GENERALIZED LIE-BÄCKLUND THEOREM FOR LIE CLASS $\omega = 1$ OVERDETERMINED SYSTEMS

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Abstract. In this paper we prove a version of Lie-Bäcklund theorem for overdetermined systems of scalar PDEs, whose general solution depends on 1 function of 1 variable. This generalizes the case of involutive system of the second order on the plane treated by E.Cartan in 1910. The approach is based on the geometric theory of PDEs and Tanaka theory. Many examples are provided.

INTRODUCTION

Consider the space $J^k\pi$ of $k$-jets of sections of a vector bundle $\pi : E \to M$. This manifold carries the canonical Cartan distribution $C_k$ (also known as higher contact vector distribution). Diffeomorphisms of $J^k\pi$ preserving the Cartan distributions are called Lie transformations.

The classical Lie-Bäcklund theorem states that a Lie transformation $F : J^k\pi \to J^k\pi$ is the prolongation of a Lie transformation $f$ of $J^1\pi$: $F = f^{(k-\epsilon)}$, where $\epsilon = 1$ and $f$ is a contact transformation of $J^1\pi$ in the case $\text{rank } \pi = 1$, while $\epsilon = 0$ and $f$ is a diffeomorphism of $J^0\pi$ (point transformation) for $\text{rank } \pi > 1$.

Various generalizations of this to Lie transformations of differential equations $E \subset J^k\pi$ have been discovered since, and they became known as Lie-Bäcklund (type) theorems. Namely an internal transformation is a symmetry of the induced distribution $C_E = C \cap TE$ on $E$. Lie-Bäcklund theorem holds for $E$ if the internal symmetries coincide with the external symmetries, which are (restrictions of) those Lie transformations of the ambient jet-space that preserve $E$.

An instance of such theorem for scalar second order equations was proved in [4] (parabolic equation) and [7] (hyperbolic and elliptic PDE). For many classes of equations ($C$-general, normal) this phenomenon was established in [14, 2].

But Lie-Bäcklund theorem does not always hold. It is easily seen to fail for a single scalar PDE of 1st order $F(x^1, \ldots, x^n, u, u_1, \ldots, u_n) = 0$

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for which the internal symmetry group is infinite-dimensional. Another important counter-example is the Hilbert-Cartan equation $y' = (z'')^2$, for which the external symmetry group is 9-dimensional\footnote{This is the parabolic subgroup $P_2 \subset G_2$ giving the contact gradation.} vs. 14-dimensional internal symmetry group, though the latter coincides with the space of generalized Lie-Bäcklund symmetries \[2, 8\].

More general Monge equations $y^{(m)} = F(x, y, \ldots, y^{(m-1)}, z, \ldots, z^{(n)})$ were considered in \[1\]. If $m = 1, n > 2$ or $m > 1$ then Lie-Bäcklund theorem holds provided the equation is non-degenerate (this condition meaning that $F$ is not affine in $z^{(n)}$ cannot be dropped) and the external symmetries concern the ambient mixed jet-space $J^{m,n}(\mathbb{R}, \mathbb{R} \times \mathbb{R})$.

The Monge equations are included into a more general class of Monge systems (systems of ODEs with the minimal degree under-determinacy), which are the natural target of reduction for so-called Lie class $\omega = 1$ compatible overdetermined PDE systems. These are the systems $E$ whose general solution depends on 1 function of 1 variable.

Lie class $\omega = 1$ PDE systems are integrable by ODE methods \[15\] \[11\], and they often arise as symmetry reductions of more complicated PDEs (a Darboux integrable or semi-integrable equation coupled with one intermediate integral is a particular case of class $\omega = 1$ system). Lie-Bäcklund theorem clearly fails for Lie class $\omega = 1$ overdetermined PDE systems, since the internal symmetry group is always infinite-dimensional while the external group is usually not.

An important example of such $E$ constitute involutive overdetermined systems of 2nd order scalar PDEs on the plane considered by E.Cartan (“le théorème imporant” \[4\] §26). He indicated a generalization of Lie-Bäcklund theorem to this case, namely the internal group is changed to the symmetry group of the reduction $(M, \Delta)$ of $(E, C_E)$ by the Cauchy characteristic $\Pi \subset C_E$ and then it is bijective with the external symmetry group.

This was generalized to involutive 2nd order PDE systems in $n$-dimensions in \[18\] \[19\]. The purpose of this paper is to prove

**Theorem 1.** Let $E$ be a Lie class $\omega = 1$ overdetermined compatible system of PDEs of orders greater than 1. If $E$ is sufficiently non-degenerate, then the symmetry algebra of the reduction $(M, \Delta)$ is equal to the Lie algebra of external symmetries of $E$.

The requirement of sufficient non-degeneracy is technical and will be formulated, after we introduce some preliminary material, in Section 4. Then in Section 5 we will prove the main result.

We will apply it to calculate the group of contact transformations of some model overdetermined systems of PDEs. To find the complete
symmetry group we will elaborate upon the Tanaka theory of symmetries of vector distributions.

We shall also discuss limitations of the theorem, by showing examples of degenerate systems when the external and internal symmetries are different. Here is the version of Theorem 1 for two different equations (by micro-local we mean in a neighborhood of a point on the equation).

**Theorem 2.** Two Lie class $\omega = 1$ overdetermined compatible sufficiently non-degenerate systems ${\mathcal E}, {\mathcal E}'$ of orders greater than 1 are micro-locally equivalent if and only if their reductions $(M, \Delta), (M', \Delta')$ are.

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1. **Symmetries of differential equations**

A differential equation ${\mathcal E}$ is often treated geometrically as a submanifold in the space of jets $J^k(B, F)$, where $B$ is the space of independent variables $x = (x^1, \ldots, x^n)$ and $F$ is the space of dependent variables $u = (u^1, \ldots, u^m)$, which we write as $u = u(x)$. A choice of these coordinates on $J^0 = B \times F$ yields canonical coordinates on the space $J^k$: $(u^j_\sigma)$, where multiindex $\sigma = (i_1, \ldots, i_n)$ has length $|\sigma| = \sum_{a=1}^n i_a \leq k$.

We will assume regularity, i.e. that the projection maps $\pi_{k,l} : J^k \to J^l$ have constant ranks when restricted to ${\mathcal E}$ and its prolongations. Usually it is assumed that the equation has pure order $k$ and then $\pi_{k,k-1} : {\mathcal E} \to J^{k-1}$ is surjective. But we will allow PDEs in the system ${\mathcal E}$ to have different orders (of which $k$ is the maximal). For the setup of this theory we refer to [13].

The Cartan distribution $C_k$ on $J^k$ is given as the kernel of the forms

$$\theta^j_\sigma = du^j_{\sigma} - \sum_{i=1}^n u^j_{\sigma+1,i} dx^i$$

for all $j \in \{1, \ldots, m\}$ and $|\sigma| < k$. It is generated as follows:

$$C_k = \left\langle D_x^i = \partial_x^i + \sum_{j:\, |\tau| < k} u^j_{\tau+1,i} \partial_{u^j_{\tau}}, \partial_{u^i_{\sigma}} : 1 \leq i \leq n, |\sigma| = k \right\rangle$$

Cartan distribution $C_l$ on $J^l$ can be lifted to $J^k$ via $\pi_{k,l}^{-1}$. In particular $C_1$ is the classical contact distribution.

These distributions for different $l$ are related by the formula\footnote{For brevity sake we will write $[\nabla, \Delta]$ instead of more appropriate ”distribution whose module of sections equals $[\Gamma(\nabla), \Gamma(\Delta)]”$ (regularity assumed). Notice that we always have $[\nabla, \Delta] \supset \nabla + \Delta$.}

$$[C_l, C_l] = C_{l-1}.$$
The classical Lie-Bäcklund theorem claims that a symmetry of distribution \( C_k \) on \( J^k \) is necessarily the lift of a contact transformation from \( J^1 \) if \( m = 1 \) or a point transformation of \( J^0 \) if \( m > 1 \). The proof follows almost immediately from the above display formula.

A symmetry of \( (J^k, C_k) \), preserving \( E \), is called an external symmetry of the system. The Lie algebra of all external symmetries is denoted by \( \text{Sym}(E) \). The internal symmetries \( \text{Sym}_{\text{int}}(E) \) are by definition the symmetries of the induced distribution \( C_E = C_k \cap T_E \) on \( E \).

Restriction obviously gives the homomorphism \( \text{Sym}(E) \to \text{Sym}_{\text{int}}(E) \). Generalized Lie-Bäcklund theorems address the occasions when it is an isomorphism.

