High orders of Weyl series for the heat content

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This article concerns the Weyl series of spectral functions associated with the Dirichlet Laplacian in a $d$-dimensional domain with a smooth boundary. In the case of the heat kernel, Berry and Howls predicted the asymptotic form of the Weyl series characterized by a set of parameters. Here, we concentrate on another spectral function, the (normalized) heat content. We show on several exactly solvable examples that, for even $d$, the same asymptotic formula is valid with different values of the parameters. The considered domains are $d$-dimensional balls and two limiting cases of the elliptic domain with eccentricity $\varepsilon$: a slightly deformed disk ($\varepsilon \to 0$) and an extremely prolonged ellipse ($\varepsilon \to 1$). These cases include two-dimensional domains with circular symmetry and those with only one shortest periodic orbit for the classical billiard. We also analyse the heat content for the balls in odd dimensions $d$ for which the asymptotic form of the Weyl series changes significantly.

Keywords: heat kernel; heat content; Weyl series

1. Introduction

We consider two types of spectral functions—the heat kernel and the (normalized) heat content—also known as the survival probability in the physical literature. They are associated with the spectrum of the Laplacian in a bounded domain. Let $\Omega$ be the domain in the $d$-dimensional flat space, with a smooth boundary $\partial \Omega$. The spectrum of the Laplacian, say with the Dirichlet boundary condition (BC), is given by

$$-
\Delta \phi(r) = \lambda \phi(r) \quad r \in \Omega,$$

$$\phi(r) = 0 \quad r \in \partial \Omega. \quad (1.1)$$

The eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots$ form a discrete set (Courant & Hilbert 1953). The associated eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots, \phi_j, \ldots$ form an orthonormalized basis of real functions, $\int_{\Omega} dr \phi_i(r) \phi_j(r) = \delta_{ij}$.

It is instructive to formulate the spectral problem of the Dirichlet Laplacian in the context of the diffusion (probability) theory. The conditional probability

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\(\rho(\mathbf{r}, t|\mathbf{r}_0, 0)\) of finding a particle at a point \(\mathbf{r} \in \Omega\) at time \(t > 0\), if it started from \(\mathbf{r}_0 \in \Omega\) at \(t_0 = 0\), is governed by the diffusion (heat) equation

\[
\frac{\partial \rho(\mathbf{r}, t|\mathbf{r}_0, 0)}{\partial t} = \Delta \rho(\mathbf{r}, t|\mathbf{r}_0, 0).
\]

(1.2)

This equation has to be supplemented by the initial condition \(\rho(\mathbf{r}, t = 0|\mathbf{r}_0, 0) = \delta(\mathbf{r} - \mathbf{r}_0)\) and by the BC \(\rho(\mathbf{r}, t|\mathbf{r}_0, 0) = 0\) for \(\mathbf{r} \in \partial \Omega\), which reflects the absorption of the particle hitting the boundary. The conditional probability can be expressed in terms of the eigenvalues and eigenfunctions of the Dirichlet Laplacian as follows:

\[
\rho(\mathbf{r}, t|\mathbf{r}_0, 0) = \sum_j \phi_j(\mathbf{r}_0) \phi_j(\mathbf{r}) e^{-\lambda_j t}.
\]

(1.3)

Various quantities can be constructed from the conditional probability.

Historically, the most studied quantity was the heat kernel, defined as the trace

\[
K(t) = \int_{\Omega} \int_{\Omega} \rho(\mathbf{r}, t|\mathbf{r}_0, 0) \delta(\mathbf{r} - \mathbf{r}_0) = \int_{\Omega} \rho(\mathbf{r}, t|\mathbf{r}_0, 0) = \sum_j e^{-\lambda_j t}.
\]

(1.4)

Another commonly studied quantity is the heat content, in the chemical physics also called the survival probability. The ‘local’ heat content is defined as:

\[
H(t; \mathbf{r}_0) = \int_{\Omega} \mathbf{d}\rho(\mathbf{r}, t|\mathbf{r}_0, 0).
\]

(1.5)

It represents the probability that the particle, localized at a point \(\mathbf{r}_0 \in \Omega\) at \(t_0 = 0\), remains still diffusing in the domain \(\Omega\) at time \(t > 0\), unabsorbed by the boundary. The BC is again the Dirichlet one: \(H(t, \mathbf{r}_0) = 0\) for \(\mathbf{r}_0 \in \partial \Omega\). The normalized heat content is defined as the average of the local one over the whole domain,

\[
H(t) = \frac{1}{|\Omega|} \int_{\Omega} \mathbf{d}\rho(\mathbf{r}, t|\mathbf{r}_0, 0) = \sum_j \gamma_j^2 e^{-\lambda_j t} \quad \text{and} \quad \gamma_j = \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} \mathbf{d}\phi_j(\mathbf{r}).
\]

(1.6)

It represents the probability of finding the particle in \(\Omega\) at time \(t > 0\), if it was distributed uniformly with the density \(1/|\Omega|\) over the whole domain at \(t_0 = 0\). The normalization by \(1/|\Omega|\) allows us to study infinite domains also. In the mathematical literature, \(H(t)\) is interpreted as the amount of heat at time \(t\) inside the domain \(\Omega\) with the boundary \(\partial \Omega\) held at zero temperature, provided that at initial time \(0\), the heat was distributed uniformly over \(\Omega\).

The heat kernel and the heat content have many similar properties. The most important results concern the small-\(t\) expansion of both quantities. The first three terms of these series are proportional to the volume \(|\Omega|\), surface \(|\partial \Omega|\) and the integrated mean curvature, respectively. The heat kernel \(K(t)\) was intensively studied by, e.g. Weyl (1911), Pleijel (1954), Kac (1966) and Stewartson & Waechtner (1971). The heat content was introduced by Birkhoff & Kotik (1954) who analysed its one-dimensional version. The first few coefficients of the small-\(t\) expansion of \(H(t)\) for Riemannian domains with both Dirichlet and Neumann BC were derived by van den Berg et al. (1993), van den Berg & Gilkey (1994) and DesJardins (1998). An important progress has been made by Savo (1998a,b) who derived a recurrence scheme for the coefficients of the small-\(t\) series.
As concerns the heat kernel, it proves useful to apply the Laplace transform with respect to \( t \). For dimensions \( d > 1 \), the heat kernel has to be regularized by subtracting the leading terms of the \( t \)-expansion that would diverge under the Laplace integral. Thus, we get the regularized resolvent \( \tilde{K}(s) \) (e.g. Stewartson & Waechter 1971; Berry & Howls 1994). For example, in the two-dimensional case, we have

\[
\tilde{K}(s) = \int_0^\infty dt \, e^{-s^2t} \left( K(t) - \frac{|Q|}{4\pi t} \right),
\]

(1.7)

The small-\( t \) expansion of \( K(t) \) corresponds to the large-\( s \) expansion of \( \tilde{K}(s) \):

\[
\tilde{K}(s) \sim \sum_{n=1}^{\infty} \frac{c_n^{(K)}}{s^n}.
\]

(1.8)

This is the so-called Weyl series.

It turns out that not only the small-\( n \) coefficients \( c_n^{(K)} \) contain the information about the domain characteristics. Based on a general Borel-transformed theory (Balian & Bloch 1972; Voros 1983), Berry & Howls (1994) developed a formalism for calculating the large-\( n \) coefficients of the Weyl series in two-dimensional domains. They conjectured that

\[
c_n^{(K)} \sim \alpha \frac{\Gamma(n - \beta + 1)}{l^n}, \quad n \to \infty,
\]

(1.9)

where \( \alpha \) and \( \beta \) are some parameters, \( l \) is the shortest periodic (stable) geodesics for a classical billiard in \( Q \). Note that the factorial nature of the coefficients makes the Weyl series divergent. In the case of the disk domain of radius \( R \), for which the shortest periodic orbits with length \( l = 4R \) form a continuous family, the authors predicted and tested numerically the parameter values, \( \alpha = 1/(8\sqrt{2\pi}) \) and \( \beta = 3/2 \). In the case of domains with only one (isolated) shortest periodic orbit, like the ellipse, the authors conjectured the increase of the parameter \( \beta \) by \( 1/2 \), i.e. \( \beta = 2 \).

