An Efficient ADMM Algorithm for Structural Break Detection in Multivariate Time Series

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Abstract

We present an efficient alternating direction method of multipliers (ADMM) algorithm for segmenting a multivariate non-stationary time series with structural breaks into stationary regions. We draw from recent work where the series is assumed to follow a vector autoregressive model within segments and a convex estimation procedure may be formulated using group fused lasso penalties. Our ADMM approach first splits the convex problem into a global quadratic program and a simple group lasso proximal update. We show that the global problem may be parallelized over rows of the time dependent transition matrices and furthermore that each subproblem may be rewritten in a form identical to the log-likelihood of a Gaussian state space model. Consequently, we develop a Kalman smoothing algorithm to solve the global update in time linear in the length of the series.

1 Introduction

In many applied fields, such as neuroscience and economics, it is necessary to segment a non-stationary and multivariate signal into stationary regimes. Many methods have been proposed to accomplish this important challenge. Bayesian approaches typically define a generative model, like a vector autoregressive model (VAR), for each stationary regime and a switching Markov process to model switches between regimes. More nonparametric methods directly analyze jumps in the spectral density of the process over time.

Recently, many authors have explored segmentation procedures based on convex optimization. Similar to Bayesian methods, convex approaches model each regime using an autoregressive model. Furthermore, fused group lasso penalties enforce the constraint that the autoregressive parameters of the process tend to stay constant over time, and only rarely switch to new parameter values. In practice, segmentation methodologies using fused group lasso penalties have relied on an approximate group least angle regression solver for optimization. While an intuitive and widely used algorithm, group least angle regression does not provide any guarantees for returning the optimal solution for the convex segmentation problem.

Instead, we develop an efficient alternating direction method of multipliers (ADMM) algorithm that directly solves the convex segmentation problem with group fused lasso penalties. Our ADMM approach splits the convex problem into a global quadratic program that may be solved in time linear with the series length and a simple group lasso proximal update.

Both code for the ADMM algorithm and code to reproduce our experiments may be found at bitbucket.org/atank/convex_tar.
2 Background

Let \( x_t \in \mathbb{R}^p \) be a \( p \)-dimensional multivariate time series. We assume that \( x_t \) follows a structural break vector autoregressive model. Specifically, let \( L \) be the number of break points occurring at times \((t_1, \ldots, t_L)\). For each \( t \in (t_i, t_{i+1}] \), \( x_t \) follows a stationary vector autoregressive model (VAR) of lag order \( K \)

\[
x_t = \sum_{k=1}^{K} A^{ik} x_{t-k} + e_t,
\]

where \( (A^{i1}, \ldots, A^{iK}) \) are the \( K \times p \) matrices of the \( i \)th VAR process and \( e_t \in \mathbb{R}^p \) is mean zero noise.

Given an observed time series at \( N \) time points, \((x_1, \ldots, x_N)\), the goal of estimation is to segment the series into \( L + 1 \) stationary blocks, where \( L \) is the estimated number of change points. To do this, estimates of the breakpoints, \((\hat{t}_1, \ldots, \hat{t}_L)\), and estimates of the autoregressive VAR parameters, \( \hat{A}^{ik} \) for \( k \in (1, \ldots, K) \) and \( i \in (1, \ldots, \hat{L}) \), in each stationary segment must be determined.

3 Estimation

We follow previous work and formulate structural break estimation in autoregressive models via a convex optimization problem with fused group lasso penalties. First, we introduce local autoregressive parameters \( A^t = (A^{t1}, \ldots, A^{tK}) \) active at each time point. We then solve the following penalized least squares optimization problem

\[
\min_{A^1, \ldots, A^N} \sum_{t=1}^{N} \|x_t - A^t \hat{x}_t\|_F^2 + \lambda \sum_{t=2}^{N} \|A^t - A^{t-1}\|_F^2,
\]

where \( \hat{x}_t = (x_{t-1}^T, \ldots, x_{t-K}^T)^T \), \( \| \cdot \|_F \) is the Frobenius norm that acts as a group lasso penalty and \( \lambda > 0 \) is a tuning parameter that controls the number of estimated break points. In this setting, the fused group lasso penalty shrinks the \( A^t \) and \( A^{t-1} \) parameter estimates to be identical. The change point estimates \((\hat{t}_1, \ldots, \hat{t}_L)\) are those times \( t \) whenever \( \hat{A}^t \neq \hat{A}^{t+1} \), where \( (\hat{A}^1, \ldots, \hat{A}^N) \) is the solution to Problem (2), and \( \hat{L} \) is the number of such time points.

