Almost Everywhere Convergence of Inverse Dunkl Transform on the Real Line

J. El Kamel and Ch. Yacoub

Department of Mathematics, Faculty of Sciences of Monastir, 5019 Monastir, TUNISIA

jamel.elkamel@fsm.rnu.tn; chokri.yacoub@fsm.rnu.tn

Abstract

In this paper, we will first show that the maximal operator $S^*_\alpha$ of spherical partial sums $S^\alpha_R$, associated to Dunkl transform on $\mathbb{R}$, is bounded on $L^p\left(\mathbb{R}, |x|^{2\alpha+1} \right)$ functions when $\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}$, and it implies that, for every $L^p\left(\mathbb{R}, |x|^{2\alpha+1} \right)$ function $f(x)$, $S^\alpha_R f(x)$ converges to $f(x)$ almost everywhere as $R \to \infty$. On the other hand we obtain a sharp version by showing that $S^*_\alpha$ is bounded from the Lorentz space $L^{p_i,1}\left(\mathbb{R}, |x|^{2\alpha+1} \right)$ into $L^{p_i,\infty}\left(\mathbb{R}, |x|^{2\alpha+1} \right)$, $i = 0, 1$ where $p_0 = \frac{4(\alpha+1)}{2\alpha+3}$ and $p_1 = \frac{4(\alpha+1)}{2\alpha+1}$.

Keywords: Dunkl transform, maximal function, almost everywhere convergence, Lorentz space.

1 Introduction and preliminaries

Given $\alpha \geq -\frac{1}{2}$ and a suitable function $f$ on $\mathbb{R}$, its Dunkl transform $D_\alpha$ is defined by

$$D_\alpha f(y) = \int_\mathbb{R} f(x) E_\alpha(-ixy) d\mu_\alpha(x), \quad y \in \mathbb{R};$$

(1)
Here
\[ d\mu_\alpha(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} |x|^{2\alpha+1} dx \] (2)

\[ E_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \left\{ \frac{J_\alpha(iz)}{(iz)^\alpha} + \frac{J_{\alpha+1}(iz)}{(iz)^{\alpha+1}} \right\}, \] (3)

where \( J_\alpha \) denotes the Bessel function of the first kind of order \( \alpha \). The inverse Dunkl transform \( \check{D}_\alpha \) is given by
\[ \check{D}_\alpha f(\lambda) = D_\alpha f(-\lambda). \]

In this paper, we are interested in the almost everywhere convergence as \( R \to \infty \) of the partial sums \( S_\alpha^\alpha f(x) \) where
\[ S_\alpha^\alpha f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{|y| \leq R} D_\alpha f(y) E_\alpha(ify) |y|^{2\alpha+1} dy. \]

Recall that given \( \beta \geq -\frac{1}{2} \), the Hankel transform of order \( \beta \) of a suitable function \( g \) on \((0, \infty)\) is defined by:
\[ H_\beta g(y) = \int_0^\infty g(x) \frac{J_\beta(xy)}{(yx)^\beta} x^{2\beta+1} dx, \quad y > 0. \] (4)

Nowak and Stempak ([3]), found an expression of the Dunkl transform \( D_\alpha \) in terms of Hankel transform of orders \( \alpha \) and \( \alpha + 1 \).

**Lemma 1.1** (see ([3])) Given \( \alpha \geq -\frac{1}{2} \), we have:
\[ D_\alpha f(y) = H_\alpha(f_e)(|y|) - iy H_{\alpha+1}\left(\frac{f_o(x)}{x}\right)(|y|), \] (5)

where for a function \( f \) on \( \mathbb{R} \), we denote by \( f_e \) and \( f_o \) the restrictions to \((0, \infty)\) of its even and odd parts, respectively, i.e. the functions on \((0, \infty)\) defined by
\[ f_e(x) = \frac{1}{2} (f(x) + f(-x)), \quad f_o(x) = \frac{1}{2} (f(x) - f(-x)), \quad x > 0. \]

Define, the partial sums \( s_\beta^R g(x) \) by:
\[ s_\beta^R g(x) = \int_0^R H_\beta g(y) \frac{J_\beta(xy)}{(xy)^\beta} y^{2\beta+1} dy, \quad x > 0 \] (6)
and

\[ s^\beta g(x) = \sup_{R > 0} |s^\beta_R g(x)|. \quad (7) \]

In 1988, Y. Kanjin (2) and E. Prestini (4) proved, independently, the following:

**Theorem 1.2** Let \( \beta \geq -\frac{1}{2} \).

- If \( \frac{4(\beta + 1)}{2\beta + 3} < p < \frac{4(\beta + 1)}{2\beta + 1} \) then \( s^\beta \) is bounded on \( L^p((0, \infty), x^{2\beta + 1}) \) functions.
- If \( p \leq \frac{4(\beta + 1)}{2\beta + 3} \) or \( p \geq \frac{4(\beta + 1)}{2\beta + 1} \) then \( s^\beta \) is not bounded on \( L^p((0, \infty), x^{2\beta + 1}) \) functions.

