A NOTE ON THE UNIFORMIZATION OF GRADIENT KÄHLER RICCI SOLITONS

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Abstract. Applying a well known result for attracting fixed points of biholomorphisms [3-6], we observe that one immediately obtains the following result: if \((M^n, g)\) is a complete non-compact gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then \(M\) is biholomorphic to \(\mathbb{C}^n\).

We will show the following:

**Theorem 1.** If \((M^n, g)\) is a complete non-compact gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then \(M\) is biholomorphic to \(\mathbb{C}^n\).

Recall that a Kähler manifold \((M, g_{ij}(x))\) is said to be a Kähler-Ricci soliton if there is a family of biholomorphisms \(\phi_t\) on \(M\), given by a holomorphic vector field \(V\), such that \(g_{ij}(x, t) = \phi_t^*(g_{ij}(x))\) is a solution of the Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} g_{ij} = -R_{ij} - 2\rho g_{ij}
\]

\(g_{ij}(x, 0) = g_{ij}(x)\)

for \(0 \leq t < \infty\), where \(R_{ij}\) denotes the Ricci tensor at time \(t\) and \(\rho\) is a constant. If \(\rho = 0\), then the Kähler-Ricci soliton is said to be of **steady type** and if \(\rho > 0\) then the Kähler-Ricci soliton is said to be of **expanding type**. We always assume that \(g\) is complete and \(M\) is non-compact. If in addition, the holomorphic vector field is given by the gradient of a real valued function \(f\), then it is called a gradient Kähler-Ricci soliton.

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Note that in this case, we have that
\[ f_{ij} = R_{ij} + 2\rho g_{ij}, \]
\[ f_{ij} = 0. \] (0.2)

If \((M, g)\) is a gradient Kähler-Ricci soliton (of steady or expanding type) which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then one can show that \(\phi_t\), the flow on \(M\) along the vector field \(\nabla f\), satisfies:

(i) \(\phi_t\) is a biholomorphism from \(M\) to \(M\) for all \(t \geq 0\),
(ii) \(\phi_t\) has a unique fixed point \(p\), i.e. \(\phi_t(p) = p\) for all \(t \geq 0\),
(iii) \(M\) is attracted to \(p\) under \(\phi_t\) in the sense that for any open neighborhood \(U\) of \(p\) and for any compact subset \(W\) of \(M\), there exists \(T > 0\) such that \(\phi_t(W) \subset U\) for all \(t \geq T\).

Condition (i) is clear. Condition (ii) is shown in [2, 3]. To see that condition (iii) holds, we consider any \(R > 0\) and let \(B(R)\) be the geodesic ball of radius \(R\) with center at \(p\) with respect to the metric \(g(0)\). From the proof of Lemma 3.2 in [2], there exists \(C_R > 0\) such that for any \(q \in B(R)\) and for any \(v \in T^{1,0}(M)\) at \(q\),
\[ ||v||_{\phi_t(g)} \leq \exp(-C_R t)||v||_g. \]

Since \(\phi_t(p) = p\), it is easy to see that given any open set \(U \subset M\) containing \(p\), we have \(\phi_t(B(R)) \subset U\) provided \(t\) is large, and thus condition (iii) is satisfied.

The following theorem was proved for the case \(M = \mathbb{C}^n\) in [4], and was later observed to be true on a general complex manifold \(M\) in [6].

**Theorem 2.** Let \(F\) be a biholomorphism from a complex manifold \(M^n\) to itself and let \(p \in M^n\) be a fixed point for \(F\). Fix a complete Riemannian metric \(g\) on \(M\) and define
\[ \Omega := \{x \in M : \lim_{k \to \infty} \text{dist}_g(F^k(x), p) = 0\} \]
where \(F^k = F \circ F^{k-1}, F^1 = F\). Then \(\Omega\) is biholomorphic to \(\mathbb{C}^n\) provided \(\Omega\) contains an open neighborhood around \(p\).

**Proof of Theorem 1.** By conditions (i)-(iii) we may apply Theorem 2 to the biholomorphism \(\phi_1 : M \to M\) to conclude that \(M\) is biholomorphic to \(\mathbb{C}^n\). \(\square\)

**Remark 1.** In the first version of this article we proved Theorem 2 in a special case. We would like to thank Dror Varolin for pointing out to us that what we proved had been known earlier [4, 6].
Remark 2. After posting the first version of this article we learned that Theorem 1 in the case of a steady gradient Kähler Ricci soliton had been known independently to Robert Bryant [1].

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