ON PRIMITIVE IDEALS

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ABSTRACT. We extend two well-known results on primitive ideals in enveloping algebras of semisimple Lie algebras, the Irreducibility theorem for associated varieties and Duflo theorem on primitive ideals, to much wider classes of algebras. Our general version of the Irreducibility Theorem says that if \( A \) is a positively filtered associative algebra such that \( \text{gr} A \) is a commutative Poisson algebra with \( \text{finitely many} \) symplectic leaves, then the associated variety of any primitive ideal in \( A \) is the closure of a single connected symplectic leaf. Our general version of the Duflo theorem says that if \( A \) is an algebra with a “triangular structure”, then any primitive ideal in \( A \) is the annihilator of a simple highest weight module. Applications to symplectic reflection algebras and Cherednik algebras are discussed.

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1. INTRODUCTION

1.1. Let \( A \) be an associative, not necessarily commutative, algebra. Recall that a two-sided ideal \( I \subset A \) is called primitive if it is the annihilator of a simple left \( A \)-module. Primitive ideals often play a role similar to the one played by maximal ideals in the case of commutative algebras. The set of all primitive ideals in \( A \), equipped with Jacobson topology, is called the primitive spectrum of \( A \). It is a fundamental invariant of \( A \) analogous to the affine scheme \( \text{Spec} A \) in the commutative case.

A complete understanding of the primitive spectrum is only available in some “very special” cases. Our motivation in this paper comes from one such case where \( A = \mathcal{U}_g \) is the universal enveloping algebra of a complex semisimple Lie algebra \( g \). Classification of primitive ideals in \( \mathcal{U}_g \) was one of the central themes of representation theory during the 70–80’s, and it is nowadays well understood, cf. [J1], [J2] and references therein. Yet, details of the classification and the methods involved still appear to be quite complicated.

Our goal is to show that, quite surprisingly, two very important results on primitive ideals in \( \mathcal{U}_g \) have natural generalizations to much wider classes of algebras. The first result is the Duflo Theorem [Du], stating that any primitive ideal in \( \mathcal{U}_g \) is the annihilator of a simple highest weight module. We generalize this to arbitrary algebras \( A \) that have a triangular structure, i.e., a pair of “opposite” subalgebras \( A^\pm \) such that \( A \) is a finite type \( (A^+,-A^-) \)-bimodule, satisfying certain additional technical conditions, see \( \S 2 \). Our approach is based on the notion of Jacquet functor borrowed from representation theory, and on Gabber’s Separation Theorem [Ga].
The second result is a generalization of the *Irreducibility Theorem* saying that the associated variety of a primitive ideal in $U\mathfrak{g}$ is irreducible, specifically, it is the closure of a nilpotent conjugacy class in $\mathfrak{g}$. The latter theorem was first partially proved (by a case-by-case argument) in [BoBr], and in a more conceptual way in [KT] and [J2] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others. Our argument is an adaptation of a much more direct proof discovered subsequently by Vogan [Vo], combined with some recent results by Brown–Gordon [BrGo].

1.2. We got involved in these matters because of our interest in primitive ideals in symplectic reflection algebras, a new important class of associative algebras introduced recently in [EG] in connection with various works in combinatorics, completely integrable systems, and generalized McKay correspondence for orbifolds $\mathbb{C}^n/\Gamma$. In §6, we will describe all possible subvarieties in $\mathbb{C}^n/\Gamma$ that may arise as associated varieties of primitive ideals in a symplectic reflection algebra.

An important special class of symplectic reflection algebras comes from Coxeter groups of finite root systems. The algebras in question have been called *rational Cherednik algebras*, see [EG], [DO], [BEG], since they appeared as a “rational” degeneration of *double affine Hecke algebras* introduced by Cherednik [Ch]. Thus, the latter may (and should) be thought of as a deformation of the former. From this point of view, representation theory of the rational Cherednik algebra is perhaps “more fundamental” than (or at least should be studied before) that of the double affine Hecke algebra in the same sense as the representation theory of semisimple Lie algebras is “more fundamental” than that of the corresponding quantum groups.

“Experimental evidence” suggests that although rational Cherednik algebras are quite far from enveloping algebras, their representation theory shares many features of the representation theory of semisimple Lie algebras, see [BEG]. Thus, classification of primitive ideals in Cherednik algebras seems to be a challenging problem. The results of the present paper provide a first step in this direction.

In more detail, let $H_c$ be the Cherednik algebra associated to the Weyl group $W$ of a root system in a vector space $\mathfrak{h}$. There is an analogue of the Bernstein–Gelfand–Gelfand category of highest weight modules for $H_c$, to be denoted $\mathcal{O}_c$, see [BEG]. The isomorphism classes of simple objects in $\mathcal{O}_c$ are known to be parametrized by the set $\text{Irrep}(W)$ of irreducible representations of the group $W$. Given $\sigma \in \text{Irrep}(W)$, let $L_\sigma$ denote the corresponding simple object of $\mathcal{O}_c$, and let $I_\sigma := \text{Ann}(L_\sigma) \subset H_c$ denote its annihilator. The assignment $\Theta : \sigma \mapsto I_\sigma$ gives a map from $\text{Irrep}(W)$ to the set $\text{Prim}(H_c)$ of primitive ideals in $H_c$, and our general version of the Duflo theorem implies that the map $\Theta$ is surjective, in particular, $\text{Prim}(H_c)$ is a finite set. Moreover, our irreducibility theorem implies that $\mathcal{V}(I_\sigma)$, the associated variety of $I_\sigma$, is the closure of $W$-saturation of a certain symplectic leaf in $\mathfrak{h} \times \mathfrak{h}^*$. This leads to two important open problems.

**Problem 1.2.1** (Classification of primitive ideals). (i) Describe the fibers of the map $\Theta : \text{Irrep}(W) \rightarrow \text{Prim}(H_c)$ in a more direct way.

(ii) For each $\sigma \in \text{Irrep}(W)$, describe the associated variety of $I_\sigma$, in particular, for which pairs $\sigma, \tau \in \text{Irrep}(W)$ does one have $\mathcal{V}(I_\sigma) = \mathcal{V}(I_\tau)$?

**Remark.** The reader should keep in mind that the Cherednik algebra $H_c$ depends on complex parameter(s) “$c$”, which is an analogue of “central character” in the Lie algebra case. The solution to Problem [1.2.1](ii) should depend on the value of “$c$” in an essential way.

As a step towards Problem [1.2.1](ii), we will compare, in §6, the associated variety of $L_\sigma$ with that of its annihilator $I_\sigma$. Specifically, we show (Theorem 6.3.4) that the associated variety of any simple highest weight $H_c$-module $L$ equals $\mathcal{V}(\text{Ann } L) \cap (\mathfrak{h} \times \{0\})$, the intersection of the associated variety of the annihilator of $L$ (in $H_c$) with the Lagrangian subspace $\mathfrak{h} \times \{0\} \subset \mathfrak{h} \times \mathfrak{h}^*$. 


Acknowledgments. It is a pleasure to acknowledge my debt to Ofer Gabber for many extremely helpful remarks, and for a very careful reading of the manuscript.

2. MAIN RESULTS

2.1. Throughout the paper we work with associative algebras over \( \mathbb{C} \), the field of complex numbers. We write \( \otimes = \otimes \mathbb{C} \).

Let \( A \) be a unital \( \mathbb{C} \)-algebra equipped with a multiplicative increasing filtration: \( 0 = A_{-1} \subset A_0 \subset A_1 \subset \ldots \), such that \( A_i \cdot A_j \subset A_{i+j} \) and \( \bigcup_{j \geq 1} A_j = A \), \( 1 \in A_0 \). Furthermore, we assume that \( \text{gr} \, A = \bigoplus_j A_j/A_{j-1} \), the associated graded algebra, is a finitely generated commutative algebra without zero-divisors, in other words \( \text{gr} \, A \cong \mathbb{C}[X] \) is the coordinate ring of a (reduced) irreducible affine variety \( X \). For any (say left) ideal \( I \subset A \), the filtration on \( A \) induces one on \( I \), so that \( \text{gr} \, I \) becomes an ideal in \( \text{gr} \, A \). We write \( \mathcal{V}(I) \subset X \) for the zero variety of the ideal \( \text{gr} \, I \), usually referred to as the associated variety of \( I \).

Let \( \ell \) be the maximal integer (or \( \infty \), if \( A \) is commutative) such that, for all \( i, j \geq 0 \) and \( a_i \in A_i, a_j \in A_j \), one has \( a_i \cdot a_j - a_j \cdot a_i \in A_{i+j-\ell} \). (Note that \( \ell \geq 1 \) since \( \text{gr} \, A \) is commutative.) It is well known that the assignment \( a_i, a_j \mapsto [a_i, a_j] \text{ mod } A_{i+j-\ell} \) descends to a canonical Poisson bracket on \( \text{gr} \, A \) that makes \( X \) a Poisson algebraic variety. If \( X \) is smooth, then one may view \( X \) as a complex-analytic manifold equipped with a holomorphic Poisson structure. For each point \( x \in X \) one defines \( \mathcal{S}_x \), the symplectic leaf through \( x \), to be the set of points that could be reached from \( x \) by going along Hamiltonian flows.

If \( X \) is not necessarily smooth, let \( \text{Sing}(X) \) denote the singular locus of \( X \), and for any \( k \geq 1 \) define inductively \( \text{Sing}^k(X) := \text{Sing}(\text{Sing}^{k-1}(X)) \). We get a finite partition \( X = \bigsqcup_{k=0}^n X^k \), where the strata \( X^k := \text{Sing}^{k-1}(X) \setminus \text{Sing}^k(X) \) are smooth analytic varieties (by definition we put \( X^0 = X \setminus \text{Sing}(X) \)). It is known, cf. e.g., [BrGo], [Po], that each \( X^k \) inherits a Poisson structure, so for any point \( x \in X^k \) there is a well defined symplectic leaf \( \mathcal{S}_x \subset X^k \). This way one defines symplectic leaves on an arbitrary Poisson algebraic variety.

In general, each symplectic leaf is a connected smooth analytic (but not necessarily algebraic) subset in \( X \). However, if the algebraic variety \( X \) consists of finitely many symplectic leaves only, then it was shown in [BrGo] that each leaf is a smooth irreducible locally-closed algebraic subvariety in \( X \), and partition into symplectic leaves gives an algebraic stratification of \( X \).

Our first main result, to be proved in \( \S 3 \), reads

**Theorem 2.1.1 (Irreducibility theorem).** Assume that the Poisson variety \( \text{Spec}(\text{gr} \, A) \) has only finitely many symplectic leaves. Then, for any primitive ideal \( I \subset A \), the variety \( \mathcal{V}(I) \) is the closure of a single (connected) symplectic leaf.

In the classical case of a semisimple Lie algebra \( \mathfrak{g} \), given a primitive ideal \( I \subset U\mathfrak{g} \), let \( \mathcal{I} := I \cap Z \) be the intersection of \( I \) with \( Z \), the center of \( U\mathfrak{g} \). A standard argument based on a version of the Schur Lemma, see e.g. [Di], implies that \( \mathcal{I} \) is a maximal ideal in \( Z \). We let \( A := U\mathfrak{g}/\mathcal{I} \cdot U\mathfrak{g} \) be equipped with the filtration induced by the standard increasing filtration on \( U\mathfrak{g} \). The algebraic variety \( \text{Spec}(\text{gr} \, A) \) is known to coincide with the nilpotent variety in \( \mathfrak{g} \). The latter variety is partitioned into finitely many conjugacy classes. These turn out to be exactly the symplectic leaves. Thus, our theorem becomes the well known result first proved in [BoBr, Theorem 6.5] for classical simple Lie algebras, and in [J2, Theorem 3.10] and [KT, Proposition 11] in general (in the Lie algebra case our argument reduces to the proof in [Vo], and does not give anything new).

2.2. To formulate our second result, let \( A = \bigoplus_{m \in \mathbb{Z}} A(m) \) be a \( \mathbb{Z} \)-graded unital associative \( \mathbb{C} \)-algebra. We fix a pair \( A^\pm = \bigoplus_{m \in \mathbb{Z}} A^\pm(m) \subset A \) of graded unital subalgebras.

The above data is assumed to satisfy the following three conditions:
A is finitely generated as an \((A^+\cdot A^-)\)-bimodule, i.e., as an \(A^+ \otimes (A^-)_{\text{op}}\)-module.

(T1) The grading on \(A\) is inner, i.e., there is an element \(\delta \in A\) (to be fixed from now on) such that, for any \(m \in \mathbb{Z}\) and \(a \in A(m)\), one has \([\delta, a] = m \cdot a\).

(T2) Each of the two algebras \(A^\pm\) is finitely-generated and we have \(A^+(m) = 0\) for all \(m < 0\), resp. \(A^-(m) = 0\) for all \(m > 0\). Furthermore, \(A^+(0) = \mathbb{C} = A^-(0)\).

Let \(A^\pm := \oplus_{m>0} A^\pm(\pm m)\) be the augmentation ideal of the algebra \(A^\pm\).

We say that a \(\delta\)-action on a vector space \(M\) is locally-finite if any element \(m \in M\) is contained in a \(\delta\)-stable finite-dimensional vector subspace. Given a nonunital algebra \(A\) and an \(A\)-module \(M\), we say that \(A\)-action on \(M\) is locally-nilpotent if, for any \(m \in M\), there exists a large enough positive integer \(n = n(m)\) such that \(a_1 \cdot \ldots \cdot a_n \cdot m = 0\), \(\forall a_1, \ldots, a_n \in A\), and similarly for right \(A\)-modules.

**Definition 2.2.1.** Given \((A, A^\pm, \delta)\) as above, let \(\mathcal{O}\) be a full subcategory of the category of left \(A\)-modules whose objects are finitely-generated \(A\)-modules \(M\) such that the \(A^-\)-action on \(M\) is locally-nilpotent.

