Role reversal in a Bose-condensed optomechanical system

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We analyze the optomechanical properties of a Bose-Einstein condensate trapped inside an optical resonator and driven by both a classical and a quantized light field. We find that this system is characterized by a role reversal between the matter-wave field and the quantized optical field, which behaves formally like an “optical mirror”, in contrast to the familiar situation in optomechanics\textsuperscript{[1]}. We demonstrate that this system can lead to the creation of a variety of non-classical matter-wave fields, in particular Schrödinger cat states, and discuss several possible protocols to measure their Wigner function.

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Optomechanics is a fast-progressing area of research that merges techniques and approaches from fields ranging from atomic, molecular and optical physics to nanoscience and to condensed matter physics. There are two major approaches to this field: a top-down approach, exploiting a range of resources from nanoscience, advanced materials, and cryogenics, and a bottom-up approach that relies largely on developments in ultracold atomic science and cavity QED. Cavity optomechanics finds its origin in ideas developed by Braginski and coworkers in the context of the interferometric detection of gravitational waves \textsuperscript{[2]}. Its more recent focus is directed in large part to the challenge of operating mechanical oscillators deep in the quantum regime, with a motivation ranging from fundamental physics tests to high-precision quantum metrology, and from the understanding of the quantum-classical interface to the realization of interfaces for quantum information networks \textsuperscript{[3]}. Recent highlights include the successful cooling of optomechanical systems to within a fraction of their quantum mechanical ground state \textsuperscript{[4]}.

Parallel developments in the bottom-up approach have used a cloud of ultracold atoms as a mechanical element interacting with light \textsuperscript{[1,5]}. In these realizations, the role of the movable mirror is played by a centroid or internal motion of an ultracold gas, for instance a Bose-Einstein condensate (BEC). A number of fundamental effects have already been demonstrated, including the onset of several quantum phase transitions as well as the quantum back-action of a single optical photon. Hybrid optical systems that couple nanoscale systems to atomic systems are also of much interest, as they combine relatively robust mechanical devices with the remarkable precision measurements available in AMO science.

This paper considers a new bottom-up realization of a cavity optomechanical system where the roles of the optical and matter-wave fields are reversed: the role of the effective oscillating mirror is now played by an excitation of a mode of an optical cavity, while the usual role of light is now played by a trapped Bose condensate. In other words, the dynamics of an “optical mirror” is governed by the “radiation pressure” from a matter-wave field. We discuss a number of properties of that system, demonstrating that it can be utilized to generate Schrödinger cat states and near-number states of the matter-wave field. We also draw an analogy between this system and a cavity-QED situation to show how it can be used to nondestructively detect the matter-wave field and reconstruct its Wigner function.

We consider a scalar BEC confined in a three-dimensional trap located inside a single-mode unidirectional ring high-Q optical resonator. It is driven by a classical laser field of frequency $\Omega_p$ and wave vector $k_p$, and the scattered light of frequency $\omega_c$ and wave vector $k_c$ is collected along the axis of the resonator, see Fig. 1. Such a system has been realized and manipulated in recent experiments on BEC superradiance in a cavity \textsuperscript{[6]}.

Treating the incident laser field classically and the scattered field quantum mechanically, their interaction with the condensate is described by the Hamiltonian

$$H = \hbar \int \! d^3r \left[ \frac{\Omega_p}{2} \hat{\Psi}_g^\dagger \hat{\Psi}_g + g_c \hat{c} \hat{\Psi}_g \hat{\Psi}_r \right] + \text{h.c.}$$

(1)

Here $\hat{\Psi}_c$ and $\hat{\Psi}_g$ are the field annihilation operators for atoms in the excited and ground electronic state, respectively, $\Omega_p$ is the Rabi frequency of the pumping laser, and $\hat{c}$ is the photon annihilation operator for the cavity mode, with vacuum Rabi frequency $g_c$.

For large atom-field detunings $\Delta$ it is possible to eliminate adiabatically the upper electronic state. Keeping only the lowest-order term in the scattering field, and in the rotating wave approximation, the atom-filed interaction reduces then to

$$H \approx \frac{\hbar |\Omega_p|^2}{2\Delta} \int \! d^3r \hat{\Psi}_g^\dagger \hat{\Psi}_g + \frac{\hbar \Omega_p g_c^2}{2\Delta} \int \! d^3r \left[ \hat{\Psi}_g^\dagger \hat{\Psi}_g \hat{\Psi}_r \hat{\Psi}_r^\dagger e^{i\Delta k \cdot r} + \text{h.c.} \right],$$

(2)

where $\Delta k = k_p - k_c$. 

