Intrinsic entropy for generalized quasimetric semilattices

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Dedicated to the memory of Silvana Rinauro

Abstract

We introduce the notion of intrinsic semilattice entropy \( \tilde{h} \) in the category \( \mathcal{L}_{qm} \) of generalized quasimetric semilattices and contractive homomorphisms. By using appropriate categories \( \mathcal{X} \) and functors \( F : \mathcal{X} \to \mathcal{L}_{qm} \), we find specific known entropies \( \tilde{h}_X \) on \( \mathcal{X} \) as intrinsic functorial entropies, that is, as \( \tilde{h}_X = \tilde{h} \circ F \). These entropies are the intrinsic algebraic entropy, the algebraic and the topological entropies for locally linearly compact vector spaces, the topological entropy for locally compact totally disconnected groups and the algebraic entropy for locally compact compactly covered abelian groups.

Keywords: intrinsic entropy, quasimetric semilattice, normed semigroup, functorial entropy, algebraic entropy, topological entropy, abelian group, vector space, locally compact abelian group, endomorphisms, algebraic dynamical system.

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1 Introduction

Entropy has been intensively studied in ergodic theory and topological dynamics since the introduction of the measure entropy \( h_{mes} \) and the topological entropy \( h_{top} \) for single selfmaps roughly sixty years ago (see [2, 5, 43, 49]). In connection with the topological entropy, the algebraic entropy \( h_{alg} \) of group endomorphisms was introduced somewhat later (see [2, 22, 37, 46, 47, 52]), and the adjoint algebraic entropy \( h^{*}_{alg} \) more recently (see [24, 40]). Moreover, the set-theoretic entropy \( h_{set} \) of selfmaps of a set provided with no further structure was defined in [3] (see also [28, 33, 38]), and used for computing the topological entropy of generalized shifts. For the details about the origin of all these entropies as well as the connections between them see the surveys [19, 31].

In the presence of such a wealth of entropies, it gradually became clear that a common approach covering all (or at least, most) of them could be very helpful. Such a common approach was proposed in [18] aiming at a uniform argument for the basic properties of the above-mentioned entropies. This argument was elaborated, partially in collaboration with Simone Virili, in full detail and proofs in [20, 23].

Recall that an entropy over a category \( \mathcal{X} \) is an invariant \( h_X : \text{Flow}_X \to \mathbb{R}_{\geq 0} \cup \{\infty\} \) of the category \( \text{Flow}_X \) of all flows of \( \mathcal{X} \): a flow of \( \mathcal{X} \) is a pair \( (X, \phi) \) consisting of an object \( X \) of \( \mathcal{X} \) and an endomorphism \( \phi : X \to X \), whereas a morphism between flows, say \( (X, \phi) \) and \( (Y, \psi) \), is given by a morphism \( \alpha : X \to Y \) of \( \mathcal{X} \) such that \( \alpha \circ \phi = \psi \circ \alpha \). Usually, one denotes \( h_X((X, \phi)) \) simply by \( h_X(\phi) \) for a flow \( (X, \phi) \) of \( \mathcal{X} \).

The main idea of the unifying approach from [20, 23] was to define the semigroup entropy \( h_{\mathcal{S}} : \text{Flow}_{\mathcal{S}} \to \mathbb{R}_{\geq 0} \cup \{\infty\} \), where \( \mathcal{S} \) is the category of normed semigroups \( (S, v) \) whose morphisms are all semigroup homomorphisms that are “contractive” with respect to the norm (see ¶ 2.4 for the rigorous definitions). In this way, whenever a category \( \mathcal{X} \) allows for a functor \( F : \text{Flow}_X \to \text{Flow}_{\mathcal{S}} \), one can obtain an entropy \( h_F \) over \( \mathcal{X} \) by defining \( h_F = h_{\mathcal{S}} \circ F : \text{Flow}_X \to \mathbb{R}_{\geq 0} \cup \{\infty\} \). The entropy \( h_F \) was called functorial entropy in [23]. As shown in [20, 23], all entropies listed above (measure entropy, topological entropy, algebraic entropy,
adjoin algebraic entropy, set-theoretic entropy) can be obtained as functorial entropies for appropriate functors $F : \text{Flow}_X \to \text{Flow}_\mathcal{G}$, where $X$ ranges among categories such as the category of sets, the category of groups, the category of compact spaces and the category of locally compact groups. In all specific cases the functors $F : \text{Flow}_X \to \text{Flow}_\mathcal{G}$ are induced from functors $\mathcal{X} \to \mathcal{G}$ in the obvious way.

Meanwhile, the intrinsic algebraic entropy for endomorphisms of abelian groups was introduced in [27]. Its definition, for a specific endomorphism $\phi : G \to G$ of an abelian group $G$, is based on the subtle notion of $\phi$-inert subgroup inspired by the well-known notion of inert subgroup in the non-abelian context (see [13] for further details). Later on, the algebraic entropy and the topological entropy of continuous endomorphisms of locally linearly compact vector spaces were defined in [11, 12], respectively (see also [7, 8]). In these cases, the computation of the entropy of an endomorphism depends on the behavior of some subgroups that turn out to be again $\phi$-inert. So in a purely informal way we call those “intrinsic-like” entropies.

Moreover, the general definitions of the topological entropy $h_{\text{top}}$ (see [31, 39]) and the algebraic entropy $h_{\text{alg}}$ (see [51]) for locally compact groups, involving Haar measure, are not “intrinsic” – they are covered by a suitable generalization of the scheme in [29] with normed semigroups. Nevertheless, for totally disconnected locally compact groups and for locally compact strongly compactly covered groups, respectively, $h_{\text{top}}$ and $h_{\text{alg}}$ allow for an alternative “intrinsic” description, which is handler since it avoids the use of Haar measure, and the limit superior in the general definition becomes a limit (see [39, 40] respectively).

As pointed out in [24], the unifying approach from [20, 23] does not (and cannot) cover these intrinsic-like entropies. So, the aim of this paper is to elaborate a common approach to them. A careful analysis shows that the common feature of all of them is the presence of a semilattice $(S, +)$ provided with a kind of “non-symmetric distance” which may take also value $\infty$, namely a generalized quasimetric (rather than a norm as one had so far in [24]). We develop the necessary machinery regarding generalized quasimetric semilattices in the forthcoming project [10], starting from the seminal work by Nakamura [45] and from similar structures used in topological algebra (see [11]) and in computer science (see [48]).

Here we introduce and study the notion of $\phi$-inert element of a generalized quasimetric semilattice $S$ with respect to a contractive endomorphism $\phi : S \to S$. By analogy with the approach in [24], we define the intrinsic semilattice entropy $\tilde{h} : \text{Flow}_{\mathcal{L}_{qm}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where $\mathcal{L}_{qm}$ denotes the category of generalized quasimetric semilattices and their contractive homomorphisms. Moreover, for a category $\mathcal{X}$ and a functor $F : \text{Flow}_X \to \text{Flow}_{\mathcal{L}_{qm}}$, we define the intrinsic functorial entropy $\tilde{h}_F : \text{Flow}_X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by $\tilde{h}_F = \tilde{h} \circ F$, and we show how the above-mentioned specific intrinsic-like entropies can be obtained from this general scheme as intrinsic functorial entropies. In almost all cases the functor $F : \text{Flow}_X \to \text{Flow}_{\mathcal{L}_{qm}}$ is induced by a functor $\mathcal{X} \to \mathcal{L}_{qm}$.

The paper is organized as follows. In Section 2 we introduce the category $\mathcal{L}_{qm}$, we are mainly interested in, giving basic properties and examples. For the sake of completeness, in §2.2 we recall the notion of (generalized) normed semigroup from [23]; here we allow the norm to take the value $\infty$, in contrast with the initial setting in [20, 24].

In Section 3 we start studying the dynamics of a generalized quasimetric semilattice $(S, d) \in \mathcal{L}_{qm}$. First, in §3.1 we investigate the behavior of elements of $(S, d)$ under the action of a single contractive endomorphism $\phi$ and we define $\phi$-invariant and $\phi$-inert elements. Then, in §3.2 we introduce fully invariant, fully inert and uniformly fully inert elements of $(S, d)$ by analogy with [11, 13, 14]. In §3.3 we examine the properties of the trajectories of $\phi$-inert elements in order to introduce the intrinsic semilattice entropy in §3.4.

Section 4 is devoted to the study of the intrinsic semilattice entropy $\tilde{h}$. In §4.1 we propose some basic properties of $\tilde{h}$ and we show that it is actually an invariant of the category $\text{Flow}_{\mathcal{L}_{qm}}$ (see Corollary 4.3). The whole §4.2 is dedicated to the so-called logarithmic law, that is, we try to answer the following question: given a contractive endomorphism $\phi : S \to S$ of a generalized quasimetric semilattice $S$ and $k \in \mathbb{N}$, is it true that $\tilde{h}(\phi^k) = k \cdot \tilde{h}(\phi)$? The inequality $\tilde{h}(\phi^k) \geq k \cdot \tilde{h}(\phi)$ is proved in Corollary 4.6, while the opposite one is proved only under some additional restraints. Trying to carry over to this framework the proof of the logarithmic law stated in [29] for the intrinsic algebraic entropy, an error was found in one of the steps of the argument in [29], and that proof has been corrected in [50]. Nevertheless,
the argument used in [50] cannot be extended to our current setting. We expect that the answer to the above general question is negative, but we did not find a counterexample yet.

In [14] we compare the intrinsic semilattice entropy with the semigroup entropy when the semigroup is a semilattice and the generalized quasimetric is induced by the generalized norm of the semilattice. The case of the dimension entropy for discrete vector spaces (see Remark 2.2) led us to realize a sufficient condition under which the intrinsic entropy coincides with the semigroup entropy (see Corollary 3.14). An application of this result is given by Corollary 5.2, where we show that the set-theoretic entropy coincides with its intrinsic counterpart, that is, the intrinsic set-theoretic entropy.

In the final Section 5 we put the general scheme to work and we show how the above-mentioned specific intrinsic-like entropies can be recovered as intrinsic functorial entropies.

This paper is dedicated to the memory of our friend and colleague Silvana Rinauro, whose contributions towards inertial properties in groups, obtained jointly with U. Dardano ([13] [15] [17] [16]), triggered the key notion of $\phi$-inert subgroup, which is the core of the notion of intrinsic entropy.

Notation and terminology

We denote by $\mathbb{Z}$ the integers, by $\mathbb{N}$ the natural numbers and by $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ the positive integers. Moreover, $\mathbb{R}$ is the set of reals and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$.

Let $\mathcal{X}$ be a category. With some abuse of notation we write $X \in \mathcal{X}$ to say that $X \in \text{Ob}(\mathcal{X})$. If $\mathcal{Y}$ is a full subcategory of $\mathcal{X}$, we briefly write $\mathcal{Y} \subseteq \mathcal{X}$.

A flow of $\mathcal{X}$ is a pair $(X, \phi)$, where $X$ is an object of $\mathcal{X}$ and $\phi : X \to X$ is an endomorphism in $\mathcal{X}$. A morphism between two flows $(X, \phi)$ and $(Y, \psi)$ of $\mathcal{X}$ is a morphism $\alpha : X \to Y$ in $\mathcal{X}$ such that $\psi \circ \alpha = \alpha \circ \phi$. This defines the category $\text{Flow}_X$ of flows of $\mathcal{X}$.

Clearly, in case $F : \mathcal{X} \to \mathcal{Y}$ is a functor, it induces a functor $\overline{F} : \text{Flow}_X \to \text{Flow}_\mathcal{Y}$ by letting $\overline{F}(X, \phi) = (F(X), F(\phi))$ for every $(X, \phi) \in \text{Flow}_X$ and $\overline{F}(\alpha) = F(\alpha)$ in case $\alpha : (X, \phi) \to (X', \phi')$ is a morphism in $\text{Flow}_X$.

2 Generalized quasimetric semilattices and generalized normed semigroups

2.1 Semilattices with a generalized quasimetric

Here we follow the approach from [10].

