On classes of globally smooth solutions to the Euler equations in several dimensions

The aim of the present work is the investigation of solutions to the system of the Euler equations, including the systems with the right-hand sides describing various interior forces:

\[ \partial_t \rho + \text{div} (\rho \mathbf{V}) = 0, \]

\[ \partial_t (\rho \mathbf{V}) + (\rho \mathbf{V}, \nabla) \mathbf{V} + \nabla p = \rho \mathbf{f}(\mathbf{x}, t, \mathbf{V}, \rho, S), \]

\[ \partial_t S + (\mathbf{V}, \nabla S) = 0 \]

with the state equation \( p = e^S \rho^\gamma, \quad \gamma = \text{const} > 1 \). We suppose that \( \mathbf{f} \) is the smooth function of all its arguments. Here \( \rho(t, \mathbf{x}), \mathbf{V}(t, \mathbf{x}), S(t, \mathbf{x}) \) are the components of solution, corresponding to the density, the velocity and the entropy, given in \( \mathbb{R} \times \mathbb{R}^n, \quad n \geq 1 \). Equations (1–3) describe the balance of mass, momentum and entropy, correspondingly.

Set the Cauchy problem for (1 – 3):

\[ \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \geq 0, \quad \mathbf{V}(0, \mathbf{x}) = \mathbf{V}_0(\mathbf{x}), \quad S(0, \mathbf{x}) = S_0(\mathbf{x}). \]  

We deal with the classical solutions to system (1–3) with the density so quickly decreasing as \( |\mathbf{x}| \to \infty \), to guarantee the convergence of the integral \( \int_{\mathbb{R}^n} \rho |\mathbf{x}|^2 d\mathbf{x} \) (so called solutions with the finite momentum of inertia).

As well known, the solution to the Cauchy problem for system (1-3) may lose the initial smoothness in a finite time, sometimes there is a possibility to estimate the time of singularity formation from above (see, f.e., [1] and references therein). Moreover, in the case more frequently investigated \( \mathbf{f} = 0 \) the compactly supported initial data are sufficient for the further singularity formation. (f.e., [2]).

At the same time it is interesting that there are some nontrivial classes of globally smooth solutions. Note that if \( \mathbf{f} = 0 \), the components of such solutions do not belong to the Sobolev class.

The paper is organized as follows. Firstly supposing the existence of globally in time smooth solution having concrete properties (denote it \( U_0 \)), we show that if we choose some initial data \( (CD_1) \) close in the Sobolev norm to initial data of such solution \( (CD_0) \), then the corresponding solution to the Cauchy problem \( (U_1) \) occurs globally smooth as well. Then we show that the solutions \( U_0 \) with such kind of properties exists, moreover, at \( n = 2 \) we construct some of them.

The present work may be considered as a continuation of papers [3], [4], and their generalization in some sense.

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1 Symmetrization and the result on a local-in-time smoothness

The result on a local in time existence of smooth solution $U(t, x)$ for symmetric hyperbolic systems, i.e. systems of the form

$$A^0(t, x, U) \frac{\partial U}{\partial t} + \sum_{k=1}^{n} A^k(t, x, U) \frac{\partial U}{\partial x^k} = g(t, x, U),$$  \hspace{1cm} (1.1)

where the matrices $A^j(t, x, U)$ are symmetric, and additionally the matrix $A^0(t, x, U)$ is positive definite, is well known ([5], [6], the particular cases in [7], [8], [9]).

Namely, if the matrices $A^j(t, x, p), g(t, x, p)$ depending smoothly on their arguments, have continuous and bounded derivatives with respect to the variables $(x, p)$ up to order $m+1$ under bounded $p$, the initial data $U_0 = U(0, x)$ and the function $g(t, x, 0)$ belong to the Sobolev class $H^m(\mathbb{R}^n)$, $m > 1 + n/2$, for any fixed $t \geq 0$, then locally in time the corresponding Cauchy problem has a unique solution from the class $\cap_{j=0}^{1} C^j([0, T); H^{m-j}(\mathbb{R}^n))$, $T > 0$. Moreover,

$$\lim_{t \rightarrow T^{-}} \sup_{L^\infty} (\|U\|_{L^\infty} + \|\nabla_x U\|_{L^\infty}) = +\infty,$$ \hspace{1cm} (1.2)

$T = T(U_0)$ and $\lim_{\|U_0\|_{H^m} \rightarrow 0} T(U_0) = +\infty$.  

(Note that less rigid requirements may be imposed on the initial data for the existence of smooth solution locally in time, i.e. local (in space), but uniform belonging to $H^m(\mathbb{R}^n)$. [3], [10].)

In many problems having the origin in physics the coefficients of system (1.1) depend only on solution, this simplifies significantly the formulation of the result. In the case for the local in time existence of the Cauchy problem in the Sobolev class described above, it is sufficient, beside of the coefficients smoothness, to require the implementation of the condition $g(0) = 0$.

As we deals with the solutions to system (1–3) such that $\inf \rho = \inf p = 0$, we have to use the symmetrizartion firstly proposed in [11].

Involving the variable $\Pi = \kappa (p/2)^{\frac{\gamma - 1}{\gamma}}$, $\kappa = \frac{2\sqrt{\gamma}}{\gamma - 1}$, we get the symmetric form as follows:

$$\exp\left(\frac{S}{\gamma}\right)(\partial_t + V, \nabla)\Pi + \frac{\gamma - 1}{2} \exp\left(\frac{S}{\gamma}\right)\Pi \text{div} V = 0,$$ \hspace{1cm} (1.3)

$$(\partial_t + (V, \nabla))V + \frac{\gamma - 1}{2} \exp\left(\frac{S}{\gamma}\right)\Pi \nabla \Pi = f_1(t, x, \Pi, V, S) = f(t, x, e^{-\frac{\Pi}{\kappa}} \left(\frac{\Pi}{\kappa}\right)^{\frac{\gamma - 1}{\gamma - 2}}, V, S), \hspace{1cm} (1.4)$$

$$(\partial_t + (V, \nabla))S = 0.$$ \hspace{1cm} (1.5)

Denote $U = (\Pi, V, S)$. As an immediate corollary of the general result on the symmetric hyperbolic systems we obtain the following

**Theorem 1.1** Let the initial data $U_0 = (\Pi_0, V_0, S_0)$, belong to the class $H^m(\mathbb{R}^n)$, $m > 1 + n/2$. Suppose the function $f_1(t, x, U)$ have continuous and bounded derivatives with respect to space variables and the solution components up to order $m + 1$ for bounded $U$, and $f_1(t, x, 0) \in H^m(\mathbb{R}^n)$ for any fixed $t \geq 0$. 

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Then the Cauchy problem for system (1–3) has locally in time a unique solution such that
\[
(\Pi, V, S) \in \cap_{j=0}^1 C^j([0, T); H^{m-j}(\mathbb{R}^n),
\]
moreover
\[
\lim_{\|U_0\|_H^m \to 0} T = +\infty.
\]

2 The interior solution

We call an interior solution globally in time smooth solution \((\bar{\rho}(t, x), \bar{V}(t, x), \bar{S}(t, x))\) to system (1–3) having the following property: another solution \((\rho, V, S)\) to the Cauchy problem (4) having sufficiently small norm
\[
\|(\bar{\rho}^{(\gamma-1)/2}(x) - \rho^{(\gamma-1)/2}(0, x), V_0(x) - \bar{V}(0, x), S_0(x) - \bar{S}(0, x))\|_{H^m(\mathbb{R}^n)}
\]
is smooth globally in time as well, moreover
\[
(\rho^{(\gamma-1)/2} - \bar{\rho}^{(\gamma-1)/2}, V - \bar{V}, S - \bar{S}) \in \cap_{j=0}^1 C^j([0, \infty); H^{m-j}(\mathbb{R}^n)), m > 1 + n/2.
\]

Note that the trivial solution is not interior if \(f = 0\).

The set of interior solutions is not empty. In the paper [3] (the generalization of [3]) for \(f = 0\) it is shown that the solution to system (1–3) \((0, \bar{V}(t, x), \text{const})\) is interior, if \(V(t, x)\) is the solution to equation \(\partial_t V + (V, V)V = 0\) such that \(SpecD\bar{V}(0, x)\) is separated from the semi-axis \(R_-\) and \(D\bar{V}(0, x) \in L^\infty(\mathbb{R}^n), D^2\bar{V}(0, x) \in H^{m-1}(\mathbb{R}^n)\). (We denote \(D^k\) the vector of all spatial derivatives of order \(k\).)

