DIRAC OPERATORS ON NON–COMPACT ORBIFOLDS

CARLA FARSI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, 395
UCB, BOULDER, CO 80309–0395, USA. E-MAIL: FARSI@EUCLID.COLORADO.EDU

Abstract. In this paper we prove that Dirac operators on non–compact almost complex, complete orbifolds
which are sufficiently regular at infinity, admit a unique extension. Additionally, we prove a generalized orbifold
Stokes'/Divergence theorem.

0. Introduction.

Orbifolds, generalized manifolds that are locally the quotient of an euclidean space modulo a finite
group of isometries, were first introduced first by Satake. In the late seventies, Kawasaki proved an orbifold
signature formula, together with more general index theorems, see [Kw1], [Kw2], [Kw3]. In [Fa1] we proved
a $K$–theoretical index theorem for orbifolds with operator algebraic means, and in [Fa2], [Fa3] we studied
compact orbifold spectral theory and defined orbifold eta invariants. Other orbifold index formulas were
proved in [Du], [V]. In [Ch] Chiang studied compact orbifold heat kernels and harmonic maps, while in
[Stan], Stanhope established some interesting geometrical applications of orbifold spectral theory.

Here we will continue the orbifold spectral analysis started in [Fa2] and [Fa3]. In particular we show
that on a non–compact complete almost complex $Spin^c$ orbifold which is sufficiently regular at infinity (see
Definition 2.1), generalized Dirac operators are closed. This extends to orbifolds theorems of Gaffney [Gn1],
Yau [Y], and Wolf [W], whose ideas are used in our proofs, together with more orbifold-specific techniques.
In particular, our first main result, Theorem 3.1, asserts that

**Theorem 3.1.** Let $X$ be an even–dimensional non–compact complete Hermitian $Spin^c$ almost complex
orbifold which is sufficiently regular at infinity. Assume that a Hermitian connection is chosen on the dual
of its canonical line bundle $K^*$. Let $E$ be a proper Hermitian orbibundle (with connection $\nabla^E$) over $X$, and
let $D_E$ be the generalized Dirac operator with coefficients in $E$. Let $\mathcal{D}(D_E^{MIN})$ be the domain of the min
extension of $D_E$, and $\mathcal{D}(D_E^{MAX})$ be the domain of the max extension of $D_E$, see the end of Section 2 for

1991 Mathematics Subject Classification. Primary 58G03, Secondary 58G10.

Key words and phrases. Orbifold, Non–Compact, Elliptic Self-Adjoint Operator.
details. Then

\[ \mathcal{D}(D_E^{MIN}) = \mathcal{D}(D_E^{MAX}). \]

We also prove the following Stokes’/Divergence theorem, Theorem 5.1, which generalizes to orbifolds results of Gaffney [Gn2], Karp [K], and Yau [Y].

**Theorem 5.1.** Let \( X \) be an even–dimensional non–compact complete \( \text{Spin}^c \) almost complex orbifold which is sufficiently regular at infinity. Assume that a connection is chosen on the dual of its canonical line bundle. Let \( V \) be a vector field on \( X \) such that

\[
\lim_{k \to +\infty} \inf \frac{1}{k} \int_{B_{2k} - B_k} \|V\| \, dv = 0,
\]

where \( \|V\| \) denotes the length of \( V \), and \( B_k = \{ y \in X \mid \rho(y) = d(y, y_0) \leq k \} \) for a fixed \( y_0 \in X - \Sigma(X) \), where \( \Sigma(X) \) is the singular locus of \( X \). Then if either \( (\text{div} V)^+ \) or \( (\text{div} V)^- \) is integrable on \( X \), we have

\[
\int_X \text{div} (V) \, dv = 0.
\]

In a sequel to this paper [Fa4], we use the results we proved here to establish an orbifold Gromov-Lawson relative index theorem, c.f. [GL] for the manifold case. More in detail, the contents of this paper are as follows. In Section 1 we recall the definition of orbifold, orbibundles, and introduce orbifold Dirac operators. In Section 2, we study Dirac operators on non–compact orbifolds from a local viewpoint. In Section 3, we state and prove our first main result, Theorem 3.1. In Section 4 we prove that, if \( D \) is a Dirac operator, and \( D^2 \sigma = 0 \), also \( D \sigma = 0 \). In Section 5 we finally prove our Stokes’/Divergence theorem, Theorem 5.1. Vanishing results are considered in Section 6.

In the sequel, all orbifolds and manifolds are even–dimensional, smooth, Hermitian, \( \text{Spin}^c \), connected, and almost complex unless otherwise specified. All vector and orbibundles are assumed to be smooth and proper. We also assume that all of our orbifolds/manifolds are endowed with a fixed Hermitian connection on the dual of their canonical line bundle \( K^* \). This latter hypotheses allows us to define a ‘canonical’ \( \text{Spin}^c \) Dirac operator and, given a Hermitian orbibundle \( E \) with a chosen connection \( \nabla^E \), the ‘canonical’ \( \text{Spin}^c \) Dirac operator with coefficients in \( E \). Both of these operators depend, in the \( \text{Spin}^c \) case, on the choice of the selected connections, see [Du; Chapter 14], and [LM; Appendix D]. For the \( \text{Spin} \) or complex case, the choice of the connection on \( K^* \) is canonical.
I would like to thank the sabbatical program of the University of Colorado/Boulder, and the Mathematics Department of the University of Florence, Italy, for their warm hospitality during the period this paper was written. We also thanks the referee for useful suggestions.

1. Orbifolds, Orbibundles and Dirac Operators.

In this section we will review some definitions and results that we will use throughout this paper. For generalities on orbifolds and operators on orbifolds, see [Kw1], [Kw2], [Kw3], [Ch], [Du], [V].

An orbifold is a Hausdorff second countable topological space $X$ together with an atlas of charts $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$, with $\tilde{U}_i/G_i = U_i$ open, and with projection $\pi_i : \tilde{U}_i \to U_i$, $i \in I$, satisfying the following properties

1. If two charts $U_1$ and $U_2$ associated to pairs $(\tilde{U}_1, G_1), (\tilde{U}_2, G_2)$ of $\mathcal{U}$, are such that $U_1 \subseteq U_2$, then there exists a smooth open embedding $\lambda : \tilde{U}_1 \to \tilde{U}_2$ and a homomorphism $\mu : G_1 \to G_2$ such that $\pi_1 = \pi_2 \circ \lambda$ and $\lambda \circ \gamma = \mu(\gamma) \circ \lambda$, $\forall \gamma \in G_1$.

2. The collection of the open charts $U_i$, $i \in I$, belonging to the atlas $\mathcal{U}$ forms a basis for the topology on $X$.

We will call an orbifold atlas as above a standard orbifold atlas.

For any $x$ point of $X$, the isotropy $G_x$ of $x$ is well defined, up to conjugacy, by using any local coordinate chart. The set of all points $x \in X$ with non-trivial stabilizer, $\Sigma(X)$, is called the singular locus of $X$, see e.g. [Ch]. Note that $X - \Sigma(X)$ is a smooth manifold.

