The Optimality of Partial Clique Covering for Index Coding

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Abstract—Partial clique covering is one of the most basic coding schemes for index coding problems, generalizing clique and cycle covering on the side information digraph and further reducing the achievable broadcast rate. In this paper, we start with partition multicast, a special case of partial clique covering with cover number 1, and show that partition multicast achieves the optimal broadcast rate of the multiple-unicast index coding if and only if the side information digraph is partially acyclic. A digraph is said to be partially acyclic if its sub-digraph induced by the vertex with maximum in-degree and its incoming neighbors in the complementary digraph is acyclic. We further extend to the general partial clique covering, offering sufficient conditions of its optimality and sub-optimality with the aid of strong connectivity decomposition. In addition, for some digraph classes, we also prove that the optimal broadcast rate can be approximated by partial clique covering (as well as by other basic schemes) within either a constant factor, or a multiplicative factor of \(O\left(\frac{n}{\log n}\right)\), or \(O(n^\epsilon)\) for some \(\epsilon \in (0, 1)\).

Index Terms—Index Coding, Partial Clique Covering, Partition Multicast, Confusion Graph

I. INTRODUCTION

The index coding problem is one of the most intriguing open problems in network information theory and theoretic computer science, because of its rich connections to distributed storage [1], [2], coded caching [3], network coding [4], [5], topological interference management [6], and hat guessing games [7]. The multiple-unicast index coding problem considers a noiseless broadcast transmission, where each receiver wants one unique message from the transmitter and holds some other receivers’ desired messages as side information. The goal is to characterize the minimum number of transmissions, namely the broadcast rate, such that all receivers are able to decode their desired messages synchronously. Index coding has attracted extensive research efforts from various fields using tools such as graph theory [8]–[14], interference alignment [6], [15], linear programming [16], [17], matrix completion [18], [19], random coding [20] as well as source coding [21], to name just a few.

Although the index coding problem is very simple to describe, it is very hard to solve in general. The difficulty lies in the necessity of not only nonlinear coding schemes [22] but also non-Shannon information inequalities [23], neither of which is well understood so far. The majority of the progress made in the past decade consists of improving over the basic coding schemes (e.g., clique covering, cycle covering, and partial clique covering). Among various advanced achievability schemes, by composite coding [20] as well as linear coding schemes [24], all index coding instances up to five messages were fully characterized, and by one-to-one/subspace interference alignment, the broadcast rate of a family of index coding problems was identified under the equivalent topological interference management (TIM) setting [6]. Other attempts using linear programming formulation were also made to improve the basic achievability schemes through fractionalization, localization, and generalization, such as fractional clique covering [16], (fractional) local graph coloring [11], [25] as well as the recent development of fractional local partial clique covering [26] and interlinked cycle covering [27], among many others. While the boundary of our knowledge towards the best possible achievability schemes is pushed ahead, it is perhaps hopeless to expect that a single scheme achieves the optimal broadcast rate for all index coding instances.

Another line of work consists of evaluating the optimality of the most basic achievability schemes at hand, by identifying the fundamental structure for which such schemes are sufficient and/or necessary to achieve the optimal broadcast rate. In doing so, we could understand better the pros and cons of the available tools, and at the same time reduce the unsolved problem space. In particular, [28] showed that chordal network topology is the fundamental topological structure that determines the optimality of fractional clique covering on the side information digraph (or fractional vertex coloring on the underlying undirected graph of its complement). Specifically, fractional clique covering achieves the all-unicast capacity region of index coding problems if and only if the bipartite network topology (see Definition 1) is chordal. Such an all-unicast setting includes all possible unicast messages. The characterization of the all-unicast capacity region automatically yields the characterization of the capacity region for any arbitrary subset of unicast messages, as well as other traditional metrics such as sum or symmetric capacity (c.f. broadcast rate).

So, when fractional clique covering achieves the all-unicast capacity region, it achieves the capacity region of any arbitrary subset of messages as well. With respect to side information digraphs, the chordal network topology implies the absence of directed cycle of length no less than 3, which turns out to be one of the fundamental topological structures that determine the optimality of clique covering.

Beyond this, however, the fundamental topological structure that determines the optimality of off-the-shelf achievability schemes is still far less understood. On one hand, while the sufficiency of the optimality of some sophisticated schemes, such as interference alignment [6], [15], composite coding [20],
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start with partition multicast, and characterize the sufficient and
concepts related to partial clique covering. In Section III-A, we
lem, followed by the definition of confusion graph, and some
partial clique. Note that strong connectivity of a digraph is
every vertex has a directed path to any other vertices in this
partial clique covering is optimal, if every strongly-connected
characterization leads to sufficient conditions for the optimality
partial acyclicity in digraphs indicates the absence of directed
A digraph, as a single partial clique, is said to be
partial acyclic (see Definition 5) if the sub-digraph induced by
complementary digraph is acyclic. In other words, partial acyclicity in digraphs indicates the absence of directed
cycles in the dominant partial clique (see Definition 4). This
characterization leads to sufficient conditions for the optimality
and the sub-optimality of partial clique covering. On one hand,
partial clique covering is optimal, if every strongly-connected
component (see Appendix) of the side information digraph is
partially acyclic. On the other hand, partial clique covering is
suboptimal if neither every strongly-connected component is
partially acyclic, nor the strongly-connected components that are not partially acyclic are divisible (see Definition 6).
A strongly-connected component is a partial clique, such that
every vertex has a directed path to any other vertices in this
partial clique. Note that strong connectivity of a digraph is
an equivalence relation of the set of its vertices. As such, the
partitioning of a digraph into strongly-connected components
(i.e., equivalent classes) is unique, in the sense that any vertex belongs uniquely to one component. A digraph is said to be
divisible if there exists a further partition such that the sum of the minimum in-degree of each portion is strictly increased.
The rest of the paper is organized as follows. In the next
section, we introduce the multiple-unicast index coding problem,
followed by the definition of confusion graph, and some
concepts related to partial clique covering. In Section III-A, we
start with partition multicast, and characterize the sufficient
and necessary condition under which partition multicast achieves
the optimal broadcast rate. That is, the dominant partial clique
(see Definition 4) in the side information digraph is acyclic.
We show that partial acyclicity is the fundamental topological
structure that yields the optimality of partition multicast.
Further, we extend in Section III-B to the general partial clique
covering, identifying sufficient conditions for which partial clique covering can or cannot achieve the optimal broadcast rate,
with the aid of strong connectivity decomposition. We also show
in Section III-C that (partial) clique covering approximates the
broadcast rate for certain classes of directed graphs within
either a constant factor, or a multiplicative factor of \( O\left(\frac{n}{\log n}\right) \), or \( O(n^\epsilon) \) for some \( \epsilon \in (0, 1) \), with the aid of the connection
between the broadcast rate of index coding and the acyclic set
coloring of side information digraphs.

II. BACKGROUND

In what follows, some preliminaries pertaining to index
coding [8], confusion graph [9], [12], and partial clique
covering [8] are briefly recalled. The relevant graph theoretic
definitions can be found in Appendix.

A. Index Coding

Consider a message set \( \{0,1\}^t \), from which a set of \( n \)
messages is uniformly and randomly chosen. Let \( \{1,\ldots,n\} \) be
the message index set. Let \( x(S) \) be the set of messages indexed by
the subset \( S \), where \( S \subseteq \{1 : n\} \triangleq \{1,\ldots,n\} \). We use the
shorthand term \( x_j \) to represent the individual message \( x(\{j\}) \),
and the \( n \)-tuple \( x^n \) to represent the whole set of message
\( x(\{1 : n\}) \). The network topology of index coding is formed
by a transmitter and \( n \) receivers, connected by a unit-capacity
shared noiseless link. Each receiver \( j \) has a side information set
\( S_j \), i.e., it already knows the messages \( x(S_j) \). For the multiple-unicast
index coding problem, each receiver \( j \) wishes to receive
message \( x_j \), aiming at the minimum number of transmission
(i.e., the minimum number of channel use of the unit-capacity
noiseless link). Obviously, \( j \notin S_j \), otherwise the demand is
trivial. The goal consists of the construction of a \((t,r)\) index
code \( C \) based on the side information index sets \( \{S_1,\ldots,S_n\} \),
including the following encoding/decoding functions:

- An encoding function, \( \phi : \{0,1\}^t x \{2^{[1:n]}\}^n \rightarrow \{0,1\}^r \)
at the transmitter that encodes \( n \)-tuple of messages \( x^n \) to
a length-\( r \) index coding codeword, i.e.,

\[
\phi(x^n, S_1, \ldots, S_n) \mapsto \{0,1\}^r;
\]

- A decoding function at each receiver \( j \), \( \psi_j : \{0,1\}^r \times \{0,1\}^{||S_j||} \rightarrow \{0,1\}^t \), \( \forall j \) that decodes the received index
coding codeword back to \( x_j \) based on the side information
\( x(S_j) \) held at the receiver \( j \). That is, an index code is
feasible if

\[
\psi_j(\phi(x^n, S_1, \ldots, S_n), x(S_j)) = x_j,
\]
for all \( j = \{1,\ldots,n\} \).

