Genus one super-Green function revisited and superstring amplitudes with non-maximal supersymmetry

H. Itoyama¹,²,* and Kohei Yano¹,*

¹Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
²Osaka City University Advanced Mathematical Institute (OCAMI), 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan
*E-mail: itoyama@sci.osaka-cu.ac.jp, kyano@sci.osaka-cu.ac.jp

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We reexamine genus one super-Green functions with general boundary conditions twisted by \((\alpha, \beta)\) for \((\sigma, \tau)\) directions in the eigenmode expansion and derive expressions as infinite series of hypergeometric functions. Using these, we compute one-loop superstring amplitudes with non-maximal supersymmetry, taking the example of massless vector emissions of open string type \(I\mathbb{Z}_2\) orbifold.

1. Introduction

The study of one-loop superstring amplitudes [1–5] with bosonic predecessors [6–8] and of the attendant genus one Green functions have a long history. Major results and useful formulas have already been incorporated in standard textbooks [9,10] and we think that the study of this subject has in that sense been trivialized. The number of articles devoted to computations and (phenomenological) applications of superstring amplitudes in recent years is, however, relatively small and the generalized Green functions that do not satisfy ordinary periodicity or anti-periodicity on the genus one Riemann surfaces in the \(\sigma\) or \(\tau\) directions do not seem to have been systematically studied, according to our search [11], despite that they are after all two point functions of QFT free fields. These Green functions are needed in order to study scattering properties of particles in superstring compactifications [12–18] that carry non-maximal supersymmetry and are soluble by free fields.

In the first half of this paper, we study these bosonic and fermionic Green functions as the inverse of Laplacian and Dirac operators respectively, exploiting the elementary method of eigenmode expansion. In the known special cases, our computation boils down to a formula expressible in terms of the theta functions. In general, our final expression is given by an infinite series consisting of a hypergeometric function (with its argument successively shifted), which is relevant to the genus zero Green function.¹ This is in accord with the picture that the genus one Green functions can be obtained from those of genus zero by putting an infinite number of image charges. Our result can be represented

¹ Such a Green function in fact appears in string theory under a constant \(B\) field [19–27].
as super-Green functions with worldsheet supersymmetry broken by the boundary condition. In the latter half of this paper, we demonstrate the use of these Green functions by a simple yet nontrivial example that has non-maximal supersymmetry.

In the next section, we compute the genus one Green functions with the twist angles \((\alpha, \beta)\) in \((\sigma, \tau)\) directions, respectively, using the eigenmode expansion. We exploit partial fractions. In Sect. 3, we recall superstring one-loop vacuum amplitudes in the worldsheet covariant path integrals and recast them with those of the light-cone operator formulation in order to circumvent the nuisance of the overall normalization. We review the case of the \(T^4/\mathbb{Z}_2\) orbifold.

In Sect. 4, we demonstrate the use of the super-Green functions derived in Sect. 2 by computing the superannulus contributions to massless vector emission amplitudes, taking an example from the latter half of this paper, we demonstrate the use of these Green functions by a simple yet nontrivial example that has non-maximal supersymmetry.

In Appendices A–H, we give some details of the computation and background materials quoted in the text.

2. Genus one Green functions with \((\alpha, \beta)\) boundary condition

In this section, we compute the genus one Green functions with general boundary condition to be designated by \((\alpha, \beta)\), using the eigenmode expansion. We mainly consider the case of a torus here. The other one-loop geometries, Klein bottle, annulus, and Möbius band, can be constructed by involution (or the image method), as seen, for example, in [28–30]

Let \(z \equiv \sigma^1 + \tau \sigma^2\) and \(\bar{z} = \sigma^1 - \bar{\tau} \sigma^2\) \((0 \leq \sigma_1, \sigma_2 \leq 1)\) be the complex coordinates on the worldsheet torus with modular parameter \(\tau \equiv \tau_1 + i \tau_2\). The Laplacian is defined by \(\Delta \equiv 4 \partial_z \partial_{\bar{z}}\).

We use the plane wave bases

\[
\Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right) \equiv \frac{1}{\sqrt{\tau_2}} e^{2\pi i (n_1 + \alpha) \sigma_1} e^{2\pi i (n_2 + \beta) \sigma_2} e^{\frac{2\pi i}{\tau} \left[(n_2 + \beta) - (n_1 + \alpha) \tau\right] \bar{z}}
\]

as our eigenfunctions, where \(n_1, n_2 \in \mathbb{Z}, 0 \leq \alpha, \beta < 1\). We have imposed the orthonormality on \(\Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right)\) to determine the normalization factor \(\frac{1}{\sqrt{\tau_2}}\) (see Appendix B). This function possesses the following quasi-periodicities:

\[
\Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1 + 1, \sigma^2\right) = e^{2\pi i \alpha} \Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right)
\]
\[
\Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2 + 1\right) = e^{2\pi i \beta} \Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right).
\]

In the subsequent subsections, we will first consider the bosonic and fermionic components and then use these components to provide the super-torus Green function. We will also consider the superannulus Neumann function as the involution of the super-torus Green function.

2.1. Bosonic part

Since the eigenequation is

\[
\Delta \Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right) = \Lambda_{n_1, n_2}^{(\alpha, \beta)} \Phi_{n_1, n_2}^\alpha \beta \left(\sigma^1, \sigma^2\right),
\]

(2.3)
the eigenvalue reads

\[ \Lambda_{n_1,n_2}^{(\alpha,\beta)} = \frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} |(n_2 + \beta) - (n_1 + \alpha)\tau|^2 \]

\[ = -\frac{(2\pi)^2}{\tau_2^2} \left[ (n_2 + \beta) - (n_1 + \alpha)\tau_1 \right]^2 + \left[ (n_1 + \alpha)\tau_2 \right]^2. \]  

(2.4)

Note that \( \Lambda_{0,0}^{(0,0)} = 0 \). In the following, we consider the cases of \( \alpha \neq 0 \) and \( (\alpha, \beta) = (0, 0), (0, \frac{1}{2}) \).

2.1.1. \( \alpha \neq 0 \)

Now we would like to compute the Green function

\[ G \left[ \alpha, \beta \right](z, \bar{z}|0, 0) = \sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{\Lambda_{n_1,n_2}^{(\alpha,\beta)}} \Phi_{n_1,n_2}^{(\alpha)}(\sigma, \sigma) \Phi_{n_1,n_2}^{*}(\beta, 0) \]

\[ = \frac{1}{\tau_2} \sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{\Lambda_{n_1,n_2}^{(\alpha,\beta)}} e^{2\pi i(n_1+\alpha)\sigma} e^{2\pi i(n_2+\beta)\sigma^2}. \]  

(2.5)

By translational invariance, we have chosen 0 in the second set of arguments.

Exploiting the partial fraction, we decompose \( \frac{1}{\Lambda_{n_1,n_2}^{(\alpha,\beta)}} \) into

\[ \frac{1}{\Lambda_{n_1,n_2}^{(\alpha,\beta)}} = \frac{\tau - \bar{\tau}}{4(2\pi)^2} \frac{1}{n_1 + \alpha} \left\{ \frac{1}{(n_2 + \beta) - (n_1 + \alpha)\tau} - \frac{1}{(n_2 + \beta) - (n_1 + \alpha)\bar{\tau}} \right\}. \]  

(2.6)

which is permissible even for \( n_1 = 0, \alpha \neq 0 \). Using

\[ \int_0^1 d\sigma e^{-2\pi i(n_2+\beta)-(n_1+\alpha)\tau}\sigma = \frac{1 - e^{-2\pi i(n_2+\beta)-(n_1+\alpha)\tau}}{2\pi i(n_2+\beta)-(n_1+\alpha)\tau}, \]

(2.7)

we obtain

\[ \frac{1}{\Lambda_{n_1,n_2}^{(\alpha,\beta)}} = \frac{i(\tau - \bar{\tau})}{4(2\pi)} \frac{1}{n_1 + \alpha} \left[ \frac{1}{1 - e^{-2\pi i\beta q^{n_1+\alpha}}} \int_0^1 d\sigma e^{-2\pi i(n_2+\beta)-(n_1+\alpha)\tau}\sigma \right] \]

\[ - \frac{1}{1 - e^{-2\pi i\beta q^{n_1+\alpha}}} \int_0^1 d\sigma e^{-2\pi i(n_2+\beta)-(n_1+\alpha)\tau}\sigma \right] \]

(2.8)

where \( q = e^{2\pi i\tau} \). According to Eq. (C5), we have the following manipulation:

\[ \sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{n_1 + \alpha} \frac{1}{1 - e^{-2\pi i\beta q^{n_1+\alpha}}} \int_0^1 d\sigma e^{-2\pi i(n_2+\beta)-(n_1+\alpha)\tau}\sigma e^{2\pi i(n_1+\alpha)\sigma} e^{2\pi i(n_2+\beta)\sigma^2} \]

\[ = \sum_{n_1=-\infty}^{\infty} \frac{1}{n_1 + \alpha} \frac{1}{1 - e^{-2\pi i\beta q^{n_1+\alpha}}} \int_0^1 d\sigma \delta \left( \sigma^2 - \sigma \right) e^{-2\pi i\beta \left( \sigma^2 - \sigma \right) + 2\pi i(n_1+\alpha)\left( \sigma^2 + \tau \sigma \right)} \]

\[ = \sum_{n_1=-\infty}^{\infty} \frac{1}{n_1 + \alpha} \frac{\zeta^{n_1+\alpha}}{1 - e^{-2\pi i\beta q^{n_1+\alpha}}} \]

(3.40)
\[
\begin{align*}
\sum_{m=0}^{\infty} \frac{1}{m + \alpha} & e^{2\pi i \beta \left( \frac{\xi}{\bar{\xi}} \right)^{m+\alpha}} + \sum_{m=1}^{\infty} \frac{1}{m - \alpha} e^{2\pi i \beta \left( \frac{\xi}{\bar{\xi}} \right)^{m-\alpha}} \\
\sum_{m=0}^{\infty} \frac{1}{m + \alpha} & e^{2\pi i \beta \left( \frac{\xi}{\bar{\xi}} \right)^{m+\alpha}} + e^{2\pi i \beta} \sum_{m'=0}^{\infty} \frac{1}{m' + \alpha'} \left( \frac{\xi}{\bar{\xi}} \right)^{m'+\alpha'}
\end{align*}
\]

where \( \zeta \equiv e^{2\pi i z} = e^{2\pi i \left( \sigma + \tau \sigma' \right)} \) and \( \alpha' \equiv 1 - \alpha \). Similarly,

\[
\begin{align*}
\sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{n_1 + \alpha} & e^{-2\pi i \beta \bar{\xi}} e^{2\pi i \sigma (n_1 + \alpha)} e^{2\pi i (n_1 + \alpha) \sigma'} e^{2\pi i (n_1 + \alpha) \sigma^2} \\
\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} & \sum_{n=0}^{\infty} \left( e^{2\pi i \beta} \right)^{n+1} \left( \frac{\xi}{\bar{\xi}} \right)^{n+1} F \left( 1, \alpha + \alpha'; \frac{\xi}{\bar{\xi}} \right)
\end{align*}
\]

Substituting Eqs. (2.9) and (2.10) into Eq. (2.5), we obtain

\[
G \left[ \frac{\alpha}{\beta} \right] (z, \bar{z}|0, 0) = -\frac{1}{2(2\pi)^2} \left[ \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left( e^{2\pi i \beta} \right)^{n+1} \left( \frac{\xi}{\bar{\xi}} \right)^{n+1} F \left( 1, \alpha + \alpha'; \frac{\xi}{\bar{\xi}} \right)
\right.
\]

\[
-\frac{\Gamma(\alpha')}{\Gamma(1+\alpha')} \sum_{n=0}^{\infty} \left( e^{2\pi i \beta} \right)^{n+1} \left( \frac{\xi}{\bar{\xi}} \right)^{n+1} F \left( 1, \alpha + \alpha'; \frac{\xi}{\bar{\xi}} \right)
\]

\[
+ \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left( e^{2\pi i \beta} \right)^{n+1} \left( \frac{\xi}{\bar{\xi}} \right)^{n+1} F \left( 1, \alpha + \alpha'; \frac{\xi}{\bar{\xi}} \right)
\]

\[
+ \frac{\Gamma(\alpha')}{\Gamma(1+\alpha')} \sum_{n=0}^{\infty} \left( e^{2\pi i \beta} \right)^{n+1} \left( \frac{\xi}{\bar{\xi}} \right)^{n+1} F \left( 1, \alpha + \alpha'; \frac{\xi}{\bar{\xi}} \right).
\]

2.1.2. \((\alpha, \beta) = (0, 0)\)

In this case, it is necessary to exclude \((n_1, n_2) = (0, 0)\) from the sum in Eq. (2.5):

\[
G_{++}(z, \bar{z}|0, 0) \equiv G \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (z, \bar{z}|0, 0) = \frac{1}{\tau^2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{|n_2 - n_1 \tau|^2} e^{2\pi i n_1 \sigma} e^{2\pi i n_2 \sigma^2}.
\]

(2.12)
As the result of the calculation in Appendix D1, we obtain

\[ G_{++}(z, \bar{z}|0, 0) \equiv \frac{1}{2\pi} \ln \left| \frac{\vartheta \left[ \frac{1}{2} \right]}{\vartheta' \left[ \frac{1}{2} \right]} (0) \right| - \frac{1}{2} (\text{Im}z)^2 \tau_2 \]

\[ + \left[ \frac{1}{2\pi} 2 \sum_{n=1}^{\infty} \ln \left| 1 - q^n \right| - \frac{1}{2} (\text{Im}z) + \frac{1}{2\pi} \ln(2\pi) + 2\tau_2 \cdot \frac{\pi^2}{6} \right]. \quad (2.13) \]

The terms in the bracket \[ \ldots \] vanish when acting on \( \Delta = 4\partial_z \partial_{\bar{z}} \).

2.1.3. \((\alpha, \beta) = (0, \frac{1}{2})\)

Here we consider

\[ G_{+\bar{+}}(z, \bar{z}|0, 0) \equiv G \left[ \frac{0}{\frac{1}{2}} \right] (z, \bar{z}|0, 0) \]

\[ = \frac{1}{\tau_2} \sum_{n_1, n_2=-\infty}^{\infty} \frac{1}{4(2\pi)^2} (n_2 + \frac{1}{2}) - n_1 \tau \right|^2 e^{2\pi i n_1 \sigma} e^{2\pi i (n_2 + \frac{1}{2}) \sigma^2}. \quad (2.14) \]

Now we divide this sum into an \( n_1 \neq 0 \) part and an \( n_1 = 0 \) part to use the partial fraction decomposition in Eq. (2.6).

