Multiple positive solutions for a class of quasilinear singular elliptic systems

Hana Didi1 · Abdelkrim Moussaoui2

Received: 5 June 2019 / Accepted: 28 August 2019 / Published online: 4 September 2019
© Springer-Verlag Italia S.r.l., part of Springer Nature 2019

Abstract
In this paper we establish the existence of two positive solutions for a class of quasilinear singular elliptic systems. The main tools are sub and supersolution method and Leray–Schauder Topological degree.

Keywords Singular system · p-Laplacian · Leray–Schauder degree · Regularity

Mathematics Subject Classification 35J75 · 35J48 · 35J92

1 Introduction

We consider the following system of quasilinear elliptic equations:

\[
\begin{cases}
-\Delta_p u = f(u, v) & \text{in } \Omega, \\
-\Delta_q v = g(u, v) & \text{in } \Omega, \\
u, v > 0 & \text{in } \Omega, \\
u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

\[ (P) \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (\( N \geq 2 \)) with \( C^{1,\alpha} \) boundary \( \partial \Omega \), \( \alpha \in (0, 1) \), \( \Delta_p \) and \( \Delta_q \), \( 1 < p, q < N \), are the \( p \)-Laplacian and \( q \)-Laplacian operators, respectively, that is, \( \Delta_p u = div(|\nabla u|^{p-2}\nabla u) \) and \( \Delta_q v = div(|\nabla v|^{q-2}\nabla v) \). The nonlinearities \( f, g : (0, +\infty) \times (0, +\infty) \to (0, +\infty) \) are continuous functions satisfying the growth condition:

(H.1) For every \( \bar{L} > 0 \), there are constants \( m_i, M_i > 0 \) (\( i = 1, 2 \)) such that

\[
m_1 s^{\alpha_1} t^{\beta_1} \leq f(s, t) \leq M_1 s^{\alpha_1} t^{\beta_1}, \quad \text{for all } 0 < s < \bar{L}, \text{ and all } t > 0, \\
m_2 s^{\alpha_2} t^{\beta_2} \leq g(s, t) \leq M_2 s^{\alpha_2} t^{\beta_2}, \quad \text{for all } 0 < t < \bar{L}, \text{ and all } s > 0,
\]
with
\[
\begin{cases}
-1 < \alpha_1 < 0 < \beta_1 < p - 1 \\
-1 < \beta_2 < 0 < \alpha_2 < q - 1.
\end{cases}
\tag{1.1}
\]

\textbf{(H.2)} For every $\bar{L}^* > 0$ there exist constants $J_1 > \lambda_1, p$ and $J_2 > \lambda_1, q$ such that
\[
\lim_{s \to \infty} \frac{f(s, t)}{s^{p-1}} = J_1 \quad \text{for all } 0 < t < \bar{L}^*,
\]
\[
\lim_{t \to \infty} \frac{g(s, t)}{t^{q-1}} = J_2 \quad \text{for all } 0 < s < \bar{L}^*.
\]

We provide an example where (H.1) and (H.2) are fulfilled. Notice that under the above assumptions system $(P)$ is cooperative, that is, for $u$ (resp. $v$) fixed the right term in the first (resp. second) equation of $(P)$ is increasing in $v$ (resp. $u$).

\textbf{Example 1} Let $\theta \in C_c(\mathbb{R})$ with $\theta(s) = 1$ on bounded sets. Consider the functions $f, g : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ defined by the following:
\[
f(s, t) = \theta(s)s^{\alpha_1}t^{\beta_1} + (1 - \theta(s))J_1s^{p-1}, \quad \text{for } s, t > 0
\]
and
\[
g(s, t) = \theta(t)s^{\alpha_2}t^{\beta_2} + (1 - \theta(t))J_2t^{q-1}, \quad \text{for } s, t > 0,
\]
which clearly verify assumptions (H.1) and (H.2).

The study of singular elliptic problems is greatly justified because they arise in several physical situations such as fluid mechanics pseudoplastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation and so on. In Fulks and Maybee [10], the reader can find a very nice physical illustration of a practical problem which leads to singular problem.

With respect to singular system it is worth to cite, among others, the important Gierer–Meinhardt system which is the stationary counterpart of a parabolic system proposed by Gierer–Meinhardt (see [8,15]) which occurs in the study of morphogenesis on experiments on hydra, an animal of a few millimeters in length.

Besides the importance of the physical application above mentioned, we would like to mention that from a mathematical point of view the singular problems are also interesting because to solve some of them are necessary nontrivial mathematical techniques, which involve Topological degree, Bifurcation theory, Fixed point theorems, sub and supersolution Method, Pseudomonotone Operator theory and Variational Methods. Here, it is impossible to cite all papers in the literature which use the above techniques, however the reader can find the applications of the above mentioned methods in Alves and Moussaoui [3], Hai [16], Ghergu and Radulescu [13], Giacomoni et al. [11], Giacomoni, Hernandez and Sauvy [14], Hernandez et al. [17], Khodja and Moussaoui [18], Zhang [27], Zhang and Yu [28], Diaz et al. [9], Alves et al. [2], Crandall et al. [7], Taliaferro [26], Lunning and Perry [20], Motreanu and Moussaoui [21–23], Moussaoui et al. [24], Agarwall and O’Regan [5], Stuart [25] and their references.

After a review bibliography, we did not find any paper where the existence of multiple solutions have been considered for a singular system. Motivated by this fact, we prove in the present paper the existence of at least two positive solutions for system $(P)$. Our main result has the following statement:
Theorem 1 Under assumptions (H.1) and (H.2) problem (P) possesses at least two (positive) solutions in $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$, for certain $\gamma \in (0, 1)$.

In the proof of the above theorem, we will use sub and supersolution method combined with Leray–Schauder Topological degree. However, before proving that theorem it was necessary to get some informations about the regularity of the solutions. To this end, the below result was crucial in our approach.

Theorem 2 Assume (H.1) holds. Then, system (P) has a positive solution $(u, v)$ in $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. Moreover, there exist a sub-supersolution $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in C^{1}(\overline{\Omega}) \times C^{1}(\overline{\Omega})$ for (P) such that

$$u(x) \leq \underline{u}(x) \leq \overline{u}(x) \text{ and } v(x) \leq \overline{v}(x) \text{ for all } x \in \overline{\Omega}. \tag{1.2}$$

In the present paper, a solution of (P) is understood in the weak sense, that is, a pair $(u, v) \in W_{0}^{1,p}(\Omega) \times W_{0}^{1,q}(\Omega)$, with $u, v$ positive a.e. in $\Omega$, satisfying

$$\begin{cases}
  \int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx = \int_{\Omega} f(u, v)\varphi \, dx, \\
  \int_{\Omega} |\nabla v|^{q-2}\nabla v \nabla \psi \, dx = \int_{\Omega} g(u, v)\psi \, dx,
\end{cases} \tag{1.3}$$

for all $(\varphi, \psi) \in W_{0}^{1,p}(\Omega) \times W_{0}^{1,q}(\Omega)$.

The Proof of Theorem 2 is done in Sect. 2. The main technical difficulty consists in the presence of singular terms in system (P) under condition (H.1). Our approach is based on the sub-supersolution method in its version for systems [18, Theorem 2]. We show the existence of a (positive) solution $(u, v) \in C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$, for certain $\gamma \in (0, 1)$, of problem (P).

The Proof of Theorem 1 is done in Sect. 3. It is based on topological degree theory with suitable truncations. Here, it suffices to show the existence of a second (positive) solution for problem (P). The first one is given by Theorem 2 which is located in a rectangle formed by the sub-supersolutions. However, due to the singular terms in system (P), the degree theory cannot be directly implemented. To handle this difficulty, the degree calculation is applied for the regularized problem $(P_{\varepsilon})$ for $\varepsilon > 0$. Under assumption (H.1), Theorem 2 ensures the existence of a smooth solution for (P). This gives rise to the possible existence a constant $R > 0$ such that all solutions $(u, v)$ with $C^{1,\gamma}$-regularity satisfy $\|u\|_{C^{1,\gamma}}, \|v\|_{C^{1,\gamma}} < R$.

On the basis of this, we show that the degree of an operator corresponding to system $(P_{\varepsilon})$ on a larger set is 0. Another hand, we show that the degree of an operator corresponding to the system $(P_{\varepsilon})$ is 1 on an appropriate set. This leads to the existence of a second solution for $(P_{\varepsilon})$ by using the excision property of Leray–Schauder degree. Then the existence of a second solution for (P) is derived by passing to the limit as $\varepsilon \to 0$.

