Contributions to the study of Anosov Geodesic Flows in Non-Compact Manifolds

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Abstract
In this paper we prove that when the geodesic flow of a compact or non-compact complete manifold without conjugate points is of the Anosov type, then the average along of the sectional curvature in planes tangent to the geodesic is negative away from zero for some uniform time. Moreover, in dimension two, if the manifold has no focal points, then the latter condition is sufficient to obtain that the geodesic flow is of Anosov type.

1 Introduction
Let \((M, \langle , \rangle)\) be a complete Riemannian manifold and \(SM\) the unitary tangent bundle. Let \(\phi^t : SM \to SM\) be the geodesic flow and suppose that \(\phi^t\) is Anosov. This means that \(T(SM)\) have a splitting \(T(SM) = E^s \oplus \langle G \rangle \oplus E^u\) such that
\[
\begin{align*}
    d\phi^t_{\theta}(E^s(\theta)) &= E^s(\phi^t(\theta)), \\
    d\phi^t_{\theta}(E^u(\theta)) &= E^u(\phi^t(\theta)), \\
    ||d\phi^t_{\theta}||_{E^s} &\leq C\lambda^t, \\
    ||d\phi^{-t}_{\theta}||_{E^u} &\leq C\lambda^t,
\end{align*}
\]
for all \(t \geq 0\) with \(C > 0\) and \(0 < \lambda < 1\), where \(G\) is the vector field derivative of the geodesic vector flow.

For any \(\theta = (p, v) \in SM\), we will denoted by \(\gamma_\theta(t)\) the unique geodesic with initial conditions \(\gamma_\theta(0) = p\) and \(\gamma'_\theta(0) = v\). Let \(V(t)\) be a nonzero perpendicular parallel vector field along \(\gamma_\theta(t)\). We denote by \(k(V(t))\) the sectional curvature of the subspace spanned by \(\gamma'_\theta(t)\) and \(V(t)\).

Now we present our first result,

**Theorem 1.1.** Let \(M\) be a complete manifold with curvature bounded below by \(-c^2\) without conjugate points and whose geodesic flow is Anosov. Then, there are two positive constants \(B\) and \(t_0\) such that, for all \(\theta \in SM\) and for any unit perpendicular parallel vector field \(V(t)\) along \(\gamma_\theta(t)\) we have that
\[
    \frac{1}{t} \int_0^t k(V(r)) \, dr \leq -B,
\]
whenever if \(t > t_0\).
When the manifold $M$ is compact or compactly homogeneous, i.e., the isometry group of its universal cover acts co-compactly, Eberlien in [Ebe73] (see Corollary 3.4 and Corollary 3.5) proved the following result:

**Theorem** [Ebe73] Assume that $M$ has no conjugate points, then

(i) If the geodesic flow is Anosov, then for all $\theta \in SM$ and for any nonzero perpendicular parallel vector field $V(t)$ along $\gamma_\theta(t)$ there is $t$ such that $k(V(t)) < 0$.

(ii) If $M$ has no focal points and satisfies the condition of (i), then geodesic flow is of Anosov type.

Thus, Theorem 1.1 generalizes the result (i) of the above theorem. It is worth emphasizing that its result can be applied in non-compact manifolds.

Some immediate consequence of Theorem 1.1 are:

**Corollary 1.** Let $M$ be a complete manifold with curvature bounded below by $-c^2$, whose geodesic flow is Anosov. Then, if $M$ has finite volume, we get

$$\int_{SM} \text{Ric} \, d\mu < -B \cdot \mu(SM) < 0,$$

where Ric is the Ricci curvature and $\mu$ is the Liouville measure on $SM$.

As the Gauss-Bonnet theorem holds for surfaces of finite volume (see [Ros82]), then the above corollary implies that, neither surface with Euler Characteristic zero and finite volume admits a geodesic flow of Anosov type.

**Corollary 2.** No manifold $M$ that admits a geodesic $\gamma(t)$ and nonzero perpendicular parallel vector field $V(t)$ with $k(V(t)) \geq 0$ for any $t \geq t_1$, has geodesic flow of Anosov type.

In particular, if $M$ is the product of two manifolds, with the product metric, then the geodesic flow is never an Anosov flow.

It is worth noting that, the second part of this corollary was well known for product of compact manifolds or product of compactly homogeneous manifolds (cf. [Ebe73]). Therefore, ours is a more general result.

Thus, when we have the product of two manifolds, the last result leads us to think that if it is possible to change the product metric in such a way that the geodesic flow becomes Anosov. In fact, in the Section 5, using the Theorem 1.2 and a “Warped Product” to construct a metric in $\mathbb{R} \times S^1$ whose geodesic flow is Anosov (see Section 5).

Our third corollary stated that the world of the compact manifold and non-compact manifold are, in some way, very different.

Before we present the corollary, we consider the following function $K(t)$, associated to the sectional curvatures of the a complete non-compact manifold $M$, as the follows:

For each $x \in M$ and $P \subset T_xM$, we denotes by $k(P)$ the sectional curvature of plane $P$. Thus, we defines $k(x) = \sup_{P \subset T_xM} k(P)$. Fixed a point $O \in M$, we define

$$K(t) := \sup_{x \in M \setminus B(t)} k(x),$$
where $B_t(0)$ is the open ball of center $O$ and radius $t$.

We say that a complete non-compact manifold $M$ is \textit{asymptotically flat} if
\[
\lim_{t \to +\infty} K(t) = 0.
\]

**Corollary 3.** Let $M$ be a asymptotically flat manifold, assume that $M$ has no conjugate points and curvature bounded below. Then its geodesic flow is not an Anosov flow.

This corollary allows us to construct a large category of non-compact manifold whose geodesic flow is not Anosov.

Note that for manifold of negative curvature bounded between two negative constants (this condition is called “pinched”), its geodesic flow is Anosov (cf. [Ano69]). However, in the non-compact case, the negative sign of the curvature does not implies that we have a geodesic flow of Anosov type. In fact, in Section 5, we construct a non-compact surface of negative curvature whose geodesic is not Anosov. In other words, from the point view of the dynamic, a manifold of negative curvature and a manifold of pinched negative curvature are totally different.

The second result of this paper is a more general version, in dimension two, of (ii) at Theorem [Ebe73] above. In fact, if our manifold has dimension two with the additional condition of does not have focal points, then the conclusion of the Theorem 1.1 is a necessary condition to the geodesic flow to be an Anosov flow. More precisely,

**Theorem 1.2.** Let $M$ be a complete surface with curvature bounded below by $-c^2$ without focal points. Assume that, there are two constant $B, t_0 > 0$ such that for all geodesic $\gamma(t)$
\[
\frac{1}{t} \int_0^t k(\gamma(s)) \, ds \leq -B \text{ whenever } t > t_0,
\]
then the geodesic flow is an Anosov flow.

In the Subsection 5.2, we use the Theorem 1.2 to construct a family of non-compact surface with non-positive curvature, which is non compactly homogeneous. Therefore, the Eberlein result does not apply, but its geodesic flow is of Anosov type. To be more specific, we make a \textit{Warped Product} of the circle $S^1$ and the line $\mathbb{R}$ to construct such surfaces (cf. Section 5).

We take Theorem 1.1 and Theorem 1.2 to present the following corollary,

**Corollary 4.** Let $M$ be a complete surface with curvature bounded below by $-c^2$ without focal points. Then the geodesic flow is Anosov if and only if there are two constants $B, t_0 > 0$ such that for all geodesic $\gamma(t)$
\[
\frac{1}{t} \int_0^t k(\gamma(s)) \, ds \leq -B \text{ whenever } t > t_0.
\]

We believe that, in greater dimension, it is possible to obtain a similar result of Theorem 1.2. In fact:

Suppose that $M$ has dimension $n$. Then consider for each $\theta \in SM$ a family of $n - 1$
perpendicular parallel vector field $V_j(t)$ along $\gamma_\theta$, $j = 1, \ldots, n - 1$. Then, we present the following conjecture,

**Conjecture:** Let $M$ be a complete manifold with curvature bounded below, without focal points. Assume that, there are two positive constants $B, t_0$ such that for all geodesic $\gamma_\theta(t)$ and all $j = 1, \ldots, n - 1$ we have

$$\frac{1}{t} \int_0^t k(V_j(s))ds \leq -B$$ 

whenever $t > t_0$.

then the geodesic flow is an Anosov flow.

