Two-fluid hydrodynamic modes in a trapped superfluid gas

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I. INTRODUCTION

In 1938, Tisza [1] first suggested that the classic features of superfluid helium were a manifestation of a two-fluid hydrodynamics, originating from a Bose-Einstein condensate. The correct two-fluid equations in the non-dissipative limit were formulated and solved by Landau a few years later [2, 3]. Two-fluid hydrodynamics describes the coupled dynamics of the superfluid and normal fluid components when collisions produce a state of local thermodynamic equilibrium. This collision-induced state is described by the usual thermodynamic variables, which are now dependent on position and time. The two-fluid equations are essentially the same as the fluid dynamics of a normal fluid, but with additional equations describing the new superfluid component. In simple terms they describe the collective oscillations of frequency $\omega$ such that $\omega \tau_R \ll 1$, where $\tau_R$ describes the time it takes for a nonequilibrium state to reach local equilibrium (see, for example, Ref. [4]). For brief accounts of two-fluid hydrodynamics in the context of uniform quantum gases, see Refs. [3, 4].

The two-fluid regime was difficult to reach in the first wave of experiments on trapped Bose-condensed gases after 1995. The typical collisional cross-section and the achievable densities were not sufficient to reach the region described by Landau’s two-fluid hydrodynamics. An exception was the pioneering experiment by Ketterle and coworkers [5], which later theoretical work showed was well within the two-fluid region [6]. However, there is new interest in the study of the two-fluid collisional dynamics in trapped gases. One reason is that with atom chips, one can now produce tightly localized high density atomic Bose condensates. In addition, in a trapped two-component Fermi gas, one can use a Feshbach resonance to adjust the magnitude and sign of the $s$-wave scattering length $a$ between Fermi atoms prepared in two different hyperfine states. This allows one to study the collective modes across the BCS-BEC crossover region in great detail (see, for example, Refs. [6, 7]), with extremely large values of $|a|$. One also expects that even at the unitarity limit ($|a| \to \infty$) of trapped Fermi superfluids, one is dealing with a region where the Landau two-fluid equations are now the appropriate description of the dynamics at finite temperatures. Several recent papers [10, 11] have considered first and second sound in a uniform two-component Fermi gas in the unitarity region.

So far, discussions of collective modes in a trapped Fermi gas with strong interactions have been limited to either $T = 0$, where one is dealing with a pure superfluid [12, 13, 14, 15, 16, 17, 18], or to $T > T_c$, where one is dealing with the hydrodynamics of a normal Fermi liquid [14, 20, 21]. The much richer collective mode spectrum at intermediate temperatures described by Landau two-fluid hydrodynamics awaits exploration. In particular, the BEC side of the BCS-BEC crossover is well described as a strongly interacting Bose gas of very stable molecules [22]. This should be an ideal system to study the analogue of first and second sound in a uniform Bose superfluid predicted by the Landau two-fluid equations.

The Landau two-fluid equations describe all superfluids with a two-component order parameter in terms of coupled differential equations for the local hydrodynamic variables and the velocity fields of the two fluids. The coefficients in these equations are given in terms of equilibrium thermodynamic functions. Thus, the only differences between the hydrodynamics of various superfluids are the values of these thermodynamic functions. These depend on the thermal excitations which are appropriate for different systems (superfluid helium, BCS superconductors, trapped Bose gases, etc.). The two-fluid equations for a BCS superfluid have been derived by Galasiewicz [22].

The usual derivation [8] of the Landau two-fluid hydrodynamic equations is based on conservation laws and
macroscopic considerations, with key equations describing the irrotational superfluid velocity field \( \mathbf{v}_s(\mathbf{r}, t) \) being postulated. There is a considerable literature on this approach, which makes little contact with any microscopic theory of the origin of the superfluid component. Since the 1960s, however, it has generally been accepted that this originates from Bose-Einstein condensation (for a review, see Ref. [24]).

In the context of trapped atomic gas superfluids, an important development was the work of Zaremba, Nikuni, and Griffin (ZNG) in 1999 [27], who discussed a simple microscopic model of a condensate (described by a generalized Gross-Pitaevskii equation) coupled to a thermal gas of atoms moving in a self-consistent Hartree-Fock (HF) field (described by a kinetic Boltzmann equation generalized to include the Bose statistical factors). The ZNG coupled equations for the condensate (superfluid) and non-condensate (normal fluid) have been used to discuss the collisionless (see, for example, Ref. [26] and references therein) as well as the collision-dominated regions. In the latter region, ZNG showed that in the limit that collisions establish complete local equilibrium \( (\omega_T \tau \to 0) \), their equations can be reduced precisely to the Landau two-fluid equations, with all thermodynamic functions being given explicitly for a trapped gas of thermal atoms with a HF spectrum. This formalism is immediately applicable to a molecular Bose condensate at finite \( T \), on the BEC side of a Feshbach resonance [8].

For a uniform superfluid, the linearized Landau equations (without dissipation due to transport processes) are solved in the standard texts [3, 5, 6, 27, 28] and predict two sound modes. These are the in-phase (first sound) and out-of-phase (second sound) oscillations of the two fluids, familiar in studies of bulk superfluid helium. The mode frequencies have been calculated for a uniform weakly-interacting Bose-condensed gas (see Ref. [29] and Appendix C in Ref. [30]). For a trapped superfluid, in contrast, it is very difficult to directly solve the Landau equations for the normal modes of the gas [27, 31]. The present paper deals with a new approach for calculating the frequencies of the normal modes of a trapped superfluid.

We develop a variational approach of the kind first used by ZNG (for a slightly modified version of the two-fluid equations reviewed in Appendix B). Our discussion is based on an action given in terms of the velocity and density fields of the superfluid and normal fluid, first proposed by Zilisel in 1950 [22]. The stationary point of the action corresponds to the solutions of the Landau two-fluid differential equations. Considering fluctuations of this action about an equilibrium state, we introduce simple variational ansatzes for the velocity fields. The end result is that the two-fluid hydrodynamic modes can be described in terms of two coupled harmonic oscillators. The effective masses and spring constants of the two oscillators are given explicitly in terms of spatial integrals over various static equilibrium thermodynamic functions and their derivatives. The latter can be computed in a straightforward manner, once we have a specific model for the thermal excitations of the trapped superfluid.

We illustrate our formalism in Section VI by using it to derive first and second sound modes in a uniform superfluid as well as the frequencies of the dipole and breathing modes of a trapped superfluid.

There is an extensive older literature [32, 33, 34, 35, 36, 37, 38] on a variational formulation of the Landau two-fluid equations, with little or no reference to an underlying Bose condensate. Indeed, the kind of variational formulation we use was originally inspired by the goal of giving a clear derivation of the two-fluid equations for superfluid helium in the absence of a more microscopic theory based on BEC. In contrast, our interest in such a variational formulation lies in its usefulness in finding explicit solutions of the linearized two-fluid equations corresponding to collective modes in non-uniform trapped superfluid gases. However, in Appendix A, we briefly review some subtle aspects of the variational formulation of the two-fluid equations.

Our discussion of the two-fluid hydrodynamic equations using a variational approach is a natural extension of the approach recently used to deal with a pure superfluid in trapped gases at \( T = 0 \) [14, 16, 18]. In Section VII, we consider the \( T = 0 \) limit of our formalism to make contact with this “quantum hydrodynamic” theory, originally pioneered by Pitaevskii and Stringari [39] to extend the mean-field Gross-Pitaevskii theory of collective modes [40]. A hydrodynamic description means that a few local variables are sufficient to describe the dynamics. While such a description requires rapid collisions and local equilibrium to be valid in the case of a normal fluid, a hydrodynamic description of a superfluid is always correct. Thus it is no surprise that the \( T \to 0 \) limit of two-fluid hydrodynamics gives the correct quantum description of the pure superfluid (see discussion on p. 170 of Ref. [41]).

The discussion in Sections II-IV is fairly technical. Readers primarily interested in the final results can immediately go to Section V and the examples in Section VI. Detailed numerical predictions for the hydrodynamic mode frequencies at finite \( T \) based on our formal expressions will be presented in future papers.

II. LANDAU-KHALATNIKOV TWO-FLUID THERMODYNAMICS FOR A TRAPPED GAS

We note that since two-fluid hydrodynamics describes a system in local equilibrium, all thermodynamic quantities we discuss are functions of position and time. This dependence will usually be left implicit. Even in static equilibrium in the presence of a trapping potential, most thermodynamic quantities will still be position dependent.

