SPACE QUASICONFORMAL COMPOSITION OPERATORS
WITH APPLICATIONS TO NEUMANN EIGENVALUES

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Abstract. In this article we obtain estimates of Neumann eigenvalues of $p$-
Laplace operators in a large class of space domains satisfying quasihyperbolic
boundary conditions. The suggested method is based on composition opera-
tors generated by quasiconformal mappings and their applications to Sobolev-
Poincaré-inequalities. By using a sharp version of the inverse Hölder inequality
we refine our estimates for quasi-balls, that is, images of balls under quasicon-
formal mappings of the whole space.

1. Introduction

The article is devoted to applications of the space quasiconformal mappings the-
ory to the spectral theory of elliptic operators. Applications are based on the geo-
metric theory of composition operators on Sobolev spaces [20, 34, 38] in the special
case of operators generated by quasiconformal mappings. Composition operators
on Sobolev spaces permit us to give estimates of norms for embedding operators
of Sobolev spaces into Lebesgue spaces in a large class of space domains that in-
cludes domains with Hölder singularities [22, 23]. Quasiconformal mappings allow
us to describe the important subclass of these embedding domains in the terms of
quasihyperbolic geometry. It permit us to obtain estimates of Neumann eigenval-
ues of the $p$-Laplace operator, $p > n$, in domains with quasihyperbolic boundary
conditions.

These estimates are refined for $K$-quasi-balls, that is, images of the ball $B \subset \mathbb{R}^n$
under $K$-quasiconformal mappings $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with the help of a sharp (with
constant estimates) reverse Hölder inequality. We prove the constant estimates in
the reverse Hölder inequality on the base of the quasiconformal mapping theory [4]
and the non-linear potential theory [24] [31].

Recall that in the terms of embedding operators (Min-Max Principle) the first
non-trivial Neumann eigenvalue $\mu_p(\Omega)$ of the $p$-Laplace operator can be charac-
terized as

$$\mu_p(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u(x)|^p \, dx}{\int_\Omega |u(x)|^p \, dx} : u \in W^{1,p}_0(\Omega) \setminus \{0\}, \int_\Omega |u|^{p-2}u \, dx = 0 \right\}.$$ 

In the article we prove that if a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, satisfying the $\gamma$-
quasi hyperbolic boundary condition, is a $K$-quasi-ball, then for $p > n$

$$\mu_p(\Omega) \geq \frac{M_p(K, \gamma)}{R^p},$$

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where \( R^* \) is a radius of a ball \( \Omega^* \) of the same measure as \( \Omega \) and \( M_p(K, \gamma) \) depends only on \( p, \gamma \) and a quasiconformity coefficient \( K \) of \( \Omega \). The exact value of \( M_p(K, \gamma) \) is given in Theorem 4.9.

Estimates of the first non-trivial Neumann eigenvalue of the \( p \)-Laplace operator, \( p > 2 \), are known for convex domains \( \Omega \subset \mathbb{R}^n \) [8]:

\[
\mu_p(\Omega) \geq \left( \frac{\pi_p}{d(\Omega)} \right)^p,
\]

where \( d(\Omega) \) is a diameter of a convex domain \( \Omega \) and

\[
\pi_p = 2 \int_0^{(p-1)/p} \frac{dt}{(1 - t^p/(p-1))^{1/p}} = 2\pi \frac{(p-1)^{1/p}}{p \sin(\pi/p)}.
\]

Unfortunately in non-convex domains \( \mu_p(\Omega) \) cannot be characterized in the terms of Euclidean diameters. This can be seen by considering a domain consisting of two identical squares connected by a thin corridor [5]. The method of the composition operators on Sobolev spaces in connection with embedding theorems in convex domains [13, 19] allows us to obtain estimates of Neumann eigenvalues of the \( p \)-Laplace operator in non-convex domains including some domains with Hölder singularities and some fractal type domains [21, 22].

In the case of composition operators generated by quasiconformal mappings the main estimates of norms of embedding operators from Sobolev spaces with first derivatives into Lebesgue spaces were reformulated in the terms of integrals of quasiconformal derivatives [14, 15, 17]. This type of global integrability of quasiconformal derivatives depends on the quasiconformal geometry of domains [9]. One of the possible geometric reinterpretation of the quasiconformal geometry is the growth condition for quasihyperbolic metric [3, 27].

Recall that a domain \( \Omega \) satisfies the \( \gamma \)-quasihyperbolic boundary condition with some \( \gamma > 0 \), if the growth condition on the quasihyperbolic metric

\[
k_\Omega(x_0, x) \leq \frac{1}{\gamma} \log \frac{\text{dist}(x_0, \partial \Omega)}{\text{dist}(x, \partial \Omega)} + C_0
\]

is satisfied for all \( x \in \Omega \), where \( x_0 \in \Omega \) is a fixed base point and \( C_0 = C_0(x_0) < \infty \), [10, 25, 26, 30].

In [3] it was proved that Jacobians \( J_\varphi \) of quasiconformal mappings \( \varphi : \mathbb{B} \to \Omega \) belong to \( L_\beta(\mathbb{B}) \) for some \( \beta > 1 \) if and only if \( \Omega \) satisfy to a \( \gamma \)-quasihyperbolic boundary conditions for some \( \gamma \).

Hence the (quasi)conformal mapping theory allows to obtain spectral estimates in domains with quasihyperbolic boundary conditions. Because we need the exact value of the integrability exponent \( \beta \) for quasiconformal Jacobians, we consider an equivalent class of \( \beta \)-quasiconformal regular domains [15], namely the class of domains

\[
\{ \Omega \subset \mathbb{R}^n : \Omega = \varphi(\mathbb{B}) \text{ with } J_\varphi \in L_\beta(\mathbb{B}) \}.
\]

An important subclass of \( \beta \)-quasiconformal regular domains are quasi-balls, because Jacobians of quasiconformal mappings are \( A_p \)-weights [33] and they satisfy the reverse Hölder inequality [4]. Methods of the harmonic analysis allow us to refine estimates of Neumann eigenvalues in the case of quasi-balls. The main technical difficulties are calculation (estimation) of exact constants in the reverse Hölder
inequality. We solve this problem using capacitary (moduli) estimates. These estimates permit us to obtain integral estimates of quasiconformal derivatives (Jacobians) in the unit ball \( \mathbb{B} \). Hence we can refine estimates of the Neumann eigenvalues for quasi-balls. Note that quasi-balls include some fractal type domains.

In the two-dimensional case \( \mathbb{R}^2 \) this approach is more accurate \([16, 17]\) because exact exponents of local integrability of planar quasiconformal Jacobians are known \([2, 12]\).

