Osmotic Properties of Charged Cylinders: Critical Evaluation of Counterion Condensation Theory

Per Lyngs Hansen, Rudi Podgornik §, V. Adrian Parsegian

Laboratory of Physical and Structural Biology
National Institute of Child Health and Human Development
National Institutes of Health, Bethesda, MD 20892-5626

§ Department of Physics
Faculty of Mathematics and Physics
University of Ljubljana, SI-1000 Ljubljana, Slovenia

The osmotic coefficient of B-DNA in water may, in moderately dilute solutions, deviate as much as 100 % from predictions based on a simple ‘counterion condensation’ theory. We determine the results for osmotic properties via a cell model description of the ionic atmosphere near a cylindrical polyelectrolyte. The cell model predictions for the osmotic properties disagree with predictions based on simple condensation theory, but are in surprisingly good harmony with experimental findings. We argue that the neglect of finite-radius effects makes simple condensation theory inapplicable at all but impractically low polyelectrolyte concentrations and, unable to reproduce osmotic properties of polyelectrolytes such as DNA.
Ionic screening of charge interactions remains one of the most vigorously discussed properties of polyelectrolyte solutions [1, 2]. Recently measured osmotic coefficients of B-DNA in dilute aqueous solutions [3], show factor-of-two deviations from predictions based on the influential counterion condensation theory of Oosawa and Manning [4, 5, 6]. We shall argue that an earlier theory pioneered by Lifson and Katchalsky [7, 8], based on the cell model formulation of the full nonlinear Poisson-Boltzmann equation, provides a more successful starting point than simple condensation theory for examining not only the osmotic properties of cylindrical polyelectrolytes in low salt conditions but also the local organization of the ionic atmosphere around them. By explicitly including the finite size of the cylinder, the cell model description fundamentally disagrees with the line charge picture of the simplest condensation theory. It is well known [4, 5] that in the absence of added salt, the simple condensation theory is a special limiting case of the cell model only when the cylinder radius is sent to zero or the cell radius is sent to infinity. We find here that neglect of finite radius is the main reason for the failure to describe DNA osmotic properties.

The osmotic pressure of a polyelectrolyte (e.g., DNA) in an aqueous solution is obtained from the equation of state that recognizes the degrees of freedom of both the counterion and the macroion. The Oosawa-Manning limit [1, 3] which decouples the counterion atmosphere from the density of the macroion can thus only make sense at effectively infinite dilution. Osmotic pressure experiments however are performed at finite macroion concentrations. In these experiments [3] the osmotic coefficient of DNA in 2 and 10 mM salt solutions is seen to be twice that predicted by the counterion condensation theory, while the Lifson - Katchalsky cell model accurately predicts its magnitude. The experimental osmotic coefficient varies weakly with the macroion concentration in the general direction predicted by the cell model.

The cell model considered here involves a rigid (hollow or solid), charged cylindrical polymer of radius $a$, coaxially enclosed in a cylindrical (Wigner-Seitz-like) cell of radius $R_0$, corresponding to the total system volume per polymer length, see Fig. 1. The cell acts as a neutralization volume for the counterions; consequently the electric field vanishes at the cell wall. Counterions organize within the cell according to the nonlinear Poisson-Boltzmann equation for the double layer electric potential, $u$. In units of $k_BT/e$, in the absence of added salt,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{du_*(r)}{dr} \right) = -\frac{\kappa_*^2}{2} e^{-u_*(r)} . \quad (1)$$

Here $\kappa_*^{-1}$ denotes a formal ‘screening length’ to be determined later by the average density of counterions. The subscript * stands for the electrostatic potential inside and outside of the cylinder The potentials in turn determine the charge densities,

$$n_*(r) = n_{*,0} e^{-u_*(r)} . \quad (2)$$

Here $n_{*,0} = \kappa_*^2/(8\pi l_{Bj})$, and $l_{Bj} = e^2/(\epsilon k_BT)$ is the Bjerrum length.

Because of the major and minor grooves in DNA [3], it is desirable to treat the cylinder at least as partly hollow and hence to give solvent and ions access to the space within the grooves. We shall compute here, for simplicity, the results for cylinders that are either solid or completely hollow to make the maximum possible ‘non-specific’ accumulation of countercharge near the
cylinder. Counterion accumulation is determined by the solutions to the PB equation for which the density variation across the boundary at \( r = a \) is continuous. \( u_o(r \geq a) \) and \( u_i(r \leq a) \) are both expressed with respect to a zero at the cell wall \( (r = R_0) \) and \( \kappa_i^2 = \kappa_o^2 = 8\pi l_{Bj} n(R_0) \), where \( n(R_0) \) is the density of counterions at the cell wall.

