CONVERGENCE OF A CLASS OF FULLY NON-LINEAR PARABOLIC EQUATIONS ON HERMITIAN MANIFOLDS

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ABSTRACT. We consider a class of fully non-linear parabolic equations on compact Hermitian manifolds involving symmetric functions of partial Laplacian $s$. Under fairly general assumptions, we show the long time existence and convergence of solutions. We also derive a Harnack inequality for the linearized equation which is used in the proof of convergence.

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1. Introduction

In this paper we study the following parabolic equation on an $n$-dimensional compact Hermitian manifold $(M, \omega)$.

\[
\frac{\partial \phi}{\partial t} = f(\Lambda(\sqrt{-1} \partial \bar{\partial} \phi + X[\phi])) - \psi[\phi]
\]

\[
\phi(x, 0) = \phi_0 \in C^\infty(M)
\]

where $f(\Lambda)$ is a symmetric function of $\Lambda_i$ which denotes a partial sum of eigenvalues of $\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]$. More precisely, let $K \leq n$ be a fixed positive integer. Set

$$
\mathcal{J}_K = \{(i_1, \ldots, i_K) : 1 \leq i_1 < \cdots < i_K \leq n, \ i_j \in \mathbb{N}\}
$$

Denote the elements of $\mathcal{J}_K$ by $\{I_1, \ldots, I_N\}$ after fixing an order. Then

$$
\Lambda(\lambda) = (\Lambda_1(\lambda), \ldots, \Lambda_N(\lambda)) := (\Lambda_{I_1}(\lambda), \ldots, \Lambda_{I_N}(\lambda))
$$

where

$$
\Lambda_{I}(\lambda) = \sum_{i \in I} \lambda_i = \lambda_{i_1} + \cdots + \lambda_{i_K}, \ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.
$$

We also write

$$
\Lambda(\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]) := \Lambda(\lambda(\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]))
$$

where $\lambda(\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of $\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]$ with respect to $\omega$.

This form of $\Lambda$ was introduced in [3] as a generalization of $\lambda(\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u + X[u])$ which is obtained in the case when $K = n-1$. The $(1, 1)$-form $X[\phi] = X(z, \phi, \partial \phi, \bar{\partial} \phi)$
and the function \( \psi[\phi] = \psi(z, \phi, \partial \phi, \bar{\partial} \phi) \) depend on \( \phi \) and its first order derivatives as well. Throughout this article \( f \), \( X \) and \( \psi \) are assumed to be smooth.

Equation (1.1) is the parabolic counterpart of an elliptic equation studied by the author with Guan and Qiu in [3] on Hermitian manifolds. Such equations are of interest, for example in the proof of Gauduchon conjecture by Székelyhidi-Tosatti-Weinkove [12] and in the work of Guan-Qiu-Yuan [7] involving the study of conformal deformations of mixed Chern-Ricci forms.

To state the main theorem we make the following set of assumptions on \( f \), \( X \) and \( \psi \).

**Assumptions on \( f \):** \( f \) is a symmetric function of \( N \) variables defined in a symmetric open convex cone \( \Gamma \subset \mathbb{R}^N \) with vertex at the origin with

\[
\Gamma_N = \{ \Lambda \in \mathbb{R}^N : \Lambda_i > 0 \} \subset \Gamma,
\]

and satisfies the conditions

\[
(1.2) \quad f_i \equiv \frac{\partial f}{\partial \Lambda_i} \geq 0 \text{ in } \Gamma, \quad 1 \leq i \leq N,
\]

(1.4) \( f \) is a concave function in \( \Gamma \),

(1.5) \( \sup_{\partial \Gamma} f < \inf_{M} \psi \)

and,

(1.6) \( \lim_{t \to \infty} f(t\Lambda) = \sup_{\Gamma} f, \quad \forall \Lambda \in \Gamma \)

We will say that \( \phi \) is an admissible function if \( \Lambda(\sqrt{-1} \partial \bar{\partial} \phi + X[\phi]) \in \Gamma \), for all \( t \).

**Remark 1.1.** With strict inequality in (1.3), conditions (1.2) to (1.6) are the structure conditions of Caffarelli-Nirenberg-Spruck [1].

For deriving first and second order estimates, we will make the following additional assumptions on \( f \).

\[
(1.7) \quad \sum f_i \Lambda_i \geq -C_0 \sum f_i \text{ in } \Gamma
\]

for some constant \( C_0 > 0 \).
(1.8) \[ \text{rank of } C^{\sigma}_\sigma \geq \frac{N(n-K)}{n} + 1, \forall \inf_{\Gamma} f \leq \sigma \leq \sup_{\partial \Gamma} f \]

and,

(1.9) \[ \lim_{t \to +\infty} f(t1) - \sup_M \psi[\phi] \geq c_0 > 0. \]

It is worth emphasizing that assumption (1.8) is the critical ingredient used in the derivative estimations. Condition (1.7) means that at any point \( \Lambda_0 \in \Gamma \), the distance from the origin to the tangent plane at \( \Lambda_0 \) of the level hypersurface \( \partial \Gamma f(\Lambda_0) \) has a uniform bound \( C_0 \). It is satisfied in most applications and is weaker than assumption (1.6) which implies

\[ \sum f_i \Lambda_i \geq 0 \text{ in } \Gamma \]

**Assumptions on } \psi \text{ and } X :** Some growth assumptions must be made on \( \psi \) and \( X \). Here we impose the following conditions.

(1.10) \[ G^{ij} X_{ij,\phi} - \psi_\phi \leq 0 \]

where \( G^{ij} \) are the coefficients of second-order terms in the linearized equation (see section 3). This is only used for estimating \( \sup |\phi_t| \) by the maximum principle. In section 8, we will assume that \( X \) and \( \psi \) are independent of \( \phi \) for proving the convergence in Theorem 1.2.

For deriving gradient estimates, the following conditions are assumed.

(1.11) \[ |D_\zeta X(z, \phi, \zeta, \bar{\zeta})| \leq \varrho_0|\zeta|, \quad D_\phi X \leq (\varrho_0 |\zeta|^2 + \varrho_1)\omega \]

where \( \varrho_1 = \varrho_1(z, \phi) \) and \( \varrho_0 = \varrho_0(z, \phi, |\zeta|) \to 0^+ \text{ as } |\zeta| \to \infty; \) we may assume \( t\varrho_0(z, \phi, t) \) to be increasing in \( t > 0 \). It follows that \( |X| \leq C\varrho_0 |\zeta|^2 + \varrho_1(z, \phi) \), for some function \( \varrho_1; \) we shall only need

(1.12) \[ X \leq (\varrho_0 |\zeta|^2 + \varrho_1)\omega \]

On \( \psi \) we impose similar constraints, but also depending on the growth of \( f \).

(1.13) \[ |D_\zeta \psi(z, \phi, \zeta, \bar{\zeta})| \leq \varrho_0 f(|\zeta|^21)/|\zeta|, \quad -D_\phi \psi \leq \varrho_0 f(|\zeta|^21) + \varrho_1(z, \phi) \]

which implies
\[ \psi \leq \varrho_0 f(|\zeta|^2) + \varrho_1(z, \phi) \]

We will also assume that

\[ |\nabla_z X| \leq |\zeta| (\varrho_0 f(|\zeta|^2) + \varrho_1(z, \phi)) \]

and,

\[ |\nabla_z \psi| \leq |\zeta| (\varrho_0 |\zeta|^2 + \varrho_1(z, \phi)) \]

We will assume for convenience that \( \int_M \omega^n = 1 \). Then the main result is stated as follows.

**Theorem 1.2.** Let \( f, X \) and \( \psi \) satisfy (1.2)-(1.6), (1.8)-(1.11), (1.13), (1.15) and (1.16). Then equation (1.1) has an admissible solution \( \phi \) for all time \( t \in [0, \infty) \). In addition, the normalized function of \( \phi \) defined by

\[ \bar{\phi} := \phi - \int_M \phi \omega^n \]

has the following uniform estimate

\[ \| \bar{\phi} \|_{C^\infty_{x,t}} \leq C \]

for a constant \( C \) that depends only on \((M, \omega)\) and \( \phi_0 \). If \( X \) and \( \psi \) are independent of \( \phi \), then \( \bar{\phi} \) converges in \( C^\infty \) to a smooth function \( \bar{\phi}_\infty \) as \( t \to \infty \), where \( \bar{\phi}_\infty \) is a solution of the elliptic equation

\[ f(\Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u])) = \psi[u] + a \]

for some constant \( a \).

