The purpose of this note is to show that unlike for set forcing, an inner model of a class-generic extension need not itself be a class-generic extension. Our counterexample is of the form $L[R]$, where $R$ is a real both generic over $L$ and constructible from $O^\#$.

**Definition** $(M, A)$, $M$ transitive is a **ground model** if $A \subseteq M$, $M \models ZFC + A$ Replacement and $M$ is the smallest model with this property of ordinal height $ORD(M)$. $G$ is **literally generic** over $(M, A)$ if for some partial-ordering $P$ definable over $(M, A)$, $G$ is $P$-generic over $(M, A)$ and $(M[H], A, H) \models ZFC + (A, H)$-Replacement for all $P$-generic $H$. $S$ is **generic** over $M$ if for some $A$, $S$ is definable over $(M[G], A, G)$ for some $G$ which is literally generic over $(M, A)$, and $S$ is **strictly generic** over $M$ if we also require that $G$ is definable over $(M[S], A, S)$.

The following is a classic application of Boolean-valued forcing and can be found in Jech [?], page ?.

**Proposition 1.** If $G$ is $P$-generic over $(M, A)$ where $P$ is an element of $M$, $S$ definable over $(M[G], A)$ then $S$ is strictly generic over $M$.

**Proof Sketch.** We can assume that $P$ is a complete Boolean algebra in $M$ and that $S \subseteq \alpha$ for some ordinal $\alpha \in M$. Then $H = G \cap P_0$ is $P_0$-generic over $M$, where $P_0$ is complete subalgebra of $P$ generated by the Boolean values of the sentences “$\hat{\beta} \in \sigma$”, where $\beta < \alpha$ and $\sigma$ is a $P$-name for $S$. Then $H$ witnesses the strict genericity of $S$. ⊥

Now we specialize to the ground model $(L, \phi)$, under the assumption that $O^\#$ exists.

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Theorem 2. There is a real $R \in L[O^\#]$ which is generic but not strictly generic over $L$.

Our strategy for proving Theorem 2 comes from the following observation.

Proposition 3. If $R$ is a real strictly generic over $L$ then for some $L$-amenable $A$, $\text{Sat}(L[R])$ is definable from $R, A$, where $\text{Sat}$ denotes the Satisfaction relation.

Proof. Suppose that $A, G$ witness that $R$ is strictly generic over $L$. Let $G$ be $P$-generic over $\langle L, A \rangle$, $P$ definable over $\langle L, A \rangle$, $R \in L[G]$, $G$ definable over $\langle L[R], A \rangle$. Also assume that $\langle L[H], A, H \rangle \models ZFC + (A, H)$-replacement for all $P$-generic $H$. The latter implies that the Truth and Definability Lemmas hold for $P$-forcing, by a result of M. Stanley (See Stanley [?] or Friedman[??]). Then we have: $L[R] \models \varphi$ iff $\exists p \in G(p \models \varphi$ holds in $L[\sigma])$ where $\sigma$ is a $P$-name for $R$ and therefore $\text{Sat}(L[R])$ is definable from $R, \text{Sat}(L, A)$. As $A$ is $L$-amenable and $O^\#$ exists, $\text{Sat}(L, A)$ is also $L$-amenable. $\dashv$

Remarks (a) $\text{Sat}(L[R])$ could be replaced by $\text{Sat}(\langle L[R], A \rangle)$ in Proposition 3, however we have no need here for this stronger conclusion. (b) A real violating the conclusion of Proposition 3 was constructed in Friedman [??], however the real constructed there was not generic over $L$.

Thus to prove Theorem 2 it will suffice to find a generic $R \in L[O^\#]$ such that for each $L$-amenable $A$, $\text{Sat}(L[R])$ is not definable (with parameters) over $\langle L[R], A \rangle$. First we do this not with a real $R$ but with a generic class $S$, and afterwards indicate how to obtain $R$ by coding $S$.

