A TIGHTER BOUND FOR THE NUMBER OF WORDS
OF MINIMUM LENGTH IN AN AUTOMORPHIC ORBIT

Donghi Lee

Abstract. Let $u$ be a cyclic word in a free group $F_n$ of finite rank $n$ that has the minimum length over all cyclic words in its automorphic orbit, and let $N(u)$ be the cardinality of the set \{ $v : |v| = |u|$ and $v = \phi(u)$ for some $\phi \in \text{Aut} F_n$ \}. In this paper, we prove that $N(u)$ is bounded by a polynomial function of degree $2n - 3$ in $|u|$ under the hypothesis that if two letters $x, y$ with $x \neq y \pm 1$ occur in $u$, then the total number of $x \pm 1$ occurring in $u$ is not equal to the total number of $y \pm 1$ occurring in $u$. We also prove that $2n - 3$ is the sharp bound for the degree of polynomials bounding $N(u)$. As a special case, we deal with $N(u)$ in $F_2$ under the same hypothesis.

1. Introduction

Let $F_n$ be the free group of a finite rank $n$ on the set \{ $x_1, x_2, \ldots, x_n$ \}. We denote by $\Sigma$ the set of letters of $F_n$, that is, $\Sigma = \{ x_1, x_2, \ldots, x_n \}^{\pm 1}$. As in [1, 6], we define a cyclic word to be a cyclically ordered set of letters with no pair of inverses adjacent. The length $|w|$ of a cyclic word $w$ is the number of elements in the cyclically ordered set. For a cyclic word $w$ in $F_n$, we denote the automorphic orbit \{ $\psi(w) : \psi \in \text{Aut} F_n$ \} by $\text{Orb}_{\text{Aut} F_n}(w)$.

The purpose of this paper is to present a partial solution of the following conjecture proposed by Myasnikov–Shpilrain [7]:

Conjecture. Let $u$ be a cyclic word in $F_n$ which has the minimum length over all cyclic words in its automorphic orbit $\text{Orb}_{\text{Aut} F_n}(u)$, and let $N(u)$ be the cardinality of the set \{ $v \in \text{Orb}_{\text{Aut} F_n}(u) : |v| = |u|$ \}. Then $N(u)$ is bounded by a polynomial function of degree $2n - 3$ in $|u|$.

This conjecture was motivated by the complexity of Whitehead’s algorithm which decides whether, for given two elements in $F_n$, there is an automorphism of $F_n$ that takes one element to the other. Indeed, proving that $N(u)$ is bounded by a polynomial function in $|u|$ would yield that Whitehead’s...
algorithm terminates in polynomial time with respect to the maximum length of the two words in question (see [7, Proposition 3.1]).

Proposing this conjecture, Myasnikov–Shpilrain [7] proved that $N(u)$ is bounded by a polynomial in $|u|$ in $F_2$. Later, Khan [3] improved their result by showing that $N(u)$ has the sharp bound of $8|u| - 40$ for $|u| \geq 9$ in $F_2$, by which the conjecture was settled in the affirmative for $F_2$. For a free group of bigger rank, Kapovich–Schupp–Shpilrain [2] showed that $N(u)$ is bounded by a constant depending only on $n$ for $u$ contained in an exponentially generic subset of $F_n$, and the author [4] recently proved that $N(u)$ is bounded by a polynomial function of degree $n(5n - 7)/2$ in $|u|$ under the following

**Hypothesis 1.1.** (i) A cyclic word $u$ has the minimum length over all cyclic words in its automorphic orbit $\text{Orb}_{\text{Aut} F_n}(u)$.

(ii) If two letters $x_i$ (or $x_i^{-1}$) and $x_j$ (or $x_j^{-1}$) with $i < j$ occur in $u$, then the total number of $x_i^{\pm 1}$ occurring in $u$ is strictly less than the total number of $x_j^{\pm 1}$ occurring in $u$.

In the present paper, we prove under the same hypothesis that $N(u)$ is bounded by a polynomial function of degree $2n - 3$ in $|u|$, and that $2n - 3$ is the sharp bound for the degree of polynomials bounding $N(u)$:

**Theorem 1.2.** Let $u$ be a cyclic word in $F_n$ that satisfies Hypothesis 1.1. Then $N(u)$ is bounded by a polynomial function of degree $2n - 3$ in $|u|$.