\\
The first term in Eq. (2) is a constant ac Stark shift caused by the pump field. In the following we renormalize it into the definition of the frequency of the trap potential of the ground-state atoms. The second term describes the absorption of laser light followed by scattering into the cavity mode and the reverse process, with a recoil of wave vector \( \Delta k \). Expanding the field operator \( \Psi_g(r) \) in terms of the eigenstates of the three-dimensional trap, 
\[
\Psi_g(r) = \sum_n \hat{a}_n \phi_n(r),
\]
where \( \hat{a}_n \) are the corresponding boson annihilation operators for atoms and \( n \equiv (n_x, n_y, n_z) \) is a triple index, that part of the interaction can be expressed as
\[
V = \hbar \sum_{n,m} G_{n,m} \hat{a}_n \hat{a}_m c^\dagger + h.c.,
\]
where
\[
G_{n,m} = (\Omega_p g^*_{c}/2\Delta) \int dr \phi_n^* (r) \phi_m (r) c^\dagger \Delta k \cdot r.
\]
are center of mass (CM) motional state dipole transition moments, some of which are illustrated in Fig. 2.

As discussed in Ref. [2], for atoms initially in the trap ground state and large scattering angles, \( \Delta k \cdot L \gg 1 \) where \( L \) is the characteristic scale of the condensate in the direction of \( \Delta k \), one finds that for short times the transition \( m = 0 \rightarrow n > 0 \) dominates and triggers a superradiant scattering of the condensate. In contrast, the present work considers a situation where the angle between the pump and cavity fields is such that \( \Delta k \) coincides with the tight axis of the BEC, so that \( \Delta k \cdot L \ll 1 \). That condition can be easily realized on atom chips, where the scale of the tight axis of the condensate is of the order of \( 10^{-7} \) m [3]. In that case \( G_{n,m} \) is sharply peaked around the transition \( m = 0 \rightarrow n = 0 \), a situation similar to the Lamb-Dicke limit for trapped ions. This approximation is analogous to the single-mode approximation in conventional optics, and allows us to approximate \( V \) as
\[
V \approx \hbar G (c^\dagger + \hat{c}) \hat{a}^\dagger \hat{a},
\]
where \( G \equiv G_{0,0} \) and \( \hat{a}^\dagger \hat{a} \equiv \hat{a}_0^\dagger \hat{a}_0 \). This is the same form as in the familiar optomechanical interaction in ultracold atomic systems [4], except that the roles of photons and atoms are exchanged. The optomechanical interaction is now proportional to the intensity of matter wave \( \hat{a}^\dagger \hat{a} \) and drives the position quadrature \( c^\dagger + \hat{c} \) of an effective “photon mirror”. As with the usual optomechanical interaction, the sign of the optomechanical coupling \( G \) can be changed by changing the sign of a detuning, in this case the atom-field detuning \( \Delta \).

The effective total Hamiltonian of the light-BEC system reduces then to
\[
H = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \hbar \omega_m c^\dagger \hat{c} + \hbar G (c^\dagger + \hat{c}) \hat{a}^\dagger \hat{a}
\]
where \( \omega_m \equiv \omega_c - \omega_p \) plays the role of the natural frequency of the “photon mirror” and \( \omega_0 \) is the renormalized ground-state CM frequency of the trapped BEC.

In the usual optomechanics situation the movable mirror or trapped ultracold atomic system acts as a Kerr-type medium for the cavity field, resulting in a series of nonlinear optical effects that include optical bistability [4], chaos [10], and squeezing [11]. More generally, Ref. [12] showed that a broad variety of nonclassical states of the cavity field can be prepared by means of the optomechanical coupling. In the present situation with its reversal of roles between light and the mechanical oscillator, one finds similarly that a rich variety of nonclassical states can be generated in the BEC. In particular, in case the cavity field is initially in a vacuum state \( |0\rangle_c \) and initially uncorrelated with the BEC in a coherent state \( |\alpha\rangle_a \) we find that in the interaction picture the final state of the system is
\[
|\Psi(t)\rangle = e^{-\frac{i}{\hbar} \omega_m t} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-G^2/\omega_m^2} \sin(\omega_m t) |n\rangle_a \otimes |\Lambda \eta\rangle_c,
\]
where \( |n\rangle_a \) are number states of the BEC, \( |\Lambda \eta\rangle_c \) are coherent states of the cavity field of complex amplitude \( \Lambda \eta \), with \( \eta = 1 - \exp(-i\omega_m t) \), and we have introduced the coefficient \( \Lambda \equiv -G/\omega_m \). In the resolved sideband limit of cavity optomechanics, \( \omega_m \gg \kappa \) where \( \kappa \)
is the resonator width, \( \Lambda > 1 \) would correspond to the strong-coupling regime, where a single photon substantially modifies the resonator properties. In the present case this parameter is likewise a measure of the nonlinearity of the dynamics, as we shall see. (But in contrast to the usual situation, the “strong coupling regime” can now be easily reached by simply increasing the Rabi frequency \( \Omega_p \) of the classical driving field.)

