Special Geometry and Mirror Symmetry for Open String Backgrounds with N=1 Supersymmetry

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Abstract

We review an approach for computing non-perturbative, exact superpotentials for Type II strings compactified on Calabi-Yau manifolds, with extra fluxes and $D$-branes on top. The method is based on an open string generalization of mirror symmetry, and takes care of the relevant sphere and disk instanton contributions. We formulate a framework based on relative (co)homology that uniformly treats the flux and brane sectors on a similar footing. However, one important difference is that the brane induced potentials are of much larger functional diversity than the flux induced ones, which have a hidden $N=2$ structure and depend only on the bulk geometry.

This introductory lecture is meant for an audience unfamiliar with mirror symmetry. The transparencies are available at: http://wwwth.cern.ch/~lerche/papers

Keywords: string theory, Calabi-Yau manifold, mirror symmetry, supersymmetry, superpotential, D-branes.
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References
1 Introduction

Most exactly computable quantities in string compactifications have some underlying “BPS-property”, which leads to special features of the relevant part of the effective lagrangian. Typically, these quantities are holomorphic functions\(^1\) depending on chiral superfields, and the rules of chiral superspace integration imply that such quantities enjoy very special non-renormalization properties. Either such quantities are not corrected at all at the quantum level, or only in a very controlled way, for example perturbative corrections may occur only up to one-loop order. A closely related feature is that the corrections to these special terms are only due to BPS saturated intermediate states; these are under tight analytical control even at the non-perturbative quantum level, and this is what underlies the exact computability of these terms of the effective theory.

The prime example of such a holomorphic BPS quantity is the prepotential \(F\), which figures as (the two-derivative part of) the gauge field effective action of \(N = 2\) supersymmetric string compactifications, and their various field theory limits. It gets radiative corrections to one-loop order only, plus an infinite series of non-perturbative corrections. From the point of view of the string world-sheet, these non-perturbative corrections are due to world-sheet instantons, but on the other hand from the point of view of the effective target space theory, they can often be interpreted terms of gauge theory instantons.

It is known since ten years or so how to compute exact, non-perturbative quantum corrections to the \(N = 2\) prepotential \([1]\), and this has been mainly done by using the methods of mirror symmetry. As we will briefly review in the following, the main ingredients that go into this framework are notions such as Calabi-Yau manifolds, topological field theory, period integrals, the variation of Hodge structures and ”Special Geometry” \([2–6]\); for in-depth references on mirror symmetry, see for example refs. \([7–12]\).

More recently there has been dramatic progress in computing analogous holomorphic quantities also for \(N = 1\) supersymmetric string and field theories. The most important such quantity is, of course, the effective superpotential \(W\), which determines the vacuum structure of the theory. Another important holomorphic quantity is the gauge coupling function \(\tau\), which multiplies the gauge kinetic term: \(\mathcal{L} \sim \tau \text{Tr} W^\alpha W_\alpha\). The main new physical ingredient are \(D\)-branes and background fluxes, whose presence in a Type II string compactification on a Calabi-Yau space reduces the supersymmetry from \(N = 2\) to \(N = 1\).

Such “open-closed Type II string backgrounds” can be crudely labelled by triples \((X, N_a; \hat{N}_a)\), where \(X\) is some compact or non-compact Calabi-Yau manifold, \(N_a\) denotes the flux numbers and \(\hat{N}_a\) denotes the brane numbers; the first two entries pertain to the closed string sector while the last entry pertains to the open string sector. In addition to these discrete data there will in general be a number of moduli from both of the closed and open string sectors.

These backgrounds may be viewed as specific parametrizations of an extremely large \([13]\) class of four-dimensional string theories with \(N = 1\) supersymmetry, and represent toy models that combine interesting physics, characteristic predictions and computability of the effective superpotential and gauge couplings. They are certainly much more complicated as compared to the ordinary Calabi-Yau backgrounds \((X, 0; 0)\) with \(N = 2\) supersymmetry, and this reflects the growing complexity of theories with less supersymmetry. This is the reason why it took quite some time until we could quantitatively access these phenomenologically

\(^1\)Rather: sections, as they are multi-valued.
much more important $N = 1$ supersymmetric theories. By “quantitatively” we mean that we now have good analytical control over the vacuum structure, and the ability to exactly and systematically compute effects such as non-perturbative supersymmetry breaking\(^2\) for many choices of $(X, N_a, \tilde{N}_a)$.

There are by now several techniques available for computing exact non-perturbative effective superpotentials (and other holomorphic quantities related to higher genus world-sheets) in the presence of D-branes,\(^3\) which are interrelated and each of which has certain merits and limitations. For example, the approaches based on boundary superconformal field theory \([15]\) and on the mathematics of coherent sheaves \([16]\) have provided important insights into the non-perturbative regime of D-brane configurations. The approach via large $N$ transitions \([17–19]\) and Chern-Simons theory \([20–23]\) has proven to be extremely powerful for a certain class of Calabi-Yau string compactifications with branes, while matrix models have been especially useful for describing $N = 1$ gauge theories \([24–26]\). Very recently, a powerful framework was developed \([27, 28]\), which unifies these and other viewpoints, and this gives an entirely new way \([29]\) to access the strong coupling regime of $N = 1$ supersymmetric theories.

Less widely known among physicists but powerful as well is a mathematically inspired method based on the localization of the path integral on certain fixed points of torus actions; for some expositions, see e.g. \([30, 31]\) (for a different perspective on localization, see: \([32, 33]\)). Moreover there are geometrical approaches involving mirror symmetry \([34–37]\), and in particular there is also an approach based on a direct generalization of closed string mirror symmetry to open string backgrounds \([38–41]\) - it is this latter approach what we will review in the present lectures.

Before ending these introductory remarks, we like to add that it is of course not just computing superpotentials what we may want to do with these techniques. A conceptually more interesting use is to get a better understanding of the stringy quantum geometry of D-branes; that is, the way classical geometrical notions such as manifold, curvature and gauge field configurations, are modified due to non-perturbative quantum corrections. Sometimes such corrections can completely blur out classical notions such as volume and dimension.

More specifically, it is important to realize that the notions of classical geometry, such as a D-brane wrapping a $p$-dimensional cycle in a manifold, apply generically only to situations with weak string coupling, small curvatures and large volumes. If we move away from the large volume limit of the compactification manifold, the effects of world-sheet instantons become stronger and stronger until the classical picture is lost. Similar to what happens in $N = 2$ Yang-Mills theory where we find massless monopoles at strong coupling \([42]\), novel phenomena can become visible in the non-classical regime, which may be better understood by switching to a more suitable description of them in terms of new, weakly coupled physical degrees of freedom. If we move to weak coupling again, we may end up, due to monodromy, with a completely “different” classical brane configuration that wraps a collection of cycles with dimensions different to the ones we had before; this shows that in stringy geometry there is no absolute and invariant notion of the dimension of a $p$-cycle.

Note however, that there is an important difference as compared to the well-understood

\(^2\)For an example, see e.g., \([14]\).

\(^3\)There are many works dealing with compactifications of $M$ and $F$ theory, as well brane constructions, but we will discuss here only Type II strings on Calabi-Yau manifolds with D-branes on top.
theories with $N = 2$ supersymmetry; for these, there is a continuous manifold of vacuum states, the moduli space, and it makes good sense to consider continuous deformations and perform analytic continuation into regimes with strong coupling. On the other hand, for the $N = 1$ theories the very presence of a superpotential represents an obstruction to continuous deformations, and this is often a direct consequence of mathematical obstruction properties of $p$-cycles. Thus it is a priori not clear to what extent arguments that rely on continuous deformations can be trusted - even more so because the presence of a superpotential renders the theory off-shell (unless we happen to sit at a critical point), for which instance the usual formulation of string theory based on conformal field theory is not well suited. However, we will not consider ordinary strings but rather topological strings, for which being off-shell is not so problematic and for which the computation of correlators just boils down to certain computations in algebraic geometry - namely to more or less exactly the ones that expose the mathematical obstruction properties of $p$-cycles. One expects that the topological strings are equivalent to the untwisted ones as far as the BPS sector (e.g., the holomorphic superpotential in the effective action) is concerned, and that’s anyway all what we expect to be able to compute exactly.

2 Recapitulation of $N = 2$ “closed string” mirror symmetry

Since the approach for computing $N = 1$ superpotentials we want to explain is an open string generalization of the well-known approach for computing $N = 2$ prepotentials via mirror symmetry, we first need to review the latter. However, because this is a subject that is well-understood since quite some time and about which there exists a collection of excellent review papers and books [7–12] (and especially also because topological field theories were lectured upon in great detail in Hiroshi Ooguri’s lectures in this school), we will be brief and just quickly go over the key points, with emphasis on the topics that we will need to know later.

2.1 Type II strings on Calabi-Yau manifolds

A Type II string compactification to four dimensions with $N = 2$ space-time supersymmetry is described at large radius by a two-dimensional sigma model with $c = 9$ and $(2, 2)$ world-sheet supersymmetry, with target space $X$ (this is the internal, compact part of the theory only; of course, we implicitly need to add the non-compact space-time sector plus the appropriate ghost fields). In order to preserve $N = 2$ space-time supersymmetry, the target space $X$ must be a three complex-dimensional Calabi-Yau manifold, which more or less by definition has the special property that it preserves a covariantly constant spinor which can serve as the supercharge. More precisely, a Calabi-Yau manifold has the following equivalent properties:

$$
\text{Calabi – Yau manifold} \iff \begin{cases}
    c_1(R) = 0 \\
    \text{holonomy group } = SU(3) \\
    \exists \text{ unique global holomorphic form } \Omega^{3,0} \in H^{3,0}(X)
\end{cases}
$$
besides being a Kähler manifold. The latter means that the metric can be written as derivative of a generating function, the Kähler potential, $g_{ij} = \partial_i \bar{\partial}_j K$. Associated with the metric is the Kähler form, $J^{1,1} = ig_{ij} dz^i \bar{d}z^j \in H^{1,1}(X)$, which as we will see, plays an important role in the deformation theory of Calabi-Yau spaces.

The $N = 2$ effective action in four dimensions contains various massless fields, notably hyper- and vector-supermultiplets (apart from fields of the gravitational sector that we will neglect). These fields correspond to the deformation parameters (moduli) of $X$, which are one-to-one to certain differential $p, q$-forms on $X$:

$$\omega^{p,q} \equiv \omega_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge \bar{d}z^{\bar{j}_1} \wedge \ldots \wedge d\bar{z}^{\bar{j}_q}. \quad (1)$$

Here, $p$ and $q$ count the holomorphic and anti-holomorphic tensor degrees with respect to the Dolbeault operator $\bar{\partial}$.

The massless fields in four dimensions correspond to the zero modes of the laplacian, $\Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^t + \bar{\partial}^t \bar{\partial}$, and thus are given by the closed but not exact differential forms, ie., by the cohomology of $X$:

$$H^{p,q}_{\bar{\partial}}(X, \mathbb{C}) \equiv \left\{ \omega^{p,q} | \bar{\partial} \omega^{p,q} = 0 \right\} / \left\{ \eta^{p,q} | \eta^{p,q} = \bar{\partial} \rho^{p,q} - 1 \right\}.$$

On a Calabi-Yau threefold $X$, there are two fundamentally different kinds of such moduli, namely the

- Kähler moduli (size parameters): $t_i \sim \omega^{1,1}_i \in H^{1,1}(X), i = 1, \ldots, h^{1,1}$,
- Complex structure moduli (shape parameters): $z_a \sim \omega^{2,1}_a \in H^{2,1}(X), a = 1, \ldots, h^{2,1}$.

The integers $h^{p,q} \equiv \dim H^{p,q}(X)$ are topological invariants of $X$ and are called Hodge numbers.

