A holographic model of $d$-wave superconductor vortices with Lifshitz scaling

Hong Guo$^1$, Fu-Wen Shu$^{1,2}$, Jing-He Chen$^1$, Hui Li$^1$, and Ze Yu$^1$

$^1$Department of Physics, Nanchang University, Nanchang, 330031, China

$^2$Center for Relativistic Astrophysics and High Energy Physics, Nanchang University, Nanchang, 330031, China

Abstract

By making use of the matching methods, we study analytically the $d$-wave holographic superconductors with Lifshitz scaling in the presence of external magnetic field. The vortex lattice solutions of the model have also been obtained with different Lifshitz scaling. Our results imply that holographic $d$-wave superconductor is indeed a type II one even for different Lifshitz scaling. This is the same as the conventional $d$-wave superconductors in the Ginzburg-Landau theory. Our results also indicate that the dynamical exponent $z$, although, has no effect to the form of the vortex lattice solutions, it has effects on the upper critical magnetic field $B_{c_2}$ through the fact that a larger $z$ results in a smaller $B_{c_2}$ and therefore influences the structure of the vortex lattices. Furthermore, close comparisons between our results and those of the Ginzburg-Landau theory reveal the fact that the upper critical magnetic field $B_{c_2}$ is inversely proportional to the square of the superconducting coherence length $\xi$, regardless of the anisotropy between space and time.

Keywords: AdS/CFT correspondence, Holographic superconductor, Lifshitz gravity

*E-mail address: shufuwen@ncu.edu.cn
I. INTRODUCTION

The gauge-gravity duality [1–3] offers a very promising way to explore the possible dynamics of strongly interacting matter in field theory. It provides a well-established method for calculating correlation functions in a strongly interacting field theory in terms of a dual classical gravity description. While its relevance to any specific strong coupling system that can realized in the laboratory is not well-understood, it nonetheless provides a window through which we might hope to obtain insight into the properties of some condensed matter systems that defy description by traditional approaches. One of the unsolved mysteries in modern condensed matter physics is the mechanism of the high temperature superconductors cuprates (HTSC). These materials are layered compounds with copper-oxygen planes and are doped Mott insulators with strong electronic correlations which the pairing symmetry is unconventional and there is a strong experimental evidence showing that it is the $d$-wave superconductor. This makes the $d$-wave superconductor particularly attractive for physicists.

It was Gubser who first noticed that by coupling the Abelian Higgs model to gravity with a negative cosmological constant, one can find solutions that spontaneously break the Abelian gauge symmetry via a charged complex scalar condensate near the horizon of the black hole[4, 5]. This model exhibits the key properties of superconductivity: a phase transition at a critical temperature, where a spontaneous symmetry breaking of a $U(1)$ gauge symmetry in the bulk gravitational theory corresponds to a broken global $U(1)$ symmetry on the boundary, and the formation of a charged condensate. Based on this observation, Hartnoll et al proposed a holographic model for $s$-wave superconductors by considering a neutral black hole with a charged scalar and the Maxwell field[6]. Since then this correspondence has been widely explored in order to understand several crucial properties of these holographic superconductors (see Ref. [7] for reviews). The gravitational model that dual to the $d$-wave superconductors was proposed in [8, 9] where the complex scalar field for the $s$-wave model is replaced by a symmetric traceless tensor.

One of the major characteristic properties of superconductors is that they expel magnetic fields as the temperature is lowered through the critical temperature. In the presence of an external magnetic field, ordinary superconductors may be classified into two categories, namely type I and type II. It was found that at $T < T_c$, magnetic field expels the wave
condensation for holographic $s$-wave superconductor\cite{9, 10}, holographic $p$-wave superconductor \cite{11}, and holographic $d$-wave superconductor\cite{12} as well, along with the formation of Abrikosov vortices. This indicates that these holographic models of superconductor belong to type II ones. However, all these holographic models were constructed only in the relativistic spacetimes. Thus we wonder whether the holographic $d$-wave superconductor still be the type II one in non-relativistic spacetimes, for example, Lifshitz spacetime, which is our main motivation in this paper.

It is often observed that the behaviors of many condensed matter systems are governed by Lifshitz-like fixed points. These fixed points are characterized by the anisotropic scaling symmetry

\[ t \rightarrow \lambda^z t, \quad x^i \rightarrow \lambda x^i, \]

where $z$ is called the dynamical critical exponent and it describes the degree of anisotropy between space and time. The nonrelativistic nature of these systems makes the dual description different and a gravity dual for such systems can be realized by nonrelativistic CFTs \cite{14–17}. Recently, Bu used the nonrelativistic AdS/CFT correspondence to study the holographic superconductors in the Lifshitz black hole geometry for $z = 2$ in order to explore the effects of the dynamical exponent and distinguish some universal properties of holographic superconductors \cite{18}. It is found that the Lifshitz black hole geometry results in different asymptotic behaviors of temporal and spatial components of gauge fields compared to those in the Schwarzschild-AdS black hole, which brings some new features of holographic superconductor models. More recently, Lu et al. discussed the effects of the Lifshitz dynamical exponent $z$ on holographic superconductors and gave some different results from the Schwarzschild-AdS background \cite{19}. To date, there have attracted considerable interest to generalize the holographic superconducting models to nonrelativistic situations \cite{20–27}.