**Remark 1.3.** If \( \phi \) is a solution of (1.1), then \( \bar{\phi} \) solves

\[ \frac{\partial \bar{\phi}}{\partial t} = f(\Lambda(\sqrt{-1} \partial \bar{\partial} \bar{\phi} + X[\phi])) - \psi[\phi] - \int_M \frac{\partial \phi}{\partial t} \omega^n \]

\[ \bar{\phi}(x, 0) = \phi_0 - \int_M \phi_0 \omega^n \]

**Remark 1.4.** It would be interesting to investigate whether the convergence in Theorem (1.2) holds with \( X \) and \( \psi \) depending on \( \phi \).
The organization of the paper is as follows. To prove the long-time existence of solutions, we will derive apriori estimates for the $C^{2,\alpha}(M)$ norm of $\phi$. The second-order and gradient estimates for $\phi$ will be derived in sections 4 and 5 respectively. This will be done by applying maximum principle to test functions similar to the elliptic case. These estimates will depend on $\sup |\phi_t|$ which is easily bounded by applying maximum principle to the linearized equation. This is done in section 3.

Gradient estimates in section 5 will in turn provide a uniform estimate for $\phi$. All of this combined with Evans-Krylov theorem and a standard bootstrapping argument will imply the apriori estimate for $\phi$ in Theorem 1.2. Now the solution can be extended to $T = \infty$, which will be shown in section 6.

For proving the convergence, we will derive a Harnack inequality for positive solutions of equations of the form,

$$\frac{\partial u}{\partial t} = G^{ij} \partial_i \partial_j u + \chi_k u_k + \chi_{0} u$$

This is similar to the results of Li-Yau [8] and Gill [4], but on Hermitian manifolds and with lower order terms. Li and Yau considered parabolic equations associated to the Schrödinger operator given by

$$u_t = \Delta u - q(x,t) u$$

on a Riemannian manifold $M$ with $q \in C^{2,1}_{\alpha}(M, [0,T])$. On the other hand, Gill derived Harnack inequality for

$$u_t = g^{ij} u_{ij}$$

on compact Hermitian manifolds, for proving the convergence of Chern-Ricci flow in the case when $c_{1}^{BM}(M) = 0$. This is also related to the work of H.D. Cao [2] who considered the Kähler version of the same equation. Our result uses similar techniques as the above equations, but now the estimation is more complicated because of the additional terms involved.

The Harnack inequality can further be used to derive an exponential decay for the oscillation $\omega(t)$ of $\phi_t$. From there, the convergence follows by standard arguments as detailed in section 8.

2. Preliminaries

Now we shall introduce the basic notations and state some key lemmas that will be used in the subsequent sections. Let $C^{k,p}_{x,t}(M \times I)$ be the set of functions defined on $M \times I$ whose derivatives of orders up to $(k,p)$ in $(x,t)$ variables exist and are continuous.
We recall some notions from [3], [5] and [6]. For a fixed real number \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \) define

\[
\Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}.
\]

**Lemma 2.1** ([3]). *Under conditions (1.3) and (1.4), the level hypersurface of \( f \)

\[
\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \},
\]

which is the boundary of \( \Gamma^\sigma \), is smooth and convex.*

This is clearly true with strict inequality in (1.3), but still remains valid under the slightly weaker hypothesis.

Define for \( \lambda \in \partial \Gamma^\sigma \),

\[
\nu_\lambda = \frac{Df(\lambda)}{|Df(\lambda)|}
\]

\( \nu_\lambda \) is the unit normal vector to \( \partial \Gamma^\sigma \) at \( \lambda \). The key ingredient used in finding apriori estimates later on is obtained by studying the tangent cone at infinity to the level sets of \( f \).

**Definition 2.2** ([5]). For \( \mu \in \mathbb{R}^n \) let

\[
S_\mu^\sigma = \{ \lambda \in \partial \Gamma^\sigma : \nu_\lambda \cdot (\mu - \lambda) \leq 0 \}.
\]

The *tangent cone at infinity* to \( \Gamma^\sigma \) is defined as

\[
C_\sigma^+ = \{ \mu \in \mathbb{R}^n : S_\mu^\sigma \text{ is compact} \}.
\]

Clearly \( C_\sigma^+ \) is a symmetric convex cone. As in [5] one can show that \( C_\sigma^+ \) is open.

**Definition 2.3** ([6]). The *rank* of \( C_\sigma^+ \) is defined to be

\[
\min \{ r(\nu) : \nu \text{ is the unit normal vector of a supporting plane to } C_\sigma^+ \}
\]

where \( r(\nu) \) denotes the number of non-zero components of \( \nu \).

Under the assumptions (1.3), (1.4) and that \( f \) satisfies,

\[
(2.1) \quad \sum f_i \Lambda_i \geq -C_0 \sum f_i \text{ in } \Gamma,
\]

we have the following results from [3].

**Lemma 2.4.** Let \( P = \{ \mu \in \mathbb{R}^N : \nu \cdot \mu = c \} \) be a hyperplane, where \( \nu \) is a unit vector. Suppose that there exists a sequence \( \{ \Lambda_k \} \) in \( \partial \Gamma^\sigma \) with

\[
(2.2) \quad \lim_{k \to +\infty} \nu_{\Lambda_k} = \nu, \quad \lim_{k \to +\infty} \nu_{\Lambda_k} \cdot \Lambda_k = c, \quad \lim_{k \to +\infty} |\Lambda_k| = +\infty.
\]

Then \( P \) is a supporting hyperplane to \( C_\sigma^+ \) at a non-vertex point.
Lemma 2.5. Suppose that the rank of $C^+_\sigma$ is $r$. There exists $c_0 > 0$ such that at any point $\Lambda \in \partial \Gamma^\sigma$ where without loss of generality we assume $f_1 \leq \cdots \leq f_n$,

$$\sum_{i \leq N-r+1} f_i \geq c_0 \sum f_i.$$  

Proofs of these statements can be found in [3].

Remark 2.6. An inequality of Lin-Trudinger [10] shows that the rank of $C^+_\sigma$ is $N-k+1$ for $f = \sigma^+_k$ and $\sigma > 0$, where

$$\sigma_k(\Lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \Lambda_{i_1} \cdots \Lambda_{i_k}$$

is the $k$-th elementary symmetric function defined on the Garding cone

$$\Gamma_k = \{ \Lambda \in \mathbb{R}^N : \sigma_j(\Lambda) > 0, \text{ for } 1 \leq j \leq k \}.$$ 

3. Estimate for $\phi_t$

To estimate $\sup |\phi_t|$, we will apply maximum principle to the linearization of (1.1).

Proposition 3.1. Let $\phi \in C^{2,1}_{x,t}(M \times [0,T])$ be a solution of (1.1). Then under the assumption (1.10),

$$\sup |\phi_t| \leq C$$

for some constant $C$ only depending on the initial data.

Proof. Denote

$$g[\phi] := \sqrt{-1} \partial \bar{\partial} \phi + X[\phi],$$

and define $G$ by

$$G(g[\phi]) = f(\Lambda(g[\phi]))$$

Consequently, we can write (1.1) as

(3.1) \[ \frac{\partial \phi}{\partial t} = G(g) - \psi[\phi] \]

Differentiate the equation wrt. $t$ and denote $\phi_t \equiv u$.

(3.2) \[ \frac{\partial u}{\partial t} = G^{ij} \partial_i g_{ij} - \partial_t \psi[\phi] \]

where $G^{ij} = \frac{\partial G}{\partial g_{ij}}(g[\phi])$. Expanding the last two terms by chain rule gives,
for any two smooth sections \( s \) onto \( T \) of \( \omega \), note the Chern connection with respect to the metric and, respectively. Note that the infimum of \( \text{(4.1)} \) is uniformly bounded. We introduce some geometric preliminaries first. Throughout this article \( \nabla \) will denote the Chern connection with respect to the metric \( \omega \). This is the unique connection on \( TM \) defined by

\[
d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle \quad \text{and} \quad \nabla^{0,1} = \bar{\partial}
\]

for any two smooth sections \( s_1 \) and \( s_2 \) of \( TM \) and \( \nabla^{0,1} \) denotes the projection of \( \nabla \) onto \( T^{0,1}M \). In local coordinates this can be written as

\[
\nabla_i \partial_j := \Gamma^k_{ij} \partial_k, \quad \text{where} \quad \Gamma^k_{ij} = g^{kl} \partial_i g_{jl}
\]

The torsion and the curvature tensors are

\[
T^l_{ij} = \Gamma^l_{ij} - \Gamma^l_{ji}
\]

and,

\[
R_{ijkl} = -\partial_j \partial_i g_{kl} + g^{pq} \partial_i g_{kq} \partial_j g_{pl}
\]

respectively.

