THE KALMAN–YAKUBOVICH–POPOV INEQUALITY FOR PASSIVE DISCRETE TIME-INVARIANT SYSTEMS

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Abstract. We consider the Kalman - Yakubovich - Popov (KYP) inequality

\[
\begin{pmatrix}
X - A^*XA - C^*C - A^*XB - C^*D \\
- B^*XA - D^*C - B^*XB - D^*D
\end{pmatrix} \geq 0
\]

for contractive operator matrices \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\mathcal{H}_M) \to (\mathcal{N}_N) \), where \( \mathcal{H} \), \( \mathcal{M} \), and \( \mathcal{N} \) are separable Hilbert spaces. We restrict ourselves to the solutions \( X \) from the operator interval \([0, I]_H\). Several equivalent forms of KYP are obtained. Using the parametrization of the blocks of contractive operator matrices, the Kreǐn shorted operator, and the Möbius representation of the Schur class operator-valued function we find several equivalent forms of the KYP inequality. Properties of solutions are established and it is proved that the minimal solution of the KYP inequality satisfies the corresponding algebraic Riccati equation and can be obtained by the iterative procedure with the special choice of the initial point. In terms of the Kreǐn shorted operators a necessary condition and some sufficient conditions for uniqueness of the solution are established.

Contents

1. Introduction 1
2. Shorted operators 8
3. Parametrization of contractive block-operator matrices 10
4. The Möbius representations 13
5. The Kalman–Yakubovich–Popov inequality and Riccati equation 17
6. Equivalent forms of the KYP inequality and Riccati equation for a passive system 19
7. Properties of solutions of the KYP inequality and Riccati equation 23
8. Approximation of the minimal solution 30
References 32

1. INTRODUCTION

The system of equations

\[
\begin{cases}
h_{k+1} = Ah_k + B\xi_k, \\
\sigma_k = Ch_k + D\xi_k
\end{cases}, \quad k \geq 0
\]

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describes the evolution of a linear discrete time-invariant system \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \)
with bounded linear operators \( A, B, C, D \) and separable Hilbert spaces \( \mathcal{H} \) (state space), \( \mathcal{M} \) (input space), and \( \mathcal{N} \) (output space). If the linear operator \( T_\tau \) operator by the block-matrix
\[
T_\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{N} \end{pmatrix}
\]
is contractive, then the corresponding discrete-time system is said to be passive. If the block-operator matrix \( T_\tau \) is isometric (co-isometric, unitary) then the corresponding system is called isometric (co-isometric, conservative). Isometric and co-isometric systems were studied by L. de Branges and J. Rovnyak [19, 20] and by T. Ando [4], conservative systems are studied by L. de Branges and J. Rovnyak [19, 20] and by T. Ando [4], passive systems are studied by D.Z. Arov et al [6, 7, 8, 9, 11, 12, 13]. The subspaces
\[
(1.1) \quad \mathcal{H}_\tau^c := \text{span} \{ A^n B \mathcal{M} : n = 0, 1, \ldots \} \quad \text{and} \quad \mathcal{H}_\tau^o = \text{span} \{ A^n C^* \mathcal{N} : n = 0, 1, \ldots \}
\]
are called the controllable and observable subspaces of the system
\[
\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\},
\]
respectively. If \( \mathcal{H}_\tau^c = \mathcal{H} \) (\( \mathcal{H}_\tau^o = \mathcal{H} \)) then the system \( \tau \) is said to be controllable (observable), and minimal if \( \tau \) is both controllable and observable. If \( \mathcal{H} = \text{closure} \{ \mathcal{H}_\tau^c + \mathcal{H}_\tau^o \} \) then the system \( \tau \) is said to be a simple. Note that from (1.1) it follows that
\[
(\mathcal{H}_\tau^c)^\perp = \bigcap_{n=0}^{\infty} \ker (B^* A^n), \quad (\mathcal{H}_\tau^o)^\perp = \bigcap_{n=0}^{\infty} \ker (C A^n).
\]
Therefore
\[
(1) \quad \text{the system } \tau \text{ is controllable } \iff \bigcap_{n=0}^{\infty} \ker (B^* A^n) = \{0\};
\]
\[
(2) \quad \text{the system } \tau \text{ is observable } \iff \bigcap_{n=0}^{\infty} \ker (C A^n) = \{0\};
\]
\[
(3) \quad \text{the system } \tau \text{ is simple } \iff \left( \bigcap_{n=0}^{\infty} \ker (B^* A^n) \right) \cap \left( \bigcap_{n=0}^{\infty} \ker (C A^n) \right) = \{0\}.
\]
The function
\[
\Theta_\tau(\lambda) := D + \lambda C (I_\mathcal{H} - \lambda A)^{-1} B, \quad \lambda \in \mathbb{D},
\]
is called transfer function of the system \( \tau \).

The result of D.Z. Arov [6] states that two minimal systems \( \tau_1 \) and \( \tau_2 \) with the same transfer function \( \Theta(\lambda) \) are pseudo-similar, i.e. there is a closed densely defined operator \( Z : \mathcal{H}_1 \to \mathcal{H}_2 \) such that \( Z \) is invertible, \( Z^{-1} \) is densely defined, and
\[
ZA_1 f = A_2 Z f, \quad C_1 f = C_2 Z f, \quad f \in \text{dom } Z, \quad \text{and} \quad ZB_1 = B_2.
\]
If the system \( \tau \) is passive then \( \Theta_\tau \) belongs to the Schur class \( S(\mathcal{M}, \mathcal{N}) \), i.e., \( \Theta_\tau(\lambda) \) is holomorphic in the unit disk \( \mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \) and its values are contractive linear operators from \( \mathcal{M} \) into \( \mathcal{N} \). It is well known [20, 38, 1, 6, 8] that every operator-valued function \( \Theta(\lambda) \) of the Schur class \( S(\mathcal{M}, \mathcal{N}) \) can be realized as the transfer function of some passive system, which can be chosen as a simple conservative (isometric controllable, co-isometric
controllable, co-isometric observable) systems

\[
\tau_1 = \left\{ \begin{pmatrix} A_1 & B_1 \\ C_1 & D \end{pmatrix} ; \mathcal{H}_1, \mathcal{M}, \mathcal{N} \right\} \quad \text{and} \quad \tau_2 = \left\{ \begin{pmatrix} A_2 & B_2 \\ C_2 & D \end{pmatrix} ; \mathcal{H}_2, \mathcal{M}, \mathcal{N} \right\}
\]

having the same transfer function are unitarily similar \cite{19}, \cite{20}, \cite{21}, \cite{4}, i.e. there exists a unitary operator \( U \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that

\[
A_1 = U^{-1} A_2 U, \quad B_1 = U^{-1} B_2, \quad C_1 = C_2 U.
\]

In \cite{11}, \cite{12} necessary and sufficient conditions on operator-valued function \( \Theta(\lambda) \) from the Schur class \( \mathcal{S}(\mathcal{M}, \mathcal{N}) \) have been established in order that all minimal passive systems having the transfer function \( \Theta(\lambda) \) be unitarily similar or similar.

A system \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is called \( X \)-passive with respect to the supply rate function \( w(u, v) = ||u||_{\mathcal{M}}^2 - ||v||_{\mathcal{N}}^2, \ u \in \mathcal{M}, \ v \in \mathcal{N} \) if there exists a positive selfadjoint operator \( X \) in \( \mathcal{H} \), possibly unbounded, such that

\[
A \text{ dom } X^{1/2} \subseteq \text{ dom } X^{1/2}, \quad \text{ran } B \subseteq \text{ dom } X^{1/2}
\]

and

\[
||X^{1/2}(Ax + Bu)||_{\mathcal{H}}^2 - ||X^{1/2}x||_{\mathcal{H}}^2 \leq ||u||_{\mathcal{M}}^2 - ||C \tau x + Dv||_{\mathcal{N}}^2, \ x \in \text{ dom } X^{1/2}, \ u \in \mathcal{M}.
\]

The condition (1.2) is equivalent to

\[
\left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathcal{N}} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathcal{N}} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 \geq 0
\]

for all \( x \in \text{ dom } X^{1/2}, \ u \in \mathcal{M} \).

If \( X \) is bounded then (1.3) becomes to the Kalman – Yakubovich – Popov inequality (for short, KYP inequality)

\[
L_{\tau}(X) = \begin{pmatrix} X - A^*XA - C^*C & -A^*XB - C^*D \\ -B^*XA - D^*C & I - B^*XB - D^*D \end{pmatrix} \geq 0.
\]

The classical Kalman-Yakubovich-Popov lemma states that if \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a minimal system with finite dimensional state space \( \mathcal{H} \) then the set of the solutions of (1.4) is non-empty if and only if the transfer function \( \Theta_{\tau} \) belongs to the Schur class. If this is a case then the set of all solutions of (1.4) contains the minimal and maximal elements.

For the case \( \dim \mathcal{H} = \infty \) the theory of generalized KYP inequality (1.3) is developed in \cite{9}. It is proved that if \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a minimal system then the KYP inequality (1.3) for \( \tau \) has a solution \( X \) if and only if the transfer function \( \Theta_{\tau} \) coincides with a Schur class function in a neighborhood of the origin. Moreover, there are maximal \( X_{\max} \) and minimal \( X_{\min} \) solutions in the sense of quadratic forms:

if \( X \) is a solution of (1.3) then \( \text{dom } X_{\min}^{1/2} \supseteq \text{ dom } X^{1/2} \supseteq \text{ dom } X_{\max}^{1/2} \) and

\[
||X_{\min}^{1/2}u||^2 \leq ||X^{1/2}u||^2 \quad \text{for all} \ u \in \text{ dom } X^{1/2},
\]

\[
||X^{1/2}v||^2 \leq ||X_{\max}^{1/2}v||^2 \quad \text{for all} \ v \in \text{ dom } X_{\max}^{1/2}.
\]
Let $\Theta(\lambda) \in S(M, N)$. A passive system

$$\dot{\tau} = \left\{ \begin{pmatrix} \dot{A} & \dot{B} \\ \dot{C} & \dot{D} \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$$

with the transfer function $\Theta(\lambda)$ is called optimal realization of $\Theta(\lambda)$ [7], [8] if for each passive system $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$ with transfer function $\Theta(\lambda)$ and for each input sequence $u_0, u_1, u_2, ... \in M$ the inequalities

$$\left\| \sum_{k=0}^{n} \dot{A}^k \dot{B} u_k \right\|_{\mathcal{H}_0} \leq \left\| \sum_{k=0}^{n} A^k B u_k \right\|_{\mathcal{H}}$$

hold for all $n = 0, 1, ...$

An observable passive system

$$\dot{\tau}_* = \left\{ \begin{pmatrix} \dot{A}_* & \dot{B}_* \\ \dot{C}_* & \dot{D} \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$$

is called $(\ast)$-optimal realization of the function $\Theta(\lambda)$ [7], [8] if for each observable passive system $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$ with transfer function $\Theta(\lambda)$ hold

$$\left\| \sum_{k=0}^{n} \dot{A}_*^k \dot{B}_* u_k \right\|_{\mathcal{H}_0} \geq \left\| \sum_{k=0}^{n} A^k B u_k \right\|_{\mathcal{H}}$$

for all $n = 0, 1, ...$ and every choice of vectors $u_0, u_1, u_2, ..., u_n \in M$.

Two minimal and optimal $(\ast)$-optimal passive realizations of a function from the Schur class are unitary similar [8]. In addition the system

$$\dot{\tau}_* = \left\{ \begin{pmatrix} \dot{A}_* & \dot{B}_* \\ \dot{C}_* & \dot{D} \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$$

is $(\ast)$-optimal passive minimal realization of the function $\Theta(\lambda)$ if and only if the system

$$\dot{\tau}_*^{\ast} = \left\{ \begin{pmatrix} \dot{A}_*^{\ast} & \dot{B}_* \\ \dot{C}_*^{\ast} & D^{\ast} \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$$

is optimal passive minimal realization of the function $\Theta^{\ast}(\lambda)$ [8].

Let $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$ be a minimal system and let the transfer function $\Theta$ coincides with a Schur class function in a neighborhood of the origin. Let $X_{\min}$ and $X_{\max}$ be the minimal and maximal solutions of the KYP inequality (1.3). It is proved in [8] that the systems

$$\dot{\tau} = \left\{ \begin{pmatrix} X_{\min}^{1/2} A X_{\min}^{-1/2} & X_{\min}^{1/2} B \\ C X_{\min}^{-1/2} & D \end{pmatrix} ; \mathcal{H}_0, M, N \right\},$$

$$\dot{\tau}_* = \left\{ \begin{pmatrix} X_{\max}^{1/2} A X_{\max}^{-1/2} & X_{\max}^{1/2} B \\ C X_{\max}^{-1/2} & D \end{pmatrix} ; \mathcal{H}_0, M, N \right\}$$
are minimal optimal and minimal (\(\ast\))-optimal realizations of \(\Theta\), respectively. The contractive operators 
\[
T_1 = \begin{pmatrix}
X_{\min}^{1/2}AX_{\min}^{-1/2} & X_{\min}^{1/2}B \\
CX_{\min}^{-1/2} & D
\end{pmatrix}
\]
and 
\[
T_2 = \begin{pmatrix}
X_{\max}^{1/2}AX_{\max}^{-1/2} & X_{\max}^{1/2}B \\
CX_{\max}^{-1/2} & D
\end{pmatrix}
\]
are defined on \(\text{dom } X_{\min}^{-1/2} \oplus \mathcal{M}\) and \(\text{dom } X_{\max}^{-1/2} \oplus \mathcal{M}\), respectively.

Let \(\mathbb{T}\) be a unit circle. As it is well known \([38]\) a function \(\Theta(\lambda)\) from the Schur class \(\mathcal{S}(\mathcal{M}, \mathcal{N})\) has almost everywhere non-tangential strong limit values \(\Theta(\xi), \xi \in \mathbb{T}\). Denote by \(\varphi_\Theta (\psi_\Theta)\) the outer (co-outer) function which is a solution of the following factorization problem \([38]\)

\[
\begin{align*}
(i) \quad & \varphi_\Theta^*(\xi)\varphi_\Theta(\xi) \leq I_{\mathcal{M}} - \Theta^*(\xi)\Theta(\xi) \\
& (\psi_\Theta(\xi)\psi_\Theta^*(\xi) \leq I_{\mathcal{M}} - \Theta(\xi)\Theta^*(\xi)) \text{ almost everywhere on } \mathbb{T}; \\
(ii) \quad & \text{if } \tilde{\varphi}(\lambda) (\tilde{\psi}(\lambda)) \text{ is a Schur class function such that almost everywhere on } \mathbb{T} \text{ holds} \\
& \tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq I_{\mathcal{M}} - \Theta^*(\xi)\Theta(\xi) (\tilde{\psi}(\xi)\tilde{\psi}^*(\xi) \leq I_{\mathcal{M}} - \Theta(\xi)\Theta^*(\xi)) \text{ then} \\
& \tilde{\varphi}^*(\xi)\tilde{\varphi}(\xi) \leq \varphi_\Theta^*(\xi)\varphi_\Theta(\xi) (\tilde{\psi}(\xi)\tilde{\psi}^*(\xi) \leq \psi_\Theta(\xi)\psi_\Theta^*(\xi)) \text{ almost everywhere on } \mathbb{T}.
\end{align*}
\]

The function \(\varphi_\Theta (\psi_\Theta)\) is uniquely defined up to a left (right) constant unitary factor. The functions \(\varphi_\Theta (\psi_\Theta)\) are called the right (left) defect function of the simple conservative system with transfer function \(\Theta\). By means of the right (left) defect function the construction of the minimal and optimal (\(\ast\)-optimal) realization of the function \(\Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N})\) is given by D.Z. Arov in \([7]\). In \([8]\) the construction of the optimal (\(\ast\)-optimal) realization is given as the first (second) restriction of the simple passive system with transfer function \(\Theta\).

