Abstract. Quadric complexes are square complexes satisfying a certain combinatorial nonpositive curvature condition. These complexes generalize 2-dimensional CAT(0) cube complexes and are a square analog of systolic complexes. We introduce and study the basic properties of these complexes. Using a form of dismantlability for the 1-skeleta of finite quadric complexes we show that every finite group acting on a quadric complex stabilizes a complete bipartite subgraph of its 1-skeleton. Finally, we prove that C(4)-T(4) small cancellation groups act on quadric complexes.

The study of groups acting on combinatorially nonpositively curved spaces has been an ongoing theme in group theory tracing its origins to Dehn’s study of the fundamental groups of closed hyperbolic surfaces [9], continuing with small cancellation theory [13] and reaching more recent developments after the advent of geometric group theory [14]. One such development is the introduction of systolic complexes by Januszkiewicz and Świątkowski [25] and independently by Haglund [16]. This class of simplicial complexes first arose years earlier in the form of bridged graphs defined by Soltan and Chepoi [36, 7] in the context of metric graph theory. The flag completions of bridged graphs are precisely the systolic complexes so these are essentially the same objects. The development of systolic complexes represents a simplicial version of the cubical combinatorial nonpositive curvature theory of CAT(0) cube complexes which were introduced by Gromov [14] but which can be traced back to median graphs studied in metric graph theory [3, 29, 26, 34, 11, 7]. These two theories have since been given a common generalization in the form of the bucolic complexes [5].

In this paper we introduce the combinatorial nonpositive curvature theory of quadric complexes. Quadric complexes are closely related to systolic complexes.
but have square rather than triangular 2-cells, as in the case of CAT(0) cube complexes. We emphasize that although 2-dimensional CAT(0) cube complexes are quadric, the same does not hold in higher dimensions where these theories differ strikingly. Quadric complexes are defined by a disc diagrammatic nonpositive curvature condition, similar to that described in Wise’s presentation of systolic complexes [37]. Essentially, the local condition satisfied by quadric complexes is that the “star” of any “positively curved” vertex of a disc diagram can be replaced by a quadragulation with no internal vertices while the local condition satisfied by systolic complexes is that the star of any positively curved vertex of a disc diagram can be replaced by a triangulation with no internal vertices [37]. Despite these connections, there exist quadric groups that are not virtually systolic (e.g. Example 1.3) and quadric groups that are not virtually cocompactly cubulated (e.g. Example 1.5).

As in the case of systolic complexes, the 1-skeleta of quadric complexes can be characterized by forbidden isometric subgraph conditions. Moreover, the cell structure of a quadric complex can be recovered from its 1-skeleton. We thus find that the 1-skeleta of quadric complexes are precisely the hereditary modular graphs studied in metric graph theory [4]. Hence, as for systolic complexes and CAT(0) cube complexes, a theory arising naturally in geometric group theory has a precursor in metric graph theory. By some doubling, subdivision of squares and identification of cells, quadric complexes can also be viewed as right-angled triangle complexes. They can thus be viewed as a generalization of the folder complexes of Chepoi [7], whose leg graphs satisfy our forbidden subgraph conditions but which are further restricted. Huang and Osajda have introduced a common generalization of systolic and quadric groups called metrically systolic groups, which are essentially groups acting on metric simplicial complexes whose disc diagrams are CAT(0) [23].

Our main results are the following two theorems.

**Theorem A** (Theorem 2.9, Invariant Biclique Theorem). Let $G$ be a finite group acting on a locally finite quadric complex $X$, which is not equal to a single vertex. Then $G$ stabilizes a biclique of $X$.

In order to prove the Invariant Biclique Theorem finite quadric complexes we use the fact, first proved by Bandelt [4, Theorem 2], that the 1-skeleta of finite quadric complexes satisfy a form of dismantlability. Our proof follows that of Hensel et al. [20] and Chepoi [6] for finite systolic complexes. We then apply a theorem of Hanlon and Martinez-Pedroza [18] to lift this result to locally finite quadric complexes.

The bi-dismantlability of 1-skeleta of quadric complexes also plays an essential role in an upcoming proof of the contractibility of quadric complexes after the addition of certain higher dimensional cells [21].

**Theorem B** (Corollary 3.12). Let $G$ be a group admitting a finite C(4)-T(4) presentation $\langle S \mid R \rangle$. Then $G$ acts properly and cocompactly on a quadric complex.

We call a group acting properly and cocompactly on a quadric complex a quadric group. The proof that finitely presented C(4)-T(4) groups are quadric uses the construction of a square complex $X_Y$ from a given 2-complex $Y$ with embedded 2-cells. We show that this square complex $X_Y$ is simply connected when $Y$ is and that it is quadric when $Y$ is additionally C(4)-T(4).
0.1. Structure of the Text. The rest of this section gives some basic definitions used throughout the text and states conventions followed in the remaining sections. Section 1 defines our main objects of study, quadric complexes and quadric groups, and gives some of their basic properties. Section 2 defines bi-dismantlability for bipartite graphs and uses this property to prove the Invariant Biclique Theorem. Finally, Section 3 recalls the definition and basic properties of C(4)-T(4) complexes and proves that C(4)-T(4) groups are quadric.

0.2. Basic Definitions. For fundamental notions such as that of CW-complexes and the fundamental group see Hatcher’s textbook on algebraic topology [19]. Let \( X \) and \( Y \) be 2-dimensional CW-complexes. A combinatorial map from \( X \) to \( Y \) is a continuous map whose restriction to every open cell \( e \) of \( X \) is a homeomorphism from \( e \) to an open cell of \( Y \). Two such maps are considered the same if they are homotopic via a homotopy that is a combinatorial map at each instant of time. (Such a homotopy necessarily restricts to an isotopy on each cell.) A 2-complex is combinatorial if the attaching map of each of its 2-cells is a combinatorial map from the circle \( S^1 \) endowed with the structure of a 1-dimensional CW-complex (i.e. a cycle graph). A combinatorial 2-complex \( X \) is locally finite if every cell of \( X \) intersects finitely many other cells.

A graph is a 1-dimensional CW complex \( \Gamma \). Every such complex is combinatorial. The valence of a 0-cell of \( \Gamma \) is the number of ends of 1-cells incident to it. If no 1-cell of \( \Gamma \) has both of its endpoints attached to the same 0-cell and no two 1-cells of \( \Gamma \) have their endpoints attached to the same unordered pair of 0-cells then \( \Gamma \) is simplicial. The vertex set of any connected graph has a natural metric, the standard graph metric, defined for a pair of vertices \( u \) and \( v \) by the number of edges in the shortest path connecting \( u \) and \( v \). A simplicial graph \( \Gamma \) is bipartite if its 0-cells can be partitioned into two nonempty sets such that no 1-cell has both of its endpoints in the same part. If every pair of 0-cells from different parts is joined by a 1-cell then \( \Gamma \) is a complete bipartite graph or a biclique. It is a fact that a simplicial graph is bipartite if and only if it has no cycles of odd length, where a cycle is a closed path. A square complex is a combinatorial 2-complex whose 2-cells are squares, that is its 2-cell boundaries are endowed with the structure of 4-cycles.

A disc diagram \( D \) is a compact contractible subspace of the 2-sphere \( S^2 \) with the structure of a combinatorial 2-complex. A disc diagram \( D \) is nonsingular if it is homeomorphic to a closed 2-cell and is otherwise singular. The topological boundary of \( D \) is denoted \( \partial D \). The boundary \( \partial D \) is always a subgraph of the 1-skeleton \( D^1 \) of \( D \). A disc diagram \( D \subseteq S^2 \) induces the structure of a combinatorial 2-complex on \( S^2 \) with \( D \) a subcomplex and \( S^2 \setminus D \) an open 2-cell. The attaching map \( S^1 \to \partial D \) of this 2-cell can be made combinatorial by pulling back the cell structure of \( \partial D \). This turns \( S^1 \) into a cycle denoted \( \partial_c D \). The resulting combinatorial map \( \partial_c D \to \partial D \) is the boundary path of \( D \).

Let \( X \) be a combinatorial complex. If \( \mathcal{C} \) is a type of combinatorial complex then a \( \mathcal{C} \) in \( X \) is a \( \mathcal{C} \) along with a combinatorial map from \( \mathcal{C} \) to \( X \). When \( D \) is a disc diagram in \( X \), we abuse notation by also referring to the concatenation \( \partial_c D \to \partial D \to X \) as the boundary path of \( D \).

0.3. Conventions Followed in the Text. We use the following conventions throughout the text unless otherwise stated. Maps and complexes are combinatorial. Complexes are connected. Simply connected means connected and having
trivial fundamental group. Distances between vertices in graphs are always measured by the standard graph metric. The notation $|\cdot|, |\cdot|$ is used to denote distance. For graphs we use the terms vertex and edge in place of 0-cell and 1-cell. For square complexes we use the terms vertex, edge and square. For more general 2-complexes we use 0-cell, 1-cell and 2-cell.

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1. Quadric Complexes

We now define our main object of study, quadric complexes. Section 1.2 describes locally minimal disc diagrams in quadric complexes and shows that they are CAT(0) square complexes. In Section 1.2.1 we recall properties of such disc diagrams that are needed throughout the rest of the text. Section 1.4 characterizes the 1-skeleta of quadric complexes as those graphs whose every isometrically embedded cycle is a 4-cycle. By a theorem of Bandelt [4] these graphs are precisely those known as hereditary modular graphs in the metric graph theory literature. Finally, in Section 1.6 we state a general theorem of Hanlon and Martinez-Pedroza that implies that finitely presented subgroups of quadric groups are quadric and state another theorem of theirs needed in the proof of the Invariant Biclique Theorem.

Definition 1.1. A locally quadric complex is a square complex $X$ satisfying the following conditions.

1. The attaching map of every square is an immersion.
2. Any disc diagram in $X$ of the form of the domain of the fold map factors through the fold map. The fold map is described in Figure 1.
3. For any disc diagram in $X$ of the form of the left-hand side of Figure 2a with immersed boundary, there is a disc diagram in $X$ of the form on the right with the same boundary path.
4. For any disc diagram in $X$ of the form of the left-hand side of Figure 2b with immersed boundary, there is a disc diagram in $X$ of one of the forms on the right with the same boundary path.
A *quadric complex* is a simply connected locally quadric complex.

Condition 1.1.2 implies that no two squares have the same attaching map. Conditions 1.1.3 and 1.1.4 are nonpositive curvature requirements having important consequences for disc diagrams in locally quadric complexes.