In order to match these two types of deformation parameters to the hyper- and vector-supersmultiplets in the effective lagrangian, we have to specify about which particular Type II string compactification we actually talk about. This issue is intimately tied to the notion of mirror symmetry, which can be phrased in a simple way by saying that for “every” Calabi-Yau space $X$, there exists a mirror partner $\tilde{X}$ for which the Kähler and complex structure sectors are exchanged; ie.,

$$H^{1,1}(X) \cong H^{2,1}(\tilde{X})$$
$$H^{2,1}(X) \cong H^{1,1}(\tilde{X}). \quad (2)$$

The physical meaning of the mirror relation is that the Type IIA string compactified on $X$ is indistinguishable of the Type IIB string compactified on the mirror, $\tilde{X}$. In perturbation theory this statement boils down to a simple sign flip of one of the left- and right-moving $U(1)$ currents of the 2d $N = (2, 2)$ superconformal algebra. However mirror symmetry is a stronger statement than that, in that it is supposed to hold also at the non-perturbative level.

\[4\]While this seems to be generically true, this statement can be subtle in special cases, for example for rigid Calabi-Yau’s for which there is no obvious mirror. For such cases the notion of a mirror manifold must be appropriately generalized.
An important feature is that Kähler and complex structure moduli fields do not mix in the effective lagrangian, at the two-derivative level that we consider. Accordingly the scalar sector of the low energy effective field theory is described by a sigma model whose target space factorizes:

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_H \times \mathcal{M}_V,$$

(3)

where $\mathcal{M}_{H,V}$ denote the spaces of the VEV’s of the hyper(H)- and vector(V) multiplets, respectively. This vacuum manifold can then be matched to the Kähler and complex structure moduli spaces $\mathcal{M}_{K,CS}$ as follows:\footnote{The star on $\mathcal{M}_H$ means that the dilaton field, which is a non-geometric "universal" extra hyper-multiplet not related in any intrinsic way to the Calabi-Yau manifold, is omitted in the table.}

$$\begin{array}{c|c|c}
\text{Type IIA}/X & \leftrightarrow & \text{Type IIB}/\tilde{X} \\
\mathcal{M}_H^* : & \mathcal{M}_{CS}(X) & = \mathcal{M}_K(\tilde{X}) \\
\mathcal{M}_V : & \mathcal{M}_K(X) & = \mathcal{M}_{CS}(\tilde{X}) \\
\end{array}$$

In the following, we will consider only the manifold of the vector-multiplet VEV’s, i.e., the lower row of this table. It is only this sector of the theory that can be solved for, because the dilaton field belongs to the hyper-multiplets, and by the above-mentioned factorization theorem it cannot mix with the vector multiplets. Since its exponential corresponds to the string coupling, this means that there are no quantum corrections in the space-time sense to the vector-multiplet sector of the theory.

However, there are in general world-sheet instanton corrections, due to 1+1 dimensional string world-sheets that wrap the various 2-cycles $\gamma_{(2)} \in H_2(X)$ of the Calabi-Yau threefold. The volumes of these cycles are measured by the Kähler parameters $t_i$, $i = 1, \ldots, h^{1,1}$, via\footnote{More precisely, the $t_i$ should be considered as a complex variables whose additional imaginary part is provided by the integral over the $B$-field; this will always be implicitly assumed in the following.}

$$t_i = \int_{\gamma_{(2)}} J^{1,1}. \quad (4)$$

The instantons thus lead to classical sectors in the path integral weighted by

$$e^{-S_{\text{sphereinst}}} \sim e^{-\text{Volume}(\gamma_{(2)})} = e^{-t_i} \equiv q_i. \quad (5)$$

Mathematically, the world-sheet instantons correspond to maps from the string world-sheet into the Calabi-Yau manifold $X$, and thus determining the instanton corrections amounts to count all those maps in an appropriate way - which is a priori a quite non-trivial problem.\footnote{Note that the string world sheets can have any number of holes. We will restrict ourselves here to genus zero (spherical) world-sheets, which correspond to instanton corrections to string tree level amplitudes.}

This is where mirror symmetry comes to the rescue, as it implies among other things that

$$H_{\text{even}}(X) \cong H_3(\tilde{X}). \quad (6)$$

This in particular means that the 2-cycles, around which the Type IIA world-sheet instantons wrap, map into 3-cycles in the mirror Calabi-Yau; however, since there are no 2-branes in the Type IIB string whose 2+1 dimensional world-sheets could possibly wrap those 3-cycles,
there cannot be any instanton corrections on the mirror Type IIB side and hence, the classical computation must be exact.

In order to be more concrete, we need to specify what physical quantities we actually talk about. Some of the most basic and important objects are appropriate integrals \( \Pi^\alpha \) over the relevant even (Type IIA) or three (Type IIB) dimensional homology cycles, which physically may be viewed as “quantum volumes” and which mathematically are known as *period integrals*. Mirror symmetry implies that the following two kinds of integrals must represent the same physical quantities:

\[
\text{IIA}/X : \leftrightarrow \text{IIB}/\tilde{X}:
\]

\[
\Pi^\alpha = \int_{\gamma^{(2k)}} (\Lambda J^{1,1})^k + \text{inst. corr} = \int_{\gamma^{(3)}} \Omega^{3,0} \sim (t_i)^k + O(e^{-t}) \quad \sim (\ln z_a)^k + O(z). \tag{7}
\]

The key point is that the quantum volumes of the even dimensional cycles of \( X \) are corrected by world-sheet instantons, while their images on the mirror side (given by integrals over 3-cycles of \( \tilde{X} \)) are not corrected. Equating both sides thus allows to determine the instanton corrections exactly.

The identity (7) contains as special case \((k = 1)\) the map between the Kähler moduli \( t_i \) of \( X \) and the complex structure moduli \( z_a \) of the mirror \( \tilde{X} \), commonly called the “mirror map”:

\[
t_i = -\ln z_a + O(z). \tag{8}
\]

The period integrals, viewed as quantum corrected volume integrals, are important physical objects for a variety of reasons, and their detailed study has provided a lot of physical insight. In particular, they play a dual rôle from the perspective of BPS configurations: at the one hand, they give the exact BPS masses \( m_\alpha \sim |\Pi^\alpha| \) of p-branes wrapped around homology p-cycles \( \gamma^{(p)} \). On the other hand, as mentioned above, they determine the effect of BPS world-sheet instantons corresponding to \((1+1)\)-dimensional world-sheets wrapped around 2-cycles.

Moreover, the periods form the building blocks out of which the holomorphic \( N = 2 \) prepotential \( \mathcal{F} \) can be constructed (later we will see how also the \( N = 1 \) superpotential can be written in terms of period integrals and generalizations of them). In order to do so, one first of all needs to pick an integral basis of the homology 3-cycles and group them into two mutually intersecting sets, \( \{ \gamma^3_A \} \rightarrow \{ \gamma^3_A, \gamma^3_B \} \), \( A, B = 1, ..., h^{2,1}(\tilde{X}) + 1 \). That is, one chooses a basis of \( A \)- and \( B \)-cycles such that their intersection matrix takes the form

\[
\Sigma = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

which reflects the symplectic structure\(^9\) of the 3-homology. Accordingly, one can write

\[
\Pi^\alpha(z) = \left( X_A, \mathcal{F}^B \right) = \left( \int_{\gamma^{(3)}_A} \Omega^{3,0}, \int_{\gamma^{(3)}_B} \Omega^{3,0} \right)(z),
\]

\(^8a, i = 1, ..., h^{1,1}(X) = h^{2,1}(\tilde{X}), \alpha = 1, ..., 2h^{2,1}(\tilde{X}) + 2, k = 0, ..., 3\)

\(^9\)This carries over to the periods and one may say that the periods are sections of an \( Sp(2h^{2,1} + 2, \mathbb{Z}) \) bundle over the moduli space. This means that when varying the moduli, there will be an \( Sp(2h^{2,1} + 2, \mathbb{Z}) \)-valued monodromy action on the periods so that they are defined only up to such transformations. This ambiguity is a generalization of \( T \)-duality.
and in terms of these “electric” and “magnetic” type of periods the prepotential takes the following simple form:

\[ \mathcal{F}(z) = \frac{1}{2} X_A \mathcal{F}^A(z) . \]  

(9)

This expression is exact, as there are no instantons of the right kind to possibly correct this formula.

The more interesting part of the story is when we map back to the Type IIA side, and all what is necessary for doing so is to determine the mirror map \([9]\), invert it and substitute \(z = z(t)\) in \(\mathcal{F}(z)\). This then gives the prepotential in terms of the Type IIA Kähler moduli of \(X\), including all the quantum corrections due to (genus zero) world-sheet instantons. It has the following general form \((r = h^{1,1}(X))\):

\[ \mathcal{F}(t) = \frac{1}{3!} c^0_{ijk} t_i t_j t_k + \sum_{n_1 \ldots n_r} d_{n_1 \ldots n_r} \text{Li}_3(q_1^{n_1} \ldots q_r^{n_r}) . \]  

(10)

Here the first term is the classical contribution, which is given by the triple intersections of (dual) 2-cycles in \(X\) (actually there are further lower order terms in the \(t\)'s that we suppress). The second term is more interesting, in that the integers \(d_{n_1 \ldots n_r}\) describe the contributions of the various world-sheet instantons, ie., they count maps \(\mathbb{P}^1 \to X\) with multi-degrees \(n_1 \ldots n_r\).\(^{10}\) Moreover \(\text{Li}_s(q) \equiv \sum_k k^{-s} q^k\) is the polylogarithm function which universally takes a certain multi-covering (many-to-one instanton maps) into account. Note that the \(d\)'s obtained in this way are predictions for highly non-trivial quantities, and indeed the mathematicians were subsequently able to verify them for a large class of Calabi-Yau manifolds, \(X\).

2.2 The Special Geometry of the N=2 vector-multiplet moduli space

As is well-known, the prepotential \(\mathcal{F}\) can be understood from three interrelated viewpoints: namely as

- **A)** \(d = 4\) \(N = 2\) space-time effective lagrangian \([2,3]\) of the vector-supermultiplets.
  It gives rise to \(U(1)\) gauge couplings: \(\tau_{ij}(t) = \partial_i \partial_j \mathcal{F}(t)\), to “Yukawa” couplings:\(^{11}\) \(C_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t)\), and to the Kähler potential: \(K(t, \bar{t}) = -\ln[\bar{X}_A \mathcal{F}^A - X_A \bar{\mathcal{F}}^A]\).

- **B)** Generating function of tree-level topological field theory (TFT) correlators:

  \[ \langle O_i O_j O_k \rangle(t) = \partial_i \partial_j \partial_k \mathcal{F}(t) \equiv C_{ijk}(t) . \]  

(11)

These three-point functions are essentially the same (up to raising the index with the constant topological metric: \(C_{ijk} = \eta_{kl} C_{ij}^l\)) as the OPE coefficients, or structure

\(^{10}\)If the instantons are not isolated but rather come in continuous families, then the appropriate interpretation of the \(d\)'s is in terms of Euler numbers of the instanton moduli spaces.

\(^{11}\)This is loosely speaking - for the \(N = 2\) compactifications there are no Yukawa couplings, but rather the \(C_{ijk}\) describe certain magnetic couplings. The traditionally used term “Yukawa coupling” refers to the fact that the \(C_{ijk}\) become honest Yukawa couplings when the Calabi-Yau in question is used for an \(N = 1\) compactification of the heterotic string.
constants of the *chiral ring* [43]:

\[
\mathcal{R} : \quad O_i \cdot O_j = \sum_k C_{ij}^k(t)O_k .
\] (12)

The elements \(O_i\) of the chiral ring are nothing but the operators of the 2d world-sheet theory which are simultaneously primary and chiral: \(G_{1/2}^+ O_i|0\rangle_{NS} = G_{1/2}^- O_i|0\rangle_{NS} = 0\). Their space-time significance is, when we talk about a nonlinear sigma model whose target space is the Calabi-Yau manifold \(X\), that they correspond to the non-trivial cohomology elements of \(X\) (and thus, partly, to the massless moduli fields in the effective action); this is because the 2d supercharge can be associated with the \(d\) operator acting on differential forms on \(X\).

