Boolean Perspectives of Idioms and the Boyle Derivative

Jaime Castro Pérez1 · Mauricio Medina Bárcenas2 · José Ríos Montes3 · Angel Zaldívar Corichi4

Received: 25 August 2017 / Accepted: 5 September 2018 / Published online: 15 September 2018
© Springer Nature B.V. 2018

Abstract
We are concerned with the boolean or more generally with the complemented properties of idioms (complete upper-continuous modular lattices). Simmons (Cantor–Bendixson, socle, and atomicity. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/002-Atom.pdf, 2014) introduces a device which captures in some informal speaking how far the idiom is from being complemented, this device is the Cantor-Bendixson derivative. There exists another device that captures some boolean properties, the so-called Boyle-derivative, this derivative is an operator on the assembly (the frame of nuclei) of the idiom. The Boyle-derivative has its origins in module theory. In this investigation we produce an idiomatic analysis of the boolean properties of any idiom using the Boyle-derivative and we give conditions on a nucleus $j$ such that $[j, tp]$ is a complete boolean algebra. We also explore some properties of nuclei $j$ such that $A_j$ is a complemented idiom.

Keywords Complete boolean algebras · Lattices · Frames · Modules · Rings

Mathematics Subject Classification Primary 06Cxx; Secondary 16S90

1 Introduction

Frames (locales, complete Heyting algebras) as algebraic analogues of topological spaces, emerge naturally in many situations. For example consider any associative ring with unit $R$ and the category of left $R$-modules, $R$-Mod. It is known that a localization of $R$-Mod is given by an hereditary torsion class $T$, that is, a class of modules closed under isomorphism, quotients, subobjects, extensions and arbitrary coproducts. All these localizations or in a more amenable way all this classes are organized as a complete lattice that results to be a
frame. This frame is called $R$-tors. For years it has been seen that a decent analysis of the categorical behaviour of $R$-Mod can be done via $R$-tors (see [4]).

In many other algebraic-like-situations frames appear as a manifestation of a localization process, as in a topos $E$ the localizations are exactly the Lawvere–Tierney topologies on the subobject classifier $\Omega_E$, and it is known that the former of all Lawvere–Tierney topologies constitute a frame.

The external notion of a Lawvere–Tierney topology is a nucleus on a frame $A$, that is, a function, $j : A \rightarrow A$ such that

1. $a \leq b \Rightarrow j(a) \leq j(b)$.
2. $a \leq j(a)$.
3. $j(a \land b) = j(a) \land j(b)$.
4. $j^2 = j$.

Denote by $N(A)$ the set of all nuclei on a frame $A$. An important result states that $N(A)$ is a frame, thus many properties of $A$ are captured by the frame of all its nuclei. In fact, as frame, $N(A)$ has its own frame of nuclei $N^2(A)$ and so on. Doing this through the ordinals we obtain the assembly tower of $A$. The idea of this tower is to control the boolean behaviour of $A$ ([26]), in the sense that $A$ is a complete boolean algebra if and only if $N(A) \cong A$ (Theorem 3.9 of [13]). This extreme case also occurs in the module theoretic realm:

$$R$$-tors is boolean if and only if $R$ is a semiartinian ring

if and only if $R$-tors $\cong N(R$-tors).

These extreme cases show us that having a complete boolean algebra much of the theory is simplified or become trivial. Nevertheless, a frame $A$ is not boolean in general. We can subtract a Boolean part of $A$, this procedure is measured by the Cantor–Bendixson derivative. The author in [18,24] introduces the different interactions of the Cantor–Bendixson derivatives on every level in the assembly tower and the boolean consequences in $A$. In [19] the author observes that the Cantor–Bendixson analysis can be done for more general structures called idioms, that is, upper-continuous modular lattices. Every frame is a distributive idiom, thus idioms are a generalization of frames.

The archetypal example of an idiom is the lattice of submodules of any module, in particular the lattice of left (right) ideals of a ring. Thus the Cantor–Bendixson process can be done for this kind of lattices and in fact one can consider the frame of nuclei on a general idiom $A$. Then the Cantor–Bendixson derivate of this frame is related in somehow with the Cantor–Bendixson derivative of $A$, to see this interaction, the author in [20] introduces the idiomatic counterpart of the Boyle dimension on module categories (see [2,3]). For any idiom $A$, the Boyle derivative exists at the level of $N(A)$, that is, $\text{Boy} : N(A) \rightarrow N(A)$ as an inflator with some extra properties.

The principal idea of this manuscript is to perform a Boyle-analysis for idioms in the same path as in [18,24]. We obtain conditions on an idiom $A$ to determine when $N(A)$ is a complete boolean algebra, in fact we will do this in a more general way, we will give conditions on a nucleus $j$ to have Boy-dimension and the frame $[j, tp] \cong N(A_j)$ to be a complete boolean algebra.

Let us briefly describe the organization of this manuscript, Sect. 1 is this Introduction and Sect. 2 summarizes the required material for the rest of the investigation.

Section 3 can be understood as the idiom facet of the Boyle derivative in module categories, we will give equivalent conditions on a nucleus $j$ to have Boyle-dimension (3.1). In Sect. 3.2 we compare this dimension with the Gabriel-dimension for idioms and we generalize some results of [19]. The last section is the idiom counterpart of the theory of spectral Grothendieck
categories, we will use several facts of the previous sections to give generalizations of the module theory realm into the idiomatic view.

2 Preliminaries

In this section we recollect the necessary material for the development of all the investigation, in particular some facts about the Boyle derivative.

We recall some of the idiom theory that we will need, first let us begin with an example:

Given a module \( M \in R\text{-Mod} \), denote by \( \Lambda(M) \) the set of all submodules of \( M \). It is clear that \( \Lambda(M) \) constitutes a complete lattice where suprema are not unions, moreover the following distributive laws hold:

\[
N \cap \left( \bigcup X \right) = \bigcup \{ N \cap K \mid K \in X \}
\]

for any \( N \in \Lambda(M) \) and \( X \subseteq \Lambda(M) \) directed; and

\[
K \leq N \Rightarrow (K + L) \cap N = K + (L \cap N)
\]

for any \( L \in \Lambda(M) \). Thus the lattice of any module is an upper-continuous and modular lattice. This is the idea behind idioms:

**Definition 2.1** An idiom \((A, \leq, \lor, \land, 1, 0)\) is a complete, upper-continuous, modular lattice, that is, \( A \) is a complete lattice that satisfies the following distributive laws:

\[
a \land \left( \bigvee X \right) = \bigvee \{ a \land x \mid x \in X \} \quad \text{(IDL)}
\]

holds for all \( a \in A \) and \( X \subseteq A \) directed; and

\[
a \leq b \Rightarrow (a \lor c) \land b = a \lor (c \land b) \quad \text{(ML)}
\]

for all \( a, b, c \in A \). These are the Idiom distributive law and the modular law respectively.

A good account of the many uses of these lattices can be found in [21]. A remarkable class of idioms are the distributive ones:

**Definition 2.2** A frame \((A, \leq, \lor, \land, 1, 0)\) is a complete lattice that satisfies

\[
a \land \left( \bigvee X \right) = \bigvee \{ a \land x \mid x \in X \} \quad \text{(FDL)}
\]

for all \( a \in A \) and \( X \subseteq A \).

That is, a distributive idiom is exactly a frame.

Frames are the algebraic version of a topological space. Indeed, if \( S \) is a topological space then its topology \( \mathcal{O}(S) \) is a frame.

There exists an important characterization of frames in terms of an (Heyting) implication. Recall that in any lattice \( A \), an implication in \( A \) is an operation \( (\_ \searrow \_ ) \) given by

\[
x \leq (a \searrow b) \iff x \land a \leq b
\]

for all \( a, b \in A \). When the lattice \( A \) has an implication then \( A \) is a distributive lattice, in the context of complete lattices we have the following:

**Proposition 2.3** A complete lattice \( A \) is a frame if and only if \( A \) has an implication.
For a proof, see Lemma 1.7 of [14] or Theorem I4.2 of [8]. We require some point-free techniques.

**Definition 2.4** (1) An **inflator** on an idiom $A$ is a function $d : A \to A$ such that $x \leq d(x)$ and $x \leq y$ $\Rightarrow$ $d(x) \leq d(y)$ for all $x, y \in A$.