In what follows we will consider an overdetermined system of PDEs \( E \). It will be assumed formally integrable, meaning that for all prolongations \( E_{k+t} = E^{(t)} \) (given as the locus in \( J^{k+t} \) of defining relations for \( E \) differentiated \( \leq t \) times) the projections \( \pi_{k+t,k+t-1} : E_{k+t} \to E_{k+t-1} \) are affine bundles. In other words, all the compatibility conditions vanish.

For technical reasons we will need to prolong to the level \( s = k + t \), when \( E_s \) is involutive. This means vanishing of the Spencer cohomology \( H^{i,j}(E) = 0 \), \( \forall i \geq s, j \geq 0 \), or equivalently fulfilment of the Cartan test [5, 14]. Then for \( i \geq s \) the symbol

\[
g_i = \ker(d\pi_{i,i-1} : TE_i \to TE_{i-1})
\]

has dimension growth in accordance with Hilbert polynomial [13]. If \( x_i \in E_i \) and \( \pi_{i,0}(x_i) = (x, y) \), we can identify \( g_i(x_i) \subset S^iT_xB \otimes T_yF \).

2. Lie class \( \omega = 1 \) compatible systems

The (complex) characteristic variety can be defined as projectivization of the set of complex characteristic covectors

\[
\text{Char}^C(x_i) = \mathbb{P}\{p \in T_x^*B \otimes \mathbb{C} : p^k \otimes T_yF \cap g^C_k \neq 0\}.
\]

If \( \text{Char}^C(E) = \emptyset \) then the system is of finite type, i.e. its solution space is finite-dimensional. The next by complication case is when \( \text{Char}^C(E) \) consists of one point counted with multiplicity (then the characteristic variety must be real). These are the systems of Lie class \( \omega = 1 \) and their solutions are parametrized 1 function of 1 argument[3].

Another way to describe such systems is the following. For large \( i \) (precisely starting from the level when \( E_i \) becomes involutive) the symbol of the system stabilizes: \( \dim g_i = 1 \). We refer to [10] for a detailed discussion of these systems.

By a theorem of S. Lie compatible PDE systems of class \( \omega = 1 \) are integrable via ODEs [15]. The proof given in [11] uses the following

\[3\]In terminology of Elie Cartan \( \omega \) is the Cartan integer \( s_1 \) (provided the Cartan character is 1: \( s_2 = 0 \)).
observation: For such systems $\mathcal{E}$, starting from the involutivity jet-level, the Cartan distribution $\mathcal{C}_\mathcal{E}$ has rank equal to $(n+1)$ and it contains the subspace $\Pi$ of Cauchy characteristics of rank $(n-1)$. Since the prolongation, required to achieve involutivity, changes neither the class $\omega$ nor the external symmetry group, we will assume that already $\mathcal{E}$ is involutive (this does not restrict generality of the results).

The (local) quotient by the foliation tangent to $\Pi$ maps $(\mathcal{E}, \mathcal{C}_\mathcal{E})$ to a manifold with rank 2 distribution. Let us call this pair $(M, \Delta)$ the reduction of our system.

Shifts along Cauchy characteristic direction are trivial symmetries of $\mathcal{E}$, so the space of internal symmetries is always infinite (while this for external is usually not). To compensate this we consider the symmetry algebra of the reduction $\text{Sym}(M, \Delta)$ (all our considerations are local). Since the subalgebra of trivial symmetries is an ideal in $\text{Sym}_{\text{int}}(\mathcal{E})$, we have the homomorphism induced by the restriction

$$\text{Sym}(\mathcal{E}) \to \text{Sym}(M, \Delta).$$

The goal of the paper is to demonstrate that this is an isomorphism (under certain "general position" assumptions).

For simplicity of the exposition we restrict in the next section to the case of scalar PDEs ($m=1$) on the plane ($n=2$). We will denote a system of differential equations of orders $k_1, \ldots, k_r$ by $\sum E_{k_i}$.

A zoo of such systems is given in [10]. In this reference and [11] it is shown that internal geometry of linearizable systems is quite simple: their reduction correspond to Goursat distributions.

Recall that for a distribution $\Delta$ the weak derived flag is defined by $\{\Delta_1 = \Delta, \Delta_{i+1} = [\Delta, \Delta_i]\}$, and the strong derived flag is given by the formula $\{\nabla_1 = \Delta, \nabla_{i+1} = [\nabla_i, \nabla_i]\}$. The sequences $\{\dim(\Delta_{i+1}/\Delta_i)\}$ and $\{\dim(\nabla_{i+1}/\nabla_i)\}$ will be called weak resp. strong growth vectors.

Goursat distributions have both weak and strong growth vectors $(2, 1, 1, \ldots, 1)$ and are isomorphic to the Cartan distribution on the jet space $J^k(R, R)$. Thus the internal geometry of the linear class $\omega = 1$ overdetermined compatible systems is trivial.

On the other hand the external geometry (which is governed by the pseudogroup of point triangular transformations) is rich and is characterized by differential invariants.

An interesting example of systems of type $E_2 + E_3$ can be found already in Cartan ([4], p.147). Modifying his PDEs a bit we get:

$$\mathcal{E} : \{u_{xy} = 0, u_{yyy} = 0\}, \quad \bar{\mathcal{E}} : \{u_{yy} = 0, u_{xxy} = 0\}.$$

Internally $\mathcal{E} \simeq \bar{\mathcal{E}}$ since the growth vectors of both reductions are $(2, 1, 1, 1)$, but the systems are not equivalent externally because the second order equations are hyperbolic and parabolic respectively. Thus
generalized Lie-Bäcklund theorem fails in this case. By \cite{4} such situation is not possible for systems of type $2E_2$.

3. Two examples of class $\omega = 1$

Let us study some partial cases with $n = 2$ independent variables. We will use the classical notations $p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}, \alpha = u_{xxx}, \beta = u_{xxy}, \gamma = u_{xyy}, \delta = u_{yyy}$.

Consider at first the system $E_2 + E_3$. Without loss of generality we can assume that the characteristic\footnote{This vector belongs to the annihilator of the characteristic covector $dx + A dy$.} is $\partial_y - A \partial_x$, where $A$ is a function on 2-jets (more precisely on equation $E_2$). Thus the system $\mathcal{E}$ writes

$$t = F(r, s, \ldots), \quad \beta = A(r, s, \ldots)\alpha + B(r, s, \ldots),$$

where dots mean terms of order 1 (and $A$ satisfies $A^2 = F_s A + F_r$). The Cauchy characteristic vector field for the distribution $\mathcal{C}_\xi$ is $\xi = \mathcal{D}_y - A \mathcal{D}_x + \varrho \partial_a$ for some function $\varrho$, and it is transversal to $\Sigma : y = \text{const}$.

Consequently the reduced rank 2 distribution (on the quotient $M$) can be interpreted as the following distribution on $\Sigma$: $\Delta^2 = \mathcal{C}_\xi \cap T \Sigma$ (the value of constant plays no role). In the canonical coordinates

$$\Delta = \langle \mathcal{D}_x = \partial_x + \rho \partial_u + r \partial_p + s \partial_q + \alpha \partial_r + (A \alpha + B) \partial_s, \partial_a \rangle.$$  

This distribution has growth vector $(2, 3, 4, \ldots)$ and so is de-prolongable, i.e. locally $\Delta = \mathbb{P}(\Delta)$ for some rank 2 distribution $\Delta$ on 6-dimensional manifold $M$. Indeed, $\partial_a$ is the Cauchy characteristic for the derived distribution $\Delta_2$.

