Analytic continuation of harmonic sums with purely imaginary indices near the integer values

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Abstract

We present a simple algebraic method for the analytic continuation of harmonic sums with integer real or purely imaginary indices near negative and positive integers. We provide a MATHEMATICA code for exact expansion of harmonic sums in a small parameter near these integers. As an application, we consider the analytic continuation of the anomalous dimension of twist-1 operators in ABJM model, which contains the nested harmonic sums with purely imaginary indices. We found that in the BFKL-like limit the result has the same single-logarithmic behavior as in $\mathcal{N} = 4$ SYM and QCD, however, we did not find a general expression for the “BFKL Pomeron” eigenvalue in this model. For the slope function, we found full agreement with the expansion of the known general result and give predictions for the first three perturbative terms in the expansion of the next-to-slope function. The proposed method of analytic continuation can also be used for other generalization of the nested harmonic sums.

1 Introduction

The analytic continuation of the harmonic sums plays an important role in the study of a deep-inelastic process in the framework of the operator product expansion, where the
nested harmonic sums enter into expressions for the coefficient functions and anomalous
dimensions.

The harmonic sums defined recurrently as

\[ S_{a_1}(N) = \sum_{i_1=1}^{N} \frac{(\text{sign}(a_1))^{i_1}}{i_1^{a_1}} , \quad S_{a_1,a_2,a_3,...}(N) = \sum_{i_1=1}^{N} \frac{(\text{sign}(a_1))^{i_1}}{i_1^{a_1}} S_{a_2,a_3,...}(i_1) , \quad (1) \]

for a positive integer \( N \) can be extended to a complex argument in the same way that
the simplest harmonic sum \( S_1 \) is related to the digamma function \( \Psi \)

\[ S_1(N) = \sum_{i=1}^{N} \frac{1}{i} = \sum_{i=1}^{\infty} \frac{1}{i} - \sum_{i=N+1}^{\infty} \frac{1}{i} = \sum_{i=1}^{\infty} \frac{1}{i} - \sum_{i=1}^{\infty} \frac{1}{i+N} \]
\[ = \sum_{i=0}^{\infty} \frac{1}{i+1} - \sum_{i=0}^{\infty} \frac{1}{i+N+1} = \Psi(N+1) - \Psi(1), \quad (2) \]

where \( \Psi(z) \) is defined as the logarithmic derivative of the gamma function

\[ \Psi(z) = \frac{d}{dz} \ln\left(\Gamma(z)\right) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (3) \]

which has the single poles at \( z = 0, -1, -2, \ldots \). Such a procedure is usually called the
analytic continuation and is described in details in Refs. \[2, 3\]. A general property of the
analytically continued harmonic sum is the presence of poles in negative integer values of
its argument. These poles are related to the evaluation equations for parton distributions,
mostly to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation \[4, 5, 6\] and the double-
logarithmic equation \[7, 8\]. Knowledge of the analytic continuation of nested harmonic
sums made it possible to find higher order corrections to the BFKL equation \[9, 10, 11\]
and to the double-logarithmic equation \[12, 13, 14\].

Analytic continuation of nested harmonic sums can be performed directly by following
Refs. \[2, 3\], as was done to compute the BFKL Pomeron eigenvalue in third order in Ref. \[9\]
or as was used in Ref. \[15\] to effective calculate the values of the nested harmonic sums
for any complex argument (except negative integers).

Following another approach from Ref. \[16\], one can consider mappings the nested
harmonic sums into their related Mellin transform functions and, using the relations
between harmonic sums with a given weight, express them all in terms of several such
basis functions that are analytic at \( x = 1 \). Performing an asymptotic expansion of the
basis functions for \( N \to \infty \) it is possible to obtain the numerical result for any complex
value of \( N \).

In our previous work \[17\], we proposed a more effective way to find the expansion of
the nested harmonic sums near the negative integers by extractinf \( \ln x \)-terms in the inverse
Mellin transform for the corresponding harmonic sums using \texttt{summer} \[1\] and \texttt{harmpol} \[18\].
packages for FORM [19, 20, 21] along with database [22]. The obtained database for analytic continuation of nested harmonic sums near negative and positive integers allowed us to generalised the double-logarithmic equation [14]. Knowing the pole expressions for the nested harmonic sums, one can use the dispersion representation [23] to obtain the value for any complex argument.

In this paper, we present a simple algebraic method that can be used for nested harmonic sums with real and purely imaginary indices. Such sums appear in the results for the anomalous dimension of composite operators in ABJM model [24, 25] and during the computations of the BFKL Pomeron eigenvalue in $\mathcal{N} = 4$ SYM theory in the fourth order [11]. The proposed method also works for generalisations of usual harmonic sums such as the so-called SSum [26] or cyclotomic sums [27].

## 2 Analytic continuation

We start our consideration with harmonic sum with one index $S_a$, which can be positive, negative or purely imaginary

$$S_a(N) = \sum_{i=1}^{N} \frac{[a]_i}{i!} \rightarrow \sum_{i=1}^{\infty} \frac{[a]_i}{i!} - \sum_{i=1}^{\infty} \frac{[a]_{i+N}}{(i+N)!}, \quad [a] \equiv \text{sign}(a) = \frac{a}{|a|}$$  \hspace{1cm} (4)

where, for brevity, we used square brackets to denote the sign of index $a$. The above expression is correct for positive integers $N$. To make this expression correct for any complex $N$, we must leave $N$ only in the denominator, while in the numerator $N$, which we will denote as $N_0$ from now, is related to the initial positive integer value and gives in the case of usual alternating harmonic sums with the real indices [2, 3] a common plus/minus sign for even/odd $N_0$ for the second term in Eq. (4). The following expression

$$S_a(N, N_0) = \sum_{i=1}^{\infty} \frac{[a]_i}{i!} - [a]^{N_0} \sum_{i=1}^{\infty} \frac{[a]_i}{(i+N)!}$$  \hspace{1cm} (5)

is correct now for any complex $N$.

