Quantum control of transmon superconducting qubits

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In this work we analyze the implementation of a control-phase gate through the resonance between the $|11\rangle$ and $|20\rangle$ states of two statically coupled transmons. We find that there are many different controls for the transmon frequency that implement the same gate with fidelities around 99.8% ($T_1 = 15 \mu s$) and 99.97% ($T_1 = 100 \mu s$) within a time that approaches the theoretical limit. All controls can be brought to this accuracy by calibrating the waiting time and the destination frequency near the $|11\rangle - |20\rangle$ resonance. However, some controls, such as those based on the theory of dynamical invariants, are particularly attractive due to reduced leakage and their limited bandwidth.

I. INTRODUCTION

Transmon qubits presently dominate the quantum computation and quantum simulation landscape. They are mildly anharmonic qubits, a fact that restricts speed of operations and the strength interactions that can be used in single and two-qubit gates. Within this platform, we find a great variety of two-qubit gates, which include gates assisted by microwave pulses [1], parametrically modulated couplers [2, 3], parametrically modulated qubits [4], gates implemented with tuneable-frequency qubit-qubit resonances [5–7], and gates implemented with tuneable couplings [8].

Out of this list, the last two paradigms include some of the experiments with greatest fidelities, including 99.1% in the case of tuneable-frequency gates and 99.41% for tuneable couplers [8], values which slowly approach the 99.9(1)% record fidelities of trapped ions. In this work we study the possibility of improving these metrics, optimizing superconducting qubit gates to reduce errors down to the $10^{-3} - 10^{-4}$ range. This reduction would be a dramatic increase in quantum volume [9], increasing the power of NISQ computations [10], and opening the door to scalable error correction and fault-tolerant quantum computation.

Our research focuses on the resonant CZ gate demonstrated by DiCarlo et al. [5], and later on scaled up by Barends et al. [6] to setups with up to 9 qubits. This gate uses qubits that are parked at different frequencies, $\omega_1 > \omega_2$, so that under normal conditions their interaction is suppressed. To make a two-qubit gate, the frequency of the high-laying qubit $\omega_1$ is brought down to a resonant condition between the transmon states that have two excitations, $|11\rangle$ and $|20\rangle$. An adiabatic or quasiadiabatic ramp [3, 7] guarantees that the transmons are returned to their original conditions, with eigenstates suffering only phase shifts

$$|ss\rangle \rightarrow \exp(-i\phi_s - i\phi_s - i\phi_{11}|11\rangle\langle11|)|ss\rangle.$$ (1)

Our study focuses on different choices for ramping down the frequency of the control qubit $\omega_1(t)$. We will show that, provided that the ramps are slower than the anharmonicity, errors can be brought below $10^{-4}$ by tuning the waiting time and the distance from perfect resonance. Moreover, we engineer controls based on variational methods that are bandwidth limited, demand a smoother change in the flux applied to the qubit and minimize leakage errors $10^{-2}$ times below quasiadiabatic protocols. Finally, our research shows that using quasiadiabatic controls does not improve the resilience against spontaneous emission errors.

This work shows that there is great potential for implementing high-fidelity quantum gates in existing setups [7, 11], with speeds that are competitive, with little to no changes to the setups. This should help improving the quality of ongoing applications of these gate, as well as inspire similar studies for other gate paradigms [8].

The paper is structured as follows. In Sects. II A and II B we introduce the quantum description of one and two coupled transmon qubits. In Sect. II C we explain how the energy level structure of the transmons supports a phase or CZ gate, by bringing the qubits close to the $|11\rangle - |20\rangle$ resonance. Section III introduces three approaches to the design of the qubit ramp $\omega_{11}(t)$ using fast-quasiadiabatic techniques (Sects. III A and III B), the invariants method (Sect. III C) and a variational approximation to the transmon dynamics (Sect. III D). In Sect. IV we study the performance of these protocols and variations thereof. In Sect. IV E we show that just ramping down and up the frequency of the transmon produces rather large errors, all of which can be corrected by (i) slowing the ramp, (ii) tuning the destination frequency and (iii) the waiting time at the middle of the ramp. Sect. IV C illustrates how these simple tweaks can bring the errors down to $10^{-6}$ within realistic times for an ideal qubit. Moreover, even for moderate qubit lifetimes, of $T_1 = 15$ or $100 \mu s$, gate errors of $2 \times 10^{-3}$ and $3 \times 10^{-4}$ are feasible.

