A POSITIVE ANSWER TO AMBROSETTI-MALCHIODI CONJECTURE IN FRACTIONAL SCHRÖDINGER EQUATIONS

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ABSTRACT. We study the following fractional Schrödinger equation
\[ \epsilon^{2s}(-\Delta)^s u + Vu = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \]
where \( s \in (0, 1), \) \( p \in (2+2s/(N-2s), 2^*_s), \) \( 2^*_s = 2N/(N-2s), \) \( N > 2s, \) \( V \in C(\mathbb{R}^N; [0, \infty)). \)
We use penalized technique to show that the problem has a family of solutions concentrating at a local minimum of \( V. \) Our results solve the fractional version of Ambrosetti-Malchiodi conjecture ([4]) completely since the potential can decay arbitrarily. The argument used in this paper also works well to problems above with more general nonlinear terms.

Key words: fractional Schrödinger; local minimum; vanishing potential; Ambrosetti-Malchiodi conjecture; penalized technique; variational methods.

1. Introduction and main results

In this paper, we consider the fractional Schrödinger equation
\[ \epsilon^{2s}(-\Delta)^s u + Vu = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \]  
(1.1)
where \( N > 2s, \) \( s \in (0, 1), \) \( V \) is continuous function, \( \epsilon > 0 \) is a small parameter, \( p \in (2, 2^*_s), \) \( 2^*_s = 2N/(N-2s). \) Problem (1.1) is from the study of time-independent waves \( \psi(x,t) = e^{-iEt/\epsilon}u(x) \) of the following nonlinear fractional Schrödinger equation
\[ i\epsilon \frac{\partial \psi}{\partial t} = \epsilon^{2s}(-\Delta)^s \psi + U(x)\psi - f(\psi) \quad x \in \mathbb{R}^N. \]  
(1.2)
This equation has a wide application in Physics, for example, the Einstein’s theory of relativity, phase transition, conservation laws and fractional quantum mechanics, for more physical background, we refer the readers to [18, 22, 23].

Equation (1.1) in the local case \( s = 1 \) has been studied extensively in recent years, see [1, 8, 10, 11, 14, 15, 17, 24, 25] and their references therein for example. It is worth mentioning that the Ambrosetti-Malchiodi conjecture in [4] which asks about the existence of solutions to (1.1) in the local case \( s = 1 \) when
\[ \lim_{|x| \to \infty} V(x)|x|^2 = 0. \]

In [7, 9], this conjecture was solved partially and, in [27], it was solved completely. Considering the nonlocal case \( 0 < s < 1, \) a natural conjecture analogue to Ambrosetti-Malchiodi conjecture then arise:

Is there a solution to (1.1) when \( \lim_{|x| \to \infty} V(x)|x|^{2s} = 0 \) with \( 0 < s < 1? \)  
(C)
To our best knowledge, there is no any answer to the problem (C) up to now. Our aim of this paper is to settle the problem (C) completely, i.e., to find a solution \( u_\epsilon \) to (1.1) for all potentials that decay faster than \(|x|^{-2s}\).

For \( s \in (0, 1) \), the fractional Sobolev space \( H^s(\mathbb{R}^N) \) is defined as
\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x - y|^{N+2s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},
\]
endowed with the norm
\[
\|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \frac{|(-\Delta)^{s/2}u|^2 + u^2}{|x - y|^{N+2s}} \right)^{1/2} dx.
\]

Like the classical case, we define the space \( H^s(\mathbb{R}^N) \) as
\[
H^s(\mathbb{R}^N) = \left\{ u : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) \right\}.
\]

Generally, the fractional Laplacian (see [18] for example) is defined as
\[
(-\Delta)^{s}u(x) = C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = C(N, s) \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.
\]

but, for the sake of simplicity, we define for every \( u \in H^s(\mathbb{R}^N) \) the fractional \((-\Delta)^{s}\) as
\[
(-\Delta)^{s}u(x) = \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy.
\]

Our work will use the following weighted Hilbert space:
\[
H^s_{V, \epsilon}(\mathbb{R}^N) = \left\{ (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) : u \in L^2(\mathbb{R}^N, V(x) dx) \right\},
\]
endowed with the norm
\[
\|u\|_{H^s_{V, \epsilon}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \epsilon^{2s}(-\Delta)^{s/2}u |u|^2 + V u^2 dx \right)^{1/2}.
\]

Assuming that \( V \in C(\mathbb{R}^N, [0, \infty)) \) and
\[(A) \text{ There exist open bounded sets } \Lambda \subset \subset U \text{ with smooth boundaries } \partial \Lambda, \partial U, \text{ such that } 0 < \lambda = \inf_{\Lambda} V < \inf_{U \setminus \Lambda} V, \tag{1.3}\]
We have the following main result:

**Theorem 1.1.** Let \( N > 2s, s \in (0, 1), p \in (2 + \frac{2s}{N-2s}, 2^*_s) \). Then problem (1.1) has a positive solution \( u_\epsilon \in H^s_{V, \epsilon}(\mathbb{R}^N) \) if \( \epsilon > 0 \) is small enough. Moreover, suppose that \( x_\epsilon \in \overline{\Lambda} \) such that \( u_\epsilon(x_\epsilon) = \sup_{x \in \Lambda} u_\epsilon(x) \), then
\[
\lim_{\epsilon \to 0} V(x_\epsilon) = \min_{\Lambda} V(x)
\]
and there exists an $\alpha \in (2s/(p-2), N-2s)$ such that

$$u_\epsilon(x) \leq \frac{C \epsilon^\alpha}{\epsilon^\alpha + |x - x_\epsilon|^{\alpha}},$$

where $C$ is positive constant.

Our argument in proving Theorem 1.1 also works well to problem 1.1 with the nonlinear term $"|u|^{p-2}u"$ being replaced by more general nonlinear term "$f(u)"$, see Section 5 for details.

Note that any decay rate of $V$ even the case that $V$ is compact supported are admissible in our results. Hence we solve the fractional Ambrosetti-Malchiodi conjecture (C) completely.

A difficulty of this paper is that $H_{V,\epsilon}^s(\mathbb{R}^N) \not\subset L^p(\mathbb{R}^N)$ when $V$ has compact support. To overcome it, the usual method is to take the penalized ideas to cut off the nonlinearity. The first creation of this method is in [14–16], where equations (1.1) with local case $s = 1$ and nonvanishing case $\inf_{\mathbb{R}^N} V > 0$ were considered. Successively, also in the local case $s = 1$, it was developed into the vanishing case $\lim_{|x| \to \infty} V(x) = 0$, see [6, 27, 28] and the references therein for example. Recently, this method in the nonlocal case $0 < s < 1$ and vanishing case $\lim_{|x| \to \infty} V(x)|x|^{2s} > 0$ was established in [5].

