2D mode equations with the resonant mode interaction

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Abstract

A system of mode equations for resonant interacted modes conserving the acoustic energy flux was obtained by the multiscale expansions method. The testing calculations with COUPLE program give a good agreement.

1 Introduction

The normal mode method is often used in problems of acoustics, electromagnetism, and generally in all wave problems where it occurs waveguide propagation. It consists in the (local) separation of variables of the original boundary value problem in such way that on cross waveguide direction we have the spectral problem, giving the normal modes and along waveguide direction we have the initial boundary value problem, determining the amplitudes of the normal modes. This initial boundary value problem restricted to a finite number of modes as a rule, is well-posed, whereas the original one is not. In the adiabatic mode approximation the independent propagation for each mode is assumed. For the coupled mode propagation the derivation of the amplitude equations is not so obvious, that is produced here for the acoustic case.

Adiabatic and coupled mode acoustic equations appeared as a convenient tool for solving problems of ocean acoustics since the works of A. D. Pierce [1], R. Burridge & H. Weinberg [2], J. A. Fawcett [3], and M. B. Porter [4, 5].

In all these papers except [2] a method of multiscale expansions was not used [6 7]. In this work we show that the systematic use of this method allows to obtain unidirectional equations that give the same results as the COUPLE 2 way equations [8]. For the obtained equations a conservation of the acoustic energy flux is proved.

The application of the corresponding computer code to the ASA wedge benchmark problem gives good results.
2 Formulation of problem

We consider the propagation of time-harmonic sound in the axially symmetric three-dimensional waveguide \( \Omega = \{(r, \phi, z)|0 \leq r < \infty, 0 \leq \phi < 2\pi, -H \leq z \leq 0\} \) (z-axis is directed upward), described by the acoustic Helmholtz equation

\[
(\gamma P_r)_r + \frac{1}{r} \gamma P_r + (\gamma P_z)_z + \gamma \kappa^2 P = \frac{-\gamma \delta(z - z_0) \delta(r)}{2\pi r},
\]

where \( \gamma = 1/\rho \), \( \rho = \rho(r, z) \) is the density, \( \kappa(r, z) \) is the wavenumber. We assume the appropriate radiation conditions at infinity in \( r, \phi \) plane, the pressure-release boundary condition at \( z = 0 \)

\[ P = 0 \quad \text{at} \quad z = 0, \tag{2} \]

and rigid boundary condition \( \partial u/\partial z = 0 \) at \( z = -H \). The parameters of medium may be discontinuous at the nonintersecting smooth interfaces \( z = h_1(r), \ldots, h_m(r) \), where the usual interface conditions

\[
\gamma_+ \left( \frac{\partial P}{\partial z} - h_r \frac{\partial P}{\partial r} \right)_+ = \gamma_- \left( \frac{\partial P}{\partial z} - h_r \frac{\partial P}{\partial r} \right)_-,
\]

are imposed. Hereafter we use the denotations \( f(z_0, r)_+ = \lim_{z \uparrow z_0} f(z, r) \) and \( f(z_0, r)_- = \lim_{z \downarrow z_0} f(z, r) \). Without loss of generality we may consider the case \( m = 1 \), so we set \( m = 1 \) and denote \( h_1 \) by \( h \).

We introduce a small parameter \( \epsilon \) (the ratio of the typical wavelength to the typical size of medium inhomogeneities), the slow variable \( R = \epsilon r \) and postulate the following expansions for the parameters \( \kappa^2, \gamma \) and \( h \):

\[
\kappa^2 = \nu_0^2(R, z) + \epsilon \nu(R, z),
\]

\[
\gamma = \gamma(R, z),
\]

\[
h = h(R). \tag{4}
\]

To model the attenuation effects we admit \( \nu \) to be complex. Namely, we take \( \text{Im} \, \nu = 2\eta \beta n_0 \), where \( \eta = (40\pi \log_{10} \epsilon)^{-1} \) and \( \beta \) is the attenuation in decibels per wavelength. This implies that \( \text{Im} \, \nu \geq 0 \).

