Quantum lower and upper speed limits using reference evolutions

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Abstract

We derive generalized quantum speed limit inequalities that represent limitations on the time evolution of quantum states. They are extensions of the original inequality and are applied to the overlap between the time-evolved state and an arbitrary state. We can discuss the lower limit of the Bures angle, in addition to the upper limit as in the original inequality, which allows us to evaluate the lower and upper bounds of processing time for the evolution toward a target state. The inequalities are written by using an arbitrary reference state and are flexibly used to obtain a tight bound. We demonstrate these properties by using the twisted Landau–Zener model, the Grover Hamiltonian, and a periodically-oscillating Hamiltonian.

1. Introduction

The quantum speed limit is an inequality applied to a distance measure of quantum states. In the Mandelstam–Tamm relation [1–4], the distance measure is bounded from above by the square root of the energy variance. Basically, we can write an inequality relation between the distance \( D \), the processing time \( t \), and the square root of the energy variance \( \sigma \) as

\[
\frac{D}{\sigma} \leq t.
\]

Here, and throughout the paper, we omit the Planck constant \( \hbar \) to set the time scale to be equal to the inverse energy scale. The right-hand side is interpreted as a minimum time for the system to evolve into a different state with the distance \( D \) from the initial state. In the Margolus–Levitin relation [5], a similar relation holds by replacing \( \sigma \) with the average energy. The inequalities are interpreted as time-energy uncertainty relations and are closely related to the geometric structure of Hilbert space.

The original inequalities applied to quantum pure states can be generalized to other problems such as quantum mixed states with dissipative environments and probability distributions in classical stochastic processes. Different forms of the speed limit are obtained in different situations, but they are written by the corresponding metric and can be treated in a unified way. They are applied to various problems including quantum control, quantum computing, information processing, and stochastic thermodynamics. We can find many applications in literature [6–15].

Although the inequality is applied to a broad range of systems, it gives a trivial relation in most of realistic systems. The right-hand side of equation (1) linearly grows while the left-hand side is basically independent of \( t \). In addition, \( \sigma \) is typically proportional to the system size and the left-hand side of equation (1) takes a small value for many-body systems.

The Mandelstam–Tamm relation can be applied even when the system Hamiltonian changes as a function of time. The state of the system by the time-dependent Hamiltonian changes in a complicated way. For example, when the Hamiltonian is a periodically-oscillating one, the state changes back and forth. Then, it is not appropriate to characterize the distance between the initial state and the time-evolved state.
The idea of treating the distance between the time-evolved state and a reference state has been discussed in several works [16–19]. For example, when the system Hamiltonian is changed slowly in time, the state is approximately given by the adiabatic state. Then, it is appropriate to use the adiabatic state as a reference one. In [17], an inequality was derived for the overlap between the time-evolved state and the adiabatic state. The idea can be applied to arbitrary reference states [18] and some applications were discussed in [20, 21]. The use of the adiabatic state was also discussed in [16, 22] to derive a nontrivial relation for the overlap. An inequality using the reference time evolutions was proposed as a conjecture [23].

The quantum speed limit is basically represents a minimum time for the system to evolve into a different state. It is based on the geometric property of the system. As a different interesting relation derived from the geometric property, we can discuss an upper limit of the time evolution of a quantum state is characterized in many different ways. In this paper, we pursue this problem and derive quantum speed limit inequalities satisfied among three quantum states. The result is a generalization of the original Mandelstam–Tamm relation and is applicable in section 3, the Grover Hamiltonian in section 4, and a periodically-oscillating Hamiltonian in section 5. The result is summarized in section 6.

