Effective Linear Two-Body Method for Many-Body Problems
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Abstract

This paper reports a detailed description of the equivalent linear two-body method for the many body problem, which is based on an approximate reduction of the many-body Schrödinger equation by the use of a variational principle. To test the accuracy of the method it has been applied to the one-dimensional $N$-body problem with pair-wise contact interactions (McGurie-Yang $N$-body problem) and to the dilute Bose-Einstein condensation (BEC) of atoms in harmonic traps at zero temperature. For both cases, it is shown that the method gives excellent results for large $N$. 

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I. Introduction

In this paper we present an approximate method of obtaining the eigenvalue solutions of the system of interacting $N$ bosons using an equivalent two-body method similar to that used by Feshbach and Rubinov [1] for the triton ($^3H$) three-body ($N=3$) bound state. They [1] used both the variational principle and a reduced coordinate variable (not the hyperradius) to obtain an equivalent two-body equation for the three-body bound state ($^3H$). For many-body problems, use of one reduced coordinate variable (the hyperradius [2]) was made to obtain equivalent two-body equations by keeping only a finite sum of terms of the hyperspherical expansion with $K = K_{\text{min}}$ ($K$ is the global angular momentum). This method has been applied to the ground state of the $N$-body system composed of distinguishable particles or of bosons and also to nuclear bound states [3,4]. It was shown that the method leads to the correctly behaved nuclear bound states in the limit of large $A$ ($A$ is the nucleon number) [4]. Recently, it has been used to describe the Bose-Einstein condensation (BEC) of atoms in isotropic harmonic traps [5]. We note that Morse and Feshbah [6] have used hyperspherical coordinates for solving the problem of two interacting particles in a central field of force. More references can be found in [7,8].

For the $N$-body system, our method for obtaining the equivalent linear two-body (ELTB) equation consists of two steps. The first step is to give the $N$-body wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots)$ a particular functional form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots) \approx \tilde{\Psi}(\zeta_1, \zeta_2, \zeta_3),$$

(1)

where $\zeta_1, \zeta_2,$ and $\zeta_3$ are known functions. We limit $\zeta$’s to three variables in order to obtain the ELTB equation. The second step is to derive an equation
for $\tilde{\Psi}(\zeta_1, \zeta_2, \zeta_3)$ by requiring that $\tilde{\Psi}$ must satisfy a variational principle

$$\delta \int \tilde{\Psi}^* \tilde{\Psi} d\tau = 0$$

with a subsidiary condition $\int \tilde{\Psi}^* \tilde{\Psi} d\tau = 1$. This leads to a linear two-body equation from which both eigenvalues and eigenfunctions can be obtained. The lowest eigenvalue is an upper bound of the lowest eigenvalue of the original N-body problem.

In Section II, we apply the method to the one-dimensional $N$-body problem with pair-wise contact interactions (the McGuire-Yang $N$-body problem [9, 10]) and demonstrate that our method is a very good approximation for the case of large $N$. In Section III, we consider the Bose-Einstein condensation (BEC) of atoms in isotropic harmonic traps. In Sections IV and V we apply the method to the dilute BEC of atoms in anisotropic harmonic traps. It was shown the method gives excellent results for large $N$. In Section VI we investigate the stability of the BEC for the case of atoms with negative scattering length at zero temperature using our method. A summary and conclusions are given in Sec. VII.

II. McGuire-Yang N-body Problem

There are only several known cases of exactly solvable three-body and four-body problems. For the $N = 3$ case it was shown [11] that the Faddeev equations [12] for one-dimensional three-body problem with pair-wise contact interactions are exactly solvable. For the one-dimensional $N = 4$ case, analytical solutions of the four-body Faddeev-Yakubovsky were obtained in [13]. We note that for nuclear three-body systems with short-range interactions, the Schrödinger equation in three dimensions is reformulated into
the Faddeev equations [12] which have been solved numerically after making partial wave expansion [14] or without partial wave expansion [15]. In the following, we consider an exactly solvable one-dimensional N-body system as a test case for our method.

For the one-dimensional N-body problem with the Hamiltonian

\[ H = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + c \sum_{i<j} \delta(x_i - x_j), \]

the Schrödinger equation

\[ H \Psi = E \Psi \]

is exactly solvable. The bound and scattering states for this system have been found by McGuire [9] and by Yang [10].

For the case \( c < 0 \), there are bound states [9] for the system of \( N \) bosons with the wave function of the following form

\[ \Psi = \exp\left[\frac{c}{2} \sum_{i<j} |x_i - x_j| \right], \]

and the energy of this bound state is given by

\[ E = -c^2 N(N^2 - 1)/24. \]  (3)

The McGuire-Yang (MY) N-body problem provides a unique possibility of checking the validity and accuracy of various approximations made for the Schrödinger equation describing \( N \) particles interacting via short range potential. In the following, we describe the equivalent linear two-body equation (ELTBE) method and compare its solution with the exact solution, Eq. (3), of the MY problem.

For this case, we seek for eigenfunction \( \Psi \) of \( H \) in the form of

\[ \Psi \approx \tilde{\Psi}(\rho), \]
where
\[ \rho = \sqrt{\frac{1}{N} \sum_{i<j} (x_i - x_j)^2}. \]

We now derive an ELTB equation for \( \tilde{\Psi}(\rho) \) by requiring that \( \tilde{\Psi} \) must satisfy a variational principle (2). This requirement leads to the equation
\[ \left[ -\frac{d^2}{d\rho^2} + \frac{(N - 2)}{\rho} \frac{d}{d\rho} + 2V(\rho) \right] \tilde{\Psi}(\rho) = 2E\tilde{\Psi}(\rho), \tag{4} \]
where
\[ V(\rho) = \frac{\tilde{g}}{\rho}, \tag{5} \]
with
\[ \tilde{g} = cN(N - 1) \frac{\Gamma(N/2 - 1/2)}{2\sqrt{2\pi}\Gamma(N/2 - 1)} \tag{6} \]
(see Appendix A for details). Eq. (4) with the Coulomb like potential \( V(\rho) = \frac{\tilde{g}}{\rho} \), Eq. (5), can be solved analytically.

