EQUICONTINUOUS ACTIONS OF SEMISIMPLE GROUPS

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ABSTRACT. We study equicontinuous actions of semisimple groups and some generalizations. We prove that any such action is universally closed, and in particular proper. We give various applications including closedness of continuous homomorphisms, metric ergodicity of transitive actions and vanishing of matrix coefficients for reflexive (more generally: WAP) representations.

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1. Introduction

This work emerged from an attempt to establish a version of the classical Howe–Moore theorem [HM79] for representations of a semisimple Lie group $G$ on reflexive Banach spaces (see Corollary 10.5). We then realized that the same properties of $G$ that are responsible for the classical Howe–Moore theorem lead towards a much more general phenomena. We prove that every continuous equicontinuous action of $G$ on a space $X$ equipped with a uniform
structure $S$ and a compatible topology $T$ is universally closed, and in particular proper (see Section 2.3) — the most general result of this paper is Theorem 4.1. This applies for instance to isometric actions, or more generally to uniformly bounded Lipschitz actions on metric spaces equipped with the metric as well as some related topologies (e.g. a Banach space with the weak or weak* topology).

The first half of the paper is devoted to the formulation and proof of the main Theorem 4.1, including a presentation of the basic notions and necessary background. The second half (Section 5 and further) is dedicated to various applications of Theorem 4.1, including closedness of continuous homomorphisms, metric ergodicity of transitive actions and vanishing of matrix coefficients for reflexive and, more generally, weakly almost periodic representations.

2. Preliminaries

2.1. On nets convergence. Recall that a net in a topological space is a map to the space from a directed set, where a directed set is a pre-ordered set in which every two elements have an upper bound. Typically we denote a directed set by the symbol $(\alpha)$ where $\alpha$ denotes a generic element in the directed set, and for a net in the topological space $X$ we use symbols as $(x_\alpha)$, representing the map $\alpha \mapsto x_\alpha$.

The net $x_\alpha$ converges to $x$, to be denoted $x_\alpha \to x$, if for every neighbourhood $U$ of $x$ there exists $\alpha_0 \in (\alpha)$ such that for every $\alpha \geq \alpha_0$, $x_\alpha \in U$.

A net $(x_\beta)$ is said to be a subnet of the net $(x_\alpha)$ if it is obtained as the composition of an order preserving cofinal map $(\beta) \to (\alpha)$ with the map $\alpha \mapsto x_\alpha$. It is well known and easy to check that a net converges if and only if all of its subnets converge and to the same point. Less well known is the fact that every net which majorizes a subnet of a converging net converges as well.

**Lemma 2.1.** Let $n : (\alpha) \to X$ be a net converging to $x$. Let $f : (\beta) \to (\alpha)$ be an ordered preserving cofinal map. Let $f'(\beta) \geq f(\beta)$. Then the net $n \circ f'$ converges to $x$.

**Proof.** Fixing a neighborhood $U$ of $x$ we need to show that there exists $\beta_0 \in (\beta)$ such that for every $\beta \geq \beta_0$, $x_{f'(\beta)} \in U$. Indeed, by the convergence of the net $x_\alpha$ there exists $\alpha_0 \in (\alpha)$ such that for every $\alpha \geq \alpha_0$, $x_\alpha \in U$, and by the cofinality of $f$, there exists $\beta_0 \in (\beta)$ such that $f(\beta_0) \geq \alpha_0$. Then for every $\beta \geq \beta_0$, $f'(\beta) \geq f(\beta) \geq \alpha_0$ implies $x_{f'(\beta)} \in U$. □

In a locally compact space $X$, a net is said to converge to infinity if for every compact subset $K$ there exists $\alpha_0$ such that for every $\alpha \geq \alpha_0$, $x_\alpha \notin K$. The following technical lemma will be of use.

**Lemma 2.2.** Let $G$ be a locally compact group acting on a topological space $X$. Let $g_\alpha$ be a net in $G$ converging to infinity and assume that for some $x, y \in X$, the net $(g_\alpha x)$ converges to $y$ in $X$. Then there exists a directed set $(\beta)$ and two nets $n, n' : (\beta) \to G$ satisfying $n(\beta)x \to y$ and $n'(\beta)x \to y$ in $X$ and $n(\beta)^{-1}n'(\beta) \to \infty$ in $G$.
Proof. We let $\mathcal{C}$ be the directed set of compact subsets of $G$, ordered by inclusion, and set $(\beta) = (\alpha) \times \mathcal{C}$ endowed with the product order. We let $f : (\beta) \to (\alpha)$ be the projection on the first variable. This is obviously an order preserving cofinal map. For every $(\alpha_0, K) \in (\beta)$ we use the fact that the subnet $(g_{\alpha})_{\alpha \geq \alpha_0}$ converges to infinity in $G$ to find an element $\alpha_1 \geq \alpha_0$ satisfying $g_{\alpha_1} \notin g_{\alpha_0} K$. We denote $\alpha_1 = f'(\alpha_0, K)$. The lemma now follows from Lemma 2.1, setting $n(\beta) = g_f(\beta)$ and $n'(\beta) = g'_{f'}(\beta)$. □

2.2. Uniform structures and compatible topologies. Recall that a uniform structure on a set $X$ is a symmetric filter $S$ of relations on $X$ containing the diagonal $D = \{ (x, x) : x \in X \}$ such that for every $U \in S$ there is $U' \in S$ with $U U' \subset U$. Here

$$U_1 U_2 = \{ (u_1, u_2) : \exists u_3, (u_1, u_3) \in U_1, (u_3, u_2) \in U \}.$$ 

Let $(X, S)$ be a uniform space.

Definition 2.3. We will say that a topology $T$ on $X$ is $S$-compatible if for every $V \in T$ and a point $y \in V$, there exists $y' \in V' \in T$ and $U \in S$ such that $UV' \subset V$, where

$$UV' = \{ v : \exists v' \in V', (v, v') \in U \}.$$ 

We shall denote by $T_S$ the $S$-topology on $X$, i.e. the topology generated by the sets $U(x) := U\{x\}, x \in X, U \in S$.

Obviously, we have:

Example 2.4. The $S$-topology $T_S$ is $S$-compatible.

A topological group action on a topological space $G \curvearrowright (X, T)$ is said to be jointly continuous or simply continuous if the action map $G \times X \to X$ is continuous as a function of two variables.

Example 2.5. Given an action of a topological group $G$ on a set $X$ we define the action uniform structure $S_G$ on $X$ to be the uniform structure generated by the images of the sets $U \times X$ under the map

$$G \times X \to X \times X, \ (g, x) \mapsto (x, gx),$$

where $U$ runs over the identity neighbourhoods in $G$. A topology $T$ on $X$ is $S_G$-compatible if and only if the action of $G$ on $(X, T)$ is continuous.

A group action on a uniform space $G \curvearrowright X$ is said to be equicontinuous (or sometimes uniformly continuous) if for every $U \in S$, also the set $\{(u, v) : \forall g \in G, (gu, gv) \in U\}$ is in $S$. This means that $S$ has a basis consisting of $G$ invariant uniformities.

Example 2.6. For a topological group $G$, setting $X = G$, the left regular action defines a uniform structure on $G$, as in Example 2.5. This structure is called the right uniform structure. Note that the right regular action is equicontinuous with respect to that structure.
Lemma 2.7. Assume $G$ acts on $(X,S)$ uniformly. Denote by $X/G$ the space of orbits and denote by $\pi : X \to X/G$ the natural quotient map. Then the collection $\{((\pi \times \pi)(U) \mid U \in X\}$ defines a uniform structure on $X/G$, to be denoted $\pi_*S$, and the associated topology on $X/G$, $T_{\pi_*S}$ coincides with the quotient topology $\pi_*T_S$.

Proof. Left to the reader. $\square$

Lemma 2.8. An equicontinuous action of a topological group is (jointly) continuous with respect to the $S$-topology if (and only if) the orbit maps are continuous.

Proof. For any $y \in X$ and a neighborhood of the form $U(y)$ associated with a uniformity $U \in S$, there exists a $G$-invariant uniformity $U'$ such that $U'U' \subseteq U$. For any $(g,x)$ with $gx = y$, let $\Omega \subseteq G$ be the pre-image of $U'(y)$ under the $x$-orbit map. Then $\Omega \times U'(x)$ is a neighborhood of $(g,x)$ in $G \times X$ whose image under the action map is contained in $U(y)$. Indeed, for $(g',x') \in \Omega \times U'(x)$, $(x',x) \in U'$ implies that $(g'x',g'x) \in U'$ which together with $(g'x, y) \in U$ gives $(g'x, y) \in U$.