II. TRANSMON MODEL

A. Bare transmon

Our starting point is the standard transmon qubit model [12], a circuit that consists on a large capacitor that shunts a nonlinear inductance, which is implemented by a Josephson junction or a SQUID. In the number-
phase representation, the Hamiltonian for this circuit reads
\[ \hat{H}_T = 4E_C \hat{n}^2 - E_J \cos(\hat{\varphi}), \]
with canonical operators \([\hat{n}, \exp(i\hat{\varphi})] = \exp(i).\) The bare transmon Hamiltonian can be approximately solved in the number basis, using states \(\hat{n}|m\rangle = m|m\rangle,\) for \(m \in \mathbb{Z},\) with the representation \(\exp(i\hat{\varphi}) = \sum_{m+1} |m+1\rangle \langle m|,\) and a moderate cut-off \(|m| \leq 10 - 20.\)

In the limit \(E_J/E_C \gtrsim 50,\) the transmon behaves as a weakly nonlinear harmonic oscillator
\[ \hat{H}_T \simeq 4E_C \hat{n}^2 + \frac{1}{2} E_J \hat{\varphi}^2 - \frac{1}{24} E_J \hat{\varphi}^4, \]
and can be solved analytically \([12].\) Identifying \(4E_C \sim \hbar^2/2m\) and \(E_J \sim m\omega^2,\) the model reads
\[ \hat{H} = \hbar \omega_{01} \hat{a}^\dagger \hat{a} + \hbar \dot{\alpha} \hat{a}^\dagger \hat{a}^2. \]

The frequency \(\hbar \omega_{01} \approx \sqrt{8E_CC}/E_J\) denotes the splitting between the two lowest energy states, \(|0\rangle\) and \(|1\rangle \approx \hat{a}^\dagger |0\rangle\), which we use to encode a qubit. The anharmonicity \(\alpha = -E_C\) is small but allows us to detune all higher energy states, \(|2\rangle, |3\rangle \ldots\) Note that the Fock operators \(\hat{a}^\dagger = \frac{\sqrt{E_J}}{8E_C} \left(\frac{\hat{a}^\dagger + \hat{a}}{2}\right)^{1/4}, \hat{a} = 2 \sqrt{E_J} \left(\frac{\hat{a}^\dagger - \hat{a}}{2}\right)^{1/4},\) but have an implicit dependency on the transmon parameters.

In this work we are concerned with processes where we tune the transmon gap \(\omega_{01}\) by manipulating the Josephson inductance \(E_J.\) This tuning is facilitated by replacing the Josephson junction in the transmon with a SQUID: the magnetic flux that threads this loop determines its effective inductance \(E_J(\phi) \sim E_J(0) \cos(2\phi/\Phi_0)\) and the properties of the qubit. Changing \(E_J\) is equivalent to \(squeezing\) the harmonic oscillator, an unitary process that can introduce decoherence, through leakage—transmon states \(|0\rangle, |1\rangle\) of the computational basis are mapped to excited states \(|2\rangle, |3\rangle \ldots\) or unwanted transitions between the computational states. One goal in the following sections will be to minimize the errors in these processes, preserving the transmon eigenstates to implement useful quantum operations.

### B. Coupled transmons

This work focuses on a setup with two capacitively coupled transmons that are detuned from each other. We want to design quantum controls where one of the qubit is ramped down in frequency, brought close to resonance, so as to implement a two-qubit quantum gate. The joint qubit model can be written as
\[ \hat{H} = \hat{H}_{T,a}(\phi) + \hat{H}_{T,b} + \frac{1}{2} g_C (\hat{n}_a - \hat{n}_b)^2. \]

where the coupling constant \(g_C\) embodies the capacitive interaction, and \(\hat{H}_{T,a}(\phi)\) and \(\hat{H}_{T,b}\) are the Hamiltonians of the tuneable and the parked qubits.

