Characteristic epsilon cycles of $\ell$-adic sheaves on varieties

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Abstract

Let $X$ be a smooth variety over a finite field $\mathbb{F}_q$. Let $\ell$ be a rational prime number invertible in $\mathbb{F}_q$. For an $\ell$-adic sheaf $\mathcal{F}$ on $X$, we construct a cycle supported on the singular support of $\mathcal{F}$ whose coefficients are $\ell$-adic numbers modulo roots of unity. It is a refinement of the characteristic cycle $CC(\mathcal{F})$, in the sense that it satisfies a Milnor-type formula for local epsilon factors. After establishing fundamental results on the cycles, we prove a product formula of global epsilon factors modulo roots of unity. We also give a generalization of the results to varieties over general perfect fields.

1 Introduction

Let $k$ be a perfect field of characteristic $p \geq 0$. For an $\ell$-adic sheaf on a variety over $k$, it is a central subject of ramification theory to calculate its global invariants by certain invariants which can be defined locally on the variety.

One of important invariants of an $\ell$-adic sheaf is the $L$-function. Let $X$ be a projective smooth variety over a finite field $\mathbb{F}_q$ and $\mathcal{F}$ be an $\ell$-adic sheaf on $X$. The $L$-function $L(X, \mathcal{F}; t)$ is defined as the infinite product

$$L(X, \mathcal{F}; t) = \prod_x \frac{1}{\det(1 - \text{Frob}_x t^{\deg(x/\mathbb{F}_q)}, \mathcal{F}_x)},$$

where $x$ runs through closed points of $X$. It admits a functional equation

$$L(X, \mathcal{F}; t) = \varepsilon(X, \mathcal{F}) t^{-\chi(X_{\overline{\mathbb{F}}_q}, \mathcal{F})} L(X, \mathcal{D}_X \mathcal{F}; t^{-1}).$$

Here $\mathcal{D}_X$ is the Verdier-dual functor. The invariant $\chi(X_{\overline{\mathbb{F}}_q}, \mathcal{F})$ is the Euler-Poincaré characteristic of $\mathcal{F}$. In [30], Saito gives a formula, called the index formula, which expresses the Euler-Poincaré characteristic as the intersection number $(CC(\mathcal{F}), T^*_X X)_{T^*X}$. Here $CC(\mathcal{F})$ is the characteristic cycle, which is constructed also in [30]. The index formula can be regarded as a higher dimensional generalization of the Grothendieck-Ogg-Shafarevich formula.

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The main subject of this paper is a more refined invariant \( \varepsilon(X, F) \); the global epsilon factor. This invariant is defined as, up to sign, the alternating product of the determinants of the actions of the Frobenius automorphisms on the cohomology groups of \( F \), i.e.

\[
\varepsilon(X, F) = \prod_i \det(-\text{Frob}_q^i, H^i(X_{\bar{k}}, F))(-1)^{i+1}.
\]

Let \( X \) be a curve. In this case, local counterparts of global epsilon factors are local epsilon factors, which are defined in [5]. For \( \ell \)-adic sheaves of rank 1, they are essentially Gauss sums. In the higher rank case, they can be calculated using the case of rank 1 and Brauer induction technique. Laumon [23] gives a formula which expresses global epsilon factors as the products of local epsilon factors, so called the product formula.

When the base field is a general perfect field \( k \), global epsilon factors should be defined as (the inverse of) the alternating products of the determinant representations of the cohomology groups. They are characters of the absolute Galois group of \( k \). The aim of this paper is to construct a micro-local theory of epsilon factors, as a refinement of that of characteristic cycles, which gives a product formula of global epsilon factors modulo roots of unity similarly as the index formula of the Euler-Poincaré characteristics.

We sketch the main results of this paper.

**Theorem 1.1.** (Theorem 4.9) Let \( X \) be a smooth variety over a finite field \( k \). Let \( F \) be a constructible complex of \( \mathbb{Q}_\ell \)-sheaves on \( X \). Then, there exists a unique cycle \( E(F) \), the epsilon cycle, supported on \( SS(F) \) with \( \mathbb{Q}_\ell \otimes \mathbb{Z} \mathbb{Q} \)-coefficients which satisfies a Milnor-type formula for local epsilon factors. Namely, for an (at most) isolated \( SS(F) \)-characteristic point (Definition 2.10.1) \( x \in X \) of a morphism \( f : X \to Y \) to a smooth curve, the equality

\[(E(F), df_x)^{\deg(x/k)} = \varepsilon_0(Y(x), R\Phi_f(F)_x, dt)^{-1}\]

of elements of \( \mathbb{Q}_\ell^\times \otimes \mathbb{Z} \mathbb{Q} \) holds. Here \( t \) is a local parameter of \( Y \) around \( f(x) \).

Strictly speaking, we need to start with \( \mathbb{Z}_\ell \)-sheaves instead of \( \mathbb{Q}_\ell \)-sheaves because of the lack of a good theory of local acyclicity for \( \mathbb{Q}_\ell \)-sheaves over general schemes.

We briefly review the construction of epsilon cycles. Similarly as that of characteristic cycles, the key ingredient is the "semi-continuity" of local epsilon factors ([33, Theorem 4.8.2]). Here we interpret the semi-continuity as the existence of a 1-dimensional representation whose values at geometric Frobeniiuses equal to local epsilon factors. For the precise statement, see [33, Theorem 4.8.2]. According to the finiteness theorem of Katz-Lang [20], the representation gives a flat function (Definition 2.9) after taking modulo roots of unity, which ensures the existence of the cycles \( E(F) \) with the desired property (Proposition 2.12).

After establishing basic properties of epsilon cycles, we prove a formula for the pullback by properly transversal morphism by a similar method developed by Beilinson [30, Section 7].

**Theorem 1.2.** (Theorem 4.24) Let \( k \) be a finite field. Let \( h : W \to X \) be a morphism of smooth \( k \)-schemes. Let \( F \) be a constructible complex of \( \mathbb{Q}_\ell \)-sheaves on \( X \). Assume that \( h \) is properly \( SS(F) \)-transversal (Definition 2.1.2). Then, we have

\[E(h^*F) = h^!(E(F)(\frac{\dim X - \dim W}{2})).\]

Here we use a notation \( E(F)(r) \) of Tate twists of epsilon cycles (Definition 4.11), which is defined as follows. Let \( X \) be a smooth scheme over a finite field \( k \) with \( q \) elements. For an
ℓ-adic sheaf $F$ on $X$ and a rational number $r$, define the $r$-twisted epsilon cycle $E(F)(r)$ to be $q^{-r \cdot CC(F)} \cdot E(F)$, where the product is understood as follows. Let $CC(F) = \sum a_m [C_a]$ and $E(F) = \sum a_\xi [C_a]$. Then, we define $q^{-r \cdot CC(F)} \cdot E(F) = \sum a_m^{-r} a_\xi [C_a]$. As $Q_\ell \otimes Z Q$ is a quotient of the multiplicative group $Q \times \ell$, we write the group law multiplicatively and also write $ab$ for $a \otimes b$. When $r = n$ is an integer, it coincides with the epsilon cycle $E(F(n))$ of the Tate twist $F(n)$ (Lemma 4.13.1) and reflects the (unramified) twist formula of local epsilon factors. See Lemma 4.13.1 for a more general result.

Finally we state and prove a product formula of global epsilon factors.

**Theorem 1.3.** (Theorem 5.7) Let $X$ be a projective smooth variety over a finite field $k$. Then, for $F \in D^b c(X, \ell)$, we have

$$\varepsilon(X, F)^{-1} = (E(F), T_X^* X)_{T^*_X}$$

as elements of $Q_\ell^\times \otimes Z Q$.

As a consequence, we get a formula which expresses the $p$-adic valuations of global epsilon factors as the products of those of local epsilon factors (Example 5.9).

In [34], N. Umezaki, E. Yang, and Y. Zhao prove the twist formula of global epsilon factors [34, Theorem 5.23.]. A weaker version modulo roots of unity can be also deduced from the theorem above and Lemma 4.13.1.

Actually we construct a theory of epsilon cycles for general perfect base field cases, under some mild assumption (cf. Definition 4.4). The assumption is always satisfied if the base field is the perfection of a finitely generated field over its prime field. When the characteristic of the base field $k$ is positive, the construction goes quite similarly as the case of finite fields using the results of Q. Guignard [12], a theory of local epsilon factors over general perfect fields of positive characteristic. In Section 3 we summarize his results which we will need. When the characteristic is 0, we define local epsilon factors of the representations $V$ with unramified determinant using Jacobi sum characters constructed in [28]. For general $V$, we take a direct sum of copies of $V$ so that the determinant becomes unramified, hence we can only construct local epsilon factors modulo roots of unity.

We also give an axiomatic description of epsilon cycles (Theorem 5.10). See Sections 4, 5 for the details.

We give notation which we use throughout this paper.

- We denote by $G_k$ the absolute Galois group of a field $k$.
- We denote by $\chi_{cyc} : G_k \to Z_\ell^\times$ the $\ell$-adic cyclotomic character.
- For a finite separable extension $k'/k$ of fields, we denote by $tr_{k'/k} : G_{k'}^b \to G_k^b$ the transfer morphism induced by the inclusion $G_{k'} \hookrightarrow G_k$. The determinant character of the induced representation $\text{Ind}_{G_{k'}}^{G_k} Q_\ell$ of the trivial representation is denoted by $\delta_{k'/k}$.
- For a scheme $X$ and its point $x$, $k(x)$ is the residue field of $X$ at $x$.
- For a finite extension $x'/x$ of the spectra of fields, we denote by $\text{deg}(x'/x)$ the degree of the extension. When $x = \text{Spec}(k)$ and $x' = \text{Spec}(k')$, we also denote it by $\text{deg}(k'/k)$.
Let $x$ be a geometric point on a scheme $X$. We denote the strict henselization of $X$ at $x$ by $X_{(x)}$. On the other hand, we denote the henselization at a point $x \in X$ by $X_{(y)}$. More generally, for a finite separable extension $y$ of $x \in X$, we denote the henselization of $X$ at $y$ by $X_{(y)}$.

For a scheme $X$ of finite type over $S$, we say that $X$ is of relative dimension $n$ if all fibers of $X \to S$ are equidimensional and of dimension $n$.

We fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let $\mu$ be the group of roots of unity in $\overline{\mathbb{Q}}_\ell$. Let $\mu_p$ denote the group of $p$-th roots of unity in $\overline{\mathbb{Q}}_\ell$. For a finite extension $E/\mathbb{Q}_\ell$, the ring of integers of $E$ is denoted by $\mathcal{O}_E$.

For the $\ell$-adic formalism of a noetherian topos $T$, we refer to [7], which we review in the appendix. The derived category of constructible complexes of $E$-sheaves (resp. $\mathcal{O}_E$-sheaves) on $T$ is denoted by $D^b_{c}(T, E)$ (resp. $D^b_{c}(T, \mathcal{O}_E)$). We put $0$ for objects of $D^b_{c}(T, \mathcal{O}_E)$ (i.e. $\mathcal{F}_0 \in D^b_{c}(T, \mathcal{O}_E)$) and denote $\mathcal{F} := \mathcal{F}_0 \otimes_{\mathcal{O}_E} E$.

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2 Relative Singular Supports and Characteristic Cycles ([1], [13], [30])

In this section, we recall the theory of singular supports and characteristic cycles. Let $S$ be a noetherian scheme. For the theory of relative singular supports over $S$, we refer to [13]. When $S$ is the spectrum of a field, we use the notation given in [1], [30].
2.1 Relative singular support

For a smooth scheme $X$ over $S$, we denote by $T^*(X/S)$ the cotangent bundle of $X$ relative to $S$. We denote by $T^*_X(X/S)$ the 0-section. For a morphism $x \to X$ from the spectrum of a field, we denote by $T^*_x(X/S)$ the base change $T^*(X/S) \times_X x$. We say that a closed subset $C$ of $T^*(X/S)$ is conical if $C$ is stable under the action of $\mathbb{G}_m$.

**Definition 2.1.** Let $X$ be a smooth scheme over $S$ and $C$ be a closed conical subset of $T^*(X/S)$.

1. We say that an $S$-morphism $h: W \to X$ from a smooth $S$-scheme $W$ is $C$-transversal if, for every geometric point $w$ of $W$, non-zero elements of $C_{h(w)}$ map to non-zero elements of $T^*_w(W/S)$ via $dh_w$.

2. Assume that $X$ and $C$ is of relative dimension $n$. Let $W$ be a smooth scheme over $S$ of relative dimension $m$. We say that an $S$-morphism $h: W \to X$ is properly $C$-transversal if $h$ is $C$-transversal and $W \times_X C$ is of relative dimension $m$.

3. We say that an $S$-morphism $f: X \to Y$ to a smooth $S$-scheme $Y$ is $C$-transversal if, for every geometric point $x$ of $X$, no non-zero elements of $T^*_{f(x)}(Y/S)$ map into $C_x$ via $df_x$.

**Lemma 2.2.** (cf. [1, 1.2]) Let $h: W \to X$ be a morphism of smooth $S$-schemes, and $C$ be a closed conical subset of $T^*(X/S)$. If $h$ is $C$-transversal, the map $dh: C \times_X W \to T^*(W/S)$ is finite.

**Definition 2.3.** Let $X$ and $C$ be as in Definition 2.1. Let $W$ and $Y$ be smooth $S$-schemes.

1. Let $h: W \to X$ be a morphism of $S$-schemes. If $h$ is $C$-transversal, we define $h^0C$ to be the image of $dh: C \times_X W \to T^*(W/S)$. This is a closed conical subset of $T^*(W/S)$.

2. Let $f: X \to Y$ be a morphism of $S$-schemes. Assume that $f$ is proper on the base of $C$, i.e. $C \cap T^*_X(X/S)$. We define $f_C$ to be the image by the projection $T^*(Y/S) \times_Y X \to T^*(Y/S)$ of the inverse image of $C$ by $df: T^*(Y/S) \times_Y X \to T^*(X/S)$.

3. Let $(h, f)$ be a pair of $S$-morphisms

$$X \xleftarrow{h} W \xrightarrow{f} Y.$$ We say that $(h, f)$ is $C$-transversal if $h$ is $C$-transversal and $f$ is $h^0C$-transversal.

Let $\Lambda$ be a finite local ring with residue characteristic invertible in $S$. For an $S$-scheme $X$, we denote by $D_{\text{ctf}}(X, \Lambda)$ the full subcategory of $D(X, \Lambda)$ consisting of constructible complexes of finite tor-dimension.

**Definition 2.4.** Let $X$ and $C$ be as in Definition 2.1. Let $K \in D_{\text{ctf}}(X, \Lambda)$. We say that $K$ is micro-supported on $C$ if, for any pair $(h, f)$ as in Definition 2.3 which is $C$-transversal, $f$ is locally acyclic relatively to $h^*K$. 

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Lemma 2.5. Let $X$ and $C$ be as in Definition 2.1. Let $K \in D_{ctf}(X, \Lambda)$. Assume that $K$ is micro-supported on $C$.

1. ([[13]], Lemma 4.7(ii)), ([1], Lemma 2.1.(ii)) Let $(h, f)$ be as in Definition 2.3 which is $C$-transversal. Then, $f$ is universally locally acyclic relatively to $h^*K$.

2. Let $h : X' \to X$ be a morphism of smooth $S$-schemes. If $h$ is $C$-transversal, then $h^*K$ is micro-supported on $h^0C$.

Proof. 2. Let $X' \xleftarrow{h'} W \xrightarrow{f} Y$ be an $h^0C$-transversal pair. Then, the pair $(h \circ h', f)$ is $C$-transversal. Thus, $f$ is locally acyclic relatively to $h^*h^*K$. 

\[\Box\]

Theorem 2.6. ([[13]], Theorem 5.2,5.3]) Let $X$ be a smooth $S$-scheme of finite type. Let $K$ be a complex in $D_{ctf}(X, \Lambda)$. After replacing $S$ by an open dense subset, the following hold.

1. There is the smallest closed conical subset $C$ of $T^*X/S$ on which $K$ is micro-supported. We call this $C$ the relative singular support and denote it by $SS(K, X/S)$.

2. For a morphism $s \to S$ from the spectrum of a field, we have

\[SS(K|_{X_s}) = SS(K, X/S) \times_S s.\]

Remark 2.7. If the relative singular support $SS(K, X/S)$ exists, the structure morphism $X \to S$ is universally locally acyclic relatively to $K$, since $X \xleftarrow{id} X \to S$ is $SS(K, X/S)$-transversal. If further $X$ is projective over $S$, the existence of $SS(K, X/S)$ is equivalent to the universal local acyclicity of $X \to S$ relative to $K$ ([[13]], Theorem 5.2).

We give some examples of singular supports.

Example 2.8. Suppose that $S = \text{Spec}(k)$ is the spectrum of a field $k$.

1. Let $X$ be a smooth curve over $k$. Let $K \in D_{ctf}(X, \Lambda)$. Then we have

\[SS(K) \subset T^*_XX \cup \bigcup_x T^*_xX,\]

where $x$ runs through the closed points at which $K$ is not locally constant. The equality holds if and only if the generic fiber of $K$ is not acyclic.

2. Let $X_1$ and $X_2$ be smooth schemes over $k$. Let $K_i \in D_{ctf}(X_i, \Lambda)$ for each $i = 1, 2$. Then,

\[SS(K_1 \boxtimes L K_2) = SS(K_1) \times SS(K_2) \subset T^*X_1 \times T^*X_2 \cong T^*(X_1 \times X_2).\]

This is proved using the Thom-Sebastiani theorem ([[20]], Theorem 2.2.3]).
2.2 Flat function and characteristic cycle

Next we recall the notion of characteristic cycles. Let $k$ be a perfect field. Before giving the definition of characteristic cycles, we recall general results on isolated $C$-characteristic points (cf. [30]). For a scheme $X$, we denote by $|X|$ the set of closed points of $X$. We fix an abelian group $A$.

**Definition 2.9.** (cf. [30, Definition 5.5]) Let $Z$ be a scheme locally of finite type over $k$. Let $\varphi: |Z| \to A$ be a function.

1. For a morphism of finite type $f: Z' \to Z$, define a function $f^*\varphi: |Z'| \to A$ by $f^*\varphi(z') := \deg(z'/f(z'))\varphi(f(z'))$. If no confusions occur, we also write $\varphi|_{Z'} = f^*\varphi$.

2. The function $\varphi$ is said to be constant if there exists a function $\psi: |\text{Spec}(k)| \to A$ such that the equality $\psi|_{Z} = \varphi$ holds.

3. Let $g: Z \to S$ be a quasi-finite morphism of schemes locally of finite type over $k$. We say that $\varphi$ is flat over $S$ if the following condition holds:
   
   For every closed point $z \in Z$, There exists a commutative diagram
   
   $$(2.1) \quad \begin{array}{ccc}
   U & \xrightarrow{c} & V \times_S Z \\
   \downarrow & & \downarrow \quad g \\
   V & \longrightarrow & S
   \end{array}$$
   
   of $k$-schemes satisfying:

   (a) $V \to S$ is étale and there exists a closed point $v \in V$ whose image in $S$ coincides with $g(z)$. The map $v \mapsto g(z)$ is an isomorphism.

   (b) $U$ is an open neighborhood of $(v, z)$ in $V \times_S Z$.

   (c) $U$ is finite over $V$. The fiber of $g$ over $v$ only consists of $(v, z)$.

   (d) The function $\tilde{g}_*\varphi|_{U}: |V| \to A$ defined by
   
   $$\tilde{g}_*\varphi|_{U}(x) = \sum_{y \in \tilde{g}^{-1}(x)} \varphi|_{U}(y)$$
   
   is constant in the sense of 2.

Let $X$ be a smooth $k$-scheme and $C$ be a closed conical subset of $T^*X$. Let $f: X \to Y$ be a $k$-morphism to a smooth curve $Y$ over $k$. Let $x \in X$ be a closed point.

**Definition 2.10.** Let the notation be as above.

1. We say that $x$ is an at most isolated $C$-characteristic point of $f$ if there exists an open neighborhood $U$ of $x$ such that the restriction $f|_{U \setminus x}$ is $C$-transversal.

2. Suppose that $X$ is purely of dimension $n$ and every irreducible component $C_a$ of $C$ is of dimension $n$. Let $\alpha = \sum_a \beta_a \otimes [C_a]$ be a cycle with $A$-coefficient and supported on $C$. Assume that $x \in X$ is an at most isolated $C$-characteristic point of $f$. We define the intersection number $(\alpha, df)_{T^*X, x} \in A$, or simply $(\alpha, df)_x$, as
   
   $$(\alpha, df)_{T^*X, x} = \sum_{a} (C_a, df)_{T^*X, x} \cdot \beta_a.$$
where \((C_a, df)_{T^*X, x}\) is the intersection number, supported on the fiber of \(x\), of \(C_a\) and the section \(f^*\omega\) of \(T^*X\) defined by the pull-back of a basis \(\omega\) of \(T^*Y\) on a neighborhood of \(f(x) \in Y\).

**Definition 2.11.** Let \(X\) and \(C\) be as above.

1. \(\varphi\) is said to be an \(A\)-valued function on isolated \(C\)-characteristic points if, for every diagram

\[
\begin{array}{c}
  U \\
  \downarrow \phi \\
  X
\end{array} \rightarrow \begin{array}{c}
  Y \\
  \downarrow j \\
  X
\end{array}
\]

of \(k\)-schemes and a point \(u \in |U|\) such that \(Y\) is a smooth curve over \(k\), \(U\) is \(\text{étale}\) over \(X\), and \(u \in U\) is an at most isolated \(C\)-characteristic point of \(f\), an element \(\varphi(f, u) \in A\) is given. Further this assignment should satisfy the following conditions:

- \(\varphi(f, u)\) is 0 when \(u\) is not an isolated \(C\)-characteristic point.
- For every commutative diagram

\[
\begin{array}{c}
  U' \\
  \downarrow \phi' \\
  X
\end{array} \rightarrow \begin{array}{c}
  Y' \\
  \downarrow j' \\
  X
\end{array}
\]

of \(k\)-schemes such that the vertical arrows are \(\text{étale}\) and \(Y, Y'\) are smooth curves over \(k\), and a closed point \(u' \in U'\) which is an at most isolated \(C\)-characteristic point of \(f'\), we have \(\varphi(f', u') = \deg(u'/u) \cdot \varphi(f, u)\) where \(u\) is the image of \(u'\) by \(U' \rightarrow U\).

2. Let \(\varphi\) be an \(A\)-valued function on isolated \(C\)-characteristic points. \(\varphi\) is said to be flat if, for every commutative diagram

\[
\begin{array}{c}
  Z \\
  \downarrow \phi_1 \\
  X
\end{array} \rightarrow \begin{array}{c}
  U \\
  \downarrow \phi \\\n  Y \\
  \downarrow \phi_2 \\
  S \\
  \downarrow \phi_3 \\
  X
\end{array}
\]

of \(k\)-schemes such that

- \(S\) is a smooth scheme over \(k\).
- \(Y \rightarrow S\) is a relative smooth curve.
- The map \(U \rightarrow X \times_k S\) is \(\text{étale}\).
- \(Z\) is a closed subscheme of \(U\) quasi-finite over \(S\).
- The pair \((\phi_1, f)\) is \(C\)-transversal outside \(Z\).
the function $\phi_f: |Z| \to A$ defined by $\phi_f(z) = \phi(f_s, z)$, where $s$ is the image of $z$ by $Z \to S$ and $f_s: U_s \to Y_s$ is the base change of $f$ by $s \to S$, is flat over $S$ in the sense of Definition 2.9.3.

**Proposition 2.12.** ([30, Proposition 5.8]) Assume that $A$ is uniquely divisible (i.e. $A \to A \otimes \mathbb{Z} \mathbb{Q}$ is an isomorphism). Let $X$ be a smooth scheme purely of dimension $n$ over $k$. Let $C$ be a closed conical subset of $T^*X$. Assume that every irreducible component $C_a$ of $C$ is of dimension $n$. Let $\phi$ be an $A$-valued function on isolated $C$-characteristic points. The following conditions are equivalent.

1. $\phi$ is flat.

2. There exists a cycle $\alpha = \sum a \beta_a \otimes [C_a] \in A \otimes \mathbb{Z} Z_n(T^*X)$ with $A$-coefficient and supported on $C$ such that

\[(2.5) \quad \phi(f, u) = \deg(u/k)(j^*\alpha, df)_{T^*U,u}\]

holds for every diagram (2.2) and every at most isolated $C$-characteristic point $u \in U$ of $f$.

Further, if these conditions hold, the cycle $\alpha$ in 2 is unique.

**Proof.** Since the proof is completely similar to [30, Proposition 5.8] and we only use the implication $1 \Rightarrow 2$ below, we sketch the proof of $1 \Rightarrow 2$.

First we consider the case when $k$ is algebraically closed. By the similar argument in [30, Proposition 5.8], we find a unique cycle $\alpha_X \in A \otimes \mathbb{Z} Z_n(T^*X)$ satisfying $\phi(f, u) = (\alpha_X, df)_{T^*X,u}$ for every diagram (2.2) such that $U \to X$ is an open immersion. Let $j: W \to X$ be an étale morphism. Restricting $\phi$ to $W$, we have an $A$-valued function on isolated $j^*C$-characteristic points. Since this is also flat, we find a cycle $\alpha_W \in A \otimes \mathbb{Z} Z_n(T^*W)$ satisfying $\phi(f, u) = (\alpha_W, df)_{T^*W,u}$ for every diagram (2.2) replaced $X$ by $W$. We need to show the equality $\alpha_W = j^*\alpha_X$, which is a consequence of [30, Proposition 5.8.2].

Next we consider the general case. Take an algebraic closure $\bar{k}$ of $k$. We put the letter $\bar{k}$ to mean the base change by $k \to \bar{k}$. Using $\phi$, we define an $A$-valued function $\phi_{\bar{k}}$ on isolated $C_{\bar{k}}$-characteristic points as follows. Let

\[(2.6) \quad U \xrightarrow{f} Y \xrightarrow{X_{\bar{k}}} \]

be a diagram as in (2.2), and $u \in U$ be an at most isolated $C_{\bar{k}}$-characteristic point of $f$. We assume that $U$ and $Y$ are quasi-compact. Take a finite subextension $k'/k$ in $\bar{k}$ such that there exists a diagram

\[U' \xrightarrow{f'} Y' \xrightarrow{X_{\bar{k}'}} \]
Let \( L \) be a field and \( \phi \) be a \( \mathbb{Q} \)-valued function on isolated \( C_k \)-characteristic points. Since \( \phi \) is flat, we find a cycle \( \alpha_k \) satisfying (2.5). From the construction of \( \phi_k \), \( \alpha_k \) is stable under the action of the Galois group \( \text{Gal}(\bar{k}/k) \). By étale descent, we get a cycle \( \alpha \) which satisfies the condition.

Let \( X \xleftarrow{f} U \xrightarrow{j} Y \) be as in (2.2). Take a closed point \( u \in U \). Let \( \eta \) and \( \eta_u \) be the generic points of the henselizations of \( Y \) at \( f(u) \) and \( u \) respectively. Let \( K \in D_{\text{ctf}}(X, \Lambda) \) be a constructible complex on \( X \). Suppose that \( u \) is an at most isolated \( j^* \text{SS}(K) \)-characteristic point of \( f \). Then, there is an open neighborhood \( V \) of \( u \) such that the restriction of the vanishing cycles complex \( R\Phi_f(j^*K) \) to \( V \times_X Y \) is supported on \( u \times_{Y(u)} \eta \). Define \( \dim_{\text{tot}} \) for \( M \in D_{\text{ctf}}(\eta_u, \Lambda) \) to be \( \dim_{\text{tot}} M := \text{rk} M + \text{Sw} M \).

