Trigonometry of ‘complex Hermitian’ type
homogeneous symmetric spaces.

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Abstract

This paper contains a thorough study of the trigonometry of the homogeneous symmetric spaces in the Cayley-Klein-Dickson family of spaces of ‘complex Hermitian’ type and rank-one. The complex Hermitian elliptic $\mathbb{C}P^N \equiv SU(N+1)/(U(1) \otimes SU(N))$ and hyperbolic $\mathbb{C}H^N \equiv SU(N,1)/(U(1) \otimes SU(N))$ spaces, their analogues with indefinite Hermitian metric $SU(p+1,q)/(U(1) \otimes SU(p,q))$ and the non-compact symmetric space $SL(N+1,\mathbb{R})/(SO(1,1) \otimes SL(N,\mathbb{R}))$ are the generic members in this family; the remaining spaces are some contractions of the former.

The method encapsulates trigonometry for this whole family of spaces into a single basic trigonometric group equation, and has ‘universality’ and ‘(self)-duality’ as its distinctive traits. All previously known results on the trigonometry of $\mathbb{C}P^N$ and $\mathbb{C}H^N$ follow as particular cases of our general equations.

The following topics are covered rather explicitly: 0) Description of the complete Cayley-Klein-Dickson family of rank-one spaces of ‘complex type’, 1) Derivation of the single basic group trigonometric equation, 2) Translation to the basic ‘complex Hermitian’ cosine, sine and dual cosine laws, 3) Comprehensive exploration of the bestiary of ‘complex Hermitian’ trigonometric equations, 4) Uncovering of a ‘Cartan’ sector of Hermitian trigonometry, related with triangle symplectic area and coarea, 5) Existence conditions for a triangle in these spaces as inequalities and 6) Restriction to the two special cases of ‘complex’ collinear and purely real triangles.

The physical Quantum Space of States of any quantum system belongs, as the complex Hermitian space member, to this parametrised family; hence its trigonometry appears as a rather particular case of the equations we obtain.
1 Introduction

In a previous paper [1] the trigonometry of the complete family of symmetric spaces of *rank one and real type* was studied. These spaces are also called Cayley-Klein (hereafter CK) real spaces and were first discussed by Klein extending the Cayley idea of ‘projective metrics’. In two dimensions there are *nine* real spaces with a real quadratic symmetric metric of any (positive, zero, negative) constant curvature, and any (positive definite, degenerate, indefinite) signature [2]. Further to this, the paper [1] had a long run aim towards opening an avenue for exploring the trigonometry of general symmetric homogeneous spaces.

Next to the spaces of real type, there are spaces of ‘complex’ type. In ‘complex’ dimension $N$ there are $3^{N+1}$ such geometries [3]; ‘complex’ type means here that these spaces are coordinatised by elements of a one-step extension of $\mathbb{R}$ through a labelled Cayley-Dickson procedure $\mathbb{R} \to \eta \mathbb{R}$ which adjoins a imaginary unit $i$ with $i^2 = -\eta$ to $\mathbb{R}$, producing either the complex, dual or split complex numbers according as $\eta > 0, = 0, < 0$. We will term Cayley-Klein-Dickson (CKD) these spaces. They are ‘Hermitian’, since they are related to a scalar product with ‘complex’ values and hermitian-like symmetry.

Within this family, only the spaces coordinatised by ordinary complex numbers (where $\eta > 0$, which can be rescaled to $\eta = 1$ and $i^2 = -1$) are actually *complex* spaces; after this restriction, there are only $3^N$ CK complex type geometries in complex dimension $N$ [4]. For $N = 2$ there are nine 2D complex Hermitian CK spaces, with constant holomorphic curvature (either $K_{hol} > 0, = 0, < 0$) and a Hermitian metric of either definite, degenerate or indefinite signature. All these are Hermitian symmetric spaces with a complex structure, hence Kählerian, but only the three spaces with a definite positive Hermitian metric belong to the restricted family of the so-called *two-point homogeneous spaces* [6]. These are the elliptic Hermitian space, i.e. the complex projective space $\mathbb{C}P^2$ with the Fubini-Study metric, the hermitian hyperbolic space $\mathbb{C}H^2$ which can be realized in the interior of a Hermitian quadric in $\mathbb{C}P^2$ like its real analogue, and the Hermitian euclidean space $\mathbb{C}R^2$, a two-dimensional Hilbert space, as the common ‘limiting’ space.

If degenerate and indefinite Hermitian products are allowed, a CK family of nine complex 2D spaces is obtained: the complex elliptic, euclidean and hyperbolic Hermitian planes with a definite positive metric, the complex co-euclidean, galilean and co-minkowskian (or Anti Newton-Hooke, Galilean and Newton-Hooke) Hermitian planes with a degenerate metric and finally the indefinite metric complex co-hyperbolic, minkowskian and doubly hyperbolic (or Anti De Sitter, Minkowskian and De Sitter) Hermitian planes. The last six spaces are the complex Hermitian analogues of the three non-relativistic and three relativistic space-times. In group theoretical terms, all these nine Hermitian spaces appear as *four* generic cases and *five* non-generic ones within the complex CK family. The generic cases are $SU(3)/(U(1) \otimes SU(2))$, $SU(2, 1)/(U(1) \otimes SU(2))$, $SU(2, 1)/(U(1) \otimes SU(1, 1))$, the last hosting two different spaces with ‘time like’ and ‘space-like’ complex lines interchanged. Non-generic cases are contractions of these four generic ones, with either curvature vanishing, metric degenerating or both. Within the full complex CK family of spaces of complex type, results on trigonometry are only available, as far as we know, for the Hermitian spaces with definite positive metric and constant (either positive or negative) holomorphic curvature [5, 7, 8, 9, 10, 11, 12, 13]; a review of these results is included in Section 2.

The spaces in the subfamily $\eta < 0$ are much less known; they have not a complex struc-
ture, but its ‘split complex’ analogue. Its generic members correspond to the symmetric homogeneous space $SL(3, \mathbb{R})/(SO(1, 1) \otimes SL(2, \mathbb{R}))$, and the non-generic ones to some of its contractions. The trigonometry for such spaces has apparently not been studied. The spaces with $\eta = 0$ appear as common contractions from the spaces with $\eta > 0$ and $\eta < 0$.

In this paper we set out the task of studying in full detail the trigonometry of the complete family of spaces of ‘complex’ type. It clearly suffices to consider the two-dimensional case, as a triangle in any CKD such ‘complex’-type space in ‘complex’ dimension $N$ is fully contained in a totally geodesic subspace with ‘complex’ dimension 2.

The approach distinctive traits are: First, it covers at once the trigonometry in the whole family of $3^3 = 27$ such geometries, parametrized by three real labels $\eta; \kappa_1, \kappa_2$. The study of the trigonometry in the family as a whole is in fact easier than its study for just one space at a time. Second, it gives a clear view of several duality relationships between the trigonometry of these different spaces. In particular it explicitly displays the self-duality of the Hermitian elliptic space (analogous to the self-duality of the real sphere $S^2$) that is completely hidden in the trigonometric equations derived for this space in [9, 10, 11, 13].

Third, it gives more than previously known: all previously known equations appear as a rather small subset of the equations to be derived here. Fourth, it deals uniformly with the (contracted) non-generic cases which correspond to curvature vanishing ($\kappa_1 = 0$) and/or (‘Fubini-Study’) metric degenerating ($\kappa_2 = 0$ or $\eta = 0$). ‘Hermitian’ analogues of the real angular and lateral excesses vanish in these limits, and are related to triangle symplectic area and its dual quantity. And fifth, the presence of the additional Cayley-Dickson type label $\eta$ makes it possible the consideration of a new type of contractions, encompassed by the limit $\eta \to 0$ whose physical meaning is worth exploring.

The paper is intended to be self-contained, but reference to [1] may be helpful, specially for motivations and general background. A condensed review of already known results on the trigonometry of both the Hermitian elliptic and hyperbolic spaces [7, 8, 9, 10, 11, 12, 13] is given in Section 2. Here we display the basic equations which in some cases were originally given in terms of angular invariants not considered in this paper (e.g. the Fubini-Study angles and the holomorphy inclinations at each vertex), and will be rewritten here so they can be easily identified with the ones we obtain. The choice of basic invariants adopted here affords the equations in a form which we believe simpler than using any other choice.

Information on CKD spaces of ‘complex’ type is given in Sections 3 and 4. Section 3 deals with the ordinary complex case, and therefore refers to a much more familiar situation. Section 4 comments the main new traits appearing when complex numbers are replaced by their ‘complex’ parabolic and split (hyperbolic) versions.

In Section 5 the approach to the trigonometry proposed in [1] is developed in depth for the complete CKD family. The whole of trigonometry for all these spaces is encapsulated in a single basic trigonometric group equation, involving sides, angles, and lateral and angular ‘phases’ and exhibiting explicitly (self)-duality in the whole family. This is mainly achieved due to a choice of triangle invariants as the canonical parameters of two pairs of commuting isometries, a choice which should ring a bell to any physicist educated in Quantum Mechanics. Dealing with many spaces at once, this equation gives a perspective on some relationships going far beyond any treatment devoted to the study of a single space. The behaviour of trigonometry when either the curvature vanishes or the metric degenerates is explicitly described through the CKD constants $\eta; \kappa_1, \kappa_2$. Duality is the
main structural backbone in our approach, and the requirement to explicitly maintaining duality in all expressions and at all stages acts as a kind of method ‘fingerprint’. Cartan duality for symmetric spaces [14] appears here as the change of sign in either $\eta, \kappa_1$ or $\kappa_2$.

The basic trigonometric group equation is an equation for the parameter-dependent group of motions. By writing it in the fundamental ‘complex’ representation, a set of nine ‘complex’ equations follows. With these equations as starting point, we will explore in section 6 the rather unknown territory of ‘complex hermitian’ trigonometric equations. The background provided by real trigonometry makes this exploration easier by deliberately pursuing the analogies, while at the same time the relevant differences stand out clearly. The most interesting difference with real trigonometry is the natural splitting of the equations into two ‘sectors’. The first involves quantities linked to Cartan generators of the motion group, where two new triangle invariants appear in a rather natural way; they play a specially important role since they are proportional to the symplectic area and coarea of the triangle. For the hermitian elliptic space $\mathbb{C}P^2$, these quantities were first found by Blaschke and Terheggen [7, 8]. The other ‘sector’ is the ‘complex’ analogue of the whole real trigonometric set of equations. Most results in the real case have (sometimes several and/or partly) analogous in this ‘complex’ trigonometry. The family form of all previously known equations is obtained here, together with a large number of new ones. In the $\mathbb{C}P^2$ case ($\eta = 1; \kappa_1 = 1, \kappa_2 = 1$) all trigonometric functions of sides, angles, lateral and angular phases are the ordinary circular ones, and at a first look the whole paper can be read by restricting to this case; this may help to grasp the key ideas, while not losing view of the increased scope afforded by the possibility of zero or negative $\eta, \kappa_1$ or $\kappa_2$.

The basic trigonometric identity for the family of ‘complex Hermitian’ spaces is also directly linked to other product formulas which we believe new. They can be considered as a kind of ‘complex Hermitian’ Gauss-Bonnet formulas, and contain the totality of ‘complex Hermitian’ trigonometry in a nutshell, as they are equivalent to the basic trigonometric identity. The subject of such ‘exponential product formulas’ appears as an step in our derivation (Section 5.1) but it can be further developed by itself, and affords a number of new identities; this will be discussed elsewhere.

There is actually a strong link between this study, which superficially seems like a work in geometry, and physics: the mathematical structure underlying the Quantum State Space belongs to the ‘complex hermitian’ CKD family. So as a byproduct of this work we obtain the basic equations of the ‘Trigonometry of the Quantum State Space’ [15, 16]. Any Hilbert space appears in the CKD family as a Hermitian euclidean space (thus with labels $\eta = 1; \kappa_1 = 0, \kappa_2 = \kappa_3 = \ldots = 1$) and its projective Hilbert space which plays in any Quantum Theory the role of space of states, appears as the Hermitian elliptic space ($\eta = 1; \kappa_1 = \kappa > 0, \kappa_2 = \kappa_3 = \ldots = 1$). Geometric phases are related to trigonometric quantities: for the simplest ‘triangle type’ loop in the Quantum state space, the Anandan-Aharonov phase appears intriguingly as one of the triangle invariants introduced by Blaschke and Terheggen sixty years ago. The paper by Sudarshan, Anandan and Govindarajan [17] gives a group theoretical derivation of the Anandan-Aharonov phase (equal to triangle symplectic area) for an infinitesimal triangle loop in $\mathbb{C}P^N$; this result appears as a particular case of our exact expressions linking triangle elements for any finite triangle. The role of symplectic area for geodesic triangles in connection with coherent states and geometric phases has also been recently discussed by Berceanu [18] and Boya, Perelomov and Santander [19]. A separate, more physically oriented paper [20] will be devoted to the trigonometry of the Quantum space of states, in relation with geometric phases and in general, with the view
towards a more geometrical formulation of Quantum Mechanics [21].

2 A review on Hermitian trigonometry of $\mathbb{C}P^2$ and $\mathbb{C}H^2$

The hermitian elliptic space, i.e., $\mathbb{C}P^N$ endowed with the natural Fubini-Study (FS) metric induced by the real part of the hermitian canonical flat product in $\mathbb{C}^{N+1}$, is an homogeneous hermitian symmetric space. It has a natural complex structure, and the FS metric is kählerian and has constant holomorphic curvature; the Kähler form is induced by the imaginary part of the hermitian canonical metric in $\mathbb{C}^{N+1}$ [22]. The standard choice of scale in the metric makes the maximum distance in $\mathbb{C}P^N$ equal to $\pi/2$, and the total length of any (closed) geodesic equal to $\pi$. With this choice the constant holomorphic curvature of $\mathbb{C}P^N$ is $K_{hol} = 4$; the ordinary sectional curvature $K$ of the FS metric in $\mathbb{C}P^N$ seen as a riemannian space of real dimension $2N$ is not constant, and lies in the interval $1 \leq K \leq 4$. Complex projective geometry was studied by Cartan [23], building over the works by Study [24] and Fubini [25].

For the real projective space $\mathbb{R}P^N$, trigonometry essentially reduces to spherical trigonometry [26]). The homogeneous symmetric character of $\mathbb{C}P^N$ makes also possible an explicit study of its trigonometry, which is however much more complicated than for $\mathbb{R}P^N$. A common trait in most of the previous works on hermitian trigonometry is to introduce a single real invariant for each side (which seems natural as $\mathbb{C}P^N$ is a rank one space), but two real invariants for each vertex, which also seems natural due to presence of two commuting factors in the group theoretical description of the hermitian elliptic space as the homogeneous space $SU(N + 1)/U(1) \otimes SU(N)$.

For side invariants the canonical choice are the distances in the FS metric $a$ (resp. $b, c$) between vertices $BC$ (resp. $CA, AB$). To avoid non-generic special cases all papers quoted before enforce the restrictions $a, b, c < \pi/2$; this means that each pair of sides does not meet the cut locus of the common vertex. In both the elliptic hermitian space $\mathbb{C}P^N$, and the hermitian hyperbolic space $\mathbb{C}H^N$, identified with a suitable bounded domain of $\mathbb{C}P^N$ with the hyperbolic FS metric, each point $[z]$ is a ray in the linear ambient space $\mathbb{C}^{N+1}$ [11]. Even if we assume normalized the ambient position vectors, $(z, z) = 1$, every ray in $\mathbb{C}^{N+1}$ still contains infinitely many normalized vectors differing only by a phase factor, $[z] \equiv \{e^{i\epsilon}z\}$. Let $z^A, z^B, z^C$ denote arbitrarily chosen normalized position vectors in $\mathbb{C}^{N+1}$ (defined up to a phase factor) for the three vertices $A \equiv [z^A]$, $B \equiv [z^B]$, $C \equiv [z^C]$. Then the length $a$ of the side $a \equiv BC$ can be obtained from the hermitian product of the two (normalized) vectors $\langle z^B, z^C \rangle$ in the ambient linear space through $\cos a \exp(ie_a) := \langle z^B, z^C \rangle$. The phase $e_a$ is not a triangle invariant, as the vectors representing the vertices can be still modified by arbitrary phase factors.

Vertex invariants are defined in terms of the tangent space to the hermitian space which will be considered here as a real vector space with a complex structure. At each point a vector tangent to a geodesic $g$ is defined only up to a nonzero real factor. The tangent space to a complex (projective) line $l$ at a point $O$ is a real 2D subspace of the tangent space invariant under the complex structure and can be thus identified to a complex 1D subspace; this subspace contains a one-parameter family of real 1D subspaces, corresponding to a one-parameter family of FS geodesics through $O$ and contained in $l$. For vertex invariants several real quantities can be used. In terms of the tangent vectors $u, v$ to two (real one-dimensional) FS geodesic sides at the vertex $C$, these are: 1) the
hermitian angle between the sides seen as complex projective lines, denoted \( C \); 2) the ordinary or FS angle \( \Lambda \) between \( u, v \) computed as usual in the natural riemannian FS metric in \( \mathbb{C}P^N \) or \( \mathbb{C}H^N \) [27], \( g(,):=\text{Re}\langle , \rangle \); 3) the FS angle between \( iu, v \), denoted here \( \Theta \). These are defined as:

\[
C := \arccos \left( \frac{|\langle u, v \rangle|}{\|u\| \cdot \|v\|} \right) \quad \Lambda := \arccos \left( \frac{\text{Re}\langle u, v \rangle}{\|u\| \cdot \|v\|} \right) \quad \Theta := \arccos \left( \frac{\text{Im}\langle u, v \rangle}{\|u\| \cdot \|v\|} \right).
\]

(2.1)

Note \( \Psi = \pi/2 - \Theta \) is the minimum value of the riemannian FS angle between the tangent vector \( u \) and a totally geodesic \( \mathbb{R}P^2 \) containing the geodesic with tangent vector \( v \).

In addition to, yet not independent from these, one can consider also 4) the holomorphy ‘inclination’ \( \Upsilon \) of the real 2-flat tangent to the triangle at the vertex \( C \), also called Kähler angle, inclination angle, holomorphy angle, slant angle, etc. between \( u \) and \( v \) [28]. This quantity depends only on the real 2-flat determined by \( u \) and \( v \), and not on \( u, v \) separately, thus the names holomorphy inclination or Kähler inclination seem more appropriate. In terms of two vectors \( u, t \) which span the given 2-plane and are furthermore FS orthogonal, the holomorphy inclination \( \Upsilon \) is given by:

\[
\Upsilon = \arccos \left( \frac{\text{Re}\langle iu, t \rangle}{\|u\| \cdot \|t\|} \right) \quad \text{where} \quad t = \alpha u + \beta v, \quad \alpha, \beta \in \mathbb{R}, \quad \text{Re}\langle u, t \rangle = 0. \quad (2.2)
\]

The holomorphy inclination measures how this real 2-flat separates from the unique real 2-flat \( \mathbb{C}u \) containing \( u \) and \( v \) and invariant under the complex structure; the FS angle between \( iu \) and \( t \) is a natural measure of the separation between these two 2-planes, since \( \mathbb{C}u = \langle u, iu \rangle \) and the given 2-plane is spanned by \( u, t \) (both pairs are FS orthogonal). Finally, 5) another angular invariant \( \Phi \) of the pair of tangent vectors \( u, v \), its pseudoangle or Kasner angle [28] is:

\[
\langle u, v \rangle = |\langle u, v \rangle| e^{i\Phi} \quad (2.3)
\]

This angle has not been explicitly used in previous works on trigonometry on \( \mathbb{C}P^N \) or \( \mathbb{C}H^N \); it is generically well defined for any two vectors in the tangent space at each point of \( \mathbb{C}P^N \) or \( \mathbb{C}H^N \) (i.e. between two intersecting FS geodesics with tangent vectors \( u, v \) at the intersection point) but becomes indeterminate when \( u, v \) are FS orthogonal. The angular invariant \( \Phi \) is obviously meaningless between complex lines in these spaces.

