On the Conformal Geometry of Transverse Riemann-Lorentz Manifolds

E. Aguirre, V. Fernández, J. Lafuente.

Dept. Geometría y Topología, Fac. CC. Matemáticas, UCM.
Plaza de las Ciencias 3, Madrid, Spain.
Mathematics Subject Classification: 53C50, 53B30, 53C15.

March 30, 2022

Abstract

Physical reasons suggested in [2] for the Quantum Gravity Problem lead us to study type-changing metrics on a manifold. The most interesting cases are Transverse Riemann-Lorentz Manifolds. Here we study the conformal geometry of such manifolds.

1 Preliminaries

Let $M$ be a connected manifold, dim $M = m \geq 2$, and let $g$ be a symmetric covariant tensor field of order 2 on $M$. Assume that the set $\Sigma$ of points where $g$ degenerates is not empty. Consider $p \in \Sigma$ and $(U, x)$ a coordinate system around $p$. We say that $g$ is a transverse type-changing metric on $p$ if $d_p(\det (g_{ab})) \neq 0$ (this condition does not depend on the choice of the coordinates). We call $(M, g)$ transverse type-changing pseudoriemannian manifold if $g$ is transverse type-changing on every point of $\Sigma$. In this case, $\Sigma$ is a hypersurface of $M$. Moreover, at every point of $\Sigma$ there exists a one-dimensional radical, that is the subspace $Rad_p(M)$ of $T_pM$ which is $g$-ortogonal to the whole $T_pM$ (and it can be transverse or tangent to the hypersurface $\Sigma$). The index of $g$ is constant on every connected component of $\tilde{M} = M - \Sigma$, thus $\tilde{M}$ is a union of connected pseudoriemannian manifolds. Locally, $\Sigma$ separates two pseudoriemannian manifolds whose indices differ in one unit (so we call
The most interesting cases are those in which $\Sigma$ separates a riemannian part from a lorentzian one. We call these cases transverse Riemann-Lorentz manifolds.

Let $\tau \in C^\infty(M)$ be such that $\tau|_{\Sigma} = 0$ and $d\tau|_{\Sigma} \neq 0$. We say that (locally, around $\Sigma$) $\tau = 0$ is an equation for $\Sigma$. Given $f \in C^\infty(M)$, it holds: $\tau|_{\Sigma} = 0 \iff f = k\tau$, for some $k \in C^\infty(M)$. In what follows we shall use this fact extensively.

On $M$ we have naturally defined all the objects associated to pseudoriemannian geometry, derived from the Levi-Civita connection. In [4], [5], [6], [7] and [1], the extendibility of geodesics, parallel transport and curvatures have been studied. Our aim in the present paper is to study the conformal geometry of transverse Riemann-Lorentz manifolds, including criteria for the extendibility of the Weyl conformal curvature.

Let $(M, g)$ be a transverse Riemann-Lorentz manifold. First of all, note that we do not have any Levi-Civita connection $\nabla$ defined on the whole $M$. However we have ([4]) a unique torsion-free metric dual connection $\Box: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ on $M$ defined by a Koszul-like formula. On $M$ it holds $\Box_X Y (Z) = g(\nabla_X Y, Z)$, and thus the concepts derived from Levi-Civita connection $\nabla$ (on $M$) coincide with those derived from the dual connection $\Box$.

We say that a vectorfield $R \in \mathfrak{X}(M)$ is radical if $R_p \in \text{Rad}_p(M) - \{0\}$ for all $p \in \Sigma$. Given a radical vectorfield $R \in \mathfrak{X}(M)$, $\Box_X Y (R)|_{\Sigma}$ only depends on $X|_{\Sigma}$ and $Y|_{\Sigma}$, thus we obtain the following well-defined map $II^R: \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma)$, $(X, Y) \mapsto \Box_X Y (R)$

Note that the $II^R$-orthogonal complement to $\text{Rad}_p(M)$ is $T_p\Sigma$ ([7], 1(a)), thus $X \in \mathfrak{X}_\Sigma$ is tangent to $\Sigma$ if and only if $II^R (X, R) = 0$.

