Spin on a 4D Feynman Checkerboard

Brendan Z. Foster∗ and Ted Jacobson†

Abstract

We discretize the Weyl equation for a massless, spin-1/2 particle on a time-diagonal, hypercubic spacetime lattice with null faces. The amplitude for a step of right-handed chirality is proportional to the spin projection operator in the step direction, while for left-handed it is the orthogonal projector. Iteration yields a path integral for the retarded propagator, with matrix path amplitude proportional to the product of projection operators. This assigns the amplitude $i^{\pm T} 3^{-B/2} 2^{-N}$ to a path with $N$ steps, $B$ bends, and $T$ right-handed minus left-handed bends, where the sign corresponds to the chirality. Fermion doubling does not occur in this discrete scheme. A Dirac mass $m$ introduces the amplitude $i\epsilon m$ to flip chirality in any given time step $\epsilon$, and a Majorana mass similarly introduces a charge conjugation amplitude.

1 Dedication

To break the spacetime code was the aim of much of David Finkelstein’s work [1, 2]. David’s approach was influenced, inter alia, by von Neumann’s quantum logic, Feynman’s operator calculus [3], and his own deep insight into quantum mechanics as a physics of processes, not things. He envisioned spacetime as an autogenerated algebra, a quantization of the natural numbers, possessing enough structure and depth to serve as the cosmological process and to generate the local symmetries of particle physics [4, 5]. He proposed that the continuum is a coherent quantum phenomenon [4]:

Vacuum I, the present ambient space-time, with its Minkowski chronometry, is a critical phenomenon, a Bose condensation of chronon pairs into a hypercubical lattice of basic states.

Though his vision did not guide him all the way to a complete new physics, he may well have been on the true path. He was nearly alone in his quest, but we imagine that others will eventually catch up, catch on, and carry it onward. The present paper meets David on the time-diagonal hypercubical lattice. But while for him it represented a coherent quantum process, for us it is but a classical scaffold on which the propagation of spin-1/2 particles plays out. We dedicate this paper to David’s memory, with gratitude for his insight and inspiration.

∗bzfoster@bzfoster.com
†jacobson@umd.edu, Maryland Center for Fundamental Physics, University of Maryland, College Park, MD 20742-4111
2 Introduction

“And, so I dreamed that if I were clever, I would find a formula for the amplitude of a path that was beautiful and simple for three dimensions of space and one of time, which would be equivalent to the Dirac equation, and for which the four components, matrices, and all those other mathematical funny things would come out as a simple consequence - I have never succeeded in that either.”

This quote is taken from Feynman’s Nobel lecture [6], where it appears just after he describes his sum over checkerboard paths for the Dirac propagator in 1+1 dimensions (see also [7]). A path is composed of steps on a square lattice of null links, and its amplitude is \( (i\epsilon m)^R \), where \( m \) is the particle mass, \( R \) is the number of direction reversals, and \( \epsilon \) is the time duration of a lattice step (and we use units with \( \hbar = c = 1 \)). If the mass vanishes the particle moves only to the left or to the right, at the speed of light, and these two motions correspond to the two chiralities for a Dirac spinor in 1+1 dimensions.

This checkerboard path integral intriguingly accounts for relativistic propagation and Dirac matrices with nothing more than a simple factor of \( i \) associated with a geometric property—a bend—in a piecewise lightlike path. Here we show that one can actually come very close to preserving these striking features in a path integral for Dirac particles in 3 + 1 dimensions.

Most work on lattice formulations of spinor propagation has been directed at lattice field theory calculations, and thus involves Grassmann variables and “path” integrals over field configurations in a spacetime of Euclidean signature [8]. Here we pursue instead a bosonic (i.e. non-Grassmanian) formulation in terms of a sum over particle paths on a Minkowski signature lattice. Such formulations have been studied previously in [9, 10, 11] and references therein. What is new here is the interesting structure of the lattice employed, and the fact that it allows for a particularly simple rule for the amplitudes.

The massless case in 3+1 dimensions is far more interesting than in 1+1 dimensions. Chiral, right-handed two-component spinors satisfy the Weyl equation,

\[
\sigma^\mu \partial_\mu \Psi = 0,
\]

where \( \sigma^\mu = (1, \vec{\sigma}) \) with \( \vec{\sigma} \) the Pauli matrices, and left handed spinors satisfy the analogous equation with \( \bar{\sigma}^\mu = (1, -\vec{\sigma}) \). As in 1+1 dimensions, the mass introduces an amplitude \( iem \) to flip chirality at each step. The key point is that the velocity operator is \( \pm \) the spin operator for a right- or left-handed Weyl particle. As we shall see, this implies that displacement amplitudes are determined by spin transition amplitudes.

3 Lattice

We discretize spacetime with a hypercubical lattice, oriented so one diagonal of the hypercube lies in the time direction, and with the step speed chosen three times the speed of light, so that the discrete causal cone just encloses the continuum one (see Fig. 1). The edges from one vertex lie in the directions of the four spacetime vectors

\[
n_i^\mu = (1, \alpha \hat{n}_i).
\]
Figure 1: Unit cell of a time-diagonal cubic spacetime lattice in $2 + 1$ dimensions. The cube faces are null planes, so that the continuum lightcone nestles tangent to the cube, and the edges are spacelike with speed 2 relative to the diagonal.

The four spatial unit vectors $\hat{n}_i$ point to the vertices of a tetrahedron, and $\alpha$ is the step speed. The $\hat{n}_i$ sum to zero, and the inner product or angle between any distinct two is the same. Hence for $i \neq j$ we have $\hat{n}_i \cdot \hat{n}_j = -1/3$, and the angle is equal to $\cos^{-1}(-1/3) \approx 109^\circ$. An explicit expression for the components of the $\hat{n}_i$ in a particular basis is

$$
\begin{align*}
\hat{n}_1 &= \frac{1}{3}(2\sqrt{2}, 0, -1) \\
\hat{n}_2 &= \frac{1}{3}(-\sqrt{2}, \sqrt{6}, -1) \\
\hat{n}_3 &= \frac{1}{3}(-\sqrt{2}, -\sqrt{6}, -1) \\
\hat{n}_4 &= (0, 0, 1).
\end{align*}
$$

(3)

It might seem natural to choose the step speed $\alpha = 1$, so that the links $n^{\mu}_i$ of the lattice would be null, as envisaged in [5], and as implemented in a lattice formulation of general relativity in [12]. However, in this case our retarded lattice propagator would fail to converge at all in the continuum limit. The reason is that such a spacetime lattice violates the well-known “Courant condition” for stability: the discrete region of causal influence must contain the continuum one.

