THE POSTERIOR DISTRIBUTION OF \sin(i) VALUES FOR EXOPLANETS WITH M_T \sin(i) DETERMINED FROM RADIAL VELOCITY DATA

SHIRLEY HO\textsuperscript{1,2} AND EDWIN L. TURNER\textsuperscript{3,4}

\textsuperscript{1} Lawrence Berkeley National Laboratory, 1 Cyclotron Road, MS 50R-5045, Berkeley, CA 94720, USA; cwho@lbl.gov
\textsuperscript{2} Berkeley Center for Cosmological Physics, University of California Berkeley, Berkeley, CA 94720, USA
\textsuperscript{3} Department of Astrophysical Sciences, Peyton Hall, Princeton University, Princeton, NJ 08544, USA
\textsuperscript{4} Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Kashiwa 227-8568, Japan

Received 2010 July 11; accepted 2011 May 20; published 2011 September 1

ABSTRACT

Radial velocity (RV) observations of an exoplanet system giving a value of \( M_T \sin(i) \) condition (i.e., give information about) not only the planet’s true mass \( M_T \) but also the value of \( \sin(i) \) for that system (where \( i \) is the orbital inclination angle). Thus, the value of \( \sin(i) \) for a system with any particular observed value of \( M_T \sin(i) \) cannot be assumed to be drawn randomly from a distribution corresponding to an isotropic distribution, i.e., the presumptive prior distribution. Rather, the posterior distribution from which it is drawn depends on the intrinsic distribution of \( M_T \) for the exoplanet population being studied. We give a simple Bayesian derivation of this relationship and apply it to several “toy models” for the intrinsic distribution of \( M_T \), on which we have significant information from available RV data in some mass ranges but little or none in others. The results show that the effect can be an important one. For example, even for simple power-law distributions of \( M_T \), the median value of \( \sin(i) \) in an observed RV sample can vary between 0.860 and 0.023 (as compared to the 0.866 value for an isotropic distribution) for indices of the power law in the range between \(-2 \) and \(+1 \), respectively. Over the same range of indices, the 95\% confidence interval on \( M_T \) varies from 1.001–2.405 (\( \alpha = -2 \)) to 1.13–94.34 (\( \alpha = +2 \)) times larger than \( M_T \sin(i) \) due to \( \sin(i) \) uncertainty alone. More complex, but still simple and plausible, distributions of \( M_T \) yield more complicated and somewhat unintuitive posterior \( \sin(i) \) distributions. In particular, if the \( M_T \) distribution contains any characteristic mass scale \( M_c \), the posterior \( \sin(i) \) distribution will depend on the ratio of \( M_T \sin(i) \) to \( M_c \), often in a non-trivial way. Our qualitative conclusion is that RV studies of exoplanets, both individual objects and statistical samples, should regard the \( \sin(i) \) factor as more than a “numerical constant of order unity” with simple and well-understood statistical properties. We argue that reports of \( M_T \sin(i) \) determinations should be accompanied by a statement of the corresponding confidence bounds on \( M_T \) at, say, the 95\% level based on an explicitly stated assumed form of the true \( M_T \) distribution in order to reflect more accurately the mass uncertainties associated with RV studies.

Key words: methods: statistical – planetary systems – techniques: radial velocities

1. INTRODUCTION

As is well known the observational study of exoplanets began with, and in large part has been based on, radial velocity (RV) data which allow a measurement of the planet’s orbital parameters plus a value of \( M_T \sin(i) \), where \( M_T \) is its true mass and \( \sin(i) \) is the angle between the direction normal to the planet’s orbital plane and the observer’s sight line (see Marcy et al. 2003, 2005; Cumming et al. 2008; Johnson 2009). Indeed, the very classification of an unseen stellar companion as an exoplanet is normally made based on the value of \( M_T \sin(i) \), hereinafter designated as \( M_0 \), the observed or indicative mass.