Finally we give the definition of characteristic cycles.

**Theorem 2.14.** (30 Theorem 5.9, 5.18) Let \( X \) be a smooth scheme over \( k \) and \( K \in D_{\text{ctf}}(X, \Lambda) \) be a constructible complex on \( X \). Let \( C \) be a closed conical subset of \( T^*X \) on which \( K \) is micro-supported. Assume that each irreducible component of \( X \) and that of \( C \) is of dimension \( n \). Then, there exists a cycle \( \text{CC}(K) \) in \( \mathbb{Q} \otimes \mathbb{Z}_n(T^*X) \), supported on \( C \), admitting the following property:

For every diagram as (2.2) and an at most isolated \( C \)-characteristic point \( u \in U \) of \( f \), we have

\[
-\dim_{\text{tot}} R\Phi_f(K)_{\text{u}} = \langle \text{CC}(K), df \rangle_u.
\]

Moreover, \( \text{CC}(K) \) is unique and independent of the choice \( C \) on which \( K \) is micro-supported. \( \text{CC}(K) \) is \( \mathbb{Z} \)-coefficient.

**Proof.** The first assertion is a direct consequence of Proposition 2.12 if one knows the \( \mathbb{Q} \)-valued function on isolated \( C \)-characteristic points defined by \( \varphi(f, u) = -\deg(u/k) \cdot \dim_{\text{tot}} R\Phi_f(K)_{\text{u}} \) is flat. This is proved in (30 Proposition 2.16), and we give another proof when \( k \) is of positive characteristic (Theorem ??). The integrality is proved in (30 Theorem 5.18).

We recall the theory of the universal hyperplane sections (30 Section 3.2) and the notion of good pencils (32). Let \( X \) be a quasi-projective smooth scheme over a field \( k \). Let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( E \) be a \( k \)-vector space of finite dimension and

\[
U' \xrightarrow{f'} Y' \\
\xrightarrow{} X' \xrightarrow{} X.
\]

This is independent of the choice of \( (k', f') \) and defines an \( A \)-valued function on isolated \( C_k \)-characteristic points. Since \( \varphi_k \) is flat, we find a cycle \( \alpha_k \) satisfying (2.5). From the construction of \( \varphi_k \), \( \alpha_k \) is stable under the action of the Galois group \( \text{Gal}(\bar{k}/k) \). By étale descent, we get a cycle \( \alpha \) which satisfies the condition. 

\[ \blacksquare \]
$E \to \Gamma(X, \mathcal{L})$ be a $k$-linear mapping inducing a surjection $E \otimes_k \mathcal{O}_X \to \mathcal{L}$. Suppose that this induces an immersion $h : X \to \mathbb{P} = \mathbb{P}(E^\vee)$. Here we use a contra-Grothendieck notation for a projective space, i.e. $\mathbb{P}(E^\vee)$ parametrizes sub line bundles of $E^\vee$. Let $\mathbb{P}^\vee := \mathbb{P}(E)$ be the dual projective space. The universal hyperplane $Q \subset \mathbb{P} \times \mathbb{P}^\vee$ parametrizes pairs $(x, H)$ consisting of points $x \in \mathbb{P}$ and hyperplanes $H \in \mathbb{P}^\vee$ which contain $x$. Since the kernel of the tautological surjection $E \otimes_k \mathcal{O}_\mathbb{P}(-1) \to \mathcal{O}_\mathbb{P}$ is canonically isomorphic to the cotangent bundle $\Omega^1_\mathbb{P}$, $Q$ is identified with the projective space bundle $\mathbb{P}(T^*\mathbb{P})$. The composition $T^*_Q(\mathbb{P} \times \mathbb{P}^\vee) \to Q \times_{\mathbb{P} \times \mathbb{P}^\vee} T^*(\mathbb{P} \times \mathbb{P}^\vee) \to Q \times_{\mathbb{P}} T^*\mathbb{P}$ is the universal sub line bundle on $Q$.

Consider the following diagram

(2.7)

We have $X \times_{\mathbb{P}} Q = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P})$.

Let $L \subset \mathbb{P}^\vee$ be a line in $\mathbb{P}^\vee$. We have a commutative diagram

(2.8)

where the right square is cartesian. Denote by $A_L$ the axis of $L$ in $\mathbb{P}$. This is a subspace of $\mathbb{P}$ of codimension 2. The $\mathbb{P}$-scheme $\mathbb{P}_L = Q \times_{\mathbb{P}^\vee} L$ is the blow-up of $\mathbb{P}$ along $A_L$. Hence, if $X$ and $A_L$ meet transversally, $X_L$ is the blow-up of $X$ along the smooth subvariety $X \cap A_L$.

**Definition 2.15.** Let $X \subset \mathbb{P}$ be a closed smooth subvariety purely of dimension $n$ over $k$. Let $C$ be a closed conical subset of $T^*X$ whose irreducible components are of dimension $n$. We call the pair $(\pi, f)$ as in (2.3) a good pencil with respect to $C$ if the following conditions hold.

1. $X$ and $A_L$ meet transversally.
2. The morphism $\pi$ is properly $C$-transversal.
3. The morphism $f$ has at most isolated $\pi^\circ C$-characteristic points.
4. For every closed point $y \in L$, there exists at most one $\pi^\circ C$-characteristic point on the fiber $f^{-1}(y)$.
5. No isolated characteristic points of $f$ are contained in the exceptional locus of $\pi$.
6. For every irreducible component $C_a$ of $C$, there is an isolated $\pi^\circ C$-characteristic point $x \in X_L$ such that $df$ only meets $C_a$ at $x$.
7. For every isolated $\pi^\circ C$-characteristic point $x \in X_L$ of $f$, the morphism $x \to f(x)$ of the spectra of fields is purely inseparable.

The existence of good pencils is proved in [34, Lemma 4.9.] using [32, Lemma 2.3.].
Lemma 2.16. ([74, Lemma 4.9], [72, Lemma 2.3]) Let $X$ and $C$ be as in Definition 2.16. Let $\text{Gr}(1, \mathbb{P}^V)$ be the Grassmannian variety parametrizing lines in $\mathbb{P}^V$. Then, after composing $X \hookrightarrow \mathbb{P}$ and the Veronese embedding $\mathbb{P} \hookrightarrow \mathbb{P}'$ of deg $\geq 3$ if necessary, there exists a dense open subset $U \subset \text{Gr}(1, \mathbb{P}^V)$ such that, for every $k$-rational point $L \in U(k)$, the pair $(\pi, f)$ in (2.13) is a good pencil.

At the end of this section, we give definitions of the local acyclicity, singular supports, and characteristic cycles for $\mathbb{Z}_\ell$-sheaves. For the $\ell$-adic formalism, see Section 6.

Lemma 2.17. Let $f : X \to Y$ be a morphism of finite type of schemes. Let $\Lambda$ be a finite local ring with the residue field $\Lambda_0$ whose characteristic is invertible in $Y$. Let $K \in D_{\text{ctf}}(X, \Lambda)$ be a constructible complex of finite tor-dimension. Then, $f$ is (resp. universally) locally acyclic relatively to $K$ if and only if so is $f$ relatively to $K \otimes_{\Lambda} \Lambda_0$.

Proof. Let $x$ be a geometric point of $X$ and $y$ be a geometric point of $Y$ which is a generalization of $f(x)$. Since the functor $\Gamma(X(x) \times_{Y(f(x))} y, -)$ is of finite cohomological dimension, we have $R \Gamma(X(x) \times_{Y(f(x))} y, K) \otimes_{\Lambda} \Lambda_0 \cong R \Gamma(X(x) \times_{Y(f(x))} y, K \otimes_{\Lambda} \Lambda_0)$. Since $\Lambda$ is an extension of finite free $\Lambda_0$-modules, the assertion follows.

Lemma 2.18. Let $X$ be a smooth scheme of finite type over a noetherian scheme $S$. Let $\Lambda$ and $\Lambda_0$ be as above. Let $K \in D_{\text{ctf}}(X, \Lambda)$. The following hold.

1. The relative singular support $SS(K, X/S)$ exists if and only if $SS(K \otimes_{\Lambda} \Lambda_0, X/S)$ exists. In this case, we have $SS(K, X/S) = SS(K \otimes_{\Lambda} \Lambda_0, X/S)$.

2. Suppose that $S$ is the spectrum of a perfect field. We have $CC(K) = CC(K \otimes_{\Lambda_0} \Lambda_0)$.

Proof. 1. It follows from Lemma 2.17

2. Let $X \leftarrow U \xrightarrow{f} Y$ be as (2.2). The assertion follows from $R \Phi_f(K) \otimes_{\Lambda} \Lambda_0 \cong R \Phi_f(K \otimes_{\Lambda_0} \Lambda_0)$ and the equality $\text{dimtot} R \Phi_f(K) = \text{dimtot}(R \Phi_f(K) \otimes_{\Lambda_0} \Lambda_0)$.

Definition 2.19. Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Assume that the prime number $\ell$ is invertible in $Y$. Let $\Lambda$ be either $\mathcal{O}_E$ or $\mathbb{Z}_\ell$. For an element $\mathcal{F}_0 \in D^b_c(X, \Lambda)$, we say that $f$ is (resp. universally) locally acyclic relatively to $\mathcal{F}_0$ if, for some (hence all) $n \geq 0$, $f$ is (resp. universally) locally acyclic relatively to $\mathcal{F}_0 \otimes_{\Lambda} \Lambda/\ell^{n+1}$.

Definition 2.20. Let $X$ be a smooth scheme of finite type over a noetherian scheme $S$. Let $\Lambda$ be $\mathcal{O}_E$ or $\mathbb{Z}_\ell$. Let $\mathcal{F}_0 \in D^b_c(X, \Lambda)$.

1. If $\Lambda = \mathcal{O}_E$, we define $SS(\mathcal{F}_0, X/S) : = SS(\mathcal{F}_0 \otimes_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1}, X/S)$ and, when $S$ is the spectrum of a perfect field, $CC(\mathcal{F}_0) := CC(\mathcal{F}_0 \otimes_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1})$ for some (hence all) $n \geq 0$.

2. If $\Lambda = \mathbb{Z}_\ell$, take a finite extension $E/\mathbb{Q}_\ell$ so that there is $\mathcal{F}_{0,E} \in D^b_c(X, \mathcal{O}_E)$ with $\mathcal{F}_{0,E} \otimes_{\mathcal{O}_E} \mathbb{Z}_\ell \cong F_0$. We define $SS(\mathcal{F}_0, X/S) := SS(\mathcal{F}_{0,E} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1}, X/S)$ and, when $S$ is the spectrum of a perfect field, $CC(\mathcal{F}_0) := CC(\mathcal{F}_{0,E} \otimes_{\mathcal{O}_E} \mathcal{O}_E/\ell^{n+1})$ for some (hence all) $n \geq 0$. These are independent of the choice of $E$. 
3 Local Epsilon Factors (cf. [5], [12], [23])

In this preliminary section, we review theories of local epsilon factors for henselian traits of equal-characteristic.

3.1 Generalities on local epsilon factors

Let $k$ be a perfect field of characteristic $p > 0$. Let $T$ be a henselian trait of equal-characteristic with residue field $k$. We write $s$ and $η$ for the closed and generic points respectively. We fix a non-trivial character $F_p \to Λ^x$ where $Λ$ is a finite local ring in which $p$ is invertible. For a $Λ$-representation $V$ of the absolute Galois group $G_η$ and a non-zero rational 1-form $ω ∈ Ω^1_η$, Yasuda [37], [36] defines a continuous character $ε_{0,Λ}(T, V, ω) : G_k^a → Λ^x$, as a generalization of the theory local epsilon factors due to Langlands-Deligne [5], [23].

**Theorem 3.1. ([37], [36, 4.12])** Let the notation be as above. For a triple $(T, (ρ, V), ω)$ where $V$ is a finite free $Λ$-module with a continuous group homomorphism $ρ : G_η → GL(V)$ and $ω ∈ Ω^1_η$ is a non-zero rational 1-form, there is a canonical way to attach a continuous character $ε_{0,Λ}(T, V, ω) : G_k^a → Λ^x$, called the local epsilon factor, with the following properties.

1. The character only depends on the isomorphism class of $(T, (ρ, V), ω)$.

2. For a short exact sequence $0 → V' → V → V'' → 0$ of representations of $G_η$, we have

$$ε_{0,Λ}(T, V, ω) = ε_{0,Λ}(T, V', ω) : ε_{0,Λ}(T, V'', ω).$$

3. For a local ring homomorphism $f : Λ → Λ'$, we have

$$f ∘ ε_{0,Λ}(T, V, ω) = ε_{0,Λ'}(T, V ⊗_Λ Λ', ω)$$

as characters $G_k → Λ'^x$.

4. We have

$$ε_{0,Λ}(T, V, ω) : ε_{0,Λ}(T, V, ω')^{-1} = det(V_{|ψ})χ_{ψ}^{ord(ω') - ord(ω)}rkV.$$

Here $k(η)^x × H^1(η, Λ^x) → H^1(k, Λ^x)$, $(a, χ) ↦ χ_{[a]}$ is the pairing defined in [36, 4.2], [33, Definition 3.10].

5. Let $W$ be an unramified representation of $G_η$ on a finite free $Λ$-module. We have

$$ε_{0,Λ}(T, V ⊗ W, ω) = det(W)^{⊗a(T, V, ω)} : ε_{0,Λ}(T, V, ω)^{rkW}.$$

Here $a(T, F, ω) := SwV + rkV(ordω + 1)$.

6. Assume that the residue field $k$ of $T$ is finite and that there exists a local ring morphism $f : O_E → Λ$ from the ring of integers of a finite extension $E/ℚ_τ$ such that $V$ comes from a representation on $O_E$, i.e. there is a representation $V'$ of $G_η$ on a finite free $O_E$-module such that $V' ⊗_{O_E} Λ ≡ V$. Then we have

$$ε_{0,Λ}(T, V, ω)(Frob_k) = (-1)^{rkV + SwV}f(ε_{0}(T, V' ⊗_{O_E} E, ω)).$$

Here the local epsilon factor in the right hand side is the one in [23, Théorème (3.1.5.4)].
If no confusions occur, we omit the subscript $\Lambda$ in $\varepsilon_{0,\Lambda}(T, F, \omega)$. By the multiplicativity in Theorem 3.1.2, we also define $\varepsilon_{0,\Lambda}(T, K, \omega)$ for a constructible complex $K \in D_{ctf}(\eta, \Lambda)$.

Let $\mathcal{O}_E$ be the ring of integers of a finite extension $E/\mathbb{Q}_\ell$. Take and fix a non-trivial character $\psi : \mathbb{F}_p \to \mathcal{O}_E^\times$. Using the property in Theorem 3.1.1 we define a local epsilon factor $\varepsilon_{0,\Lambda}(T, F, \omega) : G_k \to \mathcal{O}_E^\times$ for $F \in D_c^b(\eta, \mathcal{O}_E)$ as follows: Write $\Lambda_n := \mathcal{O}_E/\ell^n+1$. Then the reduction $F_n := F \otimes_{\mathcal{O}_E} \mathcal{O}_n$ belongs to $D_{ctf}(\eta, \Lambda_n)$ and $\psi$ induces a non-trivial character $\mathbb{F}_p \to \Lambda_n$. Hence we have a character $\varepsilon_{0,\Lambda_n}(T, F_n, \omega) : G_k \to \Lambda_n^\times$. By Theorem 3.1.3, the characters $\varepsilon_{0,\Lambda_n}(T, F_n, \omega), \varepsilon_{0,\Lambda_{n+1}}(T, F_n, \omega)$ are compatible with the quotient map $\Lambda_{n+1} \to \Lambda_n$. We define $\varepsilon(0,\mathcal{O}_E)(T, F, \omega) := \lim_{\leftarrow n} \varepsilon_{0,\Lambda_n}(T, F_n, \omega)$. Finally we explain the definitions of local epsilon factors for $\mathbb{Z}_\ell$-sheaves.

**Definition 3.2.** Let the notation be as above. Let $\mathbb{Z}_\ell$ be the integral closure of $\mathbb{Z}_\ell$ in an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

1. For a complex $F \in D_c^b(\eta, \mathbb{Z}_\ell)$, we define $\varepsilon_{0,\mathbb{Z}_\ell}(T, F, \omega)$ as follows. By definition, there exists a finite subextension $E$ of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ such that $F$ is defined over $\mathcal{O}_E$, i.e. there exists a complex $G \in D_c^b(\eta, \mathcal{O}_E)$ with $G \otimes_{\mathcal{O}_E} \mathbb{Z}_\ell \cong F$. We define $\varepsilon_{0,\mathbb{Z}_\ell}(T, F, \omega)$ to be the composition

$$G_k \xrightarrow{\varepsilon_{0,\mathcal{O}_E}(T, G, \omega)} \mathcal{O}_E^\times \to \mathbb{Z}_\ell^\times.$$ 

This does not depend on the choices of $E$ and $\mathcal{G}$.

2. For $F \in D_c^b(T, \mathbb{Z}_\ell)$, we define $\varepsilon(T, F, \omega)$ to be the product

$$\varepsilon(0,T,F,\omega) \cdot \det(F)^{-1}.$$

**Remark 3.3.** Recently, Q. Guignard gives another definition and construction of local epsilon factors [12], using Gabber-Katz canonical extension. The local epsilon factors given in Definition 3.2 are the same as his, because both of them coincide with the one defined from Laumon’s local Fourier transform. See [12, Theorem 11.8], [36, Proposition 8.3].

**Theorem 3.4.** ([12, Theorem 11.1], [23, Théorème (3.2.1.1)], [36, Theorem 4.50]) Let $X$ be a connected projective smooth curve over a perfect field $k$ of characteristic $p > 0$. Let $F \in D_c^b(X, \mathbb{Z}_\ell)$ be a constructible complex on $X$. Fix a non-zero rational 1-form $\omega$ on $X$.

Then, we have

$$\det(R\Gamma(X, F))^{-1} = \chi_{cyc}(X) \cdot \text{rk} F \prod_x \delta_{\omega}(X(x), F) \varepsilon(X(x), F, \omega) \circ \text{tr}_{x/k}$$

as a representation of the absolute Galois group $G_k$ of $k$. Here $\chi(X) = \sum_{i}(-1)^i \dim H^i(X, \mathbb{Z}_\ell)$ is the Euler-Poincaré characteristic, $\text{rk} F$ is the generic rank of $F$, $a(X(x), F) = \text{rk} F_{\eta} + \text{Sw}_x F - \text{rk} F_x$ is the Artin conductor, and $x$ runs through closed points of $X$.

Next we define local epsilon factors of tamely ramified representations in the case of characteristic $\neq \ell$.

Let $S$ be an affine (not necessarily noetherian) normal scheme in which $\ell$ is invertible. Consider a pair $(T, \chi)$ such that $T = (T_i)_i$ is a finite family of finite étale coverings of $S$ and $\chi = (\chi_i)_i$ is a family of characters $\chi_i : \mathbb{Z}/d_i \mathbb{Z}(1) \to \overline{\mathbb{Q}}_\ell^\times$ of étale sheaves on $T_i$ where $d_i$ are integers $\geq 1$ invertible in $S$ such that $\mathbb{Z}/d_i \mathbb{Z}(1) \cong \mathbb{Z}/d_i \mathbb{Z}$ as étale sheaves on $T_i$. Denote by
$N_{T_i/S}(\chi_i)$ the character $\mathbb{Z}/d_i\mathbb{Z}(1) \to \overline{\mathbb{Q}}_\ell^\times$ of étale sheaves on $S$ defined by the composition $\mathbb{Z}/d_i\mathbb{Z}(1) \to f_!\mathbb{Z}/d_i\mathbb{Z}(1) \xrightarrow{f_!\chi} f_!\mathbb{Q}_\ell^\times \xrightarrow{\rho_i} \overline{\mathbb{Q}}_\ell^\times$ where $f_i: T_i \to S$ is the structure morphism. For an integer $N \geq 1$ which is a multiple of $d_i$, we regard $N_{T_i/S}(\chi_i)$ as a character of $\mathbb{Z}/NZ(1)$ via the surjection $\mathbb{Z}/NZ(1) \to \mathbb{Z}/d_i\mathbb{Z}(1)$, $a \mapsto a^N$.

Assume that $\prod_i N_{T_i/S}(\chi_i)$ is trivial where the product is taken as characters of $\mathbb{Z}/NZ(1)$ for some common multiple $N$ of $d_i$. In this case, $(T, \chi)$ is called a Jacobi datum in [28, Section 1]. When $S$ is the spectrum of a finite field $\mathbb{F}_q$ with $q$ elements, Saito attaches a Jacobi sum $j_\chi \in \overline{\mathbb{Q}}_\ell^\times$ to a Jacobi datum $(T, \chi) = ((T_i)_i, (\chi_i)_i)$ in [28, Section 2] as follows:

$$j_\chi := \prod_i \left( \prod_j \tau_{k_{ij}}(\check{\chi}_{ij}, \psi_0 \circ \text{Tr}_{k_{ij}/\mathbb{F}_q}) \right).$$

Here $T_i = \coprod_j \text{Spec}(k_{ij})$ for finite fields $k_{ij}$ with $q_{ij}$ elements, $\check{\chi}_{ij}: k_{ij}^\times \to \overline{\mathbb{Q}}_\ell^\times$ is defined by $\alpha \mapsto \chi_i(a^{(q_{ij} - 1)/d_i})$, and $\psi_0: k \to \overline{\mathbb{Q}}_\ell^\times$ is a nontrivial character. The Gauss sums are defined by $\tau_k(\chi, \psi) = -\sum_{a \in k} \chi(a) \psi(a)$. Since $\prod_i N_{T_i/S}(\chi_i)$ is trivial, the Jacobi sum $j_\chi$ is independent of the choice of $\psi_0$.

Let $(T, \chi)$ be a Jacobi datum on an affine normal scheme $S$. In [28, Proposition 2.1, Saito constructed a smooth $\overline{\mathbb{Q}}_\ell^\times$-sheaf $J_\chi$ of rank 1 on $S$ from the Jacobi datum, which is called a Jacobi sum character. This is characterized by the following properties.

- For every morphism $f: S' \to S$ of an affine normal schemes, $f^*J_\chi \cong J_{f^*\chi}$.
- If $S$ is the spectrum of a finite field $\mathbb{F}_q$, the action of the geometric Frobenius on $J_\chi$ is the multiplication by $j_\chi$ (3.1).

Let $k$ be a perfect field of characteristic $p \neq \ell$. Take and fix an algebraic closure $\overline{k}$ of $k$ and let $I := \varprojlim_{n \nmid p} \mu_n(\overline{k})$, where $n$ runs through integers $\geq 1$ prime to $p$ and $\mu_n(\overline{k})$ is the group of $n$-th roots of unity in $\overline{k}$. The group $I$ admits an action of $\text{Gal}(\overline{k}/k)$.

Let $V$ be a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space and let $\rho: I \to \text{GL}(V)$ be a continuous representation. For an element $\sigma \in \text{Gal}(\overline{k}/k)$, we denote by $\sigma^*V$ the representation of $I$ defined by $\rho \circ \sigma$. When $\rho$ factors through the quotient $I \to \mu(k')$ for a finite Galois subextension $k'$ of $\overline{k}/k$, the twist $\sigma^*V$ only depends on the image of $\sigma$ in $\text{Gal}(k'/k)$. In this case, for $\tau \in \text{Gal}(k'/k)$, we denote by $\tau^*V$ the twist $\sigma^*V$ for any lift $\sigma \in \text{Gal}(\overline{k}/k)$ of $\tau$.

Assume that, for each $\sigma \in \text{Gal}(\overline{k}/k)$, we have $\sigma^*V \cong V$ and that $V$ is potentially unipotent, i.e. there exists an open subgroup $I' \subset I$ such that the action of $I'$ on $V$ is unipotent. Then, the semi-simplification $V^{ss}$ decomposes into a direct sum

$$V^{ss} \cong \bigoplus_{\tau \in \text{Gal}(k'/k)} \bigoplus_{\chi_i} \tau^*\chi_i$$

where $\chi_i: \mu_{d_i}(\overline{k}) \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$ is an injective character and $k_i$ is the subextension in $\overline{k}$ generated by $k$ and $\mu_{d_i}(\overline{k})$. Such a decomposition is unique up to permutation. Note that the determinant $\det(V)$ equals to $\prod_i N_{k_i/k}(\chi_i)$.

**Definition 3.5.** Let the notation be as above. Assume that $\sigma^*V$ is isomorphic to $V$ for each $\sigma \in \text{Gal}(\overline{k}/k)$ and that $V$ is potentially unipotent.
1. When the determinant \( \det(V) \) is the trivial character of \( I \), we denote by \( J(V) \) the Jacobi sum character of the Jacobi datum \( ((\text{Spec}(k_i))_i, (\chi_i)_i) \). This is a character \( \text{Gal}(\overline{k}/k) \to \mathbb{Z}_\ell^\times \).

2. In general, we define \( J(V) \) to be \( (J(V^{\otimes n}))^{1/n} \) where \( n \) is an integer \( \geq 1 \) such that \( V^{\otimes n} \) has the trivial determinant and \((-)^{1/n} \) is taken as a character to \( \mathbb{Z}_\ell^\times /\mu \). This is a character \( \text{Gal}(\overline{k}/k) \to \mathbb{Z}_\ell^\times /\mu \) and independent of the choice of \( n \).

Let \( k \) be a perfect field of characteristic \( p \neq \ell \). Let \( T \) be the henselization of \( \mathbb{A}^1_k \) at the origin. Let \( \eta \) be the generic point of \( T \) and fix a separable closure \( \overline{k(\eta)} \) of \( k(\eta) \). We take \( \overline{k} \) as the algebraic closure of \( k \) in \( \overline{k(\eta)} \). Let \( I \) be the tame inertia group of \( \text{Gal}(k(\eta)/k(\eta)) \). This is canonically isomorphic to \( \lim_{\leftarrow n \neq p} \mu_n(\overline{k}) \). Let \( V \) be a smooth \( \mathbb{Q}_\ell \)-sheaf on \( \eta \) which is tamely ramified. Then, as a representation of \( I \), \( V_\eta \) is isomorphic to \( \sigma^*V_\eta \) for \( \sigma \in \text{Gal}(\overline{k}/k) \).

To give a definition of local epsilon factors modulo roots of unity, we need to recall the construction in [33, 2.2].

