THE MAXIMAL D=5 SUPERGRAVITIES

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Abstract

The general Lagrangian for maximal supergravity in five spacetime dimensions is presented with vector potentials in the $\mathbf{27}$ and tensor fields in the $\mathbf{27}$ representation of $\text{E}_{6(6)}$. This novel tensor-vector system is subject to an intricate set of gauge transformations, describing $3(27 - t)$ massless helicity degrees of freedom for the vector fields and $3t$ massive spin degrees of freedom for the tensor fields, where the (even) value of $t$ depends on the gauging. The kinetic term of the tensor fields is accompanied by a unique Chern-Simons coupling which involves both vector and tensor fields. The Lagrangians are completely encoded in terms of the embedding tensor which defines the $\text{E}_{6(6)}$ subgroup that is gauged by the vectors. The embedding tensor is subject to two constraints which ensure the consistency of the combined vector-tensor gauge transformations and the supersymmetry of the full Lagrangian. This new formulation encompasses all possible gaugings.
1 Introduction

Maximal supergravities without gauging can be formulated on the basis of different field representations via tensor dualities which convert antisymmetric tensor fields of rank $p$ into tensor fields of rank $D - p - 2$. The choice of the field representation has implications for the symmetry group of the Lagrangian, but not of the field equations. Therefore this issue has a bearing on the introduction of possible gauge interactions associated with a nontrivial gauge group, as this group should be embedded into the symmetry group of the Lagrangian. In fact the situation is even more subtle, as gauge interactions with charged ‘matter’ fields may be incompatible with other independent gauge invariances that these fields may be subject to.

In five spacetime dimensions vector and tensor gauge fields are dual to one another in the absence of charges. The ungauged maximal supergravity Lagrangian is described in terms of vector fields transforming according to the $\overline{27}$ representation of $E_{6(6)}$, \cite{1}, where the presence of an abelian Chern-Simons term forms an obstacle to dualizing all vectors into tensor gauge fields. Obviously, a partial dualization destroys the manifest $E_{6(6)}$ invariance of the Lagrangian. When switching on a gauging, the corresponding gauge fields transform according to the adjoint representation of the gauge group. The dimension of the gauge group is usually less than 27, so that there are vector fields that do not belong to this adjoint representation. When these gauge fields carry charges that cannot be incorporated into a central extension of the gauge algebra, then the gauging can only exist provided these fields can be converted to charged tensor fields. The corresponding tensor field Lagrangians have a kinetic term linear in spacetime derivatives and proportional to a five-dimensional Levi-Civita tensor, which allows a minimal coupling to gauge fields and a possible mass term. Indeed, this option was exploited in \cite{2}, where the gauging of maximal five-dimensional supergravity with the 15-dimensional gauge groups $SO(q,6 - q)$ was constructed. In that case there are 12 massive charged tensor fields, transforming in the $(6,2)$ representation of $SO(q,6 - q) \times SL(2,\mathbb{R})$. Prior to that work a similar situation had already been noted in seven spacetime dimensions \cite{3} (see also \cite{4}).

The above seems to imply that, in five spacetime dimensions, a gauging cannot just be effected by switching on the gauge charges, as the field representation must first be suitably adapted. This feature has hampered a general analysis of all possible gaugings. One of the central results of this paper is a new formulation of five-dimensional maximal supergravity that is sufficiently flexible to incorporate all necessary field representations from the start, thus enabling a general analysis of all possible gaugings. So far, the known gaugings \cite{2,5,6,7} comprise the $SO(q,6 - q)$ gaugings, contractions thereof, and gaugings induced by reduction from higher-dimensional supergravities (although few of those have been discussed in detail). In the context of the AdS/CFT correspondence
the SO(6) gauging received most attention.

Beyond these typical five-dimensional issues, one must address the modification of the Lagrangian with masslike terms and a scalar potential. These new couplings are encoded in the so-called $T$-tensor [4], and recently it was demonstrated how viable gaugings can be investigated by means of a group-theoretical analysis of this $T$-tensor [3, 4]. The purpose of this paper is to demonstrate how this analysis leads to a completely general treatment of all maximal gauged supergravities, using the new formulation of the Lagrangian mentioned above. This formulation is based on vector fields and tensor fields transforming in the $\overline{27}$ and $27$ representations of $\text{E}_{6(6)}$, respectively, and this combined system of vector and tensor fields is subject to both vector and tensor gauge invariances encoded in the same embedding tensor that determines the gauge group and the $T$-tensor. This intricate gauge invariance guarantees that the combined system describes always 81 degrees of freedom, as required by supersymmetry. The embedding tensor is treated as a spurionic object which transforms under $\text{E}_{6(6)}$, which makes it amenable to a group-theoretical analysis. The Lagrangian remains formally $\text{E}_{6(6)}$ invariant until the embedding tensor is frozen to a constant. The embedding tensor is subject to two constraints: it must belong to the $351$ representation of $\text{E}_{6(6)}$, and the $\overline{27} + 1728$ representation contained in its square should cancel. For any embedding tensor that satisfies these constraints there exists a consistent supersymmetric and gauge invariant Lagrangian.

The relation between the embedding tensor and the $T$-tensor involves the representative of the $\text{E}_{6(6)}/\text{USp}(8)$ coset space that is parametrized by the scalar fields. Here we use the standard treatment of gauged nonlinear sigma models in which the group $\text{USp}(8)$ is realized as a local invariance which acts on the spinor fields and the scalars; the corresponding connections are composite fields. A gauging is based on a group $G_g \subset \text{E}_{6(6)}$ whose connections are (some of the) elementary vector gauge fields of the supergravity theory. The coupling constant associated with the gauge group $G_g$ will be denoted by $g$. One can impose a gauge condition with respect to the local $\text{USp}(8)$ invariance which amounts to fixing a coset representative for the coset space. In that case the $\text{E}_{6(6)}$-symmetries will act nonlinearly on the fields and these nonlinearities make many calculations intractable or, at best, very cumbersome. Because it is much more convenient to work with symmetries that are realized linearly, the best strategy is therefore to postpone the gauge fixing till the end.

The new Lagrangian based on the combined vector-tensor gauge invariance and its supersymmetry transformations are universal in the sense that they will take the same form irrespective of the gauging. All the details of the gauging are encoded into the embedding tensor and the quantities related to it. Once the group-theoretical constraints on the embedding tensor are satisfied, it is guaranteed that the gauging
is consistent with supersymmetry. Hence our results encompass all possible maximal supergravity theories in five dimensions.

This paper is organized as follows. In section 2 we discuss the embedding tensor and the constraints it must satisfy. This analysis motivates the new formulation of the Lagrangian with the combined vector-tensor system, which is presented in section 3. In section 4 we define the $T$-tensor and derive the consequences of the group-theoretical constraints. This requires a detailed discussion of the characteristic features of the $E_{6(6)}/USp(8)$ coset space. In section 5 we discuss the Lagrangian and the supersymmetry transformation rules, up to higher-order fermion terms. Finally, in section 6 we analyze a number of examples pertaining to known and new gaugings, and in section 7 we present our concluding remarks.

2 The embedding tensor

The (abelian) vector fields $A^M_\mu$ transform in a representation $\mathbf{27}$ of $E_{6(6)}$ with generators denoted by $(t_\alpha)_M^N$, so that $\delta A^M_\mu = -\Lambda^\alpha (t_\alpha)_N^M A^N_\mu$. The gauge group is a subgroup of $E_{6(6)}$ so that the generators $X_M$ are decomposable in terms of the 78 independent $E_{6(6)}$-generators $t_\alpha$, i.e.,

$$X_M = \Theta^\alpha_M t_\alpha,$$

where $\alpha = 1, 2, \ldots, 78$ and $M = 1, 2, \ldots, 27$. The gauging is thus encoded in a real embedding tensor $\Theta^\alpha_M$ assigned to the $\mathbf{27} \times \mathbf{78}$ representation of $E_{6(6)}$. The embedding tensor acts as a projector whose rank $s$ equals the dimension of the gauge group (not counting abelian gauge fields corresponding to a central extension of the gauge algebra, as we will discuss in due course). The strategy of this paper is to treat the embedding tensor as a spurionic object that transforms under $E_{6(6)}$, so that the Lagrangian and transformation rules remain formally $E_{6(6)}$ covariant. The embedding tensor can then be characterized group-theoretically. When freezing $\Theta^\alpha_M$ to a constant, the $E_{6(6)}$-invariance is broken. An admissible embedding tensor is subject to a linear and a quadratic constraint, which ensure that one is dealing with a proper subgroup of $E_{6(6)}$ and that the corresponding supergravity action remains supersymmetric. These constraints are derived in the first subsection. The second subsection describes a number of implications of these constraints, while a third subsection presents some of the results in a convenient basis.

2.1 The constraints on the embedding tensor

The fact that the $X_M$ generate a group and thus define a Lie algebra,

$$[X_M, X_N] = f_{MN}^P X_P,$$

(2.2)
with \( f_{MN}^P \) the as yet unknown structure constants of the gauge group, implies that the embedding tensor must satisfy the closure condition,

\[
\Theta_M^\alpha \Theta_N^\beta f_{\alpha \beta}^\gamma = f_{MN}^P \Theta_P^\gamma. \tag{2.3}
\]

Here the \( f_{\alpha \beta}^\gamma \) denote the structure constants of \( E_{6(6)} \), according to \([t_\alpha, t_\beta] = f_{\alpha \beta}^\gamma t_\gamma\). The closure condition implies that the structure constants \( f_{MN}^P \) satisfy the Jacobi identities in the subspace projected by the embedding tensor,

\[
f_{[MN}^Q f_{P]}^R \Theta_R^\alpha = 0. \tag{2.4}
\]

Once the gauge group is specified, one introduces covariant derivatives given by

\[
D_\mu = \partial_\mu - g A_\mu^M X_M, \tag{2.5}
\]

where \( g \) denotes the gauge coupling constant. They lead to the covariant field strengths,

\[
\Theta_M^\alpha F_{\mu \nu}^M = \Theta_M^\alpha (\partial_\mu A_\nu^M - \partial_\nu A_\mu^M - g f_{NP}^M A_\mu^N A_\nu^P). \tag{2.6}
\]

The gauge field transformations are given by

\[
\Theta_M^\alpha \delta A_\mu^M = \Theta_M^\alpha (\partial_\mu \Lambda^M - g f_{NP}^M A_\mu^N \Lambda^P). \tag{2.7}
\]

Because of the contraction with the embedding tensor, the above results only apply to an \( s \)-dimensional subset of the gauge fields; the remaining ones do not appear in the covariant derivatives and are not directly involved in the gauging. However, the \( s \) gauge fields that do appear in the covariant derivatives, are only determined up to additive terms linear in the \( 27 - s \) gauge fields that vanish upon contraction with \( \Theta_M^\alpha \).

While the gauge fields involved in the gauging should transform in the adjoint representation of the gauge group, the gauge field charges should also coincide with \( X_M \) in the \( 27 \) representation. Therefore \( (X_M)_N^P \) must decompose into the adjoint representation of the gauge group plus possible extra terms which vanish upon contraction with \( \Theta_M^\alpha \).

Note that (2.8) is the analogue of (2.3) in the \( 27 \) representation. The combined conditions (2.3) and (2.8) imply that \( \Theta \) is invariant under the gauge group and yield the \( E_{6(6)} \)-covariant condition

\[
C_{MN}^\alpha \equiv f_{\beta \gamma}^\alpha \Theta_{M}^\beta \Theta_{N}^\gamma + t_{\beta N}^P \Theta_{M}^\beta \Theta_{P}^\alpha = 0. \tag{2.9}
\]

Obviously \( C_{MN}^\alpha \) can be assigned to irreducible \( E_{6(6)} \) representations contained in the \( 27 \times 27 \times 78 \) representation. The condition (2.9) encompasses all previous results: it implies that

\[
[X_M, X_N] = -X_{MN}^P X_P. \tag{2.10}
\]
so that \((2.9)\) implies a closed gauge algebra, whose structure constants, related to \(X_{MN}^P\) in accord with \((2.8)\), have the required antisymmetry. Hence \((2.9)\) is indeed sufficient for defining a proper subgroup embedding.\(^1\)

The embedding tensor satisfies a second constraint, which is required by supersymmetry. This constraint is linear and restricts the embedding tensor to the \(351\) representation [6]. From

\[
27 \times 78 = 27 + 351 + 1728,
\]

(2.11)

one shows that this condition on the representation implies the equations,

\[
t_{\alpha M}^N \Theta_{N}^\alpha = 0, \quad (t_\beta t^\alpha)_M^N \Theta_N^\beta = -\frac{2}{3} \Theta_M^\alpha,
\]

(2.12)

where the index \(\alpha\) is raised by the inverse of the \(E_{6(6)}\)-invariant metric \(\eta_{\alpha\beta} = \text{tr}(t_\alpha t_\beta)\).

As a result of this constraint, the representation content of \(C_{MN}^\alpha\) can be further restricted. From \((2.12)\) one can derive the following equations,

\[
t_{\alpha N}^P C_{MP}^\alpha = 0, \quad (t_\beta t^\alpha)_N^P C_{MP}^\beta = -\frac{2}{3} C_{MN}^\alpha, \quad t_{\alpha M}^P C_{PN}^\alpha = t_{\alpha N}^P C_{PM}^\alpha.
\]

(2.13)

They imply that \(C_{MN}^\alpha\) belongs to representations contained in \(27 \times 351\). On the other hand, the product of two \(\Theta\)-tensors belongs to the symmetric product of two \(351\) representations. Comparing the decomposition of these two products\(^2\),

\[
(351 \times 351)_s = 27 + 1728 + 351' + 7722 + 17550 + 34398,
\]

\[
27 \times 351 = 27 + 1728 + 351 + 7371,
\]

(2.14)

one deduces that \(C_{MN}^\alpha\) belongs to the \(27 + 1728\) representation.

