Discriminant of the ordinary transversal singularity type. The local aspects.

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Abstract. Consider a space \(X\) with the singular locus, \(Z = \text{Sing}(X)\), of positive dimension. Suppose both \(Z\) and \(X\) are locally complete intersections. The transversal type of \(X\) along \(Z\) is generically constant but at some points of \(Z\) it degenerates. We introduce (under certain conditions) the discriminant of the transversal type, \(\Delta^\perp\), a subscheme of \(Z\), that reflects these degenerations whenever the generic transversal type is ‘ordinary’.

The scheme structure of \(\Delta^\perp\) is imposed by various compatibility properties and is often non-reduced. We establish the basic properties of \(\Delta^\perp\): it is a Cartier divisor in \(Z\), functorial under base change, flat under some deformations of \((X, Z)\), and compatible with pullback under some morphisms, etc. Furthermore, we study the local geometry of \(\Delta^\perp\), e.g. we compute its multiplicity at a point, and we obtain the resolution of \(\mathcal{O}_{\Delta^\perp}\) (as \(\mathcal{O}_Z\)-module) and study the locally defining equation.

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1. Introduction

1.1. The setup. Let \(k\) be an algebraically closed field of zero characteristic, e.g. \(k = \mathbb{C}\). Let \(M\) be either a smooth irreducible algebraic variety (over \(k\)), or, for \(k = \mathbb{C}\), a complex-analytic connected manifold. Let \(X \subset M\) be a reduced subscheme with non-isolated singularities. We assume that \(Z := \text{Sing}(X)\) is connected, otherwise one fixes a connected component \(Z \subset \text{Sing}(X)\) and replaces \(X\) by some neighborhood of \(Z\). We always take \(Z\) with its reduced structure.

In many examples of non-isolated singularities one observes the following pattern. For each smooth point \(o \in Z\) consider a smooth germ, \((L^\perp, o) \subset (M, o)\), transversal to \((Z, o)\), such that \((L^\perp, o) \cap (Z, o) = \{o\}\). The singularity \((L^\perp \cap X, o)\) is usually isolated and its type is in some sense generically constant along \(Z\) (thus it is called the “transversal singularity type”). The points where the transversal singularity type degenerates usually form a subset of codimension 1 in \(Z\). It is natural to call this subset the discriminant of the transversal type, \(\Delta^\perp \subset Z\).

This is the target of our work.

With the following examples we try to give some intuition and the guiding principles. The precise discussion will be given later.

First we show that at some points the transversal type is not well defined.

Example 1.1. Consider the singular surface \(X = \{x^2z = y^2\} \subset k^3\). Its (reduced) singular locus is the line \(Z = \{x = y = 0\} \subset k^3\). This is the classical Whitney umbrella/pinch point/D\(\infty\) point. For the generic point \(o \in \text{Sing}(X)\), i.e. for \(z \neq 0\), the transversal singularity, \((X, o) \cap (L^\perp, o)\), is the plane curve singularity of type \(A_1\), i.e. two smooth non-tangent branches. As \(z \to 0\) the transversal singularity degenerates, at the origin the transversal type is not well defined. Indeed, we choose the transversal section \((L^\perp, o)\) among those defined by equation \(z = ax + by + (\text{higher order terms})\).
For \( a \neq 0 \) the intersection \( (X, o) \cap (L^\perp, o) \) is a cusp, \( A_2 \).

* For \( (L^\perp, o) = \{ z = x^{n-1} \} \) the intersection \( (X, o) \cap (L^\perp, o) \) is \( A_n \)-singularity.

* For \( (L^\perp, o) = \{ z = 0 \} \) the intersection \( (X, o) \cap (L^\perp, o) \) is a double line, a non-isolated singularity.

Therefore the expectation is that the point \((0, 0, 0)\) belongs to the discriminant \( \Delta^\perp \).

The following example suggests that sometimes the scheme structure on \( \Delta^\perp \) should be taken non-reduced.

**Example 1.2.** Let \( X = \{ x^2z^q = y^2 + x^3 \} \subset k^3 \) for \( q \geq 1 \). As before, the singular locus is \( Z = \{ x = y = 0 \} \) and the transversal type degenerates as \( z \neq 0 \). Consider the deformation: \( X_t = \{ x^2(z^q - t) = y^2 + x^3 \} \subset k^3 \) for \( t \in (k, o) \). It preserves the singular set: \( \text{Sing}(X_t) = \{ x = 0 = y \} \). For \( t \neq 0 \) the discriminantal point \((0, 0, 0)\) splits into \( q \) points \( \{ x = y = z^q - t = 0 \} \), each of them being of \( D_\infty \)-type. Thus, for \( t = 0 \), it is natural to consider the point \((0, 0, 0) \in \Delta^\perp \) with multiplicity \( q \) (or a multiple of \( q \)). One can say roughly that for \( q > 1 \) the transversal type degenerates (as \( z \to 0 \)) ‘faster’. (In examples of [3] we give other reasons for non-reducedness of \( \Delta^\perp \).)

We remark that the naive geometric consideration of the transversal section, \( (X, o) \cap (L^\perp, o) \), does not work at the singular points of \( Z \). Hence it should be replaced by an algebraic counterpart.

### 1.2. Assumptions

The definition of transversal type and its discriminant in the full generality seems out of reach at the present stage. Indeed, this would use the equisingularity theory in arbitrary dimension and codimension for arbitrary classes of singularities (e.g. whenever \( (Z, o) \) is not necessarily Gorenstein or Cohen-Macaulay). Thus we work under the following assumptions. (The precise definition, examples and properties are in [2] )

* The (reduced) singular locus, \( Z = \text{Sing}(X) \) or \((Z, o) = \text{Sing}(X, o)\), is a locally complete intersection at each point (l.c.i.).

* For each point \( o \in Z \) the germ \((X, o) \) is a strictly complete intersection over \((Z, o) \) (s.c.i.). This is a strengthening of the notion of complete intersection, needed to ensure that the strict transform under blowup along \( Z \) is again a complete intersection. In particular, if \( Z \) has several irreducible components then the multidegree of \( \mathbb{P}T((L^\perp \cap X, o)) \) at generic points of each component is the same.

* The transversal type of \( X \) along \( Z \) is generically ‘ordinary’. Namely, for sufficiently generic point \( o \in Z \), the projectivized tangent cone, \( \mathbb{P}T((L^\perp \cap X, o)) \), is a smooth complete intersection of expected dimension.

Under these assumptions we define the discriminant of transversal type, \( \Delta^\perp = \Delta^\perp_{X/Z} \), (with the natural scheme structure), and establish some local and global properties. (The further global properties are established in [K.K.N.). For the history of the question and some known results see [1],[3].)

### 1.3. On the choice of scheme structure of the discriminant

In simple cases, like that of examples [1],[2] it is obvious which points belong to \( \Delta^\perp \). This determines \( \Delta^\perp \subset Z \) as a subset, not as a scheme. On the other hand, it is less obvious whether/when the singular points of \( Z \) belong to \( \Delta^\perp \).

Our definition of the subscheme \( (\Delta^\perp, o) \subset (Z, o) \) is guided by the wish-list of the following natural properties:

1. **(Normalization)** For the classical Whitney umbrella, \( \{ x^2z = y^2 \} \subset k^3 \), the discriminant is the reduced point \((0, 0, 0) \). More generally, for a \( D_\infty \) point, \( \{ x_0x_1^n + \sum_{i=2}^n x_i^2 = 0 \} \subset (k^{n+1}, 0) \), the discriminant is the reduced point \((0, \ldots, 0) \in (Z, o) = \{ x_1 = \cdots = x_n = 0 \} \subset k^{n+1} \). Even more generally, suppose the germ \((Z, o) \) is smooth and the multiplicity of \( X \) along \( Z \) is locally constant at \( o \). Take the generic section \((L^\perp \cap X, o) \), suppose the projectivization of the tangent cone, \( \mathbb{P}T((L^\perp \cap X, o)) \), has just one \( A_1 \) singularity. Then \( \Delta^\perp \subset Z \) is reduced at \( o \).

2. **(Behaviour in families).** Note that the family \( \{ \Delta^\perp(X_t) \} \) of example [1],[2] is flat.) Suppose that a flat family \( \mathcal{X} = \{ X_t \}_{t \in (k^1, o)} \rightarrow (k^1, o) \) satisfies:

   * the family \( Z \) is \( \{ Z_t = \text{Sing}(X_t) \}_{t \in (k_1, o)} \rightarrow (k^1, o) \) is flat;
   * the generic multiplicity of \( X_t \) along \( Z_t \) does not vary with \( t \);
   * for any \( t \) the transversal type of \( X_t \) along \( Z_t \) is generically ordinary (see [1],[2].

   Then the family \( \{ \Delta^\perp(X_t) \}_{t \in (k_1, o)} \) is flat.

3. **(Pullback of the classical discriminant)** Suppose \( Z \) (or its germ at a point) is smooth. Take the strict transform under blowup, \( B_{\mathbb{P}^1}(M) \supset \tilde{X} \). The exceptional divisor, \( E \subset B_{\mathbb{P}^1}(M) \) induces the family of projective complete intersections, \( \tilde{X} \cap E \rightarrow Z \). Thus one has a (rational) map from \( Z \) to the parameter space of projective complete intersections. (In the hypersurface case this parameter space is \( |O|_{\mathbb{P}^1}(d)| \).) In this parameter space we have the classical discriminant \( \Delta \). Then \( \Delta^\perp \) should be the pullback of \( \Delta \).

4. **(The image of the critical locus)** Suppose the fibres of the projection \( \tilde{X} \cap E \rightarrow Z \) are (generically) of dimension \( d \). The critical locus, \( \text{Crit}(\pi) \subseteq \tilde{X} \cap E \) is defined via the relative cotangent sheaf, \( \Omega^1_{\tilde{X} \cap E/Z} \).
by the Fitting ideal $\text{Fitt}_d(\Omega^1_{X \cap E/Z}) \subseteq \mathcal{O}_{X \cap E}$. Then $\Delta^\perp$ is the image of $\text{Crit}(\pi)$, with the Fitting scheme structure, $I_{\Delta^\perp/Z} = \text{Fitt}_0(\pi_*\mathcal{O}_{\text{Crit}\pi})$.

We define the subscheme $\Delta^\perp \subset Z$ in [11] it has all these properties.

1.4. Additional basic properties of $\Delta^\perp$. Besides the minimal requirements listed above, the discriminant of transversal singularity type possesses (as a scheme) various other nice/natural properties.

1. The scheme structure of $\Delta^\perp$ is completely determined by the ‘infinitesimal neighborhood’ of $\text{Sing}(X)$ in $X$, more precisely, by the exceptional divisor of blowup: $(E, \hat{X} \cap E) \subset (Bl_Z M, \hat{X})$, see [§5.2]. In this way it is independent of those ‘higher-order’ degenerations of $(L^\perp \cap X, o)$ that preserve the tangent cone. (In particular we do not see any direct relation of $\Delta^\perp$ to the Lê cycles of [Massey], see [§5.4].)

2. (The discriminant pulls back.) Given a morphism $M_1 \xrightarrow{\phi} M_2$, inducing $X_1 = \phi^*(X_2)$ and $Z_1 = \phi^*(Z_2)$. Suppose $Z_i$ are reduced l.c.i. and are connected components of $\text{Sing}(X_i)$. Suppose $X_i$ are s.c.i. over $Z_i$ at each point and $X_i$ are generically ordinary along $Z_i$, with the same multiplicity sequences. Then $\Delta^\perp_{X_1/Z_1} = \phi^*\Delta^\perp_{X_2/Z_2}$, see [§5.1].

A particular case of the statement is the following: given a smooth hypersurface germ $(M_1, o) \subset (M_2, o)$, such that the tangent cones intersection $T_{(M_1, o)} \cap T_{(X, o)}$ is generic enough, then $\Delta^\perp_{M_1 \cap X} = \Delta^\perp_{M_2 \cap X} \cap (M_1, o)$.

3. (The (local) defining equation of $\Delta^\perp$ is obtained by elimination procedure and thus cannot be written explicitly in the full generality. Yet, following the tradition, we present the discriminant as the determinant of a matrix. More precisely, we establish the (traditional) free resolution of $\mathcal{O}_{(\Delta^\perp,o)}$, as a module over $\mathcal{O}_{(Z,o)}$, see [§5.6]. We use this resolution to get some information about the monomials of the defining equation of $\Delta^\perp$. In particular, in the weighted-homogeneous case, we compute the total (weighted) degree of monomials that occur in the discriminant polynomial, see Proposition [5.16].

4. Flatness of a deformation of $\Delta^\perp$ (under the deformation of $X$, see (2) of [§13]) means the following: the sheaves $\mathcal{O}_{Z_i}(-\Delta_{i})$ glue to a locally free sheaf of ideals $I_{\Delta_{i}}$ on $Z = \{Z_i\}$ and the schemes $\Delta_{i}$ glue to a Cartier divisor on $Z$. If $\{X_i\}$ are not equimultiple along $\{Z_i\}$ or the induced deformation $\{Z_i\}$ is not flat then the family $\Delta_{i}$ is not flat and in general is not semi-continuous in any sense, see [§5.3].

5. The multiplicity of $\Delta^\perp$ at a point.) Given the projection $X \cap E \xrightarrow{\pi} Z$ suppose the fibre $\pi^{-1}(o)$ has only isolated singularities. Then $(\Delta^\perp, o) = \sum(\Delta_{i}, o)$, the sum of Cartier divisors corresponding to the singular points of $\pi^{-1}(o)$. Thus it is enough to assume that $\pi^{-1}(o)$ has only one singular point. In the hypersurface case, suppose in some local coordinates $x$ on $(Z,o)$ and $\bar{x}$ on $\pi^{-1}(o)$ the locally defining equation of $\pi^{-1}(o)$ is $f(x) + \bar{g}(\bar{x})$. Then $\text{mult}(\Delta^\perp, o) = \mu(\pi^{-1}(o)) \cdot \text{mult}(\bar{g}(\bar{x}))$. For complete intersection we obtain a similar result, using the Lê-Greuel formula, see [§5.5].

6. In [41.3] we define a further stratification of $\Delta^\perp$, corresponding to the higher degenerations of transversal type.

We emphasize that in the hypersurface case most statements of our paper appear in the standard literature. But the case of complete intersections is less known.

1.5. History and motivation.

- The discriminant of transversal singularity type appears naturally in geometry and singularity theory and in some particular cases was considered already by Salmon, Cayley, Noether and Zeuthen, see [Piene1977]. One context where it appears is the image of the generic map from a smooth $n$–fold into $\mathbb{P}^{n+1}$. The image has non-isolated ordinary singularities, [Mond-Pellikaan page 111], (not to be confused with the ‘ordinary transversal type’ used in this paper). The natural question is to understand their degenerations, as one runs along the singular locus.

- The class of $\Delta^\perp$ for projective surface, $X \subset \mathbb{P}^3$, with ordinary singularities goes back (probably) to the early history. For a computation see [Piene1977] (among various other invariants).

- The case of one-dimensional singular locus, i.e. $Z$ is a curve, with the generic transversal type $A_1$, was thoroughly studied by Siersma, see e.g. [Siersma2000]. The local degree of the discriminant, called also ‘the virtual number of $D_\infty$ points’ was studied in [Pellikaan1989, Pellikaan1990 and de Jong1990]. In particular, the authors show pathological behavior when $Z$ is not a locally complete intersection. In [de Jong-de Jong1990] the degree of $[\Delta^\perp]$ is computed for the case $X \subset M$ is a projective hypersurface, $Z = \text{Sing}(X)$ is of (pure) dimension one and the generic transversal type is $A_1$. For the review of various related result see [AGLV-book2 §1.4.6]. For the recent results and applications to real singularities see [van Straten2011].

We emphasize that in Pelikaan-de Jong’s approach the scheme structure on the discriminant is compatible with flat deformations, [de Jong1990 §2.5], and the discriminant is reduced for Whitney umbrella. These
two conditions determine the scheme structure uniquely, therefore their and our scheme structures (for non-isolated singularities of surfaces) coincide. In example 4.2, we show this directly.

- One often considers the singular locus with the scheme structure defined by Jacobian ideal, \( \text{Sing}(X)^{(f)} \), [Aluffi-1995], [Aluffi-2005]. The scheme \( \text{Sing}(X)^{(f)} \) also reflects the degenerations of transversal type. We emphasize, that this Jacobian scheme structure is incompatible with flat deformations and it differs from the scheme structure of our paper.

2. Preliminaries

For the general introduction to singularities see [AGLV-book1], [Dimca-book], [Looijenga-book] and [Seade-book].

2.1. Local neighborhoods. Working locally, we consider germs of spaces, \((Z, o) \subset (X, o) \subset (k^N, o)\). These germs can be algebraic, analytic (for \( k = \mathbb{C} \)), formal, etc., the category is specified by the (local) ring of regular functions.

The ring \( \mathcal{O}_{(k^N, o)} \) is a regular (Noetherian) local ring over a field of zero characteristic, e.g. one of the following: \( k[x_1, \ldots, x_n][m] \) (localization of the affine ring), or \( k\{x_1, \ldots, x_n\} \) (the ring of analytic power series, for \( k \subseteq \mathbb{C} \)), or \( k[x_1, \ldots, x_n] \) (the ring of algebraic power series), or \( k[[x_1, \ldots, x_n]] \). For a subgerm \((X, o) \subset (k^N, o)\) the local ring is the quotient by the defining ideal, \( \mathcal{O}_{(X,o)} = \mathcal{O}_{(k^N,o)}/I_{(X,o)} \). In many cases the algebraic germs are ‘too large and rigid’, e.g. when speaking of irreducible components or rectifying locally a smooth variety. In such cases we take henselization or completion (i.e. we pass to henselian or formal germs).

