Dense nuclear Fréchet ideals in $C^*$-algebras

Larry B. Schweitzer

University of California, San Francisco

June 2013

Abstract

We show that a $C^*$-algebra $B$ contains a dense left or right Fréchet ideal $A$, with $A$ a nuclear locally convex space, if and only if the primitive ideal space Prim($B$) of $B$ is discrete and countable, and $B/I$ is finite dimensional for each $I \in \text{Prim}(B)$. Here $\{\| \cdot \|_n\}_{n=0}^{\infty}$ denotes a family of increasing norms topologizing $A$. We show the forward implication holds for a general Banach algebra $B$, if the ideal is assumed two-sided. For $C^*$-algebras, we construct dense nuclear ideals by defining a set of matrix-valued Schwartz functions on the countable discrete space Prim($B$). AMS Subject Classification 2010: 46H20 (structure, classification of topological algebras), 46H10 (ideals and subalgebras). Keywords: Nuclear Fréchet space, $C^*$-algebra, discrete spectrum, dense ideal.

Contents

1 Introduction 3
1 Introduction

Dense subalgebras of $C^*$-algebras are well-known to be useful in the study of $C^*$-algebras. The dense subalgebra can be viewed as $C^\infty$ functions on a manifold, where instead of a manifold we have an underlying “noncommutative space”. To be more useful, the dense subalgebra often has a Fréchet topology and is spectral invariant in the $C^*$-algebra (see Remark 3.2). While some $C^*$-algebras have nice dense subalgebras, others don’t seem to.

If we insist that the dense subalgebra be an ideal in the $C^*$-algebra, we get a much stronger condition than spectral invariance. Few $C^*$-algebras can have a “smooth” dense ideal. For example, any compact manifold $M$ without boundary has a spectral invariant Fréchet algebra of smooth functions $C^\infty(M)$, dense in the $C^*$-algebra of continuous functions $C(M)$. But $C^\infty(M)$ is an ideal in $C(M)$, if and only if $M$ is discrete (and therefore finite by compactness).

In this paper, I will classify which $C^*$-algebras have dense nuclear Fréchet ideals. The nuclearity property plays the role of making elements of the subalgebra “smooth” or “differentiable”. For example, $C^\infty(M)$ is a nuclear Fréchet space [Treves, 1967], Chapters 10 and 51.

The notion of nuclearity for a locally convex space is different than nuclearity for a $C^*$-algebra [Kad Ring II, 1997], Chapter 11. If a $C^*$-algebra were nuclear as a locally convex space, it would be finite dimensional (Proposition 2.1 (b) below). However, it is reasonable that a dense Fréchet subalgebra of an infinite dimensional nuclear $C^*$-algebra be a nuclear locally convex space (see Corollary 4.7 below).

In §2, we recall definitions and properties of nuclear locally convex spaces and Fréchet
spaces from the literature. We define Schwartz functions on a countable set. Every nuclear Fréchet space with basis is one of these.

In §3, we go over definitions and basic lemmas on dense Fréchet ideals of Banach algebras, and state the example of $l^p(X)$ and Schwartz functions $S(X)$ as dense ideals of the commutative pointwise-multiplication $C^*$-algebra $c_0(X)$. We show the property of having a dense nuclear ideal is preserved by taking quotients and subalgebras formed using idempotents.

In §4, we apply nuclearity and results of §3 to see that a $C^*$-algebra has a dense nuclear left or right ideal only if its spectrum is discrete and countable. We do this by showing that the “finite socle” is dense. In the commutative case, a shorter proof is given.

In §5, we generalize our results on discrete spectrum to the case of an arbitrary Banach algebra. We prove that any Banach algebra with a dense nuclear two-sided ideal is both left and right completely continuous. We also show the primitive ideal space is countable, and that primitive quotients are finite dimensional. Complete continuity on both sides of a Banach algebra is already known to imply discrete spectrum and finite dimensional primitive quotients [Kaplansky, 1949], [Kaplansky, 1948], Lemma 4.

In §6, we construct dense nuclear two-sided ideals for every $C^*$-algebra $B$ which is the direct sum of simple finite dimensional $C^*$-algebras (i.e. full matrix algebras). We define our dense ideal to be matrix-valued Schwartz functions on the countable discrete spectrum of $B$. In many cases, the underlying Fréchet space structure of the dense ideal
is the standard Schwartz functions on a countably infinite set $X$, namely

$$S_\gamma(X) = \{ \varphi: X \to \mathbb{C} \mid \text{for each } n = 0, 1, 2 \ldots, \sup_{x \in X} \gamma(x)^n |\varphi(x)| < \infty \},$$

(1)

where $\gamma$ is an enumeration of $X$.

In §7, we give examples of dense nuclear ideals, including the convolution algebra of $C^\infty$ functions on a compact Lie group, and $C^\infty$ functions on the Cantor set.

In Appendix A, we show that the constants $m_n$ and $C_n$ (in the dense ideal inequality (12)) can be made to satisfy $m_n = n$ and $C_n = 1$ by choosing an equivalent family of increasing norms $\{ \| \cdot \|_n \}_{n=0}^\infty$ for the topology of the dense Fréchet ideal.

In Appendix B, we give counterexamples to our theorems, when various hypotheses are dropped.

Throughout the paper, $X$ will denote a countable set (with discrete topology), usually infinite. All algebras in this paper are over the field of complex numbers $\mathbb{C}$. The set of natural numbers $\{0, 1, 2, \ldots \}$ is denoted by $\mathbb{N}$, and $\mathbb{N}^+ = \{1, 2, 3, \ldots \}$. The set $X$ is often paired with an enumeration $\gamma$ of $X$, which is a bijection of $X$ with $\mathbb{N}^+$.

2 Nuclearity and Fréchet Spaces with Basis

We state some basics facts about nuclear Fréchet spaces used throughout the paper. The reader unfamiliar with these concepts can consult standard references, such as [Pietsch, 1972], [Treves, 1967].
Proposition 2.1. Nuclearity Facts.

(a) A nuclear Fréchet space is separable.

(b) A nuclear Banach space must be finite dimensional.

(c) If a Fréchet space $F$ contains an infinite dimensional Banach space $E$, and the topology on $E$ inherited from $F$ is stronger than the Banach space topology, then $F$ is not nuclear.

(d) A bounded set in a nuclear Fréchet space is relatively compact.

Proof: For separability, see [Pietsch, 1972], Theorem 4.4.10. For finite dimensionality of a nuclear normed space, see [Pietsch, 1972], Theorem 4.4.14, or [Treves, 1967], Chapter 50-12, Corollary 2.

Let $F$ be a Fréchet space containing an infinite dimensional Banach space $E$. Since the Banach space topology on $E$ is assumed weaker than the Fréchet space topology inherited from $F$, these two topologies agree on $E$ [Treves, 1967], Chapter 17-7, Corollary 2. But a subspace of a nuclear Fréchet space is nuclear [Pietsch, 1972], Proposition 5.1.5, or [Treves, 1967], Proposition 50.1 (50.3). $E$ is not nuclear by (b), so therefore $F$ cannot be nuclear.

A bounded set in a nuclear locally convex space is precompact by [Pietsch, 1972], 4.4.7, or [Treves, 1967], Proposition 50.2 (50.12).

Remark 2.2. The hypotheses of Proposition 2.1 (c) imply that $E$ is a closed subspace of $F$, namely $\overline{E}^F = E$. Here is a counterexample when the Banach space is not a closed subspace of the Fréchet space. Let $F = l_u(X)$ be the nuclear Fréchet space of complex-valued sequences on $X$ (allowed to be unbounded), topologized by the seminorms
\[ \|\varphi\|_n = \max_{\gamma(x) \leq n} |\varphi(x)| \] (see [Pietsch, 1972], 4.3.4 or [Treves, 1967], Theorem 51.1), and let \( E \) be the Banach space \( l^p(X) \) for any \( 1 \leq p \leq \infty \).

We will often cite the following obvious Corollary to Proposition 2.1 (a).

**Corollary 2.3.** A Banach space with a dense nuclear Fréchet subspace is separable.

For any locally convex space \( E \), recall that the polar \( M^0 \) of a set \( M \subseteq E \) is defined to be the set

\[ \{ \varphi \in E' \mid |\varphi(m)| \leq 1, \ m \in M \}. \]  (2)

Here \( E' \) denotes the topological dual of \( E \), the set of linear functionals on \( E \) which are continuous. We will use the following consequence of the Uniform Boundedness Theorem in \( \S 5 \).

**Theorem 2.4. Uniform Boundedness.** Let \( E \) be a Banach space with unit ball \( U \). If \( \{\varphi_n\}_{n=0}^\infty \subseteq E' \) is a sequence of continuous linear functionals converging to zero pointwise, then \( \sup \{ \|\varphi_n\|_{U^0} \mid n \in \mathbb{N} \} < \infty \).

Proof: See [Rudin, 1973], Theorem 2.6. \qed

**Definition 2.5. Fréchet Space with Basis.** A sequence \( \{e_k\}_{k=0}^\infty \) in a Fréchet space \( F \) is a basis if every \( f \in F \) has a unique series expansion \( f = \sum_{k=0}^\infty f_k e_k \) which converges in \( F \), where each \( f_k \) is a complex number. By [Husain, 1991], Chapter I, Theorem 4.3, any basis in a Fréchet space is a Schauder basis, meaning that the coordinate functional \( f \mapsto f_k \) is a continuous linear map from \( F \) to \( \mathbb{C} \) for each \( k \in \mathbb{N} \). The basis is unconditional.
if the series expansion converges to \( f \) in any order. If for every \( n \in \mathbb{N} \), there is some \( C > 0 \) and \( q \in \mathbb{N} \) such that \( |f_k|\|e_k\|_n = \|f_k e_k\|_n \leq C\|f\|_q \), for all \( f \in F \) and \( k \in \mathbb{N} \), then the basis is *equicontinuous* [Pietsch, 1972], 10.1.2. According to [Pietsch, 1972], Theorem 10.1.2, every Schauder basis in a Fréchet space is equicontinuous. If \( F \) is nuclear, an equicontinuous basis is *absolute* [Pietsch, 1972], 10.2.1. This means that for each \( n \in \mathbb{N} \) there exists \( C > 0 \) and \( q \in \mathbb{N} \) for which \( \sum_{k=0}^\infty |f_k|\|e_k\|_n \leq C\|f\|_q \), for all \( f \in F \). Then the series \( \sum_{k=0}^\infty f_k e_k \) converges absolutely and unconditionally to \( f \). For each \( e_k \), let \( e'_k : F \to \mathbb{C} \) denote the coordinate functional \( e'_k(f) = f_k, f \in F \).

**Definition 2.6. Basic Schwartz Spaces.** Let \( X \) be a countable set. A function \( \sigma : X \to [1, \infty) \) is a *scale* on \( X \). If \( \sigma = \{\sigma_n\}_{n=0}^\infty \) is a family of scales on \( X \), define the Fréchet space

\[
S^1_\sigma(X) = \left\{ \varphi : X \to \mathbb{C} \left| \|\varphi\|_n^1 < \infty, n \in \mathbb{N} \right. \right\},
\]

where

\[
\|\varphi\|_n^1 = \sum_{x \in X} \sigma_n(x)|\varphi(x)|.
\]

Similarly, define \( S^\infty_\sigma(X) \) using sup-norms in place of \( \ell^1 \) norms:

\[
\|\varphi\|_n^\infty = \sup_{x \in X} \left( \sigma_n(x)|\varphi(x)| \right).
\]

We call \( S^1_\sigma(X) \) the *\( \ell^1 \)-norm \( \sigma \)-rapidly vanishing functions on \( X \), and \( S^\infty_\sigma(X) \) the *sup-norm \( \sigma \)-rapidly vanishing functions on \( X \). Usually the family \( \sigma \) will satisfy \( \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_n \leq \ldots \), so that the families of norms \( \{\|\cdot\|_n^1\}_{n=0}^\infty \) and \( \{\|\cdot\|_n^\infty\}_{n=0}^\infty \) are increasing.

We often adopt the convention \( \sigma_0 \equiv 1 \). Let \( \delta_x \) denote the step function at \( x \in X \). Then span\( \{\delta_x \mid x \in X \} = c_f(X) \) is the dense subspace of finite support functions in \( S^1_\sigma(X) \) and
\( S^\infty_\sigma(X) \) \footnote{We define \( S^\infty_\sigma(X) \) as the completion of \( c_f(X) \) in the sup norms \([5]\), or, alternatively, restrict to functions vanishing at infinity when defining \( S^\infty_\sigma(X) \). This happens automatically when \( \sigma_n \) is \textit{proper} for sufficiently large \( n \). In either case \( c_f(X) \) is dense.} The set of functions \( \{ \delta_x \}_{x \in X} \) is an equicontinuous unconditional Schauder basis for \( S^\infty_\sigma(X) \) since \( |\varphi(x)||\delta_x|_\infty \leq \|\varphi\|_\infty \), and an equicontinuous absolute basis for \( S^1_\sigma(X) \) since \( \sum_{x \in X} |\varphi(x)||\delta_x|_1 = \|\varphi\|_1 \).

A scale \( \sigma \) dominates another scale \( \tau \) if there exists \( C > 0 \) such that \( \tau(x) \leq C\sigma(x) \) holds for all \( x \in X \). We use the notation \( \tau \lesssim \sigma \). If \( \tau \) also dominates \( \sigma \), we say that \( \sigma \) and \( \tau \) are equivalent, and write \( \sigma \sim \tau \). If \( \sigma \) is a scale, an associated family is given by the powers \( \sigma_n = \sigma^n \). A family \( \sigma \) dominates another family \( \tau \) if for every \( n \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that \( \tau_n \lesssim \sigma_m \). One can easily verify that the identity map \( \text{id} : c_f(X) \rightarrow c_f(X) \) extends to an isomorphism of Fréchet spaces \( S^1_\sigma(X) \cong S^1_\tau(X) \) or \( S^\infty_\sigma(X) \cong S^\infty_\tau(X) \) if and only if \( \sigma \sim \tau \).

**Theorem 2.7. Nuclear Fréchet Spaces with Basis.** Let \( X \) be a countable set, and \( F \) a Fréchet space with absolute basis \( \{ e_x \}_{x \in X} \). Assume there exists a continuous norm \( \| \cdot \|_0 \) on \( F \) for which \( \| e_x \|_0 = 1 \), \( x \in X \). (This happens, for example, if \( F \) is a dense Fréchet subspace of a \( C^* \)-algebra, with continuous inclusion, and each \( e_x \) is a partial isometry.)

Let \( \{ \| \cdot \|_n \}_{n=0}^\infty \) be an increasing family of norms dominating \( \| \cdot \|_0 \) and topologizing \( F \).

Then \( \sigma_n(x) = \| e_x \|_n \) defines a family of scales on \( X \), and the Fréchet spaces \( F \cong S^1_\sigma(X) \) are naturally isomorphic.

Moreover, \( F \) is nuclear if and only if \( \sigma \) satisfies the summability condition

\[
(\forall n \in \mathbb{N}) \ (\exists m > n ) \sum_{x \in X} \frac{\sigma_n(x)}{\sigma_m(x)} < \infty, \quad (6)
\]
and if and only if the Fréchet spaces $F \cong S_1^1(X) \cong S_\infty^\infty(X)$ are naturally isomorphic.

Proof: The isomorphism $F \cong S_1^1(X)$ is [Pietsch, 1972], Theorem 10.1.4. The second paragraph is [Pietsch, 1972], Theorems 6.1.2 and 6.1.3.

**Definition 2.8. Standard Scales.** We call $\gamma$ an *enumeration* of a countably infinite set $X$ if $\gamma: X \cong \mathbb{N}^+$ is a bijection. An enumeration $\gamma$ is also a scale, which we call a *standard scale* on $X$. If $\varphi: X \to \mathbb{C}$ is a function, we write $\varphi = (\varphi_1, \varphi_2, \varphi_3, \ldots \varphi_k, \ldots)$, where $\varphi_k$ is short for $\varphi(\gamma^{-1}(k))$. The step functions $\delta_x$ may be written $\delta_k$, where $k = \gamma(x)$. We may write $(1, 2, \ldots k, \ldots)$ in place of $(x_1, x_2, \ldots x_k, \ldots)$, when $k = \gamma(x_k)$.

If $\sigma$ is a family of scales on $X$, then $\gamma\sigma$ is a new family of scales on $X$, given by $\{\gamma^n\sigma_n\}_{n=0}^\infty$.