2. Composition operators on Sobolev spaces

2.1. Sobolev spaces. Let \( \Omega \) is an open subset of \( \mathbb{R}^n \), \( n \geq 2 \). The Sobolev space \( W^1_p(\Omega) \), \( 1 \leq p \leq \infty \) is defined as a Banach space of locally integrable weakly differentiable functions \( f : \Omega \rightarrow \mathbb{R} \) equipped with the following norm:

\[
\| f \|_{W^1_p(\Omega)} = \| f \|_{L^p(\Omega)} + \| \nabla f \|_{L^p(\Omega)},
\]

where \( \nabla f \) is the weak gradient of the function \( f \). The homogeneous seminormed Sobolev space \( L^1_p(\Omega) \), \( 1 \leq p \leq \infty \) is equipped with the seminorm:

\[
\| f \|_{L^1_p(\Omega)} = \| \nabla f \|_{L^p(\Omega)}.
\]

We consider the Sobolev spaces as Banach spaces of equivalence classes of functions up to a set of \( p \)-capacity zero \([31]\).

The following theorem gives the analytic description of composition operators on Sobolev spaces:

**Theorem 2.1.** \([33, 38]\) A homeomorphism \( \varphi : \Omega \rightarrow \Omega' \) between two domains \( \Omega \) and \( \Omega' \) induces a bounded composition operator

\[
\varphi^* : L^1_p(\Omega') \rightarrow L^1_q(\Omega), \ 1 \leq q < p < \infty,
\]

if and only if \( \varphi \in W^1_{1, \text{loc}}(\Omega) \), has finite distortion, and

\[
K_{p,q}(\Omega) = \left( \int_{\Omega} \left( \frac{|D\varphi(x)|^p}{|J(x, \varphi)|} \right)^{\frac{n}{p-q}} \, dx \right)^{\frac{p-q}{np}} < \infty.
\]

Recall that a mapping \( \varphi : \Omega \rightarrow \Omega' \) is called a mapping of finite distortion, if, \( D\varphi(x) = 0 \) for almost all points \( x \) that belongs to set \( Z = \{ x \in \Omega : J(x, \varphi) = 0 \} \).

Mappings that generate bounded composition operators on Sobolev spaces

\[
\varphi^* : L^1_p(\Omega') \rightarrow L^1_q(\Omega), \ 1 \leq q \leq p \leq \infty
\]

are called weak \((p, q)\)-quasiconformal mappings and generalize Sobolev mappings \((p = \infty, 20, 35)\) and quasiconformal mappings \((p = q = n, 36)\).

Recall that a homeomorphism \( \varphi : \Omega \rightarrow \Omega' \) is called a \( K \)-quasiconformal mapping if \( \varphi \in W^1_{n, \text{loc}}(\Omega) \) and there exists a constant \( 1 \leq K < \infty \) such that

\[
|D\varphi(x)|^n \leq K |J(x, \varphi)| \text{ for almost all } x \in \Omega.
\]

The quasiconformal mappings are mappings of finite distortion and possess the Luzin \( N \)-property, that is, an image of a set of measure zero has measure zero.

Let \( \Omega \) and \( \Omega' \) be domains in \( \mathbb{R}^n \), that is, open and connected sets. Then \( \Omega \) and \( \Omega' \) are called \( K \)-quasiconformal equivalent domains if there exists a \( K \)-quasiconformal homeomorphism \( \varphi : \Omega \rightarrow \Omega' \). Note that in \( \mathbb{R}^2 \) any two simply connected domains are quasiconformal equivalent domains \([1]\).
In the following theorems we obtain the estimate of the norm of the composition operator on Sobolev spaces in quasiconformal equivalent domains with finite measure. The case of planar conformal mappings was considered in [17].

**Theorem 2.2.** Let $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, be $K$-quasiconformal equivalent domains with finite measure. Then a $K$-quasiconformal mapping $\varphi : \Omega \to \Omega'$ generates a bounded composition operator

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$$

for any $p \in (n, +\infty)$ and $q = [1, n]$ with $K_{p,q}(\Omega) = K^{\frac{1}{p}} |\Omega'|^{\frac{n-n}{np-q}} |\Omega|^{\frac{n-q}{np-q}}$.

**Proof.** By Theorem 2.1 a homeomorphism $\varphi : \Omega \to \Omega'$ generates a bounded composition operator

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega), \quad 1 \leq q < p < \infty,$$

if and only if $\varphi \in W_{1,\text{loc}}(\Omega)$, has finite distortion and

$$K_{p,q}(\Omega) = \left( \int_{\Omega} \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^{\frac{np-q}{q(p-q)}}} \, dx \right)^{\frac{q}{np-q}} \leq \infty.$$

Since $\varphi$ is a quasiconformal mapping, then $|J(x, \varphi)| > 0$ for almost all $x \in \Omega$.

By using the quasiconformal inequality $|D\varphi(x)|^n \leq K|J(x, \varphi)|$ for almost all $x \in \Omega$ we obtain

$$K_{p,q}(\Omega) = \left( \int_{\Omega} \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^{\frac{np-q}{q(p-q)}}} \, dx \right)^{\frac{q}{np-q}} \leq K^{\frac{1}{p}} \left( \int_{\Omega} |D\varphi(x)| \frac{(p-n)q}{q(p-q)} \, dx \right)^{\frac{p-q}{np-q}}.$$

Note that if $q \leq n$ then the quality $(p-n)q/(p-q) \leq n$. Hence applying the Hölder inequality to the last integral we have

$$\left( \int_{\Omega} |D\varphi(x)| \frac{(p-n)q}{q(p-q)} \, dx \right)^{\frac{p-q}{np-q}} \leq \left[ \left( \int_{\Omega} |D\varphi(x)|^n \, dx \right)^{\frac{p-n}{n(p-q)}} \left( \int_{\Omega} |dx|^{\frac{n(p-q)}{n(p-q)}} \right)^{\frac{n(q-n)}{p-n}} \right].$$

By the condition of the theorem, $\Omega$ and $\Omega'$ are Euclidean domains with finite measure and therefore

$$K_{p,q}(\Omega) \leq K^{\frac{1}{p}} K^{\frac{n-n}{np-q}} |\Omega'|^{\frac{n-n}{np-q}} |\Omega|^{\frac{n-q}{np-q}} = K^{\frac{1}{p}} |\Omega'|^{\frac{n-n}{np-q}} |\Omega|^{\frac{n-q}{np-q}}.$$

We have proved that a composition operator

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$$

is bounded for any $p \in (n, +\infty)$ and $q = [1, n]$. \qed
Let $\Omega$ and $\Omega'$ be domains in $\mathbb{R}^n$, $n \geq 2$. Then a domain $\Omega'$ is called a $K$-quasiconformal $\beta$-regular domain about a domain $\Omega$ if there exists a $K$-quasiconformal mapping $\varphi : \Omega \to \Omega'$ such that

$$\int_{\Omega} |J(x, \varphi)|^\beta \, dx < \infty$$

for some $\beta > 1$,

where $J(x, \varphi)$ is a Jacobian of a $K$-quasiconformal mapping $\varphi : \Omega \to \Omega'$. The domain $\Omega' \subset \mathbb{R}^n$ is called a quasiconformal regular domain if it is a $K$-quasiconformal $\beta$-regular domain for some $\beta > 1$.

