DISJOINTLY HOMOGENEOUS REARRANGEMENT INVARIANT SPACES VIA INTERPOLATION

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Abstract. A Banach lattice $E$ is called $p$-disjointly homogeneous, $1 \leq p \leq \infty$, when every sequence of pairwise disjoint normalized elements in $E$ has a subsequence equivalent to the unit vector basis of $\ell_p$. Employing methods from interpolation theory, we clarify which r.i. spaces on $[0, 1]$ are $p$-disjointly homogeneous. In particular, for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0, 1]$, which is not equivalent neither $1$ nor $t$, there exists a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$. Moreover, in the class of all interpolation r.i. spaces with respect to the Banach couple of Lorentz and Marcinkiewicz spaces with the same fundamental function, dilation indices of which are non-trivial, for every $1 < p < \infty$, there is only a unique $p$-disjointly homogeneous space.

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1. Introduction

A Banach lattice $E$ is called disjointly homogeneous if any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of pairwise disjoint normalized elements from $E$ contain subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$, respectively, which are equivalent in $E$. Similarly, $E$ will be called $p$-disjointly homogeneous, $1 \leq p \leq \infty$, when every sequence of pairwise disjoint normalized elements in $E$ has a subsequence equivalent to the unit vector basis of $\ell_p$. The importance of disjointly homogeneous spaces introduced in [16] is based, mainly, on close connections between the classes of compact and strictly singular operators acting in such spaces. Recall that a linear operator between Banach spaces is strictly singular if it is not an isomorphism when restricted to any infinite-dimensional subspace. Recently, it was obtained a series of interesting results showing that a strictly singular operator has some compact power whenever the Banach lattice in which it is bounded is disjointly homogeneous (see [14], [15], [16], [17]); in particular, in the paper [14], it is proved that every strictly singular operator in a $p$-disjointly homogeneous rearrangement invariant (r.i.) space with lower Boyd index $\alpha_X > 0$ has compact square and if $p = 2$ such a operator even is compact itself. For this reason it is important to know how wide the class of disjointly homogeneous Banach lattices. As is shown in the above cited papers it contains $L_p(\mu)$-spaces, $1 \leq p \leq \infty$, Lorentz function spaces $L_{q,p}$ and $\Lambda(W,p)$, certain classes of Orlicz function spaces and also some discrete spaces such as the Tsirelson space.

The main aim of this paper is to clarify which r.i. spaces on $[0, 1]$ are $p$-disjointly homogeneous. We focused on the more interesting reflexive case, when $1 < p < \infty$. Our approach to this problem is based on using tools from interpolation theory, especially, the real and complex methods of interpolation. By the complex method
of interpolation, we prove that for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0, 1]$, which is not equivalent neither 1 nor $t$, there exists a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$ (Corollary 1). Note that there is the only r.i. space on $[0, 1]$, $L_\infty$ (resp. $L_1$), having the fundamental function equivalent to 1 (resp. $t$). This result is new even for the power functions $\varphi(t) = t^\alpha$, $0 < \alpha < 1$. Moreover, in the class of all interpolation r.i. spaces with respect to the Banach couple of Lorentz and Marcinkiewicz spaces with the same fundamental function $\varphi$, dilation indices of which are non-trivial, for every $1 < p < \infty$, there is only a unique $p$-disjointly homogeneous space, namely, the Lorentz space $\Lambda_{p, \varphi}$ with the quasi-norm $\|x\|_{p, \varphi} = (\int_0^1 |x^*(t)\varphi(t)|^p \, dt/t)^{1/p}$ (Theorem 2). At the same time, in Section 4, for every $1 < p < \infty$ and any increasing concave function $\varphi$ on $[0, 1]$ such that $\lim_{t \to 0} \varphi(t) = 0$ and upper dilation index $\beta_\varphi < 1$, we construct a $p$-disjointly homogeneous r.i. space with the fundamental function $\varphi$, which is not interpolation with respect to the corresponding couple of Lorentz and Marcinkiewicz spaces (Theorem 3). Finally, in Section 5 we investigate some properties of sequences of pairwise disjoint functions in the real interpolation spaces $(X_0, X_1)_{\alpha, p}$ ($0 < \theta < 1$, $1 \leq p < \infty$) provided that $X_1 \subset X_0$.

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2. Preliminaries

2.1. Rearrangement invariant spaces. In this subsection we present some definitions and auxiliary results from the theory of rearrangement invariant spaces. For more details on the latter theory we refer to [8, 19, 21].

A Banach function space $X = (X, \| \cdot \|)$ of (classes of) real measurable functions $x(t)$ defined on the interval $[0, 1]$ is said to be rearrangement invariant (r.i.) space if the conditions $x^*(t) \leq y^*(t)$ a.e. on $[0, 1]$ and $y \in X$ imply $x \in X$ and $\|x\|_X \leq \|y\|_X$. Here, $x^*$ denotes the non-increasing right-continuous rearrangement of $|x(s)|$ given by

$$x^*(t) = \inf \{ \tau \geq 0 : m\{s \in [0, 1] : |x(s)| > \tau\} \leq t \}, \quad 0 \leq t \leq 1,$$

where $m$ is the Lebesgue measure.

For every r.i. space $X$ on $[0, 1]$ we have the continuous embeddings $L_\infty[0, 1] \subset X \subset L_1[0, 1]$. The fundamental function of an r.i. space $X$ is given by $\varphi_X(t) := \|\chi_{[0, t]}\|_X$, $0 \leq t \leq 1$, where $\chi_A$ denotes the characteristic function of a measurable set $A \subset [0, 1]$. It is well known that every fundamental function is quasi-concave on $[0, 1]$, i.e., it is non-decreasing and the function $\varphi_X(t)/t$ is non-increasing on $(0, 1)$. Each quasi-concave function $\varphi$ on $[0, 1]$ is equivalent to its least concave majorant $\tilde{\varphi}$, more exactly, $\frac{1}{t} \tilde{\varphi}(t) \leq \varphi(t) \leq \tilde{\varphi}(t)$ ($0 \leq t \leq 1$) [19, Theorem 2.1.1].

If $X$ is an r.i. space on $[0, 1]$, then the Köthe dual space $X'$ consists of all measurable functions $y$ such that

$$\|y\|_{X'} = \sup \left\{ \int_0^1 x(t)y(t) \, dt : \|x\|_X \leq 1 \right\} < \infty.$$

The space $X'$ is r.i. as well; it is embedded into the dual space $X^*$ of $X$ isometrically, and $X' = X^*$ if and only if $X$ is separable. An r.i. space $X$ is said to have the Fatou property if the conditions $x_n \in X$ ($n = 1, 2, \ldots$), $\sup_{n=1,2,\ldots} \|x_n\|_X < \infty$, and $x_n \to x$ a.e. imply that $x \in X$ and $\|x\|_X \leq \lim inf_{n \to \infty} \|x_n\|_X$. $X$ has the Fatou
property if and only if the natural embedding of $X$ into its second Köthe dual $X''$ is an isometric surjection.

For a given $t > 0$ the dilation operator $\sigma_t$ defined by $\sigma_t x(s) = x(s/t)\chi_{[0,1]}(s/t)$, $s \in [0,1]$, is bounded in every r.i. space $X$ and $\|\sigma_t\|_{X \to X} \leq \max(1,t)$. The lower and upper Boyd indices of $X$ are defined by

$$\alpha_X = \lim_{t \to 0^+} \frac{\log \|\sigma_t\|_{X \to X}}{\log t}, \quad \beta_X = \lim_{t \to \infty} \frac{\log \|\sigma_t\|_{X \to X}}{\log t},$$

respectively. In general, $0 \leq \alpha_X \leq \beta_X \leq 1$.

The lower and upper dilation indices of an increasing concave function $\psi : [0,1] \to [0, \infty)$ are defined as

$$\gamma_\psi = \lim_{t \to 0^+} \frac{\log m_\psi(t)}{\log t}, \quad \delta_\psi = \lim_{t \to \infty} \frac{\log m_\psi(t)}{\log t},$$

where $m_\psi(t) = \sup_{0 < s_1 < 1, 0 < s_2 < 1} \frac{\psi(st)}{\psi(s)}$ respectively. We have $0 \leq \gamma_\psi \leq \delta_\psi \leq 1$ and from the inequality $m_{\psi X}(t) \leq \|\sigma_t\|_{X \to X}$, $t > 0$, it follows that $\alpha_X \leq \gamma_{\psi_X} \leq \delta_{\psi_X} \leq \beta_X$.

Most important examples of r.i. spaces are the $L_p$-spaces (1 $\leq p < \infty$) and their natural generalization, the Orlicz spaces. The Orlicz space $L_M$ on $[0,1]$ is generated by the Luxemburg-Nakano norm

$$\|x\|_{L_M} = \inf \left\{ \lambda > 0 : \int_0^1 M \left( \frac{|x(t)|}{\lambda} \right) dt \leq 1 \right\},$$

where $M$ is an Orlicz function, that is, an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. The fundamental function of $L_M$ is $\varphi_{L_M}(t) = 1/M^{-1}(1/t)$.

An important role will be played throughout the paper by Lorentz and Marcinkiewicz spaces. Let $\varphi$ be an increasing concave function on $[0,1]$. The norms of the Lorentz $\Lambda(\varphi)$ and Marcinkiewicz spaces $M(\varphi)$ are defined by the functionals

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) \varphi(t) \, dt \quad \text{and} \quad \|x\|_{M(\varphi)} = \sup_{0 < t \leq 1} \frac{1}{\varphi(t)} \int_0^t x^*(s) \, ds,$$

respectively. The spaces $\Lambda(\varphi)$ and $M(\varphi)$, where $\tilde{\varphi}(t) := t/\varphi(t)$, have the same fundamental function equal to $\varphi$. Moreover, these spaces are extreme in the class of all r.i. spaces with the fundamental function $\varphi$, i.e., if $\varphi_X(t) = \varphi(t)$, $0 \leq t \leq 1$, then $\Lambda(\varphi) \subset X \subset M(\varphi)$ and

$$\|x\|_{M(\varphi)} \leq \|x\|_X \leq \|x\|_{\Lambda(\varphi)} \quad (x \in \Lambda(\varphi)) \quad (1)$$

(see e.g. [19] Theorems 2.5.5 and 2.5.7).