We can express Hamiltonian \((5)\) in the basis of eigenstates of the uncoupled problem. Since we are focused on manipulating qubits, we can focus on the subspace with up to two excitations. Assuming \(\omega_a \geq \omega_b,\) this subspace is formed by states \(|00\rangle, |01\rangle, |10\rangle, |02\rangle, |11\rangle\) and \(|20\rangle,\) in order of increasing energy. In this basis, the model is very well approximated by a Hamiltonian matrix of the form
\[ \hat{H}_{\text{eff}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_b & J_1 & 0 & 0 \\ 0 & 0 & \omega_a & 0 & 0 \\ 0 & 0 & J_2 & \omega_a + \omega_b & J_2 \\ 0 & 0 & 0 & J_2 & 2\omega_a - \alpha_a \end{pmatrix}. \]

Here the frequency of the first qubit \(\omega_a\) is the only tuneable parameter, depending on the control flux. The anharmonicities \(\alpha_a\) and \(\alpha_b\) are approximately constant, as they only depend on the capacitive energy. Finally, we have that \(J_2 \approx \sqrt{2} J_1 \propto g_C,\) but we cannot rely on this if we want to have accurate gates with precisions below 1%.

In the simulations that follow, without loss of generality, we will use the parameters from Ref. \([7].\) This implies qubits with parameters
\[ \omega_a = 2\pi \times 6.91 \text{GHz}, \quad \alpha_a = -2\pi \times 0.331 \text{GHz}, \]
\[ \omega_b = 2\pi \times 5.69 \text{GHz}, \quad \alpha_b = -2\pi \times 0.300 \text{GHz}, \]
\[ J_1 = 2\pi \times 14.3 \text{MHz}, \quad J_2 = 2\pi \times 20.2 \text{MHz}. \]

### C. Resonant CZ gate

Assuming that \(\omega_b\) is the smallest frequency and that \(\omega_a\) can be tuned, the effective Hamiltonian \((6)\) has two avoided crossings. One at \(\omega_a = \omega_b\) enables coherent exchange of excitations between the \(|01\rangle\) and \(|10\rangle\) qubit states. The second crossing, sketched in Fig. \([4]\) happens at \(\omega_a = \omega_b + \alpha_a\) and is a result of the interaction between the qubit state \(|11\rangle\) and a state \(|20\rangle\) outside the computational basis.

We will use this second avoided crossing to model a controlled-Z gate demonstrated in various experiments with transmon qubits \([4,7].\) Following the literature, we regard the subspace \(|11⟩, |20⟩\) as an effective pseudospin
\[ \hat{H}_{11,20} = \frac{1}{2} J_2 \hat{\sigma}_z + J_2 \hat{\sigma}_x + \mathcal{O} \left(\frac{J_2^2}{\alpha_a + \alpha_b}\right) \]
where we have full control of the longitudinal magnetic field \(\delta(\phi) = \omega_a(\phi) - \omega_b - \alpha_a,\) with fixed transverse field \(J_2.\) We will control \(\delta(\phi)\) following the protocol from Fig. \([4]\) bringing the qubits in and out of resonance. In the adiabatic limit, where \(\omega_a\) changes much slower than the
As mentioned above, a perfectly adiabatic gate can be a prohibitively demand for a realistic NISQ device. Fortunately there are many control designs that are robust and which allow us to implement the CZ gate in a time that approaches the ideal limit $\pi/J_2$ of instantaneous quenches—i.e. $T = 0$ in Fig. 1(b). The controls are divided into two families. The FAQUAD and Slepian pulses aim at preserving the instantaneous eigenstates of the problem, minimizing the non-adiabatic corrections. The invariants and variational methods, on the other hand, aim at producing the right final state, allowing for high-order excitations that are self-corrected at the end of the process.

