AN ORBIFOLD APPROACH TO SEVERI INEQUALITY

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Abstract: For a smooth minimal surface of general type $S$ with $\text{Albdim}(S) = 2$, Severi inequality says that $K_S^2 \geq 4\chi(S)$, which was proved by Pardini (cf. [7]). It is expected that when the equality is attained, $S$ is birational to a double cover over an Abelian surface branched along a divisor having at most negligible singularities. This was proved when $K_S$ is ample by Manetti (cf. [4]). In this paper, we applied Manetti’s method to the canonical model of $S$, with some additional assumptions we proved Severi inequality and characterized the surfaces with $K_S^2 = 4\chi(S)$ (cf. Theorem 1.2). One assumption is an algebraic problem (cf. Conj. 4.6), which implies Severi inequality and was proved in a special case. In addition, we gave a characterization of the double cover over an Abelian surface via the ramification divisor (cf. Theorem 5.1).

1. Introduction

We work over complex numbers. Severi inequality is proposed by Severi in ([Sev32]), which states that:

**Theorem 1.1** (Severi inequality). Let $S$ be a smooth minimal surface of general type with $\text{Albdim}(S) = 2$. Then

$$K_S^2 \geq 4\chi(S)$$

But Severi provided a wrong proof, and Catanese pointed it out and posed the inequality as a conjecture (cf. [2]). The conjecture was proved by Manetti when $K_S$ is ample (cf. [4]), what’s more he showed that when the equality is attained, then $S$ is birational to a double cover of an Abelian surface. Using completely different method, Pardini finally proved the conjecture (cf. [7]). However, her method, heavily relying on Xiao’s inequality, gave little geometric information. In particular, when the equality is attained, i.e., $K_S^2 = 4\chi(S)$, it is conjectured that the canonical model of $S$ is a double cover over an Abelian surface branched along a divisor having at most negligible singularities.

Before we explain our idea, let’s recall Manetti’s method: Let $S$ be a smooth minimal surface of general type with $\text{Albdim}(S) = 2$. A curve $C$ on $S$ is called of

- type 0: if it is contracted by the Albanese map;
- type 1: if at a general point of $C$ the differential of the Albanese map has rank 1;
- type 2: if at a general point of $C$ the differential of the Albanese map has rank 2.

Let $p : \mathbb{P}(T_S) \rightarrow S$ be the natural projection, $L$ the anti-tautological line bundle. Then we have

$$(p^*K_S + L)L^2 = 3(K_S^2 - 4\chi(S))$$
It suffices to prove

\[(p^*K_S + L)^2 \geq 0\]

Take general \(L_1, L_2 \in |L|\), and write \(L_1 \cap L_2 = V + \Gamma_0 + \Gamma_1 + \Gamma_2\) where \(V\) is the vertical part, \(\Gamma_i\) is composed with some curves which are mapped to curves of type \(i\). The term \(\Gamma_0(p^*K_S + L)\) may be \(<0\). Manetti decomposed \((p^*K_S + L)L_1L_2\) into the summation of terms depending on certain points and curves. If \(p(\Gamma_0)\) contains no \((-2)\)-curves, by careful analysis, Manetti showed that \((p^*K_S + L)L_1L_2 \geq 0\) and the equality is attained if and only if the Albanese map is a smooth double cover over an Abelian surface. Using Manetti’s idea, Mendes Lopes and Pardini proved some stronger inequalities (cf. \([6]\)).

We will apply Manetti’s argument to the canonical model \(\bar{S}\) of \(S\), which comes from contracting all the \((-2)\)-curves of \(S\). \(\bar{S}\) has at most quotient singularities (cf. \([5]\), Chap. 4, Sec. 6), hence has an orbifold structure, which is associated with a stack \(S\). Similarly we define curves of type \(0, 1, 2\) on \(\bar{S}\). We will do intersections on \(S\) and \(\mathbb{P}_S(T_S)\) which is also a stack induced by an orbifold. If the curves of type 0 do not pass through the singularities of \(S\), we reduced Severi inequality to an interesting and seemingly easy algebraic problem. However, due to lack of capability, the author failed in solving that problem except for a special case, so there arises Conjecture 4.6. Granted this conjecture, we proved

**Theorem 1.2.** Let \(S\) be the canonical model of a smooth surface of general type and of maximal Albanese dimension, and denote by \(\text{alb}_S : S \to A\) the Albanese map. Assume that no curves of type 0 pass through the singularities of \(S\). If Conjectures 4.6 is true, then

\[K_S^2 \geq 4\chi(S)\]

moreover the equality is attained if and only if \(\text{alb}_S : S \to A\) is a double cover over an Abelian surface branched along a curve with at most negligible singularities.

In particular if \(q(S) = 2\) and for two elements \(\alpha, \beta \in H^0(S, \Omega^1_S)\) the divisor given by \(\alpha \wedge \beta \in H^0(S, K_S)\) is reduced, then the assertion above is true.

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2. Preliminaries

As preliminaries, let’s recall the results about orbifolds, stacks, V-free sheaves, and the intersection theory on stacks. We will omit the detailed proof involved, for which the readers can refer to \([1]\) and \([9]\).

2.1. Orbifold (V-manifold) and V-free sheaves.

**Definition 2.1.** Let \(X\) be a variety. \(X\) is called an orbifold (or V-manifold), if every point \(x \in X\) is attached a pair \(((Y, y), G_x)\) where \(G_x\) is a finite group acting faithfully on the smooth germ \((Y, y)\) such that the germ \((X, x) \cong (Y, y)/G_x\). Naturally, we can associate \(X\) with a smooth Deligne-Mumford stack \(X\). In the following, we usually denote by \(\pi : X \to X\) the natural morphism, and if no confusion occurs, the local quotient map \((Y, y) \to (X, x)\) is also denoted by \(\pi\).

**Definition-Proposition 2.1** (\([1]\)). Let \(X\) be an orbifold. For \(x \in X\), assume it is attached the pair \(((Y, y), G_x)\), and denote by \(\pi : (Y, y) \to (X, x)\) the quotient map.
A coherent sheaf $\mathcal{F}$ on $X$ is called $V$-free at $x$, if it satisfies one of the following equivalent conditions:

(i) there exists a free $G_x$-invariant sheaf $\mathcal{F}$ on $(Y, y)$ such that $\mathcal{F} = \pi^{*}G_x \mathcal{F}$;

(ii) $\mathcal{F}$ is reflexive and $\mathcal{F} = (\pi^{*}\mathcal{F})^{**}$ is free.