**Theorem 1.3.** Let $n \geq 2$ be arbitrary. Then there exist a polynomial $p_n(t)$ of degree exactly $2n - 3$ in $t$ and a sequence $(u_l)$ of cyclic words in $F_n$ satisfying Hypothesis 1.1 such that $|u_l| \to \infty$ as $l \to \infty$ and such that $N(u_l) \geq p_n(|u_l|)$. Thus $2n - 3$ is a sharp bound for the degree of a polynomial in $|u|$ bounding $N(u)$ from above, provided $u$ is a cyclic word in $F_n$ that satisfies Hypothesis 1.1.

As a special case, we deal with $N(u)$ in $F_2$:

**Theorem 1.4.** Let $u$ be a cyclic word in $F_2$ that satisfies Hypothesis 1.1. Then $N(u) \leq 8|u| - 40$. 
Moreover there exists a sequence \((u_l)\) of cyclic words in \(F_2\) satisfying Hypothesis 1.1 such that
\[ |u_l| \geq 9, \quad |u_l| \to \infty \text{ as } l \to \infty \quad \text{and such that } \quad N(u_l) = 8|u_l| - 40. \]
Thus \(N(u)\) has the sharp bound of \(8|u| - 40\) for \(|u| \geq 9\).

The same technique as used in [4] is applied to the proofs of these theorems. The proofs will appear in Sections 3–5. In Section 2, we will establish a couple of technical lemmas which play an important role in the proof of Theorem 1.2.

Now we would like to recall several definitions. As in [4], a Whitehead automorphism \(\sigma\) of \(F_n\) is defined to be an automorphism of one of the following two types (cf. [5, 8]):

(W1) \(\sigma\) permutes elements in \(\Sigma\).

(W2) \(\sigma\) is defined by a set \(A \subset \Sigma\) and a letter \(a \in \Sigma\) with both \(a, a^{-1} \notin A\) in such a way that if \(x \in \Sigma\) then (a) \(\sigma(x) = xa\) provided \(x \in A\) and \(x^{-1} \notin A\); (b) \(\sigma(x) = a^{-1}xa\) provided both \(x, x^{-1} \in A\); (c) \(\sigma(x) = x\) provided both \(x, x^{-1} \notin A\).

If \(\sigma\) is of type (W2), we write \(\sigma = (A, a)\). By \((\bar{A}, a^{-1})\), we mean a Whitehead automorphism \((\Sigma - A - a \pm 1, a^{-1})\). It is then easy to see that \((A, a)(w) = (\bar{A}, a^{-1})(w)\) for any cyclic word \(w\) in \(F_n\).

We also recall the definition of the degree of a Whitehead automorphism of the second type (see [4]):

**Definition 1.5.** Let \(\sigma = (A, a)\) be a Whitehead automorphism of \(F_n\) of the second type. Put \(A' = \{i : \text{either } x_i \in A \text{ or } x_i^{-1} \in A, \text{ but not both}\}\). Then the degree of \(\sigma\) is defined to be \(\max A'\). If \(A' = \emptyset\), then the degree of \(\sigma\) is defined to be zero.

Let \(w\) be a fixed cyclic word in \(F_n\) that satisfies Hypothesis 1.1 (i). For two letters \(x, y \in \Sigma\), we say that \(x\) depends on \(y\) with respect to \(w\) if, for every Whitehead automorphism \((A, a)\) of \(F_n\) such that
\[ a \notin \{x^\pm 1, y^\pm 1\}, \quad \{y^\pm 1\} \cap A \neq \emptyset, \quad \text{and } \exists v \in \text{Orb}_{\text{Aut}\, F_n}(w) : |(A, a)(v)| = |v| = |w|, \]
we have \(\{x^\pm 1\} \subseteq A\). Then, as shown in [4], if \(x\) depends on \(y\) with respect to \(w\), then \(y\) depends
on $x$ with respect to $w$.