Instead of the parameter \( E \) and variable \( \rho \) in the Schrödinger equation (4), we introduce the new quantities
\[ z = \frac{2\tilde{g}}{\eta} \rho, \quad \eta = \frac{\tilde{g}}{\rho} \sqrt{\frac{1}{-2E}} \tag{7} \]
For negative \( \tilde{g} \) and negative energies, \( \eta \) is real negative number. On making the substitutions (7), Eq. (4) becomes
\[ \frac{d^2\tilde{\Psi}}{dz^2} + \frac{(N - 2)}{z} \frac{d\tilde{\Psi}}{dz} + \left[ -\frac{1}{4} - \frac{\eta}{z} \right] \tilde{\Psi} = 0. \tag{8} \]
To calculate the asymptotic behavior of \( \tilde{\Psi} \) for large \( z \), we omit from Eq. (8) the terms in \( 1/z \), and \( 1/z^2 \), and obtain the equation
\[ \frac{d^2\tilde{\Psi}}{dz^2} = \frac{1}{4} \tilde{\Psi}, \]
which shows that \( \tilde{\Psi} \) behaves as \( \exp[-z/2] \), a correct asymptotic form.

After substitution
\[
\tilde{\Psi}(z) = e^{-z/2}y(z)
\]

Eq. (8) becomes
\[
z \frac{d^2y}{dz^2} + (N - 2 - z) \frac{dy}{dz} - (\eta + \frac{N - 2}{2})y = 0. \tag{9}
\]

The solution of this equation (finite for \( z = 0 \)) is the confluent hypergeometric function
\[
y = F(\eta + \frac{N - 2}{2}, N - 2, z). \tag{10}
\]

A solution which satisfies the condition at infinity is obtained only for negative integer (or zero) values of \( \eta + \frac{N - 2}{2} \),
\[
\eta + \frac{N - 2}{2} = -n, (n = 0, 1, ...). \tag{11}
\]

From the definition, Eq.(7), of the parameter \( \eta \), we find
\[
E_n(N) = -2 \frac{\tilde{g}^2}{(N - 2 + 2n)^2}. \tag{12}
\]

Using Eq.(6), we have the following expression for ground state energy
\[
E_0(N) = -\frac{e^2}{4\pi} \frac{N(N - 1)\Gamma(N/2 - 1/2)}{(N - 2)\Gamma(N/2 - 1)}^2. \tag{13}
\]

We note that our system has only one bound state, \( n = 0 \), consistent with the exact solution, Eq.(3). States with \( n \neq 0 \) are not bound states because \(| E_{n\neq0}(N) | < | E_0(N - 1) | \).

In the case of large \( N \), using the asymptotic formulas for \( \Gamma \) function,
\[
\lim_{|z| \to \infty} \frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} (1 + O(\frac{1}{z})),
\]
we obtain
\[ E_0(N) = -\frac{c^2}{8\pi}N^3 \]
for the leading term of Eq. (13). On the other hand we have for the large \( N \) case from Eq. (3),
\[ E = -\frac{c^2}{24}N^3(1 + O\left(\frac{1}{N^2}\right)). \]

Therefore, for the McGuire-Yang N-body problem, we have demonstrated that the ELTBE method, Eqs. (4 - 6), is a very good approximation for the case of large \( N \) (the relative error for binding energy is about 4.5 %). Furthermore, our approximation, Eq. (13), agrees remarkably well with the exact value, Eq. (3), for any \( N \) (the maximum value of relative error for the binding energy occurs for \( N = 3 \) and is about 10 %).

\textbf{III. Isotropic Trap}

In this section, we consider \( N \) identical bosonic atoms confined in a harmonic isotropic trap with the following Hamiltonian
\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \frac{1}{2}m\omega^2 \sum_{i=1}^{N} r_i^2 + \sum_{i<j} V_{int}(r_i - r_j).
\]

(14)

For the eigenfunction \( \Psi \) of \( H \), we assume that the solution for \( \Psi \) has the following form
\[
\Psi(\vec{r}_1, \ldots, \vec{r}_N) \approx \frac{\tilde{\Psi}(\rho)}{\rho^{(3N-1)/2}},
\]

(15)

where
\[
\rho = \sum_{i=1}^{N} r_i^2.
\]

(16)
We now derive an ELTB equation for \( \tilde{\Psi} \) by requiring that \( \tilde{\Psi} \) must satisfy the variational principle (2). This requirement leads to the equation

\[
\tilde{H}\tilde{\Psi} = E\tilde{\Psi},
\]

where

\[
\tilde{H} = -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \frac{m}{2} \tilde{\omega}^2 \rho^2 + \frac{\hbar^2}{2m} \frac{(3N-1)(3N-3)}{4\rho^2} + V(\rho),
\]

with

\[
V(\rho) = \frac{N(N-1)}{\sqrt{2\pi}} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{1}{\rho^3} \int_0^{\sqrt{2}\rho} V_{\text{int}}(r) (1 - \frac{r^2}{2\rho^2})^{(3N/2-5/2)} r^2 dr,
\]

(see Appendix B for details).

We note that Eq. (17) is exactly the form of the Schrödinger two-body equation in which a centrifugal potential energy is given by \((N - 1)(N - 3)/(4\rho^2)\) with identification of angular momentum quantum number \(l = (N - 1)/2\).

In the dilute condensate case

\[
\rho \gg r_A,
\]

where \(r_A\) is an atom-atom interaction range. Hence we can use for \(V(\rho)\), Eq.(19), the following approximation

\[
V(\rho) = \frac{N(N-1)}{\sqrt{2\pi}} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{1}{\rho^3} \int_0^{\infty} V_{\text{int}}(r) r^2 dr,
\]

which is proportional to the scattering length \(a_B\) in the Born approximation for binary collisions

\[
a_B = \frac{m}{4\pi\hbar^2} \int_0^{\infty} V_{\text{int}}(r) r^2 dr.
\]
Since the actual atom-atom interaction in condensate is much larger than the scattering energy, it is not possible to use perturbation theory to describe the scattering. However, since large changes in the wave function only occur over very small distances and since the wave function outside the range of interaction is only slightly changed by the interaction, Fermi [16] realized that it is possible to introduce a pseudopotential, which can be used to calculate small changes in the wave function outside the range of interaction by perturbation theory.

