Deconstructing Functions on Quadratic Surfaces into Multipoles

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Abstract

Any homogeneous polynomial $P(x, y, z)$ of degree $d$, being restricted to a unit sphere $S^2$, admits essentially a unique representation of the form $\lambda_0 + \sum_{k=1}^{d} \lambda_k [\prod_{j=1}^{k} L_{kj}]$, where $L_{kj}$'s are linear forms in $x, y$ and $z$ and $\lambda_k$'s are real numbers. The coefficients of these linear forms, viewed as 3D vectors, are called multipole vectors of $P$. In this paper we consider similar multipole representations of polynomial and analytic functions on other quadratic surfaces $Q(x, y, z) = c$, real and complex. Over the complex numbers, the above representation is not unique, although the ambiguity is essentially finite. We investigate the combinatorics that depicts this ambiguity. We link these results with some classical theorems of harmonic analysis, theorems that describe decompositions of functions into sums of spherical harmonics. We extend these classical theorems (which rely on our understanding of the Laplace operator $\Delta_{S^2}$) to more general differential operators $\Delta_Q$ that are constructed with the help of the quadratic form $Q(x, y, z)$. Then we introduce modular spaces of multipoles. We study their intricate geometry and topology using methods of algebraic geometry and singularity theory. The multipole spaces are ramified over vector or projective spaces, and the compliments to the ramification sets give rise to a rich family of $K(\pi, 1)$-spaces, where $\pi$ runs over a variety of modified braid groups.

1 Introduction: the cosmic background motivation

This paper aims to generalize and extend some results in [KW] and [W] about deconstruction of cosmic microwave background radiation (CMBR) into multipole vectors and explain these results to a mathematically inclined audience. Our exposition, to a degree, is self-sufficient.

Cosmologists who study the CMBR routinely decompose it into a sum of spherical harmonics—the eigenfunctions of the Laplace operator on the 2-sphere. The summand that corresponds to the eigenvalue $-d(d+1)$ of the Laplace operator is called a $d$-th spherical harmonic. This decomposition helps to analyze the correlations between the magnitude and geometry of various harmonics. The ultimate goal of these investigations is to understand the geometry of the visible universe and the physics of the Big Bang.

The data obtained by Wilkinson Microwave Anisotropy Probe (WMAP) reveal puzzling correlations between the low-$d$ portion of the harmonic decomposition of the cosmic microwave background
Multipole vectors provide a convenient means to study the isotropic properties of CMBR. In particular, they help to quantify the planarity of a given multipole, as well as to compare the alignment of two different multipoles [CHS]. Jeff Weeks and I, coming from a background in pure mathematics, were unable to decipher the formalism and terminology of [CHS] and chose instead to re-create the multipole vector concept from scratch (see [KW]). Later on, as we read the papers [L] and [D], we realized that we managed to reinvent the wheel twice: our key algebraic lemma turns out to be a classical theorem of Sylvester [S], [S1], while the application of that lemma to harmonic polynomials on sphere implicitly was known to Maxwell [M].

The real-valued spherical harmonics of order $d$ are precisely the homogeneous harmonic polynomials of degree $d$ in the variables $x$, $y$, and $z$. Thus, we sought to understand the multipole vectors of Copi, Hutener and Starkman [CHS] from a polynomial point of view. As in [CH], this calls for using some elementary tools from the field of algebraic geometry. Translated to the language of polynomials, [CHS]'s motivating goal was to factor every homogeneous harmonic polynomial $P$ of degree $d$ on the sphere into a product of linear factors

$$P(x, y, z) = (a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) \ldots (a_d x + b_d y + c_d z).$$

Such a factorization is impossible in general, as [CHS] implicitly acknowledge by their introduction of suitable error terms. The correct statement is given by the Sylvester Theorem [S], [S1]:

**Theorem 1** Every real homogeneous polynomial $P$ of degree $d$ in $x$, $y$, and $z$ may be written as

$$P(x, y, z) = (a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) \ldots (a_d x + b_d y + c_d z) + (x^2 + y^2 + z^2) \cdot R(x, y, z),$$

where the remainder term $R$ is a homogeneous polynomial of degree $d - 2$. The decomposition is unique up to reordering and rescaling of the linear factors in the product.

For a harmonic homogeneous $P$, a somewhat similar representation was discovered by James Clerk Maxwell in his 1873 Treatise on Electricity and Magnetism [M]. Its relation to algebraic geometry in general, and to the Sylvester theorem, in particular, is well-explained by Courant and Hilbert in [CH] VII, Sec. 5. We will describe the Maxwell representation later in the paper.

Sylvester’s Theorem 1 (which is just an application of Bézout’s theorem) has a pleasing corollary:

**Theorem 2** [KW] When restricted to a unit sphere $S^2$, every real polynomial $P$ in $x$, $y$, and $z$ of degree $d$ can be written in the form

$$P(x, y, z) = \lambda_0 + \lambda_1 (a_{11} x + b_{11} y + c_{11} z) + \lambda_2 (a_{21} x + b_{21} y + c_{21} z)(a_{22} x + b_{22} y + c_{22} z) + \ldots$$

$$\ldots + \lambda_d (a_{dd} x + b_{dd} y + c_{dd} z)(a_{dd} x + b_{dd} y + c_{dd} z) \ldots (a_{dd} x + b_{dd} y + c_{dd} z)$$

1For example, the spherical harmonic $Y_d^m$ is the polynomial $x^m + y^m - 2z^2$, up to normalization.

2rediscovered in [WK] by the authors.
The decomposition is unique up to reordering and rescaling of the linear factors in each of the products. In particular, one can pick each \((a_{jk}, b_{jk}, c_{jk})\) to be a unit vector and, for \(j > 0\), each \(\lambda_j \geq 0\).

It is well-known that spherical harmonics form a basis in the space \(L_2(S^2)\) of \(L_2\)-integrable functions on a unit sphere (see [CH]). As a result, an infinite Maxwell-type decomposition is available for any function \(f \in L_2(S^2)\) (cf. [L], formula (26)).

**Theorem 3** Any real function \(f \in L_2(S^2)\) has a representation in the form

\[
f(x, y, z) = \sum_{d=1}^{\infty} \left\{ \lambda_{d,0} + \sum_{k=1}^{d} \lambda_{d,k} \prod_{j=1}^{k} (a_{dj}x + b_{dj}y + c_{dj}z) \right\}
\]

where the series converges in the \(L_2\)-norm, and the mutually orthogonal polynomials in the figure brackets are among the \(d\)-th spherical harmonics. This representation is unique, up to reordering and rescaling of the linear factors in each of the products.

In the space of real continuous functions \(f\) on \(S^2\) there is a dense (in the sup-norm) and \(O(3)\)-invariant subset \(\mathcal{F}(S^2)\) comprised of functions that admit a representation in the form of a series

\[
\lambda_0 + \sum_{d=1}^{\infty} \lambda_d \prod_{k=1}^{d} (a_kx + b_ky + c_kz)
\]

that uniformly converges on the sphere. Each functions from \(\mathcal{F}(S^2)\) admits a canonical extension into the interior of the unit ball \(D^3\) where it produces a real analytic function.

This theorem, claiming a form of stability of decomposition (3) under the polynomial approximations, is a useful tool for analyzing patterns in the spherical sky [W].

At the first glance, the sphere seems to occupy a special place in these results. However, what is really important, is the quadratic nature of the surface. This paper is concerned with similar decompositions of functions on other quadratic algebraic skies (real and complex). For instance, it is natural to wonder: Can one deconstruct in a similar fashion polynomial or real analytic functions on a hyperboloid? The short answer is “Yes, one can”, but the uniqueness of the representation is lost.

The focal point of this paper is to describe rich and interesting algebro-geometrical structures and topology of modular spaces of multipoles. In the process, we bring under the same roof a variety of mathematical techniques and constructions that belong to the fields of algebraic geometry, singularity theory, algebraic topology, harmonic analysis, and the polynomial approximation theory. None of our techniques is very advanced, but their natural appearance within the context of studying quadratic surfaces is pleasing and somewhat surprising...

The reader can follow two distinct treads in the core of the paper. The first unifies results that are linear in nature and are based on the classical theory of harmonic analysis on quadratic surfaces. These results are valid for quadratic hypersurfaces in any dimension. The second tread winds through the results that require methods of algebraic geometry. These results are uniquely three-dimensional, that is, applicable only to quadratic surfaces.
2 Multipoles and polynomials on complex quadratic surfaces

We denote by $\mathbb{C}[x, y, z]$ the ring of complex polynomials in the variables $x, y,$ and $z$. Let $Q(x, y, z)$ be an irreducible quadratic form over the complex numbers $\mathbb{C}$. From a strictly algebraic perspective, one can interpret this section as describing the ways in which the quotient ring $\mathbb{C}[x, y, z]/(Q - \lambda)$, $\lambda \in \mathbb{C}$, fails to be a Unique Factorization Domain. However, the flavor of our approach to this problem is more geometrical and combinatorial.

The reader will be well-advised to ignore for a while the asterisks in our notations: they are there to distinguish between vector spaces and their duals. Let $S = \{Q(x, y, z) = 1\}$ be a complex algebraic surface in $\mathbb{C}^3$. At the same time, $\{Q(x, y, z) = 0\}$ gives rise to a complex projective curve $Q \subset \mathbb{C}P^2$, where $\mathbb{C}P^2$ stands for the projectivization of the 3-space with coordinates $x, y,$ and $z$.

As in Theorem 2, we aim to decompose any polynomial $P(x, y, z)$ restricted to $S$ as an “economic” sum of products of homogeneous linear polynomials.

First, consider the case of an homogeneous $P$ and the corresponding complex projective quadratic curve $P$ in $\mathbb{C}P^2$ it generates. Denote by $Z(P, Q)$ the intersection $P \cap Q$. We assume that $P$ and $Q$ do not share a common component, so that $Z(P, Q)$ is a finite set. When $Z(P, Q)$ is a complete intersection, it consists of exactly $2d$ points, where $d = \deg(P)$.

**Definition 1** Let $Q(x, y, z)$ be an homogeneous irreducible quadratic polynomial, and $P(x, y, z)$ an homogeneous polynomial of degree $d$. Assume that $Z(P, Q)$ is a complete intersection. A parcelling of the set $Z(P, Q)$ is comprised of a number of subsets $\{Z_\nu \subset Z(P, Q)\}_\nu$ so that:

- $Z(P, Q) = \bigsqcup_\nu Z_\nu$
- $Z_\nu \cap Z_{\nu'} = \emptyset$, provided $\nu \neq \nu'$
- the cardinality of each subset $Z_\nu$ is equal to 2.

When $Q$ is irreducible and $P$ is not divisible by $Q$, the intersection set $Z(P, Q)$ is equipped with a function $\mu$ which assigns to each point $p \in Z(P, Q)$ its multiplicity $\mu(p)$.

We denote by $\mathbb{Z}_+$ the set of non-negative integers and by $\mathbb{N}$ the set of positive integers.

**Definition 2** Let $Z$ be a finite set equipped with a multiplicity function $\mu : Z \to \mathbb{N}$ whose $l_1$-norm $\|\mu\|_1$ is $2d$. A generalized parcelling of $(Z, \mu)$ is a collection of functions $\mu_\nu : Z \to \{0, 1, 2\}$, such that

- $\sum_\nu \mu_\nu = \mu$
- $\|\mu_\nu\|_1 = 2$

Of course, each parcelling of $Z(P, Q)$ is also a generalized parcelling, where the roles of the functions $\mu_\nu$ are played by the characteristic functions of the parcels $Z_\nu$. 
In the case of $Q = x^2 + y^2 + z^2$, the following lemma is due to Sylvester [S], [S1].

Lemma 1 Let $Q(x, y, z)$ be an irreducible homogeneous quadratic polynomial and let $P(x, y, z)$ be a homogeneous polynomial of degree $d$ which is not divisible by $Q$. Consider a generalized parcelling

$$\mu = \sum_\nu \mu_\nu$$

of the multiplicity function $\mu : Z(P, Q) \to \mathbb{N}$.

Then the homogeneous polynomial $P(x, y, z)$ admits a representation of the form

$$Q(x, y, z) \cdot R(x, y, z) + \prod_\nu L_\nu(x, y, z),$$

where $L_\nu$ denotes a linear homogeneous polynomial that vanishes at each point $p \in Z(P, Q) \subset Q$ with the multiplicity $\mu_\nu(p)$.

Proof There exist a linear polynomial $L_\nu$ and a corresponding line $L_\nu \subset \mathbb{CP}^2_\nu$ such that the multiplicity of $L_\nu \cap Q$ at each point $p \in Z(P, L_\nu)$ equals $\mu_\nu(p)$. When $\mu_\nu(p) = 2$, the line must be tangent to $Q$ at $p$. When the support of $\mu_\nu$—the parcel $Z_\nu \subset Z(P, Q)$—is comprised of two points, the line $L_\nu$ is chosen to contain $Z_\nu$.

Let us compare the restrictions of $P$ and $\prod_\nu L_\nu$ to the curve $Q$. Both polynomials are of the same degree $d$. Moreover, the curve $L := \{\prod_\nu L_\nu = 0\}$ intersects with $Q$ at the points of $Z(P, Q)$, where it realizes the multiplicities prescribed by the function $\mu$. It follows that the restrictions of $P$ and $\prod_\nu L_\nu$ to $Q$ are proportional, that is, for an appropriate choice of scalar $\lambda$, $P|_Q = \lambda \cdot \prod_\nu L_\nu|_Q$. Just take $\lambda = P(q)/\prod_\nu L_\nu(q)$ where $q \in Q \setminus Z(P, Q)$. With this choice, the curves $\{P - \lambda \prod_\nu L_\nu = 0\}$ and $Q$ intersect so that the total multiplicity of the intersection is at least $2d + 1$. By the Bézout theorem, this is possible only when $Z(P - \lambda \prod_\nu L_\nu) \supset Q$. Employing the irreducibility of $Q$, we get that $P - \lambda \cdot \prod_\nu L_\nu$ must be divisible by $Q$. This completes the proof. □

Corollary 4 Any effective divisor $D$ on the complex quadratic surface $S = \{Q(x, y, z) = 0\}$ that is the zero set of a homogeneous polynomial $P(x, y, z)$ can be represented as a sum of lines $D_\nu$. □

In general, the representation (6) of $P$ depends on a generalized parcelling, and thus, is not unique.

We denote by $V(d)$ the complex vector space of homogeneous polynomials in $x, y, z$ of degree $d$. Its dimension is $(d^2 + 3d + 2)/2$. Consider a vector subspace $V_Q(d) \subset V(d)$ comprised of polynomials divisible by $Q$. There is a canonical imbedding $\beta_Q : V(d - 2) \to V(d)$ whose image is $V_Q(d)$. It is produced by multiplying polynomials of degree $d - 2$ by $Q$. Thus, any space $V(d)$ is equipped with a natural filtration $F_Q(d) = \{V_Q^k(d)\}_k$ by subspaces of polynomials divisible by various powers $\{Q^k\}$ of $Q$. We equip each $V(d)$ with an Hermitian inner product. For various $d$’s, we insist that these inner products are synchronized by the requirement: all the imbeddings $\beta_Q$ must be isometries. Denote by $V_Q^\perp(d)$ the subspace of $V(d)$ orthogonal to $V_Q(d)$. It can be identified with the quotient space $V(d)/V_Q(d)$—the $d$-graded part of the algebra of regular functions on the complex cone $\{Q(x, y, z) = 0\}$. The dimension of $V_Q^\perp(d)$ is equal to $(2d + 1)$.

For a real form $Q$, similar spaces based on real homogeneous polynomials of a degree $d$ make sense. We denote them $V(d; \mathbb{R}), V_Q(d; \mathbb{R})$, and $V_Q^\perp(d; \mathbb{R})$.  

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When $Q = x^2 + y^2 + z^2$, there is a classical interpretation for the quotient $V(d; \mathbb{R})/V_Q(d; \mathbb{R})$. It can be identified with the set of harmonic homogeneous polynomials of degree $d$ (cf. [CH] and [M]). In other words, the kernel $Har(d; \mathbb{R})$ of the Laplace operator $\Delta : V(d; \mathbb{R}) \to V(d-2; \mathbb{R})$ is complementary to $V_Q(d; \mathbb{R})$ in $V(d; \mathbb{R})$. This leads to a decomposition similar to the one in (6) (cf. [Sh, Theorem 22.2], [L], and [W]):

$$V(d; \mathbb{R}) \approx Har(d; \mathbb{R}) \oplus V_Q(d; \mathbb{R}) \quad (7)$$

The direct summands in (7) are orthogonal with respect to the inner product in $V(d; \mathbb{R})$ defined by the formula $\langle f, g \rangle = \int_{S^2} f \cdot g \, dm$. The measure $dm$ on the sphere $S^2$ is the standard one.

Moreover, according to Maxwell, any polynomial $P \in Har(d; \mathbb{R})$ admits a beautiful representation of the form

$$P(x, y, z) = r^{2d+1} \nabla_{v_1} \nabla_{v_2} \cdots \nabla_{v_d} \left( \frac{1}{r} \right), \quad (8)$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, $\{v_j \in \mathbb{R}^3\}$ are some vectors, and $\nabla_{v_j}$ stands for the directional derivative operator.

Physics behind Maxwell's representation is quite transparent: $1/r$ is a potential of a single electrical charge, $\nabla_{v_1}(1/r)$ is a potential of a “virtual” (that is, very small) dipole formed by two opposite charges, $\nabla_{v_2} \nabla_{v_1}(1/r)$ is a potential of a “virtual” quadropole—a close pair of $v_1$-oriented dipoles merging along the direction of $v_2$—, $\nabla_{v_3} \nabla_{v_2} \nabla_{v_1}(1/r)$ is a potential of a “virtual” octopole, and so on...

Note that, the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ can act on complex polynomials as well. Moreover, if $\Delta(f) = 0$ for some analytic function $f$, then its real and imaginary parts are also harmonic: $\Delta(\text{Re}f) = 0$ and $\Delta(\text{Im}f) = 0$.

For any invertible complex $(3 \times 3)$-matrix $A = (a_{jk})$, consider a symmetric matrix $B = (b_{jk}) = A \cdot A^T$ and the corresponding quadratic form $Q(v) = \langle vB, v \rangle$, where $v = (x, y, z)$, $\langle \cdot, \cdot \rangle$ stands for the inner product $xx' + yy' + zz'$ in $\mathbb{C}^3$, and the upper script $T$ denotes the matrix transposition. Employing $Q$, we can form a second order differential operator $\Delta_Q = \sum_{1 \leq i,j \leq 3} b_{ij} \partial_i \partial_j$ acting on holomorphic functions $f$ in the complex variables $\{x_1 = x, x_2 = y, x_3 = z\}$. Here $\{b_{ij}\}$ denote elements of the inverse matrix $B^{-1}$. Formally, $\Delta_Q = [\partial] \cdot B^{-1} \cdot [\partial]^T$ where $[\partial] := (\partial_x, \partial_y, \partial_z)$.

Given a non-degenerated quadratic form $Q$, there is a change of complex coordinates $(x', y', z') = (x, y, z) \cdot A$ that reduces it to the canonical form $Q' = x'^2 + y'^2 + z'^2$. Consider the complex Laplace operator $\Delta' = \partial_{x'}^2 + \partial_{y'}^2 + \partial_{z'}^2$, in the new coordinates $(x', y', z')$. Then, for any holomorphic function $f(x', y', z')$, we have

$$[\Delta' f]( (x, y, z) \cdot A) = \Delta_Q [f((x, y, z) \cdot A)]$$

Thus, there is a 1-to-1 correspondence between harmonic homogeneous polynomials $f(x', y', z')$ of degree $d$ and degree $d$ homogeneous polynomial solutions $g(x, y, z)$ of the equation

$$\Delta_Q(g(x, y, z)) = 0.$$ 

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Let us examine the intersection $\text{Ker}(\Delta') \cap V_Q'(d)$. For any homogeneous polynomial $T$ of degree $d - 2$, we get $\Delta'(Q' \cdot T) = Q' \cdot \Delta'(T) + \Delta'(Q') \cdot T + 2(\nabla T, \nabla Q') = Q' \cdot \Delta'(T) + (4d - 6)T$. Therefore, if $Q' \cdot T \in \text{Ker}(\Delta')$, then $T$ must be divisible by $Q'$ (note that $\Delta'(T) = 0$ implies $T = 0$). Put $T = Q' \cdot T_1$ and $\kappa(d) = 4d + 2$. The equation $0 = \Delta'(Q' \cdot T) = Q' \cdot \Delta'(Q' \cdot T_1) + \kappa(d - 2) \cdot Q' \cdot T_1$ is equivalent to the equation $0 = \Delta'(Q' \cdot T_1) + \kappa(d - 2) \cdot T_1 = Q' \cdot \Delta'(T_1) + [\kappa(d - 2) + \kappa(d - 4)] \cdot T_1$. Again, it follows that $T_1$ must be divisible by $Q'$. Continuing inductively this kind of reasoning, we see that $\text{Ker}(\Delta') \cap V_Q'(d) = \{0\}$. On the other hand, one can verify that $\dim [\text{Ker}(\Delta')] + \dim [V_Q'(d)] = \dim [V(d)]$. Thus, $V(d) = \text{Ker}(\Delta') \oplus V_Q'(d)$.