Because of the properties of $\Box$, the restriction of $II^R$ to vectorfields in $\mathfrak{X}(\Sigma)$ is a well-defined $(0, 2)$ symmetric tensor field $II^R_\Sigma \in S^2(\Sigma)$. Furthermore, since $\Box_X Y$ is a one-form on $M$ and the radical is one-dimensional, the condition $II^R_\Sigma = 0$ does not depend on the radical vectorfield $R$. A transverse Riemann-Lorentz manifold is said to be $II$-flat if $II^R_\Sigma = 0$, for some (and thus, for any) radical vectorfield $R$. It turns out ([7] for transverse, [1] for tangent radical) that $M$ is $II$-flat if and only if all covariant derivatives $\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$ tangent to $\Sigma$, smoothly extend to $M$. Moreover,
in that case, $\nabla_X Y|_{\Sigma}$ only depends on $X|_{\Sigma}$ and $Y|_{\Sigma}$, thus we obtain another well-defined map
\[
III^R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma), (X,Y) \mapsto II^R(\nabla_X Y, R)
\]
which is a $(0,2)$ symmetric tensorfield on $\Sigma$. A transverse Riemann-Lorentz $II$-flat metric is said to be $III$-flat if $III^R = 0$. If the radical is tangent, $\nabla_R R$ becomes transverse ([1]); therefore, in order that a $II$-flat metric becomes $III$-flat, the radical must be transverse. And we have the following result ([7]), concerning the extendibility of curvature tensors:

**Theorem 1** The covariant curvature $K$ smoothly extends to $M$ if and only if the radical is transverse and $g$ is $II$-flat, while the Ricci tensor $\text{Ric}$ smoothly extends to $M$ if and only if the radical is transverse and $g$ is $III$-flat.

## 2 A Gauss formula for Transverse Riemann-Lorentz Manifolds

Let $(M, g)$ be a transverse Riemann-Lorentz manifold with transverse radical.

**Lemma 2** There exists a unique (canonically defined) radical vectorfield $R$ such that $II^R(R,R) = 1$.

**Proof:** Given a radical vectorfield $U$, consider $R = (II^U(U,U))^{-\frac{1}{3}} \cdot U$, which is a well-defined radical vectorfield (since the radical is transverse). Thus $II^R(R,R) = 1$. Furthermore, if $Z = fR$ is another radical vectorfield such that $II^Z(Z,Z) = 1$, then $1 = II^Z(Z,Z) = f^3 II^R(R,R) = f^3$, and consequently $f = 1$ ♠

Suppose that $(M, g)$ is $II$-flat. As we said before, given $X, Y \in \mathfrak{X}(\Sigma)$, $\nabla_X Y$ is well-defined. Moreover, $\tan(\nabla_X Y) := \nabla_X Y - III^R(X,Y) \cdot R$ is indeed tangent to $\Sigma$, since
\[
II^R(R, \tan(\nabla_X Y)) = III^R(X,Y) - III^R(X,Y) II^R(R,R) = 0
\]

**Lemma 3** If $X, Y \in \mathfrak{X}(\Sigma)$ and $\nabla^\Sigma$ is the Levi-Civita connection of $(\Sigma, g_\Sigma)$, it holds
\[
\nabla_X Y = \nabla_X^\Sigma Y + III^R(X,Y) \cdot R
\]
Proof: Let be $Z \in \mathfrak{X}(\Sigma)$. Since $(M, g)$ is $\text{II}$-flat, $\nabla_X Y$ is well defined and it must hold $\Box_X Y (Z) = g(\nabla_X Y, Z) = g_{\Sigma} (\tan(\nabla_X Y), Z)$. On the other hand, $\Box$ has always a good restriction $\Box : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^*(\Sigma)$, which must coincide with $\Box^\Sigma$, the unique torsion-free metric dual connection on $(\Sigma, g_{\Sigma})$. Since $(\Sigma, g_{\Sigma})$ is riemannian, it must hold $\Box^\Sigma_X Y (Z) = g_{\Sigma} (\nabla^\Sigma_X Y, Z)$, and the result follows ♣

The existence of a canonical radical vectorfield leads to the following Gauss formula.

Proposition 4 Let $(M, g)$ be a transverse Riemann-Lorentz manifold with transverse radical and II-flat. Then $\Sigma$ is "totally geodesic" in the sense that, if $X, Y, Z, T \in \mathfrak{X}(\Sigma)$ it holds

$$K(X, Y, Z, T) = K^\Sigma(X, Y, Z, T)$$

where $K^\Sigma$ is the covariant curvature of $\Sigma$.

Proof: As we said in the proof of previous lemma we have, for $X, Y, Z, T \in \mathfrak{X}(\Sigma)$: $\Box_X Y (Z) = \Box^\Sigma_X Y (Z)$, where $\Box^\Sigma$ is the dual connection of $(\Sigma, g_{\Sigma})$. Moreover, since $\Box_X R(T) = -\Box_X T (R) = -\text{II}^R (X, T) = 0$, again previous lemma leads to

$$\Box_X (\nabla Y Z) (T) = \Box_X (\nabla^\Sigma_Y Z + \text{III}^R (Y, Z) R) (T) = \Box^\Sigma_X (\nabla^\Sigma_Y Z) (T)$$

what gives the result ♣

Corollary 5 Let $(M, g)$ be a transverse Riemann-Lorentz manifold with transverse radical. If $(M, g)$ is flat, then $(M, g)$ is III-flat and $\Sigma$ is flat.

