On Limits to the Scope of the Extended Formulations “Barriers”

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Abstract: In this paper, we introduce the notion of augmentation for polytopes and use it to show the error in two presumptions that have been key in arriving at over-reaching/over-scoped claims of “impossibility” in recent extended formulations (EF) developments. One of these presumption is that: “If Polytopes $P$ and $Q$ are described in the spaces of variables $x$ and $y$ respectively, and there exists a linear map $x = Ay$ between the feasible sets of $P$ and $Q$, then $Q$ is an EF of $P$”. The other is: “(An augmentation of Polytope $A$ projects to Polytope $B$) $\implies$ (The external descriptions of $A$ and $B$ are related)”. We provide counter-examples to these presumptions, and show that in general: (1) If polytopes can always be arbitrarily augmented for the purpose of establishing EF relations, then the notion of EF becomes degenerate/meaningless in some cases, and that: (2) The statement: “(Polytope $B$ is the projection of an augmentation of Polytope $A$) $\implies$ (Polytope $B$ is the projection of Polytope $A$)” is not true in general (although, as we show, the converse statement, “($B$ is the projection of $A$) $\implies$ ($B$ is the projection of every augmentation of $A$)”, is true in general). We illustrate some of the ideas using the minimum spanning tree problem, as well as the “lower bounds” developments in Fiorini et al. (2011; 2012), in particular.

Keywords: Linear Programming; Combinatorial Optimization; Traveling Salesman Problem; TSP; Computational Complexity, Extended Formulations.

1 Introduction

There has been a renewed interest in Extended Formulations (EF’s) over the past 3 years (see Conforti et al. (2010), Vanderbeck and Wolsey (2010), Fiorini et al. (2011; 2012), and Kaibel (2011), for example). Despite the great importance of the EF paradigm in the analysis of linear programming (LP) and integer programming (IP) models of combinatorial optimization problems (COP’s), the clear definition of its scope of applicability has been largely an overlooked issue. The purpose of this paper is to make a contribution towards addressing this issue. Specifically, we will show that the notion of an EF can become ill-defined and degenerate (and thereby lose its meaningfulness) when it is being used to relate polytopes involved in alternate abstractions of a given optimization problem. Because most of the papers on EF’s focus on the TSP specifically, we will center our discussion on the TSP. However, the substance of the paper is applicable for other NP-Complete problems.

It should be noted that, in this paper, we are not concerned with and do not claim the correctness or incorrectness of any particular model that may have been developed in trying to address the “$P = NP$” question. Our aim is, strictly, to bring attention to limits to the scope within which EF Theory is applicable when attempting to derive bounds on the size of the description of a polytope.

The plan of the paper is as follows. First, in section 2 we will review the basic definitions and notation and show, in particular, the error of any notion there may be, suggesting an “impossibility”
of abstracting the TSP optimization problem over a polytope of polynomial size, by showing that TSP tours can be represented (by inference) independently of the (standard) TSP polytope. Then, we will introduce the notion of “polyhedron augmentation” in section 3.1 and use it (in section 3.2) to develop our results on the condition about EF’s becoming ill-defined. Finally, we will offer some concluding remarks in section 5.

The general notation we will use is as follows.

Notation 1

1. $\mathbb{R}$: Set of real numbers;
2. $\mathbb{R}_{\geq 0}$: Set of non-negative real numbers;
3. $\mathbb{N}$: Set of natural number;
4. $\mathbb{N}_+$: Set of positive natural numbers;
5. “0” : Column vector that has every entry equal to 0;
6. “1” : Column vector that has every entry equal to 1;
7. $\text{Conv}(\cdot)$: Convex hull of $(\cdot)$.

2 Background overview

2.1 Basic definitions

Definition 2 (“traditional $x$-variables”) We will generically refer to 2-indexed variables that have been used in traditional IP formulations of the TSP to represent inter-city travels as “traditional $x$-variables.” In other words, $\forall (i, j) \in \{1, ..., n\}^2 : i \neq j$, we will refer to the 0/1 decision variables that are such that the $(i, j)^{th}$ entry of their vector is equal to “1” iff there is travel from city $i$ to city $j$, as the “traditional $x$-variables,” irrespective of the symbol used to denote them.

Definition 3 (“Standard TSP Polytope”) Let $A := \{(i, j) \in \Omega^2 : i \neq j\}$ denote the set of arcs of the complete digraph on $\Omega$. Denote the characteristic vector associated with any $F \subseteq A$, by $x^F$ (i.e., $x^F_{ij} \in \{0, 1\}$ is equal to 1 iff $(i, j) \in F$). Assume (w.l.o.g.) that the TSP tours (defined in terms of the arcs) have been ordered, and let $T_k \subset A$ denote the $k^{th}$ tour. The “Standard TSP Polytope” (in the asymmetric case) is defined as $\text{Conv}\left\{x^{T_k} \in \mathbb{R}^{n(n-1)}, k = 1, \ldots, n!\right\}$ (see Lawler et al. (1985, pp. 257-258), and Yannakakis (1991, p. 441), among others).

Definition 4 (“Standard EF Definition” (Yannakakis (1991); Conforti et al. (2010; 2013))) An “extended formulation” for a polytope $X \subseteq \mathbb{R}^p$ is a polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g\}$ the projection, $\varphi_x(U) := \{x \in \mathbb{R}^p : \exists w \in \mathbb{R}^q : (x, w) \in U\}$, of which onto $x$-space is equal to $X$ (where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).

Definition 5 (“Alternate EF Definition #1” (Kaibel (2011); Fiorini et al. (2011; 2012))) A polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g\}$ is an “extended formulation” of a polytope $X \subseteq \mathbb{R}^p$ if there exists a linear map $\pi : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^p$ such that $X$ is the image of $Q$ under $\pi$ (i.e., $X = \pi(Q)$; where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).
Definition 6 ("Alternate EF Definition #2" (Fiorini et al. (2012))) An “extended formulation” of a polytope \( X \subseteq \mathbb{R}^p \) is a linear system \( U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g \} \) such that \( x \in X \) if and only if there exists \( w \in \mathbb{R}^q \) such that \( (x, w) \in U \). (In other words, \( U \) is an EF of \( X \) if \((x \in X \iff (\exists w \in \mathbb{R}^q : (x, w) \in U))\) (where \( G \in \mathbb{R}^{m \times p} \), \( H \in \mathbb{R}^{m \times q} \), and \( g \in \mathbb{R}^m \)).

Remark 7 The following observations are in order with respect to Definitions 4, 5, and 6:

1. The statement of \( U \) in terms of inequality constraints only does not cause any loss of generality, since each equality constraint can be replaced by a pair of inequality constraints. (Yannakakis (1991, p. 442), for example) just says that \( U \) is a set of linear constraints.

2. The statements “\( U \) is an extended formulation of \( X \)” and “\( U \) expresses \( X \)” (see Yannakakis (1991)) are equivalent.

3. The system of linear equations which specify \( \pi \) in Definition 5 must be valid constraints for \( X \) and \( U \). Hence, \( X \) and \( U \) can be respectively extended by adding those constraints to them, when trying to relate \( X \) and \( U \) using Definition 5. In that sense, Definition 5 “extends” Definitions 4 and 6.

4. All three definitions are equivalent when \( G \neq 0 \) and \( U \) is minimally-described. However, this is not true when \( G = 0 \), as we will show in section 3.2 of this paper, causing a condition of ill-definition.

5. In the remainder of this paper, we will use the following terminologies interchangeably: “\( A \) is an extended formulation of \( B \)”; “\( A \) is an extension of \( B \)”; “\( A \) extends \( B \)”; “\( B \) is extended by \( A \)”.

\[ \square \]

Remark 8 With respect to Definition 4 the following alternate definition of a projection is provided by Conforti et al. (2010; 2013):

Given a polyhedron \( U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g \} \), its projection onto the \( x \)-space is \( \varphi_x(U) = \{x \in \mathbb{R}^p : uGx \leq ug \text{ for all } u \in C_Q\} \), where \( C_Q := \{u \in \mathbb{R}^m : uH = 0, u \geq 0\} \).

