An analysis on the stability of a state dependent delay differential equation

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Abstract: In this paper, we present an analysis for the stability of a differential equation with state-dependent delay. We establish existence and uniqueness of solutions of differential equation with delay term \( \tau(u(t)) = \frac{u + bu(t)}{c + du(t)} \). Moreover, we put the some restrictions for the positivity of delay term \( \tau(u(t)) \). Based on the boundedness of delay term, we obtain stability criterion in terms of the parameters of the equation.

Keywords: Asymptotic stability, state depended delay, delay differential equation

1 Introduction

Delay differential equations (DDE) have been used in many fields for a long time. However, state-dependent delay differential equations (SDDE) are used to make more realistic modelling in the systems whose delay varies according to the internal effects of the system. For example, the length of time to maturity is taken as constant delay in a simple population dynamics model, see in [1]. In [2], it was observed that the length of time to maturity of Antarctic whales and seals alter according to the state of the population and it was analyzed by using a mathematical model with SDDE in [3]. In addition, mathematical models with SDDE appear in many fields such as physics, control theory, neural network, medicine, biology etc., see Section 2 in [4] as a review.

Researchers have investigated SDDE for the last 50 years. Driver [5, 6] and Driver and Norris [7] developed a fundamental theory and proved local existence and uniqueness theorem for SDDE having Lipschitz continuous initial functions. Winston [8] showed that SDDE has a unique solution under some conditions in addition to continuous initial function. There are some of the earliest studies on SDDE in [9–11]. Moreover, many researches on stability, bifurcations and existence of solutions of SDDE have been done so far, for example, [3, 4, 12–31]. Especially, [32] can be seen as a detailed review on DDE and SDDE and related studies.

In this paper, we consider the following type of SDDE

\[
u'(t) = -A_0 u(t) - A_1 u(t - \tau(u(t))) \tag{1}\]

where \( A_0, A_1 \in \mathbb{R} \) and \( \tau(u(t)) > 0 \) for all \( t \in \mathbb{R}^+ \). To analyze stability of solution of equation (1), we use the following characteristic equation

\[
g(\lambda) = \lambda + A_0 + A_1 e^{-\lambda h} = 0 \tag{2}\]

where \( h \) is an independent real valued parameter which is in the range of \( \tau(u(t)) \). By this way, we analyze the stability of equation (1) by using the stability analysis of certain linear delay differential equations with constant delay which has the characteristic equation (2).

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In the general case, the characteristic roots $\lambda_j, j = 1, 2, \cdots$, of equation (1) are obtained by solving the characteristic equation (2) where $\lambda_j$ is a complex number. If the characteristic roots have negative real parts, i.e., \( \text{Re}(\lambda_j) < 0 \) for all $j = 1, 2, \cdots$, then the solution of (1) is asymptotically stable and if at least one of the characteristic roots have positive real parts, i.e., \( \text{Re}(\lambda_j) > 0 \) for some $j = 1, 2, \cdots$, then the solution of (1) is unstable.

We attempt to determine the stability and instability regions of the system in parameter space \((A_0, A_1)\) by using D-partition method. The method is originated from paper [33]. It is well explained in [34–38] and analysis are conducted. Let’s consider the characteristic equation $g(\lambda, A_0, A_1)$ in two parameters for equation (1). D-partition method is based on fact that the roots of the characteristic equation are continuous functions of the parameters $A_0$ and $A_1$. When varying the parameters, $\lambda_j$ change continuously in complex plane and at the point where the stability changes, one $\lambda_j$ crosses the imaginary axis. In this method parameter space is divided into regions with the hypersurfaces. These hypersurfaces are called the D-curves. The points of the D-curves correspond to pure imaginary roots or zero root of the characteristic equation. Moreover, in each region in the parameter space determined by the D-curves, the characteristic equation has the same number of roots with positive real part. Thus, finding the number of roots with positive real parts for specific point is enough to find the number of roots with positive real parts the region including this specific point.

In order to obtain D-curves, pure imaginary number $\lambda = i\omega$ is substituted in characteristic equation $g(\lambda, A_0, A_1)$. Equating to zero the real and imaginary parts, we have
\[
U(\omega, A_0, A_1) = \text{Re}(g(i\omega, A_0, A_1)) = 0
\]
\[
V(\omega, A_0, A_1) = \text{Im}(g(i\omega, A_0, A_1)) = 0
\]
Hence, by making use of (3) and (4), parametric equations can be written as
\[
A_0 = A_0(\omega) \quad A_1 = A_1(\omega)
\]
where $\omega$ is a parameter and ranges from $-\infty$ to $\infty$. These curves and singular solutions of equations (3) and (4) constitute D-curves.

