Probabilistic representations of solutions to the heat equation

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Abstract. In this paper we provide a new (probabilistic) proof of a classical result in partial differential equations, viz. if $\varphi$ is a tempered distribution, then the solution of the heat equation for the Laplacian, with initial condition $\varphi$, is given by the convolution of $\varphi$ with the heat kernel (Gaussian density). Our results also extend the probabilistic representation of solutions of the heat equation to initial conditions that are arbitrary tempered distributions.

Keywords. Brownian motion; heat equation; translation operators; infinite dimensional stochastic differential equations.

1. Introduction

Let $(X_t)_{t \geq 0}$ be a $d$-dimensional Brownian motion, with $X_0 \equiv 0$. Let $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions. Let $\varphi_t$ represent the unique solution to the heat equation with initial value $\varphi$, viz.

$$\partial_t \varphi_t = \frac{1}{2} \Delta \varphi_t \quad 0 \leq t \leq T; \quad \varphi_0 = \varphi.$$ 

It is well-known that $\varphi_t = \varphi * p_t$, where $p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}$ and `$*$' denotes convolution. When $\varphi$ is smooth, say $\varphi \in \mathcal{S}$, the space of rapidly decreasing smooth functions, then the probabilistic representation of the solution is given by the equality $\varphi(t, x) = E\varphi(X_t + x)$ and is obtained by taking expectations in the Ito formula

$$\varphi(X_t + x) = \varphi(x) + \int_0^t \nabla \varphi(X_s + x) \cdot dX_s + \frac{1}{2} \int_0^t \Delta \varphi(X_s + x) \, ds.$$ 

Such representations are well-known (see [1,2,3,4]) and extend to a large class of initial value problems, with the Laplacian $\Delta$ replaced by a suitable (elliptic) differential operator $L$ and $(X_t)$ being replaced by the diffusion generated by $L$. A basic problem here is to extend the representation to situations where $\varphi$ is not smooth.

The main contribution of this paper is to give a probabilistic representation of solutions to the initial value problem for the Laplacian with an arbitrary initial value $\varphi \in \mathcal{S}'$. This representation follows from the Ito formula developed in [9], for the $\mathcal{S}'$-valued process $(\tau_x \varphi)$, where $\tau_x \varphi$ is the translation of $\varphi$ by $x \in \mathbb{R}^d$. Our representation (Theorem 2.4) then reads, $\varphi_t = E\tau_{X_t} \varphi$ where of course $\varphi_t$ is the solution of the initial value problem for the Laplacian, with initial value $\varphi \in \mathcal{S}'$. In particular, the fundamental solution $p_t(x - \cdot)$
has the representation, 
\[ p_i(x - i) = E \tau_i \delta_i. \]
However, the results of [9] only show that if \( \varphi \in \mathcal{S}'_p \), then there exists \( q > p \) such that the process \( (\tau_i \varphi) \) takes values in \( \mathcal{S}'_q \). Here for each real \( p \), the \( \mathcal{S}'_p \)’s are the ‘Sobolev spaces’ associated with the spectral decomposition of the operator \( |x|^2 - \Delta \) or equivalently they are the Hilbert spaces defining the countable Hilbertian structure of \( \mathcal{S}' \) (see [4]). \( \mathcal{S}'_p \), the dual of \( \mathcal{S}_p \), is the same as \( \mathcal{S}_{-p} \). Clearly it would be desirable to have the process \( (\tau_i \varphi) \) take values in \( \mathcal{S}'_p \), whenever \( \varphi \in \mathcal{S}_p \).

Such a result also has implications for the semi-martingale structure of the process \( (\tau_i \varphi) \) – it is a semi-martingale in \( \mathcal{S}'_{p+1} \) (Corollary 2.2) and fails to have this property in \( \mathcal{S}'_q \) for \( q < p + 1 \) (see Remark 5.2 of [5]).

Given the above remarks and the results of [9], the properties of the translation operators become significant. We show in Theorem 2.1 that the operators \( \tau_i : \mathcal{S}'_p \rightarrow \mathcal{S}'_p \) for \( i \in \mathbb{R}^d \), are indeed bounded operators, for any real \( p \), with the operator norms being bounded above by a polynomial in \(|x|\). The proof uses interpolation techniques well-known to analysts. Theorem 2.4 then gives a comprehensive treatment of the initial value problem for the Laplacian from a probabilistic point of view.

2. Statements of the main results

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a filtered probability space with a filtration \((\mathcal{F}_t)\) satisfying usual conditions: \(\mathcal{F}_t = \bigcap_{s \geq t} \mathcal{F}_s \) and \(\mathcal{F}_0 \) contains all \(P\)-null sets. Let \((X_t)_{t \geq 0}\) be a \(d\)-dimensional, \((\mathcal{F}_t)\)-Brownian motion with \(X_0 \equiv 0\).

