On Induced Colourful Paths in Triangle-free Graphs

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Abstract

Given a graph $G = (V, E)$ whose vertices have been properly coloured, we say that a path in $G$ is colourful if no two vertices in the path have the same colour. It is a corollary of the Gallai-Roy Theorem that every properly coloured graph contains a colourful path on $\chi(G)$ vertices. It is interesting to think of what analogous result one could obtain if one considers induced colourful paths instead of just colourful paths. We explore a conjecture that states that every properly coloured triangle-free graph $G$ contains an induced colourful path on $\chi(G)$ vertices. As proving this conjecture in its fullest generality seems to be difficult, we study a special case of the conjecture. We show that the conjecture is true when the girth of $G$ is equal to $\chi(G)$. Even this special case of the conjecture does not seem to have an easy proof: our method involves a detailed analysis of a special kind of greedy colouring algorithm. This result settles the conjecture for every properly coloured triangle-free graph $G$ with girth at least $\chi(G)$.

Keywords: Induced Path, Colourful Path, Triangle-free Graph

1. Introduction

All graphs considered in this paper are simple, undirected and finite. For a graph $G = (V, E)$, we denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. A function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ is said to be a proper $k$-colouring of $G$ if for any edge $uv \in E(G)$, we have $c(u) \neq c(v)$. A graph is properly coloured, if it has an associated proper $k$-colouring $c$ specified (for some $k$). The minimum integer $k$ for which a graph $G$ has a proper $k$-colouring is the chromatic number of $G$, denoted by $\chi(G)$. A subgraph $H$ of a properly coloured graph $G$ is said to be colourful if no two vertices of $H$ have the same colour. If a colourful subgraph $H$ of $G$ is also an induced subgraph, then we say that $H$ is an induced colourful subgraph of $G$.

It is a corollary of the classic Gallai-Roy Theorem [3] that every (not necessarily optimally) properly coloured graph $G$ has a colourful path on $\chi(G)$ vertices (an alternative proof for this is given in Theorem 4). We are interested in the question of when one can find colourful paths that are also induced in a given properly coloured graph. Note that the colourful path on $\chi(G)$ vertices that should exist in any properly coloured graph $G$ may not always be an induced path. In fact, when $G$ is a complete graph, there is no induced path on more than two vertices in the graph. The following hitherto unpublished conjecture is due to N. R. Aravind.

Conjecture 1 ([2]). Let $G$ be a triangle-free graph that is properly coloured. Then there is an induced colourful path on $\chi(G)$ vertices in $G$.

Surprisingly, despite being known to many researchers for well over two years, the conjecture has remained open even for the special case when $\chi(G) = 4$. Note that Conjecture 1 is readily seen to be true for any triangle-free graph $G$ with $\chi(G) = 3$, because the colourful path guaranteed to exist in $G$ by the Gallai-Roy Theorem is also an induced path in $G$. In this paper, we first prove Conjecture 1 for the case when $\chi(G) = 4$. We then extend this proof to show that the conjecture holds for any triangle-free graph $G$ with $g(G) \geq \chi(G)$, where $g(G)$ is the girth of $G$, or the length of the smallest cycle in $G$.

A necessary condition for Conjecture 1 to hold is the presence of an induced path on $\chi(G)$ vertices in any triangle free graph $G$. Indeed something stronger is known to be true: each vertex in a triangle-free graph $G$ is the starting point of an induced path on $\chi(G)$ vertices [4]. There have been several investigations on variants of the Gallai-Roy Theorem [1, 7]. Every connected graph $G$ other than $C_7$ admits a proper $\chi(G)$-colouring such that every vertex of $G$ is the beginning of a (not necessarily induced) colourful path on $\chi(G) - 1$ vertices [1]. Concerning induced trees, Győrfi conjectured that there exists an integer-valued function $f$ defined on the finite trees with the property that...

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every triangle-free graph $G$ with $\chi(G) = f(T)$ contains $T$ as an induced subgraph. This was proven true for trees of radius two by Gyárfás, Szemerédi, and Tuza [5]. A stronger version of the Gallai-Roy Theorem that guarantees an induced directed path on $\chi(G)$ vertices in any directed graph $G$ would have easily implied Conjecture 1. Clearly, such a theorem cannot be true for every directed graph. But Kierstead and Trotter [6] show that no such result can be obtained even if the underlying undirected graph of $G$ is triangle-free. They show that for every natural number $k$, there exists a digraph $G$ such that its underlying undirected graph is triangle-free and has chromatic number $k$, but $G$ has no induced directed path on 4 vertices.

2. Preliminaries

Notation used in this paper is the standard notation used in graph theory (see e.g. [3]). We shall now describe a special greedy colouring procedure for an already coloured graph that will later help us in proving our main result.

**The refined greedy algorithm.** Given a properly coloured graph $G$ with the colouring $\beta$, we will construct a new proper colouring $\alpha : V(G) \to \mathbb{N}^{\geq 0}$ of $G$, using the algorithm given below. Let $b_1 < b_2 < \cdots < b_t$ be the colours used by $\beta$.

For every vertex $v \in V(G)$, set $\alpha(v) \leftarrow 0$.

for $i$ from 1 to $t$ do

for vertex $v$ with $\beta(v) = b_i$ and $\alpha(v) = 0$ do

Colour $v$ with the least positive integer not present in its neighbourhood, i.e., set $\alpha(v) \leftarrow \min(\mathbb{N}^{\geq 0} \setminus \{\alpha(u) : u \in N(v)\})$.

end for

end for

**Definition 2** (Decreasing path). A path $u_1u_2\ldots u_l$ in $G$ is said to be a “decreasing path” if for $2 \leq i \leq l$, $\alpha(u_i) < \alpha(u_{i-1})$ and $\beta(u_i) < \beta(u_{i-1})$.

**Lemma 3.** Let $v \in V(G)$ and $X \subseteq \{1, 2, \ldots, \alpha(v) - 1\}$. Then there is a decreasing path $vu_1|u_1|u_2|u_2|\ldots |u_l$ in $G$ such that for $1 \leq i \leq |X|$, $\alpha(u_i) \in X$.

This lemma directly shows that there is a colourful path on $\chi(G)$ vertices in every properly coloured graph $G$ (without using the Gallai-Roy Theorem).

**Theorem 4.** If $G$ is any graph whose vertices are properly coloured, then there is a colourful path on $\chi(G)$ vertices in $G$.

**Proof.** Let $\beta$ denote the proper colouring of $G$. Run the refined greedy algorithm on $G$ to generate the colouring $\alpha$. Clearly, the algorithm will use at least $\chi(G)$ colours as the colouring $\alpha$ generated by the algorithm is also a proper colouring of $G$. Let $v$ be any vertex in $G$ with $\alpha(v) = \chi(G)$. Now consider the set $X = \{1, 2, \ldots, \chi(G) - 1\}$. By Lemma 3, there is a path on $\chi(G)$ vertices starting at $v$ on which the colours in the colouring $\beta$ are strictly decreasing. This path is a colourful path on $\chi(G)$ vertices in $G$.

**Corollary 5.** Any properly coloured graph $G$ with $g(G) > \chi(G)$ has an induced colourful path on $\chi(G)$ vertices.

**Proof.** If $g(G) > \chi(G)$, then the colourful path given by Theorem 4 is an induced path in $G$.

This implies that the conjecture is true for all triangle-free graphs with chromatic number at most 3. It also implies that in order to prove Conjecture 1, one only has to consider graphs $G$ with $g(G) \leq \chi(G)$. The main result of this paper is that Conjecture 1 holds true for all triangle-free graphs $G$ with $g(G) = \chi(G)$.

