Abstract: The purpose of this paper is fourfold: (i) to introduce and study the Euler–Lagrange prolongations of flatness PDEs solutions (best approximation of flatness) via associated least squares Lagrangian densities and integral functionals on Riemannian manifolds; (ii) to analyze some decomposable multivariate dynamics represented by Euler–Lagrange PDEs of least squares Lagrangians generated by flatness PDEs and Riemannian metrics; (iii) to give examples of explicit flat extremals and non-flat approximations; (iv) to find some relations between geometric least squares Lagrangian densities.

Keywords: geometric flatness; least squares Lagrangian densities; adapted metrics and connections

MSC: 58J99; 53C44; 53C21

1. Introduction and Contributions

Least squares Lagrangians on Riemannian manifolds and the problem of best approximation of flatness have gained much attention lately [1], especially when they are involved in optimization problems whose objectives are integral functionals. Combining this theory with decomposable multivariate dynamics [2], we get new results in differential geometry and global analysis.

Section 1 outlines the ground material regarding PDEs in differential geometry, least squares Lagrangian densities, dual variational principle, Riemannian volume form, and positive definite differential operators. Section 2 recalls the basic properties of $\nabla$-flatness, introduces the crucial notion of least squares Lagrangian density attached to $\nabla$-flatness and underlines that the Euclidean metrics extremals are stable with respect to conformal changing. In Section 3 comes the heart of the paper. Detailing the Riemann-flatness, we introduce the least squares Lagrangian density attached to Riemann-flatness and best approximations of Riemann-flatness solutions. Then the non-flatness extremals are analyzed in detail. Section 4 shows how Ricci-flatness implies a least squares Lagrangian density and best approximations of Riemann-flatness solutions. Section 5 lists some analogues of least squares Lagrangian density attached to scalar curvature-flatness and confirms again that Einstein PDEs are extremals. Section 6 proves some inequalities between our least squares densities. Section 7 underlines that the least squares technique is suitable for solving some problems in differential geometry.

All Lagrangians we use are written in a local, version which is of special interest for geometers and nonlinear analysts. Their explicit formulas reflect the properties usually needed for differential
geometric constructions. In order to make the techniques in this paper available for a broad mathematical audience, we have tried to make the article as much self-contained as possible.

1.1. PDEs in Differential Geometry

The behavior of many different systems in nature and science are governed by a PDEs system. Usually such a system is thought of in terms of coordinates in order to prove the existence of solutions or to find concrete ones. However, tensorial PDEs in differential geometry also contain information that is independent of the choice of coordinates. This is actually the most important information, as it is independent of any external structure artificially added to the PDE, and in this sense is genuine. That is why differential geometry is often considered as an “art of manipulating PDEs” [3–10].

The most important geometric PDEs are those producing flatness (e.g., connection-flatness, curvature-flatness, Ricci-flatness, scalar curvature-flatness) and those producing constant curvature (−1, 0, 1). The connection-flatness PDEs system is non-tensorial, while curvature-flatness, Ricci-flatness, and scalar curvature-flatness PDEs systems are tensorial. Our ideas come also from the papers [11–19].

The connection-flatness and the curvature-flatness are interconnected.

In this paper, we present some specific features: (i) introducing those differential geometric structures needed to define and study geometric PDEs (some of them in a manifestly coordinate-independent way); (ii) defining PDEs and their signification within differential geometry and global analysis; (iii) developing techniques to find intrinsic properties of PDEs; (iv) discussing explicit examples to illustrate the importance of the choice of an appropriate context and language.

1.2. Riemannian Volume Form

Suppose \((M, g = (g_{ij}))\) is a smooth oriented Riemannian manifold. Then there is a consistent way to choose the sign of the square root \(\sqrt{\det(g_{ij})}\) and define a volume form \(d\mu = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n\). We call it the Riemannian volume form of \((M, g)\). Having a volume form allows us to integrate functions on \(M\). In particular, \(\text{vol}(M) = \int_M d\mu\) is an important invariant of \((M, g)\). It also allows us to define an inner product \(\langle \phi, \psi \rangle = \int_M \langle \phi(x), \psi(x) \rangle_g \, d\mu\) on the space of differential forms and other tensors or objects on \(M\), using the metric \(g\) and its inverse \(g^{-1}\). This inner product induces the square of the norm \(\|\phi\|^2 = \int_M |\phi(x)|_g^2 \, d\mu\).

1.3. Least Squares Lagrangian Densities

Having in mind the so-called variational approach [1,20,21], in this Subsection we add typical functional that appear in the theory of geometric and physical fields [2].

Let \(M\) be an oriented manifold of dimension \(n\). Any differential operator (of vectorial form, tensorial or not) on the Riemannian manifold \((M, g = (g_{ij}))\) and the metric (geometric structure) \(g\) generate a least squares Lagrangian density \(L\). The extremals of the Lagrangian \(L = L(\sqrt{\det(g_{ij})})\), described by Euler–Lagrange PDEs, include the solutions of initial PDEs and other solutions that we call “Euler–Lagrange prolongations” of those solutions (best approximation of initial PDEs solutions).

Generally, the Euler–Lagrange equation provides the equation of motion for the dynamical field specified in the Lagrangian. If the Lagrangian attached to a PDE is that of the smallest squares, then the extremals give the best approximation of the PDE solutions.

The Euler–Lagrange PDEs are indexed related to the chosen fibered chart \((\mathbb{R}^n, \Psi)\), \(\Psi = (f^1, x^1)\). However, since the Euler–Lagrange expressions are components of a global differential form (the Euler–Lagrange form), the solutions are independent of fibered charts [21].

Example 1. (Compare with the paper [2]) Let \((M, g = (g_{ij}))\) be an \(n\)-dimensional Riemannian manifold, with local coordinates \(x = (x^1, \ldots, x^n)\), and \(\Omega \subset M\) be a compact subset. Let \(I, J\) be multi-indices, with each subindex running in \(I, n\). When given \(I \times J\) Lagrangians \(L_1 (x, f(x), f_x(x))\), where \(f(x)\) has multi-components,
the associated least squares Lagrangian with respect to the Riemannian metrics \( G^I_j(x)G^K_L(x) \), induced by the Riemannian metric \( g_{ij} \), is:

\[
\mathcal{L} = \frac{1}{2} G^I_j(x)G^K_L(x) L^I_H(x, f(x), f_x(x)) L_H^J(x, f(x), f_x(x)) \sqrt{\det(g_{ij})}.
\]

The extremals are solutions of the Euler–Lagrange PDE system:

\[
\left( \frac{1}{2} \frac{\partial(G^I_jG^K_L)}{\partial x^m} L^I_HL_H^J + G^I_jG^K_L \frac{\partial L_H^J}{\partial x^m} \right) \sqrt{\det(g_{ij})} + \frac{1}{2} G^I_jG^K_L L_H^J \frac{\partial}{\partial x^m} \sqrt{\det(g_{ij})} - D_I \left( G^I_jG^K_L L_H^J \sqrt{\det(g_{ij})} \frac{\partial L_H^J}{\partial f^m} \right) = 0.
\]

If the Lagrangian \( L^I_H \) is associated to the PDEs system \( L^I_H(x, f(x), f_x(x)) = 0 \), then the extremals contain the solutions of that system and the Euler–Lagrange dynamics are decomposable.

The best example is the least squares Lagrangian with respect to the Riemannian metrics \( G^I_j(x)G^K_L(x) \).

**Remark 1.** If we need to subject the Euler–Lagrange PDEs to boundary conditions, then instead of \( M \) we use \( \Omega \) as a compact, \( n \)-dimensional submanifold of \( M \) with a boundary (a piece of \( M \)).

### 1.4. Dual Variational Principle

Let \((M, g)\) be a Riemannian manifold. Usually, the local components of the metric \( g \) are denoted by \( g_{ij} \) and the components of the inverse \( g^{-1} \) are denoted by \( g^{ij} \). Due to the musical isomorphism between the tangent bundle \( TM \) and the cotangent bundle \( T^*M \) of a Riemannian manifold induced by its metric tensor \( g \), the arbitrary variations of \( g_{ij} \) are equivalent to the arbitrary variations of \( g^{ij} \), and any Lagrangian with respect to \( g_{ij} \) can be regarded as a Lagrangian in relation to \( g^{ij} \), but the differential orders are different. Consequently, a functional depending on the metric is stationary with respect to variations of the metric if and only if the same functional depending on the metric is stationary with respect to variations of the inverse metric. Noting this fact, we see that we can choose to perform either variations with respect to the metric or its inverse depending on which is more convenient.

When calculating the variation with respect to \( g^{ij} \), certain terms may appear whose integral over any domain \( \Omega \) can be reduced via the divergence theorem (integration by parts) to an integral over the boundary \( \partial \Omega \), which vanish (variations vanish on boundary). Modulo this statement, and the Euler–Lagrange PDEs are reduced to \( \frac{\partial \mathcal{L}}{\partial g^{ij}} = 0 \) (the formal partial derivatives are equal to zero).

### 1.5. Positive Definite Differential Operator

For an \( n \times n \) matrix of numbers or functions, positive definiteness is equivalent to the fact that its leading principal minors are all positive (\( n \) inequalities).

