ON DEFORMATIONS OF GORENSTEIN-PROJECTIVE MODULES OVER MONOMIAL ALGEBRAS WITH NO OVERLAPS

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Abstract. Let \( k \) be a field of arbitrary characteristic, let \( \Lambda \) be a finite dimensional \( k \)-algebra, and let \( V \) be an indecomposable Gorenstein-projective \( \Lambda \)-module with finite dimension over \( k \). It follows that \( V \) has a well-defined versal deformation ring \( R(\Lambda, V) \), which is complete local commutative Noetherian \( k \)-algebra with residue field \( k \), and which is universal provided that the stable endomorphism ring of \( V \) is isomorphic to \( k \). We prove that if \( \Lambda \) is a monomial algebra without overlaps, then \( R(\Lambda, V) \) is universal and isomorphic either to \( k \) or to \( k[[t]]/(t^2) \).

1. Introduction and Preliminary Results

Let \( k \) be a field of arbitrary characteristic and denote by \( \hat{C} \) the category of all complete local commutative Noetherian \( k \)-algebras with residue field \( k \). In particular, the morphisms in \( \hat{C} \) are continuous \( k \)-algebra homomorphisms that induce the identity map on \( k \). Let \( \Lambda \) be a finite dimensional \( k \)-algebra. We denote by \( \Lambda \)-mod the abelian category of left \( \Lambda \)-modules with finite dimension over \( k \), and for all objects \( V \) in \( \Lambda \)-mod, we denote by \( \Omega V \) the first syzygy of \( V \), i.e., the kernel of a projective cover \( \pi V : P(V) \to V \). For all objects \( R \) in \( \hat{C} \), we denote by \( R \otimes k \Lambda \) the tensor product of \( k \)-algebras \( R \otimes k \Lambda \). Let \( V \) be a fixed finitely generated left \( \Lambda \)-module. We denote by \( \text{End}_\Lambda(V) \) (resp., by \( \text{End}_\Lambda(V) \)) the endomorphism ring (resp., the stable endomorphism ring) of \( V \). Let \( \hat{R} \) be an arbitrary object in \( \hat{C} \). Following [12], a lift \((M, \phi)\) of \( V \) over \( \hat{R} \) is a finitely generated left \( \hat{R} \Lambda \)-module \( M \) that is free over \( \hat{R} \) together with an isomorphism of \( \Lambda \)-modules \( \phi : k \otimes_R M \to V \). Two lifts \((M, \phi)\) and \((M', \phi')\) over \( \hat{R} \) are isomorphic if there exists an \( \hat{R} \Lambda \)-module isomorphism \( f : M \to M' \) such that \( \phi' \circ (k \otimes_R f) = \phi \). If \((M, \phi)\) is a lift of \( V \) over \( \hat{R} \), then we denote by \([M, \phi]\) its isomorphism class and say that \([M, \phi]\) is a deformation of \( V \) over \( \hat{R} \). We denote by \( \text{Def}_\Lambda(V, R) \) the set of all deformations of \( V \) over \( R \). The deformation functor over \( V \) is the covariant functor \( \hat{F}_V : \hat{C} \to \text{Sets} \) defined as follows: for all objects \( R \) in \( \hat{C} \) define \( \hat{F}_V(R) = \text{Def}_\Lambda(V, R) \), and for all morphisms \( \alpha : R \to R' \) in \( \hat{C} \), let \( \hat{F}_V(\alpha) : \text{Def}_\Lambda(V, R) \to \text{Def}_\Lambda(V, R') \) be defined as \( \hat{F}_V(\alpha)([M, \phi]) = [R' \otimes_{R, \alpha} M, \phi_\alpha] \), where \( \phi_\alpha : k \otimes_R (R' \otimes_{R, \alpha} M) \to V \) is the composition of \( \Lambda \)-module isomorphisms \( k \otimes_R (R' \otimes_{R, \alpha} M) \cong k \otimes_R M \xrightarrow{\phi} V \). Suppose
there exists an object \( R(\Lambda, V) \) in \( \hat{\mathcal{C}} \) and a deformation \([U(\Lambda, V), \phi_{U(\Lambda, V)}]\) of \( V \) over \( R(\Lambda, V) \) with the following property. For each object \( R \) in \( \hat{\mathcal{C}} \) and for all lifts \((M, \phi)\) of \( V \) over \( R \) there exists a morphism \( \psi_{R(\Lambda, V)} : R(\Lambda, V) \to R \) in \( \hat{\mathcal{C}} \) such that

