COMBINATORIAL PROPERTIES ON NODEC COUNTABLE SPACES
WITH ANALYTIC TOPOLOGY

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Abstract. We study some variations of the product topology on families of clopen subsets of $2^\mathbb{N} \times \mathbb{N}$ in order to construct countable nodec regular spaces (i.e. in which every nowhere dense set is closed) with analytic topology which in addition are not selectively separable and do not satisfy the combinatorial principle $q^+$.

Keywords: nodec countable spaces; analytic sets, selective separability, $q^+$
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1. Introduction

A topological space $X$ is selectively separable (SS), if for any sequence $(D_n)_n$ of dense subsets of $X$ there is a finite set $F_n \subseteq D_n$, for $n \in \mathbb{N}$, such that $\bigcup_n F_n$ is dense in $X$. This notion was introduced by Scheepers [13] and has received a lot of attention ever since (see for instance [1, 2, 3, 4, 5, 6, 8, 12]). Bella et al. [4] showed that every separable space with countable fan tightness is SS. On the other hand, Barman and Dow [1] showed that every separable Fréchet space is also SS (see also [6]).

A topological space is maximal if it is a dense-in-itself regular space such that any strictly finer topology has an isolated point. It was shown by van Douwen [17] that a space is maximal if, and only if, it is extremely disconnected (i.e. the closure of every open set is closed), nodec (i.e. every nowhere dense set is closed) and every open set is irresolvable (i.e. if $U$ is open and $D \subseteq U$ is dense in $U$, then $U \setminus D$ not dense in $U$). He constructed a countable maximal regular space.

A countable space $X$ is $q^+$ at a point $x \in X$, if given any collection of finite sets $F_n \subseteq X$ such that $x \in \bigcup_n F_n$, there is $S \subseteq \bigcup_n F_n$ such that $x \in \overline{S}$ and $S \cap F_n$ has at most one point for each $n$. We say that $X$ is a $q^+$-space if it is $q^+$ at every point. Every countable sequential space is $q^+$ (see [15, Proposition 3.3]). The collection of clopen subsets of $2^\mathbb{N}$ with the product topology is not $q^+$ at any point. This notion is motivated by the analogous concept of a $q^+$ filter (or ideal) from Ramsey theory.

A problem stated in [4] was to analyze the behavior of selective separability on maximal spaces. The existence of a maximal regular SS space is independent of ZFC. In fact, in ZFC there is a maximal non SS space [1] and it is consistent with ZFC that no countable maximal space is SS [1, 12]. On the other hand, it is also consistent that there is a maximal, countable, SS regular space [1].

In this paper we are interested in these properties on countable spaces with an analytic topology (i.e. the topology of the space $X$ is an analytic set as a subset of $2^X$ [14]). Maximal topologies are not analytic. In fact, in [16] it was shown that there are neither extremely

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disconnected nor irresolvable analytic topologies, nevertheless there are nodec regular spaces with analytic topology. In view of the above mentioned results about maximal spaces, it seems natural to wonder about the behavior of selective separability on nodec spaces with an analytic topology. Nodec regular spaces are not easy to construct. We continue the study of the method introduced in [16] in order to construct similar nodec regular spaces with analytic topology. Nodec regular spaces are not easy to construct. We will always assume that an ideal contains all finite subsets of $X$. If $I$ is an ideal on $X$, then $I^+ = \{ A \subseteq X : A \notin I \}$. $\text{Fin}$ denotes the ideal of finite subsets of the non negative integers $\mathbb{N}$. An ideal $I$ on $X$ is called \textit{tall}, if for every $A \subseteq X$ infinite, there is $B \subseteq A$ infinite with $B \notin I$. We denote by $A^{<\omega}$ the collection of finite sequences of elements of $A$. If $s$ is a finite sequence on $A$ and $i \in A$, $|s|$ denotes its length and $s \upharpoonright i$ the sequence obtained concatenating $s$ with $i$. For $s \in 2^{<\omega}$ and $\alpha \in 2^\mathbb{N}$, let $s < \alpha$ if $\alpha(i) = s(i)$ for all $i < |s|$ and
\[
|s| = \{ \alpha \in 2^\mathbb{N} : s < \alpha \}.
\]
If $\alpha \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, we denote by $\alpha \upharpoonright n$ the finite sequence $(\alpha(0), \cdots, \alpha(n-1))$ if $n > 0$ and $\alpha \upharpoonright 0$ is the empty sequence. The collection of all $|s|$ with $s \in 2^{<\omega}$ is a basis of clopen sets for $2^\mathbb{N}$. As usual we identify each $n \in \mathbb{N}$ with $\{0, \cdots, n-1\}$.

The ideal of nowhere dense subsets of $X$ is denoted by $\text{nw}(X)$. Now we recall some combinatorial properties of ideals. We put $A \subseteq^* B$ if $A \setminus B$ is finite.

\begin{itemize}
  \item[(p$^+$)] $I$ is p$^+$, if for every decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of sets in $I^+$, there is $A \in I^+$ such that $A \subseteq^* A_n$ for all $n \in \mathbb{N}$. Following [9], we say that $I$ is p$^+$, if for every decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of sets in $I^+$ such that $A_n \setminus A_{n+1} \in I$, there is $B \in I^+$ such that $B \subseteq^* A_n$ for all $n$.
  \item[(q$^+$)] $I$ is q$^+$, if for every $A \in I^+$ and every partition $(F_n)_{n \in \mathbb{N}}$ of $A$ into finite sets, there is $S \in I^+$ such that $S \subseteq A$ and $S \cap F_n$ has at most one element for each $n$. Such sets $S$ are called (partial) selectors for the partition.
\end{itemize}

A point $x$ of a topological space $X$ is called a \textit{Fréchet point}, if for every $A$ with $x \in \overline{A}$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ converging to $x$. We will say that $x$ is a q$^+$-point, if $I_x$ is q$^+$. We say that a space is a q$^+$-space, if every point is q$^+$. We define analogously the notion of a p$^+$ and p$^-$ points. Notice that if $x$ is isolated, then $I_x$ is trivially q$^+$ as $I_x^+$ is empty. Thus a space is q$^+$ if, and only if, $I_x$ is q$^+$ for every non isolated point $x$. The same occurs with the other combinatorial properties defined in terms of $I_x$.

We say that a space $Z$ is wSS if for every sequence $(D_n)_{n \in \mathbb{N}}$ of dense subsets of $Z$, there is $F_n \subseteq F_n$ a finite set, for each $n$, such that $\bigcup_n F_n$ is not nowhere dense in $Z$. In the terminology of selection principles [13], wSS corresponds to $S_{fin}(\mathcal{D}, \mathcal{B})$ where $\mathcal{D}$ is the collection of dense subsets and $\mathcal{B}$ the collection of non nowhere dense sets. Seemingly this notion has not been considered before. Notice that if $Z$ is SS and $W$ is not SS, then the direct sum of $Z$ and $W$ is wSS but not SS.

2. Preliminaries

An \textit{ideal} on a set $X$ is a collection $I$ of subsets of $X$ satisfying: (i) $A \subseteq B$ and $B \in I$, then $A \in I$. (ii) If $A, B \in I$, then $A \cup B \in I$. (iii) $\emptyset \notin I$. We will always assume that an ideal contains all finite subsets of $X$. If $I$ is an ideal on $X$, then $I^+ = \{ A \subseteq X : A \notin I \}$. $\text{Fin}$ denotes the ideal of finite subsets of the non negative integers $\mathbb{N}$. An ideal $I$ on $X$ is called \textit{tall}, if for every $A \subseteq X$ infinite, there is $B \subseteq A$ infinite with $B \notin I$. We denote by $A^{<\omega}$ the collection of finite sequences of elements of $A$. If $s$ is a finite sequence on $A$ and $i \in A$, $|s|$ denotes its length and $s \upharpoonright i$ the sequence obtained concatenating $s$ with $i$. For $s \in 2^{<\omega}$ and $\alpha \in 2^\mathbb{N}$, let $s < \alpha$ if $\alpha(i) = s(i)$ for all $i < |s|$ and

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If $\alpha \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, we denote by $\alpha \upharpoonright n$ the finite sequence $(\alpha(0), \cdots, \alpha(n-1))$ if $n > 0$ and $\alpha \upharpoonright 0$ is the empty sequence. The collection of all $|s|$ with $s \in 2^{<\omega}$ is a basis of clopen sets for $2^\mathbb{N}$. As usual we identify each $n \in \mathbb{N}$ with $\{0, \cdots, n-1\}$.