The next theorem summarizes some results established in \([7], [8], [12], [16], [17], [18]\).

**Theorem 1.1.** Let \(\Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N})\) and let

\[
\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}
\]

be a simple conservative system with transfer function \(\Theta\). Then

1. the subspace \((\mathcal{H}_0)^\perp ((\mathcal{H}_0)^\perp)\) is invariant with respect to \(A, (A^*)\) and the restriction 
\(A\upharpoonright ((\mathcal{H}_0)^\perp (A^*)\upharpoonright (\mathcal{H}_0)^\perp)\) is a unilateral shift;
2. the functions \(\varphi_\Theta(\lambda)\) and \(\psi_\Theta(\lambda)\) take the form

\[
\begin{align*}
\varphi_\Theta(\lambda) &= P_{\Omega}(I_{\mathcal{H}_0} - \lambda A)^{-1}B, \\
\psi_\Theta(\lambda) &= C(I_{\mathcal{H}_0} - \lambda A)^{-1}\Omega^*,
\end{align*}
\]

(1.5)

where

\[
\Omega = (\mathcal{H}_0)^\perp \ominus A(\mathcal{H}_0)^\perp, \quad \Omega^* = (\mathcal{H}_0)^\perp \ominus A^*(\mathcal{H}_0)^\perp;
\]

3. \(\varphi_\Theta(\lambda) = 0 (\psi_\Theta(\lambda) = 0)\) if and only if the system \(\tau\) is observable (controllable).

The proof of next theorem is based on the concept and properties of optimal and \(\ast\)-optimal passive systems.

**Theorem 1.2.** \([7], [8], [12]\). Let \(\Theta(\lambda) \in \mathcal{S}(\mathcal{M}, \mathcal{N})\). Then

1. if \(\Theta\) is bi-inner and \(\tau\) is a simple passive system with transfer function \(\Theta\) then \(\tau\) is conservative system;
2. if \(\varphi_\Theta(\lambda) = 0\) or \(\psi_\Theta(\lambda) = 0\) then all passive minimal systems with transfer function \(\Theta(\lambda)\) are unitary equivalent and if \(\varphi_\Theta(\lambda) = 0\) and \(\psi_\Theta(\lambda) = 0\) then they are in addition conservative.
In [5] the following strengthening has been proved using the parametrization of contractive block-operator matrices.

**Theorem 1.3.** Let \( \Theta(\lambda) \in S(\mathcal{M}, \mathcal{N}) \) be a not constant.

1. Suppose that \( \varphi_\Theta(\lambda) = 0, \psi_\Theta(\lambda) = 0, \) and \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a simple passive system with transfer function \( \Theta \). Then the system \( \tau \) is conservative and minimal.

   If \( \Theta(\lambda) \) is bi-inner then in addition the operator \( A \) belongs to the class \( C_{00} \).

2. Suppose that \( \varphi_\Theta(\lambda) = 0 (\psi_\Theta(\lambda) = 0) \). If \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a controllable (observable) passive system with transfer function \( \Theta \) then the system \( \tau \) is isometric (co-isometric) and minimal.

   If \( \Theta(\lambda) \) is inner (co-inner) then in addition the operator \( A \) belongs to the class \( C_{00} \).

In this paper we consider the KYP inequality for the case of contractive operator \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). Because in this case the set of solutions contains the identity operator, the minimal solution \( X_{\text{min}} \) is a positive contraction. That’s why we are interested in only contractive positive solutions \( X \) of the KYP inequality.

We will keep the following notations. The class of all continuous linear operators defined on a complex Hilbert space \( \mathcal{H}_1 \) and taking values in a complex Hilbert space \( \mathcal{H}_2 \) is denoted by \( L(\mathcal{H}_1, \mathcal{H}_2) \). The domain, the range, and the null-space of a linear operator \( T \) are denoted by \( \text{dom} T, \text{ran} T, \) and \( \text{ker} T \). For a contraction \( T \in L(\mathcal{H}_1, \mathcal{H}_2) \) the nonnegative square root \( D_T = (I - T^*T)^{1/2} \) is called the defect operator of \( T \) and \( D_T \) stands for the closure of the range \( \text{ran} D_T \). It is well known that the defect operators satisfy the commutation relation \( TD_T = D_T^*T \) and \( T D_T \subset D_T^* \) cf. [38]. The set of all regular points of a closed operator \( T \) is denoted by \( \rho(T) \) and its spectrum by \( \sigma(T) \). The identity operator in a Hilbert space \( \mathcal{H} \) we denote by \( I_\mathcal{H} \) and by \( P_L \) we denote the orthogonal projection onto the subspace \( L \). We essentially use the following tools.

1. The parametrization of the \( 2 \times 2 \) contractive block-operator matrix \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \): \( \left( \begin{array}{c} \mathcal{H} \\ \mathcal{M} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathcal{H} \\ \mathcal{N} \end{array} \right) \) \[14], \[22], \[35]:

   \[B = FD_D, \quad C = D_D^*G,\]
   \[A = -FD_D^*G + D_D^*LD_G,\]

   where the operators \( F \in L(\mathcal{D}_D, \mathcal{K}), \) \( G \in L(\mathcal{H}, \mathcal{D}_D^*) \) and \( L \in L(\mathcal{D}_G, \mathcal{D}_F^*) \) are contractions.

2. The notion of the shorted operator introduced by M.G. Krein in \[27]:

   \[S_K = \max \{ Z \in L(\mathcal{H}) : 0 \leq Z \leq S, \text{ran} Z \subseteq K \},\]

   where \( S \) is a bounded nonnegative selfadjoint operator in the Hilbert space \( \mathcal{H} \) and \( K \) is a subspace in \( \mathcal{H} \).
(3) The M"obius representation

\begin{equation}
\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)}Z(\lambda)(I_{\mathcal{D}\Theta(0)} + \Theta^*(0)Z(\lambda))^{-1}D_{\Theta(0)}, \ \lambda \in \mathbb{D}
\end{equation}

of the Schur class operator-valued function $\Theta(\lambda)$ by means of the operator-valued parameter $Z(\lambda)$ from the Schur class $\mathcal{S}(\mathcal{D}\Theta(0), \mathcal{D}\Theta^*(0))$.

Such a kind representation was studied in [37], [32], [15]. The operator $\Gamma_1 = Z'(0)$ is called the first Schur parameter and the function $\Theta_1(\lambda) = \lambda^{-1}Z(\lambda)$ is called the first Schur iterate of the function $\Theta$ [15]. In this paper a more simple and algebraic proof of the representation (1.7) is given using the equalities (1.6). In particular, we establish that if $\Theta(\lambda)$ is the transfer function of the passive system

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}$$

with entries $A$, $B$, and $C$ given by (1.6) then the parameter $Z(\lambda)$ is the transfer function of the passive system

$$\nu = \left\{ \begin{pmatrix} D_{F^*L}D_{G} & F \\ G & 0 \end{pmatrix} ; \mathcal{H}, \mathcal{D}_D, \mathcal{D}_{D^*} \right\}.$$ 

Moreover, we prove that

(i) the correspondence

$$\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \longleftrightarrow \nu = \left\{ \begin{pmatrix} D_{F^*L}D_{G} & F \\ G & 0 \end{pmatrix} ; \mathcal{H}, \mathcal{D}_D, \mathcal{D}_{D^*} \right\}$$

preserves the properties of the system to be isometric, co-isometric, conservative, controllable, observable, simple, optimal, and $(\ast)$-optimal,

(ii) $\varphi_Z(\lambda) = 0 \iff \varphi_\Theta(\lambda) = 0, \psi_Z(\lambda) = 0 \iff \psi_\Theta(\lambda) = 0$,

(iii) the KYP inequalities for the systems $\tau$ and $\nu$ are equivalent,

(iv) the inequality

\begin{equation}
\begin{cases}
(I_\mathcal{H} - X)P_\mathcal{H} \leq (D_\mathcal{T}^2 + T^*(I_\mathcal{H} - X)P_\mathcal{H}^T)_{\mathcal{H}} \\
0 < X \leq I_\mathcal{H}
\end{cases}
\end{equation}

is equivalent to the KYP inequality (1.4), here $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $P_{\mathcal{H}}$ ($P_{\mathcal{H}}'$) is the orthogonal projection onto $\mathcal{H}$ in the Hilbert space $\mathcal{H} \oplus \mathcal{M}$ ($\mathcal{H} \oplus \mathcal{N}$).

We give several equivalent forms for the KYP inequality and establish that the minimal solution $X_{\text{min}}$ satisfies the algebraic Riccati equation

$$(I_\mathcal{H} - X)P_\mathcal{H} = (D_\mathcal{T}^2 + T^*(I_\mathcal{H} - X)P_\mathcal{H}^T)_{\mathcal{H}}.$$ 

We prove that the condition

$$(D_\mathcal{T}^2)_{\mathcal{H}} = 0 \ (\iff \text{ran} \ D_T \cap \mathcal{H} = \{0\})$$

is a necessary for the uniqueness of the solutions of (1.8). In an example it is shown that this condition is not sufficient. Some sufficient conditions for the uniqueness are obtained. We show that a nondecreasing sequence

$$X(0) = 0, \ X(n+1) = I_\mathcal{H} - (D_\mathcal{T}^2 + T^*(I_\mathcal{H} - X^{(n)})P_\mathcal{H}^T)_{\mathcal{H}} | \mathcal{H}$$

of nonnegative selfadjoint contractions strongly converges to the minimal solution $X_{\text{min}}$ of the KYP inequality.
2. Shorted operators

For every nonnegative bounded operator \( S \) in the Hilbert space \( \mathcal{H} \) and every subspace \( \mathcal{K} \subseteq \mathcal{H} \) M.G. Krein [27] defined the operator \( S_{\mathcal{K}} \) by the formula
\[
S_{\mathcal{K}} = \max \{ Z \in L(\mathcal{H}) : 0 \leq Z \leq S, \text{ ran } Z \subseteq \mathcal{K} \}.
\]
The equivalent definition
\[
(2.1) \quad (S_{\mathcal{K}} f, f) = \inf_{\varphi \in \mathcal{H} \ominus \mathcal{K}} \{(S(f + \varphi), f + \varphi)\}, \quad f \in \mathcal{H}.
\]
The properties of \( S_{\mathcal{K}} \), were studied in [1, 2, 3, 25, 29, 30, 32]. \( S_{\mathcal{K}} \) is called the shorted operator (see [1], [2]). It is proved in [27] that \( S_{\mathcal{K}} \) takes the form
\[
S_{\mathcal{K}} = S^{1/2} P_\Omega S^{1/2},
\]
where \( P_\Omega \) is the orthogonal projection in \( \mathcal{H} \) onto the subspace
\[
\Omega = \{ f \in \overline{\text{ran } S} : S^{1/2} f \in \mathcal{K} \} = \overline{\text{ran } S \ominus S^{1/2} (\mathcal{H} \ominus \mathcal{K})}.
\]
Hence (see [27]),
\[
\text{ran } S^{1/2} = \text{ran } S^{1/2} P_\Omega = \text{ran } S^{1/2} \cap \mathcal{K}.
\]
It follows that
\[
S_{\mathcal{K}} = 0 \iff \text{ran } S^{1/2} \cap \mathcal{K} = \{0\}.
\]
The shortening operation possess the following properties.

**Proposition 2.1.** [2]. Let \( \mathcal{K} \) be a subspace in \( \mathcal{H} \). Then
\begin{enumerate}
\item if \( S_1 \) and \( S_2 \) are nonnegative selfadjoint operators then
\[
(S_1 + S_2)_\mathcal{K} \geq (S_1)_\mathcal{K} + (S_2)_\mathcal{K};
\]
\item \( S_1 \geq S_2 \geq 0 \Rightarrow (S_1)_\mathcal{K} \geq (S_2)_\mathcal{K}; \)
\item if \( \{S_n\} \) is a nonincreasing sequence of nonnegative bounded selfadjoint operators and \( S = s - \lim_{n \to \infty} S_n \) then
\[
s - \lim_{n \to \infty} (S_n)_\mathcal{K} = S_{\mathcal{K}}.
\]
\end{enumerate}

Let \( \mathcal{K}^\perp = \mathcal{H} \ominus \mathcal{K} \). Then a bounded selfadjoint operator \( S \) has the block-matrix form
\[
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{12}^* & S_{22}
\end{pmatrix} : \begin{pmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K} \\ \mathcal{K}^\perp \end{pmatrix}.
\]
According to Sylvester’s criteria the operator \( S \) is nonnegative if and only if
\[
S_{22} \geq 0, \quad \text{ran } S_{12}^* \subseteq \text{ran } S_{22}^{1/2}, \quad S_{11} \geq \begin{pmatrix} S_{22}^{-1/2} S_{12}^* \\ S_{22}^{-1/2} S_{12} \end{pmatrix}^* \begin{pmatrix} S_{22}^{-1/2} S_{12}^* \\ S_{22}^{-1/2} S_{12} \end{pmatrix},
\]
where \( S_{22}^{-1/2} \) is the Moore – Penrose pseudoinverse operator. Moreover, if \( S \geq 0 \) then the operator \( S_{\mathcal{K}} \) is given by the relation
\[
(2.2) \quad S_{\mathcal{K}} = \begin{pmatrix}
S_{11} - \begin{pmatrix} S_{22}^{-1/2} S_{12}^* \\ S_{22}^{-1/2} S_{12} \end{pmatrix}^* \begin{pmatrix} S_{22}^{-1/2} S_{12}^* \\ S_{22}^{-1/2} S_{12} \end{pmatrix} & 0 \\
0 & 0
\end{pmatrix}.
\]
If \( S_{22} \) has a bounded inverse then \(2.2\) takes the form
\[
(2.3) \quad S_{\mathcal{K}} = \begin{pmatrix}
S_{11} - S_{12} S_{22}^{-1} S_{12}^* & 0 \\
0 & 0
\end{pmatrix}.
\]
As is well known, the right hand side of (2.3) is called the Schur complement of the matrix $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix}$. From (2.2) it follows that

$$S_K = 0 \iff \text{ran} \ S_{12}^* \subset \text{ran} \ S_{22}^{1/2} \quad \text{and} \quad S_{11} = \left( S_{22}^{-1/2} S_{12}^* \right)^* \left( S_{22}^{-1/2} S_{12} \right).$$

The next representation of the shorted operator is new.

