Research article

Weighted boundedness for Toeplitz type operator related to singular integral transform with variable Calderón-Zygmund kernel

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Abstract: In the paper, some weighted maximal inequalities for the Toeplitz operator related to the singular integral transform with variable Calderón-Zygmund kernel are proved. As an application, the boundedness of the operator on weighted Lebesgue space are obtained.

Keywords: Toeplitz operator; variable Calderón-Zygmund kernel; singular integral transform; weighted BMO function; weighted Lipschitz function

Mathematics Subject Classification: 42B20, 42B25

1. Introduction

Suppose $b$ is a locally integrable function on $\mathbb{R}^n$ and $T$ is an integral operator. The principle model of commutator generated by $b$ and $T$ is Calderón commutator $[T, M_b](f) = T(bf) - bT(f)$ (see [5]). The boundedness of commutator characterizes some function spaces (see [2, 10, 21]). In the mid seventies, Coifman, Rochberg and Weiss showed that the commutator is bounded on Lebesgue space. In fact they even proved that this property characterizes $BMO$ functions. As the development of singular integrals (see [7, 21]), the commutator has been well studied. In [5, 19, 20], the authors proved that the commutators of $BMO$ functions and the singular integral are bounded on Lebesgue space. In [3], the author proved a similar result where singular integral is replaced by fractional integral. In [10, 18], the boundedness of the commutator of the Lipschitz function and singular integral on Triebel-Lizorkin and Lebesgue spaces are gained. In [1, 9], the boundedness for the commutator by the weighted $BMO$ and Lipschitz functions and singular integral on Lebesgue spaces are gained (also see [8]). In [2], the authors introduced certain singular integral operator with variable kernel and obtained its boundedness. In [13–15], the boundedness for the commutator by the $BMO$ function and operator is obtained. In [17], the authors proved the boundedness of the multilinear oscillatory singular integral by $BMO$ function and the operator. In [11, 12, 16], certain Toeplitz operator related to the strongly singular integral
is studied.

Motivated by these, in the paper, certain Toeplitz operator of the weighted \(BMO\) and Lipschitz functions with the singular integral transform with variable Calderón-Zygmund kernel are studied.

2. Preliminaries and notations

In the paper, we will study following singular integral transforms (see [2])

**Definition.** Let \(K(x, \cdot)\) be a variable Calderón-Zygmund kernel for a.e. \(x \in \mathbb{R}^n\) as [2] and for a locally integrable function \(b\) on \(\mathbb{R}^n\) and the singular integral transform \(T\) with variable Calderón-Zygmund kernel as

\[
T(f)(x) = \int_{\mathbb{R}^n} \Omega(x, x-y)f(y)dy.
\]

The Toeplitz operator related to \(T\) is defined as

\[
T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},
\]

where \(T^{k,1}\) are the \(\pm I\) (the identity operator) or singular integral transform with variable Calderón-Zygmund kernel, and \(T^{k,2}\) are the linear operators for \(k = 1, \ldots, m\), \(M_b(f) = bf\).

Now, we introduce some notations. In the paper, \(Q\) will denote a cube of \(\mathbb{R}^n\). For a weight function \(\omega\) (i.e. \(\omega\) is a nonnegative locally integrable function), let \(\omega(Q) = \int_Q \omega(x)dx\) and \(\omega_Q = |Q|^{-1} \int_Q \omega(x)dx\).

For a locally integrable function \(b\), the maximal sharp function of \(b\) is defined by

\[
M^#(b)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q|dy.
\]

We know that (see [7])

\[
M^#(b)(x) \approx \sup_{Q\ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |b(y) - c|dy.
\]

Let

\[
M(b)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |b(y)|dy.
\]

For \(\eta > 0\), let \(M^\eta(b)(x) = M^\eta(|b|^\eta)^{1/\eta}(x)\) and \(M_\eta(b)(x) = M(|b|^\eta)^{1/\eta}(x)\).

