Numerical solution of a parabolic optimal control problem in economics

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Abstract. In this paper, we construct and study finite difference approximations of optimal control problems arising in the mathematical modeling of certain management and financial problems. For the definiteness we consider two-dimensional in space variables problems, although all the theoretical results remain valid for any dimension. The numerical calculation results are presented for 1D and 2D problems. To approximate the state equations we use implicit (backward Euler) and fractional steps (operator splitting) methods. The solutions of the constructed mesh state equations are strictly positive and keep an analogue of the mass balance condition. We provide the existence results and first order optimality conditions for the corresponding mesh optimal control problems. We use several iterative methods to implement the constructed nonlinear optimization problems and compare their effectiveness.

1. Differential problem

Let $\Omega = (0, l_1) \times (0, l_2)$, $\vec{n} = (n_1, n_2)$ be unit outward normal vector to $\partial \Omega$, $\sigma_i$ are positive constants. We consider the pairs $(m(x, t), \tilde{\alpha}(x, t))$ such that

$$\frac{\partial m}{\partial t} - \sigma_1 \frac{\partial^2 m}{\partial x_1^2} - \sigma_2 \frac{\partial^2 m}{\partial x_2^2} - \text{div}(\tilde{\alpha} m) = 0 \text{ for } (x, t) \in Q_T = \Omega \times (0, T],$$

$$\frac{\partial m}{\partial \vec{n}} = 0 \text{ for } x \in \partial \Omega \times (0, T], \quad m(x, 0) = m_0(x) \geq 0 \text{ for } x \in \Omega,$$

$$\tilde{\alpha} \cdot \vec{n} = 0 \text{ for } (x, t) \in \partial \Omega \times (0, T],$$

and solve the optimal control problem

$$\text{find } \min_{(m, \tilde{\alpha})} \left\{ J(m, \tilde{\alpha}) = \int_{Q_T} \left( g(m) + mA(\tilde{\alpha}) \right) \, dx dt \right\}$$

(3)
on the pairs \((m, \bar{\alpha})\) satisfying (1),(2). The functions \(g(m)\) and \(\mathcal{A}(\bar{\alpha})\) are supposed to satisfy the following assumptions for all \((x, t) \in Q_T\) and \(m \geq 0\):

\[
g(m) = g(x, t, m) \text{ is continuous and } |g(x, t, m)| \leq d_1 + d_2 m^2, \quad d_1, d_2 \geq 0,\]

\[
\mathcal{A}(\bar{\alpha}) = \sum_{i,j=1}^{2} a_{ij}(x, t) \alpha_i \alpha_j \text{ is a positive definite form with continuous } a_{ij}(x, t).
\]

To define a weak solution of problem (3) we use several function spaces (cf., e.g. [4] - [6]): Sobolev space \(H^1(\Omega)\) with the dual \((H^1(\Omega))^*\), the space \(H^{1/2}(\partial \Omega)\) of the traces of the functions from \(H^1(\Omega)\) and its dual \(H^{-1/2}(\partial \Omega)\), the space \(H(\text{div}, \Omega) = \{\bar{\alpha} \in L^2(\Omega), \text{div}\bar{\alpha} \in L^2(\Omega)\}\) and its subspace \(H_0(\text{div}, \Omega)\) of functions satisfying boundary condition \(\bar{\alpha} \cdot n = 0\) in \(H^{-1/2}(\partial \Omega)\).

Let also \(V_{\alpha} = (L^4(Q_T))^2 \cap L^2(0, T; H_0(\text{div}, \Omega))\) and \(W(0, T) = \{u \in L^2(0, T; H^1(\Omega)) : \frac{\partial u}{\partial t} \in L^2(0, T; (H^1(\Omega))^*)\}\).