Later Howls & Trasler (1999) extended these results to higher dimensions. They derived exact \( \alpha \), \( \beta \) and the generalized interpretation of \( l \) in the case of \( d \)-dimensional balls of radius \( R \). They had to distinguish between dimensions \( d \) according to their parity. For odd \( d \), the contribution of the shortest periodic orbit vanishes and the next shortest one, i.e. that with three bounces, becomes leading. Thus, \( l \) becomes the perimeter of an equilateral triangle:

\[
l = \begin{cases} 
4R & d \text{ even}, \\
3\sqrt{3}R & d \text{ odd}.
\end{cases}
\]

(1.10)

The parameter \( \beta \) depends on \( d \) as follows:

\[
\beta = \begin{cases} 
(5 - d)/2 & d \text{ even}, \\
7/2 - d & d \text{ odd}.
\end{cases}
\]

(1.11)
Explicit formulas for $\alpha(d)$ were also given by Howls & Trasler (1998, 1999). Further, they proposed a generalization of the asymptotic formula (1.9), which involves all periodic orbits $l_j$ on the domain $\Omega$:

$$c_n^{(K)} \sim \sum_j \sum_{k=0}^{\infty} \alpha_{kj} \frac{\Gamma(n - \beta_j + 1 - k)}{l_j^n}, \quad n \to \infty. \quad (1.12)$$

Howls (2001) studied quantum balls threaded by a single magnetic flux at their centre. While in two dimensions, the parameters $l$ and $\beta$ are insensitive to the periodic orbits arising from the diffractive flux line; for a spherical domain, these parameters are modified by diffractive orbits, in particular $l = 2R$ and $\beta = 3$.

No regularization is needed in the case of the Laplace transform of the heat content (van den Berg & Gilkey 1994). We can introduce the Laplace transforms for both the local heat content

$$\tilde{H}(s; r_0) = \int_0^{\infty} dt e^{-s^2 t} H(t, r_0), \quad (1.13)$$

and the heat content

$$\tilde{H}(s) = \int_0^{\infty} dt e^{-s^2 t} H(t) = \sum_{j=1}^{\infty} \frac{\gamma_j^2}{s^2 + \lambda_j}. \quad (1.14)$$

These Laplace transforms are related by

$$\tilde{H}(s) = \frac{1}{|\Omega|} \int_\Omega dr_0 \tilde{H}(s; r_0). \quad (1.15)$$

The main goal of this paper is the analysis of the counterpart of the Weyl series (1.8) for the heat content. The Laplace transform of the heat content has the large-$s$ expansion of the form

$$\tilde{H}(s) \sim \sum_{n=2}^{\infty} \frac{c_n}{s^n}. \quad (1.16)$$

We obtain the coefficients $\{c_n\}$ for few exactly solvable domains. For even dimensions $d$, the coefficients $\{c_n\}$ fulfil at asymptotically large $n$ an analogy of the equation (1.9),

$$c_n \sim \alpha \frac{\Gamma(n - \beta + 1)}{l_n^{-2}}, \quad n \to \infty, \quad (1.17)$$

where the power $n - 2$ in the denominator ensures that $\alpha$ remains dimensionless.

The parameters $\alpha$, $\beta$ and $l$ differ from those for the heat kernel. Our task is to determine these parameters for the studied domains and to point out their possible step-wise modifications under a symmetry change of the domain. A typical example of the symmetry change is a transition between a disc, possessing an infinite number of the shortest periodic orbits for the billiard, and an ellipse, with only one shortest periodic orbit. For odd dimensions $d$, the Gamma function in the asymptotic equation (1.17) has to be replaced by a bounded oscillating function (at least for the studied $d$-balls).
In connection with the Weyl series (1.16), for the heat content, we mention the paper of van den Berg (2004) who showed that the relation between the coefficients $c_n$ and the shortest periodic orbit does not hold in general. In particular, two domains with different shortest orbits can have the same $c_n$ coefficients; the shorter of the two periodic orbits is determined by the difference of the two (exact) heat contents. This ambiguity of the Weyl series for the heat content inspires us to investigate the last for simple domains like $d$-dimensional balls and the ellipse. To our surprise, while a general theory seems to be more complicated for the heat content when compared with the heat kernel, the asymptotic Weyl series are explicitly available from the exact results and expansions for our simple domains. Similarities and differences between Weyl series for the heat content and the heat kernel are pointed out.

The paper is organized as follows. In §2, we derive a differential equation for $\tilde{H}(s; r)$. In §3, we write its exact solution and $\tilde{H}(s)$ for $d$-balls; even dimensions are analysed in §3a and odd dimensions in §3b. Section 4 is devoted to the two-dimensional ellipse domain of eccentricity $\epsilon$ and the small-$s$ expansion of $\tilde{H}(s)$. In §5, for an arbitrary value of $s$, we solve exactly the leading terms for two special cases: a slightly deformed disc ($\epsilon \to 0$) (§5a, b) and an extremely prolonged ellipse ($\epsilon \to 1$) (§5c). We extract the large-$n$ coefficients of the Weyl series from all exact solutions. Section 6 concludes the article.

2. Differential equation

The Laplace transform of the local heat content (1.13) satisfies a differential equation, which is derived by the following procedure. Applying the conjugated Laplacian $\Delta^+$ (acting upon the coordinates of $r_0$) to the representation (1.13) and using the conjugate of the diffusion equation (1.2), we obtain

$$\Delta^+ \tilde{H}(s; r_0) = \int_0^\infty dt e^{-s^2t} \int_{\Omega} d\mathbf{r} \Delta^+ \rho(\mathbf{r}, t|\mathbf{r}_0, 0) = \int_{\Omega} d\mathbf{r} \int_0^\infty dt e^{-s^2t} \partial_t \rho(\mathbf{r}, t|\mathbf{r}_0, 0).$$

Integration by parts in $t$ then implies the desired equation

$$\Delta \tilde{H}(s; \mathbf{r}) - s^2 \tilde{H}(s; \mathbf{r}) = -1,$$  \hspace{1cm} (2.2)

where we abandon the subscript 0 and replace $\Delta^+$ by $\Delta$. This differential equation is supplemented by the Dirichlet BC $\tilde{H}(s; \mathbf{r}) = 0$ for $\mathbf{r} \in \partial \Omega$. A similar differential equation was derived by van den Berg & Gilkey (1994).

3. $d$-Balls

For $d$-balls, the Laplacian in equation (2.2) can be expressed in terms of $d$-dimensional spherical coordinates. As $\tilde{H}^{(d)}(s; \mathbf{r})$ does not depend on angle coordinates, we end up with the ordinary differential equation in $r = |\mathbf{r}|$ (see also van den Berg & Gilkey (1994)). The solution reads

$$\tilde{H}^{(d)}(s; \mathbf{r}) = \frac{1}{s^2} \left[ 1 - \frac{r^{1-d/2} I_{d/2-1}(sr)}{R^{1-d/2} I_{d/2-1}(sR)} \right],$$  \hspace{1cm} (3.1)
where \( I_p(x) \) are modified Bessel functions. Integrating over the domain according to equation (1.15), we get

\[
\tilde{H}^{(d)}(s) = \frac{1}{s^2} \left[ 1 - \frac{dI_{d/2}(u)}{uI_{d/2-1}(u)} \right] = \frac{1}{s^2} I_{d/2+1}(u),
\]

(3.2)

where we introduced the scaled variable \( u = Rs \). This result can also be found in the paper of van den Berg & Gilkey (1994), although in a slightly more complicated form. Its asymptotic analysis depends on the parity of \( d \).