For a function \( f \), denote \( f_{ij} = \nabla_j \nabla_i f \), \( f_{ij} = \nabla_j \nabla_i f \) etc. Then we have the following equations for commuting covariant derivatives,

\[
\begin{align*}
\text{(4.1)} \\
f_{ijk} - f_{ikj} &= -g^{lm} R_{kljm} f_l, \\
f_{ijk} - f_{ikj} &= T^l_{jk} f_{il}, \\
f_{ijk} - f_{ikj} &= -g^{lm} R_{ijkm} f_l + T^l_{ik} f_{ij}, \\
f_{ijkl} - f_{klij} &= g^{pq} (R_{klpq} f_{pj} - R_{ijpq} f_{pl}) + T^p_{ik} f_{pj} + T^p_{ij} f_{pk} - T^p_{ik} T^p_{ij} f_{pq}
\end{align*}
\]
As in section 3, we write (1.1) as
\[
\frac{\partial \phi}{\partial t} = G(\mathbf{g}) - \psi[\phi]
\]
Differentiating this equation with respect to \(z_i\) first and then wrt. \(z_j\) gives,
\[
\begin{align*}
\partial_i \phi_t &= G^p q \nabla_i g_{pq} - \nabla_i \psi[\phi] \\
\partial_i \partial_j \phi_t &= G^{p, s l} \nabla_j g_{s l} \nabla_i g_{pq} + G^p q \nabla_j \nabla_i g_{pq} - \nabla_j \nabla_i \psi[\phi] \\
&\leq G^p q \nabla_j \nabla_i g_{pq} - \nabla_j \nabla_i \psi[\phi]
\end{align*}
\]
where the last inequality follows from concavity of \(f\), and
\[
G^{p, s l} := \frac{\partial^2 G}{\partial g_{pq} \partial g_{st}}(\mathbf{g}[\phi])
\]
From now on we assume that \(f\) satisfies (1.7), (1.8) and (1.9), where \(1 = (1, \ldots, 1) \in \Gamma\) and \(c_0\) may depend on \(|\phi|_{C^1(M)}\).

We estimate \(|\partial \phi_t|\) by following [3] which uses ideas of Tossati-Weinkove [13] and consider the test function which is given in local coordinates by
\[
A := \sup_{(z, t) \in M \times [0, T]} \max_{\xi \in T^{1,0}_z M} e^{(1+\gamma) \eta} g_{pq} \xi_p \xi_q (g^{kl} g_{ij} \xi_i \xi_j)^2 / |\xi|^{2+\gamma}
\]
where \(\eta\) is a function depending on \(|\nabla \phi|\), and \(\gamma > 0\) is a small constant to be chosen later.

**Theorem 4.1.** Let \(\phi \in C^{4,1}_{x, t}(M \times [0, T])\) be a solution to (1.1). Then
\[
\sup_{M \times [0, T]} |\partial \phi_t| \leq C
\]
where \(C\) depends on \((M, \omega), \sup |\nabla \phi|\) and \(\sup |\phi_t|\).

**Proof.** Assume that \(A\) is achieved at a point \((z_0, t_0) \in M \times [0, T)\) for some \(\xi \in T^{1,0}_z M\). We choose local coordinates around \(z_0\) such that \(g_{ij} = \delta_{ij}\) and \(T^k_{ij} = 2 \Gamma^k_{ij}\) using the lemma of Streets and Tian [11], and that \(g_{ij}\) is diagonal at \(z_0\) with \(g_{11} \geq g_{22} \geq \ldots \geq g_{nn}\).

The maximum \(A\) is achieved for \(\xi = \partial_1\) at \((z_0, t_0)\) when \(\gamma > 0\) is sufficiently small (see [13] and [6]). We can assume \(g_{11} > 1\); otherwise we are done.

Let \(W = g^{-1}_{11} g^{kj} g_{ij} g_{k1}\). We see that the function \(Q = (1+\gamma) \eta + \log g^{-1}_{11} g_{11} + \frac{\gamma}{2} \log W\) which is locally well defined attains a maximum \((1+\gamma) \eta + (1+\gamma) \log g_{11}\) at \((z_0, t_0)\)
where $W = g_{ii}^2$ and the following equations are obtained.

\[
\frac{\partial_i (g_{ii}^{-1} g_{1i})}{g_{1i}} + \frac{\gamma \partial_i W}{2W} + (1 + \gamma) \partial_i \eta = 0,
\]

\[
\frac{\bar{\partial}_i (g_{ii}^{-1} g_{1i})}{g_{1i}} + \frac{\gamma \bar{\partial}_i W}{2W} + (1 + \gamma) \bar{\partial}_i \eta = 0
\]

for each $1 \leq i \leq n$, and

\[
0 \geq \frac{1}{g_{1i}} G^{\bar{a}i} \partial_{\bar{a}} \partial_i (g_{ii}^{-1} g_{1i}) - \frac{1}{g_{1i}} G^{\bar{a}i} \partial_i (g_{ii}^{-1} g_{1i}) \bar{\partial}_i (g_{ii}^{-1} g_{1i})
\]

\[
+ \frac{\gamma}{2W} G^{\bar{a}i} \partial_{\bar{a}} \partial_i W - \frac{\gamma}{2W^2} G^{\bar{a}i} \partial_i W \bar{\partial}_i W + (1 + \gamma) G^{\bar{a}i} \bar{\partial}_i \partial_i \eta.
\]

The following identities can be derived by direct calculation.

\[
\partial_i (g_{ii}^{-1} g_{1i}) = \nabla_i g_{1i}, \quad \partial_i W = 2g_{ii} \nabla_i g_{1i},
\]

\[
\bar{\partial}_j \partial_i (g_{ii}^{-1} g_{1i}) = \nabla_j \nabla_i g_{1i} + (\Gamma_{j1}^m \nabla_i g_{1i} - \Gamma_{j1}^l \nabla_i g_{1i})
\]

\[
+ (\Gamma_{i1}^m \nabla_j g_{1i} - \Gamma_{i1}^l \nabla_j g_{1i}) + (\Gamma_{i1}^l \Gamma_{j1}^m - \Gamma_{i1}^m \Gamma_{j1}^l) g_{1i},
\]

and

\[
\bar{\partial}_j \partial_i W = 2g_{1i} \nabla_j \nabla_i g_{1i} + 2 \nabla_i g_{1i} \nabla_j g_{1i} + \sum_{l>1} \nabla_i g_{1i} \nabla_j g_{1i}
\]

\[
+ \sum_{l>1} (\nabla_i g_{1i} + \Gamma_{i1}^l \nabla_i g_{1i} + \Gamma_{j1}^l \nabla_i g_{1i})
\]

\[
+ g_{1i} \sum_{l>1} (\Gamma_{j1}^l \nabla_i g_{1i} + \Gamma_{i1}^l \nabla_j g_{1i})
\]

\[
- g_{1i} \sum_{l>1} \Gamma_{i1}^m \Gamma_{j1}^m (g_{1i} + g_{1i}).
\]

It follows that

\[
G^{\bar{a}i} \partial_i W \bar{\partial}_i W = 4g_{1i}^2 G^{\bar{a}i} \nabla_i g_{1i} \nabla_i g_{1i},
\]

\[
G^{\bar{a}i} \partial_i (g_{ii}^{-1} g_{1i}) \bar{\partial}_i (g_{ii}^{-1} g_{1i}) = G^{\bar{a}i} \nabla_i g_{1i} \nabla_i g_{1i}
\]

and by Cauchy-Schwarz inequality,

\[
G^{\bar{a}i} \bar{\partial}_i \partial_i W \geq 2g_{ii} G^{\bar{a}i} \nabla_i g_{1i} + 2G^{\bar{a}i} \nabla_i g_{1i} \nabla_i g_{1i}
\]

\[
+ \sum_{l>1} G^{\bar{a}i} \nabla_i g_{1i} \nabla_i g_{1i} + \frac{1}{2} \sum_{l>1} G^{\bar{a}i} \nabla_i g_{1i} \nabla_i g_{1i} - C g_{1i}^2 \sum G^{\bar{a}i},
\]
It follows using (4.14) that
\begin{equation}
\nabla_i g_{11} + g_{11} \partial_i \eta = 0, \quad \nabla_i g_{11} + g_{11} \bar{\partial}_i \eta = 0
\end{equation}
and
\begin{equation}
0 \geq \frac{1}{g_{11}} G^{\bar{a}} \bar{\partial}_i \nabla_i g_{11} - \frac{1}{2} \frac{G^{\bar{a}} \nabla_i g_{11} \nabla_i g_{11} + G^{\bar{a}} \bar{\partial}_i \bar{\partial}_i \eta}{g_{11}} + \frac{\gamma}{16 g_{11}^2} \sum_{l>1} G^{\bar{a}} \nabla_i g_{11} \nabla_i g_{11} - C \sum G^{\bar{a}}.
\end{equation}

Now using (4.1),
\begin{equation}
\nabla_i \nabla_i g_{11} - \nabla_1 \nabla_1 g_{11} = R_{111} g_{11} - R_{111} g_{11} - T_{11} \nabla_i g_{11} - T_{11} \nabla_i g_{11} - T_{11} \nabla_i g_{11} + H_{\bar{a}}
\end{equation}
where
\[ H_{\bar{a}} = \nabla_i \nabla_i X_{11} - \nabla_1 \nabla_1 X_{\bar{a}} - 2 \Re \{ T_{11} \nabla_i X_{11} \} + R_{111} X_{11} - R_{111} X_{11} - T_{11} \bar{T}_{11} X_{\bar{a}}. \]