In section 4, we will prove the following generalization of a classical result of J. Bernstein, S. Gelfand, and I. Gelfand [BGG].

**Theorem 2.2.2.** Let \((A, A^\pm, \delta)\) be a data satisfying conditions (T0)-(T2). Then, one has

(i) The \(\delta\)-action on any object \(M \in \mathcal{O}\) is locally-finite and all (generalized) eigenspaces of this \(\delta\)-action are finite dimensional.

(ii) The category \(\mathcal{O}\) is abelian and any object of \(\mathcal{O}\) has finite length.

(iii) The category \(\mathcal{O}\) has only finitely many (isomorphism classes of) simple objects.

(iv) The category \(\mathcal{O}\) has enough projective and injective objects.

2.3. We call the data \((A, A^\pm, \delta)\) a commutative triangular structure on \(A\) if conditions (T0)–(T2) hold and, moreover, both subalgebras \(A^\pm\) are commutative. We will also consider noncommutative subalgebras \(A^\pm\). In that case, we will assume further that the algebra \(A\) is equipped with a multiplicative increasing filtration \(0 = A_{-1} \subset A_0 \subset A_1 \subset \ldots\). We write \(\text{gr} A\) for an associated graded algebra (which is not assumed to be commutative, in general), and endow \(A^\pm\) with induced filtrations.

We will say that the data \((A^\pm, \delta)\) gives a noncommutative triangular structure on \(A\) if, in addition to (T0)–(T2), the following holds:

(T0') \(\text{gr} A\) is a finitely generated \((\text{gr} A^+ \cdot \text{gr} A^-)\)-bimodule.

(T1') We have \(\delta \in A_1\), moreover, the image of \(\delta\) in \(A_1/A_0\) is a central element in \(\text{gr} A\).

(T3) The algebras \(\text{gr} A^\pm\) are commutative.

(T4) The algebras \(\text{gr} A^\pm\) are generated by the subspace \(\text{gr}_0 A^\pm \oplus \text{gr}_1 A^\pm\), the corresponding degree \(\leq 1\) components. Moreover, \(\dim(\text{gr}_i A^\pm) < \infty\), for any \(i \geq 0\).

Clearly, \((T0')\) implies \((T0)\). Conditions \((T3)-(T4)\) imply that the subspaces \(A^\pm_1\) form finite-dimensional Lie subalgebras in \(A\) (with respect to the commutator bracket). We denote them by \(n^\pm := A^\pm_1\). It follows, since \(\text{gr} A^\pm\) is generated by the corresponding degree \(\leq 1\) component, that the algebra \(A^\pm\) is generated by the subspace \(A^\pm_1\); hence it is the quotient of the augmentation ideal in \(\mathcal{U}n^\pm\) by some other ad \(\delta\)-stable two-sided ideal in the enveloping algebra \(\mathcal{U}n^\pm\). Condition \((T1')\) implies that, for each \(i \geq 1\), the space \(A_i\) is ad \(\delta\)-stable. Further, the Lie algebras \(n^\pm\) are necessarily nilpotent, and also \(1 \notin A^\pm\) (unless \(A = 0\)), because of \((T2)\).

Recall that a two-sided ideal \(I \subseteq A\) is called prime, if for any two-sided ideals \(J_1, J_2 \subseteq A\) we have \(J_1 \cdot J_2 \subseteq I \implies J_1 \subseteq I\) or \(J_2 \subseteq I\). Our final important result, proved in §5, reads

**Theorem 2.3.1** (Generalized Duflo theorem). Let \((A^\pm, \delta)\) be either a commutative or a noncommutative triangular structure on \(A\). Then, the following conditions on a two-sided ideal \(I \subseteq A\) are equivalent:
In the classical case, one considers a semisimple Lie algebra $\mathfrak{g}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Given a primitive ideal $I \subset U\mathfrak{g}$, we define $A := U\mathfrak{g}/I \cdot U\mathfrak{g}$, where $I := I \cap Z$ as above. Let $A^\pm$ be the image in $A$ of $\mathfrak{n}^\pm \cdot U\mathfrak{n}^\pm$. Using the Harish-Chandra isomorphism and the Poincaré–Birkhoff–Witt theorem, it is not difficult to show that $A$ is a finitely generated $(A^+\cdot A^-)$-bimodule, see e.g. [BL] or [Di]. We equip $A$ with an increasing filtration induced by the standard increasing PBW-filtration on $U\mathfrak{g}$. Let $\delta \in \mathfrak{h}$ be the half-sum of positive coroots. Then the adjoint action of $\delta$ on $U\mathfrak{g}$ is known to be diagonal with integral eigenvalues. Furthermore, the data $(A^\pm, \delta)$ gives a noncommutative triangular structure on $A$. The categories $^\circ \mathcal{O}$ and $\mathcal{O}$ are both equivalent, in this case, to the standard category $\mathcal{O}$ (with fixed central character corresponding to $I$) as defined by Bernstein–Gelfand–Gelfand. Thus, our Theorem 2.3.1 applies and it provides a new proof (cf. also [J2, n. 2.4] for a similar idea) of the original Duflo theorem [Du].

Remark. In the special case $A = U\mathfrak{g}$ above, our proof of the Duflo theorem is perhaps not much simpler than the earlier proofs, see [Du], [BG], [J1], because of its heavy dependence on the difficult separation theorem due to Gabber [Ga]. One advantage of our approach however is that, unlike all other proofs, it requires very little information about the representation theory of the algebra $A$.

3. Proof of Theorem 2.1.1

3.1. We need to recall some results due to K. Brown and I. Gordon [BrGo].

Let $B$ be a finitely generated Poisson algebra, with Poisson bracket $\{-,-\}$. An ideal $J \subset B$ (with respect to the commutative product) is called a Poisson ideal if it is also an ideal with respect to the Lie bracket $\{-,-\}$, i.e., if $\{J, B\} \subset J$. For the following standard result of commutative algebra, see e.g. [CO, Theorem 4.5], or [Di] §3.3.2.

**Lemma 3.1.1.** If $J$ is a Poisson ideal, then so is its radical and also all the associated prime ideals. 

Write $m_x$ for the maximal ideal in $B$ corresponding to a closed point $x \in \text{Spec } B$, and recall the notation $S_x \subset \text{Spec } B$ for the symplectic leaf through $x$. Following [BrGo], given any ideal $J \subset B$ we let $\mathcal{P}(J)$ denote the maximal Poisson ideal contained in $J$. If $J$ is prime, then so is $\mathcal{P}(J)$, and one has

**Proposition 3.1.2 ([BrGo], Prop. 3.7).** Assume the variety $\text{Spec } B$ consists of finitely many symplectic leaves. Then, for any closed point $x \in \text{Spec } B$, the leaf $S_x$ coincides with the regular locus of the zero variety of the ideal $\mathcal{P}(m_x)$. Thus, $S_x$ is a smooth connected locally-closed subvariety in $\text{Spec } B$. 

Recall that the Poisson structure on $B$ restricts to a (nondegenerate) symplectic structure on each symplectic leaf. In particular, all symplectic leaves have even (complex) dimension. Hence, from Lemma 3.1.1 and Proposition 3.1.2 we deduce

**Corollary 3.1.3.** If the variety $\text{Spec } B$ consists of finitely many symplectic leaves, then every irreducible component of the zero variety of any Poisson ideal $J \subset B$ is the closure of a symplectic leaf, in particular, it is even dimensional. 

3.2. Let $A$ be an associative filtered $\mathbb{C}$-algebra as in Theorem 2.1.1. We recall

**Lemma 3.2.1 ([BoKr], Korollar 3.6).** If $I \subset J \subset A$ are two-sided ideals, $I \neq J$ and moreover $I$ is prime, then $\dim \mathbb{V}(J) < \dim \mathbb{V}(I)$. 

Given a finitely generated \( A \)-module \( M \), choose a good filtration (cf. e.g. [Be]) on \( M \) and write \( \text{Supp}(\text{gr } M) \) for the support of the corresponding associated graded \( \text{gr } A \)-module, a reduced algebraic subvariety in \( \text{Spec}(\text{gr } A) \). It is well known (due to Bernstein [Be]) that this subvariety is independent of the choice of a good filtration on \( M \), and it will be denoted \( \text{Supp } M \) below. Note that if \( I \subset A \) is a left ideal, then in the special case \( M = A/I \) we have by definition \( \text{Supp}(A/I) = \mathcal{V}(I) \).

Assume from now on that \( \text{gr } A \) is a Poisson algebra with finitely many symplectic leaves. It is straightforward to verify that, for any two-sided ideal \( I \subset A \), the associated graded ideal, \( \text{gr } I \), is a Poisson ideal in \( \text{gr } A \). Thus by Corollary 3.1.3 every irreducible component of \( \mathcal{V}(I) \), the zero variety of \( \text{gr } I \), is the closure of a symplectic leaf.

**Proof of Theorem 2.1.1.** Let \( I \) be a primitive ideal. We have just seen that proving the theorem amounts to showing that \( \mathcal{V}(I) \) is irreducible. To prove this, we mimic the argument in ([Vo], §§3-4). Put \( M = A/I \), viewed as an \( A \)-bimodule, and let \( X := \text{Spec}(\text{gr } A) \). The filtration on \( A \) induces a filtration on \( M \), and we view \( M = \text{gr } A/\text{gr } I \) as a finitely generated \( A \)-module. Let \( M \) denote the coherent sheaf on \( X \) corresponding to this \( \text{gr } A \)-module. By definition we have \( \text{Supp } M = \mathcal{V}(I) \), and we must show that this variety is irreducible. Pick an irreducible component of \( \text{Supp } M \) of maximal dimension, \( \dim(\text{Supp } M) \). Corollary 3.1.3 says that there exists a symplectic leaf \( S \subset X \) such that this irreducible component is \( S \), the Zariski closure of \( S \). Furthermore, by Lemma 3.1.1 any imbedded component of \( \mathcal{V}(I) \) (corresponding to an associated prime of \( \text{gr } I \)) that has nonempty intersection with \( S \) must contain the whole of \( S \).

A key point is that since all symplectic leaves are even dimensional we have

\[
\dim(\overline{S \setminus S}) \leq \dim S - 2 = \dim(\text{Supp } M) - 2.
\] (3.2.2)

The idea of the proof of Irreducibility Theorem given in ([Vo], §4), that we will follow closely, is to introduce a certain new filtration, \( F^* \), on \( M = A/I \) defined "locally" in terms of the stratum \( S \), and such that \( \text{gr } F^* M \) gives rise to a coherent sheaf on \( S \). Then, since the boundary of \( S \) has codimension greater than or equal to two by (3.2.2), the global sections of \( \text{gr } F^* M \) over \( S \) necessarily form a finitely generated \( \text{gr } A \)-module which is, moreover, automatically supported on \( \overline{S} \). Hence, the filtration \( F \) is good, and one obtains \( \mathcal{V}(I) = \text{Supp}(A/I) = \text{Supp}(\text{gr } F^* A/I) = \overline{S} \).

We proceed to a more detailed exposition. First, Proposition 3.1.2 implies that \( S \) is Zariski-open in \( \overline{S} \), hence also in \( \text{Supp } M \). Therefore, \( Z := (\text{Supp } M) \setminus S \) is a Zariski closed subset in \( X \). Thus \( X \setminus Z \) is a Zariski open subset in \( X \). We may choose a smaller affine Zariski open subset \( U \subset X \setminus Z \) such that \( (\text{Supp } M) \cap U = \emptyset \). Write \( j : U \hookrightarrow X \) for the corresponding open imbedding. Set \( M_U := M|_U \), a coherent sheaf on \( U \). By construction, the leaf \( S \) is closed in \( U \), and \( \text{Supp } M_U = \overline{S} \). Hence, \( \text{Supp}(j_* M_U) = \overline{S} \). Therefore, (3.2.2) yields

\[
\dim(\text{Supp}(j_* M_U) \cap (X \setminus U)) \leq \dim(\text{Supp } M_U) - 2.
\]

A standard result of algebraic geometry now says that \( j_* M_U \) is a coherent sheaf on \( X \). Therefore, we have

\[
\Gamma(U, M_U) = \Gamma(X, j_* M_U) \text{ is a finitely generated } \text{gr } A\text{-module.} \] (3.2.3)

We are going to apply microlocal techniques – pioneered by Gabber [Ga1] – equivalent to the argument in ([Vo], §3), that has been also inspired by Gabber. To this end, observe that the variety \( X \) has a cone-structure, that is, a contracting \( \mathbb{C}^* \)-action induced by the (nonnegative) grading on \( \text{gr } A \). Clearly, \( \mathcal{V}(I) \), hence also \( S \) and \( Z \), are \( \mathbb{C}^* \)-stable subvarieties in \( X \).

For any nonzero homogeneous element \( f \in \text{gr } A \), let \( U_f \) denote the complement in \( X \) of the divisor \( f = 0 \). Thus, \( U_f \) is an affine Zariski open \( \mathbb{C}^* \)-stable subset of \( X \). Let \( \tilde{A}_{U_f} \) be the formal micro-localization of \( A \) at \( U_f \), see e.g. ([Gi], §1.3). Thus, \( \tilde{A}_{U_f} \) is a complete \( \mathbb{Z} \)-filtered ring such\footnote{Here \( \mathcal{V}(I) \) is viewed as a scheme rather than as a reduced variety.}
that \( \text{gr}(\hat{A}_{U_j}) = \mathbb{C}[U_j] \), the ring of regular functions on the affine open set \( U_j \). The assignment \( U_j \mapsto \hat{A}_{U_j} \) extends to a sheaf \( \mathcal{U} \mapsto \hat{A}_{\mathcal{U}} \) in the topology of Zariski open cone-subsets in \( X \setminus \{ 0 \} \).