Using the above argument, let us introduce a pseudopotential $U$, so that the two-body Schrödinger equation becomes

$$\frac{\hbar^2}{m} \Delta \psi + U \psi = E \Psi,$$  \hspace{1cm} (23)

with

$$U = \begin{cases} 
U_0 & \text{if } r < \eta, \\
0 & \text{otherwise,}
\end{cases}$$  \hspace{1cm} (24)

where $\eta$ is some distance chosen so that $\eta \gg |a|, \eta \gg r_A$, and $\eta \ll \tilde{\rho}$, with the scattering length $a$ and range of condensate $\tilde{\rho}$.

We hope to find $U$ so that the exact atom-atom wave function for $r > \eta$ will be given by the solution of Eq. (23) with amplitude $f(\theta)$ given by the Born approximation applied to $U$

$$f_B(\theta) = -\frac{m}{4\pi\hbar^2} \int d^3r U(r) e^{i(k_i - k_f) \cdot \vec{r}}.$$  \hspace{1cm} (25)

Since we know that the exact $f(\theta)$ is independent of $\theta$ for our very low energy case, the range $\eta$ of the pseudopotential $U(r)$ must satisfy

$$k\eta \ll 1,$$  \hspace{1cm} (26)
where \( k = \sqrt{mE/\hbar^2} \). In this case Eq. (25) reduces to

\[
f_B(\theta) = -a_B = -\frac{m}{3\hbar^2}U_0\eta^3.
\]  

(27)

In order for the Born approximation to be valid we require the following relation

\[
\frac{a - a_B}{a} \ll 1,
\]

(28)

where \( a \) is the exact scattering length with pseudopotential \( U(r) \).

Combining Eqs. (26-28) we find

\[
\eta_0 \ll \eta \ll \frac{1}{k},
\]

(29)

where \( \eta_0 \) is fixed from Eq.(28).

Therefore, the concept of pseudopotential may be used as long as the energy of relative motion, \( E \), is small

\[
E \ll \frac{\hbar^2}{m\eta_0^2}
\]

(30)

where \( \eta_0 \) is fixed from the condition \( \frac{a - a_B}{a} \leq 0.1 \). For the \( ^7\text{Li} \) case we have

\[
\eta_0 = 0.2 \times 10^{-5} \text{cm},
\]

(31)

and for the case of \( ^{87}\text{Rb} \) atoms we have

\[
\eta_0 = 0.7 \times 10^{-5} \text{cm}.
\]

(32)

Eqs.(30-32) imply that the temperature of the condensate must be considerably lower than \( 10^{-4}K \) for the \( ^7\text{Li} \) condensate and \( 7.6 \times 10^{-7}K \) for the \( ^{87}\text{Rb} \) condensate.
IV. Anisotropic Trap

In this section, we consider \( N \) identical bosonic atoms confined in a harmonic anisotropic trap with the following Hamiltonian

\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \frac{1}{2} \sum_{i=1}^{N} m (\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2) + \sum_{i<j} V_{\text{int}}(\vec{r}_i - \vec{r}_j),
\]  

(33)

For eigenfunction \( \Psi \) of \( H \), we assume the solution for \( \Psi \) has the following form

\[
\Psi(\vec{r}_1, \ldots, \vec{r}_N) \approx \tilde{\Psi}(x, y, z) \frac{(xyz)^{(N-1)/2}}{(1 x^2 + 1 y^2 + 1 z^2)^{(N-1)/2}},
\]

(34)

where

\[
x^2 = \sum_{i=1}^{N} x_i^2, \quad y^2 = \sum_{i=1}^{N} y_i^2, \quad z^2 = \sum_{i=1}^{N} z_i^2.
\]

(35)

We now derive an equation for \( \tilde{\Psi}(x, y, z) \) by requiring that \( \tilde{\Psi}(x, y, z) \) must satisfy the variational principle (2). This requirement leads to the equation

\[
\tilde{H} \tilde{\Psi} = E \tilde{\Psi},
\]

(36)

where

\[
\tilde{H} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)
\]

\[
+ \frac{\hbar^2 (N-1)(N-3)}{8m} \left( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) + V(x, y, z),
\]

(37)

with

\[
V(x, y, z) = \frac{N(N-1)}{2(2\pi)^{3/2}} \left( \frac{\Gamma(N/2)}{\Gamma((N-1)/2)} \right)^3 \frac{1}{xyz} G(x, y, z),
\]

(38)
\[ G(x, y, z) = \int_{-\sqrt{2}x}^{\sqrt{2}x} dx' \int_{-\sqrt{2}y}^{\sqrt{2}y} dy' \int_{-\sqrt{2}z}^{\sqrt{2}z} dz' V_{\text{int}}(\sqrt{(x')^2 + (y')^2 + (z')^2}) \]
\[ \times ((1 - (x')^2)(1 - (y')^2)(1 - (z')^2))^{\frac{N-3}{2}} \]

(39)

(see Appendix C for details). To the best of our knowledge, Eqs. (38) and (39) have not been discussed in the literature.