Our variational principle will make use of a Lagrangian density of the form

\[
\mathcal{L} = T - U, \tag{1}
\]
where \( T \) is the kinetic energy density of the fluid and \( U \) is the internal energy density. Thus, we need to formulate the thermodynamics in terms of the internal energy density \( U \) rather than the total energy \( E = T + U \), as is normally done in two-fluid hydrodynamics (see, for example, p. 521 in Ref. [27]). Our approach will be to obtain \( U \) from the total energy density used by Landau and Khalatnikov (LK) to derive the Landau two-fluid (LTF) equations [3]. For a trapped two-fluid system, we define the total energy density \( E \) as the sum of the kinetic and potential energy densities,

\[
E = \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_s v_s^2 + U + \rho U_{\text{ext}}, \tag{2}
\]

where \( \rho_n \) and \( \rho_s \) are the normal fluid and superfluid densities, \( v_n \) and \( v_s \) are the normal fluid and superfluid velocities, and \( \rho = \rho_n + \rho_s \) is the total density. Also,

\[
U_{\text{ext}}(r) = \frac{1}{2} (\omega_n^2 x^2 + \omega_s^2 y^2 + \omega_s^2 z^2) \tag{3}
\]

is the harmonic trapping potential divided by the particle mass, as follows from our use of the mass density \( \rho \), rather than the number density \( n \).

Following LK, the total energy density \( E_0(r,t) \) as measured in a frame of reference moving with the local superfluid velocity is related to \( E \) by

\[
E_0 = E - \frac{1}{2} \rho_n v_n^2 - v_s \cdot [\rho_n (v_n - v_s)]. \tag{4}
\]

Using the LK identity for local equilibrium

\[
dE_0 = T dS + \mu d\rho + (v_n - v_s) \cdot d [\rho_n (v_n - v_s)], \tag{5}
\]

with Eqs. (2) and (4), we obtain the following identity for the internal energy density [41]:

\[
dU = (\mu - U_{\text{ext}}) d\rho + T dS + \frac{1}{2} (v_n - v_s)^2 d\rho_n. \tag{6}
\]

In similar fashion, the LK expression for the pressure \( P = \partial(E_0V)/\partial V \) including the effects of a trapping potential can be recast in terms of the internal energy density, giving

\[
P = -U - \rho U_{\text{ext}} + TS + \rho \mu + \frac{1}{2} \rho_n (v_n - v_s)^2. \tag{7}
\]

The two definitions given by Eqs. (6) and (7) can be combined to give

\[
\rho \mu = dP - S dT - \rho_n (v_n - v_s) \cdot d(v_n - v_s). \tag{8}
\]

The three thermodynamic identities given by Eqs. (6), (7), and (8) define all thermodynamic properties that we will require in developing our variational approach. We note that Eq. (8) implies that

\[
\mu = \left( \frac{\partial U}{\partial \rho} \right)_{S,\rho_n} + U_{\text{ext}}. \tag{9}
\]

### III. ZILSEL'S HYDRODYNAMICS

In both classical and quantum mechanics, one can obtain dynamical equations of motion by equating to zero the variation of some action, usually expressed in the form \( A = \int dt (T - U) \), where \( T \) and \( U \) are the kinetic and potential energies, respectively. To derive the LTF equations, we take an action of the form [32]

\[
A = \int d^3x \left[ \frac{1}{2} (\rho - \rho_n) v_s^2 + \frac{1}{2} \rho_n v_n^2 - U(\rho, \rho_n, S) - \rho U_{\text{ext}} \right]. \tag{10}
\]

Here \( d^3x = d^3 r \, dt \), while \( U \) is the internal energy defined by Eq. (10) and \( U_{\text{ext}} \) is the harmonic trapping potential given by Eq. (3). An action similar to Eq. (10) was originally used by Zilsel in 1950 to derive the LTF equations. In this section, we largely follow Zilsel's approach, with some differences noted in Appendix A.

In taking the variation of the action in Eq. (10), the variables \( v_n, v_s, \rho, \rho_n, \) and \( S \) will be treated as independent so that, for instance,

\[
\frac{\delta A}{\delta \rho} = \left( \frac{\partial U}{\partial \rho} \right)_{S,\rho_n}. \tag{11}
\]

Two important conservation laws not incorporated into the action given by Eq. (10) are the conservation of mass and entropy, described by:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot [(\rho - \rho_n) v_s + \rho_n v_n] = 0, \tag{12}
\]

\[
\frac{\partial S}{\partial t} + \nabla \cdot (S v_n) = 0. \tag{13}
\]

The variation of the action in Eq. (10) must be taken subject to the constraints given by Eqs. (12) and (13). An additional constraint not employed by Zilsel describes conservation of circulation in the normal fluid and allows for the possibility of vorticity. This constraint takes the form [32, 56, 37]

\[
\frac{\partial S \eta}{\partial t} + \nabla \cdot (S \eta v_n) = 0, \tag{14}
\]

where \( \eta \) depends on \( r \) and \( t \). Even though the inclusion of this constraint has no effect on the final form of the LTF equations, it eliminates the restrictions on the normal fluid velocity field that were present in Zilsel's original work. A more complete discussion of this point is given in Appendix A. Following the approach pioneered by Eckart [42], this constraint, along with entropy and mass conservation, can be incorporated into the variational principle by introducing Lagrange multipliers \( \phi, \alpha, \) and \( \gamma \) (all dependent on \( r \) and \( t \)) so that the action becomes

\[
A = \int d^3x \left[ \frac{1}{2} (\rho - \rho_n) v_s^2 + \frac{1}{2} \rho_n v_n^2 - U(\rho, \rho_n, S) \right.
\]

\[
+ \phi \rho + \alpha \rho_n + \gamma S \].
\[-\rho U_{ext} + \phi \left\{ \frac{\partial}{\partial t} + \nabla \cdot \left[ (\rho - \rho_n)\mathbf{v}_s + \rho_n \mathbf{v}_n \right] \right\} \]
\[+ \alpha \left\{ \frac{\partial S}{\partial t} + \nabla \cdot (S \mathbf{v}_n) \right\} \]
\[+ \gamma \left\{ \frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot (\mathbf{S} \mathbf{v}_n) \right\} \]
\[= 0. \tag{15} \]

It is understood that the time and spatial integration in Eq. (15) is done between two fixed points with the fluctuations of the variables appearing in the action vanishing at both.

Setting the variation of the action given by Eq. (15) with respect to \(\rho, S, \eta, \mathbf{v}_s, \) and \(\mathbf{v}_n\) equal to zero, and making use of the thermodynamic identities implied by Eq. (6), we obtain the following relations:

\[\frac{\delta A}{\delta \rho} = \frac{1}{2} \mathbf{v}_s^2 - \frac{\partial \phi}{\partial t} - \mathbf{v}_s \cdot \nabla \phi - \mu = 0, \tag{16} \]

\[\frac{\delta A}{\delta S} = -\frac{\partial \alpha}{\partial t} - \mathbf{v}_n \cdot \nabla \alpha - T - \eta \left( \frac{\partial \gamma}{\partial t} + \mathbf{v}_n \cdot \nabla \gamma \right) = 0, \tag{17} \]

\[\frac{\delta A}{\delta \eta} = -S \left( \frac{\partial \gamma}{\partial t} + \mathbf{v}_n \cdot \nabla \gamma \right) = 0, \tag{18} \]

\[\frac{\delta A}{\delta \mathbf{v}_s} = \rho_s (\mathbf{v}_s - \nabla \phi) = 0, \tag{19} \]

and

\[\frac{\delta A}{\delta \mathbf{v}_n} = \rho_n (\mathbf{v}_n - \nabla \phi) - S (\nabla \alpha + \eta \nabla \gamma) = 0. \tag{20} \]

Taking the variation of the action with respect to \(\rho_n\) and using Eq. (19), one recovers a thermodynamic identity already known from Eq. (6):

\[\left( \frac{\partial U}{\partial \rho} \right)_{S, \rho} = \frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2. \tag{21} \]

Additionally, taking the variation of the action with respect to \(\phi, \alpha, \) and \(\gamma\) recovers the continuity equation given by Eq. (12), as well as the entropy and circulation conservation laws given by Eqs. (13) and (14).