We say $\mathcal{F}$ is $V$-free if it is $V$-free at every point of $X$.

The natural correspondence $\mathcal{F} \mapsto (\pi^{*}\mathcal{F})^{**}$ gives an 1 : 1 correspondence between the $V$-free sheaves on $X$ and the vector bundles on $X$.

**Example 2.1.** Let $S$ be a normal surface with at most canonical singularities. Then $S$ has at most isolated quotient singularities of type $A_i, i = 1, 2, \ldots, D_i, i \geq 4, E_6, E_7, E_8$, so for every $x \in S$, in the usual way we can assume the germ $(S, x) \cong (\mathbb{C}^2, 0)/G_x$ where $G_x \subset SL(2, \mathbb{C})$ is finite group and the action $G_x$ on $(\mathbb{C}^2, 0)$ is induced by its natural representation on $\mathbb{C}^2$. By attaching $x \in S$ the pair $((\mathbb{C}^2, 0), G_x)$, we get an orbifold structure on $S$. Let $i : S^{sm} \subset S$ be the smooth part. Then $\Omega^1_S := i_*\Omega^1_{S^{sm}}$ is a $V$-free sheaf since $(\pi^1\Omega^1_S)^{**} \cong \Omega^1_S$.

### 2.2. Intersection theory and Chern classes

Let $X$ be an orbifold of dimension $n$ associated with a smooth Deligne-Mumford stack $\mathcal{X}$. In [9], the author built an intersection theory on such $\mathcal{X}$. By that theory, if $D_i, i = 1, 2, \ldots, n$ are $(n-1)$-cycles on $X$, then $\pi^{*}D_1 \cdot \pi^{*}D_2 \cdots \pi^{*}D_n = D_1 \cdot D_2 \cdots D_n$, precisely for $x \in X$ and $\pi : (Y, y) \to (X, x)$ the local covering, if $D_1, D_2, \ldots, D_n$ intersect properly, then $(\pi^{*}D_1 \cdot \pi^{*}D_2 \cdots \cdot \pi^{*}D_n)y)/G_x = (D_1 \cdot D_2 \cdots D_n)x$.

Let $\mathcal{L}$ be a $V$-free sheaf on $X$ of rank $r$. First we introduce the definition of Chern classes by means of metric (cf. [1]).

**Definition 2.2.** A $V$-metric is a hermitian metric $h$ defined on $\mathcal{F}|_{X^{sm}}$ such that $\pi^*h$ extends to a hermitian metric on $(\pi^{*}\mathcal{F})^{**}$ locally. In turn, we can associate the $V$-metric $h$ the $i$th Chern class $c_i(X, \mathcal{F}, h) \in H^i(X, \mathbb{Q})$ by means of the curvature matrix of $\pi^*h$ on $(\pi^{*}\mathcal{F})^{**}$.

Another definition is purely algebraic. Let $P(X, \mathcal{F}) = Proj \oplus_i \mathcal{O}_{X}(S^i(\mathcal{F})^{**)$. Consider the local quotient map $\pi : (Y, y) \to (X, x)$. $G_x$ acts naturally on $Proj \oplus_i \mathcal{O}_{Y}(S^i((\pi^{*}\mathcal{F})^{**)$. Since $(\pi^{*}\mathcal{F})^{**}$ is a $G_x$-invariant sheaf, and one can prove that $P((X, \mathcal{F})) := Proj \oplus_i \mathcal{O}_{(Y, y)}(S^i((\pi^{*}\mathcal{F})^{**) \cong Proj \oplus_i \mathcal{O}_{(Y, y)}(S^i((\pi^{*}\mathcal{F})^{**)\bigg)/G_x$. In this way, we attach $P(X, \mathcal{F})$ an orbifold structure with Deligne-Mumford stack $P(X, (\pi^{*}\mathcal{F})^{**})$. Let $L$ be the $V$-line bundle $\mathcal{O}_{P(X, \mathcal{F})}(1)$, and $\rho : P(X, \mathcal{F}) \to X$ the natural map. We can inductively define the $\hat{c}_i(X, \mathcal{F})$'s by the following equations (cf. [8]):

\[
(2.1) \quad \mathcal{L}^r - \mathcal{L}^{r-1}\rho^*\hat{c}_1(X, \mathcal{F}) + \cdots + (-1)^r\rho^*\hat{c}_r(X, \mathcal{F}) = 0, r = 1, 2, \ldots
\]

In the paper [8], Y. Kawamata proved that

**Proposition 2.2.** $\hat{c}_i(X, \mathcal{F})$ coincides with $c_i(X, \mathcal{F}, h)$.

**Example 2.3.** Fix a point $p \in \mathbb{P}^1$. Associate an orbifold structure on $\mathbb{P}^1$ by attaching a point $x \neq p$ the pair $((\mathbb{C}, 0), \{1\})$ and the point $p$ the pair $((\mathbb{C}, 0), \{1, \sigma\})$ where $\sigma$ is an involution given by $\sigma(t) = -t$. We get an orbifold curve $\mathcal{C}$ with Deligne-Mumford stack $\mathcal{C}$. Its cotangent bundle $\Omega^1_{\mathcal{C}}$ gives an orbifold line bundle $L$ on $\mathcal{C}$. Since $2\Omega^1_{\mathcal{C}} = \pi^*(2\Omega^1_{\mathbb{P}^1} + p)$, so $\deg(\hat{c}_i(L)) = -\frac{1}{2}$. 
2.3. The curves on an orbifold surface. Let $S$ be a surface with at most canonical singularities. We attach $S$ an orbifold structure with Deligne-Mumford stack $S$ as in Example 2.1. Let $D \subset S$ be an irreducible curve on $S$, and $\sigma : \tilde{D} \to D$ the normalization. Note that $\mathcal{O}_S(D) := i_*\mathcal{O}_{\sigma^{-1}(D)}$ is a V-free sheaf on $S$, since $\mathcal{O}_S(\pi^*D)$ is a line bundle on $S$.

