Lieb-Thirring inequalities on the torus

A. A. Ilyin and A. A. Laptev

Abstract. We consider the Lieb-Thirring inequalities on the $d$-dimensional torus with arbitrary periods. In the space of functions with zero average with respect to the shortest coordinate we prove the Lieb-Thirring inequalities for the $\gamma$-moments of the negative eigenvalues with constants independent of ratio of the periods. Applications to the attractors of the damped Navier-Stokes system are given.

Bibliography: 33 titles.

Keywords: Lieb-Thirring inequalities, Schrödinger operators, interpolation inequalities, attractors, fractal dimension.

§ 1. Introduction

The Lieb-Thirring inequalities (see [1]) give estimates for the $\gamma$-moments of the negative eigenvalues of the Schrödinger operator

$$-\Delta - V$$

in $L_2(\mathbb{R}^d)$:

$$\sum_{\nu_i \leq 0} |\nu_i|^\gamma \leq L_{\gamma,d} \int V(x)^{\gamma+d/2} \, dx.$$  (1.2)

Here $V \geq 0$ is a real-valued potential sufficiently fast decaying at infinity and $\gamma > \max\{0, 1 - d/2\}$.

Sharp values of the constants $L_{\gamma,d}$ for all $\gamma \geq 3/2$ and all $d$ were found in [2]:

$$L_{\gamma,d} = L_{\gamma,d}^{\text{cl}},$$

where

$$L_{\gamma,d}^{\text{cl}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|^2)^{\gamma} \, d\xi = \frac{\Gamma(\gamma + 1)}{2^d\pi^{d/2}\Gamma(\gamma + d/2 + 1)}$$  (1.3)

is the semiclassical constant. For $1 \leq \gamma < 3/2$ the best known bounds for $L_{\gamma,d}$ were found in [3]:

$$L_{\gamma,d} \leq \frac{\pi}{\sqrt{3}} L_{\gamma,d}^{\text{cl}}.$$  (1.4)

The research of the first-named author was carried out at the expense of the Russian Science Foundation (project no. 14-21-00025).

AMS 2010 Mathematics Subject Classification. Primary 35J10, 35P15; Secondary 37L30, 76D05.

© 2016 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
Both in [2] and [3] (see also [4] and [5]) the key role is played by the corresponding one-dimensional Lieb-Thirring estimates for matrix-valued potentials.

If we consider the Lieb-Thirring inequalities on a compact manifold $M$, then we have to take care of the simple eigenvalue $0$ of the Laplacian, so that instead of (1.1) we consider the Schrödinger operator

$$-\Delta - \Pi(V \Pi \cdot),$$

where $\Pi$ is the orthoprojection

$$\Pi \varphi(x) = \varphi(x) - \frac{1}{|M|} \int_M \varphi(x) \, dM,$$

and $|M|$ denotes the measure of $M$. Then estimate (1.2) still holds on $M$ with constant $L_{\gamma,d} = L_{\gamma,d}(M)$ depending on the geometric properties of $M$.

The spectral inequality (1.2) for 1-moments is equivalent to the inequality for orthonormal families. Let $\{\varphi_j\}_{j=1}^N \in H^1(\mathbb{R}^d)$ be an orthonormal family in $L_2(\mathbb{R}^d)$.

In the case of a manifold we have to assume that $\int_M \varphi_j(x) \, dM = 0$. Then $\rho_\varphi(x) := \sum_{j=1}^N \varphi_j(x)^2$ satisfies the inequality

$$\int \rho_\varphi(x)^{1+2/d} \, dx \leq k_d \sum_{j=1}^N \|\nabla \varphi_j\|^2,$$

(1.5)

where the best constants $k_d$ and $L_{1,d}$ satisfy the following equality (see [1], [6]):

$$k_d = \frac{2}{d} \left(1 + \frac{d}{2}\right)^{1+2/d} L_{1,d}^{2/d}.$$  

(1.6)

In addition to the initial quantum mechanical applications, inequality (1.5) is essential for finding good estimates for the dimension of the attractors in the theory of infinite-dimensional dissipative dynamical systems, especially for the attractors of the Navier-Stokes equations (see, for instance, [6]–[11] and the references therein). Lieb-Thirring inequalities (1.5) were generalized to higher-order elliptic operators on domains with various boundary conditions and Riemannian manifolds (see [9], [10]), however with no information on the corresponding constants. A different approach based on the methods of trigonometric series was proposed in [12].

For the two-dimensional square torus $\mathbb{T}^2$ an explicit bound for the Lieb-Thirring constant $L_{1,2}(\mathbb{T}^2)$ was obtained in [13]:

$$L_{1,2}(\mathbb{T}^2) \leq \frac{3}{8}.$$  

(1.7)

Following the original approach in [1], the Birman-Schwinger kernel was used in [13] and the bound $3/8$ is the same as that for $\mathbb{R}^2$ in [1].

The Lieb-Thirring constants on the torus depend only on the ratio of the periods and on the elongated torus $\mathbb{T}_\alpha^2 = (0, 2\pi/\alpha) \times (0, 2\pi)$ are unbounded as $\alpha \to 0$. This
is due to the fact that the space \( L_2(\mathbb{T}_\alpha^2) \) contains the subspace of (periodic) functions depending only on the long coordinate \( x_1 \), which gives that, for example, for the 1-moments we have

\[
L_{1,2}(\mathbb{T}_\alpha^2) = \frac{L_{1,2}(\mathbb{T}^2)}{\alpha}.
\]

To see this we extend the functions in the direction \( x_2 \) by periodicity \( \alpha \)-times (assuming that \( \alpha \) is an integer) to reduce the treatment to the square torus (see [14]).

The orthogonal complement to the subspace \( L_2(0, \frac{2\pi}{\alpha}) \) of functions depending only on \( x_1 \) consists of functions \( \varphi(x_1, x_2) \) that have zero average with respect to \( x_2 \) for all \( x_1 \):

\[
\left\{ \varphi(x), \int_0^{2\pi} \varphi(x_1, t) \, dt = 0 \quad \forall x_1 \in \left(0, \frac{2\pi}{\alpha}\right) \right\}.
\]

Let \( P \) denote the corresponding orthoprojection. Then the Lieb-Thirring constants for the operator

\[
\mathcal{H} = -\Delta - P(V(x) P \cdot)
\]

on \( \mathbb{T}_\alpha^2 \) are independent of \( \alpha \) (more precisely, have \( \alpha \)-independent upper bounds). This was first observed in [15], and explicit estimates were obtained in [11].

In this work we consider Lieb-Thirring inequalities for the operator (1.8) on the \( d \)-dimensional torus

\[
\mathbb{T}_\alpha^d = (0, L_1) \times \cdots \times (0, L_{d-1}) \times (0, 2\pi),
\]

where \( L_i = 2\pi/\alpha_i \), and the lengths of the periods are arranged in nonincreasing order

\[
\alpha_1 \leq \cdots \leq \alpha_{d-1} \leq \alpha_d = 1.
\]

Here \( P \) is the orthoprojection

\[
(P\psi)(x_1, \ldots, x_d) = \psi(x_1, \ldots, x_d) - \frac{1}{2\pi} \int_0^{2\pi} \psi(x_1, \ldots, x_{d-1}, t) \, dt,
\]

so that the resulting function has zero average with respect to the shortest coordinate \( x_d \) for all \( x_1, \ldots, x_{d-1} \).

We can now state our main result.

**Theorem 1.** Let \( d \leq 19 \). Then for any \( \alpha \) satisfying (1.10) the negative eigenvalues of the operator (1.8) on the torus \( \mathbb{T}_\alpha^d \) satisfy the bound (1.2) for \( \gamma \geq 1 \) with

\[
L_{\gamma,d} \leq \left( \frac{\pi}{\sqrt{3}} \right)^d L_{\gamma,d}^{cl},
\]

where \( L_{\gamma,d}^{cl} \) is the semiclassical constant (1.3).

The main idea is to write the Laplacian in (1.8) in the form

\[
-\Delta = \left( -\frac{d^2}{dx_1^2} + \alpha_1^2 \beta_1 \right) + \cdots + \left( -\frac{d^2}{dx_{d-1}^2} + \alpha_{d-1}^2 \beta_{d-1} \right) + \left( -\frac{d^2}{dx_d^2} - \delta \right),
\]

where

\[
\delta = \alpha_1^2 \beta_1 + \cdots + \alpha_{d-1}^2 \beta_{d-1},
\]

and where the \( \beta_j > 0 \) are chosen so that \( \delta < 1 \).
For \( j = 1, \ldots, d - 1 \) each operator
\[
- \frac{d^2}{dx^2_j} + \alpha^2 \beta_j, \quad j = 1, \ldots, d - 1,
\]
is invertible in \( L_2(0, 2\pi/\alpha_j)_{\text{per}} \), while for \( \delta < 1 \) the operator
\[
- \frac{d^2}{dx^2} - \delta
\]
is invertible in
\[
\dot{L}_2(0, 2\pi) = L_2(0, 2\pi) \cap \left\{ \psi : \int_0^{2\pi} \psi(t) \, dt = 0 \right\}.
\]

Accordingly, in §2 we consider for these two types of operators the interpolation inequalities of \( L_\infty - L_2 - L_2 \)-type:
\[
\|u\|_\infty^2 \leq K_1(\beta) \left( \int_0^{2\pi/\alpha} \left( u'(x)^2 + \alpha^2 \beta u(x)^2 \right) \, dx \right)^{1/2} \left( \int_0^{2\pi/\alpha} u(x)^2 \, dx \right)^{1/2}, \quad (1.13)
\]
where \( \beta > 0 \) and \( u \in H^1(0, 2\pi/\alpha)_{\text{per}} \); and
\[
\|u\|_\infty^2 \leq K_2(\beta) \left( \int_0^{2\pi} \left( u'(x)^2 - \beta u(x)^2 \right) \, dx \right)^{1/2} \left( \int_0^{2\pi} u(x)^2 \, dx \right)^{1/2}, \quad (1.14)
\]
where \( \beta < 1 \) and
\[
u \in \dot{H}^1(0, 2\pi)_{\text{per}} = H^1(0, 2\pi) \cap \int_0^{2\pi} u(x) \, dx = 0.
\]

We find sharp constants in these inequalities and, in particular, show that \( K_1(\beta) = 1 \) for \( \beta \geq \beta_* = 0.045 \ldots \) and \( K_2(\beta) = 1 \) for \( \beta \leq \beta^{**} = 0.839 \ldots \).

