On the satiability of floating bodies

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Abstract

The potential energy of a system in stable equilibrium has a minimum value. This property is used to derive a formula that is useful in determination of stability of a floating body. It is found that a floating body is in stable equilibrium if its center of gravity has a minimum height with respect to its related center of buoyancy.

keywords: stability, floating body, center of buoyancy, center of gravity, potential energy

1 Introduction

Study of stability of floating bodies is a conventional subject in fluid mechanics. Two forces act on a floating object; weight, that acts on the center of gravity $G$, and buoyancy force that acts on the center of buoyancy $B$ (the centroid of the displaced volume of fluid) [1]. The equilibrium of a body requires that these two forces be equal and opposite and the joint line of these two points has to be in vertical direction too.

Stability of a floating body is divided in two different types, vertical and rotational. A floating body has vertical stability but its rotational stability depends upon the positions of $G$ and $B$. If $G$ is below $B$ the equilibrium is stable. But if $G$ is above $B$ the equilibrium may or may not be stable. The usual method in specification of stability of a floating body is finding the metacenter point and then comparing its position with $G$. The equilibrium is stable if the metacenter lies above $G$ [2,3].

Another approach to study this subject is using the energy principle. Seemingly this idea first has been suggested by Chr. Huygens in more than three centuries ago. P. Erdős et al have used this approach in solving some problems [4] but they have made some nonessential assumptions. In this paper stability of floating bodies is studied by this method without restrictions assumed in Ref. [4].

Consider a system consisting of a liquid and a floating object. If the floating object is in stable equilibrium, the gravitational potential energy of the system
must be a minimum. Therefore, when the object configuration is changed, the center of gravities of the liquid and of the object will change; then the center of gravity of the system raises and consequently the gravitational potential energy of the system increases.

Inversely, if by changing the object configuration the gravitational potential energy of the system increases, the object is in stable equilibrium.

2 Theoretical approach

Consider a conservative system, with independent coordinates \( q_1, q_2, \ldots, q_n \), that is held in a configuration and then released. It generally changes and gets a new configuration. But, if in special cases, it does not change it will be in equilibrium and its potential energy, \( U \), is extremum with respect to all independent coordinates of the system at its equilibrium configuration, \( q_{01}, q_{02}, \ldots, q_{0n} \) \[6\], that is

\[
\left( \frac{\partial U}{\partial q_k} \right)_0 = 0; \quad k = 0, 1, 2, \ldots, n. \quad (1)
\]

Figure 1 shows an object floating in a liquid. Masses of object and liquid are \( m \) and \( M \) respectively. \( \mathbf{R}_m, \mathbf{R}_M \) and \( \mathbf{R}_{m+M} \) are the position vectors of center of gravity of object, liquid and system (object + liquid) respectively with respect to an arbitrary origin \( O \). The gravitational potential energy of the system is given by

\[
U = -(m + M)g \mathbf{R}_{m+M}, \quad (2)
\]

where \( g \) is acceleration of gravity. \( \mathbf{R}_{m+M} \) can be written in term of \( \mathbf{R}_m \) and \( \mathbf{R}_M \) \[6\]

![Figure 1: A floating object of mass \( m \) is partially immersed in a liquid of mass \( M \). \( \mathbf{R}_m, \mathbf{R}_M \) and \( \mathbf{R}_{m+M} \) are the position vectors of the center of gravities of the object, liquid and system respectively.](attachment:image.png)
Thus\[ U = -(mg \cdot R_m + M g \cdot R_M). \] (4)

Assume \( y \) axis be in upward vertical direction, therefore
\[ U = mgy_m + Mgy_M, \] (5)
where \( y_m \) and \( y_M \) are \( y \) coordinates of \( R_m \) and \( R_M \) respectively.