### A. Generalized FAQUAD

The fast quasiadiabatic dynamics method, is a technique that aims at preserving the adiabatic condition locally in time, to create fast and robust controls. We have extended this technique to consider excited states and problems with accidental degeneracies. Let us assume that we have a controlled Hamiltonian

$$\hat{H} = \hat{H}_0 + \epsilon(t)\hat{H}_1. \quad (10)$$

We wish to engineer a quasiadiabatic passage $\epsilon(t)$ that preserves a subset of eigenstates $N = \{ |\psi_n(\epsilon)\rangle \}$. We will construct a larger set $\bar{N}$ that includes $N$ and all states that are spectral neighbors along the evolution—i.e. all states with energies immediately above $E_{n_{\text{max}}}$ or below $E_{n_{\text{min}}}$ of those of $N$, as well as all eigenstates in between—and which are potentially connected via $\hat{H}_0$ or $\hat{H}_1$.

Using these definitions, we now introduce an adiabaticity parameter

$$\mu(t) = \max_{r,n \in \bar{N}} \left| \frac{E_r(t) - E_n(t)}{\hbar \langle \psi_n(t) | \partial_t \psi_r(t) \rangle} \right|. \quad (11)$$

This value estimates the rate of transition from $N$ to all other states. Imposing a small and constant transition rate $\mu(t) = c \ll 1$ we delocalize the transition probability along the whole interval and creates an equation for the control $\epsilon(t) = \tilde{\epsilon}(t/T)$,

$$\tilde{\epsilon}(s) = \pm \frac{c}{\hbar} \int_0^{s=t/T} \max_{n,r \in \bar{N}} \left| \frac{E_n - E_r}{\langle \psi_n | \partial_t \psi_r \rangle} \right| ds, \quad (12)$$

that leads to the same control profile for any $T$ value.

### B. Slepian pulses

Martinis and Geller [13] have provided an alternative derivation of fast quasiadiabatic protocols that focus on the shape of the control, providing conditions to reduce...
the non-adiabatic corrections. Essentially, the control works with the pseudospin model \[8\], introducing the instantaneous angle

\[ \theta(t) = \arctan(2J_2/\delta(t)). \]  

(13)

The bandwidth limited controls assume a ramp from \( \theta_i = \theta(0) = \theta(2T) \) to \( \theta_f = \theta(T) \) and back, with no waiting time \( t_{\text{wait}} = 0 \). The controls are designed as

\[ \theta[s(t)] = \theta_i + \sum_{n=1}^{N} \lambda_n \left[ 1 - \cos \left( \frac{2\pi n s(t)}{2T} \right) \right], \]  

(14)

where the proper time \( s(t) \) is obtained by solving

\[ t = \int_0^s \sin[\theta(s)]ds. \]  

(15)

In order to ensure the condition \( \theta(T) = \theta_f \), we have to impose

\[ \theta_f = \theta_i + \sum_n 2\lambda_n, \]  

(16)

which leaves \( N - 1 \) free parameters to optimize.

C. Invariants

FAQUAD is an effective method to implement a diagonal transformation, but the restriction of preserving the instantaneous eigenstates limits the maximal speed. There is a broad family of shortcuts to adiabaticity \[15, 16\] that ignore this restriction. The method of scaling laws or invariants relies on an operator \( \hat{I}(t) \), that is preserved by the evolution \[17\]

\[ \frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{i}{\hbar} [\hat{I}, \hat{H}], \]  

(17)

and which has imposed common eigenstates with the Hamiltonian at the beginning and end of evolution \( t = 0 \) and \( t = T \)

\[ [\hat{H}(0), \hat{I}(0)] = [\hat{H}(T), \hat{I}(T)] = 0. \]  

(18)

This property is enough to ensure that the eigenstates of the initial problem \( \hat{H}(0) \) are mapped to the corresponding eigenstates of \( \hat{H}(T) \) as in the case of a harmonic oscillator.

