Three spheres inequalities and unique continuation for a three-dimensional Lamé system of elasticity with $C^1$ coefficients

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ABSTRACT

In this paper, a quantitative estimate of unique continuation is proved for a three-dimensional Lamé system with $C^1$ coefficients in the form of three spheres inequalities. The property of the non faster than exponential vanishing of nonzero local solutions is also given as an application of the three spheres inequality.

1. Introduction

We study the three spheres inequality for Lamé systems of elasticity with $C^1$ coefficients. It is a quantitative estimate of weak unique continuation. Firstly, let us introduce the Lamé system. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$, the Lamé moduli $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are $C^1(\overline{\Omega})$ and satisfy the strong ellipticity conditions

$$\mu \geq \alpha_0 > 0, \quad 2\mu + \lambda \geq \beta_0 > 0,$$

for given positive constants $\alpha_0$ and $\beta_0$. Generally, it can be assumed that $\Omega$ contains the origin and $B_R \subset \subset \Omega$ for some $R > 0$ where $B_R$ is an open ball centered at the origin with radius $R$. The Lamé system is given by

$$\text{div}(\mu(\nabla u + (\nabla u)^\top)) + \nabla(\lambda \text{div } u) = 0,$$

(1.1)

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector and

$$\nabla u = \begin{pmatrix}
\partial_{x_1} u_1 & \partial_{x_2} u_1 & \partial_{x_3} u_1 \\
\partial_{x_1} u_2 & \partial_{x_2} u_2 & \partial_{x_3} u_2 \\
\partial_{x_1} u_3 & \partial_{x_2} u_3 & \partial_{x_3} u_3
\end{pmatrix}$$

is the gradient matrix of $u$.

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The history of three spheres inequalities is closely related to the strong continuation uniqueness principle (SUCP). The three spheres type inequality for scalar elliptic equations is a rather classical result, dated back to Landis [14], who generalized the famous Hadamard’s three circles theorem to solutions of elliptic equations. The three spheres inequality with integral norms for scalar elliptic equations was originally introduced by Garofalo and Lin [10], [11] and later developed by Kukavica [13]. Their proof is based on the monotonicity property of the frequency function [10], [13]. Recently, Alessandrini and Morassi [1] have obtained the three spheres inequality for the isotropic elasticity system. Their method can be stated as follows. Set a \((n + 1)\)-vector valued function

\[ U = \left( \begin{array}{c} u \\ \text{div } u \end{array} \right) \]

where \( u \) satisfies (1.1). The system (1.1) with \( C^{1,1} \) coefficients can be reduced to a weakly coupled elliptic system with Laplacian principal part [1], [3], [8]:

\[ -\Delta U + B(\nabla U) + VU = 0, \quad \text{in } D, \]

where \( D \) is a bounded domain in \( \mathbb{R}^n \). The coefficient tensors \( B \) and \( V \) uniquely depending on \( \mu \) and \( \lambda \) are bounded measurable. Then, using Rellich’s identity [17], one can prove that its corresponding frequency function is monotonous.

Unfortunately, the method mentioned above cannot be used when the Lamé coefficients are \( C^1 \), because one cannot apply divergence to (1.1) to diagonalize the system in this case. Unlike the approach used in [1], C.-L. Lin et al. derived the three spheres inequality for a 2-dimensional elliptic system with \( W^{1,\infty} \) coefficients by another type of reduction of the Lamé system and Carleman estimates [15]. It is carried out by using an auxiliary function \( \partial_{x_1} u + T\partial_{x_2} u \) with an appropriate matrix \( T \). The key point is that their new system contains only first order derivative of the Lamé coefficients. However such a reduction may not be applied to higher dimensions.

Eller proposed another way to reduce the Lamé system for the case of three dimensions [7]. Set

\[ A(\partial)(u_1, u_2) = (\nabla \times u_1 + \nabla u_2, -\nabla \cdot u_1), \]

\[ A_\alpha(x, \partial)(u_1, u_2) = (\nabla \times u_1 + \alpha \nabla u_2, -\nabla \cdot u_1), \]

where \( u_1 \) is a vector-valued function with three components, \( u_2 \) and \( \alpha \) are scalar-valued functions. Choosing \( \alpha = (2\mu + \lambda)/\mu \) and \( u_2 = 0 \), we then have

\[ (\mu \Delta u_1 + (\lambda + \mu)\nabla \text{div } u_1, 0) = -\mu A_\alpha(x, \partial)A(\partial)(u_1, 0). \]

By transforming the Lamé system into the composition of two first order elliptic operators, the Carleman estimate of the Lamé operator with \( C^1 \) coefficients can be given by the Carleman estimates of \( A(\partial) \) and \( A_\alpha(x, \partial) \) (Eller [7]). Then the three spheres inequality for three dimensional Lamé system with \( C^1 \) coefficients can be proved accordingly.

We state one of the main results of the paper as follows.

