Information-Theoretic Measure of Genuine Multi-Qubit Entanglement

Jian-Ming Cai\textsuperscript{1}, Zheng-Wei Zhou\textsuperscript{1} and Xing-Xiang Zhou\textsuperscript{2}, and Guang-Can Guo\textsuperscript{1}

\textsuperscript{1}Laboratory of Quantum Information, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China

\textsuperscript{2}Physics Department, Pennsylvania State University, University Park, Pennsylvania 16802, USA

We consider pure quantum states of $N$ qubits and study the genuine $N$-qubit entanglement that is shared among all the $N$ qubits. We introduce an information-theoretic measure of genuine $N$-qubit entanglement based on bipartite partitions. When $N$ is an even number, this measure is presented in a simple formula, which depends only on the purities of the partially reduced density matrices. It can be easily computed theoretically and measured experimentally. When $N$ is an odd number, the measure can also be obtained in principle.

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The nature of quantum entanglement is a fascinating topic in quantum mechanics since the famous Einstein-Podolsky-Rosen paper \cite{1} in 1935. Recently, much interest has been focused on entanglement in quantum systems containing a large number of particles. On one hand, multipartite entanglement is valuable physical resource in large-scale quantum information processing \cite{2,3}. On the other hand, multipartite entanglement seems to play an important role in condensed matter physics \cite{4}, such as quantum phase transitions (QPT) \cite{5,6} and high temperature superconductivity \cite{7}. Therefore, how to characterize and quantify multipartite entanglement remains one of the central issues in quantum information theory.

In the present literature, there exist very few measures of multipartite entanglement with a clear physical meaning \cite{8,9,10,11,12,13,14,15}. Because of this, most research in quantum entanglement and QPT focused on bipartite entanglement, for which there have been several well defined measures \cite{16,17,18,19,20,21,22}. However, bipartite entanglement can not characterize the global quantum correlations among all parties in a multiparticle system. Since the correlation length diverges at the critical points, multipartite entanglement plays an essential role in QPT. Though localizable entanglement (LE) \cite{23} can be used to describe long-range quantum correlations, its determination is a formidable task for generic pure states. Therefore, computable measure of multipartite entanglement with clear physical meanings are highly desired \cite{24,25}.

In this paper, we define a new measure of genuine $N$-qubit entanglement based on different bipartite partitions of the qubits and existing measures for mutual information. The central idea is that, through bipartite partitions, we can get information about the genuine multi-qubit entanglement. Our measure is a polynomial SLOCC (stochastic local operations and classical communication) invariant \cite{26} and is unchanged under permutations of qubits. When $N$ is an even number, we derive a simple formula for this measure, which is determined by the purity of partially reduced density matrices only. It can be computed through experimentally observable quantities \cite{11,27}. Therefore, it is easy to obtain not only theoretically but also experimentally. For $N = 4$, we show that this measure satisfies all the necessary conditions required for a natural entanglement measure \cite{20} exactly. When $N$ is an odd number, the measure for genuine $N$-qubit entanglement is defined on the basis of the measure for even number of qubits. This measure will definitely extend the research in the field of multipartite entanglement and condensed matter systems.