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A subset $A$ of a Polish space is called analytic, if it is a continuous image of a Polish space. Equivalently, if there is a continuous function $f : \mathbb{N}^\mathbb{N} \to X$ with range $A$, where $\mathbb{N}^\mathbb{N}$ is the space of irrationals. For instance, every Borel subset of a Polish space is analytic. A general reference for all descriptive set theoretic notions used in this paper is [10]. We say that a topology $\tau$ over a countable set $X$ is analytic, if $\tau$ is analytic as a subset of the cantor cube $2^X$ (identifying subsets of $X$ with characteristic functions) [14, 15, 16], in this case we will say that $X$ is an analytic space. A regular countable space is analytic if, and only if, it is homeomorphic to a subspace of $C_p(\mathbb{N}^\mathbb{N})$ (see [14]). If there is a base $B$ of $X$ such that $B$ is an $F_\sigma$ (Borel) subset of $2^X$, then we say that $X$ has an $F_\sigma$ (Borel) base. In general, if $X$ has a Borel base, then the topology of $X$ is analytic.

We end this section recalling some results about countable spaces that will be used in the sequel.

**Theorem 2.1.** [6, Corollary 3.8] Let $X$ be a countable space with an $F_\sigma$ base, then $X$ is $p^+$. Next result is essentially Lemma 4.6 of [16].

**Lemma 2.2.** Let $X$ be a $\sigma$-compact space and $W$ a countable collection of clopen subsets of $X$. Then $W$, as a subspace of $2^X$, has an $F_\sigma$ base.

**Theorem 2.3.** [6, Theorem 3.5] Let $X$ be a countable space. If $X$ is $p^-$, then $X$ is SS. In particular, if $X$ has an $F_\sigma$ base, then $X$ is SS.

A space $X$ is discretely generated (DG) if for every $A \subseteq X$ and $x \in \overline{A}$, there is $E \subseteq A$ discrete such that $x \in \overline{E}$. This notion was introduced by Dow et al. in [7]. It is not easy to construct spaces which are not DG, the typical examples are maximal spaces (which are nodec).

**Theorem 2.4.** Let $X$ be a regular countable space. Suppose every non isolated point is $p^-$, then $X$ is discretely generated.

**Proof.** Let $A \subset X$ with $x \in \overline{A}$. Fix a maximal family $(O_n)_n$ of relatively open disjoint subsets of $A$ such that $x \not\in \overline{O_n}$. Let $B = \bigcup_n O_n$. From the maximality we get that $x \in \overline{B}$. Since each $O_n$ does not accumulate to $x$ and $x$ is a $p^-$-point, there is $E$ such that $x \in \overline{E}$ and $E \cap O_n$ is finite for every $n$. Clearly $E$ is a discrete subset of $A$. □

**Theorem 2.5.** (Dow et al [7, Theorem 3.9]) Every Hausdorff sequential space is discretely generated.

In summary, we have the following implications for countable regular spaces (see [6]).
2.1. A SS, q\(^{+}\) nodec analytic non regular topology. As we said in the introduction, nodec regular spaces are not easy to construct. However, non regular nodec spaces are fairly easy to define. We recall a well known construction given in [11]. Let \( \tau \) be a topology and define

\[
\tau^\alpha = \{ V \setminus N : V \in \tau \text{ and } N \in \text{nwd}(\tau) \}.
\]

Then \( \tau^\alpha \) is a topology finer than \( \tau \) (see [11]).

**Lemma 2.6.** [11] Let \((X, \tau)\) be a space.

(i) \( V \in \tau^\alpha \) iff \( V \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(V))) \).

(ii) Let \( A \subseteq X \) and \( x \notin A \). Then \( x \in \text{cl}_\tau(A) \) if, and only if, \( x \in \text{cl}_\tau(\text{int}_\tau(\text{cl}_\tau(A))) \).

(iii) \((X, \tau^\alpha)\) is a nodec space.

**Proposition 2.7.** Let \((X, \tau)\) be a countable space.

(i) If \((X, \tau)\) is Fréchet, then \((X, \tau^\alpha)\) is a q\(^{+}\)-space.

(ii) \((X, \tau)\) is SS if, and only if, \((X, \tau^\alpha)\) is SS.

**Proof.** (i) Suppose \( x \in \text{cl}_\alpha(A) \setminus A \) and \((F_n)\) is a partition of \( A \) with each \( F_n \) finite. Let \( V = \text{int}_\tau(\text{cl}_\tau(A)) \). By Lemma 2.6, we have that \( x \in \text{cl}_\tau(V) \). Let \((y_m)_m\) be an enumeration of \( V \). Since \( A \) is \( \tau \)-dense in \( V \), for every \( m \) there is a sequence \((x^m_i)_i \) in \( A \) such that \( x^m_i \to y_m \) when \( i \to \infty \) (with respect to \( \tau \)). Since each \( F_n \) is finite, we can assume (by passing to a subsequence if necessary) that each \((x^m_i)_i \) is a selector for the partition \((F_n)_n \). Let \( S_m \) be the range of \((x^m_i)_i \). Notice that \( x \notin \text{cl}_\tau(S_m) \) and every infinite subset of \( S_m \) is also a selector for \((F_n)_n \). By a straightforward diagonalization, for each \( m \), there is \( T_m \subseteq S_m \) such that each \( T_m \) is a selector and moreover \( \bigcup_m T_m \) is also a selector. Hence we can assume that \( S = \bigcup_m \{ x^m_i : i \in \mathbb{N} \} \) is a selector for the partition. But clearly \( S \) is \( \tau \)-dense in \( V \) and thus \( V \subseteq \text{int}_\tau(\text{cl}_\tau(S)) \). Hence \( x \in \text{cl}_\alpha(S) \) (by Lemma 2.6(i)).

(ii) By Lemma 2.6(ii), a set is \( \tau \)-dense iff it is \( \tau^\alpha \)-dense. \( \square \)

Let \( \tau \) be the usual metric topology on the rational \( \mathbb{Q} \). It is not difficult to verify that \( \tau^\alpha \) is analytic (in fact, it is Borel) and non regular (see [16]). Thus \((\mathbb{Q}, \tau^\alpha)\) is a SS, q\(^{+}\) and nodec non regular space with analytic topology. It is not known if there is a regular space with the same properties.

3. The spaces \( X(\mathcal{I}) \) and \( Y(\mathcal{I}) \)

We recall the definitions of the spaces \( X(\mathcal{I}) \) and \( Y(\mathcal{I}) \) for an ideal \( \mathcal{I} \), which were introduced in [16].

For each non empty \( \mathcal{A} \subseteq 2^\mathbb{N} \), let \( \rho_\mathcal{A} \) be the topology on \( 2^{2^\mathbb{N} \times \mathbb{N}} \) generated by the following sets:

\[
(\alpha, p)^+ = \{ \theta \in 2^{2^\mathbb{N} \times \mathbb{N}} : \theta(\alpha, p) = 1 \}, \quad (\alpha, p)^- = \{ \theta \in 2^{2^\mathbb{N} \times \mathbb{N}} : \theta(\alpha, p) = 0 \},
\]

with \( \alpha \in \mathcal{A} \). A basic \( \rho_\mathcal{A} \)-open set is as follows:

\[
V = \bigcap_{i=1}^m (\alpha_i, p_i)^+ \cap \bigcap_{i=1}^n (\beta_i, q_i)^-.
\]
for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in A$, $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{N}$. We always assume that $(\alpha_i, p_i) \neq (\beta_j, q_j)$ for all $i$ and $j$, which is equivalent to saying that any set $V$ as above is not empty.

Let $X$ be the collection of all finite unions of clopen sets of the form $[s] \times \{n\}$ with $n \in \mathbb{N}$ and $s \in 2^{<\omega}$. We also include $\emptyset$ as an element of $X$. As usual, we regard $X$ as a subset of $2^{\mathbb{N}}$. Let $\{\varphi_n : n \in \mathbb{N}\}$ be an enumeration of $X$ and for convenience we assume that $\varphi_0$ is $\emptyset$. Each $\varphi_n$, regarded as a function from $2^{\mathbb{N}} \times \mathbb{N}$ to $\{0, 1\}$, is continuous. Notice that $X$ is a group with the symmetric difference as operation.

Let $\psi_n : 2^{\mathbb{N}} \times \mathbb{N} \to \{0, 1\}$ be defined by

$$
\psi_n(\alpha, m) = \begin{cases}
\varphi_n(\alpha, m), & \text{if } \alpha(n) = 0.
1, & \text{if } \alpha(n) = 1.
\end{cases}
$$

Then $\psi_n$ is a continuous function. Let

$$
\mathcal{Y} = \{\psi_n : n \in \mathbb{N}\}.
$$

Given $\mathcal{I} \subseteq 2^{\mathbb{N}}$, we define

$$
X(\mathcal{I}) = (X, \rho_\mathcal{I}),
$$

$$
\mathcal{Y}(\mathcal{I}) = (\mathcal{Y}, \rho_\mathcal{I}).
$$

Also notice that $X(\mathcal{I})$ is a topological group.