**Structure of the Paper:** In Section 2, we present the notation and geometry setting of the paper. In Section 3, we present the proof of Theorem 1.1 and its consequences. In Section 4, we present the proof of Theorem 1.2 and some results addressed to the conjecture. Finally, in Section 5, we apply the Theorem 1.2 to construct a family of non-compact surfaces with Anosov geodesic flow. At the end of Section 5, we also show a example of a non-compact surface of negative curvature whose geodesic flow is not Anosov.

# 2 Notation and Preliminaries

Through rest of this paper, $M = (M, g)$ will denote a complete Riemannian manifold without boundary of dimension $n \geq 2$, $TM$ its the tangent bundle, $SM$ its unit tangent bundle, $\pi: TM \to M$ will denote the canonical projections, and $\mu$ the Liouville measure of $SM$ (see [Pat99]).

## 2.1 Geodesic flow

For $\theta = (p, v)$ a point of $SM$. Let $\gamma_\theta(t)$ denote the unique geodesic with initial conditions $\gamma_\theta(0) = p$ and $\gamma'_\theta(0) = v$. For a given $t \in \mathbb{R}$, let $\phi^t: SM \to SM$ be the diffeomorphism given by $\phi^t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$. Recall that this family is a flow (called the geodesic flow) in the sense that $\phi^{t+s} = \phi^t \circ \phi^s$ for all $t, s \in \mathbb{R}$.

Let $V := \ker D\pi$ denote the vertical sub-bundle of $TSM$ (tangent bundle of $SM$).

Let $\alpha: TTM \to TM$ the Levi-Civita connection map of $M$. Let $H := \ker \alpha$ be horizontal sub-bundle. Recall $\alpha$ is defined as follow: Let $\xi \in T_\theta TM$ and $z : (-\epsilon, \epsilon) \to TM$ be a curve adapted to $\xi$, i.e., $z(0) = \theta$ and $z'(0) = \xi$, where $z(t) = (\alpha(t), Z(t))$, then

$$\alpha_\theta(\xi) = \nabla_{\frac{\partial}{\partial t}} Z(t)|_{t=0}.$$

For each $\theta$, the maps $d_\theta \pi|_{H(\theta)} : H(\theta) \to T_p M$ and $K_\theta|_{V(\theta)} : V(\theta) \to T_p M$ are linear isomorphisms. Furthermore, $T_\theta TM = H(\theta) \oplus V(\theta)$ and the map $j_\theta : T_\theta TM \to T_p M \times T_p M$ given by

$$j_\theta(\xi) = (d_\theta \pi(\xi), K_\theta(\xi))$$

is a linear isomorphism.
Using the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, we can define in a natural way a Riemannian metric on $TM$ that makes $H(\theta)$ and $V(\theta)$ orthogonal. This metric is called the Sasaki metric and is given by

$$g_\theta^S(\xi, \eta) = \langle d_\theta \pi(\xi), d_\theta \pi(\eta) \rangle + \langle K_\theta(\xi), K_\theta(\eta) \rangle.$$

From now on, we consider the Sasaki metric restricted to the unit tangent bundle $SM$. It is easy to prove that the geodesic flow preserves the volume measure generated by this Riemannian metric in $SM$. However, this volume measure in $SM$ coincides with the Liouville measure $m$ up to a constant. When $M$ has finite volume the Liouville measure is finite.

Consider the one-form $\beta$ in $TM$ defined for $\theta = (p, v)$ by

$$\beta_\theta(\xi) = g_\theta^S(\xi, G(\theta)) = \langle D_\theta \pi(\xi), v \rangle_p.$$

Observe that $ker \beta_\theta \supset V(\theta)$. It is possible to prove that a vector $\xi \in T_\theta TM$ lies in $T_\theta SM$ with $\theta = (p, v)$ if and only if $\langle \alpha_\theta(\xi), v \rangle = 0$. Furthermore, when restricted to $SM$ the one-form $\beta$ becomes a contact form invariant by the geodesic flow whose Reeb vector field is the geodesic vector field $G$. However, the sub-bundle $S = ker \beta$ is the orthogonal complement of the spanned $G$. Since $\beta$ is invariant by the geodesic flow, then the sub-bundle $S$ is invariant by $\phi^t$, i.e., $\phi^t(S(\theta)) = S(\phi^t(\theta))$ for all $\theta \in SM$ and for all $t \in \mathbb{R}$.

To understand the behavior of $d\phi^t$ let us introduce the definition of Jacobi field. A vector field $J$ along of a geodesic $\gamma_\theta$ is called the Jacobi field if it satisfies the equation

$$J'' + R(\gamma_\theta', J)\gamma_\theta' = 0,$$

where $R$ is the Riemann curvature tensor of $M$ and $''$ denotes the covariant derivative along $\gamma_\theta$. Note that, for $\xi = (w_1, w_2) \in T_\theta SM$, (the horizontal and vertical decomposition) with $w_1, w_2 \in T_p M$ and $\langle v, w_2 \rangle = 0$, it is known that $d\phi^t_\theta(\xi) = (J_\xi(t), J'_\xi(t))$, where $J_\xi$ denotes the unique Jacobi vector field along $\gamma_\theta$ such that $J_\xi(0) = w_1$ and $J'_\xi(0) = w_2$. For more details see [Pat99].

### 2.2 No conjugate points

Suppose $p$ and $q$ are points on a Riemannian manifold, we say that $p$ and $q$ are conjugates if there is a geodesic $\gamma$ that connects $p$ and $q$ and a non-zero Jacobi field along $\gamma$ that vanishes at $p$ and $q$. When neither two points in $M$ are conjugated, we say the manifold $M$ has no conjugate points. Another important kind of manifolds for this paper are the manifolds without focal points, we say that a manifold $M$ has no focal point, if for any unit speed geodesic $\gamma$ in $M$ and for any Jacobi vector field $Y$ on $\gamma$ such that $Y(0) = 0$ and $Y'(0) \neq 0$ we have $(||Y||^2)'(t) > 0$, for any $t > 0$. It is clear that if a manifold has no focal points, then also has no conjugate points.

The more classical example of manifolds without focal points and therefore without conjugate points, are the manifold of non-positive curvature. It is possible to construct a manifold having positive curvature in somewhere, and without conjugate points. There is many examples of manifold without conjugate points. We emphasize here, for example, in [Mn87] Mañé proved that, when volume is finite and the geodesic flow is Anosov, then
the manifold has no conjugate points. This latter had been proved by Klingenberg (cf. [Kli74]) in the compact case. In the case of infinite volume the result by Mañé [Mn87] is an open problem:

When the geodesic flow is Anosov, then the manifold has no conjugate points?

The last fact showed that, if we would like to work with geodesic flow of Anosov type, we assume then that our manifold has no conjugate points (condition superfluous in finite volume via Mañé result). Therefore, from now on, we can assume that the manifold $M$ has no conjugate points.

Now suppose that $M$ has no conjugate points and its sectional curvatures are bounded below by $-c^2$. In this case, if the geodesic flow $\phi^t : SM \to SM$ is Anosov, then in [Bol79], Bolton showed that there exists a positive constant $\delta$ such that for all $\theta \in SM$, the angle between $E^s(\theta)$ and $E^u(\theta)$ is greater than $\delta$. Furthermore, if $J$ is a perpendicular Jacobi vector field along $\gamma_0$ such that $J(0) = 0$ then there exists $A > 0$ and $s_0 \in \mathbb{R}$ such that $\| J(t) \| \geq A \| J(s) \|$ for $t \geq s \geq s_0$. Therefore, for $\xi \in E^s(\theta)$ and $\eta \in E^u(\theta)$ since $\| J_\xi(t) \| \to 0$ as $t \to +\infty$ and $\| J_\eta(t) \| \to 0$ as $t \to -\infty$ follows that $J_\xi(0) \neq 0$ and $J_\eta(0) \neq 0$. In particular, $E^s(\theta) \cap V(\theta) = \{0\}$ and $E^u(\theta) \cap V(\theta) = \{0\}$ for all $\theta \in SM$.

