NEF CONE OF FLAG BUNDLES OVER A CURVE

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Abstract. Let $X$ be a smooth projective curve defined over an algebraically closed field $k$, and let $E$ be a vector bundle on $X$. Let $\mathcal{O}_{\text{Gr}_r(E)}(1)$ be the tautological line bundle over the Grassmann bundle $\text{Gr}_r(E)$ parametrizing all the $r$ dimensional quotients of the fibers of $E$. We give necessary and sufficient conditions for $\mathcal{O}_{\text{Gr}_r(E)}(1)$ to be ample and nef respectively. As an application, we compute the nef cone of $\text{Gr}_r(E)$. This yields a description of the nef cone of any flag bundle over $X$ associated to $E$.

1. Introduction

Let $E$ be a semistable vector bundle over a smooth projective curve defined over an algebraically closed field of characteristic zero. Miyaoka computed the nef cone of $\mathbb{P}(E)$ [Mi, p. 456, Theorem 3.1]. Our aim here is to compute the nef cone of the flag bundles associated to vector bundles over curves.

Let $X$ be an irreducible smooth projective curve defined over an algebraically closed field $k$ (the characteristic is not necessarily zero). If the characteristic of $k$ is positive, the absolute Frobenius morphism of $X$ will be denoted by $F_X$. A vector bundle $E$ on $X$ is called strongly semistable if all the pullbacks of $E$ by the iterations of $F_X$ are semistable.

Let $E$ be a vector bundle on $X$. Let
\begin{equation}
E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_m = E
\end{equation}
be the Harder–Narasimhan filtration of $E$. If the characteristic of $k$ is zero, and
\[ f : Y \longrightarrow X \]
is a nonconstant morphism, where $Y$ is an irreducible smooth projective curve, then the pulled back filtration
\[ f^*E_1 \subset f^*E_2 \subset \cdots \subset f^*E_{m-1} \subset f^*E_m = f^*E \]
coincides with the Harder–Narasimhan filtration of $f^*E$. If the characteristic of $k$ is positive, then this is not true in general. However, there is an integer $n_E$, that depends on $E$, such that the Harder–Narasimhan filtration of $(F_X^n)^*E$ has this property if $n \geq n_E$, meaning the Harder–Narasimhan filtration of $f^*(F_X^n)^*E$ is the pullback, by $f$, of the Harder–Narasimhan filtration of $(F_X^n)^*E$, where $f$ is any nonconstant morphism to $X$ from an irreducible smooth projective curve.

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Fix an integer \( r \in [1, \text{rank}(E) - 1] \). Let \( \text{Gr}_r(E) \) be the Grassmann bundle on \( X \) parametrizing all the \( r \) dimensional quotients of the fibers of \( E \). The tautological line bundle on \( \text{Gr}_r(E) \) will be denoted by \( \mathcal{O}_{\text{Gr}_r(E)}(1) \).

If the characteristic of \( k \) is positive, consider the Harder–Narasimhan filtration of \( (F_X^{n_E})^*E \)

\[
0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{d-1} \subset V_d = (F_X^{n_E})^*E,
\]

where \( n_E \) is as above; if the characteristic of \( k \) is zero, then simply take the Harder–Narasimhan filtration of \( E \). So \( V_i \) is \( E_i \) in (1.1) if the characteristic of \( k \) is zero. Using only the numerical data associated to this filtration, we can compute a rational number \( \theta_{E,r} \) (see (3.5)). The following theorem shows that \( \theta_{E,r} \) controls the positivity of the tautological line bundle \( \mathcal{O}_{\text{Gr}_r(E)}(1) \) on \( \text{Gr}_r(E) \).

**Theorem 1.1.** If \( \theta_{E,r} > 0 \), then the tautological line bundle \( \mathcal{O}_{\text{Gr}_r(E)}(1) \) is ample.

If \( \theta_{E,r} = 0 \), then \( \mathcal{O}_{\text{Gr}_r(E)}(1) \) is nef but not ample.

If \( \theta_{E,r} < 0 \), then \( \mathcal{O}_{\text{Gr}_r(E)}(1) \) is not nef.

(See Theorem 3.4 for a proof of the above theorem.)

As an application of Theorem 1.1, we compute the nef cone of \( \text{Gr}_r(E) \) (this is done in Section 4).

In order to know the nef cone of a flag bundle over \( X \) associated to \( E \), it is enough to know the nef cones of the corresponding Grassmann bundles associated to \( E \). Therefore, using our description of the nef cone of the Grassmann bundles we obtain a description of the nef cone of any flag bundle over \( X \) associated to \( E \); see Theorem 5.1.

Let \( K^{-1}_{\varphi} := K^{-1}_{\text{Gr}_r(E)} \otimes \varphi^*K_X \) be the relative anti-canonical line bundle for the natural projection \( \varphi : \text{Gr}_r(E) \to X \). It is known that \( K^{-1}_{\varphi} \) is never ample. If the characteristic of \( k \) is zero, then \( K^{-1}_{\varphi} \) is nef if and only if \( E \) is semistable [BB]; if the characteristic of \( k \) is positive, then \( K^{-1}_{\varphi} \) is nef if and only if \( E \) is strongly semistable [BH]. These criteria for semistability and strong semistability follow from the description of the nef cone of \( \text{Gr}_r(E) \) given in Proposition 4.1 and Proposition 4.3.

