On the Local Separation of $E$ and $B$ Polarization Patterns in CMBR

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This work examines the small-scale $E$ and $B$ patterns of the Cosmic Microwave Background Radiation (CMBR) polarization anisotropy. Particularly, we address the topological natures of the $E$ and $B$ modes, and how one may make use of the local measurements of Stokes parameters to separate these two modes in the real space. The analysis of a local map in the Fourier space for separating the $E$ and $B$ modes can be an ill-posed problem, due to the non-periodic boundary condition in the map for each mode. A strategy for the $E - B$ separation in a local map is through an appropriate projection of the polarization tensor field into a unique vector field, which naturally contains a curl-free $E$ component and a divergence-free $B$ component. An integral method, equivalent to the real-space top-hat filtering, is proposed for extracting the averaged "charge" and "current" of the $E$ and $B$ modes separately. The real-space top-hat filter function exhibits, in its power spectrum, an oscillation of a well-defined frequency comparable to the filter size. It is pointed out that when the inverse filter size is chosen properly, matching the oscillation period in the $E$-mode power spectrum, the sensitivity of the $E$-mode detection can be made to be significantly improved.

Subject headings: Cosmic Microwave Background — Polarization

1. Introduction

Anisotropy of the CMBR is linearly polarized (Rees 1968; Bond and Efstathiou 1984; Polnarev 1985). The linear polarization was produced by the Thomson scattering of the local temperature quadrupole anisotropy off the uniform background electrons at the last scattering surface, where the optical depth was about $1/2$. The angular pattern of the
CMBR linear polarization therefore contains the imprints of the temperature anisotropy. However, unlike the temperature anisotropy, which is a scalar, the polarization is a tensor. The angular pattern of tensor fluctuations can generally be decomposed into an electric-type pattern ($E$-mode) and a magnetic-type pattern ($B$-mode), according to their distinct global parity symmetries (Kamionkowsky et al. 1997; Zaldarriaga and Seljak 1997).

Despite a handful of past works that have been devoted to the detailed analyses of CMBR polarization for the $E$ and $B$ modes in the Fourier space (Kamionkowsky et al. 1997; Zaldarriaga and Seljak 1997; Hu and White 1997a; Hu and White 1997b), few of them gave clear descriptions of what these unfamiliar $E$ and $B$ polarization patterns really are in the real space. These works followed the standard paradigm that the initial fluctuations are Gaussian with no feature, and hence the polarization Fourier spectra were the sole relevant quantities of investigations. Hu and White (1997b) gave, by far, the most intuitive picture of what these two types of polarization patterns are on the large scale, but their work on the subject of how the $E$ and $B$ modes may be separated is still confined to the power-spectrum method and still under the assumption of Gaussian fluctuations.

In the coming dedicated experiments for CMBR polarization (Staggs et al. 1999; Lo et al. 2000), deep maps of finite patches of local sky are to be constructed. Though the $E$ and $B$ modes have conventionally be defined in the spin harmonics space (Goldberg et al. 1967) or the Fourier space, it is in fact not desirable to analyze a map of finite sky coverage in such a space. This is because, unlike the global map, the local map does not satisfy the periodic boundary condition and the image has to be distorted in order for the Fourier transformation to be carried out, thereby making the detection of the predicted small $B$ mode to be unreliable. Merely for this reason, it has already had the real need to call for the real-space analysis of the CMBR polarization maps.

The real-space analysis also has the advantage of detecting localized non-Gaussian
features, which may lead to discovery of the cosmic objects not predicted by the standard inflationary cosmology, such as the cosmic strings (Seljek et al. 1997). Even in the framework of the standard cold-dark-matter paradigm, where the CMBR polarization is produced almost instantaneously at the last scattering surface, the polarized sky should contain a clean signature of the coherent patches of Hubble spheres at the epoch of photon-electron decoupling. This expectation is signified by the large-amplitude oscillation of a well-defined period in the polarization power spectrum. The real-space analysis can thus be potentially better methodology than the Fourier (l) space analysis to capture a large fraction of the total power within the l-space oscillation by using some properly tailored real-space filter functions, a subject that will be addressed in the last section.