Note that the class of quasiconformal regular domains includes the class of Gehring domains [3, 27] and can be described in terms of quasi hyperbolic geometry [10, 25, 30].

**Remark 2.3.** Because $\varphi : \Omega \to \Omega'$, $n \geq 2$, is a quasiconformal mapping, then integrability of the derivative is equivalent to integrability of the Jacobian:

$$\int_{\Omega} |J(x, \varphi)|^\beta \, d\leq \int |D\varphi(x)|^{n\beta} \, dx \leq K^\beta \int_{\Omega} |J(x, \varphi)|^\beta \, dx.$$
Poincaré inequalities in quasiconformal equivalent bounded space domains. Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain and let \( h : \Omega \to \mathbb{R} \) be a real valued locally integrable function such that \( h(x) > 0 \) a.e. in \( \Omega \). We consider the weighted Lebesgue space \( L_p(\Omega, h) \), \( 1 \leq p < \infty \), as the space of measurable functions \( f : \Omega \to \mathbb{R} \) with the finite norm

\[
\| f \|_{L_p(\Omega, h)} := \left( \int_{\Omega} |f(x)|^p h(x) \, dx \right)^{\frac{1}{p}} < \infty.
\]

It is a Banach space for the norm \( \| f \|_{L_p(\Omega, h)} \).

On the basis of Theorem 2.1 we prove the existence of two-weight Sobolev-Poincaré inequalities in quasiconformal equivalent bounded space domains.

Recall that a bounded domain \( \Omega \subset \mathbb{R}^n \) is called a \((r, q)\)-Sobolev-Poincaré domain, \( 1 \leq r, q \leq \infty \), if for any function \( f \in L^1_r(\Omega) \), the \((r, q)\)-Sobolev-Poincaré inequality

\[
\inf_{c \in \mathbb{R}} \| f - c \|_{L^q_r(\Omega)} \leq B_{r,q}(\Omega) \| \nabla f \|_{L^q(\Omega)}
\]

3. Sobolev-Poincaré inequalities

3.1. Two-weight Sobolev-Poincaré inequalities. Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain and let \( h : \Omega \to \mathbb{R} \) be a real valued locally integrable function such that \( h(x) > 0 \) a.e. in \( \Omega \). We consider the weighted Lebesgue space \( L_p(\Omega, h) \), \( 1 \leq p < \infty \), as the space of measurable functions \( f : \Omega \to \mathbb{R} \) with the finite norm

\[
\| f \|_{L_p(\Omega, h)} := \left( \int_{\Omega} |f(x)|^p h(x) \, dx \right)^{\frac{1}{p}} < \infty.
\]

Taking into account the quasiconformal inequality \( |D\varphi(x)|^n \leq K|J(x, \varphi)| \) for almost all \( x \in \Omega \) we obtain

\[
K_{p,q}^{\frac{1}{p}}(\Omega) = \left( \int_{\Omega} \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^q} \, dx \right)^{\frac{1}{p}} \leq K \left( \int_{\Omega} \frac{|D\varphi(x)|^{p-n}}{|J(x, \varphi)|^q} \, dx \right)^{\frac{1}{p}} \leq K \left( \int_{\Omega} |D\varphi(x)|^\beta \, dx \right)^{\frac{1}{p}} \leq K \left( \int_{\Omega} |D\varphi(x)|^\beta \, dx \right)^{\frac{1}{p}} < \infty,
\]

for \( \beta = (p-n)q/(p-q) \). Hence we have a bounded composition operator

\[
\varphi^* : L^1_p(\Omega) \to L^1_q(\Omega)
\]

for any \( p \in (n, +\infty) \) and \( q = p\beta/(p + \beta - n) \).

Let us check that \( q < p \). Because \( p > n \) we have that \( p + \beta - n > \beta > 1 \) and so \( \beta/(p + \beta - n) < 1 \). Hence we obtain \( q < p \).

Assume that the composition operator

\[
\varphi^* : L^1_p(\Omega) \to L^1_q(\Omega), \quad q < p,
\]

is bounded for any \( p \in (n, +\infty) \) and \( q = p\beta/(p + \beta - n) \). Then, given the Hadamard inequality:

\[
|J(x, \varphi)| \leq |D\varphi(x)|^n \quad \text{for almost all } x \in \Omega,
\]

and Theorem 2.1, we have

\[
\int_{\Omega} |D\varphi(x)|^\beta \, dx = \int_{\Omega} |D\varphi(x)|^{(p-n)q} \, dx \leq \int_{\Omega} \left( \frac{|D\varphi(x)|^p}{|J(x, \varphi)|^q} \right)^{\frac{p}{p-n}} \, dx < +\infty.
\]

\( \square \)

Remark 2.5. In the case of bounded domains \( \Omega, \Omega' \subset \mathbb{R}^n \), \( n \geq 2 \), Theorem 2.4 is correct for any \( p \in (n, +\infty) \) and any \( q \leq p\beta/(p + \beta - n) \).

If \( q > n - 1 \), then by the duality composition theorem [34], the inverse mapping \( \varphi^{-1} \) induces a bounded composition operator from \( L^1_q(\Omega') \) to \( L^1_{q'}(\Omega) \), where \( p' = \frac{p}{p-(n-1)} \) and \( q' = \frac{q}{q-(n-1)} \).
Theorem 3.1. Let $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, be $K$-quasiconformal equivalent bounded domains and let $h(y) = |J(y, \varphi^{-1})|$ be the quasiconformal weight defined by a $K$-quasiconformal mapping $\varphi : \Omega \to \Omega'$. Suppose that $\Omega$ be a $(r, q)$-Sobolev-Poincaré domain, then for any function $f \in W^1_p(\Omega')$, $p > n$, the inequality

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^r h(y) \, dy \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega', h) \left( \int_{\Omega'} |\nabla f(y)|^p \, dy \right)^{\frac{1}{p}}$$

holds for any $1 \leq r \leq nq/(n - q)$ with the constant

$$B_{r,p}(\Omega, h) \leq \inf_{q \in [1, n]} \left\{ B_{r,q}(\Omega)|\Omega|^{\frac{q}{nq}} \right\} K^{\frac{1}{p}} |\Omega|^\frac{p-n}{np}.$$

Here $B_{r,q}(\Omega)$ is the best constant in the (unweighted) $(r, q)$-Sobolev-Poincaré inequality in the domain $\Omega$.