Lemma 2.9. Let $G \acts (X,S)$ be an equicontinuous action. Let $T$ be an $S$-compatible topology on $X$. Let $(\alpha)$ be a directed set. Assume that $x_\alpha$ is a $T_S$-converging net in $X$ with $T_S$-$\lim x_\alpha = x$, and that $g_\alpha$ is a net in $G$. Then $T$-$\lim g_\alpha x_\alpha$ exists if and only if $T$-$\lim g_\alpha x$ exists, in which case they are equal.

Proof. Let $x'_\alpha$ be an arbitrary net in $X$ which $T_S$-converges to $x$. Suppose that $T$-$\lim g_\alpha x_\alpha$ exists and denote it by $y$. Let $V \in T$ be a neighborhood of $y$. We will show that there exists $\alpha_0$ such that $\alpha \geq \alpha_0$ implies $g_\alpha x'_\alpha \in V$. Fix $V' \in T$ around $y$ and a $G$-invariant uniformity $U \in S$ so that $UV' \subseteq V$. Let $U' \in S$ be a symmetric uniformity with $UV' \subseteq U$. By the assumptions there exists $\alpha_0$ such that for every $\alpha \geq \alpha_0$,

$$g_\alpha x_\alpha \in V', \quad (x_\alpha,x) \in U' \quad \text{and} \quad (x'_\alpha,x) \in U'.$$

Thus $(x_\alpha,x'_\alpha) \in U$ and, by the $G$-invariance of $U$, also $(g_\alpha x'_\alpha,g_\alpha x_\alpha) \in U$. It follows that $g_\alpha x'_\alpha \in UV' \subseteq V$.

By switching the roles of $x_\alpha$ and $x'_\alpha$ we deduce that $\lim_T g_\alpha x_\alpha$ exists if and only if $\lim_T g_\alpha x'_\alpha$ exists, in which case they are equal. The lemma follows by specializing to the constant net $x'_\alpha \equiv x$. $\square$

Lemma 2.10 (Mautner). Let $G$ be a topological group. Let $X$ be a $G$-space equipped with a uniform structure $S$ and an $S$-compatible topology $T$. Assume that the action is continuous with respect to both topologies $T$ and $T_S$ and equicontinuous with respect to $S$. Let $g_\alpha$ be a net in $G$ and assume for some points $x,y \in X$, $y = T$-$\lim g_\alpha x$. Assume $g \in G$ satisfies $\lim g^{\alpha^n} = 1$. Then $gy = y$.

Proof. By continuity of the action $G \acts (X,T_S)$ we have $(S$-$\lim) g^{\alpha^{-1}} x = x$. Applying Lemma 2.9 to the net $g_\alpha$ in $G$ and the net $g^{\alpha^{-1}} x$ in $X$, we deduce that indeed

$$gy = g(T$-$\lim) g_\alpha x = (T$-$\lim) gg_\alpha x = (T$-$\lim) g_\alpha \cdot g^{\alpha^{-1}} x = (T$-$\lim) g_\alpha x = y.$$
Lemma 2.11. Let \( G \curvearrowright (X, S) \) be an equicontinuous action. Assume that for some net \((g_\alpha) \) in \( G \) and \( x, y \in X \), \( g_\alpha x \to y \). Then \( g_\alpha^{-1} y \to x \).

Proof. For every neighborhood \( V \) of \( x \) there exists a \( G \)-invariant uniformity \( U \) with \( U(x) \subset V \). By \( g_\alpha x \to y \) there exists \( \alpha_0 \) such that for every \( \alpha \geq \alpha_0 \), \( g_\alpha x \in U(y) \), that is \((g_\alpha x, y) \in U \). By \( G \)-invariance we get \((x, g^{-1}_\alpha y) \in U \) and by symmetricity \((g^{-1}_\alpha y, x) \in U \). Therefore for every \( \alpha \geq \alpha_0 \), \( g^{-1}_\alpha y \in U(x) \subset V \). \(\square\)

2.3. Universally closed maps and actions. Recall that a map \( \pi : X \to Y \) between topological spaces is called proper if the preimage of a compact set is compact, and closed if the image \( \pi(A) \) of every closed set \( A \subset X \) is closed in \( Y \). Under mild assumptions on \( Y \), it is automatic that a proper map is closed. This is the case if \( Y \) is a k-space, e.g. when \( Y \) is either locally compact or satisfies the first axiom of countability, see [Pa70]. In general however, a proper map is not necessarily closed. The current section deals with the general case. Recall the following classical Theorem:

Theorem 2.12. A topological space \( K \) is compact if and only if for every topological space \( Z \), the projection maps \( K \times Z \to Z \) is closed.

Note that we do not assume any separation property from the topological spaces involved. Since we are not aware of a reference for 2.12 in this generality, we add a proof for the convenience of the reader.

Proof. The fact that if \( K \) is compact then for every \( Z \), \( K \times Z \to Z \) is closed is standard and easy. Assume now \( K \) is not compact and pick a directed set \((\alpha)\) and a net \((x_\alpha)\) in \( K \) which has no converging subnet. For every \( x \in K \) we can find a neighborhood \( U_x \) and \( \alpha_x \) such that for every \( \alpha \geq \alpha_x \), \( x_\alpha \notin U_x \). Consider the poset obtained by adding to \((\alpha)\) a maximal element, \( \infty \). Observe that the collection consisting of all intervals in \((\alpha)\) of the form \([\alpha, \infty]\) forms a base for a topology. Let \( Z \) be the topological space thus obtained. Check that \( \infty \in Z \) is not isolated. Let \( A \subset X \times Z \) be the complement of the open set \( \cup_x(U_x \times [\alpha_x, \infty]) \). Observe that \( A \cap X \times \{\infty\} = \emptyset \) and for each \( \alpha, (x_\alpha, \alpha) \in A \), the projection of \( A \) to \( Z \) consists of the subset \( Z - \{\infty\} \), which is not closed. \(\square\)

Here is another basic result of point-set topology for which we couldn’t find a proper reference.

Theorem 2.13. Let \( \pi : X \to Y \) be a continuous map between topological spaces. The following are equivalent.

1. For every topological space \( Z \), the map \( \pi \times id_Z : X \times Z \to Y \times Z \) is a closed map.
2. \( \pi \) is closed and proper.
3. For every net \((x_\alpha)\) in \( X \) which has no converging subnet, the net \((\pi(x_\alpha))\) has no converging subnet in \( Y \).
Proof. (1) ⇒ (2): By taking \( Z \) to be a point we see that \( \pi \) is closed. In order to see that \( \pi \) is proper, consider an arbitrary compact subset \( K \subset Y \) and an arbitrary topological space \( Z \). The projection map \( \pi^{-1}(K) \times Z \rightarrow Z \) is closed, being the composition of the closed maps \( \pi^{-1}(K) \times Z \rightarrow K \times Z \rightarrow Z \). Thus by Theorem 2.12 \( \pi^{-1}(K) \) is compact.

(2) ⇒ (3): Assume by contradiction that \((x_\alpha)\) is a net in \( X \) which has no converging subnet and \( \pi(x_\alpha) \rightarrow y \in Y \). Denote \( X_y = \pi^{-1}(\{y\}) \). Since \( \pi \) is closed and proper, \( X_y \) is non-empty and compact. For every \( x \in X_y \) we can find an open neighborhood \( U_x \) of \( x \) and \( \alpha_x \) such that \( \alpha \geq \alpha_x \Rightarrow x_\alpha \notin U_x \). By compactness of \( X_y \) we can find a finite set \( F \subset X_y \) such that \( X_y \subset \cup_{x \in F} U_x \). We let

\[
V = Y \setminus \pi(X \setminus \cup_{x \in F} U_x).
\]

Since \( \pi \) is closed \( V \) is an open neighborhood of \( y \) in \( Y \). Note that \( U = \pi^{-1}(V) \subset \cup_{x \in F} U_x \). Let \( \alpha_0 \) be an index satisfying \( \alpha_0 \geq \alpha_x \) for every \( x \in F \). Then for every \( \alpha \geq \alpha_0 \), \( x_\alpha \notin U \) and thus \( \pi(x_\alpha) \notin V \), contradicting the assumption that \( \pi(x_\alpha) \rightarrow y \).