Now, assume \( G = 0 \) in this and Definition 4. Then, we would have:

\[ \varphi_x(U) = \{x \in \mathbb{R}^p : uGx \leq ug \text{ for all } u \in C_Q\} = \{x \in \mathbb{R}^p : ug \geq 0 \text{ for all } u \in C_Q\}. \]

Hence, exactly one of the following would be true:

\[ \varphi_x(U) = \emptyset \text{ (if } ug < 0 \text{ for some } u \in C_Q\), or \]
\[ \varphi_x(U) = \mathbb{R}^p \text{ (if } ug \geq 0 \text{ for all } u \in C_Q\). \]

Hence, \( \varphi_x(U) \) could not be equal to a nonempty polytope.

\[ \square \]

Definition 9 ("Row-redundancy") Let \( P := \{x \in \mathbb{R}^p : Ax \leq a\} \), where \( A \in \mathbb{R}^{m \times p} \) and \( a \in \mathbb{R}^m \).
1. We say that $P$ has “row-redundancy” if there exists a (non-trivial) row partitioning of $P$ with $A = \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix}$, and $a = \begin{bmatrix} \overline{a}_1 \\ \overline{a}_2 \end{bmatrix}$ (where $\overline{A}_1 \in \mathbb{R}^{n \times p}$, $\overline{A}_2 \in \mathbb{R}^{(m-n) \times p}$, $\overline{a}_1 \in \mathbb{R}^n$ and $\overline{a}_2 \in \mathbb{R}^{(m-n)}$) such that one of the following conditions is true:

(a) $(\overline{x} \in P) \iff (\overline{x} \in \{ x \in \mathbb{R}^p : \overline{A}_1 x \leq \overline{a}_1 \})$, or

(b) $(\overline{x} \in P) \iff (\overline{x} \in \{ x \in \mathbb{R}^p : \overline{A}_2 x \leq \overline{a}_2 \})$.

2. We say that the constraints $\overline{A}_3 x \leq \overline{a}_2$ are “redundant” for $\{ x \in \mathbb{R}^p : \overline{A}_1 x \leq \overline{a}_1 \}$ if Condition (1.a) is true. Similarly, we say that the constraints $\overline{A}_1 x \leq \overline{a}_1$ are “redundant” for $\{ x \in \mathbb{R}^p : \overline{A}_2 x \leq \overline{a}_2 \}$ if Condition (1.b) is true.

**Definition 10 (“Column-redundancy”)** Let $P := \{ x \in \mathbb{R}^p : Ax \leq a \}$, where $A \in \mathbb{R}^{n \times p}$ and $a \in \mathbb{R}^m$. Let $x$ denote the descriptive variables of $P$. Let $[\begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \end{array}]$ be a (non-trivial) partitioning of $x$, where $\overline{x}_1 \in \mathbb{R}^q$, and $\overline{x}_2 \in \mathbb{R}^{(p-q)}$.

1. We say that $P$ has “column-redundancy” if one of the following conditions is true:

(a) $\exists (B_1, b_1) \in \mathbb{R}^{n \times q} \times \mathbb{R}^n :$

$$\left( \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \end{array} \right) \in Ext(P) \implies \overline{x}_1 \in Ext(\{ x \in \mathbb{R}^q : B_1 x \leq b_1 \}) \text{, and}$$

$$\overline{x}_1 \in Ext(\{ x \in \mathbb{R}^q : B_1 x \leq b_1 \}) \implies \exists \overline{x}_2 \in \mathbb{R}^{(p-q)} : \left( \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \end{array} \right) \in Ext(P)$$

(where $1 \leq n \leq m$), or

(b) $\exists (B_2, b_2) \in \mathbb{R}^{n \times q} \times \mathbb{R}^n :$

$$\left( \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \end{array} \right) \in Ext(P) \implies \overline{x}_2 \in Ext(\{ x \in \mathbb{R}^{(p-q)} : B_2 x \leq b_2 \}) \text{, and}$$

$$\overline{x}_2 \in Ext(\{ x \in \mathbb{R}^{(p-q)} : B_2 x \leq b_2 \}) \implies \exists \overline{x}_1 \in \mathbb{R}^q : \left( \begin{array}{c} \overline{x}_1 \\ \overline{x}_2 \end{array} \right) \in Ext(P)$$

(where $1 \leq n \leq m$).

2. We say that the variables $\overline{x}_2$ are “redundant” for $\{ x \in \mathbb{R}^q : B_1 x \leq b_1 \}$ when Condition (1.a) is true. Similarly, we say that variables $\overline{x}_1$ are “redundant” for $\{ x \in \mathbb{R}^{(p-q)} : B_2 x \leq b_2 \}$ when Condition (1.b) is true.

**Definition 11 (“Minimally-described” polytope)** We say that a polyhedron $P$ is “minimally-described,” or that (the statement of) $P$ is “minimal,” if $P$ has no row-redundancy and no column-redundancy.
Assumption 12 In the remainder of this paper, with respect to Definitions 4, 5, and 6, we will assume (implicitly) that $U$ is minimally-described whenever we will be considering (or referring to) the case in which $G \neq 0$.

Observation 7.4 and Remark 8 above are the key point in the concept of ill-definition of an extended formulation which occurs in the special of $G = 0$ in Definitions 4, 5, and 6. This allows for a barrier to be removed by using an alternate formulation for a given combinatorial optimization problem (COP) at hand.

2.2 The “Alternate TSP Polytope”: An example of non-exponential abstraction of TSP tours

We now introduce a non-exponential LP model which correctly abstracts TSP tours. We will use this model as an illustrative example for our discussions in section 3 addressing extension relations to the Standard TSP Polytope, after we have formalized our ill-definition conditions in terms of the differences between Definitions 4, 5, and 6.

Theorem 13 Consider the TSP defined on the set of cities $\Omega := \{1, \ldots, n\}$. Assume city “1” has been designated as the starting and ending point of the travels. Let $S := \{1, \ldots, n - 1\}$ denote the times-of-travel to cities “2” through “n.” Then, there exists a one-to-one correspondence between TSP tours and extreme points of

$$AP := \left\{ w \in \mathbb{R}^{(n-1)^2} : \sum_{t \in S} w_{i,t} = 1 \quad \forall i \in (\Omega \setminus \{1\}); \quad \sum_{i \in (\Omega \setminus \{1\})} w_{i,t} = 1 \quad \forall t \in S \right\}.$$

Proof. Using the assumption that node 1 is the starting and ending point of travel, it is trivial to construct a unique TSP tour from a given extreme point of $AP$, and vice versa (i.e., it is trivial to construct a unique extreme point of $AP$ from a given TSP tour).

Corollary 14 It follows directly from Theorem 13 that $AP$ is a contradiction of any notion whereby it is “impossible” to abstract the TSP polytope into a linear program of polynomial size, since $AP$ clearly has polynomial (linear) size and it is a well-established fact that $AP$ is integral (see Burkard et al. (2009)).

Remark 15 Hence, alternate abstractions of the TSP optimization problem which may or may not involve the Standard TSP Polytope are possible. For example, in the “standard” (i.e., Standard TSP Polytope-based) abstraction of the TSP optimization problem, the cost function is trivial to develop. The challenge is to come up with linear constraints so that the extreme points of the induced polytope are TSP tours. On the other hand, in an abstraction based on $AP$, the representation of the TSP tours is a straightforward matter (since the tours are abstracted into assignment problem (bipartite matching) solutions). The challenge is to find appropriate costs to apply for these thus-abstracted TSP tours. Clearly, this challenge of coming up with a cost function is not within the scope of EF developments for the Standard TSP Polytope, since it does not involve that polytope. Examples of how this challenge can be met are described in Diaby (2007) and in Diaby and Karwan (2012), respectively. Also, clearly, from an overall perspective, one cannot reasonably equate an “impossibility” of meeting the challenge in one of the two abstractions (i.e., the “Standard TSP Polytope-based” and “$AP$-based” abstractions) with an “impossibility” of meeting the challenge in the other.
Remark 16 More formally, clearly, \( AP \) is also a TSP polytope, since its extreme points correspond to TSP tours. According to the Minkowski-Weyl Theorem (Minkowski (1910); Weyl (1935); see also Rockafellar (1997, pp.153-172)), every polytope can be equivalently described as the intersection of hyperplanes (\( H \)-representation/external description) or as a convex combination of (a finite number of) vertices (\( V \)-representation/internal description). The Standard TSP Polytope is stated in terms of its \( V \)-representation. No polynomial-sized \( H \)-representation of it is known. On the other hand, \( AP \) is stated in terms of its \( H \)-representation (which is well-known to be of (low-degree) polynomial size (see Burkard et al. (2009)), but it is trivial to state its \( V \)-representation also. The vertices of \( AP \) are assignment problem solutions, whereas the vertices of the Standard TSP Polytope are Hamiltonian cycles. Hence, even though the extreme points of \( AP \) and those of the Standard TSP Polytope respectively correspond to TSP tours, the two sets of extreme points are different kinds of mathematical objects, with unrelated mathematical characterizations. Hence, there does not exist any \textit{a priori} mathematical relation between \( AP \) and the Standard TSP Polytope. In other words, \( AP \) and the Standard TSP Polytope are simply alternate abstractions of TSP tours. Or, put another way, \( AP \) is (simply) an alternate TSP polytope from the Standard TSP Polytope, and vice versa (i.e., that the Standard TSP Polytope is (simply) an alternate TSP polytope from \( AP \)). \hfill \Box

Definition 17 ("Alternate TSP Polytope") We refer to \( AP \) as the "Alternate TSP Polytope."

