Abstract. The phase transition of the electroweak vacuum induced by a strong magnetic field is examined, and a connection is made with the Ginzburg-Landau theory of type-II superconductivity. For solutions of the exact nonlinear field equations of the electroweak theory with lattice periodicity in directions perpendicular to the magnetic field, it is proven that, likewise, each lattice cell must enclose an integer number of quanta of magnetic flux. Close to the lower critical magnetic field, a perturbative method developed by MacDowell and the author is used to study properties of the lattice solutions. Analytical expressions for observables are obtained in terms of a complex parameter $\tau$ specifying the lattice and it is shown that the triangular Abrikosov solution constitutes a local minimum of the energy provided $M_H > M_Z$.

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I Introduction

In the cosmological scenario, the electroweak transition between the symmetric $SU(2) \otimes U(1)_Y$ and the broken $U(1)_{EM}$ phases results from a temperature dependence in the coefficients of the Higgs potential. The transition occurs at a critical temperature $T_c$ and may involve the coexistence of phases, depending on the order of the transition.

It has been shown by Ambjørn and Olesen [1, 2] that the phase transition can be induced at zero temperature by a large magnetic field. Their analysis was done with particular values of the coupling constants, for which the nonlinear field equations simplify, corresponding to $M_H = M_Z$ where $M_H$ is the Higgs-boson mass. The transition from the broken phase with Higgs field $\phi \equiv \phi_0$ to the symmetric phase with $\phi \equiv 0$ was found to take place gradually as the magnetic field increases from $B_{c2} = M_W^2/e$ to $B_{c1} = M_Z^2/e$.\footnote{With this choice of labels for the critical magnetic fields, the phase transition at $B_{c2}$ is qualitatively similar to the well studied transition at $B_{c2}$ in a type-II superconductor. In Refs. [1, 2] the opposite labels were used.} Between these values the field equations for $A_\mu$, $Z_\mu$, $W_\mu$, and $\phi$ admit solutions with lattice periodicity in directions perpendicular to the magnetic field [3]. The solutions have been obtained numerically and studied in the range $B_{c2} < B < B_{c1}$ [4, 5]. The emerging vacuum structure resembles that of the mixed state in a type-II superconductor [6] with the order parameter given by a density of W-boson pairs forming a zero-charge condensate. The distinctive feature is that the electroweak vacuum is paramagnetic, i.e. the magnetic field is enhanced by the W condensate, while a superconductor is diamagnetic.

The phase transition resolves an old problem of the vacuum instability [3, 7] of the Weinberg-Salam model for magnetic fields of magnitude $B_{c2} = M_W^2/e \approx 10^{20}$ Tesla and beyond. The instability is a consequence of the large magnetic moment of the W bosons,
which, when aligned with the magnetic field, can lower the energy sufficiently to make the W condensate energetically more favorable than the trivial vacuum. The W pairs also couple to and act as sources for nontrivial Z and Higgs fields.

Explicit analytical solutions to the full, nonlinear problem are unknown, even for the case $M_H = M_Z$ considered in Refs. [1, 2]. Approximate solutions can be obtained, however, in certain limits. For $B$ near the upper critical field $B_{c1}$ ($\phi \approx 0$) a perturbative solution was recently given by Olesen [3]. In the vicinity of the lower critical field $B_{c2}$ ($\phi \approx \phi_0$) a perturbative method was first suggested by Skalozub [4], who found that, in this regime, the solutions with lattice symmetry coincide with the Abrikosov solutions [5] for a type-II superconductor near $B_{c2}$. In Skalozub’s derivation, the interactions involving the Z and $\phi$ fields were approximated with a local quartic interaction.

In a recent paper [6], MacDowell and Törnkvist applied a perturbative method to the full Weinberg–Salam model for general values of the coupling constants and solved the field equations exactly to lowest order in $(B - B_{c2})$. In this approach, the interactions mediated by Z and $\phi$ were accounted for in an effective, nonlocal quartic interaction. An investigation showed that, for $M_H > M_Z$, the triangular$^2$ Abrikosov lattice solution represents the energetically most favorable configuration.

In this paper we shall derive analytical expressions for physical observables, such as the energy density, as a function of the geometry of the lattice solutions and of $(B - B_{c2})$. The geometry is specified by a complex parameter $\tau$. In the complex plane, $\tau$ and the real number 1 span a fundamental parallelogram for an arbitrary two-dimensional simple lattice.

$^2$In Ref. [6] it was denoted “hexagonal”.