In §3 we use the information on the sharp constants \( K_1(\beta) \) and \( K_2(\beta) \) to obtain, following [16], Theorem 6.1, one-dimensional inequalities of the type (1.5) for traces of matrices built from orthonormal families of \textit{vector} functions. Then, using the equivalence (1.6), we recast these results into the Lieb-Thirring inequalities for the negative eigenvalues \( \{-\lambda_j\} \) and \( \{-\mu_j\} \) of the one-dimensional operators with \textit{matrix-valued} potentials
\[
H_1(\beta) = - \frac{d^2}{dx^2} + \alpha^2 \beta - V(x), \quad \beta > 0, \quad x \in \left( 0, \frac{2\pi}{\alpha} \right), \quad (1.15)
\]
\[
H_2(\delta) = - \frac{d^2}{dx^2} - \delta - P(V(x)P \cdot), \quad \delta \in [0, 1), \quad x \in (0, 2\pi), \quad (1.16)
\]
acting in \( L_2(0, 2\pi/\alpha) \) and \( \dot{L}_2(0, 2\pi) = \left\{ f \in L_2(0, 2\pi), \int_0^{2\pi} f(x) \, dx = 0 \right\} \), respectively.
For a nonnegative \((M \times M)\)-matrix \(V\) we have
\[
\sum_j \lambda_j \leq \frac{2}{3\sqrt{3}} K_1(\beta) \int_0^{2\pi/\alpha} \text{Tr}[V(x)^{3/2}] \, dx, \tag{1.17}
\]
\[
\sum_j \mu_j \leq \frac{2}{3\sqrt{3}} K_2(\delta) \int_0^{2\pi} \text{Tr}[V(x)^{3/2}] \, dx. \tag{1.18}
\]

Then we use the Aizenman-Lieb argument (see [17]) to obtain estimates for the Riesz means of order \(\gamma \geq 1\).

In §4 we use the lifting argument with respect to dimensions from [2] and the one-dimensional Lieb-Thirring inequalities for matrix-valued potentials from §3 to prove Theorem 6 (to which Theorem 1 is a corollary).

Finally, in §4 we give applications to the attractors of the damped-driven Navier-Stokes equations on the square and elongated torus and improve the estimates of the dimension of the attractor obtained earlier in [18] and [19].

Remark 1. We point out that the condition \(d < 19\) (that is, \(d \leq [\beta^{**}/\beta_*] + 1\)) makes it possible to choose \(\beta_j \geq \beta_*\) and \(\delta \leq \beta^{**}\), so that in (1.17), (1.18) we have \(K_1(\beta_j) = 1\) and \(K_2(\delta) = 1\). For \(d > 19\), for example, in the case of equal periods there is an additional factor greater than 1 (see (4.3)). Apparently this is due to the method of proof. Furthermore, estimate (1.12) contains the factor \((\pi/\sqrt{3})^d\) growing with dimension. On the contrary, in estimate (1.4) for \(R^d\) this factor appears only once, since in the one-dimensional case \(x \in \mathbb{R}\) one has sharp estimates for the moments of order \(\gamma \geq 3/2\) for the matrix potentials, which are not known in the periodic case even for scalar potentials.

\[
\section{2. Two interpolation inequalities}
\]

First inequality. Let \(x \in [0, 2\pi/\alpha]_{\text{per}}\) and let \(\alpha > 0\). We consider the interpolation inequality (1.13), where \(u \in H^1(0, 2\pi/\alpha)_{\text{per}}, \beta > 0\) (and no orthogonality to the constants is assumed). More precisely, we are interested in the value of the sharp constant \(K_1(\beta)\) in this inequality.

The system
\[
\left\{ \sqrt{\frac{\alpha}{2\pi}} e^{ik\alpha x} \right\}_{k \in \mathbb{Z}}
\]
is a complete orthonormal system of eigenfunctions of the operator \(-\frac{d^2}{dx^2}\) with periodic boundary conditions. Therefore the Green’s function \(G_\lambda(x, \xi)\), that is, the solution of the equation
\[
\mathbb{A}(\lambda)G_\lambda(x, \xi) = \delta(x - \xi),
\]
where
\[
\mathbb{A}(\lambda) = -\frac{d^2}{dx^2} + \alpha^2 \beta + \lambda, \quad \lambda > 0,
\]
is given by the series
\[
G_\lambda(x, \xi) = \frac{\alpha}{2\pi} \sum_{k \in \mathbb{Z}} \frac{e^{ik\alpha(x-\xi)}}{\alpha^2 k^2 + \alpha^2 \beta + \lambda}.
\]
On the diagonal
\[ G_\lambda(\xi, \xi) = \frac{\alpha}{2\pi} \sum_{k \in \mathbb{Z}} \frac{1}{\alpha^2 k^2 + \alpha^2 \beta + \lambda} =: g_\beta(\lambda). \] (2.1)

Here we omit the subscript $\alpha$ on the right because $\alpha$ is fixed and, secondly, the sharp constant $K_1(\beta)$ is independent of $\alpha$, as our final result shows. Using the general result (Theorem 2.2 in [20] with $\theta = 1/2$) we have the following expression for the sharp constant $K_1(\beta)$:
\[ K_1(\beta) = \frac{1}{\theta^\theta (1 - \theta)^{1 - \theta}} \sup_{\lambda > 0} \lambda^\theta g_\beta(\lambda) = 2 \sup_{\lambda > 0} \sqrt{\lambda} g_\beta(\lambda). \] (2.2)

(We point out that for the proof of (2.2) we can use a direct argument similar to the one used in Remark 3.1 in [16].) Next, using the formula
\[ \sum_{k \in \mathbb{Z}} \frac{1}{k^2 + \lambda} = \frac{\pi \coth(\pi \sqrt{\lambda})}{\sqrt{\lambda}} \] (2.3)
in (2.1), we find that
\[ g_\beta(\lambda) = \frac{1}{2\alpha} \frac{\coth(\pi \sqrt{\beta + \lambda/\alpha^2})}{\sqrt{\beta + \lambda/\alpha^2}}. \]
Since $\alpha > 0$ is fixed, we can replace the variable $\lambda$ in (2.2) by $\alpha^2 \lambda$, which finally gives
\[ K_1(\beta) = \sup_{\lambda > 0} \sqrt{\lambda} \frac{\coth(\pi \sqrt{\beta + \lambda})}{\sqrt{\beta + \lambda}}. \] (2.4)
Since for every fixed $\beta > 0$
\[ \lim_{\lambda \to \infty} \sqrt{\lambda} \frac{\coth(\pi \sqrt{\beta + \lambda})}{\sqrt{\beta + \lambda}} = 1, \]
it follows that
\[ K_1(\beta) \geq 1. \]
Next, (2.1) and (2.2) show that $K_1(\beta)$ is monotone nonincreasing:
\[ \beta_1 < \beta_2 \implies g_{\beta_1}(\lambda) > g_{\beta_2}(\lambda) \implies K_1(\beta_1) \geq K_1(\beta_2), \]
and, finally,
\[ K_1(\beta_0) = 1 \implies K_1(\beta) = 1 \text{ for } \beta \geq \beta_0. \]
The graphs of the function $\lambda \to \sqrt{\lambda} \coth(\pi \sqrt{\beta + \lambda})/\sqrt{\beta + \lambda}$ are shown in Fig. 1. For very small $\beta$ the graphs have a sharp peak near the origin. For $\beta_* = 0.045\ldots$ (found numerically) the supremum is equal to 1. Hence $K_1(\beta) = 1$ for all $\beta \geq \beta_*$. Thus, we have proved the following theorem.

**Theorem 2.** The sharp constant $K_1(\beta)$ in inequality (1.13) is given by (2.4). Furthermore, $K_1(\beta) = 1$ for all $\beta \geq \beta_* = 0.045\ldots$
Figure 1. Graphs of the function $\lambda \to \sqrt{\lambda} \frac{\coth(\pi \sqrt{\beta + \lambda})}{\sqrt{\beta + \lambda}}$ for $\beta = 0.01, \beta_*, 0.5$.

**Second inequality.** Now let $x \in [0, 2\pi]$ and consider inequality (1.14), where

$$u \in \dot{H}^1(0, 2\pi)_{\text{per}} = H^1(0, 2\pi)_{\text{per}} \cap \int_0^{2\pi} u(x) \, dx = 0$$

and $0 \leq \beta < 1$.