If the germ \((X, o)\) is not algebraic/analytic then one cannot take its “small enough representative”, e.g. a formal germ has no closed points besides the base point. Yet, using the standard algebra-geometry dictionary the ideas/notions of “working near the origin” are applicable. One just translates a geometric statement/condition into the algebraic one, e.g.: 

- “the points of the subgerm \((Z, o) \subset (X, o)\) satisfy ...” is replaced by “the ideal \( I_{(Z,o)} \subset \mathcal{O}_{(X,o)} \) satisfies ...
- “generic points of \((Z, o) \subset (X, o)\) satisfy ...” is replaced by “the localization of \( \mathcal{O}_{(Z,o)} \), \( \mathcal{O}_{(X,o)} \), at the prime components of \( I_{(Z,o)} \) satisfies ...

We denote the maximal ideal in the local ring \( R \) by \( \mathfrak{m}_R \) (e.g. \( \mathfrak{m}_{(X,o)}, \mathfrak{m}_{(Z,o)} \)) or just by \( \mathfrak{m} \).

2.2. Multiplicity at a point, generic vanishing order and symbolic powers of ideals. The (Taylor) order or multiplicity of an element \( f \) in a local ring \((R, \mathfrak{m})\) is defined as usual: \( \text{mult}_R(f) = \max\{k| f \in \mathfrak{m}^k\} \). More generally, the order of \( f \) with respect to an ideal \( J \subset R \) is \( \text{ord}_J(f) = \max\{k| f \in \mathfrak{m}^k\} \).

The multiplicity of a germ \((X, o) \subset (k^N, o)\) of pure dimension \( n \) is defined as \( \text{dim}_k \mathcal{O}_{(X,o)}\langle l_1, \ldots, l_n \rangle \), where \( \langle l_1, \ldots, l_n \rangle \) is the ideal generated by any \( n \)-tuple of generic elements of \( \mathfrak{m} \).

Let the germ \((Z, o) \subset (k^N, o)\) be reduced. An element \( f \in \mathcal{O}_{(k^N, o)} \) has generic order \( \geq m \) along \((Z, o)\) if its (Taylor) order at smooth points of \((Z, o)\) is \( \geq m \). The general definition of this property goes via the notion of symbolic powers, as follows. (We replace \( \mathcal{O}_{(k^N, o)} \) by \( R \) and \( I_{(Z,o)} \) by \( J \).)

Let \( R \) be a Noetherian ring and \( J \subset R \) a primary ideal, whose corresponding prime is \( p \). The \( m \)-th symbolic power is defined as

\[
J^{(m)} := (J^m \cdot R_p) \cap R = \{ f \in R | \exists s \notin p : sf \in J^m \}.
\]

If the ideal \( J \) is not primary but radical, one takes the primary decomposition \( J = J_1 \cap \cdots \cap J_m \) and defines \( J^{(m)} = \cap J_i^{(m)} \).

In the most general case the definition goes as follows (see definition 3.5.1 of [Vasconcelos]). For any ideal in a Noetherian ring, \( J \subset R \), take the decomposition: \( J = J' \cap L \), where \( J' \) is the intersection of the primary ideals associated with the minimal primes of \( J \), while \( L \) is the intersection of primary ideals corresponding to embedded primes of \( J \). Then \( J^{(m)} := (J')^m \).

**Definition 2.1.** We say that \( f \) is generically of order \( \geq m \) on all the components of \( V(J) \) if \( f \in J^{(m)} \).

For the explanation that \( f \in J^{(m)} \) means this geometric condition see [Eisenbud-book §3.9].

One has the obvious inclusion \( f_{(Z,o)}^{(m)} \supseteq I_{(Z,o)}^{(m)} \) and this inclusion can be proper. 

**Example 2.2.** Let \((Z, o) = \{xy = yz = xz = 0\} \subset (k^3, o)\), thus \( I_{(Z,o)} = (x, y) \cap (y, z) \cap (x, z)\). The generic order of \( f = xyz \) along \((Z, o)\) is 2, thus \( xyz \in I_{(Z,o)}^{(2)} \), but \( xyz \notin I_{(Z,o)}^{2} \). In fact we have the primary decomposition: \( I_{(Z,o)}^{2} = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2 \), thus \( I_{(Z,o)}^{(2)} = (x, y)^2 \cap (y, z)^2 \cap (x, z)^2 \).

Such pathologies do not occur when \((Z, o)\) is a complete intersection.
Lemma 2.3. If \((Z, o) \subset (k^N, o)\) is a complete intersection (not necessarily reduced) then \(I_{(Z, o)}^m = I_{(Z, o)}^m\) for any \(m \in \mathbb{Z}_{>0}\).

**Proof.** Let \(J \subset R\) be the defining ideal of \((Z, o) \subset (k^N, o)\).

- If \(J \subset R\) is prime then we can use the general proposition 3.5.12 of [Vasconcelos]:
  \((2)\) if \(R\) is Cohen-Macaulay and \(J \subset R\) is a prime complete intersection then \(J^m = J^m\) for any \(m \in \mathbb{Z}_{>0}\).
- If \(J\) is not prime, but \(R\) is regular and in the primary decomposition, \(J = \cap p_i\), all the minimal primes \(p_i\) are complete intersections, then one can use:
  \((3)\)

\[
J^m = \cap p_i^m = \cap p_i^m = J^m.
\]

- In general, the minimal primes \(p_i\) are not complete intersections, then one argues as follows. Suppose \(R\) is a regular local ring and \(J\) is a complete intersection, with \(\sqrt{J} \neq m\). Consider the ideal \((J^m)/J^m \subset \mathbb{R}f^m\). By the definition of symbolic powers, for any \(p_i\), the localization vanishes: \(\mathbb{J}^m/J^m\) is a torsion. But, as \(J \subset R\) is a complete intersection, the ring \(\mathbb{R}f^m\) has no torsion. Thus \(J^m/J^m = 0\).

2.3. The functor of associated graded modules. Fix a (commutative, associative) ring \(R\), and an ideal \(I \subset R\). This ideal induces the filtration, \(R = I^0 \supset I^1 \supset \cdots\). Take the associated graded ring, \(\mathbb{gr}_1(R) = \bigoplus_{j \geq 0} I^j/I^{j+1}\).

Explicitly, fix some generators, \(\{g_i\}\), of \(I\), and its module of relations, \(\text{Syz}(g) = \{\{a_i\} \sum a_i g_i = 0 \in R\} \). Then

\[
\mathbb{gr}_1 R = R/I \left[ \mathbb{Z}/ \mathbb{Z} \right]_{\{a_i y_i. a_i \in \text{Syz}(g)\}.}
\]

Thus \(\mathbb{gr}_1 R\) is a graded algebra over \(R/I\), and \(\text{Spec}(\mathbb{gr}_1 R)\) is an affine scheme over \(\text{Spec}(R/I)\).

Consider the category \(\text{mod}_{\text{filt}}(R)\), of (finitely generated) filtered \(R\)-modules,

\[
\cdots \supset M_{i-1} \supset M_i \supset M_{i+1} \supset \cdots, \quad I^j M_i \subset M_{i+j}.
\]

The morphisms here are the filtered homomorphisms:

\[
\text{Hom}_{\text{filt}}(M_i, \{N_i\}) = \{\phi \in \text{Hom}_R(M, N) | \phi(M_i) \subset N_i\}.
\]

To each filtered \(R\)-module one associates a graded module over \(\mathbb{gr}_1 R\), by \(\mathbb{gr}(M) := \bigoplus_{j \geq 0} M^j/M^{j+1}\). The filtered morphisms of \(\text{mod}_{\text{filt}}(R)\) are then sent to the graded morphisms of \(\text{mod}_{\text{gr}}(\mathbb{gr}_1 R)\). This defines the “associated graded” functor

\[
\text{mod}_{\text{filt}}(R) \xrightarrow{\text{gr}} \text{mod}_{\text{gr}}(\mathbb{gr}_1 R).
\]

In our case \(R\) is Noetherian and all the filtration are exhaustive (\(\bigcap M_i = M\)) and separated (\(\cap M_i = \{0\}\)), in particular \(\cap I^j = \{0\}\). Therefore this functor is faithful, i.e. \(\mathbb{gr}(M) = \{0\}\) implies \(M = \{0\}\), [Năstăsescu-Van Oystaeyen, Proposition I.4.1].

This functor is not exact, however it preserves exactness of strictly filtered sequences. In more detail, take a filtered morphism \(M \xrightarrow{\phi} N\), i.e. \(\phi(M_i) \subset N_i\). This morphism is called strictly filtered if \(\phi^{-1}(N_i) = M_i\).

**Theorem 2.4.** [Năstăsescu-Van Oystaeyen, Theorem I.4.4] Consider a filtered sequence in \(\text{mod}_{\text{filt}} - R\) and the associated sequence in \(\text{mod}_{\text{gr}}(\mathbb{gr}_1(R))\):

\[
* : L \rightarrow M \rightarrow N \rightarrow \mathbb{gr}(*) : \mathbb{gr}(L) \rightarrow \mathbb{gr}(M) \rightarrow \mathbb{gr}(N).
\]

Suppose all the filtrations are exhaustive and complete. Then \(\mathbb{gr}(*)\) is exact iff \(*\) is exact and strictly filtered.

2.4. The normal cone. Given a filtration \(R = I^0 \supset I^1 \supset \cdots\) and an element \(f \in R\), fix the order \(p = \text{ord}_I(f)\), i.e. \(p\) with \(f \in I^p \setminus I^{p+1}\). The leading term of \(f\) is defined as the residue class \(l.t.(f) = I^p/I^{p+1} \subset \mathbb{gr}_1(R)\). We associate with an ideal \(J \subset R\) the ideal \(\mathbb{gr}_1(J) \subset \mathbb{gr}_1 R\), generated by the leading terms of all the elements of \(J\).

In our case, for a triple of germs, \((Z, o) \subset (X, o) \subset \text{Spec}(R)\), we have the diagram:

\[
R \twoheadrightarrow \mathbb{gr}_{I(Z, o)} R = \bigoplus_{j \geq 0} (I_{(Z, o)})^j/(I_{(Z, o)})^{j+1}
\]

\[
\cup
\]

\[
I_{(X, o)} \twoheadrightarrow \mathbb{gr}_{I(X, o)} I_{(X, o)}
\]

One can write explicitly: \(\mathbb{gr}_{I(Z, o)} I_{(X, o)} = \bigoplus_{j \geq 0} I_{(X, o)} \cap (I_{(Z, o)})^j + (I_{(Z, o)})^{j+1}/(I_{(Z, o)})^{j+1}\).

Note that the transition \(f \rightarrow l.t.(f), I_{(X, o)} \rightarrow \mathbb{gr}_I(X, o)\), is not a homomorphism and is never injective/surjective. (Its image is the disjoint union of all the homogeneous components of \(\mathbb{gr}_{I(X, o)}\).)
Definition 2.5. 1. The ideal \(gr_{I(X,o)}I(X,o) \subset gr_{I(Z,o)}R\) is called the co-normal ideal.

2. The normal cone of \((X,o)\) along \((Z,o)\) is the scheme \(N_{(X,o)/(Z,o)} = V(gr_{I(Z,o)}I(X,o)) \subset Spec(gr_{I(Z,o)}R)\).

2.4.1. Example: \((Z,o) \subset Spec(R)\) is a complete intersection. Let \((Z,o)\) be a complete intersection (not necessarily reduced) and fix some regular sequence of generators \(I_{(Z,o)} = \langle g_1, \ldots, g_k \rangle\). Then the only relations among \(\{g_i\}\) are the Koszul relations, therefore equation (11) gives:

\[
\begin{align*}
gr_{I(Z,o)}R &\approx \mathcal{O}_{(Z,o)}[y] \quad \text{with } y = (y_1, \ldots, y_k), \text{ and } Spec(gr_{I(Z,o)}R) = (Z,o) \times \mathbb{k}^k. 
\end{align*}
\]

(Here the isomorphism is defined by the choice of the generators \(\{g_i\}\). This ambiguity results in the action \(GL_{(Z,o)}(k) \subset (Z,o) \times \mathbb{k}^k\), see also below.

For any \(f \in P^p_{(Z,o)} \setminus P^p_{I(Z,o)} \subset R\) we have:

\[
\begin{align*}
f &\approx \sum g_{m_1} \ldots g_{m_k} a_{m_1} \ldots a_{m_k}, \text{ for some elements } \{a_{m_1} \ldots a_{m_k} \in R\}.
\end{align*}
\]

Using this expansion we write down the leading term of \(f\) explicitly:

\[
\begin{align*}
\tilde{f} &\approx \sum y_{ij}^{m_j} a_{m_i} |(Z,o) \in \mathcal{O}_{(Z,o)}[y], \text{ where } \mathcal{O}_{(Z,o)} = R/I(Z,o) \subset gr_{I(Z,o)}R \\
\text{By the construction: } \tilde{f} &\neq 0 \in \mathcal{O}_{(Z,o)}[y], \text{ therefore ord}_{(Z,o)}(\tilde{f}) = p \text{ too.)}
\end{align*}
\]

The coefficients \(\{a_{m_1} \ldots a_{m_k}\}\) are not unique, because of the Koszul relations. But the restrictions \(\{a_{m_i} |(Z,o) \in \mathcal{O}_{(Z,o)}\}\) are defined uniquely. (Indeed, if \(\sum y_{ij}^{m_j} a_{m_i} |(Z,o) = \sum y_{ij}^{m_j} a_{m_i} |(Z,o) \) then \(\sum g_{ij}^{m_j} (a_{m_i} - b_{m_i}) = 0 \in R/P^p_{I(Z,o)}\), which means a syzygy between \(g_i\)'s in \(R/P^p_{I(Z,o)}\). It lifts to a syzygy in \(R\), with some contribution of term from \(P^p_{I(Z,o)}\). But, \(\{g_i\}\) being a regular sequence, all the syzygies are linear combinations of the Koszul ones, and this forces \(a_{m_i} - b_{m_i} \) to belong to \(I(Z,o)\), which implies \(a_{m_i} |(Z,o) - b_{m_i} |(Z,o) = 0 \in \mathcal{O}_{(Z,o)}\).)

The set of all such leading terms, \(I(X,o) \ni f \rightsquigarrow \tilde{f} := l.t.(f)\), generates the co-normal ideal \(gr_{I(Z,o)}I(X,o) \subset \mathcal{O}_{(Z,o)}[y_1, \ldots, y_k]\). Again, the transition \(I(X,o) \ni gr_{I(X,o)}I(X,o)\) is not a homomorphism and is never injective/surjective. The ideal \(gr_{I(Z,o)}I(X,o)\) is (graded by construction).

Example 2.6. Suppose \(R = \mathcal{O}_{(k^n,o)}(Z,o) \subset (k^n,o)\) is just a reduced point, then \(I(Z,o) = \mathfrak{m} \subset \mathcal{O}_{(k^n,o)}\) and \(gr_{I(Z,o)}\mathcal{O}_{(k^n,o)} = \oplus_{j \geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}\). The transition \(I(X,o) \ni gr_{I(Z,o)}I(X,o)\) takes the leading term of Taylor expansion, \(f = f_p + f_{>p} \ni \tilde{f}\). The normal cone is just the tangent cone, \(T_{(X,o)} \subset T_{(k^n,o)} = Spec(k[y_1, \ldots, y_n])\).

To identify \(gr_{I(Z,o)}R \approx \mathcal{O}_{(Z,o)}[y]\) we have chosen a set of generators of \(I(Z,o)\), but the dependence of the image of \(gr_{I(Z,o)}I(X,o)\) in \(\mathcal{O}_{(Z,o)}[y]\) on this particular choice is non-essential:

Lemma 2.7. The ideal \(gr_{I(Z,o)}I(X,o) \subset \mathcal{O}_{(Z,o)}[y]\) is well defined up to the action \(GL_{(Z,o)}(k) \subset (Z,o) \times \mathbb{k}^k\), in particular the subscheme \(N_{(X,o)/(Z,o)} \subset (Z,o) \times \mathbb{k}^k\) is defined up to an isomorphism.

Proof. Fix some other set of generators \(\{g'_i\}\) of \(I(Z,o)\), then \(g'_j = \sum \phi_{ij} g_j\), where the matrix \(\{\phi_{ij}\} \in Mat_{k \times k}(R)\) is invertible. This matrix induces an automorphism of \(\mathcal{O}_{(Z,o)}[y]\), denote it by \(\phi\). Then any homogeneous expansion \(\tilde{f} = \sum y_{ij}^{m_j} a_{m_i}\) transforms to \(\tilde{f} = \sum \phi(y_{ij}^{m_j}) a_{m_i}\). Thus \(gr_{I(Z,o)}I(X,o)(\{g'_i\})\) and \(gr_{I(Z,o)}I(X,o)(\{g'_i\})\) differ by an element of \(GL_{(Z,o)}(k)\).

2.5. Strictly complete intersections. The tangent cone of a hypersurface germ is a hypersurface, but the tangent cone to a complete intersection is not necessarily a complete intersection.

Example 2.8. For the complete intersection \((X,o) = \{x^2 + zy^3 = xy + z^3 = 0\} \subset (k^3,o)\) the tangent cone is \(T_{(X,o)} = \{x^2 = xy = xz^3 = z^6 - z^5 = 0\}\). Indeed, \cite{Eisenbud-book} §15.10.3, it is enough to check the Groebner basis of the homogenized ideal, \(\{w^2x^2 + zy^3, wxy + z^3\}\), with respect to any monomial ordering. For the ordering \(x > y > z > w\) the Groebner basis is:

\[
\begin{align*}
w^2x^2 + zy^3, \\
wxxy + z^3, \\
wxxz^3 - zy^4, \\
z^6 - y^5z.
\end{align*}
\]

By sending \(w \to 1\) and taking the leading terms we get the projectivized tangent cone. Now, by direct check, this projectivization \(\mathbb{P}T_{(X,o)} \subset \mathbb{P}^2\) is a collection of smooth (!) points, whose defining ideal is not a complete intersection.

For various other pathologies of tangent cone and conditions to prevent them see [Heinzer-Kim-Ulrich].

Definition 2.9. The germ \((X,o) \subset Spec(R)\) is called a strictly complete intersection (s.c.i.) over \((Z,o)\) if the normal cone, \(N_{(X,o)/(Z,o)}\), is a complete intersection over \((Z,o)\). Algebraically: the ideal \(gr_{I(Z,o)}I(X,o) \subset gr_{I(Z,o)}(R)\) is generated by a regular sequence.
Many results around this notion are scattered in the literature. We collect here the relevant results and examples.

