ARITHMETIC OF CURVES OVER TWO DIMENSIONAL LOCAL FIELD

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ABSTRACT. We study the class field theory of curve defined over two dimensional local field. The approach used here is a combination of the work of Kato-Saito, and Yoshida where the base field is one dimensional

1. Introduction

Let \( k_1 \) be a local field with finite residue field and let \( X \) be a proper smooth geometrically irreducible curve over \( k_1 \). To study the fundamental group \( \pi_{1}^{ab}(X) \), Saito in [8], introduced the groups \( SK_1(X) \) and \( V(X) \) and construct the maps \( \sigma : SK_1(X) \rightarrow \pi_{1}^{ab}(X) \) and \( \tau : V(X) \rightarrow \pi_{1}^{ab}(X)^{g_{\text{co}}} \) where \( \pi_{1}^{ab}(X)^{g_{\text{co}}} \) is defined by the exact sequence

\[
0 \rightarrow \pi_{1}^{ab}(X)^{g_{\text{co}}} \rightarrow \pi_{1}^{ab}(X) \rightarrow \text{Gal}(k_1^{ab}/k_1) \rightarrow 0
\]

The most important results in this context are:

1) The quotient of \( \pi_{1}^{ab}(X) \) by the closure of the image of \( \sigma \) and the cokernel of \( \tau \) are both isomorphic to \( \hat{\mathbb{Z}}^r \) where \( r \) is the rank of the curve.

2) For this integer \( r \), there is an exact sequence

\[
0 \rightarrow (\mathbb{Q}/\mathbb{Z})^r \rightarrow H^2(K, \mathbb{Q}/\mathbb{Z}(2)) \oplus \bigoplus_{v \in P} \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0
\]

where \( K = K(X) \) is the function field of \( X \) and \( P \) designates the set of closed points of \( X \).

These results are obtained by Saito in [8] generalizing the previous work of Bloch where he is reduced to the good reduction case [8, Introduction]. The method of Saito depends on class field theory for two-dimensional local ring having finite residue field. He shows these results for general curve except for the \( p \)-primary part in \( \text{char} k = p > 0 \) case [8, Section II-4]. The remaining \( p \)-primary part had been proved by Yoshida in [11].

There is another direction for proving these results pointed out by Douai in [3]. It consists to consider for all \( l \) prime to the residual characteristic, the group \( \text{Coker} \sigma \) as the dual of the group \( W_0 \) of the monodromy weight filtration of \( H^1(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l) \)

\[
H^1(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0
\]

where \( \overline{X} = X \otimes_{k_1} \overline{k_1} \) and \( \overline{k_1} \) is an algebraic closure of \( k_1 \). This allow him to extend the precedent results to projective smooth surfaces [3].

The aim of this paper is to use a combination of this approach and the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields explained by Yoshida in his paper [11], to study curves over two-dimensional local fields (section 3).

Let \( X \) be a projective smooth curve defined over two dimensional local field \( k \). Let \( K \) be its function field and \( P \) denotes the set of closed points of \( X \). For each \( v \in P \), \( k(v) \) denotes the
residue field at \( v \in P.A \) finite etale covering of \( Z \rightarrow X \) of \( X \) is called a c.s covering, if for any closed point \( x \) of \( X \), \( x \times X Z \) is isomorphic to a finite sum of \( x \). We denote by \( \pi_1^{cs}(X) \) the quotient group of \( \pi_1(X) \) which classifies abelian c.s coverings of \( X \).