For $\theta = (p,v) \in SM$, we denote by $N(\theta) := \{w \in T_xM : (w,v) = 0\}$. Moreover, by the identification of the Subsection 2.1, the horizontal space can be write as $H(\theta) = \{0\} \times N(\theta)$ and the vertical space as $V(\theta) = N(\theta) \times \{0\}$. Thus, if $E \subset S(\theta) := \ker \beta = N(\theta) \times N(\theta)$ is a subspace, $\dim E = n - 1$, and $E \cap (V(\theta) \cap S(\theta)) = \{0\}$ then $E \cap (H(\theta) \cap S(\theta))^\perp = \{0\}$. Hence, there exists a unique linear map $T : H(\theta) \cap S(\theta) \to V(\theta) \cap S(\theta)$ such that $E$ is the graph of $T$. In other words, there exists a unique linear map $T : N(\theta) \to N(\theta)$ such that $E = \{(v,Tv) : v \in N(\theta)\}$. Furthermore, the linear map $T$ is symmetric if and only if $E$ is Lagrangian (see [Pat99]).

It is known that if the geodesic flow is Anosov, then for each $\theta \in SM$, the sub-bundles $E^s(\theta)$ and $E^u(\theta)$ are Lagrangian. Therefore, for each $t \in \mathbb{R}$, we can write $d\phi^t (E^s(\theta)) = E^s(\phi^t(\theta)) = \text{graph} U_s(t)$ and $d\phi^t (E^u(\theta)) = E^u(\phi^t(\theta)) = \text{graph} U_u(t)$, where $U_s(t) : N(\phi^t(\theta)) \to N(\phi^t(\theta))$ and $U_u(t) : N(\phi^t(\theta)) \to N(\phi^t(\theta))$ are symmetric maps.

Now we describe a useful method of L. Green (cf. [Gre58]), to see what properties the maps $U_s(t)$ and $U_u(t)$ satisfies.

Let $\gamma_0$ be a geodesic, and consider $V_1, \ldots, V_n$ a system of parallel orthonormal vector fields along $\gamma_0$ with $V_n(t) = \gamma_0(t)$. If $Z(t)$ is a perpendicular vector field along $\gamma_0(t)$, we can write

$$Z(t) = \sum_{i=1}^{n-1} y_i(t)V_i(t).$$

Note that the covariant derivative $Z'(s)$ is identified with the curve $\alpha'(s) = (y'_1(s), \ldots, y'_{n-1}(s))$. Conversely, any curve in $\mathbb{R}^{n-1}$ can be identified with a perpendicular vector field on $\gamma_0(t)$.

For each $t \in \mathbb{R}$, consider the symmetric matrix $R(t) = \{R_{i,j}(t)\}$, where $1 \leq i, j \leq n-1$, $R_{i,j} = \langle R(\gamma_0(t), V_i(t)), \gamma_0(t), V_j(t) \rangle$ and $R$ is the curvature tensor of $M$. Consider the $(n-1) \times (n-1)$ matrix Jacobi equation

$$Y''(t) + R(t)Y(t) = 0. \quad (5)$$

If $Y(t)$ is solution of (5) then for each $x \in \mathbb{R}^{n-1}$, the curve $\beta(t) = Y(t)x$ corresponds to a Jacobi perpendicular vector on $\gamma_0(t)$. For $\theta \in SM$, $r \in \mathbb{R}$, we consider $Y_{\theta,r}(t)$ be the
unique solution of (5) satisfying $Y_{\theta,r}(0) = I$ and $Y_{\theta,r}(r) = 0$. In [Gre58], Green proved that $\lim_{r \to -\infty} Y_{\theta,r}(t)$ exists for all $\theta \in SM$ (see also [Ebe73] Sect. 2). Moreover, if we define:

$$Y_{\theta,u}(t) := \lim_{r \to -\infty} Y_{\theta,r}(t),$$

we obtain a solution of Jacobi equation (5) such that $\det Y_{\theta,u}(t) \neq 0$. Furthermore, it is proved in [Gre58] (see also [FM82] and [Ebe73]) that $\frac{DY_{\theta,r}}{Dt}(t) = \lim_{r \to -\infty} \frac{DY_{\theta,r}}{Dt}(t)$. However, if

$$U_r(\theta) = \frac{DY_{\theta,r}}{Dt}(0); \quad U_u(\theta) = \frac{DY_{\theta,u}}{Dt}(0),$$

then

$$U_u(\theta) = \lim_{r \to -\infty} U_r(\theta).$$

It is easy to prove that (see [FM82])

$$U_u(\phi^t(\theta)) = \frac{DY_{\theta,u}}{Dt}(t)Y_{\theta,u}^{-1}(t)$$

for every $t \in \mathbb{R}$. It follows that $U_u$ is a symmetric solution of the Ricatti equation

$$U'(t) + U^2(t) + R(t) = 0.$$  \hfill (7)

Analogously, taking the limit when $r \to +\infty$, we have defined $U_s(\theta)$, that also satisfies the Ricatti equation (7). Furthermore, in [Gre58], Green also showed that, in the case of curvature bounded below by $-c^2$, symmetric solutions of the Ricatti equation which are defined for all $t \in \mathbb{R}$ are bounded by $c$, i.e.,

$$\sup_t \| U_s(t) \| \leq c \quad \text{and} \quad \sup_t \| U_u(t) \| \leq c.$$ \hfill (8)

3 Proof of Theorem 1.1 and its Consequences

In this section, we prove the Theorem 1.1 and the Corollaries 1, 2 and 3.

In this direction, we prove the Lemma 3.1 which use a Bolton’s result (see [Bol79]), to have some control on the sum $\|U_s\|^2 + \|U_u\|^2$. More specifically,

**Lemma 3.1.** Let $M$ be a complete manifold with curvature bounded below by $-c^2$ without conjugate points and whose geodesic flow is Anosov. Then, there is a constant $D > 0$, such that for any $\theta \in SM$ and for any unit perpendicular parallel vector field $V(t)$ along $\gamma_\theta$ we have

$$\|U_s(V(t))\|^2 + \|U_u(V(t))\|^2 \geq D, \quad \text{for all} \quad t \in \mathbb{R}.$$  

**Proof.** By Bolton’s results (see [Bol79]) there exists a positive constant $\delta$ such that the angle between $E^s(\phi^t(\theta))$ and $E^u(\phi^t(\theta))$ is greater than $\delta$. Observe that $(V(t), U_s(V(t)) \in E^s(\phi^t(\theta))$ and $(V(t), U_u(V(t)) \in E^s(\phi^t(\theta))$ (see Subsection 2.2). Thus, using the definition of the Sakaki metric and definition of angle, we have for each $t \in \mathbb{R}$

$$|1 + \langle U^s(V(t)), U^u(V(t))\rangle| \leq \sqrt{1 + \|U_s(V(t))\|^2} \sqrt{1 + \|U_u(V(t))\|^2} \cos \delta.$$ \hfill (9)

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Fix $0 < D < 1$ such that $\frac{1-D}{1+D} > \cos \delta$, we claim that
\[ ||U_s(V(t))||^2 + ||U_u(V(t))||^2 \geq D, \text{ for all } t \in \mathbb{R}. \]
In fact, by contradiction, suppose that $||U_s(V(t))||^2 + ||U_u(V(t))||^2 < D$, then
- $||U_s(V(t))||^2 < D$,
- $||U_u(V(t))||^2 < D$,
- $||U_s(V(t))|| \cdot ||U_u(V(t))|| < D$.
Thus, from (9)
\[ 1 - |\langle U_s(V(t)), U_u(V(t)) \rangle| \leq (1 + D) \cos \delta. \]
By Cauchy-Schwarz inequality it follows that
\[ \frac{1-D}{1+D} \leq \cos \delta, \]
which is a contradiction by the choice of $\delta$. Therefore,
\[ ||U_s(V(t))||^2 + ||U_u(V(t))||^2 \geq D. \]

We use the Lemma 3.1 to show the Theorem 1.1. Before, we set the notation use in the Theorem 1.1.

**Remark 1.** For each $\theta \in SM$ and for any unit perpendicular parallel vector field $V(t)$ along $\gamma_\theta$, we denote by $k(V(t))$ the sectional curvature of the subspace spanned by $\gamma'_\theta(t)$ and $V(t)$.

