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Sigma Coloring and Edge Deletions

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Abstract: A vertex coloring \( c : V(G) \rightarrow \mathbb{N} \) of a non-trivial graph \( G \) is called a sigma coloring if \( \sigma(u) \neq \sigma(v) \) for any pair of adjacent vertices \( u \) and \( v \). Here, \( \sigma(v) \) denotes the sum of the colors assigned to vertices adjacent to \( v \). The sigma chromatic number of \( G \), denoted by \( \sigma(G) \), is defined as the fewest number of colors needed to construct a sigma coloring of \( G \). In this paper, we consider the sigma chromatic number of graphs obtained by deleting one or more of its edges. In particular, we study the difference \( \sigma(G) - \sigma(G-e) \) in general as well as in restricted scenarios; here, \( G-e \) is the graph obtained by deleting an edge \( e \) from \( G \). Furthermore, we study the sigma chromatic number of graphs obtained via multiple edge deletions in complete graphs by considering the complements of paths and cycles.

Keywords: sigma coloring, edge deletion, neighbor-distinguishing coloring, complement

1. Introduction

A neighbor-distinguishing graph coloring is a coloring of the vertices and/or edges of a graph that induces a vertex labelling under which any pair of adjacent vertices is assigned different labels. The most studied example of a neighbor-distinguishing coloring is the well-studied proper vertex coloring. Several neighbor-distinguishing colorings have been introduced and studied in the literature such as in Refs. [2] and [5]. In Ref. [4], Chartrand, Okamoto, and Zhang introduced a new kind of neighbor-distinguishing vertex coloring defined as follows.

Definition 1 (Chartrand et al. [4]). For a non-trivial connected graph \( G \), let \( c : V(G) \rightarrow \mathbb{N} \) be a vertex coloring of \( G \). For each \( v \in V(G) \), the color sum of \( v \), denoted by \( \sigma(v) \), is defined to be the sum of the colors of the vertices adjacent to \( v \). If \( \sigma(u) \neq \sigma(v) \) for every two adjacent \( u, v \in V(G) \), then \( c \) is called a sigma coloring of \( G \). The minimum number of colors required in a sigma coloring of \( G \) is called its sigma chromatic number and is denoted by \( \sigma(G) \).

The notion of sigma coloring is related to the vertex colorings/labellings discussed in Refs. [1], [8], [11]. These colorings/labellings also use the sum of the colors/labels of a vertex’s neighbors. Sigma colorings of different families of graphs have already been studied in Refs. [4], [6], and [9].

In this paper, we study the sigma chromatic number in relation to edge deletion. Let \( G = (V,E) \) be a graph. Let \( V' \subseteq V \) and \( E' \subseteq E \). We denote by \( G - V' \) the graph obtained by deleting from \( G \) all vertices in \( V' \) and all edges with at least one end vertex in \( V' \). Moreover, we denote by \( G - E' \) the graph obtained by deleting from \( G \) all edges in \( E' \). For simplicity, when \( V' \) or \( E' \) is a singleton, say \( k \), we denote \( G - V' \) or \( G - E' \) simply by \( G - k \).

Previous work has been done on chromatic numbers in relation to edge deletion. For instance, it is well-known that \( 0 \leq \chi(G) - \chi(G-e) \leq 1 \). In Ref. [10], the notion of critical edges (and vertices) was considered and defined as follows: An edge (or vertex) in a graph is critical if its deletion reduces the chromatic number of the graph by one. The paper studied the complexity of the problem of testing for the existence of critical vertices and edges in \( H \)-free graphs and showed that an edge in a graph is critical if and only if its contraction reduces the chromatic number by one.

In Ref. [7], \( b \)-colorings were studied in relation to edge-deleted subgraphs. A \( b \)-coloring of a graph \( G \) with \( k \) colors is a proper coloring of \( G \) that uses \( k \) colors such that for each color class, there is a vertex that has a neighbor in each of the other color classes. The \( b \)-chromatic number of \( G \), denoted by \( b(G) \), is the largest positive integer \( k \) for which \( G \) has a \( b \)-coloring using \( k \) colors. In Ref. [7], it was shown that \( b(G) - b(G-e) \geq 2 - \left\lfloor \frac{1}{2} \right\rfloor \).

