HERGLOTZ-NEVANLINNA MATRIX FUNCTIONS AND HURWITZ STABILITY OF MATRIX POLYNOMIALS

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ABSTRACT. This paper elaborates on a relationship between Herglotz-Nevanlinna matrix functions and Hurwitz stable matrix polynomials, which generalizes the corresponding classical stability criterion. Influenced by coprimeness and spectral features with regard to matrix polynomials, the “only if” and “if” implication of this generalized stability criterion becomes respectively different and much more complicated than the classical formalism. The main motivation comes from the author’s recent stability studies linked with matricial Markov parameters. To fulfill our goals, certain matrix extension is also established to two classical theorems by Chebotarev and Grommer respectively.

Keywords: Herglotz-Nevanlinna functions, stability, matrix polynomials, rational matrix functions

1. INTRODUCTION

The Herglotz-Nevanlinna (scalar) functions, analytic maps of the open upper half-plane of the complex plane \( \mathbb{C}_+ \) into itself, serve as fundamental roles in the theory of complex analysis. This classical concept is getting a growing interest recently owing to the emergence and development of its extensions or variants in various directions: It can be found in the formulations of matrix or operator valued functions [15, 14], noncommutative functions [36], multivariate functions [35] and so on.

Typically, their matrix valued analogue usually refers to matrix functions analytic on \( \mathbb{C}_+ \) of which the imaginary part are nonnegative definite Hermitian matrices. These so-called Herglotz-Nevanlinna matrix functions have important applications in system and control theory, interpolation problems, and so on.

Some important contributions of Herglotz-Nevanlinna functions to stability theory can be traced to their intimate connection with the classical theory for Hurwitz (stable) polynomials, i.e., all the roots of the polynomials are located in the open left half-plane of the complex plane:

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1Another few names as “Nevanlinna, Pick, Nevanlinna-Pick, Herglotz, and R-functions” are frequently employed as well.
Theorem (Stability criterion via Herglotz-Nevanlinna functions). \[13, 5\]

A real polynomial \( f(z) \) of degree \( n \) is a Hurwitz polynomial if and only if the associated rational function \( r(z) = -\frac{f(z)}{f(e(z))} \), formed by its even part \( f_e(z) \) and its odd part \( f_o(z) \) satisfying \( f_e(z^2) + zf_o(z^2) = f(z) \), is a Herglotz-Nevanlinna function with exact \( \left\lfloor \frac{n}{2} \right\rfloor \) poles, all of which are negative real, and, additionally for the case that \( n \) is odd, the limit

\[
\lim_{z \to \infty} r(z) > 0.
\]

This criterion is of theoretical and applied importance to the stability study for continuous time LTI systems. For more complex high-order MIMO nonlinear systems, efficient methods can be implemented with the Jacobian linearization at the equilibrium point. Via this linearization, the behaviour of nonlinear systems can be approximated by that of linear differential equations with the general form

\[
A_0 y^{(n)}(t) + A_1 y^{(n-1)}(t) + \cdots + A_n y(t) = u(t),
\]

where \( A_0, \ldots, A_n \) are complex matrices, \( y(t) \) is the output vector and \( u(t) \) denotes the control input vector. In this regard, the asymptotic stability of this linearized system (1.1) describes the local stability of nonlinear system at the equilibrium point. Moreover, it can be determined by typical zero distribution of the characteristic matrix polynomial related to (1.1)

\[
F(z) = A_0 z^n + A_1 z^{n-1} + \cdots + A_n
\]

that all roots of \( \det F(z) \) lie in the open left half-plane \( \Re z < 0 \). In correspondence with the scalar case, the matrix polynomial \( F(z) \) with this property is usually called Hurwitz stable matrix polynomials.

There are many algebraic techniques for testing the Hurwitz stability of matrix polynomials avoiding computing its determinant or zeros: the LMI approach [20, 21, 31, 32], the Anderson-Jury Bezoutian [33, 34], matrix Cauchy indices [6], lossless positive real property [3], block Hurwitz matrix [27], extended Routh-Hurwitz array [12], argument principle [24], matricial Markov parameters [39], matricial continued fractions [39] and so on.

However, the relations remain unclear between the Hurwitz stability for matrix polynomials and Nevanlinna-Herglotz matrix functions. In fact, it can be much more difficult to attain than the scalar formalism. The key point is, certain basic scalar techniques may be totally unavailable in the matrix case: The zeros and poles of a rational matrix function, related to the determinants of its matrix fraction, can not be fully reflected through its representations or derivatives.

To overcome these obstacles, otherwise we follow alternative lines to deal with the matrix extension. To notice that, although the proof for the scalar version is usually independent of some other classical stability criteria, some interconnections still can be found among them. For example, \[13, \text{Theorem 17, Chapter XV}\]
check the Hurwitz stability via positive definite Hankel matrices built from Markov parameters:

**Theorem (Stability criterion via Markov parameters).** Given a real polynomial \( f(z) \) of degree \( n = 2m \) or \( n = 2m + 1 \), define its Markov parameters \((s_k)_{k=-1}^{\infty}\) as the coefficients in the expansion of the ratio

\[
\frac{f_o(z)}{f_e(z)} = s_{-1} + \sum_{k=0}^{\infty} \frac{(-1)^k s_k}{z^{k+1}},
\]

where \( f_e(z) \) and \( f_o(z) \) are as above. Then \( f(z) \) is a Hurwitz polynomial if and only if both Hankel matrices \([s_{j+k}]_{j,k=0}^{m-1}\) and \([s_{j+k+1}]_{j,k=0}^{m-1}\) are positive definite and, for \( n = 2m + 1 \), additionally \( s_{-1} > 0 \).

On the other hand, recall the so-called Grommer theorem for rational functions:

**Theorem (Grommer theorem for rational functions).** [2,7,18] [22, Theorem 3.4] Let \( p(z) \) and \( q(z) \) be two coprime real polynomials satisfying that \( |\deg p - \deg q| \leq 1 \). Suppose that \( r(z) := \frac{q(z)}{p(z)} \) is a real rational function. \( r(z) \) is a Herglotz-Nevanlinna function if and only if \( r(z) \) can be represented by the Laurent series

\[
r(z) = s_{-2}z + s_{-1} + \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}
\]

with the conditions that \( s_{-2} \geq 0 \) and the Hankel matrix \([s_{j+k}]_{j,k=0}^{\deg p - 1}\) is negative definite.

With the help of the above theorem, to convert stability criterion via Markov parameters to that via Nevanlinna-Herglotz functions becomes apparent to some extent, and vice versa.

Our formulation for stability criterion via Herglotz-Nevanlinna matrix functions resembles this classical routes, which is greatly motivated by the recent study [39]. Based on the Anderson-Jury Bezoutian techniques, [39] Theorem 4.4] associates a class of Hurwitz matrix polynomials with Stieltjes positive definite matrix sequences. This connection leads to tests for Hurwitz stability via matricial Markov parameters (see [39, Theorem 4.10]) and via matricial Stieltjes continued fractions (see [39, Theorem 4.12]). Therefore, our key step is to give a matrix generalization of Grommer theorem for rational functions. This generalization will be established in a much more complicated form than the scalar case, which is influenced by the following factors:

(F1) the difference between coprimeness and zero distraction of matrix fraction of rational matrix functions;

(F2) the spectral features of the denominator matrix polynomials of matrix fraction of rational matrix functions.
Specifically, the former factor results in the sightly difference between respective
generalization to the “only if” and “if” implication of Grommer theorem for rational
functions, and consequently, that of stability criterion via Herglotz-Nevanlinna
functions.

We conclude the introduction with the outline of the paper. In Section 2 we
describes under which representation a rational matrix function can become a
Herglotz-Nevanlinna matrix function. This can be viewed as a matrix analogue
of the Chebotarev theorem, whereas its full matrix extension is included in next
section. Section 3 is the key step to fulfill our main goals. Subsection 3.1 and 3.2,
respectively, give a detailed explanation of two factors (F1) and (F2). The “only
if” and “if” implication of Grommer theorem is extended in Subsection 3.3 and 3.4,
respectively, to the corresponding result for rational matrix functions. Our main
results are provided in Section 4. Based on the generalized Grommer theorem in
Section 3 we convert the stability criterion for matrix polynomials via matricial
Markov parameters to that via Nevanlinna-Herglotz matrix functions.