(2) A **pre-nucleus** $d$ on $A$ is an inflator such that $d(x \land y) = d(x) \land d(y)$ for all $x, y \in A$.

(3) A **stable inflator** $d$ on $A$ is an inflator such that $d(x) \land y \leq d(x \land y)$ for all $x, y \in A$.

(4) A **closure operator** is an idempotent inflator $c$ on $A$, that is, is an inflator such that $c^2 = c$.

(5) A nucleus on $A$ is an idempotent pre-nucleus.

Let $I(A)$ denote the set of all inflators on $A$, $P(A)$ the set of all prenuclei, and $S(A)$ the set of all stable inflators. Clearly, $P(A) \subseteq S(A) \subseteq I(A)$. Let $C(A)$ be the set of all closure operators in $A$. Let $N(A)$ be the set of all nuclei on $A$. All these sets are partially ordered by $d \leq f \iff d(a) \leq f(a)$ for all $a \in A$. Note that the identity function $id_A$ and the constant function $tp(a) = 1$ for all $a \in A$ (where $1$ is the top of $A$) are inflators. These two inflators are the bottom and the top in all these partially ordered sets.

Given an inflator $d \in I(A)$ we can construct a closure operator as follows: $d^0 := id_A$, $d^{α+1} := d \circ d^α$ for a non-limit ordinal $α$, and $d^λ := \bigvee \{d^α \mid α < λ\}$ for a limit ordinal $λ$. These are inflators, ordered in a chain $d \leq d^2 \leq d^3 \leq \ldots \leq d^α \leq \ldots$.

By a cardinality argument, there exists an ordinal $γ$ such that $d^α = d^γ$, for all $α \geq γ$. In fact, we can choose $γ$ the least of these ordinals, say $∞$. Thus, $d^∞$ is an inflator such that $d \leq d^∞$, but more important this inflator satisfies $d^∞d^∞ = d^∞$, that is, $d^∞$ is a closure operator on $A$. Also this construction gives a way to obtain nuclei on an idiom $A$.

**Theorem 2.5** Let $A$ be an idiom then:

(1) For every stable inflator $s$ on $A$, the closure $s^∞$ is a nucleus.

(2) In particular for every pre-nucleus $p$ on $A$, the closure $p^∞$ is a nucleus.

The following theorem is one of the most important results in idiom theory.

**Theorem 2.6** For any idiom $A$, the complete lattice of all nuclei $N(A)$ on $A$ is a frame.
Definition 2.7  With the same notation as above:

(1) We say that a set of intervals $A \subseteq \mathcal{I}(A)$ is abstract if it is not empty and it is closed under $\sim$, that is,

$$J \sim I \in A \Rightarrow J \in A.$$  

(2) An abstract set $B$ is a basic set if it is closed under subintervals, that is,

$$J \subseteq I \in B \Rightarrow J \in B.$$ 

(3) A set of intervals $C$ is a congruence set if it is basic and closed under abutting intervals, that is,

$$[a, b][b, c] \in C \Rightarrow [a, c] \in C$$

for elements $a, b, c \in A$.

(4) A basic set of intervals $B$ is a pre-division set if

$$\forall x \in X \ [a, x] \in B \Rightarrow [a, \bigvee X] \in B$$

for each $a \in A$ and $X \subseteq [a, 1]$.

(5) A set of intervals $D$ is a division set if it is a congruence set and a pre-division set.

Denote $\mathcal{D}(A) \subseteq \mathfrak{C}(A) \subseteq \mathcal{B}(A) \subseteq \mathfrak{A}(A)$ the set of all division, congruence, basic and abstract intervals in $A$, respectively. These gadgets can be understood like certain classes of modules in a module category $R$-Mod, that is, classes closed under isomorphism, subobjects, extensions and coproducts. From this point of view $\mathfrak{C}(A)$ and $\mathcal{D}(A)$ are the idiom analogues of the Serre classes and the torsion (localizations) classes in module categories. It is not hard to see that $\mathcal{B}(A)$ is a frame, called the base frame of the idiom $A$. The top of this frame is $\mathcal{I}(A)$ and the bottom is the set of all trivial intervals of $A$, denoted by $\emptyset(A)$. Also, the family $\mathfrak{C}(A)$ is a frame and a proof of this fact can be found in [19].

For any $B \in \mathcal{B}(A)$ we can describe the least division set that contains it, denoted by $\mathcal{D}_v(B)$. In [19] it is proved that $\mathcal{D}_v(\_)$ is a nucleus on $\mathcal{B}(A)$ and the quotient of this nucleus is $\mathcal{D}(A)$. In fact, there is a way to connect this frame to the frame $N(A)$:

Definition 2.8  For each $a \in A$ and $B \in \mathcal{B}(A)$ let $|B|: A \to A$ be the function given by

$$|B|(a) = \bigvee X$$

where $x \in X \Leftrightarrow [a, x] \in B$.

This produces the associated inflator of $B$. Moreover, if the basic set $B$ is a congruence set, then $|B|$ is a pre-nucleus on $A$, and if it is a division set, then $|B|$ is a nucleus. In this way, we have for every division set a nucleus. Now, given a nucleus $j$ we can construct a division set $\mathcal{D}_j$ as follows,

$$[a, b] \in \mathcal{D}_j \Leftrightarrow j(a) = j(b) \text{ and } a \leq b.$$ 

These correspondences are bijections and they define an isomorphism between $\mathcal{D}(A)$ and $N(A)$. With this we have:

Theorem 2.9  If $A$ is an idiom, then there is an isomorphism of frames

$$N(A) \leftrightarrow \mathcal{D}(A) \quad j \leftrightarrow \mathcal{D}$$
Theorem 2.10 For every \(B \in \mathcal{B}(A)\)
\[
[a, b] \in \mathcal{D}(B) \iff \forall a \leq x < b) (\exists x < y \leq b) ([x, y] \in B).
\]

The \(\mathcal{D}(B)\)-construction can be described in a useful way:

**Theorem 2.10** For every \(B \in \mathcal{B}(A)\)
\[
[a, b] \in \mathcal{D}(B) \iff \forall a \leq x < b) (\exists x < y \leq b) ([x, y] \in B).
\]

The details can be found in [19,20]. This result shows that \(\mathcal{D}(A)\) is a frame thus it has an implication, the following gives a description of it (for a proof see [19] Lemma 4.6).

**Lemma 2.11** Let \(A\) be an idiom then
\[
I \in (A > B) \iff (\forall J \subseteq I) [J \in A \Rightarrow J \in B]
\]

for any \(A \in \mathcal{B}(A)\) and \(B \in \mathcal{D}(A)\).

As we mentioned in the introduction we are concerned with the boolean properties of modules categories and idioms, this boolean properties are measured by some special inflators that we will introduce:

**Definition 2.12** Let \(A\) be an idiom, consider the following sets of intervals:

1. An interval \([a, b]\) is **simple** if there is no \(a < x < b\), that is, \([a, b] = \{a, b\}\). Denote by \(\mathcal{S}mp\) the set of all simple intervals.
2. An interval \([a, b]\) of \(A\) is **complemented** if it is a complemented lattice, that is, for each \(a \leq x \leq b\) there exist \(a \leq y \leq b\) such that \(a = x \land y\) and \(b = x \lor y\). Let \(\mathcal{C}mp\) be the set of all complemented intervals.
3. We can relativize this notion, for each \(B \in \mathcal{B}(A)\), let \(\mathcal{S}mp(B)\) be the set of all \([a, b]\) such that for each \(a \leq x \leq b\), \([a, x] \in B\) or \([x, b] \in B\)

This is the set of \(B\)-simple intervals.
4. Let \(\mathcal{C}mp(B)\) be the set of all intervals \([a, b]\) such that:

\[
\forall a \leq x \leq b \text{ exists } a \leq y \leq b \text{ such that } [a, x \land y] \in B \text{ and } [x \lor y, b] \in B.
\]

This is the set of \(B\)-complemented intervals. With this, we have that \(\mathcal{S}mp = \mathcal{S}mp(\emptyset)\) and \(\mathcal{C}mp = \mathcal{C}mp(\emptyset)\).
5. Given any \(B \in \mathcal{B}(A)\) denote by \(\mathcal{F}ll(B)\) the set of all intervals \([a, b]\) such that,

\[
\forall a \leq x \leq b \text{ exists } a \leq y \leq b \text{ with } a = x \land y \text{ and } [x \lor y, b] \in B
\]

this is the set of all \(B\)-**full** intervals. Note that \(\mathcal{C}mp(\emptyset) = \mathcal{F}ll(\emptyset)\).
6. Let \(\mathcal{C}rt(B)\) be the set of intervals \([a, b]\) such that

\[
\forall a \leq x \leq b \text{ we have } a = x \text{ or } [x, b] \in B,
\]

this is the set of all \(B\)-**critical** intervals. Note that \(\mathcal{S}mp(\emptyset) = \mathcal{C}rt(\emptyset)\).