More precisely, we need to specify here whether we talk about Type IIA strings on \(X\) or about Type IIB strings on the mirror, \(\hat{X}\), and how the different combinations of chiral and anti-chiral rings that one can assemble out of the left- and right-moving sectors, map to the respective Kähler and complex structure moduli. This requires to refine the de Rham cohomology (of \(d\)) into the Dolbeault cohomology (of \(\bar{\partial}\)), for which there is the added notion of holomorphicity. Specifically, a differential form of (anti-holomorphic,holomorphic) degrees equal to \(p,q\) will correspond to a chiral ring element with (left-moving, right-moving) \(U(1)\) charges given by \(p,q\). In terms of the left- and right-moving fermions \(\lambda\) and \(\psi\), of the non-linear sigma model on \(X\), an explicit expression can easily be obtained from (11) by substituting \(dz^i \rightarrow \lambda^i, d\bar{z}^\bar{j} \rightarrow \psi^\bar{j}\), plus furthermore one has: \(d \lambda^i \rightarrow \lambda_i \equiv g_{ij} \lambda^j\).

If we talk about the Type IIA string, then one considers a topological twist [44] of “type A” of the 2d \(N = (2,2)\) non-linear sigma-model, to the effect that the operators that remain non-trivial BRST classes correspond to the even-dimensional cohomology, \(H^{i,i}(X)\); the complex structure type operators become BRST trivial and decouple (at tree level, that is). Thus, the chiral ring is of (chiral,chiral)-type and is generated by

\[
O_{A,i}^{1,1} = \omega_{k\bar{j}}^{1,1(i)} \lambda^k\psi^\bar{j} \in H^{1,1}(X) ,
\]

which correspond to the Kähler deformations. The OP algebra looks:

\[
\mathcal{R}^{(c,c)} : \quad O_{A,i}^{p,p} \cdot O_{A,j}^{q,q} = \sum_k C_{ij}^k \ O_{A,k}^{p+q,p+q} ,
\]

which at first sight one would tend to identify with the cohomology ring

\[
H^{i,i} : \quad H^{p,p}(X) \cup H^{q,q}(X) \rightarrow H^{p+q,p+q}(X) ,
\]

where the cup product is defined via wedging of forms. However, while the charge (degree) structure of these two rings is the same, the numerical values of the structure constants do in general not coincide, the reason being the corrections from the world-sheet instantons. One may rather say that the \((c,c)\) chiral ring is a *quantum deformation* of the classical cohomology ring: \(\mathcal{R}^{(c,c)} \cong QH^{i,i}(X)\).

On the other hand, if we talk about the Type IIB mirror theory on \(\hat{X}\), then one can implement the “B type” topological twist, to the effect that only the complex
structure type of operators survive as physical operators; the Kähler type operators become BRST trivial and decouple. The chiral ring

\[ \mathcal{R}^{(a,c)} : \quad O_{B,a}^{-p,p} \cdot O_{B,b}^{-q,q} = \sum_c c^{B}_{ab} O_{B,c}^{-p-q,p+q} \]  

is of (anti-chiral, chiral)-type\(^{12}\) and is generated by:

\[ O_{B,a}^{-1,1} = \omega^{-1,1(i)}_{j} \lambda_i \psi^j \in H^{-1,1}(\tilde{X}) \cong H^{2,1}(\tilde{X}) , \]

which correspond to the complex structure deformations. Due to the absence of any world-sheet instanton corrections in the \(B\)-model, the \((a, c)\) ring OPE coefficients \(c^{B}_{ab} c\) are identical to the ring structure constants of the classical cohomology ring:

\[ H^{-p,p}(X) \cup H^{-q,q}(X) \rightarrow H^{-p-q,p+q}(X) . \]

Now recall that, by mirror symmetry, the \(A\)-model on \(X\) is equivalent to the \(B\)-model on \(\tilde{X}\), and thus we can equate the quantum deformed cohomology ring on \(X\) with the classical cohomology ring on \(\tilde{X}\), ie.,

\[ \mathcal{R}^{(c,c)}(X) \equiv QH^{i,i}_\partial(X) \cong H^{-j,j}_\partial(\tilde{X}) \equiv \mathcal{R}^{(a,c)}(\tilde{X}). \]

This exhibits in mathematical terms how we can determine the quantum corrected pre-potential on the Type IIA side: namely by relating its triple derivatives (the quantum deformed \((c, c)\) ring structure constants) to the three-point correlators (classical \((a, c)\) ring structure constants) on the Type IIB side

\[ \partial_i \partial_j \partial_k \mathcal{F}(t) \equiv C^{A}_{ijk}(t) = \sum_{a,b,c} \frac{\partial z_a \partial z_b \partial z_c}{\partial t_i \partial t_j \partial t_k} C^{B}_{abc}(z(t)). \]

The latter are given by the following classical integral:

\[ C^{B}_{abc}(z) = \int_X \Omega^{3,0}_X(z) \wedge \partial_a \partial_b \partial_c \Omega^{3,0}_X(z) . \]

This is closely related to the third viewpoint.

- **C)** The viewpoint of variation of Hodge structures.

It is instructive to contemplate upon the previous formula - if there were no derivatives, then the resulting integral would vanish, because we need to have a \((3, 3)\) volume form to integrate over. The point is that the missing \((0, 3)\)-component is generated by taking three derivatives of \(\Omega^{3,0}_X\). This is because the very definition of what constitutes a holomorphic form (or more generally, a \((p, q)\)-form) within the full \(H^3\) changes smoothly when we vary the complex structure moduli. In particular, upon a first order variation,\(^{12}\)

\[ \text{Note that a negative form degree (charge) can be converted to a positive one by contraction with the holomorphic 3-form: } \Omega^{3,0} : \omega^{p,q} \rightarrow \omega^{3-p,q}. \]

\[ \sum_c c^{B}_{ab} c \]
[101x680]become BRST trivial and decouple. The chiral ring
[208x652]R[(a,c)]:
[218x657]O−p,pB,a·O−q,qB,b=
[340x663]∑ccBabcO−p−q,p+qB,c(13)
[101x615]is of (anti-chiral, chiral)-type
12and is generated by:
[197x585]O−1,1B,a=ω−1,1(i)jλiψj∈H−1,1(˜X)≃H2,1(˜X),
[249x619]which correspond to the complex structure deformations. Due to the absence of any
world-sheet instanton corrections in the B-model, the (a, c) ring OPE coefficients cBabc
are identical to the ring structure constants of the classical cohomology ring:
H−p,p(X)∪H−q,q(X)→H−p−q,p+q(X).
[101x475]Now recall that, by mirror symmetry, the A-model on X is equivalent to the
B-model on ˜X, and thus we can equate the quantum deformed cohomology ring on
X with the classical cohomology ring on ˜X, ie.,
R(c,c)(X)≡QHi,i∂(X)≃H−j,j∂(˜X)≡R(a,c)(˜X).
This exhibits in mathematical terms how we can determine the quantum corrected pre-
potential on the Type IIA side: namely by relating its triple derivatives (the quantum
deformed (c, c) ring structure constants) to the three-point correlators (classical (a, c)
ring structure constants) on the Type IIB side
∂i∂j∂kF(t)≡CAijk(t)=∑abc∂za∂zb∂zc∂ti∂tj∂tkCabcB(z(t)).
The latter are given by the following classical integral:
CabcB(z)=∫XΩ3,0X(z)∧∂a∂b∂cΩ3,0X(z). (14)
This is closely related to the third viewpoint.
• C) The viewpoint of variation of Hodge structures.
It is instructive to contemplate upon the previous formula - if there were no derivatives,
then the resulting integral would vanish, because we need to have a (3, 3) volume form to integrate over. The point is that the missing (0, 3)-component is generated by
taking three derivatives of Ω3,0. This is because the very definition of what constitutes a
holomorphic form (or more generally, a (p, q)-form) within the full H3 changes smoothly
when we vary the complex structure moduli. In particular, upon a first order variation,
the holomorphic \((3,0)\) form gains a component in \(H^{2,1}\), and so on. Altogether one can write this "variation of Hodge structures" schematically as follows:

\[
\begin{align*}
\Omega^{3,0}(z) & \in H^{3,0} \\
\delta_z \Omega^{3,0}(z) & \in H^{3,0} \oplus H^{2,1} \\
(\delta_z)^2 \Omega^{3,0}(z) & \in H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \\
(\delta_z)^3 \Omega^{3,0}(z) & \in H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} .
\end{align*}
\]

Note that after three variations there is indeed an anti-holomorphic \((0,3)\) component generated, and it is precisely this component what then leads to a non-vanishing integral in (14).

Note also that the sequence of variations terminates after the third step, in that the complete third cohomology, \(H^3\), has been generated and no further different \(p,q\) forms can be generated by doing further variations. This means that the fourth derivative of \(\Omega^{3,0}\) must be expressible in terms of the same set of differential forms, and thus must be expressible in terms of lower order variations - which just means that \(\Omega^{3,0}\) must obey some linear differential equation.

Specifically, fixing an ordered, \(2h^{2,1} + 2\) dimensional basis for \(H^3\):

\[
\vec{\omega} = \left( \Omega^{3,0}, \Omega^{(2,1)}_a, \Omega^{(1,2)}_a, \Omega^{(0,3)}_a \right)^t , \quad a = 1, ..., h^{2,1}(\tilde{X}) ,
\]

(16)

(where \(\Omega^{(2,1)}_a\) correspond to the \((a,c)\) ring elements \(O^{(-1,1)}_{B,a}\) in the topological field theory), we can translate the above pattern of Hodge variations into a matrix differential equation of the following form:

\[
\nabla_a \cdot \vec{\omega}(z) = [\partial_a - A_\alpha(z)] \cdot \vec{\omega}(z) = 0 .
\]

(17)

This holds only up to exact pieces that can be freely added to differential forms. However, the exact pieces drop out once we consider instead integrals over an appropriate fixed basis of 3-cycles, i.e.:

\[
\Pi^\alpha_\beta(z) = \int_{\gamma^{(3)}_\alpha} \omega_\beta(z) , \quad \gamma^{(3)}_\alpha \in H_3(\tilde{X}) , \omega_\beta \in H^3(\tilde{X}) .
\]

(18)

This "period matrix" then satisfies \(\nabla_a \cdot \Pi^\alpha_\beta = 0\) provided the 3-cycles have no boundaries (which is the case for the closed string theories at hand, and one of the new ingredients that we will find later in the case of open string backgrounds with \(D\)-branes, is that there will be extra contributions from non-zero boundaries of 3-cycles).

The nice thing about these differential equations is that it is known from general mathematical theorems about the variation of complex structures, that:

\[
[ \nabla_a, \nabla_b ] = 0 .
\]

(19)

Thus the complex moduli space is "flat" - meaning that one can view this equation as a zero curvature, or integrability property of the matrix system (17). From this viewpoint
one may split the covariant derivative into a "Gauß-Manin connection" piece plus a remainder, $C$:

$$\nabla_a \equiv \partial_{z_a} - A_a(z) = \partial_{z_a} - \Gamma_a(z) - C_a(z),$$

the two pieces being distinguished by the type of non-zero entries they have, i.e.:

$$\Gamma_a = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad (C_a)_\beta^\gamma = \begin{pmatrix} 1 \\ (C_a)_b^c \end{pmatrix}.$$

Note that the matrices $C_a(z)$, which correspond to the coefficients of the rightmost terms in the list (15) of Hodge variations, are nothing but the cohomology ring structure constants of $H^3(\tilde{X})$, and physically they are precisely the OPE coefficients (13) of the $B$-model chiral ring, $\mathcal{R}^{(a,c)}(\tilde{X})$.