Thus the weak derived flag $\{\Delta_i\}$ differs drastically from the strong derived flag $\{\nabla_i\}$. One possibility is to perform de-prolongation (in this case it is easy - to restrict $\Delta_2$ to the transversal $M : \alpha = \text{const}$), but we will instead consider the strong derived flag. It has the following generators, provided the strong growth vector is $(2, 1, 1, 2, 1)$\footnote{This is the case of general position; elsewise there are more de-prolongations or there exists a first integral of the distribution.}:

\begin{align*}
\nabla_1 &= \langle e_1, \partial_\alpha \rangle, & e_1 &= -\mathcal{D}_x \\
\nabla_2 &= \nabla_1 + \langle e'_1 \rangle, & e'_1 &= [e_1, \partial_\alpha] = \partial_r + A \partial_s \\
\nabla_3 &= \nabla_2 + \langle e_2 \rangle, & e_2 &= [e_1, e'_1] = \partial_p + A \partial_q + (e'_1(B) - \mathcal{D}_x(A)) \partial_s \\
\nabla_4 &= \nabla_3 + \langle e_3, e'_3 \rangle, & e_3 &= [e_1, e_2] = \partial_u + \ldots, & e'_3 &= [e'_1, e_2] = e'_1(A) \partial_q + C \partial_s, \\
\nabla_5 &= \nabla_4 + \langle e_4 \rangle = TM, & e_4 &= [e_1, e_3], & [e_1, e'_3] &= [e'_1, e_3] \text{ or } [e'_1, e'_3],
\end{align*}

where the coefficients, like $C$ or dots, will not be indicated explicitly (the growth vector of the reduction on $M$ is $(2, 1, 2, 1)$ with the flag generated by vectors $\langle e_1, e'_1; e_2; e_3, e'_3; e_4 \rangle$).
If we know the vertical distribution\(^6\nabla_2 = \langle e'_1, \partial_\alpha \rangle\), we can construct the distribution \(\nabla'_4 = [\nabla'_2, \nabla_3] = \langle e_1, e'_1, e_2, e'_3, \partial_\alpha \rangle\).

Distribution \(\Delta\) will be called sufficiently non-degenerate if \(\nabla'_5 = [\nabla'_2, \nabla'_4]\) has rank 6 (i.e., we can use \([e'_1, e'_3]\) for \(e_4\)). In this case it contains the fiber \(\langle \partial_p, \partial_q, \partial_r, \partial_s, \partial_\alpha \rangle\) of the projection \(d\pi_{3,0} : T\mathcal{E} \to TJ^0\) as a codimension 1 subspace, so we recover the contact distribution on \(J^1\) (lifted to \(\mathcal{E}\)): \(\mathcal{C}_1 = \nabla'_5 + \langle \xi \rangle\).

The second particular type we want to examine is the system \(3E_3\), which writes

\[
\beta = F(\alpha, \ldots), \quad \gamma = G(\alpha, \ldots), \quad \delta = H(\alpha, \ldots),
\]

now dots mean terms of order 2. The condition of 1 common characteristic implies \(G_\alpha = F'^2_2\), \(H_\alpha = F'^3_3\) and we assume \(\mathcal{E}\) fully nonlinear \(F_{\alpha\alpha} \neq 0\) (this is sufficient non-degeneracy; the quasi-linear case yields formulae similar to the first example, so most of them also work).

The characteristic is then \(\partial_y - F_\alpha \partial_x\), whence the Cauchy characteristic of \(\mathcal{C}_5\) is \(\xi = D_y - F_\alpha D_x + g \partial_\alpha\) for some function \(g\) on \(\mathcal{E}\) and we again choose the transversal as \(\Sigma : y = \text{const.}\)

This will be identified with the quotient \(M\) and the induced rank 2 distribution is

\[
\Delta = \langle D_x = \partial_x + p \partial_u + r \partial_p + s \partial_q + \alpha \partial_r + F_\partial_s + G_\partial_t, \partial_\alpha \rangle.
\]

Now due to full nonlinearity the square of this distribution does not have Cauchy characteristics (i.e., it is not de-prolongable). This implies that the first 3 elements of the weak and strong derived flags are the same, and so in our case\(^7\) both growth vectors are \((2, 1, 2, 3)\).

The flags are given by

\[
\nabla_1 = \langle e_1, e'_1 \rangle, \quad e_1 = -D_x, e'_1 = \partial_\alpha \\
\nabla_2 = \nabla_1 + \langle e_2 \rangle, \quad e_2 = [e_1, e'_1] = \partial_r + F_\alpha \partial_s + F'^2_2 \partial_t \\
\nabla_3 = \nabla_2 + \langle e_3, e'_3 \rangle, \quad e_3 = [e_1, e_2] = \partial_p + F_\alpha \partial_q + C \partial_s + D \partial_t, \quad e'_3 = [e'_1, e_2]/F_{\alpha\alpha} = \partial_s + 2F'_a \partial_t, \\
\nabla_4 = \nabla_3 + \langle e_4, e'_4, e''_4 \rangle = TM, \quad e_4 = [e_1, e_3] = \partial_u + \ldots, \quad e'_4 = [e_1, e'_3] = \partial_q + \tilde{C} \partial_s + \tilde{D} \partial_t, \quad e''_4 = [e'_1, e'_3]/(2F_{\alpha\alpha}) = \partial_\xi.
\]

Thus if we know the vertical distribution \(\langle e'_1 \rangle \subset \Delta\) we can re-cover the Cartan distribution on \(J^1\): \(\mathcal{C}_1 = [e'_1, \nabla_3] + \langle \xi \rangle\) (here we use the fact that \(e'_4 = [e'_1, e_3]/F_{\alpha\alpha} \mod \nabla_3\).

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\(^6\)Notice that it is in the kernel of the bracket-map \(\Lambda^2 \nabla_2 \to \nabla_3\) since \(\partial_\alpha\) is the Cauchy characteristic for \(\nabla_2\).

\(^7\)This equality of weak and strong derived flags can fail already when the length of the weak growth vector is 5, even in fully nonlinear case.
Another way is to re-cover the contact distribution on $J^2$: $C_2 = [e_1', [e_1', \nabla_2]] + \langle \xi \rangle$ and then get $C_1 = [C_2, C_2]$.

Of course, in both cases we do not know the vertical distributions from internal viewpoint. However generically (if the system is sufficiently non-linear) they rotate inside the corresponding internally-canonical distributions when moving along the Cauchy characteristic. This crucial observation will allow us to prove the equivalence.

4. SUFFICIENT NON-DEGENERACY

We need to specify what conditions on $E$ should be imposed to obtain the generalized Lie-Bäcklund theorem. The first one is:

\[(N) \quad \text{Nonlinearity of } E \iff \Delta \text{ is not Goursat.}\]

Thus the strong growth vector of $\Delta$ is $(2, 1, \ldots, 1, 2, \ldots)$, i.e. there exists $s > 2$ such that $\dim(\nabla_s/\nabla_{s-1}) = 2$ and this is the first 2 after a sequence of 1 ($s$ is minimal).

Then $\dim(\nabla_s) = s + 2$ and this distribution has $(s - 3)$-dimensional sub-distribution $\nabla_{s-3} \subset \nabla_{s-3}$ of Cauchy characteristics for $\nabla_s$ (that are also Cauchy characteristics for $\nabla_{s-2}$).

Denote the projection along Cauchy characteristic space $\Pi \subset C_\varepsilon$ by $\varpi: \varepsilon \to M$; it maps $C_\varepsilon \to \Delta$. The strong growth vector of $C_\varepsilon$ is $(n + 1, 1, \ldots, 1, 1, 2, \ldots)$, where the first dimension 2 occurs at the place $s$. The strong derived flag is $\hat{\nabla}_i = \varpi^{-1}(\nabla_i)$. Let $\hat{\nabla}_{s-3} = \varpi^{-1}(\nabla_{s-3}) \subset \hat{\nabla}_{s-3}$ be the space of Cauchy characteristics for $\hat{\nabla}_s$ (and $\hat{\nabla}_{s-2}$).

Let $v_{s-2} = \hat{\nabla}_{s-2} \cap \ker(d\pi)$, where $\pi: \varepsilon \to B$ is the natural projection in jets; $\text{rank}(v_{s-2}) = s - 2$. Then

\[
\nabla_{s-3} \subset v_{s-2} + \Pi \subset v_{s-2} + C_\varepsilon = \hat{\nabla}_{s-2},
\]

and both inclusions have codimension 1: $\text{rank}(\nabla_{s-3}) = n + s - 4$, $\text{rank}(v_{s-2} + \Pi) = n + s - 3$, $\text{rank}(\hat{\nabla}_{s-2}) = n + s - 2$.

Since $\Pi$ is the space of Cauchy characteristics, we have $[\Pi, \hat{\nabla}_{s-2}] = \hat{\nabla}_{s-2}$. Our second requirement is that the space $v_{s-2}$ rotates along $\Pi$:

\[(R) \quad [\Pi, v_{s-2}] = \hat{\nabla}_{s-2}.\]

This condition holds if the system exhibits some degree of non-linearity; for instance, a system of type $kE_k$ with $n = 2$ satisfies $(R)$ whenever the equations in it are not all quasi-linear.