**Theorem 2.2.** Let $X$ be a nonnegative selfadjoint contraction in the Hilbert space $\mathcal{H}$ and let $K$ be a subspace in $\mathcal{H}$. Then holds the following equality

$$\tag{2.4} (I - X)_{K} = P_K - P_K \left( (I - X^{1/2} P_{K\perp} X^{1/2})^{-1/2} X^{1/2} P_K \right)^* \left( I - X^{1/2} P_{K\perp} X^{1/2} \right)^{-1/2} X^{1/2} P_K.$$

**Proof.** Let us prove (2.4) for the case $\|X\| < 1$. In this case the operator $I - X^{1/2} P_{K\perp} X^{1/2}$ has bounded inverse and

$$P_K X^{1/2} \left( I - X^{1/2} P_{K\perp} X^{1/2} \right)^{-1} X^{1/2} P_K = P_K (I - XP_{K\perp})^{-1} XP_K.$$ 

Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$ be the block-matrix representation of the operator $X$ with respect to the decomposition $\mathcal{H} = K \oplus K^\perp$. Then

$$\left( I - XP_{K\perp} \right)^{-1} = \begin{pmatrix} I & X_{12} (I - X_{22})^{-1} \\ 0 & (I - X_{22})^{-1} \end{pmatrix}. $$

Hence

$$P_K - P_K (I - XP_{K\perp})^{-1} XP_K = \begin{pmatrix} I - X_{11} - X_{12} (I - X_{22})^{-1} X_{12}^* & 0 \\ 0 & 0 \end{pmatrix},$$

and from (2.3) we get (2.4).

Now suppose that $\|X\| = 1$. Then (2.4) holds for the operator $\alpha X$, where $\alpha \in (0, 1)$. Let $\alpha_n = 1 - 1/n$, $n = 1, 2, \ldots$. The sequence of nonnegative selfadjoint contractions $\{I - \alpha_n X\}$ is nonincreasing and $\lim_{n \to \infty} (I - \alpha_n X) = I - X$. From Proposition 2.1 it follows that $\lim_{n \to \infty} (I - \alpha_n X)_{K} = (I - X)_{K}$. Since $I - X^{1/2} P_{K\perp} X^{1/2} = I - X + X^{1/2} P_K X^{1/2}$, we have

$$\|((I - X^{1/2} P_{K\perp} X^{1/2})^{1/2} f\|^2 \geq \|P_K X^{1/2} f\|^2, \ f \in \mathcal{H}.$$ 

The equality

$$\lim_{z \to 0} \left( (B - zI)^{-1} g, g \right) = \begin{cases} \|B^{-1/2} g\|^2, & g \in \text{ran} B^{1/2}, \\ +\infty, & g \in \mathcal{H} \setminus \text{ran} B^{1/2}, \end{cases}$$

holds for a bounded selfadjoint nonnegative operator $B$ (see [28]). From R. Douglas theorem [23], [25] it follows that

$$\text{ran} \ X^{1/2} P_K \subset \text{ran} \ (I - X^{1/2} P_{K\perp} X^{1/2})^{1/2}.$$ 

Hence

$$\lim_{n \to \infty} \left( (I - \alpha_n X^{1/2} P_{K\perp} X^{1/2})^{1/2} X^{1/2} P_K f, X^{1/2} P_K f \right) = \|((I - X^{1/2} P_{K\perp} X^{1/2})^{1/2} X^{1/2} P_K f\|, \ f \in \mathcal{H}.$$ 

Now we arrive to (2.4).
3. Parametrization of contractive block-operator matrices

Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ be Hilbert spaces and let $T$ be a contraction from $\mathcal{H} \oplus \mathcal{M}$ into $\mathcal{K} \oplus \mathcal{N}$. The following well known result gives the parametrization of the corresponding representation of $T$ in a block operator matrix form.

**Theorem 3.1.** [14, 22, 35]. The operator matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{K} \\ \mathcal{N} \end{pmatrix}$$

is a contraction if and only if $D \in \mathcal{L}(\mathcal{M}, \mathcal{N})$ is a contraction and the entries $A, B,$ and $C$ take the form

$$B = FD_D, \quad C = D_D \cdot G,$$

$$A = -FD^*G + D_F \cdot LD_G,$$

where the operators $F \in \mathcal{L}(\mathcal{D}_D, \mathcal{K}), \quad G \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{D^*})$ and $L \in \mathcal{L}(\mathcal{D}_G, \mathcal{D}_{F^*})$ are contractions. Moreover, operators $F, G,$ and $L$ are uniquely determined.

Next we derive expressions for the shorted operators $(D_T^2)_\mathcal{H}, (D_{PmT}^2)_\mathcal{H}, (D_T^2)_\mathcal{K},$ and $(D_{PmT}^2)_\mathcal{K}$ for a contraction $T$ given by a block matrix form

$$T = \begin{pmatrix} -FD^*G + D_{F^*} \cdot LD_G \\ FD_D \\ D_{D^*} \cdot G \\ D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{K} \\ \mathcal{N} \end{pmatrix}.$$

By calculations from Theorem 3.1 we obtain for all $\begin{pmatrix} f \\ h \end{pmatrix}$ and $\begin{pmatrix} g \\ \varphi \end{pmatrix}$, where $f \in \mathcal{H}, \ h \in \mathcal{M}, \ g \in \mathcal{K}, \ \varphi \in \mathcal{N}$

$$(3.1) \quad \left\| D_T \left( \begin{array}{c} f \\ h \end{array} \right) \right\|^2 = \| FD (D_D h - D^* G f) - F^* LD_G f \|^2 + \| DL D_G f \|^2,$$

$$(3.2) \quad \left\| D_{T^*} \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right\|^2 = \| D_G^* (D_D \cdot \varphi - D F^* g) - GL^* D_{F^*} g \|^2 + \| D_L \cdot D_{F^*} g \|^2,$$

$$(3.3) \quad \left\| D_{PmT} \left( \begin{array}{c} f \\ h \end{array} \right) \right\|^2 = \| D_D h - D^* G f \|^2 + \| D_G f \|^2, \quad f \in \mathcal{H}, \ h \in \mathcal{M},$$

and

$$(3.4) \quad \left\| D_{PmT^*} \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right\|^2 = \| D_D \cdot \varphi - D F^* g \|^2 + \| D_{F^*} g \|^2, \quad g \in \mathcal{K}, \ \varphi \in \mathcal{N}.$$

It follows from (3.1)–(3.4) that

$$\inf_{h \in \mathcal{M}} \left\{ \left\| D_T^2 \left( \begin{array}{c} f \\ h \end{array} \right) \right\|^2 \right\} = \| DL D_G f \|^2, \quad \inf_{h \in \mathcal{M}} \left\{ \left\| D_{PmT}^2 \left( \begin{array}{c} f \\ h \end{array} \right) \right\|^2 \right\} = \| D_G f \|^2,$$

$$\inf_{\varphi \in \mathcal{N}} \left\{ \left\| D_{T^*}^2 \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right\|^2 \right\} = \| DL \cdot D_{F^*} g \|^2, \quad \inf_{\varphi \in \mathcal{N}} \left\{ \left\| D_{PmT^*}^2 \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right\|^2 \right\} = \| D_{F^*} g \|^2.
Now (2.1) yields the following equalities for shorted operators
\[
\begin{align*}
\left( (D^2_T)_S \left( \begin{array}{c} f \\ h \end{array} \right), \left( \begin{array}{c} f \\ h \end{array} \right) \right) &= \| D_L D_G f \|^2, \\
\left( (D^2_{PmT})_S \left( \begin{array}{c} f \\ h \end{array} \right), \left( \begin{array}{c} f \\ h \end{array} \right) \right) &= \| D_G f \|^2, f \in \mathcal{H}, h \in \mathcal{M}, \\
\left( (D^2_{T^*})_R \left( \begin{array}{c} g \\ \varphi \end{array} \right), \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right) &= \| D_{L^*} D_{F^*} g \|^2, \\
\left( (D^2_{PmT^*})_R \left( \begin{array}{c} g \\ \varphi \end{array} \right), \left( \begin{array}{c} g \\ \varphi \end{array} \right) \right) &= \| D_{F^*} g \|^2, g \in \mathcal{H}, \varphi \in \mathcal{M}.
\end{align*}
\]

From (3.5) it follows that
\[
(D^2_T)_S = (D^2_{PmT})_S \iff (D^2_{T^*})_R = (D^2_{PmT^*})_R \\
\iff T = \begin{pmatrix} -FD^*G & FD_D \\ 0 & D \end{pmatrix}.
\]

Let \( D \in \mathcal{L}(\mathcal{M}, \mathcal{N}) \) be a contraction with nonzero defect operators and let \( Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : (\mathcal{H} \oplus \mathcal{D}_D) \to (\mathcal{K} \oplus \mathcal{D}_{D^*}) \) be a bounded operator. Define the transformation
\[
\mathcal{M}_D(Q) = \begin{pmatrix} -FD^*G & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} I_R & 0 \\ 0 & D_{D^*} \end{pmatrix} \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} \begin{pmatrix} I_S & 0 \\ 0 & D \end{pmatrix}.
\]

Clearly, the operator \( T = \mathcal{M}_D(Q) \) has the following matrix form
\[
T = \begin{pmatrix} S - FD^*G & FD_D \\ D_D^* G & D \end{pmatrix} : (\mathcal{H} \oplus \mathcal{M}) \to (\mathcal{K} \oplus \mathcal{N}).
\]

**Proposition 3.2.** Let \( \mathcal{H}, \mathcal{M}, \mathcal{N} \) be separable Hilbert spaces, \( D \in \mathcal{L}(\mathcal{M}, \mathcal{N}) \) be a contraction with nonzero defect operators, \( Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : (\mathcal{H} \oplus \mathcal{D}_D) \to (\mathcal{K} \oplus \mathcal{D}_{D^*}) \) be a bounded operator, and let
\[
T = \mathcal{M}_D(Q) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\mathcal{H} \oplus \mathcal{M}) \to (\mathcal{K} \oplus \mathcal{N}).
\]

Then
1. hold the equalities
\[
\bigcap_{n=0}^{\infty} \ker (B^* A^n) = \bigcap_{n=0}^{\infty} \ker (F^* S^n), \\
\bigcap_{n=0}^{\infty} \ker (C A^n) = \bigcap_{n=0}^{\infty} \ker (G S^n),
\]
(2) T is a contraction if and only if Q is a contraction. T is isometric (co-isometric) if and only if Q is isometric (co-isometric). Moreover, hold the equalities
\[
(D_Q^2)_R = (D_T^2)_R, \quad \left( D_{P_{D^*}} Q \right)_R = \left( D_{P_{D^*}} T \right)_R.
\]

Proof. Observe that

\[
A = -FD^*G + S, \quad B = FD, \quad C = D^*G.
\]

Let Ω be a neighborhood of the origin and such that the resolvents \((I_\overline{H} - \lambda A^*)^{-1}, (I_\overline{H} - \lambda A)^{-1}, (I_\overline{H} - \lambda S^*)^{-1}, (I_\overline{H} - \lambda S)^{-1}\) exist. Then
\[
F^* (I_\overline{H} - \lambda A^*)^{-1} = F^* (I_\overline{H} - \lambda S^* + \lambda G^* DF^*)^{-1} = \lambda S^* (I_\overline{H} - \lambda S^* + \lambda G^* DF^*)^{-1},
\]
\[
= \lambda S^* (I_\overline{H} - \lambda S^* + \lambda G^* DF^*)^{-1} = (I_\overline{H} + \lambda F^* (I_\overline{H} - \lambda S^*)^{-1} G^* D)^{-1} F^* (I_\overline{H} - \lambda S^*)^{-1}.
\]

It follows that
\[
\bigcap_{\lambda \in \Omega} \ker (F^* (I_\overline{H} - \lambda A^*)^{-1}) = \bigcap_{\lambda \in \Omega} \ker (F^* (I_\overline{H} - \lambda S^*)^{-1}).
\]

Similarly
\[
\bigcap_{\lambda \in \Omega} \ker (G (I_\overline{H} - \lambda A)^{-1}) = \bigcap_{\lambda \in \Omega} \ker (G (I_\overline{H} - \lambda S)^{-1}).
\]

Since \(B^* = D^*F^*\) and \(C = D^*G\), we get (3.8).

The statement (2) is a consequence of Theorem 3.1 and formulas (3.1)–(3.5). \(\square\)

**Proposition 3.3.** Let \(D \in L(\mathcal{M}, \mathcal{N})\) be a contraction with nonzero defect operators, \(Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix}: \begin{pmatrix} \mathcal{H} \\ \mathcal{D} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{D}^* \end{pmatrix}\) be a contraction and let \(T = M_D(Q)\). Then for every non-negative selfadjoint contraction \(X\) in \(\mathcal{H}\) hold the following equalities
\[
(D_T^2 + T^* (I_\overline{H} - X) P_\mathcal{H}^* T)_\mathcal{H} = (D_Q^2 + Q^* (I_\overline{H} - X) P_\mathcal{H}^* Q)_\mathcal{H}, \tag{3.9}
\]
\[
(D_Q^2 + Q^* (I_\overline{H} - X) P_\mathcal{H}^* Q)_\mathcal{H} = (D_G^2 - D_G L^* D_F^* X^{1/2} (I_\overline{H} - X^{1/2} F F^* X^{1/2})^{-1} X^{1/2} D_F^* L D_G) P_\mathcal{H}, \tag{3.10}
\]
where \(P_\mathcal{H} (P_\mathcal{H}^*)\) is the orthogonal projection in \(\mathcal{H} = \mathcal{H} \oplus \mathcal{M}\) \((\mathcal{H}' = \mathcal{H} \oplus \mathcal{M})\) onto \(\mathcal{H}\), and
\[
D_F^* X^{1/2} (I_\overline{H} - X^{1/2} F F^* X^{1/2})^{-1} X^{1/2} D_F^* := \left( (I_\overline{H} - X^{1/2} F F^* X^{1/2})^{-1/2} X^{1/2} D_F^* \right)^* (I_\overline{H} - X^{1/2} F F^* X^{1/2})^{-1/2} X^{1/2} D_F^*.
\]

Proof. Define the contraction
\[
\tilde{Q} = \begin{pmatrix} X^{1/2} S & X^{1/2} F \\ G & 0 \end{pmatrix}: \begin{pmatrix} \mathcal{H} \\ \mathcal{D} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{D}^* \end{pmatrix}. \tag{3.11}
\]

Then
\[
\tilde{D}_Q^2 = D_Q^2 + Q^* (I_\overline{H} - X) P_\mathcal{H}^* Q.
\]
By Proposition 3.2 the operator \( \tilde{T} = M_D(\bar{Q}) \) is a contraction as well. Clearly,

\[
D_f^2 = D_f^2 + T^*(I_\beta - X)P'_\beta T
\]

Applying once again Proposition 3.2 we arrive to (3.9).

Since \( Q \) is a contraction, by Theorem 3.1 the operator \( S \) takes the form \( S = D_FLD_G \), where \( L \in L(\mathfrak{D}_G, \mathfrak{D}_F^*) \) is a contraction. Because \( \bar{Q} \) given by (3.11) is a contraction, we get \( X^{1/2}S = D_FLD_G \), where \( \bar{F} = X^{1/2}F \) and \( \bar{L} \in L(\mathfrak{D}_G, \mathfrak{D}_F^*) \) is a contraction. Since

\[
D_{F^*}^2 = I_\beta - \bar{F}\bar{F}^* = I_\beta - X^{1/2}FF^*X^{1/2} = I_\beta - X + X - X^{1/2}FF^*X^{1/2} =
\]

we have \( \|D_{F^*}f\| \geq \|D_FX^{1/2}f\| \) for all \( f \in \mathcal{D}_\beta \). Using R. Douglas theorem [23], [25] we conclude that \( \text{ran } D_{F^*} \supset \text{ran } (X^{1/2}D_{F^*}) \). Let \( D_{F^*}^{-1} = (I_\beta - X^{1/2}FF^*X^{1/2})^{-1/2} \) be the Moore-Penrose inverse for \( D_{F^*} \). Then we obtain the equality

\[
\tilde{L} = \bar{D}_{F^*}^{-1}(X^{1/2}D_{F^*})L.
\]

The first equality in (3.5) yields \( (D_Q^2)_{\mathcal{D}_\beta} = D_GD_L^2D_GP_{\mathcal{D}_\beta} \). Since

\[
(D_Q^2)_{\mathcal{D}_\beta} = (D_Q^2 + Q^*(I_\beta - X)P'_\beta Q)_{\mathcal{D}_\beta},
\]

we get (3.10).

\[\square\]

4. The Möbius representations

Let \( T : H_1 \rightarrow H_2 \) be a contraction and let \( \mathcal{V}_{T^*} \) be the set of all contractions \( Z \in L(\mathfrak{D}_T, \mathfrak{D}_T^*) \) such that \(-1 \in \rho(T^*Z)\). In [33] the fractional-linear transformations

(4.1)

\[\mathcal{V}_{T^*} \ni Z \rightarrow Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T\]

was studied and the following result has been established.

