A Generalization of Threshold Saturation:
Application to Spatially Coupled BICM-ID

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Abstract—Spatial coupling was proved to improve the belief-propagation (BP) performance up to the maximum-a-posteriori (MAP) performance. This paper addresses an extended class of spatially coupled (SC) systems. A potential function is derived for characterizing a lower bound on the BP performance for the extended SC systems, and shown to be different from the potential for the conventional SC systems. This may imply that the BP performance for the extended SC systems does not coincide with the MAP performance for the corresponding uncoupled system. SC bit-interleaved coded modulation with iterative decoding (BICM-ID) is also investigated as an application of the extended SC systems.

I. INTRODUCTION

Kudekar et al. [1] proved that spatial coupling can improve the belief-propagation (BP) threshold up to the maximum-a-posteriori (MAP) threshold. Since the original proof of this threshold saturation was complicated, several simpler proofs have been developed in [2]–[6]. In this paper we generalize the methodology in [4] to characterize the BP performance for extended spatially-coupled (SC) systems.

Consider the density-evolution (DE) equations of an extended SC system with the number of sections \( L \) and coupling width \( W \)

\[
 u_l(i + 1) = \frac{1}{W_d} \sum_{w_d \in W^d} \varphi(v_{l+w_d}(i)), \quad l \in \mathcal{L},
\]

\[
 v_l(i) = \frac{1}{W^d} \sum_{w_d \in W^d} \psi(u_{l-w_d}(i)), \quad l \in \{W - 1, \ldots, 1\},
\]

with \( \mathcal{L} = \{0, \ldots, L - 1\} \) and \( W = \{0, \ldots, W - 1\} \). For notational convenience, we have used the notation \( v_{l+w_d}(i) = (v_{l+w_1}(i), \ldots, v_{l+w_d}(i)) \) and \( u_{l-w_d}(i) = (u_{l-w_1}(i), \ldots, u_{l-w_d}(i)) \), with \( w_k = (w_1, \ldots, w_d) \). The notation \( l + w \) should be interpreted as \((l, \ldots, l) + w\). In [1] and [2], the state \( (u_l(i), v_l(i)) \in \mathcal{D} \times \mathcal{D} \subset \mathbb{R}^2 \) represents performance of a BP-based algorithm for section \( l \in \mathcal{L} \) in iteration \( i \). The two functions \( \varphi: \mathcal{D}^d \rightarrow \mathcal{D} \) and \( \psi: \mathcal{D}^d \rightarrow \mathcal{D} \) characterize the properties of the BP algorithm. The parameters \( d + 1 \) and \( d + 1 \) correspond to the degrees of check and variable nodes in low-density parity-check (LDPC) codes. In bit-interleaved coded modulation, \( d + 1 \) is equal to the modulation rate, whereas \( d + 1 \) is used.

Without loss of generality, we postulate that larger variables \( u_l(i) \) and \( v_l(i) \) imply better performance. Let \((u_{\text{opt}}, v_{\text{opt}})\) denote a fixed-point (FP) that has the largest \( u \) among all FPs of the DE equations for the uncoupled case \( W = 1 \). Thus, \((u_{\text{opt}}, v_{\text{opt}})\) is a solution \((u, v)\) to the following FP equations:

\[
 u = \varphi_0(v), \quad v = \psi_0(u),
\]

with \( \varphi_0(v) = \varphi(v, \ldots, v) \) and \( \psi_0(u) = \psi(u, \ldots, u) \). The FP \((u_{\text{opt}}, v_{\text{opt}})\) corresponds to the best possible performance achieved by the BP algorithm.

