The $\Box_b$ Heat Equation and Multipliers via the Wave Equation

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Abstract

Recently, Nagel and Stein studied the $\Box_b$-heat equation, where $\Box_b$ is the Kohn Laplacian on the boundary of a weakly-pseudoconvex domain of finite type in $\mathbb{C}^2$. They showed that the Schwartz kernel of $e^{-t\Box_b}$ satisfies good “off-diagonal” estimates, while that of $e^{-t\Box_b} - \pi$ satisfies good “on-diagonal” estimates, where $\pi$ denotes the Szegö projection. We offer a simple proof of these results, which easily generalizes to other, similar situations. Our methods involve adapting the well-known relationship between the heat equation and the finite propagation speed of the wave equation to this situation. In addition, we apply these methods to study multipliers of the form $m(\Box_b)$. In particular, we show that $m(\Box_b)$ is an NIS operator, where $m$ satisfies an appropriate Mihlin-Hörmander condition.

1 Introduction

In [NS01b], Nagel and Stein study the heat operator $e^{-s\Box_b}$, where $\Box_b$ is the Kohn Laplacian (acting on functions) on the boundary $M$, of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$ (or with $\Box_b$ on a polynomial model domain in $\mathbb{C}^2$). Let $\pi$ be the Szegö projection, $e^{-s\Box_b} = (1 - \pi)e^{-s\Box_b}$, and for any operator $T$, let $K_T$ denote the Schwartz kernel of $T$. The bounds in [NS01b] were in terms of a Carnot-Carathéodory distance $\rho$ on $M$ (see Section 2.1). In [NS01b] it is shown that:

$$|K_{e^{-s\Box_b}}(x,y)| \lesssim \frac{1}{V(x, \rho(x,y))} \left( \frac{s^N}{s^N + \rho(x,y)^{2N}} \right)$$

(1)

for every $N > 0$, and,

$$|K_{e^{-s\Box_b}}(x,y)| \lesssim \frac{1}{V(x, \rho(x,y) \vee \sqrt{s})}$$

(2)

with appropriate estimates for the derivatives in each variable as well (see Theorem 2.3). Here, $V(x,\delta)$ denotes the volume of the ball of radius $\delta$ in the $\rho$ metric, centered at $x$. In an unpublished result of Nagel and Müller, using the
methods of [JSC86] the bounds in (1) are improved to:
\[ |K_{e^{-\Box_b}}(x,y)| \lesssim \frac{1}{V(x,\rho(x,y))} e^{-c\rho(x,y)^2} \] (3)
for some \( c > 0 \). The main insight of [NS01b] was that one needs to prove off-diagonal estimates (ie, (1),(3)) for \( e^{-\Box_b} \) and on diagonal estimates (ie, (2)) for \( e^{-s\Box_b} \). The main goal of this paper is to reprove these results, keeping this insight in mind, using well-known methods for the classical heat equation as can be found, for instance, in [Sik04].

The novelty of our approach is that we shall use only two estimates specific to \( \Box_b \). Namely,

- There is a relative fundamental solution \( \Box_b^{-1} \) (ie, \( \Box_b\Box_b^{-1} = \Box_b^{-1}\Box_b = 1-\pi, \pi\Box_b^{-1} = 0 = \Box_b^{-1}\pi \)) which is an NIS operator of order 2 (see Definition 3.3 and Theorem 3.4 for the related estimate).
- The \( \Box_b \) wave equation has finite propagation speed (see Theorem 2.3).

and the rest of the proofs follows completely formally, from a modified version of the proofs in [Sik04]. In particular, we do not need any of the new bounds that were used in [NS01b].

That \( \Box_b^{-1} \) is an NIS operator of order 2 is well known (see [NRSW89, CNS92]), while the finite propagation speed is a result of Melrose [Mel86]. Because of this, essentially no new estimates need to be proven to achieve the main results of this paper: all of the work is completely formal use of the spectral theorem.

In fact, one may consider the methods of this paper as a (quite simple) generalization of the methods in [Sik04] where the heat equation \( e^{-t\mathcal{L}} \) is studied for some positive semi-definite operator \( \mathcal{L} \), whose wave equation has finite speed of propagation. The methods in this paper allow one to consider the case when the \( L^2 \) kernel of \( \mathcal{L} \) is non-trivial and the Schwartz kernel of the orthogonal projection onto the \( L^2 \) kernel of \( \mathcal{L} \) satisfies appropriate estimates (see Example 8.1).

After we study the \( \Box_b \) heat equation, in Section 7 we turn to studying multipliers \( m(\Box_b) \). We show (using essentially the same methods that we use for the heat equation) that \( m(\Box_b) \) is an NIS operator of order 0, provided \( m \) satisfies an appropriate Mihlin-Hörmander condition (see Theorem 2.7). In addition, we prove \( m(\Box_b) \) is bounded on \( L^p \) for \( m \) that satisfy only a finite level of smoothness (see Theorem 2.8).

Finally, since we use only the two above basic assumptions on \( \Box_b \), in Section 8 we present a few other examples where the same methods yield analogous results. We hope this will convince the reader that these methods are easily adapted to other situations.

\footnote{Here, “trivial” does not necessarily mean 0 dimensional. It could, for instance when working on a compact manifold, mean a finite dimensional space of smooth functions.}
All of the results in this paper are well known when \( \Box_b \) is replaced by, for instance, the sublaplacian on a compact manifold, defined in terms of vector fields \( X_1, \ldots, X_m \) which satisfy Hörmander’s condition, or the sublaplacian on a stratified group. See Examples 8.4 and 8.5, and the references there.

2 Setup, Notation, and Statement of Results

Despite the fact that the methods of this paper work in more general situations (see Section 8, in particular, Examples 8.1 and 8.2) it seems difficult to devise an appropriate abstract setting in which the entirety of this paper will go through, without needlessly complicating matters (in particular, the obvious generalization (Example 8.4) doesn’t contain Example 8.2). Because of this, we prove our results, in detail, in the simplest setting (as discussed in the introduction) and mention a few other settings in which these methods work, with only minor modifications, in Section 8. Throughout the paper we will use \( A \lesssim B \) to denote \( A \leq CB \) where \( C \) is a constant independent of any relevant parameters.

Let \( M \subset \mathbb{C}^2 \) be a \( C^\infty \) pseudoconvex hypersurface and assume that \( M \) is the boundary of a bounded pseudoconvex domain \( \Omega \subset \mathbb{C}^2 \) of finite type \( m \). Let \( \Box_b \) denote the Kohn-Laplacian acting on functions in \( C^\infty (M) \). Let \( \pi : L^2(M) \to L^2(M) \) be the orthogonal projection on to the \( L^2 \) kernel of \( \Box_b \) (ie, the Szegö projection).

We write \( L^p_s(M) \) for the usual (isotropic) \( L^p \) Sobolev spaces of order \( s \). Also, if \( \zeta, \zeta' \in C_c^\infty (M) \) we write \( \zeta \prec \zeta' \) if \( \zeta' \equiv 1 \) on the support of \( \zeta \). Finally, for an operator \( T : C^\infty_0(M) \to C^\infty_0(M') \), we write \( K_T(x,y) \in C^\infty_0(M \times M') \) for the Schwartz kernel of the operator \( T \).

2.1 Geometry of \( M \)

Choose real vector fields \( X_1, X_2 \) so that we can identify \( \overline{\partial}_b f \) with \( (X_1 + iX_2) f \) by identifying functions and \((0,1)\) forms, see [NS01b] for details. That \( \Omega \) is of finite type \( m \) means that \( X_1, X_2 \) and their commutators up to order \( m \) span the tangent space \( TM \) at each point (ie, \( X_1 \) and \( X_2 \) satisfy Hörmander’s condition).

There is a natural metric defined in terms of these vector fields, called the control metric, or Carnot-Carathéodory metric, and is defined by:

\[
\rho (x,y) = \inf \left\{ T > 0 : \exists \gamma : [0,T] \to M, \ \gamma \text{ piecewise } C^1, \gamma(0) = x, \gamma(T) = y, \gamma'(t) = c_1(t)X_1 + c_2(t)X_2 \text{ for a.e. } t, \text{ and } |c_1(t)|^2 + |c_2(t)|^2 \leq 1 \right\}
\]

We define \( B(x,\delta) = \{ y \in M : \rho(x,y) < \delta \} \) and let \( V(x,\delta) \) denote the volume of \( B(x,\delta) \). The following result is contained in [NSW85].
Proposition 2.1. There is a $Q$ such that $V(x, \gamma \delta) \lesssim \gamma^{Q} V(x, \delta)$ for all $\gamma \geq 1$. For the rest of the paper, $Q$ will denote this number. In addition, there is a $q > 0$ and $\delta_0 > 0$ such that for all $\delta \leq \delta_0$ and all $\gamma < 1$, we have $V(x, \gamma \delta) \lesssim \gamma^{q} V(x, \delta)$.

Remark 2.2. It is not hard to see, from the results of [NSW85], that $Q = m + 2$, where $M$ was of finite type $m$.

We write $D = (X_1, X_2)$ and use order multi-index notation: $D^\alpha$ where $\alpha$ is a sequences of 1s and 2s, and $|\alpha|$ denotes the length of that sequence. So that, for instance, $D^{(1, 2, 1, 1)} = X_1 X_2 X_1 X_1$ and $|(1, 2, 1, 1)| = 4$.

2.2 The Operator $\Box_b$

As in [NS01b], we have identified $\partial_b$ with a linear first order partial differential operator (by identifying functions and $(0, 1)$ forms), and $\partial^*_b$ with its adjoint (also a linear first order partial differential operator). Then, as in [NS01b] we may define $\Box_b = \partial_b \partial^*_b$, and see that with an appropriate definition of domain, $\Box_b$ is a self-adjoint operator (we refer the reader to [NS01b] for the details of the Hilbert space theory).