In this paper, we analytically study the spatially dependent equations of motion for the $d$-wave holographic superconductor with Lifshitz scaling when the added magnetic field is slightly below the upper critical magnetic field. We want to distinguish the effects of the dynamical exponent to the vortex lattice and explore the behavior of the upper critical magnetic field. In particular, according to the Ginzburg-Landau (GL) theory, it should be noted that the upper critical magnetic field has the well-known relation $B_c \propto (1 - T/T_c)$ \cite{28}. A number of attempts have been made to investigate the effects of applying an external magnetic field to holographic dual models \cite{29–41}. All these papers are made in relativistic
situations. It is therefore very natural to consider the nonrelativistic situations, such as Lifshitz black hole. Furthermore, we constructed the vortex lattice solution, or the Abrikosov lattice which is characterized by two lattice parameters, \( a_1 \) and \( a_2 \), perturbatively near the second-order phase transition.

The organization of this paper is as follows. In section 2, we will study the \( d \)-wave holographic superconductors with Lifshitz scaling. In section 3 we investigate the properties of the holographic superconductors with Lifshitz scaling in an external magnetic field. Section 4 is devoted to the construction of triangle vortex solution of the \( d \)-wave model. And we will conclude in the last section of our main results.

II. THE \( d \)-WAVE HOLOGRAPHIC SUPERCONDUCTOR MODELS WITH LIFSHITZ SCALING: A BRIEF REVIEW

In this section we first give the spatial dependent equations of motion for the \( d \)-wave model in the presence of a uniform magnetic field, then we will study the condensate solution and discuss the critical temperature.

A. Holographic \( d \)-wave superconductor: the model

The action of the \( d \)-wave superconductor in 4 dimensions is the following\(^1\)[8]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ (R - 2\Lambda) + \mathcal{L}_m \right\},
\]

\[
\mathcal{L}_m = -\frac{L^2}{q^2} \left[ (D_\mu B_{\nu\gamma})^* D^\mu B^{\nu\gamma} + m^2 B^*_{\mu\nu} B^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]
\]

where \( B_{\mu\nu} \) is a symmetric traceless tensor, \( R \) is the Ricci scalar, \( \Lambda = -\frac{3}{L^2} \) is the negative cosmological constant with \( L \) the AdS radius, and \( \kappa^2 = 8\pi G_N \) is the gravitational coupling. \( D_\mu = \partial_\mu + iA_\mu \) is the covariant derivative, \( q \) and \( m^2 \) are the charge and mass squared of \( B_{\mu\nu} \), respectively.

Working in the probe limit in which the matter fields do not backreact on the metric and taking the planar Lifshitz-AdS ansatz, the black hole metric reads:

\[
ds^2 = \frac{L^2}{r^2} \left( -r^2 h(r) dt^2 + \frac{1}{r^2 h(r)} dr^2 + r^2 dx^2 + r^2 dy^2 \right)
\]

\(^1\) In principle, it is possible to generalize our analysis to higher dimensions.
where the metric coefficient

\[ h(r) = 1 - \frac{r_+^{z+2}}{r^{z+2}}, \]

and \( r_+ \) is the horizon radius of the black hole. The Hawking temperature of the black hole is \( T = \frac{(z+2)r_+^z}{4\pi L^2} \). Setting \( u = \frac{r_+}{r} \), the metric can be rewritten in the form

\[ ds^2 = L^2 \left( -\frac{r_+^{2z}}{u^{2z}} h(u) dt^2 + \frac{du^2}{u^2 h(u)} + \frac{r_+^2}{u^2} dx^2 + \frac{r_+^2}{u^2} dy^2 \right) \]

in which \( h(z) = 1 - u^{z+2} \).

The \( d \)-wave superconductors is translational invariant and condensate on the boundary, while the rotational symmetry is broken down to \( Z(2) \) due to the condensate change its sign under a \( \pi/2 \) rotation on the \( x - y \) plane. To fulfill these features we take the following ansatz\[8\]

\[ B_{\mu\nu} = \text{diagonal} \ (0, 0, f(u, x, y), -f(u, x, y)), \quad A = \phi(u, x, y) dt + A_y(u, x, y) dy. \]  

After variation of the action with this ansatz, the equations of motion for the tensor field \( B_{\mu\nu} \), the gauge field components \( A_t \) and \( A_y \) are given, respectively, by

\[
\begin{align*}
    h\partial_u^2 f + \left( \partial_u h + 3 - \frac{z}{u} \right) \partial_u f + \frac{1}{r_+^2} \left( \partial_u^2 f + \partial_y^2 f \right) + \frac{2iA_y}{r_+^2} \partial_y f + \frac{i\partial_y A_y}{r_+^2} f + \frac{2\partial_u h}{u} f \\
    + \frac{u^{2z-2} \phi^2}{r_+^{2z} h} f - \frac{2(z + 1)h}{u^2} f - \frac{A_y^2}{r_+^2} f - \frac{L^2 m^2}{u^2} f = 0, \quad (7) \\
    h\partial_u^2 \phi + \frac{1}{r_+^2} \left( \partial_x^2 + \partial_y^2 \right) \phi + \frac{(z - 1)h}{u} \partial_u \phi - \frac{4u^2 |f|^2 \phi}{r_+^4 L^2} = 0, \quad (8) \\
    h\partial_u^2 A_y + \left( \partial_u h - \frac{z - 1}{u} h \right) \partial_u A_y + \frac{1}{r_+^2} \partial_y^2 A_y + \frac{2iu^2 f^* \partial_y f}{r_+^4 L^2} - \frac{2iu^2 f \partial_y f^*}{r_+^4 L^2} - \frac{4u^2 A_y |f|^2}{r_+^4 L^2} = 0. \quad (9)
\end{align*}
\]

