DEPENDENCE OF THE GAUSS-CODAZZI EQUATIONS AND THE RICCI EQUATION OF LORENTZ SURFACES

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Abstract. The fundamental equations of Gauss, Codazzi and Ricci provide the conditions for local isometric embeddability. In general, the three fundamental equations are independent for surfaces in Riemannian 4-manifolds. In contrast, we prove in this article that for arbitrary Lorentz surfaces in Lorentzian Kaehler surfaces the equation of Ricci is a consequence of the equations of Gauss and Codazzi.

1. Introduction.

Let $\tilde{M}^n$ be a complex $n$-dimensional indefinite Kaehler manifold, that means $\tilde{M}^n$ is endowed with an almost complex structure $J$ and with an indefinite Riemannian metric $\tilde{g}$, which is $J$-Hermitian, i.e., for all $p \in \tilde{M}^n$, we have

\begin{align*}
\tilde{g}(JX, JY) &= \tilde{g}(X, Y), \quad \forall X, Y \in T_p\tilde{M}^n, \\
\tilde{\nabla}J &= 0,
\end{align*}

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. It follows that $J$ is integrable.

The complex index of $\tilde{M}^n$ is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. When the complex index is one, we denote the indefinite Kaehler manifold by $\tilde{M}^n_1$, which is called a Lorentzian Kaehler manifold (cf. [1]).

The curvature tensor $\tilde{R}$ of an indefinite Kaehler manifold $\tilde{M}^n$ satisfies

\begin{align*}
\tilde{R}(X, Y; Z, W) &= -\tilde{R}(Y, X; Z, W), \\
\tilde{R}(X, Y; Z, W) &= \tilde{R}(Z, W; X, Y), \\
\tilde{R}(X, Y; JZ, W) &= -\tilde{R}(X, Y; Z, JW),
\end{align*}

where $\tilde{R}(X, Y; Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W)$.
It is well-known that the three fundamental equations of Gauss, Codazzi and Ricci play fundamental roles in the theory of submanifolds. For surfaces in Riemannian 4-manifolds, the three equations of Gauss, Codazzi and Ricci are independent in general.

On the other hand, we prove in this article a fundamental result for Lorentz surfaces; namely, for any Lorentz surface in any Lorentzian Kaehler surface the equation of Ricci is a consequence of the equations of Gauss and Codazzi.

2. Basic formulas and fundamental equations

Let $M^2_1$ be a Lorentz surface in a Lorentzian Kaehler surface $\tilde{M}^2$ with an almost complex structure $J$ and Lorentzian Kaehler metric $\tilde{g}$. Let $g$ denote the induced metric on $M^2_1$. Denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connection on $g$ and $\tilde{g}$, respectively; and by $\mathcal{R}$ the curvature tensor of $M$.

The formulas of Gauss and Weingarten are given respectively by (cf. [2, 9])

\begin{align}
(2.1) & \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \\
(2.2) & \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi
\end{align}

for vector fields $X, Y$ tangent to $M^2_1$ and $\xi$ normal to $M$, where $h, A$ and $D$ are the second fundamental form, the shape operator and the normal connection.

For a normal vector $\xi$ of $M^2_1$ at $x \in M^2_1$, the shape operator $A_\xi$ is a symmetric endomorphism of the tangent space $T_x M^2_1$. The shape operator and the second fundamental form are related by

\begin{align}
(2.3) & \quad \tilde{g}(h(X, Y), \xi) = g(A_\xi X, Y)
\end{align}

for $X, Y$ tangent to $M^2_1$.

The three fundamental equations of Gauss, Codazzi and Ricci are given by

\begin{align}
(2.4) & \quad R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle h(X, W), h(Y, Z) \rangle \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - \langle h(X, Z), h(Y, W) \rangle, \\
(2.5) & \quad (\tilde{R}(X, Y)Z)^\perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z), \\
(2.6) & \quad \tilde{g}(R^D(X, Y)\xi, \eta) = \tilde{R}(X, Y; \xi, \eta) + g([A_\xi, A_\eta]X, Y),
\end{align}
where $X, Y, Z, W$ are vector tangent to $M^2$, and $\bar{\nabla}h$ is defined by

$$\bar{\nabla}_X h(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The following lemma is an easy consequence of a result of [7].

**Lemma 2.1.** Locally there exists a coordinate system \( \{x, y\} \) on a Lorentz surface \( M^2 \) such that the metric tensor is given by

$$g = -m^2(x, y)^2(dx \otimes dy + dy \otimes dx)$$

for some positive function \( m(x, y) \).