The achievable broadcast rate of the index code \( C_t \) is
\( \beta(C_t) = \frac{t}{r} \). A broadcast rate \( \beta(C) \) is achievable for a family
of index codes \( C \) given a configuration of the side information
sets \( \{S_1,\ldots,S_n\} \), if there exists a sequence of feasible codes
\( C = \{C_t, t \in \mathbb{N}\} \), such that \( \inf_t \beta(C_t) \leq \beta(C) \). The optimal
broadcast rate (for given configuration of side information) is
the infimum of all achievable rates, defined as
\[ \beta = \inf \inf_{i_1} \beta(C_t) \]
where the second infimum is over all \((i, r)\) index codes \(C_t\). Sometimes, \(\frac{1}{2}\) is also referred to as the symmetric capacity, or the optimal symmetric degrees-of-freedom in the TIM setting.

**Definition 1 (Graphic Representations).** The index coding problem with message index set \(V\) and side information index sets \(\{S_j, \forall j\}\) can be fully represented by any of the following three graphs:

- **Conflict digraph** \(D = (V, A)\): A directed graph with message index set \(V\) being vertices and an arc \((i, j) \in A(D)\) if and only if \(i \notin S_j\); and
- **Side information digraph** \(D = (V, \bar{A})\): The complement of the conflict digraph, with an arc \((i, j) \in \bar{A}(D)\) if and only if \(i \in S_j\); and
- **Network topology** \(H = (U, V, E)\): An undirected bipartite graph with \(|U| = |V|\), and an edge \((u_i, v_j) \in E(H)\) if and only if \(i = j\) or \((i, j) \in A(D)\). The network topology can also be represented instead by the sets \(T_j, v_j \in V\), where \(T_j \triangleq \{i : (u_i, v_j) \in E\} = N^{-}_{\bar{u}}(D)\).

If the arcs in digraphs \(D\) and \(\bar{D}\) are bi-directed (see Appendix), we also use respectively the undirected graphs \(G\) and \(\bar{G}\) with the arc direction ignored to represent the conflict graph and the side information graph.

**Example 1.** In this example, we introduce a typical structure, referred to as the triangular network, which will be useful later. The triangular network topology \(H = \text{Tri}(K)\) refers to a bipartite graph with \(|U| = |V| = K\) and \((i, j) \in E(H)\) for all \(i \leq j\), corresponding to the conflict digraph where \((i, j) \in A(D)\) if and only if \(i < j\). A triangular network \(\text{Tri}(4)\) with above three graphic representations is depicted in Fig. 1.

![Fig. 1: Three graphic representations of an index coding instance.](image)

**Remark 1.** We adopt the more intuitive representation driven by the interference management perspective, such that the conflict and side information digraphs have arcs with directions opposite to what was originally used in [8].

Information graphs \(G\) are considered, using respectively the acyclic set number \(\alpha(G)\) and the independent set number \(\alpha(\bar{G})\). The definition of these two parameters can be found in Appendix.

**Lemma 1 (Lower Bounds [8]).** Consider an instance of the index coding problem with the undirected side information graph \(G\) and the side information digraph \(D\). Then, the broadcast rate satisfies the Maximum Independent Set (MIS) bound and the Maximal Acyclic Induced Subgraph (MAIS) bound, given respectively by
\[ \beta \geq \alpha(G), \quad \beta \geq \alpha(\bar{G}). \]  

**B. Confusion Graph**

The confusion graph, first introduced in [9], is an undirected graph with vertices being the message tuples, and with an edge between two vertices if the two corresponding message tuples are confusable. Given the \(n\)-tuple messages \(x^n, z^n \in \{0, 1\}^n\), they are said to be confusable at node \(j\) if \(x_j \neq z_j\) and \(x_i = z_i\) for all \(i \in S_j\). Intuitively, two message tuples are confusable at node \(j\) if the receiver \(j\) cannot distinguish them only based on its side information \(S_j\). This means that in order to determine the message, some additional information (a codeword) must be sent by the transmitter for disambiguation. Two message tuples \(x^n, z^n\) are confusable if they are confusable at some node \(j\).

Here we present an alternative construction w.r.t. conflict digraphs, which will be very useful for our proof. Given two \(n\)-tuples \(x^n\) and \(z^n\), let \(Q(x^n, z^n) \triangleq \{i : x_i \neq z_i, \forall i \}\) be the set of indices of the difference, and \(Q\) be the set of vertices indexed by \(Q(x^n, z^n)\). Here, \(N^-_{\bar{u}}(D)\) is the closed in-neighborhood (see Appendix) of the vertex \(v\) in the digraph \(D\).

**Definition 2 (Confusion Graph).** For the conflict digraph \(D = (V, A)\) and an integer \(t\), the confusion graph \(\Upsilon_t(D) = (V, A)\) is an undirected graph with \(2^{|V|}\) vertices (i.e., the \(n\)-tuple messages), two of which are connected with an edge if and only if the corresponding messages are confusable.

Given two message tuples \(x^n\) and \(z^n\), they are confusable if and only if there exists \(j \in Q(x^n, z^n)\), such that \(Q(x^n, z^n) \subseteq S_j\), where \(S_j\) is the set of nodes of \(D\). To be more specific, if and only if one of the following two conditions is satisfied:

1. \(|Q(x^n, z^n)| = 1\);
2. \(|Q(x^n, z^n)| \geq 2\), and there exists \(j \in Q(x^n, z^n)\) such that \(\forall i \in Q(x^n, z^n) \setminus \{j\}, (i, j) \in A(D)\).

Alternatively, two message tuples \(x^n\) and \(z^n\) are non-confusable if and only if \(\forall j \in Q(x^n, z^n), Q(x^n, z^n) \cap S_j \neq \emptyset\), or specifically if and only if all vertices in the induced subgraph \(\Upsilon_2(D)\) have positive in-degree.

**Remark 1.** The above alternative construction of confusion graphs is equivalent to the classical definition in [9]. When \(Q(x^n, z^n) = \{1\}\), \(x^n\) and \(z^n\) are confusable at \(i\) regardless of the side information. When \(|Q(x^n, z^n)| \geq 2\), there exists \(j \in Q(x^n, z^n)\) such that \(\forall i \in Q(x^n, z^n) \setminus \{j\}, (i, j) \in A(D)\), meaning that for all \(i \in Q(x^n, z^n) \setminus \{j\}, i \notin S_j\) and then for all \(k \in S_j\), \(x_k = z_k\). Thus, \(x^n\) and \(z^n\) are confusable at \(j\).
According to Definition 2, we have the following three important observations for any two message tuples $x^n$ and $z^n$, which are very useful for the proof of our main theorems. Here, $\Delta^-(D)$ denotes the maximum in-degree of the vertex in $D$ (see Appendix).

- **Observation 1:** If $|Q(x^n, z^n)| \geq \Delta^-(D) + 2$, then $x^n$ and $z^n$ are non-confusable, because there does not exist any $j$ such that $Q(x^n, z^n) \subseteq N_j(D)$, with $|N_j(D)| \leq \Delta^-(D) + 1$.

- **Observation 2:** If $|Q(x^n, z^n)| = \Delta^-(D) + 1$, then $x^n$ and $z^n$ are confusable if and only if $Q(x^n, z^n) = N^*_{v^*}(D)$, where $v^* = \arg\max_{v \in V}[\max_{\bar{v} \in V}[\Delta^-(\bar{D}, v)]]$, because otherwise there does not exist any $j$ such that $Q(x^n, z^n) \subseteq N_j(D)$.

- **Observation 3:** If $\bar{D}[Q]$ is exactly a directed cycle, then $x^n$ and $z^n$ are non-confusable, because directed cycles have positive in-degree for all nodes.

As an example, a conflict digraph $D$ of an index coding problem and its corresponding confusion graph $\Gamma_t(D)$ with $t = 1$ are given in Fig. 2(a) and Fig. 2(b), respectively.

![Fig. 2: (a) The conflict digraph $D$ of an index coding instance with $\Delta^-(D) = 1$, and (b) its confusion graph corresponding $\Gamma(D)$.](image)

**Lemma 2** (Broadcast Rate via Confusion Graph Construction [9], [12]). The broadcast rate of the index coding problem with confusion graph $\Gamma_t(D)$ is

$$\beta = \frac{\log(\chi_t(\Gamma_t(D)))}{t},$$

for some $t$.

**C. Partial Clique Covering**

Let us first introduce the notion of partial cliques defined in [30] dedicated to side information digraphs. Here, $d^-(D, v)$ is the in-degree of the vertex $v$ in the digraph $D$ (see Appendix).

**Definition 3** (Partial Clique). A side information sub-digraph $\bar{D}[Q]$ induced by the vertex set $Q$ is a $k$-partial clique if and only if $\forall v_q \in Q$, $d^-(\bar{D}[Q], v_q) \geq |Q| - k - 1$, and $\exists v^*_q \in Q$, $d^-(\bar{D}[Q], v^*_q) = |Q| - k - 1$.

