WAVEPACKET SOLUTIONS OF THE KLEIN-GORDON EQUATION

Shaun N Mosley,∗ Sunnybank, Albert Road, Nottingham NG3 4JD, UK

Abstract

We present dispersion-free wavepacket solutions $\Psi_v$ to the Klein-Gordon equation, with the only free parameter being the wavepacket velocity $v$. The $\Psi_v$ are eigenvectors of a velocity operator $\tilde{v}$ with commuting components, which is symmetric in a certain scalar product space. We show that $\tilde{v}$ corresponds to a classical generator $v(x, k)$ which with its conjugate $z(x, k)$ may be obtained from $x, k$ by a canonical transformation.

1 Introduction

A particle with momentum $(m\gamma v)$ in the $z$ direction is usually represented by the plane wave

$$\psi = A \exp[-i m\gamma(t - vz)] , \quad |v| < 1, \quad \gamma \equiv (1 - v^2)^{-1/2}$$

solution of the Klein-Gordon (KG) equation

$$(\partial^2_t - \nabla^2 + m^2) \psi = 0$$

(we use natural units with $\hbar = c = 1$). The plane wave is spread over all space, contrary to our usual experience. We explore a class of solutions to the KG equation that are localised, their magnitude approximately inversely proportional to the distance from the centre of the wavepacket; they travel without dispersion, and they are eigenfunctions of a particular velocity operator found below. The aim of this paper is to see if one can replace the usual plane wave basis states with those discussed here.

The KG equation admits the following solutions (considering the wavepacket velocity $v$ to be in the $z$ direction for the moment)

$$\Psi_v(x, t) = \frac{\sin(m\gamma |v| \tilde{r})}{m\gamma |v| \tilde{r}} \exp[-i m\gamma^2(t - vz)], \quad |v| < 1, \quad \gamma \equiv (1 - v^2)^{-1/2}$$

$$\equiv j_0(m\gamma |v| \tilde{r}) \exp[-i m\gamma^2(t - vz)] , \quad (2)$$

where

$$\tilde{r} \equiv \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2} ,$$

and $j_0$ is the spherical Bessel function of order zero. The wavepacket (2) can be thought of as a plane wave of wavelength $(2\pi/m\gamma^2 |v|)$, bounded by the envelope function $j_0(m\gamma |v| \tilde{r})$ which has an extent of order one wavelength in the $z$ direction (for non-relativistic velocities the envelope factor $j_0(m\gamma |v| \tilde{r})$ is nearly spherically symmetric about the point $(0, 0, vt)$). The wavepacket moves with velocity $v$ in the $z$ direction without dispersion. The $\Psi_v$ solutions are a special case of the wavepackets derived by MacKinnon. [1, 2] That (2) satisfies the KG equation follows trivially by applying the Lorentz coordinate transformation

$$z \rightarrow \gamma(z - vt), \quad x \rightarrow x, \quad y \rightarrow y, \quad t \rightarrow \gamma(t - vz)$$

(3)

to the stationary KG solution

$$\psi_v = j_0(m\gamma |v| r) \exp[-i m\gamma t] . \quad (4)$$

The Lorentz transformation is such that it exactly cancels out the negative $z$ momentum components in the superposition making up (4), as will be seen from the Fourier transform of $\Psi_v$ discussed in the next section.

* E-mail: shaun.mosley@ntlworld.com
Further solutions $\Psi_{t,v}$ of the KG equation with spin $l$ can be generated from $\Psi_v$:

$$\Psi_{t,v} = (\partial_x + i \partial_y)^l \Psi_v = j_0(m\gamma|v|\tilde{r}) \hat{Y}_{l0}(\theta, \phi) \exp[-i m\gamma^2(t - vt)]$$

where $\hat{Y}_{l0}(\theta, \phi) \propto (x + iy/r)^l$ is the spherical harmonic. For wavepackets travelling in an arbitrary direction with velocity $v$ then the $\Psi_v$ of (2) is

$$\Psi_v = j_0(m\gamma|v|\tilde{r}) \exp[-i m\gamma^2(t - v \cdot x)]$$

where now

$$\tilde{r} = \sqrt{x_+^2 + \gamma^2(x_\perp - vt)^2}, \quad x_\perp = x - (\hat{v} \cdot \hat{x}) \hat{v}, \quad x_\perp = (\hat{v} \cdot x) \hat{v}.$$  

