An Extension of Birkhoff’s Theorem to a Class of 2-d Gravity Theories Containing Black Holes

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Abstract

A class of 2-dimensional models including 2-d dilaton gravity, spherically symmetric reduction of $d$-dimensional Einstein gravity and other related theories are classically analyzed. The general analytic solutions in the absence of matter fields other than a U(1) gauge field are obtained under a new gauge choice and recast in the conventional conformal gauge. These solutions imply that Birkhoff’s theorem, originally applied to spherically symmetric 4-d Einstein gravity, can be applied to all models we consider. Some issues related to the coupling of massless scalar fields and the quantization are briefly discussed.
1. Introduction

The quantum physics of black holes has always been an exciting problem with a lot of controversies. In recent years, some of the major intricacies related to the information loss problem via the Hawking radiation process have been addressed within the framework of two-dimensional dilaton gravity model, proposed by Callan, Giddings, Harvey and Strominger (CGHS) \[3\]. This model is believed to capture many essential features of the conventional four dimensional black hole theory and, at the same time, is analytically manageable to allow more detailed calculations. Dilaton gravity is a special case of a more general family of two-dimensional models that in particular includes the spherically symmetric reduction of Einstein gravity. From a phenomenological point of view, it is therefore important to understand to what extent the results about dilaton gravity that have been obtained so far are independent of the specific choice of parameters.

The purpose of this work is to understand the relation between the different two-dimensional models at the classical level. We find that they can be given a unified treatment classically and, in the absence of matter fields other than a U(1) gauge field, all of them satisfies Birkhoff’s theorem, originally applied to spherically symmetric Einstein gravity. This theorem is a direct manifestation of the fact that there are no propagating degrees of freedom in these theories. We establish this result by getting the general solutions of the equations of motion under a particular choice of gauge. This gauge, generalized from an approach found in Ref.\[7\], enables one to obtain explicit general solutions. Following the detailed analysis of these solutions, we will discuss the issues of introducing matter fields and the quantization of models we consider.

2. The Extension of Birkhoff’s Theorem

The model we consider is given by the following action;

\[
I = \int d^2x \sqrt{-g}e^{-2\phi}[R + \gamma g^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi + V(e^{-\phi}) - \frac{1}{4}e^{\phi}F^2].
\]  

\[1\]
$R$ denotes the scalar curvature and $F$, the curvature 2-form for an Abelian gauge field. The field $\phi$ is the dilaton field and the parameters $\gamma$ and $\epsilon$ above are assumed to be arbitrary real parameters. $V(e^{-\phi})$ is a generic real function of the dilaton field. The action introduced above can be considered as a 2-d target space effective action resulting from string theory. In view of this aspect, the loop corrections from string theory can give non-trivial contribution to $V$. Eq.(1) also contains many models of interest as its special cases. For example, setting $\gamma = 2$ and $V(x) = constant/x^2$ gives the spherically symmetric reduction of 4-d Einstein gravity coupled with electromagnetic fields. As is well known, the static solution of this case is given by Reissner-Nordström blackholes and the dilaton field can be related to the usual radial coordinate of the metric. Another important example is $\gamma = 4$ and $V = constant$ case that corresponds to two-dimensional dilaton gravity discussed in Ref.[3]. As can be seen in the calculations in this section, the interpolating theories connecting these two important theories continuously show very similar behavior as far as the classical analysis is concerned. A model proposed by Teitelboim [5] can also be described by the action, if we set $\gamma = 0$ and $V = constant$.