The Green’s function $G_\lambda(x, \xi)$ of the operator

$$A_\lambda(x) = -\frac{d^2}{dx^2} - \beta + \lambda, \quad \lambda > 0,$$

in the space of functions with mean value zero is as follows

$$G_\lambda(x, \xi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0} \frac{e^{ik(x-\xi)}}{k^2 - \beta + \lambda}, \quad \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\},$$

and

$$G_\lambda(\xi, \xi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}_0} \frac{1}{k^2 - \beta + \lambda} =: f_\beta(\lambda). \quad (2.5)$$

As before we have the following expression for the sharp constant $K_2(\beta)$:

$$K_2(\beta) = 2 \sup_{\lambda > 0} \sqrt{\lambda} f_\beta(\lambda).$$

Next, from (2.3) we have

$$\sum_{k \in \mathbb{Z}_0} \frac{1}{k^2 + \lambda} = \frac{\pi \sqrt{\lambda} \coth(\pi \sqrt{\lambda}) - 1}{\lambda},$$

giving

$$f_\beta(\lambda) = \frac{1}{2} \frac{\sqrt{\lambda - \beta} \coth(\pi \sqrt{\lambda - \beta}) - 1/\pi}{\lambda - \beta},$$

and, finally,

$$K_2(\beta) = \sup_{\lambda > 0} \sqrt{\lambda} \frac{\sqrt{\lambda - \beta} \coth(\pi \sqrt{\lambda - \beta}) - 1/\pi}{\lambda - \beta}. \quad (2.6)$$
The existence of the limit
\[
\lim_{\lambda \to \infty} \sqrt{\lambda} \frac{\sqrt{\lambda - \beta} \coth(\pi \sqrt{\lambda - \beta}) - 1/\pi}{\lambda - \beta} = 1
\]
implies that
\[K_2(\beta) \geq 1.\]
This time \(K_2(\beta)\) is monotone nondecreasing:
\[\beta_1 < \beta_2 \implies f_{\beta_1}(\lambda) < f_{\beta_2}(\lambda) \implies K_1(\beta_1) \leq K_1(\beta_2),\]
and, finally,
\[K_2(\beta_0) = 1, \ 0 < \beta_0 < 1 \implies K_2(\beta) = 1 \text{ for } 0 \leq \beta \leq \beta_0.\]

The graphs of the function
\[\lambda \to \sqrt{\lambda} \frac{\sqrt{\lambda - \beta} \coth(\pi \sqrt{\lambda - \beta}) - 1/\pi}{\lambda - \beta}\]
are shown in Fig. 2, which is somewhat symmetrical to Fig. 1. For \(\beta\) close to 1 the graphs have a sharp peak near the origin. For \(\beta^{**} = 0.839\ldots\) (again found numerically) the supremum is equal to 1 and is attained both at a finite \(\lambda\) and at infinity. Hence \(K_1(\beta) = 1\) for all \(\beta \geq \beta^{**}\).

\[\begin{array}{c@{\quad}c@{\quad}c}
0 & 0.1 & 0.2 & 0.3 & 0.4 & \lambda \\
3 & 2 & 1 & 0.5 & 0.2 & 0.3 & 0.4 & \lambda \\
0 & 1 & 0.5 & 0.2 & 0.3 & 0.4 & 0.5 & \lambda \\
0 & 1 & 0.5 & 0.2 & 0.3 & 0.4 & 0.5 & \lambda \\
\end{array}\]

Figure 2. Graphs of \(\lambda \to \sqrt{\lambda} \frac{\sqrt{\lambda - \beta} \coth(\pi \sqrt{\lambda - \beta}) - 1/\pi}{\lambda - \beta}\) for \(\beta = 0.99, \beta^{**}, 0.5\).

Thus, we have proved the following theorem.

**Theorem 3.** The sharp constant \(K_2(\beta)\) in inequality (1.14) is given by (2.6). Furthermore, \(K_2(\beta) = 1\) for all \(\beta \leq \beta^{**} = 0.839\ldots\).

**Remark 2.** Inequality (1.14) with \(\beta = 0\) and \(K_2(0) = 1\) goes back to [21] where it was used for the proof of the Carlson inequality. Sharp constants in the higher-order inequalities were found in [22] for \(x \in \mathbb{R}\) and in [23] for \(x \in \mathbb{S}^1\). Various refinements and improvements of this type of inequalities were recently obtained in [16], [20] and [24].
Remark 3. By the definition of $K_1(\beta)$ and $K_2(\beta)$ we have
\[
\sum_{k \in \mathbb{Z}} \frac{1}{k^2 + \beta + \lambda} \leq K_1(\beta) \frac{\pi}{\sqrt{\lambda}}, \quad \beta > 0,
\]
\[
\sum_{k \in \mathbb{Z}_0} \frac{1}{k^2 - \beta + \lambda} \leq K_2(\beta) \frac{\pi}{\sqrt{\lambda}}, \quad \beta \in [0, 1).
\] (2.7)

§ 3. One-dimensional Sobolev inequalities for traces of matrices and associated Lieb-Thirring inequalities

Let $\{\phi_n\}_{n=1}^N$ be an orthonormal family of periodic vector functions
\[
\phi_n(x) = (\phi_n(x, 1), \ldots, \phi_n(x, M))^T,
\]
defined for $x \in [0, 2\pi/\alpha]_{\text{per}}$:
\[
(\phi_n, \phi_m) = \sum_{j=1}^M \int_0^L \phi_n(x, j) \overline{\phi_m(x, j)} \, dx = \int_0^L \phi_n(x)^T \phi_m(x) \, dx = \delta_{nm},
\]
where we set for brevity
\[
L := \frac{2\pi}{\alpha}.
\]

We consider the $(M \times M)$-matrix $U(x, y)$
\[
U(x, y) = \sum_{n=1}^N \phi_n(x) \overline{\phi_n(y)}^T
\] (3.1)
with entries $[U(x, y)]_{jk} = \sum_{n=1}^N \phi_n(x, j) \overline{\phi_n(y, k)}$. Clearly,
\[
U(x, y)^* = U(y, x)
\]
and by orthonormality
\[
\int_0^L U(x, y)U(y, z) \, dy = \sum_{n,n'=1}^N \int_0^L \phi_n(x) \overline{\phi_n(y)}^T \phi_{n'}(y) \overline{\phi_{n'}(z)}^T \, dy
\]
\[
= \sum_{n=1}^N \phi_n(x) \overline{\phi_n(z)}^T = U(x, z).
\]

Theorem 4. Let $\phi_n(x, j) \in H^1(0, L)_{\text{per}}$. Then
\[
\int_0^L \text{Tr}[U(x, x)^3] \, dx \leq K_1(\beta)^2 \sum_{n=1}^N \sum_{j=1}^M \int_0^L (|\phi_n'(x, j)|^2 + \alpha^2 |\phi_n(x, j)|^2) \, dx, \quad (3.2)
\]
where $K_1(\beta)$ is defined in (1.13) and $\beta > 0$. 
If $\alpha = 1$, $L = 2\pi$ and $\int_0^{2\pi} \phi_n(x, j) = 0$ for all $n$ and $j$, then

$$\int_0^{2\pi} \text{Tr}[U(x, x)^3] \, dx \leq K_2(\beta)^2 \sum_{n=1}^{N} \sum_{j=1}^{M} \int_0^{2\pi} (|\phi'_n(x, j)|^2 - \beta|\phi_n(x, j)|^2) \, dx,$$  

(3.3)

where $K_2(\beta)$ is defined in (1.14) and $\beta \in [0, 1)$.

**Proof.** Both inequalities are proved similarly. The argument, in turn, follows very closely the proof of Theorem 6.1 in [16]. Let us prove, for example, the second inequality (3.3). We have

$$\tilde{U}(n, x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-iyn} U(y, x) \, dy, \quad n \in \mathbb{Z}_0,$$

so that

$$U(y, x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}_0} e^{iyk} \tilde{U}(k, x).$$

Next,

$$\sum_{k \in \mathbb{Z}_0} \tilde{U}(k, x)^* \tilde{U}(k, x) = \int_0^{2\pi} U(y, x)^* U(y, x) \, dy = U(x, x)$$  

(3.4)

and

$$\sum_{k \in \mathbb{Z}_0} (k^2 - \beta) \tilde{U}(k, x)^* \tilde{U}(k, x)$$

$$= \int_0^{2\pi} [\partial_y U(y, x)]^* \partial_y U(y, x) \, dy - \beta \int_0^{L} U(y, x)^* U(y, x) \, dy,$$

where the last term is equal to $-\beta U(x, x)$. Now, by orthonormality

$$\text{Tr} \left[ \int_0^{2\pi} \sum_{k \in \mathbb{Z}_0} (k^2 - \beta) \tilde{U}(k, x)^* \tilde{U}(k, x) \, dx \right]$$

$$= \text{Tr} \left[ \int_0^{2\pi} \int_0^{2\pi} \sum_{n,n' = 1}^{N} \phi'_n(y) \phi_{n'}(x)^T \phi_n(x) \phi_{n'}(y)^T \, dx \, dy - \beta \int_0^{2\pi} U(x, x) \, dx \right]$$

$$= \text{Tr} \left[ \int_0^{2\pi} \sum_{n=1}^{N} \left[ \phi'_n(y) \phi_{n'}(y)^T - \beta \phi_n(y) \phi_{n'}(y)^T \right] \, dy \right]$$

$$= \sum_{n=1}^{N} \sum_{j=1}^{M} \int_0^{2\pi} (|\phi'_n(x, j)|^2 - \beta|\phi_n(x, j)|^2) \, dx.$$  

(3.5)

Now consider

$$\text{Tr}[U(x, x)^3] = \sum_{k \in \mathbb{Z}_0} \text{Tr}[U(x, x)^2 \tilde{U}(k, x)] \frac{e^{ixk}}{\sqrt{2\pi}}$$

$$= \sum_{k \in \mathbb{Z}_0} \text{Tr}[[k^2 - \beta I + \Lambda(x)]^{-1/2} U(x, x)^2 \tilde{U}(k, x)[(k^2 - \beta I + \Lambda(x)]^{1/2}] \frac{e^{ixk}}{\sqrt{2\pi}},$$
where $\Lambda(x)$ is an arbitrary positive definite matrix. Using below the Cauchy-Schwartz inequality for matrices and the cyclic property of the trace we get the upper bounds

\[
\sqrt{2\pi} \text{Tr}[U(x, x)^3] \\
\leq \sum_{k \in \mathbb{Z}_0} |\text{Tr}[[((k^2 - \beta)I + \Lambda(x)^2)^{-1/2}U(x, x)^2\tilde{U}(k, x)[(k^2 - \beta)I + \Lambda(x)^2]^{1/2}]| \\
\leq \sum_{k \in \mathbb{Z}_0} \left(\text{Tr}[U(x, x)^2[(k^2 - \beta)I + \Lambda(x)^2]^{-1}U(x, x)^2]\right)^{1/2} \\
\times \left(\text{Tr}[[((k^2 - \beta)I + \Lambda(x)^2)\tilde{U}(k, x)^*\tilde{U}(k, x)]\right)^{1/2} \\
\leq \left(\sum_{k \in \mathbb{Z}_0} \text{Tr}[U(x, x)^2[(k^2 - \beta)I + \Lambda(x)^2]^{-1}U(x, x)^2]\right)^{1/2} \\
\times \left(\sum_{k \in \mathbb{Z}_0} \text{Tr}[[((k^2 - \beta)I + \Lambda(x)^2)\tilde{U}(k, x)^*\tilde{U}(k, x)]\right)^{1/2}.
\]