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The paper is organized as follows. In section II, the perturbative method developed in Ref. [11] is reviewed. Physical quantities are shown to depend on the function $|W|^2$ where $W$ is the perturbative solution for the W field. A general theorem of magnetic flux quantization is presented. For any solution of the nonlinear electroweak field equations where $U(1)_{EM}$ gauge-invariant quantities possess a lattice symmetry in directions perpendicular to the magnetic field, it is shown that each lattice cell must enclose an integer number of quanta of abelian magnetic flux.

Section III is devoted to mathematical properties of the Abrikosov solutions. A Fourier expansion of $|W|^2$ is derived and provides a fast-convergent representation with a simple dependence on the lattice parameter $\tau$. This representation is used in section IV to derive explicit expressions for physical observables in terms of $\tau$. It is shown analytically that the triangular lattice solution constitutes a local minimum of the energy, provided $M_H > M_Z$.

II Electroweak Ginzburg-Landau Theory

In the unitary gauge, the Weinberg-Salam lagrangean density [12] leads to coupled equations for $A_\mu$, $Z_\mu$, $W_\mu$ and the real Higgs field $\phi$ (see Ref. [11] for details). The equations for $W_\mu$ and $A_\mu$ are

$$D^\mu F_{\mu\nu} = ig \left[ \cos \theta Z_{\mu\nu} + \sin \theta f_{\mu\nu} - ig (W^\dagger_{\mu} W_{\nu} - W^\dagger_{\nu} W_{\mu}) \right] W^\mu - \frac{1}{2} g^2 \phi^2 W_\nu$$, \hspace{1cm} (1)

$$\partial^\mu f_{\mu\nu} = ie \left[ \partial^\mu (W^\dagger_{\mu} W_{\nu} - W^\dagger_{\nu} W_{\mu}) + W^{\mu\dagger} F_{\mu\nu} - W^{\nu\dagger} F_{\nu\mu} \right]$$, \hspace{1cm} (2)

where $D^\mu = \partial^\mu + ig (A^\mu \sin \theta + Z^\mu \cos \theta)$, $F_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$, $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$ and $e = g \sin \theta$. The field $W^\mu$ is subject to the constraint

$$D^\nu W_\nu = W_\nu \left( \frac{ig}{\cos \theta} Z^\nu - \partial^\nu \ln \phi^2 \right)$$, \hspace{1cm} (3)
We shall investigate static configurations of fields which include a magnetic field \( \vec{B} \) in the \( \hat{z} \) direction and where all fields are independent of the \( z \) coordinate. One can show that it is sufficient to consider components of the vector potentials in the \( \hat{x} \) and \( \hat{y} \) directions and that no inconsistent \( \hat{z} \) or time components develop dynamically. The resulting problem is then two-dimensional.

Define the spin projection states
\[
W_{\pm} = \left( W_1 \mp iW_2 \right)/\sqrt{2}.
\]
For \( B \leq B_{c2} = M_W^2/e \), Eqs. (1) and (2) have the trivial solution \( W_\mu = 0 \), \( B = f_{12} \equiv \text{const.} \), \( \vec{A} = (\vec{B} \times \vec{r})/2 \), \( \vec{Z} = 0 \) and \( \phi \equiv \phi_0 \). When the magnetic field \( B \) exceeds \( B_{c2} \), perturbations about zero in the linear combination \( W_+ \) appear to become tachyonic in the field equation, Eq. (1), because of the spin interaction term \( ig \sin \theta f_{\mu\nu}W^\mu \). The trivial vacuum thus becomes unstable with respect to the production of pairs of \( W \) bosons with magnetic moments oriented along the magnetic field. Stability is restored by the cubic term in Eq. (1) and by the back reactions \( \vec{Z} \), \( \phi - \phi_0 \), \( B - B_{c2} \), and \( \vec{A} - \vec{A}_{c2} \) for which \( W_+ \) acts as a source. In Ref. [11] it was shown that these back reactions are all of order \( |W_+|^2 \). In particular, \( B \) in this order obeys the linear relation
\[
B - e|W_+|^2 \equiv H ,
\]
where \( H \) is a spatially uniform field.