\[
\hat{F}_V(\psi_{R(\Lambda, V)})[U(\Lambda, V), \phi_{U(\Lambda, V)}] = [M, \phi],
\]

and moreover \( \psi_{R(\Lambda, V)} \) is unique if \( R \) is the ring of dual numbers \( k[[t]]/(t^2) \). Then \( R(\Lambda, V) \) and \([U(\Lambda, V), \phi_{U(\Lambda, V)}]\) are respectively called the \textit{versal deformation ring} and \textit{versal deformation} of \( V \). If the morphism \( \psi_{R(\Lambda, V)} \) is unique for all objects \( R \) in \( \hat{\mathcal{C}} \) and all lifts \((M, \phi)\) of \( V \) over \( R \), then \( R(\Lambda, V) \) and \([U(\Lambda, V), \phi_{U(\Lambda, V)}]\) are respectively called the \textit{universal deformation ring} and the \textit{universal deformation} of \( V \). In other words, the universal deformation ring \( R(\Lambda, V) \) represents the deformation functor \( \hat{F}_V \) in the sense that \( \hat{F}_V \) is naturally isomorphic to the Hom functor \( \text{Hom}_V(R(\Lambda, V), -) \). Using Schlessinger’s criteria [36, Thm. 2.11] and using methods similar to those in [31], it is straightforward to prove that the deformation functor \( \hat{F}_V \) is continuous (see [31, §14] for the definition), that every finitely generated \( \Lambda \)-module \( V \) has a versal deformation ring, and that this versal deformation is universal provided that the endomorphism ring \( \text{End}_\Lambda(V) \) is isomorphic to \( k \) (see [12, Prop. 2.1]). Moreover, it follows from [12, Prop. 2.5] that Morita equivalence preserve versal deformation rings.

Following [21, 22], we say that a (not necessarily finitely generated) left \( \Lambda \)-module \( M \) is \textit{Gorenstein-projective} provided that there exists an acyclic complex of (not necessarily finitely generated) projective left \( \Lambda \)-modules

\[
P^\bullet : \cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \xrightarrow{f^2} \cdots
\]

such that \( \text{Hom}_\Lambda(P^\bullet, \Lambda) \) is also acyclic and \( M = \text{coker} f^0 \).

We denote by \( \Lambda\text{-Gproj} \) the full subcategory of \( \Lambda\text{-mod} \) whose objects are Gorenstein-projective \( \Lambda \)-modules. Note that all projective modules in \( \Lambda\text{-mod} \) are also Gorenstein-projective. Since \( \Lambda\text{-Gproj} \) is closed under extensions, it becomes naturally an exact category in the sense of [34]. Moreover, it is also a Frobenius category, i.e., it has enough relatively projective and injective objects, which also coincide. As a matter of fact, the relatively projective-injective objects in \( \Lambda\text{-Gproj} \) are the projective objects in \( \Lambda\text{-mod} \). We denote by \( \Lambda\text{-Gproj} \) the stable category of \( \Lambda\text{-Gproj} \). It follows from [25, §I.2] that \( \Lambda\text{-Gproj} \) is a triangulated category. From the observations in the paragraph after the proof of [17, Lemma 2.1] and [17, Lemma 2.2], it follows that \( \Omega \) induces a self-equivalence of \( \Lambda\text{-Gproj} \). On the other hand, if \( V \) is an object of \( \Lambda\text{-Gproj} \), then \( \text{Ext}_\Lambda^i(V, V) = \text{Hom}_\Lambda(\Omega^iV, V) \) for all \( i \geq 1 \) (see e.g. [16, Lemma 2.1.8]).

Following [17], we say that \( \Lambda \) is CM-finite provided that there are at most finitely many isomorphism classes of indecomposable Gorenstein-projective \( \Lambda \)-modules. We say that \( \Lambda \) is CM-free provided that all Gorenstein-projective \( \Lambda \)-modules are projective.

If \( \Lambda \) has finite global dimension, then every Gorenstein-projective module is projective, i.e., \( \Lambda \) is CM-free. If \( \Lambda \) is self-injective, then every finitely generated left \( \Lambda \)-module is Gorenstein-projective. Thus the study of Gorenstein-projective modules is dedicated mostly to non-self-injective algebras with infinite global dimension. However, explicit classifications of finitely generated Gorenstein-projective modules have been found for a very few number of classes of such algebras. More precisely,
in [35], C.M. Ringel classified all objects in $\Lambda$-Gproj for when $\Lambda$ is a Nakayama algebra. In [30], M. Kalck obtained also such classification for when $\Lambda$ is a gentle algebra (as introduced in [3]).