4 ADMM Algorithm

To solve Problem (2) exactly we develop an efficient ADMM algorithm that takes advantage of the time series structure. First, we introduce a change of variables parameterization \( \theta^1 = A^1 \) and \( \theta^t = A^t - A^{t+1} \) for \( t > 1 \). The reparameterization lets us rewrite Problem (2) as

\[
\min_{\theta^1, \ldots, \theta^N} \|Y - X\theta\|_F^2 + \lambda \sum_{t=2}^{N} \|\theta^t\|_F^2,
\]

where \( \theta = (\theta^1, \ldots, \theta^N)^T \), \( Y = (x_1, \ldots, x_N)^T \) and

\[
X = \begin{pmatrix}
x_1^T & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
x_{t-1}^T & x_t^T \& x_{t+1}^T & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
x_N^T & x_N^T \& x_N^T & \ldots & x_N^T
\end{pmatrix}
\]

Since Problem (3) takes the form of a group lasso regression problem, approximate solvers like group least angle regression may be used. However, we instead develop an efficient ADMM algorithm to solve Problem (3) exactly.

As is standard in ADMM, we introduce the parameter \( W = (W^1, \ldots, W^N)^T \) and the constraint \( W = \theta \) to break apart the least squares term and the group lasso penalty in Problem (3).
We present the specific form for solving Problems (7) and (8) in Sections 4.1 and 4.2 below.

Although the global ADMM subproblem given in Eq. (7) is a quadratic program with $p$ variables, we develop an efficient classical Rauch-Tung-Streibel smoother [4]. In our case, this smoothing algorithm reduces to first parallel for each row of $A$.

The augmented Lagrangian for Problem (5) is given by

$$\min_{W, \theta} \|Y - X\theta\|_F^2 + \lambda \sum_{t=2}^{N} \|W_t\|_F^2 + \frac{\rho}{2} \|\theta - W\|_F^2 + \text{trace}(\Omega^T (\theta - W)),$$

where $\Omega \in \mathbb{R}^{p \times pKN}$ are Lagrange multipliers and $\rho > 0$. The scaled ADMM steps for solving the augmented Lagrangian are given by

$$\theta^{(l+1)} = \min_{\theta} \|Y - X\theta\|_F^2 + \frac{\rho}{2} \|\theta - W^{(l)} + \Omega^{(l)}\|_F^2,$$

$$W^{(l+1)} = \min_{W} \lambda \sum_{t=2}^{N} \|W_t\|_F^2 + \frac{\rho}{2} \|\theta^{(l+1)} - W + \Omega^{(l)}\|_F^2,$$

$$\Omega^{(l+1)} = \Omega^{(l)} + \theta^{(l+1)} - W^{(l+1)}.$$

We present the specific form for solving Problems (7) and (8) in Sections 4.1 and 4.2 below.