Summarizing, a consistent gauging is defined by an embedding tensor \(\Theta_M^\alpha\) satisfying the linear constraint \((2.12)\) together with the quadratic constraint \((2.9)\) with the \(E_{6(6)}\) representation content,

\[
(\mathbb{P}_{27} + \mathbb{P}_{1728}) \Theta = 0,
\]

\[
(\mathbb{P}_{27} + \mathbb{P}_{1728}) \Theta \Theta = 0.
\]

(2.15)

2.2 Some implications of the embedding tensor constraints

Because \((t_\alpha)_M^N\) is an \(E_{6(6)}\)-invariant tensor, it follows that \(X_{MN}^P\) transforms in the \(351\) representation of \(E_{6(6)}\), just as the embedding tensor. Furthermore the product of

\(^1\)Note that for an abelian gauge group we have \(X_{MN}^P \Theta_P^\alpha = 0\). Using \((2.12)\) this leads to \(\text{tr}(X_M X_N) = 0\).

\(^2\)We used the LiE package [10] for computing the decompositions of tensor products and the branching of representations.
three 27 representations contains a singlet representation, associated with a symmetric $E_{6(6)}$-invariant tensor $d_{MNP}$. The same is true for the conjugate representation, so that there exists also a symmetric invariant tensor $d^{MNP}$. Hence it follows that $X_{MN}^P$ satisfies the following properties,

$$X_{MN}^N = X_{NM}^N = 0, \quad X_{M(N}^R d_{PQ)R} = 0 = X_{MN}^{(P} d^{QR)N}. \quad (2.16)$$

By writing $X_{MN}^P = X_{(MN)}^P + X_{[MN]}^P$, it seems that one can decompose the tensor $X_{MN}^P$ into two representations, while, on the other hand, we know that $X_{MN}^P$ must belong to a single irreducible representation. Therefore both the symmetric and the antisymmetric components should be proportional to the same tensor transforming in the 351 representation. This is confirmed by the fact that the 27 representation yields a 351 representation when multiplied with either the symmetric or the antisymmetric product of two 27 representations,

$$27 \times (27 \times 27)_s = 27 \times (27 + 351^\prime) = 351 + 27 + 351 + 1728 + 7722,$n
$$27 \times (27 \times 27)_a = 27 \times 351 = 351 + 27 + 1728 + 7371. \quad (2.17)$$

Therefore we can construct two contractions of $X_{MN}^P$ with invariant tensors yielding a tensor $Z^{MN}$ that must be antisymmetric so that it transforms in the 351 representation. In both cases we should find the same tensor, i.e.,

$$X_{PQ}^M d^{NPQ} = Z^{MN}, \quad 2 X_{PQ}^T d^{PRM} d^{QSN} d_{RST} = Z^{MN}. \quad (2.18)$$

We observe that the antisymmetry in $[MN]$ of the first equation of (2.18) follows also from (2.16). Possible additional proportionality factors in (2.18) can be absorbed into the invariant tensors $d_{MNP}$ and $d^{MNP}$. Using (2.16) can show that the factors in (2.18) are consistent provided that we choose the relative normalizations of the two tensors such that

$$d_{MPQ} d^{NPQ} = \delta^N_M. \quad (2.19)$$

Because the symmetric product of four 27 representations contains precisely one 27 representation, we deduce another identity,

$$d_{S(MN} d_{PQ)T} d^{STR} = \frac{2}{15} \delta^R_M d_{NPQ}. \quad (2.20)$$
Contraction with additional invariant tensors yields a number of other useful identities.\footnote{The following identities proved convenient:}

\[ X_{(MN)}^P = d_{MNQ} Z^{PQ}, \quad X_{[MN]}^P = 10 d_{MQS} d_{NRT} d^{PQR} Z^{ST}. \]  

(2.21)

A number of important identities quadratic in the embedding tensor can also be derived. The first one concerns the expression \( Z^{MN} \Theta N^\alpha \). This tensor transforms in the representation, \( \overline{27} \times 78 = \overline{27} + \overline{351} + 1728 \), and can be compared to the square of the embedding tensor, which yields the representations listed in the first equation (2.14). The only representation they have in common, however, is the \( \overline{27} + 1728 \), which vanishes because of the constraint (2.9). Therefore, we conclude,

\[ Z^{MN} \Theta N^\alpha = 0, \quad Z^{MN} X_N = 0, \]  

(2.22)

where, in the second equation, \( X_M \) is taken in an arbitrary representation. Along the same lines, we can consider the contraction \( X_{MN} [P \ ZQ]N \). From the second branching of (2.17), we readily deduce that this tensor should belong to the \( \overline{351} + \overline{27} + 1728 + 7371 \) representation. Comparing these representations to those generated by the square of the embedding tensor (c.f. the first equation (2.14)) we note that the only representations they have in common are again the ones which are set to zero by the constraint (2.9). Hence the tensor \( Z^{MN} \) is invariant under the gauge group,

\[ X_{MN} [P \ ZQ]N = 0. \]  

(2.23)

The reasoning that led to (2.22) and (2.23) can be applied to show that we are in fact dealing with equivalent forms of the quadratic closure constraint (2.9), at least for embedding tensors that are restricted to the \( \overline{351} \) representation. We list three equivalent forms of the quadratic constraint,

\[ X_{MP}^R X_{NR}^Q - X_{NP}^R X_{MR}^Q + X_{MN}^R X_{RP}^Q = 0, \]

\[ Z^{MN} X_N = 0, \]

\[ X_{MN} [P \ ZQ]N = 0. \]  

(2.24)

In the next section we will use the tensor \( X_{[MN]}^P \) as an extension of the gauge group structure constants \( f_{MN}^P \), which satisfies the Jacobi identity up to terms proportional
to \( Z \). From (2.10) and some of the previous identities one derives,

\[
X_{[MN]}^P X_{[QP]}^R + X_{[QM]}^P X_{[NP]}^R + X_{[NQ]}^P X_{[MP]}^R
= \left\{ d_{SPQ} X_{[MN]}^P + 2 d_{SP[M} d_{N]QO} Z_{OP}^{SR} \right\} Z_{SR}^{SR}
= d_{SP[Q} X_{MN]}^P Z_{SR}^{SR}.
\]  

(2.25)

As the right-hand side vanishes upon contraction with the embedding tensor \( \Theta_R^\alpha \), we see that the \( X_{[MN]}^P \) satisfy the Jacobi identity in the subspace projected by the embedding tensor, just as the gauge group structure constants (c.f. (2.4)).

2.3 A special \( E_6(6) \) basis

Let us now consider a basis for the vector fields such that all the nonzero components of the 78 vectors \( \Theta^\alpha \) cover an \( s \)-dimensional subspace parametrized by the gauge fields \( A_\mu^M \) with \( M = 1, \ldots, s \) and \( s \leq 27 \). In this basis \( X_{MN}^P \) can be written in triangular form,

\[
X_M = \begin{pmatrix}
-f_M & a_M \\
0 & b_M
\end{pmatrix},
\]

(2.26)

where the \( s \times s \) upper-left diagonal block coincides with the gauge group structure constants and the contribution of the submatrices \( a_M \) and \( b_M \) vanish in the product \( (X_M)_N^P \Theta_P^\alpha \). The lower-left \( s \times (27 - s) \) block vanishes as a result of (2.8). It is easy to see that \( a_M \) and \( b_M \) cannot both be zero. If that were the case, we would have \( f_{MN}^P = -X_{MN}^P \), which is antisymmetric in \( M \) and \( N \). Hence,

\[
\Theta_{N}^\alpha t_{\alpha M}^P = -\Theta_{M}^\alpha t_{\alpha N}^P.
\]

(2.27)

Contracting this result by \( (t^\beta)_P^M \) leads to \( t_{\alpha} t^\beta \Theta^\alpha = -\Theta^\beta \) which is in contradiction with the representation constraint (2.12).

Let us now refine this choice of basis and consider some of the results of the previous subsection. In this special \( E_6(6) \) basis the components of a vector \( V_M \) belonging to the \( 27 \) representation are decomposed as \( V_M = (V_A, V_a, V_u) \), where \( A = 1, \ldots, s \), \( a = s + 1, \ldots, 27 - t \) and \( u = 28 - t, \ldots, 27 \), where the only nonvanishing components of \( \Theta_M^\alpha \) are \( \Theta_A^\alpha \) and therefore the \( X_{MN}^P \) are nonzero only if \( M = A \). Correspondingly we decompose the vector \( V^M \) transforming in the \( 27 \) representation as \( V^M = (V^A, V^a, V^u) \). Obviously the tensor \( Z^{MN} \) vanishes whenever \( M \) or \( N \) are equal to \( A, B, \ldots \) in view of (2.22) and the distinction between indices \( a, b, \ldots \) and \( u, v, \ldots \) is due to the fact that we assume that only \( Z^{uv} \) is nonvanishing. Hence, \( V^a \) and \( V^u \) span the subspace orthogonal to \( \Theta_M^\alpha \) and \( V_A \) and \( V_a \) span the subspace orthogonal to \( Z^{uv} \). Consequently, the number \( t \) of indices \( u, v \) must be even. From (2.21), (2.22) and
it follows that $X_{AN}^P$ has a block decomposition, which goes beyond (2.26) (row and column indices are denoted by $B, b, v$ and $C, c, w$, respectively),

$$X_{AN}^P = \begin{pmatrix} -f_{AB}^C & h_{AB}^c & C_{AB}^w \\ 0 & 0 & C_{AB}^w \\ 0 & 0 & D_{Av}^w \end{pmatrix}, \quad (2.28)$$

where we note the following relations,

$$h_{(AB)}^c = f_{(AB)}^C = f_{AB}^B = D_{Au}^u = Z^w[u] D_{Aw}^v = 0, \quad (2.29)$$

which imply that the gauge group is unimodular. Observe that the basis choice for $V^M$ and $V_M$ is not unambiguous. The $V^A$ can still be modified by linear combinations of $V^a$ and $V^u$, and likewise, the $V_u$ can be modified by terms linear in $V_A$ and $V_a$. These redefinitions do not alter the general form of (2.28) but they affect the expressions for the nondiagonal blocks.

From other relations derived above (in particular from the first equation (2.21)), we can establish a variety of results,

$$C_{(AB)}^u = d_{ABv} Z^{uv}, \quad C_{Aa}^u = 2 d_{Avu} Z^{uw}, \quad D_{Av}^u = 2 d_{Avu} Z^{uw}. \quad (2.30)$$

Furthermore one derives that $d_{uvw} = d_{uva} = d_{uab} = 0$. Other identities follow from the invariance of $d_{MNP}$, such as

$$f_{AB}^C d_{abc} + 4 Z^{wx} d_{Aw(a} d_{b)xB} = 0, \quad f_{AB}^C d_{avc} + 4 Z^{wx} d_{Aw(u} d_{v)xB} = 0, \quad f_{AB}^C d_{auc} + 2 Z^{wx} d_{Awa} d_{uxB} + 2 Z^{wx} d_{Awu} d_{axB} = 0. \quad (2.31)$$

Likewise, the invariance of the $d^{MNP}$ tensor leads to a large variety of equations, of which we present the following two,

$$f_{AE}^{(B} d^{CD)E} = 0, \quad h_{AD}^a d^{BCD} - 2 f_{AD}^{(B} d^{C)Da} = 0. \quad (2.32)$$

The closure relations (2.2) imply three additional identities,

$$f_{[AB}^D f_{C]DE} = 0, \quad f_{[AB}^D h_{C]Da} = 0, \quad 2 f_{C[A}^D C_{B]D}^u - f_{AB}^D C_{DC}^u - 2 h_{C[A}^a C_{B]a}^u = 0. \quad (2.33)$$
From the first two equations it follows that the upper-left submatrix of (2.28) parametrized in terms of $f_{AB}^C$ and $h_{AB}^a$ closes under commutation and defines consistent gauge transformation rules for the gauge fields $A^A_\mu$ and $A^a_\mu$,

$$\delta A^A_\mu = \partial_\mu A^A - g f_{BC}^A A^B_\mu A^C,$$
$$\delta A^a_\mu = \partial_\mu A^a + g h_{BC}^a A^B_\mu A^C.$$  

(2.34)

The corresponding field strengths read,

$$\mathcal{F}^A_{\mu\nu} = \partial_\nu A^A_\mu - \partial_\mu A^A_\nu - g f_{BC}^A A^B_\mu A^C,$$
$$\mathcal{F}^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu + g h_{BC}^a A^B_\mu A^C.$$  

(2.35)

The only gauge fields that appear in the covariant derivatives are the fields $A^A_\mu$, so no other gauge fields couple to charges that act on the matter fields. However, to write consistent transformation rules for the gauge fields, one must incorporate the abelian gauge fields $A^a_\mu$ into the gauge algebra. These gauge fields couple to charges that are central in the gauge algebra so that the gauge algebra is a central extension of the algebra (2.2). Introducing formal generators $\tilde{X}_A$ and $\tilde{X}_a$, it reads,

$$[\tilde{X}_A, \tilde{X}_B] = f_{AB}^C \tilde{X}_C - h_{AB}^a \tilde{X}_a.$$  

(2.36)

On the matter fields the charges $\tilde{X}_a$ vanish and the gauge algebra coincides with (2.2).

The remaining gauge fields $A^a_\mu$ carry charges related to the last column in (2.28). Since these charges cannot be incorporated in the gauge transformations on the vector fields and lead to inconsistent couplings, these gauge fields must be dualized to charged tensor fields. This is a well-known feature in gauged supergravities in odd dimensions [2, 3, 4], as we already discussed in the introduction. Therefore the physically relevant gauge group is the one associated with the gauge fields $A^A_\mu$ and $A^a_\mu$.