The (time dependant) distance $\tilde{r}$ is zero at the wavepacket centre $(t_0, vt)$. The $\Psi_v$ satisfy the identity

$$i (\partial_t + v \cdot \nabla) \Psi_v = m \Psi_v$$

noting that the operator $(\partial_t + v \cdot \nabla)$ commutes with $\tilde{r}$ in (6).

We are curious as to whether $\Psi_v$, $\Psi_{l,v}$ are orthogonal, the natural approach to this question is to enquire if $\Psi_v$ is an eigenfunction of some operator $\hat{v}$ which has commuting components. It turns out that this operator is easier to spot in Fourier transform space.

2 The Fourier transform of $\Psi_v$

Positive energy solutions of the KG equation are of the form (see for example Schweber [3])

$$\psi(x, t) = \frac{2}{(2\pi)^{3/2}} \int \phi(k^0, \mathbf{k}) e^{-ik^0t + ik \cdot x} \delta(k^0 - k^2 - m^2) \theta(k^0) \, dk^0 \, dk$$

$$= \frac{1}{(2\pi)^{3/2}} \int \phi(k) e^{-i\sqrt{m^2 + k^2} t + ik \cdot x} \frac{dk}{\sqrt{m^2 + k^2}}, \quad k \equiv |\mathbf{k}|,$n

$$F_k \left[ \phi(k) e^{-i\sqrt{m^2 + k^2} t} \right](x) = \sqrt{\frac{\pi}{2}} \frac{1}{f} \int \phi(k) \sin(kr) k \, dk,$n

or using the delta function identity $\delta(k - m\gamma|v|) = |v| \delta(\sqrt{m^2 + k^2} - m\gamma)$

$$F_k \left[ \left\{ \sqrt{\frac{\pi}{2}} \delta(\sqrt{m^2 + k^2} - m\gamma) \right\} e^{-i\sqrt{m^2 + k^2} t} \right](x) = j_0(m\gamma|v|) \exp[-i m\gamma t]$$

which RHS is (4), so that the quantity in braces is the momentum space counterpart $\phi(k)$ to (4). Next we substitute into the

$$\phi(k) = \sqrt{\frac{\pi}{2}} \delta(\sqrt{m^2 + k^2} - m\gamma) \quad \text{m}\gamma|v|$$

the transformations

$$k_3 \to \gamma k_3 - \gamma v\sqrt{m^2 + k^2}, \quad k_1 \to k_1, \quad k_2 \to k_2,$n

$$\sqrt{m^2 + k^2} \to \gamma \sqrt{m^2 + k^2} + \gamma v k_3$$

$$\sqrt{m^2 + k^2}$$

2
corresponding to the Lorentz transformations (3), then we find

\[ \phi(k) \rightarrow \phi_v(k) = \sqrt{\frac{\pi}{2}} \\delta(\gamma\sqrt{m^2 + k^2} - m\gamma - \gamma v k_3) \]

\[ = \sqrt{\frac{\pi}{2}} \\delta(\sqrt{m^2 + k^2} - m - v k_3) \]

and so the \( \Psi_v(x,t) \) of (6) is

\[ \Psi_v(x,t) = F_k \left[ \phi_v(k) e^{-i\sqrt{m^2 + k^2}t} \right](x) \]