For an \( n \times n \) matrix of partial derivatives operators, positive definiteness is equivalent to the fact that its leading principal minors are all positive (\( n \) partial differential inequalities). For differential inequalities, see also [22].

### 2. Least Squares Lagrangian Density Attached to \( \nabla \)-Flatness

Let \((M, g)\) be a smooth oriented Riemannian manifold. The Riemannian metric \( g \) of components \( g_{ij} \) and its inverse \( g^{-1} \) of components \( g^{ij} \) determine (locally) the Christoffel symbols of the second kind:

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) = \frac{1}{2} g^{jl} (\delta^i_l \delta^k_j + \delta^i_k \delta^j_l - \delta^i_j \delta^k_l) \frac{\partial g_{rs}}{\partial x^l},
\]

\(i,j,k,l = 1, n\) (overdetermined elliptic partial differential operator).
From a physical point of view, the Riemannian metric is the gravitational potential and the connection reflects the gravitational field. The $\nabla$-flatness PDEs system $\Gamma_{jk}^i = 0$ is

$$\frac{1}{2} \delta_{ij}^l \left( \delta^j_k \delta^l_i \delta^1_m - \delta^j_i \delta^l_k \delta^1_m \right) \frac{\partial g_{rs}}{\partial x^l} = 0 \iff \frac{\partial g_{rs}}{\partial x^l} = 0$$

on the space of Riemannian metrics $\mathcal{S}_q^2 T^* M$, i.e., $\frac{n^2(n+1)}{2}$ consists of distinct first order non-linear nonhomogeneous PDEs whose unknowns are $n^{n+1}$ functions $g_{ij}$ (positive definite tensor); $n > 1$ indicates an overdetermined system of PDEs; $n = 1$ indicates a determined system. This PDEs system is symmetric in $j,k$. Imposing the initial condition $g_{ij}(0) = \delta_{ij}$, we find the solution $g_{ij}(x) = \delta_{ij}$ (Euclidean manifold).

The square of the norm $L = ||\nabla||^2 = g_{ip}g^{ip} g^{kr} \Gamma_{jk}^r$ is a Lagrangian density of first order with respect to $g_{ij}$ and of order zero with respect to $g_{ij}$. The functional describing $\nabla$-flatness deviation is $I = \int_M ||\nabla||^2 d\mu$. This can be considered as a functional of $g$, i.e., $I = I(g)$, and then we consider variations with respect to $g$, or a functional of $g^{-1}$, i.e., $I = I(g^{-1})$, and variations in relation to $g^{-1}$. Though the second is more simple, from variational point of view, let us begin the study with $I(g)$, whose associated Lagrangian $L = ||\nabla||^2 \sqrt{\det(g_{ij})}$ is of first order in $g_{ij}$.

**Theorem 1.** The extremals $g$ of $I(g)$, i.e., the solutions of PDEs

$$\Gamma_{jk}^r g_{ip} g^{kr} \left[ g^{lp} \delta^m_p \delta^m_k g_{ij} + g_{lm} g^m_{jk} + \frac{1}{2} g_{lm} g^m_{ij} \right] \sqrt{\det(g_{ij})} = 0$$

split into two categories: $g_{ij}(x) = \delta_{ij}$ (global minimum points, i.e., solutions of $\nabla$-flatness) and local minimum points of $I(g)$.

**Proof.** The extremals $g_{mn}$ of the Lagrangian $L$ are solutions of Euler–Lagrange PDEs:

$$\frac{\partial L}{\partial g_{mn}} - D_x \left[ \frac{\partial L}{\partial (\partial_x g_{mn})} \right] = 0.$$

These critical points are global (when $L = 0$) or local (when $L \neq 0$). Suppose $L \neq 0$. Based on obvious formulas:

$$\frac{\partial g_{ij}}{\partial g_{mn}} = \delta^m_j \delta^n_i, \quad \frac{\partial g^{ij}}{\partial g_{mn}} = -g^{mj} g^{nk}, \quad \frac{\partial}{\partial g_{mn}} \det(g_{ij}) = \det(g_{ij}) g_{mn}, \quad \frac{\partial (\partial_x g_{rs})}{\partial (\partial_x g_{mn})} = \delta^r_i \delta^s_j \delta^m_n,$$

we obtain:

$$\frac{\partial L}{\partial g_{mn}} = \left[ g^{lp} \delta^m_p \delta^m_k g_{ij} + g_{lm} g^m_{jk} + \frac{1}{2} g_{lm} g^m_{ij} \right] \sqrt{\det(g_{ij})} = 0,$$

$$\frac{\partial L}{\partial (\partial_x g_{mn})} = 2 g_{ip} g^{ip} g_{iq} g^{qr} \sqrt{\det(g_{ij})} \frac{\partial \Gamma_{jk}^r}{\partial (\partial_x g_{mn})} \frac{\partial g_{mp}}{\partial g_{mn}} \frac{\partial \Gamma_{jk}^r}{\partial (\partial_x g_{mn})} \sqrt{\det(g_{ij})} = 0,$$

$$= g_{ip} g^{ip} g_{iq} g^{qr} \left( \delta^m_j \delta^m_k + \delta^m_i \delta^m_k - \delta^m_i \delta^m_j \right) \frac{\partial \Gamma_{jk}^r}{\partial (\partial_x g_{mn})} \frac{\partial g_{mp}}{\partial g_{mn}} \frac{\partial \Gamma_{jk}^r}{\partial (\partial_x g_{mn})} \sqrt{\det(g_{ij})}$$

$$= g_{ip} g^{ip} g_{iq} g^{qr} \left( \delta^m_j \delta^m_k + \delta^m_i \delta^m_k - \delta^m_i \delta^m_j \right) \delta^m_l \delta^m_n \sqrt{\det(g_{ij})}$$
\[ = g^{ij} g^{kr} \left( \delta^m \delta^n \delta^l + \delta^m \delta^n \delta^l - \delta^m \delta^n \delta^k \right) \Gamma^s_{qr} \sqrt{\det(g_{ij})}. \]

The explicit Euler–Lagrange PDEs are those in the Theorem. Now let us compute the Hessian matrix of components:

\[ H_{(mn)(abc)} = \frac{\partial^2 L}{\partial (\partial_{ij} g_{mn}) \partial (\partial_{kl} g_{bc})} = g^{ij} g^{kr} \left( \delta^m \delta^n \delta^l + \delta^m \delta^n \delta^l - \delta^m \delta^n \delta^k \right) \frac{\partial \Gamma^s_{qr}}{\partial (\partial_{kl} g_{bc})} \]

\[ = \frac{1}{2} g^{ij} g^{kr} \left( \delta^m \delta^n \delta^l + \delta^m \delta^n \delta^l - \delta^m \delta^n \delta^k \right) g^{uv} \left( \delta^q \delta^r \delta^s + \delta^q \delta^r \delta^s - \delta^q \delta^s \delta^r \right) \frac{\partial (\partial_{kl} g_{bc} \delta_v)}{\partial (\partial_{ij} g_{mn})}. \]

This matrix is invariant if one interchanges \( l \) with \( a \) and the (un-ordered) pair \( m, n \) with the (un-ordered) pair \( b, c \), which must happen with a mixed derivative. Since the matrix \( H \) is positive and definite, all extremals are minimum points (Legendre–Jacobi criterion).

2.1. Homothetic Flat Extremals

The extremals \( g \) of \( I(g) \) are Euler–Lagrange prolongations (the best approximations) of the flat solutions \( g_{ij}(x) = \delta_{ij} \). Let us show that the Euclidean metrics extremals are stable with respect to conformal changing.

To simplify the problem, we consider a two-dimensional manifold with the Riemannian metric \( g_{11} = f, g_{22} = h, g_{12} = 0 \). Then the least squares Lagrangian is:

\[ \mathcal{L}(g) = g_{ip} g^{ij} \Gamma^q_{ij} \Gamma^p_{qr} \sqrt{\det(g_{ij})} = L \sqrt{\det(g_{ij})}, \]

and the Euler–Lagrange PDEs are:

\[ \frac{\partial \mathcal{L}}{\partial g_{mn}} - D_{xi} \frac{\partial \mathcal{L}}{\partial (\partial_{xi} g_{mn})} = 0. \]

We find:

\[ \frac{\partial \mathcal{L}}{\partial g_{mn}} = \sqrt{\det(g_{ij})} \left( \frac{\partial L}{\partial g_{mn}} + \frac{1}{2} L g^{mn} \right). \]

The Lagrangian density

\[ L = 2g_{11}g^{11}g^{22}(\Gamma^1_{12})^2 + g_{11}(g^{11})^2(\Gamma^1_{11})^2 + g_{11}(g^{22})^2(\Gamma^1_{22})^2 + g_{22}(g^{11})^2(\Gamma^2_{11})^2 + 2g_{22}g^{11}g^{22}(\Gamma^2_{12})^2 + g_{22}(g^{22})^2(\Gamma^2_{22})^2 \]

becomes

\[ L = \frac{3}{4f^2h} g^{11} + \frac{3}{4f^2h} g^{22} + \frac{1}{4f^2h} g^{11} + \frac{1}{4f^2h} g^{22} \cdot 2. \]