Equations (16)-20 can be rearranged to yield the useful expressions

\[\mathbf{v}_s = \nabla \phi, \tag{22} \]

\[\frac{\rho_n}{S} (\mathbf{v}_n - \mathbf{v}_s) = \nabla \alpha + \eta \nabla \gamma, \tag{23} \]

\[\frac{\partial \alpha}{\partial t} + \eta \frac{\partial \gamma}{\partial t} = -T - \frac{\rho_n}{S} \mathbf{v}_n \cdot (\mathbf{v}_n - \mathbf{v}_s), \tag{24} \]

\[\frac{\partial \gamma}{\partial t} + \mathbf{v}_n \cdot \nabla \gamma = 0, \tag{25} \]

and

\[\frac{\partial \phi}{\partial t} = -\left( \mu + \frac{1}{2} \mathbf{v}_s^2 \right). \tag{26} \]

Equation (24) shows that the superfluid velocity is irrotational. Furthermore, comparing Eqs. (13) and (14), we find

\[\frac{\partial \eta}{\partial t} + \mathbf{v}_n \cdot \nabla \eta = 0. \tag{27} \]

Another relationship that will be helpful in deriving the equation of motion for the normal fluid, found from Eqs. (23), (25), and (27), is

\[\frac{\partial \eta}{\partial t} \nabla \gamma - \frac{\partial \gamma}{\partial t} \nabla \eta = \mathbf{v}_n \times \left( \nabla \times \left[ \frac{\rho_n}{S} (\mathbf{v}_n - \mathbf{v}_s) \right] \right). \tag{28} \]

Taking the time derivative of Eq. (22) and the gradient of Eq. (26), we find the following equation for the superfluid velocity:

\[\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left( \mu + \frac{1}{2} \mathbf{v}_s^2 \right). \tag{29} \]

Rearranging Eq. (4) to obtain an equation for \(\mu, \) and making use of the result

\[\mathbf{\nabla} U = \left( \frac{\partial U}{\partial \rho} \right)_{\rho_n, S} \rho + \left( \frac{\partial U}{\partial \rho} \right)_{S, \rho} \rho_n + \left( \frac{\partial U}{\partial S} \right)_{\rho, \rho_n} \mathbf{\nabla} S, \]

we obtain the expression

\[\mathbf{\nabla} \mu = \frac{1}{\rho} \mathbf{\nabla} P - \frac{S}{\rho} \mathbf{\nabla} T \]

\[+ \mathbf{\nabla} U_{ext} - \frac{1}{\rho} \frac{\rho_n}{\rho} \mathbf{\nabla} \left[ (\mathbf{v}_n - \mathbf{v}_s)^2 \right]. \tag{31} \]

Thus, Eq. (20) for the superfluid velocity can be rewritten as

\[\left( \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla \right) \mathbf{v}_s = \frac{S}{\rho} \mathbf{\nabla} T \]

\[+ \frac{1}{\rho} \mathbf{\nabla} P - \mathbf{\nabla} U_{ext} + \frac{1}{2} \frac{\rho_n}{\rho} \mathbf{\nabla} \left[ (\mathbf{v}_n - \mathbf{v}_s)^2 \right]. \tag{32} \]

To determine the analogous equation for the velocity of the normal fluid, we take the time-derivative of Eq. (24) and the gradient of Eq. (24). Then, using Eqs. (12), (13), (14), and (25), as well as the identity \(\mathbf{v} \cdot \nabla \mathbf{v} = (1/2) \nabla (\mathbf{v}^2) - \mathbf{v} \times (\nabla \times \mathbf{v}),\) after some labourious algebra, we find

\[\left( \frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) \mathbf{v}_n = -\frac{\rho_n}{\rho} \frac{S}{\rho_n} \mathbf{\nabla} T \]

\[- \frac{1}{\rho} \mathbf{\nabla} P - \mathbf{\nabla} U_{ext} - \frac{1}{2} \frac{\rho_n}{\rho} \mathbf{\nabla} \left[ (\mathbf{v}_n - \mathbf{v}_s)^2 \right] \]

\[- \frac{1}{\rho_n} (\mathbf{v}_n - \mathbf{v}_s), \tag{33} \]
where the “source function” \( \Gamma \) is defined by
\[
\Gamma = \frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n v_n). \tag{34}
\]
From the continuity equation given by Eq. (12), Eq. (34) implies that
\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s v_s) = -\Gamma. \tag{35}
\]
Combining Eqs. (32) and (35) to obtain an equation of motion for the current, \( j = \rho_s v_s + \rho_n v_n \), the source terms of the two components cancel and we find
\[
\frac{\partial j_i}{\partial t} = -\nabla P - \rho \nabla U_{\text{ext}} - \rho_s v_s \cdot \nabla v_s - \rho_n v_n \cdot \nabla v_n - v_s \nabla \cdot (\rho_s v_s) - v_n \nabla \cdot (\rho_n v_n). \tag{36}
\]
More familiarly, in component form, this equation can be written as
\[
\frac{\partial j_i}{\partial t} = -\delta \rho \frac{\partial \rho}{\partial x_j} [P \delta_{ij} + \rho_s v_{s i} v_{s j} + \rho_n v_{n i} v_{n j}] - \rho \frac{\partial U_{\text{ext}}}{\partial x_i}, \tag{37}
\]
where the index \( j \) is summed over.

IV. ACTION FOR LINEARIZED LANDAU TWO-FLUID HYDRODYNAMICS

As was shown in the previous section, variation of the action given by Eq. (15) with respect to \( \rho, S, \eta, v_s, v_n, \phi, \alpha, \text{ and } \gamma \) leads to the non-dissipative LT2F hydrodynamic equations (24). To determine the low-energy collective modes of this system given by the solutions of the linearized hydrodynamic equations, we Taylor-expand the action about the equilibrium values of these variables. In principle, one could then take the variation of the resulting action with respect to the fluctuations of its variables, i.e., \( \delta \rho, \delta S, \delta \eta, \delta v_s, \delta v_n, \delta \phi, \delta \alpha, \text{ and } \delta \gamma \), to obtain the linearized hydrodynamic equations. These could then be solved to find the collective mode frequencies. In practice, rather than dealing with the Lagrange multipliers \( \phi, \alpha, \text{ and } \gamma \) in Eq. (15), it is more convenient to introduce displacement fields for the two velocities. This allows one to incorporate the conservation laws directly into expressions for \( \delta \rho \) and \( \delta S \), thereby eliminating the need for Lagrange multipliers. We will employ a simplified Rayleigh-Ritz method in conjunction with our variational approach to obtain estimates of the collective mode frequencies by using physically reasonable ansatzes for the displacement fields of the superfluid and normal fluid.

Recalling that the conservation of circulation constraint does not affect the form of the equations resulting from the variational principle, we will omit this term in the action for the sake of clarity. However, if we were interested in collective oscillations of the normal fluid with non-zero circulation, we would need to incorporate this term and allow for fluctuations of \( \eta \) and \( \gamma \).

Considering fluctuations of the action about equilibrium we set \( \rho = \rho_0 + \delta \rho, \rho_n = \rho_{n0} + \delta \rho_n, S = S_0 + \delta S \),
\[ \phi = \phi_0 + \delta \phi, \ v_n = v_{n0} + \delta v_n, \ v_s = v_{s0} + \delta v_s, \] and expand the action given by Eq. (15) up to quadratic order in these fluctuations. As in the above, we assume that the two fluid components are stationary in equilibrium, so that \( \delta v_n = \delta v_n^0 \) and \( \delta v_s = \delta v_s^0 \). Writing \( A = A^{(0)} + A^{(1)} + A^{(2)} + \cdots \), the contribution to the action from terms which are quadratic in the fluctuations is

\[
A^{(2)} = \int d^4x \left[ \frac{1}{2} \rho_{s0} v_s^2 + \frac{1}{2} \rho_{n0} v_n^2 - \frac{1}{2} \left( \frac{\partial \mu}{\partial \rho} \right)_{s,\rho_n} (\delta \rho)^2 \right. \\
- \left. \left( \frac{\partial T}{\partial \rho} \right)_{s,\rho_n} \delta S \delta \rho - \frac{1}{2} \left( \frac{\partial T}{\partial S} \right)_{\rho,\rho_n} (\delta S)^2 \right] \\
+ \delta \phi \left\{ \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_{s0} v_s + \rho_{n0} v_n) \right\} \\
+ \phi_0 \left\{ \nabla \cdot [\{\delta \rho - \rho_n^0\} v_s + \delta \rho_n^0 v_n] \right\} \\
+ \delta \alpha \left\{ \frac{\partial \delta S}{\partial t} + \nabla \cdot (S_0 v_n) \right\} \\
- \alpha_0 \left\{ \nabla \cdot (\delta S v_n) \right\}, \tag{44} \]

where we have made use of Eq. (10) to rewrite the coefficients in terms of thermodynamic derivatives. Note that there is an alternative expression for the coefficient multiplying \( \delta S \delta \rho \), given by the Maxwell relation

\[
(\partial T/\partial \rho)_S = (\partial \mu/\partial S)_\rho. \tag{45} \]

Contributions to the fluctuations of the internal energy arising from fluctuations of \( \rho_n \) have vanished since, as can be seen from Eq. (21), the equilibrium value \( (\partial U/\partial \rho_n)_S,\rho \) is zero when \( v_{n0} = v_{s0} \). We do not consider the contributions which are linear in fluctuations since we know from the stationary action principle that \( A^{(1)} = 0 \).