We assume that \( \varphi(v_1, \ldots, v_d) \) and \( \psi(u_1, \ldots, u_d) \) are bounded, smooth\(^1\), and strictly increasing in all arguments everywhere. The monotonicity implies that the performance of the BP algorithm improves monotonically. We impose the worse initial condition \( u_l(0) = u_{\text{min}} = \inf \mathcal{D} \) for all \( l \in \mathcal{L} \) and the best boundary conditions \( v_l(i) = v_{\text{opt}} \) for any \( l \notin \{W - 1, \ldots, L - 1\} \) and \( i \). The aim of spatial coupling is to let the state \((u_l(i), v_l(i))\) converge toward \((u_{\text{opt}}, v_{\text{opt}})\) for all sections \( l \in \mathcal{L} \) after sufficiently many iterations via coupling.

We consider the continuum limit in which \( L \) and \( W \) tend to infinity while the ratio \( \alpha = W/L \) is kept constant. The BP performance for the SC systems [1] and [2] is characterized by a potential function for the uncoupled system

\[
 V(u) = \int \left\{ u - \varphi_0(\psi_0(u)) \right\} \psi_0'(u) \cdot \exp \{ D(u; \psi) + D(\psi_0(u); \varphi) \} dv,
\]

with

\[
 D(u; \psi) = \int \frac{\Delta \psi'(u)}{\psi_0'(u)} du - \ln \psi_0'(u),
\]

where the single-variate Laplacian \( \Delta \psi'(u) \) is given by \( \Delta \psi'(u) = \sum_j \partial^2 \psi/\partial u_j^2(u, \ldots, u) \). The goal of this paper is to prove the following statement:

**Theorem 1.** Take the continuum limit \( W = \alpha L \rightarrow \infty \), the infinite-iteration limit \( i \rightarrow \infty \), and finally the limit \( \alpha \rightarrow 0 \). If \( u_{\text{opt}} \) is the unique global stable solution (global minimizer) of the potential [4], the state \((u_l(i), v_l(i))\) convergences to the target solution \((u_{\text{opt}}, v_{\text{opt}})\) in the limits above.

**Theorem 1** is a generalization of previous works [2]–[6] for \( d = d = 1 \), and implies that the qualitative shape of the potential [4] determines whether the BP algorithm can achieve the best possible performance point \((u_{\text{opt}}, v_{\text{opt}})\), whereas the potential [4] is a generalization of previous works [2]–[6].

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\(^1\) A function is said to be smooth if it is infinitely differentiable.

\(^2\) One should not regard that only degree \( d + 1 = 2 \) was considered in [2]–[6], although \( d + 1 \) and \( d + 1 \) are interpreted as the degrees of nodes for factor graphs in this paper. Since \( \varphi \) and \( \psi \) have a special structure for SC LDPC codes, the DE equations [1] and [4] reduce to those with \( d = d = 1 \).
uniqueness of solutions to the potential \( \mathbf{4} \) does for the uncoupled case \( W = 1 \). The potential \( \mathbf{4} \) reduces to the conventional one defined in \( \mathbf{2} \) for \( d = d = 1 \), and coincides with the conventional one for \( d, d > 1 \) if \( D(u; \psi) + D(\psi(u); \varphi) \) is independent of \( u \). The latter observation may imply that for \( d, d > 1 \) the BP threshold does not coincide with the MAP threshold for the corresponding uncoupled system, since the potential for \( d = d = 1 \) is used to characterize the MAP threshold.

Theorem \( \mathbf{1} \) is useful when no analytical formulas of the multi-variate functions \( \varphi \) and \( \psi \) in \( \mathbf{1} \) and \( \mathbf{2} \) are available. In this case, the cost for calculating the two functions numerically increases exponentially as \( d \) and \( d \) grow. Theorem \( \mathbf{1} \) implies that we can know the BP performance just by estimating six single-variate functions \( \varphi_0, \varphi'_0, \varphi, \psi_0, \psi'_0, \) and \( \Delta \psi \) via numerical integration or sampling.

This paper is organized as follows: In Section II we shall present an application of Theorem \( \mathbf{1} \) to SC bit-interleaved coded modulation with iterative decoding (BICM-ID). Theorem \( \mathbf{1} \) is proved in Section III.