Recall also the definition of some generalizations of the classical Lorentz spaces $\Lambda(\varphi)$. Let us begin with the most important for us Lorentz space $\Lambda_{p,\varphi}$, which will arise later as a $p$-disjointly homogeneous space. For $1 \leq p < \infty$ and any increasing concave function $\varphi$ on $[0,1]$ this space is generated by the functional

$$\|x\|_{\Lambda_{p,\varphi}} = \left( \int_0^1 [x^*(t)\varphi(t)]^p \frac{dt}{t} \right)^{1/p}.$$

The spaces $\Lambda(\varphi)$ were investigated by Sharpley [29] and Raynaud [28]. If $\gamma_\varphi > 0$, then

$$\varphi(t) \leq \int_0^t \frac{\varphi(s)}{s} ds \leq C\varphi(t), \quad 0 < t \leq 1.$$
with respect to it. One of the most important interpolation methods is the
which assigns to every Banach couple \( \overline{X} \) the
with natural norms. A Banach space \( X \) is called an intermediate space between \( X_0 \) and \( X_1 \) if
\( X_0 \cap X_1 \subset X \subset X_0 + X_1 \). Such a space \( X \) is called an interpolation space with
the couple \((X_0, X_1)\) (we write: \( X \in \text{Int}(X_0, X_1) \)) if, for any bounded
linear operator \( T : X_0 + X_1 \to X_0 + X_1 \) such that the restriction \( T|_{X_i} : X_i \to X_i \)
is bounded for \( i = 0, 1 \), the restriction \( T|_{X} : X \to X \) is also bounded and \( \|T\|_{X \to X} \leq C \max \{\|T\|_{X_0 \to X_0}, \|T\|_{X_1 \to X_1}\} \) for some \( C > 0 \geq 1 \).

An interpolation method or interpolation functor \( \mathcal{F} \) is a construction (a rule)
which assigns to every Banach couple \( X = (X_0, X_1) \) an interpolation space \( \mathcal{F}(X) \)
with respect to it. One of the most important interpolation methods is the K-
method known also as the real Lions-Peetre interpolation method. Throughout
the paper we will consider only embedded Banach couples \((X_0, X_1)\) or, more precisely,
satisfying the condition: \( X_1 \subset X_0 \) and \( \|x\|_{X_0} \leq \|x\|_{X_1} \) for all \( x \in X_1 \). For this
reason, we adopt our definition of the real K-method to this special case (for general
situation we refer the reader to monograph [10]). For a Banach couple \( \overline{X} = (X_0, X_1) \)
(cf. [19], Lemma 2.1.4), and therefore \( \Lambda_{1, \varphi} \) coincides with the Lorentz space \( \Lambda(\varphi) \).
Moreover, \( \Lambda_{p, \varphi} \) is an r.i. space on \([0, 1]\) with the equivalent norm
\[
\|x\|_{\Lambda_{p, \varphi}} = \left( \int_0^1 [x^{**}(t)\varphi(t)]^p \frac{dt}{t} \right)^{1/p}
\]
whenever \( 0 < \gamma \varphi \leq \delta \varphi < 1 \) (cf. [29], Lemma 3.1), where \( x^{**}(t) = \frac{1}{t} \int_0^t x^*(s) \, ds \), and
its fundamental function \( \varphi_{\Lambda_{p, \varphi}} \) is equivalent to \( \varphi \). If \( \varphi(t) = t^{1/q}, 1 < q < \infty, \)
and \( 1 \leq p < \infty \) we obtain the well-known spaces \( L_{q,p} \),
\[
\|x\|_{L_{q,p}} = \left( \int_0^1 \left( t^{1/q} x^*(t) \right)^p \frac{dt}{t} \right)^{1/p},
\]
which are very important in interpolation theory of operators (see e.g. [3] and [19]).

Much earlier, in [22], Lorentz introduced the another space of such a type,
\( \Lambda(W, p) \), generated by the functional
\[
\|x\|_{\Lambda(W, p)} = \left( \int_0^1 [x^*(t)]^p W(t) \, dt \right)^{1/p},
\]
where \( 1 \leq p < \infty \) and \( W \) is a positive, non-increasing function on \([0, 1]\) such that
\( \lim_{t \to 0} W(t) = +\infty \) and \( \int_0^1 W(t) \, dt = 1 \). This space is also \( p \)-disjointly homogeneous (see Introduction) but, unlike the spaces \( \Lambda_{p, \varphi} \), for a fixed \( 1 < p < \infty \) the
fundamental function of \( \Lambda(W, p) \) cannot be equivalent to arbitrary increasing quasi-
concave function. Indeed, \( \varphi_{\Lambda(W, p)}(t) = \left( \int_0^t W(s) \, ds \right)^{1/p}, 0 \leq t \leq 1 \),
and hence for every function \( W \) with the above properties we have the following restriction on its
upper dilation index: \( \delta_{\varphi_{\Lambda(W, p)}} \leq 1/p \). In particular, this class of spaces does not
embrace the spaces \( \Lambda_{1, \varphi} \) if \( 1 < q < p < \infty \).

2.2. Interpolation spaces and functors. Let us recall some definitions from the
theory of interpolation of operators; more detailed information see in the mono-
graphs [3] [4] [10] [19] [22].

Say that two Banach spaces \( X_0 \) and \( X_1 \) form a Banach couple \( \overline{X} = (X_0, X_1) \) if
they linearly and continuously embedded into a Hausdorff topological linear space;
then we can define their sum \( X_0 + X_1 \) and intersection \( X_0 \cap X_1 \) with natural norms. A Banach space \( X \) is called an intermediate space between \( X_0 \) and \( X_1 \) if
\( X_0 \cap X_1 \subset X \subset X_0 + X_1 \). Such a space \( X \) is called an interpolation space with
respect to the couple \((X_0, X_1)\) (we write: \( X \in \text{Int}(X_0, X_1) \)) if, for any bounded linear
operator \( T : X_0 + X_1 \to X_0 + X_1 \) such that the restriction \( T|_{X_i} : X_i \to X_i \)
is bounded for \( i = 0, 1 \), the restriction \( T|_{X} : X \to X \) is also bounded and \( \|T\|_{X \to X} \leq C \max \{\|T\|_{X_0 \to X_0}, \|T\|_{X_1 \to X_1}\} \) for some \( C > 0 \geq 1 \).

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satisfying the condition: \( X_1 \subset X_0 \) and \( \|x\|_{X_0} \leq \|x\|_{X_1} \) for all \( x \in X_1 \). For this
reason, we adopt our definition of the real K-method to this special case (for general
situation we refer the reader to monograph [10]). For a Banach couple \( \overline{X} = (X_0, X_1) \)
the Peetre $K$-functional of an element $f \in X_0 + X_1$ is defined by

$$K(t, f; X_0, X_1) = \inf \{ \| f_0 \|_{X_0} + t \| f_1 \|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1 \}, \ t > 0.$$  

Let $X_1 \subset X_0$ and $\| x \|_{X_0} \leq \| x \|_{X_1}, x \in X_1$. Then, if $E$ is a Banach lattice of sequences $(a_k)_{k=1}^{\infty}$ such that $l_t \subset E \subset l_{\infty}$ and $\varphi$ is a continuous increasing concave function on $[0, 1]$ such that $\varphi(0) = 0$, the space $(X_0, X_1)^{K_{E(1/\varphi(2^{-n}))}}$ of the $K$-method of interpolation consists of all $f \in X_0 + X_1$ such that

$$(K(2^{-n}, f; X_0, X_1))_{n=1}^{\infty} \in E(1/\varphi(2^{-n})) \text{ and } \| f \|_{\varphi, E} := \| (K(2^{-n}, f; X_0, X_1))_{E(1/\varphi(2^{-n}))} \|	ext{.}$$

In what follows, if $E$ is a Banach sequence lattice and $b_0 \geq 0$, by $E(b_0)$ we denote the Banach lattice consisting of all sequences $(x_n)_{n=1}^{\infty}$ such that $(x_n b_n)_{n=1}^{\infty} \in E$ with the norm $\| (x_n) \|_{E(b_0)} := \| (x_n b_n) \|_E$. It is easy to see that the space $(X_0, X_1)^{K_{E(1/\varphi(2^{-n}))}}$ belongs to the class $\text{Int}(X_0, X_1)$ for every Banach couple $(X_0, X_1)$. In particular, if $E = l_t$, $1 \leq p < \infty$, and $\varphi(t) = t^\theta$, $0 < \theta < 1$, we obtain classical Lions–Peetre interpolation spaces $(X_0, X_1)_{\theta, p}$ with the norm

$$\| f \|_{\theta, p} = \left( \sum_{n=1}^{\infty} (2^{n\theta} K(2^{-n}, f; X_0, X_1))^p \right)^{1/p} \text{.}$$

The complex method of interpolation introduced by Calderon and Lions (see e.g. [9, Chapter 4]) may be regarded as another important way of constructing interpolation spaces. It associates to any Banach couple $(X_0, X_1)$ the family of spaces $[X_0, X_1]_\theta (0 \leq \theta \leq 1)$ interpolation with respect to this couple. Next, we will consider this method only for couples of the form $(M(\tilde{\varphi}), \Lambda(\varphi))$, where $\varphi$ is an increasing concave function on $[0, 1]$, using the fact that, for every $0 < \theta < 1$, the space $[M(\tilde{\varphi}), \Lambda(\varphi)]_\theta$ isometrically coincides with the Calderón–Lozanovskii space $M(\tilde{\varphi})^{-\theta}\Lambda(\varphi)^\theta$ [13, Theorem 4.1.4]. The latter one can be defined as follows: if $(X_0, X_1)$ is a couple of r.i. spaces on $[0, 1]$, the space $X_0^{-\theta}X_1^\theta$ consists of all measurable functions $x(t)$ on $[0, 1]$ such that

$$|x(t)| \leq C|y(t)|^{1-\theta}|z(t)|^\theta, \ 0 \leq t \leq 1,$$

for some $C > 0$, $y \in X_0$ and $z \in X_1$, with $\| y \|_{X_0} \leq 1$, $\| z \|_{X_1} \leq 1$. The space $X_0^{-\theta}X_1^\theta$ is endowed with the norm $\| x \| := \inf C$, where infimum is taken over all $C$ satisfying the preceding inequality together with some $y$ and $z$.

### 2.3. Ultrasymmetric spaces

The following concept from the paper [25] will be useful for us. An r.i. space $X$ on $[0, 1]$ is called ultrasmetric if there exists an increasing concave function $\varphi$ on $[0, 1]$ such that $X \subset \text{Int}(M(\tilde{\varphi}), \Lambda(\varphi))$. It is clear that the latter assumption implies immediately that the fundamental function of $X$ is equivalent to $\varphi$, that is, there are constants $c_0, c_1 > 0$ such that $c_0 \varphi(t) \leq \varphi(t) \leq c_1 \varphi(t)$ for all $t \in [0, 1]$. Ultrasymmetric spaces embrace all possible generalizations of Lorentz-Zygmund spaces and this class of spaces is useful in many applications (see, for example, [26] and [27]). We mention also that the natural question, which Orlicz spaces are ultrasmetric, was answered in the paper [6].

In Section 4 we apply the following characterization theorem of ultrasmetric spaces (cf. [25, Theorem 2.1]). Let us write it in the form used further: An r.i. space $X$ such that $\gamma_{\varphi, X} > 0$ is ultrasmetric if and only if

$$X = (L_1, L_\infty)^{K_{E(\varphi X(2^{-n})2^n)}}, \text{where } E \in \text{Int}(l_{\infty}, l_1).$$
In particular, clearly, if an r.i. space $X$ is ultrasyymmetric, the Banach lattice $E$ in
the preceding formula should be symmetric.