As a side remark, note also that the differential equations have a concrete meaning also completely within topological field theory. An important property of the TFT is the existence of a flat connection on the bundle of Ramond-Ramond vacuum states, fibered over the moduli space. From this viewpoint, $[\partial_a - C_a(z)] \simeq 0$ expresses that a derivative with respect to a deformation parameter amounts to an operator insertion in the path integral, and this insertion is represented by the ring structure constant matrix, $(C_a)_\beta^\gamma$. Indeed, $(C_a)_\beta^\gamma$ indeed represents the chiral primary field $O_a$ when acting on a given basis of the chiral ring. For further reading about the rich "tt* geometry" of $N = (2,2)$ superconformal field theories we refer to [6, 45].

Now, the zero curvature condition (19) implies that there exist "flat coordinates" $t_i(z_a)$, in terms of which the connection piece $\Gamma$ vanishes:

$$[\partial_{t_i} - C_i(t)] \cdot \Pi_\beta^\alpha(z(t)) = 0,$$

in conjunction with $[\partial_{t_i}, C_j(t)] = [C_i(t), C_j(t)] = 0$.

As the reader might have guessed from the notation, the flat coordinates $t_i = t_i(z)$ are precisely the Kähler parameters of the $A$-model on the threefold $X$, that we had defined in and above of (3). The make this more concrete, note that due to the triangular structure of the matrix differential equation, we can recursively solve for the higher rows of the period matrix $\Pi_{\beta}^\alpha$ in favor of its first row, which is nothing but the vector of quantum volumes we introduced earlier: $\Pi_1^\alpha \equiv \Pi^\alpha = \int_{\gamma_0(3)} \Omega^{3,0}$. By this iterative procedure the first order matrix system (17) turns into an equivalent system of higher order differential equations for the period vector:

$$\mathcal{L}_a(z, \partial_z) \cdot \Pi^\alpha(z) = 0, \quad a = 1, ..., h^{2,1}(\tilde{X}).$$

This system is commonly called "Picard-Fuchs system"; in order to determine the periods, or quantum volumes, it turns out that it is often easier to solve these differential equations rather than performing complicated multi-dimensional integrals.

It is a general property of the period integrals that $h^{2,1}(\tilde{X}) = h^{1,1}(X)$ of them behave near the limit of large complex structures (where $z_a \sim 0$) like $\ln z_a$ (c.f., (3)). These
are the ones to be associated with the flat coordinates, or Kähler parameters of $X$: $t_i \sim \ln z_a + \mathcal{O}(z)$. Actually there is one unique period (denoted by $X_0$) that is a pure series in $z_a$ without a logarithm, and one can use this to normalize the period integrals by dividing them by it. This means that after normalization the unique special period will turn into a constant, so that the period vector, when expressed in terms of the flat coordinates, looks schematically as follows:

$$
\Pi^\alpha(z(t)) = (X_A, \mathcal{F}^B) = (X_0, X_a, \mathcal{F}^b, \mathcal{F}^0) \longrightarrow (1, t_i, \partial_i \mathcal{F}, 2 \mathcal{F} - t^j \partial_j \mathcal{F}) \simeq (1, t, t^2 + \mathcal{O}(e^{-t}), t^3 + \mathcal{O}(e^{-t})).
$$

(22)

This reproduces the prepotential as promised in [11], ie., $\mathcal{F}(t) = \frac{1}{2} X_A \mathcal{F}^A(z(t))$. Moreover, the complete period matrix takes the form:

$$
\Pi_{\beta}^\alpha \equiv \int_{\gamma_{\alpha}} \omega_\beta = \begin{pmatrix} 1 & t_a & \mathcal{F}^a & 2 \mathcal{F} - t_a \mathcal{F}^a \\ 0 & \delta^a_b & \mathcal{F}^b & \mathcal{F}^b - t_a \mathcal{F}^ab \\ 0 & 0 & \mathcal{F}^{abc} & -t_a \mathcal{F}^{abc} \\ 0 & 0 & ... & ... \end{pmatrix},
$$

(23)

which solves (20) upon substituting $c_{ijk} \rightarrow \mathcal{F}^{abc}$.

Now, having reviewed some of the main aspects of $N = 2$ Special Geometry, we are prepared to proceed into new territory by incorporating fluxes and $D$-branes.

### 3 Open/closed string backgrounds with $N = 1$ supersymmetry

So far we considered a closed string background resulting in an $N = 2$ supersymmetric theory in four dimensions, namely Type II strings on a Calabi-Yau manifold $X$. We now seek to reduce the supersymmetries to $N = 1$. More precisely, what we mean is that we seek a modification of this $N = 2$ supersymmetric background that can be described in terms of an effective $N = 1$ supersymmetric lagrangian with a holomorphic superpotential; this superpotential may or may not break supersymmetry even further. In fact there are two sorts of such modifications, which we will discuss in turn.

#### 3.1 Adding background fluxes

The first modification is a deformation in the closed string theory, obtained by ”switching on” certain background fluxes. This means that we take $\int_{\gamma_{(p)}} H^{(p)} \neq 0$, where $H^{(p)} = dC^{(p-1)}$ are the field strengths of the Type II anti-symmetric tensor fields. Such backgrounds will generically induce non-zero potentials [46] and trigger subsequent supersymmetry breaking. Which fluxes one can switch on depends of course on which Type II string we consider; more precisely, for the Type IIA string we can have $p = 3, 7$ in the NSNS- and $p = 2, 4, 6, 8$ in the RR-sector, and for the Type IIB string we can have $p = 1, 3, 5, 7$ in the NSNS- and
Let us first consider the Type IIB string compactified on a threefold \( \tilde{X} \), and focus on the fluxes associated with the 2-form gauge fields. It is well known that the RR and NSNS fluxes fit nicely together in the form 

\[
H^{(3)} = H^{(3)}_{RR} + \tau H^{(3)}_{NS},
\]

where \( \tau = C^{(0)} + ie^{-\varphi} \) is the complexified Type IIB coupling field (\( \varphi \) is the dilaton). In the following we take \( H^{(3)}_{NS} = 0 \), so that we are left with RR fluxes only which are quantized.

It is known [14, 47] that such fluxes lead to an \( N = 1 \) superpotential of the form:

\[
W_{IIB/\tilde{X}}(z) = \int_{\tilde{X}} H^{(3)}_{RR} \wedge \Omega^{3,0} = \sum_{\alpha} N_{\alpha} \cdot \Pi^\alpha(z), \quad N_{\alpha} \in \mathbb{Z},
\]

(24)

which can be non-vanishing only if the \((0, 3)\) component of \( H^{(3)}_{RR} \) is non-zero; general conditions on fluxes are discussed in [48]. For us, the main point is that the 3-form flux numbers \( N_{\alpha} \) simply multiply the periods \( \Pi^\alpha(z_a) \) of the holomorphic \((3, 0)\)-form \( \Omega^{3,0}(\tilde{X}) \).

Note that this modification is purely within the closed string background, and may be viewed as a spontaneous breaking of \( N = 2 \) to \( N = 1 \) supersymmetry. That is, apart from the choice of flux numbers, the theory depends only on the intrinsic "bulk" geometrical data of the Calabi-Yau threefold \( \tilde{X} \). Moreover, it depends only on the complex structure deformations \( z_a \) from the closed string sector which enter via \( \Omega^{3,0}(\tilde{X}) \), and it specifically does not depend on the Kähler deformations. This decoupling property is inherited from \( N = 2 \) supersymmetry, and is why one can make use of mirror symmetry also for \( N = 1 \) supersymmetric flux compactifications.

The Type IIA picture of the above superpotential is obtained via mirror symmetry, and given, essentially, by replacing in the above formula the period by volume integrals, i.e.,

\[
W_{IIA/X}(t) = \int_{X} \sum_{k=0}^{3} H^{(2k)}_{RR} (\wedge J^{1,1})^{3-k} + \ldots \quad (25)
\]

Here \( N^{(2k)} \) denote the RR-fluxes through the \( 0, 2, 4, 6 \)-cycles, and the dots indicate that the classical integrals get corrected by contributions from the world-sheet instantons. A priori there would be little guidance of how to compute these - were it not for mirror symmetry [21], which via [22] implies that the non-perturbative completion of (25) must be given by:

\[
W_{IIA/X}(t) = W_{IIB/\tilde{X}}(z(t)) = N^{(6)} + N^{(4)} t + N^{(2)} t^2 + N^{(0)} t^4 + O(e^{-t}).
\]

One may wonder how we could dare to make statements like those above, despite they deal with non-trivial RR backgrounds for which there is no good description in terms of a world-sheet theory, and which in addition may involve some intractable back-reaction on the Calabi-Yau geometry. The point is that the topological field theory setup, where we focus only on holomorphic quantities at string tree level, is insensitive to such back-reaction, and there is a simple argument [19] in terms of the effective action that shows that the superpotential indeed arises as advertised, irrespective of how the world-sheet theory is
precisely defined. For this, let us for simplicity focus only on the 2-flux and first state that while the corresponding CFT vertex operator is not given by a marginal operator, it is rather given by a massive, "unphysical" vertex operator which corresponds to an auxiliary field. Schematically, we can write the modulus superfield as follows: $\Phi = t + \theta^2 H^{(2)}_{RR}$. Thus, if the flux is non-zero: $\langle H^{(2)}_{RR} \rangle = N^{(2)}$, then indeed: $\int d^4 \theta F(\Phi) \rightarrow \int d^2 \theta N^{(2)} \partial_\Phi F(\Phi) \equiv \int d^2 \theta \mathcal{W}(t)$.

3.2 Adding D-branes

The other possibility to reduce the $N = 2$ supersymmetry to $N = 1$ is to add a D-brane background (a "D-manifold"), which introduces an open string sector into the theory. A D-brane configuration will in general contribute its own moduli (i.e., related to location and Wilson line and gauge bundle data), on top of the moduli that are intrinsic to the embedding bulk (Calabi-Yau) geometry. It is thus an important question to ask about the structure of the combined open plus closed string parameter space, and how the open and closed string moduli fields enter in the effective lagrangian.

A D-brane background consists of one or several D-branes that wrap $p$-cycles $\gamma^{(p)} \in H_p(X)$, and otherwise span the non-compact 3+1 dimensional space-time; the world-volume of a single brane will describe a 3+1 dimensional theory with $N = 1$ supersymmetries if the cycle is supersymmetric. This amounts to the existence of a covariantly constant spinor $\eta$, which can serve as an unbroken supercharge. The relevant equations boil down to: $(1 - \Gamma) \eta = 0$, where $\Gamma = \frac{1}{\sqrt{h}} e^{\alpha_1 ... \alpha_{p+1}} \partial_{\alpha_1} x^{m_1} ... \partial_{\alpha_{p+1}} x^{m_{p+1}} \Gamma_{m_1 ... m_{p+1}}$. Here, $h$ is the induced metric on the world-volume, the derivatives of the coordinates $x$ describe a pull-back to the world-volume, and $\Gamma_{m_1 ... m_{p+1}}$ denote the 10d gamma matrices.