Using Eq. (3), one can also show [13] that the suppressed component \( W_- \) is of order \( |W_+|^3 \). Therefore, for an average magnetic field \( \bar{B} \) above and near \( B_{c2} \), we can write the equation for zero-energy eigenstates \( W_+ \) as follows:
\[
\left[ -(\nabla - ie\vec{A})^2 + M_W^2 - 2e\bar{B} + O(|W_+|^2) \right]W_+ = E^2W_+ = 0 ,
\]
where \( \vec{A} \) is merely a notation for a vector potential such that \( \nabla \times \vec{A} = \vec{B}\hat{z} \).
We recognize Eq. (5) as a generalization of the first Ginzburg-Landau equation of type-II superconductivity. That equation is the special case where $O(|W_+|^2)$ is a positive constant times $|W_+|^2$ and corresponds to a Hamiltonian density with an effective local quartic interaction. In contrast, the effective Hamiltonian of the electroweak problem is nonlocal. It was derived in Ref. [11] by expanding the exact Hamiltonian up to second order in $|W_+|^2$ including back reactions on the fields. The resulting effective Hamiltonian is

$$\mathcal{E}(\vec{r}) = \frac{1}{2} B^2 - (eB - M_W^2)|W_+|^2 + \frac{1}{2} g^2 \left[ \sin^2 \theta |W_+|^4 + M_W^2 U(\vec{r}) |W_+|^2 \right],$$

where

$$U(\vec{r}) = \frac{1}{2\pi} \int d^2 r' \left[ K_0(M_Z |\vec{r} - \vec{r}'|) - K_0(M_H |\vec{r} - \vec{r}'|) \right] |W_+(\vec{r}')|^2,$$

(7)

$K_0$ is a Bessel function and $M_Z, M_H$ are the masses of the Z and Higgs bosons.

The interaction involving $U(\vec{r})$ can be interpreted as an effective long-range interaction between W pairs, mediated by Z and Higgs bosons. For $B > B_c$, Eq. (6) shows that the minimal energy occurs for a non-zero $W_+$ field. In the case $M_H < M_Z$, one can find configurations which yield a negative quartic term. It is then necessary to go to higher orders of perturbation theory, which is beyond the scope of this paper. There are indications that an electroweak generalization of the Nielsen-Olesen vortex may be the stable solution in this mass regime [14]. This cannot be verified with a perturbative approach.

If $M_H \geq M_Z$, stability is ensured by the quartic interaction. Then there exist perturbative solutions of Eq. (5) for which $|W_+|^2$ is periodic on a fine-grain lattice of parallelograms and uniform on a macroscopic scale. They are known, from previous work in superconductivity, as Abrikosov flux lattice solutions [10].

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In this paper, the solutions will be reconsidered in the context of the electroweak theory and physical quantities will be expressed as functions of the geometry of the lattice. This geometry is specified by the two lattice vectors $a\hat{x}$ and $a\hat{\tau}$ or, equivalently, by the complex numbers $a$ and $a\tau$, $a \in \mathbb{R}$, with the correspondence $\hat{\tau} = \text{Re} \tau \hat{x} + \text{Im} \tau \hat{y}$. In the complex picture, $|W_+|^2$ is referred to as doubly periodic with periods $a$ and $a\tau$.

Introduce the notation $\tau_R = \text{Re} \tau$ and $\tau_I = \text{Im} \tau$. For a given lattice, the cell side $a$ and area $A = a^2\tau_I$ are dynamically determined by the value of the average magnetic field $\bar{B}$ through the following theorem, which is valid also for nonperturbative solutions.

**Theorem (Flux Quantization Condition)** For field configurations where $U(1)_{EM}$ gauge-invariant quantities, such as $f_{12}$, $Z_\nu$, $\phi$, $|W_+|$, $|W_-|$ and $W_-/W_+$, are doubly periodic, the abelian magnetic flux that penetrates each lattice cell of area $A$ is quantized and restricted to the values $\bar{B}A = 2\pi k/e$, where the integer $k$ is the common winding number of the phases of $W_+$ and $W_-$.  

Proof: Through integration by parts, Eq. (2) can be written

$$\partial^\mu f_{\mu\nu} = 2ie\partial^\mu (W^\dagger_\mu W_\nu - W^\dagger_\nu W_\mu)$$

$$- ie \left[(D^\mu W_\mu)\dagger W_\nu - W_\nu (D^\mu W_\mu)\right]$$

$$+ \underbrace{ie \left[(D_\nu W^\mu)\dagger W_\mu - W^\mu (D_\nu W_\mu)\right]}_{j^\text{top}_\nu}.$$  

Let $W_+ = |W_+|e^{i\chi_+}$ and $W_- = |W_-|e^{i\chi_-}$. The last term on the right hand side then

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3The theorem has previously been shown to hold for perturbative solutions [11] and, nonperturbatively, for the special case of the electroweak theory where $M_Z = M_H$ [15]. This proof was constructed in collaboration with MacDowell.
becomes

\[ j_{\nu}^{\text{top}} = -2e^2(|W_+|^2 + |W_-|^2) \left[ A_\nu + \frac{1}{2e} \partial_\nu (\chi_+ + \chi_-) \right] \]

\[ - 2eg \cos \theta(|W_+|^2 + |W_-|^2) Z_\nu - e(|W_+|^2 - |W_-|^2) \partial_\nu (\chi_+ - \chi_-) . \]  \quad (9)

