REMARK ABOUT HEAT DIFFUSION ON PERIODIC SPACES

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Abstract. Let $M$ be a complete Riemannian manifold with a free cocompact $\mathbb{Z}^k$-action. Let $k(t, m_1, m_2)$ be the heat kernel on $M$. We compute the asymptotics of $k(t, m_1, m_2)$ in the limit in which $t \to \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. We show that in this limit, the heat diffusion is governed by an effective Euclidean metric on $R^k$ coming from the Hodge inner product on $H^1(M/\mathbb{Z}^k; \mathbb{R})$.

1. Introduction

Let $M$ be a complete connected oriented $n$-dimensional Riemannian manifold. Let $k(t, m_1, m_2)$ be the time-$t$ heat kernel on $M$. The usual ansatz to approximate $k(t, m_1, m_2)$ is to say that

$$k(t, m_1, m_2) \sim P(t, m_1, m_2) e^{-\frac{d(m_1, m_2)^2}{4t}}$$ \hspace{1cm} (1.1)

where $e^{-\frac{d(m_1, m_2)^2}{4t}}$ is considered to be the leading term and $P(t, m_1, m_2)$ is a correction term which can be computed iteratively. There are results which make this precise. For example [1], if $m_1$ and $m_2$ are nonconjugate, then as $t \to 0$,

$$k(t, m_1, m_2) = \sum_{\gamma} \frac{(\det d(\exp_{m_1} sv_{\gamma}))^{-1/2}}{(4\pi t)^{n/2}} e^{-\frac{d(m_1, m_2)^2}{4t}} (1 + O(t)).$$ \hspace{1cm} (1.2)

Here the sum is over minimal geodesics $\gamma : [0, 1] \to M$ joining $m_1$ to $m_2$ of the form $\gamma(s) = \exp_{m_1}(sv_{\gamma})$. For another example, if $M$ has bounded geometry, then lower and upper heat kernel bounds [4], [5] imply that (1.1) is a good approximation if $d(m_1, m_2) \gg t$, in the sense that $-\ln(k(t, m_1, m_2))$ is well-approximated by $\frac{d(m_1, m_2)^2}{4t}$.

One can ask if the ansatz (1.1) is relevant for other asymptotic regimes. In this paper we look at the case when $M$ has a periodic metric, meaning that $\mathbb{Z}^k$ acts freely by orientation-preserving isometries on $M$, with $X = M/\mathbb{Z}^k$ compact. We consider the asymptotic regime in which $t \to \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. As the typical time-$t$ Brownian path will travel a distance comparable to $\sqrt{t}$, this is the regime which contains the bulk of the diffusing heat. We show that, in this regime, (1.1) is no longer a valid approximation. Instead, the heat diffusion is governed by...
an effective Euclidean metric on \( \mathbb{R}^k \). This metric is constructed using the Hodge inner product on \( H^1(X; \mathbb{R}) \).

To state the precise result, let \( \mathcal{F} \) be a fundamental domain in \( M \) for the \( \mathbb{Z}^k \)-action. Given \( \mathbf{v} \in \mathbb{Z}^k \), put
\[
(1.3) \quad k(t, \mathbf{v}) = \int_{\mathcal{F}} k(t, m, \mathbf{v} \cdot m) \, d\text{vol}(m).
\]
This is independent of the choice of fundamental domain \( \mathcal{F} \).

The covering \( M \to X \) is classified by a map \( \nu : X \to B\mathbb{Z}^k \), defined up to homotopy, which is \( \pi_1 \)-surjective. It induces a surjection \( \nu_* : H_1(X; \mathbb{R}) \to \mathbb{R}^k \) and an injection \( \nu^* : (\mathbb{R}^k)^* \to H^1(X; \mathbb{R}) \). Let \( \langle \cdot, \cdot \rangle_{H^1(X; \mathbb{R})} \) be the Hodge inner product on \( H^1(X; \mathbb{R}) \).