Since \( v_{n0} = 0, \) Eq. (22) gives us \( \nabla \cdot \delta \phi_0 = 0 \). Also, with \( v_{n0} = v_{s0} = 0, \) from Eq. (23) we find \( \nabla \cdot \delta \alpha_0 = 0 \) since \( \eta \) is an independent and arbitrary variable. With these relations, having integrated by parts the terms in Eq. (44) which contain \( \delta \phi_0 \) and \( \delta \alpha_0 \), the quadratic action simplifies to

\[
A^{(2)} = \int d^4x \left[ \frac{1}{2} \rho_{s0} v_s^2 + \frac{1}{2} \rho_{n0} v_n^2 - \frac{1}{2} \left( \frac{\partial \mu}{\partial \rho} \right)_{s,\rho_n} (\delta \rho)^2 \right. \\
- \left. \left( \frac{\partial T}{\partial \rho} \right)_{s,\rho_n} \delta S \delta \rho - \frac{1}{2} \left( \frac{\partial T}{\partial S} \right)_{\rho,\rho_n} (\delta S)^2 \right] \\
+ \delta \phi \left\{ \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_{s0} v_s + \rho_{n0} v_n) \right\} \\
+ \phi_0 \left\{ \nabla \cdot [\{\delta \rho - \rho_n^0\} v_s + \delta \rho_n^0 v_n] \right\} \\
+ \delta \alpha \left\{ \frac{\partial \delta S}{\partial t} + \nabla \cdot (S_0 v_n) \right\} \tag{46} \]

In the present form, the variational principle has been reduced to equating to zero the variation of the action given by the first two lines in Eq. (46), subject to the constraints given by the linearized continuity and entropy conservation equations,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 v_s + \rho_0 v_n) = 0 \tag{47} \]

and

\[
\frac{\partial \delta S}{\partial t} + \nabla \cdot (S_0 v_n) = 0. \tag{48} \]

For our purposes, it is more convenient to incorporate these two constraints into expressions for \( \delta \rho \) and \( \delta S \) than to use the Lagrange multipliers \( \delta \phi \) and \( \delta \alpha \). Introducing the displacement fields

\[
v_s(r, t) \equiv \frac{\partial u_s(r, t)}{\partial t}, \quad v_n(r, t) \equiv \frac{\partial u_n(r, t)}{\partial t}, \tag{49} \]

the linearized continuity and entropy conservation equations can be expressed in terms of these fields as

\[
\delta \rho(r, t) = -\nabla \cdot [\rho_{s0}(r) u_s(r, t) + \rho_{n0}(r) u_n(r, t)] \tag{50} \]

and

\[
\delta S(r, t) = -\nabla \cdot [S_0(r) u_n(r, t)]. \tag{51} \]

Removing the terms that contain the Lagrange multipliers from Eq. (46) and substituting these expressions for \( \delta \rho \) and \( \delta S \), we find our central result for the action that is second order in the fluctuations \( u_s \) and \( u_n \):

\[
A^{(2)} = \int d^4x \left\{ \frac{1}{2} \rho_{s0} u_s^2 + \frac{1}{2} \rho_{n0} u_n^2 - \frac{1}{2} \left( \frac{\partial \mu}{\partial \rho} \right)_{s,\rho_n} (\delta \rho)^2 \right. \\
- \left. \left( \frac{\partial T}{\partial \rho} \right)_{s,\rho_n} \delta S \delta \rho - \frac{1}{2} \left( \frac{\partial T}{\partial S} \right)_{\rho,\rho_n} (\delta S)^2 \right] \\
+ \delta \phi \left\{ \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_{s0} u_s + \rho_{n0} u_n) \right\} \\
+ \phi_0 \left\{ \nabla \cdot [\{\delta \rho - \rho_n^0\} u_s + \delta \rho_n^0 u_n] \right\} \\
+ \delta \alpha \left\{ \frac{\partial \delta S}{\partial t} + \nabla \cdot (S_0 u_n) \right\} \tag{52} \]

To make our expressions as compact as possible, in what follows we will drop the \( \rho_n \) term constraining the thermodynamic derivatives in Eq. (52). However, it should be remembered that in evaluating the various local thermodynamic derivatives, \( \rho_n \) is always fixed.

V. HYDRODYNAMIC MODES DESCRIBING FLUCTUATIONS

The fluctuating quantities \( \delta \phi, \delta \alpha, \delta S, \delta \rho, v_s, \) and \( v_n \) appearing in Eq. (46), have been replaced in Eq. (52) by the two displacement fields \( u_s \) and \( u_n \). Thus, the variational principle has been reduced to taking the variation of the quadratic action given by Eq. (52) with respect to
only \(u_s\) and \(u_n\). The linearized LTF hydrodynamic equations and hence, low-energy collective modes of the system, are completely determined by the variational equations

\[
\frac{\delta A^{(2)}}{\delta u_s(r, t)} = 0, \quad \frac{\delta A^{(2)}}{\delta u_n(r, t)} = 0.
\]  

(53)

Using Eq. (52) as well as the identities given by Eqs. (50) and (51), Eq. (53) gives

\[
\ddot{u}_s = -\nabla\left[\delta \rho \left(\frac{\partial \mu}{\partial \rho}\right)_s + \delta S \left(\frac{\partial \mu}{\partial S}\right)_\rho\right]
\]

and

\[
\ddot{u}_n = -\nabla\left[\delta \rho \left(\frac{\partial \mu}{\partial \rho}\right)_n + \delta S \left(\frac{\partial \mu}{\partial S}\right)_\rho\right] - \frac{S_0}{\rho \alpha_0} \nabla\left[\delta \left(\frac{\partial T}{\partial \rho}\right)_s + \delta S \left(\frac{\partial T}{\partial S}\right)_\rho\right],
\]

(54)

(55)

where we have used the Maxwell relation given in Eq. (45). Taking \(\delta F = \delta S (\partial F/\partial S)_\rho + \delta \rho (\partial F/\partial \rho)_s\), where \(F\) denotes either one of \(\mu\) or \(T\), with the displacement fields defined by Eqs. (49), it is apparent that Eqs. (54) and (55) are equivalent to the linearized two-fluid hydrodynamic equations given by Eqs. (53) and (54). This verifies that the variational principle given by Eqs. (52) and (53) is the correct one.

Solutions of Eq. (53) corresponding to collective modes will have harmonic time-dependence of the form

\[
u_s(r, t) = u_s(r) \cos(\omega t), \quad u_n(r, t) = u_n(r) \cos(\omega t).
\]

(56)

Because these are exact solutions of the variational equations, we can insert these expressions into the action directly. Substituting Eq. (56) into Eq. (52) and performing the time-integration, we obtain, apart from an irrelevant constant factor, the following Lagrangian:

\[L^{(2)} = K[u_s, u_n] \omega^2 - U[u_s, u_n],\]

(57)

where

\[
K[u_s, u_n] = \frac{1}{2} \int d^3r \left\{ \rho \alpha_0 u_s^2 + \rho \alpha_0 u_n^2 \right\}
\]

(58)

and

\[
U[u_s, u_n] = \frac{1}{2} \int d^3r \left\{ \left(\frac{\partial \mu}{\partial \rho}\right)_s \nabla \cdot (\rho \alpha_0 u_s + \rho \alpha_0 u_n) \right\}^2
+ 2 \left(\frac{\partial T}{\partial \rho}\right)_s \nabla \cdot (S_0 u_s) \nabla \cdot (\rho \alpha_0 u_s + \rho \alpha_0 u_n))
+ \left(\frac{\partial T}{\partial S}\right)_\rho \nabla \cdot (S_0 u_n)^2 \right\}.
\]

(59)

Since the fields \(u_s(r)\) and \(u_n(r)\) are no longer time-dependent, it suffices to consider variations of the Lagrangian given by Eq. (57), and the variational equations now become

\[
\frac{\delta L^{(2)}}{\delta u_s(r)} = 0, \quad \frac{\delta L^{(2)}}{\delta u_n(r)} = 0.
\]

(60)

Solving Eq. (60) is still equivalent to solving the linearized hydrodynamic equations, with the only simplification being that we have assumed harmonic time-dependence for the solutions. Fortunately, one can easily obtain approximate expressions for the collective mode frequencies within a variational approach. Following Ref. [22], we use a simplified Rayleigh-Ritz method and make an ansatz for each Cartesian component of the displacement fields of the form

\[u_{si}(r) = a_{si} f_i(r), \quad u_{ni}(r) = a_{ni} g_i(r),\]

(61)

where the constant coefficients \(a_{si}\) and \(a_{ni}\) are variational parameters. Substituting this ansatz into Eq. (57), and equating to zero the variation of the resulting expression with respect to these parameters, we have the six variational equations

\[
\frac{\delta L^{(2)}}{\delta a_{si}} = 0, \quad \frac{\delta L^{(2)}}{\delta a_{ni}} = 0.
\]

(62)

In practice, the symmetry of the problem usually allows us to reduce the number of equations. From these equations, one obtains a rigorous upper bound for the collective mode frequencies \(\omega\).