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2. **Preliminaries**

Let \( k \) be an algebraically closed field. Let \( X \) be an irreducible smooth projective curve defined over \( k \). If the characteristic of \( k \) is positive, then we have the absolute Frobenius morphism

\[
F_X : X \to X.
\]
For convenience, if the characteristic of \( k \) is zero, by \( F_X \) we will denote the identity morphism of \( X \). For any integer \( m \geq 1 \), let
\[
F_X^m := F_X \circ \cdots \circ F_X : X \to X
\]
be the \( m \)-fold iteration of \( F_X \). For notational convenience, by \( F_X^0 \) we will denote the identity morphism of \( X \).

For a vector bundle \( E \) over \( X \) of positive rank, define the number
\[
\mu(E) := \frac{\text{degree}(E)}{\text{rank}(E)} \in \mathbb{Q}.
\]

A vector bundle \( E \) over \( X \) is called semistable if for every nonzero subbundle \( V \subset E \), the inequality
\[
\mu(V) \leq \mu(E)
\]
holds. The vector bundle \( E \) is called strongly semistable if the pullback \( (F_X^m)^*E \) is semistable for all \( m \geq 0 \).

For every vector bundle \( E \) on \( X \), there is a unique filtration of subbundles
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{d_E-1} \subset E_{d_E} = E
\]
such that \( E_i/E_{i-1} \) is semistable for each \( i \in [1,d_E] \), and \( \mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i) \) for all \( i \in [1,d_E-1] \). It is known as the Harder–Narasimhan filtration of \( E \). If \( E \) is semistable, then \( d_E = 1 \).

Given any \( E \), there is a nonnegative integer \( \delta \) satisfying the condition that for all \( i \geq 1 \),
\begin{equation}
0 = (F_X^i)^*V_0 \subset (F_X^i)^*V_1 \subset \cdots \subset (F_X^i)^*V_{d_E-1} \subset (F_X^i)^*V_d = (F_X^{i+\delta})^*E
\end{equation}
is the Harder–Narasimhan filtration of \( (F_X^{i+\delta})^*E \), where
\begin{equation}
0 = V_0 \subset V_1 \subset \cdots \subset V_{d_E-1} \subset V_d = (F_X^\delta)^*E
\end{equation}
is the Harder–Narasimhan filtration of \( (F_X^\delta)^*E \) [Lan, p. 259, Theorem 2.7] (this is vacuously true if the characteristic of \( k \) is zero). It should be emphasized that \( \delta \) in (2.1) depends on \( E \).

Note that the quotient \( V_i/V_{i-1} \) in the filtration in (2.2) is strongly semistable for all \( i \in [1,d] \). If \( \delta \) satisfies the above condition, then clearly \( \delta + j \) also satisfies the above condition for all \( j \geq 0 \).

For a vector bundle \( E \) on \( X \), let \( \mathbb{P}(E) \) denote the projective bundle over \( X \) parametrizing all the hyperplanes in the fibers of \( E \). The vector bundle \( E \) is called ample if the tautological line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \) on \( \mathbb{P}(E) \) is ample (see [Ha] for properties of ample bundles).

A line bundle \( L \) over an irreducible projective variety \( Z \) defined over \( k \) is called numerically effective ("nef" for short) if for all pairs of the form \((C,f)\), where \( C \) is a smooth projective curve, and \( f \) is a morphism from \( C \) to \( Z \), the inequality
\[
\text{degree}(f^*L) \geq 0
\]
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holds. A vector bundle $E$ is called nef if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is nef.

The following lemma is well-known.

**Lemma 2.1.** Let $0 \to W \to E \to Q \to 0$ be a short exact sequence of vector bundles. If both $W$ and $Q$ are ample (respectively, nef), then $E$ is ample (respectively, nef).

See [Ha, p. 71, Corollary 3.4] for the case of ample bundles, and [DPS, p. 308, Proposition 1.15(ii)] for the case of nef vector bundles.

3. (Semi)positivity criterion

Let $E$ be a vector bundle over $X$ of rank at least two. Fix an integer $r \in [1, \text{rank}(E) - 1]$. Let

$$\varphi : \text{Gr}_r(E) \to X$$

be the Grassmann bundle over $X$ parametrizing all the quotients, of dimension $r$, of the fibers of $E$. Let

$$\mathcal{O}_{\text{Gr}_r(E)}(1) \to \text{Gr}_r(E)$$

be the tautological line bundle; the fiber of $\mathcal{O}_{\text{Gr}_r(E)}(1)$ over any quotient $Q$ of $E_x$ is $\bigwedge^r Q$. So the line bundle $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is relatively ample.

Take any $\delta$ satisfying the condition in (2.1). Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = (F^\delta_X)^*E$$

be the Harder–Narasimhan filtration of $(F^\delta_X)^*E$. We recall that $V_i/V_{i-1}$ is strongly semistable for all $i \in [1, d]$. Let

$$t \in [1, d]$$

be the unique largest integer such that

$$\sum_{i=t}^{d} \text{rank}(V_i/V_{i-1}) \geq r;$$

so either $t = d$, or $t$ is the smallest integer with

$$\sum_{i=t+1}^{d} \text{rank}(V_i/V_{i-1}) = \text{rank}((F^\delta_X)^*E)/V_i < r.$$

Define

$$\theta_{E,r} := (r - \text{rank}(((F^\delta_X)^*E)/V_i)) \cdot \mu(V_i/V_{i-1}) + \text{degree}(((F^\delta_X)^*E)/V_i),$$

where $t$ is defined above using (3.4). If $E$ is strongly semistable, then we may take $\delta = 1$; in that case, $\theta_{E,r} = r \cdot \mu(E)$. Note that the condition that $\theta_{E,r}$ is nonzero, or the condition that $\theta_{E,r}$ is positive, does not depend on the choice of the integer $\delta$ in (3.3).