Since $\Box_b$ is a self-adjoint operator, it admits a spectral decomposition $E(\lambda)$; so that, in particular, $\pi = E(0)$. Hence, for any bounded Borel measurable function $F : [0, \infty) \to \mathbb{C}$, we may define:

$$F(\Box_b) = \int_{[0, \infty)} F(\lambda) dE(\lambda)$$

and with an abuse of notation, we define:

$$F(\Box_b) = \int_{[0, \infty)} F(\lambda) dE(\lambda)$$

So that

$$F(\Box_b) = (1 - \pi) F(\Box_b) = F(\Box_b) - F(0) \pi$$

2.3 Statement of Results

Theorem 2.3 ([Mel86]). There exists a constant $\kappa > 0$ such that:

$$\text{supp} \left( K_{\cos(t \sqrt{\kappa})} \right) \subseteq \{(x, y) : \rho(x, y) \leq \kappa t\}$$

See Section 2 for a discussion of this result and for another proof.

We will fix the constant $\kappa$ as in Theorem 2.3 and all of our other results will be in terms of this $\kappa$. Our main results are now as follows:

As in [NS01b], we study the operators $e^{-t\Box_b}$ and $e^{-t\Box_b}$ which satisfy:

$$e^{-t\Box_b} = e^{-t\Box_b} + \pi$$

(4)

We have:
Theorem 2.4. \( K_{e^{-t\Box_b}} \in C^\infty((0,\infty) \times M \times M) \). Moreover, for every integer \( j \) and ordered multi-indices \( \alpha \) and \( \beta \), there is a constant \( C = C(\alpha, \beta, j) \) such that:

\[
\left| D_x^\alpha D_y^\beta \partial_t^j K_{e^{-t\Box_b}} (x, y) \right| \leq C \frac{(\rho(x, y) \vee \sqrt{t})^{-2j-|\alpha|-|\beta|}}{V(x, \rho(x, y) \vee \sqrt{t})}
\]

and

\[
\left| D_x^\alpha D_y^\beta \partial_t^j K_{e^{-t\Box_b}} (x, y) \right| \leq \begin{cases} 
CV(x, \rho(x, y))^{-1} \left( \frac{\rho(x, y)}{t} \right)^{|\alpha|+|\beta|+2j} \left( \frac{\rho(x, y)^2}{t} \right)^{\frac{4-j}{2}} e^{-\frac{\rho(x, y)^2}{4t}} & \text{if } t < \frac{\rho(x, y)^2}{c_1}, \\
CV(x, \rho(x, y))^{-1} \rho(x, y)^{-2j-|\alpha|-|\beta|} & \text{if } t \geq \frac{\rho(x, y)^2}{c_1}
\end{cases}
\]

Corollary 2.5. Fix \( c < \frac{1}{4\kappa^2} \). Then, for \( 0 < t < \frac{\kappa(x, y)^2}{c} \), we have:

\[
\left| D_x^\alpha D_y^\beta \partial_t^j K_{e^{-t\Box_b}} (x, y) \right| \leq V(x, \rho(x, y))^{-1} \rho(x, y)^{-2j-|\alpha|-|\beta|} e^{-\frac{\kappa(x, y)^2}{t}}
\]

Theorem 2.6. \((\text{NS01b})\). \( e^{-t\Box_b} \) and \( e^{-t\Box_b} \) are NIS operators of order 0 uniformly in \( t > 0 \). We offer a new proof of this result. See Definition [3,7] for the definition of NIS operators.

Theorem 2.7. Suppose \( m : [0, \infty) \to \mathbb{C} \) and \( m \big|_{(0,\infty)} \) satisfies a Mihlin-Hörmander condition of the form:

\[
| (\lambda \partial \lambda)^a m(\lambda) | \leq C_a
\]

for every \( a > 0 \), then \( m(\Box_b) \) is an NIS operator of order 0.

In light of Theorem 2.7, we see that \( m(\Box_b) \) is bounded on \( L^p \) \((1 < p < \infty)\). One expects that we do not need an infinite amount of smoothness for \( m \) to achieve this \( L^p \) boundedness. Indeed, fix \( \eta \in C_0^\infty \left( \left( \frac{1}{4}, 2 \right) \right) \), with \( \eta = 1 \) on a neighborhood of 1, and define:

\[
\| m \|_{L^2_{a, \text{loc}}} = \sup_{t \geq 0} \| \eta(\cdot) m(t \cdot) \|_{L^2(R)}
\]

Where \( \| \cdot \|_{L^2_{a, \text{loc}}} \) denotes the usual a \( L^2 \) Sobolev space. (One gets essentially the same norm with any non-trivial choice of \( \eta \in C_0^\infty \left( (0, \infty) \right) \). See [Chr91].)

Theorem 2.8. Suppose \( a > \frac{Q+1}{4} \), and that \( \| m \|_{L^2_{a, \text{loc}}} < \infty \). Then, \( m(\Box_b) \) is bounded on \( L^p \), \( 1 < p < \infty \).

Remark 2.9. Note that, by the Sobolev embedding theorem, \( m|_{(0,\infty)} \) in the statement of Theorem 2.8 is equal to a continuous function. \( m(\Box_b) \) is defined in terms of this continuous version of \( m \). That is, we are not allowed to change \( m \) on a set of measure 0.

For results similar to Theorem 2.8 see [Ale94] and references therein. Indeed, the methods in that reference are related to the methods in this paper.
Remark 2.10. Due to the results in [Chr91], one expects a stronger version of Theorem 2.8, with \( a > \frac{Q}{2} \). This does not seem to follow directly from our methods. This is due to the fact that all of the proofs we know of for multipliers that yield this sharper result take place on groups, and use a Plancheral type theorem, for which we do not seem to have a convenient analog.

3 Background

In this section, we review the theory of NIS operators. In addition, we discuss the main inequality that we will use throughout the paper.

NIS operators were first studied in [NRSW89] and the definition we use is from [Koe02].

Definition 3.1. Let \( T : C_0^\infty (M) \to C_0^\infty (M) \) be a linear operator, with Schwartz kernel \( K_T (x, y) \). We say \( T \) is an NIS operator, smoothing of order \( r \), if for all \( f \in C_0^\infty (M) \),

\[
\| \zeta f \|_{L^2(M)} \leq C \left( \| \zeta' f \|_{L^2(M)} + \| f \|_{L^2(M)} \right)
\]

1. For \( s \geq 0 \), there exist parameters \( a(s) < \infty \) and \( b < \infty \) such that if \( \zeta, \zeta' \in C_0^\infty (M) \) with \( \zeta \prec \zeta' \), then there exists \( C = C (s, \zeta, \zeta') \) such that

\[
\| \zeta' f \|_{L^2(M)} \leq C \left( \| \zeta f \|_{L^2(M)} + \| f \|_{L^2(M)} \right)
\]

2. There exist constants \( C_{a,\beta} \) such that for \( x \neq y \),

\[
| D_x^\alpha D_y^\beta K_T (x, y) | \leq C_{a,\beta} \frac{\rho (x, y)^{r-|\alpha|-|\beta|}}{\rho (x)}
\]

3. For each integer \( l \geq 0 \), there is an integer \( N = N (l) \geq 0 \) and a constant \( C = C (l) \) such that if \( \phi \in C_0^\infty (B (x, \delta)) \), then

\[
\sum_{|\alpha|=l} | D^\alpha T (\phi) (x) | \leq C \delta^{l-1} \sup_{y \in M} \sum_{|\beta| \leq N} \delta^{\beta} | D^\beta \phi (y) |
\]

4. The above conditions also hold for the adjoint operator \( T^* \).

The following results about NIS operators are well-known (see [Koe02, NS01b] and references therein):

- \( \pi \) is an NIS operator of order 0.
- There is a self-adjoint NIS operator \( \overline{\Box}_b^{-1} \) of order 2 such that \( \Box_b \overline{\Box}_b^{-1} = 1 - \pi = \overline{\Box}_b^{-1} \overline{\Box}_b \) and \( \pi \overline{\Box}_b^{-1} = 0 = \overline{\Box}_b^{-1} \pi \).
- If \( T \) is an NIS operator of order \( m \), then \( D^\alpha T \) and \( T D^\alpha \) are NIS operators of order \( m - \alpha \).
• NIS operators of order \( \geq 0 \) are bounded on \( L^p \) (\( 1 < p < \infty \)). For NIS operators of order > 0 this is related to the fact that \( M \) is compact.

• NIS operators of order 0 form an algebra.

• If \( T \) is an NIS operator of order 0, and \( \alpha \) is a fixed ordered multi-index, then there exist NIS operators \( T_\beta \) of order 0 such that:

\[
D^\alpha T = \sum_{|\beta| \leq |\alpha|} T_\beta D^\beta
\]

• If \( S \) is an NIS operator of order 0 and \( \alpha \) is a fixed ordered multi-index, then there exist NIS operators \( S_\beta \) of order 0 such that:

\[
SD^\alpha = \sum_{|\beta| = |\alpha|} D^\beta S_\beta
\]

Remark 3.2. In Examples 8.3 and 8.4 below, we actually have that:

\[
D^\alpha T = \sum_{|\beta| = |\alpha|} T_\beta D^\beta
\]

For NIS operators \( T \) of order 0.

Remark 3.3. Property 1 of Definition 3.1 was originally (in [NRSW89]) replaced with that there existed functions \( K_j^r \in C^\infty (M \times M) \) satisfying properties 2-4 uniformly such that \( K_j^r \rightarrow K_T \) in \( C^\infty_0 (M \times M) \). These two definitions (at least when, say, \( r = 0 \)) turn out to be equivalent, as was remarked to us by Ken Koenig. Indeed, it was shown [NS01b] that the identity satisfied both definitions. Let \( K_j \in C^\infty (M \times M) \) be an approximation of \( \delta_{x-y} \) satisfying properties 2-4 uniformly in \( j \). Let \( T_j \) be the operator with Schwartz kernel \( K_j \). Then, if \( S \) is an operator of order 0 in the sense of Definition 3.1, \( ST_j \) will be an appropriate smooth approximation. The other direction is easy.

