Stress formulation and duality approach in periodic homogenization

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Abstract

This paper describes several different variational formulations of the so-called “cellular problem” which is a system of partial differential equations arising in the theory of homogenization, subject to periodicity boundary conditions. These variational formulations of the cellular problems have as main unknown the displacement, the stress or the strain, respectively. For each of these three cases, an equivalent minimization problem is introduced. The variational formulation in stress proves to have a distinguished role and it gives rise to two dual formulations, one in displacement-stress and another one in strain-stress. The corresponding Lagrangians may be used in numerical optimization for developing algorithms based on alternated directions, of Uzawa type.

Keywords: periodic homogenization, cellular problem, formulation in stress, dual formulation, Lagrangian, displacement-stress formulation, strain-stress formulation

Introduction

In engineering, it is common practice to approach linearized elasticity problems through formulations in displacement and in stress. Using the stress as unknown is convenient because, above a critical stress value, the material’s behaviour gets out from the linearized elasticity framework. For an optimal design the principal stresses form naturally a flow chart between the load points, similar to the velocity of a fluid. This observation follows from analysing the famous Michell trusses [14]. The concept of fully stressed design [13, Section 9.1] is also aligned with this observation, although it is still an empirical principle.

This is why, for optimality purposes, we need to understand the stress formulation in relation with the formulation in displacement and in strain, as dual formulations to the primal formulation in stress, according to the Legendre-Fenchel duality theory. The two lagrangians thus obtained have a saddle-point; this unifying theory was developed by Ciarlet in [10] where a new duality approach to elasticity is stated and proven. In the present work we extend this duality approach to periodic problems which is the natural framework of the cellular problems that define the homogenized elastic tensor according to the theory of homogenization.

We believe that the obtained Lagrangians may play a role in numerical methods for optimizing the properties of the homogenized elastic tensor, namely in an alternated directions algorithm of Uzawa type, as in [8, subsection 8.1].

1 Periodic functions

We begin by describing the general periodicity notion that we have been using since [5].

Consider a lattice of vectors in $\mathbb{R}^3$ (an additive group $\mathcal{G}$ generated by three independent vectors $\vec{g}_1, \vec{g}_2, \vec{g}_3$). Define the parallelepiped $Y \subset \mathbb{R}^3$:

$$Y = \{s_1 \vec{g}_1 + s_1 \vec{g}_2 + s_1 \vec{g}_3 : s_1, s_2, s_3 \in [0, 1]\}.$$
Y is called “periodicity cell”.

A function \( \varphi \) is said to be \( \mathcal{G} \)-periodic (or \( Y \)-periodic) if it is invariant to translations with vectors in \( \mathcal{G} \).

**Remark 1.** Let \( \psi \) be a \( \mathcal{G} \)-periodic function; let \( \tau \) be a translation of \( \mathbb{R}^3 \) (not necessarily belonging to \( \mathcal{G} \)). Then

\[
\int_{\tau(Y)} \psi = \int_Y \psi.
\]

In a periodic microstructure, the rigidity is a fourth order tensor field \( C : \mathbb{R}^3 \to \mathbb{R}^{81} \) which is \( \mathcal{G} \)-periodic, that is, it varies according to a periodic pattern. Denote by \( D \) the compliance tensor, that is, the inverse tensor of \( C \).

According to the homogenization theory, the macroscopic behaviour of a body with periodic structure is described by a constant homogenized elastic tensor \( C^H \). It is possible to define \( C^H \) in terms of the solutions of the cellular problems which are PDEs subject to periodicity conditions:

\[
\begin{aligned}
\text{div}(Ce(u_A)) &= 0 \quad \text{in } \mathbb{R}^3 \\
\phi_A(y) &= Ay + \varphi_A(y), \quad \text{with } \varphi_A \ \mathcal{G} - \text{periodic},
\end{aligned}
\]  

(1)

where \( A \) is a given macroscopic strain (a \( 3 \times 3 \) symmetric matrix) and \( e \) represents the corresponding microscopic strain, that is, the symmetric part of the gradient of \( u_A \). In the sequel, for historical reasons, we shall use also the notation \( \nabla_s \) for the symmetric part of the gradient, when the intention is to focus on the displacement \( u_A \).

The solution \( u_A \) depends linearly on the matrix \( A \) and verifies

\[
A = \int_Y e(u_A) = \frac{1}{|Y|} \int_Y e(u_A). 
\]

(2)

The macroscopic stress associated to \( u_A \) is defined by

\[
S = \int_Y Ce(u_A) = \frac{1}{|Y|} \int_Y Ce(u_A)
\]

(3)

and consequently depends linearly on \( A \). The symbol \( \int_Y \) denotes the average value of the integrand on the periodicity cell \( Y \).

The homogenized elastic tensor is then defined, for all macroscopic strains \( A \), through the equality

\[
C^H A = S, \quad \text{that is,}
\]

\[
C^H A = \int_Y Ce(u_A).
\]

(4)

\( C^H \) can be also defined, equivalently, in terms of energy products:

\[
\langle C^H A, B \rangle = \int_Y \langle Ce(u_A), e(u_B) \rangle,
\]

(5)

where \( A \) and \( B \) are any two symmetric matrices (strains). See formula (14) below for a better comprehension.

Denote by \( D^H \) the homogenized compliance tensor, that is, \( D^H = (C^H)^{-1} \) is the inverse of the homogenized elastic tensor.

### 2 Fundamental ingredients

We begin this section by presenting some notation used throughout the paper.

Denote by \( L^2_\text{loc}(\mathbb{R}^3, \mathbb{R}^3) \) the space of vector fields in \( L^2_\text{loc}(\mathbb{R}^3, \mathbb{R}^3) \) which are \( \mathcal{G} \)-periodic, endowed with the \( L^2(Y, \mathbb{R}^3) \) norm. Denote by \( H^1_\#(\mathbb{R}^3, \mathbb{R}^3) \) the space of vector fields in \( H^1_\text{loc}(\mathbb{R}^3, \mathbb{R}^3) \) which are \( \mathcal{G} \)-periodic, endowed with the \( H^1(Y, \mathbb{R}^3) \) norm. Equivalently, \( H^1_\#(\mathbb{R}^3, \mathbb{R}^3) \) can be viewed as the completion in the norm of \( H^1(Y, \mathbb{R}^3) \) of the space of functions in \( C^\infty(\mathbb{R}^3, \mathbb{R}^3) \) which are \( \mathcal{G} \)-periodic. We also define
the space $H^1_{loc}(\mathbb{R}^3, \mathbb{R}^3) = \{\varphi \in H^1_{#}(\mathbb{R}^3, \mathbb{R}^3) \mid \int_Y \varphi = 0\}$. Denote by $H^{-1}_{#}(\mathbb{R}^3, \mathbb{R}^3)$ the dual space of $H^1_{#}(\mathbb{R}^3, \mathbb{R}^3)$.

For the sake of abbreviation we shall use the notations $L^2_{#}$ for $L^2(\mathbb{R}^3, \mathbb{R}^3)$, $H^1_{#}$ for $H^1_{#}(\mathbb{R}^3, \mathbb{R}^3)$ and $H^{-1}_{#}$ for $H^{-1}_{#}(\mathbb{R}^3, \mathbb{R}^3)$.

Denote by $\mathbb{L}_{#}^2$ the space of matrix fields in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$ which are $G$-periodic, endowed with the $L^2(Y, \mathbb{R}^3)$ norm; we add the subscript $s$ for symmetric matrices: $L_{#s}^2$. Denote by $\mathbb{H}_{#}^1$ the space of matrix fields in $H^1_{loc}(\mathbb{R}^3, \mathbb{R}^9)$ which are $G$-periodic, endowed with the $H^1(Y, \mathbb{R}^9)$ norm; we add the subscript $s$ for symmetric matrices: $\mathbb{H}_{#s}^1$.