Let $D \to \pi^*D$ be the normalization. Then we get a natural map $D \to \tilde{D}$ which induces an orbifold structure on $\tilde{D}$ in the following way. For $x \in D$, assume it is attached the pair $((\mathbb{C}^2,0), G_x)$ and denote by $\pi : (\mathbb{C}^2,0) \to (S,x)$ the quotient map. Let $\Pi_i(D_i, y_i) \to (\pi^*D,0)$ be the normalization. The action of $G_x$ on $\pi^*D$ induces an action on $\Pi_i(D_i, y_i)$ naturally. Write that $\sigma^{-1}(D,x) = \Pi_j(\tilde{D}, x_j)$. For every $j$, we find a $(\mathcal{D}_i, y_i)$ and a subgroup $G_{ij} \subset G_x$ which fixes $\mathcal{D}_i$; such that $(\tilde{D}, x_j) \cong (\mathcal{D}_i, y_i)/G_{ij}$, up to an isomorphism the quotient map is independent of the choices of $i$. Therefore, we get a natural map $D \to \tilde{D}$, and it induces an orbifold structure on $\tilde{D}$ with Deligne-Mumford stack $D$.

3. Characterization of the double cover over an Abelian surface

The following theorem gives a characterization of the double cover over an Abelian surface by the ramification divisor. The idea is from [3], in particular if $R$ is irreducible, our theorem follows from applying Prop. 3.1 in [3] directly.

**Theorem 3.1.** Let $X$ be a normal surface, $A$ an Abelian surface, $f : X \to A$ a finite surjective morphism, $R$ the ramification divisor of $f$. Assume that

- $R$ is reduced and ample; and
- for every $r \in R$, the restriction map to the germ $f|_{(R,r)}$ is an isomorphism to its image.

Then there exists an Abelian surface $T$ such that $f$ factors through a double cover $g : X \to T$ which is ramified along $R$.

**Proof.** Consider the map $p : X \times R \to A$ given by $(x,r) \mapsto f(x) - f(r)$. $p$ has a factorization

$$p = \pi \circ q : X \times R \to T \to A$$

where $T = \text{spec}_{\mathcal{O}_A}(p_*\mathcal{O}_{X \times R})$, hence $q$ is a connected morphism and $\pi$ is finite. Our theorem follows from the following lemma.

**Lemma 3.2.** The map $\pi : T \to A$ is an étale cover, hence $T$ is an Abelian surface. And there exists a morphism $g : X \to T$ such that $f = \pi \circ g + a_0$ for some $a_0 \in A$ and $g(x,r) = g(x) - g(r)$, in particular the fiber $q^{-1}(0) \cong R \times_T R$ is connected.

Now we begin to prove this lemma.

Step 1: For every $a \in A$, we can find an analytic open neighborhood $a \in U \subset A$ such that:

- $f^{-1}(U) = V_1 \cup V_2 \cup \ldots \cup V_k$ where $V_i \cap V_j = \emptyset, i \neq j$;
- $f|_{V_i \cap R}$ is an isomorphism if $V_i \cap R \neq \emptyset$;
- $f|_{V_i} : V_i \to U$ is a double cover ramified along $V_i \cap R$ if $V_i \cap R \neq \emptyset$ and is an isomorphism otherwise.

We say an open set $V \subset X$ is good if $f|_{V}$ is either an isomorphism or a standard double cover ramified along $V \cap R$. So for every $x \in X$, there exists a good neighborhood containing $x$. 

Let $B = f(R)$. For $a' \in A$, denote by $\Delta_{a'} \subset (B + a') \times B$ the set of the points like $(b + a', b)$. Then $p^{-1}(a') = F^{-1}(\Delta_{a'})$ where $F = f \times f|_R : X \times R \to A \times B$. For a good open set $V \subset X$ and an open subset $R_s$ of $R$ such that $f|_{R_s}$ is an isomorphism, putting $U = f(V), R_V = V \cap R, B_V = f(R_V), B_s = f(R_s)$, we have the following two facts:

(i) $p^{-1}(a') \cap (V \times R_s) = F|_{(V \times R_s)}^{-1}(\Delta_{a'}) \cong f^{-1}_V(B_s + a')$;

(ii) $p^{-1}(a') \cap (V \times R_s)$ is connected if and only if $(B_s + a') \cap U$ is connected and in addition $(B_s + a') \cap B_V \neq \emptyset$ if $f|_V$ is a double cover;

where (i) follows from the fact that $R_s \to B_s$ is an isomorphism, and (ii) is from (i) and the fact that $V$ is good.

Step 2: $\pi$ is unramified if seen as a map between two analytically topological spaces.

We argue by contrary, so assume that $\pi$ is ramified at $t \in T$ and let $a = \pi(t)$. Then we can find a neighborhood $U_a$ of $a$ and a connected component $D_a$ of $p^{-1}(a)$, such that for every $(x, r) \in D_a$, there exists a neighborhood $V_x \times R_r$ of $(x, r)$ such that

(c1) $V_x$ is a good open neighborhood of $x$, and $f|_{R_r}$ is an isomorphism;

(c2) $p^{-1}(a) \cap (V_x \times R_r) = D_a \cap (V_x \times R_r)$ is connected and for general $a' \in U_a$

$p^{-1}(a') \cap (V_x \times R_r)$ is not connected.

Put

$U_x = f(V_x), \ B_r = f(R_r), \ R_x = V_x \cap R, \ B_x = f(R_x)$

Since $(B_r + a) \cap U_x$ is connected, shrinking $V_x, U_x$ and $U_a$ if necessary, we can assume

(c3) $(B_r + a') \cap U_x$ is connected for $a' \in U_a$.

Then (ii) in Step 1 implies that $f|_{V_x}$ is a double cover, $(B_r + a) \cap B_x \neq \emptyset$ and $(B_r + a') \cap B_x = \emptyset$ for general $a' \in A$. However, we will show it is impossible below.

First remark that

(a) if $B_r + a$ and $B_x$ intersect properly or one of them is reducible, then shrinking $U_a$ if necessary, for every $a' \in U_a$, $(B_r + a') \cap B_x \neq \emptyset$, so we conclude that both $B_r + a$ and $B_x$ are irreducible and $B_r + a = B_x$;

(b) since $R$ is ample it is connected by Hodge Index Theorem, so from (a) and

Step 1 (i), we conclude that $D_a \cap (X \times R)$ is mapped onto $R$ via the natural projection $X \times R \to X$.