We truncate the nonlinearities by a special function (see (2.1)) and then get a penalized solution $u_\epsilon$ (Lemma 2.4). There are two difficulties in proving $u_\epsilon$ solves the origin problem (1.1). Firstly, we need to linearize penalized equation (2.2) that comes from the concentration phenomenon in Lemma 3.3. But, for the concentration phenomenon, the most important thing is establishing the lower bounds of energy (Lemma 3.1). Actually, the problem is still nonlocal after truncating, which makes us have to the the global $L^2$-norm information of $u_\epsilon$. Secondly, we need to construct super-solutions to the linearized equation (4.1). This steps is very difficult since for a function $f$ one can not compute $(-\Delta)^s f(0 < s < 1)$ as precise as $-\Delta f$. The global $L^2$-norm information of $u_\epsilon$ can not be obtained by using the term $\int_{\mathbb{R}^N} V(x)|u_\epsilon|^2 < +\infty$ like before (see [5] for example). Skillfully, we obtain it by using the fractional Hardy inequality in [21], which says that there exists a positive constant $C_{N,s}$ such that

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \leq C_{N,s} |(-\Delta)^{s/2}|^2 \quad \forall u \in \dot{H}^s(\mathbb{R}^N),$$

(1.4)

see Remark 3.2 for more details. The solvability of the second difficulty above is also based on inequality (1.4), see the construction in Section 4 for more details.

The paper is organized as follows: in Section 2 we establish the penalized scheme and obtain a penalized solution $u_\epsilon$. In Section 3 we study the concentration phenomenon of $u_\epsilon$. In Section 4, we prove the penalized solution $u_\epsilon$ solves the origin problem by constructing a special penalized function. In Section 5, we give a short proof to Theorem 1.1 that with more general nonlinear term.
2. The penalized problem

In this section, we establish the penalized scheme, which is to cut off the nonlinear term “$|u|^{p-2}u$” by a suitable function.

We first introduce the fractional version of Sobolev embedding theorem.

**Proposition 2.1.** (Fractional version of the Gagliardo–Nirenberg inequality) For every $u \in H^s(\mathbb{R}^N)$,

$$
\|u\| \leq C\|(-\Delta)^{s/2} u\|^\beta_2 \|u\|^{1-\beta},
$$

where $q \in [2, 2_s^*)$ and $\beta$ satisfies $\frac{\beta}{2} + \frac{(1-\beta)}{2} = \frac{1}{q}$.

**Remark 2.2.** The above inequality implies that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$. Moreover, on bounded set, the embedding is compact (see [18]), i.e.,

$$
H^s(\mathbb{R}^N) \subset \subset L^q_{\text{loc}}(\mathbb{R}^N) \text{ compactly, if } q \in [1, 2_s^*].
$$

According to the fractional Hardy inequality [14], we choose a family of penalized potentials $\mathcal{P}_\epsilon \in L^\infty(\mathbb{R}^N, [0, \infty))$ for $\epsilon > 0$ small in such a way that

$$
\mathcal{P}_\epsilon(x) = 0 \text{ for all } x \in \Lambda \text{ and } \lim_{\epsilon \to 0} \sup_{x \in \mathbb{R}^N \setminus \Lambda} \mathcal{P}_\epsilon(x) \epsilon^{-(2s+3s/2)} |x|^{2s+\kappa} = 0.
$$

(2.1)

where $\kappa > 0$ is a small parameter. Note that when $\epsilon > 0$ is small enough, we will have

$$
\int_{\mathbb{R}^N} |(-\Delta)^s u|^2 - \int_{\mathbb{R}^N} \epsilon^{-2s} \mathcal{P}_\epsilon(x)|u|^2
\geq \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 - \epsilon^{-2s} \sup_{x \in \mathbb{R}^N \setminus \Lambda} (\mathcal{P}_\epsilon(x)|x|^{2s}) \int_{\mathbb{R}^N} |u|^2/|x|^{2s}
\geq \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}, \quad \forall u \in \dot{H}^s(\mathbb{R}^N),
$$

where $C_{N,s}$ is the constant in (1.4). Such kinds of estimate will be largely involved in subsequent proofs.

With the prescribed penalized function $\mathcal{P}_\epsilon$ at hand, we define the penalized nonlinearity $g_\epsilon : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$
g_\epsilon(x, s) := \chi_\Lambda(x) s_+^{p-1} + \chi_{\mathbb{R}^N \setminus \Lambda}(x) \min(s_+^{p-1}, \mathcal{P}_\epsilon(x)s_+).
$$

We denote $G_\epsilon(x, t) = \int_0^t g_\epsilon(x, s) ds$.

Following, we define the penalized superposition operators $g_\epsilon$ and $G_\epsilon$ as

$$
g_\epsilon(u)(x) = g_\epsilon(x, u(x)) \text{ and } G_\epsilon(u)(x) = G_\epsilon(x, u(x)).
$$

Following, we define the penalized functional $J_\epsilon : H^s_{V,\epsilon}(\mathbb{R}^N) \to \mathbb{R}$ as

$$
J_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (e^{2s}|(-\Delta)^{s/2} u|^2 + V(x)|u|^2) - \int_{\mathbb{R}^N} G_\epsilon(u).
$$

The Hardy inequality [14] implies that $J_\epsilon$ is well-defined on $H^s_{V,\epsilon}(\mathbb{R}^N)$. Now we prove that $J_\epsilon$ is $C^1$ and satisfies the (P.S.) condition.
Lemma 2.3. $J_\epsilon : H^s_{V,\epsilon}(\mathbb{R}^N) \to \mathbb{R}$ is $C^1$ and satisfies (P.S.) condition.