Proof. By the conditions of the theorem there exists a $K$-quasiconformal mapping $\varphi : \Omega \to \Omega'$. Denote by $h(y) = |J(y, \varphi^{-1})|$ the quasiconformal weight in $\Omega'$.

Let $f \in L^1_+(\Omega')$ be a smooth function. Then the composition $g = f \circ \varphi^{-1}$ is well defined almost everywhere in $\Omega$ and belongs to the Sobolev space $L^1_+(\Omega)$ [37]. Hence, by the Sobolev embedding theorem $g = f \circ \varphi^{-1} \in W^1_q(\Omega)$ [31] and the unweighted Poincaré-Sobolev inequality

$$\inf_{c \in \mathbb{R}} ||f \circ \varphi^{-1} - c||_{L^r(\Omega)} \leq B_{r,q}(\Omega)||\nabla(f \circ \varphi^{-1})||_{L^q(\Omega)}$$

holds for any $1 \leq r \leq nq/(n - q)$.

Taking into account the change of variable formula for quasiconformal mappings [37], the Poincaré-Sobolev inequality (3.1) and Theorem 2.2 we obtain for a smooth function $f \in W^1_p(\Omega')$

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^r h(y) \, dy \right)^{\frac{1}{r}} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^r |J(y, \varphi^{-1})| \, dy \right)^{\frac{1}{r}}$$

$$= \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r \, dx \right)^{\frac{1}{r}} \leq B_{r,q}(\Omega) \left( \int_{\Omega} |\nabla g(x)|^q \, dx \right)^{\frac{1}{q}}$$

$$\leq B_{r,q}(\Omega) K^{\frac{1}{p}} |\Omega|^\frac{q-n}{np} |\Omega|^\frac{p-n}{np} \left( \int_{\Omega'} |\nabla f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Approximating an arbitrary function $f \in W^1_p(\Omega')$ by smooth functions we have

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^r h(y) \, dy \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega', h) \left( \int_{\Omega'} |\nabla f(y)|^p \, dy \right)^{\frac{1}{p}}.$$
with the constant

\[ B_{r,p}(\Omega, h) \leq \inf_{q \in [1,n]} \left\{ B_{r,q}(\Omega) \right\} K^\frac{\beta}{\beta - 1} |\Omega'\|^{\frac{\beta}{\beta - 1}}. \]

The property of the \( K \)-quasiconformal \( \beta \)-regularity implies the integrability of a Jacobian of quasiconformal mappings and therefore for any \( K \)-quasiconformal \( \beta \)-regular domain we have the embedding of weighted Lebesgue spaces \( L_r(\Omega, h) \) into non-weight Lebesgue spaces \( L_s(\Omega) \) for \( s = \frac{\beta - 1}{\beta} r \).

**Lemma 3.2.** Let \( \Omega' \) be a \( K \)-quasiconformal \( \beta \)-regular domain about a domain \( \Omega \). Then for any function \( f \in L_r(\Omega, h) \), \( \beta/(\beta - 1) \leq r < \infty \), the inequality

\[ \| f \|_{L_s(\Omega')} \leq \left( \int_\Omega |J(x, \varphi)|^\beta \, dx \right)^\frac{1}{\beta} \left( \int_{\Omega'} |f(y)|^s \, dy \right)^\frac{1}{s} \| f \|_{L_r(\Omega, h)} \]

holds for \( s = \frac{\beta - 1}{\beta} r \).

**Proof.** By the assumptions of the lemma there exists a \( K \)-quasiconformal mapping \( \varphi : \Omega \to \Omega' \) such that

\[ \int_\Omega |J(x, \varphi)|^\beta \, dx < \infty. \]

Let \( s = \frac{\beta - 1}{\beta} r \). Then using the change of variable formula for quasiconformal mappings [37], Hölder’s inequality with exponents \( (r/s, r/(r - s)) \) and equality \( |J(y, \varphi^{-1})| = h(y) \), we obtain

\[ \| f \|_{L_s(\Omega')} = \left( \int_{\Omega'} |f(y)|^s \, dy \right)^\frac{1}{s} = \left( \int_{\Omega'} |f(y)|^s |J(y, \varphi^{-1})|^\frac{\beta}{\beta - 1} |J(y, \varphi^{-1})|^{-\frac{\beta}{\beta - 1}} \, dy \right)^\frac{1}{s} \]

\[ \leq \left( \int_{\Omega'} |f(y)|^r |J(y, \varphi^{-1})| \, dy \right)^\frac{1}{\beta} \left( \int_{\Omega'} |J(y, \varphi^{-1})|^{-\frac{\beta}{\beta - 1}} \, dx \right)^\frac{\beta - 1}{\beta} \]

\[ = \left( \int_{\Omega'} |f(y)|^r h(y) \, dy \right)^\frac{1}{\beta} \left( \int_{\Omega} |J(x, \varphi)| \, dx \right)^\frac{\beta - 1}{\beta} \]

\[ = \left( \int_{\Omega'} |f(y)|^r h(y) \, dy \right)^\frac{1}{\beta} \left( \int_{\Omega} |J(x, \varphi)|^\beta \, dx \right)^\frac{\beta}{\beta - 1}. \]

\[ \square \]

According to Theorem 3.1 and Lemma 3.2 we obtain an upper estimate of the Poincaré constant in quasiconformal regular domains.
Theorem 3.3. Let $\Omega'$ be a $K$-quasiconformal $\beta$-regular domain about a $(r, q)$-Sobolev-Poincaré domain $\Omega$. Then for any function $f \in W^1_{p}(\Omega')$, $p > n$, the Poincaré-Sobolev inequality

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^s h(y) dy \right)^{\frac{1}{s}} \leq B_{s,p}(\Omega') \left( \int_{\Omega'} |\nabla f(y)|^p dy \right)^{\frac{1}{p}}$$

holds for any $1 \leq s \leq nq/(n - q) \cdot (\beta - 1)/\beta$ with the constant

$$B_{s,p}(\Omega') \leq \inf_{q \in (q^*, n]} \left\{ B_{r,q}(\Omega) \left( \frac{n-q}{n} \right)^{\frac{\beta}{r}} \right\} K^\frac{r}{r-p} \left( \frac{n-q}{n} \right) \cdot \|J\varphi\|_{L^\beta(\Omega)}^{\frac{r}{r}}$$

where $q^* = \beta n s / (\beta s + \beta(n - 1))$

Proof. Let $f \in W^1_p(\Omega')$, $p > n$. Then by Theorem 3.1 and Lemma 3.2 we obtain

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega'} |f(y) - c|^s dy \right)^{\frac{1}{s}} \leq \left( \int_{\Omega} |J(x, \varphi)|^\beta dx \right)^{\frac{1}{\beta}} B_{r,p}(\Omega, h)$$

$$\leq \inf_{q \in (q^*, n]} \left\{ B_{r,q}(\Omega) \left( \frac{n-q}{n} \right)^{\frac{\beta}{r}} \right\} K^\frac{r}{r-p} \left( \frac{n-q}{n} \right) \cdot \|J\varphi\|_{L^\beta(\Omega)}^{\frac{r}{r}}$$

for $1 \leq s \leq nq/(n - q) \cdot (\beta - 1)/\beta$.

