Hamilton-Jacobi treatment of fields with constraints

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Abstract

In this paper the Güler’s formalism for the systems with finite degrees of freedom is applied to the field theories with constraints. The integrability conditions are investigated and the path integral quantization is performed using the action given by Hamilton-Jacobi formulation. The Proca’s model is investigated in details.

1 Introduction

The most common method for investigating the Hamiltonian treatment of constrained systems was initiated by Dirac[1]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Hence, equations of motion are obtained in terms of arbitrary parameters.

The starting point of the Güler’s method[2,3,4,5,6,7,8] is the variational principle. The Hamiltonian treatment of constrained systems leads us to total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system.

Recently Güler has presented a treatment of classical fields as constrained systems[9]. Then Hamilton-Jacobi quantization of finite dimensional system with constraints was investigated in[10]. The purpose of this paper is to

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generalize the Güler's formalism for systems with finite degrees of freedom in order to include the field theories with constraints also.

The plan of this paper is the following: In Sec.2 the Güler's formalism for the theories of fields with constraints is presented and the integrability conditions are investigated. In Sec.3 the quantization of Proca’s model is analyzed. In Sec.4 we present our conclusions.

2 Güler's formalism for the field theories with constraints

We generalize the Güler’s formalism for systems finite dimensions and constraints [10] in order to describe the field theories with constraints. Let suppose that the local field theories are described by the Lagrangean

\[ L(\phi_i, \frac{\partial \phi_i}{\partial x_\mu}), i = 1, \cdots, n \] (1)

The canonical formulation gives the set of Hamilton-Jacobi partial-differential equation as

\[ H'_\alpha(\chi_\beta, \phi_\alpha, \frac{\partial S}{\partial \phi_\alpha}, \frac{\partial S}{\partial \chi_\alpha}) = 0, \alpha, \beta = 0, n - r + 1, \cdots, n, a = 1, \cdots, n - r, \] (2)

where

\[ H'_\alpha = H_\alpha(\chi_\beta, \phi_\alpha, \pi_a) + \pi_\alpha \] (3)

and \( H_\alpha \) is the canonical hamiltonian. The equations of motion are obtained as total differential equations in many variables as follows

\[ d\phi_a = \frac{\partial H'_\alpha}{\partial \pi_a} d\chi_\alpha, d\pi_a = -\frac{\partial H'_\alpha}{\partial \phi_\alpha} d\chi_\alpha, d\pi_\mu = -\frac{\partial H'_\alpha}{\partial \chi_\mu} d\chi_\alpha, \mu = 1, \cdots, r \] (4)

\[ dz = (-H_\alpha + \pi_a \frac{\partial H'_\alpha}{\partial \pi_a})d\chi_\alpha \] (5)

where \( z = S(\chi_\alpha, \phi_\alpha) \). The set of equations (3,4) is integrable if

\[ dH'_\alpha = 0, dH'_\mu = 0, \mu = 1, \cdots r \] (6)

If conditions (3,4) are not satisfied identically, one considers them as new constraints and again tests the consistency conditions. Thus repeating this procedure one may obtain a set of conditions.
2.1 Integrability conditions

If eqs. (4) are integrable, then the solutions of eqs. (5) are obtained by a quadrature. Hence, the investigation of the integrability condition of eqs. (4) is sufficient. As it was discussed before in [2] equations of motion of a singular system are total differential equations. As it is well known, to any set of total differential equations

\[dx_i = b_{i\alpha}(t_\beta, x_j) dt_\alpha, i, j = 1, \ldots, n, \alpha, \beta = 0, 1 \ldots, p < n\]  

(7)

there are corresponding set of partial differential equations in the form

\[b_{i\alpha} \frac{\partial f}{\partial x_i} = 0\]  

(8)

Now we may investigate the integrability conditions of the equations (4).

To achieve this aim we define the linear operators \(X_\alpha\) as

\[X_\alpha f = b_{i\alpha} \frac{\partial f}{\partial x_i}\]  

(9)

For the field theory we will define the linear operators \(X_\alpha\) as

\[X_\alpha f(\chi_\beta, \phi_a, \pi_a) = [f, H_\alpha'] = \frac{\delta f}{\delta \phi_a} \frac{\delta H_\alpha'}{\delta \pi_a} - \frac{\delta f}{\delta \pi_a} \frac{\delta H_\alpha'}{\delta \phi_a} + \frac{\delta f}{\delta \chi_\alpha}\]  

(10)

Lemma

A system of differential equations (4) is integrable iff

\[[H_\alpha', H_\beta'] = 0\]  

(11)

Proof.