Proof. We only need to show the nonlinear term

$$I_\epsilon(u) = \int_{\mathbb{R}^N} G_\epsilon(u)$$

is $C^1$. Noting that for every $\varphi \in C^\infty_c(\mathbb{R}^N)$ and $0 < |t| < 1$, we have

$$|G_\epsilon(u + t\varphi) - G_\epsilon(u)|/t \leq C \left( (|u|^p + |\varphi|^p)\chi_A + |u|^2 + |\varphi|^2 \right) \in L^1(\mathbb{R}^N),$$

then the existence of first order Gateaux derivative follows by Dominated Convergence Theorems. Let $(u_n)$, $u \in H^s_{V,\epsilon}(\mathbb{R}^N)$ satisfy $u_n \to u \in H^s_{V,\epsilon}(\mathbb{R}^N)$. By Remark 2.2, the construction of $P_\epsilon$, (1.4) and Dominated Convergence Theorem, we deduce that for $\varphi \in H^s_{V,\epsilon}(\mathbb{R}^N)$ with $||\varphi||_{H^s_{V,\epsilon}(\mathbb{R}^N)} \leq 1$,

$$|\langle I_\epsilon'(u_n) - I_\epsilon'(u), \varphi \rangle|$$

$$\leq C||u_n - u||^p_{H^s_{V,\epsilon}(\mathbb{R}^N)} + \int_{\mathbb{R}^N \setminus A} \left| \min\{P_\epsilon, (u_n)_{p-2}^+\}u_n \varphi - \min\{P_\epsilon, u^p_{+}\}u \varphi \right| dx$$

$$\leq o_n(1) + \int_{\mathbb{R}^N \setminus A} \left| \min\{P_\epsilon, (u_n)_{p-2}^+\}(u_n - u) \varphi \right|$$

$$+ \left| \min\{P_\epsilon, (u_n)_{p-2}^+\} - \min\{P_\epsilon, u^p_{+}\} \right| |u\varphi| dx$$

$$\leq o_n(1) + \int_{\mathbb{R}^N \setminus A} \left| \min\{P_\epsilon, (u_n)_{p-2}^+\} - \min\{P_\epsilon, u^p_{+}\} \right| |u|^2 dx$$

$$= o_n(1).$$

Now we prove that $J_\epsilon$ satisfies (P.S.) condition, i.e., to prove that any sequence $(u_n) \in H^s_{V,\epsilon}(\mathbb{R}^N)$ satisfying

$$J_\epsilon(u_n) \to c, \quad J_\epsilon'(u_n) \to 0$$

is relatively compact.

It is standard to verify using the fact $p > 2$ and the construction of $P_\epsilon$ that $(u_n)$ is bounded in $H^s_{V,\epsilon}(\mathbb{R}^N)$. By Remark 2.2 one has thus $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$.

On the other hand, one has, for every $\sigma > 0$, by the fractional Hardy inequality in (1.4), for $R > 0$ large enough,

$$\int_{\mathbb{R}^N \setminus B_R(0)} |P_\epsilon(x)||u_n|^2 = \sup_{x \in \mathbb{R}^N \setminus B_R(0)} (C_{N,s}^{-1} P_\epsilon(x)||x|^{2s}C_{N,s}) \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|u_n|^2}{|x|^{2s}}$$

$$\leq \sup_{x \in \mathbb{R}^N \setminus B_R(0)} (P_\epsilon(x)||x|^{2s}|(-\Delta)^{s/2}u_n|^2)$$

$$\leq \frac{C}{R^\kappa}$$

$$< \sigma.$$
Hence
\[
\limsup_{n \to \infty} \|u_n - u\|_{H^s_{V,\epsilon}(\mathbb{R}^N)}^2 = \limsup \left( \langle J'_\epsilon(u_n) - J'_\epsilon(u), u_n - u \rangle \right)
+ \int_{\mathbb{R}^N} (g_\epsilon(u_n) - g_\epsilon(u))(u_n(x) - u(x))dx
\leq \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N \setminus B_R(0)} \mathcal{P}_\epsilon(|u_n| + |u|^2)dx \right)^{\frac{1}{2}}
\leq \sigma.
\]
Since \(\sigma > 0\) is arbitrary, this proves the lemma. \(\square\)

From Lemma 2.3, we have

**Lemma 2.4.** The Mountain Pass value
\[
c_\epsilon := \inf_{\gamma \in \Gamma_\epsilon} \max_{t \in [0, 1]} J_\epsilon(\gamma(t))
\]
can be achieved by a positive function \(u_\epsilon\) satisfying the following penalized problem:
\[
\epsilon^{2s}(-\Delta)^s u + Vu = g_\epsilon(u),
\]
where
\[
\Gamma_\epsilon := \{ \gamma \in C([0, 1], H^s_{V,\epsilon}(\mathbb{R}^N)) \mid \gamma(0) = 0, J_\epsilon(\gamma(1)) < 0 \}.
\]

**Proof.** It is easy to check that \(J_\epsilon\) owns Mountain Pass geometry, this and lemma 2.3 imply \(c_\epsilon\) can be achieved by a nonnegative function \(u_\epsilon\). By the regularity argument in [20], \(u_\epsilon\) is \(C^{1,\alpha}\) for some \(\alpha \in (0, 1)\). Hence, we can conclude by contradiction that \(u_\epsilon\) is positive. \(\square\)

## 3. Concentration phenomena of penalized solutions

In this section, we prove that the penalized solutions \(u_\epsilon\) obtained by Lemma 2.4 will concentrate at the local minimum of \(\Lambda\) as \(\epsilon \to 0\). This plays an essential role in linearizing the penalized equation (2.2) see (4.1) below.

We first give the lower estimate on energy.

**Lemma 3.1.** Let \((\epsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers converging to 0, \((u_n)\) be a sequence of critical points given by Lemma 2.4 and for \(j \in 1, 2, \cdots, k\), \((x_n^j)\) be a sequence in \(\mathbb{R}^N\) converging to \(x_i^j \in \mathbb{R}^N\). If
\[
\limsup_{n \to \infty} \frac{1}{\epsilon_n^N} \int_{\mathbb{R}^N} \epsilon_n^{2s}|(-\Delta)^{s/2} u_n|^2 + |V u_n|^2 < \infty,
\]
\[
V(x_i^j) > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{|x_i^j - x_j^j|}{\epsilon_n} = \infty \quad i \neq j \quad \text{for all} \ i, j = 1, 2, \cdots, k,
\]

and for some $\rho > 0$, 
\[
\liminf_{n \to \infty} \| u_n \|_{L^\infty(B_{\epsilon_n}(x_n^j))} > 0 \quad j = 1, 2, \ldots k,
\]
then $x_*^j \in \Lambda$ and 
\[
\liminf_{n \to \infty} \frac{J_{\epsilon_n}(u_n)}{\epsilon_n^N} \geq \sum_{j=1}^k C(V(x_*^j)),
\]
where $C(a)$ with $a > 0$ is the ground energy of the equation $(-\Delta)^s u + au = |u|^{p-2}u$.

Note that $C(\cdot) : (0, +\infty) \to (0, +\infty)$ is continuous and increasing.

In the proof of this lemma, we will omit some tedious details that are similar to Proposition 3.4 in [5]. We will pay more attention on the difference caused by the nonlocal operator $(-\Delta)^s$ and the vanishing of $V$ (compact support), see (3.3) for more details.