\(\mathcal{S}'\) denotes the space of rapidly decreasing smooth functions on \(\mathbb{R}^d\) (real valued) and \(\mathcal{S}'^t\) its dual, the space of tempered distributions. We refer to [11] for formal definitions. For \(x \in \mathbb{R}^d\), \(\delta_x \in \mathcal{S}'\) will denote the Dirac distribution at \(x\). Let \(\{\tau_x : x \in \mathbb{R}^d\}\) denote the translation operators defined on functions by the formula \(\tau_x f(y) = f(y - x)\) and let \(\tau_x : \mathcal{S}' \rightarrow \mathcal{S}'\) act on distributions by

\[ \langle \tau_x \varphi, f \rangle = \langle \varphi, \tau_{-x} f \rangle. \]

The nuclear space structure of \(\mathcal{S}'\) is given by the family of Hilbert spaces \(\mathcal{S}'_p, p \in \mathbb{R}\), obtained as the completion of \(\mathcal{S}\) under the Hilbertian norms \(\| \cdot \|_p\) defined by

\[ \| \varphi \|^2_p = \sum_k (2|k| + d)^{2p} \langle \varphi, h_k \rangle^2, \]

where \(\varphi \in \mathcal{S}\), and the sum is taken over \(k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d, \|k\| = k_1 + \cdots + k_d\) denotes the inner product in \(L^2(\mathbb{R}^d)\) and \(\{h_k : k \in \mathbb{Z}_+^d\}\) is the ONB in \(L^2(\mathbb{R}^d)\), constructed as follows: for \(x = (x_1, \ldots, x_d)\), \(h_k(x) = h_{k_1}(x_1) \cdots h_{k_d}(x_d)\). The one-dimensional Hermite functions are given by \(h_k(s) = \frac{1}{\sqrt{(2\pi)^{d/2}}} e^{-s^2/2} H_k(s), \) where \(H_k(s) = (-1)^k e^{s^2} \frac{d^k}{ds^k} e^{-s^2}\) are the Hermite polynomials. While we mainly deal with real valued functions, at times we need to use complex valued functions. In such cases, the spaces \(\mathcal{S}'_p\) are defined in a similar fashion as above, i.e. as the completion of \(\mathcal{S}'\) with respect to \(\| \cdot \|_p\). However, in the definition of \(\| \varphi \|^2_p\) above we need to replace the real \(L^2\) inner product \(\langle \varphi, h_k \rangle\) by the one for complex valued functions, viz. \(\langle \varphi, \psi \rangle = \int_{\mathbb{R}^d} \varphi(x) \overline{\psi}(x) dx\) and \(\langle \varphi, h_k \rangle^2\) is replaced by \(\| \varphi, h_k \|^2\). It is well-known (see [6]) that \(\mathcal{S} = \bigcap_p \mathcal{S}'_p, \mathcal{S}' = \bigcup_p \mathcal{S}'_p\) and \(\mathcal{S}'_p = \text{dual of } \mathcal{S}'_{p-1/2}\). We will denote by \(\langle \cdot, \cdot \rangle_p\) the inner product corresponding to the norm \(\| \cdot \|_p\). Let \(\{Y_t\}_{t \geq 0}\) be an \(\mathcal{S}'_p\)-valued, locally bounded, previsible process, for some \(p \in \mathbb{R}\). Let \(\partial_t : \mathcal{S}'_p \rightarrow \mathcal{S}'_{p-1/2}\) be the partial derivatives, \(1 \leq i \leq d\), in the sense of distributions. Then
Probabilistic representation of the heat equation 323

since $\partial_t, 1 \leq i \leq d$ are bounded linear operators it follows that $(\partial_i Y_t)_{t \geq 0}$ is an $\mathcal{S}_{p-1/2}$-valued, locally bounded, predictable process. From the theory of stochastic integration in Hilbert spaces $[8]$, it follows that the processes

$$\left( \int_0^t Y_s dX^i_s \right)_{t \geq 0}, \left( \int_0^t \partial_i(Y_s) dX^i_s \right)_{t \geq 0}$$

are continuous $\mathcal{F}_t$ local martingales for $1 \leq i \leq d$, with values in $\mathcal{F}_p$ and $\mathcal{F}_{p-1/2}$ respectively. If $X_t = (X^1_t, \ldots, X^d_t)$ is a continuous $\mathbb{R}^d$-valued, $\mathcal{F}_t$-semi-martingale, it follows from the general theory that the above processes too are continuous $\mathcal{F}_t$-semi-martingales with values in $\mathcal{F}_p$ and $\mathcal{F}_{p-1/2}$ respectively.

**Theorem 2.1.** Let $p \in \mathbb{R}$. There exists a polynomial $P_k(\cdot)$ of degree $k = 2(\|p\| + 1)$ such that the following holds: For $x \in \mathbb{R}^d$, $\tau_x : \mathcal{F}_p \to \mathcal{F}_p$ is a bounded linear map and we have

$$\|\tau_x \varphi\|_p \leq P_k(|x|) \|\varphi\|_p$$

for all $\varphi \in \mathcal{F}_p$.

In (9), Theorem 2.3 we showed that if $(X_t)_{t \geq 0}$ is a continuous, $d$-dimensional, $\mathcal{F}_t$-semi-martingale and $\varphi \in \mathcal{F}_p \subset \mathcal{F}'$, then the process $(\tau_X \varphi)_{t \geq 0}$ is an $\mathcal{F}_q$-valued continuous semi-martingale for some $q < p$. Corollary 2.2 below says that we can take $q = p - 1$.