B. Condensate in holographic \( d \)-wave superconductors

The equations of motion (7)-(9) of the \( d \)-wave superconductors are very similar to the \( s \)-wave model and the matching method should be very efficient. In order to solve the above equations, let us impose the boundary condition near the horizon and in the asymptotic AdS region, respectively:

1). On the horizon \( z = 1 \), the scalar potential must be vanishing \( \phi = 0 \) since the \( \phi dt \) must be well defined and the other fields should be regular.
2). On the boundary $z = 0$, the solution of fields behaves like

$$ f(u) = J_- u^\Delta_- + J_+ u^\Delta_+, \quad (10) $$

$$ \phi(u) = \mu - \rho \frac{u^{2-z}}{r_+^{2-z}} + \cdots, \quad (11) $$

$$ B(x) = \partial_x A_y - \partial_y A_x, \quad (12) $$

where $\Delta_\pm = -\frac{(2-z)\pm \sqrt{(2-z)^2+8(z+1)+4m^2L^2}}{2}$. The coefficients $J_-$ represents as the source of the dual operator and $J_+$ correspond to the vacuum expectation values of the operator that couples to $B_{\mu\nu}$ at the boundary theory. BF bound requires $m^2L^2 \geq -\frac{(2-z)^2+8(z+1)}{4}$ (thus $\Delta_+ \geq 0$) such that the $J_+$ term is a constant or vanishes on the boundary.

Then, in order to solve the critical temperature with the spatial dependent equations of motion, we need to ignore the influence of the external field so as to get the equations of motion only with the reaction of radial coordinates:

$$ f'' + \left( \frac{h'}{h} + \frac{3-z}{u} \right) f' + \left( \frac{2h'}{uh} + \frac{u^{2-z-2}\phi^2}{r_+^{2z}h^2} - \frac{2(z+1)}{u^2} - \frac{m^2L^2}{u^2h} \right) f = 0, \quad (13) $$

$$ \phi'' + \frac{2}{u} \phi' - \frac{4u^2 |f|^2}{r_+^4 L^2 h^2} \phi = 0. \quad (14) $$

As we can see, the change of the equations does not affect the boundary conditions. We impose boundary condition $J_- = 0$ in the following discussion. For clarify, we set $J = J_+$ and $\Delta = \Delta_+$ in this work.

It should be noted that Frobenius analysis of the equations of motion near the boundary reveals that $\phi(u) = \rho - \mu \log u$ for the case $z = d$. For simplicity, we will not consider this case in the following studies.

Following the matching method applied in [42], which expands the fields $f$ and $\phi$ near the horizon $u = 1$, reads off the expanded solutions from the equations of motion with the above boundary conditions, then matches the asymptotic solutions at some intermediate point $u = u_m$, in the end we obtain

$$ J = \frac{u_m^{1-\Delta} \left[ 2(z+2)(2-u_m) + m^2L^2(1-u_m) \right]}{(z+2) \left[ (1-u_m)\Delta + 2u_m \right]} f(1), \quad (15) $$

where

$$ f^2(1) = \frac{(z+2) \left[ 1 + (1-z)(1-u_m) \right]}{4(1-u_m)} \left( \frac{4\pi T}{z+2} \right)^\frac{4}{3} \left( \frac{T_c}{T} \right)^\frac{2}{3} \left[ 1 - \left( \frac{T}{T_c} \right)^\frac{2}{3} \right], \quad (16) $$
and we have defined the critical temperature $T_c$ as

$$T_c = \frac{z + 2}{4\pi} \left[ \frac{(2-z)\rho u_m^{1-z}}{\alpha [1 + (1-z)(1-u_m)]} \right]^\frac{1}{2}.$$  \hspace{1cm} (17)

The parameter $\alpha$ in (17) is given by

$$\alpha^2 = \frac{14(z+2)^2 + 10(z+2)m^2L^2 + m^4L^4}{(1-u_m)^2} + \frac{[8(z+2)^2 + 4(z+2)m^2L^2]}{(1-u_m)^2} \left[ u_m + (1-u_m)\Delta \right] + 4(z+2)^2\Delta.$$  \hspace{1cm} (18)

In order to avoid a breakdown of the matching method, we take the value of $m^2$ to ensure that $\alpha$ is real and find the range of the matching point

$$0 < u_m < 1, \text{ for } -(7 - \sqrt{23})(z+2) < m^2 < 0.$$  \hspace{1cm} (19)

It is interesting to observe that the value of $-(7 - \sqrt{23})(z+2)$ is smaller than $-\frac{(2-z)^2 + 8(z+1)}{4}$ when $z < 2$, which means that the value of $\alpha$ is real all the while when we set the range of $m^2$

$$-\frac{1}{4} \left[ (z-2)^2 + 8(z+1) \right] < m^2 < 0,$$  \hspace{1cm} (20)

so it is convenient to use the range $0 < u_m < 1$ in this work. In addition, Eq. (17) implies that the larger dynamical exponent $z$ makes the condensation harder to form.