It turns out [49], by analyzing the solutions to this condition, that there are two kinds of supersymmetric $p$-cycles on a Calabi-Yau space, and this reflects the decoupled Kähler and complex structure sectors of the moduli space. Correspondingly, D-branes wrapped on such cycles are called A- or B-type of D-branes:

- **A-type branes** wrap special lagrangian (SL) cycles. These are generally middle dimensional, which means three dimensional for Calabi-Yau threefolds: $\gamma^{(3)} \in H_3(X)$. There are some extra conditions, like that the Kähler and holomorphic three-forms vanish when pulled back to the cycle, but this won’t be important for our discussion. As far as the moduli of the brane configuration go, we know a priori that there should be $\dim_{\mathbb{R}}(\mathcal{M}_{\gamma^{(3)}}) = b_1(\mathcal{M}_{\gamma^{(3)}}) \equiv \hat{r}$ deformations of a SL cycle. This could be odd, but certainly we need complex moduli fields in a supersymmetric theory, so at first sight this seems to pose a problem. The resolution is that these moduli pair up with extra Wilson line moduli, which come from a flat $U(1)$ gauge field on the world-volume. In total we get a doubling of real scalar fields to yield complex ones, and we denote these by $\hat{t}_k$, $k = 1, ..., \hat{r}$. The hat signifies the open string sector, and the letter $t$ reflects that these moduli are the analogs of the Kähler moduli of the Calabi-Yau space.

- **B-type branes** wrap holomorphic cycles, which can be 0,2,4,6-dimensional on a Calabi-Yau threefold. Besides the holomorphic embedding geometry, there appears...
more structure - as is well-known, for \( n \) branes there is a \( U(n) \) gauge symmetry from the open string sector, and the topological sectors of the gauge field configurations are extra important data of the brane configuration. In fact, due to the anomalous WZ couplings \( \int C \wedge Tr[e^F] \wedge \sqrt{\hat{A}(R)} \) on the world-volume of \( B \)-type of branes, such gauge field configurations (instantons or some more general, non-trivial gauge bundles) of a \( p \)-brane correspond to bound states of this \( D \)-brane with lower dimensional ones; however, delving into this interesting issue would lead us too far away, and we refer the reader to the literature [50, 51].

We should be careful to add that notions such as “branes wrapping \( p \)-cycles and gauge configurations on top of them” make sense only in the semi-classical regime, where curvatures are small and the string coupling small. Away from this regime there will be quantum corrections that can wipe out the original meaning of these notions, and one should adopt a better-suited language of quantum geometry that takes over in the strong coupling regime. As we will see, a useful tool for accessing this regime is provided by the open string analog of mirror symmetry.

For the time being, let us consider only branes with trivial bundle structure, and moreover restrict to a class of certain specific \( D \)-brane geometries. These geometries were introduced first in refs. [36,37] and are given by certain non-compact \( D \)-branes on non-compact Calabi-Yau manifolds. As these are best understood in terms of toric geometry, which is a broad field outside the scope of these lectures, we refrain from explaining them in more depth but rather refer the interested reader to these references for more details. For our purposes suffice it to summarize a few key features of the \( D \)-brane setup.

For the \( A \)-type of branes we consider \( D6 \) branes (which have a \( 6+1 \) dimensional world-volume) that wrap special lagrangian 3-cycles of a threefold \( X \) and otherwise stretch over the remaining \( 3+1 \) dimensional space-time. We parametrize the relevant open string deformations by the sizes of disks \( D^{(2)} = \hat{\gamma}^{(2)} \) whose boundaries lie on the given 3-cycle:

\[
\hat{t}_k = \int_{\hat{\gamma}^{(2)}_k} J^{1,1}.
\]

(26)

This is entirely analogous to the definition of the closed string, “bulk” Kähler parameters \( t_i \) defined in [4], the only difference being that the open string Kähler moduli are not integrated over holomorphic spheres but over holomorphic disks with a boundary (and similarly, as discussed above, the moduli get complexified by adding the Wilson line moduli as imaginary components). Obviously, when we move a given SL \( D \)-brane within its homology class, the size of the disks attached to it will change, which then corresponds to different values of the \( \hat{t}_k \).

The open string Kähler moduli are analogous to the closed string ones also with respect to world-sheet instantons: if we consider open string tree-level amplitudes which are given by CFT correlators on world-sheets with disk topology, then there will be extra non-perturbative sectors in the path integral that arise from holomorphic maps from the disk-like world-sheets to disks within the Calabi-Yau whose boundaries sit on the given 3-cycle [34–36] - exactly the same kind of disks that are integrated over in (26). In other words, there will be open string instanton contributions to tree-level amplitudes which are weighted by

\[
e^{-S_{\text{disk inst}}^{(k)}} \sim e^{-\text{Volume}(\hat{\gamma}^{(2)}_k)} = e^{-\hat{t}_k} = \hat{q}_k ,
\]

(27)
besides the well-known closed string world-sheet instantons with spherical world-sheets.\textsuperscript{14}

Specifically, the disk partition function, which is none other than the superpotential, has the schematic form:

\[ W(t, \hat{t}) = 0 \cdot \hat{t} + \sum_{n_i, \hat{n}_k} d_{n_1, \ldots, n_r; \hat{n}_1, \ldots, \hat{n}_r} \text{Li}_2(q_1^{n_1} \ldots q_r^{n_r} \hat{q}_1^{\hat{n}_1} \ldots \hat{q}_r^{\hat{n}_r}). \] (28)

The first term symbolically stands for the classical perturbative contribution, and the coefficient equal to zero signifies that classically, there is no superpotential - the reflects a mathematical theorem that there is no obstruction in deforming a SL 3-cycle. On the other hand, the presence of the instanton sum, which represents a non-perturbative obstruction to deformations, signifies that the mathematical theorem is violated at the quantum level - this may be seen as another manifestation of "stringy quantum geometry".

The dilogarithm function takes the multi-coverings (many-to-one instanton maps) appropriately into account, and when the instanton sum is parametrized in this way, the coefficients \( d_{n_1, \ldots, n_r; \hat{n}_1, \ldots, \hat{n}_r} \) are integers. Their indices label "relative" homology classes of the instanton maps, i.e., they correspond to 2-cycles that are closed only up to boundaries lying on the SL 3-cycle, \( \gamma^{(3)} \). That is, while \( n_i \in H_2(X) \) label the usual homology classes of spheres in \( X \), the \( \hat{n}_k \in H_1(\gamma^{(3)}) \) label homology classes of the boundaries of the disks lying on \( \gamma^{(3)} \). To \textit{a priori} determine these numbers is a formidable mathematical problem, and we will show later how they can be efficiently computed for sufficiently simple geometries via mirror symmetry.\textsuperscript{15}

Let is now consider \( B \)-type branes that wrap holomorphic cycles. It is known [52] that for branes wrapping the whole Calabi-Yau manifold (i.e., a 6-cycle), the open string disk partition function is given by holomorphic Chern-Simons theory:

\[ \mathcal{W} = \int_X \Omega^{3,0} \wedge \text{Tr} [A \wedge \bar{\partial}A + \frac{2}{3} A \wedge A \wedge A] , \] (29)

where \( A \) is the gauge field on the world-volume of the D6-brane (or a stack of several D6-branes). For lower dimensional branes wrapping holomorphic submanifolds, one needs to appropriately dimensionally reduce this expression, and this just amounts to replacing the relevant components of \( A \) by scalar fields \( \phi \). We will specifically consider only \( B \)-type of branes wrapped around holomorphic 2-cycles \( \gamma^{(2)} \), for which dimensional reduction gives [36]

\[ \mathcal{W} = \int_{\gamma^{(2)}(\hat{\varepsilon})} \Omega^{3,0}_{ij} \phi^i \partial_v \phi^j d\hat{z} d\bar{z} . \]

This can be locally rewritten as

\[ \mathcal{W}(z, \hat{z}) = \int_{\gamma^{(3)}(\hat{\varepsilon})} \Omega^{3,0}(z) , \] (30)

where \( \gamma^{(3)}(\hat{\varepsilon}) \) is a 3-chain, whose boundary consists of the 2-cycle \( \gamma^{(2)} \) around which we wrap the brane. Moreover, \( \hat{z} \) governs the size of the 3-chain, by parametrizing the location of

\textsuperscript{14}And more generally, to string amplitudes on world-sheets with genus \( g \) and \( h \) boundaries, there will be contributions of instantons of the corresponding topology. In this lecture we will stick to open string tree level, with \( g = 0, h = 1 \).

\textsuperscript{15}Very recently, novel methods have been devised [22, 27, 28] that are even more efficient.
An important point to note is that (30) is quite similar to the usual period integral (7), the only difference being that it is not a closed 3-cycle over which we integrate but rather a 3-chain, i.e., a 3-cycle with boundary. And analogously, upon analyzing zero mode structures, one finds that there are no possible corrections of world-sheet instantons to (30), so that it is an exact result - recall however, that despite the similarity, there is an important physical difference, in that now we talk about a quantity of an $N = 1$ supersymmetric theory rather than about one with $N = 2$ supersymmetry.

In summary, we have seen that the extra Kähler- and complex structure-like moduli $\hat{t}, \hat{z}$ coming from the $D$-brane geometry are quite similar to the closed string, “bulk” moduli $t, z$ – this suggests that mirror symmetry might be similarly successfully applied as before to the closed string.

### 3.3 Mirror symmetry of $D$-brane configurations

Recall that in the closed string, “bulk” sector, an important feature as to why mirror symmetry was at all useful, was the fact that the Kähler and the complex structure sectors are decoupled (up to the two-derivative level of the effective lagrangian). That is, the total moduli space was a direct product: $\mathcal{M}_{\text{tot}} = \mathcal{M}_{\text{KS}}(t) \times \mathcal{M}_{\text{CS}}(z)$.

In order to successfully apply mirror symmetry to the open string sector as well, we need to have analogous properties to hold for the open string Kähler and the complex structure type moduli, $\hat{t}, \hat{z}$. In fact it can be argued [15] with methods of boundary CFT that the requisite decoupling properties indeed hold (at least at tree level). Specifically, the dependence of D- and F-terms in the effective lagrangian on Kähler and complex structure type of moduli is as follows:

- **$A$-branes**:
  \[
  \begin{align*}
  \mathcal{W}(t, \hat{t}) &\quad \text{holom. F - term potential} \\
  D(z, z^*, \hat{\hat{z}}, \hat{\hat{z}}^*) &\quad \text{Fayet – Iliopoulos D - term}
  \end{align*}
  \]

- **$B$-branes**:
  \[
  \begin{align*}
  \mathcal{W}(z, \hat{z}) &\quad \text{holom. F - term potential} \\
  D(t, t^*, \hat{\hat{t}}, \hat{\hat{t}}^*) &\quad \text{Fayet – Iliopoulos D - term}
  \end{align*}
  \]

From this we see that there is indeed a chance to exactly determine the instanton corrections to the superpotential (and also to other holomorphic quantities), by using mirror symmetry for mapping $\mathcal{W}(t, \hat{t})$ on the Type IIA string side to the non-corrected superpotential $\mathcal{W}(z, \hat{z})$ on the Type IIB side.

The important question is how mirror symmetry acts on the open/closed string background in the Type IIA string picture. In fact it so happens for the specific geometries we consider, that the Type IIB mirror of the SL 3-cycle of the Type IIA picture is a holomorphic 2-cycle of the kind we considered in the previous section; see Fig[11] for a visualization of the geometry. Thus we can impose mirror symmetry by requiring the two expressions (28) and (30) for the superpotential to be equal:

\[
\mathcal{W}_{\text{IIA/II}}(t, \hat{t}) \equiv \sum_{n, \bar{n}} d_{n, \bar{n}} \text{Li}_2[q^n \bar{q}^{\bar{n}}] \equiv \frac{1}{\Omega(\hat{z})} \int_{\gamma^{(2)}} \Omega^3 \mathcal{W}_{\text{IIB/\bar{\Xi}}}(z, \hat{z}).
\]
As discussed before, the LHS is corrected by sphere and disk instantons, while there are no instantons that could possibly correct the RHS. Thus we have managed to represent the exact $N = 1$ $D$-brane superpotential in terms of a period-like integral. There goes quite a bit more into this construction, and the precise details of the geometric setup can be found in [36, 37, 41].