Denote by $\mathbb{H}_{#s}(\text{div})$ the space

$$\mathbb{H}_{#s}(\text{div}) = \{\boldsymbol{\mu} \in \mathbb{L}_{#s}^2 \mid \text{div}\boldsymbol{\mu} \in L_{#s}^2\},$$

where, with Einstein’s repeated index notation, the operator $\text{div} : L^2_{loc}(\mathbb{R}^3, \mathbb{R}^9) \to H^{-1}(\mathbb{R}^3, \mathbb{R}^3)$ is defined by $\text{div}\boldsymbol{\mu} = (\mu_{ij,j})_{1 \leq i \leq 3}$.

Consider $LP$ the space of linear plus periodic displacements defined in $\mathbb{R}^3$:

$$LP = \{u : \mathbb{R}^3 \to \mathbb{R}^3 \mid u(y) = Ay + \varphi(y), A \in \mathcal{M}^3_s(\mathbb{R}), \varphi \in H^1_{#}\},$$

where $\mathcal{M}^3_s(\mathbb{R})$ denotes the space of symmetric $3 \times 3$ matrices with real elements. $LP$ is a Hilbert space since it is a direct sum between a finite dimensional space and $H^1_{#}$.

For an arbitrarily fixed strain matrix $A \in \mathcal{M}^3_s(\mathbb{R})$, denote by $LP(A)$ the set of linear plus periodic displacements having the linear part $Ay$:

$$LP(A) = \{u : \mathbb{R}^3 \to \mathbb{R}^3 \mid u(y) = Ay + \varphi(y), \varphi \in H^1_{#}\} = \{Ay\} + H^1_{#}.$$

Thus the last equation in (1) is equivalent to

$$\text{div} \boldsymbol{\mu} = 0 \quad \text{in} \quad \mathbb{L}_{#s}^2.$$

Remark 2. For a given symmetric matrix $A$, a function $v \in LP$ belongs to $LP(A)$ if and only if

$$\int_Y e(v) = A.$$
In the remaining of this section we present some ingredients needed to prove the results in sections 3, 4 and 5. Among the stated results are the Korn inequality, the Green’s formula, extensions of the Donati representation Theorem, all adapted to the periodic framework. The proofs of these results may be found in [6] jointly with other fundamental results adapted to periodic context in relation to the homogenization theory.

We begin by stating the Korn inequality in the periodic context, which stays behind the isomorphisms in Theorems 9 and 10.

**Theorem 3.** The Korn inequality below holds for a positive constant $C$ and for all $v$ in $H^1_\#$.

$$\|v\|_{H^1_\#} \leq C (\|v\|_{L^2_\#} + \|\nabla_s v\|_{L^2_\#}) \quad (10)$$

**Theorem 4.** The Green’s formula

$$\int_Y \langle \mu, \nabla_s v \rangle \, dy + \int_Y (\text{div} \, \mu) \cdot v \, dy = 0 \quad (11)$$

holds for all $\mu \in H^1_\#(\text{div})$ and all $v \in H^1_\#(\mathbb{R}^3, \mathbb{R}^3)$.

The proof of Green’s formula above is based on the notion of trace and its properties in the context of periodic functions, see [6, Section 4].

Theorem 5 below is a direct consequence of Green’s formula. But it also has far-reaching connections with the compensated compactness and the div-curl Lemma introduced by L. Tartar in [18], as pointed out in [6, Section 5].

**Theorem 5.** Given an arbitrary $v \in L^p$ and an arbitrary $\sigma \in T$, the following equality holds

$$\int_Y \langle \nabla_s v, \sigma \rangle \, dy = \langle \int_Y \nabla_s v \, dy, \int_Y \sigma \, dy \rangle. \quad (12)$$

The above Theorem 5 may be stated in the following form that is due to P. Suquet, see [17]: If $v \in L^p(A)$ and $\sigma \in T(S)$,

$$\int_Y \langle \nabla_s v, \sigma \rangle \, dy = \langle A, S \rangle. \quad (13)$$

Indeed, for a given macroscopic stress $S$, the space $T(S)$ is defined as the set of stress matrix fields in $T$ whose mean is $S$. Similarly, for given macroscopic strain $A$, the space $L^p(A)$ is equal to the set of strain matrix fields in $L^p$ whose mean is $A$. The notation $L^p$ has been used by the authors in [5, Lemma 2], the only difference is that in the present paper the symmetry of the matrix $A$ is imposed in the very definition of the space $L^p$.

For two arbitrary symmetric matrices (strains) $A$ and $B$, the equality (13) implies that the homogenized tensor $C^H$ can be defined, according to [5], by

$$\langle C^H A, B \rangle = \langle \int_Y C \, e(w_A), B \rangle = \int_Y \langle C \, e(w_A), B \rangle = \int_Y \langle C \, e(w_A), e(w_B) \rangle. \quad (14)$$

The following three results, Theorems 6, 7 and 8, are extensions of Donati’s representation Theorem and generalise to the periodic framework Theorems 4.2 and 4.3 in [2]. Their proofs and some specific remarks may be found in [6].

**Theorem 6.** Consider $e \in L^2_\#$. Then there exists $v \in H^1_\#$ such that $e = \nabla_s v$ in $L^2_\#$, if and only if

$$\int_Y (e, s) \, dy = 0, \text{ for all } s \in L^2_\#, \text{ such that } \text{div} \, s = 0 \text{ in } H^{-1}_\#. \quad (15)$$

In this case $v$ is unique up to an additive constant vector. If we add the zero-average hypothesis, $v \in H^1_{\#0}$, then $v$ is unique.
Theorem 7. Let \( e \in \mathbb{L}^2_{\#s} \). Then
\[
\int_Y (e, \mu) = 0 \text{ for all } \mu \in \mathbb{T}(0)
\] (16)
if and only if there exists \( v \in LP \) such that \( e = \nabla_s v \). In this case \( v \) is unique up to an additive constant vector.

While in Theorem 6 the mean of \( e \) is zero, in Theorem 7 the mean of \( \mu \) is zero. We can let both \( e \) and \( \mu \) to have non-zero average and state the following Theorem 8 below that join together Theorems 6 and 7. Moreover the following result contains both Theorem 5 and its reciprocal.

Theorem 8. Consider \( e \in \mathbb{L}^2_{\#s} \). Then there exists \( v \in LP \) such that \( e = \nabla_s v \in \mathbb{L}^2_{\#s} \) if and only if
\[
\int_Y (e, s) \, dy = \left( \int_Y \nabla_s v \, dy, \int_Y s \, dy \right) \text{ for all } s \in \mathbb{L}^2_{\#s} \text{ such that } \text{div } s = 0.
\] (17)

The following two results allow to conclude the equivalence between the formulation in displacement and the formulation in strain in the next Section 4.

Theorem 9 (\( H^1_{\#0} \) is isomorphic to \( T^\perp \)). The operator \( \nabla_s : H^1_{\#0} \mapsto \mathbb{L}^2_{\#s} \) is an isomorphism from the space
\[
H^1_{\#0} := \{ v \in H^1_{\#}(\mathbb{R}^3, \mathbb{R}^3) : \int_Y v \, dy = 0 \}
\]
onto \( \text{Im} \nabla_s \). Consequently \( \text{Im} \nabla_s \) is closed in \( \mathbb{L}^2_{\#s} \), more precisely \( \text{Im} \nabla_s = T^\perp \).

Moreover, the dual operator of \( \nabla_s : H^1_{\#0} \mapsto \mathbb{L}^2_{\#s} \) is \( -\text{div} : \mathbb{L}^2_{\#s} \mapsto H^{-1}_{\#0} \).

The above result is a consequence of Theorems 22, 26 and 27 in [6] and is an ingredient of the proof of Theorem 10. Jointly with the next theorem, it gives a more profound knowledge on the space of strain fields \( T^\perp \).

The following result states an isomorphism between the space \( LP_0 \) and the orthogonal complement of \( \mathbb{T}(0) \) by mean of the inverse operator of \( \nabla_s \). It is an important ingredient of the formulation of strain obtained in Theorem 24 from the next Section 4.