**Theorem 4.1.** [33] Let \( T \in L(H_1, H_2) \) be a contraction and let \( Z \in \mathcal{V}_{T^*} \). Then \( Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T \) is a contraction,

\[
\|D_Qf\|^2 = \|D_{T}(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_Tf\|, \quad f \in H_1,
\]

ran \( D_Q \subseteq \text{ran } D_T \), and ran \( D_Q = \text{ran } D_T \) if and only if \( \|Z\| < 1 \). Moreover, if \( Q \in L(H_1, H_2) \) is a contraction and \( Q = T + D_{T^*}XD_T \), where \( X \in L(\mathfrak{D}_T, \mathfrak{D}_T^*) \) then

\[
2 \text{ Re } ((I_{\mathfrak{D}_T} - T^*X)f, f) \geq \|f\|^2
\]

for all \( f \in \mathfrak{D}_T^* \), the operator \( Z = X(I_{\mathfrak{D}_T} - T^*X)^{-1} \) belongs to \( \mathcal{V}_{T^*} \), and

\[Q = T + D_{T^*}Z(I_{\mathfrak{D}_T} + T^*Z)^{-1}D_T \]

The transformation (4.1) is called in [33] the unitary linear-fractional transformation. It is not difficult to see that if \( \|T\| < 1 \) then the closed unit operator ball in \( L(H_1, H_2) \) belongs to the set \( \mathcal{V}_{T^*} \) and, moreover

\[
T + D_{T^*}Z(I_{H_1} + T^*Z)^{-1}D_T = D_{T^*}^{-1}(Z + T)(I_{H_1} + T^*Z)^{-1}D_T =
\]

\[
= D_{T^*}(I_{H_2} + ZT^*)^{-1}(Z + T)D_{T^*}^{-1}
\]
for all $Z \in L(H_1, H_2)$, $\|Z\| \leq 1$. Thus, the transformation \eqref{1} is an operator analog of a well known Möbius transformation of the complex plane

$$z \rightarrow \frac{z + t}{1 + tz}, \quad |t| \leq 1.$$ 

The next theorem is a version of the more general result established by Yu.L. Shmul’yan in [33].

**Theorem 4.2.** Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert spaces and let the function $\Theta(\lambda)$ be from the Schur class $S(\mathcal{M}, \mathcal{N})$. Then

1. the linear manifolds $\text{ran} D_{\Theta(\lambda)}$ and $\text{ran} D_{\Theta^*(\lambda)}$ do not depend on $\lambda \in \mathbb{D}$,
2. for arbitrary $\lambda_1$, $\lambda_2$, $\lambda_3$ in $\mathbb{D}$ the function $\Theta(\lambda)$ admits the representation

$$\Theta(\lambda) = \Theta(\lambda_1) + D_{\Theta^*(\lambda_2)} \Psi(\lambda) D_{\Theta(\lambda_3)},$$

where $\Psi(\lambda)$ is $L(\mathcal{D}_{\Theta(\lambda_3)}, \mathcal{D}_{\Theta^*(\lambda_2)})$-valued function holomorphic in $\mathbb{D}$.

Now using Theorems 4.1 and 4.2 we obtain the following result (cf. [15]).

**Theorem 4.3.** Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert spaces and let the function $\Theta(\lambda)$ be from the Schur class $S(\mathcal{N}, \mathcal{N})$. Then there exists a unique function $Z(\lambda)$ from the Schur class $S(\mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)})$ such that

$$\Theta(\lambda) = \Theta(0) + D_{\Theta^*(0)} Z(\lambda)(I_{\mathcal{D}_{\Theta(0)}} + \Theta^*(0)Z(\lambda))^{-1} D_{\Theta(0)}, \quad \lambda \in \mathbb{D}.$$ 

In what follows we will say that the right hand side of the above equality is the Möbius representation and the function $Z(\lambda)$ is the the Möbius parameter of $\Theta(\lambda)$. Clearly, $Z(0) = 0$ and by Schwartz’s lemma we obtain that

$$\|Z(\lambda)\| \leq |\lambda|, \quad \lambda \in \mathbb{D}.$$ 

The next result provides connections between the realizations of $\Theta(\lambda)$ and $Z(\lambda)$.

**Theorem 4.4.**

1. Let $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}$ be a passive system and let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LDG & FD_D \\ D_{D^*}G & D_D \end{pmatrix} : \left( \begin{array}{c} \mathcal{H} \\ \mathcal{M} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathcal{H} \\ \mathcal{N} \end{array} \right)$.

   Let $\Theta(\lambda)$ be the transfer function of $\tau$. Then
   
   a) the Möbius parameter $Z(\lambda)$ of the function $\Theta(\lambda)$ is the transfer function of the passive system

   $$\nu = \left\{ \begin{pmatrix} D_{F^*}LDG & F \\ G & 0 \end{pmatrix} : \mathcal{H}, \mathcal{D}_D, \mathcal{D}_D^* \right\};$$

   b) the system $\tau$ isometric (co-isometric) $\Rightarrow$ the system $\nu$ isometric (co-isometric);

   c) the equalities $\mathcal{H}_\nu = \mathcal{H}_\tau$, $\mathcal{H}_\nu^* = \mathcal{H}_\tau^*$ hold and hence the system $\tau$ is controllable (observable) $\Rightarrow$ the system $\nu$ is controllable (observable), the system $\tau$ is simple (minimal) $\Rightarrow$ the system $\nu$ is simple (minimal).

2. Let a nonconstant function $\Theta(\lambda)$ belongs to the Schur class $S(\mathcal{M}, \mathcal{N})$ and let $Z(\lambda)$ be the Möbius parameter of the function $\Theta(\lambda)$. Suppose that the transfer function of the linear system

   $$\nu' = \left\{ \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \mathcal{H}, \mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)} \right\}$$
coincides with \( Z(\lambda) \) in a neighborhood of the origin. Then the transfer function of the linear system

\[
\tau' = \left\{ \begin{pmatrix} -F\Theta^*(0)G + S & FD\Theta(0) \\ D\Theta^*(0)G & \Theta(0) \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}
\]

coincides with \( \Theta(\lambda) \) in a neighborhood of the origin. Moreover

(a) the equalities \( \mathcal{H} = \mathcal{V} \), \( \mathcal{M} = \mathcal{V} \) hold, and hence the system \( \nu' \) is controllable (observable) \( \Rightarrow \) the system \( \tau' \) is controllable (observable), the system \( \nu' \) is simple (minimal) \( \Rightarrow \) the system \( \tau' \) is simple (minimal),

(b) the system \( \nu' \) is passive \( \Rightarrow \) the system \( \tau' \) is passive,

(c) the system \( \nu' \) isometric (co-isometric) \( \Rightarrow \) the system \( \tau' \) isometric (co-isometric).

**Proof.** Suppose that \( D \in \mathcal{L}(\mathcal{M}, \mathcal{N}) \) is a contraction with nonzero defects. Given the operator matrices \( Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \mathcal{H}_D \to \mathcal{H}_D^* \). Let

\[
T = \mathcal{M}_D(Q) = \begin{pmatrix} S - FD^*G & FD_D \\ D_D^*G & D \end{pmatrix} : \mathcal{H}_M \to \mathcal{H}_N
\]

and let \( \Omega \) be a sufficiently small neighborhood of the origin. Consider the linear systems

\[
\left\{ \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \mathcal{H}_D, \mathcal{D}_D, \mathcal{D}_D^* \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} -FD^*G + S & FD_D \\ D_D^*G & D \end{pmatrix} ; \mathcal{H}_M, \mathcal{M}, \mathcal{N} \right\}.
\]

Define the transfer functions

\[
Z(\lambda) = \lambda G(I_H - \lambda S)^{-1} \quad \text{and} \quad \Theta(\lambda) = D + \lambda D_D^*G(I_H - \lambda(S - FD^*G))^{-1} FD_D
\]

Since \( \Theta(0) = D \), we have for \( \lambda \in \Omega \)

\[
Z(\lambda)(I_{\mathcal{D}_D} + \Theta^*(0)Z(\lambda))^{-1} = \\
= \lambda G(I_H - \lambda S)^{-1}F(I_{\mathcal{D}_D} + \lambda D^*G(I_H - \lambda S)^{-1}F)^{-1} = \\
= \lambda G(I_H - \lambda S)^{-1}(I_H + \lambda FD^*G(I_H - \lambda S)^{-1}F)^{-1} F = \\
= \lambda G(I_H - \lambda S + \lambda FD^*G)^{-1} F.
\]

Hence

\[
\Theta(\lambda) = \Theta(0) + D\Theta^*(0)Z(\lambda)(I_{\mathcal{D}_\Theta(0)} + \Theta^*(0)Z(\lambda))^{-1}D\Theta(0), \quad \lambda \in \Omega.
\]

According to Theorem 3.1 the operator \( Q \) is a contraction if and only if \( F, G \) are contractions and \( S = D_F^*LD_G \), where \( L \in \mathcal{L}(\mathcal{D}_G, \mathcal{D}_F^*) \) is a contractions. Now from Proposition 3.2 we get that all statements of Theorem 4.4 hold true. \( \square \)

**Corollary 4.5.** The equivalences

\[
\varphi_\Theta(\lambda) = 0 \iff \varphi_Z(\lambda) = 0,
\]

\[
\psi_\Theta(\lambda) = 0 \iff \psi_Z(\lambda) = 0
\]

hold.
Proof. Let \( \varphi_\Theta(\lambda) = 0 \) (\( \psi_\Theta(\lambda) = 0 \)) and let \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) be a simple conservative system with transfer function \( \Theta(\lambda) \). By Theorem 4.1 the system \( \tau \) is observable (controllable). By Theorem 4.4 the corresponding system \( \nu \) with transfer function \( Z(\lambda) \) is conservative and observable (controllable). Theorem 1.1 yields that \( \varphi_Z(\lambda) = 0 \) (\( \psi_Z(\lambda) = 0 \)).

Conversely, let \( \varphi_Z(\lambda) = 0 \) (\( \psi_Z(\lambda) = 0 \)) and let \( \nu' \) be a simple conservative system with transfer function \( Z(\lambda) \). Again by Theorem 1.1 the system \( \nu' \) is observable (controllable). By Theorem 4.4 the corresponding system \( \tau' \) with transfer function \( \Theta(\lambda) \) is conservative and observable (controllable) as well and Theorem 1.1 yields that \( \varphi_\Theta(\lambda) = 0 \) (\( \psi_\Theta(\lambda) = 0 \)). \( \square \)

The next statement follows from Theorem 1.3.

Corollary 4.6. Let \( \Theta(\lambda) \in S(\mathcal{M}, \mathcal{N}) \).

1. Suppose that \( \varphi_\Theta(\lambda) = 0 \) (\( \psi_\Theta(\lambda) = 0 \)). Then every passive controllable (observable) realization of the M{"o}bius parameter \( Z(\lambda) \) of \( \Theta(\lambda) \) is isometric (co-isometric) and minimal system;
2. Suppose that \( \varphi_\Theta(\lambda) = 0 \) and \( \psi_\Theta(\lambda) = 0 \). Then every simple and passive realization of the M{"o}bius parameter \( Z(\lambda) \) of \( \Theta(\lambda) \) is conservative and minimal.

Proposition 4.7. Let \( \Theta(\lambda) \) be a function from the Schur class \( S(\mathcal{M}, \mathcal{N}) \). Suppose that the M{"o}bius parameter \( Z(\lambda) \) of \( \Theta(\lambda) \) is a linear function of the form \( Z(\lambda) = \lambda K \), \( ||K|| \leq 1 \). Then there exists a passive realization \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) such that

\[
(D^2_{F^*_N T})_{\mathcal{S}} = (D^2_T)_{\mathcal{S}},
\]

where \( T = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \).

Conversely, if a passive system \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) possess the property \( (4.2) \) then the M{"o}bius parameter \( Z(\lambda) \) of the transfer function \( \Theta(\lambda) \) of \( \tau \) is a linear function of the form \( \lambda K \).

Proof. Let \( Z(\lambda) = \lambda K \), \( \lambda \in \mathbb{D} \), where \( K \in L(\mathfrak{D}(\Theta(0)), \mathfrak{D}(\Theta^*(0))) \) is a contraction. Then \( Z(\lambda) \) can be realized as the transfer function of a passive system of the form

\[
\nu = \left\{ \left( \begin{array}{cc} 0 & F' \\ G & 0 \end{array} \right); \mathcal{H}, \mathfrak{D}(\Theta(0)), \mathfrak{D}(\Theta^*(0)) \right\}.
\]

Actually, take

\[
\mathcal{H} = \text{ran} K, \; F = K, \; G = j,
\]

where \( j \) is the embedding of \( \text{ran} K \) into \( \mathfrak{D}(\Theta^*(0)) \). It follows that \( GF = K \). By Theorem 4.4 the system \( \tau = \left\{ \left( \begin{array}{cc} -F(\Theta^*(0))G & FD_\Theta(0) \\ D_\Theta(0)G & \Theta(0) \end{array} \right); \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a passive realization of the function \( \Theta(\lambda) \). From (3.6) it follows that \( (D^2_{F^*_N T})_{\mathcal{S}} = (D^2_T)_{\mathcal{S}} \) for \( T = \left( \begin{array}{cc} -F(\Theta^*(0))G & FD_\Theta(0) \\ D_\Theta(0)G & \Theta(0) \end{array} \right) \).
Assume now \((D_{P_0T})_\mathcal{H} = (D_T^2)_{\mathcal{H}}\), where \(T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix}\) is a contraction and let \(\Theta(\lambda)\) be the transfer function of the system \(\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N}\). Then the entries \(A, B,\) and \(C\) takes the form \((1.6)\), and \(D = \Theta(0).\) According to \((3.6)\) we have \(D_{F^*}LD_G = 0, \) i.e. \(T = \begin{pmatrix} -F\Theta^*(0)G & FD\Theta^*(0) \\ D\Theta^*(0)G & \Theta(0) \end{pmatrix}\). By Theorem 4.4 the Möbius parameter \(Z(\lambda)\) of \(\Theta(\lambda)\) takes the form \(Z(\lambda) = \lambda GF\). 

