Remarks on the Decay Rate of the Energy for Damped Modified Boussinesq-Beam Equations on the 1-D Half Line

By

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Abstract. We consider the mixed problem for weakly damped modified Boussinesq-Beam equations on the one dimensional half line \( (0, +\infty) \). We shall derive fast decay results of the total energy and \( L^2 \)-norm of solutions based on the idea due to [7], which is an essential modification of that developed by Morawetz [15]. In order to apply that idea due to [7] to the one dimensional exterior mixed problem, one also constructs an important Hardy-Sobolev type inequality, which holds only in the 1-D half line case.

Key Words and Phrases. Damping term, Modified Boussinesq equation, Beam equation, 1-D Half line, Multiplier method, Total energy decay, Weighted initial data.

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1. Introduction

We consider the mixed problem for the following linear modified Boussinesq Equation with constant damping in the half line \( (0, \infty) \):

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(t, x) - \gamma \frac{\partial^4 u}{\partial x^4}(t, x) + \frac{\partial^4 u}{\partial x^4}(t, x) + \frac{\partial u}{\partial t}(t, x) &= 0, \\
(t, x) &\in (0, \infty) \times (0, +\infty), \\
\end{align*}
\]

\[
\begin{align*}
u(0, x) &= u_0(x), \\
u_t(0, x) &= u_1(x), \\
\frac{\partial u}{\partial x}(t, 0) &= 0, \\
t &> 0,
\end{align*}
\]

where the initial data \([u_0, u_1]\) are chosen as

\[
u_0 \in \{ v \in H^3(0, \infty) \mid v(0) = v_{xx}(0) = 0 \}, \quad u_1 \in H^2(0, \infty) \cap H^1_0(0, \infty)
\]

in order to get a unique existence result of the weak solution. The constant coefficient \( \gamma \) is non negative. For \( \gamma > 0 \) the equation (1.1) is related to the modified Boussinesq equation. When \( \gamma = 0 \) the equation (1.1) is the well-known Beam equation.

Let \( H := L^2(0, \infty) \), and we define the operator \( A : D(A) \subset H \to H \) by

\[
D(A) := H^2(0, \infty) \cap H^1_0(0, \infty), \quad Au := -u_{xx} \quad (u \in D(A)).
\]
It is well-known that the operator $A$ is nonnegative and self-adjoint in $H$, and the fractional Laplacian $A^s (s \geq 0)$ can be well-defined in $H$, and also becomes nonnegative and self-adjoint. Furthermore, it is also known (see Fujiwara [4] and/or Grisvard [5]) that

$$D(A^{3/2}) = \{ v \in H^3(0, \infty) \mid v(0) = v_{xx}(0) = 0 \},$$

and

$$D(A^2) = \{ v \in H^4(0, \infty) \mid v(0) = v_{xx}(0) = 0 \}.$$

Under these preparations, by using a semi-group theory as in Amari-Damassi-Zerzeri [1] and especially in Luz-Charão [12] we may obtain a unique existence of the weak solution $u = u(t, x)$ for each initial data $[u_0, u_1] \in D(A^{3/2}) \times D(A)$, to problem $(1.1)-(1.3)$ in the class

$$u \in X(0, +\infty) := C([0, +\infty); D(A^{3/2})) \cap C^1([0, +\infty); D(A))$$

$$\cap C^2([0, +\infty); D(A^{1/2}))$$

satisfying the variational form:

$$\ll(u_{tt}(t, \cdot), \psi) + \gamma(u_{ttt}(t, \cdot), \psi_x) + (u_{xx}(t, \cdot), \psi_{xx}) + (u_t(t, \cdot), \psi) = 0, \quad t > 0$$

for all $\psi \in D(A)$.

For more regular initial data $[u_0, u_1] \in D(A^2) \times D(A^{3/2})$ the problem admits also a unique strong solution in the class

$$X_1(0, +\infty) := C([0, +\infty); D(A^2)) \cap C^1([0, +\infty); D(A^{3/2}))$$

$$\cap C^2([0, +\infty); D(A)).$$

The purpose in this remark is to introduce new decay estimates for the $L^2$-norm and the total energy of solutions to the mixed problem $(1.1)-(1.3)$, which is considered in the 1-D half line $(0, \infty)$. For this end, we have developed a new Hardy-Sobolev type inequality as stated in Lemma 3.1, which is true only for the 1-dimensional half space case. This inequality stated in Lemma 3.1 seems to be unknown until now. Based on a multiplier method developed in [7], which is a modification of the one due to Morawetz [15] combined with this Lemma 3.1, one can derive the desired fast decay estimates of solutions to problem $(1.1)-(1.3)$. We note that due to our method in this article we obtain decay estimates for the $L^2$-norm of solutions to a problem in the one dimensional case. A generalization of the equation $(1.1)$ to the higher dimensional case is the plate equation. For example, in Luz-Charão [12], Charão-Luz-Ikehata [3] and Luz-Ikehata-Charão [13] the authors obtained several decay estimates for the $L^2$-norm of solutions to the plate equation.
in the unbounded domain only for the higher dimensional case strictly bigger than 4. Especially, in [12, Theorem 3.10] they studied a fast decay property of the solution to the Cauchy problem of the plate equation in the whole space $\mathbb{R}^n$ only for $n \geq 5$:

$$u_{tt}(t, x) - \gamma Au(t, x) + \Delta^2 u(t, x) + u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.$$  

In fact, Luz-Charão [12] considered the corresponding semi-linear problem, and applied the obtained results for the linear equation to the corresponding semi-linear problems. Anyway, the low dimensional case was handled by Sugitani-Kawashima [19] and they obtained several types of estimates such as $\|u(\cdot, t)\|^2 \sim t^{-1/4}$, $\|u_x(\cdot, t)\|^2 \sim t^{-3/4}$ and $E(t) \sim t^{-5/4}$ (as $t \to +\infty$) for the one dimensional case. So, our results in Section 2 improve these and this fact was one of our motivations to deal with at least one dimensional unbounded domain case $(0, \infty)$ as one of examples of the important low dimensional case.

