Finitely-additive measures on the asymptotic foliations of a Markov compactum.

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1 Introduction.

1.1 Hölder cocycles over translation flows.

Let \( \rho \geq 2 \) be an integer, let \( M \) be a compact orientable surface of genus \( \rho \), and let \( \omega \) be a holomorphic one-form on \( M \). Denote by \( m = (\omega \wedge \bar{\omega})/2i \) the area form induced by \( \omega \) and assume that \( m(M) = 1 \).

Let \( h^+_t \) be the vertical flow on \( M \) (i.e., the flow corresponding to \( \Re(\omega) \)); let \( h^-_t \) be the horizontal flow on \( M \) (i.e., the flow corresponding to \( \Im(\omega) \)). The flows \( h^+_t, h^-_t \) preserve the area \( m \) and are uniquely ergodic.

Take \( x \in M \), \( t_1, t_2 \in \mathbb{R}_+ \) and assume that the closure of the set \( \{h^+_t h^-_s x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\} \) does not contain zeros of the form \( \omega \). Then the set \( \mathbb{P} \) is called an admissible rectangle and denoted \( \Pi(x, t_1, t_2) \). Let \( \mathcal{C} \) be the semi-ring of admissible rectangles.

Consider the linear space \( \mathcal{Y}^+ \) of Hölder cocycles \( \Phi^+(x, t) \) over the vertical flow \( h^+_t \) which are invariant under horizontal holonomy. More precisely, a function \( \Phi^+(x, t) : M \times \mathbb{R} \to \mathbb{C} \) belongs to the space \( \mathcal{Y}^+ \) if it satisfies:

1. \( \Phi^+(x, t + s) = \Phi^+(x, t) + \Phi^+(h^+_s x, s) \);

2. There exists \( t_0 > 0, \theta > 0 \) such that \( |\Phi^+(x, t)| \leq t^\theta \) for all \( x \in M \) and all \( t \in \mathbb{R} \) satisfying \( |t| < t_0 \);

3. If \( \Pi(x, t_1, t_2) \) is an admissible rectangle, then \( \Phi^+(x, t_1) = \Phi^+(h^+_x, t_1) \).

For example, if a cocycle \( \Phi_1^+ \) is defined by \( \Phi_1^+(x, t) = t \), then clearly \( \Phi_1^+ \in \mathcal{Y}^+ \).

In the same way define the space of \( \mathcal{Y}^- \) of Hölder cocycles \( \Phi^-(x, t) \) over the horizontal flow \( h^-_t \) which are invariant under vertical holonomy, and set \( \Phi_1^- (x, t) = t \).

Given \( \Phi^+ \in \mathcal{Y}^+, \Phi^- \in \mathcal{Y}^- \), a finitely additive measure \( \Phi^+ \times \Phi^- \) on the semi-ring \( \mathcal{C} \) of admissible rectangles is introduced by the formula

\[
\Phi^+ \times \Phi^- (\Pi(x, t_1, t_2)) = \Phi^+(x, t_1) \cdot \Phi^-(x, t_2) \tag{2}
\]
In particular, for $\Phi^- \in \mathcal{Y}^-$, set $m_{\Phi^-} = \Phi_1^+ \times \Phi^-:

\begin{equation}
\Phi_1^+ \times \Phi^-(x, t_2) = m_{\Phi^-}(\Pi(x, t_1, t_2)).
\end{equation}

For any $\Phi^- \in \mathcal{Y}^-$ the measure $m_{\Phi^-}$ satisfies $(h_t^+)_* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni \cite{Forni2005}, \cite{Forni2007}. For instance, $m_{\Phi_0^+} = \mu$.

A $C$-linear pairing between $\mathcal{Y}^+$ and $\mathcal{Y}^-$ is given, for $\Phi^+ \in \mathcal{Y}^+, \Phi^- \in \mathcal{Y}^-$, by the formula

\begin{equation}
\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(M)
\end{equation}

The space of Lipschitz functions is not invariant under $h_t^+$, and a larger function space $\text{Lip}^+_w(M, \omega)$ of weakly Lipschitz functions is introduced as follows. A bounded measurable function $f$ belongs to $\text{Lip}^+_w(M, \omega)$ if there exists a constant $C$, depending only on $f$, such that for any admissible rectangle $\Pi(x, t_1, t_2)$ we have

\begin{equation}
|\int_0^{t_1} f(h_t^+ x) dt - \int_0^{t_1} f(h_t^+ x) dt| \leq C.
\end{equation}

Let $C_f$ be the infimum of all $C$ satisfying (5). We norm $\text{Lip}^+_w(X)$ by setting

\begin{equation}
\|f\|_{\text{Lip}^+_w} = \sup_X f + C_f.
\end{equation}

By definition, the space $\text{Lip}^+_w(M, \omega)$ contains all Lipschitz functions on $M$ and is invariant under $h_t^+$. We denote by $\text{Lip}^+_w,\omega(M, \omega)$ the subspace of $\text{Lip}^+_w(M, \omega)$ of functions whose integral with respect to $\mu$ is 0.

1.2 Flows along the stable foliation of a pseudo-Anosov diffeomorphism.

Assume that $\theta_1 > 0$ and a diffeomorphism $g : M \to M$ are such that

\begin{equation}
g^* (\mathcal{R}(\omega)) = \exp(\theta_1)\mathcal{R}(\omega); \quad g^* (\mathcal{S}(\omega)) = \exp(-\theta_1)\mathcal{S}(\omega).
\end{equation}

The diffeomorphism $g$ induces a linear automorphism $g^*$ of the cohomology space $H^1(M, \mathbb{C})$. Denote by $E^+$ the expanding subspace of $g^*$ (in other words, $E^+$ is the subspace spanned by vectors corresponding to Jordan cells of $g^*$ with eigenvalues exceeding 1 in absolute value). The action of $g$ on $\mathcal{Y}^+$ is given by $g^* \Phi^+(x, t) = \Phi^+(gx, \exp(\theta_1)t)$.

Proposition 1 There exists a $g^*$-equivariant isomorphism between $E^+$ and $\mathcal{Y}^+$.

Theorem 1 There exists a continuous mapping $\Xi^+ : \text{Lip}^+_w(M, \omega) \to \mathcal{Y}^+$ such that for any $f \in \text{Lip}^+_w(M, \omega)$, any $x \in X$ and any $T > 0$ we have

\begin{equation}
|\int_0^T f \circ h_t^+(x) dt - \Xi^+(f)(x, T)| < C_{\varepsilon} \|f\|_{\text{Lip}^+_w} (1 + \log(1 + T))^{2\rho+1}.
\end{equation}

The mapping $\Xi^+$ satisfies $\Xi^+(f \circ h_t^+) = \Xi^+(f)$ and $\Xi^+(f \circ g) = g^* \Xi^+(f)$.
The mapping $\Xi^+$ is constructed as follows. By Proposition 1 applied to the flow $h_t^+$, there exists a $g$-equivariant isomorphism between $\mathcal{Y}^+$ and the contracting space for the action of $g^*$ on $H^1(M, \mathbb{C})$ (in other words, the subspace spanned by vectors corresponding to Jordan cells with eigenvalues strictly less than 1 in absolute value).

**Proposition 2** The pairing $\langle \cdot, \cdot \rangle$ given by (4) is nondegenerate and $g^*$-invariant.

**Remark.** Under the identification of $\mathcal{Y}^+$ and $\mathcal{Y}^-$ with respective subspaces of $H^1(M, \mathbb{C})$, the pairing $\langle \cdot, \cdot \rangle$ is taken to the cup-product on $H^1(M, \mathbb{C})$ (see Proposition 4.19 in Veech [14]).

If $f \in \text{Lip}_+^w(M, \omega)$, then $f$ is Riemann-integrable with respect to $m_{\Phi^-}$ for any $\Phi^- \in \mathcal{Y}^-$ (see (30) for a precise definition of the integral). Assign to $f$ a cocycle $\Phi^+_f$ in such a way that for all $\Phi^- \in \mathcal{Y}^-$ we have

$$\langle \Phi^+_f, \Phi^- \rangle = \int_M f \, dm_{\Phi^-}. \quad (7)$$

By definition, $\Phi^+_f \circ h_t^+ = \Phi^+_f$. The mapping $\Xi^+$ of Theorem 1 is given by the formula

$$\Xi^+(f) = \Phi^+_f. \quad (8)$$

The first eigenvalue for the action of $g^*$ on $E^+$ is $\exp(\theta_1)$ and is always simple. If its second eigenvalue has the form $\exp(\theta_2)$, where $\theta_2 > 0$, and is simple as well, then the following limit theorem holds for $h_t^+$.

**Theorem 2** If $g^*|_{E^+}$ has a simple, real second eigenvalue $\exp(\theta_2)$, $\theta_2 > 0$, then there exists a continuous functional $\alpha : \text{Lip}_+^w(M, \omega) \to \mathbb{R}$ and a compactly supported non-degenerate measure $\eta$ on $C[0, 1]$ such that for any $f \in \text{Lip}_+^w(M, \omega)$ satisfying $\alpha(f) \neq 0$ the sequence of random variables

$$\frac{\mathcal{S}_n[f, x]}{\alpha(f) \exp(n\theta_2)}$$

converges in distribution to $\eta$ as $n \to \infty$.

The functional $\alpha$ is constructed explicitly as follows. Under the assumptions of the theorem the action of $g^*$ on $E^-$ has a simple eigenvalue $\exp(-\theta_2)$; let $v(2)$ be the eigenvector with eigenvalue $\exp(-\theta_2)$, let $\Phi^-_2 \in \mathcal{Y}^-$ correspond to $v(2)$ by Proposition 1 and $m_{\Phi^-_2}$ be given by (3); then

$$\alpha(f) = \int f \, dm_{\Phi^-_2}.$$
1.3 Generic translation flows.

Let \( \rho \geq 2 \) and let \( \kappa = (\kappa_1, \ldots, \kappa_\sigma) \) be a nonnegative integer vector such that \( \kappa_1 + \cdots + \kappa_\sigma = 2\rho - 2 \). Denote by \( \mathcal{M}_\rho \) the moduli space of Riemann surfaces of genus \( \rho \) endowed with a holomorphic differential of area 1 with singularities of orders \( k_1, \ldots, k_\sigma \) (the stratum in the moduli space of holomorphic differentials), and let \( \mathcal{H} \) be a connected component of \( \mathcal{M}_\rho \). Denote by \( g_t \) the Teichmüller flow on \( \mathcal{H} \) (see [6, 8]), and let \( \Lambda(t, X) \) be the Kontsevich-Zorich cocycle over \( g_t \) [8].

Let \( \mathbb{P} \) be a \( g_t \)-invariant ergodic probability measure on \( \mathcal{H} \). For \( X \in \mathcal{H} \), \( X = (M, \omega) \), let \( Y^+_X, Y^-_X \) be the corresponding spaces of Hölder cocycles. Denote by \( E^+_X \) the space spanned by the positive Lyapunov exponents of the Kontsevich-Zorich cocycle.

**Proposition 3** For \( \mathbb{P} \)-almost all \( X \in \mathcal{H} \), we have
\[
\dim Y^+_X = \dim Y^-_X = \dim E^+_X,
\]
and the pairing \( <, > \) between \( Y^+_X \) and \( Y^-_X \) is non-degenerate.