To marginally satisfy the Courant condition, the tetrahedral cone, formed by the four hyperplanes spanned by three of the $n^{\mu}_i$’s, must barely enclose the spherical, continuum light cone. That is, the faces of the hypercube must be null. Equivalently, the dual lattice must be generated by null covectors.\footnote{See [13] for a general investigation of spacetime polytopes with null faces.} Each of these hyperplanes must therefore contain one and only one null direction. By symmetry this null direction $k^{\mu}$ must coincide with the sum of the three link vectors, e.g. $k^{\mu} = (3, \alpha(\hat{n}_1 + \hat{n}_2 + \hat{n}_3))$. The Minkowski norm of this vector is $9 - \alpha^2$, hence we must choose $\alpha = 3$ if it is to be null. Moreover, $k^\mu n^{\mu}_1 = 3 - (\alpha^2/3)$, so if $\alpha = 3$ the null vector $k^\mu$ is orthogonal to all vectors in the hyperplane, confirming that the hyperplane is indeed null.

The spatial lattice at one time is a face-centered cubic (fcc) lattice. A way to see this is to begin with the tetrahedron of points that lies at one time...
Figure 2: Face-centered cubic lattice of points at one time step. The tetrahedron (dotted lines) is comprised of the four points reached from the center of the small cube in one time step (dashed lines). The continuum sphere of light is enclosed by and tangent to the tetrahedron. The distance from the center to a tetrahedron vertex is three times the radius of the sphere. The step length $a$ and cube edge length $L$ are shown.

step to the future of a given spacetime point $p$. The four dimensional lattice has translation symmetries that map any point to any other point, and the spatial lattice at one time must share this property. Hence it can be grown from this tetrahedral seed by translation along the edges of the tetrahedron, which produces the fcc lattice shown in Fig. 2.

Evolving the spatial lattice one time step to the future amounts to shifting it along the displacement from the center of one tetrahedron to one of its vertices, yielding a distinct but equivalent spatial lattice. After four such steps the original spatial lattice is recovered. One can visualize these steps all in the same direction, each one extending for half a diagonal of one of the constituent cubes in Fig. 2. Alternatively, one can think of them as successively along four distinct edges of the hypercube, which makes it obvious that four steps brings one back to the original lattice.

It will be important later to know the spatial volume per point at one time step, expressed in terms of the step length. It is clear from Fig. 2 that each point can be associated uniquely with a pair of constituent cubes of edge length $L$. The volume of these cubes is $2L^3$. The step length $a$ is half the diagonal of one of these cubes, hence $a = \sqrt{3}L/2$, so the volume per point $V_p$ is given by

$$V_p = \frac{16}{3\sqrt{3}}a^3. \quad (4)$$

We will take the time step size to be $\epsilon$, so the spatial step size is $a = \alpha\epsilon$.

4 Discrete Weyl equation

Now consider the tetrahedral quartet of unit vectors $\hat{n}_i$ defined below (2). The sums $\sum_i \hat{n}_i^a$ and $\sum_i \hat{n}_i^a \hat{n}_i^b$ (with ''$i^a$'' denoting the 'a' component of $\hat{n}_i$) are invariant under the symmetries of the tetrahedron, hence the first sum must vanish and the second sum must be proportional to the Euclidean metric $\delta^{ab}$. Since the trace is equal to four this yields the relation

$$\sum_i \hat{n}_i^a \hat{n}_i^b = \frac{4}{3} \delta^{ab}. \quad (5)$$
Using these identities and the definition (2) of the 4-vectors $n_i^\mu$, the matrix 4-vector $\sigma^\mu = (1, \vec{\sigma})$ can be expressed as

$$\sigma^\mu = \frac{1}{2} \sum_i \frac{1}{2} \left( 1 + \frac{3}{\alpha} \hat{n}_i \cdot \vec{\sigma} \right) n_i^\mu. \quad (6)$$

In the special case $\alpha = 3$, for which the polyhedral light cone consists of null hyperplanes, this becomes just

$$\sigma^\mu = \frac{1}{2} \sum_i P_i n_i^\mu, \quad (7)$$

where

$$P_i = \frac{1}{2} (1 + \hat{n}_i \cdot \vec{\sigma}) \quad (8)$$

is the projector for spin up in the direction $\hat{n}_i$. Note in particular that $\frac{1}{2} \sum_i P_i = I$. (A version of this construction involving an integral over the sphere of directions was developed in [14].)

Using the identity (7), the Weyl equation (1) for right-handed spinors takes the form

$$\frac{1}{2} \sum_i P_i n_i^\mu \partial_\mu \Psi = 0, \quad (9)$$

If we approximate the directional derivatives by finite differences,

$$\epsilon n_i^\mu \partial_\mu \Psi(x) \cong \Psi(x) - \Psi(x - \epsilon n_i), \quad (10)$$

(9) yields a formula for $\Psi(x)$ on the lattice in terms of the values $\Psi(x - \epsilon n_i)$ at the immediately preceding points,

$$\Psi(x) = \frac{1}{2} \sum_i P_i \Psi(x - \epsilon n_i). \quad (11)$$

This is our one-step evolution rule. It states that a prior amplitude contributes to $\Psi(x)$ precisely to the extent that its spin is parallel to the spatial step arriving at $x$.