A more precise Rayleigh-Ritz method would require us to expand each component of the displacement fields in terms of \(n > 1\) elements of some complete set of functions. As \(n\) is increased, solving the resulting \(6n\) variational equations, one iteratively generates successively better approximations for the collective mode frequencies. For harmonically-confined gases, however, there exist simple trial functions for \(f_i(r)\) and \(g_i(r)\) which are sufficiently close to the exact solutions that excellent results are obtained by considering only a single expansion term as in Eq. (61). Our choices of ansatzes for the displacement fields at finite temperatures will be guided by the known exact solutions at \(T = 0\) and \(T > T_c\) (for bosons, see Ref. [23]; for fermions, see Ref. [24]) for modes of experimental interest, such as the dipole and breathing modes.

Using the ansatz given by Eq. (61), the Lagrangian given by Eqs. (57)-(59) describes the dynamics of a pair of coupled harmonic oscillators, with \(a_{si}\) and \(a_{ni}\) representing the displacements of the two oscillators from equilibrium. The effective spring constants are determined by the equilibrium thermodynamic quantities of the system. This is a useful picture to have when envisioning the low-energy dynamics of the two fluids. It immediately implies, for instance, the existence of in-phase as well as out-of-phase oscillation modes of the two fluids.
VI. EXAMPLES OF COLLECTIVE MODES

A. Uniform Gas

As a simple example of our formalism, we consider first a uniform gas. In this case, \( \rho_{s0}, \rho_{n0} \), and the equilibrium thermodynamic derivatives appearing in Eqs. (58) and (60) are all independent of position. Anticipating sound modes with a dispersion relation of the form \( \omega = \frac{u}{k} \), where \( u \) is the sound speed, we use the plane-wave ansatz

\[
\begin{align*}
\mathbf{u}_s(r) &= \hat{x} N a_s \cos(\omega x/u), \\
\mathbf{u}_n(r) &= \hat{x} N a_n \cos(\omega x/u),
\end{align*}
\]

(63)

where \( a_s \) and \( a_n \) are the variational parameters. The normalization constant \( N \) is chosen so that \( \int d^3r \mathbf{u}_s^2 = a_s^2 \). Inserting Eq. (63) into Eqs. (58) and (59), we find

\[
K[a_s, a_n] = \frac{1}{2} \rho_{s0} a_s^2 + \frac{1}{2} \rho_{n0} a_n^2
\]

(64)

and

\[
U[a_s, a_n] = \frac{\omega^2}{u^2} \left\{ \frac{a_s^2}{2} \left[ (\rho_{s0})^2 \left( \frac{\partial \mu}{\partial \rho} \right)_s \right] + a_s a_n \left[ \rho_{s0} \rho_{n0} \left( \frac{\partial \mu}{\partial \rho} \right)_s + S_0 \rho_{s0} \left( \frac{\partial T}{\partial \rho} \right)_s \right] + \frac{a_s^2}{2} \left[ (\rho_{n0})^2 \left( \frac{\partial \mu}{\partial \rho} \right)_s + 2S_0 \rho_{n0} \left( \frac{\partial T}{\partial \rho} \right)_s \right] + (S_0)^2 \left( \frac{\partial T}{\partial S} \right)_s \right\}.
\]

(65)

The trial functions given by Eq. (63) are, in fact, exact. Thus, the variational equations given by Eq. (62), which in the present case take the form

\[
\frac{\delta L^{(2)}}{\delta a_s} = 0, \quad \frac{\delta L^{(2)}}{\delta a_n} = 0,
\]

(66)

will give the exact eigenvalues \( \omega \) or, equivalently, the exact sound speeds \( u \). Using the above expressions for \( K \) and \( U \), Eqs. (57) and (60) give the following quadratic equation for \( u^2 \):

\[
\begin{align*}
\frac{\partial \mu}{\partial \rho} - 2s_0 \frac{\partial T}{\partial \rho} + \frac{\rho_{s0} (S_0)^2}{\rho_{n0}} \frac{\partial T}{\partial S} &= 0.
\end{align*}
\]

(67)

Here \( \partial (\mu, T)/\partial (\rho, S) \) denotes the Jacobian of the transformation between \( \mu, T \) and \( \rho, S \).

In the usual textbook discussions, one works with the entropy density \( \sigma = S/\rho \), rather than the entropy \( S \). To express Eq. (67) in terms of \( \sigma \), we use the following identities which are found from the transformation properties of thermodynamic derivatives (see, for example, chap. 16 in Ref. 45):

\[
\begin{align*}
\left( \frac{\partial \mu}{\partial \rho} \right)_s &= -2\sigma \left( \frac{\partial T}{\partial \rho} \right)_s + \frac{\sigma^2}{\rho} \left( \frac{\partial T}{\partial \sigma} \right)_s + \frac{1}{\rho} \left( \frac{\partial P}{\partial \rho} \right)_s, \\
\left( \frac{\partial T}{\partial \rho} \right)_s &= \left( \frac{\partial T}{\partial \rho} \right)_\sigma - \frac{\sigma}{\rho} \left( \frac{\partial T}{\partial \sigma} \right)_\rho, \\
\left( \frac{\partial (\mu, T)}{\partial (\rho, S)} \right)_s &= \frac{1}{\rho^2} \left( \frac{\partial P}{\partial \rho} \right)_T \left( \frac{\partial T}{\partial \sigma} \right)_\rho,
\end{align*}
\]

(68-70)

and

\[
\left( \frac{\partial T}{\partial S} \right)_\rho = \frac{1}{\rho} \left( \frac{\partial T}{\partial \sigma} \right)_\rho.
\]

(71)

With these relations, Eq. (67) for the sound speed can be rewritten as

\[
u^4 - u^2 \left[ \frac{\partial P}{\partial \rho} \right]_\sigma + \frac{\rho_{s0}}{\rho_{n0}} (S_0)^2 \left( \frac{\partial T}{\partial \sigma} \right)_\rho + \frac{\rho_{s0}}{\rho_{n0}} (S_0)^2 \frac{\partial (\mu, T)}{\partial (\rho, S)} = 0.
\]

(72)

This is the standard equation giving the first and second sound speeds in a uniform superfluid.\[ \text{[add references here]} \]

B. Dipole Mode

We next consider a dipole mode in a harmonically confined gas. This mode is characterized by displacements of the centre-of-masses of the two fluids along one of the axes of the harmonic trap, \( x_i \). In this case, we use the following ansatz for the displacement fields:

\[
\begin{align*}
\mathbf{u}_s &= \hat{x}_i a_s, \\
\mathbf{u}_n &= \hat{x}_i a_n,
\end{align*}
\]

(73)

where \( a_s \) and \( a_n \) describe the displacements of the centre-of-masses of the two fluids from the trap centre. Substituting this ansatz into Eqs. (58) and (60), we find

\[
K[a_s, a_n] = \frac{1}{2} M_s a_s^2 + \frac{1}{2} M_n a_n^2
\]

(74)

and

\[
U[a_s, a_n] = \frac{1}{2} k_s a_s^2 + \frac{1}{2} k_n a_n^2 + \frac{1}{2} k_{sn} (a_s - a_n)^2,
\]

(75)

where \( M_s \) and \( M_n \) are the masses of the superfluid and normal fluids, respectively, given by

\[
M_s = \int d^3r \rho_{s0}, \quad M_n = \int d^3r \rho_{n0}.
\]

(76)
The spring constants \( k_s, k_n, \) and \( k_{sn} \) are given by

\[
k_s = \int d^3r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial \mu}{\partial S} \right)_\rho \frac{\partial S_0}{\partial x_i} \right\},
\]

(77)

\[
k_n = \int d^3r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial \mu}{\partial S} \right)_\rho \frac{\partial S_0}{\partial x_i} \right\} + \left[ \left( \frac{\partial T}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial T}{\partial S} \right)_\rho \frac{\partial S_0}{\partial x_i} \right] \frac{\partial \rho_0}{\partial x_i},
\]

(78)

and

\[
k_{sn} = -\int d^3r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial \rho_{n0}}{\partial x_i} \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial T}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} \right\}.
\]