Defnition 2.1. A generalized (or extended) quasimetric on a non-empty set $S$ is a function $d : S \times S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that:

(QM1) for $x, y \in S$, $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

(QM2) for every $x, y, z \in S$, $d(x, z) \leq d(x, y) + d(y, z)$; with the standard convention that $r < r + \infty = \infty + \infty = \infty$ for every $r \in \mathbb{R}_{\geq 0}$.

The pair $(S, d)$ is called generalized quasimetric space.

By analogy with the classical case of quasimetrics, we give the following natural definition.

Definition 2.2. Let $(S_1, d_1)$ and $(S_2, d_2)$ be generalized quasimetric spaces. Then a map $\alpha : S_1 \to S_2$ is an isometry if $d_2(\alpha(x), \alpha(y)) = d_1(x, y)$ for every $x, y \in S_1$.

For a generalized quasimetric space $(S, d)$, let $\leq_d$ be the partial order on $(S, d)$ defined by letting, for $x, y \in S$, $x \leq_d y$ if and only if $d(x, y) = 0$ (this is the dual of the specialization order of $d$).

Definition 2.3 (See [10]). A generalized quasimetric space $(S, d)$ is a generalized quasimetric semilattice if $(S, \leq_d)$ is a join-semilattice; the semilattice operation is denoted by $\cdot$. Moreover, $(S, d)$ is invariant if

(QM3) $d(x, y) = d(x, x + y)$ for all $x, y \in S$. 

3
It is natural to define the morphisms between invariant generalized quasimetric semilattices as follows.

**Definition 2.4.** A homomorphism \( \phi : (S, d) \to (S', d') \) between two invariant generalized quasimetric semilattices is contractive if \( d'(\phi(x), \phi(y)) \leq d(x, y) \) for every \( x, y \in S \).

Let \( \mathcal{L}_{qm} \) denote the category of all invariant generalized quasimetric semilattices (i.e., satisfying \([QM1]\) [QM2] [QM3]) and their contractive homomorphisms.

If \( (S, d) \in \mathcal{L}_{qm} \), then a simple application of \([QM2]\) and \([QM3]\) shows that the function \( d(-, y) : S \to \mathbb{R}_{\geq 0} \) is decreasing for every \( y \in S \), while \( d(x, -) : S \to \mathbb{R}_{\geq 0} \) is increasing for every \( x \in S \), that is:

1. **(M1)** if \( x, x', y \in S \) and \( x \leq x' \), then \( d(x', y) \leq d(x, y) \);
2. **(M2)** if \( x, y, y' \in S \) and \( y \leq y' \), then \( d(x, y) \leq d(x, y') \).

As further examples show, it is useful to allow the objects \((S, d)\) of \( \mathcal{L}_{qm} \), to satisfy some additional properties such as:

1. **(L1)** if \( x, y, y' \in S \), then \( d(x, y + y') \leq d(x, y) + d(x, y') \), (i.e., the function \( d(x, -) : S \to S \) is subadditive);
2. **(L2)** if \( x, y, z \in S \) and \( y \leq z \), then \( d(x, z) = d(x, y) + d(y, z) \).

One can see that \([L2]\) is equivalent to \( d(x, y + y') = d(x, y) + d(x + y, y') \) for all triples \( x, y, y' \in S \) (see [10]). Moreover, making use of \([M1]\) one can show that \([L2]\) implies \([L1]\) while an example witnessing that \([L1]\) is strictly weaker than \([L2]\) can be found in [10]. Finally, \([L1]\) implies

\[
d(x + x', y + y') \leq d(x, y) + d(x', y') \quad \text{for every} \quad x, x', y, y' \in S.
\] (2.1)

The above properties define the following full subcategories of \( \mathcal{L}_{qm} \):

- \( \mathcal{L}'_{qm} \) with objects all \( S \in \mathcal{L}_{qm} \) satisfying \([L1]\)
- \( \mathcal{L}''_{qm} \) with objects all \( S \in \mathcal{L}_{qm} \) satisfying \([L2]\)

Clearly, we can write \( \mathcal{L}''_{qm} \subseteq \mathcal{L}'_{qm} \subseteq \mathcal{L}_{qm} \).

### 2.2 The closeness relation

**Definition 2.5.** Let \( S \in \mathcal{L}_{qm} \). Two elements \( x, y \in S \) are close, denoted by \( x \sim y \), if \( d(x, y) < \infty \) and \( d(y, x) < \infty \).

It is easy to see that \( \sim \) is an equivalence relation on \( S \in \mathcal{L}_{qm} \) (the transitivity property holds by \([QM2]\)).

Let \( (S, d) \in \mathcal{L}_{qm} \) with zero element 0 and let

\[
\mathcal{F}_d(S) = \{ x \in S \mid d(0, x) < \infty \} \subseteq S.
\]

Since by definition \( d(x, 0) = 0 \) for every \( x \in S \), clearly \( \mathcal{F}_d(S) = \{0\} \sim \).

**Remark 2.6.** Let \( S \in \mathcal{L}'_{qm} \). Then \( \sim \) is a congruence on \( S \). In fact, for \( x, x', y, y' \in S \), if \( x' \sim x \) and \( y' \sim y \), then also \( x' + y' \sim x + y \) by (2.1).

Therefore, if \( H \) is a subsemilattice of \( S \), then so is

\[
H^\sim = \{ x \in S \mid \exists y \in H, x \sim y \} = \bigcup_{y \in H} [y]_\sim.
\]

In particular, if \( S \in \mathcal{L}'_{qm} \) has zero element 0, then \( \mathcal{F}_d(S) = \{0\} \sim \) is a subsemilattice of \( S \).
2.3 Examples of generalized quasimetric semilattices

Here we collect some examples that are used in Section 5 (see also [10]).

**Example 2.7.** Let \( X \) be a non-empty set and \( S = (\mathcal{P}(X), \cup, \subseteq) \) its powerset considered as a semilattice. Define, for every \( A, B \in \mathcal{P}(X) \),
\[
d(A, B) = |(A \cup B) \setminus A| = |B \setminus A|.
\]
Then \( (S, d) \in \mathcal{T}_{qm} \) and \( \mathcal{F}_d(S) = \mathcal{P}_{fin}(X) \) is the subsemilattice of all finite subsets of \( X \).

One can consider the reverse order in \( \mathcal{P}(X) \), obtaining \( S^* = (\mathcal{P}(X), \cup, \supseteq) \) with generalized quasimetric
\[
d^*(A, B) = |A \setminus (A \cap B)| = |A \setminus B|.
\]
Then again \( (S^*, d^*) \in \mathcal{T}_{qm} \) and \( \mathcal{F}_d(S^*) = \mathcal{P}_{co-fin}(X) \) is the subsemilattice of all co-finite subsets of \( X \). Moreover, the objects \( (S, d) \) and \( (S^*, d^*) \) of \( \mathcal{T}_{qm} \) are isomorphic with isomorphism defined by \( \phi : A \mapsto X \setminus A \), which is also an isometry, and \( \phi(\mathcal{F}_d(S)) = \mathcal{F}_{d^*}(S) \).

**Example 2.8.** Let \( G \) be a group and denote by \( S(G) \) the family of all subgroups of \( G \). For \( H, H' \in S(G) \) with \( H \subseteq H' \), the index of \( H \) in \( H' \) is denoted by \( [H : H'] \).

(a) If \( G \) is abelian, the lattice \( S(G) \) can be considered as a semilattice whose elements are partially ordered by inclusion and join-operation \( H + H' \) for \( H, H' \in S(G) \). This gives a semilattice \( S = (S(G), +, \subseteq) \) with generalized quasimetric defined by
\[
d_{1, \subseteq}(H, H') = \log[H + H' : H] \text{ for } H, H' \in S(G).
\]

(b) The set \( S(G) \) can be partially ordered by inverse inclusion even when \( G \) is not necessarily abelian. Hence \( S^* = (S(G), \cap, \supseteq) \) can be regarded as a semilattice with the operation \( H \cap H' \) for \( H, H' \in S(G) \). In such a case, one has the generalized quasimetric defined by
\[
d_{1, \supseteq}^*(H, H') = \log[H : H \cap H'] \text{ for } H, H' \in S(G).
\]

The generalized quasimetries \( d_{1, \subseteq} \) and \( d_{1, \supseteq}^* \) satisfy all the properties (QM1), (QM2), (QM3), (L2) so \( (S, d_{1, \subseteq}) \in \mathcal{T}_{qm} \) and \( (S^*, d_{1, \supseteq}^*) \in \mathcal{T}_{qm} \). Clearly, \( d_{1, \subseteq}^*(H, H') = d_{1, \subseteq}(H', H) \) for all \( H, H' \in S(G) \) when \( G \) is abelian, that is, \( d_{1, \subseteq} \) coincides with the dual metric of \( d_{1, \supseteq}^* \).

In both cases the closeness relation is known under the name *commensurability*, that is, \( H, H' \in S(G) \) are commensurable if \( [H : H \cap H'] \) and \( [H' : H \cap H'] \) are finite. Moreover, \( \mathcal{F}_{d_{1, \subseteq}}(S) \) is the family of all finite subgroups and \( \mathcal{F}_{d_{1, \supseteq}^*}(S^*) \) is the family of all finite-index subgroups of \( G \).

The next obviously generalizes the previous example with \( i(G) = \log |G| \).

**Example 2.9.** Let \( M \) be a unitary \( R \)-module, where \( R \) is a unitary commutative ring. Now let \( S = \mathcal{L}(M) \) be the lattice of all submodules of \( M \), considered as a semilattice with operation \( H + H' \) for \( H, H' \in S \). Fix a module invariant \( i \), that is, \( i(M) \in \mathbb{R}_{\geq 0} \cup \{ \infty \} \) and \( i(M) = i(N) \) whenever \( M \cong N \). Moreover, assume that \( i \) is subadditive, that is, \( i(M) \leq i(N) + i(M/N) \) when \( N \) is a submodule of \( M \).

Define the generalized quasimetries \( d_i \) and \( d_i^* \) by
\[
d_i(H, H') = i((H + H')/H) \text{ and } d_i^*(H, H') = i(H/(H \cap H')) \text{ for } H, H' \in S.
\]

If \( R \) is a field, then one is left with the only possible invariant \( i = \dim_R \) and \( M \) is a vector space over \( R \). Moreover, \( d_i \) and \( d_i^* \) satisfy all the properties (QM1), (QM2), (QM3), (L2) and so \( (\mathcal{L}(M), d_{\dim_M}) \in \mathcal{T}_{qm} \) and \( (\mathcal{L}(M), d_i^*) \in \mathcal{T}_{qm} \). Clearly, \( \mathcal{F}_{d_{\dim_M}}(M) \) is the family of all finite-dimensional subspaces of \( M \) and \( \mathcal{F}_{d_i^*}(M) \) is the family of all subspaces of \( M \) with finite co-dimension.

**Remark 2.10.** In all cases considered above we have a concrete category \( \mathfrak{X} \) with a forgetful functor \( U : \mathfrak{X} \rightarrow \text{Set} \) with plenty of nice properties. For example, for \( X \in \mathfrak{X} \), the poset \( \mathcal{L}(X) \) of all subobjects of \( X \) in \( \mathfrak{X} \) is obtained from the lifting of subsets of \( \mathcal{P}(U(X)) \) along \( U \). Hence, the meet in \( \mathcal{L}(X) \) is simply the subobject with underlying set the intersection.

In the above examples \( \mathcal{L}(X) \) is a complete lattice, so it has two semilattice structures which are related by an isomorphism or anti-isomorphism.
2.4 Generalized normed semigroups

Here we recall the notion of normed semigroup from [23]. By analogy with generalized quasi-metric semilattices, now we allow the norm to take also the value $\infty$, so we introduce generalized norms:

**Definition 2.11.** A generalized norm $v$ on a semigroup $S$ is a function $v: S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ and the pair $(S,v)$ is called generalized normed semigroup. If $v(x) < \infty$ for every $x \in S$, then $(S,v)$ is called normed semigroup.