Suppose that (11) is satisfied. Then

\[(X_\alpha, X_\beta) = (X_\alpha X_\beta - X_\beta X_\alpha) f = X_\alpha [f, H_\beta'] - X_\beta [f, H_\alpha']\]  

(12)

we get after using Jacobi's identity

\[(X_\alpha, X_\beta) f = [f, [H_\beta', H_\alpha']]\]  

(13)

From (11) and (13) we conclude that

\[(X_\alpha, X_\beta) = 0\]  

(14)

Conversely, if the system is complete, then (14) is satisfied for any \(\alpha\) and \(\beta\) and we get

\[[H_\alpha', H_\beta'] = 0\]  

(15)

Q.E.D.
2.2 Quantization of field theories with constraints

In this section we will investigate the quantization of the fields with constraints using G"uler's formalism.

Let us suppose that for a field with constraints we found all independent hamiltonians \( H'_\mu \) using the calculus of variations[2],[3],[6]. At this stage we can use Dirac's procedure of quantization[1]. We have

\[ H'_\mu \Psi = 0, \mu = 0, n - r + 1, \ldots, n \]  

(16)

where \( \Psi \) is the wave function. The consistency conditions are

\[ [H'_\mu, H'_\nu] \Psi = 0, \mu, \nu = 1, \ldots r \]  

(17)

where \([,] \) is the commutator. If the hamiltonians \( H'_\mu \) satisfy

\[ [H'_\mu, H'_\nu] = C_{\mu\nu} H'_\alpha \]  

(18)

then we have a theory with a first class constraints.

In the case when the hamiltonians \( H'_\mu \) satisfy

\[ [H'_\mu, H'_\nu] = C_{\mu\nu} \]  

(19)

with \( C_{\mu\nu} \) not depending on \( \phi_i \) and \( \pi_i \), from(17) there arises naturally Dirac’s brackets and the canonical quantization will be performed taking Dirac’s brackets into commutators.

On the other hand G"uler's formalism gives an action when all hamiltonians \( H'_\mu \) are in involution. Because in G"uler's formalism we work from the beginning in the extended space we suppose that variables \( \chi_\alpha \) depend of \( \tau \).Here \( \tau \) is canonical conjugate with \( p_0 \).

If we are able , for a given system with constraints, to find the independent hamiltonians \( H'_\mu \) in involution then we can perform the quantization of this system using the path integral quantization method with the action given by(3). After some calculations we found that the action \( z \) has the following form

\[ z = \int \left( -H_\alpha + \pi_a \frac{\partial H'_\alpha}{\partial \pi_a} \right) \dot{\chi}_\alpha d\tau \]  

(20)

where \( \dot{\chi}_\alpha = \frac{d\chi_\alpha}{d\tau} \).

3 The Proca’s model

The Proca’s model is described by the Lagrangian

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu \]  

(21)
The form of the hamiltonian in Güler’s formalism is:

\[ H'_{0} = p_{0} + \int \left[ \frac{\pi^{2}}{2} + \frac{1}{4} F_{ij}^{2} + \frac{m^{2}}{2} \left( A_{0}^{2} + A_{i}^{2} \right)^{2} - A_{0} \phi_{2} \right] d^{3}x \]  

(22)

where

\[ \phi_{2} = \partial_{i} \pi^{i} + m^{2} A_{0} \]  

(23)

The system possesses the primary constraint

\[ H'_{1} = \pi_{0} \]  

(24)

Imposing

\[ dH'_{0} = 0 \]  

(25)

we get another constraint

\[ H_{2} = \partial_{i} \pi^{i} + m^{2} A_{0} \]  

(26)

Then in the Güler’s formalism we have three hamiltonians. The hamiltonian \( H_{2} \) is not yet in the form (3). Since

\[ [H'_{1}, H_{2}] = -m^{2} \delta (x - y) \]  

(27)

the hamiltonians are not in involution. At this stage we can investigate the canonical quantization method using Dirac’s formalism. From (27) we conclude that the system have two second class constraints in Dirac’s classification and for canonical quantization we need Dirac’s brackets

\[ \{ F, G \}_{D.B.} = \{ F, G \} - \{ F, H_{2} \} C^{21} \{ H_{1}', G \} - \{ F, H_{1}' \} C^{12} \{ H_{2}', G \} \]  

(28)

where

\[ \{ , \} \]  

are the Poisson-brackets and the matrix \( C^{\alpha \beta} \) is the inverse of the matrix

\[ C_{\alpha \beta} = \begin{pmatrix} 0 & -m^{2} \delta (x - y) \\ m^{2} \delta (x - y) & 0 \end{pmatrix} \]  

