Path integrals and low-dimensional topology

Bogusław Broda

Department of Theoretical Physics, University of Łódź,
Pomorska 149/153, PL–90-236 Łódź, Poland
e-mail: bobroda@krysia.uni.lodz.pl

1 Introduction

The aim of our talk is to present a specific, non-perturbative, path-integral approach to topological invariants of knots/links and manifolds of dimension three and four. The technique is not rigorous but very intuitive and strongly motivated by physics. An exception is the four-dimensional case in Sect. 4, which is rather rigorous but less intuitive from physical point of view.

The plan of the paper is as follows. In Sect. 2, we will give an account of a non-perturbative, path-integral derivation of standard (quantum-group) knot and link invariants in the spirit of an original idea of Witten. We will also present two higher-dimensional generalisations. Sect. 3 is devoted to a Chern-Simons approach to topological invariants of three-dimensional manifolds of Reshetikhin, Turaev and Witten. Sect. 4 presents a four-dimensional version of the three-manifold invariant.

2 Knot and link invariants

According to Witten [1] topological invariants of knots and links can be described as expectation values of Wilson loop observables in the framework of three-dimensional Chern-Simons gauge theory.

2.1 Chern-Simons theory

The classical action of Chern-Simons theory is given by an appropriately normalized secondary characteristic class

$$S_{CS}(A) = \frac{1}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$  (1)

where $A$ is the gauge-potential (connection) for the simple, compact Lie group $G$. Since a knot (link) is a simple loop (collection of loops), we can associate an observable to it: Wilson loop(s). The Wilson loop is defined as

$$W^L_C(A) = \text{Tr}_l \text{P \, exp} \int_C A,$$  (2)
where \( \mathcal{C} \) is a loop (knot) in three-dimensional space, and \( l \) labels irrep’s of \( G \). Since the action and Wilson loops are gauge-invariant and metric independent, we can claim that expectation values \( \langle W_{\mathcal{C}}(A) \rangle = \int W_{\mathcal{C}}(A) \exp \left[ ikS_{CS}(A) \right] DA \) should be metric independent as well, i.e. they are topological invariants. \( k \) is an integer (coupling constant or conformal weight). To determine the corresponding topological invariant, one should derive the, so-called, skein relation, allowing to recursively compute the topological invariant. An example of a three-component skein relation is given below

\[
\alpha \bigg/ \bigg/ + \beta \bigg/ \bigg/ + \gamma \bigg/ \bigg/ = 0.
\]

To perform a concrete calculation we translate the operator form of the Wilson loop (2) to a path-integral one, and apply the Stokes theorem transforming a line integral into a surface one [2]. If one of the lines pierces the surface, there is a contribution to the path integral we should calculate [3]. As a result, we obtain all well-known link invariants [4]:

| \( SU(2) \) | Fundamental representation | \( SU(n) \) | Higher representations | \( SO(n) \) | \begin{tabular}{c} \begin{tabular}{c} \begin{tabular}{c} Jones, Kauffman \end{tabular} \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} Jones, Kauffman \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} HOMFLY \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} Dubrovnik Kauffman \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} Akutsu-Wadati \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} \ldots \end{tabular} \end{tabular} | \begin{tabular}{c} \begin{tabular}{c} \ldots \end{tabular} \end{tabular} |
\hline
2.2 Higher dimensions

It is not difficult, at least formally, to generalise our considerations to the case of higher dimensions. In principle, there are two possibilities we will very shortly describe.

**Inhomogeneous Chern-Simons theory**  One can consistently define, in arbitrary dimension, an action of the form (1), with \( A = \sum_{i=\text{odd}} A_i \) inhomogeneous form, where \( A_i \) is a \( i \)-form. Corresponding generalisation of observables is straightforward [5]. This way one can describe invariants of links analogous to the three-dimensional case.