2. Speed limits from triangle inequality

We consider quantum states described by the density operator \( \hat{\rho} \). Different quantum states are distinguished by introducing a distance measure \( D(\hat{\rho}_1, \hat{\rho}_2) \). Its properties are prescribed by the standard requirements: nonnegativity, symmetry, and triangle inequality

\[
D(\hat{\rho}_1, \hat{\rho}_2) \leq D(\hat{\rho}_2, \hat{\rho}_3) + D(\hat{\rho}_3, \hat{\rho}_1).
\]

It is well-known that the trace distance and the Bures angle satisfy the requirements and are used in many applications [27]. In the present paper, we mainly use the Bures angle defined from the fidelity

\[
\Theta(\hat{\rho}_1, \hat{\rho}_2) = \arccos \operatorname{Tr}\left( \sqrt{\frac{1}{2} \hat{\rho}_1^{1/2} \hat{\rho}_2 \hat{\rho}_1^{1/2}} \right)^{1/2}.
\]

When the system is described by the pure state \( \hat{\rho} = |\psi\rangle \langle \psi| \), the Bures angle is written by the state overlap

\[
\Theta_{\psi_1, \psi_2} = \arccos |\langle \psi_1 | \psi_2 \rangle|.
\]

Then, the triangle inequality reads

\[
|\Theta_{\psi_1, \psi_3} - \Theta_{\psi_2, \psi_3}| \leq \Theta_{\psi_1, \psi_2} \leq \Theta_{\psi_1, \psi_3} + \Theta_{\psi_2, \psi_3}.
\]

We can use this inequality to find a lower limit and an upper limit for \( \Theta_{\psi_1, \psi_2} \). The triangle inequality among three quantum states was used to derive a novel type of speed limit [16, 22]. Here, we derive a different relation by using the method in [17, 18].

In most of the applications, we are interested in the fidelity between the time-evolved state \( |\psi(t)\rangle \) and a target state \( |\psi_{\text{target}}\rangle \). The former state is obtained from the Schrödinger equation \( i\hbar \partial_t |\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \) with the Hamiltonian \( \hat{H}(t) \). The latter is set to be the initial state in the standard method. Here, we can take it as an arbitrary state. We also introduce a reference state \( |\psi_{\text{ref}}(t)\rangle \) to write the relation

\[
\Theta_{\psi_{\text{target}}, \psi_{\text{ref}}(t)} - \Theta_{\psi_{\text{ref}}(t), \psi(t)} \leq \Theta_{\psi_{\text{target}}, \psi(t)} \leq \Theta_{\psi_{\text{target}}, \psi_{\text{ref}}(t)} + \Theta_{\psi_{\text{ref}}(t), \psi(t)}.
\]

In the standard quantum speed limit inequality, the Bures angle \( \Theta_{\psi(0), \psi(t)} \) is bounded from above as

\[
\Theta_{\psi(0), \psi(t)} \leq \int_0^t dt \sigma[\hat{H}(t'), |\psi(t')\rangle] \text{ where } \sigma[\hat{H}, |\psi\rangle] = \langle \psi| \hat{H}^2|\psi\rangle - \langle \psi| \hat{H}|\psi\rangle^2/2.
\]

The main idea of the present paper is to use the inequality for the Bures angle \( \Theta_{\psi_{\text{ref}}(t), \psi(t)} \) developed in [17, 18]. To apply the standard Mandelstam–Tamm relation, we introduce the Hermitian operator \( \hat{H}_{\text{ref}}(t) \) from the relation

\[
i\hbar \partial_t |\psi_{\text{ref}}(t)\rangle = \hat{H}_{\text{ref}}(t)|\psi_{\text{ref}}(t)\rangle \text{ with } |\psi_{\text{ref}}(0)\rangle = |\psi(0)\rangle.
\]

We write the overlap \( \langle \psi_{\text{ref}}(t) | \psi(t) \rangle \) as \( \langle \psi(0) | \tilde{\psi}(t) \rangle \) where \( |\tilde{\psi}(t)\rangle \) satisfies the Schrödinger equation with an effective Hamiltonian comprised of \( \hat{H}(t) \) and \( \hat{H}_{\text{ref}}(t) \) [17, 18]. Then, we obtain

\[
\Theta_1 \leq \Theta_{\psi_{\text{target}}, \psi(t)} \leq \Theta_a,
\]
where

\[ \Theta_{\text{f}} = \Theta_{\psi_{\text{target}}, \psi_{\text{ref}}} - \int_{0}^{t} \text{d}t' \sigma [\hat{H}(t') - \hat{H}_{\text{ref}}(t'), \psi_{\text{ref}}(t')] \tag{8} \]

\[ \Theta_{\text{u}} = \Theta_{\psi_{\text{target}}, \psi_{\text{ref}}} + \int_{0}^{t} \text{d}t' \sigma [\hat{H}(t') - \hat{H}_{\text{ref}}(t'), \psi_{\text{ref}}(t')] \tag{9} \]

This is the main result of the present paper.