Theorem 10 (\( LP_0 \) is isomorphic to \( \mathbb{T}(0)^\perp \)). For each \( e \in \mathbb{T}(0)^\perp \), denote by \( F(e) \) the unique element in \( LP_0 \) that satisfies \( \nabla_s F(e) = e \) (according to Theorem 5). Then the mapping \( F : \mathbb{T}(0)^\perp \mapsto LP_0 \) defines an isomorphism between the Hilbert spaces \( \mathbb{T}(0)^\perp \) and \( LP_0 \).

The above two isomorphisms allow us to introduce two variational formulations in strain, see Propositions 11 and 12 in the next section.

3 Variational formulations of the cellular problem

In this section we state and prove several equivalent variational formulations where the main unknown is the displacement, the strain and the stress, respectively. This approach gives a novel perspective in the field of periodic homogenization showing that each of these quantities – displacement, strain or stress – completely characterize the properties of the homogenized material. In these formulations, either a macroscopic strain \( A \) or a macroscopic stress \( S \) is given as data; the equivalence between these formulations is guaranteed by the subjacent relation \( S = C^H A \), equivalent to \( A = D^H S \).

Proposition 11. The variational formulation of the cellular problem \( (\mathbb{T}) \), when the macroscopic strain \( A \in M_A^0(\mathbb{R}) \) is given, has the following form:
\[
\begin{cases}
\text{find } u_A \in LP(A) \text{ such that } \\
\int_Y (C e(u_A), e(v)) \, dy = 0 \quad \forall v \in H^1_{\#0}.
\end{cases}
\] (18)
Proof. The second line in (13) says that $u_A \in LP(A)$. To prove that the first line in (13) implies the second line in (15), it suffices to apply Green’s formula (Theorem 3). The difficult part is to prove that the second line in (13) implies the first line in (1).

Suppose $u_A$ satisfies (13). By choosing a test function $v \in \mathcal{D}(Y)$, the space of $C^\infty$ functions with compact support, we conclude that $\text{div}(C e(u_A)) = 0$ almost everywhere for the Lebesgue measure in $Y$. However, this does not imply $\text{div}(C e(u_A)) = 0$ almost everywhere in $\mathbb{R}^3$; $\text{div}(C e(u_A))$ could be a measure (or a distribution) concentrated on the interface between different translations of $Y$ (with vectors in the periodicity group $\mathcal{G}$); see Remark 12.

However, the periodic character of $C$, $e(u_A)$ and $e(v)$ ensures that $u_A$ satisfies not only (13) but also
\[
\int_{\gamma(Y)} \langle C e(u_A), e(v) \rangle \, dy = 0 \quad \forall v \in H^1_{\#},
\]
for any translation $\tau$ of $\mathbb{R}^3$ (see Remark 12). We stress that $\tau$ does not have to belong to the periodicity group. Thus, $\text{div}(C e(u_A)) = 0$ almost everywhere for the Lebesgue measure in $\gamma(Y)$. Since the translation $\tau$ is arbitrary, we conclude that $\text{div}(C e(u_A)) = 0$ almost everywhere in $\mathbb{R}^3$.

\[
\int_{\gamma(Y)} \langle C e(u_A), e(v) \rangle \, dy = 0 \quad \forall v \in H^1_{\#}.
\]

Figure 1: A periodic function with $u'' = -2$ in $[0, 1]$

Remark 12. A function $u \in H^1_{\#}$ may satisfy the equality $\text{div}(C e(u)) = 0$ (or some other differential equality) in the periodicity cell $Y$, considered as an open set, without satisfying the same property in the entire space. Figure 1 presents a somewhat similar situation in one variable only. It shows a $Z$-periodic function $u$ such that $u'' = -2$ in $Y = [0, 1]$ but $u'' \neq -2$ in $\mathbb{R}$; $u'' + 2$ is a sum of Dirac measures supported at $Z$.

Remark 13. The solution of problem (13) is not unique; it contains an arbitrary additive constant vector. We need to tighten the space in order to obtain a unique solution. We can do this by imposing a zero-average condition: $u_A \in LP_0 \cap LP(A)$. In this case, problem (13) becomes
\[
\begin{cases}
\text{find } u_A \in LP_0 \cap LP(A) \text{ such that } \\
\int_Y \langle C e(u_A), e(v) \rangle \, dy = 0 \quad \forall v \in H^1_{\#0}.
\end{cases}
\]

A roughly equivalent approach is to use a quotient space: $LP(A)$ divided by the subspace of constant vectors.

Note that the classical Lax Milgram lemma does not apply to the above formulation since the space $LP(A)$ where the solution is to be found is a translation of the space $H^1_{\#0}$ of test functions. An adapted version of the Lax Milgram lemma was proven in [19]; for the sake of self containedness we also state it as Lemma 14 below. It ensures the existence and the uniqueness of the solution of the above formulation by taking $V = LP_0$, $V_0 = H^1_{\#0}$ and $K = LP_0 \cap LP(A)$.

Lemma 14. Let $V$ be a fixed Hilbert space, let $V_0$ be a closed subspace of $V$ and let $K = \gamma + V_0$ be a translation of $V_0$ (a closed affine manifold in $V$) where $\gamma$ is a fixed element of $V$. Consider $a : V \times V \mapsto \mathbb{R}$ a bilinear symmetric continuous form on $V$ which is coercive only on $V_0$, and consider
\[
l : V \mapsto \mathbb{R} \text{ a linear continuous form on } V.
\]
Then the problem
\[
\begin{cases}
\text{find } u \in K \text{ such that } \\
a(u, v) = l(v), \quad \forall v \in V_0,
\end{cases}
\]
has a unique solution.
Another variational formulation of the cellular problem is

\[
\begin{aligned}
\text{find } \varphi_A \in H^1_{\#0} \text{ such that } \\
\int_Y \langle C \mathbf{e}(\varphi_A), \mathbf{e}(v) \rangle \, dy = -\langle A, \int_Y C \mathbf{e}(v) \, dy \rangle, \quad \forall v \in H^1_{\#0}.
\end{aligned}
\]  

(20)

If we add a zero-average condition on \( \varphi_A \), the solution is unique:

\[
\begin{aligned}
\text{find } \varphi_A \in H^1_{\#0} \text{ such that } \\
\int_Y \langle C \mathbf{e}(\varphi_A), \mathbf{e}(v) \rangle \, dy = -\langle A, \int_Y C \mathbf{e}(v) \, dy \rangle, \quad \forall v \in H^1_{\#0}.
\end{aligned}
\]  

(21)

The classical Lax Milgram Lemma ensures the existence of a solution \( \varphi_A \).

Formulation (21) is frequently encountered in the literature, often stated on the periodicity cell \( Y \) rather than on the entire space \( \mathbb{R}^3 \). See e.g. [7, Chapter I, section 2.2], [16, Chapter I, section 6.1], [1, Definition 2.1].

A variational formulation in strain arises naturally (recall that the strain is the symmetric part of the gradient of the displacement):

**Proposition 15.** The variational formulation in strain of the cellular problem

\[
\begin{aligned}
\text{find } \mathbf{e} \in \mathbb{T}^\perp \text{ such that } \\
\int_Y \langle C \mathbf{e}, \mathbf{\mu} \rangle \, dy = -\langle A, \int_Y C \mathbf{\mu} \, dy \rangle, \quad \forall \mathbf{\mu} \in \mathbb{T}^\perp.
\end{aligned}
\]

(22)

is equivalent to (21).

**Proof.** Follows from the isomorphism between \( \mathbb{T}^\perp \) and \( H^1_{\#0} \) (see Theorem 9).

The proof of Proposition 16 below is a mere exercise.

**Proposition 16.** Formulations (18) and (20) are equivalent, the solutions \( u_A \) and \( \varphi_A \) being related by \( u_A(y) = Ay + \varphi_A(y) \). Also, formulations (19) and (21) are equivalent.