**Proof of Theorem 1.1.** Fix $\theta \in SM$, and consider an unit perpendicular parallel vector field $V(t)$ along $\gamma_\theta$. Since the operators $U_s$ and $U_u$ are symmetric and satisfy the equation (7) follows that
\[ \langle U_s(V(t)), V(t) \rangle' + ||U_s(V(t))||^2 + k(V(t)) = 0, \]
\[ \langle U_u(V(t)), V(t) \rangle' + ||U_u(V(t))||^2 + k(V(t)) = 0. \]
Integrating the sum of the above equations, we get
\[ \sum_{s=u} (\langle U_s(V(t)), V(t) \rangle - \langle U_s(V(0)), V(0) \rangle) + \int_0^t ||U_s(V(r))||^2 + ||U_u(V(r))||^2 \, dr \]
\[ + 2 \int_0^t k(V(r)) \, dr = 0. \]
Observe that by inequality (8), $\| U_s(t) \| \leq c$ and $\| U_u(t) \| \leq c$ for all $t \in \mathbb{R}$, which allows to state that
\[ \lim_{t \to +\infty} \frac{1}{t} \sum_{s=u} (\langle U_s(V(t)), V(t) \rangle - \langle U_s(V(0)), V(0) \rangle) = 0. \]
Thus, from the Lemma 3.1 follows that, there exists $t_0 > 0$ then
\[ \frac{1}{t} \int_0^t k(V(r)) \, dr \leq -\frac{D}{2}, \text{ whenever } t > t_0. \]
Taking $B = \frac{D}{2}$, we concludes the proof of theorem. \qed
3.1 Consequences of Theorem 1.1

In the follows, we prove some important consequences of Theorem 1.1. We denote by $Ric_p(v)$ the Ricci curvature in the direction $v$, which is defined in the follows way: consider $\{v, v_1, v_2, \ldots, v_{n-1}\}$ a orthogonal basis of $T_pM$, then

$$Ric_p(v) = \frac{1}{n-1} \sum_{i=1}^{n-1} < R(v, v_j)v, v_j >.$$ 

In other words, $Ric_p(v)$ is the average of the sectional curvature in planes generated by $v$ and $v_j$. In particular, as $R_p(v)$ is the trace of the matrix of curvature, then $R_p(v)$ does not depend of the orthonormal set $\{v_1, v_2, \ldots, v_{n-1}\}$. Thus, for $(p, v) \in SM$, we denote by $Ric(p, v) = Ric_p(v)$ the Ricci curvature, which is a function of $SM$ on the real line. As an immediate consequence of Theorem 1.1, we have

**Corollary 5.** Let $M$ be a complete manifold with curvature bounded below by $-c^2$ without conjugate points and whose geodesic flow is Anosov. Then there are $B, t_0 > 0$ such that for any $\theta \in SM$ we have

$$\frac{1}{t} \int_0^t Ric_{\gamma_\theta(r)}(\gamma_\theta'(r)) dr \leq -B,$$  \hspace{1cm} (10)

whenever $t > t_0$.

Let us to prove Corollary 1 using Corollary 5 and Birkhoff’s ergodic theorem.

**Proof of Corollary 1.** As $M$ is complete with curvature bounded below by $-c^2$ whose geodesic flow is Anosov, then the condition of finite volume gives us, thanks to Mañe result(cf. [Mn87]), that $M$ has no conjugate points. Therefore, $M$ is a manifold without conjugate points and whose negative part of the Ricci curvature is integrable (with respect to Liouville measure) on $SM$, then a Guimarães result (cf. [Ga92]) implies that the Ricci curvature is integrable on $SM$. Moreover, for each $\theta \in SM$ the equation (10) can be written as

$$\frac{1}{t} \int_0^t Ric(\phi^t(\theta)) dr \leq -B,$$  \hspace{1cm} whenever $t > t_0$.

As the Liouville measure, is invariant by the geodesic flow, then the Birkhoff ergodic theorem, applied to the Ricci curvature, provides us

$$-c^2 \cdot \mu(SM) < \int_{SM} Ric d\mu < -B \cdot \mu(SM) < 0,$$

which concludes the proof of Corollary 1. \hfill $\Box$

**Remark 2.** In [Ga92], Guimarães proved that: If $M$ manifold without conjugate points with the positive or negative part of the Ricci curvature is integrable, then

$$\int_{SM} Ric d\mu \leq 0,$$

where the equality holds only if the curvature tensor of $M$ is identically zero $SM$. Thus, by Theorem 1.1, manifold of zero curvature has no geodesic flow of Anosov type, which implies that in the Anosov case should be $\int_{SM} Ric d\mu < 0$. Obtaining another proof of Corollary 1.
It is well known that compact $M$ surface with non-negative Euler characteristic $\chi(M)$, does not admit Rieamannian metric whose geodesic flow is of Anosov Type. In fact, if the geodesic flow of a compact surface is Anosov, then the surface has no conjugate points (cf. [Kli74]). Thus, by a Hopf’s result (cf. [Hop48]) the integral of the Gaussian curvature is non-positive and zero in the case of zero curvature. Therefore, as in the Anosov case the curvature is not zero everywhere (cf. [Ebe73]), then the Gauss-Bonnet implies that if the geodesic flow is Anosov, then $\chi(M) < 0$.

For non-compact surface of finite volume the Gauss-Bonnet theorem holds (cf. [Ros82]), then as a consequence of Corollary 1 we have

**Corollary 6.** Complete surfaces of finite volume, curvature bounded below and non-negative Euler characteristic does not admit a Riemannian metric whose geodesic flow is Anosov.

In particular, if $S$ is a sphere or a torus, which has non-negative Euler Characteristic, then no complete surface $M$ of finite volume homeomorphic to $S$ with a finite points deleted, has Anosov geodesic flow.

**Proof of Corollary 2** Let us to prove by contradiction. Assume that the geodesic flow is an Anosov flow, then by Theorem 1.1 there are $B, t_0 > 0$ which satisfies (1). Let $t > t_1$, where $t_1$ is as our hypotheses, therefore,

$$\frac{1}{t} \int_0^t k(V(t)) dt = \frac{1}{t} \int_0^{t_1} k(V(t)) dt + \frac{1}{t} \int_{t_1}^t k(V(t)) dt. \quad (11)$$

Given any $\delta < B$, we taken $t > \max\{t_0, t_1\}$ large enough such that 

$$-\delta < \frac{1}{t} \int_0^{t_1} k(V(t)) dt < \delta.$$ 

Thus by our hypotheses, $\frac{1}{t} \int_{t_1}^t k(V(t)) dt \geq 0$. Therefore, by equation (11) and the chosen of $t$, we have $-B > -\delta$, which is a contradiction. Thus, we concludes the proof of the first part of corollary. To prove, the second part, note that in a manifold $M = N \times O$ endowed with the product metric, it is possible to construct a parallel perpendicular vector field along of a geodesic $\gamma$, totally contains in $N$ or $M$. Follows then, by the first part of corollary, that the geodesic flow of $M$ is not Anosov.

We end this section with the proof of Corollary 3.

**Proof of Corollary 3** It is known that non-compact manifold has rays, $i.e.$, there exists $\theta = (p, v) \in SM$ such that the geodesic $\gamma_\theta : [0, \infty) \to M$ satisfies $d(\gamma_\theta(t), \gamma_\theta(s)) = |s - t|$. Since $K(t) \to 0$ as $t \to \infty$, then $Ric_{\gamma_\theta(t)}(\gamma_\theta'(t)) \to 0$ as $t \to \infty$. In particular,

$$\lim_{t \to +\infty} \left| \frac{1}{t} \int_0^t Ric_{\gamma_\theta(r)}(\gamma_\theta'(r)) \ dr \right| = 0.$$ 

Therefore, by Corollary 5 follows that the geodesic flow of $M$ is not Anosov.

At the end of Section 5 we construct a non-compact surface of negative curvature whose geodesic is not Anosov, showing that, from the point view of the dynamic, manifold of negative curvature and manifold of pinched negative curvature are totally different.
4 Proof of Theorem 1.2

The main goal of this section is to prove Theorem 1.2. The idea of the proof is to use the hypothesis of Theorem 1.2 to show that the stable Jacobi field has norm exponential decreasing for the future, and unstable Jacobi field has norm exponential increasing for the future (see Proposition 1). Then, we use a strategy similar to Eberlein at [Ebe73], to show the uniform contraction of the stable and unstable bundle.

**Theorem 4.1.** Let $M$ be a complete surface with curvature bounded below by $-c^2$ without focal points. Assume that, there are two constants $B, t_0 > 0$ such that for all geodesic $\gamma(t)$

\[
\frac{1}{t} \int_0^t k(\gamma(s)) \, ds \leq -B \quad \text{whenever} \quad t > t_0,
\]

then the geodesic flow is an Anosov flow.

The following proposition gives us control of the determinant of the matrix of stable and unstable Jacobi fields.