If $v(x+y) \leq v(x)+v(y)$ for all $x, y \in S$, then the generalized norm $v$ is called subadditive. In case $S$ is preordered, $v$ is said to be monotone if $v(x) \leq v(y)$ whenever $x \leq y$.

**Definition 2.12.** A semigroup homomorphism $\phi: (S,v) \rightarrow (S,v')$ is norm-contractive if $v'(\phi(x)) \leq v(x)$ for every $x \in S$.

Following [23], let $\mathcal{S}$ be the category of all generalized normed semigroups and norm-contractive homomorphisms.

For $(S,v) \in \mathcal{S}$, let $F_v(S) = \{x \in S \mid v(x) < \infty\}$. In case $v$ is subadditive, $F_v(S)$ is a normed subsemigroup of $S$.

**Remark 2.13.** For every object $(S,d) \in \mathcal{L}_{qm}$ with zero element 0, one can define on $(S,\leq_d)$ the generalized norm $v_d: S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by $v_d(x) = d(0,x)$ for every $x \in S$.

For the precise relation between invariant generalized quasi-metrics and generalized norms see [10]. In particular $v_d$ is always monotone and it is also subadditive when $(S,d) \in \mathcal{L}_{qm}$. Moreover, $F_d(S) = F_{v_d}(S)$.

3 Dynamics in $\mathcal{L}_{qm}$

3.1 The $\phi$-invariant and $\phi$-inert elements

In this section we study the interaction of single elements of some $(S,d) \in \mathcal{L}_{qm}$ with endomorphisms of $(S,d)$ in $\mathcal{L}_{qm}$.

**Definition 3.1.** Let $((S,d),\phi) \in \text{Flow}_{\mathcal{L}_{qm}}$. An element $x \in S$ is called:

(i) $\phi$-invariant if $d(x,\phi(x)) = 0$, (i.e., if $\phi(x) \leq x$);

(ii) $\phi$-inert if $d(x,\phi(x)) < \infty$.

We denote respectively by $\text{Inv}_\phi(S)$ and $\mathcal{I}_\phi(S)$ the subsets of the $\phi$-invariant and the $\phi$-inert elements of $S$ (we shall see below that these are actually subsemilattices of $S$). Obviously, $\text{Inv}_\phi(S) \subseteq \mathcal{I}_\phi(S)$.

Next we see that if $S$ has a zero element 0, then a large supply of $\phi$-inert elements is provided by the elements of $S$ close to 0, shortly, $F_d(S) \subseteq \mathcal{I}_\phi(S)$.

**Remark 3.2.** Let $((S,d),\phi) \in \text{Flow}_{\mathcal{L}_{qm}}$ and $x \in S$.

(a) If $x \in F_d(S)$, then $\phi(x) \in F_d(S)$ and $x \in \mathcal{I}_\phi(S)$. In fact, $d(0,x) < \infty$ implies $d(0,\phi(x)) = d(\phi(0),\phi(x)) \leq d(0,x) < \infty$. Then $d(x,\phi(x)) \leq d(0,\phi(x))$ by [M1].

(b) The element $x$ is $\phi$-invariant precisely when $x = x + \phi(x)$.

We show some properties of the $\phi$-inert elements, starting with the verification that $\mathcal{I}_\phi(S)$ is $\phi$-invariant.

**Lemma 3.3.** Let $((S,d),\phi) \in \text{Flow}_{\mathcal{L}_{qm}}$. Then

(a) $\phi^n(\mathcal{I}_\phi(S)) \subseteq \mathcal{I}_\phi(S)$ for all $n \in \mathbb{N}$, in particular, if $x \in S$ is $\phi$-inert, then $\phi(x)$ is $\phi$-inert;

(b) $\text{Inv}_\phi(S)$ is a subsemilattice of $S$;

(c) if in addition $((S,d),\phi) \in \text{Flow}_{\mathcal{L}_{qm}}$, then $\mathcal{I}_\phi(S)$ is a subsemilattice of $S$.

**Proof.** (a) It suffices to observe that $d(\phi^n(x),\phi^{n+1}(x)) \leq d(x,\phi(x))$.

(b) If $x, y \in S$ are $\phi$-inert, then $x + y$ is $\phi$-invariant by Remark 3.2(b).

(c) By [21], one has $0 \leq d(x+y,\phi(x+y)) = d(x+y,\phi(x)+\phi(y)) \leq d(x,\phi(x))+d(y,\phi(y))$ for every $x,y \in S$. Therefore, if $x,y \in S$ are $\phi$-inert, then $x+y$ is $\phi$-inert. □
3.2 Fully invariant, fully inert and uniformly fully inert elements

Inspired by the notions introduced and studied in [13] [14], we give the following.

**Definition 3.4.** Let \((S,d) \in \mathcal{L}_{qm}\). An element \(x \in S\) is called:

(i) **fully invariant** if \(x\) is \(\phi\)-invariant for every contractive endomorphism \(\phi\) of \(S\);

(ii) **fully inert** if \(x\) is \(\phi\)-inert for every contractive endomorphism \(\phi\) of \(S\);

(iii) **uniformly fully inert** if there exists \(C > 0\) such that \(d(x,\phi(x)) \leq C\) for every contractive endomorphism \(\phi\) of \(S\).

In the sequel, given \((S,d) \in \mathcal{L}_{qm}\), we denote by:

(i) \(\mathcal{I}(S)\) the set of all fully inert elements of \(S\);

(ii) \(\text{Inv}(S)\) the set of all fully invariant elements of \(S\);

(iii) \(\mathcal{I}_u(S)\) the set of all uniformly fully inert elements of \(S\).

Clearly, \(\text{Inv}(S) \subseteq \mathcal{I}_u(S) \subseteq \mathcal{I}(S)\) and

\[
\mathcal{I}(S) = \bigcap_{\phi \in \text{End}(S)} \mathcal{I}_\phi(S), \quad \text{Inv}(S) = \bigcap_{\phi \in \text{End}(S)} \text{Inv}_\phi(S).
\]

By Lemma 3.3(b), \(\text{Inv}(S)\) is a subsemilattice of \(S\). If in addition \((S,d) \in \mathcal{L}_{qm}\) also \(\mathcal{I}(S)\) is a subsemilattice of \(S\) by Lemma 3.3(c), and a similar argument shows that also \(\mathcal{I}_u(S)\) is a subsemilattice of \(S\).

**Lemma 3.5.** Let \((S,d) \in \mathcal{L}_{qm}\), and \(x,y \in S\) with \(x \sim y\).

(a) If \(\phi\) is a contractive endomorphism of \(S\) and \(x \in \mathcal{I}_\phi(S)\), then \(y \in \mathcal{I}_\phi(S)\).

(b) If \(x \in \mathcal{I}_u(S)\), then \(y \in \mathcal{I}_u(S)\) (in particular, if \(x \in \text{Inv}(S)\), then \(y \in \mathcal{I}_u(S)\)).

**Proof.** (a) Using [QM2] and the fact that \(\phi\) is contractive, we have

\[
d(y,\phi(y)) \leq d(y,x) + d(x,\phi(x)) + d(\phi(x),\phi(y)) \leq d(y,x) + d(x,\phi(x)) + d(x,y) < \infty.
\]

(b) Similarly, if for some \(C\) one has \(d(x,\phi(x)) \leq C\) for all contractive endomorphisms \(\phi\) of \(S\), then using the fact that \(\phi\) is contractive and [QM2] we have

\[
d(y,\phi(y)) \leq d(y,x) + d(x,\phi(x)) + d(\phi(x),\phi(y)) \leq d(y,x) + d(x,\phi(x)) + d(x,y) \leq d(y,x) + d(x,y) + C. \quad \square
\]

Let us consider the set \(\text{Inv}(S)^\sim\) of all elements \(x \in S\) which are close to some \(y \in \text{Inv}(S)\). In Lemma 3.3(b) we showed that \(\text{Inv}(S)^\sim \subseteq \mathcal{I}_u(S)\). It is not clear whether one can invert this inclusion, namely:

**Question 3.6.** Let \((S,d) \in \mathcal{L}_{qm}\). If \(y \in \mathcal{I}_u(S)\), does there exist \(x \in \text{Inv}(S)\) such that \(x \sim y\)? In other words, does the equality \(\text{Inv}(S)^\sim = \mathcal{I}_u(S)\) hold?

**Remark 3.7.** The above notions come from the case of groups, that is, in case \(G\) is a group, one considers the lattice \(S(G)\) of all its subgroups with the generalized quasi-metric discussed in Example 2.8(b).

(a) Since fully invariant subgroups are usually hard to come by, one relaxes the property of fully invariance defining a subgroup \(H\) of \(G\) to be characteristic in \(G\) if \(H\) is \(\phi\)-invariant for every \(\phi \in \text{Aut}(G)\).

In this case one may also choose some other subgroup of \(\text{Aut}(G)\); in particular, if this subgroup of \(\text{Aut}(G)\) is \(\text{Inn}(G)\), then the subgroups \(H\) of \(G\) with \(\phi(H) \leq H\) for every \(\phi \in \text{Inn}(G)\) are obviously the normal ones.

(b) More in general, for an operator group or \(\Omega\)-group \(G\), that is, a group \(G\) equipped with a family \(\Omega\) of endomorphisms of \(G\), a subgroup \(H\) of \(G\) is called \(\Omega\)-invariant or \(\Omega\)-admissible if \(\phi(H) \subseteq H\) for every \(\phi \in \Omega\). In particular, with \(\Omega = \text{Inn}(G)\) (respectively, \(\Omega = \text{End}(G)\)), the \(\Omega\)-admissible subgroups of \(G\) are precisely the normal (respectively, the fully invariant, the characteristic) ones.
(c) Analogously to the discussion in item (a), the subgroups $H$ of $G$ that are “fully inert with respect to $\text{Inn}(G)$”, that is, those $H$ that are $\phi$-inert for every $\phi \in \text{Inn}(G)$ where studied under the name inert subgroups in the nineties. Clearly, fully inert subgroups in the above sense are inert. This triggered the introduction of fully inert subgroups of abelian groups in [26].

(d) Bergman and Lenstra [3] introduced the notion of uniformly inert subgroups of $G$. These are the subgroups $H$ of $G$ that are “uniformly inert with respect to $\text{Inn}(G)$”, that is, those $H$ such that for some constant $C > 0$, $[H : \phi(H) \cap H] \leq C$ for every $\phi \in \text{Inn}(G)$. Clearly, every uniformly fully inert subgroup is uniformly inert.

It is known from [4, Theorem 3] that a subgroup of a group $G$ is uniformly inert if and only if it is commensurable with a normal subgroup of $G$. Nevertheless, Question 3.6 is still open; it was raised in [13, 14] in the case of a group $G$ and semilattice $S = S(G)$ with the generalized quasimetric described in Example 2.8(b).

3.3 Trajectories and their properties

In this subsection we investigate the properties of the $\phi$-trajectories of $\phi$-inert elements. In particular, the $\phi$-trajectories of $\phi$-inert elements turn out to be $\psi$-inert elements (see Lemma 3.12).

**Definition 3.8.** Let $((S, d), \phi) \in \text{Flow}_{\text{qm}}$ and $x \in S$. For $n \in \mathbb{N}_+$, the $n$-th $\phi$-trajectory of $x$ is $T_n(\phi, x) = x + \phi(x) + \ldots + \phi^{n-1}(x) \in S$. In case $S$ has a zero element 0, let $T_0(\phi, x) = 0$.

In the sequel we simply write $T_n$ in place of $T_n(\phi, x)$, when $\phi$ and $x$ are clear from the context.

**Remark 3.9.** Let $((S, d), \phi) \in \text{Flow}_{\text{qm}}$ and $x \in S$. For every $n, m, i \in \mathbb{N}_+$, $i \leq m$, it is straightforward to see that $T_n(\phi^i, T_m(\phi, x)) = T_{n+i}(\phi, x)$.