The last term near the negative integer $N = -r + \omega$ is rewritten as follows:

$$\sum_{i=1}^{\infty} \frac{[a]_{i+N}}{(i-r+\omega)!} = \sum_{i=1}^{r-1} \frac{[a]_{i+N}}{(i-r+\omega)!} + \frac{[a]^{r+N}}{\omega!} + \sum_{i=r+1}^{\infty} \frac{[a]_{i+N}}{(i-r+\omega)!}.$$

(6)

The first term is related to harmonic sums (actually to Euler-Zagier sums, see below) due to the following property for integers $r$ and real or purely imaginary $a$:

$$\sum_{i=1}^{r-1} \frac{[a]_i}{(i-r)!} = (-1)^[a] \times \sum_{i=1}^{r} \frac{[a]_i}{(i-r)!}$$  \hspace{1cm} (7)

The first term is related to harmonic sums (actually to Euler-Zagier sums, see below) due to the following property for integers $r$ and real or purely imaginary $a$:
where \( a^* \) is complex conjugation. When expanding Eq. (6) in \( \omega \), the series will be sign-
alternating, but for the first term, the sign will be compensated due to Eq. (7).

The general formula for the \( \omega \)-expansion near negative \( N = -r + \omega \) from even or odd
\( N_0 \) looks like:

\[
S_a(N, N_0) = \zeta_a - [a]^N \sum_{k=0}^{\infty} \left( \sum_{i=1}^{k} (i^* + k - 1) \left( S_{a+i^*} + (-1)^k \zeta_a \right) \omega^k \right) \frac{\omega^k}{k!}.
\]

For the expansion near positive \( N \) we obtain directly from Eq. (5) returning back to \( S_a \)

\[
S_a(N, N_0) = \zeta_a - \frac{[a]_{N_0}}{[a]^N} \sum_{k=0}^{\infty} (-1)^k \prod_{i=1}^{k} (i^* + k - 1) \left( -S_{a+i^*} + \zeta_a \right) \omega^k \frac{\omega^k}{k!}.
\]

For the harmonic sums with two indices, generalizing Eq. (4), we obtain

\[
S_{a_1,a_2}(N, N_0) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{[a_1]^{k+N_0}}{(k+N)^{a_1}} \frac{[a_2]^{j+k+N_0}}{(j+k+N)^{a_2}} \frac{[a_1]^{k+N_0}}{(k+N)^{a_1}} \frac{[a_2]^{j}}{(j+k+N)^{a_2}} \frac{[a_1]^{k}}{k^{a_1}} \frac{[a_2]^{j}}{j^{a_2}}
\]

\[
- \frac{[a_1]^{k}}{k^{a_1}} \frac{[a_2]^{j+k}}{(j+k+N)^{a_2}} \frac{[a_1]^{k}}{k^{a_1}} \frac{[a_2]^{j}}{j^{a_2}} \sum_{k=1}^{\infty} \frac{[a_1]^{k+N_0}}{(k+N)^{a_1}} - \zeta_{a_1} \sum_{k=1}^{\infty} \frac{[a_1]^{k+N_0}}{(k+N)^{a_1}} - \zeta_{a_2,a_1} + \zeta_{a_1} \zeta_{a_2}.
\]

First term in the above equation, which we will call the generalised \( \Psi \)-function following
Ref. [3], can be rewritten near negative integers \( N = -r + \omega \) as:

\[
\frac{\Psi_{a_1,a_2}(r, N_0)}{([a_1][a_2])_{N_0}} = \sum_{i_1=1}^{\infty} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} \sum_{i_2=1}^{\infty} \frac{[a_2]^{i_1+i_2}}{(i_2 - r + \omega)^{a_2}} =
\]

\[
= \left( \sum_{i_1=1}^{r-1} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} + \frac{[a_1]^r}{\omega^{a_1}} \right) \sum_{i_1=r+1}^{\infty} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} \times
\]

\[
\times \left( \sum_{i_2=1}^{r-1-i_1} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}} + \frac{[a_2]^r}{\omega^{a_2}} \right) \sum_{i_2=r-i_1+1}^{\infty} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}}
\]

\[
= \frac{[a_2]^r}{\omega^{a_2}} \sum_{i_1=1}^{r-1} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} \sum_{i_2=1}^{r-1-i_1} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}}
\]

\[
+ \sum_{i_1=1}^{r-1} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} \sum_{i_2=1}^{r-1-i_1} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}}
\]

\[
+ \sum_{i_1=1}^{r-1} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{a_1}} \sum_{i_2=1}^{\infty} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}} + \sum_{i_2=1}^{\infty} \frac{[a_2]^{i_1+i_2}}{(i_1+i_2 - r + \omega)^{a_2}}
\]
and the following relation between the vectors \( \vec{N} \) near positive integers we can use the existing database for the harmonic polylogarithms at unity with harmonic polylogarithms \([18]\) or with multiple polylogarithms \([28]\). In the first case, combination of \( \Psi_{\vec{a}} \) this is a general properties due to transformation from the usual harmonic sum to the harmonic sum with one index or, in general, with an odd number of in dices and we define as

\[
\tilde{\omega}_{\vec{a},1,2} = \sum_{r=1}^{\infty} \frac{[a_1][a_2][a_3] \ldots [a_n]}{(r+\omega)^{|a_1|} |a_2| |a_3| \ldots |a_n|} \cdot (-1)^{|a_1|} \zeta_{\vec{a},1,2} + \ldots
\]