Bipartite partition and genuine multi-qubit entanglement For multi-qubit pure states, there exist local information, and nonlocal information which is related to quantum correlations \cite{28}. In a closed two- and three-qubit system, i.e. the system state is pure and its evolution is unitary, we have shown that \cite{24} entanglement is relevant to some kind of nonlocal information which contributes to the conserved total information. Consider a pure state $|\psi\rangle$ of $N$ qubits, labelled as $1, 2, \cdots, N$, generally we can write $|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_M\rangle$ \cite{29}, where $|\psi_m\rangle$ are non-product pure states, $m = 1, 2, \cdots, M$, and the qubits of different $|\psi_m\rangle$ have no intersection. If $M = 1$, $|\psi\rangle$ itself is a non-product pure state, otherwise $|\psi\rangle$ is a product pure state. A bipartite partition $P$ will divide the qubits of $|\psi_m\rangle$ into two subsets $A_m$ and $B_m$, then all the $N$ qubits are divided into $A = \cup_{m=1}^{M} A_m$ and $B = \cup_{m=1}^{M} B_m$, e.g. see Fig.1(a). In this paper, we use the linear entropy \cite{30}, then the mutual information between $A_m$ and $B_m$ is $I_{A_m B_m} = S_{A_m} + S_{B_m} - S_{A_m B_m}$, where $S_Y = 1 - \text{Tr} \rho_Y^2$, $Y = A_m, B_m, A_m B_m$. Since $|\psi_m\rangle$ is pure, we can write $I_{A_m B_m} = 2(1 - \text{Tr} \rho_{A_m}^2) = 2(1 - \text{Tr} \rho_{B_m}^2)$, where $\rho_{A_m}$ and $\rho_{B_m}$ are the reduced density matrices. The bipartite nonlocal information between $A$ and $B$, denoted as $S_{A|B}$, is defined as the sum of mutual information between $A_m$ and $B_m$,

$$S_{A|B} := \sum_{m=1}^{M} I_{A_m B_m}$$  \hspace{1cm} (1)

If $A_m$ or $B_m$ is empty we set $I_{A_m B_m} = 0$.

The information diagram for two- and three-qubit pure

\textsuperscript{*}Electronic address: zwzhou@ustc.edu.cn
the bipartite nonlocal information for the other seven partitions, denoted as $S_{1|234}$, $S_{2|34}$, $S_{1|23}$, $S_{123|4}$, $S_{13|24}$, and $S_{14|23}$. It can be seen that $S_T = \sum_{P \in P_T} S_P = 2I_{12} + 2I_{13} + 2I_{14} + 2I_{23} + 2I_{24} + 2I_{34} + 3I_{123} + 3I_{124} + 3I_{134} + 3I_{234} + 4I_{1234}$. In the same way, we can get $S_{IT} = \sum_{P \in P_{IT}} S_P = S_T - I_{1234}$. Therefore, genuine four-qubit entanglement $\mathcal{E}_{1234}$ can be naturally measured by the nonlocal information $I_{1234}$, i.e. the difference between $S_T$ and $S_{IT}$.

$$\mathcal{E}_{1234} = S_T - S_{IT}$$  \hspace{1cm} (2)

The above definition of the measure for genuine four-qubit entanglement must satisfy the following conditions in order to be a natural entanglement measure for pure states: (1) $\mathcal{E}_{1234}$ is invariant under local unitary operations. (2) $\mathcal{E}_{1234} \geq 0$ for all pure states. (3) $\mathcal{E}_{1234}$ is an entanglement monotone, i.e., $\mathcal{E}_{1234}$ does not increase on average under local quantum operations assisted with classical communication (LOCC).

It is obvious that the mutual information $I_{A_{\alpha}B_{\alpha}}$ is determined by the eigenvalues of the partially reduced density matrices. Therefore, $\mathcal{E}_{1234}$ is invariant under local unitary operations. Moreover, according to the above definitions in Eqs.(1-2), it is easy to verify that $\mathcal{E}_{1234} = 0$ for product pure states.