### 4.1 Dispersion relation

We can find the lattice dispersion relation by considering a wave function with plane wave form,

$$\Psi(x, 0) = e^{-ik_\mu x^\mu} \Psi_0, \quad (12)$$

where $\Psi_0$ is a constant spinor. This satisfies (11) if and only if

$$\Psi_0 = \frac{1}{2} \sum_j P_j e^{i k_\mu n_j^\mu} \Psi_0. \quad (13)$$

For small $\epsilon k$, we may expand to first order, and using equation (7) we find the standard Weyl equation for momentum eigenstates, $\sigma^\mu k_\mu \Psi_0 = 0$. These solutions obey the relativistic dispersion relation, $k_\mu k^\mu = 0$, at $O(\epsilon)$. For generic $\epsilon k$, (13) imposes the condition that the matrix multiplying $\Psi_0$ must have a unit eigenvalue, $\Psi_0$ being the corresponding eigenvector. This condition is the exact dispersion relation.
To extract its physical meaning, we assume that the spatial components $k_a$ of $k_\mu$ are real, and solve for the frequency $\omega = k_0$, which in general will be complex. First we recast (13) in the form

$$A(\theta)\Psi_0 = e^{-i\omega \epsilon}\Psi_0,$$

with

$$A(\theta) = \frac{1}{2} \sum_j P_j e^{i\theta_j},$$

and $\theta_j := \epsilon k_a n_{a,j}$. Since $\sum_j n_{a,j}^2 = 0$, we also have $\sum_j \theta_j = 0$. So the problem is to find the eigenvalues of $A(\theta)$, when the four $\theta_j$’s sum to zero. Let us begin by finding when the frequency $\omega$ is real, i.e. when the eigenvalue has unit modulus.

It is shown in the Appendix that the eigenvalues of $A(\theta)$ have less than unit modulus except when at least three of the $\theta_j$ coincide. Hence it suffices to consider the case when $\theta_1 = \theta_2 = \theta_3$, and $\theta_4 = -3\theta_1$. Then $A(\theta)$ takes the form

$$A(\theta) = \cos 2\theta_1 e^{-i\theta_1} P_4 + e^{i\theta_1} \tilde{P}_4,$$

where $\tilde{P}_4 = 1 - P_4$ is the projector orthogonal to $P_4$. The only nontrivial eigenvector solution with real frequency thus has spin in the direction $-\hat{n}_4$, and frequency $\omega = -\theta_1/\epsilon$. The wavevector generally satisfies $k_\mu n_{a,j}^\mu = \omega + k_a n_{a,j} = (-\theta_1 + \theta_j)/\epsilon$. For this solution we thus have $k_\mu n_{1,2,3}^\mu = 0$, which implies that $k_\mu$ is precisely the null normal to the 123 (null) hyperplane. The only solutions to the lattice dispersion relation with real frequency are thus exactly the same as the continuum ones, but restricted to the four null directions that generate the reciprocal lattice.

4.1.1 Fermion doubling?

Most lattice discretizations of the Dirac or Weyl equations exhibit fermion doubling, which means that they have zero frequency solutions with large spatial wavevectors. As just seen, that does not occur for equation (11) if we restrict to real frequency. However, decaying modes with zero real part of $\omega$ and small negative imaginary part would contribute to a quantity — and thus effectively behave as doublers — if the decay $\exp(-|\omega| \Delta t)$ is negligible over the time interval $\Delta t$ relevant to that quantity. To check for these we need to find the complex frequency roots of the dispersion relation.

It seems complicated to develop much analytic understanding of the spectrum, but a numerical computation easily yields the portrait in Fig. 3, which depicts the eigenvalues as points on and inside the unit circle in the complex plane. The spectrum has fourfold rotation symmetry since the replacement $\theta_j \rightarrow \theta_j + \pi/2$, which preserves the condition $\sum_j \theta_j = 0$ modulo $2\pi$, just multiplies the eigenvalue by $e^{i\pi/2}$. The roots $e^{-i\omega \epsilon}$ with pure negative imaginary frequency $\omega$ lie on the positive real axis in the figure. We learn from the figure that there is a gap in the spectrum of these roots. Numerical experiment reveals that the maximum value on the real axis is $1/\sqrt{3}$, which corresponds to the case $\theta_1 = \theta_2 = 0, \theta_3 = \theta_4 = \pi$. A solution with this eigenvalue would be damped by a factor $1/\sqrt{3}$ for each evolution time step. Moreover, the figure on the right shows that if we remove a neighborhood around the zero wavevector, then we

\[\text{The eigenvalues of } A(\theta) \text{ have modulus less than or equal to unity if and only if the step speed } \alpha \text{ is } \geq 3. \text{ Our choice } \alpha = 3 \text{ is thus the marginal value for a convergent finite difference scheme.}\]
remove a neighborhood around zero frequency, so none of the solutions with nearly zero frequency correspond to large wavevectors. We therefore expect that the lattice equation (11) would not exhibit fermion doubling phenomena in any low energy observables. This expectation is further corroborated in Section 6, where we demonstrate that the continuum limit of the lattice propagator is precisely the continuum propagator.

This raises the question of how the Nielsen-Ninomiya theorem [8] is evaded. The theorem shows that, under certain general conditions, fermion doubling is unavoidable. One of the assumptions is that the equation of motion is derived from a local lattice action, but on account of the time-asymmetric finite difference, we did not succeed in deriving Eq. (11) from any local action without introducing further fields. At present, the lack of a local action is our proposed explanation for how the fermion doubling is evaded.

5 Path integral

Iterating Eq. (11) backwards in time, we see that the retarded propagator between two points can be written as a sum over paths involving steps in the directions \( \hat{n}_{i_1} \ldots \hat{n}_{i_N} \) at speed 3c, with the amplitude for such a path given by the product of projection operators,

\[
2^{-N} P_{i_N} \ldots P_{i_1}.
\] (17)

The propagator is then the sum of these operators over all paths that connect the points. Note that for fixed initial and final points the number of each of the four step types is fixed, so one sums only over the order in which the steps are taken.