When the object is partially immersed in the liquid, some part of the liquid is displaced by the object. We name mass of the displaced liquid by \( M' \) and the \( y \) coordinate of its center of gravity, which is buoyancy center, by \( y_{M'} \). Figure 2 shows a hole in the liquid formerly occupied by the immersed part of the object. If this hole to be filled by the same type of liquid, so the mass of resulted liquid is \( M + M' \) and \( y \) coordinate of its center of gravity is \( y_{M+M'} \). But, according to the statement quoted before Eq. (3),
\[ (M + M')y_{M+M'} = My_M + M' y_{M'}. \] (6)

By eliminating \( y_M \) between Eqs. (5) and (6), we have
\[ (m + M)R_{m+M} = mR_m + M R_M. \] (3)

Figure 2: Initial liquid and the hole in the liquid formerly occupied by the immersed part of the object. The hole is filled by the same type of liquid. The mass of added liquid is \( M' \). \( R_M, \ R_{M'}, \) and \( R_{M+M'} \) are center of gravities of initial, added and resulted liquids respectively.

\[ U = [m y_m + (M + M') y_{M+M'} - M' y_{M'}] g. \] (7)

This object has six degrees of freedom and then is described by six independent coordinates; for example, three coordinates of its center of mass (or gravity) and three angles that describe its orientation. It is evident that the potential energy of the system is independent of the object rotation around any vertical axis, and the two coordinates that describe its center of mass in horizontal plane. An equilibrium condition is
\[
\frac{\partial U}{\partial y_m} = 0. \tag{8}
\]

Now I calculate \(\frac{\partial U}{\partial y_m}\) explicitly.

\[
\frac{\partial U}{\partial y_m} = m + \frac{\partial M'}{\partial y_m} y_{M+M'} + (M + M') \frac{\partial y_{M+M'}}{\partial y_m} - y_{M'} \frac{\partial M'}{\partial y_m} - M' \frac{\partial y_{M'}}{\partial y_m}. \tag{9}
\]

Figure 3 shows two configurations of the system. Assume areas of liquid surface and cross section of object in liquid surface level are \(S\) and \(s\) respectively. In the first case liquid surface is at \(y_0\). In the second case object has been raised by \(\Delta y_m\) in vertical direction and the liquid surface level has lowered by \(|\Delta y_0|\) (dashed shape). By inspection in this Figure we find

\[
\Delta M' = -\rho s (\Delta y_m - \Delta y_0), \tag{10}
\]

where \(\rho\) is liquid mass density, and

Figure 3: Two configurations of the system; before raising (dashed), where the liquid surface level is at \(y_0\), and after raising the object by \(\Delta y_m\) in vertical direction (solid). In the latter case the liquid surface level lower by \(|\Delta y_0|\).

Figure 4: Added liquid in two configurations, before (dashed) and after (solid) raising the object.
\[
\Delta y_0 = -\frac{s}{S - s} \Delta y_m, \quad (11)
\]

Therefore
\[
\frac{\partial M'}{\partial y_m} = -\frac{\rho s}{S - s}, \quad (12)
\]

By raising the object, both \(M'\) and \(y_{M'}\) will change. This is equivalent to raising \(M'\) by \(\Delta y_m\) and then shearing its top by \(\Delta y_m - \Delta y_0\), Figure 4. Relation between these two parts according to center of mass theorem is
\[
M'(y_{M'} + \Delta y_m) = (M' + \Delta M')(y_{M'} + \Delta y_{M'}) - \Delta M'(y_0 + \frac{1}{2} \Delta y_m).
\]

By ignoring second order differential terms and introducing \(\frac{\partial M'}{\partial y_m}\) given by (12) into (13), it follows that
\[
\frac{\partial y_{M'}}{\partial y_m} = 1 + \frac{\rho}{M'} \frac{Ss}{S - s}(y_{M'} - y_0). \quad (14)
\]

Now by referring to Figure 5, that shows resulted liquid in two cases, and repeating the procedure in writing Eq. (13) we have

\[
(M + M')(y_{M + M'} + \Delta y_{M + M'}) = (M + M' + \Delta M')(y_{M + M'} + \Delta y_{M + M'}) - \Delta M'(y_0 + \frac{1}{2} \Delta y_0).
\]

By ignoring second order differential terms and using Eq. (12) we have
\[
\frac{\partial y_{M + M'}}{\partial y_m} = \frac{\rho}{M + M'} \frac{Ss}{S - s}(y_{M + M'} - y_0). \quad (16)
\]