We then construct the dependence graph $\Gamma_w$ of $w$ as follows: Take the vertex set as $\Sigma$, and connect two distinct vertices $x, y \in \Sigma$ by a non-oriented edge if either $y = x^{-1}$ or $y$ depends on $x$ with respect to $w$. Let $C_i$ be the connected component of $\Gamma_w$ containing $x_i$. Clearly there exists a unique factorization

$$w = v_1 v_2 \cdots v_t \quad \text{(without cancellation)},$$

where each $v_i$ is a non-empty (non-cyclic) word consisting of letters in $C_j$, with $C_j \neq C_{j+t} \mod t$. The subword $v_i$ is called a $C_j$-syllable of $w$. By the $C_k$-syllable length of $w$ denoted by $|w|_{C_k}$, we mean the total number of $C_k$-syllables of $w$. We also define $|w|_s$ as $|w|_s = \sum_{k=1}^n |w|_{C_k}$.

**Example 1.6.** Consider the cyclic word $u = x_1^2x_2^3x_3^4x_4^5$ in $F_4$. Letting $v = (\{x_2^{\pm 1}\}, x_1)(u) = x_1x_2^2x_1x_3^4x_4^5$, $v$ is an automorphic image of $u$ with $|v| = |u|$ (hence $\Gamma_u = \Gamma_v$). This implies that both $x_3^{\pm 1}$ and $x_4^{\pm 1}$ do not depend on $x_2^{\pm 1}$. Also putting $v' = (\{x_2^{\pm 1}\}, x_3^{-1})(u)$, we have $|v'| = |u|$, so that $x_1^{\pm 1}$ does not depend on $x_2^{\pm 1}$. Hence the connected component $C_2$ of $\Gamma_u$ containing $x_2$ consists of only $x_2^{\pm 1}$. This way we can show that the dependence graph $\Gamma_u = \Gamma_v$ has four distinct connected components, each $C_i$ of which contains only $x_i^{\pm 1}$. Thus $|u|_{C_i} = 1$ for each $1 \leq i \leq 4$ and so $|u|_s = 4$, whereas $|v|_{C_i} = 2$, $|v|_{C_j} = 1$ for each $2 \leq j \leq 4$ and so $|v|_s = 5$.

**Example 1.7.** Consider the cyclic word $u = x_1^2x_2^3x_3^2x_4^4x_5^{-1}x_4x_3x_4^3$ in $F_4$, of which the dependence graph $\Gamma_u$ has three distinct connected components $C_1, C_2, C_3 = C_4$. Putting $v = (\{x_2^{\pm 1}\}, x_3^{-1})(u) = x_1^2x_3^2x_2^3x_4x_3^{-1}x_4x_3x_4^3$, $v$ is an automorphic image of $u$ with $|v| = |u|$, so $\Gamma_u = \Gamma_v$. While $|u|_{C_i} = 1$ for each $1 \leq i \leq 4$ and so $|u|_s = 4$, $|v|_{C_1} = |v|_{C_2} = 1$, $|v|_{C_3} = |v|_{C_4} = 2$ and so $|v|_s = 6$.

2. Preliminary Lemmas

Throughout this section, when we say that $\sigma = (A,a)$ is a Whitehead automorphism of $F_n$ of degree $i$, the following restriction is additionally imposed:

$$a = x_j^{\pm 1} \quad \text{with } j > i.$$
For two automorphisms $\phi$ and $\psi$ of $F_n$, by writing $\phi \equiv \psi$ we mean the equality of $\phi$ and $\psi$ over all cyclic words in $F_n$, that is, $\phi(v) = \psi(v)$ for any cyclic word $v$ in $F_n$. For a cyclic word $v$ in $F_n$, we define $M_k(v)$, for $k = 0, 1, \ldots, n - 1$, to be the cardinality of the set $\Omega_k(v) = \{\phi(v) : \phi$ can be represented as a composition $\phi = \alpha_t \cdots \alpha_1$ $(t \in \mathbb{N})$ of Whitehead automorphisms $\alpha_i$ of $F_n$ of the second type such that $k = \deg \alpha_t \geq \deg \alpha_{t-1} \geq \cdots \geq \deg \alpha_1$ and $|\alpha_t \cdots \alpha_1(v)| = |v|$ for all $i = 1, \ldots, t\}.$

**Lemma 2.1.** Under the foregoing notation, $M_1(v)$ is bounded by a polynomial function of degree $n - 1$ in $|v|$.

**Proof.** Let $\ell_i$ be the number of occurrences of $x_i^{\pm 1}$ in $v$ for $i = 1, \ldots, n$. Clearly

$$M_1(v) \leq M_1(x_1^{\ell_2} x_2^{\ell_3} \cdots x_{n-1}^{\ell_n} x_n^{\ell_1 + \ell_2 - 2}).$$