Throughout this paper we use the convention that \( c_\alpha \) denotes a constant, depending on \( \alpha \) and \( p \), its value may change from line to line.

## 2 Almost everywhere convergence

Define linear operators \( S^\alpha_R, R > 0 \) and \( S^\alpha_* \) on the Schwartz space \( S(\mathbb{R}) \) by

\[ S^\alpha_R f(x) = \frac{1}{2^{\alpha + 1} \Gamma(\alpha + 1)} \int_{|y| \leq R} D_\alpha f(y) E_\alpha (ixy) |y|^{2\alpha + 1} dy \quad (8) \]

and

\[ S^\alpha_* f(x) = \sup_{R > 0} |S^\alpha_R f(x)|, \quad x \in \mathbb{R}. \quad (9) \]

**Lemma 2.1** Given \( \alpha \geq -\frac{1}{2} \), we have

\[ S^\alpha_R(f)(x) = s^\alpha_R (f_\infty)(|x|) + xs^\alpha_R \left( \frac{f_\infty(r)}{r} \right) (|x|), \quad (10) \]

\[ S^\alpha_* f(x) \leq s^\alpha_* (f_\infty)(|x|) + |x| s^\alpha_* \left( \frac{f_\infty(r)}{r} \right) (|x|). \quad (11) \]
Proof. Let $x \in \mathbb{R}$. By (3), (8) and lemma 1.1, we have

$$S_{R}^{\alpha}f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha + 1)} \int_{|y| \leq R} \left[ \mathcal{H}_{\alpha}(f_{e})(|y|) - iy\mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|y|) \right]$$

$$\left[ 2^{\alpha}\Gamma(\alpha + 1) \left\{ \frac{J_{\alpha}(yx)}{(yx)^{\alpha}} + ixy \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} \right\} \right] |y|^{2\alpha+1} dy$$

$$= \frac{1}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha}(f_{e})(|y|) \frac{J_{\alpha}(yx)}{(yx)^{\alpha}} |y|^{2\alpha+1} dy$$

$$+ \frac{i}{2} \int_{|y| \leq R} y\mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+1} dy$$

$$- \frac{i}{2} \int_{|y| \leq R} y\mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|y|) \frac{J_{\alpha}(yx)}{(yx)^{\alpha}} |y|^{2\alpha+1} dy$$

$$+ \frac{x}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+3} dy$$

We note that the second and the third integrals are equal to zero. So

$$S_{R}^{\alpha}f(x) = \frac{1}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha}(f_{e})(|y|) \frac{J_{\alpha}(yx)}{(yx)^{\alpha}} |y|^{2\alpha+1} dy$$

$$+ \frac{x}{2} \int_{|y| \leq R} \mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|y|) \frac{J_{\alpha+1}(yx)}{(yx)^{\alpha+1}} |y|^{2\alpha+3} dy$$

$$= \int_{0}^{R} \mathcal{H}_{\alpha}(f_{e})(y) \frac{J_{\alpha}(|x| y)}{(|x| y)^{\alpha}} y^{2\alpha+1} dy + x \int_{0}^{R} \mathcal{H}_{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(y) \frac{J_{\alpha+1}(|x| y)}{(|x| y)^{\alpha+1}} y^{2\alpha+3} dy$$

$$= s_{R}^{\alpha}(f_{e})(|x|) + x s_{R}^{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|x|).$$

Thus

$$S_{R}^{\alpha}f(x) \leq s_{R}^{\alpha}(f_{e})(|x|) + |x| s_{R}^{\alpha+1} \left( \frac{f_{o}(r)}{r} \right)(|x|).$$
Almost Everywhere Convergence 5

**Proposition 2.2** Let \( \alpha \geq -\frac{1}{2} \).

- If \( \frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1} \) then \( S^\alpha_\ast \) is bounded on \( L^p(\mathbb{R}, |x|^{2\alpha + 1} \, dx) \) functions.
- If \( p \leq \frac{4(\alpha + 1)}{2\alpha + 3} \) or \( p \geq \frac{4(\alpha + 1)}{2\alpha + 1} \) then \( S^\alpha_\ast \) is not bounded on \( L^p(\mathbb{R}, |x|^{2\alpha + 1} \, dx) \) functions.