Let $V$ be an object in $\Lambda$-Gproj such that $\text{End}_\Lambda(V) = k$. By [9, Thm. 1.2 (ii)], it follows that the versal deformation ring $R(\Lambda, V)$ of $V$ is universal. Following [5], we say that $\Lambda$ is a Gorenstein $k$-algebra provided that $\Lambda$ has finite injective dimension as a left and right $\Lambda$-module. In particular, algebras of finite global dimension as well as self-injective algebras are Gorenstein. By [9, Thm. 1.2 (iii)] that the isomorphism class of versal deformation rings of finitely generated Gorenstein-projective modules is preserved by singular equivalences of Morita type (as introduced in [20] and discussed in [38]) between Gorenstein $k$-algebras.

It is important to mention that $\Lambda$-Gproj is closely related to the singularity category of $\Lambda$ (see e.g. [13, 14, 15, 18, 26, 30] and [32, 33] in the context of algebraic geometry and mathematical physics).

Recall that a quiver $Q$ is a directed graph with a set of vertices $Q_0$, a set of arrows $Q_1$ and two functions $s, t: Q_1 \to Q_0$, where for all $\alpha \in Q_1$, $s\alpha$ (resp. $t\alpha$) denotes the vertex where $\alpha$ starts (resp. ends). A path in $Q$ is either an ordered sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $t\alpha_j = s\alpha_{j+1}$ for $1 \leq j < n$ (in this situation we say that $p$ has length $n$), or for each $i \in Q_0$, the symbol $e_i$ such that $s e_i = i = t e_i$. We call the symbols $e_i$ the trivial paths, which have length zero. For a nontrivial path $p = \alpha_n \cdots \alpha_1$ we define $sp = s\alpha_1$ and $tp = t\alpha_n$. A non-trivial path $p$ in $Q$ is said to be an oriented cycle provided that $sp = tp$. The path algebra $kQ$ of a quiver $Q$ is the $k$-vector space whose basis consists in all the paths in $Q$, and for two paths $p$ and $q$, their multiplication is given by the concatenation $pq$ provided that $sp = tq$, or zero otherwise. Let $J$ be the two-sided ideal of $kQ$ generated by all the arrows in $Q$. We say that an ideal $I$ of $kQ$ is admissible if there exists $d \geq 2$ such that $J^d \subseteq I \subseteq J^2$. In this situation, the quotient $kQ/I$ is a finite dimensional $k$-algebra. If $p$ is a path in $Q$, we denote also by $p$ is equivalence class a call it a path in $kQ/I$. In particular, a path $p$ in $kQ/I$ is zero if and only if $p$ belongs to $I$. Recall that an admissible ideal $I$ of $kQ$ is said to be monomial if it is generated by paths of length at least two. In this situation we say that the quotient $kQ/I$ is a monomial algebra.

In [19], X.-W. Chen et al. obtained a classification of all indecomposable objects in $\Lambda$-Gproj for when $\Lambda$ is a monomial algebra, which generalizes the classification provided in [30]. Moreover, they also proved that every monomial algebra is CM-finite.

Using the equivalence [19, Prop. 5.9], we say that a monomial algebra $\Lambda = kQ/I$ has no overlaps provided that the stable category $\Lambda$-Gproj is semi-simple, i.e., $\Lambda$-Gproj is triangle equivalent to a semi-simple abelian category in the sense of e.g. [27, §5]. In particular, gentle algebras as well as quadratic monomial algebras are also monomial algebras with no overlaps (see [19, §5] for more details).

The main goal of this article is to prove the following result.

**Theorem 1.1.** Let $\Lambda = kQ/I$ be a monomial algebra with no overlaps and let $V$ be an indecomposable Gorenstein-projective left $\Lambda$-module with finite dimension over $k$. Then the versal deformation ring $R(\Lambda, V)$ of $V$ is universal and isomorphic either to $k$ or to $k[[t]]/(t^2)$.