On the usual damped wave equation case, which was considered in the 1-D half line, Ikehata [6] derived fast decay estimates of the $L^2$-norm and total energy of solutions to the mixed problem of the equation:

$$(1.6) \quad u_{tt}(t, x) - u_{xx}(t, x) + u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, +\infty).$$

In order to obtain the desired decay estimates of solutions of (1.6) in the 1-D half line he constructed a Hardy-Sobolev type inequality such as

$$(1.7) \quad \sup_{x \geq 0} \frac{|u(x)|}{(1 + x)^{1/2}} \leq \|u_x\|, \quad u \in H^1_0(0, \infty).$$

So, we shall employ the same story as in [6] to obtain decay estimates of the solution to the beam equation by replacing the inequality (1.7) by the one stated in Lemma 3.1.

Moreover, for the Cauchy or exterior mixed problems in $\mathbb{R}^n$ or in the outside of some obstacle respectively to the plate and/or beam equations several mathematicians have already studied some decay estimates of the linear and nonlinear problems, and for these topics and the related results one can cite Jiang [9], Lin [11], Pausader [16], Racke [18], Takeda [20], Takeda-Yoshikawa [21], Ikehata-Soga [8] and the references therein. About the generalized Bq and IBq equations with some damping Wang-Xu [23], Wang [22] and Wang-Xue [24] investigated decay estimates of some norms of solutions and its application to nonlinear problems.

This paper is organized as follows. In section 2 we shall state our main results as Theorems 2.1 and 2.2. In section 3 we shall introduce a new Hardy-Sobolev type inequality as Lemma 3.1 and the proof of Theorem 2.1 will be
also given by relying only on a multiplier method which was introduced by the author’s collaborative work [7]. The result of Theorem 2.2 is a direct consequence of Theorem 2.1.

2. Notation and results

Throughout this paper we use the standard notation of Sobolev Spaces. For example, \( \| \cdot \|_q \) stands for the usual \( L^q(0, \infty) \)-norm. For simplicity of notation, in particular we use \( \| \cdot \| \) instead of \( \| \cdot \|_2 \). The inner product in \( L^2(0, \infty) \) is denoted by \( (\cdot, \cdot) \).

The total energy \( E(t) \) for the solution \( u(t, x) \) of (1.1) is denoted by

\[
E(t) = E(u(t, \cdot)) := \frac{1}{2} \left( \| u_t(t, \cdot) \|^2 + \gamma \| u_{xt}(t, \cdot) \|^2 + \| u_{xx}(t, \cdot) \|^2 \right)
\]

and the high order energies are defined by

\[
E_1(t) = E_1(u(t, \cdot)) := \frac{1}{2} \left( \| u_{xt}(t, \cdot) \|^2 + \gamma \| u_{xxt}(t, \cdot) \|^2 + \| u_{xxx}(t, \cdot) \|^2 \right)
\]

\[
E_2(t) = E_2(u(t, \cdot)) := \frac{1}{2} \left( \| u_{xxx}(t, \cdot) \|^2 + \gamma \| u_{xxxx}(t, \cdot) \|^2 + \| u_{xxxxx}(t, \cdot) \|^2 \right)
\]

and one sets

\[
\| u \|_{1, x} := \int_0^\infty (1 + x)^\gamma |u(x)|dx,
\]

\[
u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2},
\]

\[
u_{xxxx} = \frac{\partial^4 u}{\partial x^4}, \quad u_{xt} = \frac{\partial^3 u}{\partial x^2 \partial t}, \quad u_{xxt} = \frac{\partial^4 u}{\partial x^2 \partial t^2}.
\]

We also use the following numbers depending on the initial data

\[
K_0 := I_0 + \frac{1}{2} \| u_0 + u_1 - \gamma(u_1)_{xx} \|_{1, 3/2}^2.
\]

Furthermore, one define a series of constants:

\[
K_1 := \frac{3}{2} E(0) + \frac{\gamma}{2} E_1(0) + \frac{1}{2} I_1,
\]

\[
K_2 := \frac{3}{2} E_1(0) + \frac{\gamma}{2} E_2(0) + \frac{1}{2} I_2,
\]

\[
K_3 := I_3 + \gamma^2 K_1.
\]
Our main result reads as follows.

**Theorem 2.1.** If the initial data \([u_0, u_1] \in D(A^2) \times D(A^{3/2})\) further satisfies

\[
\int_0^{+\infty} (1 + x)^{3/2}|u_0(x) + u_1(x) - \gamma(u_1)_{xx}(x)|\,dx < +\infty,
\]

then the unique solution \(u(t, x) \in X_1(0, +\infty)\) to problem (1.1)–(1.3) satisfies

\[
\|u(t, \cdot)\|^2 \leq I_{0, \gamma}^2(1 + t)^{-1}, \quad E(t) \leq J_{0, \gamma}^2(1 + t)^{-2}
\]

for \(t > 0\), where

\[
I_{0, \gamma}^2 := 8K_3,
\]

and

\[
J_{0, \gamma}^2 := E(0) + K_1 + \gamma K_2 + K_3.
\]

**Corollary 2.1.** Under the hypothesis as in Theorem 2.1 one has

\[
E_1(t) \leq K_2(1 + t)^{-1}, \quad \|u_x(\cdot, t)\|^2 \leq (4K_3 + K_1)(1 + t)^{-1}
\]

for \(t > 0\).