(3) ⇒ (1): Let \( A \subset X \times Z \) be a closed set. Assume, by way of contradiction, that \((\pi \times \text{id}_Z)(A)\) is not closed in \( Y \times Z \) and pick a net \((y_\alpha, z_\alpha) \in (\pi \times \text{id}_Z)(A)\) converging to a point \((y, z) \notin (\pi \times \text{id}_Z)(A)\). Pick lifts \((x_\alpha)\) of \((y_\alpha)\) such that \((x_\alpha, z_\alpha) \in A\). By our assumption, since \((y_\alpha)\) converges, \((x_\alpha)\) has a converging subnet. Abusing the notation we assume that \((x_\alpha) \rightarrow x\). It follows that \((x_\alpha, z_\alpha) \rightarrow (x, z)\). Since \( A \) is closed, \((x, z) \in A\) and thus \((y, z) = (\pi \times \text{id}_Z)(x, z) \in (\pi \times \text{id}_Z)(A)\) a contradiction. \(\square\)

Definition 2.14. A map satisfying the above properties is called "universally closed".

Recall that a continuous action of \( G \) on \( X \) is called a proper action if the map

\[(1)\quad G \times X \rightarrow X \times X, \ (g, x) \mapsto (x, gx)\]

is a proper map. Similarly, we say that the action is universally closed is the map (1) is universally closed. Every universally closed action is proper.

Proposition 2.15. If \( G \) acts on \( X \) and the action is universally closed then the point stabilizers are all compact and the quotient topology on the orbit space \( X/G \) is Hausdorff. In particular, every orbit is closed.

Proof. The fact that stabilizers are compact follows from the properness of the action. To show that \( X/G \) is Hausdorff, observe that the set \( X \times X \setminus \text{Im}(G \times X) \) is open in \( X \times X \) and hence its image under the open map to \( X/G \times X/G \) is open. Thus its complement, the diagonal of \( X/G \times X/G \), is closed. \(\square\)

The following is a useful variant.

Proposition 2.16. Suppose a topological group \( G \) acts on \( X \) and \( T, T' \) are two topologies on \( X \) such that that map

\[
G \times (X, T) \rightarrow (X, T) \times (X, T'), \quad (g, x) \mapsto (x, gx)
\]

is universally closed. Assume that points in \( X \) are \( T \)-closed. Then the stabilizers are compact and the \( G \)-orbits in \( X \) are \( T' \)-closed.
Proof. Again, compactness of the stabilizers follows from properness. Given a point \( x_0 \in X \), the image of \( G \times \{ x_0 \} \), that is \( \{ x_0 \} \times Gx_0 \), is a closed subset of \( (X, T) \times (X, T') \) and its preimage in \( X \) under the continuous map \( (X, T') \rightarrow (X, T) \times (X, T') \), \( x \mapsto (x_0, x) \) is the orbit \( Gx_0 \).

\[ \square \]

3. Quasi-semi-simple groups

The main objects of this paper are semisimple Lie groups over local fields. However, much of the things we prove are based on two specific properties, namely:

- the existence of a Cartan KAK decomposition for \( G \), and that
- for every \( a \in A \), the group \( G \) is generated by elements \( g \) with the following property:

\[
\lim_{n \to \infty} a^n g a^{-n} = 1, \quad \lim_{n \to -\infty} a^n g a^{-n} = 1 \quad \text{or} \quad \sup_{n \in \mathbb{Z}} \| a^n g a^{-n} \| < \infty. \quad (1)
\]

This observation encourages us to introduce an axiomatic approach. Indeed, formulating (variants of) the above as axioms will, on one hand, make our future arguments cleaner and more transparent, while on the other hand, our results will be more general, and apply for other classes of groups. Our axiomatic approach is influenced by [Ci].

Given a topological group \( G \) and a net \( g_{\alpha} \) in \( G \) we define the following three groups:

\[
U^+(g_{\alpha}) = \{ x \in G \mid g_{\alpha}^{-1} x g_{\alpha} \to e \}, \quad U^-(g_{\alpha}) = \{ x \in G \mid g_{\alpha} x g_{\alpha}^{-1} \to e \} \quad \text{and}
\]

\[
U^0(g_{\alpha}) = \{ x \in G \mid \text{every subnet of both nets} \ g_{\alpha}^{-1} x g_{\alpha} \text{and} \ g_{\alpha} x g_{\alpha}^{-1} \text{admit a converging subnet} \}.
\]

The following lemma is obvious and left as an exercise for the reader.

**Lemma 3.1.** Let \( G \) be a topological group and \( g_{\alpha} \) a net in \( G \). The \( U^+(g_{\alpha}) \), \( U^-(g_{\alpha}) \) and \( U^0(g_{\alpha}) \) defined above are indeed groups and the group \( U^0(g_{\alpha}) \) normalizes both groups \( U^+(g_{\alpha}) \) and \( U^-(g_{\alpha}) \).

**Definition 3.2.** A locally compact topological group \( G \) is said to be quasi-semi-simple (qss, for short) if there exists a closed subgroup \( A < G \) satisfying the following axioms:

- There exists a compact subset \( C \subset G \) such that \( G = CAC \).
- For every net \( a_{\alpha} \) in \( A \) with \( a_{\alpha} \to \infty \), there exists a subnet \( a_{\beta} \) such that \( U^+(g_{\beta}) \) is not precompact and the group generated by the three groups \( U^+(g_{\beta}), U^-(g_{\beta}) \) and \( U^0(g_{\beta}) \) is dense in \( G \).

**Remark 3.3.** The class QSS of quasi-semi-simple groups is closed under finite direct product. Every compact group is qss and in addition if \( H = G/O \) where \( O \triangleleft G \) is a compact normal subgroup, then \( H \) is qss iff \( G \) is qss.

The following theorem is well known:

**Theorem 3.4.** Let \( k \) be a local field and \( G \) a connected semi-simple groups defined over \( k \). Then \( G(k) \) is qss. In particular every connected semis-simple Lie group with finite center is qss.

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\[ ^1 \text{In the classical case one can deduce this property using a root space decomposition of the Lie algebra.} \]
Remark 3.5 (Adelic groups are qss). Let $K$ be a global field and $G$ a semisimple $K$ algebraic group. Let $\mathbb{A} = \mathbb{A}_K$ be the associated ring of adels. Then $\mathbb{G}(\mathbb{A})$ is qss. To see this recall that $\mathbb{G}(\mathbb{A})$ is the restricted topological product of $\mathbb{G}(K_v)$ relative to the open compact subgroups $\mathbb{G}(\mathcal{O}_v) < \mathbb{G}(K_v)$ where $v$ runs over the finite valuations, and $\mathcal{O}_v$ is the local ring of $K_v$. The reason that $\mathbb{G}(\mathbb{A})$ is qss is that the same subgroups $\mathbb{G}(\mathcal{O}_v)$ used in the construction of restricted topological product can be used in the associated $\text{CAC}$ (or rather $KAK$) decomposition of the corresponding factors $\mathbb{G}(K_v)$. It is easy to verify the details.

Another family of qss groups is given by the following (see [Ci] and [CC, Proposition 3.6]):

Theorem 3.6. Let $G$ be a group acting strongly transitive on an affine building. Then $G$ is qss. In particular every group of automorphism of a simplicial tree which action on the boundary of the tree is 2-transitive is qss.

We note that the first ones to implicitly use the qss axioms for a tree group are Burger and Mozes, in their proof of the Howe-Moore property for such groups in [BM00].

4. The main theorem

Most of the results that will appear in consecutive sections can be considered as consequences of the following general statement:

Theorem 4.1. Let $G$ be a quasi-semi-simple group. Let $X$ be a $G$-space equipped with a uniform structure $S$ and an $S$-compatible topology $T$. Assume that the action is continuous with respect to both topologies $T$ and $T_S$ and equicontinuous with respect to $S$. Suppose that no non-compact normal subgroup of $G$ admits a global fixed point in $X$. Then the map

$$\phi : G \times (X,T_S) \to (X,T_S) \times (X,T), \quad (g,x) \mapsto (x,gx)$$

is universally closed. In particular, it is proper.

Applying Proposition 2.16 we get the following.

Corollary 4.2. Under the conditions of Theorem 4.1 we have

1. the stabilisers in $G$ of every point in $X$ is compact, and
2. the $G$-orbits in $X$ are $T$-closed.

Moreover, in the special case where $T = T_S$ we also have:

Corollary 4.3. With respect to the quotient topology induced from $T_S$, the orbit space $X/G$ is Hausdorff and completely regular.

