Unification of Mixed Hilbert-Space Representations in Condensed Matter Physics and Quantum Field Theory

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Abstract

We present a unification of mixed-space quantum representations in Condensed Matter Physics (CMP) and Quantum Field Theory (QFT). The unifying formalism is based on being able to expand any quantum operator, for bosons, fermions, and spin systems, using a universal basis operator ˆY(u, v) involving mixed Hilbert spaces of ˆP and ˆQ, respectively, where ˆP and ˆQ are momentum and position operators in CMP (which can be considered as a bozonization of free Bloch electrons which incorporates the Pauli exclusion principle and Fermi-Dirac distribution), whereas these are related to the creation and annihilation operators in QFT, where ˆψ† = −i ˆP and ˆψ = ˆQ. The expansion coefficient is the Fourier transform of the Wigner quantum distribution function (lattice Weyl transform) otherwise known as the characteristic distribution function.

Thus, in principle, fermionization via Jordan-Wigner for spin systems, as well as the Holstein-Primakoff transformation from boson to the spin operators can be performed depending on the ease of the calculations. Unitary transformation on the creation and annihilation operators themselves is also employed, as exemplified by the Bogoliubov transformation. Moreover, whenever ˆY(u, v) is already expressed in matrix form, M_{ij}, e.g. the Pauli spin matrices, the Jordan-Schwinger transformation is a map to bilinear expressions of creation and annihilation operators which expedites computation of representations.

We show that the well-known coherent states formulation of quantum physics is a special case of the present unification. A new formulation of QFT based on Q-distribution of functional-field variables is suggested. The case of nonequilibrium quantum transport physics, which not only involves non-Hermitian operators but also time-reversal symmetry breaking, is discussed in the Appendix.
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1 Introduction

The canonical conjugate variables formulation of quantum physics has been thoroughly established and seems to characterize classical limits and quantum dynamics. Indeed, the discrete quantum mechanics and mixed representations, namely, the lattice Weyl-Wigner (W-W) quantum physics has been developed in condensed matter physics with its various successful applications [1–13]. Here we extend the formalism to non-Hermitian operators, where translation operators or quantum state generators are still well-defined, as well as transition function between the dual Hilbert spaces. The resulting general formalism unifies all mixed-space representations on quantum mechanics, Hermitian and non-Hermitian operators and includes coherent state representation [14] as a special case. It has been employed by one of the authors in calculating the magnetic susceptibility of interacting many-body Bloch electrons [15], and in direct construction of fermionic path integrals [10]. First, we give an introduction on the physical basis of the discrete W-W nonequilibrium quantum transport theory [16]. One of the important aspects of the formalism is the bijective pairing of the elements of the dual spaces, known in mathematics as Pontryagin duality, which allows generalization to other dual spaces. The present unification naturally leads us to propose a new formulation of QFT based on the
Q-distribution of functional-field conjugate variables, which is expected to avoid infinities and the need of artificially imposed cut-off ubiquitous in QFT. Consequently, time-dependent problems can then be dealt as a nonequilibrium QFT transport physics in place of S-matrix theory.

The case of nonequilibrium quantum transport physics of many-body in CMP, which not only involves non-Hermitian operators but also time-reversal symmetry breaking, deserves a separate treatment. This is given in the Appendices. By virtue of the doubling of degrees of freedom, there it is expected that the dual spaces consist of chronological and anti-chronological solutions of the quantum Liouville equation, instead of dual spaces made through ordinary operator conjugation in many-body physics or quantum field theory.

1.1 Physical models for discrete quantum mechanics

Historically, there are already long-standing existing quantum models in physics that have physically guided the discrete phase-space physics of the lattice W-W formulation, namely, (a) Localized Wannier function and extended Bloch function for discrete lattices in solid state physics, obeying the Born-von Karman boundary condition (strictly speaking, obeying modular arithmetic based on finite fields, akin to a group theory of integers), (b) Dirac delta function and plane waves in the continuum limit, i.e., only for continuous coordinate space can one have continuous momentum space (this bijective quantum-mechanical canonical variables is disregarded in some of the W-W formalism of lattice models, where discrete lattice coordinates is unphysically paired with compact continuous momentum space [17–19]).

In both (a) and (b), we have the eigenvector for positions (or discrete lattice position), \(|q\rangle\), and eigenvector for momentum (or discrete crystal momentum), \(|p\rangle\). Of course, all respective eigenspaces go to continuum spaces in the limit of lattice constant goes to zero as in (b).

These respective eigenvectors, \(|q\rangle\) and \(|p\rangle\), in (a) and (b) are bijectively related by Fourier transformation, \(|q\rangle\) to \(|p\rangle\) via the transition function \(\langle p | q \rangle\), and often produces results akin to quantum uncertainty principle in their probabilistic continuous coordinate components. Thus, to construct a physically-based discrete phase space or discrete W-W quantum physics one must be guided by the following observations.

1.2 Discrete phase space on finite fields

The original formulation of lattice-space discrete Weyl transform and discrete WDF\(^1\) is based on crystalline solid with inversion symmetry, and hence based on an odd number of discrete lattice points, \((q, p)\), obeying the Born-von Karman boundary condition. Thus, this formulation is generally based on a finite field represented by a finite prime number, \(N\), of lattice points obeying modular arithmetic.

\(^1\)The original formulation is entitled, "Method for Calculating \(Tr\mathcal{H}\) in Solid State Theory", Phys. Rev., \textbf{B10}, 3700-3705 (1974).
arithmetic, closed under addition and multiplication. The presence of multiplicative inverses in the formulation assumes that the finite number of lattice points is a prime number, since primality is required for the nonzero elements to have multiplicative inverses. Indeed, in the limit that the number of lattice points is very large, the lattice point coordinates obeying the Born-von Karman boundary condition, assumes the field of prime integers.

1.3 Transition functions between dual Hilbert spaces

The existence of transition function between dual spaces can be easily established for Hermitian operators leading to bijective discrete Fourier transformation. The physically-based construction of W-W formalism in condensed-matter physics is guided by well-known aspects of solid-state physics, namely, (a) The invariance of this physical and canonical scheme of complete and orthogonal set of $\{|q\rangle\}$ and complete and orthogonal set of $\{|p\rangle\}$ in going from discrete to continuum physics, and (b) in the compactification of the lattice, the number of discrete lattice points (and hence the number of discrete momentum points) must be an odd prime number for obvious inversion symmetry reason. Moreover, all arithmetic operations on this group of numbers must be closed, i.e., all arithmetic operation must be a modular arithmetic with prime number modulus (akin to a group operation on prime number of integers). In short, all arithmetic operation is a modular arithmetic based on finite fields, since only for finite fields with prime number modulus does every nonzero element have well-defined multiplicative inverse, and hence modular division operation also provides closure. Thus, these dual set of spaces, $\{|q\rangle\}$ and $\{|p\rangle\}$, is connected by transition function which defines the lattice bijective Fourier transform.

1.4 The direct and reciprocal lattice in condensed-matter physics

We first summarize the Bravais lattice vectors and their corresponding reciprocal lattice vectors, since this points to symmetry properties of the discrete W-W formulation in condensed matter physics. In 2-D lattice, we have the reciprocal lattice vectors, $\vec{b}_1$ and $\vec{b}_2$ given by the matrix,

$$
\begin{pmatrix}
\vec{b}_1 & \vec{b}_2
\end{pmatrix}
= \frac{1}{\hat{n} \cdot (\vec{a}_1 \times \vec{a}_2)}
\begin{pmatrix}
\vec{a}_2 \times \hat{n} & \vec{n} \times \vec{a}_1
\end{pmatrix}
$$

which geometrically means that $\vec{b}_1$ is perpendicular to $\vec{a}_2$ and $\vec{b}_2$ perpendicular to $\vec{a}_1$. The $\hat{n}$ is the unit vector normal to the 2-D lattice plane.

---

2There are infinitely many prime numbers. Examples of prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, and 229, etc. When written in base 10, all prime numbers except 2 and 5 end in 1, 3, 7 or 9.

3A review article by Kasperkovitz and Peev, Ann. Phys. 230, 21 (1994) and more recently by Fialkovsky et al [17] makes several misleading statements about the discrete phase-space formulation of the quantum theory of solids by failing to recognize the finite-field aspects of the theory.
In 3-D lattice, the reciprocal lattice vectors, \( \vec{b}_1, \vec{b}_2, \) and \( \vec{b}_3 \) are given by,

\[
\begin{pmatrix}
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{pmatrix} = \begin{pmatrix}
\frac{\vec{a}_2 \times \vec{a}_3}{(\vec{a}_2 \times \vec{a}_3) \cdot \vec{a}_1} & \frac{\vec{a}_3 \times \vec{a}_1}{(\vec{a}_3 \times \vec{a}_1) \cdot \vec{a}_2} & \frac{\vec{a}_1 \times \vec{a}_2}{(\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3}
\end{pmatrix}
\]

(2)

The above formulas are independent of any chosen coordinate system. We have

\[
\begin{pmatrix}
\vec{a}_1 & \vec{a}_2 & \vec{a}_3
\end{pmatrix}
\begin{pmatrix}
\vec{b}_1 & \vec{b}_2 & \vec{b}_3
\end{pmatrix}^T = I
\]

(3)

1.5 Translation symmetry

The one eclectic concept in solid state physics is the concept of translation symmetry along any symmetry directions of the lattice. Inversion symmetry of the lattice structure is also among the common symmetry properties. Thus, the discrete phase space W-W formalism in condensed matter physics [20] is compatible with any of the lattice structures defined by Eq. (1) – (2), i.e., not limited to cubic lattice structures only.

1.5.1 Generalization to other discrete quantum and classical systems

The guidance of (a) and (b) allow us to generalize discrete phase space based on finite fields to be useful when the quantum numbers specifying the quantum states, are discrete configurations other than the particle position and momentum [9], even to a system describable by a smallest prime number 2. A simplest example is that of quantum bit or qubit, like a two-level atomic systems. Another example lies in the two-state entangled diagrams of multi-qubit systems [21], where the entangled basis states are connected by the Hadamard lattice transform.

1.5.2 Generalized operator basis

Another generalization has to do in the construction of the Wigner distribution function. This is done through the generalized projection operators, sometimes referred to as the generalized Pauli spin operators for both orthogonal and biorthogonal spaces. This will be made clear later. Indeed, the construction can be generalized to the spaces of non-Hermitian operators or biorthogonal systems. The power of using finite fields is that one can also generalized the discrete Wigner distribution construction based on the algebraic concept of finite fields, which are extension of prime fields, where \( \text{qand pare field elements (modirreducible polynomial)} \), useful in quantum computing, visualization, and communication/information sciences. Here we have \( p^n \) elements for some prime \( p \) and some integer \( n > 1 \) useful for constructing the Wigner distribution function for spin-\( \frac{1}{2} \) systems [9][22].
1.5.3 One-to-one mapping of discrete momentum and lattice position spaces

Each energy band corresponds to the splitting of the energy levels of one atomic site into $N$ levels where $N$ is the number of lattice sites in a compactified Bravais lattice obeying the Born-von Karman boundary condition in a given symmetry direction. This is the basis of energy band quantum dynamics. Therefore, the number of crystal momentum states in each band (in Brillouin zone) is exactly equal to the number of lattice sites. Hence, there has to be a bijective mapping between number of discrete lattice sites and the number of discrete crystal momentum states. This is the essence of the powerful theoretical concept of localized function around each lattice site, the Wannier function $|q\rangle$, and the extended function over all lattice points, the Bloch function $|p\rangle$, related through the bijective discrete Fourier transformation. The bijectivity and finite fields aspects are crucial in the mathematical manipulations to avoid ambiguities.

It is worth cautioning that the use of discrete lattice models coupled with the compact continuous momentum space in some W-W formulations renders an ill-defined transition function, by incurring a non-bijective canonical conjugate dynamical variables, e.g., absence of bijective transformation between momentum $|p\rangle$ and coordinate $|q\rangle$. This, and together with the lack of modular arithmetic on finite fields not invoked at all is adding to several more ambiguities.

1.5.4 Bijective Fourier transformation (Pontryagin duality)

In short, the crystal momentum space is essentially discrete and yield a bijective mapping to the discrete lattice sites through a discrete Fourier transformation. Moreover, dual Hilbert spaces related by generalized Fourier transform also endows quantum uncertainty relations, well known in classical probability theory [23].

The Bloch function space is orthogonal and complete and so is the Wannier function space. Observe that these eigenfunctions of phase space operators are well established for gapped energy band structures, or energy band far removed from the other energy bands. Generalized Wannier function can also be defined for coupled energy bands, using decoupling scheme like the Foldy-Wouthysen transformation for relativistic Dirac electrons. Indeed, it has been shown that counterparts of Wanner function and Bloch function exist for the decoupled positive energy states of relativistic electrons, with the 'Dirac-Wannier function' localization about the size of Compton wavelength [3,4]. Electric Wannier function and magnetic Wannier function also exist, as well as their respective Bloch functions, for uniform external electromagnetic fields [24]. This physical idea has been extended to formally construct the discrete phase space quantum mechanics based on finite fields for cases where $q$ and $p$ are not position and momentum variables, useful for quantum computing [9].

In general, the discrete phase space in condensed matter physics is based
on the mathematics of finite fields and holds for any prime number of lattice points obeying the Born-von Karman boundary condition. I.e., with prime modulus. It even holds for the most elementary prime number of lattice points, to yield the 2 × 2 Pauli spin matrices and the well-known Hadamard transformation between “Wannier function” and “Bloch function”, i.e., discrete Fourier transformation of two points in “phase space” leading to transformation of qubits. Here the “Wannier function” and “Bloch function” has acquired the status of a simple theoretical device for discrete quantum physics. Indeed, the Buot discrete phase-space formalism also gives the generalized Pauli spin operators for any given prime number of lattice points, which can be generalized to spaces of non-Hermitian operators in coherent state representation. It has also yielded all the entangled basis states, e.g., for two, three and four qubits, crucial to the physics of quantum teleportation. In other words, bijective pairing of different Hilbert spaces is crucial to various generalization discussed in this paper.

2 Mixed Space Representations in CMP

The mixed q-p representations is very useful in making analogy with classical dynamics in terms of position and momentum eigenvalues. This has also ushered a better understanding of the path integral formulations of quantum field theory to make analogy with classical partition function of statistical physics, especially with the compactification of time by the aid of Matsubara technique. This quantum field theory aspects will be discussed later in this paper.

2.1 The q-p representation

The mixed q-p representation basically start by expanding any quantum operator, $\hat{A}$, in terms of mutually unbiased basis states, namely the eigenvector of

\[\text{Although every finite field, with } p^n\text{ elements for some prime } p\text{ and integer } n \geq 1, \text{ often deals with irreducible polynomials over ring } \mathbb{Z}\text{ of integers, or over field } \mathbb{Q}\text{ of rational numbers, or over field } \mathbb{R}\text{ of real numbers, or over field } \mathbb{C}\text{ of complex numbers, the role of irreducible polynomials can be played by prime numbers themselves for } n = 1: \text{ prime numbers (together with the corresponding negative numbers of equal modulus) are the irreducible integers. They exhibit many of the general properties of the concept 'irreducibility' that equally apply to irreducible polynomials, such as the essentially unique factorization into prime or irreducible factors. Every polynomial } p(x)\text{in ring of polynomials with coefficients in } F, \text{ denoted by } F[x], \text{ can be factorized into polynomials that are irreducible over } F. \text{ This factorization is unique up to permutation of the factors and the multiplication of constants from } F\text{ to the factors.}

The simplest case of interest in Buot discrete W-W formulation is when } n = 1. \text{ In this case the finite field } GF(p)\text{ is the ring } \frac{\mathbb{Z}}{p}\text{. This is a finite field with } p\text{ elements, usually labelled } 0, 1, 2, \ldots, p - 1, \text{ where arithmetic is performed modulo } p, \text{ where nonzero elements have multiplicative inverses.}
position operator, $\hat{Q}$, and the eigenvector of momentum operator, $\hat{P}$. We have

$$\hat{A} = \sum_{p,q} |q\rangle \langle q| \hat{A} |p\rangle \langle p|$$

$$= \sum_{p,q} \langle q| \hat{A} |p\rangle \langle p|$$

(4)

### 2.1.1 The completeness of mixed-space projector

Thus, the set $\{|q\rangle \langle p|\}$ is the basis operator for the mixed $q$-$p$ representation. From the completeness relations of the unbiased basis states, $\{|q\rangle\}$ and $\{|p\rangle\}$, the set $\{|q\rangle \langle p|\}$ obeys the completeness relation

$$\sum_{q,p} \langle q| \langle p\rangle |q\rangle \langle p| = 1$$

(5)

Substituting the expression for $\langle q| \langle p\rangle$,

$$\langle q| \langle p\rangle = C_o e^{\frac{i}{\hbar} \vec{p} \cdot \vec{q}}$$

(6)

$$\langle p| \langle q\rangle = C_o^* e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{q}}$$

(7)

we obtained, for the completeness relation,

$$\left(\frac{\hbar}{N}\right)^{\frac{1}{2}} \sum_{q,p} e^{\frac{i}{\hbar} \vec{p} \cdot \vec{q}} |q\rangle \langle p| = 1$$

(8)

where $C_o$ is chosen as

$$C_o = \left(\frac{\hbar}{N}\right)^{-\frac{1}{2}}$$

Equation (8) can be rewritten as

$$\left(\frac{\hbar}{N}\right)^{-\frac{1}{2}} \sum_{q,p} |q\rangle \langle p| \langle q| \langle p\rangle = 1$$

(9)

Here we use the transformation identities in the mixed $q$-$p$ representations,

$$|p\rangle = \sum_q \langle q| \langle p\rangle |q\rangle$$

(10)

$$\langle p\rangle = \sum_q \langle q| \langle p| \langle q\rangle$$

(11)

$$|q\rangle = \sum_p \langle p| \langle q| \langle p\rangle$$

(12)

$$\langle q\rangle = \sum_p \langle p| \langle q| \langle p\rangle$$

(13)

$$\langle p| \langle q\rangle = \exp \left( -\frac{i}{\hbar} \vec{p} \cdot \vec{q} \right)$$

(14)
2.1.2 Covariant state vector and contravariant components

The position operator, $\hat{Q}$, and momentum operator, $\hat{P}$, obey the following commutation relations. When operating on components, we have,

$$\left[\hat{Q}, \hat{P}\right] \langle x | q \rangle = i\hbar \langle x | q \rangle = i\hbar \psi(x, q)$$

On the other hand, when operating on the eigenvector, $|q\rangle$,

$$\left[\hat{Q}, \hat{P}\right] |q\rangle = -i\hbar \ |q\rangle$$

The translation operator, $T(q')$, is defined as

$$\exp\left(-\frac{i}{\hbar} q' \cdot \hat{P}\right) |q\rangle = |q + q'\rangle$$

(15)

where

$$\hat{P} |p\rangle = p \ |p\rangle$$

$$\hat{P} |q\rangle = i\hbar \frac{\partial}{\partial q} |q\rangle$$

whereas,

$$\exp\left(i p' \cdot \hat{Q}\right) |p\rangle = |p + p'\rangle$$

(16)

where

$$\hat{Q} |q\rangle = q \ |q\rangle$$

$$\hat{Q} |p\rangle = -i\hbar \frac{\partial}{\partial p} |p\rangle$$

Equations (15) and (16) can be deduced from Eqs. (10) - (14).

3 Physics of Commutators and Anti-Commutators

Here, we proposed a couple of simple physical interpretation of commutators and anti-commutators in quantum physics. To the author’s knowledge, only Schwinger seems to have highlighted the physical implications for bosons and fermions.

3.1 Conjugate field operators and quantum state generators

Indeed, conjugate operators in CMP and QFT serve to generate new quantum states.
3.1.1 Space translation, time evolution, and state generation

It is not often emphasized in the literature, as well as in physics textbooks, that commutation and anti-commutation relation of canonically conjugate operators do carry very important physical meanings \[25\], i.e., quantum state generation, and imply the concept of either space translation (Wannier state generation) or time evolution, and in quantum field theory it implies energy or excitation-number state generation.