Substituting from Eqs. (12), (14) and (16) into (9) it reduces to
\[
\frac{\partial U}{\partial y_m} = m - M'.
\]  
(17)

Combination of (8) and (17) gives

\[
M' = m.
\]  
(18)

This is Archimedes’ principle, which requires that in equilibrium the mass of the displaced liquid \(M'\) is equal to the mass of the floating object \(m\), independent of orientation of the object. Therefore in vertical equilibrium \(M'\) and also \(y_{M+M'}\) are constants. It is easy to derive this relation by another method but this method is instructive and may be useful in studying dynamics of floating bodies. Furthermore,

\[
\frac{\partial^2 U}{\partial y_m^2} = -\frac{\partial M'}{\partial y_m} = \frac{\rho_s S s}{\rho s - s}
\]  
(19)

is always positive. Thus, equilibrium is always stable against vertical displacements. By applying this result to Eq. (7) it becomes

\[
U = [(M + m)y_{M+M'} + m(y_m - y_{M'})]g.
\]  
(20)

The first term on the right side of Eq. (20) is a constant. Therefore gravitational potential energy of the system is a function of \(\Delta y = y_m - y_{M'}\). In case of \(\Delta y\) is an extremum, the object is in equilibrium. When \(\Delta y\) is minimum (maximum), \(U\) is minimum (maximum) and therefore object is in stable (unstable) equilibrium. Similar to this result \((BG\) instead of \(\Delta y\)) has been quoted in Ref. [7] but has not been derived by present method.

Another equilibrium condition is extremizing the potential energy of system with respect to rotation of object around any axes that is parallel to the liquid surface. This is equivalent to two equations. The following example clarifies how this method can be used in specification of stability of a floating body.

### 3  example

A long uniform bar of square cross section each sides \(a\), floats in liquid with its longitudinal axis parallel to the liquid surface. Ratio of the specific masses of the solid \((\rho')\)and liquid \((\rho)\) is denoted by \(r = \frac{\rho'}{\rho}\). Bar has vertical stability and we wish to determine its configurations in equilibrium condition. For this purpose \(\Delta y\) must be determined as a function of the variable \(\theta\), which is the angle between the symmetry plane of the bar shown in Figure 6 and the vertical plane passing through the longitudinal axis of the bar. \(\theta\) is restricted to the range \([0, \pi/4]\), because increasing \(\theta\) over \(\pi/4\) the bar practically gets one of the old configurations.

An interchange \(r \leftrightarrow (1 - r)\) corresponds to \(v_1 \leftrightarrow v_2\), where \(v_1\) denotes the immersed volume and \(v_2\) the exposed volume. Hence to each equilibrium configuration there is a dual one where \(r \rightarrow (1 - r)\) and the body is rotated through
180°. In fact it is proved that duality preserves stability [4, 7]. Therefore I shall restrict the discussion to \( r \leq \frac{1}{2} \). For difference in geometries, calculating of \( \Delta y \) is made in two stages.

3.1 Configuration with two corners immersed

Figure 6a shows the bar with its two corners immersed. In this case we have

\[
y_m = a \left( \frac{1}{2} - r \right) \cos \theta \tag{21}
\]

and

\[
y_M' = -a \left( \frac{1}{2} r \cos \theta + \frac{1}{24} \sin \theta \tan \theta \right), \tag{22}
\]

then

\[
\Delta y = a \left[ \frac{1}{2} (1 - r) \cos \theta + \frac{1}{24} \sin \theta \tan \theta \right]. \tag{23}
\]

By taking the derivative of \( \Delta y \) with respect to \( \theta \) and then equating to zero we find

\[
\sin \theta [\tan^2 \theta - 12r(1 - r) + 2] = 0. \tag{24}
\]
There are two solutions (a) and (b) of (24):

(a) \[ \sin \theta = 0. \quad (25) \]

Since

\[ \frac{\partial^2 \Delta y}{\partial \theta^2} \bigg|_{\theta=0} = \frac{a}{24r} [12r(r-1) + 2], \quad (26) \]

this equilibrium position is stable as long as \( r < \frac{3 - \sqrt{3}}{6} \), but is unstable for \( \frac{3 - \sqrt{3}}{6} < r \leq \frac{1}{2} \).