The main advantage of the relation in equation (7) is that the inequalities hold for arbitrary choices of a target state \(|\psi_{\text{target}}\rangle\) and a reference state \(|\psi_{\text{ref}}(t)\rangle\). By using the reference state \(|\psi_{\text{ref}}(t)\rangle\), we can derive a lower limit of \(\Theta_{\psi_{\text{target}}, \psi_{\text{ref}}}\) in addition to the upper limit. The bounds can be estimated without knowing the time-evolved state \(|\psi(t)\rangle\).

We note that the inequalities also hold when we use \(|\psi(t')\rangle\) in place of \(|\psi_{\text{ref}}(t)\rangle\) for \(\sigma\) [17]. The standard upper speed limit is obtained by setting \(|\psi_{\text{target}}\rangle = |\psi(0)\rangle\), \(\hat{H}_{\text{ref}}(t) = 0\), and \(\sigma[H(t') - \hat{H}_{\text{ref}}(t'), \psi_{\text{ref}}(t')] \to \sigma[H(t'), \psi(t')]\). We can also find the result in [19] by setting \(|\psi_{\text{target}}\rangle = |\psi(0)\rangle\), \(\hat{H}_{\text{ref}}(t) = 0\), \(\hat{H}(t) = \hat{H}\), and \(\sigma[H(t') - \hat{H}_{\text{ref}}(t'), \psi_{\text{ref}}(t')] \to \sigma[H, \psi(t')]\). We also note that the reverse quantum speed limit discussed in [15] is different from the present result since the reverse limit denotes an upper bound of equation (1). The present method treats a lower bound of equation (1).

Since the Bures angle satisfies \(0 \leq \Theta \leq \pi/2\), the bound is useful only when it is within the domain of definition. When we choose \(|\psi_{\text{target}}\rangle = |\psi(0)\rangle\), \(\Theta_{\psi(0), \psi(t)}\) is upper-bounded for a short period of time and we can discuss how fast the time-evolved state deviates from the initial state. This gives a result similar to the standard speed limit. By choosing the reference state in a proper way, we can obtain a tight bound. On the other hand, when the state evolves toward a target state \(|\psi_{\text{target}}\rangle\) with \(|\psi_{\text{target}}|\psi(0)\rangle = 0\), we can estimate a minimum time \(t\) for the time-evolved state \(|\psi(t)\rangle\) to reach the target state as \(|\langle\psi_{\text{target}}|\psi(t)\rangle| = 1\).

There are several ways to utilize the bounds. In the following sections, we discuss possible applications by using several model Hamiltonians.

3. Twisted Landau–Zener model: tight bound at large time

When we exactly know the state evolution \(|\psi_{\text{ref}}(t)\rangle\) under the Hamiltonian \(\hat{H}_{\text{ref}}(t)\), we can estimate the bound on the unknown state \(|\psi(t)\rangle\) under the Hamiltonian \(\hat{H}(t)\). Since the bound is represented by the time integration of \(\sigma\), the bound becomes tight when \(\sigma\) takes nonzero values only for a finite interval of \(t\).