If we now endow $X$ with a countable locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_i, G_i)|i \in \mathbb{N}\}$, then by standard theory there exists a smooth partition of unity $\eta = \{\eta_i\}_{i \in \mathbb{N}}$ subordinated to $\mathcal{F}$, [Ch]. This in particular means that, for any $i \in \mathbb{N}$, $\eta_i$ is a smooth function on $U_i$ (i.e., its lift to any chart of a standard orbifold atlas is smooth), the support of $\eta_i$ is included in an open subset $U'_i$ of $U_i$, and $\cup U'_i = X$. We will call any $\eta$ as above an $\mathcal{F}$-partition of unity. Let $E$ be an orbibundle over the orbifold $X$. (For the precise definition see [Kw1], [Kw2], [Kw3], [Ch].) In particular $E$ is an orbifold in its own right; on an orbifold chart $U_1$ associated to a pair $(\tilde{U}_1, G_1)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$ of $X$, $E$ lifts to a $G_1$-equivariant bundle. Standard orbifold atlases on $X$ can be used to provide standard orbifold atlases on $E$.

If $E$ is an orbibundle over the orbifold $X$, a section $s : X \to E$ is called a smooth orbifold section if for each chart $U_i$ associated to a pair $(\tilde{U}_i, G_i)$ of a standard orbifold atlas $\mathcal{U} = \{(\tilde{U}_i, G_i)|i \in I\}$ of $X$, we have
that \(s|_{U_i} : U_i \to E|_{U_i}\) is covered by a smooth \(G_i\)-invariant section \(\tilde{s}|_{\tilde{U}_i} : \tilde{U}_i \to \tilde{E}|_{\tilde{U}_i}\). Given a Hermitian orbibundle \(E\) over \(X\), we will denote by \(C^\infty(X, E)\) the space of all smooth sections of \(E\), and by \(C^\infty_c(X, E)\) the space of all smooth sections with compact support. Classical orbibundles over \(X\) are the tangent bundle \(TX\), and the cotangent bundle \(T^*X\) of \(X\). We can form orbibundles tensor products by taking the tensor products of their local expressions in the charts of a standard orbifold atlas.

Define an inner product between sections of \(C^\infty(X, E)\) (or \(C^\infty_c(X, E)\)) by the following formula (c.f., [Ch; 2.2a])

\[
(\sigma_1, \sigma_2) = \sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{\tilde{U}_i} \tilde{\eta}(\tilde{x}_i) < \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) > dv(\tilde{x}_i),
\]

where \(\eta = \{\eta_i\}_{i \in \mathbb{N}}\), is a \(\mathcal{F}\)-partition of unity subordinated to the locally finite orbifold cover \(\mathcal{F} = \{(\tilde{U}_i, G_i)| i \in \mathbb{N}\}\), and \(<, >\) is a \(G_i\)-invariant product on \(\tilde{E}\). (Note that, with slight abuse of notation, we used \(\tilde{\cdot}\) to denote lift to \(\tilde{U}_i\).)

We will now review the construction of the Dirac operators with coefficients in an Hermitian orbibundle \(E\) endowed with a connection \(\nabla E\), over an orbifold \(X\) satisfying our hypotheses, [Du; Sections 5 and 12], [Kw2], [BGV], [LM; Appendix D]. First of all, \(X\) admits a \(Spin^c\)-principal tangent orbibundle, \(Spin^c(TX)\), with, in our hypotheses, canonical \(Spin^c\) orbifold connection \(\nabla c\). Let \(\Delta^{\pm, c}\) be the half \(Spin^c\) representations (recall that the \(X\) is even dimensional). Then we have two orbibundles

\[
\Delta^{\pm, c}(TX) = Spin^c(TX) \times_{Spin^c} \Delta^{\pm, c},
\]

with induced connections \(\nabla^{\pm, c}\), from \(\nabla c\); \(\nabla^{\pm, c} : C^\infty_c(X, \Delta^{\mp, c}(TX)) \to C^\infty_c(X, T^*X \otimes \Delta^{\mp, c}(TX))\). The Clifford module structure on \(\Delta^{\pm, c}\) defines Clifford multiplications

\[
m_{\pm} : TX \otimes_{\mathbb{R}} \Delta^{\pm, c}(TX) \to \Delta^{\mp, c}(TX)
\]

On \(E\) we have the connection \(\nabla E\). Then the generalized \(\pm\) Dirac operator with coefficients in \(E\), \(d^{\pm, c}_E\),

\[
d^{\pm, c}_E : C^\infty_c(X, \Delta^{\pm, c}(TX) \otimes_{\mathbb{C}} E) \to C^\infty_c(X, \Delta^{\mp, c}(TX) \otimes_{\mathbb{C}} E)
\]

is defined by

\[
d^{\pm, c}_E = M \circ (\nabla^{\pm, c} \otimes Id + Id \otimes \nabla E),
\]

where \(M\) denotes the map induced by Clifford multiplication and \(TX\) has been identified with \(T^*X\) via the orbifold metric. We will also use the notation \(S\) for the orbifold \(Spin^c\) bundle \((\Delta^{+c} \oplus \Delta^{-c})(TX)\), and \(S \otimes E\)
or $E$ for $(\Delta^{+,c} \oplus \Delta^{-,c})(TX) \otimes \mathbb{C} E$ throughout this paper. We will define $D_E$, the generalized Dirac operator on $X$ with coefficient in $E$, to be $(d_{E}^{+,c} + d_{E}^{-,c})$.

2. Dirac Operators on Complete Orbifolds.

On an orbifold $X$ (not necessarily compact), the generalized Dirac operator with coefficients in the orbibundle $E$ (with connection $\nabla^{E}$), $D_{E}$, as defined in Section 1, is given by

$$D_{E} : \mathcal{C}_{c}^{\infty}(X, (\Delta^{+,c} \oplus \Delta^{-,c})(TX) \otimes \mathbb{C} E) \to \mathcal{C}_{c}^{\infty}(X, (\Delta^{-,c} \oplus \Delta^{+,c})(TX) \otimes \mathbb{C} E)$$

$$D_{E} = M \circ \left( (\nabla^{+,c} + \nabla^{-,c}) \otimes \text{Id} + \text{Id} \otimes \nabla^{E} \right).$$

On orbifold charts, the Dirac operator $D_{E}$ with coefficients in the Hermitian orbibundle $E$ (with connection $\nabla^{E}$), has the following local expression $\tilde{D}_{E}$. Let $\mathcal{U} = \{(\tilde{U}_{i}, G_{i}) | i \in I\}$, with $\tilde{U}_{i}/G_{i} = U_{i}$ be a standard orbifold atlas. On a local chart $\tilde{U}_{i}$, $i \in I$ fixed, we have

$$\Delta^{\pm,c}(T \tilde{U}_{i}) = \text{Spin}^{c}(T \tilde{U}_{i}) \times_{\text{Spin}^{c}} \Delta^{\pm,c},$$

with induced $G_{i}$–invariant connections $\nabla^{\pm,c}$, from $\nabla^{c}$. The Clifford module structure on $\Delta^{\pm,c}$ defines Clifford multiplications

$$m_{\pm} : T \tilde{U}_{i} \otimes_{\mathbb{R}} \Delta^{\pm,c}(T \tilde{U}_{i}) \to \Delta^{\mp,c}(T \tilde{U}_{i}).$$