In terms of the unit eigenspinor \(|i\rangle\) of \(P_i = |i\rangle\langle i|\) the amplitude (17) becomes

\[
2^{-N} |i_N\rangle\langle i_N| i_{N-1}\rangle\langle i_{N-1}| \cdots |i_2\rangle\langle i_2| i_1\rangle\langle i_1| i\rangle.
\] (18)

To obtain a scalar amplitude we may consider the matrix element of the propagator between fixed ingoing and outgoing spinors \(|i\rangle\) and \(|o\rangle\) selected from the set (20). These specify directions of arrival at the initial point and departure from the final point. Then, apart from the factor \(2^{-N}\), the amplitude becomes nothing but a product of spin transition amplitudes,

\[
2^{-N} \langle o| i_N\rangle \langle i_N| i_{N-1}\rangle \cdots \langle i_2| i_1\rangle \langle i_1| i\rangle.
\] (19)
This is a lattice version of the continuum spinor chain path integral of [14], which involved integration over the sphere of spin states at each step, rather than a sum over the four states. (A cubic spatial lattice version was developed in [10].)

The amplitude matrix (18) is independent of the choice of phases for the spinors, so we are free to adjust those phases to produce a nice result if possible. Let us see what can be done. The unit spinor corresponding to a unit vector with spherical angles $(\theta, \phi)$ is

$$\left( \cos \left( \frac{\theta}{2} \right), \sin \left( \frac{\theta}{2} \right) \exp(\imath \phi) \right),$$

and an arbitrary overall phase. If we align the $z$ axis with one of the tetrahedral directions, and the $x$ axis with another, then the spinors $|i\rangle$ take the form

$$|4\rangle = e^{\imath \delta_4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = e^{\imath \delta_1} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

$$|2\rangle = e^{\imath \delta_2} \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{\imath \frac{2\pi}{3}} \end{pmatrix}, \quad |3\rangle = e^{\imath \delta_3} \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} 1 \\ e^{\imath \frac{4\pi}{3}} \end{pmatrix},$$

(20)
corresponding to the four unit vectors (3). We have identified a rather nice choice for the phases, although it requires that the tetrahedral symmetry be broken by singling out one preferred axis. This axis may be chosen to be the one determined by the ingoing spinor, $\vec{n}_i = \langle i | \vec{\sigma} | i \rangle$, and identified with the $z$ axis in the above parameterization, so that the preferred spinor is labeled $|4\rangle$.

If we choose $\delta_4 = \pi/2$ and $\delta_1 = \delta_2 = \delta_3 = 0$, then the inner products between the four spinors become identical up to a sign that depends only on the order:

$$\langle 1, 2, 3 | 4 \rangle = \langle 3 | 1 \rangle = \langle 2 | 3 \rangle = \langle 1 | 2 \rangle = \frac{i}{\sqrt{3}},$$

(21)

and the complex conjugates correspond to the opposite order. The $+i$ goes with bends away from the $z$ axis, or with bends between 1, 2, and 3 that are right-handed with respect to the $-z$ axis. Alternatively, we may set also $\delta_4$ to zero, so that bends away from the $z$ axis have unit phase. For this second rule, the amplitude for an $N$ step path with $B$ bends and a net number $T$ of right-handed bends minus left-handed bends is

$$i^T 3^{-B/2} 2^{-N}.$$  

(22)

We wonder whether Feynman would have considered this beautiful and simple.

To treat left-handed Weyl spinors we need only replace the spin projections $P_i$ by the orthogonal projections $\bar{P}_i = \frac{1}{2} (1 - \hat{n}_i \cdot \sigma)$, which differ just by the minus sign in place of the plus sign. Since the eigenspinors of $\bar{P}_i$ are the charge conjugates $i \sigma_y | i \rangle^* \rangle$ of the four spinors (20), we conclude that, for left-handed spinors, $T$ in the amplitude (22) is replaced by $-T$. Note that we cannot interchange the right- and left-handed rules by a change of phase choice. This would require choosing $e^{\imath (\delta_1 - \delta_2)} = e^{\imath (\delta_2 - \delta_3)} = e^{\imath (\delta_3 - \delta_1)} = -1$. Multiplication of these three phase factors together reveals that this is impossible.

5.1 2+1 dimensions

The entire discussion so far can easily be adjusted to apply in the case of 2+1 spacetime dimensions. The Weyl equation can be adapted to 2+1 dimensions simply by omitting the $z$ direction. The resulting equation is also the full massless Dirac equation. Instead of the four unit step vectors we now have
three, which can be taken to lie in the $xy$ plane separated by $2\pi/3$ radians, the step speed becomes $2c$, and the identity (7) becomes

$$\sigma^\mu = \frac{2}{3} \sum_i P_i n_i^\mu. \quad (23)$$

The spinors whose polarization vector lies in the $xy$ plane have components $(1,e^{i\phi})/\sqrt{2}$, up to an independent overall phase for each one. Hence for the spinors corresponding to the three steps, in the directions $\phi = 0, 2\pi/3, -2\pi/3$, we can adopt $(1, q)/\sqrt{2}$, where $q = 1, e^{i2\pi/3}, e^{-i2\pi/3}$ are the cube roots of unity. The inner product of two of these is $e^{\pm i\pi/3}/2$, where the sign corresponds to clockwise or counterclockwise turns. For example, $\langle (1,1) | (1, e^{i2\pi/3}) \rangle = 1 + e^{i2\pi/3} = e^{i\pi/3}$.

Since the 2+1 dimensional Weyl or massless Dirac equation does not break parity symmetry, it should also be possible to choose the phases of the three spinors so that the amplitude rule is the conjugate one. Indeed, this results from the spinors $(1, 1), (e^{-i2\pi/3}, 1), (e^{i2\pi/3}, 1)$, all divided by $\sqrt{2}$. In addition, there is one more symmetric choice, which yields inner product $-1/2$ for both left and right turns! This corresponds to the choice of spinors $(1, 1), (e^{i2\pi/3}, e^{-i2\pi/3}), (e^{-i2\pi/3}, e^{i2\pi/3})$. All three of these rules have the same amplitude, proportional to $(-1)^n$, for a path that winds completely around in direction $n$ times. This last rule assigns to any path of $N$ steps the simple amplitude

$$(-2)^{-B} 2^{-N}, \quad (24)$$

where $B$ is the total number of bends.