To each $F \subseteq \mathbb{N}$, we associate two sets $F' \subseteq X$ and $\widehat{F} \subseteq \mathcal{Y}$:

$$
F' := \{\varphi_n : n \in F\},
$$

$$
\widehat{F} := \{\psi_n : n \in F\}.
$$

The topological similarities between $F'$ and $\widehat{F}$ are crucial to establish some properties of $\mathcal{Y}(\mathcal{I})$.

As usual, we identify a subset $A \subseteq \mathbb{N}$ with its characteristic function. So from now on, an ideal $\mathcal{I}$ over $\mathbb{N}$ will be also viewed as a subset of $2^{\mathbb{N}}$. The properties of $\mathcal{Y}(\mathcal{I})$ naturally depend on the ideal $\mathcal{I}$.

**Lemma 3.1.** If $\mathcal{I}$ is analytic, then $X(\mathcal{I})$ and $\mathcal{Y}(\mathcal{I})$ have analytic topologies.

**Proof.** It is easy to see that the standard subspace subbases for $X(\mathcal{I})$ and $\mathcal{Y}(\mathcal{I})$ are also analytic when $\mathcal{I}$ is analytic. Thus the topology is analytic (see [14, Proposition 3.2]). \hfill $\Box$

**Theorem 3.2.** If $\mathcal{I}$ is an $F_\sigma$ ideal over $\mathbb{N}$, then $\mathcal{Y}(\mathcal{I})$ has an $F_\sigma$ base and thus it is SS and DG.

**Proof.** It follows from Lemma 2.2 and Theorems 2.3 and 2.4. \hfill $\Box$

The reason to study the space $\mathcal{Y}(\mathcal{I})$ is the following theorem. Let

$$
\mathcal{I}_{nd} := \{F \subseteq \mathbb{N} : \{\varphi_n : n \in F\} \text{ is nowhere dense in } X\}.
$$

**Theorem 3.3.** [16] $\mathcal{Y}(\mathcal{I}_{nd})$ is a nodec regular space without isolated points and with an analytic topology.

$\mathcal{Y}(\mathcal{I}_{nd})$ was so far the only space we knew with the properties stated above. We will present a generalization of this theorem showing other ideals $\mathcal{I}$ such that $\mathcal{Y}(\mathcal{I})$ has the same properties.
3.1. The space $\mathbb{X}(\mathcal{I})$. We present some properties of the space $\mathbb{X}(\mathcal{I})$ that will be needed later. We are interested in whether $\mathbb{X}(\mathcal{I})$ is DG, SS or $q^+$. We start with a general result which is proven as Theorem 3.2.

**Theorem 3.4.** If $\mathcal{I}$ is an $F_\sigma$ ideal over $\mathbb{N}$, then $\mathbb{X}(\mathcal{I})$ has an $F_\sigma$ base, and thus it is SS and DG.

We will show that $\mathbb{X}(\mathcal{I})$ is not $q^+$ except in the extreme case when $\mathcal{I}$ is Fin. The key lemma to show this is the following result.

**Lemma 3.5.** There is a pairwise disjoint family $\{A_n : n \in \mathbb{N}\}$ of finite subsets of $\mathbb{X}$ such that $\bigcup_{k \in E} A_k$ is dense in $\mathbb{X}$ (with the product topology) for any infinite $E \subseteq \mathbb{N}$. Moreover, for each infinite set $E \subseteq \mathbb{N}$, each selector $S$ for the family $\{A_n : n \in E\}$ and each $\varphi \notin S \cup \{\emptyset\}$, there is $p \in \mathbb{N}$ and $\alpha \in 2^{\mathbb{N}}$ such that $\alpha^{-1}(1) \subseteq^* E$, $\varphi \in (\alpha, p)^+$ and $(\alpha, p)^+ \cap S$ is finite.

**Proof.** We say that a $\varphi \in \mathbb{X}$ has the property $(*^m)$, for $m \in \mathbb{N}$, if there are $k \in \mathbb{N}$ and infinite sequences $s_i$, for $i = 1, \ldots, k$, of length $m + 1$ such that $\varphi = \bigcup_{i=1}^k [s_i] \times \{m_i\}$, $m_i \leq m$ and $s_i \cap m \neq s_j \cap m$, whenever $m_i = m_j$ (i.e. $[s_j] \cup [s_i]$ is not a basic clopen set). Let $A_m = \{\varphi \in \mathbb{X} : \varphi$ has the property $(*^m)\}$.

Let $E \subseteq \mathbb{N}$ be an infinite set. We will show that $A := \bigcup_{k \in E} A_k$ is dense in $2^{2k \times \mathbb{N}}$. Let $V$ be a basic open set of $2^{2k \times \mathbb{N}}$, let us say

$$V = \bigcap_{i=1}^m (\alpha_i, p_i)^+ \cap \bigcap_{i=1}^n (\beta_i, q_i)^-$$

for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in 2^{\mathbb{N}}$, $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{N}$. We need to show that $V \cap A$ is not empty. Pick $l$ large enough such that $l + 1 \in E$, $l + 1 > \max\{p_i, q_j : i \leq m, j \leq n\}$, $\alpha_i \uparrow l \neq \alpha_j \uparrow l$ for all $i$ and $j$ such that $\alpha_i \neq \alpha_j$, $\beta_i \uparrow l \neq \beta_j \uparrow l$ for all $i$ and $j$ such that $\beta_i \neq \beta_j$ and $\alpha_i \uparrow l \neq \beta_j \uparrow l$ for all $i$ and $j$ such that $\alpha_i \neq \beta_j$. Let $\varphi = \bigcup_{i=1}^m [\alpha_i \uparrow (l+2)] \times \{p_i\}$. Then $\varphi$ belongs to $A_{l+1} \cap V$.

To see the second claim, let $E \subseteq \mathbb{N}$ be an infinite set and let $S = \{z_n : n \in E\}$ be a selector, that is, $z_n \in A_n$ for all $n \in E$. Fix $\varphi \notin S \cup \{\emptyset\}$, say $\varphi = \bigcup_{i=1}^l [t_i] \times \{p_i\}$ for some $t_i \in 2^{<\omega}$ and $p_i \in \mathbb{N}$. The required $\alpha$ is recursively defined as follows:

$$\alpha(n) = \begin{cases} t_1(n), & \text{if } n < |t_1|, \\ 1, & \text{if } n \geq |t_1|, n \in E \text{ and } [(\alpha \uparrow n)^\sim 0] \times \{p_1\} \subseteq z_n, \\ 0, & \text{otherwise}. \end{cases}$$

From the definition of the sets $A_m$, it is easily shown that $(\alpha, p_1) \notin \bigcup\{z_k : k \geq |t_1| \text{ and } k \in E\}$. Clearly $(\alpha, p_1)^+ \cap S \subseteq \{z_k : k < |t_1| \text{ and } k \in E\}$ is finite and $\varphi \in (\alpha, p_1)^+$. Finally, it is also clear from the definition of $\alpha$ that $\alpha^{-1}(1) \subseteq^* E$.

**Theorem 3.6.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then $\mathbb{X}(\mathcal{I})$ is $q^+$ at some (every) point if, and only if, $\mathcal{I} = \text{Fin}$.

**Proof.** If $\mathcal{I} = \text{Fin}$, then $\mathbb{X}(\mathcal{I})$ has a countable basis and thus it is $q^+$ at every point. Since $\mathbb{X}(\mathcal{I})$ is homogeneous (as it is a topological group), then if $\mathbb{X}(\mathcal{I})$ is $q^+$ at some point, then it is $q^+$ at every point. Suppose now that there is $E \in \mathcal{I} \setminus \text{Fin}$. We will show that $\mathcal{I}$ is not $q^+$ at some point. Let $\{A_n : n \in \mathbb{N}\}$ be the sequence, given by Lemma 3.5, of pairwise disjoint
finite subsets of $X$ such that $A := \bigcup_{k \in E} A_k$ is dense in $X$. Since the topology of $X$ is finer than the topology of $\bar{X}(I)$, then $A$ is dense in $\bar{X}(I)$. Let $\varphi \notin A \cup \{0\}$. We will show that $\bar{X}(I)$ fails the property $q^+$ at $\varphi$. Let $S$ be a selector of $\{A_n : n \in E\}$. Let $\alpha \in 2^\mathbb{N}$ and $p \in \mathbb{N}$ be as in the conclusion of Lemma 3.5, that is, $\alpha^{-1}(1) \subseteq E$, $\varphi \in (\alpha,p)^+$ and $(\alpha,p)^+ \cap S$ is finite. Notice that $\alpha \in I$ and hence $\varphi$ is not in the $\rho_I$-closure of $S$. Hence $\bar{X}(I)$ is not $q^+$ at $\varphi$.