(a) Asymptotic Weyl series for the balls in even \( d \)

Our aim is to find the asymptotic large-\( n \) form of the coefficients \( c_n \) of the Weyl series (1.16) resulting from equation (3.2). It is useful to rewrite the ratio of Bessel functions with the help of a logarithmic derivative:

\[
\frac{dI_{d/2}(u)}{uI_{d/2-1}(u)} = \frac{d}{2u^2} - \frac{d}{us^2} \frac{d}{du} \ln I_{d/2-1}(u).
\]

(3.3)

From Gradshteyn & Ryzhik (2007), we use the large-\( u \) expansion

\[
I_p(u) \sim \frac{e^u}{\sqrt{2\pi u}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2u)^j} \frac{\Gamma(v + j + 1/2)}{\Gamma(j + 1)\Gamma(v - j + 1/2)},
\]

(3.4)

plus exponentially small terms; for the present \( v = d/2 - 1 \) being an integer, this is an infinite series. After straightforward calculation outlined in appendix A, for asymptotically large \( n \), we get

\[
c_n^{(d)} = (-1)^{d/2-1} \frac{4d}{\pi} \frac{\Gamma(n-3) + \frac{(d-1)(d-3)}{2} \Gamma(n-4) + \cdots}{(2R)^{n-2}}.
\]

(3.5)

Comparing with the asymptotic representation (1.17), we see that the length \( l = 2R \) is now one-half of the shortest periodic orbit \( l^{(K)} = 4R \) appearing in the case of the heat kernel. The symmetry parameter also changes significantly, \( \beta = 4 \) regardless of the (even) dimension. Contrary to the heat kernel, the whole dependence on dimension concentrates in the prefactor \( a^{(d)} = (-1)^{d/2-1}4d/\pi \). The next-to-leading term in equation (3.5) is analogous to the one in the heat-kernel series (1.12).

(b) Asymptotic Weyl series for the balls in odd \( d \)

\( \tilde{H}^{(d)}(s) \) in the formula (3.2) with odd \( d \) involves Bessel functions \( I_p(u) \) with half-integer \( v \). The asymptotic series (3.4) of such functions terminate,

\[
I_p(u) \sim \frac{e^u}{\sqrt{2\pi u}} \sum_{j=0}^{v-1/2} \frac{(-1)^j}{(2u)^j} \frac{\Gamma(v + j + (1/2))}{\Gamma(j + 1)\Gamma(v - j + (1/2))}, \quad u \to \infty.
\]

(3.6)
Weyl series for the heat content

For \( d = 1 \), we have

\[
\tilde{H}^{(1)}(s) = \frac{1}{s^2} \left( 1 - \frac{1}{u} \right) + \mathcal{O}\left( \frac{1}{u^\infty} \right).
\]

Consequently, \( c_n = 0 \) for \( n > 3 \). An analogous result holds for the Weyl series of the heat kernel.

For \( d = 3 \), we also find a finite series

\[
\tilde{H}^{(3)}(s) = \frac{1}{s^2} \left( 1 - \frac{3}{u} + \frac{3}{u^2} \right) + \mathcal{O}\left( \frac{1}{u^\infty} \right),
\]

so \( c_n = 0 \) for \( n > 4 \). This differs from the heat kernel which has an infinite number of non-zero terms in Weyl series for all \( d \geq 2 \) (e.g. Howls \\& Trasler 1999).

In the case of \( d = 5 \), the number of non-zero coefficients is infinite:

\[
\tilde{H}^{(5)}(s) \sim \frac{1}{s^2} \frac{(1 - 6u^{-1} + 15u^{-2} - 15u^{-3})}{1 - u^{-1}} = \frac{1}{s^2} \left( 1 - \frac{5}{u} + \frac{10}{u^2} - 5 \sum_{k=3}^{\infty} \frac{1}{u^k} \right),
\]

so that \( c_n = -5/R^{n-2} \) for \( n \geq 5 \). We see that the asymptotic formula (1.17) no longer holds: there is no \( \Gamma \)-function (thus no \( \beta \)) in the expression and the length \( l = R \).

For odd \( d \geq 5 \), the asymptotic formula for \( c_n \) acquires a new \( n \)-dependent term in the numerator. In contradiction to the function \( \Gamma(n - \beta + 1) \), this term exhibits bounded oscillations. The convergence of the series \( \sum_n c_n/s^n \) will depend on \( R \).

For \( d = 7 \), these oscillations become periodic. Using the asymptotic expansion (3.6) for \( \tilde{H}_{1/2}(u) \) in the denominator of the representation (3.2) and applying the partial fraction decomposition

\[
\frac{1}{1 - 3v + 3v^2} = \frac{1}{3(v_1 - v_2)} \left( \frac{1}{v - v_1} - \frac{1}{v - v_2} \right) = \sum_{k=0}^{\infty} \frac{u^k}{3(v_1 - v_2)} \left( \frac{1}{v_1^{k+1}} - \frac{1}{v_2^{k+1}} \right),
\]

where \( v = 1/u = 1/(Rs) \) and \( v_1 = 3^{-1/2} \exp(i\pi/6) \), \( v_2 = 3^{-1/2} \exp(-i\pi/6) \), we get

\[
c_n = \frac{7}{R^{n-2}} 2^{n-2} \left[ \cos \left( \frac{n\pi}{6} \right) - \sqrt{3} \sin \left( \frac{n\pi}{6} \right) \right], \quad n \geq 5.
\]

The characteristic length is now \( l = R|v_1| = R/\sqrt{3} \). The quasi-periodicity \( c_{n+12} = 3^6 R^{-12} c_n \) is due to the commensurability of the phase \( \pm \pi/6 \) of the complex conjugate roots \( v_{1,2} \) with \( 2\pi \).

For \( d = 9 \), we have the polynomial \( 1 - 6v + 15v^2 - 15v^3 \) in the denominator. It has two complex conjugate roots \( v_1, \ v_2 \) and one real root \( v_3 \), expressible in terms of Cardano formulas. Numerically, \( |v_1| = |v_2| \sim 0.39346201 \), with phases \( \pm 0.2425136494068\ldots \pi \), and \( v_3 \sim 0.4306828846 \). Since the complex phases are no more commensurate with \( 2\pi \), the oscillations of \( c_n \) are not periodic. The reciprocal of the polynomial \( 1 - 6v + 15v^2 - 15v^3 \) has three summands of the type \( v^k/v_1^{k+1} \), \( m = 1, 2, 3 \) as in equation (3.10). The coefficients \( |c_n| \) with large \( n \) are dominated by the complex roots with the lowest absolute value \( |v_1| = |v_2| < v_3 \); thus we get \( l = R|v_1| \). A formal comparison with equation (1.12) leads to the introduction of two lengths, \( l_1 = R|v_1| \) and \( l_2 = Rv_3 \), but we did not find any geometric interpretation of these lengths.

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Figure 1. Complex poles of $\tilde{H}^{(d)}(s)$ in the $v = 1/(Rs)$ plane. Circles, $d = 23$; squares, $d = 29$; diamonds, $d = 33$; triangles, $d = 39$; inverted triangles, $d = 43$; plus symbols, $d = 49$; asterisks, $d = 53$.

For dimensions $d > 9$, we get more poles and more summands of type $v^k/v^{k+1}_m$. The leading term of $|c_n|$ is still given by the pair of conjugate poles with the lowest absolute value $v_1$, $v_2 = v_1^*$ and the relation $l = R|v_1|$ still holds. In figure 1, in the units of $R = 1$, the roots of the denominator are plotted in the complex $v$-plane for larger dimensions $d$. The roots with negative imaginary part, placed symmetrically below the real axis, are not shown. The root $v_1$ with the lowest absolute value is always the leftmost point of the set.

To conclude, the consideration of the large-$s$ asymptotic for balls in odd $d$ leads to $\tilde{H}(s)$, which are the ratios of two polynomials and as such have a non-zero radius of convergence in $s$. On the other hand, the Berry and Howls conjecture (1.9) as well as our conjecture (1.17) have zero radius of convergence in $s$ and no longer hold. It is also questionable whether to expand the ratio of two polynomials into an infinite Weyl series, may be the geometric information about the domain is hidden in the polynomial coefficients themselves.