We use Rekasius transform
\[
e^{-i\omega h} = \frac{1-i\omega T}{1+i\omega T} \quad \text{for} \quad h = \frac{2}{\omega} (\arctan(\omega T) + p\pi)
\]
where $h, T \in \mathbb{R}$ and for $p \in \mathbb{Z}$ in addition to D-partition method. In 1980, Rekasius [39] proposed the transformation (5) for DDEs. Later, Thowsen [40] did exact calculations by taking the square of right hand side of (5) since the transformation (5) transform a circle to a semi-circle which leads some mistakes. However, Hertz et al. [41] did exact calculations by considering two singular cases:
(i) $e^{-i\omega h} = -1$ for $T = \pm \infty$
(ii) $e^{-i\omega h} = 1$ for $T = 0$
Olgaç and Sipahi [42, 43] studied a method using Rekasius transform for DDE with constant delay.

We establish the existence and uniqueness of solutions of equation (1) with delay term $\tau(u(t)) = \frac{a+bu(t)}{c+du(t)}$ where $a, b, c, d \in \mathbb{R}$ such that $a$ and $c$ are nonzero and at least one of $b$ or $d$ is nonzero, under certain condition in Section 2. In Section 3, by using D-partition method and transformation (5), conditions for the stability of equation (1) are presented.

### 2 Existence and uniqueness of solution

In this section, we consider the following type of SDDE
\[
u'(t) = -A_0 u(t) - A_1 (u(t) - \tau(u(t)))
\]
\[
\tau(u(t)) = \frac{a+bu(t)}{c+du(t)} > 0, \forall t \in \mathbb{R}^+
\]
where $A_0, A_1 \in \mathbb{R}^+, a, b, c, d \in \mathbb{R}$ such that $a$ and $c$ are nonzero and at least one of $b$ or $d$ is nonzero.
Let \( P(u(t)) = a + bu(t) \) and \( Q(u(t)) = c + du(t) \). \( \sigma = \frac{a}{b}, \ \mu = \frac{c}{d} \) be the roots of \( P(x) \) and \( Q(x) \) respectively. If one of the following conditions

i) \( u(t) > \sigma \) or \( u(t) < \mu \) when \( \text{sign}(b) \text{sign}(d) = 1 \) and \( \mu < \sigma \),

ii) \( u(t) > \mu \) or \( u(t) < \sigma \) when \( \text{sign}(b) \text{sign}(d) = 1 \) and \( \sigma < \mu \),

iii) \( \mu < u(t) < \sigma \) when \( \text{sign}(b) \text{sign}(d) = -1 \) and \( \mu < \sigma \),

iv) \( \sigma < u(t) < \mu \) when \( \text{sign}(b) \text{sign}(d) = -1 \) and \( \sigma < \mu \),

is satisfied then \( \tau(u(t)) \geq 0 \).

In order to guarantee the positivity of delay term \( \tau(u(t)) \), we need to put some restrictions on the range of parameter values under consideration.

**Theorem 2.1.** Let \( A_0, \ A_1 \in \mathbb{R}^+ \) and \( \tau(u(t)) \) be delay function of equation (6). The delay differential equation

\[
\begin{align*}
    u'(t) &= -A_0 u(t) - A_1 u(t - \tau(u(t))) \\
    \tau(u(t)) &= \max\{0, \tau(u(t))\} \geq 0 \text{ for all } t \in \mathbb{R}^+
\end{align*}
\]  

(7)

has a unique solution \( u(t) \in C^1([0, \infty) \rightarrow (L_0, M_0)) \) if Lipschitz history function \( u_0(t) : [-\tau, 0] \rightarrow (L_0, M_0) \) exists such that

\[
(L_0, M_0) = \begin{cases}
    (\sigma, -\frac{\sigma A_1}{A_0}); & \mu < \sigma < 0 \\
    (-\frac{\sigma A_1}{A_0}, \sigma); & 0 < \sigma < \mu \\
    (\sigma, \mu); & \sigma < 0, -\frac{\sigma A_1}{A_0} < \mu \\
    (\mu, \sigma); & \mu < 0, \mu < -\frac{\sigma A_1}{A_0}
\end{cases}
\]  

(8)

and \( \tau = \max_{u(t) \in (L_0, M_0)} (\tau(u(t))). \)

**Proof.** For the proof, the following four cases are considered.

Case 1: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( \mu < \sigma < 0 \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0, M_0) \) for all \( t < t_0 \) but \( u(t_0) = L_0 \) or \( u(t_0) = M_0 \). First assume that \( u(t_0) = L_0 \) which implies \( u'(t_0) \leq 0 \). On the other hand

\[ u(t_0 - \tau(u(t_0))) = u(t_0) = \sigma \]

and

\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) = -\sigma (A_1 + A_0) > 0 \]

which is a contradiction. In a similar way, if \( u(t_0) = M_0 \), then \( u'(t_0) \geq 0 \). On the other hand

\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) < A_0 \frac{\sigma A_1}{A_0} - A_1 \sigma = 0 \]

which is a contradiction.