Closely related to NIS operators is the main inequality that we shall use (it is essentially contained in Theorem 3.4.2 of [NS01b]):

**Theorem 3.4.** There is a constant \( R_0 > 0 \) such that for all \( R \leq R_0 \) and all \( f \in C^\infty (M) \) such that \( \pi f = 0 \), we have that for every \( \alpha \), there exists an \( L = L (\alpha) \) such that:

\[
\sup_{B(x,R)} |D^\alpha f| \lesssim V (x,R)^{-\frac{\alpha}{2}} \sum_{j=0}^L R^{2j-|\alpha|} \|\Box_b^j f\|_{L^2(M)}
\]

**Proof.** This theorem is closely tied to the fact that \( \Box_b^{-1} \) is an NIS operator of order 2. In fact, one way of showing that \( \Box_b^{-1} \) is an NIS operator of order 2 is to prove something like the above theorem. Conversely, assuming \( \Box_b^{-1} \) is an NIS operator of order 2, the above theorem follows. Indeed, the theorem is well
known for all $f \in C^\infty$ if $\Box_b$ is replaced by the sublaplacian $L := X_1^* X_1 + X_2^* X_2$, and is proven with standard scaling arguments. Now the theorem follows easily:

$$\sup_{B(x,R)} |D^\alpha f| \lesssim V(x,R)^{-\frac{j}{2}} \sum_{j=0}^L R^{2j-|\alpha|} \|L^j f\|_{L^2(M)}$$

But, $\|L^j f\|_{L^2(M)}$ is a linear combination of terms of the form $\|D^\beta f\|_{L^2(M)}$ where $|\beta| \leq 2j$. And we see, for $f \in C^\infty$ such that $\pi f = 0$:

$$\|D^\beta f\|_{L^2(M)} = \|D^\beta \Box_b^{-1} \Box_b f\|_{L^2(M)}$$

The result now follows by induction.

Remark 3.5. In Examples 8.3 and 8.5 the same proof works, since (due to Remark 3.2) only the $L^2$ boundedness of NIS operators of order 0 is needed. Moreover, in these cases we may take $R_0 = \infty$.

4 On Diagonal Bounds

In this section we present the bounds for $K_{e^{-t\Box_b}}$, which we call “on diagonal bounds,” due to the fact that they are analogous to the on diagonal bounds for the classical heat operator. This is all essentially contained in [NS01b], however we include it here as we will need some of the side results later on, and we wish to emphasize a particular approach, so as to make it clear how to generalize these results. We close the section with the main part of the proof of Theorem 2.6.

First, we note that since $\partial_t e^{-t\Box_b} = -\Box_b e^{-t\Box_b}$ and $\partial_t e^{-t\Box_b} = -\Box_b e^{-t\Box_b}$ and since $\Box_b$ is a polynomial of degree 2 in $X_1, X_2$ it follows that the results of Theorem 2.4 when $j \neq 0$ follow from the case when $j = 0$. For this reason, we focus only on the case $j = 0$.

Second, we note that the bounds in Theorem 2.4 for $K_{e^{-t\Box_b}}$ when $t \geq \frac{\rho(x,y)^2}{\kappa}$ follow from those for $K_{e^{-t\Box_b}}$, the fact that $\pi$ is an NIS operator of order 0, and (4). Similarly, the bounds in Theorem 2.4 for $K_{e^{-t\Box_b}}$ when $t < \frac{\rho(x,y)^2}{\kappa}$ follow from those for $K_{e^{-t\Box_b}}$, the fact that $\pi$ is an NIS operator of order 0, and (4). Hence, in this section, we are only concerned with the bounds for $K_{e^{-t\Box_b}}$ when $t \geq \frac{\rho(x,y)^2}{\kappa}$.

In this section, and in the rest of the paper, we will need some elementary inequalities that are essentially contained in [Sik04] (see Equations (2.7) and...
Proposition 2.1 and the fact that \( \sup \) Here, we have implicitly used that \( K \) this case we are concerned with, \( V \).

Lemma 4.1. Suppose \( S_1, S_2 : L^2(M) \to L^2(M) \). Fix an ordered multi-index \( \alpha \). Suppose that for some open set \( U \), \( D^\alpha_x K_{S_1 S_2}(x, y) \in L^1_\text{loc}(U \times M) \), and that \( \sup_{x \in U} \| D^\alpha_x K_{S_1}(x, \cdot) \|_{L^2(M)} < \infty \). Then, for \( x \in U \),

\[
\| D^\alpha_x K_{S_1 S_2}(x, \cdot) \|_{L^2(M)} \leq \| S_2 \|_{L^2(M)} \circ \| D^\alpha_x K_{S_1}(x, \cdot) \|_{L^2(M)}
\]

(5)

Furthermore, if instead we have two neighborhoods, \( U, V \subseteq M \), and if \( D^\alpha_x K_{S_1}(x, y), D^\beta_y K_{S_2}(x, y) \in L^1_\text{loc}(M \times M) \)

with

\[
\sup_{x \in U} \| D^\alpha_x K_{S_1}(x, \cdot) \|_{L^2(M)} + \sup_{y \in V} \| D^\beta_y K_{S_2}(\cdot, y) \|_{L^2(M)} < \infty
\]

then for \( x \in U, y \in V \),

\[
| D^\alpha_x D^\beta_y K_{S_1 S_2}(x, y) | \leq \| D^\alpha_x K_{S_1}(x, \cdot) \|_{L^2(M)} \circ \| D^\beta_y K_{S_2}(\cdot, y) \|_{L^2(M)}
\]

(6)

Remark 4.2. The main point of (6) is that:

\[
K_{S_1 S_2}(x, y) = \int_M K_{S_1}(x, z) K_{S_2}(z, y) \, dz
\]

Note that, by (6), and using the fact that \( e^{-t\Box_h} \) is self-adjoint:

\[
| D^\alpha_x D^\beta_y K_{e^{-t\Box_h}}(x, y) | \leq \| D^\alpha_x K_{e^{-t\Box_h}}(x, \cdot) \|_{L^2(M)} \circ \| D^\beta_y K_{e^{-t\Box_h}}(\cdot, y) \|_{L^2(M)}
\]

(7)

Here, we have implicitly used that \( K_{e^{-t\Box_h}} \in C^\infty(M \times M) \), which follows easily from Theorem 3.4, since \( \Box^j e^{-t\Box_h} \Box^k_h \) is bounded on \( L^2(M) \) for every \( j, k \), and \( \pi e^{-t\Box_h} = 0 = e^{-t\Box_h} \pi \). (7) shows that to prove the on diagonal estimates for \( e^{-t\Box_h} \), the following proposition will be sufficient:

Proposition 4.3. For \( t > 0 \),

\[
\| D^\alpha_x K_{e^{-t\Box_h}}(x, \cdot) \|_{L^2(M)}^2 \leq \frac{\sqrt{t}^{-2|\alpha|}}{V(x, \sqrt{t})}
\]

Remark 4.4. To see that Proposition 4.3 is sufficient, we must use that, in this case we are concerned with, \( V(x, \sqrt{t}) \approx V(y, \sqrt{t}) \), which follows from Proposition 2.4 and the fact that \( \sqrt{t} \geq \rho(x, y) \).
Proof of Proposition 4.3. Fix $x$ and $\alpha$, and recall the number $R_0$ from Theorem 3.3. There are two cases: $\sqrt{t} \leq R_0$ and $\sqrt{t} > R_0$. We first investigate the case $\sqrt{t} \leq R_0$. Let $\phi \in C^\infty (M)$. We apply Theorem 3.4 to see that there exists an $L$ such that:

$$
\left| \left( D^a x e^{-t L_h} \phi \right)(x) \right| \lesssim V \left( x, \sqrt{t} \right)^{-\frac{1}{2}} \sum_{j=0}^{L} \sqrt{t}^{2j-|\alpha|} \left\| \Box^j_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

where in the last line, we have used that $(t \Box_b) e^{-t L_h}$ is bounded on $L^2$ uniformly in $t$. Taking the supremum over all $\parallel \phi \parallel_{L^2(M)} = 1$, we see that the statement of the proposition follows in this case.

If $R_0 = \infty$, we would be done. Since in this case $R_0$ may not equal $\infty$, we use the fact the $\Box^{-1}_b : L^2 (M) \rightarrow L^2 (M)$ (a fact we do not have in all of the examples we consider in Section 3, however, in those examples where we do not have it, we instead have $R_0 = \infty$).

We now assume $\sqrt{t} > R_0$. We use that $V (x, R_0) \approx 1 \approx V (x, \sqrt{t})$ for all $x$. We apply the above proof with $R_0$ in place of $\sqrt{t}$ to see:

$$
\left| \left( D^a x e^{-t L_h} \phi \right)(x) \right| \lesssim V \left( x, R_0 \right)^{-\frac{1}{2}} \sum_{j=0}^{L} R_0^{2j-|\alpha|} \left\| \Box^j_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

$$
= \sum_{j=0}^{L} \left\| \Box^j_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

$$
\lesssim \sum_{j=0}^{L} \left\| \Box^{j+1}_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

$$
\lesssim \sum_{j=0}^{L} \left\| \Box^{j+1}_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

$$
\lesssim \sum_{j=0}^{L} \left\| \Box^{j+N}_b e^{-t L_h} \phi \right\|_{L^2(M)}
$$

$$
\lesssim V \left( x, \sqrt{t} \right)^{-\frac{1}{2}} \sqrt{t}^{-|\alpha|} \parallel \phi \parallel_{L^2(M)}
$$

provided $2N \geq |\alpha|$, which completes the proof. \hfill \Box

Remark 4.5. One might note that following the same method we used when
For all $N \geq |\alpha|$, we have:

$$\left\| D^\alpha K e^{-t\Box_b} (x, \cdot) \right\|_{L^2(M)} \lesssim \frac{\sqrt{t}^{-|\alpha|}}{V(x, \sqrt{t})^{\frac{1}{2}}}$$

The case when $\delta > R_0$ follows just as above, but by using $R_0$ in place of $\delta$.

Once we have established the off diagonal bounds for $e^{-t\Box_b}$, the remainder of Theorem 2.6 will follow immediately. We leave the details to the reader.