On $\tilde{E}$, the lift of $E$, we have the $G_{i}$–invariant connection $\nabla^{\tilde{E}}$. Then the generalized $\pm$ Dirac operators with coefficients in $E$, $\tilde{d}_{E}^{\pm,c}$,

$$\tilde{d}_{E}^{\pm,c} : \mathcal{C}_{c}^{\infty}(\tilde{U}_{i}, \Delta^{\pm,c}(T \tilde{U}_{i}) \otimes \mathbb{C} \tilde{E}) \to \mathcal{C}_{c}^{\infty}(\tilde{U}_{i}, \Delta^{\mp,c}(T \tilde{U}_{i}) \otimes \mathbb{C} E)$$

is given by

$$\tilde{d}_{E}^{\pm,c} = M \circ \left( \nabla^{\pm,c} \otimes \text{Id} + \text{Id} \otimes \nabla^{\tilde{E}} \right),$$

where $M$ is induced by Clifford multiplication and $T \tilde{U}_{i}$ has been identified with $T^{*}\tilde{U}_{i}$ via the $G_{i}$–invariant metric. Also, $\tilde{D}_{E}$, the generalized Dirac operator on $X$ with coefficient in $E$, is given by $\tilde{d}_{E}^{+,c} + \tilde{d}_{E}^{-,c}$ on $\tilde{U}_{i}$.

If $e_{1}, \ldots, e_{n}$ is an orthonormal local basis for the space $T \tilde{U}_{i}$ at a point $\tilde{x}$, then $\tilde{D}_{E}$ has local expression

$$\tilde{D}_{E}^{\tilde{x}} = \sum_{k=1}^{n} e_{k} \tilde{\nabla}_{e_{k}}^{E},$$

where

$$\tilde{\nabla}^{E} = (\nabla^{+,c} + \nabla^{-,c}) \otimes 1 + 1 \otimes \nabla^{\tilde{E}}.$$

Now, in analogy with the manifold case, see [GL], [W], [Gn1], [LM], [Y], we will show that $D_{E}$ is symmetric, whenever $X$ is a sufficiently regular at infinity.
**Definition 2.1.** Let $X$ be a non-compact complete orbifold. Then we say that $X$ is sufficiently regular at infinity if, for any neighborhood $\Omega \subseteq X$ of infinity, there exists a compact domain $K_\Omega$ with $\Omega \cup K_\Omega = X$ and with boundary strictly included in $\Omega$, on which the Divergence and Stokes’ Theorems hold.

For a compact orbifold without boundary, the Divergence Theorem holds, [Ch]. See also [C] for other results. Sufficient regularity also holds in the case of a product end, by an adaptation of Chiang’s method, [Ch], and in the case of finite volume hyperbolic orbifolds because of the structure of the cusps cross sections, [LoR]. Also, geometrically finite orbifolds with pinched negative sectional curvature satisfy this hypothesis, [AX]. In general, ours seems to be a very reasonable assumption to make, which will be certainly satisfied in many cases of interest, see above examples. For Sobolev inequalities of Gallot type involving domains, see [N].

**Theorem 2.2.** Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle (with connection $\nabla^E$) over $X$. Let $D_E$ be the generalized Dirac operator with coefficients in $E$, as defined above. Then $D_E$ is symmetric, i.e.,

$$(D_E \sigma_1, \sigma_2) = (\sigma_1, D_E \sigma_2), \quad \forall \sigma_1, \sigma_2 \in C_c^\infty(X, S \otimes \mathcal{C} E),$$

where $(, )$ denotes the inner product defined earlier.

**Proof.** Let $\mathcal{E} = S \otimes E$, $D = D_E$. Let $\eta = \{\eta_i\}_{i \in \mathbb{N}}$ be a $\mathcal{F}$-partition of unity subordinated to the locally finite orbifold cover $\mathcal{F} = \{(\tilde{U}_i, G_i)|i \in \mathbb{N}\}$. (Note that, as before, we are using $\tilde{\cdot}$ to denote lift to $\tilde{U}_i$.) Then

$$(\sigma_1, \sigma_2) = \sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{\tilde{U}_i} \tilde{\eta}_i(\tilde{x}_i) \langle \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) \rangle d\nu(\tilde{x}_i), \quad \forall \sigma_1, \sigma_2 \in C_c^\infty(U_i, \mathcal{E}).$$

Since $T(\tilde{U}_i)$ is parallelizable, we can choose a local orthonormal basis $e_1, \ldots, e_n$ for the space $T\tilde{U}_i$ at any point $\tilde{x}_i$; thus, if we set $\nabla = \tilde{\nabla}^E$,

$$< \tilde{\nabla} e_k \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) > = \sum_{k=1}^{n} < e_k \tilde{\nabla} e_k \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) >$$

$$- \sum_{k=1}^{n} \langle \nabla e_k \tilde{\sigma}_1(\tilde{x}_i), e_k \tilde{\sigma}_2(\tilde{x}_i) \rangle$$

$$= - \sum_{k=1}^{n} \left\{ \langle \nabla e_k \tilde{\sigma}_1(\tilde{x}_i), e_k \tilde{\sigma}_2(\tilde{x}_i) \rangle - \langle \tilde{\sigma}_1(\tilde{x}_i), (\nabla e_k) \tilde{\sigma}_2(\tilde{x}_i) + e_k \tilde{\nabla} e_k \sigma_2(\tilde{x}_i) \rangle \right\},$$

where $<,>$ is a $G_i$-invariant inner product on $\mathcal{S}$. 


If we define the $G_i$–invariant vector field $V_i$ on $\tilde{U}_i$ by

$$< V_i, W >= - < \tilde{\sigma}_1, W \circ \tilde{\sigma}_2 >,$$

for any vector field $W$, we have that the above expression can be rewritten as

(2.1) $$< \tilde{D}, \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) >= \text{div}(V_i(\tilde{x}_i)) + < \tilde{\sigma}_1(\tilde{x}_i), < \tilde{\sigma}_2(\tilde{x}_i) >$$

Now integrate (2.1) (multiplied by $\tilde{\eta}_i$ and divided by $|G_i|$) over $\tilde{U}_i$. Then by using the Divergence Theorem, we are done. □

**Remark 2.3.** Theorem 2.2 is also valid when only one of the two sections $\sigma_1$, $\sigma_2$ has compact support.

Now complete the space $C^\infty_c(X, E)$, $E = S \otimes_C E$, $S \text{Spin}^c$ bundle on $X$, $E$ Hermitian orbibundle over $X$, with respect to the norm

$$||\sigma||_X = \sqrt{< \sigma, \sigma >} = \left( \sum_{i=1}^{+\infty} \frac{1}{|G_i|} \int_{\tilde{U}_i} \tilde{\eta}_i(\tilde{x}_i) < \tilde{\sigma}_1(\tilde{x}_i), \tilde{\sigma}_2(\tilde{x}_i) > dv(\tilde{x}_i) \right)^{\frac{1}{2}}.$$