If $\vec{\gamma} = \{\gamma_n\}_{n=1}^\infty$ is a sequence of enumerations on $X$, we define the $\vec{\gamma}$-*standard family of scales on* $X$ by $\sigma_{\vec{\gamma}} = \{\gamma_1 \cdots \gamma_n\}_{n=0}^\infty$. Also define $\sigma_{\vec{\gamma}}^2$ and $\sqrt{\sigma_{\vec{\gamma}}}$, the $\vec{\gamma}$-*standard family of squared and square-root scales*, in the obvious way.

**Proposition 2.9. Standard Scales and Summability.** Let $\vec{\gamma}$ be a sequence of enumerations of $X$. The families $\sigma_{\vec{\gamma}}$, $\sigma_{\vec{\gamma}}^2$, and $\sqrt{\sigma_{\vec{\gamma}}}$ satisfy summability with $m = n + 2$, $m = n + 1$, and $m = n + 4$, respectively.

A family of scales $\sigma$ is summable if and only if for every $n \in \mathbb{N}$ there exists an enumeration $\gamma$ of $X$ such that $\sigma$ dominates the scale $\sigma_n\sqrt{\gamma}$. In particular, if $\sigma$ is summable, then $\sigma$ dominates $\sqrt{\sigma_{\vec{\gamma}}}$ for some sequence of enumerations $\vec{\gamma}$. 

10
Proof: First note that
\[ \frac{\sigma_{\gamma n}^2}{\sigma_{\gamma n+1}^2} = \frac{1}{\gamma_{n+1}}. \] (7)
Then \( \sum_{x \in X} 1/\gamma_{n+1}(x)^2 = \sum_{k=1}^{\infty} 1/k^2 = \pi^2/6 < \infty \) shows that \( \sigma_{\gamma} \) satisfies summability with \( m = n + 1 \).

Next note that
\[ \frac{\sigma_{\gamma n}}{\sigma_{\gamma n+2}} = \frac{1}{\gamma_{n+1}\gamma_{n+2}}. \] (8)
Then \( \sum_{x \in X} 1/\gamma_{n+1}(x)\gamma_{n+2}(x) \leq \|1/\gamma_{n+1}\|_2 \|1/\gamma_{n+2}\|_2 = \pi^2/6 < \infty \) shows that \( \sigma_{\gamma} \) satisfies summability with \( m = n + 2 \), where we used the Cauchy-Schwartz inequality. Similarly \( \sqrt{\sigma_{\gamma}} \) satisfies summability with \( m = n + 4 \).

Next let \( \sigma \) be a family satisfying the condition of the second paragraph. Apply the condition four times to get four enumerations \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) of \( X \) such that \( \sqrt{\gamma_1} \leq C_1\sigma_m/\sigma_n, \sqrt{\gamma_2} \leq C_2\sigma_p/\sigma_m, \sqrt{\gamma_3} \leq C_3\sigma_q/\sigma_p, \) and \( \sqrt{\gamma_4} \leq C_4\sigma_r/\sigma_q. \) Then, using the Cauchy-Schwartz inequality and setting \( C = C_1C_2C_3C_4, \) we have
\[
\sum_{x \in X} \frac{\sigma_n(x)}{\sigma_r(x)} = \sum_{x \in X} \frac{\sigma_n \sigma_m \sigma_p \sigma_q}{\sigma_m \sigma_p \sigma_q \sigma_r}(x)
\leq C \sum_{x \in X} \left( \frac{1}{\gamma_1\gamma_2\gamma_3\gamma_4}(x) \right)^{1/2}
\leq C \left( \|1/\gamma_1\|_2 \|1/\gamma_2\|_2 \|1/\gamma_3\|_2 \|1/\gamma_4\|_2 \right)^{1/2}
= C \left( \sqrt{\frac{\pi^2}{6}} \sqrt{\frac{\pi^2}{6}} \sqrt{\frac{\pi^2}{6}} \sqrt{\frac{\pi^2}{6}} \right)^{1/2} = \frac{C \pi^2}{6} < \infty, \] (9)
so \( \sigma \) is summable.

Conversely, find an enumeration \( \gamma \) of \( X \) such that \( (\sigma_n/\sigma_m) \circ \mu \) is non-increasing, where \( \mu = \gamma^{-1}. \) Apply the Cauchy Condensation Test [Mars Hoff, 1999], p. 194, for convergence of the series \( \sum_{k=1}^{\infty} (\sigma_n/\sigma_m) \circ \mu(k), \) to see that \( \sum_{i=0}^{\infty} 2^i(\sigma_n/\sigma_m) \circ \mu(2^i) < \infty. \)
Let $C > 0$ be such that $\sigma_n/\sigma_m \circ \mu(2^i) \leq C/2^i$, $i \in \mathbb{N}$. Since the terms are non-increasing, $\sigma_n/\sigma_m \circ \mu(k) \leq C/2^i = C/\sqrt{2^i}$ for $2^i \leq k < 2^{2i}$, so $\sigma_n/\sigma_m \circ \mu(k) \leq C/\sqrt{k}$, $k \in \mathbb{N}^+$. Replacing $k$ with $\gamma(x)$, we see $\sigma_n(x)/\sqrt{\gamma(x)} \leq C\sigma_m(x)$, $x \in X$.

Finally, let $\tilde{\gamma} = \gamma_1, \gamma_1, \ldots, \gamma_n, \ldots$ be a sequence of enumerations, and $\{m_n\}_{n=0}^\infty$ a sequence of natural numbers, with the property that $\sigma_1\sqrt{\gamma_1} \lesssim \sigma_{m_1}$, $\sigma_{m_1}\sqrt{\gamma_2} \lesssim \sigma_{m_2}$, $\cdots$, $\sigma_{m_{n-1}}\sqrt{\gamma_n} \lesssim \sigma_{m_n}$, and so on. Then $\sqrt{\sigma_{\tilde{\gamma}n}} = \sqrt{\gamma_1 \cdots \gamma_n} \leq \sigma_1\sqrt{\gamma_1 \cdots \gamma_n} \lesssim \sigma_{m_1}\sqrt{\gamma_1 \cdots \gamma_n} \lesssim \sigma_{m_2}\sqrt{\gamma_1 \cdots \gamma_n} \lesssim \cdots \lesssim \sigma_{m_{n-1}}\sqrt{\gamma_n} \lesssim \sigma_{m_n}$, so $\sigma$ dominates $\sqrt{\sigma_{\tilde{\gamma}}}$.

**Corollary 2.10. Single Scales and Summability.** The family of scales associated with a single scale $\sigma$ is summable if and only if some power of $\sigma$ dominates an enumeration $\gamma$ of $X$.

Proof: If $\sigma$ is summable, then by Proposition 2.9 there exists an enumeration $\gamma$ of $X$ and $m \in \mathbb{N}$ such that $\sqrt{\gamma} = \sigma^0 \sqrt{\gamma} \lesssim \sigma^m$, so $\gamma \lesssim \sigma^{2m}$. On the other hand, if there exists $C, d$ such that $\gamma(x) \leq C\sigma(x)^d$, $x \in X$, then

$$\frac{1}{C^2} \sum_{x \in X} \frac{1}{\sigma(x)^{2d}} \leq \sum_{x \in X} \frac{1}{\gamma(x)^2} = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6} < \infty,$$

and $\sigma$ is summable.

**3 Fréchet Ideals**

**Definition 3.1. Fréchet Ideals and Continuous Inclusion.** Let $B$ be a Banach algebra, with norm $\| \cdot \|_B$, and $A$ a subalgebra of $B$. The algebra $A$ is called a Fréchet
algebra when endowed with a Fréchet space topology for which multiplication is jointly continuous. Let \( \{\| \cdot \|_n\}_{n=0}^{\infty} \) be an increasing family of seminorms giving the topology for \( A \). When we say that \( A \) is a Fréchet subalgebra of \( B \), we require the inclusion map \( A \hookrightarrow B \) be continuous. In terms of seminorms, this means that if \( n \in \mathbb{N} \) is sufficiently large, there exists a constant \( C > 0 \) such that

\[
\| a \|_B \leq C \| a \|_n, \tag{11}
\]

for \( a \in A \). Define new norms \( \{\| \cdot \|'_n\}_{n=0}^{\infty} \) on \( A \) by \( \| a \|'_0 = \| a \|_B \) and \( \| a \|'_{n+1} = \max\{\| a \|_n, \| a \|_B\} \).

By continuous inclusion (11), the “primed” norms also topologize \( A \), and we will use them in place of our original family, so that we may work with a countable family of norms for \( A \), with zeroth norm equal to \( \| \cdot \|_B \).

We say that \( A \) is a right Fréchet ideal in \( B \) if \( a \in A \) and \( b \in B \) implies \( ab \in A \), and this multiplication operation is continuous for the respective topologies. In terms of norms, this means that for each \( n \in \mathbb{N} \) there exists an integer \( m \geq n \) and constant \( C_n > 0 \) such that the inequality

\[
\| ab \|_n \leq C_n \| a \|_m \| b \|_B \tag{12}
\]

holds for all \( a \in A \) and \( b \in B \). Similarly, a left Fréchet ideal and two-sided Fréchet ideal is defined.

If \( \{\| \cdot \|'_n\}_{n=0}^{\infty} \) is an equivalent increasing family of seminorms for \( A \), then the inequality (12) is still satisfied but with adjusted constants \( C'_n \) and integers \( m'_n \). If \( \| \cdot \|'_B \) is an equivalent Banach algebra norm on \( B \), then the constants \( C_n \) will scale uniformly in \( n \), with each \( m_n \) staying the same for a given \( n \) in (12).
Remark 3.2. Spectral Invariance. A subalgebra $A$ of an algebra $B$ is *spectral invariant* in $B$ if every element $a \in A$ is quasi invertible in $B$ if and only if it is quasi invertible in $A$. An element $x \in B$ is a *quasi-inverse* for $y \in B$ if $x \circ y = y \circ x = 0$, where $x \circ y$ is defined as $x + y + xy$ for any $x, y \in B$. If $A$ is a left or right ideal in an algebra $B$, then $A$ is spectral invariant in $B$. For let $x \in A$ have quasi-inverse $y \in B$. If $A$ is a right ideal, then $xy \in A$. So $0 = x \circ y = x + y + xy$ and $y = -x - xy \in A$. For left ideals, apply the same argument with $yx$ in place of $xy$.

We say that a scale $\ell$ on $X$ is *proper* if the inverse map $\ell^{-1}$ takes bounded subsets of $[1, \infty)$ to finite subsets of $X$.

Example 3.3. Let $B = c_0(X)$ be the commutative $C^*$-algebra of complex-valued sequences which vanish at infinity, with pointwise multiplication and sup-norm $\| \cdot \|_B = \| \cdot \|_\infty$.

(a) Let $A$ be the dense Banach subalgebra $l^p(X)$ for some $1 \leq p < \infty$, with pointwise multiplication. The inequality $\|fg\|_p \leq \|f\|_p \|g\|_\infty$ is satisfied for all $f, g \in A$, so $A$ is a dense Banach ideal in $B$. $A$ is not nuclear by Proposition 2.1 (b).

(b) Let $\ell$ be a proper scale on $X$, and let $A$ be the Fréchet space of $\ell$-rapidly vanishing sequences $S^{\infty}_\ell(X)$, topologized by the sup-norms $\|f\|_n = \|\ell^n f\|_\infty$ (see Definition 2.6 with $\sigma_n = \ell^n$). The inequalities $\|fg\|_n \leq \|f\|_n \|g\|_\infty$ are all satisfied, so $A$ is a dense Fréchet ideal in $B$. $A$ is nuclear if and only if there exists a $p \in \mathbb{N}^+$ for which $\sum_{x \in X} \frac{1}{\ell(x)^p} < \infty$ (Theorem 2.7). For example, this sum is bounded with $p = 2$ for standard Schwartz functions on $X$ (where $\ell$ is an enumeration of $X$ - see Equation (I)).
Proposition 3.4. Unital Banach Algebras. Let $A$ be a subalgebra of a Banach algebra $B$. If $A$ is dense in $B$, then a (left or right) unit for $A$ is also a (left or right) unit for $B$.

If $B$ has a two-sided unit or left (right) unit, and $A$ is a dense right (left) ideal, then $A$ contains the same unit. If $B$ has a left (right) unit $1_B$, and $A$ is a dense left (right) ideal, then $A$ contains a left (right) unit, possibly different than $1_B$.

If $A$ and $B$ have the same left (right) unit, and $A$ is a right (left) ideal in $B$, then $A = B$. If $A$ is a right (left) Fréchet ideal, the equality $A = B$ is topological, and $A$ is a Banach algebra exactly equal to $B$.

In other words, dense ideals can only be proper when the Banach algebra $B$ is non-unital.

Remark 3.5. One-sided Units Not Unique. If $B$ has a left unit $1_L$, and $A$ is a left ideal, then $A$ will contain a left unit, but maybe different than $1_L$. (A similar statement holds for right units.) Let $B = \ell^2(X)$ with multiplication $\chi \ast \eta = \chi_1\eta$. Then $1_L = (1, 0, \ldots, 0, \ldots)$ is a natural left unit to take for $B$. Let $\xi = (1, 1/2, 1/3, \ldots, 1/k, \ldots) \in \ell^2(X)$, and take $A = \mathbb{C}\xi \oplus (\xi^\perp \cap c_f(X))$. Note that $A$ is a left ideal in $B$, but not a right ideal, and $\xi \in A$ is a left unit for $B$, but $1_L \not\in A$. To see that $A$ is dense in $B$, let $\chi \in \xi^\perp$ and $\epsilon > 0$. Let $\eta \in c_f(X)$ satisfy $\|\eta - \chi\|_2 < \epsilon$. Then $\eta' = \eta - <\eta, \xi> \delta_1$ is in $\xi^\perp \cap c_f(X)$, and $\|\eta' - \chi\|_2 \leq \epsilon + |<\eta, \xi>| = \epsilon + |<\eta - \chi, \xi>| \leq \epsilon + \|\eta - \chi\|_2\|\xi\|_2 \leq \epsilon + \epsilon\|\xi\|_2$, using the Cauchy-Schwartz inequality. To make $A$ a nuclear left Fréchet ideal, replace $\xi^\perp \cap c_f(X)$ with $\xi^\perp \cap S_\gamma(X)$. 

15
Proof of Proposition 3.4: Let $A$ be a dense subalgebra of $B$, and $1_L$ a left unit in $A$, satisfying $1_L a = a$ for every $a \in A$. Let $b \in B$, and $a_\epsilon \in A$ satisfy $\|a_\epsilon - b\|_B < \epsilon$. Then $\|1_L b - b\|_B \leq \|1_L b - 1_L a_\epsilon\|_B + \|1_L a_\epsilon - b\|_B \leq \|1_L\|_B \epsilon + \epsilon$. Letting $\epsilon \to 0$, we see that $1_L b = b$ and $1_L$ is a left unit for $B$. The same argument works for a right unit.

Let $1_B$ be a two-sided unit in $B$, and let $a \in A$ be close to $1_B$ in $\| \cdot \|_B$, so that $b = a^{-1}$ exists in $B$. If $A$ is a right ideal, then $1_B = ab \in A$, so $A$ is unital with the same unit as $B$. The same argument holds if $A$ is a left ideal.

Let $1_L \in B$ be a left unit for $B$. Then $A1_L$ is a dense subalgebra of the closed subalgebra $B1_L$ of $B$, and $1_L$ is a two-sided unit for $B1_L$, so $1_L \in A1_L$ by the previous paragraph. If $A$ is a right ideal of $B$, then $A1_L \subseteq A$ so $1_L \in A$. If $A$ is a left ideal, let $a_0 \in A$ be such that $1_L = a_0 1_L$. Then $a_0$ is a left unit for $B$ contained in $A$. A similar argument applies to right units.

Next assume that $A$ is a right ideal and $1_L \in A$ is a left unit for $A$ and $B$. Then for every $b \in B$, $b = 1_L b \in A$, so $A = B$. If $A$ is a right Fréchet ideal, then $\|a\|_n = \|1_L a\|_n \leq C_n \|1_L\|_m \|a\|_B = C'_n \|a\|_B$ for all $a \in A$, so each seminorm for $A$ is bounded by $\| \cdot \|_B$. By continuous inclusion $A \hookrightarrow B$ \cite{[11]}, $A$ has a topology equivalent to $B$’s. \hfill \qed

Proposition 3.6. Quotients and Idempotent Subalgebras. Let $A$ be a Fréchet subalgebra of a Banach algebra $B$.

(a) If $J$ is a closed two-sided ideal of $B$, then the image of $A$ in the quotient Banach algebra $B/J$ is a Fréchet subalgebra. The image of $A$ in the quotient $B/J$ is a left (right) Fréchet ideal if $A$ is a left (right) Fréchet ideal in $B$. The image is nuclear if $A$ is nuclear, and dense if $A$ is dense in $B$. 