**Remark.** In particular, if \( \mathbb{P} \) is the Masur-Veech “smooth” measure [10, 12], then \( \dim Y^+_X = \dim Y^-_X = \rho \).

Assign to \( f \in \text{Lip}^+_w(M, \omega) \) a cocycle \( \Phi^+_f \) by (7).

**Theorem 3** For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) depending only on \( \mathbb{P} \) such that for \( \mathbb{P} \)-almost every \( X \in \mathcal{H} \), any \( f \in \text{Lip}^+_w(X) \), any \( x \in X \) and any \( T > 0 \) we have
\[
| \int_0^T f \circ h^+_t(x)dt - \Phi^+_f(x, T) | < C_\varepsilon \| f \|_{\text{Lip}^+_w}(1 + T^\varepsilon).
\]

If both the first and the second Lyapunov exponent of the measure \( \mathbb{P} \) are positive and simple (as, by the Avila-Viana Theorem [2], is the case with the Masur-Veech “smooth” measure on \( \mathcal{H} \)), then the following limit theorem holds.

As before, consider a \( C[0,1] \)-valued random variable \( S_t[f, x] \) on \( (M, m) \) defined by the formula
\[
S_t[f, x](\tau) = \int_0^\tau \exp(s) f \circ h^+_t(x)dt.
\]

Let \( ||v|| \) be the Hodge norm in \( H^1(M, \mathbb{R}) \). Let \( \theta_2 > 0 \) be the second Lyapunov exponent of the Kontsevich-Zorich cocycle and let \( v_2(X) \) be a Lyapunov vector corresponding to \( \theta_2 \) (by our assumption, such a vector is unique up to scalar multiplication). Introduce a real-valued multiplicative cocycle \( H_2(t, X) \) over \( g_t \) by the formula
\[
H_2(t, X) = \frac{||A(t, X)v_2(X)||}{||v_2(X)||}.
\]

**Theorem 4** Assume that both the first and the second Lyapunov exponent of the Kontsevich-Zorich cocycle with respect to the measure \( \mathbb{P} \) are positive and simple. Then for \( \mathbb{P} \)-almost any \( X' \in \mathcal{H} \) there exists a non-degenerate compactly supported measure \( \eta_{X'} \) on \( C[0,1] \) and, for \( \mathbb{P} \)-almost all \( X, X' \in \mathcal{H} \), there exists a
sequence of moments \( s_n = s_n(X, X') \) such that the following holds. For \( \mathbb{P} \)-almost every \( X \in \mathcal{H} \) there exists a continuous functional

\[
a^{(X)} : \text{Lip}^+_w(X) \to \mathbb{R}
\]
such that for \( \mathbb{P} \)-almost every \( X' \) and for any real-valued \( f \in \text{Lip}^+_w(X) \) satisfying \( a^{(X)}(f) \neq 0 \), the sequence of \( C[0,1] \)-valued random variables

\[
\mathcal{G}_{s_n}[f, x](\tau) \frac{a^{(f)}(X)}{(a^{(X)}(f)) H_2(s_n, X)}
\]
converges in distribution to \( \eta_{X'} \), as \( n \to \infty \).

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2 Asymptotic foliations of a Markov compactum.

2.1 Definitions and notation.

Let \( m \in \mathbb{N} \) and let \( \Gamma \) be an oriented graph with \( m \) vertices \( \{1, \ldots, m\} \) and possibly multiple edges. We assume that that for each vertex there is an edge starting from it and an edge ending in it.

Let \( \mathcal{E}(\Gamma) \) be the set of edges of \( \Gamma \). For \( e \in \mathcal{E}(\Gamma) \) we denote by \( I(e) \) its initial vertex and by \( F(e) \) its terminal vertex. Let \( Q \) be the incidence matrix of \( \Gamma \) defined by the formula

\[
Q_{ij} = \#\{ e \in \mathcal{E}(\Gamma) : I(e) = i, F(e) = j \}.
\]

By assumption, all entries of the matrix \( Q \) are positive. A finite word \( e_1 \ldots e_k, \ e_i \in \mathcal{E}(\Gamma), \) will be called admissible if \( F(e_{i+1}) = I(e_i), \ i = 1, \ldots, k. \)

To the graph \( \Gamma \) we assign a Markov compactum \( X_\Gamma \), the space of bi-infinite paths along the edges:

\[
X_\Gamma = \{ x = \ldots x_{-n} \ldots x_0 \ldots x_n \ldots, \ x_n \in \mathcal{E}(\Gamma), F(x_{n+1}) = I(x_n) \}.
\]

**Remark.** As \( \Gamma \) will be fixed throughout this section, we shall often omit the subscript \( \Gamma \) from notation and only insert it when the dependence on \( \Gamma \) is underlined.
Cylinders in $X^1$ are subsets of the form $\{x : x_{n+1} = e_1, \ldots, x_{n+k} = e_k\}$, where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $e_1 \ldots e_k$ is an admissible word. The family of all cylinders forms a semi-ring which we denote by $\mathcal{C}$.

For $x \in X$, $n \in \mathbb{Z}$, introduce the sets

$$\gamma^+_n(x) = \{x' \in X^1 : x'_t = x_t, t \geq n\}; \quad \gamma^-_n(x) = \{x' \in X^1 : x'_t = x_t, t \leq n\};$$

$$\gamma^+_\infty(x) = \bigcup_{n \in \mathbb{Z}} \gamma^+_n(x); \quad \gamma^-_\infty(x) = \bigcup_{n \in \mathbb{Z}} \gamma^-_n(x).$$

The sets $\gamma^+_\infty(x)$ are leaves of the asymptotic foliation $\mathcal{F}^+$ on the space $X^1$; the sets $\gamma^-_\infty(x)$ are leaves of the asymptotic foliation $\mathcal{F}^-$ on $X^1$.

For $n \in \mathbb{Z}$ let $\mathcal{C}^+_n$ be the collection of all subsets of $X^1$ of the form $\gamma^+_n(x)$, $n \in \mathbb{Z}$, $x \in X$; similarly, $\mathcal{C}^-_n$ is the collection of all subsets of the form $\gamma^-_n(x)$. Set

$$\mathcal{C}^+_n = \bigcup_{n \in \mathbb{Z}} \mathcal{C}^+_n; \quad \mathcal{C}^-_n = \bigcup_{n \in \mathbb{Z}} \mathcal{C}^-_n. \quad (11)$$

The collection $\mathcal{C}^+_n$ is a semi-ring for any $n \in \mathbb{Z}$. Since every element of $\mathcal{C}^+_n$ is a disjoint union of elements of $\mathcal{C}^+_{n+1}$, the collection $\mathcal{C}^+$ is a semi-ring as well. The same statements hold for $\mathcal{C}^-_n$ and $\mathcal{C}^-$.

Let $\exp(\theta_1)$ be the spectral radius of the matrix $Q$, and let $h = (h_1, \ldots, h_m)$ be the unique positive eigenvector of $Q$: we thus have $Q h = \exp(\theta_1) h$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be the positive eigenvector of the transpose matrix $Q^t$: we have $Q^t \lambda = \exp(\theta_1) \lambda$. The vectors $\lambda, h$ are normalized as follows:

$$\sum_{i=1}^m \lambda_i = 1; \quad \sum_{i=1}^m \lambda_i h_i = 1. \quad (12)$$

Introduce a sigma-additive positive measure $\Phi^+_1$ on the semi-ring $\mathcal{C}^+_1$ by the formula

$$\Phi^+_1(\gamma^+_n(x)) = h_{F(x_n)} \exp((n-1)\theta_1) \quad (13)$$

and a sigma-additive positive measure $\Phi^-_1$ on the semi-ring $\mathcal{C}^-_1$ by the formula

$$\Phi^-_1(\gamma^-_n(x)) = \lambda_{I(x_n)} \exp(-n\theta_1). \quad (14)$$

Let $n \in \mathbb{Z}$, $k \in \mathbb{N}$, and let $e_1 \ldots e_k$ be an admissible word. The Parry measure $\nu$ on $X^1$ is defined by the formula

$$\nu(\{x : x_{n+1} = e_1, \ldots, x_{n+k} = e_k\}) = \lambda_{I(e_k)} h_{F(e_1)} \exp(-k\theta_1). \quad (15)$$

The measures $\Phi^+_1$, $\Phi^-_1$ are conditional measures of the Parry measure $\nu$ in the following sense. If $C \in \mathcal{C}$, then $\gamma^+_\infty(x) \cap C \in \mathcal{C}^+$, $\gamma^-_\infty(x) \cap C \in \mathcal{C}^-$ for any $x \in C$, and we have

$$\nu(C) = \Phi^+_1(\gamma^+_\infty(x) \cap C) \cdot \Phi^-_1(\gamma^-_\infty(x) \cap C). \quad (16)$$
2.2 Finitely-additive measures on leaves of asymptotic foliations.

Given $v \in \mathbb{C}^m$, write
\[
|v| = \sum_{i=1}^{m} |v_i|.
\]

(17)

The norms of all matrices in this paper are understood with respect to this norm. Consider the direct-sum decomposition

\[
\mathbb{C}^m = E^+ \oplus E^-,
\]

where $E^+$ is spanned by Jordan cells of eigenvalues of $Q$ with absolute value exceeding 1, and $E^-$ is spanned by Jordan cells corresponding to eigenvalues of $Q$ with absolute value at most 1. Let $v \in E^+$ and for all $n \in \mathbb{Z}$ set $v^{(n)} = Q^n v$ (note that $Q|_{E^+}$ is by definition invertible). Introduce a finitely-additive complex-valued measure $\Phi^+_v$ on the semi-ring $\mathcal{E}^+$ (defined in (11)) by the formula

\[
\Phi^+_v(\gamma_{n+1}(x)) = (v^{(n)})_{F(x_{n+1})}.
\]

(18)

The measure $\Phi^+_v$ is invariant under holonomy along $F^-$: by definition, we have the following

**Proposition 4** If $I(x_n) = I(x'_n)$, then $\Phi^+_v(\gamma_{n+1}(x)) = \Phi^+_v(\gamma_{n+1}(x'))$.

The measures $\Phi^+_v$ span a complex linear space, which we denote $\mathcal{Y}^+$ (or, sometimes, $\mathcal{Y}^+_1$, when dependence on $\Gamma$ is stressed.) The map

\[
\mathcal{I} : v \rightarrow \Phi^+_v
\]

(19)

is an isomorphism between $E^+$ and $\mathcal{Y}^+_1$.

For $Q^t$, we have the direct-sum decomposition

\[
\mathbb{C}^m = \tilde{E}^+ \oplus \tilde{E}^-,
\]

where $\tilde{E}^+$ is spanned by Jordan cells of eigenvalues of $Q^t$ with absolute value exceeding 1, and $\tilde{E}^-$ is spanned by Jordan cells corresponding to eigenvalues of $Q^t$ with absolute value at most 1. As before, for $\tilde{v} \in \tilde{E}^+$ set $\tilde{v}^{(n)} = (Q^t)^n \tilde{v}$ for all $n \in \mathbb{Z}$, and introduce a finitely-additive complex-valued measure $\Phi^-_{\tilde{v}}$ on the semi-ring $\mathcal{E}^-$ (defined in (11)) by the formula

\[
\Phi^-_{\tilde{v}}(\gamma_{n}(x)) = (\tilde{v}^{(-n)})_{I(x_n)}.
\]

(20)

By definition, the measure $\Phi^-_{\tilde{v}}$ is invariant under holonomy along $F^+$: more precisely, we have the following

**Proposition 5** If $I(x_n) = I(x'_n)$, then $\Phi^-_{\tilde{v}}(\gamma_{n}(x)) = \Phi^-_{\tilde{v}}(\gamma_{n}(x'))$. 

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Let $Y^-_\Gamma$ be the space spanned by the measures $\Phi^-_v$, $v \in \tilde{E}^+$. The map

$$\tilde{I} : v \mapsto \Phi^-_v$$

(21)
is an isomorphism between $\tilde{E}^+$ and $Y^-_\Gamma$.