As the result of the calculation in Appendix D2, we obtain

\[ G_{+\bar{+}}(z, \bar{z}|0, 0) \equiv \frac{1}{2\pi} \left[ \ln \left| 1 - \xi \right| + \sum_{m=1}^{\infty} (-1)^m \ln \left| 1 - \xi q^m \right| \right] + \pi^2 \tau_2. \quad (2.15) \]

2.2. Fermionic part

The eigenequations are

\[ (-i) \partial_z \Phi_{n_1, n_2} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( \sigma^1, \sigma^2 \right) = \kappa_{n_1, n_2}^{(\alpha, \beta)} \Phi_{n_1, n_2} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( \sigma^1, \sigma^2 \right) \]

\[ (-i) \partial_{\bar{z}} \Phi_{n_1, n_2} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( \sigma^1, \sigma^2 \right) = \bar{\kappa}_{n_1, n_2}^{(\alpha, \beta)} \Phi_{n_1, n_2} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( \sigma^1, \sigma^2 \right). \quad (2.16) \]

The eigenvalues can be written as

\[ \kappa_{n_1, n_2}^{(\alpha, \beta)} = -\frac{2\pi}{\tau - \bar{\tau}} \left\{ (n_2 + \beta) - (n_1 + \alpha)\tau \right\} + \frac{\pi i}{\tau_2} \left\{ (n_2 + \beta) - (n_1 + \alpha) \bar{\tau} \right\} \]

\[ \kappa_{n_1, n_2}^{(\alpha, \beta)} = +\frac{2\pi}{\tau - \bar{\tau}} \left\{ (n_2 + \beta) - (n_1 + \alpha) \bar{\tau} \right\} - \frac{\pi i}{\tau_2} \left\{ (n_2 + \beta) - (n_1 + \alpha)\tau \right\}. \quad (2.17) \]

Note that \( \kappa_{0,0}^{(0,0)} = \bar{\kappa}_{0,0}^{(0,0)} = 0 \).

2.2.1. \((\alpha, \beta) \neq (0, 0)\)

Here we calculate the Green function

\[ S \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{n_1, n_2=-\infty}^{\infty} \frac{1}{\kappa_{n_1, n_2}^{(\alpha, \beta)}} e^{2\pi i (n_1 + \alpha)\sigma^1} e^{2\pi i (n_2 + \beta)\sigma^2} \]

\[ S \bar{z} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{n_1, n_2=-\infty}^{\infty} \frac{1}{\kappa_{n_1, n_2}^{(\alpha, \beta)}} e^{-2\pi i (n_1 + \alpha)\sigma^1} e^{-2\pi i (n_2 + \beta)\sigma^2}. \quad (2.18) \]
Similarly, using

\[
\sum_{n_1=-\infty}^{\infty} \frac{x}{1 - e^{-2\pi i \beta q n_1 + \alpha}} = -\frac{(\tau - \bar{\tau}) i}{\tau_2} \sum_{n_1=-\infty}^{\infty} \frac{x}{1 - e^{-2\pi i \beta q n_1 + \alpha}} = -\frac{(\tau - \bar{\tau}) i}{\tau_2} \sum_{n_1=-\infty}^{\infty} \zeta_{n_1 + \alpha} \frac{\vartheta_{\frac{\alpha}{2} - \frac{i}{2}}}{\vartheta_{\frac{i}{2}}} (\vartheta_0) \frac{\vartheta_{\frac{\alpha}{2} - \frac{i}{2}}}{\vartheta_{\frac{i}{2}}} (\vartheta_0)
\]

(2.19)

This time, we have used

\[
\int_0^1 d\sigma e^{2\pi i ((n_2 + \beta) - (n_1 + \alpha)\sigma)} = -\frac{1 - e^{2\pi i \beta q n_1 + \alpha}}{2\pi i ((n_2 + \beta) - (n_1 + \alpha)\bar{\sigma})}
\]

(2.22)

instead of Eq. (2.7), avoiding getting \(\delta (\sigma^2 + \sigma)\) which vanishes in the original domain.

In particular, when \((\alpha, \beta) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0), (0, \frac{1}{2})\), we obtain

\[
S_{--}(z, \bar{z}|0, 0) \equiv S\left[ \frac{1}{2}, \frac{1}{2} \right] (z, \bar{z}|0, 0) = \frac{i}{\pi} \frac{\vartheta_{\frac{1}{2}}}{\vartheta_{\frac{0}{2}}} (\vartheta_0) \frac{\vartheta_{\frac{1}{2}}}{\vartheta_{\frac{1}{2}}} (\vartheta_0)
\]

(2.23)

and their complex conjugates.
2.2.2. \((\alpha, \beta) = (0, 0)\)
In this case, we need to exclude the zero mode \((n_1, n_2) = (0, 0)\) in the sum:

\[
S_{++}(z, \bar{z}|0, 0) = \mathcal{S}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{n_1, n_2}^{\infty} \frac{1}{(n_2 - n_1 \tau)} e^{2\pi i n_1 \sigma^1} e^{2\pi i n_2 \sigma^2}
\]

\[
\mathcal{S}_{++}(z, \bar{z}|0, 0) = \mathcal{S}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] (z, \bar{z}|0, 0) = \frac{1}{\tau_2} \sum_{n_1, n_2}^{\infty} \frac{1}{(n_2 - n_1 \tau)} e^{-2\pi i n_1 \sigma^1} e^{-2\pi i n_2 \sigma^2}.
\]

(2.24)

Here we use the relation

\[
S_{++}(z, \bar{z}|0, 0) \overset{\text{Eqs. (2.12), (2.24)}}{=} 4i \frac{\partial}{\partial z} G_{++}(z, \bar{z}|0, 0)
\]

to calculate Eq. (2.24), because Eq. (C22) appears not to work well when \((\alpha, \beta) = (0, 0)\).

Equation (2.25) can be easily understood by using the last line in Eq. (2.1). From Eqs. (2.25) and (2.13), we obtain

\[
S_{++}(z, \bar{z}|0, 0) = \frac{i}{\pi} \theta'\left[\begin{array}{c} 1 \\ 2 \end{array}\right] (z) - 2 \left(\frac{\text{Im}z}{\tau_2}\right) - 1.
\]

(2.26)

In addition,

\[
\mathcal{S}_{++}(z, \bar{z}|0, 0) = \mathcal{S}_{++}(z, \bar{z}|0, 0) \overset{\text{Eqs. (2.26), (2.27)}}{=} - \frac{i}{\pi} \theta'\left[\begin{array}{c} 1 \\ 2 \end{array}\right] (z) - 2 \left(\frac{\text{Im}z}{\tau_2}\right) - 1.
\]

(2.27)

The last term, i.e. \(-1\) in Eqs. (2.26) and (2.27), vanishes when acting with \((-i)\partial \bar{z}\) or \((-i)\partial z\).

2.3. **Supertorus Green function and superannulus Neumann function**

2.3.1. **Supertorus Green function**
We define the supertorus Green function \([\nu_f = (-, -)\) or \((-, +)\) or \((+, -)\)] by

\[
G_{\nu_f}^{\text{supertorus}}(z_I, \bar{z}_I|z_J, \bar{z}_J) \equiv G_{++}(z_I, \bar{z}_I|z_J, \bar{z}_J) + \frac{\theta_I \tilde{\theta}_J}{4} S_{\nu_f}(z_I, \bar{z}_I|z_J, \bar{z}_J)
\]

\[
- \frac{\theta_I \tilde{\theta}_J}{4} \mathcal{S}_{\nu_f}(z_I, \bar{z}_I|z_J, \bar{z}_J).
\]

(2.28)

where \(\theta, \tilde{\theta}\) are Grassmann coordinates and \(G_{++}, G_{+-},\) and \(S_{\nu_f}\) are given in Eqs. (2.13), (2.15), and (2.23), respectively. According to Appendix E, we can see that \(G_{\nu_f}^{\text{supertorus}} \sim G^{\text{supersphere}}\) when \(z_I \sim z_J\), where \(G^{\text{supersphere}}\) is the supersphere Green function. The worldsheet supersymmetry is broken in general by the boundary condition, but it is still useful to consider this object, which we demonstrate in Sect. 4.
2.3.2. Superannulus Neumann function

Using the image method as in [31] (Appendix G), the superannulus Neumann function can be written as

\[
N^{\text{superannulus}}_{+\pm \nu_{\Gamma}} \left( z, z', z, \tilde{z} \middle| i\tau_2 \right) = \frac{1}{2} \left\{ G^{\text{supertorus}}_{+\pm \nu_{\Gamma}} \left( \frac{z}{2}, \frac{z'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| i\tau_2 \right) + G^{\text{supertorus}}_{+\pm \nu_{\Gamma}} \left( \frac{\tilde{z}}{2}, \frac{\tilde{z}'}{2}; \frac{-\theta}{\sqrt{2}}, \frac{-\theta'}{\sqrt{2}} \middle| i\tau_2 \right) \right\} \\
+ \left\{ G^{\text{supertorus}}_{+\pm \nu_{\Gamma}} \left( \frac{z}{2}, \frac{z'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| i\tau_2 \right) + G^{\text{supertorus}}_{+\pm \nu_{\Gamma}} \left( \frac{\tilde{z}}{2}, \frac{\tilde{z}'}{2}; \frac{-\theta}{\sqrt{2}}, \frac{-\theta'}{\sqrt{2}} \middle| i\tau_2 \right) \right\} \\
= \frac{1}{2} \left\{ \mathcal{N}^{\text{supertorus}} \left( \frac{z}{2}, \frac{z'}{2}; \frac{\theta}{\sqrt{2}}, \frac{\theta'}{\sqrt{2}} \middle| i\tau_2 \right) + \mathcal{N}^{\text{supertorus}} \left( \frac{\tilde{z}}{2}, \frac{\tilde{z}'}{2}; \frac{-\theta}{\sqrt{2}}, \frac{-\theta'}{\sqrt{2}} \middle| i\tau_2 \right) \right\}
\]

(2.29)

where \( \tilde{z}, \tilde{z}', \tilde{\theta} \), and \( \tilde{\theta} \) denote respectively the conjugate points of \( z, z', \theta \), and \( \theta' \).

3. Path integral of an NSR fermionic string and genus one vacuum amplitudes

In this section, quoting the formula (F8) in Appendix F for the path integral of a Neveu–Schwarz–Ramond (NSR) fermionic string valid for any genus, we briefly review genus one vacuum amplitudes.

Introducing notation to represent the contributions of the path integrals from a worldsheet chiral boson and a fermion obeying the general boundary condition specified by \( \left( \frac{a_0, b_0}{a_1, b_1} \right) \), we formulate our discussion to cover a large class of cases with toroidal compactification and its orbifolding. In notation, we have labelled the bosonic case by \( b \) and the fermionic case by \( f \).

3.1. Path integral formula for an NSR fermionic string

Let the bosonic coordinates, the fermionic coordinates, the zweibein, and the Rarita–Schwinger field be \( X^M, \psi_{\text{Maj}}, \epsilon_a^m \), and \( \chi_a^m \) respectively. We denote by \( M, N, \ldots, \alpha, \beta, \ldots, m, n, \ldots, \) and \( a, b, \ldots \), ten-dimensional vector indices, two-dimensional spinor indices, two-dimensional worldsheet indices, and two-dimensional local Lorentz indices respectively. According to Appendix F, the path integral formula can be written as

\[
\left\{ \prod_i O_I \right\} = \sum_{\text{top.}} \sum_{\text{s.s.}} \int \frac{D\epsilon_a^m}{\Omega(D)\Omega(W)\Omega(L)} \int \frac{D\chi_a^m}{\Omega(S)\Omega(SW)} \int DX^M \int D\psi_{\text{Maj}}^M e^{-S} \prod_i O_I \\
= \sum_{\text{top.}} \sum_{\text{s.s.}} \int \prod_i dt_i \frac{1}{\Omega(\text{CKV})} \det' \left( P_{ij}^+ P_i^+ \right)^{\frac{1}{2}} \det \left( \psi_i | \psi_j \right)^{-\frac{1}{2}} \det \left( \psi_i | \frac{\partial \epsilon_a^m}{\partial t_i} \right) \\
\times \int \prod_i da_i \frac{1}{\Omega(\text{CKS})} \det' \left( P_{ij}^+ P_{ij}^+ \right)^{-\frac{1}{2}} \det \left( \psi_i | \psi_j \right) \frac{1}{2} \det \left( \psi_i | \Phi_j \right)^{-1} \\
\times \int DX^M \int D\psi_{\text{Maj}}^M e^{-S} \prod_i O_I. \tag{3.1}
\]
where the action is

$$S = \frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{g} \left\{ \frac{1}{2} g^{\mu \nu} \partial_{\mu} X^M \partial_{\nu} X_M - \frac{i}{2} \psi_{\text{Maj}}^M \gamma^a \nabla_a \psi_{\text{Maj}}^M \right\},$$

and

$$\chi_a = e_a^m \chi_m, \quad \partial_b = e_b^m \partial_m$$

$$\nabla_a = e_a^m \left( \partial_m - \omega_m \frac{1}{2} \gamma^5 \right)$$

$$\omega_m = e_m^a e_p^q \partial_p e_q^b \delta_{ab}.$$ (3.3)

For the notation in Eqs. (3.1), (3.2), and (3.3), please refer to Appendix F. Below, at one-loop, we will recast the expressions Eq. (3.1) into that from the light-cone operator formalism.

### 3.2. Superstring genus one vacuum amplitudes in flat ten dimensions

#### 3.2.1. Torus

In the case of a torus, the boundary conditions for fermions in flat ten dimensions are specified by + (periodic) or − (antiperiodic) for both \( \sigma^1 \) and \( \sigma^2 \) directions:

$$\psi \left( \sigma^1 + 1, \sigma^2 \right) = r \psi \left( \sigma^1, \sigma^2 \right), \quad r = \pm 1$$ (3.4)

$$\psi \left( \sigma^1, \sigma^2 + 1 \right) = s \psi \left( \sigma^1, \sigma^2 \right), \quad s = \pm 1.$$ (3.5)

Similar expressions hold for \( \tilde{\psi} \). In the notation of Sect. 2, \( r = e^{2\pi i \alpha}, s = e^{2\pi i \beta} \), so that

$$r = +1 \leftrightarrow \alpha = 0, \quad r = -1 \leftrightarrow \alpha = \frac{1}{2}$$

$$s = +1 \leftrightarrow \beta = 0, \quad s = -1 \leftrightarrow \beta = \frac{1}{2} \text{ modulo 1.}$$ (3.6)

Except for the (+, +) spin structure, there is neither conformal killing spinor, nor supermoduli. For the (+, +) spin structure, its contribution to the vacuum amplitude vanishes due to the integrations of \( \psi, \tilde{\psi} \) fermionic zero modes. The torus vacuum amplitude for IIB/IIA in flat ten dimensions is, therefore, simply written as

$$Z_{\text{flat}}^{\text{IIB/IIA}} = V_E \sum_{(r,s)} \sum_{(r',s')} C_{rs} \tilde{C}_{r's'} \frac{1}{2} \left[ \int_{\mathcal{F}} \prod_{i=1,2} d\tau_i \det \left( P_{1/2}^+ P_1 \right) \right] \left[ \det \Delta_{\tilde{g}} \right]^{-5/2}$$

$$\times \left[ \det \left( \frac{\partial \bar{\epsilon}_m^a}{\partial \tau_j} \right) \right]^{-1} \left( \int d^2 \sigma \sqrt{\tilde{g}} \right)^5$$

$$\times \left[ \det \left( P_{1/2}^+ P_{1/2} \right) \right]^{-1/2} \det (\gamma \cdot \partial)^5 \left( r, s, r', s' \right),$$ (3.7)

where \( \tilde{g}_{mn} = \left[ \frac{1}{\tau_1 \tau_2 + \tau_2^2} \right] \), and we have chosen \( C_{++} = -C_{--} = -C_{-+} = \frac{1}{2}, \tilde{C}_{-+} = -\tilde{C}_{++} = -\frac{1}{2}, C_{++} = \tilde{C}_{++} = 0 \) in accordance with the GSO projection \([32,33]\) of the IIB superstring that implements the modular invariance. We have denoted by \( \mathcal{F} \) the fundamental region of the torus, and the factor \( \frac{1}{2} \) is accounted for by \( SL(2, \mathbb{Z}) / PSL(2, \mathbb{Z}) = \{ \pm \mathbf{1}_2 \} \). The Euclidean volume
is denoted by \( V_E \). Omitting the calculations of the Weil–Petersson measure factor and those of the functional determinants, we obtain

\[
Z_{\text{flat}}^{\text{IIB/IIA}} = K V_E \frac{1}{2} \int_F \frac{d^2 \tau}{(\tau_2)^2} \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} |T(\tau)|^2, \tag{3.8}
\]

where

\[
T(\tau) = \frac{1}{\eta(\tau)^4} \left( C_{--} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 + C_{-+} \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}^4 + C_{+-} \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}^4 \right) \tag{3.9}
\]

and \( \eta(\tau) = e^{\frac{i \pi}{12}} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi in\tau} \right) \) is the Dedekind eta function.