Now take finite $(x_i, r_i)$ and finite open sets $\{V_{x_i} \times R_{r_i}\}_i$ satisfying (c1–3) and covering $D_a$. Shrinking $U_a$ if necessary, for $a' \in U_a$, the union of $\{V_{x_i} \times R_{r_i}\}_i$ covers a component $D_{a'}$ of $p^{-1}(a')$. Let $I_{a'}$ denote the image of $D_{a'}$ via the natural projection $X \times R \to X$. Since $R$ is ample, $I_{a'} \cap R \neq \emptyset$. By (b) the $V_{x_i}$'s cover $R$, so we can find a $V_{x_i}$ such that $I_{a'} \cap R_{x_i} \neq \emptyset$, hence $f(I_{a'} \cap R_{x_i}) \subset (B_r + a') \cap B_x \neq \emptyset$, and a contradiction follows.

Step 3: $T$ is smooth, hence $\pi : T \to A$ is an étale cover, and we can assume $\pi : T \to A$ is an isogeny between two Abelian surfaces.

Considering the normalization $\hat{T}$ of $T$, by Step 2 it is topologically isomorphic to $T$ and the composed map $\hat{T} \to A$ is étale, hence $\hat{T}$ is smooth. For a point $t \in T$, denote by $\hat{t} \in \hat{T}$ the point over $t$ and $a \in A$ the image of $t$ via $\pi$. Then the composed map of germs $(\hat{T}, \hat{t}) \to (T, t) \to (A, a)$ is an isomorphism, so $(T, t)$ must be a smooth germ, and our assertion follows.

Step 4: $q^{-1}(0) \cong R \times_T R$ is connected.
Take \( r_0 \in R \), and define \( g : X \to T \) by \( x \mapsto q(x, r_0) \). Then \( g \) is ramified along \( R \) since \( \pi \) is étale. Consider the map \( q' : X \times R \to T \) defined by \( (x, r) \mapsto g(x) - g(r) \). Since \( \pi \) is an isogeny, check that
\[
\pi \circ q'(x, r) = \pi(g'(x) - g'(r)) = \pi(q(x, r_0) - q(r, r_0)) = \pi \circ q(x, r_0) - \pi \circ q(r, r_0) = \pi(q(x, r) - q(r)) = \pi(q(x, r))
\]
i.e., \( \pi \circ q = \pi \circ q' \). We conclude that \( q' \) is connected, so there exists isomorphism \( \sigma : T \to T \) such that \( q = \sigma \circ q' \). Replacing \( \pi \) by \( \pi \circ \sigma \), we can assume \( q = q' \). It follows that that \( q(x, r) = g(x) - g(r) \) and \( f(x) = \pi \circ g(x) + f(r_0) \). By Step 1 (i), \( R \times_T R \) is composed with some connected components of \( q^{-1}(0) \), hence \( q^{-1}(0) = R \times_T R \) is connected, and we are done.

\[\square\]

4. Proof of Theorem 1.2

4.1. Reduce to the intersections of cycles. Let \( S \) be assumed as in Theorem 1.2, and \( \sigma : \tilde{S} \to S \) the minimal resolution. As in Example 2.1, we give \( S \) an orbifold structure with Deligne-Mumford stack \( S \). For a point \( x \in S \), we assume it is attached the pair \( ((\mathbb{C}^2, 0), G_x) \), and denote by \( \pi : (\mathbb{C}^2, 0) \to (S, x) \) the quotient map and by \( (u, v) \) a coordinate of \( (\mathbb{C}^2, 0) \). Let \( \Omega_S = i_* \Omega_{S^{sm}} \). Note that \( \Omega_S \cong \sigma_* \Omega_{\tilde{S}} \), and then \( \sigma_* : H^0(\tilde{S}, \Omega_{\tilde{S}}) \cong H^0(S, \Omega_S) \). For \( \alpha \in H^0(\tilde{S}, \Omega_{\tilde{S}}) \), \( \pi^* \sigma_* \alpha |_{(\mathbb{C}^2, 0)} \) is a closed holomorphic 1-form, hence exact, i.e., \( \exists f \in \mathbb{C}[[u, v]]^{G_x} \) such that \( \pi^* \sigma_* \alpha = df \) locally.

By Definition 2.13 and the theorem in [1] P.25, we have
\[
c_2(\Omega_S) = e_{arb}(S) = e(S) - \sum_{x \in \text{Sing}(S)} (1 - 1/|G_x|)
\]
and
\[
c_1(\Omega_S) = K_S
\]
Let \( X = P(S, \Omega_S^1) \) and \( L = \mathcal{O}_X(1) \). Using Eq. 2.1, we have
\[
(K_S + L)L^2 = 2c_1(\Omega_S)^2 - c_2(\Omega_S) = 2K_S^2 - e(S) + \sum_{x \in \text{Sing}(S)} (1 - 1/|G_x|)
\]
By \( e(\tilde{S}) = e(S) + \sum_{x \in \text{Sing}(S)} (\# \text{conj. of } G_x - 1) \), it follows that
\[
(K_S + L)L^2 = 2K_S^2 - e(\tilde{S}) + \sum_{x \in \text{Sing}(S)} (\# \text{conj. of } G_x - 1/|G_x|)
\]
By Neother’s formula, we have \( K_S^2 - 4\chi(\tilde{S}) = \frac{1}{3}(2K_S^2 - e(\tilde{S})) \), so we reduce Severi inequality to
\[
(K_S + L)L^2 \geq \sum_{x \in \text{Sing}(S)} (\# \text{conj. of } G_x - 1/|G_x|)
\]
If the Albanese image is of dimension 2, then there exist two elements \( \alpha, \beta \in H^0(S, \Omega_S^1) \) such that \( \alpha \wedge \beta \neq 0 \) as a nonzero element in \( H^0(S, K_S) \). Note that \( \alpha \)
and β define two hyperplanes \( L_\alpha \) and \( L_\beta \) in \( X \). Therefore, the Severi inequality is equivalent to

\[
(\rho^* K + L)L_\alpha L_\beta \geq \sum_{x \in \text{Sing}(S)} (\#\{\text{conjugencies of } G_x\} - 1/|G_x|)
\]

We will divide the left hand side into the summation of some terms in local form.