Proof: If $K = 0$ then $\text{Ric} = 0$. In particular, $\text{Ric}$ extends to $M$, thus by Theorem 1 $(M, g)$ is III-flat. By Proposition 4 $\Sigma$ is flat ♣

We now restate Theorem 9 of [5] in the following terms (the flatness of $\Sigma$, being a consequence of the Collorary, needs not be included as an extra hypothesis):

Theorem 6 Let $(M, g)$ be a transverse Riemann-Lorentz manifold. Then, $M$ is locally flat around $\Sigma$ if and only if, around every singular point $p \in \Sigma$, there exists a coordinate system $(U, x)$ such that $g = \sum_{i=0}^{m-1} (dx^i)^2 + \tau (dx^m)^2$, where $\tau = 0$ is a local equation for $\Sigma$. 

4
3 Conformal geometry and the extendibility of Weyl curvature

Let us consider a transverse Riemann-Lorentz manifold \((M, g)\) and the family \(\mathcal{C} = \{e^{2f} g : f \in C^\infty (M)\}\). Take \(\overline{g} = e^{2f} g \in \mathcal{C}\). Then \((M, \overline{g})\) is also a transverse Riemann-Lorentz manifold, and \(\Sigma = \Sigma\). Moreover, for each singular point \(p \in \Sigma\) the radical subspaces are the same: \(\text{Rad}_p(M) = \text{Rad}_p(M)\)

We say that \((M, \mathcal{C})\) is a transverse Riemann-Lorentz conformal manifold if some (and thus any) \(g \in \mathcal{C}\) is transverse Riemann-Lorentz. Let \((M, \mathcal{C})\) be a transverse Riemann-Lorentz conformal manifold. We say that \(g \in \mathcal{C}\) is conformally II-flat if \(\mathcal{II} \Sigma = h g_{\Sigma}\), for some radical vectorfield \(R\) and some \(h \in C^\infty (\Sigma)\). This definition does not depend on \(R\) and, even more, it is conformal: if \(\overline{g} = e^{2f} g \in \mathcal{C}\), then it holds

\[
\mathcal{II} \Sigma = e^{2f} \left\{ \mathcal{II} \Sigma - Rf|_{\Sigma} g_{\Sigma} \right\}
\]

(1)

Thus we say that \((M, \mathcal{C})\) is conformally II-flat if some (and thus, any) metric \(g \in \mathcal{C}\) is conformally II-flat.

**Proposition 7** A transverse Riemann-Lorentz conformal manifold \((M, \mathcal{C})\) is conformally II-flat if and only if around every singular point \(p \in \Sigma\) there exist an open neighbourhood \(U\) in \(M\) and a metric \(g \in \mathcal{C}\) which is II-flat on \(U\), that is \(\mathcal{II} \Sigma \cap U = 0\).

**Proof:** Let \((U, E)\) be an adapted orthonormal frame near \(p \in \Sigma\) (that is, \(E_m\) is radical and \((E_1, ..., E_{m-1})\) are orthonormal) and \(g \in \mathcal{C}\). If \(\mathcal{C}\) is conformally II-flat, then there exists \(h \in C^\infty (\Sigma)\) such that \(\mathcal{II}^E_{\Sigma} = h g_{\Sigma}\). Take \(\hat{h} \in C^\infty (U)\) any local extension of \(h\) (shrinking \(U\) if necessary). There exists \(f \in C^\infty (U)\) (shrinking again \(U\) if necessary) satisfying \(E_m f = \hat{h}\) (since it is locally a first order linear equation), what gives on \(U\): \(\mathcal{II}^E_{\Sigma} = (E_m f)|_{\Sigma} g_{\Sigma}\). Let \(\hat{f} \in C^\infty (M)\) be any extension of (possibly a restriction of) \(f\). Applying \(\mathcal{II}\) to \(g\) and \(\overline{g} := e^{2\hat{f}} g \in \mathcal{C}\) we have \(\mathcal{II}^E_{\Sigma} = 0\).

To show the converse we start considering \(g \in \mathcal{C}\). Since conformally II-flatness is a local condition, it suffices to take an arbitrary \(p \in \Sigma\) and \(\overline{g} = e^{2\hat{f}} g \in \mathcal{C}\) such that \(\overline{g}\) is II-flat around \(p\). Then, formula \(\mathcal{II}\) applied to \(g\) and \(\overline{g}\) shows that \(\mathcal{II}^f_p = (\xi f) g_p\), where \(\xi \in \text{Rad}_p(M) - \{0\}\).