The static solutions of Eq.(1) were obtained in Ref.[2]. When it comes to spherically symmetric four dimensional gravity in the absence of matter fields, we have more general results, namely Birkhoff’s theorem. This theorem essentially asserts that the general solutions of this problem are just static Schwarzschild solution in each local coordinate patch. In this note we wish to extend this result to all models described by Eq.(1). To this end we have to solve the equations of motion

$$D_\alpha D_\beta \Omega - g_{\alpha\beta} D \cdot D \Omega + \frac{\gamma}{8} (g_{\alpha\beta} \frac{(D\Omega)^2}{\Omega} - 2 \frac{D_\alpha \Omega D_\beta \Omega}{\Omega}) + \frac{1}{2} g_{\alpha\beta} \Omega V(\Omega) = 0$$

$$R + \frac{\gamma}{4} \left( \frac{(D\Omega)^2}{\Omega^2} - 2 \frac{D \cdot D \Omega}{\Omega} \right) + \frac{d}{d\Omega} (V(\Omega)\Omega) - \frac{1}{4} (1 - \frac{\epsilon}{2}) \Omega^{-\epsilon/2} F^2 = 0,$$

where we define $\Omega = \exp(-2\phi)$ and $D$ denotes the covariant derivative. The first of the above equations is obtained by varying the action with respect to the metric tensor and the second, with respect to the dilaton field.

The well known conformal gauge is not so convenient for the classical analysis we are interested in. Instead we choose a different gauge where the metric tensor is of the form

$$g_{\alpha\beta} = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}.$$
Furthermore, we choose coordinates in such a way that $x^1 \equiv r \equiv \exp(-\phi)$ and require $[\partial_0, \partial_1] = 0$. This procedure defines a coordinate system up to the diffeomorphisms of the time coordinate. The scalar curvature in this coordinate system has the following form:

$$\sqrt{-g}R = 2[\partial_0(\frac{\partial_0 \beta}{\alpha}) - \partial_1(\frac{\partial_1 \alpha}{\beta})].$$

The resulting Christoffel symbols are calculated to be

$$\Gamma^0_{11} = \frac{\beta \partial_0 \beta}{\alpha^2}, \quad \Gamma^0_{01} = \frac{\partial_1 \alpha}{\alpha}, \quad \Gamma^0_{00} = \frac{\partial_0 \alpha}{\alpha},$$

$$\Gamma^1_{11} = \frac{\partial_1 \beta}{\beta}, \quad \Gamma^1_{01} = \frac{\partial_0 \beta}{\beta}, \quad \Gamma^1_{00} = \frac{\alpha \partial_1 \alpha}{\beta^2},$$

which is the standard result for a diagonal metric. We note that in case of the 4-d spherically symmetric Einstein gravity $x^1$ reduces to the usual radial coordinate of the Schwarzschild geometry.

Modulo total derivatives, the action in this gauge is simplified to yield

$$I = \int d^2 x [4r \frac{\partial_1 \alpha}{\beta} + \gamma \frac{\alpha}{\beta} + r^2 V(r) \alpha \beta + r^{2-\epsilon} \frac{f^2}{2\alpha \beta}],$$

where the function $f \equiv \partial_0 A_r - \partial_r A_0$ and satisfies $F^2 = -\frac{2}{\alpha \beta} f^2$. We immediately see that the dynamics of the original action looks greatly simplified in this gauge. First of all, the derivatives with respect to $x^0$ that, in spherically symmetric 4-d gravity, corresponds to the time derivatives appear only as total derivatives and, consequently, can be completely thrown away. Secondly, since there is only one linear first-order $r$-derivative, the resulting equations of motion in this gauge are first order differential equations, not the generic second order differential equations. The equations of motions from the action (7) are

$$4r \frac{\partial_1 \alpha}{\beta^2} + \gamma \frac{\alpha}{\beta^2} - r^2 V(r) \alpha + r^{2-\epsilon} \frac{f^2}{2\alpha \beta^2} = 0$$

$$4r \frac{\partial_1 \beta}{\beta^2} + (\gamma - 4) \frac{1}{\beta} + r^2 V(r) \beta - r^{2-\epsilon} \frac{f^2}{2\alpha^2 \beta} = 0,$$

along with the equations for gauge fields,

$$\partial_0 \left( r^{2-\epsilon} \frac{f}{\alpha \beta} \right) = 0$$
\[ \partial_1 \left( \frac{r^{2-\epsilon}}{\alpha \beta} f \right) = 0. \] (11)

The equations for the abelian gauge field can be solved to give

\[ f = \alpha \beta r^{-2+\epsilon} f_0, \] (12)

where \( f_0 \) is a constant. The trivial nature of the solution is understandable since the purely radial motion of the system can not generate physical (transversal) polarization states of photons.

