On orthomorphism elements in ordered algebra

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Abstract: Let $C$ be an ordered algebra with a unit $e$. The class of orthomorphism elements $\text{Orthe}(C)$ of $C$ was introduced and studied by Alekhno in "The order continuity in ordered algebras". If $C = L(G)$, where $G$ is a Dedekind complete Riesz space, this class coincides with the band $\text{Orth}(G)$ of all orthomorphism operators on $G$. In this study, the properties of orthomorphism elements similar to properties of orthomorphism operators are obtained. Firstly, it is shown that if $C$ is an ordered algebra such that $C_r$, the set of all regular elements of $C$, is a Riesz space with the principal projection property and $\text{Orthe}(C)$ is topologically full with respect to $I_e$, then $B_e = \text{Orthe}(C)$ holds, where $B_e$ is the band generated by $e$ in $C_r$. Then, under the same hypotheses, it is obtained that $\text{Orthe}(C)$ is an $f$-algebra with a unit $e$.

Key words: Ordered algebra, orthomorphism elements, orthomorphism, $f$-algebra

1. Introduction

All vector spaces are considered over the reals only. An ordered vector space (Riesz space) $C$ under an associative multiplication is said to be an ordered algebra (Riesz algebra) whenever the multiplication makes $C$ an algebra, and in addition it satisfies the following property: $a,b \in C^+$ implies $ab \in C^+$. A Riesz algebra $C$ is called an $f$-algebra if $C$ has the additional property that $a \land b = 0$ implies $ac \land b = ca \land b = 0$ for each $c \in C^+$. Throughout the study, we will assume $C \neq \{0\}$ and $C$ has a unit element $e > 0$. An element $a \in C$ is called a regular element if $a = b - c$ with $b$ and $c$ positive, the space of all regular elements of $C$ will be denoted by $C_r$. Obviously, $C_r$ is a real ordered algebra. Let $C$ be an ordered vector space and an element $a \in C^+$, the order ideal $I_a$ generated by $a$ is the set $I_a = \{b \in C : -\lambda a \leq b \leq \lambda a \text{ for some } \lambda \in \mathbb{R}^+\}$. Under the algebraic operations and the ordering induced by $C$, $I_a$ is an ordered vector subspace of $C$. Moreover, $I_e$ is an ordered algebra [1].

An element $q \in C$ is said to be an order idempotent whenever $0 \leq q \leq e$ and $q^2 = q$. Under the partial ordering induced by $C$, the set of all order idempotents $\text{OI}(C)$ of $C$ is a Boolean algebra and its lattice operations satisfy the identities $p \land q = pq$ and $p \lor q = p + q - pq$ for all $p, q \in \text{OI}(C)$. If $c \in C$ and the modulus $|c|$ of $c$ exists, then $q|c| = |qc|$ and $|c|q = |cq|$ for all $q \in \text{OI}(C)$ [2].

Definition 1.1 [1] Let $C$ be an ordered algebra, an element $a \in C$ is said to be an order idempotent preserving element whenever $(e - q)aq = 0$ for all $q \in \text{OI}(C)$. An element $a$ is said to be an orthomorphism element of

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an ordered algebra $C$ whenever $a$ is an order idempotent preserving element that is also regular.

The collection of all orthomorphism elements of an ordered algebra $C$ will be denoted by $\text{Orthe}(C)$. An operator $\pi : G \to G$ on a Riesz space $G$ is said to be band preserving whenever $\pi(B) \subseteq B$ holds for each band $B$ of $G$. $\pi$ is a band preserving operator if and only if $\pi(x) \perp y$ whenever $x \perp y$ in $G$. A band preserving and order bounded operator $\pi$ is called orthomorphism of $G$ and the set of all orthomorphisms of $G$ is denoted by $\text{Orth}(G)$. If $G$ has the principal projection property, then an operator $\pi : G \to G$ is band preserving if and only if $\pi p = p\pi$ (or $(I - p)p\pi = 0$) for every order projection $p$ on $G$ [3, Theorem 8.3]. If $C = L(G)$ is taken, where $G$ is a Dedekind complete Riesz space, then the set of all order idempotents $OI(C)$ of $C$ is the set of all order projections on $G$ [3, Theorem 3.10] and the band $B_e$ generated by $e$ in $C_r$ is equal to $\text{Orth}(G) = \text{Orthe}(C)$ [3, Theorem 8.11]. In general, the equality $B_e = \text{Orthe}(C)$ does not hold in the case of an arbitrary ordered algebra $C$. Therefore, the following question might come into mind. Under what condition $\text{Orthe}(C)$ could be identified to $B_e$? In this work, we try to provide an answer to this question. Moreover, we will show that, under the same hypothesis, $\text{Orthe}(C)$ has the similar properties of orthomorphisms.