\[ = F_k \left[ \sqrt{\frac{\pi}{2}} \frac{1}{m\gamma^2 |v|} \delta(\sqrt{m^2 + k^2} - m - v k_3) e^{-i\sqrt{m^2 + k^2}t} \sqrt{m^2 + k^2} \right](x) \]  \( \text{(10)} \)

and the quantity in square brackets is the desired Fourier transform of \( \Psi_v \) which we will call \( \alpha_v(k) \):

\[ \alpha_v(k) = \sqrt{\frac{\pi}{2}} \frac{1}{m\gamma^2 |v|} \delta(\sqrt{m^2 + k^2} - m - v k_3) e^{-i\sqrt{m^2 + k^2}t} \sqrt{m^2 + k^2} \]

The delta function \( \delta(\sqrt{m^2 + k^2} - m - v k_3) \) within \( \alpha_v(k) \) is non-zero on the surface

\[ k_1^2 + k_2^2 + \gamma^2[k_3 - mv\gamma^2]^2 = m^2v^2\gamma^2, \]

which is an ellipsoid tangential to the \( k_3 = 0 \) plane at the origin, with centre \((0,0, mv\gamma^2)\) and elongated in the \( k_3 \) direction. As already mentioned, the \( \alpha_v(k) \) has no negative \( k_3 \) component (the ellipsoid surface has \( k_3 \geq 0 \)). For the general case when the velocity \( v \) is in an arbitrary direction then \( \alpha_v(k) \) is

\[ \alpha_v(k) = \sqrt{\frac{\pi}{2}} \frac{1}{m\gamma^2 |v|} \delta(\sqrt{m^2 + k^2} - m - v \cdot k) e^{-i\sqrt{m^2 + k^2}t} \sqrt{m^2 + k^2} \]  \( \text{(11)} \)

Note that due to the delta function

\[ \sqrt{m^2 + k^2} = m + v \cdot k \]

which corresponds to (7), and the time evolution operator is \( e^{-i\sqrt{m^2 + k^2}t} = e^{-i(m + v \cdot k)t} \) where the \( e^{-i\mathbf{v} \cdot \mathbf{k}t} \) factor generates the translation \( \mathbf{v}t \) as expected.
3 The velocity operator

Our goal is to find the velocity operator $\tilde{v}$ in Fourier transform space such that $\tilde{v} \alpha_v(k) = v \alpha(k)$. Once we have found the operator at time $t = 0$ then the operator $\tilde{v}_t$ for time $t$ will be

$$\tilde{v}_t = e^{-i\sqrt{m^2+k^2}t} \tilde{v}_0 e^{i\sqrt{m^2+k^2}t}.$$  

At time $t = 0$ then $\alpha_v(k)$ is (disregarding a factor)

$$\alpha(k) = \frac{\delta(\sqrt{m^2+k^2} - m - v \cdot k)}{\sqrt{m^2+k^2}}.$$  

(12)

Clearly the operator $\tilde{v}$ cannot be simply multiplication by a function of $k$ as the delta function is a 2-D surface, not a point. Differentiating $\alpha_v(k)$ with respect to $k$ yields a term $v$ multiplied by the derivative of the delta function. We can avoid this complication by considering first the ‘anti-derivative’ of the delta function which is a unit step function: we define $E(\chi) = -1/2, \chi < 0$; $E(\chi) = 1/2, \chi > 0$. Then

$$E(\sqrt{m^2+k^2} - m - v \cdot k)$$

has the value $-1/2$ within the delta function ellipsoid, and is $1/2$ outside. Then

$$\nabla_k E(\sqrt{m^2+k^2} - m - v \cdot k) = \left(\frac{k}{\sqrt{m^2+k^2}} - v\right) \delta(\sqrt{m^2+k^2} - m - v \cdot k) = (k - \sqrt{m^2+k^2}v) \alpha(k)$$  