4. Ellipse, small-$s$ expansion

Now, in two dimensions, we shall pass from the disc to an ellipse which is an example of the domain possessing only one shortest periodic orbit. It will allow us to reveal a discontinuous change of the $\beta$ parameter in equation (1.17).

Let us consider the elliptic domain $\Omega$ centred at the origin, with major and minor semiaxes $a$ and $b$, respectively. In the Cartesian coordinates $(x, y)$, its boundary is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \geq b.$$  

(4.1)
The eccentricity is defined by \( e = \sqrt{1 - (b/a)^2} \in (0, 1) \). The extreme cases \( e = 0 \) and \( e \to 1 \) correspond to the disc and the infinitely prolonged (locally strip-like) ellipse, respectively.

There is little hope to obtain explicitly the heat content for the ellipse domain with general \( e \). Nevertheless, we are able to construct systematically several types of expansions for \( \tilde{H}(s; a, b) \). We derive the small-\( s \) expansion for arbitrary values of \( a, b \) in the present section. Expansions with respect to \( e \), around \( e = 0 \) and 1, for arbitrary \( s \) will be analysed in §5.

To find the small-\( s \) expansion of \( \tilde{H}(s) \) for the elliptic domain, we first formally expand the local \( \tilde{H}(s; r) \) in powers of \( s^2 \),

\[
\tilde{H}(s; r) = \sum_{j=0}^{\infty} \tilde{H}^{(s)}_j(r) s^2.
\]  

Inserting this expansion into the differential equation (2.2) implies an infinite sequence of coupled equations obeyed by the unknown functions \( \{\tilde{H}^{(s)}_j(r)\} \):

\[
\Delta \tilde{H}^{(s)}_0(r) = -1 \quad \text{(4.3a)}
\]

and

\[
\Delta \tilde{H}^{(s)}_j(r) = \tilde{H}^{(s)}_{j-1}(r) \quad \text{for } j \geq 1. \quad \text{(4.3b)}
\]

Each of these functions must satisfy the Dirichlet BC

\[
\tilde{H}^{(s)}_j(r) = 0 \quad \text{for } r \in \partial \Omega, \quad j = 0, 1, \ldots.
\]  

It is convenient to work with complex coordinates \( z = x + iy \) and \( \bar{z} = x - iy \), in which the elliptic boundary (4.1) becomes

\[
\frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) z\bar{z} + \frac{1}{4} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) (z^2 + \bar{z}^2) - 1 = 0.
\]  

The Laplacian in complex coordinates has the form \( \Delta = 4\partial_z \partial_{\bar{z}} \). We solve successively the set of equations (4.3a) and (4.3b) integrating their right-hand side (r.h.s.) in \( z \) and \( \bar{z} \). Adding general solutions of the homogeneous equation \( \Delta f = 0 \),

\[
f(z, \bar{z}) = \sum_{j=0}^{\infty} (a_j z^j + b_j \bar{z}^j), \quad a_j = b_j,
\]  

will permit us to fulfil the Dirichlet BC at the boundary (4.5).

Starting with equation (4.3a), we have

\[
\tilde{H}^{(s)}_0(z, \bar{z}) = -\frac{1}{4} z\bar{z} + c_0^{(0)} + c_0^{(1)} (z^2 + \bar{z}^2).
\]  

The coefficients \( c_0^{(0)} \) and \( c_0^{(1)} \) follow from the condition \( \tilde{H}^{(s)}_0(z, \bar{z}) = 0 \) at the boundary (4.5):

\[
c_0^{(0)} = \frac{(ab)^2}{2(a^2 + b^2)} \quad \text{and} \quad c_0^{(1)} = \frac{a^2 - b^2}{8(a^2 + b^2)}.
\]  

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The result (4.7) with (4.8) is substituted into equation (4.3b) for \( j = 1 \). After integrating and adding the homogeneous solution, we obtain

\[
\tilde{H}_1^{(s)}(z, \bar{z}) = -\frac{1}{64} (z \bar{z})^2 + \frac{1}{4} c_0^{(0)} z \bar{z} + \frac{1}{12} c_0^{(1)} (z^3 \bar{z} + z \bar{z}^3) + c_1^{(0)} (z^2 + \bar{z}^2) + c_1^{(2)} (z^4 + \bar{z}^4).
\] (4.9)

The coefficients \( c_1^{(0)}, c_1^{(1)} \) and \( c_1^{(2)} \) are again fixed to satisfy the Dirichlet BC for \( \tilde{H}_1^{(s)}(z, \bar{z}) \) at the elliptic boundary. We proceed analogously in higher orders. The desired expansion of \( \tilde{H}^{(s)}(s) \) in powers of \( s^2 \) is obtained by averaging the relation (4.2) over the ellipse surface:

\[
\tilde{H}(s) = \sum_{j=0}^{\infty} \tilde{H}_j^{(s)} s^{2j} \quad \text{and} \quad \tilde{H}_j^{(s)} = \frac{1}{|\Omega|} \int_\Omega \text{d}r \tilde{H}_j^{(s)}(r),
\] (4.10)

where \( |\Omega| = \pi ab \). The coefficients of the small-\( s \) expansion are obtained in the form

\[
\tilde{H}_0^{(s)} = \frac{(ab)^2}{4(a^2 + b^2)},
\]

\[
\tilde{H}_1^{(s)} = -\frac{(ab)^4}{12(a^2 + b^2)^2},
\]

\[
\tilde{H}_2^{(s)} = \frac{(ab)^6[17(a^4 + b^4) + 98(ab)^2]}{576(a^2 + b^2)^4[(a^4 + b^4) + 6(ab)^2]},
\]

and

\[
\tilde{H}_3^{(s)} = -\frac{(ab)^8[93(a^8 + b^8) + 1048a^2b^2(a^4 + b^4) + 3190(ab)^4]}{8640(a^2 + b^2)^4[(a^4 + b^4) + 6(ab)^2]^2},
\] etc. (4.15)

For future purposes, we expand these coefficients around the limit \( \varepsilon \to 1 \), taken as \( a \to \infty \) with \( b \) fixed. We include also the subleading term in \( b^2/a^2 = 1 - \varepsilon^2 \):

\[
\tilde{H}_0^{(s)} = \frac{b^2}{4} - \frac{b^2}{4} \left( \frac{b^2}{a^2} \right)^2 + O \left( \frac{b^4}{a^4} \right),
\]

\[
\tilde{H}_1^{(s)} = -\frac{b^4}{12} + \frac{b^4}{6} \left( \frac{b^2}{a^2} \right)^2 + O \left( \frac{b^4}{a^4} \right),
\]

\[
\tilde{H}_2^{(s)} = \frac{17}{576} b^6 - \frac{55b^6}{576} \left( \frac{b^2}{a^2} \right)^2 + O \left( \frac{b^4}{a^4} \right),
\]

and

\[
\tilde{H}_3^{(s)} = -\frac{31}{2880} b^8 + \frac{11b^8}{216} \left( \frac{b^2}{a^2} \right)^2 + O \left( \frac{b^4}{a^4} \right).
\] (4.15)

5. Ellipse, eccentricity expansions

We perform a change of variables \( x' = (b/a)x, \ y' = y \); the Jacobian of this transformation is \( J = a/b \). The boundary then becomes \( x'^2 + y'^2 = b^2 \), i.e. the

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transformed domain $\mathcal{Q}'$ is the disc of radius $b$. The differential equation (2.2) modifies to
\[
\left[ \frac{\partial^2}{\partial y'^2} + (1 - \epsilon^2) \frac{\partial^2}{\partial x'^2} \right] \tilde{H}(s; r') - s^2 \tilde{H}(s; r') = -1. \tag{5.1}
\]
Within the probabilistic context explained in §1, instead of an isotropic diffusion in the ‘anisotropic’ ellipse, we get an anisotropic diffusion in the isotropic disc. This approach was inspired by the work of Kalnay & Percus (2006).

