On global absence of Lavrentiev gap for functionals with $(p, q)$-growth

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Abstract

We prove that for convex vectorial functionals with $(p, q)$-growth the Lavrentiev phenomenon does not occur up to the boundary when $(p, q)$ are suitably restricted. Under minimal assumptions on the boundary data, we obtain results for autonomous and non-autonomous functionals, under natural, controlled and controlled duality growth bounds.

1 Introduction and results

We consider minimisation problems of the form

$$\min_{u \in g + W^{1,p}_0(\Omega, \mathbb{R}^m)} \mathcal{F}(u) \quad \text{where} \quad \mathcal{F}(u) = \int_{\Omega} F(x, Du) \, dx.$$  \hfill (1.1)

Here $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, $g \in W^{1,q}(\Omega, \mathbb{R}^m)$ is the Dirichlet boundary condition and $F$ is an integrand satisfying $(p, q)$-growth conditions in the gradient variable.

It is well known that the Lavrentiev phenomenon provides a fundamental obstruction to the regularity theory regarding (1.1). The Lavrentiev phenomenon describes the fact that it may occur that

$$\min_{u \in g + W^{1,p}_0(\Omega, \mathbb{R}^m)} \mathcal{F}(u) < \min_{u \in g + W^{1,q}_0(\Omega, \mathbb{R}^m)} \mathcal{F}(u).$$

This observation was first made in [33]. With regards to (1.1), the theory was further developed in [41], [39] and [40]. In this paper, we adopt the viewpoint and terminology of [8] and view the Lavrentiev phenomenon through the so-called Lavrentiev gap.

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Suppose $X$ is a topological space of weakly differentiable functions and $Y \subset X$. Introduce

$$
\mathcal{F}_X = \sup\{ \mathcal{G} : X \to [0, \infty] : \mathcal{G} \mathrm{slsc}, \mathcal{G} \leq \mathcal{F} \text{ on } X \} \\
\mathcal{F}_Y = \sup\{ \mathcal{G} : X \to [0, \infty] : \mathcal{G} \mathrm{slsc}, \mathcal{G} \leq \mathcal{F} \text{ on } Y \}.
$$

The Lavrentiev gap functional is then defined for $u \in X$ as

$$
\mathcal{L}(u, X, Y) = \begin{cases} \\
\mathcal{F}_Y(u) - \mathcal{F}_X(u) & \text{if } \mathcal{F}_X(u) < \infty \\
0 & \text{else.}
\end{cases}
$$

Note that $L(u, X, Y) \geq 0$ and that $L(u, X, Y) > 0$ for some $u \in X$ when the Lavrentiev phenomenon occurs. However, in general, it could be that $L(u, X, Y) > 0$ for some $u \in X$, but the Lavrentiev phenomenon does not occur. There is an extensive literature on the Lavrentiev phenomenon and gap functional in this abstract set-up, an overview of which can be found in [7], [20] to which we also refer for further references.

We remark that under these assumptions the existence of a solution to (1.1) follows from the direct method. (H2)–(H4) are increasingly strong assumptions, that are referred to as natural growth, controlled growth and controlled duality growth assumptions, respectively.

In order to compare our assumptions and results to the literature, we state them precisely. Let $1 < p \leq q$. Suppose $F(\cdot, z)$ is continuous for any $z \in \mathbb{R}^{n \times m}$ and $F(x, \cdot)$ is $C^1(\Omega)$ for almost every $x \in \Omega$. We will always assume a strict convexity assumption on $F$: There is $\nu > 0$ and $\mu \in [0, 1]$ such that for almost every $x \in \Omega$ and every $z \in \mathbb{R}^{n \times m}$,

$$
\nu(\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} \leq \frac{F(x, z) - F(x, w) - \langle \partial_z F(x, w), z - w \rangle}{|z - w|^2}. \tag{H1}
$$

We quantify the growth of $F$ through one of the following growth conditions: There is $\Lambda > 0$ such that for almost every $x \in \Omega$ and every $w, z \in \mathbb{R}^{n \times m}$,

$$
|F(x, z)| \leq \Lambda (1 + |z|^2)^{\frac{q}{2}} \tag{H2}
$$

$$
\frac{F(x, z) - F(x, w) - \langle \partial_z F(x, w), z - w \rangle}{|z - w|^2} \leq \Lambda (1 + |z|^2 + |w|^2)^{\frac{2q}{2q-2}}. \tag{H3}
$$

$$
\frac{F(x, z) - F(x, w) - \langle \partial_z F(x, w), z - w \rangle}{|z - w|^2} \leq \Lambda (1 + |\partial_z F(z)|^2 + |\partial_z F(w)|^2)^{\frac{q-2}{2q-2}}. \tag{H4}
$$

We remark that under these assumptions the existence of a solution to (1.1) follows from the direct method. (H2)–(H4) are increasingly strong assumptions, that are referred to as natural growth, controlled growth and controlled duality growth assumptions, respectively.

Finally, we assume that $F(\cdot, z)$ is Hölder-continuous, that is for almost every $x, y \in \Omega$ and every $z \in \mathbb{R}^{n \times m}$,

$$
|F(x, z) - F(y, z)| \leq \Lambda |x - y|^\alpha \left(1 + |z|^2\right)^{\frac{q}{2}}. \tag{H5}
$$
In the case of controlled growth conditions, we need to make the strong assumption that $F$ is doubling, that is there is $s_0 > 0$ such that for $s \in [1, s_0)$,
\[ F(sz) \lesssim 1 + F(z). \quad (H6) \]
It would be desirable to remove this assumption.

We remark that a consequence of any of (H2)–(H4) in combination with the convexity of $F(x, \cdot)$ expressed in (H1) is the bound:
\[ |F(x, z) - F(x, w)| \lesssim |z - w|(1 + |z| + |w|)^{q-1} \quad (1.2) \]
for almost every $x \in \Omega$ and every $z, w \in \mathbb{R}^{n \times m}$.

For technical reasons, we additionally require the following condition, which we will discuss in more detail soon: there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in \Omega$ there is $\hat{y} \in B_\varepsilon(x) \cap \Omega$ such that
\[ F(\hat{y}, z) \leq F(y, z) \quad \forall y \in B_\varepsilon(x) \cap \Omega, \quad z \in \mathbb{R}^{n \times m}. \quad (H6) \]

Introduce
\[ \mathcal{F}(u) = \inf \{ \liminf (\mathcal{F}(u_j) : (u_j) \subset Y, u_j \rightharpoonup u \text{ weakly in } X \}, \]
where $X = g + W_{0}^{1,p}(\Omega, \mathbb{R}^m)$ and $Y = g + W_{0}^{1,q}(\Omega, \mathbb{R}^m)$. Since $F(x, z)$ is convex due to (H1), with this choice of $X$ and $Y$, standard methods show that $\mathcal{F}_X(\cdot) = \mathcal{F}(\cdot)$, see [22, Chapter 4]. Further $\mathcal{F}_Y(\cdot) = \mathcal{F}(\cdot)$.

We then prove the following result:

**Theorem 1.** Let $1 < p \leq q$ and suppose $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Assume $g \in W_{0}^{1,q}(\Omega, \mathbb{R}^m)$ and take $\alpha \in (0, 1)$. Let $u \in g + W_{0}^{1,p}(\Omega, \mathbb{R}^m)$. Suppose $F$ satisfies (H1) and one of the following conditions hold:

(i) $q < \frac{(n + \alpha)p}{n}$ and $F$ satisfies (H2), (H5) and (H6),

(ii) $q < \min \left( p + 1, \frac{np}{n-1} \right)$ and $F \equiv F(z)$ satisfies (H2),

(iii) $p \leq q < p + \max \left( 1, \frac{p}{n} \right)$, $u \in L^{\infty}(\Omega)$ and $F \equiv F(z)$ satisfies (H2),

(iv) $p \leq q < p + \max \left( 1, \frac{p}{n} \right)$, $m = 1$ and $F \equiv F(z)$ satisfies (H2),

Then
\[ \mathcal{F}(u) = \mathcal{F}(u). \]

If $F$ satisfies (H1), $u \in g + W_{0}^{1,p}(\Omega)$ solves (1.1) and one of the following conditions holds:

(v) $2 \leq p \leq q < \min \left( p + 2, p \left( 1 + \frac{2}{n-1} \right) \right)$ and $F \equiv F(z)$ satisfies (H3), (H6)

(vi) $2 \leq p \leq q < \frac{np}{n-p}$ and $F \equiv F(z)$ satisfies (H4),

(vii) $2 < p \leq q < p + \max \left( 2, \frac{2q}{n} \right)$, $u \in L^{\infty}(\Omega)$ and $F \equiv F(z)$ satisfies (H3),

(viii) $2 \leq p \leq q < p + \max \left( 2, \frac{2q}{n} \right)$, $m = 1$ and $F \equiv F(z)$ satisfies (H3), (H6),
then
\[ \mathcal{F}(u) = \inf_{v \in g + W^{1,q}_0(\Omega)} \mathcal{F}(v). \]

Integrands with \((p, q)\)-growth, such as the ones we consider in this paper, have been studied since the seminal papers [34], [35]. There is by now an extensive literature regarding the regularity theory. We don’t aim to give a complete overview of this theory here, but refer to [36, 37] for a good overview and further references. We remark that the first step in the regularity theory is to prove that minimisers of \(\mathcal{F}\) are \(W^{1,q}\)-regular. As soon as the Lavrentiev gap is excluded, this proves that minimisers of \(\mathcal{F}\) are \(W^{1,q}\)-regular. Hence, it is useful to compare the range of \((p, q)\) in Theorem 1 to the range in available (local) \(W^{1,q}\)-regularity results.

Under the assumptions of (i), minimisers of \(\mathcal{F}\) are \(W^{1,q}_{loc}\)-regular and this assumption is sharp [16, 15], see also [4] regarding the sharpness. If \(g \in W^{1+\alpha,q}(\Omega, \mathbb{R}^m)\) and \(\Omega\) is Lipschitz, minimisers of \(\mathcal{F}\) are even \(W^{1,q}\)-regular [30]. In the setting of (iii) (assuming only \(q < p + 1\)) and (vii) (assuming only \(q < p + 2\)), \(W^{1,q}_{loc}\)-regularity is due to [9].