(13)

and

$$(k \cdot \nabla_k) E(\sqrt{m^2+k^2} - m - v \cdot k) = \left(k^2 - (v \cdot k) \sqrt{m^2+k^2}\right) \alpha(k)$$

$$= m \left(\sqrt{m^2+k^2} - m\right) \alpha(k)$$

so that

$$E(\sqrt{m^2+k^2} - m - v \cdot k) = m (k \cdot \nabla_k)^{-1}(\sqrt{m^2+k^2} - m) \alpha(k).$$  

(14)

Now we substitute this last result into (13) obtaining

$$\left\{k - m \nabla_k(k \cdot \nabla_k)^{-1}(\sqrt{m^2+k^2} - m)\right\} \alpha(k) = \sqrt{m^2+k^2}v \alpha(k)$$

or

$$\tilde{v}_k \alpha(k) \equiv \left\{\frac{k}{\sqrt{m^2+k^2}} - m \frac{1}{\sqrt{m^2+k^2}} \nabla_k(k \cdot \nabla_k)^{-1}(\sqrt{m^2+k^2} - m)\right\} \alpha(k) = v \alpha(k).$$  

(15)

The operator $\nabla_k(k \cdot \nabla_k)^{-1}$ within $\tilde{v}_k$ is defined as follows:

$$\nabla_k(k \cdot \nabla_k)^{-1} = \nabla_k(k \partial_k)^{-1} = (-i \nabla_k k) (-i \partial_k k)^{-1} \frac{1}{k} \equiv (-i \nabla_k k) D_k^{-1} \frac{1}{k}$$  

(16)

where $D_k$ is the dilation operator

$$D_k \equiv -i (\partial_k k)$$

which commutes with any operator of homogeneity degree zero (i.e. any operator invariant under a dilation of $k$). And

$$D_k^{-1} \phi(k) \equiv i (\partial_k k)^{-1} \phi(k) \equiv \frac{i}{2} \left[ \int_0^1 \phi(\lambda k) d\lambda - \int_1^\infty \phi(\lambda k) d\lambda \right]$$  

(17)
which is a right and left inverse of \( D_k \). The operator from (16) which is
\[-i \nabla_k D_k^{-1} \frac{1}{\sqrt{x}} = D_k^{-1} (-i \nabla_k)\]
can be shown to be symmetric in the usual 3-D inner product space, as
\[
(-i \nabla_k)^\dagger = (-i \nabla_k), \quad D_k^\dagger = kD_k \frac{1}{\sqrt{k}}, \quad (D_k^{-1})^\dagger = kD_k \frac{1}{k},
\]
\[
[D_k^{-1}(-i \nabla_k)]^\dagger = (-i \nabla_k)^\dagger (D_k^{-1})^\dagger = (-i \nabla_k)kD_k \frac{1}{k} = D_k^{-1}(-i \nabla_k).
\]  

(18)

From (15,18) it follows that \( \tilde{v}_k \) is symmetric in the scalar product space
\[
\langle \alpha_1(k) , \alpha_2(k) \rangle_w \equiv \langle \alpha_1(k) \mid (\sqrt{m^2 + k^2} - m) \sqrt{m^2 + k^2} \alpha_2(k) \rangle
\]
instead of the usual Klein-Gordon scalar product space \( \langle \alpha_1(k) \mid \sqrt{m^2 + k^2} \alpha_2(k) \rangle \). As the components of \( \tilde{v}_k \) which are
\[
\{ \tilde{v}_k^1, \tilde{v}_k^2, \tilde{v}_k^3 \}
\]
have a common eigenvector \( \alpha(k) \), it also follows that these components commute: we explicitly calculate the commutator \([\tilde{v}_k^1, \tilde{v}_k^2] \) in the Appendix and check that it is zero. It follows that eigenvectors with different eigenvalues of \( \tilde{v}_k \) are orthogonal to each other with respect to the scalar product space (19), i.e if we label \( \alpha_{v_1}(k) \) with the property \( \tilde{v}_k \alpha_{v_1} = v_1 \alpha_{v_1} \), etc, then
\[
\langle \alpha_{v_1}(k), \alpha_{v_2}(k) \rangle_w = 0, \quad (v_1 \neq v_2).
\]