We will use the invariants method as it was designed for the harmonic oscillator \[18, 19\], ignoring the weak nonlinearity of our transmon \( \alpha \). Let us define \( \omega(t) = \sqrt{2E_C E_1(t)/\hbar} \) as the instantaneous frequency of the transmon model. The invariant associated with the single transmon Hamiltonian \[8\] becomes \[20\]

\[ \hat{I}(t) = \frac{4E_C}{\hbar^2} \left[ \rho(t)\hat{n} - \frac{\hbar^2 \dot{\rho}(t)}{8E_C} \right]^2 + \frac{\hbar^2}{16E_C \rho^2(t)} c^2, \]  

(19)

where \( c \) is a constant that we take as the initial gap of the problem \( c = \omega(0) \), for convenience, and \( \rho \equiv \rho(t) \) satisfies the equation

\[ \ddot{\rho} + \omega^2(t)\rho = \frac{\omega^2(0)}{\rho^3}, \]  

(20)

\[ \rho(0) = 1, \ \rho(T) = \sqrt{\omega(T)/\omega(0)} =: \gamma, \]  

\[ \dot{\rho}(0) = \dot{\rho}(T) = \ddot{\rho}(0) = \ddot{\rho}(T) = 0. \]

Our goal is now to design an appropriate \( \rho(t) \) and infer the control \( \omega(t) \) from the equation for the frequency

\[ \omega^2(t) = \frac{\omega^2(0)}{\rho^4} - \frac{\ddot{\rho}}{\rho^3}. \]  

(21)

In our work we have adopted a polynomial form that satisfies the boundary conditions,

\[ \rho(t) = \gamma + (1 - \gamma)(1 - t/T) \sum_{n=0}^{n_{\text{max}}} c_n (t/T)^n, \]  

(22)

with \( c_0 = 1, \ c_1 = 2, \ c_3 = 3 \) and the condition \( \sum_n c_n = 0 \). Already the fourth-order solution \( n_{\text{max}} = 4 \), with no free parameters, provides a very good control, but global searchers over various cost-functions—e.g. nonlinear energy, fidelities, leakage, etc—can also be implemented.

D. Variational ansatz

The variational method is an alternative technique, in which we approximate the evolution of a state by a manually crafted ansatz, and then design the control to ensure that the initial and final form of our ansatz match the preserved eigenstates. In our particular model, we just aim at preserving the vacuum state,

\[ \phi(x; \sigma, \beta) = \frac{1}{\sqrt{\sqrt{\pi} \sigma}} \exp \left( -\frac{x^2}{2\sigma^2} - i\frac{\beta}{2} x \right). \]  

(23)

Using the Lagrangian associated with the Schrödinger equation

\[ \mathcal{L}[\psi] := \frac{1}{2} (\psi | i\hbar \partial_t | \psi) - \frac{1}{2} (i\hbar \partial_t | \psi) - (\psi | \hat{H}(t) | \psi), \]  

(24)

we construct a new Lagrangian for the variational parameters as \( L(\sigma, \beta) = \mathcal{L}[\phi(x; \sigma, \beta)] \)

\[ L(\sigma, \beta) = -\frac{\hbar}{4} \dot{\beta} \sigma^2 + E_C e^{-\sigma^2/4} - 2 \frac{E_C}{\sigma^2} - 2E_C \beta^2 \sigma^2, \]  

(25)

and find the optimal approximation to the evolution using the Lagrange equations,

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\sigma}} = \frac{\partial L}{\partial \sigma}, \]  

(26)

The only relevant equation is that of the radius \( \sigma \equiv \sigma(t) \)

\[ \hbar^2 \frac{\ddot{\sigma}}{2} + 8E_C E_1 e^{-\sigma^2/4} \sigma = \frac{(8E_C)^2}{\sigma^3}. \]  

(27)
As before, we solve for the frequency and impose boundary conditions

$$E_f(t) = \frac{e^{\sigma^2/4}}{8E_C} \left[ \frac{(8E_C)^2}{\sigma^4} - \frac{\hbar^2 \ddot{\sigma}}{\sigma} \right],$$