**Theorem 1.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) and the Lamé moduli \( \mu, \lambda \in C^1(\Omega) \) satisfy the strong elliptic conditions. For any \( R_1, R_2, 0 < R_1 < R_2 < R \),

\[ \int_{B_{R_2}} |u|^2 \, dx \leq C \int_{B_{R_1}} |u|^2 \, dx \sigma \left( \int_{B_R} |u|^2 \, dx \right)^{1-\sigma} \quad (1.2) \]
holds for \( u \in H^2(\Omega) \) being a solution of (1.1), where \( C \) and \( \sigma \in (0,1) \) are two constants depending on \( \frac{B_0}{\mathbb{R}}, \frac{B_0}{\mathbb{R}} \), \( \|\lambda\|_{C^1(\Omega)} \) and \( \|\mu\|_{C^1(\Omega)} \).

Quantitative estimates like (1.2) have been shown to be extremely useful in the treatment of the unique continuation principle and the inverse boundary value problems [2], [12], [13], [15]. Another result of our paper is related to the strong unique continuation. Before stating it we recall a relevant definition.

**Definition 1.1.** A function \( u \in L^2_{\text{loc}}(\Omega) \) is said to vanish of infinite order at \( x_0 \in \Omega \) if for every \( K \in \mathbb{N} \),

\[
\int_{|x-x_0|<r} |u|^2 \, dx = O(r^K), \quad \text{as } r \to 0^+.
\]  

**Definition 1.2.** Let \( u \in H^2(\Omega) \) be a solution of (1.1). The Lamé system (1.1) is said to have the strong unique continuation property if \( u \) satisfies the property that if there exists a point \( x_0 \in \Omega \) such that \( u \) vanishes of infinite order at \( x_0 \), then \( u \equiv 0 \) in \( \Omega \).

The result of the weak unique continuation for the Lamé system was first given by Dehman and Robbiano for \( \lambda, \mu \in C^\infty(\mathbb{R}^n) \) [9]. They proved the Carleman estimate by pseudodifferential calculus. Then Ang, Ikehata, Trong and Yamamoto gave a result for \( \lambda \in C^2(\mathbb{R}^n), \mu \in C^3(\mathbb{R}^n) \) [3]; Weck proved a result for \( \lambda, \mu \in C^2(\mathbb{R}^n) \) [13], [19]. On the other hand, the result on the strong unique continuation (SUCP) for the Lamé system was first obtained by Alessandrini and Morassi for \( n \geq 2, \lambda, \mu \in C^{1,1}(\mathbb{R}^n) \) [1]. Then Lin and Wang studied the SUCP in the case of \( n = 2, \lambda, \mu \in W^{1,\infty}(\mathbb{R}^n) \) [16]. Their proof relies on reducing the Lamé system to a first order elliptic system and on some suitable Carleman estimates with polynomial weights. Recently, Escauriaza [9] has proved the SUCP in the case of \( n = 2, \lambda \) being measurable and \( \mu \) being Lipschitz by a similar method as the one proposed by Lin and Wang.

In this paper, the UCP for the Lamé system of elasticity will be proved for \( n = 3 \) and \( \lambda, \mu \in C^1(\Omega) \). The following theorem is stronger than the weak unique continuation property but a little weaker than the strong unique continuation property.

**Theorem 1.2.** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \), the Lamé moduli \( \mu, \lambda \) satisfy the strong elliptic conditions and \( u \in H^2(\Omega) \) be a solution to (1.1).

(i) Let \( \lambda, \mu \in C^1(\Omega) \). If there is a point \( x_0 \in \Omega \) and \( \varepsilon > 0 \) such that,

\[
\int_{|x-x_0|<r} |u|^2 \, dx = O(e^{-r^{-\varepsilon}}), \quad \text{as } r \to 0^+.
\]  

then \( u \equiv 0 \), in \( \Omega \).

(ii) Let \( \lambda, \mu \in C^2(\Omega) \). If there is a point \( x_0 \in \Omega \) and \( \varepsilon > 0 \) such that,

\[
\int_{|x-x_0|<r} |\nabla u|^2 \, dx = O(e^{-r^{-\varepsilon}}), \quad \text{as } r \to 0^+.
\]  

then \( u \equiv \text{const.} \), in \( \Omega \).

The plan of this paper is as follows. In Section 2, we will show the conditionally stability estimate in the Cauchy problem for (1.1). In Section 3, the three spheres inequality will be proved based on the results in Section 2. The unique continuation will be given in Section 4 as an application of the three spheres inequality. Throughout the paper, \( C \) stands for a generic constant and its value may vary from line to line.
2. Conditional stability

The Carleman estimate is a powerful technique not only for the unique continuation, but also for solving the exact controllability, stability and the inverse problems. Carleman estimates are available for scalar elliptic operators with $C^1$-coefficients whereas many of the results for elliptic systems require coefficients with higher regularity.