**Example 2.10.** Let \((X, o) \subset (k^N, o)\) be a hypersurface with \((Z, o) \subset (X, o)\) a complete intersection.

i. Suppose \((Z, o)\) is irreducible and reduced, then \((X, o)\) is s.c.i. over \((Z, o)\). Indeed: if \(I_{(X, o)} = (f)\) then \(gr_I_{(Z, o)} I_{(X, o)} = (\tilde{f})\), and \(\tilde{f} \in O_{(Z, o)}[y]\) is regular, i.e. not a zero divisor.

ii. If \((Z, o)\) is reduced but reducible then \(gr_I_{(Z, o)} I_{(X, o)}\) is still generated by one element, but this might be not a regular sequence, since this element can be a zero divisor. More precisely, let \(I_{(X, o)} = (f)\) with \((Z, o) = \bigcup (Z_i, o)\) and \(f \in \mathbb{C} (\{p_1^i \cap Z_i, o\} \setminus \{p_{i+1}^i \cap Z_i, o\})\). Then \((X, o)\) is s.c.i. over \((Z, o)\) iff \(p_1 = \cdots = p_n\).

iii. Similarly for the case: \((Z, o)\) is a multiple of an irreducible germ, e.g. \(I_{(Z, o)} = (x^2, y^2) \subset k[[x_1, x_2, x_3]]\) and \(I_{(X, o)} = (x^2, y^2) \subset k[[x_1, x_2, x_3]]\). Here the ideal \(gr_I_{(Z, o)} I_{(X, o)} = (x_1 y_1 + x_2 y_2) \subset O_{(Z, o)}[y]\) is principal. But its generator is not regular, being a zero divisor.

**Example 2.11.** Suppose \((Z, o)\) is just a reduced point, then definition 2.3 reads as follows: the germ \((X, o)\) is called a **strictly complete intersection**, s.c.i. at \(o\), if it is a complete intersection and its tangent cone is a complete intersection too. (The condition \((X, o)\) is a complete intersection at \(o\)" is redundant here, as we show below.) Thus a hypersurface germ is always a s.c.i. at the origin. The name “strict complete intersection” seems to be coined by [Bennett, 1974] pp.31. The name “strong complete intersection” is used in commutative algebra to denote “geometric” complete intersections, i.e. rings of the form \(S/(f_1, \ldots, f_r)\), where \(S\) is a regular local ring and \(\{f_i\}\) is a regular sequence, [Heitmann-Jorgensen].

The name “absolute complete intersection” would suggest that both the germ and all its proper transforms and exceptional loci in the resolution are locally complete intersections.

**Proposition 2.12.** If \((X, o)\) is a strictly complete intersection over \((Z, o)\) then \((X, o)\) is a complete intersection, i.e. \(I_{(X, o)}\) is generated by a regular sequence. Moreover, there exists a choice of generators, \(I_{(X, o)} = (f_1, \ldots, f_r)\), such that the leading terms, \(\{f_i\}\), form a regular sequence that generates \(gr_I_{(Z, o)} I_{(X, o)}\).

(For the proof see Corollary 2.4 of [Valabrega-Valla]. Following that paper the sequence \(\{f_i\}\) is often called a “super-regular” sequence.)

This proposition, together with example 2.8 show that the condition “\((X, o)\) is a s.c.i. over \((Z, o)\)” is stronger than the condition “\((X, o)\) is a complete intersection as a scheme over \((Z, o)\)”.

**Example 2.13.** Suppose \((Z, o) \subset (k^N, o)\) is a reduced complete intersection such that \((Z, o) \subset (X, o)\) is also a complete intersection, and \((X, o) \subset (k^N, o)\) is also a complete intersection. (Equivalently, \(I_{(X, o)} \subset O_{(k^N, o)}\) has a basis that can be extended to a basis of \(I_{(Z, o)}\). Indeed, choose a regular sequence that generates the defining ideal of \((Z, o) \subset (X, o)\), take some representatives \(g_1, \ldots, g_r, y_1, \ldots, y_k \in O_{(k^N, o)}\). Take some generators \(f_1, \ldots, f_r\) of \(I_{(X, o)} \subset O_{(k^N, o)}\), then \(I_{(Z, o)} \subset O_{(k^N, o)}\) is generated by \(\{f_i\}, \{g_i\}\). And this is a regular sequence.) Then \((X, o)\) is s.c.i. over \((Z, o)\). Indeed, take a basis of \(I_{(Z, o)} \subset O_{(k^N, o)}\) as above, \(\{f_j\}, \{g_i\}\). Then \(\tilde{f}_1, \ldots, \tilde{f}_r \subset O_{(Z, o)}[y_1, \ldots, y_k]\) that generates \(gr_I_{(Z, o)} I_{(X, o)}\).

If \((X, o)\) is smooth and \((Z, o) \subset (k^N, o)\) is a reduced complete intersection the condition “\((Z, o) \subset (X, o)\) is also a complete intersection” holds automatically. More generally, this often holds when \((Z, o) \subset \text{Sing}(X, o)\).

Usually we assume \((Z, o) \subset \text{Sing}(X, o)\).

**Example 2.14.** Let \(I_{(X, o)} = (x^2 y + y^2 + z^4, y^2 w + x^2)\) and \(I_{(Z, o)} = (x, y, z)\). Here \((Z, o) = \text{Sing}(X, o)\). Then \(gr_I_{(Z, o)} I_{(X, o)} = (x^2 y + y^2 w, x^2 + y^2) \subset O_{(k^4, o)[x, y, z]}\) is not a complete intersection. Note though that \(gr_I_{(Z, o)} I_{(X, o)}\) is generically complete intersection along \((Z, o)\).

Example 2.10 shows that “being s.c.i. at \(o\)" does not imply “being s.c.i. over \((Z, o)\)”. The converse implication does not hold either:

**Example 2.15.** Fix any complete intersection \((Z, o) \subset (k^N, o)\) and some basis \(I_{(Z, o)} = (g_1, \ldots, g_k)\). Let \(f_1, g_1, \ldots, f_r, g_r \in (y_1, \ldots, y_k)\) be some homogeneous polynomials, \(r \leq k\), such that \((f_1, \ldots, f_r)\) is a regular sequence in \(k[y_1, \ldots, y_k]\). Consider the ideal \(I_{(X, o)} = (f_1, g_1, \ldots, f_r, g_r, \ldots, f_r, g_r)\).

Then \((X, o)\) is a complete intersection at \(o\) and s.c.i. over \((Z, o)\). But in general \((X, o)\) is not s.c.i. at \(o\). As a particular example, let \(I_{(Z, o)} = (x^2 + z^3, x y + z^3)\), see example 2.8 and take \(f_1 = t_1, f_2 = t_2\), i.e. \(I_{(X, o)} = I_{(Z, o)}\). Then \(gr_I_{(Z, o)} I_{(X, o)} = (y_1, y_2) \subset O_{(Z, o)[g_1, y_2]}\), thus \((X, o)\) is s.c.i. over \((Z, o)\). But \((X, o)\) is not s.c.i. at the origin.

**Example 2.16.** A warning: even if \((Z, o)\) is a s.c.i. at \(o\), \((Y, o)\) is smooth, and the intersection \((X \cap Y, o)\) is proper, it can happen that \((X \cap Y, o)\) is in general not a s.c.i. at \(o\). For example, let \((X, o) = \{x^2 y + z^3 + h.o.t. = 0 = y^3 + z^3 + h.o.t.\} \subset (k^3, o)\), where h.o.t. denote some elements of \((x, y, z)^4\). Let \((Y, o) = \{z = 0\}\), then \((X \cap Y, o) = \{z = x^2 y + h.o.t. = 0 = y^3 + h.o.t.\} \subset (k^3, o)\) is not a s.c.i.
2.6. Good bases and multiplicity sequences. Let \((Z, o)\) be a reduced complete intersection and \((X, o)\) be s.c.i. over \((Z, o)\). By Proposition 2.12 we can choose some regular sequence of generators, \(f_1, \ldots, f_r \in I_{(X, o)}\), whose leading terms form a (graded) basis \(\{f_i\}\) of \(gr(I_{(Z, o)})I_{(X, o)}\). We can assume in addition:

\[
\text{ord}_{(Z, o)}(f_1) \leq \text{ord}_{(Z, o)}(f_2) \leq \cdots \leq \text{ord}_{(Z, o)}(f_r).
\]

By the construction: \(\text{ord}_{(Z, o)}(f_i) = \text{ord}_{(Z, o)}(f_i)\). Recall that for complete intersections the ordinary and symbolic powers coincide, \(I_{(Z, o)}^{m} = I_{(Z, o)}^{m}\). Then \(\text{ord}_{(Z, o)}(f_i)\) is the generic order of vanishing of \(f_i\) along \((Z, o)\), in particular it is the same on all the components of \((Z, o)\). Indeed, (cf. example 2.10) let \((Z, o) = \bigcup J, o\) be the irreducible decomposition, so that \(f_i \in \bigcap I_{(Z, o)}^{(\text{ord}_{(f_i)})}\). If \(f_i \in I_{(Z, o)}^{(\text{ord}_{(f_i)})+1}\) holds for some \(j\), then \(\tilde{f}_i \in I_{(Z, o)}^{(\text{ord}_{(f_i)})}/I_{(Z, o)}^{(\text{ord}_{(f_i)})+1} \subset gr(I_{(Z, o)}(R))\) is a zero divisor, contradicting regularity of the sequence \(\tilde{f}_1, \ldots, \tilde{f}_r\).

**Definition 2.17.** The sequence \(f_1, \ldots, f_r\), as in equation (15), is called a good basis of \(I_{(X, o)}\). The sequence of integers \(\text{ord}_{(Z, o)}(f_1), \text{ord}_{(Z, o)}(f_2), \ldots, \text{ord}_{(Z, o)}(f_r)\) is called the multiplicity sequence of \((X, o)\) along \((Z, o)\), associated with \(\{f_i\}\).

**Example 2.18.** (The case: \((Z, o)\) is a point.) Let \((X, o) \subset (k^{n+r}, o)\) be a s.c.i. with multiplicity sequence \((p_1, \ldots, p_r)\). Blowup at \(o\), see the diagram, then \(\tilde{X} \cap E \subset E \approx C_{0,1} = (k^{n+r}, o) \cap E \subset (k^{n+r}, o)\).

Even though the good basis is never unique, the multiplicity sequence is well defined.

**Proposition 2.19.** 1. The multiplicity sequence of \((X, o)\) along \((Z, o)\) does not depend on the choice of bases of \(I_{(X, o)}, \tilde{I}_{(X, o)}\).

2. The product \(\prod \text{ord}(f_i)\) equals the generic multiplicity of \((X, o)\) along \((Z, o)\). In particular, it is the same on all the components of \((Z, o)\).

**Proof.** We decompose \(I_{(Z, o)} = \cap p_i\) and localize at \(p_i\). We get the local ring \((O_{(k^{n+r}, o)})_{p_i}\), with the maximal ideal \((I_{(Z, o)})_{p_i}\) and elements \(\{f_j \in (I_{(Z, o)})_{p_i}^{\text{ord}(f_j)}\}\). So the situation is reduced to the case when \((Z, o)\) is a point. Suppose \((Z, o) = \emptyset\), then \(\{f_i\}\) is the graded basis of the tangent cone, \(I_{(X, o)}\). Thus the uniqueness of the multiplicity sequence (up to permutation) follows from the fact that any two graded bases of the graded ideal \(I_{(X, o)}\) are related by a graded(!) invertible linear map.

To compute the multiplicity of \((X, o)\), at \(o\), let \(l_1, \ldots, l_n \in m \subset (O_{(k^{n+r}, o)})\) be some generic elements. Geometrically they define a smooth subspace transversal and complementary to \((X, o)\). The quotient ring \((O_{(k^{n+r}, o)})_{(l_1, \ldots, l_n, f_1, \ldots, f_r)}\) is still a strictly complete intersection. Therefore we can assume from the beginning \(n = 0\), i.e. \((X, o)\) is a one-point scheme and \(\text{mult}(X, o) = \dim (O_{(k^{n+r}, o)})_{(f_1, \ldots, f_r)}\). But then the multiplicity can be computed by blowing up. And the total transform of \(o\) is a complete intersection of multidegree \(m_1, \ldots, m_r\), thus of degree \(\prod_{i=1}^{r} m_i\).

**Example 2.20.** (Behavior of multiplicity sequence in a family.) Let \((X, o)\) be a s.c.i. over \((Z, o)\) with a good basis \(\{f_1, \ldots, f_r\}\). Consider a deformation \(\{f_1 + \epsilon g_1, \ldots, f_r + \epsilon g_r\}_{\epsilon \in (k, 0)}\) that preserves the multiplicity sequence (generic multiplicity over \((Z, o)\), \(\text{mult}(g_i) \geq \text{mult}(f_i)\). Then, by the openness of regularity in deformations, the generic member of this family is a s.c.i. over \((Z, o)\). (Indeed, any relation \(\sum r_i(f_i + \epsilon g_i) = 0\) leads to the relation \(\sum r_i(f_i + \epsilon g_i) = 0\), which is necessarily Koszul. Subtract this relation from the initial one, to get \(r_i|_{\epsilon = 0} = 0\), i.e. \(r_i\) is divisible by \(\epsilon\). Divide all \(r_i\) by \(\epsilon\) and proceed in the same way. The statement then follows by Nakayama-type argument.)

If the multiplicities are not preserved then a flat deformation of s.c.i. is not s.c.i. For example, the family of ideals \(I = (x^3, tx^2 + y^4) \subset (k[x, y, t])\) defines s.c.i. at the origin for \(t = 0\) but not for \(t \neq 0\).

2.7. Singularities generically ordinary along \((Z, o)\). Recall that an isolated hypersurface singularity, \(\{f_p + f_{p'} = 0\} \subset (k^{n+1}, o)\), is called an ordinary multiple point if its projectivized tangent cone, \(\{f_p = 0\} \subset P^n\), is smooth. In the case \(k = \mathbb{C}\) this can also be stated as follows: the hypersurface germ is topologically equisingular to \(\{\sum x_i^p = 0\} \subset (\mathbb{C}^{n+1}, o)\). This is “the lowest/simplest” hypersurface singularity of a given multiplicity. Similarly, among the s.c.i. germs of a given multiplicity at \(o\), the “lowest” is the one whose projectivized tangent cone is a smooth complete intersection.

Let \((Z, o)\) be reduced, \(dim(Z, o) > 0\), though \((Z, o)\) is not necessarily a complete intersection or pure dimensional. Then the primary decomposition contains only prime ideals, \(I_{(Z, o)} = \cap p_i\).
Definition 2.21. \((X, o)\) is called generically ordinary along \((Z, o)\) if for any \(i\) the localization \((gr_{(Z, o)}I_{(X, o)})(p_i)\) is a complete intersection over \(\text{Spec}(\mathcal{O}_{(Z, o)}(p_i))\), whose projectivization is smooth.

Example 2.22. If \((Z, o) \not\subseteq \text{Sing}(X, o)\) and is irreducible then \((X, o)\) is generically smooth along \((Z, o)\), in particular \((X, o)\) is generically ordinary along \((Z, o)\). However, in this case \(\text{ord}_{(Z, o)}(f_i) = 1\), as will be shown in example 5.23 the discriminant is empty. Therefore we assume \((Z, o) \subseteq \text{Sing}(X, o)\).

Example 2.23. \(\bullet\) Let \(I_{(X, o)} = (x^p z + y^{p+1}), p \geq 2,\) and \(I_{(Z, o)} = (x, y)\). Then \(gr_{(Z, o)}I_{(X, o)} = (x^p z) \subset \mathcal{O}_{(Z, o)}[x, y]\) and \((gr_{(Z, o)}I_{(X, o)})(x, y) = (x^p) \subset (\mathcal{O}_{(Z, o)}[x, y])(x, y)\). Thus \((X, o)\) is not generically ordinary along \((Z, o)\).

\(\bullet\) The hypersurface \(x^p z^q + y^p + x^{p+1} = 0\) is generically ordinary along \((Z, o) = \{x = 0 = y\}\).

\(\bullet\) Let \(I_{(X, o)} = (x^2 z + y^2 w + x^4 + y^4)\) and \(I_{(Z, o)} = (x, y)\). Then \(gr_{(Z, o)}I_{(X, o)} = (x^2 z + y^2 w)\) and
\[
(\text{gr}_{(Z, o)}I_{(X, o)})(x, y) = (x^2 z + y^2 w) \subset (\mathcal{O}_{(Z, o)}[x, y])(x, y).
\]

This is a hypersurface and its projectivization is smooth. Thus \((X, o)\) is generically ordinary along \((Z, o)\).

Example 2.24. \(\bullet\) Let \(I_{(X, o)} = (x^2 w + y^2 + z^4, y^2 w + x^4)\) thus \(I_{(Z, o)} = (x, y, z)\). Then \(gr_{(Z, o)}I_{(X, o)} = (x^2 w, y^2 w, x^6 - y^2(y^4 + z^4))\) and \((gr_{(Z, o)}I_{(X, o)})(x, y, z) = (x^2, y^2) \subset (\mathcal{O}_{(Z, o)}[x, y, z])(x, y, z)\). This is a complete intersection, but its projectivization is not a smooth subscheme of \(\mathbb{P}^2_{K}\), here \(K\) is the fraction field of \(\mathcal{O}_{(Z, o)}\).

