MINIMAL AND CHARACTERISTIC POLYNOMIALS OF SYMMETRIC MATRICES IN CHARACTERISTIC TWO

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Abstract. Let \( k \) be a field of characteristic two. We prove that a monic polynomial \( f \in k[X] \) of degree \( n \geq 1 \) is the minimal/characteristic polynomial of a symmetric matrix with entries in \( k \) if and only if it is not the product of pairwise distinct inseparable irreducible polynomials. In this case, we prove that \( f \) is the minimal polynomial of a symmetric matrix of size \( n \). We also prove that any element \( \alpha \in k_{\text{alg}} \) of degree \( n \geq 1 \) is the eigenvalue of a symmetric matrix of size \( n \) or \( n + 1 \), the first case happening if and only if the minimal polynomial of \( \alpha \) is separable.

Keywords: Symmetric matrices, Minimal polynomial, Characteristic polynomial, Eigenvalues, Symmetric bilinear forms, Transfer

2020 MSC Codes: 11C20, 15A15, 15A18, 11E39

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Notation. The following notation will be used throughout this paper.
If $k$ is a field, $k_{\text{alg}}$ will denote a fixed algebraic closure of $k$, and $k_s$ will denote the separable closure of $k$ in $k_{\text{alg}}$.

Let $V$ be a finite dimensional $k$-vector space, and let $B$ be a basis of $V$. If $u : V \rightarrow V$ is an endomorphism of $V$, its matrix representation with respect to the basis $B$ will be denoted by $\text{Mat}(u;B)$.

Similarly, the Gram matrix of a symmetric $k$-bilinear form $b : V \times V \rightarrow k$ with respect to the basis $B$ will be denoted by $\text{Mat}(b;B)$.

For $n \geq 1$, $M_n(k)$ will denote the ring of $n \times n$ matrices with entries in $k$. If $M \in M_n(k)$, the minimal polynomial and the characteristic polynomial of $M$ will be denoted respectively by $\mu_M$ and $\chi_M$.

1. Introduction

The problem of determining the minimal and characteristic polynomials of symmetric matrices with entries in a field has a long history, that we briefly sketch. The first major result on this question is due to Krakowski. If $k$ is a formally real field, we say that $f \in k[X]$ is totally real if it splits over every real closure of $k$. Krakowski proved ([4]) that if $k$ is a formally real field, a non constant monic polynomial $f \in k[X]$ is the minimal polynomial of a symmetric matrix with entries in $k$ if and only if it is separable and totally real. He also proved that if $k$ is not formally real and $\text{char}(k) \neq 2$, any non constant monic polynomial $f \in k[X]$ is the minimal polynomial of a symmetric matrix with entries in $k$. In particular, any totally real algebraic number (resp. any algebraic number) $\alpha \in k_{\text{alg}}$ is the eigenvalue of a symmetric matrix with entries in $k$ if $k$ is formally real (resp. if $k$ is not formally real of characteristic different from two). However, the size of the symmetric matrices constructed by Krakowski was huge compared to the degree of $f$, and the question of finding the minimal size of such symmetric matrices was still open. This question was solved by Bender. Concerning the eigenvalues of symmetric matrices, he proved ([2]) that, if $k$ is a number field, any totally real number of degree $n$ is the eigenvalue of a symmetric matrix of $M_n(k)$ or $M_{n+1}(k)$. When $\text{char}(k) \neq 2$, given a monic polynomial $f \in k[X]$, (where $f$ is totally real if $k$ is formally real), Bender ([3]) computed the smallest integer $r$ such that there exists a symmetric matrix with entries in $k$ satisfying $\mu_M = f$ and $\chi_M = f^r$. In particular, if $-1$ is a square in $k$, Bender’s results imply that any monic polynomial $f \in k[X]$ of degree $n \geq 1$ is the minimal/characteristic polynomial of a symmetric matrix of $M_n(k)$. 
When \( \text{char}(k) = 2 \), much less is known. In [3], Bender stated without proof the following result: if \( k \) has characteristic two, and if \( f \in k[X] \) is a monic polynomial of degree \( n \) which has at least one separable irreducible divisor or such that all the valuations corresponding to inseparable irreducible divisors are even, then \( f \) is the minimal polynomial of a symmetric matrix of \( M_n(k) \).

Finally, results of Bass, Estes and Guralnick (see [1], (9.4)) imply that, if \( k \) has characteristic two, any element \( \alpha \in k_{\text{alg}} \) of degree \( n \) is the eigenvalue of a symmetric matrix with entries in \( k \) of size at most \( 2n + 1 \).

In this paper, we will determine the minimal and characteristic polynomials of symmetric matrices with entries in a field of characteristic two. More precisely, we will prove the following results.

**Theorem 1.1.** Let \( k \) be a field of characteristic two, and let \( f \in k[X] \) be a monic polynomial of degree \( n \geq 1 \). Then \( f \) is the minimal polynomial of a symmetric matrix with entries in \( k \) if and only if \( f \) is not the product of pairwise distinct monic irreducible inseparable polynomials.

In this case, \( f \) is the minimal polynomial of a symmetric matrix of \( M_n(k) \).

In particular, if \( k \) is perfect, any monic polynomial \( f \in k[X] \) of degree \( n \geq 1 \) is the minimal polynomial of a symmetric matrix of \( M_n(k) \).

**Corollary 1.2.** Let \( k \) be a field of characteristic two, and let \( f \in k[X] \) be a monic polynomial of degree \( n \geq 1 \). Then \( f \) is the characteristic polynomial of a symmetric matrix of \( M_n(k) \) if and only if it is not the product of pairwise distinct monic irreducible inseparable polynomials.

In particular, if \( k \) is perfect, any monic polynomial \( f \in k[X] \) of degree \( n \geq 1 \) is the characteristic polynomial of a symmetric matrix of \( M_n(k) \).

**Theorem 1.3.** Let \( k \) be a field of characteristic two, and let \( \alpha \in k_{\text{alg}} \) be an algebraic element of degree \( n \), with minimal polynomial \( f \). Then :

1. if \( f \) is separable, \( \alpha \) is the eigenvalue of a symmetric matrix of \( M_n(k) \);
2. if \( f \) is inseparable, \( \alpha \) is the eigenvalue of a symmetric matrix of \( M_{n+1}(k) \), but not of any symmetric matrix of \( M_n(k) \).

In particular, if \( k \) is perfect, any algebraic element of degree \( n \) is the eigenvalue of a symmetric matrix of \( M_n(k) \).
The proofs rely on a technique introduced by Bender, which consists in constructing $k$-linear forms $k[X]/(f) \to k$ such that the corresponding transfer is isomorphic to the unit form (see Lemma 2.1).

2. BILINEAR FORMS AND SYMMETRIC MATRICES

We first recall some well-known definitions on bilinear forms.

The **unit form of rank** $n$ is the symmetric $k$-bilinear form

$$k^n \times k^n \to k$$

$$(x, y) \mapsto x^t y.$$

A symmetric $k$-bilinear form $b : V \times V \to k$ on a finite dimensional $k$-vector space will then be isomorphic to the unit form (of rank $\dim_k(V)$) if and only if $V$ has an orthonormal basis with respect to $b$.

If $E$ is a commutative $k$-algebra and $s : E \to k$ is a $k$-linear form, the **transfer** associated to $s$ is the symmetric $k$-bilinear form

$$E \times E \to k$$

$$s((1)) : (x, y) \mapsto s(xy).$$

We now prove a slight variation of a lemma of Bender, which relates the theory of symmetric matrices and the notion of transfer.

Before stating it, recall that if $E$ is a finite dimensional commutative $k$-algebra, the **minimal polynomial** of $x \in E$ is the unique monic generator $\mu_x$ of the ideal $I_x$ of $k[X]$ defined by

$$I_x = \{ P \in k[X] \mid P(x) = 0 \}.$$

**Lemma 2.1.** Let $k$ be an arbitrary field, and let $E$ be a commutative $k$-algebra of dimension $n \geq 1$. Assume that there exists a $k$-linear form $s : E \to k$ such that $s_*((1))$ is isomorphic to the unit form, and let $B$ be an orthonormal basis of $E$. Then for all $x \in E$, the matrix $M_x = \text{Mat}(\ell_x; B)$ is a symmetric matrix of $M_n(k)$ such that $\mu_{M_x} = \mu_x$, where $\ell_x : E \to E$ denotes the endomorphism of left multiplication by $x$.

**Proof.** Assume that there exists a $k$-linear form $s : E \to k$ such that $s_*((1))$ is isomorphic to the unit form, and let $B$ be an orthonormal basis of $E$ with respect to $s_*((1))$. Then for all $x, x_1, x_2 \in E$, we have

$$s(\ell_x(x_1)x_2) = s(x_1xx_2) = s(x_1xx_2) = s(x_1\ell_x(x_2)).$$
Hence $\ell_x$ is self-adjoint with respect to $s((1))$. It follows that $M_x = \text{Mat}(\ell_x; \mathcal{B})$ is symmetric. Now for all $P \in k[X]$, we have

$$P(\ell_x) = 0 \iff \ell_{P(x)} = 0 \iff P(x) = 0 \iff \mu_x \mid P.$$ 

This implies that the minimal polynomial of $M_x$ is $\mu_x$. \qed

For the rest of the paper, we will assume that $k$ is a field of characteristic two. Recall that a $k$-bilinear form $b : V \times V \to k$ is alternating if $b(x,x) = 0$ for all $x \in V$. Alternating forms are necessarily symmetric. If $V$ is finite dimensional, non-degenerate alternating $k$-bilinear forms are hyperbolic, that is, isomorphic to an orthogonal sum of hyperbolic planes $\mathbb{H}$, where

$$\mathbb{H} : k^2 \times k^2 \to k, \quad (x, y) \mapsto x_1y_2 + x_2y_1.$$ 

The following lemma gives a nice characterization of the unit form in characteristic two.