**Lemma 1.** For given functions \(\bar{\alpha} \in V_{\alpha}\) and \(m_0 \in L^2(\Omega), m_0 \geq 0\), problem (1) has a unique weak solution \(u \in W(0, T), \) which is non-negative: \(m(x, t) \geq 0\) a.e. \(Q_T\), and satisfies mass conservation property: \(\int_{\Omega} m(x, t)dx = \int_{\Omega} m_0(x)dx\) for all \(\bar{\alpha}\) a.e. \(t \in (0, T]\).

**Theorem 1.** Let \(C_\alpha\) be a positive constant and \(K = \{(m, \bar{\alpha}) \in W(0, T) \times V_{\alpha} : (m, \bar{\alpha})\) satisfies (1) and \(\|\bar{\alpha}\|_{V_{\alpha}} \leq C_\alpha\}. For any \(m_0 \in L^2(\Omega), m_0 \geq 0\), there exists a solution to the minimization problem

\[
\min_{(m, \bar{\alpha}) \in K} J(m, \bar{\alpha}).
\]

**Remark 1.** We impose the additional constraint \(\|\bar{\alpha}\|_{V_{\alpha}} \leq C_\alpha\) to ensure the boundedness of the set \(K\), because the objective functional \(J(m, \bar{\alpha})\) is not coercive.

2. Approximation

We approximate the optimal control problem using finite difference method on the uniform meshes \(\omega_x = \{x_{ij} = (ih_1, jh_2), 0 \leq i \leq N_1, 0 \leq j \leq N_2\}\) and \(\omega_t = \{t_j = j\tau, j = 0, \ldots, M, M\tau = T\}.\) Let \(V_x\) be the vector space of mesh functions defined on \(\omega_x\) and \((., .)\) means Euclidian scalar product in \(V_x\). The space \(V^0_{x_1} \times V^0_{x_2} = \{\bar{\alpha} \in V_x \times V_x : \alpha_{1,ij} = 0 \text{ for } i = 0, i = N_1, \quad \alpha_{2,ij} = 0 \text{ for } j = 0, i = N_2\}\) is a discrete counterpart of \(V(\alpha)\).

Let us define the mesh operators \(A_{01}\) and \(A_{11}(\beta)\) for a \(\beta \in V^0_{x_1}\) by the following equalities (for all \(j = 0, 1, \ldots, N_2\)):

\[
(A_{01}m)_{ij} = \frac{\sigma_1}{h_1^2} \begin{cases} 
2m_{0,j} - 2m_{1,j}, & i = 0, \\
-m_{i+1,j} + 2m_{i,j} - m_{i-1,j}, & 1 \leq i \leq N_1 - 1, \\
2m_{N_1,j} - 2m_{N_1-1,j}, & i = N_1,
\end{cases}
\]

\[
(A_{11}(\beta)m)_{ij} = \frac{1}{h_1} \begin{cases} 
-2\beta^{+}_{i+1,j}m_{1,j}, & i = 0, \\
-\beta^{+}_{i+1,j}m_{i+1,j} + |\beta_{i,j}| m_{i,j} - \beta^{-}_{i-1,j}m_{i-1,j}, & 1 \leq i \leq N_1 - 1, \\
-2\beta^{-}_{N_1-1,j}m_{N_1-1,j}, & i = N_1.
\end{cases}
\]