We consider the equilibrium system

$$\tilde{L}u(x) = \text{div}(\mu(x)(\nabla u(x) + (\nabla u(x))^\top)) + \nabla(\lambda(x)\text{div } u(x)) = 0.$$  \hfill (2.1)

Define the Lamé operator as follows

$$L = \mu(x)\Delta + (\lambda(x) + \mu(x))\nabla\text{div}.$$  \hfill (2.2)

Then

$$\tilde{L}u = Lu + (\nabla u + (\nabla u)^\top)\nabla\mu + (\nabla u)\nabla\lambda.$$  \hfill (2.3)

The Carleman estimate of operator $L$ was first given by Dehman and Robbiano \[6\] when $\lambda$ and $\mu$ are infinite differentiable. In this section, we introduce a Carleman estimate given by Eller \[7\] at first, which plays an essential role in proving the conditional stability of the Cauchy problem for system (1.1). In his recent work, Eller proved a Carleman estimate for a certain first order elliptic system which can be used to prove the Carleman estimate for the isotropic Lamé system with $C^1$-coefficients.

**Theorem 2.1** (Eller). Let $\psi \in C^2(\overline{\Omega})$ have non-vanishing gradient and set $\varphi = e^{s\psi} - 1$ for some $s > 0$. Furthermore, assume that $\mu, \lambda \in C^1(\Omega)$ satisfy the strong elliptic conditions. Then there exist positive constants $s_0$ and $C$ such that for $s > s_0$, $\tau > \tau_0(s)$

$$\tau^2 s^4 \int_\Omega e^{2s\psi} e^{2\tau\varphi} |u|^2 dx + s^2 \int_\Omega e^{2\tau\varphi} |\nabla u|^2 dx + \frac{s}{\tau^2} \int_\Omega e^{-2s\psi} e^{2\tau\varphi} |\nabla^2 u|^2 dx \leq C \int_\Omega e^{2\tau\varphi} |Lu|^2 dx$$  \hfill (2.4)

for $u \in H^2(\Omega)$ with compact support in $\Omega$.

The proof of the above theorem can be seen in \[7\]. Compared to Dehman and Robbiano’s method, Eller’s proof is quite simple since no pseudo-differential calculus is used.

**Remark 2.1.** From the proof of the above theorem, we know that the constant $C$ depends on $|\mu|$, $|\lambda + \mu|$, $\|\mu\|_{C^1(\Omega)}$, $\|\lambda\|_{C^1(\Omega)}$ and the weight function $\psi$’s $C^2$ norm in $\Omega$, but not on $s$ and $\tau$.

Now we can get a theorem of the conditional stability.

**Theorem 2.2.** Assume that $\mu, \lambda \in C^1(\Omega)$ satisfy the strong elliptic conditions and $B_\theta \subset B_R$ for some $\theta \in (0, R)$. Let $\gamma = \partial B_\theta$ and $G = B_R \setminus \overline{B_\theta}$. Suppose that $u \in H^2(G)$ solves the Cauchy problem

$$\begin{cases}
\tilde{L}u = 0, & \text{in } G, \\
\partial^\alpha u|_\gamma = f_\alpha, & |\alpha| \leq 1
\end{cases}$$  \hfill (2.5)
with \( f_\alpha \in H^{3-|\alpha|}(\gamma) \). Then there exist a sub-domain \( \omega \subset G \) with \( \gamma \subset \partial \omega \) and constants \( C > 0, \ 0 < \epsilon < 1 \) such that
\[
\|u\|_{L^2(\omega)} \leq CM_0^{1-\epsilon} \zeta_0
\]  
(2.6)
where \( M_0 := \|u\|_{H^1(G)} \), \( \zeta_0 := \sum_{|\alpha| \leq 1} \|f_\alpha\|_{H^{3-|\alpha|}(\gamma)} \) and the constant \( C \) only depends on \( R, \theta, s, \gamma, G, \|\psi\|_{C^2(\overline{G})}, \|\mu\|_{C^1(B_R)} \) and \( \|\lambda\|_{C^1(B_R)} \).

**Proof.** By inverse trace theorem, there exists a \( u^* \in H^2(G) \), such that,
\[
\partial^\alpha u^*|_\gamma = f_\alpha, \quad |\alpha| \leq 1
\]
and
\[
\|u^*\|_{H^2(G)} \leq C \zeta_0
\]  
(2.7)
for some constant \( C \) depending on \( G \) and \( \gamma \). Hence
\[
\|\tilde{L}u^*\|_{L^2(G)} \leq C \zeta_0.
\]  
(2.8)

We set \( v = u - u^* \). By (2.5), \( v \) satisfies
\[
\partial^\alpha v|_\gamma = 0, \quad |\alpha| \leq 1
\]  
(2.9)
and
\[
\tilde{L}v = -\tilde{L}u^*, \text{ in } G.
\]

Set
\[
\psi(x) = R^2 - |x|^2, \text{ in } G,
\]
and
\[
\varphi = e^{s\psi} - 1.
\]

Obviously, \( \psi \in C^2(\overline{G}) \) and
\[
\nabla \psi = -\begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} \neq 0, \text{ in } G.
\]  
(2.10)

We have
\[
\min_{x \in \overline{G}} \varphi(x) = 0,
\]
\[
\varphi^* := \max_{x \in \overline{G}} \varphi(x) = e^{s(R^2-\theta^2)} - 1 > 0.
\]  
(2.11)