**Example 4.8.** Let \(A\) be completely non-unitary contractions in the Hilbert space \(\mathcal{H}\) and let \(\Phi(\lambda)\) be the Sz.-Nagy–Foias characteristic function of \(A^* \mathcal{M}\):

\[
\Phi(\lambda) = (-A^* + \lambda DA(I_\mathcal{H} - \lambda A)^{-1}DA^*) | \mathcal{D}_{A^*} : \mathcal{D}_{A^*} \to \mathcal{D}_A, \ |\lambda| < 1.
\]

The system

\[
\tau = \begin{cases} (A & D_{A^*} \\ D_A & -A^*) ; \mathcal{H}, \mathcal{D}_{A^*}, \mathcal{D}_A \end{cases}
\]

is conservative and simple. Let

\[
\Phi(\lambda) = \Phi(0) + D_{\Phi^*(0)}Z(\lambda)(I_{\mathcal{D}_{\Phi(0)}} + \Phi^*(0)Z(\lambda))^{-1}D\Phi(0), \ \lambda \in \mathbb{D}
\]

be the Möbius representation of the function \(\Phi(\lambda)\). Since \(F\) and \(G^*\) are imbedding of the subspaces \(\mathcal{D}_{A^*}\) and \(\mathcal{D}_A\) into \(\mathcal{H}\), we get that

\[
D_{F^*} = P_{\ker D_{A^*}}, \ D_G = P_{\ker D_A}
\]

and \(L = A|_{\ker \mathcal{D}_A}\) is isometric operator. Let

\[
\nu = \begin{cases} (AP_{\ker D_A} & I \\ P_{\mathcal{D}_A} & 0) ; \mathcal{H}, \mathcal{D}_{A^*}, \mathcal{D}_A \end{cases}
\]

By Theorem 4.4

\[
Z(\lambda) = \lambda P_{\mathcal{D}_A} (I_{\mathcal{H}} - \lambda AP_{\ker D_A})^{-1} | \mathcal{D}_{A^*}, \ |\lambda| < 1
\]

and this function is transfer function of \(\nu\). Note that this function is the Sz.-Nagy–Foias characteristic function of the partial isometry \(A^*P_{\ker \mathcal{D}_{A^*}}\).

5. **The Kalman–Yakubovich–Popov Inequality and Riccati Equation**

Let \(\mathcal{H}, \mathcal{M}\) and \(\mathcal{N}\) be Hilbert spaces and let \(T\) be a bounded linear operator from the Hilbert space \(\mathcal{H} = \mathcal{H} \oplus \mathcal{M}\) into the Hilbert space \(\mathcal{H}' = \mathcal{H} \oplus \mathcal{N}\) given by the block matrix

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{N} \end{pmatrix}.
\]

Suppose that \(X\) is a positive selfadjoint operator in the Hilbert space \(\mathcal{H}\) such that

\[
A \text{dom} X^{1/2} \subset \text{dom} X^{1/2}, \ \text{ran} B \subset \text{dom} X^{1/2}.
\]

As was mentioned in Introduction the inequality \((1.3)\)

\[
\left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_\mathcal{M} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_\mathcal{N} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 \geq 0
\]

for all \(x \in \text{dom} X^{1/2}, \ u \in \mathcal{M}\).
is called the generalized KYP inequality with respect to $X$ \cite{8,9}. For a bounded solution $X$ the KYP inequality (1.3) takes the form (1.4).

Put

$$
\hat{X} := \begin{pmatrix} X & 0 \\ 0 & I_{2n} \end{pmatrix}, \quad \bar{X} := \begin{pmatrix} X & 0 \\ 0 & I_n \end{pmatrix}.
$$

Operators $\hat{X}$ and $\bar{X}$ are positive selfadjoint operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively, dom $\hat{X} = \text{dom} X \oplus \mathfrak{M}$, dom $\bar{X} = \text{dom} X \oplus \mathfrak{N}$. Let the operator $X$ satisfies the KYP inequality. Let us define the operator $T_1$:

$$
\text{dom} T_1 := \text{ran} \hat{X}^{1/2} = \text{ran} X^{1/2} \oplus \mathfrak{M},
$$

\begin{align*}
(5.1) \\
T_1 := \hat{X}^{1/2} \hat{X}^{-1/2} &= \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{2n} \end{pmatrix} T \begin{pmatrix} X^{-1/2} & 0 \\ 0 & I_n \end{pmatrix} = \\
&= \begin{pmatrix} X^{1/2} A X^{-1/2} & X^{1/2} B \\ C X^{-1/2} & D \end{pmatrix}.
\end{align*}

Clearly, the following statements are equivalent:

1. $X$ is a solution of the KYP inequality (1.3);
2. the operator $T_1$ is densely defined contraction, i.e.

$$
\left\| \begin{pmatrix} X^{1/2} x \\ u \end{pmatrix} \right\|^2 - \left\| T_1 \begin{pmatrix} X^{1/2} x \\ u \end{pmatrix} \right\|^2 \geq 0, \quad x \in \text{dom} X^{1/2}, u \in \mathfrak{M};
$$

3. the operator $T$ is a contraction acting from a pre-Hilbert space dom $\hat{X}$ into a pre-

- Hilbert space dom $\bar{X}$ equipped by the inner products

$$
\left( \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} \right) = (X^{1/2} x_1, X^{1/2} x_2)_{\mathcal{H}} + (u_1, u_2)_{\mathfrak{M}},
$$

$$
\left( \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ v_2 \end{pmatrix} \right) = (X^{1/2} x_1, X^{1/2} x_2)_{\mathcal{H}} + (v_1, v_2)_{\mathfrak{N}},
$$

$x_1, x_2 \in \text{dom} X^{1/2}, u_1, u_2 \in \mathfrak{M}$, $v_1, v_2 \in \mathfrak{N}$;

4. $Z = X^{-1}$ is the solution of the generalized KYP inequality for the adjoint operator

$$
\left( \begin{pmatrix} Z^{1/2} & 0 \\ 0 & I_{2n} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \right) - \left( \begin{pmatrix} Z^{1/2} & 0 \\ 0 & I_{2n} \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \right) \geq 0
$$

for all $x \in \text{dom} Z^{1/2}, v \in \mathfrak{N}$.

Let the positive selfadjoint operator $X$ in $\mathcal{H}$ satisfies the KYP inequality. If

$$
(5.3) \inf_{u \in \mathfrak{M}} \left\{ \left\| \hat{X}^{1/2} \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 - \left\| \hat{X}^{1/2} T \begin{pmatrix} x \\ u \end{pmatrix} \right\|^2 \right\} = 0
$$

for all $x \in \text{dom} X^{1/2}$

we will say that the operator $X$ satisfies the Riccati equation.

**Proposition 5.1.** If the positive selfadjoint operator $X$ satisfies the Riccati equation then the continuation of the operator $T_1$ defined by (5.1) meets the condition

$$
(\mathcal{D}^2 T_1)_\mathcal{H} = 0.
$$
Proof. By (5.1) we get for all $\vec{f} \in \text{dom} \, \hat{X}^{1/2}$

$$\left\| D_{T_1} \hat{X}^{1/2} \vec{f} \right\|^2 = \left\| \hat{X}^{1/2} \vec{f} \right\|^2 - \left\| T_1 \hat{X}^{1/2} \vec{f} \right\|^2 = \left\| \hat{X}^{1/2} \vec{f} \right\|^2 - \left\| \tilde{X}^{1/2} \vec{T} \vec{f} \right\|^2.$$

Since (5.3) holds, we get

$$\inf_{u \in \mathfrak{M}} \left\{ \left\| D_{T_1} \left( X^{1/2} x \right) \right\|^2 \right\} = 0$$

for all $x \in \text{dom} \, X^{1/2}$ and all $u \in \mathfrak{M}$. Because $\text{ran} \, X^{1/2}$ is dense in $\hat{H}$, we obtain $(D_{T_1}^2)_{\hat{H}} = 0$. □

**Proposition 5.2.** Let $D \in \mathcal{L}(\mathfrak{M}, \mathfrak{N})$ be a contraction with nonzero defect operators. Let

$$Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \left( \hat{H} \mathfrak{D}_D \right) \rightarrow \left( \hat{H} \mathfrak{N} \right)$$

and let

$$T = \mathcal{M}_D(Q) : \left( \mathfrak{H} \mathfrak{M} \right) \rightarrow \left( \mathfrak{H} \mathfrak{N} \right).$$

Then

1. the KYP inequality (1.3) and the KYP inequality

$$\left\| \left( \begin{array}{cc} X^{1/2} & 0 \\ 0 & I_{\mathfrak{D}_D^*} \end{array} \right) \left( \begin{array}{c} x \\ u \end{array} \right) \right\|^2 - \left\| \left( \begin{array}{cc} X^{1/2} & 0 \\ 0 & I_{\mathfrak{D}_D^*} \end{array} \right) \begin{pmatrix} S & G \\ F & 0 \end{pmatrix} \left( \begin{array}{c} x \\ u \end{array} \right) \right\|^2 \geq 0$$

for all $x \in \text{dom} \, X^{1/2}$, $u \in \mathfrak{D}_D$ are equivalent,

2. the Riccati equation (5.3) and the Riccati equation

$$\inf_{u \in \mathfrak{D}_D} \left\{ \left\| \left( \begin{array}{c} X^{1/2} x \\ u \end{array} \right) \right\|^2 - \left\| \left( \begin{array}{c} X^{1/2} x \\ u \end{array} \right) \begin{pmatrix} S & G \\ F & 0 \end{pmatrix} \left( \begin{array}{c} x \\ u \end{array} \right) \right\|^2 \right\} = 0$$

are equivalent.

Proof. From (3.7) we have the relation

$$\mathcal{M}_D \left( \begin{array}{cc} X^{1/2} & 0 \\ 0 & I_{\mathfrak{D}_D^*} \end{array} \right) Q \left( \begin{array}{cc} X^{-1/2} & 0 \\ 0 & I_{\mathfrak{D}_D} \end{array} \right) = \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix} \mathcal{M}_D(Q) \begin{pmatrix} X^{-1/2} & 0 \\ 0 & I_{\mathfrak{M}} \end{pmatrix}. $$

Now the result follows from Propositions 3.2 and 5.1. □

6. Equivalent forms of the KYP inequality and Riccati equation for a passive system

**Theorem 6.1.** Let

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LD_G & FD_D \\ D_{D^*}G & D \end{pmatrix} : \left( \mathfrak{H} \mathfrak{M} \right) \rightarrow \left( \mathfrak{H} \mathfrak{N} \right)$$
be a contraction and let
\[ Q = \begin{pmatrix} D_F L D_G & F \\ G & 0 \end{pmatrix} : \mathcal{D}_D \rightarrow \mathcal{D}_{D^*}. \]

Then the following inequalities are equivalent
\begin{equation}
\begin{aligned}
&\begin{cases}
(X - A^* X A - C^* C & -A^* X B - C^* D \\
-B^* X A - D^* C & I_{2n} - B^* X B - D^* D \end{cases} \geq 0, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.1}
\end{equation}
\begin{equation}
\begin{aligned}
&\begin{cases}
(I_{\delta} - X) P_{\delta} \leq (D_T^2 + T^*(I_{\delta} - X) P_{\delta} T)_{\delta}, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.2}
\end{equation}
\begin{equation}
\begin{aligned}
&\begin{cases}
(X - G^* G & D_G L^* D_F^* X^{1/2} \\
X^{1/2} D_F^* L D_G & I_{\delta} - X^{1/2} F F^* X^{1/2} \end{cases} \geq 0, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.3}
\end{equation}
\begin{equation}
\begin{aligned}
&\begin{cases}
\begin{pmatrix} X & 0 \\ 0 & I_{2D} \end{pmatrix} - Q^* \begin{pmatrix} X & 0 \\ 0 & I_{2D^*} \end{pmatrix} Q \geq 0, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.5}
\end{equation}
\begin{equation}
\begin{aligned}
&\begin{cases}
\begin{pmatrix} X - G^* G - D_G L^* D_F^* X D_F^* L D_G & -D_G L^* D_F^* X F \\
-F^* X D_F^* L D_G & I_{2D} - F^* X F \end{pmatrix} \geq 0, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.6}
\end{equation}
\begin{equation}
\begin{aligned}
&\begin{cases}
(I_{\delta} - X) P_{\delta} \leq (D_Q^2 + Q^*(I_{\delta} - X) P_{\delta} Q)_{\delta}, \\
0 < X \leq I_{\delta}
\end{cases}
\end{aligned}
\tag{6.7}
\end{equation}

**Proof.** Note that (6.6) are (6.5) written in terms of the entries. By Proposition 5.2 the inequalities (6.1) and (6.5) are equivalent. Let us prove the equivalence of (6.1) and (6.2). Suppose that \( X \) satisfies (6.1) and put \( Y = I_{\delta} - X \). The operator \( Y \) belongs to the operator interval \([0, I_{\delta}]\) and \( \ker (I_{\delta} - Y) = \{0\} \). In terms of the operator \( Y \) we have
\[
0 \leq \begin{pmatrix} X & 0 \\ 0 & I_{2m} \end{pmatrix} - T^* \begin{pmatrix} X & 0 \\ 0 & I_{2m} \end{pmatrix} = \begin{pmatrix} I_{\delta} - Y & 0 \\ 0 & I_{2m} \end{pmatrix} - T^* \begin{pmatrix} I_{\delta} - Y & 0 \\ 0 & I_{2m} \end{pmatrix} T =
\]
\[
= I - T^* T + T^* Y P_{\delta} T - Y P_{\delta},
\]
The weak form of the above inequality is the following
\begin{equation}
(Y x, x) \leq \left( D_T^2 + T^* Y P_{\delta} T \right) \begin{pmatrix} x \\ u \end{pmatrix}, \begin{pmatrix} x \\ u \end{pmatrix}, \ x \in \mathcal{H}, \ u \in \mathcal{M}.
\end{equation}
The equality (2.1) for the shorted operator yields that the operator \( Y \) is a solution of the system
\[
\begin{align*}
YP_S & \leq \left( D_T^2 + T^* Y P_S T \right)_S, \\
0 & \leq Y < I_S
\end{align*}
\]
(6.9)
If \( X \) is a solution of the system (6.2) then \( Y = I_S - X \) satisfies (6.8) and therefore \( X \) satisfies (6.1).

Similarly (6.5) is equivalent to (6.7). Note that by Proposition 3.3 the right hand sides of (6.2) and (6.7) are equal. Using (3.10) we get that (6.7) is equivalent to (6.4). By Sylvester’s criteria (6.3) is equivalent to (6.4).

**Proposition 6.2.** Let the function \( \Theta(\lambda) \) belongs to the Schur class \( S(M, N) \) and let \( Z(\lambda) \) be its Möbius parameter. Then the passive minimal realization
\[
\nu = \left\{ \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} ; \mathcal{H}, \mathcal{D}_{\Theta(0)}, \mathcal{D}_{\Theta^*(0)} \right\}
\]
of \( Z(\lambda) \) is optimal ((*)-optimal) if and only if the passive minimal realization
\[
\tau = \left\{ \begin{pmatrix} -F\Theta^*(0)G + S & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}
\]
of \( \Theta(\lambda) \) is optimal ((*)-optimal).