4.2. **Reduce to a local inequality.** Assume \( L_\alpha \cap L_\beta = \sum_i n_i C_i + \sum_{q \in S} m_q F_q + \sum_j l_j E_j \) where every \( C_i \) is a reduced and irreducible horizontal curve, \( F_k \) a vertical curve and \( E_j \) the pull-back \( \rho^* e_j \) of a curve \( e_j \subset S \). Put \( D_i = \rho_\ast C_i \). Then it follows that \( (\alpha \wedge \beta) = \sum_i n_i D_i + \sum_j l_j e_j \).

For a \( D_i \) not of type 0, we calculate \( LC_i \). We will consider the intersections of some cycles on the stack \( X = P(S, \Omega^1_S) \) and \( S \). We also denote by \( \pi : X \to X \) the natural map and by \( L \) the pull-back \( \pi^* L \).

Let \( \tilde{D}_i \) be the normalization of \( D_i \) and denote by \( j \) the composed map \( \tilde{D}_i \to D_i \to S \). As in Section 2.3 attach \( \tilde{D}_i \) an orbifold structure with Deligne-Mumford stack \( \tilde{D}_i \to D_i \). The natural morphism \( D_i \to S \) is also denoted by \( j \) if no confusion occurs. Then there exists a divisor \( R_i \) on \( D_i \) fitting into the following exact sequence

\[
\mu : j^* \Omega^1_S \to \Omega^1_{\tilde{D}_i}(-R_i) \to 0
\]

Set \( \mathcal{Y} = P(\mathcal{D}_i, j^* \Omega^1_S) \) which is a \( \mathbb{P}^1 \) bundle over \( D_i \). The kernel of \( \mu \) defines a section \( B_i \subset \mathcal{Y} \). By abuse of notations, we also denote by \( L \) the line bundle \( \mathcal{O}_\mathcal{Y}(1) \) and by \( j \) the natural map from \( \mathcal{Y} \) to \( X \) if no confusion occurs.

In the following we assume \( \alpha, \beta \) are general, so we can assume

- \( e_j \) is of type 0;
- \( \Diamond \) \( \alpha \) is mapped to a nonzero element in \( \Omega^1_{\tilde{D}_i}(-R_i) \) via \( \mu \), hence it defines the section \( C_i = j^* \pi^* C_i \subset \mathcal{Y} \) which is different from \( B_i \).

Then we have

\[
\deg(\Omega^1_{\tilde{D}_i}(-R_i)) = \langle C_i + \sum_j \beta_j F_j \rangle B_i
\]

thus

(4.1)

\[
LC_i = j^* L(j^* L - \sum_j \beta_j F_j)
\]

\[
= j^* L^2 - \deg(\Omega^1_{\tilde{D}_i}(-R_i)) + \sum_i C_i B_i
\]

\[
= K_S \pi^* D_i - (K_S + \pi^* D_i) \pi^* D_i + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{\tilde{D}_i}) + \deg(R_i) + C_i B_i
\]

\[
= -\pi^* D_i^2 + ((K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{\tilde{D}_i}) + C_i B_i) + \deg(R_i)
\]

Following the calculation of [4], we have

(4.2)

\[
(\rho^* K_S + L)L_\alpha L_\beta = (\rho^* K_S + L)(\sum n_i C_i + \sum m_p F_p + \sum l_j E_j)
\]

\[
= (\rho^* K_S + L)(\Sigma_{D_i} \text{ is not of type } \alpha n_i C_i) + \Sigma_{p \text{ is not on a curve of type } \alpha m_p F_p} + l_0^p
\]

\[
= l_1^p + l_0^p
\]
Now we focus on $I'_1$, so assume that none of the $D_i$’s appearing in the following equations are of type 0 and the point $p$ is not on a curve of type 0. Writing $K_S \equiv \sum_j n_j \pi^* D_i + \pi^* D'$, then by (4.1) we have

\begin{equation}
I'_1 = \sum_i n_i (K_S \pi^* D_i - \pi^* D'_i + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p \\
= \sum_i (n_i - 1) K_S \pi^* D_i \\
+ \sum_i (K_S - n_i \pi^* D_i) \pi^* D_i + n_i (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p \\
= \sum_i (n_i - 1) K_S \pi^* D_i + \sum_i D' D_i \\
+ \sum_i n_i (\sum_{j \neq i} \pi^* D_j \pi^* D_i + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p.
\end{equation}

Let $I_0 = I'_0 + \sum_i D' D_i$, and $I_1 = I'_1 - \sum_i D' D_i = \sum_i (n_i - 1) K_S \pi^* D_i + \sum_i n_i (\sum_{j \neq i} \pi^* D_j \pi^* D_i + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p$. By assumption, curves of type 0 do not pass through singularities of $S$, completely the same argument as in \[\text{4.3}\] shows that

$\spadesuit$ $I_0 \geq 0$, and $I_0 = 0$ holds only if the divisor $(\alpha \land \beta)$ contains no curves of type 0, i.e., $D' = 0$.

We will try to show the following more precise inequality

\begin{equation}
I_1 - \sum_i (n_i - 1) K_S \pi^* D_i \\
= \sum_i n_i (\sum_{j \neq i} \pi^* D_i \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p \\
\geq \sum_{x \in \text{Sing}(S)} \left( \#\{\text{conjugencies of } G_x\} - 1/|G_x| \right)
\end{equation}

If Inequality (4.4) were proved to be true and the equality in Severi inequality were attained, since $K_S$ is ample we could conclude that

$\heartsuit$ $K_S \equiv (\alpha \land \beta) = \sum_i D_i$ where $D_i$ are distinct reduced and irreducible curves of type 1 or 2.