The variational formulation of the cellular problem when the macroscopic stress \( S \) is given in \( \mathcal{M}^*_3(\mathbb{R}) \) is the following

\[
\begin{aligned}
\text{find } w_S \in L^P_1 \text{ such that } \\
\int_Y \langle C \mathbf{e}(w_S), \mathbf{e}(v) \rangle \, dy = 0, \quad \forall v \in H^1_{\#0}.
\end{aligned}
\]

(23)

Once again, one can apply Lemma 14, the adapted version of the Lax Milgram Lemma, in order to ensure the existence and the uniqueness of the solution \( w_S \); here, \( V = L^P_0, V_0 = H^1_{\#0} \) and \( K \) is the set of functions in \( L^P_0 \) satisfying \( \int_Y C \mathbf{e}(w_S) \, dy = S \).

An equivalent approach is to implement the average condition in the linear form:

\[
\begin{aligned}
\text{find } w_S \in L^P_1 \text{ such that } \\
\int_Y \langle C \mathbf{e}(w_S), \mathbf{e}(v) \rangle \, dy = \langle S, \int_Y \mathbf{e}(v) \, dy \rangle, \quad \forall v \in L^P_0.
\end{aligned}
\]

(24)

**Proposition 17.** The variational formulations (23) and (24) are equivalent.

**Proof.** Take \( w_S \) satisfying (23) and let \( v \in L^P_0 \); we want to prove the equality in (24). Any \( v \in L^P_0 \) can be written as a sum \( v(y) = By + \bar{v}(y) - \int_Y By \, dy \) for some symmetric matrix \( B \) and for some \( \bar{v} \in H^1_{\#0} \).

Then

\[
\int_Y \langle C \mathbf{e}(w_S), \mathbf{e}(v) \rangle \, dy = \int_Y \langle C \mathbf{e}(w_S), B \rangle \, dy + \int_Y \langle C \mathbf{e}(w_S), \mathbf{e}(\bar{v}) \rangle \, dy = \langle B, S \rangle
\]
which proves $w_S$ satisfies (24).

Conversely, take $w_S$ satisfying (24). We begin by choosing the test function $v(y) = By - \int_Y By \, dy$ for an arbitrarily chosen symmetric matrix $B$. Then the equality in (24) implies $\int_Y C e(w_S) \, dy = S$. Now choose $v \in H^1_{\#0}$ and, since the gradient of any periodic function has zero average, we obtain the third line in (24).

Again, a variational formulation in strain arises naturally:

**Proposition 18.** The variational formulation in strain of the cellular problem

\[
\begin{align*}
\text{find } e \in \mathbb{T}(0)^1 \text{ such that } \\
\int_Y (Ce, \mu) \, dy = \langle S, \int_Y \mu \, dy \rangle, \quad \forall \mu \in \mathbb{T}(0)^1.
\end{align*}
\]

is equivalent to (24).

**Proof.** Follows from the isomorphism between $\mathbb{T}(0)^1$ and $LP_0$ (see Theorem 10).

**Proposition 19.** The variational formulations (19) and (23) are equivalent, in the following sense: Consider a macroscopic strain $A$; suppose $u_A$ satisfies formulation (19); define $S = \int_Y C e(u_A)$ (or, equivalently, $S = C^H A$); then $u_A$ is a solution of (23). Conversely, let $S$ be a macroscopic stress; suppose $w_S$ satisfies formulation (23); define $A = \int_Y e(w_S)$ (or, equivalently, $A = D^H S$); then $w_S$ is a solution of problem (19).

**Proof.** Both implications stated above are rather obvious due to the choice of the matrices $A$ and $S$. The second implication relies also on Remark 2.

Problem (19) is still equivalent with the following formulation in stress, provided $S = C^H A$.

\[
\begin{align*}
\text{find } \sigma_S \in \mathbb{T}(S) \text{ such that } \\
\int_Y \langle D\sigma_S, \mu \rangle \, dy = 0, \quad \forall \mu \in \mathbb{T}(0),
\end{align*}
\]

where the existence and the uniqueness of the solution $\sigma_S$ is guaranteed by the adapted version of the Lax-Milgram Lemma [14] above, applied with $V = \mathbb{T}$, $V_0 = \mathbb{T}(0)$ and $K = \mathbb{T}(S)$.

**Proposition 20.** The variational formulations (23) and (26) are equivalent.

**Proof.** Take $w_S$ satisfying (23) and define $\sigma_S = Ce(w_S)$. Then $\sigma_S$ is periodic and its average is $S$. Also, $\text{div} \sigma_S = 0$ since the formulations (1), (18) versus (19) and (23) are equivalent up to an additive constant vector (see Propositions 11 and 18 above). Therefore, $\sigma_S$ belongs to $\mathbb{T}(S)$. On the other hand, recalling that $D$ is the inverse of the rigidity tensor $C$, one has $\int_Y \langle D\sigma_S, \mu \rangle \, dy = \int_Y \langle e(w_S), \mu \rangle \, dy$. Due to Theorem 5 the above quantity is equal to $\langle \int_Y e(w_S) \, dy, \int_Y \mu \, dy \rangle$ which is zero because $\mu \in \mathbb{T}(0)$.

Conversely, let $\sigma_S$ be the solution of (26). Due to Theorem 7 there exists $w_S \in LP$ such that $D\sigma_S = e(w_S)$ thus $\sigma_S = Ce(w_S)$. We can use the arbitrary additive constant vector mentioned in Theorem 7 and choose a function $w_S$ having zero average. On the other hand, since $Ce(w_S) \in \mathbb{T}_S \subset \mathbb{T}$ and taking a test function $v \in H^1_{\#0} \subset LP_0 \subset LP$, Theorem 5 implies $\int_Y \langle Ce(w_S), e(v) \rangle = \int_Y Ce(w_S) \, dy, \int_Y e(v) \, dy$. Again, because the gradient of any periodic function has zero average, $w_S$ satisfies (23).
Figure 2 shows a diagram of the equivalences between the variational formulations presented in the current section. Problems (18) and (20) have non-unique solutions; by adding a zero-average condition we obtain problems (19) and (21) with unique solution. Joint to each equivalence sign, the proposition stating it is specified. Formulations (18), (19), (20), (21) and (22) have the macroscopic strain $A$ as datum. Formulations (23), (24), (25) and (26) have the macroscopic stress $S$ as datum. In formulations (18), (19), (23) and (24) the unknown is the microscopic displacement. In formulations (20) and (21) the unknown is $\varphi$ which is the periodic part of the microscopic displacement. In formulations (22) and (25) the unknown is the microscopic strain. Finally, in formulation (26) the unknown is the microscopic stress.

Figure 2: Relationships between different variational formulations of the cellular problem

4 The cellular problem formulated as minimization problem

In the sequel we extend to the periodic framework the results in [10, sections 4 and 5], in order to obtain formulations of the cellular problem (1) in displacement, stress and strain. We relate these formulations to the variational formulations (18), (20), (23), (24) and (26) in section 3. Recall that these variational formulations are all equivalent in the sense that $u_A = w_S$ and $\sigma_S = C e(w_S)$, provided $S = C^H A$.

**Theorem 21** (formulation in displacement). Given a macroscopic strain $A$, there exists a unique vector field $u_A$ in $L^p_0 \cap L^p(A)$ that satisfies

$$\mathcal{J}(u_A) = \inf_{u \in L^p_0 \cap L^p(A)} \mathcal{J}(u)$$

where

$$\mathcal{J}(u) = \frac{1}{2} \int_Y \langle Ce(u), e(u) \rangle \, dy$$

for all $u \in L^p_0 \cap L^p(A)$.

Moreover, $u_A$ is the solution of the variational formulation (19).

**Proof.** Given $u \in L^p(A)$ the Gâteaux derivative of $\mathcal{J}$ is $D\mathcal{J}(u)(v) = \int_Y \langle Ce(u), e(v) \rangle \, dy$ for all $v \in H^1_{\#0}$. The condition $D\mathcal{J}(u)(v) = 0 \forall v \in H^1_{\#0}$ is equivalent to the variational formulation (19).
Consequently there exists a unique \( u_A \in LP_0 \cap LP(A) \) with the property that \( D\mathcal{J}(u_A)(v) = 0 \). The functional \( \mathcal{J} \) attains its infimum in \( u_A \).

