Traversable Wormholes
in Geometries of Charged Shells

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Abstract

We construct a static axisymmetric wormhole from the gravitational field of two charged shells which are kept in equilibrium by their electromagnetic repulsion. For large separations the exterior tends to the Majumdar-Papapetrou spacetime of two charged particles. The interior of the wormhole is a Reissner-Nordström black hole matching to the two shells. The wormhole is traversable and connects to the same asymptotics without violation of energy conditions. However, every point in the Majumdar-Papapetrou region lies on a closed timelike curve.

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I. INTRODUCTION

When discussing the Reissner-Nordström - solution, Hawking and Ellis in their book [1] mention "the intriguing possibility to travel to other universes passing through the wormholes made by charges. Unfortunately it seems that one would not be able to get back again to our universe..."

Carter, in 1966 [2], addressed this question by saying "Since the complete manifold consists of an infinite chain of universes connected successively in time by wormholes, one has apparently two possibilities: either to regard each particle as being connected to an infinite set of distinct universes, or else to devise some scheme whereby some of these universes are identified with other."

The latter possibility was also considered by Morris and Thorne [3], Visser [4] and Hawking [5]. However, no explicit identification was given. The problem is not as trivial as one may think at first sight, since a solution where the wormhole connects to the same asymptotic region can at best be axially (rather than spherically) symmetric.

In this note we give an explicit construction of a charged wormhole which does connect to the same asymptotic region. The exterior spacetime is that of two shells held in static equilibrium by their electric repulsion while the interior is a Reissner-Nordström black hole. The transition between the exterior and interior spacetime is achieved by introducing two shells of charged matter. The matching can be made exact by making use of the image method in analogy to the construction given by R.W. Lindquist [6] who considered the time-symmetric initial value problem for Einstein-Rosen manifolds.

Not only is the wormhole traversable and allows one to travel back to the same universe, but closed timelike curves exist. Moreover, no violation of energy conditions is necessary. However, it faces the problem of the instability of the inner (Cauchy) horizon.

A similar construction for uncharged shells held in static equilibrium by strings has recently be given by W. Israel and the authors [7], but at the price of introducing exotic matter.
II. MAJUMDAR-PAPAPETROU SOLUTION

The exterior field of a system of charged bodies which are held in equilibrium by a balance between electrostatic repulsion and gravitational attraction is given by the Majumdar-Papapetrou solution of Einstein-Maxwell equations. In cartesian coordinates this solution has the form

\[ ds^2 = -V^{-2}dT^2 + V^2(dx^2 + dy^2 + dz^2) \]  

(1)

where the function \( V(x,y,z) \) satisfies Laplace’s equation,

\[ \Delta V(x, y, z) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x, y, z) = 0. \]  

(2)

The fact that for the Majumdar-Papapetrou metric (1) Einstein’s equations reduce to the Laplace equation (2) offers the possibility of constructing an exact wormhole solution. We cut out from the Majumdar-Papapetrou spacetime the interior of two (non-intersecting) spheres \( S_i^+ \) \((i=1,2)\) and require the potential function \( V \) to be constant on the surfaces. The problem is analogous to that of finding the electric potential outside two charged metal spheres. Such solutions can be found for any location and radii and arbitrary values of the potential on the spheres [8].

In what follows we give an explicit construction for the symmetric two-body problem. We choose the z-axis to point along the line of symmetry joining the two spheres \( S_i^+ \) with radii \( R \) and center them at \( z = \pm d_1 \). Moreover, we fix the value of the potential function on the spheres to

\[ V|_{S_i^+} = V_0 = 1 + \frac{m_1}{R}. \]  

(3)

This choice ensures that for large distances of the two spheres the field is that of two particles with \textit{mass} = \textit{charge} = \( m_1 \). The image masses \( m_n \) to make \( V \) constant on \( S_i^+ \) have to be located on the z-axis at \( z = \pm d_n \), where

\[ d_n = d_1 - \frac{R^2}{d_1 + d_{n-1}} \quad (n > 1) \]  

(4)

\[ m_n = -\frac{m_{n-1}R}{d_1 + d_{n-1}} \quad (n > 1) \]  

(5)
The resulting expression for the metric potential $V(\vec{x})$, $\vec{x} = (x, y, z)$, is

$$V(\vec{x}) = 1 + \sum_{n=1}^{\infty} \frac{m_n}{|\vec{x} + \vec{d}_n|}. \tag{6}$$

Following the work of R. Lindquist [6] we define a new pair of parameters $c, \mu_0$ by

$$R = \frac{c}{\sinh \mu_0} \quad d_1 = c \coth \mu_0 \tag{7}$$

which allow us to solve the recursion formulas (4) and (5):

$$d_n = c \coth n\mu_0 \quad (n \geq 1) \quad m_n = (-1)^n \frac{\sinh \mu_0}{\sinh n\mu_0} m_1 \quad (n \geq 1) \tag{8}$$