We remark that $(K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i})$ only depends on the singularities of $\pi^* D_i$, i.e., the singularities of $S$ and $D_i$, so for every $x \in S$, we can calculated its contributions to $(K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i})$ (cf. Lemma \[\text{4.3}\]). And since $\pi^* D_i$ and $\pi^* D_j$ $i \neq j$, $C_i$ and $B_j$ intersect properly, the divisor $R_i$ is effective, for a point $x \in S$, we can naively define the contributions of $x$ to $\sum_i n_i (\sum_{j \neq i} \pi^* D_i \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p$, which we denote by

\[\sum_i n_i (\sum_{j \neq i} \pi^* D_i \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) + C_i B_i + \deg(R_i)) + \sum_p m_p]_x\]
We will calculate the contributions of the singularities on $S$ in next section. To show Inequality 4.4, it suffices to show that for every $x \in \text{Sing}(S)$

$$
\sum_i n_i \left( \pi^* D_i \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_j - \deg(\Omega^1_{\mathcal{B}_i}) + C_i B_i + \sum_p m_p \right) x 
\geq \#\{\text{conjugencies of } G_x\} - 1/|G_x|
$$

(4.5)

In a special case, we proved this inequality.

**Lemma 4.1.** Suppose that the divisor given by $(\alpha \wedge \beta)$ is reduced near a singularity $x \in S$. Then Inequality 4.4 holds, and the equality is attained only when the local Albanese map $a_S : (S, x) \to (\mathbb{C}^2, 0)$ given by path integral via $\alpha$ and $\beta$, is a standard double cover branched along a curve with a negligible singularity at 0.

We postpone the proof to Sec. 4.5.

**Remark 4.2.** The proof of Lemma 4.1 also applies for a smooth point not lying on a curve of type 0, so we conclude that

- ♦ for a smooth point $x \in S$ not lying on a curve of type 0, if $(\alpha \wedge \beta)$ is reduced near $x$, then Inequality 4.4 holds, and the equality is attained if and only if $(\alpha \wedge \beta)$ is smooth at $x$, and the local Albanese map defined by $\alpha$ and $\beta$ is a standard double cover branched along a smooth curve.

4.3. **Proof of Theorem 1.2.** By assumption Conjecture 4.0 holds, which implies Inequality 4.4 by the argument in Sec. 4.3. Hence Severi inequality is true.

Now we assume the equality in Severi inequality is attained. Then

- ♦ tells that there exist no curves of type 0 on $S$;
- ♢ tells that $(\alpha \wedge \beta)$ is reduced;
- ♣ $q = 2$, because otherwise we can find $\alpha, \beta$ satisfying assumptions ♦ and ♢ and that the divisor $(\alpha \wedge \beta)$ is singular at a smooth point, which contradicts ♦.

So the Albanese map $a_S : S \to A$ is either an isomorphism or a standard double cover locally, finally our theorem is an easy consequence of Theorem 3.1.

The converse is from direct calculation.

If $q = 2$ and $(\alpha \wedge \beta)$ is reduced, our assertion follows from applying Lemma 4.1 and Theorem 3.1.

4.4. **Local calculation.**

**Notation 4.3.** Let $(S, x) = (\mathbb{C}^2, 0)/G_x$. Assume $\pi^* \alpha, \pi^* \beta$ are defined by $df_x, dg_x$ where $f_x, g_x \in \mathbb{k}[u, v]^{G_x}$ whose constant term is assumed to be 0, and assume $\pi^* D_i$ is defined by the equation $h_{xi} \in \mathbb{k}[u, v]$, let $h_x = h_{x_1} \cdots h_{x_l}$. By assumption of the theorem, none of the $h_{xi}$’s divides $f_x$. Denote by $L_{dh_{xi}}$ the hyperplane determined by $dh_{xi}$ in $P((\mathbb{C}^2, 0), \Omega^1_{(\mathbb{C}^2, 0)})$, and write that

$$
J^* L_{dh_{xi}} = B_i + \sum_j \gamma_{xij} F_{xj}, \quad J^* L_{df_x} = C_i + \sum_j \beta_{xij} F_{xj} \quad \text{and} \quad L_{df_x} \cap L_{dg_x} = \sum_i n_i C_i + m_0 F_0
$$

where $j : D_i \to \pi^* D_i \to (\mathbb{C}^2, 0)$ the normalization of $\pi^* D_i$ and the $x_j$’s are the points on $D_i$ over 0. For $f \in \mathbb{k}[u, v]$, $(f)$ denotes the divisor given by $f = 0$; and for a 2-form $fdu \wedge dv$, $(fdu \wedge dv)$ denotes the divisor $(f)$. In this section, for $f, g \in \mathbb{k}[u, v]$, $(f)(g)$ means the intersection number of the divisors $(f)$ and $(g)$ at 0.

Now the key point is the following lemma.
Lemma 4.4.

\[(K_S + \pi^* D_i) \pi^* D_i - \text{deg}(\Omega^1_{D_i}) + C_i B_i + \sum_p m_p|_x = \frac{(df_x \wedge dh_x)(h_{xij}) - \sum_j \beta_{xij} + m_0}{|G_x|}\]

Proof. Note that the pull-back of residue of \(\frac{du \wedge dw}{dh_x}\) is a local generator of the line bundle \(j^*(K_S + \pi^* D_i)\) on \(\mathcal{D}_i\), and \(dt\) is a local generator of \(\Omega^1_{\mathcal{D}_i}\) where \(t\) is a parameter of a component of \(\mathcal{D}_i\). Comparing the two generators, we get

\[[(K_S + \pi^* D_i) \pi^* D_i - \text{deg}(\Omega^1_{D_i})]_x = \sum_j \gamma_{xij}/|G_x|\]

Then the lemma follows from the calculation \((df_x \wedge dh_x)(h_{xij}) = j^* L_{dh_x}; j^* L_{df_x} = C_i B_i + \sum_j(\gamma_{xij} + \beta_{xij})\) and the facts that \([C_i B_i]_x = \frac{C_i B_i}{|G_x|}\) and \([\sum_p m_p]_x = \frac{m_0}{|G_x|}\). □

Claim 4.5. With the notations as in Lemma 4.4, we have

(i) \(df_x \wedge dg_x = c \cdot h_x^n du \wedge dv\) with \(c \in k[[u, v]]\) such that \(c(y) \neq 0\);
(ii) \((df_x \wedge dh_x)(df_x \wedge dg_x) = \sum_i n_i((df_x \wedge dh_x)(h_{xij}) + \sum_{i \neq j}(h_{xij})(h_{xij}))\);
(iii) \(L_{dh_x} L_{df_x} L_{dg_x} = (df_x \wedge dh_x)(df_x \wedge dg_x) - \sum_i \sum_j n_i \beta_{xij} + m_0\) for \(H_x \in k[[u, v]]\) such that \(H_x\) has no common divisor with \(f_x\).