(29)

On the other hand in the Güler’s formalism we have an action which is well defined when the hamiltonians are in involution. In our case we found the hamiltonians \( \tau'_{0}, \tau'_{1}, \tau'_{2} \) in involution in the following form

\[ \tau'_{1} = H'_{1} + m^{2} \rho = \pi_{0} + \tau_{1} \]

\[ \tau'_{2} = H_{2} + \pi_{\rho} = \pi_{\rho} + \tau_{2} \]

\[ \tau'_{0} = p_{0} + H^{(1)} + H^{(2)} = p_{0} + \tau_{0} \]  

(30)

where

\[ H^{(1)} = \int d^{3}x \left[ \left( \partial_{i} A^{i} \right) m^{2} \rho - \frac{\pi_{\rho}}{m^{2}} \left( \partial_{i} \pi^{i} + m^{2} A_{0} \right) \right] \]  

(31)
\[ H^{(2)} = \int d^3x \left[ \frac{-1}{2m^2} \pi_\rho^2 - \frac{m^2}{2} (\partial_i \rho) (\partial^i \rho) \right] \] (32)

and \( \rho, \pi \) are the extra fields satisfying \( \{ \rho, \pi \} = 1 \) all the other commutation relations become zero.

The action \( z \) has the following form

\[ dz = \left[ -\tau_0 + \int (\pi_i^2 + \partial_i A_0 + \frac{\partial_i \pi \rho}{m^2}) \right] d^3x d\tau + \int (-m^2 \rho) d^3x dA_0 + \int (-\partial_i \pi^i + m^2 A_0) d^3x d\rho \] (33)

or

\[ z = \int d\tau d^3x \left[ -\tau_0 + \int (\pi_i^2 + \partial_i A_0 + \frac{\partial_i \pi \rho}{m^2}) \right] + \int (-m^2 \rho) A_0 + \int (-\partial_i \pi^i + m^2 A_0) \dot{\rho} \] (34)

Here \( \dot{A}_0 = \frac{dA_0}{d\tau} \) and \( \dot{\rho} = \frac{d\rho}{d\tau} \).

For a system with \( r \) first class-constraints \( \psi^\alpha \) the path integral representation is given as

\[ <\phi' | e^{i(t' - t)\hat{H}_0} | \phi> = \int \prod d\mu(\phi_\mu, \pi_\mu) e^{i\int_{-\infty}^{+\infty} dt (\pi_\mu \dot{\phi}_\mu - H_0)} \] (35)

where the measure of integration is given as

\[ d\mu(\phi, \pi) = \text{det} \{ \psi^\alpha, \psi^\beta \} | \prod \delta(\chi^\alpha) \delta(\phi^\alpha) \prod d\phi^\mu d\pi_\mu \] (36)

and \( \psi^\alpha \) are \( r \)- gauge constraints.

We found after some calculations that the action (34) gives us the same result as (35) for the Proca’s model when \( \tau = t \) but using different gauge conditions.

4 Concluding remarks

Güler has initiated a new formalism for quantization of systems with constraints [2],[3],[4],[5],[6],[7],[8]. In this paper we have generalized the Güler’s formalism for the system with finite degrees of freedom [10] to the field theories with constraints. Our formalism is completely different from the formalism presented in [9].

An interesting case appears when a theory has secondary constraints and the constraints are of second class in the Dirac’s classification. We found that for the system with second class constraints the Dirac’s brackets arise naturally in the process of quantization for field with constraints in the Güler’s formalism. The Dirac’s brackets are defined on the extended space.

When the system with constraints has only primary constraints in involution Güler’s formalism gives us exactly the action which has the same expression as obtained in the path integral quantization after performing all calculations. In this case we do not need any gauge conditions. If the system
has secondary constraints or second class constraints this result is not valid since the hamiltonians are not in involution. To obtain the system in involution we need to extend the system. Because in the Güler's formalism we have the freedom to choose the dependence of gauge variables $\chi_\alpha$ we will choose $\chi_\alpha = \chi_\alpha(\tau)$. In this case the action of Güler's formalism gives the same results as path integral formulation for the system with constraints. For the Proca's model we extend the system and we found three hamiltonians in involution. The path integral quantization was performed with an action given by Güler's formalism and the results are in agreement with those obtained by other methods.

5 Acknowledgements

One of the authors (D.B.) would like to thank TUBITAK and NATO for financial support and METU for the hospitality during his working stage at Department of Physics.

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