We get:

\[ \frac{\partial L}{\partial g_{11}} = \frac{3f_2^2}{2f^2h} - \frac{3h_1^2}{4f^2h^2} - \frac{3f_2^2}{4f^2h^2} - \frac{3h_1^2}{4f^2h^2} - \frac{3h_2^2}{4f^2h^2} + \frac{1}{4f^2h} \]

\[ \frac{\partial L}{\partial g_{11}} = \sqrt{f^2h} \frac{f_1}{2f^2h}, \quad \frac{\partial L}{\partial g_{11}} = \sqrt{f^2h} \frac{f_2}{2f^2h}. \]

\[ \frac{\partial L}{\partial g_{22}} = \sqrt{f^2h} \frac{h_1}{2f^2h}, \quad \frac{\partial L}{\partial g_{22}} = \sqrt{f^2h} \frac{h_2}{2f^2h}. \]
It follows the Euler–Lagrange PDEs system:

\[
\sqrt{\mathcal{h}} \left( -\frac{9f_1^2}{8f^3} \frac{3h_1^2}{8f^2 h^2} - \frac{5f_1^2}{8f^3} + \frac{h_1^2}{8fh^3} \right) - D_{x^3} \left( \sqrt{\mathcal{h}} \frac{f_1}{2f^3} \right) - D_{x^2} \left( \sqrt{\mathcal{h}} \frac{3f_2}{2fh} \right) = 0.
\]

\[
\sqrt{\mathcal{h}} \left( -\frac{3f_2^2}{8f^2 h^2} + \frac{9h_1^2}{8fh^3} - \frac{5h_1^2}{8fh^3} + \frac{f_1^2}{8f^3 h} \right) - D_{x^4} \left( \sqrt{\mathcal{h}} \frac{3h_1}{2fh^2} \right) - D_{x^2} \left( \sqrt{\mathcal{h}} \frac{h_2}{2h^3} \right) = 0.
\]

**Remark 2.** If \( f = f(x^1), h = h(x^2) \), then the previous PDEs system is reduced to:

\[
\sqrt{\mathcal{h}} \left( -\frac{5f_1^2}{8f^4} + \frac{f_1^2}{8fh^2} \right) - D_{x^3} \left( \sqrt{\mathcal{h}} \frac{f_1}{2f^3} \right) = 0,
\]

\[
\sqrt{\mathcal{h}} \left( -\frac{5h_1^2}{8fh^4} + \frac{h_1^2}{8fh^3} \right) - D_{x^2} \left( \sqrt{\mathcal{h}} \frac{h_2}{2h^3} \right) = 0.
\]

**Remark 3.** If \( f = h \) (conformal case), then one gets the PDEs system:

\[
2f(f_1^2 + f_2^2) + (ff_{11} - 2f_1^2) + 3(ff_{22} - 2f_2^2) = 0,
\]

\[
2f(f_1^2 + f_2^2) + 3(ff_{11} - 2f_1^2) + (ff_{22} - 2f_2^2) = 0,
\]

which is equivalent to:

\[
f(f_1^2 + f_2^2) + 2(ff_{22} - 2f_2^2) = 0, \quad f(f_1^2 + f_2^2) + 2(ff_{11} - 2f_1^2) = 0.
\]

Since \( f \) must be positive throughout, this system of PDEs has only solutions of the form \( f(x) = c > 0 \) (see Maple (pde, pdsolve(pde))). The metrics with \( c > 0, c \neq 1 \) are homothetic to \( \delta_{ij} \). Consequently, the Euclidean metrics extremals are stable with respect to conformal changing.

For comparison we use \( \mathcal{L}(g^{-1}) = g^{ij} s^{ij} \sqrt{\mathcal{g}} g^{mn} \sqrt{\det(g_{ij})} \), the variations with respect to \( g^{mn} \), and the general form of Euler–Lagrange PDEs \( \frac{d}{\delta g_{ij}} = 0 \) (equation of motion for the metric tensor field), and we formulate the following:

**Theorem 2.** The extremals \( g = (g_{ij}) \) of \( \mathcal{L}(g^{-1}) \) are solutions of the PDEs system:

\[
g^{ij} \left( -2g_{mp} g_{ni} s^{ij} + g_{mi} g_{np} s^{ij} - 2g_{ip} \delta^j_m \delta^l_p + \frac{1}{2} g_{ip} s^{ij} g_{mn} \right) g^{ij} g^{mn} g^{ij} = 0.
\]

For calculus of the matrix \( H_{(ab);(mn)} = \frac{\partial \mathcal{L}}{\partial g_{ab} \partial g_{mn}} \), we need \( \frac{\partial \Delta^i}{\partial g_{ab}} = \delta^i_a \delta^i_b \) and \( \frac{\partial \Delta^i}{\partial g_{mn}} = -g_{ma} g_{nb} \). We find:

\[
H_{(ab);(mn)} = 2g_{ip} g_{ma} g_{nb} + g^{ij} \left[ 2g_{ap} g_{bi} g_{mn} + (4g_{mp} g_{ni} - g_{ip} g_{mn} - 2g_{mi} g_{np}) g_{im} g_{mn} g_{ij} \right]
\]

\[
+ \frac{1}{2} g^{ij} (g_{ip} g_{ma} g_{nb} - g_{ap} g_{mn} g_{ib}) g_{im} g_{mn} g_{ij}.
\]

This matrix is not definite (neither positive nor negative), since it vanishes in the center of normal coordinates. This is why this matrix is of no help in determining what extremals could be extremum points.
2.2. Homothetic Flatness Extremals

General dimension: Let \((M, g_{ij})\) be a Riemannian manifold of dimension \(n\). Are there are extremals of the type \(g_{ij}(x) = f(x)\delta_{ij}, \ f(x) > 0\)?

Since
\[
\Gamma^i_{jk} = \frac{1}{2}f \left( f_{ii} \delta^i_k + f_k \delta^i_j - f_{jk} \right), \ i, j, k = 1, \ldots, n,
\]
the Euler–Lagrange PDEs are reduced to:
\[
\sum_t f_t^2 \delta_{ij} \left( \frac{3n}{2} - 7 \right) + f_t f_t (6 - 3n) = 0.
\]

It follows that \(f_k = 0, \forall k = 1, \ldots, n\). Therefore \(f(x) = c\) and \(g_{ij} = c\delta_{ij}\). The metrics with \(c > 0, c \neq 1\) are homothetic to \(\delta_{ij}\). Consequently, the Euclidean metric extremals are stable with respect to conformal changing.

Bidimensional case: Let us consider a two-dimensional Riemannian manifold with the metric \(g_{11} = 1, g_{22} = h, g_{12} = 0\). Let us show again that the Euclidean metric extremals are stable with respect to conformal changing.

In this case:
\[
\mathcal{L}(g^{-1}) = g_{ip}g_{jq}g_{kr}g^p_{kl}g^r_{mn}\sqrt{\det(g_{ij})} = L\sqrt{\det(g_{ij})}
\]
and the general form of Euler–Lagrange PDEs system is:
\[
\frac{\partial \mathcal{L}}{\partial g^{mn}} = \sqrt{\det(g_{ij})} \left( \frac{\partial L}{\partial g^{mn}} - \frac{1}{2}Lg_{mn} \right) = 0.
\]

Since:
\[
L = \frac{3}{4f^2h}g_{12}^2 + \frac{3}{4fh^2}g_{22}^2 + \frac{1}{4f^3}g_{11}^2 + \frac{1}{4h^3}g_{22}^2,
\]
\[
\frac{\partial L}{\partial g^{11}} = \frac{3f_2^2}{2fh} + \frac{3h_1^2}{4h^2} + \frac{3f_1^2}{4f^2}, \quad \frac{\partial L}{\partial g^{22}} = \frac{3f_2^2}{4f^2} + \frac{3h_1^2}{2fh} + \frac{3h_2^2}{4h^2},
\]
the Euler–Lagrange PDEs system becomes:
\[
12fh^2f_2^2 + 6f^2h^2 - 3fh^2f_2^2 - 3f^2h^2 - h^3f_1^2 - f^3f_2^2 = 0,
\]
\[
6fh^2f_2^2 + 12f^2h^2 - 3fh^2f_2^2 - 3f^2h^2 - h^3f_1^2 - f^3f_2^2 = 0.
\]

Remark 4. If \(f = f(x^1), h = h(x^2)\), then the Euler–Lagrange PDEs are reduced to \(h^3f_1^2 = 0, f^3h_2^2 = 0, i.e., f_1 = 0, h_2 = 0\) (Euclidean case).

Remark 5. The conformal case \(f = h\) leads to \(f_1^2 + f_2^2 = 0\), and we get \(f_1 = f_2 = 0, i.e., f\) is constant (confirming the general case).