In [20] is proved that for any \(B \in \mathcal{B}(A)\), \(\mathcal{F}ll(B) \leq \mathcal{C}mp(B)\). Moreover, one can show that for any \(B \in \mathcal{B}(A)\) the sets \(\mathcal{C}mp(B)\) and \(\mathcal{F}ll(B)\) are basic.

The item 4 is the main object of study in Sects. 3.2 and 4.
Let $A$ be an idiom. Given elements $a, b \in A$, we say that $b$ is essentially above $a$

$$a \ll b$$

if $a \leq b$ and for every $y \in A$ such that

$$b \wedge y \leq a \Rightarrow y \leq a$$

If the idiom is distributive, that is, a frame then this notion is equivalent to $(b \succ a) = a$, and this is the central relation of the investigation in [15, 18].

Also observe that if $a \ll a$ then $a = 1$. The following lemma will be useful and a proof can be found in [23].

**Lemma 2.14** Let $A$ be an idiom and consider any basic set $B \in \mathcal{B}(A)$. For each interval $[a, b]$ the following are equivalent:

1. $[a, b] \in \mathcal{I}(\mathcal{B})$.
2. $(\forall x \in A)[a \ll x \Rightarrow [x \land b, b] \in B]$

$\mathcal{I}(\mathcal{B})$ and $\mathcal{C}rt(\mathcal{B})$ are basic sets for any basic set $B$ in particular for any nucleus $j$, we can consider $\text{Boy}(\mathcal{D}_j) := \mathcal{D}vs(\mathcal{I}(\mathcal{D}_j))$ and $\text{Gab}(\mathcal{D}_j) := \mathcal{D}vs(\mathcal{C}rt(\mathcal{D}_j))$. By Theorem 2.9 we denote the corresponding nuclei as $\text{Boy}(j)$ and $\text{Gab}(j)$ respectively. The associations $j \mapsto \text{Boy}(j)$ and $j \mapsto \text{Gab}(j)$ set up two prenuclei on $N(A)$ called the Boyle and Gabriel derivative respectively.

The details of these facts are not straightforward, the reader must see [20, 23].

Let us summarize some facts about this construction

**Remark 2.15** If we consider the simple intervals and the complement intervals as in Definition 2.12 then we can associate the corresponding inflators, these are the socle derivative, $\text{soc}$ and the Cantor–Bendixson derivative, $\text{cbd}$ respectively (these two are stable inflators). The idea is that we want the relative version of these derivatives with respect to the basic set given by any nucleus $j \leftarrow \mathcal{D}_j$, that is why the author in [23] introduces $\mathcal{I}(\mathcal{D}_j)$ and $\mathcal{C}rt(\mathcal{D}_j)$

1. Let $cbd_j$ be the corresponding inflator of $\mathcal{I}(\mathcal{D}_j)$.
2. Let $soc_j$ be the corresponding inflator of $\mathcal{C}rt(\mathcal{D}_j)$.
3. These two inflators are stable, then by Theorem 2.5 their closures $cbd_j^\infty$ and $soc_j^\infty$ are nuclei on $A$. The corresponding division sets are $\text{Boy}(\mathcal{D}_j)$ and $\text{Gab}(\mathcal{D}_j)$ respectively, thus $\text{Boy}(j) = cbd_j^\infty$ and $\text{Gab}(j) = soc_j^\infty$.
4. Note that if $[a, b] \in \mathcal{I}(\mathcal{D}_j)$ then this interval is complemented in $A_j$.

There exists another construction for $soc_j$. Define the set of $j$-semicritical intervals as the set of intervals $[a, b]$ such that there exists $X \subseteq [a, b]$ with $[a, x] \in \mathcal{C}rt(\mathcal{D}_j)$ for each $x \in X$. Denote the set of all this intervals by $\mathcal{S}ct(\mathcal{D}_j)$, note that, if $j = id$ then $\mathcal{S}rt(\mathcal{D}_j) = \mathcal{S}Sp$ the set of semi-simple intervals. The set of semi-critical intervals is characterized by:

$$[a, b] \in \mathcal{S}ct(\mathcal{D}_j) \Leftrightarrow b \leq soc_j(a) \text{ and } a \leq b.$$ 

Then

$$soc_j \leftrightarrow \mathcal{S}ct(\mathcal{D}_j) \quad \text{cbd}_j \leftrightarrow \mathcal{I}(\mathcal{D}_j)$$

The relation of these basic sets is described in the following:
**Lemma 2.16** For any nucleus \( j \) with division set \( D_j \),
\[
Sct(D_j) = \mathcal{S}_{ab}(D_j) \cap \mathcal{F}H(D_j).
\]

The proof of this lemma is in [23](Lemma 6.4).

Now for last we will discuss the dimension facts about this theory. First since \( N(A) \) is a frame then it has its own inflators, in particular it has its socle derivative \( Soc \) and its Cantor–Bendixson derivative \( Cbd \).

**Definition 2.17** Let \( A \) be an idiom and \( S \leq Cbd \) any stable inflator on the frame \( N(A) \). For each \( j \in N(A) \) we set
\[
LS(j) = (RS(j) \succ j) \quad RS(j) = (S(j) \succ j)
\]
where \((\_ \succ \_)_\) is the implication of \( N(A) \).

This two operators are studied in [20]. In particular the following theorem ([20] Theorem 5.5) is proved.

**Theorem 2.18** Let \( A \) be an idiom and consider any stable inflator \( S \leq Cbd \) on the frame \( N(A) \), then
\[
S = S^\infty \land Cbd = LS \land Cbd.
\]
In particular \( S = Cbd \) if \( LS = Tp \) (the top of \( N^2(A) \)).

**Definition 2.19** Let \( A \) be an idiom and \( S \) an stable inflator with \( S \leq Cbd \) on the frame \( N(A) \). Consider any nucleus \( j \in N(A) \), we say that \( j \) has:

- **S-dimension** if \( S^\infty(j) = tp \)

and

- **weak S-dimension** if \( LS(j) = tp \).

It follows that if \( j \) has S-dimension then \( S = Cbd \).

Recall that in any idiom the Cantor–Bendixson derivative \( cbd \) produce the largest complemented interval \([a, cbd(a)]\) above \( a \). This investigation particularizes in the Boyle-derivative so we state an important fact about it.

**Theorem 2.20** Let \( j \) be a nucleus on an idiom \( A \). Then the interval \([j, Boy(j)]\) is boolean.

Therefore in any idiom \( Boy \leq Cbd \). In particular if \( j \) has the property that \( Boy(j) = tp \) then the upper section \( \uparrow (j) \) is boolean and this upper section is isomorphic to \( N(A_j) \) as frames, we will use this fact in several parts of the investigation.

### 3 Boyle Dimension for Idioms

First we are going to prove a slight modification of theorem 4.10 of [18] and theorems 5.13 and 5.14 of [15]. Then we will connect these ideas with the Boyle dimension. In the first theorem we use the fact (theorem 6.5 of [23]) that in any idiom
\[
cbd_j(a) = \bigwedge \{j(x) \mid j(a) \leq x\}.
\]
Theorem 3.1 For an idiom $A$ and a nucleus $j \in N(A)$ the following are equivalent:

1. $\text{Boy}(j) = tp$.
2. $a \preceq \text{cbd}_j(a)$ for all $a \in A_j$.

Proof Suppose (1), let $a, b \in A_j$ such that $b \land \text{cbd}_j(a) \leq a$. By an induction argument it follows that $b \land \text{cbd}_j(a) \leq a$. By Remark 2.15 item 3, $\text{cbd}_j(a) = \text{Boy}(j)(a)$ thus $\text{cbd}_j(a) = \text{Boy}(j)(a) = 1$, that is, $a \preceq \text{cbd}_j(a)$. Thus, $\text{Boy}(j)(0) = \text{cbd}_j(0) = 1$.