The differential equation (5.1) can be formally expressed as follows:
\[
(\mathcal{A} + \lambda \mathcal{B}) \tilde{H}(s; r') - s^2 \tilde{H}(s; r') = -1, \tag{5.2}
\]
where $\lambda$ is a smallness parameter and $\mathcal{A}, \mathcal{B}$ are the corresponding operators. There are two natural choices of the smallness parameter $\lambda$. In §5a, we shall set $\lambda = \epsilon^2$, whereas in §5c, we shall choose $\lambda = 1 - \epsilon^2$. In the case of $\lambda = \epsilon^2$, we have
\[
\mathcal{A} = \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial x'^2} \quad \text{and} \quad \mathcal{B} = -\frac{\partial^2}{\partial x'^2}. \tag{5.3}
\]
For $\lambda = 1 - \epsilon^2$, we have
\[
\mathcal{A} = \frac{\partial^2}{\partial y'^2} \quad \text{and} \quad \mathcal{B} = \frac{\partial^2}{\partial x'^2}. \tag{5.4}
\]
We look for the solution of equation (5.2) perturbatively as an infinite series in the smallness parameter $\lambda$:
\[
\tilde{H}(s; r') = \sum_{n=0}^{\infty} \tilde{H}^{(e)}_n(s; r') \lambda^n. \tag{5.5}
\]
Inserting this expansion into equation (5.2) and collecting terms of the same powers of $\lambda$, we get a coupled set of differential equations
\[
\mathcal{A} \tilde{H}^{(e)}_0(s; r') - s^2 \tilde{H}^{(e)}_0(s; r') = -1 \tag{5.6}
\]
and
\[
\mathcal{A} \tilde{H}^{(e)}_n(s; r') - s^2 \tilde{H}^{(e)}_n(s; r') = -\mathcal{B} \tilde{H}^{(e)}_{n-1}(s; r'), \quad n = 1, 2, \ldots. \tag{5.7}
\]
All $\tilde{H}^{(e)}_n(s; r')$ satisfy the Dirichlet BC on the disc domain $\mathcal{Q}'$. The quantity of interest $\tilde{H}(s)$ is given by $\tilde{H}(s) = \sum_{n=0}^{\infty} \tilde{H}^{(e)}_n(s) \lambda^n$, where
\[
\tilde{H}^{(e)}_n(s) = \frac{J}{|\mathcal{Q}|} \int_{\mathcal{Q}'} \frac{\partial}{\partial r'} \tilde{H}^{(e)}_n(s; r') = \frac{1}{\pi b^2} \int_{\mathcal{Q}'} \partial' \tilde{H}^{(e)}_n(s; r'). \tag{5.8}
\]

(a) Ellipse, small-$\epsilon$ expansion

We first choose $\lambda = \epsilon^2$ as the smallness parameter. Equation (5.6) then becomes
\[
\left[ \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial x'^2} \right] \tilde{H}^{(0)}_0(s; r') - s^2 \tilde{H}^{(0)}_0(s; r') = -1, \tag{5.9}
\]
where the upper index $(0)$ refers to the $\epsilon \to 0$ limit. The BC is $\tilde{H}^{(0)}_0(s; r') = 0$ for $|r'| = b$. The solution in polar coordinates $(r', \phi')$ reads $\tilde{H}^{(0)}_0(s; r', \phi') = [1 - I_0(sr')/I_0(sb)]/s^2$, which is the special case of equation (3.1) for $d = 2$ and $R = b$. 

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Applying relation (5.8) and using the notation \( u = sb \), we find the 2-ball version of equation (3.2)

\[
\tilde{H}_0^{(0)} = \frac{1}{s^2} \left[ 1 - \frac{2I_1(u)}{uI_0(u)} \right].
\] (5.10)

Equation (5.7) with \( n = 1 \) takes the form

\[
\left[ \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial x'^2} \right] \tilde{H}_1^{(0)}(s; r', \varphi') - s^2 \tilde{H}_1^{(0)}(s; r') = \frac{\partial^2}{\partial x'^2} \tilde{H}_0^{(0)}(s; r').
\] (5.11)

In polar coordinates, the r.h.s. becomes

\[
\frac{\partial^2}{\partial x'^2} \tilde{H}_0^{(0)}(s; r', \varphi') = -\frac{I_0(sr')}{2I_0(sb)} - \frac{I_2(sr')}{4I_0(sb)}(e^{2i\varphi'} + e^{-2i\varphi'}).
\] (5.12)

The general solution has the form

\[
\tilde{H}_1^{(0)}(s; r', \varphi') = \tilde{h}_0(s; r') + \tilde{h}_1(s; r')(e^{2i\varphi'} + e^{-2i\varphi'}).
\] (5.13)

\( \tilde{h}_0(s; r') \) is determined by the differential equation

\[
\frac{dl^2}{dr'^2} \tilde{h}_0(s; r') + \frac{1}{r'} \frac{dl}{dr'} \tilde{h}_0(s; r') - s^2 \tilde{h}_0(s; r') = -\frac{I_0(sr')}{2I_0(sb)}.
\] (5.14)

The homogeneous solutions are \( I_0(sr') \) and \( K_0(sr') \), their Wronskian is \( W = -1/r' \). Using indefinite integrals like \( \int zI_0(z) \, dz = z^2[K_1(z)I_1(z) + K_0(z)I_0(z)]/2 \) that can be found in Bateman & Erdélyi (1953), the solution can be simplified to

\[
\tilde{h}_0(s; r') = -\frac{r'I_1(sr')}{4sI_0(sb)} + c_I I_0(sr') + c_K K_0(sr').
\] (5.15)

We set \( c_K = 0 \), as we demand regular solution inside the domain, including the origin \( r' = 0 \). The constant \( c_I \) is fixed by the BC \( \tilde{h}_0(s; b) = 0 \). We thus get

\[
\tilde{h}_0(s; r') = -\frac{r'I_1(sr')}{4sI_0(sb)} + \frac{bI_1(sb)I_0(sr')}{4sI_0^2(sb)}.
\] (5.16)

Since the integral \( \int_0^{2\pi} e^{2i\varphi'} \, d\varphi' = 0 \) unless \( l = 0 \), only \( \tilde{h}_0(s; r') \) contributes to \( \tilde{H}_1^{(0)} \) and we finally obtain

\[
\tilde{H}_1^{(0)} = -\frac{1}{2s^2} \left[ 1 - \frac{2I_1(u)}{uI_0(u)} - \frac{I_2^2(u)}{I_0^2(u)} \right].
\] (5.17)
Proceeding analogously in higher orders, we find

$$\tilde{H}_2^{(0)} = \frac{1}{16s^2 u^3 I_3^0(u) I_2^0(u)} [-2u^3 I_0^5(u) + 3u^2(4 + u^2) I_0^4(u) I_1(u) - u(24 + 11u^2) I_0^3(u) I_1^2(u) + (16 + 8u^2 - 3u^4) I_0^2(u) I_1^3(u) + 2u(2 + 5u^2) I_0(u) I_1^4(u) - 8u^2 I_1^5(u)]$$

and

$$\tilde{H}_3^{(0)} = \frac{1}{192s^2 u^3 I_0^4(u) I_2^0(u)} [u^3(-12 + 5u^2) I_0^6(u) + 2u^2(36 + 7u^2) I_0^5(u) I_1(u) - 2u(72 + 57u^2 + 10u^4) I_0^4(u) I_1^2(u) + 32(3 + 4u^2 + u^4) I_0^3(u) I_1^3(u) + (8 + 52u^2 + 15u^4) I_0^2(u) I_1^4(u) - 36u^2(2 + u^2) I_0(u) I_1^5(u) + 24u^3 I_1^6(u)]$$

and so on. Recall that $\tilde{H}(s) = \sum_{j=0}^{\infty} \tilde{H}_j^{(0)}(s) \varepsilon^{2j}$.