In this paper, we take a short cut to proceed further and to determine the normalization factor \( K \) by comparing the last expression of Eq. (3.8) with the vacuum amplitude evaluated in the light-cone gauge operator formalism, written in terms of the \( \text{so}(8) \) characters. (The overall normalization can also be seen by the one-loop free energy in local field theory.)

\[
1_{\text{flat}}^{\text{IIB/IIA}} = -\frac{V_E}{2(4\pi^2 \alpha')^5} \int_F \frac{d^2 \tau}{\tau_2^2} (\bar{\chi} \chi), \tag{3.10}
\]

where

\[
(\bar{\chi} \chi) \equiv \sum_{i,j=\text{NS,R}} \bar{\chi}_i X_{ij} \chi_j = \left| V_8 - S_8/C_8 \right|^2 \frac{1}{\tau_2^4 |\eta(\tau)|^{16}} \tag{3.11}
\]

\[
V_8 = \frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 - \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}^4}{2\eta^4}, \quad S_8/C_8 = \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^4 \pm \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^4}{2\eta^4} = \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^4}{2\eta^4}. \tag{3.12}
\]

Identifying Eq. (3.8) with Eq. (3.10), we obtain

\[
K = -\frac{1}{(4\pi^2 \alpha')^5}. \tag{3.13}
\]

Note that, from the point of view of one-loop free energy in local field theory, \( \frac{1}{(4\pi^2 \alpha' \tau_2)^2} \) comes from a Gaussian integration over one momentum, and \( -\frac{1}{2} \int \frac{d^2 \tau}{\tau_2} \cdots \) comes from a proper time representation of \( \log \text{Det} \).

### 3.2.2. Klein bottle, annulus, Möbius strip

Let us write Eq. (3.8) as

\[
Z_{\text{torus}}^{\text{IIB}} = \frac{1}{2} K V_E T \tag{3.14}
\]

\[
T = \int_F \frac{d^2 \tau}{\tau_2^2} |\chi(\tau)|^2_{\text{torus}}, \quad |\chi(\tau)|^2_{\text{torus}} = \text{Tr} q_0^{(\text{cyl})} \bar{q}_0^{(\text{cyl})}, \tag{3.15}
\]

where \( q_0^{(\text{cyl})} \) and \( \bar{q}_0^{(\text{cyl})} \) are the right and left Hamiltonian of the closed fermionic string on the cylinder in the light-cone gauge operator formalism.
To construct an unoriented string, namely the type I superstring, one first makes the closed string sector by the $\Omega$ (twist) projection: $\text{Tr} q^{L_{0}(\text{cyl})} \bar{q}^{L_{0}(\text{cyl})} \rightarrow \text{Tr} \left( \frac{1}{2}(1+\Omega) q^{L_{0}(\text{cyl})} \bar{q}^{L_{0}(\text{cyl})} \right)$. We obtain

$$Z_{\text{closed, one-loop}}^{I} = \frac{1}{2} K V E \frac{T + K}{2} \quad \kappa = \int_{0}^{\infty} \frac{d \tau_2}{\tau_2^2} \sum_{i=NS,R} \chi_i(2i \tau_2).$$  

(3.16)  

(3.17)

By a similar procedure, the vacuum amplitude of the open string sector reads

$$Z_{\text{open, one-loop}}^{I} = \frac{1}{2} K V E \frac{A + M}{2}$$

(3.18)

$$A = \int_{0}^{\infty} \frac{d \tau_2}{\tau_2^2} \sum_{i=NS,R} \chi_i \left( \frac{i}{2} \tau_2 \right) (\text{cpf})^2$$

(3.19)

$$M = \int_{0}^{\infty} \frac{d \tau_2}{\tau_2^2} \sum_{i=NS,R} \chi_i \left( \frac{i}{2} \tau_2 + \frac{1}{2} \right) (\text{cpf}) \epsilon.$$  

(3.20)

Here, ${\text{cpf}}$ denotes the Chan–Paton factor, $\epsilon = \pm 1$, and $\bar{\chi} \left( \frac{i}{2} \tau_2 + \frac{1}{2} \right)$ indicates that the replacement by $\tau \rightarrow \frac{i}{2} \tau_2 + \frac{1}{2}$ in the argument is to be made only for the oscillator part. These replacements $\kappa : \tau \rightarrow 2i \tau_2, A : \tau \rightarrow \frac{i}{2} \tau_2$, and $M : \tau \rightarrow \frac{i}{2} \tau_2 + \frac{1}{2}$ are understood both by the twist projection in the operator formalism and by the worldsheet involutions of the worldsheet path integrals with torus as the double of the respective open Riemann surfaces.

Finally, the infrared stability seen as the cancellation of the massless poles in $Z_{\text{closed, one-loop}}^{I} + Z_{\text{open, one-loop}}^{I}$ in the transverse channel (or equivalently the cancellation of dilaton tadpoles [31,34–39] or infinity cancellation [31,40]) selects cpf = 2$^5 = 32$, $\epsilon = -1$, and the gauge group $SO(32)$ [41].

### 3.3 Generalization to cases of toroidal compactification and their orbifolding

In order to proceed even further and to prepare for calculation of string scattering amplitudes in Sect. 4, we will introduce notation for the integrand of the string one-loop partition function. Let us, in particular, write $(\bar{\chi} X \chi)_{\text{IB/IIA flat}}$ as

$$(\bar{\chi} X \chi)_{\text{IB/IIA flat}} = \frac{1}{2} \left( \begin{array}{c|c} + & + \\ - & + \\ + & - \\ + & + \\ + & + \\ + & + \\ + & + \\ + & + \\ \end{array} \right)$$

(3.21)

Here we have introduced

$$\left( \begin{array}{c|c} \alpha_b & \beta_b \\ \alpha_f & \beta_f \\ \end{array} \right)$$

(3.22)

in order to represent the contribution from a single chiral boson and fermion obeying the boundary conditions ($\alpha_b$, $\beta_b$) and ($\alpha_f$, $\beta_f$) respectively:

$$\left( \begin{array}{c|c} + & + \\ r_i & s_f \\ \end{array} \right) = \left[ \begin{array}{c|c} 0 & 0 \\ \alpha_f & \beta_f \\ \end{array} \right]$$

(3.23)
The power $8 = 10 - 2$ seen in Eq. (3.21) permits covariant interpretation as the 2d metric and 2d gravitino fields obey the same boundary condition as the worldsheet bosons and fermions do respectively. As a simple prototypical example, let us consider IIB string on $T^4 \left( = (S^1)^4 \right)/\mathbb{Z}_2$ with radii of $S^1$ being $R_I$, $I = 5, 6, 7, 8$.

$$
(\tilde{\chi} \chi)_{\text{IIB}, T^4/\mathbb{Z}_2} = \frac{1}{2} \left( \prod_{I=5,6,7,8} F_2(a_I, \tau) \right) (\tilde{\chi} \chi)_{\text{IIB, flat}} + \frac{1}{2} \left( \prod_{I=5,6,7,8} a_I \sqrt{\tau_2} \right) \\
\times \frac{1}{2} \left( \prod_{I=5,6,7,8} a_I \right) \\
\times \frac{1}{2} \left( \prod_{I=5,6,7,8} a_I \right) \\
\times \frac{1}{2} \left( \prod_{I=5,6,7,8} a_I \right) \\
\times \frac{1}{2} \left( \prod_{I=5,6,7,8} a_I \right) \\
\equiv \sum_{\nu,\bar{\nu}} J_{\nu, \bar{\nu}, \text{IIB}, T^4/\mathbb{Z}_2},
$$

(3.24)

where $F_2(a_I, \tau) = a_I \sqrt{\tau_2} \sum_{\ell, m \in \mathbb{Z}} q^{\frac{1}{4} (m a_I + \ell^2)} \tilde{q}^{\frac{1}{4} (m a_I - \ell^2)}$ and $a_I = \frac{\sqrt{\tau_2}}{R_I}$. The first line represents the contribution from the $T^4$ compactification without $\mathbb{Z}_2$ insertion; the second, third, and fourth lines represent the contributions from the untwisted sector with $\mathbb{Z}_2$ insertion, the $\mathbb{Z}_2$ twisted sector, and the $\mathbb{Z}_2$ twisted sector with the $\mathbb{Z}_2$ insertion, respectively. In each term inside the bracket, the first bin represents the spacetime part and the second bin the internal part. Referring to the character of $c = 1$, $\mathbb{Z}_2$ orbifold, we are able to see

$$
\begin{align*}
\epsilon_{\bar{\eta}} & = \left[ \frac{\alpha_b \beta_b}{\alpha_I \beta_I} \right] \\
& = \sqrt{\frac{2\eta}{\vartheta \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0)}} \\
& \times \sqrt{\frac{2\eta}{\vartheta \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0)}} \\
& \times \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0) \\
& \times \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0) \\
& \times \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0) \\
& \times \left[ \frac{1}{2} + \alpha_b \right] \left[ \frac{1}{2} + \beta_b \right] (0) \\
& \text{for } (\alpha_b, \beta_b) = \left( 0, \frac{1}{2} \right), \left( \frac{1}{2}, 0 \right), \left( \frac{1}{2}, \frac{1}{2} \right).
\end{align*}
$$

(3.25)
Here, the arguments of the theta constants are modulo 1 and the non-integer parts are understood to be taken. Note that, in this notation, we have included the contribution from the $2^4 = 16$ fixed points in the twisted sector in Eq. (3.25).

### 3.4. Open superstring on $T^4/Z_2$

Another prototypical example which we will consider in the next section is the open string sector in the type I superstring on $T^4$ ($= (S^1)^4$)/$Z_2$. The partition function is

$$Z_{I,T^4/Z_2} = -\frac{V_E}{2} \frac{1}{(4\pi^2\alpha')^5} A_{I,T^4/Z_2} + M_{I,T^4/Z_2}$$

(3.26)

$$A_{I,T^4/Z_2} = \int_0^\infty \frac{d\tau_2}{\tau_2} \frac{1}{\tau_2^5} J_{I,T^4/Z_2}|_{\tau = \frac{i}{2}\tau_2}$$

(3.27)

$$M_{I,T^4/Z_2} = \int_0^\infty \frac{d\tau_2}{\tau_2} \frac{1}{\tau_2^5} \tilde{J}_{I,T^4/Z_2}|_{\tau = \frac{i}{2}\tau_2 + \frac{1}{2}}.$$  

(3.28)

Among the many possibilities discussed in [38,39,42], where the dilaton tadpoles cancel, we will consider the simplest case where the gauge group is $U(n=16)_{(9)} \times U(d=16)_{(5)}$ with all of the D5 branes at the same fixed point and the first and the second subscripts indicate D9 and D5 brane respectively.\(^2\)

$$J_{I,T^4/Z_2} = \frac{1}{2} J_{I,T^4} + \frac{1}{2} J_{I,Z_2}$$

$$= (2n)^\frac{1}{2} \left( \prod_{I=5,6,7,8} F_1(a_I, \tau_2) \right) \frac{1}{2} \left( \begin{array}{c} + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \end{array} \right)$$

$$- (2d)^\frac{1}{2} \left( \prod_{I=5,6,7,8} (a_I \sqrt{\tau_2}) \right)$$

$$\times \frac{1}{2} \left( \begin{array}{c} + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \\ + & + & + & + & - & - & - & - \end{array} \right)$$

$$\equiv \frac{1}{2} \sum_v J_v + \frac{1}{2} \sum_v J_v(Z_2),$$

(3.29)

where $F_1(a_I, \tau_2) = a_I \sqrt{\tau_2} \sum_p e^{-ip_1 p_1'}, \pi \tau_2 \equiv \frac{1}{a}$. See also [59–63].

### 4. One-loop superstring amplitudes with non-maximal supersymmetry

In this section, we apply the genus one super Green function constructed under the general twists in the $(\sigma, \tau)$ directions to superstring amplitudes. For simplicity, we illustrate this by the annulus contribution to the open superstring amplitudes of the compactification in Sect. 3.4, but our procedure is applicable to a large class of toroidal models and their orbifolding of closed and open superstrings including heterotic string [64,65] compactifications.