To study the class field theory of the curve \( X \), we construct the generalized reciprocity map

\[
\sigma/\ell : SK_2(X)/\ell \rightarrow \pi_1^{ab}(X)/\ell
\]

where \( SK_2(X)/\ell = \text{Coker} \left\{ K_2(K) / \ell \oplus v \in P K_2(k(v))/\ell \right\} \) and \( \tau/\ell : V(X)/\ell \rightarrow \pi_1^{ab}(X)^{\text{c.s}}/\ell \) for all \( \ell \) prime to residual characteristic. The group \( V(X) \) is defined to be the kernel of the norm map \( N : SK_2(X) \rightarrow K_2(k) \) induced by the norm map \( N_{k(v)/k} : K_2(k(v)) \rightarrow K_2(k) \) for all \( v \) and \( \pi_1^{ab}(X)^{\text{c.s}}/\ell \) by the exact sequence

\[
0 \rightarrow \pi_1^{ab}(X)^{\text{c.s}} \rightarrow \pi_1^{ab}(X) \rightarrow \text{Gal}(k^{ab}/k) \rightarrow 0
\]

The cokernel of \( \sigma/\ell \) is the quotient group of \( \pi_1^{ab}(X)/\ell \) that classifies completely split coverings of \( X \); that is; \( \pi_1^{cs}(X)/\ell \).

We begin by proving the exactness of the Kato-Saito sequence (Proposition 4.2):

\[
0 \rightarrow \pi_1^{cs}(X)/\ell \rightarrow H_1(K, \mathbb{Z}/\ell(3)) \rightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) \rightarrow \mathbb{Z}/\ell \rightarrow 0
\]

To determinate the group \( \pi_1^{cs}(X)/\ell \), we need to consider a semi stable model of the curve \( X \) (see Section 5) and the weight filtration on its special fiber. In fact, we will prove in (Proposition 5.1) that \( \pi_1^{cs}(X) \otimes \mathbb{Q}_\ell \) admits a quotient of type \( \mathbb{Q}_{\ell}^r \) where \( r \) is the rank of the first crane of this filtration.

Now, to investigate the group \( \pi_1^{ab}(X)^{\text{c.s}} \), we use class field theory of two-dimensional local field and prove the vanishing of the group \( H^2(k, \mathbb{Q}/\mathbb{Z}) \) (theorem 3.1). This yields the isomorphism

\[
\pi_1^{ab}(X)^{\text{c.s}} \simeq \pi_1^{ab}(\overline{X})_{G_k}
\]

Finally, by the Grothendieck weight filtration on the group \( \pi_1^{ab}(\overline{X})_{G_k} \) and assuming the semi-stable reduction, we obtain the structure of the group \( \pi_1^{ab}(X)^{\text{c.s}} \) and information about the map \( \tau : V(X) \rightarrow \pi_1^{ab}(X)^{\text{c.s}} \).

Our paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the proprieties which we need concerning two-dimensional local field: duality and the vanishing of the second cohomology group. In section 4, we construct the generalized reciprocity map and study the Bloch-Ogus complex associated to \( X \). In section 5, we investigate the group \( \pi_1^{cs}(X) \).

2. Notations

For an abelian group \( M \), and a positive integer \( n \geq 1, M/nM \) denotes the group \( M/nM \).

For a scheme \( Z \), and a sheaf \( \mathcal{F} \) over the étale site of \( Z \), \( H^1(Z, \mathcal{F}) \) denotes the i-th étale cohomology group. The group \( H^1(Z, \mathbb{Z}/\ell) \) is identified with the group of all continues homomorphisms \( \pi_1^{ab}(Z) \rightarrow \mathbb{Z}/\ell \). If \( \ell \) is invertible on \( \mathbb{Z}/\ell(1) \) denotes the sheaf of \( l \)-th root of unity and for any integer \( i \), we denote \( \mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i} \).

For a field \( L \), \( K_i(L) \) is the i-th Milnor group. It coincides with the i-th Quillen group for \( i \leq 2 \). For \( \ell \) prime to \( \text{char} \ L \), there is a Galois symbol

\[
h_{\ell, L} : K_iL/\ell \rightarrow H^i(L, \mathbb{Z}/\ell(i))
\]

which is an isomorphism for \( i = 0, 1, 2 \) (\( i = 2 \) is Merkur’jev-Suslin).
A local field $k$ is said to be $n$-dimensional local if there exists the following sequence of fields $k_i$ $(1 \leq i \leq n)$ such that

(i) each $k_i$ is a complete discrete valuation field having $k_{i-1}$ as the residue field of the valuation ring $O_{k_i}$ of $k_i$, and

(ii) $k_0$ is a finite field.