\(\bullet\) Let \(I_{(X, o)} = (x^2 + y^2 + z^2 + h_1(x, y, z), xyw + h_2(x, y, z))\), here the orders of \(h_1, h_2\) are \(\geq 4\), and \(I_{(Z, o)} = (x, y, z)\). Then
\[
(14) \quad \text{gr}_{(Z, o)}I_{(X, o)} = (x^2 + y^2 + z^2)w, xyw, xyh_1 - (x^2 + y^2 + z^2)h_2),
\]
and
\[
(\text{gr}_{(Z, o)}I_{(X, o)})(x, y, z) = (x^2 + y^2 + z^2)w \subset (\mathcal{O}_{(Z, o)}[x, y, z])(x, y, z).
\]

This is a complete intersection and its projectivization is a smooth subscheme of \(\mathbb{P}^2_{K}\). Thus \((X, o)\) is generically ordinary over \((Z, o)\). Note that \((X, o)\) is not s.c.i. at the origin.

3. The classical discriminant of projective complete intersections

While there are many extensive treatments of the discriminant of projective hypersurfaces in \(\mathbb{P}^N\), see e.g. [G.K.Z.], we do not know any textbook or lecture notes on the discriminant of projective complete intersections.

However, for some particular results on the classical discriminant of projective complete intersections see [Esterov2011], [Benoist2012], [C.C.D.R.S.2011]. In particular, in many cases the multi-degrees were computed.

In [Teissier1976], [Looijenga-book] one treats mostly the local case. See also [AGLV-book2, §1.2.2] for a collection of known local facts.

In this section we (re)prove some of the standard needed results.

3.1. The critical locus and the discriminant of a map. Let \(X \xrightarrow{\pi} S\) be a flat map of (algebraic/analytic/formal) spaces, with fibres of pure dimension \(d\). Then \(\text{Crit}(\pi)\) is defined, see e.g. [Teissier1976] pg.587, by the (coherent) sheaf of ideals \(I_{\text{Crit}(\pi)} := \text{Fitt}_d(\Omega^1_{X/S})\). Here \(\Omega^1_{X/S} = \Omega^1_X/\pi^*\Omega^1_S\) is the sheaf of relative differentials, while \(\text{Fitt}_d(\ldots)\) is the \(d\)th Fitting ideal of an \(\mathcal{O}_X\)-module, [Eisenbud-book, §20].

Suppose the restriction \(\pi|_{\text{Crit}(\pi)}\) is finite. The discriminant of \(\pi\) is defined as the image, \(\Delta_\pi := \pi(\text{Crit}(\pi)) \subset S\), with the Fitting scheme structure, \(I_{\Delta_\pi} := \text{Fitt}_0(\pi_*\mathcal{O}_{\text{Crit}(\pi)})\). [Teissier1976] pg. 588. Here \(\pi_*\mathcal{O}_{\text{Crit}(\pi)}\) is the pushforward of the \(\mathcal{O}_X\)-module \(\mathcal{O}_{\text{Crit}(\pi)}\), while \(\text{Fitt}_0(\ldots)\) is the minimal Fitting ideal of a module, as an \(\mathcal{O}_S\) module, i.e. the ideal of maximal minors of a presentation matrix of the module.

3.2. Assumptions on the base of the family. Consider the family of complete intersections, \(X = \{F_1(x, s) = F_2(x, s) = \cdots = F_r(x, s) = 0\} \subset \mathbb{P}^{n+r} \times S\).

We denote the fiber in \(X\) over the point \(s \in S\) by \(X_s \subset \mathbb{P}^{n+r}\). We have the natural projection \(X \xrightarrow{\pi} S\).

We assume:

\(\bullet\) \(S\) is quasi-projective, smooth, connected.

\(\bullet\) The generic fibre over \(S\) is a smooth complete intersection in \(\mathbb{P}^{n+r}\) and the family \(X \subset S \times \mathbb{P}^{n+r}\) is smooth.

\(\bullet\) Denote by \(\Delta \subset S\) the subset of points whose fibres are singular or not of expected dimension. Denote by \(\Delta_{A_i} \subseteq \Delta \subset S\) the subset of points corresponding to fibres with just one node. Then we assume that \(\Delta_{A_i}\) is dense in \(\Delta\) and is connected in Zariski topology.

For hypersurfaces of degree \(p\) in \(\mathbb{P}^{n+1}\) the standard parameter space is \(|\mathcal{O}_{\mathbb{P}^{n+1}}(p)|\). For complete intersections of multi-degree \((p_1, \ldots, p_r)\) in \(\mathbb{P}^{n+r}\) one can consider the multi-projective space \(\prod_i |\mathcal{O}_{\mathbb{P}^{n+r}}(p_i)|\). To a point of this space, \((f_1, \ldots, f_r) \in \prod_i |\mathcal{O}_{\mathbb{P}^{n+r}}(p_i)|\), corresponds the subscheme \(X_{\underline{f}} = \{f_1 = \cdots = f_r = 0\} \subset \mathbb{P}^{n+r}\). These subschemes are projective complete intersections when the polynomials \(\{f_i\}\) form a regular sequence. Thus there exists a
Zariski open subset \( U \subset \prod_i |\mathcal{O}_{\mathbb{P}^{n+r}}(p_i)| \), whose points correspond to projective complete intersections. (Note that the complement, \( \prod_i |\mathcal{O}_{\mathbb{P}^{n+r}}(p_i)| \setminus U \), is of high codimension.) This is the reason to consider \( S \) as a parameter space for globally complete intersections, even though for \( r > 1 \) the correspondence \( S \ni f \mapsto X_f \subset \mathbb{P}^{n+r} \) is far from being injective.

3.3. An example: the critical locus in the case \( S = \prod_i |\mathcal{O}_{\mathbb{P}^{n+r}}(p_i)| \). The critical locus of \( \pi \) is defined (as in §3.1) by the sheaf of ideals \( \text{Fitt}_n(\Omega^1_{X/\mathcal{S}}) \subset \mathcal{O}_X \). We write down the generators \( \text{Fitt}_n(\Omega^1_{X/\mathcal{S}}) \) explicitly.

Fix some points \( x \in \mathbb{P}^{n+r} \), \( s \in S \) and work locally near these points, with the local coordinates \( x = (x_1, \ldots, x_{n+r}) \), \( s = (s_1, \ldots, s_t) \). We work with modules and then glue them to sheaves. One has

\[
\Omega^1(X/(x,s)) = \mathcal{O}(x,s)((dx_1, \ldots, dx_{n+r}, ds_1, \ldots, ds_r))/dF, \quad \pi^*\Omega^1(S,s) = \mathcal{O}(x,s)(dx),
\]

where the differentials in \( dF \) are taken with respect to both variables, \( (x, s) \). Therefore

\[
\Omega^1(X/S,(x,s)) = \mathcal{O}(x,s)((dx_1, \ldots, dx_{n+r}))/\{\sum_i \partial F_i/dx_i\}_{j=1,\ldots,r}.
\]

The \( \mathcal{O}(x,s) \)-resolution of this module begins as

\[
\to \mathcal{O}^{(n+r)}(x,s)((\partial F_i)_{ij}) \mathcal{O}^{(n+r)}(x,s) \to \Omega^1(X/S) \to 0.
\]

Therefore the ideal \( \text{Fitt}_n(\Omega^1_{X/\mathcal{S}}) \subset \mathcal{O}(x,s) \) is defined by all the \( r \times r \)-minors of the matrix

\[
\begin{pmatrix}
\partial_1 F_1 & \ldots & \partial_{n+r} F_1 \\
\ldots & \ldots & \ldots \\
\partial_1 F_r & \ldots & \partial_{n+r} F_r
\end{pmatrix}.
\]

To get the sheaf of ideals \( I_{\text{Crit}(\pi)} \subset \mathcal{O}_X \) we pass from the local coordinates of \( \mathbb{P}^{n+r} \) to the homogeneous coordinates \([x_0 : \cdots : x_{n+r}] \in \mathbb{P}^{n+r} \). Using Euler's formula \( \sum_{i=0}^{n+r} x_i \partial_i F_j = p_j F_j \) we get on \( X \): the rows of the matrix in equation (17) are linearly dependent iff the extended rows of derivatives in homogeneous coordinates, \( \partial_0 F_1 \ldots \partial_{n+r} F_j \), are linearly dependent. Therefore the explicit equations of the critical locus are:

\[
\text{Crit}(\pi) = \{ F_1(x) = \cdots = F_r(x) = 0, \text{ rank } \begin{pmatrix}
\partial_0 F_1 & \ldots & \partial_{n+r} F_1 \\
\ldots & \ldots & \ldots \\
\partial_0 F_r & \ldots & \partial_{n+r} F_r
\end{pmatrix} < r \} \subset \mathbb{P}^{n+r} \times S.
\]

3.4. The discriminant as the pushforward of the critical locus. Usually the projection \( \text{Crit}(\pi) \to S \) is not flat over its image. More precisely, it is generically finite over its image (with varying degrees of fibres) but the fibres over some points can be of positive dimension (when \( S \) contains points whose fibres \( X_s \) have non-isolated singularities or are not of expected dimension). Yet this projection is proper everywhere, thus the pushforward \( \pi_*(\mathcal{O}_{\text{Crit}(\pi)}) \) is a coherent sheaf of \( \mathcal{O}_S \) modules.

We work in the assumptions of §3.2.

Definition 3.1. The (classical) discriminant \( \Delta \subset S \) of complete intersections is the closure of a (algebraic) subscheme, which is defined by the zero Fitting ideal, \( \text{Fitt}(\pi_*(\mathcal{O}_{\text{Crit}(\pi)})) \) at points where \( \pi \) is finite.

Example 3.2. Suppose the multi-degree is \( p_1 = \cdots = p_r = 1 \). Then every fiber of \( X \to S \) is smooth, in particular \( \Delta_{A_1} = \emptyset \). Therefore, according to our definition, \( \Delta = \emptyset \).

Proposition 3.3. 1. (Set theoretically) A point \( s \in S \) belongs to \( \Delta \) iff the subscheme \( X_s \subset \mathbb{P}^{n+r} \) is singular or not of expected dimension.

2. \( \Delta \subset S \) is a reduced irreducible Cartier divisor. The germ \( (\Delta, s) \) is smooth iff the fibre \( X_s \) has just one singularity of type \( A_1 \).

Proof. 1. It is enough to check only the points of \( \Delta \) over which \( \pi \) is finite. (Indeed, the fibre over a point of \( \Delta \) added by the closure procedure is the limit of singular fibres, hence cannot be a smooth variety of expected dimension.)

For the points where \( \pi \) is finite, it is enough to check the support of the module \( \Omega^1_{X/S} \). Note that the presentation of equation (10) holds locally for any \( S \), and for a fixed \( s \in S \) the minors of (17) define the singular set of \( X_s \). This proves the statement.

2. Recall that \( S \) is smooth and \( \Delta \) is defined as the closure of a scheme well-defined on an open set above which \( \pi \) is finite. Thus, to establish that \( \Delta \) is a reduced Cartier divisor, it is enough to check only those points, where \( \pi \) is finite.

We should prove that the defining ideal \( I_\Delta = \text{Fitt}(\pi_*(\mathcal{O}_{\text{Crit}(\pi)})) \subset \mathcal{O}_S \) is locally principal at each point.
Note that $\mathcal{X}$ is smooth and $\text{Crit}(\pi) \subset \mathcal{X}$ is a determinantal subscheme. Therefore $\text{Crit}(\pi)$ is a Cohen-Macaulay subscheme, [Eisenbud-book Theorem 18.18]. As $(S, s)$ is smooth, the module $\pi_*\langle O_{\text{Crit}(\pi)} \rangle$ has a finite projective dimension and we use the Auslander-Buchsbaum formula, [Eisenbud-book theorem 19.9]:

$$\text{proj.dim}(M) + \text{depth}(M) = \text{dim}(R).$$

Since $\pi$ is finite, $M = \pi_*\langle O_{\text{Crit}(\pi)} \rangle$ is a Cohen-Macaulay module over $O_{(S, s)}$, i.e. $\text{depth}(M) = \text{dim}(R) - 1$. Therefore the minimal resolution of $\pi_*\langle O_{\text{Crit}(\pi)} \rangle$ is of length one,

$$0 \to O_{(S, s)}^\oplus N \to \pi_*\langle O_{\text{Crit}(\pi)} \rangle \to 0.$$

Thus the presentation matrix is square and its Fitting ideal is principal.

As $\Delta \subset S$ is Cartier, to prove reducedness it is enough to find just one reduced point. Suppose $X_{s_o}$ has $A_1$-singularity at a point $x_o \in \mathbb{P}^{n+r}$, here $s_o \in S$. Then in some local coordinates the defining equations of $(\mathcal{X}, (x_o, s_o))$ are: $F_1 = x_1, \ldots, F_{r-1} = x_{r-1}, F_r = \sum_{i=r}^{n+r} x_i^2 - g(s),$ where $g(s_o) = 0$. As $(\mathcal{X}, (x_o, s_o))$ is smooth, $s_o$ is not a critical point of $g(s)$. Thus, we can choose $g(s)$ as one of the local coordinates, denote it $s_1$.

Then using (16) we get: $\text{Fit}_0(\langle \Omega^1_{\mathcal{X}/S} \rangle) = (x_r, \ldots, x_n) \subset \langle O_{(x_r, s_o)} \rangle,$ i.e. in the chosen coordinates $\text{Crit}(\pi) = V(x_1, \ldots, x_n, s_1) \subset (k^{n+r}, x_0) \times (S, s_o).$ Therefore $\langle O_{\text{Crit}(\pi)} \rangle \cong k[[s_1, \ldots, s_r]]/(s_1)$ and $\text{Fit}_0(\pi_*\langle O_{\text{Crit}(\pi)} \rangle) = (s_1).$ This ideal defines a reduced, smooth germ $(\Delta, s) \subset (S, s)$. Suppose $(X_1)$ has a singularity other than $A_1$ then the local length of $\langle O_{\text{Crit}(\pi)} \rangle$ at this point is at least two. Thus the germ $(\Delta, s)$ is singular.

Suppose the singular points of $X_s$ are $\{p_i\}$, then $\text{Fit}_0(\pi_*\langle O_{\text{Crit}(\pi)} \rangle) = \prod \text{Fit}_0(\pi_*\langle O_{\text{Crit}(\pi), p_i} \rangle),$ hence the local multiplicity of $(\Delta, s)$ is at least the number of these points. Thus, if $X_s$ has more than one singular point then $(\Delta, s)$ is singular.

Finally, we know that $\Delta$ is a reduced divisor, thus to prove the irreducibility it is enough to check that the space $\Delta \setminus \text{Sing}(\Delta)$ is connected. But $\Delta \setminus \text{Sing}(\Delta) = \Delta_{A_1}$, the open set of points corresponding to complete intersections with just one node and $\Delta_{A_1}$ is connected.

3.5. **Discriminant as a dual variety.** For a fixed tuple $p_1, \ldots, p_r$ consider the multi-Veronese embedding,

$$\mathbb{P}^{n+r} \overset{\nu_{j}}{\rightarrow} \prod_{j=1}^{r} \langle O_{\mathbb{P}^{n+r}}(p_j) \rangle^\vee, \quad \nu_j(x_0, \ldots, x_{n+r}) = \{x_0^{m_{j,0}} \cdots x_{n+r}^{m_{j,n+r}} \sum_{m_{j,i} \geq 0} m_{j,i} = p_j \} \subset \langle O_{\mathbb{P}^{n+r}}(p_j) \rangle^\vee.$$ 

A hyperplane in $\langle O_{\mathbb{P}^{n+r}}(p_j) \rangle^\vee$ corresponds to a choice of coefficients $\{a_{j, m_{j,i}} \}$ with $\sum_{m_{j,i} \geq 0} m_{j,i} = p_j$, (up to $k^*$-action), i.e. to a choice of the hypersurface $\{f_j(x) = 0\} \subset \mathbb{P}^{n+r}$. Pullback this hyperplane under the projection $\prod \langle O_{\mathbb{P}^{n+r}}(p_j) \rangle^\vee \to \langle O_{\mathbb{P}^{n+r}}(p_j) \rangle^\vee$ and denote the resulting hyperplane by $L_j$. Thus we have $r$ hyperplanes and the intersection $L_1 \cap \cdots \cap L_r \cap \{\nu_1, \ldots, \nu_r\}(\mathbb{P}^{n+r})$ defines the subscheme $\cap \{f_j(x) = 0\} \subset \mathbb{P}^{n+r}$.

This subscheme is a smooth complete intersection (of codimension $r$) iff the intersection is transversal. Thus $(f_1, \ldots, f_r) \in \prod \{O_{\mathbb{P}^{n+r}}(p_j)\}^\vee$ belongs to the discriminant iff $\cap L_j$ is either tangent to $(\nu_1, \ldots, \nu_r)(\mathbb{P}^{n+r})$ or intersects it non-properly, i.e. the resulting codimension is smaller than $r$. Thus $\Delta$ is the dual variety of the embedding $(\nu_1, \ldots, \nu_r)(\mathbb{P}^{n+r})$. In particular it is a hypersurface, i.e. a Cartier divisor.

To relate this definition to the definition in [4,4], we note that for a regular sequence $(f_1, \ldots, f_r)$ the deformation $(f_1(\epsilon), \ldots, f_r(\epsilon))$ is flat iff each $f_j(\epsilon)$ is flat. And any tuple can be deformed to a tuple $(f_1(\epsilon), \ldots, f_r(\epsilon))$ defining a complete intersection with isolated singularities. Therefore the full projective discriminant can be obtained as the (Zariski) closure:

$$\Delta = \{ \text{tuples } (f_1, \ldots, f_r), \text{defining complete intersections of expected codimension, with isolated singularities} \} \subset \prod \{O_{\mathbb{P}^{n+r}}(p_j)\}.$$

In particular it is irreducible and reduced and coincides with the discriminant of [4,4].