Regularity in the range given in (iii) and (vii), respectively, follows from combining these results with those in [16, 15], that is corresponding to the set-up of (i) with the choice \(\alpha = 1\). In particular, we note that in these settings we show that there is no Lavrentiev gap precisely in the regime where \(W^{1,q}_{loc}\)-regularity is known. We remark that if the a-priori assumption on \(u\) is strengthened beyond \(u \in L^\infty\), \(W^{1,q}_{loc}\)-regularity of minimisers can be obtained under weaker assumptions than those of (iii) or (vii), c.f. [12], but we do not pursue this direction here.

In the setting of (ii), \(W^{1,q}_{loc}\)-regularity of minimisers of \(\mathcal{F}\) if \(1 < p < \frac{np}{n-1}\) is due to [10]. If \(g \in W^{2,q}(\Omega, \mathbb{R}^m)\), \(\Omega\) is Lipschitz and \(q < \min \left(p + 1, \frac{np}{n-1}\right)\), relaxed minimisers are even \(W^{1,q}\)-regular [31]. Thus, we recover this range of \((p, q)\) in (ii). \(W^{1,q}_{loc}\) regularity in the setting of (v) with \(q < p \left(1 + \min \left(\frac{2}{n-1}, 1\right)\right)\) is due to [38]. Again, we note that we recover this range of \((p, q)\) if \(n \geq 3\) and \(p \leq n - 1\) in (v). In the scalar case \(m = 1\) stronger results are known. In particular, if \(|z|^p \lesssim F(z) \lesssim (1 + |z|^q)\) and \(F\) is convex, it suffices to assume \(\frac{q}{p} \leq 1 + \frac{q}{n-1}\) in order to ensure local boundedness of minimisers of \(\mathcal{F}\), [26]. This restriction is sharp [35, 27]. Finally, we remark that the case of controlled-duality growth is treated in the upcoming [11], assuming \(q \leq \frac{np}{n-2}\).

In order to explain our proof, it is instructive to first consider the interior case when \(F \equiv F(z)\) is autonomous: Let \(\{\rho_\varepsilon\}\) be a standard family of mollifiers. Given \(u \in g + W^{1,q}_0(\Omega, \mathbb{R}^m)\) and \(\omega \in \Omega\), for \(\varepsilon > 0\) sufficiently small, \(u_\varepsilon = u \star \rho_\varepsilon \in W^{1,q}_{loc}(\Omega, \mathbb{R}^m)\). Moreover, due to Jensens’ inequality and the convexity of \(F\), i.e. (H1),
\[ \int_{\omega} F(Du_\varepsilon) \, dx \leq \int_{\omega} F(Du) \star \rho_\varepsilon \, dx \xrightarrow{\varepsilon \to 0} \int_{\omega} F(Du) \, dx. \]

Thus, in the local convex autonomous case, there is no Lavrentiev gap.

From this calculation it is clear, that there are two substantial difficulties in proving Theorem 1. We need to adapt the mollification near the boundary and the application of Jensen’s inequality needs to be justified in the non-autonomous setting. To our knowledge, Theorem 1 is the first result excluding the Lavrentiev phenomenon globally in the vectorial setting without assuming structure conditions on \(F\) going beyond \(p\)-ellipticity and \(q\)-growth.
There are various approaches towards dealing with the non-autonomous setting. Condition (H6) holds for many of the commonly considered examples, see [16, 30]. It is very similar to [40, Assumption 2.3] and has also appeared in the context of lower semi-continuity in [2]. In the context of functionals with Orlicz growth, similar assumptions have been used, see for example [6], [24]. Moreover, recently in [13] the Lavrentiev gap has been excluded for non-autonomous integrands in the setting of Theorem 1(i) with (H6) replaced by the assumption that the integrand $F$ can be approximated from below by integrands $F_k$ satisfying

$$|\partial_z F_k(x, z) - \partial_z F_k(x, z)| \lesssim c_k |x - y|^{\alpha} (1 + |z|^{p-1}).$$

Moreover, for special cases, such as the double-phase functional $F(x, z) = |z|^p + a(x)|z|^q$, the equality case $q = \left(\frac{n+\alpha}{n}\right)\frac{p}{q}$ has been settled [5]. However, at the moment there seems to be no condition available that covers all cases in which the (local) Lavrentiev phenomenon is known not to occur. Finally, we would like to contrast our results, in particular the cases (iii) and (iv) of Theorem 1 with those obtained in [6]. The results in [6] concern integrands with Orlicz-growth, a direction we do not pursue here. When restricted to integrands with polynomial $(p,q)$-growth, we recover the results of [6] for a larger class of integrands $F$. Further, we remark that the results of [6] have been extended to anisotropic functionals in [3].

Regarding the mollification near the boundary, in this paper, we expand a technique already utilised in [30], [31]. We emphasize that in the analysis of [30], [31], the Lavrentiev phenomenon was excluded, but no result was proven regarding the Lavrentiev gap. Moreover, only the setting of cases (i) and (ii) in Theorem 1 was studied and stronger assumptions on the regularity of $g$ were imposed. The key idea is to use a two-parameter mollification $u \ast \phi_{\delta(x)}$ where $\delta(x) \sim d(x, \partial \Omega)$. This idea will be implemented using a Whitney-Besicovitch covering of $\Omega$. Similar ideas have been used in [19] in relation with studying measure representations of $\overline{F}$ and more recently in [14] in order to prove density of smooth functions for weighted fractional Sobolev spaces on open sets.

The rest of this paper is structured as follows. In Section 2, we introduce our notation and collect a number of preliminary results. In Section 4, we construct for $u \in g + W^{1,p}_0(\Omega, \mathbb{R}^m)$ a sequence of regular approximations $u_\varepsilon \in g + W^{1,q}_0(\Omega, \mathbb{R}^m)$. In Section 5, we prove under various assumptions that $F(u_\varepsilon) \to F(u)$. Finally, we prove Theorem 1 in Section 6.

### 2 Preliminaries

Throughout $\Omega$ is a Lipschitz domain of $\mathbb{R}^n$. Denote by $\overline{\Omega}$ the closure of $\Omega$. Given $\varepsilon > 0$, set $\Omega_\varepsilon = \{x \in \Omega : d(x, \partial \Omega) < \varepsilon\}$. For two sets $A, B \subset \mathbb{R}^n$ we denote $A + B = \{a + b : x \in A, b \in B\}$. We write $B_r(x)$ for the usual open Euclidean ball of radius $r$ in $\mathbb{R}^n$. For readability, we adopt the non-standard notation $sB_r(x) = B_{sr}(x)$ for $s > 0$. $|\cdot|$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$ and likewise the Euclidean norm of a matrix $A \in \mathbb{R}^{n \times n}$.

If $p \in [1, \infty]$ denote by $p' = \frac{p}{p-1}$ its Hölder conjugate. The symbols $a \sim b$ and $a \lesssim b$ mean that there exists some constant $C > 0$, depending only on $n, m, p, q, \Omega, \mu, \lambda$ and $\Lambda$, and independent of $a$ and $b$ such that $C^{-1}a \leq b \leq Ca$ and $a \leq Cb$, respectively.
Let $1 < p \leq \infty$. We denote by $L^p(\Omega) = L^p(\Omega, \mathbb{R}^m)$ the usual Lebesgue space on $\Omega$. Since we do not make a regularity assumption on $\Omega$, we need to be precise in our definition of Sobolev spaces. Denote by $C^\infty(\Omega) = C^\infty_c(\Omega, \mathbb{R}^m)$ the set of smooth $\mathbb{R}^m$-valued functions in $\Omega$ and by $C^\infty_c(\Omega) = C^\infty_c(\Omega, \mathbb{R}^m)$, the set of smooth $\mathbb{R}^m$-valued functions compactly supported in $\Omega$. Then we consider $W^{1,p}(\Omega, \mathbb{R}^m) = W^{1,p}(\Omega)$ to be the closure of $C^\infty_c(\Omega)$ in $\Omega$ with respect to $\| \cdot \|_{W^{1,p}(\Omega)}$. Here $\| \cdot \|_{W^{1,p}(\Omega)}$ denotes the usual Sobolev norm. $W^{1,p}_0(\Omega) = W^{1,p}_0(\Omega, \mathbb{R}^m)$ denotes the closure of functions compactly supported in $\Omega$.

Whitney-Besicovitch coverings will be a key tool in our construction. They combine properties of Whitney and Besicovitch coverings and were introduced in [28]. A nice presentation of the theory is given in [29]. To be precise:

**Definition 1.** A family of balls $\{ B_i \}_{i \in I}$ is called a Whitney-Besicovitch-covering (WB-covering) of $\Omega$ if there is a triple $(\delta, M, \varepsilon)$ of positive numbers such that

\[
\bigcup_{i \in I} B_i = \bigcup_{i \in I} (1 + \delta) B_i = \Omega \tag{2.1}
\]

\[
\sum_{i \in I} \chi_{(1+\delta)B_i} \leq M \tag{2.2}
\]

\[
B_i \cap B_j \neq \emptyset \Rightarrow |B_i \cap B_j| \geq \varepsilon \max(|B_i|, |B_j|). \tag{2.3}
\]

For any open set $\Omega \subset \mathbb{R}^n$, a WB-covering of $\Omega$ exists due to [29, Theorem 3.1]. We note that we can furthermore ensure that there is $C > 0$ such that, for all $i \in I$,

\[
\frac{1}{C} d(B_i, \partial \Omega) \leq |B_i|^{\frac{1}{n}} \leq C d(B_i, \partial \Omega). \tag{2.4}
\]

Finally, a consequence of (2.1) and (2.3) is that

\[
(1 + \delta/2) B_i \cap (1 + \delta/2) B_j \neq \emptyset \Rightarrow |B_i| \sim |B_j|. \tag{2.5}
\]

We will require a partition of unity related to a WB-covering.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ be open. Suppose the balls $\{ B_i \}_{i \in I}$ form a WB-covering of $\Omega$. Let $B_i$ have radius $r_i$. Suppose $\{ \phi_i \}_{i \in I} \subset C^\infty_c(\Omega)$ is such that

\[
\text{supp } \phi_i \subset (1 + \delta/2) B_i, \quad 0 \leq \phi_i \text{ and } \phi_i = 1 \text{ on } B_i \tag{2.6}
\]

for all $i \in I$. Then there is a family family $\{ \psi_i \}_{i \in I}$ of infinitely differentiable functions that form a partition of unity on $\Omega$ with the following properties:

\[
\supp (\psi_i) \subset (1 + \delta/2) B_i, \quad 0 \leq \psi_i \leq 1 \quad \text{and} \quad \psi_i(x) \geq \frac{1}{M} \text{ for } x \in B_i \tag{2.7}
\]

for all $i \in I$. Further,

\[
|D\psi_i| \lesssim \sum_{\{j : (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} |D\phi_j|. \tag{2.8}
\]

Moreover, it is possible to choose $\{ \phi_i \}_{i \in I}$ such that

\[
|D\psi_i| \leq cr_i^{-1}. \tag{2.9}
\]
Proof. Suppose \( \{ \phi_i \} \) satisfies (2.5). Note that then

\[
\psi_i = \sum \phi_i
\]
is a smooth partition of unity of \( \Omega \) satisfying (2.6) due to (2.2). We further note that

\[
D\psi_i = \frac{D\phi_i}{\sum_j \phi_j} + \frac{\phi_i}{(\sum \phi_j)^2} \sum_j D\phi_j.
\]

In particular,

\[
|D\psi_i| \lesssim \sum_{\{j: (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} |D\phi_j| \quad (2.8)
\]
as \( \sum_j \phi_j \geq 1 \).

