SU(2)–invariant reduction of the 3+1 dimensional Ashtekar’s gravity

Sergei Alexandrov†, Ignati Grigentch‡ and Dmitri Vassilevich†

†Department of Theoretical Physics, St. Petersburg University 198904, St. Petersburg, Russia
‡Department of Physics, Old Dominion University, Norfolk, VA 23529 and Jefferson Lab, 12000 Jefferson Avenue, Newport News, VA 23606, USA

Abstract

We consider a space–time with spatial sections isomorphic to the group manifold of SU(2). Triad and connection fluctuations are assumed to be SU(2)-invariant. Thus, they form a finite dimensional phase space. We perform non–perturbative path integral quantization of the model. Contrary to previous claims the path integral measure appeared to be non–singular near configurations admitting additional Killing vectors. In this model we are able to calculate the generating functional of Green functions of the reduced phase space variables exactly.

PACS: 04.60.+n; 04.20.Fy
1 Introduction

Ashtekar’s variables \[1\] simplify considerably the algebraic structure of constraints in general relativity. These variables give rise to even simpler models \[2\], which can be solved completely. Such models are useful for a non-perturbative study of quantum gravitational effects.

In the present paper we start with a 3 + 1-dimensional Ashtekar’s gravity. We assume that spatial sections are topologically \(S^3\), which is the group manifold of \(SU(2)\). Next we reduce the phase space to \(SU(2)\) invariant fields. This gives a finite dimensional (complex) phase space. By solving constraints and fixing gauge freedom we arrive at a four dimensional (real) reduced phase space\[1\]. We also construct the path integral for this model and calculate Green functions at all orders of the perturbation theory. The corresponding quantum effective action coincides with the classical one. Our technique is similar to that used in the 2D dilatonic gravity \[3\].

Our primary aim is to study the behavior of the path integral measure near the points in the phase space admitting additional Killing vectors. Mottola \[4\] argued that due to the presence of zero modes in the Faddeev–Popov determinant on a background with Killing vectors the path integral measure become zero, and hence symmetric configurations (as e.g. de Sitter space) do not contribute to the path integral. Our model appeared to be useful for a non-perturbative study of this effect. Among \(SU(2)\)-invariant configurations it contains also \(SU(2) \times U(1)\) and \(SU(2) \times SU(2)\)-invariant configurations. We found that the Faddeev–Popov determinant has zeros at symmetric configurations. However, these zeros are cancelled by the contribution of the delta functions of the constraints. The resulting path integral measure is regular.

2 The reduced action

We begin with a complex gravitational action in 3 + 1 dimensions,

\[
S = \int d^4x \left( i E^i_i \partial_t A_i^a - N^a G_a - N^i G_i - NG_0 \right),
\]

\(1\)

\(^1\)In our model the phase space of Ashtekar’s gravity undergoes two subsequent reductions. We hope, it is clear from the context, which one is meant in any particular case.
where, as usual, the densitized triad $E_a^i$ and connection $A^a_i$ are the canonical variables; $G_a, G_i$ and $G_0$ are the Gauss law, the vector and the scalar constraints respectively:

$$G_a = D_i E_a^i = 0, \quad \text{(2)}$$

$$G_i = F^a_{ij} E_a^j = 0, \quad \text{(3)}$$

$$G_0 = \varepsilon^{abc} E_a^i E_b^j F^c_{ij} + \frac{\Lambda}{3} \varepsilon^{abc} \varepsilon_{ijk} E_a^i E_b^j E_c^k = 0, \quad \text{(4)}$$

where $\Lambda$ is the cosmological constant; $N^a, N^i$ and $N$ are the Lagrange multipliers, $D_i$ is the covariant derivative with respect to the connection $A^a_i$, $F^a_{ij}$ is the field strength.