\(^2\) Other aspects of this series of models are discussed in [43–58].
4.1. Neumann functions with arguments on the boundary

In order to proceed to the computation, we need the Neumann function for the superannulus under a variety of boundary conditions for a worldsheet boson and a worldsheet fermion specified by \( \left( \frac{v_b}{v_f} \right) \) and with the arguments set on the same boundary. The Neumann function for the Möbius strip case can be read off from the annulus case by the change of arguments in the theta functions and will not be discussed explicitly here. For the case of \( \left( \frac{++}{v_f} \right) \), which is always needed,

\[
N_{++}^{\text{superannulus}} \left( z_I, \bar{z}_J; z_J, \bar{z}_J \middle| \frac{i \tau_2}{2} \right)
\]

on \( z = \bar{z}, \theta = \bar{\theta} \)

\[
= \frac{1}{2} \cdot 4 \mathcal{G}_{++}^{\text{superannulus}} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \frac{\theta_I}{\sqrt{2 \tau}} \frac{\theta_J}{\sqrt{2 \tau}} \middle| \frac{i \tau_2}{2} \right) \right)_{(z, \theta) = (\bar{z}, \bar{\theta}) = (-\bar{z}, \pm i \bar{\theta})}
\]

\[
= \frac{4}{2} \left[ G_{++} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \middle| \frac{i \tau_2}{2} \right) \right) \right]_{z = \bar{z} = -\bar{z}} + \frac{\frac{\theta_I}{\sqrt{2 \tau}} \frac{\theta_J}{\sqrt{2 \tau}}}{4} \mathcal{S}_{v_f} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \middle| \frac{i \tau_2}{2} \right) \right)_{z = \bar{z} = -\bar{z}}
\]

Eqs. (C8), (C9), (2.20)

\[
= \frac{1}{2 \pi} \ln \left| \frac{1}{\frac{1}{i} \theta} \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \left| \frac{i \tau_2}{2} \right) \right| + \frac{(z_I - z_J)^2}{4 \tau_2}
\]

\[
+ \frac{\theta_I}{\sqrt{2 \tau}} \frac{\theta_J}{\sqrt{2 \tau}} \mathcal{S}_{v_f} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \middle| \frac{i \tau_2}{2} \right) \right) + \frac{\theta_I}{\sqrt{2 \tau}} \frac{\theta_J}{\sqrt{2 \tau}} \mathcal{S}_{v_f} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \middle| \frac{i \tau_2}{2} \right) \right)
\]

\[
= \frac{2}{2 \pi} \ln \left| \frac{1}{\frac{1}{i} \theta} \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \left| \frac{i \tau_2}{2} \right) \right| + \frac{2(z_I - z_J)^2}{4 \tau_2} + \frac{2 \cdot 2 \theta_I}{\sqrt{2 \tau}} \frac{\theta_J}{\sqrt{2 \tau}} \mathcal{S}_{v_f} \left( \left( \frac{z_I}{2} \frac{z_J}{2} \middle| \frac{i \tau_2}{2} \right) \right)
\]

\[
+ 2 \left\{ \frac{1}{2 \pi} \sum_{n=1}^{\infty} \ln \left| 1 - \frac{z_I}{2} \frac{z_J}{2} \right| + \frac{1}{2 \pi} \ln(2\pi) + \frac{1}{2 \tau} \pi^2 \right\}
\]
Let \( \Pi \) be the product of \( N \) vertex operators \( \zeta_j^{(P)} \cdot \int dz J \int d\theta J \), \( I = 1, \ldots, N \), for massless vector emission of an open superstring. It can be written as

\[
\prod_{I=1}^{N} O_I = \prod_{J=1}^{N} \left( \zeta_j^{(P)} \cdot \int dz J \int d\theta J \right) \times \exp \left[ i \int d^2 z \int d\theta J(z, \bar{z}, \theta, \bar{\theta}) \cdot X(z, \bar{z}, \theta, \bar{\theta}) \right].
\]

Here we have introduced the Grassmann source \( \eta_I, J = 1, 2, 3, \ldots, N \), for this representation. Following Appendix F, we carry out the Gaussian integration\(^3\) and the sum \( S \) over the boundary conditions.

Let \( S = S' \oplus S_{(+)} \), where \( S_{(+)} \) is the part of the sum which contains \( (++) \) to some power in \( \nu_f \). For these parity-violating cases [71], it is well known that the amplitudes for lower \( N \) vanish.

\(^3\) See [31]. See also [66–70] for different computational approaches.
Ignoring these cases in this paper, let us denote the remaining part of the $N$ point amplitude for the case labelled by $\bullet$ by

$$A_N^\bullet = -\frac{1}{2} \left( \frac{V_{E}\delta}{(4\pi^2\alpha')^5} \right) A_N^\bullet + M_N^\bullet. \quad (4.5)$$

Here, we have denoted by $(V_{E}\delta)$ a product of the momentum-conserving delta functions and $(\sum_{k_l})$, which appear from the integrations of the zero modes of the bosonic coordinates, and the volume of the compactification $V_c$. The annulus and the Möbius strip contributions are denoted by $A_N^\bullet$ and $M_N^\bullet$, respectively, and

$$A_N^\bullet = \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{v\in S'} J^\bullet_v \prod_{J=1}^N \left( \zeta_j^{(P)} \cdot \int d\eta J \int dz J \int d\theta J \right)$$

$$\times \exp \left[ \pi\alpha' \sum_{I,J=1}^N (k_I - i\eta_I D_I)^M (k_J - i\eta_J D_J)^L N_{ML,v}^\nu \right]_{\tau=\frac{1}{2}\tau_2} \quad (4.6)$$

$$M_N^\bullet = \int_0^\infty \frac{d\tau_2}{\tau_2^2} \sum_{v\in S'} J^\bullet_v \prod_{J=1}^N \left( \zeta_j^{(P)} \cdot \int d\eta J \int dz J \int d\theta J \right)$$

$$\times \exp \left[ \pi\alpha' \sum_{I,J=1}^N (k_I - i\eta_I D_I)^M (k_J - i\eta_J D_J)^L N_{ML,v}^\nu \right]_{\tau=\frac{1}{2}\tau_2+\frac{1}{2}} \quad (4.7)$$

in accordance with Eqs. (3.20) and (3.28). We will restrict our attention to the annulus case from now on.

We have denoted by $\sum_{v\in S'} J^\bullet_v$ the part of the integrand which has appeared in the vacuum amplitude [for instance, Eq. (3.29)], and $N_{ML,v}^\nu$ indicates the superannulus Neumann function specified by the boundary condition $v = \left( \frac{v_B}{v_f} \right)$ determined by the spacetime indices $M, L$. The prime $'$ indicates the omission of the bosonic and fermionic zero modes.

Let us analyze the exponential part of the integrand in Eq. (4.6):

$$\exp[\cdots] = \exp \left[ 2\alpha' \sum_{1\leq I < J \leq N} k_I^M k_J^L \pi N_{ML,v_b}^{IJ} \right]$$

$$\times \exp \left[ 2\alpha' \sum_{1\leq I < J \leq N} \left( ik_I^M \theta_I k_J^L \theta_J B_{ML,v_i}^{IJ} + \left( -\eta_I^M k_J^L \theta_I + \eta_J^L k_I^M \theta_I \right) B_{ML,v_i}^{IJ} \right. \right.$$  

$$\left. + \left( \eta_I^M \theta_I k_J^L - \eta_J^L \theta_I k_I^M \right) C_{ML,v_b}^{IJ} - i\eta_I^M \eta_J^L B_{ML,v_i}^{IJ} + \eta_I^M \eta_J^L \theta_I \theta_J E_{ML,v_b}^{IJ} \right]. \quad (4.8)$$

where the $I = J$ part vanishes by the on-shell condition. Following [31], let us label the first, the second, the third, and the fourth lines of the exponent by the number of $\eta$s and by the number of $\theta$s, namely, $[0, 2], [1, 1], [2, 0], [2, 2]$, respectively. Also, upon compactification, namely the division of the pair of indices $(M, L)$ into the spacetime part $(\mu, \lambda)$ and the internal part $(\ell, \ell')$, we set the internal part of the momenta $k_I^\ell = 0$ for simplicity. The index structure of $\pi N_{ML,++}^{IJ}, \pi N_{ML,+-}^{IJ},$
\[ B_{\nu}^{++} = \frac{1}{2} \pi \frac{i}{\nu} S_{\nu} \left( \frac{z_I}{2} - \frac{z_J}{2} \right) = \frac{1}{2} \frac{\partial_{\nu} \left( \frac{z_I}{2} - \frac{z_J}{2} \right)}{\partial_{\nu} \left( \frac{i \tau_2}{2} \right)} \frac{\partial_{\nu} \left( \frac{i \tau_2}{2} \right)}{\partial_{\nu} \left( \frac{z_I}{2} - \frac{z_J}{2} \right)} \]

\[ C_{++}^{II} = \frac{\partial}{\partial z_I} \left[ N_{++}^{II} \right]_{z = \bar{z} = \tilde{z}} = \frac{1}{2} \frac{\partial}{\partial z_I} \left( \frac{z_I - z_J}{2} \right) = \frac{1}{2} \frac{\partial}{\partial z_J} \left( \frac{z_I - z_J}{2} \right) \]

\[ E_{++}^{II} = \frac{\partial}{\partial z_I} \left[ N_{++}^{II} \right]_{z = \bar{z} = \tilde{z}} \]

\[ = \ln \left| \sqrt{\xi_I} - \sqrt{\xi_J} \right| + \sum_{m=1}^{\infty} (-1)^m \ln \left| 1 - \frac{\sqrt{\xi_I} \left( \sqrt{\xi_J} \right)^m}{\sqrt{\xi_J}} \right| \left| 1 - \frac{\sqrt{\xi_J} \left( \sqrt{\xi_I} \right)^m}{\sqrt{\xi_I}} \right| \]

See Appendix H for these properties.

### 4.3. Analysis and evaluation of \( N = 1, 2, 3 \) cases

We will now analyze a few of the simplest cases. Let us first obtain a few generic features of the amplitudes from the integral representation. First, in order to obtain a non-vanishing amplitude, all Grassmann integrations must be saturated. Also, under the assumption made in the last
subsection, \( \sum_j k_I^M = 0 \) for \( M = 0, 1, \ldots, 9 \). Note that the zero mode is absent in the expansion of \( X^\ell, \ell = 5, 6, 7, 8 \), and that momentum conservation is not ensured.

I) \( N = 1 \); the amplitude vanishes generally and trivially such as such a case is absent in the integrand.

II) \( N = 2 \); there is no contribution from \([0, 2][2, 0]\) or from \([1, 1]^2\) by \( k_I \cdot k_J = k_I \cdot \xi^{(p)} = 0 \) for \( I, J = 1, 2 \). Neither is there any contribution from \([2, 2]\) which does not involve \( \nu_I \) by the same reason that the vacuum amplitude vanishes by the Jacobi identity or supersymmetry.

III) \( N = 3 \); this case poses the general question of the presence or absence of the vertex correction.

There is no contribution from the parts in which \([0, 2]\) is involved as \( k_I \cdot k_J = 0 \) for \( I, J = 1, 2, 3 \). The remaining possibilities for a non-vanishing amplitude are \([1, 1]^3\) and \([1, 1][2, 2]\). Among them, the parts which do not involve \( B^H_{B_{\nu_I}, \nu_J} \) vanish after the summation over \( \nu_I \) by the Jacobi identity or supersymmetry. We conclude that the possibilities are contained in \([1, 1]\) of \( B^3_{\nu_I \nu_J} \), \( B^3_{\nu_I \nu_J} \), \( B^3_{\nu_I \nu_J} \) type and of \( B^2_{\nu_I \nu_J} \), \( C_{\nu_I \nu_J} \) type.

4.3.1. Maximal supersymmetry

In this case, namely, in the case of flat 10d and its toroidal compactifications, it is well known that the vanishing of these two types after the summation over \( \nu_I \) is established, (see, for example, [72]) by the Riemann identity Eq. (C11). In fact,

\[
\mathcal{J}_I = (2n)^2 \left( \prod F_1 (a_I, \tau_2) \right) C_{\nu} \frac{\partial \nu(0)^4}{\eta^{1/2}}
\]

according to Eqs. (3.23) and (3.29), and

\[
\frac{\eta^{12}}{(2n)^2 \left( \prod F_1 (a_I, \tau_2) \right)} \sum_{\nu_I} \mathcal{J}_{\nu_I} \left( B^H_{\nu_I} \right)^2 = \sum_{\nu_I} C_{\nu_I} \frac{\partial \nu_I(0)^4}{\eta^{1/2}} \left( B^H_{\nu_I} \right)^2
\]

\[
= \sum_{\nu_I} C_{\nu_I} \frac{\partial \nu_I(0)^4}{\eta^{1/2}} \left( \frac{1}{2} \partial \nu_I \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \partial \nu_I (0) \right)^2
\]

\[
\tag{4.11}
\]

\[
\text{Eq. (C11)} = \frac{1}{2} \left( \frac{1}{2} \partial \left[ \frac{1}{2} \right] \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \right)^2 \partial \left[ \frac{1}{2} \right] \left( \frac{z_I}{2} - \frac{z_J}{2} \right)
\]

\[
\times \partial \left[ \frac{1}{2} \right] \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \partial \left[ \frac{1}{2} \right] (0) \partial \left[ \frac{1}{2} \right] (0)
\]

\[
= 0,
\tag{4.12}
\]

where

\[
x_1 = \frac{1}{2} \left\{ 0 + \left( \frac{z_1}{2} - \frac{z_2}{2} \right) + \left( \frac{z_1}{2} - \frac{z_2}{2} \right) \right\} = \frac{z_1}{2} - \frac{z_2}{2}
\]

\[
y_1 = \frac{1}{2} \left\{ 0 - \left( \frac{z_1}{2} - \frac{z_2}{2} \right) - \left( \frac{z_1}{2} - \frac{z_2}{2} \right) \right\} = - \left( \frac{z_1}{2} - \frac{z_2}{2} \right)
\]

\[
u_1 = \frac{1}{2} \left\{ 0 - \left( \frac{z_1}{2} - \frac{z_2}{2} \right) + \left( \frac{z_1}{2} - \frac{z_2}{2} \right) \right\} = 0.
\tag{4.13}
\]
and we have used $\vartheta \left[ \frac{1}{2} \right] (0) = 0$. Similarly,

\[
\begin{align*}
\frac{\eta^{12}}{(2n)^2 (\prod_{i} F_i (a_i, \tau_2))} & \sum_{\nu_f} J_{\nu_f} B_{\nu_f}^{12} B_{\nu_f}^{23} B_{\nu_f}^{13} \\
& = \sum_{\nu_f} C_{\nu_f} \vartheta_{\nu_f} (0)^4 B_{\nu_f}^{12} B_{\nu_f}^{23} B_{\nu_f}^{13} \\
& = \sum_{\nu_f} C_{\nu_f} \vartheta_{\nu_f} (0)^4 \left( \frac{1}{2} \right)^3 \left( \frac{\vartheta_{\nu_f} \left( \frac{z_1}{2} - \frac{z_2}{2} \right)}{\vartheta_{\nu_f} (0)} \frac{\vartheta \left[ \frac{1}{2} \right] (0)}{\vartheta \left[ \frac{1}{2} \right] \left( \frac{z_1}{2} - \frac{z_2}{2} \right)} \right) \\
& \quad \times \left( \frac{\vartheta_{\nu_f} \left( \frac{z_1}{2} - \frac{z_3}{2} \right)}{\vartheta_{\nu_f} (0)} \frac{\vartheta \left[ \frac{1}{2} \right] (0)}{\vartheta \left[ \frac{1}{2} \right] \left( \frac{z_1}{2} - \frac{z_3}{2} \right)} \right) \\
& \quad \times \left( \frac{\vartheta_{\nu_f} \left( \frac{z_2}{2} - \frac{z_3}{2} \right)}{\vartheta_{\nu_f} (0)} \frac{\vartheta \left[ \frac{1}{2} \right] (0)}{\vartheta \left[ \frac{1}{2} \right] \left( \frac{z_2}{2} - \frac{z_3}{2} \right)} \right) \\
& = 0,
\end{align*}
\]

where

\[
\begin{align*}
x_1 &= \frac{1}{2} \left\{ 0 + \left( \frac{z_1}{2} - \frac{z_2}{2} \right) + \left( \frac{z_2}{2} - \frac{z_3}{2} \right) + \left( \frac{z_1}{2} - \frac{z_3}{2} \right) \right\} = \left( \frac{z_1}{2} - \frac{z_3}{2} \right) \\
y_1 &= \frac{1}{2} \left\{ 0 + \left( \frac{z_1}{2} - \frac{z_2}{2} \right) - \left( \frac{z_2}{2} - \frac{z_3}{2} \right) - \left( \frac{z_1}{2} - \frac{z_3}{2} \right) \right\} = \left( \frac{z_3}{2} - \frac{z_2}{2} \right) \\
u_1 &= \frac{1}{2} \left\{ 0 - \left( \frac{z_1}{2} - \frac{z_2}{2} \right) + \left( \frac{z_2}{2} - \frac{z_3}{2} \right) - \left( \frac{z_1}{2} - \frac{z_3}{2} \right) \right\} = \left( \frac{z_2}{2} - \frac{z_1}{2} \right) \\
v_1 &= \frac{1}{2} \left\{ 0 - \left( \frac{z_1}{2} - \frac{z_2}{2} \right) - \left( \frac{z_2}{2} - \frac{z_3}{2} \right) + \left( \frac{z_1}{2} - \frac{z_3}{2} \right) \right\} = 0.
\end{align*}
\]