In order to prove that $\mathcal{E}_{1234}$ satisfies the other two conditions, we first investigate how $\mathcal{E}_{1234}$ will change under deterministic 1 SLOCC operations. Determinant 1 SLOCC operations $23, 32$ are local operations which transform a pure state $|\psi\rangle$ to $|\psi'\rangle = A_1 \otimes \cdots \otimes A_n |\psi\rangle/Q$ with $Q = Tr(A_1 \otimes \cdots \otimes A_n |\psi\rangle \langle \psi|A_1' \otimes \cdots \otimes A_n'|)$, where $A_i \in SL(2, C)$ is an operator on the $i$th qubit. Without loss of generality, we assume that $|\psi\rangle$ is a non-product pure state, and the determinant 1 SLOCC operation is only performed on the 1st qubit. The determinant 1 SLOCC operation $A_1$ can be written as $U \cdot \text{diag}(d, 1/d) \cdot V$, where $U, V \in SU(2)$ and $d$ is a positive real number. Since the mutual information is invariant under local unitary operations, we do not have to consider the unitary operations $U$ and $V$. Let us write the four-qubit state $|\psi\rangle = \sqrt{p_0} |\varphi_0\rangle + \sqrt{p_1} |\varphi_1\rangle$, where $p_0 + p_1 = 1$. $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are pure states of qubits 2, 3, 4. After the operation $A_1$, $|\psi'\rangle = A_1 |\psi\rangle = \sqrt{\frac{Q}{Q_0}} |\varphi_0\rangle + \sqrt{\frac{1}{Q_0}} |\varphi_1\rangle$, where $Q = (p_0d^2 + p_1/d^2)^{1/2}$. Since $p_{124} = (p_0d^2 |\varphi_0\rangle \langle \varphi_0| + p_1/d^2 |\varphi_1\rangle \langle \varphi_1|)/Q^2$, $Tr(\rho_{124}^Q) = Tr(\rho_{124}) = (p_0d^2 + p_1/d^2 + f_1)/Q^4$, $f_1 = 2p_0p_1 Tr(|\varphi_0\rangle \langle \varphi_0| |\varphi_1\rangle \langle \varphi_1|)$ is independent on the value of $d$. Then we can get the bipartite nonlocal information for partition 1/234 as $S_{1\mid234} = 2 - 2(p_0d^2 + p_1/d^2 + f_1)/Q^4$. In the same way, after straightforward calculation, we can get the other bipartite nonlocal information $S_{ij\mid\bar{i}j} = 2 - 2(p_0d^2 + p_1/d^2 + f_{ij})/Q^4$ and $S_{1\midij\bar{i}j} = 2 - 2(p_0d^2 + p_1/d^2 + f_{ij})/Q^4$. Therefore, genuine four-qubit entanglement $\mathcal{E}_{1234}$ is naturally measured by the nonlocal information $I_{1234}$, i.e. the difference between $S_T$ and $S_{IT}$.\hspace{1cm} (2)
are the reduce density matrix of the \( i \)th and \( j \)th qubit from \(|\phi_k\rangle\) \((k = 0, 1)\). \( f_i \) and \( f_{ij} \) are all independent on the value of \( d \). We note that \( Tr\rho_3^{(k)} = Tr\rho_3^{(k)}(k), Tr\rho_3^{(k)} = Tr\rho_3^{(k)}(k) \) and \( Tr\rho_3^{(k)} = Tr\rho_3^{(k)}(k) \). It can be seen that \( \mathcal{E}_{1234}(|\psi\rangle\langle\psi|) = 2(2p_0p_1 - f_1 - f_2 - f_3 - f_4 + f_{23} + f_{24} + f_{34})/Q^4 \), i.e.

\[
\mathcal{E}_{1234}(|\psi\rangle\langle\psi|) = \mathcal{E}_{1234}(|\psi\rangle\langle\psi|)/Q^4
\]  

(3)

If \(|\psi\rangle\) is a product pure state, \( \mathcal{E}_{1234} \) is always zero, i.e. it satisfies the above Eq.(3) too. Therefore, if we take into account the fact, \( \mathcal{E}_{1234} \) is SLOCC invariant.\[33\]

We now prove that \( \mathcal{E}_{1234} \) satisfies the above condition (2), i.e. \( \mathcal{E}_{1234} \geq 0 \). It has been shown that \( \mathcal{E}_{1234} \) is a product state, \( \mathcal{E}_{1234} = 0 \). Again we assume \( G_{abcd} \) is a non-product state in the following. The bipartite nonlocal information of different partitions is given by \( S_{12} = S_{34} = S_{14} = 0 \). \( S_{1234} \) and \( S_{1234} \) can be obtained from \( Tr\rho_3^{(k)} = Tr\rho_3^{(k)}(k) \) and \( Tr\rho_3^{(k)} = Tr\rho_3^{(k)}(k) \), where \( \mathcal{T}(A, B, C, D) \) is the general two-outcome POVMs performed on the 1-2-3-4 system.

\[
\mathcal{E}_{1234}(G_{abcd}) \geq 0
\]  

(4)

Together with Eq.(3), we obtain that \( \mathcal{E}_{1234}(|\psi\rangle\langle\psi|) \geq 0 \) for any four-qubit pure state.