The vector \( \partial_\nu (\chi_+ - \chi_-) \) is invariant under lattice translation by virtue of the periodicity of \( W_-/W_+ \). Consequently, its line integral around the boundary of a parallelogram vanishes and, with the requirement that fields be single valued,

\[ \oint \partial_\nu \chi_+ dx^{\nu} = \oint \partial_\nu \chi_- dx^{\nu} = 2\pi k, \quad k \text{ integer} . \]  \quad (10)

Thus the phases of \( W_+ \) and \( W_- \) have the same winding number \( k \). Using Eq. (3) and, again, the periodicity of \( W_-/W_+ \), one can show that the middle term of Eq. (8) is an invariant vector under translation. When the expression for \( j_{\nu}^{\text{top}} \), Eq. (9), is substituted into Eq. (8), the latter divided by \( 2e^2(|W_+|^2 + |W_-|^2) \) can be rearranged in the form

\[ - A_\nu = \frac{1}{2e} \partial_\nu (\chi_+ + \chi_-) + j_{\nu}^{\text{inv}} , \]  \quad (11)

where \( j_{\nu}^{\text{inv}} \) is an invariant vector under lattice translation. Therefore the integral \( \oint j_{\nu}^{\text{inv}} dx^{\nu} \) around the boundary of a parallelogram vanishes, and we get

\[ \text{flux} = - \oint A_\nu dx^{\nu} = \frac{1}{2e} \oint \partial_\nu (\chi_+ + \chi_-) dx^{\nu} = \frac{2\pi k}{e} . \]  \quad (12)

The periodicity condition on \( W_-/W_+ \) is required for the theorem to hold. It emerges naturally, if one assumes that the phases acquired under a lattice translation correspond to a gauge transformation, where the vector potential is form-invariant but expressed about the translated origin \[13\]. We remind ourselves that \( W_- \) and \( W_+ \) have the same \( U(1)_{EM} \) charge and that \( W_-/W_+ \) therefore is a gauge invariant quantity.
From the flux quantization condition we find that the side \( a \) is determined by the relation
\[
a^2 = \frac{2\pi k}{eB\tau I} .
\]
(13)

It is convenient to redefine the problem in terms of coordinates, where the sides of a lattice cell have lengths 1 and \( |\tau| \). The lattice is then specified by the sole parameter \( \tau \), and we can impose \( \tau_I > 0 \) with no lack of generality.

Define the dimensionless quantities \( \mathcal{B}, \bar{\rho}, V(\bar{\rho}), \mathcal{W}, \varepsilon, \) and \( \kappa \) by
\[
\mathcal{B} = \frac{eB}{M^2_W}, \quad \bar{r} = a\bar{\rho}, \quad V(\bar{\rho}) = g^2U(\bar{r}), \quad W_+ = \frac{M_W}{e}\mathcal{W}, \quad \mathcal{E} = \frac{M^4_W}{e^2}\varepsilon, \quad \kappa = \frac{k\pi}{\tau_I} .
\]
(14)

With these units, the effective hamiltonian, Eq. (6), becomes
\[
\varepsilon(\bar{\rho}) = \frac{1}{2}\mathcal{B}^2 - (\mathcal{B} - 1)|\mathcal{W}|^2 + \frac{1}{2}\left[ |\mathcal{W}|^4 + V(\bar{\rho})|\mathcal{W}|^2 \right]
\]
(15)

and the critical magnetic field is \( \mathcal{B}_{c2} = 1 \). From Eq. (4) it follows that, in the new units, \( \mathcal{B} - |\mathcal{W}|^2 \equiv \overline{\mathcal{B}} - |\overline{\mathcal{W}}|^2 \equiv \hbar \) is a uniform field. Substituting this relation into the effective hamiltonian we obtain the space-averaged energy density in terms of the average magnetic field \( \overline{\mathcal{B}} \).
\[
\overline{\varepsilon} = \frac{1}{2}\overline{\mathcal{B}}^2 - (\overline{\mathcal{B}} - 1)|\overline{\mathcal{W}}|^2 + \frac{1}{2}\left[ |\overline{\mathcal{W}}|^4 + V(\bar{\rho})|\overline{\mathcal{W}}|^2 \right]
\]
(16)