We produce $S$ using the Reverse Easton iteration $P = \langle P_\alpha | \alpha \leq \omega \rangle$, defined as follows. $P_0 =$trivial forcing and for limit $\lambda \leq \omega$, Easton support is used to define $P_\lambda$ (as a direct limit for $\lambda$ regular, inverse limit otherwise). For singular $\alpha$, $P_{\alpha+1} = P_\alpha \cdot Q(\alpha)$ where $Q(\alpha)$ is the trivial forcing and finally for regular $\alpha$, $P_{\alpha+1} = P_\alpha \ast Q(\alpha)$ where $Q(\alpha)$ is defined as follows: let $\langle b_\gamma | \gamma < \alpha \rangle$ be the $L$-least partition of the odd ordinals $< \alpha$ into $\alpha$-many disjoint pieces of size $\alpha$ and we take a condition in $Q(\alpha)$
to be $p = \langle p(0), p(1), \ldots \rangle$ where for some $\alpha(p) < \alpha$, $p(n) : \alpha(p) \rightarrow 2$ for each $n$.

Extension is defined by: $p \leq q$ iff $\alpha(p) \geq \alpha(q)$, $p(n)$ extends $q(n)$ for each $n$ and $q(n+1)(\gamma) = 1$, $\delta \in b_\gamma \cap [\alpha(q), \alpha(p)) \rightarrow p(n)(\delta) = 0$. Thus if $G$ is $Q(\alpha)$-generic and $S_n = \bigcup\{p(n)|p \in G\}$ then $S_{n+1}(\gamma) = 1$ iff $S_n(\delta) = 0$ for sufficiently large $\delta \in b_\gamma$.

Now we build a special $P$-generic $G(\leq \infty)$, definably over $L[O^\#]$. The desired generic but not strictly generic class is $S_0 = \bigcup\{p(0)|p \in G(\infty)\}$. We define $G(\leq i_\alpha)$ by induction on $A \in \text{ORD}$, where $\langle i_\alpha|\alpha \in \text{ORD} \rangle$ is the increasing enumeration of $I \cup \{0\}$, $I = \text{Silver Indiscernibles}$. $G(\leq i_0)$ is trivial and for limit $\lambda \leq \infty$, $G(< i_\lambda) = \bigcup\{G(< i_\alpha)|\alpha < \lambda\}$, $G(i_\lambda) = \bigcup\{G(i_2\alpha)|\alpha < \lambda\}$ (where $i_\infty = \infty$).

Suppose that $G(\leq i_\lambda)$ is defined, $\lambda$ limit or 0, and we wish to define $G(\leq i_{\lambda+n})$ for $0 < n < \omega$. If $n$ is even and $G(\leq i_{\lambda+n})$ has been defined then we define $G(\leq i_{\lambda+n+1})$ as follows: $G(< i_{\lambda+n+1})$ is the $L[O^\#]$-least generic extending $G(\leq i_{\lambda+n})$. To define $G(i_{\lambda+n+1})$ first form the condition $p \in Q(i_{\lambda+n+1})$ defined by: $\alpha(p) = i_{\lambda+n} + 1$, $p(m) \upharpoonright i_{\lambda+n} = G(i_{\lambda+n})(m) = \bigcup\{q(m)|q \in G(i_{\lambda+n})\}$ for all $m$, $p(m)(i_{\lambda+n}) = 1$ iff $m > n$. Then $G(i_{\lambda+n+1})$ is the $L[O^\#]$-least $Q(i_{\lambda+n+1})$-generic (over $L[G(< i_{\lambda+n+1})]$) containing the condition $p$. If $n$ is odd and $G(\leq i_{\lambda+n})$ has been defined then we define $G(\leq i_{\lambda+n+1})$ as follows: $G(< i_{\lambda+n+1})$ is the $L[O^\#]$-least generic extending $G(\leq i_{\lambda+n})$. To define $G(i_{\lambda+n+1})$, first form the condition $p \in Q(i_{\lambda+n+1})$ by: $\alpha(p) = i_{\lambda+n}$, $p(m)(\gamma) = G(i_{\lambda+n})(m)(\gamma)$ for $\gamma \neq i_{\lambda+n-1}$ and $p(m)(i_{\lambda+n-1}) = 0$ for all $m$. Then $G(i_{\lambda+n+1})$ is the $L[O^\#]$-least $Q(i_{\lambda+n+1})$-generic (over $L[G(< i_{\lambda+n+1})]$) containing the condition $p$. This completes the definition of $G(\leq \infty)$.

