ASSOCIATED PRIME SUBMODULES OF FINITELY
GENERATED MODULES

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Abstract. Let \( R \) be a commutative ring with identity. For a finitely generated
\( R \)-module \( M \), the notion of associated prime submodules of \( M \) is defined. It is
shown that this notion inherits most of essential properties of the usual notion of
associated prime ideals. In particular, it is proved that for a Noetherian multiply-

cation module \( M \), the set of associated prime submodules of \( M \) coincides with
the set of \( M \)-radicals of primary submodules of \( M \) which appear in a minimal
primary decomposition of the zero submodule of \( M \). Also, Anderson’s theorem
\([2]\) is extended to minimal prime submodules in a certain type of modules.

1. Introduction

Recently, extensive research has been done on prime submodules. Let \( R \) be a
commutative ring with identity and \( M \) an \( R \)-module. A proper submodule \( N \) of \( M \)
is said to be prime (or \( p \)-prime) if \( re \in N \) for \( r \in R \) and \( e \in M \), implies that either
\( e \in N \) or \( r \in p = N : M \). A general theme in the studying of prime submodules
is to extend results concerning prime ideals to prime submodules. For example,
Cohen’s Theorem, Prime Avoidance Theorem and Krull’s Principal Ideal Theorem
are generalized to prime submodules in \([13, \text{Theorem 5}], [8, \text{Theorem 2.3}] \) and \([6,\
\text{Theorem 11}] \) respectively.

The use of the notion of associated prime ideals has found substantial applications
in commutative algebra. In fact, the set of associated prime ideals of a module
contains a lot of information about the module itself. It is natural to expect that,
if there is an appropriate definition of associated prime submodules, then many
results concerning associated prime ideals can be generalized to associated prime
submodules.

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modules.
The main goal of this paper is to define the concept of associated prime submodules of finitely generated modules and to investigate their properties. Surely, there is much more work to be done. Let $M$ be a finitely generated $R$-module and $\mathfrak{p}$ a prime ideal of $R$. Following [11], we set

$$M(\mathfrak{p}) = \{ x \in M : sx \in \mathfrak{p}M, \text{ for some } s \in R \setminus \mathfrak{p} \}.$$ 

It is easy to see that, if $\mathfrak{p}$ contains the annihilator of $M$, then $M(\mathfrak{p})$ is a $\mathfrak{p}$-prime submodule of $M$. We define the set of associated prime submodules of $M$ as

$$\text{Ass}_\mathfrak{p} M = \{ M(\mathfrak{p}) : \mathfrak{p} \in \text{Ass}_R M \},$$

where $\text{Ass}_R M$ denotes the set of associated prime ideals of $M$.

In section 2, some results related to associated prime ideals are extended to associated prime submodules. In particular, it is shown that, if $0 = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition of the zero submodule of the Noetherian multiplication module $M$, then $\text{Ass}_\mathfrak{p} M = \{ \text{rad}(Q_i) : i = 1, 2, \ldots, n \}$. Recall that an $R$-module $M$ is called multiplication, if every submodule $N$ of $M$ is of the form $aM$, for some ideal $a$ of $R$. Also, for a submodule $N$ of the $R$-module $M$, $\text{rad}(N)$, the $M$-radical of $N$, is defined as the intersection of the prime submodules of $M$ containing $N$ (see [10]).

In section 3, we study the set of minimal associated prime submodules of finitely generated modules. Firstly, we introduce the class of quasi multiplication modules. This class of modules contains multiplication, finitely generated weak multiplication and flat modules. Then, we extend Anderson’s theorem to minimal prime submodules of finitely generated quasi multiplication modules to the effect that if all minimal prime submodules of a finitely generated quasi multiplication $R$-module $M$ are finitely generated, then their number is finite.

Throughout this paper, $R$ is a commutative ring with identity and all modules are assumed to be unitary.

2. Associated prime submodules

Recall that the sets of associated and of supported prime ideals of a given $R$-module $M$ are defined respectively as:

$$\text{Ass}_R M = \{ \mathfrak{p} \in \text{Spec } R : \mathfrak{p} = (0 : x), \text{for some nonzero element } x \text{ of } R \},$$

and

$$\text{Supp}_R M = \{ \mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supseteq (0 : x), \text{for some nonzero element } x \text{ of } R \}.$$
It is well known that, if $M$ is finitely generated, then $\text{Supp}_R M$ is the set of all prime ideals of $R$ which contain $\text{Ann}_R M$.