3.6. **The transversal multiplicity of the discriminant.** Given a complete intersection germ $(X, o) \subset (k^{n+r}, o)$ (with isolated singularity), choose some generic basis $f_1, \ldots, f_r \in I(X, o)$. Then, besides the ordinary Milnor number, $\mu(X, o)$, we define the auxiliary number $\mu'(X, o) = \mu(f_2 = 0 = \cdots = f_r)$. (For $r = 1$ we put $\mu' = 0$.) This $\mu'$ is well defined and depends on $(X, o)$ only. (By the genericity of the basis one could omit any $f_i$ instead of $f_1$.) Accordingly for any (global) variety $X$ with ICIS, we define the total Milnor/auxiliary numbers,

$$\mu_{\text{total}} = \sum_{o \in \text{Sing}(X)} \mu(X, o), \quad \mu'_{\text{total}} = \sum_{o \in \text{Sing}(X)} \mu'(X, o).$$
Proposition 3.4. Suppose the projection \((X, (x, s)) \supset \text{Crit}(\pi) \to \Delta \subset (S, s)\) is finite at \(s \in \Delta\) and \((S, s)\) contains the (germ of) miniversal deformation of \(X_s\). Then \(\text{mult}(\Delta, s) = \mu_{\text{total}}(X_s) + \mu'_{\text{total}}(X_s)\).

Proof. Fix a smooth curve germ, \((C, s) \subset (S, s)\), whose tangent line does not belong to the tangent cone \(T_{(\Delta, s)}\), then \(\text{mult}(\Delta, s) = \deg((C, s) \cap (\Delta, s))\). The later degree is computed by restriction of \((\Delta, s)\) onto \((C, s)\). By the base-change properties of Fitting ideals (i.e. the right exactness of tensor product) we have

\[
\left( \text{Fitto}(\pi_* \mathcal{O}_{\text{Crit}(\pi)}) \right)_{|(C, 0)} = \text{Fitto}(\pi_* (\mathcal{O}_{\text{Crit}(\pi)})_{|(C, 0)}).
\]

Furthermore, \(\pi_* (\mathcal{O}_{\text{Crit}(\pi)})_{|(C, 0)} = \pi_* (\mathcal{O}_{\text{Crit}(\pi) \times^{-1}(C, 0)})\). Therefore we can assume \((S, s)\) a smooth curve-germ. The module \(\pi_* (\mathcal{O}_{\text{Crit}(\pi)})\) is then a skyscraper at \(s\) and the degree of \(\text{Fitto}(\pi_* \mathcal{O}_{\text{Crit}(\pi)})\) equals the length of the module \(\pi_* (\mathcal{O}_{\text{Crit}(\pi)})\). Finally,

\[
\text{length}(\pi_* (\mathcal{O}_{\text{Crit}(\pi)})) = h^0(\mathcal{O}_{\text{Crit}(\pi)}, \text{Crit}(\pi)) = \sum_{pt \in \text{Sing}(X_s)} h^0(\mathcal{O}_{\text{Crit}(\pi), pt}).
\]

To compute \(h^0(\mathcal{O}_{\text{Crit}(\pi), pt})\) we write the local presentation:

\[
\mathcal{O}_{(\text{Crit}(\pi), pt)} = \mathcal{O}_{(S, o)}[[x]]/\{F_i(x, s) + a_i s \}_{i=1, \ldots, r}, \text{Fitto}(dF_i)\).
\]

Here \(s\) is the local parameter of the curve \((S, s)\), the constants \(a_i \in k\) are generic because the curve \((C, s)\) is not tangent to the discriminant. The differentials \(dF_i\) are taken with respect to \(x\)-variables only.

Write \(F_1(x, s) = h_1(x) + s(a_1 + G(x, s))\), with \(G(x, s) \in (x)\). Then, by Nakayama lemma, we can eliminate \(s\) and write:

\[
\mathcal{O}_{(\text{Crit}(\pi), pt)} = k[[x]]/(h_2, \ldots, h_r, \text{Fitto}
\]

\[
\begin{pmatrix}
  dh_1 \\
  \vdots \\
  dh_r
\end{pmatrix}.
\]

Here \(h_1, \ldots, h_r\) are some generators of \(I_{(X, o)} \subset k[[x]]\). They are generic, as the constants \(\{a_i\}\) are generic. Finally we use the Lé-Greuel formula, \cite{Greuel}:

\[
\mu(h_1, \ldots, h_r) + \mu(h_2, \ldots, h_r) = \text{dim } k[[x]]/(h_2, \ldots, h_r, \text{Fitto}
\]

\[
\begin{pmatrix}
  dh_1 \\
  \vdots \\
  dh_r
\end{pmatrix}.
\]

Therefore

\[
\sum_{pt \in \text{Sing}(X_s)} h^0(\mathcal{O}_{\text{Crit}(\pi), pt}) = \sum_{pt \in \text{Sing}(X_s)} (\mu(X_s, pt) + \mu'(X_s, pt)) = \mu_{\text{total}}(X_s) + \mu'_{\text{total}}(X_s). \quad \blacksquare
\]

Remark 3.5. In the last proposition \((S, s)\) was assumed “large enough”. Often the deformation space is rather small then the statement should be corrected. Consider a particular case, \((S, s)\) being just one dimensional, with \(X \subset (k^{n+r}, o) \times (S, s)\) defined by \(I_X = \{f_1(x), \ldots, f_j-1(x), f_j(x) + s, f_j+1(x), \ldots, f_r(x)\}\). Here we do not assume \(\{f_i\}\) to be generic, but we assume that both \(\{f_j\}_{j \neq 1}\) and \(\{f_j\}_{j \neq i}\) define isolated (complete intersection) singularities. In this case, instead of the invariant \(\mu'(X, o)\), we define the invariant

\[
\mu_j := \mu(f_1(x), \ldots, f_j-1(x), f_j(x) + s, f_j+1(x), \ldots, f_r(x)).
\]

Then the same proof of proposition \cite[3.14]{Teissier} gives: \(\text{mult}(\Delta, s) = \mu(X_s, o) + \mu_j(X_s, o)\). This formula is well known, see e.g. \cite[page 589]{Teissier}.

4. The discriminant of transversal singularity type, \(\Delta^\perp\)

4.1. The definition of \(\Delta^\perp \subset Z\) as a pullback of the classical discriminant.

4.1.1. The local case. Let \((X, o) \subset (k^{n+r}, o)\) with \(\text{Sing}(X, o) = (Z, o)\) a reduced complete intersection. Suppose \((X, o)\) is s.c.i. over \((Z, o)\). Fix a good basis, \(I_{(X, o)} = \{f_1, \ldots, f_r\}\), such that the leading terms of \((f_i)\) form a basis of \(\text{gr}(Z, o) I_{(X, o)}\), as in \cite[2.6]{Kerner}. Let the generic multiplicity of \(f_j\) along \((Z, o)\) be \(p_j\). Fix some basis \(I_{(Z, o)} = \{g_1, \ldots, g_k\}\). Projectivize the normal cone to get the family:

\[
\mathbb{P}(\mathcal{N}_{(X, o)}/(Z, o)) = \prod_{j=1}^r \{ \sum_{m_1 + \cdots + m_k = p_j} g_1^{m_1} \cdots g_k^{m_k} a^{(j)}_{m_1, \ldots, m_k} = 0 \} \subset (Z, o) \times \text{Proj}(k[y_1, \ldots, y_k]).
\]
Let $\Delta$ be the classical discriminant in the parameter space of projective complete intersections in $\mathbb{P}^{k-1}$ of codimension $r$ and multidegree $(p_1, \ldots, p_r)$, see [33]. It is a hypersurface, defined by one equation, $\Delta = \{D = 0\}$. We assume that $(X, o)$ is generically ordinary along $(Z, o)$, see [27], thus $D(\{a_{m_1, \ldots, m_k}^{(j)}\sum_{1 \leq j \leq r} m_i = p_j\} \neq 0$.

We define the Cartier divisor $(\Delta^\perp, o) \subset (Z, o)$ by the principal ideal
\[(\Delta^\perp, o) := \{D(\{a_{m_1, \ldots, m_k}^{(j)}\sum_{1 \leq j \leq r} m_i = p_j\}) = 0\} \subset (Z, o).
\]

By the definition of $D$: $\Delta' \in \Delta^\perp$ if and only if $(X, \Delta')$ is not ordinary along $(Z, o')$. Note that $D$ is a polynomial, therefore this construction “preserves the category”: if the germs $(Z, o) \subset (X, o)$ are algebraic/analytic/formal/etc. then so is the subgerm $(\Delta^\perp, o) \subset (Z, o)$.

This definition can be restated more geometrically as follows. The choice of a good basis of $I(X, o)$ near $(Z, o)$, defines a rational map from $(Z, o)$ to the parameter space of complete intersections:

\[(Z, o) \ni o' \mapsto \left(\{a_{m_1, \ldots, m_k}^{(j)}\sum_{1 \leq j \leq r} m_i = p_j\}\right) \in \prod_{1 \leq j \leq r} |\mathcal{O}_{\mathbb{P}^{k-1}}(p_j)|.
\]

As $(X, o)$ is generically ordinary along $(Z, o)$ this map is generically well defined. Its indeterminacy locus consists of those points $o' \in Z$ where at least one of the collections of coefficients vanishes, i.e. the multiplicity of some $f_i$ jumps. The discriminant of transverse type is the pullback: $\Delta^\perp = \phi^\ast(\Delta)$.

**Proposition 4.1.** The defining ideal $I(\Delta^\perp, o) \subset \mathcal{O}(Z, o)$ is independent of all the choices made (the local coordinates in $(k^{n+1}, o)$, the basis of $I(Z, o)$, the good basis of $I(X, o)$).

Indeed, as is proved in proposition [27], the change of bases/coordinates results in the action $\mathbb{P}GL(\mathcal{O}(Z, o))(k) \subset (Z, o) \times \mathbb{P}^{k-1}$. This action is linear, it preserves the classical discriminant. Thus it does not change the defining ideal of $(\Delta^\perp, o) \subset (Z, o)$.

**Example 4.2.** In the simplest case let $(X, o) = \{(f(z) = 0) \subset (k^{n+1}, o)$ be a hypersurface singularity, with $(Z, o) = Sing(X, o)$ a complete intersection. Suppose the generic transversal type of $(X, o)$ along $(Z, o)$ is ordinary of multiplicity two. (In Siersma’s notations this is called: $A_1$-transversal type.) Then $f = \sum_{i,j=1}^n a_{ij} g_i g_j$, and we can assume that the matrix $\{a_{ij}\}_{ij}$ is symmetric. The discriminant is then $\Delta^\perp = \{det\left(\{a_{ij}\}_{ij}|(Z, o)\right) = 0\}$. Suppose $(Z, o)$ is smooth and $dim(Z, o) = 1$, i.e. $k = n$, then the generic singularity type of $(X, o)$ along $(Z, o)$ is $A_{\infty}$. Take a deformation of $(X, o)$ that preserves $Sing(X, o)$ and splits $\Delta^\perp$ into a few reduced points. Near such points the local equation of $(X, o)$ can be brought to the form $\{\sum_{i=1}^{n-2} x_i^2 + x_{n-1} x_n^2 = 0\}$, the standard notation for this singularity type is $D_{\infty}$. Then we get: the number of these $D_{\infty}$ points is the degree of the scheme $\{det\left(\{a_{ij}\}_{ij}|(Z, o)\right) = 0\}$. This recovers [Pelissian1985] theorem 7.18, see also [de Jong1990] page 176.

**Example 4.3.** Let $(X, o) = \{(xy)^p a = z^p b\} \subset (k^3, 0)$, where $a, b \in \mathcal{O}(k^3, 0)$ are invertible. Here $I(Z, o) = (xy, z)$ and $\mathbb{P}(N(X, o)/(Z, o)) = \{y_i^p a|_{(Z, o)} = y_j^p b|_{(Z, o)}\} \subset \mathbb{P}^1 \times (Z, o)$ is a hypersurface smooth over $(Z, o)$. Thus $\Delta^\perp$ is not supported at the origin, even though $(Z, o)$ is singular. Note also that the flat deformation $\{(xy + t)^p a = z^p b\} \subset (k^3, 0) \times (k^1, o)$ induces a (flat) smoothing $Z = \{Z_t\}$ of $(Z, o)$, while preserving the generic vanishing order. And for $t \neq 0$ all the fibres of $\mathbb{P}N(X_t, Z_t)$ are smooth, thus $\Delta_{t \neq 0} = \emptyset$. Compare to the flatness of $\Delta^\perp$ in deformations, proposition 5.6.

4.1.2. **The global case.** Suppose we begin from quasi-projective or analytic (for $k = \mathbb{C}$) spaces $Z \subset X$. Then the local/poinwise definition of germs $(\Delta^\perp, o) \subset (Z, o)$ globalizes. Here the germs are algebraic/analytic (i.e. all the local rings are either localizations of affine or analytic), thus we can take the representatives/open neighborhoods and glue along them.

**Proposition 4.4.** The local divisors $(\{\Delta^\perp, o \subset (Z, o)\}_{o \in Z}$ glue to the global effective Cartier divisor $\Delta^\perp \subset Z$.

**Proof.** The defining ideal of each germ $(\Delta^\perp, o)$ is principal. Thus it is enough to prove that these ideals glue to a coherent sheaf of ideals, $\Delta^\perp \subset \mathcal{O}_Z$. Namely, we should check compatibility: given a germ $(Z, o)$ with some representatives $\mathcal{U}_1 \overset{i_1}{\rightarrow} \mathcal{U}_2$, the identification of sheaves $\mathcal{O}_{\mathcal{U}_1} \cong i_1^\ast \mathcal{O}_{\mathcal{U}_2}$ induces the identity isomorphism $I_{\Delta^\perp}(\mathcal{U}_1) \cong i_1^\ast (I_{\Delta^\perp}(\mathcal{U}_2))$. And this follows as $\Delta^\perp$ does not depend on the choice of coordinates/representatives/bases of ideals, see Proposition 4.1. ■
4.2. The defining ideal of $(\Delta^\perp, o) \subset (Z, o)$. The discriminant of transversal type is defined in the last section as the pullback of the classical discriminant. It is often useful to work directly with the ideal $I_{(\Delta^\perp, o)} \subset \mathcal{O}(Z, o)$ or the sheaf $I_{\Delta^\perp} \subset \mathcal{O}_Z$. These are directly obtained using \[ \text{[3,3]} \]

Fix the complete intersections $\text{Sing}(X, o) = (Z, o) \subset (X, o)$, suppose $(X, o)$ is s.c.i. over $(Z, o)$. Fix a basis $I(Z, o) = (g_1, \ldots, g_k)$ and a good basis $I(X, o) = (f_1, \ldots, f_r)$. Then $\mathbb{P}N_{(Z, o)/(X, o)} = \{ \tilde{f_1} = \cdots = \tilde{f_r} = 0 \} \subset (Z, o) \times \mathbb{P}^{r-1}$, where $\tilde{f_j} \in \mathcal{O}_{(Z, o)}/(g_1, \ldots, g_k)$ is obtained as the leading term of $f_j$. The projection $\mathbb{P}N_{(Z, o)} \to (Z, o)$ is precisely the projection $\pi$ of \[ \text{[3,3]} \]

Thus equation \[ \text{[3,3]} \] and definition \[ \text{[3,3]} \] give us:

**Corollary 4.5.** 1. The critical locus of the projection is the subscheme:

\[ \text{Crit}(\pi) := \{ \tilde{f_1} = \cdots = \tilde{f_r} = 0, \quad \text{rank} \left( \begin{array}{c} d\tilde{f_1}, \ldots, d\tilde{f_r} \end{array} \right) < r \} \subset \mathbb{P}^{r-1} \times (Z, o), \]

where $\{d\tilde{f_i}\}$ are $k \times r$ columns of partial derivatives of $\{\tilde{f_j}\}$, taken with respect to the homogeneous coordinates in $\mathbb{P}^{r-1}$.

2. Suppose the restriction $\text{Crit}(\pi) \subset (Z, o)$ is a finite map. Then the defining ideal of $\Delta^\perp \subset (Z, o)$ is: $\mathcal{Fitt}_0(\pi_* \mathcal{O}_{\text{Crit}(\pi)})$.

4.3. Further stratifications of the singular locus. Recall that at some points of $(Z, o) = \text{Sing}(X, o)$, the $(\mu = \text{const})$ singularity type of $(L^\perp \cap X, o)$ depends on the choice of the section $L^\perp$, see example \[ \text{[1,1]} \]

Therefore we make the stratification according to the singularities of the fibers of the projectivized normal cone, $\mathbb{P}N_{(X, o)/(Z, o)}$.

Any stratification of the parameter space, $|\mathcal{O}_{p_{k-1}}(p)|$ or $\prod_1 |\mathcal{O}_{p_{k-1}}(p_i)|$, e.g. by singularity type for some equivalence relation, induces a stratification of $\text{Sing}(X)$. More precisely, using the map $\phi$ of equation \[ \text{[3,3]} \], we get the following:

if (the closure of) some stratum $\Sigma \subset \prod_1 |\mathcal{O}_{p_{k-1}}(p_i)|$ is defined by an ideal $I_{\Sigma}$, then

the ideal $\phi^*(I_{\Sigma})$ defines the corresponding stratum on the singular locus.

**Example 4.6.** Consider the $\mu = \text{const}$ stratification of $|\mathcal{O}_{p_{k-1}}(p)|$: the points of a stratum correspond to all the hypersurfaces that can be deformed to a given hypersurface in a $\mu = \text{const}$ way. This defines the strata:

\[ \text{Sing}(X) \supset \Delta^\perp = \Sigma_{A_1} \supset \Sigma_{A_2} \supset \Sigma_{A_1, A_1} \supset \Sigma_{A_3} \supset \Sigma_{A_1, A_2} \supset \Sigma_{A_1, A_1, A_1} \supset \cdots \Sigma_{D_4} \supset \cdots \]

5. Some general properties of $\Delta^\perp$

The definition of $\Delta^\perp$ as the pullback of the classical discriminant is somewhat theoretical, as in most cases it is extremely difficult to write down the classical discriminant explicitly. (Recall that even in the hypersurface case, $r = 1$, $\mathcal{D}$ is a polynomial of degree $k(p-1)k^{-1}$ in $k$ variables.) Also the computation of the Fitting ideal $\mathcal{Fitt}_0(\pi_* \mathcal{O}_{\text{Crit}(\pi)})$ is, in general, an involved procedure. Yet, some consequences are obtained immediately.