Now for each $i \in I \cup \{\infty\}$ and $n \in \omega$ let $S_n(i) = \bigcup\{p(n)|p \in G(i)\}$ and $S(i) = S_0(i)$, $S = S(\infty)$. We now proceed to show that $S$ is not strictly-generic over $L$.

**Definition.** For $X \subseteq \text{ORD}$, $\alpha \in \text{ORD}$ and $n \in \omega$ we say that $\alpha$ is $X - \Sigma_n$ stable if $\langle L_\alpha[X], X \cap \alpha \rangle$ is $\Sigma_n$-elementary in $\langle L[X], X \rangle$. $\alpha$ is $X$-stable if $\alpha$ is $X - \Sigma_n$ stable for all $n$.

**Lemma 4.** For $\lambda$ limit or 0, $n$ even, $I_{\lambda+n+1}$ is not $S$-stable.
Proof. Let \( i = i_{\lambda+n} \) and \( j = i_{\lambda+n+1} \). Note that \( S_m(j) \) is defined from \( S(j) \) just as \( S_m(\infty) \) is defined from \( S(\infty) = S \). But \( S(j) = S \cap j \) and for \( M > n \), \( S_m(j) \neq S_m(\infty) \) since \( i \in S_m(j), i \notin S_m(\infty) \). So \( j \) is not \( S \)-stable. \( \dashv \)

Lemma 5. For \( L \)-amenable \( A \subseteq \text{ORD} \), \( i_{\lambda+n+1} \) is \( (S,A) - \Sigma_n \) stable for sufficiently large limit \( \lambda \), all \( n \in \omega \).

Proof. Let \( i = i_{\lambda+n+1} \) where \( \lambda \) is large enough to guarantee that \( i \) is \( A \)-stable. For \( p \in P_{i+1} = P_i \ast Q(i) \) and \( m \in \omega \), we let \( (p)_m \) be obtained from \( p \) by redefining \( p(i)(\bar{m}) = \phi \) for \( \bar{m} > m \) and otherwise leaving \( p \) unchanged.

Claim. Suppose \( \varphi \) is \( \Pi_m \) relative to \( S(i), B \) where \( B \subseteq i, B \in L \). If \( p \in P_{i+1}, p \models \varphi \) then \( (p)_m \models \varphi \).

Proof of Claim. By induction on \( m \geq 1 \). For \( m = 1 \), if the conclusion failed then we could choose \( q \leq (p)_1, q(<i) \models \sim \varphi \) holds of \( q(0), B \); then clearly \( (q)_0 \models \sim \varphi \), \( (q)_0 \) is compatible with \( p \), which contradicts the hypothesis that \( p \models \varphi \). Given the result for \( m \), if the conclusion failed for \( m+1 \) then we could choose \( q \leq (p)_{m+1}, q \models \sim \varphi \). Now write \( \sim \varphi \) as \( \exists x \psi, \psi \Pi_m \) and we see that by induction we may assume that \( (q)_m \models \psi(\hat{x}) \) for some \( x \). But \( (q)_m, p \) are compatible and \( p \models \sim \exists x \psi \), contradiction. \( \dashv \)

(Proof.)

Now we prove the lemma. Suppose \( \varphi \) is \( \Pi_n \) and true of \( (S(i), A \cap i) \). Choose \( p \in G(\leq i), p \models \varphi \). Then by the Claim, \( (p)_n \models \varphi \). As \( i \) is \( A \)-stable, \( (p)_n \models \varphi \) in \( P(\leq \infty) \). By construction \( (p)_n \) belongs to \( G(\leq \infty) \), in the sense that \( (p)_n (<i) \in G(<i) \subseteq G(<\infty) \) and \( (p)_n(i) \in G(\infty) \). So \( \varphi \) is true of \( (S,A) \). \( \dashv \)

Theorem 6. \( S \) is generic, but not strictly generic, over \( L \).