For completeness, we collect some important known properties of these notions in the following lemma.

**Lemma 2.1.**

i) If $R$ is Noetherian, then $M$ is zero if and only if $\text{Ass}_R M$ is empty.

ii) If $R$ is Noetherian, then the set of minimal elements of $\text{Ass}_R M$ and that of $\text{Supp}_R M$ are equal.

iii) If $M$ is Noetherian, then $\text{Ass}_R M$ is finite.

iv) If $M$ is Artinian, then $\text{Ass}_R M = \text{Supp}_R M$, and this set consists of finitely many maximal ideals.

v) If $R$ is Noetherian and $S$ a multiplicatively closed subset of $R$, then $\text{Ass}_{S^{-1}R} S^{-1}M = \{ S^{-1}p : p \in \text{Ass}_R M and p \cap S = \phi \}$.

vi) If $R$ is Noetherian and $0 = \bigcap_{i=1}^n Q_i$ is a minimal primary decomposition of the zero submodule of $M$, then $\text{Ass}_R M = \{ \text{rad}(Q_i : M) : i = 1, 2, \ldots, n \}$.

**Proof.** (i) and (ii) hold respectively, by [9, 7.B Corollary 1] and [9, Theorem 9]. Also, (v) and (vi) follow respectively, by [9, 7.C Lemma] and [9, 8.E Lemma]. It is easy to deduce (iv) from [12, Exercise 8.49]. Finally, (iii) is clear by (vi). □

In the sequel, we generalize the above mentioned properties of associated prime ideals to associated prime submodules. Let $M$ be an $R$-module and $p$ a prime ideal of $R$. Following [11], we denote the set $\{ x \in M : sx \in pM, for some s \in R \setminus p \}$, by $M(p)$. We summarize some important properties of this notion in the following lemma and in the sequel, we may use them without further comment.

**Lemma 2.2.** Let $M$ be an $R$-module and $p$ a prime ideal of $R$.

i) $M = M(p)$ or $M(p)$ is a $p$-prime submodule of $M$.

ii) If $M$ is either finitely generated or multiplication, then $M(p)$ is a $p$-prime submodule of $M$ if and only if $p \in \text{Supp}_R M$.

iii) Every $p$-prime submodule of $M$ contains $M(p)$.

**Proof.** (i) holds by [11, Proposition 1.7], while (iii) is true by [11, Lemma 1.6]. Now, we show (ii). Suppose $M$ is either finitely generated or multiplication. It is easy to check that $M = M(p)$ if and only if $M_p = (pR_p)M_p$. By [3, Proposition 1], the assertion of Nakayama’s Lemma holds also for multiplication modules. Thus, in
both cases, it follows that \( M_p = (pR_p)M_p \) if and only if \( M_p = 0 \). Therefore, it turns out by (i), that \( M(p) \) is a \( p \)-prime submodule of \( M \) if and only if \( p \in \text{Supp}_R M \). □

Now, we are ready to present the definitions of associated and of supported prime submodules of a finitely generated module.

**Definition 2.3.**

(i) Let \( M \) be an \( R \)-module. We say \( M \) is **weakly finitely generated**, if for any \( p \in \text{Supp}_R M \), the submodule \( M(p) \) of \( M \) is proper.

(ii) Let \( M \) be a weakly finitely generated \( R \)-module. We define the sets of associated and of supported prime submodules of \( M \), respectively as:

\[
\text{Ass}_p M = \{ M(p) : p \in \text{Ass}_R M \},
\]

and

\[
\text{Supp}_p M = \{ M(p) : p \in \text{Supp}_R M \}.
\]

**Example 2.4.** By Lemma 2.2(ii), it becomes clear that if the \( R \)-module \( M \) is either finitely generated or multiplication, then \( M \) is weakly finitely generated. It is worthy to mention that, by [5, Corollary 3.9] over a Noetherian ring, any multiplication module is finitely generated.