It would, of course, be preferable to determine \( M_T \) itself and avoid the degeneracy with the largely uninteresting random variable \( \sin(i) \), and our understanding of exoplanet systems has been greatly advanced by the relatively few cases in which the observations of transit events allows the two parameters to be measured separately (see Charbonneau et al. 2007 and references therein; Winn 2010 and references therein).

Nevertheless, the \( M_T \sin(i) \) degeneracy does not seem too serious because it appears to be so simple and well understood. In particular, it seems extremely safe to assume that \( i \) is randomly and isotropically distributed or, in other words, that the orientation of the orbital plane of an exoplanet in space is independent of the direction from which we observe it. However, this isotropic distribution only describes the prior distribution of \( i \), not its posterior one, i.e., not the relevant distribution after it is conditioned by the measurement of an \( M_T \sin(i) \) value.

In order to specify this isotropic prior distribution of \( \sin(i) \), we consider a longitudinal strip of a sphere between \( \theta = i \) and \( \theta = i + d\theta \) in polar coordinates. The strip extends around \( 2\pi \) in \( \phi \) (the azimuthal angle), and the surface area of the strip is just \( 2\pi r^2 \sin(i)d\theta \). The probability of a randomly oriented vector piercing that area is then just its fractional area of the hemispherical surface of the sphere

\[
f_i = \frac{2\pi r^2 \sin(i)d\theta}{2\pi r^2}.
\]

The pdf of the inclination angle (assuming random orientation) is thus \( \sin(i) \). In other words, the probability distribution function of the inclination angle of the exoplanet falls into this range \( i \) to \( i + d\theta \) is just \( \sin(i) \). (Note that we need only consider a single hemisphere since the distribution of \( i \) will be the same in both.)

In order to determine the prior probability of \( \sin(i) \), simply consider

\[
f_idi = f_jdj,
\]

where \( j = \sin(i) \). After some algebraic manipulation, it is easy to show that \( f_j \) (the pdf of \( \sin(i) \) falling into a range of \( \sin(i) - d\sin(i) \) and \( \sin(i) + d\sin(i) \)) is \( j\sqrt{1 - j^2} \tan(i) \).

Up to this point, the analysis is straightforward. However, complications arise at the next step, the derivation of the posterior distribution of \( \sin(i) \), because it depends on the
prior or true distribution of $M_f$. As we do not yet know the $M_f$ distribution with high precision in any mass range (see Jorissen et al. 2001; Zucker & Mazeh 2001; Marcy et al. 2005; Udry & Santos 2007; Brown 2011) and have little or no empirical information constraining it in others, this consideration is an important one not only in principle but perhaps also be in practice. The present paper is primarily intended to investigate this issue, the posterior distribution of sin$\i$ given an observation of $M_0$, in some detail.

Before presenting a Bayesian analysis in the next sections, it may be helpful to note that the issue resembles familiar complications in interpreting photometric data that are conventionally called Malmquist-type biases (see Malmquist 1920; Eddington 1913; Hogg & Turner 1998) in some respects. Namely, even if the measurement errors are symmetric and unbiased (and, in the simple cases most often analyzed, also Gaussian distributed—but that is not essential), the true brightness of an astronomical object is normally more likely to be fainter than its measured brightness than it is to be brighter. The well-known reason is that there are usually a larger number of fainter objects than brighter ones on which the (symmetrical) measurement errors may act to produce the observed brightness.

However, the considerations for sin$\i$ which we investigate in this paper are not related to measurement errors. It would be unchanged even if all of the observations in question were perfect and ideal. Neither is it a selection bias on sin$\i$ of the sort that was briefly considered as an explanation of exoplanet RV discoveries in their earliest days (see Black 1997; Gray 1997). Rather, we are considering the unavoidable consequences of the combination of a physical variable, $M_f$ with an unobservable stochastic one, sin$\i$, when conditioned by a measurement of their product. This is, of course, a classical issue in Bayesian statistics.