However, to dualize vectors to tensors affects the manifest $E_{6(6)}$ covariance of our results, and requires a case-by-case dualization for every separate gauging. Therefore, rather than to perform this dualization, we will keep all 27 gauge fields and introduce 27 tensor fields as well. At the same time we introduce an extended tensor-vector gauge invariance in order to balance the degrees of freedom. Upon a suitable gauge condition one can remove the gauge fields $A^u_\mu$ whose degrees of freedom are then carried by the tensor fields. This implies that we must introduce a new gauge transformation for the vector fields of the form $\delta A^M_\mu \propto Z^{MN} \Xi^{\mu N}$ where $\Xi^{\mu N}$ is the gauge parameter, that enables to remove the gauge fields so that they are effectively replaced by the tensor fields. When the gauge-fixing is postponed until the end the Lagrangian for this vector-tensor system takes a unique and $E_{6(6)}$-covariant form. The gauging is encoded in terms of the embedding tensor, or equivalently, in terms of the tensor $Z^{MN}$. This formulation is discussed in the next section.
3 Tensor and vector gauge fields

As explained in the introduction we will set up a formulation based on vector and tensor fields, \( A_\mu^M \) and \( B_{\mu\nu}^M \), which transform in the \( \overline{27} \) and \( 27 \) representation of \( E_6(6) \), respectively. The combined vector and tensor gauge transformations will ensure that the number of physical degrees of freedom will remain independent of the embedding tensor. The latter will only determine how the degrees of freedom are shared between the vector and tensor fields. Two ingredients play a crucial role in order to accomplish this. First of all the vector fields transform under tensor gauge transformations proportional to \( Z_{MN} \) and secondly, the tensor fields in the Lagrangian will always appear multiplied by \( Z_{MN} \). The identities proven in the previous section are essential in what follows.

To see how this works we first consider the gauge transformations of the vector fields,

\[
\delta A_\mu^M = \partial_\mu \Lambda^M - g X_{[PQ]}^M \Lambda^P A_\mu^Q - g Z^{MN} \Xi_{\mu N},
\]

(3.1)

where \( \Lambda^M \) and \( \Xi_{\mu M} \) denote the parameters of the vector and tensor gauge transformations. Observe that \( X_{[MN]}^P \) play the role of generalized structure constants of the gauge group. Obviously, a number of vector fields can be set to zero by a gauge choice. This number \( t \) equals the rank of the matrix \( Z^{MN} \). As was explained in the previous section, we refrain from doing this in order to remain independent of the specific choice for the embedding tensor.

Because the Jacobi identity does not hold for the \( X_{[MN]}^P \), the would-be covariant field strength,

\[
F_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M + g X_{[NP]}^M A_\mu^N A_\nu^P,
\]

(3.2)

does not transform covariantly,

\[
\delta F_{\mu\nu}^M = -g X_{[NP]}^M \Lambda^N F_{\mu\nu}^P - g^2 Z^{MN} d_{NP[R} X_{ST]P} \Lambda^R A_\mu^S A_\nu^T
+ g Z^{MN} \left( 2 \partial_\mu \Xi_{\nu|N} - g X_{PNQ} A_\mu^P A_\nu^{Q} \right),
\]

(3.3)

Up to terms proportional to \( Z^{MN} \) the field strengths \( F^M \) transform covariantly under the gauge group. To regain full covariance, we introduce tensor fields \( B_{\mu\nu}^M \) such that the modified field strengths,

\[
\mathcal{H}_{\mu\nu}^M = F_{\mu\nu}^M + g Z^{MN} B_{\mu\nu N},
\]

(3.4)
transform covariantly under the gauge group,
\[ \delta \mathcal{H}_{\mu \nu}^M = -g X_P^M A^P \mathcal{H}_{\mu \nu}^N , \] (3.5)
and are invariant under the tensor gauge transformations. This implies that the transformations of the fields \( B_{\mu \nu}^M \) are as follows,
\[ Z^{MN} \delta B_{\mu \nu}^N = Z^{MN} \left( 2 \partial_\mu \Xi_{\nu}^N - g X_P^Q A_{[\mu}^P \Xi_{\nu]}^Q \right) + g Z^{MN} \Lambda^P X_P^Q B_{\mu \nu}^Q \]
\[ - Z^{MN} \left( 2 d_{NPQ} \partial_\mu A_{\nu]}^P - g X_R^P d_{PS} A_{[\mu}^R A_{\nu]}^S \right) \Lambda^Q . \] (3.6)

Obviously, the symmetry transformations on the tensor fields are only determined modulo terms that vanish under contraction with \( Z^{MN} \); this poses no problem as, in the Lagrangian, the tensor fields \( B_{\mu \nu}^M \) will always be contracted with \( Z^{MN} \).

It is important to note that the covariant derivative, \( D_\mu = \partial_\mu - g A_\mu^M X_M \), does not transform under tensor gauge transformations, by virtue of (2.22). Furthermore, we note the validity of the Ricci identity,
\[ [D_\mu , D_\nu] = -g F_{\mu \nu}^M X_M . \] (3.7)

To verify the consistency of the above transformation rules, one may consider the commutator algebra of the vector and tensor gauge transformations. The tensor transformations commute,
\[ [\delta (\Xi_1) , \delta (\Xi_2)] = 0 . \] (3.8)

The commutator of a vector and a tensor gauge transformation gives rise to a tensor gauge transformation,
\[ [\delta (\Xi) , \delta (\Lambda)] = \delta (\tilde{\Xi}) , \] (3.9)
with \( \tilde{\Xi}_{\mu M} = \frac{1}{2} g X_{PM}^N \Lambda^P \Xi_{\mu N} \). Finally, the commutator of two vector gauge transformations gives rise to a vector gauge transformation and a tensor gauge transformation,
\[ [\delta (\Lambda_1) , \delta (\Lambda_2)] = \delta (\tilde{\Lambda}) + \delta (\tilde{\Xi}) , \] (3.10)
where \( \tilde{\Lambda}^M = g X_{[NP]}^M A_1^N A_2^P \) and \( \tilde{\Xi}_{\mu M} = -g d_{MN[\rho} X_{QR]}^N A_1^P A_2^Q A_{\rho}^R \).

There exists a kinetic term for the tensor fields which is of first-order in derivatives, which is modified by Chern-Simons-like terms in order to be fully gauge invariant under the combined vector and tensor gauge transformations. It reads as follows,
\[ \mathcal{L}_{VT} = \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} \left\{ g Z^{MN} B_{\mu \nu}^M \left[ D_\rho B_{\sigma \tau}^N + 4 d_{NPQ} A_{[\rho}^P \left( \partial_\sigma A_{\tau]}^Q + \frac{1}{3} g X_{[SR]}^Q A_{\tau]}^R A_{\sigma}^S \right) \right] \right. \\
\left. \right. \\
- \frac{2}{3} d_{MN} d_{NPQ} A_{[\mu}^M A_{\nu]}^N A_{\rho}^P \partial_\sigma A_{\tau]}^P \\
+ \frac{2}{3} g X_{[QR]}^M A_1^N A_2^Q A_{[\rho}^P \left( \partial_\sigma A_{\tau]}^P + \frac{1}{5} g X_{[ST]}^P A_{\sigma}^S A_{\tau]}^T \right) \right\} , \] (3.11)

\[ ^4 \text{Note that the term } \partial_\mu \Xi_{\nu}^N \text{ in } (3.6) \text{ is not properly covariantized by the next term proportional to } X_{PN}^Q; \text{ this would require an additional factor } 2. \]
where
\[ D_\mu B_{\nu\rho M} = \partial_\mu B_{\nu\rho M} - g X_{PM}^N A^P_\mu B_{\nu\rho N}. \] (3.12)

Under variations \( A^M_\mu \rightarrow A^M_\mu + \delta A^M_\mu \) and \( B^M_{\mu\nu} \rightarrow B^M_{\mu\nu} + \delta B^M_{\mu\nu} \) this Lagrangian changes as follows,
\[
\delta \mathcal{L}_{VT} = i \varepsilon^{\mu\nu\rho\sigma\tau} \left\{ \left( \delta B^M_{\mu\nu} + 2 d_{MPQ} \delta A^P_\mu A^Q_\nu \right) D_\rho \mathcal{H}_{\sigma\tau}^M - \delta A^M_\mu d_{MNP} \mathcal{H}_{\nu\rho}^N \mathcal{H}_{\sigma\tau}^P \right\} + \text{total derivatives} \cdots ,
\] (3.13)

where we note the identity,
\[
\varepsilon^{\mu\nu\rho\sigma\tau} D^M_\mu \mathcal{H}_{\nu\rho}^M = g \varepsilon^{\mu\nu\rho\sigma\tau} Z^{MN} \left[ D_\mu B_{\nu\rho N} + 2 d_{NPQ} A_\mu^P \left( \partial_\nu A_\rho^Q + \frac{1}{3} g X_{[RS]}^Q A^R_\rho A^S_\sigma \right) \right].
\]

Clearly the terms \( \varepsilon^{\mu\nu\rho\sigma\tau} D_\mu \mathcal{H}_{\sigma\tau}^M \) and \( \varepsilon^{\mu\nu\rho\sigma\tau} d_{MNP} \mathcal{H}_{\nu\rho}^N \mathcal{H}_{\sigma\tau}^P \) will appear as part of the equations of motion for the vector and tensor fields.

As we shall see later there is a second term in the supergravity Lagrangian, quadratic in \( \mathcal{H}^M_{\mu\nu} \). It requires an \( E_6(6) \)-covariant metric \( \mathcal{M}_{MN} \) that will also be discussed in due course. Here we just note that the variation of this term in the Lagrangian under changes of the vector and tensor fields yields,
\[
\delta \left( \mathcal{M}^M_{MN} \mathcal{H}^M_{\mu\nu} \mathcal{H}^{\mu\nu} \right) = -2 \left( \delta B^M_{\mu\nu} - 2 d_{MPQ} A^P_\mu \delta A^Q_\nu \right) Z^{MN} \mathcal{M}^R_{NR} \mathcal{H}^{\mu\nu R} + 4 \delta A^M_\mu D_\nu \left( \mathcal{M}^M_{MN} \mathcal{H}^{\mu\nu} \right),
\] (3.14)

which shows the same combinations of field variations as in (3.13) so that the two variations can be combined without difficulty.

At this point one has the option of removing \( t \) of the vector fields by a gauge choice. To do this one employs the special \( E_6(6) \) basis of subsection 2.3. We will return to this in subsection 6.1 where we will also exhibit the consequences of this gauge choice for the supersymmetry transformations. In the absence of a gauging the embedding tensor (or, equivalently, the gauge coupling constant \( g \)) vanishes, \( \mathcal{H}^M_{\mu\nu} \) coincides with the abelian field strengths \( F^M_{\mu\nu} \) and \( \mathcal{L}_{VT} \) reduces to an abelian Chern-Simons term,
\[
\mathcal{L}_{VT} \longrightarrow - \frac{4}{3} i \varepsilon^{\mu\nu\rho\sigma\tau} d_{MNP} A^M_\mu \partial_\nu A^N_\rho \partial_\sigma A^P_\tau.
\] (3.15)

4  \( E_6(6)/USp(8) \) and the \( T \)-tensor

We already stressed in the introduction that the scalar fields parametrize the \( E_6(6)/USp(8) \) coset space. These fields are described by a matrix \( V(x) \in E_6(6) \) (taken in the fundamental 27 representation) which transforms from the right under local USp(8) and
from the left under rigid $E_6(6)$. The matrix $V$ can be used to elevate the embedding tensor to the so-called $T$-tensor, which is the USp(8)-covariant, field-dependent, tensor that appears in the masslike terms and the scalar potential. The $T$-tensor is thus defined by,

$$T^M_{\alpha}[\Theta, \phi] t_\alpha = V^{-1/2}_M \Theta^\alpha_N (V^{-1} t_\alpha V), \quad (4.1)$$

where the underlined indices refer to local USp(8). The appropriate representation is the $27$, so that we can write

$$T^M_{MN} E[\Theta, \phi] = V^{-1/2}_M V^{-1/2}_N \varphi_{\alpha\beta} X_{MN}^{\alpha\beta}. \quad (4.2)$$

When treating the embedding tensor as a spurionic object that transforms under the duality group, the Lagrangian and transformation rules remain formally invariant under $E_6(6)$. Under such a transformation $\Theta$ would transform as $\Theta_M^\alpha t_\alpha \rightarrow g_M^N \Theta_N^\alpha (g t_\alpha g^{-1})$, with $g \in E_6(6)$. Of course, when freezing $\Theta_M^\alpha$ to a constant, the $E_6(6)$-invariance is broken.

It is clear that the $E_6(6)$-covariant constraints on the embedding tensor are in direct correspondence to a set of USp(8)-covariant constraints on the $T$-tensor, as their assignment to representations of $E_6(6)$ and/or its subgroups are directly related. To make this more explicit we note that every variation of the coset representative can be expressed as a (possibly field-dependent) $E_6(6)$ transformation acting on $V$ from the right. For example, a rigid $E_6(6)$ transformation acting from the left can be rewritten as a field-dependent transformation from the right,

$$V \rightarrow V' = g V = V \sigma^{-1}, \quad (4.3)$$

with $\sigma^{-1} = V^{-1} g V \in E_6(6)$, but also a supersymmetry transformation can be written in this form. Consequently, these variations of $V$ induce the following transformation of the $T$-tensor,

$$T^M_{MN} \rightarrow T'^M_{MN} = \sigma_M^Q \sigma_N^R (\sigma^{-1})_S^P T^P_{QK}. \quad (4.4)$$

This implies that the $T$-tensor constitutes a representation of $E_6(6)$. Observe that this is not an invariance statement; rather it means that the $T$-tensor (irrespective of the choice for the corresponding embedding tensor) varies under supersymmetry or any other transformation in a way that can be written as a (possibly field-dependent) $E_6(6)$-transformation. Note also that the transformation assignment of the embedding tensor and the $T$-tensor are opposite in view of the relationship between $g$ and $\sigma$, something that is important in practical applications.