In particular, as it is noted in [3] in the role of such solution may be played by the solution with linear profile of velocity \(V = A(t)r\), where \(r\) is the radius-vector of point, \(A(t)\) is the matrix with the coefficients depending on time such that \(SpecA(0) \notin R_+\). The solution may be found explicitly.

Note that the given interior solution (and the close solution as well) does not belong to the Sobolev space, since the velocity components grow at infinity. The result is clear from the physical point of view: the velocity field having the positive divergence ”spreads out” the initially concentrated sufficiently small mass, that prevent the singularity formation.

But the given solution with the zero density is not physical. It follows, in particular, in the case of the velocity Jacobian having initially negative spectrum at some value of \(x\), the appearance during a finite time of points with infinite negative divergence. If the solution with non-zero velocity, initially close to the described above, would be have like this, it would follow (according to (1)) the infinite value of density in some points.

The question arises: can one construct the interior solution with the density not close to zero in the Sobolev norm? Or to be interior the solution must describe the scattering of sufficiently small mass?

If from some reasons we succeed to find the velocity field, then from the linear with respect to \(\rho\) and \(S\) equations (1), (3) we can find the solution to the Cauchy problem (4), moreover, if the velocity field occurs smooth, then the rest of the components are smooth.

It needs to note, that the values \(\rho\) and \(p\) can become unbounded in some point of trajectory in a finite time \(T\) if and only if \(\int_0^T \text{div} V dt = -\infty\) on the trajectory (see (1)).

Before formulating the theorem we do some transformation under supposition that \((\bar{\Pi}, \bar{V} = A(t)r, \bar{S})\) is a globally in time smooth solution to system (1.3–1.5).
According to (1.3–1.5) the vector-function \( \mathbf{u} = \mathbf{U} - \bar{\mathbf{U}} \) (here \( \mathbf{u} = (\pi, \mathbf{v}, s) \), \( \mathbf{U} = (\Pi, \mathbf{V}, S) \), \( \bar{\mathbf{U}} = (\bar{\Pi}, \bar{\mathbf{V}}, \bar{S}) \)) satisfies the system of equations

\[
(\partial_t + \mathbf{V}, \nabla)\mathbf{u} + (\mathbf{v}, \nabla)\mathbf{u} + \frac{\gamma - 1}{2}(\Pi + \bar{\Pi})\text{div}\mathbf{V} = \\
-(\mathbf{v}, \nabla)\bar{\Pi} - \frac{\gamma - 1}{2}\text{div}\bar{\mathbf{V}},
\]

(2.1)

\[
(\partial_t + (\mathbf{V}, \nabla))\mathbf{v} + (\mathbf{v}, \nabla)\mathbf{v} + \frac{\gamma - 1}{2}\exp(\frac{S}{\gamma})(\Pi + \bar{\Pi})\nabla\pi = \\
-(\mathbf{v}, \nabla)\bar{\mathbf{V}} - \frac{\gamma - 1}{2}\exp(\frac{S}{\gamma})(\Pi + \bar{\Pi})\nabla\bar{\Pi} + \frac{\gamma - 1}{2}\exp(\frac{S}{\gamma})\bar{\Pi}\nabla\bar{\Pi} + \\
+f(t, \mathbf{x}, (\mathbf{V} + \mathbf{v}), (\bar{S} + s), (\bar{\Pi} + \pi)) - f(t, \mathbf{x}, \bar{\mathbf{V}}, \bar{S}, \bar{\Pi}),
\]

(2.2)

\[
(\partial_t + \mathbf{V}, \nabla)s + (\mathbf{v}, \nabla)s = -(\mathbf{v}, \nabla)\bar{S}.
\]

(2.3)

Further, following to [3], we carry out the nondegenerate change of variables such that the infinite semi-axis of time turns out to semi-interval \([0, \sigma_\infty)\), to obtain the symmetric hyperbolic system with the coefficients allowing to apply the theorem on a local existence of smooth solution on the interval \([0, \sigma_*)\), \(\sigma_* \leq \sigma_\infty\).

So, let \((t_1, \mathbf{y}) := (t, A(t)x)\) be the new variables \((A(t) = \|a_{ij}(t)\| \text{ is a quadratic } (n \times n) \text{ matrix})\). Then \(\nabla_x = A^*\nabla_y, \text{div}_x \mathbf{V} = \text{div}_y A \mathbf{V}, \partial_{t_1} = \partial_t + A \nabla_y\). Choose the nonnegative function \(\lambda(t)\) such that the integral \(\int_0^\infty \lambda(t)dt = \sigma_\infty < \infty\) converges and set \(\sigma(t_1) = \int_0^{t_1} \lambda(t)dt\). In that way, the semi-infinite axis of time goes to the semi-interval \([0, \sigma_\infty)\).

Further, involve the variables

\[
\mathbf{W} = A(t)\mathbf{v} \lambda^{-1}(t), \quad P = \lambda^{-q}(t)\pi,
\]

the constant \(q\) will be defined below.

After the transformation we get the system

\[
(\partial_\sigma + (\mathbf{W}, \nabla_y))P + \frac{\gamma - 1}{2}(P + \bar{P})\text{div}_y \mathbf{W} = \\
-(\mathbf{W}, \nabla_y)\bar{P} - PQ_1(\sigma),
\]

(2.4)

\[
(\partial_\sigma + (\mathbf{W}, \nabla_y))\mathbf{W} + \frac{\gamma - 1}{2}\Psi(S, \sigma)(P + \bar{P})\nabla_y P = \\
-\frac{\gamma - 1}{2}\Psi(S, \sigma)(P + \bar{P})\nabla_y \bar{P} + \frac{\gamma - 1}{2}\Psi(\bar{S}, \sigma)\bar{P}\nabla_y \bar{P} - \mathbf{W}Q_2(\sigma) + G,
\]

(2.5)

\[
(\partial_\sigma + \mathbf{W}, \nabla_y)s = -(\mathbf{W}, \nabla_y)\bar{S}.
\]

(2.6)

Here we denote

\[
Q_1(t(\sigma)) = \lambda(t)^{-1}(\frac{\gamma - 1}{2}\text{tr}A(t) + q(\ln \lambda(t))'),
\]

\[
Q_2(t(\sigma)) = \lambda(t)^{-1}((\ln \lambda(t))'E + A(t)((A^{-1})'(t) + E)),
\]

\[
\Psi(S, \sigma) = \exp(S)R(\sigma)B(\sigma),
\]
derivatives up to order \( r \)

Theorem 2.1

Let the function

\[ f(t, A(t)x, \lambda^{-q}(t)\bar{P}, \lambda(t)A^{-1}(t)(\bar{W} + W), (\bar{S} + s)) - f_1(t, A(t)x, \lambda^{-q}(t)\bar{P}, \lambda(t)A^{-1}(t)W, \bar{S})) \]

taking into account that \( t = t(\sigma) \).

Note that \( B \) is a bounded invertible matrix.

After multiplying (2.5) by \( \Psi^{-1}(S, \sigma) \) the system become symmetric hyperbolic, however, generally speaking, \( \Psi^{-1}(S, \sigma) \to 0 \) as \( t \to \infty \) or will be unbounded owing to the behaviour of \( R(\sigma(t)) \).

In another words, we can increase the time of existence of smooth solution to system (1–3) as long as we wish due to choosing the initial data small in the Sobolev \( H^m \) norm, but we cannot guarantee the existence of the solution during the infinite time. Let

\[ D^a = \left( \frac{\partial}{\partial y_1} \right)^{\alpha_1} \ldots \left( \frac{\partial}{\partial y_n} \right)^{\alpha_n}, \sum_i \alpha_i = \alpha. \]

Denote \( C^*_r(\mathbb{R}^n) \) the space of continuous functions having continuous bounded in \( \mathbb{R}^n \) derivatives up to order \( r = 0, 1, \ldots \). Further, denote \( f_V \) the matrix \( \frac{\partial f}{\partial y} \).

Now formulate the theorem taking into account all denotation involved.

