STEP MULTIPLIERS, FOURIER STEP MULTIPLIERS AND
MULTIPLICATIONS ON QUASI-BANACH MODULATION
SPACES

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Abstract. We prove the boundedness of a general class of multipliers and Fourier multipliers, in particular of the Hilbert transform, on quasi-Banach modulation spaces. We also deduce boundedness for multiplications and convolutions for elements in such spaces.

0. Introduction

In the paper we deduce mapping properties of step multipliers and Fourier step multipliers when acting on quasi-Banach modulation spaces. Some parts of our investigations are based on certain continuity properties for multiplications and convolutions for elements in such spaces, deduced in Section 3 and which might be of independent interests.

The Hilbert transform, i.e. multiplication by the signum function on the Fourier transform side, is frequently used in mathematics, science and technology. In physics it can be used to secure causality. For example, in optics, the refractive index of a material is the frequency response of a causal system whose real part gives the phase shift of the penetrating light and the imaginary part gives the attenuation. The relationship between the two are given by the Hilbert transform. Consequently, knowledge of one is sufficient to retrieve the other.

An inconveniently property with the Hilbert transform concerns lack of continuity when acting on commonly used spaces. For example, it is well-known that the Hilbert transform is continuous on $L^2$, but fails to be continuous on $L^p$ for $p \neq 2$ as well as on $\mathcal{S}$. (See [22] and Section 1 for notations.) A pioneering contribution which drastically improve the situation concerns [23], where K. Okoudjou already in his thesis showed that the Hilbert transform is continuous on the modulation space $M^{p,q}$ when $p \in (1, \infty)$ and $q \in [1, \infty]$. The result is surprising because $M^{p,q}$ is rather close to $L^p$ when $q$ stays between $p$ and $p'$ (see e.g. [8, 29]).

Okoudjou’s result in [23] was extended in [3], where Bényi, Grafakos, Gröchenig and Okoudjou show that Fourier step multipliers, i.e. Fourier multipliers of the form

$$f \mapsto \mathcal{F}^{-1} \left( \sum_{j \in \mathbb{Z}} a_0(j) \chi_{j+[0,b)} \hat{f} \right), \quad a_0 \in \ell^\infty(b\mathbb{Z}),$$

(0.1)
are continuous on the modulation space $M^{p,q}(\mathbb{R}^d)$, when $p \in (1, \infty)$ and $q \in [1, \infty]$. (See [3] Theorem 1.)

Recall that modulation spaces is a family of function and distribution spaces introduced by Feichtinger in [8] and further developed by Feichtinger and Göckenig in [10–13,17]. In particular, the modulation spaces $M^{p,q}_{\omega}(\mathbb{R}^d)$ and $W^{p,q}_{\omega}(\mathbb{R}^d)$ are the set of tempered (or Gelfand-Shilov) distributions whose short-time Fourier transforms belong to the weighted and mixed Lebesgue spaces $L^{p,q}_{\omega}(\mathbb{R}^{2d})$ respectively $L^{p,q}_{*,\omega}(\mathbb{R}^d)$. Here $\omega$ is a weight function on phase (or time-frequency shift) space and $p,q \in (0, \infty]$. Note that $W^{p,q}_{\omega}(\mathbb{R}^d)$ is also an example on Wiener-amalgam spaces (cf. [11]).

There are several convenient characterizations of modulation spaces. For example, in [9,13,17,18], it is shown that modulation spaces admit reconstructible sequence space representations using Gabor frames.

In Section 2 we extend [3, Theorem 1] in several ways (see Theorems 2.1 and 2.3).

(1) The condition $q \in [1, \infty]$ is relaxed into $q \in (0, \infty]$.

(2) We allow weighted modulation spaces $M^{p,q}_{\omega}(\mathbb{R}^d)$, where the weight $\omega$ only depends on the momentum or frequency variable $\xi$, i.e. $\omega(x,\xi) = \omega(\xi)$. These weights are allowed to grow or decay at infinity, faster than polynomial growth.

(3) Our analysis also include continuity properties for the modulation spaces $W^{p,q}_{\omega}(\mathbb{R}^d)$.

In similar ways as in [3], we use Gabor analysis for modulation spaces to show these properties. In [3], the continuity for Fourier step multipliers are obtained by a convenient choice of Gabor atoms in terms of Fourier transforms of second order B-splines. This essentially transfer the critical continuity questions to a finite set of discrete convolution operators acting on $\ell^p$, with dominating operator being the discrete Hilbert transform. The choice of Gabor atoms then admit precise estimates of the appeared convolution operators.

In our situation the B-splines above are insufficient, because B-splines lack in regularity, and when $p$ approaches 0, unbounded regularity on the Fourier transform of the Gabor atoms are required. In fact, in order to obtain continuity for weighted modulation spaces with general moderate weights in the momentum variables, it is required that the Fourier transform of Gabor atoms obey even stronger regularities of Gevrey types.

In Section 4 we obtain some further extensions and deduce precise estimates of the Fourier multipliers in (0.1), where more restrictive $a_0$ should belong to $\ell^q(b\mathbb{Z})$ for some $q \in (0, \infty]$. In the end we are able to prove that the Fourier multiplier in (0.1) is continuous from $M^{p,q_1}$ to $M^{p,q_2}$ when $p \in (1, \infty)$ and $q_1,q_2 \in (0, \infty]$ satisfy

$$\frac{1}{q_2} - \frac{1}{q_1} \leq \frac{1}{q}$$

More generally, in Section 4 we generalize the continuity properties for the step and Fourier step multiplier results in Section 2 with more general slope step multiplier and Fourier slope step multipliers.
An important ingredient for the proofs of the latter extension is multiplication and convolution properties for $M_{p,q}^{0,q_0} \subseteq S(R^d)$ and $W_{p,q}^{0,q_0}$ spaces, given in Section 3.

**Proposition 0.1.** Let $p_j, q_j \in (0, \infty], j = 0, 1, 2,$

$$\theta_1 = \max \left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2} \right) \quad \text{and} \quad \theta_2 = \max \left(1, \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{q_0} \right).$$

Then

$$M^{p_1,q_1} \cdot M^{p_2,q_2} \subseteq M^{p_0,q_0}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \theta_1 + \frac{1}{q_0}. $$

$$M^{p_1,q_1} \ast M^{p_2,q_2} \subseteq M^{p_0,q_0}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \theta_2 + \frac{1}{p_0}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0}. $$

Similar result holds for $W_{p,q}^{0,q_0}$ spaces. The general multiplication and convolution properties in Section 3 also overlap with results by Bastianoni, Cordero and Nicola in [1], by Bastianoni and Teofanov in [2], and by Guo, Chen, Fan and Zhao in [21].

In Section 3 we extend the multiplication and convolution results in [1, 2, 21] to allow more general weights as well as finding multi-linear versions. We stress that the results in Section 3 hold true for general moderate weights, while corresponding results in [21] are formulated only for polynomially moderate weights which also should be split, i.e. of the form $\omega(x, \xi) = \omega_1(x) \omega_2(\xi)$. In Section 3 we also carry out questions on uniqueness for extensions of multiplications and convolutions from the Gelfand-Shilov space $\Sigma_1(R^d)$, to the involved modulation spaces. Note that $\Sigma_1(R^d)$ is dense in $\mathcal{S}(R^d)$ and is contained in all modulation spaces with moderate weights.
The analysis to show Proposition 0.1 is more complex compared to the restricted case when \( p_j, q_j \geq 1 \), because of absence of local-convexity of involved spaces when some of the Lebesgue exponents are smaller than one. In fact, the desired estimates when \( p_j, q_j \geq 1 \) can be achieved by straightforward applications of Hölder’s and Young’s inequalities. For corresponding estimates in Proposition 0.1', some additional arguments seems to be needed. In our situation we discretize the situations in similar ways as in [1] by using Gabor analysis for modulation spaces, and then apply some further arguments, valid in non-convex analysis. This approach is slightly different compared to what is used in [21] which follows the discretization technique introduced in [36], and which has some traces of Gabor analysis.

A non-trivial question concerns whether the multiplications and convolutions in Propositions 0.1 and 0.1' are uniquely defined or not. If \( p_j, q_j < \infty \), \( j = 1, 2 \), then the uniqueness is evident because the Schwartz space is dense in \( M^{p_j, q_j} \). In the case \( p_1, q_1 < \infty \) or \( p_2, q_2 < \infty \), the uniqueness in Proposition 0.1 follows from the first case, duality and embedding properties for quasi-Banach modulation spaces into Banach modulation spaces. The uniqueness in 0.1' then follows from the uniqueness in Proposition 0.1 and the fact that \( M^{p,q} \) increases with \( p \) and \( q \).

A critical situation appear when \( p_1 + q_1 = p_2 + q_2 = \infty \). Then \( \mathcal{S} \) is neither dense in \( M^{p_1,q_1} \) nor in \( M^{p_2,q_2} \). For the multiplications in Propositions 0.1, the uniqueness can be obtained by suitable approaches based on the so-called narrow convergence, which is a weaker form of convergence compared to norm convergence (see [28,29,31]). However, for the convolution in Propositions 0.1, we are not able to show any uniqueness of these extensions in this critical situation.

The paper is organized as follows. In Section 1 we present well-known properties of Gelfand-Shilov spaces, modulation spaces, multipliers and Fourier multipliers. In Section 2 we deduce continuity properties for step and Fourier step multipliers when acting on (quasi-Banach) modulation spaces. Then we establish convolution and continuity properties for quasi-Banach modulation spaces in Section 3. In Section 4 we show how the multiplication and convolution results in Section 3 can be used to generalize the continuity results in Section 2 to more general slope step multiplier and Fourier slope step multipliers. Finally we present a proof of a multi-linear convolution result in Appendix A.

**Acknowledgement**

The idea of the paper appeared when I supervised Nils Zandler-Andersson for his bachelor degree (see [37]). In those thesis, Mr. Andersson deduced some extensions of the multiplier results in [3] to certain quasi-Banach modulation spaces (see [37, Theorem 4.16]). I am also grateful to Elena Cordero and Nenad Teofanov for reading the paper and giving valuable comments, leading to improvements of the content.
1. Preliminaries

In this section we recall some facts on Gelfand-Shilov spaces, modulation spaces, discrete convolutions, step and Fourier step multipliers. After explaining some properties of the Gelfand-Shilov spaces and their distribution spaces, we consider a suitable twisted convolution and recall some facts on weight functions and mixed norm spaces. Thereafter we consider classical modulation spaces, which are more general compared Feichtinger in [8] in the sense of more general weights as well as we permit the Lebesgue exponents to belong to the full interval \((0, \infty]\) instead of \([1, \infty]\). Here we also recall some facts on Gabor expansions for modulation spaces. Then we collect some facts on discrete convolution estimates on weighted \(\ell^p\) spaces with the exponents in the full interval \((0, \infty]\). We finish the section by giving the definition of step and Fourier step multipliers.

1.1. Gelfand-Shilov spaces and their distribution spaces. For any \(0 < h, s, \sigma \in \mathbb{R}\), \(S^\sigma_{s,h}(\mathbb{R}^d)\) consists of all \(f \in C^\infty(\mathbb{R}^d)\) such that

\[
\|f\|_{S^\sigma_{s,h}} = \sup_{|\alpha| + |\beta| \leq \sigma_0} \frac{|\partial^\alpha \partial^\beta f(x)|}{h^{\alpha + \beta}} s^{\sigma_0} \|s,h\|,
\]

is finite. Then \(S^\sigma_{s,h}(\mathbb{R}^d)\) is a Banach space with norm \(\| \cdot \|_{S^\sigma_{s,h}}\). The Gelfand-Shilov spaces \(S^\sigma_{s,h}(\mathbb{R}^d)\) and \(\Sigma^\sigma_{s,h}(\mathbb{R}^d)\), of Roumieu and Beurling types respectively, are the inductive and projective limits of \(S^\sigma_{s,h}(\mathbb{R}^d)\) with respect to \(h > 0\) (see e.g. [15]). It follows that

\[
S^\sigma_s(\mathbb{R}^d) = \bigcup_{h > 0} S^\sigma_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma^\sigma_s(\mathbb{R}^d) = \bigcap_{h > 0} S^\sigma_{s,h}(\mathbb{R}^d) \quad (1.2)
\]

We remark that \(\Sigma^\sigma_s(\mathbb{R}^d) \neq \{0\}\), if and only if \(s + \sigma > 1\), and \(S^\sigma_s(\mathbb{R}^d) \neq \{0\}\), if and only if \(s + \sigma > 1\), and that

\[
S^\sigma_{s_1}(\mathbb{R}^d) \subseteq \Sigma^\sigma_{s_2}(\mathbb{R}^d) \subseteq S^\sigma_{s_2}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d), \quad s_1 < s_2, \ \sigma_1 < \sigma_2.
\]

The Gelfand-Shilov distribution spaces \((S^\sigma_s)'(\mathbb{R}^d)\) and \((\Sigma^\sigma_s)'(\mathbb{R}^d)\), of Roumieu and Beurling types respectively, are the (strong) duals of \(S^\sigma_s(\mathbb{R}^d)\) and \(\Sigma^\sigma_s(\mathbb{R}^d)\), respectively. It follows that if \((S^\sigma_{s,h})'(\mathbb{R}^d)\) is the \(L^2\)-dual of \(S^\sigma_{s,h}(\mathbb{R}^d)\) and \(s + \sigma \geq 1\) \((s + \sigma > 1)\), then \((S^\sigma_{s,h})'(\mathbb{R}^d)\) \((\Sigma^\sigma_{s,h})'(\mathbb{R}^d)\) can be identified with the projective limit (inductive limit) of \((S^\sigma_{s,h})'(\mathbb{R}^d)\) with respect to \(h > 0\). It follows that

\[
(S^\sigma_s)'(\mathbb{R}^d) = \bigcap_{h > 0} (S^\sigma_{s,h})'(\mathbb{R}^d) \quad \text{and} \quad (\Sigma^\sigma_s)'(\mathbb{R}^d) = \bigcup_{h > 0} (S^\sigma_{s,h})'(\mathbb{R}^d) \quad (1.3)
\]

for such choices of \(s\) and \(\sigma\). (See [24].) We remark that

\[
\mathcal{S}'(\mathbb{R}^d) \subseteq (S^\sigma_{s_2})'(\mathbb{R}^d) \subseteq (\Sigma^\sigma_{s_2})'(\mathbb{R}^d) \subseteq (S^\sigma_{s_1})'(\mathbb{R}^d),
\]

when

\[
s_1 < s_2, \ \sigma_1 < \sigma_2 \quad \text{and} \quad s_1 + \sigma_1 \geq 1.
\]

For convenience we set \(S_s = S^s_s\) and \(\Sigma_s = \Sigma^s_s\).

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations and under
(partial) Fourier transformations. In fact, let $\mathcal{F}$ be the Fourier transform which takes the form

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i(x,\xi)} \, dx$$

when $f \in L^1(\mathbb{R}^d)$. Here $(\cdot, \cdot)$ denotes the usual scalar product on $\mathbb{R}^d$. The map $\mathcal{F}$ extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbb{R}^d)$, from $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ to $(\mathcal{S}_s^\sigma)'(\mathbb{R}^d)$ and from $(\Sigma_s^\sigma)'(\mathbb{R}^d)$ to $(\Sigma_s^\sigma)'(\mathbb{R}^d)$. Then the map $\mathcal{F}$ restricts to homeomorphisms on $\mathcal{S}'(\mathbb{R}^d)$, from $\mathcal{S}_s^\sigma(\mathbb{R}^d)$ to $\mathcal{S}_s^\sigma(\mathbb{R}^d)$, from $\Sigma_s^\sigma(\mathbb{R}^d)$ to $\Sigma_s^\sigma(\mathbb{R}^d)$, and to a unitary operator on $L^2(\mathbb{R}^d)$.

There are several characterizations of Gelfand-Shilov spaces and their distribution spaces (cf. [34] and the references therein). For example, it follows from [6,7] that the following is true. Here $g(\theta) \lesssim h(\theta)$, $\theta \in \Omega$, means that there is a constant $c > 0$ such that $g(\theta) \leq c h(\theta)$ for all $\theta \in \Omega$.

**Proposition 1.1.** Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $s, \sigma > 0$. Then the following conditions are equivalent:

1. $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ ($f \in \Sigma_s^\sigma(\mathbb{R}^d)$);
2. $|f(x)| \lesssim e^{-r|x|^\frac{1}{2}}$ and $|\hat{f}(\xi)| \lesssim e^{-r|\xi|^\frac{1}{2}}$ for some $r > 0$ (for every $r > 0$);
3. $f \in C^\infty(\mathbb{R}^d)$ and $|\partial^\alpha f(x)| \lesssim h|\alpha|! \sigma e^{-r|x|^\frac{1}{2}}$ for some $r, h > 0$ (for every $r, h > 0$).