Case 2: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( 0 < \sigma < \mu \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0, M_0) \) for all \( t < t_0 \) but \( u(t_0) = L_0 \) or \( u(t_0) = M_0 \). First assume that \( u(t_0) = L_0 \) which implies \( u'(t_0) \leq 0 \). On the other hand

\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) > A_0 \frac{\sigma A_1}{A_0} - A_1 \sigma = 0 \]

which is a contradiction. In a similar way, if \( u(t_0) = M_0 \), then \( u'(t_0) \geq 0 \). On the other hand

\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) = -\sigma (A_1 + A_0) < 0 \]

which is a contradiction.

Case 3: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( \sigma < 0 < \mu, -\frac{\sigma A_1}{A_0} < \mu \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0, M_0 - \varepsilon) \) for all \( t < t_0 \) but \( u(t_0) = L_0 \) or \( u(t_0) = M_0 - \varepsilon \) for \( \varepsilon > 0 \). First assume that \( u(t_0) = L_0 \) which implies \( u'(t_0) \leq 0 \). On the other hand

\[ u(t_0 - \tau(u(t_0))) = u(t_0) = \sigma \]
and
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) = -\sigma (A_1 + A_0) > 0 \]
which is a contradiction. In a similar way, if \( u(t_0) = M_0 - \varepsilon \) for \( \varepsilon > 0 \), then \( u'(t_0) \geq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) < -A_0 (\mu - \varepsilon) - A_1 \sigma < A_0 \frac{\sigma A_1}{A_0} - A_0 \varepsilon - A_1 \sigma = -A_0 \varepsilon \]
which is a contradiction since \( u(t_0) \) and \( A_0 \varepsilon \) tend to \( \mu \) and 0 respectively when \( \varepsilon \) tends to 0.

Case 4: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( \mu < 0 < \sigma, \mu < -\frac{\sigma A_1}{A_0} \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0 + \varepsilon, M_0) \) for all \( t < t_0 \) but \( u(t_0) = L_0 + \varepsilon \) for \( \varepsilon > 0 \) or \( u(t_0) = M_0 \). First assume that \( u(t_0) = L_0 + \varepsilon \) for \( \varepsilon > 0 \) which implies \( u'(t_0) \leq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) > -A_0 (\mu + \varepsilon) - A_1 \sigma > A_0 \frac{\sigma A_1}{A_0} - A_0 \varepsilon - A_1 \sigma = -A_0 \varepsilon \]
which is a contradiction since \( u(t_0) \) and \( -A_0 \varepsilon \) tend to \( \mu \) and 0 respectively when \( \varepsilon \) tends to 0. In a similar way, if \( u(t_0) = M_0 \), then \( u'(t_0) \geq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) = -\sigma (A_1 + A_0) < 0 \]
which is a contradiction. As a result, there is no such \( t_0 \in \mathbb{R}^+ \) and (8) holds.

Since
\[ f(t, u, v) = -A_0 u(t) - A_1 v(t), \alpha(t, u) = t - \frac{a+b u(t)}{c+\sigma u(t)} \]
are Lipschitz with respect to each of their argument, local existence and uniqueness of the solution \( u(t) \) follows from Driver [5].

**Theorem 2.2.** Let \( A_0, A_1 \in \mathbb{R}^+ \) such that \( A_0^2 - A_1^2 > 0 \) and \( \tau(u(t)) \) be delay function of equation (6). The delay differential equation (7) has a unique solution \( u(t) \in C^1([0, \infty) \rightarrow (L_0, M_0)) \), if Lipschitz history function \( u_0(t) : [-\tau, 0] \rightarrow (L_0, M_0) \) exists such that
\[
(L_0, M_0) = \begin{cases} 
(-\frac{u A_0}{A_1}, \mu) ; & 0 < \mu \leq \sigma \text{ or } \sigma < 0 < \mu, -\frac{\sigma A_1}{A_0} \geq \mu \\
(\mu, u A_0/A_1) ; & \sigma \leq 0 < \mu < 0 < \mu, \mu \geq -\frac{\sigma A_1}{A_0}
\end{cases}
\]
and \( \tau = \max_{u(t) \in (L_0, M_0)} (\tau(u(t))) \).