A polynomial $P(x', y', z')$ is divisible by $x'^2 + y'^2 + z'^2$, if and only if, $P((x, y, z) \cdot A)$ is divisible by $Q(x, y, z)$. Therefore, $V(d) = \text{Ker}(\Delta_Q) \oplus V_Q(d)$ as well.

Let $O_Q(3; \mathbb{C})$ denote a subgroup of the general linear group $GL(3; \mathbb{C})$ that preserves the quadratic form $Q$. A matrix $U \in O_Q(3; \mathbb{C})$ if and only if $U B U^T = B$. The natural $O_Q(3; \mathbb{C})$-action on the space of $x, y,$ and $z$-variables induces an action on the polynomial space $V(d)$. Evidently, $V_Q(d)$ is invariant under this action. On the other hand, for any polynomial $P(x, y, z)$,

$$\nabla P((x, y, z) \cdot U) = [\Delta_Q P]((x, y, z) \cdot U),$$

where the operator $\Delta_Q := [\tilde{\partial}] \cdot U B^{-1} U \cdot [\tilde{\partial}]^T$. Since $U B U^T = B$, we get $(U^{-1})^T B^{-1} U^{-1} = B^{-1}$. By a simple algebraic trick, it follows that $U^T B^{-1} U = B^{-1}$. As a result, both the quadratic form $Q$ and the kernel $\text{Ker}(\Delta_Q)$ are invariant under the $O_Q(3; \mathbb{C})$-action.

Consider Maxwell’s representation (8) of a real homogeneous harmonic polynomial $P(x', y', z')$ of degree $d$. It gives rise to a map $\Xi$ that takes the sets of vectors $v_1, v_2, \ldots, v_d$ to elements of $\text{Har}(d, \mathbb{R})$. The map $\Xi$ is evidently linear in each of the $v_j$’s, and thus it is a real polynomial map with a vector space of real dimension $2d + 1$ for its target. By [CH], [L] and [W], $\Xi$ is onto the vector space $\text{Har}(d, \mathbb{C}) \approx \text{Har}(d, \mathbb{R}) \oplus i \text{Har}(d, \mathbb{R})$ (indeed, the image of $\Xi^C$ in $\text{Har}(d, \mathbb{C})$ is a complex algebraic set containing a totally real vector subspace $\text{Har}(d, \mathbb{R})$ of a maximal dimension). In other words, formula (8) must be valid for any complex homogeneous harmonic polynomial $P(x', y', z')$ of degree $d$ and appropriate complex vectors $u_1, u_2, \ldots, u_d \in \mathbb{C}^3$. In fact, each vector $u_j = v_j \cdot A^{-1}$. Formula (8) describes any homogeneous complex polynomial solution of the equation $\Delta_Q(f) = 0$ in the $(x', y', z')$ coordinates. Translating them back to the $(x, y, z)$-coordinates with the help of the identity $[\nabla_{x'} f'][(x \cdot A) = \nabla_{x'} A^{-1} [f'(x \cdot A)]$ leads to a formula (10) below. Thus, we have established the following proposition:

**Lemma 2** The decomposition

$$V(d) = \text{Ker}(\Delta_Q) \oplus V_Q(d)$$

holds for any non-degenerated complex quadratic form $Q$. It is invariant under the natural $O_Q(3; \mathbb{C})$-action on $V(d)$. Moreover, for any $P \in \text{Ker}(\Delta_Q)$ of degree $d$, there exist vectors $\{u_k \in \mathbb{C}^3\}_{1 \leq k \leq d}$ so that the generalized Maxwell formula

$$P(x, y, z) = Q(x, y, z)^{d + \frac{1}{2}} \cdot \nabla u_1 \nabla u_2 \ldots \nabla u_d \left( Q(x, y, z)^{-\frac{1}{2}} \right)$$

is valid. The real part of the LHS of (10) generates all real homogeneous polynomial solutions $P$ of the equation $\Delta_Q(P) = 0$. 

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In particular, with $Q = x^2 + y^2 - z^2$, any degree $d$ homogeneous complex polynomial solution $P$ of the wave equation $(\partial_x^2 + \partial_y^2)P = \partial_z^2 P$ is given by the formula (10). □

Formula (10) might pose a slight challenge: after all, square roots are multivalued analytic functions. However, the $\pm$-ambiguities in picking a single-valued branch cancel each other. We shall see that, even up to reordering and rescaling of the multipole vectors $\{w_k\}$, the complex Maxwell representation (10) of a given “$Q$-harmonic” $P$ is not unique.

**Remark.** In fact, by a similar argument, the decomposition (9) is available for homogeneous polynomials in any number of variables and for any non-degenerated quadratic form $Q$ (see [Sh], Theorem 22.2, for the proof). At the same time, the representation (10) seems to be a 3-dimensional phenomenon. It looks like that not any $Q$-harmonic polynomial in $n > 3$ variables can be expressed in terms of $d$ directional derivatives sequentially applied to the $Q$-harmonic potential $Q^{1-n/2}$. This seems to be related to the failure of the Sylvester-type formula (6) for $n > 3$. When I raised this issue with Michael Shubin, he proposed a nice conjecture in the flavor of Maxwell’s representation (although, not a direct generalization of it). In the Maxwell representation, one employs a differential operator which is a monomial in $d$ directional derivatives, while in the conjecture below one invokes a differential operator which is a polynomial in $d$ directional derivatives.

**Shubin’s Conjecture** (An $n$-dimensional variation on the theme of Maxwell’s representation).

Let $n > 2$. For any real homogeneous and harmonic polynomial $P$ of degree $d$ in $n$ variables $\{x_j\}$, there exists a unique real homogeneous and harmonic polynomial $P^\bullet$ also in $n$ variables such that

$$P(x_1, \ldots, x_n) = r^{2d+n-2} \left[ P^\bullet(\partial x_1, \ldots, \partial x_n) r^{-2-n} \right],$$

where $r = (\sum_{j=1}^n x_j^2)^{1/2}$, and the differential operator $P^\bullet(\partial x_1, \ldots, \partial x_n)$ being applied to the potential function $r^{-2-n}$. Moreover, the polynomials $P$ and $P^\bullet$ are proportional with the coefficient of proportionality depending only on $d$ and $n$. □

Given a vector space $V$, we denote by $V^\circ$ the space $V \setminus \{0\}$. We consider the 3-space $V(1) \approx \mathbb{C}^3$ of linear forms $L = ax + by + cz$ (which we identify with the space of their coefficients $\{(a, b, c)\}$) and the 3-space $V_3(1) \approx \mathbb{C}^4$ with the coordinates $x, y, z$. This calls for a distinction in notations for their projectivizations: $\mathbb{C}P^2$ and $\mathbb{C}P^2_*$. Hence, each point $(a, b, c) \in V(1)^\circ$ determines a point $l = [a : b : c]$ in $\mathbb{C}P^2$ and a complex line $\mathcal{L}$ in $\mathbb{C}P^2_*$—the zero set of the form $L$.

Now, let us return to the decomposition (6). Each polynomial $L_\nu$ from (6) can be viewed as a vector in $V(1) = V_Q^\perp(1)$. Therefore, a given homogeneous polynomial $P$ of degree $d$ which is not divisible by $Q$, together with the appropriate generalized parcelling, produces a collection of non-zero vectors $\{w_\nu \in V(1)\}_\nu$. We shall call this unordered set of vectors $\{w_\nu\}$ the leading multipole vectors of $P$ with respect to the corresponding generalized parcelling of $\mu : Z(P, Q) \to \mathbb{Z}_+$.\footnote{These vectors are not necessarily distinct.}

**Lemma 3** For a given generalized parcelling $\mu = \sum_\nu \mu_\nu$ of $\mu : Z(P, Q) \to \mathbb{Z}_+$, the subordinate leading multipole vectors $\{w_\nu \in V(1)\}_\nu$ are unique up to reordering and rescaling of the $L_\nu$’s (equivalently, the polynomial $R$ in (6) is unique).
We use the same notations as in the proof of Lemma 1. By an argument as in that proof, any linear polynomial $L'_{\nu}$ which defines a line $L'_{\nu}$ with the property $L'_{\nu} \cap \mathcal{Q} = L_{\nu} \cap \mathcal{Q}$ (the points of intersection have multiplicities prescribed by $\mu_{\nu}$) is of the form $\lambda L_{\nu}$. Here $\lambda \in \mathbb{C}^*$ and $L_{\nu}$ is a preferred polynomial corresponding to a vector $w_{\nu} \in V(1)$. Indeed, as in the proof of Lemma 1, for an appropriate $\lambda$, by the Bézout Theorem, $\lambda L_{\nu} - L'_{\nu}$ must be divisible by $\mathcal{Q}$. □

As a result, the multipole is well defined by the parcelling of $\mu$ modulo some rescaling. Clearly, one can always replace each $L_{\nu}$ in (6) with $\lambda_{\nu} L_{\nu}$ as long as $\prod_{\nu} \lambda_{\nu} = 1$. Next, we analyze the exact meaning of the “reordering and rescaling” ambiguity.

Let $H_q$ be an abelian subgroup of $(\mathbb{C}^*)^q$ formed by vectors $\{\lambda_{\nu}\}_{1 \leq \nu \leq q}$ subject to the restriction $\prod_{\nu} \lambda_{\nu} = 1$. Its rank is $q - 1$. Let $S_q$ stand for the symmetric group in $q$ symbols $\{1, 2, \ldots, q\}$. We denote by $\Sigma_q$ an extension $1 \to H_q \to \Sigma_q \to S_q \to 1$ of $S_q$. This group is generated by the obvious actions of $S_q$ and $H_q$ on $(\mathbb{C}^*)^q$.

We introduce an orbit-space

$$\mathcal{M}(k) := [(V(1)^{\circ})^k] / \Sigma_k$$

whose points encode the products of linear forms as in the decomposition (6). Here the group $\Sigma_k$ acts on the products of spaces by permuting them and by rescaling their vectors. Its subgroup $H_k$ acts freely.

Because of the uniqueness of the prime factorization in the polynomial ring $\mathbb{C}[x, y, z]$, the space $\mathcal{M}(k)$ provides us with a 1-to-1 parametrization of the space of homogeneous degree $k$ polynomials that are products of linear forms. As an abstract space, (11) can be expressed as

$$\mathcal{M}(k) = \left\{ [\mathbb{C}^3^{\circ}]^k \right\} / \Sigma_k$$

(12)

**Definition 3** The elements of orbit-space $\mathcal{M}(k)$ will be called $k$-poles, or simply, multipoles.

We introduce groups $\Gamma_k$ in a manner similar to the introduction of the groups $\Sigma_k$. The group $\Gamma_k$ is an extension of the permutation group $S_k$ by the group $(\mathbb{C}^*)^k \supset H_k$. Thus, $\Gamma_k / \Sigma_k \approx \mathbb{C}^*$.

Therefore, the space $\mathcal{M}(k)$ in (11), (12) fibers over the projective variety

$$\mathcal{B}(k) := \prod_{\nu=1}^k V(1)^{\circ} / \Gamma_k$$

$$= \text{Sym}^k \mathbb{C} P[V(1)] \approx \text{Sym}^k (\mathbb{C} P)^2$$

(13)

with the fiber $\mathbb{C}^*$. Here $\text{Sym}^t(X) := X^t / S_t$ denotes the $t$-th symmetric power of a space $X$. The natural map $\mathcal{M}(k) \to \mathcal{B}(k)$ is a principle $\mathbb{C}^*$-fibration which gives rise to a line bundle $\eta(k) := \{ \mathcal{M}(k) \times_{\mathbb{C}^*} \mathbb{C} \to \mathcal{B}(k) \}$. By shrinking its zero section to a point, we form a quotient space

$$\overline{\mathcal{M}}(k) := [\mathcal{M}(k) \times_{\mathbb{C}^*} \mathbb{C}] / \mathcal{B}(k).$$

(14)

It differs from $\mathcal{M}(k)$ by a single new point $0$—a point which will represent the zero multipole. Evidently, $\overline{\mathcal{M}}(k)$ is a contractible space.
Given a collection of vector spaces \( \{ V_\alpha \} \), their wedge product \( \wedge_\alpha V_\alpha \) (not to be mixed with the exterior product!) is the quotient of the Cartesian product \( \times_\alpha V_\alpha \) by the subspace comprised of sequences \( \{ v_\alpha \in V_\alpha \} \) such that at least one vector from the sequence is zero. Thus, topologically,

\[
\overline{\mathcal{M}}(k) := [\wedge^k(V(1))/\Sigma_k].
\]

Lemma 3 leads to the following proposition.

**Corollary 5** Consider an homogeneous polynomial \( P(x, y, z) \) of degree \( d \) which is not divisible by an irreducible homogeneous quadratic polynomial \( Q(x, y, z) \). Then any generalized parcelling \( \mu = \sum_\nu \mu_\nu \) of the multiplicity function \( \mu : Z(P, Q) \to \mathbb{Z}_+ \) uniquely determines a multipole in the space \( \mathcal{M}(d) \) introduced in (11) or (12). \( \square \)

The space of multipoles has singularities. Its singular set \( \text{sing}(\mathcal{M}(k)) \) arises from the sets points in \( [\mathbb{C}^3]^k \) fixed by various non-trivial subgroups of \( \Sigma_k \). These subgroups all are the conjugates (in \( \Sigma_k \)) of certain non-trivial subgroups of \( S_k \) (recall that \( H_k \subset \Sigma_k \) acts freely). The partially ordered set of the orbit-types give rise to a natural stratification of the multipole space \( \mathcal{M}(k) \). The space \( \text{sing}(\mathcal{M}(k)) \) is of complex codimension two in \( \mathcal{M}(k) \). Its top strata corresponds to transpositions from \( S_k \). Thus, a generic point from \( \text{sing}(\mathcal{M}(k)) \) has a normal slice in \( \mathcal{M}(k) \) which is diffeomorphic to a cone over the real projective space \( S^3/\mathbb{Z}_2 = \mathbb{R}P^3 \). The larger stabilizers of vectors from the space \( \mathbb{C}^k \) of the obvious \( S_k \)-representation correspond to smaller strata of more complex geometry. Evidently, \( \text{sing}(\mathcal{M}(k)) \) is invariant under the diagonal action of \( \mathbb{C}^* \approx \Gamma_k/\Sigma_k \). These observations are summarized in the lemma below.

**Lemma 4** The singular set \( \text{sing}(\mathcal{M}(k)) \subset \mathcal{M}(k) \) is of codimension two. It is invariant under the \( \mathbb{C}^* \)-action. Therefore, it is a principle \( \mathbb{C}^* \)-fibration over the singular set \( \text{sing}(\text{Sym}^k(\mathbb{C}P^2)) \subset \text{Sym}^k(\mathbb{C}P^2) \). A generic point of \( \text{sing}(\mathcal{M}(k)) \) has \( \mathbb{R}P^3 \) for its normal link. Points of \( \text{sing}(\mathcal{M}(k)) \) correspond to completely factorable polynomials \( L = \prod_\nu L_\nu \) with at least two proportional linear factors (in other words, to weighted collections of lines in \( \mathbb{C}P^2 \) that contain at least one line of multiplicity greater than one). \( \square \)

Let \( \text{Fact}(d) \subset V(d) \) denote the variety of homogeneous polynomials of degree \( d \) in \( x, y, \) and \( z \) that are products of linear forms.

Given a multipole \( w \in \mathcal{M}(d) \) one can construct the corresponding completely factorable polynomial \( L(w) = \prod_\nu L_\nu \in V(d) \). Note that, due to the uniqueness of the prime factorization in the polynomial ring and in view of our definition of multipoles, the correspondence \( w \Rightarrow L(w) \) gives rise to a 1-to-1 map \( \Theta : \mathcal{M}(d) \xrightarrow{\approx} \text{Fact}(d) \). Consider a “Vi`e¨te-type” algebraic map

\[
\Phi_Q : \mathcal{M}(d) \xrightarrow{\Theta} \text{Fact}(d) \xrightarrow{\Pi_Q} \text{Fact}_Q(d),
\]

where \( \Pi_Q \) is induced by restricting polynomials in \( x, y, \) and \( z \) to the surface \( \{ Q(x, y, z) = 0 \} \). The symbol \( \text{Fact}_Q(d) \subset V(d)/V_Q(d) \approx V_Q^*(d) \) denotes the variety of homogeneous polynomial functions on the surface \( \{ Q = 0 \} \) that also decompose into products of linear forms. Due to formula (6) in Lemma 1, any non-zero homogeneous polynomial on the surface \( \{ Q = 0 \} \) admits a linear
factorization. Hence, $\Phi_Q$ is onto and $\mathcal{F}act_Q(d)$ can be identified with the space $[V(d)/V_Q(d)]^\circ \approx V_Q^\perp(d)^\circ$.

The map $\Phi_Q$, extends to an algebraic map

$$\tilde{\Phi}_Q : E\eta(d) \rightarrow V_Q^\perp(d) \quad (17)$$

defined on the space $E\eta(d)$ of the line bundle $\eta(d)$. It sends the zero section $B(d)$ of $\eta(d)$ to the zero vector $0 \in V_Q^\perp(d)$ and each fiber of $\eta(d)$ isomorphically to a line in $V_Q^\perp(d)$ passing through the origin. In fact, $\tilde{\Phi}_Q|_{E\eta(d) \setminus B(d)} = \Phi_Q$. Evidently, $\Phi_Q$ gives rise to a continuous map

$$\Phi_Q : \mathcal{M}(d) \rightarrow V_Q^\perp(d) \quad (18)$$

It turns out that $\Phi_Q$ has finite fibers. We need some combinatorial constructions which will help us to prove this claim and to describe the cardinality of the $\Phi_Q$-fibers.

With every natural $d$ we associate an integer $\kappa(d)$ that counts the number of distinct parcellings in a finite set of cardinality $2d$. Any parcelling is obtained by breaking a set $Z$ of cardinality $2d$ into disjoint subsets of cardinality $2$. Thus, $\kappa(d) = (2d - 1)!! := (2d - 1)(2d - 3)(2d - 5) \ldots 3 \cdot 1$ is the number of possible handshakes among a company of $2d$ friends. There is an alternative way to compute this number: consider the standard action of the permutation group $S_{2d}$ on the set of $2d$ elements. The action induces a transitive action on the set of all parcellings. Under this action, the subgroup $S_{2d}^#$ that preserves the parcelling $\{\{1, 2\}, \{3, 4\}, \ldots, \{2d - 1, 2d\}\}$ is an extension

$$1 \rightarrow (S_2)^d \rightarrow S_{2d}^# \rightarrow S_d \rightarrow 1 \quad (19)$$

of the permutation group $S_d$ that acts naturally on the pairs by the group $(S_2)^d$ that exchanges the elements in each pair. As a result, we get an identity

$$\kappa(d) = (2d - 1)!! = (2d)!/(2^d \cdot d!)$$

As we deform a polynomial $P$ into a polynomial $P_1$, two or more points in $Z(P, Q)$ can merge into a single point of $Z(P_1, Q)$. Its multiplicity is equal to the sum of multiplicities of the points forming the merging group. Through this process, any generalized parcelling $p$ of the original $\mu : Z(P, Q) \rightarrow \mathbb{N}$ gives rise to a new and unique generalized parcelling $p_1$ of $\mu_1 : Z(P_1, Q) \rightarrow \mathbb{N}$. Thus, we can define a partial order in the set of all generalized parcellings of effective divisors on $Q$ of degree $2d$ by setting $\mu \gg \mu_1$ and $p \gg p_1$ (see Figures 1 and 3).

