EXPLICIT APPROXIMATE CONTROLLABILITY OF THE
SCHRÖDINGER EQUATION WITH A POLARIZABILITY TERM

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ABSTRACT

The problem discussed in this talk is the one of the controllability of a quantum particle in a potential. The control used is the real amplitude of an external electric field and we assume that the dipolar approximation is not valid. This imposes to consider a polarizability term. The global strategy is inspired from previous work in finite dimension by Coron, Grigoriu, Lefter and Turinici. Thus we consider highly oscillating controls and prove the semi-global stabilization of the averaged system towards the ground state using a Lyapunov function introduced by Nersesyan. Then we prove that the solutions of the averaged system approximate the solutions of the polarizability system on every finite time interval. This leads to approximate controllability of our system in large time with explicit controls.

1. INTRODUCTION

The evolution of the wave function is given by a Schrödinger equation. Usually the interaction between the external electric field and the particle is described in the dipolar approximation leading to a bilinear term with respect to the control and the state. Here we consider an expansion of this Hamiltonian to the second order. Thus we study the following control system in a smooth bounded domain of $\mathbb{R}^n$

$$\left\{ \begin{array}{l}
i\partial_t\psi = (-\Delta + V(x))\psi + u(t)Q_1(x)\psi + u(t)^2Q_2(x)\psi, \\
\psi|_{\partial D} = 0.
\end{array} \right. $$

The functions $V, Q_1, Q_2 \in C^\infty(D, \mathbb{R})$ are given, $V$ is the potential, $Q_1$ is the dipolar moment and $Q_2$ the polarizability moment. This model is of physical interest when the dipolar approximation is not valid.

We consider the operator $(-\Delta + V)$ of the uncontrolled system with domain $H_0^1 \cap H^2(D, \mathbb{C})$ and denote by $(\lambda_k)_{k \in \mathbb{N}}$ the non decreasing sequence of its eigenvalues and by $(\phi_k)_{k \in \mathbb{N}}$ the associated eigenvectors in $S$, the unit sphere of $L^2(D, \mathbb{C})$. The target state for the controllability problem is $\phi_1$ called the ground state or equivalently

$$\mathcal{C} := \{c\phi_1 ; c \in \mathbb{C}, |c| = 1\},$$

as the global phase is meaningless when modelling quantum particles.

A finite dimension approximation of this model was studied in [1]. In the dipolar approximation ($Q_2 \equiv 0$), the question of stabilization was studied in [2, 3]. Not neglecting the polarizability term also appears to be useful in mathematical proofs as we can weaken the assumptions on the dipolar moment made in [3].

2. MAIN RESULT

2.1. Study of the averaged system

One of the main idea, inspired from [1], to study this system is to consider oscillating controls of the form

$$u(t, \psi) := \alpha(\psi) + \beta(\psi) \sin \left( \frac{t}{\varepsilon} \right).$$

Following classical techniques of dynamical systems in finite dimension we can define an averaged system by computing the time average of the oscillating terms. We thus study the following Schrödinger equation with feedback controls

$$\left\{ \begin{array}{l}
i\partial_t\psi_{av} = \left(-\Delta + V(x)\right)\psi_{av} + \alpha(\psi_{av})Q_1(x)\psi_{av} \\
+ \left(\alpha(\psi_{av})^2 + \frac{1}{2}\beta(\psi_{av})^2\right)Q_2(x)\psi_{av}, \\
\psi_{av|\partial D} = 0.
\end{array} \right. $$

We consider the Lyapunov function used in [2],

$$\mathcal{L}(\psi) := \gamma ||(-\Delta + V)P\psi||_{L^2}^2 + 1 - ||\psi, \phi_1||^2, $$

for $\gamma > 0$.
where $P$ is the orthogonal projection in $L^2$ onto the closure of Span $\{\phi_k; k \geq 2\}$ and $\gamma$ a positive constant. We design the feedback controls $\alpha$ and $\beta$ to ensure that this Lyapunov function is decreasing along the trajectories of (2). This leads to the choice

$$
\begin{align*}
\alpha(\psi_{av}(t, \cdot)) & := -k_1 I_1(\psi_{av}(t, \cdot)), \\
\beta(\psi_{av}(t, \cdot)) & := g(I_2(\psi_{av}(t, \cdot))),
\end{align*}
$$