Finally, due to (2.3), it is possible to choose \( \{ \phi_i \} \) such that

\[
|D\phi_i| \lesssim |B_i|^{-\frac{n}{p}}. \quad (2.8)
\]
and (2.4) then give (2.7).

We further record a number of well-known estimates concerning rate of convergence of mollifications. We fix a family \( \{ \rho_\varepsilon \} \) of radially symmetric, non-negative mollifiers of unitary mass. We denote convolution with \( \rho_\varepsilon \) as

\[
u \ast \rho_\varepsilon (x) = \hat{R} \int R u(y) \phi_\varepsilon (x-y) \, dy.
\]

Lemma 2. Let \( 1 < p < \infty \) and \( 0 < r \leq 1 \). Suppose \( u: R^n \to R^m \). Let \( B = B_r(x_0) \subset R^n \).

For \( \varepsilon > 0 \) denote \( u_\varepsilon = u \ast \rho_\varepsilon \). Then, the following estimates hold (with implicit constants independent of \( r \)):

(i) \( \| u_\varepsilon \|_{L^p(B)} \lesssim \| u \|_{L^p(B+\varepsilon B_1(0))} \),

(ii) \( \| u_\varepsilon \|_{L^\infty(B)} \lesssim \| u \|_{L^\infty(B+\varepsilon B_1(0))} \),

(iii) \( \| u_\varepsilon \|_{L^\infty(B)} \lesssim \varepsilon^{-\frac{n}{p}} \| u \|_{L^p(B+\varepsilon B_1(0))} \),

(iv) \( \| u_\varepsilon - u \|_{L^p(B)} \lesssim \varepsilon \| u \|_{W^{1,p}(B+\varepsilon B_1(0))} \).

Moreover,

(i) for \( p \leq q \leq \frac{np}{n+p} \), \( \| u_\varepsilon - u \|_{L^q(B)} \lesssim \varepsilon^{1+n\left(\frac{1}{p} - \frac{1}{q}\right)} \| u \|_{W^{1,p}(B+\varepsilon B_1(0))} \),

(ii) for \( p \leq q < \infty \), \( \| u_\varepsilon - u \|_{L^q(B)} \lesssim \varepsilon^{\frac{n}{p}} \| u \|_{W^{1,p}(B+\varepsilon B_1(0))} \| u \|_{L^\infty(B+\varepsilon B_1(0))}^{1-\frac{p}{q}} \).

Proof. The estimates are standard. Proofs for the first part can be found in [17]. The moreover part follows by interpolation from the first part.

Next, we present two key technical tools. First, we require a lemma that essentially allows us to work on spheres with regards to certain estimates. This is an observation, originally proven in [38]. We state a slightly modified version as in [31]:
Lemma 3. Fix \( n \geq 2 \) and let \( t_1 > t_2 \geq 1 \). For given \( 0 < \rho < \sigma < \infty \) with \( \sigma - \rho < 1 \) and \( w \in L^1(B_\sigma) \), consider

\[
J(\rho, \sigma, w) = \inf \left\{ \int_{B_\sigma} |w|(|D\phi|^{t_1} + |D\phi|^{t_2}) \, dx : \phi \in C_0^1(B_\sigma), \phi \geq 0, \phi = 1 \text{ in } B_\rho \right\}.
\]

Then, for every \( \delta \in (0,1) \), we have

\[
J(\rho, \sigma, w) \leq (\sigma - \rho)^{-t_1 - 1/\delta} \left( \int_{\rho}^{\sigma} \left( \int_{\partial B_r} |w| \, d\sigma \right)^{\delta} \, dr \right)^{1/\delta}.
\]

Second, we recall a lemma from [15], which will be crucial for dealing with non-autonomous integrands:

Lemma 4. Assume \( 1 < p \leq q < \frac{(n+\alpha)p}{n} \). Suppose \( F \) satisfies (H1), (H2), (H5) and (H6). Then for \( x \in \Omega \) and \( \varepsilon \leq \min(\varepsilon_0, d(x, \partial \Omega)) \),

\[
F(x, Du(\cdot) \star \phi_\varepsilon(x)) \lesssim 1 + \left( F(\cdot, Du(\cdot) \star \phi_\varepsilon) \right)(x).
\]

Moreover, if \( |z| \leq C\varepsilon^{-\frac{\alpha}{p}} \), it holds

\[
\sup_{y \in B_\varepsilon(x)} F(y, z) \leq 1 + \inf_{y \in B_\varepsilon(x)} F(y, z)
\]

Finally, we recall the following lemma, which is a combination of [30, Lemma 10] and [32, Theorem 5].

Lemma 5. Let \( 1 < p \). Suppose \( F \) satisfies (H1) and \( F(x, z) \lesssim 1 + |z|^q \). Then, if \( u \in g + W^{1,p}_0(\Omega) \) solves (1.1),

\[
\int_{\Omega} \partial_\varepsilon F(x, Du) \cdot D\phi \, dx = 0 \text{ for all } \phi \in W^{1,q}_0(\Omega).
\]

and

\[
\int_{\Omega} \partial_\varepsilon F(x, Du) \cdot (Du - Dg) \, dx \leq 0.
\]

Moreover, if \( q \leq \frac{np}{n-p} \),

\[
\partial_\varepsilon F(x, Du) \in L^q(\Omega).
\]

3 A simplification

We claim that in order to prove Theorem 1, we may make the following additional assumption:

Claim 6. Under the assumptions of Theorem 1, it suffices to construct \((u_j) \subset g + W^{1,p}_0(\Omega) \cap W^{1,q}_{loc}(\Omega)\) such that \( u_j \to u \) in \( W^{1,p}(\Omega) \).
We first focus on the case where \( f \equiv f(z) \). Since \( \Omega \) is a Lipschitz domain, there exists a finite family of domains \( x_i + \Omega_i, i = 1, \ldots m \) such that \( \Omega_i \) is strongly star-shaped with respect to \( 0 \) and \( \Omega = \cup(x_i + \Omega_i) \). We may also ensure that there exist \( \omega_i \in \Omega_i \) with respect to \( \Omega \), strongly star-shaped with respect to \( x_i \) and such that still \( \Omega = \cup(x_i + \Omega_i) \). Extend \( g \) to a \( W^{1,q} \)-function on \( \mathbb{R}^n \) and extend \( u \) by \( g \) outside of \( \Omega \). We want to introduce an approximation to \( u \) on \( x_i + \Omega_i \). After a translation, we may assume \( x_i = 0 \). Then consider for \( s > 1 \) and \( t = t(s) < 1 \) to be chosen at a later point,

\[
u^s_i(x) = t \left( \frac{1}{s}(u - g)(sx) + g(x) \right).
\]

Clearly \( u^s_i \to u \) in \( W^{1,p}(\Omega_i) \) and \( u^s_i = g \) on \( \partial \Omega_i \). Further \( u^s_i \) is \( W^{1,q} \) in an open neighbourhood of \( (\partial \omega \cap \partial \Omega_i) \). Moreover, using the convexity of \( F \),

\[
\int_{\Omega_i} F(Du^s_i) \, dx = \int_{\Omega_i} F(t D(u - g)(sx) + Dg(sx) - tDg(sx) + tDg(x)) \, dx \\
\leq t\int_{\Omega_i} F(D(u - g)(sx) + Dg(sx)) \, dx + (1 - t)\int_{\Omega_i} F \left( \frac{t}{1 - t} (Dg(x) - Dg(sx)) \right) \, dx.
\]

On the one hand,

\[
\int_{\Omega_i} F(Du(sx)) \, dx = \int_{s\Omega_i \cap \Omega} F(Du)s^{-d} \, dx + \int_{s\Omega_i \setminus (\Omega \cap \Omega)} F(Dg)s^{-d} \, dx \to \int_{\Omega_i} F(Du) \, dx.
\]

On the other hand,

\[
(1 - t)\int_{\Omega_i} F \left( \frac{t}{1 - t} (Dg(x) - Dg(sx)) \right) \, dx \leq \frac{t^q}{(1 - t)^{q-1}} \int_{\Omega_i} |Dg(x) - Dg(sx)|^q \, dx.
\]

Note that this tends to 0 as \( s \to 1 \). In particular, we may choose \( t(s) \) such that by a version of the dominated convergence theorem,

\[
\int_{\Omega_i} F(Du^s_i) \, dx \to \int_{\Omega_i} F(Du) \, dx.
\]

Let \( \{ \eta_i \} \) be a smooth partition of unity such that \( \text{supp } \eta_i \subset x_i + \omega_i \). Then define,

\[
u^s(x) = \eta_i u^s_i(x).
\]

Note that clearly \( u^s(x) \to u \) in \( W^{1,p}(\Omega) \) as \( i \to \infty \), \( u^s \in g + W^{1,p}_0(\Omega) \) and moreover \( u^s \) is \( W^{1,q} \) in an open neighbourhood of \( \partial \Omega \). Further, ensuring also that \( \sup_{x \in \text{supp } \eta_i} \frac{|D\eta|}{(1 - \eta_i)^{p-1}} \leq C < \infty \),

\[
\int_{\Omega} F(Du^s(x)) \, dx = \sum_i \int_{\Omega_i} F(\eta_i Du^s_i(x) + u^s_i \otimes D\eta_i) \\
\leq \sum_i \int_{\text{supp } \eta_i} \eta_i F(Du^s_i(x)) \, dx + \sum_i \int_{\text{supp } \eta_i} (1 - \eta_i) F \left( \frac{1}{1 - \eta_i} (u^s_i - u) \otimes D\eta_i \right) \, dx \\
\leq \sum_i \int_{\Omega_i} F(Du^s_i(x)) \eta_i \, dx + \sum_i \int_{\text{supp } \eta_i} |D\eta_i| \frac{|u^s_i - u|^q}{(1 - \eta_i)^{q-1}} \, dx
\]

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\[
\leq \int_{\Omega} F(Du) \, dx + \int_{\Omega_i} |u_i^s - u|^q \, dx.
\]

Now if \( q \leq \frac{np}{n-p} \), the second term is estimated using Sobolev embedding by

\[
\int_{\Omega_i} |u_i^s - u|^q \, dx \lesssim \sum_i \|u_i^s - u\|^q_{W^{1,q}(\Omega_i)} \to 0
\]
as \( s \to 1 \). In particular, by a version of dominated convergence and passing to a suitable subsequence, letting first \( s \to 1 \), then \( t \to 1 \), \( \mathcal{F}(u^{t,s}) \to \mathcal{F}(u) \).