Now we look at the SS property. The following result provides a method to construct dense subsets of $\bar{X}(I)$.

**Lemma 3.7.** For each $A \subseteq \mathbb{N}$ infinite, let $D(A)$ be the following subset of $X$:

$$
\left\{ \bigcup_{i=0}^k [s_i] \times \{m_i\} \in X : A \cap s_i^{-1}(0) \neq \emptyset \quad \text{for all } i \in \{0, ..., k\}, \ k \in \mathbb{N}, \ s_i \in 2^{<\omega} \right\} \cup \{0\}.
$$

Then $A \in I$ if, and only if, $D(A)$ is not dense in $\bar{X}(I)$ if, and only if, $D(A)$ is nowhere dense and closed in $\bar{X}(I)$.

**Proof.** We first show that $D(A)$ is closed for every $A \in I$. We shall show that the complement of $D(A)$ is open in $\bar{X}(I)$. Let $\varphi \in X \setminus D(A)$. Since $\varphi \neq \emptyset$, we have that $\varphi = \bigcup_{i=1}^k [s_i] \times \{m_i\}$ and we can assume that $A \cap s_i^{-1}(0) = \emptyset$. Let $B = A \cup s_i^{-1}(1)$. Notice that $B \in I$. Let $\beta$ be the characteristic function of $B$. Clearly $\beta \in [s_1]$ and thus $\varphi \in (\beta,m_1)^+$. On the other hand, suppose that $\varphi' = \bigcup_{i=1}^k [t_i] \times \{p_i\} \in (\beta,m_1)^+$. Assume that $\beta \in [t_i]$ and $p_i = m_1$, then $t_i^{-1}(0) \subset \beta_i^{-1}(0)$ and hence $t_i^{-1}(0) \cap A = \emptyset$. This shows that $\varphi' \notin D(A)$ and thus $(\beta,m_1)^+ \cap D(A) = \emptyset$.

Now we show that if $A \in I$, then $D(A)$ is nowhere dense. Since $D(A)$ is closed, it suffices to show that it has empty interior. Let $V$ be a basic $\rho_I$-open set. Let us say

$$V = \bigcap_{i=1}^m (\alpha_i, p_i)^+ \cap \bigcap_{i=1}^n (\beta_i, q_i)^-$$

for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in I$, $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{N}$. Recall that $(\alpha_i, p_i) \neq (\beta_j, q_j)$ for all $i \neq j$. Since $\beta_i \in I$, then $\beta_i^{-1}(0) \neq \emptyset$ for all $i$. Let $l = \max\{\min(\beta_i^{-1}(0)) : 1 \leq i \leq n\}$ and $l$ be the constant sequence $1$ of length $l$. Since $X$ is clearly $\rho_I$-dense, let $\varphi \in V \cap X$. Then $\varphi \cup ([l] \times \{0\}) \in V \setminus D(A)$.

Finally, we show that if $A \notin I$, then $D(A)$ is dense. Let $V$ be a basic $\rho_I$-open set as given by (1). Pick $l$ large enough such that $\alpha_i \uparrow l \neq \alpha_j \uparrow l$ for $i \neq j$, $\beta_i \uparrow l \neq \beta_j \uparrow l$ for $i \neq j$ and $\alpha_i \uparrow l \neq \beta_j \uparrow l$ for all $i$ and $j$ such that $\alpha_i \neq \beta_j$. Then pick $k \geq l$ such that $k \geq \min(\alpha_i^{-1}(0) \cap A)$ for all $i \leq m$ (notice that $\alpha_i^{-1}(0) \cap A \neq \emptyset$ as $A \notin I$ and $\alpha_i \in I$). Let $s_i = \alpha_i \uparrow k$ for $i \leq m$ and $\varphi = \bigcup_{i=1}^m [s_i] \times \{p_i\}$. Then $\varphi \in V \cap D(A)$.

We remind the reader that $F'$ denotes the set $\{\varphi_n : n \in F\}$ for each $F \subseteq \mathbb{N}$.

**Theorem 3.8.** Let $I$ be an ideal over $\mathbb{N}$. If $I$ is not $p^+$, then $X(I)$ is not wSS.

**Proof.** Suppose that $I$ is not $p^+$ and fix a sequence $(A_n)_{n\in \mathbb{N}}$ of subsets of $\mathbb{N}$ such that $A_n \notin I$, $n \in \mathbb{N}$, and $\bigcup_{n\in \mathbb{N}} F_n \in I$ for all $F_n \subseteq A_n$ finite.
Let \( D_n = D(A_n) \) as in Lemma 3.7. We show that the property \( wSS \) fails at the sequence \((D_n)_n\). Let \( K_n \subseteq D_n \) be a finite set for each \( n \), we need to show that \( \bigcup_n K_n \) is nowhere dense in \( X(\mathcal{I}) \). Let us enumerate each \( K_n \) as follows:

\[
K_n = \left\{ \bigcup_{i=0}^{k_{n,l}} [s_i^{n,l}] \times \{ p_i^{n,l} \} : l < |K_n| \right\}.
\]

Let \( q_n > \max \{|s_i^{n,l}| : l < |K_n|, i \leq k_{n,l} \} \). By hypothesis, \( B = \bigcup_{n \in \mathbb{N}} (A_n \cap \{0, \ldots, q_n\}) \in \mathcal{I} \).

Let \( \beta \) be the characteristic function of \( B \). We claim that for all \( m \in \mathbb{N} \)

\[
(\beta, m)^+ \cap \left( \bigcup_{n \in \mathbb{N}} K_n \right) = \emptyset.
\]

Otherwise, there are \( n \in \mathbb{N} \), \( l < |K_n| \) and \( i \leq k_{n,l} \) such that \( \beta \in [s_i^{n,l}] \), that is, \( s_i^{n,l} \leq \beta \). But this contradicts the fact that \( (A_n \cap \{0, \ldots, q_n\}) \cap (s_i^{n,l})^{-1}(0) \neq \emptyset \) for all \( i \) and \( l \) (recall that \( D_n = D(A_n) \)). Thus \( (\bigcup_{n \in \mathbb{N}} K_n) \cap (\bigcup_m (\beta, m)^+) = \emptyset \). Since \( \bigcup_m (\beta, m)^+ \) is \( \rho_\mathcal{I} \)-open dense, \( \bigcup_n K_n \) is \( \rho_\mathcal{I} \)-nowhere dense. \( \square \)

**Proposition 3.9.** Let \( \mathcal{I} \) be an ideal over \( \mathbb{N} \). Any element of \( X(\mathcal{I}) \) is a limit of a non trivial sequence.

**Proof.** Since \( X(\mathcal{I}) \) is a topological group, it suffices to show that there is a sequence converging to \( \emptyset \) (i.e. to \( \varphi_0 \)).

Let \( (\alpha_n)_{n \in \mathbb{N}} \) be a sequence in \( 2^\mathbb{N} \) such that \( \alpha_k \upharpoonright (k + 1) \neq \alpha_l \upharpoonright (k + 1) \) for each \( k < l \). Let \( (x_n) \) be defined by \( x_n = [\alpha_n \upharpoonright (n + 1)] \times \{0\} \). Let \( V \) be a neighborhood of \( \emptyset \), namely, \( V = \bigcap_{i=1}^m (\beta_i, n_i)^- \) for some \( \beta_i \in \mathcal{I} \) and \( n_i \in \mathbb{N} \). We have that \( \alpha_n \upharpoonright (n + 1) \notin \beta_i \) for almost every \( n \), therefore \( x_n \in V \) and \( x_n \to \emptyset \).

\( \square \)

**Question 3.10.** When is \( X(\mathcal{I}) \) discretely generated?

3.2. The space \( c(\mathcal{I}) \). It is natural to wonder what can be said if instead of \( X \) we use the more familiar space \( CL(2^\mathbb{N}) \) of all clopen subsets of \( 2^\mathbb{N} \).

Exactly as before we can define a space \( c(\mathcal{I}) \) as follows.

**Definition 3.11.** Let \( \mathcal{I} \) be an ideal over \( \mathbb{N} \) and \( c(\mathcal{I}) \) be \( (CL(2^\mathbb{N}), \tau_\mathcal{I}) \), where \( \tau_\mathcal{I} \) is generated by the following subbasis:

\[
\alpha^+ = \{ x \in CL(2^\mathbb{N}) : \alpha \subseteq x \} \quad \text{and} \quad \alpha^- = \{ x \in CL(2^\mathbb{N}) : \alpha \nsubseteq x \},
\]

where \( \alpha \in \mathcal{I} \).

In fact, it is easy to see that \( c(\mathcal{I}) \) is homeomorphic to \( \bigcup_{i=0}^{k}[s_i] \times \{0\} \in X : k \in \mathbb{N}, s_i \in 2^{<\omega} \) and by a simple modification of the proofs above we have the following.