We refer to [3, 5, 7, 9] for definitions and notations which are not explained here. All Riesz spaces in this paper are assumed to be Archimedean.

2. Ortomorphism elements

**Proposition 2.1** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space. Then, $\text{Orthe}(C)$ is a band in $C_r$ so that $B_e \subseteq \text{Orthe}(C)$ where $B_e$ is the band generated by $e$ in $C_r$.

**Proof** Since $q|a| = |qa|$ and $|a|q = |aq|$ for all $q \in OI(C)$ and $a \in C_r$, it is easy to show that $\text{Orthe}(C)$ is an order ideal. To see that $\text{Orthe}(C)$ is a band in $C_r$, let $0 \leq (b_\alpha) \uparrow b$ in $C_r$ with $(b_\alpha) \subseteq \text{Orthe}(C)$. Then, for all $\alpha$ we have

$$0 \leq (e - q)bq = (e - q)(b - b_\alpha)q + (e - q)b_\alpha q = (e - q)(b - b_\alpha)q \leq (b - b_\alpha).$$

Thus, $b - b_\alpha \downarrow 0$ implies $(e - q)bq = 0$ and $b \in \text{Orthe}(C)$. $B_e \subseteq \text{Orthe}(C)$ is obtained from the definition of $B_e$. \hfill \Box

**Lemma 2.2** Let $C$ be an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $b \in C_r$. Then, $b \in \text{Orthe}(C)$ if and only if $ba = ab$ for all $a \in I_e$.

**Proof** Let $b \in C_r$. If $ba = ab$ for all $a \in I_e$ then $b \in \text{Orthe}(C)$ as $OI(C) \subseteq I_e$. Now, let $b \in \text{Orthe}(C)$. From Freudenthal’s Spectral Theorem [3, Theorem 6.8], there exists a sequence $(u_n)$ of $e$-step function satisfying

$$0 \leq a - u_n \leq n^{-1}e$$

for each $a \in I_e$. As $u_n$ $e$-step function, there exist $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ and $p_1, p_2, \ldots, p_k \in OI(C)$ such that $u_n = \sum_{i=1}^{k} \lambda_i p_i$. Thus, we have $bu_n = u_n b$ for each $n$. This yields

$$0 \leq |ab - ba| = |ab - u_n b + u_n b - ba| \leq |ab - u_n b| + |bu_n - ba| \leq n^{-1}b + n^{-1}b$$

for each $n$. Since $C$ is Archimedean, we have $ab = ba$ for every $a \in I_e$. \hfill \Box
If \( C = L(G) \), where \( G \) is a Dedekind complete Riesz space, then \( \text{Orth}(G) = \text{Orth}(C) = B_I \) where \( B_I \) is the generated by the identity operator \( I \) in \( C_r \). In general, the equality \( B_c = \text{Orth}(C) \) does not hold in the case of an ordered algebra \( C \).

**Example 2.3** Let \( G \) be the Riesz space of all continuous piecewise linear functions on \([0,1]\), then \( \text{Orth}(G) = \langle \lambda I : \lambda \in \mathbb{R} \rangle \) by the Problem 7 in [3, p. 124]. If we take \( C = L(G) \), then we have \( OI(C) = \{0, I\} \) as \( OI(C) \subseteq \text{Orth}(G) \) holds. As a result of these simple observations we obtain that \( \text{Orth}(C) = L_r(G) \neq B_I \).

Now, we will investigate when \( B_c = \text{Orth}(C) \) holds.

**Definition 2.4** Let \( C \) be an ordered algebra such that \( C_r \) is a Riesz space and \( \text{Orth}(C) \) has separating order dual. Let \( b,c \in \text{Orth}(C) \) be arbitrary and \( 0 \leq b \leq c \). \( \text{Orth}(C) \) is said to be topologically full with respect to \( I_e \) if there exists a net \( 0 \leq a_\alpha \leq e \) with \( a_\alpha c \to b \) in \( \sigma(\text{Orth}(C), \text{Orth}(C)^\sim) \).