16
(b) Let \( e \) be an idempotent in \( B \). Then \( Be \) and \( eB \) are closed subalgebras of \( B \). If \( A \) is a
left (right) Fréchet ideal in \( B \), then \( Ae \) (\( eA \)) inherits a natural quotient Fréchet topology
from \( A \), and is a left (right) Fréchet ideal in \( B \). Both \( Ae \) and \( eA \) are nuclear if \( A \) is, and
\( Ae \) (\( eA \)) is dense in \( Be \) (\( eB \)) if \( A \) is dense in \( B \). If \( e \in A \), then \( Ae \) and \( eA \) are closed
subalgebras of \( A \).

Note that Remark 3.5 gives an example when \( e \notin A \), and \( Ae \) is not contained in \( A \).

Proof of (a): Let \( \pi: B \rightarrow B/J \) be the canonical quotient map. The Fréchet algebra \( A \)
maps into \( B/J \) by a composition of continuous maps \( \pi \circ \iota: A \rightarrow B/J \), where \( \iota: A \hookrightarrow B \)
is the inclusion map. So the kernel \( I = J \cap A \) is a closed two-sided ideal in \( A \), and we
identify \( A/I \) with a Fréchet subalgebra of \( B/J \).

Assume that \( A \) is a right Fréchet ideal in \( B \), and let \( a \in A \) and \( b \in B \). Then \( ab \in A \)
so \( \pi(a)\pi(b) \in A/I \). Let \( \{ \| \cdot \|_n \}_{n=0}^\infty \) be increasing seminorms for \( A \), and \( \| \cdot \|_B \) the norm
on \( B \). The \( n \)th quotient seminorm for \( A/I \) is

\[
\| \pi(a) \|_n = \inf_{i \in I} \| a + i \|_n,
\]

and similarly \( \inf_{j \in J} \| b + j \|_B \) is the norm on \( B/J \). Using the seminorm inequality \( \| x y \| \leq \| x \| \| y \| \)

\[\tag{12} \]

17
for right ideals,

\[ \|\pi(a)\pi(b)\|_n = \inf_{i \in I} \|ab+i\|_n \]

\[ = \inf_{i \in I} \inf_{j \in J} \|a(b+j)+i\|_n \quad \text{since } AJ \subseteq A \cap J = I \]

\[ \leq \inf_{i \in I} \inf_{j \in J} \|(a+i)(b+j)\|_n \quad \text{since } IB \subseteq A \cap J = I \]

\[ \leq \inf_{i \in I} \inf_{j \in J} C_n \|a+i\|_m \|b+j\|_B \quad \text{by inequality (12)} \]

\[ = C_n \|\pi(a)\|_m \|\pi(b)\|_B, \quad (13) \]

so \( A/I \) is a right Fréchet ideal in \( B/J \). The “left” case is handled in the same way.

If \( A \) is nuclear, then \( A/I \) is nuclear, since quotients by closed linear subspaces preserve nuclearity [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4). If \( a_k \in A \) is a sequence approaching \( b \) in \( B \), then by continuity \( \pi(a_k) \) approaches \( \pi(b) \) in \( B/J \). So \( A/I \) is dense in \( B/J \) whenever \( A \) is dense in \( B \).

Proof of (b): Let \( e \) be an idempotent in \( B \), and consider the linear subspace \( Be = \{be \mid b \in B\} \) of \( B \). If \( \{b_k\}_{k \in \mathbb{N}} \) is a sequence in the Banach algebra \( B \), such that \( b_k e \) converges in \( B \) to a limit \( b_0 \), then \( b_0 e = b_0 \), since multiplication is continuous. So \( Be \) is a closed linear subspace of \( B \), and clearly a subalgebra. Similarly for \( eB \). If \( e \in A \), the same argument shows \( Ae \) and \( eA \) are closed subalgebras of the Fréchet algebra \( A \).

Assume that \( A \) is a left Fréchet ideal in \( B \). Then \( Ae \) is a left ideal in \( B \) since \( b(ae) \in B(Ae) \subseteq (BA)e \subseteq Ae \), for \( a \in A \) and \( b \in B \). Note that \( I = \{a \in A \mid ae = 0\} \) is a closed left ideal in \( A \), with quotient \( A/I = Ae \). The inherited topology on \( Ae \) is given by the quotient seminorms \( \|ae\|'_n = \inf_{i \in I} \|a+i\|_n \), for \( ae \in Ae \). For \( b \in B \), \( ae \in Ae \), we
have

\[ \|b(\alpha e)\|'_n = \inf_{i \in I} \|ba + i\|_n \]
\[ \leq \inf_{i \in I} \|b(a + i)\|_n \]
\[ \leq C_n \|b\|_B \inf_{i \in I} \|a + i\|_m \]
\[ = C_n \|b\|_B \|ae\|'_m, \]

(14)

so \( Ae \) is a left Fréchet ideal in \( B \).

If \( A \) is nuclear, then \( Ae \) is nuclear, since quotients by closed linear subspaces preserve nuclearity [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4). If \( be \in B \) and \( a_k \to b \) in \( B \), then \( a_k e \to be \) in \( Be \), by continuity of multiplication, so \( Ae \) is dense in \( Be \) if \( A \) is dense in \( B \).

Remark 3.7. Note that the quotient Fréchet ideal \( A/I \subseteq B/J \) in Proposition 3.6 (a) satisfies inequality \([12]\) with the same integers \( m_n \) and constants \( C_n \) as the original Fréchet ideal \( A \subseteq B \). The same is true for the ideals \( Ae \subseteq B \) (or \( eA \subseteq B \)).

Proposition 3.8. Algebraic Ideals. Let \( A \) be a Fréchet subspace of a Banach algebra \( B \), with continuous inclusion \( \iota: A \hookrightarrow B \). If \( A \) is a left (right) ideal in the purely algebraic sense, then \( A \) is a left (right) Fréchet ideal in \( B \).

Proof: Let \( L_b: A \to A \) denote left multiplication by some \( b \in B \). Assume \( a_\alpha \to 0 \) and \( L_b(a_\alpha) \to a_0 \) in \( A \). Then \( a_\alpha \to 0 \) and \( L_b(a_\alpha) \to a_0 \) in \( B \), by continuous inclusion. But since \( B \) is a Banach algebra, the multiplication is continuous, and \( a_0 = 0 \). Apply the closed graph theorem to see that \( L_b \) is a continuous linear map from \( A \) to \( A \).
Let $R_a : B \to A$ denote right multiplication by some $a \in A$. Let $b_\alpha \to 0$ in $B$ and $b_\alpha a \to a_0$ in $A$. Then $b_\alpha a \to a_0$ in $B$, by continuous inclusion, and so $a_0 = 0$. Apply the closed graph theorem again to see that $R_a$ is continuous from $B$ to $A$.

We have shown that the bilinear map of multiplication $M : B \times A \to A$ is separately continuous. By [Rudin, 1973], Theorem 2.17, $M$ is jointly continuous. \hfill \qed

### 4 Dense Nuclear Ideals in $C^*$-Algebras

In this section and the next, we apply the dense ideal condition together with nuclearity, to get results about the structure of the Banach algebra. Here we obtain complete results for $C^*$-algebras, but wait until §5 for the general Banach algebra case.

An algebra is *semiprime* if it has no non-zero nilpotent ideal. (One can use two-sided, left, or right ideals to define semiprime; the resulting definitions are equivalent [Palmer, 1994], Proposition 4.4.2 (d).)

**Proposition 4.1.** Let $A$ be a dense nuclear left (right) Fréchet ideal of a Banach algebra $B$. Let $e \in B$ be an idempotent. Then $Ae$ ($eA$) is a finite dimensional Banach algebra equal to $Be$ ($eB$).

Assume further that $B$ is semiprime. Then $Be$, $eB$, and $BeB$ are all finite dimensional Banach algebras equal to $Ae$, $eA$, and $AeA$ respectively. The Fréchet ideal $A$ can be either left or right for this to work.
Remark 4.2. If $B$ is not semiprime, it may happen that $A$ is a dense nuclear left Fréchet ideal, but $eB$ and $BeB$ are not finite dimensional, for some idempotent $e \in A$. Let $B$ be the Hilbert space $l^2(X)$ and $A$ be Schwartz functions $S_\gamma(X)$, with the natural inclusion map $A \hookrightarrow B$. For $\chi, \eta \in B$, define multiplication by $\chi \ast \eta = \chi_1 \eta$. Then $B$ is a Banach algebra and $A$ is a Fréchet algebra for this multiplication. $B$ is not semiprime since the (two-sided) ideal $I = \{ \chi \in B \mid \chi_1 = 0 \}$ satisfies $I^2 = 0$. The Fréchet algebra $A$ is a dense nuclear left Fréchet ideal in $B$, but not a right ideal since $AB = B$. Let $e = (1, 0, 0, \ldots) \in A$. Note that $e^2 = e$, $Be = \mathbb{C}e$, $eB = B$ and $BeB = B$.

Proof of Proposition 4.1: By Proposition 3.6 (b), $Ae$ is a dense nuclear left Fréchet ideal in $Be$. Since $e$ is a right unit for $Be$, Proposition 3.4 tells us $e \in Ae$. So by the last part of Proposition 3.4, $Ae$ is a Banach algebra exactly equal to $Be$. Being a nuclear Banach space, $Ae = Be$ is finite dimensional (Proposition 2.1 (b)).

Next, I will imitate the proof of the first Lemma of [Smyth, 1980], to show that $eB$ is also finite dimensional, with the added assumption that $B$ is semiprime. We just proved that $Be$ is finite dimensional, so $eBe$ must also be finite dimensional. Assume the dimension is some positive integer $N$, and let $\theta: eBe \to \mathbb{C}^N$ be a linear bijective map. For each $y \in eB$, define a linear map $\varphi_y: Be \to \mathbb{C}^N$ by $\varphi_y(x) = \theta(yx)$, $x \in eB$. The map $y \mapsto \varphi_y$ from $eB$ to the (finite dimensional) space of linear maps $\mathcal{L}(Be, \mathbb{C}^N)$ is linear. If $\varphi_{y_0} = 0$ for some $y_0 \in eB$, then $y_0Be = 0$, and $(By_0)^2 = (Be\theta)(By_0)$ (since $y_0 \in eB$) = $Be(\theta(\theta(y_0)))y_0 = 0$. So the left ideal $By_0$ is nilpotent with order 2, contradicting our assumption that $B$ is semiprime. Therefore the mapping $y \in eB \mapsto \varphi_y \in \mathcal{L}(Be, \mathbb{C}^N)$ is one-to-one, and hence $\dim(eB) < \dim(\mathcal{L}(Be, \mathbb{C}^N)) = \dim(Be) \ast N < \infty$. 

21
Now we know that both $eB$ and $Be$ are finite dimensional. Let $b_1, \ldots, b_K \in B$ and $c_1, \ldots, c_L \in B$ satisfy $eB = \mathbb{C}\text{span}\{eb_i | i = 1, \ldots, K\}$ and $Be = \mathbb{C}\text{span}\{c_ie | i = 1, \ldots, L\}$. Then $BeB$ has dimension at most $KL$ since $BeB = \mathbb{C}\text{span}\{b_i ec_j | i = 1, \ldots, K, j = 1, \ldots, L\}$.

**Theorem 4.3.** Let $B$ be a commutative $C^\ast$-algebra, with maximal ideal space $M$. Then $B$ has a dense nuclear Fréchet ideal if and only if $M$ is discrete and countable.

Proof: The maximal ideal space $M$ is locally compact, and $B$ is isomorphic to the $C^\ast$-algebra $C_0(M)$ of continuous functions vanishing at infinity on $M$ [Dixmier, 1982], §1.4.1.

Let $m \in M$, and let $U \subseteq M$ be an open, relatively compact set about $m$. The set of functions $f \in B$ which vanish on the closure $\overline{U}$, is a closed ideal $I_\overline{U}$ in $B$. Let $A$ be a dense nuclear Fréchet ideal in $B$. Applying Proposition 3.6 (a), we find the image $\pi(A)$ of $A$ in the quotient $B/I_\overline{U}$ is again a dense nuclear Fréchet ideal. Since $B/I_\overline{U} \cong C(\overline{U})$ is unital by the compactness of $\overline{U}$, Proposition 4.1 (with $e = 1_{C(\overline{U})}$) tells us that $\pi(A)$ is equal to $B/I_\overline{U}$ and finite dimensional. But $C(\overline{U})$ can only be finite dimensional if $\overline{U}$ is a finite set of points. Thus every point $m \in M$ has a finite neighborhood. This proves that $M$ is discrete.

To see that $M$ is countable, let $\delta_m$ denote the unit step function at $m \in M$. An element $b \in B$ can satisfy $\|b - \delta_m\|_B < 1/4$ for at most one $m \in M$, since $\|\delta_{m_1} - \delta_{m_2}\|_B = 1$ for distinct $m_1, m_2 \in M$. By Corollary 2.3, $B$ has a countable dense set $S$. The correspondence $m \mapsto \text{"choose } b \in S \text{ within distance } 1/4 \text{ of } \delta_m$ gives an injective map from $M$ into $S$, so $M$ is countable.

Since $M$ is discrete and countable, it must be either finite or countably infinite. For
the infinite case, $B \cong c_0(X)$, and the standard set of Schwartz functions $A = S_\gamma(X)$ is a dense nuclear Fréchet ideal (see Equation (1) or Example 3.3 (b)).

**Lemma 4.4. Existence of Projections.** Let $B$ be a $C^*$-algebra with a dense nuclear left or right Fréchet ideal $A$. Let $I$ be any proper closed two-sided ideal in $B$. Then $A$ contains a nontrivial projection which does not lie in $I$.

Proof: Let $a$ be an element of $A - I$, with image $[a]$ in the quotient $B/I$. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate identity for the ideal $I$. Assume $A$ is a right Fréchet ideal, and note that $\|a - ae_\lambda\|_B$ approaches $\|[a]\|_{B/I}$. By the Fréchet ideal condition, $\|ae_\lambda\|_n \leq C_n\|a\|_m e_\lambda\|_B = C_n\|a\|_m$, $n \in \mathbb{N}, \lambda \in \Lambda$. Since a bounded set in a nuclear locally convex space is relatively compact (Proposition 2.1 (d)), there is a cluster point $i_0 \in I \cap A$ for the net $\{ae_\lambda\}_{\lambda \in \Lambda}$, converging in the Fréchet topology of $A$. Replace $a$ with $a - i_0$ so now we have $\|a\|_B = \|[a]\|_{B/I} \neq 0$. Replace $a$ with $aa^*/\|a\|_B^2$. This new $a$ remains in $A$, is positive, and satisfies $\|a\|_B = \|[a]\|_{B/I} = 1$.

For each $k \in \mathbb{N}$, $a^{2k}$ has $C^*$-norm equal to one, by applying the $C^*$-identity $\|a^{2k+1}\|_B = \|a^{2k}\|_B^2$ repeatedly. By the Fréchet ideal condition, $\|a^{2k}\|_n \leq C_n\|a^{2k-1}\|_B\|a\|_m \leq C_n\|a\|_m$, $k, n \in \mathbb{N}$. Since a bounded set in a nuclear locally convex space is relatively compact (Proposition 2.1 (d)), there is a subsequence $\{a^{k_i}\}_{i=0}^\infty$, converging in the Fréchet topology of $A$. The limit point, $a_0 \in A$, must also have unit norm in $B$. Moreover, the image $[a_0]$ of $a_0$ in $B/I$ also has norm 1, since $\|[a_{k_i}]\|_{B/I} = 1$ for each $k_i$.

Consider the commutative $C^*$-subalgebra $C^*(a)$ of $B$ generated by $a$. This algebra must be isomorphic to continuous functions on a locally compact space $M$, vanishing at infinity. We may think of $a$ and $a_0$ as real-valued functions on $M$, with range in
[0,1], both taking on the value 1 for at least one point of \( M \). Since \( a^{k_i} \to a_0 \) in the sup-norm, \( a^{k_i}(m) \to a_0(m) \) for each \( m \in M \). It follows that \( a_0(m) \in \{0,1\} \), and \( a_0 \) is a projection.

**Definition 4.5. The Finite Socle.** Let \( A \) be any algebra. The *left* (right) *finite socle* of \( A \) is the sum of all minimal left (right) finite dimensional ideals of \( A \). If the left and right finite socles are equal, their common value is the *finite socle* of \( A \), denoted by \( A_{\text{fin}} \).

For \( a \in A \) and minimal left ideal \( L \) of \( A \), \( La \) is a minimal left ideal or \( \{0\} \) [Palmer, 1994], Proposition 8.2.8. If \( L \) is finite dimensional, so is \( La \). Hence the left finite socle is a right, and therefore two-sided, ideal of \( A \). Similarly the right finite socle is a two-sided ideal of \( A \).