With respect to the conflict sub-digraphs $\bar{D}[Q]$, the above condition is equivalent to that if and only if $\forall v_q \in Q$, $d^-(\bar{D}[Q], v_q) \leq k$, and $\exists v^*_q \in Q$, $d^-(\bar{D}[Q], v^*_q) = k$. With respect to the network topology $\mathcal{H}[Q] = (\mathcal{U}[Q], \mathcal{V}[Q], \mathcal{E})$, the above condition is equivalent to that and only if $\forall v_q \in \mathcal{V}[Q]$, $|T_{v_q}| \leq k + 1$ and $\exists v^*_q \in \mathcal{V}[Q]$, $|T_{v^*_q}| = k + 1$. The directed acyclic graph is a $(n-1)$-partial clique, the directed cycle is a $(n-2)$-partial clique, and the clique is a 0-partial clique.

**Lemma 3** (Achievable Broadcast Rate via Partial Clique Covering [30]). The broadcast rate of the index coding problem with side information digraph $D = (V, A)$ satisfies

$$\beta \leq \min_{\{v_1, \ldots, v_s\}} \sum_{i=1}^{s} (k_i + 1),$$

where the upper bound is achievable by partial clique covering, the minimum is over all possible partitions of $V = \{V_1, \ldots, V_s\}$, and $\bar{D}[V_i]$ is a $k_i$-partial clique for $i = 1, \ldots, s$.

For each $k_i$-partial clique, a $k_i + 1$ Maximum Distance Separable (MDS) code can make all the desired messages in this partial clique recoverable. In this paper, we only consider non-fractional covering, where partial clique covering boils down to graph partition into partial cliques. Still, characterizing the achievable broadcast rate via partial clique covering is challenging, as the best partial clique partition to minimize the sum is a complex combinatorial optimization problem.

The partition multicast [31] is a special case of partial clique covering when $s = 1$. Roughly speaking, partition multicast consists of sending linearly independent combinations of messages such that the receiver which collects enough number of linear combinations can recover its desired message. From a topological interference management perspective [6], partition multicast is also referred to as Code Division Multiple Access (CDMA).

**Lemma 4** (Achievable Broadcast Rate via Partition Multicast). The broadcast rate of the index coding problem with conflict digraph $D$ satisfies

$$\beta \leq \Delta^-(D) + 1,$$

where the upper bound is achievable by partition multicast, and $\Delta^-(D)$ is the maximum number of incoming arcs of any vertex in $D$ (see Appendix).

From Definition 3 and Lemma 3, we find that the vertex with the minimum incoming arc in the side information digraph $D$ (i.e., the vertex with maximum incoming arcs in the conflict digraph $D$) dominates the achievable broadcast rate. We introduce the following definition to represent this dominant vertex and its neighbors.

**Definition 4** (Dominant Partial Clique). The dominant partial clique in a digraph $D$ is the induced sub-digraph $\bar{D}[N^*_v(D)]$, where $N^*_v(D)$ is the largest closed in-neighborhood in its complement digraph $\bar{D}$, i.e., $v^* = \arg\max_{v \in V}[\max_{\bar{v} \in V}[\Delta^-(\bar{D}, v)]]$.

In words, the dominant partial clique of a digraph is the sub-digraph induced by the vertex with maximum in-degree and its incoming neighbors in the complementary digraph. The dominant partial clique in $D$ is a $\Delta^-(D)$-partial clique with $\Delta^-(D) + 1$ vertices. The side information digraph in Fig. 1(b) is a dominant partial clique. A side information digraph can
have multiple dominant partial cliques, when there are multiple vertices \( v^* \) with the same maximum in-degree in \( \mathcal{D} \).

**Definition 5 (Partial Acyclicity).** A partial clique is said to be partially acyclic in a digraph \( \mathcal{D} \) if there exists at least one dominant partial clique \( \mathcal{D}[\mathcal{N}^+_{\mathcal{D}}(\mathcal{D})] \) that is acyclic.

**Remark 2.** For the case with multiple dominant partial cliques in a digraph, as long as one of them is acyclic, then the digraph is partially acyclic. The property of partial acyclicity is also referred to as the absence of directed cycles in at least one of the dominant partial cliques. 

**Definition 6 (Divisibility).** Given a digraph \( \mathcal{D} = (\mathcal{V}, \mathcal{A}) \), it is said to be divisible if there exists a further partition \( \mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_s\} \) with \( \mathcal{V}_i \cap \mathcal{V}_j = \emptyset, \forall i \neq j \) such that
\[
\delta^-(\mathcal{D}[\mathcal{V}]) < \sum_{i=1}^s \delta^-(\mathcal{D}[\mathcal{V}_i]),
\]
where \( \delta^-(\mathcal{D}) \) is the minimum number of incoming arcs over all vertices in \( \mathcal{D} \).

According to the fact that
\[
\min_{\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}} \sum_{i=1}^s (k_i + 1) = \min_{\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}} \sum_{i=1}^s (\delta^-(\mathcal{D}[\mathcal{V}_i]) + 1)
\]
\[
= |\mathcal{V}| - \max_{\{\mathcal{V}_1, \ldots, \mathcal{V}_s\}} \sum_{i=1}^s \delta^-(\mathcal{D}[\mathcal{V}_i]),
\]
we conclude that a partial clique is divisible indicates that there exists a further partition by which partial clique covering further decreases the broadcast rate. It is easy to verify that, if \( \mathcal{D} \) is partially acyclic, then \( \mathcal{D} \) is not divisible. So, directed acyclic graphs, directed cycles, and cliques are not divisible.

### III. MAIN RESULTS

#### A. The Optimality of Partition Multicast

The main result is a sufficient and necessary condition when partition multicast achieves the optimal broadcast rate.

**Theorem 1 (Optimality of Partition Multicast).** Partition multicast achieves the optimal broadcast rate, if and only if, the side information digraph \( \mathcal{D} \) is partially acyclic. The optimal broadcast rate of such a family of index coding problems is
\[
\beta = \Delta^- \mathcal{D} + 1.
\]

**Proof:** The sufficiency is straightforward, while the necessity is non-trivial. Here we present a sketch of the proof, and relegate the detailed proof to Section IV.

**Sufficiency:** For the “if” part, we need to show that, if there exists one dominant partial clique in the side information digraph \( \mathcal{D} \) that is acyclic, then partition multicast achieves the optimal broadcast rate. As the size of the acyclic dominant partial clique is \( \Delta^- \mathcal{D} + 1 \), by Lemmas 1 and 4, the achievable broadcast rate by partition multicast matches the MAIS lower bound. This completes the proof of the sufficiency.

**Necessity:** For the “only if” part, we need to show that, partition multicast achieves the optimal broadcast rate, only if the side information digraph \( \mathcal{D} \) is partially acyclic. By contraposition, we need to prove that, if the side information digraph \( \mathcal{D} \) is not partially acyclic, then partition multicast cannot achieve the optimal broadcast rate. To prove this, we show that, if every dominant partial clique contains a directed cycle, there exist coding schemes that achieve strictly less broadcast rate. Without knowing the structure of other parts of the side information digraph, none of the coding schemes that we know so far can guarantee this. So, we resort to the construction of the confusion graph, showing that such a coding scheme does exist, although it is not explicitly constructed. By the construction of the confusion graph \( \Gamma_i(\mathcal{D}) \), we show that, if every dominant partial clique contains a directed cycle, the fractional chromatic number of the confusion graph satisfies \( \chi_f(\Gamma_i(\mathcal{D})) < 2^{(\Delta^- \mathcal{D}) + 1} \). By Lemma 2, we conclude that there exists a coding scheme with \( \beta < \Delta^- \mathcal{D} + 1 \), making partition multicast strictly suboptimal. This completes the proof of the necessity.

**Remark 3.** The condition that a dominant partial clique \( \mathcal{D}[\mathcal{N}^+_{\mathcal{D}}(\mathcal{D})] \) is acyclic is equivalent to that the network topology contains a triangular network \( \text{Tri}(\Delta^- \mathcal{D} + 1) \) as a subnetwork. As long as the above condition is satisfied, more conflict for \( \mathcal{D} \), less side information for \( \mathcal{D} \), or more interference for \( \mathcal{H} \) does not increase the broadcast rate. The underlying undirected conflict graph is a clique with size \( \Delta^- \mathcal{D} + 1 \), which is not changed by these operations.

Through the proof of Theorem 1 in Section IV, we obtain immediately the following corollary.

**Corollary 1.** If the dominant partial clique \( \mathcal{D}[\mathcal{N}^+_{\mathcal{D}}(\mathcal{D})] \) contains exactly one directed cycle, then the broadcast rate satisfies \( \Delta^- \mathcal{D} \leq \beta < \Delta^- \mathcal{D} + 1 \).