The lattice geometry does not fix the overall normalization of the solution \( \mathcal{W} \). For a given geometry, the normalization can be determined by minimizing the energy with a fixed average magnetic field \( \overline{\mathcal{B}} \). The resulting condition is
\[
- (\overline{\mathcal{B}} - 1)|\overline{\mathcal{W}}|^2 + \left[ (|\overline{\mathcal{W}}|^2)^2 + V(\bar{\rho})|\overline{\mathcal{W}}|^2 \right] = 0 .
\]
(17)
Physical quantities can then be expressed in terms of $\mathcal{B}$ and the quantity

$$\beta = \frac{V(\vec{\rho}) |\mathcal{W}|^2}{(|\mathcal{W}|)^2},$$

(18)

which is independent of the $\mathcal{W}$ normalization. One finds

$$|\mathcal{W}|^2 = \frac{\mathcal{B} - 1}{1 + \beta},$$

(19)

$$\varepsilon = \frac{1 - \beta}{2} - \frac{1}{2} \frac{(\mathcal{B} - 1)^2}{1 + \beta},$$

(20)

and

$$h = 1 + (\mathcal{B} - 1) \frac{\beta}{1 + \beta}.$$  

(21)

Using Eqs. (7), (14), and (13) we can write

$$V(\vec{\rho}) |\mathcal{W}|^2 = V(\vec{\rho}, m_Z) |\mathcal{W}|^2 - V(\vec{\rho}, m_H) |\mathcal{W}|^2,$$

where

$$V(\vec{\rho}, m) = \frac{1}{2\pi} \int d^2 \rho' \frac{2\pi k}{\tau_l} K_0(m \rho' \sqrt{\frac{2\pi k}{\tau_l}}) |\mathcal{W}(\vec{\rho} + \vec{\rho}')|^2$$

(23)

and

$$m_X = \frac{1}{\sqrt{\mathcal{B}}} \frac{M_X}{M_W} \quad (X = Z, H).$$

(24)

For $M_H \geq M_Z$ we have $\beta \geq 0$ and therefore $1 \leq h < \mathcal{B}$. The uniform field $h$ will remain frozen at $\mathcal{B}_c = 1$ if and only if $M_H/M_Z = 1$, which is in agreement with the result of Ref. [15] for that mass ratio.

This concludes the general description of electroweak Ginzburg-Landau theory. We have identified the physical quantities of interest. The purpose of this paper is to express them, analytically, as a function of the lattice geometry or, more precisely, of the lattice parameter $\tau$. In order to do so, we shall have to find a representation of the lattice solutions that will make possible an evaluation of the integral and averages in Eqs. (23) and (22).
III Expansions of Abrikosov Solutions

The Abrikosov flux lattice solutions of Eq. (5) were first derived \[10\] in the gauge \( \vec{A} = B \hat{z} \times \hat{r} \). We prefer the cylindrically symmetric gauge \( \vec{A} = \left( \vec{B} \hat{z} \times \hat{r} \right) / 2 \), as it leads to a quicker and more elegant derivation.

The lowest order perturbative solutions of Eq. (5) are obtained by replacing \( O(|W|^2) \) with an effective mass term \( M_C^2 \) (see Ref. \[11\]). After transforming to the rescaled units defined by Eq. (14), and after expressing the vector \( \vec{\rho} \) in cylindrical coordinates \((\rho, \varphi)\), the equation becomes

\[
- \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + 2i \kappa \frac{\partial}{\partial \varphi} + \kappa^2 \rho^2 + \frac{2\kappa}{\mathcal{B}} \left( 1 + \frac{M_C^2}{M_W^2} \right) - 4\kappa \right] W = 0 .
\]

(25)

It is easily checked that the angular momentum eigenstates with eigenvalues \( m : m \geq 0 \)

\[
W_m(\rho, \varphi) = \frac{1}{\sqrt{\pi m!}} \kappa^{m+1} \rho^m e^{im\varphi} \exp\left( - \frac{1}{2} \kappa \rho^2 \right)
\]

(26)

and orthonormality condition

\[
\int d^2 \rho \ W_{m'}(\rho, \varphi)^* W_m(\rho, \varphi) = \delta_{m'm}
\]

(27)

are infinitely degenerate solutions of Eq. (25), satisfying the eigenvalue relation

\[
\mathcal{B} = 1 + \frac{M_C^2}{M_W^2} .
\]

(28)

If we introduce the complex variable \( z = \rho e^{i\varphi} \), the most general solution is

\[
W(\rho, \varphi) = \sum_{m=0}^{\infty} c_m W_m(\rho, \varphi) = \exp\left( - \frac{1}{2} \kappa zz^* \right) f(z) ,
\]

(29)

where \( f(z) \) is an arbitrary analytic function.
We are interested in solutions with $|\mathcal{W}|^2$ invariant under the lattice translations $z \to z + 1$ and $z \to z + \tau$. Consider therefore the transformation properties of the theta function.