3. ULTRASYMMETRIC $p$-DISJOINTLY HOMOGENEOUS R.I. SPACES

**Theorem 1.** Let $\varphi$ be an increasing concave function on $[0,1]$ such that $\varphi(0) = 0$ and

$$\lim_{t \to 0} \varphi(t) = \lim_{t \to 0} \frac{t}{\varphi(t)} = 0.$$  

Then, for every $1 < p < \infty$, the space $\mathcal{L}_{\theta} := [M(\tilde{\varphi}), \Lambda(\varphi)]_{\theta}$, with $\theta := 1/p$, is $p$-disjointly homogeneous.

**Proof.** Let $x_n \in \mathcal{L}_{\theta}$ be pairwise disjoint functions, $x_n \geq 0$, and $\|x_n\|_{\theta} = 1$ ($n = 1, 2, \ldots$). We begin with proving the upper estimate showing that some subsequence of $\{x_n\}$ (we will denote it, as above, by $\{x_n\}$) satisfies the inequality

$$\|\sum_{k=1}^{\infty} a_k x_k\|_{\theta} \leq C \| (a_k) \|_{l_p}$$

for any sequence $(a_k) \in l_p$.

Firstly, since the space $\mathcal{L}_{\theta}$ coincides isometrically with the Calderón-Lozanovskii space $M(\tilde{\varphi})^{1-\theta} \Lambda(\varphi)^{\theta}$, $0 < \theta < 1$ (see [19, Theorem 4.1.4] or Preliminaries), for arbitrary $n \in \mathbb{N}$ and $\varepsilon > 0$ there are $y_n \in \Lambda(\varphi)$, $z_n \in M(\tilde{\varphi})$, $y_n \geq 0$, $z_n \geq 0$, supp $y_n \subset$ supp $x_n$, supp $z_n \subset$ supp $x_n$, such that $x_n \leq (1 + \varepsilon) y_n \varphi(z_n)\Lambda(\varphi)$ and $\|y_n\|_{\Lambda(\varphi)} = \|z_n\|_{M(\tilde{\varphi})} = 1$. Taking into account (3) and (13) Theorem 5.1] (see also [31, Proposition 1]), we can assume (passing to a subsequence if necessary) that

$$\|\sum_{k=1}^{\infty} b_k y_k\|_{\Lambda(\varphi)} \simeq \|(b_k)\|_{l_1} \text{ and } \|\sum_{k=1}^{\infty} c_k z_k\|_{M(\tilde{\varphi})} \simeq \|(c_k)\|_{l_\infty}$$

for all $(b_k) \in l_1$ and $(c_k) \in l_\infty$.

Next, let $(a_k) \in l_p$, $a_k \geq 0$ ($k \in \mathbb{N}$), be arbitrary. Since $\theta = 1/p$, we have $l_{1-\theta}^{\theta} = l_p$ (see e.g. [19, Ch. IV, the end of §1]). Hence, there exist $(b_k) \in l_1$ and $(c_k) \in l_\infty$, $b_k \geq 0$, $c_k \geq 0$, satisfying the conditions: $\|(b_k)\|_{l_1} \leq \|(a_k)\|_{l_p}$, $\|(c_k)\|_{l_\infty} \leq \|(a_k)\|_{l_p}$, and $a_k \leq (1 + \varepsilon) b_k^\theta c_k^{1-\theta}$. Then, from the inequality

$$\sum_{k=1}^{\infty} a_k x_k \leq (1 + \varepsilon)^2 \left( \sum_{k=1}^{\infty} b_k y_k \right)^\theta \left( \sum_{k=1}^{\infty} c_k z_k \right)^{1-\theta},$$

and [22] it follows that

$$\|\sum_{k=1}^{\infty} a_k x_k\|_{\theta} \leq (1 + \varepsilon)^2 \left\| \sum_{k=1}^{\infty} b_k y_k \right\|_{\Lambda(\varphi)} \left( \sum_{k=1}^{\infty} c_k z_k \right)^{1-\theta} \simeq \|(b_k)\|_{l_1}^\theta \|(c_k)\|_{l_\infty}^{1-\theta} \leq \|(a_k)\|_{l_p},$$

and (4) is proved.

The reverse inequality we prove by using duality. Let us find $x'_n \in \mathcal{L}_{\theta}'$ such that $\|x'_n\|_{\mathcal{L}_{\theta}'} \simeq 1$, supp $x'_n \subset$ supp $x_n$, and

$$\int_0^1 x_n x'_n \, dt = 1, \quad n \in \mathbb{N}.$$
From the Köthe duality theorem (see e.g. [23] Theorem 2) or [30] and the equalities
\( M(\psi)' = \Lambda(\psi) \) and \( \Lambda(\psi)' = M(\psi) \), which hold for every increasing concave function \( \psi \) [19] Theorems 4.5.2 and 4.5.4, it follows that
\[
\mathcal{L}_m' = [M(\tilde{\varphi}), \Lambda(\tilde{\varphi})]' = [M(\varphi), \Lambda(\varphi)]_{[1, q]}.
\]
If \( 1/p + 1/q = 1 \), then \( 1 - \theta = 1/q \). Therefore, as is shown above, there is a subsequence of the sequence \( \{x_n'\} \) (we keep the same notation), which satisfies the inequality:
\[
\left\| \sum_{k=1}^{\infty} b_k x_k \right\|_{\mathcal{L}_m'} \leq C \| (b_k) \|_{l_q}
\]
for every \( (b_k) \in l_q \). Equivalently, the operator \( T(b_k) := \sum_{k=1}^{\infty} b_k x_k' \) acts boundedly from \( l_q \) into \( \mathcal{L}_m' \). Since for arbitrary \( (b_k) \in l_q \) and \( x \in \mathcal{L}_m'' = \mathcal{L}_m' \) we have
\[
\int_0^1 x(t)T(b_k)(t) \, dt = \sum_{k=1}^{\infty} b_k \int_0^1 x(t)x_k'(t) \, dt,
\]
the Köthe dual operator to \( T \) is defined by the formula: \( T' x := \left( \int_0^1 x(t)x_k'(t) \, dt \right)_{k=1}^{\infty} \), and \( T' \) is bounded from \( \mathcal{L}_m' \) into \( l_p \). By [9], we have \( T' (\sum_{k=1}^{\infty} a_k x_k) = (a_k) \). Hence,
\[
\| (a_k) \|_{l_p} \leq \| T' \| \sum_{k=1}^{\infty} \| a_k x_k \|_{\mathcal{L}_m'},
\]
and desired result follows. \( \square \)

Suppose that an increasing concave function \( \varphi \) on \([0, 1] \) does not satisfy the condition [8]. Obviously, then either \( \varphi(t) \asymp 1 \) or \( \varphi(t) \asymp t \) on \([0, 1] \). A unique (up to equivalence of norms) r.i. space with the fundamental function 1 (resp. \( L_\infty \)) is \( L_1 \) (resp. \( L_1 \)). Since the first of them is \( \infty \)-disjointly homogeneous and the second one is 1-disjointly homogeneous, we obtain the following assertion.

**Corollary 1.** Let \( 1 < p < \infty \) and \( \varphi \) be an increasing concave function on \([0, 1] \), \( \varphi(0) = 0 \). Then, there exists a \( p \)-disjointly homogeneous r.i. space \( X \) with the fundamental function \( \varphi_X = \varphi \) if and only if \( \varphi(t) \) is equivalent neither 1 nor \( t \).

Clearly, \( [M(\tilde{\varphi}), \Lambda(\varphi)]_q \) is an interpolation space with respect to the couple \( (M(\tilde{\varphi}), \Lambda(\varphi)) \), i.e., is ultrasymetric. Let us show that if the dilation indices of an increasing concave function \( \varphi \) are non-trivial, i.e., \( 0 < \gamma_\varphi \leq \delta_\varphi < 1 \), for every \( 1 < p < \infty \), there is a unique ultrasymetric \( p \)-disjointly homogeneous r.i. space \( X \) with the fundamental function \( \varphi \).

**Theorem 2.** Let \( 1 < p < \infty \) and let \( \varphi \) be an increasing concave function on \([0, 1] \) such that \( 0 < \gamma_\varphi \leq \delta_\varphi < 1 \). Then the Lorentz space \( \Lambda_{p, \varphi} \) with the quasi-norm
\[
\| x \|_{\Lambda_{p, \varphi}} = \left( \int_0^1 [x^*(t)\varphi(t)]^p \frac{dt}{t} \right)^{1/p}
\]
is a unique \( p \)-disjointly homogeneous r.i. space from the set \( \text{Int}(M(\tilde{\varphi}), \Lambda(\varphi)) \).

To prove this result we will need Theorem 2 from [4] restated below as Proposition 1.
Proposition 1. Suppose that sets $e_k \subset [0, 1], k = 1, 2, \ldots$, satisfy the conditions:

$$0 < m(e_{k+1}) \leq m(e_k) \ (k = 1, 2, \ldots), \quad \sum_{k=1}^{\infty} m(e_k) \leq 1,$$

and

$$\sup_{k=1,2,\ldots} \frac{1}{m(e_k)} \sum_{i=k}^{\infty} m(e_i) < \infty.$$

Then, if $X_1$ and $X_2$ are r.i. spaces, which are interpolation with respect to the couple $(M(\tilde{\varphi}), \Lambda(\varphi))$, with $0 < \gamma \varphi \leq \delta \varphi < 1$, then from the equivalence

$$\left\| \sum_{k=1}^{\infty} c_k \chi_{e_k} \right\|_{X_1} \asymp \left\| \sum_{k=1}^{\infty} c_k \chi_{e_k} \right\|_{X_2}, \quad \text{for all } c_k \in \mathbb{R},$$

it follows that $X_1 = X_2$ (with equivalence of norms).

Proof of Theorem 2. Since $0 < \gamma \varphi \leq \delta \varphi < 1$, we have

$$\Lambda(\varphi) = (L_{1, \infty})_{1, (2^k \varphi(2^{-k}))}^{K}, \quad M(\tilde{\varphi}) = (L_{1, \infty})_{l_{\infty, 1}(2^k \varphi(2^{-k}))}^{K}$$

(see e.g. [12]). For the same reason, the discrete Calderon operator is bounded in the space $l_p(2^k \varphi(2^{-k}))$ for every $1 \leq p \leq \infty$ (see e.g. [3, Theorem 3]). Therefore, by the Brudnyi theorem [10, Theorem 4.3.1], we obtain

$$\mathcal{L}_{[\theta]} = [M(\tilde{\varphi}), \Lambda(\varphi)]_{\theta} = (L_{1, \infty})_{l_{\infty, 1}(2^k \varphi(2^{-k}))}^{K}.$$

If $\theta = 1/p$, as above, we have $[l_{\infty, 1}]_{\theta} = l_{\infty, 1}^{1-\theta} p_1 = l_p$. Hence,

$$\mathcal{L}_{[\theta]} = (L_{1, \infty})_{l_p(2^k \varphi(2^{-k}))}^{K}$$

and, taking into account the equality

$$(7) \quad K(t, f; L_{1, \infty}) = \int_{0}^{t} f^*(s) \, ds, \quad t > 0$$

[9, Theorem 5.2.1], we infer that

$$\left\| x \right\|_{\mathcal{L}_{[\theta]}} \asymp \left( \int_{0}^{1} \left| x^{**}(t) \varphi(t) \right|^p \frac{dt}{t} \right)^{1/p}$$

(as above, $x^{**}(t) := \frac{1}{t} \int_{0}^{t} x^*(s) \, ds$). Since $0 < \gamma \varphi \leq \delta \varphi < 1$, from the Hardy inequality [15, Theorem 2.6.6] (see also Preliminaries) it follows that $\mathcal{L}_{[\theta]} = \Lambda_{p, \varphi}$. By Theorem [11, Theorem 5.2.1], $\Lambda_{p, \varphi}$ is a $p$-disjointly homogeneous space. Moreover, its uniqueness is an immediate consequence of Proposition [11] and the definition of $p$-disjointly homogeneous space.