**Proof.** According to Theorem 6.1 the set of all solutions of the KYP inequality (6.1) for \( T = \left( \begin{pmatrix} -F\Theta^*(0)G + S & FD_{\Theta(0)} \\ D_{\Theta^*(0)}G & \Theta(0) \end{pmatrix} \right) \) coincides with the set of all solutions of the KYP inequality (6.5) for \( Q = \left( \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} \right) \). If the system \( \nu (\tau) \) is optimal realization of \( Z(\lambda) (\Theta(\lambda)) \) then the minimal solution of (6.5) (6.1) is \( X_0 = I_S \). Therefore, the minimal solution of (6.1) (6.5) is \( X_0 = I_S \) as well. Thus, the system \( \tau (\nu) \) is optimal realization of \( \Theta(\lambda) (Z(\lambda)) \). Passing to the adjoint systems
\[
\nu^* = \left\{ \begin{pmatrix} S^* & G^* \\ F^* & 0 \end{pmatrix} ; \mathcal{H}, \mathcal{D}_{\Theta^*(0)}, \mathcal{D}_{\Theta(0)} \right\}
\]
and
\[
\tau^* = \left\{ \begin{pmatrix} -G^*\Theta(0)F^* + S^* & G^*D_{\Theta^*(0)} \\ D_{\Theta(0)}F^* & \Theta^*(0) \end{pmatrix} ; \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}
\]
and their transfer functions \( Z^*(\lambda) \) and \( \Theta^*(\lambda) \), respectively, we get that \( \nu^* \) is (*)-optimal iff \( \tau^* \) is (*)-optimal.

The next theorem is an immediate consequence of Theorem 6.1

**Theorem 6.3.** Let
\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -FD^*G + D_{F^*}LD_{G} & FD_{D} \\ D_{D^*}G & D \end{pmatrix} : \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \right) \to \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{N} \end{pmatrix} \right)
\]
be a contraction and let \( Q := \begin{pmatrix} D_{F^*}LD_{G} & F \\ G & 0 \end{pmatrix} : \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{D}_{D^*} \end{pmatrix} \right) \to \left( \begin{pmatrix} \mathcal{H} \\ \mathcal{D}_{D^*} \end{pmatrix} \right) \). Then the following equations are equivalent on the operator interval \((0, I_S)\):
\[
X - A^*XA - C^*C - (A^*XB + C^*D)(I_{2R} - B^*XB - D^*D)^{-1}(B^*XA + D^*C) = 0,
\]
(6.10)
(6.11) \( (I_\delta - X)P_\delta = (D_T^2 + T^*(I_\delta - X)P_\delta^r T)_\delta \),

(6.12) \( (I_\delta - X)P_\delta = (D_Q + Q^*(I_\delta - X)P_\delta^r Q)_\delta \),

(6.13) \( X - G^*G - S^*XS - S^*XF(I_\delta - F^*XF)^{-1}F^*XS = 0 \),

(6.14) \( X = G^*G + S^*X^{1/2}(I_\delta - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}S \),

where \( S = D_F LD_G \),

\[ S^*XF(I_\delta - F^*XS)^{-1}F^*XS := \]
\[ = ((I_\delta - FXF^*)^{-1/2}F^*XS)^* \left( (I_\delta - FXF^*)^{-1/2}F^*XS \right) , \]

and

\[ S^*X^{1/2}(I_\delta - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}S := \]
\[ = D_G L^* ((I_\delta - X^{1/2}FF^*X^{1/2})^{-1/2}X^{1/2}D_{F^*})^* ((I_\delta - X^{1/2}FF^*X^{1/2})^{-1/2}X^{1/2}D_{F^*}) \] \( LD_G \).

Moreover, the equations (6.11) – (6.14) are equivalent to the equation (5.13).

The equivalent equations (6.11) – (6.14) will be called the Riccati equations.

**Remark 6.4.** Let \( Q = \begin{pmatrix} S & F \\ G & 0 \end{pmatrix} : \left( \begin{array}{c} \mathfrak{H} \\ \mathfrak{M} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathfrak{H} \\ \mathfrak{N} \end{array} \right) \) be an isometric operator. Then \( F \) is isometry, \( D_{F^*} \) is the orthogonal projection, \( S = D_{F^*} LD_G \), and \( D_L = 0 \). Denote \( \mathcal{K} = \ker F^* = \text{ran} D_{F^*} \). Since \( D_{F^*} = P_{\mathcal{K}} \), we get

\[ D_G^2 - D_G L^* D_{F^*} X^{1/2}(I_\delta - X^{1/2}FF^*X^{1/2})^{-1}X^{1/2}D_{F^*} LD_G = \]
\[ = S^* \left( P_{\mathcal{K}} - P_{\mathcal{K}} X^{1/2}(I_\delta - X^{1/2}P_{\mathcal{K}^\perp} X^{1/2})^{-1}X^{1/2}P_{\mathcal{K}} \right) S. \]

Taking into account Theorem 2.7 and (6.4) we get the corresponding KYP inequality

\[ I_\delta - X \leq S^* (I_\delta - X)_{\mathcal{K}} S. \]

**Example 6.5.** Let \( \mathfrak{H}, \mathfrak{M}, \) and \( \mathfrak{N} \) be separable Hilbert spaces. Suppose that \( G \in \mathbf{L}(\mathfrak{H}, \mathfrak{M}) \) and \( F \in \mathbf{L}(\mathfrak{M}, \mathfrak{N}) \) are such that \( G^*G = FF^* = \alpha I_\delta \), where \( \alpha \in (0, 1) \). Then \( D_G = D_{F^*} = (1 - \alpha)^{1/2} I_\delta \). Let \( L \) be a unitary operator in \( \mathfrak{H} \). By Theorem 3.1 the operator

\[ Q = \begin{pmatrix} (1 - \alpha) L & F \\ G & 0 \end{pmatrix} : \left( \begin{array}{c} \mathfrak{H} \\ \mathfrak{M} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathfrak{H} \\ \mathfrak{N} \end{array} \right) \]

is a contraction. Consider the passive system

\[ \nu = \left\{ \begin{pmatrix} (1 - \alpha) L & F \\ G & 0 \end{pmatrix} : \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\}. \]

Because \( \text{ran} F = \text{ran} G^* = \mathfrak{H} \), the system \( \nu \) is minimal. The corresponding Riccati equation (6.14) takes the form

\[ (6.15) \quad X = \alpha I_\delta + (1 - \alpha)^2 L^* X (I_\delta - \alpha X)^{-1} L, \quad 0 < X \leq I_\delta. \]

We will prove that this equation has a unique solution \( X = I_\delta \).

Put \( W = (1 - \alpha)(I_\delta - \alpha X)^{-1} \). Then \( (1 - \alpha) I_\delta < W \leq I_\delta \). From (6.15) we obtain the equation

\[ (6.16) \quad L^* WL + W^{-1} = 2I_\delta, \]
Clearly, (6.16) has a solution \( W = I_\mathcal{S} \). Let \( W \) be any solution of (6.16) such that \((1 - \alpha)I_\mathcal{S} < W \leq I_\mathcal{S}\), i.e., \( \sigma(W) \subset [1 - \alpha, 1] \). Since \( L \) is unitary operator, from (6.16) it follows that \( \sigma(2I_\mathcal{S} - W^{-1}) = \sigma(W) \). Let \( \lambda_0 \in \sigma(W) \) then \( 2 - \lambda_0^{-1} \in \sigma(2I_\mathcal{S} - W^{-1}) = \sigma(W) \). Since \( 2 - \lambda_0^{-1} > 0 \), we get that

\[
\frac{1}{2} < \lambda_0 \leq 1.
\]

Because \( \mu_0 = 2 - \lambda_0^{-1} \in \sigma(W) \) and \( 1/2 < \mu_0 \leq 1 \), we get

\[
\frac{2}{3} < \lambda_0 \leq 1.
\]

Thus

\[
\frac{2}{3} < 2 - \lambda_0^{-1} \leq 1.
\]

It follows that

\[
\frac{3}{4} < \lambda_0 \leq 1.
\]

Continuing these reasonings, we get

\[
\frac{n}{n + 1} < \lambda_0 \leq 1, \quad n = 1, 2, \ldots.
\]

And now we get that \( \lambda_0 = 1 \), i.e. \( \sigma(W) = \{1\} \). Because \( W \) is selfadjoint operator we have \( W = I_\mathcal{S} \). Thus, the equation (6.15) has a unique solution \( X = I_\mathcal{S} \).

7. Properties of solutions of the KYP inequality and Riccati equation

**Proposition 7.1.** Let \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{S}, \mathcal{M}, \mathcal{N} \right\} \) be a passive observable system. If a non-negative contraction \( X \) in \( \mathcal{S} \) is a solution of the inequality

\[
(I_\mathcal{S} - X)P_\mathcal{S} \leq \left( D_T^2 + T^*(I_\mathcal{S} - X)P'_\mathcal{S}T \right)_{I_\mathcal{S}}
\]

then \( \ker X = \{0\} \).

**Proof.** Suppose that \( X \) satisfies (7.1) and \( \ker X \neq \{0\} \). Then there is a nonzero vector \( x \) in \( \mathcal{S} \) such that \((I_\mathcal{S} - X)x = x \). Since \( D_T^2 + T^*(I_\mathcal{S} - X)P'_\mathcal{S}T \) is a contraction, we obtain \((D_T^2 + T^*(I_\mathcal{S} - X)P'_\mathcal{S}T)_{I_\mathcal{S}} = x \) and hence \( D_T^2 + T^*(I_\mathcal{S} - X)P'_\mathcal{S}T = x \). It follows that

\[
P_\mathcal{S}Tx = 0, \quad XP'_\mathcal{S}Tx = 0.
\]

This means that \( Cx = 0 \) and \( XAx = 0 \). Replacing \( x \) by \( Ax \) we get \( CAx = 0 \) and \( XA^2x = 0 \). By induction \( CA^nx = 0 \) for all \( n = 0, 1, \ldots \). Since the system \( \tau \) is observable, we get \( x = 0 \). \( \square \)

**Theorem 7.2.** Let \( T = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{S}, \mathcal{M} \mapsto \left( \begin{array}{cc} \mathcal{S} \\ \mathcal{M} \end{array} \right) \) be a contraction. Then every solution \( Y \) of (6.9) satisfies the estimate

\[
Y \leq (D_T^2)_{I_\mathcal{S}} \upharpoonright \mathcal{S}.
\]

If the system \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathcal{S}, \mathcal{M}, \mathcal{N} \right\} \) is observable then every \( Y \) from the operator interval \([0, (D_T^2)_{I_\mathcal{S}} \upharpoonright \mathcal{S}]\) is a solution of (6.9).
Proof. If \( Y \) is a solution of (6.9) then in view of
\[
D_T^2 + T^*YP_T^2T \leq D_T^2 + T^*P_T^2T = I - T^*P_T^2T = D_{P_T}^2
\]
we get \( Y \leq (D_{P_T}^2)_{\delta} \vert \mathcal{H} \).

Moreover, if \( I \) is a solution of (6.11) then\( I \) is a solution of the inequality
\[
YP = (D_T^2 + T^*P_T^2T)_{\delta}.
\]
It follows that the shorted operator \( Y = (D_T^2)_{\delta} \vert \mathcal{H} \) is a solution of the inequality
\[
YP \leq (D_T^2 + T^*P_T^2T)_{\delta}.
\]
Moreover, if \( Y \in [0, (D_T^2)_{\delta} \vert \mathcal{H}] \) then
\[
YP \leq (D_T^2)_{\delta} P = (D_T^2 + T^*P_T^2T)_{\delta}.
\]
By Proposition 7.1 we have \( \ker(I_{\delta} - Y) = \{0\} \).

Corollary 7.3. Suppose that \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a passive observable system.
Then every \( X \) from the operator interval \([I_{\delta} - (D_T^2)_{\delta} \vert \mathcal{H}, I_{\delta}]\) is a solution of (6.11) and if \( X \) is a solution of (6.11) then \( X \geq I_{\delta} - (D_{P_T}^2)_{\delta} \vert \mathcal{H} \).
Assume
\[
(D_T^2)_{\delta} = (D_{P_T}^2)_{\delta}.
\]
Then the operator \( X_0 = I_{\delta} - (D_{P_T}^2)_{\delta} \vert \mathcal{H} \) is the minimal solution of KYP inequality (6.1).

Corollary 7.4. Let \( \Theta(\lambda) \) belongs to the Schur class \( \mathcal{S}(\mathcal{M}, \mathcal{N}) \). If the system
\[
\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}
\]
is a passive minimal and optimal realization of \( \Theta(\lambda) \) then
\[
(D_T^2)_{\delta} = 0.
\]
Proof. If the system \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) is a passive minimal and optimal realization of \( \Theta(\lambda) \) then the unique solution \( X \) from the operator interval \([0, I_{\delta}]\) of the KYP inequality (1.4) is the identity operator \( I_{\delta} \). If \((D_T^2)_{\delta} \neq 0\) then according to Corollary 7.3 the operator \( I_{\delta} - (D_T^2)_{\delta} \vert \mathcal{H} \) is a solution of (1.4). It follows that \((D_T^2)_{\delta} = 0\).

Corollary 7.5. Let \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) be a passive minimal system. Then the minimal \( X_0 \) solution of the KYP inequality (6.1) satisfies the Riccati equations (6.10) – (6.14).

Proof. Since the system \( \tau \) is passive and minimal, the minimal solution \( X_0 \) of (1.3) satisfies \( 0 < X_0 \leq I_{\delta} \). Let the operator \( T_0 \) be defined on the domain \( \text{ran} X_0^{1/2} \plusslash \mathcal{M} \) by the equality
\[
T_0 = \begin{pmatrix} X_0^{1/2} & 0 \\ 0 & I_{\mathcal{M}} \end{pmatrix} T \begin{pmatrix} X_0^{-1/2} & 0 \\ 0 & I_{\mathcal{M}} \end{pmatrix}.
\]
The operator \( T_0 \) is a contraction and has contractive continuation on \( \mathcal{H} \plusslash \mathcal{M} \). We preserve the notation \( T_0 \) for this continuation. The system \( \tau_0 = \{T_0, \mathcal{H}, \mathcal{M}, \mathcal{N}\} \) is passive minimal.
and optimal realization of the transfer function $\Theta(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$, $|\lambda| < 1$ for the system $\tau$. According to Corollary 7.4, the operator $T_0$ satisfies the condition $(D^2_{T_0})_\delta = 0$. It follows that

$$\inf_{u \in \mathfrak{g}} \left\{ \left\| D_{T_0} \left( \begin{array}{c} g \\ u \end{array} \right) \right\|^2 \right\} = 0$$

for all $g \in \mathfrak{g}$. In particular

$$\inf_{u \in \mathfrak{g}} \left\{ \left\| D_{T_0} \left( \begin{array}{c} X_0^{1/2} \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) \left( \begin{array}{c} g \\ u \end{array} \right) \right\|^2 \right\} = 0, \quad g \in \mathfrak{g}.$$  

Since

$$\left( \begin{array}{c} X_0 \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) - T^* \left( \begin{array}{c} X_0 \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) T = \left( \begin{array}{c} X_0^{1/2} \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) D^2_{T_0} \left( \begin{array}{c} X_0^{1/2} \\ 0 \\ I_{\mathfrak{g}} \end{array} \right),$$

we get

$$\left( \left( \begin{array}{c} X_0 \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) - T^* \left( \begin{array}{c} X_0 \\ 0 \\ I_{\mathfrak{g}} \end{array} \right) \right)_\delta = 0.$$  

For $Y_0 = I_{\delta} - X_0$ we have

$$\left( I_{\delta} - Y_0 \\ 0 \\ I_{\mathfrak{g}} \right) - T^* \left( I_{\delta} - Y_0 \\ 0 \\ I_{\mathfrak{g}} \right) T = D^2_{T} + T^* Y_0 P_{\delta}^T - Y_0 P_{\delta}.$$  

Since

$$\left( \left( I_{\delta} - Y_0 \\ 0 \\ I_{\mathfrak{g}} \right) - T^* \left( I_{\delta} - Y_0 \\ 0 \\ I_{\mathfrak{g}} \right) \right)_\delta = 0,$$

we get

$$0 = (D^2_T + T^* Y_0 P_{\delta}^T - Y_0 P_{\delta})_\delta = (D^2_T + T^* Y_0 P_{\delta}^T)_\delta - Y_0 P_{\delta}.$$

Thus, $Y_0$ satisfies the equation

$$YP_{\delta} = (I - T^* T + T^* Y P_{\delta}^T)_\delta$$

and $X_0 = I_{\delta} - Y_0$ satisfies the equation (6.11). \hfill \Box

**Corollary 7.6.** Let $\Theta(\lambda) \in \mathfrak{S}(\mathfrak{M}, \mathfrak{N})$ and let $\tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \right\}$ be a passive minimal realization of $\Theta$. If $(D^2_{\tau})_\delta = 0$ then the system $\tau$ is optimal.