We thus obtain the $L^2$–space $L^2(X, E)$. The Dirac operator

$$D_E : C^\infty_c(X, E) \to C^\infty_c(X, E)$$

has two natural extensions, min and max listed below, as an unbounded operator

$$D_E : L^2(X, E) \to L^2(X, E).$$

1. **Minimal Extension $D^{MIN}_E$.** The minimal extension of $D_E$, $D^{MIN}_E$, is obtained by taking the graph closure of the graph of $D_E$, i.e.,

$$D^{MIN}_E : D(D^{MIN}_E) \to L^2(X, E),$$

where $D(D^{MIN}_E)$, the domain of $D^{MIN}_E$, is defined to be the set of $\sigma \in L^2(X, E)$ for which there exists a sequence $\sigma_k \in C^\infty_c(X, E)$ such that $\sigma_k \to \sigma$ and $D_E \sigma_k \to \tau$ in $L^2(X, E)$, for some $\sigma, \tau \in L^2(X, E)$. Set $D^{MIN}_E(\sigma) = \tau$.

2. **Maximal Extension $D^{MAX}_E$.** The maximal extension of $D_E$, $D^{MAX}_E$, is obtained by taking its domain to be the set of all $\sigma \in L^2(X, E)$ such that the distributional image of $D_E(\sigma)$ is still in $L^2(X, E)$. More precisely,

$$D^{MAX}_E : D(D^{MAX}_E) \to L^2(X, E).$$
where \( \mathcal{D}(D_{E}^{\text{MAX}}) \), the domain of \( D_{E}^{\text{MAX}} \), is defined to be the set of \( \sigma \in \mathcal{L}^{2}(X, \mathcal{E}) \) such that the linear functional \( L(\sigma_{2}) = (\sigma, D_{E}(\sigma_{2})) \) on \( \mathcal{C}^{\infty}(X, \mathcal{E}) \) is bounded in the \( \mathcal{L}^{2}(X, \mathcal{E}) \) norm. Note that the boundedness of \( L \) implies that there exists an element \( \tau \in \mathcal{L}^{2}(X, \mathcal{E}) \), such that

\[
(\tau, \sigma_{2}) = (\sigma, D_{E}\sigma_{2}), \quad \forall \sigma_{2} \in \mathcal{C}^{\infty}(X, \mathcal{E}).
\]

Define the above \( \tau \) to be \( D_{E}^{\text{MAX}}(\sigma) \).

**Remark 2.4.** Since \((\cdot, \cdot)\) is continuous in the \( \mathcal{L}^{2}(X, \mathcal{E}) \) norm, we have that

\[
\mathcal{D}(D_{E}^{\text{MIN}}) \subseteq \mathcal{D}(D_{E}^{\text{MAX}}).
\]

3. **Generalized Dirac Operators on Non–Compact Complete Orbifolds are Closed.**

In this Section we will prove that generalized Dirac operators on complete orbifolds which are sufficiently regular at infinity, are closed operators. This theorem generalizes to orbifolds [GL; Theorem 1.17], and [W; Theorem 5.1].

**Theorem 3.1.** Let \( X \) be an even–dimensional non–compact complete Hermitian Spin\(^{c} \) almost complex orbifold which is sufficiently regular at infinity. Assume that a Hermitian connection is chosen on the dual of its canonical line bundle \( K^{*} \). Let \( E \) be a proper Hermitian orbibundle (with connection \( \nabla^{E} \)) over \( X \), and let \( D_{E} \) be the generalized Dirac operator with coefficients in \( E \). Let \( \mathcal{D}(D_{E}^{\text{MIN}}) \) be the domain of the min extension of \( D_{E} \), and \( \mathcal{D}(D_{E}^{\text{MAX}}) \) be the domain of the max extension of \( D_{E} \), see the end of Section 2 for details. Then

\[
\mathcal{D}(D_{E}^{\text{MIN}}) = \mathcal{D}(D_{E}^{\text{MAX}}).
\]

Our proof of Theorem 3.1, which will occupy the remaining of this section, will be an adaptation of [W; Proof of Theorem 5.1]. In particular, suitable modifications to Wolf's proof for manifolds will be mostly needed to deal with orbifold distance functions.

**Proof.** Firstly, recall that we denoted by \( \Sigma(X) \) the singular locus of \( X \). Then \( X - \Sigma(X) \) is a convex manifold. In particular any two points of \( X - \Sigma(X) \) can be connected by a geodesic arc lying entirely in \( X - \Sigma(X) \), see [Stan; Section 4]. Because of Remark 2.4, to prove Theorem 3.1 it is enough to show that

\[
(3.1) \quad \mathcal{D}(D_{E}^{\text{MAX}}) \subseteq \mathcal{D}(D_{E}^{\text{MIN}}).
\]
Note that $\mathcal{D}(D_{E}^{\text{MAX}})$ carries the norm

$$N(\sigma) = \left\{ \| \sigma \|_{X}^{2} + \| D_{E}(\sigma) \|_{X}^{2} \right\}^{+}, \quad \forall \sigma \in \mathcal{D}(D_{E}^{\text{MAX}}),$$

where $\| \cdot \|_{Y}$ denotes the $L^{2}(Y, \mathcal{E})$ norm, for $Y \subseteq X$, c.f. Section 2. But (3.1) is equivalent to

$$(3.2) \quad C_{c}^{\infty}(X, \mathcal{E}) \text{ is dense in } \mathcal{D}(D_{E}^{\text{MAX}}) \text{ in the norm } N.$$

Thus Theorem 3.1 will clearly follow once we have proven the Lemmas 3.2 and 3.3 below. □.

**Lemma 3.2.** Let $X$ be a non–compact complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle (with connection $\nabla^{E}$) over $X$. Let $D_{E} : C_{c}^{\infty}(X, \mathcal{E}) \to C_{c}^{\infty}(X, \mathcal{E})$, be the generalized Dirac operator on $X$ with coefficients in $E$ as in Theorem 3.1. Then

$$C_{c}^{\infty}(X, \mathcal{E}) \text{ is dense in } \mathcal{D}_{c}(D_{E}^{\text{MAX}}) \text{ in the norm } N,$$

where we set $\mathcal{D}_{c}(D_{E}^{\text{MAX}})$ to be the subset of the elements of $\mathcal{D}(D_{E}^{\text{MAX}})$ with compact support, and where

$$N(\sigma) = \left\{ \| \sigma \|_{X}^{2} + \| D_{E}(\sigma) \|_{X}^{2} \right\}^{+}, \quad \forall \sigma \in \mathcal{D}(D_{E}^{\text{MAX}}).$$