2. MATRICIAL ANALOGUE OF CHEBOTAREV THEOREM

To identify a real rational function as a Nevanlinna function, [22, Theorem 3.4]
seeks a series of equivalent conditions of rational Nevanlinna functions, among
which there is

**Theorem (Chebotarev theorem).** [7, 22, Theorem 3.4] \[38\] Given two coprime
real polynomials \( p(z) \) and \( q(z) \), let \( r(z) := \frac{q(z)}{p(z)} \) be a real rational function. \( r(z) \)
is a Nevanlinna-Herglotz function if and only if \( r(z) \) can be represented in the form

\[
r(z) = cz + d + \sum_{j=1}^{r} \frac{e_j}{\lambda_j - z},
\]

where \( c \geq 0, \ d \in \mathbb{R} \) and \( e_j = -\frac{q(\lambda_j)}{p'(\lambda_j)} > 0 \).

The main object of this section is to provide a matricial analogue of Chebotarev
theorem. We begin with some basic notation. Throughout the rest of this pa-
der, we denote by \( \mathbb{C}, \mathbb{R} \) and \( \mathbb{N} \), respectively, the sets of all complex, real, and
positive integer numbers. Unless explicitly noted, we assume in this paper that
\( p,q,r,m,n \in \mathbb{N} \). Let \( \mathbb{C}^{p \times q} \) stand for the set of all complex \( p \times q \) matrices. Let
also \( 0_p \) and \( I_p \) be, respectively, the zero and the identity \( p \times p \) matrices. Given a
matrix \( A \in \mathbb{C}^{p \times p} \) we denote its transpose by \( A^T \) and its conjugate transpose by \( A^* \),
and write

\[
\begin{cases}
A \succ 0, & \text{if } A \text{ is positive definite;} \\
A \succeq 0, & \text{if } A \text{ is nonnegative definite;} \\
A \prec 0, & \text{if } A \text{ is negative definite.}
\end{cases}
\]
It is well known that each Herglotz-Nevanlinna matrix function $R(z) : \mathbb{C}_+ \rightarrow \mathbb{C}^{p \times p}$ admits the so-called Nevanlinna-Riesz-Herglotz integral representation as

$$R(z) = Az + B + \int_\mathbb{R} \left( \frac{1}{u - z} - \frac{u}{1 + u} \right) d\Omega(u), \quad z \in \mathbb{C}_+, \quad (2.1)$$

where $A \succeq 0$, $B = B^*$ and $\Omega$ is a $p \times p$ positive semi-definite Borel matrix measure on $\mathbb{R}$ such that

$$\text{trace} \int_\mathbb{R} \frac{1}{1 + |u|} d\Omega(u) < +\infty,$$

which is called spectral measure of $R(z)$.

For a $p \times p$ matrix function function $R$ defined in $\mathbb{C}_+$ and $\lambda \in \mathbb{R} \cup \{\infty\}$, by the notation $\angle \lim_{z \to \lambda} R(z) = A$ we means that $R(z)$ has a nontangential limit $A$ at $\lambda$, that is, $R(z)$ tends to $A$ as $z$ tends to $\lambda$ along a nontangential path in $\mathbb{C}_+$.

Given a $p \times p$ Herglotz-Nevanlinna matrix function $R(z)$, its coefficient $A$ in the form (2.1) has a relation to the nontangential limit $\angle \lim_{z \to \infty} R(z) = A \succeq 0$. (2.2)

Given a $\lambda \in \mathbb{R}$, the matrix mass of the spectral measure $\Omega$ assigned at $\lambda$, denoted by $\Omega(\{\lambda\})$, can be represented as follows:

**Lemma 2.1.** [23, Lemma 3.1] Let $R(z) : \mathbb{C}_+ \rightarrow \mathbb{C}^{p \times p}$ be a Herglotz-Nevanlinna matrix function with an integral representation as in (2.1). Then for $\lambda \in \mathbb{R}$,

$$\Omega(\{\lambda\}) = \angle \lim_{z \to \lambda} (\lambda - z) R(z).$$

Let $R(z)$ is a $p \times p$ matrix whose entries are complex-valued rational functions in $z$, that is, a $p \times p$ rational matrix function. In the following we turn to see under which representation the restriction of $R(z)$ to $\mathbb{C}_+$ can become a Herglotz-Nevanlinna matrix function.

Let $F(z)$ be a $p \times p$ matrix whose entries are scalar polynomials with complex coefficients in $z$, that is, to say, a $p \times p$ matrix polynomials. $F(z)$ may be written as

$$F(z) = \sum_{k=0}^n A_k z^{n-k}, \quad \text{with} \quad A_0, \ldots, A_n \in \mathbb{C}^{p \times p}, \quad A_0 \neq 0 \quad (2.3)$$

for certain $n \in \mathbb{N}_0$, which is called the degree of $F(z)$ and denoted by $\deg F$. In particular, $F(z)$ is said to be regular if $\det F(z)$ is not identically zero and it is monic if $A_0 = I_p$. It is clear that all monic matrix polynomials are regular. Given a regular matrix polynomial $F(z)$, we say that $\lambda \in \mathbb{C}$ is a zero of $F(z)$ if $\det F(\lambda) = 0$. Its multiplicity is the multiplicity of $\lambda$ as a zero of $\det F(z)$. The spectrum $\sigma(F)$ of $F(z)$ is the set of all zeros of $F(z)$.

Given $A, B \in \mathbb{C}^{p \times p}$ such that $B$ is nonsingular, denote

$$\frac{A}{B} := A \cdot B^{-1}.$$
In view of [25, Section 6.1], one can write each $p \times p$ rational matrix function $R(z)$ as a fraction of a $p \times p$ matrix polynomial $Q(z)$ and a monic $p \times p$ matrix polynomial $P(z)$

$$R(z) := \frac{Q(z)}{P(z)}, \quad z \notin \sigma(P).$$

(2.4)

**Proposition 2.2.** Suppose that $R(z)$ is a $p \times p$ rational matrix function with the matrix fraction (2.4). $R(z)$ is a Herglotz-Nevanlinna matrix function if and only if $R|_{\mathbb{C}^+}$ can be represented in the form

$$R(z) = Cz + D + \sum_{j=1}^{r} E_j \frac{\lambda_j}{\lambda_j - z}, \quad z \in \mathbb{C}^+, \quad (2.5)$$

where $C \succeq 0$, $D = D^*$, $E_j \succeq 0$ and $\{\lambda_j\}_{j=1}^{r} \subseteq \sigma(P) \cap \mathbb{R}$.

**Proof.** The proof for “only if” implication: Suppose that $R(z)$ is a Herglotz-Nevanlinna matrix function with the form (2.1). Lemma 2.1 implies the mass representation of $\Omega$ assigned at $\lambda \in \mathbb{R}$

$$\Omega(\{\lambda\}) = \angle \lim_{z \to \lambda} (\lambda - z)R(z) = \angle \lim_{z \to \lambda} (\lambda - z)Q(z)(P(z))^{-1}.$$

If $\lambda \notin \sigma(P)$, obviously $\Omega(\{\lambda\}) = 0_p$. So $\text{supp}(\Omega) \subseteq \sigma(P) \cap \mathbb{R}$. Let $\text{supp}(\Omega) =: \{\lambda_j\}_{j=1}^{r}$. Then we can rewrite (2.1) into

$$R(z) = Az + (B - \sum_{j=1}^{r} \Omega(\{\lambda_j\}) \frac{\lambda_j}{1 + \lambda_j}) + \sum_{j=1}^{r} \frac{\Omega(\{\lambda_j\})}{\lambda_j - z}, \quad z \in \mathbb{C}^+, \quad (2.5)$$

and, subsequently, into (2.5).

The proof for “if” implication: If $R(z)$ has the form (2.5), then $R(z)$ is analytic on $\mathbb{C}^+$ and

$$\frac{R(z) - R(z)^*}{z - \bar{z}} = C + \sum_{j=1}^{r} \frac{E_j}{|\lambda_j - z|^2} \succeq 0, \quad z \in \mathbb{C}^+.$$

Thus $R(z)$ is a Herglotz-Nevanlinna matrix function. $\square$

This theorem can be viewed as a matricial analogue to the Chebotarev theorem, whereas the former, when reduced to the scalar case, can not cover the full information of the Chebotarev theorem: the representation of each $E_j$ is still unknown. Our solution to this problem will be included in next section.