Then by Theorem 2.1 and the fact of \( C^\infty_c(G) \) being dense in \( H^2_0(G) \), we know that there exist two positive constants \( s_0 \) and \( C \) such that for \( s > s_0, \tau > \tau_0(s) \)
\[
\tau^2 s^4 \int_G e^{2s\psi} e^{2\tau\varphi} |w|^2 dx + s^2 \int_G e^{2\tau\varphi} |\nabla w|^2 dx
\]
\[
+ \frac{1}{\tau^2} \int_G e^{-2s\psi} e^{2\tau\varphi} |\nabla^2 w|^2 dx \leq C \int_G e^{2\tau\varphi} |Lw|^2 dx
\]  
(2.12)
for any \( w \in H^2_0(G) \).
We define the family \( \{ \omega(\delta) \}_{0 < \delta < \varphi^*} \) of subsets of \( G \) by
\[
\omega(\delta) = \{ x \in G : \varphi(x) > \delta \}. \tag{2.13}
\]
Then the family satisfies
\[
\emptyset = \omega(\varphi^*) \subset \omega(\delta') \subset \omega(\delta) \subset \omega(0) = G
\]
for \( 0 < \delta < \delta' < \varphi^* \). Moreover, it is easy to see that \( \omega(\delta) \) is a sub-domain of \( G \) and we have
\[
\partial \omega(\delta) \supset \gamma,
\]
for each \( 0 < \delta < \varphi^* \). Let
\[
\eta > 2.
\]
Note
\[
0 < \mu = \frac{1 - \frac{1}{\eta-1})\varphi^*}{2} < \frac{\varphi^*}{2},
\]
and
\[
\omega(\mu) \subset \omega\left(\frac{\mu}{2}\right).
\]

Let \( \chi \in C^\infty_c(B_R) \) be a cut off function satisfying
\[
0 \leq \chi \leq 1, \quad \text{in } B_R,
\]
\[
\chi = 1 \quad \text{in } \omega(\mu) \cup \overline{B_\theta},
\]
\[
\chi = 0 \quad \text{in } G \setminus \omega\left(\frac{\mu}{2}\right),
\]
\[
|\partial^\alpha \chi| < C_1, \quad \text{in } B_R, \quad |\alpha| \leq 1,
\]
where \( C_1 \) is a constant depending on \( R \) and the radius of the support of \( \chi \). Then by (2.9) and (2.14), we have \( \chi v \in H^2_0(G) \). Putting \( \chi v \) into (2.12) yields
\[
\tau^2 s^4 \int_G e^{2s\psi} e^{2r\varphi} |\nabla \chi v|^2 dx + s^2 \int_G e^{2r\varphi} |\nabla^2 \chi v|^2 dx
\]
\[
+ \frac{1}{\tau^2} \int_G e^{-2s\psi} e^{2r\varphi} |\nabla^2 \chi v|^2 dx \leq C \int_G e^{2r\varphi} |\nabla (\chi v)|^2 dx
\]
where \( C \) depends on \( \|\psi\|_{C^2(G)}, \|\lambda\|_{C^1(G)} \) and \( \|\mu\|_{C^1(G)} \). Ignoring the second order term of the left, one has
\[
\tau^2 s^4 \int_G e^{2s\psi} e^{2r\varphi} |\chi v|^2 dx + s^2 \int_G e^{2r\varphi} |\nabla \chi v|^2 dx \leq C \int_G e^{2r\varphi} |\nabla (\chi v)|^2 dx. \tag{2.15}
\]
Note that by (2.3),
\[
L(\chi v) = \chi Lv + [L, \chi]v
\]
\[ = \chi \left( \tilde{L}v - (\nabla v + (\nabla v)^T) \nabla \mu - \text{div} \, v \nabla \lambda \right) + [L, \chi]v \]
\[ = \chi \left( - \tilde{L}u^* - (\nabla v + (\nabla v)^T) \nabla \mu - \text{div} \, v \nabla \lambda \right) + [L, \chi]v, \tag{2.16} \]

where the communicator \([L, \chi]v = L(\chi v) - \chi Lv\) is a system of first order operators whose coefficients vanish on \((\omega(\mu) \cup \overline{B}_\delta) \cup (G \setminus \omega(\frac{\mu}{2}))\). And the coefficients of \([L, \chi]\) are bounded by a constant depending only on \(\lambda, \mu\) and \(C_1\).