The implication (a)$\Rightarrow$(c) in the next result is in Lemma 3.3(a).

**Proposition 3.10.** Let $((S, d), \phi) \in \text{Flow}_{\text{qm}}$ and $x \in S$. Then the following conditions are equivalent:

(a) $x$ is $\phi$-inert (i.e., $d(x, \phi(x)) < \infty$);
(b) $d(x, T_n(\phi, x)) < \infty$ for every $n \in \mathbb{N}_+$;
(c) $x$ is $\phi^n$-inert (i.e., $d(x, \phi^n(x)) < \infty$) for every $n \in \mathbb{N}_+$.

**Proof.** Let $n \in \mathbb{N}_+$. From [M2] we have $d(x, \phi^{n-1}(x)) \leq d(x, T_n)$. Moreover,
\[
d(x, T_n) \leq d(x, \phi(x)) + d(\phi(x), \phi(T_{n-1})) \leq d(x, \phi(x)) + d(x, T_{n-1}) \leq (n-1)d(x, \phi(x)).\]

Hence, $d(x, \phi^{n-1}(x)) \leq (n-1)d(x, \phi(x))$, and this gives the implications (a)$\Rightarrow$(b)$\Rightarrow$(c), while (c)$\Rightarrow$(a) is trivial.

The above proposition implies in particular that $\mathcal{I}_\phi(S) = \bigcap_{n \in \mathbb{N}_+} \mathcal{I}_{\phi^n}(S)$.

**Lemma 3.11.** Let $((S, d), \phi) \in \text{Flow}_{\text{qm}}$ and let $x \in S$ be $\phi$-inert. Then, for every $n \in \mathbb{N}_+$,
\[
d(T_n(\phi, x), T_{n+1}(\phi, x)) \leq d(T_{n-1}(\phi, x), T_n(\phi, x)).\]

So the sequence $\{d(T_n, T_{n+1})\}_{n \in \mathbb{N}}$ of non-negative reals is decreasing.

**Proof.** Fix $n \in \mathbb{N}_+$. Since $\phi(T_{n-1}) \leq x + \phi(T_{n-1}) = T_n$, (QM3) gives
\[
d(T_{n+1}, T_n) = d(x + \phi(T_{n-1}), x + \phi(T_{n-1}) + \phi(T_n)) = d(x + \phi(T_{n-1}), \phi(T_n)) \leq d(T_{n-1}, \phi(T_n)) \leq d(T_{n-1}, T_n).\]

The next result is useful in [11.2] about the so-called logarithmic law.

**Lemma 3.12.** Let $((S, d), \phi) \in \text{Flow}_{\text{qm}}$. If $x$ is $\phi^k$-inert for some $k \in \mathbb{N}$, then $T_k(\phi, x)$ is $\phi$-inert and so $\phi^k$-inert.

In particular, if $x$ is $\phi$-inert, then $T_n(\phi, x)$ is $\phi$-inert for all $n \in \mathbb{N}$.
Proof. Let \( k \in \mathbb{N} \) and \( x \) be \( \phi^k \)-inert. Since

\[
T_k(\phi, x) + \phi(T_k(\phi, x)) = T_{k+1}(\phi, x) = T_k(\phi, x) + \phi^k(x),
\]

[QM3] implies that

\[
d(T_k(\phi, x), \phi(T_k(\phi, x))) = d(T_k(\phi, x), T_k(\phi, x) + \phi^k(x)) = d(T_k(\phi, x), \phi^k(x)).
\]

Then \( d(T_k(\phi, x), \phi(T_k(\phi, x))) \leq d(x, \phi^k(x)) \) by [M1], so \( T_k(\phi, x) \) is \( \phi \)-inert.

The remaining part is a consequence of Proposition 3.10. \( \square \)

We see some more properties in the smaller categories \( L'_{qm} \) and \( T_{qm} \).

Remark 3.13. Let \( ((S, d), \phi) \in \text{Flow}_{L'_{qm}} \). If \( x, y \in S \) are \( \phi \)-inert and \( n \in \mathbb{N} \), then

\[
d(T_n(\phi, x), T_n(\phi, y)) \leq nd(x, y) \text{ by (2.1)}. \text{ Consequently, for every } m \in \mathbb{N}, \text{ since } T_{n+m}(\phi, x) = T_n(\phi, T_{m+1}(\phi, x)) \text{ by Remark 3.9,}
\]

\[
d(T_n(\phi, x), T_{n+m}(\phi, x)) \leq nd(x, T_{m+1}(\phi, x)).
\]

Lemma 3.14. Let \( ((S, d), \phi) \in \text{Flow}_{L'_{qm}} \) and \( x \in S \). Then, for every \( n, m \in \mathbb{N} \), we have

\[
d(x, T_{n+m}) = d(x, T_n) + d(T_n, T_{n+m}).
\]

Proof. Since \( x \leq T_n \leq T_{n+m} \), the assertion follows from [L.2]. \( \square \)

3.4 The intrinsic semilattice entropy

Thanks to the next result, we are now in position to introduce the fundamental notion for this paper, that is, the notion of intrinsic semilattice entropy of a contractive endomorphism \( \phi \) of an object \((S, d)\) of \( L'_{qm} \).

Theorem 3.15. Let \( ((S, d), \phi) \in \text{Flow}_{L'_{qm}} \). The following limit exists for every \( x \in I_{\phi}(S) \):

\[
\tilde{h}(\phi, x) = \lim_{n \to \infty} \frac{d(x, T_n(\phi, x))}{n}.
\]

This important result is a consequence of the following proposition and Fekete Lemma (see [32]).

Proposition 3.16. Let \( ((S, d), \phi) \in \text{Flow}_{L'_{qm}} \) and \( x \in I_{\phi}(S) \). Then \( \{d(x, T_n(\phi, x))\}_{n \in \mathbb{N}} \) is subadditive.

Proof. For \( n \in \mathbb{N} \) let \( c_n := d(x, T_{n+1}(\phi, x)) \). We have to prove that \( c_{m+n} \leq c_m + c_n \) for every \( m, n \in \mathbb{N} \). One has

\[
c_{m+n} = d(x, T_{m+n+1}(\phi, x)) \leq c_n + d(T_{n+1}(\phi, x), T_{m+n+1}(\phi, x))
\]

by [QM2]. Hence, to conclude that \( c_{m+n} \leq c_m + c_n \), it suffices to compute

\[
d(T_{n+1}(\phi, x), T_{m+n+1}(\phi, x)) = d(T_{n+1}(\phi, x), T_{n+1}(\phi, x) + \phi^{n+1}(T_m(\phi, x)))
\]

\[
= d(T_{n+1}(\phi, x), \phi^{n+1}(T_m(\phi, x)))
\]

\[
\leq d(\phi^n(x), \phi^{n+1}(T_m(\phi, x)))
\]

\[
\leq d(x, \phi(T_m(\phi, x)))
\]

\[
\leq d(x, T_{m+n}(\phi, x)) = c_m,
\]

where the first equality holds by definition, the second by [QM3], the first inequality by [M1] since \( \phi^n(x) \leq T_{n+1}(\phi, x) \), the second inequality because \( \phi \) is contractive, and the last inequality by [M2] since \( \phi(T_m(\phi, x)) \leq T_{m+n}(\phi, x) \).

Theorem 3.16 allows us to give main definition of this paper.

Definition 3.17. Let \( ((S, d), \phi) \in \text{Flow}_{L'_{qm}} \). The intrinsic semilattice entropy of \( \phi \) with respect to \( x \in I_{\phi}(S) \) is the value \( \tilde{h}(\phi, x) \) introduced in Theorem 3.15.

The intrinsic semilattice entropy of \( \phi \) is \( \tilde{h}(\phi) = \sup \{\tilde{h}(\phi, x) \mid x \in I_{\phi}(S)\} \).
Due to Lemma 3.14 we see now that a stronger result with respect to Theorem 3.15 holds for flows in $\mathcal{L}_{qm}$.

**Proposition 3.18.** Let $((S,d),\phi) \in \text{Flow}_{\mathcal{L}_{qm}}$ and the value set $d(S \times S)$ be well-ordered subset of the range. If $x \in S$ is $\phi$-inert, then

$$\tilde{h}(\phi, x) = \inf_{n \in \mathbb{N}} d(T_n(\phi, x), T_{n+1}(\phi, x)) \in \mathbb{R}_{\geq 0}.$$

**Proof.** By Lemma 3.11 the sequence $\{d(T_n, T_{n+1})\}_{n \in \mathbb{N}}$ is decreasing, so it stabilizes, according to our hypothesis. Let $\alpha = \inf\{d(T_n, T_{n+1}) \mid n \in \mathbb{N}\}$. There exists $n_0 \in \mathbb{N}$ such that $d(T_n, T_{n+1}) = \alpha$ for every $n \in \mathbb{N}$ with $n \geq n_0$. By Lemma 3.14 we have that $d(x, T_{n_0+m}) = d(x, T_{n_0}) + ma$ for every $m \in \mathbb{N}$; therefore,

$$\tilde{h}(\phi, x) = \lim_{m \to \infty} \frac{d(x, T_{n_0+m}(\phi, x))}{n_0+m} = \lim_{m \to \infty} \frac{d(x, T_{n_0}) + ma}{n_0+m} = \alpha. \quad \Box$$

### 4 Basic properties of the intrinsic semilattice entropy

In this section we investigate several properties of the map $\tilde{h}: \text{Flow}_{\mathcal{L}_{qm}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where $\tilde{h}((S,d),\phi)) = \tilde{h}(\phi)$ for every $((S,d),\phi)) \in \text{Flow}_{\mathcal{L}_{qm}}$.

#### 4.1 The intrinsic semilattice entropy is an invariant

We start by showing that the identity map has zero intrinsic semilattice entropy.

**Example 4.1.** If $(S,d) \in \mathcal{L}_{qm}$, then $\tilde{h}(id_S) = 0$. Indeed, every $x \in S$ is $id_S$-inert, and $T_n(id_S, x) = x$ for every $n \in \mathbb{N}$, so $\tilde{h}(id_S, x) = 0$.

The condition needed in item (a) of the next result seems to be different from the surjectivity of $\alpha: S_1 \to S_2$.

**Proposition 4.2.** Let $\alpha: ((S_1,d_1),\phi_1) \to ((S_2,d_2),\phi_2)$ be a morphism in $\text{Flow}_{\mathcal{L}_{qm}}$. Then $\alpha(\mathcal{I}_{\phi_1}(S_1)) \subseteq \mathcal{I}_{\phi_2}(S_2)$ and $T_n(\phi_2, \alpha(x)) = \alpha(T_n(\phi_1, x))$ for every $x \in S_1$ and $n \in \mathbb{N}$. Moreover:

(a) if $\alpha(\mathcal{I}_{\phi_1}(S_1)) = \mathcal{I}_{\phi_2}(S_2)$, then $\tilde{h}(\phi_2) \leq \tilde{h}(\phi_1)$;

(b) if $\alpha$ is an injective isometry, then $\tilde{h}(\phi_2) \geq \tilde{h}(\phi_1)$.

**Proof.** Since $\alpha$ is a contractive semilattice homomorphism such that $\alpha\phi_1 = \phi_2\alpha$, one has $d_2(\alpha(x), \phi_2(\alpha(x))) = d_2(\alpha(x), \alpha(\phi_1(x))) \leq d_1(x, \phi_1(x)) < \infty$. Then $\alpha(x) \in S_2$ is $\phi_2$-inert whenever $x \in S_1$ is $\phi_1$-inert.