3. **Generalized Grommer theorem for rational matrix functions**

3.1. **Coprimeness and zero disctraction of matrix polynomials.** First we give the definition of matrix analogue of common divisors and coprimeness.
Definition 3.1. Given two $p \times p$ matrix polynomials $F(z)$ and $L(z)$, $L(z)$ is called a right divisor of $F(z)$ if there exists a $p \times p$ matrix polynomial $M(z)$ such that

$$F(z) = M(z)L(z).$$

Let additionally $\tilde{F}(z)$ be a $p \times p$ matrix polynomial. Then

- $L(z)$ is called a right common divisor of $F(z)$ and $\tilde{F}(z)$ if $L(z)$ is a right divisor of $F(z)$ and also a right divisor of $\tilde{F}(z)$.
- $L(z)$ is called a greatest right common divisor (GRCD) of $F(z)$ and $\tilde{F}(z)$ if any other right common divisor is a right divisor of $L(z)$.
- $F(z)$ and $\tilde{F}(z)$ are said to be right coprime if any GRCD of $F(z)$ and $\tilde{F}(z)$ is unimodular (i.e., det $F(z)$ never vanishes in $\mathbb{C}$).

Lemma 6.3-6 in [25] provides a rank criterion for right coprimeness.

**Proposition 3.2.** Given two $p \times p$ matrix polynomials $Q(z)$ and $P(z)$, $Q(z)$ and $P(z)$ are right coprime if and only if the matrix

$$\begin{bmatrix}
Q(z) \\
P(z)
\end{bmatrix}$$

has full column rank for any $z \in \mathbb{C}$.

The right coprimeness of matrix polynomials can be obtained via the condition that no common zeros exist between them, whereas the converse implication does not hold in general:

**Proposition 3.3.** Let $Q(z)$ and $P(z)$ be two $p \times p$ matrix polynomials. If $Q(z)$ and $P(z)$ have no common zeros, then $Q(z)$ and $P(z)$ are right coprime.

**Proof.** Assume that $Q(z)$ and $P(z)$ have no common zeros, that is to say, either $Q(z)$ or $P(z)$ is nonsingular for each $z \in \mathbb{C}$. In this case, $
\begin{bmatrix}
Q(z) \\
P(z)
\end{bmatrix}$

has full column rank. From Proposition 3.2, $Q(z)$ and $P(z)$ are right coprime. \qed

**Example 3.4.** Let two matrix polynomials

$$P(z) := \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, \quad Q(z) := \begin{bmatrix} 1 & 1 \\ 0 & z \end{bmatrix}.$$  

$Q(z)$ and $P(z)$ are right coprime while 0 is their unique shared zero.

Hence, zero distraction is a stronger condition than right coprimeness for matrix polynomials. This gap between scalar and matrix polynomials can make the matrix generalization of Grommer theorem more complicated. We illustrate this fact with the following example:

**Example 3.5.** Let two $p \times p$ matrix polynomials

$$P(z) := \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \quad Q(z) := \begin{bmatrix} 2z - 1 & z + 1 \\ z + 1 & z - 1 \end{bmatrix}.$$  


The $p \times p$ rational matrix function $R(z) := Q(z)(P(z))^{-1}$ has the form

$$R(z) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{z}, \quad z \neq 0.$$  

We can check that $R(z)$ is a Herglotz-Nevanlinna matrix function and $Q(z)$ and $P(z)$ are right coprime, while $\mathcal{H}_0(R) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is a singular matrix.

This example shows that regarding matrix polynomials, the statement directly resembling the “only if” implication of Grommer theorem can be false. To check, $Q(z)$ and $P(z)$ are right coprime while $Q(z)$ and $P(z)$ have a common zero $0$. This motivates us to replace the coprimeness with zero distraction in seeking the matrix extension of this implication. To verify it, extra conditions for the spectral structure of $P(z)$ have to be added, as the following subsection shows.

### 3.2. Simple matrix polynomials and orthogonal matrix polynomials.

**Definition 3.6.** [30, P. 42] Let $F(z)$ be a $p \times p$ matrix polynomial. $F(z)$ is said to be simple if $F(z)$ is regular and for any zero $\lambda$ of $F(z)$, the multiplicity of $\lambda$ coincides with the degeneracy of $F(z)$ evaluated at $\lambda$, i.e., the nullity of the matrix $F(\lambda)$.

The reason why a matrix polynomial as in above definition is called “simple” can be seen from its Smith form. Recall that for a $p \times p$ regular matrix polynomial $F(z)$ of degree $n$, $F(z)$ is equivalent to the following Smith form:

$$E_L(z)F(z)E_R(z) = \begin{bmatrix} f_1(z) & 0_p & \cdots & 0_p \\ 0_p & f_2(z) & \cdots & 0_p \\ \vdots & \vdots & \ddots & \vdots \\ 0_p & 0_p & \cdots & f_p(z) \end{bmatrix}$$

such that

- $f_1(z), \ldots, f_p(z)$ are monic scalar polynomials and uniquely determined by $F(z)$;
- For $j = 1, \ldots, p - 1$, $f_{j+1}(z)$ is divisible by $f_j(z)$.

All the factors $f_j(z)$ are called invariant polynomials of $F(z)$. Moreover, if $F(z)$ has $r$ distinct zeros $\lambda_1, \lambda_2, \ldots, \lambda_r$, write each invariant polynomial $f_j(z)$ as

$$f_j(z) = (z - \lambda_1)^{l_{j,1}}(z - \lambda_2)^{l_{j,2}} \cdots (z - \lambda_r)^{l_{j,r}}$$

Then all the factor $(z - \lambda_k)^{l_{j,k}}$ are called the elementary divisors of $F(z)$.

**Proposition 3.7.** [30, Corollary 1, P. 46] A $p \times p$ matrix polynomial is simple if and only if all its elementary divisors are linear in $\mathbb{C}$, or equivalently, all zeros of its invariant polynomials are simple.
Certain features of a simple matrix polynomial can also be found via its adjoint matrix:

**Proposition 3.8.** [9, Lemma 2.2] Let \( F(z) \) be a \( p \times p \) matrix polynomial and let \( \lambda \) be a zero of \( F(z) \) with multiplicity \( l \). If \( F(z) \) is simple, then \((\text{adj} \ F)^{(k)}(\lambda) = 0_p\) for \( k = 0, 1, \ldots, l - 2 \) and \((\text{adj} \ F)^{(l-1)}(\lambda) \neq 0_p\). Moreover, \( \text{rank}(\text{adj} \ F)^{(l-1)}(\lambda) = l \).

In the following we see a typical class of simple matrix polynomials.

Given a \( p \times p \) matrix polynomial \( F(z) \) written as in \([2.3]\), define a matrix polynomial \( F^\lor(z) \) by

\[
F^\lor(z) := \sum_{k=0}^{n} A^*_k z^{n-k}.
\]

**Definition 3.9.** Let \( \Omega \) be a \( \mathbb{C}^{p \times p} \)-valued positive semi-definite Borel matrix measure on \( \mathbb{R} \). A sequence of matrix polynomials \( (P_k(z))_{k=0}^{m} \) is called a sequence of **monic right orthogonal matrix polynomials with respect to** \( \Omega \) if for \( k = 0, \ldots, m \), \( P_k(z) \) is monic, \( \text{deg} \ P_k(z) = k \) and

\[
\int_{\mathbb{R}} P_k^\lor(u) d\Omega(u) P_j(u) = \delta_{jk} I_p, \quad j, k = 0, \ldots, m,
\]

where \( \delta_{jk} \) stands for the Kronecker symbol.

The theory of orthogonal matrix polynomials originates from two outstanding papers by Krein [28, 29] and now has various applications in telecommunication (see e.g. [19]), information theory (see e.g. [37, 11]), matricial interpolation and moment problems (see e.g. [40, 23]) and so on.