3. Induced colourful paths in graphs with girth equal to chromatic number

In this section, we shall prove our main result, given by the theorem below.

**Theorem 6.** Let $G$ be a graph with $g(G) = \chi(G) = k$, where $k \geq 4$, and whose vertices have been properly coloured. Then there exists an induced colourful path on $k$ vertices in $G$. 

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Note that we can assume that $G$ is connected, because if the theorem holds for connected graphs, then it will hold for the connected component of $G$ with chromatic number equal to $k$ and hence also for $G$. Let $\beta : V(G) \rightarrow \{1, 2, \ldots, t\}$ denote the proper colouring of $G$ that is given.

A $k$-cycle in $G$ in which no colour repeats is said to be a colourful $k$-cycle, sometimes shortened to just “colourful cycle”. Notice that every colourful cycle in $G$ is also an induced cycle as $g(G) = k$. From here onwards, we shorten “colourful path on $k$ vertices” to just “colourful path”.

Suppose that there is no induced colourful path on $k$ vertices in $G$.

**Observation 7.** Since $g(G) = k$, if $y_1y_2 \ldots y_k$ is a colourful path on $k$ vertices in $G$, then the edge $y_1y_k \in E(G)$. Thus, $y_1y_2 \ldots y_ky_1$ is a colourful $k$-cycle in $G$.

Let $\alpha$ be a proper colouring of $G$ generated by running the refined greedy algorithm on $G$. We shall refer to the colours of the colouring $\alpha$ as “labels”. From here onwards, we shall reserve the word “colour” to refer to a colour in the colouring $\beta$. As before, whenever we say that a path or a cycle is “colourful”, we are actually saying that it is colourful in the colouring $\beta$.

We say that a path with no repeating colours is an “almost decreasing path” if the subpath induced by the vertices other than the starting vertex is a decreasing path. Note that any decreasing path is also an almost decreasing path.

The proof of Theorem 6 is split into two cases: when $k = 4$ and when $k > 4$.

### 3.1. Case when $k = 4$

In this case, we have $\chi(G) = g(G) = 4$.

As $\alpha$ is also a proper colouring of $G$, we know that there exists a vertex $v$ in $G$ with label 4. By Lemma 3, there exists a decreasing path $v_4v_3v_2v_1$ where $v_4 = v$ and for $1 \leq i \leq 3$, we have $\beta(v_i) < \beta(v_{i+1})$ and $\alpha(v_i) = i$. Again by Lemma 3, we have a path $v_iv'_i$ in which we have $\beta(v'_i) < \beta(v_i') < \beta(v_i)$, $\alpha(v'_i) = 2$ and $\alpha(v_i') = 1$. Note that $v'_i \neq v_2$ and $v'_i \neq v_1$ (as otherwise $v_4v'_1v_1$ would be a triangle in $G$). This means that the vertices in $\{v_4, v_3, v_2, v_1, v'_1, v'_2\}$ are all pairwise distinct. Let $\beta(v_i) = b_i$ for each $i$, where $1 \leq i \leq 4$. We shall call the colours $b_1, b_2, b_3, b_4$ “primary colours”.

Clearly, as $v_4v_3v_2v_1$ is a decreasing and hence colourful path, by Observation 7, we have $v_1v_4 \in E(G)$.

**Claim 8.** $\beta(v'_1) = b_2$ and $\beta(v'_1) = b_1$.

**Proof.** Suppose that $\beta(v'_1) \neq b_2$. Then we have that either the path $v'_1v_4v_3v_2$ or the path $v'_1v_4v_1v_2$ is colourful, which implies that $v'_1v_2 \in E(G)$, a contradiction since $\alpha(v'_1) = \alpha(v_2)$. Therefore we have $\beta(v'_2) = b_2$. Similarly if $\beta(v'_1) \neq b_1$, then the path $v'_1v_2v_1v_3$ is colourful, which implies that $v'_1v_3 \in E(G)$, a contradiction since $\alpha(v'_1) = \alpha(v_1)$. Thus we have $\beta(v'_1) = b_1$.

Now notice that the path $v'_1v_4v_3v_2$ is colourful and hence we have that $v'_1v_3 \in E(G)$. We call the vertices in the set $\{v_4, v_3, v_2, v_1, v'_1, v'_2\}$ “forced vertices”. Any other vertex in the graph will be called an “optional vertex”. The following observations about forced vertices are easy to verify.

**Observation 9.** For any forced vertex $w$, we have $\beta(w) = b_{\text{col}w}$.

**Observation 10.** Every forced vertex is in a colourful cycle containing only forced vertices and which also contains the vertices $v_3$ and $v_4$.

**Observation 11.** For any forced vertex $w$ and set of primary colours $X \subseteq \{b_i \mid b_i < \beta(w)\}$, there exists a decreasing path starting from $w$ having only forced vertices and whose vertices other than $w$ have exactly the colours in $X$.

**Observation 12.** For $w \in \{v_3, v_4\}$ and for any set $X$ of primary colours, there exists an almost decreasing path containing only forced vertices and whose vertices other than $w$ see exactly the colours in $X \setminus \{\beta(w)\}$.

For a vertex $w$, we define $\rho(w) = v_1$ if $\beta(w) = b_4$ and $\rho(w) = v_4$ otherwise.

**Lemma 13.** If an optional vertex is adjacent to a forced vertex, then it is adjacent to at least two forced vertices. Moreover, there is a colourful cycle containing the optional vertex in which every other vertex is a forced vertex.
Proof. Let \( w \) be an optional vertex that is adjacent to a forced vertex \( u \). From Observation 10, there is a colourful cycle \( C \) that contains \( u \) and \( \rho(w) \). Let \( P \) be a subpath of \( C \) with endvertices \( \rho(w) \) and \( u \) which does not contain the colour \( \beta(w) \) (note that if \( \rho(w) = u \), then \( P \) consists of just the single vertex \( u = \rho(w) \)). From Observation 12, there exists an almost decreasing path \( P' \) from \( \rho(w) \) whose vertices other than \( \rho(w) \) see \( 4 - |V(wu \cup P)| \) primary colours not present in the path \( wu \cup P \). Let \( x \) be the endpoint of this path (note that \( P' \) can be the single vertex \( x = \rho(w) \) in case the path \( wu \cup P \) already contains 4 vertices). Clearly, \( wu \cup P \cup P' \) is a colourful path and hence \( wx \in E(G) \). As it can be easily seen that \( x \) is a forced vertex that is different from \( u \), we now have at least two forced vertices in \( N(w) \). Also, \( wu \cup P \cup P' \cup xw \) is a colourful cycle with the required properties.

Lemma 14. All optional vertices are adjacent to a forced vertex.