Despite these advantages, the direct means for measuring the $E$ and $B$ modes in the real-space maps has not been available. It is in this context that the CMBR polarization in the real space is addressed in this report. To begin, I shall briefly review the CMBR polarization, followed by a more rigorous description of the $E$ and $B$ modes in the real space. After that, a direct means for disentangling these two modes is proposed.

## 2. Spatial Symmetry

Unlike the brightness, which is a scalar, the linear polarization cannot distribute itself uniformly on the celestial sphere, due to the curvature of a sphere. In other words, one can never arrange line segments to be uniformly distributed on a spherical surface. This statement echoes the more familiar notation that never can two families of lines lay out a globally uniform coordinate on a spherical surface; the best one can do is the longitude-latitude coordinate, which has quadrupole inhomogeneity. For vector fields, the highest spatial symmetry has a dipole pattern, and for the polarization, a tensor field, the highest symmetry has a quadrupole pattern. Therefore the CMBR linear polarization must
be angular dependent and must be a field with angular patterns, at scales smaller than or equal to the quadrupole, on the sky.

Though the instantaneous electric field is indeed a vector, the measured electric field of a polarized light is not a vector, since the measured electric field can be in either direction, a ”bi-vector”. With an 180-degree rotation of the ”bi-vector”, the polarization can recover itself but a true vector can not. Such a peculiar bi-vector can be constructed from a second-rank polarization tensor, as shown below. Similar to a two-dimensional vector, the polarization tensor can also be described by two parameters — the magnitude and orientation of the bi-vector. To distinguish a vector \( \mathbf{V} \) from a polarization tensor \( \mathbf{P} \), the former is usually expressed as a two-component array, or \( \mathbf{V} = V_x \hat{x} + V_y \hat{y} \), and the latter as a traceless \( 2 \times 2 \) matrix, or \( \mathbf{P} = P_x \sigma_1 + P_y \sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) are respectively the \( z \) and \( x \) components of the Pauli matrices. (Assume the line of sight to be along the \( z \) direction.)

The components \( P_x \) and \( P_y \) are the Stokes parameters, \( Q \) and \( U \), respectively, and the ratio \( U/Q \) is related to the orientation of polarization. The angle between the polarized electric field and the \( x \) axis is \( \theta = (1/2) \tan^{-1}(U/Q) \), which is bi-directional and has an 180-degree directional ambiguity, and \( (Q^2 + U^2)^{1/4} \) is the amplitude of the polarized electric field.

As the polarization field \( \mathbf{P}(z, x) \) must have spatial patterns on the sky, much like the more familiar vector field, the polarization field can be decomposed into two coordinate-independent components of different topologies. For the small-scale angular pattern, one may adopt the sky-flat approximation. The two topologically distinct components of a two-dimensional vector field are the curl-free \( \nabla \phi(x, y) \) and divergence-free \( \nabla \times (\psi(x, y)\hat{z}) \) vector fields. However, the decomposition for the polarization field is not obvious, and it needs to be guided by some principles of spatial symmetry.

We shall first examine the familiar vector field in an attempt to extracting some guiding principles. Consider first the parity symmetry. The spatial inversion \( (x, y) \rightarrow (-x, -y) \)
yields $\nabla \phi \rightarrow -\nabla \phi$ and $\nabla \times (\psi \hat{z}) \rightarrow -\nabla \times (\psi \hat{z})$, and hence such an operation can not distinguish the two components. In fact, the visual distinction of a diverging pattern from a swirling pattern arises from partial spatial inversion, $x \rightarrow -x$ or $y \rightarrow -y$. The $x$ and $y$ components of a diverging vector field transform in accordance with how the two coordinates change, but those of a swirling vector field transform just oppositely; that is, flipping the sign of the $x$ coordinate changes only the sign of the $y$ component of the swirling vector and flipping the sign of the $y$ coordinate changes only the sign of the $x$ component. The partial space symmetry clearly shows the difference of a two-dimensional vector (the gradient) from a two-dimensional pseudo-vector (the curl).