$$\vartheta_1(\pi(z + 1)|\tau) = -\vartheta_1(\pi z|\tau),$$

$$\vartheta_1(\pi(z + \tau)|\tau) = -\exp(-i\pi \tau - 2i\pi z) \vartheta_1(\pi z|\tau).$$

The change of modulus under the second translation can be compensated for by attaching a prefactor with suitable transformation properties. If we make the choice

$$f(z) = (2\tau)\frac{1}{2} \exp\left(\frac{1}{2\tau} \kappa z^2\right) \vartheta_1(\pi z|\tau)$$

for $f$ in Eq. (29), it is easily seen that $\mathcal{W}$ will transform by at most a phase under the two distinct translations, provided $\kappa = \pi/\tau$. According to Eq. (14), this is the solution corresponding to a single quantum ($k = 1$) of magnetic flux per lattice cell:

$$\mathcal{W}(\rho, \varphi) = (2\tau)^{\frac{1}{4}} \exp\left[\frac{\pi}{2\tau}(z^2 - zz^*)\right] \vartheta_1(\pi z|\tau).$$

The normalization is chosen so that, as we shall see, $|\mathcal{W}|^2$ is coordinate covariant under the modular group and the spatial average $|\mathcal{W}|^2$ is equal to one. The solutions with higher $k$ are given simply by $|\mathcal{W}(\rho, \varphi)|^k$.

Since $|\mathcal{W}|^2$ is doubly periodic, it can be expanded in a Fourier series in the coordinates $(u, v)$ defined by $z = u + v\tau$. From Eq. (32) and the series representation

$$\vartheta_1(\pi z|\tau) = \frac{1}{i} \sum_{n = -\infty}^{\infty} (-1)^n q^{(n + \frac{1}{2})^2} e^{i(2n + 1)\pi z},$$

where $q = e^{i\pi \tau}$, $|\mathcal{W}|^2$ can be written

$$|\mathcal{W}(u, v)|^2 = (2\tau)^{\frac{1}{2}} \sum_{n = -\infty}^{\infty} \sum_{n' = -\infty}^{\infty} (-1)^{n + n'} e^{i\pi \tau \left[v + (n + \frac{1}{2})\right]^2} e^{-i\pi \tau \left[v - (n' + \frac{1}{2})\right]^2} e^{2\pi (n - n') u}.$$

\[\text{[A proof of uniqueness of solutions was provided in Ref. [1].]}\]
By trading the dummy index $n$ for $k = n - n'$, the Fourier components in the $u$ coordinate are already explicit. Since the expression is not termwise periodic in $v$, integration in $v$ over merely a period will not help. Consider instead the continuous Fourier transform with respect to the $v$ coordinate,

\[
|\mathcal{W}(u; p)|^2 = \int dv \, e^{-i2\pi pv} |\mathcal{W}(u, v)|^2 = \sum_{k=-\infty}^{\infty} (-1)^k e^{i2\pi ku} \exp \left( -\frac{\pi|k\tau - p|^2}{2\tau} \right) e^{i\pi (k+1)} \sum_{n'=-\infty}^{\infty} e^{i2\pi pn'} .
\] (35)

From the inverse Fourier transform and the Poisson formula

\[
\sum_{n'=-\infty}^{\infty} e^{i2\pi pn'} = \sum_{l=-\infty}^{\infty} \delta(p - l) ,
\] (36)

one then obtains the result

\[
|\mathcal{W}(u, v)|^2 = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (-1)^{kl+k+l} \exp \left( -\frac{\pi|k\tau - l|^2}{2\tau} \right) e^{i2\pi (ku+lv)} .
\] (37)

This representation converges extremely fast and uniformly on the plane. We remark that the expression is invariant under the modular group generated by the transformations $\tau \to \tau' = \tau + 1$ and $\tau \to \tau' = -1/\tau$, and under reflexion in the imaginary axis $\tau \to \tau' = -\tau^*$, provided $u$ and $v$ transform covariantly, i.e. $u \to u'$, $v \to v'$ where $u + v\tau = u' + v'\tau'$ \[1\].

With the chosen normalization, the constant term 1 can be identified with the average value $|\mathcal{W}|^2$.

The Fourier expansion of $|\mathcal{W}|^2$, Eq. (37), facilitates considerably the evaluation of integrals which occur in expressions for physical quantities in theories which allow Abrikosov flux lattice solutions, such as Type-II superconductivity or the electroweak theory. In particular it will enable us to write down an analytical expression for the parameter $\beta$ [see Eq. (18)] in terms of the lattice parameter $\tau$. 