Section 2 defines the basic question addressed by this paper and gives a very simple illustrative example of why it can be an important issue. Section 3 presents a Bayesian derivation of the equations needed to answer the question for any given distribution of masses for a population of exoplanets, and Section 4 presents the results of the analysis obtained by assuming various “toy models” for the true exoplanet distribution of masses. We then discuss observational selection effects briefly in Section 5 and conclude in Section 6 with a discussion of its practical implications for RV studies of exoplanets.

2. ILLUSTRATIVE EXAMPLE

The question we wish to analyze can be formulated in two equivalent but slightly different forms, one describing the $M_f$ distribution and one the sin$\i$ distribution.

1. What is the probability that $M_f$ is less than $X$, given that RV data yield $M_0$ (= $M_f$ sin$\i$)? The answer may be written as $P(M_f < X|M_0)$ and depends on $P(M_f)$, the intrinsic distribution of exoplanet masses.

2. What is the probability that sin$\i$ is less than $Z$, given that RV data yield $M_0$ (= $M_f$ sin$\i$)? This answer may be written as $P(\text{sin}(\i) < Z|M_0)$ and also depends on $P(M_f)$.

To relate the sin$\i$ probability distribution and the true mass distribution, it is simply

$$P(M_f < X|M_0) = 1 - P(\text{sin}(\i) < Z|M_0)$$

for $Z = \frac{M_0}{M_f}$.

To illustrate the fundamental issue, we consider the following toy model: suppose that all exoplanets have a true mass of either 1.0 $M_J$ or 2.0 $M_J$, where $M_J$ is the mass of Jupiter and that there are an equal number of exoplanets with each of these masses. If an exoplanet is determined to have $M_0 = M_f$ sin$\i = 0.5 M_f$, the value of sin$\i$ is obviously either 0.5 or 0.25 depending on whether it is one of the low or high true mass exoplanets, respectively. Moreover, since a sin$\i$ value of 0.5 is about 2.236 times more likely than one of 0.25 for the prior (isotropically) distribution of sin$\i$, it follows that the posterior distribution of sin$\i$ for this system consists of two $\delta$ functions, one at 0.5 and one at 0.25 with the former having an amplitude 2.236 times that of the latter.

Since any intrinsic exoplanet mass distribution could be arbitrarily well approximated by a series of $\delta$ functions, we can conclude that the intrinsic mass distribution affects the posterior distribution of the sin$\i$, for any particular observed value of $M_0$.

3. BAYESIAN DERIVATION OF THE POSTERIOR DISTRIBUTIONS

Consider a planet at any mass $M_f$. Given that the inclination angle $\i$ is randomly (isotropically) distributed, we know that (from the previous sections) the pdf of sin$\i$ falling into a range of sin$\i$ - d sin$\i$ and sin$\i$ + d sin$\i$ is sin$\i$/$\sqrt{1 - \sin^2(\i)}$. The cumulative probability of sin$\i$ < Z will then be

$$P(\text{sin}(\i) < Z) = \int_0^Z \frac{\text{sin}(\i)}{\sqrt{1 - \sin^2(\i)}} d\text{sin}(\i),$$

which is simply

$$P(\text{sin}(\i) < Z) = 1 - \cos(\text{arcsin}(Z)).$$

This is however not surprising, since we know (from trigonometric argument) that

$$P(\i < x) = 1 - \cos(x),$$

and therefore, we know that the prior probability of finding sin$\i$ less than Z, which is equivalent to the prior probability of observed mass $M_0$ given $M_f$ is just

$$P(\text{sin}(\i) < Z) = 1 - \cos(\text{arcsin}(Z)).$$

This, of course, is simply the prior pdf of sin$\i$, derived in Section 1. This distribution function is plotted in the upper panel of Figure 1 in cumulative form; we can also look at the probability of observed mass $M_0$ given $M_f$, which is simply the following:

$$P(M_0|M_f) = \frac{(M_0/M_f)^2}{\sqrt{1 - \left(\frac{M_0}{M_f}\right)^2}}.$$
Assigning $A = M_T$ and $B = M_0$, we directly obtain