**Definition 1.** The inner product \( \langle \cdot, \cdot \rangle_{(\mathbb{R}^k)^*} \) on \( (\mathbb{R}^k)^* \) is given by
\[
(1.4) \quad \langle \cdot, \cdot \rangle_{(\mathbb{R}^k)^*} = \frac{(\nu^*)^*(\cdot, \cdot)_{H^1(X; \mathbb{R})}}{\text{vol}(X)}.
\]
The inner product \( \langle \cdot, \cdot \rangle_{\mathbb{R}^k} \) is the dual inner product on \( \mathbb{R}^k \).

Let \( \text{vol}(\mathbb{R}^k/\mathbb{Z}^k) \) be the volume of a lattice cell in \( \mathbb{R}^k \), measured with \( \langle \cdot, \cdot \rangle_{\mathbb{R}^k} \).

**Proposition 1.** Fix \( C > 0 \). Then in the region \( \{(t, \mathbf{v}) \in \mathbb{R}^+ \times \mathbb{Z}^k : \langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k} \leq Ct\} \), as \( t \to \infty \) we have
\[
(1.5) \quad k(t, \mathbf{v}) = \frac{\text{vol}(\mathbb{R}^k/\mathbb{Z}^k)}{(4\pi t)^{k/2}} e^{-\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k}/(4t)} + O(t^{-\frac{k+1}{2}})
\]
uniformly in \( \mathbf{v} \).

**Example.** 1. If \( M = \mathbb{R}^k \) with a flat metric \( \langle \cdot, \cdot \rangle_{\text{flat}} \), then one can check that \( \langle \cdot, \cdot \rangle_{\mathbb{R}^k} = \langle \cdot, \cdot \rangle_{\text{flat}} \), so one recovers the standard flat-space heat kernel.

2. If \( n = 2 \), then \( \langle \cdot, \cdot \rangle_{H^1(X; \mathbb{R})} \) is conformally-invariant. Hence in this case, the heat kernel asymptotics only depend on \( \text{vol}(X) \) and the induced complex structure on \( X \).

One can get similar pointwise estimates on \( k(t, m_1, m_2) \) by the same methods. We omit the details.

The result of Proposition 1 is an example of the phenomenon of “homogenization”, which has been much-studied for differential operators on \( \mathbb{R}^n \). Homogenization means that in an appropriate scaling limit, the solution to a problem is governed by the solution to a spatially homogeneous problem; see [2] and references therein. Thus it is not surprising that the answer in Proposition 1 has a homogeneous form. The point of the present paper is to show how one can compute the exact asymptotics in the general geometric setting.

We remark that when \( t \to \infty \) and \( d(m_1, m_2) \gg t \), the asymptotic expression (1.1) also shows homogenization. This follows from the result of D. Burago [3] that there are a Banach norm \( \| \cdot \| \) on \( \mathbb{R}^k \) and a constant \( c > 0 \) such that if \( m \in M \) and \( \mathbf{v} \in \mathbb{Z}^k \), then \( |d(m, \mathbf{v} \cdot m) - \| \mathbf{v} \| \| \leq c \). Thus as \( t \to \infty \), if \( d(m_1, m_2) \sim \sqrt{t} \), then the effective geometry is \( (\mathbb{R}^k, \langle \cdot, \cdot \rangle_{\mathbb{R}^k}) \), while if \( d(m_1, m_2) \gg t \), then the effective geometry is \( (\mathbb{R}^k, \| \cdot \|) \).

It would be interesting if one could extend the results of this paper to the setting in which \( \Gamma \) is a nonabelian discrete group, such as the fundamental group of a closed hyperbolic surface. In this case, the relevant scaling regime should be \( t \to \infty \) and
d(m_1, m_2) \sim t$, as the typical time-$t$ Brownian path on the hyperbolic plane travels a distance comparable to $t$.

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2. Proof of Proposition 1

We first recall some basic facts about the eigenvalues of a parametrized family of operators [7, Chapter XII].