In the dilute condensate case

\[ x \gg r_A, \quad y \gg r_A, \quad z \gg r_A \quad \text{(40)} \]

we can use for \( V(x, y, z) \) the following approximation

\[ V(x, y, z) = \frac{N(N - 1)}{2(2\pi)^{3/2}} (\frac{\Gamma(N/2)}{\Gamma((N - 1)/2)})^3 \frac{1}{xyz} \]

\[ \times \int_{-\sqrt{2}x}^{\sqrt{2}x} dx' \int_{-\sqrt{2}y}^{\sqrt{2}y} dy' \int_{-\sqrt{2}z}^{\sqrt{2}z} dz' V_{\text{int}}(\sqrt{(x')^2 + (y')^2 + (z')^2}). \]

(41)

Since we have the following relation for the case of large \( x, y, \) and \( z, \)

\[ \int_{-\sqrt{2}x}^{\sqrt{2}x} dx' \int_{-\sqrt{2}y}^{\sqrt{2}y} dy' \int_{-\sqrt{2}z}^{\sqrt{2}z} dz' V_{\text{int}}(\sqrt{(x')^2 + (y')^2 + (z')^2}) = 4\pi \int_0^\infty V_{\text{int}}(r)r^2 dr \]

(42)

\( V(x, y, z) \) is proportional to the scattering length in the Born approximation for binary collisions

\[ V(x, y, z) = \frac{g}{xyz}, \]

(43)

with

\[ g = \frac{a_B h^2 N(N - 1)}{\sqrt{2\pi m}} (\frac{\Gamma(N/2)}{\Gamma((N - 1)/2)})^3. \]

(44)
Therefore, it is reasonable to replace expression (44) with a corresponding expression proportional to the exact scattering length $a$ for binary collisions (Landau replacement [17]).

$$g = \frac{ah^2N(N-1)}{\sqrt{2\pi m}} \left( \frac{\Gamma(N/2)}{\Gamma((N-1)/2)} \right)^3.$$

This approximation is equivalent to the following approximation for $V_{int}$

$$V_{int}(\vec{r}_i - \vec{r}_j) = \frac{4\pi h^2a}{m} \delta(\vec{r}_i - \vec{r}_j),$$

which is the Fermi pseudopotential [16].

For the positive scattering length case, $a > 0$, we look for the solution of Eq. (36) of the form

$$\tilde{\Psi}(x, y, z) = \sum_{i,j,k} c_{ijk} \Phi^{(1)}_i(x) \Phi^{(2)}_j(y) \Phi^{(3)}_k(z),$$

where $c_{ijk}$ are solutions of the following equations

$$\sum_{l,m,n} H_{ijk,lmn} c_{lmn} = E \sum_{l,m,n} \lambda_{ijk,lmn} c_{lmn}$$

with

$$H_{ijk,lmn} = <\Phi^{(1)}_i \Phi^{(2)}_j \Phi^{(3)}_k | \tilde{H} | \Phi^{(1)}_l \Phi^{(2)}_m \Phi^{(3)}_n >,$$

and

$$\lambda_{ijk,lmn} = <\Phi^{(1)}_i \Phi^{(2)}_j \Phi^{(3)}_k | \Phi^{(1)}_l \Phi^{(2)}_m \Phi^{(3)}_n > .$$

Using

$$\Phi^{(1)}_i(x) = x^{(N-1)/2} \exp[-m\tilde{\omega}(x/\alpha_i)^2/(2\hbar)],$$

$$\Phi^{(2)}_j(y) = y^{(N-1)/2} \exp[-m\tilde{\omega}(y/\beta_j)^2/(2\hbar)],$$

and

$$\Phi^{(3)}_k(z) = z^{(N-1)/2} \exp[-m\tilde{\omega}(z/\gamma_k)^2/(2\hbar)]$$
we have

\[
H_{ijk,lmn} = \frac{\hbar \omega N \lambda_{ijk,lmn}}{2} \left[ 1 + \alpha_i^2 \alpha_j^2 \alpha_k^2 \right] \left[ 1 + \beta_j^2 \beta_m^2 \gamma_n^2 \right] + \frac{1}{\alpha_i^2 + \alpha_i^2} \left[ 1 + \beta_j^2 \beta_m^2 \gamma_n^2 \right] + \frac{1}{\gamma_k^2 + \gamma_n^2} \gamma_n \gamma_n \right],
\]

and

\[
\lambda_{ijk,lmn} = \left[ \frac{8 \alpha_i \alpha_j \beta_m \gamma_k \gamma_n}{(\alpha_i^2 + \alpha_i^2)(\beta_j^2 + \beta_m^2)(\gamma_k^2 + \gamma_n^2)} \right]^{N/2}.
\]

with \( \tilde{g} = \frac{(N-1)}{2\sqrt{2N}} \tilde{n} \), \( \tilde{n} = 2\sqrt{\omega m/(2\pi \hbar)} N a \), \( \bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3} \), \( \alpha_x = \omega_x/\bar{\omega} \), \( \alpha_y = \omega_y/\bar{\omega} \), and \( \alpha_z = \omega_z/\bar{\omega} \).

For the case of large \( N \), \( \lambda_{ijk,lmn} \) reduces to the \( \delta \)-function

\[
\lambda_{ijk,lmn} \approx \delta_{il} \delta_{jm} \delta_{kn},
\]

and hence

\[
H_{ijk,lmn} \approx E \delta_{il} \delta_{jm} \delta_{kn}.
\]

Using Eq. (54) we have for the ground state energy

\[
E = \frac{\hbar \omega N}{2} \left[ 1 + \alpha_i^2 \alpha_j^2 \alpha_k^2 \right] \left[ 1 + \beta_j^2 \beta_m^2 \gamma_n^2 \right] + \frac{1}{\alpha_i^2 + \alpha_i^2} \left[ 1 + \beta_j^2 \beta_m^2 \gamma_n^2 \right] + \frac{1}{\gamma_k^2 + \gamma_n^2} \gamma_n \gamma_n \right],
\]

where parameters \( \alpha, \beta, \) and \( \gamma \) are solutions of the following equations

\[
\frac{\partial E}{\partial \alpha} = \frac{\partial E}{\partial \beta} = \frac{\partial E}{\partial \gamma} = 0.
\]

For the case of large \( N \) we can neglect the kinetic energy term in Eq. (55)

\[
E = \frac{\hbar \omega N}{2} \left[ \alpha_x^2 + \alpha_y^2 \beta^2 + \alpha_z^2 \gamma^2 + \frac{\tilde{g} \sqrt{2}}{\alpha \beta \gamma} \right]
\]
Substitution Eq. (57) into Eq. (56) gives

\[ \alpha^2 \alpha^2 = \frac{2\sqrt{2}\tilde{g}}{\alpha \beta \gamma} \]

\[ \alpha^2 \beta^2 = \frac{2\sqrt{2}\tilde{g}}{\alpha \beta \gamma} \]

\[ \alpha^2 \gamma = \frac{2\sqrt{2}\tilde{g}}{\alpha \beta \gamma} \]  