In what follows, we study conformally II-flat Riemann-Lorentz conformal structures with transverse radical. Let \(g\) and \(\overline{g} = e^{2f} g \in \mathcal{C}\) be two
transverse Riemann-Lorentz metrics which are $II$-flat. Formula (1) shows that $(Rf)|_{\Sigma} = 0$. The expression of $grad_{g}(f)$ in an adapted orthonormal frame such that $R = E_{m}$ is $grad_{g}(f) = \sum_{i=1}^{m-1} (E_{i}f) E_{i} + \tau^{-1} (Rf) R$, thus $grad_{g}(f)$ extends to the whole $M$. Now a simple computation gives

$$III^{R} = e^{2f} \left\{ III^{R} - II^{R} (grad_{g}(f), R) g_{\Sigma} \right\}$$  \hspace{1cm} (2)

We say that $g \in \mathcal{C}$ is conformally $III$-flat if it is $II$-flat (in order that $III^{R}$ exists) and it holds $III^{R} = k g_{\Sigma}$, for some radical vectorfield $R$ and some $k \in C^{\infty}(\Sigma)$. Since $II$-flatness is not conformal, the above definition, although independent of $R$, cannot be conformal. However, it is conformal in the subset of $II$-flat metrics.

**Definition 8** We say that a transverse Riemann-Lorentz conformal manifold $(M,\mathcal{C})$ with transverse radical is conformally $III$-flat if it is conformally $II$-flat and every $g \in \mathcal{C}$ which is $II$-flat on some open $U$ of $M$ is also conformally $III$-flat on $U$.

Note that there may exist no conformally $III$-flat metrics on a conformally $III$-flat manifold, simply because there may exist no $II$-flat metric there. However, since a conformally $III$-flat space is conformally $II$-flat, we deduce from Proposition 7 that there always exist locally $II$-flat metrics. Let us show that in fact there also exist locally $III$-flat metrics:

**Proposition 9** A transverse Riemann-Lorentz conformal manifold $(M,\mathcal{C})$ with transverse radical is conformally $III$-flat if and only if around every singular point $p \in \Sigma$ there exist an open neighbourhood $U$ in $M$ and a metric $g \in \mathcal{C}$ which is $III$-flat on $U$, that is $III_{\Sigma \cap U} = 0$.

**Proof:** Consider $p \in \Sigma$ and $(U, E)$ a completely adapted orthonormal frame (i.e., $E_{m}$ is radical and $(E_{1},...,E_{m-1})$ are orthonormal and tangent to $\Sigma$). If $(M,\mathcal{C})$ is conformally $III$-flat, there exist $g \in \mathcal{C}$ which is $II$-flat on $\Sigma \cap U$ (without loss of generality) and $k \in C^{\infty}(\Sigma \cap U)$, such that $III^{E_{m}} = kg_{\Sigma}$. Since the radical is transverse, we have $II^{E_{mm}} \neq 0$, thus $k_{1} := \frac{k}{II^{E_{mm}}} \in C^{\infty}$ on $\Sigma \cap U$. As in Proposition 7 we can obtain $f \in C^{\infty}(U)$ such that $E_{m} f = \tau \hat{k}_{1}$, where $\tau = g(E_{m},E_{m})$ and $\hat{k}_{1} \in C^{\infty}(U)$ is any local extension of $k_{1}$. Since $(E_{m} f)|_{\Sigma} = 0$, we get $grad_{g}(f) \in \mathcal{X}(U)$ and we have $II^{E_{m}} (grad_{g}(f), E_{m}) = (\tau^{-1}E_{m} f)_{\Sigma} II^{E_{mm}} = k$. Now, take any extension $\hat{f} \in C^{\infty}(M)$ of (possibly a
restriction of) \( f \). Since \( g \) is \( II \)-flat, we deduce from (1) that \( \overline{g} = e^{2f} g \in C \) is also \( II \)-flat on \( U \). We also deduce that \( \overline{g} \) is \( III \)-flat on \( U \).