For such a field, and for $\ell$ prime to $\text{Char}(k)$, the well-known isomorphism

$$H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell$$

(3.1)

and for each $i \in \{0, \ldots, n+1\}$ a perfect duality

$$H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j)) \longrightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell(3.2)$$

hold. The class field theory for such fields is summarized as follows: There is a map $h : K_2(k) \longrightarrow \text{Gal}(k_{ab}/k)$ which generalizes the classical reciprocity map for usually local fields. This map induces an isomorphism $K_2(k)/N_{L/k}K_2(L) \simeq \text{Gal}(L/k)$ for each finite abelian extension $L$ of $k$. Furthermore, the canonical pairing

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \times K_2(k) \longrightarrow H^3(k, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathbb{Q}_l/\mathbb{Z}_l$$

(3.3)

induces an injective homomorphism

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \text{Hom}(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

(3.4)

It is well-known that the group $H^2(M, \mathbb{Q}/\mathbb{Z})$ vanishes when $M$ is a finite field or usually local field. Next, we prove the same result for two-dimensional local field.

**Theorem 3.1.** If $k$ is a two-dimensional local field of characteristic zero, then the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ vanishes.

**Proof.** We proceed as in the proof of theorem 4 of [10]. It is enough to prove that $H^2(k, \mathbb{Q}_l/\mathbb{Z}_l)$ vanishes for all $l$ and when $k$ contains the group $\mu_l$ of $l$-th roots of unity. For this, we prove that multiplication by $l$ is injective. That is, we have to show that the coboundary map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta} H^2(k, \mathbb{Z}/l\mathbb{Z})$$

is injective.

By assumption on $k$, we have

$$H^2(k, \mathbb{Z}/l\mathbb{Z}) \simeq H^2(k, \mu_l) \simeq \mathbb{Z}/l$$

The last isomorphism is well-known for one-dimensional local field and was generalized to non archimedian and locally compact fields by Shatz in [6]. The proof is now reduced to the fact that $\delta \neq 0$;

By class field theory of two dimensional local field, the cohomology group $H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$ may be identified with the group of continuous homomorphisms $K_2(k) \xrightarrow{\Phi} \mathbb{Q}_l/\mathbb{Z}_l$.

Now, $\delta(\Phi) = 0$ if and only if $\Phi$ is a $l$-th power, and $\Phi$ is a $l$-th power if and only if $\Phi$ is trivial on $\mu_l$. Thus, it is sufficient to construct an homomorphism $K_2(k) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$ which is non trivial on $\mu_l$.

Let $i$ be the maximal natural number such that $k$ contains a primitive $l^i$-th root of unity. Then, the image $\xi$ of a primitive $l^i$-th root of unity under the composite map
\[
\frac{k^e}{k^{zl}} \simeq H^1(k, \mu_l) \simeq H^1(k, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)
\]
is not zero. Thus, the injectivity of the map

\[
H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow \text{Hom}(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l)
\]
gives rise to a character which is non trivial on \(\mu_l\). \(\square\)

4. **Curves over two dimensional local field**

Let \(k\) be a two dimensional local field of characteristic zero and \(X\) a smooth projective curve defined over \(k\).

We recall that we denote:

- \(K = K(X)\) its function field,
- \(P\) : set of closed points of \(X\), and for \(v \in P\),
  \(k(v)\) : the residue field at \(v \in P\).

The residue field of \(k\) is one-dimensional local field. It is denoted by \(k_1\).