According to the AdS/CFT dictionary, near the critical temperature $T \sim T_c$ we can express the relation for the condensation operator $\langle O \rangle = J r^\Delta_{++}$ as

$$\langle O \rangle = \left( \frac{4\pi T_c}{z + 2} \right)^{\frac{2+\Delta}{2}} \left\{ \frac{u_m^{1-\Delta} [2(z+2)(2-u_m) + m^2L^2(1-u_m)]}{(1-u_m)^2 + 2u_m \Delta} \right\} \cdot \left[ 1 + (z-1)(1-u_m) \right]^{\frac{1}{2}} \left[ 1 - \left( \frac{T}{T_c} \right)^{\frac{2}{z}} \right]^{\frac{1}{2}}.$$  \hspace{1cm} (21)

The analytic result shows that the phase transition of holographic superconductors with Lifshitz scaling belongs to the second order. It also indicates that condensation versus temperature have a square root behavior near $T_c$, which suggests that the critical exponent is $1/2$, as expected from the mean field theory. The Lifshitz scaling and spacetime dimension will not influence the result[19].

In Fig.1, we visualize the condensate of the operator $\langle O \rangle$ as a function of temperature with different dynamical exponent $z$ for the mass of the traceless tensor field $m^2L^2 = -1/4$. It is observed that corresponding to the lower critical temperature, the gap becomes increasingly smaller as $z$ increases than the results in [42].
FIG. 1: The condensation operator as a function of $T/T_c$ obtained by using the analytic matching method. We choose $m^2L^2 = -1/4$ and $u_m = 1/2$. The top point line corresponds to $z = 1$ and bottom one is $z = 3/2$.

III. EFFECTS OF EXTERNAL MAGNETIC FIELD ON THE HOLOGRAPHIC $d$-WAVE SUPERCONDUCTOR

In this section we would like to study the effect of external magnetic field on the holographic superconductors with Lifshitz scaling. From the gauge/gravity correspondence, the asymptotic value of the magnetic field corresponds to a magnetic field added to the boundary field theory. Near the upper critical magnetic field $B_{c2}$, the tensor field $f$ can be regarded as a perturbation.

A. Perturbative expansion of the equations of motion

To exactly solve the above nonlinear coupled partial differential equations is a difficult task. But we can perturbatively solve these equations when the magnetic field is slightly below the upper critical field $B_{c2}$ [11]. First we define a small parameter $\epsilon = \frac{B_{c2} - B}{B_{c2}}$, then
we can expand the fields as

\[ f(x, u) = \epsilon^{1/2} f_1(x, u) + \epsilon^{3/2} f_2(x, u) + \cdots, \quad \text{(22a)} \]

\[ A_y(x, u) = A_y^{(0)}(x, u) + \epsilon A_y^{(1)}(x, u) + \cdots, \quad \text{(22b)} \]

\[ \phi(x, u) = \phi^{(0)}(x, u) + \epsilon \phi^{(1)}(x, u) + \cdots \quad \text{(22c)} \]

in which \( x = (x, y) \). The zeroth order solution corresponding to the normal state is

\[ f = 0, \quad \phi = \mu - \rho \left( \frac{u}{r^+} \right)^{2-z}, \quad A_y^0 = B_{c_2}x. \quad \text{(23)} \]

We can see clearly that the magnetic field on the boundary is \( B_{c_2} \). Substituting Eq. (23) into the equations of motion (7), with the following ansatz \( f_1(x, u) = e^{ipy} F(x, u; p) / L \) (\( p \) is a constant), the equation of motion for \( F \) is

\[
\begin{align*}
\left[ h \partial_u^2 + \left( \frac{3 - z}{u} \right) \partial_u + \frac{2 \partial_u h}{u} + \frac{u^{2z-2} \phi^2}{r^+_2 h} - \frac{2(z+1)h}{u^2} - \frac{L^2 m^2}{u^2} \right] F(x, u; p) = \frac{1}{r^+} \left[ -\partial_x^2 + (p + B_{c_2}x)^2 \right] F(x, u; p). \quad \text{(24)}
\end{align*}
\]

Then we separate the \( F \) as \( F_n(x, u; p) = \rho_n(u) \gamma_n(x; p)/L \), where \( \lambda_n \) is a constant. \( \rho_n \) and \( \gamma_n \) admit the following equations:

\[
\begin{align*}
\left( -\frac{\partial^2}{\partial X^2} + \frac{X^2}{4} \right) \gamma_n(x; p) &= \frac{\lambda_n}{2} \gamma_n(x; p), \quad \text{(25a)}
\end{align*}
\]

\[
\begin{align*}
\left[ h \partial_u^2 + \left( \frac{3 - z}{u} \right) \partial_u \right] \rho_n(u) &= \left( \frac{m^2 L^2}{u^2} + \frac{2(z+1)h}{u^2} - \frac{2 \partial_u h}{u} - \frac{u^{2z-2} \phi^2}{r^+_2 h} + \frac{B_{c_2} \lambda_n}{r^+_2} \right) \rho_n(u), \quad \text{(25b)}
\end{align*}
\]

where \( X := \sqrt{2B_{c_2}}(x + p/B_{c_2}) \) in our calculation. Eq. (25a) determines the distribution of the order parameter on the \( x-y \) plane, while Eq. (25b) determines when a superconducting phase transition will occur.