To prove the converse, first observe that the hypothesis implies in particular that \((M, C)\) is conformally \( II \)-flat. Consider \( p \in \Sigma \) and \( g \in C \), \( II \)-flat on a neighbourhood of \( p \). By hypothesis, there exists \( \overline{g} = e^{2f} g \in C \) which is \( III \)-flat around \( p \). Thus we deduce from (2) that \( III^R = II^R (\text{grad}_g (f), R) g \Sigma \), so \( g \) is conformally \( III \)-flat ♣

In what follows we shall assume that \( \dim M = m \geq 4 \). We now study the extendibility of the Weyl tensor, naturally defined on \((M, C)\). It is well-known that this tensor plays a main role in deciding when \( M \) is (locally) conformally flat, according to Weyl Theorem: a pseudoriemannian conformal manifold is (locally) conformally flat if and only if the Weyl tensor vanishes identically (see for instance the preliminaries of [3]). At the end of the paper we discuss the problem of establish a modified version of Weyl Theorem for transverse Riemann-Lorentz conformal manifolds.

The Weyl tensor \( W \) on \((M, g_M)\) can be defined as

\[
W := K - h \bullet g \in \mathcal{I}_1^0 (M),
\]

where \( h = \frac{1}{m-2} \left\{ \text{Ric} - \frac{Sc}{2(m-1)} g \right\} \) is the Schouten tensor, \( \text{Ric} \) is the Ricci tensor and \( Sc \) is the scalar curvature associated to \((M, g_M)\), and where

\[
\bullet : S^2 (M) \times S^2 (M) \to \mathcal{I}_4^0 (M)
\]

is the so-called Kulkarni-Nomizu product, given by

\[
\theta \bullet \omega (x, y, z, t) := \det \left( \begin{array}{cc} \theta (x, z) & \omega (x, t) \\ \theta (y, z) & \omega (y, t) \end{array} \right) + \det \left( \begin{array}{cc} \omega (x, z) & \theta (x, t) \\ \omega (y, z) & \theta (y, t) \end{array} \right)
\]

If we pick \( \overline{g} = e^{2f} g \in C \), then the Weyl tensor associated to \((M, \overline{g}_M)\) satisfies \( \overline{W} = e^{2f} W \), thus the Weyl conformal curvature \( W := \uparrow_2^1 W \in \mathcal{I}_3^1 (M) \) becomes a conformal invariant. Notice that the extendibility of \( W \) (which is equivalent to the extendibility of \( W \)) is a conformal condition, therefore it should be stated in terms of the conformal structure. In fact, we prove that it is equivalent to conformal \( III \)-flatness.

**Theorem 10** Let \((M, C)\) be a transverse Riemann-Lorentz conformal manifold, with \( \dim M = m \geq 4 \). Then \( W \) (smoothly) extends to the whole \( M \) if and only if the radical is transverse and \( C \) is conformally \( III \)-flat.
**Proof:** If \((M, C)\) has transverse radical and is conformally III-flat, there exist (Proposition \(\Box\)) a \(M\)-open covering \(\{U_\alpha\}\) of \(\Sigma\) and a family of metrics \(\{g_\alpha\}\) in \(C\) such that \(g_\alpha\) is III-flat on \(U_\alpha\). By Theorem \(\Box\) the covariant curvature \(K_\alpha\), the Ricci tensor \(Ric_\alpha\) and the scalar curvature \(S_\alpha\) associated to \(g_\alpha\) extend to \(\Sigma \cap U_\alpha\), therefore the Weyl tensor \(W_\alpha\) also extends to \(\Sigma \cap U_\alpha\). Since this is a conformal condition, \(W_\alpha\) extends to \(\Sigma \cap \cup_\beta\) for all \(\beta\), and thus \(W_\alpha\) extends to the whole \(M\).

To show the converse we start picking an adapted orthonormal frame \((U, E)\). Then, we can express the functions \(W_{abcd} = W(E_a, E_b, E_c, E_d)\) as second order polynomials in \(\tau^{-1} = (g(E_m, E_m))\). Let us call \((W_{abcd})_0, (W_{abcd})_1\) and \((W_{abcd})_2\) the differentiable coefficients of the terms of order 0, 1 and 2. Since \(\tau = 0\) is a local equation for \(\Sigma\), \(W\) extends to \(U\) if and only if the restricted functions \((W_{abcd})_2|_\Sigma\) and \((W_{abcd})_1 + \tau^{-1}(W_{abcd})_2|_\Sigma\) identically vanish.

Suppose the radical is tangent to \(\Sigma\) at a singular point \(p \in \Sigma\). We can choose the frame such that \(E_1(p), E_2(p) \in T_pM - T_p\Sigma\). But then, using that \(II^{E_m}(E_m, E_m)(p) = 0\) (because the radical is tangent), we obtain \((W_{1323}(p))_2 = \frac{\varepsilon}{m-2}II^{E_m}_p(\varepsilon E_1, E_m)II^{E_m}_p(\varepsilon E_2, E_m)\). Since \(E_1\) and \(E_2\) are transverse to \(\Sigma\) at \(p\), \((W_{1323}(p))_2 \neq 0\), hence \(W\) cannot be extended. Therefore the radical must be transverse to \(\Sigma\).