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IV Analytical Expressions for Physical Quantities

In the lowest order of perturbation theory, physical quantities depend on the lattice geometry through dimensionless parameters which are specific to the Abrikosov solutions but independent of their overall normalization. In the theory of superconductivity, the naturally arising geometrical quantity is the Abrikosov parameter

$$\beta_A = \frac{|W|^4}{(|W|^2)^2} = \sum_{k, l=-\infty}^{\infty} \exp \left( -\frac{\pi |k\tau - l|^2}{\tau_I} \right).$$

(38)

The last equality was obtained by extracting the constant term in the Fourier expansion of $|W|^4$ that resulted from squaring Eq. (37). We remark that the right-hand side of Eq. (38), and in fact any function of $|k\tau - l|^2/\tau_I$, summed over all integers $k$ and $l$, is modular invariant.

According to section II, the corresponding quantity in the electroweak theory is

$$\beta = \frac{V(\vec{\rho})|W|^2}{(|W|^2)^2}.$$  

The new feature here is the nonlocal quartic interaction in the numerator. It is described, as shown in Eqs. (22) and (23), by an integral kernel.

In order to evaluate $\beta$, we first find the Fourier expansion of the function $V(\vec{\rho}, m)$ defined by Eq. (23) and restrict ourselves to the case $k = 1$. With the representation

$$K_0(x) = \frac{1}{2} \int_0^\infty dt \left( -t - \frac{x^2}{4t} \right),$$

(39)

integrations are straightforward, and one finds

$$V(u, v; m) = \sum_{k, l=-\infty}^{\infty} (-1)^{kl+k+l} \left[ m^2 + 2 \frac{\pi |k\tau - l|^2}{\tau_I} \right]^{-1}\exp \left( -\frac{\pi |k\tau - l|^2}{2\tau_I} \right) e^{2\pi i (ku+lv)}.$$  

(40)
The average $\langle V(\vec{\rho}, m) |W|^2 \rangle$ is then obtained by multiplying Eqs. (40) and (37) together and extracting the constant term. With $m_Z$ and $m_H$ defined by Eq. (24), the resulting expression for $\beta$ is

$$\beta = \frac{b(\tau, m_Z) - b(\tau, m_H)}{B \sin^2 \theta},$$

(41)

where

$$b(\tau, m) = \frac{\langle V(\vec{\rho}, m) |W|^2 \rangle}{\langle |W|^2 \rangle^2} = \sum_{k, l = -\infty}^{\infty} \left[ m^2 + 2 \frac{\pi |k\tau - l|^2}{\tau_I} \right]^{-1} \exp \left( -\frac{\pi |k\tau - l|^2}{\tau_I} \right).$$

(42)

The quantity $\beta$ depends on, besides $\tau$, the masses of the two bosons that mediate the interaction and the redefined magnetic field $B$. It is therefore not a scale independent geometric parameter in the same sense as the Abrikosov number $\beta_A$. The dependence on $B$ enters through the size of the flux lattice, which becomes significant with the introduction of interaction scales $M_Z^{-1}$ and $M_H^{-1}$ in the nonlocal kernel. We can, however, consistently set $B = 1$ in Eq. (41) within the order of perturbation theory we are considering.

The above calculation can be done also for $k > 1$. Although the results are not as elegant and multiply convoluted sums abound, considerable computation time can be gained versus numerical integration. It has been shown numerically [11, 5] that the solutions with $k > 1$ have higher energy, and for this reason they are not the focus of this paper.

The behavior of $\beta$ in terms of the lattice parameter $\tau$ has been investigated numerically in Ref. [11], where it was found that, for $M_H > M_Z$, the global minimum of $\beta$ occurs at $\tau = e^{i\pi/3}$ corresponding to a triangular lattice. With the above results, it is now possible to show analytically that $\tau = e^{i\pi/3}$ is a local minimum.
From Eqs. (41) and (42) we have

$$\beta = \frac{1}{B \sin^2 \theta} \int_{m_2}^{m_H} dm \beta_m \ ,$$  \hspace{1cm} (43)

where

$$\beta_m \equiv -\frac{d}{dm} b(\tau, m) = \sum_{k, l = -\infty}^{\infty} F_m(|\omega_{kl}|^2) \ ,$$  \hspace{1cm} (44)

and the rescaled lattice vectors $\omega_{kl}$ are given by $\omega_{kl} = \sqrt{\frac{\pi}{\tau}}(k\tau - l)$. The properties of $\beta$ can be demonstrated by analysis which holds true for each $\beta_m$ separately. Introducing

$$\frac{\partial}{\partial \tau^*} = \frac{1}{2} \left( \frac{\partial}{\partial \tau_R} + i \frac{\partial}{\partial \tau_I} \right) \ , \quad \frac{\partial}{\partial \tau} = \frac{1}{2} \left( \frac{\partial}{\partial \tau_R} - i \frac{\partial}{\partial \tau_I} \right)$$  \hspace{1cm} (46)

one finds

$$\frac{\partial \beta_m}{\partial \tau^*} = -\frac{i}{2\tau_I} \sum_{k, l = -\infty}^{\infty} \omega_{kl}^2 F_m'(|\omega_{kl}|^2) \ .$$  \hspace{1cm} (47)