3. Least Squares Lagrangian Density Attached To Riemann-Flatness

Let \(\nabla\) be a symmetric connection of components \(\Gamma^l_{jk}\) and \(g\) be a Riemannian metric of components \(g_{ij}\). We use the operator \(P^p_{jk} = \frac{1}{2} \left( \delta^p_k \delta^l_j - \delta^p_j \delta^l_k \right)\), which is a projection, i.e., \(P^2 = P\), and is covariantly constant. The Riemann-flatness PDEs system is either \(\text{Riem} \nabla = 0\) or
\[
R^l_{ijk} = \frac{\partial}{\partial x^l} \Gamma^l_{jk} - \frac{\partial}{\partial x^l} \Gamma^l_{kj} - \Gamma^l_{js} \Gamma^s_{kj} - \Gamma^s_{ks} \Gamma^s_{ij} = 2P^p_{jk} \left( \delta^p_l \Gamma^l_{ks} - \Gamma^l_{ps} \Gamma^p_{ks} \right).
\]
where $i, j, k, l, \ldots = 1, n$, and has the general solution $\Gamma_{jk}^i = 0$.

Each of the Riemann-flatness PDE systems $R_{ij}^{k} = 0$ is a system of $\frac{n^2(n^2-1)}{12}$ distinct first order linear quadratic PDEs whose unknowns are $2^{\frac{n(n+1)}{2}}$ functions $\Gamma_{jk}^i, n > 7$ indicates an overdetermined system; $n < 7$ indicates an underdetermined system; $n = 7$ indicates a determined system.

The curvature flatness was discussed in [11–15, 17] based on the idea of finding an adapted coordinate system. We bring up another point of view, looking for suitable metrics and connections, and not for adapted coordinate systems.

On the smooth oriented manifold $(M, \nabla, g)$, we introduce the square of the norm $L = \|\text{Riem}^g\|^2 = \delta_{ij}q_{ik}^j R_{ijkl}^p R_{pqrs}^p$, which is a Lagrangian density of first order in $\Gamma_{jk}^i$. It determines a functional (Riemann-flatness deviation) similar to the Yang–Mills functional, namely $I(\nabla) = \int_M \|\text{Riem}^g\|^2 d\mu$. The extremals $\nabla$ of $\mathcal{L}(\nabla, \partial \nabla) = \|\text{Riem}^g\|^2 \sqrt{\det(g_{ij})}$ are solutions of the Euler–Lagrange PDEs

$$\frac{\partial L}{\partial g_{ij}} - D_i \frac{\partial L}{\partial (\partial_j g_{ij})} = 0$$

(equation of motion for the connection).

**Theorem 3.** The explicit form of Euler–Lagrange PDEs attached to the Lagrangian $\mathcal{L}(\nabla, \partial \nabla)$ is:

$$(\partial_i q_{[ik]}^j R_{ij}^{[kl]} + \delta_{ij} q_{ik}^j R_{ijkl}^p R_{pqrs}^p) = D_i \left[ \partial_j q_{ik}^j R_{ijkl}^p q_{pqrs}^p \sqrt{\det(g_{ij})} \right] - D_j \left[ \partial_i q_{ik}^j R_{ijkl}^p q_{pqrs}^p \sqrt{\det(g_{ij})} \right] = 0.$$  

The Riemann-flatness solutions $\Gamma_{jk}^i = 0$ are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

Let $(M, g = (g_{ij}))$ be a Riemannian manifold. The Riemannian metric $(g_{ij})$ determines the Riemannian curvature tensor field $\text{Riem}^g$ of components:

$$R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} \right) + g_{mn} (\Gamma_{jk}^m \Gamma_{il}^n - \Gamma_{il}^m \Gamma_{jk}^n).$$

where:

$$\delta_{[ij]} = \delta_i \delta_j - \delta_j \delta_i,$$

$$\Gamma_{jk}^i = \frac{1}{2} \left( \delta_i \delta_j \delta_k - \delta_j \delta_i \delta_k \right) + \frac{1}{2} \left( \delta_i \delta_j \delta_k - \delta_k \delta_i \delta_j \right) \frac{\partial g_{mn}}{\partial x^l}.$$

In this case Riemannian curvature flatness condition means the tensorial PDEs system

$$\frac{1}{2} \left( \delta_i \delta_j \delta_k - \delta_k \delta_i \delta_j \right) \frac{\partial g_{mn}}{\partial x^l} = 0,$$

on $S^2 \times T^* M$, with $\frac{n^2(n^2-1)}{12}$ distinct second-order linear non-homogeneous PDEs whose unknowns are $\frac{n(n+1)}{2}$ functions $g_{ij}$ (positive definite tensor); for $n < 3$, undetermined system; $n > 3$ indicates an overdetermined system; $n = 3$ indicates a determined system. This PDEs system is parabolic since the set of eigenvalues of the matrix $\Gamma_{ijkl} = \delta^{[i}_{[j]} \delta^{k}_{[l]} \delta^{l}_{[k]}$ (tensorial product of a matrix by itself) contains the eigenvalue 0. Indeed, for all eigenvectors, the eigenvalues are: $X_{ijkl}$-symmetric in $(i, j)$ or in $(l, k)$, with $\lambda = 0$; $X_{ijkl}$-skewsymmetric in $(i, j)$ and in $(l, k)$ with $\lambda = 2$. Of course, this PDEs system has all of the properties of a curvature tensor field.

In the general relativity, the Riemann tensor field is a physical observable quantity.

On the Riemannian manifold $(M, g = (g_{ij}))$, we introduce the square of the norm $L = \|\text{Riem}^g\|^2 = \delta_{ij} q_{ik}^j R_{ijkl}^p R_{pqrs}^p$, which is of second order with respect to $g_{ij}$ and of order zero with respect to $g^j$. The Riemann-flatness deviation is described by the functional $I = \int_M \|\text{Riem}^g\|^2 d\mu$. This can be considered as a functional of $g$, and then we consider variations with respect to $g$, or a functional of $g^{-1}$, and finally variations in relation to $g^{-1}$.  


For $I(g)$ the extremals $g$ are solutions of fourth-order Euler–Lagrange PDEs
\[
\frac{\partial L}{\partial g_{mn}} - D_x \frac{\partial L}{\partial (\partial_x g_{mn})} + D_x D_x \frac{\partial L}{\partial (\partial_x^2 g_{mn})} = 0,
\]
while for $I(g^{-1})$ the Euler–Lagrange PDEs are reduced to $\frac{\partial L}{\partial g_{mn}} = 0$ (equation of motion for the metric tensor field).

**Theorem 4.** The extremals $g = (g_{ij})$ of the functional $I(g^{-1})$ are solutions of the PDEs system
\[
-2 \delta^{cd}_{[i} R_{pqrs} g^{ap} g^{bq} g^{c}^{kr} g^{d}^{ls} g_{mn} g_{pqrs} \Gamma^{0}_{bc} \Gamma^{0}_{ad} + 2 R_{ijkl} R_{pqrs} (\delta^{ij}_{m} \delta^{pq}_{n} g^{r} g^{s} + \delta^{ij}_{m} \delta^{pq}_{n} g^{r} g^{s} g_{mn} g_{pqrs} \Gamma^{0}_{bc} \Gamma^{0}_{ad}) - \frac{1}{2} R_{ijkl} R_{pqrs} (\delta^{ij}_{m} \delta^{pq}_{n} g^{r} g^{s} g_{mn} g_{pqrs} \Gamma^{0}_{bc} \Gamma^{0}_{ad}) g_{kk} g_{mm} = 0.
\]

The Riemann-flat solutions $g_{ij}(x) = \delta_{ij}$ are global minimum points. So are the metrics obtained from $\delta_{ij}$ by changing variables, such as
\[
g(x) = \text{diag} \left( \frac{1}{h_1(x)^2}, \ldots, \frac{1}{h_n(x)^2} \right).
\]
The other solutions are best approximation of flatness PDEs solutions.

### 3.1. Non-Flat Extremals

We consider a two-dimensional Riemannian manifold $(M, g)$, where $g_{11} = f, g_{22} = h, g_{12} = 0$. In this case $L = (g^{11} g^{22} R_{1212})^2 \sqrt{\det(g_{ij})}$, and
\[
R_{1212} = -\frac{1}{2} (g_{11,22} + g_{22,11}) + g_{ab} (\Gamma^a_{b1} \Gamma^b_{12} - \Gamma^a_{b2} \Gamma^b_{11}) = -\frac{1}{2} (g_{11,22} + g_{22,11}) + g_{11} (\Gamma^1_{21} \Gamma^2_{12} - \Gamma^1_{22} \Gamma^2_{11}) + g_{22} (\Gamma^2_{12} \Gamma^2_{11} - \Gamma^2_{22} \Gamma^2_{11})
\]
or
\[
R_{1212} = -\frac{1}{2} (h_{11} + f_{22}) + \frac{1}{4 f} (f_{2}^2 + f_1 h_1) + \frac{1}{4 h} (h_1^2 + f_2 h_2).
\]
We thus get:
\[
\frac{\partial R_{1212}}{\partial g_{11}} = \frac{1}{4} (f_{2}^2 + f_1 h_1), \quad \frac{\partial R_{1212}}{\partial g_{22}} = \frac{1}{4} (h_1^2 + f_2 h_2), \quad \frac{\partial R_{1212}}{\partial g_{12}} = 0.
\]