We study these properties by using the twisted Landau–Zener Hamiltonian [28]

\[ \hat{H}(t) = \begin{pmatrix} vt & \Delta e^{-i\varphi(t)} \\ \Delta e^{i\varphi(t)} & -vt \end{pmatrix}. \tag{10} \]

\(\Delta\) and \(v\) are arbitrary positive parameters and \(\varphi(t)\) represents an arbitrary function of \(t\). This Hamiltonian is compared to the standard form of the Landau–Zener Hamiltonian [29, 30]

\[ \hat{H}_{\text{ref}}(t) = \begin{pmatrix} vt & \Delta \\ \Delta & -vt \end{pmatrix}. \tag{11} \]

We start the time evolution from

\[ |\psi(-\infty)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{12} \]

and study the overlap \(\langle\psi_{\text{target}}|\psi(\infty)\rangle\) where \(|\psi_{\text{target}}\rangle = |\psi(-\infty)\rangle\). \(|\psi_{\text{ref}}(t)\rangle\) is exactly solvable and we obtain the Landau–Zener formula

\[ |\langle\psi_{\text{target}}|\psi_{\text{ref}}(\infty)\rangle| = \exp \left( -\frac{\pi \Delta^2}{2v} \right). \tag{13} \]

The twisted Landau–Zener model is equivalent to the Landau–Zener model with a nonlinear protocol. It is shown by using the transformation to the rotating frame as

\[ \exp \left[ \frac{i}{2}\varphi(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] (\hat{H}(t) - i\partial_t) \exp \left[ -\frac{i}{2}\varphi(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \]

\[ = \begin{pmatrix} vt - \frac{1}{2}\varphi(t) & \Delta \\ \Delta & -\left( vt - \frac{1}{2}\varphi(t) \right) \end{pmatrix}, \tag{14} \]
where the dot symbol denotes the time derivative. It is difficult to obtain the general solution for a given \( \varphi(t) \) and we apply the inequalities derived in the previous section.

With the use of the reference Hamiltonian in equation (11), the bounds of \( \Theta_{\text{target}, \psi(t)} \) in the twisted Landau–Zener model are written as

\[
\Theta_{\text{u}, l} = \arccos \exp \left( \frac{\pi \Delta^2}{2v^2} \right) \pm \int_0^\infty \! \! \\ dt \sigma (\hat{H}(t) - \hat{H}_{\text{ref}}(t), \psi_{\text{ref}}(t)).
\]

(15)

Using the relation \( \sigma (\hat{X}, \psi_{\text{ref}}) \leq \sqrt{\langle \psi_{\text{ref}} | \hat{X}^2 | \psi_{\text{ref}} \rangle} \), we can estimate the second term without knowing \( |\psi_{\text{ref}}(t)\rangle \) as

\[
\int_0^\infty \! \\ dt \sigma (\hat{H}(t) - \hat{H}_{\text{ref}}(t), \psi_{\text{ref}}(t)) \leq 2\Delta \int_0^\infty \! \\ dt \left| \sin \frac{\varphi(t)}{2} \right|.
\]

(16)
To obtain a finite value of the bounds, we require that \( \varphi(t) \bmod 2\pi \) is nonzero for a finite domain of \( t \). We choose two types of the protocol:

\[
\varphi(t) = \begin{cases} 
\pi \left( 1 + \tanh \frac{t}{\tau} \right) & \text{protocol 1} \\
\pi \exp \left( -\frac{t^2}{\tau^2} \right) & \text{protocol 2}
\end{cases}
\] (17)

We plot the parameter dependence of \( \Theta_{\text{charge},0(\infty)} \) in figure 1. We can find finite bounds even at the limit \( t \to \infty \). We note that the standard quantum speed limit does not give any meaningful result since the bound goes to infinity.

We also plot results for various choices of parameters in figure 2. The result shows that the bounds become tight for small \( \tau \) and large \( v \). The former condition, small \( \tau \), is reasonable since \( \sigma \) becomes small in that limit. The latter condition represents nonadiabatic regime. The opposite limit, small \( v \), is basically described by the adiabatic approximation. Our inequalities can be useful in the nonadiabatic regime rather than in the adiabatic one.

4. Grover Hamiltonian: protocol-independent bound

As we can see from the example in the previous section, the bound is in principle dependent on the Hamiltonian and on the protocol. In the present section, we discuss that a universal bound is obtained by choosing the reference state in a proper way.

One of the reasonable choice of the reference state is the adiabatic state \( |\psi_{\text{ad}}(t)\rangle \). It is written by using the instantaneous eigenstates of the Hamiltonian \( \hat{H}(t) \). It corresponds to choosing the reference Hamiltonian as the counteradiabatic Hamiltonian \( \hat{H}_{\text{ref}}(t) = \hat{H}(t) + \hat{H}_{\text{cd}}(t) \).