For any Zariski open cone-subset \( \mathcal{U} \), we have an imbedding \( \text{gr}(\hat{A}_U) \hookrightarrow \mathbb{C}[\mathcal{U}] \), which becomes an equality if \( \mathcal{U} \) is affine. There is a canonical algebra imbedding \( A \hookrightarrow \hat{A}_U \), which is strictly compatible with the filtrations and such that the induced map \( \text{gr} A \hookrightarrow \text{gr}(\hat{A}_U) \) is identified with restriction of functions \( \text{gr} A = \mathbb{C}[X] \hookrightarrow \mathbb{C}[\mathcal{U}] \).

Further, given a finitely generated \( A \)-module \( L \), we let \( \hat{L}_{U_j} = \hat{A}_{U_j} \otimes_A L \) denote its formal micro-localization, a finitely generated \( \hat{A}_{U_j} \)-module. The assignment \( U_j \mapsto \hat{L}_{U_j} \) extends to a sheaf \( \hat{L} \) in the topology of Zariski open cone-subsets in \( X \setminus \{ 0 \} \). One has a canonical sheaf isomorphism \( \hat{L} \simeq \hat{A} \otimes_A L \); hence, in particular, an isomorphism \( \Gamma(\mathcal{U}, \hat{L}) \simeq \Gamma(\mathcal{U}, \hat{A}) \otimes_A L \), for any affine open cone-subset \( \mathcal{U} \).

We now set \( \mathcal{U} := U \), the affine Zariski open subset in \( X \setminus Z \) that has been chosen earlier, cf. (3.2.3), and which we now additionally assume to be a cone-subset. We apply the formal micro-localization at \( U \) to our \( A \)-module \( M = A/I \) to get the left \( \hat{A}_U \)-module \( \hat{M}_U = \hat{A}_U \otimes_A (A/I) = \hat{A}_U / \hat{A}_U \cdot I \). We note that since \( M \) has also a right \( A \)-action (the ideal \( I \) is a two-sided ideal) the left \( \hat{A}_U \)-module \( \hat{M}_U \) acquires a canonical \( \hat{A}_U \)-bimodule structure. Equivalently, one can get the same bimodule by applying the formal micro-localization procedure to \( M \) viewed as a left \( A \otimes A^{\text{op}} \)-module (from this point of view, \( \hat{M}_U \) is “supported” on the diagonal \( U \subset U \times U \), since \( M \), viewed as a \( A \otimes A^{\text{op}} \)-module, is clearly “supported” on the diagonal \( X \subset X \times X \)). Further, we have a canonical \( A \)-bimodule map

\[
i : M \rightarrow \hat{A}_U \otimes_A M = \hat{M}_U.
\]

Let \( K \) denote the kernel of \( i \) and \( \overline{M} \) denote the image of \( i \). Both \( K \) and \( \overline{M} \) are \( A \)-bimodules again, in particular, the preimage of \( K \) under the projection \( A \rightarrow A/I = M \) is clearly a two-sided ideal, say \( J \subset A \). Thus, we have the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & J/I & \rightarrow & A/I & \rightarrow & A/J & \rightarrow & 0 \\
\contentsline {map}{0}{0} & \mapsto & \mapsto & \mapsto \\
K & \rightarrow & M & \rightarrow & \overline {M} & \hookrightarrow & \hat {M}_U = \hat {A}_U \otimes _A M,
\end{array}
\]

where we have factored the map \( i \) as a composition \( M \rightarrow \overline {M} \hookrightarrow \hat {M}_U \). The latter factorization shows that the imbedding \( i \) induces an isomorphism \( \hat {A}_U \otimes _A \overline {M} \rightarrow \hat {A}_U \otimes _A M \) of the corresponding micro-localizations. The micro-localization functor being exact, this yields \( \hat {K}_U = \hat {A}_U \otimes _A K = 0 \). Hence \( (\text{Supp} K) \cap U = \emptyset \).

We claim that \( K = 0 \). If not, then the two-sided ideal \( J \subset A \), see (3.2.4), properly contains \( I \). Hence, Lemma [3.2.1] yields \( \dim(\text{Supp} A/J) < \dim(\text{Supp} A/I) \). But since \( S \) is a component of \( \text{Supp} A/I \) of maximal dimension, we get \( \dim S > \dim(\text{Supp} A/J) \). Therefore, the equality of sets \( \text{Supp} A/I = (\text{Supp} J/I) \cup (\text{Supp} A/J) \), which follows from the short exact sequence in the top row of (3.2.4), cannot be achieved unless \( S \subset \text{Supp} J/I = \text{Supp} K \). This last inclusion contradicts the conclusion of the previous paragraph, stating that \( (\text{Supp} K) \cap U = \emptyset \). Thus, \( K = 0 \) and our claim follows.
To proceed further, observe that the module $\hat{M}_U$ comes equipped with a natural increasing filtration $F_\bullet \hat{M}_U$ obtained, essentially, by localizing the standard filtration on $M = A/I$. Moreover, for the associated graded module corresponding to the filtration $F_\bullet \hat{M}_U$, one has a canonical isomorphism $\text{gr}^F(\hat{M}_U) \cong \Gamma(U, M_U)$ since $U$ is affine. We see that $\text{gr}^F(\hat{M}_U)$ is a finitely generated $A$-module, due to (5.2.3). Further, the increasing filtration on $\hat{M}_U$ induces a filtration $F_\bullet M = M \cap F_\bullet \hat{M}_U$ via the imbedding $i : M \hookrightarrow \hat{M}_U$ (we have shown that $i$ is injective). Then $\text{gr}^F M$ is a finitely generated $A$-submodule of $\text{gr}^F(\hat{M}_U)$, hence is also finitely generated. Therefore the filtration $F_\bullet M$ is a good filtration on $M$. This yields $\text{Supp} M = \text{Supp}(\text{gr}^F M)$, and we deduce

$$S \subset V(I) = \text{Supp} M = \text{Supp}(\text{gr}^F M) \subset \text{Supp}(\text{gr}^F \hat{M}_U) \subset \text{Supp}(J, M_U) \subset \overline{S}.$$ 

It follows that $V(I) = \overline{S}$, and the theorem is proved. □

4. CATEGORY $\mathfrak{G}$

4.1. Fix a data $(A, A^\pm, \delta)$ satisfying conditions (T0)-(T2).

Given a $\delta$-action on a vector space $V$ and $\mu \in \mathbb{C}$, let $V_\mu := \{ v \in V \mid (\delta - \mu \cdot \text{Id})^N \cdot v = 0 \text{ for } N \gg 0 \}$ denote the generalized $\mu$-eigenspace of $\delta$. If the $\delta$-action on $V$ is locally-finite, then there is a (possibly infinite) direct sum decomposition $V = \bigoplus_{\mu \in \mathbb{C}} V_\mu$.

Let $M$ be an $A$-module. We consider the following three properties of $M$:

1. $M$ is finitely generated over $A^+$.
2. The $\delta$-action on $M$ is locally-finite and each generalized $\delta$-eigenspace $M_\mu$ is finite dimensional.
3. The real parts of the eigenvalues of the $\delta$-action in $M$ are all bounded from below. \hspace{1cm} (4.1.1)

For $m \geq 1$, put $(\bar{A}^-)^m := \bar{A}^- \cdot \ldots \cdot \bar{A}^-$ ($m$ factors).

**Lemma 4.1.2.** Let $M$ be an $A$-module generated by a finite dimensional subspace $M_0 \subset M$ such that, for some $m \geq 1$, we have $(\bar{A}^-)^m \cdot M_0 = 0$. Then properties (4.1.1)(1)-(3) hold.

**Proof.** We start with statement (4.1.1)(2). Property (T0) implies that there exists a graded finite dimensional $\mathbb{C}$-vector subspace $E = \bigoplus_k E(k) \subset A$ such that $A = A^+ \cdot E \cdot A^-$. Separating homogeneous components, we get

$$A(0) = \sum_{k, \ell \in \mathbb{Z}} A^+(k - \ell) \cdot E(k) \cdot A^-(\ell).$$ \hspace{1cm} (4.1.3)

Observe that since the algebra $\bar{A}^-$ is negatively graded and finitely generated, each homogeneous component of $\bar{A}^-$ is finite dimensional, furthermore, for any integer $m > 0$, the space $(\bar{A}^-)^m$ has finite codimension in $\bar{A}^-$. Similar properties hold for the algebra $A^+$. It follows that, for all $\ell \in \mathbb{C}$ except possibly a finite set, one has $\bar{A}^-(\ell) \subset (\bar{A}^-)^m$. Therefore, (4.1.3) implies that there exist finite dimensional subspaces $B_m \subset A(0)$ such that

$$A(0) = B_m + A(0) \cap (A^+ \cdot E \cdot (\bar{A}^-)^m), \quad m = 1, 2, \ldots.$$ \hspace{1cm} (4.1.4)

Hence, since $(\bar{A}^-)^m \cdot M_0 = 0$, we get $A(0) \cdot M_0 = B_m \cdot M_0$, is a finite-dimensional space. Since $\delta^m \in A(0)$ for any $n \geq 0$, we conclude that $A(0) \cdot M_0$ is a finite-dimensional vector space and that $M$ is generated by this vector space as an $A$-module. It follows that the $\delta$-action on $M$ is locally-finite since the $ad$-$\delta$-action on $A$ is semisimple.

To prove statement (4.1.1)(3), choose $B$, a complementary subspace to $(\bar{A}^-)^m$ in $\bar{A}^-$ so that $\bar{A}^- = B \oplus (\bar{A}^-)^m$. Then we have $M = A \cdot M_0 = A^+ \cdot E \cdot (B \oplus (\bar{A}^-)^m) \cdot M_0 = A^+ \cdot E \cdot B \cdot M_0$. Further, the space $E \cdot B \cdot M_0$ is finite dimensional, and the action of $\bar{A}^+$ may only increase the real parts of the eigenvalues of $\delta$-action in $M$. It follows that the real parts of the eigenvalues of $\delta$-action in $M$ are all bounded from below, and (4.1.1)(3) is proved.
Finally, since each homogeneous component of $\overline{A}^+$ is finite dimensional, we deduce from the equality $M = A^+ \cdot (E \cdot B \cdot M_0)$ that each $\delta$-eigenspace in $M$ is finite dimensional. This completes the proof of Lemma 4.1.2(2). The same equality clearly implies that $M$ is finitely generated over $A^+$.

**Corollary 4.1.5.** Let $M$ be an $A$-module. Then, $M \in \mathcal{O}$ if and only if $M$ is generated by a finite dimensional subspace $M_0 \subset M$ such that $(\overline{A}^-)^m \cdot M_0 = 0$, for some $m \geq 1$. In particular, for any $m \geq 1$, the $A$-module $A/A \cdot (\overline{A}^-)^m$ is an object of category $\mathcal{O}$.

**Proof.** Assume that $M$ is generated by a subspace $M_0$, as in the statement of the corollary. Then Lemma 4.1.2 implies that statement (4.1.1)(3) holds. Hence, since the $\overline{A}^-$-action decreases the real parts of the eigenvalues of the $\delta$-action in $M$, we conclude that the $\overline{A}^-$-action on $M$ is locally-nilpotent. This applies in particular to the module of the form $A/A \cdot (\overline{A}^-)^m$, for any $m \geq 1$.

Conversely, any object $M \in \mathcal{O}$ is generated by a finite dimensional subspace $M_0 \subset M$. The action of $\overline{A}^-$ on $M$ is locally-nilpotent. Hence, there exists $m \gg 0$ such that $(\overline{A}^-)^m \cdot M_0 = 0$.

**Corollary 4.1.6.** The category $\mathcal{O}$ is abelian, and properties (4.1.1)(1–3) hold for any object $M \in \mathcal{O}$.

**Proof.** The last claim is immediate from Lemma 4.1.2 and Corollary 4.1.5. In particular, we conclude that any object $M \in \mathcal{O}$ is finitely generated over $A^+$, due to property (4.1.1)(1). Now, the algebra $\text{gr } A^+$ is commutative and finitely generated, hence noetherian. It follows that the algebra $A^+$ is noetherian. Therefore, any $A$-submodule of a module $M \in \mathcal{O}$ is finitely-generated over $A^+$, hence also over $A$. Thus, the category $\mathcal{O}$ is abelian.

4.2 We put $A(\leq 0) := \oplus_{n \leq 0} A(n)$, resp. $A(< 0) := \oplus_{n < 0} A(n)$. Let $I := A(0) \cap A \cdot A(< 0)$. Since, $\overline{A}^- \subset A(< 0)$, equation (4.1.4) with $m = 1$ shows that $I$ is a subspace of finite codimension in $A(0)$. Furthermore, it is clear that $I$ is a two-sided ideal of the algebra $A(0)$. Thus, $H := A(0)/I$ is a finite-dimensional algebra.

Given an $H$-module $V$, we may (and will) view $V$ as an $A(\leq 0)$-module via an algebra projection defined as the following composite $A(\leq 0) \twoheadrightarrow A(0) = A(\leq 0)/A(< 0) \twoheadrightarrow H = A(0)/I$. We introduce analogues of Verma modules as follows

$$\Delta(V) := A \otimes_{A(\leq 0)} V.$$  \hspace{1cm} (4.2.1)

The following lemma is an adaptation of a classical result of J. Bernstein, S. Gelfand, and I. Gelfand [BGG] to our present setting.

**Lemma 4.2.2.** (i) For any simple $H$-module $V$, we have $\Delta(V) \in \mathcal{O}$. Furthermore, the module $\Delta(V)$ has a unique simple quotient $L(V) \in \mathcal{O}$.

(ii) Any simple object of category $\mathcal{O}$ is isomorphic to an object of the form $L(V)$ for some simple $H$-module $V$.