In the same spirit, let $\kappa(\mu)$ stand for the number of distinct generalized parcellings of a function $\mu : Z \rightarrow \mathbb{N}$ (recall that $||\mu||_1 = 2d$) on a finite set $Z$. Unless $\mu$ is identically 1 and $|Z| = 2d$, $\kappa(\mu) < \kappa(d)$. When two intersection points merge, the number of generalized parcellings drops: there are distinct original parcellings that become indistinguishable after the merge (see the left diagram in Figure 1). For example, when two simple points in a complete intersection merge, the number of generalized parcellings changes from $\kappa(d)$ to $\kappa(d - 2) + [\kappa(d) - (\kappa(d - 2))]/2 = [\kappa(d) + (\kappa(d) - 2)]/2$, that is, it drops by $[\kappa(d) - (\kappa(d - 2))]/2$. In general, $\mu \gg \mu_1$ implies $\kappa(\mu) > \kappa(\mu_1)$.

For a generic $L \in \mathcal{F}act(d)$ the set $Z(L, Q)$, as well as all the parcels $\{Z_\nu\}$ defined by the linear factors $L_\nu$, are complete intersections. For such an $L$, the number of multipoles in $\Phi_Q^{-1}(\Pi_Q(L))$ is the number $\kappa(d)$ of distinct parcellings in the set $Z(L, Q)$. For any $L \in \mathcal{F}act(d)$, the cardinality
of $\Phi_Q^{-1}(\Pi_Q(L))$ (equivalently, of $\Pi_Q^{-1}(\Pi_Q(L))$) equals to the number of generalized parcellings of the multiplicity function $\mu : Z(L, Q) \rightarrow \mathbb{N}$. Indeed, assume that two completely factorable polynomials $L, L'$ coincide when restricted to the surface $\{Q = 0\}$. Then $L - L'$ must be divisible by $Q$. Therefore, $Z(L, Q) = Z(L', Q)$, moreover, the two multiplicities of each point in the intersection (defined by the curves $L$ and $L'$) must be equal as well. Hence, $L$ and $L'$ define two parcellings of the same multiplicity function $\mu$ on the intersection set.

Let $X, Y$ be topological spaces and $f : X \rightarrow Y$ a continuous map with finite fibers. For a while, the ramification set $D(f)$ of $f$ is understood as the set $\{y_0 \in Y\}$ such that, for any open neighborhood $U$ of $y_0$, the cardinality of the fibers $\{f^{-1}(y)\}_{y \in U}$ is not constant.

![Figure 1: Two distinct ways in which parcellings degenerate.](image)

Each time $L \in \text{Fact}(d)$ produces in $\mathbb{CP}^2$ a union of lines with at least one pair of lines sharing its intersection point with the curve $Q$, the point $\Pi(L) \in D(\Pi_Q) = D(\Phi_Q)$. Moreover, as Figure 1 demonstrates, this change in cardinality of fibers occurs due to their bifurcations in the vicinity of $L$. As a result, points in $D(\Phi_Q)$ give rise to effective divisors of degree $2d$ on the curve $Q$ with at least one point in their support being of multiplicity greater than one. On the other hand, any such divisor can be generated by intersecting a weighted set of lines with $Q$ (just use any generalized parcelling). Note that any line tangent to $Q$ also contributes a point of multiplicity two. However, the lines tangent to $Q$ are not contributing to the bifurcation of the $\Pi_Q$-fibers (see the right diagram in Figure 1). Therefore, in order to conclude that any divisor on $Q$ with multiple points corresponds to a point of $D(\Phi_Q)$, we need to use generalized parcellings which favor pairs of lines that share their intersection with $Q$ to a single tangent line (both patterns produce an intersection point of multiplicity two as shown in Figure 1). Evidently, this can be done, provided $d > 1$.

The requirement that $L \in \text{Fact}(d)$ has a pair of linear forms vanishing at a point of $\{Q = 0\}$, locally, is a single algebraic condition imposed on the coefficients of of the two forms. Therefore, taking closures, it picks a codimension one subvariety $\Pi_Q^{-1}(D(\Pi_Q)) \subset \text{Fact}(d)$. Thus, $D(\Pi_Q) \subset V_Q^+(d)\circ$ is a subvariety of codimension one as well. Fortunately, since $Q$ admits a rational parameterization by a map $\alpha : \mathbb{CP}^1 \rightarrow Q$, the set $\Pi_Q^{-1}(D(\Pi_Q))$ can be described in terms of solvability of a system of two rather simple equations. In homogeneous coordinates $[u_0 : u_1]$ such a parameterization $\alpha$ can be given by the formula

$$\alpha([u_0 : u_1]) = [\alpha_0([u_0 : u_1]) : \alpha_1([u_0 : u_1]) : \alpha_2([u_0 : u_1])],$$

where $\alpha_0, \alpha_1, \alpha_2$ are some quadratic forms. For example, for $Q = x^2 + y^2 + z^2$,

$$\alpha_0 = i(u_0^2 - u_1^2), \quad \alpha_1 = 2i u_0 u_1, \quad \alpha_2 = u_0^2 + u_1^2.$$
The inverse of $\alpha$ is produced by the central projection of $Q$ onto a line in $\mathbb{C}P^2$ from a center located at $Q$. Therefore, as abstract algebraic curves, $Q$ and $\mathbb{C}P^1$ are isomorphic.

In fact, $L(x, y, z) \in \mathcal{F}act(d)$ belongs to $\Pi_Q^{-1}(\mathcal{D}(\Pi_Q))$, if and only if, the system

$$\left\{ \begin{array}{l} \partial_x L(\alpha_0, \alpha_1, \alpha_2) \partial_u \alpha_0 + \partial_y L(\alpha_0, \alpha_1, \alpha_2) \partial_u \alpha_1 + \partial_z L(\alpha_0, \alpha_1, \alpha_2) \partial_u \alpha_2 \\
(0, 0, 0) \end{array} \right\} \Rightarrow (u_0, u_1) = 0, \quad (20)$$

that guarantees an existence of a multiple zero for the polynomial $L(\alpha(u_0, u_1))$ in $\mathbb{C}P^1$, has a non-trivial solution $(u_0, u_1)$. Writing down explicitly the resultant of the two polynomials in the LHS of (20), and thus the equation of $\Pi_Q^{-1}(\mathcal{D}(\Pi_Q))$, seems to be cumbersome. Note that the Euler identity

$$2d \cdot L = (u_0 \partial_u \alpha_0 + u_1 \partial_u \alpha_0) \partial_x L + (u_0 \partial_u \alpha_1 + u_1 \partial_u \alpha_1) \partial_y L + (u_0 \partial_u \alpha_2 + u_1 \partial_u \alpha_2) \partial_z L$$

explains the asymmetry of (20) with respect to the variable $u_0$: exchanging the roles of $u_0$ and $u_1$ leads to an equivalent system of equations.

Recall that for a projective variety $X$, the set of zero-dimensional, degree $d$ effective divisors is the projective variety $Sym^d(X)$ (see [Chi]). The map $\Psi_Q$ in formula (16) induces a well-defined regular map of the varieties:

$$\Psi_Q : Sym^d(\mathbb{C}P^2) \rightarrow Sym^d(Q). \quad (21)$$

This map is produced by realizing a given multipole $w$, or rather its $\mathbb{C}^*$-orbit $\tilde{w}$, by a completely factorable polynomial $L(w)$ and then forming the intersection set $Z(L(w), Q)$ equipped with the appropriate multiplicities—the divisor $\Psi_Q(\tilde{w})$. The map $\Psi_Q$ is onto: any effective divisor of degree $2d$ on $Q$ admits a generalized parcelling, and thus is generated by intersecting $Q$ with a weighted collection of lines. Since the number of generalized parcellings of $Z(L(w), Q)$ is finite, the map $\Psi_Q$ has finite fibers. All this can be seen from a different angle. The divisor $\Psi_Q(\tilde{w})$ uniquely determines the proportionality class of the function $L(w)|_{|Q=0}$—a point in $\mathbb{C}P(V_Q^+)$. We have seen that the $\mathbb{C}^*$-equivariant map $\Phi_Q$ in (16) is onto. Thus, $\Phi_Q/\mathbb{C}^* : Sym^d(\mathbb{C}P^2) \rightarrow \mathbb{C}P(V_Q^+)$ is onto as well. Since $Q$ is a rational curve (topologically, a 2-sphere), there is a 1-to-1 correspondence between points of $\mathbb{C}P(V_Q^+) \approx Sym^{2d}(\mathbb{C}P^1_*)$ and of $Sym^{2d}(Q)$. Moreover, as abstract varieties, $\mathbb{C}P(V_Q^+)$ and of $Sym^{2d}(Q)$ are isomorphic. This isomorphism can be used to identify the maps $\Psi_Q$ and $\Phi_Q/\mathbb{C}^*$. To simplify our notations, we will use the same symbol $\Psi_Q$ for both maps; when there is a need to distinguish them, we will just indicate the relevant target space.

Our combinatorial considerations imply that the locus $\mathcal{D}(\Psi_Q)$ consists of the effective divisors of degree $2d$ on $Q$ with at least one point in their support being of multiplicity at least two. While $\Psi_Q^{-1}(\mathcal{D}(\Psi_Q))$ is also described by (20), the locus $sing(\mathcal{F}act(d))$ is the preimage of effective divisors of degree $2d$ on $Q$ with at least a pair of points in their support being of multiplicity at least two (they can generate a double line), or at least one point being of multiplicity at least four (it corresponds to a double line tangent to $Q$).

Employing Lemma 4, we have established

**Lemma 5** The ramification set $\mathcal{D}(\Phi_Q)$ for the map $\Phi_Q : M(d) \rightarrow V_Q^+(d)$ is the set $\{\Pi_Q(L)\}$ of complex codimension one, where $L \in \mathcal{F}act(d)$ defines on $Q$ an effective divisor with at least one
point in its support being of multiplicity at least two. In other words, $\Pi_Q^{-1}(D(\Phi_Q))$ is defined by the resultant of the two polynomials in the LHS of (20). The ramification set $D(\Phi_Q)$ contains the $\Phi_Q$-image of the singular set $\text{sing}(\mathcal{M}(d))$. This image is of codimension two in $V_{Q}^\perp(d)$. It can be identified with completely factorable homogeneous polynomials of degree $d$ on the the surface $\{Q = 0\}$ that have at least one pair of proportional linear factors. □

Note that the space $Z$ of simple effective degree $k$ divisors on the curve $Q$ is homeomorphic to the space of simple effective degree $k$ divisors on the sphere $S^2 = \mathbb{C}P^1_*$.

The map $\Psi_Q : \text{Sym}^d(\mathbb{C}P^2) \to \mathbb{C}P^{2d} \approx V_{Q}^\perp(d)^\circ / \mathbb{C}^*$ is a finite ramified covering with a generic fiber of cardinality $(2d - 1)!$. It is ramified over the subvariety $D(\Psi_Q)$ whose points are the proportionality classes of homogeneous polynomials of degree $d$ on the quadratic surface $\{Q(x, y, z) = 0\}$ that define there effective divisors of degree $d$ with at least one multiple line.$^5$

- The space $\mathbb{C}P^{2d} \setminus D(\Psi_Q)$ is a $K(\pi, 1)$-space with the group $\pi$ being isomorphic to the braid group $B_{2d}$ in $2d$ strings in the spherical shell $S^2 \times [0, 1]$.

- As a result, $\text{Sym}^d(\mathbb{C}P^2) \setminus \Psi_Q^{-1}(D(\Psi_Q))$ is a $K(B_{2d}, 1)$-space, where $B_{2d}$ is the braid group in $2d$ strings with coupling.

- The space $\mathcal{M}(d) \setminus \Phi_Q^{-1}(D(\Phi_Q))$ also is a $K(\pi, 1)$-space with the group $\pi$ being isomorphic to an extension of the infinite cyclic group $\mathbb{Z}$ by the group $B_{2d}^\#$. □

Now, we will investigate the ramification locus of $\Phi_Q$ from a more refined point of view characteristic to the singularity theory. First, we would like to understand when a non-zero vector $w$ from the tangent cone $T_L$ of $\text{Fact}(d)$ at a point $L = \prod_{j=1}^d L_j$ is parallel to the subspace $V_Q(d)$, in other words, when $T_L$ contains a vector $w$ that is mapped to zero under the projection $\pi : V(d) \to V(d)/V_Q(d) \approx V_{Q}^\perp(d)$. Away from the singularity set $\text{sing}(\text{Fact}(d)) \subset \text{Fact}(d)$, the $\Pi_Q$-image of such an $L$ belongs to a locus $\mathcal{E} \subset V_{Q}^\perp(d)$ over which $\text{rank}(d\pi|_{\text{Fact}(d)}) < \text{dim}(V_{Q}^\perp(d))$ at some point in $\Pi_Q^{-1}(\mathcal{E})$. By the implicit function theorem, the ramification locus $D(\Pi_Q) \subset V_{Q}^\perp(d)$ for the map $\Pi_Q : \text{Fact}(d) \to V_{Q}^\perp(d)$ (equivalently, for the map $\Phi_Q$) is contained in the union $\mathcal{E} \cup \Pi_Q(\text{sing}(\text{Fact}(d)))$. It can happen that at a singularity $L \in \text{sing}(\text{Fact}(d))$ the tangent cone does not have vectors $w \neq 0$ with $^5$Note that $B_{2d}^\#$ is not the braid group in $2d$ strings colored with $d$ colors, each color marking a pair of strings!

$^6$equivalently, that define on the curve $Q \subset \mathbb{C}P^2$ effective divisors with a point of multiplicity at least two.
the property $\pi(w) = 0$, and still $\pi$ is ramified in the vicinity of $\pi(L)$. For instance, consider the obvious projection $\pi$ of the real cone $x^2 + y^2 - z^2 = 0$ onto the $xy$-plane: $\pi$ is ramified at the origin $(0,0)$, but for any $w \neq 0$ from the tangent cone at $(0,0)$, $\pi(w) \neq 0$.

Any $w \in T_L$ is of the form $\lim_{t \to 0} \frac{\prod_{j=1}^d (L_j + tM_j) - \prod_{j=1}^d L_j}{t}$, where each $M_j$ is an appropriate linear form in $x_j, y_j, z_j$ or $M_j = 0$ identically. This limit is equal to the polynomial $P = \sum_j (\prod_{i \neq j} L_i) M_j$. The vector $P$ at $L$ is parallel to the subspace $V_Q$ if and only if the polynomial $P$ is divisible by $Q$. In other words, the vector $P$ at $L$ is parallel to $V_Q$ if and only if the polynomial $P$, being restricted to the curve $Q$, is identically zero. Thus, we are looking for the $M_j$'s subject to the constraint: the polynomial $\sum_j (\prod_{i \neq j} L_i) M_j = (\prod_i L_i)(\sum_j \frac{M_j}{L_j})$, being restricted to $Q$, is identically zero. For $L \neq 0$, this is equivalent to the constraint (22) imposed on the rational functions $\{\frac{M_j}{L_j}\}$:

$$\sum_j \frac{M_j}{L_j} \bigg|_Q = 0. \tag{22}$$

Equation (22) always have obvious solutions: $\{M_j = \alpha_j L_j\}$, where $\{\alpha_j \in \mathbb{C}\}$ and $\sum_j \alpha_j = 0$. These are exactly solutions that represent the zero tangent vector (the tip $L$ of the tangent cone). Indeed, put $\prod_{j=1}^d (L_j + tM_j) = \prod_{j=1}^d (L_j + t\alpha_j L_j)$ to conclude that $w = 0$ if and only if $\sum_j \alpha_j = 0$.

So, the proper question is how to describe all the $L \in \text{Fact}(d)$ for which (22) has a solution distinct from the set of obvious solutions $\{M_j = \alpha_j L_j\}$ with $\sum_j \alpha_j = 0$. The images of such $L$'s under the projection $\pi : V(d) \to V_Q^\perp(d)$ will generate the locus $E \subset V_Q^\perp(d)$ over which the differential $d\pi|_{\text{Fact}(d)}$ is not of the maximal rank $\dim(V_Q^\perp(d)) = 2d + 1$. Evaluating the LHS of equation (22) at $2d + 1$ generic points residing in $Q$ imposes linear constraints on $3d$ variables—the coefficients of the $M_j$'s. If these constraints are independent, the solution space of the linear system must be of dimension $3d - (2d + 1) = d - 1$, which is exactly the dimension of the space formed by the obvious solutions. This indicates that, for a generic $L$, we should not expect any non-obvious solutions. Lemma 6 below validates this guess.

Let us denote by $M_L$ the quotient of the vector space of all solutions $\{M_j\}$ of (22) by the subspace of obvious solutions, as defined above. The correspondence $L \Rightarrow \dim(M_L)$ gives rise to a new natural stratification of the space $\text{Fact}(d)$, and thus, of the space $\mathcal{M}_Q(d)$. In the new notations, $\pi(L) \in E$ if and only if $M_L \neq 0$. We suspect that this stratification is consistent with, but cruder than the stratification induced by the orbit-types of the $\Sigma_d$-action.

Lemma 6 The loci $E$ and $D(\Pi_Q)$ coincide. As a result, the existence of a non-trivial solution for the system (20) is equivalent to the existence of a non-obvious solution for the equation (22). Also, the locus $E \supset \Phi_Q(sing(\mathcal{M}(d)))$.

Proof. We notice that if two distinct lines, say $L_1$ and $L_2$, share a point $p \in Q$, then (22) has a non-obvious solution $(M_1, M_2, 0, \ldots, 0)$. Indeed, inscribe in the quadratic curve $Q$ any "quadrilateral" formed by the pair of lines $L_1, L_2$ together with a new pair of lines $M_1, M_2$. The lines $L_1, M_1$ share a point $q \in Q$, the lines $L_2, M_2$ share a point $r \in Q$, and the lines $M_1, M_2$ share a point $s \in Q$, all four points $p, q, r, s$ being distinct. Next, pick some linear forms $M_1$ and $M_2$ representing $M_1$ and $M_2$. Then one can find constants $\lambda_1, \lambda_2$ so that the polynomial $Q = \lambda_1 M_1 L_2 + \lambda_2 M_2 L_1$. The argument is very similar to the one used in the proof of Lemma 1.
Thus, $\lambda_1 M_1 L_2 + \lambda_2 M_2 L_1 |_Q = 0$, which can be written in the form $\lambda_1 M_1/L_1 + \lambda_2 M_2/L_2 |_Q = 0$ required by (22). Here, evidently, $M_1$ is not proportional to $L_1$ and $M_2$ is not proportional to $L_2$. Therefore, for any $L \in \text{Fact}(d)$ that defines on $Q$ an effective divisor containing a point of multiplicity two, $\Pi_Q(L)$ must belong to the locus $E$. It follows ”by continuity” that all the $L$’s that produce multiplicity functions $\mu : Z(L, Q) \to \mathbb{N}$, distinct from the identity function 1, project to $E$ via $\Pi$. According to Lemma 5, these are exactly the factorable polynomials that project to the ”combinatorial” ramification locus $D$. \qed

It follows from Lemma 6 that, for a generic $P \in V(d)$, the affine subspace $P + V_Q(d)$ hits transversally the subvariety $\text{Fact}(d)$ at a finite set of points whose cardinality is exactly $\kappa(d)$. Therefore, $\kappa(d)$ must be the degree of that variety.

Let $X,Y$ be quasi-affine varieties and let $F : X \to Y$ be a proper regular map. In Theorem 7 below, the ramification set $D(F)$ of a mapping $F : X \to Y$ with finite fibers is understood to be the closure of a set $D^0(F)$. By definition, $y \in D^0(F)$ when $F^{-1}(y)$ contains a point $x$ such that there is a non-zero vector $v \in T_x X$ that is mapped to zero in $T_y Y$ by the differential $DF$.

The arguments above lead to our main result:

**Theorem 7**

- The map $\Phi_Q : \mathcal{M}(d) \to \text{Fact}_Q(d) = V_Q^\perp(d)^\circ$ is onto, and its generic fiber is a finite set of cardinality $(2d-1)!!$.