If $x \in S_1$ and $n \in \mathbb{N}$, then

$$T_n(\phi_2, \alpha(x)) = \alpha(x) + \phi_2 \alpha(x) + \cdots + \phi_2^{n-1} \alpha(x) = \alpha(x) + \alpha \phi_1(\alpha(x)) + \cdots + \alpha \phi_1^{n-1}(x) = \alpha(T_n(\phi_1, x)).$$

(a) Let $y \in \mathcal{I}_{\phi_2}(S_2)$, and let $x \in \mathcal{I}_{\phi_1}(S_1)$ be such that $y = \alpha(x)$. Using the first part of the proof, we obtain

$$\tilde{h}(\phi_2, y) = \lim_{n \to \infty} \frac{d_2(\alpha(x), \alpha(T_n(\phi_1, x)))}{n} \leq \lim_{n \to \infty} \frac{d_1(x, T_n(\phi_1, x))}{n} = \tilde{h}(\phi_1, x),$$

Since $\tilde{h}(\phi_1, x) \leq \tilde{h}(\phi_1)$, taking the supremum over $y \in \mathcal{I}_{\phi_2}(S_2)$ in the above inequality we get $\tilde{h}(\phi_2) \leq \tilde{h}(\phi_1)$.

(b) Assume that $\alpha$ is injective and $d_2(\alpha(x), \alpha(y)) = d_1(x, y)$ for every $x, y \in S_1$. For a $\phi_1$-inert element $x \in S_1$, we proved already that $\alpha(x) \in S_2$ is $\phi_2$-inert. Moreover,

$$\tilde{h}(\phi_2, \alpha(x)) = \lim_{n \to \infty} \frac{d_2(\alpha(x), \alpha(T_n(\phi_1, x)))}{n} = \lim_{n \to \infty} \frac{d_1(x, T_n(\phi_1, x))}{n} = \tilde{h}(\phi_1, x).$$

Then $\tilde{h}(\phi_2) \geq \tilde{h}(\phi_2, \alpha(x)) = \tilde{h}(\phi_1, x)$ for every $\phi_1$-inert element $x$, so $\tilde{h}(\phi_2) \geq \tilde{h}(\phi_1). \quad \Box$
When \( \alpha : ((S_1, d_1), \phi_1) \rightarrow ((S_2, d_2), \phi_2) \) is an isomorphism in Flow\( L_{q\mu} \), it satisfies all the hypotheses in Proposition 4.12(a,b). Moreover, \( \phi_2 \) coincides with \( \alpha \phi_1 \alpha^{-1} \), so \( \tilde{h}(\alpha \phi_1 \alpha^{-1}) = \tilde{h}(\phi_1) \) in this case.

**Corollary 4.3** (Invariance under conjugation). Let \( \alpha : ((S_1, d_1), \phi_1) \rightarrow ((S_2, d_2), \phi_2) \) be an isomorphism in Flow\( L_{q\mu} \). Then \( \alpha(S_d(S_1)) = S_d(S_2) \) and \( \tilde{h}(\phi_2) = \tilde{h}(\phi_1) \).

This shows that \( \tilde{h} : \text{Flow}_{L_{q\mu}} \rightarrow \mathbb{R}_{\geq 0} \cup \{ \infty \} \) is an invariant of Flow\( L_{q\mu} \).

### 4.2 Towards the logarithmic law

In the following results, we compare the intrinsic semilattice entropy \( \tilde{h}(\phi) \) of a flow \((S, d), \phi) \) in Flow\( L_{q\mu} \), with the intrinsic semilattice entropy of the composition flow \((S, d), \phi^k) \).

**Lemma 4.4.** Let \((S, d), \phi) \in \text{Flow}_{L_{q\mu}} \) and \( k \in \mathbb{N} \). If \( x \) is \( \phi \)-inert, then \( \tilde{h}(\phi, T_k(x)) = \tilde{h}(\phi, x) \).

**Proof.** Let \( k \in \mathbb{N} \) and \( x \in S \) be \( \phi \)-inert. Then \( T_k(\phi, x) \) is \( \phi \)-inert by Lemma 3.12. Let \( n \in \mathbb{N}_+ \). Remark 3.9 gives

\[
\tilde{h}(\phi, T_k(x)) = \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_n(\phi, T_k(\phi, x)))}
= \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_{n+k-1}(\phi, x))}
\]

Since \( d(T_k(\phi, x), x) = 0 \), by (QM1) we obtain

\[
\tilde{h}(\phi, T_k(x)) \leq \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), x)} + \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_{n+k-1}(\phi, x))} = \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_{n+k-1}(\phi, x))} \cdot \frac{n+k-1}{n} = \tilde{h}(\phi, x).
\]

On the other hand,

\[
\tilde{h}(\phi, x) = \lim_{n \rightarrow \infty} \frac{n}{d(x, T_{n+k}(\phi, x))} \leq \lim_{n \rightarrow \infty} \frac{n+k}{d(x, T_k(\phi, x))} + \lim_{n \rightarrow \infty} \frac{n+k}{d(T_k(\phi, x), T_{n+k}(\phi, x))}.
\]

As \( d(x, T_k(\phi, x)) \in \mathbb{R} \) and does not depend on \( n \), Remark 3.9 gives

\[
\tilde{h}(\phi, x) \leq \lim_{n \rightarrow \infty} \frac{n+k}{d(T_k(\phi, x), T_{n+k}(\phi, x))} \cdot \frac{n+k-1}{n+k} = \tilde{h}(\phi, T_k(\phi, x)).
\]

Then we obtain some sort of “local” logarithmic law passing to the trajectories.

**Proposition 4.5.** Let \((S, d), \phi) \in \text{Flow}_{L_{q\mu}} \) and \( k \in \mathbb{N} \). If \( x \) is \( \phi^k \)-inert, then

\[
\tilde{h}(\phi^k, T_k(x)) = k \cdot \tilde{h}(\phi, T_k(x)).
\]

Moreover, if \( x \) is \( \phi \)-inert, then

\[
\tilde{h}(\phi^k, T_k(x)) = k \cdot \tilde{h}(\phi, T_k(x)) = k \cdot \tilde{h}(\phi, x).
\]

**Proof.** First assume that \( x \) is \( \phi^k \)-inert. Then \( T_k(\phi, x) \) is \( \phi \)-inert and \( \phi^k \)-inert by Lemma 3.12. Let \( n \in \mathbb{N}_+ \). By Remark 3.9

\[
T_{nk}(\phi, x) = T_n(\phi^k, T_k(\phi, x)) = T_{kn+k+1}(\phi, T_k(\phi, x)).
\]

Then we get (4.1) as

\[
\tilde{h}(\phi^k, T_k(x)) = \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_n(\phi^k, T_k(\phi, x)))}
= \lim_{n \rightarrow \infty} \frac{n}{d(T_k(\phi, x), T_{kn+k+1}(\phi, T_k(\phi, x)))} \cdot \frac{kn+k+1}{n}
= k \cdot \tilde{h}(\phi, T_k(x)).
\]
Now assume that $x$ is $\phi$-inert. Then $x$ is $\phi^k$-inert as well, so \((4.1)\) ensures the first equality in (4.2). Moreover, Lemma (4.4) applies to provide the second equality in (4.2).

As an immediate consequence of Proposition 4.5 we obtain:

**Corollary 4.6.** If \(((S,d),\phi) \in \text{Flow}_{\mathcal{L}_q}\) and $k \in \mathbb{N}$, then $k \cdot \tilde{h}(\phi) \leq \tilde{h}(\phi^k)$.

**Proof.** Let $x \in \mathcal{I}_\phi(S)$. By (4.2), $k \cdot \tilde{h}(\phi, x) = \tilde{h}(\phi^k, T_k(\phi, x)) \leq \tilde{h}(\phi^k)$. Thus, $k \cdot \tilde{h}(\phi) \leq \tilde{h}(\phi^k)$ by taking the supremum over all $x \in \mathcal{I}_\phi(S)$.

In the rest of this subsection we give partial results concerning the converse inequality $\tilde{h}(\phi^k) \leq k \cdot \tilde{h}(\phi)$. We start from a “local” version generalizing Proposition 4.5 where we replace the $\phi$-inert element $T_k(\phi, x)$ that appears in (4.1) with a generic $\phi$-inert element of $S$.

**Lemma 4.7.** If \(((S,d),\phi) \in \text{Flow}_{\mathcal{L}_q}\), $k \in \mathbb{N}$ and $x$ is $\phi$-inert, then $\tilde{h}(\phi^k, x) \leq k \cdot \tilde{h}(\phi, x)$.

**Proof.** Note first that (even in case $x$ is not $\phi$-inert), $T_n(\phi^k, x) \leq T_k n + k + 1(\phi, x)$. Then

$$\tilde{h}(\phi^k, x) \leq \lim_{n \to \infty} \frac{d(x, T_k n - k + 1(\phi, x))}{k n - k + 1} \frac{k n - k + 1}{n} = k \cdot \tilde{h}(\phi, x).$$

The next corollary gives a precise description of $k \cdot \tilde{h}(\phi)$ and covers, in particular, Corollary 4.8.

**Corollary 4.8.** Let \(((S,d),\phi) \in \text{Flow}_{\mathcal{L}_q}\), and $k \in \mathbb{N}$. Then

$$k \cdot \tilde{h}(\phi) = \sup \{\tilde{h}(\phi^k, x) \mid x \in \mathcal{I}_\phi(S)\} \leq \tilde{h}(\phi^k).$$

**Proof.** The second inequality follows from the fact that $\mathcal{I}_\phi(S) \subseteq \mathcal{I}_{\phi^k}(S)$. Let $x \in \mathcal{I}_\phi(S)$; then $y = T_k(\phi, x) \in \mathcal{I}_\phi(S)$ by Proposition 4.10. Respectively from Lemma 4.4 and Lemma 4.7 it follows that $\tilde{h}(\phi^k, x) \leq k \cdot \tilde{h}(\phi, x) = k \cdot \tilde{h}(\phi, y)$. So, $\sup \{\tilde{h}(\phi^k, x) \mid x \in \mathcal{I}_\phi(S)\} \leq k \cdot \tilde{h}(\phi)$. To prove the converse inequality, apply (4.2) to obtain $k \cdot \tilde{h}(\phi, x) = \tilde{h}(\phi^k, T_k(\phi, x)) \leq \sup \{\tilde{h}(\phi^k, x) \mid x \in \mathcal{I}_\phi(S)\}$. Hence, $\tilde{h}(\phi) \leq \sup \{\tilde{h}(\phi^k, x) \mid x \in \mathcal{I}_\phi(S)\}$.

Corollary 4.8 implies that the logarithmic law holds in the following special cases.

**Corollary 4.9.** Let \(((S,d),\phi) \in \text{Flow}_{\mathcal{L}_q}\), and $k \in \mathbb{N}$. If either $h(\phi^k) = 0$ or $\mathcal{I}_\phi(S) = \mathcal{I}_{\phi^k}(S)$, then $\tilde{h}(\phi^k) = k \cdot \tilde{h}(\phi)$.

Note that, as $\mathcal{I}_\phi(S) \subseteq \mathcal{I}_{\phi^k}(S)$ holds in general by Proposition 4.10, $\mathcal{I}_\phi(S) = \mathcal{I}_{\phi^k}(S)$ occurs for example when $\mathcal{I}_\phi(S) = S$. This is the case when the generalized quasimetric $d$ is a quasimetric (that is, $d$ takes only finite values), and so we obtain the following instance of the logarithmic law.

**Corollary 4.10.** Let \(((S,d),\phi) \in \text{Flow}_{\mathcal{L}_q}\) with $d$ a quasimetric, and let $k \in \mathbb{N}$. Then $\tilde{h}(\phi^k) = k \cdot \tilde{h}(\phi)$.

This is not surprising, since under the assumption that $\mathcal{F}_d(S) = S$ the intrinsic semilattice entropy coincides with the semigroup entropy (see Corollary 4.13 below), and it is known from [22] that the semigroup entropy satisfies the logarithmic law.

### 4.3 Intrinsic semilattice entropy vs semigroup entropy

We recall the main definition from [23].