Proof. Consider the set of all optional vertices that have no forced vertices as neighbours. Let \( w \) be a vertex in this set that is closest to a forced vertex. As \( G \) is connected, \( w \) has a neighbour \( w' \) such that \( N(w') \) contains a forced vertex. From Lemma 13, there is a colourful cycle \( C \) containing \( w' \) in which all other vertices are forced vertices. Let \( z \) be the vertex in \( C \) which has highest colour in \( C \) other than \( \beta(w) \) and \( \beta(w') \). From the observation in the previous sentence, we know that \( z \) is a forced vertex. Let \( P \) be a subpath of \( C \) with endvertices \( w' \) and \( z \) that does not contain \( \beta(w) \). By Observation 11, from \( z \), there is a decreasing path \( P' \) whose vertices other than \( z \) see \( 4 - |V(ww' \cup P)| \) primary colours not seen on the path \( wu \cup P \). Let \( x \) be the endpoint of \( P' \). As \( P' \) is a decreasing path starting from the forced vertex \( z \), we have that \( x \) is a forced vertex. Now, \( ww' \cup P \cup P' \) is a colourful path and hence \( wx \in E(G) \). But now \( x \) is a forced vertex in \( N(w) \), contradicting the assumption that \( w \) had no forced vertices in its neighbourhood.

Let \( S_1 \) denote the set of optional vertices adjacent to at least one of the forced vertices \( \{v_4, v_2, v_1'\} \) and let \( S_2 \) denote the set of optional vertices adjacent to at least one of the forced vertices \( \{v_3, v_1, v_2'\} \).

Lemma 15. (i) \( S_1 \) and \( S_2 \) are disjoint, and (ii) \( S_1 \) and \( S_2 \) are both independent sets.

Proof. First let us show that \( S_1 \) and \( S_2 \) are disjoint. Suppose that there is a vertex \( w \in S_1 \cap S_2 \). We know that there are two forced vertices \( x \) and \( y \) in \( N(w) \) such that \( x \in \{v_4, v_2, v_1'\} \) and \( y \in \{v_3, v_1, v_2'\} \). As \( G \) is triangle-free, we only have two possibilities \( (x = v_1' \land y = v_1) \text{ or } (x = v_2 \land y = v_2') \). Suppose that \( x = v_1' \) and \( y = v_1 \). If \( \beta(w) \neq b_k \), then as at least one of the paths \( vv_1'v_2v_1 \) or \( vv_1'v_3v_1 \) is a colourful path, we have \( vv_1 \in E(G) \). But this is a contradiction as \( vv_1v_1' \) is a triangle in \( G \). Therefore, we can conclude that \( \beta(w) = b_k \). But now, the path \( vv_1v_2v_3 \) is colourful, implying that \( vv_2 \in E(G) \). This is a contradiction as \( vv_3v_1' \) is now a triangle in \( G \). The other case when \( x = v_2 \) and \( y = v_2' \) is symmetric. This proves (i).

This tells us that for each vertex \( w \in S_1 \), the forced vertices in \( N(w) \) all lie in \( \{v_4, v_2, v_1'\} \) and for each vertex \( w' \in S_2 \), the forced vertices in \( N(w') \) all lie in \( \{v_3, v_1, v_2'\} \). Since we know from Lemma 13 and Lemma 14 that each vertex in \( S_1 \cup S_2 \) has at least two forced vertices in their neighbourhood, we can conclude that each vertex in \( S_1 \) has at least two neighbours from \( \{v_4, v_2, v_1'\} \) and each vertex in \( S_2 \) has at least two neighbours from \( \{v_3, v_1, v_2'\} \). This means that for any two \( w, w' \in S_1 \), there is at least one vertex in \( \{v_4, v_2, v_1'\} \) that is a neighbour of both \( w \) and \( w' \). As \( G \) is triangle-free, we can conclude that \( ww' \notin E(G) \). For the same reason, for any two vertices \( w, w' \in S_2 \), we have \( ww' \notin E(G) \). This proves (ii).

From Lemma 15(i), we know that there are no edges between \( S_1 \) and \( \{v_3, v_1, v_2'\} \). Similarly, there are no edges between \( S_2 \) and \( \{v_4, v_2, v_1'\} \). Now, by Lemma 15(ii), we have that \( S_1 \cup \{v_3, v_1, v_2'\} \) is an independent set and \( S_2 \cup \{v_4, v_2, v_1'\} \) is an independent set. Since from Lemma 14, we know that \( V(G) = S_1 \cup S_2 \cup \{v_4, v_3, v_2, v_1, v_2', v_1'\} \), this tells us that \( G \) is bipartite, which contradicts the assumption that \( \chi(G) = 4 \). Therefore, there can be no properly coloured graph \( G \) such that \( g(G) = \chi(G) = 4 \) with no induced colourful path on 4 vertices. This completes the proof of Theorem 6 for the case \( k = 4 \).

3.2. Case when \( k > 4 \)

The proof for this case also follows the same general pattern as the case \( k = 4 \), but more technicalities are involved.

Lemma 16. Let \( y_1y_2 \ldots y_ky_1 \) be a colourful \( k \)-cycle. Let \( z \in N(y_i) \setminus \{y_{i-1}, y_{i+1}\} \) for some \( i \in \{1, 2, \ldots, k\} \). Then \( \beta(z) \notin \{\beta(y_1), \ldots, \beta(y_k)\} \setminus \{\beta(y_{i-1}), \beta(y_i), \beta(y_{i+1})\} \). (Here we assume that \( y_{i+1} = y_1 \) when \( i = k \) and that \( y_{i-1} = y_k \) when \( i = 1 \))
Proof. Clearly, $z \notin \{y_1, y_2, \ldots, y_k\}$ as every colourful cycle is an induced cycle. Suppose $\beta(z) \notin \{\beta(y_1), \ldots, \beta(y_k)\} \setminus \{\beta(y_{i-1}), \beta(y_i), \beta(y_{i+1})\}$. Clearly, $\beta(z) \neq \beta(y_i)$. Suppose that $\beta(z) \neq \beta(y_{i+1})$. Then observe that $zy_1y_2 \ldots y_{i-1} \ldots y_i \ldots y_{i+2}$ is a colourful path on $k$ vertices and hence $zy_1y_2 \ldots y_{i-2} \in E(G)$. This implies that $zy_1y_2 \ldots y_{i-2}$ is a 4-cycle in $G$, which is a contradiction. If $\beta(z) = \beta(y_{i+1})$, then we have $\beta(z) \neq \beta(y_{i-1})$. In this case, the path $zy_1y_2 \ldots y_{i-1} \ldots y_i \ldots y_{i+2}$ is a colourful path and the same reasoning as above tells us that there is a 4-cycle $zy_1y_2z$ in $G$, which is again a contradiction. \hfill \Box

**Corollary 17.** Let $y_1y_2 \ldots y_{i-1}y_iy_1$ be a colourful $k$-cycle. Let $z \in N(y_i)$ for some $i \in \{1, 2, \ldots, k\}$. Then $\beta(z) \in \{\beta(y_1), \ldots, \beta(y_k)\}$.

**The vertex $v$:** Fix $v$ to be a vertex which has the largest label. Since $\alpha$ is also a proper vertex colouring of $G$, it should use at least $k$ labels. In other words, $\alpha(v) \geq k$.

**Primary cycle:** By applying Lemma 3 to $v$ and the set of labels $\{1, 2, \ldots, k-1\}$, we can conclude that there exists a decreasing path $y_1y_2 \ldots y_k = v$ and such that $\alpha(v_i) = i$ for all $i < k$ and $\beta(v_i) < \beta(v_{i+1})$ for all $1 \leq i \leq k - 1$. Since this path is colourful, by Observation 7, $y_1y_2 \ldots y_kv$ is a colourful cycle, which we shall call the “primary cycle”. For $1 \leq i \leq k$, we shall denote by $b_i$ the colour $\beta(v_i)$. The set of colours $\{b_1, b_2, \ldots, b_k\}$ shall be called the set of “primary colours”.