  This map takes the multipole space $\mathcal{M}(d)$—the space of a $\mathbb{C}^*$-bundle $\eta(k)$ over the variety $\text{Sym}^d(\mathbb{C}P^2)$—onto the complex vector space $V_Q^\perp(d)$ of dimension $2d+1$ with its origin being deleted. The map $\Phi_Q$ extends to a map $\Phi_Q^\infty : \overline{\mathcal{M}}(d) \to V_Q^\perp(d)$ with finite fibers.

- $\Phi_Q$ is ramified over the discriminant variety $D(\Phi_Q)$—the set of polynomials $P \in \text{Fact}_Q(d)$ whose zero sets are effective divisors on the surface $\{Q(x,y,z) = 0\}$ with at least one of their line components being of multiplicity at least two.\footnote{Equivalently, the set of polynomials $P$ for which $P \cap Q$ has a point of multiplicity at least two.}

  The subvariety $D(\Phi_Q) \subset V_Q^\perp(d)^\circ$ is of complex codimension one, and is described in terms of solvability of (20) or (22). It contains the $\Phi_Q$-image of the singular set $\text{sing}(\mathcal{M}(d))$ as a codimension one subvariety.

- In fact, $\Phi_Q, \Phi_Q^\infty$ are $\mathbb{C}^*$-equivariant maps. Therefore, $\Phi_Q$ gives rise to a surjective map

  $$\Psi_Q : \text{Sym}^d(\mathbb{C}P^2) \to \mathbb{C}P^{2d} = \mathbb{C}P(V_Q^\perp(d)^\circ)$$

  of degree $(2d-1)!!$ which is also ramified over a subvariety $D(\Psi_Q) \subset \mathbb{C}P^{2d}$ of codimension one. The complement $\mathbb{C}P^{2d} \setminus D(\Psi_Q)$ is a $K(B_{2d},1)$-space, $\text{Sym}^d(\mathbb{C}P^2) \setminus \Psi_Q^{-1}(D(\Psi_Q))$ is a $K(\mathbb{B}_{2d}^\#,1)$-space, where $\mathbb{B}_{2d}^\#$ is the braid group with coupling, while $\mathcal{M}(d) \setminus \Phi_Q^{-1}(D(\Phi_Q))$ is a $K(\pi,1)$-space, where $\pi$ an extension of $\mathbb{Z}$ by $\mathbb{B}_{2d}^\#$.

- The degree of the variety of completely factorizable homogeneous polynomials $\text{Fact}(d) \subset V(d)$ is also $(2d-1)!!$. This variety is invariant under the obvious $\mathbb{C}^*$-action on $V(d)$. \qed

Theorem 7 has a number of topological implications, some of them dealing with interesting ramifications over complex projective spaces. Our next goal is to describe these implications.
Let the space $X$ be of a homotopy type of a connected, finite-dimensional CW-complex. Recall that the Dold-Thom Theorem [DT] links the homotopy groups of $\text{Sym}^d(X)$, $d$ being large, with the integral homology of a space $X$. By picking a base point $a$ in $X$, one gets a stabilization map $\text{Sym}^d(X) \to \text{Sym}^{d+1}(X)$, well-defined by the formula $(x_1, x_2, \ldots, x_d) \to (a, x_1, x_2, \ldots, x_d)$. Here $\{x_i \in X\}$. This provides us with canonical homomorphisms $\pi_k(\text{Sym}^d(X)) \to \pi_k(\text{Sym}^{d+1}(X))$ of the $k$-th homotopy groups.

In our case, the Dold-Thom Theorem claims that $\lim_{d \to \infty} \pi_k(\text{Sym}^d(\mathbb{C}P^2)) = H_k(\mathbb{C}P^2; \mathbb{Z})$. In particular, $\lim_{d \to \infty} \pi_2(\text{Sym}^d(\mathbb{C}P^2)) = \mathbb{Z} = \lim_{d \to \infty} \pi_4(\text{Sym}^d(\mathbb{C}P^2))$. Also, $\lim_{d \to \infty} \pi_k(\text{Sym}^d(\mathbb{C}P^2)) = 0$, provided $k = 1, 3$ or $k > 4$. On the other hand, $\pi_2(\mathbb{C}P^{2d}) = \mathbb{Z}$, but $\pi_4(\mathbb{C}P^{2d}) = 0$. Therefore, at least for large $d's$, there is an infinite order element $\alpha \in \pi_4(\text{Sym}^d(\mathbb{C}P^2))$ that is mapped by $(\Psi_Q)_*$ to zero and an element $\beta \in \pi_2(\text{Sym}^d(\mathbb{C}P^2))$, so that $(\Psi_Q)_*(\beta) \in \pi_2(\mathbb{C}P^{2d})$ is a generator. It is possible to realize $\alpha$ and $\beta$ geometrically. What is clear ratherway, that the spheroids $\alpha$ and $\beta$ cannot be pushed into the aspherical portion $\text{Sym}^d(\mathbb{C}P^2) \setminus \Psi_Q^{-1}(D(\Psi_Q))$ of $\text{Sym}^d(\mathbb{C}P^2)$. The realization of $\alpha$ is based on an interesting fact that I learned from Blaine Lawson: the quotient of $\mathbb{C}P^2$ by the complex conjugation $\tau : [x : y : z] \to [\bar{x} : \bar{y} : \bar{z}]$ is homeomorphic to the sphere $S^4$. Therefore, the map $\phi : \mathbb{C}P^2 \to \mathbb{C}P^2 \times \mathbb{C}P^2$ given by the formula $\phi(p) = (p, \tau(p))$, where $p \in \mathbb{C}P^2$, is evidently $\mathbb{Z}_2$-equivariant with respect to the $\tau$-action in the domain and the symmetrizing action in the range. This gives rise to the desired quotient map $\alpha : S^4 \approx \mathbb{C}P^2/\{\tau\} \to \text{Sym}^2(\mathbb{C}P^2)$ that survives into the higher symmetric powers of $\mathbb{C}P^2$. Constructing class $\beta$ is straightforward: it is given by the obvious inclusion $S^2 \approx \mathbb{C}P^1 \subset \mathbb{C}P^2$ followed by the diagonal embedding $\Delta : \mathbb{C}P^2 \to \text{Sym}^d(\mathbb{C}P^2)$.

The existence of a non-trivial $\alpha : S^4 \to \text{Sym}^d(\mathbb{C}P^2)$ whose $\Psi_Q$-image is null-homotopic in $\mathbb{C}P^{2d}$ has a curious implication:

**Corollary 8** For any $d \geq 2$, there is a family of polynomial functions of degree $d$ on the quadratic surface $\{Q = 0\}$ that is parameterized by a 5-dimensional disk which does not admit a continuous lifting to the multipole space $M(d)$. At the same time, the functions parameterized by the 4-sphere forming the boundary of the disk can be continuously represented by multipoleos. □

Any map $f : X \to Y$ induces a natural map $f^k : \text{Sym}^k(X) \to \text{Sym}^k(Y)$ that is defined by the formula $(f_*^k)[\sum_\nu \mu_\nu x_\nu] = \sum_\nu \mu_\nu f(x_\nu)$, where $\{x_\nu \in X\}$ and $\{\mu_\nu \in \mathbb{N}\}$. When $f$ is 1-to-1 or onto, so is $f^k_*$. The construction of $f^k_*$ provides us with a rich source of interesting ramified coverings. Consider for example, a semi-free cyclic action on a sphere $S^2 \subset \mathbb{R}^3$. The group $\mathbb{Z}_l$ acts on $S^2$ by rotations around a fixed axis on the angles that are multiples of $2\pi/l$. Topologically, the orbit-space $\tilde{S}^2 := S^2/\mathbb{Z}_l$ is again a 2-sphere. Let $f : S^2 \to \tilde{S}^2$ be the orbit-map. Then $f^k_* : \text{Sym}^k(S^2) \to \text{Sym}^k(\tilde{S}^2)$ gives an example of a ramified degree $l^k$ covering map of $\mathbb{C}P^k$ over itself! For $l = 2$, we shall see later how this degree $2^k$ ramification $f^k_* : \mathbb{C}P^k \to \mathbb{C}P^k$ is linked to the multipole spaces on quadratic surfaces. On the other hand, by a simple cohomological argument, any map $F : \mathbb{C}P^k \to \mathbb{C}P^k$ has a degree that is the $k$-th power of a non-negative integer $l$. For instance, there is no ramified map from $\mathbb{C}P^2$ to itself of degrees that are not of the form $l^2$.

**Question** Given a closed oriented manifold $M$, what are possible degrees of maps from $M$ to itself? Evidently, the answer depends on the homotopy type of $M$.  

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When $Z_2$ acts freely on $S^2$ by the central symmetry, the orbit-map $f : S^2 \to \mathbb{R}P^2$ gives rise to another interesting ramified map $f_*^k : C^d \to \mathbb{R}P^{2k}$ of degree $2^k$ (its existence follows from our results in Section 3 dealing with real multipole spaces).

In order to derive next few corollaries of Theorem 7, we need to take a detour aimed at constructing (with the help of $f$) transfer maps that take effective 0-divisors on $Y$ to effective 0-divisors on $X$. These constructions are variations on the theme of the classical Hurwitz’ Theorem (see [H], pp. 299-304). Our next goal is to present constructions and arguments that lead to Theorem 10.

Let $X$ and $Y$ be smooth complex projective varieties and $f : X \to Y$ a regular surjective map with finite fibers. With any $x \in X$ we associate a multiplicity number $\mu_f(x)$. It is the multiplicity attached to the intersection of the $f$-graph $\Gamma_f$ with the subspace $X \times f(x) \subset X \times Y$ at the point $(x, f(x))$. Since all the $f$-fibers are finite, the intersection $\Gamma_f \cap (X \times f(x))$ is finite as well.

Next, with any $y \in Y$ we associate an effective 0-divisor $D_{f^{-1}(y)} := \sum_{x \in f^{-1}(y)} \mu_f(x) x$ whose degree is $y$-independent and coincides with the degree $d_f$ of the map $f$. The correspondence $y \mapsto D_{f^{-1}(y)}$ produces a regular embedding

$$f^\#: Y \to Sym^{d_f}(X)$$

(23)

For any $k$, the map $f^\#$ gives rise to an embedding

$$f_k^\#: Sym^k(Y) \to Sym^{k \cdot d_f}(X)$$

(24)

defined by the formula $f_k^\#(\sum_{\nu} \mu(y_{\nu}) y) = \sum_{\nu} \mu(y_{\nu}) D_{f^{-1}(y)}$.

Therefore, applying this construction to the setting of Theorem 7 with $X = Sym^d(\mathbb{C}P^2)$, $Y = \mathbb{C}P^{2d}$, and $f = \Psi_Q$, we get the following proposition:

**Corollary 9** Any irreducible quadratic form $Q(x, y, z)$ determines canonical embeddings

$$\Psi^\#_{Q,k} : Sym^k(\mathbb{C}P^{2d}) \to Sym^{k \cdot (2d-1)!}(Sym^d(\mathbb{C}P^2)),$$

where $d$ and $k$ are arbitrary whole numbers. In particular, $\Psi^\#_{Q} := \Psi^\#_{Q,1}$ embeds $\mathbb{C}P^{2d}$ into $Sym^{(2d-1)!}(Sym^d(\mathbb{C}P^2))$. □

For a map $f : X \to Y$ as above, one can define a map $f^\#$ that takes effective 0-divisors on $X$ into effective 0-divisors on $X$. By definition, each point $x \in X$ is mapped to the divisor $D_{f^{-1}(f(x))}$. By linearity, we have

$$f^\#(\sum_{\nu} \mu(x_{\nu}) x_{\nu}) = \sum_{\nu} \mu(x_{\nu}) D_{f^{-1}(f(x_{\nu}))}.$$  

(25)

This map transforms 0-divisors of degree $k$ into 0-divisors of degree $k \cdot d_f$:

$$f_k^\#: Sym^k(X) \to Sym^{k \cdot d_f}(X)$$

(26)
Both maps, $f_k^\#$ from (26) and $f_k^\#$ from (24), have the same targets. Moreover, their images coincide. Indeed, for each point $y \in Y$, pick any point $x \in f^{-1}(y)$. Then, $f_k^\#(y) = Df_{x^{-1}}(y) = f_k^\#(x)$. Recall, that $f_k^\#$ is a 1-to-1 map and thus is invertible over its image. Therefore, the map

$$(f_k^\#)^{-1} \circ f_k^\#: \text{Sym}^k(X) \to \text{Sym}^k(Y)$$

(27)

is well-defined.

Lemma 7 Let $X$ and $Y$ be smooth complex projective varieties and $f : X \to Y$ be a regular onto map with finite fibers. The map $(f_k^\#)^{-1} \circ f_k^\#$ in (27) coincides with the natural map $f_k^\#: \text{Sym}^k(X) \to \text{Sym}^k(Y)$. It defines a ramified covering with a generic fiber of cardinality $(d_f)^k$.

Proof The map $f_\#$ takes each point $x \in X$ to the divisor $D_{f^{-1}(f(x))}$.

At the same time, the transfer $f_\#$ takes $f(x)$ to the same divisor $D_{f^{-1}(f(x))}$. Thus, $(f_k^\#)^{-1} \circ f_\#$ maps $x$ to $f(x)$. Extending this argument by linearity proves the claim. $\square$

Now we examine how these constructions apply to projective curves in $\mathbb{C}P^2$ and eventually to the multipole spaces. The role of $X$ will be played by a curve $\mathcal{C} \subset \mathbb{C}P^2$ (most importantly, by $Q$), the role of $Y$ by a pencil of lines in $\mathbb{C}P^2$ through a point $p$. The map $f$ takes any point $q \in \mathcal{C}$ to a line $L$ passing through $q$ and $p$.

Any linear embedding $\rho : \mathbb{C}P^1 \subset \mathbb{C}P^2$ induces an embedding $\rho^d : \text{Sym}^d(\mathbb{C}P^1) \to \text{Sym}^d(\mathbb{C}P^2)$. The geometry of $\rho^d$ is tricky. Since the quotient space $\mathbb{C}P^2/\mathbb{C}P^1$ is homeomorphic to a 4-sphere $S^4$, we get a homeomorphism

$$\text{Sym}^d(\mathbb{C}P^2)/\text{Sym}^d(\mathbb{C}P^1) \approx \text{Sym}^d(S^4),$$

but even the spaces $\{\text{Sym}^d(S^4)\}$ have subtle topology. For example, $\text{Sym}^2(S^4)$ is homeomorphic to a mapping cylinder of a map $\Sigma^4(\mathbb{R}P^3) \to S^4$, where $\Sigma^4(\sim)$ denotes the fourth suspension (see [Ha], Example 4K.5).

The regular map

$$\Psi_Q : \text{Sym}^d(\mathbb{C}P^1) \overset{\rho^d}{\to} \text{Sym}^d(\mathbb{C}P^2) \overset{\Psi_Q}{\to} \text{Sym}^d(Q) \approx \mathbb{C}P^{2d}$$

(28)

describes the role and place of $\mathbb{C}$-planar multipoles (they are linked to the planarity of the quadrupoles and octapoles in the deconstruction of CMBR that was briefly mentioned in the introduction). In contrast with $\Psi_Q$, we will see that the map $\Psi_Q$ is 1-to-1. Since $\text{Sym}^d(\mathbb{C}P^1)$ is homeomorphic to $\mathbb{C}P^d$, its image in $\text{Sym}^d(Q)$ is homeomorphic to $\mathbb{C}P^d$ as well (compare this with (21)). Moreover, the 2d-cycle $\Psi_Q[\text{Sym}^d(\mathbb{C}P^1)]$ is homologous to the cycle $[\text{Sym}^d(Q)]$ (equivalently, to the cycle $[\mathbb{C}P^d]$) defined by a natural imbedding $\text{Sym}^d(Q) \subset \text{Sym}^d(Q)$ (equivalently, by a linear imbedding $\mathbb{C}P^d \subset \mathbb{C}P^{2d}$).

Recall that points $L \in \mathbb{C}P^1 \subset \mathbb{C}P^2$ correspond to a pencil of lines $L \subset \mathbb{C}P^2$ that pass through a particular point $p \in \mathbb{C}P^2$. That $p$ determines the embedding $\rho$. Each point $q \in Q$ determines a unique line $L_q$ that passes through $q$ and $p$, and therefore, a unique point $L_q$ in the dual subspace.

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8Recall that the multiplicity of $x$ in $D_{f^{-1}(f(x))}$ is $\mu_f(x)$. 19
Let $f : Q \to \mathbb{CP}^1$ be a 2-to-1 ramified map defined by the correspondence $q \mapsto L_q$. As described in (24) and (26), $f$ gives rise to maps $f^\# : \text{Sym}^d(\mathbb{CP}^1) \to \text{Sym}^{2d}(Q)$ and $f^d : \text{Sym}^d(Q) \to \text{Sym}^{2d}(Q)$ with $f^\#$ being a 1-to-1 map. In fact, examining the construction of $\Psi_Q$, we see that $\hat{\Psi}_Q = f^\#$. Moreover, using Lemma 7, the map $f^d : \text{Sym}^d(Q) \to \text{Sym}^d(\mathbb{CP}^1)$ factors as $(f^\#)^{-1} \circ f^d$. Therefore, the ramification locus $\mathcal{D}(f^\#) \subset \text{Sym}^d(\mathbb{CP}^1)$ for $f^\#$ is the $\hat{\Psi}_Q^{-1}$-image of the ramification locus $\mathcal{D}(f^d) \subset f^d(S\text{ym}^d(Q)) \subset \text{Sym}^{2d}(Q)$ for $f^d$.

Figure 2 illustrates the case $d = 3$. Passing from the first to the second column depicts the map $f^3_\#: \text{Sym}^3(Q) \to \text{Sym}^{6}(Q)$ generated by the linear projection from a center $p$ located at infinity. Passing from the first to the third column depicts the map $f^3_* : \text{Sym}^3(Q) \to \text{Sym}^3(\mathbb{CP}^1) \approx \mathbb{CP}^3$. Here $\mathbb{CP}^1$ is viewed as the pencil of lines in $\mathbb{CP}^2_*$ through $p$. Topologically, the map $f^3_*$ is a 8-to-1 ramification of $\mathbb{CP}^3$ over itself. It is described in some detail in Example 1 and is depicted in Figure 3. The passage from the second to the third column is a 1-to-1 correspondence.

When $L_q$ is not tangent to $Q$ at $q$, then let $q^* \neq q$ be “the other point” in $Q$ that belongs to the line $L_q$. When $L_q$ is tangent to $Q$ at $q$, then by definition, put $q^* = q$. The correspondence $\tau_p : q \mapsto q^*$
is an involution on $Q$ with two fixed points $a$ and $b$. Its orbit-space $Q/\{\tau_p\}$ topologically is a 2-sphere. The involution $\tau_p$ induces an involution $\tau_p^k$ that acts on the space $\text{Sym}^k(Q)$. By definition, any effective divisor $\sum_{\nu} \mu(q_{\nu})q_{\nu} \in \text{Sym}^k(Q)$ is transformed by $\tau_p^k$ into the divisor $\sum_{\nu} \mu(q_{\nu})q_{\nu}^*$. Evidently, the image of $f^d_{\#} : \text{Sym}^d(Q) \to \text{Sym}^{2d}(Q)$ is contained in the $\tau_p^{2d}$-invariant part of the space $\text{Sym}^{2d}(Q)$; however, not any invariant divisor belongs to that image. Each divisor from $\text{Im}(f^d_{\#})$ not only must be $\tau_p^d$-invariant, but in addition, its multiplicity at the fixed points $a$ and $b$ must be even. In other words, such a divisor $D_{\#}$ must be of the form

$$2\mu_a a + 2\mu_b b + \sum_{\nu} \mu(q_{\nu})[q_{\nu} + q_{\nu}^*],$$

where $q_{\nu} \neq a, b$ and $\mu_a, \mu_b \in \mathbb{Z}_+$. The cardinality of the $f^d_{\#}$-fiber over $D_{\#}$ is given by the formula

$$|{(f^d_{\#})^{-1}(D_{\#})}| = \prod_{\nu} [\mu(q_{\nu}) + 1]. \tag{29}$$

Indeed, for any $q \in Q \setminus \{a, b\}$ there are $\mu(q_{\nu}) + 1$ effective divisors of degree $\mu(q_{\nu})$ with the support in $q_{\nu} \prod q_{\nu}^*$ and whose $f^d_{\#(q_{\nu})}$-image is $\mu(q_{\nu})[q_{\nu} + q_{\nu}^*]$; at the same time, there is a unique divisor $\mu_a a$ whose $f^d_{\#(\mu_a)}$-image is $2\mu_a a$. When all $\mu_{\nu} = 1$ and $\mu_a = 0 = \mu_b$, the fiber $(f^d_{\#})^{-1}(D_{\#})$ is of cardinality $2^d$.