**Remark 4.** The necessity proof of Theorem 1 mainly focus on the existence proof of an index code that achieves broadcast rate strictly smaller than that achieved by partition multicast, if \( \mathcal{D}[\mathcal{N}^+_{\mathcal{D}}(\mathcal{D})] \) contains exactly one directed cycle. In this case, it also shows that the best improvement (reduction) of the achievable broadcast rate of such an index code over partition multicast is no better than \( \Delta^- \mathcal{D} \), although the upper bound might be not tight for some special cases.

**Example 2.** Consider the example in [30, Proposition 4] with side information digraph in Fig. 3. The partial clique covering yields achievable broadcast rate of 3, while local graph coloring achieves broadcast rate of 2. This is confirmed by the above corollary, where \( \Delta^- \mathcal{D} = 2, v^* = \{1, 2, 4\} \) and the subdigraph induced by partial cliques \( \mathcal{N}^-_1 = \{3, 4\}, \mathcal{N}^-_2 = \{3, 4\} \) and \( \mathcal{N}^-_4 = \{1, 5\} \) contain directed cycles in \( \mathcal{D} \).

The \((K, L)\) regular network refers to a family of index coding problems with network topology \( \{T_j = \{j, j + 1, \ldots, j + L - 1\} \mod K\}, \forall j \}. As the above partial acyclicity condition is satisfied, we characterize the broadcast rate as follows.

**Corollary 2.** For the \((K, L)\) regular networks, partition multicast achieves the optimal broadcast rate \( \beta = L \).

The above result confirms those in [15], revealing that partial acyclicity makes interference alignment unnecessary. The side information digraph of a \((|\mathcal{Q}|, L)\) regular network topology is
The optimal broadcast rate of such a family of index coding problems is given by

\[ \bar{\rho} \] one dominant partial clique in every SCC.

\( \{V_i\} \) is strongly-connected, and all SCCs are disjoint. As such, \( V = \{V_1, \ldots, V_s\} \) is a valid partial clique partition.

For the converse, as \( V = \{V_1, \ldots, V_s\} \) is a strong decomposition, the strong component digraph \( S(D) \) (see Appendix) is acyclic [32]. For every portion \( D[V_i] \), there exists a directed acyclic set with size \( k_i + 1 \), because the dominant partial clique in the \( k_i \)-partial clique \( D[V_i] \) is acyclic. The strong component digraph \( S(D) \) with expansion of directed acyclic sets in all the SCCs will not result in directed cycles. Otherwise, there exist directed cycles between two SCCs, and thus they should be contained in the same SCC, which contradicts with the fact that two SCCs are disjoint. Thus, there exists a directed acyclic induced sub-digraph with size at least \( \sum_{i=1}^s (k_i + 1) \), and hence we have the lower bound \( \beta \geq \sum_{i=1}^s (k_i + 1) \), according to Lemma 1. The upper bound of broadcast rate achieved by partial clique covering coincides with the lower bound, yielding the optimality. This completes the proof.

\[ \text{Proof:} \] Before the proof, we show that the strong decomposition \( V = \{V_1, \ldots, V_s\} \) is unique and any two SCCs are disjoint. As the strong connectivity of a digraph is an equivalence relation of the set of its vertices, the partitioning of a digraph into SCCs (i.e., equivalent classes) is unique. This unique strong decomposition into disjoint SCCs provides us a partition that is most suited for partial clique covering.

As such, we need to show that, if there always exists a dominant partial clique in each SCC that is acyclic, then partial clique covering achieves the optimal broadcast rate.

Given the unique strong decomposition \( V = \{V_1, \ldots, V_s\} \), for any \( i, D[V_i] \) is strongly-connected, and all SCCs are disjoint.

\[ \beta = \sum_{i=1}^s (k_i + 1) \]

\( \bar{D}[V_i] \) is a \( k_i \)-partial clique.

\[ \text{Proof:} \] Before the proof, we show that the strong decomposition \( V = \{V_1, \ldots, V_s\} \) is unique and any two SCCs are disjoint. As the strong connectivity of a digraph is an equivalence relation of the set of its vertices, the partitioning of a digraph into SCCs (i.e., equivalent classes) is unique.

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\[ \text{Proof:} \] Before the proof, we show that the strong decomposition \( V = \{V_1, \ldots, V_s\} \) is unique and any two SCCs are disjoint. As the strong connectivity of a digraph is an equivalence relation of the set of its vertices, the partitioning of a digraph into SCCs (i.e., equivalent classes) is unique.

This unique strong decomposition into disjoint SCCs provides us a partition that is most suited for partial clique covering.

\[ \beta = \sum_{i=1}^s (k_i + 1) \]

\( \bar{D}[V_i] \) is a \( k_i \)-partial clique.
Example 4. Let us take two examples shown in Fig. 5(a) [27, Fig. 13(b)] and Fig. 5(b). In Fig. 5(a), there exists a strong decomposition \( V_1 = \{1, 2\} \) and \( V_2 = \{3, 4, 5\} \), and two strong connected components \( \overline{D}[V_1] \) and \( \overline{D}[V_2] \) are directed cycles and hence are partial cliques respectively with \( k_1 = 0 \) and \( k_2 = 1 \). Theorem 3 holds if there exist some SCCs that are not partially acyclic and not divisible, then strong decomposition is the optimal (see Theorem 1). The sufficient condition of the sub-optimality of partial clique covering identified in Theorem 3 also gives a counterexample.

Fig. 5: (a) A side information digraph with strong decomposition \( \{\{1, 2\}, \{3, 4, 5\}\} \), and (b) a side information digraph with strong decomposition \( \{\{1, 2, 8\}, \{3, 4, 5\}, \{6, 7\}\} \). The dominant partial clique of every strongly-connected component is acyclic. As such, partial clique covering achieves the optimal broadcast rate for Fig. 5(a) and Fig. 5(b) respectively, with total covering number \( \beta = \sum_{i=1}^{2} (k_i + 1) = 3 \) and \( \beta = \sum_{i=1}^{3} (k_i + 1) = 5 \).

Theorem 3 (Sub-Optimality of Partial Clique Covering). Partial clique covering cannot achieve the optimal broadcast rate, if the side information digraph \( \overline{D} = (\overline{V}, \overline{A}) \) falls in neither of the following two conditions:

- Every SCC is partially acyclic.
- Those SCCs that are not partially acyclic are divisible.

**Proof:** To prove the sub-optimality, we need to show that, if there exist some SCCs that are not partially acyclic and not divisible, then partial clique covering is strictly suboptimal.

Let us divide SCCs in the side information graph into three groups: (1) SCCs that are partially acyclic, (2) SCCs that are not partially acyclic but divisible, and (3) SCCs that are not partially acyclic and not divisible. We follow the following strategy. For the SCCs in the first group, they are also not divisible, and we use partition multicast (a special case of partial clique covering), which is optimal (see Theorem 1). For those SCCs in the second group, we use partial clique covering, which may be suboptimal but it does not matter. For the SCCs in the third group, we will show that, there exists a coding scheme that is strictly better than partial clique covering. Overall, if the third group is not empty, we show that the above strategy yields partial clique covering strictly suboptimal.

Let us focus on the third group. If every SCC in the third group is not divisible, then strong decomposition is the optimal partial clique partition, because (1) further partition of each SCC does not further decrease broadcast rate by partial clique covering according to Definition 6, and (2) the capacity region of the union of two sub-digraphs from two different SCCs is achieved by the time division of two separate parts according to [12, Theorem 2], and thus the union does not decrease broadcast rate.

Given the optimal partial clique partition for the third group, we need to show that if the SCCs are not partially acyclic, partial clique covering is strictly suboptimal. To this end, we show that, similarly to Theorem 1, if for any SCC \( \overline{D}[V_i] \) of the strong decomposition, each dominant partial clique has a directed cycle, then there exists a coding scheme with achievable broadcast rate strictly less than \( \sum_{i=1}^{\ell} (k_i + 1) \). Given the unique strong decomposition \( V = \{V_1, \ldots, V_s\} \), for any two disjoint SCCs \( \overline{D}[V_i] \) and \( \overline{D}[V_j] \), there are only directed cuts (see Appendix) between them, i.e., only uni-directed arcs always from one to the other. According to [12, Theorem 2], we have the confusion graph associated with the side information sub-digraphs\(^4\)

\[
\Gamma_i(\overline{D}[V_i \cup V_j]) = \Gamma_i(\overline{D}[V_i]) \cdot \Gamma_j(\overline{D}[V_j]),
\]

for some integer \( t_i \) and \( t_j \), where \( (\cdot) \) is the lexicographic product (see Appendix) of two undirected graphs. By Lemma 8 in Appendix, we have

\[
\chi_f(\Gamma_i(\overline{D}[V_i \cup V_j])) = \chi_f(\Gamma_i(\overline{D}[V_i])) \cdot \chi_f(\Gamma_j(\overline{D}[V_j])).
\]

According to the necessity proof of Theorem 1, if there exists a directed cycle in every dominant partial clique of the strong component \( \overline{D}[V_i] \), then

\[
\chi_f(\Gamma_i(\overline{D}[V_i])) < 2^{t_i(k_i + 1)}.
\]

As such, by Lemma 2, we conclude that, there exists a coding scheme with strictly less broadcast rate than \( k_i + k_j + 2 \) for \( \overline{D}[V_i \cup V_j] \). This argument can be straightforwardly extended to the union of any number of SCCs.