Figure 1: Sketch of the mirror pair of the $D$-brane configurations we consider. In the $A$-model on the left side, disk and $S^1$ instantons deform the quantum geometry of the SL 3-cycle $\gamma^{(3)}$, around which a $D6$-brane is partially wrapped. On the other hand, for the $B$-model on the right side the brane geometry remains uncorrected. Note that this is a simplified picture, in that for the concrete physical models under consideration, the Calabi-Yau manifolds and the relevant cycles are non-compact.

Actually it is not just expressions for the superpotential on both sides what gets mapped onto each other by mirror symmetry, but also the open string modulus $\hat{t}$, and more generally there is a whole sequence of various different chain integrals (“semi-periods”) of the form (30) which on the Type IIA correspond to

$$\hat{\Pi}^{\hat{\alpha}}(t, \hat{t}) = \{ \hat{t}_k, \mathcal{W}^{\hat{\alpha}}(t, \hat{t}), ... \}.$$ \hspace{1cm} (33)

They form the open string analog of the closed string “bulk” period vector $\Pi^{\alpha}(t)$ \hspace{1cm} (7) of section 2.1.

### 3.4 Relative homology

Let us summarize what we found: in the Type IIB picture, both flux-induced and brane-induced superpotentials \hspace{1cm} (24,30) have a very similar form,

$$\mathcal{W}_{\text{flux}} = N_{\alpha} \Pi^{\alpha} \equiv N_{\alpha} \int_{\gamma^{(3)}} \Omega^{3,0}, \hspace{1cm} (34)$$

$$\mathcal{W}_{\text{brane}} = \hat{N}_{\hat{\alpha}} \hat{\Pi}^{\hat{\alpha}} \equiv \hat{N}_{\hat{\alpha}} \int_{\gamma^{(3)}} \hat{\Omega}^{3,0}. \hspace{1cm} (35)$$

This suggests formulating a framework in which the two sectors are uniformly treated, by succinctly writing:

$$\mathcal{W}_{\text{total}} = N_{A} \Pi^{A} \equiv N_{A} \int_{\Gamma^{(3)}} \Omega^{3,0}, \hspace{1cm} (36)$$
where

\[ \Gamma^{(3)}_A \in \{ \gamma^{(3)}_{\alpha}, \hat{\gamma}^{(3)}_{\hat{\alpha}} \} \cong H_3(\tilde{X}, Y, \mathbb{Z}) \]

is the set of relative homology cycles, defined to be the set of SL 3-cycles in \( \tilde{X} \) that need to be closed only up to a boundary lying on the 2-cycle \( \gamma^{(2)} \) that is wrapped by the D-brane. By definition, they belong to the indicated relative homology group, with \( Y \equiv \gamma^{(2)} \).

Correspondingly, the possible superpotentials are given by integral linear combinations of the components of the “relative period vector”:

\[ \Pi^A \equiv (\Pi^\alpha, \hat{\Pi}^{\hat{\alpha}}) = (1, t_I, W^\ell, \ldots), \text{ where} \]

\[ t_I = \{ t_i, \hat{t}_k \} \]

\[ W^\ell = \{ \partial^I \mathcal{F}(t), W^\ell(t, \hat{t}) \} , \]

which one may call the “holomorphic potentials of \( N = 1 \) Special Geometry”. This emphasizes that they are the analogs of the prepotential \( \mathcal{F} \), which is governed by \( N = 2 \) Special Geometry. Note that a subset of them, namely those which come from the closed 3-cycles, are not independent but are derivatives of the bulk prepotential \( \mathcal{F} \). This reflects that the flux-induced superpotentials arise from the spontaneous breaking of \( N = 2 \) supersymmetry to \( N = 1 \), and that they do not carry more information than what the bulk, closed string geometry already provides. On the other hand, the brane-induced potentials \( W^\ell \) are more genuine \( N = 1 \) quantities, and this is reflected by the fact that in general they do not integrate to a generating function; the index \( \ell \) labels the D-branes, and there is one value for each possible boundary component. The much larger variety of the \( W^\ell \) reflects the obvious fact that \( N = 1 \) supersymmetric lagrangians are less restricted than the \( N = 2 \) supersymmetric ones.

At this point one may ask what insight can be gained by apparently just combining the flux and brane sectors into a larger set, and attaching the label “relative” on it. The point is, and this is a priori not at all obvious, that closed and open string sectors consistently fit together into one larger moduli space. As we will point out below, the total moduli space is flat, despite not being a direct product of closed and open string moduli spaces; in fact, the open string sector is deformed by the closed one (but not vice versa), and it may be viewed as a fibration over the closed one.

In this larger, flat moduli space, we can for example study singularities that appear when appropriately adjusting both open and closed string moduli, and investigate the monodromies that are induced by encircling such singularities. Generic monodromies will mix the components of the relative period vector \( (37) \), and thus mix brane with flux numbers.\(^{17}\) This kind of quantum duality symmetries, which generalize the well-known \( Sp(2h^{2,1} + 2, \mathbb{Z}) \)-valued monodromy transformations of the bulk theory, is an example of an insight that can be gained by a uniform treatment of flux and brane induced potentials.

\(^{17}\)Even without further analysis, we can tell beforehand that such monodromies will always modify brane numbers by adding or subtracting flux numbers, but not vice versa: this follows from the fact that there is always a monodromic ambiguity in adding a closed 3-cycle to an open one, but not the other way around. This is one manifestation of the stated fact that brane quantities are deformed by bulk quantities, but not vice versa (at least in the TFT at tree level, which is insensitive to back-reaction).

Note also that this brane-flux mixing is different to the large-\( N \) brane-flux transitions of ref. [19], which can make branes disappear and turn into flux configurations; taking brane or flux numbers to be large is nowhere important in our discussion.
3.5 The Special Geometry underlying the $N = 1$ superpotential

Just like the $N = 2$ prepotential, the $N = 1$ superpotential (given by periods and semi-periods) can be interpreted from at least three interrelated viewpoints [40]:

- **A)** As part of the $d = 4 N = 1$ supersymmetric space-time effective lagrangian. From that perspective, the flux and brane induced superpotentials have a special structure as compared to an arbitrary effective superpotential: as discussed above, they have an integral instanton expansion of the form $\sum d_n \ln [q^n \bar{q}^n]$ (the flux induced ones being a small subset of this), and they have specific transformation properties under duality transformations (they are sections over the combined open/closed string moduli space).

- **B)** Generating function of open string TFT correlators on the disk. The main difference to what we discussed before is that we now deal with a CFT on world-sheets with boundaries, and the structure of the observables and their correlators are correspondingly modified. In particular, in the topological $B$-model we consider $B$-type (Dirichlet) boundary conditions along the sub-manifold $Y \equiv \gamma^{(2)}$, around which the $D$-brane is wrapped:

$$\psi^i = 0 \ (D), \quad \lambda_i = 0 \ (N),$$

where $\lambda, \psi$ are the left- and right-moving fermions of the non-linear sigma model on the Calabi-Yau threefold $\tilde{X}$. In terms of these fields, the boundary chiral ring observables $\hat{O}^{p,q}$ formally look like as in Section 2.2, but are now interpreted as elements of $H^{0,q}(Y, \wedge^p N_Y)$ rather than of $H^{0,q}(Y, \wedge^p T(\tilde{X}))$; here $N_Y$ denotes the normal bundle to the 2-cycle $Y$. The generators of the boundary ring are fermionic because they are associated with 1-forms:

$$\hat{O}^1 = \omega^{1(\bar{a})}_i \lambda_i \in H^0(Y, N_Y).$$

These open string moduli operators generate the boundary chiral ring as follows, and moreover obey mixed bulk-boundary operator products as indicated:

$$\mathcal{R}^{(b)}: \quad \hat{O}^1_a \cdot \hat{O}^1_b = \sum_{\hat{c}} C^{\hat{a} \hat{b}}_{\hat{c}} \hat{O}^2_{\hat{c}}, \quad (39)$$

while the bulk (anti-chiral, chiral) ring remains as before. All of these OPE’s have an interpretation in terms of cup products of certain cohomology groups, and this is explained in more detail in ref. [41]. Suffice it to observe from the OPE structure that

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18While this is true for the specific geometries we consider, more generally the boundary ring is given by an appropriate Ext group; see e.g., [54].

19That the boundary chiral ring has fermionic generators has first been emphasized in ref. [55]. Note that the deformations of the theory are nevertheless described by bosonic parameters $\hat{t}$, because the 1-form operators will be integrated against a supercharge; i.e., the perturbing boundary terms in the 2d lagrangian look like $\hat{t}_k \int_x G^{-1} \hat{O}^1_k$. 
the boundary ring is deformed by the bulk deformations, but not vice versa - this is another manifestation of the above-mentioned non-renormalization property.

The upshot is that we can formally pull through the program of $N = 2$ Special Geometry, but now for relative cohomology instead of the absolute one. To facilitate this, we can implement the extension of the bulk chiral ring by boundary operators by introducing formal superfields

$$\vec{O}^1_A = (O^{-1,1}_a, \hat{O}^1_a) \in H^*(\tilde{X}, Y), \quad (40)$$

that are elements of a relative cohomology group – by definition the one which is dual to the relative homology group of section 3.4. Hence the OPE’s shown above can concisely be written as:

$$R_{\text{open/closed}}: \quad \vec{O}^1_A \cdot \vec{O}^1_B = \sum_C C_{AB}^C \vec{O}^2_C. \quad (41)$$

Of course, if there are more boundary components than one, there will be correspondingly more components of the “superfields”.

The concept of relative cohomology is very natural in the context of $D$-branes, and we will see momentarily that it provides the right framework for doing explicit calculations, such as deriving the analogs of the Picard-Fuchs differential equations in the presence of $D$-branes.

- C) Viewpoint of Hodge variations in relative cohomology. Note that the notation (41) precisely mirrors the structure of differentials in relative cohomology: for $Y$ being a sub-manifold of $X$, these generically look like

$$\vec{\Theta} = (\theta_X, \theta_Y), \quad \theta_X \in H^*(X), \quad \theta_Y \in H^*(Y),$$

and their equivalence relation (modulo addition of exact forms) is:

$$\vec{\Theta} \equiv \vec{\Theta} + (d\omega, i^*\omega - d\eta). \quad (42)$$

Here, $\omega$ and $\eta$ are forms of one degree less than $\theta_X$ and $\theta_Y$, the $i^*$ is the pullback of $\omega$ onto $Y$. This relation tells that if some differential form is exact on $X$ and thus trivial in $H^*(X)$, it may actually be non-trivial on a sub-manifold $Y$. We can understand this also by loosely saying that total derivatives can become non-trivial under integrals once there are boundaries, ie.: $\int_Y d\lambda = \int_{\partial Y} \lambda$. Translated into physics terms, this means that operators that are BRST-exact in the closed string theory, can become non-trivial when $D$-branes are present.

The natural pairing between relative homology and cohomology is then given by

$$\langle \Gamma_A, \vec{\Theta}_B \rangle = \int_{\Gamma_A} \theta^{(X)}_{\beta} - \int_{\partial \Gamma_A} \theta^{(Y)}_{\beta} \equiv \Pi^A_B(t, \hat{t}) \equiv \Pi^A_B(z(t), \hat{z}(t, \hat{t})), \quad (43)$$

which is invariant under the equivalence relation (42). By definition, this yields the relative period matrix, which in the present context looks as follows:

$$\Pi^A(t, \hat{t}) = \begin{pmatrix}
1 & \{t_i, t_k\} & \{\partial_a F, \mathcal{W}^f\} & \ldots \\
0 & 1 & \partial_I \{\partial_a F, \mathcal{W}^f\} & \ldots \\
0 & \ldots & \ldots & \ldots 
\end{pmatrix}, \quad (44)$$
and whose top row $\Pi^A \equiv \Pi^A_1$ is the relative period vector containing the flux and brane superpotentials. The dots to the right indicate further unspecified terms, which would appear in a general situation but which do not appear for the non-compact geometries we consider.