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates on their short-time Fourier transforms Let $\phi \in \mathcal{S}_s(\mathbb{R}^d)$ ($\phi \in \Sigma_s(\mathbb{R}^d)$) be fixed. Then the short-time Fourier transform of $f \in \mathcal{S}_s^\sigma(\mathbb{R}^d)$ (of $f \in \Sigma_s^\sigma(\mathbb{R}^d)$) with respect to $\phi$ is defined by

$$(V_\phi f)(x, \xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i(x,\xi)} \, \phi(x \cdot - y) \, dy. \quad (1.4)$$

We observe that

$$(V_\phi f)(x, \xi) = \mathcal{F}(f \cdot \overline{\phi(x \cdot - y)})(\xi) \quad (1.4)'$$

(cf. [31]). If in addition $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then

$$(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{f}(y) \phi(y \cdot - x) e^{-i(y,\xi)} \, dy. \quad (1.4)''$$

In the next lemma we present characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of estimates on the short-time Fourier transforms of the involved elements. The proof is omitted, since the first part follows from [20], and the second part from [31,33].

**Lemma 1.2.** Let $p \in [1, \infty]$, $f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)$, $s, \sigma > 0$, $\phi \in \mathcal{S}_s^\sigma(\mathbb{R}^d) \setminus 0$ ($\phi \in \Sigma_s^\sigma(\mathbb{R}^d) \setminus 0$) and

$$v_r(x, \xi) = e^{r(|x|^\frac{1}{2} + |\xi|^\frac{1}{2})}, \quad r \geq 0.$$ 

Then the following is true:
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(1) \( f \in S^r_s(R^d) (f \in \Sigma^s_s(R^d)) \), if and only if
\[ \|V_{\phi} f \cdot v_r\|_{L^p} < \infty \] (1.5)
for some \( r > 0 \) (for every \( r > 0) \);
(2) \( f \in (S^0_s)'(R^d) (f \in (\Sigma^0_s)'(R^d)) \), if and only if
\[ \|V_{\phi} f / v_r\|_{L^p} < \infty \] (1.6)
for every \( r > 0 \) (for some \( r > 0) \).

We also need the following. Here the first part is a straight-forward conse-
quity of the definitions, and the second part follows from the first part
duality. The details are left for the reader.

**Proposition 1.3.** Let \( \phi \in \Sigma_s(R^d) \setminus 0 \). Then the following is true:

1. \( V_{\phi} \) is continuous from \( \Sigma_s(R^d) \) to \( \Sigma_s(R^{2d}) \) and from \( \Sigma'_s(R^d) \) to \( \Sigma'_s(R^{2d}) \);

2. \( V_{\phi}^* \) is continuous from \( \Sigma_s(R^{2d}) \) to \( \Sigma_s(R^d) \) and from \( \Sigma'_s(R^{2d}) \) to \( \Sigma'_s(R^d) \).

The same holds true with \( S_s \) or \( \mathcal{J} \) in place of \( \Sigma_s \) at each occurrence.

1.2. A suitable twisted convolution. Let \( f \) be a distribution on \( R^d \),
\( \phi, \phi_j, j = 1, 2, 3 \), be suitable test functions on \( R^d \), and let \( F \) and \( G \) be a pair of
suitable distribution/test function on \( R^{2d} \). Then the twisted convolution
\( F \ast_V G \) of \( F \) and \( G \) is defined by
\[
(F \ast_V G)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{R^{2d}} F(x - y, \xi - \eta) G(y, \eta) e^{-i(y,\xi,\eta)} \, dy \, d\eta,
\]
\[
= (2\pi)^{-\frac{d}{2}} \int_{R^{2d}} F(y, \eta) G(x - y, \xi - \eta) e^{-i(x,\xi,\eta)} \, dy \, d\eta. \tag{1.7}
\]
The convolution above should be interpreted as
\[
(F \ast_V G)(X) = (2\pi)^{-\frac{d}{2}} (F(X - \cdot) e^{-i\Phi(X, \cdot)}, G)
\]
\[
= (2\pi)^{-\frac{d}{2}} (F, G \ast_X e^{-i\Phi(X, X - \cdot)}), \tag{1.7}'
\]
where \( \Phi(X, Y) = \langle y, \xi - \eta \rangle \), \( X = (x, \xi) \in R^{2d}, Y = (y, \eta) \in R^{2d} \),
when \( F \) belongs to a distribution space on \( R^{2d} \) and \( G \) belongs to the corre-
sponding test function space. By straight-forward computations it follows that
\[
(F \ast_V G) \ast_V H = F \ast_V (G \ast_V H), \tag{1.8}
\]
when \( F, H \) are distributions and \( G \) is a test function, or \( F, H \) are test
functions and \( G \) is a distribution.

**Remark 1.4.** Let \( s > 0 \). An important property of \( \ast_V \) above is that if
\( f \in \Sigma'_s(R^d) \) and \( \phi_j \in \Sigma_s(R^d) \) and \( \phi \in \Sigma_s(R^d) \setminus 0, j = 1, 2, 3 \), then it follows by straight-forward applications of Parseval’s formula that
\[
((V_{\phi_3} \phi_3) \ast_V (V_{\phi_1} f))(x, \xi) = (\phi_3, \phi_1)_{L^2} \cdot (V_{\phi_2} f)(x, \xi). \tag{1.9}
\]
and that if
\[
P_{\phi} \equiv \|\phi\|_{L^2}^{-2} \cdot V_{\phi} \circ V_{\phi}^*, \tag{1.10}
\]
then

$$P_\phi F = \|\phi\|_{L^2}^2 \cdot V_\phi \phi \ast V F$$ (1.11)

when $F \in \Sigma'_s(\mathbb{R}^{2d})$. We observe that

$$P_\phi^* = P_\phi$$ and $P_\phi^2 = P_\phi$. (1.12)

(See e.g. Chapters 11 and 12 in [17].)

We also remark that if $F \in \Sigma'_s(\mathbb{R}^{2d})$, then $F = V_\phi f$ for some $f \in \Sigma'_s(\mathbb{R}^d)$, if and only if

$$F = P_\phi F.$$ (1.13)

Furthermore, if (1.13) holds, then $F = V_\phi f$ with

$$f = \|\phi\|_{L^2}^2 V_\phi^* F.$$ (1.14)

In fact, suppose that $f \in \Sigma'_s(\mathbb{R}^d)$ and let $F = V_\phi f$. Then (1.13) follows from (1.14).

On the other hand, suppose that (1.13) holds and let $f$ be given by (1.14). Then

$$V_\phi f = P_\phi F = F,$$

and the asserted equivalence follows.

We notice that the same holds true with $\mathcal{S}_s$ or $\mathcal{S}$ in place of $\Sigma_s$ at each occurrence.

1.3. Mixed norm space of Lebesgue types. A weight on $\mathbb{R}^d$ is a function $\omega_0 \in L^\infty_{loc}(\mathbb{R}^d)$ such that $1/\omega_0 \in L^\infty_{loc}(\mathbb{R}^d)$. The weight $\omega_0$ on $\mathbb{R}^d$ is called moderate, if there is an other weight $v$ on $\mathbb{R}^d$ such that

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d.$$ (1.15)

The set of moderate weights on $\mathbb{R}^d$ is denoted by $\mathcal{P}_E(\mathbb{R}^d)$, and if $s > 0$, then $\mathcal{P}_{E,s}(\mathbb{R}^d)$ is the set of all moderate weights $\omega_0$ on $\mathbb{R}^d$ such that (1.15) holds for $v(y) = e^{r|y|^{2s}}$ for some $r > 0$. We also let $\mathcal{P}_{E,s}^*(\mathbb{R}^{2d})$ be the set of all weights $\omega$ such that

$$\omega(x + y, \xi + \eta) \lesssim \omega(x, \xi) e^{r(|y|^{1s} + |\eta|^{1s})}$$

for some $r > 0$. We recall that if $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, then there is a constant $r \geq 0$ such that

$$\omega(x + y) \lesssim \omega(x)e^{r|y|}, \quad x, y \in \mathbb{R}^d.$$ In particular, $\mathcal{P}_{E,s}(\mathbb{R}^d) = \mathcal{P}_E(\mathbb{R}^d)$ when $s \leq 1$ (see [19]).

For any weight $\omega$ on $\mathbb{R}^{2d}$ and for every $p, q \in (0, \infty)$, we set

$$\|F\|_{L^p_{\omega}^{q}(\mathbb{R}^{2d})} \equiv \|G_{F,\omega,p}\|_{L^q(\mathbb{R}^d)}, \quad \text{where} \quad G_{F,\omega,p}(\xi) = \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p(\mathbb{R}^d)}$$

and

$$\|F\|_{L^p_{\omega}^{q} (\mathbb{R}^{2d})} \equiv \|H_{F,\omega,q}\|_{L^p(\mathbb{R}^d)}, \quad \text{where} \quad H_{F,\omega,q}(x) = \|F(x, \cdot)\omega(x, \cdot)\|_{L^q(\mathbb{R}^d)},$$

when $F$ is (complex-valued) measurable function on $\mathbb{R}^{2d}$. Then $L^p_{\omega}(\mathbb{R}^{2d})$ ($L^p_{\omega}^{q}(\mathbb{R}^{2d})$) consists of all measurable functions $F$ such that $\|F\|_{L^p_{\omega}} < \infty$ ($\|F\|_{L^p_{\omega}^{q}} < \infty$).
In similar ways, let \( \Omega_1, \Omega_2 \) be discrete sets and \( \ell'_0(\Omega_1 \times \Omega_2) \) consists of all formal (complex-valued) sequences \( c = \{c(j, k)\}_{j \in \Omega_1, k \in \Omega_2} \). Then the discrete Lebesgue spaces
\[
\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2) \quad \text{and} \quad \ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)
\]
of mixed (quasi-)norm types consists of all \( c \in \ell'_0(\Omega_1 \times \Omega_2) \) such that \( \|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2)} < \infty \) respectively \( \|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)} < \infty \). Here
\[
\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2)} \equiv \|G_{c,\omega,p}(k)\|_{\ell^p(\Omega_1)} \quad \text{where} \quad G_{c,\omega,p}(k) = \|F(\cdot, k)\omega(\cdot, k)\|_{\ell^p(\Omega_1)}
\]
and
\[
\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)} \equiv \|H_{c,\omega,q}(j)\|_{\ell^q(\Omega_2)} \quad \text{where} \quad H_{c,\omega,q}(j) = \|c(j, \cdot)\omega(j, \cdot)\|_{\ell^q(\Omega_2)}
\]
when \( c \in \ell'_0(\Omega_1 \times \Omega_2) \).

1.4. Modulation spaces and other Wiener type spaces. The (classical) modulation spaces, essentially introduced in [8] by Feichtinger are given in the following. (See e. g. [10] for definition of more general modulation spaces.)

Definition 1.5. Let \( p, q \in (0, \infty] \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and \( \phi \in \Sigma_1(\mathbb{R}^d) \setminus 0 \).

1. The modulation space \( M_{(\omega)}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \Sigma'_1(\mathbb{R}^d) \) such that
\[
\|f\|_{M_{(\omega)}^{p,q}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}
\]
is finite. The topology of \( M_{(\omega)}^{p,q}(\mathbb{R}^d) \) is defined by the (quasi-)norm \( \| \cdot \|_{M_{(\omega)}^{p,q}} \).

2. The modulation space (of Wiener amalgam type) \( W_{(\omega)}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \Sigma'_1(\mathbb{R}^d) \) such that
\[
\|f\|_{W_{(\omega)}^{p,q}} \equiv \|V_\phi f\|_{L^{p,q}_{*,(\omega)}}
\]
is finite. The topology of \( W_{(\omega)}^{p,q}(\mathbb{R}^d) \) is defined by the (quasi-)norm \( \| \cdot \|_{W_{(\omega)}^{p,q}} \).

Remark 1.6. Modulation spaces possess several convenient properties. In fact, let \( p, q \in (0, \infty] \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) and \( \phi \in \Sigma_1(\mathbb{R}^d) \setminus 0 \). Then the following is true (see [8,10,12,14,17] and their analyses for verifications):

- the definitions of \( M_{(\omega)}^{p,q}(\mathbb{R}^d) \) and \( W_{(\omega)}^{p,q}(\mathbb{R}^d) \) are independent of the choices of \( \phi \in \Sigma_1(\mathbb{R}^d) \setminus 0 \), and different choices give rise to equivalent quasi-norms;
- the spaces \( M_{(\omega)}^{p,q}(\mathbb{R}^d) \) and \( W_{(\omega)}^{p,q}(\mathbb{R}^d) \) are quasi-Banach spaces which increase with \( p \) and \( q \), and decrease with \( \omega \). If in addition \( p, q \geq 1 \), then they are Banach spaces.
- \( \Sigma_1(\mathbb{R}^d) \subseteq M_{(\omega)}^{p,q}(\mathbb{R}^d), W_{(\omega)}^{p,q}(\mathbb{R}^d) \subseteq \Sigma_1(\mathbb{R}^d) \);
- If in addition \( p, q \geq 1 \), then the \( L^2(\mathbb{R}^d) \) scalar product, \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \), on \( \Sigma_1(\mathbb{R}^d) \times \Sigma(\mathbb{R}^d) \) is uniquely extendable to dualities between \( M_{(\omega)}^{p,q}(\mathbb{R}^d) \) and \( M_{(1/\omega)}^{p,q}(\mathbb{R}^d) \), and between \( W_{(\omega)}^{p,q}(\mathbb{R}^d) \) and \( W_{(1/\omega)}^{p,q}(\mathbb{R}^d) \).
If in addition $p, q < \infty$, then the duals of $M_{(p, q)}^{\omega}(\mathbb{R}^d)$ and $W_{(p, q)}^{\omega}(\mathbb{R}^d)$ can be identified with $M_{p'}^{\omega}(\mathbb{R}^d)$ respectively $W_{p'}^{\omega}(\mathbb{R}^d)$, through the form $(\cdot, \cdot)_{L^2(\mathbb{R}^d)}$.

- Let $\omega_0(x, \xi) = \omega(-\xi, x)$. Then $\mathcal{F}$ on $\Sigma_1'(\mathbb{R}^d)$ restricts to a homeomorphism from $M_{(p, q)}^{\omega}(\mathbb{R}^d)$ to $W_{(p, q)}^{\omega}(\mathbb{R}^d)$.

1.5. **Gabor expansions for modulation spaces.** A fundamental property for modulation spaces is that they can be discretized in convenient ways by Gabor expansions. For fundamental contributions, see e.g. [5, 9, 11–14, 16, 17, 20] and the references therein. Here we present a straight way to obtain such expansions in the case when we may find compactly supported Gabor atoms.

Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$. Then $\mathcal{D}^{\sigma}(\mathbb{R}^d)$ is the set of all compactly supported elements in $\mathcal{S}'_s(\mathbb{R}^d)$. That is, $\mathcal{D}^{\sigma}(\mathbb{R}^d)$ consists of all $\phi \in C^\infty_0(\mathbb{R}^d)$ such that

$$\|\partial^\alpha \phi\|_{L^\infty} \lesssim h^{\|\alpha\| \sigma}$$

holds true for some $h > 0$. We recall that if $\sigma \leq 1$, then $\mathcal{D}^{\sigma}(\mathbb{R}^d)$ is trivial (i.e. $\mathcal{D}^{\sigma}(\mathbb{R}^d) = \{0\}$). If instead $\sigma > 1$, then $\mathcal{D}^{\sigma}(\mathbb{R}^d)$ is dense in $C^\infty_0(\mathbb{R}^d)$.