The counteradiabatic driving is used in the method of shortcuts to adiabaticity [31–36]. When the Hamiltonian is written by the spectral representation as

\[
\hat{H}(t) = \sum_n \epsilon_n(\lambda(t)) |n(\lambda(t))\rangle \langle n(\lambda(t))|,
\] (18)

the counteradiabatic term is given by

\[
\hat{H}_{\text{cd}}(t) = \dot{\lambda}(t) \cdot \sum_n \left( 1 - |n(\lambda(t))\rangle \langle n(\lambda(t))| \right) |\nabla_{\lambda} n(\lambda(t))\rangle \langle n(\lambda(t))|,
\] (19)

where \( \lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots) \) is a set of time-dependent parameters. The solution of the Schrödinger equation with the Hamiltonian \( \hat{H}(t) + \hat{H}_{\text{cd}}(t) \) is given by the adiabatic state

\[
|\psi_{\text{ad}}(t)\rangle = \sum_n |n(\lambda(t))\rangle \langle n(\lambda(0))|\psi(0)\rangle \\
\times \exp \left[ -i \int_0^t \! d\tau \epsilon_n(\lambda(\tau)) - \int_{\lambda(0)}^{\lambda(t)} \! d\lambda \cdot \langle n(\lambda)|\nabla_{\lambda} n(\lambda)\rangle \right].
\] (20)

The counteradiabatic term is written in a form \( \hat{H}_{\text{cd}}(t) = \dot{\lambda}(t) \cdot A(\lambda(t)) \) and the adiabatic gauge potential \( A(\lambda) \) characterizes the geometric property of the system.

We set \( |\psi_{\text{ref}}(t)\rangle = |\psi_{\text{ad}}(t)\rangle \) to find a universal bound. When we choose the initial state as one of eigenstates of \( H(0) \), and use a single parameter \( \lambda(t) \), the time integration of \( \sigma \) is written as

\[
\int_0^t \! dt' \sigma(\hat{H}_{\text{cd}}(t'), |\psi_{\text{ad}}(t')\rangle) = \int_{\lambda(0)}^{\lambda(t)} \! d\lambda \sigma_A(\lambda),
\] (21)

where \( \sigma_A(\lambda) \) represents the square root of the variance of the adiabatic gauge potential \( A(\lambda) \) with respect to the eigenstate. This representation means that the bounds are dependent only on the initial and final values of the protocol. When \( \lambda(t) \) has multiple components, the time integration of \( \sigma \) depends only on the path in parameter space \( \lambda \) and is independent of the velocities on the path.

To demonstrate the described general properties, we study the Grover Hamiltonian [37–39]

\[
\hat{H}(t) = A(t) \left( 1 - |+\rangle \langle +| \right) + B(t) \left( 1 - |0\rangle \langle 0| \right).
\] (22)
In the present choice of parameters where the Hamiltonian changes from \(|\psi\rangle\) to \(H(t_f) = 1 - |0\rangle\langle 0|\). In the adiabatic quantum computation, the Hamiltonian is changed slowly and the system goes from the initial state to the target state.

The counterdiabatic Hamiltonian is written as

\[
\hat{H}_{cd}(t) = \frac{i}{2} \sqrt{\frac{N}{N-1}} \hat{\theta}(t) \left(|0\rangle \langle +| + |+\rangle \langle 0| \right),
\]

where

\[
\hat{\theta}(t) = \arctan \frac{\sqrt{\frac{N-1}{N}}} {\frac{1}{2} \left(1 - \frac{A(t)}{N} \right) \left(A(t) - B(t)\right)}.
\]

For \(\Theta(t) = \arccos \langle 0 | \psi(t) \rangle\), the bounds are calculated as

\[
\Theta_s(t) = \frac{\pi - \theta(0)}{2},
\]

\[
\Theta_l(t) = \frac{\pi - \theta(0)}{2} - (\theta(t) - \theta(0)).
\]

The upper bound is time independent, which shows that \(\Theta\) does not exceed the initial value. The lower bound shows that we can find the minimum time for the time-evolved state to reach the target state. The minimum time \(t_{\text{min}}\) is obtained by solving \(\theta(t_{\text{min}}) = (\pi + \theta(0))/2\).