**Lemma 18.** $\alpha(v) = k$. Hence, for all $i$, $\alpha(v_i) = i$.

**Proof.** Suppose for the sake of contradiction that $\alpha(v) > k$. By Lemma 3, there exists a decreasing path $y_{i-1}y_i \ldots y_k = v$ and for $1 \leq i \leq k$, we have $\alpha(v_i) = i$ and $\beta(v_i) < \beta(v_{i+1})$. As the paths $y_{i-1}y_i \ldots y_k$ and $y_{i+1}y_{i+2} \ldots y_k$ are both colourful, it must be the case that $y_{i-1}y_iy_{i+1} \in E(G)$. But then, $y_{i-1}y_iy_{i+1}y_{i+2}$ is a cycle on four vertices in $G$, which is a contradiction. \hfill \Box

**Forced and optional vertices:** A vertex $u \in V(G)$ is said to be a “forced vertex” if there is a decreasing path from $v$ to $u$. Any vertex of $G$ that is not forced is said to be an “optional vertex”.

**Lemma 19.** For each $i \in \{1, 2, \ldots, k\}$ there is exactly one vertex $u_i$ in $N(v)$ with label $i$. Moreover, $\beta(u_i) = b_i$ and there is a colourful cycle $C_i$ containing $u_i$ and $v$ that contains only forced vertices.

**Proof.** Observe that because the refined greedy algorithm assigned $\alpha(v)$ to be $k$, it must be the case that for every $i \in \{1, 2, \ldots, k\}$, there exists a vertex $u_i$ in $N(v)$ such that $\alpha(u_i) = i$ and $\beta(u_i) < \beta(v)$. We shall choose $u_i$ to be $v_{i-1}$. Because $u_i$ is adjacent to $v$ which is on the primary cycle, by Corollary 17, we know that $\beta(u_i)$ is a primary colour.

We claim that $\beta(u_i) = b_i$ and that there is a colourful cycle containing $v$ and $u_i$ that contains only forced vertices. We shall use backward induction on $i$ to prove this. Consider the base case when $i = k - 1$. Since $u_{k-1} = v_{k-1}$, we know that $\beta(u_{k-1}) = b_{k-1}$ and that there is a colourful cycle (the primary cycle) that contains $u_{k-1}$ and $v$ and also contains only forced vertices. Thus the claim is true for the base case. Let us assume that the claim has been proved for $u_{i-1}, u_{i-2}, \ldots, u_{i+1}$. If $\beta(u_i) = b_i$, then $b_i \in \{b_{i+1}, b_{i+2}, \ldots, b_{k-1}\}$. By the induction hypothesis, we know that the vertex $u_i \in N(v)$ has $\beta(u_i) = b_i$ and that there is a colourful cycle $C_i$ containing $u_i$ and $v$. Note that $u_i \neq u_j$ (as $\alpha(u_i) \neq \alpha(u_j)$), but $\beta(u_i) = \beta(u_j) = b_i$. Therefore, as $C_i$ contains $u_i$ and is a colourful cycle, it cannot contain $u_i$. Since $u_i$ is adjacent to $v$ which is on $C_j$ and $\beta(u_i) = b_i$, we now have a contradiction to Lemma 16 (note that $u_i$ is an edge of $C_j$ as every colourful cycle is a chordless cycle). So it has to be the case that $\beta(u_i) \neq b_i$. By Lemma 3, there exists a path $y_{i-1}y_iy_{i+1} \ldots y_k$ such that for $1 \leq j \leq i-1$, $\alpha(y_j) = j$ and $\beta(y_j) < \beta(y_{j+1})$. Notice that $y_{i-1}y_iy_{i+1} \ldots y_k$ is a colourful path and therefore by Observation 7, $C_i = y_{i-1}y_iy_{i+1} \ldots y_k$ is a colourful cycle containing both $u_i$ and $v$. Since $v_i$ is adjacent to $y_{i+1}$ which is on $C_i$, by Corollary 17, we know that there is some vertex $z$ on $C_i$ such that $\beta(z) = b_i$. Clearly, $z \in \{y_i, y_{i+1}, \ldots, y_k\}$. If $z \in \{y_i, y_{i+1}, \ldots, y_k\}$, then $\beta(y_j) > b_i$, which is a contradiction to our earlier observation that $\beta(u_i) \leq b_i$. Therefore, $z = y_i$, which implies that $\beta(u_i) = b_i$. Notice that each $y_j \in \{y_i, y_{i+1}, \ldots, y_k\}$, because of the decreasing path $y_{i-1}y_iy_{i+1} \ldots y_k$, is a forced vertex. Thus, $C_i$ is a colourful cycle containing $u_i$ and $v$ that contains only forced vertices. This shows that for any $i \in \{1, 2, \ldots, k-1\}$, $\beta(u_i) = b_i$ and there is a colourful cycle containing $v$ and $u_i$ that contains only forced vertices.

We shall now show that $u_i$ is the only vertex in $N(v)$ which has the label $i$. Suppose that there is a vertex $u \in N(v)$ such that $\alpha(u) = i$ and $u \neq u_i$. Since $u$ is adjacent to a colourful cycle containing only primary colours (the
primary cycle), we can conclude from Corollary 17 that $\beta(u)$ is a primary colour. Therefore, $\beta(u) = b_j$ for some $j \in \{1,2,\ldots,k-1\}$. From what we observed above, $\beta(u_j) = b_j$ and there exists a colourful cycle $C_j$ containing the vertices $v$ and $u_j$. Note that $u_j \neq u$ since if $j \neq i$, then $u_i$ and $u$ have different labels and if $j = i$, we know that $u_i \neq u$ (as we have assumed that $u_i \neq u$). Hence $u$ is not in $C_j$ (as $C_j$ already has a vertex $u_j$ with $\beta(u_j) = b_j$) but is adjacent to it. But now $C_j$ and $u$ contradict Lemma 16 as $u,v$ is an edge of $C_j$. Therefore, $u$ cannot exist. □

Corollary 20. Let $C$ be any colourful cycle containing $v$. Then $C$ contains only primary colours.

Proof. Notice that from Lemma 19, we know that for every primary colour $b_j \in \{b_1,b_2,\ldots,b_{k-1}\}$, there is a vertex $u_j$ with $\beta(u_j) = b_j$ that is adjacent to $v$. Because $v$ is in $C$, we can apply Corollary 17 to $C$ and $u_j$ to conclude that $b_j$ is present in $C$. This means that every primary colour appears on at least one vertex of $C$. Since $C$ was a $k$-cycle, this means that $C$ contains only primary colours.

Lemma 21. If $u \in V(G)$ is a forced vertex such that $a(u) = i$, then $\beta(u) = b_i$. Moreover, if $P$ is any decreasing path from $u$ to $v$, then there is a colourful cycle which has $P$ as a subpath and contains only forced vertices and primary colours.