(28)

$$\sigma(0) = \left(\frac{8E_C}{E_J(0)}\right)^{1/4}, \quad \sigma(T) = \left(\frac{8E_C}{E_J(T)}\right)^{1/4},$$

$$\dot{\sigma}(0) = \ddot{\sigma}(0) = \dot{\sigma}(T) = \ddot{\sigma}(T) = 0.$$  

Note how in the linear limit, in which $e^{-\sigma^2/4} \simeq 1$, this control is identical to (20) with the identifications

$$\sigma(t) = \sigma(0) \rho(t).$$

(29)

### E. Error quantification

To analyze the performance of our controls, we will use two figures of merit. The first and simplest one will be the leakage of the $d = 4$ qubit states outside the computational basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, given by

$$L[\hat{U}] = \left| 1 - \frac{1}{d^2} \sum_{s,s'=1}^{d} \langle s(T)|\hat{U}(T)|s'(0)\rangle^2 \right|^2.$$  

(30)

This quantity is different from zero when, say, states such as $|11\rangle$ experience non-recoverable transitions to nearby states, such as $|02\rangle$ or $|20\rangle$.

The second figure of merit will be the average fidelity. As explained in Ref. [21], when we want to compare two unitary operators $\hat{U}_{\text{target}}$ and $\hat{U}(T)$, their average fidelity over the whole Hilbert space is given by

$$\bar{F}[\hat{U}_{\text{target}}, \hat{U}(T)] = \frac{N F_e[\hat{U}_{\text{target}}, \hat{U}(T)] + 1}{N + 1},$$

(31)

where $N$ is the dimension of the Hilbert space and

$$F_e[\hat{U}_{\text{target}}, \hat{U}(T)] = \frac{1}{N} \sum_{s,s'=1}^{N} \langle s(0)|\hat{U}_{\text{target}}^\dagger \hat{U}(T)|s'(0)\rangle^2.$$  

(32)

This formula can be extended to consider spontaneous emission, computing the entanglement fidelity with the positive map that describes this process. Finally, when computing the fidelity of a two-qubit phase gate, we will use a transformed version of the physical unitary $\hat{U}(T)$ where all locally correctable phases have been eliminated.

### IV. PERFORMANCE ANALYSIS

#### A. Ramping an isolated transmon

As warmup problem we have studied how to change the gap of an isolated transmon, implementing the protocol from Fig. 1b without interactions. Figure 2 illustrates the frequency change of the qubit for the controls from Sect. III. Note how the FAQUAD method accelerates in the regions of the passage that have a large gap, while it slows down close to the crossing. The Slepian pulses from [13] find a similar behavior through a different reasoning.

Remember, however, that in order to tune the frequency of the transmon we have to thread a flux through its SQUID. The change in flux required to implement the controls are shown in Fig. 2b. In solid blue line we draw a simple control that uses a simple ramp. The invariants...
and variational controls follow hardware friendly paths with vanishing slopes at the beginning and the end. Finally, both the FAQUAD and the Slepian controls exhibit a nasty behavior at these extremes: since \( d\omega /dt \) is finite for these methods close to the sweet spot, it requires a diverging flux derivative to implement such pulses.

Figures 2 and 3 show the average fidelity of the down-and-up ramp for the different protocols, as a function of the ramp time \( T \) for \( t_{\text{wait}} = 0 \). Remarkably, the linearly growing pulse exhibits as good a behavior as the quasiadiabatic methods, but all of them are well separated from the invariants and variational controls, which are the best performing methods.

Note how these controls provide errors below \( 10^{-6} \) for any ramp above 0.1 ns, which is on the limit of the fastest ramps available in the laboratory. These two controls perform so well because they are essentially tracking the full dynamics of the zero and one excitation subspaces, which behave like the eigenstates of the harmonic oscillator. In particular, these protocols reproduce perfectly the squeezing of the oscillator and its eigenstates, down to very high precision. Interestingly, we have attempted to create optimal control pulses using parameterized methods and global optimizations—see App. I from Ref. [22]—, but the fidelities were comparable at very large computational cost.