4.3.2. Non-maximal supersymmetry

Finally, let us consider the case of the type I superstring on $T^4/Z_2$. Among the summation over $\nu_f$, only the part belonging to $\frac{1}{2} \tau (Z_2)$ in Eq. (3.29) contributes to the $N = 3$ amplitude. So we will concentrate on this case. The integrations of the exponential factor Eq. (4.8) over the Grassmann coordinates $\prod_{j=1}^{3} \xi^{(p)}_j \int d \eta_J \int d \vartheta_J \exp \cdots$ yield

\[
(2\alpha')^3 k_3 \cdot s_1^{(p)} k_1 \cdot s_2^{(p)} k_2 \cdot s_3^{(p)} \left\{ -2B_{\nu_f}^{12} B_{\nu_f}^{23} B_{\nu_f}^{31} + \left( B_{\nu_f}^{12} \right)^2 \left( C_{++}^{13} - C_{++}^{23} \right) + \left( B_{\nu_f}^{23} \right)^2 \left( C_{++}^{32} - C_{++}^{12} \right) \right\} \times \left( C_{++}^{21} + C_{++}^{31} \right) + \left( B_{\nu_f}^{31} \right)^2 \left( C_{++}^{32} - C_{++}^{12} \right). \tag{4.16}
\]
Coming back to Eq. (4.6), we obtain the expression for $A_3^{T^4/Z_2}$:

$$A_3^{T^4/Z_2} = (2n) \text{Tr} \left( T^1 T^2 T^3 \right) (2\alpha')^3 \left( k_3 \cdot \xi_1^{(p)} \right) \left( k_1 \cdot \xi_2^{(p)} \right) \left( k_2 \cdot \xi_3^{(p)} \right)$$

$$\times \int_0^\infty \frac{d\tau_2}{\tau_2^3} \frac{1}{\tau_2^5} \left( \prod_{l=1,2,3} \int_{\nu \in S^2} \frac{1}{2} J'_l(Z_2) \right)$$

$$\times \left\{ -2B_{\nu_1^6}^{12} B_{\nu_1^6}^{22} B_{\nu_1^6}^{31} + \left( B_{\nu_1^6}^{12} \right)^2 \left( C_{++}^{13} - C_{++}^{23} \right) \right.$$

$$+ \left( B_{\nu_1^6}^{23} \right)^2 \left( C_{++}^{21} - C_{++}^{31} \right) + \left( B_{\nu_1^6}^{31} \right)^2 \left( C_{++}^{32} - C_{++}^{12} \right) \left\} \right., \quad (4.17)$$

where $J'_l(Z_2)$ has been introduced in Eq. (3.29) and $\nu_1^6$ refers to the spacetime part (bin) of $\nu_l$. Let us recall, from Eqs. (3.23), (3.25),

$$\begin{align*}
\begin{array}{c}
++ = \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0),
+- = \frac{2\partial \left[ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right]}{\eta^3}(0),
\end{array}
\end{align*}$$

$$\begin{align*}
\begin{array}{c}
++ = \frac{\partial \left[ \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0),
+- = \frac{2\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right]}{\eta^3}(0),
\end{array}
\end{align*}$$

and therefore

$$\begin{align*}
\begin{array}{c}
++ = \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0),
+- = \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right]}{\eta^3}(0),
\end{array}
\end{align*}$$

$$\begin{align*}
\begin{array}{c}
++ = \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0),
+- = \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right]}{\eta^3}(0),
\end{array}
\end{align*}$$

as well as, from Eq. (4.9),

$$B_{--}^{\nu} = \frac{1}{2} \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0) \left( \frac{\xi \xi - \xi \xi}{i \xi} \right)$$

$$B_{++}^{\nu} = \frac{1}{2} \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0) \left( \frac{\xi \xi - \xi \xi}{i \xi} \right)$$

$$C_{++}^{\nu} = \frac{1}{2} \frac{\partial \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right]}{\eta^3}(0) \left( \frac{\xi \xi - \xi \xi}{i \xi} \right) + \pi \frac{z x - z x}{\tau_2}. \quad (4.20)$$

We obtain

$$A_3^{T^4/Z_2} = (2n) \text{Tr} \left( T^1 T^2 T^3 \right) (2\alpha')^3 \left( k_3 \cdot \xi_1^{(p)} \right) \left( k_1 \cdot \xi_2^{(p)} \right) \left( k_2 \cdot \xi_3^{(p)} \right)$$
where we have introduced the shorthand notation \( \vartheta^A_{\beta} \equiv \vartheta^A_{\beta} (I - J) \). The second, third, and fourth terms in Eq. (4.21) can be further converted by using Eq. (C12):

\[
\sum_{v \in S'_{\nu \cdot A}} \frac{1}{2} \vartheta^{(Z)}_{v \cdot A} \left( B^{IJ}_{v_0} \right)^2 = \frac{1}{2} \left( \prod_{l=5,6,7,8} (a_I \sqrt{\tau_2}) \right) \frac{4}{2} \eta^6 \left( \frac{\vartheta^0_0}{\vartheta^{I/2}_0} (0) \right) \left( \frac{\vartheta^{I/2}_0}{\vartheta^{I/2}_0} (0) \right)
\]

\[
\times \left\{ \left( B^{IJ}_{v_0} \right)^2 - \left( B^{IJ}_{v_0} \right)^2 \right\}
\]

\[
= \frac{1}{2} \left( \prod_{l=5,6,7,8} (a_I \sqrt{\tau_2}) \right) \frac{4}{2} \eta^6 \left( \frac{\vartheta^0_0}{\vartheta^{I/2}_0} (0) \right) \left( \frac{\vartheta^{I/2}_0}{\vartheta^{I/2}_0} (0) \right)
\]
\[
\times \left\{ \frac{\vartheta \left[ \frac{0}{0} \right] \left( I - J \right| \frac{i\tau}{2} \right)}{2^2} \frac{\vartheta \left[ \frac{0}{0} \right] \left( 0 \right| \frac{i\tau}{2} \right)}{2} - \frac{\vartheta \left[ \frac{1}{1} \right] \left( \frac{i\tau}{2} \right)}{2} \left( I - J \left| \frac{i\tau}{2} \right. \right) \right\} \\
- \frac{1}{2^2} \frac{\vartheta \left[ \frac{0}{0} \right] \left( I - J \right| \frac{i\tau}{2} \right)}{2} \frac{\vartheta \left[ \frac{1}{1} \right] \left( 0 \right| \frac{i\tau}{2} \right)}{2} \left( I - J \left| \frac{i\tau}{2} \right. \right) \right\} \\
\text{Eq. (C12) =} - \left( \prod_{I=5,6,7,8} (a_I \sqrt{\tau_2}) \right) \frac{1}{\eta^6} \frac{1}{2^2} \vartheta \left[ \frac{1}{1} \right] \left( 0 \right| \frac{i\tau}{2} \right), \tag{4.22}
\]

where, in Eq. (C12),

\[
x_1 = \frac{1}{2} \left( \left( \frac{z_I}{2} - \frac{z_J}{2} \right) + \left( \frac{z_I}{2} - \frac{z_J}{2} \right) + 0 + 0 \right) = \frac{z_I}{2} - \frac{z_J}{2}
\]

\[
y_1 = \frac{1}{2} \left( \left( \frac{z_I}{2} - \frac{z_J}{2} \right) + \left( \frac{z_I}{2} - \frac{z_J}{2} \right) - 0 - 0 \right) = \frac{z_I}{2} - \frac{z_J}{2}
\]

\[
u_1 = \frac{1}{2} \left( \left( \frac{z_I}{2} - \frac{z_J}{2} \right) - \left( \frac{z_I}{2} - \frac{z_J}{2} \right) + 0 - 0 \right) = 0
\]

Finally, we obtain

\[
\mathcal{A}_{3}^{T_{1}T_{2}} = (2n) \text{Tr} \left( T^1 T^2 T^3 \right) (2a')^3 \left( k_3 \cdot \xi_1 (P) \right) \left( k_1 \cdot \xi_2 (P) \right) \left( k_2 \cdot \xi_3 (P) \right)
\]

\[
\times \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^5} \left( \prod_{I=1,2,3} \int dz_I \right) \left( \prod_{I=5,6,7,8} \left( a_I \sqrt{\tau_2} \right) \right) \frac{1}{\eta^6}
\]

\[
\times \left\{ \frac{\vartheta \left[ \frac{0}{0} \right] \left( 0 \right| \frac{i\tau}{2} \right)}{2} \frac{\vartheta \left[ \frac{1}{1} \right] \left( 0 \right| \frac{i\tau}{2} \right)}{2} \left( -2 \right) \vartheta \left[ \frac{1}{1} \right] \left( \frac{i\tau}{2} \right) \left( 0 \right)
\]

\[
- \left\{ \vartheta \left[ \frac{0}{0} \right] \left( 1 - 2 \right) \vartheta \left[ \frac{0}{0} \right] \left( 0 \right) \vartheta \left[ \frac{0}{0} \right] \left( 3 - 1 \right)
\]

\[
\times \vartheta \left[ \frac{0}{0} \right] \left( 0 \right)
\]

\[
\left. + 2 \left( \vartheta \left[ \frac{1}{1} \right] \left( 1 - 2 \right) \vartheta \left[ \frac{1}{1} \right] \left( 2 - 3 \right) \vartheta \left[ \frac{1}{1} \right] \left( 3 - 1 \right) \right) \vartheta \left[ \frac{1}{1} \right] \left( 2 \right) \right|_{\tau = \frac{i\tau}{2}} \right).
\tag{4.24}
\]
Unlike the case of maximal supersymmetry, after nationalizing, each term consists of the product of different θ functions and we do not find the use of the Riemann identity.

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Appendix A. Some of the notation

The complex coordinates on the worldsheet torus are denoted by

$$z \equiv \sigma^1 + \tau \sigma^2, \quad \bar{z} = \sigma^1 + \bar{\tau} \sigma^2 \quad (0 \leq \sigma_1, \sigma_2 \leq 1)$$  \hspace{1cm} (A1)

with modular parameter $\tau \equiv \tau_1 + i \tau_2$.

The Laplacian is defined by

$$\Delta \equiv 4 \partial_z \partial_{\bar{z}}.$$  \hspace{1cm} (A2)

We introduce a real superfield by

$$X^M(z, \bar{z}, \theta, \bar{\theta}) = X^M(z, \bar{z}) + \sqrt{1/2} \psi^M(z, \bar{z}) \theta + \sqrt{1/2} \bar{\psi}^M(z, \bar{z}) \bar{\theta}$$  \hspace{1cm} (A3)

where $\theta$ and $\bar{\theta}$ are Grassmann numbers, $X^M(z, \bar{z})$ and $\psi^M(z, \bar{z})$ are bosonic and fermionic fields, and $F^M(z, \bar{z})$ is an auxiliary field. The super-derivatives are defined by

$$D = -\partial_{\theta} + i \theta \partial_z, \quad \bar{D} = \partial_{\bar{\theta}} - i \bar{\theta} \partial_{\bar{z}}.$$  \hspace{1cm} (A4)

Appendix B. Normalization in Eq. (2.1)

Let us determine the normalization $N$ in

$$\Phi_{n_1, n_2}^\alpha\beta(\sigma^1, \sigma^2) = N e^{2\pi i (n_1 + \alpha) \sigma^1} e^{2\pi i (n_2 + \beta) \sigma^2}.$$  \hspace{1cm} (B1)

The inner product with functions $f$, $g$ is defined as

$$(f, g) = \int_0^1 d\sigma^1 \int_0^1 d\sigma^2 \sqrt{g} f \ast g.$$  \hspace{1cm} (B2)

On a torus geometry

$$\hat{g}_{mn}(\tau) = \begin{bmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{bmatrix}$$  \hspace{1cm} (B3)

the orthonormality of $\Phi_{n_1, n_2}^\alpha\beta(\sigma^1, \sigma^2)$ implies

$$\delta_{m_1, n_1} \delta_{m_2, n_2} = \left( \Phi_{m_1, n_2}^\alpha\beta(\sigma^1, \sigma^2), \Phi_{n_1, n_2}^\alpha\beta(\sigma^1, \sigma^2) \right) = |N|^2 \tau_2 \delta_{m_1, n_1} \delta_{m_2, n_2}.$$  \hspace{1cm} (B4)

Hence we take

$$N = \frac{1}{\sqrt{\tau_2}}.$$  \hspace{1cm} (B5)
Appendix C. Some formulae

C.1. Gauss hypergeometric function

The Gauss hypergeometric function is

\[ F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \]

where \((\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\), \((\alpha)_0 = 1\). \hfill (C2)

In order to obtain the second line in Eq. (C1), we must have

- \(\text{Re} \gamma > \text{Re} \beta > 0\),
- \(z\) cannot be a real number which is greater than 1,
- \((1-tz)^{-\alpha}\) takes the branch which goes to 1 as \(t \to 0\).

When \(b = 1, k = 0\), the formula

\[ \sum_{n=0}^{\infty} \frac{x^n}{(a+nb)(a+nb+1)\cdots(a+nb+k)} = \frac{1}{k!} \int_{0}^{1} \frac{t^{a-1}(1-t)^k}{1-xt} \, dt, \quad [a, b > 0, |x| < 1] \]

\hfill (C3)

can be written as

\[ \sum_{n=0}^{\infty} \frac{x^n}{n+a} = \int_{0}^{1} \frac{t^{a-1}}{1-t} \, dt = \frac{\Gamma(a)}{\Gamma(1+a)} F(1, a, 1+a; x). \]

\hfill (C4)

Thanks to Eq. (C4),

\[ \sum_{n=0}^{\infty} \frac{1}{n+a} \frac{x^{n+a}}{1-Cy^{n+a}} = \sum_{n=0}^{\infty} \frac{1}{n+a} x^{n+a} \sum_{m=0}^{\infty} \frac{(Cy^{n+a})^m}{m!} = \sum_{m=0}^{\infty} x^a (Cy^a)^m \sum_{n=0}^{\infty} \frac{(xy^m)^n}{n+a} \]

\hfill (C4)

\[ = \sum_{m=0}^{\infty} x^a (Cy^a)^m \frac{\Gamma(a)}{\Gamma(1+a)} F(1, a, 1+a; xy^m) \]

\[ = \frac{x^a \Gamma(a)}{\Gamma(1+a)} \sum_{m=0}^{\infty} (Cy^a)^m F(1, a, 1+a; xy^m) \]

\hfill (C5)

with \(a > 0, m \in \{0, N\}, \) and \(|xy^m| < 1\).