We would like to emphasize that the obtained $\varepsilon$-expansion of $\tilde{H}(s)$ is valid for all values of $s$. This enables us to perform a consistency check of the above results by expanding them in small $s$ and comparing with the previous small-$s$ formulas (4.11)–(4.14). Expanding our $\tilde{H}_j^{(0)}(s)$ in $1/s$, we can also test our results by comparison with the exact recurrence scheme of Savo (1998) for the small-$n$ coefficients $c_n$. Our results pass this consistency check also.

We can now analyse the large-$s$ behaviour of the set (5.17)–(5.19), in analogy with appendix A. As analytical calculations are cumbersome, they were checked numerically as well. We found that the leading large-$n$ term for the coefficient $c_n$ coming from $\tilde{H}_k^{(0)} \varepsilon^{2k}$ is proportional to $\varepsilon^{2k} [n^k + O(n^{k-1})]$. Collecting only these leading terms, we get

$$c_n(\varepsilon) = \frac{8\Gamma(n-3)}{\pi(2b)^{n-2}} \left[1 - \frac{ne^2}{4} + \frac{(ne^2)^2}{64} - \frac{5(n^2e^2)^3}{768} + O((ne^2)^4)\right].$$

(5.20)

Here $n \gg N$, where $N$ is a large number. So far, the series (5.20) is a formal expansion in $ne^2$. As we are interested in the asymptotically large $n$ for a fixed $\varepsilon$, i.e. $n\varepsilon^2 \rightarrow \infty$, we have to know all terms of the series (5.20). In §5(b), we propose another method for finding $\tilde{H}(s)$ based on plausible, but not rigorously justified arguments. This method will predict all terms of expansion (5.20), reproducing the lowest ones correctly.

(b) Renormalized small-$\varepsilon$ expansion

We return to the original (non-transformed) space $r$. The ellipse boundary $R(\varphi)$ is expressed in polar coordinates as follows:

$$R(\varphi) = \frac{b}{\sqrt{1 - \varepsilon^2 \cos^2 \varphi}}, \quad \varphi \in (0, 2\pi).$$

(5.21)

For small $\varepsilon$, the ellipse is very close to the disc. Our intuitive approach is based on the assumption that in the differential equation (2.2) for $\tilde{H}(s; r)$, we are allowed to neglect the angular part of the Laplacian, i.e. $\Delta = \partial_r^2 + (1/r)\partial_r$. The resulting
equation is equivalent to that of the disc, the dependence on the angle is included only via the elliptic BC at \( r = R(\phi) \):

\[
\tilde{H}(s; r) \sim \frac{1}{s^2} \left[ 1 - \frac{I_0(sr)}{I_0(sR(\phi))} \right].
\]  

(5.22)

In polar coordinates, the averaging over the ellipse domain is expressible as:

\[
\frac{1}{|\Omega|} \int \Omega dr \cdots \equiv \sqrt{1 - \epsilon^2} \int_0^{2\pi} d\phi \int_0^{R(\phi)} dr r \cdots.
\]  

(5.23)

After the integration over \( r \), we get

\[
\tilde{H}(s) \sim \sqrt{1 - \epsilon^2} \int_0^{2\pi} d\phi \left[ \frac{1}{2s^2(1 - \epsilon^2 \cos^2 \phi)} - \frac{2I_1(sR(\phi))}{sb\sqrt{1 - \epsilon^2 \cos^2 \phi}I_0(sR(\phi))} \right].
\]  

(5.24)

To analyse the large-\( s \) behaviour of the ratio under integration, we repeat all steps (A.2)–(A.5) of appendix A, the case \( \nu = 0 \). The only difference compared with the 2-ball consists in the replacement of the Bessel functions argument \( u \) by \( sR(\phi) \):

\[
\frac{1}{w^j} \rightarrow \frac{(1 - \epsilon^2 \cos^2 \phi)^{j/2}}{(sb)^j} = \frac{1}{(sb)^j} \exp \left[ \frac{j}{2} \ln(1 - \epsilon^2 \cos^2 \phi) \right] \sim \frac{1}{(sb)^j} e^{-\left(j\frac{\epsilon^2}{2}\right)\cos^2 \phi},
\]  

(5.25)

where we expanded the logarithm only up to the leading \( \epsilon^2 \) term. The integration over the angle gives

\[
\int_0^{2\pi} d\phi \exp \left( -\frac{j\epsilon^2}{2} \cos^2 \phi \right) = \exp \left( -\frac{j\epsilon^2}{4} \right) I_0 \left( \frac{j\epsilon^2}{4} \right).
\]  

(5.26)

Similarly as in appendix A, the \( 1/s^n \) term in the expansion of \( \tilde{H}(s) \) is identified with the substitution \( j = n - 4 \). Using that \( e^{-q}I_0(x + q) \sim I_0(x) \) for large \( x \), we finally arrive at

\[
c_n(\epsilon) \sim \frac{8\Gamma(n - 3)}{\pi(2b)^{n-2}} \frac{e^{-n\epsilon^2/4}}{I_0 \left( \frac{n\epsilon^2}{4} \right)}.
\]  

(5.27)

Let us first assume that \( n\epsilon^2 \) is finite. Then

\[
c_n(\epsilon) \sim \frac{8\Gamma(n - 3)}{\pi(2b)^{n-2}} \sum_{k=0}^{\infty} (2k - 1)!! \left( \frac{-n\epsilon^2}{4} \right)^k.
\]  

(5.28)

The first four terms of this expansion match perfectly with those in equation (5.20). In the limit of interest \( n\epsilon^2 \to \infty \), according to equation (3.4) it holds

\[
c_n(\epsilon) \sim \frac{8\Gamma(n - 3)}{\pi(2b)^{n-2}} \frac{2}{\sqrt{2\pi n \epsilon}} \frac{16\Gamma(n - 7/2)}{\sqrt{2\pi^3(2b)^{n-2}}}.
\]  

(5.29)

The last expression exploits the property \( \Gamma(n - 7/2) \sim \Gamma(n - 3)/\sqrt{n} \) for \( n \to \infty \). The comparison with equation (1.17) implies the parameter \( \beta = 9/2 \) for the ellipse with small \( \epsilon \). This value is by one-half larger than \( \beta = 4 \) of a disc, in close analogy with the asymptotic Weyl series for the heat kernel.

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For the heat kernel, Berry & Howls (1994) expected the parameters $\alpha$ and $\beta$ to be of the order 1. Our result, equation (5.29), suggests something different for the heat content. We expect the divergence $\alpha(\varepsilon) \propto 1/\varepsilon$ in the symmetry-breaking limit $\varepsilon \to 0^+$. This is an acceptable price for the step-wise change of $\beta$ when restoring the circular symmetry. But this is a minor comment, the more important statement about the universality of $\beta$ for domains with only one shortest periodic orbit still holds.

(c) Ellipse with $\varepsilon \to 1$

Now, let us consider $\lambda = 1 - \varepsilon^2$ as the smallness parameter in the formalism developed at the beginning of §5. Equation (5.6) takes the form

$$\frac{\partial^2}{\partial y^2} \tilde{H}^{(1)}_0(s;r') - s^2 H^{(1)}_0(s;r') = -1,$$

(5.30)

where the upper index (1) refers to the limit $\varepsilon \to 1$. From the homogeneous solutions, we choose only $\cosh(sy')$, as the odd function $\sinh(sy')$ cannot fulfil the BC at $y' = \pm \sqrt{b^2 - x'^2}$. Thus, we get

$$\tilde{H}^{(1)}_0(s;r') = \frac{1}{s^2} \left[ 1 - \frac{\cosh(sy')}{\cosh(s\sqrt{b^2 - x'^2})} \right].$$

(5.31)

Now we perform the averaging (5.8), considering for simplicity four times the first quadrant:

$$\tilde{H}^{(1)}_0(s) = \frac{4}{\pi b^2} \int_0^b \int_0^{\sqrt{b^2-x'^2}} dx' \frac{dy'}{s^2} \left[ 1 - \frac{\cosh(sy')}{\cosh(s\sqrt{b^2 - x'^2})} \right].$$

(5.32)

The integration over $y'$ is simple. To integrate over $x'$, we make the substitution $x' = b \sin \varphi$ and resort to the full-angle integration,

$$\tilde{H}^{(1)}_0(s) = \frac{1}{s^2} - \frac{1}{\pi bs^3} \int_0^{2\pi} \cos \varphi \tanh(bs \cos \varphi) d\varphi.$$

(5.33)

This is the exact Laplace transform of the heat content for the limiting case of an ellipse with finite width $2b$ and infinite length $2a \to \infty$, valid for any value of $s$.