In what follows, we denote by $\phi_{1,p}$ and $\phi_{1,q}$ the normalized positive eigenfunctions associated with the principal eigenvalues $\lambda_{1,p}$ and $\lambda_{1,q}$ of $-\Delta_{p}$ and $-\Delta_{q}$, respectively:

$$-\Delta_{p}\phi_{1,p} = \lambda_{1,p}|\phi_{1,p}|^{p-2}\phi_{1,p} \text{ in } \Omega, \quad \phi_{1,p} = 0 \text{ on } \partial \Omega, \quad \int_{\Omega} \phi_{1,p}^{p} = 1 \tag{1.4}$$

and

$$-\Delta_{q}\phi_{1,q} = \lambda_{1,q}|\phi_{1,q}|^{q-2}\phi_{1,q} \text{ in } \Omega, \quad \phi_{1,q} = 0 \text{ on } \partial \Omega, \quad \int_{\Omega} \phi_{1,q}^{q} = 1. \tag{1.5}$$

The strong maximum principle ensures the existence of positive constants $l_{1}$ and $l_{2}$ such that

$$l_{1}\phi_{1,p}(x) \leq \phi_{1,q}(x) \leq l_{2}\phi_{1,p}(x) \text{ for all } x \in \Omega. \tag{1.6}$$
For a later use we recall that there exists a constant \( l > 0 \) such that
\[
\phi_{1,p}(x), \phi_{1,q}(x) \geq ld(x) \quad \text{for all } x \in \Omega,
\] (1.7)
where \( d(x) := \text{dist}(x, \partial \Omega) \) (see, e.g., [12]). Moreover, since \( \phi_{1,p} \) and \( \phi_{1,q} \) belongs to \( C^1(\overline{\Omega}) \), there is \( M > 0 \) such that
\[
M = \max_{x \in \Omega} |\phi_{1,p}(x)| + |\phi_{1,q}(x)|.
\] (1.8)

2 Proof of Theorem 2: Existence of the first solution

Let us define \( w_1 \) and \( w_2 \) as the unique weak solutions of the problems
\[
\begin{align*}
-\Delta_p w_1 &= w_1^{q_1} \quad \text{in } \Omega, \\
w_1 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (2.1)
and
\[
\begin{align*}
-\Delta_q w_2 &= w_2^{p_2} \quad \text{in } \Omega, \\
w_2 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (2.2)
respectively, which are known to satisfy
\[
c_2 \phi_{1,p}(x) \leq w_1(x) \leq c_3 \phi_{1,p}(x) \quad \text{and} \quad c_2' \phi_{1,q}(x) \leq w_2(x) \leq c_3' \phi_{1,q}(x),
\] (2.2)
with positive constants \( c_2, c_3, c_2', c_3' \) (see [12]). Consider \( \xi_1, \xi_2 \in C^1(\overline{\Omega}) \) the solutions of the homogeneous Dirichlet problems:
\[
\begin{align*}
-\Delta_p \xi_1 &= \phi^{q_1}_{1,p}(x) \quad \text{in } \Omega, \\
\xi_1 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (2.3a)
and
\[
\begin{align*}
-\Delta_q \xi_2 &= \phi^{p_2}_{1,q}(x) \quad \text{in } \Omega, \\
\xi_2 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (2.3b)

The Hardy–Sobolev inequality (see, e.g., [1, Lemma 2.3]) guarantees that the right-hand side of (2.3) belongs to \( W^{-1,p'}(\Omega) \) and \( W^{-1,q'}(\Omega) \), respectively. Consequently, the Minty–Browder theorem (see [6, Theorem V.15]) implies the existence of unique \( \xi_1 \) and \( \xi_2 \) in (2.3). Moreover, (2.1), (2.2), the monotonicity of the operators \( -\Delta_p \) and \( -\Delta_q \) yield
\[
c_0 \phi_{1,p}(x) \leq \xi_1(x) \leq c_1 \phi_{1,p}(x) \quad \text{and} \quad c_0' \phi_{1,q}(x) \leq \xi_2(x) \leq c_1' \phi_{1,q}(x) \quad \text{in } \Omega,
\] (2.4)
for some positive constants \( c_0, c_1, c_0', c_1' \). Let \( z_1 \) and \( z_2 \) satisfy
\[
-\Delta_p z_1 = h_1(x), \quad z_1 = 0 \quad \text{on } \partial \Omega,
\] (2.5)
and
\[
-\Delta_q z_2 = h_2(x), \quad z_2 = 0 \quad \text{on } \partial \Omega,
\] (2.6)
where
\[
h_1(x) = \begin{cases} 
\phi^{q_1}_{1,p}(x) \quad \text{in } \Omega \setminus \overline{\Omega}_\delta, \\
-\phi^{q_1}_{1,p}(x) \quad \text{in } \overline{\Omega}_\delta,
\end{cases}
\] (2.7)
and
\[
h_2(x) = \begin{cases} 
\phi^{p_2}_{1,q}(x) \quad \text{in } \Omega \setminus \overline{\Omega}_\delta, \\
-\phi^{p_2}_{1,q}(x) \quad \text{in } \overline{\Omega}_\delta,
\end{cases}
\] (2.8)
with a fixed \( \delta > 0 \) sufficiently small and \( d(x) = d(x, \partial \Omega) \).
The Hardy–Sobolev inequality together with the Minty-Browder theorem imply the existence and uniqueness of \( z_1 \) and \( z_2 \) in (2.5) and (2.6). Moreover, (2.5) and (2.6), the monotonicity of the operators \(-\Delta_p\) and \(-\Delta_q\) and [16, Corollary 3.1] imply that

\[
\frac{c_0}{\alpha} \phi_{1,p}(x) \leq z_1(x) \leq c_1 \phi_{1,p}(x) \quad \text{and} \quad \frac{c_0}{\beta} \phi_{1,q}(x) \leq z_2(x) \leq c_1 \phi_{1,q}(x) \quad \text{in} \quad \Omega.
\]

Next, our goal is to show the existence of sub and supersolution for \((P)\).

**Existence of sub-solution**

For a constant \( C > 0 \), we have

\[
-C^{-(p-1)} \phi_{1,p}^{\alpha_1}(x) < 0 \leq m_1(C^{-1}z_1(x))^{\alpha_1}(C^{-1}z_2(x))^{\beta_1}, \quad x \in \Omega_\delta
\]

and

\[
-C^{-(q-1)} \phi_{1,q}^{\beta_2}(x) < 0 \leq m_2(C^{-1}z_1(x))^{\alpha_2}(C^{-1}z_2(x))^{\beta_2}, \quad x \in \Omega_\delta.
\]

Let \( \mu > 0 \) be a constant such that

\[
\phi_1(x), \phi_2(x) \geq \mu \quad \text{in} \quad \Omega \setminus \Omega_\delta.
\]

Then, since \( \alpha_1 < 0 < \beta_1 \), (2.9) and (2.12) lead to

\[
C^{\alpha_1 + \beta_1 - (p-1)} \phi_{1,p}^{\alpha_1}(x)(z_1(x))^{-\alpha_1} \leq C^{\alpha_1 + \beta_1 - (p-1)} \phi_{1,p}^{\alpha_1}(x)(c_1 \phi_{1,p}(x))^{-\alpha_1}
\]

\[
= C^{\alpha_1 + \beta_1 - (p-1)} (Mc_1)^{-\alpha_1} < m_1(c_0 \mu)^{\beta_1} \leq m_1(c_0 \phi_{1,q}(x))^{\beta_1}
\]

\[
\leq m_1(z_2(x))^{\beta_1}, \quad \text{for all} \quad x \in \Omega \setminus \Omega_\delta,
\]

provided \( C > 0 \) large enough. This is equivalent to

\[
C^{-(p-1)} \phi_{1,p}^{\alpha_1}(x) < m_1(C^{-1}z_1(x))^{\alpha_1}(C^{-1}z_2(x))^{\beta_1}, \quad \text{for all} \quad x \in \Omega \setminus \Omega_\delta.
\]

Similarly,

\[
C^{-(q-1)} \phi_{1,q}^{\beta_2}(x) < m_2(C^{-1}z_1(x))^{\alpha_2}(C^{-1}z_2(x))^{\beta_2}, \quad \text{for all} \quad x \in \Omega \setminus \Omega_\delta,
\]

for \( C > 0 \) large enough. The pair

\[
(u, v) = C^{-1}(z_1, z_2).
\]

is a subsolution for \((P)\). Indeed, a direct computation shows that

\[
\int_{\Omega} |u|^{p-2} \nabla u \nabla \varphi \, dx = C^{-(p-1)} \int_{\Omega \setminus \Omega_\delta} \phi_{1,p}^{\alpha_1} \varphi \, dx - C^{-(p-1)} \int_{\Omega_\delta} \phi_{1,p}^{\alpha_1} \varphi \, dx
\]

and

\[
\int_{\Omega} |u|^{q-2} \nabla u \nabla \psi \, dx = C^{-(q-1)} \int_{\Omega \setminus \Omega_\delta} \phi_{1,q}^{\beta_2} \psi \, dx - C^{-(q-1)} \int_{\Omega_\delta} \phi_{1,q}^{\beta_2} \psi \, dx,
\]

where \((\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\) with \( \varphi, \psi \geq 0 \). Combining (2.17), (2.18), (2.10), (2.11), (2.13), (2.15) and (H.1), it is readily seen that

\[
\int_{\Omega} |u|^{p-2} \nabla u \nabla \varphi \, dx \leq m_1 \int_{\Omega} u^{\alpha_1} \varphi \, dx
\]

\[
\leq m_1 \int_{\Omega} u^{\alpha_1} \omega_2^{\beta_1} \varphi \, dx \leq \int_{\Omega} f(u, \omega_2) \varphi \, dx
\]
and
\[
\int_{\Omega} |\nabla u|^{q-2} \nabla u \nabla \psi \, dx \leq m_2 \int_{\Omega} u^{q_2} v^{\beta_2} \psi \, dx
\]
\[
\leq m_2 \int_{\Omega} \omega_1^{q_2} v^{\beta_2} \psi \, dx \leq \int_{\Omega} g(\omega_1, \psi) \, dx,
\]
for all \((\varphi, \psi) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) with \(\varphi, \psi \geq 0\), all \(\omega_1 \geq u\) and \(\omega_2 \geq v\) in \(\Omega\). This proves that \((u, v)\) is a subsolution for \((P)\).