If \( M \) is a Noetherian \( R \)-module, then the ring \( T = R/\text{Ann}_R M \) is a Noetherian ring. One can check easily that \( M \) is a \( T \)-module and that \( \text{Ass}_T M = \{ p/\text{Ann}_R M : p \in \text{Ass}_R M \} \). Hence the following is immediate, by Lemma 2.1.

**Lemma 2.5.** Suppose that \( M \) is a Noetherian \( R \)-module.

i) \( \text{Ass}_p M \) is finite.

ii) \( M = 0 \) if and only if \( \text{Ass}_p M = \phi \).

The set of all prime submodules of the \( R \)-module \( M \) is denoted by \( \text{Spec} M \). Also, the set of maximal submodules of \( M \) is denoted by \( \text{Max} M \).

**Lemma 2.6.** Assume that \( M \) is an \( R \)-module of finite length.

i) \( \text{Ass}_p M = \text{Supp}_p M \) and this set is finite.

ii) Moreover, if \( M \) is a multiplication module, then \( \text{Ass}_p M = \text{Max} M = \text{Spec} M \).

**Proof.** i) follows by Lemma 2.1(iv).

ii) Let \( M(p) \) be an associated prime submodules of \( M \). Then, by Lemma 2.1(iv), \( p \) is a maximal ideal of \( R \). Let \( N \) be a proper submodule of \( M \) containing \( M(p) \). Then

\[
p = (M(p) : M) \subset (N : M) \subsetneq R.
\]
Hence \( p = (N : M) \). But \( M \) is a multiplication module and so

\[ N = (N : M)M = (M(p) : M)M = M(p). \]

Thus \( M(p) \) is a maximal submodule of \( M \).

Because any maximal submodule of \( M \) is a prime submodule, to complete the proof, it suffices to show that any prime submodule of \( M \) is an associated prime submodule. Assume that \( N \) is a prime submodule of \( M \). It follows that \( p = (N : M) \) is an element of \( \text{Supp}_R M \) and so \( p \) belongs to \( \text{Ass}_R M \), by Lemma 2.1(iv). Now, we have \( N = pM = M(p) \), and so \( N \in \text{Ass}_P M \), as required. \( \square \)

**Corollary 2.7.** Let \( M \) be a multiplication module. The following are equivalent:

i) \( M \) is Artinian.

ii) \( M \) is Noetherian and \( \text{Ass}_P M \subseteq \text{Max} M \).

**Proof.** By [5, Corollary 2.9], any Artinian multiplication module is cyclic. Hence (i) implies (ii), by Lemma 2.6, and the fact that every finitely generated Artinian module is Noetherian.

Now, assume (ii) holds. Let \( p \in \text{Ass}_R M \). Then \( M(p) \in \text{Max} M \), by the assumption. This yields that \( p \in \text{Max} R \). Using Lemma 2.1(ii), we can deduce that every prime ideal of the Noetherian ring \( T = R/\text{Ann}_R M \) is maximal, and so \( T \) is an Artinian ring. Now, because \( M \) is finitely generated, we can deduce that \( M \) is Artinian as an \( R \)-module. \( \square \)

**Lemma 2.8.** Let \( S \) be a multiplicatively closed subset of the Noetherian ring \( R \) and let \( M \) be a finitely generated \( R \)-module. Then the set of associated prime submodules of the \( S^{-1}R \)-module \( S^{-1}M \) is equal to \( \{S^{-1}P : P \in \text{Ass}_P M \text{ and } (P : M) \cap S = \phi \} \).

**Proof.** By Lemma 2.1(v), \( \text{Ass}_{S^{-1}R} S^{-1}M = \{S^{-1}p : p \in \text{Ass}_R M \text{ and } p \cap S = \phi \} \). Let \( P \in \text{Ass}_P M \) be such that \( (P : M) \cap S = \phi \). Set \( p = P : M \). Then \( p \) is an associated prime ideal of \( M \) and \( P = M(p) \). It is straightforward to see that \( S^{-1}P = S^{-1}M(S^{-1}p) \), and so \( S^{-1}P \) is an associated prime submodule of the \( S^{-1}R \)-module \( S^{-1}M \).