**Theorem 2.1** Let the function \( f_1(t, x, \Pi, V, S) = f(t, x, e^{-\frac{t}{2}}(\Pi)^{1/n}, V, S) \) have the derivatives with respect to all components of solution up to order \( m + 1 \), continuous and bounded under bounded \( (W, V, S) \), moreover \( f_1(t, x, 0, 0, 0) \in H^m(\mathbb{R}^n) \) at any fixed \( t \geq 0 \). Suppose system (1–3) has the globally in time smooth solution \( \bar{U} = (\bar{\rho}, V, \bar{S}) \) with the linear profile of velocity \( V = A(t)x \) such that

\begin{enumerate}
  \item \( \bar{\rho}^{-\frac{1}{n}} \in \cap_{j=0}^{m+1} C^j_b(\mathbb{R}^n); \ D\bar{S}(0, x) \in \cap_{j=0}^m C^j_b(\mathbb{R}^n); \)
  \item \( \xi(t) = \det A(t) > 0 \text{ for } t \geq t_0 \geq 0; \)
\end{enumerate}

and there exist a smooth real-valued function \( \lambda(t) \), a constant \( q \) and a skew symmetric matrix with the real-valued coefficients \( U_\phi(t) \) with the following properties:

\begin{enumerate}
  \item \( \int_{t_0}^{t_\infty} \lambda(\tau)d\tau < \infty, \)
  \item \( \int_{t_0}^{t_\infty} \lambda^q(\tau)\xi^{1/n}(\tau) d\tau < \infty, \)
  \item \( \lambda(t)A(t) > 0 \text{ for } t \geq t_0 > 0; \)
  \item \( Q_1(t)R(t), \)
  \item \( Q_2(t) - \lambda^{-1}(A(t)f\nu A^{-1}(t) - U_\phi(t)), \)
\end{enumerate}

are bounded on \( \mathbb{R}^+ \times \mathbb{R}^n \).

Then the solution \( \bar{U} \) is interior.
Proof. At first we consider the case of $R(\sigma)$ such that $0 < r_1 = const \leq R(\sigma) \leq r_2 = const < \infty$. If the other conditions of Theorem 2.1 hold, then we can prove Theorem 1.1 immediately. The case is realized, for example, at $\gamma = \frac{2}{n} + 1$, $f = 0$, in the situation, considered in [3] \((trA(t) \sim \frac{n}{t}, \xi(t) \sim t^{-n} (t \to \infty), \lambda(t) = (1 + t)^{-2}, q = \frac{n(n-1)}{4})\).

In the general case the proof is more complicated. Suppose $\sigma \in [0, \sigma_\infty)$ and $(X, Y, Z) : R^n \to R \times R^n \times R$ is a vector-function from $L_2$. Following [3], involve the norm $\|X, Y, Z\|_2(\sigma) := \int_{R^n} (X^2 + Z^2 + Y^* \Psi^{-1}(S, \sigma)Y) dy$.

For $p \in N$ define

$$E_p(\sigma) = \frac{1}{2} \sum_{|\alpha| = p} [D^\alpha P, D^\alpha W, D^\alpha s]^2(\sigma),$$

$$F_m(\sigma) = \sum_{p=0}^{m} E_p(\sigma).$$

If we should succeed to show that $F_m(\sigma)$ is bounded on $[0, \sigma_\infty]$ function, the theorem would be proved, as the impossibility to prolong the solution is connected with the going to the infinity of $L_\infty$ norms of the solution itself or its gradient (see (1.2)).

Showing that $H^m$ norm of the solution admits a majorization by means of some function, bounded on $[0, \sigma_\infty]$, we shall show that $\sigma_\ast = \sigma_\infty$ and thus the theorem will be proved.

After involving of the new variable $\Pi$, which is necessary for the symmetrization of the system (see above), globally smooth solution $\bar{U}$ corresponds to the globally smooth solution $(\bar{\Pi}, \bar{V}, \bar{S})$ with the initial data $(\bar{\Pi}_0, \bar{V}_0, \bar{S}_0)$. After the changing of variables it corresponds to the solution $(\bar{P}, \bar{W}, \bar{S})$ with initial data $(\bar{P}_0, \bar{W}_0, \bar{S}_0)$. The last solution is a smooth solution to system (2.4–2.6) on $[0, \sigma_\infty]$, and therefore its $H^m$ norm is bounded on the segment.

Compute

$$\frac{dE_q}{d\sigma} = \sum_{|\alpha| = q} \int_{R_n} (D^\alpha PD^\alpha \partial_\sigma P d\gamma +$$

$$\int_{R_n} D^\alpha sD^\alpha \partial_\sigma s d\gamma +$$

$$\int_{R_n} D^\alpha W \Psi^{-1}(S, \sigma)D^\alpha \partial_\sigma W d\gamma +$$

$$\int_{R_n} (D^\alpha W)^* (\frac{\partial \Psi^{-1}(S, \sigma)}{\partial \sigma} - \frac{1}{\gamma} \Psi^{-1}(S, \sigma) \partial_\sigma S)D^\alpha W d\gamma) =$$
\[ I_1 + I_2 + I_3 + I_4 \]

(further it is supposed the summation over repeated indices, the integrals \( I_k, \ k = 1, 2, 3, 4, \) are numbered in consecutive order).

Estimate all integrals. Below we denote by \( c_i \) the positive constants, depending only on initial data. Begin from \( I_4. \) Before all we stress that according to (2.9) \( R'(\sigma)/R(\sigma) \) is bounded as \( \sigma \to \infty, \) therefore

\[
\left| \frac{\partial \Psi^{-1}(S, \sigma)}{\partial \sigma} \right| = \left| -\frac{\Psi^{-2}(S, \sigma)\Psi(S, \sigma)R'}{R} + O(\Psi^{-1}(S, \sigma)) \right| \leq c_1 \Psi^{-1}(S, \sigma).
\]

Thus,

\[ I_4 \leq c_2(1 + \|W\nabla_y s\|_\infty + \|W\nabla_y \tilde{S}\|_\infty)E_p \leq (c_2 + c_3 R^{1/2} F_m + c_4 R^{1/2} F_m^{1/2}) E_p. \]

Further,

\[ I_1 = -\int_{R_n} \{ \frac{1}{2} |D^\alpha P| \text{div}_y W + D^\alpha P ((W, \nabla_y (D^\alpha P)) - D^\alpha (W, \nabla_y P)) \} \, dy \]

\[ -\frac{\gamma - 1}{2} \int_{R_n} D^\alpha P \{ (\nabla_y P, D^\alpha W) + (P \text{div}_y D^\alpha W - D^\alpha (P \text{div}_y W)) \} \, dy \] \( := I_{11} \).

Under sign of the integral there is the sum of members of the form \( \partial^\alpha U_1 \partial^\beta U_2 \partial^{\alpha+1-l} U_2, \) \( 0 \leq l \leq \alpha \) (by \( U_j, \ j = 1, 2, \) we mean the corresponding components of the solution. At first let \( \alpha \neq 0, 1. \) According to the Galiardo-Nirenberg inequality we have

\[ |\partial^\alpha U_j|_{p_i} \leq C |D U_j|^{1-2/p_i} |D^\alpha U_j|^{2/p_i} \]

for \( p_i = 2 \frac{\alpha - 1}{\alpha - 1}. \) For \( l \neq 1, \alpha \) it is true that \( \frac{1}{p_i} + \frac{1}{p_{\alpha - l + 1}} = \frac{1}{2}. \) From the Hölder inequality it follows that

\[ \int_{R_n} |\partial^\alpha U_1 \partial^\beta U_2 \partial^{\alpha+1-l} U_2| \, dy \leq \]

\[ \|D^\alpha U_1\|_2 \|\partial^\beta U_2 \partial^{\alpha+1-l} U_2\|_2 \leq \]

\[ C \|D^\alpha U_1\|_2 \|D^\alpha U_2\|_2 \|D^\alpha U_2\|_\infty. \]

For the other values of \( \alpha \) and \( l \) the last inequality is evident.