**BF-theory**  In \( d \) dimensions the, so-called, BF-theory is defined by \( S_{BF}(A, B) = \int \text{Tr}B \wedge F \), where \( F \) is the gauge strength (curvature), and \( B \) is an independent non-abelian \((d-2)\)-form field. This theory naturally describes linking phenomena between components of dimension 1 and \( d - 2 \) [6].

**Quantisation**  There is a purely technical but important issue concerning the quantisation procedure. In higher dimensions, theories under consideration are plagued by on-shell reducible gauge symmetries. To cope with this problem one should use the formalism of Batalin and Vilkovisky. This procedure introduces a host of different kinds of ghosts and auxiliary fields (and also metric) [7].
3 Three-manifold invariants

Up to the present moment, we have been interested in knot and link invariants of knots and links in $S^3$ (or $R^3$). It is time now, to extend our analysis to the case of an arbitrary closed, connected, three-dimensional manifold $M^3$. First of all, we will be interested in the extreme case of empty knot/link, i.e. in the manifold $M^3$ itself. As far as non-perturbative calculations are concerned, there is a class of basically equivalent invariants which definitions depend on topological description of $M^3$. There are the three main possibilities: (1) surgery on a link, (2) Heegaard decomposition, (3) simplicial decomposition (triangulation). Correspondingly, we have three types of invariants: (1) Reshetikhin-Turaev-Witten (RTW), (2) Kohno, (3) Turaev-Viro. In this talk, we will confine ourselves to the first type.

3.1 RTW invariant

The idea is to use the fact that it is possible to obtain an arbitrary three-dimensional manifold $M^3$ via surgery on a link, i.e. by attaching two-handles along a link $L$. In physical terms, attaching means gluing or identifying boundary values [8].

Second Kirby move In the first step, we should formally calculate the partition function of Chern-Simons theory with fixed boundary conditions (holonomies) along all components of the surgery link $L$

$$Z(g_1, \ldots, g_N) = \langle \delta(g_1, \text{Hol}_{C_1}) \cdots \delta(g_N, \text{Hol}_{C_N}) \rangle,$$

where $\delta$ is a (group-theoretic) Dirac delta-function, $\delta(g, h) = \sum \chi_l(g) \chi_l(h)$, $\chi_l$ are characters, and $\text{Hol}_{C_i}$ are holonomies around $C_i$. Then

$$Z(g_1, \ldots, g_N) = \left\langle \sum_{l_1} \chi_{l_1}(g_1) W_{l_1}^{i_1} \cdots \sum_{l_N} \chi_{l_N}(g_N) W_{l_N}^{i_N} \right\rangle.$$

In the second step, we should identify and sum up corresponding boundary values $(g_1, \ldots, g_N)$. Thus, we obtain

$$Z_{M^3} = \int dg_1 \cdots dg_N Z_O(g_1^{-1}) \cdots Z_O(g_N^{-1}) Z(g_1, \ldots, g_N) = \langle \omega_{C_1} \cdots \omega_{C_N} \rangle, \quad (3)$$

where $\omega_{C_i} = \sum \langle W_{O}^{\ell}(A) \rangle W_{C_i}^{\ell}(A)$, and “O” refers to calculations performed for an unknot. The quantity (3) is invariant with respect to the, so-called, second Kirby move.

First Kirby move To stabilise (3) to obtain a true topological invariant we should normalize it with the factor (in the denominator) $\langle \omega_+\rangle^b_+ \langle \omega_-\rangle^b_-$, where $b_+$ ($b_-$) means the number of positive (negative) eigenvalues of the liking matrix $L_k$ of $L$, and $+$ ($-$) as a subscript pertains to an unknot with positive (negative) twist, correspondingly. The normalized (3) is independent of the both Kirby moves.
3.2 Knots and links

Incorporation of knots and links is straightforward. We should modify (3) to the expression
\[ \langle W_{\ell_1}^i \cdots W_{\ell_M}^i \omega_{\ell_1} \cdots \omega_{\ell_N} \rangle, \]
where barred quantities mean components of an ordinary knot/link.