Let \(\mathcal{H}^n(U, \mathbb{Z}/\ell(3))\), \(n \geq 1\), the Zariski sheaf associated to the presheaf \(U \longrightarrow H^n(U, \mathbb{Z}/\ell(3))\). Its cohomology is calculated by the Bloch-Ogus resolution. So, we have the two exact sequences:

(4.1) \[
H^3(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))) \longrightarrow 0
\]

(4.2) \[
0 \longrightarrow H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2))
\]

4.1. **The reciprocity map.** We introduce the group \(SK_2(X)/\ell\):

\[
SK_2(X)/\ell = \text{CoKer} \left\{ K_3(K)/\ell \oplus \bigoplus_{v \in P} K_2(k(v))/\ell \right\}
\]

where \(\partial_v : K_3(K) \longrightarrow K_2(k(v))\) is the boundary map in K-Theory. It will play an important role in class field theory for \(X\) as pointed out by Saito in the introduction of [8]. In this section, we construct a map

\[
\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell
\]

which describe the class field theory of \(X\).

By definition of \(SK_2(X)/\ell\), we have the exact sequence

\[
K_3(K)/\ell \longrightarrow \bigoplus_{v \in P} K_2(k(v))/\ell \longrightarrow SK_2(X)/\ell \longrightarrow 0
\]

On the other hand, it is known that the following diagram is commutative:

\[
\begin{array}{ccc}
K_3(K)/\ell & \longrightarrow & \bigoplus_{v \in P} K_2(k(v))/\ell \\
\downarrow h^3 & & \downarrow h^2 \\
H^3(K, \mathbb{Z}/\ell(3)) & \longrightarrow & \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2))
\end{array}
\]

where \(h^2, h^3\) are the Galois symbols. This yields the existence of a morphism

\[
h : SK_2(X)/\ell \longrightarrow H^1(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2)))
\]

taking in account the exact sequence (4.1). This morphism fit in the following commutative diagram
\[ 0 \to K_3(K)/\ell \oplus K_2(k(v))/\ell \to SK_2(X)/\ell \to 0 \]

\[ 0 \to H^3(K,\mathbb{Z}/\ell(2)) \oplus v \in P H^2(k(v),\mathbb{Z}/\ell(2)) \to H^1(X_{\text{Zar}},\mathcal{H}^3(\mathbb{Z}/\ell(2))) \to 0 \]

By Merkur'jev-Suslin, the map \( h^2 \) is an isomorphism, which implies that \( h \) is surjective. On the other hand, the spectral sequence

\[
\begin{align*}
H^p(X_{\text{Zar}},\mathcal{H}^q(\mathbb{Z}/\ell(3))) &\Rightarrow H^{p+q}(X,\mathbb{Z}/\ell(3)) \\
\end{align*}
\]

induces the exact sequence

\[
\begin{align*}
0 &\to H^1(X_{\text{Zar}},\mathcal{H}^3(\mathbb{Z}/\ell(3))) \xrightarrow{e} H^4(X,\mathbb{Z}/\ell(3)) \\
&\to H^0(X_{\text{Zar}},\mathcal{H}^4(\mathbb{Z}/\ell(3))) \to H^2(X_{\text{Zar}},\mathcal{H}^3(\mathbb{Z}/\ell(3))) = 0 \\
\end{align*}
\]

Composing \( h \) and \( e \), we get the map

\[
SK_2(X)/\ell \to H^4(X,\mathbb{Z}/\ell(3))
\]

Finally the group \( H^4(X,\mathbb{Z}/\ell(3)) \) is identified with the group \( \pi_1^{ab}(X)/\ell \) by the duality [4,II, th 2.1]

\[
H^4(X,\mathbb{Z}/\ell(3)) \otimes H^1(X,\mathbb{Z}/\ell) \to H^5(X,\mathbb{Z}/\ell(3)) \simeq H^3(k,\mathbb{Z}/\ell(2)) \simeq \mathbb{Z}/\ell
\]

Hence, we obtain the map

\[
\sigma/\ell : SK_2(X)/\ell \to \pi_1^{ab}(X)/\ell
\]