By Lemma 2.7, $X/G$ admits a uniform structure, hence it is Hausdorff and completely regular given that it is $T_0$, but it is $T_1$ by the above discussion. To see directly the Hausdorff property of $X/G$, consider two points $x, y$ which do not belong to a single orbit. Since $Gy$ is closed, we have an open neighbourhood $V$ of $x$ which is disjoint from $Gy$. Consider a $G$ invariant uniformity $U$ such that $UU(x) \subset V$ and pick a symmetric uniformity $U'$ contained in $U$. It is easy to verify that the open sets $GU'(x)$ and $GU'(y)$ are disjoint.
Proof of Theorem 4.1. By way of contradiction we assume that the map \( \phi \) is not universally closed and show eventually the existence of a point fixed by some non-compact normal subgroup of \( G \). The proof consists of four steps.

Throughout the proof we let \( A < G \) be the subgroup guaranteed by the qss assumption, and let \( C \) be a compact subset of \( G \) such that \( G = CAC \).

**Step 1:** There exist points \( x, y \in X \) and a net \( a_\alpha \in A \) satisfying \( a_\alpha \to \infty \) and \( (T\text{-lim})a_\alpha x = y \).

In view of Theorem 2.13, the assumption the \( \phi \) is not universally closed is equivalent to the existence of a directed set \( (\alpha) \) and a net \( (g_\alpha, x_\alpha) \) which has no converging subnet, such that the net \( (x_\alpha, g_\alpha x_\alpha) \) converges in the \( T_S \times T \)-topology.

Let \( g_\alpha = c_\alpha a_\alpha c_\alpha' \) be a corresponding \( CAC \) expression of the elements \( g_\alpha \). Upon passing to a subnet we may assume that both \( c_\alpha \) and \( c_\alpha' \) converge in \( C \). Note that necessarily \( a_\alpha \) has no converging subnet in \( A \), that is \( a_\alpha \to \infty \).

Denote \( c = \lim c_\alpha \) and \( c' = \lim c_\alpha' \), and set
\[
x = c'(S\text{-lim})x_\alpha \quad \text{and} \quad y = c^{-1}(T\text{-lim})g_\alpha x_\alpha.
\]
Since \( G \) acts continuously on \((X, T_S)\), we have
\[
(S\text{-lim})c_\alpha' x_\alpha = x.
\]
Since \( G \) acts continuously on \((X, T)\), we have
\[
(T\text{-lim})a_\alpha c_\alpha' x_\alpha = (T\text{-lim})c_\alpha^{-1} \cdot g_\alpha x_\alpha = \lim c_\alpha^{-1} \cdot (T\text{-lim})g_\alpha x_\alpha = c^{-1}(T\text{-lim})g_\alpha x_\alpha = y.
\]
Applying Lemma 2.9 to the net \( a_\alpha \) in \( G \) and the net \( c_\alpha x_\alpha \) which \( T_S \)-converges to \( x \) in \( X \), we deduce that \( y = (T\text{-lim})a_\alpha x \).

**Step 2 (reducing to the case \( T = T_S \)): The action of \( G \) on \((X, T_S)\) is not universally closed.**

By Step 1, and by the second property in Definition 3.2, up to replacing \( (\alpha) \) by a sub-directed set \( (\beta) \), we have in addition to
- \( a_\beta \to \infty \) and
- \( (T\text{-lim})a_\beta x = y \),
that
- \( U_+^{(a_\beta)} \) is not precompact.

For \( g \in U_+^{(a_\beta)} \) we have \( \lim g_\beta^{-1} = 1 \), hence by Lemma 2.10, \( gy = y \). Thus the stabilizer of \( y \) is non-compact. By Proposition 2.15 it follows that the action of \( G \) on \((X, T_S)\) is not universally closed.

**Step 3:** There exist a point \( x \in X \) and a net \( a_{\beta'} \in A \) satisfying \( a_{\beta'} \to \infty \) and \( (S\text{-lim})a_{\beta'} x = x \).
By Step 2 we know that the map
\[ G \times (X, T_S) \to (X, T_S) \times (X, T_S), \quad (g, x) \mapsto (x, gx) \]
is not universally closed. We thus may apply Step 1 in the special case \( T = T_S \) and obtain points \( x, y \in X \) and a net \( a_\alpha \in A \) satisfying \( a_\alpha \to \infty \) and \( (S\text{-lim})a_\alpha x = y \). By Lemma 2.2, there exists a directed set \((\beta')\) and two nets \( n, n' : (\beta') \to A \) satisfying \( n(\beta')x \to y \) and \( n'(\beta')x \to y \) in \( X \) (all limits in \( X \) here are with respect to \( T_S \)) and \( n(\beta')^{-1}n'(\beta') \to \infty \) in \( A \). By Lemma 2.11, \( n(\beta)^{-1}y \to x \). Applying Lemma 2.9 (in the special case \( T = T_S \)) with respect to the directed set \((\beta')\), the net \( n'(\beta')x \) in \( X \) and the net \( n(\beta')^{-1} \) in \( A \), we conclude that \( n(\beta')^{-1}n'(\beta')x \to x \). We are done by setting \( a_{\beta'} = n(\beta'^{-1}n'(\beta')) \).

**Step 4:** There exists a point in \( X \) which is fixed by a non-compact normal subgroup of \( G \).

We let \( x \) be a point as obtained in Step 3. We will show that its stabilizer \( G_x \) contains a normal non-compact subgroup of \( G \). By replacing the net obtained in Step 3 by a subnet, using the qss second axiom we get a net \((a_{\alpha'})\) in \( A \) satisfying the following properties:

\begin{itemize}
  \item \( a_{\alpha'} \to \infty \).
  \item \( (S\text{-lim})a_{\alpha'} = x \).
  \item \( U^{(a_{\alpha'})} \) is not precompact.
  \item The group generated by the three groups \( U^{(a_{\alpha'})}_+ \), \( U^{(a_{\alpha'})}_- \), and \( U^{(a_{\alpha'})}_0 \) is dense in \( G \).
\end{itemize}

In view of Lemma 2.10, \( U^{(a_{\alpha'})}_+ < G_x \). Moreover, by Lemma 2.11 we also have \( (S\text{-lim})a_{\alpha'} = x \),

which by Lemma 2.10 gives \( U^{(a_{\alpha'})}_- < G_x \). By Lemma 3.1, the closed group generated by the subgroups \( U^{(a_{\alpha'})}_+ \) and \( U^{(a_{\alpha'})}_- \) is normal in \( G \). It is non-compact as \( U^{(a_{\alpha'})}_+ \) is not precompact. We conclude that \( G_x \) contains a normal non-compact subgroup of \( G \), completing the argument by contradiction. □

5. **Image of a homomorphism**

**Theorem 5.1.** Let \( G \) be a qss group. Let \( \phi : G \to H \) be a continuous injective homomorphism. Then \( \phi(G) \) is closed in \( H \).

**Proof.** Set \( X = H \) and consider the right \( G \) action on \( X \). Endow \( H \) with the right uniform structure described in Example 2.6. This uniform structure is invariant for the right regular action of \( H \), and in particular under the \( G \) action, thus the assumptions of Theorem 4.1 holds. By Corollary 4.2 the \( G \)-orbits are closed. Since the image of \( \phi \) coincides with the orbit of the identity \( 1_H \), the theorem is proved. □

In the special case where \( G \) is a semisimple group, we know not only that \( G \) is qss, but also that every quotient group of \( G \) is qss. Given any continuous homomorphism \( \phi : G \to H \), we may replace \( G \) by \( G/\text{Ker}(\phi) \), thus reduce to the case \( \phi \) is injective.
Corollary 5.2. Let $G$ be a semisimple analytic group with a finite center (the $F$ point of a Zariski connected semisimple algebraic group $G$, defined over a local field $F$). Let $H$ be a Hausdorff topological group. Let $\phi : G \to H$ be a continuous homomorphism. Then $\phi(G)$ is closed in $H$.

Note that a similar theorem was proven by Omori [Om66] for a class of connected Lie groups, including all connected semisimple Lie groups with finite center, under the assumption that the target group $H$ satisfies the first axiom of countability.

6. MEASURABLE METRICS AND METRIC ERGODICITY

Theorem 6.1. Let $G$ be a semisimple analytic group with a finite center (the $F$ point of a Zariski connected semisimple algebraic group $G$, defined over a local field $F$). Let $H < G$ be a closed subgroup. Suppose that $G/H$ admits a $G$-invariant, separable, measurable metric. Then $H$ contains a factor of $G$ as a cocompact subgroup.