**Lemma 2.2.** A nonzero symmetric bilinear form $b : V \times V \to k$ is isomorphic to the unit form if and only if it is non-degenerate, non-alternating, and $b(x,x)$ is a square for all $x \in V$.

In particular, if $m \geq 1$ and $h$ is a hyperbolic form, $m \times \langle 1 \rangle \perp h$ is isomorphic to the unit form.

**Proof.** Since $k$ has characteristic two, any sum of squares is a square, and the unit form satisfies the conditions of the lemma. Conversely, assume that $b : V \times V \to k$ is a nonzero symmetric bilinear form satisfying the conditions of the lemma. Then there exists $e_1 \in V$ such that $b(e_1, e_1) \neq 0$, and by assumption $b(e_1, e_1) = \lambda^2$ for some $\lambda \in k^\times$. Replacing $e_1$ by $\lambda^{-1}e_1$ if necessary, one may assume that $b(e_1, e_1) = 1$. The restriction of $b$ to $F = ke_1$ being non-degenerate, we have $V = F \oplus F^\perp$. Now, the restriction of $b$ to $F^\perp$ is also non-degenerate (if $(e_2, \ldots, e_m)$ is a basis of $F^\perp$, then the matrix of $b$ in the basis $(e_1, \ldots, e_n)$ is block-diagonal, with a 1 in the left upper corner). If this restriction is non-alternating, one may apply induction on $\dim_k(V)$ (the case $\dim_k(V) = 1$ being trivial). If it is alternating, then it is hyperbolic. Hence, it is enough to prove that the matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is congruent to the identity matrix, and apply induction. But one may
check that $P^tBP = I_3$, where $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. The last part is clear.

\[ \square \]

**Remark 2.3.** The previous lemma is a direct consequence of the following well-known fact, which may be proved using a slight modification of the arguments above: any non-alternating symmetric bilinear space $(V, b)$ has an orthogonal basis.

**Remark 2.4.** Let $b : V \times V \rightarrow k$ be a symmetric bilinear form, and let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis of $V$. If $x = \sum_{i=1}^{n} x_i \cdot e_i$, since $k$ has characteristic two, we have $b(x, x) = \sum_{i=1}^{n} x_i^2 b(e_i, e_i)$. Using the fact that a sum of squares is a square, we see that the following properties are equivalent:

1. $b(x, x)$ is a square for all $x \in V$;
2. for any basis $\mathcal{B} = (e_1, \ldots, e_n)$, $b(e_i, e_i)$ is a square for all $1 \leq i \leq n$;
3. there exists a basis $\mathcal{B} = (e_1, \ldots, e_n)$ such that $b(e_i, e_i)$ is a square for all $1 \leq i \leq n$.

The same argument shows that the following properties are equivalent:

1. $b$ is alternating;
2. for any basis $\mathcal{B} = (e_1, \ldots, e_n)$, $b(e_i, e_i) = 0$ for all $1 \leq i \leq n$;
3. there exists a basis $\mathcal{B} = (e_1, \ldots, e_n)$ such that $b(e_i, e_i) = 0$ for all $1 \leq i \leq n$.

This remark will be useful to prove that a bilinear form is isomorphic to the unit form.

For practical computations, we now explain a modified version of Gauss reduction which computes an orthonormal basis of a bilinear space $(V, b)$ satisfying the conditions of Lemma 2.2.

If $\varphi, \psi \in V^*$, we will denote by $\varphi \cdot \psi$ the bilinear form

$\varphi \cdot \psi : V \times V \rightarrow k$

$\varphi \cdot \psi : (x, y) \mapsto \varphi(x)\psi(y)$.

The proposed algorithm will rely on the following identities: if $\varphi_1, \varphi_2, \varphi_3 \in V^*$, we have
(i) \( \varphi_1 \cdot \varphi_1 + \varphi_1 \cdot \varphi_2 + \varphi_2 \cdot \varphi_2 = (\varphi_1 + \varphi_2) \cdot (\varphi_1 + \varphi_2) + \varphi_2 \cdot \varphi_2 \)

(ii) \( \varphi_1 \cdot \varphi_1 + \varphi_2 \cdot \varphi_3 + \varphi_3 \cdot \varphi_2 = (\varphi_1 + \varphi_2) \cdot (\varphi_1 + \varphi_2) + (\varphi_1 + \varphi_3) \cdot (\varphi_1 + \varphi_2 + \varphi_3) \)

Let \( b : V \times V \to k \) be a non-alternating symmetric bilinear form such that \( b(x,x) \) is a square for all \( x \in V \). We do not assume here that \( b \) is non-degenerate. Our first goal is to write \( b = \sum_{i=1}^{r} \varphi_i \cdot \varphi_i \), where \( \varphi_1, \ldots, \varphi_r \in V^* \) are \( k \)-linearly independent linear forms on \( V \).

We start with the Gram matrix \( M = (a_{ij}) \) of \( b \) in a fixed basis \( E = (e_1, \ldots, e_n) \) of \( V \).

If \( x = \sum_{i=1}^{n} x_i e_i \), \( y = \sum_{j=1}^{n} y_j e_j \), we then have

\[
b(x,y) = \sum_{i=1}^{n} a_{ii} x_i y_i + \sum_{i<j} a_{ij} (x_i y_j + x_j y_i).
\]

**Step 1.** Since \( b \) is non-alternating, it contains a term of the form \( u^2 x_i y_i \), where \( u \in k^* \). For sake of simplicity, assume that \( i = 1 \). Collecting all terms containing \( x_1 \) or \( y_1 \), write

\[
b(x,y) = (ux_1)(uy_1) + (ux_1)\varphi(y) + (uy_1)\varphi(x) + \text{ the remaining terms},
\]

where \( \varphi \) is a linear combination of \( e_2, \ldots, e_n^* \). If we set

\[
V \to k;
\]

\[
\varphi_1 : x \mapsto ux_1 + \varphi(x),
\]

using identity (i), we may write \( b = \varphi_1 \cdot \varphi_1 + b_1 \), where \( b_1 \) is a symmetric bilinear form on \( V \), whose expression only depends on \( x_2, y_2, \ldots, x_n, y_n \).

Note that \( b_1(x,x) \) is a square for all \( x \in V \), since \( b_1(x,x) = \varphi_1(x)^2 + b(x,x) \) and a sum of squares is a square.

**Step 2.** If \( b_1 = 0 \), we are done. Otherwise, if \( b_1 \) is non-alternating, we repeat Step 1 with \( b_1 \). If \( b_1 \) is alternating, it contains a nonzero term of the form \( a_{ij} (x_i y_j + x_j y_i) \). Say for example that \( (i,j) = (2,3) \). Collecting terms containing \( x_2, x_3, y_2 \) or \( y_3 \), we may write

\[
b_1(x,y) = a_{23} (x_2 y_3 + x_3 y_2) + \psi(x) y_3 + \psi(y) x_3 + \psi'(x) y_2 + \psi'(y) x_2 + \text{ the remaining terms in } x_i, y_i, i \geq 4
\]

where \( \psi, \psi' \) are linear combinations of \( e_4^*, \ldots, e_n^* \).
Now, $a_{23}(x_2y_3 + x_3y_2) + \psi(x)y_3 + \psi(y)x_3 + \psi'(x)y_2 + \psi'(y)x_2$ is equal to

$$(a_{23}x_2 + \psi(x))(y_3 + a_{23}^{-1}\psi'(y)) + (a_{23}y_2 + \psi(y))(x_3 + a_{23}^{-1}\psi'(x)) + a_{23}^{-1}(\psi(x)\psi'(y) + \psi'(x)\psi(y)).$$

Hence, we may write

$$b = \varphi_1 \cdot \varphi_1 + \psi_2 \cdot \psi_3 + \psi_3 \cdot \psi_2 + b_2,$$

where $b_2$ is a symmetric bilinear form, whose expression only depends on $x_4, y_4, \ldots, x_n, y_n$. Using identity (ii), and replacing the former $\varphi_1$ by $\varphi_1 + \psi_2$, we have $b = \sum_{j=1}^{3} \varphi_j \cdot \varphi_j + b_2$, for some $\varphi_1, \varphi_2, \varphi_3 \in V^*.$

Once again, $b_2(x, x) = 0$ for all $x \in V$. If $b_2 = 0$, we are done. Otherwise, we repeat Step 1 or Step 2 with $b_2$.

Since the number of variables decrease by one or two at each step, we will end with a form $b_1$ or $b_2$ which is identically zero in at most $r$ iterations of Steps 1 or 2.

Using induction on $n$, it is not difficult to prove, as in the classical Gauss reduction algorithm, that the linear forms $\varphi_1, \ldots, \varphi_r$ obtained at the end of this procedure are linearly independent.

**Step 3.** We now have $b = \sum_{j=1}^{r} \varphi_j \cdot \varphi_j$, where $\varphi_1, \ldots, \varphi_r \in V^*$ are linearly independent.

If $\varphi_i = \sum_{j=1}^{n} \varphi_{ij} e_j^*$, the matrix $U = (\varphi_{ij}) \in M_{r \times n}(k)$ has rank $r$. We may then add $n - r$ rows to $U$ in order to obtain a matrix $Q \in \text{GL}_n(k)$. Note that this step is not necessary if $b$ is non-degenerate, that is, if $r = n$.

The columns of $P = Q^{-1}$ then represent the coordinate vectors of the elements of an orthogonal basis $B$ in the basis $E$, for which the corresponding Gram matrix is

$$\text{Mat}(b; B) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$  

In particular, if $b$ is non-degenerate, $B$ will be an orthonormal basis of $V$ with respect to $b$.