The gauge constraints resulting from the choice of our gauge follow from Eqs. (2) and (3). They are

\[ \frac{1}{\sqrt{-g}} \frac{\delta I}{\delta \Omega} = 0 \] (13)

and

\[ \frac{1}{\sqrt{-g}} \frac{\delta I}{\delta g_{01}} = 0, \] (14)

where \( I \) represents the original action (1). Using Christoffel symbols (7) to explicitly write down covariant derivatives, we obtain

\[ \partial_0 \partial_1 \Omega - \frac{\partial_1 \alpha}{\alpha} \partial_0 \Omega - \frac{\partial_0 \beta}{\beta} \partial_1 \Omega - \frac{\gamma}{4} \partial_0 \Omega \partial_1 \Omega = 0 \] (15)

from Eq. (14). From Eq. (13), we obtain

\[ \frac{2}{\alpha \beta} (\partial_0 (\frac{\partial_0 \beta}{\alpha}) - \partial_1 (\frac{\partial_1 \alpha}{\beta})) + \frac{d}{d\Omega} (V(\Omega) \Omega) - \frac{1}{4} (1 - \epsilon/2) \Omega^{-\epsilon/2} F^2 \] (16)

\[ -\frac{\gamma}{4} \Omega^2 \left( \frac{(\partial_0 \Omega)^2}{\alpha^2} - \frac{(\partial_1 \Omega)^2}{\beta^2} \right) \]

\[ + \frac{2}{\Omega} \left\{ - \frac{\partial_2 \Omega}{\alpha^2} + \frac{\partial_1 \Omega}{\beta^2} + \frac{1}{\alpha^2} (\frac{\partial_0 \alpha}{\alpha} - \frac{\partial_0 \beta}{\beta}) \right\} \partial_0 \Omega + \frac{1}{\beta^2} \left( \frac{\partial_1 \alpha}{\alpha} - \frac{\partial_1 \beta}{\beta} \right) \partial_1 \Omega \] = 0.

The crucial conditions from the definition of our coordinate systems are \( \partial_1 \Omega = 2r \) and \( \partial_0 \Omega = 0 \). Plugging these conditions into Eq. (13) and Eq. (16) yields

\[ 2r \frac{\partial_0 \beta}{\beta} = 0 \] (17)

and

\[ \frac{2}{\alpha \beta} (\partial_0 (\frac{\partial_0 \beta}{\alpha}) - \partial_1 (\frac{\partial_1 \alpha}{\beta})) - \frac{\gamma}{\Omega^{1/2} \beta^2} \left( \frac{\partial_1 \alpha}{\alpha} - \frac{\partial_1 \beta}{\beta} \right) \] (18)
\[ + \frac{d}{d\Omega}(V(\Omega)\Omega) - \frac{1}{4}(1 - \frac{\epsilon}{2})\Omega^{1/2}F^2 = 0. \]

Since we are engaged in classical analysis, Eq. (18) can be further simplified using the equations of motion (8) and (9) along with the classical solution of \( F \) to remove \( r \)-derivatives of \( \alpha \) and \( \beta \). The result of this removal is

\[ \frac{1}{\alpha \beta} \partial_0 \left( \frac{\beta}{\alpha} \left( \partial_0 \beta \right) \right) = 0. \quad (19) \]

We now find that the constraint reduces to

\[ \partial_0 \beta = 0 \quad (20) \]

from Eq. (17) and this automatically implies that Eq. (19) is satisfied.