In case $d$ is continuous, this theorem is an immediate application of Corollary 4.2 (1). Indeed, the $d$-uniform structure on $G/H$ is $G$-invariant and continuous. Replacing $G$ by $G/N$ where $N$ is the action kernel, using the fact that $G/N$ is qss we see that $H$, being the stabiliser of a point, must be compact. The fact that the theorem applies also for measurable metrics is a consequence of the following:

Lemma 6.2. Let $G$ be a locally compact group and $H < G$ a closed subgroup. Denote by $T$ the standard topology on $G/H$. Let $d$ be a $G$-invariant, separable, measurable metric on $G/H$. Then $d$ is $T$-continuous. If further $G$ is $\sigma$-compact then $T_d = T$ where $T_d$ denotes the metric topology on $G/H$.

Proof. The fact that $T_d \subset T \Rightarrow T_d = T$ when $G$ is $\sigma$-compact is a standard application of the Baire category theorem. We will prove that $T_d \subset T$. Let $\pi : G \to G/H$ be the quotient map. By the definition of the topology $T$ on $G/H$, $\pi$ is $T$-open, so it is enough to show that $\pi$ is $T_d$-continuous. By $G$-invariance it is enough to show continuity at $e$. Denote by $B(\varepsilon)$ the $d$-ball of radius $\varepsilon$ centred at $\pi(e)$. We need to find for every $\varepsilon > 0$ an identity neighbourhood $U$ in $G$ whose image is in $B(\varepsilon)$. For a given $\varepsilon > 0$ fix a countable cover of $G/H$ by balls of radius $\varepsilon/2$. At least one of the preimages of the balls is not Haar null, hence also the set $A = \pi^{-1}(B(\varepsilon/2))$ is not null. One easily check that $A = A^{-1}$ and $\pi(AA) \subset B(\varepsilon)$. Moreover, It is well known that $AA^{-1}$ contains an identity neighbourhood $U$, as desired. □

Theorem 6.3. Let $G$ be a semisimple analytic group with a finite center (the $F$ point of a Zariski connected semisimple algebraic group $G$, defined over a local field $F$). Let $H < G$ be a closed subgroup. Assume there exists a metric $d$ on $G$ which is separable, measurable, left $G$-invariant and right $H$-invariant. Then $H$ is compact.
Proof. By Lemma 6.2, $d$ is continuous. By the $G \times H$-invariance, the formula $\bar{d}(xH, yH) = d(x, yH)$ defines a continuous metric $\bar{d}$ on $G/H$. By Theorem 6.1, $H$ contains cocompactly a factor $G_1$ of $G$. Thus we wish to show that $G_1$ must be compact. Note that as $d|_{G_1}$ is bi-invariant, it induces a metric on $Y = G_1 \times G_1/\sim$, where the relation $\sim$ is defined by

$$(y_1, y_2) \sim (y_1', y_2') \iff y_1 y_2 = y_1' y_2',$$

for $y_1, y_2, y_1', y_2' \in G_1$. Considering the $G_1 \times G_1$ action on $Y$ given by

$$(g_1, g_2) \cdot (y_1, y_2) = (g_1 y_1, g_2^{-1} y_2),$$

we see that no factor group has a fixed point while the diagonal group $\{(g, g) : g \in G_1\}$ fixes the point $(1, 1) \in Y$. In view of Corollary 4.2 (1), this implies that $G_1$ is compact. □

**Definition 6.4.** Let $G$ be a group. Let $X$ be a $G$-Lebesgue space, that is a standard Borel space endowed with a measure class, on which $G$ acts measurably, preserving the measure class. The action of $G$ on $X$ is said to be **metrically ergodic** if for every separable metric space $U$ on which $G$ acts isometrically, every $G$-equivariant measurable function from $X$ to $U$ is a.e a constant.

**Theorem 6.5.** Let $G$ be a semisimple analytic group with a finite center (the $F$ point of a Zariski connected semisimple algebraic group $G$, defined over a local field $F$). Let $H < G$ be a closed subgroup. Endow $G/H$ with the unique $G$-invariant Radon measure class. Then $G/H$ is $G$-metrically ergodic if and only if the image of $H/G_1$ is not precompact in $G/G_1$ for every proper factor group $G_1 \lhd G$.

An ergodic $G$-Lebesgue space $X$ is not metrically ergodic if and only if it is induced from an ergodic $H$-space, for some closed subgroup $H < G$ which contains cocompactly a factor group $G_1 \lhd G$ with $G/G_1$ non-compact.

Proof. Let $G_1 \lhd G$ be a proper normal subgroup and suppose that that $H' = \overline{H G_1}/G_1$ is compact in $G' = G/G_1$. Pick a positive function $f \in L^2(G')$ and average it over the right action by $H'$, using the Haar measure on $H'$. The function obtained is $H'$ invariant, but not $G'$ invariant (as $G'$ is non-compact), thus provides a non-constant $G'$-equivariant map $G'/H' \to L^2(G')$. Precomposing with the map $G/H \to G'/H'$ we disprove the metric ergodicity of $G/H$.

More generally, given a $G$-space $X$ of the form $X = \text{Ind}_{H}^{G}(X')$ where $X'$ is an $H$-space on which $H$ acts with co-compact kernel, one observes that $H$ must be unimodular and the procedure above produces a non-constant $G$-map from $X$ to $L_2(G/H)$.

Let now $X$ be an ergodic $G$-Lebesgue space which is not metrically ergodic, and let $\phi : X \to U$ be a $G$-equivariant map to a separable metric $G$-space. Let $G_1$ be the maximal factor of $G$ for which the image of $X$ is essentially contained in $U^{G_1}$ and let $G' = G/G_1$. By ergodicity of $X$ we assume as we may that $\phi(X)$ intersects nullly the fixed points set of all proper factors of $G'$ in $U^{G_1}$. Replacing $U$ with $U^{G_1}$ minus the union of these fixed points sets, we may assume that the action of $G$ on $U$ factors through $G'$ and that proper factors of $G'$ have no fixed points. By Corollary 4.3 $U/G'$ is Hausdorff. Hence by the ergodicity of $X$, $\phi(X)$ is essentially supported on a unique orbit, which we identify with $G'/H'$ for
some closed subgroup \( H' < G' \). By Corollary 4.2, \( H' \) is compact in \( G' \). Letting \( H \) be the preimage of \( H' \) in \( G \), we deduce that \( X \) is induced from \( H \).

In particular, it follows that if \( X = G/H \) is \( G \)-metrically ergodic then the image of \( H \) is not precompact in \( G/G_1 \) for every proper factor group \( G_1 < G \). \( \square \)

The fact that metric ergodicity is preserved by a restriction to a lattice is general. We record it here for reference.

**Corollary 6.6.** Let \( G \) be a semisimple analytic group with a finite center, and \( \Gamma \) a lattice in \( G \). Then every metrically ergodic \( G \)-space \( Y \) is also \( \Gamma \)-metrically ergodic.

In particular \( \Gamma \) acts metrically ergodic on \( G/H \) whenever \( H \leq G \) is a closed subgroup whose image in every proper quotient of \( G \) is not pre-compact.

**Proof.** Assume that \( \phi : Y \to U \) is a \( \Gamma \)-equivariant measurable map into a separable metric space on which \( \Gamma \) acts isometrically. Replacing if necessary the metric \( d \) on \( U \) by \( \min\{d, 1\} \) we assume that \( d \) is bounded. Consider the space of \( \Gamma \)-equivariant measurable maps, defined up to null sets, \( L(G,U)^\Gamma \), endowed with the metric

\[
D(\alpha, \beta) = \sqrt{\int_{\Gamma \backslash G} d(\alpha(x), \beta(x))^2 dx}
\]

where the integration is taken over a fundamental domain for \( \Gamma \) in \( G \). Define the map \( \psi : Y \to L(G,U)^\Gamma \) by \( \psi(y)(g) = \phi(gy) \). Note that indeed, \( \psi(y) \) is \( \Gamma \)-invariant, and further \( \psi \) intertwines the \( G \) action on \( Y \) and the \( G \) action on \( L(G,U)^\Gamma \) coming from the right regular action of \( G \). By \( G \)-metric ergodicity of \( Y \) we conclude that \( \psi \) is essentially constant. The essential image is a \( G \)-invariant function on \( G \), thus a constant function to \( U \). This constant in turn is the essential image of \( \phi \), thus \( \phi \) is essentially constant as well. \( \square \)

Recall that for probability measure preserving actions, metric ergodicity is equivalent to the weak mixing property.

**Corollary 6.7.** Let \( G \) be a semisimple analytic group with a finite center and no compact factors. Let \( \mu \) be an admissible probability measure on \( G \). Let \((X, \nu)\) be a \( G \)-Lebesgue space endowed with a \( \mu \)-stationary ergodic probability measure. Then \( X \) is metrically ergodic. In particular, if the action on \( X \) is measure preserving then \( X \) weakly mixing (and in fact it is mixing modulo the action kernel).