Let \( Y \) be a regular scheme and \( D \in Y \) be a regular divisor. Fix a global section \( \pi \in \Gamma(Y, \mathcal{O}_Y) \) which generates the ideal sheaf of \( D \). For an integer \( m \geq 1 \) invertible in \( Y \), write \( Y_m := \text{Spec}(\mathcal{O}_Y[t]/(t^m + \pi)) \). This is a tamely ramified covering of \( Y \) and has a unique lift \( D \to Y_m \) of the immersion \( D \to Y \). Let \( F \) be a locally constant constructible sheaf on the complement \( U = Y \setminus Z \) tamely ramified along \( D \). Zariski-locally on \( Y \), we can find such an \( m \) that the restriction of \( F \) to \( U_m := U \times_Y Y_m \) is unramified along \( D \). Write \( F_m \) for its extension to \( Y_m \). The restriction \( F_m |_{D} \) glues to a locally constant constructible sheaf on \( D \), which we denote by \( \langle F, \pi \rangle \). We also define \( \langle F, \pi \rangle \) for a smooth \( \mathbb{Z}_\ell \)-sheaf and a smooth \( \mathbb{Q}_\ell \)-sheaf by taking mod \( \ell^n \) reductions.

**Definition 3.6.** Consider the situation as above. For a smooth \( \mathbb{Q}_\ell \)-sheaf \( V \) on \( \eta \) which is tamely ramified and potentially unipotent, define \( \xi_0(T, V) : G_k \to \mathbb{Z}_\ell^\times /\mu \) as follows:

\[
\xi_0(T, V) := \langle \det V, \pi \rangle J(V_\eta).
\]

This is independent of the choice of \( \pi \) since \( \det(V)^n \) is unramified for some \( n \geq 1 \).

**Lemma 3.7.** Let \( V \) and \( W \) be smooth \( \mathbb{Z}_\ell \)-sheaves on \( \eta \). Assume that \( V \) is unramified and that \( W \) is tamely ramified and potentially unipotent. We have

\[
\xi_0(T, V \otimes W) = \det(V)^{\dim W} \cdot \xi_0(T, W)^{\dim V}.
\]

**Proof.** Since \( V \) is unramified, we have \( J((V \otimes W)_\eta) = J(W_\eta)^{\dim V} \). On the other hand, we have \( \langle \det(V \otimes W), \pi \rangle = \langle \det V, \pi \rangle^{\dim W} \cdot \langle \det W, \pi \rangle^{\dim V} \) by the multiplicativity of the construction \( F \mapsto \langle F, \pi \rangle \). The assertion follows as \( \langle \det V, \pi \rangle = \det V \).

**Lemma 3.8.** Assume that \( k \) is of characteristic \( p > 0 \). Let \( V \) be a tamely ramified smooth \( \mathbb{Q}_\ell \)-sheaf on \( \eta \) which is potentially unipotent. Let \( \pi \) be a uniformizer of \( T \). Then, the character \( \xi_0(T, V) \) in Definition 3.6 coincides with the character \( \varepsilon_0(T, V_0, d\pi) \) in Definition 3.2.1 followed by \( \mathbb{Z}_\ell \to \mathbb{Z}_\ell^\times /\mu \). Here \( V_0 \) is a \( G_\eta \)-stable \( \mathbb{Z}_\ell \)-lattice of \( V \).

**Proof.** We may assume that \( V \) is irreducible. Let \( \chi \) be a character of \( I \) which appears in \( V \). Let \( n \geq 1 \) be an integer such that \( \chi \) factors as \( I \to \mu_n(\overline{k}) \hookrightarrow \mathbb{Z}_\ell^\times \) and let \( \eta_n \) be the unramified extension of \( \eta \) with residue field \( k(\mu_n(\overline{k})) \). The \( \chi \)-isotypic part \( V_\chi \) of \( V \)
is stable under the action of \( G_m \) and \( V \cong \text{Ind}_{G_m}^G V_\chi \). Hence we reduce it to the case when \( \eta_n = \eta \). In this case, we also denote by \( \chi \) the character of \( G_\eta \) induced from the identification \( \text{Gal}(k(\eta)[\pi^\mathbb{A}]/k(\eta)) \cong \mu_n(k) \). Then, we have \( V \cong \chi \otimes V_0 \) where \( V_0 \) is a representation of \( G_k \). By [23, Proposition (2.5.3.1.)], we have \( F^{(0,\infty)}(V) \cong V \otimes G(\chi, \psi) \) where \( F^{(0,\infty)} \) is the local Fourier transform [23, Définition (2.4.2.3)] and \( G(\chi, \psi) \) is defined in loc. cit. Hence we have \( \det F^{(0,\infty)}(V) \cong \det(V) \otimes G(\chi, \psi)^{\dim V} \). On the other hand, for an integer \( m \geq 1 \) such that \( (\det V)^m \) is unramified, we have \( J(V^{\oplus m}) \cong G(\chi, \psi)^{\oplus m \cdot \text{rk} V_0} \). The assertion follows.

\[ \square \]

### 3.2 Reduction to the case of positive characteristic

To compute the local epsilon factors of vanishing cycles complex, we give a method to reduce it to the case of positive characteristic. This subsection is only necessary for the case of characteristic 0.

**Remark 3.9.** The following technique is needed since we treat \( \ell \)-adic sheaves. If one could develop a theory of epsilon cycles for local epsilon factors without taking modulo roots of unity and one could treat \( \Lambda \)-sheaves for a finite local ring \( \Lambda \), the technique seemed unnecessary.

We start with general lemmas.

Let \( R \) be a discrete valuation ring of residue characteristic \( \neq \ell \). Denote by \( K \) and \( F \) its function field and residue field respectively. Fix a uniformizer \( \pi \in R \) and, for an integer \( m \geq 0 \), denote by \( R_m \) the ring \( R[\pi^{1/m}] \) and by \( K_m \) the quotient field of \( R_m \). We write \( R_\infty \) and \( K_\infty \) for the unions \( \cup_{m \geq 0} R_m \) and \( \cup_{m \geq 0} K_m \) respectively. The rings \( R_m \) are valuation rings with residue field \( F \).

Let \( \mathcal{X} \) be a scheme over \( R \). Let \( m \) be an integer \( m \geq 0 \) or \( \infty \). Consider the diagram

\[ (3.3) \hspace{1cm} X_m \xrightarrow{j_m} \mathcal{X}_m \xleftarrow{i_m} \mathcal{X}_F \]

where the left arrow is the base change by \( R \to R_m \) of

\[ (3.4) \hspace{1cm} X := \mathcal{X} \times_R K \xrightarrow{j} \mathcal{X} \]

and \( i_m \) is the lift of the special fiber \( \mathcal{X}_F := \mathcal{X} \times_R F \xrightarrow{i} \mathcal{X} \).

**Lemma 3.10.** Let the notation be as above. Let \( \Lambda \) be a finite local ring of residue characteristic \( \ell \).

1. For a bounded below complex \( C \in D^+(\mathcal{X}, \Lambda) \) such that the structure morphism \( \mathcal{X} \to \text{Spec}(R) \) is locally acyclic relatively to \( C \), the canonical map \( i^*C \to i_{\infty}^*Rj_{\infty}*C|_{X_\infty} \) induced from (3.3) is an isomorphism.

2. Assume that \( \mathcal{X} \) is of finite type over \( R \). Then, the functor \( Rj_{\infty}* \) has finite cohomological dimension. For a constructible complex \( C \in D_c^b(\mathcal{X}, \Lambda) \), the complex \( i_{\infty}^*Rj_{\infty}*C|_{X_\infty} \) on \( \mathcal{X}_F \) is constructible.

In the situation of 2, for \( C \in D_c^b(\mathcal{X}, \Lambda) \), we write \( \langle C, -\pi \rangle := i_{\infty}^*Rj_{\infty}*C|_{X_\infty} \). When \( \mathcal{X} \) and the special fiber \( \mathcal{X}_F \) are regular, and \( C \) is a locally constant sheaf on which the inertia groups at generic points of the divisor \( \mathcal{X}_F \) act through \( \ell \)-groups, this notion coincides with the one given in [23, Definition 2.10].
Proof. We may assume that $R$ is strictly henselian, in particular $F$ is separably closed. Let $K$ (resp. $\overline{R}$) be a separable closure of $K$ (resp. the normalization of $R$ in $\overline{K}$). The residue field $\overline{F}$ of $\overline{R}$ is an algebraic closure of $F$. We also consider

$$\mathcal{X} \overset{j}{\to} \mathcal{X} \overset{i}{\to} \mathcal{X}_{\overline{F}},$$

where $\mathcal{X} := X \times_K \overline{K}$, $\mathcal{X} := X \times_R \overline{R}$, and $\mathcal{X}_{\overline{F}} := X \times_R \overline{F}$. We take and fix an injection $K_\infty \to \overline{K}$ of extensions of $K$. Then, they fit into the commutative diagram

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{X} \\
\downarrow{j} & & \downarrow{i} \\
\mathcal{X}_\infty & \xrightarrow{j} & \mathcal{X}_\infty \\
\end{array}$$

The two squares are cartesian if we replace $\mathcal{X}_{\overline{F}}$ by $X \times_R (\overline{R} \otimes_{R_\infty} F)$, whose étale topos is canonically isomorphic to that of $\mathcal{X}_{\overline{F}}$. Let $I' = \text{Gal}(\overline{K}/K_\infty)$ be the Galois group of $\overline{K}/K_\infty$. Note that the functor $\Gamma(I', -)$ on discrete $\Lambda[I']$-modules is exact, since all the finite quotients of $I'$ are of order prime to $\ell$.

1. By the local acyclicity, the canonical map $f^\ast i^\ast C \to \tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}}$ is an isomorphism. Taking the fixed part $R\Gamma(I', -) = \Gamma(I', -)$, we have an isomorphism $\Gamma(\overline{I}', f^\ast i^\ast C) \to \Gamma(\overline{I}', \tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}})$, which is isomorphic to $f^\ast i^\ast C$, since the action of $I'$ on $f^\ast i^\ast C$ is trivial. Since we have $\tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}} \cong f^\ast i^\ast Rj_\infty* f^\ast C|_{X_\infty}$, the target is isomorphic to

$$f^\ast i^\ast Rj_\infty* \Gamma(\overline{I}', f^\ast i^\ast Rj_\infty* f^\ast C|_{X_\infty}) \cong f^\ast i^\ast Rj_\infty* C|_{X_\infty},$$

hence the assertion 1.

2. For a sheaf $\mathcal{G}$ of $\Lambda$-modules on $X_\infty$, we show that $R^n j_\infty* \mathcal{G}$ is zero for $n > 2d(X_\infty)$. Let $x \to \mathcal{X}_{\overline{F}}$ be a geometric point. We have an isomorphism

$$(Rj_\infty* \mathcal{G})_x \cong \Gamma(\overline{I}', \Gamma((X_\infty \times_{X_\infty} X_\infty(x)) \times_{K_\infty} \overline{K}, \mathcal{G})).$$

Since $H^n((X_\infty \times_{X_\infty} X_\infty(x)) \times_{K_\infty} \overline{K}, \mathcal{G})$ is zero for $n > 2d(X_\infty)$, the first assertion follows. Let $C \in D^b_c(X, \Lambda)$. We have

$$f^\ast i^\ast Rj_\infty* C|_{X_\infty} \cong \Gamma(\overline{I}', f^\ast i^\ast Rj_\infty* f^\ast C|_{X_\infty}) \cong \Gamma(\overline{I}', \tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}}).$$

The second assertion follows from the constructibility of the nearby cycles complex $\tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}}$. Indeed, since $\Gamma(\overline{I}', -)$ is exact, cohomology sheaves $H^i(f^\ast i^\ast Rj_\infty* C|_{X_\infty})$ are subsheaves of $H^i(\tilde{t}^* R\tilde{j}_* C|_{\mathcal{X}})$.

We define the category $D_{ctf}(-, \mathbb{Z}_\ell/\ell \mathbb{Z}_\ell)$ of constructible complexes to be the 2-colimit of the categories $D_{ctf}(-, \mathcal{O}_\mathcal{E}/\ell \mathcal{O}_\mathcal{E})$ indexed by finite subextensions $E$ of $\overline{Q}_\ell/\ell \mathbb{Q}_\ell$.

**Corollary 3.11.** Let the notation be as in Lemma 3.10. Assume that $\mathcal{X}$ is of finite type over $R$. We use the notion and notation in Section 6. Let $\mathcal{F}_0 \in D(X_\text{pp}^\text{nr}, \mathbb{Z}_\ell)$ be a normalized constructible complex on $X$. Then, the complex $i^\ast_\infty R(j_\infty* \mathcal{F}_0) \in D(\mathcal{X}_\text{pp}^\text{nr}, \mathbb{Z}_\ell)$ is a normalized $\mathbb{Z}_\ell$-complex and $i^\ast_\infty R(j_\infty* \mathcal{F}_0 \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell/\ell \mathbb{Z}_\ell)$ is constructible, i.e. $i^\ast_\infty R(j_\infty* \mathcal{F}_0)$ defines a constructible complex of $\mathbb{Z}_\ell$-sheaves on $\mathcal{X}_F$, in the sense of Definition 6.7.2.
We denote this complex $i_\infty^* R j_\infty^* F_0$ by $\langle F_0, -\pi \rangle$.

**Proof.** Let $F_{0,n} := F_0 \otimes_{\ell Z} \ell [1]$. By Lemma 3.10.2, we have $R j_\infty^* F_{0,n+1} \otimes_{\ell Z} \ell [1] \cong R j_\infty^* F_{0,n}$. Hence $i_\infty^* R j_\infty^* F_0$ is a normalized $\ell [1]$-complex. Also by Lemma 3.10.2, the complex $i_\infty^* R j_\infty^* F_0 \otimes_{\ell Z} \ell [1] \cong i_\infty^* R j_\infty^* F_0$ is constructible.

**Lemma 3.12.** Let the notation be as in Lemma 3.10. Let $r \geq 0$ be an integer. Let $L, N \in D(X_r, \Lambda)$ be bounded below complexes. Assume that $i_r^* N$ is bounded and constructible and that the structure morphism $X_r \to \text{Spec}(R_r)$ is locally acyclic relatively to $L$. Let

\begin{equation}
L|_{X_r} \to M' \to N|_{X_r} \to \text{(3.5)}
\end{equation}

be a distinguished triangle on $X_r$. Then, for some integer $n \geq r$, there exists a distinguished triangle $L|_{X_n} \to M \to N|_{X_n} \to$ on $X_n$ whose pull-back to $X_n$ is isomorphic to that of (3.5).

**Proof.** Let $N|_{X_r} \to L|_{X_r}[1]$ be the morphism corresponding to (3.5). Let $C_n := i_n^* i_{n,r}^* L|_{X_n}[2]$ be the complex on $X_n$. It fits into the distinguished triangle $L|_{X_n}[1] \to R j_{n,*} L|_{X_n}[1] \to C_n \to$. We need to show that, for some $n \geq r$, the composition $N|_{X_n} \to R j_{n,*} N|_{X_n} \to R j_{n,*} L|_{X_n}[1] \to C_n$ is zero. Since $C_n$ is supported on $X_F$, it is enough to show that the restriction $i_n^* N \cong i_r^* N|_{X_n} \to i_n^* C_n$ is zero. Since $X_r \to \text{Spec}(R_r)$ is locally acyclic relatively to $L$, the colimit $\lim_{\to r} i_n^* C_n$ is acyclic by Lemma 3.10.1. Since $i_r^* N$ is constructible, the composition is zero for large $n$.

Let $S$ be a regular connected scheme of finite type over $\mathbb{Z}[1/\ell]$. Let $k$ be the perfection of the function field of $S$. Let $s \in S$ be a closed point. Let $S'$ be the blow-up of $S$ at $s$ and let $s$ be the generic point of the exceptional divisor. Let $R$ be the henselization of $O_{S',s}$. Fix a uniformizer $\pi \in R$ of $R$. We use the notation as above. That is, for an integer $m \geq 0$, we define $R_m$ to be $R[1/\ell^m]$. Let $K_m$ be the fraction field of $R_m$. We write $R_\infty := \lim_{\to m} R_m$ and $K_\infty := \lim_{\to m} K_m$. The rings $R_m$ are valuation rings whose residue fields are isomorphic to $k(s)$.

**Lemma 3.13.** The conjugates of the images in $G_k$ of $G_{K_\infty}$, for all the closed points $s \in S$ and uniformizers $\pi \in R$, topologically generate $G_k$.

**Proof.** Let $H$ be a finite quotient of $G_k$. After shrinking $S$, the quotient map $G_k \to H$ factors through $\pi_1(S)$ and $H$ is generated by the geometric Frobeniuses at closed points $s \in S$. Since the composition $G_{K_\infty} \to G_k \to \pi_1(S)$ factors as $G_{K_\infty} \to G_{k(s)} \to \pi_1(s)$ and the map $G_{K_\infty} \to \pi_1(s)$ is surjective, the assertion follows from the Chebotarev density.

Consider the commutative diagram

\begin{equation}
\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{t} \\
\text{Spec}(k) & & \mathbb{A}_k^1
\end{array}
\end{equation}

Diagram (3.6)
of $k$-schemes of finite type and a constructible complex $F_0 \in D_c^b(U, \mathbb{Z}_\ell)$ with the following properties.

1. $Y$ is a smooth separated curve. $U$ is smooth over $k$. $Z$ is a closed subscheme of $U$ finite étale over $k$.

2. The morphism $t: Y \to \mathbb{A}_k^1$ is an étale morphism.

3. $f|_{U \setminus Z}$ is $SS(F_0)$-transversal, in the sense of Definition 2.20.

Assume that the data given above except $F_0$ are defined over $S$. In other words, we have a commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{j} & U \\
\downarrow{\bar{g}} & & \downarrow{i} \\
S & & \mathbb{A}_S^1
\end{array}
\]

of $S$-schemes of finite type whose base change by $\text{Spec}(k) \to S$ is isomorphic to (3.6). We also assume that there exists $\bar{F}_0 \in D_{ctf}(U, Z_\ell)$ whose base change to $U$ is isomorphic to $F_0 \otimes^L \mathbb{Z}_\ell/\ell \mathbb{Z}_\ell$. We assume that they satisfy the following properties.

1. $Y$ is a smooth separated $S$-curve. $U$ is smooth over $S$. $Z$ is a closed subscheme of $U$ finite étale over $S$.

2. The morphism $\bar{t}: Y \to \mathbb{A}_S^1$ is an étale morphism.

3. The relative singular support $SS(\bar{F}_0, 0, U/S)$ exists and satisfies the condition 2 in Theorem 2.6. In particular, $\bar{g}$ is universally locally acyclic relatively to $\bar{F}_0$ (cf. Remark 2.7).

4. $\bar{f}|_{U \setminus Z}$ is $SS(\bar{F}_0, 0, U/S)$-transversal.

5. The restriction of the vanishing cycles complex $R\Phi_{\bar{t} \circ \bar{f}}(\bar{F}_0, 0)$ to $Z \times_{\mathbb{A}_S^1} (\mathbb{A}_S^1 \setminus \bar{t} \circ \bar{f}(Z)) \subset Z \times_{\mathbb{A}_S^1} \mathbb{A}_S^1 \cong Z \times_{\mathbb{A}_S^1} \mathbb{A}_S^1$ is locally constant. For each $i \in \mathbb{Z}$, the function on $Z$ defined by $z \mapsto \dim_{\text{tot}} R^i\Phi_{\bar{t} \circ \bar{f}}(\bar{F}_0, 0 \otimes \mathbb{Z}_\ell/\ell \mathbb{Z}_\ell)|_\eta$ is locally constant (cf. [33, Proposition 3.5.3]). Here $\bar{z}$ is the spectrum of an algebraic closure of $k(z)$ and $\eta_{\bar{z}}$ is the generic point of the strict henselization of $\mathbb{A}_k^1$ at $\bar{z}$.

We denote by

\[
\begin{array}{ccc}
Z_s & \xrightarrow{i} & U_s \\
\downarrow{\bar{i}_s} & & \downarrow{\bar{j}_s} \\
\mathcal{Y}_s & & \mathbb{A}_{k(\bar{s})}^1
\end{array}
\]

the base change of $Z \hookrightarrow U \to \mathcal{Y}$ by $s \to S$. Let $m$ be an integer $\geq 0$ or $\infty$. We define $U_m \to U_m \leftarrow U_s$ to be the base change of $U$ by $\text{Spec}(K_m) \to \text{Spec}(R_m) \leftarrow s$ over $S$.

**Proposition 3.14.** Let the notation be as above. Then, for every closed point $s \in S$ and every uniformizer $\pi \in R$, we have a commutative diagram

\[
\begin{array}{ccc}
G_k & \longrightarrow & \mathbb{Z}_\ell^x \\
\downarrow & & \downarrow \\
G_{K_\infty} & \longrightarrow & G_{k(\bar{s})}
\end{array}
\]
if \( k \) is of positive characteristic. Here the top horizontal arrow is \( \prod_{z \in Z} \varepsilon_0(Y(z), R\Phi_f(F)_z, dt) \circ \text{tr}_{z/k} \) and the right vertical arrow is \( \prod_{z \in Z} \varepsilon_0(Y_{6, z}, R\Phi_{\tilde{f}}(\langle F, -\pi \rangle)_z, dt) \circ \text{tr}_{z/s} \). Here the definition of \( \langle F, -\pi \rangle \) is given after the statement of Corollary 3.11.

When \( k \) is of characteristic 0, we have a commutative diagram (3.8) after replacing \( \varepsilon_0 \) and \( \mathbb{Z}_\ell^\ast \) by \( \varepsilon_0 \) (Definition 3.6) and \( \mathbb{Z}_\ell^\ast / \mu \).

Proof. In the course of the proof, we use the notion of oriented products and the local Fourier transforms in the relative settings. We refer to [33, Sections 2.3].

Replacing \( S \) and \( \mathcal{U} \) by \( Z \) and an open neighborhood of the graph \( Z \hookrightarrow Z \times_S \mathcal{U} \), we may assume that \( Z \rightarrow S \) is isomorphic. Composing \( \tilde{f} : \mathcal{Y} \rightarrow \mathbb{A}_S^1 \), we may replace \( \mathcal{Y} \) by \( \mathbb{A}_S^1 \).

We take \( S \cong Z \stackrel{f}{\longleftarrow} \mathbb{A}_S^1 \) as the origin. By induction on \( m \geq 0 \) and applying Lemma 3.12 to \( L = \tilde{F}_{0,m-1} \) and \( N = \tilde{F}_{0,0} \), we find a constructible complex \( \tilde{F}_{0,m} \) of \( \mathbb{Z}_\ell / \ell^{m+1} \mathbb{Z}_\ell \)-sheaves on \( \mathcal{U}_n \) for some integer \( n \geq 0 \) which fits into a distinguished triangle

\[
\tilde{F}_{0,m-1} \rightarrow \tilde{F}_{0,m} \rightarrow \tilde{F}_{0,0} \rightarrow
\]

whose restriction to \( U_n \) is isomorphic to the pull back of

\[
\mathcal{F}_{0,m-1} \rightarrow \mathcal{F}_{0,m} \rightarrow \mathcal{F}_{0,0} \rightarrow
\]

We claim that the canonical morphism \( \tilde{\phi} : \tilde{F}_{0,m} \otimes_{\mathbb{Z}_\ell / \ell^{m+1} \mathbb{Z}_\ell} \mathbb{Z}/\ell \mathbb{Z} \rightarrow \tilde{F}_{0,0} \) is an isomorphism. Indeed, the restriction \( \tilde{\phi}|_{U_n} \) is isomorphic to the canonical one \( \phi : \mathcal{F}_{0,m} \otimes_{\mathbb{Z}_\ell / \ell^{m+1} \mathbb{Z}_\ell} \mathbb{Z}/\ell \mathbb{Z} \rightarrow \mathcal{F}_{0,0} \), which is an isomorphism. On the other hand, by Lemma 3.10, the restriction \( \iota_n^* \tilde{\phi} \) to the special fiber is identified with \( \iota^* \Phi_{\tilde{f}} \mathcal{F}_{0,m} \otimes_{\mathbb{Z}_\ell / \ell^{m+1} \mathbb{Z}_\ell} \mathbb{Z}/\ell \mathbb{Z} \rightarrow \mathcal{F}_{0,0} \), which further can be identified with \( \iota^* \Phi_{\tilde{f}} \mathcal{F}_{0,0} \) since the cohomological dimension of \( \Phi_{\tilde{f}} \mathcal{F}_{0,0} \) is finite. Since \( U_n \rightarrow \text{Spec}(R_n) \) is locally acyclic relatively to \( \tilde{F}_{0,m} \), \( \tilde{F}_{0,m}|_{U_n} \) is isomorphic to \( \langle \mathcal{F}_{0,m}, -\pi \rangle := i^*_\infty \Phi_{\tilde{f}} \mathcal{F}_{0,m} \) by Lemma 3.10.

Let \( \tilde{f}_n : \mathcal{U} \times_S R_n \rightarrow \mathbb{A}_R^1 \), be the base change of \( \tilde{f} \). The restrictions of \( R\Phi_{\tilde{f}_n} \mathcal{F}_{0,m} \) to \( 0 \hookrightarrow \mathbb{G}_{m,s} \times_{\mathbb{A}_s^1} \mathbb{G}_{m,K_n} \) and \( 0\mathbb{K}_n \times_{\mathbb{A}_n^1} \mathbb{G}_{m,K_n} \) are isomorphic to \( R\Phi_{\tilde{f}_n}(\mathcal{F}_{0,m}, -\pi) \) and \( R\Phi_{\tilde{f}_n} \mathcal{F}_{0,m} \), respectively by [33, Proposition 2.2.1]. By the assumption 5, the restriction of \( R\Phi_{\tilde{f}_n} \mathcal{F}_{0,m} \) to \( 0 \hookrightarrow \mathbb{G}_{m,R_n} \times_{\mathbb{A}_R^1} \mathbb{G}_{m,R_n} \) is locally constant and its total dimension is locally constant (the condition in [33, Proposition 3.5.3]). In particular, if the generic fiber \( R\Phi_{\tilde{f}_n} \mathcal{F}_{0,m} \) is tamely ramified, so is \( R\Phi_{\tilde{f}_n}(\mathcal{F}_{0,m}, -\pi) \).

We show the assertion for the case when \( k \) is of positive characteristic. By [33, Proposition 1.3], \( \mathcal{F}^{0,\infty}(R\Phi_{\tilde{f}} \mathcal{F}_{0,m}) \) is locally constant and its restrictions to \( \infty_s \times_{\mathbb{P}_s^1} \mathbb{A}_s^1 \) and \( \infty_K \times_{\mathbb{P}_K^1} \mathbb{A}_K^1 \) are isomorphic to \( \mathcal{F}^{0,\infty}(R\Phi_{\tilde{f}}(\mathcal{F}_{0,m}, -\pi)) \) and \( \mathcal{F}^{0,\infty}(R\Phi_{\tilde{f}} \mathcal{F}_{0,m}) \), respectively. By [23, Lemme (3.4.1.2)], [33, Corollary 3.7], the determinant \( \text{det} \mathcal{F}^{0,\infty}(R\Phi_{\tilde{f}} \mathcal{F}_{0,m}) \) is tamely ramified. Applying [33, Lemma 2.11.1], we get a character \( \langle \text{det} \mathcal{F}^{0,\infty}(R\Phi_{\tilde{f}} \mathcal{F}_{0,m}), 1/x \rangle \) on \( R_n \), where \( x \) is the standard coordinate on \( \mathbb{A}_R^1 \subset \mathbb{P}^1 \). The assertion follows from [33, Lemma 2.11.2] and Laumon’s cohomological interpretation [33, Theorem 3.8] in this case.

The assertion for the case of characteristic 0 is proved as follows. Let \( x \) be the standard coordinate of \( \mathbb{A}_R^1 \). Since the construction of \( \langle -, x \rangle \) commutes with base change [33, Lemma...
We may assume that Jacobi datum on \( K_i \) constructed from \( H \) are the canonical ones.

