On the Integrability of Bianchi Cosmological Models

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Abstract

In this work, we are investigating the problem of integrability of Bianchi class A cosmological models. This class of systems is reduced to the form of Hamiltonian systems with exponential potential forms.

The dynamics of Bianchi class A models is investigated through the Euler-Lagrange equations and geodesic equations in the Jacobi metric. On this basis, we have come to some general conclusions concerning the evolution of the volume function of 3-space of constant time. The formal and general form of this function has been found. It can serve as a controller during numerical calculations of the dynamics of cosmological models.

The integrability of cosmological models is also discussed from the points of view of different integrability criterions. We show that dimension of phase space of Bianchi class A Hamiltonian systems can be reduced by two. We prove vector field of the reduced system is polynomial and it does not admit any analytic, or even formal first integral.
1 Introduction

We shall investigate the dynamics of the most interesting group of homogeneous Bianchi class A cosmological models which is described by the natural Lagrangean function

\[
\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V(q) = T - V(q) = \frac{1}{4} \sum_{i=1, i<j}^3 \frac{d \ln q^i}{dt} \frac{d \ln q^j}{dt} - \frac{1}{4} \left( 2 \sum_{i=1, i<j}^3 n_i n_j q^i q^j - 3 \sum_{i=1}^3 n_i^2 q_i^2 \right),
\]

where \( q_i \approx A_i^2 \) \((i = 1, 2, 3)\) are three squared scale factors \( A_i \) for diagonal class A Bianchi models; different Bianchi types correspond to different choices of \( n_i \in \{-1, 0, 1\} \), \( i = 1, 2, 3 \); a dot denotes differentiation with respect to cosmological time \( t \). Logarithmic time \( \tau \) is related with cosmological time \( t \) by

\[
d\tau = \frac{dt}{(q^1 q^2 q^3)^{1/6}} = \frac{dt}{\text{Vol} M^3}.
\]

Bogoyavlensky [1], proves an important property of system (1), namely the existence of the monotonic function \( F \) with the following form

\[
F = \frac{d}{dt} \left( q^1 q^2 q^3 \right)^{1/6} = \frac{d}{dt} \text{Vol} M^3,
\]

such that

\[
\frac{dF}{dt} \leq 0.
\]

Function (2) is invariant with respect to the scaling transformations and it has the sense of the speed of change of the average radius of the universe. Function \( |F| \) along any solution decreases from infinity to zero in such a way that \( F = 0 \) is reached at the moment of maximal expansion, and \( |F| = \infty \) corresponds to the initial singularity. The existence of function \( F \) allows us to define what we call the early stage of the evolution of the universe as

\[
F \gg 1.
\]

The importance of this function for numerical integration of B(IX) models has been pointed out in the work [2]. The authors have used the Rauchaudhuri equations to show the property of upper-convexity of function \( (\text{Vol} M^3)(t) \) which means that this function does not possess a local minimum (where \( F = 0 \) and \( \dot{F} > 0 \) and may possess not more
than one maximum. If $F < 0$, the volume function $(\text{Vol}M^3)(t)$ shrinks; whereas if $F > 0$ it expands. Both of the processes take place in the same region of the phase space $(p, q)$ but with reverse directions of time. In the phase space $(p, q)$ function $F$ has the following form

$$F = \frac{(q^1 q^2 q^3)^{1/6}}{3} \sum_{i=1}^{3} p_i q^i, \quad p_i = \frac{\partial L}{\partial \dot{q}^i},$$

and

$$\frac{dF}{dt} = \frac{(q^1 q^2 q^3)^{1/6}}{9} \left[ \left( \sum_{i=1}^{3} p_i q^i \right)^2 - 6V \right],$$

where $p_i$ are momenta conjugated with generalized coordinates $q^i$.

In work [2], function $F$ was used for controlling the quality of numerical integrations of the B(IX) model. In this model, the scale factors oscillate in a neighborhood of the initial and final singularity. The function $(\text{Vol}M^3)(t)$, obviously, does not possess the analogous property [2, 3].

In the present work we give some general and formal expressions for the function $(\text{Vol}M^3)(t)$. It can be used as a tool for studying the B(IX) models. Let us note recent important results of Cushman and Śniatycki [9] concerning chaos in the B(IX) system. They proved that existence of a monotonic function $F$ excludes the possibility of recurrence in the system and, thus, any form of standard deterministic chaos in the system. This illuminates previous negative results and shows that for a study of this system we have to use non-conventional methods.