In fact, in the measure preserving case, \( G' \simeq X \) is even mixing, as we shall see in 9.4. Below we sketch the proof of the corollary. Since we do not want to dive into the details of the subject here, we address interested reader to \([BF]\) for further details and clarifications. Assume that \( X \) is not metrically ergodic. By Theorem 6.5, there exists a (non-compact) quotient group \( G' \), a compact group \( H' < G' \) and an equivariant map \( \phi : X \to G'/H' \). Denote \( \nu' = \phi_*(\nu) \). Since \( \nu' \) is recurrent with respect to a random sequence in \( G \), while the action is dissipative, we get a contradiction. We further remark that by the theory Furstenberg-Poisson Boundaries, it is a general fact that the question of metric ergodicity
of a stationary measure reduces to the invariant measure case. Indeed, the Furstenberg-Poisson Boundary of a group, with respect to an admissible measure, is always a metrically ergodic action. It follows that for a stationary space \( X \) and an equivariant map into a metric space, \( X \to U \), the pushed measure is invariant: the associated boundary map from the Furstenberg-Poisson boundary to \( \text{Prob}(U) \) must be constant, due to the existence of a natural invariant metric on \( \text{Prob}(U) \). Thus the corollary above is reduced to the classical theorem of Howe-Moore, Theorem 9.4 which we will prove independently.

**Corollary 6.8.** Let \( G \) be a semisimple analytic group with a finite center. Let \( Y \) be a metrically ergodic \( G \)-space. Let \( X \) be an ergodic probability measure preserving \( G \)-Lebesgue space. Then the diagonal action of \( G \) on \( X \times Y \) is metrically ergodic.

**Proof.** Assume that \( \phi : X \times Y \to U \) is a \( G \)-equivariant measurable map into a separable metric space on which \( G \) acts isometrically. By replacing if necessary the metric \( d \) on \( U \) by \( \min\{d, 1\} \) we may assume that \( d \) is bounded. Consider the space of measurable maps, defined up to null sets, \( L(X,U) \), endowed with the metric

\[
D(\alpha, \beta) = \sqrt{\int_X d(\alpha(x), \beta(x))^2 dx}.
\]

Define the map \( \psi : Y \to L(X,U) \) by \( \psi(y)(x) = \phi(x, y) \). Note that \( \psi \) is \( G \)-equivariant. By the \( G \)-metric ergodicity of \( Y \) we conclude that \( \psi \) is essentially constant. The essential image is a \( G \)-equivariant map from \( X \) to \( U \). By Corollary 6.7, \( X \) is metrically ergodic as well, thus the latter map is also essentially constant. It follows that \( \phi \) was essentially constant to begin with. \( \Box \)

# 7. Monoid Compactifications

## 7.1. Ellis joint continuity

Let \( G \) be a topological group, \( X \) a topological space and \( G \times X \to X \) an action. We will say that the action is *separately continuous* if for every \( x_0 \in X \) and \( g_0 \in G \) both maps

\[
G \to X, \ g \mapsto gx_0 \quad \text{and} \quad X \to X, \ x \mapsto g_0x
\]

are continuous. We will say that the action is *jointly continuous* if the map

\[
G \times X \to X, \ (g, x) \mapsto gx
\]

is continuous.

**Lemma 7.1.** Let \( G \) be a topological group, \( X \) a locally compact topological space and \( G \times X \to X \) a separately continuous action. Consider the left regular action of \( G \) on \( C_0(X) \) endowed with the sup-norm topology. Then the following are equivalent:

1. The action of \( G \) on \( X \) is jointly continuous.
2. For every \( f \in C_0(X) \), the orbit map \( G \to C_0(X) \) given by \( g \mapsto f(g^{-1}x) \) is continuous.
3. The action of \( G \) on \( C_0(X) \) is jointly continuous.
Proof. The fact that (1) implies (3) is standard. Clearly (3) implies (2) (in fact, the converse implication is given by Lemma 2.8). We prove that (2) implies (1). By Urysohn’s lemma, the collection of subsets of $X$ of the form $f^{-1}(W)$ for $f \in C_0(X)$ and $W$ open in $\mathbb{C}$ is a sub-basis for the topology. Fixing $f$ and $W$, our aim is to show that for every $g \in G$ and $x \in X$ with $gx \in f^{-1}(W)$ there exists an open set $(g, x) \in U \times V \subset G \times X$ such that $U \cdot V \subset f^{-1}(W)$. Choose $\epsilon > 0$ for which the disc $B(f(gx), \epsilon) \subset W$ and let $V = (g^{-1}f)^{-1}(B(g^{-1}f(x), \epsilon/2))$.

Let $U^{-1} \subset G$ be the preimage of $B(g^{-1}f, \epsilon/2) \subset C_0(X)$ under the $f$-orbit map $G \to C_0(X)$, $h \mapsto h^{-1}f$. Then $U$ is open by our continuity assumption, and for $h \in U, y \in V$, $|f(hy) - f(gx)| \leq |(h^{-1}f - g^{-1}f)(y)| + |g^{-1}f(y) - g^{-1}f(x)| < \|h^{-1}f - g^{-1}f\| + \epsilon/2 < \epsilon$, i.e. $f(hy) \in W$. Thus, $U \cdot V \subset f^{-1}(W)$. □

**Theorem 7.2 (Ellis).** Let $G$ be a locally compact group and $X$ a locally compact space. Then every separately continuous action of $G$ on $X$ is jointly continuous.

This is a corollary of Ellis’ joint continuity theorem [El57]. We give below an independent short proof, assuming that $G$ is first countable. We will relay on the following well known fact.

**Proposition 7.3.** For a representation of a locally compact group on a Banach space by bounded operators, the following are equivalent:

- the orbit maps are weakly continuous
- the orbit maps are strongly continuous.

Proof. This is a standard approximate identity argument, see for example [LG65, Theorem 2.8]. □

**Proof of Theorem 7.2 for first countable groups.** In view of Proposition 7.3 and Lemma 7.1, it is enough to show that for $f \in C_0(X)$, the orbit map $g \mapsto gf$ is weakly continuous. By Riesz’ representation theorem every functional on $C_0(X)$ is represented by a finite complex measure and by the Hahn-Jordan decomposition it is enough to consider a positive measure $\mu$. By the first countability of $G$ it is enough to prove that for a converging sequence in $G$, $g_n \to g$, we have the convergence $\int g_nf d\mu \to \int gf d\mu$. This indeed follows from Lebesgue’s bounded convergence theorem. □

**7.2. Monoids.** Let $(X, T)$ be a compact semi-topological monoid. By this we mean that $X$ is a monoid and $T$ is a compact topology on $X$ for which the product is separately continuous — for each $y \in X$ the functions

$$X \to X, \ x \mapsto xy \quad \text{and} \quad X \to X, \ x \mapsto yx$$

are both continuous, but the map $X \times X \to X$ may not be. Note that $C(X)$ is invariant under left and right multiplication. For every $f \in C(X)$ we denote $xf(y) = f(yx)$ and let $S_f$ be the uniform structure obtained on $X$ by pulling back the sup-norm uniform structure
from \( C(X) \) via the orbit map \( X \to C(X), x \mapsto xf \). We let \( S \) be the uniform structure on \( X \) generated by all the structures \( S_f \), that is \( S = \bigvee_{f \in C(X)} S_f \).

**Lemma 7.4.** The topology \( T \) is \( S \)-compatible.

**Proof.** Note that by Urysohn’s lemma \( T \) is the weakest topology on \( X \) generated by the functions in \( C(X) \). Thus it is enough to show that for a given \( f \in C(X) \), the topology \( T_f \), generated on \( X \) by \( f \), is \( S \)-compatible. We will show that it is in fact an \( S_f \)-compatible.

Fix \( x \in X \) and \( \varepsilon > 0 \) and consider \( V = f^{-1}(B(f(x), \varepsilon)) \in T_f \). Set \( V' = f^{-1}(B(f(x), \varepsilon/2)) \in T_f \) and \( U = \{(y, z) \mid \|yf - zf\| < \varepsilon/2\} \in S_f \).

For \( y \in UV' \) there exists some \( z \in V' \) such that \( (y, z) \in U \). Therefore

\[
|f(y) - f(x)| \leq |yf(e) - zf(e)| + |f(z) - f(x)| < \|yf - zf\| + \varepsilon/2 < \varepsilon,
\]

and thus \( z \in V \). It follows that \( UV' \subset V \).