For \(0 < \eta < n\), \(1 \leq p < \infty\) and weight function \(v\), set

\[
M_{\eta,p,v}(b)(x) = \sup_{Q\ni x} \left( \frac{1}{v(Q)^{1-pq/n}} \int_Q |b(y)|^p v(y)dy \right)^{1/p}
\]

and

\[
M_v(b)(x) = \sup_{Q\ni x} \frac{1}{v(Q)} \int_Q |b(y)|v(y)dy.
\]

The \(A_p\) weight is defined by (see [7])

\[
A_p = \left\{ 0 < v \in L^1_{loc}(\mathbb{R}^n) : \sup_{Q} \left( \frac{1}{|Q|} \int_Q v(x)dx \right) \left( \frac{1}{|Q|} \int_Q v(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \ 1 < p < \infty,
\]
Given a weight function $v$, the weighted Lebesgue space $L^p(R^n, v)$ is the space of functions $b$ such that, for $1 \leq p < \infty$,

$$
\|b\|_{L^p(v)} = \left(\int_{R^n} |b(x)|^p v(x) dx\right)^{1/p} < \infty.
$$

The weighted BMO space $BMO(v)$ is the space of functions $f$ such that

$$
\|f\|_{BMO(v)} = \sup_Q \frac{1}{v(Q)} \int_Q |f(y) - f_Q| dy < \infty,
$$

For $0 < \beta < 1$, the weighted Lipschitz space $Lip\beta(v)$ is the space of functions $f$ such that

$$
\|f\|_{Lip\beta(v)} = \sup_Q \frac{1}{v(Q)\beta/n} \left(\frac{1}{v(Q)} \int_Q |f(y) - f_Q|^\beta v(x)^{1-p} dy\right)^{1/p} < \infty.
$$

Remark. (1) We know that (see [6]), for $f \in Lip\beta(v)$, $v \in A_1$ and $x \in Q$,

$$
|f_Q - f_{2Q}| \leq Ck\|f\|_{Lip\beta(v)} v(x)(2^k Q)^{\beta/n}.
$$

(2) Given $f \in Lip\beta(v)$ and $v \in A_1$. By [5], It is known that spaces $Lip\beta(v)$ coincide and the norms $\|f\|_{Lip\beta(v)}$ are equivalent for different values $1 \leq p < \infty$.

The following preliminary lemma needs.

**Lemma 1.** ([7, p.485]) Suppose $0 < p < q < \infty$ and any positive function $f$. It is defined that, for $1/r = 1/p - 1/q$,

$$
\|f\|_{WL^q} = \sup_{A \ni 0} A\|f(x) > A\|^{1/q}, \quad N_{r,q}(f) = \sup_Q \frac{\|f(x)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^q}},
$$

where the sup is taken for all measurable sets $Q$ with $0 < |Q| < \infty$. Then

$$
\|f\|_{WL^q} \leq N_{r,q}(f) \leq (q/(q-p))^{1/p}\|f\|_{WL^q}.
$$

**Lemma 2.** (see [2]) Suppose $T$ is the singular integral transform as Definition 2. Then $T$ is bounded on $L^p(R^n, v)$ for $v \in A_p$ with $1 < p < \infty$, and weak $(L^1, L^1)$ bounded.

**Lemma 3.** (see [1]) Suppose $b \in BMO(v)$. Then

$$
|b_Q - b_{2Q}| \leq C\|b\|_{BMO(v)} v_Q,
$$

where $v_Q = \max_{1 \leq j \leq |2Q|^{-1}} \int_Q v(x) dx$.

**Lemma 4.** (see [1]) Suppose $v \in A_p$, $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $v^{-r/p} \in A_{r}$ for any $p' \leq r \leq p' + \varepsilon$.

**Lemma 5.** (see [1]) Suppose $v \in BMO(v)$, $v = (\mu v^{-1})^{1/p}$, $\mu, v \in A_p$ and $p > 1$. Then there exists $\varepsilon > 0$ such that for $p' \leq r \leq p' + \varepsilon$,

$$
\int_Q |f(x) - f_Q|^{r/p} \mu(x)^{-r/p} dx \leq C\|f\|_{BMO(v)} \int_Q v(x)^{-r/p} dx.
$$
\textbf{Lemma 6.} (see [1]) Suppose $v \in A_p$, $1 < p < \infty$. Then there exists $0 < \delta < 1$ such that $v^{1-\delta}/p \in A_p/\mu(d\mu)$ for any $p' < r < p'(1 + \delta)$, where $d\mu = v^{r/p}dx$.