**B. The upper critical magnetic field**

There is critical value \( B_{c_2} \) above which Eq. (25b) only has vanishing solutions. As one lowers the magnetic field below \( B_{c_2} \), we lead to a phase transition. The maximum upper critical magnetic field is given by \( n = 0 \) where \( \lambda_n = 2n + 1 \) take the minimum value. Thus, we can express the equation of \( \rho(u) \) as

\[
\begin{align*}
\rho'' + \left[ \frac{h'}{h} + \frac{3 - z}{u} \right] \rho' + \left[ \frac{2h'}{uh} + \frac{u^{2z-2} \phi^2}{r^+_2 h^2} - \frac{2(z+1)h}{u^2} - \frac{m^2 L^2}{u^2 h} - \frac{B_{c_2}}{r^+_2} \right] \rho &= 0. \quad \text{(26)}
\end{align*}
\]
The asymptotic behavior \((u \to 0)\) for (26) can be expressed as

\[
\rho(u) = J_- u^{\Delta_-} + J_+ u^{\Delta_+}.
\] (27)

In the following calculation we still let \(J_- = 0\) and set \(J = J_+\) and \(\Delta = \Delta_+\) just as discussed in the previous section.

Using the matching method just what we did in the last section, one can get from Eq. (26)

\[
p B_{c_2}^2 + r_+^2 \left\{ [8(z + 2) + 2m^2L^2] p + 4(z + 2)q \right\} B_{c_2} + r_+^4 \left\{ [m^4L^4 + 10(z + 2)m^2L^2
+ 14(z + 2)^2] p + [8(z + 2)^2 + 4(z + 2)m^2L^2] q + 4(z + 2)^2 \Delta - p\alpha^2 \right\} = 0,
\] (28)
in which \(p = [2u_m + (1 - u_m)\Delta](1 - u_m)\) and \(q = [u_m + (1 - u_m)\Delta].\)

In order to get the external critical magnetic field which is very close to the critical magnetic field we find the solution

\[
B_{c_2} = \frac{r_+^2}{2p} \left( \sqrt{\gamma + 4\rho^2\alpha^2} - \beta \right),
\] (29)

with

\[
\gamma = 8(z + 2) [(z + 2) - m^2L^2] p^2 + 32(z + 2)^2 pq + 16(z + 2)^2q^2 - 16(z + 2)^2 \Delta p,
\] (30)

\[
\beta = \left[ 2(z + 2) + 2m^2L^2 \right] p + 4(z + 2)q.
\] (31)

As \(T = \frac{(z+2)\gamma^{\frac{3}{2}}}{4\pi L^2}\), we can express the critical magnetic field \(B_{c_2}\) as

\[
B_{c_2} = \left( \frac{4\pi T_c}{z + 2} \right)^{\frac{1}{2}} \frac{1}{2p} \left( \sqrt{\beta^2 - \gamma} \left[ 1 + (z - 1)(1 - u_m)^2 \right] u_m^{2z-2} + \gamma \left( \frac{T}{T_c} \right)^{\frac{3}{2}} - \beta \left( \frac{T}{T_c} \right)^{\frac{4}{2}} \right),
\] (32)

which has the same form as [42]. It is convenient to observe that there is a superconducting phase transition when \(B_{c_2} = 0\) at \(T = T_c\) where

\[
\gamma = \beta^2.
\] (33)

This is equivalent to

\[
u_m^{2z-2} [1 + (z - 1)(1 - u_m)]^2 = 1,
\] (34)
which is related to Lifshitz scaling but independent of the tensor field mass. In order to ensure the condition $B_{c2} = 0$ at $T = T_c$, we calculate the Eq.(33) and Eq.(34) with the requirement of $m^2 L^2 \geq - \frac{(2-z)^2 + 8(z+1)}{4}$. As a consequence, we get

$$\begin{cases} 
0 < u_m < 1, & \text{as } z = 1, \\
u_m = 1, & \text{as } z \neq 1.
\end{cases}$$

(35)

What can be noted is that all of the $u_m$ is selected in the range $0 < u_m < 1$ with the situation $z = 1$ so we can choose the matching point $u_m$ arbitrarily for this case. And the results also show that the allowable range of $u_m$ is restricted at the point $u_m = 1$ when $z = \frac{3}{2}$. So we clearly find that the range of the matching point $u_m$ depends on Lifshitz scaling $z$ and tensor field mass $m$.

FIG. 2: The critical magnetic field as a function of $T/T_c$ obtained by the analytic matching method. The left graph we set $u_m = 9998/10000$, it shows that there is a breakdown of matching method. The right figure refers to $u_m = 9/10$, which has more reasonable behavior. As before, in both graphs red point line denotes $z = 1$ and blue bold line describes $z = 3/2$.

