$$ 

Define $\Phi : \text{Ind}^{X \times G/H}_{G/H}(B, \beta) \to C_0(X, B)$ by $\Phi(f)(x) = f(x, H)$. Then $\Phi$ is onto (see, for example, the first sentence of the proof of [13, Lemma 2.6]). Since

$$\Phi(\tau \otimes \text{id}(f))(x) = \tau \otimes \text{id}(f)(x, H) = f(s^{-1} \cdot x, s^{-1} H) = \beta_{sH}(f(s^{-1} \cdot x, H))$$

$$= \tau \otimes \beta_{sH}(\Phi(f))(x),$$

$\Phi$ is equivariant, and it only remains to show that $\Phi$ induces a bijection of the quotient by $J$ with the quotient of $C_0(X, B)$ by the balancing ideal $I$.

However, if $\Phi(f) \in I$, then $f(x, H)(H \cdot x) = 0$ for all $x \in X$. But then $f(x, rH)(H r^{-1} \cdot x) = \beta_{rH}^{-1}(f(x, H))(H r^{-1} \cdot x)$, which is zero since $\beta$ covers the $G/H$-action on $X$, and $f \in J$. The argument reverses, so $\Phi(J) = I$, and the result follows. $\square$
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