We assume that the spatial sections are topologically equivalent to $S^3$. We restrict ourselves to the $SU(2)$-invariant canonical variables $A^a_i$ and $E^a_i$. They can be presented in the following form:

$$A^a_i = f_b^a(t) e^b_i, \quad \text{(5)}$$

$$E^a_i = h^b_a(t) e^b_i e, \quad \text{(6)}$$

where $e^a_i$ is an invariant triad field on “round” $S^3$, $e = \det (e^a_i)$ and $f, h$ are $3 \times 3$ (complex) matrices depending only on time. Here we used the fact that $S^3$ is the group manifold of $SU(2)$. In general, the canonical connection $A^c$ should be subtracted from the r.h.s. of (3). In the canonical coordinates on $SU(2)$ (see below) $A^c$ is zero, and $A^a_i$ behaves like a tensor with respect to $SU(2)$ transformations. For any given group $G$ the $G$-invariant tensor fields are those that have constant components in the canonical tangential basis. The case of unit matrices $f$ and $h$ corresponds to the $SU(2) \times SU(2)$-invariant configuration isometric to the “round” $S^3$ with the metric of the maximum symmetry.

Group elements of $SU(2)$ can be considered as canonical coordinates on $S^3$. We can take $g = \exp(x^j \frac{1}{2} \tau_j)$, where $\tau_j$ are the Pauli matrices. The triad one-form can be calculated from the relation $e = g^{-1} dg$. In the vicinity of the unit element we have

$$e^b_j(0) = \delta^b_j, \quad \partial_j e^b_k(0) = -\frac{1}{2} \varepsilon_{jkb}. \quad \text{(7)}$$

Regardless of the type of the indices the Levi–Civita symbol $\varepsilon$ is ±1.
By substituting the ansatz (5), (6) into the action (1) we obtain:

\[ S = \int dt \left( i h_a b \partial_t f_b a - \eta^a C_a - n^a \mathcal{H}_a - n \mathcal{H}_0 \right), \]  

(8)

where only \( SU(2) \) invariant components of the Lagrange multipliers survive:

\[ N^a = \eta^a(t), \quad N^i = e^i_a n^a(t), \quad eN = n(t). \]  

(9)

In (8) we discarded an overall constant factor equal to the volume of \( SU(2) \).

The Poisson brackets of the canonical variables become:

\[ \{ h_a^b, f_d^c \} = -i \delta_a^c \delta_b^d. \]  

(10)

The following useful identity holds on the group manifold of \( SU(2) \):

\[ e^k_a \partial_k e^j_b - e^k_b \partial_k e^j_a = [e_a^j, e_b^j] = \varepsilon_{abc} e^j_c, \]  

(11)

which enables us to express the constraints in terms of the invariant variables

\[ C_a = \varepsilon_{abc} f_d^b h_c^d \]  

(12)

\[ \mathcal{H}_a = \varepsilon_{abc} f_b^d h_a^e + \varepsilon_{ebc} f_d^b h_a^e \]  

(13)

\[ \mathcal{H}_0 = -\varepsilon_{abc} \varepsilon_{def} h_a^d h_b^e f_c^f + \varepsilon_{abc} \varepsilon_{fgc} f_d^f h_a^d f_e^g h_b^f + \frac{\Lambda}{3} \varepsilon_{abc} \varepsilon_{def} h_a^d h_b^e h_c^f. \]  

(14)

For the sake of symmetry it is more convenient, however, to consider a combination of the vector and Gauss law constraints (see [5]) instead of the vector constraint itself. The modified vector constraint reads

\[ \tilde{G}_i = G_i - A_i^a G_a. \]  

(15)

In terms of the invariant variables this constraint becomes

\[ \tilde{\mathcal{H}}_a = -\varepsilon^{abc} h_d^b f_c^d \]  

(16)

Instead of the original action (8) we can use the following action

\[ S = \int dt \left( i h_a b \partial_t f_b a - \bar{\eta}^a C_a - n^a \tilde{\mathcal{H}}_a - n \tilde{\mathcal{H}}_0 \right), \]  

(17)
which is obtained from (8) by a shift of the Lagrange multiplier $\eta_a$.