C.2. Jacobi theta function

We define the Jacobi theta function as

\[ \vartheta_{[\alpha \beta]}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+\alpha)^2 \tau} e^{2\pi i(n+\alpha)(z+\beta)} \]

\[ = e^{2\pi i\alpha(z+\beta)} e^{\pi i\alpha^2 \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \left( 1 + e^{2\pi i \left(n+\alpha - \frac{1}{2}\right)} e^{2\pi i(z+\beta)} \right) \]

\[ \times \left( 1 + e^{2\pi i \left(n-\alpha - \frac{1}{2}\right)} e^{-2\pi i(z+\beta)} \right). \]

\hfill (C6)
We also use the following notation:

\[
\vartheta_{-\pm} \equiv \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vartheta_{-+} \equiv \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \quad \vartheta_{+-} \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.
\] (C7)

This function has the following properties:

\[
\vartheta \begin{bmatrix} \alpha + 1 \\ \beta \end{bmatrix} (z|\tau) = \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau), \quad \vartheta \begin{bmatrix} \alpha \\ \beta + 1 \end{bmatrix} (z|\tau) = e^{2\pi i \alpha} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau),
\] (C8)

\[
\frac{1}{\vartheta} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \vartheta \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} (-|\tau - \bar{\tau}).
\] (C9)

The theta function satisfies the heat equation:

\[
\frac{\partial^2 \vartheta}{\partial z^2} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = 4\pi i \frac{\partial}{\partial \tau} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau).
\] (C10)

For \(\alpha, \beta = 0, \frac{1}{2}\), this function satisfies the Riemann identity \([73]\)

\[
\sum_{\alpha, \beta = 0, \frac{1}{2}} (-1)^{2\alpha+2\beta} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (x|\tau) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (y|\tau) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (u|\tau) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (v|\tau)
\]

\[
= 2\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (x_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (y_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (v_1|\tau),
\] (C11)

and also

\[
\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (x|\tau) \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (y|\tau) \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (u|\tau) \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (v|\tau)
\]

\[
- \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (x|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (y|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (v|\tau)
\]

\[
+ \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (x|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (y|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (v|\tau)
\]

\[
= -2\vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (x_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (y_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u_1|\tau) \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (v_1|\tau),
\] (C12)

where

\[
x_1 = \frac{1}{2}(x + y + u + v), \quad y_1 = \frac{1}{2}(x + y - u - v),
\]

\[
u_1 = \frac{1}{2}(x - y + u - v), \quad v_1 = \frac{1}{2}(x - y - u + v).
\] (C13)

C.3. Dedekind eta function

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\] (C14)

Using this,

\[
\vartheta \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} (0|\tau) = -2\pi \{\eta(\tau)^2\}.
\] (C15)
C.4. Ramanujan’s $\psi_1$ summation formula

The Ramanujan’s summation formula [74,75] is

$$
\sum_{n=-\infty}^{\infty} \frac{(a; q)_n z^n}{(b; q)_n} = \frac{(az; q)_{\infty}(q; q)_{\infty} \left(\frac{q}{az}; q\right)_{\infty} \left(\frac{b}{a}; q\right)_{\infty}}{(z; q)_{\infty}(b; q)_{\infty} \left(\frac{b}{az}; q\right)_{\infty} \left(\frac{q}{a}; q\right)_{\infty}},
$$

(C16)

with $|\frac{b}{a}| < |z| < 1$, $|q| < 1$. For $a, q \in \mathbb{C}$, $|q| < 1$, $n \in \mathbb{Z}$,

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \quad (C17)
$$

is the $q$-Pochhammer symbol.

From Eq. (C16), one can derive

$$
\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} = \frac{(az; q)_{\infty} \left(\frac{q}{az}; q\right)_{\infty} (q; q)_{2\infty}}{(a; q)_{\infty}(z; q)_{\infty} \left(\frac{q}{z}; q\right)_{\infty} \left(\frac{q}{a}; q\right)_{\infty}}
$$

(C18)

with $|q| < |z| < 1$.

Proof  Substituting $b = aq$ into Eq. (C16), the left hand side is

$$
\frac{(a; q)_n b=aq}{(b; q)_n} = \frac{(a; q)_n}{(aq; q)_n} = \prod_{k=0}^{n-1} \frac{(1 - aq^k)}{(1 - aq^{k+1})} = \frac{(1 - a)(1 - aq)(1 - aq^2)\cdots(1 - aq^{n-2})(1 - aq^{n-1})}{(1 - aq)(1 - aq^2)\cdots(1 - aq^{n-2})(1 - aq^{n-1})} = \frac{1 - a}{1 - aq^n}
$$

(C19)

$$
\therefore \quad \frac{1}{1 - a} \frac{(a; q)_n}{(aq; q)_n} = \frac{1}{1 - aq^n} \quad (C20)
$$

The right-hand side is

$$
\frac{1}{1 - a} \frac{(az; q)_{\infty}(q; q)_{\infty} \left(\frac{q}{az}; q\right)_{\infty} \left(\frac{b}{a}; q\right)_{\infty} b=aq}{(z; q)_{\infty}(b; q)_{\infty} \left(\frac{b}{az}; q\right)_{\infty} \left(\frac{q}{z}; q\right)_{\infty} \left(\frac{q}{a}; q\right)_{\infty}} = \frac{(az; q)_{\infty} \left(\frac{q}{az}; q\right)_{\infty} (q; q)_{2\infty}}{(z; q)_{\infty}(a; q)_{\infty} \left(\frac{q}{z}; q\right)_{\infty} \left(\frac{q}{a}; q\right)_{\infty}}.
$$

(C21)

Therefore, substituting $b = aq$ and dividing Eq. (C16) by $1 - a$ on each side, we obtain Eq. (C18).  

Using Eq. (C18), we find

$$
\sum_{n=-\infty}^{\infty} \frac{(e^{-2\pi i}\tau)^n}{1 - (e^{-2\pi i}\beta q^\alpha) q^n} = \frac{ie^{-2\pi i\alpha\beta}}{2\pi} \frac{\theta \left[ \frac{\alpha - \frac{1}{2}}{\frac{1}{2} - \beta} \right] (\tau|\tau)}{\theta \left[ \frac{\alpha - \frac{1}{2}}{\frac{1}{2} - \beta} \right] (0|\tau)} \frac{\theta \left[ \frac{1}{2} \right] (0|\tau)}{\theta \left[ \frac{1}{2} \right] (\tau|\tau)}.
$$

(C22)
Proof \quad With \ \zeta = e^{2\pi i z} \text{ and } a = e^{-2\pi i \beta q^a},
\[
\sum_{m=-\infty}^{\infty} \frac{\zeta^m}{1 - aq^m}
\]

Eq. (C18) \quad \frac{(a\zeta; q)_{\infty} \left( \theta^q; q \right)_{\infty}}{(a; q)_{\infty} (\zeta; q)_{\infty} \left( \frac{q}{a}; q \right)_{\infty}} \quad = \quad \prod_{m=0}^{\infty} \left( 1 + e^{2\pi i (z+(\frac{1}{2})-\beta) |q^{m+a+1}} \right) \prod_{m=1}^{\infty} (1 - q^m)^2
\]

= \quad \prod_{m=0}^{\infty} \left( 1 + e^{2\pi i (0+(\frac{1}{2})-\beta) |q^{m+a}} \right) \left( 1 + e^{2\pi i (\frac{1}{2}) q^m} \right) \left( 1 + e^{-2\pi i (\frac{1}{2}) q^{m+1}} \right) \left( 1 + e^{2\pi i (0+(\frac{1}{2})-\beta) |q^{m-a+1}} \right)
\]

= \quad e^{2\pi i (\frac{1}{2}) (\frac{1}{2} - \beta)} q^{-\frac{1}{2}} \prod_{m=1}^{\infty} (1 - q^m) \quad \frac{e^{2\pi i (\frac{1}{2}) (\frac{1}{2} - \beta)} q^{-\frac{1}{2}}}{\prod_{m=1}^{\infty} (1 - q^m)}
\]

= \quad \prod_{m=1}^{\infty} (1 - q^m) \left( 1 + e^{2\pi i (\frac{1}{2}) q^m (\frac{1}{2} - \beta)} \right) \left( 1 + e^{-2\pi i (\frac{1}{2}) q^{m-\frac{1}{2} - \beta}} \right) \left( 1 + e^{2\pi i (0+(\frac{1}{2})-\beta) |q^{m-\frac{1}{2} - \beta}} \right)
\]

\[
\times \quad \prod_{m=1}^{\infty} (1 - q^m) \left( 1 + e^{2\pi i (\frac{1}{2}) q^m (\frac{1}{2} - \beta)} \right) \left( 1 + e^{-2\pi i (\frac{1}{2}) q^{m-(\frac{1}{2}) - \beta}} \right)
\]

\[
\times \quad \frac{e^{2\pi i (\frac{1}{2}) (\frac{1}{2} - \beta)} q^{-\frac{1}{2}} \prod_{m=1}^{\infty} (1 - q^m)}{(-2\pi) q^{\frac{1}{2}} \prod_{m=1}^{\infty} (1 - q^m)^3}
\]

Eqs. (C6), (C14) \quad e^{-2\pi i (\frac{1}{2})} e^{-\pi i (\frac{1}{2} - \beta)} \left( z | \tau \right) = \left( \theta^{-\frac{1}{2}} \frac{a - \frac{1}{2}}{1 - \beta} \right) \left( \theta^{-\frac{k}{2}} \frac{z}{\tau} \right)
\]

= \quad \frac{1}{\sqrt{2\pi}} e^{2\pi i (\frac{1}{2} - \beta)} \left( z | \tau \right) \quad \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)
\]

Eqs. (C8), (C15) \quad i e^{-2\pi i (\frac{1}{2})} \left( z | \tau \right) \quad \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)
\]

C.5. \quad Zeta function

The zeta function is defined by
\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.
\] (C24)

For example,
\[
\zeta(0) = -\frac{1}{2}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}.
\] (C25)

In addition, the generalized zeta function is defined by
\[
\zeta(z, a) = \sum_{n=0}^{\infty} \frac{1}{(a + n)^z} \quad [a : \text{const.}, \ \Re z > 1].
\] (C26)

This function satisfies
\[
\zeta(z, 1) = \zeta(z), \quad \zeta\left(z, \frac{1}{2}\right) = (2^z - 1)\zeta(z).
\] (C27)

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Appendix D. \( G_{++} \) and \( G_{+-} \)

D.1. \( G_{++}(z, \bar{z}|0, 0) = G \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \bar{z}|0, 0) \)

\[
G_{++}(z, \bar{z}|0, 0) \equiv G \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \bar{z}|0, 0)
\]

\[
= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2 |n_2 - n_1|^2} e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2 |n_2 - n_1|^2} e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[+ \frac{1}{\tau_2} \sum_{n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2 |n_2|^2} e^{2\pi in_2\sigma^2}.
\] (D1)

Using

\[
\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1 - q^n} = -\sum_{m=0}^{\infty} \ln(1 - xq^m).
\] (D2)

the first term in the last line of Eq. (D1) can be computed as

\[
\frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2 |n_2 - n_1|^2} e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2 |n_2 - n_1|^2} e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[= \frac{i(\tau - \bar{\tau})}{4(2\pi)} \frac{1}{n_1} \left[ \int_0^1 d\sigma e^{-2\pi i(n_2-n_1)\sigma} - \int_0^1 d\sigma e^{-2\pi i(n_2-n_1)\sigma} \right] e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[= \frac{-2}{4(2\pi)} \sum_{n_1 = -\infty}^{\infty} \frac{1}{n_1} \left[ \int_0^1 d\sigma \delta(\sigma^2 - \sigma) e^{2\pi in_1(\sigma^1 + \tau)\sigma} - \int_0^1 d\sigma \delta(\sigma^2 - \sigma) e^{2\pi in_1(\sigma^1 + \bar{\tau})\sigma} \right] e^{2\pi in_1\sigma} e^{2\pi in_2\sigma^2}
\]

\[= \frac{-2}{4(2\pi)} \sum_{n_1 = -\infty}^{\infty} \left[ \frac{1}{n_1} \frac{\xi_{n_1}}{1 - \bar{q}^n_1} - \frac{1}{n_1} \frac{\xi^{-1}_{n_1}}{1 - q^{-1}n_1} \right]
\]

\[= \frac{-2}{4(2\pi)} \sum_{n_1 = -\infty}^{\infty} \left[ \frac{1}{n_1} \frac{\xi_{n_1}}{1 - \bar{q}^n_1} + \frac{1}{n_1} \left( \frac{\bar{q}}{\xi} \right)^{n_1} + \frac{1}{n_1} \left( \frac{q}{\xi} \right)^{n_1} + \frac{1}{n_1} \frac{\bar{\xi}^{n_1}}{1 - q^{n_1}} \right]
\]
Now, \( \prod_{n=1}^{\infty} \frac{1}{1 - \zeta q^n} \) \( \equiv \frac{1}{2(2\pi)} \left[ -\ln(1 - \zeta q^m) - \ln\left(1 - \frac{q}{\zeta} q^m\right) - \ln\left(1 - \frac{q}{\zeta} q^m\right) - \ln(1 - \zeta q^m) \right] \)
\[ \equiv \frac{2}{4(2\pi)} \left[ \ln|\zeta - 1|^2 + \sum_{m=1}^{\infty} \ln|1 - \zeta q^m|^2 \left| 1 - \frac{q^m}{\zeta} \right|^2 \right]. \]

Next, turning to the second term,
\[ \frac{1}{\tau_2} \sum_{n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2} |n_2|^2 e^{2\pi in_2\sigma^2} = \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} \left( \sum_{n_2 = 1}^{\infty} \frac{1}{n_2^2} e^{2\pi in_2\sigma^2} + \sum_{n_2 = 1}^{\infty} \frac{1}{n_2^2} e^{-2\pi in_2\sigma^2} \right) \]
\[ \equiv \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} F(\sigma^2). \]

Now,
\[ \frac{1}{(2\pi i)^2} \frac{d^2 F(\sigma^2)}{d(\sigma^2)^2} = \sum_{n_2 = 1}^{\infty} e^{2\pi in_2\sigma^2} + \sum_{n_2 = 1}^{\infty} e^{-2\pi in_2\sigma^2} = \frac{e^{2\pi i\sigma^2 - \sigma^2}}{1 - e^{2\pi i\sigma^2}} - \frac{1}{1 - e^{2\pi i\sigma^2}} \]
\[ = -1 \text{ modulo } \delta(\sigma^2). \]
\[ \therefore F(\sigma^2) = (2\pi i) \left( -\frac{1}{2} \sigma^2 \right)^2 + A(\sigma^2) + B. \]

Using
\[ A = F'(0) = 0, \quad B = F(0) = 2 \sum_{n_2 = 1}^{\infty} \frac{1}{n_2^2} = 2\zeta(2) = 2 \cdot \frac{\pi^2}{6}, \]
we obtain
\[ \frac{1}{\tau_2} \sum_{n_2 = -\infty}^{\infty} \frac{1}{4(2\pi)^2} |n_2|^2 e^{2\pi in_2\sigma^2} = \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} (2\pi i)^2 \left( -\frac{1}{2} \sigma^2 \right)^2 + 2 \cdot \frac{\pi^2}{6} \]
\[ = -\frac{1}{2} \frac{(\text{Im } z)^2}{\tau_2} + 2\tau_2 \cdot \frac{\pi^2}{6}. \]

As a result of Eqs. (D1), (D3), and (D8),
\[ G_{++}(z, \bar{z}|0, 0) = \frac{2}{4(2\pi)} \left[ \ln|\zeta - 1|^2 + \sum_{m=1}^{\infty} \ln|1 - \zeta q^m|^2 \left| 1 - \frac{q^m}{\zeta} \right|^2 \right] - \frac{1}{2} \frac{(\text{Im } z)^2}{\tau_2} + 2\tau_2 \cdot \frac{\pi^2}{6}. \]