Then by (2.15), (2.16) and the triangle inequality, we have

\[ \tau^2 s^4 \int_G e^{2s\psi} e^{2\tau \varphi} |\chi v|^2 dx + s^2 \int_G e^{2s\varphi} |\nabla \nabla v|^2 dx - C_2 s^2 \int_{\omega(\frac{s}{2}) \setminus \omega(\mu)} e^{2s\varphi} |v|^2 dx \]
\[ \leq C \left( \int_G e^{2s\varphi} (|\tilde{L}u^*|^2 + |\nabla \nabla v|^2) dx + \int_{\omega(\frac{s}{2}) \setminus \omega(\mu)} e^{2s\varphi} (|\nabla v|^2 + |v|^2) dx \right). \]

Choose \(s\) large enough so that \(C \int_G e^{2s\varphi} |\nabla \nabla v|^2 dx\) can be absorbed into the left side and move the last term of the left side to the right. Noting that \(\min_{G} \psi(x) = 0\), we have \(e^{2s\psi} \geq 1\) and consequently

\[ \tau^2 \int_G e^{2s\varphi} |\chi v|^2 dx + \int_G e^{2s\varphi} |\nabla \nabla v|^2 dx \]
\[ \leq C \left( \int_G e^{2s\varphi} (|\tilde{L}u^*|^2 + |\nabla \nabla v|^2) dx + \int_{\omega(\frac{s}{2}) \setminus \omega(\mu)} e^{2s\varphi} (|\nabla v|^2 + |v|^2) dx \right). \]

Noting \(\omega(\frac{s}{2}^*) \subset \omega(\mu)\) and by the definition of \(\chi\), we have

\[ \tau^2 \int_{\omega(\varphi^*/2)} e^{2s\varphi} |v|^2 dx + \int_{\omega(\varphi^*/2)} e^{2s\varphi} |\nabla \nabla v|^2 dx \]
\[ \leq C \left( \int_G e^{2s\varphi} (|\tilde{L}u^*|^2 + |\nabla \nabla v|^2) dx + \int_{\omega(\frac{s}{2}) \setminus \omega(\mu)} e^{2s\varphi} (|\nabla v|^2 + |v|^2) dx \right). \tag{2.17} \]

Furthermore, the definition of \(\{\omega(\delta)\}\) shows that

\[ \varphi \geq \frac{\varphi^*}{2} \quad \text{on} \ \omega(\frac{\varphi^*}{2}), \]
\[ \varphi \leq \varphi^* \quad \text{in} \ G, \tag{2.18} \]
\[ \frac{\mu}{2} \leq \varphi \leq \mu \quad \text{on} \ \omega(\frac{\mu}{2}) \setminus \omega(\mu). \]
We obtain from (2.17) and (2.18) that
\[ \tau^2 e^{\tau \varphi^*} \int_{\omega(\frac{\varphi^*}{2})} |v|^2 dx + e^{\tau \varphi^*} \int_{\omega(\frac{\varphi^*}{2})} |\nabla v|^2 dx \leq C \left( e^{\tau \varphi^*} \int_{G} |\tilde{L} u^*|^2 dx + e^{2 \tau \mu} \int_{\omega(\frac{\varphi^*}{2}) \setminus \omega(\mu)} (|\nabla v|^2 + |v|^2) dx \right). \] (2.19)

Dividing both side of (2.19) by $e^{\tau \varphi^*}$ and noting that $v = u - u^*$, we have
\[ \int_{\omega(\frac{\varphi^*}{2})} (|v|^2 + |\nabla v|^2) dx \leq C \left( e^{\tau \varphi^*} \int_{G} |\tilde{L} u^*|^2 dx + e^{-\frac{\tau \varphi^*}{\eta-1}} \int_{\omega(\frac{\varphi^*}{2}) \setminus \omega(\mu)} (|\nabla v|^2 + |v|^2) dx \right) \]
\[ \leq C \left( e^{\tau \varphi^*} \int_{G} |\tilde{L} u^*|^2 dx + e^{-\frac{\tau \varphi^*}{\eta-1}} \right) \left( \|u\|^2_{H^1(G)} + \|u^*\|^2_{H^1(G)} \right) \]
\[ \leq C \left( e^{\tau \varphi^*} \zeta_0^2 + e^{-\frac{\tau \varphi^*}{\eta-1}} (M_0^2 + \zeta_0^2) \right). \] (2.20)

**Case 1:** $M_0 > \zeta_0 \exp\left\{ \frac{3\tau_0(s)\varphi^*}{2(1 - \frac{1}{\eta})} \right\}$. Then (2.20) becomes
\[ \int_{\omega(\frac{\varphi^*}{2})} (|v|^2 + |\nabla v|^2) dx \leq C \left( e^{\tau \varphi^*} \zeta_0^2 + e^{-\frac{\tau \varphi^*}{\eta-1}} (1 + \exp\left\{ -\frac{3\tau_0(s)\varphi^*}{2(1 - \frac{1}{\eta})} \right\}) M_0^2 \right) \]
\[ \leq C (1 + \exp\left\{ -\frac{3\tau_0(s)\varphi^*}{2(1 - \frac{1}{\eta})} \right\}) \left( e^{\tau \varphi^*} \zeta_0^2 + e^{-\frac{\tau \varphi^*}{\eta-1}} M_0^2 \right). \] (2.21)

Noting $\varphi^* > 0$, we can put
\[ \tau = 2(1 - \frac{1}{\eta}) \frac{1}{\varphi^*} \ln \frac{M_0}{\zeta_0}. \]