Let \( \Omega \) be a \( \mathbb{C}^{p \times p} \)-valued positive semi-definite Borel matrix measure on \( \mathbb{R} \). A sequence of matrix polynomials \( (P_k(z))_{k=0}^{m} \) with respect to \( \Omega \) can be determined by the moment sequence \( (s_k)_{k=0}^{2m} \) of \( \Omega \) as

\[
\int_{\mathbb{R}} u^k d\Omega(u) = s_k, \quad k = 0, \ldots, n - 1.
\]
In fact, suppose that \( P_k(z) := \sum_{j=0}^{k} A_{k,k-j} z^j \), where \( A_{k0} = I_p \). For \( j = 0, \ldots, k \),
\[
\int_{\mathbb{R}} P_k^\vee(u) d\Omega(u) P_j(u) \]
\[
=[A^*_k, A^*_{k,k-1}, \ldots, A^*_{k0}] \int_{\mathbb{R}} \begin{bmatrix} I_p \\ uI_p \\ \vdots \\ u^k I_p \end{bmatrix} d\Omega(u) [I_p, uI_p, \ldots, u^k I_p] \begin{bmatrix} A_{jj} \\ \vdots \\ 0_p \end{bmatrix}
\]
\[
=[A^*_k, \ldots, A^*_{k0}] \begin{bmatrix} s_0 & \cdots & s_k \\ \vdots & \ddots & \vdots \\ s_k & \cdots & s_{2k} \end{bmatrix} \begin{bmatrix} A_{jj} \\ \vdots \\ 0_p \end{bmatrix},
\]
which follows that
\[
\begin{bmatrix} \int_{\mathbb{R}} P_k^\vee(u) d\Omega(u) P_{k-1}(u) \\ \vdots \\ \int_{\mathbb{R}} P_k^\vee(u) d\Omega(u) P_0(u) \end{bmatrix}
\]
\[
=[A^*_k, A^*_{k,k-1}, \ldots, A^*_{k0}] \begin{bmatrix} s_0 & \cdots & s_{k-1} \\ \vdots & \ddots & \vdots \\ s_k & \cdots & s_{2k-1} \end{bmatrix} \begin{bmatrix} A_{k-1,k-1} & A_{k-2,k-2} & \cdots & A_{00} \\ A_{k-1,k-2} & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{k-1,0} & \cdots & \cdots & A_{00} \end{bmatrix}.
\]

From these above equations one can show that

**Proposition 3.10.** Let \( \Omega \) be a \( \mathbb{C}^{p \times p} \)-valued positive semi-definite Borel matrix measure on \( \mathbb{R} \). Assume that \((s_j)_{j=0}^{2m}\) is the associated moment sequence of \( \Omega \) as in (3.1). \((P_k(z))_{k=0}^{m}\) is a sequence of monic right orthogonal matrix polynomials with respect to \( \Omega \) if and only if the following equations hold for the coefficients of \( P_k(z) \):

\[
[A^*_k, \ldots, A^*_{k0}] \begin{bmatrix} s_0 & \cdots & s_k \\ \vdots & \ddots & \vdots \\ s_k & \cdots & s_{2k} \end{bmatrix} \begin{bmatrix} A_{kk} \\ \vdots \\ A_{k0} \end{bmatrix} = I_p
\]

and

\[
[A^*_k, \ldots, A^*_{k0}] \begin{bmatrix} s_0 & \cdots & s_{k-1} \\ \vdots & \ddots & \vdots \\ s_k & \cdots & s_{2k-1} \end{bmatrix} = 0_{p \times kp},
\]

where \( P_k(z) := \sum_{j=0}^{k} A_{k,k-j} z^j \).

[9] Theorem 2.3] points it out that these orthogonal matrix polynomials are special types of simple matrix polynomials:
Proposition 3.11. Let $\Omega$ be a $\mathbb{C}^{p \times p}$-valued positive semi-definite Borel matrix measure on $\mathbb{R}$. Suppose that $(P_k(z))_{k=0}^{\infty}$ is a sequence of monic right orthogonal matrix polynomials with respect to $\Omega$. Then each $P_k(z)$ is simple and $\sigma(P_k) \subseteq \mathbb{R}$.

3.3. Generalization to the “only if” implication of Grommer theorem. [15] Lemma 5.6] discusses how a Nevanlinna-Herglotz matrix function analytically continues through an interval $(\lambda_1, \lambda_2)$ from $\mathbb{C}_+$ into $\mathbb{C}_-$. Special attention is put to the analytic continuation of $R(z)$ through an interval $(\lambda_1, \lambda_2)$ by reflection, that is,

$$R(z) = R(\bar{z})^*, \quad \forall z \in \mathbb{C}_-.$$  \hfill (3.2)

For the case when $R(z)$ is a rational Nevanlinna-Herglotz matrix function with the matrix fraction (2.4), this continuation can be established once the chosen interval $(\lambda_1, \lambda_2)$ exclude all zeros of $P(z)$. To generalize the “only if” implication of Grommer theorem, we focus on the case when a $p \times p$ rational matrix function is Hermitian (i.e., it obeys that $R(z) = R(\bar{z})^*$ for all $z$ belong to $\mathbb{C}$ except the poles of each entry of $R(z)$) and, moreover, becomes a rational Nevanlinna-Herglotz matrix function with the reflection (3.2).

We begin with a structure-preserving decomposition of Hermitian rational matrix functions. For a $p \times p$ rational matrix function $R(z)$, it can be uniquely decomposed by

$$R(z) = R_p(z) + R_{sp}(z),$$  \hfill (3.3)

where $R_p(z)$ is a $p \times p$ matrix polynomial and $R_{sp}(z)$ is a $p \times p$ strictly proper rational matrix function, i.e., $\lim_{z \to \infty} R(z) = 0$. This factorization can be conducted via a unique decomposition of each entry of $R(z)$ into a scalar polynomial and a strictly proper function, which turn out to be the entry of $R_p(z)$ and $R_{sp}(z)$, respectively.

Another way to derive this factorization is to utilize the matrix fraction of $R(z)$. Given two matrix polynomials, if they have no division relation, then matrix analogue of quotients and remainders may appear:

Definition 3.12. Given two $p \times p$ matrix polynomials $F(z)$ and $G(z)$, if there exists a pair of $p \times p$ matrix polynomials $Q(z)$ and $\tilde{F}(z)$ such that

$$F(z) = C(z)G(z) + \tilde{F}(z)$$

and $\deg \tilde{F} < \deg G$, then we call $C(z)$ and $\tilde{F}(z)$ are the right quotient and right remainder of $F(z)$ on division by $G(z)$.

Remark 3.13. In the case that $G(z)$ is monic, there exists a unique pair of right quotient and right remainder of $F(z)$ on division by $G(z)$ (see [13], P. 78]).

Suppose that $R(z)$ be a $p \times p$ rational matrix function with the matrix fraction (2.4). Then from Remark 3.13 there exists a unique pair of right quotient $C(z)$ and right remainder $Q(z)$ of $Q(z)$ on division by $P(z)$. From Lemma 6.3-11 of [25],
we can see that $\tilde{Q}(z)(P(z))^{-1}$ is strictly proper. So in this event, the factorization (3.3) of $R(z)$ can be obtained in the concrete form

$$R(z) = C(z) + \tilde{Q}(z)(P(z))^{-1}. \quad (3.4)$$

Moreover, if $R(z)$ is Hermitian, the entry relation in this decomposition indicates that $C(z) = C^\prime(z)$ and $\tilde{Q}(z)(P(z))^{-1}$ is Hermitian as well. We remark that different from [6, Theorem 4.1, pp. 666-667], this decomposition does not assure that $\tilde{Q}(z)$ is regular.

In what follows we obtain the degree relation of matrix fraction of a rational matrix function which obeys the Herglotz-Nevanlinna property. Suppose that a rational matrix function $R(z)$ admits the Laurent expansion

$$R(z) = \sum_{j=0}^{k} z^j s_{-(j+1)} + \sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}} \quad (3.5)$$

that converges for sufficiently large $|z|$. Comparing the matrix fraction (2.4) and the Laurent expression (3.5) of $R(z)$, one can find the equations

$$\begin{bmatrix} s_j & s_{j+1} & \cdots & s_{j+m} \\ s_{j+1} & s_{j+2} & \cdots & s_{j+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{j+k} & s_{j+k+1} & \cdots & s_{j+k+m} \end{bmatrix} \begin{bmatrix} A_m \\ A_{m-1} \vdots A_0 \end{bmatrix} = 0_{j \times p}, \quad j, k = 0, 1, \ldots \quad (3.6)$$

where $P(z) := \sum_{j=0}^{m} A_k z^{k-j}$.