Thus,

\[ I_{11} \leq c \|D^\beta P\|_2 (\|D^\beta P\|_2 \|\nabla_y W\|_\infty + \|D^\beta W\|_2 \|\nabla_y P\|_\infty) \leq c R^{1/2} F_m^{1/2} E_p. \]
Further, according to condition (2.7) of the Theorem

\[ \left| \int_{R_n} D^\alpha PD^\alpha (W \nabla_y P) dy \right| \leq c_5 R^{1/2} F_m, \]

\[ \left| \int_{R_n} D^\alpha PD^\alpha (PQ_1) dy \right| \leq c_6 Q_1 RF_m \leq c_7 F_m. \]

The integral \( I_2 \) can be estimated analogously:

\[ |I_2| \leq c_8 R^{1/2} F_m^{1/2} E_q + c_9 R^{1/2} F_m. \]

At last,

\[ I_3 = \int_{R_n} (D^\alpha)^* W \Psi^{-1}(S, \sigma) \{(W, \nabla_y)D^\alpha W - D^\alpha(W, \nabla_y W)\} dy + \]

\[ \frac{1}{2} \int_{R_n} \left\{ (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha W \right\} \text{div}_y W - \frac{W \nabla_y S}{\gamma} \left( (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha W \right) dy + \]

\[ \frac{\gamma - 1}{2} \int_{R_n} \left\{ (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha (P\Psi(S, \sigma)\nabla_y P + \bar{P}\Psi(S, \sigma)\nabla_y P) - \right. \]

\[ \left. (P + \bar{P})\Psi(S, \sigma)\nabla_y \bar{P} + \bar{P}\Psi(\bar{S}, \sigma)\nabla_y \bar{P} \right\} dy + \]

\[ \int_{R_n} (D^\alpha W)^* \Psi^{-1}(S, \sigma) D^\alpha (WQ_2 + G) dy. \]

The first and second integrals can be estimated from above by values

\[ c_{10} R^{-1} \|D^\alpha W\|_2^2 \|\nabla W\|_\infty + c_{11} R^{-1} \|D^\alpha W\|_2^2 \|W\|_2 \leq c_{12} R^{1/2} F_m^{1/2} E_p, \]

and the third one by the value

\[ c_{13} \|D^\alpha W\|_2^2 (\|D^\alpha P\|_2 + \|D^\alpha s\|_2 + \|D^\alpha \bar{P}\|_2 + \|D^{p-1}\nabla_y \bar{S}\|_2)(1 + \|P\|_\infty + \|\nabla_y P\|_\infty + \|\nabla_y s\|_\infty + \|\nabla_y \bar{P}\|_\infty + \|\nabla_y \bar{S}\|_\infty)^{p+1} \leq \]

\[ c_{14} R^{1/2}(c_{14} + F_m^{(p+1)/2})E_p + c_{15} R^{1/2}(c_{16} + F_m^{(p+1)/2})E_p^{1/2}, \]

and, at last, the fourth one taking into account conditions (2.8) by the value

\[ c_{17} E_p. \]

In the last case note that

\[ G = \lambda^{-1}(t) A(t) f_V(t, x, \Pi, \bar{V} + \theta v, S) A^{-1}(t), \theta \in (0, 1), \]

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as \((w, U_\phi(t)w) = 0\) for any vector \(w \in \mathbb{R}^n\).

Uniting all the estimates we get

\[
E_p' \leq c_{18}E_p + c_{19}R^{1/2}E_p + c_{20}R^{1/2}F_m^{1/2}E_p + c_{21}R^{1/2}F_mE_p + c_{22}R^{1/2}F_m^{(p+1)/2}E_p + c_{23}R^{1/2}F_m^{(p+1)/2}E_p^2,
\]

whence it follows after summation over \(p\) that

\[
F_m' \leq c_{24}F_m + c_{25}R^{1/2}(F_m^{1/2} + F_m + F_m^{3/2} + F_m^{(p+2)/2} + F_m^{(p+3)/2}).
\]

Set \(\Lambda_m(\sigma) = \exp(-c_{24}\sigma)F_m\). Then

\[
\Lambda_m' \leq c_{26}(\Lambda_m^{1/2} + \Lambda_m + \Lambda_m^{3/2} + \Lambda_m^{(p+2)/2} + \Lambda_m^{(p+3)/2})R^{1/2}, \tag{2.10}
\]

where the constant \(c_{26}\) depends on \(c_{24}, c_{25}, \sigma_\infty\).

We set

\[
\Theta(\tilde{g}) := \int_{\delta>0} \frac{dg}{g^{1/2} + g + g^{3/2} + g^{(p+2)/2} + g^{(p+3)/2}}.
\]

The integral diverges in the zero, therefore \(\Theta(0) = -\infty\).

Integrating inequalities (2.10) over \(\sigma\) we obtain that

\[
\Theta(\Lambda_m(\sigma)) \leq \Theta(\Lambda_m(t_0)) + c_{26} \int_{\sigma(t_0)}^{\sigma} R^{1/2}(\tilde{\sigma})d\tilde{\sigma},
\]

moreover, as follows from (2.8), the integral in the right-hand side of the last inequality converges as \(\sigma \to \sigma_\infty\) to the constant \(C\), depending only on the initial data.

Choosing \(\Lambda_m(t_0)\), (and together with the value of the \(H_m\)—norm of the initial data) sufficiently small, one can obtain that \(\Theta(\Lambda_m(t_0)) + C\) is later then \(\Theta(+\infty)\), it signifies that \(\Lambda_m(\sigma)\) and \(F_m(\sigma)\) are bounded from above for all \(\sigma \in [0, \sigma_\infty]\) and \(\sigma_* = \sigma_\infty\). So, Theorem 2.1 is proved.

**Remark.** The solutions with the linear profile of velocity, satisfying the theorem conditions, exist. In the case described in (3) \(f = 0\), \(\gamma \leq 1 + \frac{2}{m}\), \(A(t) = (E + A(0)t)^{-1}A(0)\), \(\text{Spec}A(0) \notin \mathbb{R}_-\), \(\text{tr}A(t) \sim \frac{n}{t}, \det A(t) \sim \frac{1}{t^2}\), \(Q_1 = 0, Q_2 \sim 2(\frac{1}{n}((\text{tr}A^{-1})E - A^{-1})\), as \(t \to \infty\). Here \(\lambda = (1 + t)^{-2}, q = \frac{n(\gamma - 1)}{4}, U_\phi(t) = 0\).

We shall get two corollaries from Theorem 2.1, basing on which we can assert that the solutions which will be constructed in the next section are interiors.

**Corollary 2.1** Let \(f = 0\). If system (1–3) has the globally smooth in time solution with the linear profile of velocity, described in the statement of Theorem 2.1 and \(A(t) \sim \frac{q}{t}E, t \to \infty\), where \(\delta\) is a positive constant, then the solution is interior.
Proof. Choose \( \lambda(t) = t^{b+1}, \; q \geq \frac{3}{2} - \frac{r+1}{2r+1}, \; U_\phi(t) = 0 \). In that way all conditions of Theorem 2.1 are satisfied.

**Corollary 2.2** Let \( f = LV \), where \( L \) is a matrix with the smooth coefficients such that \( A_0L^\alpha - U_1(t) = -\mu E \), where \( A_0 = \delta E + U_2 \) is a matrix with the positive determinant, the matrices \( U_1 \) and \( U_2 \) are skew symmetric, \( \delta \) is a constant, \( \mu \) is a positive constant. If system (1–3) has the globally smooth in time solution with the linear profile of velocity described in the statement of Theorem 2.1 and \( A(t) \sim \frac{1}{t}A_0, \; t \to \infty \), then the solution is interior.

Proof. To verify condition (2.8) one has to show the boundedness of the value

\[
\lambda^{-1}(t)((\ln \lambda(t))'E + A(t)E - A(t)LA^{-1}(t) - A^{-1}(t)A'(t) - U_\phi) \sim \lambda^{-1}(t)((\ln \lambda(t))'E + \frac{1}{t}((\delta + 1)E + U_2) - U_1(t) + \mu E - U_\phi), \; t \to \infty.
\]

Choose \( \lambda(t) = t^{b+1} \exp\{-\mu t\} \). Condition (2.7) is satisfied at \( q \geq \frac{3}{2} \), condition (2.9) can be verified elementary. It needs to choose the matrix \( U_\phi(t) = \frac{1}{t}U_2 - U_1(t) \).

### 3 Constructing of interior solutions

Limit ourselves to the important case of \( n = 2 \) and set \( f = LV \), where the matrix \( L = \begin{pmatrix} -\mu & -l \\ l & -\mu \end{pmatrix} \), \( \mu \) is a nonnegative constant, \( l \) is an arbitrary constant. Thus, we in the simplest way describe the Coriolis force and the Rayleigh friction, for example, in the meteorological model under neglecting the vertical processes and the Earth curvature.

It is easy to verify that on the smooth solutions to system (1–3), so quickly vanishing at infinity to assure the convergency of all integrals involved, there are the following conservation quantities: the mass \( m = \int_{\mathbb{R}^2} \rho \, dx = \text{const} \), for \( \mu = 0 \) the total energy \( E = \int_{\mathbb{R}^2} \left( \frac{\omega |\nabla r|^2}{2} + \frac{\rho}{\gamma - 1} \right) \, dx = E_k(t) + E_p(t) = \text{const} \) and the momentum of inertia \( J = \int_{\mathbb{R}^2} \rho((V, r) + \frac{l}{2} |x|^2) \, dx = \text{const} \). The conservation laws are true also for the solutions to system (1–3) with the density (and the pressure) quickly vanishing as \( |x| \to \infty \), and the velocity components may even increase.