The maximal compact USp(8) subgroup of $E_6(6)$ coincides with the R-symmetry group that acts on the fermion fields: the gravitino and spinor fields are symplectic Majorana spinors transforming in the $8$, and $48$ representation, respectively. A
crucial role is played here by the USp(8)-invariant skew-symmetric tensors $\Omega_{AB}$ and 
$\Omega^{AB} = (\Omega_{AB})^\ast (A, B, \ldots = 1, \ldots, 8)$ satisfying $\Omega^{AC} \Omega_{CB} = -\delta^A_B$. The presence of the representations of the spinor fields leads one to adopt a notation for $E_{6(6)}$ vectors $x_M$ and $y^M$ where indices $M$ are replaced by antisymmetric, symplectically traceless, index pairs $[AB]$. The $27$ representation is thus described by a pseudoreal, antisymmetric and symplectic traceless tensor $x_{AB}$,

$$x^{AB} \equiv (x_{AB})^\ast = \Omega^{AC} \Omega^{BD} x_{CD}, \quad \Omega^{AB} x_{AB} = 0 . \quad (4.5)$$

Raising and lowering of indices is effected by complex conjugation. Corresponding identities hold for the $y^{AB}$ transforming in the $27$ representation. The action of infinitesimal $E_{6(6)}$ transformations reads as follows,

$$\delta x_{AB} = -2 \Lambda_{\{A}^C x_{B\}C} + \Sigma_{ABCD} x^{CD},$$
$$\delta y^{AB} = 2 \Lambda_C^{\{A} y^{B\}C} - \Sigma_{ABCD} y^{CD}, \quad (4.6)$$

so that $x_{AB} y^{AB}$ is an $E_{6(6)}$ invariant. Here $\Lambda^{AB}$ parametrizes the USp(8) transformations; the fully antisymmetric pseudoreal and symplectic traceless tensors $\Sigma$ parametrize the remaining $E_{6(6)}$ transformations in accord with the following decomposition of the adjoint representation of $E_{6(6)}$: $78 \to 36 + 42$. The explicit restrictions on the parameters read,

$$\Lambda^A_B \equiv (\Lambda^A_B)^\ast = -\Lambda_B^A, \quad \Lambda_{\{A}^C \Omega_{B\}C} = 0, \quad \Lambda_A^A = 0, \quad \Sigma_{ABCD} = \Sigma_{[ABCD]},$$
$$\Sigma_{ABCD} \equiv (\Sigma^{ABCD})^\ast = \Omega_{AE} \Omega_{BF} \Omega_{CG} \Omega_{DH} \Sigma^{EFGH}, \quad \Omega^{AB} \Sigma_{ABCD} = 0 . \quad (4.7)$$

Furthermore we note the identity,

$$\Omega_{[AB} \Sigma_{CDEF]} = 0 , \quad (4.8)$$

which holds by virtue of the fact that $\Sigma$ transforms in the irreducible $42$ representation of USp(8). From it one derives that $\Omega_{AB} y^{BC} \Omega_{CD} y^{DE} \Omega_{EF} y^{FA}$ is an $E_{6(6)}$ invariant and thus represents the invariant symmetric tensor $d_{MNP}$ that we encountered in previous sections. From (4.8) we also derive the identities,

$$\Sigma_{1ACDE} \Sigma_2^{BCDE} + \Sigma_{2ACDE} \Sigma_1^{BCDE} = \frac{1}{16} \delta^B_A \Sigma_{1CDEF} \Sigma_2^{CDEF},$$
$$\Sigma_{1ABEF} \Sigma_2^{EFCD} - \Sigma_{2ABEF} \Sigma_1^{EFCD} =$$
$$\frac{2}{3} \delta_{[A}^{[C} (\Sigma_{1B|EFG} \Sigma_2^{D]|EFG} - \Sigma_{2B|EFG} \Sigma_1^{D]|EFG} . \quad (4.9)$$

The term on the right-hand side of the second equation parametrizes an infinitesimal USp(8) transformation. This identity ensures the closure of the algebra associated with $E_{6(6)}$. 

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In supergravity, which we will discuss in the next section, we will use a hybrid notation where rigid $E_{6(6)}$ indices will be denoted by $M, N, \ldots$, whereas the local $USp(8)$ indices $\underline{M}, \underline{N}, \ldots$ will be replaced by antisymmetric, symplectic traceless, index pairs $[ij]$. The indices $i, j, \ldots$ are also carried by the fermion fields; those fields carry an odd number of indices\(^5\). With these conventions the coset representative $V_M^N$ is written as $V_M^{ij}$, with $V_M^{ij} \Omega_{ij} = 0$. Furthermore $V_M^{ij}$ is pseudoreal: $V_M^{ij} \equiv (V_M^{ij})^\ast = V_M^{kl} \Omega_{ki} \Omega_{lj}$. Denoting the inverse of $V_M^{ij}$ by $V_{ij}^M$, we have\(^6\),

$$V_M^{ij} V_{ij}^N = \delta_M^N,$$

$$V_{ij}^M V_M^{kl} = \delta_{ij}^{kl} - \frac{1}{8} \Omega_{ij} \Omega_{kl}. \quad (4.10)$$

Exploiting the group property it follows that any variation $\Delta V$ takes the form,

$$\Delta V_M^{ij} - V_M^{kl} \left(2 \delta_k^i Q_l^j + P^{ijmn} \Omega_{mk} \Omega_{nl}\right) = 0,$$

(4.11)

where $Q$ and $P$ span the $E_{6(6)}$ algebra, so that they transform according to the 36 and 42 representation of $USp(8)$, respectively. Consequently $Q$ and $P$ are subject to the conditions (4.7) (but with local $USp(8)$ rather than with rigid $E_{6(6)}$ indices). It is straightforward to find explicit expressions for $Q$ and $P$ in terms of $V^{-1} \Delta V$,

$$Q_i^j = \frac{1}{3} V_{ik}^M \partial_M V_M^{jk},$$

$$P^{ijkl} = V_{mn}^M \Delta V_M^{[ij} \Omega^{k|m|} \Omega_l^n], \quad (4.12)$$

where $|m|$ indicates that the index $m$ is exempted from the antisymmetrization which pertains to $[ijkl]$. Possible variations $\Delta V$ include spacetime derivatives $\partial_{\mu} V$ or a gauge transformation with parameters $\Lambda^M$ according to $\Delta V^{ij} = \Lambda^P X_{PM}^N V_N^{ij}$. When $\Delta$ denotes the gauge-covariant derivative $\partial_{\mu} - g A_{\mu}^M X_M$, (4.11) defines the $USp(8)$ composite connection $Q_{\mu}^{i} j$ in the presence of the gauging, while $P_{\mu}^{ijkl}$ is the $USp(8)$-covariant tensor whose square will constitute the kinetic term for the scalar fields,

$$Q_{\mu i}^j = \frac{1}{3} V_{ik}^M \partial_M V_M^{jk} - g A_{\mu}^M Q_M^i j,$$

$$P_{\mu}^{ijkl} = V_{mn}^M \partial_M V_M^{[ij} \Omega^{k|m|} \Omega_l^n] - g A_{\mu}^M P_M^{ijkl}. \quad (4.13)$$

Here we used the definitions,

$$Q_M^i j = \frac{1}{2} V_{ik}^N X_{MN}^P V_P^{jk},$$

$$P_M^{ijkl} = V_{mn}^N X_{MN}^P V_P^{[ij} \Omega^{k|m|} \Omega_l^n]. \quad (4.14)$$

\(^5\)Note that only even-rank tensors can be pseudoreal, so that fermions transform in complex representations of $USp(8)$. Observe also that $\Omega$ is pseudoreal, i.e., $\Omega_{ij} = \Omega_{ik} \Omega_{jl} \Omega^{kl}$.

\(^6\)Observe that the double counting associated with summing over antisymmetric index pairs implies a change of the inner product as the direct conversion $X_M \rightarrow X_{ij}$ and $Y_M \rightarrow Y^{ij}$ yields $X_M Y_M = 2 \sum_{i>j} X_{ij} Y^{ij}$. The factor 2 can in principle be absorbed into the definitions of $X_{ij}$ and $Y^{ij}$, but we refrain from doing so.
We note that $\mathcal{P}_M^i{}^{jkl}$ represents the Killing vectors of the $E_6(6)/USp(8)$ coset space associated with its gauged isometries.

Here and henceforth we will use derivatives $D_\mu$ that are covariant with respect both local USp(8) (with composite connections $Q_{\mu i}^j$) and the subgroup of $E_6(6)$ that is gauged by (a subset of) the vectors $A_\mu^M$. Applying two such derivatives on $V$ leads to an integrability relation upon antisymmetrization,

$$
F_{\mu\nu}(Q)_i^j \equiv \partial_\mu Q_{\nu i}^j - \partial_\nu Q_{\mu i}^j + 2 Q_{[\mu i}^k Q_{\nu] k}^j = \frac{2}{3} \mathcal{P}_{[\mu i k l m} \mathcal{P}_{\nu] j k l m} - g F_{\mu\nu}^M Q_M i^j ,
$$

$$
D_{[\mu} \mathcal{P}_{\nu]}^i{}^{jkl} = -\frac{1}{2} g F_{\mu\nu}^M \mathcal{P}_M^i{}^{jkl} ,
$$

(4.15)

where we made use of (3.7). These are the Cartan-Maurer equations with extra terms of order $g$ induced by the gauging. In the Lagrangian those terms initially cause a breaking of supersymmetry that will have to be compensated by new interaction terms and variations. The order-$g$ corrections in (4.15) are in fact proportional to the two components of the $T$-tensor defined already in (4.2),

$$
T^i_{j mn} = Q_M^i j^M V_{mn}^M ,
T^{ijkl}_{mn} = \mathcal{P}_M^{ijkl} V_{mn}^M .
$$

(4.16)

Both these components are pseudoreal and have symmetry properties that should be obvious from the preceding text. In particular, note that $T_i^{jkl} = \Omega_i^m \Omega_j^n \Omega_k^p \Omega_l^q T_m^{npq}$ and $\Omega_k[T_i^{j}{}_{mn}] = \Omega_k^{[ij} T_{jknm} = 0$, so that $T_i^{jkl} = -T_i^{jmn} \Omega_{mk}^l \Omega_{nl}^j$. We note the following convenient relation,

$$
X_{MN}^P = V_M^m n \mathcal{V}_N^K l \mathcal{V}_{ij}^P \left[ 2 \delta_i^k T_{jlmn} + T_{ijpq}^m \Omega_{pk}^l \Omega_{ql}^i \right] ,
$$

(4.17)

which is just the inverse of (4.2).

Following the argument given at the beginning of the section, we consider variations of the coset representative. These can always be cast in the form of an $E_6(6)$ transformation acting on the right of $V$, which implies that any variation of the $T$-tensor is again proportional to the $T$-tensor itself. Since the variations under USp(8) are obvious, the relevant variation concerns

$$
\delta V_M^i j = -V_M^m n \Omega_{mk}^l \Omega_{nl}^j \Sigma^{ijkl} .
$$

(4.18)

It is straightforward to determine the effect of this variation on various USp(8) tensors and we present the explicit results,

$$
\delta Q_{\mu i}^j = -\frac{1}{3} \mathcal{P}_{\mu i k l m} \Sigma^{j k l m} + \frac{1}{3} \Sigma_{i k l m} \mathcal{P}_\mu^{j k l m} ,
\delta \mathcal{P}_\mu^{ijkl} = -D_\mu \Sigma^{ijkl} ,
\delta T^i_{j mn} = \frac{1}{3} \Sigma_{j p q r} T^{ipqr} mn - \frac{1}{3} \Omega_i^k \Omega_j^w \Sigma_{vpqr} T^{wpqr} mn + \Sigma_{mnpq} \Omega^{pr} \Omega^{qs} T^{ijkl} ,
\delta T^{ijkl}_{mn} = -4 T^{i}_{p m n} \Sigma^{k l p} + \Sigma_{mnpq} \Omega^{pr} \Omega^{qs} T^{ijkl} .
$$

(4.19)
Armed with these results we can now proceed and derive the constraints on the $T$-tensor that are induced by the corresponding constraints on embedding tensor discussed in section 2. First of all, the $T$-tensor will be restricted as a result of the representation constraint (2.12), according to which the embedding tensor belongs to the $351$ representation of $E_{6(6)}$. This representation branches under $\text{USp}(8)$ into $36 + 315$. According to (4.19) the $T$-tensor should therefore precisely comprise these two representations. Here it is helpful to indicate the $\text{USp}(8)$ representations that are described by the unconstrained $T$-tensor, $T_{ijkl}$:

$$
36 \times 27 = 36 + 315 + 27 + 594, \\
315 \times 27 = 315 + 36 + 315 + 792,
$$