Proof. Consider a forced vertex $u$. We shall prove the statement of the lemma for $u$ by backward induction on $a(u)$. The statement is true for $a(u) \in \{k,k-1\}$ as there is only one forced vertex each with labels $k$ and $k-1$—which are $v$ and $v_{k-1}$ respectively (recall that from Lemma 19, $u_{k-1} = v_{k-1}$ is the only vertex in $N(v)$ with label $k-1$). Also, note that they are both in a colourful cycle (the primary cycle) that satisfies the required conditions. Let us assume that the statement of the lemma has been proved for $a(u) \in \{k, k-1, \ldots, i + 1\}$. Let us look at the case when $a(u) = i$. Let $z$ be the predecessor of $u$ in the path $P$ and let $P_z$ be the subpath of $P$ that starts at $v$ and ends at $z$. Let $a(z) = j$. By the induction hypothesis, $\beta(z) = b_j$ and $z$ is in a colourful cycle $C$ which contains only primary colours. By Corollary 17, we can infer that $\beta(u)$ is a primary colour. Since $P$ was a decreasing path, $\beta(u) \in \{b_1,b_2,\ldots,b_{i-1}\}$. If $\beta(u) = b_j$ with $b_j > b_i > b_j$, then notice that there already exists a neighbour $y$ of $z$ with $a(y) = i$ and $\beta(y) < \beta(z)$, because the refined greedy algorithm set $a(z) = j$. Note that $P_z \cup Zy$ is a decreasing path from $v$ to $y$, which implies that $y$ is a forced vertex. Clearly, $u \neq y$ as $a(u) \neq a(y)$. Because of our induction hypothesis, $\beta(y) = b_i$ and there is a colourful cycle containing the path $P_z \cup Zy$ as a subpath. As $\beta(u) = \beta(y)$, $u$ is outside this cycle but is a neighbour of $z$. This contradicts Lemma 16. Therefore, $\beta(u) \leq b_i$. Consider the decreasing path $y_1y_2\ldots y_1$ where $y_1 = u$, and for $s \in \{1,2,\ldots,i-1\}, a(y_s) = s$ and $\beta(y_s) < \beta(y_{s+1})$ which exists by Lemma 3. Again by Lemma 3, there exists a decreasing path $Q$ starting from $v$ whose vertices other than $v$ have exactly the labels in $\{i+1,i+2,\ldots,k\}$ that are not seen on $P_z$. By the induction hypothesis, we can now see that every colour in $\{b_{i+1},b_{i+2},\ldots,b_k\}$ occurs exactly once in the path $Q \cup P_z$. Since $y_1y_2\ldots y_1$ is a decreasing path in which every vertex has colour at most $b_i$, we can conclude that the path $P' = Q \cup P_z \cup Zy_1y_2\ldots y_1$ is a colourful path. By Observation 7, the graph induced by $V(P')$ is a colourful cycle containing $v$, which we shall call $C'$. By Corollary 20, we know that $C'$ contains only primary colours. Now, if $\beta(u) < b_i$, then because $y_{s+1} = y_s$ was a decreasing path, it should mean that $\beta(y_s) < b_i$, which is a contradiction. Thus, $\beta(u) = b_i$ and $C'$ is a cycle containing $P$ as a subpath and which contains only forced vertices and primary colours (note that each $y_s$, for $1 \leq s \leq k-1$, is a forced vertex as there is the decreasing path $P_z \cup Zy_1y_2\ldots y_1$ from $v$ to $y_1$).

Lemma 22. If $P_1$ and $P_2$ are two decreasing paths ($P_1 \neq P_2$) that start from a forced vertex $u$ and meet at a vertex $z$, then

(i) $\beta(z) = b_1$ and $a(z) = 1$

(ii) $u = v$, and

(iii) $P_1 \cup P_2$ is a colourful cycle.

Proof. Since $u$ is a forced vertex there exists a decreasing path $P$ from $v$ to $u$.

We shall first show that the paths $P_1$ and $P_2$ cannot see exactly the same set of colours. Suppose for the sake of contradiction that they do. As the paths $P_1$ and $P_2$ are different, there is a vertex in one of these paths that is not in the other. Let us assume without loss of generality that there is a vertex in the path $P_2$ that is not present in $P_1$. We denote by $x'$ the first vertex (when walking from $u$) on $P_2$ that is not present in $P_1$. Let $x$ be the predecessor of $x'$ on $P_2$. Clearly, $x$ is also in $P_1$. Let $x''$ denote the successor of $x$ on $P_1$. As $P_1$ and $P_2$ are both decreasing paths with the same set of colours, it must be the case that $\beta(x') = \beta(x'')$. From Lemma 21, we know that there is a colourful cycle that contains $P \cup P_1$ as a subpath. Clearly, this colourful cycle contains $x''$ and as $\beta(x') = \beta(x'')$, this cycle does not contain $x'$. But now this colourful cycle and $x'$ contradict Lemma 16.

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Let $\alpha(z) = i$. By Lemma 21, we have $\beta(z) = b_i$. Let $Y$ be the decreasing path $y_i y_{i-1} \ldots y_1$ (guaranteed to exist by Lemma 3) where $y_i = z$, and for $s \in \{1,2,\ldots,i-1,i\}$, $\alpha(y_s) = s$ and $\beta(y_s) = b_i$ (by Lemma 21; note that each $y_i$ is a forced vertex because of the decreasing path $P \cup P_j \cup y_i y_{i-1} \ldots y_s$ from $v$ to it). Again by Lemma 3, for $j \in \{1,2\}$, there exists a decreasing path $Q_j$ starting from $v$ whose vertices other than $v$ have exactly the labels in $\{i+1,i+2,\ldots,k\}$ that are not seen on $P \cup P_j$. Therefore, it must be the case that $z$ is a neighbour of $Q_j$. Since all the vertices in $P$, $P_j$, $Q_j$ and $Y$ are forced vertices and therefore by Lemma 21, they will all have primary colours. Furthermore, the colours on the path $Q_j$ are exactly the primary colours greater than $b_i$ that are not seen on the path $P \cup P_j - v$ and the path $Y$ will contain all the primary colours less than or equal to $b_i$. Let $w_j$ denote the endpoint of $Q_j$ that is not $v$. Since $P \cup P_j \cup Y \cup Q_j$ is a colourful path, we have $w_j y_1 \in E(G)$ for $j \in \{1,2\}$. Observe that $P_1$ and $P_2$ are two paths such that in at least one of them, we have a vertex that is not present in the other. Since these paths also have the same endpoints, it is clear that there is a cycle in $G$ using only edges in $E(P_1) \cup E(P_2)$. Note that as $P_1$ and $P_2$ do not see exactly the same set of colours, the paths $Q_1 \cup w_1 y_1$ and $Q_2 \cup w_2 y_1$ are different. Now, following the same reasoning as above, one can see that there is at least one cycle in $G$ whose edges are from $E(Q_1 \cup w_1 y_1) \cup E(Q_2 \cup w_2 y_1)$. Since girth of $G$ is $k$, we now have the following inequalities.

$$||P_1|| + ||P_2|| \geq k$$

$$||Q_1|| + ||Q_2|| + 2 \geq k$$

As $P \cup P_j \cup Y \cup Q_j \cup w_j y_1$ is a colourful cycle for $j \in \{1,2\}$, we have the following equalities.

$$||P|| + ||P_1|| + ||Q_1|| + ||Y|| + 1 = k$$

$$||P|| + ||P_2|| + ||Q_2|| + ||Y|| + 1 = k$$

Summing the first two inequalities, we get,

$$||P_1|| + ||P_2|| + ||Q_1|| + ||Q_2|| + 2 \geq 2k$$

and by adding the third and fourth, we get,

$$2||P|| + ||P_1|| + ||P_2|| + ||Q_1|| + ||Q_2|| + 2||Y|| + 2 = 2k.$$.

Combining the previous two equations, we have,

$$2||P|| + 2||Y|| \leq 0$$

which implies that $||Y|| = 0$ and $||P|| = 0$. Therefore, we have $\alpha(z) = 1$ (and hence, $\beta(z) = b_1$) and also $u = v$. This proves (i) and (ii).