**COROLLARY 2.2.**

Let $(X_t)_{t \geq 0}$ be a continuous $d$-dimensional, $\mathcal{F}_t$-semi-martingale. Let $\varphi \in \mathcal{F}_p, p \in \mathbb{R}$. Then $(\tau_X \varphi)_{t \geq 0}$ is an $\mathcal{F}_p$-valued, continuous adapted process. Moreover it is an $\mathcal{F}_{p-1}$-valued, continuous $\mathcal{F}_t$-semi-martingale and the above formula holds in $\mathcal{F}_{p-1}$: a.s., $\forall t \geq 0$,

$$\tau_X \varphi = \tau_0 \varphi - \sum_{i=1}^d \int_0^t \partial_i(\tau_X \varphi) dX^i_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial^2_{ij}(\tau_X \varphi) d\langle X^i, X^j \rangle_s,$$

(2.1)

where $X_t = (X^1_t, \ldots, X^d_t)$ and $(\langle X^i, X^j \rangle_t)_t$ is the quadratic variation process between $(X^i_t)$ and $(X^j_t), 1 \leq i, j \leq d$.

**Proof.** From Theorem 2.1, it follows that $(\tau_X \varphi)$ is an $\mathcal{F}_p$-valued continuous adapted process. By Theorem 2.3 of (9), $\exists q < p$, such that $(\tau_X \varphi)$ is an $\mathcal{F}_q$ semi-martingale and the above equation holds in $\mathcal{F}_q$. Clearly each of the terms in the above equation is in $\mathcal{F}_{p-1}$ and the result follows.

The next corollary pertains to the case when $(X_t) = (X^1_t, \ldots, X^d_t)$ is a $d$-dimensional Brownian motion, $X_0 \equiv 0$. In (9), Definition 3.1, we introduced the notion of an $\mathcal{F}'(= \mathcal{S}_p; p > 0)$-valued strong solution of the SDE

$$dY_t = \frac{1}{2} \Delta(Y_t) dr + \nabla Y_t \cdot dX_t,$$

$$Y_0 = \varphi,$$

(2.2)

where $\nabla = (\partial_1, \ldots, \partial_d)$ and $\Delta = \sum_{i=1}^d \partial^2_i$. There we showed that if $\varphi \in \mathcal{F}_p$, then the above equation has a unique $\mathcal{F}_q$-valued solution, $q \geq p + 2$. Theorem 2.1 implies that we indeed have an (unique) $\mathcal{F}_p$-valued strong solution.
COROLLARY 2.3.
Let $\varphi \in \mathcal{S}_p$. Then, eq. (2.2) has a unique $\mathcal{S}_p$-valued strong solution on $0 \leq t \leq T$.

Proof. By Corollary 2.2, the process $(\tau_t, \varphi)$, where $(X_t)$ is a $d$-dimensional Brownian motion, $X_0 \equiv 0$, satisfies eq. (2.1). Further,

$$E \int_0^T \| \tau_t, \varphi \|_{-p}^2 \, dt = \int_0^T \int_{\mathbb{R}^d} \| \tau_t, \varphi \|_{-p}^2 \, \frac{e^{-|x|^2/2}}{(2\pi)^{d/2}} \, dx \, dt < \infty.$$ 

Uniqueness follows as in Theorem 3.3 of [5].

We now consider the heat equation for the Laplacian with initial condition $\varphi \in \mathcal{S}_p$, for some $p \in \mathbb{R}$.

$$\partial_t \varphi = \frac{1}{2} \Delta \varphi, \quad 0 \leq t \leq T,$$

$$\varphi_0 = \varphi. \quad (2.3)$$

By an $\mathcal{S}_p$-valued solution of (2.3), we mean a continuous map $t \rightarrow \varphi_t : [0, T] \rightarrow S_p$ such that the following equation holds in $\mathcal{S}_{p-1}$:

$$\varphi_t = \varphi + \int_0^t \frac{1}{2} \Delta \varphi_s \, ds. \quad (2.4)$$

Let $\{h_k^{p-1}\}$ be the ONB in $\mathcal{S}_{p-1}$ given by $h_k^{p-1} = (2|k| + d)^{-(p-1)} h_k$. We then have for $p < 0$ and $t \leq T$:

$$\| \varphi_t \|_{p-1}^2 = \sum_{|k|=0}^\infty (\varphi_t, h_k^{p-1})_{p-1}^2 = \sum_{|k|=0}^\infty \left\{ (\varphi, h_k^{p-1})_{p-1}^2 + 2 \int_0^t (\varphi_s, h_k^{p-1})_{p-1} \, ds \right\}$$

$$= \| \varphi \|_{p-1}^2 + \sum_{|k|=0}^\infty 2 \int_0^t (\varphi_s, h_k^{p-1})_{p-1} \left\langle \frac{1}{2} \Delta \varphi_s, h_k^{p-1} \right\rangle_{p-1} \, ds$$

$$= \| \varphi \|_{p-1}^2 + 2 \int_0^t \left\langle \frac{1}{2} \Delta \varphi_s, \varphi_s \right\rangle_{p-1} \, ds.$$ 