We consider another integral functionals, characterizing the average properties of solutions, namely

\[
G(t) = \frac{1}{2} \int_{\mathbb{R}^2} \rho |r|^2 \, dx, \quad F_i(t) = \int_{\mathbb{R}^2} (V, X_i) \rho \, dx, \quad i = 1, 2,
\]

where \( X_1 = r = (x, y), \; X_2 = r_\perp = (y, -x) \).

Note that \( G(t) > 0 \).

On smooth solutions to system (1–3) the following relations take place

\[
G'(t) = F_1(t), \quad (3.1)
\]

\[
F_2'(t) = lF_1(t) - \mu F_2(t), \quad (3.2)
\]
\[ F_1'(t) = 2(\gamma - 1)E_p(t) + 2E_k(t) - 2F_2(t) - \mu F_1(t), \quad (3.3) \]

\[ E'(t) = -2\mu E_k(t). \]

Remark once more that at \( \mu = 0 \) the quantity \( E(t) \) of total energy is constant. The functions \( E_p(t) \) and \( E_k(t) \) generally cannot be expressed through \( G(t), F_1(t), F_2(t) \).

We try to choose the form of velocity field in such way that from system (3.1–3.3) one could find it directly. Suppose, for example,

\[ V = A(t)r, \quad (3.4) \]

where \( A(t) \) is a \((2 \times 2)\) matrix , whose time-depending coefficients we have to find.

1) We get the simplest result if we choose the matrix \( A(t) \) such that

\[ V = \alpha(t)r + \beta(t)r_\perp, \]

It is easy to see that in the case we have

\[ F_1(t) = 2\alpha(t)G(t), \quad F_2(t) = 2\beta(t)G(t), \]

\[ E_k(t) = (\alpha^2(t) + \beta^2(t))G(t). \]

It is not difficult to obtain from the equation

\[ \partial_t p + (V, \nabla p) + \gamma p \text{div} V = 0, \quad (3.5) \]

which is a corollary of system (1–3) and the state equation, that

\[ E_p(t) = E_p(0)G^{\gamma - 1}(0)G_{\gamma - 1}(t). \]

In that way, all functions involved in system (3.1–3.3) are expressed through \( G(t), \alpha(t), \beta(t) \).

For the sake of convenience we denote \( G_1(t) = 1/G(t) \) and get the system of equations

\[ G_1'(t) = -2\alpha(t)G_1(t), \quad (3.5) \]

\[ \beta'(t) = \alpha(t)(l - 2\beta(t)) - \mu \beta(t), \quad (3.6) \]

\[ \alpha'(t) = -\alpha^2(t) + \beta^2(t) - l\beta(t) - \mu \alpha(t) + (\gamma - 1)E_p(0)G_1^{\gamma - 1}(0)G_1'(t). \quad (3.7) \]

The solutions of the system of ordinary differential equations are smooth if they are can be prolonged up to all axis of time. The non-prolongation is connected with the escape of the components of solution to infinity during a finite time.

Note that for \( \gamma > 1 \) the quantities \( \alpha(t) \) and \( \beta(t) \) are bounded, this follows immediately from the expression for the total energy of the system:

\[ E(t) = (\alpha(t)^2 + \beta(t)^2)G(t) + E_p(0)G^{\gamma - 1}(0)G_{\gamma - 1}(t) \leq E(0), \]

therefore

\[ \alpha^2(t) + \beta^2(t) \leq E(0)G_1(t) - E_p(0)G_1^{\gamma - 1}(0)G_1'(t) < +\infty, \]

thus the quantities of density and pressure remain bounded for all solution we want to construct.
At $\mu = 0$ system (3.5–3.7) can be integrated explicitly:

$$
\alpha(G_1) = \pm \sqrt{KG_1^2 - C^2G_1^2 + (E - IC)G_1 - l^2/4},
$$

$$
\beta(t) = CG_1(t) + l/2,
$$

with the constants $C = \frac{2\beta(0) - l}{2G_1(0)}$, $K = (a^2(0) + C^2G_1(0) - (E - IC)G_1(0) + l^2/4)/G_1^2(0)$.

Due to (3.8) one can reduce the number of equations in the system and consider it on the phase plane $(\alpha, \beta)$. There exists the unique singular point: at $l \neq 0$ it is a center situated on the axis $\alpha = 0$, at $l = 0$ it is a complex equilibrium in the origin, the trajectories form the elliptic saddle point. In the last case the time of movement from any point of the phase plane to the origin is infinite. The equilibria, at $l \neq 0$ lying in the plane $G_1 = 0$, namely, the points $\alpha = \beta = 0$ and $\alpha = 0, \beta = l$, are the centers under consideration in the plane $(\alpha, \beta)$.

At $\mu \neq 0$ one needs to integrate the system (3.5–3.7) numerically. However, the conclusion on the behaviour of the solution components at infinity we can do analytically as well. At $\mu > 0$ the system has a unique stable equilibrium in the origin (at $l = 0$ it is a knot, at $l \neq 0$ it is a focus.) Therefore we can find the solution as a formal asymptotic series by the negative powers of $t$. Limit ourselves to writing out the leading terms:

$$
\alpha(t) \sim a_1t^p, \beta(t) \sim b_1t^r, G_1(t) \sim c_1t^q, a_1, b_1, c_1 = \text{const} \neq 0.
$$

From (3.5) we have immediately that $q_c1t^{p-1} = -2a_1c_1t^{p+q}$, therefore $p = -1, q = -2a_1$.

From (3.6) we obtain that $rb_1t^{r-1} \sim l(a_1t^{p} - \mu b_1t^{r} - 2a_1b_1t^{p+r})$. If $l \neq 0, \mu \neq 0$, then $r = p = -1$, moreover, $l(a_1) = \mu b_1$.

From (3.7) we get analogously that $pa_1t^{p-1} = -a_1^2t^{2p} + b_1^2t^{2r} - lb_1t^{r} - \mu a_1t^{p} + Kc_1t^{\gamma q}$, where it is denoted for short $K = (\gamma - 1)E_p(0)G_1^{-\gamma}(0)$, that is $-a_1t^{-2} \sim -a_1^{-2}t^{-2} + b_1^{-2}t^{-2} - lb_1t^{-1} - \mu a_1t^{-1} + Kc_1t^{\gamma q}$.

Let $\gamma q < -1$, that is the last term does not contain the senior degree. Then the rest of members of senior order must be eliminated, and $-lb_1 = \mu a_1 = \mu^2b_1/l$, therefore $l = \mu = 0$ in spite of the supposition. Consequently, $q = -1/\gamma$.

So, at $\mu \neq 0, l \neq 0$

$$
\alpha(t) \sim \frac{1}{2\gamma}t^{-1}, \beta(t) \sim \frac{\mu}{2l\gamma}t^{-1}, G_1(t) \sim \left(\frac{\mu^2 + l^2}{2K\gamma \mu}\right)^{1/\gamma}t^{-1/\gamma}, t \to \infty.
$$

At $l = 0$, due to the existence of the integral $\beta(t) = CG_1(t)\exp\{-\mu t\}$, system (3.5–3.7) can be reduced to the nonautonomous system of two equations:

$$
G_1'(t) = -2\alpha(t)G_1(t),
$$

$$
\alpha'(t) = -\alpha^2(t) - \mu a_1t^{p} + C^2 E_p(0)\exp\{-2\mu t\}G_1^2(t) + KG_1^2(t).
$$

From (3.9) we have that $pa_1t^{p-1} \sim -a_1^{2}t^{2p} - \mu a_1t^{p} + C^2 c_1^{2}t^{2q} \exp\{-2\mu t\} + Kc_1^2 t^{\gamma q}$, where, remember, $p = -1, q = -2a_1$. To compensate the senior term $-\mu a_1t^{-1}$ it is necessary that the equality $q = -1/\gamma$ holds. Hence, we find the coefficient $c_1$. 