□

Remark 1. If $\varphi(t) = t^{1/q}, 1 < q < \infty$, and $1 \leq p < \infty$ we obtain the spaces $L_{q, p}$ (see Section 2); the fact that they are $p$-disjointly homogeneous is known for a long time (see e.g. [13] and [11]).
4. Non-ultrasymmetric \( p \)-disjointly homogeneous r.i. spaces

Regarding to the result of Theorem 2 the following rather natural question appears: If the space \( \Lambda_{p,\varphi} \), with \( 1 < p < \infty \) and \( 0 < \gamma_{\varphi} \leq \delta_{\varphi} < 1 \), is a unique \( p \)-disjointly homogeneous r.i. space with the fundamental function \( \varphi \)? In the next theorem that problem is resolved in negative even in the case of power functions.

**Theorem 3.** Let \( 1 < p < \infty \). For every increasing concave function \( \varphi \) on \([0,1]\) such that \( \lim_{t \to 0} \varphi(t) = 0 \) and \( \beta_{\varphi} < 1 \) there exists a \( p \)-disjointly homogeneous r.i. space \( Y \) with the fundamental function \( \varphi \) such that \( Y \not\in \text{Int}(M(\tilde{\varphi}), \Lambda(\varphi)) \).

For proving this theorem we need several auxiliary results. The first of them was proved in [6, Proposition 2]. In what follows we set \( \chi_k := \chi_{(2^{-k}, 2^{-k+1}]} \), \( k = 1, 2, \ldots \)

**Lemma 1.** Let \( X \) be an r.i. space on \([0,1]\) such that its upper Boyd index \( \beta_X < 1 \). Then, we have

\[
X = (L_1, L_\infty)_{E(\varphi(2^{-k})2^k)},
\]

where \( \varphi \) is the fundamental function of \( X \) and \( E \) is the Banach lattice of sequences with the norm

\[
\| (a_k) \|_E = \left\| \sum_{k=1}^{\infty} a_k \frac{\chi_k}{\varphi(2^{-k})} \right\|_X.
\]

Let us recall that an Orlicz function \( M \) on \([0,\infty)\) is called regularly varying at \( \infty \) if the limit \( \lim_{t \to \infty} \frac{M(\lambda t)}{M(t)} \) exists for every \( x > 0 \). Then, as is well known, we have

\[
\lim_{t \to \infty} \frac{M(\lambda t)}{M(t)} = x^p,
\]

with some \( 1 \leq p < \infty \), and then we say that \( M \) is regularly varying at \( \infty \) of order \( p \).

**Lemma 2.** If \( M \) is an Orlicz function regularly varying at \( \infty \) of order \( p \), the Orlicz space \( L_M \) is \( p \)-disjointly homogeneous.

**Proof.** Consider the sets

\[
E_{M,s}^\infty := \left\{ \frac{M(\lambda t)}{M(t)} : t > s \right\} \quad \text{and} \quad E_{M}^\infty := \cap_{s > 0} E_{M,s}^\infty,
\]

where the closure is taken in the space \( C[0,1] \). Since \( M \) regularly varies at \( \infty \) of order \( p \), by [13, Lemma 6.1], there is a certain constant \( C > 0 \) such that for every \( 0 < x_0 \leq 1 \) it can be found \( s_0 > 0 \) such that for all \( t \geq s_0 \) and \( x_0 \leq x \leq 1 \) we have

\[
C^{-1} x^p \leq \frac{M(\lambda t)}{M(t)} \leq C x^p.
\]

Hence, if \( s \geq s_0 \), then arbitrary function \( f \in E_{M,s}^\infty \) satisfies the inequality

\[
C^{-1} x^p \leq f(x) \leq C x^p \quad \text{for all} \quad x_0 \leq x \leq 1.
\]

Since \( x_0 > 0 \) may be chosen arbitrarily small, combining this together with the definition of the set \( E_{M,s}^\infty \), we infer that every function from \( E_{M,s}^\infty \) is equivalent to the function \( x^p \) at zero. Finally, applying [14, Theorem 4.1], we get desired result. \( \square \)
Proposition 2. Let $X$ and $Y$ be r.i. spaces on $[0,1]$, $1 \leq p < \infty$. Suppose that $Y$ is separable and $X$ is $p$-disjointly homogeneous. Then, if

$$
\left\| \sum_{k=1}^{\infty} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \right\|_Y \approx \left\| \sum_{k=1}^{\infty} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \right\|_{X'},
$$

for arbitrary $a_k \in \mathbb{R}$, then $Y$ is also $p$-disjointly homogeneous.

Proof. At first, we prove the property of $p$-disjoint homogeneity for block basic sequences of $\{\chi_k\}$ normalized in $Y$, i.e., for an arbitrary sequence $\{y_i\}$, $\|y_i\|_Y = 1$, such that

$$y_i = \sum_{k=n_i}^{n_{i+1}-1} a_k \chi_k, \quad 1 \leq n_1 < n_2 < \ldots, \quad a_k \in \mathbb{R}.$$ 

By assumption, without loss of generality, we can assume that after normalization the sequence

$$x_i := \sum_{k=n_i}^{n_{i+1}-1} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \chi_k, \quad i = 1, 2, \ldots$$

is equivalent in $X$ to the unit vector basis in $l_p$. By [8], for any $c_i \in \mathbb{R}$ we have

$$\left\| \sum_{i=1}^{\infty} c_i y_i \right\|_Y = \left\| \sum_{i=1}^{\infty} c_i \sum_{k=n_i}^{n_{i+1}-1} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \chi_k \right\|_Y$$

$$\approx \left\| \sum_{i=1}^{\infty} c_i \sum_{k=n_i}^{n_{i+1}-1} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \chi_k \right\|_{X'}$$

$$= \left\| \sum_{i=1}^{\infty} c_i x_i \right\|_{X'} \approx \left( \sum_{i=1}^{\infty} |c_i|^p \|x_i\|_X^p \right)^{1/p}.$$ 

Since

$$\|x_i\|_X = \left\| \sum_{k=n_i}^{n_{i+1}-1} a_k \frac{\varphi_Y(2^{-k})}{\varphi_X(2^{-k})} \chi_k \right\|_X$$

$$\approx \left\| \sum_{k=n_i}^{n_{i+1}-1} a_k \chi_k \right\|_Y = \|y_i\|_Y = 1,$$

we come to the equivalence

$$\left\| \sum_{i=1}^{\infty} c_i y_i \right\|_Y \approx \|(c_i)\|_{l_p},$$

and our claim is proved.

Now, show that the general case can be reduced to the previous one. A given sequence $\{y_i\}_{i=1}^{\infty} \subset Y$ of pairwise disjoint functions such that $\|y_i\|_Y = 1$, $i = 1, 2, \ldots$, we construct a normalized block basic sequence of $\{\chi_k\}$, which is equivalent to $\{y_i\}$ in $Y$.

Since $Y$ is a separable r.i. space, we can assume that the functions $y_i$ are finite-valued, $\text{supp} y_i = [\alpha_i, \beta_i]$, $y_i(t) = y_i^*(t - \alpha_i)$ if $\alpha_i \leq t \leq \beta_i$, $i = 1, 2, \ldots$ Setting

$$z_i' := \sum_{k=1}^{\infty} y_i^*(2^{-k}) \chi_k,$$
it is easily to see that
\[ \sigma_{1/2} z_1'(t) \leq y_n^*(t) \leq z_1'(t), \quad 0 \leq t \leq 1, \]
where \( \sigma_{1/2} x(t) = x(2t) \). Let \( \varepsilon_i > 0 \) and \( \varepsilon_i \to 0 \) as \( i \to \infty \). Since \( y_1 \) is a finite-valued function and \( \lim_{t \to +0} \varphi_Y(t) = 0 \) (as \( Y \) is separable), there is \( n_1 \in \mathbb{N} \) such that
\[ y_n^*(t) = y_1^*(2^{-n_1}) \text{ if } 0 \leq t \leq 2^{-n_1} \quad \text{and} \quad y_n^*(2^{-n_1}) \varphi_Y(2^{-n_1}) < \varepsilon_1. \]
Then, the functions
\[ u_1 := y_n^* \chi_{(2^{-n_1}, 1]} \quad \text{and} \quad z_1 := \sum_{k=1}^{n_1-1} y_n^*(2^{-k}) \chi_k \]
satisfy the following conditions:
\[ \| y_n^* - u_1 \|_Y < \varepsilon_1 \]
and
\[ \sigma_{1/2} z_1'(t) \leq u_1(t) \leq z_1(t) + y_n^*(2^{-n_1}) \chi_{n_1}, \quad 0 \leq t \leq 1. \]
Clearly, one may assume that \( \beta_2 - \alpha_2 < 2^{-n_1} \). Then \( y_2^*(t) = 0 \) if \( t \geq 2^{-n_1} \), whence
\[ \sigma_{1/2} z_2'(t) \leq y_2^*(t) \leq z_2'(t), \quad 0 \leq t \leq 1, \]
where we set
\[ z_2' := \sum_{k=n_1+1}^{\infty} y_n^*(2^{-k}) \chi_k. \]
As above, there exists \( n_2 > n_1 \), for which we have
\[ y_n^*(t) = y_2^*(2^{-n_2}) \text{ if } 0 \leq t \leq 2^{-n_2} \quad \text{and} \quad y_2^*(2^{-n_2}) \varphi_Y(2^{-n_2}) < \varepsilon_2. \]
Now, if the functions \( u_2 \) and \( z_2 \) are defined as follows:
\[ u_2 := y_n^* \chi_{(2^{-n_2}, 2^{-n_1}]} \quad \text{and} \quad z_2 := \sum_{k=n_1+1}^{n_2-1} y_n^*(2^{-k}) \chi_k, \]
we obtain
\[ \| y_n^* - u_2 \|_Y < \varepsilon_2 \]
and
\[ \sigma_{1/2} z_2(t) \leq u_2(t) \leq z_2(t) + y_n^*(2^{-n_2}) \chi_{n_2}, \quad 0 \leq t \leq 1. \]
Proceeding in the same way, we find the numbers \( n_0 = 1 < n_1 < n_2 < \ldots \) and the functions
\[ u_i := y_n^* \chi_{(2^{-n_i}, 2^{-n_{i-1}}]} \quad \text{and} \quad z_i := \sum_{k=n_{i+1}+1}^{n_{i+1}-1} y_n^*(2^{-k}) \chi_k \]
such that for all \( i = 1, 2, \ldots \) we have
\[ y_i^*(2^{-n_i}) \varphi_Y(2^{-n_i}) < \varepsilon_i, \]
\[ \| y_i^* - u_i \|_Y < \varepsilon_i, \]
and
\[ \sigma_{1/2} z_i(t) \leq u_i(t) \leq z_i(t) + y_i^*(2^{-n_i}) \chi_{n_i}, \quad 0 \leq t \leq 1. \]
Now, set \( \bar{u}_i(t) = 0 \) if \( 0 \leq t < \alpha_i \) or \( \beta_i < t \leq 1 \), and \( \bar{u}_i(t) = u_i(t - \alpha_i) \) if \( \alpha_i \leq t \leq \beta_i \) (\( i = 1, 2, \ldots \)). Since the functions \( u_i \) (resp. \( \bar{u}_i \)) are pairwise disjoint and, by (11),