So it is enough to prove that $M_1(x_1^{\ell_2} x_2^{\ell_3} \cdots x_{n-1}^{\ell_n} x_n^{\ell_1 + \ell_2 - 2})$ is bounded by a polynomial function in $|v|$ of degree $n - 1$. Noting that $|x_1^{\ell_2} x_2^{\ell_3} \cdots x_{n-1}^{\ell_n} x_n^{\ell_1 + \ell_2 - 2}|_s = n$, put

$$\Lambda = \{v' : |v'|_s = n \text{ and } v' \in \Omega_0(x_1^{\ell_2} x_2^{\ell_3} \cdots x_{n-1}^{\ell_n} x_n^{\ell_1 + \ell_2 - 2})\}.$$

Obviously the cardinality of the set $\Lambda$ is $(n - 1)!$.

Let $w \in \Omega_1(x_1^{\ell_2} x_2^{\ell_3} \cdots x_{n-1}^{\ell_n} x_n^{\ell_1 + \ell_2 - 2})$. Then for an appropriate $v' \in \Lambda$, there exist Whitehead automorphisms $\sigma_i$ of degree 0 and $\tau_j$ of degree 1 such that

$$w = \tau_q \cdots \tau_1 \sigma_p \cdots \sigma_1 (v'),$$

where $|\sigma_i \cdots \sigma_1 (v')| = |v'|$ and $|\sigma_i \cdots \sigma_1 (v')|_s \geq |\sigma_{i-1} \cdots \sigma_1 (v')|_s$ for all $1 \leq i \leq p$, and $|\tau_j \cdots \tau_1 \sigma_p \cdots \sigma_1 (v')| = |v'|$ for all $1 \leq j \leq q$. Here, the same reasoning as in [4, Lemma 4.1] shows that $\sigma_i \sigma_{i'} \equiv \sigma_{i'} \sigma_i$ for all $1 \leq i, i' \leq p$. Furthermore, the chain $\tau_q \cdots \tau_1$ in (2.1) can be chosen so that, for $\tau_{ij} = (A_{ij}, a_{ij})$,

$$\tau_q \cdots \tau_1 = (\tau_{rq}, \cdots \tau_{r_1}) \cdots (\tau_{2q_2}, \cdots \tau_{2_2}) (\tau_{1q_1}, \cdots \tau_{11}),$$
where $A_{ij} = A_{ij'}$ for all $1 \leq j, j' \leq q$, and $x_1 \in A_{i1} \subset A_{i+11}$.

We may assume without loss of generality that the index $r$ in (2.2) is minimum over all chains satisfying (2.1) and (2.2). Clearly in (2.1)–(2.2) the element $v'$ in $\Lambda$, the Whitehead automorphisms $\sigma_1, \ldots, \sigma_p$, and the index $r$ are determined by $w$; so we put

$$v'_w = v', \quad \psi_w = \sigma_p \cdots \sigma_1, \quad \text{and} \quad r_w = r.$$ 

It is easy to see that $r_w$ is at most $n - 1$.

For $s = 1, \ldots, n - 1$, put

$$L_s = \text{the cardinality of the set } \{\psi_w(v'_w) : w \in \Omega_1(x_1^2 x_2^2 \cdots x_{n-1}^2 x_n^2 + x_{n+1}^2) \text{ with } r_w = s\}.$$ 

Then in view of (2.1)–(2.2), we have

$$M_1(x_1^2 x_2^2 \cdots x_{n-1}^2 x_n^2 + x_{n+1}^2) \leq 2^{(n-1)}|v|L_1 + 2^{2(n-1)}|v|^2 L_2 + \cdots + 2^{(n-1)^2}|v|^{n-1} L_{n-1},$$

since the number of possible $A_{ij}$'s and the indices $q_i$'s in (2.2) are less than or equal to $2^{n-1}$ and $|v|$, respectively. Hence it is enough to prove that each $L_s$ is bounded by a polynomial function in $|v|$ of degree $n - s - 1$. Due to the result of [4, Lemma 4.1], there is nothing to prove for $s = 1$.