(12)

where \( \tilde{\omega}_{\vec{a}} \) and \( \zeta_{\vec{a}} \) denote the \( \omega \)-expansion, which generalize Eq. \([8]\)

\[
\tilde{\omega}_{\vec{a},1,2} = \zeta_{\vec{a},1,2} + \sum_{r=1}^{\infty} \frac{\tilde{\omega}_{\vec{a},1,2} \omega + \frac{1}{2} \tilde{\omega}_{\vec{a},1,2} \omega^2 + \frac{1}{3!} \tilde{\omega}_{\vec{a},1,2} \omega^3 + \frac{1}{4!} \tilde{\omega}_{\vec{a},1,2} \omega^4 + \ldots}{\prod_{i=1}^{r-1} (i_1 - r + \omega)^{|a_1|} (i_2 - r + \omega)^{|a_2|}} \]

(13)

that is, a Tailor expansion with a formal “differentiation” with respect to indices, which we define as

\[
\tilde{F}_{\vec{a},1,2,\ldots,n} = [a_1]F_{a_1+\vec{a},1,2,\ldots,n} + [a_2]F_{a_1+a_2+\vec{a},1,2,\ldots,n} + \ldots + [a_n]F_{a_1,a_2,\ldots,a_n+\vec{a}}
\]

(15)

Note, that for the expansion of the Euler-Zagier sums \( \tilde{\omega}_{\vec{a}} \) the series is not sign-alternating, as for \( \zeta_{\vec{a}} \), due to Eq. \([7]\). In Eq. \([11]\) we used the following properties for the term with finite summations

\[
\sum_{i_1=1}^{r-1} \frac{[a_1]^{i_1}}{(i_1 - r + \omega)^{|a_1|}} \sum_{i_2=1}^{r-1-i_1} \frac{[a_2]^{i_1+i_2}}{(i_2 - r + \omega)^{|a_2|}} = [a_1]^{r}[a_2]^{r}(-1)^{|a_1|}(-1)^{|a_2|} \zeta_{\vec{a},1,2}
\]

(16)

For the expansion near positive \( N \), we obtain, by generalizing Eq. \([9]\), for the first term in Eq. \([11]\)

\[
\tilde{\zeta}_{\vec{a},1,2} \equiv \zeta_{\vec{a},1,2} + \tilde{\omega}_{\vec{a},1,2} \omega - \frac{[a_1]^{N_0}}{(a_1)^N} \left( -\tilde{\omega}_{\vec{a},1,2} - \tilde{\omega}_{\vec{a},1,2} \omega \right) \zeta_{\vec{a},2}
\]

\[
+ \frac{([a_1][a_2])[N_0]}{([a_1][a_2])^N} \left( -\tilde{\omega}_{\vec{a},1,2} - \tilde{\omega}_{\vec{a},1,2} \omega \right) \zeta_{\vec{a},2,\vec{a}}
\]

(17)

where the last term without common prefactor is the analytic continuation for \( \Psi_{\vec{a}}(N, N_0) \) near positive integers \( N \). It can be seen that there is a minus sign in the brackets before the harmonic sum with one index or, in general, with an odd number of indices and this is a general properties due to transformation from the usual harmonic sum to the combination of \( \Psi_{\vec{a}} \) and \( \zeta_{\vec{a}} \) and vice versa.

The multiple zeta values \( \zeta_{\vec{a}} \) can be reduced to several basis MZV’s using their relations with harmonic polylogarithms \([18]\) or with multiple polylogarithms \([28]\). In the first case, we can use the existing database for the harmonic polylogarithms at unity \( H_{\vec{c}}(1) \) \([22]\) and the following relation between the vectors \( \vec{a} \) and \( \vec{c} \) in \( \zeta_{\vec{a}} \) and \( H_{\vec{c}}(1) \)

\[
\{a_1, a_2, a_3, \ldots, a_n\} \rightarrow \{a_1 \prod_{i=1}^{1}[a_i], a_2 \prod_{i=1}^{2}[a_i], a_3 \prod_{i=1}^{3}[a_i], \ldots, a_n \prod_{i=1}^{n}[a_i]\}
\]

(18)
and multiplying $H_C(1)$ by the additional sign-factor

$$\zeta \bar{a} = \text{Li}_{|\bar{a}|}(\bar{a}) = \prod_{i=1}^{N}[a_i] H_C(1) \quad (19)$$

However, the database \[22\] only contains results for real indices (positive and negative integers). In the case of purely imaginary indices, it was necessary to obtain such a database, which was done in [29], where we computed the numerical values of multiple polylogarithms with high accuracy using the GiNaC \[30\] implementation of their numerical evaluation \[31\] and found the relationship between multiple polylogarithms with several basis ones using PSLQ method \[32\].

One can numerically check the agreement between the first term in Eq. (11) and Eqs. (12) and (17) using the program described in the next Section.

In general, the analytic continuation can be subdivided into two steps. First of all, we should move from the usual nested harmonic sums $S_{\vec{a}}$ to the combination of the $\Psi_{\vec{a}}$ and $\zeta_{\vec{a}}$. Then we perform the expansion of $\Psi_{\vec{a}}$ as in Eq. (12) or in Eq. (17).