**Theorem 3.12.** Let \( \mathcal{I} \) be an ideal over \( \mathbb{N} \). Then \( c(\mathcal{I}) \) is \( q^+ \) at some (every) point if, and only if, \( \mathcal{I} = \text{Fin} \).

**Theorem 3.13.** Suppose that \( \mathcal{I} \) is an ideal over \( \mathbb{N} \). If \( \mathcal{I} \) is not \( p^+ \), then \( c(\mathcal{I}) \) is not \( wSS \).
3.3. The space $Y(I)$. In this section we work with the space $Y(I)$ in order to construct nodec spaces. To that end we introduce an operation $\ast$ on ideals. We remind the reader that to each $F \subseteq \mathbb{N}$ we associate the sets $F' = \{\varphi_n : n \in F\}$ and $\widehat{F} = \{\psi_n : n \in F\}$.

**Definition 3.14.** Let $I$ be a nonempty subset of $2^\mathbb{N}$. We define:

$$I^* = \{F \subseteq \mathbb{N} : F' \text{ is nowhere dense in } X(I)\}.$$  

Notice that $I^*$ is a free ideal and $I_{ad} = (2^\mathbb{N})^*$. We are going to present several results that are useful to compare $X(I)$ and $Y(I)$.

The following fact will be used several times in the sequel.

**Lemma 3.15.** Let $I$ be an ideal over $\mathbb{N}$. Let $V$ be a basic $\rho_I$-open set. Then

$$\{n \in \mathbb{N} : \varphi_n \in V\} \Delta \{n \in \mathbb{N} : \psi_n \in V\} \in I.$$  

**Proof.** Let $V$ be a nonempty basic open set, that is,

$$V = \bigcap_{i=1}^{m}(\alpha_i)\cup \bigcap_{j=1}^{l}(\beta_j)\cap (\alpha)\cap (\beta).$$

From the very definition of $\psi_n$ and viewing it as a clopen set, we have that

$$\psi_n = \varphi_n \cup \{\alpha \in 2^{2\mathbb{N}} : \alpha(n) = 1\} \times \mathbb{N}.$$  

From this we have the following:

$$\{n \in \mathbb{N} : \varphi_n \in V\} \setminus \{n \in \mathbb{N} : \psi_n \in V\} \subseteq \bigcup_{j=1}^{l} \beta_j^{-1}(1)$$

and

$$\{n \in \mathbb{N} : \psi_n \in V\} \setminus \{n \in \mathbb{N} : \varphi_n \in V\} \subseteq \bigcup_{i=1}^{m} \alpha_i^{-1}(1).$$

Thus when each $\alpha_i$ and each $\beta_j$ belongs to $I$, the unions on the right also belong to $I$. 

In the following we compare $X(I)$ and $Y(I)$ in terms of their dense and nowhere dense subsets. Some results need that the ideals $I$ and $I^*$ are comparable, i.e. $I \subseteq I^*$ or $I^* \subseteq I$, it is unclear whether this is always the case.

We are mostly interested in crowded spaces. The following fact gives a sufficient condition for $Y(I)$ to be crowded.

**Lemma 3.16.** Let $I$ be an ideal on $\mathbb{N}$. Then

1. $X$ is dense in $(2^{2^\mathbb{N} \times \mathbb{N}}, \rho_I)$.
2. $\text{int}_{X(I)}(F') = \emptyset$, for all $F \in I$ if, and only if, $Y$ is dense in $(2^{2^\mathbb{N} \times \mathbb{N}}, \rho_I)$.
3. If $I \subseteq I^*$, then $Y$ is dense in $(2^{2^\mathbb{N} \times \mathbb{N}}, \rho_I)$.
4. If $I^* \subseteq I$, then $Y$ is dense in $(2^{2^\mathbb{N} \times \mathbb{N}}, \rho_{I^*})$.

**Proof.** (1) is clear. The only if part of (2) was shown in [16, Lemma 4.2], but we include a proof for the sake of completeness. Let $V$ be a nonempty basic $\rho_I$-open set. We need to find $n$ such that $\psi_n \in V$. From Lemma 3.15 we have that

$$E = \{n \in \mathbb{N} : \varphi_n \in V \text{ and } \psi_n \not\in V\} \in I.$$
Since \( int_X(I)(E') = \emptyset \), there is \( n \) such that \( \varphi_n \in V \) and \( n \not\in E \). Therefore \( \psi_n \in V \).

For the if part, suppose that \( Y \) is dense in \((2^{2^N}, \rho_I)\) and, towards a contradiction, that there is a nonempty basic \( \rho_I \)-open set \( V \) such that \( F = \{ n \in N : \varphi_n \in V \} \) belongs to \( I \). From this and Lemma 3.15 the following set belongs to \( I \):

\[ E = F \cup \{ n \in N : \varphi_n \not\in V \text{ and } \psi_n \in V \}. \]

Let \( \beta \) be the characteristic function of \( E \). Since \( V \) is a basic open set of the form \( \{ \} \), there is \( m \) such that \( V \cap (\beta, m)^{-} \neq \emptyset \). Since \( Y \) is \( \rho_I \)-dense, there is \( n \) such that \( \psi_n \in V \cap (\beta, m)^{-} \). Hence \( \psi_n \not\in (\beta, m)^{+} \) and, by the definition of \( \psi_n \), we have that \( \beta(n) = 0 \). Therefore \( n \not\in E \) and \( \psi_n \in V \), then \( \varphi_n \in V \). Thus \( n \in F \), a contradiction.

(3) follows immediately from (2). To see (4), it suffices to show that \( int_X(I)(F') = \emptyset \), for all \( F \in I \). Let \( F \in I \). By definition, \( F' \) is nowhere dense in \( X(I) \). In particular \( int_X(I)(F') = \emptyset \), as \( I^* \subseteq I \). \( \Box \)

Now we show that the operation \( \ast \) is monotone.

**Lemma 3.17.** Let \( I \) and \( J \) be ideals over \( N \) with \( J \subseteq I \). Then

1. For every basic \( \rho_I \)-open set \( V \) of \( 2^{2^N} \) there are sets \( W, U \) such that \( V = W \cap U \), \( W \) is a \( \rho_J \)-open set and \( U \) is a basic \( \rho_I \)-open set which is also \( \rho_J \)-dense.
2. If \( A \subseteq 2^{2^N} \) is \( \rho_J \)-nowhere dense, then \( A \) is \( \rho_I \)-nowhere dense.
3. \( J^* \subseteq I^* \). Moreover, if \( J \not\subseteq I \), then \( J^* \not\subseteq I^* \).

**Proof.** (1) Let \( V \) be a basic open set, that is,

\[ V = \bigcap_{i=1}^{m} (\alpha_i, p_i) \cap \bigcap_{j=1}^{l} (\beta_j, q_j)^{-}. \]

Notice that if every \( \alpha \) and \( \beta \) belongs to \( I \setminus J \), then \( V \) is \( \rho_J \)-dense. Thus given such basic open set \( V \) where every \( \alpha \) and \( \beta \) belongs to \( I \), we can separate them and form \( W \) and \( U \) as desired: For \( W \), we use the \( \alpha \)'s and \( \beta \)'s belonging to \( J \) (put \( W = 2^{2^N} \) in case there is none in \( J \)) and for \( U \), we use the \( \alpha \)'s and \( \beta \)'s belonging to \( I \setminus J \).

(2) Let \( A \subseteq 2^{2^N} \) be a \( \rho_J \)-nowhere dense set. Let \( V \) be a basic \( \rho_I \)-open set of \( 2^{2^N} \). Then \( V = W \cap U \) where \( W \) and \( U \) are as given by part (1). As \( A \) is \( \rho_J \)-nowhere dense, there is a non empty \( \rho_J \)-open set \( W' \subseteq W \) such that \( W' \cap A = \emptyset \). Since \( W' \) is also \( \rho_I \)-open and \( U \) is \( \rho_J \)-dense, then \( U \cap W' \) is a non empty \( \rho_I \)-open set disjoint from \( A \) and contained in \( V \).

(3) Since \( X \) is dense in \( 2^{2^N} \), then \( A \in nwd(X(I)) \) if, and only if, \( A \) is nowhere dense in \( (2^{2^N}, \rho_J) \). From this and (2) we immediately get that \( J^* \subseteq I^* \). Finally, notice that from Lemma 3.17 we have that for \( A \in I \setminus J \), the set \( D(A) \) is nowhere dense in \( X(I) \) and dense in \( X(J) \). \( \Box \)

Next result gives a sufficient condition for \( \mathbb{Y}(I^*) \) to be nodec. It is a generalization of a result from [16].