(4.20)

where the underlined representations are those that are allowed by the constraint. Therefore, when subject to the constraint, $T_{ijkl}$ will exclusively belong to the $315$ representation, while $T_{ijlm}$ is decomposable into the $315$ and the $36$ representations. Describing these two representations by two pseudoreal, symplectic traceless, tensors $A_{ij}$ and $A_{ijkl}$, satisfying

$$
\delta A_{ij} = \frac{4}{9} \Omega^{p[j} A_{1p,km]} A_{2p,klm}, \\
\delta A_{ijkl} = \frac{3}{2} \left( \Omega^{m[j} \Sigma^{k]im} + \Omega^{m[j} \Sigma^{k]lm} \right) A_{1mn} \\
- \left( \Omega^{[i} \Omega^{k|m]} \Sigma^{j]npq} - 3 \Omega^{m[j} \Omega^{n[k]pq} \\
- \frac{1}{6} \Omega^{m[j} \Omega^{k]npq} + \frac{1}{6} \Omega^{m[j} \Omega^{k]pq} \right) A_{2mnpq},
$$

(4.22)

We note that these variations are consistent with the $\text{USp}(8)$ irreducibility constraints for the tensors $A_{1,2}$ themselves. Furthermore we note that the linear combination

$$
Z_{ijkl} \equiv \Omega^{i[k} A_{1}^{j]} + A_{2}^{i[j]kl},
$$

(4.23)

defines an antisymmetric tensor in the symplectic traceless index pairs $[ij]$ and $[kl]$, transforming according to

$$
\delta Z_{ijkl} = -\Sigma^{ijmn} \Omega_{mp} \Omega_{nq} Z_{pq,kl} - \Sigma^{klnm} \Omega_{mp} \Omega_{nq} Z_{ij,pq}.
$$

(4.24)
This shows (c.f. (4.6)) that $Z^{ij,kl}$ must be the dressed version of the $E_{6(6)}$ tensor $Z^{MN}$,

$$Z^{ij,kl} = \frac{1}{5} \sqrt{5} V^i_M V^j_N V^k_P V^{kl} Z^{MN}.$$  \hfill (4.25)

The proportionality constant follows from applying (2.18), with $X_{MN}^P$ expressed by (4.17), employing the following representation of the invariant symmetric three-rank tensors,

$$d_{MN} = 2 \sqrt{5} V^i_M V^j_N V^k_P V^{kl} \Omega^{ij} \Omega^{km} \Omega^{ni},$$

$$d^MNP = 2 \sqrt{5} V^i_M V^j_N V^k_P V^{kl} \Omega^{ij} \Omega^{km} \Omega^{ni}.$$  \hfill (4.26)

To derive the above representation we made use of the observation below (4.8). Note that the constancy of $d_{MNP}$ and $d^MNP$ is ensured by $E_{6(6)}$ invariance and that the normalization is in accord with (2.19).

The variations (4.22) can be used to determine the supersymmetry variations of these tensors (as we will discuss in the next section). We also note the following expressions for the USp(8)-covariant derivatives of $A_{1,2}$,

\[
D_\mu A_{1}^{ij} = -\frac{4}{9} \Omega^{i\mu(j} P_{\mu}^{jklm} A_{2p,klm}, \\
D_\mu A_{2}^{i,k,l} = -\frac{3}{2} \left( \Omega^{i\mu(j} P_{\mu}^{jklm} + \Omega^{(i} P_{\mu}^{jklm} \right) A_{1mn} + \left( \Omega^{i\mu(j} P_{\mu}^{jklm} - \frac{3}{2} \Omega^{im} \Omega^{(i} \Omega^{m] P_{\mu}^{jklm}} \right) A_{2m, npq}.
\]  \hfill (4.27)

Having determined the consequences of the representation constraint, it remains to derive the consequences of the closure constraint (2.9). This will lead to identities quadratic in the $T$-tensor. Here we have the option of using either one of the equivalent version presented in (2.24). It is convenient to choose the second one and write it in terms of the $T$-tensor and the tensor (4.23). The constraint then implies that the following products of these tensors should vanish,

$$T^{i}_{j,k,l} Z^{kl,mn} = 0 = T^{ij,kl} Z^{mn,pq}.$$  \hfill (4.28)

These two identities take the form,

\[
0 = \delta^{[i}_{(k} A_{1}^{j]m} A_{1}^{l)m} + A_{2(k,l)}^{i,mn} A_{1}^{m[i} \Omega^{j]n} + 2 A_{2}^{i,j,mn} A_{1}^{m[i} \Omega^{j]n} - A_{2}^{m,i} A_{2}^{i,j,mn}, \\
0 = -2 A_{2}^{k,pqr} \delta^{[i}_{k} A_{1}^{j]k} - 2 A_{2}^{k,pqr} \Omega^{[i} \Omega^{j]l} A_{2[k} \Omega^{i]j]l} - 3 \Omega^{[i} \Omega^{j]l} A_{2}^{r,s} A_{1}^{k[l} \Omega^{i]j]} + 4 A_{2}^{k,pqr} \Omega^{[i} A_{2}^{j]kl} + 3 \Omega^{[r} A_{2}^{p,q]mn} A_{2}^{m,nij},
\]  \hfill (4.29)
By contraction one derives three equations, which can be written as follows,

\[ 0 = A_{1kl} A_{2}^{k,lm} \Omega_{mj} + A_{2k,lmj} A_{2}^{k,lm} - \frac{1}{9} A_{2j,klm} A_{2}^{i,klm} - \frac{1}{9} \delta_j^i |A_{2}^{k,lmn}|^2, \]

\[ 0 = 3 A_{1}^{im} A_{1}^{jm} - \frac{1}{3} A_{2}^{i,klm} A_{2}^{j,klm} - \frac{1}{8} \delta_j^i (3 |A_{1}^{k}|^2 - \frac{1}{3} |A_{2}^{k,lmn}|^2), \]

\[ 0 = (\frac{1}{4} A_{1mn} \Omega_{pi} - A_{2[m,n]pi}) A_{2}^{m,np[j} \Omega^{kl]} + (\frac{1}{2} A_{1p(m\Omega_{ni})} + A_{2(m,n)pi} + A_{2i,mnp}) A_{2}^{m,n[jk} \Omega^{l]p} + \frac{1}{3} A_{1im} A_{2}^{m,jkl} + \frac{1}{2} A_{1mn} A_{2}^{m,n[jk} \delta_i^l) . \]  

(4.30)

The first equation (4.29) can be written with a similar index structure as the last equation (4.30),

\[ \frac{1}{3} A_{1im} A_{2}^{m,jkl} + (\frac{1}{2} A_{1p(m\Omega_{ni})} + A_{2(i)pmn}) A_{2}^{m,n[jk} \Omega^{l]p} = - \frac{1}{18} (\delta_j^i \Omega^{k[q} A_{2}^{l]mnp} A_{2q,mnp} - \Omega^{i[jk} A_{2}^{l]mnp} A_{2i,mnp}), \]

(4.31)

where we also made use of the second equation (4.30). The same applies to the first equation (4.30), which, when combined with the third equation of (4.30) and with (4.31), yields an identity that only involves terms quadratic in \( A_{2} \),

\[ 0 = A_{2m,npi} A_{2}^{m,n[jk} \Omega^{l]p} + \frac{1}{2} A_{2m,npq} A_{2}^{m,np[j} \Omega^{k[q} \delta_i^l] - \frac{1}{2} (A_{2m,npi} A_{2}^{m,np[j} - \frac{1}{6} A_{2m,npq} A_{2}^{m,np} \delta_i^j) \Omega^{k]l}. \]

(4.32)

Somewhat surprisingly, this identity is not implied by (4.28) as we have also been able to derive it directly, without making use of (4.28). The identity simply reflects the fact that the symmetric product of two \( 315 \) representations contains only a single \( 315 \) representation.

5 Lagrangian and supersymmetry transformations

The previous results play a crucial role in establishing the supersymmetry of the action of five-dimensional maximal gauged supergravity. The various gaugings are encoded in the embedding tensor. When treating the embedding tensor as a (spurionic) \( E_{6(6)} \)-covariant tensor, the action will be manifestly \( E_{6(6)} \) invariant, irrespective of the gauge group, provided that its corresponding embedding tensor satisfies the constraints outlined previously.

Five-dimensional world and tangent-space indices are denoted by \( \mu, \nu, \ldots \) and \( a, b, \ldots \), respectively, and take the values \( 1, 2, \ldots, 5 \). We employ hermitean \( 4 \times 4 \) gamma matrices \( \gamma_a \), which satisfy

\[ C \gamma_a C^{-1} = \gamma_a^T, \quad C^T = -C, \quad C^\dagger = C^{-1}, \quad \gamma_{abcde} = \mathbf{1} \varepsilon_{abcde}. \]

(5.1)
Here $C$ denotes the charge-conjugation matrix and gamma matrices with $k$ multiple indices denote the fully antisymmetrized product of $k$ gamma matrices in the usual fashion, so that we have, for instance, $\gamma_a \gamma_b = 1 \delta_{ab} + \gamma_{ab}$. In view of the last equation, gamma matrices with more than two multiple indices are not independent, and can be linearly expressed into the unit matrix, $\gamma_a$ and $\gamma_{ab}$. Note that $C$, $C\gamma_a$ and $C\gamma_{ab}$ constitute a complete basis of 6 antisymmetric and 10 symmetric (unitary) matrices in spinor space. The gamma matrices commute with the automorphism group of the Clifford algebra, USp(2$N$), where $N$ denotes the number of independent spinors. In the case at hand we have $N = 4$. Spinors can be described either as Dirac spinors, or as symplectic Majorana spinors. The latter description is superior in that it makes the action of the USp(8) R-symmetry group manifest. We will thus employ symplectic Majorana spinors $\psi^i$ with $i = 1, 2, \ldots, 8$, subject to the reality constraint,

$$C^{-1} \bar{\psi}_i^T = \Omega_{ij} \psi^j,$$  \hspace{1cm} (5.2)

where $\Omega$ is the symplectic USp(8)-invariant tensor introduced previously. Observe that we adhere to our convention according to which raising or lowering is effected by complex conjugation.

The gravitini $\psi_{\mu}^i$ and associated supersymmetry parameters $\epsilon^i$ transform in the 8 representation of USp(8), whereas the spinor fields $\chi^{ijk}$ transform in the 48 representation. The symplectic Majorana constraint for the latter reads,

$$C^{-1} \bar{\chi}^{ijk} = \Omega_{il} \Omega_{jm} \Omega_{kn} \chi^{lmn},$$  \hspace{1cm} (5.3)

Finally we note the following relation for fermionic bilinears, with spinor fields $\psi^i$ and $\varphi^j$,

$$\bar{\psi}_i \Gamma \varphi^j = -\Omega_{ik} \Omega^{lj} \bar{\varphi}_l (C^{-1} \Gamma^T C) \psi^k.$$  \hspace{1cm} (5.4)

Comparing this to the hermitean conjugate of these bilinears, one finds that $i \bar{\psi}_i \varphi^j$, $\bar{\psi}_i \gamma_a \varphi^j$ and $i \bar{\psi}_i \gamma_{ab} \varphi^j$ are pseudoreal.

Rather than first deriving the Lagrangian, we start by considering the transformation rules, restricted by $E_6(6)$, vector-tensor gauge invariance, and other invariances, up to terms of higher order in the fermion fields. The coefficients in these variations (apart from certain normalizations) can be fixed by requiring that the supersymmetry closes up that order,

$$\delta e_{\mu}^a = \frac{1}{2} \bar{\epsilon}^i \gamma^a \psi_{\mu}^i,$$

$$\delta V_{M}^{ij} = i V_{M}^{kl} \left[ 4 \Omega_{p[k} \bar{\chi}^{lmn]} \epsilon^p + 3 \Omega_{kl} \bar{\chi}^{mn} \epsilon^p \right] \Omega_{ij} \Omega_{mn},$$

$$\delta A_{\mu}^M = 2 \left[ i \Omega^{ik} \bar{\epsilon}_k \psi_{\mu}^j + \bar{\epsilon}_k \gamma_{\mu} \chi^{ijk} \right] V_{ij}^M,$$

$$\delta B_{\mu\nu}^M = \frac{4}{\sqrt{5}} V_{M}^{ij} \left[ 2 \bar{\psi}_{[\mu} \gamma_{\nu]} \epsilon^k \Omega_{jk} - i \bar{\chi}_{ijk} \gamma_{\mu\nu} \epsilon^k \right] + 2 d_{MNP} A_{[\mu}^N \delta A_{\nu]}^P,$$
\[ \delta \psi^i = (\partial_\mu \delta^i_j - Q_{\mu j}^i - \frac{i}{4} \omega_{\mu}^{ab} \gamma_{ab} \delta^i_j) \epsilon^j \\
+ i \left[ \frac{1}{12} \left( \gamma_{\mu \nu \rho} \mathcal{H}^{\mu \nu \rho} - 4 \gamma^\nu \mathcal{H}_{\mu \nu}^{ij} \right) - g \gamma_{\mu} A_{1}^{ij} \right] \Omega_{jk} \epsilon^k, \]

\[ \delta \chi^{ijk} = \frac{1}{2} i \gamma^\mu P_{\mu}^{ijk} \Omega_{lm} \epsilon^m - \frac{3}{16} \gamma^\mu \left[ \mathcal{H}_{\mu \nu}^{[ij} \epsilon^k] - \frac{1}{3} \Omega^{[ij} \mathcal{H}_{\mu \nu}^{k]m} \Omega_{mn} \epsilon^n \right] \\
+ g A_2^{j,ijk} \Omega_{lm} \epsilon^m, \]  

(5.5)

where

\[ \mathcal{H}_{\mu \nu}^{ij} = \mathcal{H}^M_{\mu \nu} V_M^{ij} = (F^M_{\mu \nu} + g Z^{MN} B_{\mu \nu N}) V_M^{ij}. \]  

(5.6)