**Proof.** Let $V$ be a simple $H$-module. Then, $V$ has finite dimension since $H$ is a finite-dimensional algebra. Furthermore, the image of $V$ in $\Delta(V)$ is annihilated by the algebra $A(< 0)$, by definition. Since $\overline{A}^- \subset A(< 0)$ we deduce from Corollary 4.1.5 that $\Delta(V) \in \mathcal{O}$. The proof that the module $\Delta(V)$ has a unique simple quotient repeats the classical argument word for word.

To prove part (ii), let $L$ be a (nonzero) simple object of category $\mathcal{O}$ and choose $\mu \in \mathbb{C}$ such that $L_\mu \neq 0$ and such that the real part of $\mu$ is minimal possible. Then, one must have $A(\leq 0)L_\mu = 0$. Furthermore, since $\delta$ is a central element of the algebra $A(0)$, we deduce that $L_\mu$ is an $A(0)$-stable subspace. Moreover, the space $L_\mu$ is finite dimensional and it is annihilated by the action of $I$.  


Thus, one may view \( L_\mu \) as a finite dimensional \( H \)-module. Let \( V \subset L_\mu \) be a (nonzero) simple \( H \)-submodule. The inclusion \( V \hookrightarrow L \) extends to an \( A \)-module map \( \Delta(V) = A \otimes_{A(\leq 0)} V \to L \). This map is surjective since the space \( V \) generates the \( A \)-module \( L \), the latter being a simple \( A \)-module. We conclude that \( L \) is a simple quotient of \( \Delta(V) \). □

Lemma 4.2.3. (i) The number of (isomorphism classes of) simple objects of category \( {}^\circ \mathcal{O} \) is finite.

(ii) Any object of category \( {}^\circ \mathcal{O} \) has finite length.

Proof. Part (i) follows from Lemma 4.2.2 since \( H \), the algebra involved in the definition of Verma modules, is finite dimensional, hence it has finitely many nonisomorphic simple \( H \)-modules.

To prove (ii), fix \( M \in {}^\circ \mathcal{O} \) and a simple object \( L \in {}^\circ \mathcal{O} \). We claim that there exists an integer \( a(M : L) \) such that the following holds:

\[
\text{For any descending, resp. ascending, chain of subobjects: } M = M^0 \supset M^1 \supset \ldots \supset M^N, \\
\text{let } S \text{ be the set formed by the indices } "i" \text{ such that } M^i/M^{i+1} \text{ contains } L \text{ as a subquotient. Then, one has } \#S \leq a(M : L).
\]

(4.2.4)

To prove the claim, choose \( \mu \in \mathbb{C} \) such that the eigenspace \( L_\mu \) is nonzero. Then, we clearly have

\[
\dim M_\mu = \sum_{i=1}^{N} \dim(M^i/M^{i+1})_\mu \leq \sum_{i \in S} \dim(M^i/M^{i+1})_\mu \leq \#S \cdot \dim L_\mu.
\]

Since \( \dim L_\mu \not= 0 \), and \( \dim M_\mu < \infty \) by part (i) of the theorem, we deduce \( \#S \leq \dim M_\mu / \dim L_\mu \). Thus, we can take \( a(M : L) \) to be the integral part of \( (1 + \dim M_\mu / \dim L_\mu) \).

Now, let \( \{L_1, \ldots, L_d\} \) be a complete (finite) collection of isomorphism classes of simple objects of \( {}^\circ \mathcal{O} \). To prove that an object \( M \in {}^\circ \mathcal{O} \) has finite length, one has to verify both ascending and descending chain conditions for \( M \). Consider, for instance, a descending chain of subobjects \( M = M^0 \supseteq M^1 \supseteq \ldots \supseteq M^N \). We must show that the length of this chain is bounded from above by a certain number \( a = a(M) \) which is independent of the chain. But this is clear from Claim 4.2.4 since for each \( i = 1, \ldots, N \), the nonzero object \( M^i/M^{i+1} \) must contain at least one of the simple objects \( L_r \), \( r = 1, \ldots, d \), as a subquotient. Therefore, \( N \leq \sum_{r=1}^{d} a(M : L_r) \), and we may put \( a(M) := \sum_{r=1}^{d} a(M : L_r) \). □

4.3. For an algebra \( A \) satisfying conditions (T0)-(T2), one can similarly define a category \( \mathcal{O}_+ \) to be the category of finitely-generated right \( A \)-modules \( M \) such that the \( \Delta^+ \)-action on \( M \) is locally-nilpotent. Below, we will also need to consider a bimodule setting. Specifically, let \( \mathcal{O}_+(A-A) \) be the category of finitely-generated left \( A \otimes A^{op} \)-modules \( M \) such that the action on \( M \) of the subalgebra \( A^- \otimes A^+ + A^- \otimes (A^+)^{op} \subset A \otimes A^{op} \) is locally-nilpotent.

The above results concerning category \( {}^\circ \mathcal{O} \) have counterparts for categories \( \mathcal{O}_+ \) and \( {}^\circ \mathcal{O}_+(A-A) \). In particular, we will make use of the following

Lemma 4.3.1. (i) The action of the element \( \delta \otimes 1 + 1 \otimes \delta \) on any object \( M \in \mathcal{O}_+(A-A) \) is locally-finite and each generalized eigenspace of the \( (\delta \otimes 1 + 1 \otimes \delta) \)-action in \( M \in {}^\circ \mathcal{O}_+(A-A) \) is finite-dimensional.

(ii) Any simple object of category \( {}^\circ \mathcal{O}_+(A-A) \) has the form \( L' \cong L'' \), where \( L' \) is a simple object in \( {}^\circ \mathcal{O} \) and \( L'' \) is a simple object in \( \mathcal{O}_+ \).

(iii) Category \( {}^\circ \mathcal{O}_+(A-A) \) is abelian and any object of that category has finite length.
Proof. Let $M \in \mathcal{O}_\downarrow (A\cdot A)$. Part (i) follows by an argument very similar to the proof of Lemma 4.1.2.

We now prove part (ii). Fix a simple object $L \in \mathcal{O}_\downarrow (A\cdot A)$. Choose the nonzero generalized $\delta$-eigenspace of $L$ corresponding to an eigenvalue with minimal possible real part. Commutation relations show that the action of any element of $A^- \otimes A^+ + A^- + (A^+)^{\text{op}}$ decreases the real part of the eigenvalue. Using this, one finds a simple $H \otimes H^{\text{op}}$-module $V$ inside $L$, cf. proof of Corollary 4.1.6. The algebra $H$ is finite dimensional, so the quotient of this algebra by its Jacobson radical is a direct sum of matrix algebras. It follows easily that any simple $H \otimes H^{\text{op}}$-module has the form $V = V' \boxtimes V''$, where $V'$ is a simple left, resp. $V''$ is a simple right, $H$-module. Following the proof of Corollary 4.1.6 we deduce that $L$ is a quotient of an $A \otimes A^{\text{op}}$-module of the form $\Delta(V) \boxtimes \Delta^{\text{op}}(V'')$, where $\Delta^{\text{op}}(V'') := A^{\text{op}} \otimes A^{\text{op}}(\geq 0) V''$, a 'right module' counterpart of the Verma module (4.2.1).

We know that $\Delta(V) \in \mathcal{O}$ and $\Delta^{\text{op}}(V'') \in \mathcal{O}_\downarrow$, and, moreover, these objects have finite length, cf. Lemmas 4.2.2, 4.2.3 and its $\mathcal{O}_\downarrow$-analogue. Therefore both $\Delta(V)$ and $\Delta^{\text{op}}(V'')$ have finite Jordan–Hölder series with simple subquotients, say, $L'_1, \ldots, L'_{p'}$, and $L''_1, \ldots, L''_{q''}$, respectively. It is clear from the Jacobson Density Theorem that, for any $i \in [1, p']$, $j \in [1, q'']$, the $A \otimes A^{\text{op}}$-module $L'_i \boxtimes L''_j$ is simple. Hence, the object $\Delta(V) \boxtimes \Delta^{\text{op}}(V'')$ also has a finite Jordan–Hölder series (as an $A \otimes A^{\text{op}}$-module) with simple subquotients of the form $L'_i \boxtimes L''_j$. It follows that $L$, being a simple quotient of $\Delta(V) \boxtimes \Delta^{\text{op}}(V'')$, must be isomorphic to some simple subquotient from that Jordan–Hölder series, that is, to some $L'_i \boxtimes L''_j$, and part (ii) is proved.

Thanks to part (ii), category $\mathcal{O}_\downarrow (A\cdot A)$ has finitely many nonisomorphic simple objects. Hence, the argument used in the proof of Lemma 4.2.3 shows that any object of category $\mathcal{O}_\downarrow (A\cdot A)$ has finite length. Hence, any $A \otimes A^{\text{op}}$-submodule of an object of category $\mathcal{O}_\downarrow (A\cdot A)$ has finite length. It follows in particular that $\mathcal{O}_\downarrow (A\cdot A)$ is an abelian category. \qed

Later on, we will need the following

**Lemma 4.3.2.** Let $(A^\pm, \delta)$ be a noncommutative triangular structure on the algebra $A$ and let $M$ be a finitely-generated $A \otimes A^{\text{op}}$-module equipped with a good filtration. If $M \in \mathcal{O}_\downarrow (A\cdot A)$, then $\text{gr } M$, an associated graded module, is a finitely-generated $\text{gr}(A^+) \otimes \text{gr}(A^-)^{\text{op}}$-module.

**Proof.** We recall, see e.g. [CG] Corollary 2.3.19], the following well known result: if for some good filtration on $M$, an associated graded module $\text{gr } M$ is finitely-generated over $\text{gr } A^+ \otimes \text{gr}(A^-)^{\text{op}}$, then a similar property holds for any good filtration on $M$. Therefore, it suffices to verify the statement of the lemma for a particular good filtration on $M$ of our choice. Furthermore, if $N$ be a subobject of $M$ equipped with an induced filtration then the validity of the lemma for the modules $N$ and $M/N$ implies the validity of the lemma for $M$. Since any object of category $\mathcal{O}_\downarrow (A\cdot A)$ has finite length, we are reduced to proving the lemma for simple objects. The proof of Lemma 4.3.1 shows that any such module is a quotient of an $A \otimes A^{\text{op}}$-module of the form $\Delta(V) \boxtimes \Delta^{\text{op}}(V'')$. The latter is itself a quotient of $K := (A/A \cdot A^-) \boxtimes (A^{\text{op}}/A^{\text{op}} \cdot (A^+)^{\text{op}})$. Thus, it suffices to prove the lemma for the module $K$. Furthermore, we may (and will) equip this module with a quotient filtration induced by the natural tensor product filtration on the algebra $A \otimes A^{\text{op}}$. With this choice of filtration, using that $\text{gr}(A/A \cdot A^\pm)$ is a quotient of $\text{gr } A/ \text{gr } A \cdot A^- \cdot A^\pm$, we deduce that $\text{gr } K$ is a quotient of $\mathcal{K} := (\text{gr } A/ \text{gr } A \cdot A^-) \boxtimes (\text{gr } A^{\text{op}}/ \text{gr } A^{\text{op}} \cdot \text{gr } A^+)$. We claim that $\mathcal{K}$ is a finitely generated $(\text{gr } A^+)$-$(\text{gr } A^-)$-bimodule. To see this, we use property $(T0')$, which says that $\text{gr } A$ is a finitely generated $(\text{gr } A^+)$-$(\text{gr } A^-)$-bimodule. The property implies
Our stabilization claim follows.

4.4. Given a (possibly infinite-dimensional) vector space \( V \) with locally finite \( \delta \)-action such that all generalized \( \delta \)-eigenspaces \( V_\lambda \) are finite dimensional, let \( V^* := \bigoplus_\lambda (V_\lambda)^* \subset \text{Hom}_C(V, \mathbb{C}) \) denote the direct sum of the spaces dual to \( V_\lambda \)'s. If \( V \) is a left \( A \)-module, then \( V^* \) is an \( A \)-stable subspace in the right \( A \)-module \( \text{Hom}_C(V, \mathbb{C}) \) (= linear dual of \( V \)). The right \( A \)-module \( V^* \) is called the restricted dual of \( V \). There is a canonical \( A \)-module isomorphism \( V \cong (V^*)^* \).

**Lemma 4.4.1.** Restricted duality functor \( M \mapsto M^* \) induces anti-equivalences: \( \overset{\sim}{\mathcal{O}} \overset{\sim}{\longleftarrow} \mathcal{O}_\lambda \).

**Proof.** It is clear that the real parts of the eigenvalues of the \( \delta \)-action in \( M \) are bounded from below if and only if the real parts of the eigenvalues of the \( \delta \)-action in \( M^* \) are bounded from above. Thus, it follows by Corollary 4.1.5 and Lemma 4.1.2 that \( M \in \overset{\sim}{\mathcal{O}} \) if and only if \( M^* \in \mathcal{O}_\lambda \). \( \square \)

**Proof of Theorem 2.2.2.** Part (i) of the theorem is immediate from Corollary 4.1.5 and Corollary 4.1.6. Parts (ii) and (iii) follow from Corollary 4.2.3.

To prove part (iv), let \( L_1, \ldots, L_d \) be a complete collection of representatives of simple objects of category \( \mathcal{O} \). For each \( r = 1, \ldots, d \), there is a real number \( \lambda_r \in \mathbb{R} \) such that the real parts of \( \delta \)-eigenvalues in \( L_r \) are all \( \geq \lambda_r \), by Lemma 4.1.2. It follows that, for any object \( M \in \overset{\sim}{\mathcal{O}} \), the real parts of \( \delta \)-eigenvalues in \( M \) are all \( \geq \min(\lambda_1, \ldots, \lambda_d) \).