Let $M_d(\mathbb{C})$ be the vector space of $d \times d$ complex matrices and let $M_d^{sa}(\mathbb{C})$ be the subspace of self-adjoint matrices. Let $f : \mathbb{R}^k \to M_d(\mathbb{C})$ be a real-analytic map. The eigenvalues $\{\lambda_i(x)\}_{i=1}^d$ of $f(x)$ are algebraic functions of $x$, meaning the roots of a polynomial whose coefficients are real-analytic functions of $x$, as they are given by $\text{det}(f(x) - \lambda) = 0$. If $\lambda_1(0)$ is a nondegenerate eigenvalue of $f(0)$, then it extends near $x = 0$ to a real-analytic function $\lambda_1(x)$.

If $k = 1$ and $f$ takes values in $M_d^{sa}(\mathbb{C})$, then the eigenvalues of $f$ form $d$ real-analytic functions $\{\lambda_i(x)\}_{i=1}^d$ on $\mathbb{R}$. Of course, these functions may cross. If $k > 1$ and $f$ takes values in $M_d^{sa}(\mathbb{C})$, then it may not be true that the eigenvalues form real-analytic functions on $\mathbb{R}^k$. This can be seen in the example $f(x_1, x_2) = \begin{pmatrix} 0 & x_1 - i x_2 \\ x_1 + i x_2 & 0 \end{pmatrix}$. Its eigenvalues are $\pm \sqrt{x_1^2 + x_2^2}$, which are not the union of two smooth functions on $\mathbb{R}^2$. However, if $\gamma(s)$ is a real-analytic curve in $\mathbb{R}^2$, then the eigenvalues of $f(\gamma(s))$ do form real-analytic functions in $s$.

If $f$ is instead an appropriate real-analytic family of operators on a Hilbert space, then one has similar results. We refer to [7, Chapter XII.2] for the precise requirements.

To prove Proposition 1, we use the method of [6, Section VI]. The Pontryagin dual of $\mathbb{Z}^k$ is $T^k = (\mathbb{R}^k)^*/2\pi(\mathbb{Z}^k)^*$. Given $\theta \in T^k$, let $\rho(\theta) : \mathbb{Z}^k \to U(1)$ be the corresponding representation and let $E(\theta)$ be the flat line bundle on $X$ associated to the representation $\pi_1(X) \to \mathbb{Z}^k \xrightarrow{\rho(\theta)} U(1)$. Let $\Delta_\theta$ be the Laplacian on $L^2(X; E(\theta))$. Then Fourier analysis gives

$$k(t, \nu) = \int_{T^k} e^{i \theta \cdot \nu} \text{Tr} \left( e^{-t \Delta(\theta)} \right) \frac{d^k \theta}{(2\pi)^k}. \tag{2.1}$$

Now Ker($\Delta(\theta)$) = 0 if $\theta \neq 0$ and Ker($\Delta(\theta)$) $\approx \mathbb{C}$ consists of the constant functions on $X$.

In order to write all of the operators $\Delta(\theta)$ as acting on the same Hilbert space, let $\{\tau^j\}_{j=1}^k$ be a set of harmonic 1-forms on $X$ which gives an integral basis of $\mathbb{Z}^k = (\mathbb{R}^k)^* \subseteq H^1(X; \mathbb{R})$. Let $d(\tau^j)$ denote exterior multiplication by $\tau^j$ on $C^\infty(X)$ and let $i(\tau^j)$ denote interior multiplication by $\tau^j$ on $\Omega^1(X)$. Putting