For the beam equation (1.1) with \(\gamma = 0\) we have the following result. For the result of the unique existence of the weak solution \(u(t, x)\) to problem (1.1)–(1.3) with \(\gamma = 0\), by a slight modification, one can refer the reader to [1, Proposition 1.1].

**Theorem 2.2.** Let \(\gamma = 0\) in (1.1). If the initial data \([u_0, u_1] \in (H^2(0, \infty) \cap H_0^1(0, \infty)) \times L^2(0, \infty)\), further satisfies

\[
\int_0^{+\infty} (1 + x)^{3/2}|u_0(x) + u_1(x)|\,dx < +\infty,
\]
then the unique solution \( u(t, x) \in C([0, \infty); H^2(0, \infty) \cap H^1_0(0, \infty)) \cap C^1([0, \infty); L^2(0, \infty)) \) to problem (1.1)–(1.3) satisfies

\[
\|u(t, \cdot)\|^2 \leq I_{0,0}^2 (1 + t)^{-1}, \quad E(t) \leq J_{0,0}^2 (1 + t)^{-2},
\]

where \( I_{0,0}^2 \) and \( J_{0,0}^2 \) are the numbers \( I_{0,\gamma}^2 \), \( J_{0,\gamma}^2 \) defined in Theorem 2.1 with \( \gamma = 0 \), that is

\[
I_{0,0}^2 = 2\left[8(u_1, u_0) + 16 E(0) + 6\|u_0\|^2 + 4\|u_0 + u_1\|_{1,3/2}^2\right],
\]

\[
J_{0,0}^2 = \frac{5}{2}(u_1, u_0) + 7E(0) + 2\|u_0\|^2 + \frac{5}{4}\|u_0 + u_1\|_{1,3/2}^2.
\]

**Remark 2.1.** As one of results of Takeda [20], he studied the decay property and the global existence of solutions for the Cauchy problem to the one dimensional nonlinear beam equation:

\[
u_{tt}(t, x) + uu_{xxxx}(t, x) + u_t(t, x) = f(u), \quad (t, x) \in (0, \infty) \times R.
\]

Takeda [20] derived the following decay estimate together with the global existence of solutions for the Cauchy problem of the equation above:

\[
\|u(t, \cdot)\|_{L^2(R)}^2 \leq C(1 + t)^{-1/4}
\]

in the case when the initial data \([u_0, u_1] \in (H^2(R) \cap W^{2,1}(R)) \times (L^2(R) \cap L^1(R))\) are small enough. So, the results about the decay order of Theorem 2.2 may catch a kind of property from the viewpoint of the half space case.

**Remark 2.2.** The similar result to Theorem 2.1 can be also derived with a slight modification to the linear damped modified Boussinesq equation with a variable coefficient:

\[
u_{tt}(t, x) - \gamma u_{xxx}(t, x) + uu_{xxxx}(t, x) + V(x)u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times (0, +\infty),
\]

where \( V \in L^\infty(0, \infty) \) satisfies \( V_0 \leq V(x) \ (x \in (0, +\infty)) \) with some \( V_0 > 0 \).

Throughout this paper we shall rely on the following result.

**Proposition 2.1.** The unique solution \( u(t, x) \in X(0, +\infty) \) to problem (1.1)–(1.3) satisfies

\[
E(t) + \int_{0}^{t} \|u_\xi(s, \cdot)\|^2\,ds = E(0),
\]

\[
\frac{d}{dt}(u_t(t, \cdot), u(t, \cdot)) - \|u_t(t, \cdot)\|^2 + \gamma \frac{d}{dt}(u_{xx}(t, \cdot), u_x(t, \cdot))
\]

\[
- \gamma \|u_{xx}(t, \cdot)\|^2 + \|u_{xx}(t, \cdot)\|^2 + \frac{1}{2} \frac{d}{dt}\|u(t, \cdot)\|^2 = 0.
\]
3. Energy lemmas

We first introduce the following Hardy-Sobolev type inequality, which plays an important role throughout this paper. One believes that this result itself seems new.

Lemma 3.1. It is true that

$$\sup_{x \geq 0} \frac{|v(x)|}{(1 + x)^{3/2}} \leq \|v_{xx}\|$$

for all $v \in H^2(0, \infty) \cap H^1_0(0, \infty)$.