Since by Lemma 3.2 $s = \frac{\beta-1}{\beta} r$ and by Theorem 3.1 $r \geq 1$, then $s \geq 1$ and the theorem proved.

By the generalized version of the Rellich-Kondrachov compactness theorem (see, for example, [31]) and the $(r, p)$-Sobolev-Poincaré inequality for $r > p$, it follows that the embedding operator

$$i : W^1_p(\Omega) \hookrightarrow L^p(\Omega)$$

is compact in $K$-quasiconformal $\beta$-regular domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Note that sufficient conditions for validity of the Rellich-Kondrachov theorem in non-smooth domains have been given by using a general quasihyperbolic boundary condition in [7]. In particular, for domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying a quasihyperbolic boundary condition it is proved that there exists $p_0 = p_0(\Omega) < n$ such that the embedding operator

$$i : W^1_p(\Omega) \hookrightarrow L^p(\Omega)$$

is compact for every $p > p_0$. 
So, the first non-trivial Neumann eigenvalue $\mu_p(\Omega)$ can be characterized as

$$
\mu_p(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} : u \in W^1_p(\Omega) \setminus \{0\}, \int_\Omega |u|^{p-2}u \, dx = 0 \right\}.
$$

Furthermore, $\mu_p(\Omega)^{-\frac{1}{p}}$ is equal to the best constant $B_{p,p}(\Omega)$ (see, for example, [6, 31]) in the $p$-Poincaré-Sobolev inequality

$$
\inf_{c \in \mathbb{R}} \left( \int_\Omega |f(x) - c|^p \, dx \right)^{\frac{1}{p}} \leq B_{p,p}(\Omega) \left( \int_\Omega |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad f \in W^1_p(\Omega).
$$

In the case $s = p$ Theorem 3.3 implies the lower estimates of the first non-trivial eigenvalue of the degenerate $p$-Laplace Neumann operator, $p > n$, in $K$-quasiconformal $\beta$-regular domains $\Omega' \subset \mathbb{R}^n$, $n \geq 2$.

**Theorem 3.4.** Let $\Omega'$ be a $K$-quasiconformal $\beta$-regular domain about a $(r,q)$-Sobolev-Poincaré domain $\Omega$, $r = p\beta/(\beta - 1)$, $p > n$. Then the following inequality holds

$$
\frac{1}{\mu_p(\Omega')} \leq \inf_{q \in (q^*, n]} \left\{ B_{r,q}^p(\Omega') \frac{1}{|\Omega|^{\frac{n-q}{n}}} \right\} K_{\frac{n}{\beta}}^p |\Omega'|^{\frac{n-q}{n}} \cdot |J_\varphi| L_\beta(\Omega),
$$

where $q^* = \beta np/(\beta p + n(\beta - 1))$.

In the case of $K$-quasiconformal $\infty$-regular domains, in the similar way we obtain the following assertion:

**Theorem 3.5.** Let $\Omega'$ be a $K$-quasiconformal $\infty$-regular domain about a $(r,q)$-Sobolev-Poincaré domain $\Omega$. Then for any $p > n$ the following inequality holds

$$
\frac{1}{\mu_p(\Omega')} \leq \inf_{q \in (q^*, n]} \left\{ B_{r,q}^p(\Omega') \frac{1}{|\Omega|^{\frac{n-q}{n}}} \right\} K_{\frac{n}{\beta}}^p |\Omega'|^{\frac{n-q}{n}} \cdot |J_\varphi| L_\infty(\Omega),
$$

where $q^* = np/(p + n)$.

Note that any convex domain $\Omega \subset \mathbb{R}^n$ is the Sobolev-Poincaré domain and the constant $B_{r,q}(\Omega)$ can be estimated as [21]

$$
B_{r,q}(\Omega) \leq \frac{d_{\Omega}^q}{n |\Omega|} \left( \frac{1 - \frac{1}{p} + \frac{r}{q}}{1 - \frac{1}{p} + \frac{1}{\beta}} \right)^{1 - \frac{1}{p} + \frac{1}{\beta}} \omega_n^{1 - \frac{1}{p} - \frac{1}{\beta}} |\Omega|^{\frac{1}{p} + \frac{1}{\beta}},
$$

where $\omega_n = \frac{2\pi^{n/2}}{n |\pi^{n/2}|}$ is the volume of the unit ball in $\mathbb{R}^n$ and $d_{\Omega}$ is the diameter of $\Omega$.

As examples, we consider non-convex star-shaped domains which are $K$-quasiconformal $\infty$-regular domains.

**Example 3.6.** The homeomorphism

$$
\varphi(x) = |x|^a x, \quad a > 0,
$$

is $(a + 1)$-quasiconformal and maps the $n$-dimensional cube

$$
Q := \{x_k \in \mathbb{R}^n : |x_k| < \sqrt{2}/2\}
$$
on onto non-convex star-shaped domains $\Omega_a$. 
Now we estimate the following quantities. A straightforward calculation yields

\[ ||J_\varphi| L_\infty(Q)|| = \text{ess sup}_{x \in Q} [(a + 1)|x|^{\alpha}] \leq a + 1 \]

and

\[ d_Q = 2, \quad |Q| = 2^n/2, \quad |\Omega_n| \leq \omega_n. \]

Then by Theorem 3.3 we have

\[
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, n]} \left\{ B_{p,q}(Q) \frac{p}{p - n} K_{p,n}^{\beta} |\Omega|^{\frac{\beta}{n}} \cdot ||J_\varphi| L_\infty(Q)|| \right\} \\
\leq \inf_{q \in (q^*, n]} \left( \frac{1}{n} \right) \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{p - r} \right)^{p+1} a^{p+1} \omega_n |\Omega|^{\frac{\beta}{n}} \cdot \omega_n^{p+1} \frac{1}{n},
\]

where \( q^* = np/(p+n) \).

In the case of quasiconformal mappings \( \varphi : B \rightarrow \Omega \) Theorem 3.4 can be reformulated as

**Theorem 3.7.** Let \( \Omega \) be a K-quasiconformal \( \beta \)-regular domain about the unit ball \( B \), \( p = \beta / (\beta - 1) \), \( p > n \). Then the following inequality holds

\[
\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, n]} \left\{ \frac{2np}{n^p} \left( \frac{1 - \frac{1}{q} + \frac{1}{p}}{p - r} \right)^{p+1} \omega_n^{p+1} \right\} \\
K_{p,n}^{\beta} |\Omega|^{\frac{\beta}{n}} \cdot ||J_\varphi| L_\beta(B)||,
\]

where \( q^* = \beta np/(\beta p + n(\beta - 1)) \).