**Proof.** \( S^\alpha_\ast \) cannot be bounded for \( p \leq \frac{4(\alpha + 1)}{2\alpha + 3} \) or \( p \geq \frac{4(\alpha + 1)}{2\alpha + 1} \) (see: [2], [4]).

By theorem 1, we have for \( \frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1} \)

\[
\| s_\ast^\alpha(f_e)(|x|) \|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)} = 2 \| s_\ast^\alpha(f_e) \|_{L^p((0,\infty),x^{2\alpha+1}dx)} \\
\leq c_\alpha \| f_e \|_{L^p((0,\infty),x^{2\alpha+1}dx)} \\
\leq c_\alpha \| f \|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)}.
\]

On the other hand, as in ([4], [3]), one gets

\[
|x| s_\ast^{\alpha+1} \left( \frac{f_e(r)}{r} \right)(|x|) \leq c_\alpha \left| \frac{r}{|x|^{\alpha+\frac{3}{2}}} \right| \left[ M + H + \tilde{H} + \tilde{C} \right] \left( \frac{f_e(r)}{r} r^{\alpha+\frac{3}{2}} \right)(|x|),
\]

where \( M, H, \tilde{H} \) and \( \tilde{C} \) denotes respectively, the maximal function, the Hilbert integral, the maximal Hilbert transform and the Carleson operator.

Let \( K = M + H + \tilde{H} + \tilde{C} \) and \( w \in A_p(\mathbb{R}) \), \( p > 1 \). It is well known that

\[
\| Kf \|_{L^p(\mathbb{R},w(x)dx)} \leq c_\alpha \| f \|_{L^p(\mathbb{R},w(x)dx)}.
\]

Hence

\[
\left\| x \left| s_\ast^{\alpha+1} \left( \frac{f_o(r)}{r} \right)(|x|) \right\|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)} \\
\leq c_\alpha \left\| \left| x \right|^{-\alpha-\frac{9}{2}} K \left( \frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right)(|x|) \right\|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)} \\
\leq c_\alpha \left\| K \left( \frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right)(|x|) \right\|_{L^p(\mathbb{R},w(x)dx)},
\]

where \( M, H, \tilde{H} \) and \( \tilde{C} \) denotes respectively, the maximal function, the Hilbert integral, the maximal Hilbert transform and the Carleson operator.

Let \( K = M + H + \tilde{H} + \tilde{C} \) and \( w \in A_p(\mathbb{R}) \), \( p > 1 \). It is well known that

\[
\| Kf \|_{L^p(\mathbb{R},w(x)dx)} \leq c_\alpha \| f \|_{L^p(\mathbb{R},w(x)dx)}.
\]

Hence

\[
\left\| x \left| s_\ast^{\alpha+1} \left( \frac{f_o(r)}{r} \right)(|x|) \right\|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)} \\
\leq c_\alpha \left\| \left| x \right|^{-\alpha-\frac{9}{2}} K \left( \frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right)(|x|) \right\|_{L^p(\mathbb{R},|x|^{2\alpha+1}dx)} \\
\leq c_\alpha \left\| K \left( \frac{f_o(r)}{r} r^{\alpha+\frac{3}{2}} \right)(|x|) \right\|_{L^p(\mathbb{R},w(x)dx)},
\]

where \( M, H, \tilde{H} \) and \( \tilde{C} \) denotes respectively, the maximal function, the Hilbert integral, the maximal Hilbert transform and the Carleson operator.

Let \( K = M + H + \tilde{H} + \tilde{C} \) and \( w \in A_p(\mathbb{R}) \), \( p > 1 \). It is well known that

\[
\| Kf \|_{L^p(\mathbb{R},w(x)dx)} \leq c_\alpha \| f \|_{L^p(\mathbb{R},w(x)dx)}.
\]
with \( w(x) = |x|^{2\alpha+1-p(\alpha+1/2)} \).