Quadric complexes are similar in nature to systolic complexes. This is especially apparent in the presentation given by Wise [37]. Wise also introduces “generalized $(p,q)$-complexes” which encompass systolic complexes as a subclass of generalized $(3,6)$-complexes and quadric complexes as a subclass of generalized $(4,4)$-complexes [37].

The following proposition follows immediately from Definition 1.1.

**Proposition 1.2.** *The class of locally quadric complexes is closed under the operations of taking full subcomplexes and taking covering spaces.*

A full subcomplex is one that includes any cell whose boundary is in the subcomplex.

**Definition.** A group is *quadric* if it acts properly and cocompactly on a quadric complex.

If $X$ is a compact, connected locally quadric complex, then its universal cover $\tilde{X}$ is quadric and so its fundamental group $\pi_1(X)$ is quadric.

1.1. **Examples.** We now discuss a few classes of examples of quadric complexes and groups. It follows immediately from Definition 1.1 that CAT(0) square complexes are quadric.

**Example 1.3.** The following example of Elsner and Przytycki [10] is of a non-positively curved square complex whose fundamental group does not virtually act properly and cocompactly on a systolic complex [10, Theorem 4.1].

$$\langle a, b, c \mid aba^{-1}b, c^{-1}ac = b \rangle$$

Hence, this group is quadric but not virtually systolic. This shows that though the quadric and systolic theories share many similarities, they are nevertheless distinct.
Example 1.4. Let $n$ be a positive integer. Let $\sigma$ be a permutation of $\{1, 2, \ldots, n\}$. The presentation
\[
\langle g_1, g_2, \ldots, g_n \mid g_i g_{\sigma(i)} = g_j g_{\sigma(j)}, \text{ for all } i \neq j \rangle
\]
is locally quadric and so presents a quadric group. Indeed, no disc diagram of the form on the left-hand side of Figure 2b has immersed boundary and any disc diagram of the form on the left-hand side of Figure 2a with immersed boundary is as in Figure 3 and so has boundary which bounds a square. Setting $n = 3$ and letting $\sigma$ be the permutation $(1 \ 2 \ 3)$ we obtain a presentation of the braid group $B_3$ on three strands.

Example 1.5. By Corollary 3.12, $C(4)$-T(4) small cancellation groups are quadric. In particular, the Artin group
\[
\langle a, b, c \mid ab = ba, bcb = cbc \rangle
\]
is quadric. However, by Huang, Jankiewicz and Przytycki [22, Theorem 1.2] or, independently, Haettel [15, Theorem A], this group does not virtually act properly and cocompactly on a CAT(0) cube complex. In general, Artin groups whose defining graphs are triangle free are $C(4)$-T(4) (see Pride [33]) and many such Artin groups are not virtually cocompactly cubulated [15, 22].

1.2. Nonpositive Curvature of Disc Diagrams. The nonpositive curvature conditions 1.1.3 and 1.1.4 in Definition 1.1 imply that any nullhomotopic closed path in a locally quadric complex bounds a nonpositively curved disc diagram.

A disc diagram $D$ in a 2-complex $X$ has minimal area if it contains the minimal number of 2-cells over all disc diagrams in $X$ having the same boundary path. A disc diagram $D$ in a quadric complex is locally minimal if every internal vertex of $D$ is incident to at least four squares.

Lemma 1.6. Let $X$ be a locally quadric complex and $D$ a minimal area disc diagram in $X$. Then $D$ is locally minimal.

Proof. Suppose $v$ is an internal vertex of $D$ incident to $k < 4$ squares. By Condition 1.1.1, $k \geq 2$. Let $D'$ be the union of closed 2-cells incident to $v$.

Suppose first that $k = 2$. Then $D'$ either has the form of the disc diagram in Condition 1.1.2 or that of the disc diagram in Condition 1.1.3. In the former case the exterior edges of the two squares in $D'$ map to the same edge of $X$ and with the same orientation so that $D$ can be replaced by a disc diagram obtained by cutting out $D'$ and gluing together these exterior edges. In the latter case $D'$ can

\footnote{Thanks to Jingyin Huang for bringing this class of examples to the author’s attention.}
be replaced by a single square if its boundary immerses. If the boundary of $D'$
do not immerse then we may cut $D$ along a nonimmersing path of length 2 in $\partial D'$ and glue it back together in such a way as to introduce a subdisc of the form in Condition 1.1.2. In any case we contradict the minimality of the area of $D$.

Suppose now that $k = 3$. By the $k = 2$ case we may assume that every vertex of $D$ is incident to at least three squares. Then $D'$ has the form of the disc diagram in Condition 1.1.4. If the $\partial D'$ immerses then $D'$ can be replaced by a pair of squares. If $\partial D'$ does not immerse then we may cut $D$ along a nonimmersing path of length 2 in $\partial D'$ and glue it back together in such a way as to introduce a vertex incident to two squares. In any case we again contradict the minimality of the area of $D$. □

We see from the proof of Lemma 1.6 that, given a disc diagram $D$ in a locally quadric complex, we can obtain a locally minimal disc diagram with the same boundary path by performing a finite number of replacements. Each replacement reduces the number of squares, though the locally minimal disc diagram we ultimately obtain may not be of minimal area.

1.2.1. CAT(0) Disc Diagrams. A CAT(0) cube complex is a cube complex for which the metric obtained by making each cube isometric to a standard Euclidean cube satisfies a metric nonpositive curvature condition concerning the thinness of its triangles. Such complexes have been studied extensively in the geometric group theory literature [35, 38, 39, 31, 17, 30] as well as in the metric graph theory literature, via their 1-skeleta [34, 11, 7], median graphs [3, 29, 26].

In this paper we make use of a purely combinatorial characterization of the CAT(0) condition for cube complexes which is due to Gromov [14]. In dimension 2 this characterization, which we may take as a definition, is as follows. A square complex $X$ is CAT(0) if and only if $X$ is simply connected and the shortest embedded cycle in the link of any 0-cell of $X$ has length at least 4. The link of a 0-cell $v$ of $X$ is the graph whose vertices correspond to ends of 1-cells of $X$ incident to $v$ and whose edges correspond to corners of 2-cells of $X$ incident to $v$. In the case where $X$ is a disc diagram, the CAT(0) condition is equivalent to the condition that each interior vertex of $X$ is incident to at least four squares.

By their definition, locally minimal disc diagrams in locally quadric complexes are CAT(0) square complexes. In particular, by Lemma 1.6, minimal area disc diagrams in locally quadric complexes are CAT(0). We refer to disc diagrams that are CAT(0) square complexes as CAT(0) disc diagrams for brevity. Such disc diagrams are amenable to standard arguments using a combinatorial version of the Gauss-Bonnet Theorem as well as to well-known dual curve constructions. These and other well-known properties of CAT(0) disc diagrams will be recalled in the rest of this section. Most of these results can be found in Wise’s study of minimal area cubical disc diagrams [38].

Let $v$ be a vertex of a disc diagram $D$ with square 2-cells. Let $\delta(v)$ denote the valence of $v$ and $\rho(v)$ the number of corners of squares incident to $v$. The curvature of $v$ is

$$\kappa(v) = 2\pi - \delta(v)\pi + \frac{\rho(v)}{2}\pi.$$ 

This notion of curvature may be considered as emerging from the metric obtained on $D$ by treating each square as a standard Euclidean square. Intuitively, this forces all of the curvature of $D$ to be concentrated at its vertices.
Lemma 1.7. Let $D$ be a disc diagram with square 2-cells. Assume that $D$ is not a single vertex and let $v$ be a vertex of $\partial D$. If $\kappa(v) > 0$ then $v$ is not a cutpoint of $D$ and $\kappa(v)$ is either $\pi$ or $\frac{\pi}{2}$. If $\kappa(v) = \pi$ then $v$ has valence 1 and is not incident to any squares. If $\kappa(v) = \frac{\pi}{2}$ then $v$ has valence 2 and is incident to a single corner of a square. If $\kappa(v) = 0$ then either $v$ is a cutpoint, has valence 2 and is not incident to any corners of squares or $v$ is not a cutpoint, has valence 3 and is incident to 2 corners of squares.

Proof. We first consider the case where $v$ is not a cut point of $D$. If $v$ is not a cutpoint then $\delta(v) = \rho(v) + 1$ and so

$$\kappa(v) = 2\pi - (\rho(v) + 1)\pi + \frac{\rho(v)}{2} \pi = \pi - \frac{\rho(v)}{2} \pi$$

which can only take the nonnegative values $\pi$ and $\frac{\pi}{2}$ and 0. If $\kappa(v) = \pi$ then $\rho(v) = 0$ and so $\delta(v) = 1$. If $\kappa(v) = \frac{\pi}{2}$ then $\rho(v) = 1$ and so $\delta(v) = 2$. If $\kappa(v) = 0$ then $\rho(v) = 2$ and so $\delta(v) = 3$.

If $v$ is a cut point of $D$ then consider the disc diagrams $(D_i)_{i=1}^n$ obtained as closures of the components of $D \setminus \{v\}$. Then, by the preceding paragraph, we have

$$\kappa(v) = 2\pi + \sum_{i=1}^n (\kappa_{D_i}(v) - 2\pi) \leq 2\pi - n\pi = (2 - n)\pi \leq 0$$

where $\kappa_{D_i}(v)$ is the curvature of $v$ in $D_i$. We have equality if and only if $\kappa_{D_i}(v) = \pi$, for each $i$, and $n = 2$. So, by the preceding paragraph, we have equality if and only if $v$ is not incident to any corners of squares and has valence 2 in $D$. $\square$

Proposition 1.8 (Gauss-Bonnet Theorem for CAT(0) Disc Diagrams). Let $D$ be a disc diagram with square 2-cells and assume that $D$ is not a single vertex. The sum of the curvatures of vertices of $D$ is $2\pi$, i.e.,

$$\sum_{v \text{ vertex}} \kappa(v) = 2\pi.$$

Proof. The Euler characteristic $\chi(D)$ of $D$ can be computed by subtracting the number of its edges from the number of its vertices and squares. That is, each edge contributes $-1$ to the Euler characteristic and each vertex or square contributes 1. Evenly distributing the $-1$ of each edge to its vertices and the 1 from each square to the vertices on its boundary gives the sum

$$\sum_{v \text{ vertex}} \frac{\kappa(v)}{2\pi},$$

which then must equal $\chi(D) = 1$. $\square$

Corollary 1.9 (Greendlinger’s Lemma for CAT(0) Disc Diagrams). Let $D$ be a CAT(0) disc diagram and assume that $D$ is not a single vertex. Then there are at least two positively curved vertices on the boundary $\partial D$ of $D$. If $D$ has no valence 1 vertices then there are at least four vertices on $\partial D$ with curvature $\frac{\pi}{2}$.