3 \textit{Ill-definition} condition for “Extended Formulations”

3.1 Polytope Augmentation

Definition 18 ("Class of variables") We refer to a set of variables which model a given aspect of a problem, as a "class of variables." The traditional \( x \)-variables for example, would constitute one class of variables in a TSP model, as they represent (single) “travel legs” in the TSP. Similarly, the \( y \)- and \( z \)-variables used in the models of Diaby (2007) and Diaby and Karwan (2012), respectively, would constitute two distinct classes of variables, with the \( y \)-variables modeling doublets of “travel legs” in the TSP, and the \( z \)-variables modeling triplets of “travel legs” in the TSP.

Assumption 19 In the remainder of this paper, we will assume (without loss of generality) that a given class of variables is denoted by the same symbol in all of the models in which it is used. That is, we will assume that the same notation (whatever that may be) will be used to designate the traditional \( x \)-variables for example, in every model in which these variables are used.

Definition 20 ("Independent spaces") Let \( x \in \mathbb{R}^p \ (p \in \mathbb{N}_+) \) and \( w \in \mathbb{R}^q \ (q \in \mathbb{N}_+) \) be the vectors of descriptive variables for two polyhedra in \( \mathbb{R}^p \) and \( \mathbb{R}^q \), respectively.

1. We say that \( x \) and \( w \) (or that the polyhedra are in “independent spaces”) if \( x \) and \( w \) do not have any class of variables in common. That is, we say that \( x \) and \( w \) are in “independent spaces” if the following conditions holds:

   (a) \( x \) cannot be partitioned as \( x = \begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix} \);

   (b) \( w \) cannot be partitioned as \( w = \begin{pmatrix} \bar{w} \\ \bar{x} \end{pmatrix} \); and


\( \forall m \in \mathbb{N}_+ : m < \min\{p, q\}, \overline{p(w, v)} \in \mathbb{R}^{(p-m)} \times \mathbb{R}^{(q-m)} \times \mathbb{R}^m : (x \text{ and } w \text{ can be respectively partitioned as } x = \begin{pmatrix} \overline{x} \\ v \end{pmatrix} \text{ and } w = \begin{pmatrix} \overline{w} \\ v \end{pmatrix}, \text{ where } v \text{ denotes a given class of variables for the problem at hand.} \)

2. We will say that \( x \) and \( w \) (or that the polyhedra they respectively describe) “overlap” if \( x \) and \( w \) have one or more classes of variables in common.

Regan and Lipton (2013) remarked that all polytopes may be viewed, in a degenerate way, as being part of one overall multi-dimensional space. The following alternate (and equivalent) definition of “independent spaces” is therefore useful in further clarifying the notion.

**Definition 21 (Alternate definition of “Independent spaces”)** Let \( P \) and \( Q \) be polytopes in \( \mathbb{R}^{p+q} \), with descriptive variables \( (x, y) \in \mathbb{R}^p \times \mathbb{R}^q \). We say that \( P \) and \( Q \) are in “independent spaces” iff exactly one of the following two conditions holds:

1. \( \{x \in \mathbb{R}^p : (\exists y \in \mathbb{R}^q : (x, y) \in P)\} = \mathbb{R}^p \) and \( \{y \in \mathbb{R}^q : (\exists x \in \mathbb{R}^p : (x, y) \in Q)\} = \mathbb{R}^q \);

2. \( \{y \in \mathbb{R}^q : (\exists x \in \mathbb{R}^p : (x, y) \in P)\} = \mathbb{R}^q \) and \( \{x \in \mathbb{R}^p : (\exists y \in \mathbb{R}^q : (x, y) \in Q)\} = \mathbb{R}^p \).

**Example 22** Definitions 20 and 21 can be illustrated as follows.

- Assume \( x \in \mathbb{R}^2 \) and \( y \in \mathbb{R}^2 \) refer to different classes of variables in a modeling context at hand.

- Let \( x \) and \( y \) be the descriptive variables for Polytopes \( P \) and \( Q \) respectively, with:

\[
P := \{x \in \mathbb{R}^2 : x_1 - x_2 \geq 6; \ 0 \leq x_1 \leq 6; \ 0 \leq x_2 \leq 5\};
\]

\[
Q := \{y \in \mathbb{R}^2 : y_1 + y_2 = 6; \ y_1 \geq 1.5; \ y_2 \geq 0\}.
\]

- Clearly, \( P \) and \( Q \) are in \( \mathbb{R}^4 \) in a degenerate sense, respectively.

- However:

  - \( P \) and \( Q \) are independent spaces according to Definition 20 directly;

  - \( P \) and \( Q \) can be respectively re-written as:

\[
P' = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \mathbf{A}x + 0 \cdot y \leq \mathbf{a}\}; \text{ where } \mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} -6 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \text{ and}
\]

\[
Q' = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : 0 \cdot x + \mathbf{B}y \leq \mathbf{b}\}; \text{ where } \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ -6 \\ -1.5 \\ 0 \end{bmatrix},
\]

so that:

\[
\{y \in \mathbb{R}^2 : (\exists x \in \mathbb{R}^2 : (x, y) \in P')\} = \mathbb{R}^2 \text{ and } \{x \in \mathbb{R}^2 : (\exists y \in \mathbb{R}^2 : (x, y) \in Q')\} = \mathbb{R}^2.
\]

Hence, \( P' \) and \( Q' \) (and therefore, \( P \) and \( Q \)) are independent spaces according to Definition 21.
Definition 23 ("Polyhedron augmentation") Let $X$ be a non-empty polyhedron described in terms of variables $x \in \mathbb{R}^p$. Let $\overline{X}$ be a polyhedron the description of which consists of the constraints of $X$, plus additional variables and constraints that are not used in the description of $X$. We will say that $\overline{X}$ is an “augmentation” of $X$ (or that $\overline{X}$ “augments” $X$) if the problem of optimizing any given linear function of $x$ over $X$, is equivalent to the problem of optimizing that linear function over $\overline{X}$. In other words, let $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ be vectors of variables in independent spaces. Let $X := \{ x \in \mathbb{R}^p : Ax \leq a \} \neq \emptyset$, and $\overline{X} := \{ (x, y) \in \mathbb{R}^{p+q} : Ax \leq a; Bx + Cy \leq b \} \neq \emptyset$ (where $A \in \mathbb{R}^{k \times p}$, $a \in \mathbb{R}^k$, $B \in \mathbb{R}^{m \times p}$, $C \in \mathbb{R}^{m \times q}$, and $b \in \mathbb{R}^m$). We say that $\overline{X}$ augments $X$ if $(\forall x \in X, \exists y \in \mathbb{R}^q : (x, y) \in \overline{X})$.

Remark 24 With respect to Definition 23:

1. The additional variables and constraints of $\overline{X}$ are redundant for $X$ (see Definitions 9 and 10);
2. The optimization problem over $\overline{X}$ may not be equivalent to the optimization problem over $X$, if the objective function in the problem over $\overline{X}$ is changed from that of $X$ to include non-zero terms of the new variables;
3. Every augmentation of $X$ is an extended formulation of $X$, but the converse is not true (since an extended formulation of $X$ need not include the constraints of $X$ explicitly);
4. The polyhedral set associated to an optimization problem is equivalent to all of its augmentations respectively, provided the expression of the objective function of the optimization problem is not changed;
5. In the discussions to follow we will assume (w.l.o.g.) that the objective function is not changed when new variables and constraints are added to an optimization problem. Hence, in the discussions to follow, we will not distinguish between a polyhedral set and the optimization problem to which it is associated, except for where that causes confusion.