Since \( \frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1} \) if and only if \(-1 < 2\alpha + 1 - p(\alpha+1/2) < p-1\), then \( w \in A_p(\mathbb{R}) \) and by (13)

\[
\left\| |x| \, s_r^{\alpha+1} \left( \frac{f_0(r)}{r} \right) (|x|) \right\|_{L^p(\mathbb{R}, |x|^{2\alpha+1} \, dx)} \leq c_\alpha \left\| \frac{f_0(|x|)}{|x|^\alpha} |x|^{\alpha+3/2} \right\|_{L^p(\mathbb{R}, w(x) \, dx)}
\leq c_\alpha \| f_0(x) \|_{L^p(\mathbb{R}, |x|^{2\alpha+1} \, dx)}
\leq c_\alpha \| f(x) \|_{L^p(\mathbb{R}, |x|^{2\alpha+1} \, dx)}.
\]

We conclude by lemma 2.1.

**Corollary 2.3** For every \( f \in L^p(\mathbb{R}, |x|^{2\alpha+1} \, dx) \), if \( \frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1} \), then

\( S^\alpha_R f(x) \rightarrow f(x) \) a.e. as \( R \rightarrow \infty \).

### 3 Endpoint estimates

We recall that the Lorentz space \( L^{p,q}(X, \mu) \), is the set of all measurable functions \( f \) on \( X \) satisfying

\[
\| f \|_{p,q} = \left( \frac{q}{p} \int_0^\infty \left( t^{\frac{1}{p}} \, f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty
\]

when \( 1 \leq p < \infty \), \( 1 \leq q < \infty \), and

\[
\| f \|_{p,q} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{\lambda > 0} \lambda (d_f(\lambda))^{\frac{1}{p}} < \infty
\]

when \( 1 \leq p \leq \infty \) and \( q = \infty \). Where \( f^* \) denotes the nonincreasing rearrangement of \( f \), i.e.

\[
f^*(t) = \inf \{ s > 0 / d_f(s) \leq t \}, \quad d_f(s) = \mu \{ x \in X / |f(x)| > s \}.
\]

In 1991 E. Romera and F. Soria [5] (see also L. Colzani and all [1]) proved the following:
Almost Everywhere Convergence

**Theorem 3.1** Let \( \alpha > -\frac{1}{2} \), then \( s_\alpha^* \) is bounded from the Lorentz space \( L^{p,1}((0, \infty), x^{2\alpha+1}dx) \) into \( L^{p,\infty}((0, \infty), x^{2\alpha+1}dx) \), \( i=0,1 \) when \( p_0 = \frac{4(\alpha + 1)}{2\alpha + 3} \) and \( p_1 = \frac{4(\alpha + 1)}{2\alpha + 1} \) is the index conjugate to \( p_0 \).

Using this result, we will see that proposition 2.2 can be strengthened. More precisely we obtain :

**Proposition 3.2** Let \( \alpha > -\frac{1}{2} \), then \( S_\alpha^* \) is bounded from the Lorentz space \( L^{p,1}(\mathbb{R}, |x|^{2\alpha+1}dx) \) into \( L^{p,\infty}(\mathbb{R}, |x|^{2\alpha+1}dx) \), \( i = 0,1 \).

So using the formulation of Marcinkiewicz interpolation theorem in terms of Lorentz space we retrieve Proposition 2.2 \((\alpha > -\frac{1}{2})\) as a corollary.

**Proof.** By lemma 2.1, we have

\[
\mu_\alpha \left\{ x \in \mathbb{R} : S_\alpha^* f(x) > \lambda \right\} \leq \mu_\alpha \left\{ x \in \mathbb{R} : s_\alpha^* f_e(|x|) > \frac{\lambda}{2} \right\} + \mu_\alpha \left\{ x \in \mathbb{R} : |x| s_\alpha^* f_e \left( \frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\}.
\]

\[= I + II\]

By theorem 2.4, we get :

\[
\mu_\alpha \left\{ x \in \mathbb{R} : s_\alpha^* f_e(|x|) > \frac{\lambda}{2} \right\} = 2\mu_\alpha \left\{ x \in (0, \infty) : s_\alpha^* f_e(x) > \frac{\lambda}{2} \right\}
\]

\[
\leq \frac{c_\alpha}{\lambda p_i} \| f_e \|_{p_i,1} \leq \frac{c_\alpha}{\lambda p_i} \| f \|_{p_i,1}.
\]

To estimate \( II \), we follow closely [5] and we sketch a proof for completeness. We decompose the set :

\[
\left\{ x \in \mathbb{R} : |x| s_\alpha^* f_e \left( \frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\}
\]

\[= \bigcup_{k \in \mathbb{Z}} \left\{ x \in \mathbb{R} : |x| \in I_k, |x| s_\alpha^* f_e \left( \frac{f_o(r)}{r} \right) (|x|) > \frac{\lambda}{2} \right\},\]
where $I_k = [2^k, 2^{k+1}]$.