For \( a < r < R_0 \) solution of the Poisson - Boltzmann equation yields \[ 7 \]

\[
 u_o(r) = \ln \left( \frac{(\kappa r)^2}{2z} \cos^2(2 \ln \left( \frac{r}{R_m} \right)) \right),
\]

\[ 3 \] and for \( r < a \) one finds \[ 10 \]

\[
 u_i(r) = u_0 + 2 \ln \left( 1 + cr^2 \right).
\]

The integration constants \( z, R_m, \kappa, c, \) and \( u_0 \) are obtained from boundary conditions. In addition to the requirement of a continuous variation of the potential at \( r = a \), these conditions include \( du_i(r)/dr|_0 = 0 \), \( du_o/dr|_{R_0} = 0 \), and \( (du_o(r)/dr - du_i(r)/dr)|_{a} = 2Q/a \). For DNA the dimensionless linear charge density \( Q = l_{Bj}/l_{PO4} \approx 4.353 \) is determined by the charge separation \( l_{PO4} \approx 1.7\AA \) and the Bjerrum length \( l_{Bj} \approx 7.14\AA \) (as in water at room temperature). In practice one solves the problem of fixing the parameters by an iteration that starts with a trial partitioning of counterions inside and outside the cylinder that is subsequently refined until the boundary conditions are exactly satisfied.

The osmotic properties of charged cylinders are encoded in the osmotic pressure \( \pi_{osm} = k_B T n(R_0) \) on the cell wall, which, of course, also codifies the equation of state for the macroion at the density set by \( R_0 \). (Because the electric field vanishes at the cell wall, the only contribution to the Maxwell stress tensor comes from the osmotic pressure.) The osmotic coefficient \( \phi \) is defined as the ratio of the actual osmotic pressure to the osmotic pressure of a hypothetical gas of uniformly distributed counterions or, equivalently, the number density of ions \( n(R_0) \) divided by the total density \( n_{PO4} = 1/(l_{PO4} \pi R_0^2) \).

Fig. 1. shows the results of a simple numerical calculation of the osmotic coefficient of solutions of rigid hollow cylinders as a function of the molar concentration, when the radius of the cylinder \( a = 10\AA \), the dimensionless (nominal) line charge density \( Q = 4.353 \), and the molar weight \( M \) is chosen as for DNA.

(i) The calculated osmotic coefficients vary slightly with concentration. The measured osmotic coefficients \[ 2 \] vary even more slowly. However, simple condensation theory predicts no variation at all.

(ii) In the region of significant experimental interest, i.e., for concentrations of DNA phosphates from 0.1 to 0.5 M, the calculated osmotic coefficients are close to 0.3 in reasonably close agreement with the experimental value 0.24 \[ 3 \]. Simple condensation theory predicts \( \phi = 1/(2Q) \approx 0.11 \).

(iii) As expected in the limit \( a/R_0 \rightarrow 0 \), our formulated osmotic coefficients converge to the line - charge result \( \phi(a/R_0) \rightarrow 1/(2Q) \); it is the significant deviations that occur away from this infinite dilution limit that concern us.
Specifically, consider the asymptotic expression for the osmotic coefficient in the limit where $a/R_0$ is small (the line-charge limit, or the infinite-dilution limit): The outer solution with

$$Q_o = 1 - z \tan \left( z \ln \left( \frac{a}{R_m} \right) \right)$$

(5)

$$0 = 1 - z \tan \left( z \ln \left( \frac{R_0}{R_m} \right) \right)$$

(6)

$$\kappa^2 R_0^2 = 4(1 + z^2)$$

(7)

give us $z$, as follows:

$$\ln \left( \frac{a}{R_0} \right) = \arctan \left( \frac{1 - Q_o}{z} \right) - \arctan \left( \frac{1}{z} \right).$$

(8)

When $a/R_0$ is small, $z \to 0$, the arctan(·)'s may then be replaced by, respectively, $-\pi/2$ and $\pi/2$ so that $z$ has a weak logarithmic dependence on $a/R_0$:

$$z \simeq \frac{\pi}{\ln \left( \frac{R_0}{R} \right)}. $$

(9)

The corresponding asymptotic expression for the osmotic coefficient is

$$\phi = \frac{n(R_0)}{n_{PO4}} = \frac{\kappa^2}{8\pi l_{Bj}} R_0^2 l_{PO4} = \frac{1}{2Q_o} (1 + z^2) G(a/R_0),$$

(10)

where $G(a/R_0)$ is a geometric factor, which for hollow cylinders assumes the value 1. Therefore, for small values of $a/R_0$, Eq. (10) gives

$$\phi \simeq \frac{1}{2Q_o} \left( 1 + \frac{\pi^2}{\ln^2 \left( \frac{R_0}{a} \right)} \right). $$

(11)

In other words, the osmotic coefficient develops a weak logarithmic concentration dependence. (If the cylinders were not completely hollow an extra concentration dependence would appear at high concentrations. In this case the geometric factor $G(a/R_0)$ would deviate from unity.) Any concentration dependence of the osmotic coefficient is incompatible with simple condensation theory. Only as $a/R_0 \to 0$ does the osmotic coefficient approach a concentration-independent limit $1/(2Q_o)$.