**Generator of Wannier functions** We have for localized Wannier states at lattice position \( q \), Eq. \([15]\),

\[
|q\rangle = \exp \left\{ -\frac{i}{\hbar} q \cdot P \right\} |0\rangle ,
\]

where the operator \( P = i\hbar \nabla_q \) is operating on the basis eigenvector \( |q\rangle \), since \( P = -i\hbar \nabla_x \) acting on the components behaves contravariantly. Here, we basically made an assumption in the above expression that there exist continuous function of \( q \) having an infinite radius of convergence, which are equal to \( |q\rangle \) at the lattice points, i.e., we expand the exponential in Eq. \((88)\) as a well-defined Taylor series.

**Canonical conjugate operators implies state generators** All these results are reminiscent of well-known relations with other canonical conjugate variables operating on wavefunctions, for example, we have

\[
\Psi (t + t_0) = \exp \left( -\frac{t}{\hbar} \mathcal{H} \right) \Psi (t_0)
\]

and

\[
\Psi (q + q_0) = \exp \left( i \frac{p \cdot q}{\hbar} \right) \Psi (q_0)
\]

In other words, canonical conjugate variables acts as displacement operators acting on its dual eigenvector space, respectively. This operation is also known as generation of eigenvectors. This principle is useful when we deal with quantum field theory later. In fact we shall see that the commutation relation of annihilation and creation operator given by

\[
[\hat{a}, \hat{a}^\dagger] = 1
\]

imply the generation of eigenvector of \( \hat{a} \) as follows,

\[
\exp (\alpha \hat{a}^\dagger) |0\rangle = |\alpha\rangle
\]

where \( \alpha \) is an eigenvalue of annihilation operator \( \hat{a} \). This implication towards quantum state generator holds for commutator and anti-commutator relations, either fermions or bosons \[25\].
As a side remark, it is not clear why time is not elevated to an energy
displacement operator in elementary quantum mechanics since it appears in
Poisson bracket operator on the same level as the position and momentum
variables, although energy displacement operator are ubiquitous in quantum
field theory.

3.2 Anti-commutator and commutator

3.2.1 Entanglement-induced localization and delocalization

In quantum transport theory of interacting chiral (axial or pseudovectors) de-
grees of freedom [16], e.g., Landau orbits, spin and pseudospins, the anti-
commutator of scattering terms are identified as a localization, characterized
by a fixed point in phase space, whereas commutator terms are associated with
nonlocalization or quantum diffusion. There, the concept of locality and nonlo-
cality have been extended to entangled chiral degrees of freedom in nonequilib-
rium quantum transport physics.

3.2.2 Particles and forces

Indeed, anti-commutation is associated with a wavefunction property of a rotat-
ing spinors where excitations have spatial sizes and behaving like billiard balls
upon exchange, resulting in the wavefunction rotation and phase change of π.
This is illustrated in Fig. 1.

Figure 1: A very simple intuitive explanation of spinors. The anti-commutator
represents local or non-diffusive term in quantum transport since repetition of
the anti-commutation operation just result in pure rotation of Ψ with a phase
change of π. Fermions are spinors and behave like billiard balls under exchange,
as indicated in the figure.

On the other hand, a commutator of conjugate operators implies spatial
nonlocality of excitation wavefunctions, where energy is the one quantized with
indefinite non-local spatial dimensions, as depicted in Fig. 2. Even the zero-point energy is still non-local in space, which is the main reason the lowest energy Bose condensate cannot form a solid. Clearly, commutator cannot be characterized by a point, i.e., it is nonlocal and excitation can conceptually pass through each other in the process of exchange [26, 28]. Indeed, Fig. 2 seems to suggest that only boson quasiparticles can move independently in one dimensions.

Figure 2: A qualitative explanation and schematic representation of bosons as obeying a nonlocal commutation relation. Bosons can pass through each other so the wavefunction does not have to rotate to affect an exchange of particles. Thus, there is no wavefunction phase change. The commutation terms in quantum transport usually implies unitary evolution as well as a nonlocal diffusive terms, since repeated commutation will result in diffusion as schematically indicated in the figure. Note that the boson particle can pass through the other boson particle in straight line, unlike the fermions which must go around since the cannot occupy the same state as indicated in Fig. 1. This is the reason that boson are often nonlocal excitations with small masses. There is also an implied overlap of their wavefunctions which allows for quantum diffusion. The wavefunctions maintain their phase after exchange with effective diffusion. The lower graph is a naive interpretation of zero-point dynamics of nonlocal bosons. Bosons have quantized energy and momentum and hence essentially nonlocal or cannot be localized even for Bose condensate which cannot be a solid.

3.2.3 Particles versus quantized force fields: Nambu-Goldstone bosons and the Higgs bosons

The spatial size implications of the anticommutation relations is the reason why elementary particles are generally fermions whereas quantized excitations of force fields are bosons, Figs. 3, 4. Newly discovered Higgs fields are thought to provide the short-range force field and very heavy short-range bosons. Higgs fields are also responsible for giving masses to particles [29].
3.2.4 Fermionization and bosonization: Symmetry breaking excitations, spinons, magnons

Fermionization and bosonization are discussed in several works \cite{30-34}. The Jordan-Wigner transformation is discussed in Refs. \cite{35,36}. Indeed, there is duality between free bosons and free fermions, or free Bloch fermions in CMP.

Symmetry breaking excitations of ground states can generally pass through each other, even in lower dimensions, and are characterized as bosons. In fact spinons are bosons which carry spin-$\frac{1}{2}$ in 1-D and 2-D antiferromagnets \cite{37-40,46}. Magnons are also bosons \cite{47}. Within superconductivity in condensed-matter physics and electroweak symmetry breaking in particle physics, some of the intriguing phenomena can be understood within the regime of spontaneous breaking of symmetries, i.e., the quantum mechanical scenario of the ground state of a quantum system being less symmetric than the corresponding Hamiltonian. A telltale sign of broken symmetries are massless bosonic excitations, which generally can pass through each other, known as Nambu-Goldstone bosons (NGB). The number of such distinct bosonic excitations are generally taken to be a measure of how many of the original symmetries are dynamically broken. Despite decades of research on the subject, a general formula that predicts the number of different NGBs in a given dynamical system with broken symmetries has eluded theorists \cite{48,49}.

Fermions in one dimension cannot move independently, but only collide with each other resulting in density fluctuations. An interesting instability of quasiparticles in condensed matter physics is illustrated by the transformation of an electron into spinon, holon, and orbiton in in a one-dimensional sample of strontium cuprate \cite{40,46}. Spinons and holons are illustrated in Fig. 5. Spin-charge separations occurs in one-dimensional Tomonaga-Luttinger liquids.

The following table illustrate the difference between baryons and mesons as subatomic quasiparticles in matter. The key difference between baryons and mesons is that baryons consist of a combination of three quark particles, whereas mesons consist of a pair of quark-antiquark particles.

| Baryons versus Mesons |
|------------------------|
| **Baryons** | **Mesons** |
| **Definitions** | Baryons are subatomic particles, which have three quark particles | Mesons are hadronic particles that have a pair of quark and anti-quark |
| **Category** | Fermions | Bosons |
| **Spin** | Half-integer spin | Integer spin |
| **Quarks** | Has three quarks | Has a pair of quarks anti-quarks |
| **Interactions** | Participate in strong interactions | Participate in both strong and weak interactions |
| **Examples** | Protons and Neutrons | Heavier mesons decay to lighter mesons and ultimately to stable electrons neutrinos and photons, etc. |
3.3 The Tomonaga-Luttinger liquids and Hubbard models

A prominent bosonization of a one-dimensional interacting fermi system occurs in the so-called Tomonaga-Luttinger liquids and Hubbard models. Although the bare system of electrons in a quantum wire consists of fermions, the low-energy elementary excitations in a Luttinger liquid are bosons, these are sound waves of the 1-D electron. In one dimension, a fermion or electron that tries to propagate has to push its neighbours because of electron-electron interactions and Pauli exclusion principle. No individual motion in possible. Any individual excitation has to become a collective one, e.g., density fluctuation quasiparticles. The peculiarities of a 1-D system of interacting fermions makes bosonization of low-energy excitation spectrum is so powerful. The low-energy is supposed to be universal and well described by the Tomonaga-Luttinger liquid (LL) theory.

Moreover, the one-dimensional Hubbard model in a magnetic field is equivalent under renormalization-group transformation to a multicomponent Tomonaga-Luttinger model. The numerical evaluations of the correlation functions and the analytic results indicated clearly that the 1-D Hubbard Model is a Tomonaga-Luttinger liquid (TLL).
Figure 4: All meson particles are unstable. These particles tend to decay, forming electrons and neutrinos if the meson has a charge. But uncharged mesons undergo decay forming photons. The mesons have an integer spin (baryons have half-integer spin) Reproduced from: [https://www.differencebetween.com/difference-between-baryons-and-mesons/#Mesons].

3.3.1 "Decay" of fermion into spinon, holon and orbiton in CMP

Remarkably, the Tomonaga-Luttinger model exhibits spin-charge separation, in which the spin and charge of the fermions possess independent dynamics, so the velocities of the spin and charge excitations of a Tomonaga-Luttinger liquid are distinct. These quasiparticles are the so-called spinons, holons, and in some cases orbitons, e.g. in quasi one-dimensional sample of strontium cuprate [44,45]. Since spinons are bosons, we expect spinons move faster than holons and separates.

4 The Expansion of Any Operators in CMP

Any operator, ˆA, can be expanded in terms of mixed-space projector by Eq. (4), which we rewrite as,

\[ A = \sum_{p',q'} |p'\rangle \langle p'|A|q'\rangle \langle q'| \]

\[ A = \sum_{p',q'} \langle p'|A|q'\rangle |p'\rangle \langle q'| \] (19)

We wish to express \( \langle p'|A|q'\rangle \) and \( |p'\rangle \langle q'| \) in terms of the position eigenstate matrix elements and momentum space projectors or vice versa.

4.0.1 Change of units for a unification

In preparations of our discussions using creation and annihilation operators on the unification with quantum field theory, we let \( \hbar = 1 \) in what follows. Using,
Eqs. (10)-(14), we write

\[
\langle p'' | A | q' \rangle = \frac{1}{\sqrt{N}} \sum_{q''} e^{-ip'' \cdot q''} \langle q'' | A | q' \rangle
\]

\[
= \frac{1}{\sqrt{N}} \sum_{q''} \langle q'' | A | q' \rangle \langle q'' | | p'' \rangle
\]

\[
| p'' \rangle \langle q' | = \frac{1}{\sqrt{N}} \sum_{p'} e^{ip' \cdot q'} | p' \rangle \langle q' | p'' \rangle
\]

\[
= \frac{1}{\sqrt{N}} \sum_{p'} | p' \rangle \langle p' | q' \rangle
\]  

(20)

with completeness relation, characteristic of dual Hilbert spaces,

\[
\frac{1}{\sqrt{N}} \sum_{p,q} e^{-ip \cdot q} | p \rangle \langle q | = 1
\]

\[
\frac{1}{\sqrt{N}} \sum_{p,q} | p \rangle \langle q | = 1
\]

Introducing the notation in Eq. (20),

\[
p' = \vec{p} + \hat{a}, \quad q' = \vec{q} + \hat{v},
\]

\[
p'' = \vec{p} - \hat{a}, \quad q'' = \vec{q} - \hat{v},
\]
we have, upon substituting in Eq. \([19]\),

\[
A = \sum_{p,q,u,v} \langle p'\prime' | A | q'\prime' \rangle | p'\prime' \rangle \langle q'\prime' | = \frac{1}{N} \sum_{p,q,u,v} (e^{i2\mathbf{p} \cdot \mathbf{v}} \langle \mathbf{q} - \mathbf{v} | A | \mathbf{q} + \mathbf{v} \rangle) (e^{i2\mathbf{q} \cdot \mathbf{q}'} | \mathbf{p} - \mathbf{u} \rangle | \mathbf{p} + \mathbf{u} \rangle)
\]

4.0.2 Mixed space operator basis

We write the last result as an expansion in terms of phase-space point projector, \(\Delta (p, q)\), defined as the lattice Weyl transform of a projector, by

\[
\hat{\Delta} (p, q) = \sum_{u} e^{i2\mathbf{u} \cdot \mathbf{q}} | \mathbf{p} - \mathbf{u} \rangle | \mathbf{p} + \mathbf{u} \rangle \quad (21)
\]

and the coefficient of expansion, the so-called lattice Weyl transform of matrix element of operator, \(A (p, q)\), defined by

\[
A (p, q) = \sum_{v} e^{i2\mathbf{p} \cdot \mathbf{v}} \langle \mathbf{q} - \mathbf{v} | A | \mathbf{q} + \mathbf{v} \rangle.
\]

Clearly, for a density matrix operator \(\hat{\rho}\), the lattice Weyl transform obeys,

\[
\sum_{p,q} \rho (p, q) = 1
\]

If one accounts for other extra discrete quantum labels like spin and energy-band indices, we can incorporate this in the summation in a form of a trace.

Thus, we eventually have any operator expanded in terms of \(\Delta_{\lambda'\lambda} (p, q)\),

\[
\hat{\mathcal{A}} = \sum_{p,q,\lambda,\lambda'} A_{\lambda'\lambda} (p, q) \hat{\Delta}_{\lambda'\lambda} (p, q) \quad (22)
\]

and hence it follows,

\[
A_{\lambda'\lambda} (p, q) = Tr \left( \hat{\mathcal{A}} \hat{\Delta}_{\lambda'\lambda} (p, q) \right)
\]

Upon similar procedure based on Eq. \([19]\), an equivalent expression can be obtain for \(A_{\lambda'\lambda} (p, q)\) and \(\Delta_{\lambda'\lambda} (p, q)\), namely,

\[
A_{\lambda'\lambda} (p, q) = \sum_{u} e^{2i\mathbf{q} \cdot \mathbf{u}} \langle \mathbf{p} + \mathbf{u}, \lambda | \hat{\mathcal{A}} | \mathbf{p} - \mathbf{u}, \lambda' \rangle \quad (23)
\]

\[
\hat{\Delta}_{\lambda'\lambda} (p, q) = \sum_{v} e^{2i\mathbf{p} \cdot \mathbf{v}} \langle \mathbf{q} + \mathbf{v}, \lambda | \hat{\mathcal{A}} | \mathbf{q} - \mathbf{v}, \lambda' \rangle \quad (24)
\]
4.0.3 Fully symmetric translation operators

The state

\[ |q \rangle = \exp \left\{ -i q \cdot P \right\} |0 \rangle \]

is an eigenstate of the position operator with displaced eigenvalue by \( q \). However, if the limit \( q' \to 0 \) is not taken then the state \( \exp \left\{ -i q \cdot P \right\} |q' \rangle \) is an eigenstate of the position operator with eigenvalue \( q' + q \).

We can symmetrize the translation operator by inserting \( \exp \{ i p \cdot Q \} \) in front of \(|0\rangle \) which effectively insert unity. We thus have

\[ |q \rangle = \exp \{ -i q \cdot P \} \exp \{ i p \cdot Q \} |0 \rangle \].

(25)

By the use of the Campbell-Baker-Hausdorff operator identity, we obtained

\[ \exp \{ -i q \cdot P \} \exp \{ i p \cdot Q \} = \exp \left\{ -i (q \cdot P - p \cdot Q) \right\} \exp \left\{ -ip \cdot \frac{q}{2} \right\} \].

(26)

Therefore we have the symmetric form for the displacement operator generating the state \(|q\rangle\) from \(|0\rangle\) given by

\[ T(q)^{\text{sym}} = \exp \left\{ -i \frac{p \cdot q}{2} \right\} \exp \left\{ -i (q \cdot P - p \cdot Q) \right\} \].

(27)

The displacement operator may also be interpreted as an operator for the preparation of the quantum eigenstate out of the 'vacuum', \(|0\rangle\) basis eigenstate.

4.0.4 The symmetric operator basis for mixed representations

The symmetric operator factor,

\[ Y_{p,q} = \left( \exp \left\{ -i (q \cdot P - p \cdot Q) \right\} \right) \]

\[ = \exp \left\{ -i \frac{P \cdot q}{2} \right\} \exp \left\{ -i q \cdot P \right\} \exp \{ i p \cdot Q \} , \]

(28)

is referred to here as the generalized mixed-space projector or sometimes referred to as the generalized Pauli-matrix operator \([9]\). It can be considered the universal form of projector in mixed representations of condensed matter physics. We shall see that this also holds with dual space of non-Hermitian operators in quantum field theory.

---

5Here, the concept of a vacuum state does not have a special meaning since \(|0\rangle\) represents arbitrary reference position. It is introduced simply to bring analogy with zero-eigenvalue of non-Hermitian operators in later chapters, there the state \(|0\rangle\) has a distinguished position.
Symmetric form of $\hat{\Delta}_{\lambda'}^{\lambda}(p,q)$ Consider Eq. (24). The following identities can be verified,

$$|\bar{q} + \bar{v}, \lambda\rangle = \exp \left[-2i\hat{P} \cdot \bar{v}\right] |\bar{q} - \bar{v}, \lambda\rangle$$  \hspace{1cm} (29)$$

Then

$$|\bar{q} - \bar{v}, \lambda\rangle \langle \bar{q} - \bar{v}, \lambda'| = (N)^{-1} \sum_{\bar{u}} \exp \left[-2i \left(\bar{q} - \bar{v} - \hat{Q}\right) \cdot \bar{u}\right] \sum_q |q, \lambda\rangle \langle q, \lambda'|$$  \hspace{1cm} (30)$$

$$\Omega_{\lambda\lambda'} = \sum_q |q, \lambda\rangle \langle q, \lambda'|$$

Substituting the expressions, Eqs (29) and (30) in Eq. (24), we obtain a completely symmetric expression of $\hat{\Delta}_{\lambda'}^{\lambda}(p,q)$,

$$\hat{\Delta}_{\lambda'}^{\lambda}(p,q) = \sum_{\bar{v}} e^{2i\hat{P} \cdot \bar{v}} |\bar{q} + \bar{v}, \lambda\rangle \langle \bar{q} - \bar{v}, \lambda'|$$

$$= (N)^{-1} \sum_{\bar{v}, \bar{u}} e^{\hat{P} \cdot \bar{v}} \exp \left[-2i \left(\bar{q} - \bar{v} - \hat{Q}\right) \cdot \bar{u}\right] \sum_q |q, \lambda\rangle \langle q, \lambda'|$$

$$= \exp \left\{-2i \left(\hat{P} \cdot \bar{v} - \hat{Q} \cdot \bar{u}\right)\right\} \exp \left\{2i \left(\bar{q} \cdot \bar{v} - \bar{q} \cdot \bar{u}\right)\right\}$$

Thus, we have changed the seemingly asymmetric expression of the first line into a symmetric form of the last line. We can combine the exponential operators to obtain

$$\hat{\Delta}_{\lambda'}^{\lambda}(p,q) = (N)^{-1} \sum_{\bar{v}, \bar{u}} \exp -2i \left[\left(\hat{P} - \bar{v}\right) \cdot (Q - \bar{u}) \cdot \bar{u}\right] \Omega_{\lambda\lambda'}$$

where now we can write, in general,

$$\Omega_{\lambda\lambda'} = \sum_{q'} |q', \lambda\rangle \langle q', \lambda'|$$

$$= \sum_{p'} |p', \lambda\rangle \langle p', \lambda'|$$

We therefore have,

$$A(p,q) = Tr \left(\hat{A} \hat{\Delta}\right)$$

$$= (N)^{-1} \left(\times Tr \left\{\hat{A} \exp -2i \left[\hat{P} \cdot \bar{v} - \hat{Q} \cdot \bar{u}\right] \Omega_{\lambda\lambda'}\right\} \right)$$