Proof. i) is trivial.

(ii) Notice that \(dh_x = h_{x2} \cdots h_{x1} dh_x + \cdots + h_{x1} \cdots h_{x(t-1)} dh_x\). Then we have

\[(df_x \wedge dh_x)(df_x \wedge dg_x) = \sum_i n_i(h_{x2} \cdots h_{x1} df_x \wedge dh_x + \cdots + h_{x1} \cdots h_{x(t-1)} df_x \wedge dh_x)(h_{xij})\]

\[= n_1(h_{x2} \cdots h_{x1} df_x \wedge dh_x + h_{x1}(...))(h_{x1}) + \cdots + m(h_{x1} \cdots h_{x(t-1)} df_x \wedge dh_x + h_{x1}(...))(h_{x1})\]

\[= n_1(h_{x2} \cdots h_{x1} df_x \wedge dh_x + h_{x1}(...))(h_{x1}) + \cdots + n_i(h_{x1} \cdots h_{x(t-1)} df_x \wedge dh_x)(h_{xij})\]

\[= \sum_i (df_x \wedge dh_x)(h_{xij}) + \sum_{i \neq j}(h_{xij})(h_{xij})\]

(iii) Similarly by \(j^* L_{df_x} = C_i + \sum_j \beta_{xij} F_{xij}\), we have \(L_{dh_x} C_i = (dh_x \wedge df_x)(h_{xij}) - \sum_j \beta_{xij}\). Then iii) follows easily. □

With the help of Lemma 4.4 and Claim 4.5 above, we obtain

\[\sum_i n_i(\sum_{j \neq i} \pi^* D_i \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \text{deg}(\Omega^1_{D_i}) + C_i B_i + \sum_p m_p)\]

\[(4.7) = \frac{L_{dh_x} L_{df_x} L_{dg_x}}{|G_x|}\]

So we reduce Inequality (4.7) to

\[L_{dh_x} L_{df_x} L_{dg_x} \geq \#\{\text{conjugencies of } G_x\}|G_x| - 1\]

For Inequality (4.8) we proposed a pure algebraic inequality as a conjecture.

Conjecture 4.6. Let \(G\) be a finite subgroup of \(SL(2, \mathbb{C})\) acting naturally on \((\mathbb{C}^2, 0) = \text{spec}(\mathbb{C}[[u, v]])\), and let \(f, g \in \mathbb{C}[[u, v]]\) be two \(G\)-invariant formal polynomials whose constant term is 0. Write \(df \wedge dg = h_1^{n_1} \cdots h_l^{n_l} du \wedge dv\) where the \(h_i\)’s are reduced, irreducible and distinct to each other, and put \(h = h_1 \cdots h_l\). Assume that none
of \( h_i \) divides \( f \) (which means that the curve defined by \( h_i = 0 \) is not of type 0).

Denote \( X = P((\mathbb{C}^2,0), \Omega_{\mathbb{C}^2,0}) \). For a polynomial \( H \in \mathbb{C}[u,v] \), \( L_dH \) denotes the hyperplane defined by \( dH \) in \( X \). Then

\[
L_{dh}L_{df}L_{dg} \geq \#\{\text{conjugencies of } G\}|G| - 1
\]

(4.9)

Remark 4.7. During the proof of Lemma 4.1 when \((df_x \wedge dg_x)\) is reduced near 0, we in fact proved a more precise inequality

\[
\sum n_i \left( \sum_{j \neq i} \pi^* D_j \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) \right) + \sum m_p|_x
\]

\[
\geq \#\{\text{conjugencies of } G_x\} - 1/|G_x|
\]

(4.10)

Therefore, by Eq. 4.7 Conjecture 4.6 holds when \((df \wedge dg)\) is reduced.

4.5. Proof of Lemma 4.1

For a singularity \( x \in S \), we now calculate part of its contributions \( \sum n_i \left( \sum_{j \neq i} \pi^* D_j \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) \right) + \sum m_p|_x \).

Let the notations be as in 4.3.

For simplicity, we consider the case when \( x \) is an \( A_{n-1} \)-singularity. In this case, \( G_x \cong Z_n \cong \{1, \epsilon, \epsilon^2, \ldots, \epsilon^{n-1}\} \) where \( \epsilon = e^{\frac{2\pi i}{n}} \), its action on \( \mathbb{C}[u,v] \) is given by \( \epsilon : u \mapsto \epsilon u, v \mapsto \epsilon v \), hence \( \mathbb{C}[u,v]|_{G_x} = \mathbb{C}[u^n, v^n, uv] \). Then observe that

\[
\text{mult}_0(df_x \wedge dg_x) \geq n
\]

Let \( \sigma : V \to (\mathbb{C}^2,0) \) be the blowing up map at 0, denote the exceptional (-1)-curve by \( E \), and write \( \sigma^* \pi^* D_i = \tilde{D}_i + a_i E \) where \( \tilde{D_i} \) denotes the strict transform of \( \pi^* D_i \). Then \( \left[ (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) \right]_{x} \) is no less than \((a_i - 1)a_i/n\), and the equality is attained if and only if \( \tilde{D}_i \) is smooth. Therefore, we have

\[
\sum n_i \left( \sum_{j \neq i} \pi^* D_j \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) \right)_{x}
\]

\[
\geq \left( \sum n_i (\sum_{j \neq i} a_j a_j + (a_i - 1)a_i) \right)/n
\]

(4.11)

\[
= (\sum i (a_i - 1))/n = (\text{mult}_0(df_x \wedge dg_x)(\text{mult}_0(df_x \wedge dg_x) - 1))/n
\]

\[
\geq n - 1
\]

If the equality in the first inequality is attained, then the \( \tilde{D}_i \)'s are smooth, and any two of them are disjoint.