In Ref. [2], Chartrand et al. studied edge deletion in relation to another neighbor-distinguishing coloring called set coloring. Let \( c : V(G) \rightarrow \mathbb{N} \) be a vertex coloring of a non-trivial connected graph \( G \) and denote by \( NC(x) \) the set of colors assigned to vertices adjacent to \( x \). Then \( c \) is a called a set coloring if \( NC(u) \neq NC(v) \) for any pair of adjacent vertices \( u \) and \( v \). The set chromatic number of \( G \), denoted by \( \chi_{S}(G) \), is defined as the least number of colors needed to construct a set coloring of \( G \). Since a set coloring induces a proper vertex coloring using the neighborhood of each vertex, the removal of a neighbor from two vertices) on the set chromatic number. In Ref. [2], Chartrand et al. proved the following:

Theorem 2 (Ref. [2]).

(1) If \( e \) is an edge of a graph \( G \), then

\[ |\chi_{S}(G) - \chi_{S}(G-e)| \leq 2. \]
2. Sigma Coloring and Edge Deletion

Our first result is on the bounds for $\sigma(G) - \sigma(G - e)$ for general $G$. The result is analogous to the result in Theorem 2.

Theorem 3. If $e = uv$ is an edge of a graph $G$, then

$$|\sigma(G) - \sigma(G - e)| \leq 2.$$  

Proof. We first show that $\sigma(G - e) - \sigma(G) \leq 2$. Let $c$ be a sigma coloring of $G$ that uses $\sigma(G)$ colors. We will show that $G - e$ can be sigma colored using $\sigma(G) + 2$ colors. Define the coloring $\overline{c}$ on $G - e$ as follows:

$$\overline{c}(x) = \begin{cases} c(x), & x \notin \{u, v\} \\ c(x) + S, & x \in \{u, v\}, \end{cases}$$

where $S := \sum_{e \in V(G)} c(x)$. Note that $\overline{c}$ uses at most $\sigma(G) + 2$ colors. For a vertex $x \in V(G - e)$, we denote by $\overline{c}(x)$ the color sum of $x$ with respect to $\overline{c}$. Then since $\sigma(x) \leq S - c(x) < S$ for every $x \in V(G)$, we have $\overline{c}(u) = \sigma(u) - c(e) < S$ and $\overline{c}(v) = \sigma(v) - c(u) < S$. If $y$ is adjacent to $u$ or $v$ (possibly both), then it is clear that $\overline{c}(y) = \sigma(y) + S > S$ or $\overline{c}(y) = \sigma(y) + 2S > S$; and so $\overline{c}(y) \notin \{\overline{c}(u), \overline{c}(v)\}$. Now, suppose that $x_1$ and $x_2$, where both $x_1$ and $x_2$ are not $u$ nor $v$, are adjacent in $G - e$. Then exactly one of the following holds for $x_1$ (resp. $x_2$): (1) it is not adjacent to both $u$ and $v$, (2) it is adjacent to $u$ or $v$ but not both, or (3) it is adjacent to both $u$ and $v$. Thus,

$$\overline{c}(x_1) \in \{\sigma(x_1), \sigma(x_1) + S, \sigma(x_1) + 2S\}$$

and

$$\overline{c}(x_2) \in \{\sigma(x_2), \sigma(x_2) + S, \sigma(x_2) + 2S\}.$$  

Since $\sigma(x_1) \neq \sigma(x_2)$ and by the definition of $S$, it follows that $\overline{c}(x_1) \neq \overline{c}(x_2)$. Hence, $\overline{c}$ is a sigma coloring of $G - e$ that uses at most $\sigma(G) + 2$ colors.

Now, we show that $\sigma(G) - \sigma(G - e) \leq 2$. Let $c$ be a sigma coloring of $G - e$ that uses $\sigma(G - e)$ colors. We will show that $G$ can be sigma colored at most $\sigma(G - e) + 2$ colors. Note that the addition of an edge $e$ to $G - e$ (to form $G$) changes the color sums of only $u$ and $v$. Define the coloring $\overline{c}$ on $G$ as follows:

$$\overline{c}(x) = \begin{cases} c(x), & x \notin \{u, v\} \\ c(x) + S, & x \equiv u, \\ c(x) + 2S, & x \equiv v, \end{cases}$$

where $S := \sum_{e \in V(G - e)} c(x)$. Note that $\overline{c}$ uses at most $\sigma(G - e) + 2$ colors. Again, for a vertex $x \in V(G)$, we denote by $\overline{\sigma}(x)$ the color sum of $x$ with respect to $\overline{c}$. We have $\sigma(x) < S$ for every $x \in V(G - e) = V(G)$. Also, $0 < \sigma(u) + c(v) \leq S$ and $0 < \sigma(v) + c(u) \leq S$ since $uv \notin E(G - e)$. It follows that

$$2S < \overline{\sigma}(u) = \sigma(u) + c(v) + 2S \leq 3S$$

and

$$S < \overline{\sigma}(v) = \sigma(v) + c(u) + S \leq 2S.$$  

Thus, $\overline{\sigma}(u) \neq \overline{\sigma}(v)$.