Let \( \eta \) be the function field of the henselization \( \mathbb{A}^1_{k(s)}(0) \) (resp. \( \mathbb{A}^1_{K_\infty}(0) \)). Fix geometric points \( \eta \) and \( \eta_\infty \) and also fix a specialization \( \eta_\infty \) over \( \eta \) and \( \eta_\infty \), and also fix a specialization \( \eta_\infty \) over \( \eta \) and \( \eta_\infty \). The group \( I^p \) (resp. \( I \)) can be naturally regarded as the tame inertia group \( I^t_{\eta_\infty} \) (resp. \( I^t_{\eta} \)). Let \( \pi_1 \) be the fundamental group classifying finite étale coverings of \( \mathbb{A}^1_{R_0(0)} \) \( 0 \) \( R_0(0) \) tamely ramified along \( 0 \). We also have a natural embedding \( I^p \rightarrow \pi_1 \) and a commutative diagram

\[
\begin{array}{ccc}
I & \rightarrow & I^p \\
\searrow & \downarrow & \nearrow \\
I^t_{\eta_\infty} & \rightarrow & I^t_{\eta}
\end{array}
\]

where the top horizontal arrow is the projection \( I \rightarrow I^p \) and the bottom horizontal arrows are the canonical ones.

Since the generic characteristic of \( R_n \) is zero, the restriction of \( \Phi_j \) \( F_{0,m} \) to \( \mathbb{A}^1_{R_0(0)} \) \( 0 \) \( R_0(0) \) \( \mathbb{A}^1_{R_0(0)} \mathbb{G}_{m,R_0} \) is locally constant with tamely ramified cohomology sheaves. Since it is locally constant, the specialization \( (\Phi_j \mathbb{F}_{0,m})_{\eta_\infty} \) is an isomorphism, which we regard as an isomorphism of complexes of \( I \)-representations. These isomorphisms commute with the transition maps \( \Phi_j \mathbb{F}_{0,m+1} \rightarrow \Phi_j \mathbb{F}_{0,m} \) and \( \Phi_j \mathbb{F}_{0,m+1} \rightarrow \Phi_j \mathbb{F}_{0,m} \). Hence \( C_\infty := \Phi_j \mathbb{F} \mathbb{F}_{0,m} \) and \( C_s := \Phi_j \mathbb{F}_{0,m} \) are isomorphic as complexes of \( I \)-representations on finite dimensional \( \mathbb{Q}_\ell \)-vector spaces. Let \( I \rightarrow \mu_N(\overline{K}) \) be a finite quotient through which \( I \) acts on the semi-simplifications of cohomologies of \( C_\infty \). We may assume that \( p \nmid N \). For each \( i \), the semi-simplification of \( H^i(C_\infty) \otimes N \) gives a Jacobi datum on \( K_\infty \) as in Definition \( \overline{1} \) \( \overline{3} \), which extends to a Jacobi datum on \( R_\infty \) since \( N \) is invertible in \( R_\infty \). Since this Jacobi datum on \( R_\infty \) is restricted to the one on \( k(s) \) constructed from \( H^i(C_s) \otimes N \), the assertion follows.

\[ \square \]

**Remark 3.15.** Using a similar method as above repeatedly, one can reduce several problems on \( \ell \)-adic sheaves on schemes of finite type over (the perfections of) finitely generated fields to cases over finite fields. For example, Theorem 1 in \( [23] \) can be proven unconditionally, i.e. without the assumption that the sheaf \( \mathcal{F} \) in loc. cit. is defined over a scheme of finite type over \( \mathbb{Z} \), if the function field of the base scheme \( S \) is a purely inseparable extension of a finitely generated field, although it should be proved by developing a theory of Jacobi sum characters for representations with torsion coefficient.
3.3 Local epsilon factors of convolutions

At the end of this section, we compute the local epsilon factors of the convolutions of vanishing cycles. To do so, we need to recall the Thom-Sebastiani theorem for étale sheaves proved in [15], for properties of oriented products, we refer to [17], [13] Section 1.

Let $A^1_k, A^2_k$ be the henselizations of $A^1_k, A^2_k$ at $0, (0, 0)$ respectively. Let $f_1: X_1 \to A^1_k$ and $f_2: X_2 \to A^1_k$ be two morphisms of schemes of finite type. Denote by $X := (X_1 \times X_2) \times_{A^1_k \times A^1_k} A^2_k$ and by $f: X \to A^2_k$ the projection. Let $a: A^2_k \to A^1_k$ be the map induced from the summation $A^2_k \to A^1_k$. We regard $X$ as an $A^1_k$-scheme by the composition $X \xrightarrow{f} A^2_k \xrightarrow{a} A^1_k$.

**Definition-Lemma 3.16. (13, Definition 4.1.][15, Proposition 4.3)] For each $i = 1, 2$, let $K_i$ be an object of $D_{ctf}(X_i \times A^1_k, \Lambda)$. We define the local convolution $K_1 \ast^L K_2 \in D(X \times A^1_k, \Lambda)$ of $K_1$ and $K_2$ by the following:

$$
K_1 \ast^L K_2 := R\alpha^L_*(\overleftarrow{pr}_{1*}K_1 \otimes^L \overleftarrow{pr}_{2*}K_2)[1],
$$

where $\overleftarrow{pr}_i : X \times A^1_k \to X_i \times A^1_k$ and $\overleftarrow{a} : X \times A^1_k \to X \times A^1_k$ are induced from the $i$-th projections and $a$. The complex $K_1 \ast^L K_2$ belongs to $D_{ctf}(X \times A^1_k, \Lambda)$.

We remark that this definition is slightly different from that in [15], since the complex is shifted by 1.

We define a variant of the convolution functor on the derived category $D^b_c(\eta, E)$. Let $K_1, K_2$ be objects of $D_{ctf}(\eta, \Lambda)$. Denote by $K_{1!}$ the 0-extension of $K_1$ to $A^1_k$. Since $a: A^2_k \to A^1_k$ is universally locally acyclic relatively to $pr_1^*K_{1!} \otimes^L pr_2^*K_{2!}$ outside $(0, 0)$ (cf. Examples 2.8 1,2.), we can regard the vanishing cycles complex $R\Phi_a(pr_1^*K_{1!} \otimes^L pr_2^*K_{2!})$ as a complex on $(0, 0) \times A^1_k \eta \cong \eta$. For complexes $F_1, F_2 \in D^b_c(\eta, E)$, we define the convolution $F_1 \ast F_2$ as follows. Take $O_E$-lattices $F_{i,0}$ of $F_i$ for $i = 1, 2$ and let $F_{i,n} := F_{i,0} \otimes^L_{O_E} O_E/\ell^{n+1}$. We set

$$F_1 \ast F_2 := \lim_{\leftarrow n} R\Phi_a(pr_1^*(F_{1,n})! \otimes^L_{O_E/\ell^{n+1}} pr_2^*(F_{2,n})! \otimes_{O_E} E.
$$

This is a complex on $\eta$ and independent of the choices of $O_E$-lattices $F_{i,0}$. The convolution defines a functor $\ast : D^b_c(\eta, E) \times D^b_c(\eta, E) \to D^b_c(\eta, E)$. This is isomorphic to the restriction of $pr^*F_{1!} \ast^L pr^*F_{2!}$ to $(0, 0) \times A^1_k \eta \cong \eta$, where $pr_1^*K_{1!} \otimes^L A^1_k \to A^1_k$ is the second projection.

**Theorem 3.17. (15, Theorem 4.5.)** With the notation above, let $K_1, K_2$ be objects of $D_{ctf}(X_1, \Lambda), D_{ctf}(X_2, \Lambda)$ respectively. Let $K := (K_1 \boxtimes^L K_2)|_{X}$. Then, there is a functorial isomorphism

$$(R\Phi_{f_1}(K_1)) \ast^L (R\Phi_{f_2}(K_2))|_{X_0 \times A^1_k} \cong R\Phi_{af}(K)|_{X_0 \times A^1_k}$$

in $D_{ctf}(X_0 \times A^1_k, \Lambda)$, where $X_0$ is the closed fiber of $X \to A^1_k$.

We slightly change the notation. Let $f_1: X_1 \to A^1_k$ and $f_2: X_2 \to A^1_k$ be $k$-morphisms of finite type. Let $X := X_1 \times_k X_2$ and denote by $af: X \to A^1_k$ the composition of the product $f := f_1 \times f_2: X_1 \times_k X_2 \to A^1_k$ and the summation $a: A^2_k \to A^1_k$. For each $i = 1, 2$, let $\mathcal{F}_{0,i} \in D^b_c(X_i, \mathbb{Z}_\ell)$ and $x_i \in X_i$ be an at most isolated $SS(\mathcal{F}_{0,i})$-characteristic $k$-rational point of $f_i$ such that $f_i(x_i) = 0$. 


Lemma 3.18. Let the notation be as above. Let \( x := (x_1, x_2) \in X \) be the \( k \)-rational point above \( x_1 \) and \( x_2 \). Let \( F_0 := F_{0,1} \boxtimes F_{0,2} \). Denote by \( t \) the standard coordinate of \( \mathbb{A}_k^1 \).

1. Assume that \( p > 0 \). We have the equality
\[
\varepsilon_0(A_h^1, R\Phi^f(F)_x, dt)^{-1} = \\
\varepsilon_0(A_h^1, R\Phi^f_1(F_1)_{x_1}, dt)^{\dimtot R\Phi^f_2(F_2)_{x_2}} \cdot \\
\varepsilon_0(A_h^1, R\Phi^f_2(F_2)_{x_2}, dt)^{\dimtot R\Phi^f_1(F_1)_{x_1}}.
\]

2. Assume that \( k \) is finitely generated over \( \mathbb{Q} \). We have
\[
\varepsilon_0(A_h^1, R\Phi^f_1(F_1)_{x_1})^{-1} = \\
\varepsilon_0(A_h^1, R\Phi^f_1(F_1)_{x_1})^{\dimtot R\Phi^f_2(F_2)_{x_2}} \cdot \\
\varepsilon_0(A_h^1, R\Phi^f_2(F_2)_{x_2})^{\dimtot R\Phi^f_1(F_1)_{x_1}}.
\]

Proof. 1. By Theorem 3.17, we have an isomorphism
\[
(R\Phi^f_1(F_1)_{x_1}) \ast (R\Phi^f_2(F_2)_{x_2}) \cong R\Phi^f(F)_x[1].
\]

By [23, Proposition (2.7.2.2)], we have
\[
F^{(0,\infty)}(R\Phi^f_1(F_1)_{x_1}) \otimes F^{(0,\infty)}(R\Phi^f_2(F_2)_{x_2}) \cong F^{(0,\infty)}(R\Phi^f(F)_x)[1].
\]

Using this isomorphism and [23, Théorème (3.5.1.1)], the assertion follows.

2. Apply Proposition 3.14 to the commutative diagrams

\[
\begin{array}{ccc}
\text{Spec}(k) & \xrightarrow{x_i} & X_i \\
\downarrow & & \downarrow f_i \\
\mathbb{A}_k^1 & \xrightarrow{\text{id}} & \mathbb{A}_k^1 \\
& \downarrow & \\
\text{Spec}(k)
\end{array}
\]

and \( F_{0,i} \), and the similar diagram for \( X \) and \( F_0 \). Then, the assertion follows from 1 and Lemma 3.13.

\[\square\]

4 Epsilon Cycles of \( \ell \)-adic Sheaves

In this section, we construct epsilon cycles which compute local epsilon factors modulo roots of unity.

4.1 Group of characters modulo torsion

For a field \( E \), we denote by \( \mu_E \) the group of roots of unity in \( E \).

Definition 4.1. Let \( G \) be a compact Hausdorff abelian group.

1. For a finite extension \( E \) of \( \mathbb{Q}_\ell \), define \( \Theta_{G,E} \) to be the group \( \text{Hom}_{\text{cont}}(G, \mathcal{O}_E^\times/\mu_E) \) of continuous homomorphisms.

2. Define the group \( \Theta_G \) by \( \Theta_G := \varinjlim E \Theta_{G,E} \), where \( E \) runs through finite subextensions in \( \overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell \).
3. When $G$ is the abelianization of the absolute Galois group of a field $k$, $\Theta_{G,E}$ and $\Theta_G$ are also denoted by $\Theta_{k,E}$ and $\Theta_k$.

We usually identify $\Theta_G$ with a subgroup of the group $\text{Hom}(G, \mathbb{Z}_\ell^\times / \mu)$ of group homomorphisms. A group homomorphism $G \to \mathbb{Z}_\ell^\times / \mu$ is said to be continuous if it belongs to $\Theta_G$. By Lemma 4.2, a compact subgroup of $\mathbb{Z}_\ell^\times$ (resp. $\mathbb{Z}_\ell^\times / \mu_p$) is contained in $\mathcal{O}_E^\times$ (resp. $\mathcal{O}_E^\times / \mu_p$) for some finite subextension $E$ of $\mathbb{Q}_\ell / \mathbb{Q}$, where $\mathbb{Z}_\ell^\times$ (resp. $\mathbb{Z}_\ell^\times / \mu_p$) is equipped with the topology induced from the valuation of $\mathbb{Q}_\ell$ (resp. the quotient topology of $\mathbb{Z}_\ell^\times$). Therefore continuous homomorphisms $G \to \mathbb{Z}_\ell^\times$ (resp. $G \to \mathbb{Z}_\ell^\times / \mu_p$) give continuous homomorphisms $G \to \mathbb{Z}_\ell^\times / \mu$.

Lemma 4.2. Let $K \subset \text{GL}_n(\mathbb{Q}_\ell)$ be a compact subgroup. Then, there exists a finite subextension $E$ of $\mathbb{Q}_\ell / \mathbb{Q}_\ell$ such that $K \subset \text{GL}_n(E)$.

Proof. We give a proof for completeness. Fix a bijection from the set of integers $\geq 0$ to the set of finite subextensions of $\mathbb{Q}_\ell / \mathbb{Q}_\ell$, which is denoted by $m \mapsto E_m$. For an integer $m \geq 0$, put $K_m := K \cap \text{GL}_n(E_m)$. They are closed subgroups of $K$ and cover the whole of $K$, i.e. $\cup_m K_m = K$. Since $K$ is compact Hausdorff, Baire category theorem can be applied. Hence, there exists $m \geq 0$ such that $K_m$ contains a non-empty open subset of $K$, which implies that $K_m$ is an open subgroup. Since the index $[K : K_m]$ is finite, the assertion follows.

Lemma 4.3. Let $G$ be a compact Hausdorff abelian group.

1. The group $\Theta_G$ is uniquely divisible.

2. Let $\text{Hom}_\text{cont}(G, \mathbb{Z}_\ell^\times)$ be the group of continuous group homomorphisms to $\mathbb{Z}_\ell^\times$. Then, the kernel and the cokernel of the natural map $\text{Hom}_\text{cont}(G, \mathbb{Z}_\ell^\times) \to \Theta_G$ are torsion.

Proof. 1. Since the group $\mathcal{O}_E^\times / \mu_E$ is torsion-free, so is $\Theta_{G,E}$. Hence $\Theta_G$ is torsion-free. Let $\chi \in \Theta_{G,E}$ be a continuous homomorphism. For an integer $n \geq 1$, we need to find a finite extension $E'$ of $E$ and a continuous homomorphism $\xi : G \to \mathcal{O}_E^\times / \mu_{E'}$ so that $\xi^n = \chi$. Let $E'$ be a finite extension of $E$ which contains the $n$-th roots of all elements in $\mathcal{O}_E^\times$. Since $E'$ exists since $\mathcal{O}_E^\times / (\mathcal{O}_E^\times)^n$ is finite. Then, the composition of $\chi$ and the natural inclusion $\mathcal{O}_E^\times / \mu_E \to \mathcal{O}_E^\times / \mu_{E'}$ factors through the injection $\mathcal{O}_E^\times / \mu_{E'} \to \mathcal{O}_E^\times / \mu_{E'}$ defined by $a \mapsto a^n$. Since this injection is a homeomorphism onto the image, we find a desired homomorphism $\xi$.

2. The kernel is torsion since compact subgroups of $\mu \subset \mathbb{Z}_\ell^\times$ are finite subgroups by Lemma 4.2.

Let $E$ be a finite extension of $\mathbb{Q}_\ell$ and $\chi : G \to \mathcal{O}_E^\times / \mu_E$ be a continuous homomorphism. We find a continuous homomorphism $\xi : G \to \mathcal{O}_E^\times$ and an integer $n \geq 1$ such that the composition of $\xi$ and the quotient $\mathcal{O}_E^\times \to \mathcal{O}_E^\times / \mu_E$ equals to $\chi^n$. Take an open subgroup $U \subset \mathcal{O}_E^\times$ such that $U \cap \mu_E$ is trivial. Then, the composition $U \to \mathcal{O}_E^\times \to \mathcal{O}_E^\times / \mu_E$ is an isomorphism onto an open subgroup of $\mathcal{O}_E^\times / \mu_E$, which we also denote by $U$. Let $H$ be the inverse image of $U \subset \mathcal{O}_E^\times / \mu_E$ by $\chi$. This is an open subgroup of $G$. Let $n := [G : H]$ be the index. Define $\xi$ by the composition $G \xrightarrow{\chi^n} H \xrightarrow{\chi} U \to \mathcal{O}_E^\times$. Then, $\xi$ and $n$ satisfy the condition.
Definition 4.4. Let $k$ be a field of characteristic $p \neq \ell$. For a scheme $X$ of finite type over $k$, define the subcategory $\tilde{D}(X)$ of $D^b_c(X, \mathbb{Z}_\ell)$ by declaring what the morphisms are, as follows. Let $\varphi : \mathcal{F}_0 \to \mathcal{G}_0$ be a morphism in $D^b_c(X, \mathbb{Z}_\ell)$. It belongs to $\tilde{D}(X)$ if there exist a cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & S
\end{array}
$$

of schemes and a morphism $\varphi' : \mathcal{F}'_0 \to \mathcal{G}'_0$ in $D^b_c(X', \mathbb{Z}_\ell)$ such that the vertical arrows are of finite type, $S$ is the spectrum of a finitely generated field over its prime field, and the pull-back $f^* \varphi'$ is isomorphic to $\varphi$.

The subcategory $\tilde{D}(X)$ is triangulated and stable under Grothendieck’s 6 operations, i.e. $Rf_*, Rf^!, f^*, Rf^!, \otimes, \text{and } R\text{Hom}$. When $k$ is the perfection of a finitely generated field, this is nothing but the whole of $D^b_c(X, \mathbb{Z}_\ell)$.

4.2 Constructions of epsilon cycles

To deduce the existence of epsilon cycles in the case of positive characteristic (Lemma 4.7) from Proposition 2.12, we need to consider the variation of local epsilon factors for families of isolated characteristic points, which is done in [33].

Lemma 4.5. Let $k$ be a perfect field of characteristic $p > 0$. Let $X$ be a smooth scheme of finite type over $k$. Let $\mathcal{F}_0$ be an object of $\tilde{D}(X)$. Let

$$
\begin{array}{ccc}
U & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{j} & \text{Spec}(k)
\end{array}
$$

be a diagram as (2.2). Let $u \in U$ be an at most isolated $SS(j^* \mathcal{F}_0)$-characteristic point of $f$. For two local parameters $t$ and $t'$ of $Y$ around $f(u)$, the ratio $\varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u, dt) \cdot \varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u, dt')^{-1} = (\det R\Phi_f(\mathcal{F})_u)_{\frac{dt}{dt'}}$ of the characters of $G_{k(u)}^{ab}$ in Theorem 3.1.4 is of finite order.

Proof. We may assume that $u \to \text{Spec}(k)$ is an isomorphism. Consider the diagram

$$
\begin{array}{ccc}
u & \xrightarrow{u'} & U & \xrightarrow{f} & Y \\
& & \downarrow^{g} & \swarrow & \\
& & \text{Spec}(k).
\end{array}
$$

We can find the perfection of a finitely generated subfield $k_1$ of $k$ and a diagram

$$
\begin{array}{ccc}
u_1 & \xrightarrow{u_1} & U_1 & \xrightarrow{f_1} & Y_1 \\
& & \downarrow^{g_1} & \swarrow & \\
& & \text{Spec}(k_1)
\end{array}
$$
of smooth $k_1$-schemes, $\tilde{F}_0 \in D^b_c(U_1, \mathbb{Z}_\ell)$, and $t_1, t'_1 \in \Gamma(Y_1, \mathcal{O}_{Y_1})$ such that these data are pulled-back to the data above by the morphism $\text{Spec}(k) \rightarrow \text{Spec}(k_1)$. We can assume that these data satisfy the conditions from 1 to 4 in [33, Section 4]. Hence we may assume that $k$ is the perfection of a finitely generated field over $\mathbb{F}_p$.

We show that $(\det R\Phi_f(\mathcal{F})_u)_{|dt}$ is of finite order. Let $n$ be the Swan conductor of $\det R\Phi_f(\mathcal{F})_u$. If the difference $dt - dt'$ vanishes at $f(u)$, the character is killed by the $n$-th power of $p$ ([36, Lemma 4.8]), Take $a \in k^\times$ so that $d(at) - dt'$ vanishes at $f(u)$. Consider the following diagram

$$
\begin{array}{c}
\mathbb{G}_{m,k} \to U \times_k \mathbb{G}_{m,k} \\
\downarrow \text{id} \quad \downarrow \text{id} \\
\mathbb{G}_{m,k} \end{array}
\xrightarrow{f \times \text{id}}
\begin{array}{c}
Y \times_k \mathbb{G}_{m,k} \\
\downarrow \text{g \times id} \\
\mathbb{G}_{m,k}
\end{array}
$$

which is the product of ([12] and $\mathbb{G}_{m,k}$. Let $x$ be the standard coordinate of $\mathbb{G}_{m,k}$ and let $t'' := xt$. By [33, Theorem 4.8.2], we get a continuous character $\rho_v : \pi_1(\mathbb{G}_{m,k})^{ab} \to \mathbb{Z}_\ell^\times / \mu$. By [20, Theorem 1], this character factors through $\pi_1(\mathbb{G}_{m,k})^{ab} \to G_k^{ab}$. Specializing $x \mapsto 1, a$, we obtain the assertion.

**Definition 4.6.** Let the notation be as in Lemma 4.5. We denote the composition of $\varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u, dt)$ and the quotient map $\mathbb{Z}_\ell^\times / \mu$ by $\varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u)$.

The character $\varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u)$ is independent of the choice of local parameters $t$ by Lemma 4.5. It belongs to $\Theta_{k(u)}$.

**Lemma 4.7.** Let $k$ be a perfect field of characteristic $p > 0$. Let $X$ be a smooth scheme of finite type over $k$. Let $\mathcal{F}_0$ be an object of $\mathcal{D}(X)$. Let the singular support of $\mathcal{F}_0$ be denoted by $C$. For a diagram as ([2,2]) and an at most isolated $C$-characteristic point $u \in U$ of $f$, put $\varphi(f, u) := \varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u)^{-1} \circ \text{tr}_{k(u)/k}$. This assignment defines a $\Theta_k$-valued function on isolated $C$-characteristic points and is flat, in the sense of Definition [2.7], 2.

**Proof.** First we verify that $\varphi(f, u)$ is a $\Theta_k$-valued function on isolated $C$-characteristic points.

When $u$ is not an isolated $C$-characteristic point, $\varphi(f, u)$ vanishes since $R\Phi_f(\mathcal{F})_u$ vanishes.

Consider the diagram ([2,3]) and an isolated $C$-characteristic point $u' \in U$ of $f'$. Since the restriction of $\varepsilon_0(Y(u), R\Phi_f(\mathcal{F})_u)$ to $G_{k(u)}^{ab}$ equals to $\varepsilon_0(Y'_u, R\Phi_{f'}(\mathcal{F})_u)$, the assertion follows from the fact that the composition $G_{k(u)}^{ab} \xrightarrow{\text{tr}_{k(u)/k}} G_{k(u)}^{ab} \to G_{k(u)}^{ab}$ is the multiplication by $\deg(u'/u)$.

Next we show the flatness in the sense of Definition [2,9]. Consider the diagram ([2,4]). We need to show that the function $\varphi_f : |Z| \to \Theta_k$ defined by $\varphi_f(z) = \varphi(f_s, z)$, where $s \in S$ is the image of $z$, is flat over $S$. After replacing $S$ by its étale covering, we may assume that $Z$ is finite over $S$. Further replacing $S$ and $Y$ by open coverings, we may assume that there exists $t \in \Gamma(Y, \mathcal{O}_Y)$ which defines an étale morphism $Y \to \mathbb{A}^1_S$. We also assume that $S$ is connected. We can find a commutative diagram

\[\begin{array}{ccc}
\mathbb{G}_{m,k} & \to & \mathbb{G}_{m,k} \\
\downarrow & \downarrow & \downarrow \\
\mathbb{G}_{m,k} & \to & \mathbb{G}_{m,k}
\end{array}\]
of schemes of finite type over a finitely generated subfield $k_1$ of $k$ whose pull-back by the morphism $\text{Spec}(k) \to \text{Spec}(k_1)$ is isomorphic to \([2.3]\). We may assume that all the conditions imposed on the original data are also satisfied. The subscripts $(-)_1$ mean the corresponding objects over $k_1$. By \([33, \text{Theorem 4.8.2}]\), we have a commutative diagram

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{f_1} & Y_1 \\
\pr_1 & \downarrow & \downarrow \\
X_1 & \rightarrow & S_1
\end{array}
\]

of topological groups. Let $k'_1$ (resp. $k'$) be the normalization of $k_1$ (resp. $k$) in $S_1$ (resp. $S$).

By \([20, \text{Theorem 1}]\), $\rho_{t_1}$ followed by the quotient map $\mathbb{Z}_t^\times \to \mathbb{Z}_t^\times/\mu$ factors through $G_{k'_1}^{ab}$. Hence $\rho_t$ followed by $\mathbb{Z}_t^\times \to \mathbb{Z}_t^\times/\mu$ factors through $G_{k'}^{ab}$, which we denote by $\xi: G_{k'}^{ab} \to \mathbb{Z}_t^\times/\mu$.

Then, for a closed point $s \in S$, we have $\prod_{z \in Z_s} \varphi_f(z) = \prod_{z \in Z_s} \varphi_0(Y_s(z), R\Phi_{f_s}(F_s)_z)^{-1} \circ \tr_{k(z)/k} = \xi|_{C_{k'(s)}^{ab}} \circ \tr_{k(s)/k} = (\xi \circ \tr_{k'/k})^{\deg(k(s)/k')}$, hence the assertion.

To prove the existence of epsilon cycles in the case of characteristic 0, we need the following lemma.

**Lemma 4.8.** Let $S$ be a noetherian regular scheme. Let $X$ be a smooth scheme of relative dimension $n$ over $S$. Let $Z \subset X$ be an integral closed subscheme which is flat of relative dimension $n - c$ over $S$. Let $W$ be a smooth scheme of relative dimension $m$ over $S$. Let $h: W \to X$ be an $S$-morphism. Assume that each irreducible component $C_a$ of $Z \times_X W = \cup_a C_a$, equipped with the reduced subscheme structure, is flat of relative dimension $m - c$ over $S$. Then, after shrinking $S$ to a dense open subscheme, there exists a cycle $\sum_a t_a[C_a]$ with $\mathbb{Z}$-coefficient and supported on $Z \times_X W$ such that, for every morphism $s \to S$ from the spectrum of a field, we have $h^i_s[Z_s] = \sum_a t_a[C_{a,s}]$ as cycles supported on $(Z \times_X W)_s$, where $(\cdot)_s$ means the base change $(\cdot) \times_S s$.