Remark 4.1. By the exact sequence (4.2) the group \( H^0(X_{\text{Zar}},\mathcal{H}^4(\mathbb{Z}/\ell(3))) \) coincides with the kernel of the map2

\[
H^4(K,\mathbb{Z}/\ell(3)) \to \oplus v \in P H^3(k(v),\mathbb{Z}/\ell(2))
\]

and by localization in étale cohomology

\[
\oplus v \in P H^2(k(v),\mathbb{Z}/\ell(2)) \to H^4(K,\mathbb{Z}/\ell(3)) \to H^4(X,\mathbb{Z}/\ell(3)) \to \oplus v \in P H^3(k(v),\mathbb{Z}/\ell(2))
\]

and taking in account (4.3), we see that \( H^1(X_{\text{Zar}},\mathcal{H}^4(\mathbb{Z}/\ell(3))) \) is the cokernel of the Gysin map

\[
\oplus v \in P H^2(k(v),\mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X,\mathbb{Z}/\ell(3))
\]

and consequently the morphism \( g \) factorizes through \( H^1(X_{\text{Zar}},\mathcal{H}^4(\mathbb{Z}/\ell(3))) \)

\[
\oplus v \in P H^2(k(v),\mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X,\mathbb{Z}/\ell(3)) \to H^1(X_{\text{Zar}},\mathcal{H}^4(\mathbb{Z}/\ell(3)))
\]

Then, we deduce the following commutative diagram
\[ K_3(K) / \ell \rightarrow \bigoplus_{v \in P} K_2(k(v)) / \ell \rightarrow SK_2(X) / \ell \rightarrow 0 \]
\[ H^3(K, \mathbb{Z}/\ell(3)) \rightarrow \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \rightarrow H^1(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \rightarrow 0 \]
\[ \pi_1^{ab}(X) / l = H^4(X, \mathbb{Z}/\ell(3)) \]

The surjectivity of the map \( h \) implies that the cokernel of
\[ \sigma / \ell : SK_2(X) / \ell \rightarrow \pi_1^{ab}(X) / \ell \]

coincides with the cokernel of \( e \) which is \( H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \). Hence \( \text{Coker} \sigma / \ell \) is the dual of the kernel of the map

(4.4) \[ H^1(X, \mathbb{Z}/\ell) \rightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell) \]

4.2. The Kato-Saito exact sequence.

**Definition 4.2.** Let \( Z \) be a Noetherian scheme. A finite etale covering \( f : W \rightarrow Z \) is called a c.s covering if for any closed point \( z \) of \( Z \), \( z \times_Z W \) is isomorphic to a finite scheme-theoretic sum of copies of \( Z \). We denote \( \pi_{1^{c.s}}(Z) \) the quotient group of \( \pi_{1}^{ab}(Z) \) which classifies abelian c.s coverings of \( Z \).

Hence, the group \( \pi_{1}^{c.s}(X) / \ell \) is the dual of the kernel of the map

(4.4) \[ H^1(X, \mathbb{Z}/\ell) \rightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell) \]

as in [8, section 2, definition and sentence just below]. Now, we are able to calculate the homologies of the Bloch-Ogus complex associated \( X \).

Generalizing [9,Theorem 7], we obtain :

**Proposition 4.3.** Let \( X \) be a projective smooth curve defined over \( k \). Then for all \( \ell \), we have the following exact sequence

\[ 0 \rightarrow \pi_{1}^{c.s}(X) / \ell \rightarrow H^1(K, \mathbb{Z}/\ell(3)) \rightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) \rightarrow \mathbb{Z}/\ell \rightarrow 0. \]

**Proof.** Consider the localization sequence on \( X \)

\[ \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X, \mathbb{Z}/\ell(3)) \rightarrow H^4(K, \mathbb{Z}/\ell(3)) \]
\[ \rightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) \rightarrow H^5(X, \mathbb{Z}/\ell(3)) \rightarrow 0 \]

We know that the cokernel of the Gysin map \( g \) coincides with \( \pi_{1}^{c.s}(X) / \ell \) and we use the isomorphism \( H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \) (4.4). \( \square \)
In his paper [8], Saito don’t prove the p– primary part in the char $k = p > 0$ case. This case was developed by Yoshida in [11]. His method is based on the theory of monodromy-weight filtration of degenerating abelian varieties on local fields. In this work, we use this approach to investigate the group $\pi_1^{c,s}(X)$. As mentioned by Yoshida in [11, section 2] Grothendieck’s theory of monodromy-weight filtration on Tate module of abelian varieties are valid where the residue field is arbitrary perfect field.