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Thus, at $\mu \neq 0$, $l = 0$

$$\alpha(t) \sim \frac{1}{2\gamma} t^{-\gamma}, \beta(t) \sim C \left(\frac{\mu}{2K\gamma}\right)^{1/\gamma} t^{-1/\gamma} \exp\{-\mu t\}, G_1 \sim \left(\frac{\mu}{2K\gamma}\right)^{1/\gamma} t^{-1/\gamma}, t \to \infty.$$ 

At $\mu = l = 0$, as follows from the explicit formulas,

$$\alpha(t) \sim \left(\sqrt{G_1(0)/E} + t\right)^{-1}, G_1(t) \sim \alpha^2(t), \beta(t) = CG_1(t), t \to \infty.$$ 

Thus, as follows from Corollaries 2.1 and 2.2, in the cases $\mu = l = 0$ and $\mu > 0$ we have constructed the velocity field for the interior solution. More precisely, at $\mu = l = 0$ $A(t) \sim \frac{1}{2} E$, $t \to \infty$, Corollary 2.1 can be applied with $\gamma = 1$. At $\mu > 0$ one can apply Corollary 2.2. In the case $\mu > 0$, $l = 0$ we have $U_1 = U_2 = 0$, $\delta = \frac{1}{2\gamma}$. If $\mu > 0$, $l \neq 0$, then $\delta = \frac{1}{2\gamma}, U_1 = \left(\begin{array}{cc} 0 & l \\ -l & 0 \end{array}\right), U_2 = \frac{\mu}{\sqrt{\gamma}} \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$.

As soon as $\alpha(t)$ and $\beta(t)$ are found, the rest of the components can be found elementary

$$p(t, |r|, \phi) = \exp(-2 \int_0^t \alpha(\tau)d\tau)\rho_0(|r|) \exp\left(-\int_0^t \alpha(\tau)d\tau\right), \phi + \int_0^t \beta(\tau)d\tau),$$

$$S(t, |r|, \phi) = S_0(|r|) \exp\left(-\int_0^t \alpha(\tau)d\tau\right), \phi + \int_0^t \beta(\tau)d\tau).$$

From (2) and (3.7) we get that on the classical solution of the initial system must be satisfied the relation

$$\nabla p = - (\gamma - 1) G_1^{-\gamma}(0) E_\gamma(0) G_1^\gamma(t)\rho r.$$ 

Hence it follows that the components of the initial data $\rho_0$ must be axisymmetric and compatible, i.e. connected as follows:

$$\nabla \rho_0 = - (\gamma - 1) G_1(0) E_\gamma(0) \rho_0 r.$$ 

However, in spite of the components of the pressure and the density must vanish as $|x| \to \infty$, the entropy must even increase (remember that the conditions to Theorem 2.1 require only the boundedness of the entropy gradient). However, one can choose

$$\rho_0 = \frac{1}{(1 + |x|^2)^a}, a = const > 3, \quad \rho_0 = \frac{2a}{(\gamma - 1) G_1(0) E_\gamma(0) (1 + |x|^2)^{a+1}}.$$ 

then $S_0 = const + (a(\gamma - 1) + \gamma) \ln(1 + |x|^2)$.

Remark. In the case $G(0) \neq 0$ the density cannot be compactly supported (in contrast $G(0) = 0$). Really, since $p = \pi \frac{\gamma}{\gamma - 1}, \rho = \pi \frac{1}{\gamma} e^{-\frac{\pi}{\gamma}}$, then due to the compatibility conditions $\pi \nabla \pi \sim const \cdot e^{-\frac{x}{\gamma}}, |x| \to c - 0$, where $c$ is a point of the support of $\pi$. Therefore, for $C^1$-smooth $\pi$ it occurs that $S \to +\infty, |x| \to c - 0$, and we cannot choose any smooth initial data.
It is interesting that if one requires only $C^0$-smoothness of $\pi (\pi \sim \text{const} \cdot (c - |x|)^{1/2}, |x| \to c - 0, \pi = 0, |x| \geq c > -0)$, the condition may be fulfilled. Moreover, $\rho$ and $p$ will be of the $C^1$ class of smoothness, however, neither the theorem on the local in time existence of the smooth solution, no Theorem 2.1 can be applied.

2) To consider the velocity field (3.4) with the matrix

\[ A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \]

involve the functions

\[ G_x(t) = \frac{1}{2} \int_\mathbb{R}^2 \rho x^2 dx, \quad G_y(t) = \frac{1}{2} \int_\mathbb{R}^2 \rho y^2 dx, \quad G_{xy}(t) = \frac{1}{2} \int_\mathbb{R}^2 \rho xy dx. \]

Then involve the auxiliary variables

\[ G_1(t) = G_x(t)\Delta^{-(\gamma+1)/2}(t), \quad G_2(t) = G_y(t)\Delta^{-(\gamma+1)/2}(t), \quad G_3(t) = G_{xy}\Delta^{-(\gamma+1)/2}(t), \quad (3.10) \]

where $\Delta(t) = G_x(t)G_y(t) - G_{xy}^2(t)$ is a positive function on solutions to system (1–3). Note that the behaviour of $\Delta(t)$ is governed by the equation

\[ \Delta'(t) = 2(a(t) + d(t))\Delta(t), \quad (3.11) \]

and a potential energy $E_p(t)$ is connected with $\Delta(t)$ as follows:

\[ E_p(t) = E_p(0)\Delta^{(\gamma+1)/2}(0)\Delta^{-(\gamma+1)/2}(t). \]

To find $G_1, G_2, G_3$, and also the elements of the matrix $A(t)$ we get the system of equations

\[ G_1'(t) = ((1 - \gamma)a(t) - (1 + \gamma)d(t))G_1(t) + 2b(t)G_3(t), \]
\[ G_2'(t) = ((1 - \gamma)d(t) - (1 + \gamma)a(t))G_2(t) + 2c(t)G_3(t), \]
\[ G_3'(t) = c(t)G_1(t) + b(t)G_2(t) - \gamma(a(t) + d(t))G_3(t), \]
\[ a'(t) = -a^2(t) - b(t)c(t) + lc(t) - \mu a(t) + K_1G_2(t), \]
\[ b'(t) = -b(t)(a(t) + d(t)) + ld(t) - \mu b(t) - K_1G_3(t), \]
\[ c'(t) = -c(t)(a(t) + d(t)) - la(t) - \mu c(t) - K_1G_3(t), \]
\[ d'(t) = -d^2(t) - b(t)c(t) + lb(t) - \mu d(t) + K_1G_1(t), \]

with $K_1 = 2^{-1}(E_p(0)\Delta^{(\gamma+1)/2}(0))$.

Proposition 3.1 In the case $\mu = l = 0$ any solution to system (1–3) with a linear profile of velocity, satisfying the condition $\beta$ of Theorem 2.1, is interior.
Proof. Show that at \( \mu = l = 0 \) there is the asymptotics \( A(t) \sim \frac{1}{t}E, t \to \infty \), that is according to Corollary 2.1 the corresponding solution with a linear profile of velocity is interior.

Go to the new variables \( a_1(t) = a(t) - d(t), b_1(t) = b(t) + c(t), c_1(t) = b(t) - c(t), d_1(t) = a(t) + d(t), G_4(t) = G_1(t) + G_2(t), G_5(t) = G_1(t) - G_2(t) \). In variables \( a_1, b_1, c_1, d_1, G_3, G_4, G_5 \) system (3.12) has a form:

\[
\begin{align*}
    a'_1(t) &= -a_1(t)d_1(t) - K_1G_5(t), \\
    b'_1(t) &= -b_1(t)d_1(t) - 2K_1G_3(t), \\
    c'_1(t) &= -c_1(t)d_1(t), \\
    d'_1(t) &= -\frac{1}{2}(a_1^2(t) + d_1^2(t)) - \frac{1}{2}(b_1^2(t) - c_1^2(t)) + K_1G_4(t), \\
    G'_3(t) &= -\gamma d_1(t)G_3(t) + \frac{1}{2}b_1(t)G_4(t) - \frac{1}{2}c_1(t)G_5(t), \\
    G'_4(t) &= -\gamma d_1(t)G_4(t) + a_1(t)G_5(t) + 2b_1(t)G_3(t), \\
    G'_5(t) &= -\gamma d_1(t)G_5(t) + a_1(t)G_4(t) - 2c_1(t)G_3(t).
\end{align*}
\]

Let as \( t \to \infty \) the asymptotics of functions involved in the system be the following:
\( a_1(t) \sim L_1t^{l_1}, b_1(t) \sim L_2t^{l_2}, c_1(t) \sim L_3t^{l_3}, d_1(t) \sim L_4t^{l_4}, G_3(t) \sim Nt^q, G_4(t) \sim M_1t^{p_4}, G_5(t) \sim M_2t^{p_2}, \) where \( L_i, i = 1, 2, 3, M_j, j = 1, 2, N \) are some constants not equal to zero. Note that \( p_2 \leq p_1 \) and in virtue of \( \Delta > 0 \) the estimate \( q \leq p_1 \) holds.