Note that the value of \( \mathcal{J} \) does not change if one adds a constant vector to \( u_A \), so it is easy to extend \( \mathcal{J} \) to \( LP(A) \).

Another formulation in displacement is needed in order to minimize the function over a Hilbert space \( H^1_{\#0} \), rather than over an affine space \( LP_0 \cap LP(A) \). The next theorem gives therefore a formulation in displacement, equivalent to the above one.

**Theorem 22** (equivalent formulation in displacement). 1) Given a macroscopic strain \( A \), there exists a unique vector field \( \varphi_A \) in \( H^1_{\#0} \) satisfying

\[
J(\varphi_A) = \inf_{\varphi \in H^1_{\#0}} J(\varphi)
\]

where

\[
J(\varphi) = \frac{1}{2} \int_Y \langle Ce(\varphi), e(\varphi) \rangle \, dy + \langle A, \int_Y Ce(\varphi) \, dy \rangle \quad \text{for all } \varphi \in H^1_{\#0}.
\]

Moreover, \( \varphi_A \) is the solution of the variational formulation \([21]\).

2) Given a macroscopic stress \( S \), there exists a unique vector field \( w_S \) in \( LP_0 \) that satisfies

\[
K(w_S) = \inf_{u \in LP_0} K(u)
\]

where

\[
K(u) = \frac{1}{2} \int_Y \langle C \nabla_s u, \nabla_s u \rangle \, dy - \int_Y \langle S, \nabla_s u \rangle \, dy \quad \text{for all } u \in LP_0.
\]

Moreover, \( w_S \) is the solution of the variational formulation \([24]\).

**Proof.** 1) Given \( \varphi \in H^1_{\#0} \), the Gâteaux derivative of \( J \) is \( D\mathcal{J}(\varphi)(\psi) = \int_Y \langle Ce(\varphi), e(\psi) \rangle \, dy + \langle A, \int_Y Ce(\psi) \, dy \rangle \) for all \( \psi \in H^1_{\#0} \). The derivative \( D\mathcal{J}(\varphi)(\psi) = 0 \ \forall \psi \in H^1_{\#0} \) is equivalent to the variational formulation \([21]\). Consequently there exists a unique \( \varphi \in H^1_{\#0} \) with the property that \( D\mathcal{J}(\varphi)(\psi) = 0 \). Therefore the functional \( J \) attains its infimum in \( \varphi_A \).

2) The proof is analogous with the one in 1): the Gâteaux derivative of \( K \) being zero is equivalent to the variational formulation \([24]\).

**Theorem 23** (formulation in stress). Given a macroscopic stress \( S \in M^3_+(\mathbb{R}) \), there exists a unique tensor field \( \sigma_S \in \mathbb{T}(S) \) that satisfies

\[
g(\sigma_S) = \inf_{\sigma \in \mathbb{T}(S)} g(\sigma)
\]

where

\[
g(\sigma) = \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy \quad \text{for all } \sigma \in L^2_{\#s}
\]

and \( D = C^{-1} \) is the compliance tensor.

Moreover, \( \sigma_S \) is the solution of the variational formulation \([20]\) and \( D\sigma_S = \nabla_s u_A \), where \( u_A \) is the solution of \([17]\).

**Proof.** For \( \sigma \in \mathbb{T}(S) \), the Gâteaux derivative of \( g \) is \( Dg(\sigma)(\mu) = \int_Y \langle D\sigma, \mu \rangle \, dy \) for all \( \mu \in \mathbb{T}(0) \). Then the derivative \( Dg(\sigma)(\mu) = 0 \ \forall \mu \in \mathbb{T}(0) \) is equivalent to the above variational formulation in stress of the cellular problem \([26]\) and therefore the solution exists and is unique so \( \sigma = \sigma_S \).

On the other hand, if \( \int_Y \langle D\sigma_S, \mu \rangle \, dy = 0 \ \forall \mu \in \mathbb{T}(0) \), applying the extension of the Donati’s Theorem \([7]\) it turns out that \( D\sigma_S = \nabla_s v \) for a function \( v \in LP \) which is precisely \( v = u_A \) for \( A = D^H S \); recall that \( D^H = (C^H)^{-1} \) is the homogenized compliance tensor.
Theorem 24 below states the cellular problem as a minimization problem on the space \( \mathbb{T}(0)^\perp \), defined in formula (8). It is inspired in [10] Theorem 4.3 where the minimization is performed on a similar space \( \mathcal{M}^\perp \). The definition of the space \( \mathcal{M} \), stated in [10] Theorem 3.2, is based on a space \( V \) which contains a Dirichlet boundary condition. Consequently, \( \mathcal{M} \) contains an implicit Neumann condition which casts a Dirichlet-like condition on elements of \( \mathcal{M}^\perp \). Similarly, the zero-average condition appearing in the definition (7) of the space \( \mathbb{T}(0) \) can be viewed as a homogeneous Neumann condition. This casts a condition on elements of \( \mathbb{T}(0)^\perp \) which has the nature of a Dirichlet boundary condition. See also [5, Remark 2], [19, Remark 1] and [4, end of Section 3].

**Theorem 24 (formulation in strain).** Each of the minimization problems below has a unique solution (a microscopic strain field):

1) For a given macroscopic strain \( A \), find \( \bar{e} \in \mathbb{T}^\perp \) such that

\[
j(\bar{e}) = \inf_{e \in \mathbb{T}^\perp} j(e),
\]

where \( j : L^2_{\text{div}}(\mathbb{T}) \to \mathbb{R} \) is defined by \( j(e) = \frac{1}{2} \int_Y \langle C e, e \rangle \text{dy} + \langle A, \int_Y C e \text{dy} \rangle \);

2) For a given macroscopic stress \( S \), find \( \bar{e} \in \mathbb{T}(0)^\perp \) such that

\[
k(\bar{e}) = \inf_{e \in \mathbb{T}(0)^\perp} k(e),
\]

where \( k : L^2_{\text{div}}(\mathbb{T}) \to \mathbb{R} \) is defined by \( k(e) = \frac{1}{2} \int_Y \langle C e, e \rangle \text{dy} - \langle S, \int_Y e \text{dy} \rangle \).

Moreover, the above two minimization problems are equivalent, in the following sense. Consider a macroscopic strain \( A \); take the solution \( \bar{e} \) of problem 1) above (associated to \( A \)); define \( S = \int_Y C e \) (or, equivalently, \( S = C^H A \)); take the solution \( \bar{e} \) of problem 2) above (associated to \( S \)); then \( \bar{e} = \bar{e} - A \) and \( j(\bar{e}) = k(\bar{e}) \). Conversely, let \( S \) be a macroscopic stress; take the solution \( \bar{e} \) of problem 2) above (associated to \( S \)); define \( A = \int_Y \bar{e} \) (or, equivalently, \( A = D^H S \)); take the solution \( \bar{e} \) of problem 1) above (associated to \( A \)); then \( \bar{e} = \bar{e} + A \) and \( k(\bar{e}) = j(\bar{e}) \).

**Proof.** 1) Deriving the function \( j \) one obtains \( D j(e)(\mu) = \int_Y \langle C e, \mu \rangle \text{dy} + \langle A, \int_Y C \mu \text{dy} \rangle \) for all \( \mu \in \mathbb{T}^\perp \). Then \( D j(e)(\mu) = 0 \) for all \( \mu \in \mathbb{T}^\perp \), is equivalent to the formulation [22]. From Proposition 13 it follows that the solution \( \bar{e} \) is unique and it verifies \( \bar{e} = \nabla s u_A \), thus \( \bar{e} = \nabla s u_A - A \).

2) The Gâteaux derivative of \( k \) is \( D k(e)(\mu) = \int_Y \langle C e, \mu \rangle dy - \langle S, \int_Y \mu \text{dy} \rangle \) for all \( \mu \in \mathbb{T}(0)^\perp \), and \( D k(e)(\mu) = 0 \) for all \( \mu \in \mathbb{T}(0)^\perp \), is equivalent to the formulation [25].

Thus, Proposition 13 implies that the solution \( \bar{e} \) is unique and verifies \( \bar{e} = \nabla s u_A \).