**Proof.** For $j \in \{1, \ldots, k\}$, the rescaling function $v_n^j \in H^s_{loc}(\mathbb{R}^N)$ defined by 
\[
v_n^j(y) = u_n(x_n^j + \epsilon_n y)
\]
satisfies weakly the rescaled equation 
\[
(-\Delta)^s v_n^j + V_n^j v_n^j = g_n^j(v_n^j) \text{ in } \mathbb{R}^N,
\]
where $V_n^j(y) = V(x_n^j + \epsilon_n y)$, $g_n^j(v_n^j) = g_{\epsilon_n}(x_n^j + \epsilon_n y, v_n^j)$. By the assumption that $\sup_{n \in N} \| u_n \|_{H^s_{loc}(\mathbb{R}^N)} < +\infty$, the sequence $(v_n^j)_{n \geq 1}$ is bounded in $H^s_{loc}(\mathbb{R}^N)$. By the regularity assertion in Appendix D of [2], the Liouville-type Lemma in Lemma 3.3 of [5], we conclude that there exist $x_*^j \in \Lambda$, $v_*^j \in H^s(\mathbb{R}^N) \setminus \{0\}$, $v_*^j$ is nonnegative such that $v_n^j \to v_*^j$ in $H^s_{loc}(\mathbb{R}^N) \cap C^1_{loc}(\mathbb{R}^N)$, $x_n^j \to x_*^j$ and 
\[
(-\Delta)^s v_*^j + V(x_*^j)v_*^j = (v_*^j)^{p-1}.
\]
Since $v_*^j \geq 0$, we see 
\[
\liminf_{n \to \infty} \frac{1}{\epsilon_n^N} \int_{B_{\epsilon_n}(x_n^j)} \frac{1}{2} (\epsilon_{2n}^s |(-\Delta)^s u_n|^2 + V|u_n|^2) - G_{\epsilon_n}(u_n) \geq C(V(x_*^j)) - C \int_{\mathbb{R}^N \setminus B_R} \frac{|(-\Delta)^s v_*^j|^2 + V(x_*^j)|v_*^j|^2},
\]
where $C > 0$ is a constant.

In order to study the integral outside $B_{\epsilon_n}(x_n^j)$, we choose $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta = 0$ on $B_1$ and $\eta = 1$ on $\mathbb{R}^N \setminus B_2$. Define 
\[
\psi_{n,R}(x) = \prod_{j=1}^k \eta \left( \frac{x - x_n^j}{\epsilon_n R} \right).
\]
Since $u_n$ is a solution to the penalized problem (2.2), we have by taking $\psi_{n,R} u_n$ as a test function in the penalized problem (2.2) that 
\[
\int_{\mathbb{R}^N \setminus \bigcup_{j=1}^k B_{\epsilon_n}(x_n^j)} \epsilon_{2n}^s \psi_{n,R} |(-\Delta)^s u_n|^2 + V \psi_{n,R} |u_n|^2
\]
& Xiaoming An, Lipeng Duan* and Yanfang Peng

Following, we have

\[ \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \mathbf{g}_{\epsilon_n}(u_n)u_n \psi_{n,R} \]

\[ - \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \int_{\mathbb{R}^N} u_n(y)(\psi_{n,R}(x) - \psi_{n,R}(y))(u_n(x) - u_n(y))|x - y|^{N+2s} dy \]

\[ := \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \mathbf{g}_{\epsilon_n}(u_n)u_n \psi_{n,R} + R_n. \]

Hence

\[ \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \frac{1}{2} (\epsilon_n 2^{s} |(-\Delta)^{s/2}u_n|^{2} + V|u_n|^{2}) - \mathbf{G}_{\epsilon_n} \]

\[ \geq \frac{1}{2} \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \epsilon_n^{2s} \psi_{n,R}(x) (-\Delta)^{s/2}u_n|^{2} + V \psi_{n,R}|u_n|^{2} - \mathbf{g}_{\epsilon_n}(u_n)u_n \]

\[ = - \frac{\epsilon_n^{2s}}{2} R_n + \int_{\mathbb{R}^N \setminus \bigcup_{j=1}^{k} B_{\epsilon_n R}(x'_n)} \mathbf{g}_{\epsilon_n}(u_n)(\psi_{n,R} - 1)u_n. \]

By scaling, \( B_{\epsilon_n R}(x'_n) \cap B_{\epsilon_n R}(x''_n) = \emptyset \) if \( n \) is large enough. Hence, by the fact that \( v_n^j \to v_s^j \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \), we have

\[ \lim \sup_{n \to \infty} \frac{1}{2} \sum_{j=1}^{k} \int_{B_{2R} \setminus B_{R}} \mathbf{g}_{n}^j(v_n^j)v_n^j = \frac{1}{2} \sum_{j=1}^{k} \int_{B_{2R} \setminus B_{R}} (v_s^j)^p = o_R(1). \]

Now we estimate \( R_n \). A change of variable tells us

\[ R_n = \int_{\mathbb{R}^N} u_n(y)dy \int_{\mathbb{R}^N} (u_n(x) - u_n(y))(\psi_{n,R}(x) - \psi_{n,R}(y))|x - y|^{N+2s} dx \]

\[ = \epsilon_n^{N-2s} \sum_{l=1}^{k} \int_{\mathbb{R}^N} v_n^l(y) \beta_n^l(y)dy \int_{\mathbb{R}^N} \alpha_n^l(x)(v_n^l(x) - v_n^l(y))(\eta_{R}(x) - \eta_{R}(y))|x - y|^{N+2s} dx, \]

where the function \( \beta_n^l \) and \( \alpha_n^l \) are defined skillfully as

\[ \beta_n^l(y) = \prod_{s=0}^{l-1} \eta \left( \frac{y}{\epsilon_n R} + \frac{x_s^l - x_s^s}{\epsilon_n R} \right), \quad \beta_n^l(y) \equiv 1 \]

and

\[ \alpha_n^l(x) = \prod_{s=l+1}^{k} \eta \left( \frac{x}{\epsilon_n R} + \frac{x_s^l - x_s^s}{\epsilon_n R} \right), \quad \alpha_n^l(x) \equiv 1. \]