**Proposition 1.** Let $M$ be a complete manifold with curvature bounded below by $-c^2$ without focal point. Assume that, there are two constants $B, t_0 > 0$ such that for all geodesic $\gamma(t)$ and for each unit perpendicular parallel vector field $V(t)$ on $\gamma(t)$ we have

\[
\frac{1}{t} \int_0^t k(V(r)) \, dr \leq -B \quad \text{whenever} \quad t > t_0.
\]

Then there are a constant $D > 0$ and $t_1 > 0$ such that

(a) $|\det Y_{\theta,s}(t)| \geq e^{Dt}$ for all $t > t_1$,

(b) $|\det Y_{\theta,s}(t)| \leq e^{-Dt}$ for all $t > t_1$.

**Proof.** We prove item (b). The proof of item (a) is analogue to item (b). Indeed, as $Y_{\theta,s}(t)$ is the stable tensor given by (5) and satisfies the Jacobi equation (5). Thus, $U_{\theta,s}(t) := U_\theta(\phi'(\theta)) = Y_{\theta,s}'(t) \cdot Y_{\theta,s}^{-1}(t)$ is solution of the Ricatti equation (7) (see Subsection 2.2). As $M$ has no focal points, then for each $x \in \mathbb{R}^n \setminus \{0\}$, we have that the function $t \to |Y_{\theta,s}(t)x|^2$ is decreasing (cf. [Ebe73]), i.e.,

\[
\frac{d}{dt}|Y_{\theta,s}(t)x|^2 = 2\langle Y_{\theta,s}(t)x, Y_{\theta,s}'(t)x \rangle \leq 0, \quad x \in \mathbb{R}^{n-1}.
\]

Therefore, as $Y_{\theta,s}(t)$ is invertible we have

\[
\langle y, U_{\theta,s}(t)y \rangle = \langle y, Y_{\theta,s}'(t) \cdot Y_{\theta,s}^{-1}(t)y \rangle \leq 0, \quad y \in \mathbb{R}^{n-1}.
\]

Since $U_{\theta,s}(t)$ is symmetric, the last equation implies that all eigenvalues of $U_{\theta,s}(t)$ are non-positive. Let $-\lambda_{n-1}(t) \leq -\lambda_{n-2}(t) \leq \cdots \leq -\lambda_1(t) \leq 0$ the eigenvalues of $U_{\theta,s}(t)$, then as $|U_{\theta,s}(t)| \leq c$, then $0 \leq \lambda_i(t) \leq c$, $i = 1, 2, \ldots, n - 1$ which provides

\[
\text{tr} (U_{\theta,s}(t))^2 = \lambda_1^2(t) + \lambda_2^2(t) + \cdots + \lambda_{n-1}^2(t) \\
\leq c(\lambda_1(t) + \lambda_2(t) + \cdots + \lambda_{n-1}(t)) \\
= -c \text{tr} U_{\theta,s}(t).
\]
Taking trace in the equation (7) and integrating we have
\[
0 = \frac{1}{t} \int_0^t \text{tr} U_{\theta,s}'(r) dr + \frac{1}{t} \int_0^t \text{tr} U_{\theta,s}^2(r) dr + \frac{1}{t} \int_0^t \text{tr} R(r) dr
\]
\[
= \frac{\text{tr} U_{\theta,s}(t) - \text{tr} U_{\theta,s}(0)}{t} + \frac{1}{t} \int_0^t \text{tr} U_{\theta,s}^2(r) dr + \frac{1}{t} \int_0^t \text{tr} R(r) dr.
\]  
(15)

For \( t > t_0 \), our hypothesis (equation (13)) implies that \( \frac{1}{t} \int_0^t \text{tr} R(r) dr < -(n - 1) \cdot B \), as \( |\text{tr} U_{\theta,s}(r)| \leq c \). From equation (15) there is \( t_1 > 0 \) such that
\[
\frac{1}{t} \int_0^t \text{tr} U_{\theta,s}^2(r) dr > \frac{(n - 1) \cdot B}{2}, \quad t > t_1.
\]  
(16)

The equations (14) and (16) provides
\[
\int_0^t \text{tr} U_{\theta,s}(r) dr \leq -\frac{(n - 1) \cdot B}{2c} t := -Dt, \quad t > t_1.
\]  
(17)

To conclude the argument, we remember the Liouville’s Formula or Jacob’s Formula, (see [DRn17, Lemma 4.6] or [FM82]) which states that
\[
\frac{d}{dt} \log |\det Y_{\theta,s}(r)| = \text{tr} U_{\theta,s}(r), \quad Y_{\theta,s}(0) = I.
\]  
(18)

Integrating this last equation and using (17) we obtain
\[
|\det Y_{\theta,s}(t)| \leq e^{-Dt}, \quad t > t_1,
\]  
(19)

which concludes the proof of proposition. \( \square \)

**Corollary 7.** In the same conditions of Proposition 1 there exists \( t_2 > 0 \) such that for each \( \theta \in SM \) we have
\[
\frac{1}{t} \log |\det D\phi^t|_{E^s(\theta)}| \leq -\frac{D}{2} \quad \text{and} \quad \frac{1}{t} \log |\det D\phi^t|_{E^u(\theta)}| \geq \frac{D}{2}, \quad \text{whatever} \quad t > t_2.
\]

**Proof.** Following the same lines of Lemma 4.6 from [DRn17], consider for each \( \theta \in SM \) the subspace \( N(\theta) \) of \( T_pM \) orthogonal to \( v \). Then \( E^s(u)(\theta) = \text{graph } U_{\theta,s(u)} = \{(x, U_{\theta,s(u)}x) : x \in N(\theta)\} \) and
\[
d\phi^t|_{E^s(u)(\theta)} = \pi_{\phi^t(\theta),s(u)}^{-1} \circ Y_{\theta,s(u)}(t) \circ \pi_{\theta,s(u)},
\]  
(20)

where \( \pi_{\theta,s(u)} : E^s(u)(\theta) \to N(\theta) \) is the projection in the first coordinate. This projection satisfies (see [DRn17, equation 4.12])
\[
1 \leq |\det \pi_{\phi^t(\theta)}| \leq (1 + c^2)^\frac{\alpha s}{2}.
\]  
(21)

The equations (20) and (21) and Proposition 1 provides that there exists \( t_2 > 0 \) such that
• Stable case:
\[
\frac{1}{t} \log |\det D\phi_t|_{E^*(\theta)}| = \frac{1}{t} \log |\det \pi_{\phi}(t)| + \frac{1}{t} \log |\det Y_{\theta,s}(t)| + \frac{1}{t} \log |\det \pi_\theta| \leq \frac{(1 + c^2)^{\frac{2}{n-2}}}{t} - D \leq -\frac{D}{2} \quad \text{for all } t > t_2,
\]
(22)
since $|\det \pi_\theta| \leq ||\pi_\theta||_{n-1} \leq 1$.

• Unstable case:
\[
\frac{1}{t} \log |\det D\phi_t|_{E^*(\theta)}| = \frac{1}{t} \log |\det \pi_{\phi}(t)| + \frac{1}{t} \log |\det Y_{\theta,s}(t)| + \frac{1}{t} \log |\det \pi_\theta| \geq D + \frac{(1 + c^2)^{\frac{2}{n-2}}}{t} \geq \frac{D}{2} \quad \text{for all } t > t_2,
\]
(24)
since $|\det \pi_\theta| = \frac{1}{||\pi_\theta||_{n-1}} \geq (1 + c^2)^{\frac{2}{n-2}}$. Thus, we conclude the proof of corollary.

**Remark 3.** It is worth noting that Proposition 1 and Corollary 7 hold in any dimension. Therefore, we believe that these can be used to proof the Theorem 1.2 in any dimension, in other words, it is still needed to be explored if it can help prove the conjecture given in the introduction for higher dimensions.

To make the proof of Theorem 1.2, let us use the Corollary 7 and the following lemma, which was proven in a similar version by Eberlein at [Ebe73, Lemma 3.12], but we present a proof with weaker hypotheses (see condition (ii) at Lemma 4.1).

**Lemma 4.1.** Let $f : (0, +\infty) \to (0, +\infty)$ a bounded function such that

(i) $f(t + s) \leq f(t)f(s)$ for every $s, t$;

(ii) There is $r > 0$ such that $f(r) < 1$.