**Remark 2.5.** The previous procedure may be slightly modified to produce an orthogonal basis of any non-alternating symmetric bilinear space $(V,b)$. 

We now go back to our original problem.

**Lemma 2.6.** Let \( b : V \times V \rightarrow k \) be a nonzero symmetric bilinear form on a finite dimensional \( k \)-vector space \( V \).

Assume that \((V, b) \simeq (V_1, b_1) \perp \cdots \perp (V_r, b_r)\), where \( V_1, \ldots, V_r \) are nonzero vector spaces and \( b_1, \ldots, b_r \) are symmetric bilinear forms. Then \( b \) is isomorphic to the unit form if and only if the following properties hold:

1. for all \( 1 \leq i \leq r \), \( b_i \) is either isomorphic to the unit form or hyperbolic;
2. there exists at least one \( 1 \leq j \leq r \) such that \( b_j \) is isomorphic to the unit form.

**Proof.** If \( b_1, \ldots, b_r \) satisfy the conditions of the lemma, then \( b \) is isomorphic to the unit form by Lemma 2.2, since \( b \simeq b_1 \perp \cdots \perp b_r \) is isomorphic to the orthogonal sum of a unit form and of an hyperbolic form. Conversely, assume that \( b \) is isomorphic to the unit form. Then \( b_1, \ldots, b_r \) are all non-degenerate and nonzero, since \( b \) is non-degenerate and each \( V_i \) is nonzero. If \( b_i \) is alternating, \( b_i \) is hyperbolic. Assume now that \( b_i \) is non-alternating. Since \( b(x, x) \) is a square for all \( x \in V \), the same property holds for \((V_i, b_i)\). In this case, \( b_i \) is isomorphic to the unit form by Lemma 2.2 again. Finally, if \( b_1, \ldots, b_r \) are all hyperbolic, so is \( b \). In this case, \( b \) is alternating, which contradicts the fact that \( b \) is isomorphic to the unit form. This concludes the proof. \( \square \)

The previous lemma, applied to transfers, yields the following result.

**Lemma 2.7.** Let \( E_1, \ldots, E_r \) be \( r \) nonzero finite dimensional commutative \( k \)-algebras. Then there exists a \( k \)-linear form \( s : E_1 \times \cdots \times E_r \rightarrow k \) such that \( s(\langle 1 \rangle) \) is isomorphic to the unit form if and only if there exist \( r \) \( k \)-linear forms \( s_i : E_i \rightarrow k \) such that the following properties hold:

1. for all \( 1 \leq i \leq r \), \((s_i)_*(\langle 1 \rangle)\) is either isomorphic to the unit form or hyperbolic;
2. there exists at least one \( 1 \leq j \leq r \) such that \((s_j)_*(\langle 1 \rangle)\) is isomorphic to the unit form.

**Proof.** Any \( k \)-linear form \( s : E_1 \times \cdots \times E_r \rightarrow k \) may be written in a unique way as

\[
s = s_1 \oplus \cdots \oplus s_r : E_1 \times \cdots \times E_r \rightarrow k
\]

\[
(x_1, \ldots, x_r) \mapsto s_1(x_1) + \cdots + s_r(x_r),
\]
where \( s_i : E_i \longrightarrow k \) is a \( k \)-linear form. For such a decomposition, we clearly have
\[
s_*(\langle 1 \rangle) \simeq (s_1)_*(\langle 1 \rangle) \perp \cdots \perp (s_r)_*(\langle 1 \rangle).
\]
Now apply Lemma 2.6 to conclude. \( \square \)

From this lemma, we now derive some information on the structure of minimal polynomials of symmetric matrices.

**Proposition 2.8.** Let \( \rho_1, \ldots, \rho_s \in k[X] \) be pairwise distinct monic inseparable irreducible polynomials. Then \( \rho_1 \cdots \rho_s \) is not the minimal polynomial of a symmetric matrix with entries in \( k \) of any size.

**Proof.** Assume that there exists a symmetric matrix \( M \in M_n(k) \) such that \( \mu_M = \rho_1 \cdots \rho_s \).

Write \( \rho_i = \pi_i(X^{2k_i}) \), where \( k_i \geq 1 \) and \( \pi_i \in k[X] \) is an irreducible separable polynomial. Thus, \( \rho_i = \prod_{j=1}^{d_i}(X^{2k_i} - a_{ij}) \in k_s[X] \), where \( a_{i1}, \ldots, a_{id_i} \in k_s \) are the roots of \( \pi_i \).

Hence
\[
\mu_M = \prod_{i=1}^{s} \prod_{j=1}^{d_i}(X^{2k_i} - a_{ij}) \in k_s[X],
\]
and \( k_s^n = \bigoplus_{i,j} E_{ij} \), where \( E_{ij} = \ker(M^{2k_i} - a_{ij}I_n) \).

Let us prove that the subspaces \( E_{ij} \) are mutually orthogonal with respect to the unit form on \( k_s^n \). Notice that, if \( (i, j) \neq (\ell, m) \), the polynomials \( P = X^{2k_i} - a_{ij} \) and \( Q = X^{2k_\ell} - a_{\ell m} \) are coprime: if \( i \neq \ell \), it follows from the fact that \( \rho_i \) and \( \rho_\ell \) are distinct irreducible polynomials (and therefore coprime), and if \( i = \ell \) and \( j \neq m \), this comes from the fact \( a_{ij} \neq a_{im} \) since \( \pi_i \) is separable, and from the relation \( P - Q = a_{ij} - a_{im} \).

Thus, there exists \( U, V \in k_s[X] \) such that \( UP + VQ = 1 \). Now, for all \( x \in E_{ij} = \ker(P(M)) \) and all \( y \in E_{\ell m} = \ker(Q(M)) \), since \( M \) is symmetric, we have
\[
x^t y = (I_n x)^t y = (U(M)P(M)x + V(M)Q(M)x)^t y
= (V(M)Q(M)x)^t y = x^t (Q(M)V(M)y)
= x^t (V(M)Q(M)y) = 0.
\]

Thus, the subspaces \( E_{ij} \) are mutually orthogonal and therefore, the unit form \( b_0 \) on \( k_s^n \) satisfies \( b_0 \simeq \perp_{i,j} b_{ij} \), where \( b_{ij} \) is the restriction of \( b_0 \) to \( E_{ij} \). By Lemma 2.6, there exist \( i, j \) such that \( b_{ij} \) is isomorphic
to the unit form. In other words, $E_{ij}$ has an orthonormal basis for the restriction of $b_0$ to $E_{ij}$. Now $E_{ij}$ is stable by $M$, so $M$ induces an endomorphism $u_{ij}$ on $E_{ij}$, which is self-adjoint with respect to $b_{ij}$. Hence the matrix $M'$ of $u_{ij}$ in an orthonormal basis of $E_{ij}$ is a symmetric matrix with entries in $k_s$ satisfying $(M')^{2k_i} = a_{ij}I_m$, where $m$ is the size of $M'$. If $N = (M')^{2k_i-1}$, then $N = (n_{ij})$ is symmetric and satisfies $N^2 = a_{ij}I_m$. Comparing the coefficients in position $(1, 1)$, we get

$$a_{ij} = \sum_{\ell=1}^{m} n_{1\ell}n_{\ell 1} = \sum_{\ell=1}^{m} n_{1\ell}^2 = (\sum_{\ell=1}^{m} n_{1\ell})^2.$$ 

Therefore, $a_{ij}$ is a square in $k_s$. Since the Galois group of a splitting field of $\pi_i$ acts transitively on its roots, this implies that all the roots of $\pi_i$ are squares in $k_s$. It follows that all the coefficients of $\rho_i = \pi_i(X^{2k_i})$ are squares of $k_s$ (as a sum of products of squares). Now, if $a \in k$ is a square of an element of $k_s$, it is already a square of an element of $k$. Indeed, if $\lambda \in k_s$ satisfies $\lambda^2 = a$, then $\mu_{\lambda,k} | X^2 - a = (X - \lambda)^2$. Since $\lambda$ is separable over $k$, the previous divisibility relation forces $\mu_{\lambda,k} = X - \lambda$, and $\lambda \in k$. It follows that $\rho_i$ is a square in $k[X]$, contradicting its irreducibility. This concludes the proof. □

3. Computation of various transfers

In the sequel, if $f \in k[X]$ is a monic polynomial of degree $n \geq 1$, $\alpha$ will denote the class of $X$ in $k[X]/(f)$ (unless specified otherwise), so that $(1, \alpha, \ldots, \alpha^{n-1})$ is a $k$-basis of $k[X]/(f)$.

**Lemma 3.1.** Assume that $f \in k[X^2]$ has degree $2m$, and let $s : k[X]/(f) \longrightarrow k$ be the unique $k$-linear form such that

$$s(1) = \cdots = s(\alpha^{2m-2}) = 0, s(\alpha^{2m-1}) = 1.$$ 

Then $s_*((1))$ is hyperbolic.

**Proof.** It is enough to prove that $s_*((1))$ is non-degenerate and alternating.

Let us prove it is non-degenerate. Let $x = \sum_{i=0}^{2m-1} \lambda_i \cdot \alpha^i \in \ker(s_*((1)))$.

Then $s(x1) = s(x) = \lambda_{2m-1} = 0$. Assume we proved that $\lambda_{2m-1}$ is $\lambda_{2m-2} = \cdots = \lambda_{2m-1-j} = 0$ for some $0 \leq j \leq 2m - 2$. Then,

$$0 = s(x \alpha^{j+1}) = s(\sum_{i=0}^{2m-j-2} \lambda_i \cdot \alpha^{i+j+1}) = s(\sum_{i=j+1}^{2m-1} \lambda_i \cdot \alpha^i) = \lambda_{2m-1-j-1}.$$
It follows by induction that all the $\lambda_i$’s are zero, that is $x = 0$. Hence, $s_*(\langle 1 \rangle)$ is non-degenerate.