**Lemma 3.3.** Let $X$ be a non–compact complete orbifold which is sufficiently regular at infinity, and let $E$ be a Hermitian orbibundle (with connection $\nabla^{E}$) over $X$. Let $D_{E} : C_{c}^{\infty}(X, \mathcal{E}) \to C_{c}^{\infty}(X, \mathcal{E})$, be the generalized Dirac operator on $X$ with coefficients in $E$ as in Theorem 3.1 and Lemma 3.2. Then

$$\mathcal{D}_{c}(D_{E}^{\text{MAX}}) \text{ is dense in } \mathcal{D}(D_{E}^{\text{MAX}}) \text{ in the norm } N.$$

**Proof of Lemma 3.2.** Set $\mathcal{D}_{c} = \mathcal{D}_{c}(D_{E}^{\text{MAX}})$, and let $\sigma \in \mathcal{D}_{c}$. Choose a locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_{i}, G_{i})| i \in \mathbb{N}\}$, with associated smooth partition of unity $\eta = \{\eta_{i}\}_{i \in \mathbb{N}}$. Suppose that $\text{supp}(\eta_{i}) \cap \text{supp}(\sigma) \neq \emptyset$ only for $i = 1, \ldots, \ell$. Then $\sigma = \sigma_{1} + \ldots, \sigma_{\ell}$, with $\sigma_{i} = \eta_{i}\sigma$ having support in $U_{i}$, $i = 1, \ldots, \ell$. We can lift $\sigma_{i}$ to a $G_{i}$–invariant section $\tilde{\sigma}_{i}$. By trivializing the bundle $\tilde{E}$ over $\tilde{U}_{i}$, we can assume that we are dealing with functions. Convolutions with an approximated identity and averaging, give a $G_{i}$–invariant sequence $\{\tilde{u}_{i,k}\}_{k \in \mathbb{N}}$ in $C_{c}^{\infty}(\tilde{U}_{i}, \tilde{E})$ whose image $\{u_{i,k}\}_{k \in \mathbb{N}}$ in $C_{c}^{\infty}(U_{i}, \mathcal{E})$ satisfies $N(\sigma_{i} - u_{i,k}) < \frac{1}{k}$, $i = 1, \ldots, \ell$. (The $L^{2}$–norm is computed by dividing by $|G_{i}|$ and integrating on $\tilde{U}_{i}$.) Now, if we set $u_{k} = u_{1,k} + \cdots + u_{\ell,k}$, we have $N(\sigma - u_{k}) < \frac{1}{k}$, so $u_{k} \to \sigma$ in the norm $N$. □

**Proof of Lemma 3.3.** Set $\mathcal{D}_{c} = \mathcal{D}_{c}(D_{E}^{\text{MAX}})$, $\mathcal{D} = \mathcal{D}(D_{E}^{\text{MAX}})$, and $D = D_{E}$. Let $y_{0} \in X - \Sigma(X)$ be fixed, where $\Sigma(X)$ is the singular locus of $X$. Let $y \in X$ and denote by $\rho(y)$ the orbifold distance between $y_{0}$ and
y. We will only be interested in the behaviour of $\rho$ at points of the convex manifold $X - \Sigma(X)$. (For more details on $X - \Sigma(X)$, see [Stan] and [B].) Note also that $\Sigma(X)$ has measure zero in $X$, [Ch], and so $\rho$ is a function which is differentiable on $X$ a.e. Therefore we can assume that we have

$$\|\nabla \rho\| \leq 1$$

almost everywhere on $X$,

where the above norm is the sup norm, c.f. [W; pg. 623]; we need to additionally remove the measure zero set $\Sigma(X)$. We can now proceed as in [W; Proof of (5.5)]. For completeness, we go through all the details of the proof below. If $r > 0$, let

$$B_r = \{ y \in X | \rho(y) < r \}.$$ 

Since $X$ is complete the closure of $B_r$, $\overline{B_r}$, is compact. Choose a $C^\infty$ function $a : \mathbb{R} \to [0,1]$ such that $a(-\infty, 1] = 1$, $a[2, +\infty) = 0$, and denote by $M$ the max of $a'$ on $\mathbb{R}$. If $r > 0$ as before, define

$$b_r : X \to [0,1], \quad \text{by} \quad b_r(y) = a\left(\frac{\rho(y)}{r}\right).$$

Then

$$b_r = 1 \text{ on } B_r, \quad \text{supp}(b_r) \subseteq \overline{B_{2r}}.$$ 

We have that $b_r$ is differentiable almost everywhere, and, at points of differentiability, the following inequality holds

$$\|\nabla(b_r)\|^2 = \left\| \frac{1}{r} a'\left(\frac{\rho}{r}\right) \right\|^2 \leq \frac{M^2}{r^2}. $$

Fix $\sigma \in \mathcal{D}$, and write $\sigma_s = b_s \sigma$, for $s \in \mathbb{N}$. Now $\sigma_s \in \mathcal{D}_c$, since the support of $b_s$ is contained in $\overline{B_{2s}}$, compact. Choose a locally finite orbifold atlas $\mathcal{F}$, $\mathcal{F} = \{(\tilde{U}_i, G_i) | i \in \mathbb{N}\}$, with associated smooth partition of unity $\eta = \{\eta_i\}_{i \in \mathbb{N}}$. Suppose that $supp(\eta_i) \cap \overline{B_{2k}} \neq \emptyset$ only for $i = 1, \ldots, \ell$. Then on a local chart $\tilde{U}_i$, $i = 1, \ldots, \ell$, we have, as in the proof of Theorem 2.2 (as usual denote by $\tilde{\cdot}$ the lift to $\tilde{U}_i$),

$$\tilde{D}(\tilde{\sigma}_s) = \tilde{D}(\tilde{b}_s \tilde{\sigma}) = \sum_{j=1}^{n} e_j \left( \nabla e_j (\tilde{b}_s \tilde{\sigma}) \right)$$

$$= \sum_{j=1}^{n} e_j \left( e_j (\tilde{b}_s) \tilde{\sigma} + (\tilde{b}_s \nabla e_j \tilde{\sigma}) \right)$$

$$= \nabla (\tilde{b}_s) \tilde{\sigma} + (\tilde{b}_s) \tilde{D}(\tilde{\sigma})$$

almost everywhere on $\tilde{U}_i$. Since $b_s = 1$ on $B_s$, we have

$$\|D(\sigma - \sigma_s)\|_X^2 = \|(1 - b_s)D(\sigma) + \nabla(b_s)\sigma\|_X^2.$$
\[ \leq \|D(\sigma)\|_{\mathcal{L}^2(\sim B_s)}^2 + \frac{M^2}{s^2}\|\sigma\|_{\mathcal{L}^2(\sim B_s)}^2, \]

where \( \| \cdot \|_Y \) denotes the \( \mathcal{L}^2(Y, \mathcal{E}) \) norm.

We thus obtain

\[ N(\sigma - \sigma_s)^2 \leq \|D(\sigma)\|_{\mathcal{L}^2(\sim B_s)}^2 + \|\sigma\|_{\mathcal{L}^2(\sim B_s)}^2 \frac{M^2}{s^2}\|\sigma\|_{\mathcal{L}^2(\sim B_s)}^2, \]

for any \( s = 1, \ldots, \ell \). This completes the proof of Lemma 3.3. \( \square \).