(58)

Solutions of these equations

\[ \alpha = \left( \frac{2\sqrt{2}\tilde{g}\alpha_x \alpha_y \alpha_z}{\alpha_x} \right)^{1/5} \]

\[ \beta = \left( \frac{2\sqrt{2}\tilde{g}\alpha_x \alpha_y \alpha_z}{\alpha_y} \right)^{1/5} \]

\[ \gamma = \left( \frac{2\sqrt{2}\tilde{g}\alpha_x \alpha_y \alpha_z}{\alpha_z} \right)^{1/5} \]

give for the ground state energy

\[ \frac{E}{N\hbar \omega} = \frac{5}{4}\tilde{n}^{3/5} \]  

(59)

We note that Eq.(59) is the exact ground state solution of Eq.(36) for large \( N \). For the case of large \( N \) we can obtain an essentially exact expression for the ground state energy by neglecting the kinetic energy term in the Ginzburg-Pitaevskii-Gross (GPG) equation [19] (the Thomas-Fermi approximation [18]) as

\[ \frac{E_{TF}}{N\hbar \omega} = \frac{5}{7} \left( \frac{15}{8} \sqrt{\pi} \right)^2 \tilde{n}^{2/5} \]  

(60)

Comparing Eq. (59) with Eq. (60), we can see that for the case of large \( N \), the ELTBE method is a very good approximation, with a relative error of about 8% for the binding energy.

V. Large N Limit.

After we have obtained Eq. (37) the next step is to make a proper choice for the large \( N \) limit of the Hamiltonian. To do this let us rescale variables \( x, y, z \)

\[ x = N^{1/2} \tilde{x}, \quad y = N^{1/2} \tilde{y}, \quad z = N^{1/2} \tilde{z}. \]  

(61)
We can rewrite Eq. (36) as
\[
-\bar{\hbar}^2 2mN^2 \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) \frac{\omega_x^2 \bar{x}^2 + \omega_y^2 \bar{y}^2 + \omega_z^2 \bar{z}^2}{2}
\]
\[+
\bar{\hbar}^2 \frac{(N-1)(N-3)}{4N^2} \left( \frac{1}{\bar{x}^2} + \frac{1}{\bar{y}^2} + \frac{1}{\bar{z}^2} \right) + \frac{\bar{\hbar}^2 N(N-1)}{mN\sqrt{2\pi}} \left( \frac{\Gamma(N/2)}{\Gamma((N-1)/2)N!^2} \right)^3 \frac{1}{\bar{x}^2 \bar{y}^2} \right] \tilde{\Psi} = \frac{E}{N} \tilde{\Psi}.
\]
(62)

In the large N limit, \(\frac{(N-1)(N-3)}{N^2}\) is of the order of unity and \(\frac{\Gamma(N/2)}{\Gamma((N-1)/2)N!^2}\) is of the order of \((1/2)^{3/2}\), and Eq. (62) simplifies to
\[
[-\frac{\hbar^2}{2mN^2} \left( \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) + V_{\text{eff}}(\bar{x}, \bar{y}, \bar{z})] \tilde{\Psi} = \frac{E}{N} \tilde{\Psi},
\]
(63)
where
\[
V_{\text{eff}}(\bar{x}, \bar{y}, \bar{z}) = \frac{m}{2} \left( \omega_x^2 \bar{x}^2 + \omega_y^2 \bar{y}^2 + \omega_z^2 \bar{z}^2 \right) + \frac{\hbar^2}{8m} \left( \frac{1}{\bar{x}^2} + \frac{1}{\bar{y}^2} + \frac{1}{\bar{z}^2} \right) + \frac{aN\hbar^2}{4m\sqrt{\pi \bar{x}\bar{y}\bar{z}}}. \tag{64}
\]

Equation (63) describes the motion of a particle with an effective mass \(mN^2\) in an effective potential \(V_{\text{eff}}(\bar{x}, \bar{y}, \bar{z})\). Therefore when \(N \to \infty\), the effective mass of the particle becomes infinitely large and then the particle may be assumed to remain essentially stationary at the absolute minimum of \(V_{\text{eff}}(\bar{x}, \bar{y}, \bar{z})\). Quantum fluctuations are unimportant in this limit and the most significant contribution to the ground state energy is given by
\[
E = NV_{\text{eff}}(x_0, y_0, z_0),
\]
(65)
where \(x_0, y_0, z_0\) are to be obtained from
\[
\frac{\partial V_{\text{eff}}(x_0, y_0, z_0)}{\partial x_0} = \frac{\partial V_{\text{eff}}(x_0, y_0, z_0)}{\partial y_0} = \frac{\partial V_{\text{eff}}(x_0, y_0, z_0)}{\partial z_0} = 0.
\]
(66)

Obviously Eq. (65) fails if the effective potential does not possess a minimum.

Instead of variables \(\bar{x}, \bar{y}, \bar{z}\) we introduce the new quantities
\[
x_t = \sqrt{\frac{m\bar{\omega}}{h}} \bar{x}, \quad y_t = \sqrt{\frac{m\bar{\omega}}{h}} \bar{y}, \quad z_t = \sqrt{\frac{m\bar{\omega}}{h}} \bar{z}.
\]
(67)
On making the substitutions (67), Eqs. (64) and (66) become

\[ V_{\text{eff}}(x_t, y_t, z_t) = \frac{\hbar \bar{\omega}}{2} \left[ (\alpha_x^2 x_{t}^2 + \alpha_y^2 y_{t}^2 + \alpha_z^2 z_{t}^2) + \frac{1}{4} \left( \frac{1}{x_t^2} + \frac{1}{y_t^2} + \frac{1}{z_t^2} \right) + \frac{\tilde{n}}{2^{3/2}} \frac{1}{x_t y_t z_t} \right], \]  

(68)

with

\[ 2\alpha_x^2 x_{t} + \frac{1}{2} x_{t}^3 = \frac{\tilde{n}}{x_t^2 y_t z_t}, \]

\[ 2\alpha_y^2 y_{t} + \frac{1}{2} y_{t}^3 = \frac{\tilde{n}}{y_t^2 x_t z_t}, \]  