Conversely, assume that \( Q \) is an associated prime submodule of \( S^{-1}M \) as an \( S^{-1}R \)-module. Then there exists \( p \in \text{Ass}_R M \), with \( p \cap S = \phi \), such that \( Q = S^{-1}M(S^{-1}p) \). Thus \( Q = S^{-1}(M(p)) \) and \( M(p) \in \text{Ass}_P M \). \( \square \)

Now, we are ready to prove the main result of this section.
Theorem 2.9. Let $M$ be a Noetherian multiplication $R$-module. If $0 = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition of the zero submodule of $M$, then $\text{Ass}_P M = \{\text{rad}(Q_i) : i = 1, 2, \ldots, n\}$.

Proof. Set $p_i = \text{rad}(Q_i : M)$, for $i = 1, 2, \ldots, n$. Then by Lemma 2.1(vi), $\text{Ass}_R M = \{p_i : i = 1, 2, \ldots, n\}$. Fix $1 \leq i \leq n$. It turns out by [10, Theorem 4], that $\text{rad}(Q_i) = (\text{rad}(Q_i : M))M$. Now, we have

$$M(p_i) = (M(p_i) : M)M = p_i M.$$ 

Hence $\text{rad}(Q_i) \in \text{Ass}_P M$.

Conversely, assume that $P \in \text{Ass}_P M$. Then there is $1 \leq i \leq n$, such that $P = M(p_i)$. But the assumption on $M$ implies that $M(p_i) = p_i M$. Thus

$$P = (\text{rad}(Q_i : M))M = \text{rad}(Q_i),$$

as required. □

3. Minimal associated prime submodules

An $R$-module $M$ is called a weak multiplication module if every prime submodule $P$ of $M$ is of the form $p M$, for some prime ideal $p$ of $R$ [1]. Next, we present the following definition.

Definition 3.1. An $R$-module $M$ is called a quasi multiplication module if $M(p) = p M$, for all $p \in \text{Supp}_R M$.

Example 3.2. i) Let $M$ be a weakly finitely generated $R$-module which is weak multiplication. Then $M$ is quasi multiplication.

ii) Every flat $R$-module is a quasi multiplication module. To see this, let $p \in \text{Supp}_R M$. If $p M = M$, then $M(p) = p M$, because $p M \subseteq M(p)$, as one can see clearly. Now assume that $p M$ is a proper submodule of $M$. Then $p M$ is a $p$-prime submodule of $M$, by [7, Theorem 3]. Hence $M(p) = p M$, by Lemma 2.2(iii)

iii) It turns out, by Lemma 2.2(ii), that every multiplication $R$-module is quasi multiplication.

The following is clear, by definition of quasi multiplication modules.

Lemma 3.3. Let $M$ be a weakly finitely generated module. If $M$ is quasi multiplication, then $\text{Ass}_P M = \{p M : p \in \text{Ass}_R M\}$ and $\text{Supp}_P M = \{p M : p \in \text{Supp}_R M\}$. 

Proposition 3.4. Let $M$ be a finitely generated $R$-module. Assume that $M$ is quasi multiplication. Then the following hold.

i) The set of minimal prime submodule of $M$ is equal to

$$\{pM : p \text{ is a minimal element of } \text{Supp}_R M\}.$$ 

ii) If $R$ is Noetherian, then the set of minimal elements of the Spec $M$, Supp$_P M$ and Ass$_P M$ are coincide. Consequently, the set of minimal elements of Spec $M$, Supp$_P M$ and Ass$_P M$ all coincide, whenever $M$ is a Noetherian module over an arbitrary ring.

Proof. i) Let $p$ be a minimal element of Supp$_R M$. Since $M$ is a finitely generated quasi multiplication module, it follows that $M(p) = pM$, and so $pM$ is a prime submodule of $M$. Assume that $Q$ is a prime submodule of $M$ such that $Q \subseteq pM$. Then

$$\text{Ann}_R M \subseteq Q : M \subseteq pM : M = p.$$ 

Hence $Q : M = p$, by the assumption on $p$. It turns out by Lemma 2.2(iii), that $Q = M(p) = pM$.