As a consequence, we obtain the following result, which improves [9, Cor. 5.2],
Corollary 1.2. Let $\Lambda$ be a finite dimensional gentle algebra and let $V$ be an indecomposable Gorenstein-projective left $\Lambda$-module with finite dimension over $k$. Then the versal deformation ring $R(\Lambda, V)$ of $V$ is universal and isomorphic either to $k$ or to $k[[t]]/(t^2)$.

It follows that Corollary 1.2 extends the results in [9, Prop. 5.3] to arbitrary finite dimensional gentle algebras.

These monomial algebras have been studied since a long time (see e.g. [24, 29, 28]). It is also important to mention that gentle algebras have been studied by many authors in different contexts (see e.g. [1, 10, 14, 30, 23, 37] and their references).

We refer the reader to look at [4, 6, 8, 13, 16, 21, 22] (and their references) for basic concepts concerning Gorenstein-projective modules as well as their applications in other settings. For more details about versal deformation rings in other settings, we refer the reader to [11], and for basic concepts from the representation theory of algebras, we refer the reader to [2] and [7].

2. Proof of Theorem 1.1

Let $\Lambda = \mathbb{k}Q/I$ be a monomial algebra. Following [19, Def. 3.3], we say that a pair $(p, q)$ of non-zero paths in $\Lambda$ is perfect provided that the following conditions are satisfied:

(P1) both $p$ and $q$ are non-trivial with $sp = tq$ and $pq \in I$;
(P2) if $pq' \in I$ for a non-zero path $q'$ with $tq' = sp$, then $q' = qq''$ for some path $q''$ in $\Lambda$;
(P3) if $p'q \in I$ for a non-zero path $p'$ with $tq = sp'$, then $p' = p''p$ for some path $p''$ in $\Lambda$.

Following [19, Def. 3.7], we say that a non-zero path $p$ in $\Lambda$ is perfect, provided that there exists a sequence $p = p_1, p_2, \ldots, p_n, p_{n+1} = p$ such that for all $1 \leq i \leq n$, the pair $(p_i, p_{i+1})$ is a perfect pair. It follows from [19, Thm. 4.1] that $V$ is indecomposable object in $\Lambda$-Gproj if and only if $V = \Lambda p$, where $p$ is a perfect path in $\Lambda$. In this situation, if $V = \Omega V = \Omega \Lambda p$ is the first syzygy of $V = \Lambda p$, then it follows from [19, Lemma 3.1] that there exists a non-zero path $q$ in $\Lambda$ such that $\Omega V = \Lambda q$, and thus we obtain a short exact sequence of left $\Lambda$-modules

$$0 \to \Lambda q \xrightarrow{\iota} \Lambda e \xrightarrow{\pi} \Lambda p \to 0. \quad (2.1)$$

Note that $\Lambda q$ is also a non-projective Gorenstein-projective left $\Lambda$-module. By [19, Prop. 4.3], we can assume that $q$ is also a perfect path in $\Lambda$. In this situation, by using the arguments in the proof of [19, Prop. 5.9], we obtain that if $\Lambda$ has no overlaps, then $\text{End}_{\Lambda}(V) = k$ and

$$\text{Ext}_{\Lambda}^1(V, V) = \text{Hom}_{\Lambda}(\Omega V, V) = \begin{cases} k, & \text{if } V = \Omega V, \\ 0, & \text{otherwise}. \end{cases} \quad (2.2)$$

Proof of Theorem 1.1. Assume that $\Lambda$ is a monomial algebra with no overlaps and let $V$ be an indecomposable object in $\Lambda$-Gproj. If $V$ is projective, then it follows from [9, Lemma 3.4] that $R(\Lambda, V)$ is universal and isomorphic to $\mathbb{k}$. Assume then that $V$ is non-projective. By the observations above, we have that $\text{End}_{\Lambda}(V) = k$, and by [9, Thm. 1.2 (ii)] we obtain that $R(\Lambda, V)$ is universal. If $\Omega V \neq V$, then by (2.3) we have that $\text{Ext}_{\Lambda}^1(V, V) = 0$, which implies that $R(\Lambda, V) \cong k$. Assume next
that $\Omega V = V$. Then again by (2.3), we obtain that $\Ext^1_\Lambda(V, V) \cong k$, which implies that $R(\Lambda, V)$ is isomorphic to a quotient of $k[[t]]$. Consider then the corresponding short exact sequence of left $\Lambda$-modules