\textbf{Lemma 7.} (see [1]) Suppose $\mu, v \in A_p$, $v = (\mu v^{-1})^{1/p}$, $1 < p < \infty$. Then there exists $1 < q < p$ such that

$$
\omega_\nu(v_\mu)^{1/q} \left( \frac{1}{|Q|} \int_Q v(x)^{-q'\nu} v(x)^{-q'/q} dx \right)^{1/q} \leq C.
$$

\textbf{Lemma 8.} (see [5, 6]) Suppose $0 \leq \eta < n$, $1 \leq s < n/\eta$, $1/q = 1/p - \eta/n$ and $v \in A_1$. Then

$$
\|M_{\eta, s, v}(f)\|_{L^\infty(v)} \leq C\|f\|_{L^\infty(v)}.
$$

\textbf{Lemma 9.} (see [7]). Suppose $0 < p, \eta < \infty$ and $v \in \cup_{1 \leq \eta < \infty} A_p$. Then, for any smooth function $f$,

$$
\int_{R^n} M_\eta(f)(x)^p v(x) dx \leq C \int_{R^n} M_\eta^p(f)(x)^p v(x) dx.
$$

3. Theorems and Proofs

We can prove the following theorems.

\textbf{Theorem 1.} Suppose $T$ is the singular integral transform as Definition 2, $1 < p < \infty$, $\mu, v \in A_p$, $v = (\mu v^{-1})^{1/p}$, $0 < \eta < 1$ and $b \in BMO(v)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\bar{x} \in R^n$,

$$
M_\eta^p(T_b(f))(\bar{x}) \leq C\|b\|_{BMO(v)} \sum_{k=1}^m \left( [M_{\eta,p/\nu}(vT_{k,2}^{\nu,2}(f)^{1/\nu})(\bar{x})]^{1/\nu'} + [M_{\nu}(vT_{k,2}^{\nu,2}(f)^{\nu})(\bar{x})]^{1/q} \right).
$$

\textbf{Theorem 2.} Suppose $T$ is the singular integral transform as Definition 2, $v \in A_1$, $0 < \eta < 1$, $1 < s < \infty$, $0 < \beta < 1$ and $b \in Lip_\beta(v)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\bar{x} \in R^n$,

$$
M_\eta^p(T_b(f))(\bar{x}) \leq C\|b\|_{Lip_\beta(v)} v(\bar{x}) \sum_{k=1}^m M_{\eta, s, v}(T_{k,2}^{\nu,2}(f))(\bar{x}).
$$

\textbf{Theorem 3.} Suppose $T$ is the singular integral transform as Definition 2, $1 < p < \infty$, $\mu, v \in A_p$, $v = (\mu v^{-1})^{1/p}$ and $b \in BMO(v)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T_{k,2}$ are the bounded operators on $L^p(R^n, v)$ for $1 < p < \infty$ and $v \in A_p(1 \leq k \leq m)$, then $T_b$ is bounded from $L^p(R^n, v)$ to $L^p(R^n, v)$.

\textbf{Theorem 4.} Suppose $T$ is the singular integral transform as Definition 2, $v \in A_1$, $0 < \beta < 1$, $b \in Lip_\beta(v)$, $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T_{k,2}$ are the bounded linear operators on $L^p(R^n, v)$ for $1 < p < \infty$ and $v \in A_1(1 \leq k \leq m)$, then $T_b$ is bounded from $L^p(R^n, v)$ to $L^q(R^n, v)$.
Proof of Theorem 1. It is only to prove the following inequality holds, for $f \in C^0_0(R^n)$ and some constant $C_0$:

\[
\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0|^q \, dx \right)^{1/q} \leq C|b|_{BM0(v)} \sum_{k=1}^m \left( [M_{\nu^{-r/p}}(|vT^{k,2}(f)|^r)(\bar{x})]^{1/r'} + [M_{\nu}(|vT^{k,2}(f)|^q)(\bar{x})]^{1/q} \right).
\]

We assume $T^{k,1}$ are $T(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\bar{x} \in Q$. We write, by $T_1(g) = 0,$

\[
T_b(f)(x) = T_{b-b_0}(f)(x) = T_{(b-b_0)\chi_{2Q}}(f)(x) + T_{(b-b_0)\chi_{2Q}f}(f)(x) = f_1(x) + f_2(x).
\]