To sum up, there are several different choices available for two independent vertex invariants; see the review by Scharnhorst [28]. Authors studying trigonometry have made different choices and the following relations will be useful for comparing the proposed trigonometric equations (there are of course similar relations for the corresponding ‘angular invariants’ at vertices \( A, B \)):

\[
\cos \Lambda_C = \cos C \cos \Phi_C \quad \cos \Upsilon_C \sin \Lambda_C = \cos C \sin \Phi_C \quad (2.4)
\]

\[
\sin C = \sin \Lambda_C \sin \Upsilon_C \quad \sin \Psi_C = \sin \Lambda_C \cos \Upsilon_C \quad (2.5)
\]

\[
\cos^2 C = \cos^2 \Lambda_C + \sin^2 \Lambda_C \sin^2 \Upsilon_C \quad \sin^2 \Lambda_C = \sin^2 C + \sin^2 \Psi_C \quad (2.6)
\]

In choosing symbols for these angular invariants, we have tried to conform to the majority usage, but nevertheless we have systematically changed to capital letters, which allows a clear and systematic typographic rendering of the self-duality of the equations we will propose, by means of the change upper/lower case letters.
2.1 Trigonometry in the Hermitian spaces $\mathbb{CP}^2$ and $\mathbb{CH}^2$

The oldest general result is the Coolidge’s (1921) sine theorem [5]: the sides $a, b, c$ and angles $A, B, C$ between the sides seen as complex lines are related by:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

(2.7)

The papers by Blaschke and Terheggen (1939) [7, 8] (hereafter BT) contained the first complete approach to trigonometry in the elliptic hermitian space, identified with $\mathbb{CP}^2$. Unlike the phase $\epsilon_a$ of $\langle z^B, z^C \rangle$ which is meaningless as a quantity in $\mathbb{CP}^2$, the combination $\Omega := \epsilon_a + \epsilon_b + \epsilon_c$ is a triangle invariant, as can be clearly seen in the relation $\langle z^A, z^B \rangle \langle z^B, z^C \rangle \langle z^C, z^A \rangle = \cos a \cos b \cos c \exp(i\Omega)$. BT named $\omega$ this quantity, but for the reasons explained below we will change this notation to $\Omega$. Let us now consider the (normalized) position vectors $Z^a, Z^b, Z^c$ of the poles $[Z^a], [Z^b], [Z^c]$ of the three sides $a, b, c$ defined by BT in the ambient space $\mathbb{C}^3$ through a ‘vector product’ as $Z^2 = \frac{z^b \times z^c}{\sin a}$, and cyclically, where the vector product is defined exactly as in the real case, without complex conjugation in any factor. Then the dual procedure ($\cos A \exp(i\epsilon_A) := \langle z^b, z^c \rangle$) applied to the poles of the three sides, produces four invariants; three angles $A, B, C$ between sides seen as complex lines and another quantity $\omega$ which was called $\tau$ by BT; these four quantities are dual to $a, b, c, \Omega$. BT gave a complete set of equations for the hermitian elliptic space trigonometry. One is the Coolidge’s sine law (2.7), and there are two new equations, which we will call Blaschke-Terheggen cosine theorem for sides and angles; the need of four quantities (e.g. $a, b, c, \Omega$) to determine a triangle up to isometry in the elliptic hermitian space follows from these equations:

$$\cos^2 a = \frac{\cos^2 A + \cos^2 B \cos^2 C - 2 \cos A \cos B \cos C \cos \omega}{\sin^2 B \sin^2 C}$$

(2.8)

$$\cos^2 A = \frac{\cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c \cos \Omega}{\sin^2 b \sin^2 c}$$

(2.9)

Another approach was put forward by Shirokov, in a paper published posthumously by Rosenfeld. Shirokov took two angular invariants at each vertex: the riemannian FS angle $\Lambda_C$ between the two sides $a, b$ considered as real-1D FS geodesics and the holomorphy inclination $\Upsilon_C$ of the real 2-flat spanned at the vertex $C$ by the real tangent vectors to the two sides $a, b$. The equations which we shall call Shirokov-Rosenfeld (SR) form of sine theorem, double sine theorem for sides, cosine theorem for sides and double cosine theorem for sides respectively are:

$$\frac{\sin a}{\sin \Lambda_A \sin \Upsilon_A} = \frac{\sin b}{\sin \Lambda_B \sin \Upsilon_B} = \frac{\sin c}{\sin \Lambda_C \sin \Upsilon_C}$$

(2.10)

$$\frac{\sin 2a}{\sin \Lambda_A \cos \Upsilon_A} = \frac{\sin 2b}{\sin \Lambda_B \cos \Upsilon_B} = \frac{\sin 2c}{\sin \Lambda_C \cos \Upsilon_C}$$

(2.11)

$$\cos^2 a = (\cos b \cos c + \sin b \sin c \cos \Lambda_A)^2 + \sin^2 b \sin^2 c \cos^2 \Upsilon_A \sin^2 \Lambda_A$$

(2.12)

$$\cos 2a = \cos 2b \cos 2c + \sin 2b \sin 2c \cos \Lambda_A - 2 \sin^2 b \sin^2 c \sin^2 \Upsilon_A \sin^2 \Lambda_A$$

(2.13)

as well as similar cosine and double cosine equations for the sides $b, c$. Among all these equations, only five are functionally independent (for instance (2.12) and (2.13) are equivalent). Note SR sine theorem (2.10) is equivalent to the Coolidge sine law as consequence of (2.5).
In 1989 Wu-Yi Hsiang [10] gave a new derivation valid simultaneously for the trigonometry of the two-point homogeneous rank-one spaces of real, complex, quaternionic and Cayley octonionic type, both elliptic and hyperbolic. At each vertex, say C, Hsiang uses the three invariants $C, \Lambda_C, \Theta_C$ linked by a relation (2.6) (recall $\Theta_C = \pi/2 - \Psi_C$). In the elliptic/hyperbolic case he obtained some equations which when translated to $C, \Lambda_C, \Psi_C$ are given below in (2.14), (2.15) as well as a rather complicated form of ‘cosine theorem’, not reproduced here and that should not be considered as a ‘basic’ equation.

In 1994 Hangan and Masala [29] gave an interpretation of $\Omega$ in the complex projective space $\mathbb{C}P^2$ as equal to twice the symplectic area enclosed by the triangle. Symplectic area comes from the Kähler structure of $\mathbb{C}P^2$, and is well defined by the triangle ‘skelethon’ itself, due to the closed nature of the Kähler form which makes the symplectic area of any surface with given boundary to depend only on the boundary.

The existence of two distinguished, non generic types of triangles is clear. In $\mathbb{C}P^2$, when $\Upsilon_A = 0$, then $\Upsilon_B = \Upsilon_C = 0$ follows and the trigonometry equations reduce to those of a spherical triangle in a sphere of curvature $K = 4$; this is seen in (2.11) and (2.13) and corresponds to a triangle completely contained in a complex line $\mathbb{C}P^1$. When $\Upsilon_A = \pi/2$, then $\Upsilon_B = \Upsilon_C = \pi/2$, the equations reduce (locally) to those of a spherical triangle in curvature $K = 1$; this can be seen in (2.10) and (2.12) and corresponds to a triangle completely contained in a real projective subplane $\mathbb{R}P^2$, whose trigonometry comes from the spherical one after antipodal identification. This case corresponds to real values for $\exp(i\Omega)$ and $\exp(i\omega)$, as implied by the Blaschke-Terhegen equations (2.8) and (2.9).

In these two special cases, contained in a totally geodesic submanifold, whose sectional curvature attains the extremal values 1 and 4. In all other situations, the triangle is not contained in a totally geodesic submanifold. The sectional curvature of either $\mathbb{C}P^2$ or $\mathbb{C}H^2$ along any real 2-direction depends only on its holomorphy inclination $\Upsilon$ and is:

$$K = \pm \left(4\cos^2 \Upsilon + \sin^2 \Upsilon\right).$$  (2.16)
3 The family of nine complex Hermitian Cayley–Klein 2D geometries and their spaces

3.1 The nine complex Hermitian Cayley–Klein 2D geometries

Let us consider a complex hermitian form \((z, w) \rightarrow \langle z, w \rangle = \sum_{i,j} \Lambda_{ij} w_i \) in an ambient complex linear space \(C^{N+1} = (z_0, z_1, \ldots, z_N)\), where the symmetric real matrix \(\Lambda\) is a diagonal matrix with entries \(\{1, \kappa_1, \kappa_1 \kappa_2, \ldots, \kappa_2 \kappa_N\}\), depending on \(N\) real numbers \(\kappa_i\). Linear isometries in \(C^{N+1}\) for such a hermitian product close the special unitary CK families of groups \(SU_{\kappa_1, \kappa_2, \ldots, \kappa_N}(N+1)\) with Lie algebras \(su_{\kappa_1, \kappa_2, \ldots, \kappa_N}(N+1)\). The structure of algebras in this family (any dimension) as well as the associated unitary \(U_{\kappa_1, \kappa_2, \ldots, \kappa_N}(N+1)\) and \(u_{\kappa_1, \kappa_2, \ldots, \kappa_N}(N+1)\) CK families is described in [4].

When particularised for \(N = 2\) a two-parametric family \(SU_{\kappa_1, \kappa_2}(3)\) of groups is obtained; these are the eight-dimensional linear isometry groups of a complex hermitian form \(su_{\kappa_1, \kappa_2}(3)\) with symmetric real matrix \(\Lambda = \text{diag}\{1, \kappa_1, \kappa_1 \kappa_2\}\). In the natural CK basis \(\{P_1, P_2, Q_1, Q_2; J, M, B, I\}\) the CK algebra \(su_{\kappa_1, \kappa_2}(3)\) has the following (fundamental or vectorial) 3D complex matrix representation, where \(i\) stands for the pure imaginary complex unit:

\[
P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1 \kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
Q_1 = \begin{pmatrix} 0 & i \kappa_1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & 0 & i \kappa_1 \kappa_2 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \kappa_2 \\ 0 & i & 0 \end{pmatrix}
\]

\[
B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix} \quad I = \begin{pmatrix} -\frac{2i}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}
\]

When \(\kappa_1, \kappa_2\) are different from zero, \(su_{\kappa_1, \kappa_2}(3)\) is simple. This algebra is isomorphic to \(su(3)\) when both \(\kappa_1, \kappa_2\) are positive or to \(su(2, 1)\) when at least one is negative.

The generators \(B, I\) span a two-dimensional Cartan subalgebra (regardless of the values of \(\kappa_i\)); further references to the Cartan subalgebra will mean to this fiducial subalgebra.

Let us introduce four new Cartan generators:

\[
T_1 = \frac{1}{2}(I + B) \quad T_2 = \frac{1}{2}(I - B) \quad H_1 = \frac{1}{2}(3I - B) \quad H_2 = \frac{1}{2}(3I + B)
\]

whose representing matrices in the vector fundamental representation are:

\[
T_1 = \begin{pmatrix} -\frac{i}{3} & 0 & 0 \\ 0 & -\frac{i}{3} & 0 \\ 0 & 0 & \frac{2i}{3} \end{pmatrix} \quad T_2 = \begin{pmatrix} -\frac{i}{3} & 0 & 0 \\ 0 & -\frac{i}{3} & 0 \\ 0 & 0 & -\frac{i}{3} \end{pmatrix}
\]

\[
H_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}
\]

The Lie commutators of all these generators are given in (4.1) for \(\eta = 1\). The CK algebras \(su_{\kappa_1, \kappa_2}(3)\) can be endowed with a \(\mathbb{Z}_2 \otimes \mathbb{Z}_2\) group of commuting involutive automorphisms.
generated by:

\[ \Pi_{(1)} : (P_1, P_2, Q_1, Q_2; J, M, B, I) \to (-P_1, -P_2, -Q_1, -Q_2; J, M, B, I) \]
\[ \Pi_{(2)} : (P_1, P_2, Q_1, Q_2; J, M, B, I) \to (P_1, -P_2, Q_1, -Q_2; -J, -M, B, I). \] (3.4)

The two remaining involutions are the composition \( \Pi_{(02)} = \Pi_{(1)} \cdot \Pi_{(2)} \) and the identity. Each involution \( \Pi \) determines a subalgebra of \( \mathfrak{su}_{\kappa_1, \kappa_2}(3) \), denoted \( \mathfrak{h} \), whose elements are invariant under \( \Pi \); the subgroups generated by \( \mathfrak{h} \) will be denoted \( H \), all with suitable subindices.

By direct checking one can assure that the three Lie subalgebras \( \mathfrak{h}_{(1)}, \mathfrak{h}_{(2)} \) and \( \mathfrak{h}_{(02)} \) are of unitary CK type, \( \mathfrak{u}_\kappa(2) \equiv \mathfrak{u}(1) \oplus \mathfrak{su}_\kappa(2) \) with \( \kappa = \kappa_2, \kappa_1, \kappa_1 \kappa_2 \) respectively. Namely:

- The subalgebra \( \mathfrak{h}_{(1)} \) is spanned by \( I; J, M, B \) which close an \( \mathfrak{u}_{\kappa_2}(2) \) (with \( I \) commuting with \( J, M, B \)). The group \( H_{(1)} \) they generate is isomorphic to \( U(1) \otimes SU_{\kappa_2}(2) \).
- The subalgebra \( \mathfrak{h}_{(2)} \) is spanned by \( T_1; P_1, Q_1, H_1 \) which close an \( \mathfrak{u}_{\kappa_1}(2) \) (with \( T_1 \) commuting with \( P_1, Q_1, H_1 \)). The group \( H_{(2)} \) they generate is isomorphic to \( U(1) \otimes SU_{\kappa_1}(2) \).
- The subalgebra \( \mathfrak{h}_{(02)} \) is spanned by \( T_2; P_2, Q_2, H_2 \) closing an \( \mathfrak{u}_{\kappa_1 \kappa_2}(2) \) (with \( T_2 \) commuting with \( P_2, Q_2, H_2 \)). The group \( H_{(02)} \) they generate is isomorphic to \( U(1) \otimes SU_{\kappa_1 \kappa_2}(2) \).

All these generators can be represented in a pictorial way in a block triangular diagram,

\[
\begin{array}{ccc}
  P_1 Q_1 & P_2 Q_2 \\
  T_1 H_1 & T_2 H_2 \\
  JM & I B
\end{array}
\] (3.5)

where each ‘block’ involves the four generators of the \( \mathfrak{u}_\kappa(2) \) subalgebras listed above; the generator in the \( \mathfrak{u}(1) \) subalgebra inside the center of each unitary subalgebra \( \mathfrak{u}_\kappa(2) \) appears at the left-lower corner in each block. The global block pattern reproduces the pattern

\[
P_1 P_2 \quad J \quad M \quad I \quad B
\]

made by the three generators \( P_1, P_2, J \) of a real type CK algebra \( \mathfrak{so}_{\kappa_1, \kappa_2}(3) \). This diagram will be extremely helpful for visualization of most properties discussed below.

There is a single quadratic Lie algebra Casimir in \( \mathfrak{su}_{\kappa_1, \kappa_2}(3) \). A good way of writing it, with each group of terms corresponding to one of the three \( \mathfrak{su}_{\kappa}(2) \)-like subalgebras is:

\[
C = \kappa_2((P_1^2 + Q_1^2) + \kappa_1 H_1^2) + ((P_2^2 + Q_2^2) + \kappa_1 \kappa_2 H_2^2) + \kappa_1((J^2 + M^2) + \kappa_2 B^2). \] (3.6)

The elements defining a 2D CK complex hermitian geometry are analogous to the ones in the real case [1, 30]. By a two-dimensional complex CK geometry we will understand the set of three symmetric homogeneous spaces of points and lines of first- and second-kind.

- The plane as the set of points corresponds to the symmetric homogeneous space

\[
\mathbb{C}S^2_{\kappa_1, \kappa_2} \equiv SU_{\kappa_1, \kappa_2}(3)/H_{(1)} \equiv SU_{\kappa_1, \kappa_2}(3)/(U(1) \otimes SU_{\kappa_2}(2)) \quad H_{(1)} = \langle I; J, M, B \rangle \] (3.7)

whose dimension (over \( \mathbb{C} \)) is 2. The generators \( I \) and \( J, M, B \) leave a point \( O \) (the origin) invariant, and so generate a direct product \( U(1) \otimes SU_{\kappa_2}(2) \) of ‘rotations’ about \( O \). The involution \( \Pi_{(1)} \) is the reflection around \( O \) and \( P_1, Q_1 \) (resp. \( P_2, Q_2 \)) move \( O \) and generate translations along the (complex) basic direction \( l_1 \) (resp. \( l_2 \)).

- The set of first-kind complex lines is identified to the symmetric homogeneous space

\[
\mathbb{C}S^2_{\kappa_1, \kappa_2} \equiv SU_{\kappa_1, \kappa_2}(3)/H_{(2)} \equiv SU_{\kappa_1, \kappa_2}(3)/(U(1) \otimes SU_{\kappa_1}(2)) \quad H_{(2)} = \langle T_1; P_1, Q_1, H_1 \rangle \] (3.8)
with dimension 2 over \( \mathbb{C} \). The generators \( T_1 \) and \( P_1, Q_1, H_1 \) should be interpreted in \( \mathbb{C}S_{\kappa_1, \kappa_2}^2 \) as the generators of 'rotations' about the 'origin' line \( l_1 \), which is left invariant by them. The point \( O \) is moved along two (complex) basic directions by \( J, M \) and \( P_2, Q_2 \). The reflexion in \( l_1 \) is \( \Pi(2) \). Complex lines obtained by group motions from the basic fiducial line \( l_1 \) will be called first-kind lines.

- There is another set of complex lines, the complex-2D symmetric homogeneous space

\[
SU_{\kappa_1, \kappa_2}(3)/H(02) \equiv SU_{\kappa_1, \kappa_2}(3)/(U(1) \otimes SU_{\kappa_1, \kappa_2}(2)) \quad H(02) = \langle T_2; P_2, Q_2, H_2 \rangle. \tag{3.9}
\]

In this space \( T_2 \) and \( P_2, Q_2, H_2 \) leave invariant an 'origin' line \( l_2 \) while \( J, M \) and \( P_1, Q_1 \) move it. The reflexion in \( l_2 \) is \( \Pi(02) \). These lines will be called second-kind. They are actually different from first-kind ones only when \( \kappa_2 \leq 0 \), since when \( \kappa_2 > 0 \) \( P_1, Q_1 \) and \( P_2, Q_2 \) are conjugate within \( \mathfrak{su}_{\kappa_1, \kappa_2}(3) \).

Consideration of the spaces of first and second kind lines can be bypassed, since lines can be seen not as 'points' in the spaces \( \mathbb{C}S_{\kappa_1, \kappa_2}^2 \) or \( SU_{\kappa_1, \kappa_2}(3)/H(02) \), but alternatively as 1D complex submanifolds of \( \mathbb{C}S_{[\kappa_1], \kappa_2}^2 \), and all properties of the two spaces of lines can be transcribed in terms of this space, in which \( l_1 \) and \( l_2 \) should be considered as two hermitian orthogonal complex lines intersecting at \( O \) (see figure 4.1 for \( \eta = 1 \)). This space \( \mathbb{C}S_{[\kappa_1], \kappa_2}^2 \) has a complex hermitian metric with an associated real 'Fubini-Study' metric ('FS') given by the real part of the hermitian product. This 'FS' metric can also be derived directly from the Casimir (3.6): at the origin \( O \) the hermitian product is given by the matrix \( \text{diag}(1, \kappa_2) \), and the 'FS' metric by \( \text{diag}(1, 1, \kappa_2, \kappa_2) \) (basis ordering \( P_1, Q_1, P_2, Q_2 \)); at other points they are uniquely determined by invariance. This 'FS' metric is definite positive when \( \kappa_2 > 0 \), degenerate for \( \kappa_2 = 0 \) and indefinite of real type (2, 2) for \( \kappa_2 < 0 \); when \( \kappa_2 = 1 \) it is the ordinary FS metric (elliptic or hyperbolic) with holomorphic curvature \( 4\kappa_1 \). The line \( l_1 \) (resp. \( l_2 \)) contains two 'FS' orthogonal geodesics through \( O \), the orbits of \( O \) by the one-parameter subgroups generated by \( P_1 \) and \( Q_1 \) (resp. \( P_2 \) and \( Q_2 \)).

Thus \( \kappa_1 \) is (one fourth of) the constant holomorphic curvature, and \( \kappa_2 \) determines the signature of both the hermitian metric and the 'FS' metric, hereafter denoted as FS for any space in the CKD family. The canonicalconexion of \( \mathbb{C}S_{[\kappa_1], \kappa_2}^2 \) as homogeneous symmetric space [27] is compatible with the FS metric. A suitable rescaling of generators \( P_1, J_{12} \) allows to reduce \( \kappa_1 \) (resp. \( \kappa_2 \)) to \( \pm 1 \). Thus nine 2D-complex hermitian CK geometries are obtained; their groups of motion and isotopy subgroups are displayed in table 1.

A fundamental property of the whole scheme of CK geometries is the existence of an 'automorphism' of each family, called ordinary duality \( D \). It is well defined for any dimension, and for the 2D case it is given by the following family automorphism:

\[
D : \begin{cases} (P_1, Q_1, P_2, Q_2; J, M, H_2, T_2) \to (-J, -M, -P_2, -Q_2; -P_1, -Q_1, H_2, -T_2) \\ (\kappa_1, \kappa_2) \to (\kappa_2, \kappa_1) \end{cases} \tag{3.10}
\]

Duality \( D \) leaves the general commutation rules (4.1) invariant while it interchanges the corresponding constants \( \kappa_1 \leftrightarrow \kappa_2 \) and the space of points with the space of first-kind lines, \( \mathbb{C}S_{[\kappa_1], \kappa_2}^2 \leftrightarrow \mathbb{C}S_{[\kappa_1], [\kappa_2]}^2 \) preserving the space of second-kind lines. It relates in general two different geometries placed in symmetrical positions relative to the main diagonal in table 1, just like in the real case. Duality also underlies the introduction of the Cartan generators (3.2): \( B, I \) form a natural basis for the fiducial Cartan subalgebra (\( B \) is the unique Cartan generator in the \( SU_{\kappa_2}(2) \) part, and \( I \) in the \( U(1) \) part, of the isotopy.
Table 1: The nine two-dimensional complex hermitian CK geometries. At each entry the group $G$ and the three subgroups $H_{(1)}, H_{(2)}, H_{(02)}$ are displayed

| Measure of angle $\kappa_1$ | Elliptic | Parabolic $\kappa_1 = 0$ | Hyperbolic $\kappa_1 = -1$ |
|-----------------------------|----------|------------------------|--------------------------|
| $\kappa_2 = 1$              | $SU(3)$  | $IU(2)$                | $SU(2,1)$                |
| $H_{(1)} = U(1) \otimes SU(2)$ | $H_{(1)} = U(1) \otimes SU(2)$ | $H_{(1)} = U(1) \otimes SU(2)$ |
| $H_{(2)} = U(1) \otimes SU(2)$ | $H_{(2)} = U(1) \otimes IU(1)$ | $H_{(2)} = U(1) \otimes SU(1,1)$ |
| $H_{(02)} = U(1) \otimes SU(2)$ | $H_{(02)} = U(1) \otimes IU(1)$ | $H_{(02)} = U(1) \otimes SU(1,1)$ |
| Elliptic $\kappa_2 = 0$    | $IU(2)$  | $IU(1)$                | $ISU(1,1)$               |
| $H_{(1)} = U(1) \otimes IU(1)$ | $H_{(1)} = U(1) \otimes IU(1)$ | $H_{(1)} = U(1) \otimes IU(1)$ |
| $H_{(2)} = U(1) \otimes SU(2)$ | $H_{(2)} = U(1) \otimes SU(1,1)$ | $H_{(2)} = U(1) \otimes SU(1,1)$ |
| $H_{(02)} = U(1) \otimes IU(1)$ | $H_{(02)} = U(1) \otimes IU(1)$ | $H_{(02)} = U(1) \otimes IU(1)$ |
| Subalgebra of a point in $\mathbb{C}S^2_{[\kappa_1,\kappa_2]}$, $H_1, T_1$ appear as their duals, and $H_2, T_2$ have the simplest behaviour under duality. In terms of the block-triangular arrangement (3.5), duality corresponds to a ‘block reflection’ along the secondary diagonal and an eventual sign change. For Cartan generators duality can be depicted by (figure 1), related to the $\mathfrak{su}(3)$ root diagram. More details on the geometric interpretation of the Cartan subalgebra generators $B, I, T_1, T_2, H_1, H_2$ will be given later.