Eqs. (8) and (11) can now be solved under the simple constraint (20). The general solutions are

\[ \beta^2 = \frac{2r^{2-\gamma/2}}{C + \int_{r_0}^{r} x^{(6-\gamma)/2}(V(x) - f_0^2 x^{\epsilon-4})dx} \]

\[ \alpha^2 = T^2(x^0) \frac{r^{-\gamma/2}}{2} \left( C + \int_{r_0}^{r} x^{(6-\gamma)/2}(V(x) - f_0^2 x^{\epsilon-4})dx \right), \quad (21) \]

where \( T(x^0) \) is an arbitrary function depending only on \( x^0 \). \( C \) and \( r_0 \) are arbitrary constants of integration. The presence of the arbitrary function \( T \) originates from the arbitrariness involved in our definition of \((x^0, r)\) coordinates, namely, the possible diffeomorphisms of the time coordinate. This arbitrariness can be fixed by absorbing \( T \) into \( x^0 \) by properly redefining it. We note that the constant \( C \) could have involved truly non-trivial \( x^0 \) dependence if there were no constraint (20).

We can derive the static solutions of Eq. (1) assuming all the relevant fields involved depends only on a single variable, say \( r \). The results of this calculation are the same as Eqs. (21) and (22) with the additional fact that \( T \) is strictly a constant. Thus, we have proven a general result; the general classical solutions of the action (1) are same as the static solutions of the same action in each properly defined local coordinate patch. In other words, the classical dynamics of the gravity models coupled with an Abelian gauge field in the absence of other matter fields is locally frozen.
3. Aspects of the Analytical Solutions

To better understand the connection between the results in the gauge of the previous section and the results in the conformal gauge, it is desirable to recast our solutions in conformal gauge. In terms of conformal coordinates \((x^+, x^-)\), the metric should be written as

\[
ds^2 = -\alpha^2(dx^0)^2 + \beta^2(dr)^2 = -e^\rho dx^+ dx^-.
\] (23)

This condition is equivalent to two sets of two partial differential equations

\[
\beta \partial_+ r = \pm \alpha \partial_+ x^0
\] (24)

\[
\beta \partial_- r = \mp \alpha \partial_- x^0,
\] (25)

along with an equation for the conformal factor \(\rho\)

\[
e^\rho = -4\beta^2 \partial_+ r \partial_- r.
\] (26)

We require \(\partial_+ r \partial_- r < 0\) to fix the orientation and make a choice of upper signs. The form of solutions (24) and (25) shows that the factors containing the coordinate \(r\) and the factors containing \(x^0\) are simply multiplied. Therefore, by a proper field redefinition, we can reduce the above equations into the Laplacian equations in flat space for two redefined fields. The general solutions of the PDEs obtained in this way are

\[
\int_{r_1}^{r} \frac{2rdr}{C + \int_{r_0}^{r} x(6-\gamma)/2(V(x) - f_0^2 x^{-4})dx} = X^+(x^+) - X^-(x^-)
\] (27)

\[
\int_{t_0}^{t_0} T(t)dt = X^+(x^+) + X^-(x^-),
\] (28)

where \(r_1\) and \(t_0\) are constants of integration and \(X^\pm\) are arbitrary chiral fields depending only on \(x^\pm\), respectively. Using these solutions, the conformal factor can be calculated to yield

\[
e^\rho = 2r^{-\gamma/2}[C + \int_{r_0}^{r} x(6-\gamma)/2(V(x) - f_0^2 x^{-4})dx] \partial_+ X^+ \partial_- X^-.
\] (29)

Combined with \(r = e^{-\phi}\), Eqs. (27) and (29) implicitly determine \(\rho\) and \(\phi\) via one left-moving and one right-moving field. Particularly, in the case of 2-d dilaton gravity, these relations get considerably simplified. If we set \(\gamma = 4\), \(V(r) = 4\lambda^2\),
\( C = -2\lambda^2 M, \; f_0 = 0 \) and conformally transform \( \exp(\pm 2\lambda^2 X^\pm) \to \pm X^\pm \), then the above solutions become

\begin{align}
\alpha^2 &= T^2(x^0)\left(1 - \frac{2M}{r} + \frac{f_0^2/2}{r^2}\right) \\
\beta^2 &= (1 - \frac{2M}{r} + \frac{f_0^2/2}{r^2})^{-1},
\end{align}

which are the familiar linear dilaton vacuum solution of CGHS model, where \( M \) is the mass of the resulting black hole. In retrospect, this shows why the choice of conformal gauge renders an analytically tractable approach in 2-d dilaton gravity, while this kind of approach is more difficult in other cases, the complication being the difficulty of the identification of chiral fields.