### 6 Evaluation of the path integral

We now evaluate the sum over paths for the lattice propagator $K_\epsilon(\Delta x)$ for a spacetime displacement $\Delta x$ and demonstrate that it reproduces the continuum retarded propagator in the limit $\epsilon \to 0$.

The lattice displacement $\Delta x$ can be expanded in terms of the four basis vectors:

$$\Delta x^\mu = \sum_j \Delta x^j n_j^\mu, \quad (25)$$

hence the displacement is determined by a unique set of four integers $N^j = \Delta x^j/\epsilon$. The constraint that a path makes a given displacement can be incorporated as four Kronecker deltas, which we express in a Fourier representation:

$$\prod_j \delta(N^j, \Delta x^j/\epsilon) = \int_{-\pi}^{\pi} \frac{d^4\theta_j}{(2\pi)^4} e^{i\theta_j(N^j-\Delta x^j/\epsilon)}. \quad (26)$$

Here, and in the following, summation over a repeated $j$ index is implicit. The lattice propagator is given by

$$K_\epsilon(\Delta x) = \sum_{N=0}^{\infty} \int_{-\pi}^{\pi} \frac{d^4\theta_j}{(2\pi)^4} e^{-i\theta_j(\Delta x^j/\epsilon)} [A(\theta)]^N, \quad (27)$$

where

$$A(\theta) = \frac{1}{2} \sum_j P_j e^{i\theta_j}. \quad (28)$$
The sum over \( N \) of \( [A(\theta)]^N \) produces every possible sequence of projection operators, each with the appropriate exponential factor encoding the number of steps in each direction. When the integrals over \( \theta_j \) are carried out, only those step sequences that produce the displacement \( \Delta x^\mu \) will survive. In particular, only the value of \( N \) equal to the total number of steps contributes.

As the step size \( \epsilon \) goes to zero, the number of steps \( N \) for a fixed time interval goes to infinity as \( \Delta t/\epsilon \). Convergence thus requires that the norm \( \|A(\theta)\| \) (i.e. the maximum norm of \( A(\theta) \) acting on a unit spinor) be less than or equal to unity. It is shown in the appendix that \( \|A(\theta)\| < 1 \) except when at least three of the \( \theta_j \) coincide. As \( N \) becomes larger, \( [A(\theta)]^N \) therefore converges to zero pointwise except at this set of degenerate \( \theta_j \) values, which has measure zero. It follows that only an \( O(1/N) \) neighborhood of \( \theta_j = 0 \) contributes in the \( \epsilon \rightarrow 0 \) limit.

In terms of the new variables \( k_j := \theta_j/\epsilon \), (27) takes the form

\[
K_\epsilon(\Delta x) = \epsilon^4 \sum_{N=0}^\infty \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{d^4 k_j}{2\pi} e^{-ik_j \Delta x^j} [A(\epsilon k)]^N. \tag{29}
\]

In the limit \( \epsilon \rightarrow 0 \), only an \( O(1/\Delta t) \) neighborhood of \( k_j = 0 \) contributes, so we may use the exponential approximation,

\[
[A(\epsilon k)]^N \approx e^{iNk_j P_j/2}, \tag{30}
\]

dropping the \( O(N\epsilon^2 k^2) \) correction. Moreover the limits of integration in (29) approach \( \pm \infty \), hence (29) yields

\[
K_{\epsilon \rightarrow 0}(\Delta x) = \epsilon^4 \sum_{N=0}^\infty \int_{-\infty}^{\infty} \frac{d^4 k_j}{2\pi} e^{-ik_j \Delta x^j} e^{iNk_j P_j/2} \tag{31}
\]

To render this in more familiar terms, we next change the variable of integration from the \( k_j \) to the generic wave co-vector components \( k_\mu \). These are related via

\[
k_j = k_\mu \eta_j^\mu, \tag{32}
\]

so that \( k_j \Delta x^j = k_\mu \Delta x^\mu \). Substituting (32) for \( k_j \), and using (7), the sum in the last exponent of (31) becomes \( k_\mu \sigma^\mu \). The Jacobian \( |\partial k_j/\partial k_\mu| = |\eta_j^\mu| \) can be computed using an explicit form of the tetrad of unit 3-vectors, yielding

\[
d^4 k_j = 4\sqrt{3} d^4 k_\mu. \tag{33}
\]

The final step in taking the limit is to replace the discrete variable \( N \) by a continuous one \( s = N\epsilon \), in terms of which the sum \( \sum_N \) becomes \( \int ds/\epsilon \). With this replacement, and the change of variables from \( k_j \) to \( k_\mu \), (31) becomes

\[
K_{\epsilon \rightarrow 0}(\Delta x) = 48\sqrt{3} c^3 \int_0^\infty ds \int_{-\infty}^{\infty} \frac{d^4 k}{(2\pi)^4} e^{-ik_\mu \Delta x^\mu} e^{iks_0 \sigma^\mu} \tag{34}
\]

Except for the peculiar factor in front, this is just the continuum retarded propagator. We can either first do the \( s \) integral, obtaining the spacetime Fourier transform of \( 1/\sigma^\mu k_\mu \), or first do the integral over \( k_0 \), which produces a Dirac delta function \( \delta(s - \Delta x^0) \). Then the \( s \)-integral sets \( s \) equal to \( \Delta x^0 \) (assuming \( \Delta x^0 > 0 \)), yielding the retarded propagator as a spatial Fourier transform. Had we kept the subleading terms of order \( N\epsilon^2 \) (\( = se \)) in (30) the
convergence factor for the integration limit $s \to \infty$ would presumably have been supplied much as in [14] or [15].

It remains to account for the pre-factor $48\sqrt{3} \epsilon^3$. We computed the propagator to go between two points on the lattice. In the continuum, the amplitude to arrive at one point starting from another point is zero, since only by integrating over a finite region should a nonzero amplitude arise. The prefactor is none other than the volume per point in the lattice (4), with the step length $a$ equal to $3\epsilon$. Hence what we have actually obtained is the continuum propagator integrated over the volume associated with one lattice point.