\[
\| y_i - \bar{u}_i \|_Y = \| y_i^* - u_i \|_Y < \varepsilon_i,
\]

thanks to the well-known principle of small perturbations of basis (see e.g. [1, Theorem 1.3.9]), choosing sufficiently small \( \varepsilon_i \), we have

\[
(13) \quad \left\| \sum_{i=1}^{\infty} c_i y_i \right\|_Y \asymp \left\| \sum_{i=1}^{\infty} c_i u_i \right\|_Y,
\]

for all \( c_i \in \mathbb{R} \). Moreover, taking into account inequalities (12) and the fact that the functions \( \sigma_{1/2} z_i \) (resp. \( z_i + y_i^*(2^{-n}) \chi_{n_i} \)) are pairwise disjoint, we obtain

\[
\sum_{i=1}^{\infty} c_i \sigma_{1/2} z_i(t) \leq \sum_{i=1}^{\infty} c_i u_i(t) \leq \sum_{i=1}^{\infty} c_i (z_i(t) + y_i^*(2^{-n}) \chi_{n_i}(t)), \quad 0 \leq t \leq 1,
\]

whence for any \( c_i \in \mathbb{R} \)

\[
\left\| \sigma_{1/2} \left( \sum_{i=1}^{\infty} c_i z_i \right) \right\|_Y \leq \left\| \sum_{i=1}^{\infty} c_i u_i \right\|_Y \leq \left\| \sum_{i=1}^{\infty} c_i (z_i + y_i^*(2^{-n}) \chi_{n_i}) \right\|_Y.
\]

Further, since \( \varepsilon_i > 0 \) may be chosen arbitrarily small, from inequalities (10), as above, we infer that

\[
\left\| \sum_{i=1}^{\infty} c_i (z_i + y_i^*(2^{-n}) \chi_{n_i}) \right\|_Y \asymp \left\| \sum_{i=1}^{\infty} c_i z_i \right\|_Y.
\]

Therefore, from the preceding inequality, combined together with the facts that \( z = \sigma_{2}(\sigma_{1/2} z) \) and \( \| \sigma_2 \|_{Y \to Y} \leq 2 \) [19 Corollary 1 after Theorem 2.4.5], it follows that

\[
\left\| \sum_{i=1}^{\infty} c_i u_i \right\|_Y \asymp \left\| \sum_{i=1}^{\infty} c_i z_i \right\|_Y,
\]

whence, by (13), we obtain

\[
\left\| \sum_{i=1}^{\infty} c_i y_i \right\|_Y \asymp \left\| \sum_{i=1}^{\infty} c_i z_i \right\|_Y.
\]

Since \( \{z_i\} \) is a block basic sequence of \( \{\chi_k\} \), by the first part of the proof (passing to an appropriate subsequence), we have

\[
\left\| \sum_{i=1}^{\infty} c_i z_i \right\|_Y \asymp \|(c_i)\|_{\ell_p},
\]

which completes the proof.

\[\square\]

**Lemma 3.** [18, p. 41] and [20, Example 2] [1] For every \( 1 < p < \infty \) there exists an Orlicz function \( M \), regularly varying at \( \infty \) of order \( p \), such that the Orlicz space \( L_M \not\subset \text{Int}(M(\psi), \Lambda(\psi)) \), where \( \psi(t) = 1/M^{-1}(1/t) \), \( 0 < t \leq 1 \).

**Proof.** Let \( v_n \in (0, 1) \), \( v_1 > v_2 > \cdots > 0 \) and \( v_n \to 0 \). The function \( g : \mathbb{R} \to (0, \infty) \) is defined as follows: \( g(t) = p \) for \( t \leq 1 \), and \( g(t) = p + (-1)^n v_n \) if \( 2^{n-1} < t \leq 2^n, n = 1, 2, \ldots \). Denote

\[
f(u) = \int_0^u g(t)dt \quad \text{and} \quad M(u) = e^{f(\log u)} \quad (u > 0).
\]
Then, it is not hard to check that $M$ satisfies all required conditions (see also [6, Example 2]).

Now, let $1 < p < \infty$ and let an Orlicz function $M$ satisfy the conditions of Lemma 3. Clearly, we can assume that $M(1) = 1$. It is not hard to check that $\alpha_{LM} = \beta_{LM} = \gamma_{\psi} = \delta_{\psi} = 1/p$ (see e.g. [21, p. 139]). Hence, since $p > 1$, by Lemma 1 we have

$$L_M = (L_1, L_\infty)_G^{\psi(2^{-k})2^k},$$

where $G$ is the Banach lattice of sequences with the norm

$$\| (a_k) \|_G = \left\| \sum_{k=1}^{\infty} a_k \frac{\chi_k}{\psi(2^{-k})} \right\|_{L_M}.$$ 

Denote by $e_k$ ($k = 1, 2, \ldots$) the vectors of the unit basis in sequence spaces and by $P_i$, $i = 1, 2, \ldots$, the shift operators, which are defined as follows:

$$P_i \left( \sum_{k=1}^{\infty} a_k e_k \right) = \sum_{k=1}^{\infty} a_{k+i} e_k.$$

**Lemma 4.** For arbitrary $\varepsilon > 0$ there is $B = B(\varepsilon) > 0$ such that

$$\| P_i \|_{G \to G} \leq B 2^i, \quad i = 1, 2, \ldots.$$

**Proof.** From the definition of $G$ it follows that

$$\| (a_k) \|_G = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} M\left( |a_k| M^{-1}(2^k) \right) 2^{-k} \leq 1 \right\}.$$ 

Therefore, if $\| (a_k) \|_G = 1$, we have

$$\sum_{k=1}^{\infty} M(|a_k| M^{-1}(2^k)) 2^{-k} \leq 1.$$ 

Let us estimate

$$\| P_i (a_k) \|_G = \inf \left\{ \lambda > 0 : \sum_{j=i+1}^{\infty} M\left( |a_j| M^{-1}(2^{j-i}) \right) 2^{i-j} \leq 1 \right\}$$

for each $i = 1, 2, \ldots$

First, since $\gamma_\psi = 1/p$, for arbitrary $0 < \eta < 1/p$ there is $C_1 = C_1(\eta) > 0$ such that

$$\sup_{0 < s \leq 1} \frac{\psi(st)}{\psi(s)} \leq C_1 t^{1/p-\eta}, \quad 0 < t \leq 1,$$

or equivalently (because of $\psi(t) = 1/M^{-1}(1/t)$),

$$u^{1/p-\eta} M^{-1}(v) \leq C_1 M^{-1}(uv), \quad u, v \geq 1.$$ 

Similarly, as $\delta_\psi = 1/p$, it can be found $C_2 = C_2(\eta) > 0$, for which

$$\sup_{0 < s \leq 1, t \geq 1, st \leq 1} \frac{\psi(st)}{\psi(s)} \leq C_2 t^{1/p+\eta}, \quad t \geq 1,$$

or

$$u^{1/p+\eta} M^{-1}(v) \leq C_2 M^{-1}(uv), \quad u \leq 1, v \geq 1, uv \geq 1.$$
Hence, for arbitrary $v \geq z \geq 1$
\[
\frac{M^{-1}(v)}{v^{1/p+\eta}} \leq C_2 \frac{M^{-1}(z)}{z^{1/p+\eta}},
\]
and, taking into account that $M$ increases and $M(1) = 1$, we obtain
\[
(19) \quad M(y) \leq C_2 \left(\frac{y}{x}\right)^{\frac{\eta}{p+1}} M(x) \text{ if } x \geq y \geq 1.
\]

Denote by $S$ the set of all positive integers $j \geq i+1$ such that $|a_j|M^{-1}(2^{j-i})C_1^{-1}C_2^{-1} > 1$. Then, for every $j \in S$ from (18), (19), and the convexity of the function $M$ it follows that
\[
M\left(\frac{|a_j|M^{-1}(2^{j-i})}{C_1C_2}\right) \leq M\left(\frac{|a_j|2^{(-p+1)\eta}}{C_2} M^{-1}(2^j)\right)
\]
\[
\leq C_2 2^{(-p+1)\eta} M\left(\frac{|a_j|2^{(-p+1)\eta}}{C_2} M^{-1}(2^j)\right)
\]
\[
= 2^{(\varepsilon-1)\eta} M\left(|a_j|2^{(-p+1)\eta} M^{-1}(2^j)\right),
\]
where $\varepsilon := 2p\eta/(p\eta + 1)$ can be made arbitrarily small together with $\eta$. Therefore, by (10), we have
\[
\sum_{j=i+1}^{\infty} M\left(\frac{|a_j|M^{-1}(2^{j-i})}{C_1C_2}\right) 2^{i-j} = \sum_{j \in S} M\left(\frac{|a_j|M^{-1}(2^{j-i})}{C_1C_2}\right) 2^{i-j}
\]
\[
+ \sum_{j \notin S} M\left(\frac{|a_j|M^{-1}(2^{j-i})}{C_1C_2}\right) 2^{i-j}
\]
\[
\leq 2^{\varepsilon i} \sum_{j=i+1}^{\infty} M\left(|a_j|2^{(-p+1)\eta} M^{-1}(2^j)\right) 2^{-j} + \sum_{j=1}^{\infty} 2^{-j} \leq 2 \cdot 2^{\varepsilon i}.
\]
Finally, using the convexity of $M$ once more and formula (17), we obtain
\[
\|P_i(a_k)\|_G \leq 2^{\varepsilon i} \cdot 2C_1C_2\|a_j\|_G,
\]
that is, desired estimate holds with $B := 2C_1C_2$.

\[ \square \]

Proof of Theorem 3. If $M$ is an Orlicz function satisfying the conditions of Lemma 3 the Orlicz space $L_M$ is not ultrasymmetric (see Preliminaries). Then, since $0 < \gamma_\psi = \delta_\psi = 1/p < 1$, by [25, Theorem 2.1], from (11) and (15) it follows that the Banach lattice $G$ is not symmetric. At the same time, $G$ is separable together with $L_M$. Moreover, applying Lemma 2 we see that $L_M$ is a $p$-disjointly homogeneous space.