**Existence of supersolution** Next, we prove that
\[
(u, v) = C(\xi_1, \xi_2)
\]
(2.19)
is a supersolution for problem \((P)\) for \(C > 0\) large enough. Obviously, we have \((\overline{u}, \overline{v}) \geq (u, v)\) in \(\overline{\Omega}\) for \(C\) large enough. Taking into account (2.3), (2.4), (1.8) and (1.1) we derive that in \(\overline{\Omega}\) one has the estimates
\[
\overline{u}^{-\alpha_1} \overline{v}^{-\beta_1} (-\Delta_p \overline{u}) = C^{p-1-\alpha_1-\beta_1} \xi_2^{-\beta_1} \geq C^{p-1-\alpha_1-\beta_1} (c' \phi_1, q(x))^{-\beta_1}
\]
\[
\geq C^{p-1-\alpha_1-\beta_1} (c' M)^{-\beta_1} \geq M_1 \text{ in } \overline{\Omega}
\]
and
\[
\overline{u}^{-\alpha_2} \overline{v}^{-\beta_2} (-\Delta_q \overline{v}) \geq C^{q-1-\alpha_2-\beta_2} (c_1 M)^{-\alpha_2} \geq M_2 \text{ in } \overline{\Omega},
\]
provided that \(C > 0\) is sufficiently large. Consequently, by (H.1), it turns out that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \geq M_1 \int_{\Omega} \overline{u}^{q_1} \overline{v}^{\beta_1} \varphi \, dx
\]
\[
\geq M_1 \int_{\Omega} \overline{u}^{q_1} \omega_2^{\beta_1} \varphi \, dx \geq \int_{\Omega} f(\overline{u}, \omega_2) \varphi \, dx
\]
(2.20)
and
\[
\int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \psi \, dx \geq M_2 \int_{\Omega} \overline{u}^{q_2} \overline{v}^{\beta_2} \psi \, dx
\]
\[
\geq M_2 \int_{\Omega} \omega_1^{q_2} \overline{v}^{\beta_2} \psi \, dx \geq \int_{\Omega} g(\omega_1, v) \psi \, dx,
\]
(2.21)
for all \((\varphi, \psi) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)\) with \(\varphi, \psi \geq 0\), all \((\omega_1, \omega_2)\) within \([u, \overline{u}] \times [v, \overline{v}]\). Thus, \((\overline{u}, \overline{v})\) is a supersolution for \((P)\).

**Proof of Theorem 2** (conclusion) Using and (H.1), (1.7), (1.2), (2.16), (2.19), (2.9) and (2.4), for \((u, v)\) within \([u, \overline{u}] \times [v, \overline{v}]\), one has
\[
f(u, v) \leq M_1 u^{q_1} v^{\beta_1} \leq u^{q_1} \overline{v}^{\beta_1} \leq C_1 d(x)^{q_1} \text{ for all } x \in \Omega
\]
and
\[
g(u, v) \leq M_2 u^{q_2} v^{\beta_2} \leq \overline{u}^{q_2} \overline{v}^{\beta_2} \leq C_2 d(x)^{\beta_2} \text{ for all } x \in \Omega,
\]
where \(C_1\) and \(C_2\) are positive constants. Then, owing to [18, Theorem 2] we deduce that there exists a solution \((u, v) \in C^{1, \gamma} (\overline{\Omega}) \times C^{1, \gamma} (\overline{\Omega})\), for certain \(\gamma \in (0, 1)\), of problem \((P)\) within \([u, \overline{u}] \times [v, \overline{v}]\). This completes the proof.
3 Proof of Theorem 1: existence of the second solution

According to Theorem 2 we know that problem \((P)\) possesses a (positive) solution \((u, v)\) in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\), located in the rectangle \([\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]\). Thus, to prove Theorem 1 it suffices to show the existence of a second solution for problem \((P)\). However, it is worth noting that by Theorem 2 the set of solutions \((u, v)\) in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\) for problem \((P)\) is not empty. Then, without any loss of generality, we may assume that there is a constant \(R > 0\) such that all solutions \((u, v)\) with \(C^1\)-regularity satisfy

\[
\|u\|_{C^1(\overline{\Omega})}, \|v\|_{C^1(\overline{\Omega})} < R. \tag{3.1}
\]

Otherwise, there are infinity solutions with \(C^1\)-regularity and the Proof of Theorem 1 is completed.

Hereafter, we denote

\[
B_R(0) = \{(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) : \|u\|_{C^1} + \|v\|_{C^1} < R\},
\]

\[
\mathcal{O}_R = \{(u, v) \in B_R(0) : u \ll u \ll \hat{u}_R \text{ and } v \ll v \ll \hat{v}_R\}
\]

and

\[
\mathcal{O}_\Lambda = \{(u, v) \in B_R(0) : u \ll u \ll \overline{u}_\Lambda \text{ and } v \ll v \ll \overline{v}_\Lambda\},
\]

where

\[
(\overline{u}_\Lambda, \overline{v}_\Lambda) = \Lambda(\xi_1, \xi_2) \text{ and } (\hat{u}_R, \hat{v}_R) = \Lambda_R(\xi_1, \xi_2), \tag{3.2}
\]

with \(\xi_1, \xi_2\) fixed in (2.3) and \(\Lambda_R, \Lambda > 0\) are constants which will be chosen later on. A simple computation gives that \(\mathcal{O}_R\) and \(\mathcal{O}_\Lambda\) are open sets in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\).

In what follows, we will assume without loss of generality that

\[
R > \max\{|u|_\infty, |\overline{u}|_\infty, |v|_\infty, |\overline{v}|_\infty, |\overline{u}_\Lambda|_\infty, |\overline{v}_\Lambda|_\infty\}.
\]

In the sequel, we use the notation \(u_1 \ll u_2\) when \(u_1, u_2 \in C^1(\overline{\Omega})\) satisfy:

\[
u_1(x) < u_2(x) \quad \forall x \in \Omega \quad \text{and} \quad \frac{\partial u_1}{\partial v} \leq \frac{\partial u_2}{\partial v} \quad \text{on } \partial \Omega,
\]

where \(\nu\) is the outward normal to \(\partial \Omega\).

The next proposition is useful for proving our second main result.

**Proposition 1** Assume (H.1) holds. Then all solutions \((u, v)\) of \((P)\) in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\), satisfy

\[
u(x) \ll \hat{u}_R(x) \text{ and } \nu(x) \ll \hat{v}_R(x) \quad \text{in } \Omega, \tag{3.3}
\]

whenever \(u(x) \geq \underline{u}(x)\) and \(v(x) \geq \underline{v}(x)\) in \(\Omega\). Moreover, for all solutions \((u, v)\) of \((P)\) within \([\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]\), it holds

\[
u(x) \ll \overline{u}_\Lambda(x) \text{ and } \nu(x) \ll \overline{v}_\Lambda(x) \quad \text{in } \Omega. \tag{3.4}
\]

**Proof** We prove only the first parts of inequalities (3.3) and (3.4) because the second ones can be justified similarly. Recalling that all solutions \((u, v)\) with \(C^{1,\gamma}\)-regularity satisfy (3.1), by (H.1), (2.16) and (2.9), for \(u(x) \geq \underline{u}(x)\) and \(v(x) \geq \underline{v}(x)\) in \(\Omega\), one has

\[\text{Springer}\]
−Δ_p u = f(u, v) ≤ M_1 u^{α_1} v^{β_1} ≤ M_1 u^{α_1} R^{β_1} 

≤ M_1 (C^{-1} \frac{C_0}{2} \phi_{1,p})^{α_1} R^{β_1} < \Lambda_R^{p-1} \phi_{1,p}^{α_1} 