(2.3) \[ 0 \to V \xrightarrow{\psi} P(V) \xrightarrow{\pi_V} V \to 0, \]

where $\pi_V : P(V) \to V$ is a projective $\Lambda$-module cover of $V$. It follows that $P(V)$ defines a non-trivial lift of $V$ over the ring of dual numbers $k[[t]]/(t^2)$, where the action of $t$ is given by $\nu_V \circ \pi_V$. This implies that there exists a unique surjective $k$-algebra morphism $\psi : R(\Lambda, V) \to k[[t]]/(t^2)$ in $\mathcal{C}$ corresponding to the deformation defined by $P(V)$. We need to show that $\psi$ is an isomorphism. Suppose otherwise. Then that there exists a surjective $k$-algebra homomorphism $\psi_0 : R(\Lambda, V) \to k[[t]]/(t^2)$ in $\mathcal{C}$ such that $\pi' \circ \psi_0 = \psi$, where $\pi' : k[[t]]/(t^3) \to k[[t]]/(t^2)$ is the natural projection. Let $M_0$ be a $k[[t]]/(t^3)$-$\Lambda$-module which defines a lift of $V$ over $k[[t]]/(t^3)$ corresponding to $\psi_0$. Let $(U(\Lambda, V), \phi_{U(\Lambda, V)})$ be a lift of $V$ over $R(\Lambda, V)$ that defines the universal deformation of $V$. Then $M_0 \cong k[[t]]/(t^3) \otimes_{R(\Lambda, V), \psi_0} U(\Lambda, V)$. Note that $M_0/tM_0 \cong V$ as $\Lambda$-modules. On the other hand, we also have that

\[
P(V) \cong k[[t]]/(t^2) \otimes_{R(\Lambda, V), \psi} U(\Lambda, V) \cong k[[t]]/(t^2) \otimes_{k[[t]]/(t^3), \pi'} (k[[t]]/(t^3) \otimes_{R(\Lambda, V), \psi_0} U(\Lambda, V)) \cong k[[t]]/(t^2) \otimes_{k[[t]]/(t^3)} M_0.
\]

Note that since $\ker \pi' = (t^2)/(t^3)$, we have $k[[t]]/(t^2) \otimes_{k[[t]]/(t^3), \pi'} (k[[t]]/(t^3) \otimes_{R(\Lambda, V), \psi_0} U(\Lambda, V)) \cong M_0/t^2 M_0$. Thus $P(V) \cong M_0/t^2 M_0$ as $k[[t]]/(t^3)$-$\Lambda$-modules. Consider the surjective $k[[t]]/(t^3)$-$\Lambda$-module homomorphism $g : M_0 \to t^2 M_0$ defined by $g(x) = t^2 x$ for all $x \in M_0$. Since $M_0$ is free over $k[[t]]/(t^3)$, it follows that $\ker g = tM_0$ and thus $M_0/tM_0 \cong t^2 M_0$, which implies that $V \cong t^2 M_0$. Hence we get a short exact sequence of $k[[t]]/(t^3)$-$\Lambda_0$-modules

(2.4) \[ 0 \to V \to M_0 \to P(V) \to 0. \]

Since $P(V)$ is a projective left $\Lambda$-module, it follows that (2.4) splits as a short sequence of $\Lambda$-modules. Hence $M_0 = V \oplus P(V)$ as $\Lambda$-modules. Writing elements of $M_0$ as $(u, v)$ where $u \in V$ and $v \in P(V)$, the $t$-action on $M_0$ is given as $t(u, v) = (\lambda \pi_V(v), tv)$ for some $\lambda \in k^*$. Since $\ker \lambda \pi_V = tP(V)$, then $\lambda \pi_V(tv) = 0 = t^2 v$ for all $v \in P(V)$, which implies that $t^2(u, v) = (\lambda \pi(tv), t^2 v) = (0, 0)$ for all $u \in V$ and $v \in P(V)$, which contradicts that $t^2 M_0 \cong V$. Thus $\psi$ is a $k$-algebra isomorphism and $R(\Lambda, V) \cong k[[t]]/(t^2)$. This finishes the proof of Theorem 1.1.

\section*{Acknowledgments.} This article was developed during the visit of the second author to the Instituto de Matemáticas at the Universidad de Antioquia in Medellín, Colombia during Summer 2017. The second author would like to express his gratitude to the first author, faculty members, staff and students at the Instituto de Matemáticas as well as the other people related to this work at the Universidad de Antioquia for their hospitality and support during his visit.

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