Let $\sigma : X_\Gamma \to X_\Gamma$ be the shift defined by $(\sigma x)_i = x_{i+1}$. The shift $\sigma$ naturally acts on the spaces $Y^+_\Gamma$, $Y^-_\Gamma$: given $\Phi \in Y^+_\Gamma$ (or $Y^-_\Gamma$), the measure $\sigma_* \Phi$ is defined, for $\gamma \in C^+$, by the formula

$$\sigma_* \Phi(\gamma) = \Phi(\sigma^\gamma).$$

From the definitions we obtain

**Proposition 6** The following diagrams are commutative:

$$\begin{align*}
E^+ & \xrightarrow{I} Y^+_\Gamma & E^+ & \xrightarrow{I} \tilde{Y}^+_\Gamma \\
\downarrow Q & & \downarrow \tilde{Q} & \\
E^+ & \xrightarrow{\sigma} Y^+_\Gamma & \tilde{E}^+ & \xrightarrow{\sigma} \tilde{Y}^+_\Gamma \\
\tilde{E}^+ & \xrightarrow{\tilde{\sigma}} \tilde{Y}^+_\Gamma & \tilde{E}^+ & \xrightarrow{\tilde{\sigma}} \tilde{Y}^-_\Gamma
\end{align*}$$

**2.3 Pairings.**

Given $\Phi^+ \in Y^+_\Gamma$, $\Phi^- \in Y^-_\Gamma$, introduce, in analogy with (16), a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $C$ of cylinders in $X_\Gamma$: for any $C \in C$ and $x \in C$, set

$$\Phi^+ \times \Phi^- (C) = \Phi^+ (\gamma^+_\infty (x) \cap C) \cdot \Phi^- (\gamma^-_\infty (x) \cap C).$$

(22)

Note that by Propositions 4, 5, the right-hand side in (22) does not depend on $x \in C$.

More explicitly, let $v \in E^+$, $\tilde{v} \in \tilde{E}^+$, $\Phi^+_v = I(v)$, $\Phi^-_\tilde{v} = \tilde{I}(\tilde{v})$. As above, denote $v^{(n)} = Q^n v$, $\tilde{v}^{(n)} = (Q^t)^n \tilde{v}$. Let $n \in Z$, $k \in N$ and let $e_1 \ldots e_k$ be an admissible word. Then

$$\Phi^+_v \times \Phi^-_\tilde{v} (\{ x : x_{n+1} = e_1, \ldots, x_{n+k} = e_k \}) = \left( v^{(n)}_{(e_1)} \right) F(e_1) \left( \tilde{v}^{(-n-k)}_{(e_n+k)} \right).$$

(23)

There is a natural $C$-linear pairing $<,>$ between the spaces $Y^+_\Gamma$ and $Y^-_\Gamma$: for $\Phi^+ \in Y^+_\Gamma$, $\Phi^- \in Y^-_\Gamma$, set

$$< \Phi^+, \Phi^- >= \Phi^+ \times \Phi^- (X_\Gamma).$$

(24)

From (23) we derive
Proposition 7 Let \( v \in E^+ \), \( \tilde{v} \in \tilde{E}^+ \), \( \Phi_v^+ = \mathcal{I}_\Gamma(v) \), \( \Phi_{\tilde{v}}^- = \tilde{\mathcal{I}}_\Gamma(\tilde{v}) \). Then

\[
< \Phi_v^+, \Phi_{\tilde{v}}^- > = \sum_{i=1}^m v_i \tilde{v}_i.
\]

(25)

In particular, the pairing \(<,>\) is non-degenerate and \(\sigma^*-\)invariant.

In particular, for \( \Phi^- \in \mathcal{Y}^- \) denote

\[
m_{\Phi^-} = \Phi_1^+ \times \Phi^-.
\]

(26)

2.4 Weakly Lipschitz Functions.

Introduce a function space \( \text{Lip}^+_w(X) \) in the following way. A bounded Borel-measurable function \( f : X \to \mathbb{C} \) belongs to the space \( \text{Lip}^+_w(X) \) if there exists a constant \( C > 0 \) such that for all \( n \geq 0 \) and any \( x, x' \in X \) satisfying \( F(x_{n+1}) = F(x'_{n+1}) \), we have

\[
\left| \int_{\gamma_n^+(x)} f \, d\Phi_1^+ - \int_{\gamma_n^+(x')} f \, d\Phi_1^+ \right| \leq C.
\]

(27)

If \( C_f \) be the infimum of all \( C \) satisfying (27), then we norm \( \text{Lip}^+_w(X) \) by setting

\[
\|f\|_{\text{Lip}^+_w} = \sup_X f + C_f.
\]

As before, let \( \text{Lip}^+_w,0(X) \) be the subspace of \( \text{Lip}^+_w(X) \) of functions whose integral with respect to \( \nu \) is zero.

Take \( \Phi^- \in \mathcal{Y}^- \). Any function \( f \in \text{Lip}^+_w(X) \) is integrable with respect to the measure \( m_{\Phi^-} \), defined by (20), in the following sense. Let \( \tilde{v} \in E^- \) be the vector corresponding to \( \Phi^- \) by (20) and let \( \tilde{v}^{(n)} = (Q^n)\tilde{v} \). Recall that

\[
|\tilde{v}^{(-n)}| \to 0 \text{ exponentially fast as } n \to \infty.
\]

(28)

Take arbitrary points \( x_i^{(n)} \in X, n \in \mathbb{N} \) satisfying

\[
F((x_i^{(n)})_n) = i, \ i = 1, \ldots, m.
\]

(29)

and consider the expression

\[
\sum_{i=1}^m \left( \int_{\gamma_n^+(x_i^{(n)})} f \, d\Phi_1^+ \right) \cdot (\tilde{v}^{(1-n)})_i.
\]

(30)

By (27) and (28), as \( n \to \infty \) the expression (30) tends to a limit which does not depend on the particular choice of \( x_i^{(n)} \) satisfying (29). This limit is denoted

\[
m_{\Phi^-}(f) = \int_X f \, dm_{\Phi^-}.
\]
Introduce a measure $\Phi^+_f \in \mathcal{Y}^+$ by requiring that for any $\Phi^- \in \mathcal{Y}^-$ we have
\[ < \Phi^+_f, \Phi^- > = \int_X f dm_{\Phi^-}. \tag{31} \]

Note that the mapping $\Xi^+: \text{Lip}^+_w(X) \to \mathcal{Y}^+$ given by $\Xi^+(f) = \Phi^+_f$ is continuous by definition and satisfies
\[ \Xi^+(f \circ \sigma) = \sigma^* \Xi^+(f). \tag{32} \]

From the definitions we also have

**Proposition 8** Let $\Phi^+_1, \ldots, \Phi^+_r$ be a basis in $\mathcal{Y}^+$ and let $\Phi^-_1, \ldots, \Phi^-_r$ be the dual basis in $\mathcal{Y}^-$ with respect to the pairing $<,>$. Then for any $f \in \text{Lip}^+_w(X)$ we have
\[ \Phi^+_f = \sum_{i=1}^r (m_{\Phi^-_i}(f)) \Phi^+_i. \]

### 2.5 Approximation

Let $\Theta$ be a finitely-additive complex-valued measure on the semi-ring $\mathcal{C}^+_\Gamma$. Assume that there exists a constant $\delta(\Theta)$ such that for all $x, x' \in X$ and all $n \geq 0$ we have
\[ |\Theta(\gamma^+_n(x)) - \Theta(\gamma^+_n(x'))| \leq \delta(\Theta) \text{ if } F(x_{n+1}) = F(x'_{n+1}). \tag{33} \]

In this case $\Theta$ will be called a *weakly Lipschitz measure*.

**Lemma 1** There exists a constant $C_\Gamma$ depending only on $\Gamma$ such that the following is true. Let $\Theta$ be a weakly Lipschitz finitely-additive complex-valued measure on the semi-ring $\mathcal{C}^+_\Gamma$. Then there exists a unique $\Phi^+ \in \mathcal{Y}^+_\Gamma$ such that for all $x \in X$ and all $n > 0$ we have
\[ |\Theta(\gamma^+_n(x)) - \Phi^+(\gamma^+_n(x))| \leq C_\Gamma \delta(\Theta)n^{m+1}. \tag{34} \]

Assign to the graph $\Gamma$ the Markov compactum $Y_\Gamma$ of one-sided infinite sequences of edges:
\[ Y = \{ y = y_1 \ldots y_n \cdots : y_n \in \mathcal{E}(\Gamma), F(y_{n+1}) = I(y_n) \}, \]
and, as before, let $\sigma$ be the shift on $Y_\Gamma$: $(\sigma y)_i = y_{i+1}$. For $y, y' \in Y_\Gamma$, write $y' \searrow y$ if $\sigma y' = y$.

**Lemma 2** There exists a constant $C_\Gamma$ depending only on $\Gamma$ such that the following is true. Let $\varphi_n$ be a sequence of measurable complex-valued functions on $Y_\Gamma$. Assume that there exists a constant $\delta$ such that for all $y \in Y$ and all $n \geq 0$ we have
\[ |\varphi_{n+1}(y) - \sum_{y' \searrow y} \varphi_n(y')| \leq \delta \tag{35} \]
and for all \( n \geq 0 \) and all \( y, \tilde{y} \in Y \Gamma \) satisfying \( F(y_1) = F(\tilde{y}_1) \) we have

\[
|\varphi_n(y) - \varphi_n(\tilde{y})| \leq \delta. \tag{36}
\]

Then there exists a unique \( v \in E^+ \) such that for all \( y \in Y \) and all \( n > 0 \) we have

\[
|\varphi_n(y) - (Q^n v)_{F(y_{n+1})}| \leq C_T \delta n^{m+1}. \tag{37}
\]

Proof of Lemma 2. Take arbitrary points \( y(i) \in Y \Gamma \) in such a way that

\[
F(y(i)_1) = i.
\]

Introduce a sequence of vectors \( v(n) \in \mathbb{C}^m \) by the formula

\[
v(n)_i = \varphi_n(y(i)).
\]

From (36) for any \( y \in Y \) we have

\[
|\varphi_n(y) - v(n)_{F(y_1)}| \leq \delta,
\]

and from (35), (36) we have

\[
|Qv(n) - v(n + 1)| \leq \delta \cdot ||Q||.
\]

To prove Lemma 2, it suffices now to establish the following.

**Proposition 9** Let \( V \) be a finite-dimensional complex linear space, let \( S : V \to V \) be a linear operator and let \( V^+ \subset V \) be the subspace spanned by vectors corresponding to Jordan cells of \( S \) with eigenvalues exceeding 1 in absolute value.