It is easy to check that $\tau > \tau_0(s)$, then the Carleman estimate applies. Choosing
\[ \omega = \omega\left(\frac{\varphi^*}{2}\right) \]
and putting $\tau$ into (2.21),
\[ \|v\|_{H^1(\omega)} \leq CM_0^{1 - \frac{1}{\eta}} \frac{1}{\zeta_0}. \]

Hence
\[ \|u\|_{H^1(\omega)} \leq CM_0^{1 - \frac{1}{\eta}} \frac{1}{\zeta_0} + \|u^*\|_{H^1(\omega)} \leq CM_0^{1 - \frac{1}{\eta}} \frac{1}{\zeta_0}. \]

where $C$ only depends on $R$, $\theta$, $s$, $\gamma$, $G$, $\|\psi\|_{C^2(G)}$, $\|\mu\|_{C^1(\overline{BR})}$ and $\|\lambda\|_{C^1(\overline{BR})}$.

**Case 2:** $M_0 \leq \zeta_0 \exp\left\{ \frac{3\tau_0(s)\varphi^*}{2(1 - \frac{1}{\eta})} \right\}$. Then we trivially have
\[ \|u\|_{H^1(G)} = M_0 = M_0^{1 - \frac{1}{\eta}} M_0^{\frac{1}{\eta}} \leq \exp\left\{ \frac{3\tau_0(s)\varphi^*}{2(\eta - 1)} \right\} M_0^{1 - \frac{1}{\eta}} \frac{1}{\zeta_0}, \]
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This ends the proof of Theorem 2.2.

In particular the theorem shows the (local) uniqueness of the solution to the Cauchy problem for $\bar{L}u = 0$.

3. Three spheres inequalities

We turn now to prove the three spheres inequality for the Lamé system of elasticity. To begin, we recall a rather well known interior estimate for elliptic systems (see [5], Ch. 8, Th. 2.2 for example).

**Theorem 3.1.** Assume $\lambda, \mu \in C^1(\Omega)$. Let $u \in H^2(\Omega)$ be a solution of (1.1). Then for any $B_r \subset B_R$ with $0 < r < R$, there exists a constant $C$ depending only on $\lambda, \mu, R$ and $r$ such that

$$\|u\|_{H^2(B_r)} \leq C\|u\|_{L^2(B_R)}.$$  

For proving a quantitative estimate of unique continuation in Section 4, we have to show how $C$ depends on $r$. Checking the proof of the above theorem in [5], we easily know that there exists a constant $C_0$ independent of $R$ and $r$ such that the following estimate holds.

$$\|u\|_{H^2(B_r)} \leq \frac{C_0}{(R - r)^2}\|u\|_{L^2(B_R)}.$$  

Inspired by [12] and [15], we give a proof of the three spheres inequality (Theorem 1.1) as follows.

**Proof of Theorem 1.1.** From the proof of Theorem 2.2, we can see that there exists a constant $\theta_1$ satisfying $\theta < \theta_1 < R$ such that

$$\|u\|_{H^1(B_{\theta_1} \setminus B_{\theta})} \leq C\left(\sum_{|\alpha| \leq 1} \|u\|_{H^{\frac{3}{2} - |\alpha|} \partial B_{\theta}}\right)^\epsilon\|u\|_{H^2(B_R)}^{1-\epsilon}.$$  

where $\epsilon \in (0, 1)$. Furthermore,

$$\|u\|_{H^1(B_{\theta_1})} \leq \|u\|_{H^1(B_{\theta})} \leq \|u\|_{H^1(B_{\theta_1} \setminus B_{\theta})} \leq C\|u\|_{H^2(B_{\theta})}^{\epsilon}\|u\|_{H^2(B_R)}^{1-\epsilon}.$$  

Then we have

$$\|u\|_{L^2(B_{\theta_1})} \leq \|u\|_{H^1(B_{\theta_1})} \leq C\|u\|_{H^2(B_{\theta})}^{\epsilon}\|u\|_{H^2(B_R)}^{1-\epsilon}.$$  

Setting $\theta_2 = \frac{\theta + \theta_1}{2}$ and using Theorem 3.1, we obtain

$$\|u\|_{H^2(B_{\theta_2})} \leq \frac{C}{(\theta_1 - \theta)^2}\|u\|_{L^2(B_{\theta_1})}.$$
Combing (3.2), we have
\[ \|u\|_{H^2(B_{R_2})} \leq C \|u\|_{H^2(B_R)}^{1-\epsilon} \|u\|_{H^2(B_R)}^{\epsilon}. \]  
(3.3)

Set
\[ R_0 = \frac{R + R_2}{2} \in (R_2, R), \]
\[ \theta := \frac{R_2}{R_0}, \ a := \frac{\theta_2}{\theta} > 1. \]

Claim that
\[ \|u\|_{H^2(B_{R-a})} \leq C \left( \frac{\theta^2}{r^2} \|u\|_{H^2(B_{R_0})}^{1-\epsilon} \right) \|u\|_{H^2(B_{R_0})}^{\epsilon}, \quad \text{for all } 0 < r < R_2. \]
(3.4)

where \( C \) and \( \epsilon \) depend only on \( R, R_2, \theta \) and the Lamé moduli \( \lambda, \mu \). Setting \( t := \frac{r}{\theta} < 1, \)
we have
\[ r = \theta t < ra = \theta_2 t < R_0 < R. \]
(3.5)