Proof. It suffices to prove this inequality in the case when $v \in C_0^\infty((0, \infty))$ (c.f., Pazy [17, Lemma 3.1 of section 8.3] and Kato [10, pp. 430, Example 1.11]). We can also assume that $v(0) = v'(0) = 0$, where we use the notation $v'(x) := v_x(x)$ and $v''(x) := v_{xx}(x)$. Then, it follows from the fundamental theorem of calculus and the Schwarz inequality that

$$|v'(x)| \leq \int_0^x |v''(r)|dr \leq \sqrt{1 + x}\|v''\|, \quad x \geq 0. \tag{3.1}$$

On the other hand, from the mean value theorem it follows that

$$\frac{v(x)}{x} = v'\left(\xi\right) \tag{3.2}$$

with some $\xi \in (0, x)$. Therefore, it follows from (3.1) with $x = \xi$ that

$$|v'(\xi)| \leq \sqrt{1 + \xi}\|v''\| \leq \sqrt{1 + x}\|v''\|, \quad x > 0. \tag{3.3}$$

Thus, (3.2) and (3.3) imply

$$\frac{|v(x)|}{x} = |v'(\xi)| \leq \sqrt{1 + x}\|v''\|, \quad x > 0,$$

so that one gets

$$|v(x)| \leq x\sqrt{1 + x}\|v''\| \leq \sqrt{(1 + x)^3}\|v''\|. \tag{3.4}$$

In particular, (3.4) holds good even for $x = 0$. Therefore, one can get the desired estimate for all $v \in C_0^\infty((0, \infty))$:

$$\frac{|v(x)|}{(1 + x)^{3/2}} \leq \|v''\|, \quad (x \geq 0). \tag{3.5}$$
By the density argument, (3.5) holds true for all \( v \in H^2(0, \infty) \cap H^1_0(0, \infty) \). In this connection it is well-known that \( v \in H^2(0, \infty) \cap H^1_0(0, \infty) \) implies \( v \in C((0, \infty)) \).

**Remark 3.1.** In connection with Lemma 3.1 one can present a conjecture, which is a generalization of that of Lemma 3.1: for each \( n \in \mathbb{N} \) there exists a constant \( C_n > 0 \) (possibly \( C_n = 1 \)) such that

\[
\sup_{x \geq 0} \frac{|v(x)|}{(1 + x)^{(2n-1)/2}} \leq C_n \|v^{(n)}\|,
\]

\( \forall v \in D(A^{n/2}) = \left\{ \phi \in H^n(0, \infty) \mid \phi^{(2k)}(0) = 0 \right\} \)

(\( \forall k \in \{0\} \cup \mathbb{N} \) satisfying \( 0 \leq k < \frac{2n-1}{4} \))

(for the characterization of \( D(A^{n/2}) \), see [4] and/or [5]). This is still open.

One prepares one more useful lemma, which can be derived based on Lemma 3.1 and the modified method of [15] coming from [7].

**Lemma 3.2.** Let \( u(t, x) \in X(0, +\infty) \) be the unique weak solution to problem (1.1)–(1.3) with initial data satisfying condition (2.1) in Theorem 2.1. Then it holds that for \( t > 0 \),

\[
\frac{1}{2} \|u(t, \cdot)\|^2 + \frac{\gamma}{2} \|u_x(t, \cdot)\|^2 + \int_0^t \|u(s, \cdot)\|^2 \, ds
\]

\[
\leq I_0 + \frac{1}{2} \left\{ \int_0^{+\infty} (1 + x)^{3/2} |u_0(x) + u_1(x) - \gamma(u_1)_{xx}(x)| \, dx \right\}^2 = K_0.
\]

**Proof.** We introduce an auxiliary function

\[
w(t, x) := \int_0^t u(s, x) \, ds.
\]

Then \( w_t = u \) and \( w(t, x) \) satisfies

\[
\begin{align*}
(3.6) \quad & w_{tt} - \gamma w_{xxxx} + w_{xxxxx} + w_t = u_0 + u_1 - \gamma(u_1)_{xx}, \quad (t, x) \in (0, \infty) \times (0, \infty), \\
(3.7) \quad & w(0, x) = 0, \quad w_t(0, x) = u_0(x), \quad x \in (0, \infty), \\
(3.8) \quad & w(t, 0) = w_{xx}(t, 0) = 0, \quad t \in (0, \infty).
\end{align*}
\]

Multiplying (3.6) by \( w_t \) and integrating over \( [0, t] \times [0, \infty) \), because of (3.7)–(3.8) we get
(3.9) \[
\frac{1}{2} \left( \|w(t, \cdot)\|^2 + \gamma \|w_x(t, \cdot)\|^2 + \|w_{xx}(t, \cdot)\|^2 \right) + \int_0^t \|w(s, \cdot)\|^2 ds
\]
\[
= \frac{1}{2} \left( \|u_0\|^2 + \gamma \|(u_0)_x\|^2 \right) + \int_0^t (u_0 + u_1 - \gamma (u_1)_{xx}, w(s, \cdot)) ds
\]
due to \( w_{xx}(0, x) = 0 \) and \( w_{xx}(0, x) = u_x(0, x) = (u_0)_x \).

Next step is to use Lemma 3.1 to obtain the final estimate:

(3.10) \[
\int_0^t (u_0 + u_1 - \gamma (u_1)_{xx}, w(s, \cdot)) ds
\]
\[
= \int_0^t \frac{d}{ds} (u_0 + u_1 - \gamma (u_1)_{xx}, w(s, \cdot)) ds
\]
\[
\leq \int_0^\infty \sqrt{(1 + x)^3} |u_0 + u_1 - \gamma (u_1)_{xx}| \frac{|w(t, x)|}{\sqrt{(1 + x)^3}} dx
\]
\[
\leq \left( \sup_{x \in [0, \infty)} \frac{|w(t, x)|}{\sqrt{(1 + x)^3}} \right) \|u_0 + u_1 - \gamma (u_1)_{xx}\|_{1,3/2}
\]
\[
\leq \frac{1}{2} \|u_0 + u_1 - \gamma (u_1)_{xx}\|^2 + \frac{1}{2} \|u_0 + u_1 - \gamma (u_1)_{xx}\|^2.
\]

Combining (3.9) and (3.10) we can derive

\[
\frac{1}{2} \left( \|w_t(t, \cdot)\|^2 + \gamma \|w_{xt}(t, \cdot)\|^2 + \int_0^t \|w_x(s, \cdot)\|^2 ds \right)
\]
\[
\leq \frac{1}{2} \left( \|u_0\|^2 + \gamma \|(u_0)_x\|^2 \right) + \frac{1}{2} \|u_0 + u_1 - \gamma (u_1)_{xx}\|^2,
\]

which implies the desired estimate because of the fact that \( w_t = u \).