**Proof.** Factoring $h$ as $W \hookrightarrow U \hookrightarrow W \times_S X \xrightarrow{\pr_X} X$, where $U$ is an open subscheme of $W \times_S X$ such that $W$ is closed in $U$, we may assume that $h$ is smooth or a closed immersion.

If $h$ is smooth, $C_a$ is a connected component of $Z \times_X W$ and we can take 1 as $t_a$.

Assume that $h$ is a closed immersion. Define $K$ to be the complex $\mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_W$ of coherent $\mathcal{O}_X$-modules. This is supported on $Z \times_X W$. Note that $K$ is bounded since $h: W \to X$ is a local complete intersection. Let $U \subset W$ be an open neighborhood around the generic points of $Z \times_X W$ so that $U \cap C_a$ are disjoint and $K|_{U \cap C_a}$ are extensions of finite free $\mathcal{O}_{U \cap C_a}$-modules. Let $\eta_a$ be the generic points of $C_a$. Let $t_a$ be the lengths of $K_{\eta_a}$ as complexes of $\mathcal{O}_{X,\eta_a}$-modules, i.e. the alternating sums of the lengths of $H^i(K_{\eta_a})$. Let $s \to S$ be a morphism from the spectrum of a field. Then we have

$$h^i_s[Z_s]|_{U_s} = [K \otimes_{\mathcal{O}_s} k(s)]|_{U_s} = \sum_a t_a[U_s \cap C_{a,s}].$$
Thus the cycle $\sum_a t_a [C_a]$ admits the property after shrinking $S$ so that the morphisms $C_a \cap C_b \to S$ are of relative dimension $< m - c$ for distinct indices $a, b$.

**Theorem 4.9.** Let $X$ be a smooth scheme of finite type over a perfect field $k$ of characteristic $p \neq \ell$ and let $\mathcal{F}_0$ be an object of $\tilde{\mathcal{D}}(X)$, defined in Definition 4.4. Then, there exists a unique cycle $\mathcal{E}(\mathcal{F}_0)_k = \sum_a \xi_a \otimes [C_a]$ with coefficients in $\Theta_k$ (Definition 4.1) and supported on $SS(\mathcal{F}_0) = \cup_a C_a$, satisfying the following property. For a diagram as (2.2) and an at most isolated $SS(\mathcal{F}_0)$-characteristic point $u \in U$ of $f$, we have

$$\bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F})_u)^{-1} \circ tr_{k(u)/k} = (\mathcal{E}(\mathcal{F}_0)_k, df_u)^{deg(u/k)}.$$

**Proof.** When $p > 0$, it follows from Lemma 4.7 and Proposition 2.12.

Let $p = 0$. For each irreducible component $C_a$ of $SS(\mathcal{F}_0)$, choose a diagram

$$U \xrightarrow{f_a} Y \xrightarrow{j_a} X,$$

where $j_a$ is étale and $Y_a$ is a smooth $k$-curve, and an isolated $SS(\mathcal{F}_0)$-characteristic point $u_a \in U_a$ of $f_a$ at which $df_u$ only meets $C_a$. Consider a continuous homomorphism $\xi_a \in \Theta_k$ satisfying the equality $\xi_a^{deg(u_a/k)(C_a, df_u)_{u_a}} = \bar{\varepsilon}_0(Y_{(u_a)}, R\Phi_{f_a}(\mathcal{F}_0)_{u_a})^{-1} \circ tr_{k(u_a)/k}$. We show that the cycle $\sum_a \xi_a \otimes [C_a]$ satisfies the condition. Let

$$U \xrightarrow{f} Y \xrightarrow{j} X,$$

be a diagram with $j$ étale and $Y$ a smooth $k$-curve, and $u \in U$ be an at most isolated $SS(\mathcal{F}_0)$-characteristic point of $f$. We need to show the equality

$$(4.3) \quad \bar{\varepsilon}_0(Y_{(u)}, R\Phi_f(\mathcal{F}))^{-1} \circ tr_{u/k} = \prod_a \bar{\varepsilon}_0(Y_{(u_a)}, R\Phi_{f_a}(\mathcal{F}))^{-1} \circ tr_{u_a/k}.$$

Taking finite extension of $k$, we may assume that $u_a$ and $u$ are $k$-rational. Let $k_1 \subset k$ be a finitely generated subfield over which all the data above are defined. Then, we may assume that $k$ is finitely generated.

Shrinking $Y_a$ and $Y$, we take étale $k$-morphisms $Y_a \to \mathbb{A}^1_k$ and $Y \to \mathbb{A}^1_k$. Applying Proposition 3.14 to the diagram

$$u_a \xrightarrow{f_a} U_a \xrightarrow{\cong} \Spec(k),$$

and the counterpart for $U \to Y$, we get commutative diagrams of topological groups as in the proposition. By Lemma 3.13 and Lemma 4.8, the equality (4.3) follows from the case of positive characteristic.

\[\square\]
Definition 4.10. We call the cycle $\mathcal{E}(F_0)_k$ the epsilon cycle of $F_0$. If no confusions occur, we omit the subscript $k$ and denote it by $\mathcal{E}(F_0)$.

Definition 4.11. Let $X$ be a smooth scheme of finite type over $k$. For a constructible complex $F_0 \in \bar{D}(X)$ and a rational number $r$, we define the $r$-twisted epsilon cycle $\mathcal{E}(F_0)(r)$ to be the product

$$\mathcal{E}(F_0)(r) := \chi_{\text{cyc}}^{rCC(F_0)} \cdot \mathcal{E}(F_0).$$

Here $\chi_{\text{cyc}}^{rCC(F_0)}$ means $\sum_a \chi_{\text{cyc}}^r(C_a)$ for $CC(F_0) = \sum_a m_a[C_a]$.

4.3 Properties of epsilon cycles

Definition 4.12. Let $f : X \to Y$ be a morphism of smooth $k$-schemes. Let $A$ be an abelian group. Let $C \subset T^*X$ be a closed conical subset. Assume that every irreducible component of $X$ and $C$ is of dimension $n$, and that of $Y$ is of dimension $m$. Further assume that $f$ is proper on the base of $C$ and that every irreducible component of $f_*C$ is of dimension $m$. For a cycle $\alpha \in A \otimes Z_n(T^*X)$ supported on $C$, define a cycle $f_*\alpha \in A \otimes Z_n(T^*Y)$ to be the push-forward by the projection $T^*Y \times_Y X \to T^*X$ of the pull-back of $\alpha$ by $df : T^*Y \times_Y X \to T^*X$ in the sense of intersection theory.

Lemma 4.13. Let $X$ be a smooth scheme of finite type over $k$ and $F_0 \in \bar{D}(X)$ be a constructible complex of $\mathcal{Z}_k$-sheaves on $X$.

1. Let $G_0 \in \bar{D}(X)$ be a smooth $\mathcal{Z}_k$-sheaf on $X$. Assume that $X$ is connected. Then we have an equality

$$\mathcal{E}(G_0 \otimes^L F_0) = (\det(G_0) \circ \text{tr}_{k'/k}) \cdot \chi_{\text{cyc}}^{-1} \cdot \chi_{\text{cyc}}^{CC(F_0)} \cdot \mathcal{E}(F_0) \cdot \mathcal{E}(G_0^\text{rkG}).$$

Here $k'$ is the normalization of $k$ in the function field of $X$. We regard $\det(G_0)$ as an element of $\Theta_{k'}$ as follows. Since there is a connected normal scheme $S$ whose function field is finitely generated such that $X$ and $G_0$ are defined over $S$, the determinant valued in $\mathcal{Z}_k^\times/\mu$ factors through $G_0^\text{ab}_{k'}$ by [20, Theorem 1].

In particular, we have $\mathcal{E}(F_0(n)) = \mathcal{E}(F_0)(n)$.

2. Let $k_1$ be a subfield of $k$ such that $\text{deg}(k/k_1)$ is finite. Then,

$$\mathcal{E}(F_0)_k \circ \text{tr}_{k/k_1} = \mathcal{E}(F_0^\text{deg(k/k_1)}).$$

3. Let $k'/k$ be an extension of perfect fields. Let $\mathcal{E}(F_0) = \sum_a \xi_a \otimes [C_a]$ be the epsilon cycle. Assume that, for each irreducible component $C_a$ of $SS(F_0)$, $C_a \cup \cup_{b \neq a} C_b$ has a smooth $k$-rational point. Then, we have

$$\mathcal{E}(F_0|_{X_{k'}}) = \sum_a \xi_a|_{G_{k'}^{\text{ab}}} \otimes [C_a \times_k k'].$$

The same equality holds if the extension $k'/k$ is algebraically closed or $k$ is finite.

4. Let $i : X \to X'$ be a closed immersion to a smooth $k$-scheme $X'$ of finite type. Then, we have

$$i_* \mathcal{E}(F_0) = \mathcal{E}(i_* F_0).$$
Proof. 1. Take a diagram as (2.2) and an at most isolated $SS(F_0)$-characteristic point $u \in U$ of $f$. Let $E := F_0 \otimes \mathbb{Q}_l \otimes \mathbb{Q}_l$ and $G := G_0 \otimes \mathbb{Q}_l \otimes \mathbb{Q}_l$. Then, we have

$$
\begin{align*}
(E(G_0 \otimes L F_0), df^\deg(u/k)) &= \Xi_0(Y(u), R\Phi_f(G) \otimes F_u)^{-1} \circ \text{tr}_{u/k} \\
&= \Xi_0(Y(u), G_0 \otimes R\Phi_f(F)u)^{-1} \circ \text{tr}_{u/k} \\
&= \left(\text{det} G \circ \text{tr}_{u/k}\right)^{-\text{dim}_{\text{tot}}R\Phi_f(F_u)} \Xi_0(Y(u), R\Phi_f(F_u)^{-1} \circ \text{tr}_{u/k} \\
&= \left(\text{det} G \circ \text{tr}_{u/k}\right)^{\deg(u/k)}(CC(F_0), df_u)(E(F_0), df_u)^{\deg(u/k)} \circ \text{tr}_{u/k} \\
&= \left(\text{det} G \circ \text{tr}_{u/k}\right)^{\deg(u/k)}(CC(F_0) \circ E(F_0)^{\text{rk}(G), df_u})^{\deg(u/k)}.
\end{align*}
$$

2. Consider morphisms of $k_1$-schemes $X \xrightarrow{j} U \xrightarrow{f} Y$ where $j$ is étale and $Y$ is a smooth $k_1$-curve. Replacing $Y$ by $Y \times_k k_1$ if necessary, to calculate local epsilon factors, we may assume that the diagram is defined over $k$. Then it follows from the uniqueness of epsilon cycles.

3. For each irreducible component $C_a$, take a diagram (2.2), and an isolated $SS(F_0)$-characteristic $k$-rational point $u_a \in U$ of $f$ over which $df$ only meets $C_a$ at a smooth point of $C_a \setminus \cup_{b \neq a} C_b$. We have $(E(F_0)_u, df)_u = \Xi_0(Y(u_a), R\Phi_f(F)_{u_a})^{-1} \circ \text{tr}_{u_a/k}$ and the counterpart over $k'$. Hence the coefficient of $C_a$ and that of $C_a \times_k k'$ coincide.

We show that when $k'/k$ is algebraically closed or $k$ is finite, the question can be reduced to the case treated above. First assume that $k'/k$ is algebraically closed. By 2, we may replace $k$ by a finite extension $k''$ and $k'$ by $k''' := k'' \otimes_k k'$, since the following diagram is commutative. Hence this case is reduced to the case above.

Next assume that $k$ is finite. When $k'$ is also finite, the assertion follows since in this case the composition $G_{k'}^{ab} \rightarrow G_{k}^{ab} \rightarrow G_{k''}^{ab}$ is the multiplication by $\text{deg}(k'/k)$. In general, let $k''$ be a finite extension of $k$ so that each irreducible component of $SS(F_0) \times_k k''$ has a smooth $k''$-rational point outside other irreducible components. Let $k''' := k' \cdot k''$ be a composition field. Since the cases $k''/k$ and $k'''/k''$ are treated already, we have $E(F_0)_{k'''} = E(F_0, k''')$. The assertion follows from 2.

4. Consider a commutative diagram of $k$-schemes

$$
\begin{array}{c}
X \xrightarrow{j} U \\
\downarrow i \quad \downarrow f \\
X' \xrightarrow{j'} U' \xrightarrow{f'} Y
\end{array}
$$

where the left horizontal arrows are étale, the square is cartesian, and $Y$ is a smooth curve. Since $SS(i_* F_0) = i_* SS(F_0)$, it suffices to show, for an isolated $SS(i_* F_0)$-characteristic point $u' \in U'$ of $f'$, the equality

$$
\Xi_0(Y(u'), R\Phi_{f'}(i_* F)u') = \Xi_0(Y(u'), R\Phi_{f}(F)u'),
$$

which follows from the isomorphism $R\Phi_{f'}(i_* F)u' \rightarrow i_* R\Phi_{f}(F)u'$. 

$\square$

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Proposition 4.14. Let $X_1$ and $X_2$ be smooth schemes of finite type over $k$. Take $F_{0,i} \in D(X_i)$ for each $i = 1, 2$. Then, we have an equality

$$\mathcal{E}(F_{0,1} \boxtimes F_{0,2}) = (\mathcal{E}(F_{0,1}) \boxtimes CC(F_{0,2})) \cdot (CC(F_{0,1}) \boxtimes \mathcal{E}(F_{0,2})), $$

where $\mathcal{E}(F_{0,1}) \boxtimes CC(F_{0,2})$ is defined as follows. Write $\mathcal{E}(F_{0,1}) = \sum_a \xi_a \otimes [C_a]$ and $CC(F_{0,2}) = \sum_b n_b \cdot [D_b]$. Then, $\mathcal{E}(F_{0,1}) \boxtimes CC(F_{0,2}) := \sum_{a,b} \xi_a n_b \otimes [C_a \times D_b]$. The definition of $CC(F_{0,1}) \boxtimes \mathcal{E}(F_{0,2})$ is similar. The product $\cdot$ is the group law of $\Theta_k \otimes Z_\bullet(T^*(X_1 \times_k X_2))$.

Proof. Let $C_1, C_2$ be irreducible components of $SS(F_{0,1}), SS(F_{0,2})$ respectively. By Lemma 4.13.2, after replacing $k$ by its finite extension, we may assume that, for each $i = 1, 2$, there exist a diagram

$$
\begin{array}{ccc}
U_i & \xrightarrow{f_i} & \mathbb{A}^1_k \\
\downarrow & & \\
X_i & & 
\end{array}
$$

and a $k$-rational isolated $SS(F_{0,i})$-characteristic point $u_i \in U_i$ at which the section $d f_i$ meets only $C_i$. We also assume that $C_i$ is smooth at the intersection point and that $f_i$ maps $u_i$ to 0.

By Example 2.8.2, the cycle $\mathcal{E}(F_{0,1} \boxtimes F_{0,2})$ is supported on $SS(F_{0,1}) \times SS(F_{0,2})$. Hence it suffices to compute the coefficient of $[C_1 \times C_2]$ in $\mathcal{E}(F_{0,1} \boxtimes F_{0,2})$. Let $k_1 \subset k$ be the perfection of a finitely generated subfield over which all the data above are defined. By Lemma 4.13.3, we reduce the assertion to the case when $k$ is the perfection of a finitely generated field.

Let $f: U_1 \times U_2 \to \mathbb{A}^2_k$ be the product of $f_1$ and $f_2$ and let $a: \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the summation map. Let $\xi_i$ be the coefficient of $C_i$ in $\mathcal{E}(F_{0,i})$ and $\xi$ be that of $C_1 \times C_2$ in $\mathcal{E}(F_{0,1} \boxtimes F_{0,2})$. Denote $u := (u_1, u_2) \in U_1 \times U_2$. Since $u$ is an isolated $SS(F_{0,1} \boxtimes F_{0,2})$-characteristic point of $af$, we have

$$
(\mathcal{E}(F_{0,1} \boxtimes F_{0,2}), d(af))_{T^*(U_1 \times U_2), u} = \bar{\varepsilon}_0(\mathbb{A}^1_k(0), R\Phi_{af}(F_1 \boxtimes F_2)_u)^{-1}.
$$

Since $d(af)$ only meets $C_1 \times C_2$ at $u$, the left hand side equals to $\xi^{(C_1, d f_1)_{T^*U_1, u_1} \cdot (C_2, d f_2)_{T^*U_2, u_2}}_{(C_1, d f_1, u_1) \cdot (C_2, d f_2, u_2)}$. On the other hand, by Lemma 3.18, the right hand side equals to

$$
\bar{\varepsilon}_0(\mathbb{A}^1_k(0), R\Phi_{f_1}(F_1)_{u_1})^{\dim tot R\Phi_{f_2}(F_2)_{u_2} \cdot \bar{\varepsilon}_0(\mathbb{A}^1_k(0), R\Phi_{f_2}(F_2)_{u_2})^{\dim tot R\Phi_{f_1}(F_1)_{u_1}},
$$

which equals to

$$
\xi_1^{(C_1, d f_1, u_1) \cdot (CC(F_{0,1}), d f_2, u_2)} \cdot \xi_2^{(C_2, d f_2, u_2) \cdot (CC(F_{0,2}), d f_1, u_1)}
$$

hence the assertion. 

Definition 4.15. Let $X$ and $W$ be smooth schemes over a field $k$ and let $C$ be a closed conical subset of $T^*X$. Assume that every irreducible component of $X$ and $C$ is of dimension $n$ and that every irreducible component of $W$ is of dimension $m$. Let $A$ be an abelian group. Let $h: W \to X$ be a properly $C$-transversal $k$-morphism and let

$$
T^*W \leftarrow W \times_X T^*X \to T^*X
$$

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be the canonical morphisms. Then, for an $A$-linear combination $\alpha = \sum_a \beta_a \otimes [C_a]$ of irreducible components of $C = \bigcup_a C_a$, we define $h^! \alpha$ to be $(-1)^{n-m}$-times the push-forward by the first arrow $W \times X T^*X \to T^*W$ of the pull-back of $\alpha$ by the second arrow $W \times X T^*X \to T^*X$ in the sense of intersection theory. This is a cycle supported on $h^! C$.

**Corollary 4.16.** Let $h: W \to X$ be a smooth morphism of smooth schemes of finite type over $k$. Assume that each irreducible component of $X$ and $W$ is of dimension $n$ and $m$ respectively. Let $\mathcal{F}_0 \in \widetilde{D}(X)$. Then, we have

$$\mathcal{E}(h^! \mathcal{F}_0) = h^!(\mathcal{E}(\mathcal{F}_0)(\frac{n-m}{2})) = h^!(\mathcal{E}(\mathcal{F}_0)(\frac{n-m}{2})).$$

**Proof.** Since the assertion is étale local on $W$, we may assume that $W = X \times \mathbb{A}^m_\mathbb{K}$ and $h$ is the projection. By induction on $m$, we reduce the question to the case when $W = X \times \mathbb{A}^1_\mathbb{K}$ and $h$ is the projection. By Proposition 4.14, it is enough to show, for the trivial $\mathbb{Z}_\ell$-sheaf $G_0 := \mathbb{Z}_\ell$ on $\mathbb{A}^1_\mathbb{K}$, the equality $\mathcal{E}(G_0) = \frac{1}{2} \chi_{\text{cyc}} \otimes [T^*_\mathbb{K} \mathbb{A}^1_\mathbb{K}]$.

First assume that $p > 2$. By Lemma 4.13, we may assume that $k$ is finite. Let $f: \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$ be the Kummer covering defined by $t \mapsto t^2$. In this case, the epsilon factor $\varepsilon_0(\mathbb{A}^1_{\mathbb{F}_q(0)}, R\Phi f(G)_0, dt)$ equals to the quadratic Gauss sum, which is $q^{\frac{1}{8}}$ up to roots of unity. On the other hand, the intersection number $(T^*_\mathbb{K} \mathbb{A}^1_{\mathbb{F}_q}, df)_0$ is 1.

When $p = 0$ or 2, we argue as follows. Let $S := \text{Spec}(\mathbb{Z}[rac{1}{2p}])$ and consider the following commutative diagram

$$\begin{array}{ccc}
S \xrightarrow{0} \mathbb{A}^1_S & \xrightarrow{j} & \mathbb{A}^1_S \xrightarrow{id} \mathbb{A}^1_S \\
& \searrow & \downarrow \\
& S &
\end{array}$$

where $j$ is defined by $t \mapsto t^3$. This diagram and the trivial $\mathbb{Z}_\ell$-sheaf on $\mathbb{A}^1_S$ satisfy the conditions from 1 to 5 given after the diagram (3.7). Then, the assertion follows from Lemma 3.13, Proposition 3.14, and the case when $p > 3$.

**Corollary 4.17.** Let $X$ be a connected smooth scheme of finite type over $k$. Put $n := \text{dim } X$. For a smooth $\mathbb{Z}_\ell$-sheaf $\mathcal{F}_0 \in \widetilde{D}(X)$ on $X$, we have

$$\mathcal{E}(\mathcal{F}_0) = (\det(\mathcal{F}) \circ \text{tr}_{k'/k})^{(-1)^n}_{\text{deg}(k'/k)} \cdot \chi_{\text{cyc}} \cdot \frac{(-1)^{n+1} \cdot \text{rk}_{\mathcal{F}}}{2} \otimes [T^*_X X].$$

Here $k'$ is the normalization of $k$ in $X$.

**Proof.** This follows from Lemma 4.13 and Corollary 4.16.

**Example 4.18.** Let $X$ be a smooth connected curve over $k$. Let $\mathcal{F}_0 \in \widetilde{D}(X)$ be a constructible complex of $\mathbb{Z}_\ell$-sheaves on $X$. Let $U \subset X$ be an open dense subset where $\mathcal{F}_0$ is smooth. Then, we have

$$\mathcal{E}(\mathcal{F}_0) = (\det(\mathcal{F}|_U) \circ \text{tr}_{k'/k})^{(-1)^n}_{\text{deg}(k'/k)} \cdot \chi_{\text{cyc}} \cdot \frac{\text{rk}_{\mathcal{F}|_U}}{2} \otimes [T^*_X X] + \sum_{x \in X \setminus U} (\varpi(X(x), \mathcal{F})^{-1} \circ \text{tr}_{x/k})^{(-1)^n}_{\text{deg}(k'/k)} \otimes [T^*_x X].$$

Here $\varpi(X(x), \mathcal{F}) = \varpi_0(X(x), \mathcal{F}_{U(x)}) \cdot \det(\mathcal{F}_x)^{-1}$. 

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Lemma 4.19. (cf. [23, Théorème (3.2.1.1)], [12, Theorem 11.1]) Let $X$ be a projective smooth curve over $k$ and $\mathcal{F}_0$ be a constructible complex in $\mathcal{D}(X)$. Then, the product formula

$$\det R\Gamma(X_k, \mathcal{F}) = (\mathcal{E}(\mathcal{F}_0), T^*_X X)_{T^*X}$$

as an element of $\Theta_k$ holds.

Proof. We may assume that $k$ is the perfection of a finitely generated field. When $k$ is of positive characteristic, it follows from Theorem 3.4 and Example 4.18. Let $Z$ be a closed subscheme of $X$ such that $\mathcal{F}_0$ is smooth outside $Z$. The case when $k$ is of characteristic 0 is reduced to the case of positive characteristic by applying Proposition 3.14 to the diagram

$$\begin{array}{ccc}
Z' & \rightarrow & X \\
\downarrow & & \downarrow \text{id} \\
\text{Spec}(k) & \rightarrow & X
\end{array}$$

and Lemma 3.13.

Proposition 4.20. We identify the group $\Theta_{\mathbb{F}_q}$ with $\mathbb{Z}^\times_q / \mu \subset \overline{\mathbb{Q}}_\ell^\times / \mu$ via $\xi \mapsto \xi(\text{Frob}_q)$. Let $X$ be a smooth scheme of finite type over $\mathbb{F}_q$. Let $F$ be a field of characteristic 0. Let $\mathcal{F}_0$ and $\mathcal{F}'_0$ be elements of $D^b_c(X, \mathcal{O}_X)$ and $D^b_c(X, \mathcal{O}'_X)$, where $\ell$ and $\ell'$ are prime numbers which do not divide $q$. Fix embeddings $F \rightarrow \overline{\mathbb{Q}}_\ell, F \rightarrow \overline{\mathbb{Q}}_{\ell'}$ of fields. Assume that, for all closed points $x$ of $X$, the coefficients of the characteristic polynomials $\det(T - \text{Frob}_x, \mathcal{F})$ and $\det(T - \text{Frob}_x, \mathcal{F}')$ are contained in $F$ and give the same elements of $F$. Then, the coefficients of the epsilon cycle $\mathcal{E}(\mathcal{F}_0)$ (resp. $\mathcal{E}(\mathcal{F}'_0)$) are contained in $F^\times \otimes \mathbb{Q} \subset \overline{\mathbb{Q}}_\ell^\times \otimes \mathbb{Q} \cong \overline{\mathbb{Q}}_{\ell'}^\times / \mu$ (resp. $\subset \overline{\mathbb{Q}}_{\ell'}^\times \otimes \mathbb{Q} \cong \overline{\mathbb{Q}}_{\ell'}^\times / \mu$) and give the same elements of $F^\times \otimes \mathbb{Q}$.

Proof. Since the assertion is étale local, we may assume that $X$ is affine. Taking an immersion $X \rightarrow \mathbb{P}$ and replacing $\mathcal{F}_0, \mathcal{F}'_0$ by their 0-extensions, we may assume that $X$ is projective purely of dimension $n$. Let $C = \cup_a C_a$ be a closed conical subset of $T^*X$ such that $\mathcal{F}_0$ and $\mathcal{F}'_0$ are micro-supported on $C$ and irreducible components are of dimension $n$. By Lemma 2.16 after replacing $\mathbb{F}_q$ by a finite extension, we have a good pencil

$$X \xleftarrow{\pi} X_L \xrightarrow{f} \mathbb{P}^1.$$ 

By the properties 5 and 6, the base $C_a \cap T^*_X X$ of every irreducible component $C_a$ of $SS(\mathcal{F}_0) \cup SS(\mathcal{F}'_0)$ is not contained in the exceptional locus of $\pi$. Thus it is enough to show the statement for $\pi^* \mathcal{F}_0$ and $\pi^* \mathcal{F}'_0$. Further by the properties 4, 6, Theorem 4.9, and 8, Theorem 2, we may replace them by the push-forwards $Rf_* \pi^* \mathcal{F}_0$ and $Rf_* \pi^* \mathcal{F}'_0$.

Hence we may assume that $X$ is a projective smooth curve. Let $U$ be an open dense subset of $X$ where $\mathcal{F}_0$ and $\mathcal{F}'_0$ are smooth. Let $x \in X$ be a closed point and $\omega$ be a basis of $\Omega^1_{X(x)}$. We need to show that $\varepsilon(X(x), \mathcal{F}, \omega)$ and $\varepsilon(X(x), \mathcal{F}', \omega)$ are contained in $F^\times$ and coincide. This follows from [23, Théorème (3.1.5.4)(iii)] and [5, Théorème 9.8].