The Euler–Lagrange PDEs $\frac{\partial L}{\partial g_{mn}} = 0$ become the following system of equations:
\[
2 g^{11} (g^{22})^2 R_{1212} + 2 (g^{11} g^{22})^2 R_{1212} \frac{\partial R_{1212}}{\partial g_{11}} + \frac{1}{2} (g^{11} g^{22} R_{1212})^2 g_{11} = 0,
\]
\[
2 g^{22} (g^{11})^2 R_{1212} + 2 (g^{11} g^{22})^2 R_{1212} \frac{\partial R_{1212}}{\partial g_{22}} + \frac{1}{2} (g^{11} g^{22} R_{1212})^2 g_{22} = 0.
\]

Explicitly,
\[
\frac{R_{1212}}{2 f^2 h^2} (3 f R_{1212} + f_{2}^2 + f_1 h_1) = 0, \quad \frac{R_{1212}}{2 f^2 h^2} (3 f R_{1212} + h_1^2 + f_2 h_2) = 0,
\]

**Case 1:** $R_{1212} = -\frac{1}{2} (h_{11} + f_{22}) + \frac{1}{4 f} (f_{2}^2 + f_1 h_1) + \frac{1}{4 h} (h_1^2 + f_2 h_2) = 0$ produces the Euclidean metric.
Case 2: $3fR_{1212} + f_2^2 + f_1 h_1 = 0$, $3hR_{1212} + h_1^2 + f_2 h_2 = 0$. Equivalently

$$3fR_{1212} + f_2^2 + f_1 h_1 = 0, \ h(f_2^2 + f_1 h_1) - f(h_1^2 + f_2 h_2) = 0.$$ 

The conformal case: $f = h$ becomes:

1. $R_{1212} = -\frac{1}{2}(f_{11} + f_{22}) + \frac{1}{2}(f_1^2 + f_2^2) = 0$, i.e., Euclidean space.
2. $3fR_{1212} + f_2^2 + f_2^2 = 0$. Therefore $3(f_{11} + f_{22}) = 5(f_1^2 + f_2^2)$ or $3f = 5\|\text{grad} \ f\|^2$ (Poisson PDE). Maple answer (pde, sol := pdsolve(pde)):

This PDE has solutions of the form $f(x^1, x^2) = \varphi_1(x^1)\varphi_2(x^2)$, where:

$$\frac{d^2 \varphi_1}{dx^2}(x) = \frac{1}{3} c_1 \varphi_1(x) + \frac{5}{3} \left(\frac{d \varphi_1}{dx}(x)\right)^2, \quad \frac{d^2 \varphi_2}{dy^2}(y) = \frac{1}{3} c_1 \varphi_2(y) + \frac{5}{3} \left(\frac{d \varphi_2}{dy}(y)\right)^2$$

or

$$C_1 \sin((x/3)\sqrt{2c_1}) + C_2 \cos((x/3)\sqrt{2c_1}) = \frac{\sqrt{2c_1}}{\varphi_1(x)^{2/3}} = 0,$$

$$C_3 \sin((y/3)\sqrt{2c_1}) + C_4 \cos((y/3)\sqrt{2c_1}) = \frac{\sqrt{2c_1}}{\varphi_2(y)^{2/3}} = 0.$$ 

Globally, these solutions are not convenient since they are not strictly positive.

We have two particular cases: (a) If $f_2 = 0$, then $f(x^1, x^2) = f(x^2)$ and hence $-3f_{22} + 5f_2^2 = 0$, a Liouville equation with the general solution

$$\frac{3}{2f(x^2)^{2/3}} + c_1 x^2 + c_2 = 0.$$ 

This function is strictly positive only locally. (b) If $f_2 = 0$, then $f(x^1, x^2) = f(x^1)$ and hence $-3f_{11} + 5f_1^2 = 0$, a Liouville equation with the general solution

$$\frac{3}{2f(x^1)^{2/3}} + c_1 x^1 + c_2 = 0.$$ 

This function is strictly positive only locally.

Theorem 5. The extremals $g = (g_{ij})$ of the Lagrangian

$$\mathcal{L}(g, \partial g, \partial^2 g) = ||\text{Riem} g||^2 \sqrt{\text{det}(g_{ij})}$$

are solutions of the PDEs system

$$R_{pqrs} S^{kr} \left[2(\Gamma^m_{jk} \Gamma^m_{il} - \Gamma^n_{jl} \Gamma^n_{km}) S^{lp} S^{lq} - R_{ijkl} \left(2(S^m_{ji} S^{np} S^{lq} + S^{jp} S^{qi} S^{lm}) - \frac{1}{2} S^{mn} S^{lq} S^{jp}\right)\right] \sqrt{\text{det}(g_{ij})}$$

$$+ D_{x^k} \left[\delta^i_{x^k} S^{ip} S^{lj} + \delta^j_{x^k} S^{ip} S^{li}\right] \left[\Gamma^m_{ad} (\delta^b_{x^d} S^{i} + \delta^b_{x^d} S^{j}) - \Gamma^h_{ad} S^{i} S^{j}\right] R_{pqrs} S^{ij} S^{kr} S^{ls} \sqrt{\text{det}(g_{ij})}$$

$$+ D_{x^k} S^{ij} S^{kr} S^{ls} \left[\delta^{ip} S^{lj} + \delta^{jp} S^{li}\right] R_{pqrs} \sqrt{\text{det}(g_{ij})} = 0.$$ 

The Riemann-flat solutions $g_{ij}(x) = \delta_{ij}$ are global minimum points.
3.2. Non-Flat Extremals

We consider a two-dimensional Riemannian manifold \((M, \mathcal{g})\), \(\mathcal{g}_{11} = f, \mathcal{g}_{22} = h, \mathcal{g}_{12} = 0\). Then, \(\mathcal{L} = (\mathcal{g}^{11}\mathcal{g}^{22}\mathcal{R}_{1212})^2 \sqrt{\det(\mathcal{g}_{ij})}\), where:

\[
\mathcal{R}_{1212} = -\frac{1}{2} (\mathcal{g}_{11,22} + \mathcal{g}_{22,11}) + \mathcal{g}_{ab}(\mathcal{G}^b_{21} \mathcal{G}^b_{12} - \mathcal{G}^a_{11} \mathcal{G}^a_{12}) - \frac{1}{2} (h_{11} + f_{22}) + \frac{1}{4f}(f^2 + f_1 h_1) + \frac{1}{4h}(h^2 + f_2 h_2).
\]

Since

\[
\frac{\partial \mathcal{L}}{\partial \mathcal{g}_{mn}} = 2(\mathcal{g}^{11}\mathcal{g}^{22})^2 \mathcal{R}_{1212} \sqrt{\det(\mathcal{g}_{ij})} \frac{\partial \mathcal{R}_{1212}}{\partial \mathcal{g}_{mn}} + \frac{1}{2} \mathcal{L}\mathcal{g}^{mn} - 2\mathcal{g}^{11}\mathcal{g}^{22}(\mathcal{g}^{11}\mathcal{g}^{11} \mathcal{g}^{11} + \mathcal{g}^{11}\mathcal{g}^{22} \mathcal{g}^{22}) \mathcal{R}_{1212}^2 \sqrt{\det(\mathcal{g}_{ij})},
\]

the following Euler–Lagrange PDEs follow:

\[
\frac{\partial \mathcal{L}}{\partial \mathcal{g}_{11}} - D_{x_1} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{11}, \mathcal{g}_{11,1})} - D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{11}, \mathcal{g}_{11,2})} + D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{11}, \mathcal{g}_{11,1})} + 2D_{x_1} D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{11}, \mathcal{g}_{11,2})} + D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{11}, \mathcal{g}_{11,2})} = 0,
\]

\[
\frac{\partial \mathcal{L}}{\partial \mathcal{g}_{22}} - D_{x_1} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{22}, \mathcal{g}_{22,1})} - D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{22}, \mathcal{g}_{22,2})} + D_{x_1} D_{x_1} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{22}, \mathcal{g}_{22,1})} + 2D_{x_1} D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{22}, \mathcal{g}_{22,2})} + D_{x_2} D_{x_2} \frac{\partial \mathcal{L}}{\partial (\mathcal{g}_{22}, \mathcal{g}_{22,2})} = 0.
\]

Equivalently,

\[
- \left( 3 \frac{\mathcal{R}_{1212}}{2f^3 h^2} + \frac{1}{2f^2 h^2} \right) \mathcal{R}_{1212} \sqrt{f h} - D_{x_1} \left( \frac{h_1}{2f^3 h^2} \mathcal{R}_{1212} \sqrt{f h} \right)
\]

\[
- D_{x_2} \left[ \frac{1}{2f^2 h^2} \left( \frac{2f_2}{f} + \frac{h_2}{h} \right) \mathcal{R}_{1212} \sqrt{f h} \right] + D_{x_2} \left( -\frac{1}{2f^2 h^2} \mathcal{R}_{1212} \sqrt{f h} \right) = 0,
\]

\[
- D_{x_1} \left[ \frac{1}{2f^2 h^2} \left( \frac{f_1}{f} + \frac{2h_1}{h} \right) \mathcal{R}_{1212} \sqrt{f h} \right] - D_{x_2} \left( \frac{f_2}{2f^2 h^3} \mathcal{R}_{1212} \sqrt{f h} \right) + D_{x_3} \left( -\frac{1}{f^2 h^2} \mathcal{R}_{1212} \sqrt{f h} \right) = 0.
\]