4. **The weak reverse Hölder inequality**

In this section we obtain estimates of constants in the reverse Hölder inequality. In this section we suppose that \( n \geq 3 \). The case \( n = 2 \) was considered in [16] [17]. We start from the following version of the weak reverse Hölder inequality [4].

**Proposition 4.1.** [4] Suppose that \( \varphi : \Omega \rightarrow \mathbb{R}^n \) is a K-quasiconformal mapping and \( 0 < \sigma < 1 \) is given. Then, for any ball \( B \) in \( \Omega \) there exists a constant \( C(n) \), depending only on \( n \) such that

\[
\left( \frac{1}{\sigma B} \int_{\sigma B} |D\varphi(x)|^n \right)^{\frac{1}{n}} \leq \frac{KC(n)}{\sigma(1 - \sigma)} \left( \frac{1}{B} \int_B |D\varphi(x)| \right)^{\frac{1}{1-n}}.
\]

In inequality (1.1)

\[
C(n) = 2^{n+3/2+1/n}(n/(n-2))^{1/n} \omega_n, \quad n > 2.
\]

We need the following special case of [29] when balls are used instead of cubes. This result comes from an iteration process where the exponent \( n/2 \) on the right hand side of (1.1) can be reduced to 1.

**Proposition 4.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( f \in L_{\infty}^n(\Omega) \). Suppose that there exists a constant \( C_0 \) such that for each ball \( B \) with \( 2B \subset \Omega \\
\left( \frac{1}{|B|} \int_B |f(x)|^n \right)^{\frac{1}{n}} \leq C_0 \left( \frac{1}{2|B|} \int_{2B} |f(x)|^n \right)^{\frac{1}{n}}.
\]
Then for each ball $B$ with $2B \subset \Omega$

$$\left( \frac{1}{|B|} \int_B |f(x)|^n \, dx \right)^{\frac{1}{n}} \leq C_1 \frac{1}{|2B|} \int_{2B} |f(x)| \, dx. \quad (4.3)$$

We apply the following result.

**Proposition 4.3.** Let $\Omega$ be a cube in $\mathbb{R}^n$ and let $g, h \in L^p(\Omega)$, $1 < p < \infty$, be nonnegative functions satisfying:

$$\left( \frac{1}{|Q|} \int_Q g(x)^p \, dx \right)^{\frac{1}{p}} \leq C_1 \frac{1}{|2Q|} \int_{2Q} g(x) \, dx + \left( \frac{1}{|2Q|} \int_{2Q} h(x)^p \, dx \right)^{\frac{1}{p}}$$

for all cubes $Q$ with $2Q \subset \Omega$. Then for each $0 < \sigma < 1$ and

$$p < s < p + \frac{n - 1}{10^n 4^n C_1^n}$$

we have

$$\left( \frac{1}{|\sigma\Omega|} \right)^{\frac{1}{s}} \int_{\sigma\Omega} g(x)^s \, dx \leq \frac{100^n}{\sigma^{n/p}(1 - \sigma)^n} \left[ \left( \frac{1}{|\Omega|} \right)^{\frac{1}{p}} \int_\Omega g(x)^p \, dx + \left( \frac{1}{|\Omega|} \right)^{\frac{1}{p}} \int_\Omega h(x)^p \, dx \right].$$

We state the previous proposition for balls when $g, h \in L^n(\Omega)$.

**Proposition 4.4.** Let $\Omega$ be a ball in $\mathbb{R}^n$ and let $g, h \in L^n(\Omega)$, $n \geq 2$, be nonnegative functions satisfying:

$$\left( \frac{1}{|B|} \int_B g(x)^n \, dx \right)^{\frac{1}{n}} \leq C_1 \frac{1}{|2B|} \int_{2B} g(x) \, dx + \left( \frac{1}{|2B|} \int_{2B} h(x)^n \, dx \right)^{\frac{1}{n}}$$

for all balls $B$ with $2B \subset \Omega$. Then for each $0 < \sigma < 1$ and

$$n < p < n + \frac{n - 1}{10^n 4^n C_1^n}$$

we have

$$\left( \frac{1}{|\sigma\Omega|} \right)^{\frac{1}{s}} \int_{\sigma\Omega} g(x)^s \, dx \leq \frac{100^n}{\sigma^{n/p}(1 - \sigma)^n} \left[ \left( \frac{1}{|\Omega|} \right)^{\frac{1}{p}} \int_\Omega g(x)^p \, dx + \left( \frac{1}{|\Omega|} \right)^{\frac{1}{p}} \int_\Omega h(x)^p \, dx \right].$$

(4.4)

Now we are able to obtain the weak reverse Hölder inequality with bounds for the constants.

**Theorem 4.5.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ such that $B(0, 2) \subset \Omega$. Suppose that $\varphi: \Omega \to \mathbb{R}^n$ is a $K$-quasiconformal mapping such that $C_1$ is the constant from (4.3) for $D\varphi$. Then, for any $p > n$ satisfying

$$n < p < n + \frac{n - 1}{10^n 4^n C_1^n}$$
we have
\[
\left( \frac{1}{|B(0,1)|} \int_{B(0,1)} |D\varphi(x)|^p \, dx \right)^{\frac{1}{p}} \leq C(n,p) \left( \frac{1}{|B(0,2)|} \int_{B(0,2)} |D\varphi(x)|^n \, dx \right)^{\frac{1}{n}}.
\]

**Proof.** By Proposition 4.1 the assumptions in Proposition 4.2 are valid. Hence, for all the balls $B$ with $2B \subset \Omega$ the inequality
\[
\left( \frac{1}{|B|} \int_{B} |D\varphi(x)|^n \, dx \right)^{\frac{1}{n}} \leq c_0(n)(4KC(n))^2(\frac{n-1}{n}) \frac{1}{|2B|} \int_{2B} |D\varphi(x)| \, dx
\]
holds with $C(n)$ from (4.2). We write $C_1 = c_0(n)(4KC(n))^2(\frac{n-1}{n})$. Thus the assumptions of Proposition 4.4 are valid when $h = 0$ and $\sigma = 1/2$. Hence by Proposition 4.4 for any $p > n$ satisfying
\[
n < p < n + \frac{n-1}{10^{2n+4n}C_1^2}
\]
we have
\[
\left( \frac{1}{|B(0,1)|} \int_{B(0,1)} |D\varphi(x)|^p \, dx \right)^{\frac{1}{p}} \leq C(n,p) \left( \frac{1}{|B(0,2)|} \int_{B(0,2)} |D\varphi(x)|^n \, dx \right)^{\frac{1}{n}}
\]
where $C(n, p) = 2^{1+n/p}(100)^n$. \hfill \Box

Now we need the doubling constant for the Jacobian of a given quasiconformal mapping. For this aim we use the moduli (capacity) estimates. Let $\Gamma$ be a family of curves in $\mathbb{R}^n$. Denote by $adm(\Gamma)$ the set of Borel functions (admissible functions) $\rho : \mathbb{G} \to [0, \infty]$ such that the inequality
\[
\int_{\gamma} \rho \, ds \geq 1
\]
holds for locally rectifiable curves $\gamma \in \Gamma$.