The integral conditions follow from (2) and (3). \( \Box \)

The state of a body is characterized by the internal stress and the local displacement. Its energy is the product between the stress tensor and the strain tensor, defined as the symmetric gradient of the displacement. We introduce an energy operator \( \Lambda : L^2_{\text{div}}(\mathbb{T}) \to (LP)^* \), where \( (LP)^* \) is the dual of the Hilbert space \( LP \), defined by

\[
\langle \Lambda \sigma, v \rangle = \int_Y \langle \sigma, \nabla v \rangle \text{dy}, \forall v \in LP.
\]

Consider \( S \in \mathcal{M}^\perp(\mathbb{R}) \) a stress matrix. Consider the particular energy functional defined as

\[
\eta_S(v) = \langle \Lambda S, v \rangle = \int_Y \langle S, \nabla v \rangle \text{dy}.
\]

(28)

Define the indicator function \( I_S : (LP)^* \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
I_S(\eta) = \begin{cases} 0 & \text{if } \eta = \eta_S, \\ +\infty & \text{if } \eta \neq \eta_S. \end{cases}
\]

(29)
Note that $I_S(\Lambda \sigma) = \begin{cases} 0 & \text{if } \langle \Lambda \sigma, v \rangle = \langle \Lambda S, v \rangle, \\ +\infty & \text{if } \langle \Lambda \sigma, v \rangle \neq \langle \Lambda S, v \rangle. \end{cases}$ $I_S(\Lambda \sigma)$ is zero if and only if $\int_Y \langle \sigma, \nabla_S v \rangle \, dy = \int_Y \langle S, \nabla v \rangle \, dy, \forall v \in LP.$

Theorem 25 below transforms the stress formulation in Theorem 23 in a minimization over all symmetric stresses in $L^2_{\# S}$ of a function that contains implicitly the variational formulation (26). As we shall see during the proof, the second term in (30), $I_S(\Lambda \sigma)$, ensures that the stress $\sigma$ (minimizer of $G$) belongs to $T(S)$.

**Theorem 25.** The energy operator $\Lambda$ belongs to $L(L^2_{\# S}, (LP)^*)$. The functional $I_S$ above defined is proper, convex and lower-continuous and verifies $I_S(\Lambda \sigma) = I_{T(S)}(\sigma)$ where $I_{T(S)}$ denotes the indicator function of the set $T(S)$. The function $g$ defined in (27) is proper, convex and lower-continuous. Defining the function $G : L^2_{\# S} \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$G(\sigma) = g(\sigma) + I_S(\Lambda \sigma), \quad \forall \sigma \in L^2_{\# S},$$

the minimization problem in Theorem 23 is equivalent to the following one:

$$\inf_{\sigma \in L^2_{\# S}} G(\sigma),$$

in the sense that the minimum value is the same and is attained for the same $\sigma : \inf_{\sigma \in T(S)} g(\sigma) \Leftrightarrow \inf_{\sigma \in L^2_{\# S}} G(\sigma)$.

**Proof.** For any $\sigma \in L^2_{\# S}$ the functional $\Lambda \sigma : LP \mapsto \mathbb{R}$ defined by $\langle \Lambda \sigma, v \rangle = \int_Y \langle \sigma, \nabla_S v \rangle \, dy$ is linear and continuous, therefore $\Lambda \sigma$ belongs to $(LP)^*$ and the definition of $\Lambda$ is consistent. Moreover, for all $\sigma \in L^2_{\# S}$

$$\|\Lambda \sigma\|_{(LP)^*} = \sup_{v \in LP} \frac{\int_Y \langle \sigma, \nabla_S v \rangle \, dy}{\|v\|_{H^1_{\# S}}} \leq \sup_{v \in LP} \frac{\int_Y |\Lambda \sigma, \nabla_S v \rangle \, dy}{\|\nabla_S v\|_{L^2_{\# S}}} = \|\sigma\|_{L^2_{\# S}},$$

and consequently $\Lambda \in L(L^2_{\# S}, (LP)^*)$.

The functional $I_S$ is the indicator function of $\eta_S \subset (LP)^*$ and therefore it is proper, convex and lower-continuous (since $\eta_S$ is closed in $(LP)^*$).

Let us prove that $I_S(\Lambda \sigma) = I_{T(S)}(\sigma)$, which is equivalent to $\Lambda \sigma = \eta_S$, that is,

$$\langle \Lambda \sigma, v \rangle = \langle S, \nabla_S v \rangle, \forall v \in LP \quad \Leftrightarrow \quad \langle \sigma, \nabla_S v \rangle = \langle S, \nabla_S v \rangle, \forall v \in LP \quad \Leftrightarrow \quad (\sigma - S, \nabla_S v) = 0, \forall v \in LP.$$

The above equality contains the mean condition on $\sigma$ and the zero divergence condition and is thus equivalent to $\sigma \in T(S)$. We conclude that $I_S(\Lambda \sigma) = I_{T(S)}(\sigma)$, for all $\sigma \in L^2_{\# S}$.

The function $g$ is proper, convex (quadratic in $\sigma$) and continuous for the norm $\| \cdot \|_{L^2_{\# S}}$. The minimization problem in Theorem 23 is equivalent to

$$\inf_{\sigma \in L^2_{\# S}} g(\sigma) + I_{T(S)}(\sigma)$$

and the last assertion in the theorem is a consequence of $I_S(\Lambda \sigma)$ being equal to $I_{T(S)}(\sigma)$.

**Definition 26.** Let $X$ be a normed vector space and let $\theta : X \mapsto \mathbb{R} \cup \{+\infty\}$ be a proper function. The Legendre-Fenchel transform of $\theta$ is the function $\theta^* : X^* \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$\theta^*(e) = \sup_{x \in X} \{ \langle e, x \rangle - \theta(x) \}, \quad \text{with } e \in X^*.$$
Lemma 29 below ensures that are introduced: 

Two Lagrangeans theory, see [11, Chapter 6]. A dual problem in strain and a dual problem in displacement will be presented, generalizing the results in [10, Sections 6 and 7] to the periodicity context.

\[ \eta \]

\[ \eta_S(v), \forall v \in LP, \]

where \( \eta_S \) is the functional defined in [23].

The proof is straightforward from the Definition 26 of the Legendre-Fenchel transform.

5 Dual problems for the formulation in stress

We introduce two dual problems to the minimization problem in stress, according to the duality theory, see [11] Chapter 6. A dual problem in strain and a dual problem in displacement will be presented, generalizing the results in [10] Sections 6 and 7 to the periodicity context. Two Lagrangeans are introduced: 

\[ L(\sigma, e) = \int_Y \langle \sigma, e \rangle \, dy - g^*(e) + I_S(\Lambda \sigma), \text{ for the stress-strain duality, and } \]

\[ L(\sigma, v) = \int_Y \langle \sigma, e \rangle \, dy + (\Lambda \sigma, v) - I_S(v), \text{ for the stress-displacement duality.} \]

We begin by presenting in detail the stress-strain Lagrangean \( L(\sigma, e) \) (Theorem 28 and Lemma 29) and afterwards the stress-displacement Lagrangean \( L(\sigma, v) \) (Theorem 30 and Lemma 31).

The minimization problem in stress (31) (primal problem) has a dual problem in strain, as we shall prove in the sequel. A Lagrangian associated to the dual problem is introduced in the following theorem and its saddle-point corresponds to the solutions of the primal and dual problems, respectively (31) and (33).

\[ \text{Theorem 28. Consider the Lagrangian } L : \mathbb{L}^2_{#_\sigma} \times \mathbb{L}^2_{#_e} \to \mathbb{R} \cup \{+\infty\} \text{ defined by} \]

\[ L(\sigma, e) = \int_Y \langle e, \sigma \rangle \, dy - g^*(e) + I_S(\Lambda \sigma) \quad \forall (\sigma, e) \in \mathbb{L}^2_{#_\sigma} \times \mathbb{L}^2_{#_e} \quad (32) \]

Then

\[ \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} G(\sigma) = \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} \sup_{e \in \mathbb{L}^2_{#_e}} L(\sigma, e) = L(\sigma_S, e(u_A)) = \sup_{e \in \mathbb{L}^2_{#_e}} \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} L(\sigma, e), \]

where \( \sigma_S \) is the solution of both minimization problems in Theorem 27 and Theorem 25 and \( e(u_A) = \nabla_s u_A \) is the solution in Theorem 24 provided \( S = C^{H^\alpha} A \).