**Proof.** For the proof, following two cases are considered.

Case 1: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( 0 < \mu \leq \sigma \text{ or } \sigma < 0 < \mu, -\frac{\sigma A_1}{A_0} \geq \mu \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0 + \varepsilon, M_0) \) for all \( t < t_0 \) but \( u(t_0) = L_0 \) or \( u(t_0) = M_0 - \varepsilon \) for \( \varepsilon > 0 \). First assume that \( u(t_0) = L_0 \) which implies \( u'(t_0) \leq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) \geq \frac{\mu (A_0^2 - A_1^2)}{A_1} > 0 \]
which is a contradiction. In a similar way, if \( u(t_0) = M_0 - \varepsilon \) for \( \varepsilon > 0 \), then \( u'(t_0) \geq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) < -A_0 \mu + A_0 \varepsilon + A_1 \frac{\mu A_0}{A_1} = A_0 \varepsilon \]
which is a contradiction since \( u(t_0) \) and \( A_0 \varepsilon \) tend to \( \mu \) and 0 respectively when \( \varepsilon \) tends to 0.

Case 2: Let’s prove that \( u(t) \in (L_0, M_0) \) for all \( t > 0 \) when \( \sigma \geq 0 < \mu < 0 < \mu, \mu \geq -\frac{\sigma A_1}{A_0} \). Suppose not, then there exists \( t_0 > 0 \) such that \( u(t) \in (L_0 + \varepsilon, M_0) \) for all \( t < t_0 \) but \( u(t_0) = L_0 + \varepsilon \) for \( \varepsilon > 0 \) or \( u(t_0) = M_0 \). First assume that \( u(t_0) = L_0 + \varepsilon \) for \( \varepsilon > 0 \) which implies \( u'(t_0) \leq 0 \). On the other hand
\[ u'(t_0) = -A_0 u(t_0) - A_1 u(t_0 - \tau(u(t_0))) \geq -A_0 \mu - A_0 \varepsilon + A_1 \frac{\mu A_0}{A_1} = -A_0 \varepsilon \]
which is a contradiction since \(u(t_0)\) and \(-A_0\varepsilon\) tend to \(\mu\) and 0 respectively when \(\varepsilon\) tends to 0. In a similar way, if \(u(t_0) = M_0\), then \(u'(t_0) \geq 0\). On the other hand

\[
u'(t_0) = -A_0u(t_0) - A_1u(t_0 - \tau(t_0)) < \frac{\mu(A_0^2 - A_1^2)}{A_1} < 0\]

which is a contradiction. As a result, there is no such \(t_0 \in \mathbb{R}^+\) and (9) holds.

Since

\[f(t, u, v) = -A_0u(t) - A_1v(t), \quad \alpha(t, u) = t - \frac{a + bu(t)}{c + du(t)}\]

are Lipschitz with respect to each of their argument, local existence and uniqueness of the solution \(u(t)\) follows from Driver [5].

If the delay functions \(\tau(u(t))\) and \(\tau(u(t))\) are equal to each other for \(u(t) \in (L_0, M_0)\), Theorems 2.1 and 2.2 also hold for the equation (6).

**Corollary 2.3.** Let’s be \((L_0, M_0) = (\sigma, -\frac{\sigma A_1}{A_0})\) for \(\sigma < 0 < 0\) or \((L_0, M_0) = (-\frac{\sigma A_1}{A_0}, \sigma)\) for \(0 < \sigma < \mu\) and \(\text{sign}(b) \text{sign}(d) = 1\). Equation (6) has a unique solution \(u(t) \in C^1([0, \infty) \to (L_0, M_0))\) where \(\tau\) is defined in Theorem 2.1 with Lipschitz history function \(u_0(t) : [-\tau, 0] \to (L_0, M_0)\).

**Corollary 2.4.** Let’s be \((L_0, M_0) = (-\frac{\mu A_0}{A_1}, \mu)\) for \(0 < \mu \leq \sigma\) or \((L_0, M_0) = (\mu, -\frac{\mu A_0}{A_1})\) for \(\sigma \leq \mu < 0\) and \(\text{sign}(b) \text{sign}(d) = 1\). Equation (6) has a unique solution \(u(t) \in C^1([0, \infty) \to (L_0, M_0))\) where \(\tau\) is defined in Theorem 2.2 with Lipschitz history function \(u_0(t) : [-\tau, 0] \to (L_0, M_0)\) and under the condition \(A_0^2 - A_1^2 \geq 0\).

**Corollary 2.5.** Let’s be \((L_0, M_0) = (\sigma, \mu)\) for \(\sigma < 0 < \mu, -\frac{\sigma A_1}{A_0} < \mu\) or \((L_0, M_0) = (\mu, \sigma)\) for \(\mu < 0 < \sigma, \mu < -\frac{\sigma A_1}{A_0}\) and \(\text{sign}(b) \text{sign}(d) = -1\). Equation (6) has a unique solution \(u(t) \in C^1([0, \infty) \to (L_0, M_0))\) where \(\tau\) is defined in Theorem 2.1 with Lipschitz history function \(u_0(t) : [-\tau, 0] \to (L_0, M_0)\).