From (3.15) we get immediately that
\[ l_3L_3t^{l_3 - 1} = -L_3L_4t^{l_3 + l_4}, \]
hence \( l_4 = -1, L_4 = -L_3 \).

From (3.13) we have taking it into account
\[ l_1L_1t^{l_1 - 1} = -L_1L_4t^{l_1 - 1} - 2K_1M_2t^{p_2}. \]

The following variants are possible:
\( p_2 \leq l_1 - 1, \) hence \( l_1 = -L_4, \) i.e. \( l_1 = l_3, \)
\( p_2 = l_1 - 1, \) hence
\[ l_1 = -L_4 - \frac{K_1M_2}{L_1} = l_3 - \frac{K_1M_2}{L_1}. \]

From (3.14) we have analogously
\[ l_2L_2t^{l_2 - 1} = -L_2L_4t^{l_2 - 1} - 2K_1Nt^q. \]

There are the variants:
\( q \leq l_2 - 1, \) hence \( l_2 = -L_4, \) i.e. \( l_2 = l_3, \)
\( q = l_2 - 1, \) hence
\[ l_2 = -L_4 - \frac{2K_1N}{L_2} = l_3 - \frac{2K_1N}{L_2}. \]

From (3.16) we get
\[ -L_4t^{-2} = -\frac{1}{2}(L_1^2t^{2l_1} + L_4^2t^{-2}) - \frac{1}{2}(L_2^2t^{2l_2} - L_3^2t^{2l_3}) + K_1M_1t^{p_4}. \]
As soon as \( A = l \) and (or) \( q - l \) then \( l_2 = -l_1 \leq -1 \), \( p_1 \leq -2 \). Really, in the case \( l_3 = -l_4 = -1 - \sqrt{1 + L_3^2} < -1 \) or \( l_3 = -l_4 = -1 - \sqrt{1 + L_3^2 + 2K_1M_1} < -1 \), it contradicts to the supposition.

Now let \( l_2 > -1 \). Then from (3.22) \( l_1 = l_3 \) or (and) \( l_2 = l_3 \). Suppose, for example, that \( l_1 = l_3 \). Then \( p_2 < l_1 - 1 \), from (3.19) it follows that

\[
p_2M_2t^{p_2} = -\gamma L_4M_2t^{p_2} - L_1M_1t^{p_1 + l_1} + 2L_2Nt^{q + l_2},
\]

(3.23)

and therefore \( p_1 + l_1 = q + l_2, l_1 \leq l_2, i.e. l_2 > -1 \). Besides,

\[
L_1M_1 = -2L_2N.
\]

(3.24)

From (3.17) we have

\[
qNt^{q - 1} = -\gamma L_4Nt^{q - 1} + \frac{1}{2}L_2M_1t^{p_1 + l_2} - \frac{1}{2}L_3M_2t^{p_2 + l_3}.
\]

(3.25)

As soon as \( q - 1 \leq p_1 - 1, and p_1 - 1 < l_1 + l_2, then p_1 + l_2 = p_2 + l_3 \leq p_1 + l_3, and therefore \( l_2 \leq l_3, and as l_1 = l_3, then l_1 = l_2 = l_3 \) and \( p_1 = q \). Besides,

\[
L_2M_1 = L_3M_2.
\]

(3.26)

Further, from (3.18) we obtain

\[
p_1M_1t^{p_1 - 1} = -\gamma L_4M_1t^{p_1 - 1} + L_1M_2t^{p_2 + l_3} - 2L_3Nt^{q + l_3}.
\]

(3.27)

As \( q + l_3 > p_1 - 1 \), then \( p_2 + l_1 = q + l_3, p_2 = q \),

\[
L_2M_1 = 2L_3N.
\]

(3.28)

From (3.24) and (3.28) we have \( \frac{M_1}{M_2} = -\frac{L_2}{L_3} \), and from (3.26) we obtain \( \frac{M_1}{M_2} = \frac{L_2}{L_3} \), i.e. \( L_2^2 = -L_3^2, L_2 = L_3 = 0 \) in spite of the supposition.

Suppose now that \( l_2 = l_3 \). Then \( q < l_2 - 1, p_2 \leq l_1 - 1 \). From (3.17) we have that \( q - 1 < l_2 + p_1 \), since \( l_2 > -1 \), and therefore \( l_2 + p_1 = l_3 + p_2, p_1 = p_2, L_2M_1 = L_3M_2 \).

Taking this into account from (3.23) we have that \( p_1 + l_1 = q + l_2 \leq p_1 + l_2, l_1 \leq l_2, L_2M_1 = -2L_2N \).

From (3.27) we get \( p_2 + l_1 = q + l_3 \leq p_1 + l_2, p_2 = q, l_1 = l_2, L_1M_2 = -2L_3N \). In that way, we obtain the contradiction analogous to the previous one.

Now let \( l_3 \leq -1, l_1 > -1 \) and (or) \( l_2 > -1 \). Then from (3.22) it follows that \( p_1 = 2l_1 \) and (or) \( p_1 = 2l_2 (p_1 > -2) \). For example, if \( p_1 = 2l_1 \), then from (3.24) we get \( p_2 - 1 < p_1 + l_1, p_1 + l_1 = q + l_2 \leq p_1 + l_2, l_1 \leq l_2 \). From (3.25) \( p_2 + l_3 \leq p_1 - 1, l_2 + p_1 \geq l_1 + p_1 > p_1 - 1, \)
therefore \( q - 1 = l_2 + p_1 \). But \( q - 1 \leq p_1 - 1 \), therefore \( l_2 \leq -1 \), and \( l_2 \leq -1 \) in spite of the supposition.

If \( p_1 = 2l_2 \), then from (3.25) we get \( l_3 + p_2 \leq p_1 - 1 \), therefore \( p_1 + l_2 \geq p_2 + l_3 \), \( q - 1 = p_1 + l_2 \), \( l_1 \leq l_2 \). But \( q - 1 \leq p_1 - 1 \), therefore \( l_2 \leq -1 \) in spite of the supposition.

So, it remains the unique possibility: \( l_3 \leq -1 \), \( l_1 \leq -1 \) and (or) \( l_2 \leq -1 \). In the case, as follows from (3.22)

\[
l_3 = -1 \pm \sqrt{1 - (\delta_1 L_1^2 + \delta_2 L_2^2 - \delta_3 L_3^2 - 2\delta_4 K_1 M_1)},
\]

(3.29)

where \( \delta_i = 1 \), if \( i = -1 \), and \( \delta_i = 0 \) otherwise, \( i = 1, 2, 3 \), \( \delta_4 = 1 \), if \( p_2 = -2 \) and \( \delta_4 = 0 \) otherwise. Hence

\[
\delta_1 L_1^2 + \delta_2 L_2^2 - \delta_3 L_3^2 - 2\delta_4 K_1 M_1 \leq 1,
\]

(3.30)

if inequality (3.30) is strict, then \( l_3 < -1 \).

We consider this case, that is \( l_3 < -1 \), \( l_1 = -1 \) and (or) \( l_2 = -1 \).

If \( l_1 = -1 \neq l_3 \), then \( p_2 = l_1 - 1 = -2 \). If \( l_2 = -1 \neq l_3 \), then \( q = l_2 - 1 = -2 \). In that way, in any of these cases we have \( p_1 = -2 \).

At \( l_1 = -1 \), \( l_2 < -1 \), \( l_3 < -1 \), from (3.23) we obtain \( p_2 = -\gamma L_4 + \frac{L_1 M_1}{M_2} \),

\[
\frac{L_1 M_1}{M_2} = -(2 + \gamma l_4),
\]

(3.31)

from (3.27) we get \( p_1 = -\gamma L_4 + \frac{L_1 M_2}{M_1} \),

\[
\frac{L_1 M_2}{M_1} = -(2 + \gamma l_3),
\]

(3.32)

and from (3.21)

\[
\frac{K_1 M_2}{L_1} = l_3 + 1.
\]

(3.33)

From (3.31), (3.33) after excluding \( L_1 \) and \( M_2 \) we obtain

\[
K_1 M_1 = -(2 + \gamma l_3)(l_3 + 1).
\]

(3.34)

As \( K_1 \) and \( M_1 \) are positive, and \( l_3 \leq -1 \), then

\[
l_3 < -\frac{2}{\gamma}.
\]

(3.35)

At \( \gamma > 2 \) it goes already to the contradiction.