**Lemma 3.1.** Assume that $\theta_{E,r} > 0$. Then the line bundle $\mathcal{O}_{\text{Gr}_r(E)}(1) \to \text{Gr}_r(E)$ in (3.2) is ample.
Proof. Consider the Plücker embedding
\begin{equation}
\rho : \text{Gr}_r(E) \rightarrow \mathbb{P}(\wedge^r E).
\end{equation}
We have
\begin{equation}
\rho^* \mathcal{O}_{\mathbb{P}(\wedge^r E)}(1) = \mathcal{O}_{\text{Gr}_r(E)}(1).
\end{equation}
Therefore, to prove that $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is ample, it suffices to show that the vector bundle $\wedge^r E$ is ample. Since $F_\delta^* X$ is a finite flat surjective morphism, it follows that $\wedge^r E$ is ample if and only if $(F_\delta^* X)^* \wedge^r E$ is ample [Ha, p. 73, Proposition 4.3].

Using the filtration in (3.3) it follows that the vector bundle $(F_\delta^* X)^* \wedge^r E$ admits a filtration of subbundles such that each successive quotient is of the form
\begin{equation}
V_a := \bigotimes_{i=1}^d \wedge a_i (V_i/V_{i-1})
\end{equation}
with $\sum_{i=1}^d a_i = r$; we use the standard convention that $\wedge^0 F$ is the trivial line bundle for every vector bundle $F$. Since each $V_i/V_{i-1}$ is strongly semistable, the above vector bundle $V_a$ is also strongly semistable (see [RR, p. 285, Theorem 3.18] for $\text{Char}(k) = 0$, and [RR, p. 288, Theorem 3.23] for $\text{Char}(k) > 0$). From the assumption that $\theta_{E,r} > 0$ it follows immediately that
\begin{equation}
\text{degree}(V_a) > 0.
\end{equation}
Since $V_a$ is strongly semistable of positive degree, it can be shown that $V_a$ is ample [BP]. We include the details for completeness.

To prove that $V_a$ is ample, we need to show that for any coherent sheaf $\mathcal{E}$ on $X$, there is a positive integer $b_\mathcal{E}$ such that
\begin{equation}
H^1(X, \text{Sym}^j(V_a) \otimes \mathcal{E}) = 0
\end{equation}
for all $j \geq b_\mathcal{E}$ [Ha, p. 70, Proposition 3.3]. Since $H^1(X, \text{Sym}^j(V_a) \otimes \mathcal{E}) = 0$ if $\mathcal{E}$ is a torsion sheaf, and any vector bundle on $X$ admits a filtration of subbundles such that each successive quotient is a line bundle, it is enough to prove (3.10) for all line bundles $\mathcal{E}$. Take a line bundle $\mathcal{E}$. Since $V_a$ is strongly semistable, it follows that $\text{Sym}^j(V_a)$ is semistable for all $j \geq 1$ (see [RR, p. 285, Theorem 3.18] for $\text{Char}(k) = 0$, and [RR, p. 288, Theorem 3.23] for $\text{Char}(k) > 0$). Therefore, the vector bundle $\text{Sym}^j(V_a)^* \otimes \mathcal{E}^* \otimes K_X$ is semistable. Now, from (3.9) we conclude that
\begin{equation}
\mu(\text{Sym}^j(V_a)^* \otimes \mathcal{E}^* \otimes K_X) = -j \cdot \mu(V_a) - \text{degree}(\mathcal{E}) + 2(\text{genus}(X) - 1) < 0
\end{equation}
for all $j$ sufficiently large positive. Consequently,
\begin{equation}
H^0(X, \text{Sym}^j(V_a)^* \otimes \mathcal{E}^* \otimes K_X) = 0
\end{equation}
for all $j$ sufficiently large positive. Therefore, from Serre duality,
\begin{equation}
H^1(X, \text{Sym}^j(V_a) \otimes \mathcal{E}) = 0
\end{equation}
for all $j$ sufficiently large positive. Hence $V_a$ is ample.

We note that if the characteristic of $k$ is zero, then the nef cone of the projective bundle $\mathbb{P}(V_a)$ is explicitly described in [Mi, p. 456, Theorem 3.1(4)]. It is straightforward to check
that the tautological line bundle $O_{P(V^r)}(1)$ lies in the interior of the nef cone of $P(V^r)$. This also proves that $V^r$ is ample under the assumption that the characteristic of $k$ is zero.

Since $V^r$ is ample, and $(F^{\delta}_X)^* \wedge^r E$ admits a filtration of subbundles such that each successive quotient is of the form $V^r$, using Lemma 2.1 we conclude that the vector bundle $(F^{\delta}_X)^* \wedge^r E$ is ample. We noted earlier that $O_{Gr_r(E)} (1)$ is ample if $(F^{\delta}_X)^* \wedge^r E$ is ample. □

Lemma 3.2. Assume that $\theta_{E,r}$ defined in (3.5) satisfies the inequality $\theta_{E,r} < 0$. Then $O_{Gr_r(E)}(1)$ is not nef.