There exists a constant \( C > 0 \) depending only on \( S \) such that the following is true. Assume that the vectors \( v(n) \in V, n \in \mathbb{N}, \) satisfy

\[
|Sv(n) - v(n + 1)| < \delta
\]

for all \( n \in \mathbb{N} \) and some constant \( \delta > 0 \). Then there exists a unique \( v \in V^+ \) such that for all \( n \in \mathbb{N} \) we have

\[
|S^n v - v(n)| \leq C \cdot \delta \cdot n^{\dim V - \dim V^+ + 1}. \tag{38}
\]

Proof of Proposition 9. By definition, the subspace \( V^+ \) is \( S \)-invariant and \( S \) is invertible on \( V^+ \); we have furthermore that \( |Q^{-n} v| \to 0 \) exponentially fast as \( n \to \infty \). Let \( V^- \) be the subspace spanned by Jordan cells corresponding to eigenvalues of absolute value at most 1; for \( v \in V^- \), we have \( |Q^n v| < C n^{\dim V - \dim V^+} \) as \( n \to \infty \). We have the decomposition \( V = V^+ \oplus V^- \). Let

\[
u(0) = v(0), u(n + 1) = v(n + 1) - Sv(n).
\]

Decompose \( u(n) = u^+(n) + u^-(n) \), where \( u^+(n) \in V^+, u^-(n) \in V^- \). Denote

\[
v^+(n + 1) = u^+(n + 1) + S u^+(n) + \cdots + S^n u^+(1);
\]

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\[ v^-(n + 1) = u^-(n + 1) + S^{}u^-(n) + \cdots + S^nu^-(1); \]
\[ v = u^+(0) + S^{-1}u^+(1) + \cdots + S^{-n}u^+(n) + \cdots. \]

By definition, \(|v^-(n + 1)|\) is bounded above by \(C\delta n^{\dim V - \dim V^+ + 1}\) and there exists \(\tilde{C}\) such that \(|S^n v - v^+(n)| < \tilde{C}\delta\) for all \(n \in \mathbb{N}\), whence (38) follows. Uniqueness of \(v\) follows from the fact that for any nonzero \(v' \in V^+\) the sequence \(|S^n v'|\) grows exponentially as \(n \to \infty\). Proposition 9 and Lemmas 1, 2 are proved completely.

Let \(f \in \text{Lip}_w^+(X)\). We then have a measure \(\Theta_f\) on the semi-ring \(\mathfrak{C}_0^+\) given, for \(\gamma \in \mathfrak{C}_0^+\), by the formula
\[ \Theta_f(\gamma) = \int \gamma f d\Phi^+_1. \]

By (37), the measure \(\Theta_f\) satisfies the assumptions of Lemma 1. Let \(\Xi^+_f \in \mathcal{Y}^+\) be the measure assigned to \(\Theta_f\) by Lemma 1.

**Lemma 3** Let \(f \in \text{Lip}_w^+(X)\), \(\Phi^- \in \mathcal{Y}^-\). Then
\[ < \Xi^+_f, \Phi^- > = \int_X fd\Phi^-_. \tag{39} \]

**Proof:** Choose the points \(x_i^{(n)} \in X\) satisfying (29). As above, let \(\tilde{v} \in E^-\) be the vector corresponding to \(\Phi^-\) by (20) and let \(\tilde{v}^{(n)} = (Q^t)^n\tilde{v}\), \(n \in \mathbb{Z}\). For any \(\varepsilon > 0\) and \(n > 0\) sufficiently large, by definition, we have
\[ \left| m_{\Phi^-}(f) - \sum_{i=1}^{m} \left( \int_{\gamma_i^+(x_i^{(n)})} f d\Phi^+_1 \right) \cdot (\tilde{v}^{(-n)})_i \right| < \varepsilon. \tag{40} \]

By definition of \(\Xi^+_f\) and Lemma 1 we have
\[ \left| \sum_{i=1}^{m} \left( \int_{\gamma_i^+(x_i^{(n)})} f d\Phi^+_1 \right) \cdot (\tilde{v}^{(-n)})_i - \sum_{i=1}^{m} (\Xi^+_f(\gamma_i^+(x_i^{(n)})) \cdot (\tilde{v}^{(-n)})_i \right| < C_{\Gamma} \cdot n^{m+1} |\tilde{v}^{(-n)}|, \]
and, by (28), the right-hand side tends to 0 exponentially fast as \(n \to \infty\).

It remains to notice that, by definition,
\[ \sum_{i=1}^{m} \Xi^+_f(\gamma_i^+(x_i^{(n)})) \cdot (\tilde{v}^{(-n)})_i = < \Xi^+_f, \Phi^- >, \]
and the Lemma is proved completely.

We have thus established that \(\Xi^+_f = \Phi^+_1\), where \(\Phi^+_1\) is given by (31).
2.6 Orderings.

Following S. Ito \[7\], A.M. Vershik \[15, 16\], assume that a partial order $\bullet$ is given on $E(\Gamma)$ in such a way that edges starting at a given vertex are ordered linearly, while edges starting at different vertices are not comparable. An edge will be called maximal (with respect to $\bullet$) if there does not exist a greater edge; minimal, if there does not exist a smaller edge; and an edge $e$ will be called the successor of $e'$ if $e > e'$ but there does not exist $e''$ such that $e > e'' > e'$.

The ordering $\bullet$ is extended to a partial ordering of $X_T$: we write $x < x'$ if there exists $l \in \mathbb{Z}$ such that $x_l < x'_l$ and $x_n = x'_n$ for all $n > l$. Under this ordering each leaf $\gamma_+^+$ of the foliation $\mathcal{F}^+$ is linearly ordered, while points lying on different leaves are not comparable.

Let $Max(\bullet)$ be the set of points $x \in X$, $x = (x_n)_{n \in \mathbb{Z}}$, such that each $x_n$ is a maximal edge. Similarly, $Min(\bullet)$ denotes the set of points $x \in X$, $x = (x_n)_{n \in \mathbb{Z}}$, such that each $x_n$ is a minimal edge. Since edges starting at a given vertex are ordered linearly, the cardinalities of $Max(\bullet)$ and $Min(\bullet)$ do not exceed $m$.

If a leaf $\gamma_+^+$ does not intersect $Max(\bullet)$, then it does not have a maximal element; similarly, if $\gamma_+^+$ does not intersect $Min(\bullet)$, then it does not have a minimal element.

For $x(1), x(2) \in \gamma_+^+$, let

$$ [x(1), x(2)] = \{ x' \in \gamma_+^+ : x(1) \leq x' \leq x(2) \}. $$

The sets $(x(1), x(2)), [x(1), x(2)), (x(1), x(2)]$ are defined similarly.

**Proposition 10** Let $x \in X$. If $\gamma_+^+(x) \cap Max(\bullet) = \emptyset$, then for any $t \geq 0$ there exists a point $x' \in \gamma_+^+(x)$ such that

$$ \Phi_+^+(|x, x'|) = t. \quad (41) $$

Proof. Let $V(x) = \{ t : \exists x' \geq x : \Phi_+^+(|x, x'|) = t \}$. Since $\gamma_+^+(x) \cap Max(\bullet) = \emptyset$, for any $n$ there exists $x'' \in \gamma_+^+(x)$ such that all points in $\gamma_+^+(x'')$ are greater than $x$. Since $\Phi_+^+(\gamma_+^+(x''))$ grows exponentially, uniformly in $x''$, as $n \to \infty$, the set $V(x)$ is unbounded. Furthermore, since $\Phi_+^+(\gamma_+^+(x''))$ decays exponentially, uniformly in $x''$, as $n \to -\infty$, the set $V(x)$ is dense in $\mathbb{R}$. Finally, by compactness of $X$, the set $V(x)$ is closed, which concludes the proof of the Proposition.

A similar proposition, proved in the same way, holds for negative $t$.

**Proposition 11** Let $x \in X$. If $\gamma_+^+(x) \cap Min(\bullet) = \emptyset$, then for any $t \geq 0$ there exists a point $x' \in \gamma_+^+(x)$ such that

$$ \Phi_+^+(|x', x|) = t. \quad (42) $$

Define an equivalence relation $\sim$ on $X$ by writing $x \sim x'$ if $x \in \gamma_+^+(x')$ and $\Phi_+^+(|x, x'|) = \Phi_+^+(|x', x|) = 0$. The equivalence classes admit the following explicit description, which is clear from the definitions.
Proposition 12 Let \( x, x' \in X \) be such that \( x \in \gamma_\infty^+(x'), x < x' \) and \( \Phi_1^+([x, x']) = 0 \). Then there exists \( n \in \mathbb{Z} \) such that

1. \( x'_n \) is a successor of \( x_n \);
2. \( x \) is the maximal element in \( \gamma_n(x) \);
3. \( x' \) is the minimal element in \( \gamma_n(x') \).

In other words, \( \Phi_1^+([x, x']) = 0 \) if and only if \( (x, x') = \emptyset \). In particular, equivalence classes consist at most of two points and, \( \nu \)-almost surely, of only one point.

Denote \( X_\circ = X/\sim \), let \( \pi_\circ : X \to X_\circ \) be the projection map and set \( \nu_\circ = (\pi_\circ)_*\nu \). The probability spaces \((X_\circ, \nu_\circ)\) and \((X, \nu)\) are measurably isomorphic; in what follows, we shall often omit the index \( \circ \). The foliations \( \mathcal{F}^+ \) and \( \mathcal{F}^- \) descend to the space \( X_\circ \); we shall denote their images on \( X_\circ \) by the same letters and, as before, denote by \( \gamma_\infty^+(x) \), \( \gamma_\infty^-(x) \) the leaves containing \( x \in X_\circ \).

Now let \( x \in X_\circ \) satisfy \( \gamma_\infty^+(x) \cap \text{Max}(\circ) = \emptyset \). By Proposition 10 for any \( t \geq 0 \) there exists a unique \( x' \) satisfying (41). Denote \( h_1^+(x) = x' \). Similarly, if \( x \in X_\circ \) satisfy \( \gamma_\infty^-(x) \cap \text{Min}(\circ) = \emptyset \). By Proposition 11 for any \( t \geq 0 \) there exists a unique \( x' \) satisfying (42). Denote \( h_1^-(x) = x' \).

We thus obtain a flow \( h_1^+ \), which is well-defined on the set

\[
X_\circ \setminus \left( \bigcup_{x \in \text{Max}(\circ) \cup \text{Min}(\circ)} \gamma_\infty^+(x) \right),
\]

and, in particular, \( \nu \)-almost surely on \( X_\circ \). By (16), the flow \( h_1^+ \) preserves the measure \( \nu \).

More generally, it is clear from the definitions that for any \( \Phi^- \in \mathcal{Y}^- \), the measure \( m_{\Phi^-} \), defined by (26), satisfies

\[
(h_1^+)_*m_{\Phi^-} = m_{\Phi^-},
\]

similarly to G. Forni’s invariant distributions [5], [6].

Remark. S.Ito in [7] gives a construction of a flow similar to the one above. The flow \( h_1^+ \) is a continuous-time analogue of a Vershik automorphism [15] (of which a variant also occurs in Ito’s work [7]), and, in fact, is a suspension flow over the corresponding Vershik’s automorphism, a point of view adopted in [4].

2.7 Decomposition of Arcs.

We assume that an ordering \( \circ \) is fixed on \( \Gamma \). Denote by \( \mathfrak{C}(\circ) \) the semi-ring of subsets of \( X_\Gamma \) of the form \([x, x')\), where \( x < x' \). Any measure \( \Phi^+ \in \mathcal{Y}^+ \) can be extended to \( \mathfrak{C}(\circ) \) in the following way.