In the case of three indices, the generalization of Eq. (14) for $N = -r + \omega$ looks like

$$\tilde{\Psi}_{a_1,a_2,a_3}(r, N_0) = \frac{\Psi_{a_1,a_2,a_3}(r, N_0)}{\prod_{i=1}^{3}[a_i]^r} = \tilde{Z}_{a_1,a_2,a_3} + \tilde{Z}_{a_1,a_2} \tilde{\zeta}_{a_3}
+ \tilde{Z}_{a_1} \tilde{\zeta}_{a_2,a_3} + \tilde{Z}_{a_1,a_2} \frac{[a_3]}{\omega[a_3]} \tilde{\zeta}_{\omega} + \tilde{Z}_{a_1} \frac{[a_2]}{\omega[a_2]} \tilde{\zeta}_{\omega} + \frac{[a_1]}{\omega[a_1]} \tilde{\zeta}_{a_1,a_2,a_3} \quad (20)$$

with replacement

$$\tilde{Z}_{\bar{a}} = (-1)^{w_{\bar{a}}} \tilde{Z}_{\bar{a}}, \quad (21)$$

where $w_{\bar{a}} = \sum_{i=1}^{3}|a_i|$ is the weight of Euler-Zagier sum $Z_{\bar{a}}$. While the generalization of Eq. (17) for positive $N$ is the following

$$\Psi_{a_1,a_2,a_3}(N, N_0) = \prod_{i=1}^{3}[a_i]^{-N_0} \prod_{i=1}^{3}[a_i]^{-N} \left(- \tilde{S}_{a_1,a_2,a_3} + \tilde{S}_{a_1,a_2} \tilde{\zeta}_{a_3} - \tilde{S}_{a_1} \tilde{\zeta}_{a_2,a_3} + \tilde{\zeta}_{a_1,a_2,a_3} \right) \quad (22)$$

In general case, the corresponding expansions can be written as

$$\tilde{\Psi}_{a_1,a_2,a_3,\ldots,a_n} = \tilde{Z}_{a_1,a_2,a_3,\ldots,a_n} + \tilde{Z}_{a_1} \tilde{\zeta}_{a_2,a_3,\ldots,a_n} + \tilde{Z}_{a_2} \tilde{\zeta}_{a_1,a_3,\ldots,a_n} + \cdots + \tilde{\zeta}_{a_1,a_2,a_3,\ldots,a_n} + \frac{1}{\omega[a_1]} \tilde{\zeta}_{a_2,a_3,\ldots,a_n} + \frac{1}{\omega[a_2]} \tilde{\zeta}_{a_1,a_3,\ldots,a_n} + \cdots + \frac{1}{\omega[a_{n-1}]} \tilde{\zeta}_{a_1,a_2,a_3,\ldots,a_{n-2}} + \frac{1}{\omega[a_n]} \tilde{\zeta}_{a_1,a_2,a_3,\ldots,a_{n-1}} \quad (23)$$

and

$$\Psi_{a_1,a_2,a_3,\ldots,a_n}(N, N_0) = \prod_{i=1}^{n}[a_i]^{-N_0} \prod_{i=1}^{n}[a_i]^{-N} \left(- \tilde{S}_{a_1,a_2,a_3,\ldots,a_n} + \tilde{S}_{a_1,a_2} \tilde{\zeta}_{a_3,\ldots,a_n} + \tilde{S}_{a_1,a_3} \tilde{\zeta}_{a_2,\ldots,a_n} + \tilde{S}_{a_1} \tilde{\zeta}_{a_2,a_3,\ldots,a_n} \right)$$

\[\text{We used MATHEMATICA function FindIntegerNullVector for lowest weights and code [33] for higher weights up to weight 7. In the forthcoming paper [29] we will extend this database up to weight 8.}\]
\[ + \tilde{S}_{a_1,a_2} \zeta_{a_n,a_3} + \cdots + \tilde{S}_{a_1,a_2,a_3} \]  

(24)

with replacement

\[ \tilde{S}_{\vec{a}} = (-1)^{\ell_{\vec{a}}} \tilde{S}_{\vec{a}}, \]  

(25)

where \( \ell_{\vec{a}} = a_1,a_2,...,a_n = n \) is the number of indices in nested harmonic sum \( S_{\vec{a}} \) or \( \Psi_{\vec{a}} \).

So, we have the rather simple and easy-to-program algebraic method for the analytic continuation of the nested harmonic sums, and in the next Section we provide the code that implements the proposed method.

3 Implementation in MATHEMATICA

We implemented the above described method in MATHEMATICA. All functions discussed in this Section are collected in ACHSI.m file\(^2\) and it must be preloaded in MATHEMATICA session as usual

<<"ACHSI.m";

adding the full path, if necessary.