Proof. By Proposition 3 (which also holds for classes), if \( S \) were strictly generic over \( L \) then for some \( L \)-amenable \( A \) we would have that \( \text{Sat}(L[S], S) \) would be definable over \( (L[S], S, A) \). But then for some \( n \), all sufficiently large \( (S,A) - \Sigma_n \) stables would
be $S$-stable, in contradiction to Lemmas 4, 5. $\dashv$

To prove Theorem 2 we must show that an $S$ as in Theorem 6 can be coded by a real $R$ in such a way as to preserve the properties stated in lemmas 4, 5. We must first refine the above construction:

**Theorem 7.** Let $\langle A(i) \mid i \in I \rangle$ be a sequence such that $A(i)$ is a constructible subset of $i$ for each $i \in I$. Then there exists $S$ obeying Lemmas 4, 5 such that in addition, $A(i)$ is definable over $\langle L_i[S], S \cap i \rangle$ for $i \in \text{Odd}(I) = \{i_{\lambda+n} \mid \lambda \text{ limit or 0, } n \text{ odd} \}$.

**Proof.** We use a slightly different Reverse Easton iteration: $Q(\alpha)$ specifies $n(\alpha) \leq \omega$ and if $n(\alpha) < \omega$, it also specifies a constructible $A(\alpha) \subseteq \alpha$; then conditions and extension are as before, except we now require that if $n(\alpha) < \omega$ then for $p$ to extend $q$, we must have $p(n(\alpha))(2\beta + 2) = 1$ iff $\beta \in A(\alpha)$, for $2\beta + 2 \in [\alpha(q), \alpha(p)]$. Then if $n(\alpha) < \omega$, the $Q(\alpha)$-generic will code $A(\alpha)$ definably (though the complexity of the definition increases with $n(\alpha) < \omega$).

Now in the construction of $G(\leq \iota_\alpha), \alpha \leq \infty$ we proceed as before, with the following additional specifications: $n(i_{\lambda+n}) = n$ for odd $n$ and $n(i_{\lambda+n}) = \omega$ for even $n$ ($\lambda$ limit or 0). And for odd $n$ we specify $A(i_{\lambda+n})$ to be the $A(i), i = i_{\lambda+n}$ as given in the hypothesis of the Theorem.

Lemma 4 holds as before; we need a new argument for Lemma 5. Note that for $i \in \text{Odd}(I)$ it is no longer the case that $P(< i) \Vdash Q(i) = Q(\infty) \cap L_i[G(< i)]$. Let $Q^*(i)$ denote $Q(\infty) \cap L_i[G(< i)]$, i.e., the forcing $Q(i)$ where $n(i)$ has been specified as $\omega$. Define $(p)_m$ as before for $p \in P(\leq i)$.

**Claim.** Suppose $m \leq n+1$, $n$ is even, $i = i_{\lambda+n+1}$ ($\lambda$ limit or 0) and $\varphi$ is $\Pi_m$ relative to $S(i)$, $B$ with parameters, where $B \subseteq i$, $B \in L$. If $p \in P(\leq i)$ (where $n(i) = n + 1$) then $p \Vdash \varphi$ in $P(\leq i)$ iff $(p)_m \Vdash \varphi$ in $P^*(\leq i) = P(< i) * Q^*(i)$ iff $p \Vdash \varphi$ in $P^*(\leq i)$.