Let \( S \) be the set \( \{ e \mid e \) is a minimal idempotent in \( A \} \), and assume \( A \) is semiprime. Then \( \{ Ae \mid e \in S \} \) (\( \{ eA \mid e \in S \} \)) gives all the minimal left (right) ideals of \( A \) [Palmer, 1994], Corollary 8.2.3. By the first Lemma of [Smyth, 1980], we know that \( \dim(Ae) \) is finite if and only if \( \dim(eA) \) is finite. Let \( S_{\text{fin}} \) denotes those \( e \in S \) for which \( Ae \) (or \( eA \)) is finite dimensional. Then the left (right) finite socle is equal to \( AS_{\text{fin}} \) (\( S_{\text{fin}}A \)). By the previous paragrapgh, these are both two-sided ideals in \( A \). But then they must both equal \( AS_{\text{fin}}A \), so the left and right finite socles agree, and \( A_{\text{fin}} = AS_{\text{fin}} = S_{\text{fin}}A = AS_{\text{fin}}A \). Note that \( C^* \)-algebras are semiprime.

**Theorem 4.6.** Let \( B \) be a \( C^* \)-algebra containing a dense nuclear left or right Fréchet ideal \( A \). Then the finite socle \( B_{\text{fin}} \) is dense in both \( B \) and \( A \), the primitive ideal space \( \text{Prim}(B) \) is discrete and countable, and \( B/I \) is finite dimensional for any \( I \in \text{Prim}(B) \).
is the countable direct sum, or restricted product, of finite dimensional matrix algebras.

Proof: We do the proof for a left ideal $A$. The closure $I = B_{\text{fin}}^I$ is a two-sided ideal in $B$. Assume for a contradiction that $I \neq B$. Apply Lemma 4.4 to get a nontrivial projection $p \in A - I$. By Proposition 4.1, $Ap = Bp$ is finite dimensional. If $I \cap Bp \neq \{0\}$, find only finitely many $e_1, \ldots, e_k \in S_{\text{fin}}$ such that $e_i p \neq 0$. A simple calculation shows $e' = p - e_1 - \cdots - e_k$ is an idempotent in $B$ which is orthogonal to $I$. Since $p \notin I$, $e' \neq 0$.

By Proposition 4.1, $Ae' = Be'$ is finite dimensional. This is a left ideal of $B$ whose intersection with $I$ is $\{0\}$. Let $e_{\text{min}}$ be a minimal idempotent contained therein. Then $Be_{\text{min}}$ is a minimal left finite dimensional ideal in $B$, which is not in $B_{\text{fin}}$, a contradiction.

Let $e$ be any idempotent in $B$. The second part of Proposition 4.1 tells us $eA = eB$. Since $e \in eB$, we know $e \in eA$. But $eA \subseteq A$ since $A$ is a left ideal. So $A$ contains any idempotent of $B$, $S_{\text{fin}} \subseteq A$, and $B_{\text{fin}} \subseteq A$.

The distance in $B$ between two idempotents $e, f \in S_{\text{fin}}$ is at least one, since $\|e - f\|_B \geq \|e - ef\|_B = \|e\|_B \geq 1$. By Corollary 2.3, $B$ is separable, so $S_{\text{fin}}$ is at most countable.

For $e \in S_{\text{fin}}$, $BeB$ is a finite dimensional matrix algebra with dimension $(\dim Be)^2$. So as a $C^*$-algebra, $B$ is the direct sum of finite dimensional matrix algebras, and $\text{Prim}(B)$ is well-known to be discrete [Fell Dor, 1988], Proposition 5.21. Also $B/I$ is one of the finite dimensional direct summands for each $I \in \text{Prim}(B)$.

To see that $B_{\text{fin}}$ is dense in $A$, let $P_K = \sum_{k \leq K} 1_k$ be the sum of of the first $K$ identity matrices, from the matrix algebras in the direct sum for $B$. Then $\{P_K\}_{K=0}^\infty$ is a bounded approximate unit for $B$. For any $a \in A$, the left ideal condition tells us that the set $\{P_Ka \mid K \in \mathbb{N}\}$ is bounded in $A$. By Proposition 2.1 (d), some $a_0 \in A$ is a cluster point.
But $a_0$ is also a cluster point in the topology of $B$, so since $P_K a \to a$ in $B$, we must have $a_0 = a$. Since $P_K a \in B_{\text{fin}}$, this shows that $B_{\text{fin}}$ is dense in $A$. 

There is a notion of nuclearity for $C^*$-algebras [Kad Ring II, 1997], Chapter 11, which is different from the notion of nuclearity for locally convex spaces.

**Corollary 4.7. Nuclearity of the $C^*$-algebra.** Let $B$ be a $C^*$-algebra with a dense left or right Fréchet ideal $A$. If $A$ is nuclear as a locally convex space, then $B$ is nuclear as a $C^*$-algebra.

**Proof:** By Theorem 4.6, every irreducible representation of $B$ is finite dimensional, and $B$ is separable by Corollary 2.3. Applying [Dixmier, 1982], §9.1 Theorem (iv) $\Rightarrow$ (i), we find that $B$ is a Type I $C^*$-algebra. But every Type I $C^*$-algebra is nuclear in the $C^*$-algebraic sense [Paterson, 1988], (1.31).

5 Complete Continuity and the General Banach Case

In this section we generalize Theorem 4.6 for an arbitrary Banach algebra.

For any Banach algebra $B$, we give $\text{Prim}(B)$ the Jacobson topology [Dixmier, 1982], §3.1.1. Our main goal in this section is to show that $\text{Prim}(B)$ has a discrete topology, whenever $B$ has a dense nuclear Fréchet ideal. The proof of Theorem 4.3 does not easily generalize to the noncommutative case, and the proofs of Theorem 4.6 and Lemma 4.4 rely on the theory of $C^*$-algebras. In this section, we find a different approach to the problem, with the concept of complete continuity.
If $E$ and $F$ are Banach spaces, a continuous linear map $T : E \to F$ is completely continuous if it maps weakly converging sequences in $E$ to norm converging sequences in $F$. A Banach algebra $B$ is said to be left completely continuous if for every $b \in B$, the left multiplication operator $L_b : B \to B$, given by $x \in B \mapsto bx$, is a completely continuous map from $B$ to itself. Similarly, $B$ is right completely continuous if right multiplication $R_b$ is completely continuous for every $b \in B$, and $B$ is completely continuous if it is both left and right completely continuous [Kaplansky, 1949].

**Theorem 5.1.** Let $B$ be a Banach algebra containing a dense nuclear left (right) Fréchet ideal $A$. Then $B$ is right (left) completely continuous.

Proof: We do the proof for right Fréchet ideals. Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in $B$, and $a$ a fixed element of $A$. It is not hard to show that if $x_k$ converges to a limit $x$ in the norm of $B$, then $ax_k$ converges to $ax$ in the Fréchet topology of $A$, using the right ideal inequality (12). But we must begin by assuming $x_k$ converges weakly, not in norm.

Let $\varphi$ be a continuous linear functional on $A$. Then $\varphi \circ L_a$ is continuous on $B$, since

$$|\varphi \circ L_a(x)| = |\varphi(ax)| \leq C \|ax\|_n \quad C, n \text{ exist since } \varphi \text{ is continuous on } A$$

$$\leq CC_n\|a\|_m\|x\|_B \quad \text{by the ideal inequality (12)}, \quad (15)$$

for any $x \in B$. Thus if our original sequence $\{x_k\}_{k=0}^{\infty}$ converges weakly to zero in $B$, then $ax_k$ converges weakly to zero in $A$. To simplify notation, define $y_k = ax_k$. We wish to prove that $y_k \to 0$ in the norm of $B$, by using the fact that $y_k \to 0$ weakly in $A$.

Let $U$ be the unit ball of $B$. Then $U \cap A$ is absolutely convex, and by continuous inclusion $A \hookrightarrow B$, $U \cap A$ is a zero neighborhood of $A$. Apply nuclearity [Pietsch, 1972], Proposition 4.1.4, to find an absolutely convex zero neighborhood $V$ of $A$, and sequence
of continuous linear functionals $\varphi_n \in A'$ satisfying
\begin{equation}
\sum_{n=0}^{\infty} \|\varphi_n\|_{V^0} < \infty, \tag{16}
\end{equation}

such that the inequality $\|x\|_B \leq \sum_{n=0}^{\infty} |\varphi_n(x)|$ holds for all $x \in A$. Taking $x = y_k$, we get
\begin{equation}
\|y_k\|_B \leq \sum_{n=0}^{\infty} |\varphi_n(y_k)| \tag{17}
\end{equation}
holds for every $k \in \mathbb{N}$.

**Lemma 5.2.** The sequence $\{y_k\}_{k=0}^{\infty}$ is bounded in the seminorm $\|\cdot\|_V$ on $A$.

Proof: Since $y_k \to 0$ weakly in the Fréchet space $A$, the set $\{y_k\}_{k=0}^{\infty}$ is weakly bounded in $A$. In the terminology of [Rudin, 1973], Theorem 3.18, $V$ is an original neighborhood of $A$. So $\{y_k\}_{k=0}^{\infty}$ is contained in $tV$ for some $t > 0$. \qed

We use Lemma 5.2 to show that the right hand side of inequality (17) tends to zero as $k \to \infty$. Let $\epsilon > 0$. Let $N$ be large enough so that $\sum_{n=N+1}^{\infty} \|\varphi_n\|_{V^0} < \epsilon/2M$, where $M$ is the bound on $\|y_k\|_V$ from Lemma 5.2. This is possible since the full series in (16) converges. Using weak convergence of $\{y_k\}_{k=0}^{\infty}$ to zero, find $K$ large enough so that $\sum_{n=0}^{N} |\varphi_n(y_k)| < \epsilon/2$ for $k \geq K$. So we have
\begin{align*}
\sum_{n=0}^{\infty} |\varphi_n(y_k)| &\leq \sum_{n=0}^{N} |\varphi_n(y_k)| + \sum_{n=N+1}^{\infty} |\varphi_n(y_k)| \\
&\leq \epsilon/2 + \sum_{n=N+1}^{\infty} |\varphi_n(y_k)| \quad \text{since } k \geq K \\
&\leq \epsilon/2 + \sum_{n=N+1}^{\infty} \|\varphi\|_{V^0}M \quad \text{since } \|y_k\|_V \leq M \text{ by Lemma 5.2} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon. \tag{18}
\end{align*}
Thus the right hand side of inequality (17) is less than \( \epsilon \) for \( k \geq K \), so the left hand side \( \| y_k \|_B \) is also less than \( \epsilon \) for \( k \geq K \). Thus \( y_k \to 0 \) in the norm of \( B \).

We have proved that if \( \{ x_k \}_{k=0}^\infty \) is a sequence in \( B \) converging weakly to zero, and \( a \in A \), then \( y_k = ax_k \) converges to zero in the norm of \( B \). To finish Theorem 5.1 and prove the complete continuity of \( B \), we would like to replace \( a \in A \) with an arbitrary element \( b \) of \( B \). Since \( x_k \to 0 \) weakly in \( B \), the Uniform Boundedness Principle (Theorem 2.4), gives us an \( M < \infty \) for which \( \| x_k \|_B \leq M \) for all \( k \in \mathbb{N} \). Let \( \epsilon > 0 \) and pick \( a \in A \) such that \( \| a - b \|_B < \epsilon/2M \). Let \( K \) be big enough so that \( \| ax_k \|_B < \epsilon/2 \) for \( k \geq K \).

Then
\[
\| bx_k \|_B \leq \| (a - b)x_k \|_B + \| ax_k \|_B \\
\leq \| a - b \|_B \| x_k \|_B + \epsilon/2 \quad \text{since } k \geq K \\
\leq (\epsilon/2M)M + \epsilon/2 = \epsilon,
\]
so \( bx_k \to 0 \) in the norm of \( B \). This completes the proof of Theorem 5.1.

A primitive ideal of a Banach algebra \( B \) is the kernel of a non-zero algebraically irreducible, continuous left Banach-space representation of \( B \). (This is equivalent to the purely algebraic definition of primitive ideal [Palmer, 1994], Corollary 4.2.9.) The primitive ideal space of \( B \), or \( \text{Prim}(B) \), is the set of all primitive ideals in \( B \). We also call \( \text{Prim}(B) \) the spectrum of \( B \).

**Corollary 5.3.** Let \( B \) be a Banach algebra containing a dense nuclear two-sided Fréchet ideal \( A \). Then \( B \) is completely continuous and the primitive ideal space of \( B \) is discrete.

Proof: By Theorem 5.1 just proved, containment of a dense nuclear two-sided Fréchet
ideal implies $B$ is completely continuous on both sides. By [Kaplansky, 1949], Theorem 5.1, a completely continuous Banach algebra has discrete primitive ideal space. \hfill \Box

**Proposition 5.4.** Let $B$ be a Banach algebra with a dense nuclear left or right Fréchet ideal. Then $B/I$ is finite dimensional for each $I \in \text{Prim}(B)$.

Proof: Let $V$ be an algebraically irreducible left Banach $B$-module. Then $I = \{b \in B \mid bV = 0\} \in \text{Prim}(B)$ is the primitive ideal corresponding to $V$. Let $v$ be a nonzero element of $V$, and define $M = \{b \in B \mid bv = 0\}$. Then $M$ is a maximal modular closed left ideal of $B$, and $V = B/M$ as left $B$-modules.

Let $A$ be a dense nuclear left Fréchet ideal of $B$. Then $Av$ is dense in $V$ (since $Bv = V$ and $A$ is dense in $B$), so there must be an $a_0 \in A$ such that $a_0v \neq 0$. Since $A$ is a left ideal in $B$, we have $Av \supseteq (Ba_0)v = B(a_0v) = V$. Let $N = M \cap A = \{a \in A \mid av = 0\}$. Then $N$ is a closed left ideal in $A$, $V = A/N$ as left $B$-modules, and $A/N$ is nuclear [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4).

The map $a \in A \mapsto av \in V$ is continuous, so the Fréchet topology on $A/N$ is at least as strong as the original Banach space topology on $V$. By [Treves, 1967], Chapter 17-7, Corollary 2, the topologies must agree. So $V$ is a nuclear Banach space and therefore finite dimensional (Proposition 2.1 (b)). Since $B/I$ is represented faithfully on $V$, it must also be finite dimensional. \hfill \Box

**Remark 5.5.** An alternative proof of Proposition 5.4 (assuming a dense nuclear two-sided ideal) is given by applying the complete continuity of $B$ from Theorem 5.1, and

\footnote{The same is true if we give $V$ the Banach space topology of $B/M$.}
The Jacobson radical $\mathcal{A}_J$ of an algebra $\mathcal{A}$ is the intersection of all primitive ideals of $\mathcal{A}$. $\mathcal{A}$ is semisimple if $\mathcal{A}_J = 0$, and radical if $\mathcal{A}_J = \mathcal{A}$ [Palmer, 1994], §4.3.1. A semisimple algebra is also semiprime [Palmer, 1994], Theorems 4.4.6 and 4.5.9.

**Theorem 5.6.** Let $B$ be a Banach algebra containing a dense nuclear two-sided Fréchet ideal $\mathcal{A}$. Then $\text{Prim}(B)$ is discrete and countable, and $B/I$ is finite dimensional for each $I \in \text{Prim}(B)$.

**Proof:** Proposition 3.6 (a) shows the quotient of $B$ by its Jacobson radical still has a dense nuclear Fréchet ideal. Since the Jacobson radical $\mathcal{B}_J$ is the intersection of all kernels of primitive ideals, $\text{Prim}(B) = \text{Prim}(B/\mathcal{B}_J)$. So without loss of generality, I will assume that $B$ is semisimple.

I will construct an idempotent for each primitive ideal. Let $I \in \text{Prim}(B)$ and let $\{I\}^c$ denote the complement of the singleton $\{I\}$, namely $\{J \in \text{Prim}(B) | J \neq I\}$. Discreteness (Corollary 5.3) tells us the intersection $L$ of all elements of $\{I\}^c$ is not contained in $I$. (The closure of $\{I\}^c$ in the Jacobson topology is by definition the set of primitive ideals containing $L$. By discreteness this closure must only be $\{I\}^c$ and nothing more.) Note that $L$, being the intersection of closed two-sided ideals, is itself a closed two-sided ideal in $B$. Intersecting $L$ with $I$ gives the zero ideal, since $L \cap I = \bigcap \{J | J \in \text{Prim}(B)\}$ and $B$ is semisimple. So $LI = IL \subseteq L \cap I = 0$, and $B$ is the direct sum of ideals $B = L \oplus I$. Since $\text{Prim}(B)$ is discrete, the singleton $\{I\}$ is a closed set, so $I$ is not contained in any other primitive ideal. This implies $L$ must be simple.