This means that $Y$ contains just the vertex $z$ and that $P_1$ and $P_2$ are both paths that start at $v$ and end at $z$. Now, by our earlier observation, $z$ is in a colourful cycle $C_1 = P_1 \cup Q_1 \cup w_1 z$ and also in a colourful cycle $C_2 = P_2 \cup Q_2 \cup w_2 z$. There exists a vertex $z' \in C_1$ with $\beta(z') = b_2$ and a vertex $z'' \in C_2$ such that $\beta(z'') = b_2$. Because each of $C_1$ and $C_2$ is the union of two decreasing paths from $v$, one can conclude that $z'$ is a neighbour of $z$ in $C_1$ and $z''$ is a neighbour of $z$ in $C_2$. Applying Lemma 16 to either one of the cycles $C_1$ or $C_2$, it can be seen that there can only be one neighbour of $z$ coloured $b_2$, and therefore, $z' = z''$. Thus, both $C_1$ and $C_2$ contain the vertex $z'$. Let $R_j$, for $j \in \{1,2\}$, be the subpath of $C_j$ with $v$ and $z'$ as endvertices that does not contain $z$. Clearly, $R_j$ is a decreasing path. Therefore, $R_1$ and $R_2$ are two decreasing paths that start at $v$ and meet at $z'$. If $R_1 \neq R_2$, then since $\beta(z') = b_2$, we have a contradiction to (i). Therefore, it must be the case that $R_1 = R_2$, which implies that one of $P_1 - z = P_2 - z$, $Q_1 = Q_2$, $P_1 - z = Q_2$ or $P_2 - z = Q_1$ is true. Since $P_1 \neq P_2$, we know that $P_1 - z \neq P_2 - z$ and because $P_1$ and $P_2$ do not contain the same set of colours, we also have $Q_1 \neq Q_2$. Therefore, we can conclude that either $P_1 - z = Q_2$ or $P_2 - z = Q_1$. But in either case, we have that the colours on the path $P_1 - [v,z]$ are exactly all the primary colours that are absent in $P_2$. This can only mean that $P_1 \cup P_2$ is a colourful cycle. This proves (iii).

\[\square\]

**Corollary 23.** The graph induced by all the forced vertices other than those coloured $b_1$ is a tree.
Proof. Suppose for the sake of contradiction that there is a cycle containing only forced vertices with colours other than \( b_1 \). Let \( u \) be the vertex in the cycle with the least colour in the colouring \( \beta \) and let \( u' \) and \( u'' \) be its two neighbours on the cycle. Since \( u' \) and \( u'' \) are both forced vertices there are decreasing paths \( P' \) and \( P'' \) that start at \( v \) and end at \( u' \) and \( u'' \) respectively. Let \( P_1 \) be the decreasing path obtained by adding \( u \) to \( P' \) and \( P_2 \) be the decreasing path obtained by adding \( u \) to \( P'' \). Since \( P' \neq P'' \), we clearly have \( P_1 \neq P_2 \). Thus, \( P_1 \) and \( P_2 \) are two decreasing paths starting at \( v \) and meeting at \( u \) and because \( \beta(u) \neq b_1 \), we have a contradiction to Lemma 22(i).

Given any vertex \( w \), we define \( \rho(w) = v_{k-1} \) if \( \beta(w) = b_1 \) and \( \rho(w) = v_k \) otherwise.

Let \( X \subseteq \{b_1, b_2, \ldots, b_k\} \) be any set of primary colours. Let \( L = \{i \mid b_i \in X\} \). From Lemma 3, we know that there is a decreasing path starting from \( v_k \) consisting only of forced vertices and which sees exactly the labels in \( L \cup \{k\} \).

Applying Lemma 21 to the vertices of this path, we have that this path sees exactly the colours in \( X \cup \{b_k\} \). Thus for any set of colours \( X \subseteq \{b_1, b_2, \ldots, b_k\} \), there exists a decreasing path starting from \( v_k \), made up of forced vertices and which sees exactly the colours in \( X \cup \{b_k\} \). Note that if \( b_k \notin X \), we can apply the same argument to \( v_{k-1} \) and \( X \) and conclude that there is a decreasing path starting at \( v_{k-1} \) consisting only of forced vertices and which sees exactly the colours in \( X \cup \{b_{k-1}\} \). Suppose that \( b_k \in X \). Then let \( X' = X \setminus \{b_{k-1}\} \). From our above observation, there is a decreasing path starting from \( v_k \), consisting of only forced vertices and which sees exactly the colours in \( X \cup \{b_k\} \). Thus, for any set of primary colours \( X \), we have almost decreasing paths starting from both \( v_k \) and \( v_{k-1} \) consisting of only forced vertices and which see exactly the colours in \( X \cup \{b_k\} \) and \( X \cup \{b_{k-1}\} \) respectively. This gives us the following observation.

Observation 24. For any vertex \( w \), and for any set of colours \( X \subseteq \{b_1, b_2, \ldots, b_k\} \), there exists an almost decreasing path starting from \( \rho(w) \), made up of forced vertices and which sees exactly the colours in \( X \cup \{\beta(\rho(w))\} \).

Suppose that \( P \) is the decreasing path from \( v_k \) to a forced vertex \( y \). Clearly, if \( v_{k-1} \) is on this path, then the subpath of \( P \) from \( v_{k-1} \) to \( y \) is a decreasing path from \( v_{k-1} \) to \( y \). If \( v_{k-1} \) is not on \( P \), then \( v_{k-1}, v_k \cup P \) is an almost decreasing path starting from \( v_{k-1} \) and ending at \( y \). As there are also decreasing paths from \( v_k \) to every other forced vertex, we now have the following observation.

Observation 25. Given any vertex \( w \) and any forced vertex \( y \), there exists an almost decreasing path starting from \( \rho(w) \) and ending at \( y \).

Lemma 26. If \( w \) is an optional vertex such that \( N(w) \) contains a forced vertex \( u \), then \( N(w) = \{u, y\} \) where \( \{\beta(u), \beta(y)\} = \{b_1, b_2\} \). Moreover, there are decreasing paths \( P_u \) from \( \rho(w) \) to \( u \) and \( P_y \) from \( \rho(w) \) to \( y \) such that \( P_u \cup uy \cup P_y \) is a colourful cycle.

Proof. We shall first show that \( \beta(u) \in \{b_1, b_2\} \). Suppose that \( \beta(u) \notin \{b_1, b_2\} \).

We claim that there is a forced vertex \( z \in N(w) \) with \( \beta(z) \notin \{b_1, b_2\} \) such that there is an almost decreasing path \( P_z \) from \( \rho(w) \) to \( z \) that does not contain \( \beta(w) \). Let \( P_u \) be the almost decreasing path from \( \rho(w) \) to \( u \) that exists by Observation 25. If \( \beta(w) \) is not in \( P_u \), then we can set \( z = u \) and we are done. Suppose \( \beta(w) \in P_u \). By Observation 24, there exists a path \( P'_u \) that starts from \( \rho(w) \) and whose vertices other than \( \rho(w) \) have exactly the colours that are missing in \( P_u \). Let \( u' \) be the endvertex of \( P'_u \) other than \( \rho(w) \). Clearly, \( P_u \cup P'_u \) is a colourful path and therefore, by Observation 7, we have \( uu' \in E(G) \). By Observation 24, there exists an almost decreasing path \( Q \) consisting of forced vertices starting from \( \rho(w) \) that sees exactly the colours in \( P_u \) other than \( \{\beta(w), \beta(u)\} \). Now, \( \{uu', uu'\} \cup P_u \cup Q \) is a colourful path and therefore the endvertex of \( Q \) is a forced vertex adjacent to \( w \). We claim that this endvertex of \( Q \) can be chosen as \( z \). Clearly, \( Q \) is an almost decreasing path from \( \rho(w) \) to \( z \) that does not contain \( \beta(w) \) and this can be considered to be the required path \( P_z \). Also, since \( \beta(u) \notin \{b_1, b_2\} \), no vertex in \( P_u \) has colour \( b_1 \) or \( b_2 \), implying that no vertex in the almost decreasing path \( Q \) has either of these colours. Therefore, \( \beta(z) \notin \{b_1, b_2\} \). Hence \( z \) is a neighbour of \( w \) of the required type.