Several authors tested if the last model passes the standard Painlevé integrability test (in the form of the ARS algorithm [14]). First results of Contopoulos [4] shows that B(IX) model passes this test. Next, this paper was revised [5], however, without any strict conclusions concerning integrability. It was stated also that this model passes Ziglin’s test (see [23, 24]). More careful Painlevé analysis was done by Latifi et al. in [6]. They show that B(IX) model does not passes the so called perturbative Painlevé test. Authors of this paper suggest the existence of ‘some chaotic régimes’ in the system.

The above remarks show that the notion ‘chaos’ has unclear status when dynamical systems arising from the general relativity and cosmology are studied. Moreover, for B(IX) model discrete dynamics defined in [7] shows strong ergodic properties, however, this ‘chaotic behaviour’ seems to be absent (or hidden) in the continuous dynamics. Moreover, the standard criteria of detection of chaos (Lyapunov characteristic exponents—LCE) are not invariant with respect to the time reparametrization and transformation of phase variables whereas existence of first integrals is an invariant property of the system. It is also important to note that the non-zero LCE can be used as an indicator of chaos only
when the motion take place in a compact invariant subset of the phase space, but it is not true for the B(IX) dynamical system. All these facts motivated us to study the problem of integrability of the investigated models.

The non-integrability of the system is a weaker property than chaos (in the sense of the deterministic chaos) but better described and understood. The authors believe that investigation of non-integrability in B(IX) models can contribute to a better understanding of chaos in cosmological models. Here we show that the Bianchi class A Hamiltonian system are not completely integrable in the sense of Birkhoff. This conclusion is weak as the negative answer to the question about algebraic complete integrability of B(IX) (see [21]). In order to obtain stronger result we reduce the dimension of the phase space by two. We show that the reduced system is polynomial and, what is most important, it does not admit any analytical, or even formal, first integral.

In cosmological models chaos, if properly defined and present, has some hidden character. The basic indicator of chaos in these models, the LCE, depends on the choice of the time parametrization. In the logarithmic time $\tau$, nearby trajectories diverge linearly whereas in other time parametrizations they will diverge exponentially which is characteristic for chaotic systems. The fact that the rates of separation of nearby trajectories depend on the clock used is obvious. The problem is in invariant choices of the time parameter for the invariant chaos detection. Such a role is played by Maupertuis clock (time parameter $s$ is such that $\frac{ds}{d\tau} = 2|E - V|$, where $E$ is the total energy of the system, $V$ is its potential and $\tau$ is mechanical time).

Our point of view is such that the LCE, when used in general relativity, should be defined in an invariant way. Then the results could be interpreted in a different time parametrizations. The Bianchi IX model is ‘chaotic’ in the parameter $s$ (LCE is positive), but, after transition to the parameter $\tau$, nearby trajectories diverge linearly in such a way as integrable systems. This phenomenon is called the hidden chaos. Let us note that the existence of the first integral of an autonomous system is an invariant property (with respect to time reparametrization and to transformation of phase variables).

In general relativity and cosmology, the problem of non-integrability or chaos is not only very subtle but also is strictly connected with the invariant description. One must be very careful detecting integrability in B(IX) dynamics. The problem whether chaos in the gauge theory is a physical phenomenon is, generally, open.
2 The dynamics of Bianchi class A models from the Euler-Lagrange equations

The Hamiltonian function for the system (1) has the following form

\[ H = \frac{1}{2} g^{\alpha\beta} p_{\alpha} p_{\beta} + V(q), \]

(3)

where

\[ g^{\alpha\beta} = 2 \begin{pmatrix}
-(q^1)^2 & q^1 q^2 & q^1 q^3 \\
q^2 q^1 & -(q^2)^2 & q^2 q^3 \\
q^3 q^1 & q^3 q^2 & -(q^3)^2
\end{pmatrix}, \]

\[ V(q) = \frac{1}{4} \left( 2 \sum_{i<j} n_i n_j q^i q^j - 3 \sum_{i=1} \frac{n_i^2 (q^i)^2}{2} \right), \]

\[ \mathcal{H} = 0. \]

(4)

The Euler-Lagrange equations in time \( \tau \) have the following form

\[ \frac{d^2 q^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dq^\beta}{d\tau} \frac{dq^\gamma}{d\tau} = g^{\alpha\beta} \frac{\partial V}{\partial q^\beta}, \]

(5)

where Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \) are connected with the metric defined by the kinetic energy

\[ T = \frac{1}{2} g_{\alpha\beta} q^\alpha q^\beta = \frac{1}{2} g^{\alpha\beta} p_{\alpha} p_{\beta}. \]