Therefore, the characteristic distribution of $A(p,q)$ is identically,

$$A_{\lambda\lambda'}(u,v) = Tr \left\{\hat{A} \exp \left(-2i \left[\hat{P} \cdot \bar{v} - \hat{Q} \cdot \bar{u}\right]\right) \Omega_{\lambda\lambda'}\right\}$$
By using this characteristic distribution function for $A_{\lambda\lambda'}(p, q)$

$$A_{\lambda\lambda'}(u, v) = \left(\frac{1}{N}\right)^{\frac{1}{2}} \sum_{p,q} A_{\lambda\lambda'}(p, q) e^{2i(p\cdot v - q\cdot u)}$$  \hspace{1cm} (31)

with inverse

$$A_{\lambda\lambda'}(p, q) = \left(\frac{1}{N}\right)^{\frac{1}{2}} \sum_{u,v} A_{\lambda\lambda'}(u, v) e^{-2i(p\cdot v - q\cdot u)}$$

Then we can write Eq. (22) simply like a Fourier transform (caveat: Fourier transform to operator space) of the characteristic function of the lattice Weyl transform of the operator $\hat{A}$,

$$\hat{A} = \sum_{u,v,\lambda,\lambda'} A_{\lambda\lambda'}(u, v) \exp \left[ -2i \left( \hat{P} \cdot v - \hat{Q} \cdot u \right) \right] \Omega_{\lambda\lambda'}$$  \hspace{1cm} (32)

In continuum approximation, we have,

$$\hat{\Delta}(p, q) = (2\pi)^{-3} \int du \, dv \, e^{-i(\hat{P} - \bar{p}) \cdot \bar{v} - (Q - \bar{q}) \cdot \bar{u}} \Omega_{\lambda\lambda'}$$

$$= (2\pi)^{-3} \int du \, dv \, e^{i(p\cdot v - q\cdot u)} e^{(-i)(P\cdot v - Q\cdot u)} \Omega_{\lambda\lambda'},$$  \hspace{1cm} (33)

From

$$A_{\lambda\lambda'}(p, q) = Tr \left( \hat{\Delta} \right) = (2\pi)^{-3} \left( \int du \, dv \, e^{i(p\cdot v - q\cdot u)} Tr \left( \hat{A} e^{(-i)(P\cdot v - Q\cdot u)} \right) \right)$$

$$= (2\pi)^{-3} \left( \int du \, dv \, e^{i(p\cdot v - q\cdot u)} A_{\lambda\lambda'}(u, v) \right).$$

### 4.1 Characteristic distribution of lattice Weyl transform

In general, we can have different expression for the characteristic function depending on the use of the often referred to in corresponding CS formalism as the normal and anti-normal expressions,

$$e^{(-i)(P\cdot v - Q\cdot u)} = \exp \left\{ -i \frac{u \cdot v}{2} \right\} \exp \left\{ -iv \cdot P \right\} \exp \left\{ iu \cdot Q \right\}$$

$$= \exp \left\{ i \frac{u \cdot v}{2} \right\} \exp \left\{ iu \cdot Q \right\} \exp \left\{ -iv \cdot P \right\}$$

so that with $\exp \left\{ i \frac{u \cdot v}{2} \right\} e^{(-i)(P\cdot v - Q\cdot u)}$ as our reference, we have the so-called normal and anti-normal order expressions given by,

$$\exp \left\{ -iv \cdot P \right\} \exp \left\{ iu \cdot Q \right\} = \exp \left\{ i \frac{u \cdot v}{2} \right\} e^{(-i)(P\cdot v - Q\cdot u)} ,$$  \hspace{1cm} (34)

$$\exp \left\{ iu \cdot Q \right\} \exp \left\{ -iv \cdot P \right\} = \exp \left\{ -i \frac{u \cdot v}{2} \right\} e^{(-i)(P\cdot v - Q\cdot u)} ,$$  \hspace{1cm} (35)

---

6 For creation and annihilation operators in many-body quantum physics, the proof relies on the use of normal or anti-normal ordering of canonical operators, which can then be treated like C-numbers in expansion of exponentials. The exponential in Eq. (100) is sometimes referred to as the generalized Pauli-spin operator.
respectively, yielding the following different expressions for $A_{\lambda\lambda'}(u,v)$, namely,

$$A^{w}_{\lambda\lambda'}(u,v) = \text{Tr} \left( \hat{A} \exp \left( -i (Pv - Qw) \right) \right) \quad (36)$$

$$A^{n}_{\lambda\lambda'}(u,v) = \text{Tr} \left\{ \hat{A} \exp \left( -iw \cdot P \right) \exp \left( iu \cdot Q \right) \right\} \quad (37)$$

$$A^{a}_{\lambda\lambda'}(u,v) = \text{Tr} \left\{ \hat{A} \exp \left( -iq \cdot P \right) \exp \left( ip \cdot Q \right) \right\} \quad (38)$$

Although, Eqs. (37) and (38) only amounts to difference in the phase factors, the corresponding quantities in non-Hermitian dual spaces gives a very different distributions often referred to as smooth-out distributions. Examining Eqs. (34) and (35), and the fact that in non-Hermitian mixed representation, $\{i \frac{u \cdot v}{2}\}$ is a real quantity that resembles a Gaussian function, Eqs. (34) clearly represent some smoothing of the Wigner distribution characteristic function and hence the Wigner distribution itself. In Eqs. (37) and (38) no informations are lost.

Indeed, more general phase-space distribution functions, $f^{(g)}(p,q,t)$, can be obtained from the expression

$$f^{(g)}(p,q,t) = \frac{1}{2\pi \hbar} \int dudve \ e^{-i(pu+qv)} \left\{ \text{Tr} \hat{\rho} \ \exp \left( -i (Pv - Qw) \right) \right\} g(u,v)$$

$$= \frac{1}{2\pi \hbar} \int dudve \ \exp \left( \frac{i}{\hbar} (qu + pv) \right) \left[ \mathcal{C}^{(w)}(u,v,t) \ g(u,v) \right], \quad (39)$$

where $g(u,v)$ is some chosen phase function.

### 4.1.1 Alternative symmetrization of Eq. (21) for $\hat{\Delta}_{\lambda\lambda'}(p,q)$

We can also take up Eq. (21) to symmetrize $\hat{\Delta}_{\lambda\lambda'}(p,q)$. We have

$$\hat{\Delta}(p,q) = \sum_u e^{2i\vec{u} \cdot \vec{q}} |\vec{p} - \vec{u}\rangle \langle \vec{p} + \vec{u}|$$

$$= \sum_u e^{-2i\vec{u} \cdot \vec{q}} |\vec{p} + \vec{u}\rangle \langle \vec{p} - \vec{u}|$$

$$= \sum_u e^{i2(Q-q) \cdot u} |\vec{p} - \vec{u}\rangle \langle \vec{p} - \vec{u}|$$

where

$$|\vec{p} - \vec{u}\rangle \langle \vec{p} - \vec{u}| = \sum_{\vec{v},\vec{p}'} \exp \left[ 2i \left( \vec{p} - \vec{u} - \vec{P} \right) \cdot \vec{v} \right] |\vec{p}'\rangle \langle \vec{p}'|$$

23
Therefore,
\[
\hat{\Delta} (p, q) = \sum_u e^{i2\hat{u} \cdot v} \sum_v \exp [2i (\vec{p} - \vec{u} - \vec{P}) \cdot v] |\vec{p}^\prime\rangle \langle \vec{p}^\prime| \exp [i (\vec{p}^\prime - \vec{u} - \vec{P}) \cdot v] e^{-i2Q \cdot u}
\]
\[
= \sum_{u,v} e^{i2(Q - q) \cdot u} \exp [-i2 (P - p) \cdot v] \exp [-i2\vec{u} \cdot v] |\vec{p}^\prime\rangle \langle \vec{p}^\prime| e^{-i2(Q - q) \cdot u}
\]
\[
e^{-i2Q \cdot u} e^{-i2P \cdot v} = e^{-2i(P-v-Q-u)} e^{i2[Q,P] \vec{u} \cdot v}
\]
Therefore,
\[
\hat{\Delta} (p, q) = e^{-2i(p \cdot v - q \cdot u)} \sum_{\vec{p}^\prime} |\vec{p}^\prime\rangle \langle \vec{p}^\prime|
\]

Thus,
\[
\hat{\Delta} = \sum_{p,q,\lambda,\lambda^\prime} A_{\lambda\lambda'} (p, q) \hat{\Delta}_{\lambda^\prime \lambda} (p, q)
\]
\[
= \sum_{p,q,\lambda,\lambda^\prime, u,v} A_{\lambda\lambda'} (p, q) \exp 2i (p \cdot v - q \cdot u) \exp [-2i (P \cdot v - Q \cdot u)] \sum_{\vec{p}^\prime} |\vec{p}^\prime\rangle \langle \vec{p}^\prime|
\]

with characteristic function given by,
\[
A (u, v) = \sum_{p,q} A_{\lambda\lambda'} (p, q) \exp 2i (p \cdot v - q \cdot u)
\]
and
\[
\hat{A} = \sum_{\lambda,\lambda^\prime, u,v} A (u, v) \exp [-2i (P \cdot v - Q \cdot u)] |\vec{p}_0\rangle \langle \vec{p}_0|
\]

The algebra of \(\exp \{-i (q' \cdot P - p' \cdot Q)\}\) as well as its relevance to the physics of two-state systems, spin systems, quantum computing, entanglements [21] and teleportation, are discussed in one of the author’s book [9].

4.2 Construction of path integral in CMP: Mixed space representation

We are here interested in expressing the transition amplitude as a lattice path integral in solid-state physics. A well-known procedure to implement this is to decomposed the evolution operator, \(\hat{U} (t, t_0) \equiv \exp \frac{\hat{H}}{\hbar} (t - t_0) \hat{H}\), in terms of freely wandering paths during infinitesimal time increments, \(\epsilon\). First, we decomposed in time increments as
\[
\hat{U} (t, t_0) = \lim_{n \to \infty} \prod_{j=1}^{n+1} \hat{U} (t_j, t_{j-1})
\]
where $t_{n+1} = t$. As a wandering incremental paths between $t_{n+1}$ and $t_0$ in the crystal lattice, we make use of the Wannier function wavevector and its completeness (wanderer) as,

$$
\langle q (t_{n+1}) | \hat{U} (t, t_0) | q (t_0) \rangle \\
= \langle q (t_{n+1}) | \hat{U} (t_{n+1}, t_n) \sum_{q_n} |q (t_n)\rangle \langle q (t_n) | \hat{U} (t_n, t_{n-1}) \sum_{q_{n-1}} |q (t_{n-1})\rangle \langle q (t_{n-1}) |.
$$

\[ \times \cdots \sum_{q_1} |q (t_1)\rangle \langle q (t_1) | \hat{U} (t_1, t_0) | q (t_0) \rangle \]

\[ = \sum_{q_1} \cdots \sum_{q_n} \prod_{j=1}^{n+1} \langle q (t_j) | \hat{U} (t_j, t_{j-1}) | q (t_{j-1}) \rangle \] (40)

where the end coordinates, namely, $q (t_{n+1})$ and $q (t_0)$ are fixed. By making the time intervals infinitely small or by letting $\lim_{n \to \infty}$, we can take advantage of the linearity of the small time evolution operators to calculate the matrix elements after which it can be recomposed in exponential form but now cast as $\mathbb{C}$-numbers. We have

$$
\hat{U} (t_j, t_{j-1}) \simeq 1 - \frac{i}{\hbar} (t_j - t_{j-1}) \mathcal{H} + \mathcal{O} (t_j - t_{j-1})^2 + ..
$$

Now comes our big advantage in calculating the matrix elements in mixed-space representation. We now expand the operator $\hat{U} (t_j, t_{j-1})$ in terms of the phase-space point projector $\hat{\Delta} (p, q)$. We have

$$
\mathcal{H} = \sum_{p,q} H (p, q) \hat{\Delta} (p, q)
$$

Then

$$
\langle q (t_j) | \hat{U} (t_j, t_{j-1}) | q (t_{j-1}) \rangle \\
= \langle q (t_j) | \left( 1 - \frac{i}{\hbar} (t_j - t_{j-1}) \mathcal{H} \right) | q (t_{j-1}) \rangle \\
= 1 - \frac{i}{\hbar} (t_j - t_{j-1}) \sum_{p,q} H (p, q) \left\{ \langle q (t_j) | \hat{\Delta} (p, q) | q (t_{j-1}) \rangle \right\}
$$

where for intermediate points,

$$
\left\{ \langle q (t_j) | \hat{\Delta} (p, q) | q (t_{j-1}) \rangle \right\} = \exp \left( \frac{i}{\hbar} p_j \cdot (q_j - q_{j-1}) \right) \delta (2q - [q_j + q_{j-1}])
$$

and for the end points, we have

$$
\left\{ \langle q (t_{n+1}) | \hat{\Delta} (p, q) | q (t_n) \rangle \right\} = \exp \left( \frac{i}{\hbar} p_n \cdot (q_{n+1} - q_n) \right) \delta (2q - [q_{n+1} + q_n])
$$

$$
\left\{ \langle q (t_1) | \hat{\Delta} (p, q) | q (t_0) \rangle \right\} = \exp \left( \frac{i}{\hbar} p_1 \cdot (q_1 - q_0) \right) \delta (2q - [q_1 + q_0])
$$
so that we can write to first order in time increment, \((t_j - t_{j-1})\), as,

\[
\langle q(t_j) | \hat{U}(t_j, t_{j-1}) | q(t_{j-1}) \rangle = (N\hbar^3)^{-1} \sum_{p_j, q, q_j} \exp \left( \frac{i}{\hbar} p_j \cdot (q_j - q_{j-1}) \right) \times \exp \left( -\frac{i}{\hbar} (t_j - t_{j-1}) H(p, q) \right) \delta(2q - [q_j + q_{j-1}])
\]

Substituting in Eq. (40), we obtained

\[
\langle q(t_{n+1}) | \hat{U}(t, t_0) | q(t_0) \rangle = (N\hbar^3)^{-1} \prod_{j=1}^{n} \sum_{q_j} \prod_{j=1}^{n} \sum_{p_j} \exp \left[ \frac{i}{\hbar} (t_j - t_{j-1}) \left( p_j \cdot \frac{(q_j - q_{j-1})}{(t_j - t_{j-1})} - H \left( p, \frac{q_j + q_{j-1}}{2} \right) \right) \right]
\]

Now since \(\sum_p\) is performed for each term labeled by \(j = 1, \ldots, n\), we can also label summation over \(p\) by summation over \(p_j\), i.e., \(\sum_p\) giving us a more symmetrical form,

\[
\langle q(t_{n+1}) | \hat{U}(t, t_0) | q(t_0) \rangle = (N\hbar^3)^{-1} \prod_{j=1}^{n} \sum_{q_j} \prod_{j=1}^{n} \sum_{p_j} \exp \left[ \frac{i}{\hbar} (t_j - t_{j-1}) \left( p_j \cdot \frac{(q_j - q_{j-1})}{(t_j - t_{j-1})} - H \left( p, \frac{q_j + q_{j-1}}{2} \right) \right) \right]
\]

### 4.2.1 Continuum limit

If we take the continuum limit, i.e., \(N \to \infty\), and \(\Delta t = t_j - t_{j-1} \to 0\), and defining

\[
\int \mathcal{D}p = \prod_{j=1}^{n} \sum_{p_j}, \quad q_f = q(t_n), \quad q_i = q(t_0)
\]

\[
\int \mathcal{D}q = \prod_{j=1}^{n} \sum_{q_j}
\]
We have in the exponential,

\[
S = \sum_{j=1}^{n} \left[ \frac{i}{\hbar} (t_j - t_{j-1}) \left( p_j \left( \frac{q_j - q_{j-1}}{(t_j - t_{j-1})} \right) - H \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \right) \right]
\]

\[
\Rightarrow \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ p(t) \cdot \dot{q}(t) - H(p(t), q(t)) \right]
\]

where the integral in \( S \) is identified as the Hamiltonian action. Then we end up with

\[
\langle q(t_f) | \hat{U}(t, t_0) | q(t_i) \rangle = \left( \frac{2\pi \hbar}{3} \right)^{-1} \int Dq Dp \exp(S)
\]

This last result is often called functional integral since now the \( p(t), q(t) \) defines phase-space path functions.

## 5 General Mixed Space Representation in QFT

Let the operators \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) be non-Hermitian operators. These operators obey either commutation or anti-commutation relations, i.e.,

\[
\left[ \hat{\psi}, \hat{\psi}^\dagger \right]_\eta = 1 \tag{41}
\]

where the subscript \( \eta = + \) is for the anti-commutation and \( \eta = - \) stands for commutation relation. These non-Hermitian operators have distinguished left and right eigenvectors. We have

\[
\hat{\psi} \ket{\alpha} = \alpha \ket{\alpha} \tag{42}
\]

\[
\braket{\alpha}{\hat{\psi}^\dagger} = \braket{\alpha}{\alpha^*} \tag{43}
\]

\[
\hat{\psi}^\dagger \ket{\beta} = \beta \ket{\beta} \tag{44}
\]

\[
\braket{\beta}{\hat{\psi}} = \braket{\beta}{\beta^*} \tag{45}
\]

This means the left eigenvector of \( \hat{\psi} \) is \( \braket{\beta} \), with eigenvalue \( \beta^* \), whereas the left eigenvector of \( \hat{\psi}^\dagger \) is \( \braket{\alpha} \), with eigenvalue \( \alpha^* \). We have,

\[
\braket{\beta}{\hat{\psi} \ket{\alpha}} = \alpha \braket{\beta}{\alpha} \tag{46}
\]

\[
= \beta^* \braket{\beta}{\alpha}
\]

\[
(\alpha - \beta^*) \braket{\beta}{\alpha} = 0
\]

and similarly

\[
\braket{\alpha}{\hat{\psi}^\dagger \ket{\beta}} = \alpha^* \braket{\alpha}{\beta} \tag{47}
\]

\[
= \beta \braket{\alpha}{\beta}
\]

\[
(\alpha^* - \beta) \braket{\alpha}{\beta} = 0
\]
From Eqs. (46) and (47), if \( \alpha_n \neq \beta_m^* \), \( \langle \beta_m | \alpha_n \rangle = 0 \), imitating orthogonal Hermitian Hilbert space. However, if \( \alpha_n = \beta_m^* \), then \( \langle \beta_m | \alpha_n \rangle \neq 0 \), amenable to probabilistic interpretation. Moreover, with \( |\alpha\rangle \) and \( \langle \beta| \) we have the general projection given by

\[
\sum_n \frac{|\alpha_n\rangle \langle \beta_n|}{\langle \beta_n | \alpha_n \rangle} = 1
\]

which project only within the paired eigenspace of \( \{ |\alpha\rangle , \langle \beta| \} \). Similarly,

\[
\sum_n \frac{|\beta_n\rangle \langle \alpha_n|}{\langle \alpha_n | \beta_n \rangle} = 1
\]

only projects within the paired space, \( \{ \langle \alpha| , |\beta\rangle \} \).

The general proof of Eq. (48) lies in the following expansion of \( |\Psi\rangle \), where \( |\Psi\rangle \) is a complete orthonormal eigenvector. Let,

\[
|\Psi\rangle = \sum_{n'} c_{n'} |\alpha_{n'}\rangle.
\]

Then we have

\[
\sum_n \frac{|\alpha_n\rangle \langle \beta_n| |\Psi\rangle}{\langle \beta_n | \alpha_n \rangle} = \sum_{n,n'} \frac{|\alpha_n\rangle \langle \beta_n| |\alpha_{n'}\rangle c_{n'}}{\langle \beta_n | \alpha_n \rangle} = \sum_n c_n |\alpha_n\rangle = |\Psi\rangle
\]

### 5.1 Mixed Hilbert-space construction

Thus, from Eqs. (46) and (47), we have a well-defined Hermitian-like operation in terms of the paired eigenvector set \( \{ |\alpha| , \langle \beta| \} \) and \( \{ \langle \beta| , |\alpha\rangle \} \) as dual eigenspaces. Indeed, assuming nondegenerates countable states, we can pair the states so that \( \alpha_n = \beta_m^* \) and rewrite the pairs with same quantum subscript, i.e., \( \alpha_n = \beta_n^* \), which would be consistent with the commutation relation,

\[
[\hat{\psi}_n, \hat{\psi}_n^+] = 1.
\]

Just as the dot product \( \langle p| q \rangle \neq 0 \) in condensed matter physics discussions, so \( \langle \alpha_n| \alpha_n \rangle \neq 0 \), which means that \( \langle \alpha_n| \alpha_n \rangle \) cannot be made equal to zero, since these belong to separate paired set, \( \{ |\alpha| , \langle \beta| \} \) and \( \{ \langle \beta| , |\alpha\rangle \} \), respectively.