The following lemma is useful to get energy estimates.

**Lemma 3.3.** The weak solution \( u(t, x) \in X(0, +\infty) \) to problem (1.1)–(1.3) satisfies the identities

(i) \( E_j(t) + \int_0^t \|D^j u(s, \cdot)\|^2 ds = E_j(0) \) \quad \( j = 1, 2 \),

(ii) \( (1 + t)E_j(t) + \int_0^t (1 + s) \|D^j u(s, \cdot)\|^2 ds = E_j(0) + \int_0^t E_j(s) ds \) \quad \( j = 0, 1 \),
where \( D^j := \frac{\partial^j}{\partial x^j} \) (\( j = 1, 2 \)), \( D^0 u_s := u_s \) and \( E_0(t) := E(t) \). Moreover, the identity (i) for \( j = 2 \) holds for the strong solution.

**Proof.** It is standard. Use the multiplier \((1 + t)u_t\) to get (ii) for \( j = 0 \). Take \( \psi = u_{xxx} \) in (1.4) for the case (i) with \( j = 1 \) and \( \psi = (1 + t)u_{xxx} \) for the case (ii) with \( j = 1 \). To obtain (i) for \( j = 2 \) it suffices to choose \( \psi = u_{xxxx} \) in (1.4).

**Lemma 3.4.** Let \( u(t, x) \in X(0, +\infty) \) be the weak solution to problem (1.1)–(1.3) with initial data \([u_0, u_1] \) satisfying the assumption (2.1) as in Theorem 2.1. Then it holds that

\[
\int_0^t \|u_{xx}(s, \cdot)\|^2 ds \leq I_1 \quad (t > 0).
\]

**Proof.** Integrating the identity (2.3) over \([0, t]\) we have

\[
(u_t(t, \cdot), u(t, \cdot)) - (u_1, u_0) - \int_0^t \|u_s(s, \cdot)\|^2 ds + \gamma(u_{xx}(t, \cdot), u_x(t, \cdot)) - \gamma'(u_1)(u_0)s
\]

for \( t > 0 \). Then by using the Schwarz inequality one has

\[
\frac{1}{2} \|u(t, \cdot)\|^2 + \int_0^t \|u_{xx}(s, \cdot)\|^2 ds
\]

\[
\leq (u_1, u_0) + \gamma'(u_1)(u_0)s + \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \|u_t(t, \cdot)\|^2 + \frac{1}{2} \|u(t, \cdot)\|^2
\]

\[
+ \int_0^t \|u_s(s, \cdot)\|^2 ds + \gamma \int_0^t \|u_{xx}(s, \cdot)\|^2 ds + \frac{\gamma}{2} \|u_{xx}(t, \cdot)\|^2 + \frac{\gamma}{2} \|u_x(t, \cdot)\|^2
\]

for \( t > 0 \), or equivalently,

\[
\int_0^t \|u_{xx}(s, \cdot)\|^2 ds \leq (u_1, u_0) + \gamma'(u_1)(u_0)s + \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \|u_t(t, \cdot)\|^2
\]

\[
+ \frac{\gamma}{2} \|u_{xx}(t, \cdot)\|^2 + \frac{\gamma}{2} \|u_x(t, \cdot)\|^2 + \int_0^t \|u_s(s, \cdot)\|^2 ds
\]

\[
+ \gamma \int_0^t \|u_{xx}(s, \cdot)\|^2 ds,
\]

for \( t > 0 \).

Now, we use the estimate

\[
\frac{1}{2} \|u_t(t, \cdot)\|^2 + \frac{\gamma}{2} \|u_{xx}(t, \cdot)\|^2 + \int_0^t \|u_s(s, \cdot)\|^2 ds \leq E(0)
\]
due to the identity of the energy (2.2) in Proposition 2.1. Then we have
\[ \int_0^t \| u_{xx}(s, \cdot) \|^2 ds \leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) \]
\[ + \frac{1}{2} \| u_0 \|^2 + E(0) + \frac{\gamma}{2} \| u_x(t, \cdot) \|^2 + \gamma \int_0^t \| u_{xx}(s, \cdot) \|^2 ds, \quad t > 0. \]

By Lemma 3.2 we have an estimate for the norm \((\gamma/2)\| u_x(t, \cdot) \|^2\). Thus we obtain
\[ \int_0^t \| u_{xx}(s, \cdot) \|^2 ds \leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) + \frac{1}{2} \| u_0 \|^2 + E(0) + I_0 \]
\[ + \frac{1}{2} \| u_0 + u_1 - \gamma(u_1)_{xx} \|^2 \_{1,3/2} + \gamma \int_0^t \| u_{xx}(s, \cdot) \|^2 ds, \quad t > 0. \]

The proof of lemma follows from the estimate (i) for \(j = 1\) in Lemma 3.3.