**Lemma 3.18.** Let \( I \) be an ideal over \( N \) and \( F \subseteq N \).

1. If \( F \in I \), then \( \hat{F} \) is closed discrete in \( \mathbb{Y}(I) \).
2. Let \( I \) be such that \( I^* \subseteq I \). If \( \hat{F} \) is nowhere dense in \( \mathbb{Y}(I^*) \), then \( F \in I^* \).
3. If \( I^* \subseteq I \), then \( \mathbb{Y}(I^*) \) is nodec.

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From this it follows that \( \rho \) towards a contradiction, that \( \rho \) and let \( \psi \) the definition of \( \psi \). Proof. Let \( \rho \) be \([16]\). Let \( \hat{F} \) be nowhere dense in \( \mathcal{Y}(\mathcal{I}^*) \) and suppose, towards a contradiction, that \( F \notin \mathcal{I}^* \). Let \( V \) be a basic \( \rho_\mathcal{I} \)-open set such that \( F' \cap V \) is \( \rho_\mathcal{I} \)-dense in \( V \). By Lemma \( 3.17 \) there are sets \( W \) and \( U \) such that \( V = W \cap U \), \( W \) is a \( \rho_\mathcal{I}_W \)-open set, \( U \) is a basic \( \rho_\mathcal{I}_U \)-open set and \( U \) is also \( \rho_\mathcal{I}_U \)-dense. Since \( \hat{F} \) is nowhere dense in \( \mathcal{Y}(\mathcal{I}^*) \), there is a basic \( \rho_\mathcal{I}_U \)-open set \( W' \subseteq W \) such that \( \hat{F} \cap W' = \emptyset \), that is \( F \cap \{ n \in \mathbb{N} : \psi_n \in W' \} = \emptyset \).

From Lemma \( 3.16 \) we know that \( \{ n \in \mathbb{N} : \varphi_n \in W' \} \setminus \{ n \in \mathbb{N} : \psi_n \in W' \} \in \mathcal{I}^* \).

From this and the previous fact we get \( F \cap \{ n \in \mathbb{N} : \varphi_n \in W' \} \in \mathcal{I}^* \).

This says that \( F' \cap W' \) is nowhere dense in \( \mathcal{X}(\mathcal{I}) \), which is a contradiction, as by construction, \( F' \cap V \) is \( \rho_\mathcal{I} \)-dense in \( V \) and \( W' \cap U \subseteq V \) is a non empty \( \rho_\mathcal{I} \)-open set (it is non empty as \( U \) is \( \rho_\mathcal{I}_U \)-dense).

(3) follows immediately from (1) and (2). \( \square \)

The natural bijection \( \psi_n \mapsto \varphi_n \) is not continuous (neither is its inverse), however it has some form of semi-continuity as we show below.

**Proposition 3.19.** Let \( \mathcal{I} \) be an ideal over \( \mathbb{N} \). Let \( \Gamma : \mathcal{Y} \to \mathcal{X} \) given by \( \Gamma(\psi_n) = \varphi_n \). Let \( \alpha \in \mathcal{I} \) and \( p \in \mathbb{N} \). Then \( \Gamma^{-1}((\alpha,p)^+ \cap \mathcal{X}) \) is open in \( \mathcal{Y}(\mathcal{I}) \). In general, if \( V \) is a \( \rho_\mathcal{I} \)-basic open set, then there is \( D \subseteq \mathcal{Y} \) closed discrete in \( \mathcal{Y}(\mathcal{I}) \) and an \( \rho_\mathcal{I} \)-open set \( W \) such that \( \Gamma^{-1}(V \cap \mathcal{X}) = (W \cap \mathcal{Y}) \cup D \).

**Proof.** Let \( \alpha \in \mathcal{I} \) and \( p \in \mathbb{N} \). Let \( O = \{ \psi_n : \varphi_n \in (\alpha,p)^+ \} \). We need to show that \( O \) is open in \( \mathcal{Y}(\mathcal{I}) \). Let \( F = \alpha^{-1}(1) \). Since \( ((\alpha,p)^+ \cap \mathcal{Y}) \setminus \hat{F} \subseteq O \subseteq (\alpha,p)^+ \cap \mathcal{Y} \), there is \( A \subseteq F \) such that \( O = ((\alpha,p)^+ \cap \mathcal{Y}) \setminus \hat{A} \). As \( A \in \mathcal{I} \), then by Lemma \( 3.18 \) \( \hat{A} \) is closed discrete in \( \mathcal{Y}(\mathcal{I}) \). Thus \( O \) is open in \( \mathcal{Y}(\mathcal{I}) \). On the other hand, \( \{ \psi_n : \varphi_n \in (\alpha,p)^- \} = ((\alpha,p)^- \cap \mathcal{Y}) \cup \{ \psi_n : \varphi_n \in (\alpha,p)^- \} \setminus \hat{F} \). \( \square \)

The derivative operator on \( \mathcal{Y}(\mathcal{I}) \) can be characterized as follows.

**Proposition 3.20.** Let \( \mathcal{I} \) be an ideal over \( \mathbb{N} \) and \( A \subseteq \mathbb{N} \). Then \( \psi_1 \) is a \( \rho_\mathcal{I} \)-accumulation point of \( \hat{A} \) if, and only if, for every non empty \( \rho_\mathcal{I} \)-open set \( V \) with \( \psi_1 \in V \) we have \( \{ n \in A : \varphi_n \in V \} \notin \mathcal{I} \).
Proof. Let $V$ be a $\rho_I$-open set with $\psi_i \in V$. Suppose $F = \{n \in A : \varphi_n \in V\} \subseteq I$. Then by Lemma 3.18, $\hat{F}$ is closed discrete in $\mathcal{Y}(I)$ which is a contradiction as $\psi_i$ is an accumulation point of $\hat{F}$. Conversely, let $V$ be a basic $\rho_I$-open set containing $\psi_i$. By Lemma 3.16 the following set belongs to $I$:

$$E = \{n \in \mathbb{N} : \varphi_n \in V \text{ and } \psi_n \notin V\}.$$ 

We also have

$$F = \{n \in A : \varphi_n \in V\} \subseteq \{n \in A : \varphi_n \in V \text{ and } \psi_n \in V\} \cup E.$$ 

Since $E \subseteq I$ and by hypothesis $F \not\subseteq I$, then there are infinitely many $n \in A$ such that $\psi_n \in V$ and we are done. 

Now we show that the spaces $\mathcal{X}(I)$ and $\mathcal{Y}(I)$ are not homeomorphic in general.

**Proposition 3.21.** Let $I$ be a tall ideal over $\mathbb{N}$. There are no non trivial convergent sequences in $\mathcal{Y}(I)$. In particular, $\mathcal{Y}(I)$ is not homeomorphic to $\mathcal{X}(I)$.

**Proof.** Let $A \subseteq \mathbb{N}$ be an infinite set. We will show that $\hat{A} = \{\psi_n : n \in A\}$ is not convergent in $\mathcal{Y}(I)$. Since $I$ is tall, pick $B \subseteq A$ infinite with $B \in I$. Then $\hat{B}$ is closed discrete in $\mathcal{Y}(I)$ (by Lemma 3.18). Thus $\hat{A}$ is not convergent. From this, the last claim follows since $\mathcal{X}(I)$ has plenty of convergent sequences (see Proposition 3.20).

Next result shows that our spaces are analytic.

**Lemma 3.22.** Let $I$ be an analytic ideal over $\mathbb{N}$. Then $I^*$ is analytic.

**Proof.** The argument is analogous to that of the Lemma 4.8 of [16]. We include a sketch of it for the sake of completeness. First, we recall a result from [16] (see Lemma 4.7).

**Claim:** Let $J$ be an infinite set. Then $M \subseteq 2^J$ is nowhere dense if, and only if, there is $C \subseteq J$ countable such that $M \restriction C = \{x \restriction C : x \in M\}$ is nowhere dense in $2^C$.

Let $Z$ be the set of all $z \in (2^\mathbb{N} \times \mathbb{N})^\mathbb{N}$ such that $z(k) \neq z(j)$ for all $k \neq j$ and $\{z(k) : k \in \mathbb{N}\} \subseteq I \times \mathbb{N}$. Since $I$ is an analytic set, then $Z$ is an analytic subset of $(2^\mathbb{N} \times \mathbb{N})^\mathbb{N}$.

Consider the following relation $R \subseteq \mathcal{P}(\mathbb{N}) \times (2^\mathbb{N} \times \mathbb{N})^\mathbb{N}$:

$$(F, z) \in R \iff z \in Z \text{ and } \{\varphi_n \restriction \{z(k) : k \in \mathbb{N}\} : n \in F\} \text{ is nowhere dense in } 2^{\{z(k) : k \in \mathbb{N}\}}.$$ 

Then $R$ is an analytic set. From the claim above, we have

$$F \in I^* \iff (\exists z \in (2^\mathbb{N} \times \mathbb{N})^\mathbb{N}) R(F, z).$$ 

Thus, $I^*$ is analytic. 