The remain condition is that \( \mathcal{E}_{1234} \) should be an entanglement monotone. Note that any local protocol can be decomposed into local positive-operator valued measure (POVMs) that can be implemented by a sequence of two-outcome POVMs performed by one party on the system.

\[32\] Without loss of generality, we only need to consider the general two-outcome POVMs performed on the 1st qubit \(|\Psi_1, M_1\rangle\), such that \( M_1 M_1 + M_2 M_2 = I \). Using the singular value decomposition, we can write \( M_1 = U_1 \cdot diag\{a, b\} \cdot V, M_2 = U_2 \cdot diag\{\sqrt{1-a^2}, \sqrt{1-b^2}\} \cdot V \), where \( U_1, U_2 \) and \( V \) are unitary matrices. We denote that \( M_1 = M_1(1/a^2) \), \( M_2 = M_2(1/(1-a^2)(1-b^2)) \). Note that \( det M_1 = det M_2 = 1 \). According to Eq.(3), we get

\[
\mathcal{E}_{1234}(\langle\psi|\psi\rangle) = \frac{Q_1^2}{Q^2}\mathcal{E}_{1234}(\langle\psi|\psi\rangle), \mathcal{E}_{1234}(\langle\psi|\psi\rangle) = \frac{Q_1^2}{Q^2}\mathcal{E}_{1234}(\langle\psi|\psi\rangle), \text{where} |\psi\rangle = M_1|\psi\rangle/Q_1, \text{and} Q_1 \text{are normalization factors. After simple algebra calculation and using the fact that the arithmetic mean always exceeds the geometric mean} \[32,33\] \text{we can see that}

\[
Q_1^2\mathcal{E}_{1234}(\langle\psi|\psi\rangle) + Q_2^2\mathcal{E}_{1234}(\langle\psi|\psi\rangle) \leq \mathcal{E}_{1234}(\langle\psi|\psi\rangle)
\]  

is fulfilled, i.e. \( \mathcal{E}_{1234} \) is an entanglement monotone. Therefore, it satisfies all the necessary conditions for a natural entanglement measure.

General even number qubits The above discussions can be extended to the situation when \( N \) is a generic even number. We denote the set of all genuine \( k \)-qubit entanglement that contributes to the bipartite nonlocal information of partition \( P \in P_2 \) and \( P \in P_{TT} \) as \( EC^k \), \( EC^{kk} \) respectively. If \( k = N \), the contribution is just genuine \( N \)-qubit nonlocal information \( I_{12,...,N} \) and \( |\mathcal{E}_{12,...,N} - |\mathcal{E}_{12,...,N}|| = 1 \). This can be seen from the polynomial expansion of \( (1-x)^2 |_{x=1} = (2|\mathcal{E}_{12,...,N}^2| + 2) - 2|\mathcal{E}_{12,...,N}^2| = 0 \). For \( 2 \leq k \leq N \), if the nonlocal information \( I_{a_1,a_2,...,a_k,b_2,...,b_k} \), \( \{a_1, a_2, ..., a_k \in A, b_1, b_2, ..., b_k \in B \} \) contributes to the bipartite nonlocal information of some partition \( P = A|B \in P_2 \), there must exist one maximum index, denoted as \( \lambda \), which does not belong to \( \{a_1, a_2, ..., a_k \} \). If \( \lambda \in A \) or \( B \), then we can construct a bipartite partition \( P' = A - \{\lambda\}|B + \{\lambda\} \) or \( A + \{\lambda\}|B - \{\lambda\} \in P_{TT} \). The nonlocal information \( I_{a_1,a_2,...,a_k,b_2,...,b_k} \) will also contributes to the bipartite nonlocal information of partition \( P' \). According to this one to one homologous relation, we can obtain that

\[
I_{12,...,N} = \sum_{P \in P_{TT}} S_P - \sum_{P \in P_{TT}^2} S_P
\]  

(5)

where \( S_P \) is the bipartite nonlocal information for partition \( P \).