From now on we suppose that $\sigma > 1$, giving that $\mathcal{D}^{\sigma}(\mathbb{R}^d)$ is non-trivial. In view of Sections 1.3 and 1.4 in [22], we may find $\phi, \psi \in \mathcal{D}^{\sigma}(\mathbb{R}^d)$ with values in $[0, 1]$ such that

$$\text{supp } \phi \subseteq \left[-\frac{3}{4}, \frac{3}{4}\right]^d, \quad \phi(x) = 1 \text{ when } x \in \left[-\frac{1}{4}, \frac{1}{4}\right]^d \quad (1.16)$$

$$\text{supp } \psi \subseteq [-1, 1]^d, \quad \psi(x) = 1 \text{ when } x \in \left[-\frac{3}{4}, \frac{3}{4}\right]^d \quad (1.17)$$

and

$$\sum_{j \in \mathbb{Z}^d} \phi(\cdot - j) = 1. \quad (1.18)$$

Let $f \in (\mathcal{S}^s_\sigma)'(\mathbb{R}^d)$. Then $x \mapsto f(x)\phi(x - j)$ belongs to $(\mathcal{S}^s_\sigma)'(\mathbb{R}^d)$ and is supported in $j + \left[-\frac{3}{4}, \frac{3}{4}\right]^d$. Hence, by periodization it follows from Fourier analysis that

$$f(x)\phi(x - j) = \sum_{i \in \pi \mathbb{Z}^d} c(j, i)e^{i(x, i)}, \quad x \in j + [-1, 1]^d, \quad (1.19)$$

where

$$c(j, i) = 2^{-d}(f, \phi(\cdot - j)e^{i(\cdot, i)}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} V_\phi f(j, i), \quad j \in \mathbb{Z}^d, \quad i \in \pi \mathbb{Z}^d.$$

Since $\psi = 1$ on the support of $\phi$, (1.19) gives

$$f(x)\phi(x - j) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{i \in \pi \mathbb{Z}^d} V_\phi f(j, i)\psi(x - j)e^{i(x, i)}, \quad x \in \mathbb{R}^d, \quad (1.19)'$$

By (1.18) it now follows that

$$f(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j, i) \in \Lambda} V_\phi f(j, i)\psi(x - j)e^{i(x, i)}, \quad x \in \mathbb{R}^d, \quad (1.20)$$
where

\[ \Lambda = \mathbb{Z}^d \times (\pi \mathbb{Z}^d), \]

which is the Gabor expansion of \( f \) with respect to the Gabor pair \((\phi, \psi)\) and lattice \( \Lambda \), i.e. with respect to the Gabor atom \( \phi \) and the dual Gabor atom \( \psi \). Here the series converges in \((S^p_s)'(\mathbb{R}^d)\). By duality and the fact that \( D^\sigma(\mathbb{R}^d) \) is dense in \((S^p_s)'(\mathbb{R}^d)\) we also have

\[ f(x) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} \sum_{(j,\iota) \in \Lambda} V_{\psi} f(j, \iota) \phi(x - j) e^{i(x, \iota)} , \quad x \in \mathbb{R}^d, \tag{1.22} \]

with convergence in \((S^p_s)'(\mathbb{R}^d)\).

Let \( T \) be a linear continuous operator from \( S^p_s(\mathbb{R}^d) \) to \((S^p_s)'(\mathbb{R}^d)\) and let \( f \in S^p_s(\mathbb{R}^d) \). Then it follows from (1.20) that

\[ (Tf)(x) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} \sum_{(j,\iota) \in \Lambda} V_{\phi, f}(j, \iota) T(\psi(\cdot - j)e^{i(x, \iota)})(x) \]

and

\[ T(\psi(\cdot - j)e^{i(\cdot, \iota)})(x) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} \sum_{(k,\kappa) \in \Lambda} (V_\psi T(\psi(\cdot - j)e^{i(\cdot, \iota)}))(k, \kappa) \psi(x - k)e^{i(x, \kappa)}. \]

A combination of these expansions show that

\[ (Tf)(x) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} \sum_{(j,\iota) \in \Lambda} (A \cdot V_{\psi, f})(j, \iota) \psi(x - j)e^{i(x, \iota)}, \tag{1.23} \]

where \( A = (a(j, K))_{j, K} \), \( j, K \in \Lambda \) is the \( \Lambda \times \Lambda \)-matrix, given by

\[ a(j, k) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} (T(\psi(\cdot - j)e^{i(\cdot, \iota)}), \phi(\cdot - k)e^{i(\cdot, \kappa)})_{L^2(\mathbb{R}^d)} \]

when \( j = (j, \iota) \) and \( k = (k, \kappa) \). \hspace{1cm} (1.24)

By the Gabor analysis for modulation spaces we get the following. We refer to \([11, 14, 16, 17, 32]\) for details.

**Proposition 1.7.** Let \( \sigma > 1 \), \( s \geq 1 \), \( p, q \in (0, \infty) \), \( \omega \in \mathscr{D}_E(\mathbb{R}^{2d}) \), \( \phi, \psi \in \mathscr{D}(\mathbb{R}^d; [0, 1]) \) be such that \([11, 16], [11, 17] \) and \([11, 18]\) hold true, and let \( f \in \mathscr{D}(\mathbb{R}^d) \). Then the following is true:

1. \( f \in M^{p,q}_\omega(\mathbb{R}^d) \), if and only if \( \|V_\phi f\|_{L^p_{\omega}(\mathbb{Z}^d \times \mathbb{Z}^d)} \); 
2. \( f \in M^{p,q}_\omega(\mathbb{R}^d) \), if and only if \( \|V_\psi f\|_{\ell^q_{\omega}(\mathbb{Z}^d \times \mathbb{Z}^d)} \);
3. the quasi-norms

\[ f \mapsto \|V_\phi f\|_{L^p_{\omega}(\mathbb{Z}^d \times \mathbb{Z}^d)} \quad \text{and} \quad f \mapsto \|V_\psi f\|_{\ell^q_{\omega}(\mathbb{Z}^d \times \mathbb{Z}^d)} \]

are equivalent to \( \| \cdot \|_{M^{p,q}_\omega} \).

The same holds true with \( W^{p,q}_\omega \) and \( \ell^{p,q}_{\omega} \) in place of \( M^{p,q}_\omega \) respectively \( \ell^{p,q}_{\omega} \) at each occurrence.
Remark 1.8. There are weights \( \omega \in \mathcal{P}_E(\mathbb{R}^{d}) \) such that corresponding modulation spaces \( M^{p,q}_\omega(\mathbb{R}^d) \) and \( W^{p,q}_\omega(\mathbb{R}^d) \) do not contain \( \mathcal{D}'(\mathbb{R}^d) \) for any choice of \( \sigma > 1 \). In this situation, it is not possible to find compactly supported elements in Gabor pairs which can be used for expanding all elements in \( M^{p,q}_\omega(\mathbb{R}^d) \) and \( W^{p,q}_\omega(\mathbb{R}^d) \).

For a general weight \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) which is moderated by the submultiplicative weight \( v \in \mathcal{P}_E(\mathbb{R}^{2d}) \), we may always find a lattice \( \Lambda \in \mathbb{R}^d \) and a Gabor pair \((\phi, \psi)\) such that

\[
\phi, \psi \in \bigcap_{p>0} M^p(\mathbb{R}^d)
\]

and

\[
f(x) = C \sum_{j,\ell \in \Lambda} V_\phi f(j,\ell) \psi(x-j)e^{i(x,\ell)}
= C \sum_{j,\ell \in \Lambda} V_\phi f(j,\ell) \phi(x-j)e^{i(x,\ell)}, \quad f \in M^\infty_\omega(\mathbb{R}^d),
\]

for some constant \( C \), where the series convergence with respect to the weak* topology in \( M^\infty_\omega(\mathbb{R}^d) \). (See [16, Theorem S] and some further comments in [32]. See also [11–13] for more facts.) In such approach we still have that if \( p, q \in (0, \infty) \), then

\[
f \in M^{p,q}_\omega(\mathbb{R}^d) \iff \{V_\phi f(j,\ell)\}_{j,\ell \in \Lambda} \in \ell^{p,q}_\omega(\Lambda \times \Lambda)
\]

\[
\iff \{V_\psi f(j,\ell)\}_{j,\ell \in \Lambda} \in \ell^{p,q}_\omega(\Lambda \times \Lambda),
\]

\[
f \in W^{p,q}_\omega(\mathbb{R}^d) \iff \{V_\phi f(j,\ell)\}_{j,\ell \in \Lambda} \in \ell^{p,q}_{\ast}(\Lambda \times \Lambda)
\]

\[
\iff \{V_\psi f(j,\ell)\}_{j,\ell \in \Lambda} \in \ell^{p,q}_{\ast}(\Lambda \times \Lambda),
\]

\[
\|f\|_{M^{p,q}_\omega} \asymp \|V_\phi f\|_{\ell^{p,q}_\omega(\Lambda \times \Lambda)} \asymp \|V_\psi f\|_{\ell^{p,q}_{\ast}(\Lambda \times \Lambda)}.
\]

Furthermore, if \( f \in M^{p,q}_\omega(\mathbb{R}^d) \) \( (f \in W^{p,q}_\omega(\mathbb{R}^d)) \) and in addition \( p, q < \infty \), then the series in (1.25) converges with respect to the \( M^{p,q}_\omega \) quasi-norm \( (W^{p,q}_\omega \) quasi-norm).

Remark 1.9. Let \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \), \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}), p, q, r \in (0, \infty], Q_d = [0, 1]^d \) be the unit cube, and set for measurable \( f \) on \( \mathbb{R}^d \),

\[
\|f\|_{W^r(\omega,\mathbb{R}^r)} \equiv \|a_0\|_{L^r(\mathbb{Z}^d)}
\]

when

\[
a_0(j) \equiv \|f \cdot \omega_0\|_{L^r(j+Q_d)}, \quad j \in \mathbb{Z}^d,
\]

and measurable \( F \) on \( \mathbb{R}^{2d} \),

\[
\|F\|_{W^r(\omega,\mathbb{R}^r)} \equiv \|a\|_{L^r(\mathbb{Z}^d)} \quad \text{and} \quad \|F\|_{W^r_\omega(\mathbb{R}^r)} \equiv \|a\|_{L^{r,q}_\omega(\mathbb{Z}^d)}
\]

when

\[
a(j,\ell) \equiv \|F \cdot \omega\|_{L^r(\ell+j+Q_{2d})}, \quad j, \ell \in \mathbb{Z}^d.
\]
The Wiener space
\[ W^r(\omega_0, \ell^p) = W^r(\omega_0, \ell^p(\mathbb{Z}^d)) \]
consists of all measurable \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \) such that \( \|F\|_{W^r(\omega_0, \ell^p)} \) is finite, and the Wiener spaces
\[ W^r(\omega, \ell^{p,q}) = W^r(\omega, \ell^{p,q}(\mathbb{Z}^d)) \]
consist of all measurable \( F \in L^r_{\text{loc}}(\mathbb{R}^d) \) such that \( \|F\|_{W^r(\omega, \ell^{p,q})} \) respectively \( \|F\|_{W^r(\omega, \ell^{p,q})} \) are finite. The topologies are defined through their respectively quasi-norms in (1.30) and (1.31). For convenience we set
\[ W(\omega, \ell^{p,q}) = W^\infty(\omega, \ell^{p,q}) \]
and
\[ W(\omega, \ell^{p,q}) = W^\infty(\omega, \ell^{p,q}). \]
Obviously, \( W(\omega_0, \ell^p) \) and \( W(\omega, \ell^{p,q}) \) increase with \( p, q \), decrease with \( r \), and
\[ W(\omega, \ell^{p,q}) \hookrightarrow L^p_{\omega}(\mathbb{R}^d) \cap \Sigma'_1(\mathbb{R}^d) \hookrightarrow L^p_{\omega}(\mathbb{R}^d) \hookrightarrow W^r(\omega, \ell^{p,q}) \quad (1.32) \]
and
\[ \| \cdot \|_{W^r(\omega, \ell^{p,q})} \leq \| \cdot \|_{L^p_{\omega}} \leq \| \cdot \|_{W(\omega, \ell^{p,q})}, \quad r \leq \min(1, p, q). \quad (1.33) \]
On the other hand, for modulation spaces we have
\[ f \in M^{p,q}_{\omega}(\mathbb{R}^d) \iff V_\phi f \in L^p_{\omega}(\mathbb{R}^d) \iff V_\phi f \in W(\omega, \ell^{p,q}) \quad (1.34) \]
with
\[ \|f\|_{M^{p,q}_{\omega}} = \|V_\phi f\|_{L^p_{\omega}} \asymp \|V_\phi f\|_{W(\omega, \ell^{p,q})}. \quad (1.35) \]
The same holds true with \( W^{p,q}_{\omega}, L^{p,q}_{\omega} \) and \( W(\omega, \ell^{p,q}) \) in place of \( M^{p,q}_{\omega}, L^{p,q}_{\omega} \) and \( W(\omega, \ell^{p,q}) \), respectively, at each occurrence. (For \( r = \infty \), see [17], when \( p, q \in [1, \infty], [13, 32] \) when \( p, q \in (0, \infty] \), and for \( r \in (0, \infty) \), see [35].)

1.6. Convolutions and multiplications for discrete Lebesgue spaces.
Next we discuss extended Hölder and Young relations for multiplications and convolutions on discrete Lebesgue spaces. Here the involved weights should satisfy
\[ \omega_0(x) \leq \prod_{j=1}^N \omega_j(x) \quad (1.36) \]
or
\[ \omega_0(x_1 + \cdots + x_N) \leq \prod_{j=1}^N \omega_j(x_j), \quad (1.37) \]
and it is convenient to make use of the functional
\[ R_N(p_1, \ldots, p_N) = \left( \sum_{j=1}^N \max \left( 1, \frac{1}{p_j} \right) \right) - \min_{1 \leq j \leq N} \left( \max \left( 1, \frac{1}{p_j} \right) \right). \quad (1.38) \]
The Hölder and Young conditions on Lebesgue exponent are then
\[ \frac{1}{q_0} \leq \sum_{j=1}^N \frac{1}{q_j}. \quad (1.39) \]
respectively

\[
\frac{1}{p_0} \leq \sum_{j=1}^{N} \frac{1}{p_j} - R_N(p_1, \ldots, p_N).
\]  

(1.40)

**Proposition 1.10.** Let \( p_j, q_j \in (0, \infty) \) be such that (1.38), (1.39) and (1.40) hold, \( \omega_j \in \mathcal{P}_E(\mathbb{R}^d) \), and let \( \Lambda \subseteq \mathbb{R}^d \) be a lattice containing origin. Then the following is true:

1. If (1.36) holds true, then the map \( (a_1, \ldots, a_N) \mapsto a_1 \cdot \cdots \cdot a_N \) from \( \ell_0^p(\Lambda) \times \cdots \times \ell_0^p(\Lambda) \) to \( \ell_0^p(\Lambda) \), and

\[
\|a_1 \cdots a_N\|_{\ell_0^p(\omega_0)} \leq \prod_{j=1}^{N} \|a_j\|_{\ell_0^{q_j}(\omega_j)}, \quad a_j \in \ell_0^{q_j}(\omega_j), \ j = 1, \ldots, N; \tag{1.41}
\]

(2) If (1.37) holds true, then the map \( (a_1, \ldots, a_N) \mapsto a_1 \ast \cdots \ast a_N \) from \( \ell_0^p(\Lambda) \times \cdots \times \ell_0^p(\Lambda) \) to \( \ell_0^p(\Lambda) \), and

\[
\|a_1 \ast \cdots \ast a_N\|_{\ell_0^p(\omega_0)} \leq \prod_{j=1}^{N} \|a_j\|_{\ell_0^{q_j}(\omega_j)}, \quad a_j \in \ell_0^{q_j}(\omega_j), \ j = 1, \ldots, N. \tag{1.42}
\]

The assertion (1) in Proposition 1.10 is the standard Hölder’s inequality for discrete Lebesgue spaces. The assertion (2) in that proposition is the usual Young’s inequality for Lebesgue spaces on lattices in the case when \( p_1, \ldots, p_N \in [1, \infty] \). In order to be self-contained we give a proof when \( p_1, \ldots, p_N \) are allowed to belong to the full interval \((0, \infty)\) in Appendix A.

1.7. **Step and Fourier step multipliers.** Let \( b \in \mathbb{R}_+^d \) be fixed, \( \Lambda_b \) be the lattice given by

\[
\Lambda_b = \{ (b_1n_1, \ldots, b_dn_d) \in \mathbb{R}^d; (n_1, \ldots, n_d) \in \mathbb{Z}^d \}, \tag{1.43}
\]

\( Q_b \) be the \( b \)-cube, given by

\[
Q_b = \{ (b_1x_1, \ldots, b_dx_d) \in \mathbb{R}^d; (x_1, \ldots, x_d) \in [0, 1]^d \} \tag{1.44}
\]

and \( a_0 \in \ell^\infty(\Lambda_b) \). Then we let the Fourier step multiplier \( M_{\mathcal{F}, b, a_0} \) (with respect to \( b \) and \( a_0 \)) be defined by

\[
M_{\mathcal{F}, b, a_0} \equiv \mathcal{F}^{-1} \circ M_{b, a_0} \circ \mathcal{F}, \tag{1.45}
\]

where \( M_{b, a_0} \) is the multiplier

\[
M_{b, a_0} : f \mapsto \sum_{j \in \Lambda_b} a_0(j) \chi_{Q_b} f. \tag{1.46}
\]

Here \( \chi_{\Omega} \) is the characteristic function of \( \Omega \).
2. Step and Fourier step multipliers on modulation spaces

In this section we deduce continuity properties for step and Fourier step multipliers on modulation spaces (see Theorems 2.1 and 2.3 below). In contrast to [3], the results presented here permit Lebesgue exponents to be smaller than one.