In the first step, we convert the nested harmonic sum \( S_{\vec{a}} \) into an expression containing \( \Psi_{\vec{a}} \) and \( \zeta_{\vec{a}} \). This can be done using the function SToPsi:

\[
\text{SToPsi}[S_{-1,2i,-3}] = \zeta_{-3,2i,-1} - \Psi_{-1,2i,-3} + \zeta_{-3} \Psi_{-1,2i} + \zeta_{-3} \zeta_{2i,-1} + \zeta_{-1} \zeta_{-3,2i} + \Psi_{-1} \zeta_{-3,2i} - \zeta_{-1} \zeta_{-3,2i}.
\]  

(26)

The analytic continuation of \( \Psi_{\vec{a}} \) can be performed with the function PsiAC for the \( \omega \)-expansion near negative and positive integers, which has four arguments: \( \Psi_{\vec{a}} \), final value \( N \), initial value \( N_0 \) and the order of \( \omega \)-expansion. Since the analytic continuation contains \( \zeta_{\vec{a}} \) one should load preliminary the database for the relations between \( \zeta_{\vec{a}} \) or their associated multiple polylogarithms

\[ \zeta_{\vec{a}} = \text{Li}_{|\vec{a}|}([\vec{a}]), \]  

(27)

and we use the standard definition of the multiple polylogarithms

\[ \text{Li}_{m_1,...,m_k}(x_1,\ldots,x_k) = \sum_{i_1,i_2,...,i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}. \]  

(28)

We have such database for all \( \zeta_{\vec{a}} \) with positive, negative and purely imaginary indices (or for multiple polylogarithms with positive, negative and purely imaginary arguments) up\(^2\)

\footnote{This code is available as ancillary files of arXiv-version of this paper and on GitHub https://github.com/vitvel/ACHSI}
to weight 6 and with positive, negative and only positive (or only negative related with positive by complex conjugation) purely imaginary indices for weight 7. The precomputed database should be loaded inside a Mathematica session prior to using the PsiAC function with the following code up to weight 6

\begin{verbatim}
DLiSubsRe = Get["LiSubsRew1w6.m"]//Dispatch;
DLiSubsIm = Get["LiSubsImw1w6.m"]//Dispatch;
\end{verbatim}

or up to weight 7

\begin{verbatim}
DLiSubsRe = Join[Get["LiSubsRew1w6.m"],Get["LiSubsRew7.m"]]/\text{Dispatch};
DLiSubsIm = Join[Get["LiSubsImw1w6.m"],Get["LiSubsImw7.m"]]/\text{Dispatch};
\end{verbatim}

adding the full path, if necessary.

However, \(\zeta_{\vec{a}}\) starting with index equal to 1, \(\zeta_{1,\vec{a}}\), diverge and require separate consideration. For this purpose, we use the so-called stuffle relation for multiple polylogarithms, which can be written in the case of two indices as

\begin{equation}
\text{Li}_{i_1}(x_1)\text{Li}_{i_2}(x_2) = \text{Li}_{i_1,i_2}(x_1, x_2) + \text{Li}_{i_2,i_1}(x_2, x_1) + \text{Li}_{i_1+i_2}(x_1 x_2),
\end{equation}

in the case of three indices as

\begin{equation}
\text{Li}_{i_1}(x_1)\text{Li}_{i_2,i_3}(x_2, x_3) = \text{Li}_{i_1,i_2,i_3}(x_1, x_2, x_3) + \text{Li}_{i_2,i_1,i_3}(x_2, x_1, x_3) + \text{Li}_{i_2,i_3,i_1}(x_2, x_3, x_1) \\
+ \text{Li}_{i_1 + i_2, i_3}(x_1 x_2, x_3) + \text{Li}_{i_2,i_1 + i_3}(x_2, x_1 x_3)
\end{equation}

and can be easily generalized to the case of any number of indices. Such relations allow us to extract \(\text{Li}_1(1)\), which can be performed with the function \text{Li1Extract}

\begin{verbatim}
Li1Extract[Li[1, 1, 2 I]] =
1/2*Li1^2*Li[2 I] - Li1*Li[3 I] - Li1*Li[2 I, 1] + 1/2*Li[4 I] +
1/2*Li[2 I, 2] + Li[3 I, 1] - 1/2*Li[2, 2 I] + Li[2 I, 1, 1]
\end{verbatim}

and this function is contained within PsiAC function.

After loading the precomputed database, we can compute the \(\omega\)-expansion of the \(\Psi_{\vec{a}}\) with the function PsiAC as

\begin{verbatim}
PsiAC[S[2,-1,I],-3,1,0]
\end{verbatim}

\footnote{The results for the NNNLLA BFKL Pomeron eigenvalue contains harmonic sums only with one purely imaginary index.}
which produces the following general expression for the analytic continuation of $\Psi_{2,-1,i}$ from odd positive $(N_0 \pmod{4}) = 1$ to odd negative $(N \pmod{4}) = 1$ up to $\omega^0$

\[
\frac{1}{\omega^2} \left( \frac{\pi^2}{32} + \frac{\ln^2}{8} + i \left( G - \frac{3}{8} \pi \ln 2 \right) \right) + \frac{1}{\omega} \left( S_{-3} - S_{2,-1} - \frac{1}{96} \pi^2 \ln 2 \right) \\
\frac{27}{64} \zeta_3 - \frac{1}{2} S_2 \ln 2 + i \left( \frac{3}{4} G \ln 2 - \frac{13}{384} \pi^3 - \frac{3}{2} \Im \text{Li}_{1,2} + \frac{1}{4} \pi S_2 \right) \\
-S_{2,2i} - 2S_{3,-1} + S_{2,-1,-i} + 3S_{-4} + S_{4i} - S_{-3,-i} - S_{2,-2} + \frac{1}{2} G^2 \\
+\frac{3}{16} G \pi \ln 2 - \frac{1}{384} \pi^2 \ln 2^2 + \frac{253}{46080} \pi^4 - \frac{5}{4} \zeta_3 \ln 2 - \frac{3}{8} \pi \Im \text{Li}_{1,2} \\
+\frac{45}{32} \text{Li}_{3,1} - \frac{1}{2} S_{-3} \ln 2 + \frac{1}{8} S_2 \ln 2^2 + \frac{5}{96} S_2^2 - S_3 \ln 2 + \frac{1}{2} S_{2,-1} \ln 2 \\
i \left( \frac{37}{144} G \pi^2 - \frac{1}{6} G \ln 2^2 - \frac{29}{576} \pi^3 \ln 2 - \frac{47}{384} \pi \zeta_3 - \frac{4}{3} \Im \text{Li}_{4i} \right) \ln 2 \\
+\frac{1}{3} \Im \text{Li}_{1,2} + \frac{5}{3} \Im \text{Li}_{2,1,i} + \frac{1}{4} S_{-3} \pi - \frac{3}{8} S_2 \pi \ln 2 + \frac{1}{2} S_3 \pi - \frac{1}{4} S_{2,-1} \pi \right)
\] (31)