Now, suppose $y$ is a vertex that is neither $u$ nor $v$.

- If $y$ is adjacent to $u$ but not to $v$, then $\overline{\sigma}(y) = \sigma(y) + S \leq 2S < \overline{\sigma}(u)$.
- If $y$ is adjacent to $v$ but not to $u$, then $\overline{\sigma}(y) = \sigma(y) + 2S > 2S > \overline{\sigma}(v)$.
- If $y$ is adjacent to both $u$ and $v$, then $\overline{\sigma}(y) = \sigma(y) + 3S$, which is clearly strictly greater than both $\overline{\sigma}(u)$ and $\overline{\sigma}(v)$.

Now, suppose $x_1$ and $x_2$, both not $u$ nor $v$, are adjacent in $G$, then $x_1$ and $x_2$ are also adjacent in $G - e$. Similar to the previous argument, we have

$$\overline{\sigma}(x_1) \in \{\sigma(x_1), \sigma(x_1) + S, \sigma(x_1) + 2S, \sigma(x_1) + 3S\}$$

and

$$\overline{\sigma}(x_2) \in \{\sigma(x_2), \sigma(x_2) + S, \sigma(x_2) + 2S, \sigma(x_2) + 3S\}.$$  

Since $\sigma(x_1) \neq \sigma(x_2)$ and by the definition of $S$, it follows that $\overline{\sigma}(x_1) \neq \overline{\sigma}(x_2)$. Hence, $\overline{c}$ is a sigma coloring of $G$ that uses at most $\sigma(G) + 2$ colors. \hfill $\Box$

Example 4. For all $m \geq 6$ and $k \in [-1, 0]$, there is a connected graph $G$, with order $m$, that has an edge $e$ so that $G - e$ is connected and $\sigma(G) - \sigma(G - e) = k$.

Proof. Consider the graph $G$ given below.

Clearly, $\sigma(G) = 1$. Moreover, $\sigma(G - e_1) = 1$ and $\sigma(G - e_2) = 2$.

In the above example, we considered only $-1$ and $0$ as values for $k$. The case where $k = 1$ or $k = 2$ is addressed in the following. We study the existence of sequences of edge deletions each of which decreases the sigma chromatic number of a graph by one. We consider this problem for path complements, which we define as follows:

![Fig. 1 The graph $G$ with order $4 + n$.]
Definition 5. The complement of a path $P_m$, $m \geq 2$, in the complete graph $K_n$, $n \geq m$, is the graph obtained by deleting the edges of a subgraph of $K_n$ that is isomorphic to $P_m$. This graph is denoted by $\overline{P}_m$.

As an example, the graph $\overline{P}_{3,2}$ is shown in Fig. 2 where the deleted edges are indicated using dashed segments.

Observation 6. It is easy to see that $\overline{P}_2$, $n \geq 3$, has sigma chromatic number $n - 2$; that is, deleting one edge from $K_n$ decreases the sigma chromatic number by two.

As a consequence of Proposition 3.1 in Ref. [4], it is worth noting that there is no sequence of edge deletions in $K_n$ that will decrease the sigma chromatic number to $n - 1$.

Our result on the sigma chromatic number of path complements is the following.

Proposition 7. For $n \geq 4$ and $m = 2, 3, \ldots, [n/2],\n\sigma(\overline{P}_{m,n}) = n - m.$

Proof. First, note that the graph $\overline{P}_{m,n}$ has exactly one subgraph $S$ that is isomorphic to $K_{m,n}$. Moreover, for each $s \in V(S)$, $N(s) = V(\overline{P}_{m,n})$. Hence, $\sigma(\overline{P}_{m,n}) \geq n - m$.

We are now left to show that $\overline{P}_{m,n}$ has a sigma coloring that uses $n - m$ colors. Let $c$ be a sigma coloring of $K_{m,n}$; naturally, $c$ uses $n$ colors. Moreover, by setting $d = \Delta(K_{m,n}) + 1 = n$, we can choose the colors used by $c$ to be

$$1, d, d^2, \ldots, d^{n-1}.$$ 

We proceed by considering the following cases.