7 Mass

So far we have discussed the theory of chiral fermions, i.e. fields satisfying the Weyl equation. The effect of a Dirac mass $m$ can be included by allowing for chirality flips between right- and left-handed spinor propagation at each time step, with an associated amplitude factor $iem$ [14], as on Feynman’s checkerboard. This seems to introduce new, mixed chirality spinor transition overlaps, however those can be avoided by enlarging the lattice. We discuss first a Dirac mass, and then a Majorana mass, in the following two subsections.

7.1 Dirac mass

A spin-1/2 particle with a Dirac mass is composed of two 2-component spinor fields, $L$ and $R$, which for a given spin state propagate in opposite spatial directions. Equivalently, for a given propagation direction they have opposite spins. The mass term in the continuum Dirac equation couples these two fields,

\[ \sigma^\mu \partial_\mu R = imL, \quad \bar{\sigma}^\mu \partial_\mu L = imR. \] (35)

Here $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ is the set of matrices that goes with the left handed spinor. Together with the identity $\sigma^{\mu} \bar{\sigma}^\nu = \eta^{\mu \nu}$, these equations imply that each component of both spinors satisfies the Klein-Gordon equation, $(\Box + m^2)(R, L) = 0$.

Discretization of the Dirac equation (35) on our lattice yields coupled one step evolution equations,

\[ R(x) = \frac{1}{2} \sum_i P_i (R + iem L)(x - \epsilon n_i) \] (36)

\[ L(x) = \frac{1}{2} \sum_i \bar{P}_i (L + iem R)(x - \epsilon n_i). \] (37)

If we expand these equations in powers of $\epsilon$, the $O(\epsilon^0)$ term holds identically, and the $O(\epsilon)$ terms reduce to (35). To $O(\epsilon)$, it doesn’t matter if we evaluate $R$ and $L$ in the mass term at $x$ or at $x - \epsilon n_i$, or anywhere else that differs from $x$ at $O(\epsilon)$.

Iterating these equations yields a sum over paths, with an amplitude $iem$ to swap chiralities at each step. Notice that the amplitude to continue in the same direction upon a chirality change vanishes since, for each $i$, $\bar{P}_i P_i = 0$. One can sum over the paths in spacetime, and for each path sum over the binary choices of whether the step is an $R$ step or an $L$ step. To illustrate this consider for example a three-step path, ending with $R$. The corresponding contribution to
the propagator takes the form

$$PPP R + i\epsilon m [PPP + PP\bar{P} + \bar{P}\bar{P}] L$$

$$- (\epsilon m)^2 [PP\bar{P} + P\bar{P} P + \bar{P} P\bar{P}] R - i(\epsilon m)^3 P P\bar{P} L. \quad (38)$$

Inside a string of $\bar{P}$'s the bend amplitudes are just the conjugates of what they are inside a string of $P$'s, i.e. the right- and left-handed turn rules are reversed. However, at a transition from $P$ to $\bar{P}$ or vice versa, another rule is needed, because one of the four right-handed spinors meets one of the conjugate spinors. We do not consider this beautiful and simple.

We have found a way to improve on the elegance, but at the cost of a loss of economy in the lattice structure and the need to restrict chiral spinors to certain paths on the lattice. It works as follows. Adopt now a spatial, body-centered cubic (bcc) lattice, with the distance from a cube center to a cube corner $3\epsilon$, and time steps $\epsilon$. The eight steps from each lattice point to the corners of the surrounding cube can be grouped into two tetrahedral sets of four. Call them the $R$ steps and the $L$ steps. For every $R$ step from a point there is an opposite $L$ step. Now rather than the $L$ chirality spinor taking the same spatial steps as the $R$ but with opposite spin, we have it taking the opposite steps with the same spin. This way we may work with just one set of four spin projection operators. The amplitude now depends on a pair of ingoing 2-component spinors, and a pair of outgoing ones. To express the product of projection operators in terms of bend amplitudes, as in (22), we may proceed as in the massless case, choosing, say, the incoming $R$ spinor axis as the preferred one. For example, a $RL$ bend from $\hat{n}_1$ to $-\hat{n}_3$ has the mass amplitude $(-i\epsilon m)$ and the same bend factor $i/\sqrt{3}$ as would have had an $RR$ bend from $\hat{n}_1$ to $\hat{n}_3$.

7.2 Majorana mass

A spin-1/2 particle with a Majorana mass has no more states than a massless, chiral particle. Where the Dirac equation couples the opposite chirality fields, the Majorana equation couples the field to its own orthogonal spin state. The Majorana equation in the chiral representation is obtained by replacing $mL$ on the right-hand side of the Dirac equation (35) by $mCR$, where $CR := i\sigma_2 R^*$ is—with a particular phase choice—the left-handed spinor orthogonal to $R$:

$$(C\psi)^\dagger R = (i\sigma_2 R^*)^\dagger R = \sigma_2 R = 0. \quad (39)$$

Changing notation from $R$ to $\psi$, the discrete Majorana equation can thus be chosen as

$$\psi(x) = \frac{1}{2} \sum_i P_i (1 + i\epsilon m C) \psi(x - \epsilon n_i). \quad (40)$$

In words, there is an amplitude $-i\epsilon m$ to conjugate $\psi$ before propagating. To first order in $\epsilon$, we can equivalently adopt the equation

$$\psi(x) = \frac{1}{2} \sum_i (1 + i\epsilon m C) P_i \psi(x - \epsilon n_i), \quad (41)$$

which propagates and then conjugates.

When this is iterated, we obtain a path integral in which at each step one has the amplitude $i\epsilon m$ to insert the conjugation operator. For example, if there are two conjugations, the resulting path amplitude takes the form

$$(i\epsilon m)^2 P \cdots P C P \cdots P C P \cdots P. \quad (41)$$
Using the relations

\[ C^2 = -1, \quad CP = \bar{P}C, \quad PC = C\bar{P}. \quad (42) \]

we can move the \( C \)'s to the left. When \( C \) passes \( P \) it turns it into \( \bar{P} \), and when it meets another \( C \) they annihilate to \(-1\). Thus, for example, the above amplitude is equivalent to

\[ (\epsilon m)^2 P \cdots P\bar{P} \cdots P, \quad (43) \]

where the \((-1)\) combined with the \( i^2 \) to give \( 1 \). For paths with an even number of \( C \)'s, this pattern will persist, each action of \( C \) resulting in a chirality flip. For paths with an odd number of \( C \)'s, there is a remaining factor of \( i\epsilon m C \) at the left end. Similar to the Dirac mass case, one can trade the conjugation of the spinors for a toggling to the opposite directions for the allowed spatial steps on an enlarged lattice.