**Proof.** It is known that locally there exist isothermal coordinates \((u, v)\) on a Lorentz surface \( M^2 \) such that the metric tensor takes the form:

$$g = E(u, v)^2(-du \otimes du + dv \otimes dv)$$

for some positive function \( E \) (see [7] (see also [5]). Thus, after putting

$$x = u + v, \quad y = u - v,$$

we obtain (2.8) from (2.9) with \( m(x, y) = E(x, y)/\sqrt{2} \).

\[\Box\]

3. **Main theorem.**

The main purpose of this article is prove the following fundamental result for Lorentz surfaces.

**Theorem 3.1.** The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any Lorentz surface in any Lorentzian Kaehler surface.

**Proof.** Assume that \( \phi : M^2 \to \tilde{M}^2 \) is an isometric immersion of a Lorentz surface \( M^2 \) into a Lorentzian Kaehler surface \( \tilde{M}^2 \). According to Lemma 2.1, we may assume that locally \( M^2 \) is equipped with the following Lorentzian metric:

$$g = -m^2(x, y)(dx \otimes dy + dy \otimes dx)$$

for some positive function \( m \). The Levi-Civita connection of \( g \) satisfies

$$\nabla_\frac{\partial}{\partial x} \frac{\partial}{\partial x} = \frac{2m_x}{m} \frac{\partial}{\partial x}, \quad \nabla_\frac{\partial}{\partial y} \frac{\partial}{\partial y} = 0, \quad \nabla_\frac{\partial}{\partial y} \frac{\partial}{\partial x} = \frac{2m_y}{m} \frac{\partial}{\partial y}$$

and the Gaussian curvature \( K \) is given by

$$K = \frac{2mm_{xy} - 2m_xm_y}{m^4}.$$
If we put
\begin{equation}
(3.4) \quad e_1 = \frac{1}{m} \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{m} \frac{\partial}{\partial y},
\end{equation}
then \{e_1, e_2\} is a pseudo-orthonormal frame satisfying
\begin{equation}
(3.5) \quad \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0, \quad \langle e_1, e_2 \rangle = -1.
\end{equation}

From (3.2) and (3.4) we find
\begin{equation}
(3.6) \quad \nabla e_1 e_1 = \frac{m_x}{m^2} e_1, \quad \nabla e_2 e_1 = -\frac{m_y}{m^2} e_1, \quad \nabla e_1 e_2 = -\frac{m_x}{m^2} e_2, \quad \nabla e_2 e_2 = \frac{m_y}{m^2} e_2.
\end{equation}

For each tangent vector \(X\) of \(M^2_1\), we put
\begin{equation}
(3.7) \quad JX = PX + FX,
\end{equation}
where \(PX\) and \(FX\) are the tangential and the normal components of \(JX\). For the pseudo-orthonormal frame \{e_1, e_2\} defined by (3.4), it follows from (1.1), (3.5), and (3.7) that
\begin{equation}
(3.8) \quad Pe_1 = (\sinh \alpha)e_1, \quad Pe_2 = (\sinh \alpha)e_2
\end{equation}
for some function \(\alpha\). We call this function \(\alpha\) the Wirtinger angle.

If we put
\begin{equation}
(3.9) \quad e_3 = (\text{sech} \alpha)F e_1, \quad e_4 = (\text{sech} \alpha)F e_2,
\end{equation}
then we may derive from (3.7)-(3.9) that
\begin{equation}
(3.10) \quad Je_1 = \sinh \alpha e_1 + \cosh \alpha e_3, \quad Je_2 = -\sinh \alpha e_2 + \cosh \alpha e_4,
\end{equation}
\begin{equation}
(3.11) \quad Je_3 = -\cosh \alpha e_1 - \sinh \alpha e_3, \quad Je_4 = -\cosh \alpha e_2 + \sinh \alpha e_4,
\end{equation}
\begin{equation}
(3.12) \quad \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 0, \quad \langle e_3, e_4 \rangle = -1.
\end{equation}
We call such a frame \{e_1, e_2, e_3, e_4\} an adapted pseudo-orthonormal frame for \(M^2_1\).