Next we prove a compatibility of the construction of epsilon cycles and the pull-back by properly transversal morphism. We mimic the method given in [30], due to Beilinson. We use the theory of the universal hyperplane sections and follow the notation in (2.7).

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Lemma 4.21. Let \( \mathbb{P} = \mathbb{P}^n \) be a projective space and \( \mathbb{P}^\vee \) be its dual. Let \( C^\vee \subset T^*\mathbb{P}^\vee \) be a closed conical subset whose irreducible components are of dimension \( n \). Define a closed conical subset \( C \subset T^*\mathbb{P} \) by \( C = p_*\mathbb{P}^\circ C^\vee \). Then every irreducible component of \( C \) is of dimension \( n \).

Proof. It suffices to treat the case when \( C^\vee \) is irreducible. Since the projectivization \( \mathbb{P}(C) \subset \mathbb{P}(T^*\mathbb{P}) \) coincides with \( \mathbb{P}(C^\vee) \), it is enough to show that irreducible components of \( C \) contained in the 0-section is of dimension \( n \). When the base of \( C^\vee \) is not finite, \( C \) contains \( T_F^*\mathbb{P} \). When the base of \( C^\vee \) consists of one point \( H \in \mathbb{P}^\vee \), \( C \) equals to \( T_H^*\mathbb{P} \).

\[ \Box \]

Proposition 4.22. Let \( \mathbb{P} = \mathbb{P}^n \) be a projective space, and \( \mathbb{P}^\vee \) be its dual. Let \( \mathcal{G}_0 \in \hat{D}(\mathbb{P}^\vee) \) and write \( \mathcal{F}_0 := R_{p_*}\mathbb{P}^\circ \mathcal{G}_0 \) for the naive Radon transform of \( \mathcal{G}_0 \). Let \( C^\vee \subset T^*\mathbb{P}^\vee \) be a closed conical subset whose irreducible components are of dimension \( n \). Let \( C := p_*\mathbb{P}^\circ C^\vee \subset T^*\mathbb{P} \). Assume that \( \mathcal{G}_0 \) is micro-supported on \( C^\vee \).

Let \( X \) be a smooth subscheme of \( \mathbb{P} \), and assume that the immersion \( h: X \to \mathbb{P} \) is properly \( C \)-transversal.

1. We have

\[
\mathbb{P}(\mathcal{E}(R_{p_*\mathbb{P}^\circ} \mathcal{G}_0)) = \mathbb{P}(p_!\mathcal{E}(p^\circ \mathcal{G}_0)) = \mathbb{P}(p_!p^\vee!(\mathcal{E}(\mathcal{G}_0)(\frac{1}{2} - \dim X))).
\]

In particular, we have

\[
\mathbb{P}(\mathcal{E}(R_{p_*\mathbb{P}^\circ} \mathcal{G}_0)) = \mathbb{P}(p_!\mathcal{E}(p^\circ \mathcal{G}_0)) = \mathbb{P}(p_!p^\vee!(\mathcal{E}(\mathcal{G}_0)(\frac{1}{2} - n))).
\]

2. We have

\[
\mathcal{E}(h^*\mathcal{F}_0) = h^!(\mathcal{E}(\mathcal{F}_0)(\frac{n - \dim X}{2})).
\]

Proof. Note that, by [30, Corollary 3.13.2], we have \( p_!p^\circ C^\vee = h^\circ C \). By the assumption that \( h: X \to \mathbb{P} \) is properly \( C \)-transversal, every irreducible component of \( p_!p^\circ C^\vee \) has the same dimension as \( X \). Thus the cycles \( p_!\mathcal{E}(p^\circ \mathcal{G}_0) \) and \( p_!p^\vee!(\mathcal{E}(\mathcal{G}_0)(\frac{1}{2} - \dim X)) \) are well-defined.

1. First we prove the second equality of (4.4). By [30, Corollary 3.13.2], \( \mathcal{P}^\vee: X \times_{\mathbb{P}} Q \to \mathbb{P}^\vee \) is \( C^\vee \)-transversal and hence \( p^\circ \mathcal{G}_0 \) is micro-supported on \( p^\circ C^\vee \). Since \( \mathcal{P}^\vee: X \times_{\mathbb{P}} Q \to \mathbb{P}^\vee \) is smooth outside \( \Delta_X := \mathbb{P}(T^*_X\mathbb{P}) \subset X \times_{\mathbb{P}} Q \), we have \( \mathcal{E}(p^\circ \mathcal{G}_0) = p^\vee!(\mathcal{E}(\mathcal{G}_0)(\frac{1}{2} - \dim X)) \) outside \( \Delta_X \) by Corollary 4.16. By the assumption that \( h: X \to \mathbb{P} \) is \( C \)-transversal, the pair \( (p, \mathcal{P}^\vee) \) is \( C^\vee \)-transversal around \( \Delta_X \subset X \times_{\mathbb{P}} Q \) by [30, Corollary 3.13.1]. Hence, we have the second equality in (4.4).

We prove the first equality in (4.4). Both \( \mathcal{E}(R_{p_*\mathbb{P}^\circ} \mathcal{G}_0) \) and \( p_!\mathcal{E}(p^\circ \mathcal{G}_0) \) are supported on \( h^\circ C = p_!p^\circ C^\vee \), which is purely of dimension \( \dim X \). Hence it suffices to show the equality

\[
(\mathcal{E}(R_{p_*\mathbb{P}^\circ} \mathcal{G}_0), df)_{u}^{deg(u/k)} = (p_!\mathcal{E}(p^\circ \mathcal{G}_0), df)_{u}^{deg(u/k)}
\]

for every diagram

\[
X \leftarrow U \to \mathbb{A}^1_k,
\]
where \( j \) is étale and \( f \) is smooth, and every at most isolated \( h^sC \)-characteristic point \( u \in U \) of \( f \). By Theorem 4.9 the left hand side of (4.15) equals to \( \pi_0(\mathbb{A}^1_{k(u)}, R\Phi_f(h^*F)_{u})^{-1} \circ tr_{u/k}. \) By [30] Corollary 3.15, there exist finitely many \( p^\vee C^\vee \)-characteristic points of \( fp: U \times_p Q \to \mathbb{A}_k \). Hence the right hand side of (4.5) equals to \( \prod_v(\mathcal{E}(p^\vee \mathcal{G}_0), d(fp))_{v^\deg(v/k)} \) where \( v \) runs through \( p^\vee C^\vee \)-characteristic points of \( fp \) over \( u \). Further by Theorem 4.9 this equals to \( \prod_v \pi_0(\mathbb{A}^1_{k(v)}, R\Phi_{fp}(p^\vee \mathcal{G}_v))^{-1} \circ tr_{v/k}. \) Thus the equality (4.5) follows from the isomorphism

\[
R\Phi_f(Rp_*p^\vee \mathcal{G})_u \cong \bigoplus_v \operatorname{Ind}_{G_v}^{G_u} R\Phi_{fp}(p^\vee \mathcal{G})_v.
\]

2. By the proper base change theorem, we have an isomorphism \( h^*F_0 \to Rp_*p^\vee \mathcal{G}_0. \) Hence by 1, we have

\[
\mathbb{P}(\mathcal{E}(h^*F_0)) = \mathbb{P}(p_*p^\vee l(\mathcal{E}(\mathcal{G}_0)(\frac{1 - \dim X}{2})))
= \mathbb{P}(h^!p_*p^\vee l(\mathcal{E}(\mathcal{G}_0)(\frac{1 - \dim X}{2}))) = \mathbb{P}(h^!(\mathcal{E}(F_0)(\frac{n - \dim X}{2}))).
\]

By the assumption that the immersion \( h \) is properly \( C \)-transversal, \( X \) intersects the smooth locus of \( F_0. \) Hence the coefficients of the 0-section in both \( \mathcal{E}(h^*F_0) \) and \( h^!(\mathcal{E}(F_0)(\frac{n - \dim X}{2})) \) coincide. Thus the assertion follows.

Before stating Corollary 4.23, we give definitions of the Radon transform and the Legendre transform.

Let \( F_0 \) be an element of \( \tilde{D}(\mathbb{P}). \) We define the Radon transform \( RF_0 \) of \( F_0 \) by \( RF_0 := Rp^\vee p^*F_0[n - 1] \in \tilde{D}(\mathbb{P}^\vee). \)

Let \( C \) be a closed conical subset of \( T^*\mathbb{P} \) whose irreducible components \( C_a \) are of dimension \( n = \dim \mathbb{P}. \) Let \( A := \sum_a \beta_a \otimes [C_a] \) be a cycle supported on \( C. \) We define the Legendre transform \( LA \) by \( LA := (p^\vee p^! A)^{(-1)^{n-1}}. \) Since the definition of \( p^! A \) involves the sign \((-1)^{n-1}, \) that of \( LA \) does not involve the sign.

**Corollary 4.23.** Let \( F_0 \) be an element of \( \tilde{D}(\mathbb{P}). \) We have

\[
\mathbb{P}(\mathcal{E}(RF_0)) = \mathbb{P}(L(\mathcal{E}(F_0)(\frac{1 - n}{2}))).
\]

We will show the equality \( \mathcal{E}(RF_0) = L(\mathcal{E}(F_0)(\frac{1 - n}{2})) \) in Corollary 5.6

**Proof.** We have

\[
\mathbb{P}(\mathcal{E}(RF_0)) = \mathbb{P}(p^!(\mathcal{E}(p^*F_0))^{(-1)^{n-1}}) = \mathbb{P}(L(\mathcal{E}(F_0)(\frac{1 - n}{2}))).
\]

**Theorem 4.24.** Let \( X \) be a smooth scheme of finite type over \( k. \) Let \( F_0 \in \tilde{D}(X). \) Let \( h: W \to X \) be a properly \( SS(F_0) \)-transversal \( k \)-morphism from a smooth \( k \)-scheme \( W \) of finite type. Assume that every irreducible component of \( X \) and \( W \) is of dimension \( n \) and \( m \) respectively. Then,

\[
\mathcal{E}(h^*F_0) = h^!(\mathcal{E}(F_0)(\frac{n - m}{2})).
\]
Proof. Decomposing $W \to W \times X \to X$, and by Corollary 4.16 we assume that $h$ is an immersion.

First consider the case when $X$ is a projective space $\mathbb{P}$. The case when $\mathcal{F}_0 = R\mathcal{F} \mathcal{G}_0$ is the naive Radon transform has been treated in Proposition 4.22.2. Let $\mathcal{F}_0 \in \hat{D}(\mathbb{P})$. Since $\mathcal{F}_0$ is isomorphic to a Radon transform $R\mathcal{F} \mathcal{G}_0$ up to a smooth sheaf and the assertion for smooth sheaves is proved in Corollary 4.17, it follows in the case $h$ is an immersion to $\mathbb{P}$.

We show the general case. Since the assertion is local, we may assume that $X$ is affine and take an immersion $i: X \to \mathbb{P}$. Further, we may assume that there is a smooth subscheme $V \subset \mathbb{P}$ such that $X \cap V = W$ and that the intersection is transversal. Then, the immersion $\tilde{h}: V \to \mathbb{P}$ is properly $i_*SS(\mathcal{F}_0)$-transversal around $W \subset V$. Hence, it follows from the case when $h$ is an immersion to $\mathbb{P}$.

\[ \square \]

### 4.4 Epsilon cycles for tamely ramified sheaves

Let $k$ be a perfect field of characteristic $p \neq \ell$. In this subsection, we calculate the epsilon cycles of tamely ramified $\mathbb{Z}_\ell$-sheaves.

Let $X$ be a smooth scheme of finite type over $k$ and let $D \subset X$ be a simple normal crossings divisor. Denote by $U$ the complement of $D$ in $X$. Let $(D_a)_{a \in A}$ be the irreducible components of $D$. For a subset $B \subset A$, we denote by $D_B$ the intersection $\cap_{a \in B} D_a$.

For simplicity, we assume that $X$ is connected and of dimension $n$. Then, $D_B$ is a smooth closed subscheme of $X$ and is purely of dimension $n - |B|$.

Let $\mathcal{F}_0 \neq 0$ be a non-zero smooth $\mathbb{Z}_\ell$-sheaf of free $\mathbb{Z}_\ell$-modules on $U$ which belongs to $\hat{D}(U)$ and tamely ramified along $D$. Let $j: U \to X$ be the inclusion. We have

\[ SS(j)_*\mathcal{F}_0 = \cup_B T_{D_B}^* X \]

\[ CC(j)_*\mathcal{F}_0 = \sum_B (-1)^{\text{rk} \mathcal{F}_0} T_{D_B}^* X, \]

where $B$ runs through subsets of $A$ (see [30, 4.2, 7.3]).

For each $a \in A$, let $\xi_a$ be the generic point of $D_a$ and denote by $k_a$ the normalization of $k$ in the residue field at $\xi_a$. Since $\mathcal{F}_0$ is tamely ramified, its restriction to the henselization $X_{(\xi_a)}$ gives a representation $V_{0,a}$ of the tame inertia group $I_a$ of the trait $X_{(\xi_a)}$. Let $V_a := V_{0,a} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$. Note that $I_a$ is isomorphic to $\lim_{\rightarrow} \mu_n(k)$, where $n$ runs through integers $\geq 1$ prime to $p$ and $\mu_n(k)$ is the group of $n$-th roots of unity in an algebraic closure $k$ of $k_a$, and that we have $\sigma^* V_a \cong V_a$ for each $\sigma \in \text{Gal}(k/k_a)$. Thus we get a character $J(V_a): \text{Gal}(k/k_a) \to \overline{\mathbb{Q}}_\ell/\mu$ as constructed in Definition 3.5.2. We define

\[ J_a := (J(V_a) \circ \text{tr}_{k_a/k})^{\frac{1}{\text{deg}(k/k_a)}}, \]

This is an element of $\Theta_k$.

**Proposition 4.25.** Let the notation be as above. Assume that $X$ is connected and that $\mathcal{F}_0$ is tamely ramified along $D$ and contained in $\hat{D}(U)$. For a subset $B$ of $A$, define

\[ \chi_B := (\det(\mathcal{F}_0) \circ \text{tr}_{k'/k})^{\frac{(-1)^{|B|}}{\text{deg}(k'/k)}} \cdot \frac{|B|}{n} (-1)^n \text{rk} \mathcal{F}_0 \cdot \prod_{a \in B} J_a^{(-1)^n}, \]

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where $k'$ is the normalization of $k$ in the function field of $X$. Then, we have

$$ \mathcal{E}(j_! \mathcal{F}_0) = \sum_B \chi_B \otimes [T_{D_B}^* X]. $$

Proof. Let $B \subset A$ be a subset and let $m := |B|$ be the cardinality. Let $x \in D_B$ be a closed point which is not contained in $D_a$ for any $a \in A \setminus B$. For $1 \leq i \leq n$, let $E_i \subset A^n_{k(x)}$ be the $i$-th coordinate hyperplane and define $E := \cup_{1 \leq i \leq m} E_i$ to be the union of $E_1, \ldots, E_m$. After replacing $X$ by an étale neighborhood of $x$, we find an étale morphism $f : X \to A^n_{k(x)}$ such that $x$ maps to the origin and the pull-backs of the divisors $(E_i)_{1 \leq i \leq m}$ coincide with $(D_a)_{a \in B}$ with some numbering on $B$.

Let $\pi_{1 \text{tame}}(X_{(x)} \setminus D)$ be the fundamental group which classifies finite étale coverings of $X_{(x)} \setminus D$ tamely ramified along $D$. Let $\pi^0_{1 \text{tame}}(A^n_{k(x)} \setminus E)$ be the one which classifies finite étale coverings of $A^n_{k(x)} \setminus E$ tamely ramified along $E$ and $\mathbb{P}^n_{k(x)} \setminus A^n_{k(x)}$. Then, the morphism $\pi_{1 \text{tame}}(X_{(x)} \setminus D) \to \pi^0_{1 \text{tame}}(A^n_{k(x)} \setminus E)$ induced from $f$ is an isomorphism. Thus we may assume that $X = A^n_{k'}$ for some finite extension $k'$ of $k$ and $D = E$, and that the sheaf $\mathcal{F}_0$ is also tamely ramified along $\mathbb{P}^n_{k'} \setminus A^n_{k'}$.

Fix a geometric point $\overline{\eta}$ over the generic point of $A^n_{k'}$. For $1 \leq i \leq m$, let $I_i$ be the tame inertia group of the henselization of $A^n_{k'}$ at the generic point of $E_i$. Note that $I_i$ is canonically isomorphic to a normal subgroup of $\pi_{1 \text{tame}}(A^n_{k'} \setminus E)$. Let $V_i$ be the representation $\mathcal{F}_0 \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ of $I_i$. After replacing $\mathcal{F}_0$ by its subquotients, we may assume that $I_i$ acts on $V_i$ via a character $\chi_i : I_i \to \overline{\mathbb{Q}}_\ell^\times$. By the assumption $\mathcal{F}_0 \in \tilde{D}(U)$, $\chi_i$ decomposes as $I_i \to \mu_{d_i}(\overline{k}) \to \overline{\mathbb{Q}}_\ell^\times$ for some integer $d_i \geq 1$ prime to $p$. Further extending $k'$ to a finite extension, we may assume that $\mu_{d_i}(\overline{k})$ are contained in $k'$ for all $i$. Thus there exist smooth $\mathbb{Z}_\ell$-sheaves $\mathcal{G}_{0,i}$ of rank 1 of finite order on $A^n_{k'} \setminus 0$ and a smooth $\mathbb{Z}_\ell$-sheaf $\mathcal{H}_0$ on $A^n_{k'}$ such that $\mathcal{F}_0$ is isomorphic to $\mathcal{H}_0 \otimes \bigotimes_{1 \leq i \leq m} \mathcal{G}_{0,i}$, where $pr_i : A^n \to A^1$ is the $i$-th projection. By Lemma 4.13 and Proposition 4.14, the coefficient of $[T^n_{1 \leq i \leq m} \mathcal{G}_{0,i}]$ in $\mathcal{E}(j_! \mathcal{F}_0)$ equals to

$$ (\det(\mathcal{H}_0) \circ \text{tr}_{k'/k}) \frac{1}{\deg(k'/k)} (-1)^n \cdot \chi_{\text{cyc}} \cdot \frac{m-2}{2} \cdot \text{tr}_{k'/k} \cdot \prod_{1 \leq i \leq m} (\mathcal{G}_{0,i} \circ \text{tr}_{k'/k}) , $$

where $\mathcal{G}_i = \mathcal{G}_{0,i} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$. Since we have $(\mathcal{E}(A^1_{k'}(0)), \mathcal{G}_i)^{\mathcal{F}_0} = J(V_i)$, the assertion follows.

\[ \square \]

5 Radon Transform and Product Formula

5.1 Epsilon class and product formula

In this subsection, we generalize the result in [30, Section 7.2] to epsilon cycles. Let $k$ be a perfect field of characteristic $p \geq 0$ which is different from $\ell$.

We introduce the notion of epsilon classes, an analogue of characteristic classes [30, Section 6]. Let $X$ be a smooth scheme of finite type purely of dimension $n$ over $k$. We identify $\text{CH}_n(X) = \oplus_{i=0}^n \text{CH}_i(X)$ with $\text{CH}_n(\mathbb{P}(T^*X \oplus A^1_X))$ by the canonical isomorphism

$$ \text{CH}_n(X) \to \text{CH}_n(\mathbb{P}(T^*X \oplus A^1_X)) $$

sending $(a_i)_i$ to $\sum_i c_1(O(1)^i) \cap p^* a_i$ where $p : \mathbb{P}(T^*X \oplus A^1_X) \to X$ is the projection. Tensoring $\Theta_k$ to (5.1), we also identify $\Theta_k \otimes \text{CH}_n(X)$ with $\Theta_k \otimes \text{CH}_n(\mathbb{P}(T^*X \oplus A^1_X))$.

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\textbf{Definition 5.1.} Let $X$ be a smooth scheme of finite type purely of dimension $n$ over $k$. For an element $F_0$ of $\hat{D}(X)$, define the epsilon class $\varepsilon_X(F_0)$ of $F_0$ by setting $\varepsilon_X(F_0) = \mathcal{E}(F_0) = \mathbb{P}(\mathcal{E}(F_0) \oplus \mathbb{A}_k^1) \in \Theta_k \otimes \text{CH}_n(\mathbb{P}(T^*X \oplus \mathbb{A}_k^1)) = \Theta_k \otimes \text{CH}_*(X)$.

Let $\tilde{K}(X)$ be the Grothendieck group of the triangulated category $\hat{D}(X)$. The epsilon classes define a group homomorphism

$$\varepsilon_X : \tilde{K}(X) \to \Theta_k \otimes \text{CH}_*(X).$$

\textbf{Lemma 5.2.} Let $X$ and $F_0$ be as in Definition 5.1.

1. The dimension $0$-part $\varepsilon_{X,0}(F_0) \in \Theta_k \otimes \text{CH}_0(X)$ is the intersection product $(\mathcal{E}(F_0), T^*_X X)_{T^*_X X}$ with the $0$-section.

2. Assume that $X$ is connected. Let $\text{rk}^0$ and $\det(F)^0$ be the rank and the determinant character of the restriction of $F = F_0 \otimes \mathbb{Q}_k$ to a dense open subset where $F$ is smooth. Then, the dimension $n$-part $\varepsilon_{X,n}(F_0) \in \Theta_k \otimes \text{CH}_n(X) = \Theta_k$ equals to

$$\left(\det(F)^0 \circ \text{tr}_{k'/k}\right)^{\frac{1}{\text{deg}(k'/k)}} \cdot \chi_{\text{cyc}} \cdot \frac{(-1)^n}{2 \text{rk}^0(F)}.$$ 

where $k'$ is the normalization of $k$ in $X$.

\textit{Proof.} 1. This follows from [30, Lemma 6.3.2].

2. After shrinking $X$, we may assume that $F_0$ is smooth. Then, the assertion follows from Corollary 4.17 and [30, Lemma 6.3.1].

\textbf{Lemma 5.3.} Let $X$ be a smooth scheme of finite type purely of dimension $n$ over $k$ and $F_0$ be an element of $\hat{D}(X)$. Let $h : W \to X$ be a properly $\text{SS}(F_0)$-transversal closed immersion of codimension $c$. Then, we have

$$\chi_{\text{cyc}}(h^*F_0) \cdot \varepsilon_W(h^*F_0) = c(T^*_W X)^{-1} \cap h^! \varepsilon_X(F_0)^{-1}.$$ 

\textit{Proof.} The assertion follows from Theorem 4.17 and [30, Lemma 6.5]. Note that $h^! : \text{CH}_*(X) \to \text{CH}_*(W)$ is the usual pull-back, which does not involve the sign.

We start the proof of the product formula. We follow the method by Beilinson [30, Section 7].

For the Radon transform, we use the notation in Section 4. Let $\mathbb{P} = \mathbb{P}^n$ and $\mathbb{P}^\vee$ be its dual. We identify $Q = \mathbb{P}(T^*\mathbb{P})$ and let $p : Q \to \mathbb{P}$ and $p^\vee : Q \to \mathbb{P}^\vee$ be the projections.

The Radon transforms $R = Rp^\vee p^*[n-1]$ and $R^\vee = Rp^\vee p^*[n-1](n-1)$ define morphisms

$$R : \tilde{K}(\mathbb{P}) \to \tilde{K}(\mathbb{P}^\vee), \quad R^\vee : \tilde{K}(\mathbb{P}^\vee) \to \tilde{K}(\mathbb{P}).$$ 

Let $\Theta_k := \text{Hom}_{\text{cont}}(\mathbb{C}_k, \mathbb{Z}_k)$ be the group of continuous homomorphisms. Define also morphisms $(\chi, \varepsilon^{-1}) : \tilde{K}(\mathbb{P}) \to \mathbb{Z} \times \Theta_k$ and $(\chi, \varepsilon^{-1}) : \tilde{K}(\mathbb{P}) \to \mathbb{Z} \times \Theta_k$ by $(\chi, \varepsilon^{-1})F_0 = (\chi(F_0), \det(RF_k, F_0))$ and by $(\chi, \varepsilon^{-1})G_0 = (\chi(P_k, G_0), \det(RG_k, G_0)).$

\textbf{Lemma 5.4.} Let $n \geq 1$ be an integer and $\mathbb{P} = \mathbb{P}^n$. 

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1. The diagram

\[
\begin{array}{ccc}
\tilde{K}(\mathbb{P}) & \xrightarrow{(\epsilon, \epsilon^{-1})} & \mathbb{Z} \times \tilde{\Theta}_k \\
R \downarrow & & \downarrow \\
\tilde{K}(\mathbb{P}) & \xrightarrow{(\epsilon, \epsilon^{-1})} & \mathbb{Z} \times \tilde{\Theta}_k 
\end{array}
\]

is commutative, where the right vertical arrow sends \((a, b)\) to \(((-1)^{n-1}na, \chi^{(-1)n}(-\frac{n(n-1)}{2})b(-1)^{n-1}n)\).

2. The composition \(\tilde{K}(\mathbb{P}) \xrightarrow{R^*} \tilde{K}(\mathbb{P}) \xrightarrow{(\epsilon, \epsilon^{-1})} \mathbb{Z} \times \tilde{\Theta}_k\) maps \(\mathcal{F}_0\) to \((n^2\chi(\mathcal{F}_0), \varepsilon^{-1}(\mathcal{F}_0)^n)\).

Proof. 1. For constructible complexes \(\mathcal{F}_0 \in \tilde{D}(\mathbb{P})\) on \(\mathbb{P}\), we have

\[
R\Gamma(\mathbb{P}^v_k, R\mathcal{F}_0) = R\Gamma(Q_k, p^*\mathcal{F}_0)[n - 1] = R\Gamma(\mathbb{P}_k, \mathcal{F}_0 \otimes L \cdot R\mathcal{F}_0)[n - 1]
\]

by the projection formula. Hence the assertion follows from \(R^*p^*p^*\mathcal{F}_0 = \mathbb{Z}(-q/2)\) for \(0 \leq q \leq 2(n - 1)\) even and \(q = 0\) for otherwise.

2. Similarly to 1, for constructible complexes \(\mathcal{G}_0 \in \tilde{D}(\mathbb{P}^v)\), we have \((\epsilon, \epsilon^{-1})R^*\mathcal{G}_0 = ((-1)^{n-1}n\chi(\mathcal{G}_0), \chi^{(-1)n(n-1)}\chi(\mathcal{G}_0))\varepsilon^{-1}(\mathcal{G}_0(n - 1))(-1)^{n-1}n\). Combining this, 1, and the equality \(\varepsilon(\mathbb{P}^v, \mathcal{G}_0(n - 1)) = \chi^{(1-n)\chi(\mathcal{G}_0)}\varepsilon(\mathbb{P}^v, \mathcal{G}_0)\), we get the assertion.

We define the Legendre transform

\[
L : CH_\bullet(\mathbb{P}) \rightarrow CH_\bullet(\mathbb{P}^v)
\]

by \(L(a) = p^*_v(c(T^*(Q/\mathbb{P})) \cap p^*a)\) for the projections \(p : Q \rightarrow \mathbb{P}\) and \(p^v : Q \rightarrow \mathbb{P}^v\), where \(c(T^*(Q/\mathbb{P})) \in CH_\bullet(Q)\) is the total Chern class of the relative cotangent bundle of \(Q/\mathbb{P}\).