We begin with step multipliers when acting on modulation spaces. Here involved Lebesgue exponents should fulfill

\[
\frac{1}{q_1} - \frac{1}{q_2} \geq \max \left( \frac{1}{p} - 1, 0 \right),
\]

Theorem 2.1. Let \( p \in (0, \infty], q \in (1, \infty), q_1, q_2 \in (\min(1, p), \infty) \) be such that (2.1) holds, \( b > 0, \omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \) and \( \omega(x, \xi) = \omega_0(x), x, \xi \in \mathbb{R}^d \). Let \( a_0 \in \ell^\infty(\Lambda_b) \). Then the following is true:

1. \( M_{b,a_0} \) is continuous on \( \mathcal{W}_{p,q}^0(\mathbb{R}^d) \);
2. \( M_{b,a_0} \) is continuous from \( \mathcal{M}_{p,q}(\mathbb{R}^d) \) to \( \mathcal{M}_{p,q_2}(\mathbb{R}^d) \).

We observe that the conditions on \( q_2 \) in Theorem 2.1 implies that \( q_2 > 1 \), since otherwise (2.1) should lead to \( q_1 \leq \min(p, 1) \), which contradicts the assumptions on \( q_1 \).

We need the following lemma for the proof of Theorem 2.1.

Lemma 2.2. Let \( p, q \in (1, \infty) \) and \( \theta \in (0, 1) \) be such that

\[
\theta + \frac{1}{p} = 1 + \frac{1}{q}
\]

and suppose that \( a = \{a(j)\}_{j \in \mathbb{Z}^d} \subseteq \mathbb{C} \) satisfies

\[
|a(j)| \lesssim (\langle j_1 \rangle \cdots \langle j_d \rangle)^{-\theta}.
\]

Then the map \( b \mapsto a \ast b \) from \( \ell_0(\mathbb{Z}^d) \) to \( \ell'_0(\mathbb{Z}^d) \) is uniquely extendable to a continuous mapping from \( \ell^p(\mathbb{Z}^d) \) to \( \ell^q(\mathbb{Z}^d) \).

We observe that the conditions in Lemma 2.2 implies that \( p < q \).

Lemma 2.2 is a straight-forward consequence of [22, Theorem 4.5.3]. In fact, by that theorem we have for

\[
h(x) = (|x_1| \cdots |x_d|)^{-1} \quad \text{and} \quad h_0(x) = ((x_1) \cdots (x_d))^{-1}
\]

that

\[
\|f \ast h^0\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)};
\]

when (2.2) holds. Since \( 0 < h_0(x) < h(x) \), we obtain

\[
\|f \ast h^0\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},
\]

which gives suitable boundedness properties for \( f \mapsto f \ast h^0 \).

We also have

\[
g_a(x) = \|a\|_{\ell^p(\mathbb{Z}^d)}, \quad g_a(x) = \sum_{j \in \mathbb{Z}^d} a(j) \chi_{j+[0,1]^d}(x).
\]

By a straight-forward combination of this estimate with (2.4) we obtain

\[
\|b \ast h^0\|_{\ell^p(\mathbb{Z}^d)} \lesssim \|b\|_{\ell^p(\mathbb{Z}^d)}.
\]
where now * denotes the discrete convolution. The continuity assertions in Lemma 2.2 now follows from

\[ \|b * a\|_{\ell^p(Z^d)} \lesssim \|b\|_{\ell^p(Z^d)}, \]

and the uniqueness assertions follows from the fact that \( \ell_0(Z^d) \) is dense in \( \ell^p(Z^d) \) when \( p < \infty \).

**Proof of Theorem 2.1.** By straight-forward computations it follows that if \( \omega_h(x, \xi) = \omega(bx, b^{-1}\xi) \), then \( f \in W_{\omega_h}^{\rho, q}(R^d) \), if and only if \( f(b \cdot) \in W_{\omega_h}^{\rho, q}(R^d) \), and

\[ \|f\|_{W_{\omega_h}^{\rho, q}} \asymp \|f(b \cdot)\|_{W_{\omega_h}^{\rho, q}}, \]

and similarly with \( M^{\rho, q} \) in place of \( W_{\omega_h}^{\rho, q} \) at each occurrence. This reduce ourself to the case when \( b = 1 \).

Let \( \phi, \psi \) and \( \Lambda \) be the same as in (1.16)–(1.18) and (1.21). By (1.24) we have

\[ M_{a, b, 0} f(x) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} \sum_{(j, \ell) \in \Lambda} (A \cdot V_{\phi}(j, \ell)) e^{i(x, \ell) \psi(x - j)}, \]

where \( A = (a(j, K))_{j, K} \) is the matrix with elements

\[ a(j, k) = \left( \frac{\pi}{2} \right)^{\frac{d}{2}} (M_{a, b, 0} e^{i(j, \ell) \psi(\cdot - j), e^{i(k, \kappa) \phi(\cdot - k))}, \]

when \( j = (j, \ell) \in \Lambda, k = (k, \kappa) \in \Lambda \).

Let \( Q = [0, 1]^d \) and

\[ \Omega_m = \{ j \in Z^d; |j_n| \leq m \text{ for every } n \in \{1, \ldots, d\} \}, \]

\[ m \in Z_+. \]

By the support properties of \( \phi \) and \( \psi \) we have

\[ a(j, k) = 0 \text{ when } j - k \notin \Omega_2, \]

and for \( j - k \in \Omega_2 \) we get

\[ |a(j, k)| \asymp \left| \int_{R^d} M_{b, a, 0} (e^{i(j, \kappa - l) \psi(\cdot - j))(y(\phi(y - k) dy)} \right| \]

\[ = \int_{R^d} \left( \sum_{l \in Z^d} a_0(l) \chi_Q(y - l) e^{i(j, \kappa - l) \psi(y - j) \phi(y - k) dy} \right) \]

\[ \leq \sum_{l \in Z^d} a_0(l) \left| \int_{R^d} \left( \chi_Q(y - (l - k)) e^{i(j, \kappa - l) \psi(y - (j - k)) \phi(y) dy} \right) \right| \]

\[ \leq \|a_0\|_{\ell^\infty(Z^d)} \sum_{l \in Z^d} \left| \chi_Q(\cdot - (l - k)) e^{i(j, \kappa - l) \psi(\cdot - (j - k)) \phi)_{L^2(R^d)} \right|, \]

(2.6)

where the last three sums are taken over all \( l \in Z^d \) such that \( l - (j - k) \in \Omega_3 \).

We have to estimate

\[ \left| \chi_Q(\cdot - (l - k)) e^{i(j, \kappa - l) \psi(\cdot - (j - k)) \phi)_{L^2(R^d)} \right| \]
when $j - k \in \Omega_2$ and $l - (j - k) \in \Omega_3$. By Parseval’s formula we get
\[
|\langle \chi_Q(\cdot - (l - k)), e^{-i(l - k)\cdot} \psi(\cdot - (j - k)) \phi \rangle_{L^2(\mathbb{R}^d)}| \\
= |(e^{-i(l-k)\cdot}g, V_\psi \phi(j-k, \cdot - (l - \kappa)) \rangle_{L^2(\mathbb{R}^d)}|,
\]
where
\[
g(\xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}(\xi_1 + \cdots + \xi_d)} \text{sinc}(\xi_1/2) \cdots \text{sinc}(\xi_d/2).
\]
Here
\[
\text{sinc} t = \begin{cases} \frac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0, \end{cases}
\]
is the sinc function.

Since
\[
sinc t \lesssim (t)^{-1} \quad \text{and} \quad |V_\psi \phi(j-k, \xi - (l - \kappa))| \lesssim e^{-r|\xi - (l - \kappa)|^{\frac{1}{2}}}
\]
we obtain
\[
|\langle \chi_Q(\cdot - (l - k)), e^{-i(l - k)\cdot} \psi(\cdot - (j - k)) \phi \rangle_{L^2(\mathbb{R}^d)}| \\
\leq \int_{\mathbb{R}^d} |g(\xi)| \cdot |V_\psi \phi(j-k, \xi - (l - \kappa))| \, d\xi \\
\lesssim \int_{\mathbb{R}^d} h_0(\xi) e^{-r|\xi - (l - \kappa)|^{\frac{1}{2}}} \, d\xi \\
= \int_{\mathbb{R}^d} h_0(\xi + \kappa) e^{-r|\xi|^{\frac{1}{2}}} \, d\xi \\
\leq h_0(\kappa) \int_{\mathbb{R}^d} \langle \xi_1 \rangle \cdots \langle \xi_d \rangle e^{-r|\xi|^{\frac{1}{2}}} \, d\xi \approx h_0(\kappa).
\]
Here $h_0$ is given by (2.3), and we have used
\[
h_0(\xi + \eta) = (\langle \xi_1 + \eta_1 \rangle \cdots \langle \xi_d + \eta_d \rangle)^{-1} \leq (\langle \eta_1 \rangle \cdots \langle \eta_d \rangle)^{-1} \langle \xi_1 \rangle \cdots \langle \xi_d \rangle.
\]
By inserting this into (2.6) we get
\[
|a(j, k)| \lesssim \sum_{l \in (j-k)+\Omega_3} h_0(\kappa) (\langle \xi_1 \rangle \cdots \langle \xi_d \rangle)^{-1} = 7^d h_0(\kappa).
\]
Hence,
\[
|a(j, k)| \lesssim \begin{cases} h_0(\kappa), & j - k \in \Omega_2 \\ 0, & j - k \notin \Omega_2. \end{cases} \tag{2.7}
\]
If $c(j, \iota) = |V_\phi f(j, \iota)|$, then (2.7) gives
\[
|\langle A \cdot V_\phi f(j, \iota) \rangle_{\Omega_2}| \lesssim \sum_{k \in j+\Omega_2} \left( \sum_{\kappa \in \mathbb{Z}^d} h_0(\kappa) c(k, \kappa) \right) \\
= \sum_{k \in j+\Omega_2} (h_0 * c(k, \cdot))(\iota) = \sum_{k \in \Omega_2} (h_0 * c(j+k, \cdot))(\iota), \tag{2.8}
\]
Here we have used the fact that the number of elements in $\Omega_2$ and Lemma 2.2 gives

$$\log \sum_{k \in \Omega_2} \|c(j + k, \cdot)\|_{\ell_r^r} \lesssim \sum_{k \in \Omega_2} \|c(j + k, \cdot)\|_{\ell_r^r}. \quad (2.9)$$

By applying the $\ell_{p_r}^p$ norm on the last inequality and raise it to the power $r = \min(1, p)$, we obtain

$$\|A \cdot V_0 f\|_{\ell_{p_r}^p(\Lambda)} \lesssim \sum_{k \in \Omega_2} \|c(\cdot + (k, 0))\|_{\ell_{p_r}^p(\Lambda)}$$

$$= \sum_{k \in \Omega_2} \|c\|_{\ell_{p_r}^p(\Lambda)} \lesssim \sum_{k \in \Omega_2} \|c\|_{\ell_{p_r}^p(\Lambda)} = 5^d \|c\|_{\ell_{p_r}^p(\Lambda)}. \quad (2.9)$$

Here we have used the fact that the number of elements in $\Omega_2$ is equal to $5^d$.

The asserted continuity in (1) now follows in the case when $p < \infty$ by combining (2.9) and the facts that

$$\|V_0 f\|_{\ell_{p_r}^p(\Lambda)} \approx \|f\|_{W_{p,q}^r} \quad \text{and} \quad \|A \cdot V_0 f\|_{\ell_{p_r}^p(\Lambda)} \approx \|M_{b,0} f\|_{W_{p,q}^r}.$$ The uniqueness of the map $M_{b,0}$ on $W_{p,q}^r(\mathbb{R}^d)$ follows from the fact that finite sequences in $[1, 2]$ are dense in $W_{p,q}^r(\mathbb{R}^d)$ gives. The case when $p = \infty$ now follows from the case when $p = 1$ and duality, and (1) follows.

In order to prove (2) we first consider the case when $p < \infty$. By applying the $\ell_{p_r}^p$ norm with respect to the $j$ variable in (2.8), we get

$$\|A \cdot V_0 f(\cdot, \cdot)\|_{\ell_{p_r}^p(\Omega)} \lesssim \left(\sum_{j \in \mathbb{Z}^d} \left(\sum_{k \in \Omega_2} (h_0 * (c(j + k, \cdot))\omega_0(j)\right)^p \right)^{\frac{1}{p}}$$

$$\lesssim \sum_{k \in \Omega_2} \left(h_0^r * \left(\sum_{j \in \mathbb{Z}^d} (c(j + k, \cdot)\omega_0(j + k)^p\right)^{\frac{1}{p}}\right)^{\frac{1}{p}} = \sum_{k \in \Omega_2} (h_0^r * c_0^r)(\cdot) \approx (h_0^r * c_0^r)(\cdot),$$

where $c_0(\cdot) = \|c(\cdot, \cdot)\|_{\ell_{p_r}^p(\Omega)}$.

Let $p_0 = r^{-1} q_1$, $q_0 = r^{-1} q_2$ and $u = r^{-1}$. Then (2.2) holds with $p_0$ and $q_0$ in place of $p$ and $q$, respectively. Hence by applying the $\ell_{p_0}^p$ norm on the last estimates, Lemma (2.2) gives

$$\|A \cdot V_0 f\|_{\ell_{p_0}^{q_0}(\Omega)} \lesssim \|h_0^r * c_0^r\|_{\ell_{p_0}^{q_0}} \lesssim \|c_0^r\|_{\ell_{p_0}^{q_0}} = \|c_0\|_{\ell_{p_0}^{q_0}}.$$ The asserted continuity in (2) now follows in the case when $p < \infty$ by combining (2.9) and the facts that

$$\|V_0 f\|_{\ell_{p_r}^p(\Lambda)} \approx \|f\|_{M_{p,q}^r} \quad \text{and} \quad \|A \cdot V_0 f\|_{\ell_{p_r}^p(\Lambda)} \approx \|M_{b,0} f\|_{M_{p,q}^r}.$$
The uniqueness assertions as well as the continuity in the case $p = \infty$ follow by similar arguments as in the proof of (1). The details are left for the reader.

By the links between $M^{p,q}_\omega(R^d)$ and $W^{p,q}_\omega(R^d)$ via the Fourier transform, explained in Remark 1.6 the following result follows from Theorem 2.1 and Fourier transformation. The details are left for the reader.

**Theorem 2.3.** Let $p \in (1, \infty)$, $q \in (0, \infty]$, $p_1, p_2 \in (\min(1, q), \infty]$ be such that

$$
\frac{1}{p_1} - \frac{1}{p_2} \geq \max \left( \frac{1}{q} - 1, 0 \right),
$$

$b > 0$, $a_0 \in \ell^\infty(\Lambda_b)$, $\omega_0 \in \mathcal{P}_{E,a}(R^d)$ and $\omega(x, \xi) = \omega_0(\xi)$, $x, \xi \in R^d$. Then the following is true:

1. $M_{\mathcal{F},b,a_0}$ is continuous on $M^{p,q}_\omega(R^d)$;
2. $M_{\mathcal{F},b,a_0}$ is continuous from $W^{p_1,q}_\omega(R^d)$ to $W^{p_2,q}_\omega(R^d)$.

We observe that Theorem 2.3 generalizes [3, Theorem 1] and [37, Theorem 4.16].

3. **Multiplications and convolutions of quasi-Banach modulation spaces**

In this section we extend the multiplication and convolution properties on modulation spaces in [8, 30] to allow the Lebesgue exponents to belong to the full interval $(0, \infty]$ instead of $[1, \infty]$, and to allow general moderate weights. There are several approaches in the case when the involved Lebesgue exponents belong to $[1, \infty]$ (see [4, 8, 11, 21, 27, 30]). There are also some results when such exponents belong to the full interval $(0, \infty]$ (see [1, 2, 14, 25, 26, 32]). Here we remark that our results in this section cover several of these earlier results. For example, we observe that Theorem 3.2 below extends [1, Proposition 3.1].

We recall that convolutions and multiplications on $\Sigma_1(R^d)$ are commutative and associative. That is, for any $N \geq 1$, $f_1, \ldots, f_N \in \Sigma_1(R^d)$ and $j, k \in \{1, \ldots, N\}$ one has

$$f_1 \cdots f_N = (f_1 \cdots f_j) \cdot (f_{j+1} \cdots f_N) \quad \text{and} \quad f_1 \cdots f_N = g_1 \cdots g_N$$

when

$$g_m = f_m, \quad g_j = f_k \quad \text{and} \quad g_k = f_j, \quad m \neq j, k,$$

and similarly for convolutions in place of multiplications at each occurrence.