Following, we have

\[ \epsilon_n^{2s-N} R_n = \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \beta_n^l(y)dy \int_{B_{2R}} \alpha_n^l(x)(v_n^l(x) - v_n^l(y))(\eta_{R}(x) - \eta_{R}(y))|x - y|^{N+2s} dx. \]
By the choice of $\eta$ and 
\[
\lim_{n \to \infty} \frac{\alpha_n^l - \alpha_n^s}{\epsilon_n} = \infty \text{ if } l \neq s, \text{ for } n \text{ large},
\]

\[
R_n^{(1)} = \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y)) (1 - \eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx
\]
\[
+ \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y)) (1 - \eta_R(y))}{|x - y|^{N+2s}} dx
\]
\[
= \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y)) (1 - \eta_R(y))}{|x - y|^{N+2s}} dx
\]
\[
+ \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y))}{|x - y|^{N+2s}} dx
\]
\[
:= R_n^{(11)} + R_n^{(12)}
\]

and

\[
R_n^{(2)} = \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y)) (\eta_R(x) - 1)}{|x - y|^{N+2s}} dx
\]
\[
- \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{\alpha_n^l(x)(v_n^l(x) - v_n^l(y))}{|x - y|^{N+2s}} dx
\]
\[
= \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{(v_n^l(x) - v_n^l(y)) (\eta_R(x) - 1)}{|x - y|^{N+2s}} dx
\]
\[
- \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) \alpha_n^l(y) dy \int_{B_{2R}} \frac{v_n^l(x) - v_n^l(y)}{|x - y|^{N+2s}} dx
\]
\[
:= R_n^{(21)} + R_n^{(22)}
\]

Also, for large $n$,

\[
R_n^{(3)} = \sum_{l=1}^{k} \int_{B_{2R}} v_n^l(y) dy \int_{B_{2R}} \frac{(v_n^l(x) - v_n^l(y)) (\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx
\]
+ \sum_{l=1}^{k} \int_{B^c_{2R} \setminus B_R} v^l_n(y) dy \int_{B^c_{2R} \setminus B_R} \frac{(v^l_n(x) - v^l_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx\n
+ \sum_{l=1}^{k} \int_{B^c_{2R} \setminus B_R} v^l_n(y) dy \int_{B^c_{2R} \setminus B_R} \frac{(v^l_n(x) - v^l_n(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx

:= R_n^{(31)} + R_n^{(32)} + R_n^{(33)}.

For \(|R_n^{(i2)}|, i = 1, 2\), by Remark 2.3 we have

\limsup_{n \to \infty} |R_n^{(i2)}| \leq CR^{-2s} + \limsup_{n \to \infty} 2 \sum_{l=1}^{k} \int_{B^c_{2R}} dy \int_{B_R} \frac{(v^l_n(y))^2}{|y - R|^N} dy.

Using the fractional Hardy inequality in (13) and letting \(	ilde{R} = R^{\frac{N+1}{N-1}}\), we have

\limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 dy \int_{B_R} \frac{1}{|x - y|^{N+2s}} dx

\leq C \limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 dy \frac{R^N}{|y|^{N+2s}} dy

\leq C \limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 dy \frac{R^N}{|y|^{N+2s}} dy + C \limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 \frac{R^N}{|y|^{N+2s}} dy

\leq C \limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 dy + C \limsup_{n \to \infty} \int_{B^c_{2R}} (v^l_n(y))^2 \frac{R^N}{|y|^{2s} |y|^N} dy

\leq C \int_{B^c_{2R}} (v^l_n(y))^2 dy + \frac{C}{R}

= o_R(1).

For \(R_n^{(11)}\), by the estimates of \(R_n^{(i2)}\), we have

\limsup_{n \to \infty} |R_n^{(11)}| \leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B^c_{2R} \setminus B_R} dy \int_{B^c_{2R} \setminus B_R} \frac{|v^l_n(x) - v^l_n(y)|^2}{|x - y|^{N+2s}} dx

+ \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B^c_{2R} \setminus B_R} (v^l_n(y))^2 dy \int_{B^c_{2R}} \frac{(1 - \eta_R(y))^2}{|x - y|^{N+2s}} dx

\leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B^c_{2R} \setminus B_R} dy \int_{B^c_{2R} \setminus B_R} \frac{|v^l_n(x) - v^l_n(y)|^2}{|x - y|^{N+2s}} dx
\[ + \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} dy \int_{B_{4R}^{2,2l}} \frac{|v_n^l(x) - v_n^l(y)|^2}{|x - y|^{N+2s}} dx + C \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} (v_n^l(y))^2 dy \]
\[ \leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} dy \int_{B_{4R}^{2,2l}} \frac{|v_n^l(x) - v_n^l(y)|^2}{|x - y|^{N+2s}} dx + o_R(1) \]
\[ \leq C \int_{B_{2R} \setminus B_R} dy \int_{R^N} \frac{|v_n^l(x) - v_n^l(y)|^2}{|x - y|^{N+2s}} dx + o_R(1). \]

Similarly, we get
\[ \limsup_{n \to \infty} |R_n^{(21)}| \leq o_R(1) \]
and
\[ \limsup_{n \to \infty} |R_n^{(31)}| + |R_n^{(32)}| + |R_n^{(33)}| \leq o_R(1). \]

Therefore
\[ \epsilon^{2s-N} |R_n| \leq o_R(1). \quad (3.5) \]

Letting \( R \to \infty \), we have
\[ \liminf_{n \to \infty} \frac{\mu_n(u_n)}{\epsilon_n^N} \geq \sum_{j=1}^{k} C(x_j^*). \]

**Remark 3.2.** The big difference when \( V \) has compact support embodies in the estimate of (3.3). One on one hand, the nonlocal effect make us have to obtain the \( L^2 \) estimate of \( v_n^l(y) \). But, here the potential \( V \) may has support, which makes the potential cannot offer the \( L^2 \) information anymore. Thanks to the fractional Hardy inequality in [21], we multiply \( v_n^l(y) \) by \( |y|^{-2s} \) and then obtain our estimate by choosing special \( \tilde{R} \).

With the lemma above at hand, we are going to prove that the penalized solution \( u_\epsilon \) in \( 2.4 \) concentrating at local minimum of \( V \) as \( \epsilon \to 0 \).

**Lemma 3.3.** Let \( \rho > 0 \). There exists a family of points \( \{x_\epsilon\} \subset \Lambda \) such that

(i) \( \liminf_{\epsilon \to 0} \|u_\epsilon\|_{L^\infty(B_{\rho}(x_\epsilon))} > 0 \),

(ii) \( \lim_{\epsilon \to 0} V(x_\epsilon) = \inf_{\Lambda} V(x) \),

(iii) \( \lim_{R \to \infty} \|u_\epsilon\|_{L^\infty(U \setminus B_{\epsilon R}(x_\epsilon))} = 0 \).