8 Discussion

Let us summarize what has been done in this paper. We discretized the Weyl equation for a massless, spin-1/2 particle on a time-diagonal, hypercubic spacetime lattice with null faces. The tetrahedral light cone then just encloses the spherical, continuum lightcone, ensuring fulfillment of the Courant stability condition, and the step speed is \( 3c \). With this particular step speed, the amplitude for a particle of right-handed chirality to step from a lattice point is \( \frac{1}{7} \) times the spin projection operator in the step direction, while for left-handed it is the orthogonal projector. Iteration yields a path integral for the retarded propagator, with matrix path amplitude the product of projection operators times \( 2^{-N} \). With any particular phase choices for the four eigenspinors, the product of projection matrices becomes a product of consecutive spinor inner products, \( \langle \lambda_{n+1} | \lambda_n \rangle \), and the choice can be made so that these give rise to the amplitude \( i^{kT} (3^{-B/2}) 2^{-N} \) for a path with \( N \) steps, \( B \) bends, and \( T \) right-handed minus left-handed bends, where the sign corresponds to the chirality. We evaluated the path integral and verified that it converges to the continuum one in the continuum limit, and showed that fermion doubling does not occur. Although the underlying lattice is not Lorentz invariant, that symmetry is recovered by the propagator in the continuum limit.

The Dirac equation describes a pair of Weyl particles of opposite chirality, coupled by a mass term. The mass term introduces the amplitude \( i\epsilon m \) to flip chirality in any given time step \( \epsilon \), where \( m \) is the mass. In 2D, this yields Feynman’s original checkerboard path integral. In 4D, the chiral spinor amplitudes are punctuated by chirality flips. To maintain the simplest possible bend amplitudes at the chirality flips we enlarged the lattice to a bcc one, so that opposite chirality spinors with the same spin could step in opposite directions. We also showed how a Majorana mass can be similarly introduced, with an amplitude to undergo charge conjugation.

Our motivation here has only been to explore the relation between spin and translation, and to find a pleasing path integral for spinors involving just a simple rule for bend amplitudes. Nevertheless, could it possibly be useful in a practical sense, for example in a non-perturbative, lattice computation scheme for chiral gauge theories? The first step in that direction is easy. An
external electromagnetic or non-abelian gauge field $A$ can be included simply by multiplying the one-step amplitude by $P \exp(i \int A)$, where the path-ordered integral is taken over the spacetime step. (Something like this is standard in lattice gauge theory computations, but occurs there with the finite differences in the action.) But this is where the easy answers end. Lattice gauge theory adopts Euclidean signature so that the vacuum state or its excitations can be straightforwardly selected, and the path integral over gauge fields can be evaluated by statistical methods. Another essential reason is that the symmetry of the 4D lattice theory can be sufficient to ensure 4D rotation invariance in the continuum limit. By contrast, Lorentzian lattice schemes ensuring 4D Lorentz symmetry in the continuum limit of an interacting theory are not known. This, together with the fact that our path integral constructs the retarded propagator rather than the Feynman propagator, makes any directly useful application in its present form seem unlikely.

As a model of relativistic quantum propagation in a discrete spacetime, our scheme has a serious flaw: the discrete propagator is not unitarity. This is not because discreteness and unitarity are necessarily in conflict. Indeed, as shown in [11], one can write a unitary discrete evolution rule on a body centered cubic lattice whose continuum limit is the Weyl equation. (Interestingly, unitarity and locality were shown there to imply the Weyl equation.) But consider an initial state that is non-vanishing only at one lattice point, with normalized spin state $|\psi\rangle$. At the next time step, according to (11), it has support at the four corners of a tetrahedron, with the amplitudes $\frac{1}{2}P_i|\psi\rangle$. The norm of the state after one step is then $\langle \psi | \sum_i \frac{1}{4}P_i | \psi \rangle = \frac{1}{2}$, i.e. it has decreased by a factor of two, violating unitarity. Also, the evolutions of orthogonal states are not orthogonal. Two points at one time have either one or zero common points in their one-step future. In the former case, the one-step evolutions of two orthogonal states supported on the two initial points are clearly not orthogonal, because they overlap in just one point which will make the unique non-zero (since $P_iP_j \neq 0$) contribution to the inner product. The evolutions therefore have “more overlap than they should”, which presumably counteracts the loss of norm of each individual evolution in such a way that the continuum limit is unitary.

To conclude, what looks like an innocent game of checkers has a deep and not so hidden connection with a more serious agenda. In print and in person, David Finkelstein credited conversations with Feynman for inspiring the notion, much explored in David’s work, that spin is the “growing tip” of spacetime paths. The idea is that the Dirac spin matrices $\gamma^\mu$ somehow describe quantized displacements. One can find the seed of this idea in Feynman’s “Operator calculus” paper [3]. Feynman invents the device of attaching a continuous index to operators, which keeps track of their order of multiplication. In Appendix B of his paper, he uses this method to derive a formal expression for the electron self-energy, in which the derivative operator is eliminated and the role of the position is transferred to an integral, $\int_0^w \gamma^\mu(s) ds$, over order-indexed gamma matrices. Feynman then remarks that “all reference to space coordinates have disappeared” from the expression, and he adds, parenthetically, “It is suggestive that perhaps coordinates and the space-time they represent may in some future theory be replaced completely by an analysis of ordered quantities in some hypercomplex algebra”. David sought that future theory.
Acknowledgements

We are grateful to E. Hawkins for mathematical aid, and to Yigal Shamir and Paulo Bedaque for instruction about lattice fermions. This work was supported in part by the NSF under grants PHY-9800967, PHY-0300710, PHY-0601800, PHY-0903572, PHY-1407744 at the University of Maryland, in part by the CNRS at the Insitut d’Astrophysique de Paris, and by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation.