Proof. Consider the strongly semistable vector bundle $V_t/V_{t-1}$ (see (3.5)). Given any real number $\epsilon > 0$, and any $s \in [1, \text{rank}(V_t/V_{t-1})]$, there exists an irreducible smooth projective curve $Y$, a nonconstant morphism

$$f : Y \longrightarrow X,$$

and a subbundle

(3.11) \[ W \subset f^*(V_t/V_{t-1}) \]

of rank $s$, such that

(3.12) \[ \frac{\mu(V_t/V_{t-1}) - \mu(W)}{\text{degree}(f)} = \frac{\mu(f^*(V_t/V_{t-1})) - \mu(W)}{\text{degree}(f)} < \epsilon \]

(see [PS, p. 525, Theorem 4.1]). Set

$$s = r - \text{rank}((F^{\delta}_X)^*E)/V_t \quad \text{and} \quad \epsilon = -\frac{\theta_{E,r}}{2s}.$$

Let $Q$ be the quotient of $f^*(F^{\delta}_X)^*E$ defined by the composition

$$f^*(F^{\delta}_X)^*E \longrightarrow f^*((F^{\delta}_X)^*E)/V_{t-1} \longrightarrow f^*((F^{\delta}_X)^*E)/V_{t-1}/W,$$

where $f$ and $W$ are as in (3.11) for the above choices of $s$ and $\epsilon$. Note that

$$\text{degree}(Q) = \text{degree}(f) \cdot \text{degree}(((F^{\delta}_X)^*E)/V_t) + \text{degree}(f) \cdot \text{degree}(V_t/V_{t-1} - \text{degree}(W)).$$

Hence from (3.5),

$$\text{degree}(Q) = \text{degree}(f)(\theta_{E,r} + \mu(V_t/V_{t-1}) - \frac{\mu(W)}{\text{degree}(f)}) \cdot s).$$

But from (3.12), we have $\mu(V_t/V_{t-1}) - \mu(W)/\text{degree}(f) < \epsilon$. Consequently,

(3.13) \[ \text{degree}(Q) < 0. \]

The quotient bundle $f^*(F^{\delta}_X)^*E \longrightarrow Q$ of rank $r$ defines a morphism

$$\phi : Y \longrightarrow \text{Gr}_r((F^{\delta}_X)^*E) = (F^{\delta}_X)^*\text{Gr}_r(E),$$

where $\text{Gr}_r((F^{\delta}_X)^*E)$ is the Grassmann bundle parametrizing all $r$ dimensional quotients of the fibers of $(F^{\delta}_X)^*E$, and $(F^{\delta}_X)^*\text{Gr}_r(E)$ is the pullback of the fiber bundle $\text{Gr}_r(E) \longrightarrow X$.
using the morphism $F^\delta_X$. Consider the commutative diagram
\[
\begin{array}{ccc}
(F^\delta_X)^* \text{Gr}_r(E) & \xrightarrow{\beta} & \text{Gr}_r(E) \\
\downarrow & & \downarrow \\
X & \xrightarrow{F^\delta_X} & X
\end{array}
\]
(3.14)
of morphisms. We have $\beta^* \mathcal{O}_{\text{Gr}_r(E)}(1) = \mathcal{O}_{\text{Gr}_r((F^\delta_X)^* E)}(1)$, where $\mathcal{O}_{\text{Gr}_r((F^\delta_X)^* E)}(1)$ is the tautological line bundle, and $\beta$ is the morphism in (3.14). Hence from the definition of $\phi$ it follows immediately that
\[
(\beta \circ \phi)^* \mathcal{O}_{\text{Gr}_r(E)}(1) = \bigwedge^r Q.
\]
Now from (3.13) we conclude that $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is not nef. $\square$

**Lemma 3.3.** Assume that $\theta_{E,r} = 0$ (defined in (3.5)). Then $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is nef but not ample.

**Proof.** The proof that $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is nef is very similar to the proof of Lemma 3.1.

We know that $\bigwedge^r E$ is nef if and only if $(F^\delta_X)^* \bigwedge^r E$ is nef [Fu, p. 360, Proposition 2.3] and [Fu, p. 360, Proposition 2.2]. Consider the vector bundles $V_\underline{a}$ in (3.8). We noted earlier that $V_\underline{a}$ is strongly semistable. The condition that $\theta_{E,r} = 0$ implies that $\text{degree}(V_\underline{a}) \geq 0$.

A strongly semistable vector bundle $W$ over $X$ of nonnegative degree is nef. To prove this, take any morphism
\[
\psi : Y \longrightarrow \mathbb{P}(W),
\]
where $Y$ is an irreducible smooth projective curve. Let $h : \mathbb{P}(W) \longrightarrow X$ be the natural projection. The pullback $\psi^* h^* W$ is semistable because $W$ is strongly semistable. Since $\psi^* \mathcal{O}_{\mathbb{P}(W)}(1)$ is a quotient of $\psi^* h^* W$, and $\text{degree}(\psi^* h^* W) \geq 0$, we conclude that $\text{degree}(\psi^* \mathcal{O}_{\mathbb{P}(W)}(1)) \geq 0$. Hence $\mathcal{O}_{\mathbb{P}(W)}(1)$ is nef, meaning $W$ is nef.

The above observation implies that the vector bundle $V_\underline{a}$ is nef.

Since each successive quotient of the filtration of $(F^\delta_X)^* \bigwedge^r E$ is nef (as they are of the form $V_\underline{a}$), from Lemma 2.1 we know that $(F^\delta_X)^* \bigwedge^r E$ is nef. We noted earlier that $\bigwedge^r E$ is nef if $(F^\delta_X)^* \bigwedge^r E$ is so. Now using (3.6) and (3.7) we conclude that $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is nef.