**Proof.** It is easy to verify using the construction of \( P_\epsilon \) that
\[ \liminf_{\epsilon \to 0} \|u_\epsilon\|_{L^\infty(\Lambda)} > 0. \]

Then by the regularity assertion in Appendix D of [20], we get the existence of \( x_\epsilon \in \bar{\Lambda} \). We assume that \( x_\epsilon \to x_* \).
By Lemma 3.1 it holds
\[ \liminf_{\epsilon \to 0} \frac{J_\epsilon(u_\epsilon)}{\epsilon^N} \geq C(V(x_*)), \]

But, it is easy to verify that
\[ \inf_{\Lambda} C(V(x)) \geq \limsup_{\epsilon \to 0} \frac{J_\epsilon(u_\epsilon)}{\epsilon^N}. \]

Hence by the monotonicity of \( C(\cdot) \), \( V(x_*) = \inf_{\Lambda} V(x) \).

Arguing by contradiction, if \((iii)\) is false, we will have
\[ \inf_{\Lambda} C(V(x)) \geq \liminf_{\epsilon \to 0} \frac{J_\epsilon(u_\epsilon)}{\epsilon^N} \geq 2 \inf_{\Lambda} C(V(x)), \]

which is a contradiction.

As a result, we complete the proof of this lemma.

\( \square \)

4. Back to the original problem

In this section we prove that \( u_\epsilon^p - 2 \leq P_\epsilon \) on \( \mathbb{R}^N \setminus \Lambda \) via comparison principle. Using the assumption on \( P_\epsilon \), the concentration phenomenon in Lemma 3.3 and the regularity assertion in [20], we can easily get the following linearized equation:

**Proposition 4.1.** For \( \epsilon > 0 \) small enough and \( \delta \in (0,1) \), there exist \( R > 0 \), \( x_\epsilon \in \Lambda \), such that

\[
\begin{cases}
\epsilon^2 (-\Delta)^s u_\epsilon + (1 - \delta) V u_\epsilon \leq P_\epsilon(x) u_\epsilon(x), & \text{in } \mathbb{R}^N \setminus B_R(x_\epsilon), \\
u_\epsilon \leq C_\infty & \text{in } \Lambda.
\end{cases}
\] (4.1)

Our next aim is to construct a suitable sup-solution to the linearized equation above. For the sake of intuitive, we convert the equation (4.1) as follows. Letting \( v(x) = u_\epsilon(\epsilon x + x_\epsilon) \), it is easy to check that

\[
\begin{cases}
(-\Delta)^s v_\epsilon + (1 - \delta) V_\epsilon(x) v_\epsilon \leq \tilde{P}_\epsilon(x) u_\epsilon(x), & \text{in } \mathbb{R}^N \setminus B_R(0), \\
v_\epsilon \leq C_\infty & \text{in } B_R(0),
\end{cases}
\] (4.2)

where \( V_\epsilon(\cdot) = u_\epsilon(\epsilon \cdot + x_\epsilon) \) and \( \tilde{P}_\epsilon(\cdot) = P_\epsilon(\epsilon \cdot + x_\epsilon) \).

Now we construct sup-solution to (4.2). Let \( \tilde{\eta} : \mathbb{R}^N \to [0,1] \) be a smooth decreasing function such that \( \tilde{\eta}_{\beta} \equiv 1 \) on \([-1,1]\) and \( \tilde{\eta}_{\beta} \equiv 0 \) on \((-\infty, -1 - \beta) \cup (1 + \beta, +\infty)\), where \( \beta > 0 \) is a small parameter. Define \( \eta_{\beta}(|x|) = \tilde{\eta}_{\beta}(|x|) \). Note that the special choice of \( \eta \) plays a key role in the proof of the following Proposition 4.2, see (4.4) below.

Setting \( 0 < \alpha < N - 2s \) and \( f_\alpha^\beta(x) = \eta_\beta(x) \frac{1}{|x|^\alpha} + (1 - \eta_\beta(x)) \frac{1}{|x|^\alpha} \), we have

**Proposition 4.2.** Let \( \epsilon > 0 \) be small enough. Then for every \( x \in \mathbb{R}^N \setminus B_R(0) \), it holds

\[
(-\Delta)^s f_\alpha^\beta + V_\epsilon(x) f_\alpha^\beta - \tilde{P}_\epsilon(x) f_\alpha^\beta \geq 0.
\] (4.3)
Proof. For \( x \in \mathbb{R}^N \setminus B_{2R}(0) \), letting \( \beta > 0 \) be small enough, we have

\[
(-\Delta)^{s} f_{\alpha}^\beta(x) = \int_{B_R(0)} \frac{|x|^{-\alpha} - R^{-\alpha}}{|x-y|^{N+2s}} \, dy + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|x|^{-\alpha} - |y|^{-\alpha}}{|x-y|^{N+2s}} \, dy + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{\eta(y)|y|^{-\alpha} - \eta(y)R^{-\alpha}}{|x-y|^{N+2s}} \, dy
\]

\[
= (-\Delta)^{s}((1 \cdot |^\alpha)(x) + \int_{B_R(0)} \frac{|y|^{-\alpha} - R^{-\alpha}}{|x-y|^{N+2s}} \, dy + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{\eta(y)|y|^{-\alpha} - \eta(y)R^{-\alpha}}{|x-y|^{N+2s}} \, dy
\]

\[
\geq f_{\alpha}^\beta(x) \left( \int_{(B_1(0))^c} \frac{1-|z|^{\alpha}}{|z|^{N+2s}|z-c_1|^{N+2s}} \, dz + \int_{(B_1(0))^c} \frac{|z|^{\alpha} - 1}{|z|^{N+2s}|z-c_1|^{N+2s}} \, dz \right) + R^{N-\alpha} \left( \int_{B_1(0)} \frac{|y|^{-\alpha} - 1}{|x-yR|^{N+2s}} \, dy \right)
\]

\[
\geq C_{\alpha} f_{\alpha}^\beta(x) \frac{1}{|x|^{2s}}
\]