To end this section, we would like to state separately a very useful fact shown in the proof of Lemma 3.3.

**Proposition 3.4.** Let \( X \) be a non-compact complete orbifold which is sufficiently regular at infinity, and let \( y_0 \in X - \Sigma(X) \) be a fixed point of \( X \). Then there exists a sequence of continuous functions \( b_k, k \in \mathbb{N} \), with

1. \( b_k : X \to [0, 1] \)
2. \( b_k = 1 \) on \( B_k = \{ y \in X | \rho(y) = d(y, y_0) \leq k \} \).
3. The support of \( b_k \) is contained in \( \overline{B}_{2k} \).
4. The function \( b_k \) is differentiable almost everywhere and at points of differentiability we have

\[ \|\nabla (b_k)\|^2 \leq \frac{M^2}{k^2}, \quad k \in \mathbb{N}. \]

4. **The Square of the Dirac Operator.**

As we have seen in Section 3, there is always a unique, closed, self-adjoint extension of a generalized Dirac operator \( D_E \) on a complete orbifold \( X \) which is sufficiently regular at infinity. This unique extension will still be called \( D_E \) and its domain will be denoted by \( \mathcal{D}(D_E) \). In particular, for any two sections \( \sigma_1, \sigma_2 \) of \( \mathcal{D}(D_E) \), we have

\[ (D_E\sigma_1, \sigma_2) = (\sigma_1, D_E\sigma_2). \]

From this we will derive below that if \( \sigma \in \mathcal{D}(D_E) \), then \( D_E(\sigma) = 0 \) if and only if \( D_{E}^2(\sigma) = 0 \).

**Theorem 4.1.** Let \( X \) be a non-compact complete orbifold which is sufficiently regular at infinity, and let \( E \) be a Hermitian orbibundle (with connection \( \nabla^E \)) over \( X \). Let \( D_E : \mathcal{C}_c^\infty(X, \mathcal{E}) \to \mathcal{C}_c^\infty(X, \mathcal{E}) \), be the generalized Dirac operator on \( X \) with coefficients in \( E \). Then \( D_E(\sigma) = 0 \) if and only if \( D_{E}^2(\sigma) = 0 \) for any \( \sigma \in \mathcal{D}(D_E) \).
Proof. Set $D = D_E$ and $\mathcal{D} = \mathcal{D}(D_E)$. The non-trivial part of the proof is to show that $D^2 \sigma = 0$ implies $D \sigma = 0$. Since $D^2$ is elliptic, the equation $D^2 \sigma = 0$ implies $\sigma \in C^\infty(X, \mathcal{E})$. In fact, via a partition of unity, we can consider this equation on a chart of a locally finite orbifold atlas. Then, at this level, we are dealing with a manifold elliptic operator, and therefore standard local theorems on elliptic operators apply, such as the smoothness of solutions of elliptic systems we need. Now choose a sequence $\{b_k\}, k \in \mathbb{N}$, as in the proof of Lemma 3.3 and in Proposition 3.4. Then we have, for any $\sigma \in C(X, \mathcal{E})$ with $D \sigma = 0$,

$$(D^2 \sigma, b_k^2 \sigma) = (D \sigma, D(b_k^2 \sigma))$$

$$= (D \sigma, 2b_k \nabla(b_k) \sigma + b_k^2 D \sigma) = (b_k D \sigma, 2 \nabla(b_k) \sigma + b_k D \sigma),$$

since $D(f \sigma) = (\nabla f) \sigma + f D(\sigma)$ for any $f \in C^\infty(X)$, almost everywhere. Now we have (recall that $\| \|_X$ is the $\mathcal{L}^2(X, \mathcal{E})$ norm),

$$\|b_k D(\sigma)\|_X^2 = -2(b_k D \sigma, \nabla(b_k) \sigma) \leq \frac{M}{k} (\|b_k D(\sigma)\|_X^2 + \|\sigma\|_X^2)$$

by Proposition 3.4 and the Schwartz inequality. Since the limit of $\frac{M}{k} \|\sigma\|_X^2$ tends to 0 as $k \to +\infty$, it follows that $\|b_k D(\sigma)\|_X^2$ tends to 0 as $k \to +\infty$. But as in the proof of Lemma 3.3,

$$\|D(\sigma - \sigma_k)\|_X^2 \leq \|D(\sigma)\|_X^2 - B_k + \frac{M^2}{k^2} \|\sigma\|_X^2.$$ 

Hence

$$\lim_{k \to +\infty} \|D(\sigma - \sigma_k)\|_X^2 = 0$$

as the union of all $B_k$’s is $X$, and $D(\sigma) \in \mathcal{L}^2(X, \mathcal{E})$. Then

$$\lim_{k \to +\infty} D(\sigma_k) = D(\sigma)$$

in $\mathcal{L}^2(X, \mathcal{E})$, which implies

$$\lim_{k \to +\infty} \|D(\sigma_k)\|_X = \|D(\sigma)\|_X.$$

As $D(b_k \sigma) = (\nabla b_k) \sigma + b_k D(\sigma)$ almost everywhere, $\forall k \in \mathbb{N}$, we have

$$\lim_{k \to +\infty} \|D(\sigma_k)\|_X = \lim_{k \to +\infty} \|b_k D(\sigma)\|_X$$

by Proposition 3.4. Hence

$$\|D(\sigma)\|_X = \lim_{k \to +\infty} \|D(\sigma_k)\|_X = \lim_{k \to +\infty} \|b_k D(\sigma)\|_X = 0. \quad \square$$
The statement of Theorem 4.1 is also true for sections in $L^2(X, \mathcal{E})$, as can be shown using approximation.

5. The Stokes'/Divergence Theorem on Non–Compact Orbifolds.

In this section we will state and prove a Stokes'/Divergence theorem which is a generalization of manifold results of Gaffney, Yau, and Karp, see [Gn2], [Y], [K]. Our presentation follows the outline given in [K] for the corresponding manifold case. The proof of our theorem relies heavily on the results we proved in Sections 3 and 4.