Let us summaries the conclusions of this section:

**Theorem 7.6.** Let \( G \) be a locally compact group and \((X,T)\) a compact semi-topological monoid. Suppose we are given a continuous monoid representation \( G \to (X,T) \) and let \( S \) be the associated uniform structure on \( X \). Then

- \( T \) is an \( S \)-topology,
- the left regular action \( G \curvearrowright X \) is jointly continuous with respect to both topologies \( T \) and \( T_S \), and
- \( G \curvearrowright X \) is equicontinuous with respect to \( S \).
8. Weakly almost periodic rigidity

Let $G$ be a locally compact group. By a monoid representation of $G$ we mean a continuous monoid homomorphism from $G$ into a compact semi-topological monoid.

**Example 8.1.** If $G$ is non-compact we denote by $G^*$ the one point compactification of $G$, $G \cup \{\infty\}$, with the multiplication extended from that of $G$ by

$$g\infty = \infty g = \infty \infty = \infty$$

for every $g \in G$. We let $i^* : G \to G^*$ be the obvious embedding. If $G$ is compact we set $G^* = G$ and $i^* = \text{the identity map}$. In both cases, $i^* : G \to G^*$ form a monoid representation of $G$.

We will say that a monoid representation with dense image $i : G \to X$ is a universal if for every monoid representation $j : G \to Y$ there exists a unique continuous monoid homomorphism $k : X \to Y$ such that $j = ki$. The pair $(i, X)$ will be referred as a universal system.

**Theorem 8.2.** The locally compact group $G$ admits a universal monoid representation $i : G \to X$. Every two universal systems are uniquely isomorphic. Furthermore, $i$ is a homomorphism into its image and $i(G)$ is open and dense in $X$.

*Proof.* The collection of isomorphism classes of monoid representations of $G$ with dense images forms a set; it could be described for example as a subset of the set of all norm closed subalgebras of $C_b(G)$. Pick one representative for each class and consider the product space of those, let $i$ be the diagonal morphism from $G$ to this product space and let $X$ be the closure of $i(G)$. The existence of $k$ follows immediately. The uniqueness of $k$ follows by the fact that $i(G)$ is dense in $X$, and the uniqueness of the pair $(i, X)$ is obvious. That $i(G)$ is open follows from the fact that $G^*$ is a factor of $X$. \hfill \Box

**Definition 8.3.** The representation alluded to in Theorem 8.2 is called $\text{WAP}(G)$.

**Remark 8.4.** By the Gelfand–Neumark theory, compactifications of $G$ corresponds to point separating $*$-subalgebras of $C_b(G)$, where general $*$-subalgebras of the latter correspond to compactifications of (topological) quotients of $G$, and the Stone–Čech (the largest) compactification correspond to the full algebra $C_b(G)$. Among these, the monoid representations of $G$ correspond sub-algebras carrying an additional structure, and $\text{WAP}(G)$ corresponds to the largest such algebra. It can be shown that it is the algebra of weakly almost periodic functions on $G$, hence the notation. We will not elaborate on the point of view of almost periodic functions on $G$.

**Definition 8.5.** A group $G$ will be said to be $\text{WAP-rigid}$ if $\text{WAP}(G) \simeq G^*$.

**Example 8.6.** If $G$ is compact then clearly $\text{WAP}(G) = G = G^*$ and $G$ is $\text{WAP-rigid}$.

The following theorem, which was proved first in [Ve79] and [EN89], could be seen as a special case of Theorem 8.8 below. For clarity we give a separate proof.
Theorem 8.7 ([Ve79],[EN89]). Let $G$ be an almost simple analytic group over a local field. Then $G$ is WAP-rigid.

Proof. We assume $G$ is non-compact. Let $j : G \to X$ be any monoid representation of $G$. We will construct a continuous monoid morphism $k : G^\ast \to X$. Such a morphism is clearly unique and satisfies $j = k\iota$. In view of Theorem 7.6 we are in the situation to apply Theorem 4.1 to either the left or the right actions of $G$ on $X$. Upon replacing $X$ with $X \times G^\ast$ we may assume that $j(G) = Ge = eG$ is non-compact. We therefore get by Theorem 4.1 the existence of a point $x \in j(G)$ which is right $G$-invariant and a point $y \in j(G)$ which is left $G$-invariant. By continuity of the product in $X$ we have $x = xy = y$. It follows that $x$ is the unique left $G$-invariant point in $\overline{j(G)}$. We then define $k : G^\ast \to X$ by setting $k(g) = ge$ for $g \in G$ and $k(\infty) = x$. Clearly $k$ is a continuous morphism.

We now discuss semisimple (rather than simple) groups. Let $G$ be a finite centred semisimple analytic group over a local field. Then $G = G_0G_1 \cdots G_n$ where $G_0$ is compact and $G_1, \ldots, G_n$ are the non-compact almost simple factors. For each $I \subseteq \{1, \ldots, n\}$ we let

$$G_I = \prod_{i \in I} G_i < G \text{ and } G^\dagger = G/G_I.$$ 

In particular, $G^{\{1, \ldots, n\}}$ is a quotient of $G_0$ hence a compact group. Note that for $I \subseteq J$ there is a natural homomorphisms $\phi^J_I : G^I \to G^J$. We denote $\phi^J_I = \phi^\emptyset_J : G \to G^J$.

We define $\hat{G} = \coprod_{I \subseteq \{1, \ldots, n\}} G^I$. The sets of the form

$$U \bigcup_{J' \subset \{1, \ldots, n\} \setminus J} \left( (\phi^J_{J', J^\prime})^{-1}(U) \setminus (\phi^J_{J', J^\prime})^{-1}(K_{J^\prime}) \right),$$

where $J$ is a subset of $\{1, \ldots, n\}$, $U \subset G^J$ is open, $J'$ runs over all stets disjoint from $J$ and $K \subset G^{J'}$ are compact, generate a compact Hausdorff topology on $\hat{G}$. We always refer to this topology when regarding $\hat{G}$ as a topological space. In order to understand this topology it might be helpful to note that for $I \subseteq \{1, \ldots, n\}$ and a sequence $g_n \in G^I$, $\hat{G}$-lim $g_n = \lim \phi^J_I(g_n)$ if and only if the right hand side limit, which is the standard limit in the group $G^J$, exists, where $J$ is the minimal set satisfying $I \subseteq J \subseteq \{1, \ldots, n\}$ for which $\phi^J_I(g_n)$ is bounded.

We introduce a natural monoid structure on $\hat{G}$ as follows. For $I, J \subseteq \{1, \ldots, n\}$ and $g \in G^I$, $h \in G^J$ we set $gh = \phi^I_J(g)\phi^J_I(h) \in G^{I \cap J}$. This makes $\hat{G}$ a compact semi-topological monoid.

Theorem 8.8. $\text{WAP}(G) \simeq \hat{G}$.

Proof. We prove the theorem by induction on $n$, the number of non-compact simple factors of $G$. The induction basis is the case $n = 0$, that is $G$ is compact, for which the theorem is clear. We let $j : G \to X$ be a monoid representation. For any $I \subset \{1, \ldots, n\}$ we have by our induction hypothesis $\text{WAP}(G_I) = G_I$. In particular $\text{WAP}(G_I)$ has a unique left $G_I$-fixed point which is also a unique right $G_I$-fixed point (as $G_I$ has no compact factor).
It follows that there is a unique left $G_I$-fixed point which is also a unique right $G_I$-fixed point in $j(G_I)$. We denote it by $e_I$. We define a map $\hat{G} \to X$ by sending $g \in G^I$ to $ge_I$. One checks that this is a continuous morphism.

\section{9. WAP representations and mixing}

Let $k$ be a topological field. Let $V, V'$ be $k$-vector spaces and $\langle \cdot, \cdot \rangle : V \times V' \to k$ a bilinear form. For $v \in V$ and $\phi \in V'$ we denote $\phi(v) = \langle v, \phi \rangle$. We assume that the elements of $V'$ separates points in $V$. We denote by $\text{End}(V)^{V'}$ the algebra of endomorphisms $T \in \text{End}(V)$ satisfying for every $\phi \in V'$ that $\phi \circ T$ is represented by an element (necessarily unique) of $V'$, to be denoted $T\phi$.

We endow $V$ with the weak topology, namely the weakest topology for which every $\phi \in V'$ is a continuous function to $k$. Note that the elements of $\text{End}(V)^{V'}$ are continuous functions from $V$ to $V$. Considering the Tychonoff topology on $(V, \text{weak})^{V'}$, using the embedding $\text{End}(V) \to V^{V'}, \; T \mapsto (Tv)_v$, we obtain the weak operator topology on $\text{End}(V)$, and in particular on $\text{End}(V)^{V'}$. Check that the composition operation on $\text{End}(V)^{V'}$ is continuous (separately) in each variable, thus $\text{End}(V)^{V'}$ becomes a semi-topological monoid. Note that $A \subset \text{End}(V)^{V'}$ is precompact if and only if $Av$ is precompact in $V$ for every $v$.