We also denote \(id \otimes L : \Theta_k \otimes CH_\bullet(\mathbb{P}) \rightarrow \Theta_k \otimes CH_\bullet(\mathbb{P})\) by the same letter \(L\).

**Proposition 5.5.** Let \(n \geq 1\) be an integer and \(\mathbb{P} = \mathbb{P}^n\).

1. The diagram

\[
\begin{array}{ccc}
\tilde{K}(\mathbb{P}) & \xrightarrow{(c_{ep}, \epsilon)} & (\mathbb{Z} \oplus \Theta_k) \otimes CH_\bullet(\mathbb{P}) \\
R \downarrow & & \downarrow L \\
\tilde{K}(\mathbb{P}^v) & \xrightarrow{(c_{ep}, \epsilon^v)} & (\mathbb{Z} \oplus \Theta_k) \otimes CH_\bullet(\mathbb{P}^v)
\end{array}
\]

is commutative, where \(L(a, b) = (a, L(\chi^{1-n}_\cyc \cdot b))\). The diagram replaced \(R\) by \(R^v\) and \(\tilde{L}\) by \(\tilde{L}^v : (a, b) \mapsto (a, L(\chi^{1-n}_\cyc \cdot b))\) is commutative.

2. The diagram

\[
\begin{array}{ccc}
\tilde{K}(\mathbb{P}) & \xrightarrow{(c_{ep}, \epsilon)} & (\mathbb{Z} \oplus \Theta_k) \otimes CH_\bullet(\mathbb{P}) \\
(\epsilon, \epsilon^{-1}) \downarrow & & \downarrow \deg \\
\mathbb{Z} \oplus \Theta_k
\end{array}
\]

is commutative.
Proof. We prove the assertions by induction on \( n \). If \( n = 1 \), the projections \( p : Q \to \mathbb{P} \) and \( p^\vee : Q \to \mathbb{P}^\vee \) are isomorphisms and the assertion 1 is obvious. Since \( \deg cc_{\mathbb{P}} F_0 = (CC F_0, T_0^*\mathbb{P})_{T^*p} \) and \( \deg \varepsilon_{\mathbb{P}} F_0 = (E(F_0), T_0^*\mathbb{P})_{T^*p} \), the assertion 2 for \( n = 1 \) is nothing but the Grothendieck-Ogg-Shafarevich formula and the product formula (Lemma 4.19).

We show that the assertion 2 for \( n - 1 \geq 1 \) implies the assertion 1 for \( n \). The second part of the assertion 1 follows from the first one. Hence we prove the first. We show the commutativity by using the direct sum decomposition

\[
\text{CH}_n(\mathbb{P}) = \text{CH}_n(\mathbb{P}(T^*\mathbb{P}^\vee \oplus A_1^{1/\mathbb{P}^\vee})) \longrightarrow \text{CH}_n(\mathbb{P}(T^*\mathbb{P}^\vee)) \oplus \text{CH}_n(\mathbb{P}^\vee)
\]

\[
\text{CH}_n(\mathbb{P}) = \text{CH}_n(\mathbb{P}) \oplus \mathbb{Z}.
\]

The compositions with the first projection \((\mathbb{Z} \oplus \Theta_k) \otimes \text{CH}_n(\mathbb{P}^\vee) \to (\mathbb{Z} \oplus \Theta_k) \otimes \text{CH}_n(\mathbb{P})\) are equal by Corollary 4.23 and [30, Corollary 7.5]. We show that the compositions with the second projection \( p_{2} : (\mathbb{Z} \oplus \Theta_k) \otimes \text{CH}_n(\mathbb{P}^\vee) \to \mathbb{Z} \oplus \Theta_k \) induced by the projection \( (\mathbb{P}(T^*\mathbb{P}^\vee \oplus A_1^{1/\mathbb{P}^\vee}) \to \mathbb{P}^\vee \) are same.

Let \( F_0 \) be a constructible complex of \( \mathbb{Z} \) sheaves on \( \mathbb{P} \) and \( C = SS F_0 \) be the singular support of \( F_0 \). After replacing \( k \) by its finite extension, we can take a hyperplane \( H \subset \mathbb{P} \) such that the immersion \( h : H \to \mathbb{P} \) is properly \( C \)-transversal and let \( i : \text{Spec}(k) \to \mathbb{P} \) be the immersion of the \( k \)-rational point of \( \mathbb{P}^\vee \) corresponding to \( H \). Note that \( p_{2} : \text{CH}_n(\mathbb{P}^\vee) \to \mathbb{Z} \) coincides with \( i^1 : \text{CH}_n(\mathbb{P}^\vee) \to \text{CH}_n(\text{Spec}(k)) \equiv \mathbb{Z} \).

By the hypothesis of the induction, we have \( \deg cc_H h^* F_0 = \chi h^* F \) and \( \deg \varepsilon_H h^* F_0 = \varepsilon(H, h^* F_0)^{-1} \). By [30, Proposition 7.8], we have \( cc_H h^* F = -c(O_H(-1))^{-1} \cap h^! cc_{\mathbb{P}} F \) and by Lemma 4.3 \( \chi_{\text{cyc}(h^* F_0)} \varepsilon_H(h^* F_0)^{-1} = c(O_H(-1))^{-1} \cap h^! \varepsilon_{\mathbb{P}}(F_0) \). Pulling back the short exact sequence

\[
0 \to T^*_Q(\mathbb{P} \times \mathbb{P}^\vee) \to T^*\mathbb{P} \times_{\mathbb{P}^\vee} Q \to T^*(Q/\mathbb{P}) \to 0
\]

by \( H \hookrightarrow Q \), we have

\[
c(T^*(Q/\mathbb{P}) \times H) = c(T^*_Q(\mathbb{P} \times \mathbb{P}^\vee) \times Q H)^{-1} = c(O_H(-1))^{-1}.
\]

Hence we have a commutative diagram

\[
\begin{array}{ccc}
\text{CH}_n(\mathbb{P}) & \xrightarrow{h^!} & \text{CH}_n(H) \\
\downarrow & & \downarrow \text{deg}(\varepsilon(O(-1))^{-1} \cap -) \\
\text{CH}_n(\mathbb{P}^\vee) & \xrightarrow{i^1} & \mathbb{Z}.
\end{array}
\]

Thus we get

\[
\begin{align*}
\text{pr}_2 L(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}) F_0 &= \text{pr}_2 L(cc_{\mathbb{P}}(F_0)), L(\chi_{\text{cyc}(h^* F_0)} \varepsilon_{\mathbb{P}}(F_0))) \\
&= (\text{deg}(c(O(-1))^{-1} \cap h^! cc_{\mathbb{P}} F_0), \text{deg}(c(O(-1))^{-1} \cap \chi_{\text{cyc}(h^* F_0)} h^! \varepsilon_{\mathbb{P}}(F_0))) \\
&= (-\text{deg} cc_H(h^* F_0), \text{deg}(\chi_{\text{cyc}(h^* F_0)} \cdot \chi_{\text{cyc}(h^* F_0)} \cdot \varepsilon H(h^* F_0)^{-1})) \\
&= (-\chi(h^* F_0), \chi_{\text{cyc}(h^* F_0)} \cdot \varepsilon(H, h^* F_0)).
\end{align*}
\]

On the other hand, we have

\[
\text{pr}_2(\text{cc}_{\mathbb{P}^\vee}, \varepsilon_{\mathbb{P}^\vee}) R F_0 = ((-1)^n\text{rk}^o R F_0, \text{det}(R F_0)^{n(-1)^n} \cdot \chi_{\text{cyc}}^{(-1)^{n+1} \text{rk}^o R F_0}).
\]
Hence the assertion 1 follows.

We show that the assertion 1 for \( n \geq 2 \) implies the assertion 2 for \( n \). By the commutative diagrams \((5.5)\), the endomorphism \( R^\nu R \) of \( \widetilde{K}(\mathbb{P}) \) preserves the kernel \( \widetilde{K}(\mathbb{P})^0 \) of \((cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}): \widetilde{K}(\mathbb{P}) \to (\mathbb{Z} \oplus \Theta_k) \otimes \text{CH}_*(\mathbb{P})\). Take an element \( \mathcal{F}_0 \in \widetilde{K}(\mathbb{P})^0 \). There is an element \( G_0 \in \tilde{K}(\text{Spec}(k)) \) such that \( a^* G_0 = R^\nu R \mathcal{F}_0 - \mathcal{F}_0 \), where \( a: \mathbb{P} \to \text{Spec}(k) \) is the structure morphism. Since \( cc_{\mathbb{P}}(a^* G_0) = 0 \) and \( \varepsilon_{\mathbb{P}}(a^* G_0) = 1 \), we know that \( \text{rk} G_0 = 0 \) and \( \text{det}(G_0) = 1 \). Hence we get
\[
(\chi, \varepsilon^{-1}) R^\nu R \mathcal{F}_0 = (\chi, \varepsilon^{-1}) \mathcal{F}_0,
\]
which is equivalent to \((n^2 \chi(\mathcal{F}_0), \varepsilon(\mathcal{F}_0, -n^2)) = (\chi(\mathcal{F}_0), \varepsilon(\mathcal{F}_0, -1))\) by Lemma 5.4.2. This means that \( \mathcal{F}_0 \) is contained in the kernel of \( (\chi, \varepsilon^{-1}) \). Let \( \Theta_k' \) be the subgroup of \( \Theta_k \) consisting of homomorphisms which factors through a morphism \( G_{k_1} \to G_{k_1} \) where \( k_1 \) is a finitely generated subfield of \( k \). Since the coefficients of the images of \( \varepsilon_{\mathbb{P}} \) are contained in \( \Theta_k' \), we may replace \( \Theta_k \) by \( \Theta_k' \). Then, by Lemma 4.3.2, the cokernel of \((cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}})\) is torsion. Thus there is a group homomorphism \( \text{deg}' : (\mathbb{Z} \oplus \Theta_k') \otimes \text{CH}_*(\mathbb{P}) \to \mathbb{Q} \oplus \Theta_k \) which makes the diagram \( (5.6) \) replaced \( \text{deg} \) by \( \text{deg}' \) commutative. We need to show the equality \( \text{deg} = \text{deg}' \). Let \( \mathbb{P}^a \) be a linear subspace of \( \mathbb{P} \) \((0 \leq a \leq n)\) and \( L_0 \) be a geometrically constant sheaf of rank 1 belonging to \( D(\mathbb{P}^a) \). We have
\[
\text{deg}(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}) L_0 = (\chi, \varepsilon^{-1}) L_0 = \text{deg}'(cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}) L_0,
\]
which shows the assertion since \((cc_{\mathbb{P}}, \varepsilon_{\mathbb{P}}) L_0 \) generates \((\mathbb{Z} \oplus \Theta_k') \otimes \text{CH}_*(\mathbb{P})\) (cf. 30 Lemma 7.10.1)).

\[\Box\]

**Corollary 5.6.** Let \( \mathcal{F}_0 \in \widetilde{D}(\mathbb{P}) \) be a constructible complex of \( \mathbb{Z}_\ell \)-sheaves on \( \mathbb{P} = \mathbb{P}^n \). Then, for the Radon transform \( R^\nu \mathcal{F}_0 \), we have
\[
(5.9) \quad E( R \mathcal{F}_0 ) = L( E( \mathcal{F}_0 )(1 - \frac{n}{2})).
\]

**Proof.** Except for the coefficient of the 0-section, it is proved in Corollary 14.23. Since the coefficient of the 0-section is given by \( \text{pr}_2 : \Theta_k \otimes \text{CH}_*(\mathbb{P}^\nu) \to \Theta_k \), it follows from Proposition 5.5.1.

\[\Box\]

Here is the product formula of the global epsilon factors.

**Theorem 5.7.** Let \( X \) be a projective smooth variety over \( k \). Then, for \( \mathcal{F}_0 \in \widetilde{D}(X) \), we have
\[
(5.10) \quad \text{det}(R \Theta_k(\mathcal{F}_0)) = (\mathcal{E}(\mathcal{F}_0), T^* X)_{T^* X}
\]
as elements of \( \Theta_k \).

**Proof.** Since \( X \) is assumed projective, it follows from Lemma 4.13.4 and Proposition 5.5.2.

\[\Box\]

**Corollary 5.8.** Let \( X \) be a projective smooth variety over a finite field \( \mathbb{F}_q \). Let \((K, | \cdot |)\) be a valuation field and \( \iota : \overline{\mathbb{Q}}_\ell \to K \) be a field homomorphism.
Let \( \mathcal{F}_0 \) be an element of \( D^b_c(X, \mathbb{Z}_\ell) \) and \( \mathcal{E}(\mathcal{F}_0) = \sum \beta_a \otimes [C_a] \in \mathbb{Z}_\ell^\times / \mu \otimes \mathbb{Z}_\ell(T^*X) \) be the epsilon cycle. Here we identify \( \Theta_{\mathbb{F}_q} \) and \( \mathbb{Z}_\ell^\times / \mu \) via \( \xi \mapsto \xi(\text{Frob}_q) \). We have a product formula

\[
|\iota(\varepsilon(X, \mathcal{F}_0))| = \prod_a |\iota(\beta_a)|^{-\deg(C_a, T^*_X)_{T^*X}}
\]

of the absolute value of \( \iota(\varepsilon(X, \mathcal{F}_0)) \).

\[ \square \]

**Example 5.9.** Let \( k = \mathbb{F}_q \) be a finite field with \( q \) elements. Let \( X \) be a projective smooth scheme over \( k \). Let \( \mathcal{F}_0 \in D_c^b(X, \mathbb{Z}_\ell) \) be a constructible complex on \( X \).

1. Assume that \( \mathcal{F} = \mathcal{F}_0 \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \) is \( \iota \)-pure of \( \iota \)-weight 0 ([27, II.12.7]) for an isomorphism \( \iota : \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C} \) of fields. Then, we know that the absolute values \( |\iota(\alpha)| \) of the eigenvalues \( \alpha \) of the geometric Frobenius on \( H^i(X_{\overline{\mathbb{F}_q}}, \mathcal{F}) \) equal to \( q^\frac{i}{2} \). Hence the product formula \((5.11)\) gives an expression of the weighted Euler-Poincaré characteristic \( \frac{1}{2} \sum_i (-1)^i \cdot \dim H^i(X_{\overline{\mathbb{F}_q}}, \mathcal{F}) \) as the intersection number \( (\log_q |\mathcal{E}(\mathcal{F}_0)|, T^*_X)^{T^*_X} \).

2. Let \( \iota : \overline{\mathbb{Q}_\ell} \rightarrow \overline{\mathbb{Q}_p} \) be an isomorphism of fields. Then, the product formula \((5.11)\) gives an expression of the \( p \)-adic valuation of the global epsilon factor \( \varepsilon(X, \mathcal{F}) \) using that of local epsilon factors. The \( p \)-adic valuation of the local epsilon factors of tamely ramified representations can be computed by Stickelberger’s theorem ([39, Proposition 6.13]), which is suggested to the author by N. Katz.

## 5.2 An axiomatic description of epsilon cycles

We give an axiomatic description of epsilon cycles. A similar description of characteristic cycles is considered in [29, Proposition 8].

**Theorem 5.10.** Let \( k \) be a perfect field of characteristic \( p \neq \ell \). There exists a unique way to attach, for pairs \( (X, \mathcal{F}_0) \) where \( X \) is a smooth scheme of finite type over \( k \) and \( \mathcal{F}_0 \in \hat{D}(X) \), with a cycle \( \mathcal{E}(\mathcal{F}_0) = \sum \xi_a \otimes [C_a] \) with \( \Theta_k \)-coefficient and supported on the singular support \( SS(\mathcal{F}_0) \) which should satisfy the following axioms.

1. **(Normalization)** Let \( X = \text{Spec}(k') \) be the spectrum of a finite extension \( k' \) of \( k \). Then, we have

\[
\mathcal{E}(\mathcal{F}_0) = (\det(\mathcal{F}_0) \circ \text{tr}_{k'/k})^{\frac{1}{\deg(k'/k)}} \otimes [T^*_XX].
\]

2. **(Tate Twist)** We have

\[
\mathcal{E}(\mathcal{F}_0^{(1/2)}) = \chi_{\text{CC}(\mathcal{F}_0)}^{1/2} \cdot \mathcal{E}(\mathcal{F}_0).
\]

For a half integer \( r \in \frac{1}{2}\mathbb{Z}, \) we denote \( \mathcal{E}(\mathcal{F}_0)(r) := \mathcal{E}(\mathcal{F}_0(r)). \)

3. **(Multiplicativity)** For a distinguished triangle

\[
\mathcal{F}_0' \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_0'' \rightarrow,
\]

we have \( \mathcal{E}(\mathcal{F}_0) = \mathcal{E}(\mathcal{F}_0') \cdot \mathcal{E}(\mathcal{F}_0''). \)
4. (Closed Immersion) For a closed immersion $i: X \to P$ of smooth $k$-schemes of finite type and $\mathcal{F}_0 \in \check{D}(X)$, we have $\mathcal{E}(i_*\mathcal{F}_0) = i_!\mathcal{E}(\mathcal{F}_0)$.

5. (Pull-Back) For a properly $SS(\mathcal{F}_0)$-transversal morphism $h: W \to X$ from a smooth $k$-scheme $W$ of finite type, we have

$$\mathcal{E}(h^*\mathcal{F}_0) = h^!(\mathcal{E}(\mathcal{F}_0)(\frac{\dim X - \dim W}{2})).$$

Here $\dim X$ and $\dim W$ are the locally constant function on $X$ and $W$.

6. (Radon Transform) For a constructible complex $\mathcal{F}_0 \in \check{D}(\mathbb{P})$ on a projective space $\mathbb{P} = \mathbb{P}^n$, we have

$$\mathcal{E}(R\mathcal{F}_0) = L(\mathcal{E}(\mathcal{F}_0)(\frac{1 - n}{2})).$$

7. (Same Monodromy) Let $X$ (resp. $X'$) be a smooth curve over $k$ and $x$ (resp. $x'$) be a closed point of $X$ (resp. $X'$). Let $\mathcal{F}_0$ (resp. $\mathcal{F}_0'$) be an element of $\check{D}(X)$ (resp. $\check{D}(X')$). Assume that there exists an isomorphism $f: X_{(x)} \cong X'_{(x')}$ of $k$-schemes between the henselizations such that the complexes $\mathcal{F}_0|_{X_{(x)}}$ and $f^*\mathcal{F}_0'|_{X'_{(x')}}$ are isomorphic. Then, the coefficient of $[T^*_xX]$ in $\mathcal{E}(\mathcal{F}_0)$ coincides with that of $[T^*_xX']$ in $\mathcal{E}(\mathcal{F}_0')$.

To prove the theorem, we prepare some lemmas.

**Lemma 5.11.** Let $\mathcal{E}(-)$ be an assignment as in Theorem 5.17 satisfying the axioms there. Let $X$ be a smooth curve of finite type over $k$ and $x \in X$ be a closed point. Denote by $U$ the complement of $x$ in $X$. Let $\mathcal{F}_0 \in \check{D}(U)$ be a smooth $\mathbb{Z}_l$-sheaf on $U$. Assume that $\mathcal{F} = \mathcal{F}_0 \otimes \mathbb{Q}_l$ has a unipotent monodromy at $x$. Then, the coefficient of $[T^*_xX]$ in $\mathcal{E}(j_!\mathcal{F}_0)$ equals to $(\det(\mathcal{F})^{-1} \circ \text{tr}_{k(x)/k})_{\text{fund}(x)}$ where $j: U \to X$ is the immersion. Note that we can extend $\det(\mathcal{F})$ to $X$ smoothly since this is unramified at $x$.

**Proof.** Let $k'$ be the residue field at $x$. We regard $\mathbb{A}^1_{k'}$ as a smooth $k$-scheme. Fix an isomorphism $f: X_{(x)} \cong \mathbb{A}^1_{k',(0)}$ of $k$-schemes. We claim that there exists a smooth $\mathbb{Z}_l$-sheaf $\mathcal{G}_0$ on $\mathbb{G}_{m,k'}$ such that $\mathcal{F}_0|_{\eta_x}$ and $f^*\mathcal{G}_0|_{\eta_0}$ are isomorphic and $\mathcal{G}_0$ is tamely ramified at $\infty$, where $\eta_x$ and $\eta_0$ are the generic points of $X_{(x)}$ and $\mathbb{A}^1_{k',(0)}$. When $p > 0$, this is proved in [19 THEOREM 1.5.6]. When $p = 0$, this follows since the fundamental group of $\eta_0$ is isomorphic to that of $\mathbb{G}_{m,k'}$. Since the monodromy of $\mathcal{G}$ at 0 is unipotent, the semi-simplification of $\mathcal{G}$ is unramified at 0 and $\infty$. Hence the assertion follows from the axioms (II), (3), (4), and (7).

**Lemma 5.12.** Let $\mathcal{E}(-)$ be an assignment as in Theorem 5.17 satisfying the axioms there. Let $X$ be a projective smooth scheme over $k$. Let $\mathcal{F}_0$ be an element of $\check{D}(X)$. Then, we have an equality

$$\det(R\Gamma(X, \mathcal{F})) = (\mathcal{E}(\mathcal{F}_0), T^*_X X)_{T^*_X}$$

of elements of $\Theta_k$. 

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Proof. Since $X$ is projective, we may assume that $X = \mathbb{P} = \mathbb{P}^n(n \geq 2)$ by the axiom (1). Consider the universal hyperplane section $\mathbb{P} \leftarrow Q \mathbb{P}^v \rightarrow \mathbb{P}^v$. Let $R^v = R_p^v \mathbb{P}^v[n-1](n-1)$ be the inverse Radon transform. By [21, IV. Lemma 1.4], we have a distinguished triangle

$$\mathcal{F}_0 \rightarrow R^v R\mathcal{F}_0 \rightarrow \oplus_{i=1}^{n-1} R\Gamma(\mathbb{P}^n_i, \mathcal{F}_0)|2i| \rightarrow,$$

where $R\Gamma(\mathbb{P}^n_i, \mathcal{F}_0)$ is regarded as a complex of geometrically constant sheaves. By the axioms (1), (3), and (5), we have

$$\mathcal{E}(R^v R\mathcal{F}_0)|\mathcal{E}(\mathcal{F}_0))^{-1} = \det(R\Gamma(\mathbb{P}^n_i, \mathcal{F}))(^{-1})^{n(n-1)} \otimes [T^*_p \mathbb{P}].$$

On the other hand, by the axioms (3) and (2), the left hand side of (5.12) equals to

$$L^v(\mathcal{E}(R\mathcal{F}_0(n-1 + 1 - n - \frac{n}{2})))\mathcal{E}(\mathcal{F}_0))^{-1} = L^v L\mathcal{E}(\mathcal{F}_0)\mathcal{E}(\mathcal{F}_0)^{-1} = (\mathcal{E}(\mathcal{F}_0), T^*_p \mathbb{P})^{(n-1)} \otimes [T^*_p \mathbb{P}].$$

Since $n \geq 2$, we have the assertion.

(Proof of Theorem 5.10)

First we show the uniqueness. Let $\mathcal{E}(\mathcal{F}_0)$ be an assignment which satisfies the conditions in the statement of the theorem. Let $X$ and $\mathcal{F}_0$ be as in the theorem. We need to determine the coefficients of $\mathcal{E}(\mathcal{F}_0)$ uniquely from the axioms. By the axiom (5), we may assume that $X$ is affine, and by the axiom (4) and (5), we may assume that $X$ is projective and fix an immersion $i: X \hookrightarrow \mathbb{P} = \mathbb{P}^n$.

Composing $i$ and the Veronese embedding $\mathbb{P} \hookrightarrow \mathbb{P}'$ of deg $\geq 3$ if necessary, for a finite extension $k'$ of $k$, we find a line $L \rightarrow \mathbb{P}'|_k$ such that the pair $(f, \pi)$ in the diagram (2.8) after replacing $X, X \times_p Q, \mathbb{P}^v$ by the base changes by $k \rightarrow k'$ is a good pencil (Definition 2.15). In the sequel, we regard $k'$-schemes also as $k$-schemes. Let $C_a$ be an irreducible component of $SS(\mathcal{F}_0|_x)$. By the definition of a good pencil, there exists a closed point $x \in X_{k', L}$ of the blow-up $X_{k', L}$ of $X_k$ such that $x$ is the unique isolated $SS(\mathcal{F}_0|_x)$-characteristic point on the fiber $f^{-1}(f(x))$ at which $df$ only meets $C_a$, and $x$ is not contained in the exceptional locus of the blow-up $\pi$. By the axiom (6), we have $\mathcal{E}(R(i_* \mathcal{F}_0)) = L(\mathcal{E}(i_* \mathcal{F}_0)(\frac{n}{2} - 1))$. Let $\mathbb{P}'_{k, L}$ be the blow-up of $\mathbb{P}'$ along the axis $A_L$ defined by $L$. Since $\pi: X_{k', L} \rightarrow X$ is $SS(\mathcal{F}_0)$-transversal, $\mathbb{P}_{k', L} \rightarrow \mathbb{P}$ is $SS(i_* \mathcal{F}_0)$-transversal. Let $i': L \rightarrow \mathbb{P}^v$ be the composition $L \hookrightarrow \mathbb{P}'_{k, L} \rightarrow \mathbb{P}^v$. Since $SS(R(i_* \mathcal{F}_0)) \subset LSS(i_* \mathcal{F}_0) \cup T^*_p \mathbb{P}^v$, applying [30] Lemma 3.9.3 to the cartesian diagram

$$\begin{array}{ccc}
\mathbb{P}'_{k, L} & \xrightarrow{i'} & L \\
\downarrow & & \downarrow i' \\
Q & \xrightarrow{\mathcal{E}(R(i_* \mathcal{F}_0))} & \mathbb{P}^v,
\end{array}$$

$i'$ is properly $SS(R(i_* \mathcal{F}_0))$-transversal. Hence we have

$$\mathcal{E}(i'^* R(i_* \mathcal{F}_0)) = i'^* (\mathcal{E}(R(i_* \mathcal{F}_0))(\frac{n}{2} - 1)) = i'^* L\mathcal{E}(i_* \mathcal{F}_0).$$

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Thus we may assume that \( X \) is a projective smooth curve.

By the axioms (1) and (4), we may assume \( F_0 = j_! G_0 \) where \( j: U \to X \) is an open immersion from an open dense subset \( U \) and \( G_0 \) is a smooth \( \mathbb{Z}_\ell \)-sheaf on \( U \). The coefficient of \( [T^*_X X] \) can be determined by the axioms (1) and (5). Let \( x \in X \) be a closed point not contained in \( U \). By weak approximation, we can find a finite morphism \( f: X' \to X \) from a projective smooth curve \( X' \) such that \( f \) is étale around \( f^{-1}(x) \) and \( f^* G \) has unipotent monodromy at points of \( X' \setminus f^{-1}(U \cup \{x\}) \). Then, we can determine the coefficient of \( [T^*_X X] \) by the axiom (5), Lemma 5.11, and Lemma 5.12.

We show that epsilon cycles constructed in Theorem 4.9 satisfy the axioms. The axioms (3) and (7) follow from the construction. The others are proved in Lemma 4.13, Corollary 4.17, Theorem 4.24, and Corollary 5.6.