Now, we are in position to justify previous claims about the image of the map $\hat{\Psi}_Q$ in (28). Pick a divisor $D_0$ of degree $d$ in $Q$ such that the supports of $D_0$ and $\tau_p^d(D_0)$ do not share common points. For any divisor $D$ of degree $d$ on $Q$ the operation $\text{Proj} \Rightarrow D + D_0$ gives rise to an imbedding $I_{D_0} : \text{Sym}^d(Q) \subset \text{Sym}^{2d}(Q)$. We claim that the intersection $\hat{\Psi}_Q(\text{Sym}^d(CP^1)) \cap I_{D_0}(\text{Sym}^d(Q))$ is a singleton. Indeed, $\hat{\Psi}_Q(\text{Sym}^d(CP^1))$ consists of $\tau_p^d$-invariant divisors, but the only invariant divisor of the form $D + D_0$ is the divisor $D_0 + \tau_p^d(D_0)$. Therefore, using the cohomology ring structure of $\text{Sym}^{2d}(Q) \cong \mathbb{C}P^{2d}$, the cycle $[\hat{\Psi}_Q(\text{Sym}^d(CP^1))] \in H_{2d}(\text{Sym}^{2d}(Q); \mathbb{Z})$ must be equal to the cycle $[I_{D_0}(\text{Sym}^d(Q))].$ Using the diffeomorphism $Q \cong \mathbb{C}P^1$, this translates into the generating cycle $[\mathbb{C}P^1] \in H_{2d}(\mathbb{C}P^{2d}; \mathbb{Z})$.

Examining (29), we see that the ramification set $D(f^d_{\#})$ for $f^d_{\#} : \text{Sym}^d(Q) \to \text{Sym}^{2d}(Q)$ is comprised of divisors of two kinds: 1) the ones that contain at least one summand of the form $2(q + q^*)$, where $q \neq a, b$, and 2) the ones that contain $2a$ or $2b$ as a summand. For example, if $D \in \text{Sym}^d(Q)$ contains a pair of distinct points $q_1, q_2 = q_1^*$ and the rest of points in the support of $D$ are generic (i.e., for $i, j > 2, q_i \neq q_j^*$), then $(f^d_{\#})^{-1}(f^d_{\#}(D))$ consists of $2^d - 2^d - 2 = 3 \cdot 2^{d-2}$ elements. At the same time, as we perturb $D$ in order to avoid the coincidence $q_2 = q_1^*$, $(f^d_{\#})^{-1}(f^d_{\#}(D))$ consists of $2^d$ elements.

Therefore, in view of Lemma 7, the ramification set $D(f^d_{\#})$ for $f^d_{\#} : \text{Sym}^d(Q) \to \text{Sym}^d(CP^1)$ is comprised of divisors of two kinds: 1) the ones that contain at least one summand of multiplicity at least two, and 2) the ones that contain points $L_a$ or $L_b$ giving rise to lines $L_a$ and $L_b$ passing through the point $p$ and tangent to the curve $Q$.

Given a space $X$, let us denote by $\Delta_d(X) \subset \text{Sym}^d(X)$ the discriminant set formed by the divisors containing points of multiplicity at least two. Also, for any point $a \in X$, we denote by $\text{Sym}^{d-k}_{ka}(X)$ the subset of $\text{Sym}^d(X)$ formed by the divisors containing the summand $k \cdot a$. 

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Thus,

\[ D(f_d^\#) = \Delta_d(CP^1) \cup Sym_{a}^{d-1}(CP^1) \cup Sym_{b}^{d-1}(CP^1) \]  

(30)

and

\[ D(f_d^\#) = \{ \Delta_{2d}(Q) \cup Sym_{2a}^{2d-2}(Q) \cup Sym_{2b}^{2d-2}(Q) \} \tau_p. \]  

(31)

Employing the Viète Map \( V : Sym^d(CP^1) \rightarrow CP^d \), we transplant the algebraic set \( D(f_d^\#) \) into the space \( CP^d \). The image of \( \Delta_d(CP^1) \) under the Viète Map is the classical discriminant variety in \( D_d \subset CP^d \), while the images of \( Sym_{a}^{d-1}(CP^1) \) and \( Sym_{b}^{d-1}(CP^1) \) form linear subspaces \( CP_{a}^{d-1} \) and \( CP_{b}^{d-1} \) in \( CP^d \). Thus, \( V(D(f_d^\#)) = D_d \cup CP_{a}^{d-1} \cup CP_{b}^{d-1} \). It follows from [K], Theorem 6.1, that each of the two spaces \( CP_{a}^{d-1} \) and \( CP_{b}^{d-1} \) is tangent to the discriminant variety \( D_d \) along, respectively, the linear subspaces \( CP_{a}^{d-2} \) and \( CP_{b}^{d-2} \).

Note that the complement \( Sym^d(CP^1) \setminus D(f_d^\#) \) is the configuration space of \( d \)-tuples of distinct points in the domain \( CP^1 \setminus (a^* \cup b^*) \). Here \( a^*, b^* \) stand for the two points in \( CP^1 \) that are dual to the lines passing through \( p \) and tangent to the curve \( Q \). Therefore, it is a \( K(B_{d}^{an}, 1) \)-space, where \( B_{d}^{an} \) stands for the braid group in \( d \) strings residing in a cylinder with an annulus base \( [S^2 \setminus (a^* \cup b^*)] \times [0, 1] \).

Using the birational identifications \( Sym^d(CP^1) \approx CP^d \approx Sym^d(CP^1) \), we have constructed a ramified covering \( \Gamma_Q : CP^d \rightarrow CP^d \) of degree \( 2^d \). The following proposition summarizes the conclusions of our arguments above (centered on (28) and (30), (31)). It describes an intricate stratified geometry of this ramified covering, a geometry that is “reductive” in its nature with respect to the shift \( d \Rightarrow d - 1 \).

**Theorem 10**

- Any irreducible quadratic form \( Q \) gives rise to a ramified covering \( \Gamma_Q : CP^d \rightarrow CP^d \) of degree \( 2^d \). The map \( \Gamma_Q \) is ramified over the algebraic set

\[ D(\Gamma_Q) := D_d \cup CP_{a}^{d-1} \cup CP_{b}^{d-1} \]

— the Viète image of the set \( \Delta_d(CP^1) \cup Sym_{a}^{d-1}(CP^1) \cup Sym_{b}^{d-1}(CP^1) \).

- The discriminant variety \( D_d \subset CP^d \) is \( Q \)-independent. The two linear subspaces \( CP_{a}^{d-1} \) and \( CP_{b}^{d-1} \) are tangent to the variety \( D_d \) along, respectively, subspaces \( CP_{a}^{d-2} \) and \( CP_{b}^{d-2} \). A generic point of \( D_d \) has a \( \Gamma_Q \)-fiber of cardinality \( 3 \cdot 2^{d-2} \), while a generic point of \( CP_{a}^{d-1} \) and \( CP_{b}^{d-1} \) has a fiber of cardinality \( 2^{d-1} \).

- The complement \( CP^d \setminus D(\Gamma_Q) \) to the ramification set is a \( K(B_{d}^{an}, 1) \)-space, where \( B_{d}^{an} \) denotes the annulus braid group in \( d \) strings.

- Moreover, over to each of the two subspaces \( CP_{a}^{d-1} \) and \( CP_{b}^{d-1} \), the map \( \Gamma_Q \) inherits a similar stratified structure with respect to the dimensional shift \( d \Rightarrow d - 1 \). \( \Box \)

**Example 1.** Let \( d = 2 \). Then \( \Gamma_Q : CP^2 \rightarrow CP^2 \) is of degree 4. The discriminat parabola \( D_2 \) is given by \( \{ x^2 - 4yz = 0 \} \). The ramification locus \( D(\Gamma_Q) \) consists of that parabola together

\(^{9}\) that is, \( 2^{d-2} \) less than a generic \( \Gamma_Q \)-fiber.
1. With two tangent lines $\mathbb{C}P^1_a = \{Ax + y + A^2z = 0\}$ and $\mathbb{C}P^1_b = \{Bx + y + B^2z = 0\}$ which share a point $E = \left[ -(A + B) : AB : 1 \right]$. They are tangent to the parabola at the points $F = \left[ -2A : A^2 : 1 \right]$ and $G = \left[ -2B : B^2 : 1 \right]$. Here the parameters $A$ and $B$ depend on the quadratic form $Q(x, y, z)$.

2. The $\Gamma_Q$-fibers over $E$, $F$, $G$ are singletons; the cardinality of the fiber over $\mathbb{C}P^1 \setminus (F \cup G)$ is 3; the cardinality of the fiber over $\mathbb{C}P^1 \setminus (G \cup E)$ is 2. It is still a bit mysterious how all this data manage to produce $\mathbb{C}P^2$ as a covering space! Anyway, the compliment $\mathbb{C}P^2 \setminus D(\Gamma_Q)$ is a $K(\mathbb{B}^{\text{ann}}_2, 1)$-space, where $\mathbb{B}^{\text{ann}}_2$ is the braid group with two strings in a "fat" annulus.

Now consider the case $d = 3$ depicted in Figure 3. That figure exhibits a two-fold symmetry that exchanges points $a$ and $b$ where the pencil of parallel lines is tangent to the curve $Q$. Pattern 1 in Figure 3 corresponds to the generic stratum $\mathbb{C}P^3$, while pattern 2 defines the discriminant surface $D_3 \subset \mathbb{C}P^3$ of degree 4. It is homeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Patterns 3a and 3b each corresponds to two planes $\mathbb{C}P^2_a$ and $\mathbb{C}P^2_b$. Each of the two planes is tangent to the discriminant surface $D_3$ along lines $l_a$ and $l_b$ labeled by patterns 6a and 6b, respectively. These planes intersect along another line $l$ labeled by pattern 5. Pattern 4 encodes the singular locus $C$ of the discriminant surface $D_3$. It is a rational curve of degree 3 that is homeomorphic to $S^2$. In fact, $D_3$ is the ruled surface spanned by lines tangent to $C$ ([K]). The surface $D_3$ intersects with the plane $\mathbb{C}P^2_a$ along the union of the line $l_a$ with a quadratic curve $D_{2,a}$ labeled by pattern 7a. The lines $l_a$ and $l$ are both tangent to the parabola $D_{2,a}$. This configuration is already familiar from our description of the case $d = 2$. Similarly, pattern 7b labels parabola $D_{2,b}$. Curves $C$, $D_{2,a}$, and $l_a$ are all tangent at a point $A_a$ labeled by pattern 8a. Curves $l$ and $D_{2,a}$ are tangent at a point $B_a$ labeled by pattern 9b. That
point also lies on the line \( l_b \). The labeling of points \( A_b \) and \( B_a \) is done by patterns \( 8b \) and \( 9a \), respectively.

In accordance with Theorem 10, the ramification locus \( D(T_Q) \) for the 8-to-1 map \( \Gamma_Q \) coinsides with \( D_3 \cup CP_{d}^2 \cup CP_{b}^2 \) and its complement is a \( K(B_{3}^{\text{ann}}, 1) \)-space. \( \square \)

Now, we turn to deconstructions of both homogeneous non-homogeneous polynomials on complex quadratic surfaces \( \{Q(x, y, z) = \text{const}\} \). First, consider a general homogeneous polynomial \( P \) of degree \( d \). We can apply decomposition (6) to the homogeneous term \( R \) of degree \( d-2 \) in (6). This will produce a new collection of vectors—a new multipole—\( \{w_{1, \nu}\} \) associated with an appropriate generalized parcelling of \( \mu : Z(R, Q) \to \mathbb{Z}_{+} \). This process of producing lower order multipoles \( \{w_{s, \nu} \in V(1)\}_{s, \nu} \), where \( 0 \leq s \leq [d/2] \) and \( 0 < \nu \leq d-2s \), can be repeated again and again until all the degrees are “used up”.

We notice that the leading multipole (of highest degree) is determined by this algorithm in a more “direct way” than the lower degree multipoles. Also note that the choice of a generalized parcelling \( \tau \) for \( \mu : Z(P, Q) \to \mathbb{Z}_{+} \) affects the choice of a generalized parcelling for \( \mu : Z(R, Q) \to \mathbb{Z}_{+} \), where the polynomial \( R = (P - \prod \nu L_{\nu})/Q^s \). Here the product \( \prod \nu L_{\nu} \) is determined by the \( \tau \), and \( s \) is the maximal power of \( Q \) for which the division in the ring of polynomials is possible.

**Example 2.** Let \( d = 5 \). Then any homogeneous polynomial \( P \) of degree 5 has a representation of the form

\[
P = L_{01}L_{02}L_{03}L_{04}L_{05} + Q \cdot L_{11}L_{12}L_{13} + Q^2 \cdot L_{21}
\]

(32)

where all the \( L_{ij} \)'s are linear forms (some of which might be zeros). The number of such representations does not exceed \((9!!) \times (5!!)\). \( \square \)

Any non-homogeneous polynomial \( P(x, y, z) \) of degree \( d \) can be written in the form \( P^{(0)} + P^{(1)} \) where the degrees of monomials comprising \( P^{(n)} \) are congruent to \( n \) modulo 2. Note that the decomposition \( P = P^{(0)} + P^{(1)} \) intrinsically makes sense on every quadratic surface \( Q(x, y, z) = \lambda, (\lambda \in \mathbb{C}) \): if \( P \) belongs to the principle ideal \( (Q - \lambda) \), so does each term \( P^{(n)} \), \( n = 0, 1 \). Thus, if \( P \equiv \tilde{P} \mod(Q - \lambda) \), then we have \( P^{(n)} \equiv \tilde{P}^{(n)} \mod(Q - \lambda) \).

On the surface \( \{Q(x, y, z) = 1\} \), any component \( P^{(n)} \) of the polynomial \( P^{(0)} + P^{(1)} \) can be homogenized by multiplying its terms of the same degree by an appropriate power of \( Q \). We denote by \( P^{(n)}_Q \) the appropriate homogeneous polynomial. Generically, \( \deg(P^{(n)}_Q) = d - n \).

**Example 3.** Let \( d = 5 \). Any polynomial \( P \) of degree 5 has a representation of the form

\[
P = L_{01}L_{02}L_{03}L_{04}L_{05} + Q \cdot L_{11}L_{12}L_{13} + Q^2 \cdot L_{21}
\]

\[
+ M_{01}L_{02}M_{03}M_{04} + Q \cdot M_{11}M_{12} + Q^2 \cdot \lambda
\]

(33)

where all the \( L_{ij} \)'s and \( M_{ij} \)'s are linear forms (some of which might be zeros), and \( \lambda \) is a number. The number of such representations does not exceed \((9!!) \times (5!!) \times (7!!) \times (3!!) = 9 \times 7^2 \times 5^3 \times 3^4 \). \( \square \)

Now one can apply recursively Lemmas 1, 3 and Corollary 5 to each homogeneous polynomial \( P^{(n)}_Q \), \( n = 0, 1 \). Letting \( Q = 1 \) proves formula (3) from Theorem 2. Let us restate and generalize this theorem in terms of the multipoles:
Theorem 11. Let $Q(x, y, z)$ be an irreducible quadratic form and let $P(x, y, z)$ be any complex polynomial of degree $d$. Its restriction $P|_{S}$ to the complex surface $S = \{Q(x, y, z) = 1\}$ admits a representation of the form

$$P(x, y, z) = \lambda_0 + \sum_{k=1}^{d} \prod_{l=1}^{k} L_{k,l}(x, y, z), \tag{34}$$

where the linear forms $\{L_{k,l}\}$ are chosen so that each non-zero product $\prod_{l=1}^{k} L_{k,l}(x, y, z)$ is determined, via the map $\Phi_Q$ (see (16)), by an appropriate multipole $w_k$ from the variety $\mathcal{M}(k)$.

- The representation (34) is unique, up to a finite ambiguity and up to reordering and rescaling of multipoles in each product $\prod_{l=1}^{k} L_{k,l}(x, y, z)$. In other words, the set of $P$-representing multipoles $\{w_k \in \mathcal{M}(k)\}_{1 \leq k \leq d}$ is finite. Its cardinality does not exceed $\prod_{k=1}^{d} [(2k - 1)!!]$. $\square$

We would like to end this section by establishing a few facts about alternative decompositions of polynomials that are based on formula (9). In achieving this goal we are guided by Theorem 22.2 from [Sh].

We claimed that the direct sum in (7) is orthogonal with respect to the inner product $\langle f, g \rangle = \int_{S^2} f \cdot g \, dm$, where $dm$ is the standard rotationally symmetric measure on the unit sphere $S^2$. Let us clarify this claim. Applying (7) recursively, we get that for any real homogeneous polynomial $P$ of degree $d$ can be written as $\sum_{d=2k \geq 0} Q^k \cdot P^H_k$, where $P^H_k$ is a real homogeneous harmonic polynomial of degree $d - 2k$. Letting $Q = 1$, we see that for any homogeneous $P$, there is a harmonic polynomial $P^H = \sum_{d=2k \geq 0} P^H_k$ such that $P|_{S^2} = P^H|_{S^2}$. Now, any two harmonic homogeneous polynomials $P^H_k$ and $P^H_l$ of different degrees are eigenfunctions with different eigenvalues $-(d - 2k)(d - 2k + 1)$ and $-(d - 2l)(d - 2l + 1)$ for the Laplace operator $\Delta_{S^2}$ on the sphere. Therefore, they are orthogonal, i.e. $\int_{S^2} P^H_k \cdot P^H_l \, dm = 0$. Since any homogeneous polynomial $R$ of degree $d - 2$ can be represented as $\sum_{d-2k \geq 0; \, k > 0} Q^k \cdot P^H_k$, we get the claimed orthogonality $\int_{S^2} (P^H_k)(Q \cdot R) \, dm = 0$ of the direct sum in (7).

This argument extends to complex harmonic polynomials as follows. We already remarked that the real and imaginary parts of a complex harmonic polynomial $P$ are real harmonic polynomials of the same degree. If the polynomial is homogeneous, so are its real and imaginary parts. Therefore, for any complex harmonic and homogeneous polynomial $P$ of degree $d$ and any homogeneous polynomial $R$ of degree $d - 2$, we get $\int_{S^2} P(Q \cdot R) \, dm = \int_{S^2} (\text{Re}P + i \text{Im}P)(\text{Re}R - i \text{Im}R) \, dm = \int_{S^2} (\text{Re}P \cdot \text{Re}R) \, dm + \int_{S^2} (\text{Im}P \cdot \text{Im}R) \, dm - i \int_{S^2} (\text{Re}P \cdot \text{Im}R) \, dm + i \int_{S^2} (\text{Im}P \cdot \text{Re}R) \, dm$. Each of the four integrals must vanish because each integrand is a product of a real homogeneous and harmonic polynomial of degree $d$ by a real polynomial of a lower degree.

In the spherical coordinates $\theta, \phi$, the inner product is given by an integral

$$\int_{S^2} f \cdot g \, dm = \int_{0 \leq \phi \leq \pi; \, 0 \leq \theta \leq 2\pi} f(\Lambda(\theta, \phi)) \cdot g(\Lambda(\theta, \phi)) \sin \phi \, d\theta \, d\phi$$

where $\Lambda(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$.

Now we are going to transfer this hermitian inner product from the sphere to its image $\Upsilon_Q$ in a given complex quadratic surface $S_Q := \{Q(x, y, z) = 1\}$. As before, let $A$ be a complex invertible matrix.
that reduces $Q$ to the sum of squares: $x'^2 + y'^2 + z'^2 = Q((x, y, z) \cdot A)$, where $(x', y', z') = (x, y, z) \cdot A$. Let $\Upsilon_Q = \{Q((x, y, z) \cdot A) = 1\}$, with $(x, y, z) \cdot A$ being a real vector. The ellipsoid $\Upsilon_Q$ is the image of the unit sphere under the complex linear transformation $A^{-1}$. It is a totally real 2-dimensional algebraic surface in the complex surface $S_Q$.