If \( u \in L^\infty \) and \( q > \frac{np}{n-p} \), we employ interpolation to estimate with \( \frac{1}{q} = (1-\theta)\frac{n p}{n-p} \) (note \( \theta \in (0, 1) \)),

\[
\int_{\Omega_i} |u_i^s - u|^q \, dx \lesssim \sum_i \|u_i^s - u\|_{W^{1,p}(\Omega_i)}^{1-\theta} \|u_i^s - u\|_{L^\infty(\partial \Omega_i)}^\theta + \|u_i^s - u\|_{W^{1,q}(\Omega_i)} \|g\|_{W^{1,q}(\Omega)}
\]

\[
\lesssim \sum_i \|u_i^s - u\|_{W^{1,p}(\Omega_i)}^{1-\theta} \|u_i^s - u\|_{L^\infty(\Omega_i)}^\theta \to 0
\]

Thus, we conclude also in this case, that \( \mathcal{F}(u^{t,s}) \to \mathcal{F}(u) \) after passing to a suitable subsequence, letting first \( s \to 1 \), then \( t \to 1 \).

Thus, given \( u \in W^{1,p}(\Omega) \) and \( (u_j) \subset g + W^{1,p}_0(\Omega) \cap W^{1,q}_{\text{loc}}(\Omega) \) with \( u_j \to u \) in \( W^{1,p}(\Omega) \), we can construct a diagonal subsequence \( (v_j) \) by applying the above construction to \( u_j \) that satisfies \( v_j \to u \) in \( W^{1,p}(\Omega) \), \( \mathcal{F}(v_j) \to \mathcal{F}(u) \) and \( v_j \in W^{1,q}(\Omega) \). In other words, Lavrentiev does not occur.

It remains to comment on how to adapt the argument to the non-autonomous setting with \( q < \frac{np}{n-p} \). The only part of the argument that changes concerns \( u_i^s \). We now set for \( t(s), \varepsilon(s) \) to be determined,

\[
u_i^s(x) = t(u - g) * \rho_\varepsilon(s x) + g.
\]

There is \( c_i > 0 \) such that if we ensure \( \varepsilon \leq c_i s \), then \( u_i^s \in W^{1,p}_0(\Omega) \) with \( u_i^s = g \) on \( \partial \Omega_i \cap \partial \Omega_j \). Moreover, \( u_i^s \) is \( W^{1,q} \) in an open neighbourhood of \( \partial \Omega \cap \partial \Omega_i \). We find, using convexity as before,

\[
\int_{\Omega_i} F(x, Du_i^s) \, dx 
\]

\[
\leq t \int_{\Omega_i} F(x, Du * \rho_\varepsilon(sx)) \, dx + (1-t) \int_{\Omega_i} F \left( x, \frac{t}{1-t} (Dg(sx) * \rho_\varepsilon - Dg(x)) \right) \, dx.
\]

The second term can be estimated exactly as before. For the first term, we note that \( |Du| \leq C\|u\|_{W^{1,p}(\Omega)} \varepsilon^{-\frac{n}{p}} \) by Lemma 2. Thus, using (H6) and Lemma 4, since \( \varepsilon \leq c_i s \),

\[
F(x, Du * \rho_\varepsilon(sx)) \, dx \lesssim 1 + F(sx, Du * \rho_\varepsilon(sx)) \, dx.
\]

The estimate now proceeds exactly as before.
4 Construction and convergence of approximations

Throughout this section \( \Omega \subset \mathbb{R}^n \) will be an open, bounded set and \( 1 < p \leq q \). Let \( g \in W^{1,q}(\Omega) \). Given \( u \in g + W^{1,p}_0(\Omega) \), the aim of this section is to construct \( u_\varepsilon \in g + W^{1,p}_0(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega) \) such that \( u_\varepsilon \to u \) in \( W^{1,p}(\Omega) \). The sequence \( \{u_\varepsilon\} \) will be at the heart of our proofs of non-existence of the Lavrentiev phenomenon.

Let \( \{B_i\} \) be a WB-covering of \( \Omega \), where \( B_i \) has radius \( r_i \), and \( \{\psi_i\} \) a partition of unity adapted to \( \{B_i\} \) satisfying the properties of Theorem 2, in particular also (2.7). Without loss of generality, we will always assume that all balls in the WB-covering have radius at most 1. Fix \( u \in g + W^{1,p}_0(\Omega) \). Introduce for \( \varepsilon \in (0,1) \),

\[
  u_\varepsilon = \sum_i u \ast \rho_{\delta_i,\varepsilon} \psi_i
\]

where \( \delta_i = C_0 r_i^N \) for constants \( C_0, N > 0 \) to be made precise at a later stage, but chosen so that \((1 + \delta/2)r_i + \delta_i \leq (1 + \delta)r_i \). This can be ensured if \( N \geq 1 \) and \( C_0 = c_0 \delta \) for a sufficiently small choice of \( c_0 > 0 \).

For later use, we record that

\[
  Du_\varepsilon = \sum_i Dv_\varepsilon \ast \rho_{\delta_i,\varepsilon} \psi_i + \sum_i u_\varepsilon \ast \rho_{\delta_i,\varepsilon} D\psi_i
\]

\[
  = \sum_i (Dv_\varepsilon \ast \rho_{\delta_i,\varepsilon}) \psi_i + \sum_i (v_\varepsilon \ast \rho_{\delta_i,\varepsilon} - v_\varepsilon) D\psi_i = A_1 + A_2,
\]

(4.1)
since \( \sum D\psi_i = 0 \).

We begin by showing that \( u_\varepsilon \) does indeed lie in the correct function space.

**Lemma 7.** For any \( 1 \leq p \leq q \), \( u_\varepsilon \in g + W^{1,p}_0(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega) \).

**Proof.** Consider for \( s > 0 \) the function

\[
  w_s = \sum_i \tilde{w}_s \ast \rho_{\delta_i,\varepsilon} \psi_i
\]

where

\[
  \tilde{w}_s(x) = \begin{cases} 
  0 & \text{if } d(x, \partial \Omega) < s \\
  u & \text{else} \end{cases}
\]

Note that locally \( w_s \) is a finite sum of smooth functions, compactly supported in \( \Omega \). Moreover, due to (2.3), \( w_s = 0 \) in an open neighbourhood of \( \partial \Omega \). In particular, this shows \( w_s \in C^\infty_c(\Omega) \). Thus, to prove the claim it suffices to show that \( w_s \to u_\varepsilon - g \) in \( W^{1,p}_0(\Omega) \) as \( s \to 0 \). Note that for \( s \) sufficiently small, we can guarantee that \( B_i \setminus \Omega_s \neq \emptyset \) implies \( B_i \cap \Omega_s = \emptyset \), due to (2.3). Moreover, note that due to (2.3), if \( B_i \cap \Omega_s \neq \emptyset \), then \( r_i \lesssim s \). Thus, for all sufficiently small choices of \( s \), using Lemma 2 and (2.2), there is \( c > 0 \) such that

\[
  \int_{\Omega} |u_\varepsilon - g - w_s|^q \, dx \leq \sum_i \int_{(1+\delta/2)B_i \cap \Omega_s} |g \ast \rho_{\delta_i,\varepsilon} - g|^q \, dx \\
  \lesssim \sum_i \int_{(1+\delta)B_i \cap \Omega_s} |g|^q \, dx
\]
\[ \leq \|g\|_{W^{1,q}(\Omega_{\varepsilon})} \xrightarrow{s \to 0} 0. \]

We further find, using (4.1), Lemma 2 and (2.2), for all sufficiently small \( s \) and some \( c > 0 \),
\[ \int_{\Omega} |Du_{\varepsilon} - Dw_s|^p \, dx \leq \sum_i \int_{(1+\delta/2)B_i \cap \Omega_{\varepsilon}} |Du \ast \rho_{\delta_i \varepsilon} - Du|^p + |u \ast \rho_{\delta_i \varepsilon} - g|^p |D\psi_i|^p \, dx \]
\[ \lesssim \sum_i \int_{(1+\delta)B_i \cap \Omega_{\varepsilon}} |Du|^p + (\delta_i \varepsilon)^p \rho_i^p ((|g|^p + |Dg|^p) \, dx \]
\[ \lesssim \|u\|_{W^{1,p}(\Omega_{\varepsilon})} \xrightarrow{s \to 0} 0. \]

Next, we turn to the convergence properties of \( \{u_{\varepsilon}\} \) as \( \varepsilon \to 0 \).

**Lemma 8.** Suppose \( 1 < p \leq q \). Then \( u_{\varepsilon} \to u \) in \( W^{1,p}(\Omega) \) as \( \varepsilon \to 0 \).