Note that \( \delta B_{\mu \nu M} \) is only determined up to terms that vanish upon contraction with \( Z^{MN} \). While the covariance with respect to most bosonic symmetries is straightforward, the form of \( \delta B_{\mu \nu M} \) requires further comment. On \( A_\mu^M \), the commutator of a supersymmetry transformation and a vector gauge transformation does not vanish but leads to,

\[ [\delta(\epsilon), \delta_{\text{vector}}(\Lambda^M)] = \delta_{\text{tensor}}(\Xi_{\mu M}), \]  

(5.7)

with

\[ \Xi_{\mu M} = -d_{MNP} \Lambda^N \delta(\epsilon) A_\mu^P, \]  

(5.8)

whereas supersymmetry commutes with tensor gauge transformations. The latter requires the presence of the second term in \( \delta B_{\mu \nu M} \), proportional to the invariant tensor \( d_{MNP} \). After this one verifies that the commutator \( (5.7) \) is also correctly realized on the tensor fields (up to terms that vanish upon contraction with \( Z^{MN} \)), so that all supersymmetry variations are consistent with the bosonic symmetries. What remains is to determine the various multiplicative coefficients from the requirement that the supersymmetry algebra closes on all fields. With the coefficients adopted in \( (5.9) \), the commutator of two supersymmetry transformations with parameters \( \epsilon_1 \) and \( \epsilon_2 \) closes uniformly into the bosonic symmetries,

\[ [\delta(\epsilon_1), \delta(\epsilon_2)] = \xi^\mu D_\mu + \delta_{\text{Lorentz}}(\epsilon^{ab}) + \delta_{\text{USp}(8)}(\Lambda_\mu^i) + \delta_{\text{vector}}(\Lambda^M) + \delta_{\text{tensor}}(\Xi_{\mu M}). \]  

(5.9)

Here \( \xi^\mu D_\mu \) denotes a covariant general coordinate transformation with parameter \( \xi^\mu \). Such a transformation consists of a spacetime diffeomorphism with parameter \( \xi^\mu \) combined with gauge transformations with parameters,

\[ \epsilon^{ab} = -\xi^\mu \omega^{ab}_\mu, \]
\[ \Lambda_j^i = -\xi^\mu Q_{\mu j}^i, \]
\[ \Lambda^M = -\xi^\nu A_\mu^M, \]
\[ \Xi_{\mu M} = -\xi^\nu B_{\nu \mu M}, \]  

(5.10)

where

\[ \xi^\mu = \frac{1}{2} \bar{\epsilon}_2 \gamma^\mu \epsilon_1^i. \]  

(5.11)
Beyond the covariant general coordinate transformations there are the symmetry variations indicated explicitly in (5.9) with parameters (up to higher-order fermion terms),

$$\epsilon_{ab} = \frac{1}{12} i \epsilon_2 \gamma_{abcd} \epsilon_1^k \Omega_{ijk} \mathcal{H}^{cdij} + \frac{1}{3} i \epsilon_2 \epsilon_1^k \Omega_{jk} \mathcal{H}_{ab}^{ij} - i g \bar{\epsilon}_1 \gamma_{ab} \epsilon_2^j \Omega_{jk} A_1^{ik},$$

$$\Lambda_i^j = - g T^j_i z_{kl},$$

$$\Lambda^M = z^{ij} \mathcal{V}^M_{ij},$$

$$\Xi_{\mu M} = - \frac{8}{\sqrt{5}} \xi_{\mu ij} \mathcal{V}^j M - d_{MNP} A_{\mu N} z^{ij} \mathcal{V}^P_{ij},$$

(5.12)

where we also used the bilinears $z^{ij}$ and $\xi_{\mu ij}$ which are pseudoreal and antisymmetric in $[ij]$ and under the interchange of the spinor parameters $\epsilon_1$ and $\epsilon_2$,

$$\begin{align*}
\mathcal{V}^{ij} &= -2i \Omega^{[i} \bar{\epsilon}^{j]}_2 \epsilon_1^1, \\
\xi_{\mu ij} &= \frac{1}{2} \bar{\epsilon}^{[i} \gamma_{[\mu} \epsilon_1^1 \Omega_{j]k},
\end{align*}$$

(5.13)

Observe that $\Omega^{ij} \xi_{\mu ij} = -\xi_{\mu}$.

The closure holds modulo the field equations for the tensor field (which are linear in derivatives),

$$3 D_{[\mu} \mathcal{H}_{\nu \rho]}^M - i \frac{1}{\sqrt{5}} e \epsilon_{\mu \rho \sigma \tau} Z^{MN} \mathcal{V}^N_{ij} \mathcal{V}^P_{kl} \Omega_{ik} \Omega_{jl} - \frac{2}{3} \bar{\chi}_{ijk} \partial \bar{\chi}^{ijk}$$

$$- \frac{1}{12} |\mathcal{P}_{ij}^{ijkl}|^2 + \frac{2}{3} |\mathcal{P}_{ij}^{ijkl}| \bar{\chi}_{ijk} \gamma^{[\mu} \gamma^{\rho \sigma]} \psi_{[\nu} \psi_{k]} \Omega_{lm}
+ \mathcal{H}_{\rho \sigma ij}
\left[ \frac{1}{2} i \bar{\psi}_{\mu} \gamma_{[\rho} \gamma^{[\mu]} \psi_{\sigma]} \psi_{i} \psi_{k} \Omega_{kj} - \frac{1}{4} \bar{\chi}_{ijkl} \gamma^{[\mu} \gamma_{[\rho \sigma]} \psi_{m]} \right],$$

$$- \frac{\sqrt{5}}{64} \bar{\epsilon}_{\mu \rho \sigma \tau} \left[ g Z^{MN} B_{\mu M} D_{\rho} B_{\sigma N} + 4 d_{NPQ} A_{\rho}^{P} \left( \partial_{\sigma} A_{\tau}^{Q} + \frac{1}{3} g X_{[RS]}^{Q} A_{\sigma R} A_{\tau S} \right) \right]
- \frac{8}{3} d_{MNP} \left[ A_{\mu N}^{M} \partial_{\rho} A_{\rho}^{N} \partial_{\sigma} A_{\tau}^{P} A_{\rho}^{R} \left( \partial_{\sigma} A_{\tau}^{P} + \frac{1}{3} g X_{[ST]}^{P} A_{\sigma S} A_{\tau T} \right) \right]
- \frac{3}{4} g A_{ik}^{1} \bar{\psi}_{\mu} \gamma^{\mu} \psi_{i} \psi_{j} - \frac{4}{3} g \Omega_{mk} A_{2}^{1} \bar{\chi}_{ijkl} \gamma^{k} \psi_{[m},$$

$$+ 2 g i \Omega_{kp} \Omega_{q} \left[ - 4 A_{2}^{i} j pq + A_{1}^{[i} \partial \Omega^{q]} j \right] \bar{\chi}_{ijk} \gamma^{k} \psi_{m},$$

$$+ g^{2} \left[ 3 |A_{1}^{ij}|^2 - \frac{1}{3} |A_{2}^{i} j kl|^2 \right].$$

(5.15)
where apart from the supersymmetry transformations we made use of the properties derived for the \( T \)-tensor derived in section 4. The covariant derivatives on the spinor fields are defined by

\[
D_\mu \psi^i = \partial_\mu \psi^i - Q_{\mu j}^i \psi^j - \frac{i}{4} \omega^a_{\mu} \gamma_{a b} \psi^i, \\
D_\mu \chi^{ijk} = \partial_\mu \chi^{ijk} - 3 Q_{\mu l}^{[i} \chi^{jkl]} - \frac{1}{4} \omega^a_{\mu} \gamma_{a b} \chi^{ijk}.
\]  

(5.16)

For the convenience of the reader we record the supersymmetry variations,

\[
\delta P_\mu^{ijkl} = iD_\mu \left[ 4 \bar{\epsilon}_m \chi^{ijkl} \Omega^{jm} + 3 \bar{\epsilon}_m \chi^{m[ij} \Omega^{kl]} \right] \\
- 2 g T_{ijkl}^{mn} (i \Omega_{mp} \epsilon_{n}^{\mu} + \epsilon_{\mu} \gamma_{mnp}), \\
\delta H_{\mu \nu}^{M} = 4 D_{\mu} \left[ (i \Omega_{ik} \bar{\epsilon}_k \psi_{lj}^{j} + \bar{\epsilon}_k \gamma_{lj}^{j} \chi^{ijkl}) V_{ijkl}^{M} \right] \\
+ \frac{4 g}{\sqrt{5}} Z_{MN} V_{ij}^{ij} (2 \bar{\psi}_{[\mu i} \gamma_{\nu]}^{j} \Omega_{jk} - i \bar{\chi}^{ijkl} \gamma_{\mu \nu} \epsilon^{k}),
\]  

(5.17)

which are needed for establishing the invariance of the action. The change of the scalar potential under (4.18) is also needed,

\[
\delta \left[ 3 |A_1^{ij}|^2 - \frac{1}{3} |A_2^{ijkl}|^2 \right] = \left( \frac{1}{3} A_1^{mq} A_{2m,ijk} \Omega_{lq} + 2 A_2^{mnpq} A_{2n,mij} \Omega_{pq} \Omega_{lq} \right) \Sigma_{ijkl},
\]  

(5.18)

which also reveals the condition for stationary points of the potential. Here we made use of the fact that the potential is USp(8) invariant, so that we can expand in terms of USp(8)-covariant variations of the scalar fields, using (4.11). In this way we can also express the square of the masses at the stationary point, which are then also proportional to \( g^2 \) times the square of the \( T \)-tensor. This pattern repeats itself for the other fields and all mass squares are simply determined by expressions quadratic in the \( T \)-tensor taken at the stationary point. For the fermions this is already obvious, as the masslike terms are proportional to the \( T \)-tensor. For the vector and tensor fields there is a subtlety, as there are mixing terms between these types of fields. The mass terms for these fields gives rise for the following expressions

\[
(M_{\text{vector}}^{2})_{MN} \propto g^2 V_{M}^{ij} V_{N}^{kl} T_{ijkl}^{mnpq} T_{kl}^{rstu} \Omega_{mr} \Omega_{ns} \Omega_{pt} \Omega_{qu}, \\
(M_{\text{tensor}}^{2})_{MN} \propto g^2 V_{ij}^{M} V_{kl}^{N} Z_{ijkl}^{mn} Z_{kl,pq}^{mn} \Omega_{mp} \Omega_{nq}.
\]  

(5.19)

Note that the mass term for the tensor fields should not be interpreted as the mass square, in view of the fact that the kinetic terms are linear in spacetime derivatives and proportional to \( g Z_{MN} \). But the result shows that (when properly taking into account the corresponding kinetic terms) the vector masses are encoded as eigenvalues of the matrix \( T_{ijkl}^{mnpq} T_{kl}^{rstu} \Omega_{mr} \Omega_{ns} \Omega_{pt} \Omega_{qu} \), while the tensor masses correspond to the
eigenvalues of $Z^{mn,pq}$. These mass terms are subject to an orthogonality relation in view of equation (4.28), which is crucial for obtaining the correct degrees of freedom. To deal with the mixing between vector and tensor fields, it is best to impose a suitable gauge. This will be briefly discussed in subsection 6.1.

Stationary points of the potential may lead to a (partial) breaking of supersymmetry. The residual supersymmetry of the corresponding solution (assuming maximally symmetric spacetimes) is parametrized by spinors $\epsilon^i$ satisfying the condition

$$A_2^{l,ijk} \Omega_{lm} \epsilon^m = 0.$$  \hspace{1cm} (5.20)

From the gravitino variation one derives an extra condition

$$A_1^{im} A_{1,jm} \epsilon^j = \frac{1}{8} \left| A_1^{kl} \right|^2 - \frac{1}{9} \left| A_2^{k,lmn} \right|^2 \epsilon^i,$$  \hspace{1cm} (5.21)

but the two conditions (5.20) and (5.21) are in fact equivalent by virtue of the second equation of (4.30).

6 Examples

In this section we demonstrate our method to the known and some new examples of maximal $D = 5$ supergravities. These include the original $SO(p,q)$ gaugings of [2], the CSO($p,q,r$) gaugings discussed in [5] and the Scherk-Schwarz gaugings of [6] (see also [11, 12]). In these examples the gauge group is contained either in the maximal $SL(2,\mathbb{R}) \times SL(6,\mathbb{R})$ subgroup of $E_{6(6)}$, or in a non-semisimple extension of $SO(5,5) \times SO(1,1)$, which is another maximal subgroup of $E_{6(6)}$. Our construction provides a Lagrangian formulation of all these gaugings. Before coming to the examples we will first discuss the possible gauge fixing of the tensor gauge transformations in order to make contact with previous results in the literature. For zero gauge coupling all the tensor fields disappear from the Lagrangian, and one recovers the result of [11] without further ado.