To complete the proof of the lemma we need to show that $\mathcal{O}_{\text{Gr}_r(E)}(1)$ is not ample.

Consider $V_t/V_{t-1}$ in (3.3). Let
\[
f : \text{Gr}_s(V_t/V_{t-1}) \longrightarrow X
\]
be the Grassmann bundle parametrizing quotients of the fibers of $V_t/V_{t-1}$ of dimension
\[
s := r - \text{rank}((F^\delta_X)^* E)/V_t).
\]
Let
\[
\gamma : \text{Gr}_s(V_t/V_{t-1}) \longrightarrow \text{Gr}_r((F^\delta_X)^* E)
\]
(3.17)
be the morphism of fiber bundles over $X$ that sends any quotient $q : (V_t/V_{t-1})_x \to Q$ to the quotient defined by the composition

$$(F^\delta_X)^*E_x \to (((F^\delta_X)^*E)/V_{t-1})_x \to (((((F^\delta_X)^*E)/V_{t-1})_x)/\ker(q).$$

To define $\gamma$ using the universal property of a Grassmannian, let

$$f^*(V_t/V_{t-1}) \to \tilde{q} \to Q \to 0$$

be the universal quotient bundle of rank $s$ over $Gr_s(V_t/V_{t-1})$. Now consider the diagram of homomorphisms

$$\begin{array}{cccc}
\ker(\tilde{q}) & \hookrightarrow & V_t/V_{t-1} & \xrightarrow{\tilde{q}} & Q \\
\cap & & \cap & & \\
f^*(F^\delta_X)^*E & \xrightarrow{\tilde{q}} & ((F^\delta_X)^*E)/V_{t-1} & \xrightarrow{h} & ((F^\delta_X)^*E)/V_{t-1}/\ker(\tilde{q})
\end{array}$$

Note that $\operatorname{rank}(((F^\delta_X)^*E)/V_{t-1})/\ker(\tilde{q})) = r$ by (3.16). Let

$$\tilde{\gamma} : Gr_s(V_t/V_{t-1}) \to Gr_r(F^\delta_X)^*E = Gr_s(V_t/V_{t-1}) \times_X Gr_r((F^\delta_X)^*E)$$

be the morphism representing the surjective homomorphism $h \circ \tilde{q}$ in the above diagram. The morphism $\gamma$ in (3.17) is the composition of $\tilde{\gamma}$ with the natural projection $Gr_s(V_t/V_{t-1}) \times_X Gr_r((F^\delta_X)^*E) \to Gr_r((F^\delta_X)^*E)$.

The morphism $\gamma$ in (3.17) is clearly an embedding. Define the line bundle

$$\mathcal{L} := \det(((F^\delta_X)^*E)/V_t) = \bigotimes_{i=t+1}^d \bigwedge_{\operatorname{rank}(V_t/V_{t-1})}^{\operatorname{rank}(V_t/V_{t-1})} (V_t/V_{t-1})$$

on $X$. We note that

\begin{equation}
(3.18) \quad \gamma^*\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(1) = \mathcal{O}_{Gr_s(V_t/V_{t-1})}(1) \otimes f^*\mathcal{L},
\end{equation}

where $\mathcal{O}_{Gr_s(V_t/V_{t-1})}(1) \to Gr_s(V_t/V_{t-1})$ is the tautological line bundle.

For any integer $n$, the line bundles $\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(1)\otimes^n$ and $\mathcal{O}_{Gr_s(V_t/V_{t-1})}(1)\otimes^n$ will be denoted by $\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(n)$ and $\mathcal{O}_{Gr_s(V_t/V_{t-1})}(n)$ respectively.

Assume that $\mathcal{O}_{Gr_r(E)}(1)$ is ample. Since $F^\delta_X$ is a finite morphism, this implies that $\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(1)$ is ample. Therefore, the pullback $\gamma^*\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(1)$ is ample because $\gamma$ is an embedding. Hence for sufficiently large positive $n$, we have

\begin{equation}
(3.19) \quad \dim H^0(Gr_s(V_t/V_{t-1}), \gamma^*\mathcal{O}_{Gr_r((F^\delta_X)^*E)}(n)) = cn^d_o + \sum_{j=0}^{d_o-1} a_j n^j
\end{equation}

with $c > 0$, where $d_o = \dim Gr_s(V_t/V_{t-1})$.

For convenience, the integer $\operatorname{rank}(V_t/V_{t-1})$ will be denoted by $r_t$.

Let $K_f^{-1} := K_{Gr_s(V_t/V_{t-1})}^{-1} \otimes f^*K_X$ be the relative anti-canonical line bundle for the projection $f$ in (3.15). We have,

\begin{equation}
(3.20) \quad K_f^{-1} = \mathcal{O}_{Gr_s(V_t/V_{t-1})}(r_t) \otimes ((\bigwedge^{r_t}(V_t/V_{t-1}))^{\otimes s})^*,
\end{equation}
where \( s \) is defined in (3.16). The given condition that \( \theta_{E,r} = 0 \) implies that

\[-s \cdot \text{degree}(V_i/V_{i-1}) = r_t \cdot \text{degree}(((F^s_{\mathcal{K}})^*E)/V_i)\,.