Case 1. Suppose $n = 5$ and $m = [n/2] = 3$. This case pertains to $\overline{P}_{3,5}$, for which it is easy to verify that the sigma chromatic number is 5 - 3 = 2.

Case 2. Suppose $n \geq 7$ is odd and $m = [n/2]$. Let $a$ and $b$ be the endvertices of the path $P_m$ whose edges were deleted from $K_n$ to form $\overline{P}_{m,n}$. Construct the coloring $\bar{c}$ on $\overline{P}_{m,n}$ as follows: if $x \in V(S)$, set $\bar{c}(x) = c(x)$; moreover, we define $\bar{c}$ on $V(\overline{P}_{m,n}) - V(S)$ so that

1. $\bar{c}(V(\overline{P}_{m,n}) - V(S)) \subseteq \bar{c}(S)$,
2. $\bar{c}(a) = \bar{c}(b)$,
3. $\bar{c}(x) \not= \bar{c}(a)$ for all $x \in V(\overline{P}_{m,n}) - V(S)$, and
4. $\bar{c}(x) \not= \bar{c}(y)$ for all $x, y \in V(\overline{P}_{m,n}) - V(S)$.

Note that such a coloring is possible since the vertices in $V(\overline{P}_{m,n}) - V(S)$ use only $m - 1$ colors and $m - 1 = [n/2] - 1 = n - m = |V(S)|$.

We now show that $\bar{c}$ is a sigma coloring. Suppose $x_1$ and $x_2$ are adjacent in $\overline{P}_{m,n}$.

- Case 2.1: Suppose $x_1$ and $x_2$ are both in $V(S)$. Then $\bar{c}(x_1) = \sigma(x_1)$ and $\bar{c}(x_2) = \sigma(x_2)$; hence, $\bar{c}(x_1) \not= \bar{c}(x_2)$.

- Case 2.2: Suppose $x_1$ is in $V(S)$ while $x_2$ is in $V(\overline{P}_{m,n}) - V(S)$. Then $\deg x_1 = n - 1$ while $\deg x_2 = n - 2$. By the choice of colors of $c$, $\sigma(x_1) \not= \sigma(x_2)$.

- Case 2.3: Suppose $x_1 = a$ and $x_2 = b$. Then $\deg x_1 = \deg x_2 = n - 2$. Since $m \geq 4$, then $x_1$ and $x_2$ do not have the same neighbors in $V(\overline{P}_{m,n}) - V(S)$. By the construction of $\bar{c}$, $\sigma(x_1) \not= \sigma(x_2)$.

- Case 2.4: Suppose $x_1 \in \{a, b\}$ and $x_2 \in V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$. Then $\deg x_1 = n - 2$ and $\deg x_2 = n - 3$. By the choice of colors of $c$, $\sigma(x_1) \not= \sigma(x_2)$.

- Case 2.5: Suppose $x_1$ and $x_2$ are both in $V(\overline{P}_{m,n}) - (V(S) \cup \{a, b\})$. Then $\deg x_1 = \deg x_2 = n - 3$ and $\sigma(x_1) \not= \sigma(x_2)$. Hence, $\sigma(x_1) \not= \sigma(x_2)$.

Therefore, $\bar{c}$ is a sigma coloring of $\overline{P}_{m,n}$ that uses $n - m$ colors.

Proposition 7 implies the following: Consider a subgraph of $K_n$ isomorphic to a path $P_m$ with $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m$, where each $v_i$ is a vertex of $K_n$. The deletion of edge $v_i v_{i+1}$ decreases the sigma chromatic number by two. Then in the sequence of deletions of edges $v_i v_{i+1}$, where $i$ runs from 2 to $m - 1$, each edge deletion decreases the sigma chromatic number by one. This is illustrated for $K_7$ in Fig. 3. For comparison, the same sequence of edge deletions in Fig. 3 produces the following sequence of chromatic numbers: $\chi = 6, \chi = 6, \chi = 5$.

Example 4, Observation 6, and Proposition 7 imply the following:

Corollary 8. For each $m \geq 6$ and for each $k \in \{-1, 0, 1, 2\}$, there is a connected graph $G$, with order $m$, that has an edge $e$ for which $G - e$ is connected and $\sigma(G) - \sigma(G - e) = k$.