Fig. 2 shows the osmotic pressure $\pi_{osm}$ versus the molar concentration $c_{DNA}$ of DNA or equivalently versus the radius of the cell $R_0$. Experimental data are available at low, $0.1-0.5$ M [3] and high concentrations, $1-2$ M [11]. In both ranges we find that calculated osmotic pressures are remarkably close to capturing the magnitude and variation of the measured pressures. In the low-concentration range and 2 mM salt the calculated pressures are slightly too large which might in part reflect a weak contribution from finite salt concentration that can be dealt with on the basis of a simple Donnan equilibrium picture [4, 12, 3]. This approach however fails completely at higher salts, e.g. 10 mM, where the osmotic pressure is substantially lower than
the values calculated from the Donnan equilibrium. Though one could make the calculated values of osmotic pressure in the intermediate regime of DNA densities to be even closer to the data, by choosing a somewhat smaller value for the DNA radius, it turns out that in this case one would lose the relatively good agreement in the regime of very large DNA concentrations. There appears to be no simple way of adjusting the DNA parameters to get a good quantitative fit of the calculated osmotic pressure with experiments. As pointed out many times before it appears yet again that non-electrostatic interactions at very large DNA concentrations make a significant contribution to the overall osmotic pressure in the system.

In the high-concentration range, the calculated variation of the pressure with concentration is clearly slower than what one observes for the experimental data. The (small) difference may be of non-electrostatic origin or reflect charge-discreteness effects discussed in Ref. : salt effects are likely to be unimportant in this concentration range. The renormalization effects due to chain conformational fluctuations, discussed in Ref., lead to predictions of decreasing rates of change of the pressure with concentration. Recent work on stretching of DNA at various ionic conditions also suggests that there might be an additional strong coupling between DNA elasticity and electrostatics. Though the details of this coupling are only beginning to be elucidated it is conceivable that local deformations of DNA would change the countercharge distributions at low salt conditions and thus effect also the osmotic coefficient.

To what extent do the above observations elucidate 'counterion condensation, 'simple' or 'extended' ?

In the 'simple condensation' picture highly charged and rigid cylindrical macromolecules are portrayed as line charges with explicit neglect of finite macromolecular radius, the atmosphere of counterions is divided into a condensed and osmotically inactive fraction which redefines the line charge density, and an 'unbound', osmotically active fraction. This 'two-phase coexistence' is established whenever the line-charge density (or, equivalently, the electric field at the surface of the macromolecule) becomes large enough, quantitatively whenever . For , condensation will bring down the effective (dimensionless) line-charge density from to ; a fraction , precisely the fraction of charges that are predicted to condense according to the Oosawa - Manning model. Using Eqs. in the small limit, the Manning radius disappears; the effects of the fraction of ionic atmosphere can be absorbed in a redefinition of the line charge, as in 'simple' condensation theory. However when
the Manning radius $R_M$ is finite and even diverges as one approaches the infinite-dilution limit $R_0 \rightarrow \infty$.

Is it possible to reformulate or 'extend' condensation theory to account for finite macro-molecular radius while retaining the essential idea of two-phase coexistence between free and bound fractions? If not, the organization of the counterion atmosphere around cylindrical polyelectrolytes and the electrostatic contribution to solution properties of polyelectrolytes, must be understood on the basis of a Poisson-Boltzmann or a more advanced double layer description. To the extent that experiments like [3] deal with a finite concentration of macroions that contribute in and of themselves to the equation of state, i.e., to the osmotic pressure and osmotic coefficient, the simple condensation picture appears to be too drastic an idealization.

There have been many elegant attempts to extend condensation theory [8, 17, 18, 19]. In essence these are all versions of the Poisson Boltzmann equation solved under various conditions including finite radius, explicit ion simulation, condensed layers, etc. These models create a distinction between bound and unbound ions or speak of condensation shells within the context of the continuous distributions predicted.

(i) Many have been tempted to identify, via the cell model analysis, the fraction $f = 1 - 1/Q_o$ of counterion residing in the shell $a < r < R_M$ with the condensed fraction, and so implicitly to accept that the condensed layer is a spatially extended object with a peculiar sensitivity to changes in, e.g., concentration of macroions. Generalizations of this approach have been considered where one identifies a length scale similar to $R_M$ and the accompanying 'condensation shell' from an analysis of the counterion distribution function versus linear [6] or logarithmic [18, 19] radial distance. They provide operational definitions of a 'condensed layer' but these definitions are almost a matter of terminology.