Because of possible lacks of density properties, we do not always reach the uniqueness when extending the convolutions and multiplications from the case when each $f_j$ belong to $\Sigma_1(R^d)$ to the case when each $f_j$ belong to suitable modulation spaces. In some cases we manage the uniqueness by replacing the (quasi-)norm convergence by a weaker convergence, the so-called narrow convergence (see [28,29,31]). In the other situations we define multiplications and convolutions in terms of short-time Fourier transforms, in similar ways as in [30].
Let \( \phi_0, \ldots, \phi_N \in \Sigma_1(\mathbb{R}^d) \) be fixed such that

\[
(\phi_1 \cdots \phi_N, \phi_0)_{L^2} = (2\pi)^{-(N-1)d/2} \quad (3.1)
\]

and let \( f_1, \ldots, f_N, g \in \Sigma_1(\mathbb{R}^d) \). Then the multiplication \( f_1 \cdots f_N \) can be expressed by

\[
(f_1 \cdots f_N, \varphi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left( \prod_{j=1}^N F_j(x, \xi_j) \right) \Phi(x, \xi_1 + \cdots + \xi_N) \, dx \, d\xi 
\]

for every \( \varphi \in \Sigma_1(\mathbb{R}^d) \), where

\[
F_j = V_{\phi_j} f_j \quad \text{and} \quad \Phi = V_{\phi_0} \varphi. \quad (3.3)
\]

We observe that \( (3.2) \) is the same as

\[
F_0(x, \xi) = \left( \langle \phi_1 f_1(x, \cdot) \rangle \ast \cdots \ast \langle \phi_N f_N(x, \cdot) \rangle \right)(\xi). \quad (3.2')
\]

where

\[
F_0(x, \xi) = (\|\phi_0\|_{L^2})^{-2} \cdot V_{\phi_0}(f_1 \cdots f_N)(x, \xi), \quad (3.4)
\]

and that we may extract \( f_0 = f_1 \cdots f_N \) by the formula

\[
f_0 = V_{\phi_0}^* F_0. \quad (3.5)
\]

In the same way, let \( \phi_0, \ldots, \phi_N \in \Sigma_1(\mathbb{R}^d) \) be fixed such that

\[
(\phi_1 \ast \cdots \ast \phi_N, \phi_0)_{L^2} = 1 \quad (3.6)
\]

and let \( f_1, \ldots, f_N, g \in \Sigma_1(\mathbb{R}^d) \). Then the convolution \( f_1 \ast \cdots \ast f_N \) can be expressed by

\[
(f_1 \ast \cdots \ast f_N, \varphi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{(N+1)d}} \left( \prod_{j=1}^N F_j(x_j, \xi_j) \right) \Phi(x_1 + \cdots + x_N, \xi_1 + \cdots + \xi_N) \, dx \, d\xi 
\]

for every \( \varphi \in \Sigma_1(\mathbb{R}^d) \), where \( F_j \) and \( \Phi \) are given by \( (3.3) \). We observe that

\[
(3.7) \quad F_0(x, \xi) = \left( \langle \phi_1 f_1(\cdot, \xi) \rangle \ast \cdots \ast \langle \phi_N f_N(\cdot, \xi) \rangle \right)(x). \quad (3.7')
\]

where

\[
F_0 = (\|\phi_0\|_{L^2})^{-2} V_{\phi_0}(f_1 \ast \cdots \ast f_N), \quad (3.8)
\]

and that we may extract \( f_0 = f_1 \ast \cdots \ast f_N \) from \( (3.5) \).

**Definition 3.1.** Let \( f_1, \ldots, f_N \in \Sigma_1(\mathbb{R}^d) \).
(1) Let \( \phi_0, \ldots, \phi_N \in \Sigma_1(\mathbb{R}^d) \) be fixed and such that (3.1) holds, and suppose that the integrand in (3.2) belongs to \( L^1(\mathbb{R}^{(N+1)d}) \) for every \( \varphi \in \Sigma_1(\mathbb{R}^d) \), where \( F_j = V_{\phi_j}f_j \) and \( \Phi = V_{\phi_0}\varphi \), \( j = 1, \ldots, N \). Then \( f_0 \equiv f_1 \cdots f_N \in \Sigma_1(\mathbb{R}^d) \) is defined by (3.2);

(2) Let \( \phi_0, \ldots, \phi_N \in \Sigma_1(\mathbb{R}^d) \) be fixed and such that (3.3) holds, and suppose that the integrand in (3.7) belongs to \( L^1(\mathbb{R}^{(N+1)d}) \) for every \( \varphi \in \Sigma_1(\mathbb{R}^d) \), where \( F_j = V_{\phi_j}f_j \) and \( \Phi = V_{\phi_0}\varphi \), \( j = 1, \ldots, N \). Then \( f_0 \equiv f_1 \ast \cdots \ast f_N \in \Sigma_1(\mathbb{R}^d) \) is defined by (3.7).

Next we discuss convolutions and multiplications for modulation spaces, and start with the following convolution result for modulation spaces. Here the conditions for the involved weight functions are given by

\[
\omega_0(x, \xi_1 + \cdots + \xi_N) \leq \prod_{j=1}^N \omega_j(x, \xi_j), \quad x, \xi_1, \ldots, \xi_N \in \mathbb{R}^d \tag{3.9}
\]

or by

\[
\omega_0(x_1 + \cdots + x_N, \xi) \leq \prod_{j=1}^N \omega_j(x_j, \xi), \quad x_1, \ldots, x_N, \xi \in \mathbb{R}^d. \tag{3.10}
\]

For multiplications of elements in modulation spaces we need to swap the conditions for the involved Lebesgue exponents compared to (1.39) and (1.40). That is, these conditions become

\[
\frac{1}{p_0} \leq \sum_{j=1}^N \frac{1}{p_j}, \quad \frac{1}{q_0} \leq \sum_{j=1}^N \frac{1}{q_j} - R_{p_0,N}(q_1, \ldots, q_N) \tag{3.11}
\]

or

\[
\frac{1}{p_0} \leq \sum_{j=1}^N \frac{1}{p_j}, \quad \frac{1}{q_0} \leq \sum_{j=1}^N \frac{1}{q_j} - R_N(q_1, \ldots, q_N), \tag{3.12}
\]

where

\[
R_{r,N}(q_1, \ldots, q_N) = \left( \sum_{j=1}^N \frac{1}{r_j} \right) - \min_{1 \leq j \leq N} \left( \frac{1}{r_j} \right), \quad r_j = \min(1, q_j, r) \tag{3.13}
\]

and

\[
R_N(q_1, \ldots, q_N) = R_{1,N}(q_1, \ldots, q_N). \tag{3.14}
\]

Evidently, \( R_{r,N}(q_1, \ldots, q_N) = R_N(q_1, \ldots, q_N) \) when \( r \geq 1 \).

**Theorem 3.2.** Let \( I_N = \{1, \ldots, N\} \), \( \omega_j \in \mathcal{P}_{E}(\mathbb{R}^{2d}) \) and \( p_j, q_j \in (0, \infty], j \in I_N \), be such that (3.9), (3.11), and (3.13) hold. Then \( (f_1, \ldots, f_N) \mapsto f_1 \cdots f_N \) in Definition 3.7 (1) restricts to a continuous, associative and symmetric map from \( M_{(\omega_j)}^{p_j,q_j}(\mathbb{R}^d) \times \cdots \times M_{(\omega_N)}^{p_N,q_N}(\mathbb{R}^d) \) to \( M_{(\omega_0)}^{p_0,q_0}(\mathbb{R}^d) \), and

\[
\|f_1 \cdots f_N\|_{M_{(\omega_0)}^{p_0,q_0}} \lesssim \prod_{j=1}^N \|f_j\|_{M_{(\omega_j)}^{p_j,q_j}}, \quad f_j \in M_{(\omega_j)}^{p_j,q_j}(\mathbb{R}^d), \quad j \in I_N. \tag{3.15}
\]
Moreover, $f_1 \cdots f_N$ in (3.2) is independent of the choice of $\phi_0, \ldots, \phi_N$ in Definition 3.1 (1).

**Theorem 3.3.** Let $I_N = \{1, \ldots, N\}$, $\omega_j \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $p_j, q_j \in (0, \infty]$, $j \in I_N$, be such that (3.10), (3.12) and (3.14) hold. Then $(f_1, \ldots, f_N) \mapsto f_1 \cdots f_N$ in Definition 3.1 (1) restricts to a continuous, associative and symmetric map from $W^{p_1, q_1}(\mathbb{R}^d) \times \cdots \times W^{p_N, q_N}(\mathbb{R}^d)$ to $W^{p_0, q_0}(\mathbb{R}^d)$, and

$$
\|f_1 \cdots f_N\|_{W^{p_0, q_0}(\omega)} \lesssim \prod_{j=1}^{N} \|f_j\|_{W^{p_j, q_j}(\omega_j)}, \quad f_j \in W^{p_j, q_j}(\mathbb{R}^d), \quad j \in I_N. \quad (3.16)
$$

Moreover, $f_1 \cdots f_N$ in (3.2) is independent of the choice of $\phi_0, \ldots, \phi_N$ in Definition 3.1 (1).

The corresponding results for convolutions are the following. Here the conditions on the involved Lebesgue exponents are swapped as

$$
\frac{1}{p_0} \leq \sum_{j=1}^{N} \frac{1}{p_j} - R_{q_0, N}(p_1, \ldots, p_N), \quad \frac{1}{q_0} \leq \sum_{j=1}^{N} \frac{1}{q_j} \quad (3.17)
$$

or

$$
\frac{1}{p_0} \leq \sum_{j=1}^{N} \frac{1}{p_j} - R_N(p_1, \ldots, p_N), \quad \frac{1}{q_0} \leq \sum_{j=1}^{N} \frac{1}{q_j}. \quad (3.18)
$$

**Theorem 3.4.** Let $I_N = \{1, \ldots, N\}$, $\omega_j \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $p_j, q_j \in (0, \infty]$, $j \in I_N$, be such that (3.10), (3.14) and (3.18) hold. Then $(f_1, \ldots, f_N) \mapsto f_1 \ast \cdots \ast f_N$ in Definition 3.1 (2) restricts to a continuous, associative and symmetric map from $M^{p_1, q_1}(\mathbb{R}^d) \times \cdots \times M^{p_N, q_N}(\mathbb{R}^d)$ to $M^{p_0, q_0}(\mathbb{R}^d)$, and

$$
\|f_1 \ast \cdots \ast f_N\|_{M^{p_0, q_0}(\omega)} \lesssim \prod_{j=1}^{N} \|f_j\|_{M^{p_j, q_j}(\omega_j)}, \quad f_j \in M^{p_j, q_j}(\mathbb{R}^d), \quad j \in I_N. \quad (3.19)
$$

Moreover, $f_1 \cdots f_N$ in (3.2) is independent of the choice of $\phi_0, \ldots, \phi_N$ in Definition 3.1 (2).

**Theorem 3.5.** Let $I_N = \{1, \ldots, N\}$, $\omega_j \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $p_j, q_j \in (0, \infty]$, $j \in I_N$, be such that (3.10), (3.13) and (3.17) hold. Then $(f_1, \ldots, f_N) \mapsto f_1 \ast \cdots \ast f_N$ in Definition 3.1 (2) restricts to a continuous, associative and symmetric map from $W^{p_1, q_1}(\mathbb{R}^d) \times \cdots \times W^{p_N, q_N}(\mathbb{R}^d)$ to $W^{p_0, q_0}(\mathbb{R}^d)$, and

$$
\|f_1 \ast \cdots \ast f_N\|_{W^{p_0, q_0}(\omega)} \lesssim \prod_{j=1}^{N} \|f_j\|_{W^{p_j, q_j}(\omega_j)}, \quad f_j \in W^{p_j, q_j}(\mathbb{R}^d), \quad j \in I_N. \quad (3.20)
$$

Moreover, $f_1 \cdots f_N$ in (3.2) is independent of the choice of $\phi_0, \ldots, \phi_N$ in Definition 3.1 (2).

For the proofs of Theorems 3.3–3.5 we need the following proposition. Here recall [11, 13, 17, 25, 26] and Remark 1.4 for some facts concerning the operators $P_\phi$ and $V_\phi$. 
Proposition 3.6. Let $p, q \in (0, \infty)$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$ and $P_\phi$ be the projection in Remark 1.4. Then $P_\phi$ from $\Sigma_1'(\mathbb{R}^{2d})$ to $\Sigma_1'(\mathbb{R}^{2d})$, and $V_\phi^*$ from $\Sigma_1'(\mathbb{R}^{2d})$ to $\Sigma_1'(\mathbb{R}^{2d})$ restrict to continuous mappings

$$P_\phi : \mathcal{W}(\omega, \ell^{p,q}(\mathbb{Z}^{2d})) \rightarrow \mathcal{V}(M^{p,q}(\mathbb{R}^d)) \rightarrow \mathcal{W}(\omega, \ell^{p,q}(\mathbb{Z}^{2d})), \quad (3.21)$$

$$P_\phi : \mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d})) \rightarrow \mathcal{V}(W^{p,q}(\mathbb{R}^d)) \rightarrow \mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d})), \quad (3.22)$$

$$V_\phi^* : \mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d})) \rightarrow M^{p,q}(\mathbb{R}^d) \quad (3.23)$$

and

$$V_\phi^* : \mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d})) \rightarrow W^{p,q}(\mathbb{R}^d). \quad (3.24)$$

For $p, q \geq 1$, i.e. the case when all spaces are Banach spaces, proofs of Proposition 3.6 can be found in e.g. [17] as well as in abstract forms in [11]. In the general case when $p, q > 0$, proofs of Proposition 3.6 are essentially given in [14,26]. In order to be self-contained we here present a short proof.

**Proof.** By Remark 1.4, the result follows if we prove (3.21) and (3.22), i.e.,

$$\|P_\phi F\|_{\mathcal{W}(\omega, \ell^{p,q})} \leq \|F\|_{\mathcal{W}(\omega, \ell^{p,q}(\mathbb{Z}^{2d}))} \quad (3.25)$$

and

$$\|P_\phi F\|_{\mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d}))} \leq \|F\|_{\mathcal{W}(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d}))} \quad (3.26)$$

We only prove (3.25). The estimate (3.26) follows by similar arguments and is left for the reader.

Let

$$a_j = \|F\|_{L_\infty(j+Q_{2d})} \quad \text{and} \quad b_j = \|V_\phi f\|_{L_\infty(j+Q_{2d})}.$$ 

Since $V_\phi \in \Sigma_1(\mathbb{R}^{2d})$, Proposition 1.11 gives

$$\|P_\phi F\|_{\mathcal{W}(\omega, \ell^{p,q})} \leq \|a \ast b\|_{\ell^{p,q}} \lesssim \|b\|_{\ell^{p,q}} \|a\|_{\ell^{p,q}} \asymp \|F\|_{\mathcal{W}(\omega, \ell^{p,q})}. \quad \square$$

Theorems 3.2 and 3.3 are Fourier transformations of Theorems 3.4 and 3.5. Hence it suffices to prove the last two theorems.

**Proof of Theorems 3.4 and 3.5.** First we prove (3.19). Suppose $f_1 \in M^{p,q}(\mathbb{R}^d)$, and consider the cubes

$$Q_{d,r} = [0,r]^d \quad \text{and} \quad Q = Q_{d,1} = [0,1]^d.$$ 

Then

$$0 \leq \chi_{k_1+Q} \ast \cdots \ast \chi_{k_N+Q} \leq \chi_{k_1+\cdots+k_N+Q_{d,N}} \quad k_1, \ldots, k_N \in \mathbb{Z}^d.$$ 

Let

$$G_1(x, \xi) = (V_{\phi_1} f_1(\cdot, \xi) \ast \cdots \ast V_{\phi_N} f_N(\cdot, \xi))(x),$$

$$G_2(x, \xi) = (|V_{\phi_1} f_1(\cdot, \xi)| \ast \cdots \ast |V_{\phi_N} f_N(\cdot, \xi)|)(x),$$

$$a_j(k, \kappa) = \|V_{\phi_j} f_j\|_{L_\infty((k,\kappa)+Q_{d,1})},$$
and 

\[ b(k, \kappa) = \|G_2\|_{L^\infty((k, \kappa) + Q_{2d, 1})} \]

Then

\[ \|V_{\phi_0}^* G_1\|_{M_{(\omega_0, \ell_0, q_0)}} \lesssim \|P_{\phi_0} G_1\|_{W(\omega_0, \ell_0, q_0)} \lesssim \|G_1\|_{W(\omega_0, \ell_0, q_0)} \]

\[ \leq \|G_2\|_{W(\omega_0, \ell_0, q_0)} \lesssim \|b\|_{\ell_0, q_0}(\omega_0), \tag{3.27} \]

and

\[ \|f_j\|_{M_{(\omega_j)}} \lesssim \|a_j\|_{\ell_0, q_0}(\omega_j) \tag{3.28} \]

in view of [32, Proposition 3.4] (see also Theorem 3.3 in [14]) and Proposition 3.6.