The conformal case: \(f = h\). The PDE system becomes:

\[
- \frac{3}{2f^4} \mathcal{R}^2_{1212} - \frac{1}{2f^2} (f_2^2 + f_1^2) \mathcal{R}_{1212} - D_{x_1} \left( \frac{f_1}{2f^4} \mathcal{R}_{1212} \right)
\]
\[-D_{x^2} \left( \frac{3f_2}{2f_4} R_{1212} \right) + D_{x^2x^2} \left( - \frac{1}{f_4} R_{1212} \right) = 0,\]
\[-\frac{3}{2f_4} R_{1212}^2 - \frac{1}{2f_4} (f_2^2 + f_6^2) R_{1212} - D_{x^1} \left( \frac{3f_1}{2f_4} R_{1212} \right)\]
\[-D_{x^2} \left( \frac{f_2}{2f_4} R_{1212} \right) + D_{x^1x^1} \left( - \frac{1}{f_4} R_{1212} \right) = 0.\]

The case \( R_{1212} = -\frac{3}{2}(f_{11} + f_{22}) + \frac{1}{2f_4} (f_1^2 + f_2^2) = 0 \) produces the trivial solution \( g_{11} = g_{22}, g_{12} = 0. \)

We subtract the second equation from the first one and we get:
\[D_{x^1} \left( \frac{f_1}{f_4} R_{1212} \right) - D_{x^2} \left( \frac{f_2}{f_4} R_{1212} \right) + (D_{x^2x^2} - D_{x^1x^1}) \left( - \frac{1}{f_4} R_{1212} \right) = 0.\]

Particular cases: (a) \( f_1 = 0 \) and
\[D_{x^2} \left[ \frac{f_2}{f_4} (-ff_{22} + f_2^2) \right] + D_{x^2x^2} \left[ \frac{1}{f_4} (-ff_{22} + f_2^2) \right] = 0.\]

The second PDE is equivalent to \( \frac{f_1}{f_4} (-ff_{11} + f_1^2) + D_{x^1x^1} \left[ \frac{1}{f_4} (-ff_{11} + f_1^2) \right] = 0.\)

The Liouville equation \(-ff_{22} + f_2^2 = 0\) has the solution \( f(x^2) = ae^{bx^2}, a > 0. \)
(b) \( f_2 = 0 \) and
\[D_{x^1} \left[ \frac{f_1}{f_4} (-ff_{11} + f_1^2) \right] + D_{x^1x^1} \left[ \frac{1}{f_4} (-ff_{11} + f_1^2) \right] = 0.\]

The second PDE is equivalent to \( \frac{f_1}{f_4} (-ff_{11} + f_1^2) + D_{x^1} \left[ \frac{1}{f_4} (-ff_{11} + f_1^2) \right] = 0.\)

The Liouville equation \(-ff_{11} + f_1^2 = 0\) has the solution \( f(x^1) = ce^{dx^1}, c > 0. \)

4. Least Squares Lagrangian Density Attached to Ricci-Flatness

A torsion-free affine connection \( \nabla \), of components \( \Gamma_{ik}^j \), is called (locally) equiaffine if locally there is a volume form \( \omega \) (nonvanishing \( n \)-form) \( \omega \) that is parallel with respect to \( \nabla \). An affine connection \( \nabla \) with zero torsion is Ricci-symmetric if and only if \( \nabla \) is locally equiaffine.

Let \( (M, \nabla) \) be an equiaffine manifold. The components \( R_{ik} \) of the Ricci tensor field \( \text{Ric} \nabla \) are obtained by the contraction of the first and third indices of the curvature tensor field \( R_{ijkl} \), i.e.,
\[R_{ik} = R_{ilk} = \frac{\partial}{\partial x^l} \Gamma_{ik}^j - \frac{\partial}{\partial x^l} \Gamma_{ij}^k + \Gamma_{is}^j \Gamma_{ik}^s - \Gamma_{is}^s \Gamma_{ik}^j,\]
\[= \mathcal{P}_{pq}^{ps} \left( \frac{\partial}{\partial x^p} \Gamma_{is}^q + \Gamma_{pm}^q \Gamma_{is}^m \right) = \mathcal{P}_{pq}^{ps} \left( \frac{\partial}{\partial x^p} \Gamma_{is}^q - \Gamma_{sn}^q \Gamma_{ip}^n \right),\]
\(i, j, k, l, \ldots = 1, n\). Each of the Ricci-flatness PDEs systems \( R_{ik} = 0 \) is a system of \( \frac{n(n+1)}{2} \) distinct first order divergence quadratic tensorial PDEs with \( \frac{n^2(n+1)}{2} \) unknown functions \( \Gamma_{ik}^j \), \( n > 1 \) indicates an undetermined system; \( n = 1 \) indicates a determined system. Here \( \mathcal{P}_{pq}^{ps} = \delta_{ik}^{pq} \delta_{jk}^{ps} - \delta_{jk}^{pq} \delta_{ik}^{ps} \) works like a trace...
between \( p \) and \( q \), in order to produce a divergence term. This operator is associated to the projection \( P \). Any divergence PDE represents a conservation law.

Let \( g = (g_{ij}) \) be a Riemannian metric. On the smooth oriented manifold \((M, \nabla, g)\), let us consider the Lagrangian density \( L = \|Ric^g\|^2 = g^{ik}g^{jl}R_{ij}R_{kl} \) (square of the norm, first order in \( \Gamma^i_{jk} \)) and the functional (Ricci-flatness deviation) \( I(\nabla) = \int_M \|Ric^g\|^2 d\mu \). The Euler–Lagrange PDEs are

\[
\frac{\partial L}{\partial g^{ij}} - D_{x^l} \frac{\partial L}{\partial \partial_{x^l} g^{ij}} = 0 \quad \text{(equation of motion for the connection)}.
\]

**Theorem 6.** Let \( R_{ij} = \nabla^p \left( \frac{\partial}{\partial \nabla^q} \Gamma^q_{ik} + \Gamma^q_{ip} \Gamma^p_{kj} \right) \). The extremals \( \Gamma^i_{jk} \) of the Lagrangian \( \mathcal{L}(\nabla, \partial \nabla) = g^{ik}g^{jl}R_{ij}R_{kl} \sqrt{\text{det}(g_{ij})} \) are solutions of the PDE system:

\[
[\delta^i_u \Gamma^u_{ij} - \delta^i_j \Gamma^u_{iu} + \delta^i_j (\delta^u_l \Gamma^l_{iu} - \Gamma^u_{iul})]R_{kl}g^{ik}g^{jl} \sqrt{\text{det}(g_{ij})} - D_{x^l} \left( \delta^j_l \delta^u_l R_{kl}g^{ik}g^{jl} \sqrt{\text{det}(g_{ij})} \right) = 0.
\]

The Ricci-flat solutions \( \Gamma^i_{jk} \) are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

In case that \((M, g = (g_{ij}))\) is a Riemannian manifold, the Ricci tensor field \( Ric^g \) has the components

\[
R_{ik} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^m_{jk} \Gamma^i_{ml} - \nabla_k \left( \frac{\partial}{\partial x^l} \left( \ln \sqrt{\text{det}(g_{mn})} \right) \right),
\]

where

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} (\delta^j_l \delta^k_l + \delta^k_l \delta^j_l - \delta^j_l \delta^k_l) \frac{\partial g_{ls}}{\partial x^t}.
\]

The Ricci tensor field of a connection derived from a Riemannian metric is always symmetric. In this case, a Ricci-flat manifold

\[
\frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^m_{jk} \Gamma^i_{ml} - \nabla_k \left( \frac{\partial}{\partial x^l} \left( \ln \sqrt{\text{det}(g_{mn})} \right) \right) = 0
\]

means \( \frac{n(n+1)}{2} \) distinct PDEs with \( \frac{n(n+1)}{2} \) unknown functions \( g_{ij} \), on \( S^2_+ \mathcal{T}^* M \). They are special cases of Einstein manifolds, where the cosmological constant vanishes. In Physics, Ricci-flat manifolds represent vacuum solutions to the analogues of Einstein’s equations for Riemannian manifolds of any dimension, with a vanishing cosmological constant. In this context we recall some ideas of Professor Bang-Yen Chen (April 15, 2017): “The Ricci tensor is related to the matter content of the universe via Einstein’s field equation in general relativity theory. It is the part of the curvature of spacetime that determines the degree to which matter will tend to converge or diverge in time”.

Let \((M, g)\) be a Riemannian manifold. In this case the Ricci flatness was described in the papers [16,18,19,23–25] as locally underlining the difference between an “Euclidean ball” and a “geodesic ball”. Surprisingly, there are Ricci-flatness solutions that are not Riemann-flatness solutions, for example the Schwarzchild solution.