II. APPLICATION

A. Spatially Coupled BICM-ID

We consider a BICM-ID scheme with SC interleaving \( \mathbf{7} \). One transmission consists of \( W - 1 \) binary training sequences of length \( M \) and of \( L = (W - 1) \) binary codewords of length \( M \). The training sequences are utilized to anchor the performance of the system at the boundaries to the best performance \( u_{\text{opt}} \).

After SC interleaving of length \( LM \), the obtained binary sequences are mapped to \( LM/Q \) data symbols, with \( Q \) denoting the modulation rate. The data symbols are transmitted through a memoryless time-invariant communication channel, and detected with iterative decoding at the receiver side \( \mathbf{3} \).

We shall review a construction of SC interleaving \( \mathbf{7} \). Let \( \{\pi_i^\text{in} : l \in \mathcal{L}\} \) and \( \{\pi_i^\text{out} : l \in \mathcal{L}\} \) denote 2L independent random interleavers of length \( M \) that are bijections from \( \mathcal{M} = \{0, \ldots, M - 1\} \) onto \( \mathcal{M} \). An SC interleaver \( \pi(m, l) \) is a bijection from \( \mathcal{M} \times \mathcal{L} \) onto \( \mathcal{M} \times \mathcal{L} \) that maps \( m \)th bit in section \( l \) to the pair \( \pi(m, l) \) of bit and section indices,

\[
\pi(m, l) = (\pi_l^\text{out}(\pi_l^\text{in}(m)), l'), \quad l' = (l - (\pi_l^\text{in}(m))W)_L, \tag{6}
\]

where \( (i)_n \in \{0, \ldots, n - 1\} \) denotes the remainder for the division of \( i \in \mathcal{Z} \) by \( n \in \mathbb{N} \). From the construction, \( M \) bits in section \( l \) are sent to sections \( \{l, \ldots, (l - (W - 1))_L\} \) with uniform frequency when \( M \) is a multiple of \( W \). As a result, \( m \)th bit in section \( l' \) at the output side originates from a bit in sections \( \{l', \ldots, (l' + W - 1)_L\} \) with equal probability. These properties result in the DE equations \( \mathbf{1} \) and \( \mathbf{2} \) with \( d = Q - 1 \) and \( d = 1 \) when \( M \) tends to infinity. Minus one is because iterative decoding is based on extrinsic feedback information.

B. EXIT Chart Analysis

Let us consider a mathematical model based on erasure extrinsic channels \( \mathbf{9} \) that approximates the dynamical properties of the SC BICM-ID scheme in the limit \( M \to \infty \).
find that the FP equations (3) have two stable solutions. One stable solution is the target solution \((v_{\text{opt}}, u_{\text{opt}}) = (1, f_0(1))\), and the other stable solution \((v_{\text{BP}}, u_{\text{BP}})\) is a FP to which the BP algorithm converges for the uncoupled case \(W = 1\). These observations imply that the conventional BICM-ID scheme cannot approach the target point \((1, f_0(1))\) in this case, whereas the SC BICM-ID scheme can.

It is possible to understand the performance of the SC BICM-ID scheme from the EXIT chart for the uncoupled case. Let \(S_l, S_m, S_b\) denote the three areas enclosed by the two curves in Fig. 1 from top to bottom. From the area theorems for the MAP decoder [9], [12, Corollary 5.1] and the MAP demodulator [12, Corollary 5.2], it is straightforward to find the relationship between the areas and the rate loss from the coded modulation (CM) capacity \(C_{CM, ID} \defeq C_{CM} - Qr = QS_b + Q(S_l - S_m)\),

\[
C_{CM} - Qr = QS_b + Q(S_l - S_m),
\]

with \(r\) denoting the code rate. Expression (7) implies that the rate loss from the CM capacity is characterized by \(S_b\) and \(S_l - S_m\). Since \(S_b\) is much smaller than \(S_l - S_m\), there is a gap between \(S_b\) and \(S_l - S_m\) in the gap between \(S_l\) and \(S_m\) is a dominant factor for the rate loss. In fact, the SNR that \(S_l = S_m\) holds is approximately 5.29 dB. Furthermore, the SNR corresponding to the CM capacity is approximately given by 5.12 dB. Since an SNR of 5.76 dB was considered in Fig. 1 from the losses due to \(S_l\) and \(S_l - S_m\) are given by 5.29 – 5.12 = 0.17 dB and 5.76 – 5.29 = 0.47 dB, respectively, if \(S_l\) is assumed to be identical for the two SNRs 5.29 dB and 5.76 dB.