A traceless second-rank tensor can be regarded as a bilinear combination of two vectors, and hence it is possible to construct a vector from a tensor, where the constructed vector contains the topological features of the original tensor. To see how this is possible, we now consider the polarization tensor specifically.

3. Topological Characteristics in the Polarization Tensor Field

Let the measured electric field be

$$\langle E_i(x)E_j(x) \rangle = I_0(x)\delta_{ij} + Q(x)\sigma_1 + U(x)\sigma_2,$$

(1)

where $I_0$ describes the anisotropy intensity and the traceless polarization tensor $P$ is described by the last two terms on the right. We may also express the measured electric field equally well as

$$\langle E_i(x)E_j(x) \rangle = I'_0(x)\delta_{ij} + 2p_i(x)p_j(x),$$

(2)

where $p(x)$ is a two-dimensional vector field. Similar to Eq.(1), $\langle E_iE_j \rangle$ in Eq.(2) is also characterized by three independent parameters for a general $2 \times 2$ symmetric real tensor, i.e., the unpolarized intensity $I'_0 = I_0 - (p_x^2 + p_y^2)$, and the linearly polarized components
$Q = p_x^2 - p_y^2$ and $U = 2p_x p_y$.

Though such a decomposition of a tensor field $P$ into a bilinear combination of a vector $p$ is legitimate, $p$ is not unique since $-p$ can be an equally good choice, and this reflects the bi-directional nature of the polarization of $p$. It is therefore of crucial importance to determine whether the polarization tensor $P$ is the primary field which can mathematically be decomposed into the outer product of two vectors, or the vector field $p$ is the primary field which, with a bilinear combination, can form a second-rank tensor. For the latter, the vector $p$ and the tensor $P$ are both well-defined, but for the former, $p$ is an ambiguous bi-vector. In the CMBR, it is generally believed that the polarization tensor, i.e., the Stokes $Q$ and $U$, is the primary field produced by the temperature quadrupole anisotropy. That is, the bi-vector $p$ is secondary, constructed from the primary tensor $P$. Whether the $p$ field is secondary or not can in fact be tested by examining whether there exist singularities in $p$ (Naselsky and Novikov 1998). This is because the primary tensor field $P$ vanishes locally most likely in a linear manner, i.e., non-vanishing first derivatives at the nulls; it therefore demands $p$ to vanish non-analytically due to the bilinear combination of $p$, thus resulting in the branch points in $p$.

With this understanding of the CMB polarization field $P$, it becomes clear that using the vector $p$ to describe $P$, as written in Eq.(2), can not be appropriate, and it calls for a suitable representation of the second-rank $P$ tensor field. To avoid any singularity, this representation should be linear with $P$, and the only possibility to construct a tensor via linear operations is through the spatial derivatives that act as vectors. One may construct such a second-rank tensor either by two spatial derivatives acting on a scalar field, or by one spatial derivative acting on a vector field. In two-dimensions, a vector field can always be represented by one spatial derivative on the scalar fields as shown above, and the two constructions are therefore equivalent. From now on, we adopt the former. The most
general representation of a traceless tensor field of this kind in a flat two-dimensional space can be written as:

\[ P(x, y) = \nabla \nabla - (\hat{z} \times \nabla)(\hat{z} \times \nabla) \] \( f(x, y) + [\hat{z} \times \nabla + \nabla(\hat{z} \times \nabla)] g(x, y) \). \tag{3} \]

where \( f \) and \( g \) are two independent scalar-field components. Much like the two independent topological patterns in a vector field, these two components in the tensor field \( P \) also have their own topological characteristics; the former is called the \( E \) mode and the latter the \( B \) mode. The different spatial partial symmetries are revealed in the difference in \( \nabla \) and \( \hat{z} \times \nabla \). Having this expression of \( P \), it becomes possible to construct a two-dimensional vector from the tensor \( P \) by contracting it with a vector differential operator, either with a divergence operator, a curl operator or a combination of both. However, not all vector differential operators are able to capture the topological features of the tensor; only a unique choice of vector differential operator is.