In some sense, the \( |\alpha\rangle, \langle \beta| \) respective spaces are reminiscent of the \( q-p \) phase space, where \( \langle p| q \rangle \neq 0 \) is the transition function. This conclusion can be made quite general in what follows.

### 5.2 Pairing algorithm and Hermitianization

Thus, in order to work with Hermitian-like operators, we want the eigenvalues \( \alpha = \beta^* \) and \( \alpha^* = \beta \). For nondegenerate countable finite system, this type of pairing of the different \( |\alpha\rangle \) and \( \langle \beta| \) spaces is well defined. For convenience in
what follows, we relabel the $|\alpha\rangle$ and $|\beta\rangle$ notations and their adjoints to reflect the Hermitian-like new spaces, sort of renormalize new dynamical vector spaces.

From Eqs. (46) and (47), it seems trivial just like for the Hermitian operators to prove orthogonality or more appropriately, biorthogonality, in terms of the \{\langle\alpha|, |\beta\rangle\} and \{\langle\beta|, |\alpha\rangle\} 'dual' eigenspaces. This seems to suggest the following Hermitian-like relations

\[
\langle\alpha|\hat{\psi}^\dagger|\beta\rangle \equiv \alpha^* \langle\alpha||\beta\rangle \\
\langle\beta|\hat{\psi}|\alpha\rangle \equiv \alpha \langle\beta||\alpha\rangle
\]

where $\langle\alpha|$ and $|\beta\rangle$ are the left and right eigenvectors, respectively of $\hat{\psi}^\dagger$, whereas, $\langle\beta|$ and $|\alpha\rangle$ are the left and right eigenvectors, respectively, of $\hat{\psi}$. The above pairing allows us to form dual eigenvectors to simulate the Hilbert-spaces of Hermitian operators.

We now denote the $\langle\alpha|$ and $|\beta\rangle$ left and right eigenvectors, respectively of $\hat{\psi}^\dagger$ as making up the $\alpha^\star$-Hilbert space, and we will adopt a new consistent labels, $\langle\alpha\rangle \mapsto |p\rangle$ and $|\beta\rangle \mapsto |p\rangle$. Similary, we relabel the $\langle\beta|$ and $|\alpha\rangle$ left and right eigenvectors, respectively, of $\hat{\psi}$ as making up the $\alpha$-Hilbert space, with $\langle\beta\rangle \mapsto |q\rangle$ and $|\alpha\rangle \mapsto |q\rangle$. The $q$ and $p$- eigenspaces constitute our newly-formed quantum label for Hilbert spaces for $\hat{\psi}$ and $\hat{\psi}^\dagger$, respectively.

5.3 Completeness relations

In the new dual space representation, Eq. (48) becomes simply a completeness relation,

\[
\sum_n |\alpha_n\rangle \langle\beta_n| |\alpha_n\rangle = 1 \mapsto \sum_q |q\rangle \langle q| |q\rangle = 1 \\
\quad \mapsto \quad \sum_q |q\rangle \langle q| = 1 \quad (49)
\]

which yields the completeness relation, with normalized $\langle q|q\rangle \equiv 1$. Thus, the the $|q\rangle$ and $|p\rangle$ eigenstates obey the completeness relations,

\[
\sum_q |q\rangle \langle q| = 1 \\
\sum_p |p\rangle \langle p| = 1 \quad (50)
\]

Then it becomes trivial to see the transformation between the dual spaces, $|q\rangle$ and $|p\rangle$ eigenstates,

\[
|q\rangle = \sum_\phi |p\rangle \langle p| |q\rangle \quad (51) \\
|p\rangle = \sum_\theta |q\rangle \langle q| |p\rangle \quad (52)
\]
with transformation function between elements of new dual spaces given by $\langle p | q \rangle$ and $\langle q | p \rangle$, respectively. From Eqs. (46) and (47), we have

$$\langle q_m | q_n \rangle = \delta_{m,n}$$
$$\langle p_m | p_n \rangle = \delta_{m,n}$$

firmly defining the complete and orthogonal dual Hilbert spaces, $\{ | q \rangle \}$ and $\{ | p \rangle \}$.

### 5.4 Generation of states

Equation (41), defines the generation of state $| q \rangle$,

$$| q \rangle = C_o \exp \alpha \psi^\dagger | 0 \rangle$$

$$\psi^\dagger | q \rangle = \psi^\dagger \exp \left( q \psi^\dagger | 0 \rangle \right) = \frac{\partial}{\partial q} | q \rangle$$

Inserting the term $\exp \left\{ -\alpha^* \hat{\psi} \right\}$ right in front of $| 0 \rangle$, which has the effect of multiplying by unity, we obtain a fully symmetric form as

$$| q \rangle = C_o \exp \alpha \psi^\dagger \exp \left\{ -\alpha^* \hat{\psi} \right\} | \psi_0 \rangle$$

To avoid confusion, we set the eigenvalues $\alpha = q$ and $\alpha^* = p$. We also set $| q \rangle = \exp \left( -i q \cdot \hat{P} \right) | 0 \rangle$, i.e., we have,

$$\psi^\dagger | q \rangle = \frac{\partial}{\partial q} | q \rangle = -i \hat{P} | q \rangle$$
$$\hat{P} | p \rangle = p | p \rangle$$

To calculate $\langle p | q \rangle$, we proceed as follows.

$$\langle p | \psi^\dagger | q \rangle = \langle p | \frac{\partial}{\partial q} | q \rangle$$
$$-i p \langle p | q \rangle = \frac{\partial}{\partial q} \langle p | q \rangle$$
$$\frac{\partial}{\partial q} \langle p | q \rangle = -i p$$
$$\frac{\partial}{\partial q} \ln \langle p | q \rangle = -i p$$
$$\langle p | q \rangle = \exp (-i q \cdot p)$$

\[ \text{footnote: There is arbitrariness in incorporating } \exp \{ \phi \hat{a} \} \text{, either positive or negative exponent, operating on vacuum state. To be symmetric we should use positive exponent, } \exp \{ \phi \hat{a} \}. \text{ For convenience, we want the generation os state unitary, so is advisable to use the negative exponent. We will follow this convention is what follows.} \]
Similarly, we have
\[ \hat{\psi} |p\rangle = \frac{\partial}{\partial p} |p\rangle = i\hat{Q} |p\rangle \] (62)
\[ \hat{Q} |q\rangle = q |q\rangle \] (63)
and
\[ \langle q| \hat{\psi} |p\rangle = \langle q| \frac{\partial}{\partial p} |p\rangle \]
\[ iq \langle q| |p\rangle = \frac{\partial}{\partial p} \langle q| |p\rangle \]
\[ \langle q| |p\rangle = \exp i p \cdot q \] (64)

So far all the above developments holds for fermions and bosons. However, note that for fermions the eigenvalues corresponding to q and p are elements of the Grassmann algebra. First we will treat the case of mixed space representation of boson quantum field theory.

6 Mixed Space Representation in QFT: Bosons

Since the commutator for bosons, \( [\hat{\psi}, \hat{\psi}^\dagger] \implies [\hat{Q}, -i\hat{P}] = 1 \ldots \), is a C-number (i.e., not an operator), we can readily make use of the Campbell-Baker-Hausdorff operator identity, namely, 8

\[ C_o \exp \left\{ -i q \cdot \hat{P} \right\} \exp \left\{ i p \cdot \hat{Q} \right\} = C_o \exp \left( -i \left\{ q \cdot \hat{P} - p \cdot \hat{Q} \right\} \right) \exp \left\{ -\frac{\left[ q \cdot \hat{P}, p \cdot \hat{Q} \right]}{2} \right\} \pm \]
\[ = C_o \exp \left( -i \left\{ q \cdot \hat{P} - p \cdot \hat{Q} \right\} \right) \exp \left\{ i q \cdot p \right\}. \] (65)

8Equation (65) readily follows from the symmetric form of translation operator in Eq. (26), involving the universal canonical operators, \( \hat{Q} \) and \( \hat{P} \),

\[ \exp \left\{ -\frac{i}{\hbar} q \cdot \hat{P} \right\} \exp \left\{ \frac{i}{\hbar} p \cdot \hat{Q} \right\} = \exp \left\{ -\frac{i}{\hbar} \left( q \cdot \hat{P} - p \cdot \hat{Q} \right) \right\} \exp \left\{ -\frac{i}{\hbar} \frac{p \cdot q}{2} \right\} \]

if one substitute the following relations
\[ P = i\hbar \hat{a}^\dagger \]
\[ p = i\hbar \alpha^* \]
\[ Q = \hat{a} \]
\[ q = \alpha \]
Therefore, the generator of $|q⟩$ states as,

$$
|q⟩ = C_o \exp \left\{ i \frac{q \cdot p}{2} \right\} \exp \left\{ -i \left\{ q \cdot \hat{\mathcal{P}} - p \cdot \hat{\mathcal{Q}} \right\} \right\} |0⟩ = \mathcal{D}(q)|0⟩ .
$$

(66)

Thus $\mathcal{D}(q)$ is the displacement operator that generates the state $|q⟩$ from the vacuum, $|0⟩$. Note that $\exp \left\{ i p' \cdot \hat{\mathcal{Q}} \right\} |p⟩ = \exp \left\{ p' \cdot \frac{\partial}{\partial p} \right\} |p⟩$ when operating on $|p⟩$ states.

6.1 Reformulation as mixed Space Representation

We can now formulate the non-Hermitian operators to Hermitian-like dual space representation, similar to the $q$-$p$ dual space representation of Sec. 2.1.

6.1.1 Isomorphism between $q$-$p$ and $q$-$p$ dual spaces

The development that follows basically demonstrate the remarkable isomorphism of annihilation and creation operators, $q$-$p$ or $\alpha - \alpha^*$ ($\psi - \psi^*$) and their dual Hilbert spaces with the position and momentum operators and their dual Hilbert spaces, $q$-$p$. This is clearly seen by the following bijective and homomorphic or isomorphic transformation, i.e., the harmonic oscillator transformation from position and momentum operators to annihilation and creation operators,

$$
\begin{pmatrix}
\hat{\psi} \\
\hat{\psi}^\dagger
\end{pmatrix} =
\begin{pmatrix}
\hat{Q} \\
-i \hat{P}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{Q} \\ i \hat{P} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{Q} + i \hat{P} \\ \hat{Q} - i \hat{P} \end{pmatrix}
$$

(67)

(68)

where $\hat{Q}$ and $\hat{P}$ are dimensionless canonical operators, with eigenvalues, $q$ and $p$, respectively. Then the transformation of Eq. (67) induces a transformation of Eq. (69) as

$$
\mathcal{D}_{sym}(q,p) = \exp \left\{ -i (\hat{P} \cdot q - \hat{Q} \cdot p) \right\} = \exp \left\{ -i \hat{P} \cdot q + i \hat{Q} \cdot p \right\} = \exp \left\{ \hat{\psi}^\dagger \cdot q + \hat{\psi} \cdot i p \right\}
$$

(69)

We have

$$
\begin{align*}
\hat{\psi}^\dagger \cdot q + \hat{\psi} \cdot i p &= \frac{1}{2} \left[ \left( \hat{Q} - i \hat{P} \right) \cdot (v + iu) + \left( \hat{Q} + i \hat{P} \right) \cdot (v - iu) \right] \\
&= \frac{1}{2} \left[ \left( \hat{Q} - i \hat{P} \right) \cdot (v + iu) + \left( \hat{Q} + i \hat{P} \right) \cdot (-v + iu) \right] \\
&= -i \left( \hat{P} \cdot v - \hat{Q} \cdot u \right)
\end{align*}
$$
which upon substituting in Eq. (69) gives

\[
Y(q, p) = \exp \{-i \mathbf{P} \cdot \mathbf{q} + i \mathbf{Q} \cdot \mathbf{p} \}
\]

\[
= \exp \left[ -i \left( \hat{P} \cdot v - \hat{Q} \cdot u \right) \right]
\]

where \(Y(q, p)\) is the symmetric translation operator in position-momentum, \(q-p\) dual space. Therefore the dual space \(\{q, p\}\) or \(\{\alpha, \alpha^*\}\) is isomorphic to \(q-p\) dual space.

### 6.2 The q-p representations and lattice Weyl Transform

The mixed \(q-p\) representation basically start by expanding any quantum operator, \(\hat{A}\), in terms of mutually unbiased basis states, namely the eigenvector of annihilation operator, \(\hat{\psi}\) or \(Q\), and the eigenvector of creation operator, \(\hat{\psi}^\dagger\) or \(P\). We have

\[
\hat{A} = \sum_{p, q} \langle q | \hat{A} | p \rangle |q\rangle\langle p| \tag{70}
\]

### 6.3 The completeness of dual spaces

The set \(\{|q\rangle\langle p|\}\) is the basis operators for the mixed \(q-p\) representation. From the completeness relations of the unbiased basis states, \(\{|q\rangle\}\) and \(\{|p\rangle\}\), the set \(\{|q\rangle\langle p|\}\) obeys the completeness relation

\[
\sum_{q, p} |q\rangle\langle q| |p\rangle\langle p| = 1 \tag{71}
\]

\[
\sum_{q, p} \langle q | |p\rangle\langle q| |p\rangle = 1 \tag{72}
\]

Substituting the expression for \(\langle q | |p\rangle\),

\[
\langle q | |p\rangle = \exp (i p \cdot q) \tag{73}
\]

\[
\langle p | |q\rangle = \exp (-i p \cdot q) \tag{74}
\]

we obtained, for the completeness relation,

\[
C_0 \sum_{q, p} \exp (i p \cdot q) |q\rangle\langle p| = 1 \tag{75}
\]

where \(C_0\) can be choosen as

\[
C_0 = (N)^{-\frac{1}{2}}
\]

Equation (8) can be rewritten as

\[
(N)^{-\frac{1}{2}} \sum_{q, p} |q\rangle\langle p| = 1 \tag{76}
\]
and similarly,
\[ (N)^{-\frac{1}{2}} \sum_{q,p} |p\rangle\langle q| = 1 \]

Here we use the transformation identities in the mixed q-p representation,
\[
|p\rangle = \sum_q \langle q| |p\rangle |q\rangle \quad (77)
\]
\[
\langle p| = \sum_{\theta} \langle p| |q\rangle \langle q| \quad (78)
\]
\[
|q\rangle = \sum_{\phi} \langle p| |q\rangle |p\rangle \quad (79)
\]
\[
\langle q| = \sum_{\phi} \langle q| |p\rangle \langle p| \quad (80)
\]

with transformation functions given by Eqs. (6)-(7).

6.3.1 The expansion of any operators in dual space

Any operator, \( \hat{A} \), can be expressed in terms of the unbiased eigenvector spaces, \(|q\rangle \) and \(|p''\rangle \), respectively, in a mixed representation by Eq. (4), which we rewrite as,
\[
\hat{A} = \sum_{p'',q'} |p''\rangle \langle p''| A |q'\rangle \langle q'| \quad (81)
\]

We wish to express \( \langle p''| A |q'\rangle \) and \( |p''\rangle \langle q'| \) in terms of the \(|q\rangle\)-eigenstate matrix elements and \(|p\rangle\)-space projectors, respectively. Using, Eqs. (10)-(14), we write
\[
\langle p''| A |q'\rangle = \frac{1}{\sqrt{N}} \sum_{q''} e^{-ip'' \cdot q''} \langle q''| A |q'\rangle
\]
\[
|p''\rangle \langle q'| = \frac{1}{\sqrt{N}} \sum_{p'} e^{ip' \cdot q'} |p''\rangle \langle p'|
\]

with completeness relation, using the transformation function characteristic of dual spaces,
\[
\frac{1}{\sqrt{N}} \sum_{q,p} \exp(ip \cdot q) |q\rangle \langle p| = 1 = \sum_p |p\rangle \langle p|
\]

Introducing the notation in Eq. (20),
\[
p' = p + u, \quad q' = q + v, \quad p'' = p - u, \quad q'' = q - v.
\]
Then, upon substituting in Eq. [19], we end up with

\[
A = \sum_{p'', q'} |p''\rangle \langle p'| A |q'\rangle \langle q'|
\]

\[
= \frac{1}{N} \sum_{p, q, u, v} e^{i2(p \cdot v + u \cdot q)} \langle q - v | A | q + v \rangle | p - u \rangle \langle p + u |
\]

### 6.3.2 Mixed space operator basis, \( \hat{\Delta} (p, q) \)

We write the last result as an expansion in terms of mixed-phase point projector, \( \hat{\Delta} (p, q) \), defined as the Weyl transform of a projector, by

\[
\hat{\Delta} (p, q) = \sum_u e^{-i2u \cdot q} | p + u \rangle \langle p - u |
\]

\[
= \sum_u e^{-i2u \cdot q} e^{2iQ \cdot u} | p - u \rangle \langle p - u |
\]

\[
= \sum_u e^{-i2u \cdot q} e^{2iQ \cdot u} \sum_v e^{2i(p - u \cdot P \cdot v)} | p_0 \rangle \langle p_0 |
\]

\[
= \sum_{u, v} e^{2i(Q - q) \cdot u} e^{-2i(P - p \cdot v \cdot u)} | p_0 \rangle \langle p_0 |
\]

\[
= \sum_{u, v} e^{2i(p \cdot v - q \cdot u)} e^{-2i(P \cdot u - Q \cdot u)} \sum_{p_0} | p_0 \rangle \langle p_0 | \quad (83)
\]

and the coefficient of expansion, the so-called Weyl transform of matrix element of operator, \( A (p, q) \), defined by

\[
A (p, q) = \sum_v e^{i2p \cdot v} \langle q - v | A | q + v \rangle .
\]

Clearly, for a density matrix operator \( \hat{\rho} \), the Weyl transform obeys,

\[
\sum_{p, q} \rho (p, q) = 1
\]

If one accounts for other extra discrete quantum labels like spin and energy-band indices, we can incorporate this in the summation in a form of a trace.

Thus, we eventually have any operator expanded in terms of mixed space operator basis, \( \hat{\Delta} (p, q) \),

\[
\hat{A} = \sum_{p, q} A (p, q) \hat{\Delta} (p, q) \quad (84)
\]

\[
= \sum_{u, v} \left( \sum_{p, q} A (p, q) e^{2i(p \cdot v - q \cdot u)} \right) e^{-2i(P \cdot u - Q \cdot u)} | p_0 \rangle \langle p_0 | \quad (85)
\]

35
We have

\[ A(p, q) = Tr \left( \hat{A} \hat{\Delta} (p, q) \right) \]

\[ = \sum_{u,v} e^{2i(p \cdot u - q \cdot v)} Tr \left( e^{-2i(P \cdot u - Q \cdot v)} |p_0 \rangle \langle p_0| \right) \]

\[ = \sum_{u,v} e^{2i(p \cdot u - q \cdot v)} A(u, v) \]

where \( A(u, v) \) is the characteristic function of \( A(p, q) \) distribution. Upon similar procedure based on Eq. (19), an equivalent expression can be obtain for \( A(p, q) \) and \( \hat{\Delta}(p, q) \), namely,

\[ A(p, q) = \sum_{u} e^{i2u \cdot q} (p + u \rangle \hat{A} \langle p - u) \]  

(86)

\[ \hat{\Delta}(p, q) = \sum_{v} e^{i2p \cdot v} |q + v \rangle \langle q - v| \]  

(87)

6.3.3 Symmetrization of the generator of eigenstates

Generator of eigenstates occupy a central role in non-Hermitian quantum mechanics. We can also define this in the Hermitian mixed \( q-p \) representation. We have for \( q \),

\[ |q\rangle = \exp \{-i q \cdot P\} |0\rangle , \]

(88)

where the operator \( P = i \nabla_q \) is operating on the basis eigenvector \( |q\rangle \), since \( P = -i \nabla_x \) acting on the \( x \)-components behaves contravariantly. Here, we basically made an assumption in the above expression that there exist continuous function of \( q \) having an infinite radius of convergence, which are equal to \( |q\rangle \) at the lattice points, i.e., we expand the exponential in Eq. (88) as a well-defined Taylor series.