Then

\[
\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - f_2(x_0)|^q \, dx \right)^{1/q} \leq C \left( \frac{1}{|Q|} \int_Q |f_1(x)|^q \, dx \right)^{1/q} + C \left( \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)|^q \, dx \right)^{1/q} = I_1 + I_2.
\]

For $I_1$, we know $\nu^{-r/p} \in A$, by Lemma 4, we get

\[
\left( \frac{1}{|Q|} \int_Q \nu(x)^{-r/p} \, dx \right)^{1/r} \leq C \left( \frac{1}{|Q|} \int_Q \nu(x)^{-r/p} \, dx \right)^{1/r'},
\]

then, by Lemmas 1, 2 and 5, we obtain

\[
\left( \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_0)\chi_{2Q}} T^{k,2}(f)(x)|^q \, dx \right)^{1/q} = \ \frac{|Q|^{1/r'-1} \|T^{k,1} M_{(b-b_0)\chi_{2Q}} T^{k,2}(f)\chi_Q\|_{L^q}}{|Q|^{1/q}} \leq \frac{C \|T^{k,1} M_{(b-b_0)\chi_{2Q}} T^{k,2}(f)\|_{WL^1}}{\|\chi_Q\|_{L^q}} \leq \frac{C \int_{\mathbb{R}^n} |M_{(b-b_0)\chi_{2Q}} T^{k,2}(f)(x)| \, dx}{|Q|} = \frac{C \int_{\mathbb{R}^n} |b(x) - b_{2Q}| \nu(x)^{-r/p} |T^{k,2}(f)(x)| \, dx}{|Q|} \leq \frac{C \left( \frac{1}{|Q|} \int_{2Q} |b(x) - b_{2Q}| \nu(x)^{-r/p} \, dx \right)^{1/r} \left( \frac{1}{|Q|} \int_{2Q} |T^{k,2}(f)(x)| \nu(x) \nu(x)^{1/p} \, dx \right)^{1/1'} \leq \frac{C \|b\|_{BM0(v)} \left( \frac{1}{|Q|} \int_{2Q} \nu(x)^{-r/p} \, dx \right)^{1/r} \left( \frac{1}{|Q|} \int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{1/r'} \, dx \right)^{1/1'} \leq \frac{C \|b\|_{BM0(v)} \left( \frac{1}{|Q|} \int_{2Q} \nu(x)^{-r/p} \, dx \right)^{1/r} \left( \frac{1}{|Q|} \int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{1/r'} \, dx \right)^{1/1'} \leq \frac{C \|b\|_{BM0(v)} \left( \frac{1}{\nu(2Q)^{r/p}} \int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{1/r'} \, dx \right)^{1/1'}}{\int_{2Q} \nu(x)^{1/r'}} \leq \frac{C \|b\|_{BM0(v)} \left( \frac{1}{\nu(2Q)^{r/p}} \int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{1/r'} \, dx \right)^{1/1'}}{\int_{2Q} \nu(x)^{1/r'}}.
\]
Thus
\[ I_1 \leq C \sum_{k=1}^{m} \left( \frac{1}{|Q|} \int_{Q} |T^{k,1} M_{b-b_0} T^{k,2} (f)(x)|^q dx \right)^{1/q} \]
\[ \leq C \|b\|_{BMO(v)} \sum_{k=1}^{m} \left[ M_{v'} (|vT^{k,2} (f)|') (\tilde{x}) \right]^{1/r}. \]

For \( I_2 \), by [2], we know that
\[ T(f)(x) = \sum_{k=1}^{\infty} \sum_{l=1}^{\mathbb{N}} a_{kl}(x) \int_{R^n} \frac{Y_{kl}(x-y)}{|x-y|^n} f(y) dy, \]
and for \(|x-y| > 2|x_0 - x| > 0\),
\[ \left| \frac{Y_{kl}(x-y)}{|x-y|^n} - \frac{Y_{kl}(x_0-y)}{|x_0-y|^n} \right| \leq Ck^{n/2}|x-x_0|/|x_0-y|^{n+1} \]
Thus, by the same argument of proof in [4], for \( x \in Q \), we get
\[ |T^{k,1} M_{b-b_0} T^{k,2} (f)(x) - T^{k,1} M_{b-b_0} T^{k,2} (f)(x_0)| \]
\[ \leq C \|b\|_{BMO(v)} \sum_{j=1}^{\infty} 2^{-j} \left( \frac{1}{v(2^{j+1} Q)^{r'/p}} \int_{2^{j+1} Q} |T^{k,2} (f) v(y)^r' \nu(y)^{r'/p} dy \right)^{1/r'} \]
\[ + C \|b\|_{BMO(v)} |M_{v'} (|vT^{k,2} (f)|^q) (\tilde{x})|^{1/q} \sum_{j=1}^{\infty} j 2^{-j} \]
\[ \times v_{2/Q}(v_{2/Q})^{1/q} \left( \frac{1}{|2^{j+2} Q|} \int_{2^{j+2} Q} v(y)^{-q'} v(y)^{-q'/q} dy \right)^{1/q'} \]
\[ \leq C \|b\|_{BMO(v)} |M_{v'} (|vT^{k,2} (f)|') (\tilde{x})|^{1/r'} \]
\[ + C \|b\|_{BMO(v)} |M_{v'} (|vT^{k,2} (f)|^q) (\tilde{x})|^{1/q}. \]