Let \( z \) be a forced vertex in \( N(w) \) such that there is an almost decreasing path \( P_z \) from \( \rho(w) \) to \( z \) that does not contain \( b_1, b_2 \) or \( \beta(w) \). By Observation 24, there exists an almost decreasing path \( Q \) starting from \( \rho(w) \) whose vertices other than \( \rho(w) \) have exactly the primary colours that are not in the path \( P_z \). Note that \( b_1 \) and \( b_2 \) will be present in \( Q \). Let \( u_1, u_2, u_3 \) be last three vertices of \( Q \) in that order (i.e., \( u_1 \) is the endvertex of \( Q \) other than \( \rho(w) \)). Since \( Q \) is
an almost decreasing path, we have $\beta(u_1) = b_1$ and $\beta(u_2) = b_2$. Since $wz \cup P \cup Q$ is a colourful path, we have by Observation 7 that $wu_1 \in E(G)$. By Observation 24, there exists an almost decreasing path $Q'$ that starts at $\rho(w)$ and whose vertices other than $\rho(w)$ see exactly the primary colours that are not present in $Q$. Note that the paths $Q'$ and $P \cup zw$ see exactly the same colours. Let the endpoint of $Q'$ other than $\rho(w)$ be $w'$. As $Q' \cup Q$ is a colourful path, we have $w' u_1 \in E(G)$. By applying Lemma 3 and Lemma 21 together, we know that there is a vertex $p$ with $\beta(p) = b_1$ that is adjacent to $u_1$. Let $R$ be the union of the subpath of $Q$ from $\rho(w)$ to $u_1$ and the edge $u_1 p$. By Observation 24, there exists an almost decreasing path $R'$ starting from $\rho(w)$ and whose vertices other than $\rho(w)$ have exactly the primary colours not present in $R$. Let $r$ be the end vertex of $R'$. Clearly, $\beta(r) = b_2$, and because $R \cup R'$ is a colourful path, we have $pr \in E(G)$. Now, observe that the path $zw \cup P \cup R \cup pr$ is also a colourful path, and therefore, we have $wr \in E(G)$. Similarly, the path $Q' \cup R \cup pr$ is also a colourful path, leading to the conclusion that $w' r \in E(G)$. Recall that $w \neq w'$ as one is a forced vertex while the other is an optional vertex and $u_1 \neq r$ as $\beta(u_1) \neq \beta(r)$. We now have a four cycle $w r w' u_1 w$ in $G$, which is a contradiction.

Therefore, we can conclude that $\beta(u) \in \{b_1, b_2\}$. By Observation 25, there exists an almost decreasing path $R_u$ from $\rho(w)$ to $u$. By Observation 24, there exists an almost decreasing path $R_u'$ that starts from $\rho(w')$ and ends at a vertex $w'$, and whose vertices other than $\rho(w)$ have exactly the primary colours that are not in $R_u$. By Observation 7, we know that $uu' \in E(G)$ and that $G[V(R_u) \cup V(R_u')]$ is a colourful cycle. Therefore, there exists a path $P \in \{R_u, R_u' \cup uu'\}$ such that $P$ does not contain $\beta(w)$. By Observation 24, there exists an almost decreasing path $Q$ consisting of forced vertices that starts from $\rho(w)$, ends at a vertex $y$, and whose vertices other than $\rho(w)$ see exactly the primary colours that are not in $V(P) \cup \{w\}$. Now, notice that $wu \cup P \cup Q$ is a colourful path and therefore $y$ will be adjacent to $w$. This tells us that $\beta(y) \in \{b_1, b_2\}$. If $P = R_u$, then it is clear that $P$ is an almost decreasing path from $\rho(w)$ to $u$. If, on the other hand, $P = R_u' \cup uu'$, then notice that since $wu \cup P \cup Q$ is a colourful path which contains the vertices $u, y$ with $\beta(y) = \{b_1, b_2\}$, we have $\beta(u') > \beta(u)$. Since $R_u'$ is an almost decreasing path, this tells us that $P = R_u' \cup uu'$ is also an almost decreasing path. Also, since $wu \cup P \cup Q$ is a colourful path, both the almost decreasing paths $P$ and $Q$ do not see the colour $\beta(w)$, and hence they are both decreasing paths. Thus we can set $P_u = P$ and $P_y = Q$ as the required decreasing paths. Now, the application of Lemma 16 to the colourful cycle $wu \cup P \cup Q \cup yw$ and vertex $w$ tells us that these are the only two vertices with colours $b_1$ and $b_2$ in $N(w)$. Thus, we can conclude that $N(w) = \{u, y\}$.

\begin{lemma}
Every optional vertex is adjacent to a forced vertex.
\end{lemma}

\begin{proof}
Suppose that there are optional vertices in $G$ that are not adjacent to any forced vertex. Then let $w$ be the optional vertex among them that is closest to a forced vertex. Since $G$ is connected, we can conclude that $w$ is adjacent to some optional vertex $w'$ that has a forced vertex in $N(w')$. By Lemma 26, we know that there is a colourful cycle $C$ containing $w'$ in which every vertex other than $w'$ is a forced vertex. Let $z$ be that vertex in $C$ with the highest colour in $\{b_1, b_2, \ldots, b_k\} \setminus (\beta(w), \beta(w'))$. Let $P$ be that subpath of $C$ between $w'$ and $z$ that does not contain $\beta(w)$. By Lemma 3 and Lemma 21, there exists a decreasing path $P'$ starting from $z$ whose vertices other than $z$ have exactly the primary colours that are not present in the path $P \cup w' w$. Clearly, $ww' \cup P \cup P'$ is a colourful path and therefore the endvertex $y$ of $P'$ is adjacent to $w$. Notice that $P'$ was a path consisting entirely of forced vertices and therefore $y$ is a forced vertex in the neighbourhood of $w$, which is a contradiction.
\end{proof}

\begin{lemma}
The set of optional vertices form an independent set.
\end{lemma}

\begin{proof}
We shall first observe that like forced vertices, every optional vertex also has a primary colour. Consider an optional vertex $w$. By Lemma 27, we know that there is a forced vertex $y$ in $N(w)$. Notice that by Lemma 21, the forced vertex $y$ is in a colourful cycle containing only forced vertices and primary colours. Therefore, the optional vertex $w$ is adjacent to a colourful cycle containing only primary colours. Thus, by Corollary 17, we can conclude that the $\beta(w)$ is a primary colour.