In the spherically symmetric reduction of 4-d Einstein gravity case, i.e., \( \gamma = 2, \; V(x) = 2/x^2, \; \epsilon = 0, \) and \( C = -4M \), our solutions (21) and (22) reduce to

\begin{align}
\alpha^2 &= T^2(x^0)\left(C + \frac{d-2}{2(d-3)}\mu(d)r^{2\frac{d-3}{d-2}}\right) \\
\beta^2 &= \left(C + \frac{d-2}{2(d-3)}\mu(d)r^{2\frac{d-3}{d-2}}\right)^{-1},
\end{align}

which represents the Reissner-Nordström black hole with mass \( M \) and electric charge \( f_0/\sqrt{2} \). The \( d \)-dimensional Einstein gravity with the symmetry group \( SO(d-1) \) that is the \( d \)-dimensional generalization of the 4-d spherically symmetric case can also be described by our action. After the integration over angular coordinates and the rescaling of the dilaton field which can be related to the radial coordinate, we find that the effective action for these cases corresponds to Eq. (I) with \( \gamma = 4\frac{d-3}{d-2} \) and \( V(x) = \mu(d)/x^{\frac{4}{d-2}}. \) Here, \( d \) is the dimension of the space-time and \( \mu(d) \) is a number depending on \( d \). If we set \( f_0 = 0 \), for simplicity, we get the following general solutions;

\begin{align}
\alpha^2 &= T^2(x^0)\frac{C + \frac{d-2}{2(d-3)}\mu(d)r^{2\frac{d-3}{d-2}}}{2r^{\frac{d-2}{d-2}}} \\
\beta^2 &= \frac{2r^{\frac{d-2}{d-2}}}{C + \frac{d-2}{2(d-3)}\mu(d)r^{2\frac{d-3}{d-2}}}.
\end{align}

If \( d = 4 \), this solution becomes the Schwarzschild metric.

An interesting observation is that the 2-d dilaton gravity case is same as the \( d = \infty \) limit of \( d \)-dimensional spherically symmetric Einstein gravity. In this limit, we have \( \gamma \to 4 \) along with \( V(x) \to \mu(\infty) = constant \) that we can set \( \mu(\infty) = 4\lambda^2 \). Taking \( C = -2\lambda^2 M \) as in the previous consideration of the 2-d dilaton gravity, we
can write the 2-d dilaton black hole metric with mass $M$ in a form similar to the 4-d Schwarzschild metric,
\[ ds^2 = -\lambda^2 (1 - \frac{M}{r^2}) T^2 (x^0)^2 (dx^0)^2 + \frac{1}{1 - \frac{M}{r^{d-2}}} \lambda^2 r^2 dr^2, \]  
which was obtained from Eq.(34) by taking the $d = \infty$ limit. At the level of the classical analysis, this shows that CGHS model can be identified with the leading order theory of the spherically symmetric reduction of the finite dimensional Einstein gravity in a $1/d$ expansion. In analogy to the Schwarzschild metric, 2-d dilaton black hole metric can also be maximally extended to form a space-time similar to that of the maximally extended Kruskal space-time. In the limit of the strong coupling regime where $r = e^{-\phi} \to 0$, i.e., $\phi \to +\infty$, there exists a curvature singularity as can be explicitly seen from Eq.(36).

4. Discussion

The analysis in this work provided us with the complete classical solutions for many 2 dimensional models containing black holes. The questions to pursue from now on should be at least two-fold; the coupling of matter fields other than a U(1) gauge field should be considered to study the truly dynamical formulation and Hawking evaporation of black holes. The quantization of the action considered here is another issue.