It follows from Schwarz inequality that
\begin{equation}
G^{\bar{a}} \nabla_i \nabla_i g_{11} \geq G^{\bar{a}} \nabla_1 \nabla_1 g_{11} - \frac{\gamma}{32 g_{11}} \sum_{l>1} G^{\bar{a}} \nabla_i g_{11} \nabla_i g_{11}
\end{equation}
\[- C g_{11} \sum G^{\bar{a}} + G^{\bar{a}} H_{\bar{a}}.
\]
We also have at \((z_0, t_0),\)
\begin{equation}
0 \leq \partial_t Q = (1 + \gamma) \eta + (1 + \gamma) \frac{\bar{\partial}_i g_{11}}{g_{11}}
\end{equation}
\[ \leq (1 + \gamma) \left( \eta + \frac{1}{g_{11}} \left( G^{\bar{a}} \nabla_1 \nabla_1 g_{11} - \nabla_1 \nabla_1 \psi + \partial_t X_{11} \right) \right) \]
where we used (4.3). Combining eqs. (4.15), (4.17) and (4.18),
\begin{equation}
0 \geq G^{\bar{a}} \bar{\partial}_i \partial_i \eta - \eta + \frac{G^{\bar{a}} H_{\bar{a}}}{g_{11}} - C \sum G^{\bar{a}} - \frac{G^{\bar{a}} \nabla_i g_{11} \nabla_i g_{11}}{g_{11}^2} + \frac{\nabla_1 \nabla_1 \psi}{g_{11}} - \frac{\partial_t X_{11}}{g_{11}}
\end{equation}
It follows using (4.14) that
\begin{equation}
g_{11} \left( G^{\bar{a}} \bar{\partial}_i \partial_i \eta - G^{\bar{a}} \bar{\partial}_i \bar{\partial}_i \eta - \eta \right) \leq - G^{\bar{a}} H_{\bar{a}} + \partial_t X_{11} - \nabla_1 \nabla_1 \psi + C g_{11} \sum G^{\bar{a}}
\end{equation}
By direct calculations (see e.g. [6], [7]) and using eqs. (4.3) and (4.14),

\[ G^{\bar{i}} H_{\bar{i}} \geq 2 G^{\bar{i}} \Re \{ X_{11, \zeta_\alpha} \nabla_\alpha \nabla_1 \psi \} - 2 G^{\bar{i}} \Re \{ X_{\bar{i}, \zeta_\alpha} \nabla_\alpha \nabla_1 \phi \} 
- C g_{11}^2 \sum \nabla_i \phi^i - C \sum G^{\bar{i}} |\nabla_i \nabla_k \phi|^2 - C \sum |\nabla_1 \nabla_k \phi|^2 \sum G^{\bar{i}} \]

\[ \geq 2 \Re \{ X_{11, \zeta_\alpha} (\nabla_\alpha \psi + \phi_t) \} + 2 g_{11} G^{\bar{i}} \Re \{ X_{\bar{i}, \zeta_\alpha} \nabla_\alpha \eta \} - |A|^2 \sum G^{\bar{i}} \]

where we denote

\[ |A|^2 = g_{\bar{i}i}^2 + \sum |\nabla_i \nabla_k \phi|^2, \quad |A| = \sum |A_i|^2. \]

Next,

\[ \nabla_\alpha \psi = \psi_\alpha + \psi_\phi \nabla_\alpha \phi + \psi_\zeta_\delta \partial_\alpha \partial_\beta \phi + \psi_\bar{\zeta} \partial_\alpha \bar{\partial}_\beta \phi \]

\[ \nabla_1 \nabla_1 \psi \geq \psi_\zeta_\alpha \nabla_\alpha \nabla_1 \nabla_1 \phi + \psi_\bar{\zeta} \nabla_\alpha \nabla_1 \nabla_1 \phi - |A_1|^2 \]

\[ \geq - g_{11} \psi_\zeta_\alpha \nabla_\alpha \eta - g_{11} \psi_\bar{\zeta}_\alpha \nabla_\alpha \eta - |A|^2. \]

and

\[ \partial_t X_{11} = X_{11, \phi} \phi_t + 2 \Re \{ X_{11, \zeta_\alpha} \phi_{\alpha t} \} \]

Plug in eqs. (4.21) to (4.24) in (4.20),

\[ g_{11} \left( G^{\bar{i}} \bar{\partial}_i \partial_\eta - G^{\bar{i}} \partial_\eta \bar{\partial}_i \eta - \eta_t \right) \leq - \nabla_1 \nabla_1 \psi - 2 \Re \{ X_{11, \zeta_\alpha} \nabla_\alpha \psi \} 
- 2 g_{11} G^{\bar{i}} \Re \{ X_{\bar{i}, \zeta_\alpha} \nabla_\alpha \phi \} + |A|^2 \sum G^{\bar{i}} + X_{11, \phi} \phi_t \]

\[ \leq C \left( g_{11} |\nabla \phi|^2 + |A|^2 \right) \left( 1 + \sum G^{\bar{i}} \right) \]

Let \( \eta = - \log h \), where \( h = 1 - \gamma |\nabla \phi|^2 \). We choose \( \gamma \) small enough to satisfy \( 2 \gamma |\nabla \phi|^2 \leq 1 \).

By straightforward calculations,

\[ \partial_t |\nabla \phi|^2 = \nabla_k \phi \nabla_i \nabla_k \phi + \nabla_k \phi \nabla_i \nabla_k \phi \]
and
\[
\bar{\partial}_i \bar{\partial}_i |\nabla \phi|^2 = \nabla_i \nabla_k \phi \nabla_k \nabla_i \phi + \nabla_i \nabla_k \phi \nabla_i \nabla_k \phi \\
+ \nabla_k \phi \nabla_i \nabla_k \phi + \nabla_k \phi \nabla_i \nabla_k \phi \\
= \nabla_i \nabla_k \phi \nabla_k \nabla_i \phi + \nabla_i \nabla_k \phi \nabla_i \nabla_k \phi \\
+ R_{ikl} \nabla_l \phi \nabla_k \phi - T_{ik} \nabla_l \phi \nabla_k \phi - \bar{T}_{ik} \nabla_l \phi \nabla_k \phi \\
\geq (1 - \gamma)|A_i|^2 + \nabla_k \phi \nabla_k \nabla_i \phi + \nabla_k \phi \nabla_k \nabla_i \phi - C|\nabla \phi|^2
\]

(4.28)
\[
G^{\bar{i}i}(\bar{\partial}_i \partial_i \eta - \bar{\partial}_i \eta \partial_i \eta) = \frac{\gamma}{h} G^{\bar{i}i} \bar{\partial}_i \partial_i |\nabla \phi|^2 \\
\geq \frac{G^{\bar{i}i}(1 - \gamma)}{h} |A_i|^2 + \frac{\gamma}{h} G^{\bar{i}i} \nabla_k \phi \nabla_k \nabla_i \phi + \frac{\gamma}{h} G^{\bar{i}i} \nabla_k \phi \nabla_k \nabla_i \phi \\
- C|\nabla \phi|^2 \sum G^{\bar{i}i} \\
\geq \gamma(1 - \gamma) \sum G^{\bar{i}i} |A_i|^2 + \frac{\gamma}{h} G^{\bar{i}i} \nabla_k \phi \nabla_k \nabla_i \phi \\
+ \frac{\gamma}{h} G^{\bar{i}i} \nabla_k \phi \nabla_k \nabla_i \phi - C|\nabla \phi|^2 \sum G^{\bar{i}i}
\]

We also have,

(4.29) \[
\eta_t = \gamma \frac{\partial_i |\nabla \phi|^2}{h}
\]

and,

(4.30) \[
\partial_t |\nabla u|^2 = 2\Re \{\partial_i \phi \bar{\partial}_k \phi_t \} \\
= 2\Re \{\phi_k (G^{\bar{i}i} \nabla_k \nabla_i \phi + G^{\bar{i}i} \nabla_k X_{\bar{i}i} - \partial_k \psi) \}
\]

Derive using (4.28), (4.29) and (4.30),

(4.31) \[
G^{\bar{i}i}(\bar{\partial}_i \partial_i \eta - \bar{\partial}_i \eta \partial_i \eta) - \eta_t \geq (1 - \gamma) \sum G^{\bar{i}i} |A_i|^2 - C|\nabla \phi|^2 \sum G^{\bar{i}i} \\
- \frac{2\gamma}{h} \Re \{\phi_k (G^{\bar{i}i} \nabla_k X_{\bar{i}i} - \partial_k \psi) \} \\
\geq (1 - \gamma - 4C \gamma^2) \sum G^{\bar{i}i} |A_i|^2 - C|\nabla \phi|^2 \sum G^{\bar{i}i}
\]

Further requiring that \( \gamma \) is small enough to satisfy \( \gamma \leq \frac{1}{2 + 4C} \), using (4.25)

(4.32) \[
\gamma^2 g_{11} \sum G^{\bar{i}i} |A_i|^2 \leq C|A_i|^2 \left(1 + \sum G^{\bar{i}i}\right).
\]
From Lemma 2.5 we have the following key inequality for each $i \geq 1$,

\[(4.33) \quad G_i^i = \sum_{l=1}^{N} f_{\Lambda_l} \frac{\partial \Lambda_l}{\partial \lambda_i} = \sum_{i \in I} f_{\Lambda_i} \geq c_0 \sum_{l=1}^{N} f_{\Lambda_l},\]

where the sum $\sum_{i \in I} f_{\Lambda_i}$ is taken over all $I \in \mathcal{I}_K$ with $i \in I$. Note that when $g_{ij}$ is diagonal, so is $G_{ij}$.