![Figure 1: Fiducial Cartan subalgebra generators and their behaviour under duality.](image)

### 3.2 Realization of the spaces of points in the complex Hermitian Cayley–Klein spaces

Exponentiation of the matrix representation (3.1) and (3.3) of $\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ produces a representation of $SU_{\kappa_1,\kappa_2}(3)$ as a linear transformations group in an ambient linear space $\mathbb{C}^3 = (z^0, z^1, z^2)$. The one-parametric subgroups generated by $P_1, P_2, Q_1, Q_2, J$ and $M$
the groups act on these coordinates by complex fractional linear transformations. Vector models in the $z$-plane, where the cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions with label $\kappa$ are defined by:

$$C_\kappa(x) := \begin{cases} \cos \sqrt{\kappa} x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh \sqrt{-\kappa} x & \kappa < 0 \end{cases}$$

$$S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \kappa < 0 \end{cases} \quad (3.12)$$

These functions coincide with the circular and hyperbolic trigonometric ones for $\kappa = 1$ and $\kappa = -1$; the case $\kappa = 0$ provides the so-called ‘parabolic’ or galilean functions: $C_0(x) = 1$, $S_0(x) = x$. General properties of these functions are given in the Appendix in [1].

The exponentials of the Cartan subalgebra generators $B, I, T_1, T_2, H_1, H_2$ are:

$$e^{Bx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-ix} & 0 \\ 0 & 0 & e^{ix} \end{pmatrix}, \quad e^{Ix} = \begin{pmatrix} e^{-\frac{ix}{\sqrt{\kappa}}} & 0 & 0 \\ 0 & e^{\frac{ix}{\sqrt{\kappa}}} & 0 \\ 0 & 0 & e^{ix} \end{pmatrix}$$

$$e^{T_{1x}} = \begin{pmatrix} e^{i\frac{ix}{\sqrt{\kappa}}} & 0 & 0 \\ 0 & e^{-i\frac{ix}{\sqrt{\kappa}}} & 0 \\ 0 & 0 & e^{i\frac{ix}{\sqrt{\kappa}}} \end{pmatrix}, \quad e^{T_{2x}} = \begin{pmatrix} e^{-i\frac{ix}{\sqrt{\kappa}}} & 0 & 0 \\ 0 & e^{i\frac{ix}{\sqrt{\kappa}}} & 0 \\ 0 & 0 & e^{-i\frac{ix}{\sqrt{\kappa}}} \end{pmatrix} \quad (3.13)$$

$$e^{H_{1x}} = \begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & e^{ix} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{H_{2x}} = \begin{pmatrix} e^{-ix} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{ix} \end{pmatrix}.$$

Any element $U \in SU_{\kappa_1,\kappa_2}(3)$, satisfying $U^\dagger U = \Lambda$, $\det U = 1$ where $U^\dagger = U^\dagger^\top$, can be written as a product of matrices (3.11) and two commuting ‘Cartan’ transformations in (3.13). The action of $SU_{\kappa_1,\kappa_2}(3)$ on $\mathbb{C}^3$ is linear but not transitive, since it conserves the hermitian form $|z^0|^2 + \kappa_1 |z^1|^2 + \kappa_1 \kappa_2 |z^2|^2$. The isotopy subgroup of $O = (1, 0, 0)$ is the three parameter subgroup $SU_{\kappa_2}(2)$ generated by $\langle J, M, B \rangle$, while the $U(1)$ subgroup generated by $I$ multiplies $O$ by a phase factor. Hence the homogeneous symmetric space $\mathbb{C}S^2_{[\kappa_1],\kappa_2} \equiv SU_{\kappa_1,\kappa_2}(3)/(U(1) \otimes SU_{\kappa_2}(2))$ can be identified to the orbit of the ray $[O]$ of the vector $O$ under the action of $SU_{\kappa_1,\kappa_2}(3)$. This orbit is the domain of $\mathbb{CP}^2$ determined by $|z^0|^2 + \kappa_1 |z^1|^2 + \kappa_1 \kappa_2 |z^2|^2 > 0$, and when $\kappa_1 > 0, \kappa_2 > 0$ it is the full complex projective space $\mathbb{CP}^2$. The coordinates $(z^0, z^1, z^2)$ can be called Weierstrass coordinates; they are linked by $|z^0|^2 + \kappa_1 |z^1|^2 + \kappa_1 \kappa_2 |z^2|^2 = 1$ and still are defined up to a common unimodular complex factor which can be used to make $z^0$ real and non negative; these are the natural coordinates in the vector models of the Hermitian CK spaces, since the motion groups act linearly on them. Also in analogy with the real case, $(z^1/z^0, z^2/z^0)$ are called Beltrami coordinates; the groups act on these coordinates by complex fractional linear transformations.
The non-generic situation where \( \kappa_1, \kappa_2 \) vanishes corresponds to an \( \text{In"on"u–Wigner} \) contraction [31]. The limit \( \kappa_1 \to 0 \) is a local contraction (around a point); it carries the first and third columns of table 1 to the flat middle one. The limit \( \kappa_2 \to 0 \) is an axial contraction (around a line), carrying geometries of first and third rows to the middle one.

## 4 The complete family of ‘complex Hermitian’ Cayley–Klein–Dickson 2D geometries and their spaces

The previous section has been written so it can be re-read with minimal mutatis mutandis changes to suit the description of the full family of ‘complex type’ spaces. The new fact is the explicit Cayley-Dickson (CD) label \( \eta \) in the ‘labelled’ CD doubling \( \mathbb{R} \to \eta \mathbb{R}: x \to x + iy \), where \( i^2 = -\eta \). There are three different cases: \( \eta > 1 \) can be rescaled to \( \eta = 1 \) and gives the division algebra of ordinary complex numbers \( \mathbb{C} \); \( \eta = 0 \) gives the dual or Study numbers \( \mathbb{D} \); and \( \eta < 0 \), which can be rescaled to \( \eta = -1 \) the split complex numbers \( \mathbb{C}_1 \), also called double numbers, hyperbolic complex numbers, Lorentz numbers or perplex numbers. These are three instances of a one-parameter system \( \eta \mathbb{R} \cong \mathbb{C}_\eta \), the notation duplicity stressing either the ‘complex’ nature of the numerical system here obtained \( (\mathbb{C}_\eta) \) or its character as a CD extension of \( \mathbb{R} \) \((\eta \mathbb{R})\).

### 4.1 The ‘complex Hermitian’ Cayley–Klein–Dickson 2D geometries

The groups behind these geometries are the linear isometry groups of a ‘complex hermitian’ form \( \langle z, w \rangle = \sum_{i,j} A_{ij} z_i w_j \) in the \( N+1 \) dimensional ‘complex’ ambient linear space \( \mathbb{C}_{\eta}^{N+1} \cong \eta \mathbb{R}^{N+1} = (z_0, z_1, \ldots, z_N) \) with the same \( A \) as in the complex case. For \( N = 2 \) the CKD algebra, denoted \( \eta \mathbf{su}_{\kappa_1, \kappa_2}(3) \) is eight dimensional, and its fundamental or vectorial 3D ‘complex’ representation is given by \( 3 \times 3 \) matrices (3.1, 3.3), where now entries are in \( \mathbb{C}_{\eta} \) and \( i \) stands for the pure imaginary ‘complex’ unit in \( \mathbb{C}_{\eta} \). This form has ‘hermitian’ symmetry \( \langle w, z \rangle = \overline{\langle z, w \rangle} \), with ‘complex’ conjugation in \( \mathbb{C}_{\eta} \): \( z = a + ib \to \overline{z} = a - ib \), for real \( a, b \) and the form \( \langle z, w \rangle \to \text{Im}(z, w) \) is still real and antisymmetric in \( z, w \), and therefore is a symplectic form in the real space \( \mathbb{R}^{2(N+1)} \) underlying to \( \eta \mathbb{R}^{N+1} \). Rosenfeld [13] uses the word hermitian without qualifying, but to prevent misunderstandings, we will keep the term hermitian for the truly complex case, and we will put quotes in ‘complex’ and ‘hermitian’ when referring to the general ‘complex’ numbers \( \mathbb{C}_{\eta} \) with Cayley-Dickson label \( \eta \). A simple scale change may reduce simultaneously \( \eta \) and \( \kappa_1, \kappa_2 \) to either \( 1, 0, -1 \).

When \( \eta = 1 \), the CKD algebra \( \eta \mathbf{su}_{\kappa_1, \kappa_2}(3) \) is isomorphic to the CK Lie algebra \( \mathbf{su}_{\kappa_1, \kappa_2}(3) \); it is simple when \( \kappa_1, \kappa_2 \) are different from zero. When \( \eta = -1 \) and for any non-zero values of \( \kappa_1, \kappa_2 \), the CKD algebra \( \eta \mathbf{su}_{\kappa_1, \kappa_2}(3) \) is also simple and isomorphic to the Lie algebra \( \mathbf{sl}(3, \mathbb{R}) \). Generators \( B, I, T_1, T_2, H_1, H_2 \) still belong to the fiducial two-dimensional Cartan subalgebra of \( \eta \mathbf{su}_{\kappa_1, \kappa_2}(3) \).

For any value of \( \eta \) the CKD algebras in the family \( \eta \mathbf{su}_{\kappa_1, \kappa_2}(3) \) can be endowed with a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) group of commuting involutive automorphisms generated by \( \Pi_{(1)}, \Pi_{(2)} \) (3.4); denoting everything as in the former section, the three Lie subalgebras \( \mathfrak{h}_{(1)}, \mathfrak{h}_{(2)} \) and \( \mathfrak{h}_{(02)} \) spanned by the generators with the same name as in the complex case (the Lie algebra elements invariant under the involutions with the same indices), turn out to be of CKD type, \( \eta \mathfrak{u}(1) \oplus \mathfrak{su}_{\kappa}(2) \) with \( \kappa = \kappa_2, \kappa_1, \kappa_1 \kappa_2 \) respectively. The groups they generate are isomorphic to \( \eta \mathfrak{U}(1) \oplus \eta \mathbf{SU}_{\kappa_2}(2) \), \( \eta \mathfrak{U}(1) \oplus \eta \mathbf{SU}_{\kappa_1}(2) \) and \( \eta \mathfrak{U}(1) \oplus \eta \mathbf{SU}_{\kappa_1 \kappa_2}(2) \). In all
these expressions, the Lie algebras of CKD ‘unitary type’ are $\eta\mathfrak{u}_\kappa(2) \equiv \eta\mathfrak{u}(1) \oplus \eta\mathfrak{su}_\kappa(2)$, and for the groups of ‘unimodular complex numbers’ $\eta U(1)$ we have two generic cases $\eta U(1) \equiv U(1) \equiv SO(2)$, $\eta U(1) \equiv SO(1,1)$ and one limiting case $\eta U(1) \equiv ISO(1) \equiv \mathbb{R}$.

The Lie algebra $\eta\mathfrak{su}_{\kappa_1,\kappa_2}(3)$ is given by the following Lie commutators:

$$
\begin{align*}
[P_1, P_2] &= \kappa_1 J & [P_1, J] &= -P_2 & [P_1, M] &= -Q_2 & [P_1, B] &= Q_1 & [P_1, I] &= -Q_1 & [P_1, H_1] &= -2Q_1 & [P_1, H_2] &= -2Q_2 \\
[P_2, Q_1] &= \kappa_1 M & [P_2, J] &= \kappa_2 P_1 & [P_2, M] &= -\kappa_2 Q_1 & [P_2, B] &= -\kappa_2 P_1 & [P_2, I] &= -\kappa_2 Q_1 & [P_2, H_1] &= -2\kappa_2 P_1 & [P_2, H_2] &= -2\kappa_2 Q_2 \\
[Q_1, Q_2] &= \eta\kappa_1 J & [Q_1, J] &= -Q_2 & [Q_1, M] &= \eta\kappa_2 P_1 & [Q_1, B] &= -\eta\kappa_2 P_1 & [Q_1, I] &= \eta\kappa_2 P_1 & [Q_1, H_1] &= \eta\kappa_2 P_1 & [Q_1, H_2] &= \eta\kappa_2 Q_2 \\
[Q_2, J] &= \kappa_2 Q_1 & [Q_2, M] &= \kappa_2 Q_1 & [Q_2, B] &= \kappa_2 Q_1 & [Q_2, I] &= \kappa_2 Q_1 & [Q_2, H_1] &= \kappa_2 Q_1 & [Q_2, H_2] &= \kappa_2 Q_2
\end{align*}
$$

The elements defining a 2D CK ‘complex hermitian’ geometry can be now described:

- The plane as the set of points corresponds to the symmetric homogeneous space
  $$
  \mathbb{C}\eta S^2_{\kappa_1,\kappa_2} \equiv \eta SU_{\kappa_1,\kappa_2}(3)/H(1) \equiv \eta SU_{\kappa_1,\kappa_2}(3)/(\eta U(1) \otimes \eta SU_{\kappa_2}(2)) \quad H(1) = \langle I, J, M, B \rangle.
  $$

- The set of (first-kind) ‘complex’ lines is identified to the symmetric homogeneous space
  $$
  \mathbb{C}\eta S^2_{\kappa_1,\kappa_2} \equiv \eta SU_{\kappa_1,\kappa_2}(3)/H(2) \equiv \eta SU_{\kappa_1,\kappa_2}(3)/(\eta U(1) \otimes \eta SU_{\kappa_1}(2)) \quad H(2) = \langle T_1; P_1, Q_1, H_1 \rangle.
  $$

Both spaces have again dimension 2 over $\mathbb{C}\eta$, and all comments made in the complex case can be easily rephrased. The definition of the space of second kind ‘complex’ lines can be also suitably adapted.

![Diagram](image)

Figure 2: Generators and their associated labels in a ‘complex hermitian’-2D CKD geometry. Lines $l_1$ and $l_2$ are ‘complex’, thus two-dimensional from a real point of view.

By a two-dimensional ‘complex hermitian’ CKD geometry we will mean the set of three symmetric homogeneous spaces of points, lines of first-kind and lines of second-kind. The group $\eta SU_{\kappa_1,\kappa_2}(3)$ acts transitively on each of these spaces. The fundamental ordinary duality $\mathcal{D}$ (3.10) extends, by simply assuming $\mathcal{D}: \eta \rightarrow \eta$, to an ‘automorphism’ of the complete CKD family and leaves the general commutation rules (4.1) invariant. In general
\( \mathcal{D} \) relates two different ‘complex hermitian’ CKD geometries with the same label \( \eta \) but \( \kappa_1, \kappa_2 \) interchanged. Figure 2 displays the generators (with their labels) as related to the three fiducial elements \( O, l_1, l_2 \).

The quadratic Lie algebra Casimir in \( \eta \text{su}_{\kappa_1, \kappa_2}(3) \) can be written grouping the terms which correspond to the three \( \eta \text{su}_{\kappa}(2) \)-like subalgebras:

\[
\mathcal{C} = (\eta P_2^2 + Q_2^2 + \kappa_1 \kappa_2 H_2^2) + \kappa_2((\eta P_1^2 + Q_1^2) + \kappa_1 H_1^2) + \kappa_1((\eta J^2 + M^2) + \kappa_2 B^2). \tag{4.4}
\]

From this Casimir we can easily derive the invariant FS metric in the space \( \mathbb{C}\eta S^2_{[\kappa_1, \kappa_2]} \) which is given, at the origin and in the basis \( P_1, Q_1, P_2, Q_2 \) by the matrix \( \text{diag}(1, \eta, \kappa_2, \eta \kappa_2) \), coming also as the real part of the hermitian product whose matrix at \( O \) is \( \text{diag}(1, \kappa_2) \).

We should mention that the CKD algebras in the family \( \eta \text{su}_{\kappa_1, \kappa_2}(3) \) can actually be endowed with a \( \mathbb{Z}_2 \otimes (\mathbb{Z}_2 \otimes \mathbb{Z}_2) \) group of commuting involutive automorphisms; in addition to \( \Pi_{(1)}, \Pi_{(2)} \) given by \( (3.4) \), the extra involution \( (1)\Pi \) will not play any explicit role in what follows, but it is mentioned for completeness; the homogeneous spaces \( SU(3)/SO(3) \), \( SU(2, 1)/SO(2, 1) \) and \( SL(3, \mathbb{R})/SO(3) \), \( SL(3, \mathbb{R})/SO(2, 1) \) appear by mimicking the former construction using this involution:

\[
(1)\Pi : (P_1, P_2, Q_1, Q_2; J, M, B, I) \rightarrow (P_1, P_2, -Q_1, -Q_2; J, -M, -B, -I). \tag{4.5}
\]

### 4.2 Realization of spaces of points in the ‘complex hermitian’ Cayley–Klein–Dickson spaces

When \( \eta \) is present, the fundamental 3D ‘complex’ matrix representation \( (3.1, 3.3) \) exponentiates to a representation of \( \eta SU_{\kappa_1, \kappa_2}(3) \) as a linear transformations group in the ambient linear space \( \mathbb{C}\eta^3 \). One-parametric subgroups corresponding to the generators so far considered are given again by \( (3.11, 3.13) \), where now the exponential \( e^{ix} \) is related to the sine and cosine with label \( \eta \) by a Euler-like formula:

\[
e^{ix} = C_\eta(x) + iS_\eta(x). \tag{4.6}
\]

Again the action of \( \eta SU_{\kappa_1, \kappa_2}(3) \) on \( \mathbb{C}\eta^3 \) is linear but not transitive, since it conserves the ‘hermitian’ form \( |z|^2 + \kappa_1 |z_1|^2 + \kappa_2 |z_2|^2 \). The isotopy subgroup of the point \( O \) whose position vector is \( O \equiv (1, 0, 0) \) is easily seen to be the three parameter subgroup \( \eta SU_{\kappa_2}(2) \) generated by \( J, M, B \), while the \( \eta U(1) \) subgroup generated by \( I \) multiplies this vector by a unimodular ‘complex’ phase factor. Hence the homogeneous symmetric space \( \mathbb{C}\eta S^2_{[\kappa_1, \kappa_2]} \equiv \eta SU_{\kappa_1, \kappa_2}(3)/(\eta U(1) \otimes \eta SU_{\kappa_2}(2)) \) can be identified to the orbit of the ray \( [O] \) under the action of the group \( \eta SU_{\kappa_1, \kappa_2}(3) \).

The geometry behind the case \( \eta < 0 \) differs greatly from the one in the ordinary complex case: for \( \eta < 0, \kappa_2 \neq 0 \) the FS metric is always indefinite and of \( (2, 2) \) real type, no matter of the sign of \( \kappa_2 \). The four spaces of points with \( \eta < 0, \kappa_1 \neq 0, \kappa_2 \neq 0 \) are essentially the same, though the choices of lines and FS geodesics of first- and second-kind are interchanged; this is tantamount to what happens in the real case for the 1+1 Anti-DeSitter and DeSitter spaces. These four spaces can be realized as spaces of 0-pairs in \( \mathbb{R}P^2 \) (pairs made from a point and an hyperplane (here line) in the real projective plane \( \mathbb{R}P^2 \)), and the distance between two 0-pairs \( (X; \alpha), (Y; \beta) \) is related to the cross ratio of the four points \( X, Y; Z, T \), where \( Z, T \) are the intersections of the line determined by \( X, Y \) with the hyperplanes \( \alpha, \beta \) (see Rosenfeld book [13], theorems 2.39 and 4.21).
The non-generic situation where a coefficient \( \eta; \kappa_1, \kappa_2 \) vanishes corresponds to an İnönü–Wigner contraction [31]. The limit \( \kappa_1 \to 0 \) is a local-contraction (around a point). The limit \( \kappa_2 \to 0 \) is a line-contraction (around a whole ‘complex’ line). Finally, the limit \( \eta \to 0 \) corresponds to a new kind of contraction around a purely real submanifold, the projectivized real \( \kappa_1, \kappa_2 \) CK space. Contractions are built-in in the expressions associated to the ‘complex hermitian’ CK geometries and groups, simply by making zero any of the constants \( \eta; \kappa_1, \kappa_2 \) (determining the curvature and signature of the space).

5 The compatibility conditions for a triangular loop

In this section we discuss the approach to the trigonometry of the twenty seven ‘complex hermitian type’ CKD spaces, and we introduce the ‘complex hermitian’ compatibility equations, loop equations and the basic trigonometric identity. Some general comments on this approach are given in [1] and will not be repeated here; specially we refer to the choice of ‘external angles’ at the vertex \( A \) and the fact that the standard angular excess appears without the explicit presence of the measure of twice a quadrant of angle (which equals \( \pi \) when \( \kappa_2 = 1 \)).