Now, we define the space $Y$ as follows:
\[
(20) \quad Y := \langle L_1, L_\infty \rangle_{G(\varphi, (2^{-k})2^k)}^K.
\]
Since $\|e_k\|_G = 1$, $k = 1, 2, \ldots$, we have $l_1 \subset G \subset l_\infty$, whence $\Lambda(\varphi) \subset Y \subset M(\varphi)$. Moreover, $Y \notin \text{Int}(M(\varphi), \Lambda(\varphi))$ (equivalently, $Y$ is not ultrasymmetric), because of the Banach lattice $G$ is not symmetric [25, Theorem 2.1], and $Y$ is separable together with $G$. Let us check that $\beta_Y < 1$. 

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By definition of the real interpolation spaces and formula (7), for arbitrary \( i \in \mathbb{N} \) we have
\[
\| \sigma_2 x \|_Y = \left\| \left( \int_0^{2^{-k}} x^*(2^{-i} s) ds \cdot \varphi(2^{-k}) \right)_{k=1}^{\infty} \right\|_G
= \left\| \left( 2^{i+k} \int_0^{2^{-i-k}} x^*(s) ds \cdot \varphi(2^{-k}) \right)_{k=1}^{\infty} \right\|_G.
\]
Furthermore, since \( \delta_p < 1 \), for some \( C > 0 \) and \( \eta > 0 \)
\[
\sup_{0 < s \leq 1, 0 < t \leq 1} \varphi(st) \leq C t^{1-\eta}, \quad t \geq 1.
\]
Therefore, by Lemma 4 with \( \varepsilon = \eta/2 \), we obtain
\[
\| \sigma_2 x \|_Y \leq C 2^{(1-\eta)} \left\| \left( 2^{i+k} \int_0^{2^{-(i+k)}} x^*(s) ds \cdot \varphi(2^{-(i+k)}) \right)_{k=1}^{\infty} \right\|_G
\]
\[
\leq BC 2^{(1-\eta/2)} \left\| \left( 2^{k} \int_0^{2^{-k}} x^*(s) ds \cdot \varphi(2^{-k}) \right)_{k=1}^{\infty} \right\|_G = BC 2^{(1-\eta/2)} \| x \|_Y,
\]
whence \( \beta_Y < 1 \).

Thus, we may apply Lemma 4 and thereby get
\[
(21) \quad Y = (L_1, L_\infty)_{E(\varphi(2^{-k})2^k)_{k=1}^{\infty}},
\]
where \( E \) is the Banach sequence lattice with the norm
\[
\| (a_k) \|_E = \left\| \sum_{k=1}^{\infty} a_k \frac{\chi_k}{\varphi_Y(2^{-k})} \right\|_Y.
\]
Comparing (20) with (21), by Proposition 2 from the paper 3, we conclude that
\( E = G \) (with equivalence of norms), i.e.,
\[
\left\| \sum_{k=1}^{\infty} a_k \frac{\chi_k}{\varphi_Y(2^{-k})} \right\|_Y \asymp \left\| \sum_{k=1}^{\infty} a_k \frac{\chi_k}{\varphi(2^{-k})} \right\|_{L_M}.
\]
Since \( L_M \) is a \( p \)-disjointly homogeneous, from Proposition 2 it follows that \( Y \) also has this property, and the proof is completed. \( \square \)

5. Sequences of pairwise disjoint functions in the spaces \((M(\tilde{\varphi}), \Lambda(\varphi))_{\theta, p}\)

If the dilation indices of an increasing concave function \( \varphi \) are non-trivial, i.e.,
\( 0 < \gamma_\varphi \leq \delta_\varphi < 1 \), then applying the reiteration theorem 10, Theorem 4.3.1] precisely in the same way as in the proof of Theorem 2 it can be shown that, for every \( 1 < p < \infty \), the real interpolation space \((M(\tilde{\varphi}), \Lambda(\varphi))_{1/p, p}\) coincides with the complex interpolation space \([M(\tilde{\varphi}), \Lambda(\varphi)]_{1/p}\) (and hence with \( \Lambda_{p, \varphi} \)). Thus, in this case the space \([M(\tilde{\varphi}), \Lambda(\varphi)]_{1/p}\) is \( p \)-disjointly homogeneous. In this section we investigate some properties of sequences of pairwise disjoint functions in the spaces \((M(\tilde{\varphi}), \Lambda(\varphi))_{\theta, p}\), \( 0 < \theta < 1, 1 \leq p < \infty \), for an arbitrary increasing concave function \( \varphi \).

Our first result, a version of the well-known Levy theorem on the existence of \( l_p \)-subspaces in the real interpolation spaces \((X_0, X_1)_{\theta, p}\), [20] or [10] Theorem 4.6.22] (in a more general setting, see [24] and [5]), concerns even with a more general
Hence, taking into account that \((M(\hat{\varphi}), \Lambda(\varphi))\) is replaced with a couple \((X_0, X_1)\) such that \(X_1 \subset X_0\). Given \(0 < \theta < 1\) and \(1 \leq p < \infty\), we denote \(X_{\theta, p} := (X_0, X_1)_{\theta, p}\).

**Theorem 4.** Suppose that pairwise disjoint functions \(x_n \in X_{\theta, p}\), \(\|x_n\|_{\theta, p} = 1\) \((n = 1, 2, \ldots)\) and \(\lim_{n \to \infty} \|x_n\|_{X_0} = 0\).

Then for arbitrary \(\varepsilon > 0\) there is a subsequence \(\{z_j\} \subset \{x_n\}\) such that for all \(\lambda_j \in \mathbb{R}\) \((j = 1, 2, \ldots)\) we have

\[
(1 - \varepsilon) \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^{\infty} \lambda_j z_j \right\|_{\theta, p} \leq (1 + \varepsilon) \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}.
\]

The main ingredient in the proof of Theorem 4 is the following assertion.

**Proposition 3.** Let \(0 < \theta < 1\), \(1 \leq p < \infty\) and let \(K\) be a compact subset of the unit sphere in \(X_{\theta, p}\). Moreover, suppose that \(x_k \in X_{\theta, p}\), \(\|x_k\|_{\theta, p} = 1\) \((k = 1, 2, \ldots)\) and \(\lim_{k \to \infty} \|x_k\|_{X_0} = 0\). Then, for arbitrary \(\varepsilon \in (0, 1)\) there exists \(k_0 \in \mathbb{N}\) such that

\[
(1 - \varepsilon) (|\lambda_1|^p + |\lambda_2|^p)^{1/p} \leq \|\lambda_1 x + \lambda_2 x_{k_0}\|_{\theta, p} \leq (1 + \varepsilon) (|\lambda_1|^p + |\lambda_2|^p)^{1/p}
\]

for each \(x \in K\) and \(\lambda_1, \lambda_2 \in \mathbb{R}\).

In the proof of this proposition we will make use of the following technical lemmas proved in [24].

**Lemma 5.** Let \(a \geq 0\), \(\varepsilon > 0\) and \(1 \leq p < \infty\). Then the following inequality holds:

\[
(1 + a\varepsilon)^p + (a + \varepsilon)^p \leq (1 + \varepsilon)^p (1 + a^p).
\]

**Lemma 6.** Let \(0 < \varepsilon < 1\), \(1 \leq p < \infty\) and \(c := (1 - \varepsilon^p/2^p)^{1/p}\). Then the following inequalities hold:

(a) \((1 - \varepsilon)^p (1 + a^p) \leq (ac - \varepsilon/2)^p + (c - a\varepsilon/2)^p\) for all \(a \in [\varepsilon/(2c), 2c/\varepsilon]\)

and

(b) \((1 - \varepsilon)^p (1 + a^p) \leq |1 - a|^p\) for all \(a \in [0, \varepsilon/(2c)] \cup [2c/\varepsilon, \infty)\).

**Proof of Proposition 3** Denote by \(S\) the sublinear operator defined on \(X_0\) as follows:

\[
Sx = \{Sx\}_n, \quad \{Sx\}_n := 2^{-n\theta} K(2^{-n}, x; X_0, X_1), \quad n = 1, 2, \ldots
\]

Taking into account inequalities (1) and equality (2), we see that \(S\) is an isometric embedding from \(X_{\theta, p}\) into \(l_p\). Therefore, by assumption, \(S(K)\) is a compact set in \(l_p\), and a given \(\varepsilon \in (0, 1)\) there exists \(m \in \mathbb{N}\) such that

\[
\sum_{n=m+1}^{\infty} \|Sx\|^p_n < \left( \frac{\varepsilon}{2} \right)^p \text{ for each } x \in K.
\]

Hence, taking into account that \(K\) is a subset of the unit sphere in \(X_{\theta, p}\), we obtain

\[
\sum_{n=1}^{m} \|Sx\|^p_n \geq 1 - \left( \frac{\varepsilon}{2} \right)^p \text{ for each } x \in K.
\]

Further, since the functional \(\{Sx\}_n\), for each \(n \in \mathbb{N}\), is equivalent to the norm in \(X_0\) and \(\lim_{k \to \infty} \|x_k\|_{X_0} = 0\), for every \(n \in \mathbb{N}\) we have \(\{Sx_k\}_n \to 0\) as \(k \to \infty\).
Thus, we can find $k_0 \in \mathbb{N}$ such that the element $y := x_{k_0}$ satisfies the condition:

(26) \[ \sum_{n=1}^{m} \{Sy\}^p_n < \left(\frac{\varepsilon}{2}\right)^p. \]

Combining this together with the fact that $\|y\|_{\theta, p} = 1$, we also have

(27) \[ \sum_{n=m+1}^{\infty} \{Sy\}^p_n \geq 1 - \left(\frac{\varepsilon}{2}\right)^p. \]

Now, we prove the right-hand side inequality in (23). Clearly, it suffices to consider the case when $\lambda_1 := \lambda$ is arbitrary and $\lambda_2 = 1$. Firstly, from (24) and (26) it follows that

\[ \left( \sum_{n=m+1}^{\infty} \{S(\lambda x + y)\}^p_n \right)^{1/p} \leq |\lambda| \left( \sum_{n=m+1}^{\infty} \{Sx\}^p_n \right)^{1/p} + \|y\|_{\theta, p} \leq |\lambda| \varepsilon + 1 \]

and

\[ \left( \sum_{n=0}^{m} \{S(\lambda x + y)\}^p_n \right)^{1/p} \leq \|x\|_{\theta, p} + \left( \sum_{n=0}^{m} \{Sy\}^p_n \right)^{1/p} \leq |\lambda| + \varepsilon. \]

Then,

\[ \|\lambda x + y\|_{\theta, p} = \left( \sum_{n=0}^{m} \{S(\lambda x + y)\}^p_n + \sum_{n=m+1}^{\infty} \{S(\lambda x + y)\}^p_n \right)^{1/p} \leq (|\lambda| + \varepsilon)^p + (1 + |\lambda| \varepsilon)^p, \]

whence, by Lemma 5, we obtain

\[ \|\lambda x + y\|_{\theta, p} \leq (1 + \varepsilon)(1 + |\lambda|^p)^{1/p}, \]

and the right-hand side inequality in (23) is proved.