We have seen in Section 2.2 how the variation of Hodge structures in the B-model geometry leads to a system of differential equations for the period matrix, which in turn allows to compute the prepotential $F$ explicitly. In the present context of $N = 1$ Special Geometry, we deal in addition with semi-periods based on chain integrals, and thus the question is whether we can generalize the method of variation of Hodge structures in order to find differential equations which would determine the relative period matrix $(44)$.

In fact we have already prepared the ground for doing so, and all what is necessary is to follow the same logic as before with regard to Hodge variations, but now for relative cohomology. This will automatically take care in a systematic way of boundary terms that arise from the sub-manifold $Y$. Explicitly, the way it works is easiest illustrated by the following diagram:

Here, $\delta_z$ denotes variations with respect to the complex structure moduli of the threefold $\tilde{X}$, and enter in the top row precisely as discussed before (these arrows correspond to the rightmost terms in eq. (15), the other terms are not shown). The new ingredient are the variations with respect to the brane moduli $\hat{z}$. These open up another branch in the diagram, i.e., the bottom row, which is associated with differential forms localized on the sub-manifold $Y \equiv \partial \tilde{\gamma}^{(3)}$. Evidently this diagram applies to a situation with only one boundary component, or $D$-brane; for several components $Y_i$, there will be a branch for every $i$. Moreover, for intersecting $Y_i$ there can be further sub-branches localized on the intersections, and so on.

Precisely as explained earlier, one can translate this diagram into a system of matrix differential equations of the form:

\[
\nabla_a \cdot \Pi^A_B = (\partial_{z_a} - A^A_a(z)) \cdot \Pi^A_B(z, \hat{z}) = 0 \quad (45)
\]

which represents the open string extension of the PF system.\textsuperscript{20} Again, by recursive elimination of the lower rows of $\Pi^A_B$, one can rewrite it in terms of higher order differential operators acting on the top row, i.e., the relative period vector. In order to help the reader, we will present an explicit example in the next section.

\textsuperscript{20} The derivation and structure of such extended PF systems has been discussed in [38, 39, 41, 56]; the original derivation [38] was done from the viewpoint of Calabi-Yau fourfolds, whose BPS geometry is dual to the presently discussed Calabi-Yau threefolds with wrapped $D$-branes.
However, before doing so, let us emphasize again that what we are not just simply tensoring the open and closed string sectors. Rather, the open and closed string moduli form a combined moduli space which is not a direct product, and it is thus highly non-trivial that it is flat (see the detailed discussion in [41]). This means that:

\[
[\nabla_a, \nabla_b] = [\nabla_a, \nabla_{\hat{b}}] = [\nabla_{\hat{a}}, \nabla_{\hat{b}}] = 0 ,
\]

and this allows to go to flat coordinates, which are just the Kähler-type open and closed string moduli \( \hat{t}_k, t_i \) of the \( A \)-model. For these flat coordinates the Gauß-Manin connection vanishes, so that covariant derivatives become ordinary ones and in particular, we can write for the open/closed chiral ring structure constants (39):

\[
C_{IJ}^{(t, \hat{t})} = \partial_I \partial_J W(t, \hat{t}) .
\] (46)

This is the open string analog of the well-known property (11) of the bulk chiral ring structure constants, which integrate to the prepotential \( F(t) \). However, as we already have said, in the open string sector the \( W^{\ell} \) are in general not derivatives of a single generating function \( F \), which reflects the greater freedom of \( N = 1 \) supersymmetric theories; rather, the index \( \ell \) labels the independent boundary, or \( D \)-brane sectors.

3.6 An explicit example

To illustrate our ideas and demonstrate that things work as claimed, we now work out a concrete example. Specifically we will consider a non-compact Calabi-Yau threefold \( X \) given by the canonical bundle on \( \mathbb{P}^2 \). In simple terms, it is given by taking the \( \mathbb{P}^2 \) (which is not a Calabi-Yau manifold due to its non-vanishing first Chern class), and adding an extra non-compact dimension to it such that the first Chern class is cancelled and the whole manifold is Calabi-Yau. This happens if the extra coordinate has the correct degree (or charge, in linear sigma model language) equal to \(-3\), whence one usually denotes the resulting threefold by \( X = O(-3)_{\mathbb{P}^2} \).

This non-compact Calabi-Yau manifold and its mirror geometry has been thoroughly discussed in the literature [37, 57, 58], and specifically the superpotential for a \( D6 \)-brane in this geometry has been computed before [37]. It was also used as prime example in our work [41] that we report about here; however, that discussion was based on the toric geometry of \( O(-3)_{\mathbb{P}^2} \), and this is beyond the scope of this lecture. Rather, we will present here an alternative (and somewhat over-simplified) calculation which does not rely on toric geometry, and invite the interested reader to consult ref. [41] for a more elegant and efficient derivation based on toric geometry (as well as for discussion of many subtleties and details that we drop here, such as the framing ambiguity).

As is well-known [57,58], the mirror geometry \( \tilde{X} \) of the non-compact Calabi-Yau manifold \( O(-3)_{\mathbb{P}^2} \) is characterized by a LG theory with superpotential,

\[
W(y_i, z) = y_0 + y_1 + y_2 + z \frac{y_0^3}{y_1 y_2} ,
\] (47)

where \( z \sim e^{-t_1} \) is the complex structure modulus. It is convenient to write the period integrals in the form

\[
\int_{\gamma^{(3)}} \Omega^{3,0}(z) = \int_{\gamma^{(3)}} \frac{1}{P(z)} d\mu ,
\] (48)
where
\[ P = y_0(x_1^2 + x_2^2) + y_1y_2W(y, z), \]
and where the measure is:
\[ d\mu = \frac{dy_0dy_1dy_2}{y_1y_2}dx_1dx_2. \]
From this we see that \( y_{1,2} \) are \( \mathbb{C}^* \), or exponential, variables.

As explained in [36, 37], the D-brane geometry can be specified by imposing extra linear relations between the coordinates \( y_i \). In fact there are several inequivalent possibilities ("phases") for doing so, which lead to different superpotentials. We will consider in the following the boundary condition \( y_0 = -\hat{z}y_1 \), which corresponds to the “outer phase” of ref. [37]. Actually we will perform most computations in terms of the more convenient variable \( \xi = -1/\hat{z} \), and transform back to \( \hat{z} \) at the end. Moreover we will first rescale \( y_1 \to \xi y_1 \), so that we have the simpler boundary condition
\[ H : \quad y_1 = y_0. \] (49)

By this rescaling, the dependence on the open string modulus \( \hat{z} \) has been transferred from the boundary condition to the holomorphic 3-form \( \Omega^{3,0} \), [18]. Given this \( \Omega^{3,0}(z, \hat{z}) \), we now choose a basis of differentials as follows:
\[ \vec{\omega} = \left( \Omega, \delta_z\Omega, d\eta, \frac{1}{P}\left[ \frac{1}{P}\right] d\mu, \frac{2y_0^6}{P^{13}}, -\frac{1}{\xi}y_1\partial_1\left[ \frac{y_0^3}{P}\right] d\mu \right) \] (50)
where \( \partial_i \equiv \frac{\partial}{\partial y_i} \) and \( d\eta \equiv \delta_\xi\Omega \) is a form that is exact on \( \tilde{X} \). However, under chain integrals with non-zero boundary we have \( \int_\gamma d\eta = \int_{\partial_\gamma} \eta \), reflecting that the exact form \( d\eta \) is a non-zero element in the relative cohomology, localized on the hyperplane \( H \):
\[ d\eta = \frac{1}{\xi}y_1\partial_1\left[ \frac{1}{P}\right] d\mu \quad \Rightarrow \quad \eta = -\frac{1}{\xi}P_\theta d\mu_\theta \equiv -\frac{1}{\xi}W_\theta d\nu_\theta, \] (51)
\[ \delta_z d\eta = -\frac{1}{\xi}y_1\partial_1\left[ \frac{y_0^3}{P^2}\right] d\mu \quad \Rightarrow \quad \delta_z \eta = \frac{1}{\xi}P_\theta d\mu_\theta \equiv \frac{1}{\xi}W_\theta d\nu_\theta. \]

From this we see that on \( H \), the \( (2,0) \) form \( \eta \) can formally associated with a “boundary potential”, given by
\[ P_\theta = P|_{y_1=y_0} = y_0W_\theta \] (52)
\[ W_\theta = \xi(1+\xi)y_0y_2 + (\xi y_2^2 + zy_0^2 + x_1^2 + x_2^2), \]
and with the measure:
\[ d\mu_\theta = \frac{dy_0dy_2}{y_0y_2}dx_1dx_2. \] (53)

As indicated in [51], one may alternatively consider \( W_\theta \) as a boundary potential and instead use the following measure:
\[ d\nu_\theta = \frac{dy_0dy_2}{y_0y_2}dx_1dx_2. \] (54)
Note that here $y_0$ has been turned from a $\mathbb{C}$ to a $\mathbb{C}^*$ variable.

One can then expand, as usual, derivatives of the differentials $\omega$ back into differentials modulo exact pieces, however keeping track of those exact pieces that possibly contribute under chain integrals. From partial integration follows:

$$\sum_i p_i(y) \partial_i \frac{P(y)}{P(y)^{\ell+1}} = \frac{1}{\ell} \left( \sum_i \partial_i p_i(y) \frac{P(y)}{P(y)^{\ell}} - \sum_i \partial_i \left[ \frac{p_i(y)}{P(y)^{\ell}} \right] \right)$$

(55)

For the chain integrals with boundary condition $y_1 = y_0$, the term to the right gives additional contributions:

$$\sum_i \partial_i \left[ \frac{p_i(y)}{P(y)^{\ell}} \right] = \left( \frac{1}{y_0} p_0(y) - p_1(y) \right) |_{y_1 = y_0}$$

(56)

The numerator can then be further expanded into the boundary differentials $(\eta, \delta \eta)$ plus vanishing relations of the form $\partial \eta = 0$; that is, the boundary sector behaves like a autonomous LG theory with potential $P_\partial(y)$. Equivalently, we can write everything in terms of the reduced boundary potential $W_\partial(y)$, provided we use logarithmic derivatives also with respect to $y_0$.