5 Finite Speed of Propagation

In this section, we discuss Theorem 2.3, which will be one of our main tools in proving the off diagonal bounds for $e^{-t\Box_b}$. The results in [Mel86] are stated in terms of a metric which (in this case) is defined in terms of $\Box_b$. It is easy to see that $\Box_b$ and the sublaplacian $X_1^2 X_1 + X_2^2 X_2$ give rise to the same metric. It is well known that the metric in [Mel86] induced by the sublaplacian is equivalent to the metric $\rho$. Then Theorem 2.3 follows from [Mel86].
The above outline gives rise to a somewhat round-about proof. Indeed, the result in [Mel86] follows by approximating $\Box_b$ by elliptic operators, and proving a corresponding result for the approximating elliptic operators. Then one must note the above equivalence of metrics. In the special case of $\Box_b$, however, a more direct proof will suffice. Indeed, we need only adapt the proof on page 162 of [Fol95] and the one in the appendix of [Mül04] to this situation, and we present this argument below. One benefit of this argument is that it requires essentially no work to adapt it to Examples 8.2 and 8.3 below; though these cases can also be covered by the methods of [Mel86], with a little more work.

To proceed, we need a result from [NS01a]:

**Proposition 5.1** ([NS01a]). There exists a function $d : M \times M \rightarrow \mathbb{R}^+$ such that:

$$d(x, y) \approx \rho(x, y)$$

and for $x \neq y$,

$$|D_x^a D_y^b d(x, y)| \lesssim d(x, y)^{1 - |\alpha| - |\beta|}$$

By replacing $d(x, y)$ with $d(x, y) + d(y, x)$ we may assume that $d(x, y) = d(y, x)$. By multiplying $d$ by a fixed constant, we may also assume:

$$\sup_{x \neq y} \sum_{|\alpha| = 1} |D_x^\alpha d(x, y)| \leq 1$$

**Proposition 5.2.** Suppose $u(x, t) \in C^2(M \times [0, T])$ such that $\partial_t^2 u + \Box_b u = 0$, and $u = \partial_t u = 0$ on the ball

$$\{(y, 0) : d(x_0, y) \leq t_0\}$$

where $x_0 \in M$ and $0 < t_0 \leq T$. Then $u$ vanishes in the region:

$$\Omega = \{(y, t) : 0 \leq t \leq t_0, d(x_0, y) \leq (t_0 - t)\}$$

**Proof.** Given $\delta > 0$, small, let $\chi_\delta \in C^\infty_0(\mathbb{R})$ be such that $\chi_\delta(s)$ is equal to 1 when $s \leq 1$, $\chi_\delta(s) = 0$ for $s \geq 1 + \delta$, and $\chi_\delta \leq 0$, and let $\chi_0$ be the characteristic function of $(-\infty, 1]$. Note that $\lim_{\delta \to 0} \chi_\delta = \chi_0$, pointwise. Define, for $\delta \geq 0$:

$$E_\delta(t) = \frac{1}{2} \int (|\partial_t u(y, t)|^2 + |(X_1 + iX_2) u(y, t)|^2) \chi_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) dy$$

where $X_1$ and $X_2$ are acting in the $y$ variable.

Consider, writing $\langle a, b \rangle = ab$, we have for $\delta > 0$:

$$\frac{dE_\delta}{dt}(t) = \text{Re} \int \langle u_{tt}, u_t \rangle + \langle (X_1 + iX_2) u, (X_1 + iX_2) u_t \rangle \chi_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) dy$$

$$+ \frac{1}{2} \int (|u_t|^2 + |(X_1 + iX_2) u|^2) \chi_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \frac{d(x_0, y)}{(t_0 - t)^2} dy$$

$$=: I + II$$
Since $\chi'_\delta \leq 0$, $II$ is clearly negative. We will show that (for $t < t_0$), $|I| \leq |II|$, and then it will follow that $\frac{dE_\delta}{dt} (t) \leq 0$.

We use the fact that $X_1^* = -X_1 + g$, where $g \in C^\infty (M)$ and similarly for $X_2$ to see that:

$$|I| \leq \left| \int \left( \langle u_{tt}, u_t \rangle + \langle (-X_1^* + iX_2^*) (X_1 + iX_2) u, u_t \rangle \right) \chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) dy \right|$$

$$+ \sum_{|\alpha|=1} \left| \int \left( \langle (X_1 + iX_2) u, u_t \rangle \right) D_\alpha^g \left( \chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \right) dy \right|$$

$$=: III + IV$$

Now the integrand of $III$ contains the term:

$$\langle u_{tt}, u_t \rangle + \langle (-X_1^* + iX_2^*) (X_1 + iX_2) u, u_t \rangle = \langle u_{tt} + \Box_b u, u_t \rangle = 0$$

and it follows that $III = 0$. To bound $IV$, note that:

$$\left| \sum_{|\alpha|=1} D_\alpha^g \left( \chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \right) \right| \leq -\chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \sum_{|\alpha|=1} \frac{|D_\alpha^g d(x_0, y)|}{t_0 - t}$$

$$\leq -\chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \frac{d(x_0, y)}{(t_0 - t)^2}$$

In the last line, we used that $\frac{d(x_0, y)}{t_0 - t} \geq 1$ on the support of $\chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right)$.

Thus, we have that:

$$IV \leq \int \left| \langle (X_1 + iX_2) u, u_t \rangle \right| \left( -\chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \right) \frac{d(x_0, y)}{(t_0 - t)^2}$$

$$\leq \frac{1}{2} \int \left| u_{tt} \right|^2 + \left| (X_1 + iX_2) u \right|^2 \left( -\chi'_\delta \left( \frac{d(x_0, y)}{t_0 - t} \right) \right) \frac{d(x_0, y)}{(t_0 - t)^2}$$

$$= |II|$$

Hence, $|I| \leq |II|$, and therefore $\frac{dE_\delta}{dt} (t) \leq 0$ for all $0 \leq t < t_0$. It follows that

$$E_\delta (t) \leq E_\delta (0) \quad (8)$$

for all $0 \leq t \leq t_0$. Taking the limit of both sides of (8) as $\delta \to 0$ and applying dominated convergence, we see:

$$E_0 (t) \leq E_0 (0)$$

for all $0 \leq t \leq t_0$.

However, by our assumption on $u$, $E_0 (0) = 0$. It follows that $E_0 (t) = 0$ for all $0 \leq t \leq t_0$, and in particular $\partial_t u = 0$ on $\Omega$. It follows that $u(y, t) = 0$ on $\Omega$. \qed
Now Theorem 2.3 will follow immediately from the following corollary:

**Corollary 5.3.** For \( t > 0 \),

\[
\text{supp} \left( K_{\cos(t\sqrt{\Box_b})} \right) \subseteq \{(x, y) : d(x, y) \leq t\}
\]

**Proof.** For this proof, define \( B_d(x, \delta) = \{y \in M : d(x, y) < \delta\} \). Fix \( x_0, y_0 \in M \) and \( t_1 > 0 \) such that \( d(x_0, y_0) > t_1 \). Fix \( \epsilon > 0 \) so small that for all \( x \in B_d(x_0, \epsilon) \) and \( y \in B_d(y_0, \epsilon) \), we have \( d(x, y) > t_1 + \epsilon \). We will show that for every \( \phi \in C_0^\infty (B_d(x_0, \epsilon)) \), \( \psi \in C_0^\infty (B_d(y_0, \epsilon)) \), we have:

\[
\int \psi(z) \left( \cos \left( t_1 \sqrt{\Box_b} \right) \phi \right) (z) \, dz = 0
\]

and the claim will follow.

Define \( u(x, t) = \cos \left( t\sqrt{\Box_b} \right) \phi \). We first claim that \( u \in C^\infty (M \times \mathbb{R}) \). Note that,

\[
u_1 = \pi \phi + \cos \left( t\sqrt{\Box_b} \right) \phi =: u_1 + u_2
\]

\( u_1 \) is independent of \( t \) and is \( C^\infty \) since \( \pi \) is an NIS operator. Fix \( t \) and note that \( \pi u_2 (\cdot, t) = 0 \). Thus, since

\[
\Box_b u_2 = \cos \left( t\sqrt{\Box_b} \right) \Box_b \phi \in L^2 (M)
\]

for each fixed \( t \), we have that \( u_2 (t, \cdot) \) is in \( C^\infty \) for each fixed \( t \). Moreover, since \( \partial_t^{4N} u_2 = \Box_b^{2N} u_2 \in L^2 (M) \) for every \( N \) (in distribution), the usual Fourier inversion trick now shows that \( u_2 \in C^\infty (M \times \mathbb{R}) \). It follows that \( u \in C^\infty (M \times \mathbb{R}) \).

Thus, we are in a position to apply Proposition 5.2. Note that \( u(y, 0) = 0 = \partial_t u(y, 0) \) for all \( y \in B_d(y_0, t_1 + \epsilon) \). Taking \( t_0 = t_1 + \epsilon \) in Proposition 5.2 and taking the \( t = t_1 \) slice of \( \Omega \) we see that:

\[
u(y, t_1) = 0
\]

for all \( y \in B_d(y_0, \epsilon) \). Hence,

\[
\int \psi(y) u(t_1, y) \, dy = 0
\]

completing the proof. \( \square \)

**Remark 5.4.** Note that \( \cos \left( t\sqrt{\Box_b} \right) \) does not have finite propagation speed. This is essential in understanding why we do not get off-diagonal Gaussian bounds for \( e^{-t\Box_b} \).

**Corollary 5.5.** Suppose \( \hat{F} \) is the Fourier transform of an even bounded Borel function \( F \) with \( \text{supp} \hat{F} \subseteq [-r, r] \). Then,

\[
\text{supp} \left( K_F (\sqrt{\Box_b}) \right) \subseteq \{(x, y) : \rho(x, y) \leq \kappa r\}
\]

**Proof.** This follows just as Lemma 3 of [Sik04], using Theorem 2.3. \( \square \)
6 Off Diagonal Bounds

In this section, we complete the proof of Theorem 2.4 by proving the bounds on $K_{e-\Box_b} (x, y)$ when $t < \frac{\rho(x_0, y_0)^2}{\kappa^2}$; to do so, we modify the proof of Theorem 4 of [Sik04]. We use the same notation as [Sik04] to make our modification obvious.