(69)

and

\[ 2\alpha_z^2 z_{t} + \frac{1}{2} z_{t}^3 = \frac{\tilde{n}}{z_t^2 x_t y_t}. \]

In the case of large \( \tilde{n} = 2 \sqrt{\omega m / 2\pi \hbar N a} \) we can neglect \( \frac{1}{4} \left( \frac{1}{x_t^2} + \frac{1}{y_t^2} + \frac{1}{z_t^2} \right) \). In this case, solutions of Eq. (69)

\[ x_{t}^2 = \frac{\tilde{n}^{2/5}}{2\alpha_x^2}, \quad y_{t}^2 = \frac{\tilde{n}^{2/5}}{2\alpha_y^2}, \quad z_{t}^2 = \frac{\tilde{n}^{2/5}}{2\alpha_z^2}, \]  

(70)

give for the ground state energy, Eq. (65)

\[ E/(N\hbar \omega) = \frac{5}{4} \tilde{n}^{2/5}, \]

which is identical to (59). Hence we show the semiclassical nature of the large \( N \) approximation (59). Corrections to the result of the large \( N \) limit for the finite \( N \) case may be obtained by incorporating in the theory the quantum fluctuations around the classical minimum [20].

VI. Stability of BEC

When the scattering length is negative, the effective interaction between atoms is attractive. It has been claimed that the BEC in free space is impossible [21] because the attraction makes the system tend to an ever dense phase. For \( ^7Li \), the s-wave scattering length is \( a = (-14.5 \pm 0.4) \text{Å} \) [22]. For
bosons trapped in an external potential there may exist a metastable BEC state with a number of atoms below the critical value \( N_{cr} \) [23-31].

For the \( a < 0 \) case, we can see that potential energy in Eq. (37),

\[
W(x, y, z) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) + \frac{\hbar^2}{2m} \frac{(N - 1)(N - 3)}{4} \left( \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) - \frac{|g|}{xyz},
\]

for \( N < N_{cr} \) has a single metastable minimum which leads to the metastable BEC state. We note that for the case of large \( N_{cr} \), the ELTBE method leads to the same \( N_{cr} \) as the variational GPG stationary theory [30]. To show this, let us consider an anisotropic trap, \( \omega_x = \omega_y = \omega_\perp, \omega_z = \lambda \omega_\perp \).

Local minimum conditions \( \dot{A} > 0 \), where \( \dot{A} \) is a matrix with matrix elements \( A_{ij} = \partial^2 W/\partial x_i \partial x_j \), can be written for this case as

\[
n^2/2\delta_\perp^2 \delta_z^4 - n - \lambda^2 \delta_\perp \delta_z^2/32 + O\left( \frac{1}{N} \right) < 0,
\]

where \( \delta_z = (2m\omega_\perp/\hbar N_{cr}) z^2, \delta_\perp = (2m\omega_\perp/\hbar N_{cr}) x^2, \) and

\[
n = 2(m\omega_\perp/2\pi\hbar)^{1/2} N_{cr} |a|.
\]

Setting the left-hand side of Eq. (72) to zero and neglecting \( O\left( \frac{1}{N} \right) \) terms, we obtain the following equations for \( N_{cr} \)

\[
1 - 2\delta_\perp^2 = \delta_z^2 (1 + 8 \frac{\delta_z}{\delta_\perp} \lambda^2)^{1/2},
\]

\[
1 - \lambda^2 \delta_z^2 = \delta_z \delta_\perp [1 + (1 + 8 \frac{\delta_z}{\delta_\perp} \lambda^2)^{1/2}],
\]

and

\[
n = \delta_z^{1/2} \delta_\perp [1 + (1 + 8 \frac{\delta_z}{\delta_\perp} \lambda^2)^{1/2}].
\]

Eqs. (74) are exactly the same as equations for determining \( N_{cr} \) obtained from the variational GPG approach [30]. In this reference [30], it was found
that the numerical solution of Eqs. (57) for $0 \leq \lambda \leq 1$ can be interpolated as

$$n = e^{-(\alpha + \beta \lambda^2)}$$  \hspace{1cm} (75)

with $\alpha = 0.490419$, $\beta = 0.149175$. Using Eqs. (74) and (75) we have

$$N_{cr} = \left( \frac{2\pi \hbar}{\omega_\perp^0 m} \right)^{1/2} \frac{e^{-(\alpha + \beta \lambda^2)}}{2|a|}$$ \hspace{1cm} (76)

For an isotropic trap ($\lambda = 1$) we obtain from Eqs. (57) $\delta_\perp = 5^{-1/2}$ and $n = 5^{-1/4}0.8 \approx 0.535$ which are in agreement with the results of Refs. [24, 25]. Taking the experimental values of $^7Li$ trap parameters [32], $\omega_\perp/2\pi = 152$ Hz, and $\omega_z/2\pi = 132$ Hz we obtain $N_{cr} = 1456$. This value of $N_{cr}$ is consistent with theoretical predictions [27-31] and is in agreement with those observed in a recent experiment [32].

We note that the ELTBE method for a general anisotropic trap can be improved using a generalization of the hyperspherical expansion

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{K_x, K_y, K_z, \nu_x, \nu_y, \nu_z} \Psi^{\nu_x, \nu_y, \nu_z}_{K_x, K_y, K_z}(x, y, z) Y^{\nu_x}_{K_x}(\Omega_x) Y^{\nu_y}_{K_y}(\Omega_y) Y^{\nu_z}_{K_z}(\Omega_z),$$ \hspace{1cm} (77)

where the hyperspherical harmonics $Y^{\nu_x}_{K_x}(\Omega_x)$, $Y^{\nu_y}_{K_y}(\Omega_y)$, and $Y^{\nu_z}_{K_z}(\Omega_z)$ are eigenfunctions of the angular parts of the Laplace operators $\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$, $\sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2}$, and $\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2}$, respectively. However, we do not expect a fast convergence of the expansion Eq. (77) because of nonuniformity of the convergence of the expansion of $\sum_{i<j} V_{int}(\mathbf{r}_i - \mathbf{r}_j)$ in $x$, $y$, and $z$.