**Remark 5.13.** According to the proof, we can replace the axiom (7) by Lemma 5.11.

### 6 Appendix : Complements on \( \ell \)-adic formalism

In this appendix, we review the \( \ell \)-adic formalism on a noetherian topos and give the reduction step from characteristic 0 to positive characteristic. To simplify the argument, we restrict the construction to bounded complexes.

Let \( T \) be a topos. We fix a complete discrete valuation ring \((R, m)\) and a uniformizer \( \varpi \) of \( R \). We define a category \( T^{\text{Top}} \) as follows. The objects are projective systems \((M_n, \varphi_n)_{n \in \mathbb{N}}\) indexed by \( \mathbb{N} \) where \( M_n \) are objects of \( T \) and \( \varphi_n: M_{n+1} \to M_n \) are morphisms of \( T \), called transition maps. The morphisms are families of morphisms \( M_n \to M'_n \) compatible with the transition maps. The category \( T^{\text{Top}} \) is a topos. Let

\[
\pi: T^{\text{Top}} \to T
\]

be the morphism of topoi defined by \( \pi_*(M_n, \varphi_n)_{n \in \mathbb{N}} = \lim_{\to} M_n \). The left adjoint is identified with the functor \( \pi^{-1} M = (M, \text{id})_{n \in \mathbb{N}} \). Denote \( R_n := R/m^{n+1} \). Let \( R_\bullet := (R_n, \text{proj.})_{n \in \mathbb{N}} \) be the ring object of \( T^{\text{Top}} \), where the transition maps are the natural projection \( R_{n+1} \to R_n \). We have a morphism of ringed topoi

\[
\pi: (T^{\text{Top}}, R_\bullet) \to (T, R).
\]

We denote by \( \pi^* \) the left adjoint of \( (6.2) \).

**Definition 6.1.**

1. We say that a commutative group object \((M_n, \varphi_n)_{n \in \mathbb{N}}\) is essentially zero if, for every \( n \in \mathbb{N} \), there exists \( m \geq n \) such that the transition map \( M_m \to M_n \) is zero.

2. We say that a complex \( K \in D(T^{\text{Top}}, \mathbb{Z}) \) of sheaves of abelian groups is essentially zero if each cohomology of \( K \) is essentially zero (in [13], it is called ML-zero).

3. We say that a morphism in \( D(T^{\text{Top}}, \mathbb{Z}) \) is an essential isomorphism if the mapping cone is essentially zero.

4. We say that a complex \( K \in D(T^{\text{Top}}, \mathbb{Z}) \) is essentially constant if there exist complexes \( L \in D(T^{\text{Top}}, \mathbb{Z}) \) and \( M \in D(T, \mathbb{Z}) \), and morphisms

\[
K \leftarrow L \to \pi^{-1} M
\]
Lemma 6.2. 1. Let \( M \in D^b(T, \mathbb{Z}) \) be a bounded complex. Then, the canonical morphism \( M \to R\pi_*\pi^{-1}M \) is an isomorphism.

2. (\cite{G} Lemma (1.11)) Let \( K \in D^b(T^{op}, \mathbb{Z}) \) be an essentially zero complex. Then, we have \( R\pi_*K = 0 \).

3. (\cite{G} Lemma 1.3.iv)] \( K \in D^b(T^{op}, \mathbb{Z}) \) be a bounded complex. If \( K \) is essentially constant, then \( R\pi_*K \) is bounded and the canonical morphism \( \pi^{-1}R\pi_*K \to K \) is an essential isomorphism.

4. Let \( M \) be a sheaf of \( \mathcal{R}_0 \)-modules on \( T \). Then, the morphisms \( \pi^{-1}M \to L\pi^*M \) and \( L\pi^*M \to \pi^{-1}M \) are essential isomorphisms. Here the first one is \( \pi^{-1}M \cong \pi^{-1}R \otimes^L_{\pi^{-1}R} \pi^{-1}M \to R_\pi \otimes^L_{\pi^{-1}R} \pi^{-1}M = L\pi^*M \) and the second one is \( L\pi^*M \to H^0(L\pi^*M) \cong \pi^{-1}M \).

5. Let \( K, L \in D^-(T^{op}, R_* \pi) \) be bounded above complexes. If either \( K \) or \( L \) is essentially zero, \( L \otimes^{L_\pi} R_* \pi \) is also essentially zero.

6. Let \( C \in D^-(T^{op}, R_* \pi) \) be a bounded above complex. If \( R_0 \otimes^{L_\pi} R_* \pi \) is essentially zero, \( L \otimes^{L_\pi} R_* \pi \) is acyclic, so is \( C \).

Proof. For a sheaf \( N = (N_n) \) of abelian groups on \( T^{op} \) and an object \( U \in T \), we have a short exact sequence \cite{G} Proposition (1.6)]

\[
0 \to R^1 \lim_n H^1(U, N_n) \to H^1(\pi^{-1}(U), N) \to \lim_n H^1(U, N_n) \to 0.
\]

1. We may assume that \( M \) is a sheaf. Applying (6.3) to \( N = \pi^{-1}M \), we know that \( H^1(\pi^{-1}(U), \pi^{-1}M) \) is isomorphic to \( H^1(U, M) \), hence the assertion.

2. We may assume that \( K = (K_n) \) is a sheaf. If \( K \) is essentially zero, so is \( (H^1(U, K_n))_n \) for \( U \in T \). The assertion follows from the exact sequence (6.3).

3. Take morphisms \( K \leftarrow L \to \pi^{-1}M \) as in Definition 6.1.4. Since \( K \) is bounded, we may assume that \( L \) and \( M \) are also bounded. The first assertion follows from 1 and 2.

To prove the second assertion, consider the following commutative diagram

\[
\begin{array}{cccccccccc}
\pi^{-1}R\pi_*K & \cong & \pi^{-1}R\pi_*L & \cong & \pi^{-1}R\pi_*\pi^{-1}M & \cong & \pi^{-1}M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \xrightarrow{id} & L & \to & \pi^{-1}M.
\end{array}
\]

By 1 and 2, the top horizontal arrows are isomorphisms. Then, the assertion follows from a diagram chasing.

4. Let \( L_1 = (R)_n = \pi^{-1}R \) and \( L_2 = (m^{n+1})_n \) be sheaves of \( \mathcal{R}_0 \)-modules on \( T^{op} \) of which the transition maps are the inclusions. We have a short exact sequence \( 0 \to \)}
$L_2 \to L_1 \to R_\bullet \to 0$, which defines an $R$-flat resolution of $R_\bullet$. Hence the mapping cone of $\pi^{-1}M \to L\pi^1M$ is isomorphic to $L_2 \otimes_R \pi^{-1}M$. Since the transition maps of $L_2 \otimes_R \pi^{-1}M$ is zero, the first morphism is an essential isomorphism. The assertion for the second one follows since the composition $\pi^{-1}M \to L\pi^1M \to \pi^{-1}M$ coincides with the identity.

5. This follows from the spectral sequence

$$E_2^{p,q} = \oplus_{i+j=q} \text{Tor}_p^R(\mathcal{H}^i(L), \mathcal{H}^j(K)) \Rightarrow \mathcal{H}^{p+q}(L \otimes_{R_\bullet} K).$$

6. Define $R_n' := R_\bullet / m^{n+1}R_\bullet$ for $n \geq 0$. The kernel of the natural surjection $R_{n+1} \to R_n$ is a sheaf of $R_0$-modules essentially isomorphic to $R_0$. If $R_0 \otimes_{R_\bullet} C$ is (resp. essentially zero) acyclic, so is $R_n' \otimes_{R_\bullet} C$ for each $n$. Thus, if $R_0 \otimes_{R_\bullet} C$ acyclic, so is $\mathcal{H}^i(i_n^* C) \cong \mathcal{H}^i(R_n' \otimes_{R_\bullet} C)$. If $R_0 \otimes_{R_\bullet} C$ is essentially zero, for each $n \geq 0$, there exists $m \geq n$ such that the transition map $\mathcal{H}^i(i_m^* (R_n' \otimes_{R_\bullet} C)) \to \mathcal{H}^i(i_n^* (R_n' \otimes_{R_\bullet} C)) = \mathcal{H}^i(i_n^* C)$ is zero. Hence $\mathcal{H}^i(i_m^* C) \to \mathcal{H}^i(i_n^* (R_n' \otimes_{R_\bullet} C)) \to \mathcal{H}^i(i_n^* C)$ is zero.

We define the notion of (normalized) $R$-complexes.

**Definition 6.3.** 1. We say that a complex $K \in D^b(T^{\text{opp}}, R_\bullet)$ is an $R$-complex if $K \otimes_{R_\bullet} L\pi^1 R_\bullet$ is essentially constant.

2. We say that a complex $K \in D^b(T^{\text{opp}}, R_\bullet)$ is a normalized $R$-complex if, for each $n \in \mathbb{N}$, the canonical map $i_{n+1}^* K \otimes_{R_{n+1}} R_n \to i_n^* K$ is an isomorphism.

**Lemma 6.4.** Let $K \in D^b(T^{\text{opp}}, R_\bullet)$ be a complex.

1. If $K$ is an $R$-complex, $L\pi^1 R_\pi K$ is bounded.

2. The following are equivalent.

   (a) The complex $K$ is a normalized $R$-complex.

   (b) The canonical morphism $L\pi^1 R_\pi K \to K$ is an isomorphism.

   (c) There is a complex $M \in D(T, R)$ such that $L\pi^1 M \cong K$.

3. Normalized $R$-complexes are $R$-complexes.

**Proof.** 1. It is enough to show that $R_0 \otimes_{R_\bullet} L\pi^1 R_\pi K$ is bounded. We have $R_0 \otimes_{R_\bullet} R_\pi K \cong R_\pi (L\pi^1 R_\pi \otimes_{R_\bullet} K)$. Since $L\pi^1 R_0 \otimes_{R_\bullet} K$ is bounded and essentially constant, the assertion follows from Lemma 6.2.3.

2. We show $(a) \Rightarrow (b)$. Let $K$ be a normalized $R$-complex. By Lemma 6.2.6, it suffices to show that $R_0 \otimes_{R_\bullet} L\pi^1 R_\pi K \to R_0 \otimes_{R_\bullet} K$ is an isomorphism. The former is isomorphic to $R_0 \otimes_{R_\bullet} \pi^{-1} R_\pi K \cong \pi^{-1} R_\pi (L\pi^1 R_0 \otimes_{R_\bullet} K) \cong \pi^{-1} R_\pi (\pi^{-1} R_0 \otimes_{R_\bullet} K)$.

Here the last isomorphism follows from Lemma 6.2.2, 4, 5, and the fact that $R_0 \otimes_{R_\bullet} K$ is bounded. Since $\pi^{-1} R_0 \otimes_{R_\bullet} K$ is constant, i.e. each cohomology is of the form $\pi^{-1} N$ for a sheaf on $T$, it follows from Lemma 6.2.1.

(b) $\Rightarrow$ (c) is obvious. For (c) $\Rightarrow$ (a), first we show that the boundedness of $R_0 \otimes_{R_\bullet} M \cong i_0^1 L\pi^1 M$ implies that the cohomology sheaves $H^i(M)$ are uniquely divisible by a
uniformizer \( \varpi \in R \) for any \( i \in \mathbb{Z} \) whose absolute value is large enough. Indeed, since \( R \xrightarrow{\varpi} R \) is a flat resolution of \( R_0 \), we have a distinguished triangle

\[
M \xrightarrow{\varpi} M \to R_0 \otimes^L_R M \to .
\]

Let \( n \) be a positive integer such that \( H^i(R_0 \otimes^L_R M) \) is zero for \( |i| \geq n \). Then the multiplication-by-\( \varpi \) map \( H^i(M) \to H^i(M) \) is an isomorphism when \( |i| \geq n + 1 \). Thus we have morphisms of complexes of sheaves of \( R \)-modules on \( T \)

\[
M \xrightarrow{\varpi} M' \xrightarrow{\beta} M''
\]

such that \( M'' \) is bounded and the mapping cones of \( \alpha, \beta \) have uniquely divisible cohomology sheaves. If \( N \in D(T, R) \) has uniquely divisible cohomology sheaves, the pull-back \( L\pi^*N \) is acyclic. Hence \( L\pi^*M \) is quasi-isomorphic to \( L\pi^*M'' \), which implies that we may assume that \( M \) is bounded by replacing \( M \) by \( M'' \). Then the assertion is clear.

The last assertion is verified in the course of the proof of (c) \( \Rightarrow \) (a).

3. Let \( M \in D^b(T, R) \) be a bounded complex. By 2, it suffices to show that \( L\pi^*M \) is an \( R \)-complex. This follows from the isomorphism \( L\pi^*(M \otimes^L_R R_0) \cong L\pi^*M \otimes^L_R L\pi^*R_0 \) and Lemma 6.2.4.

\[
\square
\]

Denote by \( \mathcal{A}, \mathcal{B} \), and \( D_{\text{norm}}(T^{\text{op}}, R_\bullet) \) the full subcategories of \( D^b(T^{\text{op}}, R_\bullet) \) consisting of \( R \)-complexes, essentially zero complexes, and normalized \( R \)-complexes respectively. Since \( \mathcal{B} \) is a thick triangulated subcategory of \( D^b(T^{\text{op}}, R_\bullet) \) and stable under the standard truncation functors, the quotient \( D^b(T^{\text{op}}, R_\bullet)/\mathcal{B} \) is a triangulated category admitting a \( t \)-structure, which is also called the standard \( t \)-structure.

Let \( D^b(T - R) \) be the quotient category \( \mathcal{A}/\mathcal{B} \). Since the subcategory of essentially constant complexes is stable under extensions and the shift functor, \( D^b(T - R) \) is a triangulated subcategory of \( D^b(T^{\text{op}}, R_\bullet)/\mathcal{B} \).

**Definition 6.5.** 1. We define a functor \( \mathcal{A} \to D_{\text{norm}}(T^{\text{op}}, R_\bullet) \) by \( K \mapsto \hat{K} := L\pi^*R\pi_*K \), which is well-defined by Lemma 6.4.1, 2. By Lemma 6.2.4, this induces a functor \( \Phi: D^b(T - R) \to D_{\text{norm}}(T^{\text{op}}, R_\bullet) \).

2. The functor \( \mathcal{A} \to D^b(T, R_n) \) sending \( K \) to \( R_n \otimes^L_R L\pi_*K \) induces a functor \( D^b(T - R) \to D^b(T, R_n) \) by Lemma 6.2.2, which we denote by \( R_n \otimes^L_R K \).

By Lemma 6.4.3, we can define the functor \( D_{\text{norm}}(T^{\text{op}}, R_\bullet) \to D^b(T - R) \) to be the composition \( D_{\text{norm}}(T^{\text{op}}, R_\bullet) \to \mathcal{A} \to \mathcal{A}/\mathcal{B} = D^b(T - R) \).

**Lemma 6.6.** The functor \( D_{\text{norm}}(T^{\text{op}}, R_\bullet) \to D^b(T - R) \) is an equivalence of categories with a quasi-inverse \( \Phi \).

**Proof.** We show that the compositions of the two functors are isomorphic to the identity functors. For a normalized \( R \)-complex \( K \), we know that \( L\pi^*R\pi_*K \to K \) is an isomorphism by Lemma 6.4.2.

Let \( K \) be an \( R \)-complex. Let \( C \) be the mapping cone of \( L\pi^*R\pi_*K \to K \). By Lemma 6.2.6, it suffices to show that \( R_0 \otimes^L_R L\pi^*R\pi_*K \to R_0 \otimes^L_R K \) is an essential isomorphism. By Lemma 6.2.4, 5, we show that \( L\pi^*R_0 \otimes^L_R L\pi^*R\pi_*K \to L\pi^*R_0 \otimes^L_R K \) is an essential
isomorphic to. The former complex is isomorphic to $L\pi^* R\pi_*(L\pi^* R_0 \otimes_{R^\bullet} K)$ and we have a commutative diagram

$$
\begin{array}{ccc}
L\pi^* R\pi_*(L\pi^* R_0 \otimes_{R^\bullet} K) & \rightarrow & L\pi^* R_0 \otimes_{R^\bullet} K \\
\uparrow & & \uparrow \\
\pi^{-1} R\pi_*(L\pi^* R_0 \otimes_{R^\bullet} K) & \rightarrow & \pi^{-1} R_0 \otimes_{R^\bullet} K
\end{array}
$$

of complexes in $D(T^{\text{op}}, \mathbb{Z})$, where the vertical arrow is induced from $\pi^{-1} R \rightarrow R^\bullet$ and the slant one is the adjunction. Since $L\pi^* R_0 \otimes_{R^\bullet} K$ is essentially constant, the slant one is an essential isomorphism and $R\pi_*(L\pi^* R_0 \otimes_{R^\bullet} K)$ is bounded by Lemma 6.2.3. Since the cohomologies of $R\pi_*(L\pi^* R_0 \otimes_{R^\bullet} K) \cong R_0 \otimes_{R^\bullet} R\pi_* K$ are sheaves of $R_0$-modules, the vertical one is an essential isomorphism by Lemma 6.2.4. The assertion follows.

Next we impose a finiteness condition on (normalized) $R$-complexes. From now on, we assume that $T$ is noetherian. Denote by $D^b_c(T, R_0)$ the full subcategory of $D^b(T, R_0)$ consisting of bounded complexes whose cohomologies are constructible sheaves.

**Definition 6.7.** 1. We denote by $D_{c, \text{norm}}(T^{\text{op}}, R^\bullet)$ the full subcategory of $D_{\text{norm}}(T^{\text{op}}, R^\bullet)$ consisting of $K \in D_{\text{norm}}(T^{\text{op}}, R^\bullet)$ such that $i_0^* K \in D^b_c(T, R_0)$.

2. We denote by $D^b_c(T, R)$ the full subcategory of $D^b(T-R)$ consisting of $K \in D^b(T-R)$ such that $R_0 \otimes_{R^\bullet} K \cong R_0 \otimes_{R^\bullet} R\pi_* K \in D^b_c(T, R_0)$. We call an element of $D^b_c(T, R)$ a constructible complex of $R$-sheaves.

**Lemma 6.8.** 1. The functor in Lemma 6.6 induces an equivalence $D_{c, \text{norm}}(T^{\text{op}}, R^\bullet) \cong D^b_c(T, R)$.

2. The full subcategory $D^b_c(T, R)$ of $D^b(T^{\text{op}}, R^\bullet)/\mathcal{B}$ is triangulated and stable under the truncation functors. The core of $D^b_c(T, R)$ is equivalent to the category of $\mathfrak{m}$-adic sheaves on $T$.

**Proof.** 1. It follows from Lemma 6.6 and an isomorphism $i_0^* K \cong R_0 \otimes_{R^\bullet} R\pi_* K$ for $K \in D_{\text{norm}}(T^{\text{op}}, R^\bullet)$.

2. Let $\mathcal{C}$ be the full subcategory of $\mathcal{A}$ consisting of $R$-complexes $K$ such that $R_0 \otimes_{R^\bullet} R\pi_* K$ is constructible. Note that the quotient $\mathcal{C}/\mathcal{A}$ is isomorphic to $D^b_c(T, R)$. We show the following two claims.

1. For an $R$-complex $K$, $K$ belongs to $\mathcal{C}$ if and only if all cohomologies of $K$ are essentially isomorphic to $\varpi$-adic sheaves.

2. $\varpi$-adic sheaves belong to $\mathcal{C}$.

Let $K$ be an $R$-complex. By Lemma 6.6, $K$ is essentially isomorphic to $L\pi^* R\pi_* K$. If $K$ belongs to $\mathcal{C}$, the cohomologies of $L\pi^* R\pi_* K$ are essentially isomorphic to $\varpi$-adic sheaves. Since $\mathcal{C}$ is stable under extensions and essential isomorphisms, it is enough to show the claim 2. Let $F = (F_\varpi)$ be a $\varpi$-adic sheaf. Since $\varpi$-adic sheaves are essentially isomorphic to extensions of $\varpi$-adic sheaves which are $\varpi$-torsion free or killed by $\varpi$, we treat the two cases separately. Assume that $F$ is torsion-free. The complex $L\pi^* R_0 \otimes_{R^\bullet} F$ is essentially isomorphic to $R_0 \otimes_{R^\bullet} F \cong (F_0)$. Hence the assertion follows in this case. Assume that $F$ is
killed by $\varpi$. Then $F_m \to F_0$ is isomorphic for $m \geq 0$. Using the flat resolution $R_\bullet \xrightarrow{\varpi} R_\bullet$ of $L\pi^*R_0$, we have $L\pi^*R_0 \otimes_{R_\bullet} F \cong [F_0 \to F]$. The proof is completed.

Let $E$ and $\mathcal{O}_E$ be a finite extension of $\mathbb{Q}_\ell$ and its ring of integers. Define a category $D_c^b(T, E)$ by $D_c^b(T, \mathcal{O}_E) \otimes_{\mathcal{O}_E} E$, i.e. the objects are the same as $D_c^b(T, \mathcal{O}_E)$ and the morphisms of multiplication by $\ell$ are inverted. We define a category $D_c^b(T, \mathbb{Z}_\ell)$ (resp. $D_c^b(T, \mathbb{Q}_\ell)$) by the 2-colimit $\lim_{\to E} D_c^b(T, \mathcal{O}_E)$ (resp. $\lim_{\to E} D_c^b(T, E)$) where $E$ runs through the finite subextensions of $\mathbb{Q}_\ell/\mathbb{Q}_\ell$.

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References

[1] Beilinson, A.: Constructible sheaves are holonomic, Selecta Math. (N.S.) 22 (2016), no. 4, 17971819.

[2] Deligne, P.: Applications de la formule des traces aux sommes trigonométriques, SGA 4$\frac{1}{2}$, Cohomologie étale, Lecture Notes in Mathematics, 569. Springer-Verlag, Berlin, 1977.

[3] Deligne, P.: Cohomologie à supports propres, SGA 4 Exposé XVII, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 305, 1973.

[4] Deligne, P.: Le formalisme des cycles évanescents, SGA 7 II Exposé XIII, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Mathematics Volume 340, 1973.

[5] Deligne, P.: Les constantes des quations fonctionnelles des fonctions L, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501597.

[6] Deligne, P.: La conjecture de Weil. II, Inst. Hautes tudes Sci. Publ. Math. No. 52 (1980), 137252.

[7] Ekedahl, T.: On the adic formalism, The Grothendieck Festschrift, Vol. II, 197218, Progr. Math., 87, Birkhuser Boston, Boston, MA, 1990.

[8] Fujiwara, K.: Independence of $\ell$ for intersection cohomology (after Gabber), Algebraic geometry 2000, Azumino (Hotaka), 145151, Adv. Stud. Pure Math., 36, Math. Soc. Japan, Tokyo, 2002.
[9] Grothendieck, A.: Propreté cohomologique des faisceaux d’ensembles et des faisceaux de groupes non commutatifs, SGA 1 Exposé XIII, Revetements Étales et Groupe Fondamental, Lecture Notes in Mathematics Volume 224, 1971.

[10] Grothendieck, A., Verdier, J.-L.: Topos, SGA 4 Exposé IV, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 269, 1972.

[11] Grothendieck, A., Verdier, J.-L.: Conditions de finitude. Topos et sites fibrés. Applications aux questions de passage à la limite, SGA 4 Exposé VI, Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics Volume 270, 1972.

[12] Guignard, Q.: Geometric local epsilon factors, arXiv:1902.06523.

[13] Hu, H., Yang, E.: Relative singular support and the semi-continuity of characteristic cycles for tale sheaves, Selecta Math. (N.S.) 24 (2018), no. 3, 22352273.

[14] Illusie, L.: Appendice à Théorèmes de finitude en cohomologie ℓ-adique, Cohomologie étale SGA 4 1/2, Springer Lecture Notes in Math. 569 (1977).

[15] Illusie, L.: Around the ThomSebastiani theorem, with an appendix by Weizhe Zheng, Manuscripta Math. 152 (2017), no. 1-2, 61125.

[16] Illusie, L.: Autour du théorème de monodromie locale, Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque No. 223 (1994), 957.

[17] Illusie, L.: Expos XI. Produits orientés, Travaux de Gabber sur l’uniformisation locale et la cohomologie tale des schmas quasi-excellents. Astérisque No. 363-364 (2014), 213234.

[18] Jannsen, U.: Continuous étale cohomology, Math. Ann. 280 (1988), no. 2, 207245.

[19] Katz, N.-M.: Local-to-global extensions of representations of fundamental groups, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 4, 69106.

[20] Katz, N.-M., Lang, S.: Finiteness theorems in geometric classfield theory, Enseign. Math. (2) 27 (1981), no. 3-4, 285319 (1982).

[21] Kiehl, R., Weissauer, R.: Weil Conjectures, Perverse Sheaves and ladic Fourier Transform, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 42, Springer-Verlag, Berlin, 2001.

[22] Laumon, G.: Semi-continuité du conducteur de Swan (d’après P. Deligne), Séminaire E.N.S. (1978-1979) Exposé 9, Astérisque 82-83, 173-219 (1981).

[23] Laumon, G.: Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil, Inst. Hautes tudes Sci. Publ. Math. No. 65 (1987), 131210.

[24] Lu, Q., Zheng, W.: Duality and nearby cycles over general bases, arXiv:1712.10216.

[25] Orgogozo, F.: Modifications et cycles proches sur une base générale, Int. Math. Res. Not. 2006, 1-38.
[26] Saito, T.: Characteristic cycle of the external product of constructible sheaves, Manuscripta Math. 154 (2017), no. 1-2, 112.

[27] Saito, T.: Correction to: The characteristic cycle and the singular support of a constructible sheaf, Inventiones Mathematicae, 216(3) (2019), 1005-1006.

[28] Saito, T.: Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies, J. Algebraic Geom. 3 (1994), no. 3, 411434.

[29] Saito, T.: On the proper push-forward of the characteristic cycle of a constructible sheaf, Algebraic geometry: Salt Lake City 2015, 485494, Proc. Sympos. Pure Math., 97.2, Amer. Math. Soc., Providence, RI, 2018.

[30] Saito, T.: The characteristic cycle and the singular support of a constructible sheaf, Inventiones Mathematicae, 207(2) (2017), 597-695.

[31] Saito, T.: Wild ramification and the cotangent bundle, J. Algebraic Geom. 26 (2017), no. 3, 399473.

[32] Saito, T., Yatagawa, Y.: Wild ramification determines the characteristic cycle, Ann. Sci. c. Norm. Supr. (4) 50 (2017), no. 4, 10651079.

[33] Takeuchi, D.: On continuity of local epsilon factors of ℓ-adic sheaves, preprint.

[34] Umezaki, N., Yang, E., Zhao, Y.: Characteristic class and the epsilon factor of an tame sheaf, [arXiv:1701.02841](https://arxiv.org/abs/1701.02841).

[35] Washington, L.: Introduction to cyclotomic fields, Second edition Graduate Texts in Mathematics, 83. Springer-Verlag, New York, 1997.

[36] Yasuda, S.: Local ε₀-characters in torsion rings, J. Thor. Nombres Bordeaux 19 (2007), no. 3, 763797.

[37] Yasuda, S.: Local constants in torsion rings, J. Math. Sci. Univ. Tokyo 16 (2009), no. 2, 125197.

[38] Yasuda, S.: The product formula for local constants in torsion rings, J. Math. Sci. Univ. Tokyo 16 (2009), no. 2, 199230.