Birkhoff’s theorem, extended in this work, asserts there is no truly dynamical evolution of the system described by Eq.(1). Consequently, we can say that each and every distinct classical solution of the action represents either a distinct black hole state or a vacuum state. Considering the no-hair theorem that is valid classically, we see that any additional matter coupling not shown in Eq.(1) should produce dynamic formation of black hole(s) from incoming matter fields and subsequent scattering into black hole(s) and/or outgoing matter fields. To get a proper understanding of this complicated process, it is imperative to consider additional matter couplings. The gauge choice made in this note seems advantageous even when we include non-trivial matter fields, e.g., massless scalar fields, at least classically. To be specific, we can
add to the original action
\[ -\frac{1}{2} \int d^2x \sqrt{-g} e^{-2\delta\phi} g^{\alpha\beta} \partial_\alpha f \partial_\beta f, \quad (37) \]
the action for a massless scalar field where \( \delta \) is a real number. The resulting equations of motion other than the one for the scalar field are
\[
\begin{align*}
\frac{\partial_1 \alpha}{\alpha} + \frac{\gamma}{8\Omega} - \frac{\beta^2}{2} \Omega V(\Omega) - \frac{\Omega^4}{4} \left\{ \frac{\beta^2}{\alpha^2} (\partial_0 f)^2 + (\partial_1 f)^2 \right\} &= 0, \\
\frac{\partial_1 \beta}{\beta} + \frac{\gamma}{8\Omega} + \frac{\beta^2}{2} \Omega V(\Omega) - \frac{\Omega^4}{4} \left\{ \frac{\beta^2}{\alpha^2} (\partial_0 f)^2 + (\partial_1 f)^2 \right\} &= 0, 
\end{align*}
(38) \quad (39)
\]
where, for brevity, we did not include the U(1) gauge field and used \((x^0, \Omega) = (x^0, r^2)\) coordinates. The main virtue of this gauge choice is that the resulting gauge constraint is very simple. The gauge constraint \((14)\) reduces to
\[
\frac{\partial_0 \beta}{\beta} - \frac{1}{2} \Omega^4 \partial_0 f \partial_1 f = 0, \quad (40)
\]
which shows that the time variation in the mass of a black hole results from the space and time variation of incoming and outgoing scalar fields. Just as in the case considered in this work, the remaining gauge constraint \((13)\) can be written as
\[
\frac{2}{\alpha \beta} \partial_0 \left\{ \frac{\beta}{\alpha} \left( \frac{\partial_0 \beta}{\beta} - \frac{1}{2} \Omega^4 \partial_0 f \partial_1 f \right) \right\} = 0
\]
(41)
after somewhat lengthy calculations. Thus, imposing Eq.\((40)\) automatically implies the other constraint. These relatively less complicated sets of first order partial differential equations may provide us with some further analytical understanding of the classical picture of black hole formation, clarifying the issue of gravitational back reaction.

One benefit of our gauge choice other than the simplification of classical analysis is the existence of a natural time-like coordinate \(x^0\). This can be utilized to define a natural Hamiltonian structure which, in principle, can be the basis of canonical quantization. The result of this quantization will largely be topological in the absence of additional matter fields, due to the lack of local propagating degrees of freedom. As to the quantization including additional matter fields, we also note that our solutions in conformal gauge can be very useful. In the context of two-dimensional dilaton gravity, an explicit quantization of the theory was given Ref.\([8]\). The fact that in this case the most general form of the classical solutions are known
was crucial in their analysis. The formal similarity of the class of solutions described by Eqs. (29) and (27) suggests the possibility that their analysis can be generalized to some models described by our action. The main interest, from a phenomenological viewpoint, lies in the quantization of spherically symmetric 4-d Einstein gravity. In this case there is a natural (reflecting) boundary \( r = 0 \) where left-moving and right-moving chiral fields can interact, while in case of dilaton gravity this boundary had to be introduced by hand. This distinction between the classical theories may lead to important qualitative differences at the quantum level. We plan to address the quantization of the models described by (1) in a future publication.

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