Conversely, if $a \preceq \text{cbd}_j(a)$ for all $a \in A_j$ in particular for $a = \text{cbd}_j(0)$. Then $a \leq \text{cbd}_j(a) \leq \text{cbd}_j(0) = a$, therefore $a = 1$. \qed

An immediate consequence of Theorem 3.1 is the following.

Corollary 3.2 Let $A$ be an idiom and $j \in N(A)$ suppose that the equivalent conditions of Theorem 3.1 are satisfied. Then the interval $[j, tp] \cong N(A_j)$ is a complete boolean algebra, in particular with $j = \text{id}$ we have that $N(A)$ is a complete boolean algebra.

Example 3.3 The following lattice $A$ is the only idiom which is not a frame among all lattices with $n$ points for $1 \leq n \leq 5$ (up to isomorphism).

```
0 \preceq 1 = \text{cbd}(0)

a \preceq \text{cbd}(a)

0 \preceq \text{cbd}(b)

b \preceq \text{cbd}(b)

1 \preceq \text{cbd}(c)
```

By Corollary 3.2, $N(A)$ is boolean. For the case $n = 6$ the following idiom does not satisfies Corollary 3.2.

```
0 \preceq c

0 \preceq d

0 \preceq 1
```

Hence, $\text{cbd}(0) = b$.

Note that $0$ is not essential below $b$.

Remark 3.4 Recall that the essentially above relation $2.13 a \preceq b$ on a frame is equivalent to $(b \succ a) = a$, therefore observe that in particular $R\text{Boy}(j) = j \iff j \preceq \text{Boy}(j)$. 

\[\text{Springer}\]
Theorem 3.5 For an idiom $A$ the following statements are equivalent:

1. $N^2(A)$ is boolean.
2. $R_{Boy}(j) = j$ for all $j \in N(A)$.

**Proof** Just notice that if $j \preceq Boy(j)$ then $Cbd(j) \leq Boy(j)$ from which $Cbd(j) = Boy(j)$ and in this case we can apply the argument of Theorem 4.10 in [19].

Let $j \in N(A)$ be any nucleus. Set:

$$C_j = \{a \in A_j \mid a \preceq cbd_j(a)\}.$$

Observe that, if $C_j = A_j$ then $N(A_j)$ is boolean by Theorem 3.1. In a manner of speaking $C_j$ measures the booleanness of the respective assembly. This boolean property is also captured by the following chain

$$j \leq Boy(j) \leq Boy^2(j) \leq Boy^3(j) \leq \ldots \leq Boy^\alpha(j) \leq \ldots$$

which eventually stabilizes in some ordinal, denote by $\infty$ the minimal ordinal in which the chain stabilizes.

Lemma 3.6 If $A$ is an idiom then

$$C_{Boy^\infty(j)} = \{1\}.$$

**Proof** If $a \in C_{Boy^\infty(j)}$ then

$$a \preceq cbd_{Boy^\infty(j)}(a) \leq Boy(Boy^\infty(j))(a) = Boy^\infty(j)(a) = a$$

thus $a = 1$.

With this we observe that:

Proposition 3.7 With the same notation as above a nucleus $j$ has $B$-dim if and only if $C_{Boy^\infty(j)} = A_{Boy^\infty(j)}$

3.1 Cohesive Properties for Idioms

Now we will examine the $B$-dim in idioms with ascending chain condition (ACC) on the relation $\preceq$. To do that we use cohesive subsets. This notion is introduced in [18] for frames in order to study the second level assembly of a frame. Here this notion also works fine in the idiom context.

Definition 3.8 Let $A$ be an idiom. A non-empty subset $K \subseteq A$ is cohesive if for each $a \in K$ there exists $X \subseteq K$ such that $a = \bigwedge X$ and $a \preceq x$ for each $x \in X$.

Lemma 3.9 Suppose that $A$ has ACC on $\preceq$. Then $\{1\}$ is the only cohesive subset.

**Proof** If $K \subseteq A$ is cohesive and $K \neq \{1\}$ then there exists $a \in K$ such that $a \neq 1$ and $a = \bigwedge X$ with $a \preceq x$ for all $x \in X$ for some $X \subseteq K$. Thus $X \neq \{1\}$ that is, there is some $a' \in X$ with $a' \neq 1$ such that $a \preceq a'$. This produces an ascending $\preceq$-chain thus by ACC we obtain an element $b \in K - \{1\}$ such that $b < b$, that is, $b = 1$ a contradiction.

Lemma 3.10 Let $k$ be a nucleus on $A$ such that $Boy(k) = k$. Then $A_k$ is cohesive.
Proof\ Let $a \in A_k^j$ and $X \subseteq A_k^j$ be the set of all $x$ such that $a \prec x$. Therefore
\[ a \leq \bigwedge X \leq \text{cbd}_k(a) \leq \text{Boy}(k)(a) = k(a) = a. \]
\[ \square \]

Corollary 3.11 Let $A$ be an idiom and denote $k = \text{Boy}^\infty(j)$. Then $A_k^j$ is cohesive for any $j \in N(A)$.

Lemma 3.12 If $K$ is a cohesive subset on $A$ then
\[ K \subseteq A_j^j \Rightarrow K \subseteq A_{\text{Boy}^\infty(j)} \]
for each $j \in N(A)$.

Proof\ Let $K$ be cohesive such that $K \subseteq A_j^j$. Then for all $a \in K$,
\[ \text{cbd}_j(a) = \bigwedge \{ x \in A_j^j \mid a \prec x \} \leq \bigwedge \{ x \in K \mid a \prec x \} = a. \]
\[ \square \]

Theorem 3.13 Let $A$ be an idiom and $j \in N(A)$. The following statements hold:

1. $\text{Boy}^\infty(j) = \text{tp} \iff \{1\}$ is the only cohesive subset of $A_j^j$.
2. $A_{\text{Boy}^\infty(j)}$ is the biggest cohesive subset of $A_j^j$.
3. $\text{Boy}^\infty(j) = j$ if and only if $A_j^j$ is cohesive.

Proof\ (1) Let $K \subseteq A_j^j$ be cohesive. It is enough to see that $K \subseteq A_{\text{Boy}^\infty(j)}$, if $a \in K$ then
\[ \text{cbd}_j(a) = \bigwedge \{ x \in A_j^j \mid a \prec x \} \leq \bigwedge \{ x \in K \mid a \prec x \} = a \]
because of the cohesive property. Now by induction one can show that $\text{cbd}_\alpha^j(a) = a$ for each ordinal $\alpha$, then $\text{cbd}_\alpha^j(a) = \text{Boy}(\alpha)(a) = a$. Again an induction argument leads to $\text{Boy}^\alpha(j)(a) = a$. If $k = \text{Boy}^\infty(j)$ and $k = \text{tp}$ then $K \subseteq A_k^j = \{1\}$.

Conversely, if $\{1\}$ is the only cohesive subset of $A_j^j$ we have that $A_{\text{Boy}^\infty(j)}$ is cohesive (by Corollary 3.11) thus $A_{\text{Boy}^\infty(j)} = \{1\}$, that is, $\text{Boy}^\infty(j) = \text{tp}$.
(2) This is an immediate consequence of Lemma 3.12.
(3) Put $k = \text{Boy}^\infty(j)$. If $j = k$ then $A_j^j$ is cohesive by Corollary 3.11. Reciprocally if $A_j^j$ is cohesive then $A_j^j \subseteq A_k^j$ therefore $k = j$. \[ \square \]

As a consequence of Lemma 3.10 and Theorem 3.13 we have:

Corollary 3.14 If $A_j$ satisfies ACC on $\prec$ then $j$ has Boyle-dim.

All these statements reassemble the results of [15] and the crucial fact that in a frame (a distributive idiom) the pre-nuclei $\text{Cbd}$ and $\text{Boy}$ on $N(A)$ coincide.

3.2 Boyle Dimension for Idioms

We conclude this section with some characterizations of idioms with Boyle-dimension.

Let $j$ be a nucleus on $A$. We will give a generalization of a feeble atomic idiom.