Now we are to find gauge transformations of the canonical variable $s$. In general, an infinitesimal gauge transformation generated by a constraint $G$ of a variable $Z$ is

$$\delta Z = \{G\xi, Z\},$$

where $\xi$ is the infinitesimal parameter of the transformation.

In our case the constraints (12), (14) and (16) lead to the following transformations:

Gauss law constraint:

$$\begin{align*}
\delta h^d_a &= i \varepsilon_{abc} h^b_d \xi^c \\
\delta f^a_d &= i \varepsilon_{abc} f^b_d \xi^c,
\end{align*}$$

(19)

modified vector constraint:

$$\begin{align*}
\delta h^a_d &= i \varepsilon_{abc} h^a_d \xi^c \\
\delta f^a_d &= i \varepsilon_{abc} f^a_d \xi^c,
\end{align*}$$

(20)

scalar constraint:

$$\begin{align*}
\delta h^a_d &= 2i \xi h^a_d h^c_b f^b_c - 2i \xi h^a_b f^b_a h^c_d - i \xi \varepsilon_{dgh} \varepsilon^{abc} h^b_g h^c_h \\
\delta f^a_d &= 2i \xi \varepsilon_{abc} \varepsilon_{dgh} h^b_d f^b_c - 2i \xi f^a_d f^b_a h^c_b \\
&+ 2i \xi f^a_b h^f_c f^d_f - i \Lambda \xi \varepsilon_{def} h^e_b h^f_d.
\end{align*}$$

(21)

The non-zero Poisson brackets of the constraints are:

$$\begin{align*}
\{C_a, C_b\} &= i \varepsilon_{abc} C^c \\
\{\tilde{H}_a, \tilde{H}_b\} &= i \varepsilon_{abc} \tilde{H}^c
\end{align*}$$

(22)

Thus the constraint algebra splits into a direct sum of two copies of $so(3, C)$ and a one-dimensional abelian algebra.

3 Reduced phase space quantization

To construct a reduced phase space one should solve the constraints (12), (14), (16) and fix the corresponding gauge freedom.
The symmetry between the Gauss law (12) and the modified vector constraint (16) makes it natural to eliminate the corresponding gauge freedom (19) and (20) simultaneously. One can integrate these infinitesimal transformations to finite gauge transformations. Then two of them rotate both the upper and the lower indices of $h$ by $SO(3, C)$ matrices. One can see that by means of combined action of these two transformations the matrix $h$ can be diagonalized:

$$h_a^b = \text{diag}(h_1, h_2, h_3).$$

(23)

Now we are to use this condition to solve both the Gauss law and the modified vector constraint. There are several different cases to be considered:

1. Non-degenerate fluctuation of the triad.
   All three elements of $h$ have different absolute values:
   $$|h_1| \neq |h_2| \neq |h_3|.$$  
   (24)
   In this case the solution of both (12) and (16) is the diagonal fluctuation of the connection:
   $$f_a^b = \text{diag}(f_1, f_2, f_3).$$  
   (25)

2. Once degenerate fluctuation of the triad.
   Two of the three elements of $h$ have equal absolute values:
   $$|h_1| = |h_2| \neq |h_3|.$$  
   (26)
   If $h_1 = h_2$, the solution for $f$ can be written as
   $$\begin{pmatrix}
   f_{11} & f & 0 \\
   f & f_{22} & 0 \\
   0 & 0 & f_{33}
   \end{pmatrix}.$$  
   (27)
   Note that in the case of once degenerate $h$ only five of the six parameters of the gauge transformations (19)-(20) are fixed by choosing the diagonal form of the triad fluctuation. For example, if $h_1 = h_2$ the transformation parametrized by
   $$\xi^c = \tilde{\xi}^c \propto \delta^c_3$$  
   (28)
is not involved in the diagonalization of $h$ and hence can be used to
diagonalize (27). This fixes the gauge freedom completely unless the
matrix $\begin{pmatrix} f_{11} & f \\ f & f_{22} \end{pmatrix}$ has equal eigenvalues. If the latter is the case this
part of the gauge freedom can not be eliminated by any condition on
the phase variables $f$ and $h$. A similar situation for ADM gravity was
considered in [6] and for Ashtekar’s gravity on a de Sitter background
in [7]. The proper way to fix the remaining freedom is to impose a
condition on a Lagrange multiplier.