We now use the matrix inequality

\[
\sum_{k \in \mathbb{Z}_0} [(k^2 - \beta)I + \Lambda(x)^2]^{-1} < K_2(\beta)\pi \Lambda(x)^{-1}, \quad (3.6)
\]

which follows from the scalar inequality (see (2.7))

\[
\sum_{k \in \mathbb{Z}_0} \frac{1}{k^2 - \beta + \lambda^2} < K_2(\beta)\pi \lambda^{-1},
\]

applied to each eigenvector $e = e(x)$ of $\Lambda(x)$ with eigenvalue $\lambda = \lambda(x) > 0$ from the orthonormal basis \{e_j(x), \lambda_j(x)\}_{j=1}^M. This gives for the first factor

\[
\sum_{k \in \mathbb{Z}_0} \text{Tr}[U(x, x)^2[(k^2 - \beta)I + \Lambda(x)^2]^{-1}U(x, x)^2] \\
= \text{Tr}\left[U(x, x)^2 \sum_{k \in \mathbb{Z}_0} [(k^2 - \beta)I + \Lambda(x)^2]^{-1}U(x, x)^2\right] \\
\leq K_2(\beta)\pi \text{Tr}[U(x, x)^2\Lambda(x)^{-1}U(x, x)^2].
\]

For the second factor we simply have

\[
\sum_{k \in \mathbb{Z}_0} \text{Tr}[[((k^2 - \beta)I + \Lambda(x)^2)\tilde{U}(k, x)^*\tilde{U}(k, x)] \\
= \sum_{k \in \mathbb{Z}_0} \text{Tr}[(k^2 - \beta)\tilde{U}(k, x)^*\tilde{U}(k, x)] + \sum_{k \in \mathbb{Z}_0} \text{Tr}[\Lambda(x)^2\tilde{U}(k, x)^*\tilde{U}(k, x)].
\]
If we now choose $\Lambda(x) = \gamma(U(x, x) + \varepsilon I)$, $\gamma > 0$, and let $\varepsilon \to 0$ we obtain (observing that $\lambda^4/(\lambda + \varepsilon) \to \lambda^3$ as $\varepsilon \to 0$ for $\lambda \geq 0$; this is required in the case when $U(x, x)$ is not invertible)

$$\sqrt{2\pi} \text{Tr}[U(x, x)^3] \leq \sqrt{\pi K_2(\beta)} \gamma^{-1/2} \text{Tr}[U(x, x)^3]^{1/2}$$

$$\times \left( \sum_{k \in \mathbb{Z}_0} \text{Tr}[(k^2 - \beta)\tilde{U}(k, x)^*\tilde{U}(k, x)] + \gamma^2 \text{Tr}[U(x, x)^3] \right),$$

where we have also used (3.4), or

$$\text{Tr}[U(x, x)^3] \leq \frac{K_2(\beta)}{2} \left( \gamma^{-1} \sum_{k \in \mathbb{Z}_0} \text{Tr}[(k^2 - \beta)\tilde{U}(k, x)^*\tilde{U}(k, x)] + \gamma \text{Tr}[U(x, x)^3] \right).$$

If we optimize over $\gamma$, we obtain

$$\text{Tr}[U(x, x)^3] \leq K_2(\beta) \left( \text{Tr}[U(x, x)^3] \right)^{1/2} \left( \sum_{k \in \mathbb{Z}_0} \text{Tr}[(k^2 - \beta)\tilde{U}(k, x)^*\tilde{U}(k, x)] \right)^{1/2}$$

or

$$\text{Tr}[U(x, x)^3] \leq K_2(\beta)^2 \sum_{k \in \mathbb{Z}_0} \text{Tr}[(k^2 - \beta)\tilde{U}(k, x)^*\tilde{U}(k, x)].$$

If we integrate with respect to $x$ and use (3.5), we obtain (3.3). The proof of the theorem is complete since the proof of (3.2) is totally similar.

**Remark 4.** In the scalar case $M = 1$ inequalities (3.2) and (3.3) go over to

$$\int_0^L \left( \sum_{n=1}^{N} |\phi_n(x)|^2 \right)^3 dx \leq K_1(\beta)^2 \sum_{n=1}^{N} \int_0^L \left( |\phi_n'(x)|^2 + \alpha^2 \beta |\phi_n(x)|^2 \right) dx,$$

$$\int_0^{2\pi} \left( \sum_{n=1}^{N} |\phi_n(x)|^2 \right)^3 dx \leq K_2(\beta)^2 \sum_{n=1}^{N} \int_0^{2\pi} \left( |\phi_n'(x)|^2 - \beta |\phi_n(x)|^2 \right) dx$$

and follow from the interpolation inequalities (1.13) and (1.14) by the method of [25].

Let us now consider two one-dimensional Schrödinger operators with periodic boundary conditions and matrix-valued potentials (1.15) and (1.16), where $P$ is the orthogonal projection

$$(P\psi)(x) = \psi(x) - \frac{1}{2\pi} \int_0^{2\pi} \psi(t) dt,$$

acting component-wise.

Inequalities (3.2) and (3.3) in Theorem 4 are equivalent to the estimate of the 1-moments of the negative eigenvalues $-\lambda_j \leq 0$ and $-\mu_j \leq 0$ of the Schrödinger operators (1.15) and (1.16), respectively.

**Theorem 5.** Let $V$ be a nonnegative Hermitian $(M \times M)$-matrix with $\text{Tr}V^{3/2} \in L_1$. Then the operators (1.15) and (1.16) have discrete spectrum, and their negative eigenvalues $\{-\lambda_j\}$ and $\{-\mu_j\}$ satisfy the estimates (1.17) and (1.18), respectively.
Proof. Once we have the inequalities for the traces, the proof is completely analogous to the proof of Theorem 1 in [3] (see also Theorem 6.3 in [16]), and the assertion of the theorem is just the inequality

\[ L_{1,1} \leq \frac{2}{3\sqrt{3}} \sqrt{k_1} \]

in the one-dimensional matrix case (see (1.6)). We include the proof of (1.18) for the sake of completeness (the proof of (1.17) is completely similar).

Let \( \{\phi_n\}_{n=1}^N \) be the orthonormal eigenvector functions corresponding to the negative eigenvalues \( \{-\mu_n\}_{n=1}^N \) of (1.16):

\[ -\phi''_n - \delta \phi_n - V \phi_n = -\mu_n \phi_n. \]

Using (3.3) and Hölder’s inequality for traces we obtain

\[
\sum_{n=1}^N \mu_n \leq -\sum_{n=1}^N \sum_{j=1}^M \int_0^{2\pi} \left( |\phi'_n(x,j)|^2 - \delta |\phi_n(x,j)|^2 \right) dx + \int_0^{2\pi} \text{Tr}[V(x)U(x,x)] dx \\
\leq -K_2(\delta)^{-2} X + \left( \int_0^{2\pi} \text{Tr}[V(x)^{3/2}] dx \right)^{2/3} X^{1/3},
\]

where

\[ X := \int_0^{2\pi} \text{Tr}[U(x,x)^3] dx. \]

Calculating the maximum with respect to \( X \) we obtain (1.18).

We observe that in terms of (1.3) estimates (1.17) and (1.18) can be written in the form

\[
\sum_j \lambda_j \leq \frac{\pi}{\sqrt{3}} K_1(\beta) L_{1,1}^{cl} \int_0^L \text{Tr}[V(x)^{3/2}] dx, \tag{3.9}
\]

\[
\sum_j \mu_j \leq \frac{\pi}{\sqrt{3}} K_2(\beta) L_{1,1}^{cl} \int_0^{2\pi} \text{Tr}[V(x)^{3/2}] dx. \tag{3.10}
\]

By using the Aizenman-Lieb argument (see [17]) we immediately obtain the following estimates for the Riesz means of order \( \gamma \geq 1 \) of the eigenvalues of operators (1.15) and (1.16).