We assume the semi-stable reduction and choose a regular model $X$ of $X$ over $SpecO_k$, by which we mean a two dimensional regular scheme with a proper birational morphism $f : X \rightarrow SpecO_k$ such that $X \otimes_{O_k} k \simeq X$ and if $X_\text{red}$ designates the special fiber $X \otimes_{O_k} k_1$, then $Y = (X_\text{red})$ is a curve defined over the residue field $k_1$ such that any irreducible component of $Y$ is regular and it has ordinary double points as singularity.

Let $\Gamma = Y \otimes_{k_1} k_1$, where $k_1$ is an algebraic closure of $k_1$ and $Y[p] = \bigcup_{i,j < \ldots < i_p} Y_{i_j} \cap \cdots \cap Y_{i_p}$,

$(Y)_{i \in I} =$ collection of irreducible components of $Y$.

Let $|\Gamma|$ be a realization of the dual graph $\Gamma$, then the group $H^1(|\Gamma|, \mathbb{Q}_l)$ coincides with the group $W_0(H^1(Y, \mathbb{Q}_l))$ constituted of elements of weight 0 for the filtration

$$H^1(Y, \mathbb{Q}_l) = W_1 \supseteq W_0 \supseteq 0$$

of $H^1(Y, \mathbb{Q}_l)$ deduced from the spectral sequence

$$E_1^{p,q} = H^q(Y[p], \mathbb{Q}_l) \Longrightarrow H^{p+q}(Y, \mathbb{Q}_l)$$

For details see [2], [3] and [5]

Now, if we assume further that the irreducible components and double points of $Y$ are defined over $k_1$, then the dual graph $\Gamma$ of $Y$ go down to $k_1$ and we obtain the injection

$$W_0(H^1(Y, \mathbb{Q}_l)) \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l)$$

**Proposition 5.1.** The group $\pi_1^{c,s}(X) \otimes \mathbb{Q}_l$ admits a quotient of type $\mathbb{Q}_l^r$, where $r$ is the $\mathbb{Q}_l$–rank of the group $H^1(|\Gamma|, \mathbb{Q}_l)$

**Proof.** We know (4.5) that $\pi_1^{c,s}(X) \otimes \mathbb{Q}_l$ is the dual of the kernel of the map

$$\alpha : H^1(X, \mathbb{Q}_l) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Q}_l)$$

We will prove that $W_0(H^1(Y, \mathbb{Q}_l)) \subseteq K_{\alpha}$. The group $W_0 = W_0(H^1(Y, \mathbb{Q}_l))$ is calculated as the homology of the complex

$$H^0(Y[0], \mathbb{Q}_l) \longrightarrow H^0(Y[1], \mathbb{Q}_l) \longrightarrow 0$$

Hence $W_0 = H^0(Y[1], \mathbb{Q}_l)/\text{Im}\{H^0(Y[0], \mathbb{Q}_l) \longrightarrow H^0(Y[1], \mathbb{Q}_l)\}$. Thus, it suffices to prove the vanishing of the composing map

$$H^0(Y[1], \mathbb{Q}_l) \longrightarrow W_0 \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$$

for all $v \in P$.

Let $z^v$ be the 0– cycle in $Y$ obtained by specializing $v$, which induces a map $z^v_{[1]} \longrightarrow Y^{[1]}$. Consequently, the map $H^0(Y[1], \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$ factors as follows
But the trace $z_v^{[1]}$ of $\overline{Y}$ on $z_v$ is empty. This implies the vanishing of $H^0(z_v^{[1]}, \mathbb{Q}_\ell)$. \hfill \Box