Proof. The CAT(0) property implies that no interior vertex of $D$ is positively curved. Then, by Proposition 1.8, there must be at least $2\pi$ positive curvature on $\partial D$. By Lemma 1.7, the curvature of vertices is bounded above by $\pi$ and so there must be at least two positively curved vertices of $\partial D$. $\square$
If $D$ has no valence 1 vertices then, by Lemma 1.7, the curvature of its vertices is bounded above by $\frac{\pi}{2}$ and this is the least positive curvature of a vertex of $D$. Hence $\partial D$ must have at least four vertices of $\frac{\pi}{2}$ curvature. □

**Definition.** Let $D$ be a disc diagram with square 2-cells. The *midcube* of an edge $e$ of $D$ is the midpoint of $e$. The *midcube* of a square of $D$ is the closed line segment joining the midcubes of a pair of its opposing edges. A dual curve $\alpha$ of $D$ is a minimal subspace of $D$ satisfying the following conditions.

1. Some midcube of $D$ is contained in $\alpha$.
2. If $\alpha$ intersects a midcube $\mu$ of $D$ nontrivially then it contains $\mu$.

The squares of $D$ are in one-to-one correspondence with the intersections of its dual curves.

Let $D$ be a disc diagram with square 2-cells. A nonogon of $D$ is a dual curve of $D$ homeomorphic to the circle $S^1$. A self-intersecting dual curve of $D$ forms a monogon of $D$. A pair of dual curves of $D$ that intersect twice form a bigon of $D$. Finally, three pairwise intersecting dual curves form a triangle of $D$.

**Proposition 1.10.** If $D$ is a CAT(0) disc diagram then it has no nonogons, monogons, bigons or triangles.

**Proof.** If $D$ has a nonogon, monogon, bigon or triangle, then it has a dual curve polygon with $k = 0, 1, 2, \text{ or } 3$ corners. Let $N$ be the union of the open cells intersecting the boundary of this polygon. Let $D'$ be the interior component of $D \setminus N$ and let $D''$ be the closure of $D' \cup N$. Note that $D'$ is not a single vertex $v$ since then $v$ would be incident to exactly $k$ squares and this would contradict the CAT(0) condition.

For $v \in \partial D'$ let $\rho_N(v)$ be the number of open squares of $N$ whose closures contain $v$, let $\rho'(v)$ be the number of squares of $D'$ incident to $v$, let $\delta'(v)$ be the valence of $v$ in $D'$ and let $\kappa'(v)$ be the curvature of $v$ in $D'$. By the CAT(0) property we have $\rho_N(v) + \rho'(v) \geq 4$ and so

$$\rho_N(v) - 2 \geq 2 - \rho'(v) = 4 - 2(\rho'(v) + 1) + \rho'(v) \geq 4 - 2\delta'(v) + \rho'(v) = \frac{2}{\pi} \kappa'(v)$$

where the final inequality relies on the fact that $v \in \partial D'$. But notice that

$$k = \sum_{v \in \partial D'} (\rho_N(v) - 2)$$

while the proof of Corollary 1.9 tells us that

$$\sum_{v \in \partial D'} \kappa'(v) \geq 2\pi$$

and so we have the following contradiction.

$$k = \sum_{v \in \partial D'} (\rho_N(v) - 2) \geq \frac{2}{\pi} \sum_{v \in \partial D'} \kappa'(v) \geq 4$$

□

Note that the “no nonogons” part of Proposition 1.10 implies that if $D$ is CAT(0) and has boundary path length $|\partial_v D| = n$ then it has exactly $\frac{2}{\pi}$ dual curves.

The technique of eliminating $n$-gons of dual curves for low $n$ of disc diagrams in square complexes comes from unpublished lecture notes of Casson and has been developed in the context of CAT(0) cube complexes [35, 39].
The absence of triangles in CAT(0) disc diagrams distinguishes them from general disc diagrams in CAT(0) cube complexes of dimension at least 3. Such disc diagrams may have triangles, as they contain internal vertices of valence 3, though they do not have nonogons, monogons or bigons.

**Lemma 1.11.** Let $D$ be a disc diagram with square 2-cells. Let $\zeta$ be a subpath of nonzero length of $\partial D$. Suppose the initial and terminal vertices of $\zeta$ each have curvature $\pi/2$ and that each internal vertex of $\zeta$ has curvature 0. Then the inclusion of $\zeta$ extends to a map $f : [0, 1] \times [0, n] \to D$ from a $1 \times n$ grid of squares satisfying the following conditions.

1. The restriction of $f$ to $\{1\} \times [0, n]$ is an isomorphism onto $\zeta$.
2. $f$ embeds $[0, 1] \times \{0\} \cup \{1\} \times [0, n] \cup [0, 1] \times \{n\}$ into $\partial D$.
3. The restriction of $f$ to $\left\{\frac{1}{2}\right\} \times [0, n]$ maps onto a dual curve of $D$.
4. $f$ restricts to an embedding on $(0, 1] \times [0, n]$.

**Proof.** By Lemma 1.7, the initial and terminal vertices are not cutpoints and are each incident to a square. By Lemma 1.7, for each internal vertex $v$ of $\zeta$, either $v$ is a cutpoint and $v$ is not incident to any square or $v$ is not a cutpoint and $v$ is incident to two squares. Assume for the sake of contradiction that $v$ is the first vertex of $\zeta$ which is a cutpoint of $D$. Then the preceding vertex $v'$ is not a cutpoint so is incident to a square. Hence $v$ must also be incident to a square and this is a contradiction. So no vertex of $\zeta$ is a cutpoint and so, by Lemma 1.7, each internal vertex of $\zeta$ is incident to two corners of squares and has valence 3. It follows that the inclusion of $\zeta$ extends to a map $f : [0, 1] \times [0, n] \to D$ with $\{1\} \times [0, n]$ mapping onto $\zeta$ by an isomorphism and with $[0, 1] \times \{0\} \cup \{1\} \times [0, n] \cup [0, 1] \times \{n\}$ embedding in $\partial D$. The map $f$ is also an immersion along each edge $[0, 1] \times \{k\}$ and so the dual curve $\left\{\frac{1}{2}\right\} \times [0, n]$ maps onto a dual curve of $D$. Then, by Proposition 1.10, the map $f$ is injective on $(0, 1] \times [0, n]$. \qed

**Lemma 1.12.** Let $D$ be a CAT(0) disc diagram and let $\gamma$ be a geodesic of its 1-skeleton $D^1$. Then $\gamma$ does not cross any dual curve of $D$ more than once.

**Proof.** Suppose $\gamma$ crosses the dual curve $\alpha$ at least twice. Let $e_1$ and $e_2$ be two directed edges along which $\gamma$ crosses $\alpha$ such that the subpath $\gamma'$ of $\gamma$ from the terminal point of $e_1$ to the initial point of $e_2$ does not cross $\alpha$. Let $\beta$ be the path
of \( D^1 \) from the initial point of \( e_1 \) to the terminal point of \( e_2 \) running parallel to \( \alpha \).

Let \( D' \) be the subdisc of \( D \) with boundary \( \partial D' = e_1 \gamma' e_2 \beta^{-1} \), as in Figure 4.

Now, suppose we chose \( \gamma, \alpha \) and the \( e_i \) so as to minimize the area of \( D \). Then no dual curve starts and ends on \( \gamma' \). Hence the dual curves starting on \( \gamma' \) end on \( \beta \). The subdisc \( D' \) is CAT(0) so, by the prohibition in Proposition 1.10 of bigons, every dual curve starting on \( \beta \) must end on \( \gamma' \). So the dual curves of \( \gamma' \) and \( \beta \) are one and the same. Then \( \gamma' \) and \( \beta \) have the same length and so \( \beta \) is a shorter path than \( e_1 \gamma' e_2 \). Then \( \gamma \) is not a geodesic—a contradiction. \( \square \)

**Proposition 1.13.** Let \( \alpha \) be a dual curve of a CAT(0) disc diagram \( D \). Let \([0,1] \times [0,n] \) be a \( 1 \times n \) grid of squares with \( n = |\alpha| \). Let \( f : [0,1] \times [0,n] \to D \) be the map whose restriction to \( \{1/2\} \times [0,n] \) is the inclusion of \( \alpha \). Then \( f \) is an embedding and the 1-skeleton of \([0,1] \times [0,n] \) is convex in the 1-skeleton \( D^1 \) of \( D \). We call the image of \( f \) the carrier of \( \alpha \).

**Proof.** By Proposition 1.10, the restriction of \( f \) to \((0,1) \times [0,n] \) is an embedding and the complement of \( \alpha \) in \( D \) has two components. So, it suffices to show that the restrictions of \( f \) to the parallel paths \( \{0\} \times [0,n] \) and \( \{1\} \times [0,n] \) are convex embeddings. Suppose there is a geodesic \( \gamma : P \to D \) from some \( f(0,k) \) to some \( f(0,k') \) with \( k \leq k' \), that does not coincide with \( f(\{0\} \times [k,k']) \). Choose such a \( \gamma \) minimizing \( (|\gamma|, k' - k) \) lexicographically. By minimality of \( \gamma \), the closed path given by concatenating \( \gamma \) and \( f(\{0\} \times [k,k']) \) is embedded. Hence the CAT(0) subdisc diagram \( D' \subset D \) bounded by the concatenation of \( \gamma \) and \( f(\{0\} \times [k,k']) \) contains at least one square \( s \). By Proposition 1.10, two distinct dual curves \( \alpha' \) and \( \alpha'' \) cross at \( s \). By Lemma 1.12 and Proposition 1.10, no dual curve of \( D' \) has both ends on \( P \) or both ends on \( \{0\} \times [k,k'] \). Hence \( \alpha, \alpha' \) and \( \alpha'' \) pairwise intersect, contradicting Proposition 1.10. \( \square \)

**Proposition 1.14.** Let \( D \) be a CAT(0) disc diagram and let \( u \) and \( v \) be vertices of \( D \). Then the distance between \( u \) and \( v \) in the 1-skeleton \( D^1 \) if \( D \) is equal to the number \( n \) of dual curves of \( D \) separating \( u \) from \( v \).