6.3.4 Fully symmetric translation operators

The state

\[ |q\rangle = \exp \{-i q \cdot P\} |0\rangle \]

(89)

is an eigenstate with displaced eigenvalue by \( q \). However, if the limit \( q' \rightarrow 0 \) is not taken then the state \[ \exp \{-i q \cdot P\} |q'\rangle = |q' + q\rangle \] is an eigenstate of the position operator with eigenvalue \( q' + q \).

We can symmetrize the translation operator by inserting \( \exp \{i p \cdot Q\} \) in front of \( |0\rangle \), in Eq. (89), which effectively insert unity. We thus have

\[ |q\rangle = \exp \{-i q \cdot P\} \exp \{i p \cdot Q\} |0\rangle . \]  

(90)
By the use of the Campbell-Baker-Hausdorff operator identity, we obtained
\[
\exp \{-i q \cdot P\} \exp \{i p \cdot Q\} = \exp \{ -i (q \cdot P - p \cdot Q) \} \exp \{ -i \frac{p \cdot q}{2} \}.
\]
Therefore we have the symmetric form for the displacement operator generating the state $|q\rangle$ from $|0\rangle$ given by
\[
T(q,p)_{\text{sym}} = \exp \{ -i \frac{p \cdot q}{2} \} \exp \{ -i \frac{\hbar}{\mathcal{Q}} (q \cdot P - p \cdot Q) \}.
\]
where we used the symbol $T(q,p)_{\text{sym}}$ because of isomorphism demonstrated in Eq. (87). The displacement operator may also be interpreted as an operator for the preparation of the quantum eigenstate out of the 'vacuum', $|0\rangle$ basis eigenstate.

6.3.5 The symmetric operator basis for mixed representations

The ubiquitous appearance of the symmetric operator factor,
\[
Y_{p,q} = \exp \{ -i (q \cdot P - p \cdot Q) \} \exp \{ i \frac{p \cdot q}{2} \} \exp \{ -i q \cdot P \} \exp \{ i p \cdot Q \},
\]
suggest that this operator is the basis operator for mixed space representation theory. This is referred to here as the generalized projector or as generalized Pauli-matrix operator [9]. It can be considered the universal form of projector in mixed representations of quantum physics, either dealing with Hermitian or with non-Hermitian operators.

Symmetric form of $\hat{\Delta}_{\lambda \lambda'} (p, q)$ If we consider Eq. (24) as the expression for $\hat{\Delta}_{\lambda \lambda'} (p, q)$, the following identities can be verified,
\[
|q + v, \lambda\rangle = \exp \{-2i P \cdot v\} |q - v, \lambda\rangle
\]
Then
\[
|q - v, \lambda\rangle \langle q - v, \lambda'| = (N)^{-1} \sum_{u, q_0} \exp \{-2i (q - v - Q) \cdot u\} |q_0, \lambda\rangle \langle q_0, \lambda'| = \sum_{q_0} \delta (q - v - Q) |q_0, \lambda\rangle \langle q_0, \lambda'| = \sum_{q_0} \delta (q - v - q_0) |q_0, \lambda\rangle \langle q_0, \lambda'| = \Omega_{\lambda \lambda'} |q_0, \lambda\rangle \langle q_0, \lambda'|.
\]
Here, the concept of a vacuum state does not have a special meaning since $|0\rangle$ represents an arbitrary reference position. It is introduced simply to bring analogy with zero-eigenvalue of non-Hermitian operators in later chapters, where the state $|0\rangle$ has a distinguished position.
Substituting the expressions, Eqs (29) and (30) in Eq. (24), we obtain a completely symmetric expression of \( \hat{\Delta}(p, q) \),

\[
\Delta(p, q) = \sum_v e^{i2p \cdot v} |q + v \rangle \langle q - v|
\]

\[
= (N)^{-1} \sum_{\bar{v}, \bar{u}} e^{i2p \cdot \bar{v}} \exp \{ -2i\mathcal{P} \cdot \bar{u} \} \exp \left[ -2i \left( q - v - \hat{Q} \right) \cdot u \right] \sum_{q_0, \lambda} |q_0, \lambda \rangle \langle q_0, \lambda'|
\]

Then we have

\[
\exp \{ -i(\mathcal{P} - p) \cdot 2v \} \exp \left\{ 2i \left( \hat{Q} - q \right) \cdot \bar{u} \right\} \exp 2i(\mathbf{v} \cdot \mathbf{u})
\]

which is the same as Eq. (89). Thus, we have changed the seemingly asymmetric expression of the first line into a symmetric form of the last line.

We can combine the exponential operators to obtain

\[
(N)^{-1} \sum_{\bar{v}, \bar{u}} e^{i2p \cdot \bar{v}} \exp \{ -2i\mathcal{P} \cdot \bar{u} \} \exp \left[ 2i \left( q - v - \hat{Q} \right) \cdot u \right] \sum_{q_0, \lambda} |q_0, \lambda \rangle \langle q_0, \lambda'|
\]

\[
= (N)^{-1} \sum_{\bar{v}, \bar{u}} \exp -2i [(\mathcal{P} - p) \cdot v + (\hat{Q} - q) \cdot u] \Omega_{\lambda' \lambda}
\]

where,

\[
\Omega_{\lambda' \lambda} = \sum_{q_0, \lambda} |q_0, \lambda \rangle \langle q_0, \lambda'| = \sum_{\mathbf{p}_0, \lambda} |\mathbf{p}_0, \lambda \rangle \langle \mathbf{p}_0, \lambda'|
\]

\[
A(p, q) = Tr \left( \hat{\Delta} \hat{\Delta} \right)
\]

\[
= (N)^{-1} \left( \sum_{\bar{v}, \bar{u}} \exp 2i [p \cdot v - q \cdot u] \right)
\]

\[
\times Tr \left\{ \hat{A} \exp \{ -2i \left[ \mathcal{P} \cdot v - \hat{Q} \cdot u \right] \} \Omega_{\lambda' \lambda} \right\}
\]

Therefore, the characteristic distribution for \( A(p, q) \) is identically,

\[
A_{\lambda' \lambda}(u, v) = Tr \left\{ \hat{A} \exp \{ -2i \left[ \mathcal{P} \cdot v - \hat{Q} \cdot u \right] \} \Omega_{\lambda' \lambda} \right\}
\]

as before. By using the characteristic distribution function for \( A_{\lambda' \lambda}(p, q) \)

\[
A_{\lambda' \lambda}(u, v) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{p, q} A_{\lambda' \lambda}(p, q) \exp 2i [p \cdot v - q \cdot u] \quad (97)
\]

with inverse

\[
A_{\lambda' \lambda}(p, q) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{u, v} A_{\lambda' \lambda}(u, v) \exp \{ -2i \left[ p \cdot v - q \cdot u \right] \}
\]
Then we can write Eq. (22) simply like a Fourier transform (caveat: Fourier transform to operator space) of the characteristic function of the lattice Weyl transform of the operator $\hat{A}$,

$$\hat{A} = \sum_{\tilde{u}, \tilde{v}, \lambda, \lambda'} A_{\lambda \lambda'} (u,v) \exp \{-2i [P \cdot v - Q \cdot u]\} \Omega_{\lambda' \lambda}$$  \hspace{1cm} (98)

where the inverse can be written as

$$A_{\lambda \lambda'} (u,v) = \text{Tr} \{ \hat{A} \exp \{-2i [P \cdot v - Q \cdot u]\} \} \Omega_{\lambda' \lambda}$$

In continuum approximation, we have,

$$\hat{\Delta} (p, q) = (2\pi)^{-1} \int du \wedge dv \; e^{2i[(p-P) \cdot v - q - Q \cdot u]} = (2\pi)^{-1} \int du \wedge dv \; e^{2i[p \cdot v - q \cdot u]} e^{(-2i)(P \cdot v - Q \cdot u)} \Omega_{\lambda \lambda'},$$  \hspace{1cm} (99)

6.3.6 On the counting of states: Coherent state formulation

The measure of counting of states in Eq. (99) can be shown to be related to original annihilation and creation operator in the case of harmonic oscillator. We have

$$(2\pi)^{-1} \int du \wedge dv \quad \Rightarrow \quad (2\pi)^{-1} \int -idu \wedge dv$$

$$= (2\pi)^{-1} \int \frac{1}{2} d(\bar{q} - i\bar{p}) \wedge d(\bar{q} + i\bar{p})$$

$$= (2\pi)^{-1} \int (d\bar{q} \wedge d\bar{p})$$

$$= \frac{1}{\pi} \int (d \text{Re} \alpha \wedge d \text{Im} \alpha)$$

since $d \text{Re} \alpha = \frac{d\bar{q}}{\sqrt{2}}$ and $d \text{Im} \alpha = \frac{d\bar{p}}{\sqrt{2}}$.

6.4 Characteristic distribution of lattice Weyl transform

Therefore, we have the identity for the characteristic function $A_{\lambda \lambda'} (u,v)$

$$A_{\lambda \lambda'} (u,v) = \text{Tr} \{ \hat{A} \exp \{-2i [P \cdot v - Q \cdot u]\} \} \Omega_{\lambda \lambda'}$$  \hspace{1cm} (100)

The characteristic function exist for all function of canonical quantum operators, either Hermitian or non-Hermitian, spinor (fermions) or boson operators.\(^\text{10}\)

\(^\text{10}\)For creation and annihilation operators in many-body quantum physics, the proof relies on the use of normal or anti-normal ordering of canonical operators, which can then be treated like C-numbers in expansion of exponentials. The exponential in Eq. (100) is sometimes referred to as the generalized Pauli-spin operator.
6.4.1 Implications on coherent states (CS) formulation

In what follows, we will drop the discrete indices $\lambda$ and $\lambda'$ to make contact with CS formulation of quantum physics. In general, we can have different expression for the characteristic function depending on the use of, what is often referred to in corresponding CS formalism as the normal and anti-normal expressions,

$$
\exp \{-2i [P \cdot v - Q \cdot u]\} = \exp \{iu \cdot v\} \exp \{-2iv \cdot P\} \exp \{2iu \cdot Q\}
$$

$$
= \exp \{-iu \cdot v\} \exp \{2iu \cdot Q\} \exp \{-2iv \cdot P\}
$$

so that

$$
\exp \{-2iv \cdot P\} \exp \{2iu \cdot Q\} = \exp \{-iu \cdot v\} \exp \{-2i [P \cdot v - Q \cdot u]\}
$$

$$
\exp \{2iu \cdot Q\} \exp \{-2iv \cdot P\} = \exp \{iu \cdot v\} \exp \{-2i [P \cdot v - Q \cdot u]\}
$$

(101)

yielding the following difference expressions for $A_{\lambda\lambda'}(u,v)$, namely,

$$
A^w_{\lambda\lambda'}(u,v) = Tr \left( \hat{A} \exp \{-2i (P \cdot v - Q \cdot u)\} \right)
$$

(103)

which is the characteristic function for the Wigner distribution function. We also have the so-called normal characteristic distribution function,

$$
A^n_{\lambda\lambda'}(u,v) = Tr \left[ \hat{A} \exp \{-2iv \cdot P\} \exp \{2iu \cdot Q\} \right],
$$

$$
= \exp \{-iu \cdot v\} Tr \left( \hat{A} \exp \{-2i (P \cdot v - Q \cdot u)\} \right)
$$

(104)

and the anti-normal characteristic distribution function given by

$$
A^a_{\lambda\lambda'}(u,v) = Tr \left\{ \hat{A} \exp \{2iu \cdot Q\} \exp \{-2iv \cdot P\} \right\},
$$

$$
= \exp \{iu \cdot v\} Tr \left( \hat{A} \exp \{-2i (P \cdot v - Q \cdot u)\} \right)
$$

(105)

Although, Eqs. (37) and (38) only amounts to difference in the phase factors in the canonical position-momentum $q-p$ ordinary mixed space representation, similar quantities in non-Hermitian dual spaces gives a real exponents giving very different distributions often referred to as smooth-out distributions. Examining Eqs. (34) and (35), and the fact that in non-Hermitian mixed representation,

$$
\exp \{-iu \cdot v\} = \bar{\alpha}^* \bar{\alpha} = \frac{1}{2} (\bar{q} - i\bar{p}) (\bar{q} + i\bar{p})
$$

$$
= \frac{1}{2} (\text{Re} \, \bar{\alpha}^2 + \text{Im} \, \bar{\alpha}^2), \quad (106)
$$

$$
\exp \{iu \cdot v\} = -\bar{\alpha}^* \bar{\alpha} = -\frac{1}{2} (\bar{q} - i\bar{p}) (\bar{q} + i\bar{p})
$$

$$
= -\frac{1}{2} (\text{Re} \, \bar{\alpha}^2 + \text{Im} \, \bar{\alpha}^2), \quad (107)
$$

in original notation is a real quantity that resembles a Guassian function, Eqs. (102) clearly represent some smoothing of the Wigner distribution characteristic
function and hence the Wigner distribution itself. Indeed, in Eqs. (37) and (38) no informations are lost.

Indeed, more general phase-space distribution functions, \( f^{(g)}(p, q, t) \), can be obtained from the expression

\[
f^{(g)}(p, q, t) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{u,v} A_{\lambda \lambda'}(u,v) \exp \left\{ -2i \left[ p \cdot v - q \cdot u \right] \right\} g(u,v)
\]

where \( g(u,v) \) is some chosen smoothing function.

6.4.2 P- and Q-function or Husimi distribution: smoothing

Generally all distribution function will become meaningful under the integral sign, thus these have the properties of generalized distribution functions. The distribution function \( f^{a}(p, q, t) \) that one obtain from Eq. (106) is known as the P-function and that obtain from Eq. (107) is also known as the Q-function or the Husimi distribution in quantum optics. A detailed discussion of these two distribution is given by the author’s book on the topic of coherent state formulation, and will not be repeated here.

The identical algebra of \( \exp \left\{ -i (q' \cdot P - p' \cdot Q) \right\} \) as well as its relevance to the physics of two-state systems, spin systems, quantum computing, entanglements [21] and teleportation, are discussed in one of the author’s book [9].

6.5 Path integral for bosons

The path integral for bosons straightforwardly follows from the above \( q-p \) mixed space representation. This is given by the author [10] and will not be repeated here. We will just give the result as,

\[
\langle q | U(t, t_0) | q_0 \rangle = \lim_{n \to \infty} \int \ldots \int \prod_{i=1}^{n} d^{N} q_i \prod_{i=1}^{n+1} d^{N} p_i \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{n} \epsilon \left[ p_j \cdot q_j - q_j - 1 \right] - H \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \right\}
\]

where \( \epsilon = (t_j - t_{j-1}) \) and \( H \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \) is the Weyl transform of the Hamiltonian.

7 Mixed Space Representation in QFT: Fermions

The canonical field operators satisfy the following anticommutation relations,

\[
\begin{align*}
[\psi_{\mu}^\dagger, \psi_{\nu}^\dagger]_+ & \equiv \{ \psi_{\mu}^\dagger, \psi_{\nu}^\dagger \} = 0 \\
[\bar{\psi}_{\mu}, \psi_{\nu}]_+ & \equiv \{ \bar{\psi}_{\mu}, \psi_{\nu} \} = 0 \\
[\hat{\psi}, \hat{\psi}^\dagger]_+ & \equiv \{ \hat{\psi}, \hat{\psi}^\dagger \} = 1
\end{align*}
\]
We have, as before write, \( \hat{\psi}^\dagger = -i \hat{\rho} \) as before,

\[
\left[ \hat{\psi}, \hat{\psi}^\dagger \right] \implies \left[ \hat{Q}, -i \hat{\rho} \right] = 1
\]

\[
\left[ \hat{Q}, \hat{\rho} \right] = i
\]

(109)

To avoid confusion, we set the respective eigenvalues of \( \hat{Q} \) and \( \hat{\rho} \) as \( q \) and \( p \), respectively. As before, we have the left and right eigenvector defined by

\[
\hat{\rho} |p\rangle = p |p\rangle = |p\rangle p,
\]

\[
\langle p | \hat{\rho} = \langle p | p = p \langle p | .
\]

Likewise we have

\[
\hat{Q} |q\rangle = q |q\rangle = |q\rangle q,
\]

\[
\langle q | \hat{Q} = \langle q | q = q \langle q | .
\]

(111)

Thus, although the eigenvalues and fermion operators are elements of Grassmann algebra, the eigenvectors are \( \mathbb{C} \)-vectors and commutes with Grassmann variables.

For fermions, we can with Eqs. (53) - (64) which are common to both bosons and fermions. However, for fermions, the eigenvalues are elements of the Grassmann algebra. The following eigenvector generation holds for bosons and fermions, namely,

\[
\hat{\psi}^\dagger |q\rangle = -i \hat{\rho} |q\rangle = \frac{\partial}{\partial q} |q\rangle \implies \hat{\rho} |q\rangle = i \frac{\partial}{\partial q} |q\rangle
\]

\[
|q\rangle \hat{\psi}^\dagger = \langle q | i \hat{\rho} = \langle q | \left( \frac{\partial}{\partial q} \right) \implies \langle q | \hat{\rho} = \langle q | i \frac{\partial}{\partial q}
\]

\[
\hat{\psi} |p\rangle = i \hat{Q} |p\rangle = \frac{\partial}{\partial p} |p\rangle \implies \hat{Q} |p\rangle = -i \frac{\partial}{\partial p} |p\rangle
\]

\[
\langle p | (-i \hat{Q}) = \langle p | \left( -\frac{\partial}{\partial p} \right) \implies \langle p | \hat{Q} = \langle p | \left( -i \frac{\partial}{\partial p} \right)
\]

which is compatible with Eqs. (110) - (111). We also have

\[
|q\rangle = \exp \left( -iq \hat{\rho} \right) |0\rangle \implies \hat{\rho} |q\rangle = i \frac{\partial}{\partial q} |q\rangle
\]

\[
\langle q | = \langle 0 | \exp \left( iq \hat{\rho} \right) \implies \langle q | \hat{\rho} = \langle q | i \frac{\partial}{\partial q}
\]

\[
|p\rangle = \exp \left( ip \hat{Q} \right) |0\rangle \implies \hat{Q} |p\rangle = -i \frac{\partial}{\partial p} |p\rangle
\]

\[
\langle p | = \langle 0 | \exp \left( -ip \hat{Q} \right) \implies \langle p | \hat{Q} = \langle p | \left( -i \frac{\partial}{\partial p} \right)
\]
which is again compatible with Eqs. \([110]-[111]\), where the arrows denote the left and right derivatives and the dot product is defined by

\[ \psi^\dagger \cdot q = \sum_\mu q_\mu \psi^\dagger_\mu = \sum_\mu q_\mu \left(-i \hat{P}_\mu\right) \]

Note that we retained the notation, \(p\) and \(q\) for eigenvalues, with the understanding that for fermions, \(p\) and \(q\) are elements of Grassmann algebra, not ordinary \(\mathbb{C}\)-numbers.