It is subtle for $z \neq 1$ where the matching point $u_m = 1$ leads to vanishing critical temperature $T_c$ as shown in (17) and (18), which implies a breakdown of the matching method. Our strategy is to matching the result by shifting the matching point $u_m$ to a small value $\delta$ from 1. However, $\delta$ cannot be arbitrarily small. As $u_m \rightarrow 1$, the value of $p$ approaches to zero, which causes $B_{c-2}$ divergent. As an example, we choose the Lifshitz scaling $z = 3/2$ with $L = 1, m^2 = -1/4$ and $r_+ = 1$, the left graph of Fig.2 shows that there exists a breakdown of matching method when $u_m = \frac{9998}{10000}$.
Keep this in mind, we should choose $\delta$ is large enough so as to keep $B_{c2}$ finite. As another example, we set $u_m = 9/10$ which can relax the breakdown when the matching point approaches the allowable region. The right graph of Fig.2 proves this point and shows that the critical magnetic field $B_{c2}$ decreases as we amplify $z$ which is qualitatively in good agreement with the work of [42]. When $T \sim T_c$ we can have $B_{c2} \propto (1 - T/T_c)$ for different Lifshitz scaling which agrees well with the Ginzburg-Landau theory. And it is also noted that the dynamical exponent $z$ cannot modify the relation. Thus, for the case $1 \leq z < d = 2$, the Ginzburg-Landau theory still holds in Lifshitz black hole.

**IV. CONSTRUCTION OF TRIANGLE VORTEX SOLUTION OF THE $d$-WAVE SUPERCONDUCTOR.**

Following [11], in this section we will try to construct, based on our previous observations in the last sections, the triangle vortex solution of $d$-wave superconductor model.

Our start point is Eq. (25a), whose solution of that satisfies the boundary condition and $\lim_{|x| \to \infty} |\gamma_n| < \infty$ can be expressed in terms of the Hermite functions $H_n$ as follows

$$\gamma_n(x; p) = e^{-x^2/4}H_n(x),$$

and the corresponding eigenvalue $\lambda_n$ is

$$\lambda_n = 2n + 1,$$

where $n = 0, 1, 2, 3 \cdots$. The $n = 0$ solution is the droplet solution, and the vortex solution can be constructed from the droplet solution:

$$\gamma_0(x; p) = e^{-x^2/4} = \exp \left[ -\frac{1}{2r_0^2} \left( x + pr_0^2 \right)^2 \right],$$

where $r_0 := 1/\sqrt{B_{c2}}$.

Since $\lambda_n$ is independent of $p$, a linear superposition of the solutions $e^{ipy}r_0(u)\gamma_0(x; p)$ with different $p$ is also a solution of the equation of motion for $f_1$:

$$f_1(x, u) = \frac{r_0(u)}{L} \sum_i c_i e^{ipy}\gamma_0(x; p_i).$$

Here we get the most important result in this section. When we choose a suitable configuration of $c_i$ and $p_i$, we can construct triangular lattice solutions. It is very interesting that
the result Eq. (38) is very similar to the expression of the order parameter of G-L theory for the type II superconductor in the presence of a magnetic field when \( B = B_{c_2} \), which is

\[
\psi_L = \sum_l c_l e^{ipy} \exp[-\frac{x-x_l}{2\xi^2}],
\]

(40)

where \( \xi \) is the superconducting coherence length, \( x_l = \frac{k \Phi_0}{2\pi B} \), and \( \Phi_0 \) is the flux quantum. Comparing Eq. (39) with Eq. (40), we get

\[
B_{c_2} \propto \frac{1}{\xi^2},
\]

(41)

which is also similar to the result of the GL theory. According to the behavior that \( B_{c_2} \propto (1 - T/T_c) \) near \( T_c \), we have \( \xi \propto (1 - T/T_c)^{-1/2} \). This result is also the same as that of the GL theory. We have also obtained this result by another way in Ref. [13].

Thus, the construction of triangular lattice from droplet solutions is similar to what Abrikosov did in his initial paper. This procedure has been made for the \( s \)-wave model in Ref. [11]. In the \( d \)-wave model, the construction process is the same. We briefly review the result below, considering the following form of \( p_l \) and \( c_l \):

\[
f_1(x, u) = \frac{\rho_0(u)}{L} \sum_{l=-\infty}^{\infty} c_l e^{ipy} \gamma_0(x; p_l),
\]

(42a)

\[
c_l := \exp \left( -\frac{i\pi l^2}{2} \right), \quad p_l := \frac{2\pi l}{a_1 r_0},
\]

(42b)

for arbitrary parameters \( a_1 \). The solution in Eq. (42) represents a lattice. \( \sigma(x) := |\gamma_L(x)|^2 \) in which the fundamental region \( V_0 \) is spanned by two vectors \( b_1 = a_1 r_0 \partial_y \) and \( b_2 = 2\pi r_0/a_1 \partial_y + a_1 r_0/2 \partial_y \), and the area is given by \( 2\pi r_0^2 \). Then the magnetic flux penetrating the unit cell is given by \( B_{c_2} \times (\text{Area}) = 2\pi \). This shows the quantization of the magnetic flux penetrating a vortex.