Consequently, by (4.32) we obtain

\[(4.34) \quad \frac{c_0 \gamma^2}{2} g_{ii} |A|^2 \sum f_{\Lambda_i} \leq C |A|^2\]

provided that $g_{ii}$ is large enough.

By the concavity of $f$ and using (2.1), we derive

\[
\sqrt{g_{ii}} \sum f_{\Lambda_i} = \sqrt{g_{ii}} \sum f_{\Lambda_i} - \sum f_{\Lambda_i} \Lambda_i(g) + \sum f_{\Lambda_i} \Lambda_i(g) \\
\geq f(\sqrt{g_{ii}} \mathbf{1}) - f(\Lambda(g)) - C_0 \sum f_{\Lambda_i} \\
\geq c_0 - C_0 \sum f_{\Lambda_i}
\]

by assumption (1.9), provided that $g_{ii}$ is sufficiently large. So from (4.34) we obtain

\[(4.35) \quad g_{ii} |A|^2 \sum f_{\Lambda_i} + \sqrt{g_{ii}} |A|^2 \leq C |A|^2.
\]

This gives the upper bound $g_{ii} \leq C$. To obtain a lower bound for the eigenvalues $g_{ii}$, note that $\text{tr}(g_{ii} + X) \geq 0$. This follows because the domain of $f$ is a symmetric cone in $\mathbb{R}^N$ with vertex at 0 that contains the positive cone $\Gamma_N \subset \Gamma$.

5. Gradient Estimates

In this section we assume that $X$ and $\psi$ satisfies conditions (1.11), (1.13), (1.15) and (1.16). With these assumptions in place, the gradient estimates can now be derived.

**Theorem 5.1.** Let $\phi \in C_{x,t}^{3,1}(M \times [0,T])$ be a solution of the equation (1.1) in $M \times [0,T)$. Then

\[(5.1) \quad |\nabla \phi|^2 \leq C(1 + \sup_{\Gamma} \phi - \phi)\]

for a uniform constant $C$ that depends on $\sup |\phi_t|$.

**Proof.** By adding a constant if necessary, we can assume without loss of generality that

$$\sup_{\partial \Gamma} f \leq 0 < \psi$$

Let $P = \eta + \log |\nabla \phi|^2$ where $\eta$ is a function of $\phi$ to be chosen later. Assume that $P$ attains maximum at the point $(z_0, t_0) \in M \times [0,T)$ and $|\nabla \phi| \geq 1$ at this point.
We also choose local coordinates around $z_0$ so that $g_{ij} = \delta_{ij}$, $T^{k}_{ij} = 2\Gamma^k_{ij}$ and $g_{ij}$ are diagonal at $z_0$. We have at $(z_0, t_0)$,

\begin{align}
\partial_t |\nabla \phi|^2 + |\nabla \phi|^2 \partial_t \eta &= 0 \\
\bar{\partial}_t |\nabla \phi|^2 + |\nabla \phi|^2 \bar{\partial}_t \eta &= 0
\end{align}

and,

\begin{align}
G^{\bar{i}i} \bar{\partial}_{\bar{i}} \partial_t P - \partial_t P = & G^{\bar{i}i} \bar{\partial}_{\bar{i}} |\nabla \phi|^2 + G^{\bar{i}i} \bar{\partial}_{\bar{i}} |\nabla \phi|^2 - G^{\bar{i}i} \bar{\partial}_{\bar{i}} |\nabla \phi|^2 - \bar{\partial}_t |\nabla \phi|^2 - \partial_t \eta \leq 0
\end{align}

Define $|Q_i|^2 = \nabla_i \nabla_k \phi \nabla_i \phi + \nabla_i \phi \nabla_k \phi \nabla_k \phi = \sum_k (|\nabla_i \nabla_k \phi|^2 + |\nabla_i \nabla_k \phi|^2)$. By Schwarz inequality,

\begin{align}
\bar{\partial}_t |\nabla \phi|^2 \partial_t |\nabla \phi|^2 \leq 2 |\nabla \phi|^2 |Q_i|^2
\end{align}

and,

\begin{align}
\bar{\partial}_t \partial_t |\nabla \phi|^2 = & \nabla_i \nabla_k \phi \nabla_i \nabla_k \phi + \nabla_i \nabla_k \phi \nabla_k \nabla_i \phi \\
& + \nabla_k \phi \nabla_i \nabla_k \nabla_i \phi + \nabla_k \phi \nabla_k \nabla_i \nabla_i \phi \\
& + R_{ikl} \nabla_l \phi \nabla_k \phi - T^l_{ik} \nabla_i \phi \nabla_k \phi - \bar{T}^l_{ik} \nabla_i \phi \nabla_k \phi \\
& \geq (1 - \gamma) |Q_i|^2 + \nabla_k \phi \nabla_k \nabla_i \nabla_i \phi + \nabla_k \phi \nabla_k \nabla_i \nabla_i \phi - C |\nabla \phi|^2
\end{align}

where $0 < \gamma < \frac{1}{6}$. Also,

\begin{align}
G^{\bar{i}i} \nabla_k \nabla_i \nabla_i \phi = G^{\bar{i}i} (\nabla_k \phi_{\bar{i}} - \nabla_k X_{\bar{i}}) = \nabla_k \psi + \nabla_k \phi_{\bar{t}} - G^{\bar{i}i} \nabla_k X_{\bar{i}}
\end{align}

Hence,

\begin{align}
G^{\bar{i}i} \bar{\partial}_{\bar{i}} |\nabla \phi|^2 \geq & G^{\bar{i}i} (1 - \gamma) |Q_i|^2 - C |\nabla \phi|^2 \sum G^{\bar{i}i} + R \\
& + 2 \Re \{ \nabla_k \phi_{\bar{t}} \nabla_k \phi \}
\end{align}

where $R = 2 \Re \{ (\nabla_k \psi - G^{\bar{i}i} \nabla_k X_{\bar{i}}) \nabla_k \phi \}$.

\begin{align}
\frac{\partial_t |\nabla \phi|^2}{|\nabla \phi|^2} = \frac{2}{|\nabla \phi|^2} \Re \{ \nabla_k \phi_{\bar{t}} \nabla_k \phi \}
\end{align}
Combine equations (5.2), (5.3), (5.7) and (5.8) to get cancellation of the terms involving $|Q_i|^2$ and $\nabla_k \phi_t$.

\begin{equation}
G^{\bar{i}} \partial_{\bar{i}} \partial_t \eta - \frac{1 + \gamma}{2} G^{\bar{i}} \partial_{\bar{i}} \eta \partial_t \eta \leq - \frac{R}{|\nabla \phi|^2} + C \sum G^{\bar{i}} + \eta_t
\end{equation}

Now we choose $\eta = - \log h$, where $h = 1 + \sup_{M \times [0,T]} \phi - \phi$. So,

\begin{equation}
G^{\bar{i}} \partial_{\bar{i}} \partial_t \eta = \frac{1}{h} G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi + \frac{1}{h^2} G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi \partial_t \phi
\end{equation}

and,

\begin{equation}
G^{\bar{i}} \partial_{\bar{i}} \partial_t \partial_t \eta = \frac{1}{h^2} G^{\bar{i}} \partial_{\bar{i}} \phi \partial_t \phi
\end{equation}

From (4.33) it follows that

\begin{equation}
\frac{1}{h^2} G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi \partial_t \phi - \frac{1 + \gamma}{2} G^{\bar{i}} \partial_{\bar{i}} \partial_t \eta \partial_t \eta = \frac{1 - \gamma}{2h^2} G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi \partial_t \phi \\
\geq \frac{c_1 |\nabla \phi|^2}{4h^2} \sum G^{\bar{i}}
\end{equation}

By concavity of $f$ and assumption (2.1),

\begin{equation}
|\nabla \phi|^2 \sum G^{\bar{i}} \geq f(|\nabla \phi|^2 1 - f(\Lambda) + G^{\bar{i}} g_{\bar{i}}
\end{equation}

\begin{equation}
\geq f(|\nabla \phi|^2 1 - \psi - \phi_t - C \sum G^{\bar{i}}
\end{equation}

Similarly,

\begin{equation}
G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi = G^{\bar{i}} g_{\bar{i}} - G^{\bar{i}} X_{\bar{i}} \geq -G^{\bar{i}} X_{\bar{i}} - C \sum G^{\bar{i}}
\end{equation}