To prove the left-hand side inequality in (23) assume, firstly, that $|\lambda| \in [\varepsilon/(2c), 2c/\varepsilon]$, where $c := (1 - \varepsilon^p/2p)^{1/p}$. By (25) and (26),

\[ \left( \sum_{n=0}^{m} \{S(\lambda x + y)\}^p_n \right)^{1/p} \geq |\lambda| \left( \sum_{n=0}^{m} \{Sx\}^p_n \right)^{1/p} - \left( \sum_{n=0}^{m} \{Sy\}^p_n \right)^{1/p} \geq |\lambda| c - \frac{\varepsilon}{2}, \]

and, by (24) and (27),

\[ \left( \sum_{n=m+1}^{\infty} \{S(\lambda x + y)\}^p_n \right)^{1/p} \geq \left( \sum_{n=m+1}^{\infty} \{Sy\}^p_n \right)^{1/p} - \left( \sum_{n=m+1}^{\infty} \{Sx\}^p_n \right)^{1/p} \geq c - |\lambda| \frac{\varepsilon}{2}. \]

These inequalities together with Lemma 5(a) yield

\[ \|\lambda x + y\|_{\theta, p} = \left( \sum_{n=0}^{m} \{S(\lambda x + y)\}^p_n + \sum_{n=m+1}^{\infty} \{S(\lambda x + y)\}^p_n \right)^{1/p} \geq \left( \left( |\lambda| c - \frac{\varepsilon}{2} \right)^p + \left( c - |\lambda| \frac{\varepsilon}{2} \right)^p \right)^{1/p} \geq (1 - \varepsilon)(1 + |\lambda|^p)^{1/p}. \]

If $|\lambda| \notin [\varepsilon/(2c), 2c/\varepsilon]$, then Lemma 5(b) implies that

\[ (1 - \varepsilon)(1 + |\lambda|^p)^{1/p} \leq |1 - |\lambda|| \leq \|\lambda x + y\|_{\theta, p}, \]

and the proof of the left-hand side inequality in (23) is complete. □
Proof of Theorem 4. For a given $\varepsilon > 0$, let $\theta_j \in (0, 1)$ be such that

\begin{equation}
1 - \varepsilon < \prod_{j=1}^{\infty} (1 - \theta_j) < \prod_{j=1}^{\infty} (1 + \theta_j) < 1 + \varepsilon.
\end{equation}

We construct inductively a subsequence $\{z_j\} \subset \{x_n\}$ such that, for every $l \in \mathbb{N}$ and all $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$, we have

\begin{equation}
\prod_{j=1}^{l} (1 - \theta_j) \left( \sum_{j=1}^{l} |\lambda_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^{l} \lambda_j z_j \right\|_{\theta,p} \leq \prod_{j=1}^{l} (1 + \theta_j) \left( \sum_{j=1}^{l} |\lambda_j|^p \right)^{1/p}.
\end{equation}

Let $z_1 := x_{n_1}$ be arbitrary. Then, applying Proposition 3 to the one-element set $\{z_1\}$, we can find $z_2 := x_{n_2}$, $n_2 > n_1$, such that

\begin{equation}
\prod_{j=1}^{2} (1 - \theta_j) (|\lambda_1|^p + |\lambda_2|^p)^{1/p} \leq \|\lambda_1 z_1 + \lambda_2 z_2\|_{\theta,p} \leq \prod_{j=1}^{2} (1 + \theta_j) (|\lambda_1|^p + |\lambda_2|^p)^{1/p},
\end{equation}

for every $\lambda_1, \lambda_2 \in \mathbb{R}$.

Assume that we already have chosen the elements $z_1, \ldots, z_{l-1}$ from the sequence $\{x_n\}$, which satisfy the inequality

\begin{equation}
\prod_{j=1}^{l-1} (1 - \theta_j) \left( \sum_{j=1}^{l-1} |\lambda_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^{l-1} \lambda_j z_j \right\|_{\theta,p} \leq \prod_{j=1}^{l-1} (1 + \theta_j) \left( \sum_{j=1}^{l-1} |\lambda_j|^p \right)^{1/p},
\end{equation}

for all $\lambda_1, \ldots, \lambda_{l-1} \in \mathbb{R}$. It is clear that the set $K := \{ x \in \text{span}\{z_1, \ldots, z_{l-1}\} : \|x\|_{\theta,p} = 1 \}$ is compact in $X_{\theta,p}$. Therefore, by Proposition 3, there exists a function $z_1 := x_{n_1}$, $n_1 > n_{l-1}$, such that

\begin{equation}
(1 - \theta_l) (1 + |\gamma|^p)^{1/p} \leq \|x + \gamma z_l\|_{\theta,p} \leq (1 + \theta_l) (1 + |\gamma|^p)^{1/p},
\end{equation}

for all $x \in K$ and $\gamma \in \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$ be arbitrary. Clearly, it may be assumed that $\sum_{j=1}^{l-1} \lambda_j z_j \|_{\theta,p} > 0$. Setting

\[ \gamma := \lambda_l \cdot \sum_{j=1}^{l-1} \lambda_j z_j \|_{\theta,p}^{-1}, \quad x := \sum_{j=1}^{l-1} \lambda_j z_j \cdot \| \sum_{j=1}^{l-1} \lambda_j z_j \|_{\theta,p}^{-1} \]

(note that $x \in K$) and using (31), we infer that

\begin{equation}
(1 - \theta_l) \left( \sum_{j=1}^{l-1} |\lambda_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^{l-1} \lambda_j z_j \right\|_{\theta,p} \leq (1 + \theta_l) \left( \sum_{j=1}^{l-1} |\lambda_j|^p \right)^{1/p}.
\end{equation}

Combining the latter inequality with (30), we obtain (29). Since inequality (22) is an immediate consequence (24) and (25), the proof is complete. \hfill \square

Remark 2. Repeating the arguments from the proof of Theorem 1 in [5], it can be shown also that a sequence $\{z_j\} \subset \{x_n\}$ satisfying (22), in addition, spans in $X_{\theta,p}$ a $(1 + \varepsilon)$–complemented subspace (that is, there is a projection $P$ bounded in $X_{\theta,p}$, $\|P\| \leq 1 + \varepsilon$, such that $P(X_{\theta,p}) = [z_j]$).

Now, let us consider a somewhat different situation, where we return to the special case of couples $(M(\varphi), \Lambda(\varphi))$. Let $\mathcal{L}_{\theta,p} := (M(\varphi), \Lambda(\varphi))_{\theta,p}$, $0 < \theta < 1$, $1 \leq p < \infty$. 

Proposition 4. Let $1 < p < \infty$, and let an increasing concave function $\varphi$ satisfy condition (3). Suppose that $x_n \in \Lambda(\varphi)$, $n = 1, 2, \ldots$, are pairwise disjoint functions with $\|x_n\|_{\Lambda(\varphi)} \approx \|x_n\|_{M(\varphi)} \geq 1$ ($n \in \mathbb{N}$). Then, there exists a subsequence of $\{x_n\}$ equivalent to the unit vector basis in $l_p$, which spans a subspace complemented in $L_{\theta,p}$.

Proof. As above, passing to a subsequence, we can assume that

$$\tag{32} \left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{\Lambda(\varphi)} \approx \|(a_k)\|_1 \quad \text{and} \quad \left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{M(\varphi)} \approx \|(a_k)\|_{l_\infty}.$$ 

Since $\|x_n\|_{M(\hat{\varphi})} \approx 1$ ($n \in \mathbb{N}$) and $M(\psi)' = \Lambda(\psi)$ for every increasing concave function $\psi$ [19, Theorem 4.5.4], we can find $x'_n \in \Lambda(\hat{\varphi})$ such that $\|x'_n\|_{\Lambda(\hat{\varphi})} \geq 1$, supp $x'_n \subset$ supp $x_n$, and

$$\tag{33} \int_0^1 x_n x'_n dt = 1, \quad n \in \mathbb{N}.$$ 

We claim that

$$\tag{34} \|x'_n\|_{M(\varphi)} \approx 1, \quad n \in \mathbb{N}.$$ 

In fact, on the one hand, it is clear that $\|x'_n\|_{M(\varphi)} \leq \|x'_n\|_{\Lambda(\hat{\varphi})} \leq C$, $n = 1, 2, \ldots$ for some constant $C > 0$. Further, since $\|x_n\|_{\Lambda(\varphi)} \approx 1$, by (33), we have $\inf_{n \in \mathbb{N}} \|x'_n\|_{M(\varphi)} > 0$.

Now, let us check that the projection

$$Px(t) = \sum_{k=1}^{\infty} \int_0^1 x'_n ds \cdot x_n(t), \quad 0 \leq t \leq 1,$$

is bounded both in $\Lambda(\varphi)$ and $M(\hat{\varphi})$.

First, for arbitrary set $c \subset [0,1]$, by (32), we have

$$\|Pc\|_{\Lambda(\varphi)} \leq \sum_{k=1}^{\infty} \left| \int_0^1 \chi_c x'_k ds \right| = \sum_{k=1}^{\infty} \alpha_k \int_0^1 x'_k ds,$$

where $\alpha_k := \text{sign} \int_0^1 x'_k ds$, $k = 1, 2, \ldots$. In view of (34), as above, it can be assumed that

$$\left\| \sum_{k=1}^{\infty} b_k x'_k \right\|_{M(\varphi)} \approx \|(b_k)\|_{l_\infty}.$$ 

Hence, since the functions $x'_k$, $k = 1, 2, \ldots$, are pairwise disjoint, by definition of the norm in a Marcinkiewicz space, we obtain

$$\|Pc\|_{\Lambda(\varphi)} \leq C \int_0^1 \left| \sum_{k=1}^{\infty} \alpha_k x'_k \right| ds = C \int_0^1 \left( \sum_{k=1}^{\infty} \alpha_k x'_k \right) ds \cdot \varphi(m(c))$$

$$\leq C \left( \sum_{k=1}^{\infty} \alpha_k x'_k \right) \|\chi_c\|_{\Lambda(\varphi)} \leq C \|\chi_c\|_{\Lambda(\varphi)},$$

and the boundedness of the projection $P$ in $\Lambda(\varphi)$ is proved. Furthermore, applying (32) and the fact that $\Lambda(\psi)' = M(\psi)$ [19, Theorem 4.5.2], we have

$$\|Px\|_{M(\hat{\varphi})} \approx \sup_{k=1,2,\ldots} \left| \int_0^1 x'_k ds \right| \leq \sup_{k=1,2,\ldots} \|x'_k\|_{\Lambda(\hat{\varphi})} \|x\|_{M(\hat{\varphi})} \leq C \|x\|_{M(\hat{\varphi})},$$

which implies the boundedness of $P$ in the space $M(\hat{\varphi})$. 