Fig.3 shows the configuration of \( \sigma(x) := |\gamma_L(x)|^2 \) in the \((x, y)\) plane for the triangular lattice. According to the previous section’s conclusion, in order to gain the graph we choose \( u_m = 9/10, \; m^2 L^2 = -1/4, \; r_+ = 1 \) and \( \rho = 80 \) for the Lifshitz scaling \( z = 1 \) and \( z = 3/2 \). It is interesting that from the graphs the quantity of vortex lattices decreases when we amplify \( z \) with the sizes becoming bigger although the height of \( \sigma \) is not modified. It indicates that the dynamical exponent \( z \) influences the vortex lattice solution but it can not modify the magnetic field in the holographic superconductors.
FIG. 3: The vortex lattice structure for the triangular lattice in the \((x, y)\) plane. The vertical line represents \(\sigma = |\gamma_L|^2\), and vortex cores are located at \(|\gamma_L| = 0\).

The order parameter vanishes at

\[
x_{m,n} = \left( m + \frac{1}{2} \right) b_1 + \left( n + \frac{1}{2} \right) b_2,
\]

for any integers \(m, n\). The phase of \(\langle O \rangle \propto \gamma_L(x)\) rotates by \(2\pi\) around each \(x_{m,n}\). When

\[
\frac{a_1}{2} = 3^{-1/4} \sqrt{\pi},
\]

the three adjoining vortices \(x_{m,n}\) form an equilateral triangle, which is the triangular vortex lattice solution. Here, following [8], in our calculation we only focus on the case of \(L^2m^2 = -1/4\). Since different allowed values of mass only change the dimension of the condensation operator in Eq. (10), it will not affect the Eq. (25a) and also our construction of vortex solution. So we can conclude that for other allowed values of \(m\), the vortex solutions are also expected.

V. CONCLUSIONS AND DISCUSSIONS

In this work, we have used the matching method to investigate the \(d\)-wave holographic superconductors with Lifshitz scaling in the presence of external magnetic field. Based on purely analytic methods, the vortex lattice solutions of the model have also been obtained with different Lifshitz scaling. This implies that holographic \(d\)-wave superconductor, regardless of the anisotropy between space and time, is indeed a type II one, which is the same as
the conventional $d$-wave superconductors in the GL theory. Our results also indicates that the dynamical exponent $z$, although, has no effect to the form of vortex lattice solutions as shown in (42a), it influences the structure of the vortex lattices through $B_c$. Also, close comparisons between our results and those of the GL theory reveal the fact that the upper critical magnetic field $B_{c_2}$ is inversely proportional to the square of the superconducting coherence length $\xi$.

Working in the probe limit, we obtained analytic expressions for the order parameter, the critical temperature and the upper critical magnetic field. The analytic calculation is useful for gaining insight into the strong interacting system. It is noted that the critical temperature decreases with the increase of the dynamical exponent $z$ showing that Lifshitz scaling makes the condensation harder to occur. In the section of the critical magnetic field, we also observed the behavior satisfying the relation given in the Ginzburg-Landau theory. The result shows that the dynamical exponent $z$ does have effects on the upper critical magnetic field based on the facts that a larger $z$ results in a smaller upper critical magnetic field.

Although we have performed detailed analysis on some issues of holographic Lifshitz $d$-wave superconductor in the presence of external magnetic field, it would be many more interesting outcomes that deserve further investigations. Some of these are as follows: (i) It would be natural to generalize our discussions to Fermion field and to see how the dynamic exponent $z$ influence its condensation and vortex lattice solutions (There is some related work such as Ref. [43], where the authors studied the fermionic wavefunctions for the relativistic $d$-wave superconducting background and found the formation of Fermi arcs. ). (ii) It would be possible to find analogous vortex lattice solutions for the hyperscaling violation holographic models. (iii) One significant difference between the conventional superconductors in the GL theory and the holographic superconductor hide in the free energy and the $R$-current. It is interesting to obtain these two quantities for our model and study the effects of anisotropy on them. (iv) It is also possible to consider the holographic superconductor model in the framework of modified gravity, such as the Hořava-Lifshitz gravity[44] proposed recently by Hořava. Indeed, it was found that HL gravity is a minimal holographic dual for the field with Lifshitz scaling[45]. Our recent works [46, 47] further revealed this point and found that various Lifshitz spacetimes are possible even without matter fields. It is of particular interests to see how to construct the holographic superconductor models in this framework.
Acknowledgements

We would like to thank X.-H. Ge for useful discussions. This work was supported in part by the National Natural Science Foundation of China (under Grant Nos. 11465012 and 11005165), the Natural Science Foundation of Jiangxi Province (under Grant No. 20142BAB202007) and the 555 talent project of Jiangxi Province.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, [arXiv:hep-th/9711200].
[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428 (1998) 105, [arXiv:hep-th/9802109].
[3] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, [arXiv:hep-th/9802150].
[4] S. S. Gubser, Class. Quant. Grav. 22 (2005) 5121.
[5] S. S. Gubser, Phys. Rev. D 78 (2008) 065034.
[6] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. 101 (2008) 031601 [arXiv:0803.3295[hep-th]].
[7] S. A. Hartnoll, Classical Quantum Gravity 26, 224002 (2009); C. P. Herzog, J. Phys. A 42 (2009) 343001; G. T. Horowitz, arXiv:1002.1722.
[8] J. W. Chen, Y. J. Kao, D. Maity, W. Y. Wen, C. P. Yeh, Phys. Rev. D 81 (2010) 106008 [arXiv:1003.2991[hep-th]].
[9] F. Benini, C. P. Herzog, R. Rahman, A. Yarom, J. High Energy Phys. 1011 (2010) 137 [arXiv:1007.1981[hep-th]].
[10] M. Montull, A. Pomarol, and P. J. Silva, Phys. Rev. Lett. 103 (2009) 091601.
[11] K. Maeda, M. Natsuume, T. Okamura, Phys. Rev. D 81 (2010) 026002, [arXiv:0910.4475[hep-th]].
[12] J. M. Murray and Z. Tešanović, Phy. Rev. D 83 (2011) 126011.
[13] H. Zeng, Z. Fan, Z. Ren, Phys. Rev. D 82 (2010) 126014, [arXiv:1006.5483[hep-th]].
[14] D. T. Son, Phys. Rev. D 78 (2008) 046003, [arXiv:0804.3972[hep-th]].
[15] K. Balasubramanian, J. McGreevy, Phys. Rev. Lett. 101 (2008) 061601, [arXiv:0804.4053[hep-th]].
[16] W. D. Goldberger, J. High Energy Phys. 0903 (2009) 069, [arXiv:0806.2867].
[17] M. Taylor, [arXiv:0812.0530 [hep-th]].