We now prove that it is alternating. By Remark 2.4, it is enough to show that $s(\alpha^{2i}) = 0$ for $0 \leq i \leq 2m - 1$. For, let $R$ be the remainder of the long division of $X^{2i} = (X^2)^i$ by $f$. It is clear that $R \in k[X^2]$ (since we may just think of $X^2$ as the variable when performing the long division).

Consequently, evaluation at $\alpha$ shows that $\alpha^{2i} = R(\alpha)$ is a linear combination of $1, \alpha^2, \ldots, \alpha^{2m-2}$, and $s(\alpha^{2i}) = 0$. This concludes the proof. □

Lemma 3.2. Let $L/k$ be a finite separable field extension, and let $\text{Tr}_{L/k} : L \to k$ be the corresponding trace map. Then $(\text{Tr}_{L/k})_*({\langle 1 \rangle})$ is isomorphic to the unit form.

Proof. If $X_L$ denotes the set of $k$-embeddings $L \to k_{alg}$, recall that for all $x \in L$, we have $\text{Tr}_{L/k}(x) = \sum_{\sigma \in X_L} \sigma(x)$. In particular, for all $x \in L$, we get

$$\text{Tr}_{L/k}(x^2) = \sum_{\sigma \in X_L} \sigma(x^2) = \sum_{\sigma \in X_L} \sigma(x)^2 = \text{Tr}_{L/K}(x)^2.$$ 

Since $L/k$ is separable, it is known that $\text{Tr}_{L/k}$ is nonzero, and that the corresponding transfer is non-degenerate. Since the trace map is nonzero, the previous equality shows that the corresponding transfer is non-alternating. It also implies that $\text{Tr}_{L/k}(x^2)$ is a square for all $x \in L$. Lemma 2.2 then yields the desired result. □

Lemma 3.3. Assume that $f = (X-a)^m$, $m \geq 1$, and let $s : k[X]/(f) \to k$ be the unique $k$-linear form such that

$$s(1) = s(\alpha - a) = \cdots = s((\alpha - a)^{m-1}) = 1.$$ 

Then $s_*(\langle 1 \rangle)$ is isomorphic to the unit form.

Moreover, for all $i \geq m$, we have

$$s(\alpha^i) = \sum_{j=0}^{m-1} \binom{m-1}{j} a^{i-j}.$$ 

Proof. By Lemma 2.2, it is enough to show that $s_*(\langle 1 \rangle)$ is nonzero, non-degenerate, non-alternating, and that $s(x^2)$ is a square for all $x \in k[X]/(f)$. 


It is easy to check that the Gram matrix $B$ of $s_*\langle 1 \rangle$ with respect to the basis $(1, \alpha - a, \ldots, (\alpha - a)^{m-1})$ is

$$B = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & 1 & \ddots & \vdots & \vdots \\
1 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}.$$

It follows at once that $s_*\langle 1 \rangle$ is nonzero, non-degenerate and non-alternating. The fact that $s(x^2)$ is a square for all $x \in k[X]/(f)$ follows from Remark 2.4 and the fact that the diagonal entries of $B$ are squares.

Let us prove the last part of the lemma, and let $i \geq m$. Since $(\alpha - a)^j = 0$ in $k\langle X \rangle/(X - a)^m$ for all $j \geq m$, we have

$$\alpha^i = (\alpha - a + a)^i = \sum_{j=0}^{m-1} \binom{i}{j} (\alpha - a)^j a^{i-j}.$$

Applying the definition of $s$ yields the desired equality. \qed

**Proposition 3.4.** Let $\pi \in k[X]$ be a monic irreducible separable polynomial of degree $d \geq 1$, let $a \in k_*$ be a root of $\pi$, and let $L = k(a)$. Let $m \geq 1$, and let $t : L[X]/((X - a)^m) \rightarrow L$ be the unique $L$-linear form such that

$$t(1) = t(\gamma - a) = \cdots = t((\gamma - a)^{m-1}) = 1,$$

where $\gamma$ is the class of $X$ in the quotient ring $L[X]/((X - a)^m)$.

Finally, let $s : k[X]/(\pi^m) \rightarrow k$ be the $k$-linear map defined by $s(\tilde{P}) = \text{Tr}_{L/k}(t(\tilde{P}))$ for all $\tilde{P} \in k[X]/(\pi^m)$, where $\tilde{Q}$ denotes the class of a polynomial $Q \in L[X]$ in $L[X]/((X - a)^m)$.

Then $s_*\langle 1 \rangle$ is isomorphic to the unit form.

Moreover, for all $i \geq m$, we have

$$s(\alpha^i) = \sum_{j=0}^{m-1} \binom{i}{j} \text{Tr}_{L/k}(a^{i-j}).$$

**Proof.** First, note that $s$ is well-defined. Indeed, if $P \in k[X]$ is a multiple of $\pi^m$, then it is a multiple of $(X - a)^m$ in $k_*[X]$, since $a$ is a root of $\pi$. In particular, $\tilde{P}$ only depends on the class $\tilde{P}$ (and not on the choice of a representative $P \in k[X]$).
By Lemma 3.3, we can pick an orthonormal $L$-basis $(\widetilde{Q}_1, \ldots, \widetilde{Q}_m)$ of the $L$-vector space $L[X]/((X - a)^m)$ with respect to $t_*(\langle 1 \rangle)$. By Lemma 3.2, we can also pick an orthonormal $k$-basis $(\gamma_1, \ldots, \gamma_d)$ of $L/k$ with respect to $(\text{Tr}_{L/k})_* (\langle 1 \rangle)$, where $d = [L : k] = \deg(\pi)$.

**Claim.** For all $Q \in L[X]$, there exists $P \in k[X]$ such that $\widetilde{P} = \widetilde{Q}$.

Assume the claim is proved, and let $P_{ij} \in k[X]$ be a polynomial such that $\widetilde{P}_{ij} = \gamma_i \widetilde{Q}_j$. Then for all $1 \leq i, r \leq d, 1 \leq j, s \leq m$, we have

$$s(\widetilde{P}_{ij} \widetilde{P}_{rs}) = \text{Tr}_{L/k} (t(\widetilde{P}_{ij} \widetilde{P}_{rs})) = \text{Tr}_{L/k} (t(\gamma_i \widetilde{Q}_j \gamma_r \widetilde{Q}_s)).$$

Since $t$ is $L$-linear, this yields

$$s(\widetilde{P}_{ij} \widetilde{P}_{rs}) = \text{Tr}_{L/k} (\gamma_i \gamma_r t(\widetilde{Q}_j \widetilde{Q}_s)) = \text{Tr}_{L/k} (\gamma_i \gamma_r \delta_{js}) = \delta_{js} \delta_{ir}.$$

It follows that the family $(\widetilde{P}_{ij})_{i,j}$ is orthonormal with respect to $s_* (\langle 1 \rangle)$. In particular, the $P_{ij}$’s are linearly independent over $k$. Since this family has $dm$ elements, we may conclude that $(P_{ij})_{i,j}$ is an orthonormal basis of $k[X]/(\pi^m)$ with respect to $s_* (\langle 1 \rangle)$.

It remains to prove the claim. Let $M/k$ be the Galois closure of $L$, let $G$ be the Galois group of $M/k$, and let $X_L$ be the set of $k$-embeddings $\sigma : L \rightarrow k_{\text{alg}}$. Note that elements of $X_L$ have their images contained in $M$, so $G$ acts by left composition on $X_L$.

Let $Q \in L[X]$. Since $\pi$ is separable, the polynomials $(X - \sigma(a))^m, \sigma \in X_L$ are pairwise coprime. The Chinese Remainder Theorem yields the existence of a unique polynomial $P \in M[X]$ of degree $< dm$ such that $P \equiv \sigma(Q) \mod (X - \sigma(a))^m$ for all $\sigma \in X_L$, where the congruences are viewed inside $M[X]$.

For all $\tau \in G$, we then have $\tau(P) \equiv \tau \sigma(Q) \mod (X - \tau \sigma(a))^m$ for all $\sigma \in X_L$. Since $\pi$ is irreducible, the action of $G$ on $X_L$ is transitive and we have $\tau(P) \equiv \sigma(Q) \mod (X - \sigma(a))^m$ for all $\sigma \in X_L$. Now, $\deg(\tau(P)) = \deg(P)$, so by uniqueness of $P$, we get $\tau(P) = P$ for all $\tau \in G$. Hence $P \in k[X]$. By choice of $P$, we have $(X - a)^m | (P - Q)$ in $M[X]$. Since $P \in k[X] \subset L[X]$, and $(X - a)^m \in L[X]$, the corresponding quotient also lies in $L[X]$, and we have $P \equiv Q \mod (X - a)^m$ in $L[X]$, that is $\widetilde{P} = \widetilde{Q}$.

The last part comes from Lemma 3.3. This concludes the proof. \qed

**Lemma 3.5.** Let $m \geq 2$ and $n \geq 1$. Assume that $a \in k$ is not a square, and let $f = (X^{2^n} - a)^m$. Let $s : k[X]/(f) \rightarrow k$ be the unique $k$-linear form such that

$$s(\alpha^{2^n}) = 1, s(\alpha^{2^n - 1}) = 1, s(\alpha^j) = 0 \quad \text{if} \quad j \neq 2^n, 2^n m - 1.$$
Then $s_*(\langle 1 \rangle)$ is isomorphic to the unit form.