Let \( \mathfrak{R}_\circ^+ \) be the ring generated by the semi-ring \( \mathfrak{C}_\circ^+ \). For \( \gamma \in \mathfrak{C}(\circ) \), denote by \( \gamma(n) \) the smallest (by inclusion) element of the ring \( \mathfrak{R}_\circ^+ \) containing \( \gamma \) and
let \( \hat{\gamma}(n) \) be the greatest (by inclusion) element of the ring \( R_n^+ \) contained in \( \gamma \) (possibly, \( \hat{\gamma}(n) = \emptyset \)). By definition,

\[
\hat{\gamma}(n) \subset \hat{\gamma}(n + 1) \subset \gamma(n + 1) \subset \gamma(n);
\]

where \( \hat{\gamma}^{(n)}_i \in C_{-n}^+, l_n \leq ||Q|| \), and

\[
\gamma(n) \setminus \hat{\gamma}(n) = \bigcup_{i=1}^{l_n} \gamma^{(n)}_i,
\]

(43)

where \( \gamma^{(n+1)}_i \in C_{-n-1}^+, L_n \leq 2||Q|| \).

By definition, if \( \Phi^+ \in Y^+ \), then there are only \( m \) possible values of \( \Phi^+(\gamma) \) for \( \gamma \in C_{-n}^+ \), and the maximum of these decays exponentially as \( n \to \infty \). We thus have

**Proposition 13** There exists positive constants \( C_\Gamma \), depending only on \( \Gamma \), such that the following is true. Let \( v_0 = 0, v_1, \ldots, v_l \in E^+ \), \( Qv_i = \exp(\theta)v_i + v_{i-1} \). Assume \( v \in Cv_1 \oplus \cdots \oplus Cn \) satisfies \( |v| = 1 \) and let \( \Phi^+_v = I_{\Gamma}(v) \). Then for any \( \gamma \in C(o) \) we have

\[
|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \leq C_\Gamma n^{l-1} \exp(-\Re(\theta)n);
\]

\[
|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \leq C_\Gamma n^{l-1} \exp(-\Re(\theta)n).
\]

decay exponentially as \( n \to \infty \). In particular, if \( v \in E^+ \), \( Qv = \exp(\theta)v, |v| = 1 \), then

\[
|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \leq C_\Gamma \exp(-\Re(\theta)n);
\]

\[
|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \leq C_\Gamma \exp(-\Re(\theta)n).
\]

Consequently, for any \( \Phi^+ \in Y^+ \), \( \gamma \in C(o) \), the sequence \( \Phi^+(\gamma(n)) \) converges as \( n \to \infty \), and we set

\[
\Phi^+(\gamma) = \lim_{n \to \infty} \Phi^+(\gamma(n)).
\]

By (43), we also have

\[
\Phi^+(\gamma) = \lim_{n \to \infty} \Phi^+(\gamma(n)).
\]

**Proposition 14** The measure \( \Phi^+ \) is finitely-additive on \( C(o) \).

Proof: Let \( v \in E^+ \) be such that \( \Phi^+ = \Phi^+_v \) and let \( \gamma_0, \gamma_1, \ldots, \gamma_k \in C(o) \) satisfy

\[
\gamma_0 = \bigcup_{i=1}^{k} \gamma_i.
\]

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Consider the arcs $\gamma_0(n), \gamma_1(n), \ldots, \gamma_k(n)$. We have

$$\gamma_0(n) \subset \bigcup_{i=1}^{k} \gamma_i(n). \quad (45)$$

and decompose

$$\gamma_i(n) = \bigsqcup \gamma_{ij}(n+1),$$

where $\gamma_{ij}(n+1) \in \mathcal{C}^+$.

By (45), each of the arcs $\gamma_{0j}(n+1)$ is also encountered among the arcs $\gamma_{ij}(n+1)$ (possibly, more than once, but not more than $k$ times). Consider the collection $\gamma_{ij}(n+1)$ and cross out all the arcs $\gamma_{0j}(n+1)$; by maximality, and since our ordering is linear on each leaf of the foliation $\mathcal{F}^+$, there will remain not more than $2k||Q||$ arcs, whence we obtain

$$\left| \sum_{i=1}^{k} \Phi^+(\gamma_i(n)) - \Phi^+(\gamma_0(n)) \right| \leq 2k||Q|| \cdot |Q^{n-1}v|,$$

and, since the right-hand side decays exponentially as $n \to \infty$, the Proposition is proved.

**Lemma 4** There exists a constant $C^{C^+}_\Gamma$ depending only on $\Gamma$ such that the following is true. Let $f \in \text{Lip}^w(X_\Gamma)$ and let $\Phi^+_f \in \mathcal{Y}^+$ be given by (31). For any $\gamma \in \mathcal{C}^o$ we have

$$\left| \int_{\gamma} f \, d\Phi^+_f - \Phi^+_f(\gamma) \right| \leq C^r||f||_{\text{Lip}^w}(1 + \log(1 + \Phi^+_f(\gamma)))^{m+1}. \quad (46)$$

Indeed, for $\gamma \in \mathcal{C}^+$ this follows from Lemma 1 and for all other arcs from Proposition 13.

### 2.8 Ergodic averages of the flow $h_t^+$

Let $\Phi^+ \in \mathcal{Y}^+$ and denote $\Phi^+[x,t] = \Phi^+([x,h_t^+x])$. The function $\Phi^+(x,t)$ is an additive cocycle over the flow $h_t^+$. Let $f \in \text{Lip}^w(X_\Gamma)$, and let $\Phi^+_f$ be defined by (31). By definition, $\Phi_{f \circ h_t^+} = \Phi^+_f$; recall from (32) that $\Phi^+_f = \sigma^* \Phi^+$. Lemma 4 implies

**Theorem 5** There exists a positive constant $C_\Gamma$ depending only on $\Gamma$ such that for any $f \in \text{Lip}^w(X_\Gamma)$, for all $x \in X$ and all $T > 0$ we have

$$\left| \int_{0}^{T} f \circ h_t^+(x)dt - \Phi^+_f(x,t) \right| \leq C_\Gamma||f||_{\text{Lip}}(1 + \log(1 + T))^{m+1}.$$

Given a bounded measurable function $f : X \to \mathbb{R}$ and $x \in X$, introduce a continuous function $\mathcal{S}_n[f,x]$ on the unit interval by the formula

$$\mathcal{S}_n[f,x](\tau) = \int_{0}^{\tau \exp(n\theta_1)} f \circ h_t^+(x)dt. \quad (47)$$
The functions $S_n[f, x]$ are $C[0,1]$-valued random variable on the probability space $(X, \nu)$. Theorem 6 

If $Q$ has a simple real second eigenvalue $\exp(\theta_2)$, $\theta_2 > 0$, then there exists a continuous functional $\alpha: Lip^+_w(X) \to \mathbb{R}$ and a compactly supported non-degenerate measure $\eta$ on $C[0,1]$ such that for any $f \in Lip^+_w(X)$ satisfying $\alpha(f) \neq 0$ the sequence of random variables

$$
\frac{S_n[f, x]}{\alpha(f) \exp(n\theta_2)}
$$

converges in distribution to $\eta$ as $n \to \infty$.

Remark. Compactness of the support of $\eta$ is understood in the sense of the Tchebycheff topology on $C[0,1]$. Nondegeneracy of the measure $\eta$ means that if $\varphi \in C[0,1]$ is distributed according to $\eta$, then for any $t_0 \in (0,1]$ the distribution of the real-valued random variable $\varphi(t_0)$ is not concentrated at a single point.

The measure $\eta$ is constructed as follows: let $v_2$ be an eigenvector with eigenvalue $\exp(\theta_2)$, set $\Phi_2^+ = I(v_2)$ (see (19)); then $\eta$ is the distribution of $\Phi_2^+(x, \tau)$, $0 \leq \tau \leq 1$, considered as a $C[0,1]$-valued random variable on the space $X, \nu$. The functional $\alpha(f)$ is constructed as follows: under the assumptions of Theorem 6 the matrix $Q^t$ also has the simple real second eigenvalue $\exp(\theta_2)$; let $\tilde{v}_2$ be the eigenvector with eigenvalue $\exp(\theta_2)$, normalized in such a way that $\sum_{i=1}^m (v_2)_i(\tilde{v}_2)_i = 1$; set $\Phi_2^- = I(\tilde{v}_2)$ (see (21)), and let $m_{\Phi_2^-}$ be given by (26); then

$$
\alpha(f) = \int f \, dm_{\Phi_2^-}.
$$

2.9 The diagonalizable case.

As an illustration, consider the case when $Q|_{E^+}$ is diagonalizable with eigenvalues $\exp(\theta_i)$, $i = 1, \ldots, r$, $\Re(\theta_i) > 0$. The Perron-Frobenius vector $h$ corresponds to $\exp(\theta_1)$; let $v_2, \ldots, v_r$ be eigenvectors corresponding to $\exp(\theta_i)$: thus $Qv_i = \exp(\theta_i)v_i$, $i = 2, \ldots, r$ and

$$
E^+ = Ch \oplus Cv_2 \oplus \cdots \oplus Cv_r
$$

We have a similar direct-sum representation for $Q^t$:

$$
\tilde{E}^+ = C\lambda \oplus C\tilde{v}_2 \oplus \cdots \oplus C\tilde{v}_r,
$$

where $Q^t\tilde{v}_i = \exp(\theta_i)\tilde{v}_i$, $i = 2, \ldots, r$. For $i \neq j$ we have

$$
\sum_{i=1}^m (v_i)_i(\tilde{v}_j)_i = 0, \quad (48)
$$

and, for normalization, let us assume that for all $i = 1, \ldots, r$ we have

$$
\sum_{i=1}^m (v_i)_i(\tilde{v}_i)_i = 1. \quad (49)
$$

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Let $\Phi_i^+ = I(v_i)$, $\Phi_i^- = \tilde{I}(\tilde{v}_i)$, $i = 2, \ldots, r$. Since $\Phi_i^+ = I(h)$, the measures $\Phi_i^+$, $i = 1, \ldots, r$, form a basis in $\mathcal{Y}^+$, for which the measures $\Phi_i^- = \tilde{I}(\lambda)$, $\Phi_2^-, \ldots, \Phi_r^-$ form a dual basis in $\mathcal{Y}^-$. For $i = 1, \ldots, r$, from Proposition 16 we have the measures $m_{\Phi_i^+} = \Phi_i^+ \times \Phi_i^-$. For instance, $m_{\Phi_i^-} = \nu$. Theorem 5 now implies

**Corollary 1** For any $f \in Lip^+_w(X_\Gamma)$ we have

$$
\left| \int_0^T f \circ h_t^+ (x) dt - T \int_X f d\nu - \sum_{i=2}^r \Phi_i^+ (x,T) (m_{\Phi_i^-} (f)) \right| \leq C T \| f \|_{Lip(1+\log(1+T))} m^{m+1},
$$

where $C_T$ is a constant depending only on $\Gamma$.