**Corollary 1.** Let \( V \geq 0 \) be a Hermitian \((M \times M)\)-matrix with \( \text{Tr}V^{\gamma+1/2} \in L_1 \). Then for any \( \gamma \geq 1 \) the negative eigenvalues of the operators (1.15) and (1.16) satisfy the inequalities

\[
\sum_j \lambda_j^\gamma \leq \frac{\pi}{\sqrt{3}} K_1(\beta) L_{\gamma,1}^{cl} \int_0^L \text{Tr}[V(x)^{1/2+\gamma}] dx, \tag{3.11}
\]

\[
\sum_j \mu_j^\gamma \leq \frac{\pi}{\sqrt{3}} K_2(\beta) L_{\gamma,1}^{cl} \int_0^{2\pi} \text{Tr}[V(x)^{1/2+\gamma}] dx. \tag{3.12}
\]
Proof. It is enough to prove this result for smooth matrix-valued potentials. Furthermore, we consider only (3.11), the treatment of the second operator being completely similar. Note that Theorem 5 is equivalent to

$$\sum_n \lambda_n \lesssim \frac{2}{3\sqrt{3}} K_1(\beta) \frac{1}{L_{1,1}^{cl}} \int_0^L \int_{-\infty}^{\infty} \text{Tr}[(\|\xi\|^2 - V(x)) -] \frac{d\xi \, dx}{2\pi}.$$ 

Scaling gives the simple identity for all $s \in \mathbb{R}$

$$s_\gamma = C_\gamma \int_0^\infty t^{\gamma - 2}(s + t) \, dt, \quad C_\gamma^{-1} = \mathcal{B}(\gamma - 1, 2),$$

where $\mathcal{B}$ is the Beta function. Let $\{\nu_j(x)\}_{j=1}^M$ be the eigenvalues of the matrix-function $V(x)$. Then

$$\sum_n \lambda_n^\gamma = C_\gamma \sum_n \int_0^\infty t^{\gamma - 2}(-\lambda_n + t) \, dt$$

$$\leq \frac{2K_1(\beta)}{3\sqrt{3}} \frac{C_\gamma}{L_{1,1}^{cl}} \int_0^L \int_{-\infty}^{\infty} t^{\gamma - 2} \text{Tr}[(\|\xi\|^2 - V(x) + t) -] \frac{d\xi \, dx}{2\pi} \, dt$$

$$= \frac{2K_1(\beta)}{3\sqrt{3}} \frac{C_\gamma}{L_{1,1}^{cl}} \sum_{j=1}^M \int_0^L \int_{-\infty}^{\infty} t^{\gamma - 2} \text{Tr}[(\|\xi\|^2 - \nu_j(x) + t) -] \frac{d\xi \, dx}{2\pi} \, dt$$

$$= \frac{2K_1(\beta)}{3\sqrt{3}} \frac{1}{L_{1,1}^{cl}} \int_0^L \int_{-\infty}^{\infty} \text{Tr}[(\|\xi\|^2 - V(x))_\gamma^\gamma] \frac{d\xi \, dx}{2\pi}$$

$$= \frac{2K_1(\beta)}{3\sqrt{3}} \frac{L_{\gamma,d}^{cl}}{L_{1,1}^{cl}} \int_0^L \text{Tr}[V(x)^{1/2 + \gamma}] \, dx,$$

which completes the proof since $L_{1,1}^{cl} = 2/(3\pi)$.

§ 4. Lieb-Thirring inequalities on the torus

In this section we consider Lieb-Thirring inequalities on the torus (1.9), (1.10), paying special attention to the two-dimensional case.

We recall the orthogonal projection $P$ defined in (1.11) so that the resulting function has zero average with respect to the shortest coordinate $x_d$ for all $x_1, \ldots, x_{d-1}$.

Theorem 6. Let $V \geq 0$, $\gamma \geq 1$, and let $V \in L_{\gamma + d/2}(\mathbb{T}_d^d)$. Then the negative eigenvalues $-\lambda_n \leq 0$ of the Schrödinger operator

$$\mathcal{H} = -\Delta - P(V(x)P \cdot)$$

satisfy the bound

$$\sum_n \lambda_n^\gamma \leq L_{\gamma,d} \int_{\mathbb{T}_d^d} V^{\gamma + d/2}(x) \, dx$$

where

$$L_{\gamma,d} \leq \left(\frac{\pi}{\sqrt{3}}\right)^d \prod_{j=1}^{d-1} K_1(\beta_j) K_2(\delta) L_{\gamma,d}^{cl}.$$
provided that the $\beta_j > 0$, $j = 1, \ldots, d - 1$ are chosen so small that
\[ \delta := \alpha_1^2 \beta_1 + \cdots + \alpha_{d-1}^2 \beta_{d-1} < 1. \]

If
\[ \beta_j \geq \beta_\ast \text{ and } \delta \leq \beta_\ast \ast \]
(where the numbers $\beta_\ast$ and $\beta_\ast \ast$ are defined in §2), then
\[ L_{\gamma,d} \leq \left( \frac{\pi}{\sqrt{3}} \right)^d L_{\gamma,d}^{cl}. \]

For $d \leq [\frac{\beta_\ast}{\beta_\ast \ast}] + 1 = 19$ condition (4.4) can be satisfied for any $\alpha$, so that (4.5) always holds.

Proof. We write the operator $\mathcal{H}$ in the form
\[ \left( -\frac{d^2}{dx_1^2} + \alpha_1^2 \beta_1 \right) + \cdots + \left( -\frac{d^2}{dx_{d-1}^2} + \alpha_{d-1}^2 \beta_{d-1} \right) + \left( -\frac{d^2}{dx_d^2} - \delta \right) - \text{P}(V(x)\text{P}). \]

We use the lifting argument with respect to dimensions developed in [2]. More precisely, we apply estimate (3.11) $d - 1$ times with respect to the variables $x_1, \ldots, x_{d-1}$, so that $\gamma$ is increased by $1/2$ at each step, and, finally, we use (3.12) (in the scalar case) with respect to $x_d$. Using the variational principle and denoting the negative parts of the operators by $[\cdot]_-$ we obtain

\[ \sum_n \lambda_n^\gamma(\mathcal{H}) = \sum_n \lambda_n^\gamma \left( -\partial_1^2 + \alpha_1^2 \beta_1 + \sum_{j=2}^{d-1} (-\partial_j^2 + \alpha_j^2 \beta_j) + (-\partial_2^2 - \delta - \text{P}(V(x)\text{P})) \right) \]
\[ \leq \sum_n \lambda_n^\gamma \left( -\partial_1^2 + \alpha_1^2 \beta_1 - \left[ \sum_{j=2}^{d-1} (-\partial_j^2 + \alpha_j^2 \beta_j) + (-\partial_2^2 - \delta - \text{P}(V(x)\text{P})) \right]_- \right) \]
\[ \leq \frac{\pi}{\sqrt{3}} K_1(\beta_1) L_{\gamma,1}^{cl} \int_0^{L_1} \text{Tr} \left[ \sum_{j=2}^{d-1} (-\partial_j^2 + \alpha_j^2 \beta_j) + (-\partial_2^2 - \delta - \text{P}(V(x)\text{P})) \right]_-^{\gamma+1/2} dx_1 \]
\[ \leq \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ \leq \left( \frac{\pi}{\sqrt{3}} \right)^{d-1} \prod_{j=1}^{d-1} (K_1(\beta_j)) \prod_{j=1}^{d-1} L_{\gamma+(j-1)/2,1}^{cl} \]
\[ \times \int_0^{L_1} \cdots \int_0^{L_{d-1}} \text{Tr} \left[ (-\partial_2^2 - \delta - \text{P}(V(x)\text{P})) \right]_-^{\gamma+(d-1)/2} dx_1 \cdots dx_{d-1} \]
\[ \leq \left( \frac{\pi}{\sqrt{3}} \right)^d \prod_{j=1}^d K_1(\beta_j) \prod_{j=1}^d L_{\gamma+(j-1)/2,1}^{cl} K_2(\delta) \int_{\mathbb{T}_d^\alpha} V^{\gamma+d/2}(x) \, dx, \]

which proves (4.2), (4.3) since (see (1.3))
\[ \prod_{j=1}^d L_{\gamma+(j-1)/2,1}^{cl} = L_{\gamma,d}^{cl}. \]
Finally, if $\beta_j = \beta_* = 0.045\ldots$, then $K_1(\beta_j) = 1$. Since $\alpha_j \leq 1$, it follows that

$$ \delta = \sum_{j=1}^{d-1} \alpha_j^2 \beta_j = \beta_* \sum_{j=1}^{d-1} \alpha_j^2 \leq (d-1)\beta_*.$$  

Therefore the condition $\delta \leq \beta_{**} = 0.839\ldots$ (giving $K_2(\delta) = 1$) is always satisfied if $d - 1 \leq \left[ \frac{\beta_{**}}{\beta_*} \right] = 18$.

The proof is complete.

**Remark 5.** For $d > 19$ the numbers $\beta_j$ that are at our disposal should be chosen so that the following minimum is attained:

$$ \min_{\beta_1 > 0,\ldots,\beta_{d-1} > 0, \delta < 1} \prod_{j=1}^{d-1} K_1(\beta_j) K_2(\delta) =: c_\alpha \geq 1.$$  

**Remark 6.** The complementary projection $Q = I - P$ maps $\dot{L}_2(\mathbb{T}_\alpha^d)$ onto $\dot{L}_2(\mathbb{T}_{\alpha'}^{d-1})$, where $\alpha' = (\alpha_1,\ldots,\alpha_{d-1})$ and $\dot{L}_2(\mathbb{T}_{\alpha'}^{d-1})$ is the subspace of $\dot{L}_2(\mathbb{T}_\alpha^d)$ of functions independent of $x_d$.

**Remark 7.** If we compare (1.4) and (4.5), we see that the factor $\pi/\sqrt{3}$ accumulates in (4.5) at each iteration with respect to the dimension; while in (1.4) already at the second iteration sharp bounds from [2] for the $\gamma$-moments with $\gamma \geq \frac{3}{2}$ are available, which are not known in the periodic case.

We single out the following corollary that is important for applications. Let $d = 2$ so that $\mathbb{T}_\alpha^2 = (0,2\pi/\alpha) \times (0,2\pi)$, $\alpha \leq 1$.