6.1 Gauge fixing

The five-dimensional gauged supergravities that so far have appeared in the literature, have all been formulated without the freedom of tensor gauge transformations. They are recovered from our general formulation by using the tensor gauge transformations to set some of the vector fields to zero. To describe this we employ the special basis introduced in subsection 2.3 and decompose the vector indices according to $V_M = (V_A, V_\alpha, V_u)$. By definition, the matrix $Z^{MN}$ is invertible on the space spanned by $V_u$, so that on this space we may define its inverse $Z_{uv}$ according to

$$Z_{uv} Z^{vw} \equiv \delta^v_u.$$  \hspace{1cm} (6.1)
By means of the tensor gauge transformations in (3.1) we impose the gauge condition
\( A_\mu^u = 0 \), thus breaking \( E_{6(6)} \) covariance. Therefore we have to add a compensating
term to the supersymmetry variations,
\[
\delta^{\text{new}}(\epsilon) = \delta^{\text{old}}(\epsilon) + \delta(\Xi_{\mu u}) ,
\]
with \( \Xi_{\mu u} = -2g^{-1}Z_{uv}V_{ij}(i \Omega^{ik} \bar{\epsilon}_k \psi_{ij}^j + \bar{\epsilon}_k \gamma_{ij} \chi^{ij}) \). The terms proportional to the inverse gauge coupling constant can be avoided by making a field redefinition. New
tensor fields are defined by \( B_{\mu \nu \mu} \equiv H_{\mu \nu \mu} \), which are invariant under the tensor transformations, just as the field strengths \( F_{\mu \nu A} = H_{\mu \nu A} \) and \( F_{\mu \nu a} = H_{\mu \nu a} \). Hence the new
tensor fields are
\[
B_{\mu \nu \mu} = gZ_{uv}B_{\mu \nu v} + C_{[AB]}^u A_{\mu}^A A_{\nu}^B + C_{\mu A}^u A_{\mu}^A A_{\nu}^B ,
\]
which will now appear in the Lagrangian as massive fields. Under gauge and super-
symmetry transformations they transform according to
\[
\delta(\Lambda) B_{\mu \nu \mu} = -\Lambda^A (D_{\mu}^A B_{\mu \nu v} - C_{AB}^u F_{\mu \nu v} - C_{\mu A}^u F_{\mu \nu v} ) ,
\]
\[
\delta(\epsilon) B_{\mu \nu \mu} = 4D_\mu \left[ i \Omega^{ik} \bar{\epsilon}_k \psi_{ij}^j + \bar{\epsilon}_k \gamma_{ij} \chi^{ij} \right] V_{ij}^u
+ \frac{4g}{\sqrt{5}} Z_{uv} \chi^{ij}(2 \bar{\psi}_{[\mu i} \gamma_{\nu]} \epsilon^k \Omega_{jk} - i \bar{\chi}_{ijk} \gamma_{\mu \nu} \epsilon^k ) ,
\]
where the last expression is a special case of the second equation (5.17). In the Lagrangian these tensor fields appear in the kinetic term of the modified field strength
tensor, \( H_{\mu \nu ij} = V_{A}^{ij} F_{\mu \nu A} + V_{a}^{ij} F_{\mu \nu a} + V_{i}^{ij} B_{\mu \nu} \), and in the Chern-Simons term whose
leading term now takes the form
\[
\mathcal{L}_{VT} \propto \frac{1}{2} i \varepsilon^{\mu \nu \rho \tau} g^{-1} Z_{uv} B_{\mu \nu u} D_\rho B_{\sigma \tau v} + \cdots .
\]
The appearance of the inverse coupling constant \( g^{-1} \) and the matrix \( Z_{uv} \) in this term shows that, after gauge fixing, the theory no longer possesses a smooth limit to the ungauged theory. This phenomenon has been observed in the original construction of the \( \text{SO}(p, q) \) gauged theories [2]. Note that the full Lagrangian \( \text{(5.5)} \) in contrast allows a smooth limit \( g \to 0 \).

In the gauge-fixed version there remain many more interaction terms between tensor
and vector fields than those that are known from the \( \text{SO}(q, 6-q) \) gaugings. These terms
have a similar structure as the terms that were found recently for non-maximal gauged
supergravities with eight supersymmetries [13].

### 6.2 \( \text{CSO}(p, q, r) \) gaugings

Let us first review the case of gauge groups contained in the \( \text{SL}(2, \mathbb{R}) \times \text{SL}(6, \mathbb{R}) \) maximal
subgroup of \( E_{6(6)} \). Recall that a consistent gauging is completely encoded in an embedding
tensor \( \Theta_M^a \) that satisfies the linear projection constraint \( \text{(2.12)} \) and any of the
equivalent forms of the quadratic constraint \((2.24)\). With respect to \(\text{SL}(2, \mathbb{R}) \times \text{SL}(6, \mathbb{R})\), the representations of the vector gauge fields, the \(E_{6(6)}\) generators and the embedding tensor decompose according to,

\[
\begin{align*}
\overline{27} & \rightarrow (1, 15) + (2, 6), \\
78 & \rightarrow (1, 35) + (3, 1) + (2, 20), \\
351 & \rightarrow (1, 21) + (3, 15) + (2, \overline{84}) + (2, \overline{8}) + (1, 105),
\end{align*}
\]

respectively. A generic embedding tensor \(\Theta_M^\alpha\) transforming in the \(351\) representation of \(E_{6(6)}\) thus couples vector fields to generators according to

\[
\begin{array}{|c|c|c|}
\hline
& (1, 15) & (2, 6) \\
\hline
(1, 35) & (1, 21) + (1, 105) & (2, \overline{6}) + (2, \overline{84}) \\
(3, 1) & (3, 15) & (2, \overline{6}) \\
(2, 20) & (2, \overline{6}) + (2, \overline{84}) & (3, 15) + (1, 105) \\
\hline
\end{array}
\]

Equivalent representations in the bulk of the table must be identified since all representations in the decomposition of the \(351\) representation appear with multiplicity one. We stress that the representations indicated in the first row refer to the charges, which transform in the \(27\) representation, and not to the gauge fields which transform in the \(\overline{27}\) representation. Also the first column refers to the conjugate representation of the representations into which the \(E_{6(6)}\) generators decompose, because \(\Theta\) carries upper indices \(\alpha\) unlike the \(E_{6(6)}\) generators. However, the representations in the first column happen to be self-conjugate. This will not be the case in our next example.

Searching for subgroups of \(\text{SL}(2, \mathbb{R}) \times \text{SL}(6, \mathbb{R})\) implies that the representations in the last row must be excluded. Hence the only possible representation assignment for the embedding tensor is the representation \((1, 21)\). This representation can be described by a symmetric six-by-six tensor \(\theta_{IJ}\), where the indices \(I, J = 1, \ldots, 6\) denote the vector indices of \(\text{SL}(6, \mathbb{R})\). This restricts the possible gauge groups to subgroups of \(\text{SL}(6, \mathbb{R})\) and the participating vector gauge fields to the \((1, 1\overline{5})\) representation. Denoting vector indices of \(\text{SL}(2, \mathbb{R})\) by \(\alpha = 1, 2\), the vector fields now decompose into \(A_\mu^M \rightarrow (A_\mu^{IJ}, A_\mu^{I\alpha})\). The embedding tensor is then parametrized in terms of \(\theta_{IJ}\) according to \(\Theta_{[IJ]}^K = \delta_{[I}^K \theta_{J]L}\) and all other components vanish. This leads to the gauge group generators,

\[
X_{IJ} = \theta_{L[I} t_{J]L},
\]

with \(t_{K}^L\) the \(\text{SL}(6, \mathbb{R})\)-generators. Similarly, one finds that the only nonvanishing components of the antisymmetric tensor \(Z^{MN}\) are given by

\[
Z_{I\alpha J\beta} \propto \varepsilon_{\alpha\beta} \theta_{IJ},
\]
The explicit form of $\Theta_M^\alpha$ and $Z^{MN}$ shows that $Z^{MN}\Theta_N^\alpha = 0$ for any choice of $\theta_{IJ}$. According to (2.24), the quadratic constraint is thus satisfied and every symmetric six-by-six tensor $\theta_{IJ}$ defines a viable gauging. The gauge group is contained in the subgroup of $SL(6, \mathbb{R})$ that leaves $\theta_{IJ}$ invariant. This can be verified explicitly by making use of (6.8).

The 21 representation associated with $\theta_{IJ}$ falls in 28 different conjugacy classes leading to 15 independent gaugings. The corresponding tensors take the form,

$$
\theta_{IJ} = \text{diag}(1, \ldots, -1, \ldots, 0, \ldots)
$$

with $p + q + r = 6$. The corresponding gauge group is $\text{CSO}(p, q, r)$. We have thus obtained a complete classification of possible gauge groups $G_g \subset SL(2, \mathbb{R}) \times SL(6, \mathbb{R})$. From the rank of the tensors $\Theta_M^\alpha$ and $Z^{MN}$ one determines the number of tensor fields and the number of vector fields (after an appropriate gauge choice). It follows that the number of tensor fields is equal to $t = 2(6 - r)$, and the number of vector fields that gauge the group $\text{CSO}(p, q, r)$ equals $s = \frac{1}{2}(6 - r)(5 + r)$. The latter decompose into the gauge fields associated with the subgroup $SO(p, q)$ and with $r(p + q)$ nilpotent generators. Furthermore, the number of abelian gauge fields equals $\frac{1}{2}r(r - 1)$. A Lagrangian formulation for these non-semisimple groups had not yet been obtained. It now follows directly from the universal Lagrangian (5.13).

### 6.3 Gaugings characterized by $SO(5, 5) \times SO(1, 1) \subset E_6(6)$

Another class of gaugings is based on the decomposition of $E_6(6)$ under its subgroup $SO(5, 5) \times SO(1, 1)$, where the first factor is the U-duality group of maximal supergravity in six dimensions. The decompositions of the representations of the vector gauge fields, the $E_6(6)$ generators and the embedding tensor are now given by,

$$
\begin{align*}
27 & \rightarrow \overline{16}_{-1} + 10_{+2} + 1_{-4}, \\
78 & \rightarrow 45_0 + 1_0 + 16_{-3} + \overline{16}_{+3}, \\
351 & \rightarrow 144_{+1} + 16_{+1} + 45_{+4} + 120_{-2} + 10_{-2} + \overline{16}_{-5}.
\end{align*}
$$

Hence we effect the decomposition of the vector fields by assigning vector indices $m, n = 1, \ldots, 10$ and spinor indices $\alpha = 1, \ldots, 16$, with respect to $SO(5, 5)$, respectively. The gauge fields then decompose according to $A_\mu^M \rightarrow (A_\mu^\alpha, A_\mu^m, A_\mu^0)$ whereas the $E_6(6)$ generators decompose according to $t_\alpha \rightarrow (t_{mn}, t_0, t_2, t_2)$.

Upon extending $SO(5, 5) \times SO(1, 1)$ with the 16 nilpotent generators belonging to either the $16_{-3}$ or the $\overline{16}_{+3}$ representation, the resulting non-semisimple group constitutes a maximal subgroup of $E_6(6)$. The gauge couplings induced by a generic...
embedding tensor $\Theta_M^a$ transforming in the $351$ representation of $E_{6(6)}$ are as follows,

$$
\begin{array}{|c|ccc|}
\hline
& 10_{-2} & 16_{+1} & 1_{+4} \\
\hline
16_{-3} & \underline{16}_{-5} & 120_{-2} + 10_{-2} & 16_{+1} \\
45_0 & 10_{-2} + 120_{-2} & 144_{+1} + 16_{+1} & 45_{+4} \\
1_0 & 10_{-2} & 16_{+1} \\
\underline{16}_{+3} & 144_{+1} + 16_{+1} & 45_{+4} \\
\hline
\end{array}
$$

(6.12)

Again equivalent representations for the embedding matrix are identified as they appear with multiplicity one in the decomposition of the $351$ representation. Note again that the first row denotes the representation of the charges and not of the gauge fields, whereas the assignment in the first column denotes the conjugate representation as compared to the corresponding $E_{6(6)}$ generators. In this way, the upper-left entry thus describes the coupling of the gauge fields in the $10_{+2}$ to the generators in the $\underline{16}_{+3}$ generators.

From the table one immediately concludes that no subgroup of $SO(5,5) \times SO(1,1)$ can be gauged consistently, as there is no irreducible component of the embedding tensor that appears exclusively in the two middle rows of the table. We will therefore search for gauge groups that also involve (nilpotent) generators from either the $16_{-3}$ or the $\underline{16}_{+3}$ representation.

Let us start with those gaugings that couple generators belonging to the $16_{-3}$ representation. According to table (6.12) this allows two irreducible components for the embedding tensor, namely the $45_{+4}$ and $144_{+1}$ representations. We first focus on the case of an embedding tensor in the $45_{+4}$ representation. The corresponding embedding tensor is parametrized in terms of an antisymmetric ten-by-ten tensor $\theta^{mn}$ according to

$$
\Theta^\beta_{\alpha} \propto \theta^{mn} (\Gamma_{mn})_{\alpha}^{\beta}, \quad \Theta^0_{\alpha} \propto \theta^{mn},
$$

(6.13)

where the $SO(5,5)$ generators in the (chiral) spinor representation are denoted by $(\Gamma_{mn})_{\alpha}^{\beta}$. The only nonvanishing components of the tensor $Z^{MN}$ are given by

$$
Z^{mn} \propto \theta^{mn}.
$$

(6.14)

Together, this implies that only (some of the) vector fields $A_{\mu}^{\underline{2}}$ and $A_{\mu}^0$ from the $\underline{16}_{-1} + 1_{-4}$ representation can participate in the gauging and, furthermore, that only (some of the) tensor fields $B_{\mu}^m$ from the $10_{-2}$ representation will remain in the gauge-fixed formulation. Clearly $Z^{MN} \Theta_N^a = 0$ for any choice of $\theta^{mn}$, so that the quadratic constraint $\Theta_N^a$ is satisfied and every antisymmetric ten-by-ten tensor $\theta^{mn}$ defines a viable gauging. The theories descending from $D = 6$ dimensions by Scherk-Schwarz reduction belong to this class [6], with the tensor $\theta^{mn}$ singling out the generator of $SO(5,5)$ that is associated with the compactified sixth dimension in the reduction.
On the other hand an embedding tensor $\Theta_{M}^{\alpha}$ living in the $144+1$ is parametrized by a tensor $\theta_{m}^{\underline{\alpha}}$ subject to $(\Gamma^{m})_{\underline{\alpha} \underline{\beta}} \theta_{m}^{\underline{\alpha}} = 0$, according to

$$
\Theta_{m}^{\underline{\alpha}} \propto \theta_{m}^{\underline{\alpha}}, \quad \Theta_{mn}^{\underline{\alpha} \underline{\beta}} \propto (\Gamma^{[m})_{\underline{\alpha} \underline{\beta}} \theta^{n]_{\underline{\alpha}}}, \quad Z^{m}_{\underline{\alpha}} = -Z^{m}_{\underline{\beta}} \propto \theta^{m}_{\underline{\alpha}}.
$$