= −Δ_p (\Lambda R \xi_1) = −Δ_p \bar{u}_R \in \Omega,

for \Lambda_R large enough. Furthermore, from (H.1), (2.19), (2.16), (2.4) and (2.9), for (u, v) ∈ [u, \bar{u}] × [v, \bar{v}], it follows that

\begin{align*}
- Δ_p u &= f(u, v) ≤ M_1 u^{α_1} v^{β_1} ≤ M_1 u^{α_1} v^{β_1} \\
&≤ M_1 (C^{-1} \frac{C_0}{2} \phi_{1,p})^{α_1} (C_{1,p})^{β_1} ≤ C^{-α_1 + β_1} \left( \frac{C_0}{2} \right)^{α_1} (c_1 M)^{β_1} \phi_{1,p}^{α_1} \\
&< \Lambda_R^{p-1} \phi_{1,p}^{α_1} = −Δ_p (\Lambda \xi_1) = −Δ_p \bar{u}_R \in \Omega,
\end{align*}

provided that \Lambda is large enough. Consequently, the strong comparison principle found in [4, Proposition 2.6] leads to the conclusion. This ends the proof. □

3.1 An auxiliary problem

We will make use the topological degree theory to get the second solution for system (P). However, the singular terms in system (P) prevents the degree calculation to be well defined. To overcome this difficulty, we disturb system (P) by introducing a parameter \varepsilon ∈ (0, 1).

This gives rise to a regularized system for (P) defined for \varepsilon > 0 as follows:

\begin{equation}
(P_\varepsilon) \begin{cases}
-Δ_p u = f(u + \varepsilon, v) & \text{in } \Omega, \\
-Δ_q v = g(u, v + \varepsilon) & \text{in } \Omega, \\
u, v = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

We apply the degree theory for the regularized problem (P_\varepsilon). This leads to find a positive solution for (P_\varepsilon) lying outside of the set \mathcal{O}_\Lambda. Our main result regarding problem (P_\varepsilon) is stated as follows:

**Theorem 3** Assume (H.1) and (H.2) hold. Then problem (P_\varepsilon) possesses a positive solution (\bar{u}_\varepsilon, \bar{v}_\varepsilon) ∈ C^{1,γ} (\Omega \setminus \{0\}) × C^{1,γ} (\Omega \setminus \{0\}), for certain γ ∈ (0, 1), such that (\bar{u}_\varepsilon, \bar{v}_\varepsilon) ∈ \mathcal{O}_R \setminus \mathcal{O}_\Lambda, for all \varepsilon ∈ (0, 1).

**Remark 1** It is very important to observe that the same reasoning exploited in the Proof of Theorem 2 and Proposition 1 furnishes that problem (P_\varepsilon) has a (positive) solution (u_\varepsilon, v_\varepsilon) ∈ C^{1,γ} (\Omega \setminus \{0\}) × C^{1,γ} (\Omega \setminus \{0\}), γ ∈ (0, 1), within \([u, \bar{u}] \times [v, \bar{v}],\) where functions (u, v) and (\bar{u}, \bar{v}) are sub-supersolutions of (P_\varepsilon) and (u_\varepsilon, v_\varepsilon) verifies

\begin{equation}
\begin{cases}
u_\varepsilon(x) ≪ \bar{u}_\Lambda(x) \text{ and } v_\varepsilon(x) ≪ \bar{v}_\Lambda(x) \text{ in } \Omega,
\end{cases}
\end{equation}

for all \varepsilon ∈ (0, 1).

The next lemma provides a helpful comparison property which will be used later on.

**Lemma 1** Let u_1, u_2 ∈ C^1 (\Omega \setminus \{0\}) be the solutions of the problems

\begin{equation}
T_{\varepsilon, p}(u_1) = f(x) \text{ in } \Omega, \quad T_{\varepsilon, p}(u_2) = g(x) \text{ in } \Omega, \quad u_1 = 0 \text{ on } \partial \Omega, \quad u_2 = 0 \text{ on } \partial \Omega,
\end{equation}

where

\begin{equation}
T_{\varepsilon, p}(u) = −Δ_p u + ρ(u + \varepsilon)^{α_1−(p−1)} |u + \varepsilon|^{p−2} (u + \varepsilon),
\end{equation}

\small{Springer}
with constants \( \rho, \varepsilon > 0 \) and \( f, g \in L^\infty_{loc}(\Omega) \). If \( f < g \), that is, for each compact set \( K \subset \Omega \), there is \( \tau = \tau(K) > 0 \) such that

\[
f(x) + \tau \leq g(x) \quad \text{a.e in } K,
\]

then \( u_1 \ll u_2 \).

**Proof** The proof is very similar to that of Proposition 2.6 in \([4]\). \(\square\)

### 3.2 Topological degree results

#### 3.2.1 The first estimate (the degree on \( \mathcal{O}_R \))

We transform the problem (\( \mathcal{P}_\varepsilon \)) to one with helpful monotonicity properties. To this end, let us introduce the functions

\[
\chi_\varepsilon(s) = \begin{cases} 
  s & \text{if } u \leq s, \\
  u & \text{if } s \leq u,
\end{cases} \quad \chi_\varepsilon(s) = \begin{cases} 
  s & \text{if } v \leq s, \\
  v & \text{if } s \leq v,
\end{cases}
\]

and

\[
\tilde{\phi} = \begin{cases} 
  \hat{u}_R & \text{if } \phi \geq \hat{u}_R \\
  \phi & \text{if } u \leq \phi \leq \hat{u}_R, \\
  \hat{v}_R & \text{if } \varphi \geq \hat{v}_R \\
  \varphi & \text{if } v \leq \varphi \leq \hat{v}_R \\
  \bar{u} & \text{if } \phi \leq u, \\
  \bar{v} & \text{if } \varphi \leq v,
\end{cases}
\]

where \( (u, v) \) and \( (\hat{u}_R, \hat{v}_R) \) are given by (2.16) and (3.2), respectively. We shall study the homotopy class of problem

\[
(\mathcal{P}_{\varepsilon,t}) \quad \begin{cases} 
  -\Delta_p u = F_{\varepsilon,t}(x, u, v) & \text{in } \Omega, \\
  -\Delta_q v = G_{\varepsilon,t}(x, u, v) & \text{in } \Omega, \\
  u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where functions \( F_{\varepsilon,t} \) and \( G_{\varepsilon,t} \) are defined as follows:

\[
F_{\varepsilon,t}(x, u, v) = tf(\chi_\varepsilon(u) + \varepsilon, \bar{v}) + \bar{\eta}(1-t)\chi_\varepsilon(u)^{p-1},
\]

\[
G_{\varepsilon,t}(x, u, v) = tg(\chi_\varepsilon(v) + \varepsilon) + \bar{\eta}(1-t)\chi_\varepsilon(v)^{q-1},
\]

for \( t \in [0, 1] \) and \( \varepsilon \in (0, 1) \), with constant \( \bar{\eta} > 0 \) which will be chosen later on.

The next results are crucial in our approach which establish an important prior estimate for system (\( \mathcal{P}_{\varepsilon,t} \)). Moreover, it is also shown that the solutions of problem (\( \mathcal{P}_{\varepsilon,t} \)) cannot occur outside the rectangle formed by the subsolution \( (u, v) \) and the a priori estimates of solutions of (\( \mathcal{P}_{\varepsilon,t} \)).

**Proposition 2** Under assumption (H.1), all solutions \( (u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) of (\( \mathcal{P}_{\varepsilon,t} \)) satisfy

\[
u(x) \ll u(x) \text{ and } v(x) \ll v(x) \text{ in } \Omega,
\]

for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \).