At times \( \omega_n t = 2m\pi, m \) integer, the state of the cavity field returns to the vacuum while the state of BEC can take the form of various “Schrödinger-cat” states, depending on the value of \( \Lambda \). For instance, when \( \Lambda^2 = 1/4m \), the BEC is in the two-component cat state

\[
|\phi\rangle_a = \frac{1 + i}{2}|\alpha\rangle_a + \frac{1 - i}{2}|\alpha\rangle_a.
\]

The choice of other values of \( \Lambda \) can also lead to the generation of multicomponent cat states \[12\].

Except at times \( \omega_n t = 2m\pi \), the state \[7\] is an entangled state of the cavity field and the BEC. The entanglement is most pronounced at \( \omega_n t = (2m+1)\pi \), providing a route to the preparation of nonclassical state of the condensate via conditional measurements on the cavity field. In particular, a measurement of the position quadrature \( \hat{X} = (\hat{c} + \hat{c}^\dagger)/2 \) will force the BEC into a state

\[
|\tilde{\phi}\rangle_a \propto e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{(2m+1)\Lambda^2 n^2 \pi^2} f_n(X)|n\rangle_a,
\]

where the scalar products \( f_n(X) = \langle X|2\Lambda n\rangle \) are proportional to the Gaussian distributions \( e^{-(X-2\Lambda n)^2} \), with peaks of width \( \sigma = 1/2 \) centered at \( X = 2\Lambda n \). For \( \Lambda \gg 1/4 \), these peaks are well separated, so that if the measured value of \( X \) falls near \( 2\Lambda n \) then the state \( |\tilde{\phi}\rangle_a \) is well approximated by a number state \( |n\rangle_a \).

The preparation of near number states via conditional measurements of the cavity field provides a indirect way to extract the atom number statistics of the condensate. However the number statistics alone are not sufficient to verify the emergence of Schrödinger cat states. To gain the necessary phase information, one can adapt an optical method that involves the mixing of the field to be characterized with a reference classical field with an adjustable relative phase \[13\]. This can be achieved in principle in a way recently demonstrated \[14\] for the atomic homodyne detection of entangled twin-atom states in a spinor BEC. The idea is to mix the state to be characterized with another macroscopically populated atomic state that serves as a reference, for instance via a microwave field induced transition. The relative phase between the two atomic states can be adjusted through the phase of the microwave field.

In one possible measurement protocol, the optomechanical interaction \( G \) is rapidly switched off at a time \( t \) when the cavity field is back in the vacuum and the condensate in a Schrödinger-cat state, simply by switching off the classical driving field of Rabi frequency \( \Omega_p \) in a time short compared to the BEC decoherence time. The BEC atoms are then coupled by a microwave field to another, macroscopically populated atomic state. As a result the density operator of the BEC becomes \( \hat{\rho} = \hat{D}(\beta)\hat{\rho}_a\hat{D}(-\beta) \otimes |0\rangle|0\rangle_c \), where the atomic displacement operator \( \hat{D}(\beta) \) describes the mixing with a reference field of complex amplitude \( \beta \). Turning then the optomechanical coupling back on with \( \Lambda \gg 1/4 \), we find that after a time \( \omega_n t = \pi \), the probability of getting the value \( X \) for the position quadrature of the optical field is given by

\[
P_c(X) = \langle X|\text{Tr}_a[|\tilde{\phi}\rangle\langle\tilde{\phi}|\hat{U}(\pi)^\dagger]\rangle X = \sum_n P_a(n)|f_n(X)|^2,
\]

where \( \text{Tr}_a \) is a partial trace on the atomic system, \( P_a(n) = \langle n|\hat{D}(\beta)\hat{\rho}_a\hat{D}(-\beta)|n\rangle_a \) is the atom number statistics for the displaced matter wave field, and \( \hat{U}(t) \) is the time evolution operator.