where we converted the Euler-Zagier sums to usual harmonic sums by calling the function \texttt{ZSToHS} inside \texttt{PsiAC} and in the above expression $S_{\vec{a}} = S_{\vec{a}}(|N| - 1)$, while $G$ is the Catalan constant. Note, that these are general properties of analytic continuation for harmonic sums with real and purely imaginary indices, that the same general results will be obtained for $(N_0 \pmod{4})$, for $(N \pmod{4})$ and, moreover, for $((N_0 + N) \pmod{4})$. To obtain the analytic continuation into $N = -3$, we need to put $S_{\vec{a}} = S_{\vec{a}}(|-3| - 1) = S_{\vec{a}}(2)$ and use substitution for nested harmonic sums defined in \texttt{ACHSI.m} file, e.g.

\texttt{S[2,-1,I] / S[a_,_] := HS[a, 2] = -1/16 - 9/8 I}

The analytic continuation of the nested harmonic sum can be performed with the function \texttt{HSAC}, which has the same arguments, as \texttt{PsiAC} function, and applying it to $S_{2,2i}$

\texttt{HSAC[S[2, 2 I], -3, 1, 1]}

we get the following expression

\[
\frac{S_2}{\omega^2} + \frac{3/16 \zeta_3 + 2S_3 - i \pi^3/16}{\omega} + 3S_4 - S_{-4i} + S_{2,-2i} - \frac{7}{3840} \pi^4 + 3i \Im \text{Li}_{4i} \\
+\omega \left[ 4S_5 + 2S_{2,-3i} + 2S_{3,-2i} - 4S_{-5i} + \frac{3}{16} \zeta_3 S_2 - 2\zeta_5 + \frac{1}{16} G \pi^3 + \frac{35}{768} \pi^2 \zeta_3 \right] \\
i \left( 2G \zeta_3 - \frac{1}{16} \pi^3 S_2 - \frac{7}{512} \pi^5 + \ln 2 \Im \text{Li}_{4i} + 4\Im \text{Li}_{4i,1} - 23\Im \text{Li}_{4,i} \right)
\] (32)

where again $S_{\vec{a}} = S_{\vec{a}}(|N| - 1)$.

When evaluating an expression involving several nested harmonic sums, as in real anomalous dimension, it is better to use substitution.
S[2, 2 I] /. S[a__] :> HSAC[S[a], -3, 1, 1]

with the same output as in Eq. (32).

For the numerical evaluation of the obtained results we provide the numerical values for the basis of MZV’s with an accuracy of 50 digits in the form of substitution rules LiNSubs and it can be used as

\[
\text{HSAC}[S[2, 2 I], -3, 1, 1] / . \text{LiNSubs} / . S[a__] :> \text{HS}[a, \text{Abs}[-3]-1] / \text{N}[#, 5] &
\]

with the output
\[
\frac{1.25000 + 0.10^{-6} i}{\omega^2} + \frac{2.4754 - 1.9379 i}{\omega} + (3.0099 + 2.7168 i) + (4.6486 - 4.6926 i) \omega. \quad (33)
\]

In particular, one can numerically check the agreement between the first term in Eq. (11) and Eqs. (12) and (17).

4 Application

In this Section, we provide several examples of using HSAC function to study the relevant quantities in various models. The main motivation for studying analytic continuation with the real and purely imaginary indices was the computation of the BFKL eigenvalue in the forth order [11], using the Quantum Spectral Curve (QSC) method [34, 35, 9]. During the computations we found an inconsistency when using the usual nested harmonic sums. We extended our consideration to include nested harmonic sums with purely imaginary indices, since similar objects appeared in the results for the anomalous dimensions of composite operators in ABJM model [36] calculated by the same QSC method in Refs. [24, 25]. We found that considering nested harmonic sums (more precisely, functions Ψ⃗ a) with the last purely imaginary index will considerably simplify the final result and resolve the initial inconsistency.

Another example where nested harmonic sums with real and purely imaginary indices appear is the above-mentioned anomalous dimension of composite operators in ABJM model. The available results up to third order are expressed in terms of the following nested harmonic sums [24, 25]

\[
H_a(N) = \sum_{k=1}^{N} \frac{\text{Re}(a_k)}{k|a|}, \quad H_{a_1, a_2, \ldots, a_n}(N) = \sum_{k=1}^{N} \frac{\text{Re}(a_1 k)}{k|a_1|} H_{a_2, \ldots, a_n}(k). \quad (34)
\]

with negative and positive real integer and only positive purely imaginary integer indices. To return to the usual definition of nested harmonic sums, we must also consider the
negative purely imaginary indices and replace each $i$ in $H_{\vec{a}}$ by the sum of $S_{\vec{a}}$ with $\pm i$ like

$$H_i = \frac{1}{2} \left( S_i + S_{-i} \right)$$

(35)