Definition 3.15 An interval $[a, b]$ is feeble atomic if for each complemented subinterval, say $a \leq c < d \leq b$ there is some $c < z \leq d$ with $[c, z] \in \text{Smp}$. Let $\mathcal{FA}$ denote the set of all feeble atomic intervals of $A$. The idiom $A$ is feeble atomic if $\mathcal{FA} = \mathcal{J}(A)$. Note that $\mathcal{FA}$ is a division set (for details see 7.3 of [19]).
Using the fact:

\[ \text{Sct}(\mathcal{D}_j) = \mathcal{J}ab(\mathcal{D}_j) \cap \mathcal{F}Il(\mathcal{D}_j). \]

Consider

\[ (\mathcal{F}Il(\mathcal{D}_j) \triangleright \text{Sct}(\mathcal{D}_j)) = (\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j)) \cap (\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{F}Il(\mathcal{D}_j)) = (\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j)) \]

the last equality is because in general \((a \triangleright a) = 1\) in any frame. Since \(\mathcal{J}ab(\mathcal{D}_j)\) is a division set, the set \((\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j))\) is a division set.

**Remark 3.16** We require the following facts.

(1) From Lemma 2.11, each \(I \in (\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j))\) satisfies that any \(\mathcal{D}_j\)-full sub-interval say \([a, b]\) is in \(\mathcal{J}ab(\mathcal{D}_j)\), in other words any \(j\)-full sub-interval contains \(j\)-critical intervals.

(2) The corresponding nucleus of \((\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j))\), is denoted by \(fb lj \in N(A)\). Observe that, this nucleus is exactly \((cbd j \triangleright soc j^{\infty})\).

First we prove a generalization of Theorem 7.17 of [19].

**Proposition 3.17** For every nucleus \(j \in N(A)\) we have:

\[ soc_j = fb lj \land cbd j. \]

**Proof** It is known that \(soc_j = soc_j^{\infty} \land cbd j\) (Corollary 6.3 of [23]), by construction of \(fb lj\) we have \(soc_j \leq fb lj\) and we always have that \(soc j \leq cbd j\), note that by 3.16 (2) we get

\[ fb lj \land cbd j \leq soc_j^{\infty} \]

therefore

\[ soc_j \leq fb lj \land cbd j \leq soc_j^{\infty} \land cbd j = soc_j. \]

as required. \(\square\)

**Theorem 3.18** Let \(j\) be a nucleus on an idiom \(A\). The following conditions are equivalent:

1. \(soc_j = cbd_j\)
2. \(cbd_j \leq soc_j^{\infty}\)
3. \(cbd_j \leq fb lj\)
4. \(cbd_j^{\infty} \leq soc_j^{\infty}\)
5. \(fb lj = tp\)
6. \(\text{Sct}(\mathcal{D}_j) = \mathcal{F}Il(\mathcal{D}_j)\)
7. \(\mathcal{F}Il(\mathcal{D}_j) \subseteq \mathcal{J}ab(\mathcal{D}_j)\)
8. \(\mathcal{F}Il(\mathcal{D}_j) \subseteq (\mathcal{F}Il(\mathcal{D}_j) \triangleright \mathcal{J}ab(\mathcal{D}_j))\)

**Proof** The equivalences (1) \(\Leftrightarrow\) (6), (2) \(\Leftrightarrow\) (7), (3) \(\Leftrightarrow\) (8) are immediate.

Now (1) \(\Rightarrow\) (2) is trivial, (2) \(\Rightarrow\) (3) follows from the fact that \(soc_j^{\infty} \leq fb lj\) and (3) \(\Rightarrow\) (4) is clear using Proposition 3.17 and the fact that \(fb lj\) is idempotent.

Proposition 3.17 gives (4) \(\Rightarrow\) (5), and it is obvious that (5) \(\Rightarrow\) (1).

Finally, the implications (5) \(\Rightarrow\) (6) \(\Rightarrow\) (7) are clear. \(\square\)

**Definition 3.19** Let \(j\) be a nucleus on an idiom \(A\). We say \(A\) is \(j\)-**feebly atomic** if it satisfies the conditions of Theorem 3.18.

It is clear that if a nucleus \(j\) has Gabriel dimension then it has Boyle dimension. In a feebly atomic idiom we have a partial converse.
**Proposition 3.20** Let \( j \) be a nucleus on an idiom \( A \), suppose that \( j \) has Boyle dimension and that \( A \) is \( j \)-feebly atomic. Then \( j \) has Gabriel dimension.

**Proof** This is a direct consequence of Theorem 3.18 and the fact that \( \text{Boy}^\infty(j) = tp \). \( \Box \)

One of the most important motivation in our investigation comes from ring theory and module theory. Given an associative ring \( R \) with unit, let \( R\text{-Mod} \) be the category of all unital left \( R \)-modules. There exist various ways to study \( R\text{-Mod} \), a remarkable one is via its localizations. Every localization of a Grothendieck category (and in particular for \( R\text{-Mod} \)) is given by an hereditary torsion class. If the Grothendieck category is a module category, say \( R\text{-Mod} \) we denote by \( \mathbb{D}(R) \) the set of all hereditary torsion classes. Every \( \mathbb{D}(R) \) is given by an homology torsion class. If the Grothendieck category is a module category, say \( R\text{-Mod} \) we denote by \( D(R) \) the set of all hereditary torsion classes. Every \( T \in D(R) \) determines a \( \text{Hom}(T, -) \)-orthogonal class, the torsion free class, thus a torsion free class \( F \) is a class of modules closed under isomorphisms, sub-modules, products, extensions and injective hulls (denoted by \( E(\_\_\_\_) \)). The pair \( \tau = (T, F) \) is called an hereditary torsion theory in \( R\text{-Mod} \), denote the set of all torsion theories on \( R\text{-Mod} \) by \( R\text{-tors} \).

The book [4] is devoted to the study of \( R\text{-Mod} \) via \( R\text{-tors} \). For the definitions of the \( \tau \)-Gabriel dimension and \( \tau \)-Boyle dimension in a module category the reader is referred to [3–5,7].

**Theorem 3.21** ([5]). Let \( \tau \in R\text{-tors} \). The following conditions are equivalent:

1. \( R \) has left \( \tau \)-Gabriel dimension.
2. \( R \) has left \( \tau \)-Boyle dimension and each \( M \in F_\tau \) contains an uniform submodule.

To prove the above fact in the idiomatic realm, we need the following two lemmas. Recall that a non-trivial interval \([a, b]\) is uniform if \( x \land y = a \) implies either \( x = a \) or \( y = a \) for all \( x, y \in [a, b] \).

**Lemma 3.22** Let \( j \) be a nucleus on an idiom \( A \), and consider the family of nuclei

\[
(\text{Boy}_\alpha(j) \mid \alpha)
\]

indexed over the ordinals. If \( j \leq k < k' \leq \text{Boy}^\infty(j) \) where \( k \) and \( k' \) are nuclei then there exists a \( \mathbb{D}_k \)-full interval \([u, v]\) such that \([u, v] \in \mathbb{D}_{k'} \) and \([a, b] \notin \mathbb{D}_k \).

**Proof** From the position of \( k \) and \( k' \) we can deduce that there exists an ordinal \( \alpha \) such that:

\[
k' \land \text{Boy}_\alpha(j) \nleq k.
\]

Let \( \beta \) be the least ordinal that satisfies the condition above. Observe that if \( \beta \) is a limit ordinal we have

\[
k' \land \text{Boy}_\beta(j) = k' \land \left( \bigvee \{ \text{Boy}_\lambda(j) \mid \lambda < \beta \} \right) = \bigvee \{ k' \land \text{Boy}_\lambda(j) \} \nleq k
\]

by definition of the chain in the limit case and the frame distributive law, thus \( k' \land \text{Boy}_\lambda(j) \nleq k \) for some \( \lambda < \beta \) which contradicts the choice of \( \beta \), therefore \( \beta \) is not a limit ordinal.

Then we are in the situation:

\[
k' \land \text{Boy}(\text{Boy}^{\beta-1}(j)) \nleq k
\]

which on division sets it is witnessed by a interval \([a, b]\) such that \([a, b] \in \mathbb{D}_{k' \land \text{Boy}(j)} \) and \([a, b] \notin \mathbb{D}_k \), by the other hand we have

\[
[a, b] \in \mathbb{D}_{k'}
\]
Thus \([a, b] \notin \mathcal{D}_{\text{Boy}^{\beta^{-1}}(j)}\) then the interval \([a, \text{Boy}^{\beta^{-1}}(j)\langle a \rangle \land b] \in \mathcal{D}_{\text{Boy}^{\beta^{-1}}(j)}\) and thus by Theorem 2.9 there exists a non-trivial sub-interval \([u, v] \in \mathcal{F}(\mathcal{D}_{\text{Boy}^{\beta^{-2}}(j)})\) also \([u, v] \in \mathcal{D}_k\) and then \([u, v] \in \mathcal{D}_k\).