Moreover, for all $i \geq 2^n m$, $s(\alpha^i)$ is equal to:

1. $\left( \frac{2^nu - 1}{2^n m - 1} \right)\alpha^{u-m}$ if $i = 2^nu - 1$ for some $u > m$

2. $\left( \sum_{j=0}^{2^n(m-1)-1} \binom{2^{n+1}}{j} \right)\alpha^{2u}$ if $i = 2^u(2u + 1)$ for some $u \geq \frac{m-1}{2}$

3. $0$ otherwise.

Proof. Once again, we are going to use Lemma 2.2. Note for the computations that $2^n < 2^n m - 1$ since $m \geq 2$ and $n \geq 1$.

Clearly, $s_*(\langle 1 \rangle)$ is nonzero, and non-alternating since $s(\alpha^{2^n}) = 1$. Let us prove it is non-degenerate. Let $x \in k[X]/((X^{2^n} - a)^m)$ be such that $s(xy) = 0$ for all $y \in k[X]/((X^{2^n} - a)^m)$. Assume that $x \neq 0$. Then the ideal $(x)$ is contained in $\ker(s)$. Since $\alpha$ is not a square and $k$ has characteristic two, $X^{2^n} - a$ is irreducible in $k[X]$, and the monic divisors of $(X^{2^n} - a)^m$ are the polynomials $(X^{2^n} - a)^j$, $0 \leq j \leq m$. Thus, the ideals of $k[X]/((X^{2^n} - a)^m)$ are the ideals $((\alpha^{2^n} - a)^j)$, $0 \leq j \leq m$. Since $(x)$ is nonzero and is not equal to the whole quotient ring (since $s$ is not identically zero), we have $(x) = ((\alpha^{2^n} - a)^j)$ for some $1 \leq j \leq m - 1$. In particular, $(x)$ contains $(\alpha^{2^n} - a)^m$. Hence, $(\alpha^{2^n} - a)^m \in \ker(s)$ and we have $s((\alpha^{2^n} - a)^m) = 0$. Now $(\alpha^{2^n} - a)^m \alpha^{2^n-1}$ is a polynomial in $\alpha$ of degree $2^n m - 1$, whose coefficient of $\alpha^{2^n}$ is zero and whose coefficient of $\alpha^{2^n-1}$ is 1. Therefore, $s((\alpha^{2^n} - a)^m \alpha^{2^n-1}) = 1$ and we get a contradiction.

It remains to prove that $s(\alpha^{2i})$ is a square for all $0 \leq i \leq 2^n m - 1$. This is clear if $0 \leq 2i \leq 2^n m - 2$. For the other cases, we are going to prove the formulas giving $s(\alpha^i)$ for all $i \geq 2^n m$. Let $i \geq 2^n m$, and let $\beta \in k_{alg}$ satisfying $\beta^{2^n} - a = 0$. Note that $f = (X - \beta)^{2^n m}$.

In $k_{alg}[X]$, we have

$$X^i = (X - \beta + \beta)^i \equiv \sum_{j=0}^{2^n m - 1} \binom{i}{j} (X - \beta)^j \beta^{i-j} \mod (X - \beta)^{2^n m}.$$  

Since the degree of $R_i = \sum_{j=0}^{2^n m - 1} \binom{i}{j} (X - \beta)^j \beta^{i-j}$ is $\leq 2^n m - 1$, $R_i \in k_{alg}[X]$ is the remainder of the long division of $X^i$ by $(X - \beta)^{2^n m} = f$.

Since $X^i$ and $f$ lie in $k[X]$, so is $R_i$. If $r_{j}^{(i)} \in k$ is the coefficient of $X^j$ in $R_i$, we then have $s(\alpha^i) = r_{2^n}^{(i)} + r_{2^n m - 1}^{(i)}$. 
Since $X^{2^n} - a$ is irreducible, $\beta$ has degree $2^n$. It follows that $1, \beta, \ldots, \beta^{2^n - 1}$ are $k$-linearly independent. If $j \geq 0$, a Euclidean division of $j$ by $2^n$ then shows that $\beta^j$ lies in $k$ if and only if $j$ is a multiple of $2^n$.

We clearly have $r_{2^n_m-1}^{(i)} = \left( \frac{i}{2^n_m - 1} \right) \beta^{i - (2^n_m - 1)}$. Note that $\left( \frac{i}{2^n_m - 1} \right) = 0$ or 1 in $k$. In particular, if $r_{2^n_m-1}^{(i)}$ is nonzero, it is equal to $\beta^{i - (2^n_m - 1)}$.

Since $r_{2^n_m-1}^{(i)} \in k$, we necessarily have $i \equiv -1 [2^n]$. This implies that $r_{2^n_m-1}^{(i)} = 0$ when $i$ is even. Now, if $i = 2^n u - 1$ (where $u > m$), we then get $r_{2^n_m-1}^{(i)} = \left( \frac{2^n u - 1}{2^n_m - 1} \right) a^{u-m}$.

We also have

$$r_{2^n}^{(i)} = \sum_{j=2^n}^{2^n_m-1} \left( \frac{i}{j} \right) \beta^{j - 2^n} \beta^{i-j} = \varepsilon_i \beta^{j-2^n},$$

where $\varepsilon_i = \sum_{j=2^n}^{2^n_m-1} \left( \frac{i}{j} \right) = \left( \frac{i}{2^n} \right) \left( \sum_{j=2^n}^{2^n_m-1} \left( \frac{i}{j-2^n} \right) \right)$. As before, $\varepsilon_i = 0$ or 1 in $k$. If $r_{2^n}^{(i)} \neq 0$, we then have $r_{2^n}^{(i)} = \beta^{i-2^n} \in k$, which implies that $i \equiv 0 [2^n]$.

Hence, if $r_{2^n}^{(i)} \neq 0$, there exists $\ell \geq m$ such that $i = 2^n \ell$. In particular, $r_{2^n}^{(i)} = 0$ if $i$ is odd. Now it is well-known that $\left( \frac{2^n \ell}{2^n} \right)$ and $\ell$ have same parity. Since $r_{2^n}^{(i)} \neq 0$, so is $\varepsilon_i$, and $\ell$ is necessarily odd. Thus $\ell = 2u + 1$ for some $u \geq \frac{m-1}{2}$.

Consequently, if $i = 2^n(2u + 1)$, we have $r_{2^n}^{(i)} = \varepsilon_i \beta^{2^{n+1} u} = \varepsilon_i a^{2u}$, where

$$\varepsilon_i = \sum_{j=2^n}^{2^n_m-1} \left( \frac{i - 2^n}{j - 2^n} \right) = \sum_{j=0}^{2^{n+1} - 1} \left( \frac{2^{n+1} u}{j} \right).$$

All the desired formulas follow at once. In particular, $s(\alpha^{2i})$ is a square for all $i \geq 0$. This concludes the proof. \hfill \Box

**Proposition 3.6.** Let $\rho \in k[X]$ be a monic irreducible inseparable polynomial. Write $\rho = \pi(X^{2^n})$, where $n \geq 1$ and $\pi \in k[X]$ is a monic irreducible separable polynomial of degree $d \geq 1$. Let $a \in k_*$ be a root of $\pi$, and let $L = k(a)$. Let $t : L[X]/((X^{2^n} - a)^m) \rightarrow L$ be the unique
$L$-linear form such that
\[ t(\gamma^{2^n}) = 1, t(\gamma^{2^n m - 1}) = 1, t(\gamma^j) = 0 \quad \text{if } j \neq 2^n, 2^n m - 1. \]
where $\gamma$ is the class of $X$ in the quotient ring $L[X]/((X^{2^n} - a)^m)$.
Finally, let $s : k[X]/(\pi^m) \to k$ be the $k$-linear map defined by
\[ s(\tilde{P}) = \text{Tr}_{L/k}(t(\tilde{P})) \quad \text{for all } \tilde{P} \in k[X]/(\pi^m), \]
where $\tilde{Q}$ denotes the class of a polynomial $Q \in L[X]$ in $L[X]/((X^{2^n} - a)^m)$.
Then $s_*((1))$ is isomorphic to the unit form.
Moreover, for all $i \geq 2^n m$, $s(\alpha^i)$ is equal to :
\begin{align*}
(1) & \left( \frac{2^n u - 1}{2^n m - 1} \right) \text{Tr}_{L/k}(a^{u-m}) \text{ if } i = 2^n u - 1 \text{ for some } u > m \\
(2) & \sum_{j=0}^{2^n (m-1) - 1} \left( \begin{array}{c} 2^n u + 1 \\ j \end{array} \right) \text{Tr}_{L/k}(a^{2u}) \text{ if } i = 2^n (2u + 1) \text{ for some } u \geq m - 1 \\
(3) & 0 \text{ otherwise.}
\end{align*}

Proof. If $\sigma_1, \sigma_2 \in X_L$ are two distinct $k$-embeddings of $L$ into $k_{alg}$, the polynomials $X^{2^n} - \sigma_1(a)$ and $X^{2^n} - \sigma_2(a)$ are coprime (since $\pi$ is separable), and thus so are $(X^{2^n} - \sigma_1(a))^m$ and $(X^{2^n} - \sigma_2(a))^m$.
Reasoning as in the proof of Proposition 3.4, we may show that, given $Q \in L[X]$, there exists $P \in k[X]$ such that $\tilde{P} = \tilde{Q}$. The rest of the proof is then identical to the proof of this very same proposition. \qed

4. PROOF OF THE MAIN RESULTS AND EXAMPLES

We now come to the proofs of the main results of this paper, that we state again for the convenience of the reader.

**Theorem 4.1.** Let $f \in k[X]$ be a monic polynomial of degree $n \geq 1$.
Then $f$ is the minimal polynomial of a symmetric matrix with entries in $k$ if and only if $f$ is not the product of pairwise distinct monic irreducible inseparable polynomials.

In this case, $f$ is the minimal polynomial of a symmetric matrix of $M_n(k)$.