(4)

with $k > 0$ small enough, $g \in C^2(\mathbb{R}, \mathbb{R}^+)$ satisfying $g(x) = 0$ if and only if $x \geq 0$, $g'$ bounded, and for $j \in \{1, 2\}$, and for $z \in H^2$,

$$
I_j(z) := \text{Im} \left[ \gamma (\langle -\Delta + V \rangle P(Q_j z), (-\Delta + V) P z \rangle - \langle Q_j z, \phi_1 \rangle \langle \phi_1, z \rangle \right].
$$

(5)

To obtain the stabilization of the averaged system (2) with these feedback laws we adapt the LaSalle invariance principle for weak convergence, similarly to [3], Thus, every solution of the averaged system with initial condition in $S \cap H^1_0 \cap H^2(D, \mathbb{C})$ is weakly convergent in $H^2$ towards the ground state. This is proved under the following assumptions on the potential and on the dipolar and polarizability moments:

\begin{align*}
\text{i)} & \quad \langle Q_1 \phi_1, \phi_k \rangle = 0 \implies \langle Q_2 \phi_1, \phi_k \rangle \neq 0 \text{ i.e. all coupling are realized either by } Q_1 \text{ or } Q_2, \\
\text{ii)} & \quad \text{Card}\{k \in \mathbb{N}^* : \langle Q_1 \phi_1, \phi_k \rangle = 0\} < \infty \text{ i.e. only a finite number of coupling is realized by } Q_1, \\
\text{iii)} & \quad \lambda_1 - \lambda_k \neq \lambda_p - \lambda_q \text{ for } k, p, q \geq 1 \text{ such that } \{1, k\} \neq \{p, q\} \text{ and } k \neq 1, \\
\text{iv)} & \quad \lambda_p \neq \lambda_q \text{ for } p \neq q.
\end{align*}

2.2. Approximation by averaging

Using the previous expression of the controls (4), we design explicit controls for the polarizability system. Let $X_0 := \{\psi^0 \in S \cap H^1_0 \cap H^2 ; \Delta \psi^0 \in H^1_0 \cap H^2\}$. For an initial condition $\psi^0 \in X_0$, we define the control

$$
u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin \left( \frac{t}{\varepsilon} \right),$$

(6)

where $\psi_{av}$ is the solution of (2) with initial condition $\psi_{av}(0, \cdot) = \psi^0$ and control feedback laws given by (4).

We use this explicit control in the polarizability system and prove that, on every finite time interval, the solution of the averaged system and the solution of the polarizability system initiated from the same initial condition stay as close as wished provided that the control (6) is oscillating enough. Namely, we prove

**Proposition 1.** Let $L > 0$ and $\psi^0 \in X_0$ with $0 < L(\psi^0) < 1$. For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if $\psi_z$ is the solution of (2) associated to the same initial condition $\psi_z(0, \cdot) = \psi^0$ and control $\nu^\varepsilon$ defined by (4) with $\varepsilon \in (0, \varepsilon_0)$ then

$$
\|\psi_z(t, \cdot) - \psi_{av}(t, \cdot)\|_{H^2} \leq \delta, \quad \forall t \in [0, L].
$$

2.3. Conclusion

If the previous hypotheses hold, gathering the stabilization and the approximation properties of the averaged system we get

**Theorem 1.** For any $s < 2$, for any $\psi^0 \in X_0$ with $0 < L(\psi^0) < 1$, there exist a strictly increasing time sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ tending to $+\infty$ and a decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ such that if $\psi_z$ is the solution of (2) associated to the initial condition $\psi^0$ and control $\nu^\varepsilon$ defined by (4) then for all $n \in \mathbb{N}$, $\varepsilon \in (0, \varepsilon_n)$,

$$
\text{dist}_{H^s}(\psi_z(t, \cdot), C) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].
$$

We thus have defined explicit oscillating controls that drive the solution of our system arbitrarily close to the ground state provided that the control is oscillating enough and the time is large enough. To achieve this we have used and developed tools from the theory of finite dimension dynamical systems and applied them to the considered Schrödinger equation. We managed by adding a mathematically and physically meaning term to weaken the previous assumptions on the coupling realized by this model. The assumptions that were made are proved to be generic with respect to the functions determining the system (potential, dipolar and polarizability moments). More details can be found in [4].

3. REFERENCES

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