Eventually, by iteratively applying this procedure and collecting all the terms, one can write a matrix representation of the $z, \xi$-derivatives acting on the basis (50) of forms. Transforming back to the coordinate $\hat{z} = -1/\xi$, this can be written as

$$\begin{align*}
\nabla_A \omega &= 0, \quad A = 0, 1, \\
(\nabla_A)_i^j &= \delta_i^j \frac{\partial}{\partial z_A} - (A_A)_i^j,
\end{align*}$$

(57)

where $A$ labels the open and closed string sectors ($z_0 \equiv \hat{z}, z_1 \equiv z$) and where

$$A_0 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0
\end{pmatrix} \begin{pmatrix}
\frac{(-3+\hat{z})}{\hat{z}} \\
\frac{1}{\hat{z}} \\
\frac{1}{\hat{z}} \\
\frac{1-3 \hat{z} + 3 \hat{z}^2 - 6 \hat{z}^4 + 3 \hat{z}^5 (1+10 \hat{z})}{(1+\hat{z}) \hat{z} (1-2 \hat{z} + 2 \hat{z}^2 + 4 \hat{z}^3)}
\end{pmatrix}.$$ 

(58)

It is straightforward to verify:

$$[\nabla_A, \nabla_B] = 0,$$

(59)

which expresses the advertised flatness and integrability of the combined closed and open string moduli space.
Upon iterative elimination of the higher components of $\omega$, one can reduce the first-order system $[57]$ to a system of Picard-Fuchs operators acting on $\int \Omega^{3,0}$:

\begin{align}
\mathcal{L}_1 &= \theta_1^2 (\theta_1 - \theta_0) + z (3\theta_1 - \theta_0) (1 + 3\theta_1 - \theta_0) (2 + 3\theta_1 - \theta_0), \\
\mathcal{L}_0 &= (3\theta_1 - \theta_0) \theta_0 - \hat{z} (\theta_1 - \theta_0) \theta_0,
\end{align}

where $\theta_A \equiv z_A \frac{\partial}{\partial z_A}$. This coincides with the PF system derived in $[38,40]$. The solutions of this system were discussed in $[39]$ and in particular include the mirror maps:

\begin{align}
t(z) &= \log(z) - 3A(z), \\
\hat{t}(z, \hat{z}) &= \log(\hat{z}) + A(z), \\
A(z) &= -\sum_{n>0} \frac{(-)^n (3n-1)!}{(n!)^3} z^n.
\end{align}

Note that the closed string coordinate $t$ does not depend on the open string variable $\hat{z}$, however the open one depends on both the open and closed ones. This reflects what we said before, namely that the part of the moduli space pertaining to the open string sector is deformed by the closed string sector, but not vice versa; in other words we have a fibration of the open over the closed string moduli space.

One can check that when transforming to the flat coordinates, the matrices $A_A(z, \hat{z})$ indeed turn into upper-triangular matrices $C_A(t, \hat{t})$, which amounts to a vanishing Gauß-Manin connection, $\Gamma_A$ (the vanishing of the diagonal terms also requires certain rescalings of the basis of differential forms, see below). The linear system $[57]$ then takes the form:

\[ \left( \frac{\partial}{\partial t_I} - C_I(t) \right) \cdot \Pi_B^A(t, \hat{t}) = 0. \quad I = 0, 1. \]

Its solutions form the columns of the relative period matrix $\Pi_B^A$, defined by integrating a basis of differential forms $\omega_i$ over a suitable set of 3-cycles and chains. Concretely, we choose the variations of $\Omega^{3,0}$ with respect to the flat coordinates as new basis for the differential forms (in particular, we define $d\eta = \delta_3 \Omega^{3,0}$). Then, defining the cycles $\gamma_\alpha^{(3)}$ (for $\alpha = 1, 2, 4$) and chains $\gamma_\alpha^{(3)}$ (for $\alpha = 3, 5$) such that the first row $\Pi_1^A$ yields the known solutions of the PF system $[60]$, we can write the enlarged period matrix as follows:

\[ \Pi_B^A(t, \hat{t}) = \begin{pmatrix}
\int_{\gamma_1^{(3)}} \Omega & \int_{\gamma_2^{(3)}} \Omega & \int_{\gamma_3^{(3)}} \Omega & \int_{\gamma_4^{(3)}} \Omega & \int_{\gamma_5^{(3)}} \Omega \\
0 & \int_{\gamma_2^{(3)}} \delta_1 \Omega & \int_{\gamma_3^{(3)}} \delta_1 \Omega & \int_{\gamma_4^{(3)}} \delta_1 \Omega & \int_{\gamma_5^{(3)}} \delta_1 \Omega \\
0 & 0 & \int_{\gamma_3^{(3)}} \eta & \int_{\gamma_4^{(3)}} \delta_2 \Omega & \int_{\gamma_5^{(3)}} \delta_2 \Omega \\
0 & 0 & 0 & \int_{\gamma_4^{(3)}} \delta_2 \Omega & \int_{\gamma_5^{(3)}} \delta_2 \Omega \\
0 & 0 & 0 & 0 & g \int_{\gamma_5^{(3)}} \delta_2 \eta
\end{pmatrix}. \]
structure constants look in terms of the flat coordinates:

\[
C_0(t, \hat{t}) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \partial_t \partial_{\hat{t}} \mathcal{W} & \partial_{\hat{t}}^2 \mathcal{W} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[(63)\]

\[
C_1(t, \hat{t}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial_{\hat{t}}^3 \mathcal{F} & \partial_{\hat{t}}^2 \mathcal{W} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[(64)\]

while the relative period matrix turns into:

\[
\Pi^A_B(t, \hat{t}) = \begin{pmatrix}
1 & t & \hat{t} & \partial_t \mathcal{F} & \mathcal{W} \\
0 & 1 & 0 & \partial_{\hat{t}}^3 \mathcal{F} & \partial_{\hat{t}} \mathcal{W} \\
0 & 0 & 1 & 0 & \partial_t \mathcal{W} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[(65)\]

The top row displays the components of the relative period vector, which are the building blocks for the total effective, flux and brane induced superpotential.

Above, \(\mathcal{F} = \mathcal{F}(t)\) indeed coincides with the bulk \(N = 2\) prepotential associated with \(O[-3]_{P^2}\) (which is of the form: \(\partial_{\hat{t}}^3 \mathcal{F} = -\frac{1}{3} + \sum n^3 d_n \left(\frac{q}{1-q^n}\right)\)), and \(\mathcal{W} = \mathcal{W}(t, \hat{t})\) with the superpotential on the world-volume of the \(D6\)-brane. Note, however, that the superpotential obtained in this way:

\[
\mathcal{W}(t, \hat{t}) = \frac{1}{12} (t + 3\hat{t})^2 + \mathcal{W}_0(z(t), \hat{z}(t)),
\]

\[(66)\]

\[
\mathcal{W}_0(z, \hat{z}) = \sum_{n \geq 0, \hat{n} > n} \frac{(-)^n (\hat{n} - n - 1)!}{(\hat{n} - 3n)!(n!)^2 \hat{n}} \hat{z}^n z^n
\]

\[(67)\]

has a non-vanishing classical term. This reflects a subtlety [41] in that the geometry we were studying is actually not precisely the one we claimed, and to correct for this one needs to subtract the classical term in (66). The non-perturbative piece indeed has integral expansion coefficients \(d\) when written in terms of the flat coordinates:

\[
\mathcal{W}_0(z(t), \hat{z}(\hat{t}), t) = \sum_{n, \hat{n}} d_{n, \hat{n}} \text{Li}_2(e^{-nt - \hat{n}\hat{t}}),
\]

whose explicit numerical values can be found in the literature.

The potential (66) is a generalized hypergeometric function, and can be analytically continued over the whole open/closed string moduli space. At each point one can go to suitable flat coordinates and study the physics in a local neighborhood, for example near non-perturbative critical points. In the semi-classical regime where \(z, \hat{z}\) are small, the powers of \(\hat{z}\) are always larger than the powers of \(z\), and this means that the potential can be minimized
by sending $\hat{z} \to 0$, independently of what value $z$ has - this means that the brane “runs off to infinity” while the bulk geometry stays fixed.

In contrast, this is not so in the other possible phase [37] of the theory (or patch, where we put the brane). One can compute the superpotential in a similar way and obtain [39]:

$$W_0(z, \hat{z}) = \sum_{n, \hat{n} \geq 0, n \neq \hat{n}} \frac{(-)^n (\hat{n} + 2n - 1)!}{\hat{n}!(n!)^2 (\hat{n} - n)} z^{\hat{n}} \hat{z}^n z^n,$$

and one finds that the bulk modulus $z$ appears not always multiplied with the brane modulus $\hat{z}$; this means that the presence of the brane wants to make $z$ small as well, and thus drives the Calabi-Yau to the large radius, or decompactification limit.

4 Conclusions

We have shown how the computation of superpotentials of certain $N = 1$ supersymmetric string vacua can be put on a footing very similar to the computation of the prepotential $F$ of $N = 2$ supersymmetric theories. Such $N = 1$ vacua can be very crudely labelled by $(X, N_a; \hat{N}_a)$, where $X$ is a compact or non-compact Calabi-Yau threefold and $N_a, \hat{N}_a$ denote flux and $D$-brane numbers. Switching on fluxes involves only the geometry of the bulk, i.e., closed string sector on $X$, which is governed by $N = 2$ Special Geometry. On the other hand, putting $D$-branes produces more genuine $N = 1$ theories, without hidden $N = 2$ special geometry.

Often geometries with $D$-branes are dual to closed string backgrounds, or equivalent to them to after a large-$N$ transition, the main example given by the conifold which is dual, after a large-$N$ transition, to a pure flux configuration [19] on a non-compact three-fold. However, the conifold is a very special, degenerate case, and in line of what we were just saying, in general a $D$-brane configuration is not dual to some flux configuration on a threefold - the functional diversity that can appear in the superpotential is much larger for $D$-branes as compared to flux vacua, not the least because the branes bring extra moduli into the game (such as their positions or Wilson lines), while for flux vacua only the bulk moduli enter.

Nevertheless one may ask whether there are other types of closed string backgrounds that are dual to $D$-branes on threefolds. Indeed, it has been found that the geometry we have been discussing here (relative cohomology of non-compact toric threefolds), is dual to the one of non-compact toric fourfolds with fluxes; this has been discussed in detail in refs. [38, 39]. Thus, when trying to classify $N = 1$ vacua in terms of flux backgrounds, one might better consider fourfolds rather than threefolds in order to get a more accurate counting of possibilities. Flux-induced superpotentials on fourfolds (given by certain period integrals), have been discussed first in [59, 60] and [48], and subsequently in many other papers.

It should be clear to the reader that we have barely scratched the surface of the subject; indeed we have kept under the rug all sorts of subtleties and details, which can be found in the original papers [40, 41]; see also [54] for further aspects and clarification from a more mathematical viewpoint. For example, we have neglected the so-called framing ambiguity [37], which is labelled by an integer $\nu$ and which reflects boundary conditions at infinity for non-compact branes. Explicitly, for the geometry we have been considering in the previous
section, one obtains the following infinite sequence of superpotentials:

\[ W^{(\nu)}(z, \hat{z}) = \sum_{n, \hat{n}} \frac{(-)^{n+\hat{n}} \Gamma(-n + (\nu + 1) \hat{n}) \Gamma(\hat{n})}{\Gamma(-3n + \hat{n} + 1) \Gamma(n + \nu \hat{n} + 1) \Gamma(n + 1) \Gamma(\hat{n} + 1)} z^n \hat{z}^{\hat{n}}, \quad (69) \]

and the computation we have shown just corresponds to \( \nu = 0 \). This demonstrates that when we talk about non-compact branes, we also need to specify the framing \( \nu \) besides the data \((X, N_a; \hat{N}_a)\) in order to define the \( D \)-brane geometry. And this is not the complete story, because we can put the \( D \)-branes also in different patches (“phases”) of \( X \), and this produces even further possibilities for obtaining different and inequivalent superpotentials. Moreover, recall also that we have considered only a very specific class of \( D \)-brane geometries (in \( A \)-model language: wrapped 2-cycles without extra gauge bundles).

All in all, we see that the systematic investigation on superpotentials in string theories with \( N = 1 \) supersymmetry is a problem of enormous complexity. However, there is great hope for making progress toward a deeper understanding of \( N = 1 \) vacua, as well as the quantum geometry underlying them, and this is especially due to the recent works [27–29].

Let me conclude by presenting below a picture which superficially resembles the final picture of many talks on string unification, however its meaning is totally different – it does not show the moduli space of some string theory, rather it shows various approaches for tackling \( N = 1 \) string vacua.

![Diagram](image)

**Figure 2:** There are many viewpoints from which one can tackle string vacua with \( N = 1 \) supersymmetry, each of which has its own scope, merits and limitations. What we have covered in these lectures is a small neighborhood of “\( N = 1 \) Special Geometry”.

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