Inside **MATHEMATICA** such a transformation can be done with the following code

```mathematica
H[I,2I,3I]/.H[a__]:>HS[a]//.HS[a___,b_/;Sign[b]==I, c___]:>
(HS[a,Abs[b]II,c]+HS[a,-Abs[b]II, c])/2/.II->I/.HS[a__]:>S[a]//Expand
```

The analytic continuation of the anomalous dimension for twist-1 operators in ABJM model\(^4\) from $N_0 = 2 \cdot 2$ to $N = -2 \cdot 1$, which in $\mathcal{N} = 4$ SYM theory is related to the BFKL equation, looks like

$$\gamma_{\text{ABJM}}^{N=-1,N_0=4} = h^2 \left( -\frac{16}{\omega} + 8 \ln 2 + \frac{1}{6} \pi^2 \omega - \frac{1}{4} \omega^2 \zeta_3 + \frac{1}{1440} \pi^4 \omega^3 - \frac{1}{64} \omega^4 \zeta_5 \right)$$

$$+ h^4 \left( - \frac{128 \ln 2}{\omega^2} + \frac{128 \ln 2^2 + 24 \pi^2}{\omega} - \frac{64}{3} \ln 2^3 - 24 \pi^2 \ln 2 - 120 \zeta_3 \right.$$  

$$+ \omega \left( \frac{20}{3} \pi^2 \ln 2^2 + \frac{259}{180} \pi^4 - 48 \text{Li}_{-3,1} + 156 \ln 2 \zeta_3 \right)$$

$$+ \omega^2 \left( - \frac{359}{180} \pi^4 \ln 2 - \frac{29}{12} \pi^2 \zeta_3 - \frac{893}{8} \zeta_5 + 32 \text{Li}_{-3,1} \ln 2 + 16 \text{Li}_{-3,1,1} - 54 \zeta_3 \ln 2^2 \right) \right)$$

$$+ h^6 \left( - \frac{1024}{3} \pi^2 - 2048 \ln 2^2 \right. + \frac{8192}{3} \pi^2 \ln 2 + 4224 \zeta_3 \right.$$  

$$+ \omega \left( - \frac{5704}{45} \pi^4 + 256 \text{Li}_{-3,1} - 1024 \ln 2^4 - 1152 \pi^2 \ln 2^2 - 9984 \zeta_3 \ln 2 \right)$$

$$+ \frac{512}{5} \ln 5^2 + \frac{496}{9} \pi^2 \ln 2^3 + \frac{2128}{9} \pi^4 \ln 2 + \frac{1598}{3} \pi^2 \zeta_3 + \frac{18295}{2} \zeta_5$$

$$- 1664 \text{Li}_{-3,1} \ln 2 + 9504 \zeta_3 \ln 2^2 \right),$$

(36)

where $h$ is the effective ABJM QSC coupling constant\(^5\). This result is similar to the expansion of the universal anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory, since it also has $(a/\omega)^\ell$ behavior (or single logarithms $(a \ln x)^\ell$ after inverse Mellin transform). In $\mathcal{N} = 4$ SYM theory we have for the analytic continuation of the universal anomalous dimension near $N = -1 + \omega$ we have (and similar in QCD for the gluon anomalous dimension near $j = 1 + \omega$)

$$\gamma_{\text{uni}}^{N=-1+\omega} = 2 \left( -\frac{4g^2}{\omega} \right) - 0 \left( -\frac{4g^2}{\omega} \right)^2 + 0 \left( -\frac{4g^2}{\omega} \right)^3 - 4 \zeta_3 \left( -\frac{4g^2}{\omega} \right)^4 + \cdots.$$  

(37)

\(^4\)In $\mathcal{N} = 4$ SYM theory, we perform analytic continuation for the anomalous dimension of twist-2 operators from $N_0 = 2$ to $N = -1$ to obtain results related to the BFKL equation. For the anomalous dimension of twist-1 operators in ABJM model we have to multiply both values by 2, because it depends on nested harmonic sums with double argument, $\gamma_{\text{ABJM}}(S) = \sum c_\ell H_{k\ell}(2S)$.

\(^5\)Coupling constant $h$ is a nontrivial function of 't Hooft coupling constant $\lambda$, see Refs. [37, 38, 39, 40].
This result is reproduced from the eigenvalue of the BFKL kernel in the leading-logarithm approximation
\[
\frac{\omega}{-4g^2} = \Psi \left( -\frac{\gamma}{2} \right) + \Psi \left( 1 + \frac{\gamma}{2} \right) - 2\Psi(1) \tag{38}
\]
after resolving in \( \gamma = \gamma(\omega) \) the following expansion of the above equation
\[
\frac{\omega}{-4g^2} = \frac{2}{\gamma} - 2 \sum_{k=1}^{\infty} \left( \frac{\gamma}{2} \right)^k \zeta(2k + 1) \tag{39}
\]
The same expression (39) can be obtained from the analytic continuation of the simplest harmonic sum near \( N = -1 - \omega/2 \) plus near \( N = 0 + \omega/2 \), because the analytic continuation of \( S_1(N) \) is equal to \( \Psi(N + 1) - \Psi(1) \) as in Eq. (2).