Let us put \(\nabla_X e_j = \sum_{k=1}^2 \omega^k_j(X) e_k; j, k = 1, 2\). Then we deduce from (3.5) that
\begin{equation}
(3.13) \quad \nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2, \quad \omega = \omega^1_1.
\end{equation}

Similarly, if we put \(D_X e_r = \omega^r_s(X)e_s; r, s = 3, 4\), then (3.12) yields
\begin{equation}
(3.14) \quad D_X e_3 = \Phi(X)e_3, \quad D_X e_4 = -\Phi(X)e_4, \quad \Phi = \omega^3_3.
\end{equation}
For the second fundamental form $h$, we put $h(e_i, e_j) = h_{ij}^3 e_3 + h_{ij}^4 e_4$. Then, by applying $\tilde{\nabla}_X (JY) = J\tilde{\nabla}_X Y$, (3.10)-(3.14), we may obtain the following:

\begin{align}
(3.15) \quad A_{e_3} e_j &= h_{j2}^4 e_1 + h_{1j}^4 e_2, \quad A_{e_4} e_j = h_{j2}^3 e_1 + h_{1j}^3 e_2, \\
(3.16) \quad e_j \alpha &= (\omega_j - \Phi_j) \coth \alpha - 2h_{1j}^3, \\
(3.17) \quad e_1 \alpha &= h_{12}^4 - h_{11}^3, \quad e_2 \alpha = h_{22}^4 - h_{12}^3, \\
(3.18) \quad \omega_j - \Phi_j &= (h_{1j}^2 + h_{j2}^4) \tanh \alpha,
\end{align}

where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$ for $j = 1, 2$.

For simplicity, let us put

\begin{equation}
(3.19) \quad h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_3 + \varphi e_4, \quad h(e_2, e_2) = \lambda e_3 + \mu e_4.
\end{equation}

In view of (3.12), and (3.19), equation (2.4) of Gauss can be expressed as

\begin{equation}
(3.20) \quad \gamma \lambda + \beta \mu - 2\delta \varphi = \frac{2(mm_{xy} - m_x m_y)}{m^4} - \bar{K},
\end{equation}

where $\bar{K} = -\bar{R}(e_1, e_2; e_2, e_1)$ is the sectional curvature of the ambient space $\bar{M}^2$ with respect to the 2-plane spanned by $e_1, e_2$.

By using (3.6), (3.14), and (3.18) we find

\begin{align}
(3.21) \quad D_{e_1} e_3 &= \left(\frac{m_x}{m^2} - (\beta + \varphi) \tanh \alpha\right) e_3, \\
D_{e_2} e_3 &= -\left(\frac{m_y}{m^2} + (\delta + \mu) \tanh \alpha\right) e_3, \\
D_{e_1} e_4 &= (\beta + \varphi) \tanh \alpha - \frac{m_x}{m^2} e_4, \\
D_{e_2} e_4 &= \left(\frac{m_y}{m^2} + (\delta + \mu) \tanh \alpha\right) e_4.
\end{align}
So, it follows from (3.6), (3.19) and (3.21) that

\[
(\nabla_{e_1} h)(e_1, e_1) = \left( \frac{\beta_x}{m} - \frac{\beta m_x}{m^2} - \beta(\beta + \varphi) \tanh \alpha \right) e_3 \\
+ \left( \frac{\gamma_x}{m} - \frac{3\gamma m_x}{m^2} + \gamma(\beta + \varphi) \tanh \alpha \right) e_4,
\]

\[
(\nabla_{e_1} h)(e_1, e_2) = \left( \frac{\delta_x}{m} + \frac{\delta m_x}{m^2} - \delta(\beta + \varphi) \tanh \alpha \right) e_3 \\
+ \left( \frac{\varphi_x}{m} - \frac{\varphi m_x}{m^2} + \varphi(\beta + \varphi) \tanh \alpha \right) e_4,
\]

\[
(\nabla_{e_2} h)(e_1, e_1) = \left( \frac{\beta}{m} + \frac{\beta m_y}{m^2} - \beta(\delta + \mu) \tanh \alpha \right) e_3 \\
+ \left( \frac{\gamma y}{m} + \frac{3\gamma m_y}{m^2} + \gamma(\delta + \mu) \tanh \alpha \right) e_4,
\]

\[
(\nabla_{e_1} h)(e_2, e_2) = \left( \frac{\lambda}{m} + \frac{3\lambda m_x}{m^2} - \lambda(\beta + \varphi) \tanh \alpha \right) e_3 \\
+ \left( \frac{\mu x}{m} + \frac{\mu m_x}{m^2} + \mu(\beta + \varphi) \tanh \alpha \right) e_4,
\]

\[
(\nabla_{e_2} h)(e_2, e_1) = \left( \frac{\delta y}{m} - \frac{\delta m_y}{m^2} - \delta(\delta + \mu) \tanh \alpha \right) e_3 \\
+ \left( \frac{\varphi y}{m} + \frac{\varphi m_y}{m^2} + \varphi(\delta + \mu) \tanh \alpha \right) e_4,
\]

\[
(\nabla_{e_2} h)(e_2, e_2) = \left( \frac{\lambda y}{m} - \frac{3\lambda m_y}{m^2} - \lambda(\delta + \mu) \tanh \alpha \right) e_3 \\
+ \left( \frac{\mu y}{m} - \frac{\mu m_y}{m^2} + \mu(\delta + \mu) \tanh \alpha \right) e_4.
\]