(79)

The variational equation, equivalent to the expression for the homogeneous gas given by Eq. (66) with \( K \) and \( U \) given by Eqs. (74) and (75), gives the following secular equation for the dipole frequencies:

\[
\left( M_s \omega^2 - k_s - k_{sn} \right) a_s \left( \frac{a_s}{a_n} \right) = 0, \quad \left( M_n \omega^2 - k_n - k_{sn} \right) a_n = 0.
\]

(80)

Applying some thermodynamic identities, the spring constants \( k_s, k_n, \) and \( k_{sn} \) can be simplified considerably. Specifically, with \( v_0 = 0 \), Eq. (66) can be used to find useful expressions for the gradient of various equilibrium thermodynamic quantities. One finds, for instance, from the definition \( T = \langle \partial U/\partial S \rangle_\rho \),

\[
\nabla T_0 = \left( \frac{\partial^2 U}{\partial S \partial \rho} \right)_\rho \nabla \rho_0 + \left( \frac{\partial^2 U}{\partial S^2} \right)_\rho \nabla S_0 = \left( \frac{\partial T}{\partial \rho} \right)_S \nabla \rho_0 + \left( \frac{\partial T}{\partial S} \right)_\rho \nabla S_0.
\]

(81)

Similarly, using Eq. (9),

\[
\nabla \mu_0 = \left( \frac{\partial \mu}{\partial \rho} \right)_S \nabla \rho_0 + \left( \frac{\partial \mu}{\partial S} \right)_\rho \nabla S_0 + \nabla U_{\text{ext}},
\]

(82)

which, it can be shown, is equivalent to Eq. (31) evaluated at equilibrium, when \( v_0 = 0 \). For a harmonic trapping potential given by Eq. (3), noting that \( \nabla \mu_0 = 0 \), Eq. (82) provides us with the useful expression

\[
\left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial \mu}{\partial S} \right)_\rho \frac{\partial S_0}{\partial x_i} = -\frac{\partial U}{\partial x_i},
\]

(83)

where \( \omega_i \) is the trap frequency along the \( x_i \)-axis. Also, since \( \nabla T_0 = 0 \), Eq. (31) gives us

\[
\left( \frac{\partial T}{\partial \rho} \right)_S \frac{\partial \rho_0}{\partial x_i} + \left( \frac{\partial T}{\partial S} \right)_\rho \frac{\partial S_0}{\partial x_i} = 0.
\]

(84)

Substituting Eqs. (83) and (84) into Eqs. (77) and (78) and integrating by parts, the expressions for \( k_s, k_n, k_{sn} \) readily simplify, reducing to \( k_s = \omega_i^2 M_s, k_n = \omega_i^2 M_n \). With these values, Eq. (80) becomes

\[
[M_s M_n (\omega_i^2 - \omega^2_i) - k_{sn} (M_s + M_n)] (\omega_i^2 - \omega^2_i) = 0.
\]

(85)

This gives an in-phase dipole mode with frequency

\[
\omega = \omega_i
\]

(86)

and an out-of-phase dipole mode with frequency

\[
\omega^2 = \omega_i^2 + \frac{M_s + M_n}{M_s M_n} k_{sn}.
\]

(87)

The result given by Eq. (80) is the expected Kohn mode in a harmonic trap and corresponds to the solution \( a_s = a_n \). This mode is a rigid oscillation of the superfluid and normal fluid static distributions and as a result, the interactions have no effect on the collective mode frequency. The frequency of the second mode given by Eq. (80) does depend on interactions as these determine the thermodynamic functions appearing in the \( k_{sn} \) spring constant given by Eq. (74). This mode corresponds to the solution \( M_s a_s + M_n a_n = 0 \). This is precisely the analogue in a trapped system of second sound, since the displacements of the superfluid and normal fluid have opposite signs, producing an out-of-phase oscillation of the two components.

The expression for the frequency of the out-of-phase dipole mode given by Eq. (80) agrees with that found in Refs. \( 25 \) and \( 10 \), but with a slightly different value of \( k_{sn} \) in Eq. (74) as a result of working with a modified form of two-fluid hydrodynamics (see Appendix B for further discussion).

C. Breathing Modes

For the breathing modes, we consider an ansatz of the form

\[
u_{sn} = (a_{s1} x, a_{s2} y, a_{s3} z), \quad v_n = (a_{n1} x, a_{n2} y, a_{n3} z).
\]

(88)

Substituting Eq. (88) into Eq. (74) we find

\[
K[a_s, a_n] = \frac{1}{2} \sum_i \left[ M_s a_{si}^2 + M_n a_{ni}^2 \right],
\]

(89)

and

\[
U[a_s, a_n] = \frac{1}{2} \sum_{ij} \left[ k_{sn,ij} a_{si} a_{sj} + k_{n,ij} a_{ni} a_{nj} + 2k_{sn,ij} a_{si} a_{nj} \right].
\]

(90)
where the weighted masses $M_i$ are defined by

$$M_{si} = \int d^3 r \, \rho_i(x, t)^2, \quad M_{ni} = \int d^3 r \, \rho_0(x, t)^2,$$

and the constants $k_{s,ij}, k_{n,ij},$ and $k_{sn,ij}$ (which can be related to spring constants) are

$$k_{s,ij} = \int d^3 r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial (\rho_0 x_i)}{\partial x_i} \frac{\partial (\rho_0 x_j)}{\partial x_j} \right\},$$

and

$$k_{n,ij} = \int d^3 r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial (\rho_0 x_i)}{\partial x_i} \frac{\partial (S_0 x_j)}{\partial x_j} \right\} + 2 \left( \frac{\partial T}{\partial \rho} \right)_S \frac{\partial (S_0 x_i)}{\partial x_i} \frac{\partial (S_0 x_j)}{\partial x_j} + (\frac{\partial T}{\partial S}) \frac{\partial (S_0 x_i)}{\partial x_i} \frac{\partial (S_0 x_j)}{\partial x_j} \right\},$$

and

$$k_{sn,ij} = \int d^3 r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right)_S \frac{\partial (\rho_0 x_i)}{\partial x_i} \frac{\partial (S_0 x_j)}{\partial x_j} \right\} + (\frac{\partial T}{\partial \rho}) \frac{\partial (S_0 x_i)}{\partial x_i} \frac{\partial (S_0 x_j)}{\partial x_j} \right\}. \quad (94)$$

Since we generally have six variational parameters (one for each Cartesian component of the two displacement fields), the collective mode frequencies are found from the six coupled algebraic equations, $\delta L^{(2)}/\delta a_{si} = 0$, $\delta L^{(2)}/\delta a_{ni} = 0$. We find

$$M_{si} \omega^2 a_{si} = \frac{1}{2} \sum_j \left[ (k_{s,ij} + k_{n,ij}) a_{sji} + 2 k_{sn,ij} a_{nji} \right], \quad (95)$$

$$M_{ni} \omega^2 a_{ni} = \frac{1}{2} \sum_j \left[ (k_{n,ij} + k_{n,ji}) a_{nji} + 2 k_{sn,ij} a_{nji} \right]. \quad (96)$$

In experiments performed on trapped superfluid gases, one usually has an axisymmetric trap such that, for instance, $\omega_1 = \omega_2 \equiv \omega_3 = \omega_3$, and the modes of interest are the axial and longitudinal breathing modes (see, for example, Refs. [3] and [4] which deal with Fermi gases close to unitarity). These correspond to $a_{s1} = a_{s2}$ and $a_{n1} = a_{n2}$.

**VII. HYDRODYNAMIC THEORY AT T=0**

Since the variational principle given by Eqs. (57)–(59) involves some fairly complex expressions, it is useful to consider the case of zero temperature where the Lagrangian simplifies tremendously. At $T = 0$, the Landau two-fluid equations reduce to a single hydrodynamic differential equation that was first used by Pitaevskii and Stringari [39] to derive the corrections to the mean-field Gross-Pitaevskii results [10] for the collective mode frequencies in a dilute Bose gas.