3. Twice degenerate fluctuation of the triad.

All three components of $h$ have equal absolute values:

$$|h_1| = |h_2| = |h_3|.$$  \hspace{1cm} (29)

In this case the general solution $f^b_a$ of the constraints (12) and (14)
has six independent components. For example, for $h_1 = h_2 = h_3$, any
symmetric fluctuation of the connection solves both the Gauss law and
the modified vector constraint.

Just like in the previous case the part of gauge freedom not fixed by
the condition (19) can be utilized to diagonalize $f$. Three different off-
diagonal elements of $f$ can be set to zero by the three parameter gauge
transformation of the form:

$$\xi^c = \tilde{\xi}^c.$$ \hspace{1cm} (30)

And again this works only for a nondegenerate fluctuation of the con-
nexion, i.e. when eigenvalues of $f^b_a$ are all different. Otherwise an
additional condition on a Lagrange multiplier should be imposed.

The last two cases of degenerate triad fluctuations correspond to addi-
tional invariance of the field configuration and existence of additional Killing
vectors. For example, the twice degenerate case corresponds to additional
$SO(3)$ invariance. The necessity to exclude some of the Lagrange multipliers
is natural in the context of Hamiltonian theory because some of the con-
straints become linearly dependent. Imposing a new gauge condition on a
submanifold in a phase space looks awkward from the point of view of an
ordinary gauge theory. However, this is just a manifestation of the Gribov
problem: there is no global gauge fixing in a non-abelian gauge theory. This
is especially clear in our case since gauge orbits in the phase space have
different dimensions.

Before considering the remaining constraint \((14)\), let us study the reality
conditions. In our case they read:

\[
\text{Im } h^b_a = 0, \quad \text{Re } f^b_a = -\frac{1}{2} \Gamma^b_a(h) 
\]

where \(\Gamma^b_a = e^i_a e^{bcd} \omega^c_i\), \(\omega_i\) is the spin connection compatible with the rescaled
inverse triad \(h^b_a e^i_b\). For the gauge \((23)\) the spin–connection \(\Gamma\) is diagonal,

\[
\Gamma(h) = \text{diag}(\Gamma_1, \Gamma_2, \Gamma_3),
\]

with

\[
\begin{align*}
\Gamma_1 & = \frac{h_2 h_3}{h_1} - \frac{h_2}{h_3} - \frac{h_3}{h_2} \\
\Gamma_2 & = \frac{h_1 h_3}{h_2} - \frac{h_1}{h_3} - \frac{h_3}{h_1} \\
\Gamma_3 & = \frac{h_2 h_1}{h_3} - \frac{h_2}{h_1} - \frac{h_1}{h_2}
\end{align*}
\]

We see, that the reality conditions \((31)\) are consistent with the solutions \((25)\)
of the constraint equations. Note, that the imaginary part of \(H_0\) vanishes.

We use reality in the "triad" and not in the "metric" form\(^2\). It is essential
for the path integral quantization because we wish to restrict the integration
variables rather than their composites. Imposing reality conditions restricts
the gauge group to its real form. Thus, it is important to fix all complex
gauge transformations but this should be done in a way consistent with the
reality conditions.