By (3.7) we have

\[ G_2(x, \lambda) \leq \sum_{k_1, \ldots, k_N \in \mathbb{Z}^d} \left( \prod_{j=1}^N a_j(k_j, \lambda) \right) \left( \chi_{k_1+Q} \cdots \chi_{k_N+Q} \right)(x) \]

\[ \leq \sum_{k_1, \ldots, k_N \in \mathbb{Z}^d} \left( \prod_{j=1}^N a_j(k_j, \lambda) \right) \chi_{k_1+\cdots+k_N+Q_{d,N}}(x). \tag{3.29} \]

We observe that

\[ \chi_{k_1+\cdots+k_N+Q_{d,N}}(x) = 0 \quad \text{when} \quad x \notin l + Q_d, \quad (k_1, \ldots, k_N) \notin \Omega_l, \]

where

\[ \Omega_l = \{ (k_1, \ldots, k_N) \in \mathbb{Z}^{N_d} ; l_j - N \leq k_{1,j} + \cdots + k_{N,j} \leq l_j + 1 \}, \]

and

\[ k_n = (k_{n,1}, \ldots, k_{n,d}) \in \mathbb{Z}^d \quad \text{and} \quad l = (l_1, \ldots, l_d) \in \mathbb{Z}^d, \quad n = 1, \ldots, N. \]

Hence, if \( x = l \) in (3.29), we get

\[ b(l, \lambda) \leq \sum_{(k_1, \ldots, k_N) \in \Omega_l} \left( \prod_{j=1}^N a_j(k_j, \lambda) \right) \]

\[ \leq \sum_{m \in I_{N+1}} (a_1(\cdot, \lambda) \ast \cdots \ast a_N(\cdot, \lambda))(l - N e_0 + m), \tag{3.30} \]

where \( e_0 = (1, \ldots, 1) \in \mathbb{Z}^d \) and \( I_N = \{0, \ldots, N\}^d \). By multiplying with \( \omega_0(l, \lambda) \), using (3.10), the fact that \( I_N \) is a finite set and that \( \omega_0 \) is moderate,
we obtain

\[ b(l, \lambda)\omega_0(l, \lambda) \leq \sum_{m \in I_{N+1}} (a_1(\cdot, \lambda) \ast \cdots \ast a_N(\cdot, \lambda))(l - Ne_0 + m)\omega_0(l, \lambda) \]

\[ \leq \sum_{m \in I_{N+1}} (a_1(\cdot, \lambda) \ast \cdots \ast a_N(\cdot, \lambda))(l - Ne_0 + m)\omega_0(l - Ne_0 + m, \lambda) \]

\[ \leq \sum_{m \in I_{N+1}} ((a_1(\cdot, \lambda)\omega_1(\cdot, \lambda)) \ast \cdots \ast (a_N(\cdot, \lambda)\omega_N(\cdot, \lambda)))(l - Ne_0 + m). \]

Hence (3.30) gives

\[ b_{\omega_0}(l, \lambda) \lesssim \sum_{(k_1, \ldots, k_N) \in \Omega_l} \left( \prod_{j=1}^N a_{j, \omega_j}(k_j, \lambda) \right) \]

\[ = \sum_{m \in I_{N+1}} (a_{1, \omega_1}(\cdot, \lambda) \ast \cdots \ast a_{N, \omega_N}(\cdot, \lambda))(l - Ne_0 + m), \quad (3.30)' \]

where

\[ a_{j, \omega_j}(k, \kappa) = a_j(k, \kappa)\omega_j(k, \kappa) \quad \text{and} \quad b_{\omega_0}(k, \kappa) = b(k, \kappa)\omega_0(k, \kappa). \]

If we apply the \( L^{p_0} \) quasi-norm on (3.30) with respect to the \( l \) variable, then Proposition 1.10 (2) and the fact that \( I_{N+1} \) is a finite set give

\[ \|b_{\omega_0}(\cdot, \lambda)\|_{L^{p_0}} \lesssim \left\| \sum_{m \in I_{N+1}} (a_{1, \omega_1}(\cdot, \lambda) \ast \cdots \ast a_{N, \omega_N}(\cdot, \lambda))(\cdot - Ne_0 + m) \right\|_{L^{p_0}} \]

\[ \lesssim \sum_{m \in I_{N+1}} \| (a_{1, \omega_1}(\cdot, \lambda) \ast \cdots \ast a_{N, \omega_N}(\cdot, \lambda))(\cdot - Ne_0 + m) \|_{L^{p_0}} \]

\[ \times \| a_{1, \omega_1}(\cdot, \lambda) \ast \cdots \ast a_{N, \omega_N}(\cdot, \lambda) \|_{L^{p_0}} \]

\[ \leq \| a_{1, \omega_1}(\cdot, \lambda) \|_{L^{p_1}} \cdots \| a_{N, \omega_N}(\cdot, \lambda) \|_{L^{p_N}}. \]

By applying the \( L^{p_0} \) quasi-norm and using Proposition 1.10 (1) we now get

\[ \|b_{\omega_0}\|_{L^{p_0, q_0}} \lesssim \| a_{1, \omega_1} \|_{L^{p_1}} \cdots \| a_{N, \omega_N} \|_{L^{p_N}}. \]

This is the same as

\[ \|G_2\|_{L^{p_0, q_0}} \lesssim \|F_1\|_{L^{p_1}} \cdots \|F_N\|_{L^{p_N}}. \]

A combination of this estimate with (3.27) and (3.28) gives that \( f_1 \ast \cdots \ast f_N \) is well-defined and that (3.19) holds.

Next we prove (3.20). Let \( r = \min(1, q_0) \). Then (3.30)' gives

\[ b_{\omega_0}(l, \lambda)^r \lesssim \sum_{(k_1, \ldots, k_N) \in \Omega_l} \left( \prod_{j=1}^N a_{j, \omega_j}(k_j, \lambda)^r \right) \]

\[ = \sum_{m \in I_{N+1}} (a_{1, \omega_1}(\cdot, \lambda)^r \ast \cdots \ast a_{N, \omega_N}(\cdot, \lambda)^r)(l - Ne_0 + m). \]
By applying the $\ell^{p_0/r}$ norm with respect to the $\lambda$ variable and using Minkowski’s and Hölder’s inequalities we obtain

$$
\|b_{\omega_0}(l, \cdot)\|_{\ell^{p_0/r}} = \|b_{\omega_0}(l, \cdot)^r\|_{\ell^{p_0/r}} \lesssim \sum_{(k_1, \ldots, k_N) \in \Omega_I} \left| \prod_{j=1}^N a_{j,\omega_j}(k_j, \cdot)^r \right|_{\ell^{p_0/r}}
$$

$$
\leq \sum_{(k_1, \ldots, k_N) \in \Omega_I} \left( \prod_{j=1}^N c_{j,\omega_j}(k_j)^r \right) = \sum_{m \in I_{N+1}} (c_{r_1,\omega_1} \cdots c_{r_N,\omega_N})(l - N\epsilon_0 + m),
$$

where

$$
c_{j,\omega_j}(k) = \|a_{j,\omega_j}(k, \cdot)^r\|_{\ell^{p_j/r}}^{1/r} = \|a_{j,\omega_j}(k, \cdot)\|_{\ell^{p_j}}.
$$

An application of the $\ell^{p_0/r}$ quasi-norm on the last inequality and using Proposition 1.10 (2) now gives

$$
\|b_{\omega_0}\|_{\ell^{p_0,p_0}} \lesssim \sum_{m \in I_{N+1}} \|c_{r_1,\omega_1} \cdots c_{r_N,\omega_N}(\cdot - N\epsilon_0 + m)\|_{\ell^{p_0/r}}
$$

$$
\asymp \|c_{r_1,\omega_1} \cdots c_{r_N,\omega_N}\|_{\ell^{p_0/r}} \leq \|c_{r_1,\omega_1}\|_{\ell^{p_1/r}} \cdots \|c_{r_N,\omega_N}\|_{\ell^{p_N/r}}
$$

$$
= (\|c_{1,\omega_1}\|_{\ell^{p_1}} \cdots \|c_{N,\omega_N}\|_{\ell^{p_N}})^r,
$$

which is the same as

$$
\|G_2\|_{L^{p_0,p_0}} \lesssim \|F_1\|_{L^{p_1,q_1}} \cdots \|F_N\|_{L^{p_N,q_N}},
$$

which in particular shows that $f_1 \ast \cdots \ast f_N$ is well-defined. Since

$$
\|f_1 \ast \cdots \ast f_N\|_{W^{p_0,q_0}} \asymp \|G\|_{L^{p_0,q_0}} \quad \text{and} \quad \|f_j\|_{W^{p_j,q_j}} \asymp \|F_j\|_{L^{p_j,q_j}}, \quad j = 1, \ldots, N,
$$

we get (3.20).

We need to prove the associativity, symmetry and invariance with respect to $\phi_0, \ldots, \phi_N$ in Definition 3.1. We observe that if

$$
r_j = \max(p_j, 1) \quad \text{and} \quad s_j = \frac{q_j}{q}, \quad q = \min_{0 \leq j \leq N}(q_j), \quad j = 1, \ldots, N,
$$

then $M_{(\omega,j)}^{r_j,q_j}(R^d) \subseteq M_{(\omega,j)}^{r_j,s_j}(R^d)$, $j = 1, \ldots, N$. By straight-forward computations it follows that if (3.17) or (3.18) hold, then (3.17) respectively (3.18) still hold with $r_j$ and $s_j$ in place of $p_j$ and $q_j$, respectively, $j = 1, \ldots, N$, for some $r_0, s_0 \in [1, \infty]$. This reduce ourself to the case when $p_j, q_j \in [1, \infty]$ for every $j = 0, \ldots, N$, in which case all modulation spaces are Banach spaces.

We observe that Lemmas 5.2–5.4 and their proofs in [30] still hold true when $\omega_j$ are allowed to belong to the class $\mathcal{P}_E(R^{2d})$, provided the involved window functions $\chi_j$ belong to $\Sigma_1(R^d)$, and all distributions are allowed to belong to $\Sigma'_1$ instead of $\mathcal{S}'$. The the associativity, symmetric assertions and invariant properties with respect to the choice of $\phi_0, \ldots, \phi_N$ in Definition 3.1 now follows from these modified Lemmas 5.2–5.4 in [30] and their proofs. This gives the results.

\[ \square \]

Remark 3.7. Suppose that $p_1, q_1$ and $\omega_j$ are the same as in Theorems 3.2, 3.3 and that $p_j + q_j = \infty$ for at most one $j \in \{1, \ldots, N\}$. Then it follows that extensions of the mappings $(f_1, \ldots, f_N) \mapsto f_1 \ast \cdots \ast f_N$ and $(f_1, \ldots, f_N) \mapsto \ldots \ast f_N$
Then $\Sigma_1$ from $\Sigma_1(\mathbb{R}^d) \times \cdots \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ in Theorems 3.2, 3.3 are unique.

In fact, by the proof of Theorem 3.3 below, we may assume that $p_j, q_j \geq 1$ for every $j$. If $p_j, q_j < \infty$ for every $j \in \{1, \ldots, N\}$, then the uniquenesses follow from (3.15), (3.16), (3.19), (3.20) and the fact that $\Sigma_{\omega}^N$ is dense in each $M_{(\omega)}^{p_j, q_j}(\mathbb{R}^d)$ and $W_{(\omega)}^{p_j, q_j}(\mathbb{R}^d)$ for $j \in \{1, \ldots, N\}$. For the general situation, the assertion follows from the previous case and duality.

Evidently, Theorems 3.2, 3.3 show that multiplications and convolutions on $\Sigma_1(\mathbb{R}^d)$ can be extended to involve suitable quasi-Banach modulation spaces. Remark 3.7 shows that in most situations, these extensions from products on $\Sigma_1(\mathbb{R}^d)$ are unique. For the multiplication and convolution mappings in Theorems 3.2 and 3.3 we can say more.

**Theorem 3.8.** Let $\omega_j \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $p_j, q_j \in (0, \infty)$ and $j \in \{0, \ldots, N\}$. Then the following is true:

1. If (3.10), (3.11), and (3.13) hold, then $f_1, \ldots, f_N \to f_1 \cdots f_N$ from $\Sigma_1(\mathbb{R}^d) \times \cdots \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $M_{(\omega_1)}^{p_1, q_1}(\mathbb{R}^d) \times \cdots \times M_{(\omega_N)}^{p_N, q_N}(\mathbb{R}^d)$ to $M_{(\omega)}^{p_0, q_0}(\mathbb{R}^d)$, and (3.15) holds;

2. If (3.10), (3.13), and (3.17) hold, then $f_1, \ldots, f_N \mapsto f_1 \cdots f_N$ from $\Sigma_1(\mathbb{R}^d) \times \cdots \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $W_{(\omega_1)}^{p_1, q_1}(\mathbb{R}^d) \times \cdots \times W_{(\omega_N)}^{p_N, q_N}(\mathbb{R}^d)$ to $W_{(\omega)}^{p_0, q_0}(\mathbb{R}^d)$, and (3.20) holds.

The problems with uniqueness in Theorem 3.8 appear when one or more Lebesgue exponents are equal to infinity, since $\Sigma_1(\mathbb{R}^d)$ fails to be dense in corresponding modulation spaces. In these situations we shall use narrow convergence, introduced in [28], and is a weaker form of convergence than the norm convergence.

**Definition 3.9.** Let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $p, q \in [1, \infty]$, $f, f_j \in M_{(\omega)}^{p, q}(\mathbb{R}^d)$, $j \geq 1$ and let

$$H_{f, \omega, p}(\xi) \equiv \|V_{\phi} f(\cdot, \xi) \omega(\cdot, \xi)\|_{L^p(\mathbb{R}^d)}.$$ 

Then $f_j$ is said to converge to $f$ narrowly as $j \to \infty$, if the following conditions are fulfilled:

1. $f_j \to f$ in $\Sigma_1(\mathbb{R}^d)$ as $j \to \infty$;

2. $H_{f_j, \omega, p} \to H_{f, \omega, p}$ in $L^q(\mathbb{R}^d)$ as $j \to \infty$.

The following result is a special case of Theorem 4.17 in [31]. The proof is therefore omitted.

**Proposition 3.10.** Let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $p, q \in [1, \infty]$ be such that $q < \infty$. Then $\Sigma_1(\mathbb{R}^d)$ is dense in $M_{(\omega)}^{p, q}(\mathbb{R}^d)$ with respect to the narrow convergence.

We also need the following generalization of Lebesgue’s theorem, which follows by a straight-forward application of Fatou’s lemma.

**Lemma 3.11.** Let $\mu$ be a positive measure on a measurable set $\Omega$, $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ be sequences in $L^1(d\mu)$ such that $f_j \to f$ a.e., $g_j \to g$ in $L^1(d\mu)$
as \( j \) tends to infinity, and that \(|f_j| \leq g_j \) for every \( j \in \mathbb{N} \). Then \( f_j \to f \) in \( L^1(d\mu) \) as \( j \) tends to infinity.

**Remark 3.12.** The narrow convergence is especially interesting when \( p = \infty \).

Let \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), \( q \in [1, \infty) \), \( \phi \in \Sigma_1(\mathbb{R}^d) \) and \( f \in M^{\infty,q}_\omega(\mathbb{R}^d) \), \( f_j \in \Sigma_1(\mathbb{R}^d) \) converges to \( f \) narrowly as \( j \to \infty \), and let \( H_{f,\omega,\infty} \) be the same as in Definition 3.9. Then we may choose these \( f_j \) such that

\[
\lim_{j \to \infty} V_\phi f_j(x, \xi) = V_\phi f(x, \xi), \quad |V_\phi f(x, \xi)\omega(x, \xi)| \leq H_{f,\omega,\infty}(\xi)
\]

and

\[
\lim_{j \to \infty} \|H_{f_j,\omega,\infty} - H_{f,\omega,\infty}\|_{L^q} = 0 \tag{3.31}
\]

(See [31] Theorem 4.17 and its proof.) It is then possible to apply Lemma 3.11 in integral expressions containing \( V_\phi f_j(x, \xi) \) and \( V_\phi f(x, \xi) \) and perform suitable limit processes.