Example 25 We illustrate Definition 23 and Remark 24 as follows.

Let:

(i) $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ be variables in independent spaces;
(ii) $X := \{ x \in \mathbb{R}^p : Ax \leq a \}$;
(iii) $L := \{ (x, y) \in \mathbb{R}^{p+q} : Bx + Cy \leq c \}$;
(iv) $Y := \{ y \in \mathbb{R}^q : Dy \leq d \}$;
(v) $K_1 := \left\{ (x, y) \in \mathbb{R}^{p+q} : \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} a \\ c \end{bmatrix} \right\}$;
(vi) $K_2 := \left\{ (x, y) \in \mathbb{R}^{p+q} : \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} c \\ d \end{bmatrix} \right\}$;
(vii) $K_3 := \left\{ (x, y) \in \mathbb{R}^{p+q} : \begin{bmatrix} A & 0 \\ B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} a \\ c \\ d \end{bmatrix} \right\}.

(Where: $A \in \mathbb{R}^{k \times p}; a \in \mathbb{R}^k; B \in \mathbb{R}^{m \times p}; C \in \mathbb{R}^{m \times q}; c \in \mathbb{R}^m; D \in \mathbb{R}^{l \times q}, d \in \mathbb{R}^l$).

Assume:

(vii) $A \neq 0, B \neq 0, C \neq 0, D \neq 0$;

(viii) $B$ cannot be partitioned as $B = \begin{bmatrix} A \\ B \end{bmatrix}$;

(ix) $C$ cannot be partitioned as $C = \begin{bmatrix} C \\ D \end{bmatrix}$;

(x) The constraints of $L$ are redundant for $X$ and $Y$.

Then:

(xi) $K_1$ is an augmentation of $X$, but not of $L$, nor of $Y$;

(xii) $K_2$ is an augmentation of $Y$, but not of $L$, nor of $X$;

(xiii) $K_3$ is an augmentation of $X$ and $Y$, but not of $L$;

(xiv) $K_1$ is an extended formulation of $X$, but not of $L$, and may or may not be for $Y$;

(xv) $K_2$ is an extended formulation of $Y$, but not of $L$, and may or may not be for $X$;

(xvi) $K_3$ is an extended formulation of $X$ and $Y$, but not of $L$;

(xvii) $L$ is not an augmentation of $X$ nor of $Y$;

(xviii) $L$ may or may not be an extended formulation of $X$;

(xix) $L$ may or may not be an extended formulation of $Y$.

□

Remark 26 In reference to the developments above:

1. We will refer to the constraints of $L$ as the “linking constraints” (for $X$ and $Y$) in $K_3$, regardless of whether or not the constraints of $L$ are redundant for $X$ and $Y$;

2. If $X$ and $Y$ are alternative correct abstractions of the requirements of some (same) given optimization problem, then there may or may not exist $B$ and $C$ such that $((x, y) \in K_1 \implies y \in Y)$ and $((x, y) \in K_2 \implies x \in X)$. This is exemplified by the Alternate TSP Polytope relative to the Standard TSP Polytope;
3. If there exist \(B\) and \(C\) such that \(((x, y) \in K_1 \implies y \in Y)\) and \(((x, y) \in K_2 \implies x \in X)\), then it must be possible to attach meanings to \(x\) and \(y\), so that \(X\) and \(Y\) are alternative correct abstractions of the requirements of some (same) given optimization problem. This is exemplified by the LP models of the TSP in Diaby (2007) and in Diaby and Karwan (2012), respectively, relative to theAlternate TSP Polytope, or relative to the Standard TSP Polytope;

4. The main point of our developments in section 3 below will be to show that there exists no well-defined (non-ambiguous, meaningful) extended formulation relationship between \(X\) and \(Y\).

In particular, we will show that

\[(\exists (B, C) : (x, y) \in K_1 \implies y \in Y) \not\Rightarrow (X is a well-defined extended formulation of Y),\]

and that similarly,

\[(\exists (B, C) : (x, y) \in K_2 \implies x \in X) \not\Rightarrow (Y is a well-defined extended formulation of X).\]

For example, there exist linear transformations which allow for points of the Alternate TSP Polytope to be “retrieved” from (given) solutions of the TSP LP models in Diaby [2007] and Diaby and Karwan [2012]. Note however, that the “retrieval” of points of the Standard TSP Polytope can be accomplished only through the use of “implicit” information (about TSP node “1” specifically) that is outside the scope of the TSP LP models per se. Hence, the TSP LP models can be well-defined extended formulations of the Standard TSP Polytope only if they are well-defined extended formulations of the Alternate TSP Polytope, which would seem to suggest that the Alternate TSP Polytope must be a well-defined extended formulation of the Standard TSP polytope. We do not believe such a suggestion is the intent of any extended formulations work. However, we believe the definitions of an “extended formulation” must be properly interpreted in order for them not to lead to such conclusions. Specifically, using the notion of augmentation discussed in this section, we will show in the next section (section 3) that the notion of an EF can become ill-defined (and thereby lose its meaningfulness) when the polytopes being related are expressed in coordinate systems that are independent of each other.

\[\Box\]

We will now discuss two results which will be helpful subsequently in showing the differences between the case of polytopes in overlapping spaces and the case of polytopes in independent spaces, as pertains to extension relationships.

**Theorem 27** Let \(P_1\) and \(P_2\) be non-empty, minimally-described polytopes in overlapping spaces with the set of the descriptive variables of \(P_1\) included in that of \(P_2\). An augmentation of \(P_2\) is an extended formulation of \(P_1\) if and only if \(P_2\) is an extended formulation of \(P_1\), according to Definitions 4, 5, and 6 respectively.

In other words, let:

\[P_1 := \{x \in \mathbb{R}^{q_1} : A_1 x \leq a_1\} \quad \text{where} \quad A_1 \in \mathbb{R}^{r_1 \times q_1}; \quad a_1 \in \mathbb{R}^{r_1};\]

\[P_2 := \{(x, u) \in \mathbb{R}^{q_1 + q_2} : A_2 x + B u \leq b\} \quad \text{where} \quad A_2 \in \mathbb{R}^{r_2 \times q_1}; \quad B \in \mathbb{R}^{r_2 \times q_2}; \quad b \in \mathbb{R}^{r_2}.\]
Assume $A_1 \neq 0$, $A_2 \neq 0$, $P_1 \neq \emptyset$, and $P_2 \neq \emptyset$. Then, an arbitrary augmentation, $P_3$, of $P_2$ is an extended formulation of $P_1$ if and only if $P_2$ is an extended formulation of $P_1$, according to Definitions 4, 5, and 6 respectively.

Proof. First, note that Definitions 4, 5, and 6 are equivalent to one another with respect to extension relations for $P_1$, $P_2$, and $P_3$ (see Remark 7.4). Hence, it is sufficient to prove the theorem for the standard definition (Definition 4).

$P_3$ can be written as:

$P_3 := \{(x,u,v) \in \mathbb{R}^{n_1+q_2+q_3} : A_2 x + B u \leq b; A_3 x + C u + D v \leq c\}$ (where: $A_3 \in \mathbb{R}^{r_3 \times q_1}$; $C \in \mathbb{R}^{r_3 \times q_2}$; $D \in \mathbb{R}^{r_3 \times q_2}$; $c \in \mathbb{R}^{r_3}$; $A_3 x + C u + D v \leq c$ are redundant for $P_2$).

$(A_3 x + C u + D v \leq c)$ redundant for $P_2 \Longrightarrow:

\forall (x,u) \in P_2, \exists v \in \mathbb{R}^{q_3} : (x,u,v) \in P_3.

\begin{equation}
1 \Longrightarrow:
\end{equation}

\{x \in \mathbb{R}^{n_1} : (\exists (u,v) \in \mathbb{R}^{n_1+q_2} : (x,u,v) \in P_3)\} = \{x \in \mathbb{R}^{n_1} : (\exists u \in \mathbb{R}^{q_2} : (x,u) \in P_2)\}.

\begin{equation}
2 \Longrightarrow:
\end{equation}

$(\varphi_x(P_3) = P_1) \iff (\varphi_x(P_2) = P_1)$.