Denote the operation of taking bracket with $v_{s-2}$ by $\text{ad}_{v_{s-2}}$. We have $\text{ad}_{v_{s-2}}(\hat{\nabla}_{s-1}) \subset \hat{\nabla}_s$. Let us continue taking brackets and denote the limit by $\text{ad}_{v_{s-2}}^\infty(\hat{\nabla}_{s-1})$. Our third requirement is that this latter generates the Cartan distribution:

\[(G) \quad (k - s + 1)\text{-strong derived of } \text{ad}_{v_{s-2}}^\infty(\hat{\nabla}_{s-1}) = C_1.\]
Actually, if \((k - s + 2)\) is less than the minimal order of \(E\), we can formulate the condition in a simpler form:

\[
(G') \quad \text{ad}_{\nabla s-2}^\infty (\hat{\nabla}_{s-1}) = C_{k-s+2}.
\]

This implies, in particular, that the Cartan distribution \(C_E\) on \(E\) is completely non-holonomic, i.e., its iterated brackets generate \(TE\).

Finally let us consider the condition on the space of Cauchy characteristics, which arise only in the case \(n > 2\) (when \(\dim \Pi = n - 1 > 1\)). Since the inclusion \(\Pi + \upsilon_{s-2} \subset \Pi, \nabla_{s-2} = \nabla_{s-2}\) has codimension 1, condition \((R)\) implies existence of codimension 1 sub-distribution \(\Pi_1 \subset \Pi\) such that \(\Pi_1, v_{s-2} \subset \Pi + v_{s-2}\). We assume, in addition, to \((R)\), that we have equality instead of the general inclusion.

Then there exists a codimension 1 sub-distribution \(\Pi_2 \subset \Pi_1\) such that \([\Pi_2, v_{s-2}] \subset \Pi_1 + v_{s-2}\). Again, we strengthen this to be equality and continue. To summarize we arrive to the filtration

\[
\Pi = \Pi_0 \supset \Pi_1 \supset \Pi_2 \cdots \supset \Pi_{n-2},
\]

where \(\text{rank}(\Pi_i) = n - i - 1\) and \([\Pi_i, v_{s-2}] \subset \Pi_{i-1} + v_{s-2}\). Our last condition, strengthening \((R)\), is that the filtration rotates along \(v_{s-2}\), i.e., the last defining relation is the equality:

\[
(R_+) \quad [\Pi_i, v_{s-2}] = \Pi_{i-1} + v_{s-2},
\]

where we let \(\Pi_{-1} = C_E = \omega^{-1}(\Delta)\) to include the condition \((R)\). Notice that there are no difference between \((R)\) and \((R_+)\) for \(n = 2\).

**Definition.** An overdetermined compatible system \(E\) of class \(\omega = 1\) is sufficiently non-degenerate if it satisfies conditions \((N)\), \((R_+)\) and \((G)\).

Now we can prove two theorems from the Introduction.

5. **Proof of the main results**

Let us start by considering the case, when the base \(B\) has dimension \(n = 2\), so that the space \(\Pi\) of Cauchy characteristics is 1-dimensional.

It is obvious that an external symmetry of \(E\) induces an internal symmetry of the reduction \((M, \Delta)\). We have to show that the inverse exists and is unique.

Since \(\Pi\) is the space of Cauchy characteristics, we have \((E, C_E) \simeq (M, \Delta) \times (\Pi, \Pi)\). This local diffeomorphism does not show the contact structure. The latter can be uncovered due to condition \((R)\) as follows.

By \((2)\) we can locally identify the space \((E, v_{s-2} + \Pi)\) with \(P_{s-2} = \{V^{s-2} : \Box_{s-3} \subset V^{s-2} \subset \nabla_{s-2}\}\). Since every \((s-2)\)-dimensional space \(V^{s-2} \in P_{s-2}\) is squeezed between the subspaces of \(TM\) of dimensions \((s-3)\) and \((s-1)\), this \(P_{s-2}\) is fibered over \(M\) with fibers of dimension 1, and the condition \((R)\) means that the line field \(V^{s-2}/\Box_{s-3}\) rotates along the fiber, which can be locally identified with \(\Pi\).
Indeed, the above construction corresponds to the prolongation of rank 2 distributions as follows. Recall that for a rank 2 distribution $\Delta$ on $\tilde{M}$ its prolongation $\hat{M}$ is the manifold of all 1-dimensional subspaces $\ell \subset \tilde{\Delta}$, with the natural projection $\rho : \hat{M} \to \tilde{M}$. The fiber of $\rho$ over $\tilde{x} \in \tilde{M}$ is $P_{\Delta} \tilde{x} \cong S^1$ and the natural lift of the distribution is given by the formula $\hat{\Delta}_\ell = d\rho^{-1}(\ell)$. This is a rank 2 distribution on $\hat{M}$ with the derived rank 3 distribution equal to $[\hat{\Delta}, \hat{\Delta}] = d\rho^{-1}(\Delta)$. The space of Cauchy characteristics of the latter is the fiber $T S^1$.

Now letting $(\tilde{M}, \tilde{\Delta}) = (M, \nabla_{s-2}/\square_{s-3}) = (E, \hat{\nabla}_{s-2}/\hat{\square}_{s-3})$ (this is a local construction, quotient by $\square_{s-3}$ is possible as it is the space of Cauchy characteristics), we can locally identify the distribution $\upsilon_{s-2} + \Pi$ on $E$ with the distribution $\hat{\Delta} \times \square_{s-3}$ on $\hat{M} \times \mathbb{R}^{s-3}$.

This construction allows us to uniquely (locally) recover the pair $(\hat{\nabla}_{s-2}, \upsilon_{s-2})$ on $E$ from the internal geometry of the distribution $\Delta$ on $M$. Since, by condition $(G)$, the pair $(\hat{\nabla}_{s-2}, \upsilon_{s-2})$ generates the contact distribution on the space of jets, the symmetries of $(M, \Delta)$ are bijective with external Lie infinitesimal transformations of $E$.

Consider now the case $n > 2$. By definition the bracket with $\Pi_1$ acts trivially on the distribution $\nabla_{s-2}/\square_{s-3}$. The previous construction identifies the line bundle $\Pi/\Pi_1$ with the projectivization of the rank 2 distribution $\tilde{\Delta} = \hat{\nabla}_{s-2}/\hat{\square}_{s-3}$. Thus the distribution $\tilde{\Delta} + \Pi/\Pi_1$ can be identified with the square $[\hat{\Delta}, \hat{\Delta}]$ of the prolonged distribution $\hat{\Delta}$. This latter is identified with $\upsilon_{s-2}/\square_{s-3} + \Pi/\Pi_1$, with $\Pi/\Pi_1$ corresponding to its vertical line subbundle.

By condition $(R_+)$ this line bundle rotates along $\Pi_1/\Pi_2$ and does not rotate along $\Pi_2$. Thus we can identify the line bundle $\Pi_1/\Pi_2$ with the prolongation of $\hat{\Delta}$. Continuing in the same way we identify $\Pi$ with $(n - 1)$-st iterated prolongation of the rank 2 distribution $\tilde{\Delta}$ over $M$, and so we uniquely recover the contact geometry of $E$ from the internal geometry of $(M, \Delta)$.

This proves Theorem 1; Theorem 2 follows shortly.

6. Application I

Consider the following overdetermined system $E_k \subset J^k(\mathbb{R}^2)$:

$$E_k : \left\{ u_{k-i,i} = \frac{\lambda^{i+1}}{i+1} : 0 \leq i \leq k \right\}$$

(we use jet multi-index notations $u_{10} = u_x$, $u_{01} = u_y$, $u_{20} = u_{xx}$ etc; parameter $\lambda$ is to be excluded).

For $k = 1$ this is the equation $2u_y = u_x^2$, which is equivalent via a potential change of variables to the equation of gas dynamics $u_y = u u_x$. Both have infinite-dimensional contact symmetry algebra $\text{cont}(\mathbb{R}^3)$. 
For $k = 2$ this is the celebrated involutive second order PDE system considered by E.Cartain in 1893 and 1910 [3, 4]:

$$\begin{align*}
\{u_{xx} = \lambda, \; u_{xy} = \frac{\lambda^2}{2}, \; u_{yy} = \frac{\lambda^3}{3}\}. \quad (3)
\end{align*}$$

He has shown that the contact symmetry group is the (split) exceptional Lie group $G_2$.

**Theorem 3.** The algebra of contact symmetries of the equation $\mathcal{E}_k$ for $k > 2$ has dimension $k(k+1)/2 + 6$ and is isomorphic to the semi-direct product $\mathfrak{n}_{k+1} \rtimes \mathfrak{gl}_2$, where $\mathfrak{n}_k$ is a nilpotent Lie algebra of length $k$.