**Proof of Theorem 3.8** Since (2) is the Fourier transform of (1), it suffices to prove (1).

The existence of the extension follows from Theorem 3.2. Since \( M^{p,q}_\omega(\mathbb{R}^d) \) increases with \( p \) and \( q \), we may assume that equality is attained in 3.11 and that \( p_0 = \ldots = p_N = \infty \). By replacing \( q_j \) with

\[
r_j = \max(1, q_j),
\]

it follows from 3.11 that for \( r_0 = \frac{p_0}{q} \geq 1 \) and some \( r_0 \geq 1 \),

\[
\frac{1}{r_0} \leq \sum_{j=1}^{N} \frac{1}{r_j} - N + 1,
\]

and that

\[
M^{\infty,q_j}_{\omega_j}(\mathbb{R}^d) \subseteq M^{\infty,r_j}_{\omega_j}(\mathbb{R}^d).
\]

Suppose \( g_1, g_2 \in M^{\infty,q_j}_{\omega_j}(\mathbb{R}^d) \) are such that \( g_1 \) equals \( g_2 \) as elements in \( M^{\infty,r_j}_{\omega_j}(\mathbb{R}^d) \). Then \( g_1 \) is also equal to \( g_2 \) as elements in \( M^{\infty,q_j}_{\omega_j}(\mathbb{R}^d) \). Hence it suffices to prove the uniqueness of the product \( f_1 \cdots f_N \in M^{\infty,q_0}_{\omega_0}(\mathbb{R}^d) \) of \( f_j \in M^{\infty,q_j}_{\omega_j}(\mathbb{R}^d) \), \( j = 1, \ldots, N \), when additionally \( q_j \geq 1 \), i.e.,

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_N} = N - 1 + \frac{1}{q_0}, \quad q_0, \ldots, q_N \in [1, \infty]. \tag{3.32}
\]

In particular, all involved modulation spaces are Banach spaces.

Let \( j_0 \in \{1, \ldots, N\} \) be chosen such that \( q_j \leq q_{j_0} \) for every \( j \in \{1, \ldots, N\} \).

Then \( j < \infty \) when \( j \neq j_0 \).

The product \( f_1 \cdots f_N \) is uniquely defined and can be obtained through (3.2) for every \( \varphi \in \Sigma_1(\mathbb{R}^d) \) when \( f_j \in \Sigma_1(\mathbb{R}^d) \) and \( f_{j_0} \in M^{\infty,q_{j_0}}_{\omega_0}(\mathbb{R}^d) \), \( j \neq j_0 \).

For general \( f_j \in M^{\infty,q_j}_{\omega_j}(\mathbb{R}^d) \), choose \( f_{j,k} \in \Sigma_1(\mathbb{R}^d) \), \( k = 1, 2, \ldots \) such that \( f_{j,k} \) converges to \( f_j \) narrowly as \( k \) tends to infinity, and that (3.31) holds with \( f_j \) and \( f_{j,k} \) in place of \( f \) and \( f_j \), respectively. Then it follows by replacing \( f_j \) by \( f_{j,k} \) when \( j \neq j_0 \) in (3.2) and applying Lemma 3.11 on the integral in (3.2) that

\[
\ell(\varphi) \equiv \lim_{k \to \infty} (f_{1,k} \cdots f_{N,k}) \varphi
\]
exists and defines an element in $f \in \Sigma'_1(\mathbb{R}^d)$. This shows that the only possibility to define $f_1 \cdots f_N$ in a continuous way is to put $f_1 \cdots f_N = f$, and the asserted uniqueness follows.

4. Extensions and variations

In this section we extend the results on step and Fourier step multipliers to certain so-called curve step and Fourier curve step multipliers. That is a generalized form of step and Fourier step multipliers, where the constants $a_0(j)$ in the definition of $M_{b,a_0}$ and $M_{\mathcal{F},b,a_0}$ are replaced by certain non-constant functions or even distributions. In the end we are able to generalize Theorems 2.1 and 2.3 to such multipliers. These achievements are based on Hölder-Young relations for multiplications and convolutions in Section 3. In the case of trivial weights and all modulation spaces are Banach spaces, our results are similar to [3, Theorem 6] and [27, Proposition 4.12].

The multipliers and Fourier multipliers which we consider are given in the following.

**Definition 4.1.** Let $b \in \mathbb{R}^d_+$ be fixed, $\Lambda_b$ and $Q_b$ be given by (1.43) and (1.44), and let

$$a_0 \equiv \{a_0(j, \cdot)\}_{j \in \Lambda_b} \subseteq C^\infty(\mathbb{R}^d)$$

be such that

$$\left( \sum_{j \in \Lambda_b} a_0(j, \cdot) \chi_{j+Q_b} \right) \in \Sigma'_1(\mathbb{R}^d).$$

Then the multiplier

$$M_{b,a_0} : f \mapsto \sum_{j \in \Lambda_b} a_0(j, \cdot) \chi_{j+Q_b} f,$$

from $C^\infty(\mathbb{R}^d)$ to $L^\infty_{\text{loc}}(\mathbb{R}^d)$ is called slope step multiplier with respect to $b$ and $a_0$. The Fourier multiplier

$$M_{\mathcal{F},b,a_0} \equiv \mathcal{F}^{-1} \circ M_{b,a_0} \circ \mathcal{F},$$

is called slope step Fourier multiplier with respect to $b$ and $a_0$.

First we perform some studies of

$$T_\psi a_0 \equiv \sum_{j \in \Lambda_b} a_0(j, \cdot) \psi(\cdot - j),$$

where $\psi \in \mathcal{S}(\mathbb{R}^d)$ is suitable. The conditions on the sequence (4.1) that we have in mind are that for fixed $\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)$ and $p \in (0, \infty]$, the functions

$$b_{a_0,\alpha}(x) \equiv \sup_{\beta \leq \alpha} \left( \sup_{j \in \Lambda_b} \left| \partial_\beta^x a_0(j, x) \right| \right)$$

should belong to $L^p(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}^d$, or that for some or for every $h > 0$, the function

$$b_{a_0,h}(x) \equiv \sup_{\alpha \in \mathbb{N}^d} \left( \sup_{j \in \Lambda_b} \left| \frac{\partial_\alpha^x a_0(j, x)}{h^{\mid \alpha \mid} \alpha ! \sigma} \right| \right)$$

should belong to $L^p(\mathbb{R}^d)$. 


Proposition 4.2. Let $b \in \mathbb{R}_+^d$ be fixed, $\Lambda$ be given by \((4.11)\), $s, h_0, \sigma > 0$, \((4.1)\) be a sequence of functions on $C^\infty(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$, and let $T_{\psi}a_0$, $b_{a_0,\alpha}$ and $b_{a_0,h}$ be given by \((4.2)\)–\((4.4)\) when $\alpha \in \mathbb{N}^d$ and $h > 0$. Then the following is true:

1. If $b_{a_0,\alpha} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ for $\alpha = (0, \ldots, 0) \in \mathbb{N}^d$, then the series in \((4.2)\) is locally uniformly convergent and defines an element in $C(\mathbb{R}^d)$;

2. If $b_{a_0,\alpha} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}^d$, then $T_{\psi}a_0 \in C^\infty(\mathbb{R}^d)$ and \[
|\langle \partial^\alpha T_{\psi}a_0 \rangle(x)| \lesssim b_{a_0,\alpha}(x), \quad x \in \mathbb{R}^d,
\]
for every $\alpha \in \mathbb{N}^d$;

3. If in addition $\psi \in S'_{s}(\mathbb{R}^d)$ and $b_{a_0,h_0} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, then \[
|\langle \partial^\alpha T_{\psi}a_0 \rangle(x)| \lesssim h^\alpha a^\alpha b_{a_0,h_0}(x), \quad x \in \mathbb{R}^d,
\]
for some $h > 0$;

4. If in addition $\psi \in \Sigma_{s}'(\mathbb{R}^d)$, $c > 1$ and $b_{a_0,h_0} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, then \[
|\langle \partial^\alpha T_{\psi}a_0 \rangle(x)| \lesssim (ch_0)^\alpha a^\alpha b_{a_0,h_0}(x), \quad x \in \mathbb{R}^d.
\]

Proof. We only prove (1) and (4). The other assertions follow by similar arguments and are left for the reader.

Let $\Lambda = \Lambda_0$, $\alpha = (0, \ldots, 0) \in \mathbb{N}^d$ and suppose that $b_{a_0,\alpha} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. We have
\[
\sum_{j \in \Lambda} |a_0(j,x)\psi(x-j)| \leq b_{a_0,\alpha}(x) \sum_{j \in \Lambda} |\psi(x-j)| \approx b_{a_0,\alpha}(x),
\]
which shows that \((4.2)\) is locally uniformly convergent. Since $a_0(j, \cdot)$ and $\psi(\cdot - j)$ are continuous functions, it follows that $T_{\psi}a_0$ in \((4.2)\) is continuous.

Next suppose additionally that $\psi \in \Sigma_{s}'(\mathbb{R}^d)$ and consider $f = T_{\psi}a_0$. For every $\alpha \in \mathbb{N}^d$, $\varepsilon > 0$ and $r > 0$, we have
\[
|\langle \partial^\alpha f \rangle(x)| \leq \sum_{j \in \Lambda} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\partial^{\alpha-\gamma} a_0(j,x)||\partial^\gamma \psi(x-j)|
\]
\[
\lesssim b_{a_0,h_0}(x) \sum_{j \in \Lambda} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} h_0^{\alpha-\gamma} |\alpha - \gamma|^p e^{\gamma \rho |\gamma|^q} e^{-r |x-j|^{\frac{q}{p}}}
\]
\[
\leq (h_0 + \varepsilon)^{\alpha} a^\alpha b_{a_0,h_0}(x) \sum_{j \in \Lambda} e^{-r |x-j|^{\frac{q}{p}}} \approx (h_0 + \varepsilon)^{\alpha} a^\alpha b_{a_0,h_0}(x),
\]
and the result follows.

In the next result we show that if $b_{a_0,\alpha}$ or $b_{a_0,h}$ in the previous proposition belong to $W^l(\omega_0, \ell^p)$, then for $T_{\psi}a_0$ in \((4.2)\) we have
\[
T_{\psi}a_0 \in M^{p,q}_{(\omega_r), (\omega_r)}(\mathbb{R}^d) \bigcap W^{p,q}_{(\omega_r)}(\mathbb{R}^d),
\]
and
\[
\|T_{\psi}a_0\|_{M^{p,q}_{(\omega_r)}} + \|T_{\psi}a_0\|_{W^{p,q}_{(\omega_r)}} \lesssim_{|\alpha| \leq N} \max \{ b_{a_0,\alpha} \|W^l(\omega_0, \ell^p) \}. \quad (4.6)
\]
or

\[ \|T_\psi a_0\|_{\mathcal{M}^{p,q}_{(\omega_r)}} + \|T_\psi a_0\|_{W^{p,q}_{(\omega_r)}} \lesssim \|b_{a_0,h}\|_{W^1(\omega_0,\ell^p)} \]  

(4.7)

See also Remark 1.9 for notations.

**Proposition 4.3.** Let \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \), \( \Lambda \subseteq \mathbb{R}^d \), \( s, h_0, \sigma > 0 \), \( T_\psi a_0, b_{a_0,\alpha}, b_{a_0,h} \) and \( \psi \) be the same as in Proposition 4.2 and let \( p, q \in (0, \infty] \). Then the following is true:

1. if in addition \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \), \( b_{a_0,\alpha} \in W^1(\omega_0,\ell^p) \) for every \( \alpha \in \mathbb{N}^d \), and \( \partial_r(x, \xi) = \omega_0(x)\langle \xi \rangle_r \) when \( r \geq 0 \), then (4.5) and (4.6) hold;
2. if in addition \( \omega_0 \in \mathcal{P}_{E,s}(\mathbb{R}^d) \), \( \psi \in \mathcal{S}_\omega(\mathbb{R}^d) \), \( b_{a_0,h} \in W^1(\omega_0,\ell^p) \) for some \( h > 0 \), and \( \partial_r(x, \xi) = \omega_0(x)e^{r|\xi|^b} \) when \( r \geq 0 \), then (4.5) and (4.7) hold for some \( r > 0 \);
3. if in addition \( \omega_0 \in \mathcal{P}_{E,s}(\mathbb{R}^d) \), \( \psi \in \mathcal{S}_\omega(\mathbb{R}^d) \), \( b_{a_0,h} \in W^1(\omega_0,\ell^p) \) for every \( h > 0 \), and \( \partial_r(x, \xi) = \omega_0(x)e^{r|\xi|^b} \) when \( r \geq 0 \), then (4.5) and (4.7) hold for every \( r > 0 \).

**Proof.** We only prove (2). The other assertions follow by similar arguments and are left for the reader.

Let \( f = T_\psi a_0, \psi_j = \psi(\cdot - j) \) and \( \tilde{\phi}(x) = \phi(-x) \). Since

\[
(V_\phi f)(x, \xi) = e^{-i(x, \xi)}(V_{\tilde{\phi}} f)(\xi, -x),
\]

we get

\[
(V_\phi f)(x, \xi) = e^{-i(x, \xi)} \sum_{j \in \Lambda} (V_{\tilde{\phi}} f_j)(j, \cdot))((\xi, -x)
\]

\[
= e^{-i(x, \xi)} \sum_{j \in \Lambda} \mathcal{F}^1_j \left( (\psi_j a_0(j, \cdot)) \cdot \overline{\tilde{\phi}(\cdot - \xi)} \right)(x)
\]

\[
= (2\pi)^{-d/2} e^{-i(x, \xi)} \sum_{j \in \Lambda} \left( (\psi_j a_0(j, \cdot)) \ast (\tilde{\phi} \ast e^{i(\cdot, \xi)}) \right)(x).
\]

Hence, Leibnitz rule, integrations by parts and Proposition 1.1 give

\[
|\xi^\alpha (V_\phi f)(x, \xi)| \lesssim \sum_{j \in \Lambda} \left| ((\psi_j a_0(j, \cdot)) \ast (\tilde{\phi} \ast (D^s_{\xi^\alpha} e^{i(\cdot, \xi)}))) \right|(x)
\]

\[
\leq \sum_{\gamma_1 \gamma_2 \gamma_3} \frac{\alpha!}{\gamma_1! \gamma_2! \gamma_3!} \left| (\partial^{\gamma_1} \psi_j)(\partial^{\gamma_2} a_0(j, \cdot)) \ast (\partial^{\gamma_3} \tilde{\phi}) \right|(x)
\]

\[
\lesssim \sum_{\gamma_1 \gamma_2 \gamma_3} 3^{\lvert \alpha \rvert} h^{\lvert \gamma_1 \rvert + 2\| \gamma_2 \| + \| \gamma_3 \|} \left( (e^{-r|\cdot|^{1/2}} b_{a_0,h}) \ast e^{-r|\cdot|^{1/2}} \right)(x)
\]

\[
\leq (9h)^{\lvert \alpha \rvert} \sum_{j \in \Lambda} \left( (e^{-r|\cdot|^{1/2}} b_{a_0,h}) \ast e^{-r|\cdot|^{1/2}} \right)(x)
\]

\[
\times (9h)^{\lvert \alpha \rvert} \left( b_{a_0,h} \ast e^{-r|\cdot|^{1/2}} \right)(x).
\]

Here the second and third sums are taken with respect to all \( j \in \Lambda \) and all \( \gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}^d \) such that \( \gamma_1 + \gamma_2 + \gamma_3 = \alpha \).
This implies that for some constant $C$ which is independent of $h$, $r$ and $\alpha$ we have
\[
\left(\frac{|\xi|^{\frac{d}{2}}}{(Ch)^{\frac{d}{2}}}\right)^{k} |V_{\phi} f(x, \xi)|^{\frac{2}{p}} \lesssim 2^{-k} \left((b_{a_0, h} \ast e^{-r|\cdot|^\frac{d}{2}})(x)\right)^{\frac{1}{2}},
\]
and by taking the sum over all $k \geq 0$ we land on
\[
|V_{\phi} f(x, \xi)e^{r_{\gamma}|\cdot|^\frac{d}{2}}| \lesssim (b_{a_0, h} \ast e^{-r|\cdot|^\frac{d}{2}})(x), \quad r_{h} = \frac{\sigma}{(Ch)^{\frac{d}{2}}},
\]
for some $r_{0} > 0$.