**Proof.** We note
\[ \|u - u_{\varepsilon}\|_{L^p(\Omega)} \leq \sum_i \|u - u \ast \rho_{\delta_i \varepsilon}\|_{L^p((1+\delta/2)B_i \setminus \Omega_{\varepsilon})} + \|u - g \ast \rho_{\delta_i \varepsilon}\|_{L^p((1+\delta/2)B_i \cap \Omega_{\varepsilon})}. \]

Due to (2.2) and Lemma 2,
\[ \sum_i \|u - u \ast \rho_{\delta_i \varepsilon}\|_{L^p((1+\delta/2)B_i \setminus \Omega_{\varepsilon})} \lesssim \sum_i \delta_i \varepsilon \|u\|_{W^{1,p}((1+\delta)B_i)} \]
\[ \lesssim \|u\|_{W^{1,p}(\Omega)} \xrightarrow{\varepsilon \to 0} 0. \]

Further, using (2.2) and (2.3), and noting that due to (2.3), if \( B_i \cap \Omega_{\varepsilon} \neq \emptyset, r_i \lesssim \varepsilon \),
\[ \sum_i \|u - g \ast \rho_{\delta_i \varepsilon}\|_{L^p((1+\delta/2)B_i \cap \Omega_{\varepsilon})} \lesssim \|u\|_{L^p(\Omega_{\varepsilon})} + \|g\|_{L^p(\Omega_{\varepsilon})} \xrightarrow{\varepsilon \to 0} 0, \]
for some \( c > 0 \). The convergence holds since \( u, g \in W^{1,p}(\Omega) \). In particular, we note that \( u_{\varepsilon} \to u \in L^p(\Omega) \) and hence almost everywhere.

We now turn to establishing gradient estimates. We begin by estimating \( A_1 \) (recall the definition of \( A_1 \) in (4.1)). Using (2.2) and Lemma 2, we obtain
\[ \|A_1\|_{L^p(\Omega)} \leq \sum_i \|Du_{\varepsilon} \ast \rho_{\delta_i \varepsilon}\|_{L^p((1+\delta/2)B_i)} \lesssim \sum_i \|Du_{\varepsilon}\|_{L^p((1+\delta)B_i)} \]
\[ \lesssim \|Dg\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}. \]

Further, using Lemma 2 and (2.2),
\[ \|A_2\|_{L^p(\Omega)} \leq \sum_i \|(u_{\varepsilon} \ast \rho_{\delta_i \varepsilon} - u_{\varepsilon}) \, D\psi_i\|_{L^p((1+\delta/2)B_i)} \]
\[ \lesssim \sum_i r_i^{-1} \|(u_{\varepsilon} \ast \rho_{\delta_i \varepsilon} - u_{\varepsilon})\|_{L^p((1+\delta)B_i)} \]
\[ \lesssim \sum_i r_i^{-1} \delta_i \varepsilon \|u_{\varepsilon}\|_{W^{1,p}((1+\delta)B_i)} \]
\[ \lesssim \varepsilon \left( \|u\|_{W^{1,p}(\Omega)} + \|g\|_{W^{1,p}(\Omega)} \right). \]

To obtain the last line, we also used that \( N \geq \frac{1}{\alpha} \). Thus, using (4.1) and a variant of the dominated convergence theorem, we deduce that \( u_{\varepsilon} \to u \) in \( W^{1,p}(\Omega) \). \( \square \)
5 Convergence of energies

In this section, we will show that in each of the scenarios of Theorem 1, we have \( \mathcal{F}(u_\varepsilon) \to \mathcal{F}(u) \). We establish this by showing \( \int_\Omega F(x, A_1) \, dx \to \int_\Omega F(x, Du) \, dx \), as well as \( \int_\Omega F(x, Du) - F(x, A_1) \, dx \to 0 \) as \( \varepsilon \to 0 \).

Throughout this section, we use the notation of Theorem 1 and assume that \( \Omega \) is an open, bounded set, \( 1 < p \leq q \) and \( g \in W^{1,q}(\Omega) \).

Lemma 4 is the key tool in proving convergence of \( \int_\Omega F(x, A_1) \, dx \).

**Lemma 9.** Let \( 1 \leq p \leq q < \frac{(n+\alpha)p}{n} \). Suppose \( F \) satisfies (H1), (H2), (H5) and (H6). Then

\[
\int_\Omega F(x, A_1) \to \int_\Omega F(x, Du).
\]

**Proof.** Note that, due to convexity of \( F \) and Lemma 4,

\[
\int_\Omega F(x, A_1) \leq \sum_i \int_{(1+\delta/2)B_i} F(x, Du_\varepsilon \ast \rho_{\delta_1\varepsilon}) \psi_i \, dx \\
\leq 1 + \sum_i \int_{(1+\delta/2)B_i} F(x, Du_\varepsilon) \ast \rho_{\delta_1\varepsilon} \, dx \\
= 1 + \sum_i \int_{(1+\delta)B_i \setminus \Omega_\varepsilon} F(x, Du) \ast \rho_{\delta_1\varepsilon} \, dx \\
+ \sum_i \int_{(1+\delta)B_i \cap \Omega_\varepsilon} F(x, Dg) \ast \rho_{\delta_1\varepsilon} \, dx \\
\leq 1 + \int_\Omega F(x, Du) \, dx.
\]

To obtain the last line, we used (2.2) and noted that using (2.3) and (2.2) for some \( c > 0 \),

\[
\sum_i \int_{(1+\delta)B_i \cap \Omega_\varepsilon} F(Dg) \ast \rho_{\delta_1\varepsilon} \, dx \lesssim \int_{\Omega_\varepsilon} F(Dg) \, dx \xrightarrow{\varepsilon \to 0} 0,
\]

since \( F(Dg) \in L^1(\Omega) \) due to (H2). By a variant of the dominated convergence theorem, the conclusion follows.

**Remark 10.** Note that Lemma 9 applies without restriction on \( p \leq q \) whenever \( F \) is autonomous, that is \( F \equiv F(z) \).

We now turn towards proving convergence of \( \{u_\varepsilon\} \) in the non-autonomous case.

**Lemma 11.** Let \( 1 < p \leq q < \frac{(n+\alpha)p}{n} \). Suppose \( F \) satisfies (H1), (H2), (H5) and (H6). Then

\[
\int_\Omega F(x, Du_\varepsilon) \to \int_\Omega F(x, Du).
\]

**Proof.** As a consequence of (1.2), we have

\[
\int_\Omega F(x, Du_\varepsilon) \, dx \leq \int_\Omega F(x, A_1) + \Lambda |A_2| (1 + |A_1| + |A_2|)^{q-1} \, dx.
\]
Due to Lemma 9, we only need to consider the second term. We first consider, using Lemma 2 and (2.2),

\[
\int_{\Omega} |A_2|^q \, dx \leq \sum_i \int_{(1+\delta/2)B_i} |(v_\varepsilon * \rho_{\delta_i \varepsilon} - v_\varepsilon)D\psi_i|^q \, dx \\
\leq \sum_i r_i^{-q}(\varepsilon \delta_i)^{1-n\left(\frac{1}{p} - \frac{1}{q}\right)} \|v_\varepsilon\|_{W^{1,p}(1+\delta)B_i}^p \\
\leq \varepsilon^{\frac{n}{p}} \left(\|u\|_{W^{1,p}(\Omega)} + \|g\|_{W^{1,p}(\Omega)}\right)^p \varepsilon \to 0, 0,
\]
as long as we have

\[
1 - n\left(\frac{1}{p} - \frac{1}{q}\right) > 0 \quad \iff \quad q < \frac{np}{n-p}
\]
and ensure \(N \left(1 - n\left(\frac{1}{p} - \frac{1}{q}\right)\right) \geq 1\). Further, by Hölder’s inequality, we deduce that also \(\int_{\Omega} |A_2| \, dx \to 0\). Thus, it remains to consider \(\int_{\Omega} |A_2||A_1|^{q-1} \, dx\). Here we use Lemma 2, (2.4) and (2.2) to find

\[
\int_{\Omega} |A_1||A_2|^{q-1} \, dx \\
\leq \sum_i \sum_{\{j: (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} \int_{(1+\delta/2)B_i} |(v_\varepsilon * \rho_{\delta_i \varepsilon} - v_\varepsilon)D\psi_i||Dv_\varepsilon * \rho_{\delta_j \varepsilon} \psi_j|^{q-1} \, dx \\
\leq \sum_i \sum_{\{j: (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} r_i^{-1}\|v_\varepsilon * \rho_{\delta_i \varepsilon} - v_\varepsilon\|_{L^q((1+\delta/2)B_i)} \|Dv_\varepsilon * \rho_{\delta_j \varepsilon}\|_{L^q((1+\delta/2)B_j)}^{q-1} \\
\leq \sum_i \sum_{\{j: (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} \varepsilon \delta_i^{1-n\left(\frac{1}{p} - \frac{1}{q}\right)} \|v_\varepsilon\|_{W^{1,p}(1+\delta)B_i} \\
\times (\varepsilon \delta_i)^{-\frac{n(q-1)(q-p)}{q}} \|v_\varepsilon\|_{W^{1,p}(1+\delta)B_j}^{(q-1)n} \\
\leq \varepsilon^{n+1-\frac{2nq}{p}} \left(\|u\|_{W^{1,p}(\Omega)} + \|g\|_{W^{1,p}(\Omega)}\right)^p \varepsilon \to 0, 0,
\]
as long as we have

\[
q > \frac{np}{n} \quad \iff \quad q < \frac{(n+1)p}{n}
\]
and ensure \(N(n+1-\frac{2nq}{p}) > 1\). \(\square\)

In the case of autonomous functionals, we can improve on the previous result, by choosing a partition of unity that is adapted to \(u\). Let \(\{B_i\}\) be a WB-covering of \(\Omega\), where \(B_i\) has radius \(r_i\), \(\{\phi_i\}\) a family of functions satisfying the assumptions of Theorem 2 and \(\{\psi_i\}\) the corresponding partition of unity given by Theorem 2. Note that in particular (2.7) is not necessarily satisfied. We denote by \(\{\tilde{u}_\varepsilon\}\) and \(\{\tilde{v}_\varepsilon\}\) the families of functions that are the outcome of the construction of Section 4 with respect to this partition of unity. The analogous decomposition to (4.1), we denote \(D\tilde{u}_\varepsilon = \tilde{A}_1 + \tilde{A}_2\).
Lemma 12. Let $1 < p \leq q < \min \left( p + 1, \frac{np}{n-1} \right)$. Assume $F \equiv F(z)$ satisfies (H1) and (H2). Then there is a family \{\phi_i\} satisfying the assumptions of Theorem 2, such that

\[
\int_{\Omega} F(D\tilde{u}_\varepsilon) \rightarrow \int_{\Omega} F(Du).
\]  

(5.1)

Moreover, $\tilde{u}_\varepsilon \in g + W^{1,q}_0(\Omega)$ and $\tilde{u}_\varepsilon \rightarrow u$ in $W^{1,p}(\Omega)$.