In particular, if $k$ is perfect, any monic polynomial $f \in k[X]$ of degree $n \geq 1$ is the minimal polynomial of a symmetric matrix of $M_n(k)$.
Proof. We will make use of the following easy fact: if \( E_1, \ldots, E_s \) are finite dimensional commutative \( k \)-algebras, we have
\[
\mu_x = \text{lcm}(\mu_{x_1}, \ldots, \mu_{x_s}) \quad \text{for all } x = (x_1, \ldots, x_s) \in E_1 \times \cdots \times E_s.
\]

Assume first that \( f \) has at least one separable irreducible divisor, and write \( f = \pi_1^{m_1} \cdots \pi_r^{m_r} g \), where \( r \geq 1, m_1, \ldots, m_r \geq 1 \), \( \pi_1, \ldots, \pi_r \) are monic separable irreducible polynomials, and \( g \) is the product of the inseparable irreducible divisors of \( f \) (with multiplicities). Note that \( g \) lies in \( k[X^2] \) and is isomorphic to the unit form. By Lemma 3.4, for all \( 1 \leq i \leq r \), there exists a \( k \)-linear form \( s_i : k[X]/(\pi_i^{m_i}) \rightarrow k \) such that \( (s_i)_*(\langle 1 \rangle) \) is isomorphic to the unit form. Since \( g \in k[X^2] \), by Lemma 3.1, there exists a \( k \)-linear form \( s' : k[X]/(g) \rightarrow k \) such that \( s'_*(\langle 1 \rangle) \) is hyperbolic. By Lemma 2.7, there exists a \( k \)-linear map \( s : k[X]/(\pi_1^{m_1}) \times \cdots \times k[X]/(\pi_r^{m_r}) \times k[X]/(g) \rightarrow k \) such that \( s_*(\langle 1 \rangle) \) is isomorphic to the unit form. If \( \alpha_i \) is the class of \( X \) in \( k[X]/(\pi_i^{m_i}) \) and \( \alpha' \) is the class of \( X \) in \( k[X]/(g) \), we have \( \mu_{\alpha_i} = \pi_i^{m_i} \) and \( \mu_{\alpha'} = g \). Let \( x = (\alpha_1, \ldots, \alpha_r, \alpha') \). Since \( \pi_1^{m_1}, \ldots, \pi_r^{m_r} \) and \( g \) are pairwise coprime, we have
\[
\mu_x = \text{lcm}(\pi_1^{m_1}, \ldots, \pi_r^{m_r}, g) = \pi_1^{m_1} \cdots \pi_r^{m_r} g = f.
\]

By Lemma 2.1, \( f \) is the minimal polynomial of a symmetric matrix of \( M_n(k) \).

Assume now that all irreducible divisors of \( f \) are inseparable. If \( f \) is the product of pairwise distinct monic irreducible inseparable polynomials, then \( f \) is not the minimal polynomial of a symmetric matrix of any size by Proposition 2.8.

Otherwise, we may write \( f = \rho^m g \), where \( m \geq 2 \), \( \rho \) is a monic irreducible inseparable polynomial, and \( g \) is the product of the other inseparable irreducible divisors of \( f \) (with multiplicities). Once again, \( g \) lies in \( k[X^2] \) and is coprime to \( \rho^m \). By Proposition 3.6, there exists a \( k \)-linear form \( s : k[X]/(\rho^m) \rightarrow k \) such that \( s_*(\langle 1 \rangle) \) is isomorphic to the unit form. By Lemma 3.1, there exists a \( k \)-linear form \( s' : k[X]/(g) \rightarrow k \) such that \( s'_*(\langle 1 \rangle) \) is hyperbolic. Now, we may finish the argument as before, and this concludes the proof. \( \square \)

**Corollary 4.2.** Let \( f \in k[X] \) be a monic polynomial of degree \( n \geq 1 \). Then \( f \) is the characteristic polynomial of a symmetric matrix of \( M_n(k) \) if and only if \( f \) is not the product of pairwise distinct monic irreducible inseparable polynomials.

In particular, if \( k \) is perfect, any monic polynomial \( f \in k[X] \) of degree \( n \geq 1 \) is the characteristic polynomial of a symmetric matrix of \( M_n(k) \).
Proof. Assume that \( f \) is not the product of pairwise distinct monic irreducible inseparable polynomials. Theorem 4.1 gives the existence of a symmetric matrix \( M \in M_n(k) \) such that \( \mu_M = f \). Since \( \mu_M \) and \( \chi_M \) are both monic polynomials of degree \( n \) and \( \mu_M \mid \chi_M \), we have \( \chi_M = \mu_M = f \).

Assume now that \( f \) is the product of pairwise distinct monic irreducible inseparable polynomials, and suppose that there exists a symmetric matrix \( M \in M_n(k) \) such that \( \chi_M = f \). Since \( \mu_M \) and \( \chi_M \) have the same irreducible divisors, the hypothesis on \( f \) implies that \( \mu_M = \chi_M = f \), contradicting the previous theorem. □

Theorem 4.3. Let \( \alpha \in k_{\text{alg}} \) be an algebraic element of degree \( n \), with minimal polynomial \( f \).

(1) if \( f \) is separable, \( \alpha \) is the eigenvalue of a symmetric matrix of \( M_n(k) \);
(2) if \( f \) is inseparable, \( \alpha \) is the eigenvalue of a symmetric matrix of \( M_{n+1}(k) \), but not of any symmetric matrix of \( M_n(k) \).

In particular, if \( k \) is perfect, any algebraic element of degree \( n \) is the eigenvalue of a symmetric matrix of \( M_n(k) \).

Proof. If \( f \) is separable, we may apply the previous corollary to get a symmetric matrix \( M \in M_n(k) \) such that \( \chi_M = f \). In particular, \( \alpha \) is an eigenvalue of \( M \).

Assume now that \( f \) is inseparable, and suppose that there exists a symmetric matrix \( M \in M_n(k) \) such that \( \alpha \) is an eigenvalue of \( M \). Since \( \chi_M(\alpha) = 0 \) and \( \chi_M \in k[X] \), we get \( f \mid \chi_M \). For degree reasons, we have \( \chi_M = f \), contradicting Corollary 4.2.

Since \( f \in k[X^2] \) in this case, by Lemma 3.1, there exists a \( k \)-linear form \( s' : k[X]/(f) \to k \) such that \( s'((1)) \) is hyperbolic. Note now that the \( k \)-linear form \( s_0 : k[X]/(X) \to k \) induced by evaluation at 0 obviously satisfies \( (s_0)_*((1)) \simeq (1) \). We may now conclude as in the proof of Theorem 4.1 that there exists a symmetric matrix \( M \in M_{n+1}(k) \) such that \( \mu_M = Xf \). Degree considerations show that \( \chi_M = Xf \). In particular, \( \alpha \) is an eigenvalue of \( M \). This concludes the proof. □

Remark 4.4. In all the previous results, the symmetric matrices \( M \) which we have constructed in the proofs are cyclic, that is, they satisfy \( \chi_M = \mu_M \).
We now summarize the method to compute explicitly a symmetric matrix of $M_n(k)$ of given minimal polynomial $f$ of degree $n \geq 1$, provided that we know the factorization of $f$. Of course, we assume that $f$ is not the product of pairwise distinct inseparable irreducible polynomials.

We have to consider two cases:

1. If $f$ has at least one separable irreducible divisor, write $f = \pi_1^{m_1} \cdots \pi_r^{m_r} g$, where $r \geq 1$, $m_1, \ldots, m_r \geq 1$, $\pi_1, \ldots, \pi_r$ are monic separable irreducible polynomials, and $g$ is a polynomial of $k[X^2]$ coprime to each $\pi_i$.

   Note that we can only consider the separable irreducible divisors $\pi_1, \ldots, \pi_r$ such that $m_1, \ldots, m_r$ are odd, and include the other ones in the polynomial $g$.

   For all $(x_1, \ldots, x_r, x') \in E = k[X]/(\pi_1^{m_1}) \times \cdots \times k[X]/(\pi_r^{m_r}) \times k[X]/(g)$, set
   $$s(x_1, \ldots, x_r, x') = s_1(x_1) + \cdots + s_r(x_r) + s'(x'),$$
   where $s_i : k[X]/(\pi_i^{m_i}) \to k$ is the $k$-linear map defined in Proposition 3.4 and $s' : k[X]/(g) \to k$ is the $k$-linear map defined in Lemma 3.1.

   In this case, we will denote by $E$ the $k$-basis
   $$(1, 0, \ldots, 0), \ldots, (\alpha_1^{m_1d_1-1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1), \ldots, (0, \ldots, 0, \alpha'^{d'-1}),$$
   where $\alpha_i$ is the class of $X$ in $k[X]/(\pi_i^{m_i})$, $\alpha'$ is the class of $X$ in $k[X]/(g)$, $d_i = \deg(\pi_i)$ and $d' = \deg(g)$.

   We also set $C = \begin{pmatrix} C_{\pi_1^{m_1}} & & \\ & \ddots & \\ & & C_{\pi_r^{m_r}} \end{pmatrix}$, where $C_h$ is the companion matrix of a monic polynomial $h \in k[X]$.

2. If all irreducible divisors of $f$ are inseparable, write $f = \rho^m g$, where $\rho$ is irreducible and inseparable, $m \geq 2$, and $g$ is a polynomial of $k[X^2]$ coprime to $\rho$.

   For all $(x, x') \in E = k[X]/(\rho^m) \times k[X]/(g)$, set
   $$s(x, x') = s(x) + s'(x'),$$
   where $s : k[X]/(\rho^m) \to k$ is the $k$-linear map defined in Proposition 3.6 and $s' : k[X]/(g) \to k$ is the $k$-linear map defined in Lemma 3.1.