**Lemma 3.5.** Let \(u(t, x) \in X(0, +\infty)\) be the weak solution to problem (1.1)–(1.3) with initial data \([u_0, u_1]\) satisfying the assumption (2.1) in Theorem 2.1. Then it holds that

(i) \[ \int_0^t E(s) ds \leq \frac{1}{2} E(0) + \frac{\gamma}{2} E_1(0) + \frac{1}{2} I_1, \]

(ii) \[ (1 + t) E(t) \leq \frac{3}{2} E(0) + \frac{\gamma}{2} E_1(0) + \frac{1}{2} I_1 : = K_1, \]

(iii) \[ \int_0^t (1 + s) \| u_x(s, \cdot) \|^2 ds \leq \frac{3}{2} E(0) + \frac{\gamma}{2} E_1(0) + \frac{1}{2} I_1 = K_1. \]

**Proof.** By the identity (2.2) we have
\[ \int_0^t \| u_x(s, \cdot) \|^2 ds \leq E(0). \]

The item (i) for \(j = 1\) in Lemma 3.3 says that
\[ \int_0^t \| u_{xx}(s, \cdot) \|^2 ds \leq E_1(0). \]
Finally, by Lemma 3.4 we have the estimate
\[ \int_0^t \| u_{xx}(s, \cdot) \|^2 \, ds \leq I_1. \]

Then, by observing the definition of \( E(t) \) we find the item (i) of lemma is now proved. The items (ii) and (iii) follow from the identity (ii) for \( j = 0 \) in Lemma 3.3 and the item (i) just proved.

Lemma 3.6. Let \( u(t, x) \in X(0, + \infty) \) be the weak solution to problem (1.1)–(1.3) with initial data \( u_0, u_1 \) satisfying the assumption (2.1) in Theorem 2.1. Then it holds that

\[
\frac{1 + t}{4} \| u(t, \cdot) \|^2 + \int_0^t (1 + s) \| u_{xx}(s, \cdot) \|^2 \, ds \\
\leq (u_0, u_0) + \gamma((u_1)_x, (u_0)_x) + (1 + t) \| u_t(t, \cdot) \|^2 + \frac{\gamma(1 + t)}{2} \| u_{xx}(t, \cdot) \|^2 \\
+ \frac{\gamma(1 + t)}{2} \| u_x(t, \cdot) \|^2 + K_0 + \gamma \int_0^t (1 + s) \| u_{xx}(s, \cdot) \|^2 \, ds \\
+ \int_0^t (1 + s) \| u_t(s, \cdot) \|^2 \, ds.
\]

Proof. Multiplying the identity (2.3) by \( (1 + t) \) we get for \( t > 0 \)

\[
\frac{d}{dt} [(1 + t)(u_t, u)] - (u_t, u) + \frac{d}{dt} [\gamma(1 + t)(u_{xx}, u_x)] - \gamma(u_{xt}, u_x) - \gamma(1 + t) \| u_{xt} \|^2 \\
- (1 + t) \| u_t \|^2 + (1 + t) \| u_{xx} \|^2 + \frac{d}{dt} \left[ \frac{1 + t}{2} \| u \|^2 \right] - \frac{1}{2} \| u \|^2 = 0.
\]

Integrating over \([0, t]\) it follows that

\[
(1 + t)(u_t, u) - (u_1, u_0) - \int_0^t (u_{xx}, u) ds + \gamma(1 + t)(u_{xt}, u_x) - \gamma((u_1)_x, (u_0)_x) \\
- \gamma \int_0^t (u_{xx}, u_x) ds - \gamma \int_0^t (1 + s) \| u_{xx}(s, \cdot) \|^2 ds - \int_0^t (1 + s) \| u_x(s, \cdot) \|^2 \, ds \\
+ \int_0^t (1 + s) \| u_{xx}(s, \cdot) \|^2 \, ds + \frac{1 + t}{2} \| u \|^2 - \frac{1}{2} \| u_0 \|^2 - \frac{1}{2} \int_0^t \| u(s, \cdot) \|^2 \, ds = 0,
\]

for \( t > 0 \). So, from the Schwarz inequality one has
\[
\frac{1 + t}{2} \|u\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
\leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) + \frac{1}{2} \|u_0\|^2 + (1 + t) \|u_t\|^2 + \frac{1 + t}{4} \|u\|^2 \\
+ \gamma(1 + t) \|u_x\|^2 + \frac{1}{2} \int_0^t \|u(s, \cdot)\|^2 ds + \frac{1}{2} \int_0^t (u_x(s, \cdot), u(s, \cdot)) ds \\
+ \gamma \int_0^t (u_{xx}(s, \cdot), u_x(s, \cdot)) ds + \gamma \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds + \int_0^t (1 + s) \|u_x(s, \cdot)\|^2 ds.
\]

We note that
\[
\int_0^t (u_x(s, \cdot), u(s, \cdot)) ds = \frac{1}{2} \int_0^t \frac{d}{ds} \frac{\|u(s, \cdot)\|^2}{2} ds = \frac{\|u(t, \cdot)\|^2}{2} - \frac{\|u_0\|^2}{2}
\]
and
\[
\gamma \int_0^t (u_{xx}(s, \cdot), u_x(s, \cdot)) ds = \frac{\gamma}{2} \|u_x(t, \cdot)\|^2 - \frac{\gamma}{2} \|u_0\|^2.
\]
Thus it holds
\[
\frac{1 + t}{4} \|u\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
\leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) + (1 + t) \|u_t\|^2 \\
+ \gamma(1 + t) \|u_x\|^2 + \frac{\gamma(1 + t)}{2} \|u_x\|^2 + \frac{1}{2} \int_0^t \|u(s, \cdot)\|^2 ds + \frac{\|u\|^2}{2} \\
+ \frac{\gamma}{2} \|u_x\|^2 + \gamma \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds + \int_0^t (1 + s) \|u_x(s, \cdot)\|^2 ds.
\]