$$P(M_T | M_0) = \frac{P(M_0|M_T)P(M_T)}{P(M_0)}. \quad (11)$$

Equation (8) gives the first term in the numerator of Equation (11). The second term in the numerator is an unknown function (which ultimately may be determined from observations), but it is possible to consider simple toy models, plausible guesses, and even theoretical estimates for $P(M_T)$ and thus explore their consequences. Finally, the denominator of Equation (11) can be obtained from

$$P(M_0) = \int P(M_0|M_T)P(M_T) dM_T. \quad (12)$$

Therefore,

$$P(M_T | M_0) = \frac{P(M_0|M_T)P(M_T)}{\int P(M_0|M_T)P(M_T) dM_T}. \quad (13)$$

which provides the desired posterior distribution

$$P(M_T | M_0) = \frac{\frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}}} {\int \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} P(M_T) dM_T} \quad (14)$$

for $M_0 < M_T$.

It is frequently most interesting to consider instead the cumulative probability distribution at which $M_T < X$, requiring the integration of the numerator of Equation (13) up to $X$. This gives

$$P(M_T < X | M_0) = \frac{\int_{M_0}^{X} \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} P(M_T) dM_T}{\int \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} P(M_T) dM_T}. \quad (15)$$

Since if the observed mass is $M_0$, then the true mass $M_T$ has to be larger than or equal to $M_0$, since $M_0 = M_T \sin(i)$ and $\sin(i) \leq 1$, therefore the lower integral limit is $M_0$, and the upper mass limit could be as large as physically possible for the mass of a planet ($M_{\text{max}}$).

Therefore, we have the following:

$$P(M_T < X | M_0) = \frac{\int_{M_0}^{M_{\text{max}}} \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} P(M_T) dM_T}{\int_{M_0}^{M_{\text{max}}} \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} P(M_T) dM_T}. \quad (16)$$

The upper bound on the integral in the denominator $M_{\text{max}}$ is somewhat arbitrary, corresponding to the maximum mass of any planet drawn from the $P(M_T)$ distribution. However, the value of $M_{\text{max}}$ affects only the normalization of $P(M_T < X | M_0)$, not its form.

This formulation in terms of $M_T$ most transparently displays the underlying logic of the derivation. However, the same approach can give equally well the answer to question No. 2 above, since the two are equivalent. In particular,

$$P(\sin(i) < Z | M_0) = P(M_T > X | M_0). \quad (17)$$

Thus,

$$P(\sin(i) < Z | M_0) = 1 - \frac{\int_{M_0}^{M_{\text{max}}} \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} f_{M_T}(y) dy}{\int_{M_0}^{M_{\text{max}}} \frac{(M_0/M_T^2)}{\sqrt{1-\left(\frac{M_0}{M_T}\right)^2}} f_{M_T}(y) dy}, \quad (18)$$

where $f_{M_T}(y)$ is the true mass distribution and $y$ is a dummy variable of integration. The normalization of $f_{M_T}$ cancels out of the expression.

Due to our current ignorance of the true $P(M_T)$, we cannot evaluate these expressions uniquely for the actual observed values of $M_0$ of known exoplanets. It is nevertheless instructive to do so for various assumed $P(M_T)$ distributions. We devote the remainder of the paper primarily to that exercise.

4. POSTERIOR DISTRIBUTIONS FOR ASSUMED TRUE MASS DISTRIBUTIONS

In order to investigate the size and character of the statistical effect under discussion, we will apply the formula derived in the
The Astrophysical Journal