To check this expression within the small-$s$ expansion, we apply the series representation (Gradshteyn & Ryzhik 2007)

$$\tanh z = \sum_{k=1}^{\infty} \frac{2^k(2^k - 1)}{(2k)!} B_{2k} z^{2k-1},$$

(5.34)

where $B_{2k}$ are Bernoulli numbers. Inserting this series into equation (5.33), the integration results in

$$\tilde{H}^{(1)}_0(s) = -2b^2 \sum_{k=2}^{\infty} \frac{(2^k - 1) B_{2k}}{(k!)^2} (bs)^{2k-4}$$

(5.35)
This series can be compared with the leading terms in the set (4.15) and we find a perfect agreement in all available orders. It is worth mentioning that for complex $s$, the series (5.35) converges if $|s| < \pi/2b = \sqrt{l_1}$, i.e. up to the first imaginary poles given by the lowest eigenvalue in equation (1.14).

The next-to-leading term $\tilde{H}_1^{(1)}(s; r')$ fulfils the equation (5.7) with $n = 1$:

$$
\frac{\partial^2}{\partial y'^2} \tilde{H}_1^{(1)}(s; r') - s^2 \tilde{H}_1^{(1)}(s; r') = - \frac{\partial^2}{\partial x'^2} \tilde{H}_0^{(1)}(s; r').
$$

(5.36)

With respect to equation (5.31), the r.h.s. is equal to

$$
- \frac{\partial^2}{\partial x'^2} \tilde{H}_0^{(1)}(s; r') = \cosh(sy') \frac{\partial^2}{\partial x'^2} \frac{1}{s^2 \cosh(s\sqrt{b^2 - x'^2})} \equiv \cosh(sy') A(x'),
$$

(5.37)

where we introduced $A(x')$ for brevity. The solution of equations (5.36) reads

$$
\tilde{H}_1^{(1)}(s; r') = A(x') \left[ y \sinh(sy') - \frac{A(x')}{2s} \sqrt{b^2 - x'^2} \tanh(s\sqrt{b^2 - x'^2}) \cosh(sy') \right].
$$

(5.38)

To calculate $\tilde{H}_1^{(1)}(s)$, we first integrate over $y'$ in analogy with equation (5.32), integrate by parts with respect to $x'$ and substitute $x' = b \sin \varphi$, to get

$$
\tilde{H}_1^{(1)}(s) = - \frac{2}{\pi s^2} \int_0^{\pi/2} d\varphi \cos^2 \varphi \left[ \sinh^2(b \cos \varphi) + \cosh^4(b \cos \varphi) \frac{\sinh^3(b \cos \varphi)}{b \cos \varphi} \right].
$$

(5.39)

To check this formula, we expand the integrated function in powers of $s$ to get

$$
\tilde{H}_1^{(1)}(s) = - \frac{b^2}{4} + \frac{b^4}{6} s^2 - \frac{55}{576} b^6 s^4 + \frac{11}{216} b^8 s^6 - \frac{4487}{172800} b^{10} s^8 + \cdots.
$$

(5.40)

The first four terms can be compared with the $b^2/a^2$ terms in the set of four equations (4.15) and we see the full agreement.

Now we are ready to analyse the large-$s$ expansion of the exact solutions (5.33) and (5.39). The calculations are presented in appendix B. Except for the obligatory $1/s^2$ term, only the odd powers of $1/s$ appear in the Weyl series. The results for $c_n$ with $n$ odd are summarized by the formula

$$
c_n \sim \left[ 1 + \frac{\lambda}{2} + O(\lambda^2) \right] 16 \sqrt{\frac{2}{\pi^3}} \frac{\Gamma(n - 7/2)}{(2b)^{n-2}}, \quad n \to \infty.
$$

(5.41)

Comparing with the representation (1.17), we see that $l = 2b$, i.e. one-half of the shortest periodic orbit, as was expected from the small-$\epsilon$ analysis. The symmetry parameter $\beta = 9/2$ is reproduced as well. The prefactor is non-universal, dependent on $\lambda = 1 - \epsilon^2$.

6. Conclusion

This paper concerns the asymptotic form of the Weyl series for the heat content associated with the Dirichlet Laplacian in a smooth domain $\Omega$. Using the methods developed by Balian & Bloch (1972) and Voros (1983), Berry & Howls (1994)
mapped the quantum billiard model onto the resolventa of the heat kernel and conjectured a ‘universal’ geometric interpretation of the parameters $l$ and $\beta$ in high orders of the Weyl series (1.9) for general domains. It is questionable whether an analogical approach to the heat content is possible. Some doubts come from the finding of van den Berg (2004) that two domains with different shortest periodic orbits can have the same Weyl series for the heat content. One can imagine a scenario analogous to that of the heat kernel where unstable periodic orbits are excluded from the formalism. Maybe, the accessibility conditions for orbits are even more restrictive for the heat content; they might be satisfied for even-dimensional balls and the ellipse, but no more for an annulus or twice-cut-disc from the examples of van den Berg (2004). One can also imagine a general analysis of the asymptotic Weyl coefficients starting from Savo’s recurrent scheme (Savo 1998b).

Our strategy was to analyse the asymptotic Weyl series for the heat content, conjectured in the form (1.17), from the exact results for simple domains. These results were obtained by solving the differential equation (2.2) with Dirichlet BC. For balls in even dimensions $d$, the conjecture (1.17) applies when we identify $l = 2R$ and $\beta = 4$ independently of $d$. For balls in odd dimensions, $\tilde{H}(s)$ is the ratio of two polynomials and our conjecture no longer holds. It might be that the geometric information about the domain is contained in the polynomial coefficients themselves. Another open problem is whether the symmetry of balls is more important than dimensionality, or vice versa, when adapting our results to non-ball domains.

Furthermore, we studied the ellipse, which represents domains with a single periodic orbit, in two limiting cases of eccentricity $\varepsilon \to 0$ and $\varepsilon \to 1$. In both cases, the parameter $\beta$ is shifted by $1/2$ when compared with the disc. This phenomenon is in close analogy with the heat kernel.

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Appendix A

We are interested in the asymptotic terms of the Weyl series implied by the equation (3.3). From equation (3.4), we rewrite the large-$u$ asymptotic of $I_v(u)$ with $v = d/2 - 1$ as follows:

$$I_v(u) \sim \frac{e^u}{\sqrt{2\pi u}}(1 + \Sigma) \quad \text{and} \quad \Sigma = \sum_{j=1}^{\infty} \frac{(-1)^j}{(2u)^j} \frac{\Gamma(v + j + 1/2)}{\Gamma(j + 1)\Gamma(v - j + 1/2)}. \quad (A1)$$

Since $\Sigma$ is small in the large-$u$ limit, we can expand

$$\ln I_v(u) = u - \frac{1}{2} \ln(2\pi u) + \Sigma - \frac{1}{2} \Sigma^2 + \cdots. \quad (A2)$$