When \( x \in B_{2R}(0) \setminus B_R(0) \), by the construction of \( \eta_\beta \) and the computation above, we have

\[
(-\Delta)^{s} f_{\alpha}^\beta(x) = \int_{\mathbb{R}^N} \frac{(|x|^{-\alpha} - |y|^{-\alpha}) + \eta_\beta(y)(|y|^{-\alpha} - |x|^{-\alpha})}{|x-y|^{N+2s}} \, dy
\]

\[
+ \int_{\mathbb{R}^N} \frac{(\eta_\beta(x) - \eta_\beta(y))R^{-\alpha} + (\eta_\beta(y) - \eta_\beta(x))|x|^{-\alpha}}{|x-y|^{N+2s}} \, dy
\]

\[
= (R^{-\alpha} - |x|^{-\alpha}) \int_{\mathbb{R}^N} \frac{\eta_\beta(x) - \eta_\beta(y)}{|x-y|^{N+2s}} \, dy + \int_{\mathbb{R}^N} \frac{\eta_\beta(y)(|y|^{-\alpha} - |x|^{-\alpha})}{|x-y|^{N+2s}} \, dy
\]

\[
= (R^{-\alpha} - |x|^{-\alpha}) \frac{(-\Delta)^{s} \tilde{\eta}_\beta(R/x)}{R^{2s}} + \int_{\mathbb{R}^N} \frac{\eta_\beta(y)(|y|^{-\alpha} - |x|^{-\alpha})}{|x-y|^{N+2s}} \, dy
\]

\[
\geq -C_\beta R^{-(\alpha+2s)} + K_\alpha^\beta,
\]

where \( C_\beta \) is a positive constant depending only on \( \beta \).

For \( K_\alpha^\beta \), by Change-Of-Variable Theorem and the decreasing of \( \tilde{\eta}_\beta \), we have

\[
|x|^{\alpha+2s} K_\alpha^\beta = \int_{B_1(0)} \frac{\eta_\beta(|x|)(|y|^{-\alpha} - 1)}{|y-c_1|^{N+2s}} \, dy + \int_{B_1(0) \cap B_{1+\beta}(0)} \frac{\eta_\beta(|x|)(|y|^{-\alpha} - 1)}{|y-c_1|^{N+2s}} \, dy
\]

\[
\geq \int_{B_1(0)} \frac{\eta_\beta(|x|)(|y|^{-\alpha} - 1)}{|y-c_1|^{N+2s}} \, dy + \int_{B_1(0) \cap B_{1+\beta}(0)} \frac{\eta_\beta(|x|)(|y|^{-\alpha} - 1)}{|y-c_1|^{N+2s}} \, dy
\]

\[
\geq \eta_\beta(|x|) \int_{B_1(c_1)} \frac{(|y|^{-\alpha} - 1)}{|y-c_1|^{N+2s}} \, dy
\]

\[
\geq \tilde{C}_\alpha,
\]

where \( \tilde{C}_\alpha \) is a constant depending only on \( \alpha \). Noting that in the special case that \( \alpha = N-2s \), we have \( \tilde{C}_{N-2s} > 0 \).
Now by the computation above, we conclude that if \( R \) is large enough and \( \epsilon \) is small enough, there hold
\[
(-\Delta)^s f_{\alpha}^\beta + V_\epsilon(x)f_{\alpha}^\beta - \tilde{P}_\epsilon(x)f_{\alpha}^\beta \\
\geq \left\{
\begin{array}{ll}
-C_R \alpha R^{-\alpha-2s} + \left( \inf_{x \in \Lambda} V(x) \right) f_{\alpha}^\beta(x), & x \in B_{2R}(0) \setminus B_R(0), \\
C_{\alpha |x|^{-2s}} + V_\epsilon(x)f_{\alpha}^\beta - \tilde{P}_\epsilon(x)f_{\alpha}^\beta, & x \in \mathbb{R}^N \setminus B_{2R}(0),
\end{array}
\right.
\]
\[
\geq 0
\]
for every \( x \in \mathbb{R}^N \setminus B_R(0) \), where \( \Lambda = \{ x : \epsilon x + x_\epsilon \in \Lambda \} \). This completes the proof.

At the last of this section, we prove Theorem 1.1

**Proof of Theorem 1.1**

Let
\[
\begin{align*}
\alpha & \in \left( \frac{2s}{p-2}, N-2s \right), \quad \kappa = \frac{\alpha(p-2)-2s}{4}, \\
\mathcal{P}_\epsilon(x) & = \frac{\epsilon^{2s+2s}}{|x|^{2s+2s}} x_{\mathbb{R}^N \setminus \Lambda}(x),
\end{align*}
\]
where the function \( f_{\alpha}^\beta \) is that in Proposition 1.2.

It is easy to check that \( \mathcal{P}_\epsilon \) satisfies the assumption \((2.1)\)

By Proposition 1.2, letting the constant \( C > 0 \) above be large enough and \( \tilde{v}_\epsilon(x) = \mathcal{U}(x) - v_\epsilon(x) \), we have
\[
\begin{align*}
\begin{cases}
(-\Delta)^s \tilde{v}_\epsilon(x) + V_\epsilon(x)\tilde{v}_\epsilon(x) - \tilde{P}_\epsilon(x)\tilde{v}_\epsilon(x) \geq 0, & \text{in } \mathbb{R}^N \setminus B_R(0), \\
\tilde{v}_\epsilon(x) \geq 0 & \text{in } B_R(0).
\end{cases}
\end{align*}
\]
Then, since \( \tilde{v}_\epsilon \in H^s(\mathbb{R}^N) \) (when \( \alpha \) is closed to \( N-2s \) ), testing the equation above against with \( \tilde{v}_\epsilon^-(x) \), by the fractional Hardy inequality in \((1.3)\), we find \( \tilde{v}_\epsilon^-(x) = 0, \ x \in \mathbb{R}^N \). Hence \( \tilde{v}_\epsilon(x) \geq 0, \ x \in \mathbb{R}^N \). Especially, we have
\[
u_\epsilon(x) \leq \frac{C_{\epsilon\alpha}}{\epsilon^{\alpha} + |x - x_\epsilon|^{\alpha}}, \ \forall x \in \mathbb{R}^N.
\]
Moreover, it holds
\[
(\nu_\epsilon(x))^{p-2} \leq \mathcal{P}_\epsilon(x), \ \forall x \in \mathbb{R}^N \setminus \Lambda.
\]
As a result, \( \nu_\epsilon \) solves the origin problem. This completes the proof.