**Corollary 2.** Let $V \geq 0$ and let $V \in L_2(\mathbb{T}_\alpha^2)$. Then the negative eigenvalues $-\lambda_j \leq 0$ of the operator (4.1) satisfy the following bound:

$$ \sum_j \lambda_j \leq \left( \frac{\pi \sqrt{3}}{2} \right)^2 L_{1,2}^{cl} \int_{\mathbb{T}_\alpha^2} V^2(x) \, dx = \frac{\pi}{24} \int_{\mathbb{T}_\alpha^2} V^2(x) \, dx. \tag{4.6}$$

Equivalently (see (1.5), (1.6)), if a family $\{\varphi_j\}_{j=1}^N \in PH^1(\mathbb{T}_\alpha^2)$ is orthonormal, then $\rho_\varphi(x) := \sum_{j=1}^N \varphi_j(x)^2$ satisfies

$$ \int_{\mathbb{T}_\alpha^2} \rho_\varphi(x)^2 \, dx \leq \frac{\pi}{6} \sum_{j=1}^N \|\nabla \varphi_j\|^2. \tag{4.7}$$

§ 5. Applications to attractors

Starting from the paper [6] the Lieb-Thirring inequalities have played an essential role in the theory of attractors for infinite-dimensional dissipative dynamical systems, especially for the Navier-Stokes system. They are used for the first-order moments ($\gamma = 1$) in the equivalent formulation in terms of orthonormal families.

We first consider the square torus $\mathbb{T}^2 = (0, L) \times (0, L)$. 

**Proposition 1.** If \( \{v_j\}_{j=1}^m \in \dot{H}^1(T^2) \) is an orthonormal family of vector functions and \( \text{div} \, v_j = 0 \), then \( \rho_v(x) := \sum_{j=1}^m |v_j(x)|^2 \) satisfies

\[
\int_{T^2} \rho_v(x)^2 \, dx \leq c_{LT} \sum_{j=1}^m \| \nabla v_j \|^2, \quad c_{LT} \leq \frac{3}{2}. \tag{5.1}
\]

**Proof (see [13]).** In the scalar case this follows from (1.7) and (1.5), (1.6). In two dimensions the passage to the vector case with \( \text{div} \, v_j = 0 \) does not increase the constant [11].

Turning to applications, we consider the damped and driven Navier-Stokes system

\[
\begin{cases}
\partial_t u + \sum_{i=1}^2 u^i \partial_i u = \nu \Delta u - \mu u - \nabla p + g, \\
\text{div} \, u = 0, \quad u|_{t=0} = u_0,
\end{cases} \tag{5.2}
\]

in a periodic square domain \( T^2 = (0,L) \times (0,L) \). We assume that \( g \) and \( u \) have mean value zero. The system is studied in the small viscosity limit \( \nu \to 0^+ \), while the damping coefficient \( \mu > 0 \) is arbitrary but fixed.

Using the standard notation in the theory of the Navier-Stokes equations we denote by \( H \) the closure in \( L^2(T^2) \) of the set of trigonometric polynomials with divergence and mean value zero. The norm \( \| \cdot \| \) and scalar product \( (\cdot, \cdot) \) in \( H \) are those in \( L^2(T^2) \). We project the first equation onto \( H \) and obtain the functional evolution equation

\[
\partial_t u + B(u, u) + \nu A u = -\mu u + g, \quad u(0) = u_0, \tag{5.3}
\]

where \( A \) is the Stokes operator and \( B(u, v) \) is the nonlinear term defined as follows:

\[
(\langle Au, v \rangle = (\nabla u, \nabla v), \quad u, v \in H^1 \cap H,
\]

and

\[
\langle B(v, u), w \rangle = \int_{T^2} \sum_{i,k=1}^2 v^k \partial_k u^i w^i \, dx =: b(v, u, w) \tag{5.4}
\]

for all \( u, v, w \in H^1 \cap H \).

Equation (5.3) has a unique solution \( u(t) \) and the solution semigroup \( S_t u_0 \to u(t) \) is well defined. The semigroup \( S_t \) has a global attractor \( \mathcal{A} \) which is a compact strictly invariant set in \( H \) attracting under the action of \( S_t \) all bounded sets as \( t \to \infty \). These facts are well known for the classical Navier-Stokes equations [7]–[9], [26]; the case \( \mu > 0 \) is similar. The solution semigroup \( S_t \) is uniformly differentiable in \( H \) with differential \( L(t, u_0) : \xi \to U(t) \in H \), where \( U(t) \) is the solution of the variational equation

\[
\partial_t U = -\nu A U - \mu U - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t, u_0)U, \quad U(0) = \xi. \tag{5.5}
\]

Furthermore, the differential \( L(t, u_0) \) depends continuously on the initial point \( u_0 \in \mathcal{A} \) (see [7]).
We estimate the numbers \( q(m) \), that is, the sums of the first \( m \) global Lyapunov exponents \([8], [9], [27]\):

\[
q(m) := \limsup_{t \to \infty} \sup_{u_0 \in \mathcal{A}} \sup_{\tau \in H \cap H^1} \frac{1}{t} \int_{0}^{t} \sum_{j=1}^{m} (\mathcal{L}(\tau, u_0) v_j, v_j) d\tau, \tag{5.6}
\]

where \( \{v_j\}_{j=1}^{m} \in H \cap H^1 \) is an arbitrary orthonormal system of dimension \( m \).

We first estimate the \( H^1 \)-norm of the solutions on the attractor. Taking the scalar product of (5.3) with \( Au \), using the identity \((B(u, u), Au) = 0 \) (see, for example, \([8] \) and \([9]\)) and integrating by parts we obtain

\[
\partial_t \| \nabla u \|^2 + 2\nu \| Au \|^2 + 2\mu \| \nabla u \|^2 = 2(\nabla g, \nabla u) \leq \mu \| \nabla u \|^2 + \mu^{-1} \| \nabla g \|^2.
\]

Dropping the \( \nu \)-term on the left-hand side, the Gronwall inequality gives

\[
\| \nabla u(t) \|^2 \leq \| \nabla u(0) \|^2 e^{-\mu t} + \frac{1-e^{-\mu t}}{\mu^2} \| \nabla g \|^2,
\]

so that on the attractor \( u(t) \in \mathcal{A} \) letting \( t \to \infty \) we have a \( \nu \)-independent estimate

\[
\| \nabla u(t) \|^2 \leq \frac{\| \nabla g \|^2}{\mu^2}. \tag{5.7}
\]

We now estimate the \( m \)-trace of the operator \( \mathcal{L} \) in (5.6). Integrating by parts and using the identity \((B(u(t), v_j), v_j) = 0 \) (see \([8] \) and \([9]\)) and the orthonormality of the \( v_j \), we obtain

\[
\sum_{j=1}^{m} (\mathcal{L}(t, u_0) v_j, v_j) = -\nu \sum_{j=1}^{m} \| \nabla v_j \|^2 - \mu m - \sum_{j=1}^{m} b(v_j, u(t), v_j). \tag{5.8}
\]

For the last term we use the point-wise inequality

\[
\left| \sum_{k,i=1}^{2} v^k \partial_k u^i v^i \right| \leq 2^{-1/2} |\nabla u| |v|^2, \tag{5.9}
\]

which holds for any \( v = (v^1, v^2) \) and any \((2 \times 2)\)-matrix \( \nabla u = (\partial_i u^j)_{i,j=1}^{2} \) with \( \partial_1 u^1 + \partial_2 u^2 = 0 \) (see \([28]\), Lemma 4.1). Using (5.9) and (5.1) we obtain

\[
\sum_{j=1}^{m} (\mathcal{L}(t, u_0) v_j, v_j) \leq -\nu \sum_{j=1}^{m} \| \nabla v_j \|^2 - \mu m + \frac{1}{\sqrt{2}} \int |\nabla u(t, x)| \rho_v(x) \, dx \\
\leq -\nu \sum_{j=1}^{m} \| \nabla v_j \|^2 - \mu m + \frac{1}{\sqrt{2}} \| \nabla u(t) \| \| \rho_v \| \\
\leq -\nu \sum_{j=1}^{m} \| \nabla v_j \|^2 - \mu m + \frac{1}{\sqrt{2}} \| \nabla u(t) \| \left( c_{LT} \sum_{j=1}^{m} \| \nabla v_j \|^2 \right)^{1/2} \\
\leq -\nu \sum_{j=1}^{m} \| \nabla v_j \|^2 - \mu m + \frac{c_{LT}}{8\nu} \| \nabla u(t) \|^2 + \nu \sum_{j=1}^{m} \| \nabla v_j \|^2 \\
= -\mu m + \frac{c_{LT}}{8\nu} \| \nabla u(t) \|^2.
\]
Now (5.6) and (5.7) give

\[
q(m) \leq -\mu m + \frac{c_{LT} \| \nabla g \|^2}{8 \nu \mu^2}.
\] (5.10)

We can proceed in a somewhat different way by observing that

\[
m = \int \rho_v(x) \, dx \leq \| \rho_v \| L \quad \Rightarrow \quad \sum_{j=1}^m \| \nabla v_j \|^2 \geq \frac{1}{c_{LT}} \| \rho_v \|^2 \geq \frac{1}{c_{LT}} \frac{m^2}{L^2}.
\]

Then we argue as before but in the last but one line single out the term \(\nu/2\sum_{j=1}^m \| \nabla v_j \|^2\) to absorb it into the half of the first term. We obtain

\[
q(m) \leq -\frac{\nu m^2}{2c_{LT} L^2} + \frac{c_{LT} \| \nabla g \|^2}{4 \nu \mu^2}.
\] (5.11)

Now if for an \(m^*\) we have \(q(m^*) < 0\), then both the Hausdorff dimension (see [9], [27]), and the fractal dimension (see [29], [28]) of the attractor \(\mathcal{A}\) satisfy

\[
dim_H \mathcal{A} \leq \dim_F \mathcal{A} < m^*.
\]

Therefore estimates (5.10) and (5.11) along with (5.1) show that we have proved the following theorem.