**Corollary 2.6.** Let’s be \((L_0, M_0) = (-\frac{\mu A_0}{A_1}, \mu)\) for \(\sigma < 0 < \mu, -\frac{\sigma A_1}{A_0} \geq \mu\) or \((L_0, M_0) = (\mu, -\frac{\mu A_0}{A_1})\) for \(\mu < 0 < \sigma, \mu \geq -\frac{\sigma A_1}{A_0}\) and \(\text{sign}(b) \text{sign}(d) = -1\). Equation (6) has a unique solution \(u(t) \in C^1([0, \infty) \to (L_0, M_0))\) where \(\tau\) is defined in Theorem 2.2 with Lipschitz history function \(u_0(t) : [-\tau, 0] \to (L_0, M_0)\) and under the condition \(A_0^2 - A_1^2 \geq 0\).

In the case of \(d = 0\) the Theorem 2.2 does not hold and the last two intervals in (8) in Theorem 2.1 do not longer exist. Moreover, if \(d = 0, c = 1, a > 0\) and \(b > 0\) then by Theorem 2.1, the solution exists in the interval \((\sigma, -\frac{\sigma A_1}{A_0})\) which was previously found in [22].

These results allow us to do stability analysis of solution of (1) by using the range of \(\tau(u(t))\) which is obtained by Theorems 2.1 and 2.2.

Furthermore, if the delay function \(\tau(u(t))\) has a complicated form, then \([1/1]\) Padé approximation for \(\tau(u(t))\) can be obtained and the stability analysis can be done by using rough range of \(\tau(u(t))\) which is obtained by the range of the solution \(u(t)\) approximately by the help of Theorems 2.1 and 2.2. The same can be done, if the delay function \(\tau(u(t))\) is not known exactly but some of its suitable values are obtained by some experiments or a heuristic method.

### 3 Stability analysis

In this section, we firstly consider the stability of equation (1) with the delay function \(\tau(u(t))\) which has an upper bound for all \(t \in \mathbb{R}^+\), i.e., there exist at least one \(M_1 \in \mathbb{R}\) such that \(0 < \tau(u(t)) < M_1\) for all \(t \in \mathbb{R}^+\).

In this case, the value of delay of equation (1) varies in interval \((0, M_1)\) while \(t\) is varying. The independent parameter \(h\) of the characteristic equation (2) takes values in the interval \((0, M_1)\). As a part of the D-partition method, we have

\[C_\varepsilon : A_0 + A_1 = 0 \quad \text{for} \quad \lambda = 0\]
this straight line is a line forming the boundary of the D-partition and is denoted by $C_*$. Substituting $\lambda = i\omega$ and equating to zero the real and imaginary parts in characteristic equation (2), we find the following equations

$$A_0 + A_1 \cos(\omega h) = 0 \quad (11)$$

$$\omega + A_1 \sin(\omega h) = 0. \quad (12)$$

Solving the above equations for $A_0$ and $A_1$, the following parametric curve equations are obtained

$$A_0(\omega, h) = -\frac{\omega \cos(\omega h)}{\sin(\omega h)} \quad (13)$$

$$A_1(\omega, h) = \frac{\omega}{\sin(\omega h)} \quad (14)$$

Since $A_0(\omega, h)$ and $A_1(\omega, h)$ are even with respect to $\omega$, it is sufficient to take $\omega \in (0, \infty)$. Equations (13)-(14) define a family of curves since $h$ is not a constant. Holding $h$ fixed, these define $A_0(\omega, h)$ and $A_1(\omega, h)$ as function of $\omega$, providing a parametric representation of a curve. Different values of $h$ give different curves in the family. Since equations (13)-(14) have singularity for $\omega h = k\pi$, we introduce intervals $J_k = (\frac{k\pi}{n}, \frac{(k+1)\pi}{n})$ and denote by $C_k(h)$ the curve in the parameter space $(A_0, A_1)$ for $\omega \in J_k$.