Thus, since the \( \bar{A}^\lambda \)-action decreases the real parts of \( \delta \)-eigenvalues by at least 1, we observe that, for each \( \mu \in \mathbb{C} \) there exists an integer \( n(\mu) > 0 \) such that we have

\[
(A^-)^{n(\mu)} \cdot M = 0, \quad \forall M \in \overset{\sim}{\mathcal{O}}.
\]  

(4.4.2)

Given \( \mu \in \mathbb{C} \), for each \( k = 1, 2, \ldots \), we put

\[
J(\mu, k) := A \cdot (\delta - \mu)^k + A \cdot (A^-)^{n(\mu)}.
\]

Clearly, \( J(\mu, 1) \supseteq J(\mu, 2) \supseteq \ldots \), is a descending chain of left ideals in \( A \). We claim that this chain stabilizes, i.e., there exists \( \ell \gg 0 \) such that we have \( J(\mu, \ell) = J(\mu, \ell + 1) = \ldots \). To prove the claim, observe first that the left \( A \)-module \( A/\cdot (A^-)^{n(\mu)} \) is an object of \( \overset{\sim}{\mathcal{O}} \), by Corollary 4.1.5. Hence, this object has finite length, by part (ii) of the theorem. Now, for \( k = 1, 2, \ldots \), let \( N_k \subset A/A \cdot (A^-)^{n(\mu)} \)

be an \( A \)-submodule generated by the coset \((\delta - \mu)^k \mod (A \cdot (A^-)^{n(\mu)}) \). Clearly \( N_1 \supseteq N_2 \supseteq \ldots \). We conclude that this chain of submodules must stabilize; hence there exists \( \ell \gg 0 \) such that the canonical projections \((A/A \cdot (A^-)^{n(\mu)})/N^{\ell+1} \hookrightarrow (A/A \cdot (A^-)^{n(\mu)})/N^{\ell+1} \hookrightarrow \ldots \) are all bijections. But these projections are nothing but the canonical projections \( A/J(\mu, \ell) \hookrightarrow A/J(\mu, \ell + 1) \hookrightarrow \ldots \), and our stabilization claim follows.

We put \( P(\mu) := A/J(\mu, \ell). \) The \( A \)-module \( P(\mu) \) is a quotient of \( A/A \cdot (A^-)^{n(\mu)} \), hence, an object of \( \overset{\sim}{\mathcal{O}} \). We claim\(^2\) that \( P(\mu) \) is a projective in \( \overset{\sim}{\mathcal{O}} \). To see this, let \( M \in \overset{\sim}{\mathcal{O}} \) and fix \( v \in M_\mu \). Then \((A^-)^{n(\mu)} \cdot v = 0\) due to (4.4.2). Furthermore, there exists \( k \) large enough such that \((\delta - \mu)^k \cdot v = 0\). Therefore, \( J(\mu, k) \cdot v = 0 \), hence \( J(\mu, \ell) \cdot v = 0 \), by the stabilization claim proved above. Therefore, the assignment \( a \mapsto a(v) \) descends to a well defined \( A \)-module map \( \varphi_v : P(\mu) = A/J(\mu, \ell) \to M \).

Thus, we get a canonical isomorphism

\[
M_\mu \cong \text{Hom}_A(P(\mu), M), \quad v \mapsto \varphi_v.
\]

(4.4.3)

We conclude that the functor: \( \overset{\sim}{\mathcal{O}} \to \text{Vector spaces} \), \( M \mapsto M_\mu \), which is clearly exact, is represented by the object \( P(\mu) \). Hence, \( P(\mu) \) is projective.

\(^2\) This observation has been used earlier, see [Gu].
To complete the proof, let $L \in \mathcal{O}$ be a simple object. Find $\mu \in \mathbb{C}$ such that $L_\mu \neq 0$. Then (4.4.3) implies that there is a nonzero $A$-module map $P(\mu) \rightarrow L$. This map is surjective since $L$ is simple. Thus, we have shown that any simple object in $\mathcal{O}$ is isomorphic to a quotient of $P(\mu)$, for an appropriate $\mu \in \mathbb{C}$. It follows that the category $\mathcal{O}$ has enough projectives. Similarly, one proves that the category $\mathcal{O}_\mathfrak{M}$ has enough projectives (using Lemma 4.4.1). But then the equivalence of Lemma 4.4.1 implies that category $\mathcal{O}$ has enough injectives as well. The theorem is proved. □

5. Jacquet Functor and Proof of Theorem 2.3.1

5.1. We review the construction of the Jacquet functor following [Ca] and [CO].

Let $A$ be an associative algebra and $\tau \subset A$ a finite-dimensional Lie subalgebra (with respect to the commutator bracket). Assume further that $\tau$ has the form of a semidirect product $n \times \mathbb{C}\cdot \delta$ such that

(i) The adjoint action of $n$ on $A$ is locally-nilpotent and

(ii) The $ad \, \delta$-action on $A$ is locally-finite, and all eigenvalues of $ad \, \delta$-action on $n$ are in $\mathbb{Z}_+^\star$.

Note that the second condition implies that $n$ is a nilpotent Lie algebra. Let $\mathcal{U}_+$ denote the associative subalgebra (without unit) in $A$ generated by $n$. Thus, $\mathcal{U}_+$ is a quotient of the augmentation ideal in the enveloping algebra $\mathcal{U}n$.

For each $k \geq 1$, let $\mathcal{U}_+^k$ be the $k$-th power of $\mathcal{U}_+$. These powers form a descending chain of subalgebras $\mathcal{U}_+ \supset \mathcal{U}_+^2 \supset \ldots$, such that $\cap_{k \geq 1} \mathcal{U}_+^k = 0$. Note that condition (i) above implies that, for any $a \in A$, there exists $n = n(a) \in \mathbb{Z}$ such that

$$a \cdot \mathcal{U}_+^k \subset \mathcal{U}_+^{k+n} \cdot A, \quad \forall k \gg 0.$$  \hspace{1cm} (5.1.1)

Recall further that for any left $A$-module $M$, the linear dual $M^* = \text{Hom}_\mathbb{C}(M, \mathbb{C})$ has a natural right $A$-module structure. Given a left $A$-module $M$ we define

$$\mathfrak{J}^\dagger(M) := \lim_{\longrightarrow \mathcal{K}} (M/\mathcal{U}_+^k \cdot M)^* = \{ m^* \in M^* \mid m^*(\mathcal{U}_+^k \cdot M) = 0 \text{ for some } k = k(m^*) \gg 0 \}. \hspace{1cm} (5.1.2)$$

Formula (5.1.1) insures that $\mathfrak{J}^\dagger(M)$ is an $A$-submodule of the right $A$-module $M^*$. Moreover, it is clear that the action on $\mathfrak{J}^\dagger(M)$ of the Lie subalgebra $n \subset A$ is locally nilpotent. Thus, we get a functor $\mathfrak{J}^\dagger : \text{left $A$-modules} \rightarrow \text{right $A$-modules locally-nilpotent relative to $n$}.$

It is convenient to get the following slightly different interpretation of this functor. Observe that formula (5.1.1) implies that the multiplication map: $A \xrightarrow{\mu} \mathcal{H}$ (by any element $a \in A$) is continuous in the topology on $A$ defined by the set $\{ \mathcal{U}_+^k \cdot A \}_{k \geq 1}$ of fundamental neighborhoods of zero. Thus, the completion $\widehat{A} = \lim_{\longrightarrow \mathcal{K}} A/\mathcal{U}_+^k \cdot A$ acquires the structure of a complete topological algebra. There is a canonical algebra map: $A \rightarrow \widehat{A}$ with dense image. Similarly, for any left $A$-module $M$, the completion

$$\widehat{\mathfrak{J}}(M) = \lim_{\longrightarrow \mathcal{K}} M/\mathcal{U}_+^k \cdot M$$  \hspace{1cm} (5.1.3)

acquires the structure of a complete topological left $\widehat{A}$-module. A noncommutative analogue of the standard Artin–Rees lemma implies that, on the category of finitely-generated $A$-modules, the functor $M \mapsto \widehat{\mathfrak{J}}(M)$ is exact. Furthermore, if $M$ is finitely generated over $A$, then one has a canonical surjective morphism $\widehat{A} \otimes_A M \rightarrow \widehat{\mathfrak{J}}(M)$, see [AM ch.10]. It is clear that $\mathfrak{J}^\dagger(M)$ is nothing but the continuous dual of $\widehat{\mathfrak{J}}(M)$.

The object $\widehat{\mathfrak{J}}(M)$ is too large to be finitely generated over $A$. Using the element $\delta \in A$, that has not played any role so far, one may do better provided one works with classed of $A$-modules considered below.
5.2. Let $\text{Mod}(A : U_+)$ be the category of left $A$-modules which are finitely generated as $U_+$-modules. This is an abelian category since $U_+$, being a quotient of the augmentation ideal in $U_n$, is Noetherian. Notice that for any $M \in \text{Mod}(A : U_+)$ and any $k \geq 1$, the space $M/\mathcal{U}_+^k \cdot M$ is finite dimensional. The action of $\delta \in A$ on $M$ induces a $\delta$-action on each finite-dimensional space in the inverse system

$$M/\mathcal{U}_+ \cdot M \leftarrow M/\mathcal{U}_+^2 \cdot M \leftarrow M/\mathcal{U}_+^3 \cdot M \leftarrow \ldots \quad (5.2.1)$$

One deduces from the positivity of $\text{ad}$ $\delta$-eigenvalues on $n$ that, for each $\lambda \in \mathbb{C}$, the corresponding inverse system of generalized $\lambda$-eigenspaces $[M/\mathcal{U}_+^k \cdot M]_{\lambda}$ stabilizes, i.e., the projections in (5.2.1) induce isomorphisms $[M/\mathcal{U}_+^{k+1} \cdot M]_{\lambda} \cong [M/\mathcal{U}_+^k \cdot M]_{\lambda}$ for all $k$ sufficiently large.

**Definition 5.2.2.** Let $\mathfrak{J}(M) = \bigoplus_{\lambda \in \mathbb{C}} \lim_{\downarrow k} \mathbb{M}/\mathcal{U}_+^k \cdot M$ be the direct sum of all such “stable” generalized $\delta$-eigenspaces, to be called the Jacquet module for $M$.

It is clear that $\mathfrak{J}(M)$ may be identified with the direct sum of all generalized $\delta$-eigenspaces in $\mathfrak{J}(M)$. Thus, we have:

1. $\mathfrak{J}(M)$ is an $A$-submodule in $\mathfrak{J}(M)$ which is dense in $\mathfrak{J}(M)$ in the $U_+$-adic topology.
2. $\delta$-action on $\mathfrak{J}(M)$ is locally finite, with finite dimensional generalized eigenspaces.
3. $\mathfrak{J}(M)$ is finitely-generated over $U_+$, that is, $\mathfrak{J}(M) \in \text{Mod}(A : U_+)$. 
4. The set of $\delta$-eigenvalues in $\mathfrak{J}(M)$ is bounded from below by some constant $\lambda(M) \in \mathbb{C}$.
5. $\mathfrak{J}^\dagger(\mathfrak{J}(M)) = (\mathfrak{J}(M))^\dagger$ is the restricted dual of $\mathfrak{J}(M)$.

Properties (1) and (2) above follow from stabilization of the eigenspaces in the inverse system (5.2.1). To prove (3), let $N_0 \subseteq N_1 \subseteq \ldots$ be a strictly ascending chain of $U_+$-submodules in $\mathfrak{J}(M)$. Property (2) implies that, for each $i \geq 0$, there exists $\lambda(i) \in \mathbb{C}$ such that $[N_i]_{\lambda(i)} \neq [N_{i+1}]_{\lambda(i)}$. It follows that for the corresponding closures in $\mathfrak{J}(M)$ we have $\bar{N}_i \neq \bar{N}_{i+1}$. Thus, $\bar{N}_0 \subseteq \bar{N}_1 \subseteq \ldots$ is a strictly ascending chain of $\tilde{U}_+$-submodules in $\mathfrak{J}(M)$. But this contradicts the fact that $\mathfrak{J}(M)$ is a Noetherian $\tilde{U}_+$-module (the latter holds since $M$ is finitely-generated over $U_+$, hence $\mathfrak{J}(M)$ is finitely-generated over $\tilde{U}_+$). Thus, $\mathfrak{J}(M)$ must be a finitely-generated $U_+$-module, and (3) is proved. Property (4) follows from (3), and property (5) is clear from (2) and (4).

Since taking (generalized) eigenspaces is an exact functor, we conclude that the assignment $M \mapsto \mathfrak{J}(M)$ gives an exact functor $\mathfrak{J} : \text{Mod}(A : U_+) \to \text{Mod}(A : U_+)$, called the Jacquet functor.

5.3. We say that $A$ has a triangular structure if it has either a commutative or a noncommutative triangular structure. From now one we fix a triangular structure $(A^+, \delta)$ on $A$. Thus, one has categories $\mathcal{O}_0, \mathcal{O}_+$. We also consider the category $\text{Mod}(A : U_+)$ for the subalgebra $U_+ := A^+$.

**Lemma 5.3.1.** (i) The Jacquet functor gives an exact functor $\mathfrak{J} : \text{Mod}(A : U_+) \to \mathcal{O}_0$.

(ii) The functor $\mathfrak{J}^\dagger$ gives an exact functor: $\text{Mod}(A : U_+) \to \mathcal{O}_+$ and, moreover, $\mathfrak{J}^\dagger(M) = (\mathfrak{J}(M))^\dagger$, for any $M \in \text{Mod}(A : U_+)$. 

**Proof.** To prove (i) we note that, for any $M \in \text{Mod}(A : U_+)$, the $A^-$-action on $\mathfrak{J}(M)$ is locally-nilpotent since the set of the real parts of $\delta$-eigenvalues in $\mathfrak{J}(M)$ is bounded from below by some constant $\lambda(M) \in \mathbb{R}$. We know also that $\mathfrak{J}(M)$ is finitely-generated over $A^+$ (property (3) above), hence, over $A$. Thus, $\mathfrak{J}(M) \in \mathcal{O}_0$, and (i) is proved.