Then, there are constants $C > 0$, $\lambda \in (0, 1)$ such that

\[
f(t) \leq C\lambda^t, \quad t > 0.
\]

**Proof.** Let $k \in \mathbb{N}$, then $f(kr) \leq f(r)^k$. Therefore, for each $t > 0$, we can write $t = kr + m$ for $k \in \mathbb{N}$, $m \in (0, r)$ and we have then

\[
f(t) \leq f(kr)f(m) \leq f(r)^k f(m).
\]

Let $A$ be the upper bound of $f$, and let $b = f(r) < 1$. Then, we have

\[
f(t) \leq Ab^k = Ab^\frac{m}{r}(b^\frac{1}{r})^t \leq Ab^{-1}(b^\frac{1}{r})^t.
\]

Thus, we take $C = Ab^{-1}$ and $\lambda = b^\frac{1}{r}$.

We are ready to prove of Theorem 1.2
Proof of Theorem 1.2. As \( M \) is a surface, then 
\[ |\det D\phi^t|_{E^{s(u)}(\theta)}| = |D\phi^t|_{E^{s(u)}(\theta)}. \]
By Corollary 7 for all \( \theta \in SM \) and \( t > t_2 \)
\[ |D\phi^t|_{E^s(\theta)}| \leq e^{-\frac{D^2}{2}t}, \quad t > t_0. \]

Claim: There is \( A > 0 \) such that
\[ |D\phi^t|_{E^s(\theta)}| \leq e^A \quad \text{and} \quad |D\phi^{-t}|_{E^u(\theta)}| \leq e^A, \quad t \in [0, t_0]. \]

Proof of Claim: It is sufficiently to note that, when the manifold has no focal points, then the norm of stable Jacobi fields are decreasing and \( \|U_{\theta,s}\| \leq c \). Analogue to unstable case.

To concludes the proof of Theorem 1.2 we consider the following two functions:
\[ f_s(t) = \sup_{\theta \in SM} |D\phi^t|_{E^s(\theta)}| \quad \text{and} \quad f_u(t) = \sup_{\theta \in SM} |D\phi^t|_{E^u(\theta)}|. \]
Both of the functions are bounded by the above claim and inequality (26). These functions are also sub-additive, because each \( D\phi^t|_{E^{s(u)}(\theta)} \) are linear operators, \( i.e., \) satisfies the item (i) of the Lemma 4.1. The inequalities at (26) also show that there is \( r > 0 \) such that for all \( \theta \in SM \)
\[ |D\phi^t|_{E^s(\theta)}| < 1 \quad \text{and} \quad |D\phi^{-t}|_{E^u(\theta)}| < 1, \]
which implies that \( f_{s(u)}(t) \) satisfies the item (ii) of Lemma 4.1. Thus, by Lemma 4.1 there are \( C_{s(u)} > 0, \lambda_{s(u)} \in (0, 1) \) such that
\[ f_{s(u)}(t) \leq C_{s(u)} \lambda^t_{s(u)}. \]
We taken \( C = \max\{C_s, C_u\} \) and \( \lambda = \max\{\lambda_s, \lambda_u\} \) to conclude that, for \( \theta \in SM \)
\[ |D\phi^t|_{E^s(\theta)}| \leq C\lambda^t \quad \text{and} \quad |D\phi^{-t}|_{E^u(\theta)}| \leq C\lambda^t, \quad t \geq 0. \]
The last inequalities allows us to state that the subspace \( s E^{s(u)}(\theta) \) are linearly independent. Moreover, since \( M \) has no focal points, then \( E^{s(u)}(\theta) \) are continuous in \( \theta \) (cf. [Ebe73]). Thus, we concludes that the geodesic flow is Anosov.

5 Examples of Anosov Geodesic Flows on Noncompact Surfaces

In this section, we use the Theorem 1.2 to construct a family of non-compact (non compactly homogeneous) surfaces, all diffeomorphic to the cylinder \( \mathbb{R} \times S^1 \), whose geodesic flow is Anosov (see Section 5.2). A key tool to build this family of surfaces is the “Warped product”, which will be described in the following section.
5.1 Warped Products

Let $M, N$ be Riemannian manifolds, with metrics $g_M$ and $g_N$, respectively. Let $f > 0$ be a smooth function on $M$. The warped product of $M$ and $N$, $S = M \times_f N$, is the product manifold $M \times N$ furnished with the Riemannian metric

$$g = \pi^*_M(g_M) + (f \circ \pi_M)^2 \pi^*_N(g_N),$$

where $\pi_M$ and $\pi_N$ are the projections of $M \times N$ onto $M$ and $N$, respectively.

Let $X$ be a vector field on $M$. The horizontal lift of $X$ to $M \times_f N$ is the vector field $X$ such that $d\pi_M(p, q)(X(p, q)) = X(p)$ and $d\pi_N(p, q)(X(p, q)) = 0$. If $Y$ is a vector field on $N$, the vertical lift of $Y$ to $M \times_f N$ is the vector field $Y$ such that $d\pi_M(p, q)(Y(p, q)) = Y(q)$. The set of all such lifts are denoted, as usual, by $L(M)$ and $L(N)$, respectively.

We denote by $\nabla$, $\nabla^M$ and $\nabla^N$ the Levi-Civita connections on $M \times_f N$, $N$ and $M$, respectively.

The following proposition describes the relationship between the above connections.

**Proposition 2.** [O'N83] On $S = M \times_f N$, if $\overline{X}, \overline{Y} \in L(M)$ and $\overline{U}, \overline{V} \in L(N)$ then

1. $\nabla_{\overline{X}}\overline{Y} = \nabla^N_{\overline{X}}\overline{Y}$,

2. $\nabla_{\overline{X}}\overline{Y} = \left(\frac{Xf}{f}\right)\overline{U}$,

3. $d\pi_M(\nabla_{\overline{X}}\overline{V}) = -(g(U, V)/f) \cdot \text{grad } f$,

4. $d\pi_N(\nabla_{\overline{X}}\overline{V}) = \nabla^N_U V$.

5. If $\overline{X}$ and $\overline{U}$ are unit vectors then $K(\overline{X}, \overline{U}) = -(1/f)\text{Hess}_M f(X, X)$, where $K$ denotes the sectional curvature of the plane spanned by $\overline{X}$ and $\overline{U}$.

5.2 Family of Non-Compact Surfaces with Anosov Geodesic Flow

Finally, in this section, let us to construct a surface (diffeomorphic to cylinder) of non-positive curvature with Anosov geodesic flow (see Subsection 5.3). Furthermore, using a similar arguments of Example 1, we construct, using the Warped Product and the Corollary 3, a surface of negative curvature whose geodesic is not Anosov (see Subsection 5.4).

5.3 Example 1

Consider the warped product $M = \mathbb{R} \times_f S^1$, where $f(x) = e^{g(x)}$ and $g(x)$ is a smooth function such that

(A) $g''(x) + (g'(x))^2 \geq 0$, for any $x$;
(B) \( g'' + (g')^2 \) is a periodic function with period \( T > 0 \);

(C) There are positive constants \( C_1 \) and \( C_2 \) such that \( C_1/2 < g' < C_2/2 \).

Find functions \( g \) that satisfies the above three conditions is very easy, for example, consider the one-parameter family of function \( g(x) = ax - \cos x + \sin x \), for \( a > 0 \) large enough.

Observe that from Proposition 2 and condition (A), the curvature in the point \((x, y)\) of the surface \( M \) is given by

\[
K(x, y) = K(x) = -\frac{f''}{f} = -\left(g''(x) + (g'(x))^2\right) \leq 0. \quad (27)
\]

In particular, from the condition (B), the function \( K \) is periodic with period \( T \) and \( M \) has no focal points, since the curvature is non-positive. Through the rest of this section we show that the curvature of \( M \) satisfies the equation \( \Gamma_1 \), about the average of the curvature along to geodesic. In this direction, let us understand the geodesics in \( M \), looking at the local coordinates. In fact:

Consider a geodesic \( \gamma(t) = (x(t), z(t)) \) in \( M \) with \( |\gamma'(t)| = 1 \) and a parametrization \( \varphi_{t_0} : \mathbb{R} \times (t_0, t_0 + 2\pi) \to M \) where \( \varphi_{t_0}(x, y) = (x, \cos y, \sin y) \) and

\[
\varphi_{t_0}(\mathbb{R} \times (t_0, t_0 + 2\pi)) \cap \gamma(\mathbb{R}) \neq \emptyset.
\]

Let \( (x(t), y(t)) \) be the local expression of \( \gamma(t) \). Consider \( X_1 = \frac{\partial}{\partial x} \) and \( X_2 = \frac{\partial}{\partial y} \), by the above results we have

\[
\nabla_{X_1} X_1 = 0, \quad \nabla_{X_1} X_2 = \nabla_{X_2} X_1 = g'X_2, \quad \text{and} \quad \nabla_{X_2} X_2 = -e^{2g}g'X_1.
\]

The Christoffel symbols are given by

\[
\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{21} = \Gamma^2_{22} = 0, \quad \Gamma^1_{12} = \Gamma^2_{21} = g' \quad \text{and} \quad \Gamma^1_{22} = -e^{2g}g'.
\]

Since \( \gamma \) is a geodesic with \( |\gamma'(t)| = 1 \) the functions \( x(t) \) and \( y(t) \) satisfy the following equalities,

- \( x''(t) - e^{2g(x(t))}g'(x(t))(y'(t))^2 = 0 \),
- \( y''(t) + 2g(x(t))x'(t)y'(t) = 0 \),
- \( (x'(t))^2 + e^{2g(x(t))}(y'(t))^2 = 1 \).