$C_0(h)$ contains the limit point for $\omega \to 0$

$$\lim_{\omega \to 0} A_0(\omega, h), \lim_{\omega \to 0} A_1(\omega, h) = (-\frac{1}{h}, \frac{1}{h}). \quad (15)$$

In addition, the following limits can be obtained for $k \in \mathbb{N} - \{0\}$

$$\lim_{\omega \to (\frac{2k\pi}{n} - \frac{1}{n})^{-}} A_0(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} - \frac{1}{n})^{+}} A_1(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} + \frac{1}{n})^{-}} A_0(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} + \frac{1}{n})^{+}} A_1(\omega, h) = +\infty$$

$$\lim_{\omega \to (\frac{2k\pi}{n} - \frac{1}{n})^{+}} A_0(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} + \frac{1}{n})^{-}} A_1(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} - \frac{1}{n})^{+}} A_0(\omega, h) = \lim_{\omega \to (\frac{2k\pi}{n} + \frac{1}{n})^{-}} A_1(\omega, h) = -\infty$$

**Lemma 3.1.** The curves $C_0(h)$ intersect $C_*$ exactly once at $(-\frac{1}{h}, \frac{1}{h})$ for each positive number $h$. Moreover, $C_k(h)$ do not intersect $C_*$ for $k \in \mathbb{N} - \{0\}$.

**Proof.** Intersection of $C_0(h)$ and $C_*$ is obvious from (15). For the second part of Lemma 3.1, suppose that if $C_k(h)$ and $C_*$ has intersection points there exist $\omega \in J_k$ for equations (13)-(14) which satisfies equation (10). By using equations (13)-(14) in equation (10) we have

$$\frac{\omega \cos(\omega h)}{\sin(\omega h)} = \frac{\omega}{\sin(\omega h)}.$$

There is no solution $\omega \in J_k$ for $k \in \mathbb{N} - \{0\}$ which is a contradiction. \qed

**Lemma 3.2.** The curves $C_k(h_0)$ do not intersect each other for $h_0 \in \mathbb{R}^+$.  

**Proof.** Suppose that there exist an intersection point. It means that, there exist $\omega_1 \neq \omega_2 \in \mathbb{R}^+$ such that $A_0(\omega_1, h_0) = A_0(\omega_2, h_0)$ and $A_1(\omega_1, h_0) = A_1(\omega_2, h_0)$. These equalities imply that

$$\frac{\omega_1}{\sin(\omega_1 h_0)} = \frac{\omega_2}{\sin(\omega_2 h_0)} \quad (16)$$

from equation (13) and (14). For $n \in \mathbb{N}$, $\omega_1 h_0 \neq \omega_2 h_0 \pm 2n\pi$ is obtained from the left equality in (16) because of $\omega_1 \neq \omega_2$. In addition, left and right equalities in (16) lead to $\cos(\omega_1 h_0) = \cos(\omega_2 h_0)$ which is a contradiction. \qed

**Lemma 3.3.** The curve $C_k(h_0)$ intersects the line $A_0 = 0$ exactly once for $h_0 \in \mathbb{R}^+$. Moreover, the intersection point $(0, P_k)$ satisfies the following inequalities

$$P_k < P_{k+2} \quad \text{for } k = 2n, n \in \mathbb{N}$$

$$P_{k+2} < P_k \quad \text{for } k = 2n + 1, n \in \mathbb{N}.$$
Proof. When $\omega \in J_k$, the equation $A_0(\omega, h_0) = 0$ implies $\omega = \frac{\pi + 2k\pi}{2h_0}$. Hence,

$$P_k = \begin{cases} 
\frac{\pi + 2k\pi}{2h_0} & \text{for } k = 2n, n \in \mathbb{N} \\
\frac{\pi + 2k\pi}{2h_0} & \text{for } k = 2n + 1, n \in \mathbb{N}.
\end{cases}$$

is obtained by substituting $\omega = \frac{\pi + 2k\pi}{2h_0}$ in $A_1(\omega, h_0)$. This completes the proof.

Theorem 3.4. The solution of equation

$$u'(t) = -A_0u(t) - A_1u(t - h), \quad h \in \mathbb{R}^+$$

is asymptotically stable, i.e., all the roots of equation

$$\lambda + A_0 + A_1 e^{-\lambda h} = 0$$

have negative real parts, if and only if

(a) $-\frac{1}{h} < A_0$

(b) $-A_0 < A_1 < \frac{\omega}{\sin(\omega h)}$ where $\omega$ is the root of $A_0 = -\frac{\omega \cos(\omega h)}{\sin(\omega h)}$ such that $\omega h \in (0, \pi)$

Proof. When $A_0 > 0$ and $A_1 = 0$, the solution of equation is clearly asymptotically stable. The stability region which includes half line $A_0 > 0$ and $A_1 = 0$, lies above $C_*$ and below $C_0(h)$ because of Lemmata 3.1, 3.2 and 3.3. The conditions (a)–(b) are algebraic representation of this region in parameter space $(A_0, A_1)$.