Similarly we define the mesh operators \(A_{02}\) and \(A_{12}(\beta)\) for a \(\beta \in V^0_{x_2}\). Note that the finite difference expression \(A_{01}m = (A_{01} + A_{02})m\) approximates the diffusive part of the state equation, and the finite difference expression \((A_{11}(\alpha_1) + A_{12}(\alpha_2))m\) approximates its advective part presented in the form \(-\text{div}(\bar{\alpha}^+ m) + \text{div}(\bar{\alpha}^- m)\). We consider two variants of the approximation of the state equation:
Let the function $g$ be continuously differentiable with respect to $m$ for $(x, t) \in Q_T$, $m > 0$. We use the notations $\partial_m g = \frac{\partial g}{\partial m}$ and $\partial_r A = \frac{\partial A}{\partial \alpha_r}$, $r = 1, 2$. Note that the state function $m(\alpha)$ defined from the equation (5) or (6) is Lipschitz-continuous, so the objective function $I(m(\alpha), \alpha)$ is also Lipschitz-continuous and has the generalized Clark's gradient (see [7]) $\nabla I(m(\alpha), \alpha)$. Below sign $\alpha = \{-1$ for $\alpha < 0; +1$ for $\alpha > 0; [-1, 1]$ for $\alpha = 0}\}$ is the multivalued function and we use following notations to shorten the writing: $\partial_1 \partial_1 v_{ij} = h_1^{-2}(2v_{ij} - v_{i-1,j} - v_{i+1,j})$, $\partial_1^0 v_{ij} = (2h_1)^{-1}(v_{i+1,j} - v_{i,j})$ and similarly defined $\partial_2 \partial_2 v_{ij}$ and $\partial_2^0 v_{ij}$. 

\begin{align}
\frac{m^k - m^{k-1}}{\tau} + (A_0 + A_{11}(\alpha^k) + A_{12}(\alpha^k)) m^k = 0, \quad k = 1, 2, \ldots, M, \quad (5)
\end{align}

and fractional steps scheme (cf., e.g., [3])

\begin{align}
\frac{w_{1}^{k} - m^{k-1}}{2\tau} + (A_{01} + A_{11}(\alpha^k_1)) w_{1}^{k} = 0,
\frac{w_{2}^{k} - m^{k-1}}{2\tau} + (A_{02} + A_{12}(\alpha^k_2)) w_{2}^{k} = 0,

m^k = 1/2(w_{1}^{k} + w_{2}^{k}), \quad k = 1, 2, \ldots, M, \quad (6)
\end{align}

with initial condition $m^0 = m_0 \in V_x$ for both schemes.

We approximate the objective functional $J(m, \tilde{\alpha})$ by the discrete objective function

\begin{align}
\tau h_1 h_2 I(m, \tilde{\alpha}) = \tau h_1 h_2 \sum_{k=1}^{M} (\rho, g^k(m^k) + m^k A(\tilde{\alpha}^k)), \quad (7)
\end{align}

where $\rho \in V_x$ equals to $\rho = \rho_1 \rho_2$ and the components of vectors $\rho_k$ are $\rho_k,i = \{1$ for $1 \leq i \leq N_k - 1; 1/2$ for $i = 0$ and for $i = N_k\}$.

**Theorem 2.** For any $m_0$ and any $\tilde{\alpha}(t)$ problems (5), (6) have unique solutions $m(t)$ and the following properties hold for all $k = 1, 2, \ldots, M$:  \footnote{$m \gg 0$ means that all components of $m$ are non-negative, while $m > 0$ means that all its components are positive.}

\begin{align}
m_0 \gg 0, (m_0, \rho) > 0 \Rightarrow m^k > 0, w_r^k > 0, \quad r = 1, 2; \quad (8)
\end{align}

\begin{align}(m^k, \rho) = (w_r^k, \rho) = (m_0, \rho), \quad r = 1, 2. \quad (9)
\end{align}

**Theorem 3.** Let $K$ be the set of pairs $(m, \tilde{\alpha})$ satisfying either (5) or (6). For any $m_0 \gg 0$ the mesh optimal control problem

\begin{align}
\text{find} \quad \min_{(m, \tilde{\alpha}) \in K} I(m, \tilde{\alpha}) \quad (10)
\end{align}

has at least one solution.

**Remark 2.** We emphasize that the result of theorem 3 is valid without any additional constraint for the mesh functions $m$ and $\tilde{\alpha}$ although the function $I(m, \tilde{\alpha})$ is not coercive. The properties (8) and (9) of the state function $m$ are crucial for proving this result.