Denote a finite or infinite block Hankel matrix associated with $R(z)$ by

$$\mathcal{H}_{j,k}(R) := \begin{bmatrix} s_j & s_{j+1} & \cdots & s_{j+k} \\ s_{j+1} & s_{j+2} & \cdots & s_{j+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{j+k} & s_{j+k+1} & \cdots & s_{j+2k} \end{bmatrix},$$

where $j \in \mathbb{N}_0 \cup \{-1\}$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. For simplicity we write $\mathcal{H}_k(R)$ for $\mathcal{H}_{0,k}(R)$.

**Proposition 3.14.** Let $R(z)$ be a $p \times p$ Hermitian rational matrix function with the matrix fraction (2.4). If $R(z)$ is a Herglotz-Nevanlinna matrix function, then $\deg Q - \deg P \leq 1$. Additionally if $\mathcal{H}_{\deg P-1}(R) \prec 0$, then $|\deg Q - \deg P| \leq 1$.

**Proof.** Taking Proposition 2.2 into account, one can represent $R(z)$ as the form (2.5). When a rational matrix function $R(z)$ is a Herglotz-Nevanlinna matrix function, the form (2.5) and the decomposition (3.3) of $R(z)$ are related by

$$R_p(z) = Cz + D, \quad R_{sp}(z) = \sum_{j=1}^{r} \frac{E_j}{\lambda_j - z}. \quad (3.4)$$

Combining (3.4) one can see that $Cz + D$ is the right quotient of $Q(z)$ on division by $P(z)$. It means that $\deg Q \leq \deg P + 1$. 


The remaining proof is conducted by contradiction. By assuming that \( \deg Q - \deg P < -1 \), Lemma 6.3-11 of \[25\] tells that \( R(z) \) is strictly proper and so is \( zR(z) \). However, these two statements contradict each other under our assumption: The former statement implies that 
\[
R(z) = \sum_{j=0}^{\infty} \frac{s_j}{z^j + 1}.
\]
It follows from the negative definiteness of \( \mathcal{H}_{\deg P - 1}(R) \) that
\[
\lim_{z \to \infty} zR(z) = s_0 < 0,
\]
which contradicts with the latter statement. \( \Box \)

The matrix version of Cauchy index for real rational matrix functions is provided by Bitmead and Anderson \[6\]. The formulation of this concept can be suitable to Hermitian rational matrix functions as well:

\textbf{Definition 3.15.} Suppose that \( R(z) \) is a Hermitian rational matrix function. The Cauchy index of \( R(z) \) over the real interval \( (a, b) \) \( (a, b \in \mathbb{R} \cup \{\infty\}) \), denoted by \( I_{a}^{b}R(z) \), is the number of jumps of eigenvalues of the matrix \( R(\lambda) \) from \( -\infty \) to \( +\infty \) minus the number of the opposite jumps from \( +\infty \) to \( -\infty \) as \( \lambda \) traverses the interval \( (a, b) \).

The matrix Cauchy index for a Hermitian rational matrix function can be represented in terms of the spectrum of a Hankel matrix.

For a \( p \times p \) Hermitian matrix \( A \), \( \delta(A) \) stand for the signature of \( A \), i.e., the number of positive real eigenvalues (counting algebraic multiplicities) of \( A \) minus the number of negative real eigenvalues (counting algebraic multiplicities) of \( A \).

\textbf{Lemma 3.16.} Let \( R(z) \) be a Hermitian rational matrix function. Then
\[
I_{-\infty}^{+\infty} R(z) = \delta(\mathcal{H}_{\infty}(R)).
\]

The proof of Lemma \[3.16\] which we omits, parallels that of Theorem 3.1 in \[6\]; the latter focuses on the case when \( R(z) \) is a real symmetric rational matrix function.

Now we are in a position to generalize the “only if” implication of Grommer theorem for rational functions to matrix case.

\textbf{Theorem 3.17.} Let \( R(z) \) be a \( p \times p \) Hermitian rational matrix function with the matrix fraction \( (2.4) \). Then \( \mathcal{H}_{\deg P - 1}(R) \prec 0 \) and \( |\deg Q - \deg P| \leq 1 \) if the following statements are simultaneously true:

(i) \( R(z) \) is a Herglotz-Nevanlinna matrix function;
(ii) \( P(z) \) and \( Q(z) \) have no common zeros;
(iii) \( P(z) \) is simple.

\textbf{Proof.} Taking Proposition \[2.2\] and (i) into account, one can represent \( R(z) \) as the form \( (2.5) \). Since \( R(z) \) is Hermitian and analytic on \( \mathbb{C}_+ \), \( \sigma(P) \subseteq \mathbb{R} \), i.e.,
\[ \{\lambda_j\}_{j=1}^r \subseteq \sigma(P) \]. Suppose that \( l_j \) is the multiplicity of the zero \( \lambda_j \) of \( P(z) \) for \( j = 1, \ldots, r \). By Proposition 3.8 and (iii),

\[
E_j = \angle \lim_{z \to \lambda_j} (\lambda_j - z) Q(z) \frac{\text{adj} P(z)}{\det P(z)}
\]

\[
= \angle \lim_{z \to \lambda_j} (\lambda_j - z) Q(z) \frac{1}{l_j \prod_{i \neq j} (\lambda_j - \lambda_i)^{1}} (\text{adj} P(z)) (z - \lambda_j)^{l_j - 1} + \cdots
\]

\[
= - l_j Q(\lambda_j) \frac{\text{adj} P(z) (l_j - 1)}{(\det P(z)) (l_j - 1) ! (z - \lambda_j)^{l_j - 1} + \cdots}
\]

Thus by (ii),

\[
\text{rank } E_j = \text{rank} \left( \text{adj} P(z) (l_j - 1) ! (z - \lambda_j)^{l_j - 1} \right) = l_j.
\] (3.7)

On the other hand, assume that \( m := \deg P \). From the equations (3.6) one can obtain that

\[ \mathcal{H}_{j,m-1}(R_-) = \mathcal{H}_{m-1}(R_-) \cdot C^j, \]

where \( R_-(z) := -R(z), P(z) := \sum_{j=0}^m P_j z^{m-j} \) and

\[
C := \begin{bmatrix}
0_p & \cdots & 0_p & -P_m \\
I_p & \cdots & 0_p & -P_{m-1} \\
\vdots & \ddots & \vdots & \vdots \\
0_p & \cdots & I_p & -P_1
\end{bmatrix}
\]

is the so-called companion matrix of \( -P(z) \). By denoting the following matrices

\[
C_i := -C^i \cdot [I_p \ 0_p \ \cdots \ 0_p]^T, \quad i = 1, 2, \ldots,
\]

\[
D := [C_m \ C_{m+1} \ \cdots \ C_k],
\]

the above equality implies

\[
\begin{bmatrix}
\mathcal{H}_{m-1}(R_-) & 0_{mp \times (k+1-m)p} \\
0_{(k+1-m)p \times mp} & 0_{(k+1-m)p}
\end{bmatrix}
= \begin{bmatrix}
I_{mp} & D \\
D^* & I_{(k+1-m)p}
\end{bmatrix}
\mathcal{H}_k(R_-)
\begin{bmatrix}
I_{mp} & D \\
D^* & I_{(k+1-m)p}
\end{bmatrix}
\]

(3.8)

for \( k = m, m + 1, \ldots \). Thus

\[
\delta(-\mathcal{H}_{m-1}(R)) = \delta(\mathcal{H}_\infty(R_-)) = I_{\infty}^+ R_-(z)
\]

\[
= \sum_{j=1}^r \delta(E_j) = \sum_{j=1}^r \text{rank } E_j,
\]

where the 1st and 2nd equality is due to the formula (3.8) and Lemma 3.16, respectively. This series of equalities, coupled with (3.7), yields that \( \mathcal{H}_{m-1}(R) \prec 0 \). Then it follows from Proposition 3.14 that \( |\deg Q - \deg P| \leq 1 \).
Proposition 3.18. With the notation of Theorem 3.17, if the statements (i) and (iii) in Theorem 3.17 are simultaneously true, then \( R(z) \) can be represented in the form (2.5) and, moreover,

\[
E_j = -l_j Q(\lambda_j) \frac{(\text{adj } P)^{(l_j-1)}(\lambda_j)}{(\det P)^{(l_j)}(\lambda_j)}.
\]

Additionally if the statement (ii) in Theorem 3.17 holds as well, then \( \text{rank } E_j = l_j \).

It should be pointed out that the converse statement of Theorem 3.17 is not true in general, as can be seen via the following counterexample.