The real-valued measure $dm_Q$ on $\Upsilon_Q$ is the pull-back under the map $A$ of the standard measure on the unit sphere. It is invariant under the action of the compact subgroup $A \cdot O(3; \mathbb{R}) \cdot A^{-1} \subset GL(3; \mathbb{C})$, where $O(3; \mathbb{R})$ denotes the orthogonal group.

For any pair of (homogeneous) complex functions $f, g$ on $\mathbb{C}^3$, we get

$$\int_{\Upsilon_Q} f((x, y, z)A) \cdot \bar{g}((x, y, z)A) \, dm_Q = \int_{S^2} f(x', y', z') \cdot \bar{g}(x', y', z') \, dm$$

$$= \int_{0 \leq \phi \leq \pi; 0 \leq \theta \leq 2\pi} f(\Lambda(\theta, \phi)) \cdot \bar{g}(\Lambda(\theta, \phi)) |\sin \phi| \, d\theta d\phi \quad (35)$$

Applying (9) recursively, gives the following proposition:

**Theorem 12**

- The space of complex homogeneous polynomials admits an $O_Q(3; \mathbb{C})$-invariant decomposition

$$V(d) = \bigoplus_{d-2k \geq 0} Q^k \cdot Har_Q(d - 2k) \quad (36)$$

The summands in (36) are orthogonal with respect to the hermitian inner product defined by the formula (35). In particular, the homogeneous polynomials $P \in Har_Q(d)$—the solutions of the equation $\Delta_Q(P) = 0$—are characterized by the property: for each $T \in V_Q(d)$,

$$\int_{\Upsilon_Q} P \cdot \bar{T} \, dm_Q = 0.$$

- Any polynomial function $F$ on the surface $S = \{Q(x, y, z) = \text{const}\}$ can be obtained by restricting to $S$ a polynomial $P \in Ker(\Delta_Q)$.\footnote{not necessarily homogeneous, even when $P$ is homogeneous}

- For any two polynomials $M$ and $N$, “the complex Dirichlet problem”

$$\{\Delta_Q(P) = M, \quad P|_S = N|_S\} \quad (37)$$

has a unique polynomial solution $P$.

- Any homogeneous polynomial $P$ of degree $d$ admits a Maxwell-type representation

$$P(x, y, z) = \sum_{d-2k \geq 0} Q(x, y, z)^{d-k+\frac{1}{2}} \cdot \nabla u_{1,k} \nabla u_{2,k} \cdots \nabla u_{d-2k,k} \left(Q(x, y, z)^{-\frac{1}{2}}\right) \quad (38)$$

**Remark** All the statements of Theorem 12, but the last one (dealing with the generalized Maxwell representation (38)), can be easily generalized for polynomials in any number of variables.
Proof  Decomposition (36) follows from (9), and (38) from (36) together with (10). The second bullet is obviously implied by (36) and the linearity of \( \Delta_Q \). In order to prove the third bullet, note that (9) implies that \( \Delta_Q : V(k) \to V(k - 2) \) is onto. Therefore, for any \( M \), there exists a polynomial \( T \), so that \( \Delta_Q(T) = M \). On the other hand by bullet two, there exists a harmonic polynomial \( R \in Ker(\Delta_Q) \) such that \( R|_S = (N - T)|_S \). As a result, \( P = T + R \) solves the Dirichlet problem. In order to show that \( P \) is unique, it is sufficient to prove that no polynomial of the form \((Q - 1)S\) belongs to \( Ker(\Delta_Q) \). Since \( \Delta_Q \) is homogeneous of degree \(-2\), \( \Delta_Q(Q \cdot S) \neq \Delta_Q(S) \), unless \( S = 0 \). Indeed, let \( F \) be the leading homogeneous portion of \( S \) of the degree \( deg(S) \). Then \( \Delta_Q(Q \cdot S) = \Delta_Q(S) \) implies \( \Delta_Q(Q \cdot F) = 0 \) which is, as we have shown before, impossible, unless \( F = 0 \). □

3  Deconstructing reality

We have seen that different parcellings of the intersection \( Z(P, Q) \) led to different deconstructions of polynomial functions on a quadratic surface \( \{ Q = 1 \} \). The next idea is to invoke symmetry in order to get a unique equivariant parcelling and thus, a unique deconstruction of symmetric functions on a quadratic surface supporting an appropriate group action.

We deal with a special, but important case of \( \mathbb{Z}_2 \)-action. The model example is provided by the complex conjugation \( \tau : (x, y, z) \to (\overline{x}, \overline{y}, \overline{z}) \). If \( Q(x, y, z) \) has real coefficients, both the curve \( Q \subset \mathbb{C}P^2_\tau \) and the surface \( S = \{ Q = 1 \} \subset \mathbb{C}^3 \) are invariant under the \( \tau \). When \( Q \) is a definite real form, the \( \mathbb{Z}_2 \)-action by conjugation on \( Q \) is free since the fixed point set \( Q^{\mathbb{Z}_2} = Q \cap \mathbb{R}P^2_\tau = \emptyset \). For a homogeneous polynomial \( P \) with real coefficients the intersection \( Z(P, Q) \subset \mathbb{C}P^2_\tau \) and the multiplicity function \( \mu : Z(P, Q) \to \mathbb{Z}_+ \) are \( \tau \)-invariant as well. We notice that a \( \tau \)-invariant generalized parcelling of such a \( \mu \) will produce a set of \( \tau \)-invariant lines \( L_\nu \). Thus, we can assume that all the linear forms \( L_\nu \) in (6) are with real coefficients. Indeed, given a representation \( P = \lambda \cdot \prod_\nu L_\nu + Q \cdot R \), where \( P, Q \) and \( L_\nu \)'s are real and \( \lambda \in \mathbb{C} \), we get \( P = \overline{\lambda} \cdot \prod_\nu L_\nu + Q \cdot \overline{R} \). Hence, \( 2P = (\lambda + \overline{\lambda}) \cdot \prod_\nu L_\nu + Q \cdot (R + \overline{R}) \) which implies the validity of (6) over the reals.

With this observation in mind, we introduce real versions of the multipole spaces (11), (12). Let

\[
\mathcal{M}^\mathbb{R}(k) = \left\{ [\mathbb{R}^3 \circ]^k / \Sigma_k^\mathbb{R} \right\}
\]  (39)

where the group \( \Sigma_k^\mathbb{R} \) is defined similar to its complex version. The only difference lies in the definition of \( H^\mathbb{R}_k \): it is now a subgroup of \( (\mathbb{R}^*)^k \), not of \( (\mathbb{C}^*)^k \).

The space of real multipoles \( \mathcal{M}^\mathbb{R}(k) \) is a space of a principle \( \mathbb{R}^* \)-fibration over the real variety \( B^\mathbb{R}(k) := Sym^k(\mathbb{R}^2) \). We can form an associated line bundle \( \eta^\mathbb{R}(k) = \{ \mathcal{M}^\mathbb{R}(k) \times_{\mathbb{R}} \mathbb{R} \to B^\mathbb{R}(k) \} \).

As in the complex case, put \( E\eta^\mathbb{R}(k) = \mathcal{M}^\mathbb{R}(k) \times_{\mathbb{R}} \mathbb{R} \) and form the quotient space

\[
\overline{\mathcal{M}}^\mathbb{R}(k) = E\eta^\mathbb{R}(k)/B^\mathbb{R}(k)
\]  (40)

by collapsing the zero section to a point \( 0 \). As before, \( \overline{\mathcal{M}}^\mathbb{R}(k) \) is a contractible space.

Real homogeneous polynomials of degree \( d \) form a totally real subspace \( V(d; \mathbb{R}) \) in the Hermitian space \( V(d) \). The hermitian metric on \( V(d) \) defined by (35) generates an euclidean metric on \( V(d; \mathbb{R}) \).
induced by the inner product
\[
(f, g)_{\mathbb{R}} := \text{Re}\left\{ \int_{\mathcal{X}_Q} f(x, y, z) \cdot \bar{g}(x, y, z) \, dm_Q \right\}
\]
\[
= \text{Re}\left\{ \int_{S^2} f((x, y, z)A^{-1}) \cdot \bar{g}((x, y, z)A^{-1}) \, dm \right\}
\]
(41)

Moreover, as in the complex case, the imbedding \(\beta_Q : V(d - 2; \mathbb{R}) \to V(d; \mathbb{R})\) is an isometry (recall that \(Q\) is real).

Similarly to the complex case, we introduce a variety \(\mathcal{F}act(d; \mathbb{R}) \subset V(d; \mathbb{R})\) of completely factorable real polynomials. The orthogonal projection \(V(d; \mathbb{R}) \to V_Q^+ (d; \mathbb{R})\) maps \(\mathcal{F}act(d; \mathbb{R})\) into the vector space \(V_Q^+ (d; \mathbb{R})\). Due to Theorem 12, \(V_Q^+ (d; \mathbb{R})\) can be identified with \(\text{Ker}(\Delta_{\mathbb{R}})^{\ast}\). When the quadratic form \(Q\) is not definite, we can also identify the space \(V_Q^+ (d; \mathbb{R})\) with the \(d\)-graded portion of the polynomial function ring on the real cone \(\{Q = 0\}\). In any case, we get an algebraic map
\[
\Phi_{Q, \mathbb{R}} : M^\mathbb{R}(d) \to \mathcal{F}act(d; \mathbb{R}) \to V_Q^+ (d; \mathbb{R})^{\ast}
\]
(42)

which, by Lemma 1 and the argument above, is onto. As in the complex case, this map extends to a map
\[
\Phi_{Q, \mathbb{R}} : E \eta^\mathbb{R}(d) \to V_Q^+ (d; \mathbb{R})
\]
(43)

that takes the zero section \(B^\mathbb{R}(d) = \text{Sym}^d(\mathbb{R}P^2)\) of the line bundle \(\eta^\mathbb{R}(d)\) to the origin in \(V_Q^+ (d; \mathbb{R})\), and each fiber of \(\eta^\mathbb{R}(d)\) isomorphically to a line through the origin. This generates an \(\mathbb{R}^d\)-equivariant surjection
\[
\Phi_{Q, \mathbb{R}} : \overline{M}^\mathbb{R}(d) \to V_Q^+ (d; \mathbb{R})
\]
(44)

with finite fibers, where \(\overline{M}^\mathbb{R}(d) := E \eta^\mathbb{R}(d)/B^\mathbb{R}(d)\).

The number of \(\tau\)-equivariant parcellings of \(Z(P, Q)\) is harder to determine. It depends only on the restriction of the multiplicity function \(\mu\) to the subset \(Z(P, Q; \mathbb{R}) := Z(P, Q) \cap \mathbb{R}P^2_+\) and is equal to the number of generalized parcellings subordinate to such a restriction. However, if \(Z(P, Q; \mathbb{R}) = \emptyset\), the conjugation acts freely on \(Q \subset \mathbb{C}P^2_+\) and the \(\tau\)-equivariant parcelling is unique. In such a case, we are getting a more satisfying result:

**Theorem 13** Let \(Q(x, y, z)\) be an irreducible quadratic form with real coefficients and the signature distinct from \(-3\). Then any real polynomial \(P(x, y, z)\) of degree \(d\), being restricted to the real conic \(\mathcal{S}^\mathbb{R} = \{Q(x, y, z) = 1\}\) in \(\mathbb{R}^3_+\), has a representation
\[
P(x, y, z) = \lambda_0 + \sum_{k=1}^{d} \lambda_k \left[ \prod_{j=1}^{k} (a_{k,j}x + b_{k,j}y + c_{k,j}z) \right],
\]
(45)

where each \((a_{k,j}, b_{k,j}, c_{k,j})\) is a unit vector and, for \(k > 0\), \(\lambda_k \geq 0\).

When the surface \(\mathcal{S}^\mathbb{R}\) is an ellipsoid, the representation (45) is unique, up to reordering and rescaling of the multipliers in the products. In such a case, \(P\) gives rise to a unique sequence of multiplorules \(\{w_k \in \overline{M}^\mathbb{R}(k)\}_k\). \(\Box\)
We observe that $\mathcal{M}^\mathbb{R}(k)$ happens to be a nonsingular space: locally $\mathbb{R}P^2$ and $\mathbb{C}P^1$ are diffeomorphic, and $\text{Sym}^k(\mathbb{C}P^1) \approx \mathbb{C}P^k$ is non-singular. An $\mathbb{R}$-version of the argument that follows Theorem 6 and is centered on formula (22) is valid. Therefore, an $\mathbb{R}$-version of Lemma 6 holds: the “combinatorial” and the “smooth” ramification loci coincide.

Note that one always can find a real homogeneous polynomial $P$ of degree $k$ for which the curves $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}P^2$ have an empty intersection: just take a linear form $L$ such that the line $L := \{ L = 0 \}$ misses the real quadratic curve $\mathcal{Q}$; then consider any polynomial $P$ sufficiently close to $(L)^k$. For such a $P$, the $\mathbb{Z}_2$-equivariant parcelling is unique and, thus, $(\Phi^\mathbb{R}_Q)^{-1}(P|_{Q=0})$ is a singleton. Evidently, such polynomials $P$ form an open set. Therefore, $\Psi^\mathbb{R}_Q : \text{Sym}^k(\mathbb{R}P^2) \to \mathbb{R}P(\mathcal{V}_Q(k; \mathbb{R}))$ is a map of degree one. These remarks lead to

**Theorem 14** The multipole space $\mathcal{M}^\mathbb{R}(k)$, which parametrizes completely factorable real homogeneous polynomials of degree $k$, is a space of an $\mathbb{R}^*$-bundle over the real variety $\text{Sym}^k(\mathbb{R}P^2)$. The $\mathbb{R}^*$-equivariant map $\Phi^\mathbb{R}_Q$ induces a surjective mapping $\Psi^\mathbb{R}_Q : \text{Sym}^k(\mathbb{R}P^2) \to \mathbb{R}^k(\mathcal{V}_Q(k; \mathbb{R}))$ of degree one. Unless $Q$ is a definite form, the map $\Phi^\mathbb{R}_Q$ is ramified over the discriminant variety $D(\Phi^\mathbb{R}_Q)$ of codimension one. Here $D(\Phi^\mathbb{R}_Q)$ is comprised of homogeneous polynomials $P$ of degree $k$ on the real surface $\{ Q = 0 \}$. For which the surfaces $\{ P = 0 \}$ and $\{ Q = 0 \}$ share a line of multiplicity at least two.\footnote{equivalently, the curves $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}P^2$ share a point of multiplicity at least two.} \[ \square \]

**Corollary 15** Let $Q$ be a positive definite form. Then

- the multipole space $\mathcal{M}^\mathbb{R}(d) = \left\{ [\mathbb{R}^3 \circ ]^d \right\} / \Sigma^\mathbb{R}_d$ is diffeomorphic to the space $\mathbb{R}^{2d+1}$.
- the multipole space $\bar{\mathcal{M}}^\mathbb{R}(d)$ in (40) is homeomorphic to the space $\mathbb{R}^{2d+1}$.
- the real varieties $\text{Sym}^d(\mathbb{R}P^2)$ and $\mathbb{R}P^2d$ are diffeomorphic.

**Proof** In this case, the ramification locus $D(\Phi^\mathbb{R}_Q) = \emptyset$. Since under the corollary’s hypotheses the $\tau$-equivariant parcelling is unique, every real homogeneous $P$, not divisible by $Q$, gives rise to a unique leading multipole $w(P) \in \mathcal{M}^\mathbb{R}_Q(d)$—the map $\Phi^\mathbb{R}_Q$ is 1-to-1. Recalling that $\Phi^\mathbb{R}_Q$ is onto a vector space of dimension $2d + 1$ and that the smooth ramification locus $\mathcal{E} = D(\Phi^\mathbb{R}_Q) = \emptyset$, completes the proof. \[ \square \]

The diffeomorphism $\text{Sym}^d(\mathbb{R}P^2) \to \mathbb{R}P^{2d}$ serves as yet another transparent illustration to the Dold-Thom theorem: not only it reveals $\text{Sym}^{\infty}(\mathbb{R}P^2)$ as a $K(\mathbb{Z}_2, 1)$ space, but we actually know how this stabilization of the homotopy groups occurs. In fact, $\pi_1(\text{Sym}^d(\mathbb{R}P^2)) = \mathbb{Z}_2$, and for $k > 1$, $\pi_k(\text{Sym}^d(\mathbb{R}P^2)) = \pi_k(S^{2d})$. In particular, $\{ \pi_k(\text{Sym}^d(\mathbb{R}P^2)) \}$ vanish for $1 < k < 2d$.

We have seen already the main advantages of the multipole representations for the polynomials on quadratic surfaces: such representations are independent on the choice of coordinates in $\mathbb{C}^3$ or $\mathbb{R}^3$. \[ 29 \]
This is in the sharp contrast with the classical decompositions in terms of the spherical harmonics. In the case of $Q = x^2 + y^2 + z^2$ and over the reals, the independence of the multipoles under the rotations was observed by many. When $Q$ is not positive-definite, similar observations hold.

For a non-degenerated quadratic form $Q$ with real coefficients, let $O_Q(3, \mathbb{R})$ denote the group of linear transformations from $GL(3; \mathbb{R})$ that preserve the form. When the signature $\text{sign}(Q) = 2$, then $O_Q(3, \mathbb{R})$ contains the Lorenz transformation group (equivalently, the isometry group of a hyperbolic plane) as a subgroup of index two.

The next proposition should be compared with Lemma 2 and Theorem 12. We noticed already that, for a real quadratic form $Q$, the decomposition (9) is a complexification of a similar decomposition

$$V(d; \mathbb{R}) = \text{Har}_Q(d; \mathbb{R}) \oplus V_Q(d; \mathbb{R}) \quad (46)$$

over the reals. Here $\text{Har}_Q(d; \mathbb{R}) := \ker(\Delta_Q; \mathbb{R}) \cap V(d; \mathbb{R})$. Therefore, (46) is $O_Q(3, \mathbb{R})$-equivariant with respect to the natural action on the space $V(d; \mathbb{R})$ and orthogonal with respect to the inner product (41). Furthermore, the multipole space $\mathcal{M}(d)$ is also equipped with the $O_Q(3, \mathbb{R})$-action induced by the obvious diagonal action on $V(3; \mathbb{R})^d$. Furthermore, the map $\text{Fact}(d, \mathbb{R}) \to \text{Har}_Q(d; \mathbb{R})$ induced by the projection $V(d; \mathbb{R}) \to \text{Har}_Q(d; \mathbb{R})$ defined by (46) is equivariant.

**Corollary 16** Given a real polynomial $P$ on $S^R$ together with its representation (45) and a transformation $U \in O_Q(3, \mathbb{R})$, the transformed polynomial $U^*(P)(x, y, z) := P((x, y, z) \cdot U)$ on $S^R$ acquires a representation in the form

$$U^*(P)(x, y, z) = \lambda_0 + \sum_{k=1}^d \lambda_k \left[ \prod_{j=1}^k (a'_{k,j} x + b'_{k,j} y + c'_{k,j} z) \right], \quad (47)$$

where each new multipole vector $(a'_{k,j}, b'_{k,j}, c'_{k,j}) = (a_{k,j}, b_{k,j}, c_{k,j}) \cdot U^T$. In other words, the onto map

$$\Phi_Q^R : \prod_{k=0}^d \mathcal{M}(k) \to \prod_{k=0}^d [V(k, \mathbb{R})/V_Q(k, \mathbb{R})] \approx \oplus_{k=0}^d \text{Har}_Q(d; \mathbb{R})$$

is $O_Q(3, \mathbb{R})$-equivariant. □

Combining decomposition (46) with Theorem 12 we get its real analog:

**Theorem 17** Let $Q$ be a real non-degenerated quadratic form.

- The space of real homogeneous polynomials admits an $O_Q(3; \mathbb{R})$-invariant decomposition

$$V(d; \mathbb{R}) = \bigoplus_{d-2k \geq 0} Q^k \cdot \text{Har}_Q(d - 2k; \mathbb{R}) \quad (48)$$

The summands in (48) are orthogonal with respect to the inner product defined by the formula (41).

- Any polynomial function $F$ on the surface $S^R = \{Q(x, y, z) = \text{const}\}$ can be obtained by restricting to $S^R$ a polynomial $P \in \text{Har}_Q(\mathbb{R})$.  

30
For any two polynomials \( M \) and \( N \), the Dirichlet problem

\[
\{ \Delta Q(P) = M, \quad P|_{S^k} = N|_{S^k} \}
\]

(49)

has a unique real polynomial solution \( P \).