We need a unit for $L$. By Proposition 5.4, $L$ is finite dimensional, and $L$ is semiprime.
since $B$ is [Palmer, 1994], Proposition 4.4.2 (e). By the Wedderburn Theorem [Palmer, 1994], Theorem 8.1.1, a finite dimensional semiprime algebra which is simple is isomorphic to a full matrix algebra, and therefore unital. Let $e_I$ be the unit of $L$.

Since $LI = IL = 0$, we know $e_I I = I e_I = 0$. Let $J \in \text{Prim}(B)$, $J \neq I$, with corresponding idempotent $e_J$, satisfying $e_J J = J e_J = 0$. Since the singleton $\{J\}$ is closed in $\text{Prim}(B)$, $J$ cannot be contained in $I$, so $L \cap J \neq 0$. Since $L \cap J$ is a two-sided ideal, and $L$ is simple, we have $L \subseteq J$, so $e_I \in J$. It follows that $e_I e_J = e_J e_I = 0$. The distance between these two orthogonal idempotents is at least one, because $\|e_I - e_J\|_B \|e_I\|_B \geq \|e_I - e_J e_I\|_B = \|e_I\|_B$.

An element $b \in B$ can satisfy $\|b - e_I\|_B < 1/4$ for at most one $I \in \text{Prim}(B)$, since $\|e_I - e_J\|_B = 1$ for distinct $I, J \in \text{Prim}(B)$. By Corollary 2.3, $B$ has a countable dense set $S$. The correspondence $I \mapsto \text{choose } b \in S \text{ within distance } 1/4 \text{ of } e_I$ gives an injective map from $\text{Prim}(B)$ into $S$, so $\text{Prim}(B)$ is countable.

The last statement follows from Proposition 5.4.

\[\square\]

6 Construction of Dense Nuclear Ideals for $C^*$-Algebras

Assume that a $C^*$-algebra $B$ is a countably infinite direct sum of finite dimensional matrix algebras. Let $p = \{p_z\}_{z \in Z}$ be the dimensions, with $Z$ a countably infinite set, so that $B = \bigoplus_{z \in Z} M_{p_z}(\mathbb{C})$. Note that $p$ is a scale on $Z$. We think of elements of $B$ as
matrix-valued functions $f$ on $Z$, where $f(z) \in M_{p_z}(\mathbb{C})$, $z \in Z$. The $C^*$-norm on $B$ is

$$\|f\|_B = \sup_{z \in Z} \|f(z)\|_{op},$$

and $B$ consists of those functions $f$ which vanish at $\infty$ [Dixmier, 1982], §1.9.14. In this section, we construct dense nuclear ideals in $B$.

Let $X$ be the disjoint union of finite sets

$$X = \bigcup_{z \in Z} \{z\} \times \{1, \ldots, p_z\} \times \{1, \ldots, p_z\},$$

and let $e_x = e_{z,ij}$ be matrix elements for the $p_z \times p_z$ matrices $M_{p_z}(\mathbb{C})$, for each tuple $x = \{z, i, j\} \in X$. These are partial isometries which form a Schauder basis for the $C^*$-algebra $B$ (Definition 2.5). Any $b \in B$ has a coordinate functional $b_x = <e_{z,ii}b e_{z,jj}, e_x> \in \mathbb{C}$, and unique series expansion $b = \sum_{x \in X} b_x e_x$ which converges unconditionally in $B$.

Let $c_f(X)$ be the linear span of the $e_x$’s. The finite socle $B_{\text{fin}}$ of $B$ (Definition 4.5) is identified with $c_f(X)$, and equals the algebraic direct sum of the matrix algebras $\bigoplus_{z \in Z} M_{p_z}(\mathbb{C})$.

**Definition 6.1. Socle-Specific Schwartz Spaces.** Let $\ell$ be any family of scales on $Z$, with countably infinite sets $X$ and $Z$ defined above. The Fréchet space $S_{\ell}^{\infty,\text{op}}(X)$ is defined to be the completion of $c_f(X)$ in the norms

$$\|\varphi\|_{n,\ell}^{\infty,\text{op}} = \sup_{z \in Z} \ell_n(z) \|\varphi(z)\|_{op},$$

where $\|\cdot\|_{op}$ is the operator norm on $M_{p_z}(\mathbb{C})$.

**Theorem 6.2.** Let a $C^*$-algebra $B$ be the countably infinite direct sum of finite dimen-
sional C*-algebras, with dimensions $p = \{p_z\}_{z \in Z}$. If $\ell$ is any family of scales on $Z$, then $S^{\infty, \text{op}}_{\ell}(X)$ is a dense two-sided Fréchet ideal in $B$, in which $\{e_x\}_{x \in X}$ is an equicontinuous, unconditional basis.

Define a family of scales $\sigma$ on $X$ by $\sigma_n(z,i,j) = \ell_n(z)$ for $z,i,j \in X$, $n \in \mathbb{N}$. The Fréchet ideal $S^{\infty, \text{op}}_{\ell}(X)$ is nuclear if and only if $\ell$ satisfies the $p$-summability condition

$$
(\forall n \in \mathbb{N}) \ (\exists m > n) \ \sum_{z \in Z} p^2 \frac{\ell_n(z)}{\ell_m(z)} < \infty,
$$

and if and only if $S^{\infty, \text{op}}_{\ell}(X) \cong S^{1}_{\sigma}(X) \cong S^{\infty}_{\sigma}(X)$.

It follows that a nuclear Fréchet ideal always exists, for any $C^*$-algebra $B$ satisfying the hypotheses of Theorem 6.2, since if $\vartheta$ is any enumeration of $Z$, then $\vartheta p$ gives at least one family of scales satisfying (23). (See Definition 2.8 and Proposition 6.3.)

Proof of Theorem 6.2: Since $\ell \geq 1$, $\| \cdot \|_{0, \text{op}}^\infty \geq \| \cdot \|_{B}$, and the inclusion map $S^{\infty, \text{op}}_{\ell}(X) \hookrightarrow B$ is continuous. For $f \in B$ and $\varphi \in S^{\infty, \text{op}}_{\sigma}(X)$, we have

$$
\| f \varphi \|_{n, \text{op}}^\infty = \sup_{z \in Z} \ell_n(z) \| f(z) \ast \varphi(z) \|_{\text{op}} \ \text{definition of } \| \cdot \|_{n, \text{op}}^\infty
\leq \left( \sup_{z \in Z} \| f(z) \|_{\text{op}} \right) \left( \sup_{z \in Z} \ell_n(z) \| \varphi(z) \|_{\text{op}} \right)
= \| f \|_{B} \| \varphi \|_{n, \text{op}}^\infty,
$$

so $S^{\infty, \text{op}}_{\ell}(X)$ is a left Fréchet ideal in $B$. Similarly it is a right, and therefore two-sided, Fréchet ideal in $B$.

The basis is equicontinuous since for $\varphi \in S^{\infty, \text{op}}_{\ell}(X)$, $x = (z,i,j) \in X$, and $n \in \mathbb{N}$,

$$
|\varphi(z)_{ij}| e_{x} \|_{n, \text{op}}^\infty = \| \varphi(z)_{ij} e_{x} \|_{n, \text{op}}^\infty = \ell_n(z) \| e_{z,ii} \varphi(z) e_{z,jj} \|_{\text{op}} \leq \ell_n(z) \| \varphi(z) \|_{\text{op}} \leq \| \varphi \|_{n, \text{op}}^\infty.
$$
Use the fact that $\|\varphi(z)\|_{\text{op}}$ becomes smaller if matrix entries are set to zero, to prove the basis is unconditional.

Since the operator norm on $M_{p_x}(\mathbb{C})$ is bounded by the sum of the matrix entries, and

\[\| \cdot \|_{\infty} \leq \| \cdot \|_{\text{op}} \leq \| \cdot \|_{1},\]

This proves the continuity of inclusion maps $S^1_\varphi(X) \hookrightarrow S^\infty_{\ell,\text{op}}(X) \hookrightarrow S^\infty_{\sigma}(X)$. Since $\sigma$ is constant along each matrix algebra, the \(p\)-summability condition (23) is equivalent to the summability condition (0), so Theorem 2.7 tells us the three spaces are isomorphic and nuclear if the summability condition is satisfied.

Conversely, assume $S^\infty_{\ell,\text{op}}(X)$ is a nuclear Fréchet space. We need to find a “Basic” Schwartz space (Definition 2.6) which is nuclear, to deduce the \(p\)-summability condition (23) for \(\ell\). The subspace of diagonal matrices is an obvious candidate. Let $Y$ be the countable disjoint union of diagonal sets $Y = \bigcup_{z \in \mathbb{Z}} \{z\} \times \{1, \ldots, p_x\}$. Define scales $\tau$ on $Y$ by $\tau_n(z, i) = \ell_n(z)$, and embed $\theta: S^\infty_{\tau}(Y) \hookrightarrow S^\infty_{\ell,\text{op}}(X)$, via the diagonal map $\theta(\varphi)(x) = \varphi(z, i)\delta(i - j)$. For $\varphi \in S^\infty_{\tau}(Y)$,

\[\|\theta(\varphi)\|_{\text{op}} = \sup_{z \in \mathbb{Z}} \ell_n(z)\|\theta(\varphi)(z)\|_{\text{op}} = \sup_{z \in \mathbb{Z}} \ell_n(z) \sup_{i \leq p_x} |\varphi(z, i)| = \|\varphi\|_{\infty,n},\]

since each $\theta(\varphi)(z)$ is a diagonal matrix. So $\theta$ is isometric in all the norms. Since a subspace of a nuclear Fréchet space is nuclear [Pietsch, 1972], Proposition 5.1.5, or [Treves, 1967], Proposition 50.1 (50.3), $S^\infty_{\tau}(Y)$ is nuclear.

Assume for a contradiction that every $\ell_n$ is not proper. Then there exists a sequence $z_1, z_2, \ldots, z_j, \ldots$ on which every $\ell_n$ is bounded on the tail. The sequence of basis elements $e_{z_j,0}$ is then bounded by some $C_n > 0$ in each norm $\| \cdot \|_{\infty}$. By Proposition 2.1 (d), there
would have to be a subsequence converging in \( S_r^\infty(Y) \). But this is impossible.

Notice that for each \( z \in Z \) and \( i \leq p_z \), the linear functional \( \ell_n(z)e_{zi}(\varphi) = \ell_n(z)|\varphi(z,i)| \leq \|\varphi\|_n^{\infty} \). Also, the countable set \( \{\ell_n(z)e_{zi}(\varphi)\}_{z \in Z, i \leq p_z} \) converges weakly to zero in \( S_r^\infty(Y)' \). It is an essential subset of the polar of the unit ball for \( \| \cdot \|_n^{\infty} \), in the sense of [Pietsch, 1972], §2.3.1. By nuclearity and [Pietsch, 1972], Theorem 2.3.3, for any \( n \in \mathbb{N} \), there exists an \( m \in \mathbb{N} \) and a summing sequence \( c_{zi} \) of positive numbers so that for all \( \varphi \in S_r^\infty(Y) \),

\[
\|\varphi\|_n^{\infty} \leq \sum_{z \in Z, i \leq p_z} c_{zi}|\varphi(z,i)|\ell_m(z).
\]

For each \( z_0 \) and \( i_0 \), plug in \( \varphi(z,i) = \delta(z - z_0, i - i_0)/\ell_m(z) \). The result is \( \ell_n(z_0)/\ell_m(z_0) \leq c_{z_0i_0} \). Hence we have

\[
\sum_{z \in Z} p_z\ell_n(z)/\ell_m(z) \leq \sum_{z \in Z, i \leq p_z} c_{zi} < \infty.
\]

Repeat the same argument to find a \( p \in \mathbb{N} \) such that

\[
\sum_{z \in Z} p_z\ell_m(z)/\ell_p(z) < \infty.
\]

Let \( C_1, C_2 > 0 \) be constants bounding these respective sums. Then

\[
\sum_{z \in Z} p_z^2\ell_n(z)/\ell_p(z) = \sum_{z \in Z} p_z\ell_n(z)/\ell_m(z) \cdot p_z\ell_m(z)/\ell_p(z) \\
\leq C_1 \sup_{z \in Z} p_z\ell_m(z)/\ell_p(z) < C_1C_2,
\]

which is the \( p \)-summability condition \((25)\). 

\[^3\text{Note that } S_r^\infty(Y) \text{ is contained in } c_0(Y) \text{ with continuous inclusion, and clearly } \{e_{z,j}\}_{j \in \mathbb{N}} \text{ cannot have a converging subsequence in } c_0(Y), \text{ since they are distinct projections, and separated pairwise in sup-norm by distance 1.}\]
Proposition 6.3. The family of scales associated with a single scale $\ell$ on $\mathbb{Z}$ satisfies the $p$-summability condition \((23)\) if some power of $\ell$ dominates $\vartheta p$, for some enumeration $\vartheta$ of $\mathbb{Z}$.

Proof: Assume there exists $C > 1$ and $d \in \mathbb{N}$ such that $\vartheta(z)p_z \leq C\ell(z)^d$, $z \in \mathbb{Z}$. Then

$$
\frac{1}{C^2} \sum_{z \in \mathbb{Z}} \frac{p_z^2}{\ell(z)^{2d}} \leq \sum_{z \in \mathbb{Z}} \frac{1}{\vartheta(z)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6 < \infty,
$$

and $\ell$ is $p$-summable. \(\square\)

We show that every dense nuclear two-sided Fréchet ideal is given by Theorem 6.2.

Theorem 6.4. Classification of Dense Nuclear Two-Sided Ideals. Let a $C^*$-algebra $B$ be the countable infinite direct sum of finite dimensional $C^*$-algebras, with sequence of dimensions $p = \{p_z\}_{z \in \mathbb{Z}}$. Let $A$ be any dense nuclear two-sided Fréchet ideal in $B$. Then there exists a family of scales $\ell$ on $\mathbb{Z}$ satisfying the $p$-summability condition \((23)\) such that the map $a \mapsto \{x \mapsto a_x\}$ gives an isomorphism of Fréchet ideals $A \cong S_{\ell}^{\infty, \text{op}}(X)$, where $X$ is defined by \((21)\).

Proof: First we show that $\{e_x\}_{x \in X}$ is an absolute basis for $A$. Since $A \subset B$, every element $a$ of $A$ also has a unique expansion in $\{e_x\}_{x \in X}$. We will show the series for $a$ converges in the Fréchet topology. Pick some enumeration $\vartheta$ of $\mathbb{Z}$. For $K \in \mathbb{N}$, let $P_K = \sum_{\vartheta(z) \leq K} 1_z$ be the sum of the units of the first $K$ matrix algebras which make up $B$. By Theorem 4.6, the socle $c_f(X)$ is dense in $A$. Let $\varphi \in c_f(X)$ be $\epsilon$ close to $a$ in $\| \cdot \|_m$. For large
enough $K$, $P_K \varphi = \varphi$, and we have

$$
\left\| \sum_{\theta(z) \leq K} a_x e_x - a \right\|_n \leq \left\| \sum_{\theta(z) \leq K} a_x e_x - \varphi \right\|_n + \| \varphi - a \|_n
\leq \left\| P_K(a - \varphi) \right\|_n + \| \varphi - a \|_n
\leq C_n \| a - \varphi \|_m + \| \varphi - a \|_n \leq C_n \epsilon + \epsilon,
$$

where we used the left ideal condition, $\| P_K \|_B = 1$, and $\| \cdot \|_n \leq \| \cdot \|_m$ in the last step. This shows that $\{ e_x \}_{x \in X}$ is a basis for the Fréchet algebra $A$. By the discussion in Definition 2.5, this basis is Schauder and equicontinuous, and by the nuclearity of $A$, it is also absolute.

By Appendix A, we can find an equivalent family $\{ \| \cdot \|_n \}_{n=0}^{\infty}$ of norms for the topology of $A$ which are increasing, and satisfy $\| \cdot \|_0 = \| \cdot \|_B$ and $\| ab \|_n \leq \| a \|_n \| b \|_0$, $\| ba \|_n \leq \| b \|_0 \| a \|_n$ for all $b \in B$, $a \in A$. Use these norms to define a family of scales $\sigma$ on $X$ by $\sigma_n(x) = \| e_x \|_n$. Since $\| e_x \|_B = 1$ for each $x \in X$, $\sigma_n \geq 1$. Also $\sigma_0 = 1$ and $\sigma_n \leq \sigma_{n+1}$. By the ideal condition, $\sigma_n(x) = \| e_x \|_n = \| e_{z,1} e_{z,11} e_{z,1j} \|_n \leq \| e_{z,11} \|_n = \sigma_n(z, 1, 1)$. Similarly $\sigma_n(z, 1, 1) \leq \sigma_n(x)$ for any $x \in X$ with first component $z$. Therefore the $\sigma_n$’s are constant on each matrix algebra. For each $z \in Z$, let $\ell_n(z)$ be the common value of $\sigma_n(z, i, j)$, $i, j \leq p_k$.