At $T = 0$, only the superfluid component persists and we have $S = 0$, $\rho_0 = 0$, and $\rho_0 = \rho_0$. Eqs. (57)–(59) then reduce to

$$L^{(2)}[\mathbf{u}, \mathbf{s}] = \frac{1}{2} \int d^3 r \left\{ \rho_0 \mathbf{u}^2 + \left( \frac{\partial \mu}{\partial \rho} \right) \left( \mathbf{\nabla} \cdot (\rho_0 \mathbf{u}) \right)^2 \right\}.$$ 

(97)

From $\delta L^{(2)}/\delta \mathbf{u} = 0$, one obtains

$$\omega^2 \mathbf{u} = -\mathbf{\nabla} \left[ \left( \frac{\partial \mu}{\partial \rho} \right) \mathbf{\nabla} \cdot (\rho_0 \mathbf{u}) \right]. \quad (98)$$

Using the $T = 0$ linearized continuity equation, $\delta \rho = -\mathbf{\nabla} \cdot (\rho_0 \mathbf{u})$, Eq. (98) can be rewritten as

$$\omega^2 \mathbf{u} = \mathbf{\nabla} \left[ \left( \frac{\partial \mu}{\partial \rho} \right) \delta \rho \right]. \quad (99)$$

Multiplying both sides of this expression by $\rho_0$ and taking the divergence, we obtain a closed equation for the density fluctuations,

$$\omega^2 \delta \rho = -\mathbf{\nabla} \cdot \left[ \rho_0 \mathbf{\nabla} \left[ \left( \frac{\partial \mu}{\partial \rho} \right) \delta \rho \right] \right], \quad (100)$$

that is the basis of the $T = 0$ hydrodynamic theory developed by Pitaevskii and Stringari [39]. Eq. (100) describes the low-energy collective modes of both atomic Bose and two-component Fermi superfluid gases at $T = 0$. The only difference between the two quantum gases lies in the equation of state $\mu(\rho)$.

It is evident that our variational approach will give expressions for the collective mode frequencies at $T = 0$ that agree with results derived from solving Eq. (100) directly (see, for example, Ref. [39] for Bose gases, and Refs. [12, 13, 14] for Fermi gases close to unitarity). However, it is still useful to show explicitly how our formalism reproduces the $T = 0$ results obtained in the recent literature. As a specific application, we consider the breathing modes which were discussed at finite $T$ in the previous section. The breathing modes of trapped Fermi gases close to unitarity have been studied extensively at $T = 0$ [12, 13, 14, 15, 16, 17, 18], where a simple ansatz for the density dependence of the chemical potential $\mu(\rho)$ allows one to obtain simple expressions for the frequencies of these modes. We show that in the limit of zero temperature, our general results for the breathing mode frequencies at finite $T$, given by Eqs. (95) and (96), reproduce these well-known expressions.
Since the normal fluid vanishes at zero temperature, we only have equations for the superfluid component, given by Eq. 101, which reduce to

\[ M_{s_i} \omega^2 a_{s_i} = \sum_j k_{s,i} a_{s_j} \]  

(101)

at \( T = 0 \), where \( k_n \) and \( k_{sn} \) are zero. The expression for \( k_{s,i} \), given by Eq. 102, becomes

\[ k_{s,i} = \int d^3r \left( \frac{\partial \mu}{\partial \rho} \right) \frac{\partial (\rho_0 x_i)}{\partial x_j} \frac{\partial (\rho_0 x_j)}{\partial x_i} \]  

(102)

since \( \rho_{so} = \rho_0 \) at \( T = 0 \).

To obtain a simple analytic expression for the collective mode frequencies, the following identity, found by integration by parts, will prove useful:

\[ \int d^3r \left( \frac{\partial \mu}{\partial \rho} \right) \rho_0 x_j \frac{\partial \rho_0}{\partial x_j} = -\frac{1}{2} \int d^3r \rho_0 \left\{ \frac{\partial \mu}{\partial \rho} \right\} + x_j \frac{\partial}{\partial x_j} \left( \frac{\partial \mu}{\partial \rho} \right) \]  

(103)

Expanding out the derivatives in Eq. 102 and using Eq. 103, Eq. 102 can be rewritten as

\[ k_{s,i} = \int d^3r \left\{ \left( \frac{\partial \mu}{\partial \rho} \right) \frac{\partial \rho_0}{\partial x_i x_j} \frac{\partial \rho_0}{\partial x_j} - \frac{\rho_0}{2} \left( x_j \frac{\partial}{\partial x_j} + x_i \frac{\partial}{\partial x_i} \right) \left( \frac{\partial \mu}{\partial \rho} \right) \right\} \]  

(104)

At \( T = 0 \), Eq. 103 reduces to

\( \int d^3r \left( \frac{\partial \mu}{\partial \rho} \right) \frac{\partial \rho_0}{\partial x_i} = -\omega_i^2 x_i. \)  

(105)

Using this result in Eq. 104 and integrating by parts, we find

\[ k_{s,i} = \int d^3r \rho_0 \left\{ 2 \delta_{ij} + 1 \right\} \omega_i^2 x_i^2 \]  

(106)

Assuming a so-called polytropic equation of state (see, for example, Refs. 13, 15, 16, 18, 47, 48),

\[ \mu(\rho) \propto \rho^\gamma, \]  

(107)

we find

\[ \rho_0 \frac{\partial}{\partial x_i} \left( \frac{\partial \mu}{\partial \rho} \right) = (\gamma - 1) \left( \frac{\partial \mu}{\partial \rho} \right) \rho_0 \frac{\partial}{\partial x_i} = - (\gamma - 1) \omega_i^2 x_i. \]  

(108)

Substituting this relation into Eq. 106, and making use of the definition of the weighted mass \( M_{si} \) given by Eq. 101, we obtain the following expression for the ratio of the spring constant and the weighted mass:

\[ \frac{k_{s,ij}}{M_{si}} = 2 \omega_i^2 \delta_{ij} + \left[ \frac{\gamma + 1}{2} \omega_i^2 + \chi_{ji} \frac{\gamma - 1}{2} \omega_j^2 \right], \]  

(109)

where

\[ \chi_{ji} = \frac{\int d^3r \rho_0(r) x_j^2}{\int d^3r \rho_0(r) x_i^2}. \]  

(110)

Using the Thomas-Fermi result \( \mu_0 = \mu_0(1 + U_{ext}(r)) \), the equilibrium density \( \rho_0 \) can be expressed as a function of \( U_{ext}(r) = (1/2) \left( \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 \right) \). Making a change of variables, \( x_i' = \omega_i x_i \), \( \chi_{ji} \) becomes

\[ \chi_{ji} = \frac{\omega_i^2}{\omega_j^2} \int d^3r \rho_0(x_1'^2 + x_2'^2 + x_3'^2) x_j^2 \]  

(111)

\[ \int d^3r \rho_0 [x_1'^2 + x_2'^2 + x_3'^2] x_j^2 \]  

Combining this identity with Eq. 101, we find

\[ \omega^2 a_{s_i} = \sum_j \left( 2 \omega_i^2 \delta_{ij} + \gamma \omega^2 \right) a_{s_j} \]  

(112)

which is equivalent to Eq. (3) in Ref. 15 when a polytropic equation of state is used.

For an axisymmetric trap, \( \omega_1 = \omega_2 = \omega_\perp, \omega_3 = \omega_z \), and the axial and longitudinal breathing modes are characterized by solutions of the form \( a_1 = a_2 = a \). In this case, Eq. 113 gives the normal mode frequencies found by previous authors 15, 16, 18,

\[ \omega^2 = \frac{1}{2} \left[ 2 \gamma + 1 \right] \omega_1^2 + \left[ 2 \gamma + 2 \right] \omega_2^2 \pm \sqrt{\left[ 2 \gamma + 1 \right] \omega_1^2 - \left[ 2 \gamma + 2 \right] \omega_2^2 + 8 \gamma^2 \omega_1^2 \omega_2^2}. \]  

(114)

VIII. CONCLUSIONS

More than sixty years after Tisza 1 and Landau 2 proposed two-fluid hydrodynamics, their theory turns out to be very relevant in the study of the dynamics of trapped Fermi superfluid gases with strong interactions, close to unitarity. From linearized two-fluid hydrodynamics, one can obtain collective mode frequencies in...
terms of equilibrium thermodynamic functions and thermodynamic derivatives. This is well known for uniform superfluids and still occurs in the case of trapped gases, apart from the fact that the equilibrium thermodynamic quantities are now dependent on position. This means that the two-fluid predictions are general, with a microscopic theory only being needed for the elementary excitations that determine the equilibrium thermodynamic functions.

Variational approaches have already been used extensively to obtain hydrodynamic collective mode frequencies of superfluid Fermi gases at $T = 0$ close to unitarity [14, 16, 18], as well as boson-fermion mixtures [49]. In this paper, we have extended this approach to finite temperatures to deal with the coupled hydrodynamics of a superfluid and normal fluid in a trapped gas.

We have presented a variational method to determine the collective mode frequencies of a trapped superfluid, based on an action of the type first proposed by Zilsel in 1950 to derive Landau two-fluid hydrodynamics. By making an ansatz for the superfluid and normal fluid velocity fields, in this formulation, the problem of determining the collective mode frequencies is reduced to solving a set of algebraic equations. This obviates the need to solve the Landau two-fluid differential equations numerically, a difficult task in a nonuniform trapped gas.