We will show in the next section (section 3.2) that Theorem 27 does not hold for polyhedra which are stated in independent spaces (such as $P$ and $Q$ in Example 22 or $X$ and $Y$ in Example 25), and that, as indicated in Remark 26 above, there cannot exist any well-defined (non-ambiguous, meaningful) extension relationship between such polytopes.

3.2 Ill-definition condition for EF’s

Referring back to the standard and alternate definitions of extended formulations (i.e., Definitions 4, 5, and 6 respectively), it is easy to verify (as indicated in Remark 7.4) that these three definitions are equivalent when $G \neq 0$ (with $U$ minimally-described). In other words, one can easily verify that provided $G \neq 0$, $U$ is an extended formulation of $X$ according to one of these definitions if and only if $U$ is an extended formulation of $X$ according to the other definitions. However, this is not true when $G = 0$.

A basic intuition of Definitions 4, 5, and 6 is that if the projection of $U$ onto $x$-space is equal to $X$, then the description of $X$ must be implicit in a constraint set of the form:

$Gx \leq \theta_w$.

Hence, the notion of an extended formulation can become ill-defined when $G = 0$ (i.e., when $U$ and $X$ are independent spaces). In essence, to put it roughly, there is “nothing” in the statement of $U$ for the constraints of $X$ to be “implicit in” (in $U$) when $G = 0$. Indeed, as we will show in the discussion to follow, when $G = 0$, Definition 5 can become contradictory of Definitions 4 and 6 resulting in an ill-definition (ambiguity) condition.

The theorem below shows that there exist no extension relations between polytopes stated in independent spaces according to the standard definition (Definition 4), or the second alternate definition (Definition 6) of extended formulations.
Theorem 28 Polytopes described in independent spaces cannot be extended formulations of each other according to the standard definition (Definition 4) or the second alternate definition (Definition 6) of extended formulations.

In other words, let $X := \{ x \in \mathbb{R}^p : Ax \leq a \}$ and $U := \{ w \in \mathbb{R}^q : Hw \leq h \}$ be (non-empty) polytopes in independent spaces (where: $A \in \mathbb{R}^{m \times p}$; $a \in \mathbb{R}^m$; $H \in \mathbb{R}^{n \times q}$; $h \in \mathbb{R}^n$). Then:

$(i)$ $U$ cannot be an extended formulation of $X$ (and vice versa) according to Definition 4

$(ii)$ $U$ cannot be an extended formulation of $X$ (and vice versa) according to Definition 6.

Proof. We will show that $U$ cannot be an extended formulation of $X$ according to Definitions 4 and 6, respectively. The proofs that $X$ cannot be an extended formulation of $U$ according to the definitions (Definitions 4 and 6, respectively) are similar and will be therefore omitted.

Let $U$ be re-stated in $\mathbb{R}^{p+q}$ as:

$$U' := \{ (x, w) \in \mathbb{R}^{p+q} : 0 \cdot x + Hw \leq h \}.$$  \hfill (3)

Clearly, we have:

$$(x, w) \in U' \iff w \in U.$$  \hfill (4)

Hence:

$$U \neq \emptyset \implies U' \neq \emptyset \implies (\forall x \in \mathbb{R}^p, \exists w \in \mathbb{R}^q : (x, w) \in U').$$  \hfill (5)

Now consider conditions $(i)$ and $(ii)$ of the theorem. We have the following.

$(i)$ Condition $(i)$.

Using (4) and Definition 4 we have:

$$\varphi_x(U) = \varphi_x(U') = \{ x \in \mathbb{R}^p : (\exists w \in \mathbb{R}^q : (x, w) \in U') \} = \mathbb{R}^p.$$  \hfill (6)

Because $X$ is bounded, we must have:

$$X \subset \mathbb{R}^p.$$  \hfill (7)

Combining (6) and (7) gives:

$$\varphi_x(U) = \mathbb{R}^p \neq X.$$  \hfill (8)

$(ii)$ Condition $(ii)$.

(5) $\implies$:

$$\exists x \in \mathbb{R}^d \setminus X : (\exists w \in \mathbb{R}^q : (x, w) \in U').$$  \hfill (9)

(4) and (9) $\implies$:

$$(w \in U) \iff (x, w) \in U' \not\leftrightarrow x \in X.$$  \hfill (10)

Hence, the “if and only if” condition of Definition 6 cannot be satisfied in general.  \hfill ■
Remark 29 Theorem 28 is consistent with Remark 8 (p. 3).

Corollary 30 Let \( X := \{ x \in \mathbb{R}^p : Ax \leq a \} \) and \( U := \{ w \in \mathbb{R}^q : Hw \leq h \} \) be (non-empty) polytopes in independent spaces (where: \( A \in \mathbb{R}^{m \times p}; a \in \mathbb{R}^m; H \in \mathbb{R}^{n \times q}; h \in \mathbb{R}^n \)). Then, exactly one of the following is true:

(i) There exists no extended formulation relationship between \( X \) and \( U \) (i.e., there exists no linear map \( \pi_x : \mathbb{R}^q \to \mathbb{R}^p \) such that \( \pi_x(U) = X \), and there exists no linear map \( \pi_w : \mathbb{R}^p \to \mathbb{R}^q \) such that \( \pi_w(X) = U \));

(ii) The extended formulation relationship between \( X \) and \( U \) is ill-defined due to Definition 5 being inconsistent with Definitions 4 and 6, respectively (i.e., if there exists a linear map \( \pi_x : \mathbb{R}^q \to \mathbb{R}^p \) such that \( \pi_x(U) = X \), or there exists a linear map \( \pi_w : \mathbb{R}^p \to \mathbb{R}^q \) such that \( \pi_w(X) = U \), or both).

Example 31 Corollary 30(ii) can be illustrated using the polytopes \( P \) and \( Q \) of Example 22.

We have:

(i) \( \varphi_y(P) = \mathbb{R}^2 \neq Q \) and \( \varphi_x(Q) = \mathbb{R}^2 \neq P \).

Hence, according to Definition 4 there exists no extension relationship between \( P \) and \( Q \);

(ii) \( \exists x \notin P : (\exists y \in \mathbb{R}^2 : (x, y) \in Q) \), which implies: \( (x \in P \Leftrightarrow (\exists y \in \mathbb{R}^2 : (x, y) \in Q)) \). Similarly, \( \exists y \notin Q : (\exists x \in \mathbb{R}^2 : (x, y) \in Q) \), which implies: \( (y \in Q \Leftrightarrow (\exists x \in \mathbb{R}^2 : (x, y) \in P)) \).

Hence, according to Definition 6 there exists no extension relationship between \( P \) and \( Q \);

(iii) \( (x, y) \in (P, Q) \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \).

In other words, \( (x, y) \in (P, Q) \implies x = Ay \), where \( A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) is the matrix for a linear transformation which maps \( P \) and \( Q \).

Hence, according to Definition 6 Q is an extended formulation of \( P \), which is in contradiction of (i) and (ii) above.

\( \square \)

The ill-definition condition stated in Corollary 30 will be further developed in the remainder of this section. We start with a notion which essentially generalizes the idea of the linear map \( (\pi) \) in Definition 5 with respect to the task of optimizing a linear function over a polyhedral set (since each of the linear equations that specify \( \pi \) must be valid for \( U \) and \( X \), respectively).