**Example 2.5** Let \( G \) be a Dedekind complete Riesz space with separating order dual. If we take \( C = L(G) \), then \( \text{Orth}(C) = \text{Orth}(G) \) is topologically full with respect to \( I_e = Z(G) \) from the Theorem 4.3 in [6].

Let \( C \) be a Riesz algebra such that \( C_r \) is a Riesz space. It is easy to see that \((bc)q = q(bc)\) for each \( b,c \in \text{Orth}(C) \) and \( q \in OI(C) \). Thus, \( \text{Orth}(C) \) is a Riesz algebra. For \( b \in \text{Orth}(C) \), let us define \( L_b : \text{Orth}(C) \to \text{Orth}(C) : L_b(c) = bc \) and \( R_b : \text{Orth}(C) \to \text{Orth}(C) : R_b(c) = cb \) for each \( c \in \text{Orth}(C) \). \( L_b, R_b \) are regular operators and so that the adjoint operators \( L_b^\sim, R_b^\sim \) are regular operators on \( \text{Orth}(C)^\sim \). Let us consider positive linear maps

\[
S_h : \text{Orth}(C) \to I_e^\sim, \quad b \to S_{b,h} : S_{b,h}(a) = h(ab)
\]

\[
V_h : \text{Orth}(C) \to I_e^\sim, \quad b \to V_{b,h} : V_{b,h}(a) = h(ba)
\]

for each \( b \in \text{Orth}(C) \), \( a \in I_e \) and \( h \in \text{Orth}(C)^\sim_+ \). If \( \text{Orth}(C) \) is topologically full with respect to \( I_e \), then we can say more about the positivity of the maps \( S_h \) and \( V_h \). The proof of the following Lemma is the adaptation of the Lemma in [8, p.65].

**Lemma 2.6** If \( C \) is an ordered algebra such that \( C_r \) is a Riesz space with the principal projection property and \( \text{Orth}(C) \) is topologically full with respect to \( I_e \), then \( S_h, V_h : \text{Orth}(C) \to I_e^\sim \) are lattice homomorphisms for each \( h \in \text{Orth}(C)^\sim_+ \).

**Proof** Let \( 0 \leq h \in \text{Orth}(C)^\sim_+ \). To see that \( S_h \) is a lattice homomorphism, it is enough to show that \( S_{b,h} \wedge S_{c,h} = 0 \) for each \( b,c \in \text{Orth}(C) \) satisfying \( b \wedge c = 0 \). Let \( d = b + c \) and \( I_b, I_c, I_d \) be respectively the order ideals generated by \( b, c \), and \( d \). Then \( I_d \) is actually the order direct sum of \( I_b \) and \( I_c \) by the Theorem 17.6 [5]. We denote by \( p \) the order projection of \( I_d \) onto \( I_b \). Let \( R \) be the restriction to \( I_d \) of order bounded functionals on \( \text{Orth}(C) \). Then \( R \) is an order ideal in \( I_d^\sim \) by the Theorem 2.3 in [3]. The adjoint \( p^\sim : I_d^\sim \to I_d^\sim \) of \( p \) satisfies \( 0 \leq p^\sim \leq I \) and as a consequence we obtain \( p^\sim \langle R \rangle \subseteq R \). As a result of these simple observations we obtain that the pair \( \langle I_d, R \rangle \) constitutes a Riesz pair and \( p : \langle I_d, \sigma(I_d, R) \rangle \to \langle I_d, \sigma(I_d, R) \rangle \) is continuous. Since \( 0 \leq p(d) \leq d \) there exists \( (a_\alpha) \) in \( I_e \) such that \( 0 \leq a_\alpha \leq e \) with \( a_\alpha d \to p(d) = b \) in \( \sigma(\text{Orth}(C), \text{Orth}(C)^\sim) \). As \( L_{a_\alpha} \subseteq Z(I_d) \) for each \( \alpha ) \) it is easy to see that \( a_\alpha d \to b \) in \( \sigma(I_d, R) \) and
\(a_{\alpha}p(d) = p(a_{\alpha}d)\). By the continuity of \(p\) now yields \(a_{\alpha}p(d) = a_{\alpha}b \rightarrow b\) in \(\sigma(I_d, R)\). Since \(a_{\alpha}d = a_{\alpha}b + a_{\alpha}c\) for each \(\alpha\), we have \(a_{\alpha}c \rightarrow 0\) in \(\sigma(I_d, R)\). As \((S_{b,h} \wedge S_{c,h})(a) \leq h((a - aa_{\alpha})b + (aa_{\alpha}c)\) for each \(\alpha\), we obtain
\[
0 \leq (S_{b,h} \wedge S_{c,h})(a) \leq \lim_{\alpha} h((a - aa_{\alpha})b + (aa_{\alpha}c))
\]
\[
= \lim_{\alpha} h(L_{a}(b - a_{\alpha}b + a_{\alpha}c))
\]
\[
= \lim_{\alpha} L_{a}^{\sim}(h)(b - a_{\alpha}b + a_{\alpha}c)
\]
\[
= 0
\]
as \(L_{a}^{\sim}(\text{Orthe}(C)^{\sim}) \subseteq \text{Orthe}(C)^{\sim}\), which implies that \(S_{h}\) is lattice homomorphism. On the other hand, by the Lemma 2.2 \(b\alpha_{\alpha} \rightarrow b\) and \(ca_{\alpha} \rightarrow 0\) in \(\sigma(I_d, R)\) holds. Similarly, taking \(V_{h}\) instead of \(S_{h}\) and \(R_{a}\) instead of \(L_{a}\), we get \(V_{h}\) is lattice homomorphism.