Observe that

\[
x''(t) = g'(x(t))(1 - (x'(t))^2) \quad (28)
\]

and \( |x'(t)| \leq 1 \) for every \( t \in \mathbb{R} \), since this equality does not depend of the parametrization.

Now we are going to study the geodesics of the surface \( M \). If there exists \( a \in \mathbb{R} \) such that \( |x'(a)| = 1 \), then \( z(a) = 0 \). It follows, by the uniqueness of the geodesics, that \( \gamma(t) = (x(t), z(t)) = (x(a) + t - a, y(a)) \) or \( \gamma(t) = (x(t), z(t)) = (x(0) + a - t, y(0)) \).

Assume that \( |x'(t)| < 1 \) for every \( t \in \mathbb{R} \). Set \( b(t) = x'(t) \), from (28) we have

\[
\frac{b'(t)}{1 - (b(t))^2} = g'(x(t)). \quad (29)
\]
Thus,
\[
\frac{1}{2} \left( \log \left( \frac{1 + b(t)}{1 - b(t)} \right) \right)' = g'(x(t)).
\]
Integrating, we get
\[
\log \left( \frac{1 + b(t)}{1 - b(t)} \right) - \log \left( \frac{1 + b(0)}{1 - b(0)} \right) = 2 \int_0^t g'(x(s)) \, ds.
\]
Hence, the condition (C) for \( g \) provides
\[
B_0 e^{C_1 t} < \frac{1 + b(t)}{1 - b(t)} < B_0 e^{C_2 t},
\]
where \( B_0 = \frac{1 + b(0)}{1 - b(0)} > 0 \), since \( |x'(t)| < 1 \). This implies
\[
1 - \frac{2}{B_0 e^{C_2 t} + 1} < b(t) < 1 - \frac{2}{B_0 e^{C_1 t}}.
\]
(30)

Using the above equalities and inequalities we will study the below expression
\[
\frac{1}{t} \int_0^t K(\gamma_\theta(s)) \, ds = \frac{1}{t} \int_0^t K(x(s)) \, ds.
\]
In this direction, we will divide the analysis in some cases, regarding the position of \( b(0) \) in \([-1, 1]\).

**Case 1:** \( b(0) = x'(0) = 1 \).

In this case \( x(t) = x(0) + t \). Hence,
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds = \frac{1}{t} \int_0^t K(x(0) + s) \, ds = \frac{1}{t} \int_{x(0)}^{x(0)+t} K(u) \, du.
\]

Take \( t > 2T \), where \( T \) is the period of the function \( g \). We can to write \( t = n_T + a \) where \( n_T \) is a positive integer number and \( 0 \leq a < T \). Set \( \eta := \int_0^T K(s)ds < 0 \), observe that
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds = \frac{1}{t} \int_{x(0)}^{x(0)+t} K(u) \, du
\]
\[
= \frac{1}{t} \sum_{i=1}^{n_T} \int_{x(0)+(i-1)T}^{x(0)+iT} K(u) \, du + \frac{1}{t} \int_{x(0)+n_T T}^{x(0)+t} K(u) \, du
\]
\[
\leq \frac{\eta n_T}{t} = \frac{\eta}{T} - \frac{a\eta}{tT} \leq \frac{\eta}{2T}.
\]
Case 2: \( b(0) = x'(0) = -1 \).

Proceeding in the same way, we get
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds \leq \frac{\eta}{2T},
\]
for \( t > 2T \).

Case 3: \( \frac{1}{2} \leq b(0) = x'(0) < 1 \).

Observe that by (29) and (30) \( b'(t) > 0 \) and \( b(t) < 1 \). In particular, \( b(t) \) is a strictly increasing function and \( 1/2 < b(t) < 1 \) for every \( t > 0 \). Hence, \( x(t) \) is an increasing function.

Consider the change of variable \( u = x(s) \). We have
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds = \frac{1}{t} \int_{x(0)}^{x(t)} \frac{K(u)}{x'(x^{-1}(u))} \, du \leq \frac{1}{t} \int_{x(0)}^{x(t)} K(u) \, du.
\]

Take \( t > 4T \) and write \( t/2 = n_T T + a \) where \( n_T \) is a positive integer number and \( 0 \leq a < T \). Since \( x'(t) > 1/2 \) for \( t > 0 \), it follows that \( x(t) > x(0) + 1/2t \) for \( t > 0 \). Hence,
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds \leq \frac{1}{t} \int_{x(0)}^{x(t)} K(u) \, du \\
= \frac{1}{t} \int_{x(0)}^{x(0)+t/2} K(u) \, du + \frac{1}{t} \int_{x(0)+t/2}^{x(t)} K(u) \, du \\
\leq \frac{1}{t} \int_{x(0)}^{x(0)+t/2} K(u) \, du \\
= \frac{1}{t} \int_{x(0)}^{x(0)+n_T T} K(u) \, du + \frac{1}{t} \int_{x(0)+n_T T}^{x(0)+t/2} K(u) \, du \\
\leq \frac{1}{t} \int_{x(0)}^{x(0)+n_T T} K(u) \, du.
\]

Since \( K \) is a periodic function with period \( T \), it follows that
\[
\frac{1}{t} \int_0^t K(x(s)) \, ds \leq \frac{1}{t} \sum_{i=1}^{n_T} \int_{x(0)+(i-1)T}^{x(0)+iT} K(u) \, du \\
= \frac{n_T \eta}{t} \\
= \frac{\eta}{2T} - \frac{a \eta}{t} \\
\leq \frac{\eta}{4T}.
\]
**Case 4:** $-1/2 \leq b(0) = x'(0) < 1/2$.

Now consider $-1/2 \leq b(0) = x'(0) < 1/2$. By (30), $\lim_{t \to \infty} b(t) = \lim_{t \to \infty} x'(t) = 1$, note that there is a unique $T_1 > 0$ such that $b(T_1) = 1/2$ because the function $b(t)$ is strictly increasing. Hence, by (30)

$$1 - \frac{2}{B_0 e^{C_1 T_1} + 1} < \frac{1}{2},$$

which implies, $T_1 < \frac{1}{C_1} \log \left( \frac{3}{B_0} \right) < \frac{2}{C_1} \log 3$, since $B_0 > \frac{1}{3}$.

Take $t > \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\}$, we have

$$\frac{1}{t} \int_0^t K(x(s)) \, ds = \frac{1}{t} \int_0^{T_1} K(x(s)) \, ds + \frac{1}{t} \int_{T_1}^t K(x(s)) \, ds \leq \frac{1}{t} \int_{T_1}^t K(x(s)) \, ds.$$

Now consider the geodesic $\beta(t) = \gamma(t + T_1)$ and apply the inequality in the Case 3, we have

$$\int_0^{t-T_1} K(x(s+T_1)) \, ds = \int_{T_1}^t K(x(s)) \, ds \leq \frac{\eta}{4T}(t - T_1).$$

Hence,

$$\frac{1}{t} \int_0^t K(x(s)) \, ds \leq \frac{1}{t} \int_{T_1}^t K(x(s)) \, ds \leq \frac{\eta}{4T} \left( 1 - \frac{T_1}{t} \right) \leq \frac{\eta}{8T}.$$

**Case 5:** $-1 < b(0) = x'(0) < -1/2$.

By (30), $\lim_{t \to \infty} b(t) = \lim_{t \to \infty} x'(t) = 1$, note that there is a unique $T_2 > 0$ such that $b(T_2) = -1/2$ because the function $b(t)$ is strictly increasing. Using again the inequality (30)

$$-\frac{1}{2} < 1 - \frac{2}{B_0 e^{C_2 T_2} + 1},$$

which implies,

$$T_2 > \frac{1}{C_2} \log \left( \frac{3B_0}{2} \right).$$

Note that $B_0 \to 0$ when $b(0) \to -1$. In particular, $T_2 \to +\infty$ as $b(0) \to -1$. So, let us start first suppose that $T_2 \leq 4T$. In this case, take $t > 4T + \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\}$.