Since the observed mass is set to 1, the true mass has to be larger of the models are at least qualitatively similar to the statistical effect under consideration. Nevertheless, some of the models are at least qualitatively similar to the MT distribution suggested by RV data for exoplanets with $MT \sin(i)$ values greater than one or a few tenths of Jupiter’s mass. Others are not but might well be qualitatively representative of exoplanet populations which are as yet little or not at all characterized by RV data. For example, theoretical scenarios for exoplanet populations which are as yet little or not at all characterized (Ida & Lin 2008a, 2008b; Matsumura et al. 2009; Baraffe et al. 2010 and references therein) predict a significant formation (Ida & Lin 2008a, 2008b; Matsumura et al. 2009; Baraffe et al. 2010 and references therein) predict a significant

4.1. Power-law $MT$ Distributions

Beginning with a particularly simple possibility, we now assume that the distribution of true masses of exoplanets follow a power law, thus we adopt the form $f_{MT}(y) = Ay^\alpha$, where both $A$ and $\alpha$ are constants. Then we can evaluate the main integral (hereafter $\Phi(M_0, \alpha) = \int f_{MT} (\frac{M_M}{M_0}) \frac{AM_T dM_T}{1 - (\frac{M_M}{M_0})^\alpha}$

Equation (16) for several cases ($\alpha = 2, \alpha = 1, \alpha = 0, \alpha = -1, \alpha = -2$) to obtain the following:

$$\Phi(M_0, X, \alpha) = \begin{cases} \sqrt{X^2 - 1} & \text{if } \alpha = 2, \\ \log(2\sqrt{X^2 - 1} + X) & \text{if } \alpha = 1, \\ \pi/2 - \tan^{-1}\left(\frac{1}{\sqrt{1 - X^2}}\right) & \text{if } \alpha = 0, \\ \sqrt{1 - \frac{1}{X^2}} & \text{if } \alpha = -1, \\ \frac{X^2 - X^2\sqrt{1/X^2}) - 1}{2X^2\sqrt{1 - 1/X^2}} & \text{if } \alpha = -2. \end{cases}$$

Note that we set $M_0 = 1$ and $A = 1$ to simplify the equations; this is equivalent to a simple change in the units of mass and space density.

For the distribution of $\sin(i)$, we can refer to Equation (3). We use this result to plot a few specific cases in Figure 2, assuming various values of $\alpha$.

Note from Figure 2 that $\alpha = -1$ gives a median $\sin(i)$ value of 0.860 while an equally plausible value of $\alpha = 0$ gives a median $\sin(i)$ value of 0.704. If the mass distribution is an increasing function of mass, the resulting median $\sin(i)$ value will be reduced quite dramatically; for example, $\alpha = 2$ gives a median $\sin(i)$ value of 0.02.

It is equally easy to generate the corresponding $P(M_T < X | M_0)$ distributions, using Equation (16), as shown in Figure 3. Since the observed mass is set to 1, the true mass has to be larger than 1, and as the power-law index increases (which means a larger number of high-mass planets in the true mass distribution), the probability of finding a planet below $X$ decreases (as seen in Figure 3).

4.2. Power Law plus a Delta Function $MT$ Distributions

The distribution of planetary masses in the solar system, the highly nonlinear and at least partially non-gravitational nature of planet formation as well as some specific theoretical models (see Kokubo & Ida 1996; Kokubo et al. 2006; Ida & Lin 2008a; Baraffe et al. 2010 and references therein) suggest that the $P(M_T)$ distribution might contain one or more characteristic masses, rather than being an entirely scale-free power law.

In order to investigate the implications of such a $P(M_T)$, we consider a toy model in which some of the exoplanets are distributed in an $\alpha = -1$ power-law population while the others all have the same mass $M_e$. We may then again evaluate the expressions of Section 2 directly.

Thus, we have $f_{MT}(y) = Ay^\alpha + B\delta(y - M_e)$, where $M_e$ is the critical mass scale of interest. It is convenient to introduce the dimensionless parameter $\eta$, defined by $M_0 = \eta M_e$, and to set $A$ and $B$ equal. Without loss of generality $M_1 = 1$ is adopted (i.e., $M_e$ is defined as the unit of mass, rather than $M_0$ as in the
pure power-law models analyzed in the previous subsection) for purposes of plotting and giving numerical values. We can then obtain the following:

\[ P(M_T < X|\eta) = \frac{1}{2} \sqrt{1 - \frac{\eta^2}{X^2}} + \frac{\eta}{\eta \sqrt{1 - \eta^2}} \]

(20)
given that \( M_0 < M_c < X < M_{\text{max}} \). It is easy to see that the addition of \( \frac{\eta}{\eta \sqrt{1 - \eta^2}} \) will increase the probability that \( M_T \) is smaller than \( X \).