So let $s \geq 2$ and put $E_i = A_{i1} - A_{i+11}$ for $i = 2, \ldots, s$. This can possibly happen only when $\psi_w = \sigma_p \cdots \sigma_1$ in (2.1) can be re-arranged so that, for $\sigma_j = (B_j, b_j)$,

$$(2.3) \quad \psi_w = (\sigma_{t_{s+1}} \cdots \sigma_{t_s+1}) \cdots (\sigma_{t_2} \cdots \sigma_2) \sigma_1,$$

where $b_1 \in \{x_1^{\pm 1}\}$, $b_j^{\pm 1} \in E_i$ and either $B_j \subseteq E_i$ or $B_j \cap E_i = \emptyset$ provided $t_{i-1} < j \leq t_i$ ($t_1 = 1$), and $b_j^{\pm 1} \notin (\bigcup_{i=2}^{s} E_i + x_1^{\pm 1})$ and either $B_j \subseteq (\bigcup_{i=2}^{s} E_i + x_1^{\pm 1})$ or $B_j \cap (\bigcup_{i=2}^{s} E_i + x_1^{\pm 1}) = \emptyset$ provided $t_s < j \leq t_{s+1}$.

Now, for $i = 2, \ldots, s$, let

$$h_i$$

be the half of the cardinality of the set $E_i$.  

Put \( h = \sum_{i=2}^{s} h_i \). It then follows from the result of [4, Lemma 4.1] that the number of cyclic words obtained by \( \sigma_{t_{j+1}} \cdots \sigma_{t_j+1} \) applied to \((\sigma_{t_j} \cdots \sigma_{t_{j-1}+1}) \cdots (\sigma_{t_2} \cdots \sigma_2)\sigma_1(v'_w)\) is bounded by \( |v|^{h_{j+1}-1} \) provided \( j = 1, \ldots, s-1 \) and by \( |v|^{n-(h+1)-1} \) provided \( j = s \). Moreover the number of cyclic words derived from \( \sigma_1 \) applied to \( v'_w \) is bounded by \( n - 2 \). Therefore we have from (2.3) that

\[
L_s \leq (n - 1)! (n - 2)|v|^{h_2-1} \cdots |v|^{h_s-1}|v|^{n-h-2} = (n - 1)! (n - 2)|v|^{n-s-1},
\]

which is a polynomial function in \(|v|\) of degree \( n - s - 1 \), as required. \( \square \)

**Remark.** The proof of Lemma 2.1 can be applied without further change if we replace consideration of a single cyclic word \( v \), the length \(|v|\) of \( v \), and the total number of occurrences of \( x_j^{\pm 1} \) in \( v \) by consideration of a finite sequence \((v_1, \ldots, v_m)\) of cyclic words, the sum \( \sum_{i=1}^{m} |v_i| \) of the lengths of \( v_1, \ldots, v_m \), and the total number of occurrences of \( x_j^{\pm 1} \) in \((v_1, \ldots, v_m)\), respectively.

**Lemma 2.2.** Under the foregoing notation, for each \( k = 2, \ldots, n - 1 \), \( M_k(v) \) is bounded by a polynomial function of degree \( n + k - 2 \) in \(|v|\).

**Proof.** Let \( \ell_i \) be the number of occurrences of \( x_i^{\pm 1} \) in \( v \) for \( i = 1, \ldots, n \). Since

\[
M_k(v) \leq M_k(x_1^2 \cdots x_k^{\ell_{k+1}} \cdots x_n^{\ell_{n-1}} x_n^{\ell_1+\cdots+\ell_k-2k}),
\]

it suffices to show that \( M_k(x_1^2 \cdots x_k^{\ell_{k+1}} \cdots x_n^{\ell_{n-1}} x_n^{\ell_1+\cdots+\ell_k-2k}) \) is bounded by a polynomial function in \(|v|\) of degree \( n + k - 2 \). As in the proof of Lemma 2.1, put \( \Lambda = \{v' : |v'|_s = n \) and \( v' \in \Omega_0(x_1^2 \cdots x_k^{\ell_{k+1}} \cdots x_n^{\ell_{n-1}} x_n^{\ell_1+\cdots+\ell_k-2k})\}).