We have not found a graph $G$ that has an edge $e$ for which $\sigma(G) - \sigma(G - e) = -2$. But as in Ref. [2], we have also found sufficient conditions for the inequality $\sigma(G) - \sigma(G - e) \geq -1$ to hold.

Theorem 9. Let $e = uv$ be an edge in a graph $G$. If $e$ is a bridge or $d_{G-e}(u, v) \geq 4$, then $\sigma(G) - \sigma(G - e) \geq -1$.

Proof. Let $c$ be a sigma coloring of $G$ that uses $\sigma(G)$ colors. We will show that $G - e$ can be colored using $\sigma(G) + 1$ colors. Define
create a $u - v$ path of length 2 or 3. Moreover, $\sigma(x) \neq \sigma(y)$ since $x$ and $y$ are also adjacent in $G$.

**Case 3.** Suppose $x \in N(u)$ and $y \in N(u)$. Then $\overline{\sigma}(x) = \sigma(x) - c(u) + S = \bar{\sigma}(y) - c(u) + S = \bar{\sigma}(y)$.

**Case 4.** Suppose $x \in N(u)$ and $y \notin N(u)$. Then $\overline{\sigma}(x) = \sigma(x) - c(u) + S \neq \sigma(y) = \bar{\sigma}(y)$.

**Case 5.** Suppose $x \notin N(u)$ and $y \notin N(u)$. Then $\overline{\sigma}(x) = \sigma(x) \neq \sigma(y) = \bar{\sigma}(y)$.

Therefore, $\overline{\sigma}$ is a sigma coloring of $G - e$ that uses $\sigma(G) + 1$ colors.

In the following, we consider edge deletions in regular graphs of order at least 2.

**Proposition 10.** Suppose $G$ is a connected regular graph with order at least 2.

(1) For any edge $e = uv$ in $G$, $\sigma(G - e) \leq \sigma(G)$.

(2) If $G$ is not complete and $e = uv \notin E(G)$, then $\sigma(G + e) \leq \sigma(G) + 1$.

**Proof.**

(1) Suppose $c$ is a sigma coloring of $G$ that uses $\sigma(G)$ colors. Let $\overline{\sigma}$ be the coloring of $G - e$ so that $\overline{\sigma}(x) = c(x)$ for each $x \in V(G - e) = V(G)$. We show that $\overline{\sigma}$ is a sigma coloring of $G - e$. First, $\overline{\sigma}(x) = \sigma(x)$ for each $x \notin \{u, v\}$. Let $x$ and $y$ be adjacent vertices in $G - e$. If they have different degrees, then $\overline{\sigma}(x) \neq \bar{\sigma}(y)$ (possibly needing a change of colors as in the proof of Theorem 9). If they have equal degrees, then $\overline{\sigma}(x) = \sigma(x) = \bar{\sigma}(y)$.

(2) Let $c$ be a sigma coloring of $G$ that uses $\sigma(G)$ colors. Let $\overline{\sigma}$ be the coloring of $G + e$ where $\overline{\sigma}(x) = c(x)$ if $x \neq u$ and $\overline{\sigma}(v) = \overline{\sigma}(v) = S := \sum_{x \in V(G)} \sigma(x)$. Let $x$ and $y$ be adjacent vertices of $G + e$ with equal degrees. Then $\{x, y\} \cap \{u, v\} = \emptyset$.

(a) If $x$ and $y$ are both not in $N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) = \sigma(y) = \bar{\sigma}(y)$.

(b) If $x$ and $y$ are both in $N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) = \bar{\sigma}(y)$.

(c) If exactly one of $x$ and $y$ is in $N_G(v)$, say $x \in N_G(v)$ and $y \notin N_G(v)$, then $\overline{\sigma}(x) = \sigma(x) = \bar{\sigma}(y)$.

This also covers the case where $\{x, y\} = \{u, v\}$.

\[ \square \]

### 3. On the Sigma Chromatic Number of Complements of Paths and Cycles

In this section, we determine a lower bound for the sigma chromatic number of the complement of a cycle or a path. For convenience, we introduce the following notations. For a cycle $C_n = v_1 v_2 \cdots v_{n/2} v_{n/2}$, $n \geq 3$ and for each $k = 1, 2, \ldots, [n/2]$, we denote by $A_k$ the triple of vertices $(v_{2k-1}, v_{2k}, v_{2k+1})$ and by $B_k$ the triple of vertices $(v_{2k-2}, v_{2k-1}, v_{2k})$ (Note that the subscripts are computed modulo $n$). For example, in $C_7 = v_1 v_2 v_3 v_4 v_5 v_6 v_1$, we have

$A_1 = (v_1, v_2, v_3), \quad A_2 = (v_3, v_4, v_5), \quad A_3 = (v_5, v_6, v_1)$

and

$B_1 = (v_1, v_2, v_3), \quad B_2 = (v_2, v_3, v_4), \quad B_3 = (v_4, v_5, v_6)$.