Let $\Gamma$ be a family of curves in $\overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n}$ is a one point compactification of the Euclidean space $\mathbb{R}^n$. The quantity
\[
M(\Gamma) = \inf_{\rho \in adm(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, dx
\]
is called the module of the family of curves $\Gamma$ (see, for example [39]). The infimum is taken over all admissible functions $\rho \in adm(\Gamma)$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $F_0, F_1$ be disjoint non-empty compact sets in the closure of $\Omega$, then $M(\Gamma(F_0, F_1; \Omega))$ stand for the module of a family of curves which connect $F_0$ and $F_1$ in $\Omega$.

**Theorem 4.6.** Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a $K$-quasiconformal mapping. Then for any ball $B$,
\[
\int_{2B} |J(x, \varphi)| \, dx \leq \exp \left\{ K^{1/(n-1)} 2(\log(\sqrt{3} + \sqrt{2}) + n - 1) \right\} \int_{B} |J(x, \varphi)| \, dx.
\]
Proof. When \( n \) and \( K \) are given, our goal is to find the constant \( C \) such that
\[
|\varphi(B(0, 2))| \leq C|\varphi(B(0, 1))|.
\]
That is, to find \( C \) such that
\[
\int_{B(0, 2)} |J(x, \varphi)| \, dx \leq C \int_{B(0, 1)} |J(x, \varphi)| \, dx.
\]
Let us write
\[
R = \max |\varphi(0) - y| \text{ when } y \in \partial \varphi(B(0, 2))
\]
and
\[
r = \text{dist}(\varphi(0), \partial \varphi(B(0, 1))).
\]
Let \( L_r \) be a line segment of length \( r \) joining \( \varphi(0) \) to \( \partial \varphi(B(0, 1)) \) and \( LR \) a continuum joining a point in \( \partial \varphi(B(0, 1)) \cap \partial \varphi(B(0, 2)) \) to \( \infty \) in \( \mathbb{R}^n \setminus B(\varphi(0), R) \). Then by the quasi-invariance of the modulus
\[
0 < C_1 \leq \text{mod } (\varphi^{-1}(L_r), \varphi^{-1}(LR), B(0, 10)) \leq K \text{mod } (L_r, LR, \mathbb{R}^n) \leq K \frac{\omega_{n-1}}{(\log R)^{n-1}},
\]
where \( \omega_{n-1} = \frac{2n(n/2)}{\Gamma(n/2)} \) is the hypervolume of the \( (n-1) \)-dimensional unit sphere.

We write
\[
R_T(n, t) = R([-1, 0], [t, \infty]; \mathbb{R}^n) = \mathbb{R}^n \setminus \{[-e_1, 0] \cup [te_1, \infty]\}, t > 0
\]
for the \( n \)-dimensional Teichmüller ring corresponding \( t \) and also \( \tau_n(t) = \text{mod } (R_T(n, t)) \).

We recall that
\[
\text{mod } (\Delta(E, F)) \geq \tau_n \left( \frac{|a - c|}{|a - b|} \right).
\]
The equality holds for \( E = [-e_1, 0], a = 0, b = -e_1 \) and \( F = [te_1, \infty], c = te_1, d = \infty \) (see, for example \[32\]).

Let \( R(E, F) \) be a ring with \( a, b \in E \) and \( c, \infty \in F \). Then for the Teichmüller ring \( R_T(n, t) \)
\[
\text{mod } (R(E, F)) \geq \text{mod } R_T \left( n, \frac{|a - c|}{|a - b|} \right)
\]
\[
\omega_{n-1}(\log(\lambda_n^2 t))^{1-n} \leq \tau_n(t - 1) \leq \omega_{n-1}(\log(t))^{1-n}
\]
\[
\tau_n(t) = \text{mod } (R_T(n, t)),
\]
for example \[11\].

By \[32\], based on \[39\],
\[
\tau_n(t) \geq 2^{1-n} \omega_{n-1} \left( \log \frac{\lambda_n}{2}(\sqrt{1 + t} + \sqrt{t}) \right)^{1-n}, t > 0,
\]
where \( \lambda_n \) is the Grötzsch ring constant depending only on \( n \). It is known that \( \lambda_2 = 4 \), and for \( n \geq 3 \) it is known only that \( 20.76(n-1) \leq \lambda_n \leq 2e^{n-1} \). This gives a lower bound
\[
\tau_n(t = 2) \geq 2^{1-n} \omega_{n-1} \left( \log \frac{\lambda_n}{2}(\sqrt{3} + \sqrt{2}) \right)^{1-n},
\]
where \(4 \leq \lambda_n \leq 2e^{n-1}\).

Hence we obtain
\[
C_1 = 2^{1-n} \omega_{n-1} \left( \frac{\log \frac{\lambda_n}{2}}{2} (\sqrt{3} + \sqrt{2}) \right)^{1-n} \leq K \omega_{n-1} \left( \log \frac{R}{r} \right)^{1-n}.
\]
Thus,
\[
\left( \log \frac{R}{r} \right)^{n-1} \leq \frac{K \omega_{n-1}}{2^{1-n} \omega_{n-1} \left( \frac{\log \frac{\lambda_n}{2}}{2} (\sqrt{3} + \sqrt{2}) \right)^{1-n}} \leq K \left( 2 \log \left( \frac{\lambda_n}{2} (\sqrt{3} + \sqrt{2}) \right) \right)^{n-1},
\]
and
\[
\log \frac{R}{r} \leq K^{\frac{1}{n-1}} 2 \log \left( \frac{\lambda_n}{2} (\sqrt{3} + \sqrt{2}) \right),
\]
\[
\frac{R}{r} \leq \exp \left\{ K^{\frac{1}{n-1}} 2 \log \left( \frac{\lambda_n}{2} (\sqrt{3} + \sqrt{2}) \right) \right\},
\]
Since \(\lambda_n < 2e^{n-1}\), we obtain
\[
\frac{R}{r} \leq \exp \left\{ K^{\frac{1}{n-1}} 2 \log (\sqrt{3} + \sqrt{2}) e^{n-1} \right\},
\]
that is
\[
\frac{R}{r} \leq \exp \left\{ K^{\frac{1}{n-1}} 2 (\log (\sqrt{3} + \sqrt{2}) + n - 1) \right\}.
\]

Using the previous results concerning the weak reverse Hölder inequality and the measure doubling condition we obtain an estimate of the constant in the reverse Hölder inequality for Jacobians of quasiconformal mappings.