Equations (5) after transformations to a new time parameter \( s \), called Maupertuis time, take the form of geodesic equations for the Jacobi metric

\[ \frac{d^2 q^\alpha}{ds^2} + \hat{\Gamma}^\alpha_{\beta\gamma} \frac{dq^\beta}{ds} \frac{dq^\gamma}{ds} = 0, \]

(6)

that is,
where a hat denotes that respective quantities are calculated with respect to the Jacobi metric. Christoffel symbols calculated from $g$ and $\hat{g}$ metrics are connected by relations

$$\hat{\Gamma}_{jk}^i = \Gamma_{jk}^i + A_{jk}^i,$$

(7)

where

$$A_{jk}^i = \left(\frac{\partial j}{\partial \Phi}\right)\delta_k^i + \left(\frac{\partial k}{\partial \Phi}\right)\delta_j^i - g^{ir}(\frac{\partial r}{\partial \Phi})g_{jk},$$

$$\Phi = \frac{1}{2} \ln 2W.$$

Let us note that the kinetic energy form does not depend on the Bianchi type models characterized by the set $\{n_1, n_2, n_3\}$. The only non-vanishing Christoffel symbols are

$$\Gamma_{11}^1 = -\frac{1}{q^1}, \quad \Gamma_{22}^2 = -\frac{1}{q^2}, \quad \Gamma_{33}^3 = -\frac{1}{q^3}.$$  

(8)

After substitution (8), system (5) takes the form

$$\frac{1}{q^i} \frac{d^2 q^i}{d\tau^2} - \left(\frac{d}{d\tau} \ln q^i\right)^2 = (n_j)(q^j)^2 + (n_k)(q^k)^2 - (n_i)(q^i)^2 - 2n_jn_kq^jq^k,$$

(9)

where $\{i, j, k\} \in S_3$, and $S_3$ denotes the set of even permutations of $\{1, 2, 3\}$.

The change of variables

$$q^i = e^{Q^i}, \quad i = 1, 2, 3,$$

(10)

transforms the above equations to the following form

$$\frac{d^2 Q^i}{d\tau^2} = (n_j)^2 e^{2Q^j} + (n_k)^2 e^{2Q^k} - (n_i)^2 e^{2Q^i} - 2n_jn_k e^{Q^j+Q^k},$$

(11)

where $\{i, j, k\} \in S_3$.

The system (11) is satisfied on the Hamiltonian constraint $H = 0$ which is equivalent to the condition of normalization of the tangent vector to the trajectory $u^i = dq^i/ds$, that is,

$$\|u\|^2 = 2Wg_{\alpha\beta} \frac{dq^\alpha}{ds} \frac{dq^\beta}{ds} = -\text{sgn}V,$$

(12)

or

$$g_{\alpha\beta} \frac{dq^\alpha}{d\tau} \frac{dq^\beta}{d\tau} = 2V\text{sgn}V.$$
In terms of variables $Q^i$, the constraint condition is equivalent to
\[
\sum_{i<j}^3 \frac{dQ^i}{d\tau} \frac{dQ^j}{d\tau} = -8V.
\] (13)

Adding the sides of equations (11), we obtain the following formula
\[
\sum_{i=1}^3 \frac{d^2Q^i}{d\tau^2} = -4V.
\] (14)

Equations (11), after time reparametrization $\tau \to s = s(\tau)$, take the form of geodesic equations. From (11) we obtain
\[
4W^2 \frac{d^2Q^i}{ds^2} + \frac{dQ^i}{ds} \frac{dW}{dQ^j} \frac{dQ^j}{ds} = (n_j)^2 e^{2Q^j} + (n_k)^2 e^{2Q^k} - (n_i)^2 e^{2Q^i} - 2n_j n_k e^{Q^j+Q^k},
\]
where $\{i, j, k\} \in S_3$.

The problem of investigation of Lagrange systems with the indefinite kinetic energy form is open. First steps to investigate such systems have been done in [8]. In the terminology used in [8] our system is a special case, the so called non-classical simple mechanical system. As it was established, these systems have the following fundamental property. A trajectory of the system can pass through the set $\partial D = \{q : E - V = 0\}$. During this passage the vector tangent to trajectory changes the cone sector defined by the kinetic energy form: $g_{\alpha\beta}(q_0) \xi^\alpha \xi^\beta = 0$ where $\xi^\alpha = dq^\alpha / ds$, $q_0 \in \partial D$. In our case the signature of $g_{\alpha\beta}$ is Lorentzian, i.e., $(-, +, \ldots, +)$ (for details see [8]).