**Proof** First, we transform the problem (\( \mathcal{P}_{\varepsilon,t} \)) to one with helpful monotonicity properties. To this end, let us introduce the auxiliary problem

\[
\begin{cases} 
  -\Delta_p u + \rho \mathcal{L}_{e,p}(u) = F_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{e,p}(\bar{u}) & \text{in } \Omega, \\
  -\Delta_q v + \rho \mathcal{L}_{e,q}(v) = G_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{e,q}(\bar{v}) & \text{in } \Omega, \\
  u, v = 0 & \text{on } \Omega,
\end{cases}
\]
where
\[
\begin{align*}
\mathcal{L}_{e,p}(s) &= (u + \varepsilon)^{\alpha_i - (p-1)} (s + \varepsilon)^{p-1}, \\
\mathcal{L}_{e,q}(s) &= (u + \varepsilon)^{\beta_i - (q-1)} (s + \varepsilon)^{q-1},
\end{align*}
\]
for \( s \geq 0 \) and \( \varepsilon \in (0, 1) \). Here the constant \( \rho > 0 \) is assumed sufficiently large so that the following inequalities are satisfied:
\[
\alpha_1(s_1 + \varepsilon)^{\alpha_1 - 1} \hat{v}_R(x)^{\beta_1} + \rho (p - 1) u(x)^{\alpha_1 - (p-1)} s_1^{p-2} \geq 0
\]
and
\[
\beta_2 u_R(x)^{\alpha_2} (s_2 + \varepsilon)^{\beta_2 - 1} + \rho (q - 1) v(x)^{\beta_2 - (q-1)} s_2^{q-2} \geq 0,
\]
uniformly in \( x \in \Omega \), for \( u_R \geq s_1 \geq u \) and \( \hat{v}_R \geq s_2 \geq v \). This is possible because from (2.16), (2.9), (3.2), (2.4) and (1.8) and for \( \rho > 0 \) sufficiently large, if \( p > 2 \), one has
\[
\alpha_1(s_1 + \varepsilon)^{\alpha_1 - 1} \hat{v}_R(x)^{\beta_1} + \rho (p - 1) (u + \varepsilon)^{\alpha_1 - (p-1)} (s_1 + \varepsilon)^{p-2} \geq 0
\]
and
\[
\beta_2 u_R(x)^{\alpha_2} (s_2 + \varepsilon)^{\beta_2 - 1} + \rho (q - 1) v(x)^{\beta_2 - (q-1)} (s_2 + \varepsilon)^{q-2} \geq 0,
\]
If \( p \leq 2 \), one gets
\[
\alpha_1(s_1 + \varepsilon)^{\alpha_1 - 1} \hat{v}_R(x)^{\beta_1} + \rho (p - 1) (u + \varepsilon)^{\alpha_1 - (p-1)} (s_1 + \varepsilon)^{p-2} \geq 0
\]
and
\[
\beta_2 u_R(x)^{\alpha_2} (s_2 + \varepsilon)^{\beta_2 - 1} + \rho (q - 1) v(x)^{\beta_2 - (q-1)} (s_2 + \varepsilon)^{q-2} \geq 0,
\]
where \( C_0 = \min\{1, C^{-1/2}\} 2^p \cdot \max\{1, \Lambda_R c_1\} p-2 \), provided \( \rho > 0 \) large enough. Here, it is important to observe, by the choice of \( \rho \), that functions
\[
(s_1 + \varepsilon)^{\alpha_1} s_1^{\beta_1} + \rho \mathcal{L}_{e,p}(s_1) \text{ and } s_1^{\alpha_2} (s_2 + \varepsilon)^{\beta_2} + \rho \mathcal{L}_{e,q}(s_2)
\]
increases as \( \hat{u}_R \geq s_1 \geq u \) and \( \hat{v}_R \geq s_2 \geq v \) increases, respectively, for all \( \varepsilon \in (0, 1) \).

Next, let us prove that (3.11) holds true for every solution \((u, v) \) of \( (\mathcal{P}_{e,t}) \) bounded in \( C^1(\Omega) \times C^1(\Omega) \), for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \). We only show the first inequality in (3.11) because the second one can be justified similarly. To this end, we set the functions \( X_t : \Omega \to \mathbb{R} \) given by
\[
X_1(x) = C^{-(p-1)} h_1(x) + \rho \mathcal{L}_{e,p}(u)
\]
and
\[
X_2(x) = F_{e,t}(x, u, v) + \rho \mathcal{L}_{e,p}(\tilde{u}).
\]
On account of (H.1), (2.7), (3.7)–(3.9), we have
\[
X_1(x) = -C^{-(p-1)} \phi_1^a(x) + \rho \mathcal{L}_{e,p}(u)
\]
\[
< tm_1(u + \varepsilon)^{\alpha_1} v^{\beta_1} (1 - t) \tilde{n} u^{p-1} + \rho \mathcal{L}_{e,p}(u)
\]
\[
\leq tm_1(\chi_\mu(u) + \varepsilon)^{\alpha_1} v^{\beta_1} (1 - t) \tilde{n} \chi_\mu(u)^{p-1} + \rho \mathcal{L}_{e,p}(\tilde{u})
\]
for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \).

On another hand, by (2.9), (1.1), (2.16), (2.12) and (1.8), we obtain

\[
X_1(x) = C^{-(p-1)} \phi_{1,p}^{(a)}(x) + \rho \mathcal{L}_{\varepsilon,p}(u)
\]

and

\[
m_1(u + \varepsilon) \Phi_{\varepsilon}^{(b)} + (1 - \varepsilon) \mathcal{M}_{\varepsilon,p}(u)
\]

for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \). Gathering (3.13) and (3.14) together leads to

\[
X_1(x) = C^{-(p-1)} \phi_{1,p}^{(a)}(x) + \rho \mathcal{L}_{\varepsilon,p}(u)
\]

for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \). Consequently, it follows from (3.12) and (3.15) that for each compact set \( \mathcal{K} \subset \Omega \), there is a constant \( \tau = \tau(\mathcal{K}) > 0 \) such that

\[
X_1(x) + \tau = C^{-(p-1)} \phi_{1,p}^{(a)}(x) + \rho \mathcal{L}_{\varepsilon,p}(u)
\]

and

\[
X_1(x) + \tau = C^{-(p-1)} \phi_{1,p}^{(a)}(x) + \rho \mathcal{L}_{\varepsilon,p}(u)
\]

for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \). Hence \( X_1 < X_2 \) and \( X_i \in L^\infty_{\text{loc}}(\Omega) \) and thereby, by the strong comparison principle in Lemma 1, we infer that

\[
u(x) \gtrless u(x), \quad \forall x \in \Omega.
\]

The proof of the second inequality in (3.21) is carried out in a similar way. This ends the proof.

\[\square\]

**Proposition 3** Assume (H.1) and (H.2) hold. Then all solutions \((u, v)\) of \((P_{\varepsilon,t})\) belong to \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\) and satisfy

\[
\|u\|_{C^1(\overline{\Omega})}, \|v\|_{C^1(\overline{\Omega})} < R,
\]

for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \).
Proof By contradiction suppose that for every positive integer \( n \) there exist \( t_n \in [0, 1] \) and a solution \((u_n, v_n)\) of \((P_{\varepsilon, t_n})\) such that \( t_n \to t \in [0, 1] \) and \( \|u_n\|_{\mathcal{C}^1(\Omega)}, \|v_n\|_{\mathcal{C}^1(\Omega)} \to \infty \) as \( n \to \infty \). Thanks to Proposition 2, we have

\[
\text{as } n \to \infty \quad \|u_n\|_{\mathcal{C}^1(\Omega)}, \|v_n\|_{\mathcal{C}^1(\Omega)} \to \infty
\]

Thus, \((P_{\varepsilon, t_n})\) is equivalent to

\[
\begin{aligned}
-\Delta_p u_n &= t_n f(u_n + \varepsilon, \tilde{v}_n) + \tilde{\eta}(1 - t)u_n^{p-1} \quad \text{in } \Omega, \\
-\Delta_q v_n &= t_n g(\tilde{u}_n, v_n + \varepsilon) + \tilde{\eta}(1 - t)v_n^{q-1} \quad \text{in } \Omega, \\
u_n, v_n &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Without loss of generality we may admit that

\[
\theta_n := \|u_n\|_{\mathcal{C}^1(\Omega)} \to \infty \text{ as } n \to \infty.
\]

Denote

\[
\mathcal{U}_n := \frac{1}{\theta_n} u_n \in \mathcal{C}^1(\bar{\Omega}) \text{ with } \|\mathcal{U}_n\|_{\mathcal{C}^1(\bar{\Omega})} = 1 \text{ for all } n \in \mathbb{N}.
\]

The first equation in \((P_{\varepsilon, t_n})\) results in

\[
-\Delta_p \mathcal{U}_n = \frac{1}{\theta_n^{p-1}} \left( t_n f(u_n + \varepsilon, \tilde{v}_n) + (1 - t_n)\tilde{\eta}u_n^{p-1} \right),
\]

where \(\mathcal{U}_n(x) > 0\) a.e. in \(\Omega\) because of (3.17).