Once we know that the radical must be always transverse to \(\Sigma\) (thus \(II^{E_m}_{mm} \neq 0\)), we can choose the orthonormal frame \((U, E)\) completely adapted. Thus, picking \(i, j, k\) different from \(m\), with \(i, j\) different from \(k\), and using that \(II^{E_m}_{im} = 0\), we have: if \(i \neq j\), then \(0 = (W_{ikjk})_2|_\Sigma = \frac{\varepsilon}{m-2}II^{E_m}_{ij}II^{E_m}_{mm}\). Since \(II^{E_m}_{mm} \neq 0\), we get \(II^{E_m}_{ij} = 0\). If \(i = j\) (and using \(II^{E_m}_{ii} = 0\)), the \(\left(\begin{array}{c} m-1 \\ 2 \end{array}\right)\) equalities \(0 = (W_{ikik})_2|_\Sigma\), suitably manipulated, give us \(\varepsilon_1II^{E_m}_{ii} + \varepsilon_kII^{E_m}_{kk} = \frac{2C}{m-1}\), where \(C = \sum_{i=1}^{m-1} \varepsilon_1II^{E_m}_{ii} \in C^\infty(U)\). Substracting the equation for \(i, k\) from the equation for \(k, j\), we obtain \(\varepsilon_1II^{E_m}_{ii} - \varepsilon_jII^{E_m}_{jj} = 0\), thus \(\varepsilon_1II^{E_m}_{ii} = \varepsilon_jII^{E_m}_{jj}\). Defining \(k := \varepsilon_1II^{E_m}_{11} \in C^\infty(\Sigma \cap U)\), it holds \(II^{E_m}_{ii} = \varepsilon_1II^{E_m}_{11} = kg_{ii}\) and \(II^{E_m}_{ij} = 0 = kg_{ij}\) (where \(i \neq j\)), what means \(II^{E_m}_{ij} = kg_{ij}\), that is, \(g\) is conformally II-flat on \(U\), and therefore \((M, C)\) is conformally II-flat.

Once we know that \((M, C)\) is conformally II-flat, we can choose a metric \(g \in C\) which is II-flat on \(U\) (shrinking \(U\) if necessary). By Theorem \(\Box\) the covariant curvature \(K\) associated to \(g\) extends to \(\Sigma \cap U\) and, since \(W\) also does it, necessarily \(h \bullet g\) extends to \(\Sigma \cap U\). Picking \(i, j, k\) different from \(m\), with \(i, j\) different from \(k\), we get \(\varepsilon_1II^{E_m}_{ikjk} = \varepsilon_kh_{ij} + \delta_{ij} \varepsilon_1h_{kk} = A_{ijk} + \tau^{-1}B_{ijk}\).
therefore the function

\[ B_{ijk} := \frac{1}{m-2} \left\{ \varepsilon_k K_{imjm} + \delta_{ij} \varepsilon_i K_{kmkm} - \frac{2\varepsilon_k \delta_{ij} \varepsilon_i}{m-1} \sum_{l=1}^{m-1} \varepsilon_l K_{ilm\ell} \right\} \]

must vanish on \( \Sigma \). Using the same argument as before, but with the equalities

\[ 0 = B_{ijk} |_{\Sigma}, \]

we get \( III^{E_m} = k g_\Sigma \), where \( k := \varepsilon_1 III^{E_m} |_{\Sigma} \in C^\infty (\Sigma \cap \mathbb{U}) \), that is \( g \) is conformally \( III \)-flat on \( \mathbb{U} \), and thus \( (M, C) \) is conformally \( III \)-flat ♠️

Let us consider the following conjecture:

**Conjecture 11** Let \( (M, C) \) be a transverse Riemann-Lorentz conformal manifold, with \( \dim M = m \geq 4 \). A necessary condition for being \( W = 0 \) is that, around every singular point \( p \in \Sigma \), there exist a coordinate system \((\mathbb{U}, x)\) and a metric \( g \in \mathcal{C} \) such that

\[ g = \sum_{i=0}^{m-1} (dx^i)^2 + \tau (dx^m)^2, \]

where \( \tau = 0 \) is a local equation for \( \Sigma \).

Using Theorem 3 it becomes obvious that the necessary condition stated in the conjecture is always sufficient for having \( W = 0 \) around \( \Sigma \).