Given a vector field $V$ on an orbifold $X$, choose a locally finite orbifold atlas $\mathcal{F} = \{(\tilde{U}_i, G_i) | i \in \mathbb{N}\}$, with associated smooth partition of unity $\eta = \{\eta_i\}, i \in \mathbb{N}$. Then the divergence of $V$ is given in local charts by,

$$
\text{div}(\tilde{V}) = \sum_k \frac{1}{\sqrt{g}} \frac{\partial}{\partial \tilde{x}_k} \left( \sqrt{\tilde{g}} \tilde{V}_k \right), \quad \tilde{V} = \sum_k \tilde{V}_k \frac{\partial}{\partial \tilde{x}_k} \text{ on } \tilde{U}_i.
$$

**Theorem 5.1.** Let $X$ be an even–dimensional non–compact complete Spin$^c$ almost complex orbifold which is sufficiently regular at infinity. Assume that a connection is chosen on the dual of its canonical line bundle. Let $V$ be a vector field on $X$ such that

$$
\lim_{k \to +\infty} \inf \frac{1}{k} \int_{B_{2k} - B_k} \|V\| \, dv = 0,
$$

where $\|V\|$ denotes the length of $V$, and $B_k = \{y \in X | \rho(y) = d(y, y_0) \leq k\}$ for a fixed $y_0 \in X - \Sigma(X)$, where $\Sigma(X)$ is the singular locus of $X$. Then if either $(\text{div } V)^+$ or $(\text{div } V)^-$ is integrable on $X$, we have

$$
\int_X \text{div}(V) \, dv = 0.
$$

**Proof.** Choose a sequence $\{b_k\}, k \in \mathbb{N}$, as in Proposition 3.4. Integrating $\text{div}(b_k^2 V)$ over $B_{2k}$, for a sufficiently large $k$, and applying the divergence theorem for finite domains, we obtain

$$
0 = \int_{B_{2k}} \text{div}(b_k^2 V) \, dv.
$$

Hence, by Proposition 3.4,

$$
\left| \int_{B_{2k}} b_k^2 \text{div}(V) \, dv \right| \leq M \frac{1}{k} \int_{B_{2k} - B_k} \|V\| \, dv.
$$

Thus, if we for example suppose $(\text{div } V)^-$ integrable, the above inequality implies

$$
\int_{B_k} (\text{div } V)^+ \, dv - \int_X (\text{div } V)^- \, dv \leq M \frac{1}{k} \int_{B_{2k} - B_k} \|V\| \, dv.
$$
Because of our hypothesis, we can choose a sequence \( k(j) \to +\infty \), such that

\[
\lim_{j \to +\infty} \int_{B_{2k(j)} - B_k(j)} \|V\| \, dv = 0.
\]

Consequently, \((\text{div}(V))^-\) is also integrable, and

\[
\int_X \text{div}(V) \, dv \leq 0.
\]

But now the same argument can be repeated started from \((\text{div}(V))^-\). Hence

\[
\int_X \text{div}(V) \, dv = 0. \quad \Box
\]

**Corollary 5.2.** Let \( X \) and \( V \) be as in Theorem 5.1, and also assume that \( X \) has \( q \)-th order volume growth (i.e., there exists \( c > 0 \) and \( q \geq 1 \) such that \( \text{vol}(B_k) \leq c k^q, \forall k \geq 1 \)). If \( \text{div}(V) \geq 0 \) outside of some compact set, and either

1. \( q > 1 \) and \( V \in L^p(X, \mathcal{E}) \), with \( \frac{1}{q} + \frac{1}{p} = 1 \), or
2. \( q = 1 \) and \( \|V\| \to 0 \) uniformly at \( \infty \) in \( X \), then

\[
\int_X \text{div}(V) \, dv = 0.
\]

**Proof.** Very similar to [K; Proof of Corollary 1]. \( \Box \)

### 6. Some Vanishing Theorems.

The results in this section are a generalization to orbifolds of some of the results proved for manifolds by Gromov and Lawson in [GL]. In this section, we will let \( E = \mathbb{C} \) unless otherwise noticed. We will also substitute \( S \) for \( \mathcal{E} \).

The scalar orbifold Laplacian \( \Delta \) can be defined in analogy with the Laplacian on manifolds. (c.f. [Ch] Section 2.) In fact, on an orbifold chart \( \tilde{U}_i \) of a standard orbifold atlas \( \mathcal{U} = \{ (\tilde{U}_i, G_i) | i \in I \} \) of \( X \), we define, (here \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \) denotes the coordinate in \( \tilde{U}_i \)),

\[
\Delta \tilde{u} = \sum_{k,j} \tilde{g}^{k,j} \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_k \partial \tilde{x}_j} - \sum_j \tilde{B}_j \frac{\partial \tilde{u}}{\partial \tilde{x}_j},
\]

with

\[
\tilde{B}_j = \frac{1}{2g} \sum_k \frac{\partial \tilde{g}}{\partial \tilde{x}_k} \tilde{g}^{k,j} + \sum_k \frac{\partial g_{k,j}}{\partial \tilde{x}_k}, \quad \forall u \in \mathcal{C}^\infty(U_i).
\]

In the above expression, \( \tilde{g} = \tilde{g}^{k,j}, k,j = 1, \ldots, n \), is a \( G_i \)-invariant metric. Laplacians can also be defined to act on general orbibundles such as \( S \) by using the above definition on orbisections. The following Green’s formula holds.
Proposition 6.1. Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity, and let $S$ be the Spin\(^c\) bundle of $X$. Then for any two sections $\sigma_j$, $j = 1, 2$ in $C^\infty(X, S)$, at least one of which with compact support, we have
\[
\int_X <\Delta \sigma_1, \sigma_2> \, dv = \int_X <\nabla \sigma_1, \nabla \sigma_2> \, dv
\]

Proof. Because of our hypothesis at infinity, the proof given in [Ch; Section 2] in the scalar case is also valid here. In particular, this result is an orbifold version of [Si; Proposition 1.2.2], which can be proved as in the manifold case. □

Proposition 6.1 motivates us to choose the Sobolev norm
\[
\|\sigma\|_1^2 = \int_X (\langle \sigma, \sigma \rangle + \langle \nabla \sigma, \nabla \sigma \rangle) \, dv = \int_X (\langle \sigma, \sigma \rangle + \langle \Delta \sigma, \sigma \rangle) \, dv
\]

Thus, by reasoning as in Section 5, we obtain,

Theorem 6.2. Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity. Then the domain of the unique closed self-adjoint extension of the Spin\(^c\) Laplacian $\Delta$, $\Delta : C^\infty_c(X, S) \to L^2(X, S)$, is the completion $L^{1,2}(X, S)$ of $C^\infty_c(X, S)$ in the norm $\| \cdot \|_1$. Furthermore, $\Delta(\sigma) = 0$ if and only if $\nabla(\sigma) = 0$, i.e., $\sigma$ is parallel.

Proof. We only need to prove the last claim, which follows from Proposition 6.1 in the case of sections with compact support. The general case follows from Laplacian analogs of Lemmas 3.2 and 3.3. □

The important Bochner–Weitzenböck formula, a classic result for manifolds, can also be easily extended to orbifolds, by using local coordinates.

Proposition 6.3. Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity. If $D$ is the Dirac operator on $X$ with coefficients in the Spin\(^c\) bundle $S$, and $\Delta$ is the Spin\(^c\) Laplacian, then
\[
D^2 = \Delta + R,
\]
where $R$ is given below (c.f. [Du; Theorem 6.1], [LM; Theorem D12] for the manifold case),
\[
R = \frac{1}{4} k + \frac{1}{2} c(K^*),
\]
where $k$ is the scalar curvature, and $c(K^*)$ denotes the Clifford multiplication of the curvature 2 form of the fixed connection on the line bundle $K^*$.