If \(u_n \leq \tilde{L}\) in \(\Omega\) for certain constant \(\tilde{L} > 0\), from (H.1), (3.8), (2.16), (3.17), (3.2), (2.4), (2.9) and (1.7), we have

\[
\frac{1}{\theta_n^{p-1}} \left( t_n f(u_n + \varepsilon, \tilde{v}_n) + (1 - t_n)\tilde{\eta}u_n^{p-1} \right)
\]

where a constant \(\hat{C}_1 > 0\) is independent of \(n\) and \(\varepsilon\). Otherwise, by (H.2), one has

\[
\frac{1}{\theta_n^{p-1}} \left( t_n f(u_n + \varepsilon, \tilde{v}_n) + (1 - t_n)\tilde{\eta}u_n^{p-1} \right)
\]

for some constant \(\hat{C}_2 > 0\) independent of \(n\) and \(\varepsilon\). Thus, owing to [16, Lemma 3.1] (resp. [19]) in the case of (3.20) (resp. (3.21)), one derive that \(\mathcal{U}_n\) is bounded in \(\mathcal{C}^{1,\gamma}(\bar{\Omega})\) for certain \(\gamma \in (0, 1)\). The compactness of the embedding \(\mathcal{C}^{1,\gamma}(\bar{\Omega}) \subset \mathcal{C}^{1}(\bar{\Omega})\) implies

\[
\mathcal{U}_n \to \mathcal{U} \text{ in } \mathcal{C}^1(\bar{\Omega}).
\]
From (3.19), (3.22), (H.2) and (3.17), we find
\begin{equation}
\begin{cases}
-\Delta_p U = (tJ_1 + (1-t)\tilde{\eta})U^{p-1} & \text{in } \Omega, \\
U = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{3.23}
\end{equation}
with \(U \geq 0\) in \(\Omega\). Since \(tJ_1 + (1-t)\tilde{\eta} > \lambda_{1,p_1}\), the corresponding eigenfunction \(U\) should change sign. This forces that \(U = 0\). Therefore, by (3.22), \(U_0 \to 0\) in \(C^1(\Omega)\), which contradicts (3.18).

Consequently, by increasing the constant \(R > 0\) if necessary, all solutions \((u, v) \in C^1(\Omega) \times C^1(\Omega)\) of \((P_{\varepsilon,t})\) satisfy (3.16) for all \(t \in [0, 1]\) and all \(\varepsilon \in (0, 1)\). This completes the proof. \(\Box\)

**Proposition 4** Under the assumption (1.1) problem \((P_{\varepsilon,t})\) has no solutions for \(t = 0\).

**Proof** Arguing by contradiction, let \((\hat{u}, \hat{v}) \in C^1(\Omega) \times C^1(\Omega)\) be a nontrivial solution of \((P_{\varepsilon,t})\) with
\begin{equation}
(\hat{u}, \hat{v}) \in \mathcal{O}_R \text{ and } t = 0,
\end{equation}
which reads as
\begin{equation}
\begin{cases}
-\Delta_p \hat{u} = \tilde{\eta} \hat{u}^{p-1} & \text{in } \Omega, \\
-\Delta_q \hat{v} = \tilde{\eta} \hat{v}^{q-1} & \text{in } \Omega, \\
\hat{u}, \hat{v} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
because of (3.7) and (3.11). From (2.9) and (2.16)
\[u(x) = C^{-1}z_1(x) \geq C^{-1}c_0 \phi_1(x) \text{ in } \Omega.\]
In the sequel, we fix \(u_1(x) = C^{-1}c_0 \phi_1(x) \leq u(x) \text{ in } \Omega\) and take \(\lambda_\delta = \lambda_{1,p} + \delta\) for \(\delta > 0\). Let \(u_2 \in C^1(\Omega)\) be the solution of the problem
\begin{equation}
\begin{cases}
-\Delta_p u_2 = \lambda_\delta u_1^{p-1} & \text{in } \Omega, \\
u_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
Then for \(\delta > 0\) small and \(\tilde{\eta}\) large enough, we have
\[-\Delta_p u_2 = \lambda_\delta u_1^{p-1} \leq \tilde{\eta} \hat{u}^{p-1} = -\Delta_p \hat{u} \text{ in } \Omega\]
and
\[-\Delta_p u_1 = \lambda_{1,p} u_1^{p-1} \leq \lambda_\delta u_1^{p-1} = -\Delta_p u_2 \text{ in } \Omega.\]
By weak comparison principle we get
\[u_1(x) \leq u_2(x) \leq \hat{u}(x) \text{ in } \Omega.\]
Now considering solutions of problems
\begin{equation}
\begin{cases}
-\Delta_p u_n = \lambda_\delta u_n^{p-1} & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}
we obtain an increasing sequence \(\{u_n\}_n\) such that
\[u_1(x) \leq u_{n-1}(x) \leq u_n(x) \leq \hat{u}(x) \text{ a.e. in } \Omega.\]
Passing to the limit we get a positive solution \( u \in W^{1,p}_0(\Omega) \) for problem

\[
\begin{cases}
-\Delta_p u = \lambda \delta u^{p-1} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

which is impossible for \( \delta > 0 \) small enough because the first eigenvalue for \( p \)-Laplacian is isolate. Hence, problem \((P_{\varepsilon,t})\) has no solutions for \( t = 0 \).

Define the homotopy \( \mathcal{H}_\varepsilon \) on \([0, 1] \times C^1(\bar{\Omega}) \times C^1(\bar{\Omega})\) by

\[
\mathcal{H}_\varepsilon(t, u, v) = I(u, v) - \begin{pmatrix} (-\Delta_p)^{-1} & 0 \\ 0 & (-\Delta_q)^{-1} \end{pmatrix} \times \begin{pmatrix} \mathcal{F}_{\varepsilon,t}(x, u, v) \\ \mathcal{G}_{\varepsilon,t}(x, u, v) \end{pmatrix}.
\]

Since functions \( \mathcal{F}_{\varepsilon,t} \) and \( \mathcal{G}_{\varepsilon,t} \) belong to \( C(\bar{\Omega}) \) for all \( x \in \bar{\Omega} \) and all \( \varepsilon \in (0, 1) \), \( \mathcal{H}_\varepsilon \) is well defined. Moreover, \( \mathcal{H}_\varepsilon : [0, 1] \times C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \to C(\bar{\Omega}) \times C(\bar{\Omega}) \) is completely continuous for all \( \varepsilon \in (0, 1) \). This is due to the compactness of the operators \((-\Delta_p)^{-1}, (-\Delta_q)^{-1} : C(\bar{\Omega}) \to C^1(\bar{\Omega})\). Hence, \((u, v) \in \mathcal{O}_R\) is a solution for \((P_\varepsilon)\) if, and only if,

\[
(u, v) \in \mathcal{O}_R \text{ and } \mathcal{H}_\varepsilon(1, u, v) = 0.
\]

From the previous Propositions 2 and 3 it is clear that solutions of \((P_{\varepsilon,t})\) must lie in \( \mathcal{O}_R \). Thus, the fact that problem \((P_{\varepsilon,t})\) has no solutions for \( t = 0 \) (see proposition 4) implies that

\[
\deg(\mathcal{H}_\varepsilon(0, \cdot, \cdot), \mathcal{O}_R, 0) = 0, \text{ for all } \varepsilon \in (0, 1).
\]

Consequently, from the homotopy invariance property, it follows that

\[
\deg(\mathcal{H}_\varepsilon(1, \cdot, \cdot), \mathcal{O}_R, 0) = 0, \text{ for all } \varepsilon \in (0, 1). \quad (3.25)
\]

### 3.2.2 The second estimate (the degree on \( \mathcal{O}_\Lambda \))

We show that the degree of an operator corresponding to the system \((P_\varepsilon)\) is 1 on the set \( \mathcal{O}_\Lambda \). To this end, we modify the problem to ensure that solutions cannot occur outside of the rectangle formed by \((u, v)\) and \((\bar{u}_\Lambda, \bar{v}_\Lambda)\). Set

\[
\bar{u} = \begin{cases} u_\Lambda & \text{if } u \geq u_\Lambda \\ u & \text{if } u \leq u \leq \bar{u}_\Lambda \\ \bar{u} & \text{if } u \leq u \end{cases}, \quad \bar{v} = \begin{cases} v_\Lambda & \text{if } v \geq v_\Lambda \\ v & \text{if } v \leq v \leq \bar{v}_\Lambda \\ \bar{v} & \text{if } v \leq v \end{cases}, \quad (3.26)
\]

and let us define the truncation problem

\[
(P_{\varepsilon,t}) \quad \begin{cases} -\Delta_p u = \overline{\mathcal{F}}_{\varepsilon,t}(x, u, v) & \text{in } \Omega \\ -\Delta_q v = \overline{\mathcal{G}}_{\varepsilon,t}(x, u, v) & \text{in } \Omega \\ u, v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with

\[
\overline{\mathcal{F}}_{\varepsilon,t}(x, u, v) = tf(\bar{u} + \varepsilon, \bar{v}) + (1-t)\tilde{\eta}(\phi_{1,p} + \varepsilon)^{q_1}
\]

and

\[
\overline{\mathcal{G}}_{\varepsilon,t}(x, u, v) = tg(\bar{u}, \bar{v} + \varepsilon) + (1-t)\tilde{\eta}(\phi_{1,q} + \varepsilon)^{\beta_2},
\]

for \( t \in [0, 1] \), \( \varepsilon \in (0, 1) \) and a constant \( \tilde{\eta} > 0 \).