### III. PROOF OF THEOREM 1

#### A. Sketch

The proof of Theorem 1 is a generalization of that in [4]. We shall define a differential system that approximate the dynamics of the DE equations (1) and (2).

\[
\hat{u}(x, i + 1) = \varphi_0(\psi_0(\hat{u})) + \alpha^2 \left[ A(\hat{u}) \left( \frac{d\hat{u}}{dx} \right)^2 + B(\hat{u}) \frac{d^2\hat{u}}{dx^2} \right] \tag{8}
\]

where \(\hat{u}\) is an abbreviation of \(\hat{u}(x, i)\), with

\[
A(\hat{u}) = \frac{\psi_0^\prime(\psi_0(\hat{u}))}{6}, \quad \Delta \psi_0(\hat{u}) + \frac{\Delta \varphi_0(\psi_0(\hat{u}))}{\varphi_0(\psi_0(\hat{u}))} \psi_0^\prime(\hat{u})^2 + \psi_0^\prime(\hat{u})^2 \tag{9}
\]

\[
B(\hat{u}) = \frac{1}{3} \varphi_0^\prime(\psi_0(\hat{u})) \psi_0^\prime(\hat{u}) \varphi_0(\psi_0(\hat{u})) \tag{10}
\]

We impose the boundary condition \(\hat{u}(x, I) = u_{\text{init}}(x)\) is imposed for some \(I > 0\), with a smooth function \(u_{\text{init}}(x)\). By definition, the solution \(\hat{u}(x, i)\) to the differential system (8) is smooth for all \(x \in [-1, 1], i \geq I\), and \(\alpha > 0\).

**Theorem 2.** For any \(\epsilon > 0\), there exist some \(\alpha_0 > 0\), \(I > 0\), and \(u_{\text{init}}(x)\) such that

\[
\lim_{w = \alpha \to \infty} \left| u_l(i) - \hat{u}(\frac{2I}{L} - 1, i) \right| < \epsilon, \tag{11}
\]

for all \(l \in L, \alpha \in (0, \alpha_0)\) and \(i \geq I\).

**Proof:** See Section III-B

#### Lemma 1. For any \(i\) and \(l\), \(u_l(i) \leq u_l(i + 1)\) holds.

**Proof:** The proof is by induction. The initial condition \(u_l(0) = u_{\text{min}}\) implies \(u_l(i) \leq u_l(i + 1)\) for \(i = 0\). Suppose that \(u_l(i) \leq u_l(i + 1)\) holds for some \(i\). Since \(\psi\) is increasing in all arguments, from (2) we obtain

\[
v_l(i + 1) - v_l(i) = \frac{1}{W^d} \sum_{w \in W^d} [\psi(u_l(w)(i + 1)) - \psi(u_l(w)(i))] \geq 0,
\]

(12)

for \(l \in \{W-1, \ldots, L-1\}\). Combining (12) and the boundary condition \(v_l(i) = v_l(i + 1) = v_{\text{opt}}\) for \(l \not\in \{W-1, \ldots, L-1\}\), we obtain \(v_l(i) \leq v_l(i + 1)\) for all \(l\). Repeating the same argument for (11), we arrive at \(u_l(i + 1) \leq u_l(i + 2)\) for all \(l\). Thus, Lemma 1 holds.