A triangle in a ‘complex hermitian’ CKD space can be seen either as a triangle point loop or dually as a triangle line loop (see figure (3)). In the first case a point \( C \) is considered to move to a different point \( B \) ‘translating’ either along the geodesic segment \( CB \), or along the two geodesic segments \( CA \) and \( AB \). Dually, the geodesic \( c \equiv AB \) is considered to move to a different geodesic \( b \equiv CA \) ‘rotating’ either about the vertex \( A \equiv bc \), or about the vertices \( B \equiv ca \) and then \( C \equiv ab \). There exists though a very important difference with the real 2D case: the ‘translations’ along a geodesic are not uniquely defined by the geodesic only. Thus to make out sense of the idea of triangle loop a closer analysis of the geometry is required. Any geodesic \( g \) through \( C \) determine a well defined ‘complex’ line \( C_{\eta}g \) containing \( g \). Thus for the two geodesics \( a, b \) intersecting at the vertex \( C \) there are two uniquely determined ‘complex’ lines \( C_{\eta}a, C_{\eta}b \) through \( C \). These two ‘complex’ lines will lie on a (generically) well defined line-geodesic \( G_C \) (also called a line-chain). This is dual to the determination of \( a \) (generically) well defined geodesic \( g_a \) through two different points \( C, B \). This subtlety is not necessary in the real case, as then the set of all lines through a point \( C \) is one-dimensional, while in the complex case, the set of ‘complex’ lines through \( C \) is two-dimensional.

![Figure 3: a) Triangular point loop. b) Triangular line loop.](image)

To start with the study of trigonometry we will take as ‘sides’ and ‘angles’, the canonical parameters of certain one-parameter subgroup elements associated to some algebra generators. To explain this choice, we first select a (real) flag \( O \subset g_1 \subset l_1 \subset G_1 \) as follows: \( O \) is the origin point \( O = [(1,0,0)] \), \( g_1 \) is the orbit of \( O \) under the one-dimensional subgroup generated by \( P_1 \) (thus the ‘complex’ line \( l_1 \) is the orbit of \( O \) under the subgroup generated by \( P_1, Q_1 \)) and the line-geodesic \( G_1 \), is the orbit of \( l_1 \) under the subgroup generated by \( J \); this flag is determined by singling out the generators \( P_1, Q_1, J \). Now move
the triangle to the canonical position where $C$ coincides with $O$, the side $a$ is on $g_1$ and
the side $b$ lies on a geodesic in $G_1$. This ‘canonical’ position guarantees that the side $b$
is obtained from $a$ by means of two commuting rotations generated by $J$ and $I$, where
the phase rotation generator $I$ is the unique generator in the fiducial Cartan subalgebra
commuting with $J$. As angular invariants take the canonical parameters $C$ (‘Hermitian’
or ‘pure’ angle between ‘complex’ lines) and $\Phi_C$ (angular phase between real-1D geodesics
within a ‘complex’ line) of the two ‘rotations’ whose product
\[ e^{CJ} e^{\Phi_C I} \]  
(5.1)
carries the side $a$ to coincide with $b$; this products will be called complete rotations about
the vertex $C$.

Since ‘hermitian’ CKD spaces are rank-one, and therefore each pair of points have a single
invariant, it would seem enough to consider the FS distance $a$ between the points $C, B$
as the unique moduli of sides. This is what was done in the previous works on hermitian
trigonometry [9, 10, 11, 13]. But the formal duality requirement prompts the consideration
of translation partners to both $J; I$ and since duality maps $J; I$ into $-P_1; -T_1$, this
suggests the use of the following complete translations:
\[ e^{aP_1} e^{\phi_a T_1}. \]  
(5.2)
As well as the duality requirement, there are geometrical reasons for the use of the extra
‘translation’ $e^{\phi_a T_1}$: the ‘pure’ translation $e^{aP_1}$ carries the vertex $C$ to $B$, but this alone
does not carry the unique ‘complex line’-geodesic at the vertex $C$ determined by $C_\eta a, C_\eta b$,
to the ‘complex’ line-geodesic at $B$ determined by $C_\eta b, C_\eta c$; an additional $e^{\phi_a T_1}$ is required
to bring them into each other.

If we now consider complete rotations at each vertex, and complete translations along
each side, duality is manifestly restored: at each vertex a complete rotation is required
to bring into coincidence simultaneously the sides seen both as ‘complex’ lines and as
point-geodesic sides. And along each side, a complete translation is required to bring into
coincidence simultaneously both the vertices as points and the ‘complex’ line-geodesics
determined at each vertex by the two sides.

This choice of two commuting generators is very natural from a Quantum Mechanics
viewpoint and afford six ‘vertex’ quantities (three Hermitian or pure angles $A, B, C$
and three angular phases, $\Phi_A, \Phi_B, \Phi_C$), and six ‘side’ quantities (three lengths, $a, b, c$
and three lateral phases $\phi_a, \phi_b, \phi_c$). All these invariants appear as canonical parameters
of pairs of commuting isometries, respectively generated by $J_A, J_B, J_C$; $I_A, I_B, I_C$ and
$P_a, P_b, P_c; T_a, T_b, T_c$. At each side $P_a, P_b, P_c$ are pure translation generators that perform
the canonical parallel transport along their FS geodesic axes, and $T_a, T_b, T_c$ are the only
Cartan generators in the isotopy subalgebras of the sides $a, b, c$ commuting with $P_a, P_b, P_c$
respectively.

Cartan generators exponentiate to somewhat ‘hybrid’ transformations. The Cartan
subalgebra is contained in the isotopy subalgebra of $O$, so its elements generate ‘rotations’
about $O$ and conjugates of them rotations about other points. The phase ‘rotation’
part $e^{\phi T_1}$ of the complete fiducial translation $e^{aP_1} e^{\phi_a T_1}$ apparently breaks the scheme
symmetry between rotations and translations. Nevertheless, since any Cartan transformation
as $e^{\phi T_1}$ leaves pointwise invariant the ‘complex’ line $l_1$ it should also be considered
a ‘translation’ along $l_1$. Thus Cartan transformations are both rotations about a point
three angles, the three lateral phases and the three angular phases.

From now on everything follows the real pattern [1], and the commutativity between both components of a complete transformation allows the extension of the basic real identities to ‘complex’ ones: compatibility identities, point loop and side loop equations, and basic trigonometric identity.

The generators $P_a, T_a, P_b, T_b, P_c, T_c, J_A, I_A, J_B, J_C, I_B, I_C$ are not independent. They are related by several compatibility conditions:

$$
\begin{align*}
(P_b) &= e^{C_J} e^{\Phi_C I_c} (P_a) e^{-\Phi_C I_c} e^{-C_J} \quad (J_B) = e^{C_P} e^{\phi_a T_a} (J_A) e^{-\phi_a T_a} e^{-C_P} \\
(T_b) &= e^{-A_I} e^{-\Phi_I A} (T_a) e^{\Phi_I A} e^{-A_I} \quad (J_C) = e^{-a P_a} e^{-\phi_a T_a} (J_B) e^{\phi_a T_a} e^{a P_a} \\
(P_c) &= e^{B_J} e^{\Phi_B I_B} (P_c) e^{-\Phi_B I_B} e^{-B_J} \quad (J_A) = e^{b P_b} e^{\phi_b T_b} (J_C) e^{-\phi_b T_b} e^{-b P_b}
\end{align*}
$$

which can be considered as an implicit group theoretical definition for the three sides, the three angles, the three lateral phases and the three angular phases.

All the trigonometry of the ‘complex’ CKD space is completely contained in these equations, which have as a remarkable property their explicit invariance under the duality interchange $a, b, c \leftrightarrow A, B, C$ and $\phi_a, \phi_b, \phi_c \leftrightarrow \Phi_A, \Phi_B, \Phi_C$ for triangle group theoretical invariants (sides $\leftrightarrow$ angles, lateral phases $\leftrightarrow$ angular phases) and $P \leftrightarrow J, T \leftrightarrow I$ for generators; this duality is a consequence of the fact that $D (3.10)$ is an automorphism of the family of CKD algebras which interchanges $P_1 \leftrightarrow -J$, and $T_1 \leftrightarrow -I$. These equations resemble their real analogues: the real rotation $e^{A_I} e^{\phi_a T_a}$ or translation $e^{a P_a} e^{\phi_a T_a}$ are replaced by the ‘complete’ products $e^{A_I} e^{\phi_a T_a}$ or $e^{a P_a} e^{\phi_a T_a}$.

Each equation in (5.3) is actually a pair relating both components of each ‘complete’ translation or rotation. As in [1] we will refer to them as $P_b(P_a), T_b(T_a)$, etc. (or $P_a(P_b), T_a(T_b)$ when the equation is read inversely). By cyclic substitution in the three pairs of equations $P_a(P_c), T_a(T_c); P_c(P_a), T_c(T_a)$ and $P_b(P_a), T_b(T_a)$ we find the identity

$$
e^{B_J} e^{\Phi_B I_B} e^{-A_I} e^{-\Phi_I A} e^{C_J} e^{\Phi_C I_C} (P_a) e^{-\Phi_C I_C} e^{-C_J} e^{\Phi_I A} e^{-A_I} e^{-B_J} = (P_a) T_a
$$

$P_c, T_c$, are completely parallel process gives:

$$
e^{-a P_a} e^{-\phi_a T_a} e^{C_J} e^{\phi_a T_a} e^{b P_b} e^{\phi_b T_b} (J_C) e^{-\phi_b T_b} e^{-b P_b} e^{-\phi_b T_b} e^{-c P_c} e^{\phi_c T_c} e^{a P_a} = (J_C). $$
Equations (5.4) and (5.5) can be written alternatively as:

\[ e^{BIB}e^{ΦIB}e^{-AJAe^{ΦAIC}}e^{IC} \]
\[ e^{-aPaΦaTae^{Pae^{ΦaTe}bPb}e^{ΦaTe}} \] must commute with \( P_a \) and \( T_a \),
\[ e^{-aPaΦaTae^{Pae^{ΦaTe}bPb}e^{ΦaTe}} \] must commute with \( J_C \) and \( I_C \).

5.1 Loop excesses and loop equations

Had we not considered lateral phases, the ‘loop’ product \( e^{-aPaΦaTae^{Pae^{ΦaTe}bPb}} \) would have been a natural object associated to the three ‘pure’ translations along the triangle sides \( C \rightarrow \).

This transformation is the ordinary holonomy associated to the triangle, as each factor \( e^{aPa} \) is the ordinary parallel transport operator in the canonical connection of the hermitian space \( ηU(1, \kappa)^2 \). It moves the base point \( C \) along the triangle and returns it back to its original position, so it must be a rotation about \( C \). This rotation is a product of an \( ηU(1) \) phase part and an \( ηSU(\kappa)^2 \) part, but the explicit expression for the \( ηSU(\kappa)^2 \) part is rather involved.

The use of angular and lateral phases in this self-dual approach affords some simple and apparently new results for certain similar ‘loop’ operators. The guideline is the pattern established in the real case, replacing every translation or rotation generators \( T / J \) by its ‘complete’ versions \( P, T / J, I \). We start with the equation which gives \( P_c(P_b), T_c(T_b) \) in the set (5.3), replace \( \begin{pmatrix} P_c \\ T_c \end{pmatrix} \) by \( e^{-cPeΦaTe} \begin{pmatrix} P_c \\ T_c \end{pmatrix} \) and then substitute \( P_c(P_a), T_c(T_a) \) from the compatibility equations to obtain:

\[ e^{-AJAe^{ΦAIC}} \begin{pmatrix} P_b \\ T_b \end{pmatrix} e^{ΦAIC}e^{AJA} = e^{-cPeΦaTe}BIBe^{ΦBIC} \begin{pmatrix} P_a \\ T_a \end{pmatrix} e^{ΦBIC}BIBe^{ΦaTe}e^{-cPeΦaTe}. \]

We introduce \( J_B(J_C), I_B(I_C) \) and trivially simplify and rearrange. Now we use \( J_A(J_C), I_A(I_C) \), simplify, and finally substitute \( P_b(P_a), T_b(T_a) \). This gives:

\[ e^{BIB}e^{ΦBIC}e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe}e^{-AJAe^{ΦAIC}e^{AJA} \begin{pmatrix} P_a \\ T_a \end{pmatrix} \times \]
\[ e^{-ΦCIC}e^{-CJC}e^{ΦAIC}e^{AJA}e^{-ΦaTe}bPb}e^{ΦaTe}e^{-cPeΦaTe}aPa}cPae^{ΦBIC}e^{-BIC} = \begin{pmatrix} P_a \\ T_a \end{pmatrix} \].

The three complete translations along the triangle appear in the former relation in a single piece \( e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe} \), while the three complete rotations are all about the base point \( C \). Now we can go a bit further: (5.6) implies that \( e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe} \) must commute with \( J_C \) and with \( I_C \), so it will commute with any complete rotation \( e^{-ΦXIC}e^{-ΧJC} \) about \( C \), for any values of the complete angle \( X, Φ_X \). Then we can commute the whole complete translation piece \( e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe} \) in (5.8) with the rotations about \( C \) and collect these altogether; as both components of the complete rotation do commute, we get:

\[ e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe}(-A+B+C)JC}e^{(-ΦA+ΦB+ΦC)IC} \]

must commute with \( P_a \) and \( T_a \).

We had already derived that \( e^{-aPaΦaTe}cPeΦaTe}bPb}e^{ΦaTe}XJC}e^{ΦXIC} \) must commute with \( J_C \) and \( I_C \) for any ‘complete angle’ \( X, Φ_X \) (see (5.6)). Since this expression also
commutes with $P_a, T_a$ for the special values $X = -A + B + C$, $\Phi_X = -\Phi_A + \Phi_B + \Phi_C$, we can conclude:

$$e^{-aP_a}e^{-\phi_a}T_a e^{cP_c} e^{\phi_c}T_c e^{-bP_b}e^{\phi_b}T_b e^{(-A+B+C)J_C} e^{(-\Phi_A+\Phi_B+\Phi_C)I_C} = 1 \quad (5.10)$$

because the identity is the only element of $\eta SU_{\kappa_1, \kappa_2}(3)$ commuting with two such pairs of generators as $P_a, T_a$ and $J_C, I_C$. This equation can be also written as:

$$e^{-aP_a}e^{-\phi_a}T_a e^{cP_c} e^{\phi_c}T_c e^{-bP_b}e^{\phi_b}T_b = e^{-(A+B+C)J_C} e^{-(\Phi_A+\Phi_B+\Phi_C)I_C}. \quad (5.11)$$

A similar procedure (or direct use of $(5.3)$ in $(5.11)$) allows us to derive two analogous equations:

$$e^{bP_b} e^{\phi_b} T_b e^{-aP_a} e^{-\phi_a} T_a e^{cP_c} e^{\phi_c} T_c = e^{-(A+B+C)J_A} e^{-(\Phi_A+\Phi_B+\Phi_C)I_A}$$

$$e^{cP_c} e^{\phi_c} T_c e^{-bP_b} e^{\phi_b} T_b e^{-aP_a} e^{-\phi_a} T_a = e^{-(A+B+C)J_B} e^{-(\Phi_A+\Phi_B+\Phi_C)I_B}. \quad (5.12)$$

The quantities defined as

$$\Delta := -A + B + C \quad \Delta_\Phi := -\Phi_A + \Phi_B + \Phi_C \quad (5.13)$$

will be called the (Hermitian) angular excess and the angular phase excess of the triangle loop. The complete angular excess $(\Delta, \Delta_\Phi)$ fits into the view of the geodesic line loop as a geodesic starting on $a$, and successively rotating by complete angles $(C, \Phi_C)$, $(-A, -\Phi_A)$ and $(B, \Phi_B)$ about the three vertices of the triangle; thus $(\Delta, \Delta_\Phi)$ should be looked as the (oriented) total complete angle turned by the geodesic line loop. Equations $(5.11)$ or $(5.12)$, to be called the ‘complex hermitian’ point loop equations, express the product of the three complete translations along the oriented sides of the triangle loop as a complete rotation about the loop base point. These equations are the closest ‘complex Hermitian’ analogues of the Gauss-Bonnet triangle theorem (see [1]) but we have found no reference to such a simple result in the literature.

The explicit duality of the starting equations $(5.3)$ under the interchanges $a, b, c \leftrightarrow A, B, C$ and $\phi_a, \phi_b, \phi_c \leftrightarrow \Phi_A, \Phi_B, \Phi_C$ for sides, angles and phases, and $P \leftrightarrow J, T \leftrightarrow I$ for generators immediately implies that the dual process leads to the dual partners of $(5.11)$ and $(5.12)$:

$$e^{-AJ_A}e^{-\Phi_A I_A} e^{CJ_C} e^{\Phi_C I_C} e^{BJ_B} e^{\Phi_B I_B} = e^{-(a+b+c)P_a} e^{-(\phi_a + \phi_b + \phi_c)T_c}$$

$$e^{BJ_B} e^{\Phi_B I_B} e^{-AJ_A} e^{-\Phi_A I_A} e^{CJ_C} e^{\Phi_C I_C} = e^{-(a+b+c)P_a} e^{-(\phi_a + \phi_b + \phi_c)T_a}$$

$$e^{CJ_C} e^{\Phi_C I_C} e^{BJ_B} e^{\Phi_B I_B} e^{-AJ_A} e^{-\Phi_A I_A} = e^{-(a+b+c)P_a} e^{-(\phi_a + \phi_b + \phi_c)T_b} \quad (5.14)$$

so that the two quantities

$$\delta = -a + b + c \quad \delta_\phi = -\phi_a + \phi_b + \phi_c \quad (5.15)$$

play the role of lateral excess and lateral phase excess of the triangle loop. Lateral orientations ‘phased’ along $(5.14)$ give the product of the three oriented complete rotations about the three vertices of a triangle as a complete translation along the base line of the loop.

### 5.2 The basic trigonometric identity

Each one of the equations $(5.11)$, $(5.12)$ or $(5.14)$ contains all the relationships between triangle sides, lateral phases, angles and angular phases in any CKD ‘complex hermitian’
space. However, all twelve elements appear in these equations not only explicitly as canonical parameters, but also implicitly inside the complete translation and rotation generators. This prompts the search for another relation, equivalent to the previous ones but more suitable to display the trigonometric equations; this new equation is indeed the bridge between the former equations and the trigonometry of the space.

The idea is to express all the generators as suitable conjugates of one pair of a translation and a phase translation generator and one pair of a rotation generator and a phase rotation generator, which we will take as primitive independent generators. A natural choice is to take the two pairs $P_a, T_a$ and $J_C, I_C$ as ‘basic’ independent generators. Next by using (5.3) we define the remaining triangle pairs of generators $P_b, T_b; J_A, I_A; P_c, T_c; J_B, I_B$ in terms of the previous ones and sides and angles, lateral phases and angular phases as:

\[
\begin{pmatrix}
    P_b \\
    J_A \\
    P_c \\
    J_B
\end{pmatrix} := e^{CJC} e^{ΦCIC} \begin{pmatrix}
    P_a \\
    J_C \\
    P_a \\
    J_C
\end{pmatrix} e^{-ΦCIC} e^{-CJC}
\]

which after full expansion and simplification gives:

\[
\begin{pmatrix}
    P_b \\
    J_A \\
    P_c \\
    J_B
\end{pmatrix} := e^{CJC} e^{ΦCIC} e^{bP_a} e^{pT_a} \begin{pmatrix}
    P_a \\
    J_C \\
    P_a \\
    J_C
\end{pmatrix} e^{-pT_a} e^{-bP_a} e^{-ΦCIC} e^{-CJC}
\]

(Note the highly ordered pattern in these expressions). By direct substitution in the equation (5.11) and after obvious cancellations which due to the commutativity of each member in the pairs of the complete transformations fully mimic the pattern found in the real case, we find:

\[
e^{-aP_a} e^{-φ_a T_a} e^{CJC} e^{ΦCIC} e^{bP_a} e^{pT_a} e^{-AJC} e^{-ΦAIC} e^{pT_a} e^{-bP_a} e^{-ΦCIC} e^{-CJC} = 1.
\]

The same process starting from any equation in (5.12) or (5.14) leads again to the same equation. This justifies to call (5.18) the basic trigonometric equation. We sum up in:

Theorem 1. Sides $a, b, c$, lateral phases $φ_a, φ_b, φ_c$, angles $A, B, C$ and angular phases $Φ_A, Φ_B, Φ_C$ of any triangle loop in the complex CKD space $\mathbb{C}_{η}^{S^2}$ are linked by a single group identity called the basic ‘complex hermitian’ trigonometric identity

\[
e^{-aP_a} e^{-φ_a T_a} e^{CJC} e^{ΦCIC} e^{bP_a} e^{pT_a} e^{-AJC} e^{-ΦAIC} e^{pT_a} e^{-bP_a} e^{-ΦCIC} e^{-CJC} = 1
\]

where $P, T$ are the generators of translations and phase translations along any fixed fiducial geodesic $g$, and $J, I$ are the generators of rotations and phase rotations about any fixed fiducial ‘complex’ line-geodesic $G$ containing the ‘complex’ line $l_g$ and about $O$. 

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Proof: A group motion can be used to move the triangle to a canonical position described before (5.1) for the flag \( O \subset g \subset l \subset l_g \). Then the theorem statement is simply (5.18).