**Proof.** Let \( \gamma \) be a geodesic of \( D^1 \) from \( u \) to \( v \). Then \( \gamma \) must cross the dual curves separating \( u \) from \( v \) and so must traverse at least \( n \) edges. Suppose \( \gamma \) traversed some additional edge \( e \) and let \( \alpha \) be the dual curve containing the midcube of \( e \). By Lemma 1.12, \( \alpha \) is not a dual curve separating \( u \) and \( v \). But then \( \alpha \) must be traversed a second time if \( \gamma \) is to end at \( v \). This is impossible, by Lemma 1.12 and so \( \gamma \) traverses exactly \( n \) edges and the distance \( |u,v| \) between \( u \) and \( v \) in \( D^1 \) is \( n \). \( \square \)

### 1.3. Quadratic Isoperimetric Function and Word Problem

The CAT(0) property of disc diagrams in quadric complexes implies the following proposition and corollaries.

**Proposition 1.15.** Let \( X \) be a quadric complex. Then \( X \) has quadratic isoperimetric function.

**Proof.** The number of squares in a CAT(0) disc diagram is at most quadratic in the length \( |\partial_x D| \) of its boundary path. Indeed, if \( |\partial_x D| = n \) then \( D \) has less than \( n \) dual curves. Then, by Proposition 1.10, there are less than \( \binom{n}{2} \) intersections of dual curves in \( D \) and so less than \( \binom{n}{2} \) squares. \( \square \)
**Corollary 1.16.** Let \( G \) be a quadric group. Then \( G \) has quadratic isoperimetric function.

**Proof.** Follows by quasi-isometry invariance of isoperimetric functions [1]. \( \square \)

**Corollary 1.17.** Let \( G \) be a quadric group. Then \( G \) has decidable word problem.

**Proof.** Since \( G \) has a quadratic isoperimetric function, it has a recursive isoperimetric function. It follows that \( G \) has decidable word problem [12, Theorem 2.1 and Lemma 2.2]. \( \square \)

1.4. 4-Bridged Graphs. We now use the properties of disc diagrams in quadric complexes developed in Section 1.2.1 to characterize them by their 1-skeleta.

A graph \( \Gamma \) is \( n \)-bridged if every isometrically embedded cycle of \( \Gamma \) has length \( n \).

Graphs that are 3-bridged are known as bridged graphs in the metric graph theory literature [2] and 4-bridged graphs are the same as the hereditary modular graphs of metric graph theory [4]. Note that a 4-bridged graph is simplicial and bipartite. Any immersed 4-cycle or 6-cycle in a bipartite simplicial graph is embedded. Any embedded 6-cycle in a 4-bridged graph has a diagonal, i.e., an edge joining a pair of opposing vertices.

A square complex \( X \) is 4-flag if every embedded 4-cycle in \( X \) bounds a unique square and that the boundary of every square is an embedded 4-cycle.

**Lemma 1.18.** Let \( X \) be a 4-flag square complex and suppose \( X^1 \) is 4-bridged. Then \( X \) is simply connected.

**Proof.** It suffices to construct a disc diagram for a given cycle \( \alpha : C \to X^1 \). Since the girth of \( X^1 \) is at least 4, if \( |C| < 4 \) then \( \alpha \) factors through a tree and so has a singular disc diagram with no squares. If \( |C| = 4 \) then, by 4-flagness, either \( \alpha \) factors through a tree or \( \alpha \) bounds a square whose inclusion is a disc diagram for \( \alpha \). We argue now by induction on \( |C| > 4 \). Since \( \Gamma \) is 4-bridged, our cycle \( \alpha \) is not isometrically embedded. So, for some pair of vertices \( u, v \in C^0 \), there exists a path \( \gamma : P \to X^1 \) from \( \alpha(u) \) to \( \alpha(v) \) with \( |P| < |Q| \) and \( |P| < |R| \) where \( Q \) and \( R \) are the two segments of \( C \) between \( u \) and \( v \). We glue \( \alpha \) and \( \gamma \) together to obtain a map \( \beta : (C \cup R)/\sim \to X^1 \) by identifying the initial vertex of \( R \) with \( u \) and the terminal vertex of \( R \) with \( v \). Then \( \beta|_{P\cup R} \) and \( \beta|_{Q\cup R} \) are cycles of length \( |P| + |R| < |P| + |Q| = |C| \) and \( |Q| + |R| < |Q| + |P| = |C| \). So, by induction, we have disc diagrams \( D_1 \) and \( D_2 \) for \( \beta|_{P\cup R} \) and \( \beta|_{Q\cup R} \). Gluing \( D_1 \) and \( D_2 \) together along \( R \) we obtain a disc diagram for \( C \). \( \square \)

**Proposition 1.19.** Let \( X \) be a square complex. The following are equivalent.

1. \( X \) is quadric.
2. \( X \) is 4-flag and the 1-skeleton \( X^1 \) of \( X \) is 4-bridged.
3. \( X \) is 4-flag and simply connected and the 1-skeleton \( X^1 \) of \( X \) is simplicial and every 6-cycle in \( X^1 \) has a diagonal.

**Proof.** (1) \( \Rightarrow \) (2) Suppose \( X \) is quadric. Simple connectivity of \( X \) and the fact that its 2-cells all have even boundary length imply that cycles in \( X \) have even length and so \( X^1 \) is bipartite. The CAT(0) property of minimal disc diagrams in \( X \) further implies that no embedded cycle has length 2 and that each embedded 4-cycle bounds a square: no nonsingular CAT(0) disc diagram has boundary length 2 and the only nonsingular CAT(0) disc diagram of boundary length 4 is a square.
So \( X \) is 4-flag and any embedded cycle has length at least 4. It remains to show that any isometrically embedded cycle \( \gamma \) of \( X \) has length at most 4. Suppose \( \gamma \) is isometrically embedded and take a locally minimal (and hence CAT(0)) disc diagram \( D \) in \( X \) with boundary path \( \gamma \). Let \( \alpha \) be a dual curve of \( D \) with endpoints at edges \( e \) and \( e' \) of \( \partial D \). Since \( \gamma \) is isometric, any subpath of \( \partial D \) of length at most \( \frac{1}{2} |\partial D| \) must be geodesic in \( D \) so, by Lemma 1.12, no such subpath may have initial and terminal edges \( e \) and \( e' \). Hence \( e \) and \( e' \) are antipodal in \( \partial D \) and every dual curve starting at some edge in \( \partial D \) ends at the antipodal edge in \( \partial D \). Then all such dual curves pairwise intersect and so, by Proposition 1.10, there are at most two such dual curves. This implies that \( |\gamma| \leq 4 \).

(2) \( \Rightarrow \) (3) By Lemma 1.18, our square complex \( X \) is simply connected. An embedded 6-cycle of a bipartite graph that is not isometrically embedded must have a diagonal. Hence every embedded 6-cycle in \( X^1 \) has a diagonal.

(3) \( \Rightarrow \) (1) Simple connectedness of \( X \) implies that \( X^1 \) is bipartite. Conditions 1.1.1 and 1.1.2 are satisfied by 4-flagness and because \( X^1 \) is simplicial. Since, additionally, \( X^1 \) is bipartite, any immersed 4-cycle or 6-cycle in \( X^1 \) must be embedded. Then Condition 1.1.3 follows by 4-flagness and Condition 1.1.4 follows because every embedded 6-cycle has a diagonal, which splits the 6-cycle into two embedded 4-cycles joined along the diagonal.

\[ \square \]

Compare Proposition 1.19 with the fact that a simplicial complex is systolic if and only if it is flag (in the usual sense) and its 1-skeleton is 3-bridged [7].

The 4-flag completion \( \Gamma \) of a graph \( \Gamma \) is the square complex obtained by gluing a unique square to each embedded 4-cycle of \( \Gamma \). Proposition 1.19 states that the 4-flag completion of a 4-bridged graph is quadric and that every quadric complex can be obtained in this way. In other words, the map \( X \mapsto X^1 \) is a bijection from the class of quadric complexes to the class of 4-bridged graphs with inverse \( \Gamma \mapsto \bar{\Gamma} \).

1.5. **Balls in the 1-Skeleton.** Balls in the 1-skeleton of a quadric complex are not generally convex (e.g. consider the standard square tiling of the plane). In fact, there exist quadric complexes wherein the convex hull of a ball of radius 1 is infinite, as in the following example.

**Example 1.20.** Let \( \Gamma \) be the standard Cayley graph of \( \mathbb{Z}^2 \). Let \( \bar{\Gamma} \) be the quotient of \( \Gamma \) by the \( \mathbb{Z} \) action given by \( k \cdot (m, n) = (m + 2k, n + 2k) \). Then the 4-flag completion of \( \bar{\Gamma} \) is simply connected and checking that every 6-cycle has a diagonal we see, by Proposition 1.19, that \( \bar{\Gamma} \) is 4-bridged. Then \( \bar{\Gamma} \) is the 1-skeleton of a quadric complex, by Proposition 1.19. The path of length 4 of \( \Gamma \) given by \((0, 0), (1, 0), (1, 1), (2, 1), (2, 2)\) projects to a 4-cycle in \( \bar{\Gamma} \) whose convex hull is all of \( \bar{\Gamma} \).

Example 1.20 contrasts sharply with the case of the 1-skeleton of a systolic complex, where a neighborhood of any convex subgraph is convex. Balls in the 1-skeleton of quadric complexes are isometrically embedded, however, and this fact is crucial in our proof in Section 2 of the Invariant Biclique Theorem.

A ball \( B_r(v) \) of radius \( r \) centered at a vertex \( v \) of a graph \( \Gamma \) is the full subgraph on the set of all vertices of \( \Gamma \) at distance at most \( r \) to \( v \). The following lemma is an easy corollary of a theorem of Bandelt [4, Theorem 2(ii)]. We give a disc diagrammatic proof.
Lemma 1.21. Let $X$ be a quadric complex. The balls of its 1-skeleton $X^1$ are isometrically embedded subgraphs.

Proof. Let $B_r(v)$ be the ball of radius $r$ centered at some vertex $v$ of $X$. Let $a$ and $b$ be vertices of $B_r(v)$ and take geodesics $\alpha$ from $v$ to $a$ and $\beta$ from $v$ to $b$. For each geodesic $\gamma$ of $X^1$ from $a$ to $b$ there is a minimal area disc diagram $D$ in $X$ with boundary path $\partial D = \alpha\gamma\beta^{-1}$. Choose $(\gamma,D)$ minimizing the area of $D$. Minimality implies that no interior vertex of $\gamma$ has curvature $\frac{\pi}{2}$. By Lemma 1.12, no dual curve of $D$ both starts and ends on one of $\alpha$, $\beta$, or $\gamma$.