Finally, we can show one of our main results. Let us define a sequence $(I^k)_{k \in \mathbb{N}}$ of ideals on $\mathbb{N}$ as follows:

$$I^k = \begin{cases} 2^\mathbb{N}, & \text{if } k = 0, \\ (I^{k-1})^*, & \text{if } k > 0. \end{cases}$$ 

Notice that $I^{k+1} \not\subseteq I^k$ for each $k \in \mathbb{N}$ by Lemma 3.17.

**Theorem 3.23.** For all $k > 0$, $\mathcal{Y}(I^k)$ is analytic, nodec and crowded.
Proof. That $\mathbb{Y}(\mathcal{I}^k)$ is analytic and nodec follows from Lemmas 3.22, 3.1, 3.18, and 3.17. Since $\mathcal{I}^k \subseteq \mathcal{I}_{nd}$, then by Lemma 3.16 $\mathbb{Y}(\mathcal{I}^k)$ is crowded. \hfill $\square$

Thus we do not know whether $\mathbb{Y}(\mathcal{I}^*)$ is nodec for ideals such that $\mathcal{I}^* \not\subseteq \mathcal{I}$. The reason is that it is not clear if part (2) in Lemma 3.18 holds in general without the assumption that $\mathcal{I}^* \subseteq \mathcal{I}$. In this respect, we only were able to show the following.

Lemma 3.24. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{I}^*$. Let $A \subseteq \mathbb{N}$. Then

(1) Let $V$ be a non empty $\rho_\mathcal{I}$-open set. If $A'$ is $\rho_\mathcal{I}$-dense in $V$, then $\hat{\mathcal{A}}$ is $\rho_\mathcal{I}$-dense in $V$.

(2) If $\hat{\mathcal{A}}$ is nowhere dense in $\mathbb{Y}(\mathcal{I})$, then $A'$ is nowhere dense in $\mathbb{X}(\mathcal{I})$ (i.e., $A \in \mathcal{I}^*$). In particular, if $\hat{\mathcal{A}}$ is nowhere dense in $\mathbb{Y}(\mathcal{I})$, then $\hat{\mathcal{A}}$ is closed discrete in $\mathbb{Y}(\mathcal{I})^*$.

Proof. (1) Let $V$ be a non empty $\rho_\mathcal{I}$-open set and suppose $A'$ is $\rho_\mathcal{I}$-dense in $V$. Let $W$ be a basic $\rho_\mathcal{I}$-open set with $W \subseteq V$. We need to find $n \in A$ such that $\psi_n \in W$. By Lemma 3.15 the following set belongs to $\mathcal{I}$:

$$E = \{n \in \mathbb{N} : \varphi_n \in W \text{ and } \psi_n \notin W\}.$$ 

As $\mathcal{I} \subseteq \mathcal{I}^*$, then $E'$ is nowhere dense in $\mathbb{X}(\mathcal{I})$. Since $A'$ is dense in $V$, then $A' \cap W \not\subseteq E'$. Let $n \in A \setminus E$ such that $\varphi_n \in W$. As $n \notin E$, then $\psi_n \in W$.

(2) Follows from (1) and part (1) in Lemma 3.18 \hfill $\square$

Now we compare the dense sets in $\mathbb{Y}(\mathcal{I})$ and $\mathbb{X}(\mathcal{I})$.

Lemma 3.25. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ such that $\mathbb{Y}$ is dense in $(2^{\mathbb{N}} \times \mathbb{N}, \rho_\mathcal{I})$ and $D \subseteq \mathbb{N}$. If $\hat{\mathcal{D}}$ is dense in $\mathbb{Y}(\mathcal{I})$, then $D'$ is dense in $\mathbb{X}(\mathcal{I})$.

Proof. Suppose $\hat{\mathcal{D}}$ is dense in $\mathbb{Y}(\mathcal{I})$. Let $V$ be a basic $\rho_\mathcal{I}$-open set. We need to find $n \in D$ such that $\varphi_n \in V$. By Lemma 3.15 the following set belongs to $\mathcal{I}$:

$$E = \{n \in \mathbb{N} : \varphi_n \notin V \text{ and } \psi_n \in V\}.$$ 

Let $F = \{n \in D : \psi_n \in V\}$. Since $\hat{\mathcal{D}}$ is $\rho_\mathcal{I}$-dense, then $F \not\in \mathcal{I}$ (by part (1) of Lemma 3.18 and the assumption that $\mathbb{Y}$ is dense in $(2^{\mathbb{N}} \times \mathbb{N}, \rho_\mathcal{I})$). Thus there is $n \in F \setminus E$. Then $\psi_n \in V$ and $\varphi_n \in V$. \hfill $\square$

Observe that $\text{Fin} \subseteq \text{Fin}^* \subseteq \text{Fin}^{**} \subseteq \cdots \subseteq \mathcal{I}^k$ for all $k$. Notice that $\text{Fin}^*$ is isomorphic to $\text{nwd}(\mathbb{Q})$ as $\mathbb{X}(\text{Fin})$ is homeomorphic to $\mathbb{Q}$. The following is a natural and intriguing question.

Question 3.26. Is $\mathbb{Y}(\text{Fin}^*)$ nodec?

It is unclear when an ideal $\mathcal{I}$ satisfies either $\mathcal{I} \subseteq \mathcal{I}^*$ or $\mathcal{I}^* \subseteq \mathcal{I}$. The following question asks a concrete instance of this problem.

Question 3.27. Two ideals that naturally extend $\text{Fin}$ are $\{\emptyset\} \times \text{Fin}$ and $\text{Fin} \times \{\emptyset\}$ (where $\times$ denotes the Fubini product). Let $\mathcal{I}$ be any of those two ideals. Is $\mathcal{I} \subseteq \mathcal{I}^*$?
3.4. **SS property in** $\mathbb{Y}(\mathcal{I})$. We do not know whether $\mathbb{Y}(\mathcal{I}_{nd})$ is SS. However, we show below that $\mathbb{Y}(\mathcal{I}^k)$ is not $wSS$ for all $k > 1$, this was the reason to introduce the ideals $\mathcal{I}^*$. We need an auxiliary result.

**Lemma 3.28.** Let $\mathcal{I}$ be an ideal over $\mathbb{N}$ such that $\mathcal{I}^* \subseteq \mathcal{I}$. Let $V$ be a non empty $\rho_{\mathcal{I}^*}$-open set and $D \subseteq \mathbb{N}$. If $D'$ is $\rho_{\mathcal{I}^*}$-dense in $V$, then $\hat{D}$ in $\rho_{\mathcal{I}^*}$-dense in $V$.

**Proof.** Let $V$ be a non empty $\rho_{\mathcal{I}^*}$-open set and suppose that $D'$ is $\rho_{\mathcal{I}^*}$-dense in $V$. Let $W$ be a $\rho_{\mathcal{I}^*}$-basic open set such that $W \subseteq V$. We need to show that there is $n \in D$ such that $\psi_n \in W$. By Lemma 3.15 the following set belongs to $\mathcal{I}^*$:

$$E = \{n \in \mathbb{N} : \psi_n \in W \text{ and } \psi_n \not\in W\}.$$ 

Since $W$ is also $\rho_{\mathcal{I}^*}$-open (as $\mathcal{I}^* \subseteq \mathcal{I}$) and $E'$ is $\rho_{\mathcal{I}^*}$-nowhere dense, then there is a non empty $\rho_{\mathcal{I}^*}$-open set $V_1 \subseteq W$ such that $V_1 \cap E' = \emptyset$. Since $D'$ is $\rho_{\mathcal{I}^*}$-dense in $V$, there is $n \in D$ such that $\varphi_n \in V_1$. Notice that $n \not\in E$. Since $\varphi_n \in W$, then $\psi_n \in W$. \hfill $\square$

**Theorem 3.29.** Let $\mathcal{I}$ be an ideal over $\mathbb{N}$ such that $\mathcal{I}^* \subseteq \mathcal{I}$. Then $\mathbb{Y}(\mathcal{I}^{**})$ is not $wSS$.