**Definition 4.11 (See [23]).** Let \(((S,v),\phi) \in \text{Flow}_{\mathcal{F}_d}\). The **semigroup entropy** of $\phi$ with respect to $x \in \mathcal{F}_v(S)$ is

$$h(\phi, x) = \lim_{n \to \infty} \frac{v(T_n(\phi, x))}{n}.$$  

The **semigroup entropy** of $\phi$ is $h(\phi) = \sup \{h(\phi, x) \mid x \in \mathcal{F}_v(S)\}$.  

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In case $((S,v),\phi) \in \text{Flow}_S$ and $v$ is subadditive, the limit superior in the definition of $h(\phi,x)$ is a limit (see [23]).

Now we give some relation between the semigroup entropy and the intrinsic semilattice entropy.

**Lemma 4.12.** Let $((S,d),\phi) \in \text{Flow}_{L_q^\infty}$ with a zero element $= 0$. Then:
(a) $\phi : (S,v_d) \rightarrow (S,v_d)$ is norm-contractive;
(b) $F_{v_d}(S) = F_d(S)$ is $\phi$-invariant.

**Proof.** (a) For $x \in S$, we have $v_d(\phi(x)) = d(0,\phi(x)) = d(\phi(0),\phi(x)) \leq d(0,x) = v_d(x)$.

(b) The equality is obvious (see also Remark 2.13). The inequality in the proof of item (a) shows that for $x \in S$, if $v_d(x) < \infty$, then also $v_d(\phi(x)) < \infty$; so, $\phi(F_d(S)) \subseteq F_d(S)$.

**Proposition 4.13.** Let $((S,d),\phi) \in \text{Flow}_{L_q^\infty}$ with zero element $0$. If $x \in F_d(S)$, then $h(\phi,x) = \tilde{h}(\phi,x)$. Consequently, $h(\phi) = \tilde{h}(\phi \mid F_d(S)) \leq \tilde{h}(\phi)$.

**Proof.** By (M2) we have $d(x,T_n(\phi,x)) \leq d(0,T_n(\phi,x)) = v_d(T_n(\phi,x))$ for every $x \in F_d(S)$. Then $\tilde{h}(\phi,x) \leq h(\phi,x)$. Conversely, by (QM2) $\tilde{h}(\phi,x) \leq \tilde{h}(\phi,\phi(x)) = d(0,T_n(\phi,x)) \leq d(0,x) + d(x,T_n(\phi,x))$ for every $n \in \mathbb{N}$, and so $h(\phi,x) \leq \tilde{h}(\phi,x)$. Then $h(\phi,x) = \tilde{h}(\phi,x)$. To deduce the second assertion, it suffices to note that $F_d(S) \subseteq I(S)$, by Remark 3.2.

As a consequence of Proposition 4.13 we obtain the following sufficient condition under which the intrinsic semilattice entropy coincides with the semigroup entropy. This occurs in some examples in Section 5.

**Corollary 4.14.** Let $((S,d),\phi) \in \text{Flow}_{L_q^\infty}$ with a zero element $0$. Assume that for every $x \in I(S)$ there exists $y \in F_d(S)$ such that $y \leq x$ and $x + \phi(x) = x + \phi(y)$. Then $h(\phi) = \tilde{h}(\phi)$.

**Proof.** Let $x \in I(S)$ and $y \in F_d(S)$ such that $y \leq x$ and $x + \phi(x) = x + \phi(y)$. We show by induction on $n \in \mathbb{N}_+$ that

$$T_n(\phi,x) = x + T_n(\phi,y). \quad (4.4)$$

For $n = 1$, since $y \leq x$, we have that $T_1(\phi,x) = x = x + y = x + T_1(\phi,y)$. Let $n \in \mathbb{N}_+$ and assume that (4.4) holds. Then

$$T_{n+1}(\phi,x) = x + \phi(T_n(\phi,x)) = x + \phi(x) + \phi(T_n(\phi,y)) = x + y + \phi(y) = x + y + \phi(T_n(\phi,y)) = x + y + \phi(T_n(\phi,y)) = x + T_{n+1}(\phi,y).$$

We deduce that, by (QM2) and (QM3)

$$d(x,T_n(\phi,x)) = d(x,T_n(\phi,y)) \leq d(x,0) + d(0,T_n(\phi,y)) = v_d(T_n(\phi,y)).$$

and so $\tilde{h}(\phi,x) \leq h(\phi,y) \leq h(\phi,y)$. This yields $\tilde{h}(\phi) \leq h(\phi)$, and we get the equality by applying Proposition 4.13.

## 5 Obtaining the specific entropy functions

In the next subsections of this section we use the following scheme in order to find the known intrinsic-like entropies as intrinsic functorial entropies.
5.1 Intrinsic functorial entropy

As recalled in the introduction, for $\mathcal{X}$ a category and $F : \text{Flow}_X \to \text{Flow}_{\mathcal{L}_{qm}}$ a functor, the intrinsic functorial entropy $\tilde{h}_F$ associated to $F$ is defined by letting $\tilde{h}_F = \tilde{h} \circ F$. We set $\tilde{h}_F(\phi) = \tilde{h}_F(X, \phi)$ for every $(X, \phi) \in \text{Flow}_X$ as usual.

The following shows that $\tilde{h}_F$ is an invariant of $\text{Flow}_X$.

**Proposition 5.1.** For every functor $F : \text{Flow}_X \to \text{Flow}_{\mathcal{L}_{qm}}$, the intrinsic functorial entropy $\tilde{h}_F$ is invariant under conjugation, that is, for every $(X, \phi), (Y, \psi) \in \text{Flow}_X$ such that there exists an isomorphism $\alpha : (X, \phi) \to (Y, \psi)$ one has $\tilde{h}_F(\phi) = \tilde{h}_F(\psi)$.

**Proof.** Assume that $F : \text{Flow}_X \to \text{Flow}_{\mathcal{L}_{qm}}$ is covariant. By hypothesis, $\psi = \alpha \circ \phi \circ \alpha^{-1}$. Then $F(\psi) = F(\alpha) \circ F(\phi) \circ F(\alpha)^{-1}$ in $\mathcal{L}_{qm}$. By Corollary 4.13 $\tilde{h}_F(\psi) = \tilde{h}(F(\psi)) = \tilde{h}(F(\phi)) = \tilde{h}_F(\phi)$. For a contravariant functor $F$ one can proceed analogously.

5.2 Intrinsic set-theoretic entropy

Let $\phi : X \to X$ be a selfmap of a non-empty set $X$. The set-theoretic entropy $h_{\text{set}}(\phi)$ was defined in [3] as follows. For a finite subset $F$ of $X$ let

$$H_{\text{set}}(\phi, F) = \lim_{n \to \infty} \frac{1}{n} |F \cup \phi(F) \cup \ldots \cup \phi^{n-1}(F)|$$

and $h_{\text{set}}(\lambda) = \sup \{ H_{\text{set}}(\phi, F) \mid F \in \mathcal{P}_{\text{fin}}(X) \}$.

Consider the functor $\textbf{Set} \to \mathcal{L}_{qm}$ defined by mapping a set $X$ to the quasi-metric semilattice $(\mathcal{P}(X), d)$ of Example 2.7 and such that every morphism $\lambda : X \to Y$ in $\textbf{Set}$ is sent to the morphism $\text{set}(\lambda) : \mathcal{P}(X) \to \mathcal{P}(Y)$ mapping $Z \mapsto \lambda(Z)$ for every $Z \subseteq X$. Thus, $\text{set}$ induces the functor $\text{set} : \text{Flow}_{\text{Set}} \to \text{Flow}_{\mathcal{L}_{qm}}$ and we can apply the scheme constructed above: we get an intrinsic-like entropy on $\textbf{Set}$, that is, $\tilde{h}_{\text{set}} = \tilde{h}_{\text{set}}$ is the intrinsic functorial entropy associate to $\text{set}$. In other terms, for $(X, \phi) \in \text{Flow}_{\text{Set}}$ we have

$$I_{\phi}(X) := I_{\mathcal{P}(\phi)}(\mathcal{P}(X)) = \{ F \in \mathcal{P}(X) \mid |\phi(F) \setminus F| < \infty \},$$

for $F \in I_{\phi}(X)$

$$\tilde{h}_{\text{set}}(\phi, F) = \lim_{n \to \infty} \frac{1}{n} |(F \cup \phi(F) \cup \ldots \cup \phi^{n-1}(F)) \setminus F|$$

and $\tilde{h}_{\text{set}}(\phi) = \sup \{ \tilde{h}_{\text{set}}(\phi, F) \mid F \in I_{\phi}(X) \}$.

As a consequence of Corollary 4.13 we have that this intrinsic set-theoretic entropy coincides with the set-theoretic entropy:

**Corollary 5.2.** The intrinsic set-theoretic entropy $\tilde{h}_{\text{set}}$ coincides with the set-theoretic entropy $h_{\text{set}}$, that is $\tilde{h}_{\text{set}} = h_{\text{set}}$ on $\text{Flow}_{\text{Set}}$.

**Proof.** Let $(X, \phi) \in \text{Flow}_{\text{Set}}$. For every $Z \in I_{\phi}(X)$, since $|\phi(Z) \setminus Z| < \infty$, there exists $F \in I_d(\mathcal{P}(X)) = I_{\mathcal{P}_{\text{fin}}}(X)$ such that $Z \cup \phi(Z) = Z \cup \phi(F)$. By Corollary 4.13 we conclude that $\tilde{h}_{\text{set}} = \tilde{h}_{\text{set}}$ coincides with $h_{\text{set}}$, since in this case $v_d$ coincides with the norm used in [23] to obtain $h_{\text{set}} = h_{\text{set}}$.

5.3 Intrinsic algebraic entropy and intrinsic adjoint algebraic entropy

Let $G$ be an abelian group and $f : G \to G$ an endomorphism. A subgroup $H$ of $G$ is $f$-inert if $|(H + f(H))/H|$ is finite. The family $I_f(G)$ of the $f$-inert subgroups of $G$ contains all finite subgroups, all finite-index subgroups, as well as all $f$-invariant and fully invariant subgroups of $G$. The notion of $f$-inert subgroup allowed to introduce in [25, 27] two new notions of algebraic entropy: the intrinsic algebraic entropy and the intrinsic adjoint algebraic entropy.
In detail, let \((G, f) \in \text{Flow}_{\text{Ab}}\), where we denote by \(\text{Ab}\) the category of abelian groups and their homomorphisms. Given an \(f\)-inert subgroup \(H\) of \(G\), the intrinsic algebraic entropy of \(f\) with respect to \(H\) is

\[
\text{ent}(f, H) = \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{H + f(H) + \cdots + f^{n-1}(H)}{H} \right),
\]

and the intrinsic algebraic entropy of \(f\) is \(\text{ent}(f) = \sup\{\text{ent}(f, H) \mid H \in \mathcal{I}_f(G)\}\).

On the other hand, the intrinsic adjoint algebraic entropy of \(f\) with respect to \(H\) is

\[
\text{ent}^*(f, H) = \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{H \cap f^{-1}(H) \cap \cdots \cap f^{-n+1}(H)}{H} \right),
\]

and so the intrinsic adjoint algebraic entropy of \(f\) is \(\text{ent}^*(f) = \sup\{\text{ent}^*(f, H) \mid H \in \mathcal{I}_f(G)\}\).

Next, we show that the intrinsic (respectively, intrinsic adjoint) algebraic entropy is part of the general scheme introduced in this paper, namely, we prove them to be intrinsic functorial entropies with respect to suitable functors \(\text{Flow}_{\text{Ab}} \to \text{Flow}_{\mathcal{E}_{qm}}\). Recall that the family \(\mathcal{I}_f(G)\) is a bounded sublattice of the lattice of all the subgroups of \(G\) (see \cite{27} Lemma 2.6).