**Proof.** As in the proof of the corresponding Claim in the proof of Lemma 5. If $m = 1$ and $p \Vdash \varphi$ in $P(\leq i)$, then if the conclusion failed, we could choose $q \leq (p)_1$
in \( P^*(\leq i) \), \( q \Vdash \sim \varphi \); then (we can assume) \( (q)_0 \Vdash \sim \varphi \) in \( P(\leq i) \), but \( (q)_0 \) and \( p \) are compatible. The other implications are clear, as \( P(\leq i) \subseteq P^*(\leq i) \). Given the result for \( m \leq n, \varphi \ \Pi_{m+1} \) and \( p \Vdash \varphi \) in \( P(\leq i) \), if the conclusion failed we could choose \( q \leq (p)_{m+1} \) in \( P^*(\leq i) \), \( q \Vdash \sim \varphi \) (indeed, \( q \Vdash \sim \psi(x) \) some \( x \), where \( \varphi = \forall x \psi, \psi \Sigma_m \)); then \( q \Vdash \sim \varphi \) in \( P(\leq i) \), \( (q)_m \Vdash \sim \varphi \) in \( P^*(\leq i) \), \( (q)_m \Vdash \sim \varphi \) in \( P(\leq i) \) by induction. But \( (q)_m, p \) are compatible in \( P(\leq i) \), using the fact that \( m \leq n \) and \( q \leq (p)_{m+1} \), contradiction. And again the other implications follow, as \( P(\leq i) \subseteq P^*(\leq i) \). \( \dashv \) (Claim.)

Now the proof of Lemma 5 proceeds as before, using the new version of the Claim.

The choice of \( \langle A(i) | i \in I \rangle \) that we have in mind comes from the next Proposition.

**Proposition 8.** For each \( n \) let \( A_n = \{ \alpha | \) For \( i < j_1 < \ldots < j_n \) in \( I, \alpha < i, (\alpha, j_1 \ldots j_n) \) and \( (i, j_1 \ldots j_n) \) satisfy the same formulas in \( L \) with parameters \( < \alpha \} \). Then any \( L \)-amenaable \( A \) is \( \Delta_1 \)-definable over \( \langle L, A_n \rangle \) for some \( n \).

**Proof.** For each \( i \in I \), \( A \cap L_i \) belongs to \( L \) and hence is of the form \( t(i)(\vec{j}_0(i), i, \vec{x}(n(i))) \) where \( t(i) \) is a \( \Delta_0 \)-Skolem term for \( L \), \( \vec{j}_0(i) \) is a finite sequence of indiscernibles \( < i \) and \( \vec{x}(n(i)) \) is any sequence of indiscernibles \( > i \) of length \( n(i) \in \omega \). By Fodor’s Theorem and indiscernibility we can assume that \( t(i) = t, \vec{j}_0(i) = \vec{j}_0 \) and \( n(i) = n \) are independent of \( i \). To see that \( A \) is \( \Delta_1 \)-definable over \( \langle L, A_{n+1} \rangle \) it suffices to show that for \( \vec{i} < \vec{j} \) increasing sequences from \( A_{n+1} \) of length \( n + 1 \), \( \vec{i} \) and \( \vec{j} \) satisfy the same formulas in \( L \) with parameters \( < \min(\vec{i}) \). But by definition, for \( \alpha < \min(\vec{i}) \) and \( \vec{i} = \{i_0, \ldots, i_n\}, \vec{j} = \{j_0, \ldots, j_n\} \) we get: \( L \models \varphi(\alpha, j_0 \ldots j_n) \iff \varphi(\alpha, i_0, j_1 \ldots j_n) \iff \varphi(\alpha, i_0, i_1, j_2 \ldots j_n) \iff \cdots \iff \varphi(\alpha, i_0, \ldots, i_n) \). \( \dashv \)

Now for \( i \in I \) write \( i = i_{\lambda+i_n}, \lambda \) limit or 0, \( n \in \omega \) and let \( A(i) = A_n \cap i \). Thus by Theorem 7 there is \( S \) obeying Lemmas 4,5 such that \( A_n \cap i \) is definable over \( \langle L_i[S], S \cap i \rangle \) for \( i = i_{\lambda+n+1}, n \) even.
Proof of Theorem 2 First observe that as in Friedman [?], we may build $G(\leq \infty)$ to satisfy Theorem 7 for the preceding choice of $(A(i)|i \in I)$ and in addition preserve the indiscernibility of $\text{Lim} I$. Then by the technique of Beller-Jensen-Welch [82], Theorem 0.2 we may code $(G(< \infty), S)$ by a real $R$, where $S = G_0(\infty)$. The resulting $R$ obeys Lemma 4 because $S$ is definable from $R$; to obtain Lemma 5 for $R$ we must modify the coding of $(G(< \infty), S)$ by $R$ in the following way: for inaccessible $\kappa$ we require that any coding condition with $\kappa$ in its domain reduce any dense $D \subseteq P^{< \kappa} = \{ q| \alpha(q) < \kappa \}$ strictly below $\kappa$, when $D$ is definable over $\langle L_\kappa[G(< \infty), S], G(< \kappa), S \cap \kappa \rangle$. This extra requirement does not interfere with the proofs of extendibility, distributivity for the coding conditions (see Friedman [??]).