In generic situations ($n_i \neq 0$ for $i = 1, 2, 3$) which include BVIII and BIX models (Mixmaster models), there are analytical and numerical arguments that the function of sign of the potential for a typical trajectory is an infinite subsequence that is a one sided cut of the following double infinite sequence (see [10])
\[
\text{sgn} V = \{\ldots, +1, 0, -1, 0, -1, \ldots\}.
\] (15)

If we assume that the subsequence (15) is finite, then our system reaches the state $V = 0$ ($W = 0$ in general case) which corresponds to Kasner solutions, finite number of times. During the Kasner epoch the information about the localization of the point on the interval of normal separation (modulo initial localization) grows $e$-times [11]. If the subsequence (15) is finite it means that after $\bar{n}$ epochs $\bar{n}$ bytes (i.e., finite number of bytes) of information have been lost whereas we know that our system is chaotic (the loss of infinite information is required for chaos). Let us notice that when the system goes asymptotically to the boundary sets $W = 0$ then it is asymptotically free.
3 Properties of the function of volume of the constant time 3-space

Equation (14) implies that

$$\frac{1}{2} \frac{d^2}{d\tau^2} \ln(\text{Vol}^3) = -2V = 2T. \quad (16)$$

The above relation means that, in a generic case ($\forall i n_i \neq 0$), there is an infinite number of intervals in which function $\ln(\text{Vol}^3)$ is subsequently convex up and down. These intervals are separated by an infinite number of inflexion points (which corresponds to $V = 0$) in the diagrams of the function $\ln(\text{Vol}^3)(\tau)$ and lie on the lines $\ln(\text{Vol}^3)(\tau) = \pm \tau + C$. The additional information we have about the B(IX) model is that this model has the initial and final singularities. In the following paragraphs of the work we shall concentrate on the B(IX) models.

From the $(0,0)$ components of the Einstein equation for the BIX case, we obtain that function $\ln(\text{Vol}^3)(\tau)$ cannot possess a local minimum but it can possess a single maximum. The above property suggests that for a typical trajectory in this model the qualitative diagrams of the function $\ln(\text{Vol}^3)(\tau)$ look like in Figure 1.

Fig. 1.

After integrating over $\tau$ the both sides of (14) and assuming that in the moment of maximal expansion $\tau = \tau_0$, we obtain

$$\ln(\text{Vol}^3)(\tau) \propto e^{C_1 \tau} e^{\int_{\tau_0}^{\tau} s(t) \text{sgn}(-V) dt}, \quad (17)$$

where we choose $C_1 = 1$ in the phase of expansion and $C_1 = -1$ in the phase in contraction of the volume function (if $V = 0$, $\ln(\text{Vol}^3)(\tau) \propto e^{\pm \tau}$). Finally, for any model which describes the evolution of the volume function

$$\ln(\text{Vol}^3)(\tau) \propto e^{\tau(1+\langle s \rangle)} \rightarrow_{\tau \rightarrow \infty} e^\tau e^{\langle s \rangle} \propto e^\tau, \quad (18)$$

$$\langle s \rangle = \frac{1}{\tau} \int_{\tau_0}^{\tau} s(t) \text{sgn}(-V) dt,$$

where we assume that the average value of $s(\tau)$ on the interval $(\tau, \tau_0)$ exists as $\tau \rightarrow -\infty$, and it is finite. From the formula (18) it immediately yields that in a neighborhood of the initial singularity ($\tau \rightarrow -\infty$) the volume function changes exactly as in Kasner’s models. The second observation is as follows: the volume function does not oscillate around the equilibrium positions $\ln(\text{Vol}^3)(\tau) \equiv 0$ but oscillates around Kasner’s solution. In other
words, Kasner’s solution plays the role analogical to the equilibrium positions in the small oscillation approximation.

From (17) one can obtain the following relations between a natural parameter $s$ defined along geodesics (Maupertuis time) and the volume function $\ln(\text{Vol}M^3)(\tau)$

$$s = \frac{d}{dt} \ln(\text{Vol}M^3)(\tau) \quad \text{for} \quad V < 0,$$

$$s = -\frac{d}{dt} \ln(\text{Vol}M^3)(\tau) \quad \text{for} \quad V > 0.$$  

The zero value of the parameter $s$ corresponds to the moment $\tau = \tau_0$. The relations between the parameter $s$, function $F$ and the scalar expansion function

$$\Theta \equiv \frac{d}{dt} \ln(\text{Vol}M^3)(t),$$

are as follows

$$s(\tau) = \pm 6[(\text{Vol}M^3)(\tau)]^{1/3} F(\tau),$$

$$s(\tau) = \pm [(\text{Vol}M^3)(\tau)] \Theta(t(\tau)),$$

where plus and minus sign correspond to $V < 0$, and $V > 0$, respectively.