Now, by using the estimate
\[
\frac{1}{2} \|u\|^2 + \frac{\gamma}{2} \|u_x\|^2 + \int_0^t \|u(s, \cdot)\|^2 ds \leq K_0
\]
given in Lemma 3.2, one obtains the desired estimate:
\[
\frac{1 + t}{4} \|u\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
\leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) + (1 + t) \|u_t\|^2 + \frac{\gamma(1 + t)}{2} \|u_x\|^2 + \frac{\gamma(1 + t)}{2} \|u_x\|^2 \\
+ K_0 + \gamma \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds + \int_0^t (1 + s) \|u_x(s, \cdot)\|^2 ds.
\]
Lemma 3.7. Let \( u(t, x) \in X(0, +\infty) \) be the weak solution to problem (1.1)–(1.3) with initial data \([u_0, u_1]\) satisfying the assumption (2.1) in Theorem 2.1. Then it holds that
\[
\frac{1 + t}{4} \|u(t, \cdot)\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
\leq (u_1, u_0) + \gamma(u_1, u_0) + 3E(0) + \gamma E_1(0) + I_1 \\
+ K_0 + \frac{\gamma (1 + t)}{2} \|u_x(t, \cdot)\|^2 + \gamma \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds.
\]

**Proof.** By combining (ii) for \( j = 0 \) in Lemma 3.3 with the one in Lemma 3.6 one obtains
\[
\frac{1 + t}{4} \|u(t, \cdot)\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
\leq (u_1, u_0) + \gamma((u_1)_x, (u_0)_x) + 2E(0) \\
+ 2 \int_0^t E(s) ds + \gamma \frac{1 + t}{2} \|u_x(t, \cdot)\|^2 + K_0 + \gamma \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds.
\]

By using the estimate in Lemma 3.5 for the integral \( \int_0^t E(s) ds \), the statement now follows. \( \square \)

Next, we need to get the boundedness in terms of the initial data for the integral in the right hand side of the estimate in Lemma 3.7 and the term \((1 + t)\|u_x(t, \cdot)\|^2\). For this end we prepare the following equality and inequality.

From the estimate (ii) for \( j = 1 \) in Lemma 3.3 we first have
\[
(1 + t) E_1(t) + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \\
= E_1(0) + \frac{1}{2} \int_0^t (\|u_{xx}(s, \cdot)\|^2 + \gamma \|u_{xxx}(s, \cdot)\|^2 + \|u_{xxxx}(s, \cdot)\|^2) ds.
\]

The estimate (i) for \( j = 1 \) in Lemma 3.3 gives the second estimate
\[
\int_0^t \|u_{xx}(s, \cdot)\|^2 ds \leq E_1(0).
\]

In order to obtain a complete estimate for the right hand side of (3.11) we have to prepare the next two lemmas. To prove these lemmas we need to assume additional regularity on the initial data in order to have a strong solution \( u(t, x) \).
Lemma 3.8. Let $u(t,x) \in X_1(0, +\infty)$ be the strong solution in the class (1.5). Then the following estimate holds good:

$$\int_0^t \|u_{xxx}(s, \cdot)\|^2 ds \leq I_2 \quad (t > 0).$$

Proof. In the proof of this lemma we need the strong solution $u(t,x)$ because we use the estimate (i) for $j = 2$ in Lemma 3.3. To begin with, by taking $\psi = u_{xx}$ in (1.4) it follows that

$$\frac{d}{dt}(u_{xt}, u_x) - \gamma \frac{d}{dt}(u_{xxx}, u_{xx}) - \gamma \|u_{xxx}\|^2 + \|u_{xxx}\|^2 + \frac{d}{dt} \|u_x\|^2 = 0$$

for $t > 0$. The integration of the above identity under $[0,t]$, $t > 0$, implies that

$$\frac{1}{2} \|u_x\|^2 + \int_0^t \|u_{xxx}(s, \cdot)\|^2 ds = \frac{1}{2} \|u_0\|^2 + ((u_1)_x, (u_0)_x) + \gamma ((u_1)_{xx}, (u_0)_{xx}) - (u_{xt}, u_x)$$

$$- \gamma (u_{xxx}, u_{xx}) + \int_0^t \|u_{xx}(s, \cdot)\|^2 ds + \gamma \int_0^t \|u_{xxx}(s, \cdot)\|^2 ds$$

$$\leq \frac{1}{2} \|u_0\|^2 + ((u_1)_x, (u_0)_x) + \gamma ((u_1)_{xx}, (u_0)_{xx})$$

$$+ \frac{1}{2} \|u_{xt}\|^2 + \frac{1}{2} \|u_x\|^2 + \frac{\gamma^2}{2} \|u_{xxx}\|^2 + \frac{1}{2} \|u_{xx}\|^2$$

$$+ \int_0^t \|u_{xx}(s, \cdot)\|^2 ds + \gamma \int_0^t \|u_{xxx}(s, \cdot)\|^2 ds.$$