As a typical choice of the protocol in the adiabatic quantum computation, we use

\[
A(t) = A(0)(1 - s(t/t_f)) \quad \text{and} \quad B(t) = A(0)s(t/t_f) \quad [40].
\]

The function \(s(t)\) is an increasing function satisfying \(s(0) = 0\) and \(s(1) = 1\). We choose two types of the protocol as

\[
s(t) = \begin{cases} 
1 - e^{-k\tau} & \text{protocol 1} \\
\frac{(1 - e^{-k\tau/2})((1 + e^{-k\tau/2})} {1 - (1 - e^{-k\tau/2})} & \text{protocol 2}
\end{cases}
\]

where \(k \geq 1\). In both cases, \(s(\tau) \sim \tau\) for \(k = 1\). This linear protocol is frequently used in the adiabatic quantum computation. As we increase \(k\), \(s(\tau)\) changes rapidly and the adiabaticity condition is broken. We plot the protocol 1 in the left panel of figure 3 and the protocol 2 in the left panel of figure 4.

The results are shown in the right panel of figure 3 for the protocol 1 and of figure 4 for the protocol 2. In the present choice of parameters \(N = 10\) and \(A(0)t_f = 20\), \(\Theta\) is well described by the adiabatic approximation for small \(k\) and shows a nonadiabatic behavior for large \(k\). When \(k\) is a large value for the protocol 2, the result is rapidly oscillating and the amplitude of the oscillation is tightly bounded by \(\Theta_{ud}\).

This result shows that \(\Theta_{ud}\) are not overestimation and give reasonable bounds. As in the result of the previous section, the quantum speed limit is useful when we consider nonadiabatic regime.
5. Periodically-oscillating system: bound optimization

The third example treats a periodically-oscillating Hamiltonian $\hat{H}(t)$. Generally speaking, the standard quantum speed limit bound becomes loose when the Hamiltonian changes back and forth. We discuss how this problem is improved by using a proper reference state.

As a reference Hamiltonian, we use a time-independent Hamiltonian $\hat{H}_{\text{ref}}$. This choice allows us to evaluate the bound easily. When $\hat{H}(t)$ changes rapidly, we can obtain the effective time-independent Hamiltonian from the Floquet–Magnus expansion \[41, 42\]. The effective Hamiltonian can be utilized as a reference one.

To obtain a tractable result, we use the exactly-solvable Hamiltonian

$$\hat{H}(t) = \frac{1}{2} \left( \Delta - \frac{h}{\omega} - \Delta \right),$$

where $\Delta, h, \omega$ are positive parameters. When $\omega$ is small, the adiabatic approximation gives a reasonable result and the opposite limit, large $\omega$, is described by the Floquet–Magnus expansion. Here, we consider the large $\omega$ case. The Floquet–Magnus expansion gives a time-independent Hamiltonian

$$\hat{H}_{\text{ref}} = \frac{1}{2} \left( \frac{\Delta_{\text{ref}}}{h_{\text{ref}}} - \Delta_{\text{ref}} \right).$$

Up to the second order in $1/\omega$, we obtain

$$\Delta_{\text{ref}} = \Delta - \frac{h^2}{2\omega} - \frac{\Delta h^2}{\omega^2},$$

$$h_{\text{ref}} = -\frac{\Delta h}{\omega} - \frac{\Delta^2 h}{2\omega^2} + \frac{h^3}{2\omega^2}.$$

The reference state is obtained as $|\psi_{\text{ref}}(t)\rangle = e^{-i\hat{H}_{\text{ref}}t} |\psi(0)\rangle$.