Let us see that this interval is \(\mathcal{D}_k\)-full. We will use the equivalence of 2.14.

Consider \(x \in A\) such that \(u \leq x\), then \([x \land v, v] \in \mathcal{D}_{\text{Boy}^{\beta^{-2}}(j)}\) since \(\mathcal{D}_{\text{Boy}^{\beta^{-2}}(j)} \leq \mathcal{D}_{\text{Boy}^{\beta^{-1}}(j)}\) and from \([u, v] \in \mathcal{D}_k\) we have that \([x \land v, v] \in \mathcal{D}_k\) also since \([u, v] \in \mathcal{D}_k\) we deduce \([x \land v, v] \in \mathcal{D}_k\), therefore \([u, v] \in \mathcal{F}(\mathcal{D}_k)\).

**Lemma 3.23** Let \(A\) be an idiom, consider any basic set \(\mathcal{B}\) on \(A\), and suppose that we have a uniform interval \([a, b]\) which is \(\mathcal{B}\)-full. Then \([a, b] \in \mathcal{C}(\mathcal{B})\).

**Proof** Let \([a, b] \in \mathcal{F}(\mathcal{B})\) be uniform and consider any \(a \leq x \leq b\). Then there exists \(a \leq y \leq b\) such that \(a = x \land y\) and \([x \lor y, b] \in \mathcal{B}\) since \([a, b]\) is \(\mathcal{B}\)-full. From \(a = x \land y\) we have \(a = x\) or \(a = y\) because \([a, b]\) is uniform. Therefore \(a = x\) or \([x, b] \in \mathcal{B}\), that is, \([a, b] \in \mathcal{C}(\mathcal{B})\).

With these lemmas we can give a proof of Theorem 3.21 in idiom theory.

**Theorem 3.24** For a nucleus \(j\) on an idiom \(A\) the following conditions are equivalent:

1. \(j\) has Gabriel dimension.
2. \(j\) has Boyle dimension and every interval \([a, b]\) contains a uniform sub-interval.

**Proof** (1) \(\Rightarrow\) (2) It is immediate.

(2) \(\Rightarrow\) (1) Suppose that \(j\) has \(B - \text{dim} = a\) then \(j < \text{Boy}^{\alpha}(j) = \text{tp}\). By Lemma 3.22 there exists a \(\mathcal{D}_j\)-full interval containing a proper uniform interval \([a, b]\) which is \(\mathcal{D}_j\)-full. By Lemma 3.23 \([a, b]\) is \(\mathcal{D}_j\)-critical thus \(j\) has Gabriel dimension.

**4 Spectral Aspects of Idioms**

In this section we develop a fragment of boolean aspects in idiom theory. We introduce the concept of spectral nucleus and then we mimic some spectral-Grothendieck situations into the idiomatic shape. We observe that the idiomatic facet of these objects is the external version of the Grothendieck case in particular the module category realm. First let us recall the definition of spectral category.

**Definition 4.1** A Grothendieck category \(\mathcal{C}\) is spectral if every short exact sequence in \(\mathcal{C}\) splits.

From this point, spectral category will mean a Grothendieck category which is spectral. Spectral categories are related with von Neumann regular rings, see [25, V.6.1] and [10].

Before we give the definition of spectral nucleus, we will point out a motivational situation.

**Definition 4.2** Let \(R\) be a ring. A nucleus \(j : \Lambda(R) \rightarrow \Lambda(R)\) is respectful or linear if
\[
j(I : r) = (j(I) : r)
\]
for each \(I \in \Lambda(R)\) and each \(r \in R\). Denote by \(\mathcal{E}(R)\) the set of linear nuclei on \(\Lambda(R)\).

**Definition 4.3** A global closure operator on \(R\)-Mod is a family \(j_{\bullet} = (j_M : \Lambda(M) \rightarrow \Lambda(M) \mid M \in R\text{-Mod})\) such that every \(j_M\) is a nucleus and for every morphism \(f : M \rightarrow N\),
\[
f^{-1} j_N = j_M f^{-1} : \Lambda(N) \rightarrow \Lambda(M)
\]
Theorem 4.4 Let $R$ be a ring. There is a bijective correspondence between:

1. $R$-tors.
2. $\Xi(R)$.
3. Global closure operator on $R$-Mod.
4. Gabriel Filters of $R$.
5. Left exact radicals of $R$-Mod.

A proof of this theorem can be found in [11,17]. Recall that any localization of the category of $R$-Mod is given by an element of the frame $R$-tors, hence by Theorem 4.4 an element of $\Xi(R)$ is called a localizer.

Let us recall the assignments:

$R$-tors $\leftarrow \Xi(R) \rightarrow$ Global operators on $R$-Mod.

Let $j \in \Xi(R)$.

For each module $M$ set:

$m \in j_M(K) \iff j(K : m) = R$ for every $K \in \Lambda(M)$.

This determines a nucleus on $\Lambda(M)$ and the collection $j_\bullet = (j_M \mid M \in R$-Mod) constitutes a global closure operator, in fact $j_R = j$.

On the other hand, $j$ defines a torsion class as follows:

$M \in T_j \iff j_M(0) = M$.

Definition 4.5 Given a class of modules $B$ and a module $M$, the slice of $B$ by $M$, $\langle M \rangle(B)$ is defined as:

$[K, L] \in \langle M \rangle(B) \iff L/K \in B$.

It can be seen that if $D$ is an hereditary torsion class then $\langle M \rangle(D)$ is a division set in $\Lambda(M)$ (see [22]).

We need the following lemma, a proof of it can be found in [22].

Lemma 4.6 Let $j_\bullet = (j_M \mid M \in R$-Mod) be a global inflator on a module category $R$-Mod. Then

$$j_{M/N}(\frac{K}{N}) = \frac{j_M(K)}{N}$$

for any modules $N \subseteq K \subseteq M$.

We adopt the following definition of spectral torsion theory.

Definition 4.7 A hereditary torsion theory $\tau = (T, F)$ is spectral on a module category $R$-Mod, if the quotient category $(R, \tau)$-Mod is a spectral category.

As we mentioned before spectral aspects of Grothendieck categories give rise to certain boolean aspects for example the following is a consequence of the theory developed in [10].

Proposition 4.8 Let $\tau = (T, F)$ be a spectral torsion theory in $R$-Mod. Then for any module $M$ the idiom of sub-objects of $M$ in the quotient category is a complemented idiom.

For more information in the case of module categories the author is invited to see: [1,6,9].
Theorem 4.9  Let \( j \) be a linear nucleus on \( \Lambda(R) \). Then, \( \Lambda(R)j \) is a complemented idiom if and only if the hereditary torsion theory with torsion class \( T_j \) is spectral.

Proof Suppose \( \Lambda(R)j \) is a complemented idiom. Let \( \tau_j \) be the hereditary torsion theory with hereditary torsion class \( T_j \) and \( M \) a \( \tau_j \)-torsion free module. Let \( N \in \Lambda(M) \) be an essential element. By Proposition 1.1 of \([1]\) it is enough to prove that \( M/N \in T_j \). Using Lemma 4.6 we will see that

\[
j_M(0) = M/N = \frac{j_M(N)}{N}\]

Let \( m \in M \). By hypothesis there exists \( K \in \Lambda(R)_j \) such that \( K \cap j(N : m) = 0 \) and \( K \cup j(N : m) = R \). Since \( N \) is essential in \( M \) we have that \( (N : m) \) is an essential left ideal and from \( (N : m) \cap K \leq j(N : m) \cap K = 0 \) thus \( K = 0 \). Therefore \( j(N : m) = R \). This is equivalent to say \( M = j_M(N) \) precisely when \( M/N \in T_j \). The converse follows directly from the fact that for this torsion theory \( T_j \) the induced nuclei \( j_M \) in every \( \Lambda(M) \) and the corresponding quotient \( \Lambda(M)j_M \) (which is the idiom of sub-objects of every localizing object on \( T_j \)) is complemented by Proposition 4.8, in particular \( \Lambda(R)j = j_R \) is complemented. \( \square \)

Theorem 4.9 motivates the following definition:

Definition 4.10 Let \( A \) be an idiom. A nucleus \( j \) on \( A \) is **spectral** if \( A_j \) is a complemented idiom.