Theorem 32 Any two non-empty polytopes expressed in independent spaces can be respectively augmented into being extended formulations of each other. In other words, let \( x^1 \in \mathbb{R}^{n_1} (n_1 \in \mathbb{N}_+) \) and \( x^2 \in \mathbb{R}^{n_2} (n_2 \in \mathbb{N}_+) \) be vectors of variables in two independent spaces. Then, every non-empty polytope in \( x^1 \) can be augmented into an extended formulation of every other non-empty polytope in \( x^2 \), and vice versa.
Proof. The proof is essentially by construction. Let \( P_1 \) and \( P_2 \) be polytopes specified as:

\[
P_1 = \{ x^1 \in \mathbb{R}^{n_1} : A_1 x^1 \leq a_1 \} \neq \emptyset \quad \text{(where } A_1 \in \mathbb{R}^{p_1 \times n_1}, \text{ and } a_1 \in \mathbb{R}^{p_1})
\]

\[
P_2 = \{ x^2 \in \mathbb{R}^{n_2} : A_2 x^2 \leq a_2 \} \neq \emptyset \quad \text{(where } A_2 \in \mathbb{R}^{p_2 \times n_2}, \text{ and } a_2 \in \mathbb{R}^{p_2})
\]

Clearly, \( \forall (x^1, x^2) \in P_1 \times P_2, \forall q \in \mathbb{N}_+, \forall B_1 \in \mathbb{R}^{q \times n_1}, \forall B_2 \in \mathbb{R}^{q \times n_2}, \) there exists \( u \in \mathbb{R}^q \) such that the constraints

\[
B_1 x^1 + B_2 x^2 - u \leq 0 \quad \text{(11)}
\]

are valid for \( P_1 \) and \( P_2 \), respectively (i.e., they are redundant for \( P_1 \) and \( P_2 \), respectively).

Now, consider:

\[
W := \{(x^1, x^2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^q : C_1 A_1 x^1 \leq C_1 a_1; B_1 x^2 + B_2 x^1 - u \leq 0; C_2 A_2 x^2 \leq C_2 a_2\} \quad \text{(14)}
\]

(where: \( C_1 \in \mathbb{R}^{p_1 \times p_1} \) and \( C_2 \in \mathbb{R}^{p_2 \times p_2} \) are diagonal matrices with non-zero diagonal entries).

Clearly, \( W \) augments \( P_1 \) and \( P_2 \) respectively. Hence:

\[
W \text{ is equivalent to } P_1, \quad \text{and} \quad (15)
\]

\[
W \text{ is equivalent to } P_2. \quad \text{(16)}
\]

Also clearly, we have:

\[
\varphi_{x^1}(W) = P_1 \quad \text{(since } P_2 \neq \emptyset, \text{ and (13) and (14) are redundant for } P_1), \quad \text{and} \quad (17)
\]

\[
\varphi_{x^2}(W) = P_2 \quad \text{(since } P_1 \neq \emptyset, \text{ and (12) and (13) are redundant for } P_2). \quad \text{(18)}
\]

It follows from the combination of (15) and (18) that \( P_1 \) is an extended formulation of \( P_2 \).

It follows from the combination of (16) and (17) that \( P_2 \) is an extended formulation of \( P_1 \).

Corollary 33 Provided polytopes can be arbitrarily augmented for the purpose of establishing extended formulation relationships, every two (non-empty) polytopes expressed in independent spaces are extended formulations of each other.

Theorem 32 and Corollary 33 are illustrated below.
Example 34
Let
\[ P_1 = \{ x \in \mathbb{R}_x^2 : 2x_1 + x_2 \leq 6 \}; \]
\[ P_2 = \{ w \in \mathbb{R}_w^3 : 18w_1 - w_2 \leq 23; 59w_1 + w_3 \leq 84 \}. \]
For arbitrary matrices \( B_1, B_2, C_1, \) and \( C_2 \) (of appropriate dimensions, respectively); say \( B_1 = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}, \) \( B_2 = \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix}, \) \( C_1 = [7], \) and \( C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} ; \) \( P_1 \) and \( P_2 \) can be augmented into extended formulations of each other using \( u \in \mathbb{R}_u^2 \) and \( W : \)
\[
W = \left\{ (x, w, u) \in \mathbb{R}_x^{2+3+2} : \begin{bmatrix} 7 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 42 ; \right. \\
\left. \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \right. \\
\left. \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 18 & -1 & 0 \\ 59 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \leq \begin{bmatrix} 46 \\ 42 \end{bmatrix} \right\} .
\]
\[ \square \]
Remark 35
1. According to Corollary 33, the notion of EF becomes degenerate when \( G = 0 \) in Definitions 4, 5, and 6, respectively, and one tries to apply it by augmenting one of the polytopes at hand.
2. Theorem 32 and Corollary 33 are not true for polytopes expressed in overlapping spaces, and that in fact, these two results are in contradiction of Theorem 27. Hence, whereas one can arbitrarily augment polytopes in overlapping spaces for the purpose of establishing EF relationships, such an approach is invalid (cannot produce valid results) for polytopes stated in independent spaces.

Clearly, any notion of “extension” which allows for an object to be extensions of its own extensions cannot be a well-defined one (i.e., must be an ill-defined one), unless the objects involved are indistinguishable from their respective “extensions.” For example, clearly, one cannot reasonably argue that \( P_1 \) and \( P_2 \) in Example 34 above are extended formulations of each other in a meaningful sense.

4 Redundancy matters when relating polytopes stated in independent spaces
The notion of independent spaces we have introduced in this paper is important because, as we have shown, it refines the notion of EFs by separating the case in which that notion is degenerate (with
every polytope potentially being an EF of every other polytope) from the case where the notion of EF is well-defined/meaningful. It separates the case in which the addition of redundant constraints and variables (for the purpose of establishing EF relations) matters (i.e., makes a difference to the outcome of analysis) from the case in which the addition of redundant constraints and variables does not matter.

Two key results of section 3 of this papers are that:

1. If \( U \in \mathbb{R}^p \) and \( V \in \mathbb{R}^q \) are in overlapping spaces, then an augmentation of \( V \) is an EF of \( U \) if and only if \( V \) is an EF of \( U \); (This is the case where the addition of redundant constraints does not matter, and is stated in Theorem 27 on page 10 of this paper);

2. But (1) is not true if \( U \) and \( V \) are in independent spaces. As we have shown in section 3.2 if polytopes can always be arbitrarily augmented for the purpose of establishing EF relations, then any two polytopes that are in independent spaces are EF’s of each other. (This is the case where adding redundant constraints and variables does matter, and is stated in Theorem 32 on page 13 of this paper).

Because of (2) above, the addition of redundant constraints and variables for the purpose of establishing EF relations can lead to ambiguities when applied to polytopes in independent spaces. This ambiguity would stem from the fact that one would reach contradicting conclusions depending on what we do with the redundant constraints and variables which are introduced in order to make the models overlap. To clarify this: Assume \( V \) is augmented with the variables of \( U \) plus the constraints for the linear transformation that establishes the \( 1 \rightarrow 1 \) correspondence between \( U \) and \( V \); call this augmented-\( V \), \( V’ \). In EF work it is commonly assumed/suggested that redundant constraints and variables of a model can be removed from it without any loss of generality. In the case of \( V’ \) this would lead to contradictory conclusions with respect to the question of whether or not \( V’ \) is an EF of \( U \): If the added redundant constraints and variables are kept, the answer would be “yes”; If they are removed, \( V’ \) would “revert” back to \( V \), so that the answer would be “no.” This is the inconsistency which is pointed out in Theorem 28 (p. 12) and Corollary 30 (p. 13), and illustrated in Example 31 (p. 13).

A specific implication of (2) above is that, provided polytopes can be arbitrarily augmented for the purpose of EF’s, every conceivable polytope that is non-empty and does not require the traditional \( x \)-variables (i.e., the city-to-city \( x_{i,j} \) variables of the Standard TSP Polytope; see Definitions 2 and 3 (p. 2) in its description is an EF of the Standard TSP Polytope, and vice versa. Clearly, this can only be in a degenerate/non-meaningful sense from which no valid inferences can be made.

4.1 The case of the Minimum Spanning Tree Problem

A case in point for the discussions in the introduction to section 4 above, is that of the Minimum Spanning Tree Problem (MSTP). Without the refinement that our notion of independent spaces contributes to the notion of EFs, this case (of the MSTP) would mean that it is possible to extend an exponential-sized model into a polynomial-sized one by (simply) augmenting the exponential model, which is a clearly-unreasonable/out-of-the-question proposition. This proposition would be arrived at as follows. Assume (as is normally done in EF work) that the addition of redundant constraints and variables does not matter as far EF theory is concerned. Since the constraints of Edmonds’ model (Edmonds (1970)) are redundant for the model of Martin (1991), one could augment Martin’s formulation with these constraints. The resulting model would still be considered a polynomial-sized
one. But note that this particular augmentation of Martin’s model would also be an augmentation of Edmonds’ model. Hence, the conclusion would be that Edmonds’ exponential-sized model has been augmented into a polynomial-sized one, which is an impossibility, since one cannot reduce the number of facets of a given polytope by augmenting that polytope. The refinement brought by our notion of independent spaces explains this paradox in the case of MSTP, as shown below.