Hence the two line bundles \(((\bigwedge^{r_t}(V_i/V_{i-1}))^{\otimes n})^*\) and \(\mathcal{L}^{\otimes r_t}\) differ by tensoring with a line bundle of degree zero. Therefore, from (3.20) we conclude that

\[
\gamma^* \mathcal{O}_{\text{Gr}_{r_t}(F^s_{\mathcal{K}})^*(E)}(r_t) = K_f^{-1} \otimes f^* \mathcal{L}_0,
\]

where \( \mathcal{L}_0 \) is a line bundle on \( X \) of degree zero. Now, from (3.18),

\[
(3.21)
\]

\[
\gamma^* \mathcal{O}_{\text{Gr}_{r_t}(F^s_{\mathcal{K}})^*(E)} = K_f^{-1} \otimes f^* \mathcal{L}_0.
\]

From the projection formula, and (3.21),

\[
(3.22)
H^0(\text{Gr}_{r_t}(V_i/V_{i-1}), \gamma^* \mathcal{O}_{\text{Gr}_{r_t}(F^s_{\mathcal{K}})^*(E)}(n \cdot r_t)) = H^0(X, (f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}).
\]

We will show that the line bundle \( \det(f_*(K_f^{-1})^{\otimes n}) \to X \) is trivial. For that, let \( F_{\text{Gl}_{r_t}} \) be the principal \( \text{Gl}_{r_t}(k) \)–bundle on \( X \) defined by the vector bundle \( V_i/V_{i-1} \); the fiber of \( F_{\text{Gl}_{r_t}} \) over any point \( x \in X \) is the space of all linear isomorphisms from \( k^{\otimes r_t} \) to the fiber \((V_i/V_{i-1})_x\). Let \( F_{\text{PGL}_{r_t}} := F_{\text{Gl}_{r_t}}/G_m \) be the corresponding principal \( \text{PGL}_{r_t}(k) \)–bundle. The vector bundle \( f_*(K_f^{-1})^{\otimes n} \) is the one associated to the principal \( \text{PGL}_{r_t}(k) \)–bundle \( F_{\text{PGL}_{r_t}} \) for the \( \text{PGL}_{r_t}(k) \)–module \( H^0(\text{Gr}_s(k^{\otimes r_t}), (K_{\text{Gr}_{r_t}(k^{\otimes r_t})})^{\otimes n}) \) (the action of \( \text{PGL}_{r_t}(k) \) on the space of sections is given by the standard action of \( \text{PGL}_{r_t}(k) \) on \( \text{Gr}_s(k^{\otimes r_t}) \). Since \( \text{PGL}_{r_t}(k) \) does not have any nontrivial character, the line bundle \( \det(f_*(K_f^{-1})^{\otimes n}) \) associated to \( F_{\text{PGL}_{r_t}} \) for the \( \text{PGL}_{r_t}(k) \)–module \( \bigwedge^{\text{top}} H^0(\text{Gr}_s(k^{\otimes r_t}), (K_{\text{Gr}_{r_t}(k^{\otimes r_t})})^{\otimes n}) \) is trivial.

As \( \det(f_*(K_f^{-1})^{\otimes n}) \) is trivial and degree(\( \mathcal{L} \)) = 0,

\[
\text{degree}(((f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}) = 0.
\]

Since \( V_i/V_{i-1} \) is strongly semistable, the corresponding principal \( \text{Gl}_{r_t}(k) \)–bundle \( F_{\text{Gl}_{r_t}} \) is strongly semistable. Therefore, the associated vector bundle \( f_*(K_f^{-1})^{\otimes n} \) is also semistable (see [RR] p. 285, Theorem 3.18] and [RR] p. 288, Theorem 3.23]). This implies that \( (f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n} \) is semistable.

For a semistable vector bundle \( \mathcal{V} \) on \( X \) of degree zero, any nonzero section \( \sigma : \mathcal{O}_X \to \mathcal{V} \) is nowhere vanishing. Indeed, this follows immediately from the semistability condition that the line bundle of \( \mathcal{V} \) generated by the image of \( \sigma \) is of nonpositive degree. Consequently,

\[
\dim H^0(X, \mathcal{V}) \leq \text{rank}(\mathcal{V}).
\]

Since \( (f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n} \) is semistable of degree zero, we have

\[
(3.23)
\dim H^0(X, (f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}) \leq \text{rank}((f_*(K_f^{-1})^{\otimes n}) \otimes \mathcal{L}_0^{\otimes n}) = \text{rank}((f_*(K_f^{-1})^{\otimes n}))
\]

for all \( n > 0 \).

We have \( R^j f_*(K_f^{-1})^{\otimes n} = 0 \) for \( j, n \geq 1 \). Hence from the Riemann–Roch theorem for the restriction \((K_f^{-1})^{\otimes n}|_{f^{-1}(x)}, x \in X \), we conclude that \( \text{rank}((f_*(K_f^{-1})^{\otimes n})) \) is a polynomial of degree at most \( d_0 - 1 \) (which is the dimension of the fibers of \( f \)). Therefore,
using (3.22) and (3.23) we conclude that
\[ \dim H^0(\text{Gr}_s(V_t/V_{t-1}), \gamma^*O_{\text{Gr}_s(F^N_{t+E})(n \cdot \text{rank}(V_t/V_{t-1}))}) \]
is a polynomial of degree at most \( d_0 - 1 \). But this contradicts (3.19).