If the conjecture is true, \( \Sigma \) must be (locally) conformally flat, which is well known equivalent to either \( W^{\Sigma} = 0 \) (if \( m > 4 \)) or \( \nabla_X h^{\Sigma} (Y, Z) = \nabla_Y h^{\Sigma} (X, Z) \) (if \( m = 4 \)). But the extendibility of \( W \), equivalent (Theorem 10) to conformal \( III \)-flatness, implies (Proposition 9) the existence of a metric \( g \in \mathcal{C} \) which is \( III \)-flat around \( \Sigma \), thus satisfying (Proposition 4):

\[ W |_{T\Sigma} = (K - h \cdot g) |_{T\Sigma} = K^{\Sigma} - h |_{T\Sigma} \cdot g_\Sigma = W^{\Sigma} + (h^{\Sigma} - h |_{T\Sigma}) \cdot g_\Sigma. \]

Because conditions \( W = 0 \) and \( W^{\Sigma} = 0 \) are conformal, any counterexample \((M, C)\) to the above conjecture must admit a metric \( g \in \mathcal{C} \) which is \( III \)-flat around \( \Sigma \) and satisfies either \( h^{\Sigma} \neq h |_{T\Sigma} \) (if \( m > 4 \)) or (Lemma 3) \( \nabla_X h(Y, Z) \neq \nabla_Y h(X, Z) \), for some \( X, Y, Z \in \mathfrak{X}(\Sigma) \) (if \( m = 4 \)). Now a straightforward computation for \( III \)-flat metrics, using an orthonormal completely adapted frame, leads to the following expression in terms of extendible quantities:

\[ h^{\Sigma}_{ij} - h_{ij} |_{T\Sigma} = \frac{-1}{m-2} \left\{ \frac{K_{imjm}}{\tau} - \frac{1}{m-3} \sum_{l=1}^{m-1} K_{iljl} - \frac{1}{m-1} \left[ \sum_{k=1}^{m-1} \frac{K_{kmkm}}{\tau} - \frac{1}{m-3} \sum_{k,l=1}^{m-1} K_{klkl} \right] \delta_{ij} \right\} |_{\Sigma}, \]
(i, j = 1, ..., m − 1), which shows that the construction of counterexamples is not easy.

In fact, the conjecture is true for transverse Riemann-Lorentz warped products, as we show right now. Let us consider a m-dimensional (m ≥ 4) transverse Riemann-Lorentz manifold \((M, g)\) of the form \(M = I \times S\), where \(\dim I = 1, 0 \in I\), and \(g = f (t)^2 g_s − td t^2\), where \(f \in C^\infty (I), f > 0\) and \(g_s\) is riemannian (we identify \(t\), \(f\) and \(g_s\) with the corresponding pullbacks by the canonical projections). Thus \(\Sigma = \{0\} \times S\) is homothetic to \(S\) with scale factor \(f(0)\). Calling \(U \in \mathcal{X}(M)\) the (nowhere zero) lift of the vectorfield \(\frac{d}{dt} \in \mathcal{X}(I)\), one immediately sees that \(U\) is radical and transverse to \(\Sigma\). It is not difficult to compute the curvature tensors on \(M\). Standard results on warped products (see \(\mathbb{K}\), Chapter 7) lead to (we denote by \(X, Y \in \mathcal{X}(M)\) the lifts of corresponding vectorfields \(\tilde{X}, \tilde{Y} \in \mathcal{X}(S)\)) the lifts of the corresponding vectorfields on \(S\) and \(\nabla_S \tilde{Y}\) is the lift of the corresponding vectorfield on \(S\) and also the following expressions for the curvature tensors:

\[
\begin{align*}
K &= f^2 K^S + \frac{f'^2 f^2}{2 f} g_s \cdot g_s + \frac{f'}{2} (f'_t - 2 f'') g_s \cdot dt^2 \\
Ric &= Ric^S - \left(\frac{f'}{2 f} (f'_t - 2 f'') - (m - 2) \frac{f'^2}{f^2}\right) g_s + \frac{m-1}{2f} (f'_t - 2 f'')dt^2 \\
Sc &= S_c^S - \frac{m-1}{f^2} \left(\frac{f'}{2 f} (f'_t - 2 f'') - (m - 2) \frac{f'^2}{f^2}\right) \\
h &= \frac{m-3}{m-2} h^S + \left(\frac{Sc^S}{2(m-2)(m-1)} + \frac{f'^2}{2f}\right) g_s + \\
&\quad + \left(\frac{Sc^S}{2(m-1)(m-2)} + \frac{1}{2f} \left(\frac{f'^2}{f} + f'_t - 2 f''\right)\right) dt^2 \\
W &= f^2 W^S + \frac{1}{(m-2)} \left(Ric^S - \frac{Sc^S}{m-2} g_s\right) \cdot \left(\frac{f'^2}{m-3} g_s + td t^2\right)
\end{align*}
\]

\((K^S, Ric^S, Sc^S, h^S\) and \(W^S\) denote of course the pullbacks by the projection of the corresponding tensor fields on \(S\)). It follows:

**Lemma 12** The following three conditions are equivalent: (1) \(K\) extends to \(M\), (2) \(f'(0) = 0\) and (3) \(h\) extends to \(M\). Also the following are equivalent: (1) \(Ric\) extends to \(M\), (2) \((f'/t)(0) = 0\) and (3) \(Sc\) extends to \(M\). Moreover, \(W\) extends to \(M\) in any case.