**Example 36** We show that Martin’s polynomial-sized LP model of the MSTP is not an EF (in a non-degenerate, meaningful sense) of Edmonds’s exponential LP model of the MSTP, by showing that Martin’s model can be stated in independent space relative to Edmonds’ model.

- **Using the notation in Martin(1991), i.e.:**
  - \( N := \{1, \ldots, n\} \) (Set of vertices);
  - \( E \): Set of edges;
  - \( \forall S \subseteq N, \gamma(S) \): Set of edges with both ends in \( S \).

- **Exponential-sized/“sub-tour elimination” LP formulation (Edmonds (1970))**

  \[\begin{align*}
  &\text{(P):} \\
  &\text{Minimize: } \sum_{e \in E} c_e x_e \\
  &\text{Subject To: } \sum_{e \in E} x_e = n - 1; \\
  &\quad \sum_{e \in \gamma(S)} x_e \leq |S| - 1; \quad S \subseteq E; \\
  &\quad x_e \geq 0 \text{ for all } e \in E.
  \end{align*}\]

- **Polynomial-sized LP reformulation (Martin (1991))**

  \[\begin{align*}
  &\text{(Q):} \\
  &\text{Minimize: } \sum_{e \in E} c_e x_e \\
  &\text{Subject To: } \sum_{e \in E} x_e = n - 1; \\
  &\quad z_{k,i,j} + z_{k,j,i} = x_e; \quad k = 1, \ldots, n; \quad e \in \gamma(\{i, j\}); \\
  &\quad \sum_{s > i} z_{k,i,s} + \sum_{h < i} z_{k,i,h} \leq 1; \quad k = 1, \ldots, n; \quad i \neq k; \\
  &\quad \sum_{s > k} z_{k,k,s} + \sum_{h < k} z_{k,k,h} \leq 0; \quad k = 1, \ldots, n; \\
  &\quad x_e \geq 0 \text{ for all } e \in E; \quad z_{k,i,j} \geq 0 \text{ for all } k, i, j.
  \end{align*}\]
• Re-statement of Martin’s LP model (Diaby and Karwan (2013); Regan (2013))

For each \( e \in E \):
- Denote the ends of \( e \) as \( i_e \) and \( j_e \), respectively;
- Fix an arbitrary node, \( r_e \), which is not incident on \( e \) (i.e., \( r_e \) is such that it is not an end of \( e \)).

Then, one can verify that \( Q \) is equivalent to:

\[
\begin{align*}
\text{(Q′):} & \quad \text{Minimize:} & \sum_{e \in E} c_e z_{r_e,i_e,j_e} + \sum_{e \in E} c_e z_{r_e,j_e,i_e} \\
& \quad \text{Subject To:} & \sum_{e \in E} z_{r_e,i_e,j_e} + \sum_{e \in E} z_{r_e,j_e,i_e} = n - 1; \\
& & z_{k,i_e,j_e} + z_{k,j_e,i_e} = z_{r_e,i_e,j_e} + z_{r_e,j_e,i_e}; \; k = 1, \ldots, n; \; e \in E; \\
& & \sum_{s > i} z_{k,i,s} + \sum_{h < i} z_{k,i,h} \leq 1; \; i, k = 1, \ldots, n : i \neq k; \\
& & \sum_{s > k} z_{k,k,s} + \sum_{h < k} z_{k,k,h} \leq 0; \; k = 1, \ldots, n; \\
& & z_{k,i,j} \geq 0 \text{ for all } k, i, j.
\end{align*}
\]

\( \blacksquare \)

**Remark 37** We argue that the reason that EF modeling “barriers” do not apply in the case of the MSTP is due to the fact that Martin’s formulation of the MSTP (\( Q \) in Example 36 above) is not an EF of Edmonds’ model (\( P \) in Example 36 above) in a non-degenerate/meaningful, well-defined sense. The reason for this in turn, is that Martin’s formulation can be stated in independent space relative to Edmonds’ model, due to Martin’s formulation having column-redundancy when it includes the class of variables of Edmonds’ model (see Definition 10), as shown in Example 36.

4.2 Alternate/Auxiliary Models

In this section, we provide some insights into the meaning of the existence of an affine map establishing a one-to-one correspondence between polytopes that are stated in independent spaces as brought to our attention in private discussions by Yannakakis (2013). The linear map stipulated in Definition 5 is a special case of the affine map. Referring back to Definitions 4-6, assume \( G = 0 \) in the expression of \( U \). We will show in this section, that in that case, provided the matrix of the affine transformation does not have any strictly-negative entry, \( U \) is simply an alternate model (a “reformulation”) of \( P \) which can be used, in an “auxiliary” way, in order to solve the optimization problem over \( P \) without any reference to/knowledge of the \( \mathcal{H} \)-description of \( P \) (see Remark 16 (p. 9) of this paper).

**Remark 38**
Referring back to Example 25 (p. 8), assume that the non-negativity requirements for \( x \) and \( y \) are included in the constraints of \( X \) and \( Y \), respectively, and that \( L \) has the form:

\[
L = \{(x, y) \in \mathbb{R}^{p+q}_+: x - Cy = b\} \quad (\text{where } C \in \mathbb{R}^{p \times q}, \text{ and } b \in \mathbb{R}^p).
\]

Consider the optimization problem:

**Problem LP\(_1\):**

\[
\begin{align*}
\text{Minimize:} & \quad \alpha^T x \\
\text{Subject To:} & \quad (x, y) \in L; \ y \in Y \\
& \quad (\text{where } \alpha \in \mathbb{R}^p).
\end{align*}
\]

**Problem LP\(_1\)** is equivalent to the smaller linear program:

**Problem LP\(_2\):**

\[
\begin{align*}
\text{Minimize:} & \quad (\alpha^T C) y + \alpha^T b \\
\text{Subject To:} & \quad y \in Y \\
& \quad (\text{where } \alpha \in \mathbb{R}^p).
\end{align*}
\]

Hence, if \( L \) is the graph of a one-to-one correspondence between the points of \( X \) and the points of \( Y \) (see Beachy and Blair (2006, pp. 47-59)), then, the optimization of any linear function of \( x \) over \( X \) can be done by first using **Problem LP\(_2\)** in order to get an optimal \( y \), and then using Graph \( L \) to “retrieve” the corresponding \( x \). Note that the second term of the objective function of **Problem LP\(_2\)** can be ignored in the optimization process of **Problem LP\(_2\)**, since that term is a constant.

Hence, if \( L \) is derived from knowledge of the \( V \)-representation of \( X \) only (as is the case for the TSP LP models of Diaby (2007) and Diaby and Karwan (2012) relative to the Standard TSP Polytope), then this would mean that the \( H \)-representation of \( X \) is not involved in the “two-step” solution process (of using **Problem LP\(_2\)** and then Graph \( L \)), but rather, that only the \( V \)-representation of \( X \) is involved (see Remark 16 (p. 6) of this paper). The case of the MSTP can be used to illustrate this point also. Referring back to Example 26 clearly, it is not possible to get the \( H \)-description \( P \) by simple mathematical “manipulations” of the \( H \)-description \( Qf \). One could derive \( Q \) from \( Qf \) only by establishing correspondences which are based on the knowledge of the \( V \)-descriptions of \( P \) and \( Qf \). In other words, although solutions of \( P \) can be “retrieved” from those of \( Qf \), that “retrieval” is based on knowledge of the \( V \)-descriptions involved and does not, therefore, impy any meaningful extension relationships between the \( H \)-descriptions.

Hence, in general, when \( G = 0 \) in Definition 15, the condition stipulated in that definition cannot imply (or lead to) extended formulation relations which are meaningful in relating the minimal \( H \)-representations of the polytopes involved (see Example 22 (p. 7) also).