We illustrated our formalism by considering a uniform gas, as well as the dipole and breathing modes for a harmonically confined gas. To make contact with recent discussions in the literature on trapped superfluid Fermi gases, we also used our finite $T$ variational approach to discuss the collective modes in the limiting case of a pure superfluid at $T = 0$ [12, 50].

In future work, we will apply the expressions derived in this paper and present detailed numerical predictions for the frequencies of various hydrodynamic modes in trapped superfluid gases at finite temperatures. In particular, we will discuss results in a Fermi superfluid for a strongly interacting molecular BEC at finite $T$ (the BEC side of a Feshbach resonance).

In this paper, we have only discussed how to determine the normal modes of the non-dissipative Landau two-fluid equations for a trapped (non-uniform) superfluid. The damping of these modes due to various hydrodynamic relaxation processes can be calculated using the general expressions derived by Nikuni and Griffin [50].

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**APPENDIX A: COMPARISON WITH ZILSEL’S VARIATIONAL PRINCIPLE**

As discussed in the text, our major interest in this paper is to find an efficient way of calculating the collective mode frequencies of a trapped superfluid in the region where Landau’s two-fluid hydrodynamics are valid. Following Zilsel [52], we did this in Sections II-IV by reformulating the two-fluid equations in terms of a variational theory involving the action. There is an extensive older literature on such a variational formulation in the context of superfluid helium. In this Appendix, we review in more detail some of the subtle issues which arose in connection with Zilsel’s seminal paper and their resolution.

One criticism concerned Zilsel’s use of $x = \rho_n/\rho$ as an independent variable which, it was argued, conflicted with the Landau formalism [34]. Zilsel was ultimately vindicated by Jackson who demonstrated that Zilsel’s approach was completely equivalent to those proposed by some of his critics [36].

In our approach to the two-fluid thermodynamics, using the internal energy density given by Eq. (6), we have deviated slightly from the variational method devised by Zilsel to derive the LTF equations. He considered the specific internal energy, which we denote by $U_Z$ to avoid confusion. The two energies are related by $U = \rho U_Z$. Furthermore, while we have taken the internal energy density to be a function of the entropy $S$, the total density ($\rho = \rho_n + \rho_s$), and the normal fluid density $\rho_n$, Zilsel took the specific internal energy to be a function of the entropy density $\sigma = S/\rho$, $\rho$, and the ratio $x = \rho_n/\rho$. However, one can show that the thermodynamics given in Section II, obtained from the Landau-Khalatnikov energy density, is identical to Zilsel’s and our choice of independent variables is completely consistent with Zilsel’s choice. To compare our thermodynamic identities with Zilsel’s, using Eqs. (9) and (11), with $U = \rho U_Z$ and hence, $dU = \rho dU_Z + U_Z dp$, we find

$$dU_Z = T d\sigma + \frac{P}{\rho^2} dp + \frac{1}{2} (v_n - v_s)^2 dx,$$

(A1)

where $\sigma = S/\rho$ and $x = \rho_n/\rho$. This relation gives precisely the thermodynamic relations used by Zilsel in his derivation of the LTF equations. As well, Eq. (A1) was derived by Jackson using thermodynamic arguments [36]. Zilsel chose to use $\sigma, \rho_n$, and $x$ as independent variables as these differentials appear in Eq. (A1). In contrast, with the internal energy density given by Eq. (9), the differentials of $S$, $\rho$, and $\rho_n$ are the natural choice for independent variables.

A more valid criticism directed at Zilsel’s work concerned the fact that, while he obtained the full LTF equations, his variational principle forced the normal fluid velocity to be irrotational when $S/\rho_n$ is a constant. Thus, Zilsel’s variational principle only describes a subset of all possible solutions of the LTF equations. Following an argument originally due to Lin, this deficiency can be corrected by introducing an additional constraint, given
APPENDIX B: ZNG VS. ZGN HYDRODYNAMICS

We briefly summarize some key features of the derivation of two-fluid hydrodynamics for a trapped Bose gas. As discussed in detail in Refs. [25] and [30], the ZNG microscopic coupled equations involve in a crucial way a term $\Gamma_{12}$ which is related to the collisionless transfer of atoms between the condensate and thermal cloud. In the linearized ZNG formalism, one can show that in local equilibrium,

$$\delta \Gamma_{12} = -\frac{n_{\sigma} }{k_{B} T \tau_{12} } \delta \mu_{\text{diff}}. \quad (B1)$$

This involves the difference in the chemical potentials of the condensate ($\mu_{c}$) and the thermal cloud ($\mu_{0}$)

$$\mu_{\text{diff}} = \mu_{0} - \mu_{c}. \quad (B2)$$

In static equilibrium, $\Gamma_{12}$ vanishes and $\mu_{0} = \mu_{c}$, assuming that $v_{c0} = v_{n0} = 0$. In Eq. (B1), the relaxation time $\tau_{12}$ describes collisions between atoms in the condensate and thermal cloud components. The relaxation time $\tau_{\mu}$ introduced in Ref. [25] which determines the rate at which local diffusive equilibrium is achieved ($\mu_{0} = \mu_{c}$), is given by $\tau_{\mu} = \sigma_{H} T \tau_{12}$, where $\sigma_{H}$ is a dimensionless hydrodynamic normalization factor defined in Ref. [25] involving local equilibrium thermodynamic functions.

ZNG derive the non-dissipative Landau two-fluid equations in the limit of $\omega \tau_{\mu} \ll 1$. In this limit, one finds $\delta \mu_{\text{diff}} \to 0$, but $\delta \Gamma_{12}$ in Eq. (B3) remains finite and is given by

$$\delta \Gamma_{12}(r, t) = \sigma_{H} \left\{ \mathbf{v} \cdot [n_{c}(\mathbf{v}_{c} - \mathbf{v}_{n})] + \frac{1}{3} n_{c} \mathbf{v} \cdot \mathbf{v}_{n} \right\}. \quad (B3)$$

As discussed in Ref. [25], this source term in the continuity equations Eqs. (44) and (45) plays a crucial role in establishing the equivalence between the ZNG equations (in the limit that $\omega \tau_{\mu} \to 0$) and the standard Landau two-fluid equations considered in the text of this paper. Thus, $\Gamma_{12}$ is a concrete example of the source term $\Gamma$ ($= m \Gamma_{12}$) which arises in Eqs. (63) and (64), in the context of a dilute Bose gas.

This is a convenient place to note that in Refs. [25, 24, 16], another version of the two-fluid hydrodynamics was discussed in which the collisional transfer of atoms between the condensate and thermal cloud was ignored. This corresponds to setting $\Gamma_{12} = 0$ or, equivalently, the limit $\omega \tau_{\mu} \gg 1$. This “ZGN” two-fluid hydrodynamics was used to calculate the temperature-dependence of the frequencies of first and second sound in a weakly-interacting uniform Bose gas [24]. In this case, the velocities were almost identical (see Fig. I of Ref. [25]) to those based on the Landau two-fluid hydrodynamics ($\omega \tau_{\mu} \ll 1$ limit). Expressions for the two sound velocities based on the LTF equations are given in Appendix C of Ref. [30], showing the small corrections which depend on the factor $\sigma_{H}$ in Eq. (B3).

The variational theory developed in Ref. [25] for determining the normal mode oscillations in a trapped gas was worked out originally for the ZGN two-fluid hydrodynamics [40]. The results for monopole, dipole, and quadrupole oscillations are summarized in Fig. 1 of Ref. [46] and Figs. 4 and 5 of Ref. [25]. For the dipole modes, the frequencies are identical to Eqs. (66) and (67), but with

$$k_{sn} = 2g \int \frac{d^{3}r}{\partial x_{i} } \frac{\partial n_{c0} }{\partial x_{i} } \frac{\partial n_{0} }{\partial x_{i} }, \quad (B4)$$

where $g = 4\pi \hbar^{2} a / m$. It can be shown, however, making use of approximations consistent with those used to derive ZGN hydrodynamics, that Eq. (B4) reduces to Eq. (B3). The details of this derivation will be given elsewhere.

The variational approach of Ref. [25] was formally extended to deal with the ZNG hydrodynamics in Ref. [50], but it was not used to calculate the collective mode frequencies. Instead, the out-of-phase mode was approximated by extending the $T = 0$ pure condensate mode to finite $T$ by using a reduced condensate density. The in-phase oscillation was approximated at finite $T$ by the thermal cloud oscillation above $T_{c}$, but with a reduced thermal cloud density. As discussed in Section VII of Ref. [25], these are often good first approximations.

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