Fix $x_0, y_0 \in M$, $t > 0$, with $t < \frac{\rho(x_0, y_0)^2}{\kappa^2}$.

Fix a function $\psi \in C^\infty (\mathbb{R})$, satisfying

$$\psi (x) = \begin{cases} 0 & \text{if } x \leq -1, \\ 1 & \text{if } x \geq -\frac{1}{2} \end{cases}$$

and for $s > 1$ define $\phi_s (x) = \psi (s (|x| - s))$. Define:

$$F_s (x) = \phi_s (x) \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$$

$$R_s (x) = (1 - \phi_s (x)) \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$$

so that $\hat{F}_s (\lambda) + \hat{R}_s (\lambda) = e^{-\lambda^2}$ (here $\hat{F}_s$ denotes the Fourier transform of $F_s$) and therefore $\hat{F}_s (\sqrt{t}\Box_b) + \hat{R}_s (\sqrt{t}\Box_b) = e^{-t\Box_b}$. In [Sik04] equation (5.2) it is shown that for every natural number $N$ there exists a $C$ such that for all $s > 1$,

$$|\hat{F}_s (\lambda)| \leq C \frac{1}{s (1 + \frac{s^2}{4})} e^{-s^2}$$

(9)

Note, also, that $\text{supp} (R_s) \subseteq [-s + \frac{1}{2s}, s - \frac{1}{2s}]$ and thus if we set $s_{x_0, y_0} = \frac{\rho(x_0, y_0)}{\kappa \sqrt{t}}$, Corollary [5.3] tells us that

$$D_\alpha^x D_\beta^y K_{\hat{F}_{x_0, y_0} (\sqrt{t}\Box_b)} (x, y) \bigg|_{x=x_0, y=y_0} = 0$$

Thus,

$$D_\alpha^x D_\beta^y K_{e-\Box_b} (x, y) \bigg|_{x=x_0, y=y_0} = D_\alpha^x D_\beta^y K_{\hat{F}_{x_0, y_0} (\sqrt{t}\Box_b)} (x, y) \bigg|_{x=x_0, y=y_0}$$

$$= D_\alpha^x D_\beta^y K_{\hat{F}_{x_0, y_0} (\sqrt{t}\Box_b)} (x, y) \bigg|_{x=x_0, y=y_0} + \hat{F}_{x_0, y_0} (0) D_\alpha^x D_\beta^y K (x, y) \bigg|_{x=x_0, y=y_0}$$

(10)

We bound these two terms separately. First, we start with the second term. We use the bound (9) with $\lambda = 0$ and the fact that $\pi$ is an NIS operator of

\footnote{The second term is the main difference between this proof, and the one in [Sik04].}
order 0 to see:

\[
\left| \tilde{F}_{x_{0}, y_{0}} (0) D_{x}^{a} D_{y}^{b} K_{\pi} (x, y) \right|_{x=x_{0}, y=y_{0}} \lesssim \frac{1}{s_{x_{0}, y_{0}}} e^{-\frac{\rho (x_{0}, y_{0})}{V (x_{0}, \rho (x_{0}, y_{0}))} \beta \rho (x_{0}, y_{0})^{2}}
\]

(11)

where in the last line, we have used the fact that \( \frac{1}{\rho (x_{0}, y_{0})} \lesssim \frac{\rho (x_{0}, y_{0})}{t} \).

(11) is even better than the bound in the conclusion of Theorem 2.4.

Now Theorem 2.4 will follow directly from the following lemma:

Applying Lemma 4.1 again, we see:

\[
\left| \tilde{F}_{x_{0}, y_{0}} (0) D_{x}^{a} D_{y}^{b} K_{\pi} (x, y) \right|_{x=x_{0}, y=y_{0}} \lesssim \frac{1}{s_{x_{0}, y_{0}}} e^{-\frac{\rho (x_{0}, y_{0})}{V (x_{0}, \rho (x_{0}, y_{0}))} \beta \rho (x_{0}, y_{0})^{2}}
\]

(11)

where in the last line, we have used the fact that \( \frac{1}{\rho (x_{0}, y_{0})} \lesssim \frac{\rho (x_{0}, y_{0})}{t} \).

We now turn to the first term in the last line of (10). Let \( J (\lambda) \) be a measurable function such that \( J (\lambda)^{2} = \tilde{F}_{x_{0}, y_{0}} \left( t^{2} \lambda \right) \) (we suppress \( J \)'s dependence on \( s_{x_{0}, y_{0}} \)). Note that, by (11), we have for every \( N \geq 0 \):

\[
\sup_{\lambda \geq 0} \left| J (\lambda) \left( 1 + \frac{\lambda^{2} t^{2}}{\rho (x_{0}, y_{0})^{2}} \right)^{N} \right| \lesssim \frac{1}{\rho (x_{0}, y_{0})} e^{-\frac{\rho (x_{0}, y_{0})^{2}}{s_{x_{0}, y_{0}}}}
\]

(12)

Then we have, by Lemma 4.1 and Proposition 4.3 (taking \( N \) large, depending on \( \gamma \)):

\[
\left\| D_{x}^{a} K \left( \sqrt{t} \lambda \right) (x, \cdot) \right\|_{L^{2}(M)} \lesssim \frac{e^{-\frac{\rho (x_{0}, y_{0})^{2}}{s_{x_{0}, y_{0}}}}}{\sqrt{\rho (x_{0}, y_{0}) t^{\frac{\gamma}{2}}}} \left\| D_{x}^{a} K \left( t^{\frac{\gamma}{2}} \lambda \right) (x, \cdot) \right\|_{L^{2}(M)}
\]

Applying Lemma 4.1 again, we see:

\[
\left| D_{x}^{a} D_{y}^{b} \tilde{F}_{x_{0}, y_{0}} \left( \sqrt{t} \lambda \right) (x, y) \right|_{x=x_{0}, y=y_{0}} \lesssim \left\| D_{x}^{a} K \left( \sqrt{t} \lambda \right) (x, \cdot) \right\|_{L^{2}(M)} \left\| D_{y}^{b} K \left( \sqrt{t} \lambda \right) (y, \cdot) \right\|_{L^{2}(M)}
\]

\[
\lesssim \frac{e^{-\frac{\rho (x_{0}, y_{0})^{2}}{s_{x_{0}, y_{0}}}}}{\rho (x_{0}, y_{0}) t^{-\frac{\gamma}{2}}} \left( \rho (x_{0}, y_{0}) \right)^{\frac{|\alpha| + |\beta|}{2}} V \left( x_{0}, \frac{t}{\rho (x_{0}, y_{0})} \right) V \left( y_{0}, \frac{t}{\rho (x_{0}, y_{0})} \right)^{-\frac{1}{2}}
\]

Now Theorem 2.4 will follow directly from the following lemma:
Lemma 6.1.

\[ V\left(x_0, \frac{t}{\rho(x_0, y_0)}\right)^{-1}, V\left(y_0, \frac{t}{\rho(x_0, y_0)}\right)^{-1} \lesssim \left(\frac{\rho(x_0, y_0)^2}{t}\right)^Q \frac{1}{V(x_0, \rho(x_0, y_0))} \]

Proof. For \( V\left(x_0, \frac{t}{\rho(x_0, y_0)}\right)^{-1} \) this follows directly from Proposition 2.1. For \( V\left(y_0, \frac{t}{\rho(x_0, y_0)}\right)^{-1} \) this follows from Proposition 2.1 and the fact that

\[ V(y_0, \rho(x_0, y_0)) \approx V(x_0, \rho(x_0, y_0)) \]

(Which is also a consequence of Proposition 2.1.)

Corollary 2.5 is a simple corollary of Theorem 2.4.

7 Multipliers

In this section, we prove Theorems 2.7 and 2.8. We prove the two in tandem, as some of the estimates needed for Theorem 2.8 are sharper than those needed for Theorem 2.7, however Theorem 2.7 will allow us to create a convenient Littlewood-Paley square function, with which we will complete the proof of Theorem 2.8. The arguments in this section are closely related to those of [Mü104], however, since we are not in the case of a stratified group, we are forced to deviate from those arguments (in particular, this is why we have \( Q^2 \) in Theorem 2.8, instead of \( Q^2 \)).

Proposition 7.1. Suppose \( m \) is supported in \([\frac{1}{4}, 4]\), \( r > 0 \) is fixed, \( \alpha, \beta \) are fixed ordered multi-indices, \( a > \frac{Q+1}{2} + |\alpha| \lor |\beta| \), and \( \|m\|_{L^2_\mathbb{R}} < \infty \). Then, for every \( \frac{Q}{2} + |\alpha| \lor |\beta| < b \leq a - \frac{1}{2} \), there exists a \( C = C(\alpha, \beta, a, b, \|m\|_{L^2_\mathbb{R}}) \), but not depending on \( r \), such that:

\[ |D_x^{\alpha}D_y^{\beta}K_{m(r^2 \Box_b)}(x, y)| \leq C \left(1 + \frac{\rho(x, y)}{r}\right)^{-a+b} \frac{r \lor \rho(x, y)^{-|\alpha|-|\beta|}}{V(x, \rho(x, y) + r)} \]

Proof. Fix \( x_0, y_0 \in M \). We wish to bound:

\[ D_x^{\alpha}D_y^{\beta}K_{m(r^2 \Box_b)}(x, y) \bigg|_{x=x_0, y=y_0} \]

We begin with the harder case \( \rho(x_0, y_0) \geq r \). Define \( \psi(\lambda) = m(\lambda^2) \), so that \( \|\psi\|_{L^2_\mathbb{R}} \lesssim \|m\|_{L^2_\mathbb{R}} \) (due to the support of \( m \)). Let \( \psi_r(\lambda) = \psi(r\lambda) \), so that \( \psi_r(\Box_b) = m(r^2 \Box_b) \), and \( \hat{\psi_r}(\xi) = \frac{1}{r} \hat{\psi}\left(\frac{\xi}{r}\right) \). Let \( \phi \in C^\infty(\mathbb{R}) \) be such that:

\[ \phi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq \frac{1}{4} \\ 1 & \text{if } |\xi| \geq \frac{1}{2} \end{cases} \]
For \( s > 0 \), define:
\[
\hat{F}_s (\xi) = \phi \left( \frac{\xi}{s} \right) \frac{1}{r} \tilde{\varphi} \left( \frac{\xi}{r} \right)
\]
where we have suppressed the dependence of \( F \) on \( r \). We will use the following elementary fact: for all \( 0 \leq \tilde{b} \leq a - \frac{1}{2} \),
\[
\sup_{s > 0, r > 0, \lambda > 0} (1 + \lambda s)^{\tilde{b}} \left( 1 + \frac{s}{r} \right)^{a - \frac{1}{2} - \tilde{b}} |F_s (\lambda)| \lesssim 1
\] (13)
Here, the implicit constant depends on the same parameters as in the statement of the proposition. We leave the proof of (13) to the interested reader.