Further we will explore the identity $\Gamma(1/2 - j)\Gamma(1/2 + j) = (-1)^j\pi$ for integer $j$. In what follows, we shall argue that in the series (A2), $\Sigma$ contributes to the leading and higher order terms, $\Sigma^2$ contributes to the subleading and higher order terms, etc., of the coefficients $c_n$ in the asymptotic Weyl series.
Let us first analyse just the leading term given solely by $\Sigma^1$. Substituting the truncated expansion (A2) into equation (3.3) and using standard properties of $G$-functions, we get the leading contribution

$$\tilde{H}^{(d)}(s) \sim \frac{d}{u s^2} \sum_{j=1}^{\infty} \frac{1}{2^j u^{j+1}} \frac{((d-3)/2+j) \cdots (3/2+j)(1/2+j)}{((d-5)/2-j)((d-5)/2-j) \cdots (1/2-j)} \frac{\Gamma^2(1/2+j)}{\pi \Gamma(j)}.$$  

(A3)

Furthermore, we need the large-$j$ expansion

$$\frac{\Gamma^2(1/2+j)}{\Gamma(j) \Gamma(j+1)} \sim 1 - \frac{1}{4j} + \cdots \quad j \gg 1.$$  

(A4)

Taking the limit $j \to \infty$ of the long ratio in (A3), we get $(-1)^{d/2-1}$. Considering the leading (first) term of the expansion (A4), we obtain

$$\tilde{H}^{(d)}(s) \sim (-1)^{d/2-1} \frac{d}{R s^3} \sum_{j=1}^{\infty} \frac{1}{2^j (R s)^{j+1}} \frac{\Gamma(j+1)}{\pi}.$$  

(A5)

Comparing this series with equation (1.16), we set $j = n - 4$ and find the leading term of the asymptotic formula (3.5).

The calculation of the subleading $1/n$ term in equation (3.5) is more complicated and, for simplicity, we restrict ourselves to the $d=2$ case. There are two contributions. The simpler one comes from the subleading term in equation (A4), i.e. we get the leading factor $8 \Gamma(n-3)/\pi (2R)^{n-2}$ times $(-1/4n)$. The tricky part comes from the $\Sigma^2$ term in equation (A2), which contributes to $c_{n+2}/[s^2 (2u)^n]$ by the sum

$$- \sum_{k=1}^{n-1} \frac{\Gamma^2(1/2+k)}{2 \pi^2 \Gamma(k+1)} \frac{\Gamma^2(1/2+n-k)}{\Gamma(n-k+1)} = \left[ 2 - 3 F_2 \left( \frac{1}{2}, \frac{1}{2}, -n; \frac{1}{2} - n, \frac{1}{2} - n; -1 \right) \right] \times \frac{\Gamma^2(1/2+n)}{2 \pi \Gamma(1+n)}.$$  

(A6)

Here, we introduced the hypergeometric function $3 F_2$ (e.g. Gradshteyn & Ryzhik 2007). Its large $n$ asymptotic is

$$3 F_2 \left( \frac{1}{2}, \frac{1}{2}, -n; \frac{1}{2} - n, \frac{1}{2} - n; -1 \right) \sim 2 + \frac{1}{2n} + O \left( \frac{1}{n^2} \right), \quad n \to \infty.$$  

(A7)

Inserting this into equation (A6), in equation (3.3) we get exactly the same contribution as above, i.e. the leading factor times $(-1/4n)$. Summing up two equal contributions, we get finally $8 \Gamma(n-3)/\pi (2R)^{n-2}$ times $(-1/2n)$; noting that $\Gamma(n-4) \sim \Gamma(n-3)/n$ for large $n$, this is already the subleading term of equation (3.5) for the special case $d=2$. One can show that $\Sigma^3$, etc. contribute to higher order terms.

The derivation can be generalized to higher dimensions $d$.  

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Appendix B

We aim at analysing the asymptotic Weyl expansion of equation (5.33). Let us return to the first quadrant integration and rewrite appropriately \( \tanh \),

\[
\tilde{H}_0^{(1)}(s) = \frac{1}{s^2} - \frac{4}{\pi b s^3} + \frac{8}{\pi b s^3} \int_0^{\pi/2} \frac{\cos \varphi \, d\varphi}{e^{2 b s \cos \varphi} + 1}. \tag{B1}
\]

Further, we apply the substitution \( z = 2 b s \cos \varphi \) and subsequently the series

\[
\frac{q}{\sqrt{1 - q^2}} = \sum_{k=0}^{\infty} \frac{(2k - 1)!!}{k! 2^k} q^{2k + 1}. \tag{B2}
\]

We need to calculate the integral

\[
0 \leq \int_0^{2 b s} \frac{z^{2k+1}}{e^z + 1} \, dz < \int_0^{2 b s} \frac{z^{2k+1}}{e^z} \, dz = (2k + 1)! e^{-2 b s} \sum_{m=0}^{2k+1} \frac{(2 m) m}{m!}.
\]

We can calculate the large-\( k \) asymptotic of

\[
\int_0^{\infty} \frac{z^{2k-1}}{e^z + 1} \, dz \sim (2k - 1)! \quad k \to \infty. \tag{B4}
\]

Further, we use the obvious relation \( (2k - 1)!! = (2k)!! 2^{-k} / k! \). Applying the above steps, we get for both \( s \) and \( k \) large

\[
\tilde{H}_0^{(1)}(s) \sim \frac{8}{\pi} \sum_{k} \frac{(2k)!(2k + 2)!}{2^{2k} k! (k + 1)!} \frac{1}{s^2 (2b s)^{2k+3}}, \quad k \to \infty.
\]

Considering the asymptotic behaviour of

\[
\frac{\Gamma(2k + 1) \Gamma(2k + 3)}{2^{2k} \Gamma(k + 1) \Gamma(k + 2)} \sim 2^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \left[ 1 + O\left( \frac{1}{k} \right) \right], \quad k \to \infty
\]

and identifying \( n = 2k + 5 \), we arrive at the first term in equation (5.41). Note that \( n \) is odd, the even powers of \( 1/s \) do not appear.

\( (a) \) Subleading term in \( \lambda = 1 - \varepsilon^2 \)

Let us first rewrite equation (5.39) in the following way

\[
\tilde{H}_1^{(1)}(s) = -\frac{2}{3 \pi s^2} \frac{d}{ds} F(s) \quad \text{and} \quad F(s) = s^4 \int_0^{\pi/2} d\varphi \sin^2 \varphi \frac{\tanh^3 w}{w}, \tag{B7}
\]

where \( w = b s \cos \varphi \). \( F(s) \) fulfils the relation

\[
\frac{d}{ds}[s F(s)] = 3 \int_0^{bs} \frac{dw}{bs} \sqrt{1 - \left( \frac{w}{bs} \right)^2 \sinh^2 w / \cosh^4 w}. \tag{B8}
\]
We expand
\[ \sqrt{1 - \left( \frac{w}{bs} \right)^2 \sinh^2 w} \cosh^4 w \sim - \sum_{k=0}^{\infty} \frac{(2k - 2)!}{2^{2k-1} k!(k-1)!} \left( \frac{w}{bs} \right)^{2k} [4e^{-2w} + O(e^{-4w})]. \] (B9)

The upper limit of the integration in equation (B8) can be again extended to infinity, as in the case of equation (B3). Then, we calculate \([sF(s)]'\) and after all we get
\[ F(s) \sim 12 \sum_k \frac{(2k - 2)!(2k - 1)!}{2^{2k-1} k!(k-1)!} \frac{1}{(2bs)^{2k+1}}, \] (B10)

We substitute this series into equation (B7) and use the large-\(k\) behaviour of the ratio
\[ \frac{(2k - 3)\Gamma(2k - 1)\Gamma(k + 1)}{2^{2k-1} \Gamma(k + 1)\Gamma(k)} \sim \sqrt{\frac{2}{\pi}} \Gamma\left( 2k - 1 \right), \quad k \to \infty. \] (B11)

Thus, we obtain
\[ \tilde{\tilde{H}}_1^{(1)}(s) \sim \frac{8}{\pi} \sqrt{\frac{2}{\pi}} \sum_k \frac{\Gamma(2k - 1/2)}{s^2(2bs)^{2k+1}}, \quad k \to \infty. \] (B12)

We set \(n = 2k + 3\) and finally arrive at the second term in equation (5.41). Note that only odd powers of \(1/s\) appear again.

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