Given an ordered triple $T$ of vertices (e.g., some $A_k$ or $B_k$) and a vertex coloring $c$ of $C_n$ or $\overline{C_n}$, we denote by $c(T)$ the multiset of
colors used in the vertices in \( T \). Note that \( c(T) \) is a multiset and not an ordered triple. The following is an important observation.

**Observation 11.** If \( c \) is a sigma coloring of \( C_m \), then for any triple \( T \) and \( T' \) of consecutive vertices in \( C_m \), we must have \( c(T) \neq c(T') \) if \( |T \cap T'| \leq 1 \). In particular, for any distinct \( k, j \), we must have \( c(A_k) \neq c(A_j) \) and \( c(B_k) \neq c(B_j) \).

The above observation follows from the fact that if \( v \) is the middle vertex in a triangle \( T \), then \( \sigma(v) = S - \sum_{x \in c(T)} c(x) \), where \( S \) is the sum of all colors.

**Proposition 12.** Let \( m \) be a positive integer and set \( M = \left( \frac{m+2}{3} \right) \).

Then \( \sigma(C_m) > m \) for all \( n \geq 2M + 1 \).

**Proof.** Suppose \( c \) is a vertex coloring of \( C_m \) that uses \( m \) colors. Moreover, assume that the colors are \( 1, d, d^2, \ldots, d^{m-1}, \) where \( d = n - 2 \). Then the number of 3-multisets that can be formed using these \( m \) colors (repetition of colors allowed) is \( M \). By the choice of colors, it also follows that there are \( M \) possible color sums.

Suppose \( n \geq 2M + 2 \). Then \( \left\lfloor \frac{n}{2} \right\rfloor > \frac{n}{2} - 1 \geq M \). By Observation 11, we must have \( M \geq \left\lfloor \frac{n}{2} \right\rfloor \). Therefore, \( c \) is not a sigma coloring of \( C_m \) and \( \sigma(C_m) > m \).

Now, suppose \( n = 2M + 1 \). Then \( n/2 = M \). For \( c \) to be a sigma coloring, by Observation 11, \( c(A_1), c(A_2), \ldots, c(A_M) \) must be distinct triples. Furthermore, \( c(B_1) \) must be distinct from \( c(A_2), c(A_3), \ldots, c(A_M) \). Then \( c(B_1) = c(A_1) \). Similarly, \( c(B_2) \) must be distinct from \( c(A_1), c(A_3), \ldots, c(A_M) \) and \( c(B_1) = c(A_1) \); thus, \( c(B_2) = c(A_2) \). Proceeding in this manner, we conclude that we must have \( c(A_k) = c(B_k) \) for all \( k = 1, 2, \ldots, M \). Now, consider the triple \( T = (2M, 2M+1, v_1) \). Again, for \( c \) to be a sigma coloring, we must have \( c(T) \) distinct from \( c(A_1), c(A_2), \ldots, c(A_{M-1}) \) and \( c(B_M) = c(A_M) \). But since \( T \) is a triple not in \( A_k, B_1 : k = 1, 2, \ldots, M \), \( c(T) \) will have to be one of \( c(A_1), c(A_2), \ldots, c(A_{M-1}), c(A_M) \), which implies that \( c \) is not a sigma coloring of \( C_m \). Therefore, \( \sigma(C_m) > m \).

We now turn to the complements of paths. Suppose \( P_n = v_1, v_2, \ldots, v_n, n \geq 3 \). Note that the vertices \( v_2, v_3, \ldots, v_{n-1} \), which are of degree \( n - 3 \) in \( P_n \), will also have color sums corresponding to 3-multisets of colors. Hence, by arguing in a similar manner as in Proposition 12, we obtain the following.

**Proposition 13.** Let \( m \) be a positive integer and set \( M = \left( \frac{m+2}{3} \right) \).

Then \( \sigma(P_n) > m \) for all \( n \geq 2M + 3 \).

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