Let \( \tilde{u}(y) := u(ty), \ \tilde{\lambda}(y) := \lambda(ty), \ \tilde{\mu}(y) := \mu(ty). \) Then \( \tilde{u} \) satisfies
\[ \text{div}(\tilde{\mu}(\nabla \tilde{u} + (\nabla \tilde{u})^\top)) + \nabla(\tilde{\lambda} \text{div} \tilde{u}) = 0, \]
(3.6)

where \( \tilde{\lambda}, \tilde{\mu} \in C^1(\Omega) \) satisfy strong ellipticity conditions. Repeating the same argument,
we have
\[ \|\tilde{u}\|_{H^2(B_{R-a})} \leq C \|\tilde{u}\|_{H^2(B_{R-a})} \|\tilde{u}\|_{H^2(B_{R-a})}^{1-\epsilon}, \]
(3.7)

The constant \( C \) appearing in (3.7) is independent of \( t \).

Indeed, \( \tilde{u} \) satisfies (3.6) in \( B_{R/\theta} \). Noting that \( t < 1, \) we deduce \( \tilde{u} \) also satisfies (3.6)
in \( B_R \). Then applying Theorem 2.2, we know the constant \( C \) in (3.7) depends on \( \theta, R_2, \)
\( R, \|\tilde{\lambda}\|_{C^1(\Omega)}, \|\tilde{\mu}\|_{C^1(\Omega)}, \) and \( \|\psi\|_{C^1(\Omega)} \), while the coefficients satisfy \( \tilde{\mu} \geq \alpha_0 > 0, \)
\( 2\tilde{\mu} + \tilde{\lambda} \geq \beta_0 > 0 \) and
\[ \sup_{B_R} \{ |\partial^a \tilde{\mu}|, |\partial^a \tilde{\lambda}| \} \leq \sup_{B_R} \{ |\partial^a \mu|, |\partial^a \lambda| \}. \]

By \( t < 1 \) and the change of variable \( ty = x, \) we have
\[ \|u\|_{H^2(B_{R-a})} \leq C \frac{1}{t^2} \|u\|_{H^2(B_{R-a})} \|u\|_{H^2(B_{R-a})}^{1-\epsilon}, \]
which gives (3.4).

Choose \( r = \frac{R_1}{2}. \) Then there exists a unique positive integer \( N \) such that
\[ ra^{N-1} < R_2 \leq ra^N. \]
(3.8)

Since \( ra^N < aR_2 = \frac{\theta_2}{\theta} R_2 < R_0, \) we have \( ra^N < R \) and
\[ \|u\|_{H^2(B_{rak})} \leq E_k \|u\|_{H^2(B_{rak-1})}^{\epsilon}, \]
(3.9)
for all \( k = 1, 2, \ldots, N \), where we set

\[
E_k := C \left( \frac{\theta}{ra_k-1} \right)^2 \| u \|_{H^2(B_{r(a)})}^{1-\epsilon}.
\]

Since \( a > 1 \), we have \( E_k < E_1 \) for \( k = 2, \cdots, N \). Then repeated use of (3.6) shows that

\[
\| u \|_{H^2(B_{R_2})} \leq \| u \|_{H^2(B_{ra_N})} \\
\leq E_N \| u \|_{H^2(B_{ra_{N-1}})} \\
\leq E_1 \left( E_{N-1} \| u \|_{H^2(B_{ra_{N-2}})}^{\tau_1} \right)^\epsilon \\
\leq E_1^{1-\epsilon N} \| u \|_{H^2(B_r)}^\epsilon.
\]

Setting \( \sigma = \epsilon^N < 1 \), we have

\[
\| u \|_{H^2(B_{R_2})} \leq C \| u \|_{H^2(B_r)}^\sigma \| u \|_{H^2(B_{r(a)})}^{1-\sigma}.
\]

By Theorem 3.1, we obtain

\[
\| u \|_{H^2(B_{R_2})} \leq C \frac{1}{R_1^2} \| u \|_{L^2(B_{R_1})}^\sigma \| u \|_{L^2(B_r)}^{1-\sigma}.
\]

This implies

\[
\| u \|_{L^2(B_{R_2})} \leq C \frac{1}{R_1^2} \| u \|_{L^2(B_{R_1})}^\sigma \| u \|_{L^2(B_r)}^{1-\sigma}, \quad (3.10)
\]

where \( C \) and \( \sigma \) depend on \( \frac{R_1}{R_2}, \frac{R_2}{R} \) and coefficients \( \lambda, \mu \). We thus complete the proof. \( \square \)

**Remark 3.1.** Furthermore, we can easily obtain, from (3.4) and (3.10) that

\[
C \leq \frac{\bar{C}}{R_1^3}, \quad (3.11)
\]

where \( \bar{C} \) depends on \( R_2, R, \| \lambda \|_{C^1(B_R)}, \| \mu \|_{C^1(B_R)}, \| \psi \|_{C^2(B_R \setminus B_0)}. \) By the definition of \( \psi, \)
we know that \( \| \psi \|_{C^2(B_R \setminus B_0)} \) depends on \( R \) and \( \theta(= 2 \frac{R_2 R}{R_2 + R}). \)