It follows from the results of [5] (the monotonicity condition) that for $p < 0$,

$$2 \left\langle \frac{1}{2} \Delta \varphi, \varphi \right\rangle_{p-1} + \sum_{i=1}^d \| \partial_i \varphi \|_{p-1}^2 \leq C \| \varphi \|_{p-1}^2$$

for some constant $C > 0$ for all $\varphi \in \mathcal{S}_p$. We then get

$$\| \varphi_t \|_{p-1}^2 \leq \| \varphi \|_{p-1}^2 + C \int_0^t \| \varphi_s \|_{p-1}^2 \, ds.$$ 

Hence for the case $p < 0$, uniqueness follows from the Gronwall lemma. Uniqueness for the case $p \geq 0$, follows from uniqueness for the case $p < 0$ and the inclusion $\mathcal{S}_p \subset \mathcal{S}_q$ for
q < p. It is well-known that the solutions of the initial value problem (2.3) in \( \mathscr{S}(\mathbb{R}^d) \) are

\[ q \leq p. \]

It is well-known that the solutions of the initial value problem (2.3) in \( \mathscr{S}(\mathbb{R}^d) \) are given by convolution of \( \varphi \) and \( p_t(x) \), the heat kernel. That these coincide (as they should) with the \( \mathscr{S}_p \)-valued solutions follows from the ‘probabilistic representation’ given by Theorem 2.4 below. Define the Brownian semi-group \((T_t)_{t \geq 0}\) on \( \mathscr{S} \) in the usual manner:

\[ T_t \varphi(x) = \varphi * p_t(x) \quad t > 0, \quad T_0 \varphi = \varphi \]

where \( p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}, t > 0 \) and ‘\(*\)’ denotes convolution: \( f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y) \, dy \). In the next theorem we consider standard Brownian motion \((X_t)\).

**Theorem 2.4.** (a) Let \( \varphi \in \mathscr{S}_p \). Then for \( t \geq 0 \), the \( \mathscr{S}_p \)-valued random variable \( \tau_{X_t} \varphi \) is Bochner integrable and we have

\[ E \tau_{X_t} \varphi = \varphi * p_t = T_t \varphi. \]

In particular, for every \( p \in \mathbb{R} \) and \( T > 0 \), \( \sup_{t \leq T} \|T_t\| < \infty \) where \( \|T_t\| \) is the operator norm of \( T_t : \mathscr{S}_p \to \mathscr{S}_p \).

(b) For \( \varphi \in \mathscr{S}_p \), the initial value problem (2.3) has a unique \( \mathscr{S}_p \)-valued solution \( \varphi_t \) given by

\[ \varphi_t = E \tau_{X_t} \varphi. \]

Further \( \varphi_t \to \varphi \) strongly in \( \mathscr{S}_p \) as \( t \to 0 \).

### 3. Proofs of Theorems 2.1 and 2.4

The spaces \( \mathscr{S}_p \) can be described in terms of the spectral properties of the operator \( H \) defined as follows:

\[ H f = (|x|^2 - \Delta)f, \quad f \in \mathscr{S}. \]

If \( \{h_k\} \) is the ONB in \( L^2(\mathbb{R}^d) \) consisting of Hermite functions (defined in §2), then it is well-known (see [10]) that

\[ H h_k = (2|k| + d) h_k. \]

For \( f \in \mathscr{S} \), define the operator \( H^p \) as follows:

\[ H^p f = \sum_k (2|k| + d)^p \langle f, h_k \rangle h_k. \]

Here \( p \) is any real number. For \( f \in \mathscr{S} \) and \( z = x + iy \in \mathbb{C} \) define \( H^z f = \sum_k (2|k| + d)^z \langle f, h_k \rangle h_k \) and note that, \( H^z f = H^z (H^0 f) = H^0 (H^z f) \) and \( H^0 : L^2 \to L^2 \) is an isometry. Further,

\[ \|H^z f\|_0^2 = \sum_k (2|k| + d)^{2z} \langle f, h_k \rangle^2 \]

\[ = |f|_k^2. \]

The following propositions (3.1, 3.2 and 3.3) may be well-known. We include the proofs for completeness.
PROPOSITION 3.1.

For any \( p \) and \( q, \|H^p \varphi\|_{q-p} = \|\varphi\|_q \) for \( \varphi \in \mathcal{S} \). Consequently, \( H^p : \mathcal{S}_q \rightarrow \mathcal{S}_{q-p} \) extends as a linear isometry. Moreover, this isometry is onto.

Proof. Let \( h_k^p = (2|k| + d)^{-p}h_k \). Then from the relation \( \langle \varphi, h_k \rangle_p = (2|k| + d)^2 \langle \varphi, h_k \rangle \) it follows that \( \{h_k^p\} \) is an ONB for \( \mathcal{S}_p \). Let \( \varphi \in \mathcal{S} \). Since

\[
H^p \varphi = \sum_k \langle \varphi, h_k \rangle (2|k| + d)^p h_k
\]

we get \( \|H^p \varphi\|_{q-p}^2 = \|\varphi\|_q^2 \).