From the above, we can conclude that "near the singularity" is equivalent to $s \gg (\text{Vol}M^3)^{1/3}$, i.e., $s \gg 0$.

Formula (18) implies that the characteristic time after which $(\text{Vol}M^3)(\tau)$ grows $e$-times, i.e., $(\text{Vol}M^3)(\tau) \propto e^{\tau/\tau_{\text{char}}}$ has the following form

$$\tau_{\text{char}} = (1 + \langle s \rangle)^{-1}. \quad (19)$$

This characteristic time is finite if the average $\langle s \rangle$ exists.

For a typical trajectory the function $s(\tau)$ is shown in Fig. 2.

4 The Bianchi class A models as systems with exponential potentials and Its Algebraic Non-integrability

After introducing the new variables $Q^i$ and using the definition (10), the Lagrange system (1) can be transformed to a Hamiltonian one. The Hamilton function for this system
takes the following form

\[ H(p, Q) = 2 \sum_{i<j} p_i p_j - \frac{3}{2} \sum_{i=j} p_i^2 + \frac{1}{4} \left( 2 \sum_{i<j} n_i n_j e^{Q_i + Q_j} - \sum_{i=j} n_i^2 e^{2Q_i} \right) \]

\[ = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V(Q^\alpha), \quad (20) \]

where \( p_i = \frac{1}{4} (\dot{Q}_j + \dot{Q}_k) \) for \( \{i, j, k\} \in S_3 \). The Hamilton function (20) is a special case of the Hamiltonian for the so called perturbed Toda lattice [11].

5 Analysis of the integrability of B(IX) model

There are several definitions of integrability. Generally, integrability means that the system under consideration possesses a large enough number of first integrals. Necessarily we have to specify the class of functions that contains these first integrals as well as to define the domain of their definition. Let us note here that there are examples of Hamiltonian systems that possess an first integral of class \( C^\alpha \) but does not possess an integral of class \( C^\beta \) with \( \beta > \alpha \), for \( \alpha, \beta = 1, 2, \ldots \infty, \omega \) (see [18]).

It is well known also that every system of \( n \) differential autonomous equations is locally integrable—in a neighborhood of every nonsingular point (where the right hand sides do not vanish) it possesses \( n - 1 \) first integrals. Thus, nontrivial problems are non-local or concern the existence of integrals in a neighborhood of equilibria points.

It is very difficult to prove (non)integrability of a given set of differential equations. One way to simplify the problem is to restrict a class of function where we look for integrals. As an illustration of this approach, let us consider the Birkhoff integrability (see [19]) of the Bianchi class A system in the form (20). This system belongs to the wide class of Hamiltonian systems in \( \mathbb{R}^{2n} \) equipped with the standard symplectic structure and are given by the following Hamiltonian function

\[ H = \frac{1}{2} (p, p) + \sum_{m \in \mathcal{M}} v_m \exp (c_m, q), \quad (21) \]

where

\[ (p, p) = \sum_{i,j=1}^n a^{ij} p_i p_j, \quad (c_m, q) = \sum_{i=1}^n c_m q^i, \quad m = (m_1, \ldots, m_n) \in \mathbb{Z}^n, \]
\((a_{ij}), v_m, \text{ and } c_m \text{ are constant; } \mathcal{M} \text{ is a finite subset of } \mathbb{Z}^n \)

\[ \mathcal{M} = \{ m \in \mathbb{Z}^n \mid v_m \neq 0 \} \]

We look for integrals that are polynomials with respect to \( p \), i.e.,

\[ f(q, p) = \sum_{k \in \mathcal{N}_l} f_k p^k, \]

where

\[ \mathcal{N}_l = \{ m \in \mathbb{Z}_+^n \mid |m| \leq l \}, \quad |m| = \sum_{i=1}^{n} m_i; \quad p^m = p_1^{m_1} \ldots p_n^{m_n}, \quad m \in \mathbb{Z}_+^n, \]

and coefficients \( f_k \) have the form of infinite series of exponents

\[ f_k = \sum_{m \in \mathbb{Z}^n} f_m^{(k)} \exp(c_m^{(k)}, q). \]