Consider a solution to the Helmholtz equation (1) in the form of the WKB-ansatz, where \( \{\theta_j|j = M, \ldots, N\} \) is a set of phases:

\[
P = \sum_{j=M}^{N} (u_0^{(j)}(R, z) + \epsilon u_1^{(j)}(R, z) + \ldots) \exp \left( \frac{i}{\epsilon} \theta_j \right). \tag{5}
\]
Introducing this anzatz into equation (1), boundary condition (2) and interface conditions (3), all rewritten in the slow variable, we obtain the sequence of the boundary value problems at each order of $\varepsilon$.

3 The problem at $O(\varepsilon^0)$

To obtain the normal modes we first consider the WKB anzats in the form $P = u_0^{(1)}(R) e^{\theta_j(R,z)/\varepsilon}$. Further we can omit $j$. From the equations at $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ we can conclude that $\theta$ is independent of $z$.

At $O(\varepsilon^0)$ now we have

$$\left(\gamma u_0\right)_z + \gamma n_0^2 - \gamma (\theta_R)^2 u_0 = 0 ,$$

with the interface conditions of the order $\varepsilon^0$

$$u_0^+ = u_0^-, \quad \left(\gamma \frac{\partial u_0}{\partial z}\right)_+ = \left(\gamma \frac{\partial u_0}{\partial z}\right)_- \quad \text{at} \quad z = h ,$$

and boundary conditions $u = 0$ at $z = 0$ and $\partial u / \partial z = 0$ at $z = -H$. We seek a solution to problem (6), (7) in the form

$$u_0 = A(R) \phi(R, z).$$

From eqs. (6) and (7) we obtain the following spectral problem for $\phi$ with the spectral parameter $k^2 = (\theta_X)^2$

$$\left(\gamma \phi_z\right)_z + \gamma n_0^2 \phi - \gamma k^2 \phi = 0 ,$$

$$\phi(0) = 0 ,$$

$$\left(\gamma \frac{\partial \phi}{\partial z}\right)_+ = \left(\gamma \frac{\partial \phi}{\partial z}\right)_- \quad \text{at} \quad z = h .$$

This spectral problem, considering in the Hilbert space $L_{2, \gamma_0}[-H, 0]$ with the scalar product

$$(\phi, \psi) = \int_{-H}^{0} \gamma \phi \psi \, dz ,$$

has countably many solutions $(k_j^2, \phi_j)$, $j = 1, 2, \ldots$ where the eigenfunction can be chosen as real functions. The eigenvalues $k_j^2$ are real and have $-\infty$ as a single accumulation point [9]. The normalizing condition is

$$(\phi, \phi) = \int_{-H}^{0} \gamma \phi^2 \, dz ,$$
4 The derivatives of eigenfunctions and wavenumbers with respect to \( R \)

Before considering the problem at \( O(\epsilon^1) \) we should consider the problem of calculating the derivatives of eigenfunctions and wavenumbers with respect to \( R \).

Differentiating spectral problem (9) with respect to \( R \) we obtain the boundary value problem for \( \phi_jR \)

\[
(\gamma \phi_jR_z)_z + \gamma n_0^2 \phi_jR - \gamma k_j^2 \phi_jR = - (\gamma R \phi_jz)_z - (\gamma n_0^2)_R \phi_j + 2k_jR k_j \gamma \phi_j + \gamma R k_j^2 \phi_j,
\]

\[
\phi_jR(0) = 0, \quad \phi_jRz(-H) = 0,
\]

with interface conditions at \( z = h \)

\[
\phi_{jR+} - \phi_{jR-} = -h_R(\phi_{jz+} - \phi_{jz-}),
\]

\[
\gamma_+ \phi_{jRz+} - \gamma_- \phi_{jRz-} = - (\gamma_{R+} \phi_{jz+} - \gamma_{R-} \phi_{jz-}) - h_R ((\gamma \phi_{jz})_+ - ((\gamma \phi_{jz})_-).
\]

We search a solution to problem (12), (13) in the form

\[
\phi_jR = \sum_{l=0}^\infty C_{jl} \phi_l,
\]

where

\[
C_{jl} = \int_{-H}^{0} \gamma \phi_jR \phi_l dz.
\]