Now, Lemma [4.4.1] yields $(\mathfrak{J}(M))^\dagger \in \mathcal{O}_+$. But property (5) above says that $\mathfrak{J}^\dagger(M) \cong (\mathfrak{J}(M))^\dagger$. This completes the proof of part (ii).
Next, we introduce an abelian category $\mathcal{MMod}(A \otimes A^\text{op}, \mathfrak{U})$ of left $A \otimes A^\text{op}$-modules as follows. First, we put $\mathfrak{U} := A^+ \otimes (A^-)^\text{op} + A^+ \otimes (A^-)^\text{op}$, a nonunital subalgebra of $A \otimes A^\text{op}$. In the case where the triangular structure is commutative, we let $\mathcal{MMod}(A \otimes A, \mathfrak{U})$ be the category of $A \otimes A^\text{op}$-modules which are finitely generated over the subalgebra $\mathfrak{U}$.

Now, let the triangular structure on $A$ be noncommutative. Then the filtration on $A$ induces one on $A \otimes A^\text{op}$. Restricting the latter filtration to the subalgebra $\mathfrak{U} \subset A \otimes A^\text{op}$ makes $\mathfrak{U}$ a filtered algebra. Taking associated graded algebras, one obtains a graded algebra imbedding $\text{gr} \mathfrak{U} \hookrightarrow \text{gr}(A \otimes A^\text{op})$. We define $\mathcal{MMod}(A \otimes A, \mathfrak{U})$ to be the category of finitely-generated left $A \otimes A^\text{op}$-modules $M$ such that $\text{gr} M$ is a finitely generated $\text{gr} \mathfrak{U}$-module, for any good increasing filtration on $M$. In particular, any object of $\mathcal{MMod}(A \otimes A, \mathfrak{U})$ is a finitely-generated $\mathfrak{U}$-module.

Recall that, given a noncommutative triangular structure on $A$, we have the Lie subalgebras $n^\pm \subset A^\pm$. In the case of a commutative triangular structure on $A$, we let $n^\pm$ be any $\delta$-stable finite dimensional subspace of $A^\pm$ that generates $A^\pm$ as an associative algebra. We then regard $n^\pm$ as abelian Lie subalgebras in $A$. Thus, for any (commutative or noncommutative) triangular structure on $A$, we have the Lie algebra $n^+ \otimes 1 + 1 \otimes n^- \subset A \otimes A^\text{op}$ such that $\mathfrak{U}$, an associative subalgebra of the algebra $A \otimes A^\text{op}$, is a quotient of the augmentation ideal of $\mathfrak{U}(n^+ \oplus n^-)$, the enveloping algebra of the Lie algebra $n^+ \oplus n^- \cong n^+ \oplus 1 + 1 \oplus n^-$. Thus, on the category $\mathcal{MMod}(A \otimes A, \mathfrak{U})$, there is a well defined Jacquet functor relative to the Lie algebra $n^+ \otimes 1 + 1 \otimes n^- \subset A \otimes A^\text{op}$ and the element $\delta \otimes 1 + 1 \otimes \delta$. Furthermore, one mimics the proof of Lemma 5.3.1 to show an analogue of the Lemma, saying that the Jacquet functor is exact and takes the category $\mathcal{MMod}(A \otimes A, \mathfrak{U})$ into $\hat{\mathcal{O}}_+^\text{op} \mathcal{A}(A \otimes A)$. Thus, we get an exact functor

$$\mathfrak{J} : \mathcal{MMod}(A \otimes A, \mathfrak{U}) \longrightarrow \hat{\mathcal{O}}_+^\text{op} \mathcal{A}(A \otimes A).$$

**Proposition 5.3.2.** (i) The Jacquet functor $\mathfrak{J} : \mathcal{MMod}(A \otimes A, \mathfrak{U}) \longrightarrow \hat{\mathcal{O}}_+^\text{op} \mathcal{A}(A \otimes A)$ is faithful, that is, we have

$$M \in \mathcal{MMod}(A \otimes A, \mathfrak{U}) \text{ and } M \neq 0 \implies \mathfrak{J}(M) \neq 0.$$

(ii) Any object of the category $\mathcal{MMod}(A \otimes A, \mathfrak{U})$ has finite length.

**Remark.** There is also a "one-sided" analogue of this proposition formulated as follows. Assume, for concreteness, that the triangular structure on $A$ is noncommutative. We define $\mathcal{MMod}(A, \hat{A}^+)$ to be the category of finitely-generated left $A$-modules $M$ such that $\text{gr} M$ is a finitely generated $(\text{gr} A^+)$-module for any good increasing filtration on $M$. In particular, any object of $\mathcal{MMod}(A, \hat{A}^+)$ is a finitely-generated $A^+$-module, but the category $\mathcal{MMod}(A, \hat{A}^+)$ is, in general, smaller than the category $\mathcal{MMod}(A : \mathcal{U}_+^\text{op})$ for $\mathcal{U}_+ = \hat{A}^+$. Similarly to Proposition 5.3.2 one proves

**Proposition 5.3.3.** (i) The Jacquet functor gives a faithful functor $\mathfrak{J} : \mathcal{MMod}(A, \hat{A}^+) \longrightarrow \hat{\mathcal{O}}_+^\text{op} \mathcal{A}(A \otimes A)^\text{op}$.

(ii) Any object of the category $\mathcal{MMod}(A, \hat{A}^+)$ has finite length.

5.4. Our proof of Proposition 5.3.2 (and Proposition 5.3.3) will be based on Gabber’s Separation theorem, which we now recall, cf. [Ga] for more details.

Let $\tau = n \times C : \delta$ be an arbitrary finite dimensional solvable Lie algebra such that the adjoint $\delta$-action on $n$ is diagonal with all the eigenvalues in $\mathbb{Z}_{>0}$, as at the beginning of this section. Write $\mathcal{U}_n$ and $\mathcal{U}^\text{op}n$ for the standard increasing filtrations on the corresponding enveloping algebras. Thus, $\text{gr}(\mathcal{U}^\text{op}n) = \text{Sym} n$.

One has the following result due to O. Gabber, [Ga], Theorem 1.

**Theorem 5.4.1** (Separation theorem). Let $M$ be a nonzero $\mathcal{U}_\tau$-module, and $\{ M_j \}_{j \geq 0}$ an increasing filtration on $M$ compatible with the $\mathcal{U}_\tau$-action (i.e., such that $\mathcal{U}_\tau \cdot M_j \subset M_{j+1} \otimes \mathbb{C}$, $\forall j \geq 0$). Assume, in addition, that $\text{gr} M$ is finitely generated over $\text{gr}(\mathcal{U}_\tau)$ (not only over $\text{gr}(\mathcal{U}_\tau)$). Then, we have $n \cdot M \neq M$. 

\[ \Box \]
Remark. If \( n \) is abelian, then the separation theorem simplifies, and becomes the following standard result in Commutative Algebra (cf. e.g. [SW]): If \( M \) is an \( \mathcal{U}n \)-module which is finitely-generated over the subalgebra \( \mathcal{U}n \), then \( n \cdot M \neq M \). (no filtration is needed in this case). To prove this, we claim first that the point \( 0 \in \text{Spec}(\text{Sym } n) \) belongs to \( \text{Supp}(M) \). If not, then there is a polynomial \( P \in \text{Sym } n \) that vanishes on \( \text{Supp } M \) and such that \( P(0) \neq 0 \). Replacing \( P \) by its high enough power we may achieve that \( P \) annihilates \( M \), i.e., \( P \in \text{Ann}(M) \subseteq \text{Sym } n \). But the space \( \text{Ann}(M) \) is clearly stable under the adjoint \( \delta \)-action on \( \text{Sym } n \). Moreover, since \( P = P(0) + P_1 \), where \( P_1 \in n \cdot (\text{Sym } n) \), and all weights of \( \text{ad}\delta \)-action on \( n \cdot \text{Sym } n \) are strictly positive, we deduce from \( P \in \text{Ann}(M) \) that \( P(0), P_1 \in \text{Ann}(M) \). Since \( P(0) \neq 0 \) this yields \( 1 \in \text{Ann}(M) \), a contradiction. Thus we have proved \( 0 \in \text{Supp } M \). But then \( M/n \cdot M \), the geometric fiber of \( M \) at 0, is nonzero, due to the Nakayama lemma. Thus, \( M \neq n \cdot M \). \( \square \)

Here is an example\(^3\) showing that the separation result may fail, in general, if (for \( n \) noncommutative) the assumption of Theorem 5.4.1: "\( \text{gr } M \) is finitely generated over \( \text{gr}(\mathcal{U}n) \)" is replaced by the weaker assumption: "\( M \) is finitely generated over \( \mathcal{U}n \)".

Example. Let \( n \) be the 3-dimensional Heisenberg Lie algebra with basis \( x, y, z \), where \( z \) is central and \( [x, y] = z \). Define a semidirect product \( \tau = n \rtimes \mathbb{C} \cdot \delta \) by the commutation relations \( \delta \cdot x = x, \delta \cdot y = y, \) and \( \delta \cdot z = 2z \). Further, let \( D \) be the associative algebra of polynomial differential operators on the 2-plane \( \bar{A} := \mathbb{C}^2 \), with coordinates \( y, z \). The assignment

\[
x \mapsto z \frac{\partial}{\partial y}, \quad y \mapsto y, \quad z \mapsto z, \quad \delta \mapsto 2z \frac{\partial}{\partial z} + y \frac{\partial}{\partial y}.
\]

extends to a Lie algebra imbedding: \( \tau \to D \). We set \( f(y, z) := y \cdot z - 1 \). Thus \( f \in \mathbb{C}[y, z] \) is a polynomial, and the equation \( f = 0 \) defines a hyperbola \( X \subset \bar{A} \). It is clear that

- The curve \( X \) is smooth, irreducible, and does not intersect the axis \( z = 0 \), and
- The restriction to \( X \) of the projection \( \bar{A} \to \mathbb{C} \) along the \( y \)-axis gives an étale morphism \( X \to \mathbb{C} \).

Set \( U := A \setminus X \), an affine Zariski open subset in \( \mathbb{C}^2 \). The vector space \( M := \mathbb{C}[U]/\mathbb{C}[\bar{A}] = \mathbb{C}[\bar{A}] \left[ \frac{1}{f} \right]/\mathbb{C}[\bar{A}] \) has a natural \( D \)-module structure (that makes \( M \) a simple holonomic \( D \)-module on the plane \( A \), with support \( X \)). It is straightforward to verify that the action of \( \frac{\partial}{\partial y} \) on \( M \) induces isomorphisms:

\[
\frac{1}{f^n} \cdot \mathbb{C}[y, z] \xrightarrow{\frac{1}{f^{n+1}}} \cdot \mathbb{C}[y, z] \xrightarrow{\frac{1}{f^{n+1}}} \cdot \mathbb{C}[y, z] \xrightarrow{\frac{1}{f^n}} \cdot \mathbb{C}[y, z], \quad \forall n = 1, 2, \ldots.
\]

It follows easily that \( M = \mathcal{U}n \cdot \left[ \frac{1}{f} \right] \), i.e., the class \( \left[ \frac{1}{f} \right] \in \mathbb{C}[\bar{A}] \left[ \frac{1}{f} \right]/\mathbb{C}[\bar{A}] \) generates \( M \) over the subalgebra \( \mathcal{U}n \subset D \).

On the other hand, it is clear that \( z \cdot M = M \), i.e., the separation property fails for \( M \) (in this case, \( \text{gr}(M) \) is not finitely generated over \( \text{gr}(\mathcal{U}n) \)). \( \Diamond \)

5.5. **Proof of Proposition 5.3.2.** The argument is quite standard. Set \( n = n^+ \otimes 1 \oplus 1 \otimes n^- \), a Lie subalgebra in \( A \otimes A^{\text{op}} \). Thus, the algebra \( \mathfrak{U} \) is a quotient of the augmentation ideal of the enveloping algebra of this Lie algebra. We have the Jacquet functor \( \widehat{\mathfrak{J}} \) on \( A \otimes A^{\text{op}} \)-modules defined as the completion with respect to the subalgebra \( \mathfrak{U} \). In view of the identifications above, for any \( M \in \text{Mod}(A-A, \mathfrak{U}) \), we can write: \( \widetilde{\mathfrak{J}}(M) = \lim_{\leftarrow k} M/\mathfrak{U}^k \cdot M \). The kernel of the canonical map \( M \to \widetilde{\mathfrak{J}}(M) \) is clearly equal to \( K = \bigcap_{k \geq 0} \mathfrak{U}^k \cdot M \). It is routine to verify that \( K \) is an \( A \otimes A^{\text{op}} \)-submodule in \( M \). Moreover, a noncommutative version of the Artin–Rees lemma, see [AM], [Ca], implies that \( \mathfrak{U} \cdot K = K \) or, equivalently, that \( n \cdot K = K \).

\footnote{It was kindly communicated to me by O. Gabber}
We would like to apply the separation theorem to deduce that $K = 0$. Observe that $K \in \text{Mod}(A-A, \mathcal{M})$, since $M \in \text{Mod}(A-A, \mathfrak{I})$. In the case where the triangular structure $(A^\delta, \delta)$ is commutative, it follows readily that all the conditions on $K$ required by the Separation Theorem 5.4.1 hold trivially. In the case of a noncommutative triangular structure $(A^\delta, \delta)$, choose a good filtration $K_\bullet$ on $K$. Then, since $\delta \in A_1$, for any $i \in \mathbb{Z}$, we get $(\delta \otimes 1 + 1 \otimes \delta) \cdot K_i \subset K_{i+1}$. Further, Lemma 4.3.2 guarantees that $\text{gr} K$ is a finitely-generated $\text{gr} \mathfrak{I}$-module. Thus, all the conditions of the Separation Theorem 5.4.1 hold. Thanks to the theorem, we conclude that $K = 0$.