**Theorem 2.** Let us consider a triangle loop in the ‘complex hermitian’ CKD space \( \mathbb{C}_{\eta}S^2_{[\kappa_1],\kappa_2} \), and let \( P_a,P_b,P_c;T_a,T_b,T_c \) be the generators of translations and phase translations along the three triangle geodesic sides, whose lengths and lateral phases are \( a,b,c \) and \( \phi_a,\phi_b,\phi_c \). Let \( J_A,J_B,J_C;I_A,I_B,I_C \) be the generators of rotations and phase rotations about the three geodesic line vertices of the triangle, whose angles and angular phases are \( A,B,C \), and \( \Phi_A,\Phi_B,\Phi_C \). These quantities are related by two sets of identities, (5.11, 5.12) and (5.14) called the ‘complex hermitian’ point loop and the ‘complex hermitian’ line loop triangle equations, each equation being equivalent to the identity in Theorem 1.

Several points are worth highlighting. First, each term in the basic identity is either a complete translation along a fixed geodesic \( g \) (through the fixed point \( O \)) or a complete rotation along a fixed line-geodesic \( G \) (about \( O \)). The canonical parameters of these translations or rotations are exactly the triangle sides, angles, lateral and angular phases. In the point loop or line loop equations the transformations involved are the translations along the sides or the rotations about the vertices.

Second, these equations are analogue to the ones found in the real case, with the consistent replacement of every translation or rotation by its ‘complete’ version, made up of two commuting factors. The point loop equations follow from a point travelling along the triangle according to the obvious shorthand \( a b c \) (see figure 3a). The line loop equations follow from a line looping about the triangle like \( B A C \); it starts in a base line (say \( a \)) and successively rotates by an angle \( C \), then by an angle \( -A \) and finally by an angle \( B \) about the corresponding vertices thus ending up back on the starting position \( a \) (see figure 3b). The basic equation follows from the pattern \( a C b A c B \), which keeps track of both sides and vertices found when looping around the triangle.

Third, the (three) point loop equations and the (three) line loop equations are mutually dual sets; the single basic equation is clearly self-dual. And fourth, these equations hold in the same explicit form for all the twenty seven 2D ‘complex hermitian’ CKD geometries, as neither an explicit Cayley-Dickson label \( \eta \) nor a Cayley-Klein one \( \kappa_1,\kappa_2 \) ever appears in them.

6 The basic equations of trigonometry for any ‘complex hermitian’ two-dimensional Cayley–Klein–Dickson space

To obtain the trigonometric equations for the ‘complex hermitian’ CKD space we start with the basic trigonometric identity (5.19), for the triangle in its canonical position, so \( P_a,T_a \) and \( J_C,I_C \) can be taken exactly as \( P_1,T_1 \) and \( J,I \). For notational clarity we will omit even the subindex and will denote \( P_1,T_1 \) here simply as \( P,T \). We first write (5.19) in the equivalent form \( a c b = \overline{bC}a \):

\[
e^{-AJ}e^{-\Phi_A I}e^{P}e^{\phi_c T}e^{BJ}e^{\Phi_B I}e^{P} = e^{-bP}e^{-\phi_b T}e^{-CJ}e^{-\Phi_C I}e^{aP}e^{\phi_a T} \tag{6.1}
\]

By considering this identity in the fundamental 3D vector representation of the motion group (3.11) and (3.13) we obtain an equality between \( 3 \times 3 \) complex matrices, giving rise
to nine ‘complex’ identities:

\[ \begin{align*}
C_{\kappa_1}(c) e^{i\frac{2\Phi_A+2\Phi_B+\phi_c}{3}} &= C_{\kappa_1}(a) C_{\kappa_1}(b) e^{i\frac{\phi_a-\phi_b-2\Phi_C}{3}} + \kappa_1 S_{\kappa_1}(a) S_{\kappa_1}(b) C_{\kappa_2}(C) e^{i\frac{\phi_a-\phi_b+\Phi_C}{3}}, \\
C_{\kappa_2}(C) e^{i\frac{-2\phi_a+2\phi_b+\phi_c}{3}} &= C_{\kappa_2}(A) C_{\kappa_2}(B) e^{i\frac{\phi_a+\phi_b-2\Phi_C}{3}} + \kappa_2 S_{\kappa_2}(A) S_{\kappa_2}(B) C_{\kappa_1}(c) e^{i\frac{\phi_a+\phi_b+\Phi_C}{3}}, \\
S_{\kappa_1}(a) S_{\kappa_2}(A) e^{i\frac{2\phi_a+2\phi_b+\phi_c}{3}} &= S_{\kappa_1}(a) S_{\kappa_2}(C) e^{i\frac{\phi_a+\phi_b+\Phi_C}{3}}, \\
S_{\kappa_1}(a) S_{\kappa_2}(C) e^{i\frac{-2\phi_a+2\phi_b+\phi_c}{3}} &= -S_{\kappa_1}(a) S_{\kappa_2}(b) e^{i\frac{\phi_a-\phi_b-2\Phi_C}{3}} + S_{\kappa_1}(a) C_{\kappa_2}(C) e^{i\frac{\phi_a-\phi_b+\Phi_C}{3}}, \\
S_{\kappa_1}(a) C_{\kappa_2}(B) e^{i\frac{2\phi_a+2\phi_b+\phi_c}{3}} &= -S_{\kappa_1}(a) S_{\kappa_2}(B) e^{i\frac{\phi_a-\phi_b-2\Phi_C}{3}} - S_{\kappa_1}(a) C_{\kappa_2}(C) e^{i\frac{\phi_a-\phi_b+\Phi_C}{3}}, \\
S_{\kappa_1}(a) C_{\kappa_2}(A) e^{i\frac{2\phi_a+2\phi_b+\phi_c}{3}} &= -S_{\kappa_1}(a) S_{\kappa_2}(B) e^{i\frac{\phi_a-\phi_b-2\Phi_C}{3}} + S_{\kappa_2}(A) C_{\kappa_2}(A) C_{\kappa_1}(c) e^{i\frac{\phi_A-\phi_B+\phi_c}{3}}, \\
S_{\kappa_2}(A) S_{\kappa_2}(B) e^{i\frac{-2\phi_a+2\phi_b+\phi_c}{3}} &= C_{\kappa_2}(A) C_{\kappa_2}(B) C_{\kappa_1}(c) e^{i\frac{\phi_A-\phi_B+\phi_c}{3}} + C_{\kappa_1}(a) C_{\kappa_1}(b) C_{\kappa_2}(C) e^{i\frac{\phi_a-\phi_b+\Phi_C}{3}}.
\end{align*} \]

which contain the trigonometry of the space \( \mathbb{C}_\eta S_{[\kappa_1,\kappa_2]}^2 = \eta S U_{\kappa_1,\kappa_2} / (\eta U(1) \otimes \eta S U_{\kappa_2}(2)) \).

Each equation in this set either is self-dual or appears in a mutually dual pair; this could have been expected due to the self-duality of the starting equation. We should recall again that all along this section \( i \) denotes the imaginary unit of the Cayley-Dickson ‘complex’ numbers \( \mathbb{C}_\eta \), so that \( i^2 = -\eta \); the labelled sine and cosine with label \( \eta \) are related to the exponential \( e^{ix} \) by (4.6).

The association between sides (resp. angles) and the labels \( \kappa_1 \) (resp. \( \kappa_2 \)) found in the equations of the real space \( S_{[\kappa_1,\kappa_2]}^2 = SO_{\kappa_1,\kappa_2}(3) / SO_{\kappa_2}(2) \) extends to the Hermitian complex analogues so lengths \( a, b, c \) are associated to \( \kappa_1 \), and angles \( A, B, C \) to \( \kappa_2 \). The lateral \( \phi_a, \phi_b, \phi_c \) and angular phases \( \Phi_A, \Phi_B, \Phi_C \) have \( \eta \) as its label. The elements \( (-a,-\phi_a,-A,-\Phi_A) \) always appear in the equations with a minus sign as compared with \( (b, \phi_b, B, \Phi_B), (c, \phi_c, C, \Phi_C) \); this follows from the structure of the basic equation (6.1), in which the side \( a \) and vertex \( A \) are traversed or rotated backwards.

The set (6.2) is equivalent to two other similar sets, obtained by starting with the basic identity split into the two equivalent forms symbolically denoted as \( C b A = c B \bar{A} \) and \( b A c = B \bar{C} C \). In order to present all these equations in a concise way, it proves adequate to introduce a compact notation, following the pattern explained in [1]. The sides and angles, lateral phases and angular phases will be denoted as \( x_i, X_I, \phi_i, \Phi_I, i,I = 1,2,3 \) according to

\[ \begin{align*}
x_1 &= -a, \quad x_2 = b, \quad x_3 = c, \quad X_1 = -A, \quad X_2 = B, \quad X_3 = C, \\
\phi_1 &= -\phi_a, \quad \phi_2 = \phi_b, \quad \phi_3 = \phi_c, \quad \Phi_1 = -\Phi_A, \quad \Phi_2 = \Phi_B, \quad \Phi_3 = \Phi_C.
\end{align*} \]

The built-in minus sign in \( x_i, \phi_i, X_I, \Phi_I \) when \( i = I = 1 \) is natural when the triangle is considered as a point loop with the side \( a, \phi_a \) traversed backwards, or as a side loop with the angle \( A, \Phi_A \) rotated backwards; this choice absorbs the signs related to \( a, A \) and confer an uniform appearance to the equations (6.2). In particular, the angular and lateral excesses appear in this notation as \( \Delta = X_I + X_J + X_K \) and \( \delta = x_i + x_j + x_k \). The basic
equation (5.19) now reads:
\[ e^{x_iP}e^{\phi_i}e^{X_KJ}e^{\Phi_KI}e^{x_jP}e^{\phi_j}e^{X_IJ}e^{\Phi_IJ}e^{x_kP}e^{\phi_k}e^{X_JJ}e^{\Phi_JI} = 1 \]  

(6.4)

where \( i = I, j = J, k = K \) are any cyclic permutation of 123. This basic equation can be very simply recalled: replace in the shorthand \( iKjIkJ \) each letter by the associated complete translation or rotation. From now on we will adopt this convention which makes all equations of trigonometry explicitly invariant under cyclic permutations of the ‘oriented’ complete sides \( x_i, \phi_i \) and angles \( X_I, \Phi_I \). Capital indices will also help in distinguishing between mutually dual pairs \( x_i, X_I \) and \( \phi_i, \Phi_I \).

6.1 The trigonometric equations in the ‘Cartan sector’

Each equation in (6.2) is ‘complex’, and phases, both lateral and angular, appear through unimodular ‘complex’ factors \( e^{i\phi}, e^{i\Phi} \), while ‘pure’ sides and angles \( x_i, X_I \) appear through their labelled sines or cosines, which are always real. The equations in the third line split into an equation for the modulus and another for the argument; this last part is:

\[ -\phi_j + 2\phi_k + \Phi_I = -\Phi_J + 2\Phi_K + \phi_i \]  

(6.5)

Writing the same equation for the choice of indices \( i, j, k \rightarrow j, k, i \) and comparing we get:

\[ \phi_i - \Phi_I = \phi_j - \Phi_J. \]  

(6.6)

These three equations, only two of which are independent, are self-dual and hold for all the ‘complex’ CKD spaces. A consequence of these very simple linear relations is:

\[ -\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c; \]  

(6.7)

the common value in this formula turns out to be a quantity first introduced for the complex hermitian elliptic space by Blaschke-Terheggen. We will call this \( \Omega \):

\[ \Omega := \Phi_I + \Phi_J + \phi_k. \]  

(6.8)

In a similar dual way and starting also from (6.6) we find another linear relation between phases:

\[ -\Phi_A + \phi_b + \phi_c = -\phi_a + \Phi_B + \phi_c = -\phi_a + \phi_b + \Phi_C \]  

(6.9)

whose value turns out to be the invariant dual to \( \Omega \) and which for the complex hermitian elliptic case was also introduced by Blaschke-Terheggen:

\[ \omega := \Phi_i + \phi_j + \Phi_K. \]  

(6.10)

Another quantities linked to \( \phi_i, \Phi_I \) are the angular phase excess \( \Delta_\phi \) (5.13) and lateral phase excess \( \delta_\phi \) (5.15), which appear in the hermitian analogues of Gauss-Bonnet triangle theorems. In terms of the compact notation (6.3) they are:

\[ \Delta_\phi := \Phi_I + \Phi_J + \Phi_K \quad \delta_\phi := \phi_i + \phi_j + \phi_k \]  

(6.11)

The invariants \( \Omega \) (resp. \( \omega \)) are thus two kinds of mixed phase excesses, with dominance of angular (resp. lateral) phases. The departure from the BT notation \( \omega, \tau \) to ours \( \Omega, \omega \)
conforms to the typographical convention upper/lower case in order to stress duality and to convey that each mixed excess \( \Omega, \omega \) has dominance of either angular or lateral phases. These invariants and the two phase excesses \( \Delta \phi, \delta \phi \), which appear in the point and line loop equations are related by:

\[
\Delta \phi = 2\Omega - \omega \quad \delta \phi = 2\omega - \Omega \quad (6.12)
\]

Thus there is a sector of hermitian trigonometry involving only phases and completely decoupled from ‘pure’ sides \( x_i \) and angles \( X_I \). This sector holds in exactly the same form in the twenty seven ‘complex’ CKD spaces, as no explicit labels \( \eta; \kappa_1, \kappa_2 \) appear. Since the triangle invariants \( \phi, \Phi \) are related to the Cartan subalgebra, we will call these equations the ‘Cartan’ sector of ‘complex hermitian’ trigonometry. This ‘Cartan sector’ has no analogue in the trigonometry of real spaces, and their equations are purely linear witnessing the abelian character of Cartan subalgebra.

### 6.2 The complete set of ‘complex hermitian’ trigonometric equations

Now by exploiting the ‘Cartan’ relations between phases (6.6), and introducing explicitly the invariants \( \Omega, \omega \), it turns out possible to simplify the equations (6.2) by multiplying each one of them by some suitably chosen unimodular ‘complex’ factor. This leads to the full set of trigonometric equations coming from the basic trigonometric group identity as:

\[
0ij \equiv 0IJ \quad \Phi_I - \phi_i = \Phi_J - \phi_j \quad (\Rightarrow \Omega := \Phi_I + \Phi_J + \phi_K, \quad \omega := \phi_i + \phi_j + \Phi_K)
\]

\[
1i \quad C_{\kappa_1}(x_i)e^{i\Omega} = C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 s_{\kappa_1}(x_j)s_{\kappa_1}(x_k)c_{\kappa_1}(x_I) e^{i\Phi_I}
\]

\[
1I \quad C_{\kappa_2}(X_I)e^{i\omega} = C_{\kappa_2}(X_J)C_{\kappa_2}(X_K) - \kappa_2 s_{\kappa_2}(X_J)s_{\kappa_2}(X_K)c_{\kappa_1}(x_i) e^{i\phi_i}
\]

\[
2ij \equiv 2IJ \quad \frac{s_{\kappa_1}(x_i)}{s_{\kappa_2}(X_I)} = \frac{s_{\kappa_1}(x_j)}{s_{\kappa_2}(X_J)}
\]

\[
3iJ \quad s_{\kappa_1}(x_i)c_{\kappa_2}(X_J) e^{i\phi_k} = -c_{\kappa_1}(x_j)s_{\kappa_1}(x_k) e^{-i\phi_I} - s_{\kappa_2}(x_J)c_{\kappa_1}(x_k) c_{\kappa_2}(X_I)
\]

\[
3IJ \quad s_{\kappa_2}(X_I)c_{\kappa_1}(x_j) e^{i\Phi_K} = -c_{\kappa_2}(X_J)s_{\kappa_2}(X_K) e^{-i\phi_I} - s_{\kappa_2}(X_J)c_{\kappa_2}(X_K) c_{\kappa_1}(x_i)
\]

\[
4ij \equiv 4IJ \quad -\kappa_1 s_{\kappa_1}(x_i)s_{\kappa_1}(x_j) + c_{\kappa_1}(x_i)c_{\kappa_1}(x_j)c_{\kappa_2}(X_K) e^{i\Phi_K} = -\kappa_2 s_{\kappa_2}(X_I)s_{\kappa_2}(X_J) + c_{\kappa_2}(X_I)c_{\kappa_2}(X_J)c_{\kappa_1}(x_k) e^{i\phi_k}.
\]

These equations will be referred to by a tag, and are either self-dual (for instance \( 2ij \equiv 2IJ \)) or appear in mutually dual pairs (as \( 1i, 1I \)). Equations with tag 0 allow the introduction of the ‘symmetric’ invariants \( \Omega \) and \( \omega \) and are in the ‘Cartan’ sector. The remaining tags are intentionally made to match the ones used in [1]; the equations with tags 1, 2, 3, 4 are in most respects the closest ‘complex hermitian’ analogues to the equations found in the real case, as far as their mutual relations, dependence or independence, etc. are concerned. Therefore the trigonometry of real spaces provide a rough first guide in the exploration of the whole forest of ‘complex hermitian’ trigonometric equations.

### 6.3 The ‘complex hermitian’ trigonometric bestiary

Taking the equations (6.13) as starting point, we now perform a fully explicit study of ‘complex hermitian’ trigonometry, including some comments. As the scheme enjoys self-duality, those equations which are not self-dual will have a dual partner, obtained by
exchange in capitalization of names and indices: \( x \leftrightarrow X \), \( \phi \leftrightarrow \Phi \), \( i \leftrightarrow I \) and in CK constants \( \kappa_1 \leftrightarrow \kappa_2 \); in these cases we will only sketch the derivation of one member of the dual pair, but we will write each pair together, to emphasize self-duality as the main trait of this approach. The label \( \eta \) does not change under duality.

These equations will hold for all twenty seven ‘complex’ CKD spaces with arbitrary \( \eta; \kappa_1, \kappa_2 \). In the degenerate cases \( \kappa_1 = 0 \) (flat ‘complex hermitian’ spaces) and/or \( \eta = 0 \) or \( \kappa_2 = 0 \) (degenerate ‘complex’ ‘Hermitian’ metric) some equations may collapse or even reduce to trivial identities; these cases will be discussed later but for the moment we will stay in the general situation where \( \eta; \kappa_1, \kappa_2 \) are assumed to have any values. All equations found in the literature for the elliptic (hyperbolic) complex hermitian spaces will follow from this set after we specialize \( \eta = 1; \kappa_1 = 1 (\kappa_1 = -1), \kappa_2 = 1 \); in those cases, the equations we found will be allocated a suitable name.

- The Cartan sector equations \( 0IJ \equiv 0ij \) will be called the ‘complex hermitian’ phases theorem. They are self-dual and involve only the triangle Cartan invariants. They allow the introduction of two symmetric triangle invariants \( \Omega \) and \( \omega \) after (6.8) and (6.10):

\[
0IJ \equiv 0ij \quad \phi_i - \Phi_I = \phi_j - \Phi_J = \phi_k - \Phi_K = \omega - \Omega. \tag{6.14}
\]

There are two such independent equations, thus four independent quantities among the six lateral and angular phases. This number equals the number of essential independent triangle invariants; this is not accidental (see the comment at the end of Sect. 6.5).

- The equations \( 2iJ \equiv 2jI \), taken together will be called the hermitian sine theorem.

\[
2IJ \equiv 2ij \quad \frac{S_{\kappa_1}(x_i)}{S_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(x_j)}{S_{\kappa_2}(X_J)} = \frac{S_{\kappa_1}(x_k)}{S_{\kappa_2}(X_K)}. \tag{6.15}
\]

This self-dual relation has two independent equations linking the six ‘pure’ sides and angles. The hermitian phases theorem can be written in terms of the phase factors \( e^{i\phi_i}, e^{i\Phi_I} \) and has the same form as the sine theorem; this is so because phases theorem (6.14) and sine theorem (6.15) are the modulus and argument of the same ‘complex’ equality.

- Each of the ‘complex hermitian’ cosine theorems \( 1i \) and \( 1I \) is a ‘complex’ equation. By splitting the hermitian cosine theorem \( 1i \) into real and imaginary parts, we get:

\[
1i \quad C_{\kappa_1}(x_i)C_{\eta}(\Omega) = C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)C_{\eta}(\Phi_I) \quad C_{\kappa_1}(x_i)S_{\eta}(\Omega) = -\kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)S_{\eta}(\Phi_I) \tag{6.16}
\]

called real and imaginary Hermitian cosine laws (for sides). Their duals are the real and imaginary Hermitian dual cosine laws (for angles):

\[
1I \quad C_{\kappa_2}(X_I)C_{\eta}(\omega) = C_{\kappa_2}(X_J)C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)C_{\eta}(\phi_i) \quad C_{\kappa_2}(X_I)S_{\eta}(\omega) = -\kappa_2 S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)S_{\eta}(\phi_i). \tag{6.17}
\]

- By equating the modulus of both sides of the hermitian cosine theorem \( 1i \) we get:

\[
C^2_{\kappa_1}(x_i) = \left( C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) - \kappa_1 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)C_{\eta}(\Phi_I) \right)^2 + \eta \kappa_1^2 S^2_{\kappa_1}(x_j)S^2_{\kappa_1}(x_k)C^2_{\kappa_2}(X_I)S^2_{\eta}(\Phi_I). \tag{6.18}
\]

For the complex elliptic case this is the Shirokov-Rosenfeld cosine theorem (2.12) [9] yet expressed in terms of the angular variables \( X_I \) and \( \Phi_I \), instead of the ones used in [9].