**3. First order optimality conditions and iterative solution methods**

Let the function $g(x, t, m)$ be continuously differentiable with respect to $m$ for $(x, t) \in Q_T$, $m > 0$. We use the notations $\partial_m g = \frac{\partial g}{\partial m}$ and $\partial_r A = \frac{\partial A}{\partial \alpha_r}$, $r = 1, 2$. Note that the state function $m(\alpha)$ defined from the equation (5) or (6) is Lipschitz-continuous, so the objective function $I(m(\alpha), \alpha)$ is also Lipschitz-continuous and has the generalized Clark’s gradient (see [7]) $\nabla I(m(\alpha), \alpha)$. Below sign $\alpha = \{-1$ for $\alpha < 0; +1$ for $\alpha > 0; [-1, 1]$ for $\alpha = 0\}$ is the multivalued function and we use following notations to shorten the writing: $\partial_1 \partial_1 v_{ij} = h_1^{-2}(2v_{ij} - v_{i-1,j} - v_{i+1,j})$, $\partial_1^0 v_{ij} = (2h_1)^{-1}(v_{i+1,j} - v_{i,j})$ and similarly defined $\partial_2 \partial_2 v_{ij}$ and $\partial_2^0 v_{ij}$. 

4. Numerical results

We used iterative parameter \( \mu \) multivalued function sign approximate Hessian of \( \mathbf{I} \)

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The gradient \( \nabla_{m}^{k} \mathbf{I}(m, \tilde{\alpha}) \) with respect to \( m^{k} \) equals to \( \nabla_{m}^{k} \mathbf{I}(m, \tilde{\alpha}) = \rho \Phi^{k} \), where \( \Phi^{k} = \partial_{m} g^{k}(m^{k}) + \mathbf{A}(\alpha^{k}) \).

In the case of implicit approximation (5) of the state equation the adjoint problem is given by the following system of equations:

\[
\frac{\lambda^{k} - \lambda^{k+1}}{\tau} + (A_{0} + A_{11}^{*}(\alpha_{1}^{k}) + A_{12}^{*}(\alpha_{2}^{k})) \lambda^{k} = -\Phi^{k}, \quad k = M, M - 1, \ldots, 1, \tag{11}
\]

with initial condition \( \lambda^{M+1} = 0 \).

The components of generalized gradient \( \nabla_{\alpha} \mathbf{I}(m(\tilde{\alpha}), \tilde{\alpha}) \) equal to

\[
\frac{\partial \mathbf{I}(m(\tilde{\alpha}), \tilde{\alpha})}{\partial \alpha_{r}^{k}} = m^{k} \partial_{r} \mathbf{A}(\tilde{\alpha}^{k}) + m^{k}(\alpha_{r}^{0} \lambda^{k} - \frac{h_{r}}{2} \partial_{r} \lambda^{k} \text{sign} \alpha_{r}^{k}) \tag{12}
\]

for \( r = 1, 2 \) and all \( k = 1, 2, \ldots, M \).

In the case of fractional steps approximation (6) of the state equation the adjoint problem is given by the following system of equations:

\[
\frac{\eta_{1}^{k} - 1/2(\eta_{1}^{k+1} + \eta_{2}^{k+1})}{2\tau} + (A_{01} + A_{11}^{*}(\alpha_{1}^{k})) \eta_{1}^{k} = -\frac{1}{2} \Phi^{k}, \tag{13}
\]

\[
\frac{\eta_{2}^{k} - 1/2(\eta_{1}^{k+1} + \eta_{2}^{k+1})}{2\tau} + (A_{02} + A_{12}^{*}(\alpha_{1}^{k})) \eta_{2}^{k} = -\frac{1}{2} \Phi^{k}
\]

for \( k = M, M - 1, \ldots, 1 \) with the initial conditions \( \eta_{1}^{M+1} = \eta_{2}^{M+1} = 0 \).