We start the time evolution from the eigenstate of the Hamiltonian $\hat{H}(0)$ with the positive eigenvalue. When we choose the initial state as the target state $|\psi_{\text{target}}\rangle = |\psi(0)\rangle$, $\Theta(t) = \arccos|\langle \psi(0)|\psi(t)\rangle|$ changes from the initial value $\Theta(0) = 0$ as we show crossed points in the panels (a) and (c) of figure 5. In this case, the lower bound $\Theta_l(t)$ (lower dashed line) is calculated to give negative values at any $t$ and we cannot obtain a useful result. On the other hand, the upper bound $\Theta_u(t)$ (upper dashed line) gives a reasonable result which is smaller than the bound obtained in the standard quantum speed limit (dotted line).

The negative lower bound is due to the choice of the target state. In the panels (b) and (d) of figure 5, we show the result in the case of the target state

$$|\psi_{\text{target}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$  

In this case, the lower and upper bounds are between 0 and $\pi/2$ at not too large $t$.

The use of the Floquet–Magnus expansion does not necessarily give a tight bound. Although the reference state $|\psi_{\text{ref}}(t)\rangle$ gives a good approximation to the time-evolved state $|\psi(t)\rangle$, $\sigma$ is not necessarily optimized. We choose the reference Hamiltonian in equation (30) and the parameters $\Delta_{\text{ref}}$ and $h_{\text{ref}}$ are optimized so that $\Theta_u$ is minimized and $\Theta_l$ is maximized at each $t$. The result is shown by the solid lines in
Figure 5. Quantum speed limit of the periodically-oscillating Hamiltonian in equation (29). The Bures angle \( \Theta \) (crosses) is plotted as a function of \( t \) at each panel. \( T = 2\pi / \omega \) represents the period of the oscillation and we take \( \omega / \Delta = 20 \). The parameter \( h / \Delta \) is 0.2 for the panel (a) and (b) and 0.8 for (c) and (d). We use \( |\psi_{\text{target}}\rangle = |\psi(0)\rangle \) for the panels (a) and (c), and equation (33) for (b) and (d). The bounds are estimated in three different ways. The standard quantum speed limit \( \Theta_{\text{MT}} \) is represented by the dotted line. The dashed lines represent the lower and upper bounds \( \Theta_{l,u} \) in the case that the Floquet state is chosen for the reference state. The solid lines represent the lower and upper bounds for the optimized reference state.

The bounds are useful at transient times, several periods of the oscillation. It is remarkable to notice that, in the panels (a) and (c), the lower bound gives a positive value at a small \( t \).

6. Summary and perspectives

We have discussed lower and upper quantum speed limits. The inequalities are represented by using three quantum states. When the Hamiltonian is time dependent, the quantum state changes in a complicated way as a function of time. In that case, the standard inequality mostly gives a loose bound. In the method developed in the present analysis, by setting the reference state in a proper way, the variance of the Hamiltonian is replaced by that of the difference between the Hamiltonian and the reference Hamiltonian. As a result, we can find a tight bound. Furthermore, we have an additional degrees of freedom on the choice of the target state. The choices of the reference state and the target state are arbitrary and we can considerably improve the standard bound.

We stress two important points in deriving the improved inequalities: the triangle inequality in equation (2) and the estimation of the bound by the method in [17, 18]. The triangle inequality is represented by using the density operator and the general distance measure and is generally satisfied without any approximation. Although we have treated pure quantum states in this paper, our method is applicable to broader classes of quantum and classical systems in cases that the distance measure is defined.

It is also an interesting problem to apply the present method to systems with many degrees of freedom. The system energy is typically proportional to the system size \( N \), which implies that the square root of the variance \( \sigma \) is proportional to \( N \). The overlap between two different quantum states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) has a form \( |\langle \psi_1 | \psi_2 \rangle| \sim e^{-N \sigma} \) and the quantum speed limit inequality becomes a trivial relation. Although it is possible to find a bound for the rate function \( g = -\frac{1}{2} \ln |\langle \psi_1 | \psi_2 \rangle| \) [17], \( g \) is not a distance measure and we cannot use the triangle inequality. In spite of this problem, we still have a possibility to find a meaningful result by using arbitrariness on the choice of the reference state. It will be an interesting future problem.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
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