In configuration space the scalar product corresponding to (19) is
\[
\langle \psi_1(x) \mid -\partial_t^2 - i m \partial_t \mid \psi_2(x) \rangle = \frac{1}{2} \int [\partial_t \psi_1^*(x)] [(\partial_t + i m)\psi_2(x)] \, dx + \text{s.c.}
\]
where s.c. stands for the symmetric conjugate term. The operator corresponding to \( \tilde{v}_k \) is \( \tilde{v} = \mathcal{F} \tilde{v}_k \mathcal{F}^{-1} \), found by substituting
\[
k \rightarrow -i \nabla, \quad \nabla_k \rightarrow -i \xi,
\]
\[
\sqrt{m^2 + k^2} \rightarrow \sqrt{m^2 - \xi^2} = i \partial_t,
\]
\[
(k \cdot \nabla_k) \rightarrow -(\nabla \cdot \xi) = -\frac{1}{r^2}(\partial_r r)^2 \equiv -i \frac{1}{r^2} D r^2
\]
where \( D \) is the dilation operator \( D \equiv (-i \partial r) \) into (15), obtaining
\[
\tilde{v} = -i \frac{\nabla}{\sqrt{m^2 - \xi^2}} - i m \frac{1}{\sqrt{m^2 - \xi^2} \xi} \left[ \frac{1}{r^2} (i D)^{-1} r^2 \right] \left( \sqrt{m^2 - \xi^2} - m \right)
\]
\[
= (i\partial_t)^{-1} \left( -i \nabla - m \xi \left[ \frac{1}{r^2} (D)^{-1} r^2 \right] (i\partial_t - m) \right)
\]
where \((D)^{-1}\) is defined as the space counterpart to (17). Recalling the \( \Psi_\nu \) of (6):
\[
\Psi_\nu = j_0(m\gamma |v|\bar{r}) \exp[-i m\gamma^2(t - v \cdot \xi)]
\]
then the result (22) implies that at time \( t = 0 \)
\[
(\nabla + v \partial_t) \Psi_\nu(x) = i m \xi \left[ \frac{1}{r^2} (D)^{-1} r^2 \right] (i\partial_t - m) \Psi_\nu(x),
\]
which may be verified by explicit calculation, each side having the value
\[
-m\gamma |v| \xi j_1(m\gamma |v|\bar{r}) \exp[-i m\gamma^2(t - v \cdot \xi)],
\]
where \( j_1 \) is the spherical Bessel function of order one.
4 A canonical transformation from variables \((x, k) \rightarrow (z, v)\)

The fact that the components of the velocity operator commute leads to inquire whether there is a canonical transformation underlying this property. Classically the \(\tilde{v}\) operator of (22) corresponds to the generator

\[
v(x, k) = \frac{1}{\sqrt{m^2 + k^2}} k - m \frac{(\sqrt{m^2 + k^2} - m)}{\sqrt{m^2 + k^2} (k \cdot x)} x
\]

and it may be verified that the Poisson bracket

\[
\{v_a, v_b\} = 0 \quad a, b = 1, 2, 3
\]

where the Poisson bracket (P.b.) between two variables \(f(x, k), g(x, k)\) is defined as

\[
\{f, g\} \equiv (\partial f / \partial x) \cdot (\partial g / \partial k) - (\partial f / \partial k) \cdot (\partial g / \partial x).
\]