For the action of the shift we have:

\begin{align}
(\sigma)_* \Phi_i^+ &= \exp(-\theta_i) \Phi_i^+, \quad i = 1, \ldots, r; \quad (50) \\
(\sigma)_* \Phi_i^- &= \exp(\theta_i) \Phi_i^-, \quad i = 1, \ldots, r. \quad (51)
\end{align}

Corollary 1 now yields

$$\tau \exp(\theta_n) \int_0^T f \circ h_t^+ (x) dt = \sum_{i=1}^r \exp(n \theta_i) m_{\Phi_i^-} (f) \Phi_i^+ (\sigma^n x, \tau) + O(n^{m+1}). \quad (52)$$

**2.10 The Hölder property.**

As above, we write $\Phi^+(x,t) = \Phi^+([x, h_t^+ x])$. Our next aim is to show that $\Phi^+(x,t)$ is Hölder in $t$ for any $x \in X_\phi$.

**Proposition 15** There exist positive constants $C_\Gamma$ and $t_0$, depending only on $\Gamma$ such that the following is true. Let $v \in E^+$, $Qv = \exp(\theta) v$, $|v| = 1$. Then for all $x \in X$ and positive $t < t_0$ we have

$$|\Phi^+_v (x, t)| \leq C_\Gamma t^{R\theta/\theta_1}.$$

**Proposition 16** There exist positive constants $C_\Gamma$ and $t_0$, depending only on $\Gamma$ such that the following is true. Let $v_0 = 0, v_1, \ldots, v_l \in E^+, Qv_i = \exp(\theta) v_i + v_{i-1}$. Assume $v \in C v_1 \oplus \cdots \oplus C v_l$ satisfies $|v| = 1$. Then for all $x \in X$ and positive $t < t_0$ we have

$$|\Phi^+_v (x, t)| \leq C_\Gamma \log t^{l-1} t^{R\theta/\theta_1}.$$

Proof of Propositions 15, 16. Denote $\gamma = [x, h_t^+ x]$. If $t$ is small enough, then $\hat{\gamma}(0) = \emptyset$. Let $n_0$ be the smallest positive integer such that $\hat{\gamma}(n) \neq \emptyset$. There exist positive constants $C_1, C_2$, depending only on $\Gamma$, such that

$$C_1 t \leq \exp(-\theta_1 n_0) \leq C_2 t,$$

and Propositions 15, 16 follow now from Proposition 13.
Corollary 2 There exist positive constants $\theta > 0$ and $t_0 > 0$ depending only on $Q$ such that for all $v \in E^+, |v| = 1$, all $x \in X$ and all positive $t < t_0$ we have

$$|\Phi_v^+(x, t)| \leq t^{\theta_1/\theta_v}.$$ 

For $v \in E^+, |v| = 1$ denote

$$\theta_v = \lim_{n \to \infty} \frac{\log |Q^n v|}{n}.$$ 

Corollary 3 For any $\varepsilon > 0$ there exists a constant $T_\varepsilon$ depending only on $\varepsilon$ and $\Gamma$ such that for any $v \in E^+, |v| = 1$, any $x \in X$ and any $T > T_\varepsilon$, we have

$$|\Phi_v^+(x, T)| \leq T^{\theta_v/\theta_1 + \varepsilon}.$$ 

Proof: Indeed, let $t_0$ be the constant given by Proposition 16. Let $n_0 = n_0(T)$ be the smallest such integer that $T = \tau \exp(n(T) \theta_1)$, where $\tau < t_0$. Since $\Phi_v^+(x, T) = \Phi_{Q^n v}^+(\sigma^n x, \tau)$ for all $n$, it follows from Proposition 16 that

$$|\Phi_v^+(x, T)| \leq C \Gamma n_0^m \exp(n_0 \Re(\theta_v)) \leq C_T T^{\theta_v/\theta_1 + \varepsilon}$$

if $T$ is sufficiently large (depending only on $\varepsilon$).

Corollary 4 For any $v \in E^+$ we have

$$\limsup_{T \to \infty} \frac{\log |\Phi_v^+(x, T)|}{\log T} = \frac{\theta_v}{\theta_1}. \tag{53}$$

Indeed, the upper bound for the limit superior follows from Corollary 3 and the lower bound is immediate from the relation $\Phi_v^+(\gamma_n(x)) = (Q^n v)_{F(x_{n+1})}$.

Corollary 5 For any $\tau \in \mathbb{R}$ and any $v \in E^+$ satisfying $v \neq 0$, $\sum_{i=1}^m v_i \lambda_i = 0$, the function $\Phi_v^+(x, \tau)$ is not a constant in $x$.

Proof: Indeed, assume $\Phi_v^+(x, \tau) = c$ identically. Then $\Phi_v^+(x, k\tau) = kc$, which contradicts (53): is $c = 0$, then the limit superior is 0; if $c \neq 0$, then the limit superior is 1.

2.11 Tightness.

In this subsection, we assume that $Q$ has a simple real second eigenvalue $\exp(\theta_2)$, $\theta_2 > 0$. Let $v_2$ be the corresponding eigenvector and let $\Phi_2^+ = I(v_2)$. Take $x \in X$ and consider $\Phi^+(x, \tau)$ as a continuous function of $\tau$ on the unit interval. Let $\eta$ be the distribution of $\Phi_2^+(x, \tau)$ in $C[0, 1]$. Note that by Corollary 3 for any $\tau_0$ the value of $\Phi_2^+(x, \tau)$ is not constant on $X$, so the measure $\eta$ is nondegenerate.

Let $\mathcal{S}_n[f, x]$ be defined by the equation (47). Introduce a sequence of measures $\mu_n$ on $C[0, 1]$ by the formula $\mu_n = \mathcal{S}[n, f, x] \eta$. By Theorem 8.1 in Billingsley 3, p.54, to prove Theorem 4 it suffices to establish the following two Lemmas.
Lemma 5  Finite-dimensional distributions of the measures \( \mu_n \) weakly converge to those of \( \eta \).

Lemma 6  The family \( \mu_n \) is tight in \( C[0,1] \).

Proof of Lemma 5. By Theorem 5
\[
\int_0^T f \circ h^+_n(x)dt = \Phi^+_n(x,T) + O((\log T)^{m+1}).
\]
Let \( v_2 \) be the eigenvector corresponding to the eigenvalue \( \exp(\theta_2) \), \( |v| = 1 \), and let \( \Phi^+_2 \in \mathcal{Y}^+ \) be the corresponding measure. We have
\[
E^+ = Cv_v \oplus E_3,
\]
where \( E_3 \) is spanned by Jordan cells corresponding to eigenvalues with absolute value less than \( \exp(\theta_2) \). Let \( \zeta \) be a number smaller than \( \theta_2 \) but greater than the spectral radius of \( Q \mid_{E_3} \). Write
\[
\Phi^+_n = \alpha(f)\Phi^+_2 + \beta(f)\Phi^+_v,
\]
where \( \Phi^+_v \) is the measure corresponding to \( \sigma \), \( |v| = 1 \), and \( \alpha(f) \), \( \beta(f) \) are continuous functionals on \( \text{Lip}_+^{+}(X) \), so, in particular, we have
\[
|\alpha(f)| < C_01||f||_{\text{Lip}_+^{+}}, \quad |\beta(f)| < C_02||f||_{\text{Lip}_+^{+}},
\]
where the constants \( C_01, C_02 \) only depend on \( \Gamma \).

By Corollary 2 there exists \( t_0 \) depending only on \( \Gamma \) such that for any positive \( t \) such that \( t < t_0 \), any \( x \in X \) and any \( \Phi^+_v \) satisfying \( |\Phi^+_v| = 1 \) we have
\[
|\Phi^+_v(x,t)| \leq 1.
\]
Write \( T = t \exp(n\theta_1) \), where \( t < t_0 \). Since \( \Phi^+_v(x,T) = \Phi^+_v(x,T) \), for all sufficiently large \( n \), we have \( |Q^n \mid \Phi^+_v| < \exp(\zeta n) \) and therefore
\[
|\Phi^+_v(x,\tau \exp(n\theta_1))| < \exp(n\zeta)
\]
for all \( x \in X \). By Theorem 5 we have
\[
|\int_0^{\tau \exp(n\theta_1)} f \circ h^+_n(x)dt - \Phi^+_n(x,\tau \exp(n\theta_1))| = O(n^{m+1}).
\]
Since
\[
\Phi^+_n(x,\tau \exp(n\theta_1)) = \alpha(f)\Phi^+_2((x,\tau \exp(n\theta_1)) + \beta(f)\Phi^+_v(x,\tau \exp(n\theta_1))
\]
combining the equality
\[
\Phi^+_2(x,\tau \exp(n\theta_1)) = \exp(n\theta_2)\Phi^+_2(\sigma^n x, \tau)
\]
with the bound (54), we obtain, for all large \( n \) and all \( x \in X \), uniformly in \( \tau \in [0,1] \), the estimate

\[
|\mathcal{S}_n[f, x](\tau) - \alpha(f)\Phi^+_2(\sigma^n x, \tau)| \leq C\tau \|f\|_{Lip^+}\exp((\zeta - \theta_2)n).
\]

Since \( \sigma \) preserves the measure \( \nu \), it follows that the \( k \)-dimensional distributions of \( (\mathcal{S}_n[f, x](\tau_1), \mathcal{S}_n[f, x](\tau_2), \ldots, \mathcal{S}_n[f, x](\tau_k)) \) converge to the \( k \)-dimensional distribution of \( (\Phi^+_2(x, \tau_1), \Phi^+_2(x, \tau_2), \ldots, \Phi^+_2(x, \tau_k)) \), and Lemma 5 is proved.

The argument above yields also

**Proposition 17** There exist positive constants \( C_0 = C_0(\Gamma) \) and \( T_0 = T_0(\Gamma) \) such that for any \( x \in X \), any \( f \in Lip^+_w(X) \) and any \( T > T_0 \) we have

\[
|\int_0^T f \circ h^+_t(x)dt| \leq C_0 \|f\|_{Lip^+} \cdot T^{\theta_2/\theta_1}.
\]

Indeed, for sufficiently large \( T \), \( T = t \exp(n\theta_1) \), where \( t < t_0 \), from (54) we have

\[
\Phi^+_2(x, T) = \alpha(f)\exp(n\theta_2)\Phi^+_2(\sigma^n x, t) + O(\exp(n\zeta)).
\]

Since, by (55), we have \( |\Phi^+_2(\sigma^n x, t)| \leq 1 \), Proposition 17 is established.

We proceed to the proof of Lemma 6.

**Proposition 18** There exists a constant \( C_\Gamma \) depending only on \( \Gamma \) such that for any \( f \in Lip^+_w(X) \), any \( n > 0 \), any \( x \in X \) and any \( \tau_1, \tau_2 \in [0,1] \), we have

\[
|\mathcal{S}_n[f, x](\tau_2) - \mathcal{S}_n[f, x](\tau_1)| \leq C_\Gamma \|f\|_{Lip^+} |\tau_2 - \tau_1|^{\theta_2/\theta_1}.
\]

Lemma 6 follows from Proposition 18 by the Arzelà-Ascoli theorem.

Proof of Proposition 18 Let \( \tau_1, \tau_2 \in [0,1] \), \( \tau_1 < \tau_2 \). For brevity, write \( \mathcal{S}_n = \mathcal{S}_n[f, x] \). We have then

\[
\mathcal{S}_n(\tau_2) - \mathcal{S}_n(\tau_1) = \frac{1}{\exp(n\theta_2)} \int_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h^+_t(x)dt.
\]

Let \( T_0 \) be the constant given by Proposition 17 and assume first that

\( (\tau_2 - \tau_1) \cdot \exp(n\theta_1) \geq T_0 \).