Example 3.19. Given two \( 2 \times 2 \) matrix polynomials

\[
P(z) := \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix}, \quad Q(z) := \begin{bmatrix} 4z - 2 & -z + 4 \\ -z - 1 & -z - 2 \end{bmatrix},
\]

the \( 2 \times 2 \) rational matrix function \( R(z) := \frac{Q(z)}{P(z)} \) can be represented as

\[
R(z) = \begin{bmatrix} 4 & -1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \frac{1 - z}{1 - z} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \frac{-1 - z}{1 - z} = \begin{bmatrix} 4 & -1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{z} + \cdots.
\]

It is rapidly checked that

- \( \mathcal{H}_0(R) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \) is negative definite;
- \( R|_{\mathbb{C}_+} \) is a Herglotz-Nevanlinna matrix function;
- \( Q(z) \) and \( P(z) \) are right coprime;
- the nullity of \( P(1) \) and \( P(-1) \) are both 1, i.e., \( P(z) \) is simple.

However, \( Q(z) \) and \( P(z) \) have a common zero 1.

On the other hand, this example provide a situation whenever the negative definiteness of \( \mathcal{H}_{\deg P-1}(R) \) and the coprimeness of \( Q(z) \) and \( P(z) \) simultaneously hold. So a natural question arises:

(Q1) May this relation happen by accident or in certain general way?

Our discussion to this question will be posed in next subsection.

3.4. Generalization to the “if” implication of Grommer theorem. To deal with the question (Q1), we begin with invoking the Anderson-Jury Bezoutiants associated with a quadruple of matrix polynomials.
Definition 3.20. Given a quadruple of $p \times p$ matrix polynomials $L(z), \tilde{L}(z), M(z)$ and $\tilde{M}(z)$ satisfying
\[
\tilde{M}(z)\tilde{L}(z) = M(z)L(z), \tag{3.9}
\]
the associated Anderson-Jury Bezoutian matrix $B_{\tilde{M},M}(L,\tilde{L})$ is defined via the formula
\[
[I_p \ z I_p \ \cdots \ z^{n_1-1} I_p] \cdot B_{\tilde{M},M}(L,\tilde{L}) \cdot \begin{bmatrix}
I_p \\
u I_p \\
\vdots \\
u^{n_2-1} I_p
\end{bmatrix} = \frac{1}{z-u} \left( \tilde{M}(z)\tilde{L}(u) - M(z)L(u) \right),
\]
where $n_1 := \max\{\deg M, \deg \tilde{M}\}$ and $n_2 := \max\{\deg L, \deg \tilde{L}\}$.

The Anderson-Jury Bezoutian matrix $B_{\tilde{M},M}(L,\tilde{L})$ is skew-symmetric in the sense that
\[ B_{\tilde{M},M}(L,\tilde{L}) = -B_{M,\tilde{M}}(\tilde{L},L). \]

For commuting $L(z)$ and $\tilde{L}(z)$, i.e., when $L(z)\tilde{L}(z) = \tilde{L}(z)L(z)$, it is natural to choose $\tilde{M}(z) = L(z)$ and $M(z) = \tilde{L}(z)$. For a nontrivial choice of $\tilde{M}(z)$ and $M(z)$ in the general non-commutative case, we refer the reader to the construction of the common multiples via spectral theory of matrix polynomials: see [17, Theorem 9.11] for the monic case and [16, Theorem 2.2] for the comonic case. For more comprehensive study of Anderson-Jury Bezoutians, we refer the reader to [4, 34, 9.11] for the monic case and [16, Theorem 2.2] for the comonic case. Here we only provide two interesting results for our need.

Proposition 3.21. [34, Theorem 0.2] Let $L(z), \tilde{L}(z), M(z)$ and $\tilde{M}(z)$ be a quadruple of $p \times p$ regular matrix polynomials such that (3.9) holds. Then the nullity of $B_{\tilde{M},M}(L,\tilde{L})$ is equal to the degree of the scalar polynomial $\det L_0(z)$, where $L_0(z)$ is a GRCD of $L(z)$ and $\tilde{L}(z)$.

Proposition 3.22. [4, Lemma 2.2] Let $N_R(z)$, $D_R(z)$, $N_L(z)$ and $D_L(z)$ be a quadruple of $p \times p$ regular matrix polynomials such that $\deg D_R > \deg N_R$, $\deg D_L > \deg N_L$ and, for all large enough $z \in \mathbb{C}$,
\[
(D_L(z))^{-1}N_L(z) = N_R(z)(D_R(z))^{-1} = \sum_{k=0}^{\infty} z^{-(k+1)} s_k.
\]

If $D_L(z)$ and $D_R(z)$ are written in the form
\[
D_L(z) = \sum_{k=0}^{m_L} D_{L,m_L-k} z^k \quad \text{and} \quad D_R(z) = \sum_{k=0}^{m_R} D_{R,m_R-k} z^k,
\]
then
\[ \mathbf{B}_{D_L,N_L}(D_R, N_R) = \begin{bmatrix} D_{L,m_L-1} & \cdots & D_{L,0} \\ \vdots & \ddots & \vdots \\ D_{L,0} & \cdots & \cdots \\ \end{bmatrix} \cdot \begin{bmatrix} s_0 & \cdots & s_{m_R-1} \\ \vdots & \ddots & \vdots \\ s_{m_L-1} & \cdots & s_{m_R+m_L-2} \\ \end{bmatrix} \cdot \begin{bmatrix} \mathbf{D}_{R,m_R-1} & \cdots & \mathbf{D}_{R,0} \\ \vdots & \ddots & \vdots \\ \mathbf{D}_{R,0} & \cdots & \cdots \\ \end{bmatrix}. \]

Returning to the question (Q1), now we can give an answer:

**Theorem 3.23.** Let \( R(z) \) be a \( p \times p \) Hermitian rational matrix function with the matrix fraction (2.4). If \( \deg Q - \deg P \leq 1 \) and \( \mathcal{H}_{\deg P-1}(R) \prec 0 \), then

(i) \( R(z) \) is a Herglotz-Nevanlinna matrix function;
(ii) \( Q(z) \) and \( P(z) \) are right coprime;
(iii) \( P(z) \) is simple;
(iv) \( |\deg Q - \deg P| \leq 1 \).

**Proof.** Suppose that \( R(z) \) admits the Laurent expansion (3.5) and \( m := \deg P(z) \). Let \( (\mathbf{s}_j)_{j=0}^{2m} \) be a \( p \times p \) matrix sequence given by

\[
\mathbf{s}_j := \begin{cases} 
-\mathbf{s}_j, & j = 0, \ldots, 2m - 1, \\
I + [A_k^*, \ldots, A_1^*]H_{k-1}(R) & \\
+ \sum_{j=0}^{k-1} (A_{k-j}^* \mathbf{s}_{k+j} + \mathbf{s}_{k+j} A_{k-j}), & j = 2m.
\end{cases}
\]

It is not difficult to calculate that

\[
W^* [\mathbf{s}_{j+k}]_{j,k=0}^{m} W = \begin{bmatrix} -H_{m-1}(R) \\ I_p \end{bmatrix},
\]

where \( W := \begin{bmatrix} I_p & A_k \\ \vdots & \vdots \\ I_p & A_1 \\ \vdots & \vdots \\ I_p & A_{m-1} \end{bmatrix} \). Since \( \mathcal{H}_{m-1}(R) \) is negative definite, the above equation means that \( [\mathbf{s}_{j+k}]_{j,k=0}^{m} \) is positive definite. In view of the solvability of truncated matricial Hamburger moment problems (see e.g. [26, 10, 8, 23]), there exists at least a \( p \times p \) positive semi-definite Borel matrix measure \( \Omega \) on \( \mathbb{R} \) such that

\[
\int_{\mathbb{R}} w^j d\Omega(u) = \mathbf{s}_j, \quad j = 0, \ldots, 2m.
\]

Suppose that \( (P_k(z))_{k=0}^{m} \) is a sequence of orthogonal matrix polynomials with respect to \( \Omega \). A combination of (3.10), (3.6) and Proposition 3.10 shows that \( P(z) \)
coincides with $P_m^\nu(z)$. Due to Proposition 3.11, $P(z)$ is simple and all zeros of $P(z)$ are real. The latter means that $P(z)$ is invertible on $\mathbb{C}_+$, i.e., $R(z)$ is analytic on $\mathbb{C}_+$.