We assumed that \( O_{\text{Gr}_r(E)}(1) \) is ample, and were led to the above contradiction. Therefore, we conclude that \( O_{\text{Gr}_r(E)}(1) \) is not ample. This completes the proof of the lemma.

\[ \square \]

Lemma 3.1, Lemma 3.2 and Lemma 3.3 together give the following:

\textbf{Theorem 3.4.} If \( \theta_{E,r} > 0 \), then the line bundle \( O_{\text{Gr}_r(E)}(1) \rightarrow \text{Gr}_r(E) \) in (3.2) is ample.

If \( \theta_{E,r} = 0 \), then \( O_{\text{Gr}_r(E)}(1) \) is nef but not ample.

If \( \theta_{E,r} < 0 \), then \( O_{\text{Gr}_r(E)}(1) \) is not nef.

\section{The Nef Cone of \( \text{Gr}_r(E) \)}

In this section we will compute the nef cone of \( \text{Gr}_r(E) \) using Theorem 3.4. Being a closed cone, it is generated by its boundary. For notational reasons, it will be convenient to treat the cases of characteristic zero and positive characteristic separately.

For a smooth projective variety \( Z \), the real Néron–Severi group \( \text{NS}(Z)_{\mathbb{R}} \) is defined to be
\[
(4.1) \quad \text{NS}(Z)_{\mathbb{R}} := (\text{Pic}(Z)/\text{Pic}^0(Z)) \otimes_{\mathbb{Z}} \mathbb{R},
\]
where \( \text{Pic}^0(Z) \) is the connected component, containing the identity element, of the Picard group \( \text{Pic}(Z) \) of \( Z \).

\subsection{Characteristic is zero.} In this case, the number \( \delta \) in (3.5) is zero.

As in (3.1), \( \varphi \) is the projection of \( \text{Gr}_r(E) \) to \( X \). Fix a line bundle \( L_1 \) over \( X \) of degree one. The line bundle \( \varphi^*L_1 \) will be denoted by \( \mathcal{L} \). The real Néron–Severi group \( \text{NS}(\text{Gr}_r(E))_{\mathbb{R}} \) is freely generated by \( \mathcal{L} \) and \( O_{\text{Gr}_r(E)}(1) \).

Although \( \theta_{E,r} \) in (3.5) need not be an integer, we note that \( \mathcal{L}^{\otimes -\theta_{E,r}} \) is well defined as an element of \( \text{NS}(\text{Gr}_r(E))_{\mathbb{R}} \) because \( \theta_{E,r} \in \mathbb{Q} \).

\textbf{Proposition 4.1.} The boundary of the nef cone in \( \text{NS}(\text{Gr}_r(E))_{\mathbb{R}} \) is given by \( \mathcal{L} \) and \( O_{\text{Gr}_r(E)}(1) \otimes \mathcal{L}^{\otimes -\theta_{E,r}}. \)

\textit{Proof.} We will first show that it is enough to treat the case where \( \theta_{E,r} \) is a multiple of \( r \). In fact, this argument is standard (see [Laz, p. 23, Lemma 6.2.8]). However, we describe the details for completeness.

Write
\[ \theta_{E,r} = \frac{p_1 r}{q_1}, \]
where \( p_1 \) and \( q_1 \) are integers with \( q_1 > 0 \). Take a pair \( (Y, f) \), where \( Y \) is an irreducible smooth projective curve, and \( f \) is a morphism from \( Y \) to \( X \), such that degree\((f)\) is a
multiple of $q_1$. The natural map
$$\gamma : \text{Gr}_r(f^*E) \longrightarrow \text{Gr}_r(E)$$
produces an isomorphism between $\text{NS}(\text{Gr}_r(E))_\mathbb{R}$ and $\text{NS}(\text{Gr}_r(f^*E))_\mathbb{R}$. This isomorphism preserves the nef cones. Therefore, it is enough to prove the proposition for $(Y,f^*E)$. Note that $\theta_{f^*E,r} = \frac{\deg(f)p_1r}{q_1}$ is a multiple of $r$.

Hence we can assume that $\theta_{E,r}/r$ is an integer.

Consider the vector bundle
$$F := E \otimes L^{-\theta_{E,r}}_1.$$
Note that $\text{Gr}_r(E) = \text{Gr}_r(F)$. From (3.5) and the definition of $F$ it follows immediately that
$$\theta_{F,r} = 0.$$

Since $\theta_{E,r} = 0$, from the second part of Theorem 3.4 we know that the nef cone in $\text{NS}(\text{Gr}_r(F))_\mathbb{R}$ is generated by $\mathcal{O}_{\text{Gr}_r(F)}(1)$ and $\mathcal{L}$ (it is considered as a line bundle on $\text{Gr}_r(F)$ using the identification of $\text{Gr}_r(F)$ with $\text{Gr}_r(E)$). The proposition follows immediately from this description of the nef cone in $\text{NS}(\text{Gr}_r(F))_\mathbb{R}$ using the identification of $\text{Gr}_r(F)$ with $\text{Gr}_r(E)$.

Remark 4.2. We note that the two generators of the nef cone given in Proposition 4.1 lie in the rational Néron–Severi group $\text{NS}(\text{Gr}_r(E))_\mathbb{Q} := (\text{Pic}(\mathbb{Z})/\text{Pic}^0(\mathbb{Z})) \otimes \mathbb{Q}$.

4.2. Characteristic is positive. Let $p > 0$ be the characteristic of $k$. Consider $\delta$ in (3.5). Let $\varphi_1 : \text{Gr}_r((F^\delta_X)^*E) \longrightarrow X$ be the natural projection. Define the line bundle
$$\mathcal{L}_1 := \varphi_1^*L_1 \longrightarrow X,$$
where $L_1$ is a fixed line bundle on $X$ of degree one.