By Proposition 17 we have

\[
\int_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h_t(x)dt \leq C\|f\|_{Lip^+} \cdot (\tau_2 - \tau_1)^{\theta_2/\theta_1} \exp(n\theta_2),
\]

and, consequently,

\[
|\mathcal{S}_n(\tau_2) - \mathcal{S}_n(\tau_1)| \leq C_{33} (\tau_2 - \tau_1)^{\theta_2/\theta_1},
\]

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where the constant $C_{33}$ only depends on $\Gamma$. Now let $\tau_2 - \tau_1 = \tau_0 \exp(-n\theta_1)$, $\tau_0 < T_0$. Since 
\[ \exp(-n\theta_2) = ((\tau_2 - \tau_1)/\tau_0)^{\theta_2/\theta_1}, \]
using boundedness of $f$, write
\[ \frac{1}{\exp(n\theta_2)} \int_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h_t^+ (x) dt \leq \exp(-n\theta_2) \cdot ||f||_\infty \cdot \tau_0 \leq \]
\[ \leq \tau_0^{1-\theta_2/\theta_1} ||f||_\infty (\tau_2 - \tau_1)^{\theta_2/\theta_1} \leq T_0^{1-\theta_2/\theta_1} ||f||_\infty (\tau_2 - \tau_1)^{\theta_2/\theta_1}, \]
and the Proposition is proved. Theorem 6 is proved completely.

2.12 A symbolic coding for translation flows on surfaces.

To derive Theorems 1, 2 from Theorems 5, 6, it remains to observe that the vertical flow on the stable foliation of a pseudo-Anosov diffeomorphism is isomorphic to a symbolic flow on the asymptotic foliation of a Markov compactum obtained from the decomposition of the underlying surface into Veech’s zippered rectangles, see [4], Sec. 4. The identification of $E^+$ (and, consequently, of $Y^+$) with the corresponding subspace in cohomology is given by Proposition 4.16 in Veech [14]. The fact that the pairing between cocycles corresponds to the cup-product is immediate from Proposition 4.19 in [14].

3 Spaces of Markov Compacta.

Let $\mathcal{G}$ be the set of all oriented graphs on $m$ vertices such that there is an edge starting at every vertex and an edge ending at every vertex. As before, for a graph $\Gamma \in \mathcal{G}$, we denote by $E(\Gamma)$ the set of its edges and by $A(\Gamma)$ its incidence matrix: $A_{ij}(\Gamma) = \# \{e \in E(\Gamma) : I(e) = i, F(e) = j \}$. Denote $\Omega = \mathcal{G}^\mathbb{Z}$:
\[ \Omega = \{\omega = \ldots \omega_{-n} \ldots \omega_n \ldots, \omega_i \in \mathcal{G}, i \in \mathbb{Z}\}, \]

For $\omega \in \Omega$, denote by $X(\omega)$ the corresponding Markov compactum:
\[ X(\omega) = \{x = \ldots x_{-n} \ldots x_n \ldots, x_n \in E(\omega_n), F(x_{n+1}) = I(x_n)\}. \]

For $x \in X, n \in \mathbb{Z}$, introduce the sets
\[ \gamma^+_n (x) = \{x' \in X(\omega) : x'_t = x_t, t \geq n\}; \quad \gamma^-_n (x) = \{x' \in X(\omega) : x'_t = x_t, t \leq n\}; \]
\[ \gamma^+_\infty (x) = \bigcup_{n \in \mathbb{Z}} \gamma^+_n (x); \quad \gamma^-_\infty (x) = \bigcup_{n \in \mathbb{Z}} \gamma^-_n (x). \]
The sets $\gamma^+_\infty (x)$ are leaves of the asymptotic foliation $F^+_\infty$ on $X(\omega)$; the sets $\gamma^-_\infty (x)$ are leaves of the asymptotic foliation $F^-_\infty$ on $X(\omega)$.
For $n \in \mathbb{Z}$ let $C^+_n, \omega$ be the collection of all subsets of $X(\omega)$ of the form $\gamma^+_n(x)$, $n \in \mathbb{Z}, x \in X$; similarly, $C^-_n, \omega$ is the collection of all subsets of the form $\gamma^-_n(x)$. Set

$$C^+_\omega = \bigcup_{n \in \mathbb{Z}} C^+_n, \omega; C^-_\omega = \bigcup_{n \in \mathbb{Z}} C^-_n, \omega.$$  \hspace{1cm} (58)

Just as in the periodic case, the collections $C^+_n, \omega$, $C^-_n, \omega$, $C^+_\omega$, $C^-_\omega$ are semi-rings.

**Remark.** To make notation lighter, we shall often omit the subscript $\omega$ and only include it when dependence on $\omega$ is underlined.

### 3.1 Measures and Cocycles.

Let $\sigma$ be the shift on $\Omega$ given by the formula $(\sigma \omega)_n = \omega_{n+1}$. Let $P$ be an ergodic $\sigma$-invariant probability measure on $\Omega$. We then have a natural cocycle $A$ on the system $(\Omega, \sigma, P)$ defined, for $n > 0$, by the formula

$$A(n, \omega) = A(\omega_n) \ldots A(\omega_1).$$

The cocycle $A$ will be called the renormalization cocycle.

We need the following assumptions on the measure $P$ and on the cocyle $A$.

**Assumption 1** The matrices $A(\omega_n)$ are almost surely invertible with respect to $P$. There exists $\Gamma \in \mathcal{G}$ such that $P(\Gamma) > 0$.

**Assumption 2** The logarithm of the renormalization cocycle (and of its inverse) is integrable.

For $n < 0$ set

$$A(n, \omega) = A^{-1}(\omega_{-n}) \ldots A^{-1}(\omega_0).$$

and set $A(0, \omega)$ to be the identity matrix.

The transpose cocycle $A^t$ over the dynamical system $(\Omega, \sigma^{-1}, P)$ defined, for $n > 0$, by the formula

$$A^t(n, \omega) = A^t(\omega_{-n}) \ldots A^t(\omega_0).$$

Similarly, for $n < 0$ write

$$A^t(n, \omega) = (A^t)^{-1}(\omega_{-n}) \ldots (A^t)^{-1}(\omega_1).$$

and set $A^t(0, \omega)$ to be the identity matrix.

By Assumptions 1, 2 for $P$-almost any $\omega \in \Omega$ we have the decompositions

$$\mathbb{R}^m = E^+_\omega \oplus E^-_\omega; \mathbb{R}^m = \tilde{E}^+_\omega \oplus \tilde{E}^-_\omega,$$

where $E^+$ is the Lyapunov subspace corresponding to positive Lyapunov exponents of $A$; $\tilde{E}^+$ is the Lyapunov subspace corresponding to positive Lyapunov exponents of $A^t$; $E^-$ is the Lyapunov subspace corresponding to zero and negative Lyapunov exponents of $A$; $\tilde{E}^-$ is the Lyapunov subspace corresponding to
zero and negative Lyapunov exponents of $A^t$. The standard inner product on $\mathbb{R}^m$ yields a nondegenerate pairing between the spaces $E^+_\omega$ and $\tilde{E}^+_\omega$.

In particular, by Assumption 1, the spaces $E^+_\omega$ and $\tilde{E}^+_\omega$ each contain a unique vector all whose coordinates are positive; we denote these vectors by $h^\omega$ and $\lambda^\omega$, respectively, and assume that they are normalized by (12).

Let $v \in E^+_\omega$ and for all $n \in \mathbb{Z}$ set $v^{(n)} = h^\omega(n, \omega)v$. Introduce a finitely-additive complex-valued measure $\Phi^+_v$ on the semi-ring $C^+\omega$ (defined in (58)) by the formula
\[
\Phi^+_v(\gamma^+_n(x)) = (v^{(n)}) \varphi_{\omega(x)}(x),
\]
(59)

As before, the measure $\Phi^+_v$ is invariant under holonomy along $\mathcal{F}^-$: by definition, we have the following

**Proposition 19** If $F(x_n) = F(x'_{n'})$, then $\Phi^+_v(\gamma^+_n(x)) = \Phi^+_v(\gamma^+_{n'}(x'))$.

The measures $\Phi^+_v$ span a complex linear space, which is denoted $\mathcal{Y}^+_\omega$. The map $\mathcal{I}_\omega : v \to \Phi^+_v$ is an isomorphism between $E^+_\omega$ and $\mathcal{Y}^+_\omega$. Set $\Phi^+_1 = \mathcal{I}_\omega(h^\omega)$.

Now for $\tilde{v} \in \tilde{E}^+_\omega$ and for all $n \in \mathbb{Z}$ set $\tilde{v}^{(n)} = h^\omega(n, \omega)\tilde{v}$ and introduce a finitely-additive complex-valued measure $\Phi^-_{\tilde{v}}$ on the semi-ring $C^-\omega$ (defined in (58)) by the formula
\[
\Phi^-_{\tilde{v}}(\gamma^-_n(x)) = (\tilde{v}^{(-n)}) \varphi_{\omega(x)}(x),
\]
(60)

By definition, the measure $\Phi^-_{\tilde{v}}$ is invariant under holonomy along $\mathcal{F}^+$: more precisely, we have the following

**Proposition 20** If $I(x_n) = I(x'_{n'})$, then $\Phi^-_{\tilde{v}}(\gamma^-_n(x)) = \Phi^-_{\tilde{v}}(\gamma^-_{n'}(x'))$.

Let $\mathcal{Y}^-\omega$ be the space spanned by the measures $\Phi^-_{\tilde{v}}$, $\tilde{v} \in \tilde{E}^+_\omega$. The map $\tilde{\mathcal{I}}_\omega : \tilde{v} \to \Phi^-_{\tilde{v}}$ is an isomorphism between $\tilde{E}^+_\omega$ and $\mathcal{Y}^-\omega$. Set $\Phi^-_{\tilde{1}} = \tilde{\mathcal{I}}_\omega(\lambda^\omega)$.

Define a map $t_\sigma : X_\omega \to X_{\sigma\omega}$ by $(t_\sigma x)_1 = x_{i+1}$. The map $t_\sigma$ induces a map $t_\sigma^* : \mathcal{Y}^+_{\sigma\omega} \to \mathcal{Y}^+\omega$ given, for $\Phi^+_{\sigma\omega} \in \mathcal{Y}^+_{\sigma\omega}$ and $\gamma \in C^+\omega$, by the formula
\[
t_\sigma^*\Phi^+ (\gamma) = \Phi^+_{\sigma\omega}(t_\sigma \gamma).
\]

We have the following commutative diagrams:
\[
\begin{align*}
E^+\omega & \xrightarrow{\mathcal{I}_\omega} \mathcal{Y}^+\omega \\
\bigg/ h^{1,\omega} & \xrightarrow{t_\sigma^*} \\
E^+_{\sigma\omega} & \xrightarrow{\mathcal{I}_{\sigma\omega}} \mathcal{Y}^+_{\sigma\omega} \\
\bigg/ h^{1,\sigma\omega} & \xrightarrow{t_\sigma^*} \\
\tilde{E}^+_\omega & \xrightarrow{\tilde{\mathcal{I}}_\omega} \mathcal{Y}^-\omega \\
\bigg/ \tilde{h}^{1,\omega} & \xrightarrow{\tilde{t}_\sigma^*} \\
\tilde{E}^+_{\sigma\omega} & \xrightarrow{\tilde{\mathcal{I}}_{\sigma\omega}} \mathcal{Y}^-_{\sigma\omega}
\end{align*}
\]
3.2 Pairings and weakly Lipschitz functions.