The leading poles in the analytically continued anomalous dimension of twist-1 operators in ABJM model (36) look like in the case of \( N = 4 \) SYM theory (37), but with other coefficients
\[
\gamma_{\text{ABJM}}^{N=1+\omega} = 2 \left( \frac{-8h^2}{\omega} \right) - 2 \ln 2 \left( \frac{-8h^2}{\omega} \right)^2 + 4 \left( \ln^2 + \frac{\pi^2}{6} \right) \left( \frac{-8h^2}{\omega} \right)^3 + \cdots. \tag{40}
\]
The term proportional to \( \ln 2 \) is contained in the analytic continuation of the sign-alternating analog of \( S_1 \), the harmonic sum \( S_{-1} \). Making the same combination as in Eq. (38), we get
\[
\frac{\omega}{-8h^2} = S_{-1}^{AC}(-1 - \gamma/2) + S_{-1}^{AC}(0 + \gamma/2) = \frac{2}{\gamma} + 2 \ln 2 + \frac{\pi^2}{12} \gamma + \frac{7\pi^4}{2880} \gamma^3 + \cdots \tag{41}
\]
and resolving it with respect to \( \gamma \), we obtain
\[
\gamma = 2 \left( \frac{-8h^2}{\omega} \right) + 4 \ln 2 \left( \frac{-8h^2}{\omega} \right)^2 + \left( 4 \ln^2 + \frac{\pi^2}{6} \right) \left( \frac{-8h^2}{\omega} \right)^3 + \cdots \tag{42}
\]
with the same structure as in Eq. (40), but with different coefficients and even with the opposite sign for the second term. From another side, if we resolve Eq. (40) with respect to \( \omega \), we get
\[
\frac{\omega}{-8h^2} = \frac{2}{\gamma} - \ln 2 + \left( \frac{\ln^2}{2} + \frac{\pi^2}{12} \right) \gamma + \cdots, \tag{43}
\]
which is the \( \gamma \) expansion for the analogue of the BFKL Pomeron eigenvalue. The high-energy scattering in 2 + 1-dimensional QCD and the BFKL Pomeron in such model are considered in Refs. [41, 42, 43, 44], but we did not find any results relevant to the above expressions.

For the double-logarithmic limit of \( N = 4 \) SYM theory, which corresponds to the analytical continuation of the anomalous dimension in ABJM model from \( N_0 = 2 \cdot 2 \) to \( N = -2 \cdot 2 \), we have not obtained any pole terms in the \( \omega \)-expansion, which means that there are no the double logarithms in ABJM model for twist-1 operators.
Another result that can be obtained by means of the analytic continuation is the ω-expansion of anomalous dimension near positive integers values of its argument. Such expansion corresponds to the so-called slope function, first considered in Refs. [45, 46] and computed analytically in Ref. [47] in $\mathcal{N} = 4$ SYM theory. For the three-loop anomalous dimension of twist-1 operators in ABJM model [24, 25], we found using analytic continuation from $N_0 = 0$ to $N = 0$

$$\gamma^{N_0=0}_{\text{ABJM}} = \omega \left( \frac{\pi^2}{2} h^2 - \frac{\pi^4}{4} h^4 + \frac{161\pi^6}{720} h^6 \right) + \omega^2 \left( -\frac{7\zeta_3}{4} h^2 + h^4 \left( -\frac{7\pi^2\zeta_3}{12} - \frac{\pi^4\ln 2}{4} + \frac{155\zeta_5}{8} \right) \right) + h^6 \left( \frac{\pi^6\ln 2}{48} + \frac{277\pi^4\zeta_3}{160} - \frac{341\pi^2\zeta_5}{24} - \frac{1333\zeta_7}{64} \right). \quad (44)$$

The first term of the above ω-expansion is in full agreement with the result from Ref. [40]. The second terms are predictions for the expansion of the next-to-slope function. In principle, it is possible to obtain such expansion for more terms in ω and for other values of $N$.

## 5 Conclusion

In this paper, we presented the algebraic method for the exact expansion of analytically continued nested harmonic sums with real and purely imaginary indices near integer values of their argument. Such nested harmonic sums with purely imaginary indices appeared earlier when calculating the anomalous dimension of the twist-1 operators in ABJM model [24, 25] and BFKL Pomeron eigenvalue in forth order [11]. In the last case, knowing the analytic continuation is the key point in the reconstruction of the general form of BFKL Pomeron eigenvalues, since QCS method [34, 35, 9] allows one to compute the pole expansion of the desired result, which have the nested harmonic sums in negative integers. However, the use of the usual harmonic sums with real indices caused a lot of difficulties and the resulting inconsistency, while considering, in addition to the usual sums, nested harmonic sums with purely imaginary indices, allowed to obtain the final result [11]. In the case of ABJM model, we can obtain information about the analytic properties of the anomalous dimension of twist-1 operators known up to the third order [24, 25]. We found that the expansion near negative integer related with the BFKL equation in $\mathcal{N} = 4$ SYM theory has the same single-logarithm form as in $\mathcal{N} = 4$ SYM theory and in QCD, but with substantially different coefficients. Moreover, using the proposed method, we computed the ω-expansion of the anomalous dimension of twist-1 operator in ABJM model near $N = 0 + \omega$ and found the full agreement with the perturbative expansion of the slope function from Ref. [40].

We provide the MATHEMATICA code that perform analytic continuation of nested har-

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6This code is available as ancillary files of arXiv-version of this paper and on GitHub https://github.com/vitvel/ACHSI
monic sums $S_\bar{a}$ or $\Psi_\bar{a}$ function with real and purely imaginary indices near negative and positive integers. The proposed method should also work for other generalisation of usual harmonic sums, such as the so-called SSum [26] or cyclotomic sums [27], but this would require a database for the relations between the corresponding generalization of the multiple zeta values or multiple polylogarithms.

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