From Theorem 2 and Lemma 1 it is guaranteed that \(u_l(i)\) converges to \(\hat{u}(2L/L - 1)\), in which \(\hat{u}(x)\) denotes a stationary solution of the differential system (8) that satisfies

\[
\hat{u} - \varphi_0(\psi_0(\hat{u})) = \alpha^2 \left[ A(\hat{u}) \left( \frac{d\hat{u}}{dx} \right)^2 + B(\hat{u}) \frac{d^2\hat{u}}{dx^2} \right]. \tag{13}
\]

Theorem 1 follows immediately from the following theorem.

**Theorem 3.** If and only if \(u_{\text{opt}}\) is the unique global stable solution of the potential (4), the uniform solution \(u(x) = u_{\text{opt}}\) is the unique solution of the differential system (12) in the limit \(\alpha \to 0\).

**Proof:** We first present a coordinate system that simplifies the representation of the differential system (13). Let us define the change of variables \(y = f(\hat{u})\) by

\[
f(\hat{u}) = \int e^{C(\hat{u})} d\hat{u}, \tag{14}
\]

with

\[
C(\hat{u}) = \int A(\hat{u}) \frac{d\hat{u}}{dx} + B(\hat{u}) \frac{d^2\hat{u}}{dx^2} \tag{15}
\]

Calculating \(d^2y/dx^2\) with the chain rule for partial derivative yields

\[
\frac{d^2y}{dx^2} = e^{C(\hat{u})} \left[ A(\hat{u}) \left( \frac{d\hat{u}}{dx} \right)^2 + B(\hat{u}) \frac{d^2\hat{u}}{dx^2} \right]. \tag{16}
\]

Thus, the differential equation (13) for stationary solutions reduces to

\[
\alpha^2 \frac{d^2y}{dx^2} = \hat{V}''(y(x)), \tag{17}
\]

where the derivative of a potential \(\hat{V}(y)\) is given by

\[
\hat{V}'(y) = \{ \hat{u} - \varphi_0(\psi_0(\hat{u})) \} \frac{e^{C(\hat{u})}}{B(\hat{u})}, \tag{18}
\]

with \(\hat{u} = f^{-1}(y)\). It is straightforward to confirm that \(\hat{V}'(y) = 0\) if and only if \(\hat{u} = f^{-1}(y)\) is a solution to the FP equation \(\hat{u} = \varphi_0(\psi_0(\hat{u}))\) obtained from (3) for the uncoupled system. In
Lemma 2. It can be proved that the uniform solution \( y(x) = y_{\text{opt}} \) is the unique solution to the boundary-value problem (17) with \( y(\pm(1-2\alpha)) = y_{\text{opt}} \) if and only if \( y_{\text{opt}} \) is the unique global stable solution of \( \tilde{V}(y) \). Hassani et al. [13] presented an intuitive explanation of this statement based on classical mechanics. See [4], [14] for a rigorous proof based on the intuition.

Let us prove that the potential \( \tilde{V}(y) \) defined via (18) is equivalent to (3). By definition, we use (14) and (15) to obtain

\[
\tilde{V}(y) = \int \tilde{V}'(y)dy = \int \{ \tilde{u} - \varphi_0(\psi_0(\tilde{u})) \} \frac{e^{2\alpha (u)}}{B(u)} d\tilde{u}. \tag{19}
\]

We calculate (15) with (9) and (10) to obtain

\[
2C(u) = \int \left\{ \frac{\Delta \psi(u)}{\psi_0'(u)} + \frac{\Delta \varphi(\psi(u))}{\psi_0'(\psi(u))} + \frac{\psi_0''(u)}{\psi_0'(u)} \right\} d\tilde{u} = D(\tilde{u}; \psi) + D(\psi_0(\tilde{u}); \varphi) + \ln B(\tilde{u}) + \ln \psi_0'(\tilde{u}). \tag{20}
\]

with (5). Substituting this expression into (19), we arrive at \( \tilde{V}(y) = V(f^{-1}(y)) \) given by (4). This implies that Theorem 5 holds.

B. Proof of Theorem 2

Theorem 2 is proved as follows: We first take the continuum limit to reduce the DE equations (1) and (2) to integral systems. Then, we shall derive the differential system (3) by expanding the integral systems with respect to \( \alpha \). Finally, we evaluate the difference between the states for the integral and differential systems as \( \alpha \to 0 \).