We claim that no two dual curves starting on $\gamma$ cross. Suppose there exist such pairs of dual curves and choose a pair $\delta_1$ and $\delta_2$ minimizing the number of vertices bounded by $\gamma$, $\delta_1$ and $\delta_2$. Then $\delta_1$ and $\delta_2$ start at adjacent edges $e_1$ and $e_2$ of $\gamma$. Let $s$ be the square in which the $\delta_i$ cross. Let $N_1$ and $N_2$ be the initial segments of the carriers of $\delta_1$ and $\delta_2$, starting at $\gamma$ and containing all squares up until but not including $s$. By Proposition 1.13, the parallel paths of $N_1$ and $N_2$ starting at $e_1 \cap e_2$ coincide. So if the $N_i$ have nonzero length then the final vertex of this common parallel path is an interior vertex of $D$ incident to exactly three squares: the final squares of the $N_i$ and $s$. This contradicts the CAT(0) condition. So the $N_i$ have length 0 and $s$ contains $e_1$ and $e_2$. But then the vertex $e_1 \cap e_2$ has curvature $\frac{\pi}{2}$, contradicting the minimality of $(\gamma,D)$.

Therefore, $\gamma = \gamma_\alpha \gamma_\beta$ is the disjoint union of two subpaths $\gamma_\alpha$ and $\gamma_\beta$ where all dual curves of $D$ starting on $\gamma_\alpha$ end on $\alpha$ and all those starting on $\gamma_\beta$ end on $\beta$. It follows, by Proposition 1.14, that every vertex $u$ in the interior of $\gamma$ is closer to $v$ than $a$ (if $u \in \gamma_\alpha$) or $b$ (if $u \in \gamma_\beta$). That is, $\gamma \subset B_r(v)$. \hfill $\square$

1.6. Two Results of Hanlon and Martinez-Pedroza. We end this section by stating two theorems of Hanlon and Martinez-Pedroza that apply to quadric complexes and groups.

Theorem 1.22 ([18, Theorem 1.1]). Let $C$ be any category of complexes closed under taking full subcomplexes and covering spaces. The category of groups acting properly and cocompactly on simply connected complexes in $C$ is closed under taking finitely presented subgroups.

Corollary 1.23. A finitely presented subgroup of a quadric group is quadric.

Theorem 1.22 generalizes a theorem of Wise that any finitely presented subgroup of the fundamental group of a “pure $(p,q)$-complex” is the fundamental group of a pure $(p,q)$-complex, which implies that finitely presented subgroups of torsion-free systolic groups are systolic [37].

In order to prove Theorem 1.22, Hanlon and Martinez-Pedroza use the following theorem, which we use in the proof of the Invariant Biclique Theorem.

Theorem 1.24 ([18, Theorem 4.1]). If $G$ is a group acting properly on a simply connected locally finite complex $X$ and $H$ is a finitely presented subgroup of $G$, then $H$ acts cocompactly on a simply connected complex $X'$ that maps $H$-equivariantly into $X$ through an $\mathcal{F}$-tower $X' \rightarrow X$.

An $\mathcal{F}$-tower is a composition of covering maps and inclusions of full subcomplexes and so, if $X$ is locally quadric, then $X'$ is quadric. We need only the following special case of the theorem.
Corollary 1.25. If $G$ is a finite group acting on a simply connected locally finite quadric complex $X$, then $G$ acts on a finite quadric complex $X'$ that immerses $G$-equivariantly into $X$.

2. The Invariant Biclique Property

We first prove the Invariant Biclique Theorem for finite quadric complexes and then generalize this result to locally finite quadric complexes by applying Corollary 1.25 of Hanlon and Martínez-Pedroza. We prove the finite version of the theorem by showing that finite 4-bridged graphs are bi-dismantlable—a theorem of Bandelt [4, Theorem 2]—and that each such graph has a biclique invariant under all of its automorphisms. Our proof of the latter, in Section 2.1, follows that in Hensel et al. [20] showing that dismantlable graphs have invariant cliques—original proved by Polat [32]. In Section 2.2 we apply the breadth-first search algorithm to show that 4-bridged are bi-dismantlable, a technique used by Chepoi [6] to give an alternate proof of the theorem of Anstee and Farber [2] that bridged graphs are dismantlable.

2.1. Dismantling Bipartite Graphs. A metric sphere $S_r(u)$ of radius $r$ centered at a vertex $u$ in a graph $\Gamma$ is the full subgraph on the set of vertices of $\Gamma$ at distance $r$ from $u$. If $\Gamma$ is bipartite and $S_r(u)$ is a metric sphere in $\Gamma$ then $S_r(u)$ has no edges. The neighbours of any vertex $v \in S_r(u)$ are in $S_{r-1}(u)$ and $S_{r+1}(u)$.

Definition 2.1. Let $\Gamma$ be a finite bipartite simplicial graph. If $u$ and $v$ are distinct vertices of $\Gamma$ then $u$ is bi-dominated by $v$ if every neighbour of $u$ is a neighbour of $v$, i.e., if there is a containment $S_1(u) \subset S_1(v)$ of metric spheres.

$\Gamma$ is bi-dismantlable if there exists a sequence

$\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$

of graphs ending on a biclique such that, for each $i < n$, $\Gamma_{i+1} = \Gamma_i \setminus v_i$ for some $v_i$ bi-dominated in $\Gamma_i$. In other words, $\Gamma$ is bi-dismantlable if we can obtain a biclique from $\Gamma$ by successively removing bi-dominated vertices.

These definitions are modified from the standard ones, which require a dominated vertex to be a neighbour of its dominator and require a dismantlement to end on a clique. The modification is necessary to work with bipartite graphs which, by the original definitions, are dismantlable only in the case of trees.

Note that if $u$ bi-dominates $v$ in $\Gamma$ then $u$ bi-dominates $v$ in every full subgraph of $\Gamma$ containing $u$ and $v$. The bi-domination relation is transitive. If $u$ bi-dominates $v$ which bi-dominates $w$ then $u$ bi-dominates $w$.

At each step of a bi-dismantlement of a graph, there may be several bi-dominated vertices that could potentially be removed. The following lemma and proof, which are adapted from Lemma 2.5 of Hensel et al. [20], show that these choices can be made arbitrarily.

Lemma 2.2. If $\Gamma$ is a finite bi-dismantlable simplicial graph and a vertex $v$ is bi-dominated in $\Gamma$ by a vertex $u$, then $\Gamma \setminus v$ is bi-dismantlable.

Proof. Let $\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and $v_1, v_2, \ldots, v_n$ be as in Definition 2.1. If $n = 1$ then $\Gamma$ is a biclique and then so is $\Gamma \setminus v$ so that we are done. Also, if $v_1 = u$ then $\Gamma \setminus v = \Gamma_2$, which we know is bi-dismantlable. So we may assume that $n > 1$ and that $v \neq v_1$. 
If \( v_1 \) is bi-dominated in \( \Gamma \setminus v \) then we can remove it to obtain \( \Gamma_2 \setminus v \). And if \( v \) is bi-dominated in \( \Gamma_2 \) then \( \Gamma_2 \setminus v \) is bi-dismantlable by induction on \( n \). So if both of these conditions hold, then \( \Gamma \setminus v \) is bi-dismantlable and we are done. We assume that either: (I) \( v_1 \) is not bi-dominated in \( \Gamma \setminus v \) or (II) \( v \) is not bi-dominated in \( \Gamma_2 \).

Case I: \( v_1 \) is not bi-dominated in \( \Gamma \setminus v \). Then \( v \) must have been the only bi-dominator of \( v_1 \) in \( \Gamma \). It follows that \( v_1 = u \) (otherwise \( u \) bi-dominates \( v_1 \) in \( \Gamma \setminus v \)).

Case II: \( v \) is not bi-dominated in \( \Gamma_2 \). Then \( v_1 = u \) and the bi-dominator of \( v_1 \) in \( \Gamma \) must be \( v \) (or else it bi-dominates \( v \) in \( \Gamma_2 \)).

In either case we find that \( v \) bi-dominates \( u \) in \( \Gamma \) and that \( v_1 = u \). But \( u \) bi-dominates \( v \) in \( \Gamma \) so \( u \) and \( v \) have the same set of neighbours. Hence \( \Gamma \setminus v \cong \Gamma \setminus u = \Gamma \setminus v_1 = \Gamma_2 \), which is bi-dismantlable. \( \square \)

The following theorem is the main result of this section, the invariant biclique theorem for finite bi-dismantlable graphs. Its proof is adapted from Theorem 2.4 of Hensel et al. [20].

**Theorem 2.3.** Let \( \Gamma \) be a finite bi-dismantlable simplicial graph that is not a single vertex. Then the automorphism group \( G \) of \( \Gamma \) stabilizes a biclique of \( \Gamma \).

**Proof.** The proof is by induction on the number \( \#\Gamma^0 \) of vertices of \( \Gamma \). If \( \#\Gamma^0 = 2 \) then \( \Gamma \) is a biclique. This proves the base case. For \( \#\Gamma^0 > 2 \) let \( D \) be the set of bi-dominated vertices of \( \Gamma \) and consider the following two cases.

Case I: No two vertices of \( D \) have the same set of neighbours. It follows that every bi-dominated vertex \( v \in D \) has a bi-dominator in \( \Gamma \setminus D \). Otherwise we could construct a sequence \( v = v_1, v_2, v_3, \ldots \) in \( D \) starting on \( v \) such that \( v_i \) is bi-dominated by \( v_{i+1} \), for each \( i \). But \( D \) is finite so this sequence must have repeated vertices, contradicting the assumption of this case. Now \( D \) is invariant under \( G \) and hence so is the full subgraph \( \Gamma' = \Gamma \setminus D \). But \( \Gamma' \) is bi-dismantlable, by Lemma 2.2, and so stabilizes a biclique, by induction.

Case II: There is a pair of vertices of \( D \) that have the same set of neighbours. Define a relation \( \sim \) on the vertex set of \( \Gamma \) by \( u \sim v \) if \( u \) and \( v \) have the same set of neighbours \( (S_1(u) = S_1(v)) \). This is an equivalence relation so we can take the quotient graph \( \Gamma' = \Gamma/\sim \). The quotient \( \Gamma' \) can also be obtained by successively removing bi-dominated vertices starting with \( \Gamma \), since a vertex of \( \Gamma \) is bi-dominated by every other member of its equivalence class. So, by Lemma 2.2, \( \Gamma' \) is bi-dismantlable. The action of \( G \) descends to an action on \( \Gamma' \) and the quotient map \( \Gamma \to \Gamma' \) is \( G \)-equivariant. But \( \Gamma' \) has fewer vertices than \( \Gamma \) and so, by induction, has a stable biclique \( B \). The preimage of an edge in \( \Gamma' \) is a biclique in \( \Gamma \) and so the preimage of \( B \) is an invariant biclique of \( \Gamma \). \( \square \)

### 2.2. Breadth-First Search and 4-Bridged Graphs.

In order to show that finite 4-bridged graphs are bi-dismantlable we present a well-known algorithmic tool, the breadth-first search. This algorithm gives us a spanning tree and a numbering of the vertices of a 4-bridged graph which we can use to dismantle it, as in Chepoi [6] for the case of bridged graphs. Bandelt first proved the bi-dismantlability of 4-bridged graphs [4, Theorem 2].