   In this case, we will denote by $E$ the $k$-basis
   $$(1, 0), \ldots, (\alpha^{m_1d_1-1}, 0), \ldots, (0, 1), \ldots, (0, \alpha'^{d'-1}),$$
where $\alpha$ is the class of $X$ in $k[X]/(\rho^m)$, $\alpha'$ is the class of $X$ in $k[X]/(g)$, $d = \deg(\rho)$ and $d' = \deg(g)$.

We also set $C = \begin{pmatrix} C_\rho^m & C_g \end{pmatrix}$.

Then, we may compute an orthonormal basis $B$ of $E$ with respect to $s_*((1))$, for example starting from the expression of $s(xy)$ in terms of the coordinates of $x$ and $y$ in the basis $E$, and applying the algorithm explained in Section 2. In case (a), to avoid working with too many variables, one may first compute an orthonormal basis for each of the bilinear spaces $(k[X]/(\pi_1^m), (s_i)_*(\langle 1 \rangle))$, $(1 \leq i \leq r - 1)$ and $(k[X]/(\pi_r^m) \times k[X]/(g), (s_r \oplus s'_i)_*(\langle 1 \rangle))$, and glue them into an orthonormal basis $B$ of $E$.

By choice of $E$, the representative matrix of left multiplication by $(\alpha_1, \ldots, \alpha_r, \alpha')$ in case (a), or by $(\alpha, \alpha')$ in case (b), is the matrix $C$. Hence, the matrix representation of this endomorphism with respect to the orthonormal basis $B$ is $M = P^{-1}CP = QCQ^{-1}$ (with the notation of the algorithm explained in Section 2).

The matrix $M$ is then symmetric, with minimal polynomial (and characteristic polynomial) equal to $f$.

**Example 4.5.** Let $k = \mathbb{F}_2(t)$, and let $f = (X^2 + X + t)^3$. We will use the $k$-linear map defined in Proposition 3.4. The representative matrix of $s_*((1))$ with respect to the basis $(1, \alpha, \ldots, \alpha^5)$ is

$$S = \begin{pmatrix}
0 & 1 & 1 & t + 1 & 1 & t^2 + t \\
1 & 1 & t + 1 & 1 & t^2 + t & t^3 \\
t + 1 & 1 & t^2 + t & t^2 & t^3 + 1 & 1 \\
1 & t + t^2 & t^2 & t^3 + 1 & 1 & t^3 + t^2 + t \\
t^2 + t & t^2 & t^3 + 1 & 1 & t^4 + t^2 + t & t^4 + t^2
\end{pmatrix}.$$

If $x = \sum_{i=0}^5 x_i \alpha^i$ and $y = \sum_{i=0}^5 y_i \alpha^i$, then $s(xy)$ contains the terms

$$x_1 y_1 + x_1(y_0 + (t+1)y_2 + y_3 + (t^2 + t)y_4 + t^2 y_5) + y_1(x_0 + (t+1)x_2 + x_3 + (t^2 + t)x_4 + t^2 x_5),$$

which may be rewritten as

$$(x_0 + x_1 + (t+1)x_2 + x_3 + (t^2 + t)x_4 + t^2 x_5)(y_0 + y_1 + (t+1)y_2 + y_3 + (t^2 + t)y_4 + t^2 y_5)$$

$$(x_0 + (t+1)x_2 + x_3 + (t^2 + t)x_4 + t^2 x_5)(y_0 + (t+1)y_2 + y_3 + (t^2 + t)y_4 + t^2 y_5).$$
We then have $s_*((1)) = \varphi_1 \cdot \varphi_1 + b_1$, where
\[
E \rightarrow k
\]
\[
\varphi_1: \quad x \mapsto x_0 + x_1 + (t + 1)x_2 + x_3 + (t^2 + t)x_4 + t^2 x_5.
\]
and $b_1$ is a symmetric bilinear form that we will not write here. After simplification, one can see that $b_1(x, y)$ contains the terms
\[
x_0 y_0 + x_0 (t y_2 + t y_3 + (t^2 + t + 1) y_4 + t y_5) + y_0 (t x_2 + t x_3 + (t^2 + t + 1) x_4 + t x_5),
\]
that is
\[
(x_0 + t x_2 + t x_3 + (t^2 + t + 1) x_4 + t x_5)(y_0 + t y_2 + t y_3 + (t^2 + t + 1) y_4 + t y_5)
\]
\[
+ (t x_2 + t x_3 + (t^2 + t + 1) x_4 + t x_5)(t y_2 + t y_3 + (t^2 + t + 1) y_4 + t y_5).
\]
Hence, $s_*((1)) = \varphi_1 \cdot \varphi_1 + \varphi_2 \cdot \varphi_2 + b_2$, where
\[
E \rightarrow k
\]
\[
\varphi_2: \quad x \mapsto x_0 + t x_2 + t x_3 + (t^2 + t + 1) x_4 + t x_5,
\]
and
\[
b_2(x, y) = x_2 y_3 + x_2 y_5 + x_3 y_2 + x_3 y_4 + x_3 y_5 + x_4 y_3 + x_5 y_2 + x_5 y_3.
\]
Now, $b_2(x, y)$ contains the terms
\[
x_3 y_3 + x_3 (y_2 + y_4 + y_5) + y_3 (x_2 + x_4 + x_5),
\]
so $s_*((1)) = \varphi_1 \cdot \varphi_1 + \varphi_2 \cdot \varphi_2 + \varphi_3 \cdot \varphi_3 + b_3$, where
\[
E \rightarrow k
\]
\[
\varphi_3: \quad x \mapsto x_2 + x_3 + x_4 + x_5,
\]
and
\[
b_3(x, y) = x_2 y_2 + x_2 y_4 + x_4 y_2 + x_4 y_5 + x_5 y_4 + x_5 y_5.
\]
Since $b_3(x, y)$ contains $x_2 y_2 + x_2 y_4 + x_4 y_2$, we may write $s_*((1)) = \varphi_1 \cdot \varphi_1 + \varphi_2 \cdot \varphi_2 + \varphi_3 \cdot \varphi_3 + \varphi_4 \cdot \varphi_4 + b_4$, where
\[
E \rightarrow k
\]
\[
\varphi_4: \quad x \mapsto x_2 + x_4,
\]
and
\[
b_4(x, y) = x_4 y_5 + x_5 y_4 + x_5 y_5 = (x_4 + x_5)(y_4 + y_5) + x_4 y_4.
\]
Finally, $s_*((1)) = \sum_{i=1}^6 \varphi_i \cdot \varphi_i$, where
\[
E \rightarrow k
\]
\[
\varphi_5: \quad x \mapsto x_4 + x_5,
\]
and

\[ E \rightarrow k \]

\[ \varphi_6 : x \mapsto x_4. \]

The corresponding matrix \( Q \) is then

\[
Q = \begin{pmatrix}
1 & 1 & t+1 & 1 & t^2 + t & t^2 \\
1 & 0 & t & t & t^2 + t + 1 & t \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

If \( C \) is the companion matrix of \( f \), we then get

\[
M = QCQ^{-1} = \begin{pmatrix}
t + 1 & t & 1 & 1 & 0 & 0 \\
t & t & 1 & 1 & 0 & 0 \\
1 & 1 & t & t & 1 & 1 \\
1 & 1 & t & t + 1 & 1 & 1 \\
0 & 0 & 1 & 1 & t + 1 & t \\
0 & 0 & 1 & 1 & t & t
\end{pmatrix}.
\]

**Example 4.6.** Let \( k = \mathbb{F}_2(t) \), and let \( f = (X^2 + t)^3 \). We will use the \( k \)-linear map defined in Lemma 3.5. The representative matrix of \( s_*(\langle 1 \rangle) \) with respect to the basis \((1, \alpha, \ldots, \alpha^5)\) is

\[
S = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & t^2 \\
1 & 0 & 0 & 1 & t^2 & t \\
0 & 0 & 1 & t^2 & t & 0 \\
0 & 1 & t^2 & t & 0 & 0 \\
1 & t^2 & t & 0 & 0 & t^4
\end{pmatrix}.
\]

This time, we may prove that \( s_*(\langle 1 \rangle) = \sum_{i=1}^{4} \varphi_i \cdot \varphi_i + b_4 \), where

\[
E \rightarrow k \quad \varphi_1 : x \mapsto x_1 + x_4 + t^2 x_5, \quad \varphi_2 : x \mapsto t^2 x_2 + t x_3 + x_4 + t^2 x_5,
\]

\[
E \rightarrow k \quad \varphi_3 : x \mapsto t^{-2} x_0 + t^2 x_2 + (t + t^{-2}) x_3 + (t^2 + t^{-1}) x_5,
\]

\[
E \rightarrow k \quad \varphi_4 : x \mapsto t^{-2} x_0 + (t + t^{-2}) x_3 + t^{-1} x_5,
\]