On the other hand, from (3.10) we also find

\[
(\tilde{R}(e_1, e_2)e_2)^\perp = -\sech \alpha \tilde{R}(e_1, e_2; e_2, Je_2)e_3 \\
- \{\tanh \alpha \tilde{K} + \sech \alpha \tilde{R}(e_1, e_2; e_2, Je_1)\}e_4,
\]

\[
(\tilde{R}(e_2, e_1)e_1)^\perp = \{\tanh \alpha \tilde{K} - \sech \alpha \tilde{R}(e_2, e_1; e_1, Je_1)\}e_3 \\
- \sech \alpha \tilde{R}(e_2, e_1; e_1, Je_1)e_4.
\]

By applying (3.4), (3.12), (3.22), (3.23), and the equation of Codazzi we get
\[
\lambda_x - \delta_y = (\lambda \beta + \lambda \varphi - \delta^2 - \delta \mu)m \tanh \alpha - \frac{\delta m_y + 3 \lambda m_x}{m} \\
- m \text{sech} \alpha \tilde{R}(e_1, e_2; e_2, J e_2),
\]
\[
\mu_x - \varphi_y = (\delta \varphi - \beta \mu)m \tanh \alpha + \frac{\varphi m_y - \mu m_x}{m} \\
- m \text{sech} \alpha \tilde{R}(e_1, e_2; e_2, J e_1) - m (\tanh \alpha) \tilde{K},
\]
\[
\beta_y - \delta_x = (\beta \mu - \delta \varphi)m \tanh \alpha + \frac{\delta m_x - \beta m_y}{m} \\
- m \text{sech} \alpha \tilde{R}(e_2, e_1; e_1, J e_2) + m (\tanh \alpha) \tilde{K},
\]
\[
\gamma_y - \varphi_x = (\beta \varphi + \varphi^2 - \delta \gamma - \gamma \mu)m \tanh \alpha - \frac{\varphi m_x + 3 \gamma m_y}{m} \\
- m \text{sech} \alpha \tilde{R}(e_2, e_1; e_1, J e_1).
\]
On the other hand, using (3.4) and (3.17) we find

\[(\delta + \mu)\alpha_x - (\beta + \varphi)\alpha_y = 2m(\delta \varphi - \beta \mu).\]  

(3.31)

Also, by applying (3.24), we get

\[(\delta + \mu)m_x - (\beta + \varphi)m_y + m(\delta_x + \mu_x - \beta_y - \varphi_y)\]

\[= 2(\delta \varphi - \beta \mu)m^2 \tanh \alpha - 2m^2 \tanh \alpha \tilde{K} \]

\[+ m^2 \text{sech} \alpha \left\{ R(e_2, e_1; e_1, J e_2) - \tilde{R}(e_1, e_2; e_2, J e_1) \right\}.\]  

(3.32)

Substituting (3.31) and (3.32) into equation (3.30) gives

\[\gamma \lambda + \beta \mu - 2\delta \varphi = \frac{2mm_{xy} - 2mm_y}{m^4} - \tilde{K}\]

\[- \tanh \alpha \text{sech} \alpha \left\{ \tilde{R}(e_2, e_1; e_1, J e_2) + \tilde{R}(e_1, e_2; e_2, J e_1) \right\}.\]  

(3.33)

On the other hand, by applying the curvature identities (1.3) and (1.5), we find

\[\tilde{R}(e_2, e_1; e_1, J e_2) = -\tilde{R}(e_1, e_2; e_2, J e_1).\]

Combining this with (3.33) shows that equation (3.33) becomes equation (3.20) of Gauss. Consequently, the equation of Ricci is a consequence of Gauss and Codazzi for arbitrary Lorentz surfaces in any Lorentzian Kaehler surface. \(\square\)

From the proof of Theorem 1 we also have the following.

**Theorem 3.2.** The equation of Gauss is a consequence of the equations of Codazzi and Ricci for Lorentz surfaces in Lorentzian Kaehler surfaces.

**Remark 1.** Some special cases of Theorem 1 are obtained in [3, 4].

**Remark 2.** Theorem 1 is false in general if the Lorentz surface in a Lorentzian Kaehler surface were replaced by a spatial surface in a Lorentzian Kaehler surface.

**Remark 3.** Since the three fundamental equations of Gauss, Codazzi and Ricci provide the conditions for local isometric embeddability, these equations also play some important role in physics; in particular in the Kaluza-Klein theory (cf. [6] [8] [10]).
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