Set \( s = \frac{\rho(x_0, y_0)}{r} \), so that \( \tilde{b} \gtrsim 1 \). Note that, by the definition of \( \hat{F}_s \), Theorem 2.3 and Corollary 5.5 we have:
\[
\begin{align*}
D^a_y D^\beta_y K_m (r^{2 \Delta_b} ) (x, y) &\bigg|_{x = x_0, y = y_0} = D^a_y D^\beta_y K_{F_s (\sqrt{\Delta_b} )} (x, y) \\
&= D^a_y D^\beta_y K_{F_s (\sqrt{\Delta_b} )} (x, y) + F_s (0) D^a_y D^\beta_y K_{\pi} (x, y) \\
&= \quad \text{We bound the two terms in the last line of (14) separately. For the second term, we apply (13) (with \( \tilde{b} = 0 \) and the fact that \( \pi \) is an NIS operator of order 0) to see:}
\end{align*}
\] (14)
\[
\left| F_s (0) D^a_y D^\beta_y K_{\pi} (x, y) \right|_{x = x_0, y = y_0} \lesssim \left( 1 + \frac{\rho(x_0, y_0)}{r} \right)^{-a + \frac{\tilde{b}}{2}} \frac{\rho(x_0, y_0)^{|\alpha| - |\beta|}}{V(x, \rho(x_0, y_0))}
\]
which, in light of the fact that \( \rho(x_0, y_0) \geq r \), is at least as good as the bound in the statement of the proposition. We now turn to the first term in the last line of (14). We let \( J_\alpha (\lambda) \) be a measurable function such that \( J_\alpha (\lambda)^2 = F_s (\lambda) \) (we again suppress the dependence on \( r \), and note that all of our bounds will be uniform in \( r \)); we now have, from (14), with \( \tilde{b} = b \):
\[
\sup_{s > 0, r > 0, \lambda > 0} (1 + \lambda s)^{\frac{b}{2}} \left( 1 + \frac{s}{r} \right)^{\frac{1}{2} + \frac{b}{2} - \frac{1}{2}} |J_\alpha (\lambda)| \lesssim 1
\]
Proceeding just as in the proof of off-diagonal estimates in Theorem 2.4 we see:
\[
\begin{align*}
\left\| D^\beta_y J_\alpha (\sqrt{\Delta_b} ) (x, \cdot) \right\|_{L^2(M)} &\lesssim \left( 1 + \frac{s}{r} \right)^{-\frac{3}{2} + \frac{b}{2} + \frac{\tilde{b}}{2}} \left\| D^2_y K_{(1 + s^2 \Delta_b)}^{-\frac{b}{2}} (x, \cdot) \right\|_{L^2(M)} \\
&\lesssim \left( 1 + \frac{\rho(x_0, y_0)}{r} \right)^{-\frac{3}{2} + \frac{b}{2} + \frac{1}{2}} \rho(x_0, y_0)^{|\alpha|} \frac{\rho(x_0, y_0)^{-|\alpha|}}{V(x, \rho(x_0, y_0))}
\end{align*}
\]
Thus, we have:

\[
\left| D_x^\alpha D_y^\beta K_{m(\sqrt{\cdot})}(x,y) \right|_{x=x_0,y=y_0} \approx \frac{\rho(x_0,y_0)^{-|\alpha| - |\beta|}}{V(x_0,\rho(x_0,y_0)) \frac{1}{2} V(y_0,\rho(x_0,y_0))^{\frac{1}{2}} (1 + \frac{\rho(x_0,y_0)}{r})^{-a + \frac{1}{2} + b}} \left(1 + \frac{\rho(x_0,y_0)}{r}\right)^{-a + \frac{1}{2} + b}.
\]

Using that \(V(y_0,\rho(x_0,y_0)) \approx V(x_0,\rho(x_0,y_0)) \approx V(x_0, r + \rho(x_0,y_0))\) completes the proof of the bound in the case \(\rho(x_0,y_0) \geq r\).

We now turn to the case when \(\rho(x_0,y_0) \leq r\). This will follow in much the same manner as the on-diagonal bounds in Section 4, and we merely sketch the proof. Let \(j(\lambda)\) be a measurable function such that \(j(\lambda)^2 = m(r^2\lambda)\). Note, that by the compact support of \(m\), we have for every \(N\),

\[
\sup_{\lambda > 0} (1 + r^2\lambda)^N |j(\lambda)| \leq 1
\]

with constants independent of \(r\). Here we have used that (by the Sobolev embedding theorem) \(m\) (and therefore \(j\)) is bounded. This bound is the point where Remark 2.9 is essential. Note also that \(m(0) = 0 = j(0)\), and therefore \(j(\Box_b) = j(\Box_b)\). Thus, we have:

\[
\left| D_x^\alpha D_y^\beta K_{m(\sqrt{\cdot})}(x,y) \right|_{x=x_0,y=y_0} \leq \left| D_x^\alpha K_{j(\Box_b)}(x,\cdot) \right|_{x=x_0} \left| D_y^\beta K_{j(\Box_b)}(\cdot,y) \right|_{y=y_0} \left| D_x^\alpha K_{(1+r^2\Box_b)^{-N}}(x,\cdot) \right|_{x=x_0} \left| D_y^\beta K_{(1+r^2\Box_b)^{-N}}(\cdot,y) \right|_{y=y_0} \frac{1}{V(x_0, r)^\frac{1}{2}} \frac{1}{V(y_0, r)^\frac{1}{2}} \frac{1}{V(x_0, r)}.
\]

Where in the last line, we have used that \(V(y_0, r) \approx V(x_0, r)\), since \(r \geq \rho(x_0,y_0)\). This completes the proof of the proposition, since

\[
V(x_0,r) \approx V(x_0,\rho(x_0,y_0) + r)
\]
Proposition 7.1 motivates the following definition:

**Definition 7.2.** Let $r > 0$, and $K \in C^\infty(M \times M)$. We say $K$ is a pre-$r$-elementary kernel if, for all $N > 0$:

$$|D_x^\alpha D_y^\beta K(x,y)| \leq C_{N,\alpha,\beta} \left(1 + \frac{\rho(x,y)}{r}\right)^{-N} \frac{r^{-|\alpha|-|\beta|}}{V(x,\rho(x,y)+r)}$$

If $S \subset C^\infty(M \times M) \times (0,\infty)$ is a set of pairs $(K,r)$ where $K$ is a pre-$r$-elementary kernel, we say the $K$s are uniformly pre-$r$-elementary kernels if the constants $C_{N,\alpha,\beta}$ can be chosen independently of $(K,r) \in S$.

**Remark 7.3.** The “pre” in the definition above is put there so as to not conflict with the similar definition in [Str08].

Next we prove an analog of Lemma 6.36 of [FS82]:

**Proposition 7.4.** Suppose $m \in S([0,\infty))$, $m(0) = 0$, and $r > 0$, then $K_{m(r^2\Box)}$ is a pre-$r$-elementary kernel. Moreover, as $r$ ranges over $(0,\infty)$ and $m$ ranges over a bounded subset of $S([0,\infty))$, the $K_m$ are uniformly pre-$r$-elementary kernels.

**Proof.** For $m \in C^\infty_0(\left(\frac{1}{4}, 4\right))$ (which is the only case we shall use in this paper), the result follows immediately from Proposition 7.1 (by taking $a >> b >> 0$).

For the general case, a proof similar to the one in Proposition 7.1 works, merely by keeping track of the decay of $m$ at $\infty$. Since we do not use this, we leave the details to the reader.

**Lemma 7.5.** Fix $N_0 \in \mathbb{Z}$. Suppose for each $j \in \mathbb{Z}$, $j \leq N_0$, $K_j$ is a pre-2$^j$-elementary kernel, uniformly in $j$. Then, $K(x,y) := \sum_{j \leq N_0} K_j(x,y)$ converges in $C^\infty$ off of the diagonal of $M \times M$. Moreover, the function $K$ satisfies the estimates of part 2 of Definition 3.1.

**Proof.** Fix $x, y \in M \times M$, $x \neq y$. We will show that the sum:

$$\sum_{-N_1 \leq j \leq N_0} |D_x^\alpha D_y^\beta K_j(x,y)|$$

satisfies the desired estimates uniformly in $N_1$. The result will then follow immediately.

Consider, suppressing the $-N_1 \leq j \leq N_0$, and writing $a = |\alpha| + |\beta|$, and $\delta = \rho(x,y)$,

$$\sum_j |D_x^\alpha D_y^\beta K_j(x,y)| \lesssim \sum_j \left(1 + \frac{\delta}{2^j}\right)^{-N} \frac{2^{-ja}}{V(x,\delta+2^j)}$$

(15)
We separate \( L \) into three sums. For the first we take \( N = 0 \) and recall the numbers \( \delta_0 \) and \( q \) from Proposition 2.1:

\[
\sum_{\delta_0 \geq \delta_2} \frac{2^{-ja}}{V(x, \delta + 2^j)} \lesssim \sum_{\delta_0 \geq \delta} 2^{-ja} \frac{2^{ja}}{V(x, 2^j)} \lesssim \sum_{\delta_0 \geq \delta} \left( \frac{\delta}{2^j} \right)^q \frac{2^{ja}}{V(x, \delta)} \lesssim \frac{\delta^{-a}}{V(x, \delta)}
\]

which is the desired bound. Turning to the second sum:

\[
\sum_{2^j \leq \delta} \left( 1 + \frac{\delta}{2^j} \right)^{-N} \frac{2^{-ja}}{V(x, \delta + 2^j)} \lesssim \sum_{2^j \leq \delta} \left( \frac{2^j}{\delta} \right)^N \frac{2^{ja}}{V(x, \delta)} \lesssim \frac{\delta^{-a}}{V(x, \delta)}
\]

for \( N \) sufficiently large.