$$d(\theta) = d + i \sum_{j=1}^k \theta_j e(\tau^j) \tag{2.2}$$

and

$$d^*(\theta) = d^* - i \sum_{j=1}^k \theta_j i(\tau^j), \tag{2.3}$$

$\Delta(\theta)$ is unitarily equivalent to the self-adjoint operator $d^*(\theta)d(\theta)$ (which we shall also denote by $\Delta(\theta)$) acting on $L^2(X)$. Because $\Delta(\theta)$ is quadratic in $\theta$, it is easy
to see that \( \{ \Delta(\theta) \}_{\theta \in T^k} \) is an analytic family of type (A) in the sense of [7, Chapter XII.2], so we can apply analytic eigenvalue perturbation theory. In particular, if \( \{ \lambda_i(\theta) \}_{i \in \mathbb{Z}^*} \) are the eigenvalues of \( \Delta(\theta) \), arranged in increasing order and repeated if there is a multiplicity greater than one, then \( \lambda_1(\theta) \geq 0 \) and \( \lambda_1(\theta) = 0 \) if and only if \( \theta = 0 \), in which case it is a nondegenerate eigenvalue. Thus \( \lambda_1 \) extends to a real-analytic function in a neighborhood of \( \theta = 0 \). So for sufficiently small \( \epsilon > 0 \), there is a neighborhood \( U \subseteq T^k \) of \( 0 \in T^k \) such that

1. If \( \theta \notin U \), then \( \lambda_1(\theta) > \epsilon \).

2. Restricted to \( U \), \( \lambda_1 \) is a real-analytic function which represents a nondegenerate eigenvalue and \( \lambda_2 > \epsilon \).

From (2.1), we have

\[
(2.9) \quad k(t, \psi) = \int_{T^k} e^{i\theta \cdot \psi} \sum_{i=1}^{\infty} e^{-t\lambda_i(\theta)} \frac{d^k \theta}{(2\pi)^k}.
\]

Then it is easy to show that

\[
(2.5) \quad k(t, \psi) = \int_U e^{i\theta \cdot \psi} e^{-t\lambda_1(\theta)} \frac{d^k \theta}{(2\pi)^k} + O(e^{-t/2}),
\]

uniformly in \( \psi \).

**Lemma 1.** The Taylor’s series of \( \lambda_1(\theta) \) near \( \theta = 0 \) starts off as

\[
(2.6) \quad \lambda_1(\theta) = \lambda_1(0) + (\theta^3) + O(\theta^3).
\]

**Proof.** It suffices to compute \( \frac{d\lambda_i(s\vec{w})}{ds} \big|_{s=0} \) and \( \frac{d^2\lambda_i(s\vec{w})}{ds^2} \big|_{s=0} \) for all \( s \in (\mathbb{R}^k)^* \). For simplicity, denote \( \Delta(s\vec{w}) \) by \( \Delta(s) \) and \( \lambda_1(s\vec{w}) \) by \( \lambda(s) \). As \( \lambda(s) \) is nonnegative and \( \lambda(0) = 0 \), we must have \( \lambda'(0) = 0 \). Let \( \psi(s) \) denote a nonzero eigenfunction with eigenvalue \( \lambda(s) \); we can assume that it is real-analytic in \( s \) with \( \psi(0) = 1 \). Differentiation of \( \Delta(s)\psi(s) = \lambda(s)\psi(s) \) gives

\[
(2.7) \quad \Delta'(0)\psi(0) + \Delta(0)\psi'(0) = 0
\]
and

\[
(2.8) \quad \Delta''(0)\psi(0) + 2\Delta'(0)\psi'(0) + \Delta(0)\psi''(0) = \lambda''(0)\psi(0).
\]

Taking the inner product of (2.8) with \( \psi(0) \) gives

\[
(2.9) \quad \langle \psi(0), \Delta''(0)\psi(0) \rangle + 2\langle \psi(0), \Delta'(0)\psi'(0) \rangle + \langle \psi(0), \lambda''(0)\psi(0) \rangle = \lambda''(0)\langle \psi(0), \psi(0) \rangle.
\]

Let \( G \) be the Green’s operator for \( \Delta(0) \). From (2.7),

\[
(2.10) \quad \psi'(0) = c\psi(0) - G\Delta'(0)\psi(0)
\]
for some constant \( c \). Changing \( \psi(s) \) to \( e^{-cs}\psi(s) \), we may assume that \( c = 0 \). Substituting (2.10) into (2.9) gives

\[
(2.11) \quad \langle \psi(0), \Delta''(0)\psi(0) \rangle - 2\langle \psi(0), \Delta'(0)G\Delta'(0)\psi(0) \rangle = \lambda''(0)\langle \psi(0), \psi(0) \rangle.
\]