**Theorem 7.** The fractal dimension of the attractor for the damped-driven Navier-Stokes system (5.2) satisfies the estimate

\[
\dim_F \mathcal{A} \leq \min \left( \frac{c_{LT} \| \nabla g \|^2}{8 \nu \mu^3}, \frac{c_{LT} \| \nabla g \| L}{\sqrt{2} \nu \mu} \right) \leq \min \left( \frac{3 \| \nabla g \|^2}{16 \nu \mu^3}, \frac{3 \| \nabla g \| L}{2 \sqrt{2} \nu \mu} \right).
\] (5.12)

Let us now consider the system (5.2) on the elongated torus \(T^2_\alpha = (0, L/\alpha) \times (0, L)\), where \(\alpha \leq 1\). As before we assume that both scalar and vector functions have mean value zero over \(T^2_\alpha\), and we decompose the phase space

\[
\dot{L}_2(T^2_\alpha) = L_2(T^2_\alpha) \cap \int_{T^2_\alpha} u(x) \, dx = 0
\]

into the orthogonal sum

\[
\dot{L}_2(T^2_\alpha) = P\dot{L}_2(T^2_\alpha) \oplus Q\dot{L}_2(T^2_\alpha),
\] (5.13)

where the orthogonal projection \(P\) is defined in (1.11) in the two-dimensional case, and the projection \(Q\),

\[
(Q\psi)(x_1) = \frac{1}{L} \int_0^L \psi(x_1, t) \, dt,
\]

maps \(\dot{L}_2(T^2_\alpha)\) onto \(\dot{L}_2(0, L/\alpha)\).

On the elongated torus we have the following Lieb-Thirring inequalities for orthonormal families.
Proposition 2. If \( \{v_j\}_{j=1}^m \in P \dot{L}_2^2(T^2_\alpha) \) is an orthonormal family in \( L_2^2(T^2_\alpha) \) and \( \text{div} \ v_j = 0 \), then \( \rho (x) := \sum_{j=1}^m |v_j(x)|^2 \) satisfies
\[
\int_{T^2_\alpha} \rho (x)^2 \, dx \leq c_P \sum_{j=1}^m \|\nabla v_j\|^2, \quad c_P \leq \frac{\pi}{6}.
\]
(5.14)

Accordingly, if \( \{w_j\}_{j=1}^m \in Q \dot{L}_2^2(T^2_\alpha) \) is an orthonormal family of vector functions and \( \text{div} \ w_j = 0 \), then \( \rho (x) := \sum_{j=1}^m |w_j(x)|^2 \) satisfies
\[
\int_{T^2_\alpha} \rho (x)^2 \, dx \leq \frac{c_Q}{L} \sum_{j=1}^m \|\nabla w_j\|, \quad c_Q \leq 6.
\]
(5.15)

Proof. The proof of (5.14) is the same as in Proposition 1.

Turning to (5.15) we observe that the \( w_j \) depend only on \( x_1 \) and the family \( \tilde{w}_j = \sqrt{L}w_j \) is orthonormal in \( L_2^2(0, L/\alpha) \) with mean value zero. Then we can use the one-dimensional Lieb-Thirring inequality for the operator of order one on \( \dot{L}_2^2(0, L/\alpha) \) (see [30])
\[
\int_0^{L/\alpha} \left( \sum_{j=1}^m \tilde{\psi}_j(x)^2 \right)^2 \, dx \leq 6 \sum_{j=1}^m \|\tilde{\psi}_j^{(1/2)}\|_{L_2^2(0, L/\alpha)}^2 \leq 6 \sum_{j=1}^m \|\tilde{\psi}_j\|_{L_2^2(0, L/\alpha)}^2,
\]
where the second inequality follows from the interpolation inequality
\[
\|\psi^{(1/2)}\|^2 \leq \|\psi\| \|\psi'\|
\]
and orthonormality.

The conditions \( w = Qw = w(x_1) \) and \( \text{div} \ w = 0 \) imply that \( (w^1)'_{x_1} = 0 \) and hence \( w^1 \equiv 0 \), therefore the inequality for vector functions is essentially a scalar inequality. Returning to \( \psi_j \) and \( w_j \) and to integrals over \( T^2_\alpha \), we obtain (5.15).

Finally, we observe that all the inequalities in the theorem also hold for sub-orthonormal families \( \{v_j\}_{j=1}^m \), that is, ones satisfying
\[
\sum_{i,j=1}^m \xi_i \xi_j (v_i, v_j) \leq \sum_{i=1}^m \xi_i^2 \quad \text{for every} \quad \xi \in \mathbb{R}^m,
\]
and the corresponding constants do not increase (see [11]).

We are now prepared to state the main result of this section in which we estimate the fractal dimension of the attractor \( \mathcal{A} \) of system (5.2) on the torus \( T^2_\alpha \), paying special attention to the dependence of the estimates on \( \alpha \to 0^+ \) and \( \nu \to 0^+ \).

Theorem 8. The damped Navier-Stokes system (5.2) on the torus \( T^2_\alpha \) has attractor \( \mathcal{A} \) and
\[
\dim_F \mathcal{A} \leq \left( \frac{c_P}{2} + \sqrt{c_P c_Q} \right) \frac{\|\nabla g\|^2}{\nu \mu^3} \leq \left( \frac{\pi}{12} + \sqrt{\frac{\pi}{6}} \right) \frac{\|\nabla g\|^2}{\nu \mu^3}
\]
(5.16)
for all sufficiently small \( \nu \leq 8\mu L^2 \).
Proof. As before, we have (5.8)
\[ \sum_{j=1}^{m} \left( \mathcal{L}(t, u_0)v_j, v_j \right) = -\nu \sum_{j=1}^{m} \|\nabla v_j\|^2 - \mu m - \sum_{j=1}^{m} b(v_j, u(t), v_j), \tag{5.17} \]
and our task is to estimate the last term there. The main idea \cite{15} is to decompose the solution \( u \) and the \( v_j \)'s as follows:
\[
\begin{align*}
u_j &= P^u + Q^u, \\
\end{align*}
\]
and use \( \alpha \)-independent Lieb-Thirring inequalities in Proposition 2. We note that since the \( v_j \)'s are orthonormal, both \( P^v \) and \( Q^v \) are suborthonormal.

Since \( \partial_1 Q = Q \partial_1 \) and \( \partial_2 Q = Q \partial_2 = 0 \), it follows that \( \text{div} Pw = \text{div} Qw = 0 \) if \( \text{div} w = 0 \). Since \( Qu \) and \( Qv_j \) depend only on \( x_1 \), it follows that \( Qu^1 = Qv_j^1 = 0 \) and, in addition, \( \int_0^L P^u(x_1, x_2) \, dx_2 = 0 \). Therefore,
\[
b(Qv_j, u, Qv_j) = 0, \quad b(Qv_j, Qu, P^v_j) = 0, \quad b(P^v_j, Qu, Qv_j) = 0.
\]

For example,
\[
b(Q^v, u, Q^v) = \int_0^{L/\alpha} \langle Q^v(x_1) \rangle^2 \, dx_1 \int_0^L \partial_2 u^2(x_1, x_2) \, dx_2 = 0
\]
by periodicity. Therefore,
\[
b(v_j, u, v_j) = b(Pv_j, u, Pv_j) + b(Qv_j, Pu, Pv_j) + b(Pv_j, Pu, Qv_j).
\]

Hence (5.9) gives that
\[
\sum_{j=1}^{m} b(v_j, u, v_j) \leq \frac{1}{\sqrt{2}} \|\nabla u\| \|\rho P^v\| + \sqrt{2} \|\nabla Pu\| \|\rho Q^v\|^{1/2} \|\rho P^v\|^{1/2}. \tag{5.18} \]

For the first term we use (5.14) and single out \( \nu/2 \sum_{j=1}^{m} \|\nabla Pv_j\|^2 \):
\[
\frac{1}{\sqrt{2}} \|\nabla u\| \|\rho P^v\| \leq \frac{c_P}{4\nu} \|\nabla u\|^2 + \frac{\nu}{2} \sum_{j=1}^{m} \|\nabla Pv_j\|^2.
\]

For the second term we use (5.14), estimate (5.15) in the form
\[
\|\rho Q^v\| \leq \frac{c_Q}{L} \sqrt{m} \left( \sum_{j=1}^{m} \|\nabla w_j\|^2 \right)^{1/2}
\]
and single out the terms $\nu/2 \sum_{j=1}^{m} \|\nabla P v_j\|^2$ and $\nu \sum_{j=1}^{m} \|\nabla Q v_j\|^2$:

$$\sqrt{2}\|\nabla u\| \|\rho_Q\|^{1/2} \|\rho_P\|^{1/2} \leq \sqrt{2}\|\nabla u\|(c_P c_Q)^{1/4} \left(\sum_{j=1}^{m} \|\nabla P v_j\|^2\right)^{1/4} \left(\frac{m}{L^2} \sum_{j=1}^{m} \|\nabla Q v_j\|^2\right)^{1/8}$$

$$\leq \frac{\|\nabla u\|^2 (c_P c_Q)^{1/2}}{2\nu} + \nu \left(\sum_{j=1}^{m} \|\nabla P v_j\|^2\right)^{1/2} \left(\frac{m}{L^2} \sum_{j=1}^{m} \|\nabla Q v_j\|^2\right)^{1/4}$$

$$\leq \frac{\|\nabla u\|^2 (c_P c_Q)^{1/2}}{2\nu} + \nu \sum_{j=1}^{m} \|\nabla P v_j\|^2 + \nu \sum_{j=1}^{m} \|\nabla Q v_j\|^2 + \frac{\nu m}{16L^2}.$$ 

Since $\|\nabla\|^2 = \|P \nabla v\|^2 + \|Q \nabla v\|^2$, we obtain

$$\sum_{j=1}^{m} b(v_j, u(t), v_j) \leq \frac{1}{2\nu} \left(\frac{c_P}{2} + \sqrt{c_P c_Q}\right) \|\nabla u(t)\|^2 + \nu \sum_{j=1}^{m} \|\nabla v_j\|^2 + \frac{\nu m}{16L^2}.$$ 