5.1. The discriminant pulls back. Suppose we are given morphisms of (germs of) manifolds, as on the diagram. Here $X_1 = \phi^*(Z_2)$ and $Z_1 = \phi^*(Z_2)$ are pullback of schemes/ideals. Assume $X_i$, $Z_i$ are reduced, l.c.i., $Z_i$ is a connected component of $\text{Sing}(X_i)$ and $X_i$ is s.c.i. over $Z_i$.

Suppose, moreover, $X_i$ is generically ordinary along $Z_i$ and the multiplicity sequences, of $X_1$ along $Z_1$ and $X_2$ along $Z_2$, coincide.

**Proposition 5.1.** Then $\Delta^\perp_{X_1/Z_1} = \phi^* \Delta^\perp_{X_2/Z_2}$.

**Proof.** It is enough to check the statement locally at each point. Thus we work with germs. We have: $I_{(Z_1, o_1)} = \phi^*(I_{(Z_2, o_2)}) = \langle \phi^*(g_1), \ldots, \phi^*(g_k) \rangle$ and $I_{(X_1, o_1)} = \phi^*(I_{(X_2, o_2)}) = \langle \phi^*(f_1), \ldots, \phi^*(f_r) \rangle$, and in both cases the sequences are regular.

To define $\Delta^\perp_{(X_1, o_1)/(Z_1, o_1)}$ we expand:

\[ \phi^*(f_i) = \sum_{m_j = p_i} \phi^*(g_j^{m_1}) \cdots \phi^*(g_k^{m_k}) a_{m_1 \ldots m_k}^{(i)} \]

But we can also pullback the initial expansions, $f_i = \sum_{m_j = p_i} g_j^{m_1} \cdots g_k^{m_k} a_{m_1 \ldots m_k}^{(i)}$. As $(\phi^*(g_1), \ldots, \phi^*(g_k))$ form a regular sequence, we get $a_{m_1 \ldots m_k}^{(i)} = \phi^* a_{m_1 \ldots m_k}^{(i)}$. Therefore:

\[ I_{\Delta^\perp_{(X_1, o_1)/(Z_1, o_1)}} = \left( \mathcal{O}(\{ a_{m_1 \ldots m_k}^{(i)} \sum_{m_i = p_i} \}) \right)_{1 \leq i \leq r} = \left( \mathcal{O}(\{ \phi^* a_{m_1 \ldots m_k}^{(i)} \sum_{m_i = p_i} \}) \right)_{1 \leq i \leq r} = \phi^* I_{\Delta^\perp_{(X_2, o_2)/(Z_2, o_2)}}. \]
Example 5.2. Consider the surface \( X_1 = \{ x^2z^2 = y^2 + x^3 \} \subset M_1 \approx k^3 \), cf. example [2]. Then \( X_1 \) is the pullback of \( X_2 = \{ x^2z = y^2 + x^3 \} \subset M_2 \approx k^3 \), under the covering \( (x, y, z) \overset{\phi}{\rightarrow} (x, y, z') \). Thus
\[
\Delta_{X_1/Z_1} = \phi^* (\Delta_{X_2/Z_2}) = \{ z^q = 0 \} \subset Z_1 = \{ x = 0 \}.
\]

Remark 5.3. (Importance of \( \phi^*(Z) \) being reduced.) Let \( M' = k^n \overset{\phi}{\rightarrow} M = k^n \) by \( (y_1, \ldots, y_n) \rightarrow (y_1^2, \ldots, y_n) \). Then for \( X = \{ x^2 = z^2 \} \) we have \( Z = \{ x_1 = 0 = x_2 \} \) and \( \Delta_{X/Z} = \varnothing \). But \( \phi^*(Z) = \{ y_1^2 = 0 = y_2 \} \) and \( \phi^*(X) = \{ y_1^q = y_2^q \} \), the transversal type is generically non-ordinary, i.e. \( \Delta_{X/Z} = Z' \).

5.2. The discriminant is determined by infinitesimal neighborhood of \( Z \) in \( X \). By its construction \( \Delta^\bot \) reflects degenerations of the projectivized normal cone and does not depend on the degenerations of ‘higher order terms’. This idea is made precise by a variation of the last proposition. Fix two triples (of germs) \( \{ Z_i = Sing(X_i) \subset X_i \subset (M_i) \} \). Suppose \( X_i \) is generically ordinary along \( M_1 \) and the multiplicity sequences in both cases are the same: \( p_1 \leq p_2 \leq \cdots \leq p_r \).

**Proposition 5.4.** Suppose the restriction \( Z_1 \overset{\phi^*}{\rightarrow} Z_2 \) is an isomorphism and moreover \( \phi^* (I_{X_2} \otimes \mathcal{O}_{M_2/P_{x_2}^{r+i}}) = I_{X_1} \otimes \mathcal{O}_{M_1/P_{x_1}^{r+i}}. \) Then \( \phi^* (\Delta_{X_2}^\bot) = (\Delta_{X_1}^\bot). \)

**Proof.** As before, it is enough to check the statement pointwise. As \( (\Delta^\bot, o) \) is fully determined by \( \mathbb{P} N(X_1, o)/ (Z_1, o) \), it is enough to show that \( \phi \) induces isomorphism \( \mathbb{P} N(X_1, o)/ (Z_1, o) \overset{\cong}{\rightarrow} \mathbb{P} N(X_2, o)/ (Z_2, o) \). Fix a basis \( I_{(Z_2, o)} = (g_1, \ldots, g_k) \) so that \( I_{(Z_1, o)} = (\phi^*(g_1), \ldots, \phi^*(g_k)) \). Fix a good basis \( I_{(Z_2, o)} = (f_1, \ldots, f_r) \) then the assumption reads: there exist \( t_1, \ldots, t_r \in P_{x_2}^{r+i} \) such that \( \phi^*(f_1 + t_1, \ldots, \phi^*(f_r) + t_r) \) is a basis of \( I_{(X_1, o)} \). It follows that this is a good basis. But then the expansion \( f_i = \sum_{m_j = p_i} g_1^{m_1} \ldots g_k^{m_k} a_{m_1 \ldots m_k} (i) \in \mathcal{O}(M_2, o) \) ensures the expansion:
\[
\phi^*(f_i) = \sum_{m_j = p_i} \phi^*(g_1^{m_1}) \ldots \phi^*(g_k^{m_k}) (\phi^*(a_{m_1 \ldots m_k} (i) + b_{m_1 \ldots m_k} (i)) \in \mathcal{O}(M_1, o), \text{ for some } b_{m_1 \ldots m_k} (i) \in I_{(Z_1, o)}.
\]

Thus
\[
\mathbb{P} N(X_1, o)/ (Z_1, o) = \left\{ \sum_{m_j = p_i} y_1^{m_1} \ldots y_k^{m_k} \phi^*(a_{m_1 \ldots m_k} (i) = 0, \forall i \right\} \overset{\cong}{\rightarrow} \mathbb{P} N(X_2, o)/ (Z_2, o). \]

The proposition states that \( \Delta^\bot \) is determined by the \((p_r+1)\)-infinitesimal neighborhood of \((Z, o)\) in \((X, o)\). Therefore \( \Delta^\bot \) is determined by the formal neighborhood:

**Corollary 5.5.** Given two triples \((M_1, X_1, Z_1)\) and \((M_2, X_2, Z_2)\) with \( Z_1 = Sing(X_1) \). Suppose \( Z_1 \approx Z_2 \) and the completions along the singular loci are isomorphic \((M_1, X_1) \approx (M_2, X_2) \). Then the discriminants are (embedded) isomorphic.

The converse statement to proposition 5.4 does not hold: if the map \( M_1 \overset{\phi}{\rightarrow} M_2 \) restricts to an isomorphism \( \phi|_Z \), with \( \phi^*(\Delta^\bot) = \Delta^\bot \), this does not imply much relation of \((X_1, o)\) to \((X_2, o)\). For example, compare \( I_{(X_1, o)} = (x(zx^3 + y^3) + z^5 + y^5) \) and \( I_{(X_2, o)} = (z(x^4 + y^4) + x^2y^2 + x^5 + y^5) \). In both cases \((Z, o) = Sing(X_1, o) = \{ x = 0 = y \} \) and their generic type along \((Z, o)\) is the ordinary multiple point of multiplicity 4. For \((X_1, o)\) the degeneration of transversal type at \( o \) is: 3 roots collide to a triple root. For \((X_2, o)\) the degeneration is: two pairs of roots collide to two double roots. Thus in both cases \( \Delta^\bot = (z^2 = 0) \).

5.3. Flat deformations. We prove that \( \Delta^\bot \) deforms flatly in those flat deformations of \( X \) that induce flat deformations of \((X, o)\) singular locus \( Z = Sing(X) \) and preserve the multiplicity sequence. More precisely, given a good basis \( I_{(X, o)} = (f_1(x), \ldots, f_r(x)) \), fix a flat deformation \( I_{(X, o)} = (f_1(t), \ldots, f_r(t)) \) with \( f_i(t, o) = f_i(x) \), such that the (reduced) singular locus \( Z = Sing(X) \) is flat family: \( I_{(Z, o)} = (g_1(t), \ldots, g_k(t)) \) and \( I_{(Z, o)} = (g_1(t, o), \ldots, g_k(t, o)) \).

**Proposition 5.6.** Suppose \((X, o)\) is s.c.i. over \((Z, o)\) and \((X, o)\) is s.c.i. over \((Z, o)\) and the multiplicity sequence is preserved. Then the family \( \Delta^\bot_{X/Z} \) is flat and its central fibre is \( \Delta_{X/Z} \).

**Proof.** By the assumption we can use the standard expansion \( f_j(x) = \sum g_j(x) t^{m_j} a_{m_j} (x, t) \). Thus \( \Delta^\bot_{X/Z} = \{ D(a_j (x, t)) = 0 \} \) is a flat family that specializes to \( \Delta_{X/Z} \) (Note that \( D(a_j (x, t)) \) is a power series in \( t \)). In many cases this property allows the quick computation of the transversal multiplicity of \( \Delta^\bot \).

Example [2] shows that \( \Delta^\bot \) can be non-reduced if the degeneration occurs ‘faster than normally’. Another reason for being non-reduced is when the degeneration is not ‘minimal’.
Example 5.7. Consider the surface $X = \{x^p z = y^p + x^{p+1}\} \subseteq k^3$. Its singular locus is the line $Z = \{x = 0 = y\}$. Consider the projection $k^3 \xrightarrow{\pi} Z$, $(x,y,z) \to z$, and the fibres $\pi^{-1}(z)$. Then we have a family of plane curve singularities, $\pi^{-1}(t) \cap X \subseteq \pi^{-1}(t) = (k^2,o)$, for $t \in Z$. This family is equimultiple, thus the projectivized tangent cones of these curve singularities form the flat family: $\{\sigma_{s}^p z = \sigma_{s}^p_y \subseteq \mathbb{P}_{x,s} \times k^1\}$. For each $s \neq 0$ there are $p$ distinct roots, while for $s = 0$ all these roots coincide, thus $\Delta^1$ is supported at $z = 0$. Now the multiplicity can be computed using a flat deformation or via the critical locus and the fitting ideal.

- Under a generic deformation this multiple root at $z = 0$ splits into several double roots near $z = 0$. In our case one can take $\{\sigma_{s}^p z = \sigma_{s}^p_y - \epsilon \sigma_{s}^p x \sigma_{s}^p y \subseteq \mathbb{P}_{x,s} \times k^1\}$. By direct check, for each fixed $\epsilon \neq 0$ the number of the double roots near $z = 0$ is $(p - 1)$. So, by the flatness of $\Delta^1$ in deformations, the multiplicity of $\Delta^1$ for the initial surface is $(p - 1)$.

- Blow-up $k^3$ along the line $Z = \{x = 0 = y\}$, let $E \subseteq Bl_Z k^3$ be the exceptional divisor, consider the strict transform $\tilde{X} \subseteq Bl_Z k^3$ and the projection $\tilde{X} \cap E \to Z$. Explicitly: $Z \times \mathbb{P}^1 = E \supseteq \tilde{X} \cap E = \{\sigma_{s}^p z = \sigma_{s}^p_y \subseteq \mathbb{P}_{x,s} \times k^1\}$. This is a $p : 1$ covering, totally ramified over $z = 0$ and the ramification degree is $(p - 1)$. The critical locus is (see proposition 5.3) $\{\sigma_{s}^p z = 0 = \sigma_{s}^p_y\}$. Thus $\text{O}^{\text{Crit}}(\pi) \approx k[z][z]/(z,\sigma_{s}^p-y^{-1})$ and $\pi_3(\text{O}^{\text{Crit}}(\pi)) \approx (k[z][z])^{p-1}$, as a module over $\text{O}(Z_o,k[\pi])$. Therefore $\text{Fitt}_0(\pi_3(\text{O}^{\text{Crit}}(\pi))) = (z^{p-1})$.

Example 5.8. Consider the hypersurface $X = \{x^p + y^p = x^q + y^q\} \subseteq k^3$, with $p < q, r$. Again, $\text{Sing}(X) = \{x = 0 = y\}$ and the discriminant is a point on $z$-axis, namely, $\Delta^1 = \{x = y = z = 0\}$, as a set. The deformation $\{z(x^p + (z - \epsilon)y^p = x^q + y^q\}$ splits the discriminant into two: at $z = 0$ and $z = \epsilon$. The previous example gives that both points have multiplicity $(p - 1)$, regardless of $q, r$. (Compare to proposition 5.4) Hence the multiplicity in the current case is $2(p - 1)$.

If in the family $\{X_t\}$ the generic multiplicity along $(Z_o, \epsilon)$ changes then the (non-flat) family $\Delta^1_t$ is not semi-continuous in any reasonable sense. For example, consider $(X_t, o) = \{x^p + y^p + t(x^2 + zy^2)\} \subseteq (k^3,o)$. Then $\Delta^1_{t=0} = \emptyset$, while $\Delta^1_{t\neq 0} = \{(0,0,0)\}$.  

5.4. Comparison of $\Delta^1$ to Lé numbers/cycles. We give just one example to show that the relation to Lé numbers of Massey is not at all obvious. We work in the notations of Massey, Chapter 1.

Example 5.9. For $(X,o) = \{y^p = \frac{x^{p+1} + tx^p}{p}\}$ the ideal defining the singular scheme is $(y^{p-1}, t^{q-1}x^p, x^p + t^q x^{p-1})$. Its saturation gives $\Sigma_f = V(y^{p-1}, x^{p-1})$.

Then the polar schemes are $\Gamma^2 = V(y^{p-1})$ and $\Gamma^1 = V(y^{p-1}, x^p, x^p + t^q x^{p-1}) \setminus \Sigma_f = V(y^{p-1}, x + t^q)$.

Now the Lé cycles are:

$$\begin{align*}
\Lambda^1 &= \{(y^{p-1}, x^p, t^q x^{p-1}) - (y^{p-1}, x + t^q)\} = (y^{p-1}, x^{p-1}) \quad \text{and} \\
\Lambda^0 &= \{(y^{p-1}, x + t^q, t^q x^{p-1})\} = (q-1+pq)(p-1)\{(y,x,t)\}.
\end{align*}$$

Thus $\lambda^1 = \text{dim}_k k[x,y,t](t, x^{p-1}, y^{p-1}) = (p-1)^2$, while $\lambda^0 = (q-1+pq)(p-1)$.

But $\text{deg}(|\Delta^1|) = q(p-1)$, as can be seen e.g. by deformation $(X_t, o) = \{y^p = \frac{x^{p+1}}{p+1} + \frac{(t^q-x)t^q}{p}\}$, see also example 11.

5.5. The transversal multiplicity of the discriminant. Proposition 5.2 implies the following formula: if $\pi^{-1}(o)$ has only isolated singularities and $(Z_o)$ contains the miniversal deformation of the singularities of $\pi^{-1}(o)$ then

$$\text{mult}(\Delta^1, o) = \mu_{\text{total}}(\pi^{-1}(o)) + (p-1) \mu_{\text{total}}(\pi^{-1}(o)),$$

In most cases of interest $(Z,o) = \text{Sing}(X,o)$ is of low dimension and does not contain the miniversal deformation of $\pi^{-1}(o)$. Thus the only conclusion is $\text{mult}(\Delta^1, o) \geq \mu_{\text{total}}(\pi^{-1}(o)) + (p-1) \mu_{\text{total}}(\pi^{-1}(o))$, see example 12. (Indeed, we have the usual map from $(Z,o)$ to the germ of miniversal deformation, such that the family is the pullback. And computation of $\text{mult}(\Delta^1, o)$ corresponds to the intersection of the discriminant in the miniversal deformation by a not-necessarily-transversal, not-necessarily-smooth curve-germ.)

Proposition 5.10. In the notations of 4.1 suppose the map $\text{Crit}(\pi) \xrightarrow{\pi} (Z,o)$ is finite at $o \in Z$.

1. Let $\{p_t\}_{\alpha} \subseteq \text{Crit}(\pi)$ be the points of the fibre $\pi^{-1}(o)$, let $\Delta^1_{\alpha} \subseteq (Z,o)$ be the corresponding discriminants. Then, as Cartier divisors, $\Delta^1 = \sum_{\alpha} \Delta^1_{\alpha}$.

2. Take a one dimensional complete intersection subgerm $(C,o) \subseteq (Z,o)$, such that $(C,o) \cap \Delta^1 \subseteq (C,o)$ is zero dimensional (Cartier divisor). Then $\text{deg}\left((C,o) \cap \Delta^1\right) = \text{deg}\left(\text{Crit}(\pi) \cap \pi^{-1}(1,0)\right)$.

3. Suppose $(X,o) \subseteq (k^{n+1},o)$ is a hypersurface and $(Z,o)$ is smooth. Suppose near each $p_t \in \text{Crit}(\pi)$ the defining equation of $X \cap E$ in some local coordinates has the form $f_\alpha(z) + h_\alpha(z) = 0$, here $z$ are the local coordinates of $(Z,o)$. Then $\text{mult}(\Delta^1, o) = \sum_{p_t \in \text{Crit}(\pi)} \mu(f_\alpha(z)) \cdot \text{ord}(h_\alpha(z))$. 
In part 2 on both sides we have the scheme-theoretic intersection, defined by the union of the ideals. By the degree of a zero-dimensional scheme we mean the length of its ring: \( \text{deg}(Y) = \text{dim}_k O_Y \).