It remains to compute \( \langle \psi(0), \Delta''(0)\psi(0) \rangle \) and \( \langle \psi(0), \Delta'(0)G\Delta'(0)\psi(0) \rangle \). Put \( D(s) = d_{\vec{w}}s \) and \( D^*(s) = d_{\vec{w}}^*s \). Then \( \Delta(s) = D^*(s)D(s) \). From (2.2) and (2.3), \( D(s) \) and \( D^*(s) \) are linear in \( s \), with

\[
(2.12) \quad D'(0) = i \sum_{j=1}^k w_j e(\tau^j)
\]
and

\[(D^*)'(0) = -i \sum_{j=1}^{k} w_j i^{\tau_j}.\]

Then

\[
\langle \psi(0), \triangle''(0)\psi(0) \rangle = 2\langle \psi(0), (D^*)'(0)D'(0)\psi(0) \rangle
\]

\[
= 2|D'(0)\psi(0)|_{H^1(X;\mathbb{C})}^2
\]

\[
= 2\sum_{j=1}^{k} w_j \tau_j |_{H^1(X;\mathbb{C})}.\]

Now

\[
\triangle'(0)\psi(0) = [(D^*)'(0)D(0) + D^*(0)D'(0)] \psi(0)
\]

\[
= d^* \left( -i \sum_{j=1}^{k} w_j \tau_j \right) = 0.\]

Substituting (2.14) and (2.15) into (2.11) and using the fact that \(\langle \psi(0), \psi(0) \rangle = \text{vol}(X)\), the lemma follows.

Continuing with the proof of Proposition 1, by Morse theory and Lemma 1, we can find a change of coordinates near 0 \(\in T^k\) with respect to which \(\lambda_1\) becomes quadratic. That is, if \(B_r(0)\) denotes the ball of radius \(r\) in \((\mathbb{R}^k)^*\), we can find an \(r>0\), a neighborhood \(U\) of 0 \(\in T^k\) and a diffeomorphism \(\phi: B_r(0) \to U\) such that \(\phi(0) = 0\), \(d\phi_0 = \text{Id}\) and \(\lambda_1(\phi(x)) = \langle x, x \rangle_{(\mathbb{R}^k)^*}\). Then there is some \(\alpha > 0\) such that as \(t \to \infty\),

\[
k(t, v) = \int_{B_r(0)} e^{i\phi(x) \cdot v} e^{-t \langle x, x \rangle_{(\mathbb{R}^k)^*}} \det(d\phi_x) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha t}),
\]

uniformly in \(v\). Multiplying by a cutoff function on \((\mathbb{R}^k)^*\), we can write

\[
k(t, v) = \int_{(\mathbb{R}^k)^*} e^{i\phi(x) \cdot v} e^{-t \langle x, x \rangle_{(\mathbb{R}^k)^*}} g(x) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t})
\]

\[
= t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\phi(\frac{x}{\sqrt{t}}) \cdot v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} g \left( \frac{x}{\sqrt{t}} \right) \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t})
\]

for some \(g \in C_0^\infty((\mathbb{R}^k)^*)\) with \(g(0) = 1\) and some \(\alpha' > 0\). (Here \(\phi\) has been extended to become a map \(\phi: (\mathbb{R}^k)^* \to (\mathbb{R}^k)^*\) which is the identity outside of a compact set.)

We have now reduced to a stationary-phase-type integral. Let

\[
g(x) = 1 + (\nabla g)(0) \cdot x + E(x)
\]
be the beginning of the Taylor’s expansion of \( g \). We can write

\[
(2.19) \quad t^{-\frac{1}{2}} \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} g \left( \frac{x}{\sqrt{t}} \right) \frac{d^k x}{(2\pi)^k} = t^{-\frac{1}{2}} \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \left[ 1 + (\nabla g)(0) \cdot \frac{x}{\sqrt{t}} + E \left( \frac{x}{\sqrt{t}} \right) \right] \frac{d^k x}{(2\pi)^k} + t^{-\frac{1}{2}} \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} \left[ e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} - 1 \right] e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} g \left( \frac{x}{\sqrt{t}} \right) \frac{d^k x}{(2\pi)^k}. 
\]