**Theorem 4.7.** Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) such that \(B(0,2) \subset \Omega\). Suppose that \(\varphi : \Omega \to \Omega'\) is a \(K\)-quasiconformal mapping. Then there exists a constant \(C(n,\alpha,K)\) such that
\[
\left( \int_{B(0,1)} |J(x,\varphi)|^\frac{\alpha}{2} \, dx \right)^{\frac{2}{\alpha}} \leq C(n,\alpha,K) \int_{B(0,1)} |J(x,\varphi)| \, dx
\]
for
\[
\alpha \in \left( n, n + \frac{n-1}{10^{2n} \Lambda^n C_1^n} \right)
\]
where
\[
C_1 = 2^{2n+3/2+1/n} \left( \frac{n}{(n-2)} \right)^{1/n} \omega_{n} (4K)^{2(n-1)/n}, \quad n > 2,
\]
and
\[
C(n,\alpha,K) = 2^{1-n+\alpha} (100)^n K \omega_{n}^{\alpha-1} \exp \left\{ K^{\frac{1}{n-1}} 2 (\log (\sqrt{3} + \sqrt{2}) + n - 1) \right\}.
\]
Proof. By Proposition 4.1 inequality (4.1) is valid where equation (4.2) gives a constant. Hence, by Proposition 4.2 inequality (4.3) is valid for $D\varphi$ with the constant $C_1$. Let us choose

$$\alpha \in \left(n, n + \frac{n-1}{102n^4C_1^n}\right).$$

Now for these values of $\alpha$ by Theorem 4.5 we obtain

$$\left(\int_{B(0,1)} |J(x, \varphi)|^{\frac{n}{\alpha}} dx\right)^{\frac{\alpha}{n}} \leq |B(0,1)|^{n/\alpha} \left(\frac{1}{|B(0,1)|} \int_{B(0,1)} |D\varphi(x)|^{\alpha} dx\right)^{\frac{1}{\alpha}} \leq 2^{1+n/\alpha}(100)^n \frac{|B(0,1)|^{n/\alpha}}{|B(0,2)|} \int_{B(0,2)} |D\varphi(x)|^{\alpha} dx.$$

By the definition of a quasiconformal mapping and the doubling condition, Theorem 4.6

$$\left(\int_{B(0,1)} |J(x, \varphi)|^{\frac{n}{\alpha}} dx\right)^{\frac{\alpha}{n}} \leq 2^{1-n+n/\alpha}(100)^n K\omega_n^{n/\alpha-1} \int_{B(0,2)} |J(x, \varphi)| dx \leq C(\alpha, n, K) \int_{B(0,1)} |J(x, \varphi)| dx.$$

Given this theorem, we can precise Theorem 3.7, putting $\alpha/n = \beta$.

**Theorem 4.8.** Let $\Omega$ be a $K$-quasiconformal $\beta$-regular domain about the unit ball $B$, $r = p\beta/(\beta - 1)$, $p > n$. Then

$$\frac{1}{\mu_p(\Omega)} \leq \inf_{q \in (q^*, n]} \left\{ \frac{2np}{n^p} \left(\frac{1 - \frac{1}{q} + \frac{1}{\beta}}{\frac{n}{q} - \frac{1}{\beta}}\right)^{p-rac{p}{n}+\frac{p}{n-1}} \omega_n^{\frac{p}{n-1}} \right\} K\omega_n^{\frac{p}{n-1}} C(n, \beta, K) : |\Omega|^{p/n}$$

for

$$\beta \in \left(1, 1 + \frac{n-1}{n102n^4C_1^n}\right),$$

where $q^* = \beta np/(\beta p + n(\beta - 1))$ and

$$C_1 = 2^{2n+3/2+1/n}(n/(n-2))^{1/n} 5\omega_n(4K)^{2(n-1)/n}, \quad n > 2,$$

$$C(n, \beta, K) = 2^{1-n+1/\beta}(100)^n K\omega_n^{1/\beta-1} \exp\left\{ K^{1/(n-1)} 2(\log(\sqrt{3} + \sqrt{2}) + n - 1) \right\}.$$

**Proof.** According to Theorem 3.7 we have

(4.10) $\frac{1}{\mu_p(\Omega)} \leq$

$$\inf_{q \in (q^*, n]} \left\{ \frac{2np}{n^p} \left(\frac{1 - \frac{1}{q} + \frac{1}{\beta}}{\frac{n}{q} - \frac{1}{\beta}}\right)^{p-rac{p}{n}+\frac{p}{n-1}} \omega_n^{\frac{p}{n-1}} \right\} K\omega_n^{\frac{p}{n-1}} C(n, \beta, K) : |J\varphi| L_\beta(B).$$
where \( q^* = \beta np/\(\beta p + n(\beta - 1)\) \). Now we estimate the integral from the right side of last inequality. Given Theorem 4.7 for
\[
\beta \in \left(1, 1 + \frac{n - 1}{n10^{2n}4^nC^n} \right)
\]
we get
\[
(4.11) \quad ||J_\varphi|_{L_\beta(B)}|| = \left( \int_B |J(x, \varphi)|^\beta \, dx \right)^{\frac{1}{\beta}} \leq C(n, \beta, K)|\Omega|.
\]
Combining inequalities (4.10) and (4.11) we obtain the required inequality.

This theorem immediately follows

**Theorem 4.9.** Let a domain \( \Omega \subset \mathbb{R}^n, n \geq 3 \), satisfying the \( \gamma \)-quasihyperbolic boundary condition, be a \( K \)-quasi-ball. Then
\[
\mu_p(\Omega) \geq \frac{M_p(K, \gamma)}{R_s},
\]
where \( R_s \) is a radius of a ball \( \Omega^* \) of the same measure as \( \Omega \) and \( M_p(K, \gamma) \) depends only on \( p, \gamma \) and the quasiconformity coefficient \( K \) of \( \Omega \).

**Proof.** Because the domain \( \Omega \) satisfies the \( \gamma \)-quasihyperbolic boundary condition, it is a \( K \)-quasiconformal \( \beta \)-regular domain with some \( \beta = \beta(\gamma) \). Then by the previous theorem
\[
\mu_p(\Omega) \geq \frac{M_p(K, \gamma)}{R_s},
\]
where
\[
M_p(K, \gamma) = \frac{2npK^p}{n^p} \left( \frac{1 - \frac{1}{q} + \frac{1}{r}}{\frac{1}{n} - \frac{1}{q} + \frac{1}{r}} \right)^{p - \frac{\hat{p}}{q} + \hat{p}} (\omega_n)^{\frac{\hat{p}}{q}} C(n, \beta(\gamma), K),
\]
for some \( q \in (q^*, n] \).

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