### 5.3.1 Intrinsic algebraic entropy for abelian groups

For an abelian group \(G\), denote \(S^\vee(G) = (S(G), +, \subseteq)\), that is the family \(S(G)\) of all subgroups of \(G\) partially ordered by inclusion and endowed with the ordinary sum as join-operation; the zero element of \(S^\vee(G)\) is the trivial subgroup. By Example \cite{23} we have \((S^\vee(G), d_{\lvert \cdot \rvert}) \in \text{Flow}_{\mathcal{E}_{qm}}\). In addition, for a morphism \(f : G \to G'\) in \(\text{Ab}\), let

\[ S^\vee(f) : (S^\vee(G), d_{\lvert \cdot \rvert}) \to (S^\vee(G'), d_{\lvert \cdot \rvert}); \]

mapping \(H \mapsto f(H)\). This defines the functor \(S^\vee : \text{Ab} \to \text{Flow}_{\mathcal{E}_{qm}}\), which induces a functor \(\mathfrak{S}^\vee : \text{Flow}_{\text{Ab}} \to \text{Flow}_{\mathcal{E}_{qm}}\).

**Theorem 5.3.** On \(\text{Flow}_{\text{Ab}}\), we have \(\text{ent} = \widetilde{h} \circ \mathfrak{S}^\vee\).

Indeed, \(\mathcal{I}_{S^\vee(f)}(S^\vee(G), d_{\lvert \cdot \rvert}) = \mathcal{I}_f(G)\) and \(\text{ent} = \widetilde{h} \circ \mathfrak{S}^\vee\) (i.e., the following diagram commutes).

\[
\begin{array}{ccc}
\text{Flow}_{\text{Ab}} & \xrightarrow{\mathfrak{S}^\vee} & \text{Flow}_{\mathcal{E}_{qm}} \\
\text{ent} \downarrow & & \downarrow \widetilde{h} \\
\mathbb{R}_{\geq 0} \cup \{\infty\} & & \\
\end{array}
\]

### 5.3.2 Intrinsic adjoint algebraic entropy for abelian groups

Conversely, let \(S^\wedge(G) = (S(G), \cap, \supseteq)\) denote the family \(S(G)\) partially ordered by inverse inclusion together with the intersection of subgroups as join-operation. The semilattice \(S^\wedge(G)\) has \(G\) as zero element.

By Example \cite{23} \((S^\wedge(G), d^\wedge_{\lvert \cdot \rvert}) \in \text{Flow}_{\mathcal{E}_{qm}}\). In addition, for a morphism \(f : G \to G'\) in \(\text{Ab}\), let

\[ S^\wedge(f) : (S^\wedge(G'), d^\wedge_{\lvert \cdot \rvert}) \to (S^\wedge(G), d^\wedge_{\lvert \cdot \rvert}); \]

mapping \(H \mapsto f^{-1}(H)\). This defines the functor \(S^\wedge : \text{Ab} \to \text{Flow}_{\mathcal{E}_{qm}}\), which induces a functor \(\mathfrak{S}^\wedge : \text{Flow}_{\text{Ab}} \to \text{Flow}_{\mathcal{E}_{qm}}\).

**Theorem 5.4.** On \(\text{Flow}_{\text{Ab}}\), we have \(\text{ent}^* = \widetilde{h} \circ \mathfrak{S}^\wedge\).

Indeed, \(\mathcal{I}_{S^\wedge(f)}(S^\wedge(G), d_{\lvert \cdot \rvert}) = \mathcal{I}_f(G)\) and \(\text{ent}^* = \widetilde{h} \circ \mathfrak{S}^\wedge\) (i.e., the following diagram commutes).

\[
\begin{array}{ccc}
\text{Flow}_{\text{Ab}} & \xrightarrow{\mathfrak{S}^\wedge} & \text{Flow}_{\mathcal{E}_{qm}} \\
\text{ent} \downarrow & & \downarrow \widetilde{h} \\
\mathbb{R}_{\geq 0} \cup \{\infty\} & & \\
\end{array}
\]
5.3.3 Different choice of the semilattices

In order to obtain the intrinsic algebraic entropy and the intrinsic adjoint algebraic entropy as intrinsic functorial entropies, we can also proceed as follows.

For \((G, f) \in \text{Flow}_{\text{Ab}}\) let \(I^\prime_f(G) = (I_f(G), d_{\cdot \mid \cdot}) \in \overline{\text{Z}}_{qm}\) be the subsemilattice of \(S^\prime(G)\) endowed with the generalized quasimetric induced by \(d_{\cdot \mid \cdot}\). Moreover, let

\[
\Gamma^\prime_f(I^\prime_f(G), d_{\cdot \mid \cdot}) \rightarrow \Gamma^\prime_f(I^\prime_f(G), d_{\cdot \mid \cdot}), \quad H \mapsto f(H).
\]

Consequently, the assignment \((G, f) \mapsto ((I^\prime_f(G), d_{\cdot \mid \cdot}), \Gamma^\prime_f(f))\) produces the functor \(\Gamma^\prime : \text{Flow}_{\text{Ab}} \rightarrow \text{Flow}_{\overline{\text{Z}}_{qm}}\), such that \(\text{ent} = \overline{h} \circ \Gamma^\prime\).

\[
\begin{array}{ccc}
\text{Flow}_{\text{Ab}} & \xrightarrow{\Gamma^\prime} & \text{Flow}_{\overline{\text{Z}}_{qm}} \\
\text{ent} \downarrow & & \downarrow \overline{h} \\
\mathbb{R}_{\geq 0} \cup \{\infty\} & & \\
\end{array}
\]

Analogously, let \(I^\wedge_f(G) = (I_f(G), d^\wedge_{\cdot \mid \cdot}) \in \overline{\text{Z}}_{qm}\) be the subsemilattice of \(S^\wedge(G)\) endowed with the generalized quasimetric induced by \(d^\wedge_{\cdot \mid \cdot}\). Moreover, let

\[
\Gamma^\wedge_f(I^\wedge_f(G), d^\wedge_{\cdot \mid \cdot}) \rightarrow \Gamma^\wedge_f(I^\wedge_f(G), d^\wedge_{\cdot \mid \cdot}), \quad H \mapsto f^{-1}(H).
\]

This yields the functor such that \(\text{ent}^\wedge = \overline{h}_\Gamma^\wedge\).

\[
\begin{array}{ccc}
\text{Flow}_{\text{Ab}} & \xrightarrow{\Gamma^\wedge} & \text{Flow}_{\overline{\text{Z}}_{qm}} \\
\text{ent} \downarrow & & \downarrow \overline{h} \\
\mathbb{R}_{\geq 0} \cup \{\infty\} & & \\
\end{array}
\]

5.4 Algebraic and topological entropy for locally compact groups

5.4.1 Algebraic entropy for compactly covered lca groups

A topological group \(G\) is said to be compactly covered if each element of \(G\) is contained in some compact subgroup of \(G\). Let \(\text{LCA}_{cc}\) denote the category of compactly covered locally compact abelian groups and their continuous endomorphisms. For example, the additive group \(\mathbb{Q}_p\) of \(p\)-adic rationals is an object of \(\text{LCA}_{cc}\). Compactly covered locally compact abelian groups are of great interest because they are the Pontryagin duals of abelian totally disconnected locally compact abelian groups (see the next paragraph).

Let \((G, f) \in \text{Flow}_{\text{LCA}_{cc}}\). By \([21, Proposition 2.2]\), the algebraic entropy of \(f\) with respect to \(U \in \mathcal{O}(G)\) is

\[
h_{\text{alg}}(f, U) = \lim_{n \to \infty} \frac{1}{n} \log[U + f(U) + \ldots + f^n(U) : U],
\]

and \(h_{\text{alg}}(f) = \sup\{h_{\text{alg}}(f, U) \mid U \in \mathcal{O}(G)\}\) is the algebraic entropy of \(f\).

For \(G \in \text{LCA}_{cc}\), we consider the semilattice \(\mathcal{O}^\wedge(G) = (\mathcal{O}(G) \cup \{0\}, +, \subseteq)\) seen as a subsemilattice of \(I_f(G)\) and so equipped with the generalized quasimetric \(d_{\cdot \mid \cdot}\). Then \((\mathcal{O}^\wedge(G), d_{\cdot \mid \cdot}) \in \overline{\text{Z}}_{qm}\). Subsequently, for \(f : G \rightarrow G'\) in \(\text{LCA}_{cc}\), let \(\mathcal{O}^\wedge(f) : \mathcal{O}^\wedge(G) \rightarrow \mathcal{O}^\wedge(G), U \mapsto U + f(U)\). This defines the functor \(\mathcal{O}^\wedge : \text{LCA}_{cc} \rightarrow \overline{\text{Z}}_{qm}\), and so the functor \(\overline{\mathcal{O}^\wedge} : \text{Flow}_{\text{LCA}_{cc}} \rightarrow \text{Flow}_{\overline{\text{Z}}_{qm}}\).

Remark 5.5. For every \((G, f) \in \text{Flow}_{\text{LCA}_{cc}}\) and every \(U \in \mathcal{O}(G)\), we always have \(d_{\cdot \mid \cdot}(U, \mathcal{O}^\wedge(f)(U)) = \log[U + f(U) : U] < \infty\), that is,

\[
\mathcal{O}^\wedge(G) = \mathcal{I}_{\mathcal{O}^\wedge(f)}(\mathcal{O}^\wedge(G)) \subseteq \mathcal{I}_f(G),
\]

so \(\mathcal{O}^\wedge(G)\) is a subsemilattice of \(I_f(G)\).

Theorem 5.6. On \(\text{Flow}_{\text{LCA}_{cc}}\), we have \(h_{\text{alg}} = \overline{h}_{\mathcal{O}^\wedge}\).
Indeed, the following diagram commutes by (5.3).

\[ \text{Flow}_{LCA} \xrightarrow{\text{CO}} \text{Flow}_{\text{Lqm}} \xleftarrow{\text{Flow}_{\text{Lqm}}} \]

\[ h_{\text{top}} \xrightarrow{\mathbb{R}_{\geq 0} \cup \{\infty\}} \bar{h} \]

5.4.2 Topological entropy for tdlc groups

A locally compact group \( G \) is said to be totally disconnected if the connected component of the identity \( I_G \) is reduced to the singleton \( \{1_G\} \). Discrete groups and profinite groups are example of totally disconnected locally compact groups. In particular, profinite groups are precisely the topological groups that are compact and totally disconnected. Denote by TDLC the category of totally disconnected locally compact groups and their continuous homomorphisms.

Let \( G \in \text{TDLC} \). As a consequence of van Dantzig’s theorem, the family \( CO(G) \) of all compact open subgroups of \( G \) forms a neighborhood basis at \( 1_G \). As pointed out in [29, 31], such a property allows to define the topological entropy of continuous endomorphisms of \( G \) without resorting to the Haar measure, as follows.

Let \( (G, f) \in \text{Flow}_{\text{TDLC}} \). The topological entropy of \( f \) with respect to \( U \in CO(G) \) is

\[ h_{\text{top}}(f, U) = \lim_{n \to \infty} \frac{1}{n} \log|U : U \cap f^{-1}(U) \cap \cdots \cap f^{-n+1}(U)|, \]

and \( h_{\text{top}}(f) = \sup\{h_{\text{top}}(f, U) \mid U \in CO(G)\} \) denotes the topological entropy of \( f \).

For \( G \in \text{TDLC} \) we consider the semilattice \( CO^\wedge(G) = (CO(G) \cup \{\}, \cap, \supseteq) \) equipped with the generalized quasimetric \( d_{\wedge,1}^* \). Therefore, \( (CO^\wedge(G), d_{\wedge,1}^*) \in \mathcal{L}_{\text{qm}} \). Subsequently, for \( f : G \to G' \) in \( \text{TDLC} \), let \( CO^\wedge(f) : CO^\wedge(G') \to CO^\wedge(G) \), \( U \cap f^{-1}(U) \). This defines a functor \( CO^\wedge : \text{TDLC} \to \mathcal{L}_{\text{qm}} \), which induces a functor \( \overline{CO}^\wedge : \text{Flow}_{\text{TDLC}} \to \text{Flow}_{\mathcal{L}_{\text{qm}}} \).