Thus, if any one of the SCCs is not divisible and not partially acyclic, the overall achievable broadcast rate of three groups will be strictly less than \( \sum_{i=1}^{\ell} (k_i + 1) \), which is promised by partial clique covering. This yields partial clique covering strictly suboptimal and completes the proof.

**Remark 7.** The sufficient condition of the sub-optimality of partial clique covering identified in Theorem 3 also gives a necessary condition of its optimality. That is, if partial clique covering is optimal for a certain index coding problem, then the corresponding side information digraph must fall in one of the two conditions of Theorem 3. We suspect that the conditions in Theorem 3 are both necessary and sufficient, but we do not have a proof for the sufficiency of the second condition nor a counterexample.

Example 5. Let us take partial clique covering as a two-step procedure, where the first step using strong decomposition is always globally optimal, and the second step of further partition on each SCC depends if it is partially acyclic or divisible. This categorizes the side information digraphs into three types: (1) each SCC is partially acyclic, for which partial clique covering with strong decomposition is optimal, (2) some

\(^4\)Here, with a bit abuse of the notation, we take \( \Gamma_i(\overline{D}) \) and \( \Gamma_i(\overline{D}) \) interchangeably to represent the same confusion graph corresponding to the side information digraph \( \overline{D} \) and the conflict digraph \( \overline{D} \).
SCCs are not partially acyclic but divisible, for which partial clique covering may be still optimal and further partition of these SCCs is needed, and (3) some SCCs are not partially acyclic and not divisible, for which partial clique covering is strictly suboptimal. Examples corresponding to these three categories are presented in Fig. 6.

![Fig. 6: (a) A side information digraph with strong decomposition \{1, 2\}, \{3, 4, 5\}, for which each SCC is partial acyclic and partial clique covering is optimal (\(\beta = 3\)), (b) an SCC that is not partially acyclic but divisible with two partial cliques \{1, 2\}, \{3, 4, 5\}, for which partial clique covering is still optimal (\(\beta = 3\)), and (c) an SCC that is not partially acyclic and not divisible, and fractional local coloring achieves the optimal broadcast rate \(\beta = \frac{5}{2}\) strictly less than partial clique covering with any partition.]

C. Capacity Approximation

Before proceeding to the capacity approximation results, we summarize two sandwich bounds for the side information digraphs \(\mathcal{D}\), some of which were already appeared in the literature.

For the undirected side information graphs \(\mathcal{G}\) and their conflict graphs \(\mathcal{D}\), we have the following sandwich bound.

**Lemma 5** (Sandwich Bound for Graphs).

\[
\frac{|\mathcal{V}|}{\chi(\mathcal{G})} \leq \alpha(\mathcal{G}) \leq \beta \leq \chi(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1.
\]

The first inequality is due to Lemma 7 in Appendix, the second one is the MIS bound of index coding problem in Lemma 1, the third one is due to the achievable via vertex coloring on conflict graphs, and the last inequality is due to Brooks’ theorem [33].

Similarly, for the general side information digraphs \(\mathcal{D}\) and their corresponding conflict digraphs \(\mathcal{D}\), we have an analogous sandwich bound. The definitions of the graph theoretic parameters can be found in Appendix.

**Lemma 6** (Sandwich Bound for Digraphs).

\[
\frac{|\mathcal{V}|}{\chi_A(\mathcal{D})} \leq \alpha_A(\mathcal{D}) \leq \beta \leq \chi_L(\mathcal{D}) \leq \Delta(\mathcal{D}) + 1.
\]

The first inequality is due to Lemma 7 in Appendix, the second one is the MAIS bound in Lemma 1, the third one is due to the achievable by local graph coloring [11] on conflict digraphs, and the last one is due to the fact that local chromatic number is upper bounded by the size of closed in-neighborhood [11], which is also the achievable broadcast rate via partition multicast.

By the Sandwich bound in Lemmas 5 and 6, we also identify a family of index coding problems with side information digraph \(\mathcal{D}\) where (partial) clique covering achieves the broadcast rate within a multiplicative gap from the optimal.

**Theorem 4** (Approximate Broadcast Rate with Multiplicative Gap). If the side information digraph \(\mathcal{D}\) falls into the following digraph classes, we have broadcast rate \(\beta\) lower bounds and (partial) clique covering is within a factor \(\Theta\) from the optimal.

- \(\mathcal{D}\) is oriented planar [34], \(\beta \geq \frac{n}{3}\) and \(\Theta = 3\);
- \(\mathcal{D}\) is planar with digirth 4 [35], \(\beta \geq \frac{5n}{12}\) and \(\Theta = \frac{12}{5}\);
- \(\mathcal{D}\) is planar with digirth at least 5 [36], \(\beta \geq \frac{n}{6}\), and \(\Theta = 2\);
- \(\mathcal{D}\) is a tournament [37], \(\beta \geq |\log n| + 1\), and \(\Theta = O\left(\frac{n}{\log n}\right)\);
- \(\mathcal{D}\) is oriented [37], \(\beta \geq \log n\), and \(\Theta = O\left(\frac{n}{\log n}\right)\);
- \(\mathcal{D}\) is sparse with \(m = |A(\mathcal{D})| \leq n^{1+\epsilon} - n\) [38], \(\beta \geq n^{1-\epsilon}\), and \(\Theta = O(n'^{c})\), for some \(c \in (0, 1)\), where \(n = |\mathcal{V}(\mathcal{D})|\).

**Proof:** The lower bounds of broadcast rate and the multiplicative gap results are due to the upper bounds of dichromatic number and the Sandwich bound in Lemma 6. If \(\mathcal{D}\) is oriented planar, oriented, and tournament, we have respectively \(\chi_A(\mathcal{D}) \leq \frac{n}{3}[39]\), \(\alpha_A(\mathcal{D}) \geq |\log n| + 1\), \(\chi_L(\mathcal{D}) \geq \log n\) [37]. If \(\mathcal{D}\) is planar with digirth \(g = 4\), we have \(\chi_A(\mathcal{D}) \leq \frac{5n}{12}[35]\) and further if \(g \geq 5\), a recent result showed that \(\chi_A(\mathcal{D}) \leq \frac{n}{2}[36]\). If \(\mathcal{D}\) is sparse with \(m = |A(\mathcal{D})| \leq n^{1+\epsilon} - n\), it was shown in [38] that \(\alpha_A(\mathcal{D}) \geq \frac{n}{n^{1+\epsilon}} \geq n^{1-\epsilon}\). By the Sandwich bound and the fact that \(\beta \leq n\), we obtain the results.

IV. PROOF OF THEOREM 1

There may be multiple dominant partial cliques. In the sufficiency proof, we only need to show that, if there exist at least one dominant partial clique that is acyclic, then partition multicast is optimal. In the necessity proof, we only need to show that, if every dominant partial clique contains a directed cycle, then partition multicast is strictly suboptimal.

A. Sufficiency

We focus on one of the dominant partial cliques. Let \(v^* = \arg \max_{v \in V} |\mathcal{N}^- (\mathcal{D}, v)|\) be the vertex with the maximum in-degree in the conflict digraph \(\mathcal{D}\), and \(\mathcal{N}^-\) denote its closed in-neighborhood in \(\mathcal{D}\). In what follows, we prove that if \(\mathcal{D}[\mathcal{N}^-]\) is acyclic, then partition multicast achieves the broadcast rate of the index coding problem.

By the Sandwich bound in Lemma 6, we have

\[
\alpha_A(\mathcal{D}) \leq \beta \leq \Delta(\mathcal{D}) + 1.
\]

As \(\mathcal{D}[\mathcal{N}^-]\) is acyclic, it follows that

\[
\alpha_A(\mathcal{D}) \geq |\mathcal{N}^-| = \Delta(\mathcal{D}) + 1.
\]

As such, when \(\mathcal{D}[\mathcal{N}^-]\) is acyclic, the broadcast rate of such index coding problems is exactly \(\beta = \Delta(\mathcal{D}) + 1\), which is achieved by partition multicast according to Lemma 4. This completes the sufficiency proof.
B. Necessity

To prove the necessity, we show that, if each dominant partial clique contains a directed cycle, then there exists an index code that strictly outperforms partition multicast. In what follows, we focus on one of the dominant partial cliques, and the argument applies similarly to the other dominant partial cliques.