Thus
\[ I_2 \leq \frac{1}{|Q|} \int_{Q} \sum_{k=1}^{m} \left| T^{k,1} M_{b-b_0} T^{k,2} (f)(x) - T^{k,1} M_{b-b_0} T^{k,2} (f)(x_0) \right| dx \]
\[ \leq C \|b\|_{BMO(v)} \sum_{k=1}^{m} \left[ M_{v'} (|vT^{k,2} (f)|') (\tilde{x}) \right]^{1/r'} + \left[ M_{v'} (|vT^{k,2} (f)|^q) (\tilde{x}) \right]^{1/q}. \]

Theorem 1 is proved.
Proof of Theorem 2. It only to prove the following inequality holds, for $f \in C_0^\infty (\mathbb{R}^n)$ and some constant $C_0$:

$$\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - C_0|^q \, dx \right)^{1/q} \leq C\|b\|_{Lip(p)}M_\beta (\beta) \sum_{k=1}^m M_{\beta, k} (T^{k, 2}(f))(\bar{x}).$$

We assume $T^{k, 1}$ are $T(k = 1, \ldots, m)$ and similar to Theorem 1, for a cube $Q = Q(x_0, d)$ and $\bar{x} \in Q$, we get

$$\left( \frac{1}{|Q|} \int_Q |T_b(f)(x) - f(x)|^p \, dx \right)^{1/p} \leq C\left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p} + C\left( \frac{1}{|Q|} \int_Q |f(x) - f(x_0)|^p \, dx \right)^{1/p} = I_3 + I_4.$$

For $I_3$, we have

$$I_3 \leq \frac{C}{|Q|} \int_{2Q} |b(x) - b_{2Q}|^v(x)^{1-\beta} |T^{k, 2}(f)(x)|^v(x)^{\beta} \, dx$$

$$\leq \frac{C}{|Q|} \left( \int_{2Q} |b(x) - b_{2Q}|^v(x)^{1-\beta} \, dx \right)^{1/\beta} \left( \int_{2Q} |T^{k, 2}(f)(x)|^v(x) \, dx \right)^{1/\beta}$$

$$\leq \frac{C}{|Q|} \|b\|_{Lip(v)}^v (2Q)^{1/\beta} |v(2Q)|^{1-\beta} M_{\beta, v}(T^{k, 2}(f))(\bar{x})$$

$$\leq C\|b\|_{Lip(v)}^v \frac{|2Q|}{|Q|} M_{\beta, v}(T^{k, 2}(f))(\bar{x})$$

$$\leq C\|b\|_{Lip(v)}^v (\bar{x}) M_{\beta, v}(T^{k, 2}(f))(\bar{x}),$$

thus

$$I_3 \leq C \sum_{k=1}^m \left( \frac{1}{|Q|} \int_Q |T^{k, 1}M_{\beta - bQ}^v | T^{k, 2}(f)(x)|^p \, dx \right)^{1/p}$$

$$\leq C\|b\|_{Lip(v)}^v (\bar{x}) \sum_{k=1}^m M_{\beta, v}(T^{k, 2}(f))(\bar{x}).$$