The generator conjugate to \(v\) is

\[
z(x, k) = \frac{\sqrt{m^2 + k^2} (k \cdot x)}{m (\sqrt{m^2 + k^2} - m)} k
\]

which means that

\[
\{z_a, v_b\} = \delta_{ab}, \quad \{z_a, z_b\} = 0
\]

as may be verified by labourious calculation. So then the transformation from the variables \((x, k)\) to new variables \((z, v)\) is canonical:

\[
\begin{align*}
\begin{cases}
z(x, k) = \frac{\sqrt{m^2 + k^2} (k \cdot x)}{m (\sqrt{m^2 + k^2} - m)} k \\
v(x, k) = \frac{1}{\sqrt{m^2 + k^2}} k - m \frac{(\sqrt{m^2 + k^2} - m)}{\sqrt{m^2 + k^2} (k \cdot x)} x
\end{cases} \quad \text{(23)}
\end{align*}
\]

with inverse

\[
\begin{align*}
\begin{cases}
x(z, v) = \frac{1 - (\hat{z} \cdot v)^2}{m} \left( \frac{1}{1 + (\hat{z} \cdot v)^2} z - \frac{z^2}{2 (z \cdot v)} v \right) \\
k(z, v) = \frac{2 m (\hat{z} \cdot v)}{1 - (\hat{z} \cdot v)^2} \hat{z}.
\end{cases} \quad \text{(24)}
\end{align*}
\]

where \(\hat{z} \equiv z / z\). The value of \(|v|\) from (23) is not limited to be less than unity, however the canonical transformation is meaningless if \(|v|\) is allowed to equal unity, because the denominator \((1 - (\hat{z} \cdot v)^2)\) in (24) can then be zero. The Hamiltonian in the new coordinates is

\[
\sqrt{m^2 + k^2} = m \frac{1 + (\hat{z} \cdot v)^2}{1 - (\hat{z} \cdot v)^2}.
\]
5 Orthogonality relation

We calculate orthogonality relations between coinciding wavepackets with differing velocities in the $z$ direction $\Psi_v, \Psi_{v'}$, or equivalently between their momentum space counterparts $\alpha_v, \alpha_{v'}$, recalling from (11) that at time $t = 0$

$$\alpha_v = \sqrt{\frac{\pi}{2 m \gamma^2 |v| \sqrt{m^2 + k^2}}} \delta(\sqrt{m^2 + k^2} - m - v k_3).$$

We have mentioned that when $v \neq v'$ the surface delta functions inside $\alpha_v, \alpha_{v'}$ only touch at the origin, and so whether they are orthogonal or not will depend on whether there are any factors of $k$ multiplying the delta functions. We first write the above in spherical coordinates

$$\alpha_v = \sqrt{\frac{\pi}{2 m \gamma^2 |v| \sqrt{m^2 + k^2}}} \delta(\sqrt{m^2 + k^2} - m - v k \cos \theta)$$

$$= \sqrt{\frac{\pi}{2 m \gamma^2 |v| \sqrt{m^2 + k^2}}} \delta(\cos \theta - \frac{\sqrt{m^2 + k^2} - m}{vk}).$$

Then recalling the scalar product space $\langle \cdot, \cdot \rangle_w$ of (19)

$$\langle \alpha_v, \alpha_{v'} \rangle_w = \langle \alpha_v, (\sqrt{m^2 + k^2} - m) \sqrt{m^2 + k^2} \alpha_{v'} \rangle$$

$$= \int (\sqrt{m^2 + k^2} - m) \sqrt{m^2 + k^2} \alpha_v^* \alpha_{v'} \cos \theta k^2 dk d\phi$$

$$= \frac{\pi^2}{m^2 \gamma^2 \sqrt{v v'}} \int_0^{k_{max}} \left( \frac{\sqrt{m^2 + k^2} - m}{k \sqrt{m^2 + k^2}} \right)^2 \delta(k \cos \theta - \frac{\sqrt{m^2 + k^2} - m}{vk}) k^2 dk$$