Remark 4.11 Let \( j \) be a spectral nucleus.

1. This is equivalent to \( cbd^j = tp \).
2. From Theorem 5.8 of \([23]\) we find out that \( Boy(j) = cbd_j^\infty = tp \) thus \( j \) has weak \( Boy \)-dimension.
3. Trivially every spectral nucleus satisfies Theorem 3.1. The frame \( \uparrow (j) \) is a complete boolean algebra, this is straightforward since \([j, Boy(j)]\) is complemented.
4. From \( soc_j = soc^\infty \land cbd_j \) (Corollary 6.3 of \([23]\)) we deduce that \( soc[j] = soc[j]^\infty \).
5. Let \( E(A) \) be the set of all spectral nuclei of \( A \), observe that this is an upper section of \( N(A) \).

One of the equivalent conditions of Proposition 1.1 of \([1]\) or Proposition 2.2 of \([9]\) implies that for spectral torsion theories, the torsion free modules are full modules. We will prove this fact in the idiomatic case. Recall that for any interval \([a, b]\), \( \chi(a, b) \) denotes the nucleus on \( A \) given by \( j \subseteq \chi(a, b) \iff j(a) \land b = a \). This is the idiomatic analogue of the cogenerated torsion theory for a module (see \([16]\) for details and uses of these nuclei).

Proposition 4.12 Let \( A \) be an idiom and \( j \in N(A) \) a nucleus. The following statements are equivalent:

1. \( j \) is spectral.
2. For all \([a, b]\) with \( j \leq \chi(a, b) \) we have \([a, b] \in \mathcal{H}(D) \).

Proof We will use Lemma 2.14 to show that \((1) \Rightarrow (2)\). Let \([a, b]\) be a non-trivial interval with \( j \leq \chi(a, b) \) and let \( x \in A \) be such that \( a \leq x \). From \( a \leq b \land x \leq b \) we have that \( \chi(a, b \land x)(a) \land b \land x = a \) thus \( \chi(a, b \land x)(a) = a \) (since \( b \land x \) is essential in \([a, b]\)). Therefore \( j(a) = a \) and then \( a \leq j(x) \). Now from (1) there exists \( z \in A_j \) such that \( j(x) \land z = j(0) \) and \( j(x) \lor z = 1 \). Since \( a \leq j(x) \) and \( j(0) \leq a \) then \( z \leq a \). Hence,

\[
j(x) = j(x) \lor a \geq j(x) \lor z = 1.\]
Thus \( j(b \land x) = j(b) \land j(x) = j(b) \geq b \), that is, \([b \land x, b] \in \mathcal{D}_j\).

Conversely, note that the interval \([j(0), 1]\) satisfies \( j \leq \chi(j(0), 1) \). By hypothesis, for every \( a \in A_j \) there exists \( b \in [j(0), 1] \) such that \( j(0) = a \land b = a \land j(b) \) and \([a \lor b, 1] \in \mathcal{D}_j\), this is equivalent to \( 1 = a \lor j(b) \) (where \( \lor \) is the supremum in \( A_j \)). Therefore every element in \( A_j \) has a complement as required.

Next we will see an important property of the negation of a spectral nucleus. The following Proposition uses the Lemma 2.16.

**Proposition 4.13** Let \( j \) be an spectral nucleus. For every \([a, b] \in \mathcal{D}_{\neg j} = \neg \mathcal{D}_j\) the following assertions hold:

1. \([a, b] \in \mathcal{C}mp(\emptyset)\).
2. There exist \( y \in [a, b] \) such that \([y, b] \in \mathcal{S}ct(\mathcal{D}_j)\).

**Proof** Consider any interval \([a, b] \in \mathcal{D}_{\neg j} = \neg \mathcal{D}_j\). Observe that \( j \leq \chi(a, b) \) since \([a, j(a) \land b] \in \mathcal{D}_j \land \neg \mathcal{D}_j = \emptyset\), hence \([a, b] \in \mathcal{F}ll(\mathcal{D}_j)\) by Proposition 4.12.

For (1), consider any \( a \leq x \leq b \) then there exists \( a \leq y \leq b \) such that \( a = x \land y \) and \([x \lor y, b] \in \mathcal{D}_j\) also this interval is a sub-interval of \([a, b]\) then \([x \lor y, b] \in \mathcal{D}_j \land \neg \mathcal{D}_j = \emptyset\) that is, \( x \lor y = b \).

Now for assertion (2) consider \( a \leq b \land \mathcal{S}oc^\infty_j(a) \leq b \). For this element we can find \( a \leq y \leq b \) such that \( a = y \land b \land \mathcal{S}oc^\infty_j(a) = y \land \mathcal{S}oc^\infty_j(a) \) and \([y \lor (b \land \mathcal{S}oc^\infty_j(a)), b] \in \mathcal{D}_j\).

Let us see that the element \( y \) satisfies our requirements. First this interval is a sub-interval of \([a, b]\) thus \( y \lor (b \land \mathcal{S}oc^\infty_j(a)) = b \) and by modularity we have that \( b \land (y \lor \mathcal{S}oc^\infty_j(a)) = b \), that is, \( b \leq y \lor \mathcal{S}oc^\infty_j(a) \). Therefore \( b \leq \mathcal{S}oc^\infty_j(y) \) which is equivalent to \([y, b] \in \mathcal{S}ct(\mathcal{D}_j)\). By Lemma 6.4 of [23] we have \([y, b] \in \mathcal{S}ct(\mathcal{D}_j)\), as required.

A consequence of this proposition is the following:

**Corollary 4.14** Let \( j \in \mathcal{E}(A) \). Then

\[ \mathcal{B}oy(\mathcal{D}_{\neg j}) = \mathcal{B}oy(\mathcal{C}mp). \]

which is equivalent to

\[ \mathcal{C}bd_{\neg j} = \mathcal{C}bd^\infty. \]

**Proof** This is immediate from the fact that \( \mathcal{D}_{\neg j} \subseteq \mathcal{C}mp \).

**Definition 4.15** An interval \([a, b]\) on an idiom \( A \) is weakly atomic, if for every \( a \leq c \leq d \leq b \) there exist \( c \leq x < y \leq d \) with \([x, y] \in \mathcal{S}mp\). Denote by \( \mathcal{W}A \) the set of all weakly atomic intervals.

In [19] it is proved that \( \mathcal{W}A \) is a division set and, moreover, this set [Theorem 7.9 in [19]] has the property:

\[ \mathcal{C}mp \cap \mathcal{W}A = \mathcal{S}S\mathcal{P}. \]

**Corollary 4.16** Let \( A \) be an idiom and \( j \in \mathcal{E}(A) \). Then every \([a, b] \in \mathcal{D}_{\neg j} \land \mathcal{W}A \) is semi-simple.

**Proof** From Proposition 4.13.(1), we have that any \([a, b] \in \mathcal{D}_{\neg j} \land \mathcal{W}A \) is complemented. Therefore \( \mathcal{D}_{\neg j} \land \mathcal{W}A \subseteq \mathcal{C}mp \land \mathcal{W}A = \mathcal{S}S\mathcal{P} \).
Corollary 4.17 Let $A$ be a compactly generated idiom, and $j \in \mathcal{E}(A)$. Then every interval $[a, b] \in \mathcal{D}_j$ is semi-simple.

**Proof** If $A$ is compactly generated then it is weakly atomic (see Theorem 7.8 of [19]) therefore this is a direct consequence of Proposition 4.16.

The lattice $\Lambda(M)$ is compactly generated for every module $M$, thus the above facts resemble the module theoretic environment.

The following theorem is the idiomatic analogue of Lemma 2.12 in [1].