For $I_4$, by using the same argument as in the proof of $I_2$, we have, for $x \in Q$,

$$|T^{k, 1}M_{\beta - bQ}^v | T^{k, 2}(f)(x) - T^{k, 1}M_{\beta - bQ}^v | T^{k, 2}(f)(x_0)|$$

$$\leq C \sum_{j=1}^{\infty} \frac{u_{n-2}}{d} \sum_{j=1}^{\infty} \int_{2|y| < 2^{j+1}|y|} \left| b(y) - b_{2^{j+1}} \right| \frac{|x - x_0|}{|x_0 - y|^{n+1}} |T^{k, 2}(f)(y)| \, dy$$

$$\leq C \sum_{j=1}^{\infty} \frac{d}{(2^{j+1}d)^{n+1}} \int_{2^{j+1}Q} \left| b(y) - b_{2^{j+1}Q} + b_{2^{j+1}Q} - b_{2Q} \right| v(y)^{1-\beta} |T^{k, 2}(f)(y)| v(y)^{\beta} \, dy$$

$$\leq C \sum_{j=1}^{\infty} \frac{d}{(2^{j+1}d)^{n+1}} \left( \int_{2^{j+1}Q} \left| b(y) - b_{2^{j+1}Q} \right|^v v(y)^{1-\beta} \, dy \right)^{1/\beta} \left( \int_{2^{j+1}Q} |T^{k, 2}(f)(y)|^v v(y) \, dy \right)^{1/\beta}.$$
Theorem 2 is proved.

Theorem 3 is proved.

Proof of Theorem 3. It is noticed $\nu' / p \in A_{p' + 1} \subset A_p$ and $\nu(x) dx \in A_p$. We have, by Theorem 1 and Lemma 9,

$$
\int |T_b(f)(x)|^p \nu(x) dx \leq \int |M_q(T_b(f))(x)|^p \nu(x) dx \leq C \int |M_q(T_b(f))(x)|^p \nu(x) dx
$$

Thus

$$
I_4 \leq C \sum_{j=1}^{\infty} \left( \frac{1}{|Q|} \int_Q |T^{k,1}M_{(b-b_0)\chi_{(2j+1)\nu}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_0)\chi_{2j}}T^{k,2}(f)(x_0)| dx \right)
$$

Theorem 2 is proved.

Proof of Theorem 3. It is noticed $\nu' / p \in A_{p' + 1} \subset A_p$ and $\nu(x) dx \in A_p$. We have, by Theorem 1 and Lemma 9,

$$
\int |T_b(f)(x)|^p \nu(x) dx \leq \int |M_q(T_b(f))(x)|^p \nu(x) dx \leq C \int |M_q(T_b(f))(x)|^p \nu(x) dx
$$

Thus

$$
I_4 \leq C \sum_{j=1}^{\infty} \left( \frac{1}{|Q|} \int_Q |T^{k,1}M_{(b-b_0)\chi_{(2j+1)\nu}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_0)\chi_{2j}}T^{k,2}(f)(x_0)| dx \right)
$$

Theorem 3 is proved.
Proof of Theorem 4. In Theorem 2 we choose $1 < s < p$ and by $v^{1-q} \in A_{\infty}$, we get, by Lemmas 8 and 9,

$$
|\|T_b(f)||_{L^q(v^{1-q})} \leq |\|M_q(T_b(f))||_{L^q(v^{1-q})} \leq C |\|M_q^\#(T_b(f))||_{L^q(v^{1-q})}
$$

$$
\leq C |b|_{lip(v)} \sum_{k=1}^{m} |\|vM_{\beta,s,v}(T^k(f))||_{L^q(v^{1-q})}
$$

$$
= C |b|_{lip(v)} \sum_{k=1}^{m} |\|M_{\beta,s,v}(T^k(f))||_{L^q(v)}
$$

$$
\leq C |b|_{lip(v)} \sum_{k=1}^{m} |\|T^k(f)||_{L^p(v)}
$$

$$
\leq C |b|_{lip(v)} |f|_{L^p(v)}.
$$

Theorem 4 is proved.

4. Conclusions

Some new weighted maximal inequalities for the Toeplitz operator related to the singular integral transform with variable Calderón-Zygmund kernel are proved. As an application, the boundedness of the operator on weighted Lebesgue space are obtained.

Acknowledgments

The author are very grateful to the anonymous referees for their constructive suggestions. This research was supported by the Scientific Research Funds of Hunan Provincial Education Department (No :19B509).

Conflict of interest

The author declare that he has no conflict of interest.

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