**Definition 2.4.** Breadth-first search (BFS) is an algorithm taking as input a locally finite simplicial graph \( \Gamma \) on \( n \) vertices and a starting vertex \( u \in \Gamma \). The BFS algorithm visits all of the vertices of \( \Gamma \), starting with \( u \), numbering the \( i \)th vertex it visits with the number \( i \). The algorithm proceeds as follows.
Figure 5. Figures for Lemma 2.5 and Lemma 2.6.

(1) Assign the number 1 to $u$.

(2) As long as there remains a vertex of $\Gamma$ that is not assigned a number, repeat the following two steps.

(a) Let $v$ be the least numbered vertex of $\Gamma$ that has an unnumbered neighbour.

(b) Take an arbitrary unnumbered neighbour $w$ of $v$ and assign to it the next available number (i.e. the least positive integer not yet assigned to a vertex).

The strict total order induced on the vertices of $\Gamma$ by these numbers is called the BFS order. We denote a BFS order with starting vertex $u$ by $\prec_u$ or just $\prec$.

A BFS order with starting vertex $u$ is not necessarily unique due to the arbitrary choices made during an execution of the BFS algorithm. However, the order is always compatible with the distance of vertices from $u$. Precisely, if the vertex $v$ is closer to $u$ than the vertex $w$ then $v$ precedes $w$ in any BFS order with starting vertex $u$, i.e.,

$$|v, u| < |w, u| \implies v \prec_u w.$$ 

Fix a BFS order on $\Gamma$ with starting vertex $u$. A neighbour $w$ of a vertex $v$ is called a pseudoparent of $v$ if it precedes $v$ in the BFS order. So if $v$ and $w$ are adjacent vertices of $\Gamma$ then one of $v$ or $w$ is a pseudoparent of the other. The minimal pseudoparent of $w$ in the BFS order is the parent of $w$, denoted $P_w$. Every vertex of $\Gamma$ other than the starting vertex $u$ has a parent. If $v \prec w$ for some vertices $v$ and $w$ of $\Gamma$ having parents $P_v$ and $P_w$ then $P_v \preceq P_w$. The subgraph $\Gamma'$ of $\Gamma$ defined by the parent relation is a spanning tree of $\Gamma$. If $\Gamma$ is bipartite, then the pseudoparents of a vertex $v$ are all strictly closer to $u$ than $v$ is.

**Lemma 2.5.** Let $\Gamma$ be a locally finite 4-bridged graph and run the BFS algorithm on $\Gamma$ starting at an arbitrary vertex $u$. Let $v$ and $w$ be vertices of $\Gamma$ with parents $P_v$ and $P_w$. If $w$ is a pseudoparent of $v$, then $P_w$ is a pseudoparent of $P_v$.

**Proof.** The proof is by induction on $|v, u|$, which is at least 2 since $w$ is assumed to have a parent. If $|v, u| = 2$ then $P_w = u = P^2 v$ and so the lemma holds. Suppose the lemma holds for $2 \leq |v, u| < r$. We will prove it for $|v, u| = r$. If $w = P_v$ then $P_w = P^2 v$ is the parent, and so a pseudoparent, of $P_v$. So we may assume that $w \neq P_v$. By the definition of parent, this implies that $P_v \prec w$. Since $\Gamma$ is bipartite,
it also implies that $|Pv, w| = 2$ in $\Gamma$. But the ball $B_{r-1}(u)$ contains $Pv$ and $w$ so, by Lemma 1.21, $Pv$ and $w$ have a common pseudoparent $x$.

Now, if $Pw = x$ or $Pw = P^2v$ then we are done, so assume that $Pw \neq x$ and $Pw \neq P^2v$. Then, since $Pv \prec w$, we have $P^2v \prec Pw \prec x$ and so $P^2v \neq x$. Hence the graph in Figure 5a is embedded in $\Gamma$. By induction, the parent $Px$ of $x$ is a pseudoparent of both $Pw$ and $P^2v$ as in the graph in Figure 5b which is also embedded in $\Gamma$. Then, as $\Gamma$ is 4-bridged, the 6-cycle $v, P^2v, Px, Pw, w, v$ must have a diagonal. But $Px$ cannot be adjacent to $v$, as $|Px, u| = |v, u| - 3$, and $P^2v$ cannot be adjacent to $w$, since $P^2v \prec Pw$ and $Pw$ is the parent of $w$. Hence $Pw$ and $Pv$ must be adjacent so that $Pw$ is a pseudoparent of $Pv$.

Lemma 2.6. Let $\Gamma$ be a locally finite 4-bridged graph and run the BFS algorithm on $\Gamma$ starting at some vertex $u$. Let $x$ be any vertex of $\Gamma$ at distance $r = |x, u| \geq 2$ from $u$. Then $x$ is bi-dominated in $B_r(u)$ by the parent of its $\prec$-maximal pseudoparent.

Proof. Let $w$ be the maximal pseudoparent of $v$ in the BFS order. We want to show that every neighbour of $v$ in $B_r(u)$ is also a neighbour of $Pw$. But the neighbours of $v$ in $B_r(u)$ are precisely its pseudoparents. So it suffices to show that $Pw$ is adjacent to every pseudoparent of $v$. This holds for the pseudoparents $Pv$ and $w$, by Lemma 2.5.

Let $y$ be any other pseudoparent of $v$. We will show that $Pw$ is adjacent to $y$. By Lemma 2.5, $Py$ is a pseudoparent of $Pv$. If $Py$ and $Pw$ coincide then we are done, so we may assume otherwise. Then $|Py, Pw| = 2$ in $\Gamma$ and so, by Lemma 1.21, $Py$ and $Pw$ have a common pseudoparent $x$ as shown in the graph in Figure 5c which is embedded in $\Gamma$. Then, as $\Gamma$ is 4-bridged, the 6-cycle $v, y, Py, x, Pw, w, v$ must have a diagonal. But $x$ and $v$ cannot be adjacent since $x$ is too close to $u$ relative to $v$. Nor can $Py$ and $w$ be adjacent because the facts $y \prec w$ and $Py \neq Pw$ imply $Py \prec Pw$ and yet $Pw$ is the parent of $w$. Hence $Pw$ and $y$ are adjacent.

Corollary 2.7. Let $\Gamma$ be a finite 4-bridged graph. If $\Gamma$ has at least two vertices, then $\Gamma$ is bi-dismantlable.

Proof. Take any vertex $u$ of $\Gamma$. By Lemma 2.6, for $r \geq 2$, every vertex $v$ of the metric sphere $S_r(u)$ is bi-dominated in $B_r(u)$ by some vertex $w$ in $S_{r-2}(u)$. Hence, we may successively remove vertices at maximal distance from $u$, until the biclique $B_1(u)$ is all that remains.

Corollary 2.8. A group $G$ acting on a finite quadric complex $X$ not equal to a single vertex, stabilizes a biclique of $X$.

Proof. The 1-skeleton $X^1$ of $X$ is 4-bridged, by Proposition 1.19, and every automorphism of $X$ restricts to an automorphism of $X^1$ which is bi-dismantlable, by Corollary 2.7. So by Theorem 2.3, $X$ has an invariant biclique. □

Theorem 2.9 (Invariant Biclique Theorem). Let $G$ be a finite group acting on a locally finite quadric complex $X$, which is not equal to a single vertex. Then $G$ stabilizes a biclique of $X$.

Proof. By Corollary 1.25 of Hanlon and Martinez-Pedroza, we have a finite quadric complex $X'$ mapping $G$-equivariantly into $X$. By Corollary 2.8, $G$ stabilizes a biclique of $X'$. The image of this biclique is an invariant biclique of $X$. □

As a corollary we obtain the following.
Corollary 2.10. Let $G$ be a quadric group. Then $G$ has finitely many conjugacy classes of finite groups.

Proof. Let $X$ be a quadric complex on which $G$ acts properly and cocompactly. By Theorem 2.9, each finite subgroup of $G$ is a subgroup of the stabilizer of some biclique of $X$. Since the action is proper, the stabilizer of each biclique is finite and so has finitely many finite subgroups. So it suffices to show that $X$ has finitely many orbits of bicliques. But every biclique is contained in a ball of radius 2 of $X^1$. Balls of radius 2 are finite by local finiteness and, by cocompactness, there are finitely many orbits of such balls. Hence, there are finitely many orbits of bicliques. □

3. C(4)-T(4) Groups

Small cancellation theory traces its origins to the work of Dehn on hyperbolic surface groups [9] and has played a significant role in the study of infinite groups since then. The standard textbook on the subject is Lyndon and Schupp [27]. We recall the definition of the $C(p)$ and $T(q)$ small cancellation properties in Section 3.1. Section 3.1.1 develops disc diagrammatic properties and the Strong Helly Property of C(4)-T(4) complexes. These properties are crucial in the proof that C(4)-T(4) groups are quadric, given in Section 3.2. The proof relies on a construction associating a square complex $X_Y$ to a simply connected 2-complex $Y$. We prove that this square complex, on which the automorphism group of $Y$ acts, is always simply connected and that it is quadric when $Y$ is C(4)-T(4).

3.1. The $C(p)$ and $T(q)$ Properties. Definitions of the $C(p)$ and $T(q)$ properties and disc diagrammatic consequences can be found elsewhere [27, 28]. We include them here for completeness and because we require a Greendlinger’s Lemma for C(4)-T(4) Disc Diagrams (Corollary 3.3) that gives four sites of positive curvature on the boundary.

Before defining the $C(p)$ and $T(q)$ properties we give some supporting definitions.

Definition 3.1. Let $D$ be a disc diagram. An arc $\alpha$ of $D$ is a minimal subgraph of its 1-skeleton $D^1$ satisfying the following conditions.

1. There is at least one 1-cell in $\alpha$.
2. If $\alpha$ contains a 0-cell of valence 2 of $D^1$, then it contains both of its incident 1-cells.

A 0-cell $v$ of $D$ is a node of $D$ if it has valence other than 2. If a node $v$ is incident to an arc $\alpha$ then $v$ is a node of $\alpha$.