Proof. Note that the proof of Lemma 9 did not use (2.7), so that Lemma 9 applies to $\tilde{u}_\varepsilon$. Thus as in Lemma 11, to see (5.1), it suffices to show

\[
\int_{\Omega} |\tilde{A}_2| \left( 1 + |\tilde{A}_1|^q - 1 + |\tilde{A}_2|^q - 1 \right) dx \xrightarrow{\varepsilon \to 0} 0.
\]

Let $t \in (0,1)$ be fixed at a later stage and write $s = \frac{p}{p+1-q} \in (1, \infty)$, since $p \leq q < p+1$.

Then we estimate, using Hölder’s inequality,

\[
\int_{\Omega} |\tilde{A}_2| \left( 1 + |\tilde{A}_1|^q - 1 + |\tilde{A}_2|^q - 1 \right)
\leq \left( 1 + \|\tilde{A}_1\|_{L^p(\Omega)}^{q-1} + \|\tilde{A}_2\|_{L^p(\Omega)}^{q-1} \right) \left( \int_{\Omega} \sum_i \tilde{v}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{v}_\varepsilon |D\psi_i|^q dx \right)^{\frac{1}{q}} = I \times II.
\]

Using Lemma 3, (2.2) and (2.4), we may choose $\{\phi_j\}$ so that

\[
II \lesssim \sum_j \left( \int_{(1+\delta/2)B_j \cap (1+\delta/2)B_j \neq \emptyset} |\tilde{v}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{v}_\varepsilon |^q |D\phi_j|^q dx \right)^{\frac{1}{q}}
\]

\[
\lesssim \sum_j \frac{1}{r_j} \left( \int_{r_j} \left( 1 + \delta/2 \right) \sum_{\{i: (1+\delta/2)B_i \cap (1+\delta/2)B_j \neq \emptyset\}} \|\tilde{v}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{v}_\varepsilon \|_{L^q(\partial B_{j,r})}^{\frac{q}{r_j}} \right)^{\frac{1}{q}}
\]

\[
= II_1.
\]

Here $B_{j,r}$ denotes the ball of radius $r$ with the same center as $B_j$.

Due to (2.4), we can ensure that whenever $(1 + \delta/2)B_i \cap (1 + \delta/2)B_j \neq \emptyset$, then $r_j - \delta_i > r_j/2$. Note that for $r \in (r_j, (1 + \delta/2)r_j)$, $(1 + n - 1) \frac{n-1-p}{n-1}$ needs to be replaced by $\tau$ for a sufficiently large choice of $\tau$.

\[
\|\tilde{u}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{u}_\varepsilon \|_{L^{\frac{n-1-p}{n-1}}(\partial B_{j,r})} \lesssim \|\tilde{u}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{u}_\varepsilon \|_{W^{1,p}(\partial B_{j,r})}
\]

and

\[
\|\tilde{u}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{u}_\varepsilon \|_{L^p(\partial B_{j,r})} \lesssim \delta_i \varepsilon \|\tilde{u}_\varepsilon \|_{W^{1,p}(\partial B_{j,r})},
\]

where $A_{j,\delta_i \varepsilon}$ denotes the annulus of width $2\delta_i \varepsilon$ centered around the sphere $\partial B_{j,r}$. Thus, by interpolation

\[
\|\tilde{u}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{u}_\varepsilon \|_{L^{\frac{n-p}{n-1}}} \lesssim (\delta_i \varepsilon)^\theta \left( \|\tilde{u}_\varepsilon \|_{W^{1,p}(A_{j,\delta_i \varepsilon})} + \|\tilde{u}_\varepsilon \ast \rho_{\delta_i \varepsilon} - \tilde{u}_\varepsilon \|_{W^{1,p}(\partial B_{j,r})} \right)
\]
where
\[
\frac{p + 1 - q}{p} = \frac{\theta}{p} + \frac{(1 - \theta)(n - 1 - p)}{(n - 1)p} \quad \Leftrightarrow \quad \theta = 1 + \frac{(n - 1)p - q}{p}.
\]
This is valid since
\[
\theta > 0 \quad \Leftrightarrow \quad q < \frac{np}{n - 1}.
\]
In particular, we deduce using (2.4) and with the choice \( t = p + 1 - q \in (0, 1) \) (so that \( ts = p \)),
\[
II_1 \leq \sum_j \{i: (1 + \delta/2)B_i \cap (1 + \delta/2)B_j \neq \emptyset\}
\times \int_{r_j} \|\tilde{v}_\varepsilon\|_{W^{1,p}(A_j, \delta_i, \varepsilon)}^p + \|\tilde{v}_\varepsilon \ast \rho_{\delta_i, \varepsilon} - v_\varepsilon\|_{W^{1,p}(\partial B_r)}^p \, dr
\lesssim \varepsilon^\theta \sum_j \frac{p + 1 - q}{p + 1 - q + M\theta} \|\tilde{v}_\varepsilon\|_{W^{1,p}(A_j, \delta_i, \varepsilon)}^p
\lesssim \varepsilon^\theta \|\tilde{v}_\varepsilon\|_{W^{1,p}(\Omega)},
\tag{5.2}
\]
if we choose \( N \) large enough that \(-\frac{p + 1}{p + 1 - q} + N\theta > 0\).

We now turn to estimating \( I \) and note as a consequence of our argument above, since
\[
\frac{p}{p + 1 - q} \geq p,
\]
\[
\|\tilde{A}_2\|_{L^p(\Omega)} \lesssim \|\tilde{A}_2\|_{L^p(\Omega)} \xrightarrow{\varepsilon \to 0} 0.
\]
Further, noting that our estimate on \( A_1 \) in the proof of Lemma 8 does not use (2.7), we obtain that
\[
\|\tilde{A}_1\|_{L^p(\Omega)} \lesssim \|Du\|_{L^1, p(\Omega)} + \|Dg\|_{L^p(\Omega)}.
\]
Thus, combining (5.2) with these observations, (5.1) follows.

Further, using the argument of Lemma 8, we deduce that \( \tilde{u}_\varepsilon \to u \) in \( W^{1,p}(\Omega) \). Thus, it remains to show that \( \tilde{u}_\varepsilon \in g + W^{1,q}_0(\Omega) \). Arguing as in Lemma 7, it suffices to estimate
\[
\sum_i \int_{(1 + \delta/2)B_i \cap \Omega} |\tilde{v}_\varepsilon \ast \rho_{\delta_i, \varepsilon} - \tilde{v}_\varepsilon|^q |D\psi_i|^q dx \xrightarrow{\varepsilon \to 0} 0.
\]
However, noting that \( q \leq \frac{p}{p + 1 - q} \), this again follows from our estimate on \( II \) using Hölder’s inequality.

We next consider controlled growth-conditions where we need to restrict to \( 2 \leq p \).
We prove the following:

**Lemma 13.** Let \( 2 \leq p \leq q < \min\left( p + 2, p \left( 1 + \frac{2}{n - 1} \right) \right) \). Assume \( F \equiv F(z) \) satisfies (H1) and (H3). Suppose \( u \in g + W^{1,p}_0(\Omega) \) solves (1.1). Then there is a family \( \{\phi_i\} \) satisfying the assumptions of Theorem 2, such that
\[
\int_\Omega F(D\tilde{u}_\varepsilon) \to \int_\Omega F(Du).
\]
Moreover, \( \tilde{u}_\varepsilon \in g + W^{1,q}_0(\Omega) \) and \( \tilde{u}_\varepsilon \to u \) in \( W^{1,p}(\Omega) \).
Proof. Using the same notation as in Lemma 12, we find
\[
\int_\Omega F(D\tilde{u}_\varepsilon) - \int_\Omega F(\tilde{A}_1) - \int_\Omega \partial_2 F(\tilde{A}_1) \cdot (D\tilde{u}_\varepsilon - \tilde{A}_1) \\
\lesssim \int_\Omega |\tilde{A}_2|^2(1 + |\tilde{A}_1| + |\tilde{A}_2|)^{q-2}.
\]
We first deal with the term on the right-hand side, which we’ll prove converges to 0 as \(\varepsilon \to 0\). Using Hölder, we find
\[
\int_\Omega |\tilde{A}_2|^2\left(1 + |\tilde{A}_1|^{q-2} + |\tilde{A}_2|^{q-2}\right) \, dx \\
\lesssim \left(1 + \|\tilde{A}_1\|_{L^p(\Omega)} + \|\tilde{A}_2\|_{L^p(\Omega)}\right)^{q-2}\left(\int_\Omega \left(\sum_i |\tilde{v}_\varepsilon \ast \rho_{\delta, i} \varepsilon - \tilde{v}_\varepsilon|\right)^{\frac{2p}{p + 2 - q}}\right)^{\frac{p + 2 - q}{p}}.
\]
We can now proceed exactly as in Lemma 12 as long as \(N\) is sufficiently large and \(\theta > 0\), where
\[
\frac{p + 2 - q}{2p} = \frac{\theta}{p} + \frac{(1 - \theta)(n - 1 - p)}{(n - 1)p} \iff \theta = \frac{n + 1}{2} - \frac{(n - 1)q}{2p},
\]
noting that
\[
\theta > 0 \iff q < p\left(1 + \frac{2}{n - 1}\right).
\]
Moreover, arguing as in Lemma 12 it also follows that \(\tilde{u}_\varepsilon \in g + W^{1,q}_0(\Omega)\) and \(\tilde{u}_\varepsilon \to u\) in \(W^{1,p}(\Omega)\).