Note that

$$\delta(k \cos \theta - \frac{\sqrt{m^2 + k^2} - m}{vk}) = 2 m \delta(k),$$

but this term is annihilated by the $(\sqrt{m^2 + k^2} - m)$ factor in the integrand which is equal to $k^2/2m$ as $k \to 0$. Hence we can regard $(v - v')$ as the subject of the delta function and pull out the remaining factor obtaining

$$\langle \alpha_v, \alpha_{v'} \rangle_w = \frac{\pi^2}{m^2 \gamma^2 \sqrt{v v'}} \int_0^{k_{max}} \frac{k}{\sqrt{m^2 + k^2}} \delta(v - v') dk$$

$$= \frac{\pi^2}{m^2 \gamma^2 v^2} \delta(v - v') \left| \sqrt{m^2 + k^2} \right|_{k_{max}}$$

and recalling that $k_{max} = 2mv\gamma^2$

$$\langle \alpha_v, \alpha_{v'} \rangle_w = \frac{\pi^2}{m^2 \gamma^2 v^2} \delta(v - v') \left( \frac{1}{1 - v'^2 m} - m \right)$$

$$= \frac{\pi^2}{m^2 \gamma^2 v^2} \delta(v - v') \left( 2 m v^2 \gamma^2 \right)$$

$$= \frac{2 \pi^2}{m \gamma^2} \delta(v - v').$$
6 Coordinate transformation for $\Psi_{l,v} \rightarrow \Psi_{l,v'}$

We will show that there is a coordinate transformation which changes the $\Psi_v$ of (2)

$$\Psi_v(x, t) = j_0(m\gamma|v|\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}) \exp[-im\gamma^2(t - vz)]$$

to

$$\Psi_{v'}(x, t) = j_0(m\gamma'|v'|\sqrt{x^2 + y^2 + \gamma'^2(z - v't)^2}) \exp[-im\gamma'^2(t - v'z)].$$

From inspection of the above we see that Lorentz transformations do not preserve the form of the $\Psi_v$, (as an example the spherical solution (3) is not of the form $\Psi_v$). The following is the required transformation:

$$\begin{align*}
  t &= t' \\
  z &= \frac{1}{\gamma^2 v} \left( \gamma'^2 v' z' + [\gamma^2 - \gamma'^2] t' \right) = \frac{\gamma'^2 v'}{\gamma^2 v} z' + \left( v - \frac{\gamma'^2 v'^2}{\gamma^2 v} \right) t' \\
  x &= \frac{\gamma' |v'|}{\gamma |v|} x', \\
  y &= \frac{\gamma' |v'|}{\gamma |v|} y'
\end{align*}$$

(25)

It is readily checked that (25) implies

$$\begin{align*}
  \gamma^2(t - vz) &= \gamma'^2(t' - v'z') \\
  \gamma^2 v(z - vt) &= \gamma'^2 v'(z' - v't')
\end{align*}$$

as necessary. Remarkably under (25) the time coordinate remains unchanged, but the space coordinate transformations are specific to both the original and transformed velocities $v$ and $v'$. So that if the transformation (25) is applied to, say, $\Psi_{l,v''}$ with $v'' \neq v$, the resulting function does not in general have the form $\Psi_v$.

7 Outlook

The wavepacket solutions $\Psi_v(x, t)$ of the KG equation are eigenvectors of the velocity operator (22), whose components commute and are symmetric in the inner product space (21). These wavefunctions are interesting alternative basis state to the usual plane waves.
Appendix

Recalling
\[ \dot{\psi}_k = \frac{k}{\sqrt{m^2 + k^2}} - m \frac{1}{\sqrt{m^2 + k^2}} D_k^{-1} (-i \nabla_k) (\sqrt{m^2 + k^2} - m) \]
with \( D_k \equiv -i \partial_k \) and
\[ D_k^{-1} \phi(k) \equiv \frac{i}{2} \left[ \int_{0}^{1} \phi(\lambda k) d\lambda - \int_{1}^{\infty} \phi(\lambda k) d\lambda \right] , \]
we will need the following identities:
\[ [D_k^{-1}, k \nabla_1] = 0 \]  
\[ \nabla_1 D_k^{-1} = \frac{1}{k} (k \nabla_1) D_k^{-1} = \frac{1}{k} D_k^{-1} k \nabla_1 \]  
\[ [D_k^{-1}, k \nabla_2 - k \nabla_1] = 0 \]  
\[ [D_k, \frac{k}{\sqrt{m^2 + k^2}}] = -i \frac{m^2 k}{\sqrt{m^2 + k^2}} \]  
\[ [D_k^{-1}, \frac{k}{\sqrt{m^2 + k^2}}] = i D_k^{-1} \frac{m^2 k}{\sqrt{m^2 + k^2}} D_k^{-1} \]