Multiplying (12) by \( \phi_l \) and then integrating resulting equation from \(-H\) to 0 by parts twice with the use of interface conditions (13), we obtain

\[
(k_l^2 - k_j^2) C_{jl} = \int_{-H}^{0} \gamma R \phi_jz \phi_{lz} dz + 2k_jR k_j \delta_{jl} -
\]

\[
\int_{-H}^{0} (\gamma n_0^2)_R \phi_jz \phi_l dz + k_j^2 \int_{-H}^{0} \gamma R \phi_jz \phi_l dz +
\]

\[
\left\{ h_R \gamma^2 \phi_jz \phi_{lz} \left[ \left( \frac{1}{\gamma} \right)_+ - \left( \frac{1}{\gamma} \right)_- \right] \right.
\]

\[
\left. h_R \phi_j \phi_l \left[ (\gamma (k_j^2 - n_0^2))_+ - (\gamma (k_j^2 - n_0^2))_- \right] \right|_{z=h},
\]

\[
(16)
\]
where $\delta_{jl}$ is the Kronecker delta. The coefficients $C_{jl}$ can be found from this equation when $l \neq j$ and at $l = j$ we have the formula for $k_{jR}$. Differentiating normalizing condition (11) we obtain

$$
\left( \int_{-H}^{0} \gamma \phi_j^2 dz \right)_X = \left( \int_{-H}^{h} \gamma \phi_j^2 dz + \int_{0}^{0} \gamma \phi_j^2 dz \right)_R = \int_{-H}^{0} \gamma R \phi_j^2 dz + 2 \int_{-H}^{0} \gamma \phi_j R \phi_j dz + h_R \phi_j^2 \left[ \gamma_+ - \gamma_- \right]_{z=h} = 0,
$$

which gives the equation for $C_{jj}$:

$$
2C_{jj} = -\int_{-H}^{0} \gamma R \phi_j^2 dz + h_R \phi_j^2 \left[ \gamma_+ - \gamma_- \right]_{z=h}.
$$

5 The problem at $O(\epsilon^1)$

We now seek a solution to the Helmholtz equation (1) in the form of anzats (5):

$$
P = \sum_{j=M}^{N} \left( u_0^{(j)}(R, z) + \epsilon u_1^{(j)}(R, z) + \ldots \right) \exp \left( \frac{i}{\epsilon} \theta_j \right).
$$

At $O(\epsilon^1)$ we obtain

$$
\sum_{j=M}^{N} \left( \gamma u_{1z}^{(j)} + \gamma u_0^2 u_1^{(j)} - \gamma k_j^2 u_1^{(j}) \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) =
$$

$$
\sum_{j=M}^{N} \left( -i\gamma R k_j u_0^{(j)} - 2i\gamma k_j u_0^{(j)} R - i\gamma k_j R u_0^{(j)} - ik_j \frac{1}{R} \gamma u_0^{(j)} - \nu \gamma u_0^{(j)} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right),
$$

with the boundary conditions $u_1^{(j)} = 0$ at $z = 0$, $\partial u_1^{(j)} / \partial z = 0$ at $z = -H$, and the interface conditions at $z = h(R)$:

$$
\sum_{j=M}^{N} \left( u_1^{(j)} - u_1^{(j)} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) = 0,
$$

$$
\sum_{j=M}^{N} \left[ \gamma_+ u_1^{(j)} - \gamma_- u_1^{(j)} + ik_j h_R u_0^{(j)} (\gamma_- - \gamma_+) \right] \exp \left( \frac{i}{\epsilon} \theta_j \right) = 0.
$$

5
We seek a solution to problem (20), (21) in the form
\[ u^{(j)}_1 = \sum_{l=0}^{\infty} B_{jl}(X,Y)\phi_l(z,X), \quad \text{where} \quad B_{jl} = \int_{-H}^{0} \gamma_0 u^{(j)}_1 \phi_l \, dz. \]

Multiplying (20) by \( \phi_l \) and then integrating resulting equation from \(-H\) to 0 by parts twice with the use of interface conditions (21), we obtain
\[ \sum_{j=M}^{N} \left( (k_i^2 - k_j^2)B_{jl} - A_j i k_j h R \phi_j \phi_l \left[ \gamma_+ - \gamma_- \right]_{z=h} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) = \sum_{j=M}^{N} \left( -i k_j A_j \int_{-H}^{0} \gamma_R \phi_j \phi_l \, dz - 2i k_j A_j \int_{-H}^{0} \gamma \phi_j \phi_l \, dz \right. \]
\[ \left. - 2i k_j A_{j,R} \int_{-H}^{0} \gamma \phi_j \phi_l \, dz - i k_j \frac{1}{R} A_j \int_{-H}^{0} \nu \gamma \phi_j \phi_l \, dz \right) \exp \left( \frac{i}{\epsilon} \theta_j \right). \]