Combining the above inequalities and using (5.9) we derive,

\begin{equation}
\frac{c_1 |\nabla \phi|^2}{8h^2} \sum G^{\bar{i}} + \frac{c_1}{8h^2} f(|\nabla \phi|^2 1) \leq - \frac{1}{h} G^{\bar{i}} \partial_{\bar{i}} \partial_t \phi + \frac{c_1 (\psi + \phi_t)}{8h^2} + C \sum G^{\bar{i}} - \frac{R}{|\nabla \phi|^2} + \eta \\
\leq \frac{1}{h} G^{\bar{i}} X_{\bar{i}} + \frac{c_1 (\psi + \phi_t)}{8h^2} - \frac{R}{|\nabla \phi|^2} + \frac{\phi_t}{h} + C \sum G^{\bar{i}}
\end{equation}

Using (5.2) and chain rule we obtain,
\[ \text{Re}\{\nabla_k \psi \nabla_k \phi\} = |\nabla \phi|^2 + \text{Re}\{\psi_k \nabla_k \phi + \psi_\zeta \partial_\zeta |\nabla \phi|^2 + \psi_\zeta \Gamma_{ak} \nabla_l \phi \nabla_k \phi\} \]

(5.16)

\[ = |\nabla \phi|^2 (\psi_\phi - \text{Re}\{\psi_\zeta \partial_\zeta \eta\}) + \text{Re}\{\psi_k \nabla_k \phi + \psi_\zeta \Gamma_{ak} \nabla_l \phi \nabla_k \phi\} \]

\[ = |\nabla \phi|^2 A \]

where

\[ A = \psi_\phi - \frac{1}{h} \text{Re}\{\psi_\zeta \partial_\zeta \phi\} + \frac{1}{|\nabla \phi|^2} \text{Re}\{\psi_k \nabla_k \phi + \psi_\zeta \Gamma_{ak} \nabla_l \phi \nabla_k \phi\} \]

Similarly,

(5.17)

\[ G^{\bar{i}} \text{Re}\{\nabla_k \phi \nabla_k X_{\bar{i}}\} = |\nabla \phi|^2 B \]

where

\[ B = G^{\bar{i}} X_{\bar{i}, \phi} - \frac{1}{h} G^{\bar{i}} \text{Re}\{X_{\bar{i}, \zeta} \partial_\zeta \phi\} + \frac{1}{|\nabla \phi|^2} G^{\bar{i}} \text{Re}\{(X_{\bar{i}, k} + X_{\bar{i}, \zeta} \Gamma_{ak} \nabla_l \phi) \nabla_k \phi\} \]

By assumptions (1.11) and (1.13),

(5.18)

\[ \frac{1}{h} G^{\bar{i}} X_{\bar{i}, \phi} + \frac{C_1 \psi}{8k^2} - \frac{R}{|\nabla \phi|^2} \leq CH \sum G^{\bar{i}} + CE + C \left(1 + \sum G^{\bar{i}} \right) \]

where

\[ E = |\nabla z \psi||\nabla \phi|^{-1} + (\psi_\phi) - \psi^+ \]

by (1.13), (1.14), (1.16) and,

\[ H = |\nabla X||\nabla \phi|^{-1} + \text{tr} X^+ + \text{tr}(D_\phi X)^+ + |D_\zeta X||\nabla \phi| \leq \varrho_0 |\nabla \phi|^2 + \varrho_1 \]

by (1.11), (1.12), (1.15). Use these inequalities to estimate the LHS of (5.18) and plug into (5.15) to obtain the bound \( |\nabla \phi|^2 \leq C \). From \( P(z, t) \leq P(z_0, t_0) \leq C \), the required estimate (5.1) follows.

As a consequence we can bound the oscillation of \( \phi \).

**Corollary 5.2.** For \( \phi \) as above,

\[ \left|(1 + \sup \phi - \phi(x, t))^{\frac{1}{2}} - (1 + \sup \phi - \phi(y, s))^\frac{1}{2}\right| \leq Cd \]

for any \((x, t), (y, s)\) in \( M \times [0, T)\), where \( d \) is the diameter of \( M \). In particular,

(5.19) \[ \sup \phi - \inf \phi \leq C \max\{d, d^2\} \]

**Proof.** Follows directly from the gradient estimates by using mean value theorem. \( \square \)
6. Long-time existence of solutions

We shall prove the first part of Theorem 1.2 now. Recall that the normalized solution \( \bar{\phi} \) solves,

\[
\frac{\partial \bar{\phi}}{\partial t} = f(\Lambda(\sqrt{-1}\partial \bar{\phi} + X[\phi])) - \psi[\phi] - \int_M \frac{\partial \phi}{\partial t} \omega^n
\]

(6.1)
\[
\bar{\phi}(x, 0) = \phi_0 - \int_M \phi_0 \omega^n
\]

where \( \phi \) is a solution of (1.1).

Since \( \int_M \bar{\phi} \omega^n = 0 \), there must be a \( y \in M \) such that \( \bar{\phi}(y) = 0 \). Using (5.19),

\[
|\bar{\phi}(x)| = |\bar{\phi}(x) - \bar{\phi}(y)| = |\phi(x) - \phi(y)| \leq C \max\{d, d^2\}
\]

(6.2)

Thus we obtain a uniform estimate for the normalized solution \( \bar{\phi} \).

The second order estimate derived in section 4 implies that equation (1.1) is uniformly parabolic. Hence by general parabolic theory, equation (1.1) has an admissible solution for some time \( [0, T) \), where \( T > 0 \) is the maximum time for which solution exists. Combined with the uniform apriori estimate \( |\bar{\phi}|_{2, \alpha} \leq C \) from the previous sections, it will follow that \( T = \infty \). Note that here \( C^{2, \alpha} \) estimate followed directly once the \( C^2 \) estimate is established as a consequence of the Evans-Krylov theorem for parabolic equations.

To show \( T = \infty \), assume for contradiction that \( T < \infty \). Then the solution \( \bar{\phi} \) of (6.1) can be extended to \( T \) using the apriori estimates. Now (6.1) with initial data
\(\tilde{\phi}(., T)\) is a parabolic PDE starting at time \(T\) with smooth initial data. Hence the solution can be extended to \([0, T+\varepsilon]\), for some \(\varepsilon > 0\). This contradicts the maximality of \(T\). Thus the solution exists for all time \([0, \infty)\). The long time existence of the solution \(\phi\) also follows similarly after obtaining an estimate (possibly depending on \(T\)) for \(\sup |\phi|\).

7. Harnack Inequality

In this section we will derive a Harnack inequality for the time derivative \(\phi_t\) of solutions of (1.1). For this purpose, we extend the results of Gill [4] and Li-Yau [8] to parabolic equations with lower order terms. More precisely, consider the following equation,

\[
\frac{\partial u}{\partial t} = G^{ij} \partial_i \partial_j u + \chi_k u_k + \chi_{\bar{k}} u_{\bar{k}} + \chi_0 u
\]

where \(G^{ij}, \chi_k, \chi_{\bar{k}}\) and \(\chi_0\) are time-dependent functions with \(G^{ij}\) being \(C^{3,1}_{x,t}\) and \(\chi_k, \chi_{\bar{k}}, \chi_0\) are assumed to be \(C^{1,1}_{x,t}\).

Let \(u\) be a positive solution of (7.1) in \(M \times [0, T)\) for some \(T > 0\). Define \(f = \log u\) and \(F = t(|\partial f|^2 - \alpha f_t)\), where \(|\partial f|^2 = G^{ij} f_i f_j\) and \(1 < \alpha < 2\). \(G^{ij}\) is assumed to be uniformly elliptic with \(0 < \lambda |\xi|^2 \leq G^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2\) in \(M\) for any vector \(\xi\). Also denote \(\langle X, Y \rangle = G^{ij} X_i Y_j\). All the norms and inner products in this section will be computed with respect to \(G^{ij}\).

**Lemma 7.1.** Let \(u \in C^{3,2}_{x,t}(M \times [0, T])\) be a positive solution of (7.1) in \([0, T)\). Then for \(t > 0\)

\[
|\partial f|^2 - \alpha f_t \leq C_1 + \frac{C_2}{t}
\]

for some constants \(C_1\) and \(C_2\) that depends only on the coefficient functions \(G^{ij}, \chi_k, \chi_{\bar{k}}\) and \(\chi_0\).

**Proof.** In the following calculations \(C, C_1\) and \(C_2\) will denote generic constants that may change from line to line. We will apply maximum principle to \(F\). Let \((x_0, t_0)\) be a point in \(M \times (0, T')\) where \(F\) attains maximum. Here \(0 < T' < T\) is a fixed time.