To avoid confusion in notations, in this section, we let \( D' \) denote the side information digraph whose dominant partial clique contains a directed cycle, and \( D' \) denote its corresponding confusion digraph. Let \( M = \mathcal{N}_r'(D') \) be the closed in-neighborhood of the vertex with the maximum in-degree in \( D' \) and \( \delta = \Delta^- (D') \) be the maximum in-degree in \( D' \). Assume that the dominant partial clique \( D'[M] \) contains a length-\( l \) cycle involving vertices \( \{v_1, \ldots, v_l\} \), where \( 2 \leq l \leq \delta + 1 \). Recall that \( Q(x^n, z^n) \triangleq \{ i : x_i \neq z_i, \forall i \} \), and let \( d_H (x^n, z^n) \triangleq |Q(x^n, z^n)| \) be the Hamming distance of two message tuples.

To prove the existence of a coding scheme that satisfies \( \beta < \delta + 1 \), we have to rely on confusion graphs, as we have no knowledge about the structure of other part in \( D' \). Let \( \Gamma_i(D') \) denote the corresponding confusion graph of \( D' \). In what follows, we will show that \( 2^{\delta t} \leq \chi_f (\Gamma_i(D')) \leq \sum_{j=0}^{2^\delta} 2^{jt} \), for some \( t \). The proof consists of two parts:

Claim 1 - \( \chi_f (\Gamma_i(D')) \geq 2^{\delta t} \). To show this claim, we only need to show \( \omega (\Gamma_i(D')) \geq 2^{\delta t} \). Given an arbitrary digraph \( D' \), it is not easy to count the clique number of its confusion graph \( \Gamma_i(D') \) directly.\(^5\) So, we do this in a different way.

Let \( M = \{v_0, v_1, \ldots, v_k \} \) denote the vertices in \( \mathcal{N}_r'(D') \). We introduce another digraph \( D_{k'} \), which is a copy of \( D' \) with the dominant part \( D'[M] \) replaced by the conflict digraph of the triangular network \( \text{Tri}(\delta + 1) \) (See Example 1), denoted by \( D_{k'} \), as shown in Fig. 7, in which \( \forall \ i < j, (v_i, v_j) \in \mathcal{A}(D_{k}) \).

We assume that \( D' \) is obtained by removing some arcs from adding some other arcs to \( D_{k} \).

![Fig. 7: The reference sub-digraph \( D_{k} \).](image)

First, we show that \( \omega (\Gamma_i(D_{k'}) \geq 2^{(\delta + 1)t} \), given the digraph \( D_{k'} \) and its confusion graph \( \Gamma_i(D_{k'}) \). Consider any two message tuples \( x^n \) and \( z^n \) where \( x_j = z_j, \forall j \notin M \), and \( x_j, z_j \in \mathbb{F}_{2^n}, \forall j \). If \( |Q(x^n, z^n)| = 1 \), \( x^n \) and \( z^n \) are confusable and thus adjacent in \( \Gamma_i(D_{k'}) \). If \( |Q(x^n, z^n)| \geq 2 \), as \( (v_i, v_j) \in \mathcal{A}(D_{k'}) \) for all \( i < j \), it follows by Definition 2 that \( x^n \) and \( z^n \) are confusable and thus adjacent in \( \Gamma_i(D_{k'}) \). As such, any two tuples \( x^n \) and \( z^n \) with \( x_j = z_j, \forall j \notin M \) are adjacent in \( \Gamma_i(D_{k'}) \). As there are \( 2^{(\delta + 1)t} \) such distinct tuples, we conclude that \( \omega (\Gamma_i(D_{k'}) \geq 2^{(\delta + 1)t} \).

Second, we show that \( \omega (\Gamma_i(D')) \geq \omega (\Gamma_i(D_{k'}) \) where \( D' \) is obtained by adding/removing arcs to/from \( D_{k} \) in \( D' \). Adding arcs to \( D_{k} \) does not decrease the clique number of the confusion graph, while removing any arc, e.g., \( (v_i, v_j) \), from \( D_{k} \) decreases the clique number to at most \( \frac{1}{\delta} \). We divide the message tuples considered above with difference only at \( M \) into 2\(^t\) subgroups, in each of which the messages have the same value in \( \mathbb{F}_{2^n} \) at \( v_j \). Clearly, any two tuples \( x^n \) and \( z^n \) in the same subgroup are still confusable, because (1) forcing \( x_{v_j} = z_{v_j} \) reduces the difference from \( Q(x^n, z^n) \) to \( Q(x^n, z^n) \) \( \{j\} \), and (2) removing \( (v_i, v_j) \), from \( D \) does not decrease \( \mathcal{N}_r'(D') \) for all \( v \in [1:n] \) \( \{v_j\} \). Thus, there are at least \( \frac{2^{(\delta + 1)t}}{2^t} \) tuples in each subgroup, any two of which with the same value at \( v_j \) are still confusable. It follows that \( \chi_f (\Gamma_i(D')) \geq \omega (\Gamma_i(D_{k'}) \geq \omega (\Gamma_i(D_{k'})) \geq 2^\delta t \).

Claim 2 - \( \chi_f (\Gamma_i(D')) \leq \sum_{j=0}^{2^\delta} 2^{jt} \). Given the fact that \( \chi_f (\Gamma_i(D')) \leq \chi_L (\Gamma_i(D')) \) for undirected graphs \( [40, \text{Theorem 3}] \), we need to show that there exists a proper coloring for \( \Gamma_i(D') \) such that there are at most \( \sum_{j=0}^{2^{\delta - 1}} 2^t \) colors in the closed neighborhood of any vertex, such that any two adjacent vertices in such a closed neighborhood receive distinct colors.

First, we show that any vertex is locally \( \left( \sum_{j=0}^{2^{\delta - 1}} 2^t \right) \)-colorable, that is, the closed neighborhood of any vertex can be properly assigned with at most \( \sum_{j=0}^{2^{\delta - 1}} 2^t \) colors. Before proceeding further, we introduce a general coloring function that will be repeatedly used later. Given an integer \( k \), we define a proper coloring function

\[ c : (\mathbb{F}_{2^n})^k \mapsto (\mathbb{F}_{2^n})^k, \]

(17)

which assigns a color associated with a \( k \)-tuple vector to a vertex associated with an \( n \)-tuple message in confusion graphs. Specifically, let \( \{a_i, i = \{1, \ldots, n\} \} \) be a set of distinct values in the finite field \( \mathbb{F}_{2^n} \). Let us treat each message \( x_i \) as an element of the finite filed \( \mathbb{F}_{2^n} \) such that the products with field elements \( a_i \), and the corresponding linear combinations are well-defined. For a tuple \( x^n = (x_1, \ldots, x_n) \), the coloring function is given by

\[ c(x^n) \triangleq \left( \sum_{i=1}^{n} a_i x_i, \ldots, \sum_{i=1}^{n} a_i x_i \right) \]

(18)

which maps a message \( x^n \) to a \( k \times 1 \) vector \( (c_1, \ldots, c_k) \). Taking a closer look at this coloring function, we can find that it is a syndrome computation of \((n, k, n - k + 1)\) Reed-Solomon (RS) codes, which can detect up to \( n - k \) errors. In other words, the syndromes of two distinct messages \( x^n \) and \( z^n \) with \( d_H(x^n, z^n) \leq n - k \) should not be all the same, i.e., there exist some \( j \)'s such that \( \sum_{i=1}^{n} a_i x_i \neq \sum_{i=1}^{n} a_i z_i \). Thus, it follows that any two adjacent vertices with \( d_H(x^n, z^n) \leq n - k \) receive distinct colors, i.e., \( c(x^n) \neq c(z^n) \) for all \( d_H(x^n, z^n) \leq n - k \). For each \( k \), there are \( 2^{kt} \) such RS codewords, which correspond to \( 2^{kt} \) distinct colors.

Because any two message tuples with Hamming distance no less than \( \delta + 1 \) are non-confusable (see Observation 1) and thus non-adjacent, we only have to consider the adjacent vertices whose corresponding messages have distance no more than \( \delta + 1 \). Given any message \( x^n \), without loss of generality assuming the all-zero message tuple, its confusable messages (i.e., adjacent vertices) lie in the following categories:

\[ \{x_{v_j} = z_{v_j}, \forall j \in [1:n] \} \]

where \( x_{v_j} \) and \( z_{v_j} \) are the corresponding messages at \( v_j \), and \( j \notin M \). Clearly, \( x^n \) and \( z^n \) are confusable at \( v_j \), as \( d_H(x^n, z^n) = \delta + 1 \).
Category A: This category consists of all messages $z^n$ with $d_H(x^n, z^n) \leq \delta$. According to Hamming distance between $x^n$ and $z^n$, we divide the messages in Category A into $\delta$ sub-categories, and assign colors for each sub-category. For the sub-category $j$ ($1 \leq j \leq \delta$), we consider the messages $z^n$ where $d_H(x^n, z^n) = j$ and assign such messages with a color $c(z^n)$ given by the syndrome of the $(n,k,n-k+1)$ RS codewords, where $k$ is specified by

$$k = \begin{cases} 
  j, & \text{if } 1 \leq j \leq n - \delta - 1 \\
  n - \delta - 1, & \text{if } n - \delta - 1 < j \leq \delta
\end{cases} \tag{19}$$

where $k \leq j$ for both cases. Thus, the number of codewords for the $(n,k,n-k+1)$ RS code is no more than $2^{1/2}$, corresponding to no more than $2^{1/2}$ distinct colors.