Let us define the integral systems as

\[
u(x, i + 1) = \frac{1}{(2\alpha)^d} \int_{[-\alpha, \alpha]^d} \varphi(v(x + \omega_d, i)) d\omega_d, \ |x| \leq 1, \tag{21}
\]

\[
u(x, i) = \frac{1}{(2\alpha)^d} \int_{[-\alpha, \alpha]^d} \psi(u(x - \omega_d, i)) d\omega_d, \ |x| \leq 1 - \alpha, \tag{22}
\]

where we have introduced the notation \( v(x + \omega_d, i) = (v(x + \omega_1, i), \ldots, v(x + \omega_d, i)) \) and \( u(x - \omega_d, i) = (u(x - \omega_1, i), \ldots, u(x - \omega_d, i)) \), with \( \omega_d = \omega_1, \ldots, \omega_k \). We impose the initial condition \( u(x, 0) = u_{\text{min}} \) for all \( |x| \leq 1 \). Furthermore, the boundary condition \( v(x, i) = v_{\text{opt}} \) is imposed for all \( |x| > 1 - \alpha \) and all \( i \). Since the two functions \( \varphi \) and \( \psi \) have been assumed to be bounded and continuous, the integral systems (21) and (22) are well defined for any \( i \).

Lemma 2. For any \( i \) and \( l \in L \),

\[
\lim_{W = \alpha L \to \infty} \left| u_l(i) - u \left( \frac{2l}{L} - 1, i \right) \right| = 0, \tag{23}
\]

\[
\lim_{W = \alpha L \to \infty} \left| v_l(i) - v \left( \frac{2l}{L} - 1 - \alpha, i \right) \right| = 0. \tag{24}
\]

Proof: The lemma follows immediately from the definition of the Riemann integral.

We analyze the integral systems (21) and (22) for the bulk region \( |x| < 1 - 2\alpha \) and the other region separately. The following lemma is for the neighborhood of the boundaries.

Lemma 3. For any \( \varepsilon > 0 \), there exist some \( \alpha_0 > 0 \) and \( I > 0 \) such that \( |u(x, i) - u_{\text{opt}}| < \varepsilon \) for all \( \alpha \in (0, \alpha_0) \), \( i \geq I \), and \( 1 - 2\alpha \leq |x| \leq 1 \).

Proof: Repeating the proof of Lemma 1 implies that the integral systems (21) and (22) are convergent as \( \alpha \to \infty \). From the dominated convergence theorem, a stationary solution \( u(x), v(x) \) satisfies the FP equations

\[
u(x) = \frac{1}{(2\alpha)^d} \int_{[-\alpha, \alpha]^d} \varphi(v(x + \omega_d)) d\omega_d, \ |x| \leq 1, \tag{25}
\]

\[
u(x) = \frac{1}{(2\alpha)^d} \int_{[-\alpha, \alpha]^d} \psi(u(x - \omega_d)) d\omega_d, \ |x| < 1 - \alpha, \tag{26}
\]

where \( v(x + \omega_d) \) and \( u(x - \omega_d) \) are defined in the same manner as for \( v(x + \omega_d, i) \) and \( u(x - \omega_d, i) \). Furthermore, the boundary condition \( v(x) = v_{\text{opt}} \) is imposed for \( |x| \geq 1 - \alpha \). We can prove that \( u(x) \) is a continuous even function on \([-\alpha, 1]\) and twice continuously differentiable for \( 1 - 2\alpha < |x| < 1 \).