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In the general case we will also call this equation Shirokov-Rosenfeld cosine theorem. This admits another form, starting from $C_{\kappa_1}(2x_i) + 1 = 2C_{\kappa_1}^2(x_i)$, substituting (6.18) and expanding the squared sines of sides:

$$C_{\kappa_1}(2x_i) = C_{\kappa_1}(2x_j)C_{\kappa_1}(2x_k) - \kappa_1 S_{\kappa_1}(2x_j)S_{\kappa_1}(2x_k)C_{\kappa_2}(X_I)\eta(\Phi_I) = -2\kappa_1^2\kappa_2 S_{\kappa_1}^2(x_j)S_{\kappa_1}^2(x_k)S_{\kappa_2}^2(X_I) \tag{6.19}$$

to be called Shirokov-Rosenfeld cosine double theorem because in the complex elliptic case it reduces to (2.13). The dual is the Shirokov-Rosenfeld cosine theorem (for angles):

$$C_{\kappa_2}^2(X_I) = \left(C_{\kappa_2}(X_I)C_{\kappa_2}(X_K) - \kappa_2 S_{\kappa_2}(X_I)S_{\kappa_2}(X_K)C_{\kappa_1}(x_i)\eta(\phi_i)\right)^2 \tag{6.20}$$

and Shirokov-Rosenfeld dual cosine double theorem (for angles):

$$C_{\kappa_2}(2X_I) = C_{\kappa_2}(2X_J)C_{\kappa_2}(2X_K) - \kappa_2 S_{\kappa_2}(2X_J)S_{\kappa_2}(2X_K)C_{\kappa_1}(x_i)\eta(\phi_J) = -2\kappa_1^2\kappa_2 S_{\kappa_2}^2(2X_J)S_{\kappa_2}^2(2X_K)S_{\kappa_1}^2(x_i) \tag{6.21}$$

By building up the term $\kappa_1 S_{\kappa_1}(2x_j)S_{\kappa_1}(2x_k)C_{\kappa_2}(X_I)\eta(\Phi_I)$ in (6.16) and substituting it into (6.19), expanding and simplifying we obtain:

$$C_{\kappa_1}^2(x_i) = -C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) + \kappa_1^2 S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}^2(x_I) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)\eta(\Omega) \tag{6.22}$$

which will be called the Blaschke-Terheggen cosine theorem for sides [7, 8]. Its dual is:

$$C_{\kappa_2}^2(X_I) = -C_{\kappa_2}^2(X_J)C_{\kappa_2}^2(X_K) + \kappa_2^2 S_{\kappa_2}^2(x_J)S_{\kappa_2}^2(x_K)C_{\kappa_1}^2(x_i) + 2C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_{\kappa_1}(x_k) \eta(\omega) \tag{6.23}$$

In the complex hermitian elliptic case (6.22) and (6.23) reduce directly to the Blaschke-Terheggen cosine (2.8) and dual cosine theorems (2.9).

By multiplying both sides of (6.15) by $1/\eta(\Omega)$ and using the second equation in (6.16) we obtain:

$$\frac{S_{\kappa_1}(2x_i)}{S_{\eta}(\Phi_I)C_{\kappa_2}(X_I)} = \frac{S_{\kappa_1}(2x_j)}{S_{\eta}(\Phi_J)C_{\kappa_2}(X_J)} \tag{6.24}$$

called Shirokov-Rosenfeld double sine theorem, because in the complex elliptic case reduces to the SR double sine law (2.11) after changing to the angular variables used by SR. Its dual is:

$$\frac{S_{\kappa_2}(2X_I)}{S_{\eta}(\phi_i)C_{\kappa_1}(x_i)} = \frac{S_{\kappa_2}(2X_J)}{S_{\eta}(\phi_j)C_{\kappa_1}(x_j)} \tag{6.25}$$

By multiplying (6.24) and (6.25) we get the self-dual equation:

$$\frac{S_{\kappa_1}(x_i)S_{\kappa_1}(X_I)}{S_{\eta}(\phi_i)S_{\eta}(\Phi_I)} = \frac{S_{\kappa_1}(x_j)S_{\kappa_1}(X_J)}{S_{\eta}(\phi_j)S_{\eta}(\Phi_J)} \tag{6.26}$$

By taking quotient between the double sine theorem (6.24) and the sine theorem (6.15) we get:

$$\frac{C_{\kappa_1}(x_i)T_{\kappa_2}(X_I)}{S_{\eta}(\Phi_I)} = \frac{C_{\kappa_1}(x_j)T_{\kappa_2}(X_J)}{S_{\eta}(\Phi_J)} \tag{6.27}$$

whose dual is:

$$\frac{C_{\kappa_2}(X_I)T_{\kappa_1}(x_i)}{S_{\eta}(\phi_i)} = \frac{C_{\kappa_2}(X_J)T_{\kappa_1}(x_j)}{S_{\eta}(\phi_j)} \tag{6.28}$$
• Another equations derive from the equations with tags 3iJ and 3Ij. In particular, by splitting the equations 3iJ into their real and imaginary parts we obtain:

\[ S_{\kappa_1}(x_i)C_{\kappa_2}(X_J)C_\eta(\phi_k) = -C_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_\eta(\Phi_I) - S_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_{\kappa_2}(X_I) \]
\[ S_{\kappa_1}(x_i)C_{\kappa_2}(X_J)S_\eta(\phi_k) = C_{\kappa_1}(x_j)S_{\kappa_1}(x_k)S_\eta(\Phi_I) \]

whose duals are:

\[ S_{\kappa_2}(X_I)C_{\kappa_1}(x_j)C_\eta(\Phi_K) = -S_{\kappa_2}(X_J)S_{\kappa_1}(X_K)C_\eta(\phi_i) - S_{\kappa_2}(X_J)C_{\kappa_1}(X_K)C_{\kappa_1}(x_i) \]
\[ S_{\kappa_2}(X_I)C_{\kappa_1}(x_j)S_\eta(\Phi_K) = C_{\kappa_2}(X_J)S_{\kappa_1}(X_K)S_\eta(\phi_i). \]  

(6.29)

• The same splitting for the equations 4ij \equiv 4IJ leads to the pair of self-dual equations:

\[ -\kappa_2S_{\kappa_2}(X_I)S_{\kappa_2}(X_J) + C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_1}(x_k)C_\eta(\phi_k) \]
\[ = -\kappa_1S_{\kappa_1}(x_i)S_{\kappa_1}(x_j) + C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_2}(X_K)C_\eta(\Phi_K) \]
\[ C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_1}(x_k)S_\eta(\phi_k) = C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_2}(X_K)S_\eta(\Phi_K) \]

(6.30)

• By eliminating the angles \( X_J, X_K \) between (6.27) and (6.35):

\[ C_{\kappa_1}^2(x_k) = \frac{S_\eta(\Phi_I + \phi_k)S_\eta(\Phi_J + \phi_k)}{S_\eta(\Phi_I)S_\eta(\Phi_J)} = \frac{S_\eta(\Omega - \Phi_I)S_\eta(\Omega - \Phi_J)}{S_\eta(\Phi_I)S_\eta(\Phi_J)}. \]  

(6.38)
whose dual is:

\[ C_{\kappa_2}^2(X_K) = \frac{S_\eta(\phi_i + \Phi_K)S_\eta(\phi_j + \Phi_K)}{S_\eta(\phi_i)S_\eta(\phi_j)} = \frac{S_\eta(\omega - \phi_i)S_\eta(\omega - \phi_j)}{S_\eta(\phi_i)S_\eta(\phi_j)}. \]  

These equations give the cosine of each side (angle) in terms of the angular (lateral) phases only. They somehow resemble real trigonometry Euler equations for the cosine of half the sides (angles) in terms of angles (sides). In these hermitian ‘Euler-like’ equations, pure sides (angles) are however given in terms of angular phases and \( \Omega \) (lateral phases and \( \omega \)).

- By expansion of sines of sums or differences and elementary manipulation, we finally get the expression for the squared sines of the sides:

\[ S_{\kappa_1}^2(x_k) = -\frac{S_\eta(\phi_k)S_\eta(\Omega)_{\kappa_1}}{S_\eta(\Phi_J)S_\eta(\Phi_J)}. \]

whose dual equation is:

\[ S_{\kappa_2}^2(X_K) = -\frac{S_\eta(\Phi_K)S_{\eta}(\omega)_{\kappa_2}}{S_\eta(\phi_i)S_\eta(\phi_j)}. \]

As we shall see shortly, and in spite of the presence of \( \kappa_1, \kappa_2 \) in denominators, these equations are still meaningful when \( \kappa_1 \rightarrow 0 \) or \( \kappa_2 \rightarrow 0 \).

### 6.4 Symplectic area and coarea

For real CK spaces the angular excess \( \Delta \) shares three properties: \( \Delta \) goes to zero with \( \kappa_1 \), it is proportional (coefficient \( \kappa_1 \)) to triangle area, and satisfies Gauss-Bonnet type equations. These three properties are split in the ‘complex’ case: In the hermitian point loop equations (5.11) and (5.12), the ‘complete excess’ \( (\Delta, \Delta_\Phi) \) plays a role partly analogue of the real angular excess, yet it may not vanish with \( \kappa_1 \). There are two different independent hermitian triangle quantities which vanish with \( \kappa_1 \). One of them is the Blaschke-Terheggen invariant \( \Omega \). This follows directly from the equations already derived. The situation for the other vanishing quantity is not so obvious (see however the comments in the next Section). Dually, while the real excess \( \delta \) is proportional to the coarea, vanishes with \( \kappa_2 \), and satisfies dual Gauss-Bonnet type equations, the complete lateral excess \( (\delta, \delta_\Phi) \) appears in (5.14) and may not vanish with \( \kappa_2 \), while \( \omega \) vanishes with \( \kappa_2 \).

In the real case, the three cosine equations 1i (1I) turned into trivial identities when \( \kappa_1 = 0 \) (\( \kappa_2 = 0 \)). In the ‘complex hermitian’ case, the three ‘complex’ independent equations 1i (1I), which are independent when \( \kappa_1 \neq 0 \) (\( \kappa_2 \neq 0 \)) collapse when \( \kappa_1 = 0 \) (\( \kappa_2 = 0 \)) into a single real one in the ‘Cartan’ sector, and as far as pure sides and angles are concerned become trivial:

1i when \( \kappa_1 = 0 \)
\[ e^{i\Omega} = C_\eta(\Omega) + iS_\eta(\Omega) = 1 \text{ implying } C_\eta(\Omega) = 1, S_\eta(\Omega) = 0 \]  

1I when \( \kappa_2 = 0 \)
\[ e^{i\omega} = C_\eta(\omega) + iS_\eta(\omega) = 1 \text{ implying } C_\eta(\omega) = 1, S_\eta(\omega) = 0. \]

The behaviour of the quotient \( \frac{S_\eta(\Omega)}{\kappa_1} \) as \( \kappa_1 \rightarrow 0 \) can be derived both from the imaginary part of the Hermitian cosine theorem (6.16) and from equations (6.40):

\[ \frac{S_\eta(\Omega)}{\kappa_1} = \frac{-S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)C_{\kappa_2}(X_I)S_{\eta}(\Phi_I)}{C_{\kappa_1}(x_i)} = \frac{-S_\eta(\Phi_I)S_\eta(\Phi_J)S_{\kappa_2}(x_k)}{S_\eta(\phi_k)}. \]
and since this quotient remains finite as \( \kappa_1 \to 0 \), \( \Omega \) behaves like the real case angular excess \( \Delta = -A + B + C \). Dually, \( \omega \) behaves as the real pure lateral excess \( \delta = -a + b + c \):

\[
\frac{S_\eta(\omega)}{\kappa_2} = \frac{-S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_1}(x_I)S_\eta(\phi_1)}{C_{\kappa_2}(X_I)} = \frac{S_\eta(\phi_i)S_\eta(\phi_j)S_{\kappa_2}^2(X_K)}{S_\eta(\Phi_K)} \tag{6.45}
\]

The real excesses \( \Delta, \delta \) are proportional, with coefficients \( \kappa_1 \) and \( \kappa_2 \), to the triangle area and coarea respectively. In the elliptic hermitian space \( \mathbb{CP}^2 \) Hang and Masala [29] found for the symplectic triangle area \( S \) the relation \( S = -\Omega/2 \) (the inessential minus sign comes from their definition of symplectic form). For any member of the CKD family of the ‘complex Hermitian’ spaces, the definitions for triangle symplectic area and coarea :

\[
S := \frac{\Omega}{2\kappa_1} \quad s := \frac{\omega}{2\kappa_2} \tag{6.46}
\]

(note the factor 2) are in full agreement with the standard definition of symplectic area as the integral of the symplectic form over any surface dressing the triangle; this form is closed so by the Stokes theorem the integral depends only on the boundary.

Therefore all appearances of \( \Omega \) or \( \omega \) in trigonometric the equations could be rewritten in terms of trigonometric functions of the symplectic area \( S \) with label \( \eta \kappa_1^2 \) (the symplectic area goes like the product of lengths along the geodesics generated by \( P_1 \) and \( Q_1 \), whose labels are \( \kappa_1 \) and \( \eta \kappa_1 \), and symplectic coarea \( s \), with label \( \eta \kappa_2^2 \).

\[
C_{\eta \kappa_1}(2S) = C_\eta(\Omega) \quad S_{\eta \kappa_1}(2S) = \frac{S_\eta(\Omega)}{\kappa_1} \quad C_{\eta \kappa_2}(2S) = C_\eta(\omega) \quad S_{\eta \kappa_2}(2S) = \frac{S_\eta(\omega)}{\kappa_2} \tag{6.47}
\]

When \( \kappa_1 = 0 \), \( \Omega \) vanishes but \( S \) keeps some finite value, a kind of ‘residue’ of the generically non vanishing mixed phase excess \( \Omega \). Dually, the same happens for \( \omega \) and \( s \) as \( \kappa_2 \to 0 \).

6.5 Dependence and basic equations

In the complex hermitian and hyperbolic spaces a triangle is known to be determined by four independent quantities. Since we have found eight generic independent relations between the twelve sides, angles and phases, this is still true in the generic CKD space \( \mathbb{C} \eta \kappa_1^2 \kappa_2 \). Two such relations are the hermitian phase theorems (6.14). The other six happen to be exactly twice as many independent equations as in the real case, due to their ‘complex’ nature. This allows to split the dependence discussion into the Cartan sector and the equations with a real analogue.

The Cartan sector includes six phases, to which we will add symplectic area and coarea. For any value of the labels \( \kappa_1, \kappa_2, \eta \), there are four independent equations between the eight quantities \( \phi_i, \Phi_I, S, s \):

\[
\phi_i - \Phi_I = \phi_j - \Phi_J = \phi_k - \Phi_K \quad (= \omega - \Omega) \\
(\Omega =) \quad \Phi_I + \Phi_J + \phi_k = \kappa_1 2S, \quad (\omega =) \quad \phi_i + \phi_j + \Phi_K = \kappa_2 2s \tag{6.48}
\]

so in any case there are always four independent such Cartan sector quantities. Generically the triangle is almost determined by these quantities (see 6.40, 6.41).

In order to discuss the dependence of the remaining equations, let us consider:

\[
\Delta_g := 1 - C_{\kappa_1}(x_I) - C_{\kappa_1}^2(x_J) - C_{\kappa_1}(x_K) + 2C_{\kappa_1}(x_I)C_{\kappa_1}(x_J)C_{\kappa_1}(x_K)C_\eta(\Omega) \\
\Delta_G := 1 - C_{\kappa_2}(X_I) - C_{\kappa_2}^2(X_J) - C_{\kappa_2}(X_K) + 2C_{\kappa_2}(X_I)C_{\kappa_2}(X_J)C_{\kappa_2}(X_K)C_\eta(\omega). \tag{6.49}
\]
The quantity $\triangle_g$ is the determinant of the Gramm matrix whose elements are the ‘hermitian’ products of the vectors corresponding to the vertices in the linear ambient space, and $\triangle_G$ its dual quantity. From (6.42) and (3.12) it follows that $\triangle_g$ vanishes when $\kappa_1 \to 0$; dually the same happens for $\triangle_G$ when $\kappa_2 \to 0$. Further, the quotient $\triangle_g/\kappa_1^2$ (resp. $\triangle_G/\kappa_2^2$) tends to a well defined finite limit when $\kappa_1 \to 0$ (resp. $\kappa_2 \to 0$), although still goes to zero when $\kappa_2 \to 0$ (resp. when $\kappa_1 \to 0$). To see this, simplify (6.49) by using (6.22) or (6.23) to obtain:

$$\triangle_g = \kappa_2^2 \kappa_2 S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k) \quad \triangle_G = \kappa_1 \kappa_2^2 S_{\kappa_1}(X_I) S_{\kappa_2}(x_j) S_{\kappa_2}(X_K).$$

(6.50)

This suggests to introduce two new ‘renormalized’ quantities $\gamma, \Gamma$ in a way similar to (6.46):

$$\gamma := \frac{\triangle_g}{\kappa_1^2}, \quad \Gamma := \frac{\triangle_G}{\kappa_2^2},$$

(6.51)

Relations between $\Gamma, \gamma$ and symplectic area and coarea $S, s$ holding for any $\eta; \kappa_1, \kappa_2$ follow by expressing sines of sides and angles in (6.51) by means of (6.40) and (6.41):

$$\frac{\gamma}{\kappa_2} = S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k) = -\frac{S_{\eta}(2S) S_{\eta}(2s)}{S_{\eta}(\Phi_I) S_{\eta}(\Phi_J) S_{\eta}(\Phi_K)}$$

(6.52)

$$\frac{\Gamma}{\kappa_1} = S_{\kappa_1}(X_I) S_{\kappa_2}(x_J) S_{\kappa_1}(X_K) = -\frac{S_{\eta}(2S) S_{\eta}(2s)}{S_{\eta}(\phi_I) S_{\eta}(\phi_J) S_{\eta}(\phi_K)}$$

(6.53)

By direct substitution using (6.52) and (6.53) we can also derive the following relations between $\Gamma, \gamma$ and the pure sides and angles:

$$\frac{\Gamma}{\kappa_1} = \frac{(\gamma/\kappa_2)^2}{S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k)}; \quad \frac{\gamma}{\kappa_2} = \frac{(\Gamma/\kappa_1)^2}{S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k)}$$

(6.54)

Let us now discuss the dependence issue in the generic case $\kappa_1 \neq 0, \kappa_2 \neq 0$, where all the equations with tags 2, 3 and 4 follow from 1, exactly alike in the real case. This means that a triangle is completely determined by the three sides $a, b, c$ and $\Omega$, or by the three angles $A, B, C$ and $\omega$. The proofs are also a verbatim translation of the real ones, with a single hermitian caveat: sometimes the ‘complex’ conjugate of an equation 1I or 1r should be used. For instance, let us derive the hermitian sine theorem from the dual cosine theorems 1I in the case $\kappa_2 \neq 0$. Start from the identity $S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) = S_{\kappa_1}(x_i) (1 - C_{\kappa_2}(X_J))/\kappa_2$, replace one factor $C_{\kappa_2}(X_J)$ by its expression taken from 1J, and the other $C_{\kappa_2}(X_J)$ by the complex conjugate; then expand:

$$S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) = \frac{1 - C_{\kappa_1}(x_i) - C_{\kappa_1}(x_j) - C_{\kappa_1}(x_k) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k)C_{\eta}(\Omega)}{\kappa_1^2 \kappa_2 S_{\kappa_1}(x_k)}.$$  

(6.55)

which by using (6.49) and (6.51) can be rewritten as:

$$S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k) = \frac{2}{\kappa_2}$$

(6.56)

As $\kappa_2 \neq 0$, the r.h.s of (6.56) is well defined and is clearly symmetric in the indices $ijk$ (this was also clear from (6.51)). Therefore $S_{\kappa_1}(x_i) S_{\kappa_2}(X_J) S_{\kappa_1}(x_k) = S_{\kappa_1}(x_j) S_{\kappa_2}(X_K) S_{\kappa_1}(x_i)$ leads to the sine theorem. Dually, when $\kappa_1 \neq 0$ the sine theorem follows also from the
three cosine theorems 1i. By following the real pattern, the dual hermitian cosine theorem and equations with tag 3 and 4 can also be derived.

Therefore, in the generic \( \kappa_1 \neq 0, \kappa_2 \neq 0 \) case, the three ‘complex hermitian’ cosine theorems 1i seen as six independent real equations are a set of basic equations. By duality, the same applies to the three dual cosine theorems 1I. By adding to either choice the four independent ‘Cartan sector’ phase equations relating the six phases, \( S, s \), we get a complete set of ten equations relating fourteen quantities. A triangle in the hermitian spaces \( \kappa_1 \neq 0, \kappa_2 \neq 0 \) is characterized by four independent quantities, for instance either \( a, b, c, \Omega \) or \( A, B, C, \omega \), which are a dual pair; this was already known for hermitian elliptic or hyperbolic spaces, but holds for the complete family of ‘complex’ CKD spaces as we shall see in the next section. Another choice are the six phases linked by the two relations (6.14).