Put $g(r) := \frac{f_o(r)}{r} = g^1_k(r) + g^2_k(r)$, with $g^1_k = g x^*_k$, $g^2_k = g x^*_{k+1}$, where $I^*_k = [2^{k-1}, 2^{k+2}]$.

By (12), we have:

$$|x| s^\alpha_*(g^1_k(r)) (|x|) \leq \frac{c_\alpha}{|x|^\alpha} K\left(g^1_k(r) r^\alpha (|x|)\right).$$

By (13), p: 1021, we have for $1 < p < \infty$,

$$\sum_{k \in \mathbb{Z}} \mu_\alpha \left\{ x \in \mathbb{R} / |x| \in I_k, \frac{1}{|x|^\alpha} K(r^\alpha (|x|)) > \frac{\lambda}{2}\right\} \leq \frac{c_\alpha}{\lambda^p} \|f_o\|^p_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} \leq \frac{c_\alpha}{\lambda^p} \|f\|^p_{L^p(\mathbb{R}, |x|^{2\alpha+1} dx)} \leq \frac{c_\alpha}{\lambda^p} \|f\|^p_{L^p}.$$

On the other hand as in (13), p: 1021, we have

$$|x| s^\alpha_*(g^2_k(r)) (|x|) \leq \frac{c_\alpha}{|x|^\alpha} \int_0^\infty s^{\alpha+1/2} \left| \frac{f_o(s)}{s(|x| + s)} \right| ds$$

$$\leq \frac{c_\alpha}{|x|^\alpha} \int_0^\infty |f_o(s)| s^{\alpha+1/2} ds$$

$$\leq \frac{c_\alpha}{|x|^\alpha} \int_\mathbb{R} |f_o(s)| \frac{1}{|s|^{\alpha+1/2}} |s|^{2\alpha+1} ds.$$

Remark that we have considered $f_0$ as a function defined on $\mathbb{R}$. As the same we get

$$|x| s^\alpha_*(g^2_k(r)) (|x|) \leq \frac{c_\alpha}{|x|^\alpha} \int_0^\infty |f_o(s)| s^{\alpha-1/2} ds$$

$$\leq \frac{c_\alpha}{2 |x|^\alpha} \int_\mathbb{R} |f_o(s)| \frac{1}{|s|^{\alpha+1/2}} |s|^{2\alpha+1} ds.$$

Using the following facts:

$$\frac{1}{|x|^{\alpha + \frac{\alpha}{2}}} \in L^{p_1, \infty}(\mathbb{R}, |x|^{2\alpha+1}),$$
Almost Everywhere Convergence

\[
\frac{1}{|x|^\alpha + \frac{3}{2}} \in L^{p_0, \infty} \left( \mathbb{R}, |x|^{2\alpha+1} \right),
\]

and Holder’s inequality for the Lorentz spaces, we arrive to:

\[
\mu_{\alpha}\left\{ x \in \mathbb{R}/ |x|^{\alpha+1} (g_k^2(r)) (|x|) > \frac{\lambda}{2} \right\} \leq \frac{c_{\alpha}}{\lambda^{p_i}} \|f_o\|_{p_i,1}^{p_i} \leq \frac{c_{\alpha}}{\lambda_{p_i}} \|f\|_{p_i,1}^{p_i},
\]

which completes the proof.

Acknowledgment. We are grateful to Professor K. Stempak for sending us the preprint [3].

References

[1] L. Colzani, A. Crespi, G. Travaglini and M. Vignati, Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in euclidean and noneuclidean spaces, Tran. Amer. Math. Soc. 338, N.1 (1993),43-55.

[2] Y. Kanjin, Convergence and divergence almost everywhere of spherical means for radial functions, Proc. Amer. Math. Soc. 103, N.4 (1988), 1063-1069.

[3] A. Nowak and K. Stempak, Relating transplantation and multipliers for Dunkl and Hankel transforms, To appear in Math. Nachr.

[4] E. Prestini, Almost everywhere convergence of the spherical partial sums for radial functions, Mh. Math. 105. (1988), 207-216.

[5] E. Romera and F. Soria, Endpoint estimates for the maximal operator associated to spherical partial sums on radial functions, Proc. Amer. Math. Soc. 111, N.4 (1991), 1015-1022.