Corollary 2.7 Let the hypotheses in the Lemma 2.6 hold. If \(b, c \in \text{Orthe}(C)\) and \(b \wedge c = 0\) then \(|S_{b,h}| \wedge |S_{c,t}| = 0\) for each \(h, t \in \text{Orthe}(C)^{\sim}\).

Proof Let \(b, c \in \text{Orthe}(C)\) and \(b \wedge c = 0\). From the Lemma 2.6 we have
\[
0 \leq |S_{b,h}| \wedge |S_{c,t}| \leq |S_{b,h}| \wedge |S_{c,t}| \leq |S_{b,|h\wedge|t|} \wedge |S_{c,|h\wedge|t|} = S_{b \wedge c,|h\wedge|t|} = 0.
\]

Proposition 2.8 Let \(C\) be an ordered algebra such that \(C_{r}\) is a Riesz space with the principal projection property and \(\text{Orthe}(C)\) is topologically full with respect to \(I_{C}\). Then, \(B_{e} = \text{Orthe}(C)\) holds (where \(B_{e}\) is the band generated by \(e\) in \(\text{Orthe}(C)\)).

Proof Let \(b \in \text{Orthe}(C)\) with \(|b| \wedge e = 0\). Clearly,
\[
S_{b,h}(a) = h(ab) = h(L_{b}(a)) = L_{b}^{\sim}(h)(ae) = S_{e,L_{h}^{\sim}(h)}(a)
\]
holds for each \(h \in \text{Orthe}(C)^{\sim}\). Then, it follows that
\[
0 \leq |S_{b,h}| = |S_{b,h}| \wedge |S_{b,h} \wedge S_{e,L_{h}^{\sim}(h)} = 0
\]
and so \(S_{b,h} = 0\) for each \(h \in \text{Orthe}(C)^{\sim}\). Thus, we have \(b = 0\) which implies that \(B_{e} = \{e\}^{dd} = \text{Orthe}(C)\).

Corollary 2.9 Let the hypotheses be as in the Proposition 2.8. Then, the band \(B_{e}\) generated by \(e\) in \(C_{r}\) is equal to \(\text{Orthe}(C)\).

Proof It is clear that the band generated by \(e\) in \(\text{Orthe}(C)\) is equal to the band generated by \(e\) in \(C_{r}\) as \(\text{Orthe}(C)\) is a band in \(C_{r}\).

By the Example 2.5, we have known that if \(G\) is a Dedekind complete Riesz space with separating order dual and \(C = L(G)\), then \(\text{Orthe}(C)\) has separating order dual and \(\text{Orthe}(C) = \text{Orth}(G)\) is topologically full with respect to \(I_{C} = Z(G)\). By using this observation and the above result, we can obtain the following Corollary being previously proved as a theorem in a different manner.
Corollary 2.10 Let $G$ be a Dedekind complete Riesz space and $G$ has separating order dual. Then the band $B_I$ generated by the identity operator in $L_r(G)$ is equal to $\text{Orth}(G)$.

Theorem 2.11 If $C$ is an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $\text{Orthe}(C)$ is topologically full with respect to $I_e$, then $\text{Orthe}(C)$ is an $f$-algebra. Moreover, it is a full subalgebra of $C$.