**Proof.** Since $D^2_{T} \leq D^2_{\tau} = 0$ and $(D^2_{\tau})_\delta = 0$, we obtain $(D^2_T)_\delta = 0$. By Corollary 7.3 in this case the minimal solution of (1.14) is $X_0 = I_{\delta}$. This means that $\tau$ is the optimal realization of $\Theta$. \hfill \Box

**Remark 7.7.** Let $\tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \right\}$ be a minimal passive system and let

$$T = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} -FD^*G + D_{F^*L}D_G & FD_D \\ D_{D^*G} & D \end{array} \right)$$

Then the statements of Corollaries 7.3, 7.6 can be reformulated as follows:

1. every $X$ from the operator interval $[G^*G + D_G L^* LD_G, I]$ is a solution of (6.3) and every solution of (6.3) satisfies the estimate $X \geq G^*G$;
(2) if $L = 0$ then $X_0 = G^*G$ is the minimal solution of (6.3) (cf. \[10\]);
(3) if the system $\tau$ is optimal realization of the function $\Theta(\lambda) \in S(\mathfrak{M}, \mathfrak{N})$ then $D_LD_G = 0$;
(4) if the system $\tau$ is a realization of the function $\Theta(\lambda) \in S(\mathfrak{M}, \mathfrak{N})$ and if $G$ is isometry then the system $\tau$ is optimal.

Remark 7.8. Let $\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\} : \mathfrak{H}, \mathfrak{M}, \mathfrak{N}$ be a minimal system with transfer function $\Theta(\lambda)$ from the Schur class $S(\mathfrak{M}, \mathfrak{N})$. Suppose that the bounded positive selfadjoint operator $X$ is such that the operator

$$\delta(X) = I_{2\mathfrak{N}} - D^*D - B^*XB$$

is positive definite. Then the KYP inequality

$$L(X) = \begin{pmatrix} X - A^*XA - C^*C & -A^*XB - C^*D \\ -B^*XA - D^*C & I - B^*XB - D^*D \end{pmatrix} \geq 0$$

is equivalent to the inequality $R(X) \geq 0$, where

$$R(X) = X - A^*XA - C^*C - B^*XA(I_{2\mathfrak{N}} - D^*D - B^*XB)^{-1}A^*XB$$

is the corresponding Schur complement. If there exists such $X$ that $\delta(X)$ is positive definite and $R(X) \geq 0$ then for the minimal solution $X_{\text{min}}$ of KYP inequality we have $\delta(X_{\text{min}}) \geq \delta(X)$ and $R(X_{\text{min}}) \geq 0$. For a finite dimensional $\mathfrak{H}$ it was shown in [31] that the minimal solution $X_{\text{min}}$ satisfies the algebraic Riccati equation $R(X_{\text{min}}) = 0$. Thus, the statement of Corollary 7.5 is the generalization of the result in [31] for a passive minimal system with infinite dimensional state space.

Proposition 7.9. Let $\Theta(\lambda) \in S(\mathfrak{M}, \mathfrak{N})$ and let the M"obius parameter $Z(\lambda)$ of $\Theta$ be of the form $Z(\lambda) = \lambda K$, $K \in \mathbb{L}(\mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)})$, $K \neq 0$. Then

1. the minimal passive and optimal realization $\tau$ of $\Theta$ is unitarily equivalent to the system

$$\tau = \left\{ \begin{pmatrix} -K\Theta^*(0) & KD_{\Theta(0)} \\ D_{\Theta^*(0)} & \Theta(0) \end{pmatrix} : \mathfrak{T}\mathfrak{M} K, \mathfrak{M}, \mathfrak{N} \right\};$$

2. the minimal passive and (\ast\ast\ast)- optimal realization $\tau$ of $\Theta$ is unitarily equivalent to the system

$$\eta = \left\{ \begin{pmatrix} -P_{\mathfrak{T}\mathfrak{M} K\cdot\Theta^*(0)}K & B_{\mathfrak{T}\mathfrak{M} K^*} \Theta(0) \\ D_{\Theta^*(0)}K & \mathfrak{T}\mathfrak{M} K^* \Theta(0) \end{pmatrix} : \mathfrak{T}\mathfrak{M} K^*, \mathfrak{M}, \mathfrak{N} \right\};$$

Proof. Let $j$ be the embedding of $\mathfrak{T}\mathfrak{M} K$ into $\mathfrak{D}_{\Theta^*(0)}$. Then the system

$$\nu = \left\{ \begin{pmatrix} 0 & K \\ j & 0 \end{pmatrix} : \mathfrak{T}\mathfrak{M} K, \mathfrak{D}_{\Theta(0)}, \mathfrak{D}_{\Theta^*(0)} \right\}.$$
The system
\[
\sigma = \left\{ \begin{pmatrix} 0 & P_{\text{ran}} K^* \\ K^* \text{ran} K & 0 \end{pmatrix} ; \text{ran} K^*, \mathcal{D\Theta}(0), \mathcal{D\Theta}^*(0) \right\}
\]
is the passive and minimal realization of the function \( \lambda K \). The KYP inequality (6.4) for the adjoint system \( \sigma^* \) takes the form
\[
\begin{cases}
X \geq \text{ran} K^*, \\
0 \leq X \leq \text{ran} K^*.
\end{cases}
\]
So, \( X = \text{ran} K^* \) is the minimal solution. It is the minimal solution of the generalized KYP inequality for \( \sigma^* \). Hence, \( X = \text{ran} K^* \) is the maximal solution of the generalized KYP inequality for \( \sigma \). It follows that \( \sigma \) is a \((*)\)-optimal realization of \( Z(\lambda) = \lambda K \) and by Proposition 6.2 the system \( \eta \) is a \((*)\)-optimal realization of \( \Theta(\lambda) \).

The next theorem provides sufficient uniqueness conditions for the solutions of the Riccati equation.

**Theorem 7.10.** Let \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : (\mathcal{H}_1, \mathcal{M}) \rightarrow (\mathcal{H}_2, \mathcal{N}) \) be a contraction. Suppose that
\[
(D_T^2)_{\mathcal{M}} = 0,
\]
\[
\text{ran} \left( (D_T^2)_{\mathcal{N}} \right)^{1/2} \cap \text{ran} \left( (D_T^2)_{\mathcal{N}}^* \right)^{1/2} \subset \text{ran} \left( (D_T^2)_{\mathcal{N}} \right)^{1/2}.
\]
Then the Riccati equation (6.11) has a unique solution \( X = I_{\mathcal{M}} \).

**Proof.** Let \( T \) takes the form
\[
T = \begin{pmatrix} -FD^* G + D_{F^*} L D_G & FD_D \\ D_{D^*} G & D \end{pmatrix}
\]
with contractions \( D, F, G, \) and \( L \). From (3.5) it follows that if \( (D_T^2)_{\mathcal{M}} = 0 \) then \( X = I_{\mathcal{M}} \) is a unique solution of (6.11). Assume \( (D_T^2)_{\mathcal{M}} \neq 0 \). Since \( (D_T^2)_{\mathcal{N}} = 0 \), from the equivalences (3.6) it follows that
\[
(D_T^2)_{\mathcal{N}} \neq 0.
\]
From (7.5) and (3.5) we have \( D_G \neq 0, D_{F^*} \neq 0, D_L = 0, \) and
\[
\text{ran} D_G \cap \text{ran} D_{F^*} \subset \text{ran} (D_{F^*} D_L^*).
\]
According to Proposition 6.3 the equation (6.11) is equivalent to the equation (6.14). We will prove that (6.14) has a unique solution \( X = I_{\mathcal{M}} \). Suppose that \( X \) is a solution. Define \( \Psi : I_{\mathcal{M}} - X^{1/2} F F^* X^{1/2} \). Since \( \Psi = I_{\mathcal{M}} - X + X^{1/2} D_{F^*} X^{1/2} \), we have \( \Psi \geq I_{\mathcal{M}} - X \) and \( \Psi \geq X^{1/2} D_{F^*} X^{1/2} \). Therefore
\[
(I_{\mathcal{M}} - X)^{1/2} = U \Psi^{1/2}, \quad D_{F^*} X^{1/2} = V \Psi^{1/2},
\]
where \( U : \text{ran} \Psi^{1/2} \rightarrow \text{ran} (I_{\mathcal{M}} - X)^{1/2}, \quad V : \text{ran} \Psi^{1/2} \rightarrow \text{ran} D_{F^*} = \text{ran} (D_{F^*} X^{1/2}) \), and \( U^* U + V^* V = I_{\text{ran} \Psi^{1/2}} \). Hence \( U^* U = D_{V^*}^2 \). Since \( X^{1/2} D_{F^*} = \Psi^{1/2} V^* \), we get
\[
X^{1/2} D_{F^*} D_{V^*} = \Psi^{1/2} V^* D_{V^*} = \Psi^{1/2} D_{V^*} V^*.
\]
From
\[
I_{\mathcal{M}} - X = \Psi^{1/2} U^* U \Psi^{1/2} = \Psi^{1/2} D_{V^*} \Psi^{1/2}
\]
we get that \( \text{ran} \left( I_\delta - X \right)^{1/2} = \Psi^{1/2} \text{ran} \, D_V \). Therefore,

\[(7.7) \quad \text{ran} \, X^{1/2} D_F \cdot D_{V^*} \subset \text{ran} \left( I_\delta - X \right)^{1/2}.\]

Using the well known relation

\[\text{ran} \, X^{1/2} \cap \text{ran} \left( I_\delta - X \right)^{1/2} = \text{ran} \left( X^{1/2} (I_\delta - X)^{1/2} \right)\]

for every \( X \in [0, I_\delta] \), from (7.7) we get that

\[\text{ran} \, D_F \cdot \text{ran} \, D_{V^*} \subset \text{ran} \left( I_\delta - X \right)^{1/2}.\]

The equation (6.14) can be rewritten as follows

\[X = G^* G + D_G L^* V V^* L D_G.\]

Since \( L^* L = I_{\mathcal{D}_G} \), we get \( I_\delta - X = D_G L^* D_{V^*} \cdot L D_G \). It follows that

\[\text{ran} \left( I_\delta - X \right)^{1/2} = D_G L^* \text{ran} \, D_{V^*} \subset \text{ran} \, D_G.\]

Now we obtain

\[\text{ran} \, D_F \cdot \text{ran} \, D_{V^*} \subset \text{ran} \, D_G \cap \text{ran} \, D_{F^*} \subset \text{ran} \, D_F \cdot \text{ran} \, D_{L^*}.\]

Hence \( \text{ran} \, D_{V^*} \subset \text{ran} \, D_{L^*} \). Since \( L : \mathcal{D}_G \to \mathcal{D}_{F^*} \) is isometry, we get \( \ker L^* = \text{ran} \, D_{L^*} \).

Therefore \( L^* \cdot \text{ran} \, D_{V^*} = 0 \). It follows \( \text{ran} \left( I_\delta - X \right)^{1/2} = \{0\} \), i.e., \( X = I_\delta \). \( \square \)

Observe, Example 6.5 shows that conditions (7.5) are not necessary for the uniqueness of the solutions of the KYP inequality (6.2).

**Theorem 7.11.** Let a contraction \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \) possesses the properties

\[(7.8) \quad \begin{cases} \left( D^2_T \right)_\delta = 0, \left( D^2_{T^*} \right)_\delta = 0, \\
\text{ran} \left( \left( D^2_{F^* T} \right)_\delta \right)^{1/2} \cap \text{ran} \left( \left( D^2_{P \cdot T^*} \right)_\delta \right)^{1/2} = \{0\} \end{cases}.\]

Then the generalized KYP inequality (1.3) has a unique solution \( X = I_\delta \).

**Proof.** Let \( T^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{M} \end{pmatrix} \) be the adjoint operator and let

\[(7.9) \quad \begin{cases} (I_\delta - Z) P_\delta \leq \left( D^2_{T^*} + T (I_\delta - Z) P_\delta T^* \right)_\delta, \\
0 < Z \leq I_\delta \end{cases},
\]

(7.10)

\[\begin{cases} (I_\delta - Z) P_\delta = \left( D^2_{T^*} + T (I_\delta - Z) P_\delta T^* \right)_\delta, \\
0 < Z \leq I_\delta \end{cases},\]

be the corresponding KYP inequality and Riccati equation. By Theorem 7.10 the identity operator \( I_\delta \) is a unique solution of the Riccati equations (6.11) and (7.10).

Let us show that the passive system

\[\tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H}, \mathcal{M}, \mathcal{N} \right\}\]

is minimal. Consider the parametrization of the contraction \( T \):

\[T = \begin{pmatrix} -F D^* G + D_{F^*} L D_G & F C D_D \\ D_{D^*} G \end{pmatrix}\]
and let \( Q = \begin{pmatrix} D_F^*LD_G & F \\ G & 0 \end{pmatrix} : \left( \mathcal{H}_d \right) \to \left( \mathcal{H}_{d^*} \right). \)

Assume \((D^2_F)_{\mathcal{H}_d} \neq 0\) and \((D^2_F)_{\mathcal{H}_{d^*}} \neq 0\). From (7.8) and (3.5) we get the equalities

\[
D_LD_G = D_{L^*}D_{F^*} = 0, \quad \text{ran} \, D_G \cap \text{ran} \, D_{F^*} = \{0\}.
\]

It is known that

\[
\text{ran} \, F + \text{ran} \, D_{F^*} = \mathcal{H}.
\]

Suppose that \( F^*D_GL^*D_{F^*}f = 0 \), where \( f \in \mathcal{Q}_{F^*} \). Then the vector \( D_GL^*D_{F^*}f \in \ker F^* \), hence \( D_GL^*D_{F^*}f \in \text{ran} \, D_{F^*} \). Since \( L : \mathcal{Q}_G \to \mathcal{Q}_{F^*} \) is unitary operator and \( \text{ran} \, D_G \cap \text{ran} \, D_{F^*} = \{0\} \), we get \( f = 0 \). It follows that \( \text{ran} \,(D_F^*LD_GF) = \mathcal{Q}_{F^*} \) and

\[
\overline{\text{span}} \{(D_F^*LD_G)^n F\mathcal{M}, \ n = 0, 1, 2, \ldots \} = \mathcal{H}.
\]

Thus, the system \( \nu = \left\{ \left( D_F^*LD_G, F \right ) : \mathcal{H}, \mathcal{Q}_D, \mathcal{Q}_{D^*} \right\} \) is controllable. Similarly, the system \( \nu \) is observable. By Theorem (4.4) the system \( \tau \) is minimal.