For any two adjacent messages $z^n$ and $z'^n$ in such a sub-category, we have, by Observation 1,

$$d_H(z^n, z'^n) \leq \delta + 1 \leq n - k \tag{20}$$

for both $1 \leq j \leq n - \delta - 1$ and $n - \delta - 1 < j \leq \delta$. As such, any two messages in this sub-category have Hamming distance no more than the error detection capability $n - k$ of the $(n,k,n-k+1)$ RS code. As detailed before, any two adjacent vertices $x^n$ and $z^n$ correspond exactly to the vertices of partial clique, because otherwise $x^n$ and $z^n$ are non-confusable. The indices of the length-$l$ directed cycle belong to a subset of these positions, i.e.,

$$\{w_1, \ldots, w_t\} \subseteq Q(x^n, z^n), \tag{22}$$

for sure. For these vertices, we will color them with the same colors used for Category A messages. We divide messages in Category B into subgraphs, each of which contains messages with different value only at $\{w_1, \ldots, w_t\}$. According to Observation 3, each subgraph of messages forms an independent set, and can be assigned with the same color, because the side information sub-digraph induced by $\{w_1, \ldots, w_t\}$ is a directed cycle and in turn any two messages are non-confusable. For each subgraph, there exists a message say $\tilde{z}^n$, where $Q(\tilde{z}^n, z^n) = \{w_1, \ldots, w_t\}$ for some $\tilde{z}^n$’s in this subgraph, such that there exists at least one position $w_j$ in $\{w_1, \ldots, w_t\}$ with zero value, i.e., $x_{w_j} = \tilde{z}_{w_j}$. Together with the fact that $d_H(x^n, \tilde{z}^n) = \delta + 1$, it follows that $\tilde{z}^n$ is non-confusable to all messages in such a subgraph, and belongs to Category A.

because $d_H(x^n, \tilde{z}^n) \leq \delta$. In short, each subgroup of messages in Category B can be colored with the same color assigned to a message $\tilde{z}^n$ in Category A.

Second, given the fact that $\Gamma_i(D')$ is locally $r$-colorable with $r = \left(\sum_{j=0}^{\delta-1} 2^j\right)$, there exists a homomorphism of $\Gamma_i(D')$ to the universal graph $U(m,r)$ defined in [40, Definition 6] with a sufficiently large $m$. For instance, $U(m,r)$ can be constructed as follows associated to $\Gamma_i(D')$. Each vertex of $U(m,r)$ represents a pair $(x^n, A)$ where $A \subseteq \mathcal{N}(x^n)$ is the set of vertices with distinct colors in the neighborhood of $x^n$. So, $|A| = r - 1$. Two vertices $(x^n, A)$ and $(y^n, B)$ in $U(m,r)$ are connected with an edge if $x^n \in B$ and $y^n \in A$. This happens if $x^n$ and $y^n$ are confusable in $\Gamma_i(D')$. By [40, Lemma 1], we conclude that, there exists a proper coloring for $\Gamma_i(D')$ such that the number of colors in the closed neighborhood of any vertex is at most $\sum_{j=0}^{\delta} 2^j$, and thus $\chi_f(\Gamma_i(D')) \leq \chi_L(\Gamma_i(D')) \leq \sum_{j=0}^{\delta} 2^j$. Together with $\chi_f(\Gamma_i(D')) \geq \omega(\Gamma_i(D')) \geq 2^t$, we have $2^t \leq \chi_f(\Gamma_i(D')) \leq \sum_{j=0}^{\delta} 2^j < 2^{(\delta+1)t}$, where the upper bound is strict.

In conclusion, if every dominant partial clique $\mathcal{D}[\mathcal{M}]$ contains a directed cycle, then there exists a coding scheme such that the broadcast rate satisfies

$$\beta \leq \frac{\log \chi_f(\Gamma_i(D'))}{t} < \delta + 1, \tag{23}$$

which is strictly less than that achieved by partition multicast. This completes the necessity proof.

V. CONCLUDING REMARK

For the multiple-unicast index coding problems, we have characterized the sufficient and necessary condition for which partition multicast, a special case of partial clique covering with cover number 1, achieves the optimal broadcast rate. It has been proven that partial acyclicity (i.e., the absence of directed cycles in the dominant partial clique) is the fundamental topological structure that determines the optimality of partition multicast. Such a result indicates that the benefits of sophisticated achievable schemes (such as local coloring, random coding, subspace interference alignment) should focus on the exploitation of directed cycles in the dominant partial cliques rather than those in the whole side information digraphs. On the other hand, it is also shown that maximum acyclic induced sub-digraph bound is tight for the optimality of partition multicast, in addition to the tightness for clique covering shown in [28].

We have further extended to the general partial clique covering, providing a sufficient condition for which partial clique covering is optimal, and a sufficient condition for which it is strictly suboptimal. Nevertheless, the complete characterization of the fundamental structural property of the optimality of partial clique covering is not fully understood. The sufficient and necessity condition in the most explicit form is still unknown due to the difficulty of the characterization of optimal partial clique partition, which is an NP-hard combinatorial optimization problem.

Furthermore, by the connection of the lower bound of broadcast rate and acyclic set coloring of digraphs, we have shown that, for a family of index coding instances, (partial)
 clique covering approximates the broadcast rate within a multiplicative gap from the optimal. For the future research directions, the (approximate) optimality of other achievability schemes at hand, e.g., local coloring on conflict digraphs, is also worthwhile to evaluate, which will help to reveal the fundamental limitation of one-to-one interference alignment. This will also narrow down the cases that may require advanced coding schemes, e.g., subspace alignment and nonlinear coding schemes. Besides, low-complexity algorithmic solutions for large-scale networks are also a promising research revenue for the future exploration.

APPENDIX

In what follows, some background and definitions pertaining to graph theory [32], [33], [39] are briefly recalled.

Unless otherwise stated, we use respectively the graphs \( G = (V, E) \) and the digraphs \( D = (V, A) \) to denote undirected and directed graphs, where \( E \) and \( A \) are the sets of undirected edges and directed arcs, respectively. The complement of a graph \( G = (V, E) \), denoted by \( \bar{G} = (V, \bar{E}) \), has the same vertex set \( V \) and the edge \( (u, v) \in \bar{E} \) if and only if \( (u, v) \notin E \). For the undirected graph, \( (u, v) = (v, u) \). The complement of a digraph \( D = (V, A) \), denoted by \( \bar{D} = (V, \bar{A}) \), has the same vertex set \( V \) and \( (u, v) \in \bar{A} \) if and only if \( (u, v) \notin A \). The underlying undirected graph \( \Gamma \) of \( D \) is created in such a way that any two vertices are joint with an edge in \( G \) if and only if there exists at least one arc between them in \( D \). An arc either \( (u, v) \in A \) or \( (v, u) \in \bar{A} \) but not both is referred to as a uni-directed arc, otherwise it is bi-directed. A sub-digraph of \( D \) induced by vertex set \( M \), denoted as induced sub-digraph \( D[M] \), is such that, \( \forall u, v \in M \), an arc \( (u, v) \in A(D[M]) \) if and only if \( (u, v) \in A(D) \). A digraph \( D_1 \) contains another digraph \( D_2 \) as a sub-digraph if \( V(D_2) \subseteq V(D_1) \) and \( A(D_2) \subseteq A(D_1) \). These two notions also apply to undirected graphs.

The in-neighbors and out-neighbors of a vertex \( v \) in a digraph \( D \) are respectively the vertices with an incoming arc [i.e., \( N^-(D, v) = \{ u \mid (u, v) \in A(D) \} \)] and an outgoing arc [i.e., \( N^+(D, v) = \{ u \mid (v, u) \in A(D) \} \)]. The closed in-neighborhood and out-neighborhood are respectively defined as \( N^-(D, v) = \{ v \} \cup N^-(D, v) \) and \( N^+(D, v) = \{ v \} \cup N^+(D, v) \). The in-degree and out-degree of a vertex \( v \) in \( D \) are denoted respectively by \( d^-(D, v) \triangleq |N^-(D, v)| \) and \( d^+(D, v) \triangleq |N^+(D, v)| \). The maximum in-degree \( \Delta^-(D) \) and minimum in-degree \( \delta^-(D) \) of a digraph \( D \) are respectively the maximum and minimum numbers of incoming arcs for one vertex. The maximum degree of a graph \( G \), denoted by \( \Delta(G) \), is the maximum number of edges associated with one vertex.