Here \( \mathbb{Z}_+ \) denotes non-negative integers. We say that the system \((21)\) is Birkhoff integrable if it possesses \( n \) independent integrals of the prescribed form (see note of Ziglin [25] about modification of the original definition of Kozlov). We order elements of \( \mathcal{M} \) with respect to the lexicographic order and denote by \( \alpha \) its maximal element and by \( \beta \) the maximal element of \( M \) that is not colinearly with \( \alpha \). Then, according to Theorem 3 from [20] if

\[ k(\alpha, \alpha) + (\alpha, \beta) \neq 0, \quad \text{for all } k \in \mathbb{Z}_+ \]

then the Hamiltonian system \((21)\) is not integrable in the sense of Birkhoff. We immediately have

**Theorem 1** *A generic case of the Bianchi class A system given by Hamiltonian function \((20)\) with \( n_i \neq 0 \) for \( i = 1, 2, 3 \) is not integrable in the sense of Birkhoff.*

**Proof.** For the Hamiltonian \((20)\) we have

\[ \mathcal{M} = \{(1, 1, 0), (1, 0, 1), (1, 0, 0), (0, 1, 1), (0, 1, 0), (0, 0, 1)\}, \]

and thus \( \alpha = (1, 1, 0) \) and \( \beta = (1, 0, 1) \). Metric \((a_{ij})\) has the form

\[ (a_{ij}) = 2 \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \]

and thus we have

\[ k(\alpha, \alpha) + (\alpha, \beta) = 4 \neq 0, \]

and this finishes the proof. \( \square \)

Let us remark that in a case when one of \( n_i \) is equal zero then \( k(\alpha, \alpha) + (\alpha, \beta) = 0 \) for all \( k \). In such a case the system has one additional integral, namely \( p_i \).
6 Reduction and Non-integrability of B(IX) model

The results obtained in the previous section are weak. There are two reason of it. First, we asked about the complete integrability of the system. However, the system under investigation can have only one additional integral. The most important is the fact the we pose our question for the system defined on $\mathbb{R}^6$ although we are interesting only in the system on five dimensional manifold defined by the level $\mathcal{H} = 0$. One can imagine a system that is not globally integrable although it is integrable on a one prescribed energy surface.

In this section we want to study the B(IX) Hamiltonian system just only the level $\mathcal{H} = 0$.

During investigation of a dynamical system we usually try to lower its dimension making use of its first integrals and symmetries. For Hamiltonian system (3) we know only one first integral—Hamiltonian. Thus, using it we can potentially can make reduce the dimension of the system by one however we loose the polynomial form of the system. However we do not want this side effect of reduction because it excludes possibilities of applications algebraic tools for study non-integrability.

In this section we show how to reduce the dimension of the phase space by two and preserving the polynomial form of the considered vector field. In what follows consider her the case of B(IX) ($n_1 = n_2 = n_3 = 1$).

First, we transform the Hamiltonian vector field corresponding to Hamiltonian (3), to a homogeneous polynomial form of degree two. To this end let us put

$$y_i = q^i, \quad z_i = \frac{q^i}{q^1}, \quad i = 1, 2, 3,$$

(23)

then equation of motion will have the form

$$\dot{y}_i = y_i z_i, \quad \dot{z}_i = (y_j - y_k)^2 - y_i^2, \quad \{i, j, k\} \in S_3.$$

(24)

This system has the first integral corresponding to the Hamiltonian (3). It is has the form

$$H = z_1 z_2 + z_1 z_3 + z_2 z_3 - y_1^2 + 2y_1 y_2 - y_2^2 + 2y_1 y_3 + 2y_2 y_3 - y_3^2.$$

(25)

We make change of variables

$$w_1 = y_1 + y_2, \quad w_2 = y_1 - y_2, \quad w_3 = y_3,$$

(26)
and we leave $z_i$ unchanged. In new variables system (24) has the form

\[
\begin{align*}
\dot{w}_1 &= \frac{1}{2} z_1 (w_1 + w_2) + \frac{1}{2} z_2 (w_1 - w_2), \\
\dot{w}_2 &= \frac{1}{2} z_1 (w_1 + w_2) - \frac{1}{2} z_2 (w_1 - w_2), \\
\dot{w}_3 &= z_3 w_3, \\
\dot{z}_1 &= (w_3 - w_1)(w_2 + w_3), \\
\dot{z}_2 &= (w_3 - w_2)(w_3 - w_1), \\
\dot{z}_3 &= (w_3 + w_2)(w_2 - w_3),
\end{align*}
\]

(27)

and the first integral (25) is transformed to the following form

\[
H = z_1 z_2 + z_1 z_3 + z_2 z_3 - w_2^2 + 2w_1 w_3 - w_3^2.
\]

(28)

Now, we introduce new variables

\[
u_1 = \frac{z_1}{w_3}, \quad u_2 = \frac{z_2}{w_3}, \quad u_3 = \frac{z_3}{w_3}, \quad u_4 = \frac{w_2}{w_3}, \quad u_5 = \frac{w_1}{w_3}, \quad u_6 = w_3.
\]