### 4.1 Global $\theta$ update

Although the global ADMM subproblem given in Eq. (7) is a quadratic program with $p^2 KN$ variables, we develop an efficient $O(N)$ linear time algorithm for its solution. First, we reintroduce the $\theta^t = A^{t+1} - A^t$ parameterization, which gives

$$\min_{A^1, \ldots, A^N} \sum_{t=1}^{N} \|x_t - A^t \tilde{x}_t\|_2^2 + \frac{\rho}{2} \sum_{t=1}^{N} \|A^{t+1} - A^t - W^{t(l)} + \Omega^{(l)}\|_F^2.$$

Furthermore, Problem (10) may be decomposed into $p$ independent problems which may be solved in parallel for each row of $A^t = (a^t_1, \ldots, a^t_p)^T$. The problem for each $(a^t_1, \ldots, a^t_N)$ is given by

$$\min_{a^t_1, \ldots, a^t_N} \sum_{t=1}^{N} (x_{tj} - \tilde{x}_t^T a^t_j)^2 + \frac{\rho}{2} \sum_{t=1}^{N} \|a^{t+1}_j - a^t_j - W^{t(l)}_j + \Omega^{(l)}_j\|_2^2.$$

Problem (11) may be solved efficiently by noting that it takes the same form as a canonical smoothing problem for the $(a^t_1, \ldots, a^t_N)$ in a state space model. Specifically, Problem (11) is the negative log-likelihood of a Gaussian state space model [4] of the following form:

$$a^t_j = a^{t-1}_j + \mu_t + \gamma_t$$

$$x_{tj} = \tilde{x}_t^T a^t_j + \eta_t$$

where by convention $a^0_j = 0$, $\mu_t = W^{t(l)}_j - \bar{\Omega}^{t(l)}_j$ is the bias added at each time step, $\gamma_t \sim N(0, \frac{1}{\rho} I_{pK \times pK})$ is the state evolution noise with covariance matrix $\frac{1}{\rho} I_{pK \times pK}$, and $\eta_t \sim N(0, \frac{1}{2})$ is the observation noise with variance $\frac{1}{2}$.

Inference for the maximum likelihood state sequence $(a^1_1, \ldots, a^N_N)$ in this model may be solved using a Kalman filtering-smoothing algorithm. Kalman smoothers compute the expected value of the latent sequence given the observations and due to Gaussianity this expected value is the same as the mode of the log-likelihood. Many such smoothing algorithms exist but here we employ the classical Rauch-Tung-Streibel smoother [4]. In our case, this smoothing algorithm reduces to first
We randomly generate a series with two structural break points at times $t^C$ where $N(11)$ is linear in $\Sigma^{11}$. The global step of our ADMM algorithm solves $p$ independent smoothing problems. While the Kalman filter we utilize has runtime linear in $N$, each recursive step during the backward smoothing phase requires the inverse of a $pK \times pK$ matrix. While this computation is viable for moderate sized $p$, future work aims to explore alternate Gaussian state space smoothers that scale better with $p$ for high dimensional applications.

4.2 $W$ update

The $W$ update in Problem (8) is given separately for each $W^t$. Specifically, it is given by the proximal operator for the group lasso penalty, which is a group soft threshold step

$$W^{t(t+1)} = \begin{cases} 0 & \text{if } \|\theta^{t(t+1)} + \Omega^{t(t)}\|_F < \frac{\lambda}{\rho} \\ \left(1 - \frac{\lambda}{\rho\|\theta^{t(t+1)} + \Omega^{t(t)}\|_F}\right) \left(\theta^{t(t+1)} + \Omega^{t(t)}\right) & \text{otherwise.} \end{cases}$$

(14)

5 Simulation

To test our algorithm we detect breakpoints on a $p = 10$ series with length $N = 300$ observations. We randomly generate a series with two structural break points at times $t \in (100, 200)$ for a total of three stationary regions each generated by a different VAR(1) process. We run our algorithm for three lambda settings, $\lambda \in (1, 3, 5)$, and a breakpoint is detected if $\|\theta^t\|_F > .005$. The estimated break points are shown in Figure 5. Overall, the $\lambda = 5$ case accurately detects the breakpoint times.

6 Discussion and Future Work

The global step of our ADMM algorithm solves $p$ independent smoothing problems. While the Kalman filter we utilize has runtime linear in $N$, each recursive step during the backward smoothing phase requires the inverse of a $pK \times pK$ matrix. While this computation is viable for moderate sized $p$, future work aims to explore alternate Gaussian state space smoothers that scale better with $p$ for high dimensional applications.
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