Calculate that

\[
L_{d(u^n)} \cap L_{d(v^n)} \geq (n-1)^2 F_0, L_{d(u^n)} \cap L_{d(uv)} \geq (n-1)F_0, L_{d(v^n)} \cap L_{d(uv)} \geq (n-1)F_0
\]

Since \( df_x, dg_x \) are generated by \( d(u^n), d(v^n) \) and \( d(uv) \), we conclude that \( m_0 \geq n - 1 \), and since \( m_x = m_0/n \),

\[
\sum n_i \left( \sum_{j \neq i} \pi^* D_j \pi^* D_j + (K_S + \pi^* D_i) \pi^* D_i - \deg(\Omega^1_{D_i}) \right) + \sum m_p|_x
\]

(4.12)

\[
\geq n - \frac{1}{n} = \#\{\text{conjugencies of } G_x\} - 1/|G_x|
\]

The equality above is attained if and only if

- \( \text{mult}_0(df_x \wedge dg_x) = n \);
- blowing up \((\mathbb{C}^2,0)\) at 0, the singularity \( 0 \in \sum \pi^* D_i \) is resolved, hence 0 is a simple singularity of \( \sum_i \pi^* D_i \).
Write \( df_x \wedge dg_x = hdu \wedge dv \), and \( h_x(u, v) = h_n(u, v) + h_{n+1} \) (for \( F \in \mathbb{C}[[u, v]] \) \( F_{\geq m} \) means the summation of the terms of degree \( \geq m \) appearing in \( F \)). Then the divisor \((h_n)\) has a simple singularity at \( 0 \), precisely the germ \(((h_n), 0) \cong (\sum \pi^*D_i, 0)\).

Easy calculation gives that
\[
du^n \wedge duv = nu^n du \wedge dv, \quad dv^n \wedge duv = nu^n dv \wedge du, \quad du^n \wedge dv^n = n^2(\nu(\nu-1)dv \wedge dv
\]

Up to a coordinate transform (this is needed only when \( n = 2 \)) and multiplying by a constant, we can write \( f_x = uv + f_{\geq 2n} \), and \( g_x = u^n + cv^n + uvf' + g_{\geq 2n} \) where \( c \neq 0 \). Then \( d(uv) \wedge d(u^n + cv^n) = h_n du \wedge dv \), hence \( h_n = n(u^n - cv^n) \). Replacing \( f_x \) by \( f_x - \mu g_x \) for some constant \( \mu \), we can write \( f_x = uv + c_1 u^n + uvf' + f_{\geq 2n} \), \( g_x = u^n + cv^n + uvf'' + g_{\geq 2n} \). Replacing \( v \) by \( v - c_1 u^{-1} \), we can write that
\[
f_x = uv + uvf' + f_{\geq 2n}, \quad g = u^n + c_1uv^n + uvf'' + g_{\geq 2n}
\]

The local Albanese map \( a : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0) \) defined by \( df_x, dg_x \) coincides with the map given by \( z \mapsto (f_x(z), g_x(z)) \) \( \in \mathbb{C}^2 \). We conclude that
- \( a \) is of degree \( 2n \) by the formula above;
- \( a \) factors through the quotient map \( (\mathbb{C}^2, 0) \rightarrow (S, x) \);
- \( a \) is ramified along \( \sum_i D_i \).

Then it is easy to see that the local Albanese map \( a_S \) given by path integral via \( \alpha, \beta \) is a standard double cover branched along a curve with a negligible singularity at \( 0 \).

5. Some Remarks

5.1. **About the assumption.** In Theorem 1.2 we assumed that no curves of type 0 pass through the singularities of \( S \). To eliminate this assumption, we need to form a local inequality analogous to Inequality 4.4, the left hand side of which should involve a term contributed by the curves of type 0 passing through \( x \). Due to lack of capability, the author failed in even proposing this inequality. This paper can be seen as one step to attain our goal.

5.2. **About Conjecture 4.6** Remark 4.7 says that Conjecture 4.6 is true when the divisor \((df \wedge dg)\) is reduced. When proving Lemma 4.1 we calculated part of left hand side of Inequality 4.5 instead, hence got a stronger inequality in this special case. The advantage of Conjecture 4.6 is that the left hand side of Inequality 4.9 is the intersection of 3 cycles; \( h \) is determined by \( f \) and \( g \), and if \((df \wedge dg)\) is reduced, \( h \) is also \( G \)-invariant; for explicit \( f, g \), it is easier to check Inequality 4.9 than 4.5.

On the other hand, the definition of \( h \) is very unnatural. However, we checked a lot of examples which provide evidences for Inequality 4.9 here we pose an easy example with non-reduced \((df \wedge dg)\).

**Example 5.1.** Let all the notations be as in Conjecture 4.6. We assume that \( G \) is a cyclic group of order \( n \) with an action on \( (\mathbb{C}^2, 0) \) such that \( (\mathbb{C}^2, 0)/G \) has an \( A_{n-1} \) singularity. Let
\[
f = (u + v^{n-1})^n, \quad g = uv
\]

It follows that
\[
df = n(u+v^{n-1})^{n-1}d(u+v^{n-1}) = n(u+v^{n-1})^{n-1}(du + (n-1)v^{n-2}dv), \quad dg = vdu + udv
\]

and
\[
df \wedge dg = n(u+v^{n-1})^{n-1}(u - (n-1)v^{n-1})du \wedge dv
hence

\[ h = (u + v^{n-1})(u - (n-1)v^{n-1}), \quad dh = (u + v^{n-1})d(u - (n-1)v^{n-1}) + (u - (n-1)v^{n-1})d(u + v^{n-1}) \]

Note that \( L_df = (n-1)(u + v^{n-1}) + L_d(u + v^{n-1}) \) where \((u + v^{n-1})\) denotes a divisor.

We calculate that

\[
\begin{align*}
L_dhL_dfL dg &= (n-1)((u + v^{n-1}) \cdot L_{adv + vdu} \cdot (u - (n-1)v^{n-1}) + (u + v^{n-1}) \cdot L_{adv + vdu} \cdot L_d(u + v^{n-1})) \\
&\quad + L_d(u + v^{n-1}) \cdot L_{adv + vdu} \cdot (u + v^{n-1}) + L_d(u + v^{n-1}) \cdot L_{adv + vdu} \cdot L_d(u - (n-1)v^{n-1}) \\
&= (n-1)(n-1) + (1 + (n-2)) + (1 + (n-2)) + (n-2) \\
&= n^2 - 1 + n(n-2)
\end{align*}
\]

It is easy to check that our conjecture is true for this \( f \) and \( g \).

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