The arcs of a disc diagram $D$ are, possibly closed, paths. If $D$ is a single 2-cell then its only arc has no nodes. If $D$ is a single 2-cell $F$ with a 1-cell $e$ attached, as in Figure 6 then it has two arcs: a closed arc containing the boundary of $F$ and an
arc containing only $e$. The former arc has a single node, the valence 3 0-cell of $e$. The latter has both endpoints of $e$ as nodes.

If an arc $\alpha$ of a disc diagram $D$ contains a 1-cell on the boundary $\partial D$ of $D$ then $\alpha$ lies entirely on $\partial D$. If $\alpha$ contains a 1-cell incident on its two sides to the 2-cells $F_1$ and $F_2$ (with $F_1$ and $F_2$ possibly the same 2-cell) then this holds for every 1-cell of $\alpha$.

An arc of a disc diagram $D$ is a boundary arc if it lies on the boundary $\partial D$ of $D$. A boundary arc with a valence 1 node is a spur. The valence 1 nodes of a spur are called its tips. A non-boundary arc is an internal arc. If an arc contains a 1-cell that is contained in the boundary of a 2-cell $F$, then it is an arc of the 2-cell $F$.

Let $D$ be a disc diagram in a 2-complex $Y$ and let $F_1$ and $F_2$ be two intersecting 2-cells of $D$. An arc $\alpha$ of $F_1$ and $F_2$ is foldable if, when the boundary paths of $F_1$ and $F_2$ are based and oriented so as to coincide on the length of $\alpha$, they map to the same closed path of $Y$. A disc diagram $D$ in a 2-complex is reduced if it has no foldable arcs. Given a disc diagram $D$ in a 2-complex $Y$ we can obtain a reduced disc diagram $D'$ in $Y$ with the same boundary path by performing a finite number of reductions starting from $D$. Each reduction removes a pair of 2-cells $F_1$ and $F_2$ that have a foldable arc.

**Definition.** A daisy $D$ with $p$ petals is a disc diagram satisfying the following conditions.

1. It has a central 2-cell $F$ whose boundary is covered by $p$ arcs.
2. Each arc of $F$ is given by its intersection with a 2-cell of $D$ called a petal, whose remaining boundary is a boundary arc.
3. The boundary $\partial D$ of $D$ is the concatenation of the boundary arcs of its petals.

See Figure 7a for an example of a daisy.

**Definition.** For $q \geq 3$, a jasmine $D$ with $q$ petals is a disc diagram satisfying the following conditions.

1. There is a single internal 0-cell $v$ in $D$.
2. There are $q$ 2-cells in $D$ called petals, all of which are incident to $v$. 
(3) The internal arcs of $D$ are all embedded 1-cells, each of which is incident to $v$.
(4) The boundary $\partial D$ of $D$ is the concatenation of the boundary arcs of its petals.

See Figure 7b for an example of a jasmine.

A 2-complex $Y$ satisfies the $C(p)$ property if its 2-cells are immersed and any daisy in $Y$ with fewer than $p$ petals is foldable along an arc of its central 2-cell. A 2-complex $Y$ satisfies the $T(q)$ property if its 2-cells are immersed and any jasmine in $Y$ with fewer than $q$ petals is foldable along one of its internal arcs. A group presentation $\langle S \mid R \rangle$ satisfies $C(p)$ or $T(q)$ if its associated complex does. Note that if $D$ is a reduced disc diagram in a $C(p)$ (resp. $T(q)$) complex, then $D$ is $C(p)$ (resp. $T(q)$). If $D$ is a disc diagram in a 2-complex of minimal area or one with the least number of cells of any dimension then it is reduced. If a 2-complex $Y$ is $C(p)$ or $T(q)$ then so is its universal cover $\tilde{Y}$. If a group $G$ has a finite $C(p)$ (resp. $T(q)$) presentation $Y = \langle S \mid R \rangle$ then it acts freely and cocompactly on a simply connected $C(p)$ (resp. $T(q)$) complex, namely the universal cover $\tilde{Y}$ of the presentation.

3.1.1. C(4)-T(4) Complexes. A C(4)-T(4) complex is one satisfying both the C(4) and T(4) properties. These include, for example, CAT(0) square complexes. We develop well-known disc diagrammatic tools in this section for the study of C(4)-T(4) complexes. We use these tools to present standard proofs that 2-cells of C(4)-T(4) complexes are embedded and of the Strong Helly Theorem, which is required in the proof that C(4)-T(4) groups are quadric.

Let $D$ be a C(4)-T(4) disc diagram. The curvature of a 2-cell $F$ of $D$ is

$$\kappa(F) = 2\pi - \frac{\nu(F)}{2}\pi,$$

where $\nu(F)$ is the number of nodes on the boundary of $F$. Let $\delta(v)$ denote the valence of a node $v$ and $\rho(v)$ the number of 2-cells incident to $v$. The curvature of a node $v$ of $D$ is

$$\kappa(v) = 2\pi - \delta(v)\pi + \frac{\rho(v)}{2}\pi.$$

**Proposition 3.2** (Gauss-Bonnet Theorem for C(4)-T(4) Disc Diagrams). Let $D$ be a disc diagram. The sum of the curvatures of nodes and 2-cells of $D$ is $2\pi$, i.e.,

$$\sum_{F \text{ 2-cell}} \kappa(F) + \sum_{v \text{ node}} \kappa(v) = 2\pi.$$
Proof. Let $D(N)$, $D(A)$ and $D(2)$ be the set of nodes, arcs and 2-cells of $D$. Then the proposition follows from the below computation.

\[
1 = \chi(D) = \sum_{v \in D(N)} 1 - \sum_{a \in D(A)} 1 + \sum_{F \in D(2)} 1 \\
= \sum_{v \in D(N)} \left(1 - \frac{\delta(v)}{2}\right) + \sum_{F \in D(2)} 1 \\
= \sum_{v \in D(N)} \left(1 - \frac{\delta(v)}{2} + \frac{\rho(v)}{4}\right) + \sum_{F \in D(2)} \left(1 - \frac{\nu(F)}{4}\right) \\
= \frac{1}{2\pi} \sum_{v \in D(N)} \kappa(v) + \frac{1}{2\pi} \sum_{F \in D(2)} \kappa(F)
\]

\[\square\]

Let $D$ be a disc diagram. A boundary 2-cell of $D$ is a 2-cell of $D$ with a boundary arc. A site of positive curvature on the boundary $\partial D$ of a disc diagram $D$ is the tip of a spur or a boundary 2-cell with fewer than four nodes on its boundary.

**Corollary 3.3** (Greendlinger’s Lemma for C(4)-T(4) Disc Diagrams). Let $D$ be a C(4)-T(4) disc diagram and assume that $D$ is not a single 0-cell or 2-cell. Then there are at least two positive curvature sites on the boundary $\partial D$ of $D$. If $D$ has no spurs and every boundary 2-cell of $D$ has at least three nodes on its boundary then there are at least four positive curvature sites on $\partial D$, each of which consists of a 2-cell with three nodes on its boundary.

**Proof.** An internal node $v$ of $D$ has $\rho(v) = \delta(v)$ and so has curvature

\[
\kappa(v) = 2\pi - \delta(v)\pi + \frac{\delta(v)}{2}\pi = 2\pi - \frac{\delta(v)}{2}\pi,
\]

which, by the T(4) property, is nonpositive. For a boundary node $v$, $\rho(v) < \delta(v)$ so that,

\[
\kappa(v) < 2\pi - \frac{\delta(v)}{2}\pi.
\]

But $\kappa(v)$ is an integer multiple of $\frac{\pi}{2}$, so if $\delta(v) \geq 3$ then $\kappa(v)$ is nonpositive. But $\delta(v) \neq 0$, by assumption, and $\delta(v) \neq 2$, by Definition 3.1, so only if a node $v$ has valence $\delta(v) = 1$, i.e., $v$ is the tip of a spur, can it have positive curvature and in this case

\[
\kappa(v) = \pi.
\]

By assumption, $D$ is not a single 2-cell and so $\kappa(F)$ is always less than $2\pi$ for a 2-cell $F$ of $D$. If $F$ has no boundary arc then, by the C(4) property, $\kappa(F) \leq 0$. So only boundary 2-cells can have positive curvature. Then the only positive curvature in $D$ comes from boundary 2-cells and tips of spurs, neither of which have curvature $2\pi$.

It follows, by Proposition 3.2, that there are at least two sites of positive curvature on $\partial D$.

If we add the assumption that $D$ has no spurs and all its boundary 2-cells have at least three nodes then the only sites of positive curvature on $\partial D$ are boundary 2-cells with exactly three nodes. These provide only $\frac{\pi}{2}$ curvature and so, by Proposition 3.2, there must be at least four of them. \[\square\]
**Proposition 3.4.** Let $Y$ be a simply connected C(4)-T(4) complex. Then every 2-cell $F_1$ of $Y$ is embedded.

**Proposition 3.5.** Let $Y$ be a simply connected C(4)-T(4) complex and let $F_1$ and $F_2$ be a pair of intersecting 2-cells of $Y$. Then the intersection $F_1 \cap F_2$ of these 2-cells is connected.

**Corollary 3.6.** 2-cells of a simply connected C(4)-T(4) complex $Y$ intersect along paths, possibly closed, in its 1-skeleton $Y^1$.

**Proposition 3.7 (Helly Property).** Let $F_1$, $F_2$, and $F_3$ be pairwise intersecting 2-cells of a simply connected C(4)-T(4) complex $Y$. Then $F_1 \cap F_2 \cap F_3$ is nonempty.

**Proof scheme of Propositions 3.4, 3.5 and 3.7.** Each proof follows the same pattern. We assume the statement does not hold, and let the $F_i$, $\gamma_i$ and $v_i$ be a counterexample, where $v_i$ is a 0-cell on the boundary of $F_i$ and $F_i+1$ and $\gamma_i$ is an embedded path from $v_{i-1}$ to $v_i$ on the boundary of $F_i$. Let $D$ be a disc diagram in $Y$ with boundary path the concatenation of the $\gamma_i$, as in Figure 8. Now, pick the counterexample $(F_i, \gamma_i, v_i)$ and $D$ so as to minimize the total number of cells in $D$. Let $D' = D \cup \bigcup F_i$ be disc diagram obtained from $D$ by gluing in the $F_i$ along the $\gamma_i$, as in Figure 8.