Then we have at \((x_0, t_0)\),

\[
\partial_k |\partial f|^2 = \alpha f_{kt}
\]

From (7.1) we derive
where 

\[ G^{ij} f_{ij} - f_t = -|\partial f|^2 - \chi_k f_k - \chi_k f_k - \chi_0 \]

Plugging this in \( F \) gives,

\[ F = -tG^{ij} f_{ij} + t(1 - \alpha)f_t - t\chi_k f_k - t\chi_k f_k - t\chi_0 \]

Next compute \( F_t \) and \( G^{ij} F_{ij} \).

\[ F_t = |\partial f|^2 - \alpha f_t + 2t\Re \langle \partial f, \partial f_t \rangle + t\partial_t G^{ij} f_{ij} - \alpha t f_{tt} \]

and,

\[ G^{ij} F_{ij} = tG^{ij} \left[ \partial_i \partial_j G^{kl} f_k f_l + \partial_i G^{kl} f_{kj} f_l + \partial_i G^{kl} f_{k} f_{lj} + \partial_j G^{kl} f_l f_{ki} + \partial_j G^{kl} f_k f_{li} \right. \]

\[ 
\left. + G^{kl} f_{ki} f_{lj} + G^{kl} f_{kj} f_{li} + G^{kl} f_{ki} f_{lj} - \alpha f_{tt} \right] \]

We now estimate all the terms in the above equation. Consider the first five terms in (7.7). By Cauchy-Schwarz inequality,

\[ tG^{ij} \left[ \partial_i \partial_j G^{kl} f_k f_l + \partial_i G^{kl} f_{kj} f_l + \partial_i G^{kl} f_{k} f_{lj} + \partial_j G^{kl} f_l f_{ki} + \partial_j G^{kl} f_k f_{li} \right] \]

\[
\leq C \left[ \frac{2t}{\epsilon} |\partial f|^2 + t\epsilon |\partial \bar{f}|^2 + t\epsilon |\partial f|^2 \right]
\]

where \( \epsilon > 0 \) is a small constant to be chosen later. Here \( |\partial \bar{f}|^2 = G^{ij} G^{kl} f_{ij} f_{kl} \) and \( |\partial f|^2 = G^{ij} G^{kl} f_{ij} f_{kl} \).

Write the third order terms in (7.7) using (7.5) as follows.

\[ tG^{ij} G^{kl} f_{kij} f_{l} + tG^{ij} G^{kl} f_{k} f_{lij} = 2t\Re \langle \partial f, \partial (G^{ij} f_{ij}) \rangle - tG^{kl} \partial_k G^{ij} f_{ij} f_{l} - tG^{kl} \partial_j G^{ij} f_k f_{ij} \]

\[ \geq 2t\Re \langle \partial f, \partial (G^{ij} f_{ij}) \rangle - \frac{Ct}{\epsilon} |\partial f|^2 - t\epsilon |\partial \bar{f}|^2 \]

\[ = -2t\Re \langle \partial f, \partial F \rangle + 2t(1 - \alpha)\Re \langle \partial f, \partial f_t \rangle \]

\[ \leq 2t\Re \langle \partial f, \partial [\chi_k f_k + \chi_k f_k + \chi_0] \rangle - \frac{Ct}{\epsilon} |\partial f|^2 - t\epsilon |\partial \bar{f}|^2 \]

We can write,
\[ \text{(7.10)} \]
\[ \Re\langle \partial f, \partial [\chi_k f_k + \chi_k f_k + \chi_0] \rangle = \left| \Re\langle \partial f, \partial \chi_k f_k \rangle + \Re\langle \partial f, \chi_k \partial f_k \rangle + \Re\langle \partial f, \partial \chi_k f_k \rangle \right| 
+ \Re\langle \partial f, \chi_k \partial f_k \rangle + \Re\langle \partial f, \partial \chi_0 \rangle \right| 
\leq C \left( |\partial f|^2 + |\langle \partial f, \partial f \rangle| + |\langle \partial f, \partial \bar{f} \rangle| \right) 
\leq \left( C + \frac{2C}{\epsilon} \right) |\partial f|^2 + \epsilon |\partial \bar{f}|^2 \]

Now combining (7.10), (7.9) and (7.6),

\[ \text{(7.11)} \]
\[ tG^{ij} F_{kij} f_i + tG^{ij} G^{kl} f_k f_{lij} \geq -2\Re\langle \partial f, \partial F \rangle - (\alpha - 1) F_t + (\alpha - 1)(|\partial f|^2 - \alpha f_t) 
- C_2 t |\partial f|^2 - \alpha(\alpha - 1) f_t - 2t \left( C + \frac{3C}{\epsilon} \right) |\partial f|^2 
- 2t \epsilon |\partial \bar{f}|^2 - 3t \epsilon |\partial \bar{f}|^2 \]

To estimate the last term in (7.7), we differentiate (7.5) wrt \( t \).

\[ \text{(7.12)} \]
\[ t \frac{\partial}{\partial t} (G^{ij} f_{ij}) = \frac{F}{t} - F_t + t(1 - \alpha) f_{tt} - t(\partial_t \chi_k f_k + \partial_t \chi_k \bar{f}_k + \chi_k f_{kt} + \chi_k \bar{f}_k + \partial_t \chi_0) \]

Use (7.3) to control \( f_{kt} \) and \( f_{kt} \) terms above.

\[ \text{(7.13)} \]
\[ |\chi_k f_{kt} + \chi_k \bar{f}_k| = \frac{1}{\alpha} |\chi_k \partial_k |f|^2 + \chi_k \partial_k |f|^2 | \leq \frac{C}{\alpha \epsilon} |\partial f|^2 + \frac{\epsilon}{2\alpha} |\partial f|^2 + \frac{\epsilon}{2\alpha} |\partial \bar{f}|^2 \]

Now estimate the last term in (7.7) as follows.

\[ -\alpha t G^{ij} f_{ij} = \alpha t \partial_t G^{ij} f_{ij} - \alpha t \frac{\partial}{\partial t} (G^{ij} f_{ij}) \]
\[ \geq - \frac{Ct}{\epsilon} - \epsilon t |\partial \bar{f}|^2 - \frac{\alpha}{t} F + \alpha F_t - t(1 - \alpha) f_{tt} - t(C_1 |\partial f|^2 
+ \epsilon |\partial \bar{f}|^2 + \epsilon |\partial \bar{f}|^2 + C_2) \]

where we used (7.12) and (7.13) in the last inequality. Combine eqs. (7.7), (7.8), (7.11) and (7.14) to get

\[ \text{(7.15)} \]
\[ G^{ij} F_{ij} \geq F_t - 2\Re\langle \partial f, \partial F \rangle - (|\partial f|^2 - \alpha f_t) - C_2 t |\partial f|^2 + t(1 - (5 + C) \epsilon) |\partial \bar{f}|^2 
+ t(1 - (3 + C) \epsilon) |\partial \bar{f}|^2 - C t \]
Choose \( \epsilon = \frac{1}{2(5 + C)} \). Also by (7.4),

\[
|\partial \bar{\partial} f|^2 \geq \frac{1}{n} (G^{ij} f_i f_j)^2 = \frac{1}{n} (|\partial f|^2 - f^i_i + (\chi_k f_k + \chi k f_k + \chi_k) f^i_i)^2 \\
\geq \frac{1}{2n} (|\partial f|^2 - f^i_i)^2 - C_1 |\partial f|^2 - C_2
\]

(7.16)

Plugging this above and using \( \partial F = 0 \) and \( G^{ij} F^i F^j - F_t \leq 0 \) at \((x_0, t_0)\), we get

\[
0 \geq -(|\partial f|^2 - \alpha f_t) - C t_0 |\partial f|^2 + \frac{t_0}{4n} (|\partial f|^2 - f^i_i)^2 - C t_0
\]

(7.17)

The rest of the proof can be completed by splitting into two cases when \( f_t(x_0, t_0) \) is non-negative and when it is negative, similar to [4]. For convenience, we provide the details here.

First assume that \( f_t(x_0, t_0) \geq 0 \), then we can deduce from the above equation,

\[
\frac{1}{4n} (|\partial f|^2 - f_t) \left( |\partial f|^2 - f - \frac{4n}{t_0} \right) \leq C_1 |\partial f|^2 + C_2
\]

(7.18)

So it follows that,

\[
|\partial f|^2 - f_t \leq C_1 |\partial f| + \frac{C_2}{t_0} + C_3
\]

(7.19)

Using Schwarz inequality we have,

\[
C_1 |\partial f| \leq \left( 1 - \frac{1}{\alpha} \right) |\partial f|^2 + C_4
\]

(7.20)

Plug this in (7.19) to get,

\[
\frac{1}{\alpha} |\partial f|^2 - f_t \leq C_1 + \frac{C_2}{t_0}
\]

(7.21)

For any \( x \in M \),

\[
F(x, T') \leq F(x_0, t_0) \\
\leq C_1 t_0 + C_2 \leq C_1 T' + C_5
\]

(7.22)

Now the result follows from the definition of \( F \) and taking \( T' = t \). For the case when \( f_t(x_0, t_0) < 0 \), from (7.17),

\[
\frac{t_0}{4n} |\partial f|^4 - |\partial f|^2 \leq C_1 t_0 |\partial f|^2 + C_2 t_0 - \alpha f_t
\]

(7.23)
Factor this to get,

\[(7.24) \quad |\partial f|^2 \left( \frac{1}{4nt} |\partial f|^2 - \frac{1}{t_0} - C_1 \right) \leq C_2 - \frac{\alpha}{t_0} f_t \]