4 Generalisation to four dimensions

We can easily generalise our considerations to the case of a four-dimensional (closed, connected) manifold \( M^4 \). There is an analogous possibility to construct an arbitrary topological \( M^4 \) via (generalised) surgery on a special link \( \ell \) [9]. The special link \( \ell \) consists of two kinds of components: undotted \( L \) and dotted \( \hat{L} \). The four-dimensional Kirby calculus is a little bit more complicated than in the three-dimensional case. There are three second Kirby moves now:

Nevertheless, we can find, purely combinatorically, a topological invariant [10]
\[ I_k(M^4) = \frac{\langle \omega_{\ell_1}^+ \cdots \omega_{\ell_N}^+ \omega_{\ell_1} \cdots \omega_{\ell_N} \rangle}{\langle \omega_{\ell_1}^+ \rangle^{\nu} \langle \omega_{\ell_1}^+ \rangle^{(N+N-\nu)/2}}, \]
where for simplicity we have assumed that \( G = SU(2) \). Here \( \omega^+ \) means even part of \( \omega \) (only integer spins), \( \nu \) is the nullity of the linking matrix \( \mathbf{L} \mathbf{\ell} \), and \( H, \hat{H} \) are components of the Hopf link.

5 Finishing remarks

We are able to yield all invariants of knots, links and low-dimensional manifolds (of dimension three and four) pertaining to quantum groups (and their representations) using their classical counterparts. Other interesting chapter on topological invariants could contain an approach to perturbative invariants like Vassiliev invariants of knots, Casson-Walker invariant, and its higher order generalisations, and dimensionally reduced Seiberg-Witten invariant.
Acknowledgments
I am greatly indebted to Prof. H.-D. Doebner for his kind hospitality during my stay in Clausthal. I would also like to thank the organisers, and in particular Prof. H.-D. Doebner for giving me the opportunity to present my results during the Colloquium.

The paper has been supported by Polish grant 2 P03B 094 10 and by the Humboldt Foundation.

References

[1] E. Witten, “Quantum Field Theory and the Jones Polynomial”, Commun. Math. Phys. 121, 351–399 (1989).

[2] B. Broda, “Non-Abelian Stokes theorem”, in Advanced Electromagnetism: Foundations, Theory and Applications, (Eds. T.W. Barrett and D.M. Grimes), p. 496–505, World Scientific, Singapore, 1995.

[3] B. Broda, “A path-integral approach to polynomial invariants of links”, Journ. Math. Phys. (special issue on Topology and Physics) 35, 5314–5320 (1994).

[4] B. Broda, “Skein relations for any pair of irreducible representations of any Lie group”, in Classical and Quantum Systems—Foundations and Symmetries, Proc. of the II Int. Wigner Symposium, (Eds. H.-D. Doebner, W. Scherer and F. Schroeck, Jr.), p. 273–277, World Scientific, Singapore, 1993.

[5] B. Broda, “Higher-dimensional Chern-Simons theory and link invariants”, Phys. Lett. B 280, 213–218 (1992).

[6] B. Broda, “Quantum theory of non-abelian differential forms and link polynomials”, Mod. Phys. Lett. A 9, 609–621 (1994).

[7] B. Broda, “Quantum BF-theory is topological”, Phys. Lett. B 280, 47–51 (1992).

[8] B. Broda, “Chern-Simons approach to three-manifold invariants”, Mod. Phys. Lett. A 10, 487–493 (1995).

[9] E. César de Sá, “A link calculus for 4-manifolds”, in Topology of low-dimensional manifolds, Proc. of Conf., Lect. Notes in Mathem. 722, p. 16–30, Springer, Berlin, 1979.

[10] B. Broda, “A surgical invariant of 4-manifolds”, in Proc. of Conf. on Quantum Topology (Eds. D.N. Yetter), p. 45–50, World Scientific, Singapore, 1994.