Without loss of generality, we only consider the case \( x \in [1 - \alpha, 1] \). We expand (26) with respect to \( \alpha \) to obtain \( v(x) = \varphi_0(u(x)) + O(\alpha^2) \). Similarly, we use (25) and the boundary condition \( v(x) = v_{\text{opt}} \) for \( |x| > 1 - \alpha \) to derive an approximate system

\[
u(x) = \sum_{w=0}^{d} a^w b^d - w \sum_{j=1}^{d} \varphi(V_{j1}, \ldots, V_{jd}) + O(\alpha^2)
\]

\[-\alpha a b \frac{d}{dx} \psi_0(u) \sum_{j=1}^{d} \sum_{w=0}^{d} a^w b^d - w \sum_{k=1}^{d} \frac{\partial \varphi}{\partial V_j}(V_{j1}, \ldots, V_{jd}) \tag{27}
\]

where the second summation in the first term is taken over \( \{ jk \in \{0, 1\} : \sum_{k=1}^{d} jk = w \} \), and where the summation \( \sum_{\lambda=1}^{d} \) is over \( \{ jk \in \{0, 1\} : jk = 1, \sum_{k=1}^{d} \lambda = w \} \), with

\[
a = \frac{1 - x}{2\alpha}, \quad b = \frac{x - (1 - 2\alpha)}{2\alpha}, \quad V_j = \begin{cases} \varphi_0(u(x)) & \text{for } j = 1, \\ v_{\text{opt}} & \text{for } j = 0. \end{cases} \tag{28}
\]

It is straightforward to confirm that \( u(x) = u_{\text{opt}} \) is a solution to the approximate system (27). In particular, we find \( u(1) = u_{\text{opt}} \) at the boundary.

We shall prove that \( u(x) = u_{\text{opt}} \) on \([1 - 2\alpha, 1]\) is the unique solution to the approximate system (27) as \( \alpha \to 0 \). From (27), \( u(x) = u_{\text{opt}} + O(\alpha) \) holds for \( x \in [1 - 2\alpha^2, 1] \), because of \( a = \alpha \) and \( b = 1 - \alpha \). Thus, as \( \alpha \to 0 \) we can extend the solution from \( x = 1 \) to \( x = 1 - 2\alpha^2 \) uniquely. Since the coefficient for \( du/dx \) in (27) is non-zero for \( x \in (1 - 2\alpha + 2\alpha^2, 1 - 2\alpha^2) \), we find \( u(x) = u_{\text{opt}} \) on \((1 - 2\alpha + 2\alpha^2, 1 - 2\alpha^2)\) from the uniqueness of solutions to the first-order differential equation (27) with the initial condition \( u(1 - 2\alpha^2) = u_{\text{opt}} \). Repeating the argument for the interval \((1 - 2\alpha^2, 1]\), we obtain \( u(x) = u_{\text{opt}} + O(\alpha) \) for \( x \in [1 - 2\alpha, 1 - 2\alpha + 2\alpha^2) \) from
the continuity of the solution. In summary, \( u(x) = u_{\text{opt}} \) is the unique solution on \((1 - 2\alpha, 1]\). This implies that Lemma 3 holds.

Lemma 3 is consistent with the boundary condition which we have imposed for the differential system (33). Theorem 2 follows immediately from Lemma 2, Lemma 3, and the following lemma for the bulk region.

**Lemma 4.** For any \( \epsilon > 0 \), there exist \( \alpha_0 > 0 \) and \( I > 0 \) such that

\[
\sup_{x \in (-1, 1)} |u(x, i) - \bar{u}(x, i)| < \epsilon, \tag{30}
\]

for all \( \alpha \in (0, \alpha_0) \) and \( i \geq I \).

**Proof:** Let \( I > 0 \) denote a sufficiently large number of iterations given in Lemma 3. We impose the initial condition \( \bar{u}(x, I) = u(x) \), which is a smooth function with \( \bar{u}(\pm(1 - 2\alpha)) = u_{\text{opt}} \) such that \( |\bar{u}(x) - u(x, I)| < \epsilon \) for all \( |x| \leq 1 - 2\alpha \).