(b) If \( \theta \neq 0 \), Eq. (24) leads to

\[ \tan \theta = \sqrt{12r(1-r) - 2}. \quad (27) \]

This solution has a real answer if \( \frac{3 - \sqrt{3}}{6} < r < \frac{1}{4} \). On the other hand, because both corners are immersed by assumption, there is an upper limit to \( \theta \), given by

\[ \tan \theta < 2r. \quad (28) \]

This condition with Eq. (27) leads to \( r < \frac{1}{6} \). This means that existence of this extremum requires

\[ 0.21132 = \frac{3 - \sqrt{3}}{6} < r < \frac{1}{4}. \quad (29) \]

Since

\[ \frac{\partial^2 \Delta y}{\partial \theta^2} \bigg|_{\tan \theta = \sqrt{12r(1-r) - 2}} = \frac{a}{12r} [12r(1-r) - 2] \sqrt{12r(1-r) - 1} \quad (30) \]

is positive in the range of \( r \) where second solution exists, this equilibrium position (if exists) always is stable.

### 3.2 Configuration with one corner immersed

In this case we have, Figure 6b,

\[ y_m = a \left[ \frac{1}{2} (\cos \theta + \sin \theta) - \sqrt{r} \sqrt{\sin 2\theta} \right] \quad (31) \]

and

\[ y_{M'} = -\frac{\sqrt{r}}{3} a \sqrt{\sin 2\theta}, \quad (32) \]

then

\[ \Delta y = \frac{1}{6} a \left( 3 \cos \theta + 3 \sin \theta - 4 \sqrt{r} \sqrt{\sin 2\theta} \right). \quad (33) \]
It must be noted that in this case the angle $\theta$ is restricted to
\[\tan \theta > 2r.\]  
(34)

It is more convenient for the following to express $\Delta y$ in terms of the angle $\beta$, defined by
\[\beta = \pi/4 - \theta.\]  
(35)

We have then
\[\Delta y = \frac{1}{6}a \left(3\sqrt{2} \cos \beta - 4\sqrt{r} \sqrt{\cos 2\beta}\right).\]  
(36)

The equilibrium condition yields
\[\sin \beta \left(8\sqrt{r} \frac{\cos \beta}{\sqrt{\cos 2\beta}} - 3\sqrt{2}\right) = 0.\]  
(37)

The two solutions, (c) and (d) are

(c) \[\sin \beta = 0, \ i.e., \theta = \frac{\pi}{4}.\]  
(38)

Since
\[\frac{\partial^2 \Delta y}{\partial \beta^2} \bigg|_{\beta=0} = \frac{1}{6}a(8\sqrt{r} - 3\sqrt{2}),\]  
(39)

this equilibrium position is stable as long as $\frac{9}{32} < r < \frac{1}{2}$, meanwhile in this range condition (34) is satisfied.

(d) For $\theta \neq \pi/4$
\[\cos 2\beta = \frac{16r}{9 - 16r}.\]  
(40)

The conditions (34) and $|\cos 2\beta| \leq 1$ restrict the range of $r$ to $\frac{1}{4} < r < \frac{9}{32}$. For this solution
\[\frac{\partial^2 \Delta y}{\partial \beta^2} \bigg|_{\cos 2\beta=\frac{16r}{9 - 16r}} = \frac{1}{6}a\left(\frac{512r^2 - 432r + 81}{16r\sqrt{9 - 16r}}\right),\]  
(41)

which is positive in $\frac{1}{4} < r < \frac{9}{32}$; hence this equilibrium position is stable (if exists). For more details in description of this and few other examples you can refer to Ref. [4]

### 4 Conclusion

Stability of a floating body was inspected by energy principle. This formulation enables us to solve such problems without referring to the concept of the metacenter. Archimedes’ principle was derived too by considering the energy principle. If the vertical distance between center of gravity of the body and the center of buoyancy is calculated versus a set of suitable coordinates, it is straightforward to find any information about equilibrium positions of the body.
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