Given $\Phi^+ \in \mathcal{Y}_+^\omega$, $\Phi^- \in \mathcal{Y}_-^\omega$, introduce a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\mathcal{C}$ of cylinders in $X(\omega)$: for any $C \in \mathcal{C}$ and $x \in C$, set
\begin{equation}
\Phi^+ \times \Phi^-(C) = \Phi^+(\gamma^+_\omega(x) \cap C) \cdot \Phi^-(\gamma^-_\omega(x) \cap C). \tag{61}
\end{equation}

Note that by Propositions 19, 20, the right-hand side in (61) does not depend on $x \in C$.

As above, for $\Phi^- \in \mathcal{Y}_-^\omega$, denote
\begin{equation}
m_{\Phi^-} = \Phi^+_1 \times \Phi^-_. \tag{62}
\end{equation}

In particular, we have a positive countably additive measure
\begin{equation}
\nu_\omega = \Phi^+_1 \times \Phi^-_. \tag{63}
\end{equation}

There is a natural $\mathbb{C}$-linear pairing $\langle, \rangle$ between the spaces $\mathcal{Y}_+^\omega$ and $\mathcal{Y}_-^\omega$: for $\Phi^+ \in \mathcal{Y}_+^\omega$, $\Phi^- \in \mathcal{Y}_-^\omega$, set
\begin{equation}
\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^- (X(\omega)). \tag{64}
\end{equation}

As in Sec. 2.3, we have

**Proposition 21** Let $v \in E^+_\omega$, $\tilde{v} \in \tilde{E}^+_\omega$, $\Phi^+_v = \mathcal{I}_\omega(v)$, $\Phi^-_{\tilde{v}} = \tilde{\mathcal{I}}_\omega(\tilde{v})$. Then
\begin{equation}
\langle \Phi^+_v, \Phi^-_{\tilde{v}} \rangle = \sum_{i=1}^m v_i \tilde{v}_i. \tag{65}
\end{equation}

The pairing $\langle, \rangle$ is non-degenerate and $t^*_\omega$-invariant.

The function space $\text{Lip}^+_w(X(\omega))$ is introduced in the same way as before: a bounded Borel-measurable function $f : X(\omega) \to \mathbb{C}$ belongs to the space $\text{Lip}^+_w(X)$ if there exists a constant $C > 0$ such that for all $n \geq 0$ and any $x, x' \in X$ satisfying $F(x_{n+1}) = F(x'_{n+1})$, we have
\begin{equation}
| \int_{\gamma^+_\omega(x)} f d\Phi^+_1 - \int_{\gamma^+_\omega(x')} f d\Phi^+_1 | \leq C, \tag{66}
\end{equation}
and, if $C_f$ is the infimum of all $C$ satisfying (65), then we norm $\text{Lip}^+_w(X)$ by setting
\begin{equation}
\|f\|_{\text{Lip}^+_w} = \sup_X f + C_f.
\end{equation}

As before, we denote by $\text{Lip}^+_w,0(X(\omega))$ the subspace of functions of $\nu_\omega$-integral zero.

Take $\Phi^- \in \mathcal{Y}^-$. Any function $f \in \text{Lip}^+_w(X)$ is integrable with respect to the measure $m_{\Phi^-}$ in the same sense as in Sec. 2.3 and a measure $\Phi^+_f \in \mathcal{Y}^+$ is defined by the requirement that for any $\Phi^- \in \mathcal{Y}^-$ we have
\begin{equation}
\langle \Phi^+_f, \Phi^- \rangle = \int_{X(\omega)} f d\Phi^-_. \tag{67}
\end{equation}
Note that the mapping $\Xi^+_\omega : \text{Lip}_\omega^+(X(\omega)) \rightarrow \mathcal{Y}_\omega^+$ given by $\Xi^+_\omega(f) = \Phi^+_f$ is continuous by definition and satisfies

$$\Xi^+_\omega(f \circ t_\sigma) = (t_\sigma)^* \Xi^+_\omega(f). \quad (67)$$

From the definitions we also have

**Proposition 22** Let $\Phi^+_1, \ldots, \Phi^+_r(1), \ldots, \Phi^+_r(r)$ be a basis in $\mathcal{Y}_\omega^+$ and let $\Phi^-_1, \ldots, \Phi^-_r(r)$ be the dual basis in $\mathcal{Y}_\omega^-$ with respect to the pairing $<,>$. Then for any $f \in \text{Lip}_\omega^+(X(\omega))$ we have

$$\Phi^+_f = \sum_{i=1}^r (m_{\Phi^-_i}(f)) \Phi^+_i. \quad (68)$$

### 3.3 Orderings and flows.

Assume that for $\mathbb{P}$-almost every $\omega$ a partial ordering $\sigma(\omega)$ is given on $\mathcal{E}(\omega_n)$ for all $n \in \mathbb{Z}$ in such a way that edges starting at a given vertex are ordered linearly, while edges starting at different vertices are incomparable. Assume, moreover, that the orders $\sigma(\omega)$ are $\sigma$-invariant, in the sense that the ordering $\sigma(\omega)$ on $\mathcal{E}(\omega_n)$ is the same as the ordering $\sigma(\omega)$ on $\mathcal{E}((\sigma\omega)_n)$. Similarly to the above, construct spaces $X_\sigma(\omega)$ and introduce a flow $h^{1,+}_t(\omega)$ on each $X_\sigma(\omega)$. The shift $\sigma$ renormalizes the flows $h^{1,+}_t$: if we set

$$H^{(1)}(n, \omega) = ||A(n, \omega)||, \quad (69)$$

then for any $t \in \mathbb{R}$ we have a commutative diagram

$$\begin{array}{ccc}
X(\omega) & \xrightarrow{h^{1,+}_t(\omega)} & X(\omega) \\
\downarrow t_\sigma & & \downarrow t_\sigma \\
X(\sigma) & \xrightarrow{h^{1,+}_t(\sigma)} & X(\sigma)
\end{array}$$

As before, each measure $\Phi^+ \in \mathcal{Y}_\omega^+$ yields a Hölder cocycle over the flow $h^{1,+}_t$: we shall denote the cocycle by the same letter as the measure.

Note that for any $\Phi^- \in \mathcal{Y}^-_\omega$ the measure $m_{\Phi^-}$ defined by (62) satisfies

$$(h^{1,+}_t)^* m_{\Phi^-} = m_{\Phi^-},$$

similarly to G. Forni’s invariant distributions [5], [6].

Note that the mapping $\Xi^+_\omega : \text{Lip}_\omega^+(X(\omega)) \rightarrow \mathcal{Y}_\omega^+$ given by $\Xi^+_\omega(f) = \Phi^+_f$ by definition satisfies

$$\Xi^+_\omega(f \circ h^{1,+}_t) = \Xi^+_\omega(f). \quad (69)$$

We thus have the following.
Theorem 7 Let $\mathbb{P}$ be an ergodic $\sigma$-invariant probability measure on $\Omega$ satisfying the assumptions $\text{[1.4]}$. For any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon$ depending only on $\mathbb{P}$ such that the following holds. For $\mathbb{P}$-almost any $\omega$ there exists a continuous mapping $\Xi^+_\omega : \text{Lip}^+_w(X(\omega)) \to Y^+_\omega$ such that for any $f \in \text{Lip}^+_w(X(\omega))$, any $x \in X(\omega)$ and all $T > 0$ we have

$$| \int_0^T f \circ h_t^{(+,\omega)}(x)dt - \Xi^+_\omega(f)(x, t)| \leq C_\varepsilon ||f||_{\text{Lip}_w}(1 + T^\varepsilon).$$

The mapping $\Xi^+_\omega$ satisfies the equality $\Xi^+_\omega(f \circ h_t^{(+,\omega)}) = \Xi^+_\omega(f)$. The diagram

$$\begin{array}{ccc}
\text{Lip}^+_w(X(\omega)) & \xrightarrow{\Xi^+_\omega} & Y^+_\omega \\
\downarrow^{t^*} & & \downarrow^{t^*} \\
\text{Lip}^+_w(X(\omega)) & \xrightarrow{\Xi^+_\omega} & Y^+_\omega
\end{array}$$

is commutative.

The mapping $\Xi^+_\omega$ is given by $\Xi^+_\omega(f) = \Phi^+_f$, where $\Phi^+_f$ is defined by $\text{[66]}$.

Now assume that the second Lyapunov exponent $\theta_2$ of the renormalization cocycle $A$ is positive and simple. Let $\nu_2 \in E^+_\omega$ be a Lyapunov vector corresponding to the exponent $\exp(\theta_2)$ (such a vector is defined up to multiplication by a scalar). Introduce a multiplicative cocycle $H^{(2)}(n, \omega)$ over $\sigma$ by the formula

$$H^{(2)}(n, \omega) = \left| \frac{\|A(n, \omega)\|_{\text{lip}}^{(\omega)}}{\|\nu_2^{(\omega)}\|} \right|. \tag{70}$$

Recall that the cocycle $H^{(1)}(n, \omega)$ is given by $\text{[65]}$. Similarly to the above, given a bounded measurable function $f : X(\omega) \to \mathbb{R}$ and $x \in X(\omega)$, introduce a continuous function $\mathcal{S}_n[f, x]$ on the unit interval by the formula

$$\mathcal{S}_n[f, x](\tau) = \int_0^\tau f \circ h_t^{(+,\omega)}(x)dt. \tag{71}$$

The functions $\mathcal{S}_n[f, x]$ are $C[0, 1]$-valued random variables on the probability space $(X(\omega), \nu_\omega)$.

Theorem 8 Let $\mathbb{P}$ be an ergodic $\sigma$-invariant probability measure on $\Omega$ satisfying the assumptions $\text{[1.4]}$ and such the second Lyapunov exponent of the renormalization cocycle $A$ with respect to $\mathbb{P}$ is positive and simple.

For $\mathbb{P}$-almost any $\omega' \in \Omega$ there exists a non-degenerate compactly supported measure $\eta_{\omega'}$ on $C[0, 1]$ and, for $\mathbb{P}$-almost any pair $(\omega, \omega')$ there exists a sequence of moments $l_n = l_n(\omega, \omega')$ such that the following holds.

For $\mathbb{P}$-almost any $\omega$ there exists a continuous functional

$$a^{(\omega)} : \text{Lip}^+_w(X(\omega)) \to \mathbb{R} \tag{72}$$

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such that for $\mathbb{P}$-almost any $\omega'$ and any $f \in \text{Lip}_{+,0}^+(X(\omega))$ satisfying $a^{(\omega)}(f) \neq 0$ the sequence of random variables

$$\frac{\mathbb{S}_{I_n(\omega,\omega')}[f,x]}{a^{(\omega)}(f)H^{(2)}(I_n(\omega,\omega'),\omega)}$$

converges in distribution to $\eta_{\omega'}$ as $n \to \infty$.

Theorems 7, 8 imply Theorems 3, 4. The proofs of Theorems 7, 8 follow the same pattern as those of Theorems 5, 6; detailed proofs will appear in the sequel to this paper.

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