It follows from the above that the canonical map $M \to \hat{\mathfrak{J}}(M)$ is injective. Hence, for any nonzero $M \in \text{Mod}(A-A, \mathfrak{I})$ we have $\hat{\mathfrak{J}}(M) \neq 0$. We conclude that $\hat{\mathfrak{J}}(M)$, being a dense subspace in $\hat{\mathfrak{J}}(M)$, is also nonzero, and part (i) follows.

To prove (ii), it suffices to verify the descending chain condition for any object of the category $\text{Mod}(A-A, \mathfrak{I})$. Let $M = M^0 \supseteq M^1 \supseteq M^2 \supseteq \ldots$ be a descending chain of submodules in $M$. Using the exactness of the Jacquet functor and part (i), from $M^i/M^{i+1} \neq 0$, $\forall i$, we deduce that $\hat{\mathfrak{J}}(M^i)/\hat{\mathfrak{J}}(M^{i+1}) = \hat{\mathfrak{J}}(M^i/M^{i+1}) \neq 0$. Hence, $\hat{\mathfrak{J}}(M^0) \supseteq \hat{\mathfrak{J}}(M^1) \supseteq \hat{\mathfrak{J}}(M^2) \supseteq \ldots$ is a strictly descending chain of submodules in $\hat{\mathfrak{J}}(M)$. This contradicts the fact that $\hat{\mathfrak{J}}(M)$, being an object of $\mathfrak{O}_\delta(A-A)$, must have finite length by Lemma 4.3.1 Claim (ii) follows.

5.6. **Proof of Theorem 2.3.1** First of all, we observe that conditions $(T0)$, $(T0')$ insure that $A$, the diagonal $(A-A)$-bimodule, is an object of category $\text{Mod}(A-A, \mathfrak{I})$. Thus, the last statement of Theorem 2.3.1 follows directly from Proposition 5.3.2(ii).

Now, we turn to other statements of the theorem. The implication (i) $\Longrightarrow$ (iii) is standard.

To prove (ii) $\Longrightarrow$ (iii), fix a prime ideal $I \subset A$. As we have observed above, $A \in \text{Mod}(A-A, \mathfrak{I})$, hence $A/I \in \text{Mod}(A-A, \mathfrak{I})$. Therefore, we may apply the Jacquet functor to get an object $\mathfrak{J}(A/I) \in \mathfrak{O}_\delta(A-A)$.

Given an $A \otimes A^{op}$-module, resp. $(A-A)$-bimodule, $M$, we write $\text{LAnn}(M)$ for the annihilator of $M$ in the subalgebra $A \otimes 1 \subset A \otimes A^{op}$. The proof of Proposition 5.3.2 shows that $A/I$ maps injectively into $\mathfrak{J}(A/I)$ and, moreover, the image of $A/I$ is dense in $\mathfrak{J}(A/I)$. Thus we have $I = \text{LAnn}(A/I) = \text{LAnn}(\mathfrak{J}(A/I))$. Furthermore, $\text{LAnn}(\mathfrak{J}(A/I)) = \text{LAnn}(\mathfrak{J}(A/I))$, for $\mathfrak{J}(A/I)$ is dense in $\mathfrak{J}(A/I)$. Thus, $\text{LAnn}(\mathfrak{J}(A/I)) = I$ is a prime ideal in $A$, as claimed.

Now, any object of $\mathfrak{O}_\delta(A-A)$, in particular, $M = \mathfrak{J}(A/I)$, has finite length by Lemma 4.3.1 Let $0 = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n = M$ be its Jordan–Hölder series. For each $i = 1, 2, \ldots, n$, put $J_i := \text{LAnn}(M_i/M_{i-1})$. Then, one proves by a standard argument that there exists $m \in [1, n]$ such that one has $I = J_m$, see e.g. [BG]. Below, we provide more details for the reader’s convenience.

First, since $J_i(M_i) \subset M_{i-1}$ we deduce that

$$J_1 \cdots J_{n-1} \cdot J_n(M) \subset J_1 \cdots J_{n-1}(M_{n-1}) \subset \ldots \subset J_1(M_1) = 0.$$ 

Hence, the ideal $J_1 \cdots J_n$ annihilates $M$. Therefore, we have $J_1 \cdots J_n \subset \text{LAnn}(M)$. But, $\text{LAnn}(M) = I$ being prime, the inclusion above implies that there exists an $m$ such that $J_m \subset I$. On the other hand, since $M_i/M_{i-1}$ is a subquotient of $M$, for any $i = 1, 2, \ldots, n$, one clearly has an opposite inclusion $I = \text{LAnn}(M) \subset \text{LAnn}(M_i/M_{i-1}) = J_i$. Thus, we obtain $I = J_m = \text{LAnn}(M_m/M_{m-1})$, as claimed.

By construction, $M_m/M_{m-1}$ is a simple object of $\mathfrak{O}_\delta(A-A)$. Hence, Lemma 4.3.1 yields $M_m/M_{m-1} = L' \otimes L''$, where $L' \in \mathfrak{O}$ and $L'' \in \mathfrak{O}_\delta$ are some simple objects. Further, it is clear that $\text{LAnn}(L' \otimes L'') = \text{LAnn}(L')$. Thus, we deduce that $I = \text{LAnn}(M_m/M_{m-1}) = \text{LAnn}(L' \otimes L'') = \ldots$.

\[\text{It was pointed out by a referee that a similar observation is contained in [J2, n.2.4].}\]
Let $(V, \omega)$ be a finite-dimensional symplectic vector space, and $G \subset \text{Sp}(V, \omega)$ a finite subgroup of symplectic automorphisms of $V$. An element $s \in G$ is called a symplectic reflection if $\text{rk}(\text{Id} - s) = 2$. Let $S$ denote the set of symplectic reflections in $G$. The group $G$ acts on $S$ by conjugation. For each $s \in S$, there is an $\omega$-orthogonal direct sum decomposition $V = \text{Im}(\text{Id} - s) \oplus \text{Ker}(\text{Id} - s)$, and we write $\omega_s$ for the (possibly degenerate) skew-symmetric form on $V$ which coincides with $\omega$ on $\text{Im}(\text{Id} - s)$, and has $\text{Ker}(\text{Id} - s)$ as its radical.

Write $\mathbb{C}G$ for the group algebra of $G$, and $(TV)^\#G$ for the cross product of the tensor algebra $TV$ with $\mathbb{C}G$. From now on, we identify $V$ with $V^*$ via the symplectic form. Thus, the symmetric algebra on $V$ is identified with $\mathbb{C}[V]$, the polynomial algebra on $V$.

In [EG], the authors have introduced a class of associative algebras $H_{t,c}(V, \omega, \Gamma)$, called symplectic reflection algebras, as follows. Fix an $\text{Ad} \Gamma$-invariant function $c : S \to \mathbb{C}$, $s \mapsto c_s$, and a complex number $t \in \mathbb{C}$. Let $\kappa : V \times V \to \mathbb{C}G$ be a skew-symmetric $\mathbb{C}$-bilinear pairing given by the formula

$$\kappa(x, y) = t \cdot \omega(x, y) \cdot 1 + \sum_{s \in S} c_s \cdot \omega_s(x, y) \cdot s, \quad \forall x, y \in V.$$ 

We define the symplectic reflection algebra with parameters $(t,c)$ by

$$H_{t,c}(V, \omega, \Gamma) := (TV)^\#G/I(x \otimes y - y \otimes x - \kappa(x, y)) \in T^2V \oplus \mathbb{C}G, \quad x, y \in V,$n

where $I(\ldots)$ stands for the two-sided ideal in $(TV)^\#G$ generated by the indicated set. Thus, $H_{t,c}(V, \omega, \Gamma)$ is an associative algebra which may be thought of as a deformation of $\mathbb{C}[V]^\#G$ (= cross-product of $\mathbb{C}[V]$ with $\mathbb{C}G$). Clearly, $H_{t,c}(V, \omega, \Gamma)$ contains $\mathbb{C}G$ as a subalgebra. Furthermore, there is a natural increasing filtration on $H_{t,c}(V, \omega, \Gamma)$ such that $\mathbb{C}G$ has filtration degree zero, and elements of $V \subset TV$ are assigned filtration degree one. It has been proved in [EG] that, for the corresponding associated graded algebra, there is a canonical graded algebra isomorphism

$$\text{gr} H_{t,c}(V, \omega, \Gamma) \simeq \mathbb{C}[V]^\# \Gamma \quad (\text{Poincaré–Birkhoff–Witt property).}$$

Write $e = \frac{1}{2t} \sum_{g \in G} g \in \mathbb{C}G$ for the symmetrizer idempotent, viewed as an element in $H_{t,c}(V, \omega, \Gamma)$. We let $eHe := e \cdot H_{t,c}(V, \omega, \Gamma) \cdot e$ be the spherical subalgebra in $H_{t,c}(V, \omega, \Gamma)$. The increasing filtration on $H_{t,c}(V, \omega, \Gamma)$ induces a filtration on the spherical subalgebra. From (6.1.2), one finds that $\text{gr}(eHe) = \mathbb{C}[V]^\# = \mathbb{C}[V]/\Gamma$, the algebra of $\Gamma$-invariant polynomials on $V$. The canonical Poisson structure on the commutative algebra $\text{gr}(eHe)$ makes $V/\Gamma$ a Poisson variety. It is easy to see, cf. ([EG], Theorem 1.6), that this Poisson structure on $V/\Gamma$ equals the $t$-multiple of the standard one induced by the symplectic structure on $V$ (for $t = 0$ the spherical subalgebra $eHe$ becomes commutative, and the corresponding Poisson structure on $\text{gr}(eHe) = \mathbb{C}[V]/\Gamma$ reduces to zero).

6.2. Symplectic leaves in $V/\Gamma$. The leaves of the standard Poisson structure on $\mathbb{C}[V/\Gamma]$ are described as follows, see [BrGo]. Let $\text{Isotropy}(\Gamma, V)$ denote the (finite) set of all subgroups $G \subset \Gamma$ that occur as isotropy groups of points in $V$. Given a subgroup $G \subset \Gamma$, write $V^G \subset V$ for the vector subspace of $G$-fixed points. Let $V_G$ denote the image of $V^G$ under the projection $V \to V/\Gamma$, and let $\hat{V}_G = V_G \setminus \bigcup_{G' \in \text{Isotropy}(\Gamma, V), G' \subset G} (V_G \cap V_{G'})$ be the complement in $V_G$ of the union of all proper subsets in $V_G$ of the form $V_G \cap V_{G'}$ (here the symbol $\cup$ indicates that the union is taken over proper subsets only). It is easy to see that for each subgroup $G \in \text{Isotropy}(\Gamma, V)$, the set $\hat{V}_G$ equals the image of the subset $\hat{V}_G \subset V^G$ formed by the points whose isotropy group equals $G$. If $N_G$ denotes the normalizer of $G$ in $\Gamma$, then the group $N_G/G$ acts freely on $\hat{V}_G$ and we have $\hat{V}_G \cong \hat{V}_G/(N_G/G)$. 

$\text{LAnn}(L')$ is the annihilator of a simple object of $\mathbb{O}$. Part (iii) follows. The implication (iii) $\implies$ (i) is trivial. □
It follows that each set $\hat{V}_G$ is a smooth connected locally-closed subvariety of $V/\Gamma$, and one shows that these varieties are exactly the symplectic leaves in $V/\Gamma$, see e.g., [BrGo]. Hence, there are only finitely many symplectic leaves.

From the Irreducibility Theorem\ref{thm:irreducibility} we deduce

**Corollary 6.2.1.** Let $H_{t,c}(V,\omega,\Gamma)$ be a symplectic reflection algebra, and $t \neq 0$. Then, for any primitive ideal $I$ in the spherical subalgebra $\mathfrak{e}H_{t,c}(V,\omega,\Gamma)\mathfrak{e}$, the variety $\mathcal{V}(I)$ has the form $V_G$, for a certain subgroup $G \in \text{Isotropy}(\Gamma, V)$.

We remark that Theorem\ref{thm:irreducibility} is not applicable to the symplectic reflection algebra $H_{t,c}(V,\omega,\Gamma)$ itself since $\text{gr} \ H_{t,c}(V,\omega,\Gamma) = \mathbb{C}[V]/\mathbb{C}^*\Gamma$, is a noncommutative algebra. Note however, that finitely generated $\mathbb{C}[V]/\mathbb{C}^*\Gamma$-modules may be naturally identified with $\Gamma$-equivariant coherent sheaves on $V$. The support of such a sheaf is a $\Gamma$-stable subvariety in $V$. In particular, given a two-sided ideal $I \subset H_{t,c}(V,\omega,\Gamma)$ one can view $\text{gr} \ (H_{t,c}(V,\omega,\Gamma)/I)$ as a finitely generated $\mathbb{C}[V]/\mathbb{C}^*\Gamma$-module, via (6.1.2). Let $\mathcal{V}(I) \subset V$ stand for its support.

Using the technique of Poisson orders developed in [BrGo], one can refine our argument to obtain the following result.

**Proposition 6.2.2.** For any primitive ideal $I \subset H_{t,c}(V,\omega,\Gamma)$, the variety $\mathcal{V}(I)$ is the $\Gamma$-saturation of vector subspace $V^G$, for a certain subgroup $G \in \text{Isotropy}(\Gamma, V)$. \hfill $\Box$

### 6.3. Rational Cherednik algebras.

Let $W$ be a finite Coxeter group in a complex vector space $\mathfrak{h}$. Thus, $\mathfrak{h}$ is the complexification of a real Euclidean vector space, with inner product $(\cdot,\cdot)$, and the group $W$ is generated by a finite set $S \subset W$ of reflections $s \in S$ with respect to certain hyperplanes $\{H_s\}_{s \in S}$ in that Euclidean space.