5. **Further results**

In this section, we will consider \((1.1)\) with general nonlinearity, i.e.,
\[
e^{2s}(-\Delta)^su + V(x)u = f(u),
\]
where the potential \( V(x) \) is the same as before, the nonlinearity \( f : \mathbb{R} \to \mathbb{R} \) is assumed to satisfy the following properties:
\[\begin{align*}
(f_1) & \ f \text{ is an odd function and } f(t) = o(t^{1+\tilde{c}}) \text{ as } t \to 0^+, \text{ where } \tilde{c} = \frac{\alpha + \nu}{2s + 2\kappa} > 0 \text{ with } \nu > 0 \\
& \text{ is a small parameter and } \kappa \text{ is the parameter in (4.6).}
\end{align*}\]
\((f_2)\) \(\lim_{t \to \infty} \frac{f(t)}{t^p} = 0\) for some \(1 < p < 2^*_s - 1.\)

\((f_3)\) There exists \(2 < \theta \leq p + 1\) such that

\[ 0 \leq \theta F(t) < f(t)t \text{ for all } t > 0 \]

where \(F(t) = \int_0^t f(\alpha)d\alpha.\)

\((f_4)\) The map \(t \mapsto \frac{f(t)}{t}\) is increasing on \((0, +\infty).\)

We have the following result which is same as Theorem 1.1.

**Theorem 5.1.** Let \(N > 2s, s \in (0, 1), p \in \left(2 + \frac{2s}{N-2s}, 2^*_s\right), f\) satisfy \((f_1)-(f_4)\) and \(V\) be the same as before. Then problem \((1.1)\) has a positive solution \(\hat{u}_\epsilon \in H^s_{V, \epsilon}(\mathbb{R}^N)\) if \(\epsilon > 0\) is small enough. Moreover, there exist \(\hat{x}_\epsilon \in \Lambda\) and an \(\alpha \in (\frac{2s}{p-2}, 2^*_s)\) such that

\[ \lim_{\epsilon \to 0} V(\hat{x}_\epsilon) = \min_{\Lambda} V(x) \]

and

\[ u_\epsilon(x) \leq \frac{Ce^\alpha}{e^\alpha + |x - \hat{x}_\epsilon|^{\alpha}}, \]

where \(C\) is positive constant.

**Proof.** We define the penalized nonlinearity as \(\hat{g}_\epsilon : \mathbb{R}^N \times \mathbb{R}\) as

\[ \hat{g}_\epsilon(x, s) := \chi_{\Lambda}f(s+) + \chi_{\mathbb{R}^N\setminus\Lambda} \min\{f(s+), P_\epsilon(x)s+\}. \]

In the sequel, we denote \(\hat{G}_\epsilon(x,t) = \int_0^t \hat{g}_\epsilon(x,s)ds\) and define the penalized superposition operators \(\hat{g}_\epsilon\) and \(\hat{G}_\epsilon\) as

\[ \hat{g}_\epsilon(u)(x) = \hat{g}_\epsilon(x, u(x)) \quad \text{and} \quad \hat{G}_\epsilon(u)(x) = \hat{G}_\epsilon(x, u(x)). \]

Accordingly, the penalized functional \(\hat{J}_\epsilon : H^s_{V, \epsilon}(\mathbb{R}^N) \to \mathbb{R}\) is given by

\[ \hat{J}_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (e^{2s}|(-\Delta)^{s/2}u|^2 + V(x)|u|^2) - \int_{\mathbb{R}^N} \hat{G}_\epsilon(u). \]

By conditions \((f_2)\) and \((f_3)\), we can verify using the same argument in the proof of Lemma 2.3 that \(\hat{J}_\epsilon\) is \(C^1\) and satisfies (P.S.) condition. Then by Lemma 2.4 one can show that the mountain pass value

\[ \hat{c}_\epsilon := \inf_{\gamma \in \hat{\Gamma}_\epsilon} \max_{t \in [0,1]} \hat{J}_\epsilon(\gamma(t)) \]

can be achieved by a positive function \(\hat{u}_\epsilon \in H^s_{V, \epsilon}\), where

\[ \hat{\Gamma}_\epsilon := \{ \gamma \in (C[0,1], H^s_{V, \epsilon}(\mathbb{R}^N)) : \gamma(0) = 0, \; \hat{J}_\epsilon(\gamma(1)) < 0 \}. \]

Furthermore, it holds

\[ \epsilon^{2s}(-\Delta)^s \hat{u}_\epsilon + V(x)\hat{u}_\epsilon = \hat{g}_\epsilon(\hat{u}_\epsilon). \quad (5.2) \]
From [19, 26] and the condition on $f$, we can let the ground states of the following limiting problem

$$(-\Delta)^s u + au = f(u)$$

be positive. Moreover, the least energy

$$\hat{C}(a) = \frac{1}{2} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + a|u|^2 \right) - \int_{\mathbb{R}^N} F(u)$$

is continuous and increasing. Then, by the same argument in Lemma 3.3 there exists a $\hat{x}_\epsilon \in \Lambda$ such that

$$(1') \liminf_{\epsilon \to 0} \|\hat{u}_\epsilon\|_{L^\infty(B_{\epsilon}(\hat{x}_\epsilon))} > 0,$$

$$(2') \lim_{\epsilon \to 0} V(\hat{x}_\epsilon) = \inf_{\Lambda} V,$$

$$(3') \lim_{R \to \infty} \lim_{\epsilon \to 0} \|\hat{u}_\epsilon\|_{L^\infty(U \setminus B_{\epsilon R}(\hat{x}_\epsilon))} = 0.$$  \tag{5.3}

As a result, using $(f_1)$ to linearize (5.2), we have,

$$\begin{cases}
\epsilon^{2s}(-\Delta)^s \hat{u}_\epsilon + (1 - \delta)V(\hat{x}_\epsilon) \hat{u}_\epsilon \leq P_\epsilon \hat{u}_\epsilon, & \text{in } \mathbb{R}^N \setminus B_{R_\epsilon}(x_\epsilon), \\
\hat{u}_\epsilon \leq \hat{C}_\infty & \text{in } \Lambda.
\end{cases} \tag{5.4}$$

Then, by the same argument in Section 4 we have

$$\hat{u}_\epsilon(x) \leq \frac{Ce^\alpha}{\epsilon^\alpha + |x - \hat{x}_\epsilon|^{\alpha}}, \forall x \in \mathbb{R}^N,$$

where $\alpha$ is the same as that in (4.6). Following, for every $x \in \mathbb{R}^N \setminus \Lambda$, since $\alpha \hat{\kappa} > 2s + 2\kappa$, we have

$$\frac{f(\hat{u}_\epsilon)}{\hat{u}_\epsilon} \leq (\hat{u}_\epsilon)^\hat{\kappa} \leq \frac{C_\epsilon^{\alpha \hat{\kappa}}}{|x|^{\alpha \hat{\kappa}}} \leq \frac{\epsilon^{2s + 2\kappa}}{|x|^{2s + \kappa}} = P_\epsilon(x).$$

As a result, we have $\hat{u}_\epsilon$ solves the origin problem (5.1). This completes the paper. □

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