Every contact vector field on $J^1 = J^1(\mathbb{R}^2)$ has the form (here $D_x^1 = \partial_x + u_x \partial_u$, $D_y^1 = \partial_y + u_y \partial_u$):

$$X_f = -f_{u_x} D_x^1 - f_{u_y} D_y^1 + f \partial_u + D_x^1(f) \partial_{u_x} + D_y^1(f) \partial_{u_y},$$

where $f \in C^\infty(J^1)$ is the generating function (contact Hamiltonian). Let us indicate the generating functions $f$ of a basis of $\text{Sym}(\mathcal{E}_k)$:

$$X_f = x^i y^j (i + j < k), \; u_x, \; u_y, \; u + y \cdot u_y, \; (k + 1)u - x u_x,$$

$$y \cdot u_x + x^k / k!, \; (k - 1)y \cdot u - x y \cdot u_x - y^2 \cdot u_y - x^{k+1} / (k + 1)!. \quad (3)$$

Since the prolongation of the field $X_f$ to the space $J^k$ is given by the formula ($\sigma$ is the multi-index of the derivation, $D_\sigma$ is the iterated total derivative and $D^k_\sigma = \sum_{|\sigma| \leq k} u_{\sigma+1} \partial_{u_\sigma}$ is the truncated total derivative [14])

$$\dot{X}_f = -f_{u_x} D_x^k - f_{u_y} D_y^k + \sum_{|\sigma| \leq k} D_\sigma(f) \partial_{u_\sigma},$$

we see that the first elements of the above collection act trivially on $u_{k-i,i}$. The same concerns the translations $u_x, u_y$. The next elements in the first line are scalings and it is trivial to check they preserve the system $\mathcal{E}_k$. The last elements (second line) of the above collection have the following prolongations:

$$- y \partial_x + \frac{x}{k!} \partial_u + \frac{x^{k-1}}{(k-1)!} \partial_{u_x} + u_x \partial_{u_y} + \frac{x^{k-2}}{(k-2)!} \partial_{u_{xx}} + u_{xx} \partial_{u_{xy}} + 2 u_{xy} \partial_{u_{yy}} +$$

$$+ \partial_{u_{k-0}} + u_{k-0} \partial_{u_{k-1,1}} + 2 u_{k-1,1} \partial_{u_{k-2,2}} + \cdots + k u_{1,k-1} \partial_{u_{0,k}};$$

$$x y \partial_x + y^2 \partial_y + ((k - 1)y u - \frac{x^{k+1}}{(k+1)!}) \partial_u +$$

$$+ ((k - 2)y u_x - \frac{x^k}{k!}) \partial_{u_x} + ((k - 1)u - x u_x + (k - 3)y u_x) \partial_{u_y} + \cdots -$$

$$- (x + y u_{k,0}) \partial_{u_{k,0}} - \sum_{1 \leq i \leq k} (i x u_{k-i+1,i-1} + (i + 1) y u_{k-i,i}) \partial_{u_{k-i,i}}.$$

Now it is straightforward to check these preserve the system $\mathcal{E}_k$. 
Let us consider the reduction of \( \mathcal{E}_k \) by the Cauchy characteristic, which is the vector field \( \mathcal{D}^k_y - \lambda \mathcal{D}^k_x + \varrho \partial_{\lambda}, \) where \( \lambda = u_{k,0} \) and the precise value of \( \varrho \) is not important.

The quotient \( (M, \Delta) \) is (locally) isomorphic to the intersection with the transversal \( \{y = \text{const}\} \), so that \( \Delta = \langle \mathcal{D}^k_x, \partial_{\lambda} \rangle \), where

\[
\mathcal{D}^k_x = \partial_x + u_{10} \partial_u + u_{20} \partial_{u_{10}} + u_{11} \partial_{u_{11}} + \cdots + \\
+ \lambda \partial_{u_{k-1,0}} + \frac{1}{2} \lambda^2 \partial_{u_{k-2,1}} + \cdots + \frac{1}{k} \lambda^k \partial_{u_{0,k-1}}.
\]

Denoting \( w^j_i = u_{ij}, 0 \leq i + j \leq k - 1 \), we see that the above rank 2 distribution corresponds to the following underdetermined ODE system on \( w = (w^0, \ldots, w^{k-1}) \) as a function of \( x \) (we understand \( w^j_i = \partial_x^j (w^i) \)):

\[
\mathcal{Y}_k : \{ w^0_k = \lambda, \ w^1_{k-1} = \frac{1}{2} \lambda^2, \ldots, w^{k-1}_1 = \frac{1}{k} \lambda^k \}.
\]

By Theorem 1 the algebra of external symmetries \( \text{Sym}_{\text{ext}}(\mathcal{E}_k) \) is isomorphic to the algebra of internal symmetries of the above Monge equation \( \mathcal{Y}_k \subset J^{k,k-1,\ldots,2,1}(\mathbb{R}, \mathbb{R}^k) \) (considered as a submanifold in the mixed-jet space \( \mathbb{R}^k \), which is identical with the algebra \( \text{Sym}(\Delta) \).

The latter has the following basis (we write only the point part of the transformation, from which it can be uniquely recovered by the prolongation):

\[
X = \partial_x, \quad W^j_i = \frac{1}{i!} x^i \partial_{w^j}, \quad (0 \leq i + j < k), \\
L = \frac{1}{k!} x^k \partial_{w^0} + w^0_1 \partial_{w^1} + 2w^1_1 \partial_{w^2} + \cdots + (k - 1)w^{k-2}_1 \partial_{w^{k-1}}, \\
R = \frac{1}{(k+1)!} x^{k+1} \partial_{w^0} + (xw^0_1 - (k - 1)w^0) \partial_{w^1} + 2(xw^1_1 - (k - 2)w^1) \partial_{w^2} + \\
+ 3(xw^2_1 - (k - 3)w^2) \partial_{w^3} + \cdots + (k - 1)(xw^{k-2}_1 - w^{k-2}) \partial_{w^{k-1}}, \\
S_1 = x \partial_x + kw^0 \partial_{w^0} + (k - 1)w^1 \partial_{w^1} + \cdots + 2w^{k-2} \partial_{w^{k-2}} + w^{k-1} \partial_{w^{k-1}}, \\
S_2 = w^0 \partial_{w^0} + w^1_1 \partial_{w^1} + 3w^2_1 \partial_{w^2} + \cdots + kw^{k-1} \partial_{w^{k-1}}, \\
T = \lambda \partial_x + (\lambda w^0_1 - w^1) \partial_{w^0} + (\lambda w^1_1 - w^2) \partial_{w^1} + \\
+ \cdots + (\lambda w^{k-2}_1 - w^{k-1}) \partial_{w^{k-2}} + \frac{1}{k(k+1)} \lambda^{k+1} \partial_{w^{k+1}}.
\]

Notice that \( \langle X, W^j_i, L \rangle \) is a translational part and \( \langle R, S_1, S_2, T \rangle \) is the stabilizer of the origo (all coordinates vanish) \( o \in M \) (to check this for \( L \) one has to prolong the formulae to see the term \( + \partial_{\lambda} \); recall that

\footnote{Notice that \( \mathcal{Y}_2 \subset J^{2,1}(\mathbb{R}, \mathbb{R} \times \mathbb{R}) \) is the Hilbert-Cartan equation. In \( \text{[1]} \) it was generalized in some aspects, different from the present paper.}
This algebra is graded with the following weights:

\[ w(W^j_i) = (k-i-j)W^j_i, \quad w(W^j_i, S_1, S_2) = (j+1)W^j_i, \quad w(L, S_2) = L, \]
\[ w(R, T) = -W^j_i-1, \quad w(S_1, S_2) = S_1 - S_2, \]
\[ w(R, S_1) = -R, \quad w(R, S_2) = R. \]

This algebra is graded with the following weights: \( w(X) = w(L) = -1, \)
\( w(W^j_i) = i - k - 1, \quad w(R) = w(T) = w(S_1) = w(S_2) = 0. \)

Even more, it is bi-graded: \( b(X) = (-1, 0), \quad b(L) = (0, -1), \quad b(W^j_i) = (i + j - k, -j - 1), \)
\( b(R) = (1, -1), \quad b(T) = (-1, 1), \quad b(S_1) = (0, 0), \)
\( b(S_2) = (0, 0); \) the grading \( w \) is the total weight of \( b. \)

It is rather straightforward to check that the above fields are symmetries of the distribution \( \Delta \). This gives the lower bound for \( \text{Sym}(\Delta) \).