\( \mathcal{E}_{12,...,N} \) is SLOCC invariant, and is unchanged under permutations of qubits, i.e., it represents a collective property of all the \( N \) qubits. Our measure can surely distinguish two different kinds of multi-qubit entangled states, general \( N \)-qubit GHZ states \(|GHZ\rangle_N = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle) \) and \( W \) states \(|W\rangle_N = \frac{1}{\sqrt{N}}(|00...01\rangle + |00...010\rangle + \cdots + |10...0\rangle) \), for \( \mathcal{E}_{12,...,N}(|GHZ\rangle_N) = 1 \) and \( \mathcal{E}_{12,...,N}(|W\rangle_N) = 0 \). Although
we have not proved that $\mathcal{E}_{12...N} \geq 0$ for $N > 4$, which is actually related to the intricate compatibility problem of multipartite pure states \[33\], we have calculated numerically $\mathcal{E}_{12...N}$ for more than $10^5$ arbitrarily chosen pure states of six and eight qubits. The numerical results suggest strongly that $\mathcal{E}_{12...N} \geq 0$. The calculation of $\mathcal{E}_{12...N}$ is very straightforward. It can indeed be determined from observable quantities \[11\] [27], which can be conveniently measured in experiments.

**General odd number qubits** The above results are not applicable to the situation of odd number qubits straightforwardly. However, it is easy to verify that the genuine $N$-qubit entanglement, $N \in \text{odd}$, can be characterized by the bipartite nonlocal information of different partitions together with genuine $(N-1)$-qubit entanglement, where $N-1$ is an even number. Therefore, the measure can also be obtained based on our idea in principle.

Based on the measure for pure states of $N$ qubits, the measure for $N$-qubit mixed states is defined by the convex roof extension of pure-state measure according to the standard entanglement theory \[17\], i.e.

$$\mathcal{E}_{12...N}(\rho) = \min \sum_k p_k \mathcal{E}_{12...N}(|\psi_k\rangle\langle\psi_k|)$$

where min runs through all possible decompositions of $\rho$ into pure states, i.e., $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. In conclusion, we have introduced an information-theoretic measure of genuine multi-qubit entanglement, which is the collective property of the whole state. For pure states of an even number of qubits, this measure is easily computable, and are dependent on observable quantities. Therefore the measurement in experiments is convenient. For the pure states of an odd number of qubits, the measure is defined based on the results for the states of an even number qubits. Finally, we demonstrated the usefulness of our measure in spin systems. Further generalizations and application will be presented in our future work. Our results will help gain important insight into the structure and nature of multipartite entanglement, and enlighten the research of quantum entanglement in condensed matter physics.

![FIG. 2: (Color online) Genuine $N$-qubit entanglement $\mathcal{E}_{12...N}$ for the ground state of the finite transverse field Ising model with $N = 4, 6, 8, 10$.](image)

**Genuine multi-qubit entanglement in spin systems** Our measure for genuine multi-qubit entanglement will extend the research of the relation between entanglement and quantum phase transitions. Given a quantum system with $N$ spins, one can compute the genuine $N$-qubit entanglement of the ground state for different even number $N$. In addition, the translational invariant property together with other symmetries of the system Hamiltonian will simplify the calculation of $\mathcal{E}_{12...N}$ significantly.

We plot the behavior of $\mathcal{E}_{12...N}$ for different system size $N = 4, 6, 8, 10$ (see Fig. 2). Using the standard finite-size scaling theory, it can be seen that at the quantum critical point $h_c = 1$, genuine $N$-qubit entanglement changes drastically. Compared to the results in Refs. \[23\], the behavior of $\mathcal{E}_{12...N}$ is very similar to LE, i.e. it can capture the feature of long-range quantum correlations.