**Proof. 1.** In this case \( \text{Crit}(\pi) \) is a multi-germ, thus \( \pi_*(O_{\text{Crit}(\pi)}) = \oplus \pi_*(O_{\text{Crit}(\pi), pt_n}) \). For the direct sum of modules one has: \( \text{Fitt}(M_1 \oplus M_2) = \text{Fitt}(M_1) \cdot \text{Fitt}(M_2) \). Thus \( I_{\Delta} = \prod \alpha I_{\Delta, \alpha} \).

2. We should prove the two equalities: \( \text{deg}( (C, o) \cap \Delta) = \text{deg}(\Delta_{x(C)} \cap \Delta) = \text{deg}(\text{Crit}(\pi) \cap \pi^{-1}(C, o)) \).

The left equality is immediate by base-change, as \( \Delta \subset (Z, o) \) is a Cartier divisor (in particular it is a zero dimensional subscheme). We should prove: \( \text{deg}(\Delta) = \text{deg}(\text{Crit}(\pi)) \).

By part 1 it is enough to consider the one-point critical locus, \( pt = \text{Crit}(\pi) \in \tilde{X} \cap E \).

- We start from the case: \((Z, o)\) is smooth. Note that

\[
\text{deg}(\Delta) = \text{dim}_k O_{\Delta} = \text{dim}_k O_{\text{Crit}(\pi)}/\text{Fitt}(\pi_*(O_{\text{Crit}(\pi)}))
\]

and \( \text{deg}(\text{Crit}(\pi)) = \text{dim}_k O_{\text{Crit}(\pi)} = \text{dim}_k (\pi_*(O_{\text{Crit}(\pi)})) \).

Thus, the statement to prove is: given a finite module \( M \) over a one-dimensional regular local ring, \( O_{(Z, o)} \), the length of the Fitting ideal satisfies: \( \text{colength}(O_{(Z, o)}) (\text{Fitt}(M) = \text{dim}_k M) \). This is a standard statement of commutative algebra. Take the minimal free resolution: \( O_{(Z, o)} \rightarrow O_{(Z, o)} \rightarrow M \rightarrow 0 \). As \( M \) is finite, it is supported at one point only, so \( p \geq q \). Furthermore, as the ring is local and regular, \( A \) is equivalent, by \( A \rightarrow UAV \), to a diagonal matrix. Let \( z \) be a generator of \( O_{(Z, o)} \) then \( \text{Fitt}(M) = \text{Fitt}(M) = (\Sigma, d_i) \), here \( \{d_i\} \) are the exponents of the diagonal. Thus \( M \approx \oplus_i O_{(Z, o)}/(z^{d_i}) \) and \( \text{colength}(\text{Fitt}(M) = \sum d_i = \text{dim}_k M) \). Proving that \( \text{deg}(\Delta) = \text{deg}(\text{Crit}(\pi)) \).

- Suppose \((Z, o)\) is a complete intersection of (dimension one), then it can be smoothed. Let \( \{Z_t\}_{t \in (\mathbb{C}, o)} \) be a smoothing, then we have the (flat) family of projections, \( (\tilde{X} \cap E) \rightarrow Z_t \). Explicitly, if \( \tilde{X} \cap E = \{(f_0(x, z) = 0) \} \) then \( \tilde{X} \cap E = \{(f_0(z, x) = 0) \} \).

This induces the flat family \( \Delta^t \subset Z_t \). Thus, for \( t \in (\mathbb{C}, o) \) small enough, we can fix some (small enough, Zariski open) neighborhood of \( o \in Z \) such that \( \text{deg}(\Delta^t) = \text{deg}(\Delta) \). Here the r.h.s. is the total degree of \( \Delta^t \) in the neighborhood. Note that \( \Delta^t_{\|} \rightarrow Z_t \) is a smooth scheme. Thus the statement holds for \( \Delta^t_{\|} \) and then, by flatness, for \( \Delta^t \).

3. Restrict from \((Z, o)\) to the generic curve \((C, o) \subset (Z, o)\), then

\[
\text{mult}(\Delta, o) = \text{length}(\pi_*(O_{\text{Crit}(\pi)}) = \text{length}(O_{\text{Crit}(\pi)}) = \sum_{pt_n \in \text{Crit}(\pi)} \text{length}(O_{\text{Crit}(\pi), pt_n})
\]

Thus it is enough to verify the claim for each point. In the local coordinates:

\[
(\text{Crit}(\pi), pt_n) = (\partial_x f(x), 0) = f(x) + h(z).
\]

Working locally, we can redefine \( z \) to get \( h(z) = z^p \), where \( p = \text{ord}(h) \). Then, as a \( \mathbb{R}[\mathbb{R}] \)-module, \( O_{\text{Crit}(\pi), pt_n} \approx \mathbb{R}[\mathbb{R}]/\mathbb{R} f \cdot 1, z, \ldots, z^{p-2} \). Therefore \( \text{length}(O_{\text{Crit}(\pi), pt_n}) = \mu(f(o) \cdot p) \)

**Example 5.11.**

i. Using part 3 of the proposition one gets immediately the multiplicity of the discriminant in examples [2] [3] [5] [8]. For example [3] (with non-smooth \((Z, o)\) one can use Part 2.

ii. (Extending example [5]) Consider the hypersurface singularity \( X = x^n y^n = f_p(x_1, \ldots, x_{n-1}) + g_{p-1}(x_1, \ldots, x_{n}) \subset (\mathbb{C}, o) \), where \( f_p(x_2, \ldots, x_{n-1}) \) is a homogeneous form of degree \( p \), while \( g_{p-1}(x_1, \ldots, x_{n}) \in \mathbb{R}^{\mathbb{C}} \). Suppose \( f_p \) is generic, so that \( f_p(x_2, \ldots, x_{n-1}) = 0 \in \mathbb{C}^{p-1} \) is smooth. Suppose \( g_{p-1} \) contains a monomial \( x_k^n \) for some \( k \). Then \( \text{Sing}(X) = \{x_1 = \cdots = x_{n-1} = 0 \} \subset \mathbb{C}^n \) and the generic transverse type (for \( x_n \neq 0 \)) is ordinary. The discriminant \( \Delta \subset \text{Sing}(X) \) is supported at the point \( \{x_n = 0 \} \subset \text{Sing}(X) \) and its multiplicity equals the length of the scheme \( \text{Crit}(\pi) = (\sigma^p f(x_1, \ldots, x_{n-1}) \subset \mathbb{C}^{k_1} \times \mathbb{R}^{p-1} \times \ldots \ldots \times \mathbb{R}^{p-1} \times \ldots \times \mathbb{R}^{p-1} \times \mathbb{C}^n \). As the form \( f_p \) is generic, this scheme coincides with the scheme \( \{\sigma^p = 0, (\sigma^p - \sigma^p f(x_1, \ldots, x_{n-1}) \), whose degree is \( q(p-1)^{n-2} \).

iii. Consider the hypersurface singularity \( (X, o) = \{x^n + x^n y^n g_1(z) + y^n g_2(z) + h_{p-1}(x, y, z) = 0 \} \subset (\mathbb{C}, o) \), where \( g_{p-1}(x, y, z) \in \mathbb{R}^{p-1} \). The (reduced) singular locus is of codimension two: \( (Z, o) = \{x = y = 0 \} \). The strict transform under the blow-up along \( Z, o \) is: \( X = E = \{\sigma^p + \sigma^p f(x_1, \ldots, x_{n-1}) = 0 \} \subset \mathbb{C}^{k_1} \times \mathbb{R}^{p-1} \times \ldots \ldots \times \mathbb{R}^{p-1} \times \ldots \times \mathbb{R}^{p-1} \times \mathbb{C}^n \). The critical set is: \( \text{Crit}(\pi) = \{\sigma^p - \sigma^p f(x_1, \ldots, x_{n-1}) = 0 \} \subset \mathbb{C}^{k_1} \times \mathbb{R}^{p-1} \times \ldots \ldots \times \mathbb{R}^{p-1} \times \ldots \times \mathbb{R}^{p-1} \times \mathbb{C}^n \). Therefore \( \sigma^p \in I_{\text{Crit}(\pi)} \), hence \( \text{Crit}(\pi) \) is located in the scheme \( \sigma^p = 1 \subset \mathbb{C}^{p-1} \). By part 2 of the proposition, \( \text{mult}(\Delta^t, o) \).
length(Crit(π)|(C,o)), for any generic curve germ (C,o) ⊂ (Z,o). Let ord(g₁(z)) = q₁ and ord(g₂(z)) = q₂, then

- if q₁ ≥ q₂ then mut(∆⊥, o) = q₂(p − 1).
- if q₁ < q₂ then mut(∆⊥, o) = q₁(p − 1) + (p − r)(q₂ − q₁).

Example 5.12. Take two hypersurface germs, (X₁, o) = {f₁ = 0} ⊂ (kⁿ+₁, o). Suppose they share the singular locus, Sing(X₁, o) = (Z₁, o) = Sing(X₂, o), and (Z₁, o) is a reduced complete intersection of dimension n − 1. In this case the projectivized normal cones, \( \mathbb{P}N(X₁, o)/Z₁, o \) and \( \mathbb{P}N(X₂, o)/Z₂, o \), are subschemes of \( \mathbb{P}^₁ × (Z₁, o) \). Suppose these subschemes are disjoint. Then \( ∆⊥_{f₁,f₂} = ∆⊥_{f₁} + ∆⊥_{f₂} \), i.e. \( I(∆⊥_{f₁,f₂}, o) = I(∆⊥_{f₁}, o)I(∆⊥_{f₂}, o) \).

5.6. The \( \mathcal{O}_{(Z,o)} \)-resolution of \( \mathcal{O}_{(Δ⊥, o)} \) and defining equation of \( (Δ⊥, o) \). We want to obtain some information about the local equation of the discriminant. By the previous proposition, if \( \text{Crit}(π) \) intersects \( π − 1(α) \) contains several points, then \( Δ⊥ \) is the sum of the components. Thus it is enough to consider the case of one critical point, i.e. the map of germs \( \text{Crit}(π), pt \to (Δ⊥, o) \). Let \( (X, pt) = \{\tilde{f}(x) = \tilde{a}(x, z) = 0\} \subset (\mathbb{P}N(X, o)/Z₁, o, pt), \) where \( \tilde{f}_i \in k[[x]] \) while \( a_i \in m₁[Z₁, o][z] \), i.e. \( \tilde{a}(x, 0) = 0 \), and \( \tilde{a}(0, z) ≠ 0 \). Thus, we can consider \( \tilde{f} = 0 \) as an isolated singularity and \( \tilde{a} \) as its deformation.

Note that we work locally in \( (\mathbb{P}N(X, o)/Z₁, o, pt) \), thus the singularity is not generically ordinary in any sense.

5.6.1. A free resolution of \( π_*\mathcal{O}_{(\text{Crit}(π), pt)} \) as an \( \mathcal{O}_{(Z,o)} \)-module.

Theorem 5.13. Suppose both \( (\tilde{f}_1, \ldots, \tilde{f}_r) \) and \( (\hat{f}_2, \ldots, \hat{f}_r) \) define isolated (complete intersection) singularities in \( (kⁿ+r, o) \). Fix their Milnor numbers, \( \mu = \mu(\hat{f}_1, \ldots, \hat{f}_r) \), \( \mu_1 = \mu(\tilde{f}_1, \ldots, \tilde{f}_r) \), as in \( \mathbb{R} \). Then the free resolution is

\[
0 \to \mathcal{O}_{(Z,o)}^{⊕(r+\mu)}[\tilde{f}_1+\tilde{a}_1] \to \mathcal{O}_{(Z,o)}^{⊕(r+\mu)} \to π_*\mathcal{O}_{(\text{Crit}(π), pt)} \to 0.
\]

(The linear map \( [\tilde{f}_1 + \tilde{a}_1] \) is defined during the proof.)

The locally defining ideal \( I(Δ⊥, o) \subset \mathcal{O}_{(Z,o)} \) of \( Δ⊥ \) is generated by \( \det [\tilde{f}_1 + \tilde{a}_1] \).

Proof. The hypersurface case (In this cases we follow e.g. [Teissier1976].)

Step 1. By corollary \( \mathbb{R} \), the critical locus can be presented in the form:

\[
(45) \mathcal{O}_{(\text{Crit}(π), pt)} = \mathcal{O}_{(Z,o)}[\tilde{a}] / (\tilde{f} + a_1 \tilde{f} + a_2, \ldots, a_r \tilde{f} + a_n),
\]

the derivatives are taken with respect to \( x \) coordinates.

Choose some \( k \)-basis of the the Milnor algebra \( k[[z]](\partial_1 \tilde{f}, \ldots, \partial_k \tilde{f}) \), and fix some \( k[[x]] \)-representatives of this basis, \( \{v_α\} \). Use the composition of maps \( k[[z]] \xrightarrow{π} \mathcal{O}_{(Z,o)}[z] \to \mathcal{O}_{(\text{Crit}(π), pt)} \) and denote the images of \( vα \) by \( \{vα\} \).

We claim that \( \{vα\} \) generate \( π_*\mathcal{O}_{(\text{Crit}(π), pt)} \), i.e. generate \( \mathcal{O}_{(\text{Crit}(π), pt)} \) as an \( \mathcal{O}_{(Z,o)} \)-module. Indeed, we have

\[
(46) π_*\mathcal{O}_{(\text{Crit}(π), pt)} \subseteq \text{Span}_{\mathcal{O}_{(Z,o)}} \{vα\} + m(Z,o)π_*\mathcal{O}_{(\text{Crit}(π), pt)}.
\]

Take the quotient of \( π_*\mathcal{O}_{(\text{Crit}(π), pt)} \) by \( \text{Span}_{\mathcal{O}_{(Z,o)}} \{vα\} \) and apply the Nakayama lemma.

Step 2. It remains to understand the relations among \( \{vα\} \), i.e. the kernel of the surjection

\[
(47) \mathcal{O}_{(Z,o)}^{⊕r} \to π_*\mathcal{O}_{(\text{Crit}(π), pt)} \to 0.
\]

The relations come from the ideal \( (\tilde{f} + a_1 \tilde{f} + a_2, \ldots, a_r \tilde{f} + a_n) \). We should pass from the \( \mathcal{O}_{(Z,o)}[z] \)-module structure on \( \mathcal{O}_{(\text{Crit}(π), pt)} \) to \( \mathcal{O}_{(Z,o)} \)-module structure. In other words, we should express the action of \( z \) via \( \mathcal{O}_{(Z,o)} \)-action. By the construction of \( \{vα\} \):

\[
(48) x_i vα = \sum β b^{(i,α)} β vβ + \sum j c^{(i,α)} j ∂ j \tilde{f} \in k[[z]], \>
\]

where \( b^{(i,α)} β \in k \), \( c^{(i,α)} j \in k[[z]] \).

Here the coefficients \( b^{(i,α)} β \) are defined uniquely, while the coefficients \( c^{(i,α)} j \) are unique up to the Koszul relations of the regular sequence \( \partial_1 \tilde{f}, \ldots, \partial_k \tilde{f} \). Therefore in \( \mathcal{O}_{(\text{Crit}(π), pt)} \) we have:

\[
(49) x_i vα = \sum β b^{(i,α)} β [vβ] ∈ \mathcal{O}_{(\text{Crit}(π), pt)}, \>
\]

where now \( b^{(i,α)} β (z) \in \mathcal{O}_{(Z,o)} \).

Despite the (Koszulian) non-uniqueness of \( c^{(i,α)} j \), the resulting coefficients \( b^{(i,α)} β (z) \in \mathcal{O}_{(Z,o)} \) are well defined, as any Koszul relation among \( ∂ j (\tilde{f} + a) \) induces that among \( ∂ j (\tilde{f} + a) \).
Step 3. The last equation defines the needed action \( x_i \circ O_{(\text{Crit}(\pi), pt)} \), we denote the corresponding \( O((Z, o)) \)-linear operator by \( [x_i] \). Accordingly, for any element \( h \in O((Z, o))[[\mathcal{L}]] \) we have the operator \([h] \circ O_{(\text{Crit}(\pi), pt)} \). Thus we have the operators \([\hat{f} + a], [\partial_1(\hat{f} + a)], \ldots, [\partial_k(\hat{f} + a)] \). By the construction: \([\partial_i(\hat{f} + a)] = 0 \) for all \( i \). (Here it is important to notice that the ideal \((\partial_1(\hat{f}), \ldots, \partial_k(\hat{f}))\) is a complete intersection, the only relations among its generators are Koszul and thus extend to the relations among \((\partial_1(\hat{f} + a), \ldots, \partial_k(\hat{f} + a))\). Thus the only possibly non-trivial operator is \([\hat{f} + a] = 0 \). Thus we get the presentation:

\[
O_{(Z, o)}[[\mathcal{L}]] \xrightarrow{[\hat{f} + a]} O_{(Z, o)}[[\mathcal{L}]] \rightarrow \pi_*(O_{(\text{Crit}(\pi), pt)}) \rightarrow 0.
\]

Finally we note that the module \( \pi_* O_{\text{Crit}(\pi)} \) is a torsion, of rank=0, because \( \mathcal{L}(0, z) \neq 0 \). Therefore the map \([\hat{f} + a] \) cannot have a kernel, thus is injective, i.e. we have

\[
0 \rightarrow O_{(Z, o)}[[\mathcal{L}]] \xrightarrow{[\hat{f} + a]} O_{(Z, o)}[[\mathcal{L}]] \rightarrow \pi_*(O_{(\text{Crit}(\pi), pt)}) \rightarrow 0.
\]

The case of complete intersections.

By corollary \(4.3\) the critical locus can be presented in the form:

\[
O_{(\text{Crit}(\pi), pt)} = O((Z, o))[[\mathcal{L}]] \left/ \left\langle \{\hat{f}_i + a_i\}_{i=2,\ldots,r}, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i))\right\rangle \right.,
\]

where the derivatives are taken with respect to \( \mathcal{L} \) coordinates.