Apply Theorem 2.7 to see that $A \cong S^1_\varphi(X) \cong S^\infty_\varphi(X)$, where, by nuclearity, $\sigma$ satisfies the summability condition [6], and $\ell$ satisfies the $p$-summability condition [23]. By Theorem 6.2, $S^\infty_{\ell, op}(X)$ is isomorphic to $S^1_\varphi(X) \cong S^\infty_\varphi(X)$, and hence also isomorphic to $A$. 

\footnote{Note we can apply the dense ideal condition on both sides $\| a_x \|_n = \| a_x e_x \|_n = \| e_{z,1i} e_{z,1j} \|_n \leq C \| a \|_q$, to see directly that the basis is equicontinuous.}
Remark 6.5. **Automatically Involutive.** Note that the ideals of Theorems 6.4 and 6.2 are involutive.

**Definition 6.6.** Let \( \vartheta \) be an enumeration of the set \( Z \). Define two scales on \( Z \) by

\[
\ell_{\vartheta, \min}(z) = \sum_{\vartheta(z') < \vartheta(z)} p_{z'}
\]

\[
\ell_{\vartheta, \max}(z) = \sum_{\vartheta(z') \leq \vartheta(z)} p_{z'};
\]

for each \( z \in Z \). We set \( \ell_{\min}(z_1) = 1 \).

Let \( S \) be a set and \( \mathcal{S}_\sigma(S) \) be any of the Schwartz spaces in Definition 2.6 or Definition 6.1. We say that \( \mathcal{S}_\sigma(S) \) is *standard Schwartz* if there exists an enumeration \( \gamma: S \cong \mathbb{N}^+ \), for which the identity map \( \text{id}: S \to S \) induces an isomorphism of Fréchet spaces \( \mathcal{S}_\sigma(S) \cong \mathcal{S}_\gamma(S) \). It is easy to check that \( \mathcal{S}_\sigma(S) \) is standard Schwartz if and only if \( S \) is countably infinite and \( \sigma \) is equivalent to the family of scales associated with a standard scale \( \gamma \).\(^5\) (See Definitions 2.6 and 2.8.)

**Proposition 6.7.** Let \( \vartheta \) be an enumeration of the set \( Z \). The scales \( p, \vartheta p, \) and \( \ell_{\vartheta, \min} \) are dominated by \( \ell_{\vartheta, \max} \). The following conditions are equivalent:

(i) \( p \lesssim \ell_{\vartheta, \min} \).

(ii) \( \vartheta p \lesssim \ell_{\vartheta, \min} \).

(iii) \( \ell_{\vartheta, \max} \lesssim \ell_{\vartheta, \min} \).

Let \( B \) be the \( C^* \)-algebra corresponding to \( p \). Then \( B \) contains at least one dense nuclear two-sided Fréchet ideal which is standard Schwartz, if and only if conditions (i)-(iii) are satisfied.

\(^5\)First use nuclearity and Theorem 2.7, or Theorem 6.2, to get \( \mathcal{S}_\sigma(S) \cong \mathcal{S}_\gamma(S) \) or \( \cong \mathcal{S}_\sigma(S) \).
satisfied for some enumeration \( \vartheta \) of \( Z \).

We say that \( p \) satisfies the growth condition with respect to \( \vartheta \) when (i) - (iii) are satisfied.

Proof: It is clear from definitions that \( \ell_{\vartheta, \text{min}} \leq \ell_{\vartheta, \text{max}} \) and \( p \leq \ell_{\vartheta, \text{max}} \). Since each \( p_z \) is at least 1, we know \( \vartheta \leq \ell_{\vartheta, \text{max}} \). Since both \( p \) and \( \vartheta \) are less than or equal to \( \ell_{\vartheta, \text{max}} \), we have \( \vartheta p \leq \ell_{\vartheta, \text{max}}^2 \), so \( \vartheta p \lesssim \ell_{\vartheta, \text{max}} \).

Next we prove the equivalence of (i) - (iii). Since \( p_z \geq 1 \), we have \( \vartheta(z) \leq \ell_{\vartheta, \text{min}}(z) + 1 \). Thus (i) implies \( \vartheta p \leq (\ell_{\vartheta, \text{min}} + 1)p \lesssim \ell_{\vartheta, \text{min}} \), since \( \ell_{\vartheta, \text{min}} \) dominates any linear combination of powers of \( \ell_{\vartheta, \text{min}} \). Since \( p \leq \vartheta p \), condition (ii) implies \( p \lesssim \ell_{\vartheta, \text{min}} \), so we have (i)\( \Leftrightarrow \) (ii).

The equivalence (i)\( \Leftrightarrow \) (iii) follows from \( \ell_{\vartheta, \text{max}} = \ell_{\vartheta, \text{min}} + p \).

Define a map \( \gamma : X \to \mathbb{N}^+ \) by \( \gamma(z_k, i, j) = p_1^2 + \cdots + p_{k-1}^2 + (i - 1) + (j - 1)p_k + 1 \), for \( i, j = 1, \ldots, p_k, k \in \mathbb{N}^+ \). Here \( z_k = \vartheta^{-1}(k) \) and \( p_k = p_{z_k} \). Then \( \gamma : X \to \mathbb{N}^+ \) is a bijection of sets, since for a fixed \( k \), \( \gamma \) increases, in steps of 1 if we move down rows, one column at a time, from \( p_1^2 + \cdots + p_{k-1}^2 + 1 \) (at \( i = j = 1 \)) up to \( p_1^2 + \cdots + p_{k-1}^2 + p_k^2 \) (at \( i = j = p_k \)). So \( \gamma \) is a standard scale on \( X \).

Now assume that the equivalent conditions (i) - (iii) hold. By (ii), \( \vartheta p \lesssim \ell_{\vartheta, \text{min}} \), so Theorem 6.2 with \( \ell = \ell_{\vartheta, \text{min}} \) gives us a dense nuclear two-sided Fréchet ideal \( S_{\text{min}}(X) \subset B \), where \( \sigma_{\vartheta, \text{min}}(z, i, j) = \ell_{\vartheta, \text{min}}(z) \) for \( \{z, i, j\} \in X \). But \( \sigma_{\vartheta, \text{min}} \leq \gamma \leq \sigma_{\vartheta, \text{max}}^2 \), so by condition (iii), \( \gamma \) is a scale on \( X \) equivalent to \( \sigma_{\vartheta, \text{min}} \). Since \( \gamma \) is a standard scale on \( X \), \( S_{\text{min}}(X) \) is standard Schwartz.

Conversely, assume \( B \) has a dense nuclear two-sided Fréchet ideal \( A \) which is standard Schwartz. We prove that the growth condition (i) holds. By Theorem 6.4, find a family
of scales $\ell$ on $Z$ for which $A \cong S^\infty_{\ell}(X)$. Define $\sigma_n(x) = \ell_n(z)$ as in Theorem 6.2. Let $\gamma : X \cong \mathbb{N}^+$ be an enumeration of $X$, such that $\sigma$ is equivalent to the family of scales associated with $\gamma$.

Define $\tilde{\ell}(z) = \min_{i,j} \gamma(z,i,j)$, for $z \in Z$. Let $\vartheta$ be an enumeration of $Z$ so that $\tilde{\ell}(z_1) < \tilde{\ell}(z_2) < \tilde{\ell}(z_3) < \ldots$. Since $\gamma$ maps $X$ onto $\mathbb{N}^+$, $\tilde{\ell}(z_1) = 1$. The smallest the set of values $\{\gamma(1,i,j)\}_{i,j=1,\ldots,p_1}$ could be is $\{1,2,\ldots,p_1^2\}$. So $\tilde{\ell}(z_2)$ can be no bigger than $p_1^2 + 1$. Similarly $\tilde{\ell}(z_k) \leq p_1^2 + p_2^2 + \cdots + p_{k-1}^2 + 1$. Hence $\tilde{\ell}(z_k) \leq \ell_{\vartheta,\min}(z_k)^2 + 1$.

Since $\sigma \sim \gamma$, there is some $n \in \mathbb{N}$ and $C_1 > 0$ such that $\gamma(z,i,j) \leq C_1 \ell_n(z)$, $i,j = 1,\ldots,p_z$, $z \in Z$. Since $\gamma$ is one to one, for any $z \in Z$ the set of values $\{\gamma(z,i,j)\}_{i,j=1,\ldots,p_z}$ must contain a number as big as $p_z^2$. Hence $p_z^2 \leq C_1 \ell_n(z)$. So we have

$$p_z^2 \leq C_1 \ell_n(z)$$

$$\leq C_1 C_n \tilde{\ell}(z)^m$$  \hspace{1cm} \text{since $\ell \sim \tilde{\ell}$}

$$\leq C_1 C_n (\ell_{\vartheta,\min}(z)^2 + 1)^m,$$

and $(i)$ is satisfied. \hfill \Box

7 Examples

Remark 8.1. Reordering $Z$ to satisfy the growth condition. If at least one $p_z$ is repeated infinitely many times, then we can find an enumeration $\vartheta$ of $Z$ so the growth condition of Proposition 6.7 holds. For example, the sequence $\{e^1,1,e^{2^2},1,e^{3^3},1,e^{4^4},\ldots\}$
does not satisfy the growth condition because \( p_1 + \cdots p_{2k} = k + e^1 + e^2 + \cdots e^k \leq k(1 + e^k) \)
and \( p_{2k+1} = e^{(k+1)^{k+1}} > e^{k^{k+1}} = (e^k)^k, \ k \in \mathbb{N}^+ \). Since infinitely many 1’s occur in the sequence, we can however pad as many 1’s as we like in between the elements \( e^k \), so that after reordering \( p_k < p_1 + \cdots p_{k-1} \) is satisfied.\(^6\)

If each \( p_k \) occurs only finitely many times, the growth condition can fail, even after an optimal reordering. For example, take \( p_k = e^k \). On the other hand, if the growth condition is satisfied for the sequence \( \{p_k\}_{k=1}^{\infty} \), we can always reorder so that \( p_{k-1} \leq p_k \). For let \( C, d \) be such that \( p_k \leq C(p_1 + \cdots p_{k-1})^d \). If \( p_k \) is the first out of order element of the sequence, find the smallest \( l > k \) for which \( p_l \) belongs at the \( k \)th spot. Redefine \( p_k = p_l \), and renumber \( p_{k+1} = p_k, \ldots, p_l = p_{l-1} \). Since \( p_l < p_k \leq C(p_1 + \cdots p_{k-1})^d \), the growth condition continues to hold. Continuing all the way up the sequence gives a nondecreasing ordering, which still satisfies the growth condition with the same constants \( C, d \).

**Example 8.2. Polynomial Growth.** If \( p_k \leq Ck^d, \ k \in \mathbb{N}^+ \), then since \( p_k \geq 1 \), \( p_1 + \cdots p_{k-1} \geq k - 1 \), and the growth condition is satisfied. (Use the inequality \( k < (k-1)^2 \) for \( k = 3, 4, 5, \ldots \).) The sequence \( p_k = e^k \) does not have polynomial growth, but satisfies the growth condition since \( e^1 + \cdots e^{k-1} = (e^k - e)/(e - 1) \), so \( p_k \leq (e-1)(e + p_1 + \cdots p_{k-1}) \).

**Remark 8.3. Constructing Fréchet Ideals.** A generalization of a construction used in the proof of Proposition 6.7 seems worthy of note. Let \( \ell \) and \( \sigma \) be as in Theorem 6.2, with the \( p \)-summability condition (23) satisfied. Pick a family of scales \( \beta \) on \( X \) so that

\(^6\)Note reordering the \( p_k \)’s does not change the isomorphism class of the \( C^*\)-algebra \( B \cong \bigoplus_{k=1}^{\infty} M_{p_k}(\mathbb{C}) \).
\[ \ell_n(z) \leq \beta_n(i, j, z). \] Also insure that for each \( n \in \mathbb{N} \) there is a sufficiently large \( m \in \mathbb{N} \) so that \( \beta_n(i, j, z) \leq \ell_m(z) \). Then clearly \( \sigma \sim \beta \). So \( S_\beta(X) \) is a dense nuclear two-sided Fréchet ideal in the \( C^* \)-algebra \( B \), by its isomorphism with \( S_\sigma(X) \).

**Example 8.4.** \( C^\infty(G), G \) Compact Connected Lie Group. By [Sug, 1971], Theorem 4, the Fréchet space \( C^\infty(G) \) is isomorphic to standard Schwartz functions on \( D \), where \( D \) is the set of all dominant \( G \)-integral forms on the Lie algebra of a maximal toral subgroup of \( G \). For example, consider the circle group \( G = \mathbb{T} \). The Fréchet space \( C^\infty(\mathbb{T}) \) is topologized by seminorms \( \| \varphi \|_i = \| \partial^i \varphi \|_\infty, \varphi \in C^\infty(\mathbb{T}) \), where \( \partial = \frac{d}{d\theta} \). The Fourier transform \( \hat{\cdot} \) changes \( C^\infty(\mathbb{T}) \) into \( S(\mathbb{Z}) \), \( C^*(\mathbb{T}) \) into \( c_0(\mathbb{Z}) \), and convolution multiplication of functions into pointwise multiplication. The transformed seminorms are \( \| \hat{\varphi} \|_i = \sup_{k \in \mathbb{Z}} |k^i \hat{\varphi}(k)| \).

It is well-known that \( C^\infty(G) \) is a dense two-sided Fréchet ideal in the convolution algebras \( L^1(G) \) and \( C^*(G) \).

**Example 8.5.** \( C^\infty \) functions on the Cantor group. The Cantor group \( K \) is a compact locally compact group, which is not a Lie group. It is totally disconnected and can be described as the dual group of the discrete abelian group \( \hat{K} = \mathbb{Z}[\frac{1}{2}] / \mathbb{Z} \) of dyadic rationals from 0 to 1. Define a proper scale \( \sigma(\frac{l}{2^p}) = 2^p \) on \( \hat{K} \), for any \( p \in \mathbb{N} \). Here \( l \) is any positive odd number less than \( 2^p \). The additive identity is 0, when \( p = 0 \) and \( l = 0 \). Let \( A = S^\infty_\sigma(\hat{K}) \) be \( \sigma \)-rapidly vanishing functions on \( \hat{K} \), with sup-norm. \( A \) is a dense Fréchet ideal in \( B = c_0(\hat{K}) \), the \( C^* \)-algebra of functions vanishing at \( \infty \) on \( \hat{K} \), with pointwise multiplication. (Note that convolution multiplication on \( K \)
becomes pointwise multiplication on $\hat{K}$ via the Fourier transform.) Since $\sigma$ satisfies the summability condition, $A$ is also nuclear [Sch, 1998], Lemma 1.2 with $q = 2$, Theorem 1.6.

We show that $A$ is standard Schwartz. Let $\gamma: \hat{K} \to \mathbb{N}$ be the map $\gamma(\frac{l}{2^p}) = 2^{p-1} + \lfloor l/2 \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the largest integer not greater than, and we set $\gamma(0) = 1$. For each $p \in \mathbb{N}^+$, note that $\lfloor l/2 \rfloor$ goes from 0 to $2^{p-1} - 1$, in increments of 1, and we have $\gamma \leq \sigma$. Also $\sigma \leq 2\gamma$. So $\gamma$ and $\sigma$ are equivalent scales on $\hat{K}$. Also, $\gamma$ is a bijection $\gamma: \hat{K} \cong \mathbb{N}^+$, so by Definition 6.6, $A = S^\infty_\sigma(\hat{K})$ is standard Schwartz.

**Remark 8.6. If Multiplication is Trivial, Dense Nuclear Ideals Always Exist.**

Let $B$ be any separable Banach algebra with $b_1b_2 = 0$ for all $b_1, b_2 \in B$. Let $a_0, \ldots a_n, \ldots$ be a countable sequence of elements of norm 1 of $B$, with dense linear span. Let $A_0$ be the Fréchet algebra with standard Schwartz functions $S_\gamma(X)$ as underlying Fréchet space, and zero multiplication.