**Theorem 4.18** Let $j$ be a nucleus on $A$ such that $\text{Boy}(j) = tp$. Then $\neg j \lor \neg \neg j = tp$.

**Proof** Recall that $N(A)_{\neg\neg}$ is a complete boolean algebra. Then the element

$$\neg\neg j \in N(A)_{\neg\neg}$$

has a unique complement there and this complement is $\neg j$. Recall that the supremum in the quotient $N(A)_{\neg\neg}$ is described as:

$$\neg\neg (\neg\neg k \lor \neg\neg k')$$

for any $k, k' \in N(A)$ and in our case we have

$$\neg\neg (\neg\neg j \lor \neg j) = tp$$

Now under the hypothesis $[j, \text{Boy}(j)] = [j, tp]$ is a complete boolean algebra, then for the nucleus $\neg\neg j \lor \neg j$ there exists $l \in [j, tp]$ such that $(\neg\neg j \lor \neg j) \lor l = tp$ and $(\neg\neg j \lor \neg j) \land l = j$. Applying $\neg\neg(\_)$ to the last equality

$$\neg\neg(\neg\neg j \lor \neg j) \land \neg \neg l = tp \land \neg \neg l = \neg \neg l = \neg \neg j$$

and using $(\neg\neg j \lor \neg j) \lor l = tp$ we have

$$tp = (\neg\neg j \lor \neg j) \lor l \leq (\neg\neg j \lor \neg j) \lor \neg \neg l = (\neg\neg j \lor \neg j) \lor \neg \neg j = \neg\neg j \lor \neg j = tp$$

as required.

As a consequence of the above we have:

**Corollary 4.19** Let $j \in \mathcal{E}(A)$. Then $\neg j \lor \neg \neg j = tp$.

**Corollary 4.20** Let $j$ be a nucleus such that $j, \neg j \in \mathcal{E}(A)$. Then the intervals of

$$\mathcal{D}_{\neg j} \cap \mathcal{WA}$$

and

$$\mathcal{D}_{\neg\neg j} \cap \mathcal{WA}$$

are semi-atomic.

**Proof** By Proposition 4.13.(1) we have

$$\mathcal{D}_{\neg j} \cap \mathcal{WA} \subseteq \mathcal{WA} \cap \mathcal{Cmp} = \mathcal{SS}p$$

and by the spectral property of $\neg j$ we also have

$$\mathcal{D}_{\neg\neg j} \cap \mathcal{WA} \subseteq \mathcal{WA} \cap \mathcal{Cmp} = \mathcal{SS}p$$
then

\((\mathcal{D}_{\neg j} \cup \mathcal{D}_{\neg\neg j}) \cap \mathcal{WA} \subseteq \mathcal{SS}p\)

using the frame distributivity law of \(\mathcal{B}(A)\). Taking division sets we have

\[ \mathcal{D}_{vs}(\mathcal{D}_{\neg j} \cup \mathcal{D}_{\neg\neg j}) \cap \mathcal{WA} = \mathcal{WA} \subseteq \mathcal{D}_{vs}(\mathcal{SS}p) = \mathcal{SA} \]

the first equality follows from Theorem 4.18 and the second equality from Theorem 7.11 of [19].

**Corollary 4.21** Let \(A\) be a weakly atomic idiom, that is, \(\mathcal{WA} = \mathcal{I}(A)\). Then \(A\) is semi-atomic, that is, \(\text{Gab}(id) = \text{soc}^{\infty} = \text{tp}\).

**Proof** This follows directly from Corollary 4.20.

The following appear as Lemma 2.13 in [1].

**Corollary 4.22** Let \(R\) be a ring and suppose that \(j, \neg j \in \mathcal{E}(\Xi(R))\). Then \(R\) is a left semiar-tinian ring.

**Proof** Direct from 4.21.

**Acknowledgements** We would like to thank the referee for a careful and detailed reading of the manuscript and suggestions to improve it. This work was supported by the grant UNAM-DGAPA-PAPIIT IN100517.

**References**

1. Arroyo Paniagua, M.J., Montes, J.R.: Some aspects of spectral torsion theories. Commun. Algebra 22(12), 4991–5003 (1994)
2. Boyle, A.K.: The large condition for rings with krull dimension. Proc. Am. Math. Soc. 72(1), 27–32 (1978)
3. Castro, J., González, P.M., Montes, J.: Some aspects of-full modules. East-West J. Math. 9(2), 139–159 (2007)
4. Golan, J.S.: Torsion Theories, vol. 29. Longman Scientific & Technical, Harlow (1986)
5. González, P.M.: Algunos aspectos sobre módulos \(\tau\)-plenos, Ph.D. thesis, Universidad Nacional Autónoma de México (2008)
6. Gómez Pardo, J.L.: Spectral Gabriel topologies and relative singular functors. Commun. Algebra 13(1), 21–57 (1985)
7. Golan, J.S., Simmons, H.: Derivatives, nuclei and dimensions on the frame of torsion theories. Pitman Research Notes in Mathematics Series (1988)
8. Johnstone, P.T.: Stone Spaces. Cambridge University Press, Cambridge (1986)
9. José, M., Paniagua, A., Montes, J.R., Wisbauer, R.: Spectral torsion theories in module categories. Commun. Algebra 25(7), 2249–2270 (1997)
10. Jan-Erik, R.: Locally Distributive Spectral Categories and Strongly Regular Rings, Reports of the Midwest Category Seminar, pp. 156–181. Springer, Berlin (1967)
11. Simmons, H.: The semiring of topologizing filters of a ring. Isr. J. Math. 61(3), 271–284 (1988)
12. Simmons, Harold: Near-discreteness of modules and spaces as measured by Gabriel and Cantor. J. Pure Appl. Algebra 56(2), 119–162 (1989)
13. Simmons, H.: The assembly of a frame. http://www.cs.man.ac.uk/~hsimmons/FRAMES/B-Assembly.pdf (2006)
14. Simmons, H.: The basics of frame theory. http://www.cs.man.ac.uk/~hsimmons/FRAMES/A-Basics.pdf (2006)
15. Simmons, H.: The higher level CB properties of frames. http://www.cs.man.ac.uk/~hsimmons/TEMP/HigherAssemb.pdf (2006)
16. Simmons, H.: A decomposition theory for complete modular meet-continuous lattices. Algebra Universalis 64(3–4), 349–377 (2010)
17. Simmons, H.: How to generate G-topologies for module presheaf categories. http://www.cs.man.ac.uk/~hsimmons/FOR-TOMMY/G-GTOPS.pdf (2012)
18. Simmons, H.: Cantor–Bendixson Properties of the Assembly of a Frame, Leo Esakia on Duality in Modal and Intuitionistic Logics, pp. 217–255. Springer, Berlin (2014)
19. Simmons, H.: Cantor–Bendixson, socle, and atomicity. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/002-Atom.pdf, 2 (2014)
20. Simmons, H.: The Gabriel and the Boyle derivatives for a modular idiom. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/004-GandB.pdf, 3 (2014)
21. Simmons, H.: An introduction to idioms. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/001-Idioms.pdf, 2 (2014)
22. Simmons, H.: A lattice theoretic analysis of a result due to Hopkins and Levitzki. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/007-HL.pdf, 3 (2014)
23. Simmons, H.: The relative basic derivatives for an idiom. http://www.cs.man.ac.uk/~hsimmons/00-IDSandMODS/003-Rel.pdf, 2 (2014)
24. Simmons, H.: Examples of higher level assemblies (to appear) (2017)
25. Stenström, B.: Rings of Quotients: An Introduction to Methods of Ring Theory, vol. 217. Springer, Berlin (1975)
26. Wilson, J.T.: The assembly tower and some categorical and algebraic aspects of frame theory. Ph.D. thesis, Carnegie Mellon University (1994)

Affiliations

Jaime Castro Pérez¹ · Mauricio Medina Bárcenas² · José Ríos Montes³ · Angel Zaldívar Corichi⁴

Jaime Castro Pérez
jcastrop@itesm.mx

Mauricio Medina Bárcenas
mmedina@cnu.ac.kr

José Ríos Montes
jrios@matem.unam.mx

¹ Escuela de Ingeniería y Ciencias, Instituto Tecnológico y de Estudios Superiores de Monterrey, Calle del Puente 222, Tlalpan, 14380 Mexico, D.F., Mexico
² Department of Mathematics, Chungnam National University, Yuseong-gu, Daejeon 34134, Republic of Korea
³ Instituto de Matemáticas, Universidad Nacional, Autónoma de México, Area de la Investigación Científica, Circuito Exterior, C.U., 04510 Mexico, D.F., Mexico
⁴ Departamento de Matemáticas, Centro Universitario de Ciencias Exactas e Ingenierías, Universidad de Guadalajara, Blvd. Marcelino García Barragán, 44430 Guadalajara, J.A.L., Mexico