To get the upper bound one should calculate the Tanaka algebra of \( \Delta \).

Recall [17] that with every distribution one associates the sheaf of graded nilpotent Lie algebras (called symbol or Carnot algebras) \( m = \oplus_{i<0}g_i \), \( g_i = \Delta_{-i}/\Delta_{-i-1} \), where \( \{\Delta_j\} \) is the weak derived flag of \( \Delta \).

The point-wise bracket of \( m \) is induced by the commutator of vector fields, so that for every point \( x \in M \) we get the Lie algebra \( m_x \).

The Tanaka prolongation \( \hat{m} = \oplus g_i \) is defined as the maximal graded Lie algebra with the given negative part \( m \) [17]. It can also be defined via graded Lie algebra cohomology as \( g_i = H^i_1(m, \hat{m} \oplus g_0 \cdots \oplus g_{-1}) \) for \( i \geq 0. \) In other words, \( g_i \) are constructed successively as maximal subspaces such that all possible Jacobi identities hold, see [19].

To calculate \( m \) one takes the generators \( D^1_x, \partial_\lambda \) of \( \Delta \), and computes all possible brackets. The resulting Carnot algebra is a graded nilpotent Lie algebra \( n_{k+1} \) isomorphic to \( \langle X, L, W^j_i \rangle \).

It can be described as a truncated double-graded free Lie algebra with fundamental part of grading \(-1\) and rank \( 2 \). In the appendix we demonstrate that the Tanaka prolongation of \( n_{k+1} \) is trivial in positive grading. Hence the Tanaka algebra is \( n_{k+1} \oplus g_0 \), where \( g_0 = \mathfrak{gl}(g_{-1}). \)

By the results of [17] [9] and Theorem [11] this gives the upper bound for the symmetries of both equations: \( E_k \) (PDE system) and \( \mathcal{Y}_k \) (ODE system). Since it coincides with the algebra of symmetries we already constructed, our description of \( \text{Sym}(E_k) \simeq \text{Sym}(\Delta) \) is complete. Theorem [3] is proved.

Remark. We have demonstrated that \( \Delta \) is the most symmetric distribution with the symbol \( m \) equal to the truncated double-graded free Lie algebra \( n_{k+1} \). By Theorem 4 of [11] this implies that \( \Delta \) is Tanaka-flat.
We can describe very symmetric systems, for which the corresponding distributions are non-flat. Let us start with the PDE models $\mathcal{F}_k \subset J^k(\mathbb{R}^2)$:

$$\mathcal{F}_k = \left\{ u_{k-i,i} = \frac{\lambda^{im+1}}{im+1} : 0 \leq i \leq k \right\},$$

which coincides with $\mathcal{E}_k$ for $m = 1$. For $k = 2$ this is (equivalent to) the family of involutive 2nd order PDE systems

$$\left\{ u_{xx} = \lambda, \; u_{xy} = \frac{\lambda^{m+1}}{m+1}, \; u_{yy} = \frac{\lambda^{2m+1}}{2m+1} \right\}$$

with 7-dimensional contact symmetry algebra described by Cartan in [4] (the next large after 14-dimensional $G_2$), see also [12].

By the calculation similar to the above we get the following

**Theorem 4.** The algebra of contact symmetries of the equation $\mathcal{F}_k$ for $k > 2$ and generic $m$ has dimension $k(k+1)/2 + 4$. Its basis consists of contact vector fields $X_f$ with generating functions $f$ as follows

$$x^i y^j \ (0 \leq i + j < k), \ u_x, \ u_y, \ (km + 1) u - m \ x \ u_x, \ u + m \ y \ u_y.$$ 

By Theorem 1 we can also represent this algebra as the internal symmetry of the following underdetermined ODE system obtained from $\mathcal{F}_k$ via reduction by the Cauchy characteristic:

$$w^0_k = \lambda, \ w^{1}_{k-1} = \frac{1}{m+1}\lambda^{m+1}, \ldots, \ w^{k-1}_{1} = \frac{1}{(k-1)m+1}\lambda^{(k-1)m+1}.$$ 

(here $w^j = w^j(x)$ are the dependent variables and $w^j = \partial_x^j(w^i)$).

The symmetry algebra has a filtration with the corresponding graded Lie algebra being the semi-direct product $\mathfrak{n}_{k+1} \rtimes \mathbb{R}^2$, where $\mathbb{R}^2$ is the diagonal part of $\mathfrak{g}_0$. This associated graded algebra is a subalgebra of the Tanaka algebra $\hat{\mathfrak{n}}_{k+1}$ of the flat model.

It is possible to show by the methods of [12] that the corresponding distribution $\Delta$ is sub-maximal symmetric with the Tanaka algebra $\hat{\mathfrak{n}}_{k+1}$.

### 7. Application II

The PDE system $\mathcal{E}_k$ has solution space $\text{Sol}(\mathcal{E}_k)$ that is parametrized by 1 function of 1 argument and $\dim M - 2 = \frac{k(k+1)}{2}$ constants (these are the so-called Lie class $\omega = 1$ systems [13, 10, 11]).

More general systems can be treated with the proposed technique too. Consider, for instance, the following system $\mathcal{R}_m^k \subset J^k(\mathbb{R}^2)$, $m < k$, for which the right-hand side is the $m$-th tangent cone (of $\dim = m + 1$
in \( \mathbb{R}^{k+1} \) of the normal projective curve defining \( \mathcal{E}_k = \mathcal{R}_k^0 \):

\[
\mathcal{R}_k^m = \left\{ u_{k-i,i} = \frac{\lambda i+1}{i+1} + \sum_{i=1}^{\min(m,i+1)} \frac{i!}{(i+1-j)!} \lambda i+1-j \zeta_j : 0 \leq i \leq k \right\}
\]

(the PDE system is obtained by excluding the additional parameters \( \lambda, \zeta_1, \ldots, \zeta_m \) and obtaining relations on the jet-variables \( u_{k-i,i} \)).

For instance for \( k = 3, m = 2 \) the system \( \mathcal{R}_3^2 \) looks so (this is one highly nonlinear PDE of the 3rd order)

\[
\begin{align*}
    u_{xxx} &= \lambda + \zeta_1, \\
    u_{xxy} &= \lambda^2 + \lambda \zeta_1 + \zeta_2, \\
    u_{xyy} &= \frac{\lambda^3}{3} + \lambda^2 \zeta_1 + 2\lambda \zeta_2, \\
    u_{yyy} &= \frac{\lambda^4}{4} + \lambda^3 \zeta_1 + 3\lambda^2 \zeta_2.
\end{align*}
\]

The system \( \mathcal{R}_2^1 \) is the famous Goursat parabolic PDE on the plane; its contact symmetry group is \( G_2 \) the same as for \( \mathcal{R}_3 \).

The system \( \mathcal{R}_k^m \) is involutive and its characteristic variety (both complex and real) consists of 1 point with multiplicity \( m+1 \). The solution space \( \text{Sol}(\mathcal{R}_k^m) \) is parametrized by \( \omega = m+1 \) functions of 1 argument (and some constants). There is a reduction of this parabolic system via a characteristic involutive distribution (no longer a space of Cauchy characteristics) to a distribution \( \Delta \) on the quotient that de-prolongs to a rank 2 distribution \( \Delta \) (Monge system); this generalizes \cite{6}.

This reduction makes a bijection between contact symmetries of the PDE system and the ordinary symmetries of \( \Delta \). While this idea is more general, we will indicate only how it applies to our system \( \mathcal{R}_k^m \).

**Theorem 5.** The algebra of contact symmetries of the equation \( \mathcal{R}_k^m \) for \( k > 2, 0 < m < k \), has dimension \( k(k+1)/2 + 6 \) and is isomorphic to the same Lie algebra \( \mathfrak{n}_{k+1} \rtimes \mathfrak{gl}_2 \) as in Theorem \cite{3}.

**Proof.** Let us first show how to associate a rank 2 distribution to such a parabolic system of PDEs.

Consider the Cartan distribution \( \mathcal{C} \) of \( \mathcal{R}_k^m \). It is generated by two truncated total derivatives \( \mathcal{D}_x^k, \mathcal{D}_y^k \) and the vertical fields \( \partial_{\lambda}, \partial_{\zeta_1}, \ldots, \partial_{\zeta_m} \). The sub-distribution \( \Pi^m = \langle \partial_{\zeta_1}, \ldots, \partial_{\zeta_m} \rangle \) is integrable and we would like to quotient by it.