Suppose that $C(z)$ and $\tilde{Q}(z)$ are, respectively, the right quotient and right remainder of $Q(z)$ on division by $P(z)$. Then $C(z) = C^\nu(z)$ and, by denoting that $\tilde{R}(z) := \frac{\tilde{Q}(z)}{P(z)}$, $R(z)$ is a strictly proper Hermitian rational matrix function. Since $\deg C = \deg Q - \deg P \leq 1$, $C(z)$ is a matrix pencil, i.e., $C(z) := Az + B$, where $A = A^*$ and $B = B^*$. With the help of the equation (2.2),

$$A = \angle \lim_{z \to \infty} \frac{R(z)}{z} \geq 0. \quad (3.11)$$

Then

$$P(z)^* \frac{R(z) - R(z)^*}{z - \bar{z}} P(z) = P(z)^* A P(z) = P^\nu(\bar{z}) \tilde{Q}(z) - \tilde{Q}^\nu(\bar{z}) P(z) \quad (3.12)$$

From Proposition 3.22 and the formula $R(z) = C(z) + \tilde{R}(z)$, one can deduce that

$$B_{P^\nu, \tilde{Q}^\nu}(P, \tilde{Q}) = \begin{bmatrix} D_{m-1}^* & \cdots & D_0^* \\ \vdots & \ddots & \vdots \\ D_0^* & \cdots & \vdots \end{bmatrix} \mathcal{H}_{m-1}(R) \begin{bmatrix} D_{m-1} & \cdots & D_0 \\ \vdots & \ddots & \vdots \\ D_0 & \cdots & \vdots \end{bmatrix}, \quad (3.13)$$

where $P(z)$ is written in the form $P(z) = \sum_{k=0}^m D_{m-k} z^k$. On one hand, by combining the formulas (3.11)–(3.13), we have

$$P(z)^* \frac{R(z) - R(z)^*}{z - \bar{z}} P(z) \succ 0, \quad \forall z \in \mathbb{C}_+,$$

that is, $R|_{\mathbb{C}_+}$ is a Herglotz-Nevanlinna matrix function. On the other hand, it follows from (3.13) and Proposition 3.21 that $P(z)$ and $\tilde{Q}(z)$ are right coprime. Since

$$\begin{bmatrix} P(z) \\ \tilde{Q}(z) \end{bmatrix} = \begin{bmatrix} I_p & 0_p \\ A z + B & I_p \end{bmatrix} \begin{bmatrix} \tilde{Q}(z) \\ P(z) \end{bmatrix},$$

by Proposition A.5 of [39], $P(z)$ and $Q(z)$ are right coprime as well. Then it follows from Proposition 3.14 that $|\deg Q - \deg P| \leq 1$.

The failure to establish the converse statement in Theorem 3.23 can be illustrated by Example 3.5. In this example, the nullity of $P(0)$ is 2 which means that $P(z)$ is simple, while $Q(z)$ and $P(z)$ share a unique common zero 0.
4. Stability criterion via Herglotz–Nevanlinna matrix functions

Definition 4.1. A $p \times p$ matrix polynomial $F(z)$ may be split into the even part $F_e(z)$ and the odd part $F_o(z)$ so that $F(z) = F_e(z^2) + zF_o(z^2)$. For $F(z)$ written as in (2.3), they are defined by

$$F_e(z) := \sum_{k=0}^{m} A_{2k} z^{m-k} \quad \text{and} \quad F_o(z) := \sum_{k=1}^{m} A_{2k-1} z^{m-k}$$

when $\deg F = 2m$, and by

$$F_e(z) := \sum_{k=0}^{m} A_{2k+1} z^{m-k} \quad \text{and} \quad F_o(z) := \sum_{k=0}^{m} A_{2k} z^{m-k}$$

when $\deg F = 2m + 1$.

Remark 4.2. For a monic $p \times p$ matrix polynomial of even (resp. odd) degree, its even (resp. odd) part is monic as well.

Theorem 4.4 of [39] can be rewritten as the following stability criterion of $F(z)$ in terms of the positive definiteness of block Hankel matrices related with a rational matrix function.

Lemma 4.3. Let $F(z)$ be a monic $p \times p$ matrix polynomial with the even part $F_e(z)$ and odd part $F_o(z)$.

(i) In the case when $\deg F = 2m$, Suppose that $R_F(z)$ is a $p \times p$ Hermitian rational matrix function defined by

$$R_F(z) := -\frac{F_o(z)}{F_e(z)} \quad z \in \mathbb{C} \setminus \sigma(F_e). \quad (4.1)$$

$F(z)$ is a Hurwitz matrix polynomial if and only if $\mathcal{H}_{m-1}(R_F) \prec 0$ and $\mathcal{H}_{1,m-1}(R_F) \succ 0$.

(ii) In the case when $\deg F = 2m + 1$, suppose that $F_e(z)$ is regular and $R_F(z)$ is the $p \times p$ Hermitian rational matrix function [41]. $F(z)$ is a Hurwitz matrix polynomial if and only if $\mathcal{H}_{-1,0}(R_F) \prec 0$, $\mathcal{H}_{m-1}(R_F) \prec 0$ and $\mathcal{H}_{1,m-1}(R_F) \succ 0$.

(iii) In the case when $\deg F = 2m + 1$, suppose that $R_F(z)$ is a $p \times p$ Hermitian rational matrix function defined by

$$R_F(z) := -\frac{F_e(z)}{zF_o(z)} \quad z \notin \sigma(F_o) \cup \{0\}. \quad (4.2)$$

$F(z)$ is a Hurwitz matrix polynomial if and only if $\mathcal{H}_m(R_F) \prec 0$ and $\mathcal{H}_{1,m-1}(R_F) \succ 0$.

We provide a connection between Hurwitz stable matrix polynomials of even degree and Herglotz–Nevanlinna matrix functions.
**Theorem 4.4.** Let $F(z)$ be a monic $p \times p$ matrix polynomial of even degree with the even part $F_e(z)$ and odd part $F_o(z)$. Assume that $R_F(z)$ is the Hermitian rational matrix function as (4.1).

(i) $F(z)$ is Hurwitz stable if the following statements are simultaneously true:
(a) $F_e(z)$ and $F_o(z)$ have no common zeros;
(b) $R_F(z)$ is a Herglotz-Nevanlinna matrix function;
(c) All zeros of $F_e(z)$ are negative real;
(d) $F_e(z)$ is simple.

(ii) Conversely, if $F(z)$ is Hurwitz stable, then the statements (b)–(d) hold and $F_e(z)$ and $F_o(z)$ are right coprime.

**Proof.** The proof for (i): Suppose that the statements (a)–(d) hold. Theorem 3.17 reveals that $H_{m-1}(R_F) \prec 0$. (4.3)

Proposition 2.2 shows that $R_F(z)$ obeys the form (2.5). By Lemma 6.3-11 of [25], $R_F(z)$ is strictly proper. So both coefficients $C$ and $D$ in the form (2.5) of $R_F(z)$ equal 0. By setting $R_{z,F}(z) := -zR_F(z) = zF_o(z)/F_e(z)$, $z \in \mathbb{C} \setminus \sigma(F_e)$, (4.4)

we have

$$R_{z,F}(z) = \sum_{j=1}^r \frac{-E_j z}{\lambda_j - z} = - \sum_{j=1}^r E_j + \sum_{j=1}^r \frac{-\lambda_j E_j}{\lambda_j - z}.$$ (4.5)

By the statement (c) and the fact that $\{\lambda_j\}_{j=1}^r \subseteq \sigma(F_e)$, $-\lambda_j > 0$. Using Proposition 2.2 again, we see that $R_{z,F}(z)$ is also a Herglotz-Nevanlinna matrix function. Moreover, $zF_o(z)$ and $F_e(z)$ share no common zeros due to (a) and (c). Applying Theorem 3.17 again we derive that

$$\mathcal{H}_{1,m-1}(R_F) = -\mathcal{H}_{m-1}(R_{z,F}) \succ 0.$$ (4.6)

Hence (i) of Theorem 4.3 (4.3) and (4.6) yield the assertion of Theorem 4.4.