It is convenient to fix the remaining gauge freedom \((21)\) by the condition

\[
h_1 = 1.
\]

The real part of \(H_0\) gives

\[
\phi_1 = \frac{1}{h_2 \phi_2 + h_3 \phi_3} \times \left( \frac{1}{4} \left( \frac{h_2}{h_3} \right)^2 + \left( \frac{h_3}{h_2} \right)^2 + (h_2 h_3)^2 \right) - \frac{1}{2} \left( h_2^2 + h_3^2 + 1 \right) + \Lambda h_2 h_3 - h_2 h_3 \phi_2 \phi_3 \right),
\]

\(^2\)Discussion on different types of reality conditions see e.g. in \([9]\).
where $\phi_a = \text{Im } f_a$.

Now we are able to calculate the path integral measure

$$d\mu = \prod_{i,j} (df_i^j \, dh_i^j) \det \{G_A, \chi^B\} \prod_A \delta(\chi^A) \delta(G_A),$$

where $G_A$ denotes all constraints, and $\chi^B$ stand for corresponding gauge conditions. Let us subdivide the phase space variables into two classes: \{f, h\} = \{f_2, f_3, h_2, h_3, f_A, h^A\} where the first four variables are independent, while $f_A$ and $h^A$ are fixed by means of constraints or gauge conditions. Then,

$$\{G_A, \chi^B\} = \{G_A, h^B\} = \frac{\partial G_A}{\partial f_B}.\tag{37}$$

Being combined with $\delta(G_A)$ the determinant of (37) gives $\delta(f_A - f_A^{(0)})$, where $f_A^{(0)}$ are solutions of the constraints expressed in terms of independent variables. Hence the measure (36) becomes

$$d\mu = df_2 \, df_3 \, dh_2 \, dh_3.$$

In the integrand all phase space variables should be expressed in terms of $f_2, f_3, h_2, h_3$. Since the shift of $f$ by $\frac{1}{2} \Gamma(h)$ is a canonical transformation, it does not change the measure. This means that the real integration measure will be just

$$d\mu_R = d\phi_2 \, d\phi_3 \, dh_2 \, dh_3,$$

where we now integrate over real variables.

The generating functional of Green functions is

$$Z(J, j) = \int d\mu_R \exp \int dt( - h_2 \partial_t \phi_2 - h_3 \partial_t \phi_3 + \frac{i}{2} \Gamma_2 \partial_t h_2 + \frac{i}{2} \Gamma_3 \partial_t h_3 + h_2 J^2 + h_3 J^3 + \phi_2 j^2 + \phi_3 j^3).$$

Integration over $\phi_2$ and $\phi_3$ gives delta functions

$$\delta(\partial_t h_2 + j^2) \delta(\partial_t h_3 + j^3),$$

which can be used to integrate over $h_2$ and $h_3$.

$$Z(J, j) = (\det \partial_t)^{-2} \exp \int dt(-J^2 \partial_t^{-1} j^2 - J^3 \partial_t^{-1} j^3 - \frac{i}{2} \Gamma_2 j^2 - \frac{i}{2} \Gamma_3 j^3) \tag{42}$$
where in $\Gamma_2$ and $\Gamma_3$ the triad fluctuations are expressed in terms of the currents, $h_{2,3} = -\partial_t^{-1} j^{2,3}$.

Let us now calculate the effective action

$$W_{\text{eff}}(\bar{h}, \bar{\phi}) = \frac{1}{i} \ln Z - \bar{h}_2 J^2 - \bar{h}_3 J^3 - \bar{\phi}_2 J^2 - \bar{\phi}_3 J^3.$$  \hspace{1cm} (43)

The currents should be expressed in terms of mean field by the equations

$$\bar{h}_2 = \frac{\delta}{i \delta J^2} \ln Z = -\partial_t^{-1} j^2$$
$$\bar{h}_3 = \frac{\delta}{i \delta J^3} \ln Z = -\partial_t^{-1} j^3$$  \hspace{1cm} (44)

The field independent infinite factor $(\det \partial_t)^{-2}$ does not contribute to the effective action. By substituting (42) and (44) in (43) we obtain

$$W_{\text{eff}}(\bar{h}, \bar{\phi}) = \int dt ( -\bar{h}_2 \partial_t \bar{\phi}_2 - \bar{h}_3 \partial_t \bar{\phi}_3 + \frac{i}{2} \Gamma_2 \partial_t \bar{h}_2 + \frac{i}{2} \Gamma_3 \partial_t \bar{h}_3)$$  \hspace{1cm} (45)

This means that the full effective action (45) coincides with the classical action on the reduced phase space.