We also introduce bispherical coordinates

$$\coth \mu = (x^2 + y^2 + z^2 + c^2)/(2cz) \quad \cot \eta = (x^2 + y^2 + z^2 - c^2)/(2c\sqrt{x^2 + y^2}) \quad \cot \varphi = \frac{x}{y}. \tag{10}$$

In this adapted coordinate system the equation for the two throats $S^+_1$ simply become $\mu = \pm \mu_0$ and one computes

$$\frac{m_n}{|\vec{x} + \vec{d}_n|} = (-1)^{n+1} \frac{\sqrt{\cosh \mu - \cos \eta}}{\sqrt{\cosh (\mu + 2n\mu_0) - \cos \eta}} \frac{m_1}{R}. \tag{11}$$

The metric (11) takes the form

$$ds^2 = -V^{-2}dT^2 + V^2 \frac{c^2}{(\cosh \mu - \cos \eta)^2} (d\mu^2 + d\eta^2 + \sin^2 \eta d\varphi^2), \tag{12}$$

and the solution of our boundary value problem $V(\pm \mu_0, \eta, \varphi) = V_0$ becomes

$$V(\mu, \eta, \varphi) = 1 + \frac{m_1}{R} \left[ 1 + \frac{1}{\sqrt{\cosh \mu - \cos \eta}} \sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{\cosh (\mu + 2n\mu_0) - \cos \eta}} \right]. \tag{13}$$
III. CONSTRUCTION OF THE WORMHOLE GEOMETRY

Having found the exterior solution we match a Reissner-Nordström black hole to the interior of the spheres. This requires the introduction of two infinitely thin shells of charged matter at the transition surfaces $S_i^\pm$. The wormhole is obtained by gluing different asymptotic regions of one and the same extended Reissner-Nordström spacetime to the surfaces $\mu = \pm \mu_0$. Hence, the metric interior to the shells has the form

$$ds^2 = -f(r_-)dT_-^2 + \frac{dr_-^2}{f(r_-)} + r_-^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

where

$$f(r_-) = 1 - \frac{2m}{r_-} + \frac{e^2}{r_-^2} \quad (|e| \leq m)$$

Let us take the timelike surface $S_1^-$ defined by $r_- = RV_0$ lying outside the event horizon in one asymptotically flat region, say region I (see Fig.1), of the given Reissner-Nordström spacetime and cut off the asymptotically flat part. In order to match the surface $S_1^-$ to the exterior region at the surface $S_1^+$ we have to determine the identification of points on $S_1^+$ and $S_1^-$. Therefore we introduce a spherical polar coordinate system $(T_+, r_+, \vartheta, \varphi)$ centered at $z = -d_1$ such that sphere $S_1^+$ is given by $r_+ = R$. (Note that we have the choice of taking a right or left handed coordinate system pointing to the positive or negative direction of the z-axis.) The metric (14) of the exterior region takes the form

$$ds^2 = -V^{-2}dT_+^2 + V^2(dr_+^2 + r_+^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2))$$

We have not distinguished angular components of the coordinate patches (14) and (16) because now we identify points with equal values of $\vartheta, \varphi$ and equal proper time $\tau$ on the shells $S_1^+$ and $S_1^-$, $S_1^+ \equiv S_1^- \equiv S_1$. Hence, the induced metric on the shell $S_1$ is

$$ds^2|_{S_1} = -d\tau^2 + (RV_0)^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$

To construct a traversable wormhole we repeat this procedure for a surface $S_2^-$ in the asymptotic region II of the given extended Reissner-Nordström spacetime lying
in the causal future of $S_1^-$. This leads to a second shell $S_2$. Although the coordinate system \((\text{III})\) does not cover regions I and II, it is not necessary to explicitly write down different coordinates which cover the whole spacetime. By symmetry all results such as energy density and pressures of the shells are valid for both.

### IV. ENERGY DENSITY AND PRESSURES OF THE SHELLS

Consider shell $S_1$. Denoting by $n$ the unit normal to $S_1$ (directed towards the Majumdar-Papapetrou region), and by $u = d/d\tau$ the shell’s velocity, the components of these vectors with respect to the different coordinate systems \((\text{IV})\) and \((\text{II})\) are given by

\[
\begin{align*}
    u_+^\alpha &= V_0 \left(1, 0, 0, 0\right) \quad n_+^\alpha = \frac{1}{V_0} \left(0, 1, 0, 0\right) \\
    u_-^\alpha &= \frac{1}{\sqrt{f(RV_0)}} \left(1, 0, 0, 0\right) \quad n_-^\alpha = \sqrt{f(RV_0)} \left(0, 1, 0, 0\right)
\end{align*}
\]