Furthermore, if \( M_0 < X < M_c < M_{\text{max}} \), we have

\[ P(M_T < X|\eta) = \frac{1}{2} \sqrt{1 - \frac{\eta^2}{X^2}}. \]

(21)

This makes sense as the critical mass scale is not within the boundary that we consider (\( M_T < X \)), so the probability decreases.

Finally, if \( M_0 < X < M_{\text{max}} < M_c \), then the results are similar to the original situation when \( f_{M_T}(y) = A\eta^\alpha \) except that some of the planets are in the \( \delta \) function part of the distribution, thus reducing the relative probability of sampling the power-law portion:

\[ P(M_T < X|\eta) = \frac{1}{2} \sqrt{1 - \frac{\eta^2}{X^2}}. \]

(22)

We can also obtain the distribution of \( \sin(i) \):

\[ P(\sin(i) < Z|\eta) = 1 - \frac{1}{2} \sqrt{1 - Z^2} \]

(23)

if \( 0 < \sin(i) < M_0/M_c < M_0/X \).

If we set \( Z = \frac{\eta}{X} \) and \( M_0 = \eta M_c \), while \( M_c = 1 \), then we can plot Figure 4. It illustrates the discontinuity in the probability at the delta function (i.e., when \( \eta = Z \)).

**4.3. A Solar System Like Mass Distribution**

Turning now to a more complex but also more physically plausible distribution, we analyze the case of exoplanet masses distributed in a way similar to that of solar system planets. This distribution can be modeled very roughly as two power laws separated by a gap in mass. One power law lies at a low-mass range (the terrestrial planets) while the other lies at a much higher mass range (the giant planets). We consider a toy model with two power-law mass distributions, one extending from \( 1M_c \) to \( 20M_c \), while the other power law is for \( 400M_c \) to \( 8000M_c \). There are no planets in the range between \( 20M_c \) and \( 400M_c \). We also assume that the two power laws have the same power index, and also the same coefficient (i.e., \( f_{M_T}(y) = A\eta^\alpha \) in range of \( 1M_c \) to \( 20M_c \) and \( f_{M_T}(y) = B\eta^\beta \) in the range of \( 400M_c \) to \( 8000M_c \) where \( A = B \) and \( \alpha = \beta \)). We plot the probability \( P(\sin(i) < Z|\eta) \) as \( \eta \) varies (the ratio of the observed mass \( M_0 \) to the critical mass \( M_c \)) for \( \alpha = -1 \) in Figure 5. Note that the probability \( P(\sin(i) < Z|\eta) \) can saturate very near either unity or zero over a substantial range of \( Z \) values depending on the value of \( \eta \).

**5. OBSERVATIONAL SELECTION EFFECTS**

In the preceding analysis we have consistently assumed that exoplanets discovered by the RV method uniformly (i.e., without bias) sample the distribution of \( M_T \) and \( \sin(i) \) values in nature. Obviously, this is unrealistic. In reality, both variables (and others) influence the probability that a given exoplanet system will be detected in an RV survey, and this selection bias in turn affects the likely values of both \( M_T \) and \( \sin(i) \).

Happily, this complication does not fundamentally alter our results because the basic effect discussed in this paper is a purely statistical one, independent of any observational biases. More specifically, one could conduct an exactly parallel analysis in which the true distributions of \( M_T \) and \( \sin(i) \), which appear in Equations (16)–(18), are replaced with the biased distributions which a particular RV survey samples, if its selection function can be determined reasonably accurately.