Further, from (3.32) and (3.33) we have \( \frac{L_1^2}{K_1 M_1} = -\frac{2 + \gamma l_4}{\gamma^2 - \gamma} \), this fact together with (3.34) give

\[
L_1^2 = (2 + \gamma l_3)^2.
\]

(3.36)

From (3.31), (3.34), (3.35) and (3.36) we obtain the inequality

\[
\frac{2}{\gamma} < -1 - \sqrt{1 - (2 + \gamma l_3)^2 - 2(2 + \gamma l_3)(l_3 + 1)},
\]

(3.37)

which cannot be true at \( \gamma > 1 \).
If \( l_1 < -1, l_2 = -1, l_3 < -1 \), then from (3.23) we get \( p_2 = -\gamma L_4 + \frac{2L_2N}{M_2} \),
\[
\frac{2L_2N}{M_2} = -2 - \gamma l_3,
\]
(3.38)

from (3.25) \( q = -\gamma L_4 + \frac{L_2M_1}{2N} \),
\[
\frac{L_2M_1}{2N} = -2 - \gamma l_3,
\]
(3.39)

from (3.27)
\[
p_2 = -\gamma L_4 = \gamma l_3.
\]

In that way, \( l_3 = -\frac{2}{\gamma} \), this fact together with (3.38) or (3.39) contradicts to the fact that \( L_2, N \) and \( M_1 \) are not equal to zero.

If \( l_1 < -1, l_2 = -1, l_3 = -1 \), then from (3.23), (3.25) and (3.27) we get correspondingly
\[
\frac{2L_2N}{M_2} + \frac{L_1M_1}{M_2} = -2 - \gamma l_3,
\]
(3.40)

\[
\frac{L_2M_1}{2N} = -2 - \gamma l_3,
\]
(3.41)

\[
\frac{L_1M_2}{M_1} = -2 - \gamma l_3,
\]
(3.42)

from (3.20), (3.21)
\[
\frac{K_1M_2}{L_1} = \frac{2K_1N}{L_2} = l_3 + 1.
\]
(3.43)

In that way, multiplying (3.40) by (3.42), taking into account (3.43) we get
\[
(2 + \gamma l_3)^2 = L_1^2 + \frac{M_2L_2^2}{M_1},
\]
from (3.41), (3.42), (3.43)
\[
M_1^2 = M_2^2,
\]
(3.45)

that is
\[
(2 + \gamma l_3)^2 = L_1^2 \pm L_2^2.
\]
(3.46)

As above, from (3.31), (3.34), (3.35) and (3.45) we get the inequality (3.37) which cannot hold at \( \gamma > 1 \).

It remains the unique possibility \( l_1 = l_2 = l_3 = -1 \). In the case \( p_2 < -2, q < -2 \) and, as follows from (3.23), \( p_1 = p_2 \) or \( p_1 = q \), that is \( p_1 < -2 \). Therefore, as follows from (3.29)
\[
L_1^2 + L_2^2 - L_3^2 = 1.
\]
(3.47)
where \( \lambda = q + \gamma = p_1 + \gamma \). It follows, in particular, that \( L_2^2 = -\lambda L_1, L_1 L_3 = -\lambda L_2, \lambda^2 = -L_2 L_3 \). Besides, if \( G_1(t) \sim N_1 t^{\rho_1}, G_2(t) \sim N_2 t^{\rho_2}, t \to \infty, \) where \( N_1, N_2 \) are some positive constants, \( q_1 = q_2 = p_1 = q, N_1 = N_2, M_1 = 2N_1, \) and \( N_1^2 \geq N_2^2 \) in virtue of \( G_1(t) G_2(t) - G_3^2(t) > 0 \). In such way, from (3.47) we get that \( L_2^2 \geq \lambda^2, \lambda^2 \geq L_2^2, L_3^2 \geq L_3^2 \), and taking into account (3.46), \( L_1^2 \leq 1, L_2^2 \leq 1, \lambda^2 \leq 1 \). That is if \( \lambda > 1 (\gamma > 3) \), the conditions mentioned in the paragraph cannot hold together.

At last, consider the case \( p_1 = p_2 = q \). Then from (3.23), (3.25), (3.27) we have \( \lambda = \frac{L_1 M_1}{M_2} + \frac{2 L_2 N}{M_2} = \frac{L_2 M_1}{2 M_2} - \frac{L_1 M_1}{M_2} = \frac{M_3}{M_2} - \frac{2 L_2 N}{M_2} \), that is the system of linear homogeneous equations with respect to the variables \( M_1, M_2, N \)

\[
L_1 M_1 - \lambda M_2 + 2 L_2 N = 0, \\
L_2 M_1 - L_3 M_2 - 2 AN = 0, \\
\lambda M_1 - L_1 M_2 + 2 L_3 N = 0.
\]

For the existence of its nontrivial solution the determinant of the system must be equal to zero, i.e.

\[
\lambda L_1^2 - 2 \lambda L_2 L_3 + L_1 L_2^2 + L_1 L_3^2 - \lambda^3 = 0.
\]

Taking into account (3.46), involve the function with respect to the variables \( L_2 \) and \( L_3 \), where \( \lambda \) plays the role of parameter, namely

\[
\Psi_\lambda(L_2, L_3) = \lambda(1 - L_2^2 + L_3^2) - 2 \lambda L_2 L_3 + (L_2^2 + L_3^2)\sqrt{1 - L_2^2 + L_3^2} - \lambda^3.
\]

By the standard methods one can show that the function is not equal to zero at \( \lambda > 1 \), i.e. at \( \gamma > 3 \) (remember, that \( p_1 < -2 \)).

So, it remains to investigate the case \( 1 < \gamma \leq 3 \).

Consider equation (3.11), which can be written as

\[
\Delta'(t) = 2d_1(t)\Delta(t).
\]

If we suppose that \( \Delta(t) \sim const \cdot t^m, t \to \infty, m \) is a constant, then \( m = 2L_4 = 2 \). From (3.11) we can get \( E_p(t) \sim const \cdot t^{1-\gamma}(\to 0), t \to \infty \). Therefore, if we denote \( E \) the quantity of the total energy of the system, then \( E_k(t) = E - E_p(t) \sim E(1 - C_1 t^{1-\gamma}) \), here and further \( C_i \) are some positive constants. But \( E_k(t) \sim C_2 G_1(t) t^{-2}\Delta \sim C_3 G_4 t^{1-\gamma} \), where \( G_k \) is at least one of functions \( G_1, G_2 \) or \( G_3 \). That is \( G_k \sim C_4(1 - C_1 t^{1-\gamma}) t^{1-\gamma} \sim C_5 t^{1-\gamma} \). But then \( 1 - \gamma \leq p_1 < -2, \gamma > 3, \) and \( \gamma \) do not belong to the interval under consideration.

So the proof of the proposition is over.

**Remark 3.1.** One can show more shortly, that in the physical case \( 1 < \gamma \leq 2 \) the situation \( l_1 = l_2 = l_3 = -1 \) if impossible. Taking into account (3.10) we have \( G_1 G_2 - G_3^2 = \Delta^{-\gamma} \sim const \cdot t^{-2\gamma}, t \to \infty \). But the degree of the leading term of the expression \( G_1 G_2 - G_3^2 \) is not greater than \( 2p_1 \), therefore \( p_1 \geq -\gamma \), as \( p_1 < -2 \), then \( \gamma > 2 \).

**Remark 3.2.** Actually \( l_3 = -L_4 = -2, G_1 G_2 - G_3^2 \sim const \cdot t^{-4\gamma}, t \to \infty \).

**Remark 3.3.** In the case \( \mu > 0, l = 0 \) the asymptotic \( A(t) \sim \frac{4}{H} E, t \to \infty \) holds as well, that is according to Corollary 2.1 the corresponding solution with a linear profile of velocity is interior. If \( \mu > 0, l \neq 0 \) it occurs that \( A(t) \sim A_0 \), but the matrix \( A_0 \) under arbitrary initial conditions \( A(0) \) has not to be of form \( \delta E + U_2 \) (see the denotations in the statement of Corollary 2.2), i.e. the Corollary cannot be applied in the last case.

One can write out the conditions of compatibility of initial data of the density and the pressure for the case of solution with arbitrary linear profile of velocity.
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