(29)

After this transformation we obtain the following system

\[
\begin{align*}
\dot{u}_1 &= u_6[(1 + u_4)(1 - u_5) - u_1 u_3], \\
\dot{u}_2 &= u_6[(1 - u_4)(1 - u_5) - u_2 u_3], \\
\dot{u}_3 &= u_6(u_4^2 - u_3^2 - 1) \\
\dot{u}_4 &= \frac{1}{2} u_6[u_4(u_1 + u_2 - 2u_3) + u_5(u_1 - u_2)] \\
\dot{u}_5 &= \frac{1}{2} u_6[u_4(u_1 - u_2) + u_5(u_1 + u_2 - 2u_3)u_5] \\
\dot{u}_6 &= u_3 u_6^2,
\end{align*}
\]

(30)

with the first integral

\[
H = u_6^2(u_1 u_2 + u_1 u_3 + u_2 u_3 - u_4^2 + 2u_5 - 1).
\]

(31)
Now, we make use the fact that B(IX) model is considered only on the level $H = 0$. From equation $H = 0$ we find $u_5$ as a function of $(u_1, u_2, u_3, u_4)$:

$$u_5 = \frac{1}{2}(1 + u_4^2 - u_1u_2 - u_1u_3 - u_2u_3),$$

thus we can eliminate this variable from the right hand sides of (30). Moreover if we change the independent variable according to the rule

$$\frac{d}{dt} = \frac{u_6}{2} \frac{d}{ds},$$

(note that $u_6 > 0$) than the first four equations in (30) separate from the last two. Thus, we finally obtained the following close system describing the dynamic of the B(IX) model:

$$\begin{align*}
\dot{u}_1 &= (1 + u_4)[1 + u_1u_2 + u_3(u_1 + u_2) - u_4^2] - 2u_1u_3, \\
\dot{u}_2 &= (1 - u_4)[1 + u_1u_2 + u_3(u_1 + u_2) - u_4^2] - 2u_2u_3, \\
\dot{u}_3 &= 2(u_4^2 - u_3^2 - 1) \\
\dot{u}_4 &= u_4(u_1 + u_2 - 2u_3) + \frac{1}{2}(u_1 - u_2)[1 - u_1u_2 - u_3(u_1 + u_2) + u_4^2].
\end{align*}$$

(32)

This system will be called the reduced B(IX) system. We consider this system in $\mathbb{C}^4$

**Theorem 2** The reduced B(IX) system does not have a non-trivial analytic first integral.

Our theorem will be a consequence of the following lemma.

**Lemma 1** Consider a system of differential equation

$$\dot{x} = f(x), \quad f(0) = 0, \quad f(x) = (f_1(x), \ldots, f_n(x)) \quad x \in \mathbb{C}^n,$$

(33)

with analytic right hand side, with

$$f(x) = Ax + O(|x|^2),$$

(34)

where matrix $A$ has eigenvalues $\lambda_i \in \mathbb{C}, i = 1, \ldots, n$. If the system possesses an analytical first integral $F$ then there exist non-negative integers $i_1, \ldots, i_n$ such that

$$\sum_{k=1}^n i_k \lambda_k = 0, \quad \sum_{k=1}^n i_k > 0$$

(35)
Let us assume that an analytic first integral exist and that condition (35) is not satisfied. We represent the first integral in the following form

\[ F = \sum_{l=k}^{\infty} F_l, \quad F_k \neq 0, \quad k \geq 1, \]

where \( F_l \) is a homogenous form of degree \( l \)

\[ F_l = \sum_{i_1 + \cdots + i_n = l} F_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \quad i_k \in \mathbb{Z}_+, \quad k = 1, \ldots, n. \quad (36) \]

From the equation

\[ \sum_{j=1}^{n} f_j(x) \partial_j F = 0, \]

we conclude that the form \( F_k \) is a first integral of the system \( \dot{x} = Ax \), i.e.,

\[ \sum_{j=1}^{n} l_j(x) \partial_j F_k = 0, \quad l_i(x) = \sum_{j=1}^{n} A_{ij} x_j \quad (37) \]

If matrix \( A \) is diagonalizable then we can assume that \( l_i(x) = \lambda_i x_i \), and then equation (37) reads

\[ \sum_{i_1 + \cdots + i_n = k} \sum_{l=1}^{n} i_l \lambda_l \left[ \sum_{i_1} x_1^{i_1} \cdots x_n^{i_n} = 0. \quad (38) \right. \]