Does such 'condensation' add anything essentially new either to P-B theory or to simple line-charge condensation?

None of these approaches makes it completely clear that there is a 'Debye-Hückel' cloud of counterions in the outer shell $R_M < r < R_0$, although at some distance from the polyelectrolyte potentials must weaken to a degree that a D-H description becomes accurate.

(ii) It has been argued recently [14], that the fact that in the infinite-dilution limit the contact density $n(a)$ reaches a constant limiting value $(1/ebπa^2)(Q − 1)^2/(2Q)$ leads one to expect the 'existence of a close layer that cannot be diluted away', in other words a condensed layer. This would point to the conclusion that counterion condensation has a reality going beyond that of being a mere heuristic tool. This is perhaps true but does not really prove the point, in the sense that one can not even in this case distinguish 'condensation' from mere double layer properties.

Comparison of the measured osmotic pressures for DNA at various ionic conditions suggests that at low salt concentrations the 'counterion condensation' picture does not capture the main features of the data. The discrepancy experimental and theoretical values of the osmotic coefficient, amounting to a factor of two can not be ignored. The cell-model in which the countercharge distribution is governed by the Poisson - Boltzmann equation and where the macroion is modelled as a cylinder of a finite radius produces a much better fit to data. This almost quantitative correspondence between the data and the calculations brings back into focus the counterion atmosphere without any need to invoke a 'condensation' behavior.
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**Figure 1:** Osmotic coefficient, $\phi = n(R_0)/n_{PO4}$ vs. mean counterion concentration $n_{PO4}$ ($c_{DNA}$ in molar units) and the geometry of the cell model. $n(R_0)$ is the counterion density at the cell wall. The charge density on the cylinder corresponds to one elementary charge per $1.7\AA$ contour length (or $Q=4.353$). Osmotic coefficients have been calculated for $a = 10\AA$. We have considered the consequences of allowing the counterions to enter the interior of the cylinder (dashed line), or of preventing the counterions from entering the cylinder at all (solid line). The osmotic pressure in the limit of vanishing $a$ (Manning condensation limit) is shown by the dashed line. The calculated osmotic coefficients increase as $c_{DNA}$ increases. The experimental results for DNA [3] (■) in the salt independent regime of DNA concentrations show the same trend, but the variation is slower than for any of the calculated data. The osmotic coefficient is somewhat sensitive to whether the cylinder is hollow (dotted lines) or solid (solid lines). Expectably when the cylinder is hollow, the osmotic coefficient is smaller. The hollow/solid difference increases when $R_0 \rightarrow a$. In the dilute, $a/R_0 \rightarrow 0$, limit the computed osmotic coefficient reaches the constant limiting value of $1/(2Q) \approx .11$ of counterion condensation theory.

**Figure 2:** Osmotic pressure $\pi_{osm}$ versus the molar concentration of DNA $c_{DNA}$ (upper graph) or the radius of the cell $R_0$ (lower graph). The charge density on the cylinder with radius $a = 10\AA$ corresponds to one elementary charge per $1.7\AA$ contour length (or $Q = 4.353$). Hollow cylinder, dotted line; solid cylinder, solid line. The predicted osmotic pressure in the limit of vanishing $a$ (counterion condensation limit) is shown with a dashed line. Experimental data are represented with symbols: ▼ osmotic pressure data at 2 mM salt [3]. ■ osmotic pressure data at DNA concentrations where salt does not matter, ⋄ osmotic pressure at very large DNA concentrations where again there are no salt concentration effects [11]. The experimentally determined pressures for very small values of $c_{DNA}$ can not be explained by the electrostatic cell model studied here. For $c_{DNA}$ in the range $0.1 - 0.5M$, the calculated pressures determined from the electrostatic cell model seems to capture the trend in the experimental data but the calculated pressures are slightly larger than the experimentally determined pressures, irrespective of solid-cylinder (solid line) or hollow-cylinder (dashed line) assumptions. When $c_{DNA}$ approaches $1 M$, we find that the calculated pressures and the experimentally determined pressures are not very different in magnitude, but the calculated variation of pressure with concentration does not fit the trend in the experimental data.
Osmotic coefficient ($\phi$) vs. DNA concentration ($C_{DNA}$) [M].

- $R_o$ is the outer radius of the DNA helix.
- $a$ is the inner radius of the DNA helix.

The graph shows a plot of $C_{DNA}$ [M] on the x-axis and osmotic coefficient ($\phi$) on the y-axis. The data points are scattered across the graph, indicating a trend. The solid line represents the theoretical relationship, while the dashed line shows another possible trend. The dotted line is a third theoretical relationship.