It however does not commute with \( \mathcal{C} \), so we need to add the commutators

\[
\text{ad}_{\Pi}^\omega(\mathcal{C}) = \mathcal{C} + \langle \eta_1, \ldots, \eta_m \rangle,
\]

where \( \eta_i = \sum_{i=0}^{k} \lambda^i \partial_{u_{k-i,i}}, \eta_{i+p} = \partial_{x}^p(\eta_i) \).

The quotient manifold is  \( \tilde{M} = \mathbb{R}^{k(k+1)/2 + 3}(x,y,\lambda,u_\sigma : |\sigma| < k) \) and it is equipped with the distribution \( \tilde{\Delta} = \text{ad}_{\Pi}^\omega(\mathcal{C})/\Pi \) generated by the

\footnote{This \( \mathbb{R}^{k+1} \) is the fiber of \( \pi_{k,k-1} : J^k(\mathbb{R}^2) \to J^{k-1}(\mathbb{R}^2) \).}
truncated total derivatives evaluated at $\zeta_1 = \cdots = \zeta_m = 0$, denoted $D^k_x, D^k_y$, and by the fields $\eta_1, \ldots, \eta_m, \partial_\lambda$. For instance, for $k = 3, m = 2$
\[
\begin{align*}
\bar{D}_x &= \partial_x + p \partial_u + r \partial_p + s \partial_q + \lambda \partial_r + \frac{1}{2} \lambda^2 \partial_s + \frac{1}{3} \lambda^3 \partial_t, \\
\bar{D}_y &= \partial_y + q \partial_u + s \partial_p + t \partial_q + \frac{1}{2} \lambda^2 \partial_s + \frac{1}{3} \lambda^3 \partial_p + \frac{1}{4} \lambda^4 \partial_t, \\
\eta_1 &= \partial_r + \lambda \partial_s + \lambda^2 \partial_t, \\
\eta_2 &= \partial_s + 2 \lambda \partial_t
\end{align*}
\]

The vector $\xi = \bar{D}_y - \lambda \bar{D}_x$ is the Cauchy characteristic of the distribution $\tilde{\Delta}$. Let $(M, \Delta_+) = (\bar{M}, \bar{\Delta})/\xi$ be the local quotient. It is easy to see that $\Delta_+$ is the $m$-th derived (both weak and strong) distribution of the rank 2 distribution $\Delta = \langle \bar{D}_x, \partial_\lambda \rangle$.

Thus any contact symmetry of $\mathcal{C}$ descends to an ordinary symmetry of $\Delta$. Conversely, $\text{Sym}(\Delta) = \text{Sym}(\Delta_+)$ because prolongation preserves the symmetries, and then by Theorem 1 the latter coincide with the contact symmetries of $\bar{\Delta}$. This distribution corresponds to the equation $E_x = R_0^0$, from which by taking the $m$-th tangent cone we obtain our system $R^m_x$. Thus any ordinary symmetry of $\Delta$ uniquely lifts to a contact symmetry of $\bar{\Delta}$.

The claim now follows from Theorem 3 on the symmetries of $E_x$. \qed

8. THE CASE OF $n > 2$ INDEPENDENT VARIABLES

It is easy to see that the generalized Lie-Bäcklund theorem, as it was stated in Introduction, fails if we allow equations of the first order without non-degeneracy assumptions. Indeed, this is so with any class $\omega = 1$ system of the type $2E_2 + E_1$ in $n > 2$ independent variables.

For instance, we can take (3) as $2E_2$ and trivial $E_1$:

\[
\begin{align*}
u_{xx} &= \lambda, & \nu_{xy} &= \frac{\lambda^2}{2}, & \nu_{yy} &= \frac{\lambda^3}{3}, & \nu_z &= 0.
\end{align*}
\]

Then $z \mapsto Z(x, y, z, u, u_x, u_y, u_z)$ is a contact symmetry, inducing the trivial transformation of the reduction $(M^5, \Delta_{HC})$, which is the Hilbert-Cartan equation. In fact, the general contact symmetry $X_f$ has generating function $f = f_0 + u_z \cdot \tilde{f}$, where $f_0 = f_0(x, y, u, u_x, u_y)$ is the 14-parametric generating function corresponding to $G_2$-action on (3), and $\tilde{f}$ is an arbitrary function on $J^1(\mathbb{R}^3)$. Thus the generalized Lie-Bäcklund theorem fails: the map (1) is not injective.

Let’s consider a similar system of the second order:

\[
\begin{align*}
u_{xx} &= \lambda, & \nu_{xy} &= \frac{\lambda^2}{2}, & \nu_{yy} &= \frac{\lambda^3}{3}, & \nu_{xz} &= 0, & \nu_{yz} &= 0, & \nu_{zz} &= 0.
\end{align*}
\]

Its reduction (along rank 2 distribution $\Pi$) is the following rank 2 distribution in 6D: $(M, \bar{\Delta}) = (M^6, \Delta_{HC}) \times (\mathbb{R}, 0)$. Let $t$ be the first integral of the distribution $\bar{\Delta}$ (it is not completely non-holonomic). The symmetries are $t \mapsto T(t)$ and $G_2$-transformations of the first factor, with
coefficients parametrized as functions of \( t \). Thus in total Sym(\( \bar{M}, \bar{\Delta} \))
is parametrized by 15 functions of 1 variable.

A direct calculation shows that the contact algebra is precisely the same: (\( \text{Lie}(G_2) \oplus \mathbb{R} \))\( ^{\mathbb{R}} \), so the generalized Lie-Bäcklund theorem holds.

Similarly this theorem holds for higher order equations. For instance, the system of type 3
\[ E_3 + 3E_2 \]
\[ u_{xxx} = \lambda, \; u_{xxy} = \frac{\lambda^2}{2}, \; u_{xyy} = \frac{\lambda^3}{3}, \; u_{yyy} = \frac{\lambda^4}{4}, \; u_{xz} = 0, \; u_{yz} = 0, \; u_{zz} = 0 \]
reduces to (\( \bar{M}^9, \bar{\Delta}^2 \)) = (\( M_1, \Delta_1 \)) \( \times \) \( (\mathbb{R}, 0) \), and both contact symmetry of the PDE system and the internal symmetry of the reduction are parametrized by 13 functions of 1 variable. More precisely, these algebras are both equal to (\( g \oplus \mathbb{R} \))\( ^{\mathbb{R}} \), where \( g \) is the algebra of symmetries of (\( M_1, \Delta_1 \)) equivalent to the Monge system of 2 equations on 3 unknowns (the indices denote the number of \( x \)-derivatives) studied in Section 6.

On the other hand if we consider the system of the 3rd order 9\( E_3 \)
\[ u_{xxx} = \lambda, \; u_{xxy} = \frac{\lambda^2}{2}, \; u_{xyy} = \frac{\lambda^3}{3}, \; u_{yyy} = \frac{\lambda^4}{4}, \; u_{xxz} = 0, \; u_{xyz} = 0, \; u_{yyz} = 0, \; u_{xzz} = 0, \; u_{yzz} = 0, \; u_{zzz} = 0 \]
its contact symmetry algebra is 21-dimensional, while the internal symmetry algebra of the reduction (the distribution there is not completely holonomic) is infinite-dimensional. Thus the generalized Lie-Bäcklund theorem fails: the map (\( \Phi \)) is not surjective.

All these systems have a kind of degeneracy, so let us consider a totally non-linear (and non-degenerate) system, for which our Theorems guarantee that the generalized Lie-Bäcklund theorem holds.

We start with the equations of pure order 2 and Lie class \( \omega = 1 \), for which the result, that contact external symmetries of the PDE system and the internal symmetries of the reduction coincide, follows also from [18]. Consider the following example.

\[ \mathcal{E} = \left\{ u_{ij} = \frac{\lambda^{m_i+m_j+1}}{m_i + m_j + 1} : 1 \leq i \leq j \leq n \right\}. \]

This system is involutive and, if \( m_i \neq m_j \) for \( i \neq j \), it has no first integrals. We can always achieve \( m_1 = 0 \) by re-parametrization. The reduction of \( \mathcal{E} \) is given by
\[ v_x^2 = \frac{(u_{xx})^{m_2+1}}{m_2 + 1}, \; v_x^3 = \frac{(u_{xx})^{m_3+1}}{m_3 + 1}, \; \ldots \; v_x^n = \frac{(u_{xx})^{m_n+1}}{m_n + 1} \] (4)
(the superscript numbers the unknown functions). The internal symmetry algebra is maximal for \( m_2 = 1, \; m_3 = 2, \; \ldots, \; m_n = n - 1 \), and
has dimension \((2n + 5)\); the proof of this fact is similar to the proof of Theorem 3 and so is omitted. In fact, the rank 2 distribution of ODE system \([\mathcal{I}]\) with the prescribed parameters is internally equivalent to the most symmetric Monge equation \([\mathcal{I}]\) (subscripts denote the derivatives)

\[
y_1 = (z_n)^2
\]

(its symmetry algebra with notation \(t_{1,n} = \hat{p}_{n+3}\) was studied in \([\mathcal{I}]\)).