Let $G$ be a topological group. By a continuous representation of $G$ to $V$ we mean a continuous monoid homomorphism $\rho : G \to \text{End}(V)^{V'}$. The representation $\rho$ is said to be weakly almost periodic, or WAP, if $\rho(G)$ is precompact in $\text{End}(V)^{V'}$, or equivalently, if $\rho(G)v$ is precompact in $V$ for every $v \in V$. In that case, $\rho(G)$ is a semi-topological compact monoid.

\begin{example}
Let $U$ be a Banach space, and consider a strongly continuous homomorphism $G \to \text{Iso}(U)$. Let $V = U^*$ and $V' = U$, the pairing be the usual one, and the representation $\rho$ be the contragredient representation. By Banach-Alaoglu theorem $\rho$ is a WAP representation. A special case of this example is any isometric representation on a reflexive Banach space, and in particular any unitary representation on a Hilbert space.
\end{example}

The following is an immediate application of Theorem 8.2.

\begin{corollary}
Let $G$ be a locally compact topological group and let $i : G \to X$ be its universal monoid representation into a compact semi-topological monoid. Then every WAP-representation $\rho : G \to \text{End}(V)^{V'}$ factors as a representation of $X$, that is there exists a continuous monoid homomorphism $\rho' : X \to \text{End}(V)^{V'}$ such that $\rho = \rho' \circ i$.
\end{corollary}

For a locally compact topological group $G$, $\rho : G \to \text{End}(V)^{V'}$ is said to be mixing if for every $v \in V, \phi \in V'$,

$$\lim_{g \to \infty} \langle gv, \phi \rangle = 0.$$ 

Theorem 8.8 gives a structure theorem of representation of semi-simple groups.

\begin{theorem}
Let $G$ be a semi-simple group and let $\rho : G \to V$ be a WAP representation. Then $V$ decomposes as a direct sum of representations $V = \bigoplus_{I \subseteq \{1, \ldots, n\}} V_I$ such that on $V_I$
\end{theorem}
the $G$-representation factors through $G^I$ and proper factors of $G^I$ have no fixed points in $V_I$. Furthermore, for every $I$, the representation $V_I$ is $G^I$ mixing.

A special case of Theorem 9.3 is the classical theorem of Howe-Moore [HM79].

**Theorem 9.4.** Let $G$ be a semisimple analytic group with a finite center (the $F$ point of a Zariski connected semisimple algebraic group $G$, defined over a local field $F$) and no compact factor. Then every ergodic probability preserving action is mixing modulo the action kernel.

**Proof.** Apply the last corollary to the Koopman representation. □

### 10. Banach modules

We shall now concentrate on the special case of uniformly bounded representations on Banach spaces. The main result of this section, Theorem 10.2, is a straightforward consequence of Theorem 9.3, when $G$ is a semisimple group. However, because of the importance of this special case, and for the convenience of the users, we decided to give a self contained discussion that avoids the more general notion of WAP representations. In particular, we shall provide an alternative proof for Theorem 10.2. Since we shall rely in this section only on Theorem 4.1, we can state the results for the class of quasi-semi-simple rather than semi-simple groups.

Let $V$ be a Banach space and $S$ the norm uniform structure on $V$. We denote by $\mathcal{B}(V)$ the algebra of bounded linear operators on $V$ and by $\text{GL}(V)$ the group of invertibles in $\mathcal{B}(V)$. A group representation $\rho : G \to \text{GL}(V)$ is said to be uniformly bounded if

$$\sup_{g \in G} \|\rho(g)\|_{\text{op}} < \infty,$$

i.e. if it induces a uniform action on $(V,S)$. We denote by $\rho^* : G \to \text{GL}(V^*)$ the dual (contragradient) representation. Since $\|\rho(g)^*\|_{\text{op}} = \|\rho(g)\|_{\text{op}}$, $\rho^*$ is uniformly bounded iff $\rho$ is. We will focus on the case where $G$ is a topological group and the representation $\rho$ is continuous with respect to the strong operator topology.

**Definition 10.1.** We will say that $(V, \rho)$ is a $G$-Banach module if $V$ is a Banach space, $G$ is a topological group and $\rho : G \to \text{GL}(V)$ is a uniformly bounded representation which is continuous in the sense that the map $G \times V \to V$, $(g, v) \mapsto \rho(g)(v)$ is continuous. We will say that $(V, \rho)$ is a $G$-Banach $*$-module if also the dual representation $\rho^* : G \to \text{GL}(V^*)$ is continuous in the same sense.

By Lemma 2.8 $\rho$, is continuous iff its orbit maps are continuous.

Apart from the norm topology, $V$ and $V^*$ are equipped with the weak and the weak$^*$ (hereafter $w$ and $w^*$) topologies. It is obvious that these topologies are compatible with the norm uniform structure. If $G$ is locally compact, it follows by a standard argument of approximating identity in $L^1(G)$, that a uniformly bounded representation is strongly continuous iff it is weakly continuous, see for example [LG65, Theorem 2.8]. This is also the case when $V$ is super-reflexive and $G$ is arbitrary.
Theorem 10.2. Let $G$ be a quasi-semi-simple group. Let $(V, \rho)$ be a $G$-Banach $*$-module. Assume that no point in $V^* \setminus \{0\}$ is fixed by a non-compact normal subgroup of $G$. Then for every $f \in V^*$,
$$Gf^w = Gf \cup \{0\},$$
and $\rho$ is mixing in the sense that all matrix coefficients tend to 0.

Proof. Given $f \in V^* \setminus \{0\}$, consider the space $X = \text{conv}(Gf) \setminus \{0\}$. Let $S$ be the norm uniform structure on $X$ and $T$ the weak$^*$-topology. By the Hahn–Banach and Alaoglu’s theorems $(X, T)$ is locally compact. By Corollary 4.2, $Gf$ is weak$^*$-closed in $X$ and homeomorphic to the coset space $G/G_f$, where the stabiliser $G_f$ is compact. Thus the orbit $Gf$ is non-compact. It follows that it is not weak$^*$-closed in the compact space conv($Gf$), and hence that $Gf^w = Gf \cup \{0\}$. Since the later is compact while $Gf$ is a proper $G$ space, it follows that $gf \to 0$ (in the weak$^*$ sense) when $g \to \infty$ in $G$. \qed

Remark 10.3. It follows, for instance, that for a non-compact QSS simple group $G$, the existence of a nonzero invariant vector (or more generally a vector with a non-compact stabiliser) in a Banach $*$-module $V$ implies the existence of a non-zero invariant vector in $V^*$. This property does not hold for general groups; for example consider the regular representation of a discrete non-amenable group $\Gamma$ on the space $L_\infty(\Gamma)$.

When $V$ is reflexive, the a priori weaker assumption that $G$ doesn’t fix a vector in $V$, is actually sufficient.

Lemma 10.4. Let $L$ be a group and $\rho : L \to GL(V)$ a linear representation on a reflexive Banach space $V$. If $L$ has a non-zero invariant vector in $V^*$ then it has a non-zero invariant vector in $V$.

Proof. Suppose that $f \in V^*$ is an $L$-invariant norm one functional. The invariant set of supporting unit vectors
$$S_f = \{v \in V : \langle f, v \rangle = \|v\| = 1\}$$
is non-empty by the Haan–Banach theorem and weakly compact by Alaoglu’s theorem. Hence the Ryll-Nardzewski fixed-point theorem implies that $L$ admits a fixed point in $S_f$. \qed

Corollary 10.5 (Howe-Moore’s theorem for reflexive Banach spaces). Let $G$ be a quasi-semi-simple group. Let $(V, \rho)$ be a reflexive $G$-Banach module. Assume that no point in $V \setminus \{0\}$ is fixed by a non-compact normal subgroup of $G$. Then for every $f \in V^*$,
$$Gf^w = Gf \cup \{0\},$$
and $\rho$ is mixing.

The special case of 10.5 where $V$ is uniformly convex uniformly smooth was proved in [BGFM07, Appendix].

We conclude this paper by remarking that for every group $G$, every WAP function on $G$ appears as a matrix coefficient of some reflexive representation. This result is due to [Ka81], following the important main theorem of [DFJP74]. In this regard, our results on WAP compactifications could be also deduced from the results of the current section.
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