We claim that $D'$ is reduced. If not then there would be a foldable arc $\alpha$ in some $\gamma_i$ between $F_i$ and some 0-cell $F$ of $D$. Then $\gamma_i$ is a concatenation $\beta_1 \alpha \beta_2$ for some paths $\beta_1$ and $\beta_2$. Let $\beta'_1$ be the path along $\partial F$ having the same length and terminal vertex as $\beta_1$ and having the same mapping as $\beta_1$ into $Y$. Let $\beta'_2$ be the path along $\partial F$ having the same length and initial vertex as $\beta_1$ and having the same mapping as $\beta_1$ into $Y$. We cut $D'$ open along the concatenations $\beta'_1 \beta_1^{-1}$ and $\beta_2^{-1} \beta'_2$ and glue it back together to obtain a new disc diagram $D''$ in which $F$ and $F_i$ intersect along the entirety of $\gamma_i$. But then the closure of $D'' \backslash F_i$ has fewer cells than $D'$, contradicting minimality of our choice. So $D'$ is reduced.

If $D$ is a single 0-cell then the $F_i$ are not a counterexample, which is a contradiction. Also $D$ cannot be a single 2-cell since $D'$ is reduced and $Y$ is C(4). So $D$ is not a single 0-cell or a single 2-cell. By minimality and the fact that 2-cells are immersed, $D$ has no spurs. Also, any boundary 2-cell of $D$ that has fewer than three nodes would give a foldable arc of some $F_i$ in $D'$ so all boundary 2-cells of $D$ have at least three nodes. Then, by Corollary 3.3, there are four sites of positive

![Figure 8. Disc diagrams from the proofs of various propositions.](image-url)
curvature on the boundary $\partial D$ of $D$. One of these must occur away from a $v_i$ and so give a foldable arc of an $F_i$ in $D'$, which again cannot be since $D'$ is reduced. So we have a contradiction. \qed

Proposition 3.8 (Strong Helly Property). Let $F_1$, $F_2$ and $F_3$ be pairwise intersecting 2-cells of a simply connected C(4)-T(4) complex. Then the intersection of a pair of the 2-cells is contained in the remaining 2-cell, i.e.,

$$F_{\sigma(1)} \cap F_{\sigma(2)} \subset F_{\sigma(3)}$$

for some permutation $\sigma$ of the indices.

Proof. This follows immediately from Proposition 3.4, Corollary 3.6, Proposition 3.7 and the T(4) property. \qed

3.2. Quadrization of 2-Complexes. We define now our main construction for this section, the quadrization of a 2-complex. We use the properties developed above to study the quadrization of C(4)-T(4) complexes and finally to prove that C(4)-T(4) groups are quadric.

Let $Y$ be a 2-complex whose 2-cells are embedded. Let $Y_0$ and $Y_2$ be the sets of 0-cells and 2-cells of $Y$. Let $\Gamma_Y$ be the bipartite graph on the vertex set $Y_0 \cup Y_2$ where an edge joins $v \in Y_0$ and $F \in Y_2$ whenever $v$ appears on the boundary of $F$. The quadrization $X_Y$ of $Y$ is the 4-flag completion $X_Y = \Gamma_Y$ of the graph $\Gamma_Y$. Note that if a group $G$ acts properly and cocompactly on $Y$ then it acts properly and cocompactly on the quadrization $X_Y$. If every 1-cell of $Y$ appears on the boundary of a 2-cell then the quadrization $X_Y$ of $Y$ is connected.

For the rest of this section, we will assume that every 1-cell of a 2-complex $Y$ is contained in the boundary of at least one 2-cell. This is not a serious restriction, since each 1-cell of $Y$ not appearing on the boundary of a 2-cell can be subdivided (if it is not embedded) and then thickened to a 2-cell to obtain a new 2-complex $Y'$ which deformation retracts to $Y$. The original complex $Y$ embeds $\text{Aut}(Y)$-equivariantly into $Y'$ via a continuous map $Y \to Y'$, which thus faithfully preserves group actions. Furthermore, if $Y$ is C(4)-T(4) then so is $Y'$.

Lemma 3.9. Let $Y$ be a 2-complex with embedded 2-cells. If $Y$ is simply connected then so is its quadrization $X_Y$.

Proof. Suppose $X_Y$ is not simply connected. Let $\alpha$ be any closed essential path $\alpha$ of $X_Y$. In $Y$, $\alpha$ corresponds to a cyclic sequence $F_0, v_1, F_2, v_3, \ldots, F_{2n-2}, v_{2n-1}, F_0$ of 2-cells and 0-cells such that, for every $i$, the 2-cell $F_{2i}$ contains the 0-cells $v_{2i-1}$ and $v_{2i+1}$ on its boundary (indices modulo 2n). Let $\beta_{2i}$ and $\gamma_{2i}$ be the two paths on the boundary of $F_{2i}$ from $v_{2i-1}$ to $v_{2i+1}$. Let $\delta_0, \delta_2, \ldots, \delta_{2n-2}$ be any sequence with each $\delta_{2i} \in \{\beta_{2i}, \gamma_{2i}\}$. Then the concatenation $\delta = \delta_0\delta_2 \cdots \delta_{2n-2}$ is a closed path in $Y$. So there is a disc diagram $D$ in $Y$ with boundary path $\partial \delta, D = \delta$.

Now, choose $\alpha$, the $\delta_{2i}$ and $D$ so as to minimize the total number of 0-cells, 1-cells and 2-cells of $D$. Because the 2-cells of $Y$ are embedded, any tip of a spur of $D$ must be one of the 0-cells $v_{2i+1}$. Let $u$ be the 0-cell of the spur incident to this $v_{2i+1}$. Then $F_{2i}, v_{2i+1}, F_{2i+2}, u, F_{2i}$ is a 4-cycle in $X_Y$ and so is nullhomotopic. But then we can replace $v_{2i+1}$ with $u$ in $\alpha$ and $D$ with $D \setminus e$, where $e$ is the 1-cell of the spur joining $u$ and $v_{2i+1}$. This contradicts the minimality of our choices and so $D$ has no spurs.

Suppose $D$ has a 2-cell. Then $D$ must have a 2-cell $F$ that intersects its boundary $\partial D$. Let $w$ be a 0-cell in the intersection of $F$ and $\partial D$. Then $w$ is contained in some
\[ \delta_{2i} \text{ and so some } F_{2i} \text{ in } X_Y. \] But then we can replace the subpath \( F_{2i} \) in \( \alpha \) with \( F_{2i}, w, F, w, F_{2i} \) and replace \( D \) with \( D \setminus F \), as shown in Figure 9. This contradicts the minimality of our choices. So \( D \) has no 2-cells.

Therefore, \( D \) is a spurless tree, i.e., a single 0-cell, \( x \). It follows that \( v_{2i+1} = x \), for every \( i \), so that \( \alpha \) is \( F_0, x, F_2, x, \ldots, F_{2n-2}, x, F_0 \). This path is clearly nullhomotopic, which is a contradiction.

\[ \square \]

**Lemma 3.10.** Let \( Y \) be a simply connected 2-complex. If \( Y \) is \( C(4) \)-T(4) then its quadrization \( X_Y \) is quadric.

**Proof.** By Proposition 3.4, the 2-cells of \( Y \) are embedded, so by Lemma 3.9, \( X_Y \) is simply connected. Then, by Proposition 1.19, it suffices to show that embedded 6-cycles of \( X_Y \) have diagonals. An embedded 6-cycle in \( X_Y \) corresponds to a triple of pairwise intersecting 2-cells \( F_1, F_2 \) and \( F_3 \) in \( Y \) and three 0-cells, one contained in each of the three pairwise intersections. By Proposition 3.8, \( F_1 \cap F_2 \subset F_3 \), after possibly reindexing. Then the 0-cell contained in \( F_1 \cap F_2 \) is incident to \( F_3 \). Hence the 6-cycle has a diagonal. \[ \square \]

**Theorem 3.11.** Let \( G \) be a group acting properly and cocompactly on a simply connected \( C(4) \)-T(4) 2-complex \( Y \). Then \( G \) is quadric.

**Proof.** The quadrization \( X_Y \) of \( Y \) is quadric, by Lemma 3.10. Since the action of \( G \) on the 0-cells and 2-cells of \( Y \) preserves incidences, \( G \) acts on \( X_Y \). The stabilizers of 0-cells and 2-cells are trivial in \( Y \), so then are the stabilizers of vertices and edges in \( X_Y \). Then the action must be proper since the stabilizer of a square contains the pointwise stabilizer as a finite index subgroup and the pointwise stabilizer must be trivial since it stabilizes the vertices of the square. This action is also cocompact and so we are done. \[ \square \]

**Corollary 3.12.** Let \( G \) be a group admitting a finite \( C(4) \)-T(4) presentation \( \langle S \mid R \rangle \). Then \( G \) is quadric.
Proof. The action of $G$ on the universal cover $\tilde{Y}$ of the complex $Y$ associated to the presentation $\langle S \mid R \rangle$ is free and cocompact. Hence $G$ is quadric by Theorem 3.11.

We include the following corollary which essentially presents the C(4)-T(4) case of Huebschmann’s theorem on the finite subgroups of small cancellation groups [24]. Huebschmann’s proof originally relied on a theorem of a paper of Lyndon whose correct proof was given later by Collins and Huebschmann [8]. The corollary below does not rely on cohomological arguments, as does Huebschmann’s proof, so we believe it may be of independent interest.

Corollary 3.13. Let $G$ be a finite group acting on a simply connected, locally finite C(4)-T(4) 2-complex $Y$. Then $G$ stabilizes a 0-cell, a 1-cell or the boundary of a 2-cell of $Y$.

Proof. We may assume that $Y$ is not a single 0-cell so that the quadrization $X_Y$ of $Y$ is not a single vertex. By Theorem 2.9, there is an invariant biclique of $X_Y$. Let $\{v_i\}_{i=1}^k \cup \{F_j\}_{j=1}^l$ be the vertex set of this invariant biclique with the $v_i$ corresponding to 0-cells of $Y$ and the $F_j$ corresponding to 2-cells of $Y$. Then $\bigcap_{j=1}^l F_j$ is invariant and $\{v_i\}_{i=1}^k \subset \bigcap_{j=1}^l F_j$ so $\bigcap_{j=1}^l F_j$ is nonempty. By applying Proposition 3.8 inductively to $\bigcap_{j=1}^l F_j$ we see that $\bigcap_{j=1}^l F_j = F_j \cap F_{j'}$ for some $j$ and $j'$. So, by Proposition 3.4 and Proposition 3.5, either $G$ stabilizes the boundary of $F_j$ or $G$ stabilizes an embedded subpath $P$ of the boundary of $F_j$. In the former case we are done and in the latter case $G$ stabilizes the midpoint of $P$ which, depending on the parity of $|P|$, is either contained in an invariant 0-cell or an invariant 1-cell. □

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