### 7.1 Transformation functions

Using the identity: 

\[ e^A e^B = e^B e^A [A,B] \]

for the case where the commutator \([A,B]\) is a \(\mathbb{C}\)-number, we have the following expression for the transformation functions

\[ \langle p|\langle q \rangle = \langle p = 0|\exp \left(-ip \cdot \hat{Q}\right) \exp \left(-iq \cdot \hat{P}\right) |q = 0\rangle = \langle 0|0\rangle \exp [ip \cdot \hat{Q}] = \exp [ip \cdot \hat{Q}] \]

since, we have

\[ \left[-ip \cdot \hat{Q}, -iq \cdot \hat{P}\right] = \left[\left(p \cdot \hat{Q}\right) \left(q \cdot \hat{P}\right) - \left(q \cdot \hat{P}\right) \left(p \cdot \hat{Q}\right)\right] \]

\[ = \sum_{\mu,\nu} p_\mu q_\nu \left\{ \hat{Q}_\mu \hat{P}_\nu + \hat{P}_\nu \hat{Q}_\mu \right\} = i \sum_{\mu,\nu} p_\mu q_\nu \delta_{\mu\nu} \]

\[ = i \sum_\mu p_\mu q_\mu = ip \cdot q \]

Equation \([110]\), defines the generation of state \(|q\rangle\),

\[ |q\rangle = C_o \exp \left(-iq \cdot \hat{P}\right) |0\rangle \]

Inserting the term \(\exp \left\{ip \cdot \hat{Q}\right\}\) right in front of \(|0\rangle\)\(^\text{11}\) which has the effect of multiplying by unity, we obtain a fully symmetric form as

\[ |q\rangle = C_o \exp \left(-iq \cdot \hat{P}\right) \exp \left\{ip \cdot \hat{Q}\right\} |0\rangle \]

We have,

\[ \psi^\dagger |q\rangle = \frac{\partial}{\partial q} |q\rangle = -i \hat{P} |q\rangle \]

\[ \hat{P} |p\rangle = p |p\rangle \]

\(^{11}\text{There is arbitrariness in incorporating } \exp \{\phi \hat{a}\}, \text{ either positive or negative exponent, operating on vacuum state. To be symmetric we should use positive exponent, } \exp \{\phi \hat{a}\}. \text{ For convenience, we want the generation or state unitary, so is advisable to use the negative exponent. We will follow this convention is what follows.}\)
To calculate \( \langle p | q \rangle \), on the same manner as for boson but accounting for Grassmann variables, we proceed as follows.

\[
\langle p | \hat{\psi} \rangle |q\rangle = \langle p | \frac{\partial}{\partial q} |q\rangle \tag{116}
\]

\[
-i p \langle p | q \rangle = \frac{\partial}{\partial q} \langle p | q \rangle \tag{117}
\]

\[
\frac{\partial}{\partial q} \langle p | q \rangle = -i p \tag{118}
\]

\[
\partial_q \ln \langle p | q \rangle = -i p \tag{119}
\]

\[
\langle p | q \rangle = \exp(-i q \cdot p) \tag{120}
\]

\[
= \exp(i p \cdot q) \tag{121}
\]

The last line occurs by virtue of Grassmann algebra for the eigenvalues. Similarly, we have

\[
\hat{\psi} |p\rangle = \frac{\partial}{\partial q} |p\rangle = i \hat{Q} |p\rangle \tag{122}
\]

\[
\hat{Q} |q\rangle = q |q\rangle \tag{123}
\]

and

\[
\langle q | i \hat{Q} |p\rangle = i q \langle q | p\rangle \tag{124}
\]

\[
= \frac{\partial}{\partial p} \langle q | p\rangle \tag{125}
\]

\[
i q \langle q | p\rangle = \frac{\partial}{\partial p} \langle q | p\rangle \tag{126}
\]

Note that for fermions the eigenvalues corresponding to \( q \) and \( p \) are elements of the Grassmann algebra.

From

\[
\langle p | q \rangle = \langle p = 0 \rangle \exp(-i p \cdot \hat{Q}) \exp(-i q \cdot \hat{P}) |q = 0\rangle = \langle 0 | 0\rangle \exp[ip.q] = \exp[ip.q] \tag{127}
\]

We can write

\[
\langle p = 0 \rangle \exp(-i p \cdot \hat{Q}) \exp(-i p.q) \exp(-i q \cdot \hat{P}) |q = 0\rangle \equiv 1
\]
We have from Eqs. (71) - (72)
\[ \sum_{q,p} |q\rangle\langle q| |p\rangle \langle p| = 1 \]  
(128)
\[ \sum_{q,p} |q\rangle \exp(-i q \cdot p) \langle p| = 1 \]  
(129)
\[ \sum_{q,p} \exp(i p \cdot q) |q\rangle \langle p| = 1 \]  
(130)
and fro Eqs. (77)-(80)
\[ |p\rangle = \sum_q \exp(i p \cdot q) |q\rangle \]  
(131)
\[ \langle p| = \sum_q \langle q| \exp(-i p \cdot q) \]  
(132)
\[ |q\rangle = \sum_p \exp(i p \cdot q) |p\rangle \]  
(133)
\[ \langle q| = \sum_p \langle p| \exp(-i p \cdot q) \]  
(134)
Therefore, we have also the completeness relation expressed as
\[ \sum_q |q\rangle \langle q| = 1 \]
\[ \sum_p |p\rangle \langle p| = 1 \]
\[ \sum_{q,p} \exp(i p \cdot q) |p\rangle \langle q| = 1 \]  
(135)
\[ \sum_{q,p} \exp(i p \cdot q) |q\rangle \langle p| = 1 \]  
(136)
Equations (135)-(136) are the ones used by Halperin et al [51] in following the coherent state formulation.

### 7.2 "Lattice" Weyl transform for fermions

First we will formulate the Weyl transformation based on finite set of eigenvalues discussed in Sec. 5.1. On the other hand, for continuous eigenvalues or fields, one need to use the Berezin [52] Grassman calculus involving Berezin integration of anticommuting variables.

We have for finite set of eigenvalues, expand any operator as
\[ \hat{A} = \sum_{p'',q'} |p''\rangle\langle p''| A |q'\rangle \langle q'| \]  
(137)
\[ \hat{A} = \sum_{p''', q'} |q'''angle \langle q'''| A |p'\rangle \langle p'| \]
\[ = \sum_{p''', q'} \langle q'''angle A |p'\rangle |q'''angle \langle p'| \]  
\hspace{1cm} (138)  

Using, Eqs. (10)-(14), we write
\[ \langle p'' | A | q' \rangle = \frac{1}{\sqrt{N}} \sum_{q''} e^{-ip'' \cdot q''} \langle q''| A | q' \rangle \]
\[ |p''\rangle \langle q'| = \frac{1}{\sqrt{N}} \sum_{p'} e^{ip' \cdot q'} |p''\rangle \langle p'| \]  
\hspace{1cm} (139)  

Introducing the notation in Eq. (20),
\[ p' = p + u, \quad q' = q + v, \]
\[ p'' = p - u, \quad q'' = q - v. \]

Then
\[
\begin{align*}
    p' \cdot q' &= (\vec{p} + \vec{u}) \cdot (\vec{q} + \vec{v}) = \vec{p} \cdot \vec{q} + \vec{u} \cdot \vec{v} + \vec{p} \cdot \vec{v} + \vec{u} \cdot \vec{q}, \\
    -p'' \cdot q'' &= -(\vec{p} - \vec{u}) \cdot (\vec{q} - \vec{v}) = -\vec{p} \cdot \vec{q} - \vec{u} \cdot \vec{v} + \vec{p} \cdot \vec{v} + \vec{u} \cdot \vec{q}.
\end{align*}
\]

Then, upon substituting in Eq. (19), we end up with
\[ A = \sum_{p''', q'} |p''\rangle \langle p'|| A |q'\rangle \langle q'| \]
\[ = \frac{1}{N} \sum_{p, q, u, v} e^{i2(\vec{p} \cdot \vec{v} + u \cdot q)} \langle q - v| A |q + v\rangle \langle p - u| \langle p + u| \]  

Likewise as in previous formulations, we write the last result as an expansion in terms of ”phase-space” point projector, \( \hat{\Delta} (p, q) \), defined as the fermion ”lattice” Weyl transform of a projector, by
\[ \hat{\Delta} (p, q) = \sum_{u} e^{i2\vec{p} \cdot \vec{q}} |\vec{p} - \vec{u}\rangle \langle \vec{p} + \vec{u}| \]  
\hspace{1cm} (140)  

and the coefficient of expansion, the fermion ”lattice” Weyl transform of matrix element of operator, \( A (p, q) \), defined by
\[ A (p, q) = \sum_{v} e^{i2\vec{p} \cdot \vec{q}} \langle \vec{q} - \vec{v}| A |\vec{q} + \vec{v}\rangle . \]

An equivalent expression can also be obtain for \( A (p, q)\)and \( \hat{\Delta} (p, q) \), namely,
\[ A (p, q) = \sum_{u} e^{i2u \cdot p} \langle p + u| \hat{A} |p - u\rangle \]  
\hspace{1cm} (141)  
\[ \hat{\Delta} (p, q) = \sum_{v} e^{i2p \cdot v} |q + v\rangle \langle q - v| \]  
\hspace{1cm} (142)  


7.3 Symmetric form of $\hat{\Delta}_{\lambda\lambda'}(p, q)$

In the expression, Eq. (142) for $\hat{\Delta}_{\lambda\lambda'}(p, q)$, we can use the following identities,

$$|q + v, \lambda\rangle = \exp[-2iP \cdot v]|q - v, \lambda\rangle$$  \hspace{1cm} (143)

Then

$$|q - v, \lambda\rangle \langle q - v, \lambda'| = \left((N)\right)^{-1} \sum_{u, q_0} \exp[-2i(q - v - \hat{Q}) \cdot u]|q_0, \lambda\rangle \langle q_0, \lambda'|$$  \hspace{1cm} (144)

$$\Omega_{\lambda\lambda'} = \left|q_0, \lambda\rangle \langle q_0, \lambda'|\right|$$

Substituting the expressions, Eqs (29) and (30) in Eq. (24), we obtain a completely symmetric expression of $\hat{\Delta}(p, q)$,

$$\hat{\Delta}(p, q) = \sum_{\bar{v}, \bar{u}} e^{i2\bar{p} \cdot \bar{v}} |q + v\rangle \langle q - v|$$

$$= \left((N)\right)^{-1} \sum_{\bar{v}, \bar{u}} \sum_{\bar{p}, \bar{q}} e^{i2\bar{p} \cdot \bar{v}} \exp[-2iP \cdot \bar{v}] \exp[-2i(q - v - \hat{Q}) \cdot \bar{u}] \sum_{q_0} |q_0, \lambda\rangle \langle q_0, \lambda'|$$

Then we have, even accounting for the anticommutating Grassman variables,

$$\exp\{-i(P - p) \cdot 2\bar{v}\} \exp\{2i(\hat{Q} - q) \cdot \bar{u}\} \exp\{2i(p \cdot v - q \cdot u)\} = \exp\{-i(P \cdot v - Q \cdot u)\} \exp\{2i(p \cdot v - q \cdot u)\}$$  \hspace{1cm} (145)

which is the same as Eq. (83). Thus, we have changed the seemingly asymmetric expression of the first line into a symmetric form of the last line. And again, we have

$$\hat{\Delta}(p, q) = \left((N)\right)^{-1} \sum_{\bar{v}, \bar{u}} \sum_{\bar{p}, \bar{q}} \exp\{-i(P \cdot v - Q \cdot u)\} \exp\{2i(p \cdot v - q \cdot u)\} \Omega_{\lambda\lambda'}$$  \hspace{1cm} (146)

where,

$$\Omega_{\lambda\lambda'} = \sum_{q_0} |q_0, \lambda\rangle \langle q_0, \lambda'| = \sum_{p_0} |p_0, \lambda\rangle \langle p_0, \lambda'|$$

Moreover, we also have

$$A(p, q) = Tr\left(\hat{\Delta}^2\right)$$

$$= \left((N)\right)^{-1} \left(\sum_{\bar{v}, \bar{u}} \sum_{\bar{p}, \bar{q}} \exp\{2i(p \cdot v - q \cdot u)\} \times Tr\left(A \exp\{-2i(P \cdot v - Q \cdot u)\} \Omega_{\lambda\lambda'}\right)\right)$$  \hspace{1cm} (147)
7.4 Characteristic distribution of "lattice" Weyl transform

Therefore, the characteristic distribution for \( A(p, q) \) is identically,

\[
A_{\lambda' \lambda} (u, v) = Tr \left\{ \hat{A} \exp \left\{ -2i [p \cdot v - Q \cdot u] \right\} \Omega_{\lambda' \lambda} \right\}
\]  

(148)

which is formally as before. By using the characteristic distribution function for \( A_{\lambda' \lambda} (p, q) \)

\[
A_{\lambda' \lambda} (u, v) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{p, q} A_{\lambda' \lambda} (p, q) \exp 2i [p \cdot v - q \cdot u]
\]  

(149)

with inverse

\[
A_{\lambda' \lambda} (p, q) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{u, v} A_{\lambda' \lambda} (u, v) \exp \{-2i [p \cdot v - q \cdot u]\}
\]  

(150)

Then we can write Eq. \footnote{22} simply like a Fourier transform \textit{(caveat: Fourier transform to operator space)} of the characteristic function of the lattice Weyl transform of the operator \( \hat{A} \),

\[
\hat{A} = \sum_{u, v, \lambda, \lambda'} A_{\lambda' \lambda} (u, v) \exp \{-2i [p \cdot v - Q \cdot u]\} \Omega_{\lambda' \lambda}
\]  

(151)

where the inverse can be written as

\[
A_{\lambda' \lambda} (u, v) = Tr \left\{ \hat{A} \exp \left\{ -2i [p \cdot v - Q \cdot u] \right\} \right\} \Omega_{\lambda' \lambda}.
\]

7.5 Distribution functions

\[
A_{\lambda' \lambda}^a (u, v) = \exp \{-2iu \cdot v\} Tr \left( \hat{A} \exp (-2i) (P \cdot v - Q \cdot u) \right)
\]

and the anti-normal characteristic distribution function given by

\[
A_{\lambda' \lambda}^a (u, v) = \exp \{iu \cdot v\} T r \left( \hat{A} \exp (-2i) (P \cdot v - Q \cdot u) \right)
\]

Formally again, a more general phase-space distribution functions, \( f^{(g)} (p, q, t) \), can be obtained from the expression

\[
f^{(g)} (p, q, t) = \left( \frac{1}{N} \right)^{\frac{1}{2}} \sum_{u, v} A_{\lambda' \lambda} (u, v) \exp \{-2i [p \cdot v - q \cdot u]\} g (u, v)
\]

where \( g (u, v) \) is some chosen smoothing function. Thus, for finite system, the formalism exactly maps to that of CMP since Bloch electrons are fermions.
8 Continuous Fields: Calculations of Integrals for Fermions

With the continuous annihilation and creation field operators for fermions, our pairing algorithm in Sec. 5.2 is ambiguous, but the results are assumed to be extendable to continuous fields. The major difference occurs in the functional integration of continuous Grassman variables. The reason for this is that Berezin integration [52] is actually similar to (standard) functional differentiation. Formally we can still associate a sort of unification in a symbolic sense for the Grassman integration.

A sort of justification for the readers is by considering translation invariance of a definite integral and linearity:

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} f(x + y) \, dx
\]

\(f(x)\) is at most a linear function of \(x\), a Grassman variable. Hence, we write

\[f(x) = a + bx = (a \ b \ \begin{pmatrix} 1 \\ x \end{pmatrix})\]

or more precisely

\[
f = f_{00} + f_{01}\psi^\dagger + f_{10}\psi + f_{11}\psi^\dagger\psi
\]

\[
= (f_{00} \ f_{01} \ f_{10} \ f_{11}) \begin{pmatrix} 1 \\ \psi^\dagger \\ \psi^\dagger\psi \end{pmatrix}
\]

Linearity says

\[
\int_{-\infty}^{\infty} (a + bf(x)) \, dx = \int_{-\infty}^{\infty} ax + \int_{-\infty}^{\infty} bf(x) \, dx
\]

Thus, we have

\[
\int_{-\infty}^{\infty} f(x + y) \, dx = \int_{-\infty}^{\infty} ax + \int_{-\infty}^{\infty} bx + y) \, dx
\]

\[
= \int_{-\infty}^{\infty} (a + bx) \, dx + \int_{-\infty}^{\infty} by \, dx
\]

\[
= \int_{-\infty}^{\infty} f(x) \, dx + by \int_{-\infty}^{\infty} \, dx
\]

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But by translational invariance
\[ \int_{-\infty}^{\infty} f(x+y) \, dx = \int_{-\infty}^{\infty} f(x) \, dx \]

Therefore, since by is arbitrary, one is forced to equate
\[ \int_{-\infty}^{\infty} dx = 0 \]

which does looks like a differentiation of a constant, here equals to 1. Thus, integration of Grassman numbers obeys
\[ \int dx = 0 \]
\[ \int x \, dx = 1 \]

The previous relations involving integrals must now be interpreted as standard differentiation with respect to the integration variable. We observe that like before we have the eigenvalue equations for the left and right eigenvector defined by
\[ \hat{P} |p\rangle = p |p\rangle = |p\rangle p, \]
\[ \langle p| \hat{P} = \langle p| p = p \langle p|. \quad (152) \]

Likewise we have
\[ \hat{Q} |q\rangle = q |q\rangle = |q\rangle q, \]
\[ \langle q| \hat{Q} = \langle q| q = q \langle q|. \quad (153) \]

where the eigenvalues are now continuous Grassman variables.

The completeness relation is given formally as before by
\[ 1 = \sum_{q} |q\rangle \langle q| \quad \Rightarrow \quad \int d[q] |q\rangle \langle q| \]
\[ = \prod \left( \frac{\partial}{\partial q'} + \frac{\partial}{\partial q''} \right) |q'\rangle \langle q''| \]
\[ = \int Dq \ |q\rangle \langle q| \quad (154) \]

where the multicomponent integral \( \int Dq \) is only a symbolic meaning for continuous fields, so as to formally resembles that of discrete pairing case before. Likewise
\[ 1 = \sum_{p} |p\rangle \langle p| \quad \Rightarrow \quad \int Dp \ |p\rangle \langle p| \quad (155) \]
8.1 The transformation function and resolution of identity

The transformation function is the same as Eq. (127). The fermion fourier transforms are similar as Eqs. (131) - (134) but for continuous case, the summation is now replaced by functional integral obeying the Berezin fermion field integrals. Similarly the completeness relations, often referred to as resolutions of unity, are as given by Eqs. (128) - (130) with summation replaced by Berezin fermion integrals, Eqs. (154) and (155).

8.2 Continuous Weyl transform for fermions

The discrete expression for the mixed representation of any operator given before, Eq. (137), now reads in the continuous case as

\[ \hat{A} = \sum_{p',q'} |p''\rangle \langle p'| A |q'\rangle \langle q'| \rightarrow \int \mathcal{D}p' \mathcal{D}q' \langle p'\rangle A |q'\rangle |p''\rangle \langle q'| \]

where

\[ A (p, q) = \int \mathcal{D}v e^{-2p \cdot v} \langle q + v | \hat{A} | q - v \rangle \]

\[ \hat{\Delta} (p, q) = \int \mathcal{D}u e^{2q \cdot u} |p + u \rangle \langle p - u| \]

We also have the equivalent expressions,

\[ A (p, q) = \int \mathcal{D}u e^{-i2q \cdot u} \langle q + u | \hat{A} | q - u \rangle \]

\[ \hat{\Delta} (p, q) = \int \mathcal{D}v e^{i2p \cdot v} |q + v \rangle \langle q - v| \]

in the sense of Berezin fermionic integrals.