Proof. The Fenchel-Moreau Theorem, presented in [12] and [13], states that \( g^{**} = g \), where \( g^* \) is the Legendre-Fenchel transform of \( g \). Using the definition of Legendre-Fenchel of \( g^* \), \( G \) may be written as follows:

\[ G(\sigma) = g^{**}(\sigma) + I_S(\Lambda \sigma) = \sup_{e \in \mathbb{L}^2_{#_e}} \{ \int_Y \langle e, \sigma \rangle \, dy - g^*(e) \} + I_S(\Lambda \sigma) = \sup_{e \in \mathbb{L}^2_{#_e}} L(\sigma, e), \]

for every \( \sigma \in \mathbb{L}^2_{#_\sigma} \). Defining \( G^* : \mathbb{L}^2_{#_\sigma} \to \mathbb{R} \cup \{-\infty\} \) by

\[ G^*(e) = \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} \{ \int_Y \langle e, \sigma \rangle \, dy + I_S(\Lambda \sigma) \} - g^*(e), \quad (33) \]

the dual problem of (31) is

\[ \sup_{e \in \mathbb{L}^2_{#_e}} G^*(e). \quad (34) \]

Lemma 29 below ensures that

\[ G(\sigma_S) = \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} G(\sigma) = \sup_{e \in \mathbb{L}^2_{#_e}} G^*(e) = G^*(e(u_A)) \]

\[ \sup_{e \in \mathbb{L}^2_{#_e}} G^*(e). \quad (34) \]

Lemma 29 below ensures that

\[ G(\sigma_S) = \inf_{\sigma \in \mathbb{L}^2_{#_\sigma}} G(\sigma) = \sup_{e \in \mathbb{L}^2_{#_e}} G^*(e) = G^*(e(u_A)) \]
and since one can verify that $G(\sigma_S) = L(\sigma_S, e(u_A))$, the conclusion of the theorem follows. In order to confirm the above equality, recall that $\sigma_S$ belongs to $\mathbb{T}(S)$ and according to Theorem 25 $h(\Lambda \sigma_S) = 0$. Using Theorem 27 one obtains

$$L(\sigma_S, e(u_A)) = \int_Y \langle e(u_A), \sigma_S \rangle \, dy - g^*(e(u_A)) + \mathcal{I}_S(\Lambda \sigma_S) = \int_Y \langle e(u_A), \sigma_S \rangle \, dy - \frac{1}{2} \int_Y \langle Ce(u_A), e(u_A) \rangle \, dy$$

and since from Theorem 23 $\sigma_S = Ce(u_A)$, the last expression is equal to

$$\int_Y \langle e(u_A), \sigma_S \rangle \, dy - \frac{1}{2} \int_Y \langle e(u_A), \sigma_S \rangle \, dy = \frac{1}{2} \int_Y \langle e(u_A), \sigma_S \rangle \, dy.$$ 

Moreover, from Lemma 29 below one gets

$$G(\sigma_S) = g(\sigma_S) = \frac{1}{2} \int_Y \langle D\sigma_S, \sigma_S \rangle = \frac{1}{2} \int_Y \langle e(u_A), \sigma_S \rangle.$$ 

Hence the conclusion of the Theorem follows. 

**Lemma 29.** For the function $G^* : L^2_{\mathcal{L}, \sigma} \mapsto \mathbb{R} \cup \{-\infty\}$ defined in (35) by

$$G^*(e) = \inf_{\sigma \in L^2_{\mathcal{L}, \sigma}} \left\{ \int_Y \langle e, \sigma \rangle \, dy + \mathcal{I}_S(\Lambda \sigma) \right\} - g^*(e),$$

the dual problem (37) above

$$\sup_{e \in L^2_{\mathcal{L}, \sigma}} G^*(e)$$

may be written as

$$\sup_{e \in L^2_{\mathcal{L}, \sigma}} G^*(e) = - \inf_{e \in \mathbb{T}(0)^\perp} k(e),$$

where the function $k : L^2_{\mathcal{L}, \sigma} \mapsto \mathbb{R}$ is defined in Theorem 27 by $k(e) = \frac{1}{2} \int_Y \langle Ce, e \rangle \, dy - \langle S, \int_Y e \, dy \rangle$.

Moreover, in the primal problem (37), the minimum value is equal to the maximum value in the dual problem and

$$G(\sigma_S) = \inf_{\sigma \in L^2_{\mathcal{L}, \sigma}} G(\sigma) = \sup_{e \in L^2_{\mathcal{L}, \sigma}} G^*(e) = G^*(e(u_A)).$$

**Proof.** The function $G^*$ may be written as follows

$$G^*(e) = \inf_{\sigma \in L^2_{\mathcal{L}, \sigma}} \left\{ \int_Y \langle e, \sigma \rangle \, dy + I_{\mathbb{T}(S)}(\sigma) \right\} - g^*(e) = \inf_{\sigma \in \mathbb{T}(S)} \left\{ \int_Y \langle e, \sigma \rangle \, dy \right\} - g^*(e).$$

For $\sigma \in \mathbb{T}(S)$

$$G^*(e) = \begin{cases} -\infty, & \text{if } e \not\in \mathbb{T}(0)^\perp, \\ \int_Y \langle e, \sigma \rangle \, dy - \frac{1}{2} \int_Y \langle Ce, e \rangle \, dy, & \text{if } e \in \mathbb{T}(0)^\perp, \end{cases}$$

and therefore

$$\sup_{e \in L^2_{\mathcal{L}, \sigma}} G^*(e) = \sup_{e \in \mathbb{T}(0)^\perp} G^*(e) = \sup_{e \in \mathbb{T}(0)^\perp} (-k(e)) = - \inf_{e \in \mathbb{T}(0)^\perp} k(e).$$

On the other hand

$$\inf_{\sigma \in L^2_{\mathcal{L}, \sigma}} G(\sigma) = \inf_{\sigma \in L^2_{\mathcal{L}, \sigma}} \left\{ \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy \right\}$$

according to Theorem 25

$$= \frac{1}{2} \int \langle D\sigma_S, \sigma_S \rangle = G(\sigma_S).$$
and, in view of Theorem 24 and of the variational formulation (23),
\[
\frac{1}{2} \int_Y \langle Ce(u_A), e(u_A) \rangle \, dy = \frac{1}{2} \langle S, \int_Y e(u_A) \, dy \rangle - \frac{1}{2} \int_Y \langle Ce(u_A), e(u_A) \rangle \, dy = -k(e(u_A)) = \inf_{e \in \mathcal{T}_0} k(e) = \sup_{e \in \mathcal{T}_0} G^*(e) = G^*(e(u_A)) = \sup_{e \in L^2_{sg}} G^*(e),
\]
which concludes the proof. \(\square\)

The minimization problem in stress (31) (primal problem) has also a dual problem in displacement. A Lagrangian associated to the dual problem in displacement is introduced in the next theorem. The Lagrangian has a saddle-point that corresponds to the solutions of the primal and dual problems, (31) and (34) respectively.

**Theorem 30.** Consider the Lagrangian \(\tilde{L} : L^2_{sg} \times LP_0 \rightarrow \mathbb{R} \cup \{+\infty\}\) defined by
\[
\tilde{L}(\sigma, v) = \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy + \langle \Lambda \sigma, v \rangle - I^*_S(v) \quad \forall (\sigma, v) \in L^2_{sg} \times LP_0.
\]

Then
\[
\inf_{\sigma \in L^2_{sg}} G(\sigma) = \inf_{\sigma \in L^2_{sg}, v \in LP_0} \tilde{L}(\sigma, v) = \inf_{\sigma \in LP_0} \tilde{L}(\sigma, u_A) = \inf_{\sigma \in L^2_{sg}, v \in LP_0} \sup_{v \in LP_0} \tilde{L}(\sigma, v),
\]
where \(\sigma_S\) is the solution of both minimization problems in Theorem 24 and Theorem 25 and \(e(u_A) = \nabla_s u_A\) is the solution in Theorem 24, provided \(S = C^H A\).