### B. CZ gates with simple ramps

We have studied the possibility of implementing the CZ gate using the protocol in Fig. 1, with \( t_{\text{wait}} = 0 \). In this approach, the qubit is ramped down and up and we inspect the resulting operation. This choice is very natural for the FAQUAD and Slepian protocols which, as shown in Fig. 2, have a built-in waiting time around the avoided crossing.

Figure 3a shows the average fidelity of the unitary operation acting in the qubit subspace, compared with the phase gate [1] that approximates it the best. In this figure both the FAQUAD and the Slepian pulses achieve a reasonable accuracy, with an error below 0.1% in a time around 30 ns, which is only slightly larger than the ideal limit \( \pi /J_2 \). Out of these, the Slepian pulse even reaches the desired phase \( \phi_{12} = \pi /4 \) close to this fidelity, while the FAQUAD protocol only achieves this phase in a region where the fidelity is bad again.

A naïve interpretation of these simulations would lead us to discard all protocols but the bandwidth limited controls [13]. However, if we investigate the errors further, we will find that they can be attributed to leakage from the \( |11\rangle \) state into the \( |20\rangle \). Essentially, what is happening in all controls—including the FAQUAD and Slepian method—is that the two-qubit system approaches the resonance condition \( \omega_a = \omega_b + \alpha \) a little faster than desired. This causes a Landau-Zener transition, with population that, once we ramp back, ends up in the \( |20\rangle \) state.

![Figure 3](image_url)

**FIG. 3.** CZ gate by ramping down and up the qubit with \( t_{\text{wait}} = 0 \). (a) Average gate infidelity from a generic phase gate. (b) Two-qubit phase acquired at the end. (c) Leakage of the evolved state \( U(2T)|11\rangle \) outside the \( \{|11\rangle, |20\rangle\} \) subspace. The line code follows Fig. 2a.

An even more careful inspection of the dynamics of the two-qubit system reveals that all the dynamics takes place within three separate subspaces, with different number of excitations \( S_0 := \{|00\rangle\} \), \( S_1 := \{|01\rangle, |10\rangle\} \) and \( S_2 := \{|02\rangle, |11\rangle, |20\rangle\} \). Leakage outside \( S_0 \) and \( S_1 \) is negligible for \( T \geq 0.5 \text{ns} \), while leakage of \( S_2 \) is also small for some of our control protocols, as shown in Fig. 3. Thanks to this, we can correct these errors, by just adding some wait time \( t_{\text{wait}} \), as shown below.

### C. CZ gate optimization

If we analyze the evolution of the qubit states, we find that all CZ controls suffer from the same errors: (i) when reaching the crossing point, some leakage from \( |11\rangle \) to \( |20\rangle \), \( |02\rangle \) states happens, (ii) the states \( |01\rangle \) and \( |10\rangle \) acquire some phase, making \( \phi_{12} \) deviate from \( \pi \), and (iii) there is some residual leakage states with higher number of excitations. Each source of error is best illustrated by each of the subfigures in Fig. 3 but they are all highly correctable.

As mentioned above, the dynamics takes place mostly in the zero to two excitation subspace. Moreover,
the ramp-down and ramp-up operators are related
\[ \hat{U}(2T, T)^T = \hat{U}(T, 0), \]
and they both have a simple structure when written in terms of the initial and final eigenstates
\[ \hat{U}(T, 0) \simeq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i\xi_1 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\xi_2 t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \gamma \\ 0 & 0 & 0 & \alpha \gamma \\ 0 & 0 & 0 & 0 & \beta \delta \end{pmatrix}, \quad (33) \]
As illustrated in this equation, states |00⟩, |01⟩, |10⟩ and |02⟩ are mostly mapped to eigenstates of the coupled system, modulo some phases. The last block is a 2 × 2 unitary operation that maps state |11⟩ to a combination \( \alpha |–⟩ + \beta |+⟩ \) of the pseudospin superposition \( |\pm⟩ \propto |11⟩ \pm |20⟩ \).