Thus, it remains to establish that
\[
\int_\Omega \partial_2 F(\tilde{A}_1) \cdot (D\tilde{u}_\varepsilon - \tilde{A}_1) \to 0.
\]
Note that since \(q \geq \frac{p}{p + 2 - q}\), our argument above showed that \(D\tilde{u}_\varepsilon - \tilde{A}_1 = \tilde{A}_2 \to 0\) in \(L^q(\Omega)\). In particular, using Lemma 5, it suffices to show \(\partial_2 F(\tilde{A}_1) \to \partial_2 F(Du)\) in \(L^{q'}(\Omega)\), since then by Hölder’s inequality
\[
\int_\Omega \partial_2 F(\tilde{A}_1) \cdot (D\tilde{u}_\varepsilon - \tilde{A}_1) \leq \|\partial_2 F(\tilde{A}_1)\|_{L^{q'}(\Omega)} \|\tilde{A}_2\|_{L^q(\Omega)} \\
\leq \left(1 + \|\partial_2 F(Du)\|_{L^{q'}(\Omega)}\right) \|\tilde{A}_2\|_{L^q(\Omega)} \to 0.
\]
Thus, we turn to proving \(\partial_2 F(\tilde{A}_1) \to \partial_2 F(Du)\) in \(L^{q'}(\Omega)\). Using Fenchel’s inequality, we find for any \(s > 1\),
\[
F^*(\partial_2 F(\tilde{A}_1)) = \langle \partial_2 F(\tilde{A}_1), \tilde{A}_1 \rangle - F(\tilde{A}_1) \leq \frac{1}{s} F^*(\partial_2 F(\tilde{A}_1)) + \frac{1}{s} F(s\tilde{A}_1) - F(\tilde{A}_1).
\]
Re-arranging, we find, using also that as a consequence of (H1), \(F(z) \geq -c\) for some \(c > 0\),
\[
F^*(\partial_2 F(\tilde{A}_1)) \leq c + \frac{1}{s - 1} F(s\tilde{A}_1).
\]
Re-calling the definition of $\tilde{A}_1$, using Jensen’s inequality, (2.2) and employing Lemma 2, we obtain

$$\int_{\Omega} F(s\tilde{A}_1) \, dx \leq \sum_i \int_{(1+\delta/2)B_i} F(sD\tilde{v}_\varepsilon) \ast \rho_{\delta \psi_i} \, dx$$

$$\lesssim 1 + \int_{\Omega} F(sDu) \, dx + \int_{\Omega} F(sDg) \, dx$$

$$\lesssim 1 + \int_{\Omega} F(Du) + F(Dg) \, dx < \infty.$$

To obtain the last line, we used the doubling property of $F$. Since $\tilde{A}_1 \to Du$ almost everywhere by Lemma 8, this concludes the proof, by an application of a version of dominated convergence.

We next turn to controlled-duality growth.

**Lemma 14.** Let $2 \leq p \leq q < \frac{np}{n-p}$. Assume $F \equiv F(z)$ satisfies (H1) and (H4). Suppose $u \in g + \mathcal{W}^{1,p}_0(\Omega)$ solves (1.1). Then

$$\int_{\Omega} F(Du_\varepsilon) \to \int_{\Omega} F(Du).$$

**Proof.** Arguing as in Lemma 14, it suffices to estimate

$$\int_{\Omega} |A_2|^2(1 + |\partial_z F(Du)| + |\partial_z F(A_2)|)^{\frac{q-2}{q-1}}, \quad (5.3)$$

and

$$\int_{\Omega} \partial_z F(A_1) \cdot A_2 \, dx.$$

On the one hand,

$$\int_{\Omega} |A_2|^2(1 + |\partial_z F(A_2)|)^{\frac{q-2}{q-1}} \lesssim \int_{\Omega} |A_2|^2(1 + |A_2|)^{q-2},$$

since a consequence of the convexity of $F$ and (H1) is that $|\partial_z F(z)| \lesssim 1 + |z|^{q-1}$. However, as in Lemma 11, we note that $\int_{\Omega} |A_2|^2(1 + |A_2|)^{q-2} \to 0$ as $\varepsilon \to 0$ since $2 \leq q < \frac{np}{n-p}$. On the other hand, due to Lemma 5, we may use Hölder’s inequality to estimate

$$\int_{\Omega} |A_2|^2|\partial_z F(Du)|^{\frac{q-2}{q-1}} \leq \|A_2\|_{L^q(\Omega)}^2 \|\partial_z F(Du)\|_{L^{q'}(\Omega)}^{q-2}.$$

Noting that due to Lemma 5, $\partial_z F(Du) \in L^{q'}(\Omega)$, using again that $\|A_2\|_{L^q(\Omega)} \to 0$, this concludes the proof of (5.3).

On the other hand, (H4) implies that there exists $c > 0$ such that for any $x, y \in \Omega$ and $\tau \in [0, 1],

$$F^*(\tau x + (1 - \tau y) \leq \tau F^*(x) + (1 - \tau)F^*(y) - \tau(1 - \tau)c(1 + |x| + |y|)^{q'-2}|x - y|^2.$$
For a proof of this fact, see e.g. [11] or [25]. Now, using Fenchel’s inequality and the
fact that $|z|^q - 1 \lesssim F^*(z) \lesssim 1 + |z|^p$, 

$$F^*(\partial_z F(A_1)) \lesssim 2F^*\left(\frac{1}{2} \partial_z F(A_1)\right) + 2F(A_1) \lesssim 1 + F^*(\partial_z F(A_1)) - \frac{c}{4} (1 + |\partial_z F(A_1)|)^q - 2|\partial_z F(A_1)|^2 + 2F(A_1).$$

Re-arranging and integrating, we deduce

$$\int_{\Omega} (1 + |\partial_z F(A_1)|)^{q-2}|\partial_z F(A_1)|^2 \lesssim 1 + \int_{\Omega} F(A_1) \, dx \lesssim 1 + \int_{\Omega} F(Du) \, dx.$$

To obtain the last inequality, we used Lemma 9. We now estimate for any $\tau > 0,$

$$\int_{\Omega} \partial_z F(A_1) \cdot A_2 \, dx \lesssim \tau \int_{\Omega} (1 + |\partial_z F(A_1)|)^{q-2}|\partial_z F(A_1)|^2 \, dx + C(\tau) \int_{\Omega} (1 + |A_2|)^{q-2}|A_2|^2 \, dx \lesssim \tau \left(1 + \int_{\Omega} F(Du) \, dx\right) + c(\tau) \left(\int_{\Omega} |A_2|^q \, dx + \left(\int_{\Omega} |A_2|^q \, dx\right)^{\frac{2}{q}}\right).$$

Since $A_2 \to 0$ in $L^q(\Omega),$ we deduce that

$$\int_{\Omega} \partial_z F(A_1) \cdot A_2 \, dx \to 0,$$

which concludes the proof.

Finally, we turn to the case where we make the a-priori assumption that $u \in L^\infty(\Omega)$.

**Lemma 15.** Let $1 \leq p \leq q < p + 1$. Assume $F \equiv F(z)$ satisfies (H1) and (H2). Further suppose $u \in L^\infty(\Omega)$. Then

$$\int_{\Omega} F(Du_\varepsilon) \to \int_{\Omega} F(Du).$$

If $F \equiv F(z)$ satisfies (H1), (H3), (H6) and $u$ solves (1.1), the conclusion holds under the restriction $2 \leq p \leq q < p + 2$.

**Proof.** As in Lemma 11 it suffices to deal with $\int_{\Omega} |A_2||A_1|^{q-1}$. Using (2), Hölder’s inequality (note $\frac{2}{p+q-1} \leq q$), (2.2) and (2.4) we find

$$\int_{\Omega} |A_2||A_1|^{q-1} \lesssim \|Du_\varepsilon\|^{q-1}_{L^p(\Omega)} \sum_{j \in I : (1+\delta/2)B_j \cap (1+\delta/2)B_j \neq \emptyset} \left(\int_{\Omega} |g \ast \rho_{\delta_i,\varepsilon} - g|^{\frac{p}{p+q-1}} |D\psi_i|^{\frac{p}{p+q-1}} \right)^{\frac{p+q-2}{p}} + \int_{\Omega \setminus \Omega_\varepsilon} |u \ast \rho_{\delta_i,\varepsilon} - u|^{\frac{p}{p+q-1}} |D\psi_i|^{\frac{p}{p+q-1}} \right)^{\frac{p+q-2}{p}}$$

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Consider note that due to (H1), in particular, we conclude by a variant of the dominated convergence theorem that

\[ \left\| \int_{B_i \cap \mathbb{R}^d} u \right\|_{W^{1,q}((1+\delta)B_i \cap \Omega_k)} \]

\[ + \left( \delta \varepsilon \right)^{-p} \left( u \right)_{p-n} \left( 1-\frac{1+n}{q} \right) \]

\[ \left\| u \right\|_{L^\infty((1+\delta)B_i \setminus \Omega_k)} \]

\[ \left\| u \right\|_{W^{1,q}((1+\delta)B_i \setminus \Omega_k)} \]

\[ \leq \delta \left( \frac{\varepsilon}{\varepsilon+1} \right) \left( u \right)_{p-n} \left( 1-\frac{1+n}{q} \right) \]

\[ \left\| u \right\|_{L^\infty(\Omega)} \]

\[ \left\| u \right\|_{W^{1,q}(\Omega)} \]

\[ \left\| u \right\|_{W^{1,q}(\Omega)} \]

provided \( N \) is sufficiently large that \( -1 + \frac{n}{p+1} \geq 0 \).

The proof under (H3) is nearly identical. First, it suffices to consider \( \int_{\Omega} |A_2|^2 |A_1|^q \) as in Lemma 13. This can be estimated similar to the above argument. \( \square \)

**Remark 16.** We note that we only use the fact that \( u \in L^\infty(\Omega) \) in order to establish the interpolative bound

\[ \left\| u_\varepsilon - u \right\|_{L^p(B)} \leq \varepsilon \left( \frac{\varepsilon}{\varepsilon+1} \right) \left( u \right)_{p-n} \left( 1-\frac{1+n}{q} \right) \]

Thus, \( L^\infty \) may be replaced by \( BMO(\Omega) \) in Lemma 15.

The assumption that \( u \in L^\infty(\Omega) \) is especially relevant in the scalar case \( m = 1 \). In this case, we may assume without loss of generality that \( u \in L^\infty(\Omega) \). This follows using an observation made in [6].

**Lemma 17.** Let \( 1 < p \) and \( m = 1 \). Assume \( F \) satisfies (H1) and (H2). Then there exists \( \{ u_k \} \subset g + \left( W^{1,p}_0(\Omega) \right) \) such that \( u_k \to u \in W^{1,p}(\Omega) \) and \( \mathcal{F}(u_k) \to \mathcal{F}(u) \).