Direct corollaries of the developments above in this section and in section 3 are the following,
Corollary 39 Let $P$ and $Q$ be (non-empty) polytopes stated in overlapping spaces. Assume w.l.o.g. that the set of the descriptive variables of $P$ is embedded in the set of the descriptive variables of $Q$. If $Q$ can be expressed in independent space relative to $P$ (i.e., if all of the constraints involving the variables of $P$ can be dropped from $Q$ after the variables of $P$ are substituted out of the objective function of the optimization problem over $Q$), then $Q$ is not (and cannot be augmented into) a well-defined (non-degenerate, non-ambiguous, meaningful) EF of $P$.

Corollary 40 Extended Formulations developments relating problem sizes (such as Yannakakis (1991), and Fiorini et al. (2011; 2012), in particular) are valid/applicable only when the projections involved are irredundant-component projections.

4.3 Application to the Fiorini et al. (2011; 2012) “barriers”

Having addressed the MSTP in section 4.1 we will return our focus to the TSP in this section. We will illustrate Corollaries 39 and 40 using the developments in Fiorini et al. (2011; 2012), by showing that the mathematics in those papers actually “breaks down” as one tries to apply their developments when the polytopes involved are stated in independent spaces (i.e., when $G = 0$ in Definitions 4, 5, 6, respectively). As we indicated in the Introduction section (section 1), in this paper, we are not concerned with the issue of correctness/incorrectness of any particular LP model that may have been proposed for NP-Complete problems. Rather, our aim is to show that the resolution of that issue (of correctness/incorrectness) can be beyond the scope of EF work under some conditions, such as is the case for the LP models of Diaby (2007), and Diaby and Karwan (2012), for example. In order to simplify the discussion, we will focus on the Standard TSP Polytope, and use the Alternate TSP Polytope discussed in this paper (see 17, p. 6), as well as the TSP LP models of Diaby (2007) and Diaby and Karwan (2012), respectively, as illustrations.

Fiorini et al. (2012) is a re-organized and extended version of Fiorini et al. (2011). The key extension is the addition of another alternate definition of extended formulation (page 96 of Fiorini et al. (2012)) which is recalled in this paper as Definition 6. This new alternate definition is then used to re-arrange “section 5” of Fiorini et al. (2011) into “section 2” and “section 3” of Fiorini et al. (2012). Hence, the developments in “section 5” of Fiorini et al. (2011) which depended on “Theorem 4” of that paper, are “stand-alones” (as “section 3”) in Fiorini et al. (2012), and “Theorem 4” in Fiorini et al. (2011) is relabeled as “Theorem 13” in Fiorini et al. (2012).

Our discussion of specifics why neither of the two papers are applicable when relating polytopes in independent spaces will be based on the non-validity of the proofs of “Theorem 4” of Fiorini et al. (2011) (which is “Theorem 13” of Fiorini et al. (2012), as indicated above), and of “Theorem 3” of Fiorini et al. (2012) (which is in “section 3” of that paper) when $G = 0$ in Definitions 4, 5, and 6 respectively.

Theorem 41 Let $W \subset \mathbb{R}^5$, be the polytope involved in an arbitrary abstraction of TSP tours. Assume $W$ and the Standard TSP Polytope are expressed in independent spaces (such as is the case for the Alternate TSP Polytope, or the polytopes associated with the LP models of the TSP proposed in Diaby (2007) and in Diaby and Karwan (2012), respectively). Then, the developments in Fiorini et al. (2011) are not valid (and therefore, not applicable) for relating the size of $W$ to the size of the Standard TSP Polytope.
Proof. Using the terminology and notation of Fiorini et al. (2011), the main results of section 2 of Fiorini et al. (2011) are developed in terms of $Q := \{(x, y) \in \mathbb{R}^{d+k} | Ex + Fy = g, y \in C\}$ and $P := \{x \in \mathbb{R}^d | Ax \leq b\}$.

Note that letting $Q$ (in Fiorini et al. (2011)) stand for $W$, and $P$ (in Fiorini et al. (2011)) stand for the Standard TSP Polytope respectively, $E$ would be equal to $0$ in the expression of $Q$. Hence, firstly, assume $E = 0$ in the expression of $Q$ (i.e., $Q := \{(x, y) \in \mathbb{R}^{d+k} | 0x + Fy = g, y \in C\}$). Then, secondly, consider Theorem 4 of Fiorini et al. (2011) (which is pivotal in that work). We have the following:

1. If $A \neq 0$ in the expression of $P$, then the proof of the theorem is invalid since that proof requires setting “$E := A$” (see Fiorini et al. (2011, p. 7));

2. If $A = 0$, then $P := \{x \in \mathbb{R}^d | 0x \leq b\}$. This implies that either $P = \mathbb{R}^d$ (if $b \geq 0$) or $P = \emptyset$ (if $b \neq 0$). Hence, $P$ would be either unbounded or empty. Hence, there could not exist a polytope, $\text{Conv}(V)$, such that $P = \text{Conv}(V)$ (see Bazaraa et al. (2006, pp. 39-49), or Fiorini et al. (2011, 16-17), among others). Hence, the conditions in the statement of Theorem 4 of Fiorini et al. (2011) would be ill-defined/impossible.

Hence, the developments in Fiorini et al. (2011) are not applicable for $W$.

Theorem 42 Let $W \subset \mathbb{R}^\xi$, be the polytope involved in an arbitrary abstraction of TSP tours. Assume $W$ and the Standard TSP Polytope are expressed in independent spaces (such as is the case for the Alternate TSP Polytope, or the polytopes associated with the LP models of the TSP proposed in Diaby (2007) and in Diaby and Karwan (2012), respectively). Then, the developments in Fiorini et al. (2012) are not valid (and therefore, not applicable) for relating the size of $W$ to the size of the Standard TSP Polytope.

Proof. First, note that “Theorem 13” of Fiorini et al. (2012, p. 101) is the same as “Theorem 4.” Hence, the proof of Theorem 41 above is applicable to “Theorem 13” of Fiorini et al. (2012). Hence, the developments in Fiorini et al. (2012) that hinge on this result (namely, from “section 4” of the paper, onward) are not applicable to $W$.

Now consider “Theorem 3” of Fiorini et al. (2012) (section 3, page 99). The proof of this theorem hinges on the statement that (using the terminology and notation of Fiorini et al. (2012)):

$$Ax \leq b \iff \exists y : E^\leq x + F^\leq y \leq g^\leq, E^\geq x + F^\geq y \leq g^\geq.$$  \hspace{1cm} (19)

Now, observe that if $x$ and $y$ do not overlap,

$$\exists y : 0 \cdot x + F^\leq y \leq g^\leq, \hspace{0.5cm} 0 \cdot x + F^\geq y \leq g^\geq$$

cannot imply $(Ax \leq b)$ in general.

Hence, provided $x$ and $y$ do not overlap (i.e., provided $x$ and $y$ are in independent spaces), the “if and only if” stipulation of (19) cannot be satisfied in general. Hence, Theorem 3 of Fiorini et al. (2012) is not applicable for $W$.  ■
5 Conclusions

The developments above in this paper are formalizations of the argument that the possibility of inferring solutions obtained from any given correct abstraction of a given optimization problem from those of any other correct abstraction of that (same) optimization problem is a logical necessity, and cannot systematically imply any well-defined (non-degenerate, meaningful) extension relationships between the resulting models.

We would argue that for COP’s in general, the paradox in existing extended formulations theory whereby one can extend an exponential-sized model, by augmenting it, into a polynomial-sized model (such as in the case of the spanning tree polytope), is due to the pertinent extension relationship being a ill-defined (degenerate, non-meaningful) one only. Indeed, if a given “extension” of a model has row- and/or column-redundancies, then there may exist a description of the “extension” in question in which “\( G = 0 \)” (where \( G \) is the matrix of coefficients for the variables of the “original” model, as in Definitions 4, 5, and 6). It would be possible in that case to substitute all of the variables of the “original” model out of the “extension” at hand. Hence, in that case, the “extension” at hand would simply be an alternate abstraction of the problem at hand, stated in independent space from the “original” model. Hence, fundamentally in that case, there would exist no well-defined (non-degenerate, meaningful) extension relationship between the “extension” and the “original” model, but only ill-defined ones (including the case of the “original” model being an “extension” of the “extension”), as has been discussed in this paper.

Hence, overall, we believe that the ill-definition condition we have shown in this paper, with its consequence whereby EF theory is not applicable when relating models expressed in independent spaces, constitutes a useful step towards a more complete definition of the scope of applicability for EF’s.
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