A very simple example would be a case in which the probability of an RV survey detecting an exoplanet of mass \( M_T \) is given by some selection function \( S(M_T) \), independent of \( \sin(i) \) and other properties of the system. In that case, it suffices to replace \( P(M_T) \) with \( S(M_T)P(M_T) \) everywhere it occurs in the equations and proceed as before.
The primary implication of the results presented here is that in general the value of sin(i) for a given exoplanet system will not be drawn from its prior distribution, corresponding to an isotropic distribution of i as is often assumed, at least implicitly.

The relevant, i.e., posterior, probability distribution of sin(i) depends sensitively on the distribution of true masses \( M_T \) and the observed mass \( M_0 = M_T \sin(i) \). Since the former is not well constrained, either empirically or theoretically, at present the true mass \( M_T \) of such a system cannot be trivially estimated from the value of \( M_0 \) as is also often assumed to be the case (see Butler et al. 2004; Mayor et al. 2005, 2009; Lovis et al. 2006; Wright et al. 2008).

This means, for example, that it is difficult to identify the least (or most) massive RV exoplanets discovered to date because selecting low values of \( M_T \sin(i) \) from an observed exoplanet sample is a way of picking out low sin(i) values as well as low \( M_T \) values. For some possible exoplanet mass distributions the observed objects with the lowest observed \( M_T \sin(i) \) will be dominated by systems with small sin(i) values rather than small masses.

It also implies that the distribution of true exoplanet masses is not necessarily similar in shape to the distribution of \( M_0 \), even for large samples, as is sometimes taken to be the case (see Mayor et al. 2005; Butler et al. 2006; Cumming et al. 2008). Such an approximation may be valid for some exoplanet populations or mass ranges but quite misleading for others.

The moral of the above analysis is that the sin(i) factor should be given its due in RV exoplanet studies. For example, we urge that RV observers reporting the value of \( M_T \sin(i) \), typically for a newly discovered planet, also report a confidence interval distribution of true masses for various exoplanet populations.

As a simple illustration, the 95% confidence intervals for \( M_T \) if \( M_0 = 1.0 \) are 1.0001–2.405, 1.0017–4.566, 1.005–27.02, 1.15–85.186, and 1.125–94.34 for the simple power-law \( M_T \) distributions considered in Section 4.1 with assumed power-law slopes of \( \alpha = -2, -1, 0, +1, \) and +2, respectively. Note that the \( \alpha = -1 \) case is equivalent to that of an isotropic i distribution; in other words, if the exoplanet true mass distribution is uniform in logarithmic intervals, the posterior sin(i) distribution is the same as the prior one. Moreover, if \( \alpha \) is less than \(-1 \), \( M_T \) values will be even closer to \( M_0 \) values than in the isotropic i distribution case. This means that ignoring the issue raised in this paper can lead one to over, as well as under, estimate mass uncertainties. To illustrate this dependence better, we plot the probability distribution function of the posterior probability distribution of sin(i) assuming various values of slope \( \alpha \) in Figure 6.

Although these model-dependent upper bounds may appear less impressive or exciting than the \( M_T \sin(i) \) value itself, they are a less misleading, and thus more scientifically informative indication of the actual information on any particular exoplanet’s mass provided by RV data alone.

Although this paper concerns itself only with the estimation of true masses of individual exoplanets on the basis of RV data, it is clearly relevant to the more challenging and, in some respects, more fundamental problem of estimating the distribution of true masses for various exoplanet populations. The classic work of Chandrasekhar & Münch (1950) long ago demonstrated that problems of this form are non-trivial, in particular that straightforward inversions of the data are unstable. Modern studies (Jorissen et al. 2001; Zucker & Mazeh 2001; Marcy et al. 2005; Udry & Santos 2007; Brown 2011) have addressed the issue with a variety of mathematical approaches and assumptions, some relatively sophisticated.
but it is likely that considerably more work will be required before the exoplanet mass distribution(s) can be regarded as well determined. Until that goal is accomplished, RV mass determinations of individual exoplanets will be subject to the systematic $\sin(i)$ uncertainty described in this paper.