Substituting this into (5.17) and using (5.7), we obtain the estimate for $q(m)$:

$$q(m) \leq \frac{1}{2} \left(\frac{c_P}{2} + \sqrt{c_P c_Q}\right) \frac{\|\nabla g\|^2}{\nu \mu^2} - \mu \left(1 - \frac{\nu}{16\mu L^2}\right).$$

If $\nu \leq 8\mu L^2$, then for

$$m^* = \left(\frac{c_P}{2} + \sqrt{c_P c_Q}\right) \frac{\|\nabla g\|^2}{\nu \mu^2},$$

we have $q(m^*) \leq 0$, and hence

$$\dim_{F} A \leq m^*.$$ 

In conclusion we point out that estimate (5.12) for the square torus is sharp as $\nu \to 0^+$ and estimate (5.16) is sharp as both $\nu \to 0^+$ and $\alpha \to 0^+$. The lower bounds for the dimension of the attractor are based on the characterization of the attractor as the section at any given time of the set of all complete trajectories bounded for $t \in \mathbb{R}$. Therefore all stationary solutions and their unstable manifolds lie on the attractor. Such an unstable stationary solution for the two-dimensional periodic Navier-Stokes was first constructed in [31] and is called the Kolmogorov flow. The construction was generalized in [32] to prove that the estimate for the dimension obtained in [33] is logarithmically sharp. It also applies to our periodic damped Navier-Stokes system (see [18], [19]). In particular, it was shown in [19] that the right-hand side

$$g = g_s = \begin{cases} g_1 = c_1 \nu^2 s^3 \sin sx_2, \\
g_2 = 0, \end{cases}$$

(5.19)

where $c_1$ is an absolute constant and $T_\alpha^2 = (0, 2\pi/\alpha) \times (0, 2\pi)$, produces the stationary solution with unstable manifold of dimension

$$d = c_2 \frac{s^2}{\alpha},$$
where \( s \gg 1 \). Setting \( s := \sqrt{\mu/\nu} \) we find that \( \dim \mathcal{A} \geq d = c_2 \mu/\alpha \nu \). Since

\[
\| \nabla g_s \|^2 = c_3 \frac{\nu^4 s^8}{\alpha} = c_3 \frac{\mu^4}{\alpha},
\]

it follows that for \( g = g_s \) the dimensionless number \( \| \nabla g \|^2/\nu^2 \mu^3 \) becomes

\[
\frac{\| \nabla g \|^2}{\nu^2 \mu^3} = c_3 \frac{\mu}{\alpha \nu}
\]

and the upper bound in (5.16) for \( \mathcal{A} = \mathcal{A}_s \) is supplemented with a sharp lower bound

\[
\dim \mathcal{A}_s \geq \frac{c_2}{c_3} \frac{\| \nabla g \|^2}{\nu^2 \mu^3},
\]

where \( c_2 \) and \( c_3 \) are some absolute constants (which can be calculated).

Bibliography

[1] E. H. Lieb and W. E. Thirring, “Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities”, *Studies in mathematical physics. Essays in honor of V. Bargmann*, Princeton Univ. Press, Princeton, NJ 1976, pp. 269–303.

[2] A. Laptev and T. Weidl, “Sharp Lieb-Thirring inequalities in high dimensions”, *Acta Math.* 184:1 (2000), 87–111.

[3] J. Dolbeault, A. Laptev and M. Loss, “Lieb-Thirring inequalities with improved constants”, *J. Eur. Math. Soc. (JEMS)* 10:4 (2008), 1121–1126.

[4] R. Benguria and M. Loss, “A simple proof of a theorem of Laptev and Weidl”, *Math. Res. Lett.* 7:2-3 (2000), 195–203.

[5] D. Hundertmark, A. Laptev and T. Weidl, “New bounds on the Lieb-Thirring constants”, *Inv. Math.* 140:3 (2000), 693–704.

[6] E. H. Lieb, “On characteristic exponents in turbulence”, *Comm. Math. Phys.* 92:4 (1984), 473–480.

[7] A. V. Babin and M. I. Vishik, *Attractors of evolution equations*, Nauka, Moscow 1989, 296 pp.; English transl. in Stud. Math. Appl., vol. 25, North-Holland Publishing Co., Amsterdam 1992, x+532 pp.

[8] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, IL 1988, x+190 pp.

[9] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd ed., Appl. Math. Sci., vol. 68, Springer-Verlag, New York 1997, xxii+648 pp.

[10] J.-M. Ghidaglia, M. Marion and R. Temam, “Generalization of the Sobolev-Lieb-Thirring inequalities and applications to the dimension of attractors”, *Differential Integral Equations* 1:1 (1988), 1–21.

[11] A. A. Ilyin, “Lieb-Thirring integral inequalities and their applications to attractors of the Navier-Stokes equations”, *Mat. Sb.* 196:1 (2005), 33–66; English transl. in *Sb. Math.* 196:1 (2005), 29–61.

[12] B. S. Kashin, “On a class of inequalities for orthonormal systems”, *Mat. Zametki* 80:2 (2006), 204–208; English transl. in *Math. Notes* 80:2 (2006), 199–203.

[13] A. A. Ilyin, “Lieb-Thirring inequalities on some manifolds”, *J. Spectr. Theory* 2:1 (2012), 57–78.
[14] J.-M. Ghidaglia and R. Temam, “Lower bound on the dimension of the attractor for the Navier-Stokes equations in space dimension 3”, *Mechanics, analysis and geometry: 200 years after Lagrange*, North-Holland Delta Ser., North-Holland, Amsterdam 1991, pp. 33–60.

[15] M. Ziane, “Optimal bounds on the dimension of attractor of the Navier-Stokes equations”, *Phys. D* 105:1-3 (1997), 1–19.

[16] A. Ilyin, A. Laptev, M. Loss and S. Zelik, “One-dimensional interpolation inequalities, Carlson-Landau inequalities, and magnetic Schrödinger operators”, *Int. Math. Res. Not.* 2016:4 (2016), 1190–1222.

[17] M. Aizenman and E. H. Lieb, “On semi-classical bounds for eigenvalues of Schrödinger operators”, *Phys. Lett. A* 66:6 (1978), 427–429.

[18] A. A. Ilyin, A. Miranville and E. S. Titi, “Small viscosity sharp estimates for the global attractor of the 2-D damped-driven Navier-Stokes equations”, *Commun. Math. Sci.* 2:3 (2004), 403–426.

[19] A. A. Ilyin and E. S. Titi, “The damped-driven 2D Navier-Stokes system on large elongated domains”, *J. Math. Fluid Mech.* 10:2 (2008), 159–175.

[20] S. V. Zelik and A. A. Ilyin, “Green’s function asymptotics and sharp interpolation inequalities”, *Uspekhi Mat. Nauk* 69:2 (416) (2014), 23–76; English transl. in *Russian Math. Surveys* 69:2 (2014), 209–260.

[21] G. H. Hardy, “A note on two inequalities”, *J. London Math. Soc.* (1) 11:3 (1936), 167–170.

[22] L. V. Taikov, “Kolmogorov-type inequalities and the best formulas for numerical differentiation”, *Mat. Zametki* 4:2 (1968), 233–238; English transl. in *Math. Notes* 4:2 (1968), 631–634.

[23] A. A. Ilyin, “Best constants in multiplicative inequalities for sup-norms”, *J. London Math. Soc. (2)* 58:1 (1998), 84–96.

[24] M. Bartuccelli, J. Deane and S. Zelik, “Asymptotic expansions and extremals for the critical Sobolev and Gagliardo-Nirenberg inequalities on a torus”, *Proc. Roy. Soc. Edinburgh Sect. A* 143:3 (2013), 445–482.

[25] A. Eden and C. Foias, “A simple proof of the generalized Lieb-Thirring inequalities in one-space dimension”, *J. Math. Anal. Appl.* 162:1 (1991), 250–254.

[26] O. Ladyzhenskaya, *Attractors for semigroups and evolution equations*, Lezioni Lincee, Cambridge Univ. Press, Cambridge 1991, xii+73 pp.

[27] P. Constantin and C. Foias, “Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations”, *Comm. Pure Appl. Math.* 38:1 (1985), 1–27.

[28] V. V. Chepyzhov and A. A. Ilyin, “On the fractal dimension of invariant sets: applications to Navier-Stokes equations”, *Discrete Contin. Dyn. Syst.* 10:1-2 (2004), 117–135.

[29] V. V. Chepyzhov and A. A. Ilyin, “A note on the fractal dimension of attractors of dissipative dynamical systems”, *Nonlinear Anal.* 44:6 (2001), 811–819.

[30] A. A. Ilyin, “Lieb-Thirring inequalities on the N-sphere and in the plane, and some applications”, *Proc. London Math. Soc.* (3) 67:1 (1993), 159–182.

[31] L. D. Meshalkin and Ia. G. Sinai, “Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid”, *Prikl. Mat. Mekh.* 25:6 (1961), 1140–1143; English transl. in *J. Appl. Math. Mech.* 25:6 (1961), 1700–1705.

[32] V. X. Liu, “A sharp lower bound for the Hausdorff dimension of the global attractors of the 2D Navier-Stokes equations”, *Comm. Math. Phys.* 158:2 (1993), 327–339.
[33] P. Constantin, C. Foias and R. Temam, “On the dimension of the attractors in two-dimensional turbulence”, Phys. D 30:3 (1988), 284–296.

Alexei A. Ilyin
Keldysh Institute of Applied Mathematics
of Russian Academy of Sciences, Moscow
E-mail: ilyin@keldysh.ru

Ari A. Laptev
Imperial College London, United Kingdom;
Institut Mittag-Leffler, Djursholm, Sweden
E-mail: a.laptev@imperial.ac.uk

Received 4/DEC/15 and 18/MAY/16
Translated by A. ILYIN