We have

$$\frac{1}{t} \int_0^t K(x(s)) \, ds = \frac{1}{t} \int_0^{T_2} K(x(s)) \, ds + \frac{1}{t} \int_{T_2}^t K(x(s)) \, ds \leq \frac{1}{t} \int_{T_2}^t K(x(s)) \, ds.$$

Observe that $t - T_2 > \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\}$. Now consider the geodesic $\beta(t) = \gamma(t + T_2)$ and apply the inequality in the Case 4, we have
\[
\int_{T_2}^{t} K(x(s)) \, ds \leq \frac{\eta}{8T} (t - T_2).
\]
Hence,
\[
\frac{1}{t} \int_{0}^{t} K(x(s)) \, ds \leq \frac{1}{t} \int_{T_2}^{t} K(x(s)) \, ds \\
\leq \frac{\eta}{8T \left(1 - \frac{T_2}{t}\right)}.
\]
Observe that \( t > 8T \geq 2T_2 \) thus \( T_2/t < 1/2 \). Therefore,
\[
\frac{1}{t} \int_{0}^{t} K(x(s)) \, ds < \frac{\eta}{16T}.
\]
Now suppose that \( T_2 > 4T \). For \( 4T < t \leq T_2 \) we have that \(-1 < x'(t) \leq -1/2\). In particular, \( x(t) \leq x(0) - 1/2t \) for \( 4T < t \leq T_2 \). Hence,
\[
\frac{1}{t} \int_{0}^{t} K(x(s)) \, ds = \frac{1}{t} \int_{x(0)}^{x(t)} \frac{K(u)}{x'(x^{-1}(u))} \, du \\
= \frac{1}{t} \int_{x(0)}^{x(t)} \frac{K(u)}{x'(x^{-1}(u))} \, du \\
\leq \frac{1}{t} \int_{x(0)}^{x(t)} K(u) \, du. \\
= \frac{1}{t} \int_{x(0)}^{x(t)} K(u) \, du + \frac{1}{t} \int_{x(0)-1/2t}^{x(0)} K(u) \, du. \\
\leq \frac{1}{t} \int_{x(0)-1/2t}^{x(0)} K(u) \, du.
\]
We can to write \( t/2 = n_t T + a \) where \( n_t \) is a positive integer number and \( 0 \leq a < T \). Hence,
\[
\frac{1}{t} \int_{0}^{t} K(x(s)) \, ds \leq \frac{1}{t} \int_{x(0)-1/2t}^{x(0)} K(u) \, du \\
\leq \frac{1}{t} \sum_{i=1}^{n_t} \int_{x(0)-iT}^{x(0)-(i-1)T} K(u) \, du + \frac{1}{t} \int_{x(0)-1/2t}^{x(0)-n_t T} K(u) \, du \\
\leq \frac{1}{t} \sum_{i=1}^{n_t} \int_{x(0)-iT}^{x(0)-(i-1)T} K(u) \, du \\
= \frac{\eta n_t}{t} \\
= \frac{\eta}{2T} \left( \frac{\alpha T}{t} \right) \\
< \frac{\eta}{4T},
\]
because \( t > 4T \). If \( T_2 < t \leq T_2 + \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\} \) we have

\[
\frac{1}{t} \int_0^t K(x(s)) \, ds < \frac{1}{t} \int_0^{T_2} K(x(s)) \, ds < \frac{\eta T_2}{4T t}.
\]

Set \( A = \frac{2}{C_1} \log 3 \). Since \( T_2 > 4T \) we have

\[
\frac{T_2 + 2A}{T_2} < \frac{4T + 4T + 2A}{4T},
\]

which provides

\[
\frac{T_2}{t} \geq \frac{T_2}{T_2 + \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\}} > \frac{T_2}{T_2 + 2A + 4T} > \frac{4T}{8T + 2A}.
\]

Therefore,

\[
\frac{1}{t} \int_0^t K(x(s)) \, ds < \frac{\eta}{8T + 2A}.
\]

If \( t > T_2 + \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\} \), consider the geodesic \( \beta(t) = \gamma(t + T_2) \) and apply again the inequality in the Case 4, we have

\[
\int_0^t K(x(s)) \, ds = \int_0^{T_2} K(x(s)) \, ds + \int_{T_2}^t K(x(s)) \, ds
\]

\[
< \frac{\eta T_2}{4T} + \frac{\eta(t - T_2)}{8T}
\]

\[
= \frac{\eta t}{8T} + \frac{\eta T_2}{8T} < \frac{\eta t}{8T}.
\]

Thus,

\[
\frac{1}{t} \int_0^t K(x(s)) \, ds < \frac{\eta}{8T}.
\]

Therefore, we prove that if \( t > 4T + \max \left\{ \frac{2}{C_1} \log 3 + 4T, \frac{4}{C_1} \log 3 \right\} \) then

\[
\frac{1}{t} \int_0^t K(x(s)) \, ds < \max \left\{ \frac{\eta}{16T}, \frac{\eta}{8T + 2A} \right\} < 0,
\]

for any geodesic \( \gamma(t) = (x(t), z(t)) \). Therefore by Theorem 1.2 follows that the geodesic flow of \( M = \mathbb{R} \times \mathbb{T}^1 \) is Anosov.

**Remark 4.** For the family \( g_a(x) = ax - \cos x + \sin x \), we have the new family

\[
h_a(x) = g''_a(x) + (g'_a(x))^2 = 1 + a^2 + 2a(\sin x + \cos x) + 2 \sin x \cos x.
\]
The function $h_a(x)$ are periodic with period $2\pi$. Moreover, for a large enough, we have $h_a(x) > 0$. Thus, in the interval $[0, 2\pi]$ the function $h_a$ is positive and has a minimum. So, continuity of the function family $h_a$ in the parameter $a$, there exists the parameter

$$\bar{a} = \inf\{a \in \mathbb{R}^+: h_a|_{[0,2\pi]} \geq 0 \text{ and its minimum value is } 0\}.$$  

In conclusion, the function $h_{\bar{a}}(x)$ is non-negative and by periodicity, it has an infinite many zeros.

Therefore, by (27) the surface $M = \mathbb{R} \times \mathbb{S}^1$, where $f(x) = e^{g(x)}$ has non-positive curvature with infinite many points of zero curvature and whose geodesic flow is Anosov.

### 5.4 Example 2

Finally, in this section we use the warped product to show a family of non-compact surface with negative curvature and whose geodesic flow is not Anosov. In fact, consider the surface $M = \mathbb{R} \times f\mathbb{S}^1$, where $f(x) = e^{g(x)}$ and $g(x) = \frac{1}{p(x)}$ where $p(x)$ a polynomial of degree $n$ such that $p(x) \neq 0$ for any $x \in \mathbb{R}$. Observe that the curvature in the point $(x, y)$ is given by

$$K(x, y) = K(x) = -f''/f = -(g''(x) + (g'(x))^2) = -\left(\frac{2(p'(x))^2 - p''(x)}{(p(x))^2}\right).$$

Thus, as $p(x)$ is a polynomial, then $\lim_{|x| \to +\infty} \frac{2(p'(x))^2 - p''(x)}{(p(x))^2} = 0$. Therefore, to get a surface of negative curvature, it is sufficient to choice a polynomial $p(x)$ such that $2(p'(x))^2 - p''(x) > 0$. By example, consider a one-parameter family of polynomial $p(x) = -ax^2 + x - 1$, $a > 1/4$, which satisfies the desired property.

In conclusion, $M$ is a non-compact complete surface with non-positive curvature, which is asymptotically flat surface, i.e., $K(t) \to 0$ as $t \to \infty$ (see Corollary 3). So, from Corollary 3 the geodesic flow of $M$ is not Anosov.

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