A length-\( n \) directed path \( v_0 \rightarrow v_n \) of a digraph \( D \) is a set of arcs \( \{ (v_0, v_1), \ldots, (v_{n-1}, v_n) \} \). A digraph \( D = (V, A) \) is strongly-connected (or strong) if for every two distinct vertices \( v_i, v_j \in V \), there exist directed paths \( v_i \rightarrow v_j \) and \( v_j \rightarrow v_i \) in \( D \). The directed cycles and cliques are strongly-connected. A strong component of \( D \) is a maximal induced sub-digraph which is strongly-connected. A partition \( V = \{ V_1, \ldots, V_k \} \) with \( V = \bigcup_{i=1}^k V_i \) and \( V_i \cap V_j = \emptyset \) for all \( i \neq j \) is called a strong decomposition, if every sub-digraph \( D[V_i] \) is a strong component. A strong component digraph of \( D \), denoted by \( S(D) \), is obtained by contracting strong components and deleting any parallel arcs obtained in this process. The strong component digraphs \( S(D) \) are acyclic, because otherwise a cycle containing several strongly-connected components would merge them all to a single strongly-connected component [32].

The directed cut of a digraph is a set of arcs of the form \( (\mathcal{X}, \mathcal{Y}) \), where \( \mathcal{X} \) is a non-empty proper subset of \( V \) such that there no arcs from \( \mathcal{Y} \setminus \mathcal{X} \) to \( \mathcal{X} \) [32]. The arcs between two different strongly-connected components belong to a directed cut. The strong connected components can be found in linear time with Depth-First-Search algorithms [41].

The directed cycle \( C_n \) refers to the induced sub-digraph with only the arcs \( \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\} \). The digraph of \( D \) is the number of vertices/arc of a shortest directed cycle in \( D \). A digraph is acyclic if it does not contain directed cycles. Such a digraph is also called an acyclic (arc) set. An oriented graph is a digraph with no bi-directed arcs, i.e., with digirth at least 3. A tournament is an oriented graph, whose underlying undirected graph is a complete graph that any two vertices are connected with an edge. A planar (di)graph is such that all edges/arc can be drawn at 2D plane without any crossing edges/arc.

For an undirected graph \( G \), the chromatic number \( \chi(G) \) is the minimum of colors assigned to the vertices such that any two adjacent vertices are with distinct colors, and the clique number \( \omega(G) \) is the maximum number of vertices, any two of which are adjacent. Clearly, \( \chi(G) \geq \omega(G) \). The fractional chromatic number \( \chi_f(G) \) is the linear relaxation of the parameters from \([0, 1] \) to \([0, 1] \) in the linear program formulation of \( \chi(G) \) [42]. Compared with (non-fractional) coloring where each vertex is assigned with only one color, fractional coloring allow each vertex to receive multiple colors. The local chromatic number [40] of the graphs \( G \) (resp. digraphs \( D \)), \( \chi_L(G) \) (resp. \( \chi_L(D) \)) is the maximum number of distinct colors in the closed neighborhood (resp. in-neighborhood) of any vertex, minimized over all proper color assignments on \( G \) (resp. the underlying undirected graph of \( D \)). The independent set number \( \alpha(G) \) is the size of the maximum independent set, in which any two elements have no edge between them in \( G \). For a digraph \( D \), the dichromatic number [39] \( \chi_A(D) \), is the minimum number of colors required to color the vertices of \( D \) in such a way that every set of vertices with the same color induces an acyclic sub-digraph in \( D \). The acyclic set number \( \alpha_A(D) \) is the size of the maximum directed acyclic set in \( D \). Dichromatic number \( \chi_A \) [acyclic set number \( \alpha_A \)] generalizes the notion of chromatic number \( \chi \) [independent set number \( \alpha \)] from graphs to digraphs. The subgraph induced by the vertices with the same color in graphs forms an independent set, while it forms an acyclic set in digraphs. The following lemma gives the relation between chromatic number \( \chi \) (dichromatic number \( \chi_A \)) and independent set number \( \alpha \) (acyclic set number \( \alpha_A \)) in graphs (digraphs).

**Lemma 7** ([33]). For any graph \( G = (V, E) \) and digraph \( D = (V, A) \)

\[
\alpha(G) \geq \frac{|V|}{\chi(G)} \quad \text{and} \quad \alpha_A(D) \geq \frac{|V|}{\chi_A(D)}.
\]

The lexicographic product of two undirected graphs \( \Gamma_1 \) and \( \Gamma_2 \), denoted by \( \Gamma_1 \cdot \Gamma_2 \), is such that \( \mathcal{V}(\Gamma_1 \cdot \Gamma_2) = \mathcal{V}(\Gamma_1) \times \mathcal{V}(\Gamma_2) \).
and there is an edge from \((u_1, u_2)\) to \((v_1, v_2)\) if and only if either there is an edge \((u_1, v_1)\) or there is an edge \((u_2, v_2)\) with \(u_1 = v_1\).

**Lemma 8** ([42]). For the lexicographic product of two graphs, 
\[
\chi_f(G_1 \cdot G_2) = \chi_f(G_1) \chi_f(G_2).
\]

A graph homomorphism from a graph \(G = (V, E)\) to \(G' = (V', E')\) is a mapping from the vertex set \(V\) to \(V'\) such that \((u, v) \in E\) implies \((u', v') \in E'\).

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**REFERENCES**

[1] A. Mazumdar, “Storage capacity of repairable networks,” IEEE Trans. Inf. Theory, vol. 61, no. 11, pp. 5810–5821, Nov. 2015.

[2] K. Shnamugam and A. G. Dimakis, “Bounding multiple unicasts through index coding and locally repairable codes,” in Proc. IEEE ISIT’14, 2014.

[3] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2856–2867, May 2014.

[4] S. El Rouayheb, A. Sprintson, and C. Georgiades, “On the index coding problem and its relation to network coding and matroid theory,” IEEE Trans. Inf. Theory, vol. 56, no. 7, pp. 3187–3195, Jul. 2010.

[5] M. Effros, S. El Rouayheb, and M. Langberg, “An equivalence between network coding and index coding,” IEEE Trans. Info. Theory, vol. 61, no. 5, pp. 2478–2487, May 2015.

[6] S. A. Jafar, “Topological interference management through index coding,” IEEE Trans. Inf. Theory, vol. 60, no. 1, pp. 529–568, Jan. 2014.

[7] S. Riri, “Information flows, graphs and their guessing numbers,” The Electronic Journal of Combinatorics, vol. 14, no. 1, p. R44, 2007.

[8] Z. Bar-Yossef, Y. Birk, T. S. Jayram, and T. Kol, “Index coding with side information,” IEEE Trans. Inf. Theory, vol. 57, no. 3, pp. 1479–1494, March 2011.

[9] N. Alon, E. Lubetzky, U. Stav, A. Weinstein, and A. Hassidim, “Broadcasting with side information,” in IEEE FOCS’08, 2008.

[10] M. Neely, A. Tehrani, and Z. Zhang, “Dynamic index coding for wireless broadcast networks,” IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7525–7540, Nov. 2013.

[11] K. Shnamugam, A. G. Dimakis, and M. Langberg, “Local graph coloring and index coding,” in Proc. IEEE ISIT’13, 2013.

[12] F. Arbabjolfaei and Y.-H. Kim, “Structural properties of index coding capacity using fractional graph theory,” in Proc. IEEE ISIT’15, 2015.

[13] L. Ong, C. H. Ho, and F. Lim, “The single-unipair index-coding problem: The single-sender case and the multi-sender extension,” IEEE Trans. Inf. Theory, vol. 62, no. 6, pp. 3165–3182, June 2016.

[14] F. Arbabjolfaei and Y.-H. Kim, “Structural properties of index coding capacity,” arXiv:1608.03689, Aug. 2016.

[15] H. Maleki, V. Cadambe, and S. Jafar, “Index coding: An interference alignment perspective,” IEEE Trans. Inf. Theory, vol. 60, no. 9, pp. 5402–5432, Sept 2014.

[16] A. Blasiak, R. Kleinberg, and E. Lubetzky, “Index coding via linear programming,” arXiv:1004.1379, 2010.

[17] H. Yu and M. J. Neely, “Duality codes and the integrality gap bound for index coding,” IEEE Trans. Inf. Theory, vol. 60, no. 11, pp. 7256–7268, Nov. 2014.

[18] H. Esfahanizadeh, F. Lahouti, and B. Hassibi, “A matrix completion approach to linear index coding problem,” in Proc. IEEE ITW’14, 2014.

[19] X. Huang and S. El Rouayheb, “Index coding and network coding via rank minimization,” in Proc. IEEE ITW’15, 2015.

[20] F. Arbabjolfaei, B. Bandemer, Y.-H. Kim, E. Sasoglu, and L. Wang, “On the capacity region for index coding,” in Proc. IEEE ISIT’13, July 2013.

[21] S. Unal and A. B. Wagner, “A rate-distortion approach to index coding,” in Proc. IEEE ITA’14, 2014.

[22] E. Lubetzky and U. Stav, “Nonlinear index coding outperforming the linear optimum,” IEEE Trans. Inf. Theory, vol. 55, no. 8, pp. 3544–3551, Aug. 2009.

[23] H. Sun and S. A. Jafar, “Index coding capacity: How far can one go with only Shannon inequalities?” IEEE Trans. Inf. Theory, vol. 61, no. 6, pp. 3041–3055, Jun. 2015.