A convenient and concise way to reveal such a choice of vector differential operator is in the complex representation, where

\[ P(x, y) = Q(x, y) + iU(x, y) = \frac{\partial^2}{\partial \bar{z}^2} [f(x, y) + ig(x, y)], \tag{4} \]

with \( \bar{z} \equiv x + iy \) and \( \bar{z} \equiv x - iy \). This representation agrees with the \( E \) (associated with \( f \)) and \( B \) (associated with \( g \)) modes defined above and those defined in the Fourier space (White et al. 1999). It also shows clearly how \( \partial/\partial \bar{z} \) may act onto \( P \) to form a vector field \( \mathbf{V} \):

\[ \mathbf{V} \equiv \frac{\partial}{\partial \bar{z}} \mathbf{P} = \frac{\partial}{\partial \bar{z}} [\nabla^2 (f + ig)] = (\frac{\partial Q}{\partial x} + \frac{\partial U}{\partial y}) + i(\frac{\partial U}{\partial x} - \frac{\partial Q}{\partial y}), \tag{5} \]

where \( \partial^2/\partial z \partial \bar{z} = \nabla^2 \) has been used and the last equality is given by the definition \( \mathbf{P} \equiv Q + iU \). The first equality shows clearly that when \( g = 0 \), the resulting \( \mathbf{V} \) is a diverging vector, and when \( f = 0 \), \( \mathbf{V} \) a swirling vector. Thus, the two topologically different characteristics of a vector field is also contained in the polarization tensor field through the components of \( f \) and \( g \).
4. Separation of $E$ and $B$ Modes

In conventional vector notation, Eq.(5) is re-written as

$$V(x,y) = \hat{x}(\frac{\partial Q(x,y)}{\partial x}) + \hat{y}(\frac{\partial U(x,y)}{\partial y}) = \nabla \phi(x,y) + \hat{z} \times \nabla \psi(x,y), \quad (6)$$

where $\phi \equiv \nabla^2 f$ and $\psi \equiv \nabla^2 g$. The real-space separation of the $E$ and $B$ modes may be conducted by employing suitable closed-loop integrations. Namely, using the Stokes theorem for a closed-line integral,

$$\hat{z} \cdot \int V(x,y) \times dl = \int \nabla^2 \phi(x,y) d^2 S, \quad (7)$$

it projects the $E$-mode "charge" (or $\nabla^2 \phi$) averaged over a top-hat filter enclosed by the loop. On the other hand, the closed-line integral

$$\int V(x,y) \cdot dl = \int \nabla^2 \psi(x,y) d^2 S \quad (8)$$

projects the top-hat-filter averaged $B$-mode "current" (or $\nabla^2 \psi$). The variances of the top-hat-filter averaged "charge" and "current" for various filter sizes can serve as alternative representations in place of the power spectra of $E$ and $B$ modes. This representation has an advantage over the Fourier spectral representation and will be discussed in the next section.

It is worth noting that though the construction of $V$ from the polarization tensor $P$ involves one spatial derivative, which amplifies the measurement shot noises, the proposed projection of either mode involves one spatial integration. The two operations formally compensate, though in actual operations some small level of noises can be introduced.

5. Discussions

In this work, we show the topological natures of the $E$ and $B$ modes by suitably projecting the polarization tensor into a vector. The $E$ and $B$ modes correspond to the
diverging and swirling patterns of this vector, respectively. Having them identified in the real space, an integral method then becomes a natural choice for separating these two modes, as such a method can minimize amplification of the shot noises inherent in the CMBR measurements.

Moreover, this real-space analysis of CMBR polarization can retain the characteristics of $E$ and $B$ even when the measurements are subject to the detector beam smearing, giving rise to the convolution of the sky image with the detector beam pattern. The measured quantities are the Stokes $Q$ and $U$, and when they are subject to identical beam smearing, they become $\langle Q \rangle$ and $\langle U \rangle$, where

$$\langle A \rangle(x) \equiv \int W(x + r)A(r)d^2r \quad (9)$$

with $W$ being the beam pattern. It is easy to see that if $A(r) = \partial a(r)/\partial r$, the smeared quantity $\langle A \rangle(x) = \partial \langle a \rangle/\partial x$, which retains the same differential form, and it does so even when higher derivatives are involved. That is, the tensor described by Eq.(4) can retain the same topological nature with beam smearing. Moreover since Eq.(4) is linear in $f$ and $g$ and the convolution is a linear operation, it follows that the $E$ and $B$ modes can not be mixed by beam smearing.