VII. Summary and conclusions

In summary, we have presented a method for obtaining an equivalent linear two-body equation from the Schrödinger equation for the system of
$N$ bosons, using reduced variables and variational principle. To access the accuracy of the method it has been applied to the McGuire-Yang N-body problem for which the exact solutions are known. Our method gives excellent results compared with exact solutions. The method has been applied also to the dilute Bose-Einstein condensation in anisotropic harmonic traps at zero temperature for both positive and negative scattering length. For large $N$, our method gives excellent results for all these cases.
Appendix A

In this Appendix, we derive Eqs. (5)-(6). To calculate $V(\rho)$ for McGuire-Yang problem we start from definition

$$V(\rho) = \frac{N(N-1)}{2} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dt V_{\text{int}}(\sqrt{2}x_1) e^{i \sum_{n=1}^{N-1} (x_n^2 - \rho^2)t} / \Omega, \quad (A.1)$$

where

$$\Omega = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dt e^{i \sum_{n=1}^{N-1} (x_n^2 - \rho^2)t}. \quad (A.2)$$

Using

$$\int_{-\infty}^{\infty} dx e^{ix^2t} = (-it)^{1/2} \pi^{1/2}, \quad (A.3)$$

and

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt \frac{e^{-ixt}}{(\epsilon - it)^{\nu}} = \begin{cases} \frac{2\pi x^{\nu-1}}{\Gamma(\nu)} & \text{if } x > 0, \\ 0 & \text{otherwise}, \end{cases} \quad (A.4)$$

we have from Eqs. (A.1) and (A.2)

$$\Omega = 2\pi \rho^{N-3} \frac{\Gamma(1/2)^{N-1}}{\Gamma((N-1)/2)}, \quad (A.5)$$

and

$$V(\rho) = \frac{N(N-1)}{2} \frac{\Gamma((N-1)/2)}{\Gamma(N/2 - 1) \Gamma(1/2)} \frac{1}{\rho} \int_{-\rho}^{\rho} V_{\text{int}}(\sqrt{2}x)(1 - \frac{x^2}{\rho^2})^{N/2 - 2} dx. \quad (A.6)$$

For the contact interaction $V_{\text{int}}(\sqrt{2}x) = c\delta(\sqrt{2}x)$ we have

$$V(\rho) = cN(N-1) \frac{\Gamma((N-1)/2)}{2\sqrt{2}\pi \Gamma(N/2 - 1) \rho} \quad (A.7)$$
Appendix B

In this Appendix, we present an outline for evaluating the effective potential $V(\rho)$ for the isotropic case.

Our starting formula is

$$V(\rho) = \frac{N(N - 1)}{2} \int d\vec{r} V_{\text{int}}(r) d\vec{R} d\vec{r}_3 \ldots d\vec{r}_N \int_{-\infty}^{\infty} dt e^{i(r^2/2 + R^2/2)t} e^{i(\sum_{n=3}^{N} r_n^2 - \rho^2)t} / \Omega, \quad (B.1)$$

where

$$\Omega = \int d\vec{r} d\vec{R} d\vec{r}_3 \ldots d\vec{r}_N \int_{-\infty}^{\infty} dt e^{i(r^2/2 + R^2/2)t} e^{i(\sum_{n=3}^{N} r_n^2 - \rho^2)t}. \quad (B.2)$$

Using

$$\int_0^{\infty} x^{\nu-1} \exp(-\mu x^p) dx = \frac{1}{\mu^{\nu/p}} \mu^{-\nu/p} \Gamma(\nu/p), \quad (B.3)$$

and Eq. (A.4) we have from (B.1) and (B.2)

$$\Omega = \frac{(2\pi \Gamma(3/2))^N 16\pi \rho^{3N-2}}{\Gamma(3N/2)}, \quad (B.4)$$

and

$$V(\rho) = \frac{N(N - 1)}{\sqrt{2\pi}} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{1}{\rho^2} \int_0^{\sqrt{2}\rho} r^2 dr (1 - \frac{r^2}{2\rho^2})^{3N/2-5/2} V_{\text{int}}(r). \quad (B.5)$$

Now we calculate the effective potential $V(\rho)$ for various potentials $V_{\text{int}}$.

Substitution of

$$V_{\text{int}}(r) = \lambda \delta(\vec{r}) = \frac{\lambda}{4\pi r^2} \delta(r) \quad (B.6)$$

into Eq. (B.5) gives

$$V(\rho) = \frac{N(N - 1)}{2(2\pi)^{3/2}} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{\lambda}{\rho^2}. \quad (B.7)$$

For the case of a square-well potential

$$V_{\text{int}}(r) = \begin{cases} V_0 & \text{if } r \leq b, \\ 0 & \text{otherwise}, \end{cases} \quad (B.8)$$
the calculation gives the following result

\[ V(\rho) = \begin{cases} 
  V_0/2 & \text{if } \rho \leq b/\sqrt{2}, \\
  f(\rho) & \text{otherwise},
\end{cases} \]  

(B.9)

where

\[ f(\rho) = b^3 V_0 \frac{N(N - 1)}{3\sqrt{2\pi}} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{2}{\rho^3} F_1(3/2, (5 - 3N)/2; 5/2; b^2/(2\rho^2))/\rho^3, \]

and \( _pF_q \) is the generalized hypergeometric function.