Let \( w \in \Omega_k(x_1^2 \cdots x_k^{\ell_{k+1}} \cdots x_n^{\ell_{n-1}} x_n^{\ell_1+\cdots+\ell_k-2k}) \). Then for an appropriate \( v' \in \Lambda \), there exist Whitehead automorphisms \( \gamma_i \) of \( F_n \) such that

\[
(2.4) \quad w = \gamma_q \cdots \gamma_{p+1} \gamma_p \cdots \gamma_1(v'),
\]

where the length of \( v' \) is constant throughout the chain on the right-hand side, \( \deg \gamma_i = 0 \) provided \( 1 \leq i \leq p \), \( \deg \gamma_i > 0 \) provided \( p < i \leq q \), and \( |\gamma_j \cdots \gamma_1(v')|_s \geq |\gamma_{j-1} \cdots \gamma_1(v')|_s \) for all \( 1 \leq j \leq p \).
Here, since $\gamma_i \gamma_i' \equiv \gamma_i' \gamma_i$ for all $1 \leq i, i' \leq p$ by the same reasoning as in [4, Lemma 4.1], we may assume that either none of $\gamma_i$ for $1 \leq i \leq p$ has multiplier $x_1$ or $x_1^{-1}$ or only $\gamma_1$ has multiplier $x_1$ or $x_1^{-1}$. So (2.4) can be re-written as

$$w = \gamma_q \cdots \gamma_{p+1} \gamma_p \cdots \gamma_1 \gamma_0(v'),$$

where $\gamma_0$ is either the identity or a Whitehead automorphism of $F_n$ of degree 0 with multiplier $x_1$ or $x_1^{-1}$, and none of $\gamma_j$ for $1 \leq j \leq q$ has multiplier $x_1$ or $x_1^{-1}$.

Write

(2.5) $\gamma_0(v') = x_1 u_1 x_1 u_2$ without cancellation.

(Note that $u_1$ and $u_2$ are non-cyclic subwords in $\{x_2, \ldots, x_n\}^{\pm 1}$.) Let $F_{n+1}$ be the free group on the set $\{x_1, \ldots, x_{n+1}\}$. From (2.5) we construct a pair $(v_1, v_2)$ of cyclic words $v_1, v_2$ in $F_{n+1}$ with $|v_1| + |v_2| = 2|v|$ as follows:

$$v_1 = x_1 u_1 x_{n+1} u_1^{-1} \quad \text{and} \quad v_2 = x_1 u_2 x_{n+1} u_2^{-1}.$$

For each $\gamma_j = (D_j, d_j)$ for $1 \leq j \leq q$, define a Whitehead automorphism $\varepsilon_j$ of $F_{n+1}$ as follows:

- if $x_1^{\pm 1} \in D_j$, then $\varepsilon_j = (D_j + x_1^{\pm 1}, d_j)$;
- if only $x_1 \in D_j$, then $\varepsilon_j = (D_j + x_1^{-1}, d_j)$;
- if only $x_1^{-1} \in D_j$, then $\varepsilon_j = (D_j - x_1^{-1} + x_1^{\pm 1}, d_j)$;
- if $x_1^{\pm 1} \notin D_j$, then $\varepsilon_j = (D_j, d_j)$.

Then arguing as in the proof of [4, Lemma 4.2], we have $|\varepsilon_j \cdots \varepsilon_1(v_1)| + |\varepsilon_j \cdots \varepsilon_1(v_2)| = 2|v|$ for all $1 \leq j \leq q$. Moreover, by the construction of $\varepsilon_j$, $\varepsilon_j$ is a Whitehead automorphism of $F_{n+1}$ of degree at most $k$, and the defining set of $\varepsilon_j$ contains either both of $x_1^{\pm 1}$ or none of $x_1^{\pm 1}$. This yields the same situation as for a chain of Whitehead automorphisms of $F_{n+1}$ of maximum
degree \( k - 1 \). Hence by the induction hypothesis together with the Remark after Lemma 2.1, 
\[
M_k(x_1^2 \cdots x_k^{\ell_k+1} \cdots x_{n-1}^{\ell_{n-1}+\ell_1+\cdots+\ell_k-2})
\]
is bounded by \((n - 2)\) times a polynomial function in \( 2|v| \) of degree \((n + 1) + (k - 1) - 2 = n + k - 2 \), as required.

\[\square\]

3. Proof of Theorem 1.2

Without loss of generality we may assume that \( u \) satisfies further

(i) The \( C_n \)-syllable length \( |u|_{C_n} \) of \( u \) is minimum over all cyclic words in the set \( \{ v \in \text{Orb}_{\text{Aut} F_n}(u) : |v| = |u| \} \).