Finally, the term

\[
\sum_{\delta_0 \leq \delta \leq 2N_0} K_j (x, y)
\]

is just a finite sum of \( C^\infty \) functions and so satisfies the desired bounds trivially.

Proof of Theorem 2.7. Since \( m (\Box_b) = m \left( \tilde{\Box}_b \right) + m (0) \pi \), and \( \pi \) is an NIS operator of order 0, we need only verify that \( m \left( \tilde{\Box}_b \right) \) is an NIS operator of order 0. Property 3 follows just as in the case of \( e^{-t\Box_b} \). The main point is property 2. Let \( \phi \in C^\infty_0 (\mathbb{R}) \) be such that

\[
\phi (x) = \begin{cases} 
0 & \text{if } |x| \geq 2, \\
1 & \text{if } |x| \leq 1
\end{cases}
\]

let \( \psi = \phi (x) - \phi (2x) \), and let \( m_j (\lambda) = \psi (2^j \lambda) m (\lambda) \). Thus, we have that, for \( \lambda \neq 0 \), \( m (\lambda) = \sum_{j \in \mathbb{Z}} m_j (\lambda) \), and consequently, \( \sum_{j \in \mathbb{Z}} m_j (\Box_b) = m \left( \Box_b \right) \), with convergence in the strong operator topology. Note, that since 0 is an isolated point of the spectrum of \( \Box_b \) (this follows from the easily provable fact that \( \Box_b^{-1} \) is compact), we have that there exists an \( N_0 \) such that \( m_j (\Box_b) = 0 \) for \( j > N_0 \). By Proposition 7.4, we have that \( m_j (\Box_b) \) is a \( 2^j \)-elementary operator, uniformly in \( j \). Lemma 7.5 then shows that

\[
m \left( \Box_b \right) = \sum_{j \leq N_0} m_j (\Box_b)
\]
satisfies property 2 of the definition of NIS operators. Property 1 follows from Remark 3.3 and the fact that

$$\sum_{-N \leq j \leq N_0} K_{m_j(\square_b)} \in C^\infty (M \times M)$$

for every $N$. Finally, property 4 follows immediately from what we have already done.

We now turn to constructing a Littlewood-Paley square function which will help us prove Theorem 2.8. Define $\phi, \psi \in C^\infty_0 (\mathbb{R})$ as in the proof of Theorem 2.7. That is $\phi (x) = 1$ if $|x| \leq 1$, and $\phi (x) = 0$ if $|x| \geq 2$, and $\psi (x) = \phi (x) - \phi (2x)$; furthermore, we assume that $\psi$ is real and even. Let $\psi_j (\lambda) = \psi \left( 2^j \lambda \right)$, so that $\sum_j \psi_j (\lambda) = 1$ for $\lambda \neq 0$. Define:

$$\tilde{\psi} (\lambda) = \frac{\psi (\lambda)}{\sum_{j \in \mathbb{Z}} |\psi_j (\lambda)|^2}$$

Thus, if $\tilde{\psi}_j (\lambda) = \tilde{\psi} \left( 2^j \lambda \right)$, we have:

$$\sum_j \tilde{\psi}_j (\square_b \psi_j (\square_b) = 1 - \pi$$

(16)

Hence, for $f \in C^\infty (M)$, if we define

$$\Lambda (f) = \left( \sum_j |\psi_j (\square_b f|^2 \right)^{\frac{1}{2}}, \quad \bar{\Lambda} (f) = \left( \sum_j |\tilde{\psi}_j (\square_b f|^2 \right)^{\frac{1}{2}}$$

We have, for all $1 < p < \infty$:

$$\|\Lambda (f)\|_{L^p (M)} \lesssim \|f\|_{L^p (M)}, \quad \|\bar{\Lambda} (f)\|_{L^p (M)} \lesssim \|f\|_{L^p (M)}$$

(17) follows from standard arguments. Indeed, for any sequence $\epsilon_j (j \in \mathbb{Z})$ of $-1$s and $1$s, we have that:

$$\sum_j \epsilon_j \psi_j (\square_b)$$

is bounded on $L^p$, since it is equal to an NIS operator of order 0, just as in the proof of Theorem 2.7. The result now follows from the standard trick of taking the $\epsilon_j$s to be iid random variables of mean 0 taking values of $\pm 1$. See Chapter 4, Section 5 of [Ste70] and page 267 of [Ste93].

It now follows, again from standard arguments ([Ste70, Ste93]), that for $1 < p < \infty$:

$$\|\Lambda (f)\|_{L^p (M)} + \|\pi f\|_{L^p (M)} \approx \|f\|_{L^p (M)}$$

(18) for $\lambda \neq 0$. To see this, it suffices to see that for $f$ such that $\pi f = 0$,

$$\|\Lambda (f)\|_{L^p (M)} \approx \|f\|_{L^p (M)}$$
and this follows just as in Chapter 4, Section 5.3.1 of [Ste70] by using (16).

Define the maximal function:

\[ M(f)(x) = \sup_{\delta > 0} \frac{1}{V(x, \delta)} \int_{B(x, \delta)} |f(y)| \, dy \]

We have:

**Lemma 7.6.** Suppose \( a > \frac{Q+1}{2} \), \( m \) is supported in \( [\frac{1}{4}, 4] \), \( r > 0 \) is fixed, and \( \|m\|_{L^2(\mathbb{R})} < \infty \). Then, there exists a \( C = C \left( a, \|m\|_{L^2(\mathbb{R})} \right) \), but not depending on \( r \), such that:

\[ |m(r^2 \Box_b) f(x)| \leq CM(f)(x) \]

**Proof.** Fix \( b \) such that \( \frac{Q}{2} < b < a - \frac{1}{2} \), and define \( \epsilon = a - \frac{1}{2} - b > 0 \). Applying Proposition 7.1 we have that:

\[ |K_{m(r^2 \Box_b)}(x,y)| \lesssim \left( 1 + \frac{\rho(x, y)}{r} \right)^{-\epsilon} \frac{1}{V(x, r + \rho(x, y))} \]

Hence,

\[ |m(r^2 \Box_b) f(x)| \lesssim \sum_{2^j \geq r} \int_{\rho(x, y) \leq 2^j} \left( \frac{2^j}{r} \right)^{-\epsilon} \frac{1}{V(x, 2^j)} |f(y)| \, dy \]

\[ \lesssim \sum_{2^j \geq r} \left( \frac{2^j}{r} \right)^{-\epsilon} M(f)(x) \]

\[ \lesssim M(f)(x) \]

**Proof of Theorem 2.8.** Take \( m \) as in the statement of Theorem 2.8. We know that \( m(0) \pi \) is bounded on \( L^p \), and so it suffices to show that \( m(\Box_b) \) is bounded on \( L^p \). The proof will follow from a standard Littlewood-Paley decomposition, which we sketch. Fix \( f \in C^\infty(M) \), and define \( F_j = \psi_j(\Box_b) m(\Box_b) f \). Note that, since \( \psi_j(\Box_b) \psi_k(\Box_b) = 0 \) unless \( |j - k| \leq 1 \), we have (applying Lemma 7.1):

\[ |F_j| = \left| \sum_{k=-1}^{1} \psi_j(\Box_b) m(\Box_b) \tilde{\psi}_{j+k}(\Box_b) \psi_{j+k}(\Box_b) f \right| \]

\[ \lesssim \sum_{k=-1}^{1} M(\psi_{j+k}(\Box_b) f) \]

\[ \lesssim \left( \sum_{k=-1}^{1} \left( M(\psi_{j+k}(\Box_b) f)^2 \right) \right)^{\frac{1}{2}} \]

23
And thus, since $\pi m (\bar{\square}_b) = 0$, we have:

$$
\| m (\bar{\square}_b) f \|_{L^p(M)} \approx \left\| \left( \sum_j |F_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \\
\lesssim \left\| \left( \sum_j \sum_{k=-1}^1 \mathcal{M} (\psi_j + k (\square_b) f) \right)^{\frac{1}{2}} \right\|_{L^p(M)} \\
\lesssim \left\| \left( \sum_j \mathcal{M} (\psi_j (\square_b) f)^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \\
\lesssim \| \Lambda (f) \|_{L^p(M)} \\
\lesssim \| f \|_{L^p(M)}
$$

where we have used the vector-valued inequality for $\mathcal{M}$. See Chapter 2, Section 1 of [Ste93].

\[\Box\]

8 Other Examples

In this section we state some results with $\square_b$ replaced by other operators. In each case $\pi$ will denote the orthogonal projection onto the $L^2$ kernel of the operator in question. All of the proofs of the below results are similar to the proofs above, and we therefore confine ourselves to brief comments on the necessary changes.

8.1 Example: A Generalization of Theorem 4 of [Sik04]

In this example, we discuss how the above methods may be applied in the general situation of [Sik04], except that we allow the infinitesimal generator of the heat semi-group to have non-trivial $L^2$ kernel.

We first quickly review the setup of that paper, though we refer the reader there for more rigorous details. Let $X$ be a metric measurable space with metric $\rho$, and let $\mu$ be a Borel measure on $X$. We define $B (x, \delta)$ and $V (x, \delta)$ as above, but with $\mu$ in place of Lebesgue measure. We suppose that:

$$V (x, \gamma \delta) \lesssim \gamma^Q V (x, \delta)$$

for all $\gamma \geq 1$.