So, by using the energy identity (2.2) and (i) for $j = 1, 2$ in Lemma 3.3 we get

$$\int_0^t \|u_{xxx}(s, \cdot)\|^2 ds \leq \frac{1}{2} \|u_0\|^2 + ((u_1)_x, (u_0)_x) + \gamma ((u_1)_{xx}, (u_0)_{xx})$$

$$+ E(0) + E_1(0) + \gamma E_2(0)$$

for $t > 0$, which implies the desired estimate. \qed

By combining the estimate (3.12), the estimate (i) for $j = 2$ in Lemma 3.3 and Lemma 3.8 with (3.11) one obtains the following important estimate:

$$\frac{1 + t}{2} (\|u_{xt}(t, \cdot)\|^2 + \|u_{xxx}(t, \cdot)\|^2) + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds$$

$$\leq \frac{3}{2} E_1(0) + \frac{\gamma}{2} E_2(0) + \frac{1}{2} I_2 := K_2.$$
Now, combining the estimate (3.14) above and the one in Lemma 3.7 we have the following lemma.

**Lemma 3.9.** Under the hypothesis as in Theorem 2.1 it is true that

\[
\frac{1 + t}{4} \|u(t, \cdot)\|^2 + \int_0^t (1 + s) \|u_{xx}(s, \cdot)\|^2 ds \leq I_3 + \frac{\gamma(1 + t)}{2} \|u_x(t, \cdot)\|^2.
\]

Finally, in order to prove our main result we need to get the boundedness of the quantity \((1 + t)\|u_x(t, \cdot)\|^2\). To do so we need the following elementary lemma.

**Lemma 3.10.** Let \(v \in H^2(0, \infty) \cap H^1_0(0, \infty)\). Then for any \(\varepsilon > 0\) it holds that

\[
\|v_x\|^2 \leq \frac{\varepsilon}{2} \|v\|^2 + \frac{\varepsilon^{-1}}{2} \|v_{xx}\|^2.
\]

*Proof.* By density argument it suffices to prove the inequality only for \(v = v(x) \in H^3(0, \infty) \cap H^1_0(0, \infty)\). In fact, in this case we know that that \(v, v_x\) and \(v_{xx}\) are continuous functions, and in particular (cf., Brezis [2])

\[
\lim_{x \to x_0} v(x) = \lim_{x \to x_0} v_x(x) = 0.
\]

Then it is true that

\[
\int_0^x v(s)v_{xx}(s)ds = v(x)v_x(x) - v(0)\int_0^x [v_x(s)]^2ds,
\]

which implies

\[
\int_0^x (v_x(s))^2ds = v(x)v_x(x) - \int_0^x v(s)v_{xx}(s)ds
\]

\[
\leq v(x)v_x(x) + \frac{1}{2} \int_0^x [v(s)^2 + \varepsilon^{-1}(v_{xx}(s))^2]ds
\]

\[
\leq v(x)v_x(x) + \frac{1}{2} [\varepsilon\|v\|^2 + \varepsilon^{-1}\|v_{xx}\|^2]
\]

for any \(\varepsilon > 0\), because of \(v(0) = 0\). By taking limit as \(x \to \infty\) the lemma now follows. \(\square\)

**Lemma 3.11.** Let \(u(t, x)\) be the weak solution to problem (1.1)–(1.3). Then for any \(\varepsilon > 0\) it holds that

\[
(1 + t)\|u_x(t, \cdot)\|^2 \leq \frac{\varepsilon}{2} (1 + t)\|u(t, \cdot)\|^2 + \varepsilon^{-1}K_1.
\]
Proof. The proof is a direct consequence of Lemma 3.10, and (ii) of Lemma 3.5.

Next we combine the estimates in Lemma 3.9 and Lemma 3.11 with $\varepsilon = 1/2\gamma$, to obtain the following important estimate

$$
\frac{1+t}{8} \| u(t, \cdot) \|^2 + \int_0^t (1 + s) \| u_{xx}(s, \cdot) \|^2 ds \leq K_3, \quad t > 0
$$

for the strong solution $u(t,x)$.

Now we are in a position to give a proof of Theorem 2.1.

Proof of Theorem 2.1. Because of the fact that $dE(t)/dt = -\| u(t, \cdot) \|^2$ (see (2.2)), one first has

$$
\frac{d}{dt} \{ (1 + t)^2 E(t) \} \leq 2(1 + t) E(t).
$$

Then, using (iii) in Lemma 3.5, (3.14) and (3.16) one can finally conclude the desired estimate:

$$
(1 + t)^2 E(t) \leq E(0) + \int_0^t (1 + s) \{ \| u_x(s, \cdot) \|^2 + \gamma \| u_{xx}(s, \cdot) \|^2 + \| u_{xx}(s, \cdot) \|^2 \} ds
$$

$$
\leq E(0) + K_1 + \gamma K_2 + K_3, \quad t > 0.
$$

The estimates (3.16) and (3.17) imply the desired statement of Theorem 2.1.

Proof of Corollary 2.1. The decay estimate for $E_1(t)$ follows from the estimate (3.14), and the estimate for $\| u_x(\cdot, t) \|$, which comes from Lemma 3.11 with $\varepsilon = 1$ combined with the estimate (3.16).

Remark 3.2. Once one has the Hardy-Sobolev type inequality like Lemma 3.1, one can make a similar story to this paper in order to get fast decay estimates for solutions of various evolution equations based on the method due to [7]. Recently, several mathematicians find another good applications of those method to the other types of equations, and as one of those research papers one can cite [14] and the references therein.

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