Define a linear map $\theta: A_0 \to B$ by

$$\theta(\varphi) = \sum_{x \in X} \varphi(x)a_{\gamma(x)},$$

$\varphi \in A_0$. Since $\varphi$ is a Schwartz function, and the $a_k$’s have norm 1, this series converges absolutely to a well-defined element of $B$. We have $\|\theta(\varphi)\|_B \leq \sum_{x \in X} |\varphi(x)||a_{\gamma(x)}|_B = \|\varphi\|_1$, where $\| \cdot \|_1$ is the $l^1$-norm, so $\theta$ is continuous, and trivially an algebra homomorphism. Usually $\theta$ is not 1-1, unless for example $\{a_k\}_{k=0}^\infty$ were a basis for $B$. Let $A$ be the image $\theta(A_0)$ in $B$. Then $A$ is a dense Fréchet ideal in $B$, and is nuclear since quotients by closed linear subspaces preserve nuclearity [Pietsch, 1972], Proposition 5.1.3 or [Treves, 1967],
A Appendix. Refining the Ideal Condition, and \( m \)-Convexity

We show that the constants \( C_n \) can always be taken equal to 1 in the dense ideal inequality (12), and also \( m_n = n \), by passing to an equivalent family of seminorms. We also note that any Fréchet algebra satisfying the ideal inequality is \( m \)-convex.\(^7\)

Let \( A \) be a right Fréchet ideal in a Banach algebra \( B \), and let \( \{ \| \cdot \|_n \}_{n=0}^\infty \) be an increasing family of norms giving the topology for \( A \), with \( \| \cdot \|_0 = \| \cdot \|_B \), as in Definition 3.1. Define a new family of seminorms by

\[
\|a\|_n^* = \sup \{ \|ab\|_n \mid \|b\|_0 \leq 1, b \in B \}, \tag{29}
\]

for \( a \in A \). Using the right ideal inequality (12) we see that \( \|a\|_n^* \leq C_n \|a\|_m \), so the topology given by \( \{ \| \cdot \|_n^* \}_{n=0}^\infty \) is dominated by the original topology on \( A \). These new seminorms satisfy the right ideal inequality with \( m_n = n \) and \( C_n = 1 \):

\[
\|ab\|_n^* = \sup \{ \|ab\|_n \mid \|b\|_0 \leq 1, b \in B \} \quad \text{by definition (29)}
\]

\[
\leq \sup \{ \|ab\|_n \mid \|b\|_0 \leq \|b\|_0, b \in B \} \quad b_2 = bb_1, \ \text{so} \ \|b_2\|_0 \leq \|b\|_0
\]

\[
= \|b\|_0 \sup \{ \|ab\|_n \mid \|b\|_0 \leq 1, b \in B \} \quad b_3 = b_2/\|b\|_0
\]

\[
= \|a\|_n^* \|b\|_0 \quad \text{by definition (29).} \tag{30}
\]

\(^7\)A Fréchet algebra \( A \) is \( m \)-convex if it can be topologized by a family of submultiplicative seminorms, i.e. ones that satisfy \( \|ab\|_n \leq \|a\|_n \|b\|_n \) for \( a_1, a_2 \in A \).
Note we used the submultiplicativity of the norm $\| \cdot \|_0$ on $B$ in the second step. The inequality

$$\|ab\|_n \leq \|a\|_n^* \|b\|_0$$

(31)

can also be verified from the definition [29].

Since $B$ may not be unital, the seminorms $\{\| \cdot \|_n^*\}_{n=0}^{\infty}$ are not necessarily equivalent to our original family. (For example, $B$ could be a radical Banach algebra with $ab = 0$ for all $a, b \in B$, in which case $\|a\|_n^* = 0$ for all $n$.) So define new norms by $\|a\|_n^{**} = \max\{\|a\|_n^*, \|a\|_n\}, a \in A$. Note that $\| \cdot \|_n^{**} = \| \cdot \|_0 = \| \cdot \|_B$, and $\|ab\|_n^{**} \leq \|a\|_n^* \|b\|_0$ by our estimates above. The new family $\{\| \cdot \|_n^{**}\}_{n=0}^{\infty}$ gives a topology equivalent to the original family $\{\| \cdot \|_n\}_{n=0}^{\infty}$, and satisfies the right ideal inequality $\|ab\|_n^{**} \leq \|a\|_n^* \|b\|_0$.

Hence for any Banach algebra $B$ with right Fréchet ideal $A$, it is always possible to find an equivalent family of norms giving the topology on $A$, such that the new norms satisfy the right Fréchet ideal inequality (12) with $C_n = 1$ and $m_n = n$ for every $n \in \mathbb{N}$, and the zeroth norm $\| \cdot \|_0$, which is the norm $\| \cdot \|_B$ on $B$, remains unchanged.

The new norms are submultiplicative for every $n$: $\|a_1a_2\|_n^{**} \leq \|a_1\|_n^{**} \|a_2\|_0^{**} \leq \|a_1\|_n^{**} \|a_2\|_n^{**}$, $a_1, a_2 \in A$. Hence dense Fréchet ideals are always $m$-convex Fréchet algebras.

The same results hold for left ideals, by switching the order in the above arguments. In the left case, we define $\|a\|_n^\dagger = \sup\{ \|ba\|_n : \|b\|_0 \leq 1, b \in B \}$, and $\|a\|_n^{**} = \max\{\|a\|_n^\dagger, \|a\|_n\}$.

Finally, assume that $A$ is a two-sided Fréchet ideal in the Banach algebra $B$. As before, $A$ is topologized by increasing norms $\{\| \cdot \|_n\}_{n=0}^{\infty}$, with $\| \cdot \|_0 = \| \cdot \|_B$, and left
and right ideal inequalities are satisfied: \( \|ab\|_n \leq C_n \|a\|_m \|b\|_0 \), \( \|ab\|_m \leq C_m \|a\|_k \|b\|_0 \), and \( \|ba\|_n \leq C_n \|b\|_0 \|a\|_m \), \( \|ba\|_m \leq C_m \|b\|_0 \|a\|_k \), for all \( a \in A \) and \( b \in B \). Define a new family of seminorms by

\[
\|a\|_{n}^{\text{two}} = \sup \{ \|cab\|_n : \|c\|_0, \|b\|_0 \leq 1, c, b \in B \},
\]

for \( a \in A \). Using the right and left ideal inequalities we see that \( \|a\|_{n}^{\text{two}} \leq C_n C_m \|a\|_p \), so the topology given by \( \{\| \cdot \|_{n}^{\text{two}}\}_{n=0}^{\infty} \) is dominated by the original topology on \( A \). These new seminorms satisfy the right ideal inequalities with \( m_n = n \) and \( C_n = 1 \):

\[
\|ab\|_{n}^{\text{two}} = \sup \{ \|cab\|_n : \|c\|_0, \|b\|_0 \leq 1, c, b \in B \} \tag{32}
\]

by definition. \( \leq \sup \{ \|cab\|_n : \|c\|_0 \leq \|b\|_0, \|b\|_2 \leq \|b\|_0, c, b \in B \} \) \( b_2 = bb_1 \), so \( \|b\|_2 \leq \|b\|_0 \)

\[
= \|b\|_0 \sup \{ \|cab\|_n : \|c\|_0 \leq \|b\|_0, \|b\|_2 \leq \|b\|_0, c, b \in B \} \tag{33}
\]

by definition. \( b_3 = b_2/\|b\|_0 \)

Again we used the submultiplicativity of the norm \( \| \cdot \|_0 \) on \( B \) in the second step. Similarly, the left ideal inequality holds: \( \|ba\|_{n}^{\text{two}} \leq \|b\|_0 \|a\|_{n}^{\text{two}}, a \in A, b \in B \). Since the seminorms \( \{\| \cdot \|_{n}^{\text{two}}\}_{n=0}^{\infty} \) could give a weaker topology on \( A \) (as we discussed in the right ideal case above), we define our final set of norms on \( A \) by \( \|a\|_{n}^{\text{two}+} = \max\{\|a\|_{n}^{\text{two}+}, \|a\|_n, \|a\|_n, \|a\|_n\} \). Note that \( \| \cdot \|_{0}^{\text{two}+} = \| \cdot \|_0 = \| \cdot \|_B \). We have the four
inequalities:

\[ \|ab\|_{\text{two}}^n \leq \|a\|_{\text{two}}^n \|b\|_0 \]  
by (33)

\[ \|ab\|_n^\ast \leq \|a\|_n^\ast \|b\|_0 \]  
by (30)

\[ \|ab\|_n^\dagger = \sup\{\|cab\|_n | \|c\|_0 \leq 1, c \in B\} \]  
by definition of \(\| \cdot \|_n^\dagger\) above

\[ \leq \|a\|_{\text{two}}^n \|b\|_0 \]  
by definition (32) of \(\| \cdot \|_n^{\text{two}}\)

\[ \|ab\|_n \leq \|a\|_n^\ast \|b\|_0, \]  
by (31)

for \(a \in A\) and \(b \in B\). Putting these together shows that the norms \(\{\| \cdot \|_n^{\text{two}+}\}_{n=0}^{\infty}\) satisfy the right ideal inequality with \(C_n = 1, m_n = n\) for every \(n \in \mathbb{N}\). Similarly, they satisfy the left ideal inequalities with the same constraints.

We have shown that for any Banach algebra \(B\) with two-sided Fréchet ideal \(A\), it is always possible to find an equivalent family of norms giving the topology on \(A\), such that the new norms satisfy the right and left Fréchet ideal inequalities with \(C_n = 1\) and \(m_n = n\) for every \(n \in \mathbb{N}\), and the zeroth norm \(\| \cdot \|_0\), which is the norm \(\| \cdot \|_{\text{B}}\) on \(B\), remains unchanged.

\section*{B Appendix. Counterexamples}

\textbf{Example B.1. Standard Schwartz Functions Not an Ideal.} When the sequence of dimensions \(p = \{p_k\}_{k=1}^{\infty}\) doesn’t satisfy the growth condition, Proposition 6.7 tells us a dense nuclear Fréchet ideal \(S_\sigma(X)\) given by Theorem 6.2 is never standard Schwartz.
Assume the growth condition is violated, and cannot be repaired by reordering \( p \).

Then \( p \) cannot have a bounded subsequence by Remark 8.1 and so must be proper, and we can arrange \( p_k \leq p_{k+1}, \ k \in \mathbb{N}^+ \). Since the ordering will be fixed, we identify \( Z = \mathbb{N}^+ \) throughout this example. Let \( \gamma(k, i, j) = p_1^2 + \cdots + p_{k-1}^2 + (i - 1) + (j - 1)p_k + 1 \), \( \{k, i, j\} \in X \) be as in the proof of Proposition 6.7. We saw that \( S_\gamma(X) \) is standard Schwartz. We show directly that \( S_\gamma(X) \) is not an ideal in \( B \). Define a scale \( \beta \) on \( X \) by

\[
\beta(k, i, j) = ikp_{k-1} + (j - 1)p_k.
\]

Then \( \beta \sim \gamma \), and our calculations will simplify using \( \beta \) in place of \( \gamma \). For each \( k \in \mathbb{N}^+ \), let \( c_1(k) \) be 1’s in the first column:

\[
c_1(k) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix} \in M_{p_k}(\mathbb{C}).
\]

Then \( \|c_1(k)\|_{op} = \sqrt{p_k} \), so for each \( K \in \mathbb{N}^+ \),

\[
S_K = \sum_{k=1}^{K} \frac{c_1(k)}{\sqrt{p_k}} \in c_f(X)
\]

satisfies \( \|S_K\|_B = 1 \). The \( n \)th norm in \( S_{\frac{1}{2}}(X) \) is

\[
\|S_K\|_n = \sum_{k=1}^{K} \sum_{i=1}^{p_k} \frac{\beta(k, i, 1)^n}{\sqrt{p_k}}
\]

\[
= \sum_{k=1}^{K} \frac{k^n p_{k-1}^{n-1} \sum_{i=1}^{p_k} i^n}{\sqrt{p_k}}
\]

\[
\geq \sum_{k=1}^{K} \frac{k^n p_{k-1}^{n-1} p_k^n}{\sqrt{p_k}} \geq p_K^{n-1/2}.
\]

And for each \( K \in \mathbb{N}^+ \),

\[
T_K = \sum_{k=1}^{K} e_{k,11} \in c_f(X)
\]

49
is an element of $S_\beta(X)$ with $m$th norm equal to

$$\|T_K\|_m = \sum_{k=1}^{K} \beta(k,1,1)^m$$

$$= \sum_{k=1}^{K} k^m p_{K-1}^m \leq K^{m+1} p_{K-1}^m.$$  

Since $c_1(k)e_{k,11} = c_1(k)$, $S_K T_K = S_K$. To see that the left ideal condition is violated, note that

$$\frac{\|S_K T_K\|_n}{\|S_K\|_B\|T_K\|_m} = \frac{\|S_K\|_n}{\|T_K\|_m} \geq \frac{p_{K}^{n-1/2}}{K^{m+1} p_{K-1}^m}$$

$$\geq \frac{p_{K}^{n-1/2}}{2^{m+1}(p_1 + \cdots p_{K-1})^{1+m} p_{K-1}^m},$$  

(34)

since $K - 1 \leq p_1 + \cdots p_{K-1}$ and $K \leq 2(K - 1)$ for $K \geq 2$. If for some $n \geq 1$, there were an $m > n$ for which the right hand side of (34) were bounded for all $K \in \mathbb{N}^+$, then the growth condition on the sequence $p = \{p_k\}_{k=1}^\infty$ would have to hold.

**Example B.2. Banach Algebra Not an Ideal in $c_0(X)$.** When the standard basis $\{\delta_k\}_{k=0}^\infty$ is an absolute basis for a dense Fréchet subspace $F$ of the commutative $C^*$-algebra $c_0(\mathbb{N})$, then $F$ is isomorphic to $S_\sigma^1(\mathbb{N})$ (Theorem 2.7), and is easily seen to be an ideal in $c_0(\mathbb{N})$ by the same argument as Example 3.3 (a). We exhibit a dense Banach subalgebra $A$ of $c_0(\mathbb{N})$, for which $\{\delta_k\}_{k=0}^\infty$ is not an absolute basis, and such that $A$ is not an ideal in $c_0(\mathbb{N})$.

Let $\sigma$ be a proper scale on $\mathbb{N}$. Define a norm by

$$\|f\|_{\sigma,1} = \sup_{k \in \mathbb{N}} \left( \sigma(k) \max \{ |f_+(k)|, \sigma(k) |f_-(k)| \} \right),$$  

(35)
where \( f_+(k) = f(2k) + f(2k + 1) \), \( f_-(k) = f(2k) - f(2k + 1) \), and \( f \in c_f(\mathbb{N}) \). Define a new scale \( \beta \), which is “half of \( \sigma \)” by \( \beta(2k) = \beta(2k + 1) = \sigma(k) \). Define two related norms
\[
\|f\|_\beta = \|\beta f\|_\infty \quad \text{and} \quad \|f\|_\beta^2 = \|\beta^2 f\|_\infty,
\]
which topologize the Banach algebras \( c_0(\mathbb{N}, \beta) \) and \( c_0(\mathbb{N}, \beta^2) \), respectively. Let \( A \) be the completion of \( c_f(\mathbb{N}) \) in the norm \( \| \cdot \|_{\sigma, 1} \). Then \( c_f(\mathbb{N}) \subseteq c_0(\mathbb{N}, \beta^2) \hookrightarrow A \hookrightarrow c_0(\mathbb{N}, \beta) \hookrightarrow c_0(\mathbb{N}) \), where the inclusions \( \hookrightarrow \) are continuous.

Note that \( \|\delta_{2k}\|_{\sigma, 1} = \|\delta_{2k}\|_{\beta^2} = \sigma(k)^2 \) and \( \|\delta_{2k+1}\|_{\sigma, 1} = \|\delta_{2k+1}\|_{\beta^2} = \sigma(k)^2 \). But \( \|\delta_{+, k}\|_{\sigma, 1} = \|\delta_{+, k}\|_\beta = 2\sigma(k) \) and \( \|\delta_{-, k}\|_{\sigma, 1} = \|\delta_{-, k}\|_{\beta^2} = 2\sigma(k)^2 \), where \( \delta_{+, k} \) and \( \delta_{-, k} \) are defined analogously to \( f_+ \) and \( f_- \). Then
\[
\frac{\|\delta_{+, k} * \delta_{-, k}\|_{\sigma, 1}}{\|\delta_{+, k}\|_{\sigma, 1} \|\delta_{-, k}\|_\infty} = \frac{\|\delta_{-, k}\|_{\sigma, 1}}{\|\delta_{+, k}\|_{\sigma, 1} \|\delta_{-, k}\|_\infty} = \frac{2\sigma(k)^2}{2\sigma(k) * 1} = \sigma(k),
\]
which tends to \( \infty \) as \( k \to \infty \), since \( \sigma \) was assumed proper. So \( A \) is not an ideal in \( c_0(\mathbb{N}) \).