Lemma 4.3. The nef cone in $\text{NS}(\text{Gr}_r((F^\delta_X)^*E))_\mathbb{R}$ (defined in (4.1)) is generated by $\mathcal{L}_1$ and $\mathcal{O}_{\text{Gr}_r((F^\delta_X)^*E)}(1) \otimes \mathcal{L}_1^{-\theta_{F^\delta_X,r}}$.

Proof. The proof is exactly identical to the proof of Proposition 4.1. We refrain from repeating it.

As in (3.1), the projection of $\text{Gr}_r(E)$ to $X$ will be denoted by $\varphi$. Define
$$\mathcal{L} := \varphi^*L_1.$$

Proposition 4.4. The boundary of the nef cone in $\text{NS}(\text{Gr}_r(E))_\mathbb{R}$ is given by $\mathcal{L}$ and $\mathcal{O}_{\text{Gr}_r(E)}(p^\delta) \otimes \mathcal{L}^{-\theta_{F^\delta_X,r}}$.

Proof. Consider the commutative diagram of morphisms in (3.14). The morphism $\beta$ in this diagram produces an isomorphism between $\text{NS}(\text{Gr}_r(E))_\mathbb{R}$ and $\text{NS}(\text{Gr}_r((F^\delta_X)^*E))_\mathbb{R}$. This isomorphism preserves the nef cones.

We have $\beta^*\mathcal{O}_{\text{Gr}_r(E)}(1) = \mathcal{O}_{\text{Gr}_r((F^\delta_X)^*E)}(1)$, and $(F^\delta_X)^*L_1 = L_1^{\otimes p^\delta}$. Hence the proposition follows from Lemma 4.3.

Remark 4.5. The two generators of the nef cone given in Proposition 4.4 lie in $\text{NS}(\text{Gr}_r(E))_\mathbb{Q}$. 

5. The nef cone of flag bundles

Fix integers
\[ 0 < r_1 < r_2 < \cdots < r_{\nu - 1} < r_{\nu} < \text{rank}(E). \]

Let
\[ \Phi : \text{Fl}(E) \rightarrow X \]
be the corresponding flag bundle; so for any \( x \in X \), the fiber \( \Phi^{-1}(x) \) parametrizes all filtrations of linear subspaces
\[ (5.1) \quad E_x \supset S_1 \supset S_2 \supset \cdots \supset S_{\nu - 1} \supset S_{\nu} \]
such that \( \dim E_x - \dim S_i = r_i \) for all \( i \in [1, \nu] \).

For each \( i \in [1, \nu] \), let \( \text{Gr}_{r_i}(E) \) be the Grassmann bundle over \( X \) parametrizing all the \( r_i \)-dimensional quotients of the fibers of \( E \). Let
\[ (5.2) \quad \phi_i : \text{Fl}(E) \rightarrow \text{Gr}_{r_i}(E) \]
be the natural projection that sends any filtration as in (5.1) to \( E_x/S_i \). Let
\[ \omega_i \in \text{NS(Gr}_{r_i}(E)) \]
be the element \( \mathcal{O}_{\text{Gr}_{r_i}(E)}(1) \otimes \mathcal{L}^{\otimes -\theta_{E, r_i}} \) (respectively, \( \mathcal{O}_{\text{Gr}_{r_i}(E)}(p^\delta) \otimes \mathcal{L}^{\otimes -\theta_{F, r_i}} \)) in Proposition 4.1 (respectively, Proposition 4.4) if the characteristic of \( k \) is zero (respectively, positive). Define
\[ \tilde{\omega}_i := \phi_i^* \omega_i \in \text{NS(Fl}(E)) \]
where \( \phi_i \) is the projection in (5.2).

**Theorem 5.1.** The nef cone in \( \text{NS(Fl}(E)) \) is generated by \( \{ \tilde{\omega}_i \}_{i=1}^{\nu} \bigcup \Phi^* \mathcal{L}' \), where \( \mathcal{L}' \) is a line bundle over \( X \) of degree one.

**Proof.** The dimension of the \( \mathbb{R} \)-vector space \( \text{NS(Fl}(E)) \) is \( \nu + 1 \), and the vector space is generated by \( \{ \tilde{\omega}_i \}_{i=1}^{\nu} \bigcup \Phi^* \mathcal{L}' \). We note that \( \mathcal{L}' \) and all \( \tilde{\omega}_i \) are nef.

Fix any point \( x \in X \). For each \( i \in [1, \nu] \), define
\[ \tilde{\omega}_{x,i} := \tilde{\omega}_i|_{\Phi^{-1}(x)} \in \text{NS(}\Phi^{-1}(x)\text{)}. \]

The dimension of the \( \mathbb{R} \)-vector space \( \text{NS(}\Phi^{-1}(x)\text{)} \) is \( \nu \). It is known that the nef cone of \( \text{NS(}\Phi^{-1}(x)\text{)} \) is generated by \( \{ \tilde{\omega}_{x,i} \}_{i=1}^{\nu} \) (see [B1] p. 187, Theorem 1) for a general result. In view of this, the theorem follows from Proposition 4.1 (respectively, Proposition 4.4) when the characteristic of \( k \) is zero (respectively, positive).

**Remark 5.2.** All the elements of the generating set of the nef cone in \( \text{NS(Fl}(E)) \) given in Theorem 5.1 lie in \( \text{NS(Fl}(E)) \).
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