The pixelization of image is another issue in the real-space analysis, and it involves a particular form of the window function $W(x, r) = \sum_i \delta(x - x_i)W_0(x + r)$. The Stokes $\langle Q \rangle$ and $\langle U \rangle$ can also maintain their two components of distinct topologies unmixed. However, further manipulations with differentiation, approximated by differencing, and integration, approximated by summation, should proceed with caution to keep $E$ and $B$ intact. We will leave this technical issue to a future paper.

As mentioned in the last section, the projection with a closed-loop line integral corresponds to a top-hat-filter average of the polarization signal. The Fourier spectrum of this filter function is proportional to $J_1^2(kR)$, which oscillates in the $k$ space with a
frequency comparable to the filter size $R$, where $J_1$ is the Bessel function of order one and $k$ the Fourier wavenumber. It is noted that in the CDM cosmology, with or without the cosmological constant, the power spectrum of the $E$ polarization also oscillates with a well-defined period in the $k$-space. It is therefore anticipated that by varying the radius $R$ of a circular integration loop of Eq.(7), the frequency of the filter power spectrum can be made to match that of the $E$-mode power spectrum, where the response to polarization signals is maximal. Such an analysis strategy has not been considered, or even mentioned, in the past, mainly because most attention has so far been directed toward the Fourier-space analysis of CMBR signals.

The Fourier-space analysis can be good methodology for analyzing the CMBR temperature anisotropy, since the acoustic oscillations in the spectrum have an ill-defined $k$-space frequency, due to the changing Hubble radius when the acoustic waves entered the horizon. However, the polarization in the CMBR results from a much cleaner physics than the temperature anisotropy does. The polarization involves only the Thomson scattering of anisotropic photons at the epoch when the mean-free-path was about the then Hubble radius. Any structure of size smaller than the then Hubble radius is smeared out by the large Thomson mean-free-path, and any structure of size greater than the then Hubble radius has a negligible power. This characteristic Hubble sphere enters the phase of the perturbation, and naturally gives rise an oscillation in the power spectrum with a well-defined frequency comparable to the then Hubble radius. It is therefore of little surprise that a real-space top-hat filter function can capture the Hubble spheres and be well suited for detecting the $E$ polarization signals. A detailed analysis of how the presently proposed method performs for detecting the CMBR polarization will be reported elsewhere.

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REFERENCES

Bond, J.R. & Efstathiou, G. 1984, ApJ., 285, L45

Goldberg, J.N., Macfarlane, A.J., Newman, E.T., Rohrlich, F. & Sudashan, E.C.G. 1967, Math. Phys. 8, 2155

Hu, W. & White M. 1997a, Phys. Rev. D, 56, 596

Hu, W. & White M. 1997b, New Astron., 2, 323

Kamionkowsky, M, Kosowsky, A.& Stebbins, A. 1997, Phys. Rev. D55, 7368

Lo, K.Y., Chiueh, T., Liang, H., Ma, C.P., Martin, R., Ng, K.-W., Pen, U.L. & Subramanyan, R. 2000, IAU Symp. no. 201, 31

Naselsky, P.D. & Novikov, D.I. 1998, ApJ, 507, 31

Polnarev, A.G. 1985, Sov. Astron., 29, 607

Rees, M.J. 1968, ApJ, 153, L1

Seljek, U., Pen, U.L. & Spergel D.N. 1997, Phys. Rev. Lett., 79, 1615

Staggs, S.T., Gunderson, J.O. & Church, S.E., astro-ph/9904062

White, M., Carlstrom, J. E., Dragovan, M., Holzapfel, W. L. 1999, ApJ, 514, 12

Zaldarriaga, M. & Seljek, U. 1997, Phys. Rev. D55, 1830

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