We suppose $TX$ is a continuous vector bundle with base $X$ (with fibers $\mathbb{C}^d$), and scalar product $(\cdot, \cdot)_x$. We define the space $L^2 (TX, \mu)$ of sections of $TX$ in the usual way. Now, suppose that $\mathcal{L}$ is a, possibly unbounded, self-adjoint positive semi-definite operator acting on $L^2 (TX, \mu)$. For bounded Borel measurable functions $m$, we define $m (\mathcal{L})$ and $m \left( \mathcal{L} \right)$ analogous to the definitions
earlier in the paper for $\Box_b$. Let $\pi$ be the orthogonal projection onto the $L^2$ kernel of $L$.

Suppose that, for bounded continuous sections $\phi, \psi \in L^2(TX, \mu)$ with disjoint support, we have that:

$$\langle \psi, \pi \phi \rangle = \int \langle \psi(x), K_\pi(x,y) \phi(y) \rangle_x \, dy \, dx$$

for a measurable function $K_\pi$ defined on $X \times X$ without the diagonal, and taking values in $\text{Hom}(T_yX, T_xX)$. We suppose that, for $x \neq y$:

$$|K_\pi(x,y)| \lesssim \frac{1}{V(x, \rho(x,y))}$$

where $|\cdot|$ denotes the operator norm on $\text{Hom}(T_yX, T_xX)$.

We suppose that $e^{-t\mathcal{L}}$ satisfies the on-diagonal estimate:

$$\| |K_{e^{-t\mathcal{L}}}(x, \cdot)| \|_{L^2(M)} \lesssim V(x, \sqrt{t})^{-\frac{1}{2}}$$

and that $\cos(t\sqrt{\mathcal{L}})$ has finite propagation speed, in the sense used in [Sik04]. Informally, that:

$$\text{supp} \left( \cos(t\sqrt{\mathcal{L}}) \right) \subseteq \{(x,y) : \rho(x,y) \leq t\}$$

Then, $e^{-t\mathcal{L}}$ satisfies the off diagonal estimates, for $t < \rho(x,y)^2$,

$$|K_{e^{-t\mathcal{L}}}(x,y)| \lesssim V(x, \rho(x,y))^{-1} \left( \frac{\rho(x,y)^2}{t} \right)^{Q-\frac{1}{2}} e^{-\frac{\rho(x,y)^2}{4t}}$$

The proof of this fact follows just by putting together the methods of [Sik04] and the methods of this paper.

Remark 8.1. Actually, [Sik04] has a slightly better bound. This is due to the fact that there is a slight difficulty meshing the bounds for the heat kernel with those for $\pi$.

### 8.2 Example: Pseudoconvex CR Manifolds of Finite Type

In this example, we let $M$ be a compact pseudoconvex CR manifold of dimension $2n - 1$ ($n \geq 3$), and we assume that the range of $\partial_b$ (as an operator on $L^2(M)$) is closed, and we assume that $M$ is of finite commutator type. Let $x_0 \in M$ be a fixed base point, and let $U$ be a neighborhood of $x_0$. We think of $U$ as small and may shrink it throughout the discussion. Fix a local basis $L_1, \ldots, L_{n-1}$ for $T^{1,0}$ on $U$ (which we may do by making $U$ small enough). Fix a Hermitian metric on $\mathbb{C}TM$ such that $L_1, \ldots, L_{n-1}$ are orthonormal.

Put $X_j = \text{Re}(L_j), X_{j+n-1} = \text{Im}(L_j)$, by assumption the $X_k$s along with their commutators up to a certain fixed order span the tangent space $TU$; we
use these vector fields to define a metric $\rho$ as in Section 2.1 and define $D^\alpha$ for an ordered multi-index $\alpha$ in terms of these vector fields, as well. In addition, we assume that condition $D(q)$ holds on $U$. This is the setup of [Koe02], and we refer the reader there for more details.

We use a definition from [Koe02]:

**Definition 8.2.** An operator $T$ on functions $f \in C^\infty(M)$ is said to be an NIS operator smoothing of order $r$ in $U$ if $T$ satisfies the properties of Definition 3.1 except for the following modifications:

- In property 2, we only consider $x, y \in U$.
- In property 3, we only consider $x$ and $\delta$ such that $B(x, \delta) \subset U$.

This definition extends to operators on forms in the obvious way; see [Koe02], page 158.

Consider the operator $\overline{\partial}_b$ acting on $(0, q)$ forms. We define the operator $\mathcal{L} = \partial_b \overline{\partial}_b$ acting on $(0, q)$ forms via the Hermitian product that we fixed above (here $\overline{\partial}_b$ is acting on $(0, q + 1)$ forms). Let $\pi$ be the projection onto the $L^2$ kernel of $\mathcal{L}$. It is shown in [Koe02] that $\pi$ is an NIS operator of order 0 in $U$ and that the relative fundamental solution $\tilde{L}^{-1}$ is an NIS operator of order 2 in $U$.

One may write $\overline{\partial}_b$ and $\partial_b^*$ in terms of $\overline{L}_j$ (see (2.6) and (2.7) of [Koe02]) and with this a proof almost exactly the same as the one above for Theorem 2.3 shows that

$$\supp \left( K_{\cos(t\sqrt{\mathcal{L}})} \right) \cap U \times U \subseteq \{(x, y) \in U \times U : \rho(x, y) \leq \kappa t\}$$

for some fixed constant $\kappa$. The results in Section 2.3 hold with the following modifications:

- The bounds in Theorem 2.4 and Corollary 2.5 hold for $x, y \in U$.
- In Theorem 2.6 and Theorem 2.7 “NIS operators” must be replaced with “NIS operators in $U$.”
- In Theorem 2.8 we consider $m(\mathcal{L})$ taking $L^p(U) \to L^p(U)$.

The proofs are essentially the same as the ones in this paper, however one must work with operators on forms as in [Sik04] and Example 8.1. We leave the details to the reader.

### 8.3 Example: Polynomial Model Domains

In this example, we discuss the other case treated in [NS01b]. In this case, there is a subharmonic, nonharmonic polynomial $h : \mathbb{C} \to \mathbb{R}$ such that

$$M = \{(z, w) \in \mathbb{C}^2 : \text{Im}(w) = h(z)\}$$
We define $\Box_b$ as in that reference, and refer the reader there for details. All of the results in Section 2.3 hold without any changes in the statement of the results, however, we must address a few differences in the proofs.

The analog of Theorem 2.3 follows just as before, just by using the smooth metric constructed for this case in Section 4 of [NS01a].

The main differences between this case and the case treated above are (the closely related facts) that $\Box^{-1}_b$ is not bounded on $L^2$ and that the spectrum of $\Box_b$ is not discrete. The fact that $\Box^{-1}_b$ is not bounded on $L^2$ can be worked around by using that we may take $R_0 = \infty$ in Theorem 3.4.

That the spectrum of $\Box_b$ is not discrete forces us to have a replacement for Lemma 7.5. Indeed, we replace it with the same result but with $N_0 = \infty$. This is proven in a similar manner, by using the fact that (in this case) we may take $\delta_0$ in Proposition 2.1 to be $\infty$. At this point, all of the proofs go through with only minor changes. We leave the details to the reader. For some related results, see [Rai07].

8.4 Example: Operators on a Compact Manifold, Defined by Vector Fields

For this example, let $M$ be a compact Riemannian manifold, and let $X_1, \ldots, X_n$ be vector fields satisfying Hörmander’s condition. Let $\mathcal{L}$ be an second order, self adjoint, polynomial in the vector fields $X_1, \ldots, X_n$. Using these vector fields, we obtain a metric $\rho$, as in Section 2.1. We assume that $\mathcal{L}$ has the following properties:

- The $\mathcal{L}$ wave operator, $\cos \left( t\sqrt{\mathcal{L}} \right)$, has finite propagation speed. That is, it satisfies the conclusion of Theorem 2.3
- The relative fundamental solution, $\mathcal{L}^{-1}$, of $\mathcal{L}$ is an NIS operator of order 2.

Then, all of the results in Section 2.3 remain true with $\mathcal{L}$ in place of $\Box_b$, with essentially the same proofs.

Remark 8.3. Actually, that $\mathcal{L}$ be of second order is inessential. We leave such generalizations to the reader.

In particular, all of the proofs in this paper work with $\mathcal{L}$ equal to the sublaplacian:

$$\mathcal{L} = X_1^*X_1 + \cdots + X_n^*X_n$$

where we identify $\mathcal{L}$ with its Friedrich extension. The finite propagation speed follows just as in the proof of Theorem 2.3. In this case, $\pi$ is just the projection onto the constant functions. Note that, since $e^{-t\mathcal{L}} = e^{-t\mathcal{L}} + \pi$, bounds for $e^{-t\mathcal{L}}$ and $e^{-t\mathcal{L}}$ are essentially the same. In the case of the sublaplacian, though, most of the results in this paper are quite well known. For instance, Theorem 2.3 can be found in [McS86], Corollary 2.5 can be found in [JSC86], and Theorem 2.6 was implicitly proven in [NS01a].
8.5 Example: Quasi-homogeneous Vector Fields

In this example, we let $X_1, \ldots, X_n$ be vector fields on $M = \mathbb{R}^d$ satisfying Hörmander’s condition, and which are homogeneous of degree 1 with respect to a one parameter family of dilations on $\mathbb{R}^d$. An example would be the left invariant vector fields of degree 1 on a stratified group. Let $\mathcal{L}$ be a second order, self-adjoint, homogeneous polynomial in $X_1, \ldots, X_n$, and assume that $\mathcal{L}$ satisfies the same two assumptions as in Example 8.4. Then, all of the results in Section 2.3 go through with proofs almost exactly the same as Example 8.3.

Just as in Example 8.4, the sublaplacian:

$$\mathcal{L} = -X_1^2 - \cdots - X_n^2$$

is a special case of this. The finite propagation of the wave equation may be verified for $x, y$ in a fixed compact neighborhood of 0 by the same proof as in Theorem 2.3 and then extended to all $x, y$ by homogeneity. In this case, $\pi = 0$, and so $e^{-t\mathcal{L}}$ and $e^{-t\mathcal{L}}$ satisfy the same bounds. Just as in Example 8.4 these results are well known. In particular, in the case of the sublaplacian on a stratified group, all of the results in this paper can be improved, and are quite well known. See [FS82, Chr91, Ale94], and references therein.

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28
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