**Proof.** For a given \(\sigma \in L^2_{sg}\) one has
\[
\sup_{v \in LP_0} \tilde{L}(\sigma, v) = \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy + \sup_{v \in LP_0} \{ \langle \Lambda \sigma, v \rangle - I^*_S(v) \}
\]
and using Theorem 27
\[
= \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy + \sup_{v \in LP_0} \{ \int_Y \langle \sigma - S, \nabla_s v \rangle \, dy \} = \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy + I_{\mathcal{T}(S)}(\sigma) = G(\sigma).
\]
Therefore
\[
G(\sigma) = \sup_{v \in LP_0} \tilde{L}(\sigma, v)\quad \text{and consequently}\quad \inf_{\sigma \in L^2_{sg}} G(\sigma) = \inf_{\sigma \in L^2_{sg}, v \in LP_0} \tilde{L}(\sigma, v).
\]
The infimum is attained in \(\sigma_S\):
\[
G(\sigma_S) = \inf_{\sigma \in L^2_{sg}} G(\sigma).
\]
On the other hand
\[
\tilde{L}(\sigma_S, u_A) = \frac{1}{2} \int_Y \langle D\sigma_S, \sigma_S \rangle \, dy + \langle \Lambda \sigma_S, u_A \rangle - I^*_S(u_A) = \frac{1}{2} \int_Y \langle D\sigma_S, \sigma_S \rangle \, dy + \int_Y (\sigma_S - S, \nabla_s u_A) \, dy
\]
and using Theorem 8 the last term vanishes, so
\[
= \frac{1}{2} \int_Y \langle D\sigma_S, \sigma_S \rangle \, dy = G(\sigma_S)
\]
and the inf-sup is attained in \((\sigma_S, u_A)\).

Defining \(\tilde{G}^* : LP_0 \rightarrow \mathbb{R} \cup \{+\infty\}\) by
\[
\tilde{G}^*(v) = \inf_{\sigma \in L^2_{sg}} \{ \frac{1}{2} \int_Y \langle D\sigma, \sigma \rangle \, dy + \langle \Lambda \sigma, v \rangle \} - \int_Y \langle S, \nabla_s v \rangle \, dy,
\]
the dual problem of (31) is
\[
\sup_{v \in LP_0} \tilde{G}^*(v).
\]
Using the lemma below it turns out that
\[
\sup_{v \in LP_0} \inf_{\sigma \in L^2_{\#\#}} \tilde{L}(\sigma, v) = \sup_{v \in LP_0} \tilde{G}^*(v) = \inf_{\sigma \in L^2_{\#\#}} G(\sigma) = G(\sigma_S),
\]
which concludes the proof.

**Lemma 31.** Let the function \( \tilde{G}^* : LP_0 \mapsto \mathbb{R} \cup \{-\infty\} \) be defined by
\[
\tilde{G}^*(v) = \inf_{\sigma \in L^2_{\#\#}} \left\{ \frac{1}{2} \int_Y (D\sigma, \sigma) \, dy + \langle \Lambda\sigma, v \rangle \right\} - T_S(v).
\]
Then the dual problem
\[
\sup_{v \in LP_0} \tilde{G}^*(v)
\]
may be written as
\[
\sup_{v \in LP_0} \tilde{G}^*(v) = - \inf_{v \in LP_0} K(v),
\]
where the function \( K : LP_0 \mapsto \mathbb{R} \) is defined in Theorem 22 by \( K(u) = \frac{1}{2} \int_Y \langle C\nabla_s u, \nabla_s u \rangle \, dy - \int_Y \langle S, \nabla_s u \rangle \, dy \).

Moreover, in the primal problem (31), the minimum value is equal to the maximum value in the dual problem and
\[
G(\sigma_S) = \inf_{\sigma \in L^2_{\#\#}} G(\sigma) = \sup_{v \in LP_0} \tilde{G}^*(v) = \tilde{G}^*(-u_A).
\]

**Proof.** For any \( v \in LP_0 \)
\[
\tilde{G}^*(v) = \inf_{\sigma \in L^2_{\#\#}} \left\{ \frac{1}{2} \int_Y (D\sigma, \sigma) \, dy + \langle \Lambda\sigma, v \rangle \right\} - T_S(v)
\]
and since from Theorem 22 \( T_S(v) = \eta_S(v) = \int_Y \langle S, \nabla_s v \rangle \, dy \),
\[
\tilde{G}^*(v) = \inf_{\sigma \in L^2_{\#\#}} \left\{ \frac{1}{2} \int_Y (D\sigma, \sigma) \, dy + \int_Y \langle \sigma, \nabla_s v \rangle \right\} - \int_Y \langle S, \nabla_s v \rangle \, dy.
\]
The infimum is attained in \( \sigma = -C\nabla_s v \) which implies that \( D\sigma = -\nabla_s v \), hence
\[
\tilde{G}^*(v) = \left\{ \frac{1}{2} \int_Y \langle \nabla_s v, C\nabla_s v \rangle \, dy - \int_Y \langle C\nabla_s v, \nabla_s v \rangle \, dy \right\} - \int_Y \langle S, \nabla_s v \rangle \, dy
\]
\[
= - \frac{1}{2} \int_Y \langle \nabla_s v, C\nabla_s v \rangle \, dy - \int_Y \langle S, \nabla_s v \rangle \, dy.
\]
Consequently
\[
\sup_{v \in LP_0} \tilde{G}^*(v) = \sup_{v \in LP_0} \tilde{G}^*(-v) = \sup_{v \in LP_0} \left\{ - \frac{1}{2} \int_Y \langle \nabla_s v, C\nabla_s v \rangle \, dy + \int_Y \langle S, \nabla_s v \rangle \, dy \right\} = - \inf_{v \in LP_0} K(v)
\]
On the other hand
\[
\inf_{\sigma \in L^2_{\#\#}} G(\sigma) = \inf_{\sigma \in L^2_{\#\#}} \left\{ \frac{1}{2} \int_Y (D\sigma, \sigma) \, dy \right\}
\]
according to Theorem 23
\[
= \frac{1}{2} \int_Y (D\sigma_S, \sigma_S) = G(\sigma_S)
\]
and in view of Theorem 22 2) and of the variational formulation (24)
\[
= \frac{1}{2} \int_Y \langle C\nabla_s u_A, \nabla_s u_A \rangle \, dy = \langle S, \int_Y \nabla_s u_A \, dy \rangle - \frac{1}{2} \int_Y (C\nabla_s u_A, \nabla_s u_A) \, dy = -K(u_A)
\]
\[
= \tilde{G}^*(-u_A) = - \inf_{v \in LP_0} K(v) = \sup_{v \in LP_0} \tilde{G}^*(v).
\]
6 Conclusions

Several variational formulations of the cellular problem are presented, having as unknown the displacement, the strain or the stress. Although these formulations are equivalent, the formulation in stress is special in the sense that the minimization problem in displacement and the minimization problem in strain may be viewed as dual problems for the minimization problem in stress. Thus, it is natural that in Theorems 222) (formulation in displacement) and 242) (formulation in strain) the functionals to minimize involve the macroscopic stress $S$ as datum.

Two Lagrangians are constructed based on these primal-dual problems, providing a stress-displacement approach and a stress-strain approach. Each one of these approaches gives a more complete information about the cellular problem than the stand-alone variational formulations, and are to be used according to the sought application.

Uzawa’s method may be used to obtain the solution of the saddle point problem. It uses a coupled iterative scheme along alternated directions. Other, more performant, schemes, e.g. Arrow-Hurwicz, can also be used, see [3]. For numerical solution of saddle point problems see also [8].

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