6.5.1 Alternative forms of Hermitian Cosine Equations when \( \kappa_1 = 0 \) or \( \kappa_2 = 0 \)

The collapse of the Hermitian cosine equations (6.16) to (6.42) when \( \kappa_1 = 0 \) can be circumvented by writing (6.16) in an alternative form. The imaginary part can be rewritten in terms of the symplectic area \( S \), by using (6.47). For the real part, the procedure mimics the real one [1]: write all cosines of the sides and \( \kappa_2 \) of \( \Omega \) in terms of versed sines \( V_\kappa(x) := (1 - C_\kappa(x))/\kappa \)

\[
C_{\kappa_1}(x_1) = 1 - \kappa_1 V_{\kappa_1}(x_1) \quad C_\eta(\Omega) = C_{\eta \kappa_1}(2S) = 1 - \kappa_1^2 V_{\eta \kappa_1}(2S)
\] (6.57)

and then substitute in 1i, expand, cancel a common factor \( \kappa_1 \) and use the identities for the versed sines of a sum. Thus the pair of equations (6.16) can be rewritten as:

\[
1'I \quad \begin{align*}
V_{\kappa_1}(x_1) - V_{\kappa_1}(x_j + x_k) &= S_{\kappa_1}(x_j)S_{\kappa_1}(x_k)(C_{\kappa_1}(X_I)C_\eta(\Phi_I) - 1) - \kappa_1 V_{\eta \kappa_1}(2S)C_{\kappa_1}(x_1) \\
C_{\kappa_1}(x_1)S_{\eta \kappa_1}(2S) &= -S_{\kappa_1}(x_1)S_{\kappa_1}(x_k)C_{\kappa_1}(X_I)S_\eta(\Phi_I)
\end{align*}
\] (6.58)

a form meaningful for any value of \( \kappa_1 \) (even if all other labels are equal to zero). When \( \kappa_1 = 0 \) they reduce to

\[
\begin{align*}
x_j^2 &= x_j^2 + x_k^2 + 2x_jx_k C_{\kappa_2}(X_I)C_\eta(\Phi_I) \\
2S &= -x_jx_k C_{\kappa_2}(X_I)S_\eta(\Phi_I).
\end{align*}
\] (6.59)

The dual cosine equations 1I allow a similar reformulation:

\[
1'' \quad \begin{align*}
V_{\kappa_2}(X_I) - V_{\kappa_2}(X_J + X_K) &= S_{\kappa_2}(X_I)S_{\kappa_2}(X_K)C_{\kappa_2}(x_1)C_\eta(\phi_I) - 1) - \kappa_2 V_{\eta \kappa_2}(2S)C_{\kappa_2}(X_I) \\
C_{\kappa_2}(X_I)S_{\eta \kappa_2}(2S) &= -S_{\kappa_2}(X_J)S_{\kappa_2}(X_K)C_{\kappa_2}(x_1)S_\eta(\phi_I).
\end{align*}
\] (6.60)

which is meaningful for any value of \( \kappa_2 \)

The quantities \( \gamma \) (\( \Gamma \)) can be also given in terms of sides and symplectic area, (angles and symplectic coarea) by expressions which are still meaningful when \( \kappa_1 = 0 \) (\( \kappa_2 = 0 \)):

\[
\begin{align*}
\gamma &= 2(V_{\kappa_1}(x_1)V_{\kappa_2}(x_j) + V_{\kappa_1}(x_j)V_{\kappa_2}(x_k) + V_{\kappa_1}(x_k)V_{\kappa_2}(x_i)) - V_{\kappa_1}^2(x_i) - V_{\kappa_2}^2(x_j) - 2\eta C_{\kappa_1}(x_1)C_{\kappa_2}(x_j)C_{\kappa_2}(x_k)V_{\eta \kappa_1}(2S) - 2\eta V_{\kappa_1}(x_i)V_{\kappa_1}(x_j)V_{\kappa_1}(x_k) \\
\Gamma &= 2(V_{\kappa_2}(X_I)V_{\kappa_2}(X_J) + V_{\kappa_2}(X_J)V_{\kappa_2}(X_K) + V_{\kappa_2}(X_K)V_{\kappa_2}(X_I)) - V_{\kappa_2}^2(X_I) - V_{\kappa_2}^2(X_J) - 2\kappa_2 V_{\kappa_2}(X_I)V_{\kappa_2}(X_J)V_{\kappa_2}(X_K).
\end{align*}
\] (6.61)
Thus when $\kappa_1 = 0$ but $\kappa_2 \neq 0$, the six real equations $1'i'$ are independent and all the remaining equations follow from the Cartan sector equations (6.48) and from the three pairs $1'i'$ just as everything followed from $1i$ in the case $\kappa_1 \neq 0$. Dually, mutatis mutandis from $1'I'$ when $\kappa_2 = 0$ but $\kappa_1 \neq 0$.

We finally discuss the situation when $\kappa_1 = \kappa_2 = 0$. The lateral and angular phases are equal: $\Phi_I = \phi_i$ and this provides three independent equations. Two further equations are the sine theorem which in this case cannot be derived from $1i'$ or $1'I'$. This makes five independent equations:

$$
\Phi_I = \phi_i, \quad \frac{x_i}{X_I} = \frac{x_j}{X_J} = \frac{x_k}{X_K},
$$

(6.62)

The remaining details depend on whether $\eta$ is zero or not. If $\eta \neq 0$, the equations $1'i'/1'I'$ read:

$$
\begin{align*}
\text{Re}1'i' & \quad x_i^2 = x_j^2 + x_k^2 + 2x_jx_kC_\eta(\Phi_I) \\
\text{Re}1'I' & \quad X_I^2 = X_J^2 + X_K^2 + 2X_JX_KC_\eta(\phi_i) \\
\text{Im}1'i' & \quad 2S = -x_jx_kS_\eta(\Phi_I) \\
\text{Im}1'I' & \quad 2s = -X_JX_KS_\eta(\phi_i)
\end{align*}
$$

(6.63)

Taking into account (6.62), the groups of three equations Re1'i' and Re1'I' are equivalent; either of them can be taken as three further equations. Any of these sets imply the relation (whose general form is (6.27))

$$
\frac{x_i}{S_\eta(\Phi_I)} = \frac{x_j}{S_\eta(\Phi_J)} = \frac{x_k}{S_\eta(\Phi_K)}
$$

(6.64)

which shows that the three equations either Im1'i' or Im1'I' collapse to a single equation. Taken altogether, these provide another five independent equations in (6.63). When $\eta = 0$, i.e., in the most contracted case, the three equations Re1'i' collapse to a single equation better written as $x_i + x_j + x_k = 0$ and likewise Re1'I' collapse to $X_I + X_J + X_K = 0$; these two equations are however not independent in view of the sine theorem. In this case the most contracted form of (6.27) cannot be derived from previous equations and have to be added as two further independent equations, in either of the two forms:

$$
\frac{x_i}{\Phi_I} = \frac{x_j}{\Phi_J} = \frac{x_k}{\Phi_K} \quad \text{or} \quad \frac{X_I}{\phi_i} = \frac{X_J}{\phi_j} = \frac{X_K}{\phi_k},
$$

(6.65)

using these equations, each group of three equations Im1'i' or Im1'I' collapses to a single equation. This makes again five independent additional equations altogether in:

$$
\begin{align*}
x_i + x_j + x_k &= 0, \quad X_I + X_J + X_K = 0, \\
x_i &= \frac{x_j}{\Phi_I} = \frac{x_k}{\Phi_K}, \quad 2S = -x_jx_k\Phi_I, \quad 2s = -X_JX_K\phi_i.
\end{align*}
$$

(6.66)

**Theorem 3.** The full set of equations of ‘complex Hermitian’ trigonometry linking the fourteen quantities $x_i, X_I, \phi_i, \Phi_I, S, s$ contains for any value of $\eta, \kappa_1, \kappa_2$ exactly ten independent equations. Any other equation in the set is a consequence of them. When $\kappa_1$ or $\kappa_2$ are different from zero, four such equations are the two phases equations $0ij \equiv 0IJ$ and the two relations $\Omega = \kappa_12S$, $\omega = \kappa_22s$. The remaining six independent equations are:

- When $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$, any $\eta$, either the equations $1i$ or $1I$.
- When $\kappa_1 = 0$ but $\kappa_2 \neq 0$, any $\eta$, the equations $1'i'$.
- When $\kappa_1 \neq 0$ but $\kappa_2 = 0$, any $\eta$, the equations $1I'$. 

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When both $\kappa_1 = \kappa_2 = 0$, the independent equations are:

- When $\eta \neq 0$, the ten independent equations in (6.62) and (6.63).
- When $\eta = 0$, the ten independent equations in (6.62) and (6.66).

Table 2: Complex Hermitian sine theorems and relations between symplectic area $S$, coarea $s$ and mixed phase excesses $\Omega, \omega$ for the twenty seven ‘complex Hermitian’ (‘CH’) CKD spaces. The Table is arranged with columns labeled by $\kappa_1 = 1, 0, -1$ and rows by $\kappa_2 = 1, 0, -1$; $(\eta; \kappa_1, \kappa_2)$ are explicitly displayed at each entry. All relations in this Table hold in the same form no matter of the value of $\eta$. The group description of the homogeneous spaces is given in the CKD type notation.

| 'CH' Elliptic (\(\eta; +1,1\)) | 'CH' Euclidean (\(\eta; 0, +1\)) | 'CH' Hyperbolic (\(\eta; -1, +1\)) |
|---------------------------------|---------------------------------|---------------------------------|
| $\eta SU(3)/\eta U(1) \otimes SU(2)$ | $\eta U(2)/\eta U(1) \otimes SU(2)$ | $\eta SU(2, 1)/\eta U(1) \otimes SU(2)$ |
| $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ | $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{c}{\sin C}$ | $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ |
| $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = -\omega$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ |
| $\omega = 2s$ | $\omega = 2s$ | $\omega = 2s$ |

| 'CH' Co-Euclidean (\(\eta; +1,0\)) | 'CH' Galilean (\(\eta; 0, 0\)) | 'CH' Co-Minkowskian (\(\eta; -1,0\)) |
|---------------------------------|---------------------------------|---------------------------------|
| $\eta IU(2)/\eta U(1) \otimes \eta U(1)$ | $\eta IU(1)/\eta U(1) \otimes \eta U(1)$ | $\eta IU(1, 1)/\eta U(1) \otimes \eta U(1)$ |
| $\frac{\sin a}{\sin A} = \frac{\sin b}{B} = \frac{\sin c}{C}$ | $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$ | $\frac{\sin a}{\sin A} = \frac{\sin b}{B} = \frac{\sin c}{C}$ |
| $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = 0$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ |
| $\omega = 0$ | $\omega = 0$ | $\omega = 0$ |

| 'CH' Co-Hyperbolic (\(\eta; +1, -1\)) | 'CH' Minkowskian (\(\eta; 0, -1\)) | 'CH' Doubly Hyperbolic (\(\eta; -1, -1\)) |
|---------------------------------|---------------------------------|---------------------------------|
| $\eta SU(2, 1)/\eta U(1) \otimes \eta SU(1, 1)$ | $\eta IU(1, 1)/\eta U(1) \otimes \eta SU(1, 1)$ | $\eta SU(2, 1)/\eta U(1) \otimes \eta SU(1, 1)$ |
| $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ | $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$ | $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$ |
| $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = -\omega$ | $\Phi_A - \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega$ |
| $\omega = -2s$ | $\omega = -2s$ | $\omega = -2s$ |

Tables 2 and 3,4,5 display the basic equations for the twenty seven CKD spaces, written in conventional notation. As neither the Cartan sector equations, nor the sine theorem involve the CD label $\eta$, these equations are displayed in a single Table 2, according to the values of the CK constants ($\kappa_1, \kappa_2$). The remaining basic equations, i.e. Hermitian cosine and dual cosine theorems, which involve the CD label $\eta$ are given in an Appendix threefold display (Tables 3,4,5), one table for each value of $\eta = 1, 0, -1$.

### 6.6 Symmetric invariants and existence conditions

Several Hermitian trigonometric equations are or can be rewritten as a relation belonging to one of two types. The first type has a structure similar to the sine theorem: a ‘one-element’ expression involving only one index (vertex, opposite side) has the same value
for the two remaining ones:

\[ \Phi_I - \phi_I = \Omega - \omega, \quad \frac{S_{\eta_1}(x_i)}{S_{\eta_2}(x_I)} =: \tau, \quad \frac{S_{\eta_1}(2x_i)}{S_{\eta_1}(\Phi_I)C_{\eta_2}(X_I)} =: \xi, \quad \frac{S_{\eta_2}(2X_I)}{S_{\eta_2}(\phi_I)C_{\eta_1}(x_i)} =: \Xi. \]  

(6.67)

Under duality \( \tau \leftrightarrow \frac{1}{\tau} \) and \( \xi \leftrightarrow \Xi \). Other such 'one-element' type equations have values which can be expressed in terms of the three triangle invariants \( \tau, \xi, \Xi \):

\[ \frac{S_{\eta_1}(x_i)S_{\eta_2}(X_I)}{S_{\eta_1}(\Phi_I)S_{\eta_2}(\phi_I)} = \frac{1}{4} \xi \Xi, \quad \frac{C_{\eta_1}(x_i)T_{\eta_2}(X_I)}{S_{\eta_1}(\Phi_I)} = \frac{1}{2} \xi \tau, \quad \frac{C_{\eta_2}(X_I)T_{\eta_1}(x_i)}{S_{\eta_2}(\phi_I)} = \frac{1}{2} \Xi \tau. \]  

(6.68)

The second type has a structure like formulas allowing the introduction of \( \Omega, \omega \); a 'cyclic' expression invariant under any cyclic permutation of the three indices it involves:

\[ \Phi_I + \Phi_J + \phi_K =: \Omega = \kappa_1 2S, \quad \phi_i + \phi_j + \Phi_K =: \omega = \kappa_2 2s. \]

(6.69)

\[ S_{\eta_2}(X_I)S_{\eta_2}(X_J)S_{\eta_1}(x_k) = \sqrt{\Gamma/\kappa_1}, \quad S_{\eta_1}(x_i)S_{\eta_1}(x_j)S_{\eta_2}(X_K) = \sqrt{\gamma/\kappa_2}. \]

There is no essential difference between the 'one-element' and 'cyclic' types of equations, and it turns out to be possible to express the 'one-element' invariants in an explicitly 'cyclic' form:

\[ \frac{S_{\eta_1}(x_i)}{S_{\eta_2}(X_I)} =: \tau = \frac{S_{\eta_1}(x_i)S_{\eta_1}(x_j)S_{\eta_1}(x_k)}{\sqrt{\gamma/\kappa_2}} = \frac{\sqrt{\Gamma/\kappa_1}}{\sqrt{\gamma/\kappa_2}}, \]

\[ \frac{S_{\eta_2}(X_I)}{S_{\eta_1}(x_i)} =: \tau = \frac{S_{\eta_2}(X_I)S_{\eta_2}(X_J)S_{\eta_2}(X_K)}{\sqrt{\gamma/\kappa_2}} = \frac{\sqrt{\Gamma/\kappa_1}}{\sqrt{\gamma/\kappa_2}}, \]

\[ \frac{S_{\eta_1}(2x_i)}{S_{\eta_1}(\Phi_I)C_{\eta_2}(X_I)} =: \xi = -2 \frac{S_{\eta_1}(x_i)S_{\eta_1}(x_j)S_{\eta_1}(x_k)}{S_{\eta_1}(2S)} = -2 \frac{\gamma/\kappa_2}{\sqrt{\Gamma/\kappa_1}} \frac{1}{S_{\eta_1}(2S)}, \]

\[ \frac{S_{\eta_2}(2X_I)}{S_{\eta_2}(\phi_I)C_{\eta_1}(x_i)} =: \Xi = -2 \frac{S_{\eta_2}(X_I)S_{\eta_2}(X_J)S_{\eta_2}(X_K)}{S_{\eta_2}(2S)} = -2 \frac{\Gamma/\kappa_1}{\sqrt{\gamma/\kappa_2}} \frac{1}{S_{\eta_2}(2S)}. \]  

(6.70)

The two quantities \( \sqrt{\gamma/\kappa_2} \) and \( \sqrt{\Gamma/\kappa_1} \) must be real in order the triangle to exist. Therefore, any triangle must satisfy the inequalities:

\[ \frac{\gamma}{\kappa_2} \geq 0, \quad \frac{\Gamma}{\kappa_1} \geq 0 \]  

(6.71)

which apply to any member of the CKD family of 'complex Hermitian' spaces, notwithstanding any restriction for the sides and angles. Brehm's inequalities under which a triangle with prescribed values for sides and shape invariant exists in the elliptic and hyperbolic complex hermitian spaces are simply the transcription of the condition \( \frac{\gamma}{\kappa_2} \geq 0 \) to the complex CK spaces with \( \eta = 1; \kappa_1 \neq 0, \kappa_2 = 1 \). Thus \( \gamma \geq 0 \), or, equivalently \( \Delta_g = \kappa_1^2 \gamma \geq 0 \); by using (6.49) this gives:

\[ \Delta_g = 1 - C_{\kappa_1}^2(x_i) - C_{\kappa_1}^2(x_j) - C_{\kappa_1}^2(x_k) + 2C_{\kappa_1}(x_i)C_{\kappa_1}(x_j)C_{\kappa_1}(x_k) \cos \Omega \geq 0 \]  

(6.72)

which covers simultaneously the inequalities given by Brehm for the elliptic (\( \kappa_1 > 0 \)) and hyperbolic (\( \kappa_1 < 0 \)) hermitian spaces (remark Brehm calls \( \omega \) our \( \Omega \)). Inequalities fourth element \( \omega \)

It is also worth highlighting the translation of the inequalities (6.71) by using (6.38); this brings them in terms of angular and lateral phases, and symplectic area and coarea:

\[ - \frac{S_{\eta_1}(2S)}{S_{\eta_1}(\Phi_I)S_{\eta_1}(\Phi_J)S_{\eta_1}(\Phi_K)} \geq 0, \quad - \frac{S_{\eta_2}(2S)}{S_{\eta_2}(\phi_I)S_{\eta_2}(\phi_J)S_{\eta_2}(\phi_K)} \geq 0. \]  

(6.73)
The SR double cosine for sides (6.19), and double sine for sides (6.24) become in this case:

\[
\frac{S_{\kappa_1}(x_1)}{S_{\kappa_2}(X_1)} = \tau = \frac{S_{\eta}(\phi_1)S_{\eta}(\phi_j)S_{\eta}(\phi_k)S_{\eta}(\phi_k^2)S_{\eta}(\phi_k)}{S_{\eta}(\Phi_1)S_{\eta}(\Phi_j)S_{\eta}(\Phi_K)S_{\eta}(\phi_k)},
\]

\[
S_{\kappa_2}(2x_i) = \eta = \sqrt{\frac{-4S_{\eta}(\phi_1)S_{\eta}(\phi_j)S_{\eta}(\phi_k)S_{\eta}(\phi_k^2)S_{\eta}(\phi_k)}{S_{\eta}(\Phi_1)S_{\eta}(\Phi_j)S_{\eta}(\Phi_K)S_{\eta}(\phi_k)}} \quad \text{and} \quad \Xi = \sqrt{\frac{-4S_{\eta}(\Phi_1)S_{\eta}(\Phi_j)S_{\eta}(\Phi_K)S_{\eta}(\phi_k)}{S_{\eta}(\phi_1)S_{\eta}(\phi_j)S_{\eta}(\phi_k)}},
\]

in whose form the existence inequalities (6.73) are evident.

The inequalities (6.71) are analogous to the existence conditions \( E \leq 0, \frac{E}{\delta} \leq 0 \) for the half-excesses \( E = \Delta/2, e = \delta/2 \) appearing in the trigonometry of real spaces. A way to derive such real inequalities, alternative to the one used in [1], is to introduce in the real case the determinants \( \Delta_{\eta}, \Delta_{\eta} \) of the Gramm matrices built up from the real symmetric scalar products of vectors corresponding to vertices or to poles of sides; these are given by (6.49) with \( \Omega = 0 \) appearing in the real symmetric scalar products of vectors corresponding to vertices or to poles of sides; these are given by (6.49) with \( \Omega = 0 \), and also vanish when \( \kappa_1, \kappa_2 \rightarrow 0 \). If we introduce again \( \gamma, \Gamma \) by (6.51), in absence of the factors \( C_{\eta}(\Omega), C_{\eta}(\omega) \), the identity A.30 in the appendix of [1] allow a factorization of \( \gamma, \Gamma \), translating the conditions \( \frac{E}{\kappa_1} \geq 0, \frac{E}{\kappa_2} \geq 0 \) into inequalities for angular and lateral excesses \( \frac{E}{\kappa_1} \leq 0, \frac{E}{\kappa_2} \leq 0 \). This last step cannot be done in the Hermitian case and the inequalities stay in the form (6.71).

### 6.7 The three special cases: collinear triangles, concurrent triangles and purely real triangles

Browsing through the equations we have given, we find several pairs of equations which can be stated in two similar variant forms, one involving sides (angles) and the other involving twice the sides (angles): examples of such pairs are (6.18, 6.19) or (6.15, 6.24) and their duals. This fact insinuates the existence of two special non-generic types of triangles, for which the appropriate generic equation reduces to a (known) simpler form.

#### 6.7.1 Complex Collinear triangles

The first special case corresponds to a triangle determined by three ‘complex’ collinear vertices, hence collapsing from the ‘complex’-2D CK hermitian space \( SU_{\kappa_1,\kappa_2}(3)/U(1) \) to a ‘complex’-1D subspace, which can be identified with a space \( SU_{\kappa_2}(2)/U(1) \). Depending on whether \( \kappa_1 > 0, = 0, < 0 \), this space is the elliptic, euclidean or hyperbolic hermitian ‘complex’ line. Sides are all different from zero, but angles \( X_I, X_J, X_K \) must be zero (or straight), and thus satisfy \( S_{\kappa_2}(X_I) = S_{\kappa_2}(X_J) = S_{\kappa_2}(X_K) = 0 \) (hence \( \gamma = \Gamma = 0 \)).

For these values the equations 1J reduce to \( e^{i\omega} = 1 \), thus \( \omega = 0 \) and from \( 0iI \) we get:

\[
\omega = 0, \quad \Phi_I - \phi_i = \Omega.
\]

The SR double cosine for sides (6.19), and double sine for sides (6.24) become in this case:

\[
C_{\kappa_1}(2x_j) = C_{\kappa_1}(2x_i)C_{\kappa_1}(2x_k) - \kappa_1 S_{\kappa_1}(2x_i)S_{\kappa_1}(2x_k)C_{\eta}(\Phi_J)
\]

(6.76)
so all hermitian trigonometric equations reduce in this case to the trigonometry of a triangle with sides \(2x_i, 2x_j, 2x_k\) and angles \(\Phi_I, \Phi_J, \Phi_K\) in an auxiliary real CK space with labels \(\kappa_1\) for sides and \(\eta\) for angles, or equivalently, for a triangle with sides \(x_i, x_j, x_k\) and angles \(\Phi_I, \Phi_J, \Phi_K\) in a real CK space with labels \(4\kappa_1\) for sides and \(\eta\) for angles, for which (6.76, 6.77) are the real cosine and sine theorems. The Lie algebra isomorphism \(\mathfrak{su}_{\kappa_1}(2) \simeq SO_{\kappa_1,\eta}(3)\) lies behind this.