To show that \( H^p \) is onto, consider \( \psi \in \mathcal{S}_{q-p} \). Defining \( \varphi = \sum_k \langle \psi, h_k^{q-p} \rangle_{q-p} h_k^q \), we see that \( \varphi \in \mathcal{S}_q \). Also,

\[
H^p \varphi = \sum_k \langle \varphi, h_k^{q-p} \rangle_{q-p} h_k^{q-p} = \sum_k \langle \psi, h_k^{q-p} \rangle_{q-p} h_k^{q-p} = \psi. \quad \square
\]

Let \( A_j = x_j + \partial_j \) and \( A_j^+ = x_j - \partial_j \), \( 1 \leq j \leq d \). Then it is easy to see that

\[
H = \frac{1}{2} \sum_{j=1}^d (A_j A_j^+ + A_j^+ A_j).
\]

For multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \( \beta = (\beta_1, \ldots, \beta_d) \) we define

\[
A^\alpha := A_1^{\alpha_1} \ldots A_d^{\alpha_d}, \quad (A^+)^\beta := (A_1^+)^{\beta_1} \ldots (A_d^+)^{\beta_d}.
\]

For an integer \( \ell \geq 0 \) and \( x \in \mathbb{R} \), recall that

\[
h_\ell(x) = \frac{1}{\sqrt{\pi 2^\ell \ell!}} e^{-x^2} H_\ell(x),
\]

where \( H_\ell \) is the Hermite polynomial defined by

\[
H_\ell(x) = (-1)^\ell e^{x^2} \frac{d^\ell}{dx^\ell} e^{-x^2}.
\]

It is easily verified that

\[
\left( x + \frac{d}{dx} \right) \left( e^{-x^2} H_\ell(x) \right) = 2 \ell \left( e^{-x^2} H_{\ell-1}(x) \right),
\]

\[
\left( x - \frac{d}{dx} \right) \left( e^{-x^2} H_\ell(x) \right) = e^{-x^2} H_{\ell+1}(x).
\]

It then follows that

\[
A_j^+ h_k(x_j) = \sqrt{2(k_j + 1)} h_{k_j+1}(x_j),
\]

\[
A_j h_k(x_j) = \sqrt{2k_j} h_{k_j-1}(x_j).
\]

Iterating these two formulas we get the following:
Let \( k, \beta \) and \( \alpha \) be multi-indices such that \( k_j \geq \alpha_j, j = 1, \ldots, d \). Then

\[
(A^+)^\beta h_k(x) = 2^{\beta/2} \left( \frac{(k + \beta)!}{k!} \right)^{1/2} h_{k+\beta}(x),
\]

\[
A^\alpha h_k(x) = 2^{\alpha/2} \left( \frac{k!}{(k - \alpha)!} \right)^{1/2} h_{k-\alpha}(x),
\]

where \( k! = k_1! \ldots k_d! \).

**Proposition 3.3.**

For all \( m \geq 0, \exists \) constants \( C_1 = C_1(m) \) and \( C_2 = C_2(m) \) such that the following hold:

(a) For all \( f \in \mathcal{S} \),

\[
\|f\|_m \leq C_1 \sum_{|\alpha| + |\beta| \leq 2m} \|A^\alpha (A^+)^\beta f\|_0 \leq C_2 \|f\|_m.
\]

(b) For all \( f \in \mathcal{S} \),

\[
\|f\|_m \leq C_1 \sum_{|\alpha| + |\beta| \leq 2m} \|x^\alpha \partial^\beta f\|_0 \leq C_2 \|f\|_m.
\]

**Proof.** (a) We can write

\[
H^m = \sum_{|\alpha| + |\beta| \leq 2m} C_{\alpha \beta} A^\alpha (A^+)^\beta,
\]

where \( C_{\alpha \beta} \) are constants. Since \( \|f\|_m = \|H^m f\|_0 \), the first part of the inequality follows. To show the second half of the inequality it is sufficient to show that for \( f \in \mathcal{S} \) and \( |\alpha| + |\beta| \leq 2m \),

\[
\|A^\alpha (A^+)^\beta H^{-m} f\|_0 \leq C_{\alpha \beta} \|f\|_0.
\]

Now,

\[
\|A^\alpha (A^+)^\beta H^{-m} f\|_0^2 = \sum_{\ell} \langle A^\alpha (A^+)^\beta H^{-m} f, h_{\ell} \rangle^2
\]

\[
= \sum_{\ell} \left[ \sum_k \langle 2|k| + d \rangle^{-m} \langle f, h_k \rangle \langle A^\alpha (A^+)^\beta h_k, h_{\ell} \rangle \right]^2
\]

\[
= \sum_{\ell} \left[ \sum_k \langle 2|k| + d \rangle^{-m} \langle f, h_k \rangle C_{k, \beta, \alpha} \langle h_{k+\beta-\alpha}, h_{\ell} \rangle \right]^2
\]

\[
= \sum_{\ell} (2|\ell + \alpha - \beta| + d)^{-2m} c_{\ell + \alpha - \beta, \beta, \alpha}^2
\]

where the sum is taken over \( \ell = (\ell_1, \ldots, \ell_d) \) such that \( \ell_j + \alpha_j - \beta_j \geq 0 \) for \( 1 \leq j \leq d \) and where we have used Proposition 3.2 in the last but one equality above. From the same proposition, it follows that

\[
(2|\alpha + \ell - \beta| + d)^{-2m} c_{\ell + \alpha - \beta, \beta, \alpha}^2
\]

are uniformly bounded in \( \ell \) for \( |\alpha| + |\beta| \leq 2m \) and the second inequality in (a) follows.
(b) Since $\|f\|_m = \|H^m f\|_0$ and clearly $H^m = \sum_{|\alpha| + |\beta| \leq 2m} C_{\alpha \beta} x^\alpha \partial^\beta$, the first inequality follows. To prove the second inequality, note that

$$x_j = \frac{1}{2} (A_j + A_j^+), \quad \partial_j = \frac{1}{2} (A_j - A_j^+).$$