Due to
\[ \prod_{m=1}^{\infty} \left(1 - \zeta q^m\right) \left(1 - \frac{q^m}{\zeta}\right) = \frac{i q^{\frac{1}{2}} e^{\pi i z}}{i q^{\frac{1}{2}} e^{\pi i z}} \prod_{n=1}^{\infty} (1 - q^n)^{(-2\pi)} \left\{ q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n) \right\}^3 \]
\[ \times \prod_{m=1}^{\infty} \left(1 - e^{2\pi i z q^m}\right) \left(1 - e^{-2\pi i z q^m}\right) \]
\[ \equiv \frac{(2\pi i) \prod_{n=1}^{\infty} (1 - q^n)^{\frac{1}{2}} \theta^\Gamma \left[ \frac{1}{2} \right] (z)}{\theta^\Gamma \left[ \frac{1}{2} \right] (0)}, \]

\[ \text{Eqs. (C14), (C15)} \]

\[ \text{by grant on 30 July 2018.} \]

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Hence we obtain

\[
\ln |\zeta - 1|^2 + \sum_{m=1}^{\infty} \ln |1 - \zeta q^n|^2 \left| 1 - \frac{q^m}{\zeta} \right|^2
\]

\[
\text{Eq. (D10)} \quad \ln \left| e^{2\pi i\zeta} - 1 \right|^2 \leq \frac{2(2\pi i) \prod_{n=1}^{\infty} (1 - q^n)^2 \vartheta \left[ \frac{1}{2} \right] (z)}{\vartheta' \left[ \frac{1}{2} \right] (0)}
\]

\[
= \ln \left| \vartheta \left[ \frac{1}{2} \right] (z) \right|^2 + 2 \sum_{n=1}^{\infty} \ln |1 - q^n|^2 - 2\pi (\text{Im} z) + 2 \ln(2\pi).
\]

Hence we obtain

\[
G_{++}(z, \bar{z}|0, 0) \quad \text{Eqs. (D9),(D11)} \quad \frac{1}{2\pi} \ln \left| \vartheta \left[ \frac{1}{2} \right] (z) \right| - \frac{1}{2} \frac{(\text{Im} z)^2}{\tau_2}
\]

\[
+ \left[ \frac{1}{2\pi} 2 \sum_{n=1}^{\infty} \ln |1 - q^n| - \frac{1}{2} (\text{Im} z) + \frac{1}{2\pi} \ln(2\pi) + 2\tau_2 \cdot \frac{\pi^2}{6} \right].
\]

The terms in the brackets on the last line vanish when acting \( \Delta = 4\partial_z \partial_{\bar{z}} \).

**D.2.** \( G_{+-}(z, \bar{z}|0, 0) = G \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] (z, \bar{z}|0, 0) \)

\[
G_{+-}(z, \bar{z}|0, 0) \equiv G \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] (z, \bar{z}|0, 0)
\]

\[
= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} \left( n_2 + \frac{1}{2} \right) - n_1 \tau \right|^2 e^{2\pi i n_1 \sigma_1} e^{2\pi i \left( n_2 + \frac{1}{2} \right) \sigma_2}
\]

\[
= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} \left( n_2 + \frac{1}{2} \right) - n_1 \tau \right|^2 e^{2\pi i n_1 \sigma_1} e^{2\pi i \left( n_2 + \frac{1}{2} \right) \sigma_2}
\]

\[
+ \frac{1}{\tau_2} \sum_{n_2 = -\infty}^{\infty} \frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} \left| n_2 + \frac{1}{2} \right|^2 e^{2\pi i \left( n_2 + \frac{1}{2} \right) \sigma_2}.
\]

The first term in the last line of Eq. (D13) is

\[
\frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{4(2\pi)^2}{(\tau - \bar{\tau})^2} \left( n_2 + \frac{1}{2} \right) - n_1 \tau \right|^2 e^{2\pi i n_1 \sigma_1} e^{2\pi i \left( n_2 + \frac{1}{2} \right) \sigma_2}
\]

\[
= \frac{1}{\tau_2} \sum_{n_1, n_2 = -\infty}^{\infty} \frac{\tau - \bar{\tau}}{4(2\pi)^2} n_1 \left( \frac{1}{n_2 + \frac{1}{2}} - n_1 \tau \right) - n_1 \tau \right|^2 e^{2\pi i n_1 \sigma_1} e^{2\pi i \left( n_2 + \frac{1}{2} \right) \sigma_2}
\]
where we have used

\[
\sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1 - (-y^n)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=0}^{\infty} x^n (-y)^m = \sum_{m=0}^{\infty} (-1)^m \sum_{n=1}^{\infty} \frac{1}{n} (xy^m)^n \\
= \sum_{m=0}^{\infty} (-1)^m \ln \left(1 - xy^m\right) = \sum_{m=0}^{\infty} (-1)^{m+1} \ln \left(1 - xy^m\right). \tag{D15}
\]

The second term in Eq. (D13) is

\[
\frac{1}{\tau_2} \sum_{n_2=-\infty}^{\infty} \frac{1}{4(2\pi)^2} \frac{1}{\left|n_2 + \frac{1}{2}\right|^2} e^{2\pi i\left(n_2 + \frac{1}{2}\right)\sigma^2} \\
= \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} \left( \sum_{n_2=0}^{\infty} \frac{1}{\left(n_2 + \frac{1}{2}\right)^2} e^{2\pi i\left(n_2 + \frac{1}{2}\right)\sigma^2} + \sum_{n_2=1}^{\infty} \frac{1}{\left(n_2 - \frac{1}{2}\right)^2} e^{-2\pi i\left(n_2 - \frac{1}{2}\right)\sigma^2} \right) \\
= \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} \tilde{F}(\sigma^2). \tag{D16}
\]
Then
\[
\frac{1}{(2\pi i)^2} \frac{d^2 \tilde{F}(\sigma^2)}{d(\sigma^2)^2} = \sum_{n_2=0}^{\infty} e^{2\pi i (n_2+\frac{1}{2})}\sigma^2 + \sum_{n_2=1}^{\infty} e^{-2\pi i (n_2-\frac{1}{2})}\sigma^2
\]
\[
= e^{2\pi i \frac{1}{2}\sigma^2} \left( \frac{1}{1 - e^{2\pi i \sigma^2 - \varepsilon_+}} + \frac{e^{-2\pi i \sigma^2 - \varepsilon_-}}{1 - e^{-2\pi i \sigma^2 - \varepsilon_-}} \right) = 0 \text{ modulo } \delta(\sigma^2). \quad (D17)
\]
\[
\therefore \quad \tilde{F}(\sigma^2) = (2\pi i)^2 \left( A\sigma^2 + B \right) \quad (D18)
\]
and
\[
B = \tilde{F}(0) = \sum_{n_2=0}^{\infty} \frac{1}{(n_2 + \frac{1}{2})^2} + \sum_{n_2=1}^{\infty} \frac{1}{(n_2 - \frac{1}{2})^2} \quad \text{Eq. (C26)} = \zeta \left( 2, \frac{1}{2} \right) + \left\{ \zeta \left( 2, -\frac{1}{2} \right) - \frac{1}{(0 - \frac{1}{2})^2} \right\}
\]
\[
\text{Eqs. (C25),(C27)} \quad \frac{\pi^2}{2} + \left\{ 4 + \frac{\pi^2}{2} - 4 \right\} = 2 \cdot \frac{\pi^2}{2} = \pi^2. \quad (D19)
\]
Using the formula
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a} \coth(a\pi), \quad (D20)
\]
\[
A = \tilde{F}'(0) = \sum_{n_2=0}^{\infty} \frac{2\pi i}{n_2 + \frac{1}{2}} + \sum_{n_2=1}^{\infty} \frac{-2\pi i}{n_2 - \frac{1}{2}}
\]
\[
= 2\pi i \left[ \frac{1}{0 + \frac{1}{2}} + \sum_{n_2=1}^{\infty} \left\{ \frac{1}{n_2 + \frac{1}{2}} - \frac{1}{n_2 - \frac{1}{2}} \right\} \right]
\]
\[
= 2\pi i \left[ 2 - \sum_{n_2=1}^{\infty} \frac{1}{n_2^2 + (\frac{1}{2})^2} \right]
\]
\[
eq 2\pi i \left[ 2 - \left\{ -\frac{1}{2 \left( \frac{1}{2} \right)^2} + \frac{\pi}{2\frac{1}{2}} \coth \left( \frac{i}{2} \pi \right) \right\} \right] = 0. \quad (D21)
\]
\[
\therefore \quad \tilde{F}(\sigma^2) = (2\pi i)^2 \pi^2. \quad (D22)
\]
Therefore
\[
\frac{1}{\tau_2} \sum_{n_2=-\infty}^{\infty} \frac{1}{4(2\pi)^2 n_2 + \frac{1}{2}} 2\pi i (n_2+\frac{1}{2})\sigma^2 = \frac{1}{\tau_2} \frac{(\tau - \bar{\tau})^2}{4(2\pi)^2} \tilde{F}(\sigma^2) = \frac{1}{\tau_2} \frac{(2i\tau_2)^2}{4(2\pi)^2} (2\pi i)^2 \pi^2 = \pi^2 \tau_2. \quad (D23)
\]
Finally,
\[
G_{+-}(z, \bar{z} | 0, 0) \quad \text{Eqs. (D14),(D23)} = \frac{1}{2\pi} \left[ \ln |1 - \zeta| + \sum_{m=1}^{\infty} (-1)^m \ln |1 - \zeta q^m| \frac{1 - q^m}{\zeta} \right] + \pi^2 \tau_2. \quad (D24)
\]
Appendix E. Supertorus Green function and supersphere Green function

Since

$$\frac{\partial}{\partial \frac{\tau}{2}} \left[ ^{\frac{1}{2}} \frac{(z|\tau)}{\tau} \right]^{z \sim 0} = \ln \left| 1 - e^{2\pi iz} \right|^{z \sim 0} \ln |z|$$

(E1)

$$G_{\pm}(z, z|0, 0) \sim \frac{1}{2\pi} \ln |z|$$

(E2)

and

$$S_{V} (z, z|0, 0) \sim \frac{i}{\pi z}, \quad S_{\bar{V}} (z, z|0, 0) \sim -\frac{i}{\pi \bar{z}},$$

(E3)

where \(v_{f} = (\ldots, (++, (+\ldots)) \). Using Eqs. (E2) and (E3),

$$G_{\pm}^{\text{supertorus}} (z_{I}, z_{I}|z_{J}, z_{J}) \equiv G_{\pm} (z_{I}, z_{I}|z_{J}, z_{J}) + \frac{\partial \theta_{I} \theta_{J}}{4} S_{V} (z_{I}, z_{I}|z_{J}, z_{J}) - \frac{\bar{\theta}_{I} \bar{\theta}_{J}}{4} S_{V} (z_{I}, z_{I}|z_{J}, z_{J})$$

$$\sim \frac{1}{2\pi} 1 \ln |z_{I} - z_{J}| + \frac{\theta_{I} \theta_{J}}{4} \left( \frac{i}{\pi z_{I} - z_{J}} \right) - \frac{\bar{\theta}_{I} \bar{\theta}_{J}}{4} \left( \frac{i}{\pi \bar{z}_{I} - \bar{z}_{J}} \right)$$

$$= \frac{1}{4\pi} \ln (z_{I} - z_{J} + i \theta_{I} \theta_{J}) + \frac{1}{4\pi} \ln (\bar{z}_{I} - \bar{z}_{J} + i \bar{\theta}_{I} \bar{\theta}_{J})$$

$$= \frac{1}{2\pi} 1 \ln |z_{I} - z_{J} + i \theta_{I} \theta_{J}| = G_{\pm}^{\text{supersphere}} (z_{I}, z_{I}|z_{J}, z_{J}).$$

(E4)

Appendix F. Path integral of a fermionic string at one-loop

In this appendix, the path integral of a fermionic string \([76–79]\) is briefly recalled. We present formulas in the flat ten-dimensional case but they can easily be adapted to other cases such as the \(T^{4}/Z_{2}\) orbifold given in the text. The basic variables are the bosonic coordinates (2d massless scalar fields) \(X^{\mu}\), the fermionic ones (2d massless two-component Majorana spinor fields) \(\psi_{\text{Maj}} \), the zwiebein \(e_{a}^{m}\) (or \(e_{m}^{a}\) such that \(g_{mn} = e_{m}^{a} e_{n}^{b} \delta_{ab}\)), and the Rarita–Schwinger field \(\chi_{a}^{m}\); we denote by \(\mu, \nu, \ldots, \alpha, \beta, \ldots, m, n, \ldots\) and \(a, b, \ldots\) ten-dimensional vector indices, two-dimensional spinor indices, \(\chi_{a}\) two-dimensional worldsheet indices, and two-dimensional local Lorentz indices respectively. The action \([80–83]\) is

$$S = \frac{1}{2\pi \alpha'} \int d^{2} \sigma \sqrt{g} \left\{ \frac{1}{2} g^{mn} \partial_{m} X^{\mu} \partial_{n} X_{\mu} - i \frac{1}{2} \psi_{\text{Maj}} \gamma^{a} \gamma^{b} \chi_{a} \left( \partial_{b} X_{\mu} - \frac{1}{4} \chi_{b} \psi_{\text{Maj}} \right) \right\}. \quad (F1)$$

where

$$\chi_{a} = e_{a}^{m} \chi_{m}, \quad \partial_{b} = e_{b}^{m} \partial_{m},$$

$$\nabla_{a} = e_{a}^{m} \left( \partial_{m} - \omega_{m} \frac{1}{2} \gamma^{5} \right),$$

$$\omega_{m} = e_{m}^{a} e^{pq} \partial_{p} e_{q}^{b} \delta_{ab}. \quad (F2)$$

The superstring scattering amplitudes are given in general by the functional integrals with the appropriately chosen vertex operators \(\prod_{I} O_{I}\) over these worldsheet fields with respect to this action.
modulo the local symmetries

\[ \left\langle \prod_I O_I \right\rangle = \sum_{\text{top. s.s.}} \sum \frac{\mathcal{D}e_m^a}{\Omega(D)\Omega(W)\Omega(L)} \frac{\mathcal{D}\chi_m^a}{\Omega(S)\Omega(SW)} \int \mathcal{D}X^\mu \int \mathcal{D}\psi_{\text{Maj}} \mu e^{-S} \prod_I O_I. \]  

(F3)

Here we have denoted by “top.” and “s.s.” the summation over the worldsheet topology and the summation over the spin structure, respectively. The functional integrals are (at least formally) defined through inner product and are modded out by the volumes of the two-dimensional diffeomorphisms, the local Weyl symmetry, the local Lorentz invariance, the two-dimensional local symmetry, and the local Weyl symmetry denoted respectively by \( \Omega(D) \), \( \Omega(W) \), \( \Omega(L) \), \( \Omega(S) \), and \( \Omega(SW) \).