To calculate \([\dot{v}_1, \dot{v}_2]\) we first calculate those parts of the commutator with the \( \frac{k}{\sqrt{m^2 + k^2}} \) terms, which are
\[
- \left[ \frac{k_1}{\sqrt{m^2 + k^2}} , m \frac{1}{\sqrt{m^2 + k^2}} D_k^{-1} (-i \nabla_2) (\sqrt{m^2 + k^2} - m) \right] \\
+ \left[ \frac{k_2}{\sqrt{m^2 + k^2}} , m \frac{1}{\sqrt{m^2 + k^2}} D_k^{-1} (-i \nabla_1) (\sqrt{m^2 + k^2} - m) \right] \\
= i \frac{m}{m^2 + k^2} (k_1 \nabla_2 - k_2 \nabla_1) k D_k^{-1} \frac{1}{k} (\sqrt{m^2 + k^2} - m) - i \frac{m}{m^2 + k^2} (\nabla_1 k_2 - \nabla_2 k_1) D_k^{-1} \frac{m^2 k}{\sqrt{m^2 + k^2}} \\
= i m (k_1 \nabla_2 - k_2 \nabla_1) \left( \frac{1}{\sqrt{m^2 + k^2}} \left[ \frac{k}{\sqrt{m^2 + k^2}} , D_k^{-1} \right] \frac{1}{k} (\sqrt{m^2 + k^2} - m) \right) \]
\[ = i m (k_1 \nabla_2 - k_2 \nabla_1) \left( - i D_k^{-1} \frac{m^2 k}{\sqrt{m^2 + k^2}} D_k^{-1} \right) \frac{1}{k} (\sqrt{m^2 + k^2} - m) \].  

Secondly we calculate how the second terms of \( \dot{\psi}_k \) commute with each other, yielding
\[
m^2 \frac{1}{\sqrt{m^2 + k^2}} D_k^{-1} \left( (-i \nabla_1) \frac{\sqrt{m^2 + k^2} - m}{\sqrt{m^2 + k^2}} D_k^{-1} (-i \nabla_2) - (-i \nabla_2) \frac{\sqrt{m^2 + k^2} - m}{\sqrt{m^2 + k^2}} D_k^{-1} (-i \nabla_1) \right) (\sqrt{m^2 + k^2} - m) \\
= m^2 \frac{1}{\sqrt{m^2 + k^2}} D_k^{-1} \left( - i \frac{m k_1}{\sqrt{m^2 + k^2}} (i \nabla_2) D_k^{-1} \frac{1}{k} + i \frac{m k_2}{\sqrt{m^2 + k^2}} (i \nabla_1) D_k^{-1} \frac{1}{k} \right) (\sqrt{m^2 + k^2} - m) \\
= - m^2 (k_1 \nabla_2 - k_2 \nabla_1) \frac{m k}{\sqrt{m^2 + k^2}} D_k^{-1} \frac{1}{k} (\sqrt{m^2 + k^2} - m) \]

Adding (A6) and (A7) yields zero.

References

[1] L. MacKinnon, Found. Phys. 8, 157 (1978)
[2] L. MacKinnon, Lett. Nuovo Cimento 31, 37 (1981)
[3] S. S. Schweber, An introduction to relativistic quantum field theory (Harper and Row, New York, 1961) p57