Hence, using $[A_j, A_k^+] = \delta_{jk} I$,

$$x^\alpha \partial^\beta = \sum_{|k| + |\ell| \leq |\alpha| + |\beta|} C_{k,\ell} A^k (A^+)\ell$$

and hence by part (a) we get

$$\sum_{|\alpha| + |\beta| \leq 2m} \|x^\alpha \partial^\beta f\|_0 \leq C_1 \sum_{|k| + |\ell| \leq 2m} \|A^k (A^+)\ell f\|_0 \leq C_2 \|H^m f\|_0.$$

Proof of Theorem 2.1. We first show that for an integer $m \geq 0$,

$$\|\tau_s \varphi\|_m \leq P_{2m}(|x|) \|\varphi\|_m,$$

where $P_{2m}(t)$ is a polynomial in $t \in \mathbb{R}$ of degree $2m$ with non-negative coefficients. This follows from Proposition 3.3:

$$\|\tau_s f\|_m \leq C_1 \sum_{|\alpha| + |\beta| \leq 2m} \|x^\alpha \partial^\beta \tau_s f\|_0 \leq C_1 \sum_{|\alpha| + |\beta| \leq 2m} \|(y + x)^\alpha \partial^\beta f\|_0.$$

The last sum is clearly dominated by $P_{2m}(|x|) \|f\|_m$ for some polynomial $P_{2m}$. If $m < p < m + 1$, where $m \geq 0$ is an integer, we prove the result using the 3-line lemma: for $f, g \in \mathcal{S}$, let

$$F(z) = \langle H^z \tau_s H^{-z} f, g \rangle_0.$$

Then from the expansion in $L^2$ for the RHS it is verified that $F(z)$ is analytic in $m < \text{Re } z < m + 1$ and continuous in $m \leq \text{Re } z \leq m + 1$. We will show that

$$|F(m + iy)| \leq P_{2m}(|x|) \|f\|_0 \|g\|_0,$$

$$|F(m + 1 + iy)| \leq P_{2(m+1)}(|x|) \|f\|_0 \|g\|_0$$

(3.1)

for $-\infty < y < \infty$. Hence from the 3-line lemma \[12\], it follows that

$$|F(p + iy)| \leq (P_{2m}(|x|) \|f\|_0 \|g\|_0)^{m+1-p} (P_{2(m+1)}(|x|) \|f\|_0 \|g\|_0)^{p-m} \leq P_k(|x|) \|f\|_0 \|g\|_0,$$

where $P_k(t)$ is a polynomial in $t$ of degree $k = 2(|p| + 1)$. It follows that

$$\|\tau_s f\|_p \leq P_k(|x|) \|f\|_p.$$
Using the fact that \( S_p^0 = S_p^0 \) we get \( \| \tau_x f \|_{-p} \leq P_k(|x|) \| f \|_{-p} \) for \( m \leq p \leq m + 1 \).

The following chain of inequalities establish the inequalities (3.1):

\[
|F(m + iy)| \leq \| H^{m-iy} \tau_x H^{-(m+iy)} f \|_0 \| g \|_0 \\
\leq \| H^m \tau_x H^{-(m+iy)} f \|_0 \| g \|_0 \\
= \| \tau_x H^{-(m+iy)} f \|_m \| g \|_0 \\
\leq P_{2m}(|x|) \| H^{-(m+iy)} f \|_m \| g \|_0 \\
= P_{2m}(|x|) \| H^{-iy} f \|_0 \| g \|_0 \\
= P_{2m}(|x|) \| f \|_0 \| g \|_0.
\]

This completes the proof of Theorem 2.1.

\[ \square \]

**Proof of Theorem 2.4.** (a) Let \( \varphi \in S_p, p \in \mathbb{R} \). From Theorem 2.1 we have

\[
\| \tau_x \varphi \|_p \leq P_k(|X_t|) \| \varphi \|_p,
\]

where \( P_k \) is a polynomial. Since \( EP_k(|X_t|) < \infty \), Bochner integrability follows. For \( \psi \in S, \varphi \in S \),

\[
\left\langle \psi, \int \tau_x \varphi \ p_t(x)dx \right\rangle = \int (\psi, \tau_x \varphi) p_t(x)dx \\
= \int p_t(x)dx \int \psi(y) \varphi(y-x)dy \\
= \int \psi(y)dy \int \varphi(y-x) p_t(x)dx \\
= \int \psi(y) \varphi * p_t(y)dy \\
= \langle \psi, \varphi * p_t \rangle.
\]