Any real homogeneous polynomial \( P \) of degree \( d \) admits the Maxwell-type representation

\[
P(x, y, z) = \Re\left\{ \sum_{d-2k \geq 0} Q(x, y, z)^{d-k+\frac{1}{2}} \cdot \nabla u_{1,k} \cdot \nabla u_{2,k} \ldots \nabla u_{d-2k,k} \left( Q(x, y, z)^{-\frac{1}{2}} \right) \right\}
\]

(50)

where \( \{u_{j,k}\} \) are complex 3-vectors. These vectors are real and representation (50) is unique, provided that \( Q \) is positive-definite. □

4 Why one does rarely see multipoles in non-quadratic skies?

Let \( Q(x, y, z) \) be an irreducible form of degree \( l \) over \( \mathbb{C} \). Then \( S := \{Q(x, y, z) = 1\} \) is the surface in \( \mathbb{C}^3 \) and \( Q := \{Q(x, y, z) = 0\} \) is an irreducible curve in \( \mathbb{C}P^2 \) of degree \( l \).

As before, we denote by \( V_Q(d) \) the set of homogeneous degree \( d \) complex polynomials that are divisible by \( Q \). Let \( V_Q^\perp(d) \approx V(d)/V_Q(d) \) be an orthogonal complement to \( V_Q(d) \) in \( V(d) \).

For any sequence of non-negative integers \( \{d_i\} \) so that \( \sum_{1 \leq i \leq s} d_i = d \), consider a map

\[
\eta : \prod_{i=1}^s V_Q^\perp(d_i) \to V_Q^\perp(d)
\]

(51)

which is defined by taking the product of homogeneous polynomials \( P_i \in V_Q^\perp(d_i) \) restricted to the surface \( \{Q(x, y, z) = 0\} \).

As before, the subgroup \( H_s \subset (\mathbb{C}^*)^s \) of rank \( s - 1 \) acts freely on \( \prod_{i=1}^s [V_Q^\perp(d_i)^\circ] \) by scalar multiplication. By the definition of \( H_s \), the map \( \eta \) is constant on the orbits of this action. We view the partition \( \{d = \sum_{1 \leq i \leq s} d_i\} \) as a non-increasing function \( \omega : i \to d_i \) on the index set \( \{1, 2, \ldots, s\} \).

Denote by \( S_\omega \) the subgroup of the permutation group \( S_s \) that preserves \( \omega \). Let \( \Sigma_\omega \) be an extension of \( S_\omega \) by \( H_s \) that is generated by the obvious actions of \( S_\omega \) and \( H_s \) on \( (\mathbb{C}^*)^s \approx \prod_{i=1}^s \mathbb{C}_i^* \).

Evidently, \( \eta \) is an \( \Sigma_\omega \)-equivariant map. Thus, it gives rise to a well-defined map\(^\text{12}\)

\[
\Phi_\omega : \prod_{i=1}^s V_Q^\perp(d_i)^\circ / \Sigma_\omega \to V_Q^\perp(d)^\circ
\]

(52)

Because \( Q \) is irreducible, the map \( \Phi_\omega \) has \( V_Q^\perp(d)^\circ \), and not just \( V_Q^\perp(d) \), as its target.

As before, we introduce the multipole space

\[
\mathcal{M}_Q(\omega) := \prod_{i=1}^s V_Q^\perp(d_i)^\circ / \Sigma_\omega
\]

(53)

\(^{12}\)To simplify our notations, we do not indicate (as before) the dependency of the map on \( Q \).
which is a space of a principle $\mathbb{C}^*$-fibration over the orbifold

$$
\mathcal{B}_Q(\omega) := \prod_{i=1}^{s} \mathbb{CP}(V_Q^i(d_i))/S_{\omega}
$$

As in the case of quadratic forms $Q$, one has a map $\Theta : \mathcal{M}_Q(\omega) \rightarrow \mathcal{F}act(\omega)$, where $\mathcal{F}act(\omega) \subset V(d)^\circ$ is the variety of homogeneous degree $d$ polynomials in $x, y,$ and $z$ that admit a factorization as a product of polynomials of the degrees $\{d_i\}_{1 \leq i \leq s}$ prescribed by the partition $\omega$. The map $\Theta$ takes each multipole to the product of the corresponding polynomial factors. Unlike the case of a quadratic $Q$, $\Theta$ may not be a 1-to-1 map, although, its generic fiber is a singleton. This conclusion follows from the unique factorization property for the ring $\mathbb{C}[x, y, z]$: just consider elements of $\mathcal{F}act(\omega)$ that are products of irreducible factors of the degrees prescribed by $\omega$. The same uniqueness of factorization implies that each fiber of $\Theta$ is finite: there are only finitely many ways of organizing irreducible factors, in which a polynomial $P \in \mathcal{F}act(\omega)$ decomposes, into blocks of degrees $\{d_i\}$.

The map $\Theta$ is not surjective either. However, $\Phi_\omega$ takes the multipole space onto the space $\mathcal{F}act_Q(\omega)$ of degree $d$ homogeneous polynomials on the surface $\{Q = 0\}$ that admit factorizations subordinate to $\omega$:

$$
\Phi_\omega : \mathcal{M}_Q(\omega) \rightarrow \mathcal{F}act(\omega) \rightarrow \mathcal{F}act_Q(\omega) \subset V_Q^\circ(d)^\circ
$$

**Definition 4** Let $d$ be a natural number and $\omega = \{d = \sum_{i=1}^{s} d_i\}$ its partition. Let $Z$ be a finite set equipped with a multiplicity function $\mu : Z \rightarrow \mathbb{N}$ whose $l_1$-norm $\|\mu\|_1$ is $ld$. A generalized $\omega$-parcelling of $(Z, \mu)$ is a collection of functions $\mu_i : Z \rightarrow \mathbb{N}$, such that

- $\sum_i \mu_i = \mu$
- $\|\mu_i\|_1 = ld_i$

When $Z$ is comprised of $ld$ points and each $\mu_i$ takes only two values $0, 1$, the generalized parcelling is called just an $\omega$-parcelling.

Any polynomial $P(x, y, z)$ of degree $d$ that is not divisible by $Q$ determines a multiplicity function $\mu : Z(P, Q) \rightarrow \mathbb{N}$ whose $l_1$-norm is $ld$. Here $Z(P, Q) := P \cap Q \subset \mathbb{CP}_2^s$ is a finite set. If such a polynomial $P$ is a product $\prod_i L_i$, where $\deg(L_i) = d_i$, then the $L_i$’s give rise to a unique generalized $\omega$-parcelling $\sum_i \mu_i$.

**Lemma 8** Any generalized $\omega$-parcelling $\sum_i \mu_i$ of a given multiplicity function $\mu$ on a finite set $Z \subset Q$ corresponds to at most one multipole in the space $\mathcal{M}_Q(\omega)$.

**Proof** Assume that, for each index $i$, there exist a polynomial $L_i$ that realizes $\mu_i$ on the finite intersection set $Z \subset Q$. Such polynomial is not divisible by $Q$. Put $Z_i = Z(L_i, Q) := L_i \cap Q$. Then,
employing the Bezout Theorem, any other polynomial \( M_i \) that realizes the same multiplicity on the same intersection set \( Z_i \) must be of the form \( M_i = \lambda_i L_i + Q \cdot R_i \). We notice that if \( L_i \in V_Q^\perp(d_i) \), then \( M_i \not\in V_Q^\perp(d_i) \), provided \( R_i \neq 0 \). □

**Corollary 18** The map \( \Phi_\omega \) has finite fibers over \( \text{Fact}_Q(\omega) \).

**Proof** An element of \( P \in V_Q^\perp(d)^\circ \) is determined, up to proportionality, by its multiplicity function \( \mu : Z(P, Q) \to \mathbb{N} \). Now the claim of the corollary follows from Lemma 8 and the observation that a given multiplicity function admits only finitely many generalized \( \omega \)-parcellings. □

Of course, not any generalized \( \omega \)-parceling on a given pair \((Z \subset Q, \mu)\) is realizable by a product \( \prod_i L_i \) with the properties as above. Crudely, this happens because not any \( ld_i \) points on \( Q \) can be placed on a curve \( C_i \) of degree \( d_i \) that does not contain \( Q \) as its component. A curve of degree \( d_i \) can always accommodate \((d_i^2 + 3d_i)/2\) points in \( \mathbb{C}P_s^2 \). Thus, if an inequality \((d_i^2 + 3d_i)/2 \geq d_i l \) is valid, that is, if \( d_i \geq 2l - 3 \), the right curve might be found; but it is still unclear how to avoid the very real possibility that \( C_i \supset Q \) when \( d_i \geq l \). In fact, such a possibility is a reality!

Unfortunately, unless \( l = \deg(Q) = 2 \) or \( s = 1 \), the image of \( \Phi_\omega \) is of a smaller dimension than the one of the target space \( V_Q^\perp(d)^\circ \). As a result, there is no analog of the Sylvester Theorem in the non-quadratic skies; when \( \deg(Q) > 2 \), a generic element of \( V_Q^\perp(d)^\circ \) is irreducible. Let us explain these claims.

Recall that for any \( d \geq l \), \( \dim(V_Q^\perp(d)) = \frac{1}{2}(d^2 + 3d) - [(d - l)^2 + 3(d - l)] \). Since the dimension of the group \( H_s \) is \( s - 1 \), we get

\[
\dim(V_Q^\perp(d)) - \dim(M_Q(\omega)) = \frac{l}{2}(2d - l + 3) - \sum_{i=1}^{s} \frac{l}{2}(2d_i - l + 3) + (s - 1) = \frac{s - 1}{2}[l^2 - 3l + 2],
\]

provided all \( d_i \geq l \). Under these hypotheses, the difference of the two dimensions vanishes only when \( l = 1, 2 \) or \( s = 1 \). Hence,

**Lemma 9** For \( l = \deg(Q) > 2 \) and \( \{d_i \geq l\}_{1 \leq i \leq s} \), the map \( \Phi_\omega \) is not onto, i.e. a generic polynomial from \( V_Q^\perp(d)^\circ \) is not a product of polynomials of degrees \( \geq l \). The codimension of \( \Phi_\omega(M_Q(\omega)) \) in \( V_Q^\perp(d) \) is \( \frac{s - 1}{2}[l^2 - 3l + 2] \).

For example, on a cubic surface, \( \dim(V_Q^\perp(d)) - \dim(M_Q(\omega)) = (s - 1) \), provided \( \{d_i \geq 3\}_{1 \leq i \leq s} \).

If we drop the hypotheses \( \{d_i \geq l\} \), the computation is a bit more involved:

\[
\dim(V_Q^\perp(d)) - \dim(M_Q(\omega)) = \frac{l}{2}(2d - l + 3) - \sum_{i:d_i \geq l} \frac{l}{2}(2d_i - l + 3) - \sum_{j:d_j<l} \frac{d_j}{2}(d_j + 3) + (s - 1)
\]

We conjecture that the RHS of the formula above is always positive, unless \( l = 1, 2 \) or \( s = 1 \).
5 Multipoles and function approximations on quadratic surfaces

We will be concerned with polynomial approximations of holomorphic functions \( f : \mathcal{S} \to \mathbb{C} \) on an irreducible complex quadratic surface \( \mathcal{S}_Q = \{ Q(x, y, z) = 1 \} \), as well as with polynomial approximations of continuous functions \( f : \mathcal{S}_Q^\mathbb{R} \to \mathbb{R} \) on its real version. Eventually, we would like to represent the approximating polynomials in terms of their multipoles.

As one uses polynomials of higher and higher degrees to improve approximations, the issue is stability of the multipole representations. In general, such stability is absent for several reasons: 1) the intrinsic ambiguities of the multipole representations for complex polynomials; 2) the impossibility of converting an analytic function on a surface into a “homogeneous” analytic function in the ambient space (homogenizing polynomials worked well). Even abandoning multipole representations in favor of linear methods of harmonic analysis, does not eliminate the stability issue instantly: in general, the coefficients of approximating polynomials fail to stabilize. However, if the approximating polynomials are linear combinations of mutually orthogonal and normalized polynomials (analogous to the Legendre polynomials), the coefficients of the combinations will stabilize. By introducing appropriate notions of orthogonality for polynomials on a quadratic surface, we aim to establish similar facts for polynomial approximations there (the spherical harmonics reflect a particular case of such orthogonality). Then we can combine harmonic analysis with non-linear methods of multipole representation for polynomials. We call such an approach synthetic.

Let \( C(K) \) denote the algebra of all continuous \( \mathbb{C} \)-functions on a Hausdorff compact space \( K \). Recall that a uniform algebra is a closed (in the sup-norm) subalgebra \( A \subset C(K) \) that separates points of \( K \). Such an algebra is called antisymmetric, if any real-valued function from \( A \) is constant. A subset \( Y \subset K \) is called an antisymmetry set for \( A \) if any function from \( A \), which is real-valued on \( Y \), is a constant. The Bishop Theorem about antisymmetric subdivisions (cf. \( \text{[G]}, \text{Theorem 13.1} \) claims that the maximal sets of antisymmetry \( \{ E_\alpha \} \) are closed and disjoint, and their union is \( K \). Moreover, if \( f \in C(K) \) and \( f|_{E_\alpha} \in A|_{E_\alpha} \), then \( f \in A \). In particular, if each \( E_\alpha \) is a singleton, then \( A = C(X) \).

For a space \( X \subset \mathbb{C}^n \), let us denote by \( \hat{\mathcal{P}}(X) \) the closure in the sup-norm on compacts in \( X \) of the algebra \( \mathcal{P}(X) \) generated by all complex polynomial functions. Note that if any two points in \( X \) can be separated by a real-valued polynomial, then Bishop’s Theorem implies that \( \hat{\mathcal{P}}(X) = C(X) \).

For instance, consider a section \( H \) of the complex surface \( \mathcal{S}_Q \) by a totally real subspace \( V^3 \subset \mathbb{C}^3 \) — an image of \( \mathbb{R}^3 \subset \mathbb{C}^3 \) under a complex transformation \( A \in GL(3; \mathbb{C}) \). Since any two points in \( \mathbb{R}^3 \), can be separated by a real-valued polynomial, the same property holds for any two points in \( V^3 \), and thus, in \( H \). By the Bishop Theorem, any continuous function \( f \) on \( H \) admits an approximation in the sup-norm on compacts by complex polynomials. In particular, this conclusion is valid for the real surfaces \( H = \mathcal{Y}_Q \subset (\mathbb{R}^3)A^{-1} \) and \( H = \mathcal{S}_Q^\mathbb{R} \subset \mathbb{R}^3 \) which have been employed on many occasions.

For a compact set \( K \subset \mathcal{S}_Q \), denote by \( \hat{K} \) the polynomial hull (closure) of \( K \). It consists of all points \( v \) in \( \mathbb{C}^3 \) with the property: \( |P(v)| \leq sup_{w \in K}|P(w)| \) for any complex polynomial \( P \). Because for any point \( v \notin \mathcal{S} \) and \( w \in \mathcal{S} \), we have \( |(Q - 1)(v)| > |(Q - 1)(w)| = 0 \), the polynomial closure \( \hat{K} \) must be contained in \( \mathcal{S} \). In fact, the polynomial closure \( \hat{K} \) of a compact \( K \subset \mathbb{C}^3 \) must be a polynomially convex compact set. A theorem by Oka and Weyl (cf. \( \text{[G]}, \text{Theorem 5.1} \) claims
that any complex analytic function, defined in a neighborhood of a compact polynomial convex set, admits an approximation in the sup-norm on \( K \) by complex polynomials. Thus, any analytic function \( f(x, y, z) \) defined in a neighborhood of \( \mathring{K} \subset S_Q \), \( K \) being a compact in \( S_Q \), can be uniformly approximated on \( \mathring{K} \) by complex polynomials.

When dealing with families of functions on non-compact sets, the default topology in the relevant functional spaces is defined by the uniform convergancy on compact subsets. In this topology, the subalgebra \( \mathcal{O}(S_Q) \) of holomorphic functions is closed in the algebra of all continuous functions \( C(S_Q) \) (cf. [GuR], Lemma 11). In particular, if a sequence of polynomials \((x, y, z)\) is converging in the sup-norm on every compact in \( S_Q \), then its limit is a holomorphic function on \( S_Q \). One can employ any expanding family \( \{K_r\}_{1 \leq r \leq \infty} \) (i.e., \( K_r \subset K_{r+1} \) and \( \cup_r K_r = S_Q \)) of polynomially convex compacts \( K_r \subset S_Q \) to build a sequence \( \{P_r\} \) of polynomials that will approximate a given holomorphic function \( f \in \mathcal{O}(S_Q) \).

Let \( \mathcal{F}(S_Q) \) be a subset of \( \mathcal{O}(S_Q) \) formed by functions on \( S_Q \) that admit a representation as a series

\[
\sum_{k=0}^{\infty} \prod_{j=1}^{k} L_{kj}, \tag{56}
\]

where each \( L_{kj} \) is a linear form in \( x, y, \) and \( z \). The series is required to converge uniformly on each compact \( K \subset S_Q \) (as we remarked before, any such uniformly converging series produces a holomorphic function on \( S_Q \)).

In fact, one can define a similar set \( \mathcal{F}(K) \subset C(K) \) for any closed \( K \subset \mathbb{C}^3 \). Due to Theorem 11, any polynomial on \( S_Q \) is of the form (34) (which is a special case of (56)). Therefore, when \( \mathcal{F}(K) = C(K) \) for a compact \( K \subset S_Q \), then \( \mathcal{F}(K) \) is dense in \( C(K) \) as well.

Examining (56), we observe that if this series converges at a point \( v = (x, y, z) \) it must converge absolutely at any other point \( \lambda \cdot v \), were the complex number \( \lambda \) has modulus less than 1. For any set \( Y \subset \mathbb{C}^3 \), we denote by \( Y^\star \) the set \( \{\lambda v \mid v \in Y, \lambda \in \mathbb{C}, |\lambda| < 1\} \) and call it the round hull of \( Y \).

Consider the set \( S_Q^\star \). Because any complex line through the origin that does not belong to the complex cone \( \{Q = 0\} \) hits \( S_Q \) at a pair of antipodal points, \( S_Q^\star \) is an open domain in \( \mathbb{C}^3 \), complementary to the cone (the origin belongs to \( S_Q^\star \)), whose boundary contains \( S_Q \). Therefore, any function \( f \in \mathcal{F}(S_Q) \) must be a restriction of a function which is analytic in \( S_Q^\star \) and continuous in \( S_Q^\star \cup S_Q \). I doubt that the converse statement is true. Note that a given function \( f \) on \( S_Q \) may have many analytic extensions in \( S_Q^\star \): for example, 1 extends to 1 and to \( 1/Q \) (the latter has poles along the complex cone).

If a section \( H \subset S_Q \) by a totally real subspace \( V^3 \) is an ellipsoid, then its interior in \( V^3 \) coincides with \( S_Q^\star \cap V^3 \). Thus, any real function \( f \) on \( H \) that admits a representation as in (56) must be real analytic in the interior of the ellipsoid. So, it is represented by its Taylor series at the origin; on the other hand, series (56), uniformly converging in the vicinity of the origin, is a form of a very specialized Taylor series (just count the dimensions of the coefficient spaces of each degree to see how special it is).

Since any function from \( \mathcal{O}(S_Q) \) admits a polynomial approximation on compacts, the subset \( \mathcal{F}(S_Q) \) is dense in in the space of all holomorphic functions. So, \( \mathcal{F}(S_Q) \) is squeezed between the vector
space $O(S_Q)$ and its dense subspace $P(S_Q)$. It is not even clear whether $F(S_Q)$ is a vector space. To understand the structure of the set $F(S_Q)$ seems to be an interesting and hard problem. Unfortunately, our progress towards this goal is minimal.

Note that, if a holomorphic function vanishes on a totally real analytic surface $H \subset S_Q$, it must vanish in a neighborhood of $H$ in $S_Q$.

We summarize these observations in the following proposition that, in particular, generalizes the second claim in Theorem 3.

**Theorem 19** Any holomorphic function on complex surface $S_Q$ is a limit in the topology of uniform convergence on compacts of polynomial functions in $\mathbb{C}^3$; the approximating polynomials each admit a representation as in (34).