The proof for (ii): Suppose that $F(z)$ is Hurwitz stable and $R_{z,F}(z)$ is a $p \times p$ rational matrix function as in (4.4). Then due to (i) of Lemma 4.3, both (4.3) and (4.6) hold. From Theorem 3.23, one can obtain the statements (b) and (d), the coprimeness of $F_e(z)$ and $F_o(z)$ and that $R_{z,F}(z)$ is a Nevanlinna-Herglotz matrix function.

Assume that there exists a zero $\lambda_j$ of $F_e(z)$ with multiplicity $l_j$ such that $\lambda_j \geq 0$. By the representation (4.5) and Proposition 2.2 $R_{z,F}(z)$ cannot be a Nevanlinna-Herglotz matrix function, which induces a contradiction.

□

**Theorem 4.5.** Let $F(z)$ be a monic $p \times p$ matrix polynomial of odd degree with the even part $F_e(z)$ and odd part $F_o(z)$. Assume that $R_F(z)$ is the Hermitian rational matrix function as (4.2).
(i) $F(z)$ is Hurwitz stable if the following statements are simultaneously true:
(a) $F_e(z)$ and $F_o(z)$ have no common zeros and $0$ is not a zero of $F_e(z)$;
(b) All zeros of $F_o(z)$ are negative real;
(c) $R_F(z)$ is a Hermitian rational matrix function;
(d) $F_o(z)$ is simple.

(ii) Conversely, if $F(z)$ is Hurwitz stable, then the following statements, together with (b)–(d), are true:
(e) $F_e(z)$ and $F_o(z)$ are right coprime;
(f) $F_e(z)$ and $zF_o(z)$ are right coprime.

Proof. The proof can be conducted analogously to that of Theorem 4.5, in placing the matrix function $R_F(z)$ as (4.1) with the one as (4.2): A sufficiency for Hurwitz stability of $F(z)$ can be derived from the statement (c) and

(g) $F_e(z)$ and $zF_o(z)$ have no common zeros;
(h) All zeros of $zF_o(z)$ except $0$ are negative real and $zF_o(z)$ is simple.

A necessity for Hurwitz stability of $F(z)$ can be derived from the statements (c) and (e)–(h).

Then the assertion of the theorem is obvious, since (g) is equivalent to (a), and (h) is equivalent to (b) and (d).

Theorem 4.6. Let $F(z)$ be a monic $p \times p$ matrix polynomial with the even part $F_e(z)$ and odd part $F_o(z)$. Assume that $F_e(z)$ is regular when $\text{deg} F$ is odd and $R_F(z)$ is the Hermitian rational matrix function as (4.1).

(i) $F(z)$ is Hurwitz stable if the statements (a)–(d) in Theorem 4.4 simultaneously hold and, moreover,

(e) the limit
\[
\lim_{z \to \infty} R_F(z) < 0
\]

when $\text{deg} F$ is odd.

(ii) Conversely, suppose that $F(z)$ is Hurwitz stable. The statements (b)–(d) in Theorem 4.4 coupled with (e) are true. Moreover, $F_e(z)$ and $F_o(z)$ are right coprime.

Proof. The case that $\text{deg} F$ is even reduces to Theorem 4.5, while the proof for odd case has a few differences with that of Theorem 4.5.

The proof for (i): Suppose that $\text{deg} F$ is odd and the statements (a)–(c) hold. Theorem 3.17 reveals (4.3).

Proposition 2.2 shows that $R_F(z)$ obeys the form (2.5). So $C = 0_p$ and $D = \lim_{z \to \infty} R_F(z) < 0$. We set $R_{z,F}(z)$ as in (4.4). Thus
\[
R_{z,F}(z) = -Dz + \sum_{j=1}^{r} \frac{-E_j z}{\lambda_j - z} = -Dz - \sum_{j=1}^{r} E_j + \sum_{j=1}^{r} \frac{-\lambda_j E_j}{\lambda_j - z}.
\]
With the help of the statement (c), Proposition 2.2, (4.7) and the fact that \( \{\lambda_j\}_{j=1}^r \subseteq \sigma(F_e) \), we see that \(-\lambda_j > 0 \) and \( R_{z,F}(z) \) is also a Herglotz-Nevanlinna matrix function. Moreover, \(-zF_o(z) \) and \( F_e(z) \) share no common zeros due to (a) and (c). Applying Theorem 3.17 again we derive (4.6).

Hence Theorem 4.3, (4.3) and (4.6) yield the assertion of Theorem 4.4.

The proof for (ii): Suppose that \( F_e(z) \) is Hurwitz stable and \( R_{z,F}(z) \) is a \( p \times p \) rational matrix function as in (4.4). Then due to (i) of Lemma 4.3, both (4.3) and (4.6) hold. From Proposition 3.23, one can obtain the statements (b) and (d), the coprimeness of \( F_e(z) \) and \( F_o(z) \) and that \( R_{z,F}(z) \) is a Nevanlinna-Herglotz matrix function.

Assume that there exists a zero \( \lambda_j \) of \( F_e(z) \) with multiplicity \( l_j \) such that \( \lambda_j \geq 0 \). By the representation (4.5) and Proposition 2.2, \( R_{z,F}(z) \) cannot be a Nevanlinna-Herglotz matrix function, which induces a contradiction. \( \square \)

The converse statement of (i) (resp. (ii)) in Theorems 4.4–4.6, slightly different from the statement (ii) (resp. (i)), are however not true in general. This fact can be illustrated by the following counterexamples.

**Example 4.7.** Let a monic \( 2 \times 2 \) matrix polynomial of degree 3
\[
F(z) := \begin{bmatrix}
z^3 + 205z^2 + 1006z + 1200 & 0 \\
0 & z^3 + \frac{1011}{205} z^2 + \frac{240}{41} z + \frac{304}{205}
\end{bmatrix}.
\]

By calculation, all the zeros of \( F(z) \) are \(-200\), \(-3\), \(-2\), \(-4\), \(-\frac{76}{205}\) and \(-1\), which shows that \( F(z) \) is a Hurwitz stable matrix polynomial.

On the other hand, the odd part
\[
F_o(z) = \begin{bmatrix}
z + 1006 & 0 \\
0 & z + \frac{240}{41}
\end{bmatrix}
\]
and even part
\[
F_e(z) = \begin{bmatrix}
205z + 1200 & 0 \\
0 & \frac{1011}{205} z + \frac{304}{205}
\end{bmatrix}
\]
are right coprime, so are \( F_e(z) \) and \( zF_o(z) \). Moreover, \( F_o(z) \) has two negative real zero \(-1006\) and \(-\frac{240}{41}\), and the nullity of \( F_o(-1006) \) and \( F_o(-\frac{240}{41}) \) both equals 1 so that \( F_o(z) \) is simple. The rational matrix function
\[
R_F(z) := \frac{-F_e(z)}{zF_o(z)} = \frac{600}{503} \begin{bmatrix} 0 & 0 \\ 19 & 75 \end{bmatrix} + \frac{102515}{3075} \begin{bmatrix} 0 & 0 \\ -1006 & -z \end{bmatrix} + \frac{0}{-\frac{240}{41} - z}
\]
is a Herglotz-Nevanlinna matrix function.

However, \( F_e(z) \) and \( F_o(z) \) share a common zero \(-\frac{240}{41}\).

**Example 4.8.** Let
\[
F(z) := \begin{bmatrix}
z^4 - 4z^3 + 3z^2 - 5z + 2 & z^3 + 2z \\
z^3 + 2z & z^4 - 3z^3 + 3z^2 - 4z + 2
\end{bmatrix}.
\]
The even part
\[ F_e(z) = \begin{bmatrix} z^2 + 3z + 2 & 0 \\ 0 & z^2 + 3z + 2 \end{bmatrix} \]
and the odd part
\[ F_o(z) = \begin{bmatrix} -4z - 5 & z + 2 \\ z + 2 & -3z - 4 \end{bmatrix} \]
are right coprime. \( F_e(z) \) has two negative real zero \(-1\) and \(-2\), where the former is a shared zero of \( F_o(z) \). The nullity of \( F_e(-1) \) and \( F_e(-2) \) both equals 2, which means that \( F_e(z) \) is simple. And the rational matrix function
\[
R_F(z) := -\frac{F_o(z)}{F_e(z)} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - \frac{1}{2 - z} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]
is a Herglotz-Nevanlinna matrix function.

However, \( \pm i \) are two zeros of \( F(z) \), revealing that \( F(z) \) is not a Hurwitz stable matrix polynomial.

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