Equation (146) can now be written in terms of Berezin fermion integral as

\[ \hat{\Delta} (p, q) = \int \mathcal{D}v \mathcal{D}u \exp \{ -i (\mathcal{P} \cdot v - \mathcal{Q} \cdot u) \} \exp \{ 2i (p \cdot v - q \cdot u) \} \Omega_{\lambda \lambda'} \]

and Eq. (147) is now,

\[ A (p, q) = Tr \left( \hat{A} \hat{\Delta} \right) \]

\[ = \left( \times Tr \left\{ \mathcal{D}v \mathcal{D}u \exp 2i [p \cdot v - q \cdot u] \right\} \right) \]

We also have for Eq. (150), the expression,

\[ A_{\lambda \lambda'} (u, v) = \int \mathcal{D}p \mathcal{D}q \ A_{\lambda \lambda'} (p, q) \exp 2i (p \cdot v - q \cdot u) \]
with inverse

\[ A_{\lambda \lambda'}(p, q) = \int \int \mathcal{D}v \mathcal{D}u \ A_{\lambda \lambda'}(u, v) \exp \{-2i [p \cdot v - q \cdot u]\} \quad (161) \]

### 8.3 Distribution functions

\[ A_{\lambda \lambda'}^\ast(u, v) = \exp \{-2i u \cdot v\} \text{Tr} \left( \hat{A} \exp \left(-2i (P \cdot v - Q \cdot u)\right) \right) \]

and the anti-normal characteristic distribution function given by

\[ A_{\lambda \lambda'}^\ast(u, v) = \exp \{i u \cdot v\} \text{Tr} \left( \hat{A} \exp \left(-2i (P \cdot v - Q \cdot u)\right) \right) \]

Formally, again a more general phase-space distribution functions, \( f^{(g)}(p, q, t) \), can be obtained from the expression

\[ f^{(g)}(p, q, t) = \int \int \mathcal{D}u \mathcal{D}v \ A_{\lambda \lambda'}(u, v) \exp \{-2i [p \cdot v - q \cdot u]\} g(u, v) \]

where \( g(u, v) \) is some chosen smoothing function. Finally, we can write Eq. (151) as

\[ \hat{A} = \sum_{\lambda, \lambda'} \int \int \mathcal{D}u \mathcal{D}v \ A_{\lambda \lambda'}(u, v) \exp \{-2i [P \cdot v - Q \cdot u]\} \Omega^{\lambda \lambda'} \quad (162) \]

### 8.4 Fermion path integrals

We are now equipped to calculate the path integral expression for the evolution operator, \( U(t, t_o) \),

\[ \hat{U}(t, t_o) = \exp \left( \frac{-i}{\hbar} [t - t_o] \mathcal{H} \right) = \prod_{j=1}^{n+1} \hat{U}(t_j, t_{j-1}) \quad (163) \]

We now make of the following relations for fermions in terms of Berezin Grassman integrals. Thus for any fermion operator, we have

\[ \hat{A} = \int \int \mathcal{D}q \mathcal{D}p \ A(p, q) \Delta(p, q) \]

where

\[ A(p, q) = \int \mathcal{D}u \ e^{-i2q \cdot u} (p + u| \hat{A}| p - u) \quad (164) \]

\[ \hat{\Delta}(p, q) = \int \mathcal{D}v \ e^{i2p \cdot v} (q + v| \hat{A}| q - v) \quad (165) \]
From the above relations we can transform Eq. (163) into a Grassman path integral. We have from Eq. (163)

\[ \hat{U} (t_j, t_{j-1}) = \int \mathcal{D}q \mathcal{D}p \, U_{t_j, t_{j-1}} (p, q) \Delta (p, q) \]

where

\[ U_{t_j, t_{j-1}} (p, q) = \int \mathcal{D}u \, e^{-i u q} \langle p + u | \hat{U} (t_j, t_{j-1}) | p - u \rangle = \exp \left[ -\frac{i}{\hbar} (t_j - t_{j-1}) H (p, q) \right] \]

where \( H (p, q) \) is the "lattice" Weyl transform of \( \mathcal{H} \) for the time interval \( (t_j - t_{j-1}) \).

We also have

\[ \langle q_j | \Delta (p, q) | q_{j-1} \rangle = \int \mathcal{D}v \, e^{2i p \cdot v} \langle q_j | v \rangle \langle q_j - v | q_{j-1} \rangle \]

\[ = \exp \left[ 2i p \cdot (q_j - q_{j-1}) \right] \delta \left( \frac{q_j + q_{j-1}}{2} - q \right) \]

Therefore

\[ \langle q_j | \hat{U} (t_j, t_{j-1}) | q_{j-1} \rangle = \int \mathcal{D}q \mathcal{D}p \, U_{t_j, t_{j-1}} (p, q) \Delta (p, q) \]

\[ = \int \mathcal{D}p \exp \left[ 2i p \cdot \left( \frac{q_j - q_{j-1}}{2} \right) \right] \exp \left[ -\frac{i}{\hbar} (t_j - t_{j-1}) H \left( p, \frac{q_j + q_{j-1}}{2} \right) \right] \]

\[ = \int \mathcal{D}p \exp \left[ -\frac{i}{\hbar} \left( p \cdot \left( \frac{q_j - q_{j-1}}{\epsilon} \right) - (t_j - t_{j-1}) H \left( p, \frac{q_j + q_{j-1}}{2} \right) \right) \right] \]

where \( \epsilon \) is the incremental time, \( t_j - t_{j-1} \). Therefore the transition amplitude between a state \( \Psi (t_0) \) and a state \( \Phi (t) \) is given by

\[ \langle \Phi (t) | \Psi (t_0) \rangle = \int \prod_{j=1}^{n+1} \mathcal{D}p_j \prod_{j=1}^{n+1} \mathcal{D}q_j \, \phi (q_{n+1}, t_{n+1}) \exp \left\{ -\epsilon \sum_{j=1}^{n+1} \left[ p_j \cdot \left( \frac{q_j - q_{j-1}}{\epsilon} \right) + H \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \right] \right\} \psi (q_1, t_0) \]

where \( \phi (q_{n+1}, t_{n+1}) = \langle \Phi (t) | q_{n+1} \rangle \) and \( \psi (q_1, t_0) = \langle q_1 | \Psi (t_0) \rangle \). Another useful result for the path integral is the expression for the partition function, using Matsubara imaginary time [53],

\[ \exp (-\beta \Omega) = Tr \exp \left( -\beta \left( \mathcal{H} - \mu \mathcal{N} \right) \right) = Tr \exp \left( -\beta \mathcal{K} \right) \]

which yields

\[ \exp (-\beta \Omega) = \int \prod_{j=1}^{n+1} \mathcal{D}p_j \mathcal{D}q_j \exp \left\{ -\epsilon \sum_{j=1}^{n+1} \left[ p_j \cdot \left( \frac{q_j - q_{j-1}}{\epsilon} \right) + K \left( p_j, \frac{q_j + q_{j-1}}{2} \right) \right] \right\} \]

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The anti-periodic boundary condition for \( p \) and \( q \) is precisely what is needed to preserve the antiperiodicity in each time variables of the Green’s function in the path integral formulation of statistical quantum field theory \[54\].

### 9 Unified Q-Distribution Function Theory in CMP and QFT

We have seen that both in CMP as well as in QFT, their quantum distribution functions of field variables are well-defined. This suggests that we can formulate QFT in terms of Q-distribution function of field variables, similar to those of CMP in terms of Wigner distribution functions. Some hints in this direction has been given very early on by Guttinger \[55\]. More pertinent to our work in this direction has recently been given by Drummond \[56, 57\], and also by Friederich \[58\] and by Manko et al \[59\].

Clearly, following the route to the formulation of nonequilibrium superfield theory of quantum transport \[9\], we can form a new theory of dissipative nonequilibrium quantum field theory based on the quantum transport equation of the Q-distribution function or the ‘Wigner distribution function’ of conjugate functional field variables. This is an interesting new formulation of quantum field theory which we hope to investigate in details in a separate communication. A good beginning in this direction has been given by Drummond \[56, 57\] and by Friederich \[58\].

### 10 Concluding Remarks

The unification discussed in this paper is rooted in the formally identical expression for the phase-space point projector denoted as \( \hat{\Delta}(p, q) \). Here, any quantum operator can be expanded in terms of the \( \hat{\Delta}(p, q) \) operator basis. Although \( \hat{\Delta}(p, q) \) carries two equivalent expressions as basis operators, this is made a unique and symmetrical expression, which is again formally identical in the unification discussed here, namely,

\[
\hat{\Delta}(p, q) = (N)^{-1} \sum_{v,u} \exp \{-i (P \cdot v - Q \cdot u)\} \exp \{2i (p \cdot v - q \cdot u)\} \Omega_{\lambda\lambda'}
\]

or in terms of continuous Grassman variables,

\[
\hat{\Delta}(p, q) = \int \int \mathcal{D}v \mathcal{D}u \sum_{v,u} \exp \{-i (P \cdot v - Q \cdot u)\} \exp \{2i (p \cdot v - q \cdot u)\} \Omega_{\lambda\lambda'}
\]

With the above unification all physics that follows have corresponding formally identical expressions, such as the distribution in dual space or phase-space and their path integral expressions. These results clearly demonstrate the power of the mathematical language of "lattice" Weyl-Wigner formulation of quantum
physics \[20\], as shown by the authors in previous publications. Indeed for finite fermion systems, the creation and annihilation formalism exactly maps to that of CMP since free Bloch electrons are fermions. Thus, CMP formalism maybe considered a bosonization of free Bloch electrons which incorporates the Pauli exclusion principle and Fermi-Dirac distribution.

The fact that
\[
\hat{Y}(u, v) = \exp \left[ -i \left( \hat{P} \cdot v - \hat{Q} \cdot u \right) \right] \Omega
\]
span all operators describing fermions, bosons, and spin systems suggests that in principle, bosonization, fermionization and Jordan-Wigner fermionization for spin systems, as well as the Holstein–Primakoff transformation from boson operators to the spin operator can be performed depending on the physical situations and ease in the calculations. In quantum physics, unitary transformation on the creation and annihilation operators themselves is also employed, as exemplified by the Bogoliubov transformation. Moreover, whenever \( \hat{Y}(u, v) \mapsto M_{ij} \) is already expressed in matrix form, the Jordan–Schwinger transformation is a map from matrices \( M_{ij} \) to bilinear expressions of creation and annihilation operators, e.g., of the form, \( \hat{a}_i^\dagger M_{ij} \hat{a}_j \), which expedites computation of representations.

The present unification suggests a new formulation of QFT in terms of non-perturbative 'Wigner distribution' or \( Q \)-distribution of conjugate functional-field variables quantum transport equations.

In the Appendix, we also present for completeness some of the well-known bosonization and fermionization transformations. Moreover, we mention some other very important transformations in CMP and QFT which deals with the decoupling of the different degrees of freedom. This is exemplified, e.g., by the well-known Foldy-Wouthuysen for relativistic Dirac electrons \[78\], as well as by the perturbative decoupling of energy bands to all orders in the calculation of magnetic susceptibility of Bloch fermions \[15\], etc.

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Appendices

A Non-Hermitian and time-reversal breaking symmetry in quantum transport

In nonequilibrium quantum transport physics, the doubling of degrees of freedom is what endows the dual spaces, since we have an explicit time axis to describe irreversibility in the quantum Liouville equation. We expect that the solution to the quantum Liouville equation will have as its dual the antichronological super-statevector. In what follows, we will determine this dual super-statevector by a variational technique.

A.1 Variational technique for calculating transition probability

We present here a general technique for calculating the transition probability of the super-Schrödinger equation. In CMP and QFT of closed systems, the expectation values are generally obtain as time-ordered correlations using path integrals

\[ \langle F(\phi) \rangle = \frac{\int \mathcal{D}\phi \, F(\phi) \, e^{i\frac{\hbar}{\epsilon} S(\phi)}}{\int \mathcal{D}\phi \, e^{i\frac{\hbar}{\epsilon} S(\phi)}} \]

The so-called generating functional makes use of an element of the dual space often denoted as the source \( J \). Then the generating functional denoted by \( Z(J) \) is defined by

\[ Z(J) = \int \mathcal{D}\phi \, e^{i \left[ S(\phi) + J\phi \right]} \]

and therefore,

\[ \frac{\delta^n Z(J)}{\delta J(x_1) \ldots \delta J(x_n)} = i^n Z(J) \langle \phi(x_1) \ldots \phi(x_n) \rangle \]

In what follows, we will formulate the generating functional for the Liouville equation for the density operator for open systems also termed as the generating super-functional by variational method.

A.2 Time-ordered displacement operator in Liouville space

The familiar von Neumann density-matrix operator equation in \( H \)-space given by

\[ i\hbar \frac{\partial}{\partial t} \rho(t) = [\mathcal{H}, \rho] \quad (166) \]

becomes a super-Schrödinger equation for the super-statevector in \( L \)-space expressed as

\[ i\hbar \frac{\partial}{\partial t} |\rho(t)\rangle = \mathcal{L} |\rho(t)\rangle. \quad (167) \]
Note that since in general $L$ is non-Hermitian for open and interacting systems, we expect to have a dual space of the solutions to Eq. (167). This dual space super-state eigenvector is determined by variational method in what follows.

From Eq. (167), we can formally write the solution for the super-statevector $|\rho(t)\rangle\rangle$ as

$$|\rho(t)\rangle\rangle = T \exp \left[ -\frac{i}{\hbar} \int_{t_o}^{t} L(t') dt' \right] |\rho(t_o)\rangle\rangle,$$

(168)

where $T$ is the usual real-time ordering displacement operator. We take $t_o$ as the time when the "perturbation" Liouvillian, $L^{(1)} = L - L_o$, is turned on. Then we can also write

$$|\rho(t_o)\rangle\rangle = T \exp \left[ -\frac{i}{\hbar} \int_{t_o}^{t_o} L_o(t') dt' \right] |\rho(\xi_o)\rangle\rangle,$$

(169)

where the system, during the time duration from $\xi_o$ to $t_o$, is acted on only by the "unperturbed" Liouvillian $L_o$.

A.2.1 Time evolution operator $U$ and $S$-matrix

Therefore, we can also write

$$|\rho(t)\rangle\rangle = T \exp \left[ -\frac{i}{\hbar} \int_{0}^{t} L(t') dt' \right] \mathcal{S}(t, t_o) |\rho_o(0)\rangle\rangle,$$

(170)

where

$$\mathcal{S}(t, t_o) = \mathcal{U}_o (0, t) \mathcal{U}(t, t_o) \mathcal{U}_o (t_o, 0),$$

$$\mathcal{U}_o (0, t) = T \exp \left[ \frac{i}{\hbar} \int_{0}^{t} L_o(t') dt' \right],$$

$$\mathcal{U}(t, t_o) = T \exp \left[ -\frac{i}{\hbar} \int_{t_o}^{t} L(t') dt' \right],$$

(171)

$$|\rho_o(0)\rangle\rangle = \mathcal{U}_o (0, \xi_o) |\rho(\xi_o)\rangle\rangle.$$  

(172)

If we let $t_o \Rightarrow -\infty$, and write $|\rho_o(0)\rangle\rangle = |\rho_{eq}\rangle\rangle$, then we have

$$|\rho(t)\rangle\rangle = T \exp \left[ -\frac{i}{\hbar} \int_{0}^{t} L_o(t') dt' \right] \mathcal{S}(t, -\infty) |\rho_{eq}\rangle\rangle.$$  

(173)

It follows that

$$|\rho(0)\rangle = \mathcal{S}(0, -\infty) |\rho_{eq}\rangle = |\rho_{\mathcal{L}}\rangle\rangle = |\rho_I(0)\rangle\rangle,$$

(174)

which defines the super-Heisenberg representation $|\rho_{\mathcal{L}}\rangle\rangle$ of the super-statevector $|\rho(t)\rangle\rangle$. Note that $|\rho_{\mathcal{L}}\rangle\rangle$ is independent of time. Equation (171) also leads us to define the super-interaction representation in L-space. Thus for any superoperator $O$, we define

$$O_I(t) = \mathcal{U}_o (0, t) O(t) \mathcal{U}_o (t, 0)$$

(175)

as the super-interaction representation of $O$ in L-space, and

$$O_{\mathcal{L}}(t) = \mathcal{U}(0, t) O(t) \mathcal{U}(t, 0)$$

(176)
as its super-Heisenberg representation. Therefore, we have for any superoperator $O$,

$$O_L(t) = \mathcal{S}(0,t) O(t) \mathcal{S}(t,0),$$  \hfill (177)

which give the relation between the super-Heisenberg representation and the super-interaction representation. Since $L, \tilde{H}_r,$ and $\hat{H}$ commute then we have for the "hat" and "tilde" quantum field superoperators the following relations

$$\hat{\psi}_{\hat{H}}(t) = T \exp \left[ \frac{i}{\hbar} \int_0^t \hat{H}(t') dt' \right] \hat{\psi} T \exp \left[ -\frac{i}{\hbar} \int_0^t \hat{H}(t') dt' \right]$$

$$= \hat{\psi}_L(t), \quad \hfill (178)$$

$$\tilde{\psi}_{\tilde{H}}(t) = T \exp \left[ \frac{i}{\hbar} \int_0^t \tilde{H}(t') dt' \right] \tilde{\psi} T \exp \left[ -\frac{i}{\hbar} \int_0^t \tilde{H}(t') dt' \right]$$

$$= \tilde{\psi}_L(t). \quad \hfill (179)$$

Similar relations exist for the canonically conjugate quantum field operators: $\hat{\psi}_L^\dagger(t) = \hat{\psi}_{\hat{H}}(t)$, and $\tilde{\psi}_L^\dagger(t) = \tilde{\psi}_{\tilde{H}}(t)$.

### A.3 Super S-matrix theory in L-space

The "super-Schrödinger equation" that we have developed in L-space and its corresponding formal solution given above are not complete. We have to determine the canonically conjugate counterpart of the super-statevector $|\rho(t)\rangle$. This is done in what follows by a variational technique.

### A.4 Construction of generating super-functional

For example, it is not clear what would be the canonically conjugate counterpart of the super-statevector $|\rho(t)\rangle$ for defining generating functional, analogous to the existing canonically conjugate pair of H-space states, $|\psi_+,t\rangle$ and $|\psi_-,t\rangle$, which naturally occur in zero temperature variational quantum action principle forming the basis for deriving the time-dependent Schrödinger equation, as first given by Dirac [60].

### A.5 Time-dependent variational principle in Liouville space

To complete the "super-Schrödinger quantums mechanics", we need to construct the corresponding variational principle in L-space. We follow the general construction of a variational principle [61,62] as an optimization of the estimate for
the expectation value of any operator $\mathcal{O}$ at the time $t_1$ from the knowledge of the initial condition for the density matrix super-statevector $|\rho(t_o)\rangle$, subject to the constraint $|\rho(t)\rangle$ obeys the L-space super-Schrödinger equation. Therefore, we have to optimize the following super-functional $[62]$.

$$\Phi (|\rho(t)\rangle, |\Lambda(t)\rangle) = \langle\langle \mathcal{O}(t) | \rho(t_1) \rangle \rangle - \int_{t_o}^{t_1} \langle\langle \Lambda(t) | i\hbar \frac{\partial}{\partial t} - \mathcal{L} | \rho(t) \rangle \rangle, \quad (180)$$

where we have introduced the dual space super-statevector $\langle\langle \Lambda(t) \rangle \rangle$ as a 'Lagrangian multiplier'. Upon integration by parts, $\Phi (|\rho(t)\rangle, |\Lambda(t)\rangle)$ is also equal to the following super-functional

$$\Phi (|\rho(t)\rangle, |\Lambda(t)\rangle) = \langle\langle \mathcal{O}(t) | \rho(t_1) \rangle \rangle - i\hbar \langle\langle \Lambda(t_1) | \rho(t_1) \rangle \rangle + i\hbar \langle\langle \Lambda(t_o) | \rho(t_o) \rangle \rangle + \int_{t_o}^{t_1} \left[ i\hbar \frac{\partial \Lambda(t)}{\partial t} - \mathcal{L} \Lambda(t) \right] |\rho(t)\rangle \rangle. \quad (181)$$

Thus, the optimum condition for $\Phi (|\rho(t)\rangle, |\Lambda(t)\rangle)$ occurs for $i\hbar \langle\langle \Lambda(t_1) \rangle \rangle = \langle\langle \mathcal{O}(t) \rangle \rangle$, and when the following equations of motion are obeyed for $t_o < t < t_1$,

$$\begin{cases} i\hbar \frac{\partial |\rho(t)\rangle \rangle}{\partial t} = \mathcal{L} |\rho(t)\rangle \rangle, \\ i\hbar \frac{\partial |\Lambda(t)\rangle \rangle}{\partial t} = \mathcal{L} |\Lambda(t)\rangle \rangle. \end{cases} \quad (182)$$

If we let $t_1 \Rightarrow \infty$ and $t_o \Rightarrow -\infty$, apply the boundary condition: $i\hbar \langle\langle \Lambda(t_1) \rangle \rangle = \langle\langle \mathcal{O}(t) \rangle \rangle$, and $|\rho(-\infty)\rangle = |\rho_{eq}\rangle$, then the solutions can be written as

$$\begin{cases} |\rho(t)\rangle = T exp \left[ \frac{i}{\hbar} \int_0^t \mathcal{L}_{o} (t') dt' \right] \mathcal{S}(t, -\infty) |\rho_{eq}\rangle, \\ |\Lambda(t)\rangle = T^{\ast \ast} exp \left[ \frac{i}{\hbar} \int_{-\infty}^t \mathcal{L}_{o} (t') dt' \right] \mathcal{S}(t, \infty) |\mathcal{O}\rangle, \end{cases} \quad (183)$$

where $T^{\ast \ast}$ denotes anti-chronological time ordering. Therefore, we can write a "transition probability" in L-space as

$$\langle\langle \Lambda(t) | |\rho(t)\rangle \rangle = \frac{1}{i\hbar} \langle\langle \mathcal{O} \rangle \mathcal{S}(\infty, -\infty) |\rho_{eq}\rangle = \frac{1}{i\hbar} \langle\langle \mathcal{O}_{\mathcal{L}} \rangle |\rho_{\mathcal{L}}\rangle, \quad (184)$$

where $|\rho_{\mathcal{L}}\rangle$ is the super-Heisenberg representation of the super-statevector $|\rho(t)\rangle$ defined by Eq. [174], and similarly $\langle\langle \mathcal{O}_{\mathcal{L}} \rangle = \langle\langle \mathcal{O} \rangle \mathcal{S}(\infty, 0)$.