The terms \((k_i^2 - k_j^2)B_{jl}\) in these expressions can be omitted because of the resonant condition \(|k_i - k_j| < \epsilon\). As
\[ - i k_j A_j \int_{-H}^{0} \gamma_R \phi_j \phi_l \, dz - 2i k_j A_j \int_{-H}^{0} \gamma \phi_j \phi_l \, dz = i k_j A_j (C_{ij} - C_{jl}) - i k_j A_j h R \phi_j \phi_l \left[ \gamma_+ - \gamma_- \right]_{z=h}, \]
we get, after some algebra,
\[ \sum_{j=M}^{N} \left( i k_j A_j (C_{ij} - C_{jl}) - 2i k_j A_{j,R} \delta_{jl} - i k_j R A_j \delta_{jl} - i k_j \frac{1}{R} A_j \delta_{jl} - A_j \int_{-H}^{0} \nu \gamma \phi_j \phi_l \, dz \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) = 0. \]

**Proposition 1.** The solvability condition for the problem at \(O(\epsilon^1)\) is a system of equations for \(l = M, \ldots, N\)
\[ 2i k_l A_{l,R} + i k_l R A_l + i k_l \frac{1}{R} A_l + \sum_{j=M}^{N} \alpha_{lj} A_j \exp(\theta_{lj}) = 0, \]
where \(\alpha_{lj}\) and \(\theta_{lj}\) is given by the following formulas
\[ \alpha_{lj} = \int_{-\infty}^{0} \gamma \nu \phi_j \phi_l \, dz - i k_j (C_{ij} - C_{jl}), \]
\[ \theta_{lj} = \frac{i}{\epsilon} (\theta_j - \theta_l). \]
6 Energy flux conservation for equations (24)

The acoustic energy flux is defined as

$$J(r, z) = \gamma \text{Im}((\text{grad} P(r, z)) P^*(r, z)).$$

As is well known, if $P$ is a solution of the Helmholtz equation (1) then the corresponding energy flux is conserved, that is

$$\text{div} J(r, z) = 0.$$

With our boundary conditions we have also the conservation property

$$\text{div} \int_{-H}^{0} J(r, z) \, dz = 0.$$

**Proposition 2.** Assume that $\text{Im} \, \bar{\nu} = 0$. Let $\{A_j|j = M, \ldots, N\}$ be a solution to equations (24). Then for $P = \sum_{j=M}^{N} A_j \exp \left( \frac{i}{\epsilon} \theta_j \right) \phi_j$

$$\text{div} \int_{-H}^{0} J(r, z) \, dz = O(\epsilon^2).$$

**Proof.** First calculate the divergence in general form for the anzats used:

$$\text{div} \int_{-H}^{0} J(r, z) \, dz = \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[ \sum_{j=M}^{N} k_l |A_l|^2 ight.ight.$$

$$+ \epsilon \sum_{l=M}^{N} \sum_{j=M}^{N} \text{Im} \left( C_{lj} A_l A^*_j \exp \left( \frac{i}{\epsilon} (\theta_l - \theta_j) \right) \right)$$

$$+ \epsilon \sum_{l=M}^{N} \text{Im}(A_{l,R} A^*_l) \right\}$$

$$= \epsilon \sum_{l=M}^{N} \sum_{j=M}^{N} (k_l - k_j) C_{lj} \text{Re} \left( A_j A^*_l \exp \left( \frac{i}{\epsilon} (\theta_j - \theta_l) \right) \right)$$

$$+ \epsilon \sum_{l=M}^{N} (k_l |A_l|^2)_R + \frac{1}{R} \sum_{l=M}^{N} k_l |A_l|^2 + O(\epsilon^2).$$

Consider now the sum on $l$ of the equations (24) multiplied by $A^*_l$ subtracted
conjugate equation multiplied by $A_l$

$$\sum_{l=M}^{N} \left[ \left( 2ik_lA_{l,R} + ik_l,RA_l + ik_l\frac{1}{R}A_l \right. \right.$$ 
$$+ \sum_{j=M}^{N} \alpha_{ij}A_j \exp(\theta_{lj}) \left) A_l^* - \right.$$ 
$$\left( -2ik_lA_{l,R}^* - ik_l,RA_l^* - ik_l\frac{1}{R}A_l^* \right.$$ 
$$\left. + \sum_{j=M}^{N} \alpha_{ij}^*A_j^* \exp(\theta_{lj}^*) \right) A_l \right] = 0.$$ 