**Proof.** Notice that $\mathbb{X}$ is $\rho_{\mathcal{I}}$ crowded (see Lemma 3.16). Also, observe that $\mathcal{I}^{**} \subseteq \mathcal{I}^*$ (see Lemma 3.17). Let $(U_n)_{n \in \mathbb{N}}$ be a pairwise disjoint sequence of non empty $\rho_{\mathcal{I}^{**}}$-open sets. Let $A_n = \{m \in \mathbb{N} : \varphi_m \in U_n\}$. It is clear that $A_n \not\in \mathcal{I}^*$ for each $n \in \mathbb{N}$. It is easy to verify that the sequence $(A_m)_m$ witnesses that $\mathcal{I}^*$ is not $p^+$. Let $D_n = D(A_n)$, as defined in Lemma 3.7. Let

$$E_n = \{\psi_m \in Y : \varphi_m \in D_n\}.$$ 

We claim that the sequence $(E_n)_{n \in \mathbb{N}}$ witnesses that the space $\mathbb{Y}(\mathcal{I}^{**})$ is not $wSS$. In fact, since $A_n \not\in \mathcal{I}^*$, then $D_n$ is dense in $\mathbb{X}(\mathcal{I}^*)$ (by Lemma 3.11), so $E_n$ is dense in $\mathbb{Y}(\mathcal{I}^{**})$ (by Lemma 3.28). Let $K_n \subseteq E_n$ be a finite set and $L_n = \{\varphi_m : \psi_m \in K_n\}$ for each $m \in \mathbb{N}$. Since $A_n \not\in \mathcal{I}^*$ and $\mathcal{I}^*$ is not $p^+$, then, by the proof of Theorem 3.8, $L = \bigcup_{n \in \mathbb{N}} L_n$ is nowhere dense in $\mathbb{X}(\mathcal{I}^*)$. Thus $L \in \mathcal{I}^{**}$. Therefore $\hat{L} = \bigcup_{n \in \mathbb{N}} K_n$ is closed discrete in $\mathbb{Y}(\mathcal{I}^{**})$ (by Lemma 3.18). \hfill $\square$

We have seen in Theorem 3.23 that $\mathbb{Y}(\mathcal{I}^k)$ is nodec for every $k \geq 1$. From Theorem 3.29 we have the following.

**Corollary 3.30.** $\mathbb{Y}(\mathcal{I}^k)$ is not $wSS$ for every $k > 1$.

Recall that $\mathcal{I}^1$ is $\mathcal{I}_{nd}$. We do not know whether $\mathbb{Y}(\mathcal{I}_{nd})$ is SS. We only know the following. Suppose $\hat{D}_n = \{\psi_m : m \in D_n\}$ is open dense in $\mathbb{Y}(\mathcal{I}_{nd})$, for every $n \in \mathbb{N}$. Then there is $F_n \subseteq D_n$ finite for each $n$ such that $\bigcup_{n \in \mathbb{N}} F_n$ is dense.

**Question 3.31.** Is there an ideal $\mathcal{I}$ on $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{I}^*$ and $\mathbb{Y}(\mathcal{I}^*)$ is $wSS$? In particular, is $\mathbb{Y}(\text{Fin}^*)$ $wSS$?

3.5. **$q^+$ in** $\mathbb{Y}(\mathcal{I})$. We shall prove that for certain kind of ideals, $\mathbb{Y}(\mathcal{I})$ is not $q^+$. We use a construction quite similar to that in the proof of Theorem 3.6.

We recall that in the proof of Lemma 3.5 we have introduced the following property: Let $m \in \mathbb{N}$. We say that $\varphi \in \mathbb{X}$ has the property $(s^m)$ if there are $k \in \mathbb{N}$, $s_i \in 2^{m+1}$ $(i = 1, \ldots, k)$ finite sequences and $m_i \leq m$ $(i = 1, \ldots, k)$ natural numbers such that $\varphi = \bigcup_{i=1}^{k} [s_i] \times \{m_i\}$ and if $m_i = m_j$ with $i \neq j$, then $s_i \upharpoonright m \neq s_j \upharpoonright m$. 

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Lemma 3.32. Let $\mathcal{I}$ be an ideal over $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{I}_{nd}$. Let
\[
A_m = \{ \varphi \in \mathcal{X} : \varphi \text{ has the property } (\ast^m) \}
\]
and
\[
B_m = \{ \psi_n \in \mathcal{Y} : \varphi_n \in A_m \}.
\]
Let $L = \{ n \in \mathbb{N} : \varphi_n \notin \bigcup_{m \in \mathbb{N}} A_m \}$ and suppose there is an infinite set $L' = \{ m_k : k \in \mathbb{N} \} \subseteq L$ such that $L' \in \mathcal{I}$. Let
\[
B = \bigcup_k B_{m_k}.
\]
Let $q \in \mathbb{N}$ be such that $\varphi_q = 2^\mathbb{N} \times \{0\}$. Then
1. $B$ is dense in $\mathcal{Y}(\mathcal{I})$ and, in particular, $\psi_q \not\in cl_{\mathcal{X}}(B)$. 
2. Let $S \subseteq B$ be such that $S \cap B_{m_k}$ has at most one element for each $k$, then $\psi_q \not\in cl_{\mathcal{X}}(S)$.

Proof. (1) Let $A = \bigcup_k A_{m_k}$. By Lemma 3.35, $A$ is dense in $\mathcal{X}$. Thus by Lemma 3.28, $B$ is dense in $\mathcal{Y}(\mathcal{I}_{nd})$ (recall that $\mathcal{I}_{nd} = (2^\mathbb{N})^\ast$). As $\mathcal{I} \subseteq \mathcal{I}_{nd}$, then $B$ is also dense in $\mathcal{Y}(\mathcal{I})$.

(2) Let $S = \{ \psi_{n_k} : k \in \mathbb{N} \}$ be such that $\psi_{n_k} \in B_{m_k}$ for all $k \in \mathbb{N}$. We will show that $\psi_q \not\in cl_{\mathcal{X}}(S)$.

Let $\alpha \in 2^\mathbb{N}$ be defined as follows: If $0 \in L'$ and $[\{0\}] \times \{0\} \subseteq \varphi_{n_0}$, then $\alpha(0) = 1$. Otherwise, $\alpha(0) = 0$. For $n > 1$,
\[
\alpha(n) = \begin{cases} 
1, & \text{if } n \in L', n = m_k \text{ for some } k \\
\text{and } [\langle \alpha(0), \ldots, \alpha(n-1), 0 \rangle] \times \{0\} \subseteq \varphi_{n_k}, & \text{otherwise.}
\end{cases}
\]
Observe that $\alpha \in \mathcal{I}$, as $\alpha^{-1}(1) \subseteq L' \in \mathcal{I}$.

It is clear that $\psi_q \in (\alpha, 0)^+$. To finish the proof, it suffices to show that $(\alpha, 0) \notin \bigcup_{k \in \mathbb{N}} \psi_{n_k}$. Suppose, towards a contradiction, that there is $l \in \mathbb{N}$ such that $(\alpha, 0) \in \psi_{n_l}$, that is, $(\alpha, 0) \in \varphi_{n_l} \cup ([n_l] \times \mathbb{N})$. There are two cases to be considered.

(i) Suppose $\alpha(n_l) = 1$. Then $n_l \in L'$ and thus $\varphi_{n_l} \not\in A_{m_l}$ which contradicts that $\psi_{n_l} \in B_{m_l}$.

(ii) Suppose $\alpha(n_l) = 0$ and thus $(\alpha, 0) \in \varphi_{n_l}$. Let $\varphi_{n_l} = \bigcup_{i=1}^r [s_i] \times \{p_i\}$ with $s_i \in 2^{m_l+1}$. Then $\alpha \in [s]$, where $s$ is $s_i$ for some $i$ with $p_i = 0$. Hence $\alpha(n) = \alpha(n) = 0$ for all $n \leq m_l$. We consider two cases. Suppose $\alpha(m_l) = 1$. Then $s(m_l) = 1$. Let $t$ be such that $s = [t]$. Then by the definition of $\alpha$, we have that $[\hat{t}0] \times \{0\} \subseteq \varphi_{n_1}$. But also $[s] \times \{0\} = [\hat{t}1] \times \{0\} \subseteq \varphi_{n_l}$ which contradicts that $\varphi_{n_l} \in A_{m_l}$ (i.e. that it has property ($\ast^{m_l}$)). Now suppose that $\alpha(m_l) = 0$. Then $[s] \times \{0\} = [\hat{t}0] \times \{0\} \not\subseteq \varphi_{n_1}$, but this contradicts that $[s] \times \{0\}$ is $[s_i] \times \{p_i\}$ for some $i$. 

From the previous lemma we immediately get the following.

Theorem 3.33. Let $\mathcal{I}$ be a tall ideal over $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{I}_{nd}$. Then $\mathcal{Y}(\mathcal{I})$ is not $q^+$. 

Question 3.34. Is there an ideal (necessarily non tall) different from $\text{Fin}$ such that $\mathcal{Y}(\mathcal{I})$ is $q^+$? Two natural candidates are $\{0\} \times \text{Fin}$ and $\text{Fin} \times \{0\}$.

Finally, we have the following.

Theorem 3.35. $\mathcal{Y}(\mathcal{I}^k)$ is a non SS, non $q^+$ nodec regular space with analytic topology for every $k > 1$. 

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