The result for \( \varphi \in S_p \) follows by a continuity argument: Let \( \varphi_n \in S, \varphi_n \to \varphi \) in \( S \). Hence \( \varphi_n * p_t \to \varphi * p_t \) weakly in \( S' \). Hence,

\[
\langle \psi, \varphi * p_t \rangle = \lim_{n \to \infty} \langle \psi, \varphi_n * p_t \rangle \\
= \lim_{n \to \infty} \int \psi(y) \varphi_n * p_t(y)dy \\
= \lim_{n \to \infty} \int (\psi, \tau_x \varphi_n) p_t(x)dx \\
= \int (\psi, \tau_x \varphi) p_t(x)dx \\
= \left\langle \psi, \int \tau_x \varphi \ p_t(x)dx \right\rangle,
\]

where we have used DCT in the last but one equality. That \( T_t : S_p \to S_p \) is a (uniformly) bounded operator follows:
Let $p \in \mathcal{F}_p$. Then $\|S_t \varphi\|_p \to 0$ as $t \to 0$. 

PROPOSITION 3.4.

Let $\varphi \in \mathcal{F}_p, p \in \mathbb{R}$. Then $\|S_t \varphi\|_p \to 0$ as $t \to 0$. 

Let $\varphi \in \mathcal{F}_p$. Taking expected values in (3.2) we get eq. (2.4). Hence $\varphi_t$ is the solution to the heat equation with initial value $\varphi \in \mathcal{F}_p$. The uniqueness of the solution is well-known and also follows from the remarks preceding the statement of Theorem 2.4.

To complete the proof of the theorem, we need to show that $\varphi_t \to \varphi$ in $\mathcal{F}_p$ as $t \downarrow 0$. Let $\mathcal{F}$ denote the Fourier transform, i.e. $\mathcal{F}f(\xi) = \int \overline{f(x)}e^{-i\langle x, \xi \rangle}dx$ for $f \in \mathcal{F}$. Then $\mathcal{F}$ extends to $\mathcal{F}'$ by duality, where we consider $\mathcal{F}'$ as a complex vector space. Since $\mathcal{F}(h_t) = (-\sqrt{-1})^n h_{it}$ (\cite{1}, p. 5, Lemma 1.1.3), $\mathcal{F}$ acts as a bounded operator from $\mathcal{F}_p$ to $\mathcal{F}_p$, for all $p$. Let $\varphi \in \mathcal{F}_p$. 

\[
\phi_t - \varphi = T_t \varphi - \varphi = \mathcal{F}^{-1}(S_t(\mathcal{F} \varphi)),
\]

where 

\[
S_t \varphi(x) = \mathcal{F}(T_{-t} - I) \mathcal{F}^{-1} \varphi(x) = (e^{-t/2}|x|^2 - 1)\varphi(x).
\]

Clearly, $S_t : \mathcal{F}_p \to \mathcal{F}_p$ is a bounded operator and 

\[
\|\phi_t - \varphi\|_p = \|S_t(\mathcal{F} \varphi)\|_p.
\]

The following proposition completes the proof of the theorem.

PROPOSITION 3.4.

Let $\varphi \in \mathcal{F}_p, p \in \mathbb{R}$. Then $\|S_t \varphi\|_p \to 0$ as $t \to 0$. 

\[
\|T_t \varphi\|_p = \|\varphi \ast p_t\|_p = \|E \tau_t \varphi\|_p,
\]

\[
= \left\| \int \tau_t \varphi \ p_t(x)dx \right\|_p \leq \int \|\tau_t \varphi\|_p p_t(x)dx
\]

\[
\leq \|\varphi\|_p \int P_k(|x|)p_t(x)dx \leq C \|\varphi\|_p,
\]

where $C = \sup_{x \in \mathbb{T}} \int P_k(|x|) p_t(x)dx < \infty$.

(b) Let $(X_t)$ be the standard Brownian motion so that $(X^i, X^j) \equiv 0$ for $i \neq j$. Equation (2.1) then reads, for $\varphi \in \mathcal{F}_p, p \in \mathbb{R}$,

\[
\tau_t \varphi = \varphi - \int_0^t \nabla(\tau_s \varphi) \cdot dX_s + \frac{1}{2} \int_0^t \Delta(\tau_s \varphi)ds.
\]

The stochastic integral is a martingale in $\mathcal{F}_{p-1}$:

\[
E \left\| \int_0^t \partial_i(\tau_s \varphi) dX^i_s \right\|_{p-1}^2 \leq C_1 E \int_0^t \|\partial_i(\tau_s \varphi)\|_{p-1}^2 ds
\]

\[
= C_1 \int_0^t \left( \int \|\partial_i(\tau_s \varphi)\|_{p-1}^2 p_t(x)dx \right) ds
\]

\[
\leq C_2 \int_0^t \left( \int \|\tau_s \varphi\|_p^2 p_t(x)dx \right) ds
\]

\[
\leq C_3 \|\varphi\|_p \int_0^t \left( \int P_k(|x|) p_t(x)dx \right) ds
\]

\[
< \infty.
\]

Let $\phi_t = E \tau_t \varphi$. Taking expected values in (3.2) we get eq. (2.4). Hence $\phi_t$ is the solution to the heat equation with initial value $\varphi \in \mathcal{F}_p$. The uniqueness of the solution is well-known and also follows from the remarks preceding the statement of Theorem 2.4.
Proof. We prove the proposition by showing that (i) \( S_t : \mathcal{S} \to \mathcal{S} \) are uniformly bounded, \( 0 < t \leq T \) and (ii) \( \| S_t \phi \|_p \to 0 \) for every \( \phi \in \mathcal{S} \), as \( t \to 0 \). Let us assume these results for a moment and complete the proof.