For the square ($n = 4$) and triangular ($n = 6$) lattices with $n$–fold rotational symmetry and $\tau = e^{i2\pi/n}$, all non-zero lattice vectors $\omega$ appear in $n$-tuplets $\{\omega r^r, r = 1 \ldots n\}$. Then, from the cancellation of phases within each $n$-tuplet, the right-hand side of Eq. (47) sums to zero, and therefore the square and the triangular lattices are stationary points with respect to the variables $\tau_R$ and $\tau_I$. This could have been shown directly from modular invariance \[16\].

In order to determine the type of local extremum that the triangular lattice constitutes, we must examine the eigenvalues of the Hessian matrix

$$H = \begin{pmatrix} \frac{\partial^2 \beta_m}{\partial \tau_R^2} & \frac{\partial^2 \beta_m}{\partial \tau_R \partial \tau_I} \\ \frac{\partial^2 \beta_m}{\partial \tau_I^2} & \frac{\partial^2 \beta_m}{\partial \tau_I^2} \end{pmatrix} = \frac{1}{2} U^\dagger \begin{pmatrix} \frac{\partial^2 \beta_m}{\partial \tau_R^2} & \frac{\partial^2 \beta_m}{\partial \tau_R \partial \tau_I} \\ \frac{\partial^2 \beta_m}{\partial \tau_I \partial \tau_R} & \frac{\partial^2 \beta_m}{\partial \tau_I^2} \end{pmatrix} U \ ,$$  \hspace{1cm} (48)

where $U$ is a unitary matrix.
The elements of $H$ are given by

$$\frac{\partial^2 \beta_m}{\partial \tau \partial \tau^*} = \frac{1}{4\tau^2} \sum_{k,l=-\infty}^{\infty} \left[ 2 |\omega_{kl}|^2 F_m'(|\omega_{kl}|^2) + |\omega_{kl}|^4 F_m''(|\omega_{kl}|^2) \right],$$  \hspace{1cm} (49)$$

$$\frac{\partial^2 \beta_m}{\partial \tau^2} = -\frac{1}{4\tau^2} \sum_{k,l=-\infty}^{\infty} \left[ 2 \omega_{kl}^2 F_m'(|\omega_{kl}|^2) + \omega_{kl}^4 F_m''(|\omega_{kl}|^2) \right],$$  \hspace{1cm} (50)$$

$$\frac{\partial^2 \beta_m}{\partial \tau^*^2} = \left( \frac{\partial^2 \beta_m}{\partial \tau^*^2} \right)^*.$$  \hspace{1cm} (51)$$

If the summation is performed separately over each $n$-tuple $\{\omega e^{i\pi r/3}, r = 1 \ldots 6\}$ with common modulus, we see that the right-hand sides of Eqs. (50) and (51) are zero by the cancellation of phases. The Hessian matrix is therefore diagonal with a double eigenvalue.

To show that the eigenvalue is positive, examine the expression $2xF_m'(x) + x^2F_m''(x)$ that occurs in each term of Eq. (49). It is easily shown to be positive for all $m > 0$, provided $x > 2$. Now for the triangular lattice we have

$$|\omega_{kl}|^2 \geq \frac{\pi}{\tau_1} = \frac{2\pi}{\sqrt{3}} > 2$$  \hspace{1cm} (52)$$

for each non-zero lattice vector $\omega_{kl}$. Therefore, the right-hand side of Eq. (49) is positive and, for all $m > 0$, it follows that $\beta_m$ has a local minimum at $\tau = e^{i\pi/3}$. From Eq. (43) one then concludes that $\beta$ has a local minimum for this value of $\tau$, provided $M_H > M_Z$.

The analysis of the Abrikosov parameter $\beta_A$ can be carried out similarly. An investigation of the perturbative properties of theories with more general quartic interactions is underway [16].

According to Eq. (20), the energy is a monotonically increasing function of $\beta$. If we assume that $M_H > M_Z$, the global minimum of $\beta$ occurs at $\tau = e^{i\pi/3}$ [11], and it follows that the triangular lattice solution represents the ground state of the electroweak vacuum at magnetic fields above and close to the critical field $B_{c2}$. 

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