We are now at a position to discuss SUCP. The strong unique continuation is close related with the three spheres inequality (see [3] for the case of scalar parabolic equations). However, the three spheres inequality obtained in this paper may not be used to prove SUCP of Lamé systems, since the constant \( C \) and \( \sigma \) appeared in the right hand of (1.2) both depend on \( R_1 \). Fortunately, we can get a weak sense of SUCP.
4. Unique continuation

Proof of Theorem 1.2. Part (i). Without loss of generality we may assume that $x_0 = 0$, i.e.,

$$\int_{B_r} |u|^2 dx = O(e^{-r - \epsilon}), \quad \text{as } r \to 0.$$ 

We wish to show that $u \equiv 0$ in $\Omega$.

The following proof is based on the proof of Theorem 1.1 and Theorem 2.2, from which we utilize the notation and terminology.

By (3.8), we know

$$\frac{R_1}{2} a^{N-1} < R_2 \leq \frac{R_1}{2} a^{N}.$$ 

Hence

$$(\ln a)^{-1} \ln \frac{2R_2}{R_1} \leq N < (\ln a)^{-1} \ln \frac{2R_2}{R_1} + 1 \quad (4.1)$$

where $a > 1$ and $N$ are defined in the proof of Theorem 1.1.

(1.4) implies,

$$\int_{B_{R_1}} |u|^2 dx \leq C e^{-R_1 - \epsilon}, \quad 0 < R_1 < 1. \quad (4.2)$$

In order to prove $u \equiv 0$, we need to find a proper $\epsilon$ which appears in (2.6). Let

$$\eta := \exp \left\{ \frac{1}{N} + 1 \right\}, \quad (4.3)$$

Note that $\eta > 2$.

Repeating the same discussion as the proof of Theorem 2.2, we have

$$\|u\|_{H^1(\omega)} \leq CM_0^{\frac{1}{\eta} - \frac{1}{N}} \zeta_0 \frac{1}{\eta},$$

where $C$ is independent of $R_1$. By the proof of Theorem 1.1, we know

$$\|u\|_{L^2(B_{R_2})} \leq C \|u\|^\sigma_{L^2(B_{R_1})} \|u\|^{1-\sigma}_{L^2(B_{R})},$$

where $\sigma = \frac{1}{\eta^N}$. Noting (4.1) and (4.3), we have

$$\sigma = \frac{1}{\eta^N} = e^{-(1+N)} \geq e^{-2 \left(\frac{R_1}{2R_2}\right)^{(\ln a)^{-1}}}.$$ 

By (4.1) and (4.2), we have

$$\int_{B_{R_2}} |u|^2 dx \leq C \left( \int_{B_R} |u|^2 dx \right)^{1-\sigma} \left( \int_{B_{R_1}} |u|^2 dx \right)^\sigma$$

$$\leq C e^{-\sigma R_1 - \epsilon} \left( \int_{B_R} |u|^2 dx \right)^{1-\sigma}.$$
\[ \leq \frac{\bar{C}}{R_1} \exp\left\{ e^{-2} \frac{-1}{R_1^{-(\ln a)}} \right\} \left( \int_{B_R} |u|^2 dx \right)^{1-\sigma}. \] (4.4)

Claim: \((\ln a)^{-1} < \varepsilon\).

Indeed, from the proof of Theorem 1.1 and Theorem 2.2, we know

\[ a = \frac{\theta_2}{\theta} = \frac{1}{2} (1 + \frac{\theta_1}{\theta}) \]

where \(\theta \leq \theta_1 \leq R\) such that \(B_{\theta_1} = \omega = \{x \in G : \varphi(x) > \frac{\omega^n}{2}\}\). Noting the definition of \(\varphi\):

\[ \varphi(x) = e^{s(R^2 - |x|^2)} - 1, \]

we know \(\theta_1\) does not depend on \(\theta\).

The claim follows as long as we let \(\theta\) be small enough.

Then we pass to the limit in (4.4) as \(R_1 \to 0\),

\[ \|u\|_{L^2(B_{R_2})} \leq 0. \]

This implies

\[ u \equiv 0, \text{ in } B_{R_2}. \]

Part (i) follows by standard arguments.

Part (ii). Since \(\lambda, \mu \in C^2(\Omega), \partial_s u (s = 1, \cdots, n)\) also satisfies the Lamé system (1.1), by the same argument we have

\[ \int_{B_{R_2}} |\partial_s u|^2 dx \leq C \left( \int_{B_R} |\partial_s u|^2 dx \right)^{1-\sigma} \left( \int_{B_{R_1}} |\partial_s u|^2 dx \right)^\sigma. \]

This implies \(\partial_s u = 0, (s = 1, \cdots, n)\). Then Theorem 1.2 follows.

\[ \square \]

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