Recall that the measure \( \frac{d^k x}{(2\pi)^k} \) on \((\mathbb{R}^k)^*\) derives from the product measure on \( T^k = (\mathbb{R}^*/2\pi\mathbb{Z}^*)^k \). Let \( \langle \cdot, \cdot \rangle_{prod} \) be the standard product Euclidean metric on \((\mathbb{R}^*)^k\). Let \( Q \) be the self-adjoint operator on \((\mathbb{R}^k)^*\) such that \( \langle x, x \rangle_{(\mathbb{R}^k)^*} = \langle x, Q x \rangle_{prod} \). Then a standard calculation gives

\[
(2.20) \quad t^{-\frac{1}{2}} \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k} = \frac{(\det Q)^{-1/2}}{(4\pi)^{k/2}} \frac{d^k x}{(2\pi)^k} e^{-\langle \mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^k}/(4t)}. 
\]

On the other hand,

\[
(2.21) \quad (\det Q)^{-1/2} = \text{vol}(\mathbb{R}^k/\mathbb{Z}^k). 
\]

By symmetry,

\[
(2.22) \quad t^{-\frac{1}{2}} \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} (\nabla g)(0) \cdot \frac{x}{\sqrt{t}} \frac{d^k x}{(2\pi)^k} = 0.
\]

Let \( c > 0 \) be such that \( |E(x)| \leq c \langle x, x \rangle_{(\mathbb{R}^k)^*} \) for all \( x \in (\mathbb{R}^k)^* \). Then

\[
(2.23) \quad \left| \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} E \left( \frac{x}{\sqrt{t}} \right) \frac{d^k x}{(2\pi)^k} \right| \leq c t \int_{(\mathbb{R}^k)^*} \langle x, x \rangle_{(\mathbb{R}^k)^*} \frac{d^k x}{(2\pi)^k}.
\]

Finally,

\[
(2.24) \quad \left| \int_{(\mathbb{R}^k)^*} e^{i \phi(\mathbf{x})} \mathbf{v} \left[ e^{i \phi(\mathbf{x})} - 1 \right] e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} g \left( \frac{x}{\sqrt{t}} \right) \frac{d^k x}{(2\pi)^k} \right| \leq \| g \|_{\infty} \int_{(\mathbb{R}^k)^*} 2 \left| \sin \left( \frac{1}{2} \left[ \phi \left( \frac{x}{\sqrt{t}} \right) - \frac{x}{\sqrt{t}} \right] \cdot \mathbf{v} \right) \right| e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}.
\]

We can find a constant \( c' > 0 \) such that

\[
(2.25) \quad 2 \left| \sin \left( \frac{1}{2} \left[ \phi(x) - x \right] \cdot \mathbf{v} \right) \right| \leq c' \langle x, x \rangle_{(\mathbb{R}^k)^*} \| \mathbf{v} \|_{\mathbb{R}^k}
\]
for all $x \in (\mathbb{R}^k)^*$ and $v \in \mathbb{Z}^k$. Then

\begin{equation}
\|g\|_\infty \int_{(\mathbb{R}^k)^*} 2 \left| \sin \left( \frac{1}{2} \left[ \phi \left( \frac{x}{\sqrt{t}} \right) - \frac{x}{\sqrt{t}} \right] \cdot v \right) \right| e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k} \leq c' \frac{\|v\|_{\mathbb{R}^k}}{\sqrt{t}} \left( \frac{\|g\|_\infty}{\sqrt{t}} \right) \int_{(\mathbb{R}^k)^*} \langle x, x \rangle_{(\mathbb{R}^k)^*} e^{-\langle x, x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}.
\end{equation}

By assumption,

\begin{equation}
\frac{\|v\|_{\mathbb{R}^k}}{\sqrt{t}} \leq \sqrt{C}.
\end{equation}

The proposition follows from combining equations (2.17)–(2.27).

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