For each $s \in S$, we choose a nonzero linear function $\alpha_s \in \mathfrak{h}^*$ that vanishes on $H_s$ (called the positive root corresponding to $s$), and let $\alpha_s^\vee = 2(\alpha_s,\cdot)/(\alpha_s,\alpha_s) \in \mathfrak{h}$ be the corresponding coroot. The group $W$ acts on the set $S$, and also acts diagonally on $V = \mathfrak{h} \oplus \mathfrak{h}^*$, by conjugation. We equip the space $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with the canonical symplectic structure.

Following [EG], to each $W$-invariant function $c : S \to \mathbb{C}$, $c \mapsto c_s$, one associates the symplectic reflection algebra $H_c := H_{1,c}(V,\omega,W)$, called the rational Cherednik algebra. The algebra $H_c$ is generated by the vector spaces $\mathfrak{h}$, $\mathfrak{h}^*$, and the set $W$, with defining relations (cf. formula (1.15) of [EG] for $t = 1$) given by

\begin{align}
  w \cdot x \cdot w^{-1} &= w(x), \quad w \cdot y \cdot w^{-1} = w(y), \quad \forall y \in \mathfrak{h}, x \in \mathfrak{h}^*, w \in W \\
  x_1 \cdot x_2 &= x_2 \cdot x_1, \quad y_1 \cdot y_2 = y_2 \cdot y_1, \quad \forall y_1, y_2 \in \mathfrak{h}, x_1, x_2 \in \mathfrak{h}^* \\
  y \cdot x - x \cdot y &= \langle y, x \rangle + \sum_{s \in S} c_s \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle \cdot s, \quad \forall y \in \mathfrak{h}, x \in \mathfrak{h}^*. \tag{6.3.1}
\end{align}

Thus, the elements $x \in \mathfrak{h}^*$ generate a subalgebra $\mathbb{C}[x] \subset H_c$, of polynomial functions on $\mathfrak{h}$, the elements $y \in \mathfrak{h}$ generate a subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset H_c$, of polynomial functions on the dual space, and the elements $w \in W$ span a copy of the group algebra $\mathbb{C}W$ sitting naturally inside $H_c$. Furthermore, the Poincaré–Birkhoff–Witt property yields a “triangular decomposition” $H_c \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}]$, see ([EG], Corollary 4.4). We put $A^+ := \mathbb{C}[\mathfrak{h}]^W$ and $A^- := \mathbb{C}[\mathfrak{h}^*]^W$. Further, in [BEG] we have constructed an element $\delta \in H_c$ such that the data $(A^\pm, \delta)$ gives a commutative triangular structure on the rational Cherednik algebra $H_c$.

Associated to the above data, we have categories $\mathfrak{O}, \mathfrak{O}_\delta$ (see Definition\ref{def:O_d}). Applying Theorem\ref{thm:duflo} (note that we are using here only the easy, purely commutative, part of Gabber’s separation theorem, since $A^\pm$ are commutative algebras), we obtain the following result

**Corollary 6.3.2.** Duflot theorem holds for the rational Cherednik algebra $H_c$. 

In [BEG], the authors introduced another category, \( \mathcal{O}(H_c) \), defined as the category of finitely-generated left \( H_c \)-modules \( M \) such that the \( A^- \)-action on \( M \) is locally-finite. It is clear from the triangular decomposition \( H_c \simeq \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}] \) that, in the notation of §4, we have \( \mathcal{O}(H_c) \subseteq \text{Mod}(H_c, A^+) \). Note that the inclusion \( \mathcal{O}(H_c) \subseteq \mathcal{O}(H_c) \) is strict since we do not require the locally-finite \( A^- \)-action to be locally-nilpotent. We have also introduced in [BEG] a category \( \mathcal{H}(\mathcal{C}) \) of Harish-Chandra bimodules over the Cherednik algebra \( H_c \), and showed that (in our present notation) any object of \( \mathcal{H}(\mathcal{C}) \) belongs to \( \text{Mod}(A-A, \mathcal{U}) \). Thus, from Propositions 5.3.2-5.3.3 we deduce

**Corollary 6.3.3.** Any object of the categories \( \mathcal{O}(H_c) \) and \( \mathcal{H}(\mathcal{C}) \) has finite length. □

**Remark.** In [BEG] we were only able to prove that any object of the category \( \mathcal{O}(H_c) \) has finite length, but the technique in loc. cit. was insufficient to prove Corollary 6.3.3 in full generality.

As has been explained before Proposition 6.2.2 for any \( H_c \)-module, hence for an object \( M \in \mathcal{O}(H_c) \), one can define a \( W \)-stable subvariety \( \text{Supp} M \subset \mathfrak{h} \oplus \mathfrak{h}^* \). The \( A^- \)-action on \( M \) being locally-finite, it follows, since \( A^- = \mathbb{C}[\mathfrak{h}^*]^W \), that \( \text{Supp} M \subset \mathfrak{h} \oplus \{0\} \). We identify \( \mathfrak{h}^* \) with \( \mathfrak{h} \) via the invariant form and observe that, since \( W \) is finite, the isotropy group of a point \( (y, x) \in \mathfrak{h} \oplus \mathfrak{h}^* = \mathfrak{h} \oplus \mathfrak{h} \) coincides with the isotropy group of the generic linear combination \( t_1 \cdot y + t_2 \cdot x \), \( t_1, t_2 \in \mathbb{C} \). Therefore we see that, in the notation of Theorem 6.2.1, the set \( \text{Isotropy}(W, \mathfrak{h} \oplus \mathfrak{h}^*) \) is exactly the set \( \text{Parab}(W) \) of all parabolic subgroups in \( W \), cf. [Hu].

**Theorem 6.3.4.** For any simple object \( M \in \mathcal{O}(H_c) \), the variety \( \text{Supp} M \) is the \( W \)-saturation of vector subspace \( \mathfrak{h}^G \oplus \{0\} \), for a certain subgroup \( G \in \text{Parab}(W) \).

**Sketch of Proof.** We adapt the known argument used in the case of highest weight modules over \( U_{\mathfrak{g}} \), the enveloping algebra of a semisimple Lie algebra. To this end, write \( \text{Ann} M \subset H_c \) for the annihilator of \( M \in \mathcal{O}(H_c) \). Thus, \( \text{Ann} M \) is a primitive ideal in \( H_c \).

Introduce the notation \( \Lambda := \text{Supp} M \subset \mathfrak{h} \oplus \mathfrak{h}^* \). Clearly, \( \Lambda \subset \text{Supp}(H_c/\text{Ann} M) \). Further, by Proposition 6.2.2 there exists a subgroup \( G \in \text{Parab}(W) \) such that one has \( \text{Supp}(H_c/\text{Ann} M) = W \cdot (\mathfrak{h} \oplus \mathfrak{h}^*)^G \). Therefore we get

\[
\Lambda \subset (\mathfrak{h} \oplus \{0\}) \cap W \cdot (\mathfrak{h} \oplus \mathfrak{h}^*)^G = W \cdot (\mathfrak{h}^G \oplus \{0\}).
\]

Next, we claim that

\[
\dim \Lambda = \dim (\mathfrak{h}^G).
\]

To prove this, choose a finite dimensional \( \mathbb{C}[\mathfrak{h}^*]^W \)-stable subspace \( M^0 \subset M \) that generates \( M \) over the subalgebra \( \mathbb{C}[\mathfrak{h}]^W \). For each \( i \geq 0 \), set \( M_i = F_i H_c \cdot M^0 \), where \( \{F_i H_c\}_{i \geq 0} \) is the standard increasing filtration on the Cherednik algebra \( H_c \), cf. [EG]. This gives a filtration on \( M \) compatible with that on \( H_c \) and such that \( \text{gr} M \) is finitely-generated over the subalgebra \( \text{gr} \mathbb{C}[\mathfrak{h}]^W \subset \text{gr} H_c \). Hence, the restriction to \( \text{Supp} M \subset \text{Spec}(\text{gr} H_c) \) of the canonical projection \( \text{Spec}(\text{gr} H_c) \to \text{Spec}(\mathbb{C}[\mathfrak{h}]^W) \) is a finite morphism. Therefore the dimension of the variety \( \text{Supp} M \) is unaffected by replacing the algebra \( H_c \) by the algebra \( \mathbb{C}[\mathfrak{h}]^W \). Now, following Joseph [J3], we define an increasing chain of \( \mathbb{C}[\mathfrak{h}]^W \)-submodules \( D_k(M) \subset \text{End}_{\mathbb{C}} M \), \( k = 0, 1, \ldots \), inductively as follows:

\[
D_0(M) = \mathbb{C}[\mathfrak{h}]^W, \quad \text{and} \quad D_k(M) = \{ u \in \text{End}_{\mathbb{C}} M \mid [u, \mathbb{C}[\mathfrak{h}]^W] \subset D_{k-1}(M) \}.
\]

Set \( D(M) := \bigcup_{k \geq 0} D_k(M) \). Joseph showed that \( D(M) \) is an associative subalgebra in \( \text{End}_{\mathbb{C}} M \) (which should be thought of as an algebra of "differential operators" on \( M \)). Moreover, he proved in ([J3], Lemma 2.3) that \( \dim \text{Supp} D(M) \leq 2 \dim(\text{Supp} M) \).

Observe further that the action in \( M \) of an element \( a \in H_c \) gives an endomorphism \( \hat{a} \in \text{End}_{\mathbb{C}} M \), and the assignment \( a \mapsto \hat{a} \) gives an algebra embedding \( H_c/\text{Ann} M \hookrightarrow \text{End}_{\mathbb{C}} M \). The adjoint action
of the subalgebra $\mathbb{C}[\mathfrak{h}]^W$ on $H_c$ being locally-nilpotent, see [BEG], the image of the embedding $H_c/\text{Ann } M \hookrightarrow \text{End}_c M$ is contained in $D(M)$. From this, following Joseph [J3], one derives

$$\dim \text{Supp}(H_c/\text{Ann } M) \leq \dim \text{Supp} D(M) \leq 2 \dim(\text{Supp } M).$$

(6.3.7)

Hence, using (6.3.5) and writing "$\dim(-)$" instead of "$\dim \text{Supp}(-)$", we deduce

$$\dim(\mathfrak{h} \oplus \mathfrak{h}^*)^G = \dim(H_c/\text{Ann } M) \leq 2 \dim M = 2 \dim \Lambda \leq 2 \dim \mathfrak{h}^G = \dim(\mathfrak{h} \oplus \mathfrak{h}^*)^G.$$ It follows that all the inequalities above must be equalities, and (6.3.6) is proved. In particular, there exists $\Lambda^0 \subset (\mathfrak{h}^G \oplus \{0\})$, an irreducible component of $\Lambda$, such that $\dim \Lambda^0 = \dim(\mathfrak{h}^G \oplus \{0\})$. This dimension equality yields $\Lambda^0 = \mathfrak{h}^G \oplus \{0\}$.

We claim next that all irreducible components of $\Lambda = \text{Supp } M$ have the same dimension, i.e., the following version of Gabber’s equidimensionality theorem holds for the Cherednik algebra.

**Proposition 6.3.8.** For any simple $H_c$-module $M$, the variety $\text{Supp } M$ is equidimensional.

To prove this, we recall that the equidimensionality theorem is, as explained e.g. in [Gi], a formal consequence of the Gabber–Kashiwara theorem. The proof of the latter theorem given, e.g. in [Gi, p.342–345], works for any filtered algebra $A$ such that $gr A$ is the coordinate ring of a smooth affine algebraic variety. Now, in our present situation, we have $gr H_c = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/\# W$, which is not a commutative algebra. However, the formal microlocalization construction can be carried out with respect to any multiplicative set $S \subset A$, $(0 \not\in S)$, for any algebra $A$, provided the principal symbols of the elements of $S$ belong to the center of $gr A$, see ([Gi], footnote on p. 337). In our case we have a central subalgebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W \subset gr H_c$. Furthermore, the algebra $gr H_c$ has finite homological dimension, as a cross-product of the polynomial algebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ with a finite group. Going through the proof of the Gabber–Kashiwara theorem given in ([Gi], pp. 342–345) one sees that the finiteness of homological dimension of $H_c$, plus the existence of the formal microlocalization with respect to elements whose symbols belong to $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W \setminus \{0\}$, is enough to conclude that the proof of the Gabber–Kashiwara theorem applies to the algebra $H_c$ as well. This implies Proposition 6.3.8.

From Proposition 6.3.8 we deduce that all irreducible components of the variety $\Lambda = \text{Supp } M$ have the same dimension, which is equal to $\dim \mathfrak{h}^G$. But then the inclusion in (6.3.5) forces each irreducible component to be a $W$-translate of $\mathfrak{h}^G$, and Theorem 6.3.4 follows. \qed

**Remark.** It was pointed out to us by A. Yekutieli that Gabber’s equidimensionality theorem also holds for the spherical subalgebra $eH e$ in any symplectic reflection algebra $H = H_{t,c}(V, \omega, \Gamma)$, i.e., one has

**Proposition 6.3.9.** Let $H_{t,c}(V, \omega, \Gamma)$ be a symplectic reflection algebra. Then, for any simple $eH e$-module $M$, the variety $\text{Supp } M$ is equidimensional.

**Sketch of proof.** Recall that PBW-property (6.1.2) implies that $gr(eH e) = \mathbb{C}[V]^\Gamma = \mathbb{C}[V/\Gamma]$. We see that $\text{Spec}(gr(eH e)) \simeq V/\Gamma$ is not a smooth variety. Nonetheless, $V/\Gamma$ is a Gorenstein variety, and homological duality formalism is known to work for Gorenstein varieties as nicely as for smooth varieties. In particular, it follows from [YZ] that the bimodule $eH e$ is (up to shift) a rigid dualizing complex for the algebra $eH e$ (in the sense of [YZ]). Therefore, the proof of the Gabber–Kashiwara theorem still goes through, cf. [YZ, Purity theorem]. \qed

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