On the Riemannian manifold \((M, g = (g_{ij}))\), let us consider the Lagrangian density \( L = \|Ric^g\|^2 = g^{ik}g^{jl}R_{ij}R_{kl} \) (square of the norm), which is of second order in \( g_{ij} \) and order zero in \( g^{ij} \). The Ricci-flatness deviation is described by the functional \( I = \int_M \|Ric^g\|^2 d\mu \).

For \( I(g) \), the extremals \( g \) are solutions of fourth-order Euler–Lagrange PDEs \( \frac{\partial \delta}{\partial g^{ij}} - D_{x^l} \frac{\partial \delta}{\partial \partial_{x^l} g^{ij}} + D_{x^l} D_{x^m} \frac{\partial \delta}{\partial \partial_{x^l} \partial_{x^m} g^{ij}} = 0 \) (equation of motion for the metric tensor field). To simplify, we work first with \( I(g^{-1}) \), since the Euler–Lagrange PDEs determined by \( \mathcal{L} = g^{ik}g^{jl}R_{ij}R_{kl} \sqrt{\text{det}(g_{ij})} \) are reduced to \( \frac{\partial \delta}{\partial g^{ij}} = 0 \). Also for simplification we use harmonic coordinates when Ricci’s tensor has an easily readable formula [25]:

\[
R_{ij} = g^{kl} \left( -\frac{1}{2} \partial_k \partial_l g_{ij} + g_{mn} \Gamma^m_{ik} \Gamma^l_{mj} \right).
\]
**Theorem 7.** We fix a harmonic coordinate system. The extremals \( g = (g_{ij}) \) of the functional \( I(g^{-1}) \) are solutions of the PDEs system:

\[
2g^{ik}R_{ikm} + 2g^{jk}g^{il}R_{kl} \left( -\frac{1}{2}\frac{\partial^2 \varrho_i}{\partial x^m \partial x^n} + \varrho_{cd}\Gamma^c_{im} \Gamma^d_{nj} + \varrho^{cd}g_{mc}\varrho_n\Gamma^c_{ij} \right) - \frac{1}{2}g^{ik}g^{jl}R_{ik} R_{jl} g_{mn} = 0.
\]

The Ricci-flat solutions \( g_{ij}(x) \) are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

**Theorem 8.** We fix a harmonic coordinate system. The extremals \( g = (g_{ij}) \) of the functional \( I(g) \) are solutions of the PDEs system:

\[
\sqrt{\det(g)} \left[ R_{ij} R_{kl} \left( -2g^{mi}g^{nk}g_{il} + \frac{1}{2}g^{jk}g_{mn} \right) + 2g^{ik}g^{jl} \right] \\
\times R_{ij} \left[ -g^{mp}g^{nq} \left( -\frac{1}{2}\frac{\partial^2 \varrho_i}{\partial x^p \partial x^q} + \varrho_{cd}\Gamma^c_{ip} \Gamma^d_{jq} \right) - \varrho^{mn}\Gamma^m_{ip} \Gamma^l_{jq} \right] \\
- D_{x} \left[ \sqrt{\det(g)} \left[ \left( \frac{\partial \varrho_i}{\partial x^p} \frac{\partial \varrho_j}{\partial x^q} \right) + \left( \frac{\partial \varrho_i}{\partial x^p} \frac{\partial \varrho_j}{\partial x^q} \right) \right] \right] \\
\times \frac{\partial \varrho_i}{\partial x^p} \frac{\partial \varrho_j}{\partial x^q} \right] \right] \\
+ \left( \frac{\partial \varrho_i}{\partial x^p} \frac{\partial \varrho_j}{\partial x^q} \right) \left( \frac{\partial \varrho_i}{\partial x^p} \frac{\partial \varrho_j}{\partial x^q} \right) \right] \right] \\
\times \left[ \sqrt{\det(g)} \left( \varrho_{mn} g^{ik} g^{jl} R_{ik} R_{jl} g_{mn} \right) \right] = 0.
\]

The Ricci-flat solutions \( g_{ij}(x) \) are global minimum points. The other solutions are best approximation of flatness PDEs solutions.

5. Least Squares Lagrangian Density Attached to Scalar Curvature-Flatness

Let \( \nabla \) be an equiaffine connection of components \( \Gamma^i_{jk} \) and \( g = (g_{ij}) \) be a Riemannian metric, where \( i, j, k, \ldots = 1, n \). On the manifold \( (M, \nabla, g) \), we introduce the functional (total scalar curvature)
\[ I(\nabla) = \int_M \mathcal{R} \, d\mu, \]
where \( \mathcal{R} = g^{ij} \Gamma_{ij} \), and the Lagrangian \( \mathcal{L} = \mathcal{R} \sqrt{\det(g)} \) is of first order with respect to \( \Gamma^i_{jk} \). The general Euler–Lagrange PDEs are \( \frac{\partial \mathcal{L}}{\partial \Gamma_{mn}} - \frac{\partial \mathcal{L}}{\partial \partial_x \Gamma_{mn}} = 0 \) (equation of motion for the connection).

**Theorem 9.** The Euler–Lagrange PDEs attached to the functional \( I(\nabla) \), i.e., to the Lagrangian \( \mathcal{L} = g^{ik} R_{ik} \sqrt{\det(g)} \), are:

\[
\mathcal{P}_{pk}^{pq} \left( g^{ik} \left( \delta^{q}_{p} \delta^{m}_{i} \Gamma^{l}_{j} + \delta^{i}_{k} \delta^{m}_{l} \Gamma^{q}_{j} \right) \right) \sqrt{\det(g)} - \delta^{i}_{k} \delta^{m}_{l} D_{x} \left( g^{mn} \sqrt{\det(g)} \right) = 0.
\]

**Proof.** Since \( \mathcal{L} = g^{ik} R_{ik} \sqrt{\det(g)} \), \( R_{ik} = \mathcal{P}_{pk}^{pq} \left( \frac{\partial \Gamma^{q}_{j}}{\partial x^{i}} + \Gamma^{q}_{j} \Gamma^{p}_{i} \right) \), and

\[
\frac{\partial \Gamma^{i}_{jk}}{\partial \Gamma^{m}_{mn}} = \delta^{q}_{p} \delta^{m}_{i} \delta^{n}_{k}, \quad \frac{\partial \Gamma^{i}_{jk}}{\partial \partial_x \Gamma^{m}_{mn}} = \delta^{q}_{p} \delta^{m}_{i} \delta^{n}_{k},
\]

we obtain the PDEs in the Theorem. □

On a smooth oriented Riemannian manifold \( (M, \mathcal{g} = (g_{ij})) \), we attach the functional (total scalar curvature) \( I(g) = \int_M R \, d\mu, \mathcal{R} = g^{ij} \Gamma_{ij} \). Here the Lagrangian \( \mathcal{L} = g^{ij} R_{ij} \sqrt{\det(g)} \) is of the second order with respect to \( g_{ij} \), and of order zero with respect to \( g^{ij} \). In dimension two, this is a topological quantity, namely the Euler characteristic of the Riemann surface according to the Gauss–Bonnet formula. In an \( n \geq 3 \) dimension we prefer to write the functional in the form \( I(g^{-1}) = \int_M R \, d\mu \).
Theorem 10. The Euler–Lagrange PDEs attached to the functional $I(g^{-1})$, $n \geq 3$, i.e., to the Lagrangian
$L = g^{ij}R_{ij} \sqrt{\det(g_{ij})}$, are Einstein PDEs $R_{ij} = 0$.

Proof. The Euler–Lagrange PDEs are \( \frac{\partial L}{\partial g^{mn}} = 0 \), where \( L = g^{ij}R_{ij} \sqrt{\det(g_{ij})} \). On the other hand, we have:
\[
\frac{\partial \det(g_{ij})}{\partial g^{mn}} = - \det(g_{ij})g_{mn}, \quad \frac{\partial g_{ij}}{\partial g^{mn}} = -g_{mj}g_{nl}, \quad \frac{\partial g^{ij}}{\partial g^{mn}} = \delta^i_m \delta^j_n.
\]

The term \( g^{ij} \frac{\partial R_{ij}}{\partial g^{mn}} \sqrt{\det(g_{ij})} \) is of divergence type, and it has no contribution to the Euler–Lagrange equations. Consequently \( \frac{\partial R}{\partial g^{mn}} = \frac{\partial (g^{ij}R_{ij})}{\partial g^{mn}} = R_{mn} \). Finally, we obtain the explicit Euler–Lagrange PDEs as \( R_{ij} = 0 \). \qed

Theorem 11 ([21]). The solutions of the problem \( \min_{g_{ij}} \int_M R^g \ d\mu \) subject to \( \int_M d\mu = 1, n \geq 3 \), are solutions of Einstein PDEs \( R_{ij} = \frac{R}{n} g_{ij} \).