The functional integral measure is first decomposed by noting general variation of vierbeins and gravitino variables exploiting the local transformations and the orthogonal decomposition:

\[ \delta e_m^a = \delta \sigma e_m^a + (P_1 \delta \eta)_m^a + \delta \ell e^{ab} e_m^b + \sum_i \delta c_i \psi_m^a. \]  

(F4)

Here,

\[ P_1 : (P_1 \delta \eta)_m^a = \{ \delta \eta^n \partial_n e_m^a + e_n^a \partial_m \delta \eta^n \} - \frac{1}{2} e_m^a e_n^b \{ \delta \eta^\ell \partial_\ell e_n^b + e_\ell^b \partial_n \delta \eta^\ell \} \]

\[ - \frac{1}{2} e^{ab} e_m^b \delta_{dc} e_c^a \{ \delta \eta^\ell \partial_\ell e_n^d + e_\ell^d \partial_n \delta \eta^\ell \} \]

\[ \psi^i \in \text{Ker } P_1^\dagger \]

\[ \delta \chi_m = \gamma_m \delta \rho + (P_{1/2} \delta \zeta)_m^a + \sum_i \delta c_i \psi_m^a \]

\[ (P_{1/2} \delta \zeta)_m^a = 2 \nabla_m \delta \xi - \gamma_m \gamma^n \nabla_n \delta \xi \]

\[ \psi_m^i \in \text{Ker } P_{1/2}^\dagger. \]  

(F5)

Hence,

\[ \left\langle \prod_I O_I \right\rangle = \int \frac{\mathcal{D}\sigma \mathcal{D}(P_1 \eta) \mathcal{D}\ell}{\Omega(D)\Omega(W)\Omega(L)} \prod_i dc_i \det \left\langle \psi_i^i \mid \psi_j^j \right\rangle^{1/2} \]

\[ \times \int \frac{\mathcal{D}\rho \mathcal{D}(P_{1/2} \zeta)}{\Omega(S)\Omega(SW)} \prod_i d\epsilon_i \det \left\langle \psi_i^i \mid \psi_j^j \right\rangle^{-1/2} \prod_I O_I. \]  

(F6)

This expression is further converted by referring to the global gauge slice and moduli and supermoduli parameters

\[ e_m^a = e^\Lambda (\text{Diff})_m^a (\tau_i) \]

\[ \chi_m = \gamma_m \lambda + \sum_i \alpha_i \Phi_m^i. \]  

(F7)

The vierbein variation reads

\[ \delta e_m^a = \delta \Lambda e_m^a + \delta \eta^a \partial_m e_n^a + e_n^a \partial_m \delta \eta^n + \delta L e^{ab} e_m^b + \sum_i \delta \tau_i \left( \partial e_m^a / \partial \tau_i \right) \]

\[ \begin{bmatrix} d\sigma \\ d\ell \\ P_1 \delta \eta \\ dc_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & T_{ij} \end{bmatrix} \begin{bmatrix} \delta \Lambda \\ \delta \ell \\ \delta \eta \\ \delta T_{ij} \end{bmatrix} \]

\[ \det T = \det \langle \psi_i^i \mid \psi_j^j \rangle^{-1} \det \langle \psi_i^i \mid \partial e_m^a / \partial \tau_i \rangle. \]
Hence,
\[
\frac{D\epsilon_m^a}{\Omega(D)\Omega(W)\Omega(L)} = \frac{D\Lambda DLD'\eta}{\Omega(D)\Omega(W)\Omega(L)} \prod_i d\tau_i \det \left( P_{1/2}^* P_1 \right)^{1/2} \det |\psi_i| |\psi_j|^{-1/2} \det \left( \psi_i | \frac{\partial \epsilon_m^a}{\partial \tau_i} \right) \\
= \prod_i d\tau_i \frac{1}{\Omega(CK)} \det \left( P_{1/2}^* P_1 \right)^{1/2} \det |\psi_i| |\psi_j|^{-1/2} \det \left( \psi_i | \frac{\partial \epsilon_m^a}{\partial \tau_i} \right),
\]
where \( \Omega(CK) \) is the volume of conformal Killing vectors and \( D'\eta \) indicates that ker \( P_1 \) has been excluded. The variation of the gravitinos reads
\[
\delta \chi_m = \gamma_m \delta \lambda + 2 \nabla_m \delta \xi + \sum_i dai \Phi_i.
\]
Hence,
\[
\frac{\mathcal{D}\chi_m}{\Omega(S)\Omega(SW)} = \frac{\mathcal{D}\lambda D'\xi}{\Omega(S)\Omega(SW)} \det \left( P_{1/2}^* P_1 \right)^{-1/2} \prod_i dai \det S^{-1} \det |\psi|^{-1/2} \\
= \prod_i dai \frac{1}{\Omega(CKS)} \det \left( P_{1/2}^* P_1 \right)^{-1/2} \det |\psi_i| |\psi_k|^{1/2} \det |\psi_k| |\Phi_j|^{-1},
\]
where \( \Omega(CKS) \) is the volume of the conformal Killing spinor and \( D'\xi \) indicates that ker \( P_{1/2} \) has been excluded. The final formula is
\[
\left( \prod_i O_i \right) = \sum_{\text{top. s.s.}} \sum_i \int \prod_i d\tau_i \frac{1}{\Omega(CKV)} \det \left( P_{1/2}^* P_1 \right)^{1/2} \det |\psi_i| |\psi_j|^{-1/2} \det \left( \psi_i | \frac{\partial \epsilon_m^a}{\partial \tau_i} \right) \\
\times \int \prod_i dai \frac{1}{\Omega(CKS)} \det \left( P_{1/2}^* P_1 \right)^{-1/2} \det |\psi_i| |\psi_k|^{1/2} \det |\psi_k| |\Phi_j|^{-1} \\
\times \int \mathcal{D}X^\mu \mathcal{D}\psi_M^\mu e^{-S} \prod_i O_i.
\tag{F8}
\]

Appendix G. Superannulus

In this appendix, we apply the method of images in superspace to the superannulus [29–31].

Let the conjugate point of \((z, \theta)\) be \((\tilde{z}, \tilde{\theta})\). The involution acting on \( f(z, \theta) \) associated with \((z, \theta) \rightarrow (\tilde{z}, \tilde{\theta})\) is denoted by
\[
\hat{\iota} f(z, \theta) = f(\tilde{z}, \tilde{\theta}) = \text{fn}(\tilde{z}, \text{ and } \tilde{\theta} \text{ only}).
\tag{G1}
\]
Let the supersymmetry transformation of \( f(z, \theta) \) be
\[
\delta f = (\epsilon Q - \bar{\epsilon} \bar{Q}) f(z, \theta).
\tag{G2}
\]
We require
\[
\hat{\iota} \delta f = \delta \hat{\iota} f(z, \theta) = \delta f(\tilde{z}, \tilde{\theta}) = -\bar{\epsilon} \bar{Q} f(\tilde{z}, \tilde{\theta}) \\
= \hat{\iota} \epsilon Q f = \hat{\iota} \epsilon \left( i\theta \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta} \right) f(z, \theta) = \bar{\epsilon} \bar{Q} f(\tilde{z}, \tilde{\theta}).
\tag{G3}
\]
So we conclude

\[ \hat{\epsilon} \hat{Q} = -\hat{\epsilon} \hat{Q} \]  \hspace{1cm} (G4)

\[ \delta \tilde{\theta} \left( i \tilde{\theta} \frac{\partial}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{\theta}} \right) = -\delta \tilde{\theta} \left( -i \tilde{\theta} \frac{\partial}{\partial \tilde{z}} - \frac{\partial}{\partial \tilde{\theta}} \right). \]  \hspace{1cm} (G5)

Therefore,

if \( \tilde{z} = \bar{z} \), then \( \tilde{\theta} = \pm \tilde{\theta} \) (UHP),

\( \bar{z} = -\tilde{z} \), then \( \tilde{\theta} = \pm i \tilde{\theta} \) (annulus),

and \( \tilde{z} = \frac{1}{\bar{z}} \), then \( \tilde{\theta} = \pm \frac{i \tilde{\theta}}{\bar{z}} \) (disk). \hspace{1cm} (G6)

**Appendix H. Supplement to \( N^{IJ}_{+\pm}, B^{IJ}_{\nu I}, C^{IJ}_{+\pm}, E^{IJ}_{+\pm} \)**

**H.1. Properties under \( I \leftrightarrow J \)**

Here we check the properties under \( I \leftrightarrow J \).

Due to the even/odd properties for theta functions,

\[ \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} = -\frac{\vartheta^\prime \left[ \frac{1}{2} \right] (-z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} \]  \hspace{1cm} (H1)

\[ S_{\nu I}(-z) = i \frac{\vartheta_{\nu I}(-z)}{\vartheta_{\nu I}(0)} \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (0)}{\vartheta^\prime \left[ \frac{1}{2} \right] (-z)} = i \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} \left( \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (0)}{\vartheta^\prime \left[ \frac{1}{2} \right] (z)} \right) = -S_{\nu I}(z). \]  \hspace{1cm} (H2)

In addition, by the fact that the derivative of an even/odd function becomes odd/even,

\[ \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (-z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} = \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} = -\frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} \]  \hspace{1cm} (H3)

\[ \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (-z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} = \frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} = -\frac{\vartheta^\prime \left[ \frac{1}{2} \right] (z)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} \]  \hspace{1cm} (H4)

Using these,

\[ \pi N^{IJ}_{++} \overset{\text{Eqs. (H1), (H2)}}{=} \left| \frac{\vartheta^\prime \left[ \frac{1}{2} \right] \left( \frac{z_I}{2} - \frac{z_J}{2} \right)}{\vartheta^\prime \left[ \frac{1}{2} \right] (0)} \right| \left( \frac{i \tau_2}{2} \right) + \frac{\pi (z_I - z_J)^2}{\tau_2} = \pi N^{IJ}_{++} \]

\[ B^{IJ}_{\nu I} \overset{(H2)}{=} \frac{1}{2} \frac{\pi}{i} (-1) S_{\nu I} \left( \frac{z_I}{2} - \frac{z_J}{2} \right) = -B^{IJ}_{\nu I} \]
Let us look at a singularity of \( z_I \).\n
\[ C_{++}^H = \frac{1}{2} (-1) \frac{\partial'}{\theta} \left[ \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \right] + (-1)\pi \frac{z_I - z_J}{\tau_2} = -C_{++}^H \]

\[ E_{++}^H = \frac{1}{4} \left\{ \frac{\partial''}{\theta} \left[ \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \right] - \left( \frac{\partial'}{\theta} \left[ \left( \frac{z_I}{2} - \frac{z_J}{2} \right) \right] \right) \right\} + \frac{\pi}{\tau_2} = E_{++}^H. \]

(H5)

From Eq. (4.10),

\[ \pi N_{++}^H = +\pi N_{+-}^H \]
\[ B_{++}^H = -B_{+-}^H \]
\[ C_{+-}^H = -C_{++}^H \]
\[ E_{+-}^H = +E_{++}^H. \]

(H6)

### H.2. Singularity at \( z_I \sim z_J \)

Let us look at a singularity of \( \pi N_{++}^H, B_{+-}^H, C_{++}^H, \) and \( E_{++}^H \) at \( z_I \sim z_J. \)

By Eq. (E1),

\[ \pi N_{++}^H \overset{z_I \sim z_J}{\sim} \ln \left| \frac{z_I}{2} - \frac{z_J}{2} \right| \]
\[ B_{+-}^H \overset{z_I \sim z_J}{\sim} \frac{1}{2} \frac{1}{\frac{z_I}{2} - \frac{z_J}{2}} \]
\[ C_{++}^H \overset{z_I \sim z_J}{\sim} \frac{1}{2} \frac{1}{\frac{z_I}{2} - \frac{z_J}{2}} \]
\[ E_{++}^H \overset{z_I \sim z_J}{\sim} -\frac{1}{4} \frac{1}{\left( \frac{z_I}{2} - \frac{z_J}{2} \right)^2}. \]

(H7)

where we have also used

\[ \lim_{z \to 0} \frac{\partial''}{\theta} \left[ \left( \frac{z}{\tau} \right) \right] \]
L'Hopital's rule
\[ \lim_{z \to 0} \frac{\partial''}{\theta} \left[ \left( \frac{z}{\tau} \right) \right] \]
eq \text{(C10)}
\[ \frac{\partial}{\partial \tau} \left( 4\pi i \frac{\partial}{\partial \tau} \theta \left[ \left( \frac{1}{2} \right) \right] \left( \frac{z}{\tau} \right) \right) \]
\[ = 4\pi i \lim_{z \to 0} \theta \left[ \left( \frac{z}{\tau} \right) \right] \]
\[ = 4\pi i \lim_{z \to 0} \theta \left[ \left( \frac{z}{\tau} \right) \right] \]
\[ = 4\pi i \frac{\partial}{\partial \tau} \left[ -2\pi (\eta(\tau))^3 \right] = 4\pi i \frac{\partial}{\partial \tau} \left[ \eta(\tau)^3 \right] \]
\[ = 4\pi i \frac{\partial}{\partial \tau} \ln \eta(\tau)^3 = 3 \cdot 4\pi i \frac{\partial}{\partial \tau} \ln \eta(\tau) \]

(H8)

\[ \text{to evaluate } E_{++}^H. \]
H.3. Eq. (4.8) at \( z_I \sim z_J \) in the case of maximal supersymmetry

Let us check that Eq. (4.8) reduces to that of [31] at \( z_I \sim z_J \) in the case of maximal supersymmetry. According to Eq. (H7),

\[
\exp \left[ 2\alpha' \sum_{1 \leq I < J \leq N} k_I \cdot k_J \pi N^{I J}_{++} \right]
\]

\[
\exp \left[ 2\alpha' \sum_{1 \leq I < J \leq N} \left\{ i k_I \cdot k_J \theta_I \theta_J B^{IJ}_{\nu \bar{\nu}} + (k_I \cdot \eta_I \theta_I - k_J \cdot \eta_I \theta_J) B^{IJ}_{\nu} \right. \right.
\]

\[
\left. + \left( k_J \cdot \eta_I \theta_I - k_I \cdot \eta_I \theta_J \right) C^{IJ}_{++} - i \eta_I \cdot \eta_J B^{IJ}_{\nu} + \eta_I \cdot \eta_J \theta_I \theta_J E^{IJ}_{++} \right) \right] \left[ \frac{z_I - z_J}{2} \right]
\]

\[
= \prod_{1 \leq I < J \leq N} \left| \frac{z_I - z_J}{2} \right|^{2\alpha' k_I \cdot k_J}
\]

\[
\exp \left[ \sum_{1 \leq I < J \leq N} \left\{ 2\alpha' i k_I \cdot k_J \theta_I \theta_J \frac{1}{2} \frac{1}{\frac{z_I}{2} - \frac{z_J}{2}} + 2\alpha' (k_I \cdot \eta_J + k_J \cdot \eta_I) \frac{1}{2} \frac{\theta_I - \theta_J}{\frac{z_I}{2} - \frac{z_J}{2}} \right. \right.
\]

\[
\left. - 2\alpha' \eta_I \cdot \eta_J \frac{1}{2} \frac{1}{\frac{z_I}{2} - \frac{z_J}{2}} - 2\alpha' \eta_I \cdot \eta_J \theta_I \theta_J \frac{1}{4} \frac{1}{\left( \frac{z_I}{2} - \frac{z_J}{2} \right)^2} \right] \right]. \quad (H9)
\]

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