(18)

(19)

Applying the usual formalism of thin shells the stress $p$ and surface energy density $\sigma$ of the shell can be expressed by the jump in the extrinsic curvature $[K_{ab}]$ on this surface $S_1$:

\[
\sigma = -\frac{1}{4\pi} [K_{\vartheta \vartheta}]_{S_1}
\]

(20)

\[
= -\frac{1}{4\pi V_0^2 \partial r_+} \left[1 - \left(\frac{1 - \frac{2m}{RV_0} + \frac{e^2}{(RV_0)^2}}{1 - \frac{2m}{RV_0} + \frac{e^2}{(RV_0)^2}}\right) \right] \left(1 - \frac{m}{RV_0}\right)
\]

(21)

\[
p = \frac{1}{8\pi} \left([K^\tau_\tau] + [K^\vartheta_\vartheta]\right)_{S_1}
\]

(22)

\[
= \frac{1}{8\pi RV_0} \left(\frac{1 - \frac{m}{RV_0}}{\sqrt{1 - \frac{2m}{RV_0} + \frac{e^2}{(RV_0)^2}}} - 1\right)
\]

(23)

The properties of the energy density and pressure can be inferred by decreasing the mass and charge parameters $m$ and $|e|$ (note that $|e| \leq m$ ) of the inner Reissner-Nordström region,

\[
\lim_{m, e \to 0} \sigma = -\frac{1}{4\pi V_0^2 \partial r_+} \left| \left. \frac{\partial V}{\partial r_+} \right|_{r_+ = R} = -\left(\cosh \mu_0 - \cos \eta\right) \frac{\partial V}{\partial \mu} \bigg|_{\mu = -\mu_0}
\]

(24)

\[
\lim_{m, e \to 0} p = 0
\]

(25)
We can see that the sign of the surface energy density crucially depends on the sign of the derivative of $V(\mu, \eta, \varphi)$ with respect to $\mu$,

$$\left. \frac{\partial V}{\partial \mu} \right|_{-\mu_0} = \frac{m_1}{R} \left[ \sqrt{\cosh \mu_0 - \cos \eta} \sum_{n=0}^{\infty} \frac{(-1)^{(n+1)}}{(2n+1)\mu_0} \sinh (2n+1)\mu_0 \right]$$

For $\cosh(\mu_0) \geq 3$, i.e. $\frac{R}{R_1} \leq \frac{1}{3}$, this quantity is negative on $S_1$ for arbitrary values of $\eta$ and positive mass parameter $m_1$ and hence, the energy density $\sigma$ is positive. This can be seen easily from the fact that the first term of the infinite series is negative and the absolute values of the successive terms are monotonically decreasing. Although we were unable to determine the sign of the series for smaller values of $\mu_0$ analytically, we have numerically established the existence of a critical value $\left( \frac{R}{R_1} \right)_{crit} \approx 9.993858$ at which at the inner poles ($\eta = \pi$) of the spheres $S_i$ the energy density $\sigma$ changes sign.

This proves that for $\mu_0 > \mu_{crit}$ and sufficient small values of the parameter $m$ and $|e|$ not only is the energy density $\sigma$ positive but all energy conditions are satisfied.

V. CAUSAL STRUCTURE

From the Penrose diagram of the extended Reissner-Nordström spacetime and the schematic drawing of the exterior Majumdar-Papapetrou region (Fig.1) one sees that any spacelike slice which avoids the singularities (e.g. hypersurface $\Sigma$ in Fig.1) cuts $S_1$ and $S_2$ and connects two separated asymptotic regions. Nevertheless any point in region $II$ can be connected by causal curves through the wormhole from any point in region $I$.

An observer starting from the outside region and entering the wormhole through $S_1$ is able to reemerge at $S_2$ arbitrary far in the past. If the time gap resulting from the wormhole traversal is large enough he is able to travel back to his starting point in the exterior region and meet his "former self". In this sense the wormhole is an "eternal" time machine.

Notice that the condition of continuity of the induced metric on the surfaces $S_1$ and $S_2$ does not fix the identification uniquely. There remains the possibility to introduce a constant but arbitrary shift in time. Hence, one is able to arrange the wormhole construction in a way that for example observers freely falling through
the wormhole along the z-axis (starting with a given initial velocity at $z = 0$) come back to their starting point in space and time.

Multiple traversable and non-traversable wormhole geometries may be obtained by introducing additional shells in the Majumdar-Papapetrou region and connecting them to the inner Reissner-Nordström solution.

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**Figure**

Figure 1: Wormhole geometry: (a) interior Reissner-Nordström region, (b) schematic Penrose diagram of the exterior Majumdar-Papapetrou region
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