Proof. We use the Lagrangian \( L = g^{ij}R_{ij} \sqrt{\det(g_{ij})} - \lambda \sqrt{\det(g_{ij})} \), where \( \lambda \) is a constant multiplier. Taking the variations with respect to \( g^{ij} \), we obtain:
\[
R_{ij} - \frac{R - \lambda}{2} g_{ij} = 0.
\]

The hypothesis \( n \geq 3 \) and \( \lambda = c \) implies that \( R \) is constant. We replace \( R \), respectively \( R_{ij} \), in \( \int_M R^g \ d\mu \) and we obtain \( R \ vol(M) = \int_M R^g \ d\mu = \frac{R - \lambda}{2} n \ vol(M) \). Consequently, \( \lambda = \frac{(n-2)R}{n} \) and \( R_{ij} = \frac{R}{n} g_{ij} \). \qed

The exact solutions of Einstein PDEs were discussed many times. In dimension four, there are topological obstructions to the existence of Einstein metrics.

On a smooth oriented Riemannian manifold \( (M, g = (g_{ij})) \), we attach a scalar curvature-flatness deviation either as the action \( I(g) = \int_M (R^g)^2 \ d\mu \) or as the action \( I(g^{-1}) = \int_M (R^g)^2 \ d\mu \).

Theorem 12. The Euler–Lagrange PDEs attached to functional $I(g^{-1})$, i.e., to the Lagrangian \( L = (g^{ij}R_{ij})^2 \sqrt{\det(g_{ij})} \) (zero order with respect to $g^{ij}$), are either $R = 0$ or $R_{ij} = 0$.

Corollary 1. The solutions $g_{ij}(x)$ of PDEs $R = 0$ or $R_{ij} = 0$ are Euler–Lagrange prolongations of Euclidean metrics $g_{ij}(x) = \delta_{ij}$.

Remark 6. The solutions of the problem \( \min_{g_{ij}} \frac{1}{2} \int_M (R^g)^2 \ d\mu \) subject to \( \int_M d\mu = 1, n \geq 3 \), are solutions of Einstein PDEs \( R_{ij} = \frac{R}{n} g_{ij} \).

6. Some Inequalities between Previous Least Squares Densities

Let \( (M, g_{ij}) \) be a Riemannian manifold of dimension \( n \). The Weyl tensor field is defined by:
\[
W_{iklm} = R_{iklm} - \frac{1}{n-2} (\delta_{il}g_{km} - \delta_{km}g_{il} + R_{im}g_{kl} + R_{kl}g_{im}) + \frac{R}{(n-1)(n-2)} (g_{il}g_{km} - g_{km}g_{il}).
\]

When \( n = 2 \), the Weyl tensor field is trivially defined to be 0. Additionally, when \( n = 3 \), the Weyl tensor field \( W_{ijkl} \) vanishes identically, and hence:
\[
R_{iklm} = (R_{il}g_{km} + R_{km}g_{il} - R_{im}g_{kl} - R_{kl}g_{im}) - \frac{R}{2} (g_{il}g_{km} - g_{km}g_{il}).
\]
Generally, the Weyl tensor field vanishes if and only if the manifold $M$ of dimension $n \geq 4$ is locally conformally flat.

Now let us show that the densities $\|\text{Weil}\|^2 = W_{iklm} W^{iklm}$, $\|\text{Riem}\|^2 = R_{iklm} R^{iklm}$, $\|\text{Ric}\|^2 = R g^{ik} R^k l$, $R^2$ satisfy some inequalities.

**Theorem 13.** If $n = 3$, then $\|\text{Riem}\|^2 \leq 4 \|\text{Ric}\|^2$, with equality only if $R = 0$. If $n > 3$, then $\|\text{Riem}\|^2 \geq \frac{4}{n-2} \|\text{Ric}\|^2 - \frac{2}{(n-1)(n-2)} R^2$, with equality only if the manifold is conformally flat.

**Proof.** If $n = 3$, then $R^{iklm} = (R_{ijkl} s^{km} + R_{ikl} s^{jl} - R_{ik} s^{jl} s^{kl} - R^{ijkl} g_{ik} g_{jl}) - \frac{R}{2} (s^{il} s^{jm} - s^{im} s^{jl})$. For $n > 3$, we have:

$$W^{iklm} = R^{iklm} + \frac{1}{n-2} \left( -R_{ijkl} s^{km} - R_{iklj} s^{ml} + R_{ik} s^{ml} s^{nl} + R^{ijkl} s^{km} s^{ln} \right) + \frac{R}{(n-1)(n-2)} (s^{il} s^{jm} - s^{im} s^{jl}).$$

We use direct computation or computation based on orthogonal decompositions of $W_{ijkl}$, respectively $R_{ijkl}$, concerning the scalar product $g_{ij} s^{il} s^{jl} s^{km} s^{ln}$. If $n = 3$, then $\|\text{Riem}\|^2 = 4 \|\text{Ric}\|^2 - R^2$. This is a Pythagorean theorem that shows the linear dependence of some least squares densities and implies the inequality in the theorem. If $n > 3$, then $\|\text{Riem}\|^2 = \|\text{Weyl}\|^2 + \frac{4}{n-2} \|\text{Ric}\|^2 - \frac{2}{(n-1)(n-2)} R^2$. This is a Parallelogram theorem that shows the linear dependence of some least squares densities and implies the inequality in the theorem. □

**Remark 7.** (i) For $n = 3$, the Lagrangian density $L_1 = \|\text{Riem}\|^2 = 4 \|\text{Ric}\|^2 - R^2$ is identically zero. (ii) For $n > 3$, the Lagrangian density $L_2 = \|\text{Riem}\|^2 - \|\text{Weyl}\|^2 + \frac{4}{n-2} \|\text{Ric}\|^2 - \frac{2}{(n-1)(n-2)} R^2$ is identically zero. (iii) For $n = 4$, the Gauss–Bonnet density $L = \|\text{Riem}\|^2 - 4 \|\text{Ric}\|^2 + R^2$ produces a null Lagrangian [26] (pp. 382–384) (does not contribute to the equations of motion, (locally) producing a total derivative).

7. Conclusions and Future Work

In this paper least squares Lagrangian densities attached to flatness PDEs on Riemannian manifolds were studied. Almost all Lagrangians used in Sections 3–5 are covariant generalizations of the Einstein–Hilbert Lagrangian, and their Euler–Lagrange PDEs involve higher-derivative terms and/or higher non-linearities. Such PDEs could be relevant for the early universe and are also typically predicted by quantum theories of gravity, such as string theory [26] (pp. 382–384). More generally, a density-coupling scalar curvature norm, Ricci norm, Riemann norm, or Weyl norm is $L = f(R^2, \|\text{Ric}\|^2, \|\text{Riem}\|^2)$, where $f$ is of class $C^\infty$.

The Legendre transform of a first-order regular Lagrangian density $L(g, \partial g)$ is the Hamiltonian $H = \partial_k S_{ij} \frac{\partial L}{\partial g_{ij}} - L$. Implicitly, we define the generalized momentum $p^k_{ij} = \frac{\partial L}{\partial (\partial g_{ij})}$. Additionally, reasons from physics call for introducing the energy-momentum tensor field $T^k_l = \partial_k S_{ij} \frac{\partial L}{\partial (\partial g_{ij})} - L \delta^k_l$ and reasons from differential geometry call for the $d$-tensor $T \left( l_{mn} \right) \left( i^j k \right) = \partial_k S_{ij} \frac{\partial L}{\partial (\partial g_{mn})} - \frac{1}{n^2} \partial^m \partial^l \delta^k_l$. All of these mathematical objects will be used in our future papers.

The index form technique facilitates the understanding of the significance of the geometric PDEs and of the Lagrangian densities attached to them using the Riemannian metrics. This paper is the continuation of some ideas presented in the papers [1,2].

Some of our results were proven for a wider class of manifolds and as a special case we also reproved well-known results for the Einstein PDEs. In addition, the proposed approach can be used for each least squares Lagrangian on Riemannian setting and is able to give intermediate results, which can be seen as the “best approximation” of solutions of geometric PDEs.

In the Riemannian case we have two facilities: (i) the most important Lagrange-type densities are the squares of the norms of important geometric objects: connection, curvature tensor field, Ricci tensor field, and scalar curvature field; (ii) to obtain the Euler–Lagrange PDEs, we can select alternatively the variations with respect to the metric $g$ or the variations with respect to the inverse metric $g^{-1}$. Mathematics 2020, 8, 1757 16 of 18
In light of the above discussion, if one is able to say something about the solution of a PDEs system whose solution is a Riemannian metric or an affine connection, one could perhaps say something interesting about the behavior of the manifold and its structure. Further research into the nature of the geometric extremals (metrics or connections) may yield strong theoretic results for finite dimensional Riemannian manifolds.

**Author Contributions:** Conceptualization, all authors; methodology, all authors; writing—original draft preparation, all authors; writing—review and editing, all authors. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received funding from Balkan Society of Geometers, Bucharest, Romania.

**Acknowledgments:** The authors thank the referees for their pertinent remarks and Miss Lecturer Oana-Maria Pastae, “Constantin Brancusi” University of Tg-Jiu, for the English improvement of the manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

**Review Reports:** The data used to support the findings of this paper are included within the article and are available from the corresponding author upon request.

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