Remark 5.7. For every \((G, f) \in \text{Flow}_{\text{TDLC}}\) and every \( U \in CO(G) \), we always have \( d_{\wedge,1}^*(U, CO^\wedge(f)(U)) = \log|U : U \cap f^{-1}(U)| < \infty \), that is,

\[ CO^\wedge(G) = \mathcal{I}_{CO^\wedge(f)}(CO^\wedge(G)) \subseteq \mathcal{I}_f(G), \quad (5.4) \]

and in particular \( CO^\wedge(G) \) is a subsemilattice of \( \mathcal{I}_f(G) \).

Theorem 5.8. On \( \text{Flow}_{\text{TDLC}} \), we have \( h_{\text{top}} = \overline{h}_{\mathcal{L}_{\text{qm}}} \).

Indeed, the following diagram commutes by (5.4).

\[ \text{Flow}_{\text{TDLC}} \xrightarrow{\overline{CO}^\wedge} \text{Flow}_{\mathcal{L}_{\text{qm}}} \xleftarrow{\overline{h}} \text{Flow}_{\mathcal{L}_{\text{qm}}} \]

5.5 Algebraic and topological entropy for l.l.c. vector spaces

5.5.1 Locally linearly compact vector spaces

Let \( K \) be a discrete field. A topological \( K \)-vector space \( V \) is linearly compact when:

(LC1) it is a Hausdorff space in which there is a fundamental system of neighborhoods of 0 consisting of linear subspaces of \( V \);

(LC2) any filter base on \( V \) consisting of closed linear varieties (i.e., closed cosets of linear subspaces) has a non-empty intersection.

For example, finite-dimensional discrete vector spaces are linearly compact and compact vector spaces satisfying (LC1) are linearly compact. More precisely, every linearly compact \( K \)-space is a Tychonoff product of one-dimensional \( K \)-spaces, and viceversa. Let \( \mathcal{L}_{\text{LC}} \) denote the category of linearly compact \( K \)-vector spaces and their continuous homomorphisms. We collect here a few properties of linearly compact vector spaces (see [13]) that we use further on. Let \( W \leq V \), \( U \) be \( K \)-vector spaces satisfying condition (LC1), thus:
(lc1) if \( \phi : V \to U \) is a surjective continuous homomorphism and \( V \) is linearly compact, then \( U \) is linearly compact;

(lc2) if \( V \) is linearly compact and \( W \) is closed, then \( W \) is linearly compact;

(lc3) if \( V \) is discrete, then \( V \) is linearly compact if and only if \( V \) has finite dimension over \( K \);

(lc4) if \( W \) is closed, then \( V \) is linearly compact if and only if \( W \) and \( V/W \) are linearly compact.

A topological \( K \)-vector space \( V \) is said to be \textit{locally linearly compact} if the family \( \mathcal{LCO}(V) \) of all linearly compact open linear subspaces of \( V \) is a fundamental system of neighborhoods of \( 0 \). Let \( \mathbb{g} \text{LLC} \) denote the category of locally linearly compact \( K \)-vector spaces and their continuous homomorphisms. The category \( \mathbb{g} \text{LC} \) is a full subcategory of \( \mathbb{g} \text{LLC} \), and also the category \( \mathbb{g} \text{Vect} \) of discrete \( K \)-vector spaces is a full subcategory of \( \mathbb{g} \text{LLC} \).

Remark 5.9. The partially ordered set \( (\mathcal{LCO}(V), \subseteq) \) is a lattice with join-operation given by the sum of linear subspaces (see (lc1)) and meet-operation given by the intersection (see (lc2)). The lattice \( (\mathcal{LCO}(V), \subseteq) \) is not bounded unless \( V \) has finite dimension. If \( V \) is discrete, then \( (\mathcal{LCO}(V), \subseteq, +) \) has as zero element 0. If \( V \) is linearly compact, then \( (\mathcal{LCO}(V), \subseteq, \cap) \) has as zero element \( V \).

5.5.2 Algebraic Entropy for locally linearly compact vector spaces

Following [11], for every flow \( (V, f) \) over \( \mathbb{g} \text{LLC} \), the \textit{algebraic entropy of} \( f \) \textit{with respect to} \( U \in \mathcal{LCO}(V) \) is

\[
\text{ent}(f, U) = \lim_{n \to \infty} \frac{1}{n} \dim \frac{U + f(U) + \ldots + f^{n-1}(U)}{U},
\]

and the \textit{algebraic entropy of} \( f \) is \( \text{ent}(f) = \sup \{ \text{ent}(f, U) \mid U \in \mathcal{LCO}(V) \} \).

For \( V \in \mathbb{g} \text{LLC} \), let \( \mathcal{LCO}'(V) \) denote the semilattice \( (\mathcal{LCO}(V) \cup \{0\}, \subseteq, +) \) with zero element given by the trivial subspace. Recall that the trivial subspace of \( V \) is not open unless \( V \) is discrete, and therefore we need to add it. By Example 2.9 \( \mathcal{LCO}'(V) \) inherits the generalized quasimetric \( d_{\text{dim}} \). Then \( (\mathcal{LCO}'(V), d_{\text{dim}}) \in \mathcal{T}_{\text{qm}} \).

Moreover, for a morphism \( f : V \to V' \) in \( \mathbb{g} \text{LLC} \), let \( \mathcal{LCO}'(f) : \mathcal{LCO}'(V) \to \mathcal{LCO}'(V') \), \( U \mapsto f(U) \). This gives us the functor \( \mathcal{LCO}' : \mathbb{g} \text{LLC} \to \mathcal{T}_{\text{qm}} \), which induces the functor \( \mathcal{LCO}' : \mathbb{g} \text{Vect} \to \mathcal{T}_{\text{qm}} \).

Remark 5.10. For every \( (V, f) \in \text{Flow}_{\mathbb{g} \text{LLC}} \) and every \( U \in \mathcal{LCO}(V) \) we always have \( d_{\text{dim}}(U, \mathcal{LCO}'(f)(U)) = \dim(U + f(U)/U) < \infty \) by (lc4) and (lc3) that is,

\[
\mathcal{LCO}'(V) = \mathcal{I}_{\mathcal{LCO}'(f)}(\mathcal{LCO}'(V)) \subseteq \mathcal{I}_f(V),
\]

and in particular \( \mathcal{LCO}'(V) \) is a subsemilattice of \( \mathcal{I}_f(V) \).

Theorem 5.11. On \( \text{Flow}_{\mathbb{g} \text{LLC}} \), we have \( \text{ent} = h_{\mathcal{LCO}'} \).

Indeed, in view of (5.5) the following diagram commutes.

\[
\begin{array}{ccc}
\text{Flow}_{\mathbb{g} \text{LLC}} & \xrightarrow{\mathcal{LCO}'} & \text{Flow}_{\mathcal{T}_{\text{qm}}} \\
\text{ent} & \downarrow & \text{ent} \\
\mathbb{R}_{\geq 0} \cup \{\infty\} & \xrightarrow{h} & \mathbb{R}_{\geq 0} \cup \{\infty\}
\end{array}
\]

Remark 5.12. If we restrict \text{ent} to \( \text{Flow}_{\mathbb{g} \text{ Vect}} \), then \text{ent} coincides with the dimension entropy from [35]. Indeed, for \( (V, f) \in \text{Flow}_{\mathbb{g} \text{ Vect}} \), \( \mathcal{LCO}(V) \) is the family of all finite-dimensional linear subspaces of \( V \); the \textit{dimension entropy of} \( f \) \textit{with respect to} \( F \in \mathcal{LCO}(V) \) is

\[
\text{ent}_{\text{dim}}(f, U) = \lim_{n \to \infty} \frac{1}{n} \dim(U + f(U) + \ldots + f^{n-1}(U)),
\]

and the \textit{dimension entropy of} \( f \) is \( \text{ent}_{\text{dim}}(f) = \sup \{ \text{ent}_{\text{dim}}(f, U) \mid U \in \mathcal{LCO}(V) \} \).
For every $U \in \mathcal{LCO}^\vee(V) = \mathcal{I}_{\mathcal{LCO}^\vee(f)}(\mathcal{LCO}^\vee(V))$ there exists $F \in \mathcal{F}_{\dim}(\mathcal{LCO}^\vee(V))$ with $U + f(U) = U + f(F)$; indeed from $\dim(U + f(U))/U < \infty$ it follows that there exists a finite-dimensional linear subspace $F$ of $V$ with $U + f(F)/U = U + f(U)/U$, that is, $U + f(F) = U + f(U)$. Therefore $\hat{h}(\mathcal{LCO}^\vee(f)) = h_{\mathcal{LCO}^\vee}(\mathcal{LCO}^\vee(f))$ by Corollary 5.14.

It is known from [23] that $\text{ent}_{\dim} = h_{\mathcal{LCO}^\vee}$ on $\text{Flow}_{k\text{Vect}}$, and so we conclude by Theorem 5.10 that $\text{ent} = \text{ent}_{\dim}$ on $k\text{Vect}$. This was proved directly in [6].

5.5.3 Topological Entropy for locally linearly compact vector spaces

The topological counterpart of the algebraic entropy for locally linearly compact vector spaces was introduced in [12] as follows. The topological entropy of $f$ with respect to $U \in \mathcal{LCO}(V)$ is

$$\text{ent}^*(f, U) = \lim_{n \to \infty} \frac{1}{n} \dim \frac{U}{U \cap f^{-1}(U) + \ldots + f^{-n+1}(U)},$$

and the topological entropy of $f$ is $\text{ent}^*(f) = \sup \{ \text{ent}^*(f, U) \mid U \in \mathcal{LCO}(V) \}$.

For $V \in k\text{LLC}$ consider the semilattice $\mathcal{LCO}^\wedge(V)$ given by $(\mathcal{LCO}(V) \cup \{ V \}, \supseteq, \cap)$; the semilattice $\mathcal{LCO}^\wedge(V)$ has zero element $V$. We consider on $\mathcal{LCO}^\wedge(V)$ the generalized quasi-metric $d_{\dim}$ from Example 2.9. Then $(\mathcal{LCO}^\wedge(V), d_{\dim}) \in \mathcal{T}_{\text{qmi}}$. Moreover, for a morphism $f : V \to V'$ in $k\text{LLC}$, let $\mathcal{LCO}^\wedge(f) : \mathcal{LCO}^\wedge(V) \to \mathcal{LCO}^\wedge(V)$, $U \mapsto U \cap f^{-1}(U)$. This produces the functor $\mathcal{LCO}^\wedge : k\text{LLC} \to \mathcal{T}_{\text{qmi}}$, which induces the functor $\mathcal{LCO}^\wedge : \text{Flow}_{k\text{LLC}} \to \text{Flow}_{\mathcal{T}_{\text{qmi}}}$.

**Remark 5.13.** For $(V, f) \in \text{Flow}_{k\text{LLC}}$ and $U \in \mathcal{LCO}(V)$, by [lc3] and [lc4] one has $d_{\dim}^*(U, \mathcal{LCO}^\wedge(f)(U)) = \dim(U/U \cap f^{-1}(U)) < \infty$, that is,

$$\mathcal{LCO}^\wedge(V) = \mathcal{I}_{\mathcal{LCO}^\wedge(f)}(\mathcal{LCO}^\wedge(V)) \subseteq \mathcal{I}_f(V),$$

and in particular $\mathcal{LCO}^\wedge(V)$ is a subsemilattice of $\mathcal{I}_f(V)$.

**Theorem 5.14.** On $\text{Flow}_{k\text{LLC}}$, we have $\text{ent}^* = \hat{h}_{\mathcal{LCO}^\wedge}$.

Indeed, by (5.6) the following diagram commutes.

$$
\begin{array}{ccc}
\text{Flow}_{k\text{LLC}} & \xrightarrow{\mathcal{LCO}^\wedge} & \text{Flow}_{\mathcal{T}_{\text{qmi}}} \\
\text{ent}^* & \xrightarrow{\hat{h}} & \mathbb{R}_{\geq 0} \cup \{ \infty \}
\end{array}
$$

(5.7)

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