After some transformation we have:

$$\sum_{l=M}^{N} \sum_{j=M}^{N} \left( \alpha_{ij}A_j \exp(\theta_{lj})A_l^* - \alpha_{ij}^*A_j^* \exp(\theta_{lj}^*)A_l \right)$$ 
$$+ \sum_{l=M}^{N} 2i((k_l|A_l|^2)_R + k_l\frac{1}{R}|A_l|^2) = 0,$$

then substitute for $\alpha_{ij}$ its expression \[25\]

$$\sum_{l=M}^{N} \sum_{j=M}^{N} \left( -ik_j(C_{lj} - C_{jl})A_j \exp(\frac{i}{\epsilon}(\theta_j - \theta_l))A_l^* + \right.$$ 
$$-ik_j(C_{lj} - C_{jl})A_j^* \exp(\frac{i}{\epsilon}(\theta_l - \theta_j))A_l \right.$$ 
$$+ \sum_{l=M}^{N} 2i((k_l|A_l|^2)_R + k_l\frac{1}{R}|A_l|^2) = 0,$$

and collect terms

$$\sum_{l=M}^{N} \sum_{j=M}^{N} \left( -ik_j(C_{lj} - C_{jl})2 \text{Re}(A_j \exp(\frac{i}{\epsilon}(\theta_j - \theta_l))A_l^*) \right)$$ 
$$+ \sum_{l=M}^{N} 2i((k_l|A_l|^2)_R + k_l\frac{1}{R}|A_l|^2) = 0,$$
write double sums separately for terms with $C_{lj}$ and $C_{jl}$

\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \left( -i k j C_{lj} 2 \operatorname{Re}(A_j \exp\left(\frac{i}{\epsilon}(\theta_j - \theta_l)\right)) \right)
+ \sum_{l=M}^{N} \sum_{j=M}^{N} \left( i k j C_{jl} 2 \operatorname{Re}(A_j \exp\left(\frac{i}{\epsilon}(\theta_j - \theta_l)\right)) \right)
+ \sum_{l=M}^{N} 2i((k_l|A_l|^2)_R + k_l \frac{1}{R}|A_l|^2) = 0,
\]

exchange indexes $l$ and $j$ in the second double sum and finally get

\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \left( i (k_l - k_j) C_{lj} 2 \operatorname{Re}(A_j \exp\left(\frac{i}{\epsilon}(\theta_j - \theta_l)\right)) \right)
+ \sum_{l=M}^{N} 2i((k_l|A_l|^2)_R + k_l \frac{1}{R}|A_l|^2) = 0.
\]

The last equation coincides modulo 2i with the $O(\epsilon)$-part of (26).

7 Numerical examples

To illustrate the efficiency of our equation we performed the numerical simulation of sound propagation for the standard ASA wedge benchmark with the absorbing penetrative bottom. On the figure 1 the transmission loss for the depth of receiver 30 m is presented. Comparison with the numerical solution obtained by COUPLE program [8] shows that the mean square difference between two curves is about 0.15 dB. Meanwhile adiabatic mode equation solution differs from our solution much more dramatically. The same take place on the figure 2 where transmission loss for the ASA wedge benchmark on the depth 150 m is presented. In this case the mean square difference between our curves and curve of COUPLE 2 way is about 0.4 dB.

8 Conclusion

In this article a system of mode equations for the resonant interacted modes was developed. The acoustic energy flux is conserved for the derived equations with an accuracy adequate to the used approximation. The testing calculations were done for ASA wedge benchmark and proved good agreement with COUPLE program [8].
Figure 1: The result of numerical solution of mode equation \([24]\) for the standard ASA wedge benchmark in comparison with solutions obtained by Couple program and adiabatic mode equation. The depth of receiver is 30 m, absorbing bottom.

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Figure 2: The result of numerical solution of mode equation (24) for the standard ASA wedge benchmark in comparison with solutions obtained by Couple program and adiabatic mode equation. The depth of receiver is 150 m, absorbing bottom.

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