To find the number of roots with positive real parts in each region in the parameter space determined by the D-curves, we use the following ideas from [38]. Writing $\lambda = \mu + i \omega$ with $\mu, \omega \in \mathbb{R}$ in characteristic equation $g(\lambda, A_0, A_1)$, we find two real equations

$$G_1(\mu, \omega, A_0, A_1) := \text{Re}(g(\mu, \omega, A_0, A_1)) = 0$$

$$G_2(\mu, \omega, A_0, A_1) := \text{Im}(g(\mu, \omega, A_0, A_1)) = 0$$

for the real and imaginary parts of $\lambda$. Direction of movement of an element is determined by the following proposition, using Jacobian matrix $J$ defined by

$$J = \begin{bmatrix} \frac{\partial G_1}{\partial \mu} & \frac{\partial G_1}{\partial \omega} \\
\frac{\partial G_1}{\partial A_0} & \frac{\partial G_1}{\partial A_1} \\
\frac{\partial G_2}{\partial \mu} & \frac{\partial G_2}{\partial \omega} \\
\frac{\partial G_2}{\partial A_0} & \frac{\partial G_2}{\partial A_1} \end{bmatrix}_{\mu=0}.$$
Fig. 1. The member of the D-curves family $C_k(h)$ in the parameter space $(A_0, A_1)$ for $h = 1$. The arrows along the curves refer to the direction of increasing $\omega$. The numbers $s$ in the different regions bordered by the curves indicate the number of roots in the right half plane.

Until Theorem 3.4, parameter is taken as a real constant. Now we determine how stability region varies when $h$ is varying.

Lemma 3.6. The members of family of curves $C_k(h)$ do not intersect each other for $k \in \mathbb{N}$.

Proof. Suppose that there exists an intersection point for $h_1 < h_2$. It means that, there exist $\omega_1 h_1, \omega_2 h_2 \in (k\pi, (k + 1)\pi)$ such that

\begin{align}
\frac{\omega_1 \cos(\omega_1 h_1)}{\sin(\omega_1 h_1)} &= \frac{\omega_2 \cos(\omega_2 h_2)}{\sin(\omega_2 h_2)} \tag{21} \\
\frac{\omega_1}{\sin(\omega_1 h_1)} &= \frac{\omega_2}{\sin(\omega_2 h_2)} \tag{22}
\end{align}

$\cos(\omega_1 h_1) = \cos(\omega_2 h_2)$ is obtained by using (21) and (22) which implies that $\omega_1 h_1 = \omega_2 h_2$ because of $\omega_1 h_1, \omega_2 h_2 \in (k\pi, (k + 1)\pi)$. Therefore we have $\omega_1 \neq \omega_2$ from the assumption $h_1 < h_2$ which contradicts (22).

Lemma 3.7. If $h_1 < h_2$, $C_0(h_2)$ lies below $C_0(h_1)$ in parameter space $(A_0, A_1)$.

Proof. Taking the derivative of (13) and (14) with respect to $h$, we obtain

\begin{align}
\frac{\partial A_0}{\partial h} &= \frac{\omega^2}{\sin^2(\omega h)} \\
\frac{\partial A_1}{\partial h} &= -\frac{\omega^2 \cos(\omega h)}{\sin^2(\omega h)}.
\end{align}

It implies that, $A_0(\omega, h)$ is a monotone increasing function and $A_1(\omega, h)$ is a monotone decreasing function for $\omega h \in (0, \frac{\pi}{2})$. For each $\omega h$ value, functions $A_0(\omega, h)$ and $A_1(\omega, h)$ represent coordinates of points on $C_0(h)$. Therefore, if $h_1 < h_2$, the point $(A_0(\omega, h_2), A_1(\omega, h_2))$ lies below the point $(A_0(\omega, h_1), A_1(\omega, h_1))$ for $\omega h_1, \omega h_2 \in (0, \frac{\pi}{2})$. Suppose that a part of $C_0(h_2)$ lies above $C_0(h_1)$ for $\omega h_1, \omega h_2 \in \left[\frac{\pi}{2}, \pi\right)$, then there exists at least one intersection point of $C_0(h_2)$ and $C_0(h_1)$, which contradicts Lemma 3.6.

The curves $C_0(h)$ are shown for $h = 0.25, 0.75, 1, 1.3$ in Fig. 2.
Proposition 3.8. Let’s define the set $S_h$ as follows

$$S_h = \{(A_0, A_1)|A_0, A_1 \in \mathbb{R} and satisfy the conditions (a) and (b) for h \in \mathbb{R}^+,\}$$

If $h_1 < h_2$ then $S_{h_1} \subset S_{h_2}$.