Proof Let $b,c,d \in \text{Orthe}(C)^+$ and $b \wedge c = 0$. For each $0 \leq h \in \text{Orthe}(C)^\sim$ and $a \in I_e$

$$0 \leq S_{db \wedge c,h}(a) = (S_{db,h} \wedge S_{c,h})(a) \leq S_{db,h}(a) \wedge S_{c,h}(a) = h(a(db)) \wedge S_{c,h}(a) = h(d(ab)) \wedge S_{c,h}(a) = h(L_d(ab)) \wedge S_{c,h}(a) = L_d(h)(ab) \wedge S_{c,h}(a) = S_{b,L_d^*(h)}(a) \wedge S_{c,h}(a) = 0$$

holds, which proves that $db \wedge c = 0$. Similarly, taking $V$ instead of $S$ and $R_d$ instead of $L_d$, we have $bd \wedge c = 0$.

Let $b \in \text{Orthe}(C)$ be invertible in $C$. We will show that $b^{-1} \in \text{Orthe}(C)$. As $b \in \text{Orthe}(C)$ $bq = qb$ for each $q \in OI(C)$. It is easy to see that $b^{-1}q = qb^{-1}$ for each $q \in OI(C)$. Thus, $\text{Orthe}(C)$ is a full subalgebra of $C$.

Corollary 2.12 Let $G$ be a Dedekind complete Riesz space and $G$ has separating order dual. Then, $\text{Orth}(G)$ is an $f$-algebra. Moreover, it is a full subalgebra of $L_r(G)$.

As each unital $f$-algebra $C$ with separating order dual is topologically full with respect to $I_e$ [8], we can give the following corollary.

Corollary 2.13 Let $C$ be an ordered algebra such that $C_r$ is a Riesz space with the principal projection property and $\text{Orthe}(C)$ has separating order dual. Then, $\text{Orthe}(C)$ is an $f$-algebra if and only if $\text{Orthe}(C)$ is topologically full with respect to $I_e$.

As we said before, if $G$ is a Dedekind complete Riesz space with separating order dual and $C = L(G)$ then $\text{Orthe}(C) = \text{Orth}(G)$ is topologically full with respect to $I_e = Z(G)$. However, even if $C$ is a Dedekind complete ordered algebra, $\text{Orthe}(C)$ may not be topologically full with respect to $I_e$. We now give an example of a Dedekind complete ordered algebra which is not topologically full with respect to $I_e$.

Example 2.14 Let $f$ be a multiplicative functional on $l_\infty$ satisfying $f(e_0) = 0$ and $C$ be the linear space $l_\infty \oplus \mathbb{R}$. $C$ is a Dedekind complete ordered Banach algebra with unit $(e,0)$ under the multiplication

$$(u_1, \lambda_1) \ast (u_2, \lambda_2) = (u_1 u_2, \lambda_1 f(u_2) + \lambda_2 f(u_1) + \lambda_1 \lambda_2),$$

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the norm
\[ \|(u, \lambda)\| = \|u\| + |\lambda| \]
and the order induced by the cone
\[ C^+ = \{(u, \lambda) : u \in l_\infty^+ \text{ and } \lambda \in \mathbb{R}\}. \]

Furthermore,
\[ OI(C) = \{(p, 0) : p \in OI(l_{\infty})\} \text{ and } \]
\[ Orthe(C) = \{(u, \lambda) : u \in Orthe(l_{\infty}) \text{ and } \lambda \in \mathbb{R}\} [1]. \]

Since \( C \) is Dedekind complete, \( C_r \) is a Riesz space with the principal projection property. As \( Orthe(C) \) is order closed, \( Orthe(C) \) is norm closed [9, Theorem 100.7]. This implies \( Orthe(C) \) Banach lattices, hence \( Orthe(C)^{\sim} = Orthe(C)' \) and so \( Orthe(C) \) has separating order dual. It is easy that, \((0, 1), (e, 0) \in Orthe(C)\) and \((0, 1) \perp (e, 0)\). On the other hand, we have
\[ (0, 1) * (e, 0) = (0e, 1f(e) + 0f(0) + 01) = (0, 1) \neq 0 \]
so that \( Orthe(C) \) is not an \( f \)-algebra. By the Corollary 2.13, \( Orthe(C) \) is not topologically full with respect to \( I_e \).

Since each \( f \)-algebra is commutative, we can give the following corollary.

**Corollary 2.15** Let \( C \) be an ordered algebra such that \( C_r \) is a Riesz space with the principal projection property and \( Orthe(C) \) is topologically full with respect to \( I_e \). Then, \( Orthe(C) \) is a commutative algebra.

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