For the Coulomb potential \( V_{int}(r) = \alpha/r \) we obtain

\[ V(\rho) = \frac{2N\Gamma(3N/2)}{3\sqrt{2\pi}\Gamma(3N/2 - 3/2)} \frac{\alpha}{\rho}. \]  

(B.10)

For a Gaussian potential \( V_{int}(r) = V_0 e^{-\beta^2 r^2} \)

\[ V(\rho) = \frac{V_0 N(N - 1)}{2} \Phi(3/2, 3N/2; -2\beta^2 \rho^2), \]  

(B.11)

where \( \Phi(a, b; x) \) is the confluent hypergeometric function. For \( 2\beta^2 \rho^2 \gg 1 \) Eq. (B.11) gives

\[ V(\rho) \approx \frac{V_0 N(N - 1)}{2} \frac{\Gamma(3N/2)}{\Gamma(3N/2 - 3/2)} \frac{1}{(\beta\rho)^3}, \]  

(B.12)

and for \( 2\beta^2 \rho^2 \ll 1 \)

\[ V(\rho) \approx \frac{V_0 N(N - 1)}{2} \]  

(B.13)

For the Yukawa potential \( V_{int}(r) = V_0 e^{-\mu r}/r \)

\[ V(\rho) = \frac{N(N - 1)\Gamma(3N/2)V_0}{\sqrt{2\pi}\Gamma(3N/2 - 3/2)\rho^3(6(N - 1))} \frac{4\rho^2}{1F_2(1; 1/2, 3N/2 - 1/2; \mu^2 \rho^2/2)} \]

\[ -2^{3N/4}\mu^2 - 3N/2(3N - 1)\sqrt{\pi}\rho^{4 - 3N/2} I(3N/2 - 1, \sqrt{2}\mu\rho)\Gamma(3N/2 - 3/2)], \]  

(B.14)
In the $N = 3$ case Eq.(B.14) simplifies

$$V(\rho) = \frac{N(N-1)\Gamma(3N/2)V_0}{\sqrt{2\pi}\Gamma(3N/2 - 3/2)}[6(\mu\rho)^2 + 30$$

$$-2e^{-\sqrt{2}\mu(2\sqrt{2}(\mu\rho)^3 + 12(\mu\rho)^2 + 15\sqrt{2}\mu\rho + 15)].$$

\[\text{(B.15)}\]

**Appendix C**

In this Appendix, we present an outline for obtaining the effective anisotropic potential. For an anisotropic case we introduce

$$x = \sqrt{\sum_{n=1}^{N} x_n^2}, \quad y = \sqrt{\sum_{n=1}^{N} y_n^2}, \quad z = \sqrt{\sum_{n=1}^{N} z_n^2}, \quad (C.1)$$

we can then write for effective potential $V(x, y, z)$ the following expression

$$V(x, y, z) = \frac{N(N-1)}{2} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' V_{int}(\sqrt{(x')^2 + (y')^2 + (z')^2})$$

$$\times \int_{-\infty}^{\infty} dR_x \int_{-\infty}^{\infty} dR_y \int_{-\infty}^{\infty} dR_z \int_{-\infty}^{\infty} dx_3 \ldots \int_{-\infty}^{\infty} dx_N \int_{-\infty}^{\infty} dy_3 \ldots \int_{-\infty}^{\infty} dy_N$$

$$\times \int_{-\infty}^{\infty} dz_3 \ldots \int_{-\infty}^{\infty} dz_N \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_3 e^{i[(y')^2/2 + R_x^2/2 + \sum_{n=3}^{N} x_n^2 - x_3^2]t_1}$$

$$\times e^{i[(y')^2/2 + R_y^2/2 + \sum_{n=3}^{N} y_n^2 - y_3^2]t_2} e^{i[(z')^2/2 + R_z^2/2 + \sum_{n=3}^{N} z_n^2 - z_3^2]t_3}$$

\[\text{(C.2)}\]
where
\[ \Omega = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dR_1 \int_{-\infty}^{\infty} dR_2 \int_{-\infty}^{\infty} dR_3 \]
\[ \times \int_{-\infty}^{\infty} dx_3 \ldots \int_{-\infty}^{\infty} dx_N \int_{-\infty}^{\infty} dy_3 \ldots \int_{-\infty}^{\infty} dy_N \int_{-\infty}^{\infty} dz_3 \ldots \int_{-\infty}^{\infty} dz_N \int_{-\infty}^{\infty} dt_1 \]
\[ \times e^{i[(x')^2/2 + R_1^2/2 + \sum_{n=3}^{N} x_n^2 - x^2]t_1} e^{i[(y')^2/2 + R_2^2/2 + \sum_{n=3}^{N} y_n^2 - y^2]t_2} \]
\[ \times e^{i[(z')^2/2 + R_3^2/2 + \sum_{n=3}^{N} z_n^2 - z^2]t_3}. \]

(C.3)

Using (A.4) and (B.3) we obtain
\[ \Omega = \Gamma(1/2)^{3N} (4\pi)^3 x^{N-2} y^{N-2} z^{N-2} \Gamma(N/2)^{-3}, \]  
(C.4)

and
\[ V(x, y, z) = \frac{N(N - 1)}{2(2\pi)^{3/2} \Gamma(N/2 - 1/2)} \Gamma(N/2)^{3} \frac{1}{xyz} \int_{-\sqrt{xz}}^{\sqrt{xz}} \int_{-\sqrt{yz}}^{\sqrt{yz}} \int_{-\sqrt{zx}}^{\sqrt{zx}} dx' dy' dz' \]
\[ \times V_{int}(\sqrt{(x')^2 + (y')^2 + (z')^2}) (1 - \frac{(x')^2}{2x^2})(1 - \frac{(y')^2}{2y^2})(1 - \frac{(z')^2}{2z^2})^{N/2 - 3/2} \]

(C.5)

Finally, we quote several examples of \( V(x, y, z) \) for various potentials \( V_{int} \).

For delta-potential, \( V_{int}(\vec{r}) = \lambda \delta(\vec{r}) \) we obtain
\[ V(x, y, z) = \frac{N(N - 1)}{2(2\pi)^{3/2} \Gamma(N/2 - 1/2)} \Gamma(N/2)^{3} \frac{\lambda}{xyz}, \]  
(C.6)

and for the Gaussian potential, \( V_{int}(r) = V_0 e^{-\beta r^2} \) we have
\[ V(x, y, z) = \frac{V_0 N(N - 1)}{2} \Phi(1/2, N/2; -2\beta^2 x^2) \Phi(1/2, N/2; -2\beta^2 y^2) \]
\[ \times \Phi(1/2, N/2; -2\beta^2 z^2) \]  
(C.7)
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