A Norm of the amplification matrix

In this appendix we prove that the norm of the matrix $A(\theta)$ defined in Eq. (28) is less than unity unless at least three of the $\theta_i$ coincide, in which case the norm is unity. The proof is due to Eli Hawkins.

Let $|\nu\rangle$ be any unit spinor. The squared norm of $A|\nu\rangle$ is $\|A|\nu\rangle\|^2 = \langle\nu|A^\dagger A|\nu\rangle$, whose maximum is the larger eigenvalue of $A^\dagger A$. This value defines the squared norm $\|A\|^2$.

Using the definition of the spin projection operators (8) and the inner products of the unit vectors $\hat{n}_i$ (1 if $i = j$ and $-\frac{1}{3}$ if $i \neq j$) we find

$$\text{tr}(A^\dagger A) = 1 + \frac{1}{6} \sum_{(ij)} \cos(\theta_i - \theta_j)$$

(44)

where the sum is over the 6 choices of $\{i, j\} \subset \{1, 2, 3, 4\}$. This trace is at most 2, and therefore the smaller eigenvalue of $A^\dagger A$ is less than 1 unless $A^\dagger A = 1$.

Hence

$$\Phi := \det(A^\dagger A - 1)$$

(45)

has the same sign as $1 - \|A\|^2$.

The matrix $A^\dagger A$ is linear in terms of $e^{i(\theta_i - \theta_j)}$, therefore $\Phi$ is quadratic. Because $\Phi$ is invariant under all permutations of the $\theta$’s, it can be written as a quadratic function of the cosines $\cos(\theta_i - \theta_j)$. Because $\Phi$ vanishes when the $\theta$’s are all equal, it is convenient to write it in terms of the cosines minus 1. It thus takes the form,

$$\Phi = a \sum_{(ij)} (1 - \cos[\theta_i - \theta_j]) + b \sum_{(ij)} (1 - \cos[\theta_i - \theta_j])^2 + c \sum_{(ij)(kl)} (1 - \cos[\theta_i - \theta_j]) (1 - \cos[\theta_k - \theta_l])$$

(46)

The last sum is over the 3 partitions of $\{1, 2, 3, 4\}$ into pairs. When $\theta_2 = \theta_3 = \theta_4$, $A$ takes the form (16), which obviously has an eigenvalue of unit modulus. Therefore $\Phi = 0$ in this case, which shows that $a = b = 0$. To determine the value of $c$, consider the case that $\theta_1 = \theta_2 = 0$ and $\theta_3 = \theta_4 = \pi$. Then $A = \frac{1}{2}(\hat{n}_1 + \hat{n}_2) \cdot \vec{\sigma}$, so $A^\dagger A = 1/3$, hence $\Phi = 4/9$. The last sum in (46) is 8, so $c = 1/18$.

The determinant $\Phi$ is thus given by

$$\Phi = \frac{1}{18} \sum_{(ij)(kl)} (1 - \cos[\theta_i - \theta_j]) (1 - \cos[\theta_k - \theta_l])$$

(47)
This satisfies $\Phi \geq 0$, therefore $\|A\| \leq 1$. Each term is non-negative, therefore $\Phi = 0$ only if every term vanishes. This occurs only if at least three of the $\theta$'s are equal.

References

[1] Finkelstein, D., “Space-time code,” Phys. Rev. 184, 1261 (1969).
[2] Finkelstein, D. R., “Quantum relativity: A Synthesis of the ideas of Einstein and Heisenberg,” Berlin, Germany: Springer (1996).
[3] Feynman, R.P. “An Operator calculus having applications in quantum electrodynamics,” Phys. Rev. 84, 108 (1951).
[4] Finkelstein, D., “First Flash and Second Vacuum,” Int. J. Theor. Phys. 28, 1081 (1989).
[5] Finkelstein, D. R., Saller, H., and Tang, Z., “Beneath gauge,” Class. Quant. Grav. 14, A127 (1997); [arXiv:quant-ph/9608023].
[6] Feynman, R. P., “The development of the space-time view of quantum electrodynamics,” Science 153, no. 3737, 699 (1966) [Phys. Today 19N8, no. 8, 31 (1966)] [World Sci. Ser. 20th Cent. Phys. 27, 9 (2000)] [Version at nobelprize.org]
[7] Feynman, R.P., and Hibbs, A.R., Quantum Mechanics and Path Integrals, Problem 2-6, McGraw-Hill, Inc., 1965.
[8] See e.g. Münster, G., “Lattice quantum field theory,” Scholarpedia, 5(12):8613 (2010), http://dx.doi.org/10.4249/scholarpedia.8613; Weisz, P. and Majumdar, P., “Lattice gauge theories,” Scholarpedia, 7(4):8615 (2012), http://dx.doi.org/10.4249/scholarpedia.8615.
[9] Riazanov, G.V., “The Feynman Path Integral for the Dirac Equation”, Zh. Eksp. Teor. Fiz. 33, 1437 (1957) [Sov. Phys. JETP 6, 1107 (1958)].
[10] Jacobson, T.A., Ph.D. dissertation, University of Texas-Austin, 1983.
[11] Bialynicki-Birula, I., “Dirac And Weyl Equations On A Lattice As Quantum Cellular Automata,” Phys. Rev. D 49, 6920 (1994); [arXiv:hep-th/9304070].
[12] Schaden, M., “Causal Space-Times on a Null Lattice,” arXiv:1509.03095 [gr-qc].
[13] Neiman, Y., “Causal cells: spacetime polytopes with null hyperfaces,” Geom Dedicata (2014) 168: 161; [arXiv:1212.2916 [math.CO]].
[14] Jacobson, T., “Spinor Chain Path Integral For The Dirac Equation,” J. Phys. A 17, 2433 (1984).
[15] Jacobson, T., “Feynman’s checkerboard and other games,” Lecture Notes in Physics, 226, 386 (1985) [online version].