We state the following result regarding truncation system \((P_{\varepsilon,t})\).
Proposition 5 Under assumption (H.1) every solution \((u_\varepsilon, v_\varepsilon)\) of \((\overline{P}_\varepsilon, t)\) is bounded in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\)

\[
\|u_\varepsilon\|_{C^1(\overline{\Omega})}, \|v_\varepsilon\|_{C^1(\overline{\Omega})} < R,
\]

for all \(t \in [0, 1]\) and all \(\varepsilon \in (0, 1)\). Moreover, it holds

\[
\frac{u(x)}{h_1(x)} \ll u_\varepsilon(x) \ll \bar{u}_\Lambda(x) \text{ and } \frac{v(x)}{h_1(x)} \ll v_\varepsilon(x) \ll \bar{v}_\Lambda(x), \quad \forall x \in \Omega,
\]

(3.27)

for all \(t \in [0, 1]\) and all \(\varepsilon \in (0, 1)\).

Proof By (H.1), (3.26), (2.16), (2.9), (3.2) and (2.4), one has

\[
\overline{F}_{\varepsilon,t}(x, u, v) \leq f(\bar{u} + \varepsilon, \bar{v}) + \bar{\eta}_{1,p} \leq M_1 u^{a_1} \bar{v}^{\beta_1} + \bar{\eta}_{1,p} \leq C_1 \phi_1^{\alpha_1} \text{ in } \Omega
\]

and

\[
\overline{G}_{\varepsilon,t}(x, u, v) \leq g(\bar{u} + \varepsilon, \bar{v}) + \bar{\eta}_{1,q} \leq M_2 a_2 u^{\beta_2} + \bar{\eta}_{1,q} \leq C_2 \phi_1^{\alpha_2} \text{ in } \Omega,
\]

with certain constants \(C_1, C_2 > 0\) independent of \(\varepsilon\). Then, thanks to [16, Lemma 3.1], one derive the \(C^{1,\gamma}\) -boundedness of solutions \((u_\varepsilon, v_\varepsilon)\) of \((\overline{P}_\varepsilon, t)\) for all \(\varepsilon \in (0, 1)\).

Now, let us prove (3.27). We only show the first part of inequalities in (3.27) because the second ones can be justified similarly. Let us introduce the problem

\[
\begin{cases}
-\Delta_p u + \rho \mathcal{L}_{\varepsilon, p}(u) = \overline{F}_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{\varepsilon, p}(\bar{u}) \text{ in } \Omega \\
-\Delta_q v + \rho \mathcal{L}_{\varepsilon, q}(v) = \overline{G}_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{\varepsilon, q}(\bar{v}) \text{ in } \Omega \\
u, v = 0 \text{ on } \partial \Omega,
\end{cases}
\]

for \(t \in [0, 1]\), \(\varepsilon \in (0, 1)\) and \(\bar{\eta} > 0\). The constant \(\rho > 0\) is chosen sufficiently large so that the following inequalities are satisfy:

\[
\alpha_1(s_1 + \varepsilon)^{a_1 - 1} s_2^{\beta_1} + \rho(p - 1)(u + \varepsilon)^{a_1 - (p - 1)}(s + \varepsilon)^{p - 2} \geq 0,
\]

\[
\beta_2(s_2 + \varepsilon)^{a_2 - 1} + \rho(q - 1)(\bar{v} + \varepsilon)^{\beta_2 - (q - 1)}(s + \varepsilon)^q \geq 0,
\]

uniformly in \(x \in \Omega\), for \((s_1, s_2) \in [u, \bar{u}_\Lambda] \times [v, \bar{v}_\Lambda]\), and for all \(\varepsilon \in (0, 1)\).

Set the functions \(X_i : \Omega \to \mathbb{R}\) given by

\[
X_1(x) = C^{-(p-1)} h_1(x) + \rho \mathcal{L}_{\varepsilon, p}(u)
\]

and

\[
X_2(x) = \overline{F}_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{\varepsilon, p}(\bar{u}).
\]

From (1.8) and (2.9), for all \(\varepsilon \in (0, 1)\) and all \(t \in [0, 1]\), one has

\[
(t + 1 - t)(u + \varepsilon)^{a_1} \bar{v}^{\beta_1} \leq t(u + \varepsilon)^{a_1} \bar{v}^{\beta_1} + (1 - t)(C^{-1} c_0 \phi_1, p + \varepsilon) a_1 (C^{-1} c_1 \phi_1, q)^{\beta_1} \leq t(u + \varepsilon)^{a_1} \bar{v}^{\beta_1} + (1 - t) \bar{\eta}(\phi_1, p + \varepsilon)^{a_1} \text{ in } \Omega,
\]

(3.28)

provided that \(\bar{\eta} > 0\) is sufficiently large. Then, following the quite similar argument which proves (3.11) in Proposition 2, we obtain \(X_1 < X_2\) with \(X_1, X_2 \in L^\infty_{loc}(\Omega)\). Thus, the strong comparison principle (see Lemma 1) imply

\[
u(x) \gg u(x) \quad \forall x \in \Omega.
\]
It remains to show that \( u(x) \ll \overline{u}_\Lambda(x) \), \( \forall x \in \Omega \). To do so, set functions \( \tilde{X}_1 : \Omega \to \mathbb{R} \) defined by
\[
\tilde{X}_1(x) = \overline{F}_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{\varepsilon,p}(\tilde{u})
\]
and
\[
\tilde{X}_2(x) = \Lambda^{p-1} \phi_{1,p}(x)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda).
\]
Using (2.4), (2.19) and the choice of \( \rho > 0 \) we get
\[
t M_1 (\tilde{u} + \varepsilon)^{\alpha_1} \tilde{v}^{\beta_1} + (1 - t) \tilde{\eta}(\phi_{1,p} + \varepsilon)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\tilde{u})
\leq M_1 (\overline{u}_\Lambda + \varepsilon)^{\alpha_1} \overline{v}^{\beta_1} + \tilde{\eta}(\phi_{1,p} + \varepsilon)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda)
\leq M_1 \Lambda^{\alpha_1 + \beta_1} \xi_1 \xi_2 + \tilde{\eta}(\phi_{1,p} + \varepsilon)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda)
\leq M_1 \Lambda^{\alpha_1 + \beta_1} (c_0 \phi_{1,q})^{\alpha_1}(c_1 \phi_{1,q})^{\beta_1} + \tilde{\eta}(\phi_{1,p} + \varepsilon)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda)
\leq \Lambda^{p-1} \phi_{1,p}(x)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda)\text{ in } \Omega,
\]
for \( \Lambda > 0 \) sufficiently large. Thus, for each compact set \( \mathcal{K} \subseteq \Omega \), there is a constant \( \tau = \tau(\mathcal{K}) > 0 \) such that
\[
\tilde{X}_1(x) + \tau = \overline{F}_{\varepsilon,t}(x, u, v) + \rho \mathcal{L}_{\varepsilon,p}(\tilde{u}) + \tau
\leq \Lambda^{p-1} \phi_{1,p}(x)^{\alpha_1} + \rho \mathcal{L}_{\varepsilon,p}(\overline{u}_\Lambda) = \tilde{X}_2(x) \text{ a.e. in } \mathcal{K} \cap \Omega,
\]
for all \( t \in [0, 1] \) and all \( \varepsilon \in (0, 1) \). That is, \( \tilde{X}_1 < \tilde{X}_2 \), with \( \tilde{X}_i \in L^\infty(\Omega) \) and therefore, by the strong comparison principle in Lemma 1, we infer that
\[
u_{\varepsilon}(x) \ll \overline{u}_\Lambda(x) \text{ for all } x \in \Omega \text{ and all } \varepsilon \in (0, 1).
\]
A quite similar argument provides that \( v_{\varepsilon}(x) \ll \overline{v}_\Lambda(x) \) for all \( x \in \Omega \) and all \( \varepsilon \in (0, 1) \). This ends the proof.