We already indicated that for \( \Lambda \gg 1/4 \) the integral \( \int dX|f_n(X)|^2|f_m(X)|^2 \) approaches a Kronecker delta function \( \delta_{n,m} \), so that \( P_a(n) \) can be approximated as

\[
P_a(n) \approx \int P_c(X)e^{-2(X-2\Lambda n)^2} dX.
\]

Fig. 3(a) and 3(b) show the in-phase and out-of-phase values of \( P_a(n) \) for \( |\alpha| = 2 \) and \( |\beta| = 4 \). In the in-phase case, \( P_a(n) \) is the sum of two quasi-Poissonian distributions peaked around \( |\alpha + \beta|^2 \) and \( -|\alpha + \beta|^2 \). When \( \alpha \) and \( \beta \) are \( \pi/2 \) out of phase, the interference between the two atomic fields results in \( P_a(n) \) exhibiting a Poisson envelope with superimposed modulations.

A variation on that protocol inspired by cavity QED experiments \[12\] allows to reconstruct the full Wigner function of the field. To see how this works, we start from the expression of the Wigner function in the number state basis \[16\].

\[
W_a(\beta) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n a^\dagger n\langle n|\hat{D}(\beta)|\hat{D}(\beta)|n\rangle_a \sum_{m=0}^{\infty} (-1)^m b^\dagger m\langle m|\hat{D}(\beta)|\hat{D}(\beta)|m\rangle_a,
\]
where the $(-1)^n$ factor comes from the expectation value of the number parity operator.

To determine the Wigner function \([12]\) through a measurement of the cavity field, we again assume that the system is in the uncorrelated initial state $\hat{\rho}_c \otimes \ket{0}\bra{0}_{ic}$, and perform first a displacement of the matter-wave field before switching on the classical driving field of Rabi frequency $\Omega_p$. The probability of finding the cavity field in the $n$-photon state is then found to be

$$P_c(n) = \sum_{m=0}^{\infty} \left( \frac{|\Delta m|^2}{n!} \right) e^{-|\Delta m|^2} a_m \hat{D}(\beta)a \hat{D}(\beta)|m\rangle_a.$$ \hspace{1cm} (13)

Comparing this result with the expression \([12]\) shows that by choosing $|\Delta m|^2 = \pi$ we have simply

$$W_a(\beta) = \frac{2}{\pi} \sum_{n=0}^{\infty} P_c(n)(1 + i)^n.$$ \hspace{1cm} (14)

That approach is similar in spirit to a method used in Ref. \([13]\) to measure the Wigner function of the intracavity microwave field in a micromaser via detection of the state of the atoms that drive that field. One disadvantage is that it relies on the acquisition of a large amount of experimental data as well as on the the need to resolve the exact photon number $n$.

An alternative method that yields a direct measurement of $W_a(\beta)$ can be implemented if the cavity contains a single photon, a situation that might be achieved using a photon blockade effect, maybe by including in addition to the off-resonant condensate atoms a single resonant atom coupled to the light field in the strong-coupling regime of cavity QED \([13]\), or perhaps via a movable mirror \([13]\). While this would increase significantly the complexity of the experiment, the advantage of that approach is that the Wigner function is a directly measured quantity \([14]\).

With the cavity field initially in a mixture $\hat{\rho}_c = \rho_0|0\rangle\langle 0| + \rho_1|1\rangle\langle 1|$, displacing the matter wave field as before by an adjustable amount $\beta$, and then switching on the optomechanical interaction, we find that the difference in probabilities $\Delta P_c = P_c(1) - P_c(0)$ of measuring no photon and one photon in the resonator is

$$\Delta P_c = (\rho_1 - \rho_0) \sum_{n=0}^{\infty} a_n|\hat{D}(-\beta)\hat{D}(\beta)\cos(2\Lambda|\eta|n)|n\rangle_a.$$ \hspace{1cm} (15)

By adjusting the parameter $\Lambda$ and/or the interaction time so that $2\Lambda|\eta| = \pi$, the atomic Wigner function takes the simple form

$$W_a(\beta) = \frac{2\Delta P_c}{\pi(\rho_1 - \rho_0)}.$$ \hspace{1cm} (16)

In summary we have investigated a new configuration of a BEC-based cavity optomechanical system characterized by a role reversal of the light and matter-wave fields as compared to the usual situation. We showed how this system can not only efficiently prepare nonclassical states of the BEC, but can also nondestructively characterize them and even reconstruct their Wigner function. This extends the reach of cavity optomechanics into the realm of quantum matter wave optics and opens the way to a rich new direction of investigations, with potential in quantum information processing and quantum metrology. The proposed schemes are experimentally challenging, and a number of issues need to be be addressed, from the quantum efficiency of the detectors to technical and fundamental noise issues. The most significant dephasing processes are likely to be related to the phase noise of the classical driving field, and to atomic decoherence and atom loss due to three-body collisions (s-wave collisions, which are a nonlinear coherent process in ultra cold atoms, could be included in the nonlinear coefficient $\Lambda$.) The role of these and other decoherence and noise mechanisms will be the object of a detailed future paper.

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