Let \( \varepsilon > 0 \) be given. By (i), there is a constant \( C > 0 \) such that

\[
\sup_{0 \leq t \leq T} \| S_t f \|_p \leq C \| f \|_p, f \in \mathcal{S}.
\]

Choose \( \phi \in \mathcal{S} \), so that \( \| f - \phi \|_p \leq \left( \frac{\varepsilon}{2} \right) \). Then,

\[
\| S_t f \|_p \leq \| S_t (f - \phi) \|_p + \| S_t \phi \|_p \leq \varepsilon/2 + \| S_t \phi \|_p.
\]

Now choose \( \delta > 0 \) such that \( \| S_t \phi \|_p \leq \varepsilon/2 \) for all \( 0 \leq t < \delta \), to get \( \| S_t f \|_p < \varepsilon \) for all \( 0 \leq t < \delta \).

Since \( S_t = \mathcal{F}(T_t - I) \mathcal{F}^{-1} \), (i) follows from the fact that \( T_t : \mathcal{S} \to \mathcal{S} \) are uniformly bounded (Theorem 2.4a) and \( \mathcal{F} : \mathcal{S} \to \mathcal{S} \) is a unitary operator. The proof of (ii) is by a direct calculation when \( p = m \) is a non-negative integer.

\[
\| S_t \phi \|_m = \| H^m S_t \phi \|_0 \leq C \sum_{|\alpha| + |\beta| \leq 2m} \| x^\alpha \partial^\beta S_t \phi \|_0.
\]

Since \( S_t \phi(x) = (e^{-(t/2)|x|^2} - 1) \phi(x) \), by Leibniz rule

\[
\| x^\alpha \partial^\beta S_t \phi \|_0 \leq \sum_{|\mu| + |\gamma| = |\beta|} C_{\mu \gamma} \| x^\alpha \partial^\mu (e^{-(t/2)|x|^2} - 1) \partial^\gamma \phi \|_0.
\]

When \( \mu \neq 0 \), we have

\[
\| x^\alpha \partial^\mu (e^{-(t/2)|x|^2} - 1) \partial^\gamma \phi \|_0 \leq C^2 t |\mu| \| \phi \|_m
\]

and when \( \mu = 0 \), using the elementary inequality \( |1 - e^{-u}| \leq C_3 u, u > 0 \) we get

\[
\| x^\alpha (e^{-(t/2)|x|^2} - 1) \partial^\gamma \phi \|_0 \leq C_4 t \| \phi \|_{m+1}.
\]

Therefore, \( \| S_t \phi \|_m \leq C r \| \phi \|_{m+1} \) for some constant \( C \), which shows that \( \| S_t \phi \|_m \to 0 \) as \( t \to 0 \). If \( p \) is real and \( m \) is a non-negative integer such that \( p \leq m \), we have

\[
\| S_t \phi \|_p \leq \| S_t \phi \|_m \leq C r \| \phi \|_{m+1}
\]

and so \( \| S_t \phi \|_p \to 0 \) as \( t \to 0 \) in this case as well. \( \square \)

References

[1] Bass Richard F, Diffusions and elliptic operators (Springer) (1998)
[2] Freidlin M, Functional integration and partial differential equations, Ann. Math. Stud. (NJ: Princeton University Press, Princeton) (1985) no. 109
[3] Friedman A, Stochastic differential equations and applications (New York: Academic Press) (1975) vol. 1
[4] Friedman A, Stochastic differential equations and applications (New York: Academic Press) (1976) vol. 2
[5] Gawarecki L, Mandrekar V and Rajeev B, From finite to infinite dimensional stochastic differential equations (preprint)
[6] Ito K, Foundations of stochastic differential equations in infinite dimensional spaces, Proceedings of CBMS-NSF National Conference Series in Applied Mathematics, SIAM (1984)
[7] Kallianpur G and Xiong J, Stochastic differential equations in infinite dimensional spaces, Lecture Notes, Monogr. Series (Institute of Mathematical Statistics) (1995) vol. 26
[8] Métivier M, Semi-martingales – A course on stochastic processes (Walter de Gruyter) (1982)
[9] Rajeev B, From Tanaka’s formula to Ito’s formula: Distributions, tensor products and local times, Séminaire de Probabilite’s XXXV, Lecture Notes in Math. 1755 (Springer-Verlag) (2001)
[10] Thangavelu S, Lectures on Hermite and Laguerre expansions, Math. Notes 42 (N.J: Princeton University Press, Princeton) (1993)
[11] Treves F, Topological vector spaces, distributions and kernels (New York: Academic Press) (1967)
[12] Stein E M and Weiss G, Introduction to Fourier analysis on Euclidean spaces (N.J.: Princeton University Press, Princeton) (1971)