As in the hypersurface case, choose some \( k \)-basis of the vector space \( k[[\mathcal{L}]][\hat{f}_2, \ldots, \hat{f}_r, \text{Fitt}_0(\partial_j\hat{f}_i)] \), and fix some \( k[[\mathcal{L}]] \)-representatives of this basis, \( \{v_\alpha\} \). The size of the basis is \( \mu(\hat{f}_1, \ldots, \hat{f}_r, \mu(\hat{f}_2, \ldots, \hat{f}_r)) \), by Lé-Greuel formula (see \(3.8\)). Use the composition \( k[[\mathcal{L}]] \xrightarrow{\varphi} O((Z, o))[[\mathcal{L}]] \xrightarrow{\text{Fitt}_0} O_{(\text{Crit}(\pi), pt)} \) and denote the images of \( v_\alpha \) by \( [v_\alpha] \).

As in the hypersurface case one gets: \( \{[v_\alpha]\} \) generate \( \pi_*(O_{(\text{Crit}(\pi), pt)}) \), as an \( O((Z, o)) \)-module.

It remains to understand the relations, i.e. the kernel of the surjection

\[
O_{(Z, o)}[[\mathcal{L}]] \rightarrow \pi_*(O_{(\text{Crit}(\pi), pt)}) \rightarrow 0.
\]

The relations come from the ideal \( \left\langle \{\hat{f}_i + a_i\}, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i))\right\rangle \).

We begin with the part \( \left\langle \hat{f}_2 + a_2, \ldots, \hat{f}_r + a_r, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i))\right\rangle \). Unlike the hypersurface case, this ideal is not a complete intersection. Yet, the scheme \( V(\hat{f}_2 + a_2, \ldots, \hat{f}_r + a_r, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i))) \subset (Z, o) \times \text{Spec}(k[[\mathcal{L}]]) \) is a flat family of schemes over \((Z, o)\). Therefore any syzygy of \( \left( \hat{f}_2, \ldots, \hat{f}_r, \text{Fitt}_0(\partial_j\hat{f}_i) \right) \) extends to that of \( \left( \hat{f}_2 + a_2, \ldots, \hat{f}_r + a_r, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i)) \right) \).

As in the hypersurface case, we should express the action of \( \mathcal{L} \) via \( O((Z, o)) \)-action, i.e. should define the operators \( [x_i] \circ O_{(\text{Crit}(\pi), pt)} \).

By the construction of \( \{v_\alpha\} \), as in the hypersurface case:

\[
x_i v_\alpha = \sum_\beta b^{(i, \alpha)}_\beta v_\beta + \sum_\gamma c^{(i, \alpha)}_\gamma h_\gamma \in k[[\mathcal{L}]], \quad \text{where } b^{(i, \alpha)}_\beta \in k, \quad c^{(i, \alpha)}_\gamma \in k[[\mathcal{L}]],
\]

while \( \{h_\gamma\} \) are the generators of \( (\hat{f}_2, \ldots, \hat{f}_r, \text{Fitt}_0(\partial_j\hat{f}_i)) \). Here the coefficients \( b^{(i, \alpha)}_\beta \) are defined uniquely, while \( c^{(i, \alpha)}_\gamma \) are defined up to the (not necessarily Koszul) relations in \( (\hat{f}_2, \ldots, \hat{f}_r, \text{Fitt}_0(\partial_j\hat{f}_i)) \). Therefore in \( O_{(\text{Crit}(\pi), pt)} \) we have: \( x_i[v_\alpha] = \sum_\beta b^{(i, \alpha)}_\beta [v_\beta] + \sum_\gamma c^{(i, \alpha)}_\gamma h_\gamma \). As in the hypersurface case \( c^{(i, \alpha)}_\gamma \) might still depend on \( \mathcal{L} \), more precisely: \( c^{(i, \alpha)}_\gamma \in m_{(Z, o)}[[\mathcal{L}]] \). Iterate the procedure until one gets:

\[
x_i[v_\alpha] = \sum_\beta b^{(i, \alpha)}_\beta (\mathcal{L})[v_\beta], \quad \text{where } b^{(i, \alpha)}_\beta (\mathcal{L}) \in O((Z, o)).
\]

Despite the intermediate non-uniqueness, due to relations among \( \{h_\gamma\} \), the resulting coefficients \( b^{(i, \alpha)}_\beta \) are unique. (Again, because of the flatness, any relation among \( \hat{f}_2, \ldots, \hat{f}_r, \text{Fitt}_0(\partial_j(\hat{f}_i)) \) extends to a relation among \( \hat{f}_2 + a_2, \ldots, \hat{f}_r + a_r, \text{Fitt}_0(\partial_j(\hat{f}_i + a_i)) \).

Thus we have the needed (well defined) action \( x_i \circ O_{(\text{Crit}(\pi), pt)} \), we denote the corresponding linear operator by \( [x_i] \). Accordingly, for any element \( h \in O((Z, o))[[\mathcal{L}]] \) we have the operator \([h] \circ O_{(\text{Crit}(\pi), pt)} \).
As in the hypersurface case, we get the tautology: \([\tilde{f}_2 + a_2] = 0, \ldots, [\tilde{f}_r + a_r] = 0, [\text{Fitt}_o(\partial_j(\tilde{f}_i + a_i))] = 0\). (Here the flatness of \(V(\tilde{f}_2 + a_2, \ldots, \tilde{f}_r + a_r, \text{Fitt}_o(\partial_j(\tilde{f}_i + a_i)))\) is used.) The only relation to understand is \([\tilde{f}_1 + g_1] = 0\), this operator is in general non-trivial. Thus we get the presentation of the \(O_{(Z,o)}\)-module \(\pi_*(O_{(\text{Crit}(\pi),pt)})\):

\[
O_{(Z,o)}^{E(\mu + \mu_1)} [f_1 + a_1] \rightarrow O_{(Z,o)}^{E(\mu + \mu_1)} \rightarrow \pi_*(O_{(\text{Crit}(\pi),pt)}) \rightarrow 0
\]

Finally, as in the hypersurface case, we note that the module \(\pi_*O_{(\text{Crit}(\pi))}\) is a torsion, of rank=0, because \(\mathfrak{g}(0,z) \neq 0\). Therefore the map \([\tilde{f}_1 + a_1]\) cannot have a kernel, thus is injective. \(\blacksquare\)

**Remark 5.14.** The proposed resolution is certainly not the only possible. In fact, in the hypersurface case, if \(\tilde{f}\) is not weighted homogeneous, i.e. \(\tilde{f} \neq 0 \in k[\mathfrak{z}] / (\partial_j \tilde{f})\), then \(\tilde{f}\) can be taken as one of \(\{v_i\}\). Then the matrix \([\tilde{f} + a]\) contains an entry invertible in \(O_{(Z,o)}\), i.e. the resolution is non-minimal.

In the hypersurface case one could start from a basis of the Tjurina algebra, \(k[\mathfrak{z}] / (\tilde{f}, \partial_1 \tilde{f}, \ldots, \partial_h \tilde{f})\). Lift it to \(k[\mathfrak{z}]\) and send to \(O_{(\text{Crit}(\pi),pt)}\). Then, as before, we get: \(O_{(Z,o)}^{E(\mu)} \rightarrow \pi_*O_{(\text{Crit}(\pi),pt)} \rightarrow 0\). However, the relations are more complicated now, they come from syzygies of the Tjurina algebra (which is not a complete intersection).

And, unlike the hypersurface case, the definition of \([x_i] \cap \pi_*O_{(\text{Crit}(\pi),pt)}\) is more complicated now, as the family \(V(\tilde{f} + a, \partial_1(\tilde{f} + a), \ldots, \partial_h(\tilde{f} + a)) \subset (Z,o) \times \text{Spec}(k[\mathfrak{z}])\) is not flat.

Though the defining equation, \((\Delta^+, o) = \{\text{det}[\tilde{f}_1 + a_1] = 0\} \subset (Z,o)\), cannot be written in an explicit form, below we can get some information on the participating monomials.

5.6.2. The discriminant for deformations by constant terms. Suppose in some local coordinates the universal family is \(X = \{\tilde{f}_1(\mathfrak{z}) + s_1 = 0, \ldots, \tilde{f}_r(\mathfrak{z}) + s_r = 0\} \subset (k^k, o) \times \text{Spec}(k[\{s_i\}])\). Here \(s_i\) are independent variables, the case of functions on the singular locus, \(\{a_i(\mathfrak{z})\} \in O_{(Z,o)}\), is obtained by base change. (Recall, that the singularity \(\{\tilde{f}_1\}\) is not generically ordinary.)

Denote by \(\mu\) the Milnor number of the ICIS \(\{\tilde{f}_1 = \cdots = \tilde{f}_r = 0\} \subset (k^k, o)\). Suppose for any \(1 \leq j \leq r\) the ideal \(I_j := (\tilde{f}_1, \ldots, \tilde{f}_{j-1}, \tilde{f}_{j+1}, \ldots, \tilde{f}_r)\) defines an isolated (complete intersection) singularity. Define the auxiliary Milnor number, \(\mu_j := \mu(I_j)\), as in \(\text{[3.6]}\).

**Corollary 5.15.** Then the defining equation of \(\Delta^+ \subset (Z,o)\) has the form: \(\sum_{j=1}^r c_j s_j^{\mu + \mu_j} + \{s_is_j\}_{i \neq j} = 0\).

Here \(\{c_i \in k^*\}\) are some non-zero constants, while \(\{\{s_is_j\}_{i \neq j}\}\) is a collection of monomials involving at least two distinct \(s_i\)’s.

**Proof.** By the assumption, \(\text{det}[\hat{f}_1 + s_1]\) is a polynomial in \(s_1, \ldots, s_r\). Note that \(\text{det}[\tilde{f}_1 + s_1]|_{s_i = 0} = \text{det}([\tilde{f}_1 + s_1]|_{s_i = 0})\). Put \(s_2 = \cdots = s_r = 0\), then, by remark \(\text{[5.5]}\), \(\text{det}(\tilde{f}_1 + s_1)|_{s_2 = \cdots = s_r = 0} = s_1^{\mu + \mu_1}\). Similarly,

\[
\text{det}(\tilde{f}_1 + s_1)|_{s_2 = \cdots = s_r = 0} = s_i^{\mu + \mu_i}.
\]

Hence the statement. \(\blacksquare\)

5.6.3. The discriminant for the weighted homogeneous case. Consider the family

\[
X = \{\tilde{f}_1(\mathfrak{z}) + h_1(\mathfrak{z}) s^{(1)} = 0, \ldots, \tilde{f}_r(\mathfrak{z}) + h_r(\mathfrak{z}) s^{(r)} = 0\} \subset (k^k, o) \times \text{Spec}(k[\{s^{(j)}\}]).
\]

Here, using multi-indices, \(h_j(\mathfrak{z}) s^{(j)} = \sum_{m} s_m^{(j)} z^m\).

**Proposition 5.16.** Suppose \(\tilde{f}_1, \ldots, \tilde{f}_r \in k[\mathfrak{z}]\) are weighted homogeneous, of degree \(\{w(\tilde{f}_j)\}\) with respect to the weights \(\{w(x_i)\}\). Then \(\Delta^+\) is weighted homogeneous, and the only possible monomials to appear are: \(\prod_j w(s^{(j)}_m)\), where

\[
\sum_{j=1}^r w(f_j) - \sum_i m_i w(x_i) = (\mu + \mu_1) w(\tilde{f}_1) = \sum_{j=1}^r w(f_j) \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \prod_{q=1}^r \frac{w(f_q)}{w(f_q) - w(f_q)}.
\]

**Proof.** Impose the condition “\((\tilde{f}_j + h_j)\) is weighted homogeneous, of weight \(w(\tilde{f}_j)\)”’ then the weights of the coefficients are fixed, \(w(s_m^{(j)}) = w(\tilde{f}_j) - \sum_i m_i w(x_i)\).
Then $\Delta^\perp$ is weighted homogeneous. We know that the monomials \( \{ s^{(j)}_{0,0} \} \) are present. Thus the total (weighted) degree of $\Delta^\perp$ is $(\mu + \mu_j) \cdot w(\tilde{f})$. Thus the only possible monomials in the defining equation of $\Delta^\perp$ are:

\[
\prod_j \prod_m (s^{(j)}_m)^{n^{(j)}_m}, \quad \text{where} \quad \sum_{j=1}^r \sum_m n^{(j)}_m (w(\tilde{f}_j) - \sum_i m_i w(x_i)) = (\mu + \mu_j) w(\tilde{f}_1).
\]

It remains to compute $(\mu + \mu_1)$. We use the expression for Poincaré series of weighted homogeneous complete intersection with isolated singularity, of dimension $n$ and codimension $r$ [Greuel-Hamm, Satz 3.1]:

\[
P_{\tilde{f}_1, \ldots, \tilde{f}_r}(t) = \text{res}_{\tau=0} \frac{\tau^{n-1}}{1-\tau} \left[ \prod_{i=1}^{n+r} \frac{1 + \tau w(x_i)}{1 - \tau w(x_i)} \prod_{j=1}^r \frac{1 - t w(\tilde{f}_j)}{1 + t w(\tilde{f}_j)} + 1 \right], \quad \mu = P_{\tilde{f}_1, \ldots, \tilde{f}_r}(1).
\]

So, we should extract the residue and take the limit $t \to 1$.

Consider here \( \{ w(\tilde{f}_j) \} \) as independent variables in $\mathbb{R}_{>0}$, then $P_{\tilde{f}_1, \ldots, \tilde{f}_r}(t)$ depends continuously on $\{ w(\tilde{f}_j) \}$. Therefore we can compute under the assumptions: $\{ w(\tilde{f}_j) \neq w(\tilde{f}_i) \}; i \neq j$ and $\{ w(\tilde{f}_j) \neq 0 \}$. After the computation is done, the cases $w(\tilde{f}_j) = w(\tilde{f}_i)$ are obtained by taking the limit.

The expression $R(t, \tau) := \frac{\tau^{n-1}}{1-\tau} \left[ \prod_{i=1}^{n+r} \frac{1 + \tau w(x_i)}{1 - \tau w(x_i)} \prod_{j=1}^r \frac{1 - t w(\tilde{f}_j)}{1 + t w(\tilde{f}_j)} \right]$ is a rational function in $\tau$, with poles at $\tau = 0, -1, \{-t^{-w(f_j)}\}$. For $\tau \to \infty$ this function decreases as $\frac{1}{\tau}$. Therefore

\[
\text{res}_{\tau=0}(R(t, \tau)) + \text{res}_{\tau=-1}(R(t, \tau)) + \sum_{j=1}^r \text{res}_{\tau=-t^{-w(f_j)}}(R(t, \tau)) = 0.
\]

Assuming $w(\tilde{f}_j) \neq 0$, we have $\text{res}_{\tau=-1}(R(t, \tau)) = (-1)^n 2$. Assuming $w(\tilde{f}_j) \neq w(\tilde{f}_i)$ we get:

\[
\text{res}_{\tau=-t^{-w(f_j)}}(R(t, \tau)) = \frac{(-t^{-w(f_i)})^{-n-1}}{1 - t^{-w(f_j)}} \prod_{i=1}^k \frac{1 - t^{-w(x_i)} - w(f_i)}{1 - t^{-w(x_i)}} \left( \prod_{q=1}^r \frac{1 - t^{-w(f_q)}}{1 - t^{-w(f_q)} - w(f_j)} \right) 1 - t^{-w(f_j)}
\]

Therefore

\[
\lim_{t \to 1} \left( \text{res}_{\tau=-t^{-w(f_j)}}(R(t, \tau)) \right) = \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \left( \prod_{q=1}^r \frac{w(f_q)}{w(f_j) - w(f_q)} \right).
\]

Altogether we get:

\[
\mu = P_{\tilde{f}_1, \ldots, \tilde{f}_r}(1) = (-1)^{n-2} \sum_{j=1}^k \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \left( \prod_{q=1}^r \frac{w(f_q)}{w(f_j) - w(f_q)} \right).
\]

Similarly:

\[
\mu_1 = P_{\tilde{f}_2, \ldots, \tilde{f}_r}(1) = (-1)^{n-2} \sum_{j=2}^k \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \left( \prod_{q=2}^r \frac{w(f_q)}{w(f_j) - w(f_q)} \right).
\]

Finally, combine these two to get:

\[
\mu + \mu_1 = \sum_{j=2}^r \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \left( \prod_{q=2}^r \frac{w(f_q)}{w(f_j) - w(f_q)} \right) \frac{w(f_j)}{w(f_j) - w(f_i)} + \sum_{i=1}^k \left( \prod_{q=1}^r \frac{w(f_q)}{w(f_i) - w(f_q)} \right) \frac{w(f_j)}{w(f_j) - w(f_i)} = \sum_{j=1}^r \prod_{i=1}^k \left( \frac{w(f_j)}{w(x_i)} - 1 \right) \prod_{q=1}^r \frac{w(f_q)}{w(f_j) - w(f_q)}.
\]

This finishes the proof. ■

**Example 5.17.** In the hypersurface case, $r = 1$, equation (66) gives the Milnor number of a weighted homogeneous isolated hypersurface singularity, $\mu = \prod_i \frac{w(f_i)}{w(x_i)} - 1$, cf. [Milnor-Oka]. Thus the necessary condition for a monomial
\[ \prod (s_m)^{n_m} \text{ to participate in } \Delta^\perp \text{ is:} \]
\[ \sum_m n_m (1 - \sum_i m_i \frac{w(x_i)}{w(\bar{f})}) = \prod_i \left( \frac{w(\bar{f})}{w(x_i)} - 1 \right). \]

For example, let \( \bar{f} = \sum x_i^{p_i} \) and \( h(\bar{f}, \bar{w}) = \sum_{i=1}^k s_i x_i + s_0. \) Then the only possible monomials are: \( s_0^{n_0} s_1^{n_1} \cdots s_r^{n_r}, \)

where \( n_0 + \sum_{i=1}^r n_i (1 - \frac{1}{p_i}) = \prod (p_i - 1). \)

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