Let $V(X)$ be the kernel of the norm map $N : SK_2(X) \to K_2(k)$ induced by the norm map $N_{k(v)/k^+} : K_2(k(v)) \to K_2(k)$ for all $v$. Then, we obtain a map $\tau/l : V(X)/\ell \to \pi_1^{ab}(X)^{\text{geo}}/\ell$ and a commutative diagram

\[
\begin{array}{ccc}
V(X)/\ell & \to & SK_2(X)/\ell \\
\downarrow \tau/l & & \downarrow \sigma/\ell \\
\pi_1^{ab}(X)^{\text{geo}}/\ell & \to & \pi_1^{ab}(X)/\ell & \to & Gal(k^{ab}/k)/l
\end{array}
\]

where the map $h/l : K_2(k)/l \to Gal(k^{ab}/k)/l$ is the one obtained by class field theory of $k$ (section 3). From this diagram we see that the group $\text{Coker} \tau/l$ is isomorphic to the group $\text{Coker} \sigma/l$. Next, we investigate the map $\tau/l$.

We begin by the following result which is a consequence of the structure of the two-dimensional local field $k$

**Lemma 5.2.** There is an isomorphism

\[
\pi_1^{ab}(X)^{\text{geo}} \simeq \pi_1^{ab}(X)_{G_k},
\]

where $\pi_1^{ab}(X)_{G_k}$ is the group of coinvariants under $G_k = Gal(k^{ab}/k)$.

**Proof.** As in the proof of Lemma 4.3 of [11], this is an immediate consequence of (Theorem 3.1). \hfill \Box

Finally, we are able to deduce the structure of the group $\pi_1^{ab}(X)^{\text{geo}}$

**Theorem 5.3.** The group $\pi_1^{ab}(X)^{\text{geo}} \otimes \mathbb{Q}_\ell$ is isomorphic to $\widehat{\mathbb{Q}}_\ell^r$ and the map $\tau : V(X) \to \pi_1^{ab}(X)^{\text{geo}}$ is a surjection onto $(\pi_1^{ab}(X)^{\text{geo}})_{\text{tor}}$.

**Proof.** By the preceding lemma, we have the isomorphism $\pi_1^{ab}(X)^{\text{geo}} \simeq \pi_1^{ab}(X)_{G_k}$. On the other hand the group $\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell$ admits the filtration [12, Lemma 4.1 and section 2]

\[
W_0(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell) = \pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell \supseteq W_{-1}(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell) \supseteq W_{-2}(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell)
\]

But, by assumption; the curve $X$ admits a semi-stable reduction, then the group $Gr_0(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell) = W_0(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell)/W_{-1}(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell)$ has the following structure

\[
0 \to Gr_0(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell)_{\text{tor}} \to Gr_0(\pi_1^{ab}(X)_{G_k} \otimes \mathbb{Q}_\ell) \to \widehat{\mathbb{Q}}_\ell^r \to 0
\]

where $r'$ is the $k - \text{rank}$ of $X$. This is confirmed by Yoshida [11, section 2], independently of the finitude of the residue field of $k$ considered in his paper. The integer $r'$ is equal to the integer $r = H^1(\overline{\Gamma}, \mathbb{Q}_\ell) = H^1(\overline{\Gamma}, \mathbb{Q}_\ell)$ by assuming that the irreducible components and double points of $Y$ are defined over $k_1$,.
On the other hand, the exact sequence
\[ 0 \longrightarrow W_{-1}(\pi^{ab}_1(\overline{X})_{G_k}) \longrightarrow \pi^{ab}_1(\overline{X})_{G_k} \longrightarrow Gr_0(\pi^{ab}_1(\overline{X})_{G_k}) \longrightarrow 0 \]
and (Proposition 5.1) allow us to conclude that the group \( W_{-1}(\pi^{ab}_1(\overline{X})_{G_k}) \) is finite and the map \( \tau : V(X) \longrightarrow \pi^{ab}_1(X)^{\text{galois}} \) is a surjection onto \( (\pi^{ab}_1(X)^{\text{galois}})^{\text{tor}} \) as established by Yoshida [11] \( \square \)

**Remark 5.4.** If we apply the same method of Saito to study curves over two-dimensional local fields, we need class field theory of two-dimensional local ring having one-dimensional local field as residue field. This is done by myself in [1]. Hence, one can follow Saito’s method to obtain the same results.

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