Now, assume that $P$ is a minimal prime submodule of $M$. Let $p = P : M$. Then $P = M(p)$, by Lemma 2.2(iii). Let $q$ be an element of Supp$_R M$ such that $q \subseteq p$. Then, it follows that

$$M(q) = qM \subseteq pM = M(p) = P,$$

and so $M(q) = P$. Thus $p = P : M = M(q) : M = q$, and so $p$ is minimal in Supp$_R M$.

ii) Since Ass$_P M \subseteq$ Supp$_P M \subseteq$ Spec $M$, it is enough to show that every minimal associated prime submodule of $M$ is minimal in Spec $M$. Note that in view of part (i) and Lemma 2.1(ii), it follows that any prime submodule of $M$ contains an element of Ass$_P M$. By Lemma 2.1(ii), the set of minimal elements of Ass$_R M$ and that of Supp$_R M$ are equal. Let $M(p)$ be a minimal element of Ass$_P M$ and $Q$ a minimal prime submodule of $M$, which is contained in $M(p)$. Then by (i), $Q = qM$, for some minimal element $q$ of Supp$_R M$. Hence $q \in$ Ass$_R M$, and so $Q \in$ Ass$_P M$. Therefore, $Q = M(p)$, as required.

Set $T = R/\text{Ann}_R M$. Since $M$ possesses the structure of a $T$-module in a natural way, the last assertion of part (ii) follows immediately. $\square$
Lemma 3.5. Let $M$ be a weakly finitely generated $R$-module and let $\{p_1, \ldots, p_n\}$ be a subset of minimal elements of $\text{Supp}_R M$. If $p_1 \ldots p_n M = 0$, then $p_1, \ldots, p_n$ are the only minimal elements of $\text{Supp}_R M$.

Proof. Let $p$ be a minimal element of $\text{Supp}_R M$. Then $M(p)$ is a $p$-prime submodule of $M$. Thus

$$p_1 \ldots p_n \subseteq M(p) : M = p,$$

and so $p = p_i$, for some $1 \leq i \leq n$. □

Proposition 3.6. Let $M$ be a finitely generated $R$-module. Assume that $pM$ is finitely generated for all minimal elements $p$ of $\text{Supp}_R M$. Then the number of minimal elements of $\text{Supp}_R M$ is finite.

Proof. In view of Lemma 3.5, it suffices to show that there are minimal elements $p_1, \ldots, p_n$ of $\text{Supp}_R M$, such that $p_1 \ldots p_n M = 0$. Suppose that the contrary is true and we search for a contradiction. Let $A$ denote the set of all ideals $a$ of $R$, such that $\text{Ann}_R M \subseteq a$ and $aM$ does not contain any submodule of the form $p_1 \ldots p_n M$, where $p_i$’s are minimal elements of $\text{Supp}_R M$. Then $A$ is not empty and by Zorn’s lemma, we deduce that it has a maximal element $p$, say. Note that, if $a$ and $b$ are two ideals of $R$ such that $aM$ and $bM$ are finitely generated, then the submodule $abM$ is also finitely generated.

We show that $p$ is a prime ideal. To this end, let $x$ and $y$ be two elements of $R$ such that $x \notin p$, $y \notin p$ and $xy \in p$. Then $p + Rx$ and $p + Ry$ are not in $A$. Hence there are minimal elements $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ of $\text{Supp}_R M$ such that $p_1 \ldots p_n M \subseteq (p + Rx)M$ and $q_1 \ldots q_m M \subseteq (p + Ry)M$. Then

$$(p_1 \ldots p_n q_1 \ldots q_m)M \subseteq pM,$$

which is a contradiction. Therefore $p$ is a prime ideal of $R$. Since $p \in \text{Supp}_R M$, it follows that $p$ contains a minimal element $q$ of $\text{Supp}_R M$, and so $qM \subseteq pM$. This contradicts the assumption that $p$ belongs to $A$. □

A result due to Anderson [2], asserts that, if every minimal prime ideal of $R$ is finitely generated, then the number of minimal prime ideals of $R$ is finite. Then Anderson’s result is generalized to prime submodules in multiplication modules by Behboodi and Koohy in [4, Theorem 2]. They proved that if every minimal prime submodule of the multiplication $R$-module $M$ is finitely generated, then $M$ has only
finitely many minimal prime submodules. In view of Proposition 3.4(i) and Proposition 3.6, we can extend Anderson’s theorem to finitely generated quasi multiplication modules. Namely:

**Theorem 3.7.** Let $M$ be a finitely generated quasi multiplication $R$-module. Assume that every minimal prime submodule of $M$ is finitely generated. Then $M$ has only finitely many minimal prime submodules.

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