**Proof.** Consider

\[ T_k u = \begin{cases} 
 u & \text{if } |u - g| \leq k \\
 g + \frac{u - g}{|u - g|} k & \text{else.}
\end{cases} \]

Then \( T_k u \to u \) in \( W^{1,p}(\Omega) \). Moreover,

\[ \int_{\Omega} F(x, DT_k u) = \int_{\Omega} F(x, Dg + D(u - g) 1_{|u-g| \leq k}). \]

\[ \int_{\Omega} F(x, Dg + D(u - g) 1_{|u-g| \leq k}) = \int_{\Omega \cap \{|u-g| \leq k\}} F(x, Du) + \int_{\Omega \cap \{|u-g| > k\}} F(x, Dg) \]

\[ \leq \int_{\Omega} F(x, Du) + \int_{\Omega} |Dg|^q \ dx < \infty. \]

Note that due to (H1), \( F(x, \cdot) \geq -C_0 \) for some \( C_0 \geq 0 \) and almost every \( x \in \Omega \). In particular, we conclude by a variant of the dominated convergence theorem that \( \int_{\Omega} F(x, DT_k u) \to \int_{\Omega} F(x, Du) \). This concludes the proof. \( \square \)

Having Lemma 17 at hand, we can apply Lemma 15, Remark 16 and apply a diagonal subsequence argument to obtain the following statement:

**Lemma 18.** Suppose \( u: \Omega \to \mathbb{R} \). Let \( 1 < p \leq q < p + 1 \). Suppose \( F \equiv F(z) \) satisfies (H1) and (H2) and the outcome of Lemma 9 holds. Then there exists \( u_k \in g + W^{1,q}_0(\Omega) \) such that \( u_k \to u \) in \( W^{1,p}(\Omega) \) and \( \mathcal{F}(u_k) \to \mathcal{F}(u) \) as \( k \to \infty \).

If \( F \equiv F(z) \) satisfies (H1) and (H3) it suffices to assume \( 2 \leq p \leq q < p + 2 \).

**Remark 19.** Lemma 17 and hence also the cases (iv) and (viii) in Theorem 1 easily generalise to integrands with Uhlenbeck structure \( F \equiv F(|\cdot|) \), see also [6, Section 7].
6 Proof of Theorem 1

We have now collected all ingredients we need in order to prove Theorem 1.

Proof of Theorem 1. Due to the definition in order to prove $\mathcal{F} = \overline{\mathcal{F}}$ in case (i)–(iv), it suffices for a fixed $u \in g + W^{1,p}_0(\Omega)$ to exhibit a sequence $\{u_\varepsilon\} \subset g + W^{1,q}_0(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathcal{F}(u_\varepsilon) \to \mathcal{F}(u)$. We claim that in the cases (i), (iii) and (iv) the sequence $\{u_\varepsilon\}$ constructed in Section 4 has these properties. Due to Lemma 7 and Lemma 8 the regularity and convergence properties regarding $\{u_\varepsilon\}$ are satisfied. The fact that $\mathcal{F}(u_\varepsilon) \to \mathcal{F}(u)$ follows from Lemma 11 and Lemma 15 in the case (i) and (iii), respectively. In the case (iv) it follows from combining Lemma 17 with Lemma 15. The remaining case (ii) follows from Lemma 12 using the sequence $\tilde{u}_\varepsilon$ constructed there.

In order to prove that the Lavrentiev phenomenon does not occur it suffices to prove that for $u \in g + W^{1,p}_0(\Omega)$ solving (1.1), it holds that $\mathcal{F}(u) = \overline{\mathcal{F}}(u)$. Thus it suffices to exhibit for such $u$ a sequence $\{u_\varepsilon\} \subset g + W^{1,q}_0(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $\mathcal{F}(u_\varepsilon) \to \mathcal{F}(u)$. We claim again that the sequence $\{u_\varepsilon\}$ constructed in Section 4 has these properties in the cases (vi)–(vii). As before the regularity and convergence properties regarding $\{u_\varepsilon\}$ are satisfied. The fact that $\mathcal{F}(u_\varepsilon) \to \mathcal{F}(u)$ follows from Lemma 14 in the case (vi). In the case (vii) it follows from Lemma 15, while the case (viii) follows from combining Lemma 17 and Lemma 15. The only remaining case is (v), which follows from Lemma 13.

Thus we have proven all cases in Theorem 1. □
References

[1] E. Acerbi, G. Bouchitté, and I. Fonseca, Relaxation of convex functionals: The gap problem, Ann. l’Institut Henri Poincare Anal. Non Lineare, 20 (2003), pp. 359–390.

[2] E. Acerbi and G. Dal Maso, New lower semicontinuity results for polyconvex integrals, Calc. Var. Partial Differ. Equ., 2 (1994), pp. 329–371.

[3] M. Borowski, I. Chlebicka, B. Miasojedow, Absence of Lavrentiev’s gap for anisotropic functionals, arXiv Prepr. arXiv2210:15217, (2022).

[4] A. Balci, L. Diening, and M. Surnachev, New Examples on Lavrentiev Gap Using Fractals, Calc. Var. Partial Differ. Equations2, 59 (2020).

[5] A. Balci and M. Surnachev, Lavrentiev gap for some classes of generalized Orlicz functions, Nonlinear Anal., 207 (2021), p. 112329.

[6] M. Bulíček, P. Gwiazda, and J. Skrzeczkowski, On a Range of Exponents for Absence of Lavrentiev Phenomenon for Double Phase Functionals, Arch. Ration. Mech. Anal., 246 (2022), pp. 209–240.

[7] G. Buttazo and M. Belloni, A survey of old and recent results about the gap phenomenon in the Calculus of Variations, Math. Appl., 331 (1995), pp. 1–27.

[8] G. Buttazo and V. Mizel, Interpretation of the Lavrentiev phenomenon by relaxation, J. Funct. Anal., 2 (1992), pp. 434–460.

[9] M. Carozza, J. Kristensen, and A. Passarelli di Napoli, Higher differentiability of minimizers of convex variational integrals, Ann. l’Institut Henri Poincare Anal. Non Lineare, 28 (2011), pp. 395–411.

[10] ———, Regularity of minimisers of autonomous convex variational integrals, Ann. della Scu. Norm. Sup. di Pisa, 13 (2013).

[11] C. de Filippis, L. Koch, and J. Kristensen, Regularity in relaxed convex problems, in Prep., (2022).

[12] C. de Filippis and G. Mingione, Interpolative gap bounds for nonautonomous integrals, Anal. Math. Phys., 11 (2021).

[13] F. De Filippis and F. Leonetti, No Lavrentiev gap for some double phase integrals, Adv. Calc. Var., (2022).

[14] B. Dyda and M. Klaczyk, On density of compactly supported smooth functions in fractional Sobolev spaces, Ann. di Mat. Pura Appl., 201 (2022), pp. 1855–1867.

[15] A. Esposito, F. Leonetti, and P. Petricca, Absence of Lavrentiev gap for non-autonomous functionals with (p,q)-growth, Adv. Nonlinear Anal., 8 (2019), pp. 73–78.

[16] L. Esposito, F. Leonetti, and G. Mingione, Sharp regularity for functionals with (p,q) growth, J. Differ. Equations, 204 (2004), pp. 5–55.
[17] L. Evans, *Partial Differential Equations*, American Mathematical Society, 2nd ed., 2015.

[18] I. Fonseca and J. Malý, *Relaxation of multiple integrals below the growth exponent for the energy density*, Ann. l'Institut Henri Poincare Anal. Non Lineare, 14 (1997), pp. 309–338.

[19] ———, *From Jacobian to Hessian: distributional form and relaxation*, Riv. Mat. Univ. Parma, 4 (2005), pp. 45–74.

[20] M. Foss, *On LaVrentiev’s phenomenon*, PhD thesis, Carnegie Mellon University, 2001.

[21] ———, *The LaVrentiev gaph phenomenon in nonlinear elasticity*, Arch. Ration. Mech. Anal., 167 (2003), pp. 336–365.

[22] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, 2003.

[23] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 22, Society for Industrial and Applied Mathematics, University City, Philadelphia, 1992.

[24] P. Harjulehto, P. Hästö, and A. Karppinen, *Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions*, Nonlinear Anal., 177 (2018), pp. 543–552.

[25] J. B. Hiriart-Urruty, and C. Lemar, *Convex Analysis and Minimization Algorithms II*, Springer-Verlag Berlin Heidelberg, Berlin, 1993.

[26] J. Hirsch and M. Schäffner, *Growth conditions and regularity, an optimal local boundedness result*, Commun. Contemp. Math., 23 (2020), p. 2050029.

[27] M. Hong, *Some remarks on the minimizers of variational integrals with (p,q) growth conditions*, J. Differ. Equations, 6 (1992), pp. 91–101.

[28] S. Kislyakov and N. Kruglyak, *Stability of approximation under singular integrals, and Caldéron-Zygmund type decompositions*, PDMI Prepr., (2005).

[29] ———, *Extremal Problems in Interpolation Theory, Whitney-Besicovitch Coverings and Singular Integrals*, Monogr. Mat., 74 (2013), pp. 663–714.

[30] L. Koch, *Global higher differentiability for minimisers of convex functionals with (p,q)-growth*, Calc. Var. Partial Differ. Equ., 60 (2021).

[31] ———, *Global higher integrability for minimisers of convex obstacle problems with (p,q)-growth*, Calc. Var. Partial Differ. Equ., 60 (2021).

[32] L. Koch and J. Kristensen, *On the validity of the Euler-Lagrange system without growth assumptions*, arXiv Prepr. arXiv2203.00333, (2022).

[33] M. LaVrentiev, *Sur quelques problème du calcul des variations*, Ann. di Mat. Pura Appl., 4 (1926), pp. 7–28.
[34] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions*, Arch. Ration. Mech. Anal., 105 (1989), pp. 267–284.

[35] ——, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differ. Equations, 90 (1991), pp. 1–30.

[36] G. Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math, 51 (2006), pp. 355–426.

[37] G. Mingione and V. D. Rădulescu, *Recent developments in problems with non-standard growth and nonuniform ellipticity*, J. Math. Anal. Appl., (2021).

[38] M. Schäffner, *Higher Integrability for variational integrals with non-standard growth*, Calc. Var. Partial Differ. Equ., 60 (2021).

[39] V. Zhikov, *Lavrentiev phenomenon and homogenization for some variational problems*, C. R. Acad. Sci. Paris Sér. Mat., 50 (1993), pp. 674–710.

[40] ——, *On Lavrentiev’s Phenomenon*, Russ. J. Math. Phys., 3 (1995), pp. 249–269.

[41] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Math., 29 (1987), pp. 33–66.