This equation implies that

\[ F_{i_1, \ldots, i_n} \left[ \sum_{l=1}^{n} i_l \lambda_l \right] = 0 \text{ for all } (i_1, \ldots, i_n) \in \mathbb{Z}_n^+, \sum_{l=1}^{n} i_l = k, \]

Because, \( F_k \neq 0 \) there exit indices \( (i_1, \ldots, i_n) \) such that \( F_{i_1, \ldots, i_n}^{(k)} \neq 0 \) and that for such indices we have

\[ \sum_{l=1}^{n} i_l \lambda_l = 0. \]

Contradiction with our assumption prove the Lemma for the case of a diagonalizable matrix \( A \). In the case of non-diagonalizable matrix the prove is only technically more difficult. See \[27\] and especially \[26\] where this approach is generalized.

Let us remark the the above Lemma is also true if we assume the first integral is a formal power series.
To prove our theorem let us notice that for the reduced B(IX) model point $z = (-i, -i, i, 0)$ is an equilibrium point and the matrix of the linearized system is diagonalizable and has possesses the eigenvalues $(-2i, -2i, -4i, -4i)$. For this eigenvalues condition it cannot be satisfied and imply that the system does not have an analytic first integral. In fact we prove more, namely, the system does not have a first integral that can be expanded around the point $z$ into a formal power series.

7 Conclusions

The particular integrable subclasses of the Bianchi models play an important role in the analysis of the dynamics of the cosmological models. To illustrate this fact, let us consider the phase space of the solutions of Bianchi models in the Bogoyavlensky approach. In the Bogoyavlensky methods of investigations of the corresponding dynamical systems, we glue to the phase-space the boundary $\Delta$ onto which the system prolongs almost everywhere. The systems on the boundary $\Delta$ can be integrated and, in this way, we can study basic properties of the trajectories near the singularity. From the existence of the monotonic function $F$, we obtain that in the generic situation ($\forall n_i \neq 0$) the trajectories of the Bianchi class A models close up to the boundary $\Gamma$ as $F \ll -1$.

Now, the trajectories move along the corresponding ones lying on the boundary. All the trajectories are the separatrices of the critical points. At last the trajectories reach the neighborhood of critical points $K$ (corresponding to the Kasner asymptotics of the space-time metric) and they begin to move along their separatrices. The corresponding space-time metric for the mixmaster models is the BKL approximation.

In this way the chaotic systems (so non-integrable) in Bogoyavlensky’s approach can be well approximated by an integrable system. This feature of so surprisingly good approximation is not so far understood completely.

In Bianchi models with chaos we have the infinite series of Kasner epochs (it would be good if we had a precise and exact proof of this fact) and these models do not exactly admit the Kasner asymptotics.

It may be the case when we consider the higher-dimensional generalization of the mixmaster cosmological models (or models with a massless scalar field). These systems admit exact Kasner asymptotics, so, according to our theorem they will be integrated.

In the work it has been proved that the BKL approximation is true not only for BVIII and BIX, but it exists for all Bianchi types (exceptionally BI and BV) with the movement of matter ($n^\alpha \neq \delta^\alpha_0$).

In the models I, V and also III, VII and the type II model, with the restrictions on
the velocity $u_1 = u_2 = u_3 = 0$, it is known that the Kasner solution is a general one near the initial singularity \[16\]. From the above, we can conclude that all these models are integrable.

The facts that the BKL approximation represents a typical state of the metric in a very early state of the evolution has a very simple interpretation. Let for $F = F_1 \ll -1$ there be some distribution of initial conditions (e.g. homogeneous). Approaching the initial singularity this distribution for $F = F_2$, where $F_2 < F_1$, transforms to the corresponding one concentrated in the neighborhood of the critical points. These critical points have the separatrices which move towards the physical region of the phase space. During the motion along such a separatrix the space-time metric is described by the BKL approximation.

From the fact that $F \rightarrow -\infty$ near the singularity, we can conclude the existence of a fundamental property of the system – the property of concentration of the trajectories near the boundary $\Delta$.

To finish with, we would like to make a certain suggestion more philosophical or methodological in character. Among the Bianchi class A models, the most general are BVIII and BIX. These models possess the highest dimension of space of structural constant.

So, the integrable subclass of the Bianchi models forms a set of zero measure in the space of all Bianchi class A models. This fact means that integrable cases are exceptional whereas those non-integrable ones are typical, as well as the Bianchi models which izotropize at infinite time in the full class of Bianchi models \[17\]. The establishing of the above fact for the Einstein equations in general is the challenge for the authors’ future investigations.

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