The components of generalized gradient \( \nabla_{\alpha} \mathbf{I}(m(\tilde{\alpha}), \tilde{\alpha}) \) equal to

\[
\frac{\partial \mathbf{I}(m(\tilde{\alpha}), \tilde{\alpha})}{\partial \alpha_{r}^{k}} = m^{k} \partial_{r} \mathbf{A}(\tilde{\alpha}^{k}) + w_{r}^{k}(\alpha_{r}^{0} \eta_{r}^{k} - \frac{h_{r}}{2} \partial_{r} \eta_{r}^{k} \text{sign} \alpha_{r}^{k}) \tag{14}
\]

for \( r = 1, 2 \) and all \( k = 1, 2, \ldots, M \).

We use the gradient information provided by the formulas (5), (11), (12) or (6), (13), (14)

for implementing the iterative solution methods of the form

\[
H^{(s)} \frac{\tilde{\alpha}^{(s+1)}}{\theta^{(s)}} + \nabla_{\alpha} \mathbf{I}(m(\tilde{\alpha}^{(s)}), \tilde{\alpha}^{(s)}) = 0, s = 0, 1, \ldots \tag{15}
\]

Above \( \tilde{\alpha}^{(s)} \) is \( s \)-th iteration, \( H^{(s)} \) is a preconditioning matrix and \( \theta^{(s)} \) is the iterative parameter.

We consider two variants of the preconditioners: constructed by quasi-Newton method to approximate Hessian of \( \mathbf{I}(m(\tilde{\alpha}^{(s)}), \tilde{\alpha}^{(s)}) \) and \( H^{(s)} = \mathbf{H} = \text{diag} (pm) \) (called descent method below).

Iterative parameter \( \theta^{(s)} \) is searched by line-search algorithm. In both iterative algorithms the multivalued function sign \( \alpha_{r}^{k} \) at the points \( \alpha_{r}^{k} = 0 \) was taken to be equal 0. As an initial guess we used \( \alpha^{(0)} = 0 \), which was found to be the most reasonable after numerical experiments.

4. Numerical results

We performed numerical experiments for several 1D and 2D problems. Some of them, taken

from the articles [1] and [2], are mathematical models of real problems in the field of finance and management. Other optimal control problems were chosen to demonstrate the applicability of the proposed numerical methods for the described class of objective functionals.
In 1D problems we took state problem \( \frac{\partial m}{\partial t} - 0.07 \frac{\partial^2 m}{\partial x^2} - \frac{\partial}{\partial x} (\alpha m) = 0 \), \( 0 < x < 1 \), \( 0 < t < T \) (as in [1]) and objective functional \( J(m, \alpha) = \int_0^T \int_0^1 (g(m) + m \frac{\alpha^2}{2}) dx dt \) with different functions \( g(m) \). We approximated the state equation in 1D problems by implicit finite difference scheme on the mesh with \( h = 1/64, \tau = 1/64 \) and solved by quasi-Newton and descent methods. In the figures fig.1 - fig.3 we represent the rate of decrease of the objective function in relation to the number of iterations.

In 2D problems state problem was (1) in the domain \( \Omega = (1, 5) \times (1, 3) \) and final time \( T = 1 \), with the coefficients \( \sigma_1 = 0.3 \) and \( \sigma_2 = 0.6 \); the objective functional was (3) with \( A(\alpha) = 2\alpha_1^2 + 4\alpha_2^2 \) and \( g_1(m) = e^{-0.1t}m(0.5(x_1 - x_2)^+ + 0.1 \min\{x_1; x_2\} + \frac{5x_1 - 0.5x_2^2}{10 + 0.1m}) \) (as in [1]) and \( g_2(m) = m^{3/2} \). We approximated the state equation in 2D problems by fractional steps.
Figure 3. 1D problem with $g(m) = m^{1/2}$; decay of the objective function using quasi-Newton method (left) and descent method (right)

Figure 4. 2D problem with $g(m) = g_1$ (left) and $g(m) = m^{3/2}$ (right); decay of the objective function using quasi-Newton method

finite difference scheme on the mesh with $h_1 = h_2 = 1/16, \tau = 1/16$ and solved by quasi-Newton method. In the figure fig.4 we represent the rate of decrease of the objective function in relation to the number of iterations.

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