We shall now prove the statement of the lemma. Let $w_1$ and $w_2$ be optional vertices such that $w_1 w_2 \in E(G)$. Let us assume without loss of generality that $\beta(w_1) < \beta(w_2)$. Suppose first that $\beta(w_1) = b_{k-1}$ and $\beta(w_2) = b_k$. In this case, $\rho(w_1) = v_k$ and $\rho(w_2) = v_{k-1}$. Let $y, z$ be forced vertices in $N(w_1)$ and $y', z'$ be forced vertices in $N(w_2)$ such that $\beta(y) = \beta(y') = b_1$ and $\beta(z) = \beta(z') = b_2$. Note that $y \neq y'$ and $z \neq z'$ as $G$ is triangle-free. By Lemma 26, we know that $y, z, y', z'$ exist and also that there is a colourful cycle $C$ containing vertices $w_1$, $v_k$ and the edges $w_1 y$ and $w_1 z$. Similarly, there is a colourful cycle $C'$ containing vertices $w_2$, $v_{k-1}$ and the edges $w_2 y'$ and $w_2 z'$. Let $P$, $Q$ be the subpaths of $C$ with endvertices $v_k$ and $v_{k-1}$ that contain the edges $w_1 y$, $w_1 z$ respectively. Similarly, let $P'$, $Q'$ be the subpaths of $C'$ with endvertices $v_{k-1}$ and $w_2$ that contain the edges $w_2 y'$, $w_2 z'$ respectively. Let $u$ be the vertex with
colour \( b_{k-2} \) on the cycle \( C \). Since from Lemma \( 26 \), it is clear that \( P - w_1 \) and \( Q - w_1 \) are both decreasing paths, it has to be the case that \( u \) is a neighbour of \( v_k \) on either \( P \) or \( Q \). Suppose that \( u \) is on \( P \). By Lemma \( 3 \), we know that there is a decreasing path \( R \) starting from \( u \) such that its vertices other than \( u \) see exactly the colours that are absent on the path \( P \). Notice that \( w_2v_1 \cup (P - v_k) \cup R \) is a colourful path and hence the endvertex of \( R \) is adjacent to \( w_2 \). Clearly, the endvertex of \( R \) has colour \( b_2 \) and therefore by Lemma \( 26 \), this vertex is none other than \( z' \). Now, the paths \( v_ku \cup R \) and \( v_kv_{k-1} \cup (Q' - w_2) \) are both distinct decreasing paths (note that \( v_{k-1} \) is present in one of the paths but not in the other) that meet at the vertex \( z' \). Since \( \beta(z') = b_2 \), we now have a contradiction to Lemma \((i)\). Therefore, we can conclude that \( u \) is on \( Q \). Again by Lemma \( 3 \), we know that there is a decreasing path \( R \) starting from \( u \) whose vertices other than \( u \) see exactly the colours that are absent on the path \( Q \). Notice that \( w_2v_1 \cup (Q - v_k) \cup R \) is a colourful path and hence the endvertex of \( R \) is adjacent to \( w_2 \). As it is clear that this endvertex of \( R \) has colour \( b_1 \), by Lemma \( 26 \), this vertex is none other than \( y' \). Now, the paths \( v_ku \cup R \) and \( v_kv_{k-1} \cup (P' - w_2) \) are both distinct decreasing paths that meet at the vertex \( y' \). By Lemma \((iii)\), we know that these paths together constitute a colourful cycle. But the colour \( b_2 \) is not present in either of these paths, which is a contradiction.

By the above arguments, we can assume that \( \{\beta(w_1), \beta(w_2)\} \neq \{b_{k-1}, b_k\} \). Therefore, \( \beta(w_1) < b_{k-1} \) which means that \( \rho(w_1) = v_k \). By Lemma \( 26 \), we know that there is a colourful cycle \( C \) containing \( w_2 \) and \( \rho(w_2) \). Let \( P \) be that subpath of \( C \) between \( \rho(w_2) \) and \( w_2 \) that does not contain \( \beta(w_1) \) (note that \( \beta(\rho(w_2)) \neq \beta(w_1) \), implying that such a path exists) and let \( Q \) be that subpath that does. Also, we shall denote by \( y \) the neighbour of \( w_2 \) on \( P \) and by \( z \) the neighbour of \( v_2 \) on \( Q \). By Observation \( 24 \), there exists an almost decreasing path \( P' \) starting from \( \rho(w_2) \) in which the vertices other than \( \rho(w_2) \) see exactly the colours that are not present in the path \( P \cup w_2v_1 \). Clearly, \( w_1v_2 \cup P \) is a colourful path and therefore the endvertex \( y' \) of \( P' \) is adjacent to \( w_1 \). Note that \( y' \) is different from \( y \) and \( z \) as otherwise, there would be a triangle in the graph. Let \( Q' \) be the almost decreasing path starting from \( \rho(w_2) \), that exists by Observation \( 24 \), such that its vertices other than \( \rho(w_2) \) see exactly the missing colours from \( P' \) other than \( \beta(w_1) \). Notice that \( Q' \) contains exactly the colours from \( P \). Let \( z' \) be the endvertex of \( Q' \) other than \( \rho(w_2) \). Clearly, \( z' \neq w_2 \) as one is a forced vertex while the other is an optional vertex. Observe that \( w_1y' \cup P' \cup Q' \) is a colourful path and hence \( z'w_1 \in E(G) \). As before, \( z' \) is distinct from \( y \) and \( z \) as otherwise, there would be a triangle in the graph.

Let \( Q \) be the subpath of \( Q \) from \( z \) to \( \rho(w_2) \). Recall that \( Q' \) contains exactly the colours from \( P \). As the path \( P \cup Q \), was a colourful path, we can infer that \( Q' \cup Q \) is also a colourful path. Hence there is an edge \( zz' \in E(G) \). But since \( z \in N(w_2) \) and \( z' \in N(w_1) \), we have the 4-cycle \( w_2v_1z'w_2 \) in \( G \), which is a contradiction. Therefore we conclude that for any two optional vertices \( w_1 \) and \( w_2 \), we have \( w_1w_2 \notin E(G) \). We have thus proved that the set of optional vertices is an independent set.

Let \( w \) be an optional vertex. From Lemma \( 27 \) and Lemma \( 26 \), we know that \( N(w) = \{y, z\} \) where \( \beta(y) = b_1 \), \( \beta(z) = b_2 \) and both \( y \) and \( z \) are forced vertices. This also tells us that \( \beta(w) \notin \{b_1, b_2\} \). Let \( T \) denote the graph induced in \( G \) by the forced vertices other than those coloured \( b_1 \). From Corollary \( 23 \), we know that \( T \) is a tree. By what we have observed above, it is clear that each optional vertex has exactly one neighbour in \( V(T) \). Also, Lemma \( 28 \) tells us that the optional vertices form an independent set. Therefore, \( V(T) \) and the set of optional vertices together induce a tree in \( G \). Since optional vertices do not have colour \( b_1 \) as observed above, we can conclude that the subgraph induced in \( G \) by the vertices other than those coloured \( b_1 \) is a tree. This implies that \( \chi(G) \leq 3 \), which is a contradiction to the fact that \( \chi(G) = k > 4 \). This completes the proof of Theorem \( 6 \) for the case when \( k > 4 \).

4. Conclusion

We have shown in this paper that for any properly coloured graph \( G \) with \( g(G) \geq \chi(G) \), there exists an induced colourful path on \( \chi(G) \) vertices in \( G \). The question of whether every properly coloured graph \( G \) contains an induced colourful path on \( \chi(G) \) vertices remains open for the case \( 3 < g(G) < \chi(G) \).

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