The symmetry algebra is the same for the ODE and PDE systems for all \(m_i\), but for generic parameters the algebra has smaller dimension.

Now the above PDE system \(\mathcal{E}\) can be generalized to higher orders. For the third order it writes as

\[
\{ u_{ijk} = \frac{\lambda^{m_i+m_j+m_k+1}}{m_i+m_j+m_k+1} : 1 \leq i \leq j \leq k \leq n \}.
\]

It is involutive, but to achieve complete non-holonomy (no first integrals), we have to assume that all the numbers \(m_j+m_k, 1 \leq j \leq k \leq n\), are different. The reduction can be again easily described.

Consider a particular case \(n = 3\), \(m_1 = 0\), \(m_2 = 1\), \(m_3 = 4\), satisfying the above restriction. The symmetry algebra of both ODE reduction and the PDE system is 15-dimensional.

Similarly for \(n = 4\) independent variables, and parameters \(m_1 = 0\), \(m_2 = 1\), \(m_3 = 4\), \(m_4 = 15\), we calculate both symmetry algebras to have dimension 21.

We conclude that the generalized Lie-Bäcklund theorem holds in many interesting cases. The calculations are easier for the reduced ODE models, where Tanaka theory helps to calculate symmetries by using the algebraic methods.

**Appendix A. Truncated double-graded free Lie algebra**

A double-graded free Lie algebra is such a graded Lie algebra \(\mathfrak{n}_\infty = \bigoplus_{i<0} \mathfrak{g}_i\) that its fundamental space \(\mathfrak{g}_{-1}\) is a direct sum of two subspaces \(\Pi_1, \Pi_2\), and its commutators freely generate the whole algebra with no other relations than the commutation of \(\text{ad}_{v_1}, \text{ad}_{v_2}\) for \(v_i \in \Pi_i\) (whence \(\mathbb{Z}\)-grading can be refined to \(\mathbb{Z} \oplus \mathbb{Z}\)-grading).

Let us specify this only in the case of current interest \(\dim \mathfrak{g}_{-1} = 2\), when the fundamental space has basis \(e_{10}, e_{01}\) according to the above splitting. Then \(\mathfrak{g}_{-2}\) is generated by \(e_1 = [e_{10}, e_{01}], \mathfrak{g}_{-3}\) by \(e_2 = [e_{10}, e_{11}], e_2 = [e_{01}, e_{11}]\) etc. The only relations this infinite-dimensional Lie algebra \(\mathfrak{n}_\infty\) admits are \([e_{10}, e_{i,j+1}] = [e_{01}, e_{i+1,j}] = e_{i+1,j+1}\).

Thus \(\mathfrak{g}_{-k-1} = \langle e_{k,1}, e_{k-1,2}, \ldots, e_{1,k} \rangle\) has dimension \(k\). It’s easy to check that \(\mathfrak{n}_\infty\) is a Lie algebra. Many interesting graded nilpotent Lie algebras are quotients of this algebra (see some in \([\mathcal{I}]\)).
The truncated algebra is \( n_k = g_{-k} \oplus \cdots \oplus g_{-1} \) (the brackets between \( g_{-i} \) and \( g_{-j} \) are zero if \( i + j > k \)). Our goal is to calculate the Tanaka prolongation of \( n_k \) (beware: this sub-script \( k \) is not a grading).

**Theorem 6.** For \( k > 3 \) the Tanaka prolongation of \( n_k \) is supported in non-positive grading: \( n_k = n_k \oplus g_0 \), \( g_0 = gl(g_{-1}) \).

We consider prolongation in the usual sense, not preserving double-grading (in particular \( g_0 \) consists of grading zero derivations, not necessary preserving the splitting of \( g_{-1} \)).

**Proof.** We first state that \( g_0 = gl(g_{-1}) \) is the maximal possible algebra of grading preserving derivations. The easiest way to see this is to calculate the prolongation of an element

\[
h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in g_0
\]

(the matrix in the basis \( e_{10}, e_{01} \)) and check that as derivation it preserves the relations \([ad_{e_{10}}, ad_{e_{01}}] = 0\).

This readily follows from the following formula that can be proved by induction \((1 \leq i \leq k - 1)\):

\[
h(e_{k-i,i}) = (i-1)b e_{k-i+1,i-1} + ((k-i)a + i d) e_{k-i,i} + (k-i-1)c e_{k-i-1,i+1}.
\]

Let now \( \omega \in g_1 \). Since \( g_{-1} \) is fundamental (generates \( n_k \)), this element is uniquely determined by specifying \( \omega(e_{10}) = h', \omega(e_{01}) = h'' \), where we denote \( \omega(\xi) = [\omega, \xi] \) and the elements of \( g_0 \) have the form

\[
h' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \quad h'' = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}.
\]

By the Leibniz rule we calculate:

\[
\omega(e_{11}) = (b' - a'') e_{10} + (d' - c'') e_{01},
\]

\[
\omega(e_{21}) = (a' + 2d' - c'') e_{11}, \quad \omega(e_{12}) = (2a'' + d' - b') e_{11},
\]

\[
\omega(e_{31}) = (3a' + 3d' - c'') e_{21} + c' e_{12},
\]

\[
\omega(e_{22}) = (2a'' + d'') e_{21} + (a' + 2d') e_{12},
\]

\[
\omega(e_{13}) = b '' e_{21} + (3a'' + 3d'' - b') e_{12}.
\]

In calculation of \( \omega(e_{22}) \) we can use two representations \([e_{10}, e_{12}] = [e_{01}, e_{21}] \) and this gives the same result. Thus if we truncate on the level \( k = 3 \) (so that \( e_{31} = e_{22} = e_{13} = 0 \)), then all coefficients to the right in the last three lines vanish and we obtain that \( g_1 \) is 2-dimensional (in agreement with the known grading of the exceptional Lie group \( G_2 \)).

Let now \( k > 3 \) and we calculate the action on \( g_{-5} \). We get

\[
\omega(e_{41}) = \omega([e_{10}, e_{31}]) = (6a' + 4d' - c'') e_{31} + 3c' e_{22},
\]

\[
\omega(e_{14}) = \omega([e_{01}, e_{13}]) = 3b'' e_{22} + (4a'' + 6d'' - b') e_{13}.
\]


uniquely and
\[
\omega(e_{32}) = \omega([e_{10}, e_{22}]) = (2a'' + d'' + b') e_{31} + (3a' + 4d') e_{22} + c' e_{13},
\]
\[
= \omega([e_{01}, e_{31}]) = (3a'' + d'') e_{31} + (3a' + 3d' + c'') e_{22} + c' e_{13},
\]
\[
\omega(e_{23}) = \omega([e_{10}, e_{13}]) = b'' e_{31} + (3a'' + 3d'' + b') e_{22} + (a' + 3d') e_{13},
\]
\[
= \omega([e_{01}, e_{22}]) = b'' e_{31} + (4a'' + 3d'') e_{22} + (a' + 2d' + c'') e_{13},
\]
non-uniquely, which implies \(b' = a'', c'' = d'\).

Further calculations show that the derivation respects the higher commutation relations, and we obtain by induction:

\[
\omega(e_{k-i,i+1}) = \frac{i(i-1)}{2} b'' e_{k-i+1,i-1} + \left( \frac{(k-i)(k-i-1)}{2} a' + (i+1)(k-i-1)d' \right) e_{k-i-1,i+1}
\]
\[
+ \frac{(k-i-1)(k-i-2)}{2} c' e_{k-i-2,i+2}.
\]

Now it's obvious that truncation on the level \(k\), i.e. letting \(e_{k+1-i,i} = 0\) for \(1 \leq i \leq k\), forces \(h' = h'' = 0\), so that \(g_1 = 0\).

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