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[1] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47, 777 (1935).
[2] R. Rauassendorf and H. J. Briegel, Phys. Rev. Lett. 86, 005188 (2001).
[3] Zhi Zhao, Yu-Ao Chen, An-Ning Zhang, Tao Yang, Hans Briegel, Jian-Wei Pan, Nature (London) 430, 54 (2004).
[4] C. Brukner, V. Vedral, A. Zeilinger, quant-ph/0410138.
[5] T. Roscilde, P. Verrucchi, A. Fubini, S. Haas, and V. Tognetti, Phys. Rev. Lett. 93, 167203 (2004); Phys. Rev. Lett. 94, 147208 (2005).
[6] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) 416, 608 (2002); G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003).
[7] T. J. Osborne, and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002).
[8] V. Vedral, New J. Phys. 6, 22 (2004).
[9] A. Miyake, Phys. Rev. A 67, 012108 (2003).
[10] Peter J. Love, Alec Maassen van den Brink, A. Yu. Smirnov, M. H. S. Amin, M. Grajcar, E. Il’ichev, A. Izmailov, A. M. Zagoskin, quant-ph/0602143.
[11] D. A. Meyer and N. R. Wallach, J. Math. Phys. 43, 4273 (2002); A. J. Scott, Phys. Rev. A 69, 052330 (2004); G. K. Brennen, Quant. Inf. and Comput. 3, 616 (2003).

[12] Clive Emary, quant-ph/0405049; Gerardo A. Paz-Silva, John H. Reina, quant-ph/0603102; P. Facchi, G. Florio and S. Pascazio, quant-ph/0603281.

[13] Florian Mintert, Marek Kus, Andreas Buchleitner, Phys. Rev. Lett. 95, 260502 (2005); Florian Mintert, Andreas Buchleitner, quant-ph/0411130.

[14] Howard Barnum, Emanuel Knill, Gerardo Ortiz, Rolando Somma, and Lorenza Viola, Phys. Rev. Lett. 92, 107902 (2004).

[15] D.L. Zhou, B. Zeng, Z. Xu, L. You, quant-ph/0608240.

[16] M. B. Plenio, S. Virmani, quant-ph/0504163 and reference therein.

[17] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1996).

[18] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[19] A. Peres, Phys. Rev. Lett. 76, 1413 (1996); K. Zyczkowski and P. Horodecki, Phys. Rev. A 58, 883 (1998).

[20] V. Vedral, M. B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. 78, 2275 (1997).

[21] D. P. DiVincenzo, C. A. Fuchs, H. Mabuchi, J. A. Smolin, A. Thapliyal, A. Uhlmann, quant-ph/9803033; O. Cohen, Phys. Rev. Lett. 80, 2493 (1998).

[22] Florian Mintert, Marek Kus, Andreas Buchleitner, Phys. Rev. Lett. 92, 167902 (2004).

[23] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. 92, 027901 (2004).

[24] C. Lunke, Č Brukner, and V. Vedral, Phys. Rev. Lett. 95, 030503 (2005).

[25] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A 63, 012307 (2001).

[26] M. S. Leifer and N. Linden, A. Winter, Phys. Rev. A 69, 052304 (2004).

[27] Leandro Aolita and Florian Mintert, Phys. Rev. Lett. 97, 50501 (2006); S. P. Walborn, P. H. Souto, L. Davidovich, F. Mintert, and A. Buchleitner, Nature (London) 440, 1022 (2006).

[28] M. Horodecki and R. Horodecki, Phys. Lett. A 244, 473 (1998); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Phys. Rev. Lett. 90, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A 71, 062307 (2005).

[29] J.M. Cai, Z.W. Zhou and G.C. Guo, quant-ph/0609026.

[30] E. Santos and M. Ferrero, Phys. Rev. A 62, 024101 (2000).

[31] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).

[32] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).

[33] F. Verstraete, J. Dehaene, and B. De Moor, Phys. Rev. A 68, 012103 (2003).

[34] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002).

[35] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. 89, 207901 (2002); N. Linden and W. K. Wootters, Phys. Rev. Lett. 89, 277906 (2002); A. Higuchi, A. Sudbery, and J. Szulc, Phys. Rev. Lett. 90, 107902 (2003).