Thus, \((P(M(\tilde{\varphi})), P(\Lambda(\varphi)))\) is a complemented subcouple of the Banach couple 
\((M(\tilde{\varphi}), \Lambda(\varphi))\). At the same time, by \cite{[32], Theorem 1.17.1}, the sequence 
\(\{x_k\}\) is equivalent in 
\(L^2_{\theta}, l_1\) if 
\(K\) the latter equivalence with the fact that the space 
\(L^p(36)\) is equivalent in 
\(l_1\) to the unit vector basis in the space 
\((l_\infty, l_1)_{\theta,p}\). In particular, if \(\theta = 1/p\), we have 
\((l_\infty, l_1)_{\theta,p} = l_p\) (see e.g. \cite{[9], Theorem 5.2.1}); hence, 
\(\{x_k\}\) is equivalent in 
\(L_{\theta,p}\) to the unit vector basis in 
\(l_p\). Finally, since the projection \(P\) is bounded in 
\(L_{\theta,p}\), the subspace 
\([x_k]\) is complemented in the latter space. 

\[\square\]

From Theorem 4, Remark 2 and Proposition 4 we obtain

**Corollary 2.** Let \(1 < p < \infty\) and let an increasing concave function \(\varphi\) satisfy condition (3). Suppose that 
\(x_n \in X_{1/p,p}, n = 1, 2, \ldots,\) are pairwise disjoint functions such that either \(\lim_{n \to \infty} \|x_n\|_{M(\tilde{\varphi})} = 0\) or \(\|x_n\|_{\Lambda(\varphi)} \approx \|x_n\|_{M(\tilde{\varphi})} > 1\) \((n \in \mathbb{N})\). Then, there exists a subsequence of 
\(\{x_n\}\), which is equivalent to the unit vector basis in 
\(l_p\) and spans a subspace complemented in 
\(L_{\theta,p}\).

Regarding to the latter statement the following natural question arises: Let \(1 < p < \infty\), and let an increasing concave function \(\varphi\) satisfy condition (3). Suppose that 
\(x_n \in \Lambda(\varphi), n = 1, 2, \ldots,\) are pairwise disjoint functions such that \(\|x_n\|_{\Lambda(\varphi)} \to \infty\) as 
\(n \to \infty\) and \(\|x_n\|_{M(\tilde{\varphi})} > 1\) \((n \in \mathbb{N})\). Is there a subsequence of 
\(\{x_n\}\) equivalent to the unit vector basis in 
\(l_p\)? The positive answer to this question would mean that 
each of the spaces \(X_{1/p,p} = (M(\tilde{\varphi}), \Lambda(\varphi))_{1/p,p}\), under condition (3), is \(p\)-disjointly homogeneous.

Though we are not able to resolve the above question we can present some weaker result in the positive direction.

**Proposition 5.** Let an increasing concave function \(\varphi\) satisfy condition (3). Suppose that 
\(x_n \in L_{\theta,p} = (M(\tilde{\varphi}), \Lambda(\varphi))_{\theta,p}, n = 1, 2, \ldots,\), where \(0 < \theta < 1\) and 
\(1 \leq p < \infty\), are arbitrary pairwise disjoint functions.

Then, the closed linear span \([x_n]\) in \(L_{\theta,p}\) is not closed in the Marcinkiewicz space 
\(M(\tilde{\varphi})\).

\[\text{Proof.}\] Without loss of generality, we can assume that

\[(35) \quad \|x_n\|_{M(\tilde{\varphi})} = 1, \quad n = 1, 2, \ldots\]

Then, since \(\lim_{t \to 0} t/\varphi(t) = 0\), applying once more \cite{[13], Theorem 5.1} or \cite{[31], Proposition 1} (and passing to a subsequence if necessary), we obtain

\[\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{M(\tilde{\varphi})} \approx \|a_k\|_{l_\infty}^p.\]

Assuming the contrary, let the subspace \([x_n]\) be closed in \(M(\tilde{\varphi})\). Then combining the latter equivalence with the fact that the space \(L_{\theta,p}\) has the Fatou property, we infer that

\[(36) \quad z := \sum_{k=1}^{\infty} x_k \in L_{\theta,p}.\]

In order to obtain a contradiction with \(36\), let us estimate the \(K\)-functional 
\(K(t, z; M(\tilde{\varphi}), \Lambda(\varphi))\) at points \(t = 2^{-k}, k = 1, \ldots,\), from below. By definition, for
every \( k = 1, 2, \ldots \) there exist \( u_i^k \in \Lambda(\varphi) \) and \( v_i^k \in M(\tilde{\varphi}) \) \( (i = 1, 2, \ldots) \) such that
\[
K(2^{-k}, z; M(\tilde{\varphi}), \Lambda(\varphi)) \geq \frac{1}{2}(\|v^k\|_{M(\tilde{\varphi})} + 2^{-k}\|u^k\|_{\Lambda(\varphi)}),
\]
where \( u^k := \sum_{i=1}^{\infty} u_i^k \in \Lambda(\varphi) \), \( v^k := \sum_{i=1}^{\infty} v_i^k \in M(\tilde{\varphi}) \), \( k = 1, 2, \ldots \). From the estimate
\[
\|z\|_{\theta, p}^p \geq \sum_{k=1}^{\infty} (K(2^{-k}, z; M(\tilde{\varphi}), \Lambda(\varphi))2^{k\theta})^p \geq 2^{-p} \sum_{k=1}^{\infty} (\|v^k\|_{M(\tilde{\varphi})} + 2^{-k}\|u^k\|_{\Lambda(\varphi)})^p 2^{k\theta p} \geq 2^{-p} \sum_{k=1}^{\infty} 2^{kp\theta}\|v^k\|_{M(\tilde{\varphi})}^p
\]
and (36) it follows that \( \|v^k\|_{M(\tilde{\varphi})} \to 0 \) as \( k \to \infty \). Therefore, for some \( k_0 \in \mathbb{N} \) we have
\[
\|v_i^{k_0}\|_{M(\tilde{\varphi})} \leq \|v_i^{k_0}\|_{M(\tilde{\varphi})} < \frac{1}{2}, \quad i = 1, 2, \ldots
\]
Combining the latter inequality with the equality \( u_i^k + v_i^k = x_i \) and (35), we obtain
\[
(37) \quad \|u_i^{k_0}\|_{\Lambda(\varphi)} \geq \|v_i^{k_0}\|_{M(\tilde{\varphi})} \geq 1 - \|v_i^{k_0}\|_{M(\tilde{\varphi})} > \frac{1}{2}, \quad i = 1, 2, \ldots
\]
Let \( \varepsilon_k > 0 \) \( (i = 1, 2, \ldots) \), \( \sum_{k=1}^{\infty} \varepsilon_k < \infty \). Since \( \lim_{t \to 0} \varphi(t) = 0 \), by [31, Lemma 1], there is \( \delta = \delta(\varepsilon_1) > 0 \) such that for arbitrary \( y \in \Lambda(\varphi) \), with \( m(\text{supp} y) < \delta \), we have
\[
\|u_i^{k_0} + y\|_{\Lambda(\varphi)} \geq \|y\|_{\Lambda(\varphi)} + (1 - \varepsilon_1)\|u_i^{k_0}\|_{\Lambda(\varphi)}.
\]
Then, if \( i_0 = 1 \) and \( i_1 > i_0 \) is chosen so that \( m(\text{supp} u_i^{k_0}) < \delta \), from the preceding inequality and (37) it follows that
\[
\|u_i^{k_0} + u_i^{k_1}\|_{\Lambda(\varphi)} \geq \frac{1}{2}(1 + (1 - \varepsilon_1)).
\]
Applying [31, Lemma 1] once more, but now to the function \( u_i^{k_0} + u_i^{k_0} \) and \( \varepsilon_2 \) in the similar way, we can find \( i_2 > i_1 \), for which
\[
\|u_i^{k_0} + u_i^{k_0} + u_i^{k_0}\|_{\Lambda(\varphi)} \geq \|u_i^{k_0}\|_{\Lambda(\varphi)} + \frac{1}{2}(1 - \varepsilon_2)(1 + (1 - \varepsilon_1)) \geq \frac{1}{2}(1 + (1 - \varepsilon_1) + (1 - \varepsilon_1)(1 - \varepsilon_2)).
\]
Arguing in the same way, we construct an increasing sequence of indices \( (i_j)_{j=0}^{\infty} \) such that
\[
\left\| \sum_{j=0}^{\infty} u_i^{k_j} \right\|_{\Lambda(\varphi)} \geq \frac{1}{2} \left( 1 + \sum_{r=1}^{n} \prod_{k=1}^{r} (1 - \varepsilon_k) \right), \quad n \in \mathbb{N}.
\]
Since \( \sum_{k=1}^{\infty} \varepsilon_k < \infty \), we have \( \prod_{k=1}^{\infty} (1 - \varepsilon_k) = \eta > 0 \). Therefore, from the preceding inequality it follows that
\[
\|u_i^{k_0}\|_{\Lambda(\varphi)} \geq \left\| \sum_{j=0}^{\infty} u_i^{k_j} \right\|_{\Lambda(\varphi)} \geq \frac{\eta(n + 1)}{2}, \quad \text{for every } n \in \mathbb{N}.
\]
Clearly, this contradicts to the fact that \( u_i^{k_0} \in \Lambda(\varphi) \), and the proof is completed. \( \square \)
The following theorem is an immediate consequence of Proposition 5 and arguments used in the proof of Theorem 4.

**Theorem 5.** Let an increasing concave function \( \varphi \) satisfy condition (33). Suppose that \( x_n \in L_{\theta,p} = (M(\varphi), \Lambda(\varphi))_{\theta,p}, n = 1, 2, \ldots, \) where \( 0 < \theta < 1 \) and \( 1 \leq p < \infty \), are arbitrary pairwise disjoint functions, \( ||x_n||_{\theta,p} = 1 \) \( (n = 1, 2, \ldots) \).

Then, for every \( \varepsilon > 0 \) there exists a block basis \( u_i = \sum_{j=n_i}^{n_{i+1}-1} a_j x_j, 1 = n_0 < n_1 < \ldots, \) satisfying the following conditions:

(i) for arbitrary \( \lambda_k \in \mathbb{R}, k = 1, 2, \ldots, \) we have

\[
(1 - \varepsilon) \left( \sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^{\infty} \lambda_k u_k \right\|_{\theta,p} \leq (1 + \varepsilon) \left( \sum_{k=1}^{\infty} |\lambda_k|^p \right)^{1/p};
\]

(ii) the closed linear span \([u_k]\) is \((1 + \varepsilon)\)-complemented in the space \( L_{\theta,p}\).

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