(ii) If the index \( j \) \((1 \leq j \leq n - 1)\) is such that \( C_j \neq C_k \) for all \( k > j \), then the \( C_j \)-syllable length \( |u|_{C_j} \) of \( u \) is minimum over all cyclic words in the set \( \{ v \in \text{Orb}_{\text{Aut} F_n}(u) : |v| = |u| \text{ and } |v|_{C_k} = |u|_{C_k} \text{ for all } k > j \} \).

(Namely, we may assume that \( u \) satisfies further the conditions in [4, Hypothesis 1.3].) Let \( u' \in \text{Orb}_{\text{Aut} F_n}(u) \) be such that \( |u'| = |u| \). Due to the result of [4, Theorem 1.4], there exist Whitehead automorphisms \( \pi \) of the first type and \( \tau_1, \ldots, \tau_s \) of the second type such that

\[u' = \pi \tau_s \cdots \tau_1(u),\]

where \( n - 1 \geq \deg \tau_s \geq \deg \tau_{s-1} \geq \cdots \geq \deg \tau_1 \), and \( |\tau_i \cdots \tau_1(u)| = |u| \) for all \( i = 1, \ldots, s \). This implies that

\[
N(u) \leq C(M_0(u) + M_1(u) + \cdots + M_{n-1}(u)),
\]

where \( C \) is the number of Whitehead automorphisms of \( F_n \) of the first type (which depends only on \( n \)), and \( M_k(u) \) is as defined in Section 2. The result of [4, Lemma 4.1] shows that \( M_0(u) \) is bounded by a polynomial function in \( |u| \) of degree \( n - 2 \). Also by Lemmas 2.1 and 2.2, \( M_k(u) \) for each \( k = 1, \ldots, n - 1 \) is bounded by a polynomial function in \( |u| \) of degree \( n + k - 2 \). Then the required result follows from (3.1).

\[\square\]
4. Proof of Theorem 1.3

In [7], Myasnikov–Shpilrain pointed out that experimental data provided by C. Sims show that the maximum value of \( N(u) \) in \( F_3 \) is 48\(|u|^3 - 480|u|^2 + 1140|u| - 672 \) if \(|u| \geq 11 \) and this maximum value is attained at \( u = x_1^2x_2^2x_3x_2^{-1}x_3x_2^\ell \) with \( \ell \geq 3 \). Inspired by this observation, we let

\[
u = x_1^2x_2(x_2x_nx_2^{-1}x_n)x_2x_3(x_3x_nx_3^{-1}x_n)^2x_3\cdots x_{n-1}(x_{n-1}x_nx_{n-1}^{-1}x_n)^{n-2}x_{n-1}x_n^\ell
\]

with \( \ell \gg 1 \) in \( F_n \). Note that \( u \) satisfies Hypothesis 1.1. For this \( u \), we will prove that \( N(u) \) cannot be bounded by a polynomial function in \(|u|\) of degree less than \( 2n - 3 \). For each \( i = 2, \ldots, n-1 \) and \( j = 1, \ldots, n-1 \), let

\[
\sigma_i = (\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}, x_n^{-1}) \quad \text{and} \quad \tau_j = (\{x_j, x_j^{\pm 1}, \ldots, x_{n-1}^{\pm 1}\}, x_n^{-1});
\]

then \( \sigma_i \) and \( \tau_j \) are Whitehead automorphisms of \( F_n \) of degree 0 and degree \( j \), respectively. Then the total number of cyclic words derived from automorphisms of \( F_n \) of the form \( \tau_{n-1}^{m_{n-1}} \cdots \tau_1^{m_1}\sigma_{n-1}^{k_{n-1}} \cdots \sigma_2^{k_2} \),

where \( k_i, m_j \leq \frac{\ell}{2n-3} \), applied to \( u \) is \((\frac{\ell}{2n-3})^{2n-3}\). Hence \( N(u) \) is at least \((\frac{\ell}{2n-3})^{2n-3}\), which completes the proof.

\[\square\]

5. Proof of Theorem 1.4

Let us assume that \( u \) satisfies further

(i) The \( C_2 \)-syllable length \(|u|_{C_2} \) of \( u \) is minimum over all cyclic words in the set \( \{v \in \text{Orb}_{\text{Aut}}F_n(u) : |v| = |u|\} \).