The proof is by induction. Suppose that the statement is correct for some \( i > I \). From Lemma 3 and the boundary condition \( \bar{u}(x, i) = u_{\text{opt}} \) for \( |x| \geq 1 - 2\alpha \), it is sufficient to focus on the bulk region where \( |x| < 1 - 2\alpha \). For the bulk region, the integral systems (21) depends on \( u(x, i) \) for \( |x| < 1 - \alpha \) given by (22), so that the integral systems (21) and (22) result in a single recursive formula \( u(x, i + 1) = \mathcal{L}(u(x, i)) \). We use the triangle inequality to obtain

\[
\left| u(x, i + 1) - \bar{u}(x, i + 1) \right| < |\mathcal{L}(u(x, i)) - \mathcal{L}(\bar{u}(x, i))| + \left| \mathcal{L}(\bar{u}(x, i)) - \bar{u}(x, i + 1) \right| \tag{31}
\]

We first upper-bound the first term. Since \( \varphi \) and \( \psi \) are differentiable, there exist Lipschitz constants \( L_\varphi > 0 \) and \( L_\psi > 0 \) such that \( |\varphi(v) - \varphi(\bar{v})| < L_\varphi |v - \bar{v}|_2 \) for all \( v, \bar{v} \in \mathbb{D}^d \) and that \( |\psi(u) - \psi(\bar{u})| < L_\psi |u - \bar{u}|_2 \) for all \( u, \bar{u} \in \mathbb{D}^d \). From (21) and (22), we obtain

\[
\left| \mathcal{L}(u(x, i)) - \mathcal{L}(\bar{u}(x, i)) \right| < L_\varphi L_\psi \sum_{j=1}^{d} \sum_{k=1}^{d} \int_{[-\alpha, \alpha]^2} |u(x + \omega_j - \omega_k, i) - \bar{u}(x + \omega_j - \omega_k, i)| d\omega_j d\omega_k. \tag{32}
\]

Thus, the supremum of the upper bound (32) over \( x \in \mathbb{R} \) is bounded from above by \( \int \int |1 - L_\varphi L_\psi| d\omega_j d\omega_k \).

We next evaluate the second term on the upper bound (31). Since \( \bar{u}(x, i) \) is smooth, we can expand \( \mathcal{L}(\bar{u}(x, i)) \) with respect to \( \omega \) up to the second order. Expanding the integrand in (21) with respect to \( \omega \) yields

\[
\mathcal{L}(\bar{u}) = \varphi_0(v) + \frac{\alpha^2}{6} \left( \Delta \varphi(v) \left( \frac{dv}{dx} \right)^2 + \varphi_0''(v) \frac{d^2 v}{dx^2} \right) + O(\alpha^4), \tag{33}
\]

where \( v \) is given by the right-hand side (RHS) of (22) with \( u(x, i) = \bar{u}(x, i) \). Similarly, expanding \( v \) with respect to \( \alpha \) gives

\[
v = \psi_0(\bar{u}) + \frac{\alpha^2}{6} \left( \Delta \psi(\bar{u}) \left( \frac{d\bar{u}}{dx} \right)^2 + \psi_0''(\bar{u}) \frac{d^2 \bar{u}}{dx^2} \right) + O(\alpha^4), \tag{34}
\]

where \( \bar{u} \) is an abbreviation of \( \bar{u}(x, i) \). Substituting (34) into (33) and expanding the obtained formula with respect to \( \alpha \) up to the second order, we obtain \( \mathcal{L}(\bar{u}(x, i)) = \bar{u}(x, i + 1) + O(\alpha^4) \), given by (31). In summary, it has been proved that the supremum of \( |u(x, i + 1) - \bar{u}(x, i + 1)| \) over \( x \in (-1, 1) \) tends to zero as \( \alpha \to 0 \).

**Remark 1.** Since the approximate system (33) for the bulk region is a second-order differential equation, we have to treat it not as an initial value problem but as a boundary-value problem. As a result, a careful analysis for the uniqueness of solutions is required for the bulk region as proved in Theorem 3 whereas the uniqueness for the neighborhood of the boundaries follows immediately from the uniqueness of solutions to any initial problem that satisfies a mild condition for smoothness as proved in Lemma 4.

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