By using (6.12), and recalling \(\omega = 0\), the auxiliary triangle angular excess \(\Phi_I + \Phi_J + \Phi_K\) turns out to be equal to \(2\Omega\), and thus \(\Omega\) plays the role of the angular half-excess denoted \(E\) in the previous paper on real type trigonometry [1]. It is worth remarking that the area \(A\) of this auxiliary triangle is related to its angular excess as \(A = \frac{2\Omega}{4\kappa_1} = \frac{\Omega}{2\kappa_1}\), thus coinciding with the original triangle symplectic area \(S\). The lateral phases \(\phi_i\) turn out to coincide with the three auxiliary angles denoted \(E_I\) in [1]. In terms of the symmetric invariants, the ‘collinear’ case corresponds to:

\[
\begin{align*}
\Delta &= 0 & \Delta_\Phi &= 2\Omega & \Omega &= \Omega & \gamma &= 0 & S &= S & \xi &= \xi & \tau &= \infty
\end{align*}
\] (6.78)

6.7.2 Concurrent triangles

The second special case, dual to the previous one, corresponds to a triangle determined by three different concurrent geodesic sides. Then sines of sides are equal to zero: \(S_{\kappa_1}(x_i) = S_{\kappa_1}(x_j) = S_{\kappa_1}(x_k) = 0\). Here \(\Omega = 0\), and the SR dual double cosine equation (6.21) and SR dual double sine equation (6.25) become the cosine and sine theorems for a triangle with sides \(X_I, X_J, X_K\) and angles \(\phi_i, \phi_j, \phi_k\) in a real CK space with labels \(4\kappa_2\) for sides and \(\eta\) for angles. For the angular excess of this auxiliary triangle we have \(\phi_i + \phi_j + \phi_k = 2\omega\), and thus \(\omega\) plays the role of the angular half-excess \((E\) in [1]). In terms of the symmetric invariants, this case corresponds to:

\[
\begin{align*}
\Delta &= \Delta & \Delta_\Phi &= -\omega & \Omega &= 0 & \gamma &= 0 & S &= 0 & \xi &= 0 & \tau &= 0.
\end{align*}
\] (6.79)

6.7.3 Purely Real triangles

The third special case corresponds to a triangle for which the lateral and angular phase factors \(e^{i\phi_i}\) and \(e^{i\Phi_I}\) are real, and sides and angles are different from zero; this purely real triangle is contained in a purely real totally geodesic submanifold, isometric to \(SO_{\kappa_1,\kappa_2}(3)/(O(1) \otimes SO_{\kappa_2}(2))\), and locally isometric (as \(O(1) \equiv Z_2\)) to \(SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)\); for \(\kappa_1 = 1, \kappa_2 = 1\) this is the real projective space \(\mathbb{R}P^2\). Sines of both sets of phases vanish whenever the other does (see (6.30)); this corresponds to the self-dual nature of this case. Angular and lateral phase excesses, \(\Omega\) and \(\omega\) and symplectic area and coarea have vanishing sines. Each individual phase \(\phi_i\) or \(\Phi_I\) can thus have only two values, either 0 or twice a quadrant of label \(\eta\), which have opposite cosines \(\pm 1\). In terms of the symmetric invariants, this case corresponds to the values:

\[
\begin{align*}
\Delta &= \Delta & \Delta_\Phi &= 0 & \Omega &= 0 & \gamma &= \gamma & S &= 0 & \xi &= \infty & \tau &= \tau.
\end{align*}
\] (6.80)
This reduction also provides an approach to the trigonometry of real projective planes, requiring as triangle elements, further to sides and angles, a set of discrete phases, entering the equations only through their cosines \( \varepsilon_i = C_\eta(\phi_i) = \pm 1; \varepsilon_I = C_\eta(\Phi_I) = \pm 1 \). Thus Hermitian trigonometry of the ‘complex’ spaces simultaneously afford, if we restrict phases to these two possible discrete values, the trigonometry of the real ‘projective’ CK spaces family (to which \( \mathbb{R}P^2 \) belongs). The distinction between the trigonometry of the sphere and the real projective plane is well known (e.g. in Coxeter [26]).

7 Overview and Concluding remarks

The most direct physical application of Hermitian trigonometry is to the trigonometry of the Quantum space of states, which is the elliptic member \((\kappa_1 > 0, \kappa_2 > 0)\) of the family of complex \((\eta > 0)\) CKD Hermitian spaces; geometric phases appear directly as trigonometric invariants from this point of view. This will be discussed in the companion paper [20].

There are also other possibly interesting applications of an explicit knowledge of the trigonometry of this family of spaces. The real space-time models with zero or constant space-time curvature (Minkowskian and de Sitter space-times) are superseded by a variable curvature pseudoRiemannian space-time; this is the essence of the Einsteinian interpretation of gravitation. The possibility of a kind of ‘Riemannian’ Quantum space of states, whose curvature might be not constant, cannot be precluded a priori. A good understanding of the geometry of the Hermitian constant curvature cases might be helpful to explore and figure out what consequences might follow from this idea and familiarity with their trigonometry is a first order tool in this aim.

Another physical problem where the results we have obtained could apply lies on the use of pseudo-Hilbert spaces with an indefinite Hermitian scalar product (Gupta-Bleuler type). These indefinite Quantum spaces of states are those corresponding to \( \kappa_2 > 0 \); and its hermitian trigonometry should provide the basic elementary relations in the geometry of these spaces, just as the corresponding real relations are the basic space-time relations in the de Sitter and Anti de Sitter space-times.

The identification of the Quantum space of states as a member of this complete CKD family of spaces makes it also natural to inquire about whether or not the labels \( \eta, \kappa_1, \kappa_2 \) may have any sensible physical meaning. Within the ‘kinematical’ \((\kappa_2 \leq 0)\) interpretation of the real CK spaces, \( \kappa_1 \) is the curvature of space-time and \( \kappa_2 = -1/c^2 \) is related to the relativistic constant. A natural query is: are the limits \( \eta \to 0, \kappa_1 \to 0 \) (and \( N \to \infty \)) somehow related to a ‘classical’ limit \( \hbar \to 0 \) within some sensible ‘quantum’ interpretation of the ‘complex hermitian’ spaces? This is worth exploring.

Real hyperbolic trigonometry, deeply involved in manifold classification problems, knot theory, etc., is merely a particular case of real CK trigonometry. It is not unreasonable to assume that (some instances at least) of the generic Hermitian trigonometry may be at least as relevant in the similar ‘complexified’ problems [32]. The intriguing indications for an essentially complex nature of space-time at some deep level makes also worthy the study of complex spaces in a way as explicit and visual as possible.

Aside the physical interest of particular results, another potential in the method proposed in [1] and developed in the present paper lies on the possibility of opening an
avenue for studying the trigonometry of other symmetric homogeneous spaces, most of whose trigonometries are still unknown. Very few results are known in this area; a general sine theorem is derived in Leuzinger [33] for non-compact spaces, and the trigonometry of the rank-two spaces $SU(3)$ and $SL(3, \mathbb{C})/SU(3)$ is discussed in [34, 35] heavily relying on the use of the Weyl theorems on invariant theory and characterization of invariants by means of traces of products of matrices.

The trigonometry of the rank-one ‘quaternionic hyper-hermitian’ spaces $(Sp(3)/(Sp(1) \otimes Sp(2)), Sp(2, 1)/(Sp(1) \otimes Sp(2)), Sp(2, 1)/(Sp(1) \otimes Sp(1, 1))$ or $Sp(6, \mathbb{R})/(SO(2, 1) \otimes Sp(4, \mathbb{R})))$ —which correspond to further Cayley-Dickson extensions with a new CD label $\eta_2$—, and also of the ‘octonionic type’ analogues of the Cayley plane —with another CD label $\eta_3$ altogether—, reduces in some sense to the ‘complex’ two-dimensional case, since any triangle in these spaces lies on a ‘complex’ chain; thus in a sense the study of trigonometry in rank one spaces is essentially complete with the spaces of real quadratic and ‘complex Hermitian’ type. This reduction is not natural however from a purely quaternionic or octonionic viewpoint. Perhaps quaternionic (and also the exceptional octonionic) trigonometry should be understood better. In any case, this kind of approach in a ‘$\mathbb{R}, \mathbb{C}, \mathbb{H}$ spirit’ fits into the V.I. Arnold idea of mathematical trinities; hopefully it may provide a way to the quest [32] for the quaternionic analogue of Berry’s phase.

A next natural objective along this line is the study of trigonometry of higher rank grassmannians, either real or complex. This is still largely unknown (see however [36]). Should the method outlined in this paper be able to produce in a direct form the equations of trigonometry for grassmannians which are also very relevant spaces in many physical applications, this would make a further step towards a general approach to trigonometry of any symmetric homogeneous space. This goal will require first to group all symmetric homogeneous spaces into CKD families, and then to study trigonometry for each family. Work in progress on this line [37, 3, 38] opens the possibility of realizing all simple Lie algebras (even $SL(N, \mathbb{R}), SL(N, \mathbb{C}), SO^*(2n), SU^*(2n)$ and the exceptional ones) as ‘unitary’ algebras, leaving invariant an ‘hermitian’ (relative to some antiinvolution) form over a tensor product of two pseudo-division algebras. This realization should allow a test on whether or not some extension of the ideas outlined here afford the equations of trigonometry for any homogeneous space in an explicit and simple enough way.

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Appendix

Table Captions:

Table 3 ($\eta > 0$). The Table is arranged after the values of the pair $\kappa_1, \kappa_2$, and the three labels ($\eta; \kappa_1, \kappa_2$) are explicitly displayed at each entry. The group description $G/H$ of the homogeneous space is shown only when $G$ has a standard name; this is not the case for $(\kappa_1, \kappa_2) = (0, 0)$. The two spaces of points at the two corners in the last row $\kappa_2 = -1$ are
the same, but the corresponding geometries differ by the interchange of first- and second-kind lines generated by either \( P_1 \) or \( P_2 \). Notice the sign difference in equations involving \( a, A \) and involving \( b, B; c, C \) and the relevant comments in the main text.

Table 4 (\( \eta = 0 \)). The Table is arranged after the values of the pair \( \kappa_1, \kappa_2 \), and the three labels (\( \eta; \kappa_1, \kappa_2 \)) are explicitly displayed at each entry. The group description \( G/H \) of the homogeneous space is not shown as when \( \eta = 0 \) the CKD groups are not simple and have not a standard name. The fiducial role of the trigonometry for the space (\( \eta = 0; \kappa_1 = 0, \kappa_2 = 0 \)) in the center of this Table is clear. All the trigonometries in Tables 3, 4 and 5 are deformations of this ‘purely linear’ one.

Table 5 (\( \eta < 0 \)). The Table is arranged after the values \( \kappa_1, \kappa_2 \), and the labels (\( \eta; \kappa_1, \kappa_2 \)) are explicitly displayed at each entry. The group description \( G/H \) of the homogeneous space is shown only when \( G \) has a standard name. The spaces at the four corners are equal, but the trigonometric equations in these geometries are different as they correspond to triangles with geodesics sides of the four not conjugate different possible types.
Table 3: Complex Hermitian cosine theorems and their duals for the nine complex Hermitian CKD spaces ($\eta = 1$).

| Complex Hermitian Elliptic  $\text{(}+1; +1, +1\text{)}$ | Complex Hermitian Euclidean $\text{(}+1; 0, +1\text{)}$ | Complex Hermitian Hyperbolic $\text{(}+1; -1, +1\text{)}$ |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $SU(3)/U(1) \otimes SU(2)$                      | $SU(2)/U(1) \otimes SU(2)$                      | $SU(2, 1)/U(1) \otimes SU(2)$                    |
| $\cos a \cos \Omega = \cos b \cos c - \sin b \sin c \cos A \cos \Psi_A$ | $a^2 = b^2 + c^2 + 2bc \cos A \cos \Psi_A$ | $\cosh a \cos \Omega = \cosh b \cosh c + \sinh b \sinh c \cos A \cos \Psi_A$ |
| $\cos b \cos \Omega = \cos a \cos c + \sin a \sin c \cos B \cos \Psi_B$ | $b^2 = a^2 + c^2 - 2ac \cos B \cos \Psi_B$ | $\cosh b \cos \Omega = \cosh a \cos c - \sin a \sin c \cos B \cos \Psi_B$ |
| $\cos c \cos \Omega = \cos a \cos b + \sin a \sin b \cos C \cos \Psi_C$ | $c^2 = a^2 + b^2 - 2ab \cos C \cos \Psi_C$ | $\cosh c \cos \Omega = \cosh a \cos b - \sin a \sin b \cos C \cos \Psi_C$ |
| $\cos a \sin 2S = \sin b \sin c \cos A \sin \Psi_A$ | $2S = bc \cos A \sin \Psi_A$ | $\sinh a \sin 2S = \sinh b \sinh c \cos A \sin \Psi_A$ |
| $\cos b \sin 2S = \cos c \sin a \sin B \sin \Psi_B$ | $2S = ca \sin B \sin \Psi_B$ | $\cosh b \sin 2S = \cosh a \sin c \sin B \sin \Psi_B$ |
| $\cos c \sin 2S = \sin a \sin b \sin C \sin \Psi_C$ | $2S = ab \cos C \sin \Psi_C$ | $\cosh c \sin 2S = \cosh a \sin b \cosh C \sin \Psi_C$ |
| $\cos A \sin 2s = \sin B \sinh C \cos a \sin \psi_a$ | $\cos A \sin 2s = \sin B \sinh C \cos a \sin \psi_a$ | $\cosh A \sin 2s = \sin B \sinh C \cos a \sin \psi_a$ |
| $\cos B \sin 2s = \cos C \sin a \sin b \sin \psi_b$ | $\cos B \sin 2s = \cos C \sin a \sin b \sin \psi_b$ | $\cosh B \sin 2s = \cos C \sin a \sin b \sin \psi_b$ |
| $\cos C \sin 2s = \sin a \sin b \sin B \sin \psi_c$ | $\cos C \sin 2s = \sin a \sin b \sin B \sin \psi_c$ | $\cosh C \sin 2s = \sin a \sin b \sin B \sin \psi_c$ |

Complex Hermitian Oscillating Newton-Hooke $\text{(}+1; +1, 0\text{)}$

Complex Hermitian Galilean $\text{(}+1; 0, 0\text{)}$

Complex Hermitian Co-Minkowskian $\text{(}+1; -1, 0\text{)}$

Complex Hermitian Expanding Newton-Hooke $\text{(}+1, 0\text{)}$

Complex Hermitian De Sitter $\text{(}+1, -1\text{)}$

Complex Hermitian Double Hyperbolic $\text{(}+1; -1, -1\text{)}$
Table 4: ‘Complex Hermitian’ cosine theorems and their duals for the nine parabolic complex (dual) ‘Hermitian’ CKD spaces ($\eta = 0$).

| Parabolic Complex Hermitian’ Elliptic (0; +1, +1) | Parabolic Complex Hermitian’ Euclidean (0; 0, +1) | Parabolic Complex Hermitian’ Hyperbolic (0; −1, +1) |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\cos a = \cos b \cos c - \sin b \sin c \cos A$ | $a^2 = b^2 + c^2 + 2bc \cos A$ | $\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A$ |
| $\cos b = \cos a \cos c + \sin a \sin c \cos B$ | $b^2 = a^2 + c^2 - 2ac \cos B$ | $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$ |
| $\cos c = \cos a \cos b + \sin a \sin b \cos C$ | $c^2 = a^2 + b^2 - 2ab \cos C$ | $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$ |

| Parabolic Complex Hermitian’ Co-Euclidean (0; +1, 0) | Parabolic Complex Hermitian’ Galilean (0; 0, 0) | Parabolic Complex Hermitian’ Co-Minkowskian (0; −1, 0) |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A$ | $a^2 = b^2 + c^2 + 2bc \cosh A$ | $\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A$ |
| $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$ | $b^2 = a^2 + c^2 - 2ac \cosh B$ | $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$ |
| $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$ | $c^2 = a^2 + b^2 - 2ab \cosh C$ | $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$ |

| Parabolic Complex Hermitian’ Oscillating Newton-Hooke | Parabolic Complex Hermitian’ Expanding Newton-Hooke |
|---------------------------------------------|---------------------------------------------|
| $\cos a = \cos b \cos c - \sin b \sin c \cosh A$ | $a^2 = b^2 + c^2 + 2bc \cosh A$ |
| $\cos b = \cos a \cos c + \sin a \sin c \cosh B$ | $b^2 = a^2 + c^2 - 2ac \cosh B$ |
| $\cos c = \cos a \cos b + \sin a \sin b \cosh C$ | $c^2 = a^2 + b^2 - 2ab \cosh C$ |

| Parabolic Complex Hermitian’ Co-Hyperbolic (0; +1, −1) | Parabolic Complex Hermitian’ Minkowskian (0; 0, −1) | Parabolic Complex Hermitian’ Doubly Hyperbolic (0; −1, −1) |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$ | $a^2 = b^2 + c^2 + 2bc \cosh A$ | $\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$ |
| $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cosh B$ | $b^2 = a^2 + c^2 - 2ac \cosh B$ | $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cosh B$ |
| $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C$ | $c^2 = a^2 + b^2 - 2ab \cosh C$ | $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C$ |

| Parabolic Complex Hermitian’ Anti-de Sitter | Parabolic Complex Hermitian’ De Sitter |
|---------------------------------------------|---------------------------------------------|
| $\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$ | $a^2 = b^2 + c^2 + 2bc \cosh A$ |
| $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cosh B$ | $b^2 = a^2 + c^2 - 2ac \cosh B$ |
| $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C$ | $c^2 = a^2 + b^2 - 2ab \cosh C$ |

| $\cosh A = \cosh B \cosh C + \sinh B \sinh C \cosh A$ | $A = B + C$ |
| $\cosh B = \cosh A \cosh C - \sinh A \sinh C \cosh B$ | $B = A - C$ |
| $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cosh C$ | $C = A - B$ |

| $\cosh A = \cosh B \cosh C + \sinh B \sinh C \cosh A$ | $A = B + C$ |
| $\cosh B = \cosh A \cosh C - \sinh A \sinh C \cosh B$ | $B = A - C$ |
| $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cosh C$ | $C = A - B$ |
| 'Split Complex Hermitian' Elliptic | 'Split Complex Hermitian' Euclidean | 'Split Complex Hermitian' Hyperbolic |
|------------------------------------|-------------------------------------|-------------------------------------|
| \( SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R}) \) | \( IGL(2,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R}) \) | \( SL(3,\mathbb{R})/SO(1,1)\otimes SL(2,\mathbb{R}) \) |
| \( \cos a \cosh \Omega = \cos b \cosh c - \sin b \sin c \cosh A \cos \Psi_A \) | \( a^2 = b^2 + c^2 + 2bc \cosh A \cos \Psi_A \) | \( \cosh a \cosh \Omega = \cosh b \cosh c + \sin b \sin c \cos A \cosh \Psi_A \) |
| \( \cosh b \cosh \Omega = \cos a \cosh c + \sin a \sin c \cos B \cosh \Psi_B \) | \( b^2 = a^2 + c^2 - 2ac \cosh B \cosh \Psi_B \) | \( \cosh b \cosh \Omega = \cosh a \cosh c - \sin a \sin c \cosh B \cosh \Psi_B \) |
| \( \cosh c \cosh \Omega = \cos a \cosh b + \sin a \sin b \cosh C \cosh \Psi_C \) | \( c^2 = a^2 + b^2 - 2ab \cosh C \cosh \Psi_C \) | \( \cosh c \cosh \Omega = \cosh a \cosh b - \sin a \sin b \cosh C \cosh \Psi_C \) |
| \( \cos A \sinh 2S = \sin b \sin c \cosh A \sinh \Psi_A \) | \( 2S = bc \cosh A \sinh \Psi_A \) | \( \cosh A \sinh 2S = \sin b \sin c \cosh A \sinh \Psi_A \) |
| \( \cos b \sinh 2S = \sin c \sin a \cos B \sinh \Psi_B \) | \( 2S = ca \cosh B \sinh \Psi_B \) | \( \cosh b \sinh 2S = \sin c \sin a \cos B \sinh \Psi_B \) |
| \( \cos c \sinh 2S = \sin a \sin b \cos C \cosh \Psi_C \) | \( 2S = ab \cosh C \cosh \Psi_C \) | \( \cosh c \sinh 2S = \sin a \sin b \cos C \cosh \Psi_C \) |
| \( \cosh A \sinh 2S = \sin \theta \sin \phi \) | \( 2S = \cosh A \sinh \Psi_A \) | \( \cosh A \sinh 2S = \sin \theta \sin \phi \) |
| \( \cos B \sinh 2S = \sin \theta \cos \phi \) | \( 2S = \cosh B \sinh \Psi_B \) | \( \cosh b \sinh 2S = \sin \theta \cos \phi \) |
| \( \cos C \sinh 2S = \sin \phi \cos \theta \) | \( 2S = \cosh C \cosh \Psi_C \) | \( \cosh c \sinh 2S = \sin \phi \cos \theta \) |

**Table 5:** 'Complex Hermitian' cosine theorems and their duals for the nine split complex 'Hermitian' CKD spaces \((\eta = -1)\).
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