If \((D^2_F)_{\mathcal{H}_d} = D_G = 0\) then by (3.6) also \((D^2_F)_{\mathcal{H}_{d^*}} = D_{F^*} = 0\). Therefore \( \text{ran} \, F = \text{ran} \, G^* = \mathcal{H} \) and \( Q = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix} \). It follows that in this case the systems \( \nu \) and \( \tau \) are minimal.

According to the result of [9] the KYP inequality (6.2) has a minimal solution. Since \( I_{\mathcal{H}_d} \) is a solution of (6.2) and the Riccati equation (6.11) has a unique solution \( X = I_{\mathcal{H}_d} \), the inequality (6.2) has a unique solution \( X = I_{\mathcal{H}_d} \). Similarly the KYP inequality (7.9) also has a unique solution \( Z = I_{\mathcal{H}_d} \). Hence, the minimal solution of the generalized KYP inequality (1.3)

\[
\left\| \begin{pmatrix} X^{1/2} & 0 \\ 0 & I_M \end{pmatrix} \left( \begin{array}{c} x \\ u \end{array} \right) \right\|^2 = \left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left( \begin{array}{c} x \\ u \end{array} \right) \right\|^2 \geq 0
\]

for all \( x \in \text{dom} \, X^{1/2}, \ u \in \mathcal{M} \)

is \( X = I_{\mathcal{H}_d} \). Since \( Z \) is a solution of the generalized KYP inequality for the adjoint operator \( 5.2 \iff X = Z^{-1} \) is a solution \( X \) of (1.3), the minimal solution of (5.2) is \( Z = I_{\mathcal{H}_d} \).

Hence, the identity operator \( I_{\mathcal{H}_d} \) is also the maximal solution of (1.3). So, (1.3) has a unique solution \( X = I_{\mathcal{H}_d} \).

**Corollary 7.12.** Let \( \tau = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H}, \mathcal{M}, \mathcal{N} \right\} \) be a passive system and let \( \Theta(\lambda) \) be its transfer function. If the operator \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \left( \mathcal{H} \right) \to \left( \mathcal{M} \right) \) possesses the properties (7.8) then all minimal passive realizations of \( \Theta \) are unitary equivalent.

**Remark 7.13.** The conditions (7.5) are equivalent to the following:

\[
\left\{ \begin{array}{l}
\text{ran} \, D_T \cap \mathcal{H} = \{0\}, \\
(\text{ran} \, D_{P_m} \cap \mathcal{H}) \cap (\text{ran} \, D_{P_m}^* \cap \mathcal{H}) \subset \text{ran} \, D_{T^*} \cap \mathcal{H},
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
\text{ran} \, D_T \cap \mathcal{H} = \text{ran} \, D_{T^*} \cap \mathcal{H} = \{0\}, \\
(\text{ran} \, D_{P_m} \cap \mathcal{H}) \cap (\text{ran} \, D_{P_m}^* \cap \mathcal{H}) = \{0\}.
\end{array} \right.
\]
8. Approximation of the Minimal Solution

The solutions of the Riccati equations (6.10)–(6.14) are fixed points of the corresponding maps. We will prove that extremal solutions can be obtained by iteration procedures with a special initial points.

**Theorem 8.1.** Let \( \tau = \left\{ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right); \mathfrak{H}, \mathfrak{M}, \mathfrak{N} \right\} \) be a passive observable system and let \( T = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \). Let us define the sequence of nonnegative contractions in \( \mathfrak{H} \):

\[
Y^{(0)} := I_{\mathfrak{H}}, \quad Y^{(n+1)} := \left( D_T^2 + T^* Y^{(n)} P_{\mathfrak{H}} T' \right)_{\mathfrak{H}}, \quad n = 0, 1, \ldots.
\]

Then

1. the sequence \( \{Y^{(n)}\}_{n=0}^{\infty} \) is a nonincreasing,
2. the operator

\[
Y_0 := s - \lim_{n \to \infty} Y^{(n)}
\]

satisfies the equality

\[
Y_0 P_{\mathfrak{H}} = \left( D_T^2 + T^* Y_0 P_{\mathfrak{H}} T' \right)_{\mathfrak{H}}
\]

and \( \ker (I_{\mathfrak{H}} - Y_0) = \{0\} \),
3. the operator \( Y_0 \) is a maximal solution of (6.9).

**Proof.** Let us show that the sequence defined by (8.1) is nonincreasing. Since \( (D_T^2 P_{\mathfrak{H}} T)_{\mathfrak{H}} \leq P_{\mathfrak{H}} \), we get

\[
Y^{(1)} P_{\mathfrak{H}} = \left( D_T^2 + T^* P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} = \left( D_T^2 + T^* Y_0 P'_{\mathfrak{H}} T \right)_{\mathfrak{H}}.
\]

Hence \( Y^{(1)} \leq Y^{(0)} \). Suppose that \( Y^{(n)} \leq Y^{(n-1)} \) for given \( n \geq 1 \). Then

\[
Y^{(n+1)} P_{\mathfrak{H}} = \left( D_T^2 + T^* Y^{(n)} P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} \leq \left( D_T^2 + T^* Y^{(n-1)} P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} = Y^{(n)} P_{\mathfrak{H}}.
\]

Thus, the sequence \( \{Y^{(n)}\}_{n=0}^{\infty} \) is nonincreasing. Because the operators \( Y^{(n)} \) are nonnegative, there exists a strong limit

\[
Y_0 = s - \lim_{n \to \infty} Y^{(n)}.
\]

Since

\[
Y^{(n+1)} = \left( D_T^2 + T^* Y^{(n)} P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} \quad \forall n = 0, 1, \ldots,
\]

applying Proposition 2.1 we get (8.2).

Let us show that every solution \( Y \) of (6.9) satisfies the inequality \( Y \leq Y_0 \). Suppose that \( Y \) is a solution of (6.9). Taking into account (7.2) we get \( Y \leq Y^{(1)} \). If it is proved that \( Y \leq Y^{(n)} \) for some \( n \geq 1 \) then

\[
Y \leq \left( D_T^2 + T^* Y P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} \leq \left( D_T^2 + T^* Y^{(n)} P'_{\mathfrak{H}} T \right)_{\mathfrak{H}} = Y^{(n+1)}.
\]

By induction it follows that \( Y \leq Y_0 \). Using Proposition 7.1 we get \( \ker (I - Y_0) = \{0\} \). \( \square \)

**Remark 8.2.** The nondecreasing sequence

\[
X^{(0)} = I_{\mathfrak{H}} - Y^{(0)} = 0, \quad X^{(n+1)} = I_{\mathfrak{H}} - Y^{(n+1)} = I_{\mathfrak{H}} - \left( D_T^2 + T^* (I_{\mathfrak{H}} - X^{(n)}) P'_{\mathfrak{H}} T \right)_{\mathfrak{H}}
\]

satisfies the inequality
strongly converges to the minimal solution $X_0$ of the KYP inequality (6.2) and the Riccati equations (6.10) - (6.14) (see Theorem 6.3). From (3.9) and (8.10) we get

$$X^{(n+1)} = G^*G + D_G L^*D_{F^*} (X^{(n)})^{1/2} (I_\mathcal{S} - (X^{(n)})^{1/2} F F^* (X^{(n)})^{1/2})^{-1} (X^{(n)})^{1/2} D_{F^*} L D_G.$$  

Example 8.3. Let $F \in \mathcal{L}(\mathcal{M}, \mathcal{S})$ be a strict contraction ($\|F h\|_\mathcal{S} < \|h\|_\mathcal{M}$ for all $h \in \mathcal{M} \setminus \{0\}$) and $\ker F^* = \{0\}$. Then $\text{ran} \ D_{F^*} = \mathcal{S}$. Let $\alpha \in (0, 1)$ and suppose that the operator $G \in \mathcal{L}(\mathcal{S}, \mathcal{M})$ is chosen such that $G^*G = \alpha F F^*$. Then

$$\ker G = \{0\}, \ D_G = (I_\mathcal{S} - \alpha F F^*)^{1/2}$$

and $\text{ran} \ D_G = \mathcal{S}$. Therefore $\text{ran} \ D_{F^*} \subset \text{ran} \ D_G$. Let $L = I_\mathcal{S}$. By Theorem 3.1 the operator

$$Q = \begin{pmatrix} D_{F^*} D_G & F \\ D_G & 0 \end{pmatrix} : (\mathcal{S}) \rightarrow (\mathcal{M})$$

is a contraction and from (3.5) we get that $(D_Q^2)_{\mathcal{S}} = 0$, $(D_{F^* Q}^2)_{\mathcal{S}} \neq 0$, and

$$\text{ran} \ (D_{F^* Q}^2)_{\mathcal{S}}^{1/2} \subset \text{ran} \ (D_{F^* Q}^2)_{\mathcal{S}}^{1/2}.$$  

The system

$$\nu = \left\{ \begin{pmatrix} D_{F^*} D_G & F \\ D_G & 0 \end{pmatrix} : (\mathcal{S}, \mathcal{M}, \mathcal{M}) \right\}$$

is passive. The condition $\ker F^* = \{0\}$ yields that

$$\bigcap_{n \geq 0} \ker (F^* (D_G D_{F^*})^n) = \bigcap_{n \geq 0} \ker (G (D_{F^*} D_G)^n) = \{0\}.$$  

So, the system $\nu$ is minimal. Its transfer function $Z(\lambda)$ takes the form

$$Z(\lambda) = \lambda G (I_\mathcal{S} - \lambda (I_\mathcal{S} - \alpha F F^*)^{1/2} (I_\mathcal{S} - \alpha F F^*)^{1/2})^{-1} F.$$  

The corresponding Riccati equation (6.14) takes the form

$$\begin{cases}
X = \alpha F F^* + (I_\mathcal{S} - \alpha F F^*)^{1/2} D_{F^*} X^{1/2} (I_\mathcal{S} - X^{1/2} F F^* X^{1/2})^{-1} X^{1/2} D_{F^*} (I_\mathcal{S} - \alpha F F^*)^{1/2} \\
0 < X \leq I_\mathcal{S}
\end{cases}$$

and has a solution $X_0 = \alpha I_\mathcal{S}$. Because $\alpha I_\mathcal{S} < I_\mathcal{S}$, the system $\nu$ is non-optimal realization of $Z(\lambda)$.

Let us show that $X_0 = \alpha I_\mathcal{S}$ is the minimal solution of (8.3). Note that $X_{\min} \leq X_0 = \alpha I_\mathcal{S}$. According to Remark 8.2 the sequence of operators

$$X^{(0)} = 0, \ X^{(n+1)} = \alpha F F^* + (I_\mathcal{S} - \alpha F F^*)^{1/2} D_{F^*} (X^{(n)})^{1/2} (I_\mathcal{S} - (X^{(n)})^{1/2} F F^* (X^{(n)})^{1/2})^{-1} (X^{(n)})^{1/2} D_{F^*} (I_\mathcal{S} - \alpha F F^*)^{1/2}, \ n = 0, 1, \ldots$$

is nondecreasing and strongly converges to the minimal solution $X_{\min}$ of (8.3). Hence $X^{(n)} \leq \alpha I_\mathcal{S}$ and because $X^{(1)} = \alpha F F^*$, one has $X^{(n)} F F^* = F F^* X^{(n)}$ for all $n$. It follows that
\[ X_{\text{min}} F F^* = F F^* X_{\text{min}} \] and
\[ I_{\mathcal{H}} - X_{\text{min}} = (I_{\mathcal{H}} - \alpha F F^*) (I_{\mathcal{H}} - D_{F^*}^2 X_{\text{min}} (I_{\mathcal{H}} - X_{\text{min}} F F^*)^{-1}) = \]
\[ = (I - X_{\text{min}}) (I_{\mathcal{H}} - \alpha F F^*) (I_{\mathcal{H}} - X_{\text{min}} F F^*)^{-1}. \]

Hence
\[ (I_{\mathcal{H}} - X_{\text{min}}) (\alpha I_{\mathcal{H}} - X_{\text{min}}) F F^* = 0. \]
Therefore \((\alpha I_{\mathcal{H}} - X_{\text{min}}) F F^* = 0\). Taking into account that \(\ker F^* = \{0\}\), we get \(X_{\text{min}} = \alpha I_{\mathcal{H}}\).

Note that if the orthogonal projection \(P\) in \(\mathcal{H}\) commutes with \(F F^*\) then the operator \(X = P + \alpha P^\perp\) is a solution of the Riccati equation (8.3).

Consider the adjoint system
\[
\nu^* = \left\{ \begin{pmatrix} DG D_{F^*} & G^* \\ F^* & 0 \end{pmatrix} : \mathcal{H}, \mathcal{M}, \mathcal{M} \right\}.
\]

We will show that \(X = I_{\mathcal{H}}\) is the minimal solution of the corresponding Riccati equation
\[
\begin{cases}
X = F F^* + D_{F^*} (I_{\mathcal{H}} - \alpha F F^*)^{1/2} X^{1/2} (I_{\mathcal{H}} - \alpha F F^*)^{-1} X^{1/2} (I_{\mathcal{H}} - \alpha X^{1/2} F F^* X^{1/2})^{1/2} D_{F^*}, \\
0 < X \leq I_{\mathcal{H}}
\end{cases}
\]

According to Remark 8.2, the sequence of operators
\[
X^{(0)} = 0, \quad X^{(n+1)} = F F^* + \\
+ D_{F^*} (I_{\mathcal{H}} - \alpha F F^*)^{1/2} (X^{(n)})^{1/2} (I_{\mathcal{H}} - \alpha (X^{(n)})^{1/2} F F^* (X^{(n)})^{1/2})^{-1} (X^{(n)})^{1/2} (I_{\mathcal{H}} - \alpha F F^*)^{1/2} D_{F^*},
\]
\[ n = 0, 1, \ldots \]
is nondecreasing and strongly converges to the minimal solution \(X_{\text{min}}\). It follows that \(X^{(n)} F F^* = F F^* X^{(n)}\) for all \(n\), \(X_{\text{min}} F F^* = F F^* X_{\text{min}}\) and
\[ I_{\mathcal{H}} - X_{\text{min}} = (I_{\mathcal{H}} - F F^*) (I_{\mathcal{H}} - (I_{\mathcal{H}} - \alpha F F^*) X_{\text{min}} (I_{\mathcal{H}} - \alpha X_{\text{min}} F F^*)^{-1}) = \]
\[ = (I_{\mathcal{H}} - X_{\text{min}}) (I_{\mathcal{H}} - F F^*) (I_{\mathcal{H}} - \alpha F F^* X_{\text{min}})^{-1}. \]

Hence
\[ (I_{\mathcal{H}} - X_{\text{min}}) (I_{\mathcal{H}} - \alpha X_{\text{min}}) = 0. \]

Because \(I_{\mathcal{H}} - \alpha X_{\text{min}}\) has bounded inverse, we get \(X_{\text{min}} = I_{\mathcal{H}}\). Thus, the minimal passive system \(\nu\) is \((\cdot)\)-optimal realization of the function \(Z(\lambda)\).

Observe that this example shows that the condition (7.3) for a passive minimal system is not sufficient for optimality.

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