(6.15)

Here $(\Gamma^{m})_{\underline{\alpha} \underline{\beta}}$ is symmetric in the spinor indices $\underline{\alpha}, \underline{\beta}$ and corresponds to the $SO(5,5)$ gamma matrices restricted to the chiral subspace (after multiplying with the charge conjugation matrix). In this case, the quadratic constraint (2.24) implies the nontrivial relations,

$$
\theta_{m}^{\underline{\alpha}} \theta^{m}_{\underline{\beta}} = 0, \quad (\Gamma^{[m})_{\underline{\alpha} \underline{\beta}} \theta^{n]_{\underline{\alpha}} \theta^{k}_{\underline{\beta}} = 0,
$$

(6.16)

to be satisfied by $\theta_{m}^{\underline{\alpha}}$. It is obvious that solutions to these constraints will correspond (by dimensional reduction) to maximal gauged supergravities in six dimensions $[6]$, because for that theory the embedding tensor belongs to the $144+1$ representation of the $SO(5,5)$ duality group. A particular solution is obtained by restricting $\theta_{m}^{\underline{\alpha}}$ to the unique component that is invariant under the diagonal $SO(5)$ subgroup of $SO(5) \times SO(5) \subset SO(5,5)$. The assignment of $\theta_{m}^{\underline{\alpha}}$ with respect to $SO(5)$ follows from the observation that the vector and spinor representations decompose according to $10 \rightarrow 5 + 5$ and $16 \rightarrow 1 + 5 + 10$, respectively. This leads to one singlet for $\theta_{m}^{\underline{\alpha}}$ in view of the fact that $\theta_{m}^{\underline{\alpha}}$ is traceless upon contracting with $SO(5,5)$ gamma matrices. This particular choice for the embedding tensor must thus be related by dimensional reduction to the six-dimensional $SO(5)$ gauging $[14]$ (which in turn can be obtained by dimensional reduction from a seven-dimensional gauging). It corresponds to a gauging of $CSO(5,0,1)$. Indeed, the embedding tensor in the previous subsection contains precisely one $SO(5)$ singlet (under $SO(5)$ the $21$ representation of $SO(6)$ decomposes into $15 + 5 + 1$), corresponding to $p = 5$, $q = 0$ and $r = 1$. This gauging involves 10 gauge fields associated with $SO(5)$ and 5 extra gauge fields from the $10_{+2}$ representation.

Obviously, there are also gaugings in which the embedding tensor has nonvanishing components in both the $45+4$ and the $144+1$ representations. According to (2.24) this implies the additional identities,

$$
\theta^{mn} \theta_{n}^{\underline{\alpha}} = 0, \quad \theta^{mn} \theta^{k}_{\underline{\alpha}} (\Gamma_{mn})_{\underline{\alpha} \underline{\beta}} = 0,
$$

(6.17)

among the different components of $\Theta_{M}^{\alpha}$. These gaugings will include, for example, the theories obtained by a two-fold Scherk-Schwarz reduction from seven dimensional supergravity.

Finally, let us briefly consider the class of gaugings that include gauge group generators from the $16_{+3}$ representation, so that the gauge group is contained in the conjugate extension of $SO(5,5) \times SO(1,1)$ to a maximal subgroup of $E_{6(6)}$. According to table (6.12), allowed embedding tensors have now components in the $10_{-2}$, the
and the $\mathbf{120}_{-2}$ representations. A gauging of the first type is parametrized by a constant vector $\theta_m$ with nontrivial components,
\[
\Theta_m^{kl} = \delta_m^{[k} \theta_l^{l]} , \quad \Theta_m^0 \propto \theta_m , \quad \Theta_{\underline{\alpha} \underline{\beta}} \propto \theta_m \left( \Gamma^m \right)_{\underline{\alpha} \underline{\beta}} ,
\]
\[
Z^{m0} = -Z^{0m} \propto \theta^m ,
\]
(6.18)
in $\Theta_M^{\alpha}$ and $Z^{MN}$.

The quadratic constraint (2.24) is then equivalent to the condition $\theta^m \theta_m = 0$, which does admit nontrivial real solutions. Every lightlike vector $\theta_m$ thus defines a viable gauging that involves only two tensor fields $B_{\mu \nu 0}$ and $\theta^\alpha B_{\mu \nu \alpha}$.

An embedding tensor in the $\mathbf{16}_{-5}$ representation is parametrized by a spinor $\theta^\alpha$ which induces the components
\[
\Theta_m^\underline{\alpha} = \left( \Gamma_m \right)_{\underline{\alpha} \underline{\beta}} \theta^\underline{\beta} , \quad Z^{0\underline{\alpha}} = -Z^{\underline{\alpha}0} \propto \theta^\underline{\alpha} ,
\]
(6.19)
showing that the quadratic constraint (2.24) is automatically satisfied. These gaugings constitute a new class of abelian gaugings that involve vector fields exclusively from the $\mathbf{10} + \mathbf{2}$ and only two tensor fields, $B_{\mu \nu 0}$ and $\theta^\alpha B_{\mu \nu \alpha}$. In fact, they have a geometrical interpretation originating from type-IIB RR-flux compactifications on a five-torus $T^5$. To work out this relation, representations are further decomposed under the group $\text{SL}(5, \mathbb{R}) \times \text{SO}(1, 1)$, associated to the metric moduli of $T^5$ and the Cartan subgroup of the ten-dimensional $\text{SL}(2, \mathbb{R})$ duality group, respectively. This SO(1, 1) is a combination of the two SO(1, 1) factors appearing in $\text{SL}(5, \mathbb{R}) \times \text{SO}(1, 1) \times \text{SO}(1, 1) \subset \text{SO}(5, 5) \times \text{SO}(1, 1)$. The embedding tensor $\theta^\underline{\alpha}$ then gives rise to three irreducible components $\mathbf{10}_{+1}$, $\mathbf{5}_{+2}$, $\mathbf{1}_0$, corresponding to a three-form, a one-form, and a five-form RR-flux, respectively
\[
\partial_{[\Lambda} C^{(2)}_{\Sigma \Gamma]} \propto \theta_{\Lambda \Sigma \Gamma} , \quad \partial_{\Lambda} C^{(0)} \propto \theta_{\Lambda} , \quad \partial_{[\Lambda} C^{(4)}_{\Sigma \Gamma \Delta \Pi]} \propto \epsilon_{\Lambda \Sigma \Gamma \Delta \Pi} \theta ,
\]
(6.20)
where indices $\Lambda, \Sigma, \ldots$ refer to coordinates on the five-torus and $C^{(0)}$, $C^{(2)}$, and $C^{(4)}$ denote the RR-fields in ten dimensions. After gauge fixing, the vector fields can be assigned the representations
\[
(B^{(2)}_{\mu \Lambda} , C^{(2)}_{\mu}) = \mathbf{5}_{-1} + \mathbf{5}_0 \subset \mathbf{10}_{+2} , \quad (C^{(4)}_{\mu \Lambda \Sigma \Gamma} , C^{(2)}_{\mu \Sigma}) = \mathbf{10}_0 + \mathbf{5}_{+1} \subset \mathbf{16}_{-1} .
\]
(6.21)
From (6.19) it follows, that scalars in the presence of these fluxes couple only to graviphotons $G^\Lambda_{\mu}$ and vector fields originating from the NSNS two form $B^{(2)}$. The two tensor fields in turn descend from $B^{(2)}$ and $C^{(2)}$. Details can be worked out along the lines of [9].

Note that in $\Theta_{\underline{\alpha} \underline{\beta}}$ the index $\underline{\alpha}$ couples to vector fields in the $\mathbf{16}_{-5}$ while the index $\underline{\beta}$ couples to generators in the $\mathbf{16}_{+5}$. 

31
7 Conclusions

In this paper we presented deformations of maximally supersymmetric $D = 5$ supergravity induced by gauge interactions. No other supersymmetric deformations of this theory are expected to exist. The deformed theory is described by the Lagrangian (5.15) together with the supersymmetry transformation rules (5.5). This Lagrangian gives a uniform description of all possible deformations in a manifestly $E_{6(6)}$-covariant framework. It couples vector fields in the $27$ and tensor fields in the $27$ representation of $E_{6(6)}$, which in an intricate way transform under vector and tensor gauge transformations according to (3.1) and (3.6), respectively. As a result the number of degrees of freedom is always consistent with supersymmetry.

The gauging is entirely encoded in the constant embedding tensor $\Theta^\alpha_M$ which belongs to the $351$ representation of $E_{6(6)}$ and satisfies the quadratic constraint (2.24). It describes the coupling of vector fields to gauge group generators (2.1) and implies the existence of an (antisymmetric) metric $Z^{MN}$ that serves as a metric for the first-order kinetic term of the two-form tensor fields (3.11), which is accompanied by Chern-Simons terms. Also the tensor gauge transformations depend on the tensor $Z^{MN}$. In contrast to the ungauged theory [1], the Lagrangian of the gauged supergravity combines both the vector fields and their dual tensor fields, where the embedding tensor projects out those vector and tensor fields that actually participate in the gauging. This formulation admits a smooth limit $g \to 0$ back to the ungauged theory. In section 6.1 we have discussed the form of the Lagrangian (5.15) after a specific gauge choice which fixes the freedom of tensor gauge transformations by eliminating part of the vector fields and explicitly breaks the $E_{6(6)}$ covariance. Previous constructions of gauged supergravities in five dimensions have been obtained in this special gauge [2, 5], with the exception of the work described in [15], where a variety of vector-tensor dualities is applied in the presence of Stueckelberg-type vectors.

The universal formulation of the five-dimensional gauged supergravity shows a strong similarity with the formulation of the three-dimensional gauged supergravities [16]. In three dimensions, the relevant duality relates scalar and vector fields and the ungauged theory is formulated entirely in terms of scalar fields. The general gauged theory on the other hand combines the complete scalar sector and the dual vector fields. The latter satisfy first-order field equations and do not carry additional degrees of freedom. The gauged Lagrangian is manifestly $E_8(8)$ covariant and the embedding tensor is a symmetric matrix in the $1 + 3875$ representation that projects out the vector fields that actually participate in the gauging. In close analogy to the five-dimensional case, it describes the coupling of vector fields to symmetry generators and simultaneously serves as a metric for the first-order kinetic term of the vector fields, which here is a standard Chern-Simons term.
In all spacetime dimensions the embedding tensor is subject to a linear representation constraint, required by supersymmetry, and a quadratic constraint to ensure the closure of the gauge algebra. As far as we know, there are no other conditions to ensure the consistency, irrespective of the spacetime dimension. In a forthcoming paper [17] we will analyze the four-dimensional gaugings and present a similar result. In this case there are no vector-tensor dualities, but one has to deal with electric/magnetic duality. Although the details are quite different and the group-theoretical analysis proceeds along different lines, the final result is qualitatively the same and one obtains a uniform Lagrangian with the possible gaugings encoded in an embedding tensor transforming in the $9_{12}$ representation of $E_{7(7)}$.

We expect this pattern to persist in higher spacetime dimensions as well. For higher dimensions, one has, however, to cope with a larger variety of tensor fields. While the duality group $E_{11-D(11-D)}$ becomes more simple, so that the group theory analysis becomes more straightforward, the structure of the field representation becomes more complicated. In this respect the seven-dimensional maximal supergravity theories are an interesting testing ground. Here the relevant duality relates two- and three-form tensors and the ungauged theory is formulated entirely in terms of the two-form fields [18]. In analogy to the three-dimensional scenario and the five-dimensional scenario presented here, one thus expects a universal Lagrangian for the general seven-dimensional gauged maximal supergravities that combines the two-form fields with their dual three-form tensors. Both these tensors should be subject to tensor gauge transformations to ensure the correct number of degrees of freedom. The embedding tensor in seven dimensions contains the $15$ representation of $E_{4(4)} = SL(5)$ and may act as a (symmetric) metric for a first-order kinetic term of the 3-rank tensor fields. The latter transform in the $\overline{5}$ representation. This particular embedding tensor leads to all the CSO$(p, q, r)$ gaugings with $p + q + r = 5$. Gauge-fixing the rank-3 tensor gauge invariance will reproduce the known form of the gauged theory which no longer admits a smooth limit $g \to 0$ to the ungauged theory.

However, from the existence of certain Scherk-Schwarz reductions from eight-dimensional supergravity, one deduces that the embedding tensor should in general belong to the $15 + 40$ representation, so that the assignment originally proposed in [6] will be too restrictive. This extension of the embedding tensor induces a coupling between the vector fields and 2-and 3-rank tensor fields, based on a nontrivial extension of the tensor-vector gauge invariances discussed in this paper. It should be possible to incorporate these gaugings in the context of a universal Lagrangian of the type discussed in this paper. We will report on this theory and related issues elsewhere [19].

Finally, one may wonder what the physical significance could be of the extra tensor fields that one needs for incorporating certain gaugings in a U-duality covariant
way. From an M-theory perspective the supergravity fields couple to U-duality representations of BPS states and this coupling may induce the gauging. Obviously, such couplings could involve certain supergravity fields which will not necessarily describe dynamical degrees of freedom and which could be dropped in the limit of vanishing gauge coupling constant. All of this is reminiscent of the arguments leading to BPS-extended supergravity, which were presented some time ago [20, 21]. We expect that the universal Lagrangian constructed here may well have a role to play in this context.

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