By multiplying with $\omega_{0}$ and using that $\omega_{0}(x + y) \lesssim \omega_{0}(x)e^{r_{0}|y|^\frac{d}{2}}$ for every $r > 0$, we obtain
\[
|V_{\phi} f(x, \xi)\vartheta_{r_{h}}(x, \xi)| \lesssim ((b_{a_0, h}\omega_{0}) \ast e^{-r_{0}|\cdot|^\frac{d}{2}})(x), \quad r_{h} = \frac{\sigma}{(Ch)^{\frac{d}{2}}}, \quad (4.8)
\]
for some $r_{0} > 0$.

By applying [32, Proposition 2.5] on the last inequality we obtain
\[
\|V_{\phi} f\|_{W(\vartheta_{r_{h}}, \ell^{p, \infty})} + \|V_{\phi} f\|_{W(\vartheta_{r_{h}}, \ell^{p, q})} \lesssim \|e^{-r|\cdot|^\frac{d}{2}}\|_{W(1, \ell^{\min(1, p)})}\|b_{a_0, h}\|_{W(\vartheta_{r_{h}}, \ell^{p})}.
\]
The result now follows for general $q \in (0, \infty]$ from the relations
\[
\|V_{\phi} f\|_{W(\vartheta_{r_{h}}, \ell^{p, \infty})} \asymp \|f\|_{M^{p, \infty}} \quad \text{and} \quad M_{r_{h}}^{p, \infty}(\mathbb{R}^{d}) \hookrightarrow M_{r_{h}}^{p, q}(\mathbb{R}^{d}) \hookrightarrow M_{r_{h}}^{p, \infty}(\mathbb{R}^{d}),
\]
and similarly with $W^{p, q}(\omega)$ and $\ell^{p, q}$ and $\ell^{p, q}$ spaces in place of $M^{p, q}(\omega)$ and $\ell^{p, q}$ spaces.

We have now the following extension of Theorem 24.1. Here involved Lebesgue exponents and weight functions should fulfill
\[
\frac{1}{p_{2}} - \frac{1}{p_{1}} \leq \frac{1}{p}, \quad \frac{1}{q_{1}} - \frac{1}{q_{2}} \geq \max\left(\frac{1}{p_{2}} - 1, 0\right), \quad (4.9)
\]
and
\[
\omega_{0, 2}(x) \lesssim \omega_{0, 1}(x)\omega_{0}(x), \quad (4.10)
\]

**Theorem 4.4.** Let $p, p_{1}, p_{2} \in (0, \infty]$, $q \in (1, \infty)$, $q_{1}, q_{2} \in (\min(1, p_{2}), \infty)$ be such that (4.9) holds, $b > 0$, $\omega_{0, \omega_{0}, j}$ be weights on $\mathbb{R}^{d}$ such that (4.10) holds true, $\omega(x, \xi) = \omega_{0}(x)$ and $\omega_{j}(x, \xi) = \omega_{0,j}(x)$, $j = 1, 2, x, \xi \in \mathbb{R}^{d}$. Let $a_{0}$ in (4.1) be such that $\Lambda = \Lambda_{b}$ and $a_{0}(j, \cdot) \in C^{\infty}_{0}(\mathbb{R}^{d})$ for every $j \in \Lambda_{b}$, and let $b_{a_{0}, \alpha}$ and $b_{a_{0}, h}$ be given by (1.3) and (4.4). Also suppose that one of the following conditions hold true:

(i) $b_{a_{0}, \alpha} \in W^{1}(\omega_{0}, \ell^{p})$ for every $\alpha \in \mathbb{N}^{d}$, and $\omega_{0, \omega_{0}, j} \in \mathcal{P}(\mathbb{R}^{d})$, $j = 1, 2$;

(ii) $b_{a_{0}, h} \in W^{1}(\omega_{0}, \ell^{p})$ for some $h > 0$, and $\omega_{0, \omega_{0}, j} \in \mathcal{P}_{E_{0,s}}^{0}(\mathbb{R}^{d})$, $j = 1, 2$;

(iii) $b_{a_{0}, h} \in W^{1}(\omega_{0}, \ell^{p})$ for every $h > 0$, and $\omega_{0, \omega_{0}, j} \in \mathcal{P}_{E_{0,s}}(\mathbb{R}^{d})$, $j = 1, 2$.

Then the following is true:

(1) $M_{b, a_{0}}$ is continuous from $W_{(\omega_{1})}^{p_{1}, q_{1}}(\mathbb{R}^{d})$ to $W_{(\omega_{2})}^{p_{2}, q_{2}}(\mathbb{R}^{d})$;

(2) $M_{b, a_{0}}$ is continuous from $M_{(\omega_{1})}^{p_{1}, q_{1}}(\mathbb{R}^{d})$ to $M_{(\omega_{2})}^{p_{2}, q_{2}}(\mathbb{R}^{d})$. 
Proof. We only prove the result when (iii) holds. The other cases follow by similar arguments and is left for the reader.

Let \( \psi \in \Sigma^s_q(\mathbb{R}^d) \) be such that \( \psi = 1 \) on \( Q_b \) and supported in a neighbourhood of \( Q_b \), and let \( \Lambda_1, \ldots, \Lambda_N \) be sublattices of \( \Lambda = \Lambda_b \) such that

\[
\bigcup_{j=1}^N \Lambda_j = \Lambda \quad \text{and} \quad \text{supp} \psi(\cdot - k_1) \bigcap \text{supp} \psi(\cdot - k_2) = \emptyset, \ k_1, k_2 \in \Lambda_j, \ k_1 \neq k_2,
\]

for every \( j = 1, \ldots, N \). Then

\[
M_{b,a_0} = \sum_{j=1}^N S_j,
\]

where \( S_j = S_{2,j} \circ S_{1,j} \), with \( S_{1,j} \) and \( S_{2,j} \) being the multiplication operators with the functions

\[
\varphi_{1,j} = \sum_{k \in \Lambda_j} a_0(k, \cdot) \psi(\cdot - k) \quad \text{and} \quad \varphi_{2,j} = \sum_{k \in \Lambda_j} \chi Q_b(\cdot - k),
\]

respectively. The result follows if we prove the asserted continuity properties for \( S_j \) in place of \( M_{b,a_0} \).

By Proposition 3.3 it follows that \( \varphi_{1,j} \in M^{p,r}_{\varphi,\varphi}(\mathbb{R}^d) \cap W^{p,q}_{\varphi,\varphi}(\mathbb{R}^d) \) for every \( q \in (0,1] \) and \( r > 0 \). Hence, if we choose \( q \) small enough, Theorems 3.2 and 3.3 show that \( S_{1,j} \) is continuous from \( W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \), and from \( M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \). In view of Theorem 2.1 one has that \( S_{2,j} \) is continuous on \( W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \), and from \( M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \), for every \( j \).

By combining these mapping properties it follows that \( S_j \) is continuous from \( W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \), and from \( M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \) for every \( j \), and the result follows. \( \square \)

By Fourier transforming the latter result we obtain the following extension of Theorem 2.3. The details are left for the reader. Here

\[
\frac{1}{q_2} - \frac{1}{q_1} \leq \frac{1}{q} \quad \text{and} \quad \frac{1}{p_1} - \frac{1}{p_2} \geq \max \left( \frac{1}{q_2} - 1, 0 \right).
\]  \hspace{1cm} (4.11)

Theorem 4.5. Let \( q, q_1, q_2 \in (0, \infty) \), \( p \in (1, \infty) \), \( p_1, p_2 \in (\min(1, q_2), \infty) \) be such that \( (4.11) \) holds true, \( \omega, \omega_0, j \) be weights on \( \mathbb{R}^d \) such that \( (4.11) \) holds true, \( \omega(x, \xi) = \omega_0(\xi) \) and \( \omega_j(x, \xi) = \omega_{0,j}(\xi) \); \( j = 1, 2 \), \( x, \xi \in \mathbb{R}^d \). Let \( f_0 \in L^1(\omega_0, \ell^p) \) be such that \( \Lambda = \Lambda_b \) and \( a_0(\cdot, \cdot) \in C^\infty(\mathbb{R}^d) \) for every \( j \in \Lambda_b \), and let \( b_{a_0,\alpha} \) and \( b_{a_0,h} \) be given by (4.3) and (4.4). Also suppose that one of the following conditions hold true:

(i) \( b_{a_0,\alpha} \in W^1(\omega_0, \ell^p) \) for every \( \alpha \in \mathbb{N}^d \), and \( \omega_0, \omega_0, j \in \mathcal{P}(\mathbb{R}^d) \), \( j = 1, 2 \);

(ii) \( b_{a_0,h} \in W^1(\omega_0, \ell^p) \) for some \( h > 0 \), and \( \omega_0, \omega_0, j \in \mathcal{P}^0_{E,s}(\mathbb{R}^d) \), \( j = 1, 2 \);

(iii) \( b_{a_0,h} \in W^1(\omega_0, \ell^p) \) for every \( h > 0 \), and \( \omega_0, \omega_0, j \in \mathcal{P}^0_{E,s}(\mathbb{R}^d) \), \( j = 1, 2 \).

Then the following is true:

(1) \( M_{f, b,a_0} \) is continuous from \( M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \);

(2) \( M_{f, b,a_0} \) is continuous from \( W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \) to \( W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d) \).
We observe that Theorems 4.4 and 4.5 include the following extensions of Theorems 2.1 and 2.3.

**Corollary 4.6.** Let \( p, p_1, p_2 \in (0, \infty], q \in (1, \infty), q_1, q_2 \in \min(1, p) \) be such that \((4.9)\) hold, \( b > 0 \), \( \omega_0, \omega_{0,j} \in \mathcal{P}_E(\mathbb{R}^d) \) be such that \((4.10)\), \( \omega_j(x, \xi) = \omega_{0,j}(x) \), \( j = 1, 2, x, \xi \in \mathbb{R}^d \), and let \( a_0 \in \ell^p(\omega_0)(\Lambda_b) \). Then the following is true:

1. \( M_{b,a_0} \) is continuous from \( \omega_1(q)(\mathbb{R}^d) \) to \( \omega_2(q)(\mathbb{R}^d) \);
2. \( M_{b,a_0} \) is continuous from \( \omega_1(q)(\mathbb{R}^d) \) to \( \omega_2(q)(\mathbb{R}^d) \).

**Corollary 4.7.** Let \( p \in (1, \infty), p_1, p_2 \in \min(1, q, \infty), q, q_1, q_2 \in (0, \infty) \) be such that \((4.11)\) hold, \( b > 0 \), \( \omega_0, \omega_{0,j} \in \mathcal{P}_E(\mathbb{R}^d) \) be such that \((4.10)\) holds, \( \omega_j(x, \xi) = \omega_{0,j}(\xi) \), \( j = 1, 2, x, \xi \in \mathbb{R}^d \), and let \( a_0 \in \ell^p(\omega_0)(\Lambda_b) \). Then the following is true:

1. \( M_{p,b,a_0} \) is continuous from \( \omega_1(q)(\mathbb{R}^d) \) to \( \omega_2(q)(\mathbb{R}^d) \);
2. \( M_{p,b,a_0} \) is continuous from \( \omega_1(q)(\mathbb{R}^d) \) to \( \omega_2(q)(\mathbb{R}^d) \).

**Proof of Corollaries 4.6 and 4.7.** Let \( \psi \in \Sigma_\ell^p(\mathbb{R}^d) \) be compactly supported and chosen such that \( \psi = 1 \) on \( Q_0 \). Then the results follow by letting \( a_0(j, \cdot) = a_0(j)\psi(\cdot - j) \) in Theorems 4.4 and 4.5. The details are left for the reader. \( \square \)

**Appendix A.**

In this appendix we give a proof of Proposition 1.10 (2).

**Proof of Proposition 1.10 (2).** Since \( \ell^p(\omega)(\Lambda) \) increases with \( p \) and \( \| \cdot \|_{\ell^p(\omega)} \) decreases with \( p \), we may assume that equality is attained in \((1.40)\).

By \((1.37)\) we get
\[
|[(a_1 \ast \cdots \ast a_N)(k) \cdot \omega_0(k)] - ((a_1 \cdot \omega_1) \ast \cdots \ast (a_N \cdot \omega_N))(k)|
\]
provided the left-hand side makes sense. A combination of this inequality with the fact that the map \( a_j \mapsto a_j \cdot \omega_j \) is an isometric bijection from \( \ell^p(\omega_j)(\Lambda) \) to \( \ell^p(\Lambda) \), reduces ourselves to the case when \( \omega_j = 1 \) for every \( j \).

Let \( I_N = \{ 1, \ldots, N \} \) and \( j_0 \in I_N \) be chosen such that \( p_{j_0} \geq p_j \) for every \( j \in I_N \). Then
\[
\frac{1}{p_0} = \frac{1}{p_{j_0}} + \sum_{j \neq j_0} \left( \frac{1}{p_j} - \max \left( 1, \frac{1}{p_j} \right) \right) \leq \frac{1}{p_{j_0}}, \quad (A.1)
\]
giving that \( p_j \leq p_{j_0} \) for every \( j \in I_N \).

In particular, if \( p_1, \ldots, p_N \geq 1 \), then \((1.40)\) shows that \( p_0 \geq 1 \), and the assertion agrees with the usual Young’s inequality. By \((A.1)\) it also follows that if \( p_{j_0} = \infty \), then \( p_0 = \infty \) and \( p_j \leq 1 \) when \( j \in I_N \setminus \{ j_0 \} \), since otherwise the equality in \((A.1)\) may not hold for some \( p_0 \) in \((0, \infty] \). In this case we have that if \( a_j \in \ell^{p_j}(\Lambda), j \in I_N \), then the facts \( \ell^{p_j} \subseteq \ell^{1} \) and \( \| \cdot \|_{\ell^p} \leq \| \cdot \|_{\ell^{1}} \) for \( j \neq j_0 \) give
\[
a_1 \ast \cdots \ast a_N \in \ell^1(\Lambda) \ast \cdots \ast \ell^{p_0}(\Lambda) \ast \cdots \ast \ell^{p_N}(\Lambda)
\]

\[
\subseteq \ell^1(\Lambda) \ast \cdots \ast \ell^{\infty}(\Lambda) \ast \cdots \ast \ell^1(\Lambda) = \ell^\infty(\Lambda)
\]
By the induction hypothesis we get
\[ \|a \|_{\ell^1} \leq \|a_1\|_{\ell^1} \cdot \cdots \cdot \|a_j\|_{\ell^1} \leq \|a_1\|_{\ell^1} \cdot \cdots \cdot \|a_j\|_{\ell^1} \cdot \|a_N\|_{\ell^P_N}, \]
and the result follows in this case as well.

It remains to prove the result when \( p_j < 1 \) for at least one \( j \in I_N \) and 
that \( p_{j_0} < \infty \), giving that \( \ell_0 \) is dense in \( \ell^P \) for every \( j \in I_N \). Hence the result follows if we prove \( \text{[1.42]} \) when \( a_j \in \ell_0 \).

Suppose \( N = 2 \). Then \( I = \{ j_0, j_1 \} \) for some \( j_1 \in \{ 1, 2 \} \) with \( p_{j_1} < 1 \) and 
\( j_1 \leq j_0 \). Then
\[
\frac{1}{p_0} = \frac{1}{p_{j_0}} + \frac{1}{p_{j_1}} - \frac{1}{p_{j_0} p_{j_1}} = \frac{1}{p_{j_0}},
\]
i.e. \( p_0 = p_{j_0} \). This gives
\[ \|a_{j_0} * a_{j_1}\|_{\ell^P} = \|a_{j_0} * a_{j_1}\|_{\ell^P_{j_0}} \leq \|a_{j_0}\|_{\ell^P_{j_0}} \|a_{j_1}\|_{\ell^P_{j_1}}, \]
and the result follows for \( N = 2 \).

Next suppose that \( N \geq 3 \), and that the result holds for less numbers of factors in the convolution. Since the convolution is commutative, we may assume that \( p_N < 1 \) is the smallest number in \( I_N \). Then \( p_N \leq p_0 \), and \( \text{[1.40]} \) is the same as
\[
\frac{1}{p_0} \leq \sum_{j=1}^{N-1} \frac{1}{p_j} - Q_{N-1}(p_1, \ldots, p_{N-1}).
\]
By the induction hypothesis we get
\[ \|a_1 * \cdots * a_{N-1}\|_{\ell^P_0} \leq \|a_1\|_{\ell^P_1} \cdot \cdots \cdot \|a_{N-1}\|_{\ell^P_{N-1}}. \]

Since \( p_N < 1 \) and \( p_N \leq p_0 \) we get
\[ \|a_1 * \cdots * a_N\|_{\ell^P_0} \leq \|a_1 * \cdots * a_{N-1}\|_{\ell^P_0} \|a_N\|_{\ell^P_N} \leq \|a_1\|_{\ell^P_1} \cdot \cdots \cdot \|a_N\|_{\ell^P_N}. \]

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