A subset $F(S_Q)$ of functions that admit a representation as in (56) is dense in the space of all holomorphic functions $O(S_Q)$ and invariant under multiplications by scalars and the natural $O_Q(3;\mathbb{C})$-action. Each function $f$ from $F(S_Q)$ admits a canonic holomorphic extension

$$f(\lambda x, \lambda y, \lambda z) = \sum_{k=0}^{\infty} \lambda^k \left( \prod_{j=1}^{k} L_{kj}(x, y, z) \right)$$

into the round hull $S_Q^\bullet$. Here $(x, y, z) \in S_Q$ and the module of $\lambda$ is less than one.

Any continuous function $f : H \to \mathbb{C}$ on the totally real surfaces $H = \Upsilon_Q$ or $H = S^\mathbb{R}_Q$ is a limit in the topology of uniform convergence on compacts in $H$ of polynomial functions in $\mathbb{C}^3$. As a result, the set $F(H)$ is dense in $C(H)$. Each function from $F(H)$ admits a similar analytic extension into the open portion of real cone over $H$ that is bounded by $H$ and contains the origin.

If $f = F|_H$ for a function $F$, holomorphic in a neighborhood of $H$ in $S_Q$, then such an $F$ is unique. □

Now consider the vector space $P(S_Q)$ of all polynomial functions restricted to $S_Q$ and equipped with the inner product $\langle f, g \rangle$ defined by (35). If a homogeneous polynomial $P \neq 0$, then $\langle P, P \rangle > 0$. An homogeneous polynomial is determined by its restriction to $S_Q$. Evidently, $\langle P, P \rangle = 0$ implies that the restriction of $P$ to $\Upsilon_Q$ is zero. Since $\Upsilon_Q \subset S_Q$ is a totally real analytic submanifold, $P|_{\Upsilon_Q} = 0$ implies that $P$ vanishes in the vicinity of $\Upsilon_Q$ in $S_Q$. By analyticity, $P$ must vanish everywhere in $S_Q$, and hence, $P$ is a zero polynomial. As a result, $\langle P, P \rangle$ gives rise to a norm on the space of homogeneous polynomials $V(d)$. Because any function from $P(S_Q)$ is a restriction of an homogeneous polynomial, we get a non-degenerated Hermitian inner product on the vector space $P(S_Q)$.

In view of Theorem 12 and by a similar line of arguments, $\langle P, P \rangle = 0$ implies that $P = 0$ for any complex polynomial $P \in \text{Ker}(\Delta_Q)$. Therefore, being restricted to a space of $Q$-harmonic polynomials, the inner product $\langle f, g \rangle$ in (35) gives rise to an Hermitian structure and an $L_2$-norm $\|P\|_{\Upsilon_Q}$. In particular, each space $\text{Har}_Q(k) \approx V_Q^\perp(k)$ inherits this norm, and $\text{Har}_Q(k)$ is orthogonal to $\text{Har}_Q(l)$, provided $l \neq k$. 36
Consider the vector space $\prod_{k=0}^{\infty} Har_Q(k)$ formed by finite sequences of vectors $\{P_k \in Har_Q(k)\}$ and its closure $L_{2Har}^H := \oplus_{k=0}^{\infty} Har_Q(k)$ formed by infinite sequences $\{P_k \in Har_Q(k)\}$ subject to the condition $\sum_{k=0}^{\infty} \|P_k\|_{\overline{Y_Q}}^2 < \infty$.

Let $L_2(\overline{Y_Q})$ denote the complex Hilbert space of $L_2$-integrable functions on the ellipsoid $\overline{Y_Q}$. Every function $f \in L_2(\overline{Y_Q})$ defines a unique system of its “Fourier components” $\{f_k \in Har_Q(k)\}$. Each $f_k$ is the unique polynomial from $Har_Q(k)$ that delivers the minimum $\min_{P \in Har_Q(k)} \|f - P\|_{\overline{Y_Q}}$. By Theorem 19, any $f \in C(\overline{Y_Q})$ is a limit in the sup-norm on $\overline{Y_Q}$ of harmonic polynomials, it must be also the limit of the same sequence of polynomials viewed as elements of $L_2(\overline{Y_Q})$. Therefore, if $f \in C(\overline{Y_Q})$ is orthogonal to all the subspaces $Har_Q(k)$, it must be a zero function. Hence, as an element of $L_2(\overline{Y_Q})$, $f = \sum_{k=0}^{\infty} f_k$, and $\|f\|_{\overline{Y_Q}} = \sum_{k=0}^{\infty} \|f_k\|_{\overline{Y_Q}}$. In particular, any $f \in \mathcal{O}(S_Q)$ determines a unique system of its harmonics $\{f_k \in Har_Q(k)\}$. Moreover, it is determined by $f|_{\overline{Y_Q}}$, and thus, by its harmonics $\{f_k\}$.

Evidently, the sequence of partial sums $\{f_d := \sum_{k=0}^{d} f_k \in Ker(\Delta_Q)\}_d$ converges in $L_2(\overline{Y_Q})$ to $f$.

In terms of homogeneous polynomials, we get an analogous sequence $\{f_d := \sum_{k=0}^{d} Q^{[(d-k)/2]} f_k\}_d$ converging in $L_2(\overline{Y_Q})$ to $f$.

Similar arguments can be applied, instead of the ellipsoid $\overline{Y_Q}$, to any surface $H \subset S_Q$ that is a section of $S_Q$ by a totally real subspace $V^3 \subset \mathbb{C}^3$. In particular, they are valid for $S_Q^E$. First, we pick a measure $dm$ on $H$ such that any polynomial $P|_H \in L_2(H)$ (when $H$ is compact, one can choose any measure of finite volume). For example, consider the area 2-form $\omega$ on $H$ induced by the imbedding $H \subset \mathbb{C}^3$ and multiply it by the function $\exp[-(x\overline{x} + y\overline{y} + z\overline{z})]$ to get the right $dm$. Then we define a new inner product by $(f, g)_H = \int_H f \cdot \overline{g} \, dm$. Since $H$ is totally real, this will give rise to a Hermitian structure in each space $V(k)$. Notice that the multiplication-by-$Q$ imbedding $V(k) \to V_Q(k+2)$ is an isometry. As before, we form the orthogonal complements $V_Q(k)$ to the subspaces $V^Q(k)$ (the only difference is that now the space $V_Q^Q(d)$ could be different from the space $Har_Q(d)$ of $Q$-harmonic polynomials). Then we use $\prod_{k=0}^{\infty} V_Q^Q(k)$ to construct a Hilbert space $\oplus_{k=0}^{\infty} V_Q^Q(k)$. As before, any continuous function $f$ on $H$ will acquire a unique representation $f = \sum_{k=0}^{\infty} f_k$, where $\{f_k \in V_Q^Q(k)\}$, and $\|f\|^2_H = \sum_{k=0}^{\infty} \|f_k\|^2_H$.

Thanks to Theorems 7 and 11, each complex homogeneous polynomial $f_k \in V_Q^Q(k)$ can be represented by some multipole $w_k^\ell \in \mathcal{M}(k)$. Similarly, any real homogeneous polynomial $f_k \in V_Q^Q(k; \mathbb{R})$ can be represented by some multipole $w_k^\ell \in \mathcal{M}(k)$. When $Q$ is positive definite, these real multipoles are unique. However, in general, the ambiguity of the multipole representation could cause some trouble. So, we need to choose the representing multipoles with some care.

Due to the embedding $\Theta : \overline{\mathcal{M}(k)} \to V(k)$ (see (16)) with the image $\text{Fact}(k)$, the multipole space $\overline{\mathcal{M}(k)}$ acquires a metric $\rho$ induced by the $L_2^H$-norm $\|P\|_H$ in $V(k)$.

The lemma below helps to estimate the size of the fiber $\Phi_Q^{-1}(u)$ over $u \in V_Q^Q(k)$ in terms of an universal angle $\theta_k = \theta(k, H, dm)$ and the $L_2^H$-norm of $u$:

**Lemma 10** Consider the distance function $\rho$ on $\overline{\mathcal{M}(k)}$ that is induced by the $L_2^H$-norm $\|\cdot\|_H$ in $V(k)$ via the embedding $\Theta : \overline{\mathcal{M}(k)} \to \text{Fact}(k) \subset V(k)$. Then there exist an angle $0 < \theta_k \leq \pi/2$
so that, for each \( u \in V^+_Q(k) \subset V(k) \), the distance from any \( w \in \Phi^{-1}_Q(u) \) to the zero multipole is at most \( \|u\|/\sin(\theta_k) \), and the diameter of the fiber \( \Phi^{-1}_Q(u) \) is at most \( 2\|u\|/\sin(\theta_k) \).

**Proof** Let \( S(k) \) denote a unit sphere (with respect to the \( \| \cdot \|_H \)-norm) in \( V(k) \) and centered at the origin. Because \( Q \) is irreducible, \( \mathcal{F}(k) \cap V_Q(k) = \emptyset \). Thus, the compact sets \( S(k) \cap \mathcal{F}(k) \) and \( S_Q(k) = S(k) \cap V_Q(k) \) are disjoint. Therefore, there is a number \( 0 < \theta \leq \pi/2 \) so that the angle between any two vectors \( u \in S(k) \cap \mathcal{F}(k) \) and \( v \in S_Q(k) \) is greater than or equal to \( \theta \). Note that \( S(k) \cap \mathcal{F}(k) \) and \( S_Q(k) \) are invariant under the circle action \( S^1 \subset \mathbb{C}^* \). So, \( \mathcal{F}(k) \) and \( V_Q(k) \) also are real cones with their tips at the origin and bases \( S(k) \cap \mathcal{F}(k) \) and \( S_Q(k) \). We conclude that the angle between any two vectors \( u \in \mathcal{F}(k) \) and \( v \in V_Q(k) \) has the same lower bound \( \theta > 0 \). Now consider an open real cone \( C_Q(k) \subset V(d) \) comprised of vectors that form an angle \( \phi < \theta \) with the subspace \( V_Q(k) \) and a complementary cone \( C_Q^+(k) := V(d) \setminus C_Q(k) \supset V_Q^+(k) \).

The argument above shows that \( \mathcal{F}(k) \subset C_Q^+(k) \). Hence, the distance from any \( w \in \Phi^{-1}_Q(u) \) to the zero multipole is at most \( \|u\|/\sin(\theta_k) \). As a result, the diameter of the fiber \( \Phi^{-1}_Q(u) \) is at most \( 2\|u\|/\sin(\theta_k) \). \( \square \)

**Corollary 20** Consider a continuous function \( f \in L_2(H) \) and its orthogonal decomposition

\[
 f = \sum_{k=0}^{\infty} f_k, \quad \text{where} \ f_k \in V^+_Q(k).
\]

Then, for any choice of the multipoles \( w_k \in \Phi^{-1}_Q(f_k) \),

\[
 \sum_{k=0}^{\infty} \sin^2(\theta_k) \cdot \rho(w_k,0)^2 < \infty, \quad (57)
\]

where \( \rho(w_k,0) = \|\Theta(w_k)\|_H \). \( \square \)

It seems to be far from trivial to understand the asymptotic behavior of \( \{\sin(\theta_k)\} \) as \( k \to \infty \). Perhaps, the lack of understanding of this asymptotics it is the most significant gap in our analysis.

To state the last claim in the next theorem, we need one technical definition that likely has very little to do with the essence of the statement. The set \( \mathcal{D}(\Phi_Q) \subset V_Q^+(k) \) is a complex algebraic variety that is stratified by algebraic sets \( \{\mathcal{D}_{k,\pi}\} \) which are labeled by various partitions \( \pi \) of \( 2k \). This labeling is done by attaching the divisor \( P \cap Q \in Sym^{2k}(Q) \), or rather the partition \( \pi \) of \( 2k \) defined by the multiplicity function of \( P \cap Q \), to each homogeneous polynomial \( P(x,y,z) \) restricted to the cone \( \{Q = 0\} \) and viewed as an element of \( V_Q^+(k) \). In particular, when \( \pi = \{2d = 1 + 1 + 1 + \ldots + 1\} \) or \( \{2d = 2 + 1 + 1 + \ldots + 1\} \), then \( \mathcal{D}_{k,\pi} = V_Q^+(k) \) or \( \mathcal{D}_{k,\pi} = \mathcal{D}(\Phi_Q) \), respectively. The natural partial order among partitions reflects the inclusions of the corresponding strata. If we delete all the substrata from a given stratum \( \mathcal{D}_{k,\pi} \), we get a “pure” stratum that we denote \( \mathcal{D}^e_{k,\pi} \). The variety \( \mathcal{D}(\Phi_Q) \) is a Whitney stratified space; as a result, the vicinity of every stratum \( \mathcal{D}^e_{k,\pi} \) has a structure of a bundle whose fiber is a real cone over another stratified space \( L\mathcal{D}_{k,\pi} \). We will make use of this fact together with another important feature of the stratification \( \mathcal{D}_{k,\pi} \): namely, all the strata of \( L\mathcal{D}_{k,\pi} \) have even real codimensions.

We say that a parametric curve \( \gamma : [0,1] \to V_Q^+(k) \approx Har_Q(k) \) is *tame* if it consists of a finite number of arcs, each of which has the following property: the interior of each arc is contained in some stratum \( \mathcal{D}^e_{k,\pi} \). We say that a continuous function family \( \{f_t \in L_2(T_Q)\}_{0 \leq t \leq 1} \) is *tame*, if for each \( k \), the path \( \{(f_t)_k \in Har_Q(k)\}_{0 \leq t \leq 1} \) is tame.
First, consider the functions \( f \in \mathcal{O}(\mathcal{S}_Q) \) such that, for each \( k \), the polynomial \( f_k \in \text{Har}_Q(k) \approx V_Q^k(k) \) does not belong to the ramification locus \( \mathcal{D}(\Phi_Q) \subset V_Q^k(k) \) of the map \( \Phi_Q \) (the rest of the functions form a complex codimension one subset \( \mathcal{D} \subset \mathcal{O}(\mathcal{S}_Q) \)). Over the compliment to the variety \( \mathcal{D}(\Phi_Q) \), the map \( \Phi_Q \) is a covering map. Thus, for each initial lifting, a deformation \((f_t)_k\) admits a unique lifting to the multipole space, as long as \((f_t)_k \in V_Q^k(k) \setminus \mathcal{D}(\Phi_Q)\).

Next, for any tame \( t \)-family \( \{f_t \in \mathcal{O}(\mathcal{S}_Q)\}_{0 \leq t \leq 1} \), consider the tame curve \( \{(f_t)_k \in V_Q^k(k)\}_{0 \leq t \leq 1} \) and the first arc \( \gamma \) in a finite sequence of arcs that form this curve. There are two possibilities: 1) the arc starts at a stratum \( \mathcal{D}^0_{k,\pi} \) and is confined to it for a while, 2) the arc starts at a stratum \( \mathcal{D}^0_{k,\pi} \) but moves instantly into an ambient stratum \( \mathcal{D}^0_{k,\pi'} \). In the first case, over \( \mathcal{D}^0_{k,\pi} \), \( \Phi_Q \) is a covering map and there is a unique lifting of \( \gamma \) extending each lifting \( \tilde{\gamma}(0) \) of \( \gamma(0) \). In the second case, we claim that, for any lifting \( \tilde{\gamma}(0) \) of \( \gamma(0) \), in the vicinity of \( \tilde{\gamma}(0) \) the map \( \Phi_Q \) is onto. Indeed, it is a proper holomorphic map, and thus, its image must be an analytic space (see [N], Theorem 2, page 129). Because \( \Phi_Q \) is finite, the image of a neighborhood of \( \tilde{\gamma}(0) \) under \( \Phi_Q \) must be of the maximal dimension, and hence, must contain a neighborhood of \( \gamma(0) \). As a result, the set \( \Phi_Q^{-1}(\gamma) \) (it is a finite graph) must be present in any neighborhood of \( \tilde{\gamma}(0) \); so, we can lift \( \gamma \) to an arc \( \tilde{\gamma} \) that starts at \( \tilde{\gamma}(0) \). An induction by the number of arcs in the curve \((f_t)_k\) proves the existence of its lifting to the multipole space. Note that an analogous argument fails for \( \Phi_Q^0 \); a finite image of a real analytic set is a real semi-analytic set that can miss the arc \( \gamma \). Therefore, in Theorem 22 the lifting property for tame deformations is absent.

The arguments above prove the following theorem:

**Theorem 21** Let \( Q(x, y, z) \) be an irreducible complex quadratic form, and let \( \mathcal{S}_Q \) be a complex quadratic surface defined by the equation \( \{Q = 1\} \). Denote by \( A \) a complex change of coordinates that reduces the form \( Q \) to the sum of squares. Let \( \Upsilon_Q \subset \mathcal{S}_Q \) be a totally real ellipsoid defined by the equations \( \{Q((x, y, z)A = 1), \text{Im}((x, y, z)A) = 0\} \) and equipped with the measure defined by (35). Let \( f \) be an analytic function on \( \mathcal{S}_Q \).

Then there exists sequence of multipoles \( \{w^f_k \in \overline{\mathcal{M}}(k)\}_{0 \leq k < +\infty} \) such that:

- the sequence gives rise, via the maps \( \{\overline{\mathcal{O}}_Q\} \), to complex \( Q \)-harmonic polynomials \( P^f_d(x, y, z) = \sum_{k=0}^d \overline{\mathcal{O}}_Q(w^f_k) \), where the mutually orthogonal polynomials \( \{\overline{\mathcal{O}}_Q(w^f_k) \in \text{Har}_Q(k)\} \) are uniquely determined by \( f \).

- as \( d \to \infty \), the polynomials \( \{P^f_d\} \) converge in the space \( L_2(\Upsilon_Q) \) to the function \( f|_{\Upsilon_Q} \), and therefore, uniquely determine \( f \in \mathcal{O}(\mathcal{S}_Q) \).

- for a given \( f \), there are at most \((2k - 1)!!\) choices for each multipole \( w^f_k \).

- the multipoles \( \{w^f_k\} \) satisfy property (57) from Corollary 20.

- for any tame deformation \( \{f_t \in \mathcal{O}(\mathcal{S}_Q)\}_{0 \leq t \leq 1} \) of the function \( f = f_0 \), there exists a continuous deformation \( \{w^f_t \in \overline{\mathcal{M}}(k)\} \) of the \( f_t \)-representing multipoles, such that \( \{w^f_0 = w^f_t\} \). For functions \( f \) and their continuous deformations \( f_t \) outside a subspace \( \mathcal{D} \subset \mathcal{O}(\mathcal{S}_Q) \) of complex codimension one and for each choice of the appropriate multipoles \( \{w^f_0\} \), the lifting of the deformation \( f_t \) to the multipole spaces is unique. \( \square \)
Similarly, we get

**Theorem 22** Let \( Q(x, y, z) \) be an irreducible real quadratic form. Let \( S_Q^\mathbb{R} := \{Q = 1\} \) be a real quadratic surface, equipped a measure \( dm \) for which any polynomial in \( x, y, \) and \( z \) is an \( L_2 \)-integrable function on the surface. Let \( f \) be an \( L_2 \)-integrable continuous function on \( S_Q^\mathbb{R} \).

Then there exists sequence of multipoles \( \{w^f_k \in \mathcal{M}(k)\}_{0 \leq k < +\infty} \) such that:

- the sequence gives rise, via the maps \( \{\Phi^\mathbb{R}_Q\} \), to real polynomials \( P^f_d(x, y, z) = \sum_{k=0}^d \Phi^\mathbb{R}_Q(w^f_k) \), where the mutually orthogonal polynomials \( \{\Phi^\mathbb{R}_Q(w^f_k)\} \) are uniquely determined by \( f \);
- As \( d \to \infty \), the polynomials \( \{P^f_d\} \) converge to \( f \) in the space \( L_2(S_Q^\mathbb{R}) \);
- The multipoles \( \{w^f_k\} \) satisfy property (57) from Corollary 20.

When \( Q \) is positive-definite,

- the mutually orthogonal polynomials \( \{\Phi^\mathbb{R}_Q(w^f_k) \in Har_Q(k; \mathbb{R})\} \);
- each multipole \( w^f_k \) is uniquely determined by \( f \);
- for any continuous deformation \( \{f_t \in C(S_Q^\mathbb{R})\}_{0 \leq t \leq 1} \), of the function \( f = f_0 \), there exists a unique continuous deformation \( \{w^{f_t}_k \in \mathcal{M}(k)\} \) of the \( f_t \)-representing multipoles, such that \( \{w^{f_0}_k = w^f_k\} \). \( \square \)

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