Since \( U(2T, 0) = U(T, 0)^T U(T, 0) \), if we do not wait any time and simply ramp up, this state is mapped to \( (\alpha^2 + \beta^2) |11⟩ + (\gamma \alpha + \beta \delta) (\alpha)^2 |00⟩ \) at the end, which accounts for most errors in Fig. 3.

This leakage is corrected by parking the qubits close to resonance for a certain time \( t_{\text{wait}} \). The last 2 × 2 block in \( U(T, 0) \) is an approximate unitary, which can be undone by waiting some time close to degeneracy, where states \( |\pm⟩ \) freely evolve with different energies
\[ U(T, 0)^T e^{-iH_1 T} U(T, 0) |11⟩ \simeq (\alpha^2 e^{iJ_2 t} + \beta^2 e^{-iJ_2 t}) |11⟩ + (\gamma \alpha e^{iJ_2 t} + \beta \delta e^{-iJ_2 t}) |20⟩. \]

Neglecting leakage into other states, we always find a time \( e^{iJ_2 t} = \beta/\alpha \) at which this state becomes identical to |11⟩—the contribution of |20⟩ cancels due to unitarity \((|\alpha|^2 + |\beta|^2 = 1)\) and we neglect leakage to other states—. Figure 4 illustrates this for one particular control, the dynamical invariant method with a ramp-down and up time of \( T = 2 \) ns. In this particular case, the initial leakage was about 1%, but this leakage is corrected by waiting about 27 ns.

The condition of matching perfectly the population of the |11⟩ state implies also a (-1) phase shift, caused by a 2\( \pi \) rotation of the pseudospin. However, as seen in Fig. 4, the combined nonlinear phase still deviate from \( \phi_{12} = \pi/2 \), because of dynamical phases in the |01⟩, |10⟩ and |11⟩ states. We correct these phases ramping down the qubit to a frequency that deviates slightly from the target value \( \omega_b(T) = \omega_b + \alpha_\omega \). As shown in Fig. 4, changes in the phase are linear with respect to this detuning, which will be of a few megahertz and within experimental reach.

With all these correction mechanisms—i.e. optimizing the unitary with respect to \( t_{\text{wait}} \) and the destination frequency \( \omega_b(T) \)—we obtain at least two orders of magnitude increase in gate fidelity, as seen in Fig. 5, irrespective of the control that is applied. Controls such as the invariants method perform extremely well, due to their capacity to address the oscillator squeezing and minimize leakage to states outside the computational basis.
but a trivial linear ramp of the flux reaches gate fidelities above 99.9% and 99.99% for gate durations of 25 to 40 ns. The same study can be done including losses in the superconducting qubits. As shown in Fig. [6], a decay time \( T_1 \sim 16 \mu s \), such as those in experimentally available qubits [7], dominates the errors, equalizing all ramp methods. If we increase the quality of the qubits by an order of magnitude, we find that the invariants and variational ramps become significantly better and more robust at long times, which contradicts the myth that staying close to the eigentstates leads to higher quality gates [6].

**V. SUMMARY AND DISCUSSION**

In this work we have studied the implementation of a CZ gate using the avoided crossing between the \(|11\rangle\) and \(|20\rangle\) of two statically coupled transmons. We have shown that there are many different controls all of which lead to gates with excellent fidelities within times which approach the theoretical limit \( \pi/J_2 \). For all controls, tuning the gate requires only a calibration of the waiting time and of the ramp frequency.

Given the great variety of possible controls, are all choices created equal? We argue that this is not the case. Some of these protocols, such as the invariants and variational methods, have a better performance due to their optimal control of leakage outside the computational basis, and a greater robustness against decoherence. Moreover, when we consider the physical parameters that are controlled—i.e. when we study the variation of flux that they demand—, we find that precisely those controls are the ones that have better properties of finite bandwidth and resilience to discretization [cf. Fig. [7]].

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