We thank Dan Fabrycky, Scott Gaudi, John Johnson, Geoff Marcy, Tim Morton, David Spergel, Dave Spiegel, and Jason Wright for useful comments and suggestions. S.H. acknowledges support from the Lawrence Berkeley National Laboratory Seaborg Fellowship and Chamberlain Fellowship and support from the Princeton University Department of Astrophysics as S.H. started this project when she was a graduate student at Princeton University. E.L.T. gratefully acknowledges support from a Princeton University Global Collaborative Research Fund grant and the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. E.L.T. also acknowledges the support of the Research Center for the Early Universe (RESCEU) at the University of Tokyo and the hospitality of its Department of Physics.

REFERENCES

Baraffe, I., Chabrier, G., & Barman, T. 2010, Rep. Prog. Phys., 73, 016901
Black, D. C. 1997, ApJ, 490, L171
Brown, R. A. 2011, ApJ, 733, 68
Butler, R. P., Vogt, S. S., Marcy, G. W., Fischer, D. A., Wright, J. T., Henry, G. W., Laughlin, G., & Lissauer, J. J. 2004, ApJ, 617, 580
Butler, R. P., et al. 2006, ApJ, 646, 505
Chandrasekhar, S., & Münch, G. 1950, ApJ, 111, 142
Charbonneau, D., Brown, T. M., Burrows, A., & Laughlin, G. 2007, Protostars and Planets V, ed. B. Reipurth, D. Jewitt, & K. Keil (Tucson, AZ: Univ. Arizona Press), 701
Cumming, A., Butler, R. P., Marcy, G. W., Vogt, S. S., Wright, J. T., & Fischer, D. A. 2008, PASP, 120, 531
Eddington, A. S. 1913, MNRAS, 73, 359
Gray, D. F. 1997, Nature, 385, 795
Hogg, D. W., & Turner, E. L. 1998, PASP, 110, 727
Ida, S., & Lin, D. N. C. 2008a, ApJ, 673, 487
Ida, S., & Lin, D. N. C. 2008b, ApJ, 685, 584
Johnson, J. A. 2009, PASP, 121, 309
Jorissen, A., Mayor, M., & Udry, S. 2001, A&A, 379, 992
Kokubo, E., & Ida, S. 1996, Icarus, 123, 180
Kokubo, E., Kominami, J., & Ida, S. 2006, ApJ, 642, 1131
Lovis, C., et al. 2006, Nature, 441, 305
Malmquist, K. G. 1920, Medd. Ludd Astron. Obs. Ser. 2, No. 22 (Lund: Lund Univ. Press)
Marcy, G., Butler, R. P., Fischer, D., Vogt, S., Wright, J. T., Tinney, C. G., & Jones, H. R. A. 2005, Prog. Theor. Phys. Suppl., 158, 24
Marcy, G. W., Butler, R. P., Fischer, D. A., & Vogt, S. S. 2003, in ASP Conf. Ser. 294, Scientific Frontiers in Research on Extrasolar Planets, ed. D. Deming & S. Seager (San Francisco, CA: ASP), 1
Matsumura, S., Pudritz, R. E., & Thommes, E. W. 2009, ApJ, 691, 1764
Mayor, M., Pont, F., & Vidal-Madjar, A. 2005, Prog. Theor. Phys. Suppl., 158, 43
Mayor, M., et al. 2009, A&A, 507, 487
Udry, S., & Santos, N. C. 2007, ARA&A, 45, 397
Winn, J. N. 2010, arXiv:1001.2010
Wright, J. T., Marcy, G. W., Butler, R. P., Vogt, S. S., Henry, G. W., Isaacson, H., & Howard, A. W. 2008, ApJ, 683, L63
Zucker, S., & Mazeh, T. 2001, ApJ, 562, 1038