Proof. It is clear from Lemma 3.7. \qed

Theorem 3.9. The solution of equation (1) with delay term $\tau(u(t)) > 0$ which satisfies the condition $0 < \tau(u(t)) < M_1$ for all $t \in \mathbb{R}^+$ is asymptotically stable if and only if the following conditions are satisfied:

\begin{align*}
\overline{(a)} & - \frac{1}{M_1} < A_0 \\
\overline{(b)} & A_0 < A_1 < \frac{\omega}{\sin(\omega M_1)}
\end{align*}

where $\omega$ is the root of $A_0 = -\frac{\omega \cos(\omega M_1)}{\sin(\omega M_1)}$ such that $\omega M_1 \in (0, \pi)$

Proof. It is obvious form Theorem 2.1 and Proposition 3.8 that if the conditions $\overline{(a)}$ and $\overline{(b)}$ are satisfied, all roots of characteristic equation of equation (1) have negative real parts. \qed

Definition 3.10. A delay value of DDE is called critical delay if DDE has pure imaginary or zero eigenvalues at this delay value.

Critical delays of an equation are the values at which the qualitative behavior of the system changes. Between any two successive critical values, the behavior of the solution does not change.

Now, by using transformation (5) we rewrite the critical delay values of equation (17) in terms of parameter.

Proposition 3.11. $\lambda = i \omega$ is a root of equation (18) for some $h$ if and only if $\lambda = i \omega$ is also a root of

$$T \lambda^2 + (1 + A_0 T - A_1 T)\lambda + A_0 + A_1 = 0$$

for some $T \geq 0$

Proof. Let $\lambda = i \omega$ be a root of equation (18). By using transformation (5) in equation (18), we obtain

$$i \omega + A_0 + A_1 \frac{1 - i \omega T}{1 + i \omega T} = 0.$$ 

Multiplying this equation by $1 + i \omega T$ and arranging properly, we get

$$T(i \omega)^2 + (1 + A_0 T - A_1 T)i \omega + A_0 + A_1 = 0$$

which implies that $\lambda = i \omega$ is a root of equation (23) for $h = \frac{2}{\omega}(\arctan(\omega T) + p \pi)$. Moreover, the singular cases of Rekasius transform (5) are satisfied for equation (16) and equation (21).
As a result, $T = \frac{1}{A_1 - A_0}$, $\omega = \pm \sqrt{A_1^2 - A_0^2}$ and critical delays

$$h_p = \frac{2}{\sqrt{A_1^2 - A_0^2}} \left( \text{arctan} \left( \frac{\sqrt{A_1^2 - A_0^2}}{A_1 - A_0} + p \pi \right) \right), \ p \in \mathbb{Z}$$

are obtained under the condition $|A_0| < A_1$. Let $h_n$ denote least $h_p$ value which is greater than 0. Therefore, the solution of equation (17) is stable for $h \in (0, h_n)$ when $0 < A_0 < A_1$. Hence we can state the following result about stability of the solution for the equation (1).

**Theorem 3.12.** The solution of the equation (1) with delay $\tau(u(t)) > 0$ such that $0 < \tau(u(t)) < M_1$ for all $t \in \mathbb{R}^+$ is asymptotically stable under the conditions $|A_0| < A_1$ and $M_1 < h_n$.

Now, we give a stability criterion which is independent from delay for equation (1) by using D-partition method.

**Theorem 3.13.** If $|A_1| < A_0$, then the solution of equation (1) is asymptotically stable.

**Proof.** It is obvious from (11) that, $|A_1(\omega)| \geq |A_0(\omega)|$ for all $\omega \in J_k$. Therefore, all of the D-curves is in this region, i.e., there is no D-curve in the region described by $|A_1| < |A_0|$. Moreover, the half line $A_1 > 0$ and $A_0 < 0$ on which the solution equation (1) is asymptotically stable, is in the region described by $|A_1| < A_0$.

4 Conclusion

In this study, we have analyzed the stability of a differential equation with state-dependent delay under some conditions which guarantee existence and uniqueness of solutions. For the positivity of delay term $\tau(u(t)) = a + bu(t)$, we put the some restrictions on the range of parameters values under consideration.

We consider two cases according to the boundedness of delay term. In the first case, the delay term is supposed to be bounded and we have two theorems, namely Theorems 3.9 and 3.12, related to stability of the solution. It is shown that upper bound of the delay term affects the stability region of differential equation with state-dependent delay: smaller upper bound means greater stability region. Moreover, by using the Rekasius transform, which is an analytic transformation, we get equation (23) which is equivalent to the characteristic equation of a differential equation with state-dependent delay. In the second case, the delay term is supposed to be unbounded and we have a theorem, namely Theorem 3.13. It is shown that we can find a stability region for the solution, even though the delay term is unbounded, since there is a stability region which is independent from delay term.

In the future we would like to investigate the stability of differential equation with different state-dependent delay. This would allow for a better understanding of differential equation with state-dependent delay.

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