Let us define the homotopy \( N_\varepsilon \) on \([0, 1] \times C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\) by
\[
N_\varepsilon(t, u, v) = I(u, v) - \left(\begin{array}{cc}
(-\Delta_p)^{-1} & 0 \\
0 & (-\Delta_q)^{-1}
\end{array}\right)
\times \left(\begin{array}{c}
\overline{F}_{\varepsilon,t}(x, u, v) \\
\overline{G}_{\varepsilon,t}(x, u, v)
\end{array}\right), \text{ for } \varepsilon \in (0, 1).
\]
Clearly, \( N_\varepsilon \) is well defined and completely continuous homotopy for all \( \varepsilon \in (0, 1) \) and all \( t \in [0, 1] \). Moreover, \( (u, v) \in \mathcal{O}_\Lambda \) is a solution of system \((P_\varepsilon)\) if, and only if,
\[
(u, v) \in \mathcal{O}_\Lambda \text{ and } N_\varepsilon(1, u, v) = 0 \text{ for all } \varepsilon \in (0, 1).
\]

In view of Proposition 5 and from the definition of function \( \overline{u}_\Lambda \) and \( \overline{v}_\Lambda \) it follows that all solutions of \((\overline{P}_{\varepsilon,t})\) are also solutions of \((P_\varepsilon)\). Moreover, these solutions must be in the set \( \mathcal{O}_\Lambda \). Moreover, for \( t = 0 \) in (3.29), Minty-Browder Theorem together with Hardy–Sobolev Inequality and [16, Lemma 3.1] ensure that problems
\[
\left\{\begin{array}{ll}
-\Delta_p u = \tilde{\eta}(\phi_{1,p} + \varepsilon)^{\alpha_1} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{array}\right. \text{ and } \left\{\begin{array}{ll}
-\Delta_q v = \tilde{\eta}(\phi_{1,q} + \varepsilon)^{\beta_2} & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{array}\right
\]

Springer
admit unique positive solutions $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ in $C^{1,\gamma}(\bar{\Omega})$ for certain $\gamma \in (0, 1)$ and for $\varepsilon \in (0, 1)$, respectively. Then, the homotopy invariance property of the degree gives

$$\deg(N_\varepsilon(1, \cdot, \cdot), O_\Lambda, 0) = \deg(N_\varepsilon(0, \cdot, \cdot), O_\Lambda, 0) = \deg(N_\varepsilon(0, \cdot, \cdot), B_R(0), 0) = 1.$$  \hspace{1cm} (3.30)

Since

$$H_\varepsilon(1, \cdot, \cdot) = N_\varepsilon(1, \cdot, \cdot) \text{ in } O_\Lambda,$$

it follows that

$$\deg(H_\varepsilon(1, \cdot, \cdot), O_\Lambda, 0) = 1, \text{ for all } \varepsilon \in (0, 1).$$  \hspace{1cm} (3.31)

### 3.2.3 Proof of Theorem 3

Hereafter, we will assume that

$$H_\varepsilon(1, u, v) \neq 0 \text{ for all } (u, v) \in \partial O_\Lambda,$$

otherwise we will have a solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \in \partial O_\Lambda$, which is different from the solution $(u, v)$ in Theorem 2, because $(u, v) \in O_\Lambda$. Here, we have used that $O_\Lambda$ is an open set, then $(u, v) \notin \partial O_\Lambda$.

By (3.25) and (3.31), we deduce from the excision property of Leray–Schauder degree that

$$\deg(H_\varepsilon(1, \cdot, \cdot), O_R \setminus \overline{O_\Lambda}, 0) = -1$$

and thus problem $(P_\varepsilon)$ has a solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ with

$$(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \in O_R \setminus \overline{O_\Lambda}. \hspace{1cm} (3.32)$$

Then, owing to [16, Lemma 3.1] we conclude $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \in C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$.

### 3.3 Proof of Theorem 1

Set $\varepsilon = \frac{1}{n}$ with any positive integer $n \geq 1$. From (3.32) with $\varepsilon = \frac{1}{n}$, we know that there exist $(\tilde{u}_n, \tilde{v}_n) := (\tilde{u}_{\frac{1}{n}}, \tilde{v}_{\frac{1}{n}})$ bounded in $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ such that

$$\begin{cases}
-\Delta_p \tilde{u}_n = f(\tilde{u}_n + \frac{1}{n}, \tilde{v}_n) & \text{in } \Omega, \\
-\Delta_q \tilde{v}_n = g(\tilde{u}_n, \tilde{v}_n + \frac{1}{n}) & \text{in } \Omega, \\
\tilde{u}_n = \tilde{v}_n = 0 & \text{on } \partial \Omega,
\end{cases} \hspace{1cm} (3.33)$$

satisfying

$$(\tilde{u}_n, \tilde{v}_n) \in O_R \setminus \overline{O_\Lambda} \text{ for all } n \in \mathbb{N}. \hspace{1cm} (3.34)$$

Employing Arzelà–Ascoli’s theorem, we may pass to the limit in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ and the limit functions $(\tilde{u}, \tilde{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ satisfy $(P)$ with

$$(\tilde{u}, \tilde{v}) \in O_R \setminus \overline{O_\Lambda} \hspace{1cm} (3.35)$$

Finally, on account of (3.35) and Proposition 1, we achieve that $(\tilde{u}, \tilde{v})$ is a second solution of problem $(P)$. This complete the Proof of Theorem 1.
References

1. Alves, C.O., Corrêa, F.J.S.A.: On the existence of positive solution for a class of singular systems involving quasilinear operators. Appl. Math. Comput. 185, 727–736 (2007)
2. Alves, C.O., Corrêa, F.J.S.A., Gonçalves, J.V.A.: Existence of solutions for some classes of singular Hamiltonian systems. Adv. Nonlinear Stud. 5, 265–278 (2005)
3. Alves, C.O., Moussaoui, A.: Existence of solutions for a class of singular elliptic systems with convection term. Asympt. Anal. 90, 237–248 (2014)
4. Arcoya, D., Ruiz, D.: The Ambrosetti-Prodi problem for the p-Laplace operator. Commun. Partial Differ. Equ. 31, 849–865 (2006)
5. Agarwal, R.P., O'Regan, D.: Existence theory for single and multiple solutions to singular positive boundary value problems. J. Differ. Equ. 175, 393–414 (2001)
6. Brézis, H.: Analyse Fonctionnelle Théorie et Applications. Masson, Paris (1983)
7. Crandall, M.G., Rabinowitz, P.H., Tartar, L.: On a Dirichlet problem with singular nonlinearity. Commun. Partial Differ. Equ. 2, 193–222 (1977)
8. del Pino, M.: A priori estimates applications to existence-nonexistence for a semilinear elliptic system. Indiana Univ. Math. J. 43, 77–129 (1994)
9. Diaz, I., Morel, J.M., Oswald, L.: An elliptic equation with singular nonlinearity. Commun. Partial Differ. Equ. 12, 1333–1344 (1987)
10. Fulks, W., Maybee, J.S.: A singular non-linear equation. Osaka Math. J. 12, 1–19 (1960)
11. Giacomoni, J., Hernandez, J., Moussaoui, A.: Quasilinear and Singular Systems: The Cooperative Case, Contemporary Mathematics, 540, American Mathematical Society, Providence, RI, pp. 79–94 (2011)
12. Giacomoni, J., Schindler, I., Takac, P.: Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. A. Sc. N. Sup. Pisa (5) 6, 117–158 (2007)
13. Ghergu, M., Radulescu, V.: On a class of Gierer–Meinhardt systems arising in morphogenesis. C. R. Acad. Sci. Paris, Ser. I 344, 163–168 (2007)
14. Giacomoni, J., Hernandez, J., Sauvy, P.: Quasilinear and singular elliptic systems. Adv. Nonlinear Anal. 2, 1–41 (2013)
15. Gierer, A., Meinhardt, H.: A theory of biological pattern formation. Kybernetik 12, 30–39 (1972)
16. Haï, D.D.: On a class of singular p-Laplacian boundary value problems. J. Math. Anal. Appl. 383, 619–626 (2011)
17. Hernandez, J., Mancebo, F.J., Vega, J.M.: Positive solutions for singular semilinear elliptic systems. Adv. Differ. Equ. 13, 857–880 (2008)
18. Khodja, B., Moussaoui, A.: Positive solutions for infinite semipositone/positone quasilinear elliptic systems with singular and superlinear terms. Differ. Equ. Appl. 8(4), 535–546 (2016)
19. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. 12, 1203–1219 (1988)
20. Luning, C.D., Perry, W.L.: Positive solutions of negative exponent generalized Emden–Fowler boundary value problem. SIAM J. Math. Anal. 12, 874–879 (1981)
21. Motreanu, D., Moussaoui, A.: A quasilinear singular elliptic system without cooperative structure. Act. Math. Sci. 34B(3), 905–916 (2014)
22. Motreanu, D., Moussaoui, A.: An existence result for a class of quasilinear singular competitive elliptic systems. Appl. Math. Lett. 38, 33–37 (2014)
23. Motreanu, D., Moussaoui, A.: Existence and boundedness of solutions for a singular cooperative quasilinear elliptic system. Complex Var. Elliptic Equ. 59, 285–296 (2014)
24. Moussaoui, A., Khodja, B., Tas, S.: A singular Gierer–Meinhardt system of elliptic equations in $\mathbb{R}^N$. Nonlinear Anal. 71, 708–716 (2009)
25. Stuart, C.A.: Existence and approximations of solutions of nonlinear elliptic equations. Math. Z. 147, 53–63 (1976)
26. Taliaferro, S.: A nonlinear singular boundary value problem. Nonlinear Anal. Theory Methods Appl. 3, 897–904 (1979)
27. Zhang, Z.: On a Dirichlet with a singular nonlinearity. J. Math. Anal. Appl. 194, 103–113 (1995)
28. Zhang, Z., Yu, J.: On a singular nonlinear Dirichlet problem with a convection term. SIAM J. Math. Anal. 32, 916–927 (2000)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.