It follows that,

\[(7.25) \quad |\partial f|^2 \leq C_1 + \frac{1}{t_0} + C\sqrt{-\frac{1}{t_0} f_t} \leq C_1 + \frac{C_2}{t_0} - \frac{1}{2} f_t \]

By (7.17) and using \( f_t(x_0, t_0) < 0 \) we get

\[(7.26) \quad \frac{1}{4n} (-f_t) \left( -f_t - \frac{4n\alpha}{t_0} \right) \leq C_1 |\partial f|^2 + \frac{1}{t_0} |\partial f|^2 + C_2 \]

This implies

\[(7.27) \quad -f_t \leq \frac{4n\alpha}{t_0} + C_1 |\partial f| + C_2 \sqrt{\frac{t_0}{t_0}} \]

Applying Cauchy-Schwarz inequality to the above gives,

\[(7.28) \quad -f_t \leq C_1 + \frac{C_2}{t_0} + \frac{|\partial f|^2}{2} \]

Plug (7.28) into (7.25) to get,

\[(7.29) \quad |\partial f|^2 \leq C_1 + \frac{C_2}{t_0} \]

Using this in (7.28) we deduce,

\[(7.30) \quad -\alpha f_t \leq C_1 + \frac{C_2}{t_0} \]

Adding the above two equations gives an estimate similar to (7.21).

\[(7.31) \quad |\partial f|^2 - \alpha f_t \leq C_1 + \frac{C_2}{t_0} \]

Now the proof is completed in the same way as in the first case.

\[\square\]

We use this lemma to derive a Harnack inequality along the lines of Li and Yau.
Theorem 7.2. Let $u$ be a solution of (7.1) as in Lemma 7.1. Then for $0 < t_1 < t_2,$

\begin{equation}
\sup_{x \in M} u(x,t_1) \leq C(t_1,t_2) \inf_{x \in M} u(x,t_2)
\end{equation}

for

\begin{equation}
C(t_1,t_2) = \left( \frac{t_2}{t_1} \right)^{C_2} \exp \left( \frac{C_3}{t_2 - t_1} + C_1(t_2 - t_1) \right)
\end{equation}

where $C_1,$ $C_2$ and $C_3$ are constants depending only on the $C^{3,1}_x(M \times [0,T])$ norm of $G^{ij}$ and $C^{1,1}_{x,t}(M \times [0,T])$ norms of $\chi_0,$ $\chi_k,$ $\overline{\chi}_k.$

Proof. Let $\gamma : [0,1] \to M$ be a unit speed curve such that $\gamma(0) = y$ and $\gamma(1) = x.$ Then we define a path $\eta : [0,1] \to M \times [t_1,t_2]$ joining $(y,t_2)$ to $(x,t_1)$ by $\eta(s) = (\gamma(s), (1-s)t_2 + st_1).$ We can write,

\begin{equation}
\log \frac{u(x,t_1)}{u(y,t_2)} = \int_0^1 \frac{d}{ds} f(\eta(s)) ds
\end{equation}

\begin{align*}
&= \int_0^1 \langle \dot{\gamma}, \partial f \rangle - (t_2 - t_1) f_t \ ds \\
&\leq \int_0^1 \frac{t_2 - t_1}{\alpha} \left( \frac{\alpha |\dot{\gamma}|}{t_2 - t_1} - |\partial f| \right)^2 + \frac{\alpha |\dot{\gamma}|^2}{t_2 - t_1} + \frac{t_2 - t_1}{\alpha} (|\partial f|^2 - \alpha f_t) \ ds \\
&\leq \int_0^1 \frac{C}{t_2 - t_1} + C(t_2 - t_1) \left( 1 + \frac{1}{(1-s)t_2 + st_1} \right) ds \\
&= C_1(t_2 - t_1) + C_2 \log \left( \frac{t_2}{t_1} \right) + \frac{C_3}{t_2 - t_1}
\end{align*}

where Lemma 7.1 is used in the fourth line. The final equation is obtained by taking exponentials on both sides followed by infimum in $y$ and supremum in $x$ over $M.$ \hfill \square

8. Convergence of the solution

In this section we assume that $X$ and $\psi$ are independent of $\phi$ (but still depends on $\partial \phi, \overline{\partial \phi}$). To show convergence of the solution we will use a standard iteration argument for the oscillation of the solution. Define $u = \phi_t$ as before and consider the following functions.
\[ v_n(x, t) = \sup_{y \in M} u(y, n - 1) - u(x, n - 1 + t) \]
\[ w_n(x, t) = u(x, n - 1 + t) - \inf_{y \in M} u(y, n - 1) \]

The oscillation of \( u \) is defined as a function of \( t \) by \( \omega(t) := \sup_{x \in M} u(x, t) - \inf_{x \in M} u(x, t) \). Then both \( v_n \) and \( w_n \) satisfy the following PDE.

\[ \frac{\partial \phi}{\partial t}(x, t) = \bar{G}^{ij}_x \partial_i \partial_j \phi + \chi_k(x, n - 1 + t) \partial_k \phi + \chi \bar{k}(x, n - 1 + t) \partial_k \phi \]

Note that \( \chi_0 = \bar{G}^{ij} \chi_{ij, \phi} - \psi_{\phi} \equiv 0 \) by assumption. If \( u(x, n - 1) \) is not constant then \( v_n \) is positive for some \( x \) in \( M \) at time \( t = 0 \). This implies that \( v_n \) is positive for all \( t > 0 \) by the maximum principle. Likewise for \( w_n \). So by applying Theorem 7.2 to \( v_n \) and \( w_n \) with \( t_1 = \frac{1}{2} \) and \( t_2 = 1 \),

\[ \sup_{x \in M} u(x, n - 1) - \inf_{x \in M} u \left( x, n - \frac{1}{2} \right) \leq C \left( \sup_{x \in M} u(x, n - 1) - \sup_{x \in M} u(x, n) \right) \]
\[ \sup_{x \in M} u \left( x, n - \frac{1}{2} \right) - \inf_{x \in M} u(x, n - 1) \leq C \left( \inf_{x \in M} u(x, n) - \inf_{x \in M} u(x, n - 1) \right) \]

where \( C := C(\frac{1}{2},1) \). By adding the two equations above, we see that \( \omega(t) \) satisfies the following recursion.

\[ \omega(n - 1) + \omega \left( n - \frac{1}{2} \right) \leq C(\omega(n - 1) - \omega(n)) \]

It follows that \( \omega(n) \leq \delta \omega(n - 1) \) for some \( \delta < 1 \) and by iterating we get that \( \omega(t) \leq Ce^{-\beta t} \) for \( \beta = -\log \delta \). If \( u(x, n - 1) \) is constant the same estimate holds by maximum principle applied to \( v_n \). Fix \( (x, t) \in M \times [0, \infty) \). Since \( \int_M \frac{\partial \phi}{\partial t} \omega^n = 0 \), there is a point \( y \in M \) such that \( \frac{\partial \phi}{\partial t}(y, t) = 0 \). Hence,

\[ \left| \frac{\partial \phi}{\partial t}(x, t) \right| = \left| \frac{\partial \phi}{\partial t}(x, t) - \frac{\partial \phi}{\partial t}(y, t) \right| \leq Ce^{-\beta t} \]

Now \( h(t) = \bar{\phi} + \frac{Ce^{-\beta t}}{\beta} \) satisfies \( \frac{\partial h}{\partial t} \leq 0 \). So \( h(t) \) is bounded and monotonically decreasing for each \( x \). Denote the limit function by \( \bar{\phi}_\infty \). From the definition of \( h(t) \) it is clear that \( \bar{\phi} \) converges pointwise in \( x \) to the same function \( \bar{\phi}_\infty \) as \( t \to \infty \).
To show that the convergence is smooth, we assume for contradiction that there exists a sequence of times \( \{t_l\} \) such that,

\begin{equation}
|\tilde{\phi}(., t_l) - \tilde{\phi}_\infty|_{C^k(M)} > \epsilon \forall l
\end{equation}

for some \( k \).

Using the uniform estimates on the \( C^\infty \)-norm of \( \tilde{\phi} \), we can extract a subsequence \( \{t_{l_m}\} \) along which \( \tilde{\phi} \) converges in \( C^\infty \) to some smooth function \( \hat{\phi}_\infty \). But then by pointwise convergence we have that \( \hat{\phi}_\infty \equiv \tilde{\phi}_\infty \), and hence (8.6) is not possible.

Finally we prove the convergence in Theorem 1.2. Take limit \( t \to \infty \) in (6.1). By (8.5) and the previous paragraph, it follows that

\begin{equation}
f(\Lambda(\sqrt{-1}\partial\bar{\partial}\tilde{\phi}_\infty + X[\tilde{\phi}_\infty])) = \psi[\tilde{\phi}_\infty] + a
\end{equation}

where,

\[
a = \lim_{t \to \infty} \int_M \frac{\partial \tilde{\phi}}{\partial t} \omega^n
\]
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