[18] Y. Y. Bu, Phys. Rev. D 86, 046007 (2012).

[19] J. W. Lu, Y. B. Wu, P. Qian, Y. Y. Zhao, and X. Zhang, Nucl. Phys. B 887 (2014) 112-135, [arXiv:1311.2699 [hep-th]].

[20] R. G. Cai and H. Q. Zhang, Phys. Rev. D 81 (2010) 066003.

[21] J. L. Jing, L. C. Wang, and S. B. Chen, [arXiv:1001.1472 [hep-th]].

[22] S. J. Sin, S. S. Xu, and Y. Zhou, Int. J. Mod. Phys. A 26 (2011) 4617.

[23] E. Brynjolfsson, U. H. Danielsson, L. Thorlacius, and T. Zingg, J. Phys. A 43 (2010) 065401.

[24] D. Momeni, R. Myrzakulov, L. Sebastiani, and M.R. Setare, [arXiv:1210.7965 [hep-th]].

[25] F. A. Schaposnik and G. Tallarita, Phys. Lett. B 720 (2013) 393.

[26] E. Abdalla, J. Oliveira, A. B. Pavan, and C. E. Pellicer, [arXiv:1307.1460 [hep-th]].

[27] G. Tallarita, [arXiv:1402.4691 [hep-th]].

[28] C. P. Poole, H. A. Farach and R. J. Creswick, Superconductivity, Academic Press, The Netherlands (2007).

[29] M. R. Setare and D. Momeni, Europhys. Lett. 96 (2011) 60006.

[30] X. H. Ge and H. Q. Leng, Prog. Theor. Phys. 128 (2012) 1211.

[31] M. Montull, O. Pujolas, A. Salvio, and P. J. Silva, J. High Energy Phys. 04 (2012) 135.

[32] D. F. Gao, Phys. Lett. A 376 (2012) 1705.

[33] X. H. Ge, S. F. Tu and B. Wang, J. High Energy Phys. 09 (2012) 088.

[34] D. Roychowdhury, Phys. Rev. D 86 (2012) 106009.

[35] D. Momeni, E. Nakano, M. R. Setare, and W. Y. Wen, Int. J. Mod. Phys. A 28 (2013) 1350024.

[36] D. Roychowdhury, Phys. Lett. B 718 (2013) 1089.

[37] S. L. Cui and Z. Xue, Phys. Rev. D 88 (2013) 107501; X. M. Kuang, E. Papantonopoulos, G. Siopsis and B. Wang, Phys.Rev. D 88 (2013) 086008, [arXiv:1303.2575].

[38] D. Roychowdhury, J. High Energy Phys. 05 (2013) 162; D. Roychowdhury,J. High Energy Phys. 1410 (2014) 18, [arXiv:1407.3464 [hep-th]]; D. Roychowdhury, arXiv:1403.0085 [hep-th]; N. Banerjee, S. Dutta, D. Roychowdhury, [arXiv:1311.7640 [hep-th]].

[39] R. G. Cai, S. He, L. Li, and L.F. Li, J. High Energy Phys. 12 (2013) 036, [arXiv:1309.2098 [hep-th]].

[40] S. Gangopadhyay, [arXiv:1311.4416 [hep-th]].

[41] A. Lala, Phys. Lett. B 735 (2014) 396, [arXiv:1404.2774 [hep-th]].
[42] Z. Zhao, Q. Pan, J. Jing, Phys. Lett B\textbf{735} (2014) 438-444, [arxiv:1311:6260v3 [hep-th]].

[43] F. Benini, C. P. Herzog, A. Yarom, Phys. Lett. B \textbf{701} (2011) 626-629, [arXiv: 1006.0731].

[44] P. Hořava, Phys. Rev. D \textbf{79} (2009) 084008, [arXiv:0901.3775].

[45] T. Griffin, P. Hořava, and C. Melby-Thompson, Phys. Rev. Lett. \textbf{110} (2013) 081602.

[46] F.-W. Shu, K. Lin, A. Wang, and Q. Wu, J. High Energy Phys. \textbf{04} (2014) 056.

[47] K. Lin, F.-W. Shu, A. Wang, and Q. Wu, [arXiv: 1403.3413[hep-th]].