and \( b_4(x, y) = x_3 y_5 + y_3 x_5 \).
Since $b_4$ is alternating, we replace $\varphi_4$ by
\[ E \rightarrow k \]
\[ \varphi_4: \quad x \mapsto t^{-2}x_0 + (t + t^{-2} + 1)x_3 + t^{-1}x_5, \]
and we set
\[ E \rightarrow k \]
\[ \varphi_5: \quad x \mapsto t^{-2}x_0 + (t + t^{-2})x_3 + (t^{-1} + 1)x_5, \]
and
\[ E \rightarrow k \]
\[ \varphi_6: \quad x \mapsto t^{-2}x_0 + (t + t^{-2} + 1)x_3 + (t^{-1} + 1)x_5. \]
The corresponding matrix $Q$ is
\[
Q = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & t^2 \\
0 & 0 & t^2 & t & 1 & t^2 \\
t^{-2} & 0 & t^2 & (t + t^{-2}) & 0 & t^2 + t^{-1} \\
t^{-2} & 0 & 0 & t + t^{-2} + 1 & 0 & t^{-1} \\
t^{-2} & 0 & 0 & t + t^{-2} & 0 & t^{-1} + 1 \\
t^{-2} & 0 & 0 & t + t^{-2} + 1 & 0 & t^{-1} + 1 
\end{pmatrix}.
\]
If $C$ is the companion matrix of $f$, we have $M = QCQ^{-1}$, and we finally get
\[
M = t^{-4} \begin{pmatrix}
0 & t^6 & t^6 & 0 & 0 & 0 \\
t^6 & 0 & t^3 & t^3 & t^4 + t^3 & t^4 + t^3 \\
t^6 & t^3 & 1 & t^2 + 1 & t^4 + 1 & t^4 + t^2 + 1 \\
t^6 & 0 & t^3 + 1 & t^4 + t^2 + 1 & t^5 + 1 & t^5 + t^2 + 1 \\
0 & t^4 + t^3 & t^4 + 1 & t^4 + t^2 + 1 & t^5 + t^2 + 1 & t^5 + t^4 + 1 \\
0 & t^4 + t^3 & t^4 + t^2 + 1 & 1 & t^5 + t^2 + 1 & t^5 + t^4 + 1 
\end{pmatrix}.
\]
Note that the proofs of Propositions 3.4 and 3.6 provide a way to compute an orthonormal basis of $k[X]/(h^m)$ for the appropriate $k$-linear form when $h$ is an irreducible (separable or inseparable) polynomial (with $m \geq 2$ if $h$ is inseparable), which reduces the number of variables to be manipulated at the same time:

1. If $h = \pi(X^{2^n})$ (where $n = 0$ if $h$ is separable), and if $L = k(a)$, where $a \in k_s$ is a fixed root of $\pi$, compute an orthonormal basis $(\gamma_1, \ldots, \gamma_d)$ of $L$ with respect to $(\text{Tr}_{L/k})_*((1))$, using for example the method explained in Section 2.
(2) Compute an orthonormal basis \((\tilde{Q}_1, \ldots, \tilde{Q}_{2^m})\) of \(L[X]/((X^{2^n} - a)^m)\) for \(t_*(\langle 1 \rangle)\), where \(t : L[X]/(X^{2^n} - a)^m \to L\) is the appropriate \(L\)-linear form (where \(n = 0\) if \(h\) is separable).

(3) For \(1 \leq i \leq d\) and \(1 \leq j \leq 2^m\), compute the unique polynomial \(P_{ij} \in L[X]\) of degree \(\leq 2^m d\) satisfying \(P_{ij} \equiv \gamma_i Q_j \mod (X^{2^n} - \sigma(a))^m\) for all \(\sigma \in X_L\). This polynomial lies in fact in \(k[X]\), and \((P_{ij})_{i,j}\) is the desired orthonormal basis.

**Example 4.7.** Let \(k = \mathbb{F}_2(t)\), and let \(f = (X^2 + X + t)^3\). Let \(a \in k\), be a root of \(\pi = X^2 + X + t\), and let \(L = k(a)\). It is easy to check that \(\text{Tr}_{L/k}(1) = 0\) and \(\text{Tr}_{L/k}(a) = 1\). In particular, \(\text{Tr}_{L/k}(a^2) = \text{Tr}_{L/k}(a)^2 = 1\) and \(\text{Tr}_{L/k}(a(1+a)) = \text{Tr}_{L/k}(t) = 0\). Hence, \((a, 1+a)\) is an orthonormal basis of \(L\) with respect to \((\text{Tr}_{L/k})_*(\langle 1 \rangle)\).

Now, let \(\gamma = \tilde{X} \in L[X]/((X-a)^3)\). As observed in the proof of Lemma 3.3, the representative matrix of \(t_*(\langle 1 \rangle)\) in the \(L\)-basis \((1, \gamma - a, (\gamma - a)^2)\) is

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

Hence, if \(x = x_0 + x_1(\gamma-a) + x_2(\gamma-a)^2\) and \(y = y_0 + y_1(\gamma-a) + y_2(\gamma-a)^2\), we have

\[t(xy) = x_0y_0 + x_0y_1 + x_0y_2 + y_0x_1 + y_0x_2 + x_1y_1.\]

It is not difficult to see that \(t_*(\langle 1 \rangle) = \sum_{i=1}^3 \varphi_1 \bullet \varphi_i\), where

\[
\begin{align*}
\varphi_1 : x &\mapsto x_0 + x_1 + x_2, \\
\varphi_1 : x &\mapsto x_1 + x_2, \\
\varphi_1 : x &\mapsto x_1.
\end{align*}
\]

Inverting the matrix \(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\) yields the orthonormal basis \((1, 1 + (\gamma-a)^2, (\gamma-a) + (\gamma-a)^2)\).

Set \(Q_1 = 1, Q_2 = 1 + (X-a)^2\) and \(Q_3 = (X-a) + (X-a)^2\). Using a computer algebra system, we find that the polynomials

\[aQ_1, (1-a)Q_1, aQ_2, (1-a)Q_2, aQ_3, (1-a)Q_3\]

lift to the following polynomials of \(k[X]\) respectively:

\[P_{1,1} = X^4 + (t^2 + t), \quad P_{2,1} = X^4 + (t^2 + t + 1), \quad P_{1,2} = X^5 + X^4 + X^3 + t^2 X + (t^2 + t)\]
\[ P_{2,2} = X^5 + X^3 + X^2 + t^2 X + t + 1, \quad P_{1,3} = X^4 + X^3 + t X + t^2 \]
\[ P_{2,3} = X^4 + X^3 + X^2 + (t + 1) X + t^2 + t. \]

The matrix of the orthonormal basis \((\overline{P}_{1,1}, \overline{P}_{2,1}, \ldots, \overline{P}_{2,3})\) in the basis \((1, \alpha, \ldots, \alpha^5)\) is thus
\[
P = \begin{pmatrix}
  t^2 + t & t^2 + t + 1 & t^2 + t & t + 1 & t^2 & t + t
  0 & 0 & t^2 & t & t
  0 & 0 & 0 & 1 & 0 & 1
  0 & 0 & 1 & 1 & 1 & 1
  1 & 1 & 1 & 0 & 1 & 1
  0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}.
\]

If \(C = C_f\), we get this time
\[
M = P^{-1}CP = \begin{pmatrix}
  t & t & 1 & 0 & 1 & 0
  t & t + 1 & 0 & 1 & 0 & 1
  1 & 0 & t & t & 1 & 0
  0 & 1 & t & t + 1 & 0 & 1
  1 & 0 & 1 & 0 & t + 1 & t
  0 & 1 & 0 & 1 & t & t
\end{pmatrix}.
\]

To conclude, we would like to give a result which may be useful for computations when the polynomial \(g\) in the factorization of \(f\) is a square (in this case, one may reduce the computation of an orthonormal basis on \(k[X]/(f)\) to the computation of an orthonormal basis on each factor).

**Lemma 4.8.** Assume that \(f = X^{2m} + a_m^2 X^{2m-2} + \cdots + a_2^2 X^2 + a_0^2\), where \(a_0 \neq 0\), and let \(s : k[X]/(f) \to k\) be the unique \(k\)-linear form such that
\[
s(1) = 1, s(\alpha) = \cdots = s(\alpha^{2m-1}) = 0.
\]
Then \(s_*((1))\) is isomorphic to the unit form.

**Proof.** By Lemma 2.2, it is enough to show that \(s_*((1))\) is nonzero, non-degenerate, non-alternating, and that \(s(x^2)\) is a square for all \(x \in k[X]/(f)\).

Since \(s(1) = 1, s_*((1))\) is nonzero, and non-alternating. Let us shows that it is non-degenerate.

Let \(x = \sum_{i=0}^{2m-1} \lambda_i \alpha^i \in \ker(s_*((1))\). Then \(s(x1) = s(x) = \lambda_0 = 0\). Assume we proved that \(\lambda_0 = \cdots = \lambda_j = 0\) for some \(0 \leq j \leq 2m - 2\). Since
$a_0 \neq 0$, $\alpha$ is invertible in the quotient ring $k[X]/(f)$ (since $X$ is coprime to $f$).

We then have $0 = s(x\alpha^{-j-1}) = s\left(\sum_{i=j+1}^{2m-1} \lambda_i \alpha^{i-j-1}\right) = s\left(\sum_{i=0}^{2m-j-2} \lambda_{i+j+1} \alpha^i\right) = \lambda_{j+1}$. It follows by induction that all the $\lambda_i$'s are zero, that is $x = 0$.

Let us finally prove that $s(x^2)$ is a square for all $x \in k[X]/(f)$. For $i \geq 0$, let $R$ be the remainder of the long division of $X^{2i}$ by $f$, and note that $X^{2i}$ and $f$ are squares. We claim that $R$ is a square. Indeed, since $K$ has characteristic two, $K^2$ is a field and $(K[X])^2 = K^2[X^2]$, so we may just consider $X^2$ as the variable and $K^2$ as the new field of coefficients to conclude.

If $R = S^2$, then we have $\deg(S) \leq m-1$. Writing $S = \sum_{j=0}^{m-1} s_j X^j$, we get \[ \alpha^{2i} = R(\alpha) = \sum_{j=0}^{m-1} s_j^2 \alpha^{2j}, \] and thus $s(\alpha^{2i}) = s_0^2$ is a square. By Remark 2.4, we get the desired result. \hfill \square

Acknowledgements. We would like to thank warmly the anynomous referee for his/her careful reading of a previous version of this paper and his/her insightful remarks and suggestions.

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