**Example B.3. Nilpotent Banach Algebra With No Dense Nuclear Ideal.** We exhibit a separable nilpotent Banach algebra \( B \) of order 2 (\( B^3 = \{0\} \)) with no dense nuclear Fréchet ideal. Let \( B \) be the Hilbert space direct sum of two Hilbert spaces \( B = \mathcal{H}_1 \oplus \mathcal{H}_2 \), with respective bases \( \{\alpha_k\}_{k=0}^{\infty} \), \( \{\beta_k\}_{k=0}^{\infty} \). Define multiplication by \( \alpha_0 \alpha_i = \beta_i \), and all other products zero: \( \alpha_{i+1} \alpha_j = 0, \beta_i \beta_j = 0, \alpha_i \beta_j = \beta_j \alpha_i = 0 \). Let \( b_1 = \xi_1 + \eta_1 \) and \( b_2 = \xi_2 + \eta_2 \) be arbitrary elements of \( B \). The Hilbert space norm on \( B \) is submultiplicative since
\[
\|b_1 b_2\|_B = \|\xi_1(0) \xi_2\|_{\mathcal{H}_2} = |\xi_1(0)| \|\xi_2\|_{\mathcal{H}_2} \leq \|b_1\|_B \|b_2\|_B,
\]
so \( B \) is a Banach algebra. Clearly, \( B \) is nilpotent of order 2. Let \( A \) be a dense right Fréchet ideal in \( B \). By density, we can’t have \( A \subseteq \langle \alpha_0 \rangle^\perp \). So let \( a_0 \in A \) have some component
of $\alpha_0$ in the Hilbert space $B$. Then $\mathcal{H}_2 \subset A$, since $a_0B = \mathcal{H}_2$. Rescale $a_0$ to arrange that $<a_0,\alpha_0> = 1$. If $\eta \in \mathcal{H}_2$, then $\eta = a_0\eta$ and $\|\eta\|_n = \|a_0\eta\|_n \leq C_n\|a_0\|_m\|\eta\|_B = D_n\|\eta\|_B$. So the topology on $\mathcal{H}_2$ inherited from $A$ is precisely the Hilbert space topology, and $A$ cannot be nuclear (Proposition 2.1 (c)).

Example B.4. Dense Nuclear Fréchet Subalgebra of $c_0(X)$ Not an Ideal. Let $\mathcal{A}$ be the Fréchet algebra with underlying space $S(\mathbb{N})$ and multiplication $f \ast g(r) = \sum_{s=0}^{r} f(s)g(r-s)$, $r \in \mathbb{N}$, for $f, g \in \mathcal{A}$. Then $\|f \ast g\|_d \leq C_d\|f\|_d\|g\|_d$, and $\mathcal{A}$ is a commutative $m$-convex Fréchet algebra, with unit $\delta_0$. For any $t \in \mathbb{N}$, define the closed ideal $\mathcal{A}_t = \{f \in \mathcal{A} | f(0) = f(1) = \cdots f(t) = 0\}$ of $\mathcal{A}$.

Let $X$ be a countably infinite set, and $\chi \in c_0(X)$ have range in $(0, 1)$. Define a linear map $\theta_\chi$ from $\mathcal{A}_0$ to $c_0(X)$ by

$$\theta_\chi(f)(x) = \sum_{r=1}^{\infty} f(r)^r \chi(x)^r,$$

$f \in \mathcal{A}_0$, $x \in X$. Let $\epsilon > 0$ and find finite $S \subset X$ large enough so that $|\chi(x)| < \epsilon$ if $x \in X - S$. Then $|\theta_\chi(f)(x)| \leq \epsilon\|f\|_1$, for $x \in X - S$. This proves $\theta_\chi(f) \in c_0(X)$. It is easy to see that $\|\theta_\chi(f)\|_\infty \leq \|f\|_1$, so $\theta_\chi$ is continuous. For $f, g \in \mathcal{A}_0$ we have

$$(\theta_\chi(f) \ast \theta_\chi(g))(x) = \sum_{r,s=1}^{\infty} f(r)g(s)\chi(x)^{r+s}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{r} f(s)g(r-s)\chi(x)^r$$

$$= \sum_{r=1}^{\infty} f \ast g(r)\chi(x)^r = \theta_\chi(f \ast g)(x),$$

$x \in X$. So $\theta_\chi$ is an algebra homomorphism.
Assume that $\chi$ has infinitely many values in its range. We show that $\theta_\chi$ is injective. Let $f \in A_0$, and note the power series $p(z) = \sum_{r=1}^{\infty} f(r)z^r$ converges for $|z| < 1$. If $\theta_\chi(f) = 0$, then $p(z)$ is zero on the range of $\chi$. Since $\chi$ is in $c_0(X)$ and has values in $(0, 1)$, zero is an accumulation point. By analyticity, $p(z)$ is identically zero, so $f \equiv 0$.

Moreover, the assumption implies $\theta_\chi(A_0) \cap c_f(X) = 0$. If $\theta_\chi(f)$ takes on only finitely many values $w_1, \ldots w_k$, then the power series $(p(z) - w_1) \cdots (p(z) - w_k)$ is zero for $z$ in the range of $\chi$, and therefore identically zero, by the argument of the previous paragraph. Since $p(z)$ has no constant term, the product $w_1 \ldots w_k$ is zero, so $w_i = 0$ for some $i$. If $p(z)$ is not identically zero, we may divide by it to get $(p(z) - w_1) \cdots (p(z) - w_{i-1})(p(z) - w_{i+1}) \cdots (p(z) - w_k) \equiv 0$, and we are in the same situation as before. Repeating the argument $k$ times shows $p(z) - w_j \equiv 0$ for some $j \in \{1, \ldots k\}$. Finally $w_j = 0$, contradicting our hypothesis that $p(z) \neq 0$. Hence every non-zero $\theta_\chi(f)$ takes on infinitely many values, and cannot lie in $c_f(X)$.

Assume in addition that $\chi(x) \neq \chi(y)$ if $x$ and $y$ are distinct elements of $X$. Then $\theta_\chi(A_t)$ is dense in $c_0(X)$ for every $t \geq 0$. Let $\xi$ be an element of the dual $\ell^1(X) = c_0(X)'$, and assume $\xi$ vanishes on $\theta_\chi(A_t)$. Let $s \in \mathbb{N}$ be greater than $t$. Taking $f = \delta_s \in A_t$, we see that $\xi(\chi^s) = 0$, so

$$0 = \sum_{x \in X} \xi(x)\chi(x)^s. \tag{39}$$

Find an ordering $\{x_i\}_{i=0}^{\infty}$ of $X$ for which $\chi(x_0) > \chi(x_1) > \chi(x_2) > \cdots$. Divide (39) by $\chi(x_0)^s$ and let $s \to \infty$ to see that $\xi(x_0) = 0$. Then divide (39) by $\chi(x_1)^s$ to see $\xi(x_1) = 0$, and continue this way through all the $x_i$’s to see that $\xi(x_i) = 0$ for every $i$. We have
proved that no non-zero element of $c_0(X)'$ can vanish on $\theta_\chi(A_t)$, which proves the density.

We have shown that $A_\chi = \theta_\chi(A_0)$ is a dense nuclear subalgebra of $c_0(X)$. Since $A_\chi$ does not contain $c_f(X)$, it cannot be an ideal in $c_0(X)$. Note that $A_\chi$ is not spectral invariant in $c_0(X)$. For example $\|\chi\|_\infty < 1$, so the geometric series for the quasi-inverse $-\chi/(1 + \chi)$ converges in $c_0(X)$, but not in $A_\chi$. Alternately, for any $z_0 \in \mathbb{C}$, $|z_0| < 1$, $f \mapsto \sum_{r=0}^{\infty} f(r) z_0^r$ defines a 1-dimensional representation of $A_0$, and through $\theta_\chi$ a 1-dimensional representation of $A_\chi$. But only those with $z_0$ in the range of $\chi$ can extend to a representation of $c_0(X)$.

Example B.5. Dense Nuclear Fréchet Subalgebra of $c_0(X)$ Not an Ideal and Containing $c_f(X)$. Let $\sigma$ be a family of scales on $X$ satisfying summability (\ref{eq:summability}). Then $A_\sigma = S_\sigma(X)$ is a dense nuclear two-sided Fréchet ideal in the pointwise multiplication algebra $c_0(X)$, by Theorem 6.2.

Let $\chi: X \to (0, 1)$ satisfy the criteria of the previous Example B.4. Arrange that no power of $\chi$ is in $S_\sigma(X)$. For example, we could take $\chi(x) = 1/\sigma_d(x)$ for some $d \in \mathbb{N}$, taking care to make sure $\chi(x) \neq \chi(y)$ for distinct $x, y \in X$. \footnote{\(\chi(k) = 1/(1 + k)\) works for the family $\sigma_n(k) = (1 + k)^n$.} Let $A_\chi$ be the associated Fréchet subalgebra of $c_0(X)$ from the previous Example B.4.

We show $A_\chi \cap S_\sigma(X) = 0$. Let $f \in A_0$ and let $p \in \mathbb{N}^+$ be smallest such that $f(p) \neq 0$. Let $d \in \mathbb{N}$ be such that $\sigma_d(x)\chi(x)^p$ is unbounded for $x \in X$. Let $S \subset X$ be a large
enough finite set so that \( \chi(x)\|f\|_1 < |f(p)|/2 \) for \( x \in X - S \). Then

\[
\sigma_d(x)\theta_\chi(f)(x) = \sigma_d(x)\chi(x)p|f(p)| + \sum_{q=1}^\infty f(q+p)\chi(x)^q|
\]

\[
\geq \sigma_d(x)\chi(x)p\left(|f(p)| - \left|\sum_{q=1}^\infty f(q+p)\chi(x)^q\right|\right)
\]

\[
\geq \sigma_d(x)\chi(x)p\left(|f(p)| - \|f\|_1\chi(x)\right)
\]

\[
\geq \sigma_d(x)\chi(x)p|f(p)|/2,
\]

for \( x \in X - S \). Hence by the unboundedness of \( \sigma_d\chi, \theta_\chi(f) \) is not in \( S_\sigma(X) \).

The direct sum \( A_{\text{sum}} = A_\chi \oplus A_\sigma \) is naturally a Fréchet algebra, nuclear and dense in \( c_0(X) \). It contains \( c_f(X) \) as an ideal since \( A_\sigma \) does, but not densely. In Example B.4, we noted the quasi-inverse \( \psi = -\chi/(1 - \chi) \) exists in \( c_0(X) \), but not in \( A_\chi \). If \( \psi \in A_\sigma \), then \( \chi\psi \in A_\sigma \) and \( \chi \in A_\sigma \), a contradiction. So \( \psi \notin A_{\text{sum}} \), and \( A_{\text{sum}} \) is not spectral invariant, and not an ideal, in \( c_0(X) \).

**Example B.6.** If \( \tau \precsim \sigma \), then \( \sigma \) may not satisfy the summability condition even if \( \tau \) does. For example let \( X = \mathbb{N} \), \( \tau(k) = 1 + k \), and \( \sigma_n(k) = e^k, n \in \mathbb{N} \).

**Example B.7. Standard Scales Are Not Minimal.** If \( \gamma_1 \) is an enumeration of \( X \), one can find another enumeration \( \gamma_2 \) such that \( \gamma_2 \precsim \gamma_1 \) (in fact \( \gamma_2 \leq \gamma_1 + 1 \)) but \( \gamma_1 \nprecsim \gamma_2 \). Let \( \gamma_1 \) be the identity enumeration of \( X = \mathbb{N}^+ \). Let \( \gamma_2 \) be defined by \( \gamma_2(k) = k + 1 \) for every \( k \) except \( \gamma_2(1) = 1 \) and on the set \( \{ [e^i] | i \in \mathbb{N}^+ \} \). Set \( \gamma_2([e^{(i+1)(i+1)}]) = [e^i] + 1 \),
To be clear, \( \gamma_2(1) = 1, \gamma_2(2) = 3, \gamma_2(3) = 2, \gamma_2(4) = 5, \gamma_2(5) = 6, \ldots \gamma_2(55 = [e^4]) = 4, \gamma_2(56) = 57, \ldots \gamma_2([e^{27}]) = 4, \gamma_2([e^{27}] + 1) = [e^{27}] + 2, \ldots \). The function \( e^{(i+1)^{(i+1)}} \) is not bounded by \( Ce^{di} \) for \( i \in \mathbb{N}^+ \), for any fixed power \( d \).

C  Index

\( \mathbb{N}, \mathbb{N}^+ \) ................................................. Introduction

Basis for a Fréchet space, coordinate functional ........................... Definition 2.5

Schauder, unconditional, equicontinuous, absolute basis ................. Definition 2.5

\( X \) ...................................................... Introduction and Definition 2.6

Basic Schwartz Spaces \( \mathcal{S}_\sigma^1(X), \mathcal{S}_\sigma^\infty(X) \), sup-norm and \( \ell^1 \)-norm ........ Definition 2.6

Finite support functions \( c_f(X) \) ............................................... Definition 2.6

Scale, family of scales .............................................. Definition 2.6

Family of scales given by a single scale ................................. Definition 2.6

Dominate \( \lesssim \) and equivalence \( \sim \) of scales ...................... Definition 2.6

Summability Condition, \( \sigma \) is summable ................................. Equation (6)

\( \delta_k, \varphi_k, (\varphi_1, \varphi_2, \varphi_3, \ldots) \) .................................. Definition 2.8

Enumeration, enumeration of \( X \) .................................. Introduction and Definition 2.8

Standard scale, \( \vec{\gamma} \)-standard family of scales ....................... Definition 2.8

\( \gamma\sigma, \partial p \) .................................................. Definition 2.8

\( x = \{z, i, j\} \) .................................................. Introduction to §6

\(^9\)Here set \( 0^0 = 0 \), so \( e^{0^0} = 1 \), and \( [\cdot] \) is the least integer greater than.
standard Schwartz ........................................................... Definition 6.6

D References

[Dixmier, 1982] J. Dixmier, $C^*$-algebras, North-Holland Publishing Co., Amsterdam/New York/Oxford, 1982.

[Fell Dor, 1988] J. M. G. Fell and R. S. Doran, Representations of $^*$-Algebras, Locally Compact Groups, and Banach $^*$-Algebraic Bundles, Volume I Basic Representation Theory of Groups and Algebras, Pure and Applied Mathematics 125, Academic Press, Boston, MA, 1988.

[Husain, 1991] T. Husain, Orthogonal Schauder Bases, Pure and Applied Mathematics, Volume 143, Marcel Dekker, Inc., New York, 1991.

[Kad Ring II, 1997] R. V. Kadison and J. R. Ringrose, Fundamentals of the Theory of Operator Algebras, Volume II: Advanced Theory, Graduate Studies in Mathematics, Volume 16, American Mathematical Society, 1997.

[Kaplansky, 1948] I. Kaplansky, Dual rings, Ann. of Math. 49(3) (1948), 689-701.

[Kaplansky, 1949] I. Kaplansky, Normed algebras, Duke Math. J. 16(3) (1949), 399-418.

[Mars Hoff, 1999] J.E. Marsden and M.J. Hoffman, Basic Complex Analysis, W.H. Freeman, New York, 1999.

[Palmer, 1994] T. W. Palmer, Banach Algebras and the General Theory of $^*$-algebras, Volume I: Algebras and Banach Algebras, Encyclopedia of Mathematics and its Applications, Volume 49, Cambridge University Press, New York, 1994.

[Paterson, 1988] A.L.T. Paterson, Amenability, Mathematical Surveys and Monographs, Number 29, American Mathematical Society, Providence, Rhode Island, 1988.

[Pietsch, 1972] A. Pietsch, Nuclear Locally Convex Spaces, Ergebnisse Der Mathematik und Ihrer Grenzgebiete, Volume 66, Springer-Verlag, New York/Heidelberg/Berlin, 1972.
[Rudin, 1973] W. Rudin, *Functional Analysis*, Series in Higher Mathematics, McGraw-Hill, Inc., New York, 1973.

[Sch, 1998] L. B. Schweitzer, *$C^\infty$ functions on the Cantor set, and a smooth $m$-convex Fréchet subalgebra of $O_2$*, Pac. J. Math. **184**(2) (1998), 349-365.

[Smyth, 1980] M. R. F. Smyth, *On problems of Olubummo and Alexander*, Proc. R. Ir. Acad. **80A**(1) (1980), 69-74.

[Sug, 1971] M. Sugiura, *Fourier series of smooth functions on compact Lie groups*, Osaka J. Math. **8** (1971), 33-47.

[Treves, 1967] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, Inc, San Diego, California, 1967.

Address: Larry B. Schweitzer, 600 16th Street Box 2240, University of California, San Francisco, CA 94158-2517. lsch@svpal.org.