The fact that \(W\) extends to \(M\) was obvious from the very beginning: the map \(\Psi \equiv \psi \times id : (I \setminus \{0\}) \times S \to \mathbb{R} \times S\), given by \(T \equiv \psi(t) := \int^t_0 \frac{|s|^2 ds}{f(s)}\), is
a conformal diffeomorphism onto its (non-connected) image with the metric \( g \equiv -(dT)^2 + g_S \), thus it preserves the \((\frac{1}{3})\)-Weyl tensors, and since \( g \) is regular around \( T = 0 \) and \( f(0) \neq 0 \), \( W \) (and therefore \( W \)) extends to the whole \( M \). It follows from Theorem 10 that the conformal manifold \((M, [g])\) is (in any case) conformally \( III \)-flat.

**Lemma 13** The following four conditions are equivalent: (1) \( W = 0 \), (2) \( W^S = 0 = Ric^S - \frac{Sc^S}{m-1}g_S \) and (3) \( \Sigma \) has constant (sectional) curvature.

**Proof:** (1) \( \Leftrightarrow \) (2) follows from the above formula. (2) \( \Rightarrow \) (3): \( Ric^S - \frac{Sc^S}{m-1}g_S = 0 \) implies (Schur’s lemma) \( Sc^S = (m-1)(m-2)C \) (constant), thus \( h^S = \frac{C}{2} g_S \); moreover \( W^S = 0 \) leads to \( K^S = \frac{C}{2} g_S \cdot g_S. \) (3) \( \Rightarrow \) (2): From \( K^S = \frac{C}{2} g_S \cdot g_S \), one immediately gets \( W^S = 0 = Ric^S - \frac{Sc^S}{m-1}g_S \). 

**Proposition 14** The Conjecture 11 is true for any transverse Riemann-Lorentz conformal manifold \((M, C)\) such that some \( g \in C \) is a warped product.

**Proof:** Let \( g = f(t)^2 g_s - t dt^2 \in C \) be a transverse warped product metric on \( M = I \times S \). Note that \( g = f(t)^2 \left\{ g_s - \frac{t}{f(t)} dt^2 \right\} \). From \( W = 0 \) and Lemma 13, we get, around any \( p \in \Sigma \), coordinates \((V, y)\) of \( \Sigma \) such that \( f(0)^2 g_s = g_\Sigma = e^{2h} \sum_{i=1}^{m-1} (dy^i)^2 \), for some \( h \in C^\infty(\Sigma) \). Choosing \( x^i := y^i \circ \pi \), \( x^m := t \) and \( \tau := \frac{e^{2h}}{f(t)^2} \), we get \( g = e^{2h} f(t)^2 \left\{ \sum_{i=1}^{m-1} (dx^i)^2 + \tau (dx^m)^2 \right\} \), and we are finished.

**References**

[1] E. Aguirre and J. Lafuente. *Trasverse Riemann-Lorentz metrics with tangent radical*. Diff. Geom. its App, 24, 2, 91-100, 2005.

[2] J. B. Hartle and S. W. Hawking. *Wave Function of the Universe*. Phys. Rev., D41, 1815-34, 1990.

[3] U. Hertrich-Jeromin. *Introduction to Möbius Differential Geometry*. Cambridge Univ. Press, 2003.

[4] M. Kossowski. *Fold singularities in pseudoriemannian geodesic tubes*. Proc. Amer. Math. Soc., 95, 463-469, 1985.
[5] M. Kossowski. *Pseudo-riemannian metric singularities and the extendability of parallel transport.* Proc. Amer. Math. Soc., 99, 147-154, 1987.

[6] M. Kossowski and M. Kriele. *Transverse, type changing, pseudo riemannian metrics and the extendability of geodesics.* Proc. R. Soc. Lond. A 444, 297-306, 1994.

[7] M. Kossowski and M. Kriele. *The volume blow-up and characteristic classes for transverse, type changing, pseudo-riemannian metrics.* Geom. Dedicata 64, 1-16, 1997.

[8] B. O’Neill. *Semi-riemannian Geometry.* Academic Press, 1983.