As a consequence of the above formula, we obtain, as in [GL; Theorem 2.8],
Theorem 6.4. Let $X$ be a non-compact complete orbifold which is sufficiently regular at infinity. If $D$ is the Dirac operator on $X$ with coefficients in the $\text{Spin}^c$ bundle $S$, then the domain $\mathcal{D}$ of the unique self-adjoint extension of $D$ is exactly

$$\mathcal{L}^{1,2}(X, S),$$

that is, the completion of $\mathcal{C}_{c}^\infty(X, S)$ in the norm

$$\|\sigma\|_1^2 = \int_X (\langle \sigma, \sigma \rangle + \langle \nabla \sigma, \nabla \sigma \rangle) \, dv = \int_X (\langle \sigma, \sigma \rangle + \langle \Delta_\sigma, \sigma \rangle) \, dv$$

Moreover, for every $\sigma \in \mathcal{D}$,

$$\|D\sigma\|_X^2 = \|\nabla \sigma\|_X^2 + (\mathcal{R}\sigma, \sigma),$$

where $\| \cdot \|_X$ denotes the $L^2$ norm, $\mathcal{R}$ is as in Theorem 6.3, and $(\cdot, \cdot)$ the $L^2$ inner product.

Proof. For sections with compact support Theorem 6.4 follows directly from the Bochner–Weitzenböck formula and the self-adjointness of the Dirac operator. More in general, approximate a section $\sigma \in \mathcal{D}$, via the $L^2$ norm, by a sequence $\sigma_k$ of compact support such that $D\sigma_k \to D\sigma$. (This is possible because $\mathcal{D}$ is in particular equal to the minimal domain of $D$.) By passing to the limit, we obtain,

$$\|D\sigma\|_X^2 = \|\nabla \sigma\|_X^2 + (\mathcal{R}\sigma, \sigma),$$

since $\mathcal{R}$ is bounded. \qed

The following corollaries can be derived as in the manifold case (see [GL; Section 2]). We will thus leave their proofs to the reader.

Corollary 6.5. Let $X$ be a non-compact, complete orbifold which is sufficiently regular at infinity. Let $D$ and $\mathcal{R}$ be as in Theorem 6.4. Suppose that $\mathcal{R} > 0$ pointwise on $X$. Then

$$\ker(D) = \operatorname{coker}(D) = 0.$$

If furthermore, $\mathcal{R} \geq c Id$, for some constant $c > 0$, then $D : \mathcal{L}^{1,2}(X, S) \to \mathcal{L}^2(X, S)$ is an isomorphism of Hilbert spaces. In this case, $D^{-1} : \mathcal{L}^{1,2}(X, S) \to \mathcal{L}^2(X, S)$ is also a bounded operator.

Corollary 6.6. Let $X$, $D$ and $\mathcal{R}$ be as in Corollary 6.5, with $\mathcal{R} \geq c Id$, for some constant $c > 0$. Since $X$ is even-dimensional, $D : \mathcal{L}^{1,2}(X, S) \to \mathcal{L}^2(X, S)$ splits into its $\pm$ decomposition, $D^+$ and $D^-$, see Section 1. Then both $D^+$ and $D^-$ have bounded inverses.
Remark 6.7. The results of this section can also be proved, with suitable modifications for generalized Dirac operators with coefficients in any Hermitian orbibundle $E$ (with connection $\nabla^E$).

References

[At1] M.F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Colloque "Analyse et Topologie" en l’honneur de Henri Cartan (Orsay, 1974), pp. 43–72. Asterisque, No. 32-33, Soc. Math. France, Paris, 1976.

[AX] B. Apanasov and X. Xie, Discrete actions on nilpotent Lie groups and negatively curved spaces, Differential Geom. Appl. 20 (2004), 11–29.

[B] J. Borzelliino, Orbifolds of maximal diameter, Indiana Univ. Math. J. 42 (1993), 37–53.

[BGV] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators, Grundleheren der Mathematischen Wissenschaften 298, Springer–Verlag, Berlin, 1992.

[C] G. Chen, Calculus on orbifolds, Sichuan Daxue Xuebao 41 (2004), 931–939.

[Ch] Y.-C. Chiang, Harmonic maps of $V$-manifolds, Ann. Global Anal. Geom. 8 (1990), 315-344.

[Du] J. J. Duistermaat, The heat kernel Lefschetz fixed point formula for the $Spin^c$ Dirac operator, Progress in Nonlinear Differential Equations and their Applications, 18. Birkhäuser, Inc., Boston, MA, 1996.

[Fa1] C. Farsi, $K$-theoretical index theorems for orbifolds, Quat. J. Math. 43 (92), 183–200.

[Fa2] Orbifold spectral theory, Rocky Mtn. J. Math. 31(2001), 215–235.

[Fa3] Orbifold $\eta$-invariants, Indiana Math. Journal, Indiana Math. J. 35 (2007), 501-521.

[Fa4] A relative orbifold index theorem, J. Geom Phys. 8 (2007), 1653-1668.

[GL] M. Gromov and M. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Inst. Hautes Etudes Sci. Publ. Math. No. 58, (1983), 83–196.

[Gn1] M. Gaffney, The harmonic operator for exterior differential forms, Proc. Nat. Acad. Sci. U. S. A. 37 (1951), 48–50.

[Gn2] M. Gaffney, A special Stokes’s theorem for complete Riemannian manifolds, Ann. of Math. (2) 60 (1954), 140–145.

[K] L. Karp, On Stokes’ theorem for noncompact manifolds, Proc. Amer. Math. Soc. 82 (1981), 487–490.
[Kw1] T. Kawasaki, The signature theorem for $V$–manifolds, Topology 17 (1978), 75–83.

[Kw2] The Riemann Roch theorem for complex $V$–manifolds, Osaka J. Math. 16 (1979), 151–159.

[Kw3] The index of elliptic operators over $V$–manifolds, Nagoya Math. J. 84 (1981), 135–157.

[LM] H. B. Lawson, Jr, and M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, New Jersey, 1989.

[LR] W. Lück and J. Rosenberg, Equivariant Euler characteristics and $K$–homology Euler classes for proper cocompact $G$–manifolds, Geom. Topol. 7 (2003), 569–613.

[LoR] D. D. Long, and A. W. Reid, All flat manifolds are cusps of hyperbolic orbifolds, Algebr. Geom. Topol. 2 (2002), 285–296.

[N] Y. Nakagawa, An isoperimetric inequality for orbifolds, Osaka J. Math. 30 (1993), 733–739.

[Si] J. Simons, Minimal varieties in Riemannian manifolds, Ann. of Math. (2) 88 (1968), 62–105.

[Stan] E. Stanhope, Spectral bounds on orbifold isotropy, Ann. Global Anal. Geom. 27 (2005), 355–375.

[V] M. Vergne, Equivariant index formulas for orbifolds. Duke Math. J. 82 (1996), no. 3, 637–652.

[W] J. Wolf, Essential self-adjointness for the Dirac operator and its square, Indiana Univ. Math. J. 22 (1972/73), 611–640.

[Y] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J. 25 (1976), 659–670.