Let us consider now the case of triad fluctuations with coinciding eigenvalues. The Faddeev–Popov determinant is

$$\det \{ G_A, \chi^B \} = -2 (h_2 \phi_2 + h_3 \phi_3) h_1 (h_1^2 - h_2^2)(h_2^2 - h_3^2)(h_1^2 - h_3^2).$$ \hspace{1cm} (46)

According to our gauge condition $h_1 = 1$ but we prefer to keep more symmetric notations in (46). In the case of the increased symmetry at least two eigenvalues among $h_1, h_2, h_3$ coincide, and the determinant (46) is zero. This effect was noted by Mottola [4] in a different context. He claimed that the path integral measure is zero on symmetric configurations and thus they do not contribute to the path integral. We see that only the first part of this conjecture is true. The zero in (46) is cancelled by the contribution of the delta function of the constraints. Hence the path integral measure is regular near symmetric configurations.

Note, that singularities in the Faddeev–Popov determinant can disappear if one uses the gauge fixing approach instead of the reduced phase space one. For example, if one replaces the condition (23) by $\tilde{\eta}^a = n^a = 0$ the Faddeev–Popov determinant (calculated e.g. in the BFV approach [10]) will contain $(\det \partial_0)^0$. This multiplier does not depend on a symmetry of the three space geometry.
4 Conclusions

In the present paper we suggested an $SU(2)$ invariant reduction of the $3+1$-dimensional Ashtekar gravity. We assumed that spatial sections are isomorphic to $S^3$, which is the group manifold of $SU(2)$. The invariant fields have constant components in the canonical tangential basis. We constructed a reduced phase space quantization of this model. We observe that near the triad configurations admitting additional Killing vectors the Faddeev–Popov determinant is zero. This zero is cancelled by the contribution of the delta functions of the constraints. The resulting path integral measure is regular near symmetric points in the reduced phase space. Moreover, in our simple model we are able to calculate generating functional of Green functions of the reduced phase space variables. It occurs that at all orders of the perturbation theory this functional contains tree–level diagrams only. The corresponding quantum effective action coincides with the classical action calculated on the reduced phase space.

5 Acknowledgments

This work was partially supported by GRACENAS and Russian Foundation for Fundamental Research, grant 97-01-01186.
References

[1] Ashtekar A 1987 *Phys. Rev. D* **36** 1587

[2] Witten E 1988 *Nucl. Phys. B* **311** 46
   Hussain V and Kuchar K 1990 *Phys. Rev. D* **42** 4070
   Thiemann T and Kastrup H A 1993 *Nucl. Phys. B* **399** 211
   Thiemann T 1995 *Class. Quantum Grav.* **12** 59

[3] Kummer W, Liebl H and Vassilevich D V 1997 Nucl. Phys. B, to appear

[4] Mottola E 1995 *J. Math. Phys.* **36** 2470

[5] Peldan P 1994 *Class. Quantum Grav.* **11** 1087

[6] Vassilevich D V 1993 *Int. J. Mod. Phys.* **A8** 1637

[7] Grigentch I and Vassilevich D 1995 *Int. J. Mod. Phys.* **D4** 581

[8] Faddeev L D and Slavnov A A 1980 Gauge Fields: Introduction to Quantum Theory, Benjamin/Cummings

[9] Yoneda G and Shinkai H 1996 *Class. Quantum Grav.* **13** 783

[10] Henneaux M 1985 Phys. Rep. **126** 1