Now to obtain Lemma 5 for $R$ argue as follows: Given $L$-amenable $A$, choose $n$ and $\lambda$ large enough so that $A$ is $\Delta_1$-definable from $A_n$ with parameters $< i_\lambda$. Then $i_{\lambda+n+1}$ is $(G(< \infty), S, A) - \Sigma_n$ stable. And also $A \cap i_{\lambda+n+1}$ is definable over $\langle L_i[G(< i), S \cap i], G(< i), S \cap i \rangle$ where $i = i_{\lambda+n+1}$. Thus if $\varphi$ is $\Pi_n$ and true of $G(< i), S \cap i, A \cap i$ then $\varphi$ is forced by some coding condition $p \in P^{< i}$ ($p$ in the generic determined by $R$) and hence by the $(G(< \infty), S, A) - \Sigma_n$ stability of $i$, we get that $\varphi$ is true of $G(< \infty), S, A$.

We built $R$ as in Theorem 2 by perturbing the indiscernibles. However with extra care we can in fact obtain indiscernible preservation.

Theorem 9. There is a real $R \in L[O^\#]$ such that $R$ is generic but not strictly generic over $L$, $L$-cofinalities equal $L[R]$-cofinalities and $I^R = I$.

Proof. Instead of using the $i_{\lambda+n}, n \in \omega$ (\lambda limit or 0) use the $i^n_\alpha, n \in \omega$ where $i^n_\alpha = \text{least element of } A_n \text{ greater than } i_\alpha$. Thus $\bigcup \{ i^n_\alpha| n \in \omega \} = i_{\alpha+1}$ and as above we can construct $S$ to preserve indiscernibles and $L$-cofinalities and satisfy that no $i^n_\alpha, n$ odd is $S$-stable, $i^{n+1}_\alpha$ is $(S, A) - \Sigma_n$ stable for large enough $\alpha, n$ (given any $L$-amenable $A$) and $A_n \cap i^{n+1}_\alpha$ is definable over $\langle L_i[S], S \cap i \rangle$ for $i = i^{n+1}_\alpha, n$ even. Then code $(G(< \infty), S)$ by a real, preserving indiscernibles and cofinalities, requiring as
before that for inaccessible $\kappa$, any coding condition with $\kappa$ in its domain reduces dense $D \subseteq p^{<\kappa}$ strictly below $\kappa$, when $D$ is definable over $\langle L_\kappa[G(<\kappa), S \cap \kappa], G(<\kappa), S \cap \kappa] \rangle$. Then for any $L$-amenable $A$, $i_\alpha^{n+1}$ will be $(R, A) - \Sigma_n$ stable for sufficiently large $\alpha, n$. This implies as before that $R$ is not strictly generic. \(\dashv\)

**Remark 1.** A similar argument shows: For any $n \in \omega$ there is a real $R \in L[O^\#]$ which is strictly generic over $L$, yet $G$ is not $\Sigma_n(L[R], R, A)$ whenever $R \in L[G], G$ literally generic over $(L, A)$. Thus there is a strict hierarchy within strict genericity, given by the level of definability of the literally generic $G$ from the strictly generic real.

**Remark 2.** The nongeneric real $R$ constructed in Friedman [??] is strictly generic over some $L[S]$ where $R \notin L[S]$. The same is true of the real $R$ constructed here to satisfy Theorem 2. This leads to:

**Questions** (a) Is there a real $R \in L[O^\#], R$ not strictly generic over any $L[S], R \notin L[S]$? (b) Suppose $R$ is strictly generic over $L[S], S$ generic over $L$. Then is $R$ generic over $L$?

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