(ii) If \( C_1 \neq C_2 \), then the \( C_1 \)-syllable length \(|u|_{C_1} \) of \( u \) is minimum over all cyclic words in the set \( \{v \in \text{Orb}_{\text{Aut}}F_n(u) : |v| = |u| \text{ and } |v|_{C_2} = |u|_{C_2}\} \).

(Namely, assume that \( u \) satisfies further the conditions in [4, Hypothesis 1.3].) Note that \( M_0(u) = 1 \) in \( F_2 \), where \( M_0(u) \) is as defined in Section 2. Also every Whitehead automorphism of \( F_2 \) of degree 1 is equal to either \( \{x_1\}, x_2 \) or \( \{x_1, x_2^{-1}\} \) over all cyclic words in \( F_2 \). Hence, in view of
[4, Theorem 1.4], \( N(u) \) is the same as the cardinality of the set \( \{ v : v = \pi \tau^k(u) (k \geq 0) \} \), where \( \pi \) is a permutation on \( \Sigma \) and \( \tau \) is either \( (\{x_1\}, x_2) \) or \( (\{x_1\}, x_2^{-1}) \) such that \( |\tau^i(u)| = |u| \) for all \( i = 1, \ldots, k \). Let

\[
\Lambda(u) = \{ v : v = \tau^k(u) (k \geq 0) \}, \text{ where } \tau \text{ is as above}. \]

Let \( m \) be the number of occurrences of \( x_1^{\pm 1} \) in \( u \). First consider the maximum value \( N(u) \) over all \( u \) with \( m = 2 \). If \( m = 2 \), then \( u \) is of the form either \( x_1 x_2 x_1^{-1} x_2^{-1} \) or \( x_1^2 x_2^2 \). Then the cardinality of \( \Lambda(x_1 x_2 x_1^{-1} x_2^{-1}) \) equals 1 and that of \( \Lambda(x_1^2 x_2^2) \) equals \( |u| - 1 \). Hence \( N(u) \) has the maximum value at \( u = x_1^2 x_2^2 \). For \( u = x_1^2 x_2^2 \) with \( \ell \geq 3 \), \( N(u) = 4(|u| - 1) \), since there are 8 permutations on \( \Sigma \) and \( \tau^j(x_1^2 x_2^2) = \pi \tau^{j-\ell}(x_1^2 x_2^2) \) for \( j \geq \ell/2 \), where \( \tau = (\{x_1\}, x_2^{-1}) \) and \( \pi \) is the permutation that fixes \( x_1 \) and maps \( x_2 \) to \( x_2^{-1} \).

Next consider the maximum value of \( N(u) \) over all \( u \) with \( m = 4 \). (Here note that if \( m \) is odd, then any Whitehead automorphism of degree 1 cannot be applied to \( u \) without increasing \( |u| \); hence the cardinality of \( \Lambda(u) \) equals 1.) It is not hard to see that \( \Lambda(u) \) has the maximum cardinality \( |u| - 5 \) at \( u = x_1^2 x_2 x_1^{-1} x_2 x_1 x_2^\ell \). For \( u = x_1^2 x_2 x_1^{-1} x_2 x_1 x_2^\ell \) with \( \ell \geq 3 \), \( N(u) = 8(|u| - 5) \), since 8 permutations on \( \Sigma \) applied to the elements of \( \Lambda(x_1^2 x_2 x_1^{-1} x_2 x_1 x_2^\ell) \) induce all different cyclic words. Obviously this is the maximum value of \( N(u) \) over all \( u \) with \( m = 4 \).

Finally note that the cardinality of \( \Lambda(u) \) cannot be greater than nor equal to \( |u| - 5 \) for any \( u \) with \( m > 4 \). This means that \( N(u) < 8(|u| - 5) \) for every \( u \) with \( m > 4 \). Therefore, the maximum value of \( N(u) \) over all \( u \) is \( 8(|u| - 5) \), which is attained at \( u = x_1^2 x_2 x_1^{-1} x_2 x_1 x_2^\ell \) with \( \ell \geq 3 \). \( \square \)

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*Department of Mathematics, Pusan National University, Jangjeon-Dong, Geumjung-Gu, Pusan 609-735, Korea*

*E-mail address: donghi@pusan.ac.kr*