Note that in terms of evaluating the expectation value of the superoperator at $t = t_1 = \infty$, we can explicitly write this as a time-ordered expectation value by rewriting Eq. [184] as

$$\langle\langle \Lambda(t) | |\rho(t)\rangle \rangle = \frac{1}{i\hbar} \langle\langle \mathcal{O} \rangle \mathcal{S}(\infty, -\infty) |\rho_{eq}\rangle = \frac{1}{i\hbar} \langle\langle 1 | \mathcal{O} \mathcal{S}(\infty, -\infty) |\rho_{eq}\rangle \rangle. \quad (185)$$

We are particularly interested in $\mathcal{O} = i\hbar \mathcal{S}$, where $\mathcal{S}$ is the identity operator corresponding to an asymptotic state of maximum entropy at $t = \infty$ (corresponding to
formulate the constrained stationary value \[\langle\langle \Lambda(t) | \rho(t) \rangle \rangle = \langle\langle \Sigma | \rho_{eq} \rangle \rangle = \langle\langle \mathcal{S}_\mathcal{L} | \rho_{eq} \rangle \rangle, \] where \(\langle\langle \mathcal{S}_\mathcal{L} \rangle \rangle\) is given by

\[
\langle\langle \mathcal{S}_\mathcal{L} \rangle \rangle = \langle\langle \Sigma | \mathcal{S} \rangle \rangle, \]

From the time-ordered expectation value deduced by Eq. (185), we see that indeed \(\langle\langle \Sigma | \mathcal{S} \rangle \rangle\) is the analog to the transition amplitude occurring in zero temperature time-dependent quantum mechanics [60].

A.6 The effective action and generating super-functional

In order to gain further insights into the S-matrix formalism in L-space, we formulate the constrained stationary value [63] of the "super-action" given by

\[
\int dt \ \langle\langle \Lambda(t) | \rho(t) \rangle \rangle = \left\{\begin{array}{l}
\langle\langle \Lambda(t) | \rho(t) \rangle \rangle = 1 \\
\langle\langle \Lambda(t) | \Psi \rho(t) \rangle \rangle = \langle\langle \Psi \rangle \rangle \end{array}\right.,
\]

where \(\Psi\) stands for an arbitrary quantum field superoperator (the extension to several species of quantum field superoperators is straightforward). Thus, we consider the stationary variation of the super-functional

\[
\Omega (|\rho(t)\rangle, |\Lambda(t)\rangle) = \int dt \ \langle\langle \Lambda(t) | \rho(t) \rangle \rangle
- \int dx \ \langle\langle \Lambda(t) | \eta(x) \cdot \Psi |\rho(t)\rangle \rangle
- \int dt \ \langle\langle \Lambda(t) | \rho(t) \rangle \rangle, \]

where \(\eta(x)\) and \(w(t)\) are introduced as \(\mathbb{C}\)-number Lagrange multipliers. Note that \(\eta(x)\) is also playing the role of the Schwinger source field.

Carrying out the variation with respect to \(|\Lambda(t)\rangle\) and \(|\rho(t)\rangle\), and enforcing the stationarity of \(\Omega (|\rho(t)\rangle, |\Lambda(t)\rangle)\), we obtain the following equations

\[
\begin{align*}
\frac{i\hbar}{\partial |\rho(t)\rangle} - \mathcal{L} |\rho(t)\rangle - \int dx \ \eta(x) \cdot \Psi |\rho(t)\rangle - w(t) |\rho(t)\rangle &= 0, \\
\frac{i\hbar}{\partial |\Lambda(t)\rangle} - \mathcal{L} |\Lambda(t)\rangle - \int dx \ \eta(x) \cdot \Psi |\Lambda(t)\rangle - w^* (t) |\Lambda(t)\rangle &= 0.
\end{align*}
\]

We write the solutions to these equations as

\[
\begin{align*}
|\rho(t)\rangle_{\eta,w} &= \exp \left[ \frac{-i}{\hbar} \int_{-\infty}^{t} dt' \ w(t') \right] \ |\rho(t)\rangle_{\eta}, \\
|\Lambda(t)\rangle_{\eta,w} &= \exp \left[ \frac{i}{\hbar} \int_{-\infty}^{t} dt' \ w^* (t') \right] \ |\Lambda(t)\rangle_{\eta},
\end{align*}
\]

where \(|\rho(t)\rangle_{\eta}\) and \(|\Lambda(t)\rangle_{\eta}\) obey the equations

\[
\begin{align*}
\frac{i\hbar}{\partial |\rho(t)\rangle_{\eta}} - \mathcal{L} |\rho(t)\rangle_{\eta} - \int dx \ \eta(x) \cdot \Psi |\rho(t)\rangle_{\eta} &= 0, \\
\frac{i\hbar}{\partial |\Lambda(t)\rangle_{\eta}} - \mathcal{L} |\Lambda(t)\rangle_{\eta} - \int dx \ \eta(x) \cdot \Psi |\Lambda(t)\rangle_{\eta} &= 0.
\end{align*}
\]
Therefore, we can calculate the "transition probability" in L-space from the solutions of Eq. (190), which is given by Eq. (191). We obtain the following relation
\[
\langle\langle \Lambda (t) \mid \rho (t) \rangle\rangle_{\eta,w} = 1 = \exp \left[ -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt' w (t') \langle\langle \Lambda (t) \mid \rho (t) \rangle\rangle_{\eta,w} \right],
\]
which leads to the "transition probability" in the presence of the Schwinger source term given by
\[
\langle\langle \Lambda (t) \mid \rho (t) \rangle\rangle_{\eta} = \exp \left[ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' w (t') \right] = \exp \left[ \frac{i}{\hbar} W \right].
\]
This is the same "transition probability" as that obtained in Eq. (186). Therefore, we obtain the equality
\[
\langle\langle 1 \mid \bar{S} (\infty, -\infty) \mid \rho_{eq} \rangle\rangle = \exp \left[ \frac{i}{\hbar} W \right].
\]

Analogous to the zero temperature S-matrix formalism, we identify \( W \) as the generating super-functional for connected \( n \)-point super-Green’s functions. This is supported by the time-ordered way of taking the average value, shown by Eq. (185). To have a deeper appreciation of this analogy, we take the scalar product with \( \langle\langle \Lambda (t) \mid \rho (t) \rangle\rangle_{\eta,w} \) of both sides of the first line of Eq. (190) and integrate over time. We obtain
\[
W = \int dt' w (t') = \int dt \langle\langle \Lambda (t) \mid \left\{ \frac{i\hbar}{\partial t} \rho (t) \right\} - \mathcal{L} \mid \rho (t) \rangle\rangle_{\eta,w}.
\]
from which we deduced the variational derivative
\[
\frac{\delta}{\delta \eta} W = -\langle\langle \Lambda (t) \mid \Psi \mid \rho (t) \rangle\rangle_{\eta,w} = \langle \Psi \rangle \quad \text{(at the optimum condition)}.
\]
Therefore, we have the effective super-action given by
\[
A_{\text{eff}} = \int_{-\infty}^{\infty} dt \langle\langle \Lambda (t) \mid i\hbar \frac{\partial}{\partial t} - \mathcal{L} \mid \rho (t) \rangle\rangle_{\eta,w} = W + \int dx \eta (x) \cdot \langle \Psi \rangle
\]
from which we also deduced the variational derivative
\[
\frac{\delta}{\delta \langle \Psi \rangle} A_{\text{eff}} = \eta.
\]

Hence, the effective super-action is stationary with respect to the variation of \( \langle \Psi \rangle \) when \( \eta (x) = 0 \), which is the physical situation, i.e., \( W = A_{\text{eff}} \). Thus for \( \eta (x) = 0 \), we have
\[
W = -i\hbar \ln \langle\langle 1 \mid \bar{S} (\infty, -\infty) \mid \rho_{eq} \rangle\rangle = \int_{-\infty}^{\infty} dt \langle\langle \Lambda (t) \mid i\hbar \frac{\partial}{\partial t} - \mathcal{L} \mid \rho (t) \rangle\rangle.
\]
The relation derived here for nonequilibrium quantum-field theory between super $S$-matrix and the effective super-action provides a rigorous basis supporting the functional theory of time-dependent many-body quantum mechanics discussed by Rajagopal and Buot in a series of papers \[64–66\].

We will refer to $S(\infty, -\infty)$ as the complete $S$-matrix superoperator for quantum-field theoretical methods of nonequilibrium systems. Equation (171) allows us to write the evolution equation for the $S$-matrix operator as

$$i\hbar \frac{\partial}{\partial t} S(t, t_0) = \mathcal{L}_I^{(1)}(t) S(t, t_0),$$

(201)

where

$$\mathcal{L}_I^{(1)}(t) = \mathcal{U}_o(0, t) (\mathcal{L} - \mathcal{L}_o) \mathcal{U}_o(t, 0),$$

(202)

and

$$(\mathcal{L} - \mathcal{L}_o) = [v(1) + u(1)] \Psi(1) + [v(1, 2) + u(1, 2)] \Psi(1) \Psi(2).$$

(203)

The Schwinger external sources \[67\] indicated by $u(1)$ and $u(1, 2)$ contain all the time-dependent part of $v(1)$ and $v(1, 2)$ respectively. These Schwinger source terms are to be set equal to zero at the end of all calculations. The solution to Eq. (201) immediately yields

$$S(t, t_0) = 1 - i \frac{\hbar}{\mathcal{L}_I^{(1)}(t')} \int_{t_0}^t S(t', t_0) \, dt',$$

(204)

which upon iteration yields

$$S(t, t_0) = T \exp \left[ -i \frac{\hbar}{\mathcal{L}_I^{(1)}(t')} \int_{t_0}^t \mathcal{L}_I^{(1)}(t') \, dt' \right].$$

(205)

Thus, we can explicitly exhibit $\langle 1 | S(\infty, -\infty) | \rho_{eq} \rangle$ as

$$\langle 1 | S(\infty, -\infty) | \rho_{eq} \rangle = \langle 1 | T \exp \left[ -i \frac{\hbar}{\mathcal{L}_I^{(1)}(t')} \int_{t_0}^\infty \mathcal{L}_I^{(1)}(t') \, dt' \right] | \rho_{eq} \rangle$$

$$= \exp \left[ i \frac{\hbar}{\mathcal{L}_I^{(1)}(t')} \int_{t_0}^\infty \mathcal{L}_I^{(1)}(t') \, dt' \right],$$

(206)

where $W$ is the effective action. Equation (206) shows the relation of Eq. (195) to $L_1^{(1)}(t)$ similar to zero-temperature many-body quantum-field theory.

### B  Some well-known fermionization and bosonization transformations

Aside from the well-known bosonization in one-dimensional Tomanaga-Luttinger and Hubbard model, there are other very highly-utilized fermionization and bosonization transformations in CMP. Moreover, some decoupling transformations are used in both QFT and CMP. For completeness, we give these here.
B.1 Holstein-Primakoff transformation

The Holstein–Primakoff transformation in quantum mechanics is a mapping to the spin operators from boson creation and annihilation operators [68]. This involves redefinition of angular momentum states $|j, m⟩ \rightarrow |n⟩$, $n = 1, 2, ..., 2j$, where the new eigenvalues are the values of $n$ reminiscent of the harmonic oscillator problem. This entails transformation of the creation and annihilation operators and corresponding spin vector operators. This is also reminiscent of the renormalization of position and momentum operators, lattice-position eigenvectors and crystal-momentum eigenvectors, in energy-band dynamics leading to discrete quantum mechanics on discrete finite fields. In HP transformation, $S^+$ and $S^-$ are deduced from the theory of angular momentum.

From the theory of angular momentum

$$S^z |n⟩ = \hbar (j - n) |n⟩$$

$$b^\dagger b |n⟩ = n |n⟩$$

$$S^z = \hbar (j - b^\dagger b)$$

Holstein-Primakoff transformation is

$$S_i^- = a_i^\dagger \left( \sqrt{2S - a_i^\dagger a_i} \right)$$

$$S_i^+ = \left( \sqrt{2S - a_i^\dagger a_i} \right) a_i$$

$$S_i^z = S - a_i^\dagger a_i$$

$$a_i^\dagger a_i \leq 2S$$

Note: instead of the pre-Bogoliubov transformation $u \mapsto \sqrt{2S - a_i^\dagger a_i}$, we have,

$$S_i^+ S_i^- = \left( \sqrt{2S - a_i^\dagger a_i} \right) \left( a_i^\dagger \sqrt{2S - a_i^\dagger a_i} \right)$$

$$S_i^- S_i^+ = \left( a_i^\dagger \sqrt{2S - a_i^\dagger a_i} \right) \left( \sqrt{2S - a_i^\dagger a_i} a_i \right)$$

$$= a_i^\dagger \left( 2S - a_i^\dagger a_i \right) a_i$$
B.2 Dyson–Maleev transformation

A non-Hermitian Dyson–Maleev variant realization $J$ is related to the above and valid for all spins,

\[

g_{+} = h a^+ \\
g_{-} = S_{-} \sqrt{2s - a^+ a} = h a^+ (2s - a^+ a) \\
g_{z} = S_{z} = h (s - a^+ a)
\]

satisfying the same commutation relations and characterized by the same Casimir invariant with Casimir operators, $S^2$ and $S_z$.

B.3 Jordan-Wigner transformation

The Jordan–Wigner transformation maps spin operators onto fermionic creation and annihilation operators \[69–73\].

B.3.1 Derivation of Jordan-Wigner transformation

We will show how to map a 1-D spin chain of spin-1/2 particles to fermions. Take spin-1/2 Pauli operators acting on a site $j$ of a 1-D chain, $\sigma^{+}_{j}$, $\sigma^{-}_{j}$ and $\sigma^{z}_{j}$.

The anticommutator of $\sigma^{+}_{j}$, $\sigma^{-}_{j}$, is $\{ \sigma^{+}_{j}, \sigma^{-}_{j} \} = 1$, as would be expected from fermionic creation and annihilation operators. We are tempted to set

\[
\sigma^{+}_{j} = \frac{(\sigma^{x}_{j} + i\sigma^{y}_{j})}{2} = f_{j}^+ \\
\sigma^{-}_{j} = \frac{(\sigma^{x}_{j} - i\sigma^{y}_{j})}{2} = f_{j} \\
\sigma^{z}_{j} = 2 f_{j}^+ f_{j} - 1
\]

(207)

We now have the site anticommutation $\{ f_{j}^+, f_{j} \} = 1$; however commutation enters for different sites, i.e., $[f_{j}^+, f_{k}] = 0$, for $j \neq k$ instead of the desired anticommutation for fermions. In 1928, Jordan and Wigner invented a transformation which renders $\{ f_{j}^+, f_{k} \} = 0$, for $j \neq k$. The J-W transformation is given by,

\[
\begin{align*}
\tilde{a}_{j} &= \exp \left( +i\eta \sum_{k=1}^{j-1} f_{k} f_{k} \right) \cdot f_{j}^+ \\
\tilde{a}_{j} &= \exp \left( -i\eta \sum_{k=1}^{j-1} f_{k} f_{k} \right) \cdot f_{j} \\
\tilde{a}_{j}^+ a_{j} &= f_{j}^+ f_{j}
\end{align*}
\]
which can be written as,

$$\exp \left( \pm i \pi \sum_{k=1}^{j-1} f_k^\dagger f_k \right) = \prod_{k=1}^{j-1} e^{i \pi f_k^\dagger f_k}$$

$$= \prod_{k=1}^{j-1} \left( 1 - 2 f_k^\dagger f_k \right)$$

$$= \prod_{k=1}^{j-1} \left( - \sigma^z \right)$$

They differ from the Eq. (207) only by a phase \( \exp \left( \pm i \pi \sum_{k=1}^{j-1} f_k^\dagger f_k \right) \) with \( k = 1, 2, ..., j - 1 \). The domain of \( f_k^\dagger f_k \in \{0, 1\} \). So that

$$\exp \left( \pm i \pi \sum_{k=1}^{j-1} f_k^\dagger f_k \right) = \prod_{k=1}^{j-1} e^{i \pi f_k^\dagger f_k} = \left\{ \begin{array}{ll} 1 & \text{occupied } j \text{ is even} \\ -1 & \text{occupied } j \text{ is odd} \end{array} \right.$$  

The number of occupied modes in \( k = 1, 2, ..., j - 1 \) determine the sign of \( \exp \left( \pm i \pi \sum_{k=1}^{j-1} f_k^\dagger f_k \right) \). We now have

$$\left\{ a_i^\dagger, a_j \right\} = a_i^\dagger a_j + a_j a_i^\dagger$$

$$= \prod_{k=1}^{i-1} \left( 1 - 2 f_k^\dagger f_k \right) \cdot f_j^\dagger \prod_{k'=1}^{j-1} \left( 1 - 2 f_{k'}^\dagger f_{k'} \right) \cdot f_j$$

$$+ \prod_{k'=1}^{j-1} \left( 1 - 2 f_{k'}^\dagger f_{k'} \right) \cdot f_j \prod_{k=1}^{i-1} \left( 1 - 2 f_k^\dagger f_k \right) \cdot f_j^\dagger$$

which yields,

$$\left( 1 - 2 f_k^\dagger f_k \cdot f_j^\dagger \right) \left( 1 - 2 f_{k'}^\dagger f_{k'} \cdot f_j \right)$$

$$= f_j^\dagger \cdot \left( 1 - 2 f_k^\dagger f_k \right) \left( 1 - 2 f_{k'}^\dagger f_{k'} \cdot f_j \right)$$

$$= f_j^\dagger \cdot \left( 1 - 2 f_k^\dagger f_k - 2 f_k^\dagger f_{k'} + 4 f_k^\dagger f_k f_{k'}^\dagger f_{k'} \right) \cdot f_j$$

The method of representing one type of operator (e.g. spin, boson, fermions) in terms of another is one of the fundamental aspects of theoretical quantum physics.

Moreover, in one dimension, there is a close connection between the physics of fermions, bosons, and spins which is lacking in higher dimensions. The most important aspect of one dimensional physics that distinguishes it from higher dimensional ones lies in particle statistics, where particle exchanges are only possible if particle pass through each other (i.e., collide), which is not true in higher
dimension. One dimensional systems are now commonplace with nanofabricated materials, e.g. quantum nanowires, nanotubes, and some organic compounds. The physics of one-dimensional systems is important because some higher dimensional problems can be reduced to one-dimensional, e.g. in Kondo effect the low-energy physics is basically dealt with spherically symmetric $s$ channel so that the problem is effectively radial and provides a solvable system of interacting quantum problem.

The Jordan-Wigner (J-W) transformation is the simplest statistics-changing transformation in one dimension. The $S_z^i$ essentially initiate the J-W transformation\[^{12}\] and is local in the site index. We have

$$S_z^i = \frac{2c_i^\dagger c_i}{2}$$

To effect self-induced propagation, we need a nonlocal string of lowering and raising operators, whereby spin at different sites commute while fermions anticommute.

We use the following the following representation,

$$S_i^+ = c_i^\dagger \prod_{j<i} \left(1 - 2c_j^\dagger c_j\right)$$
$$S_i^- = \prod_{j<i} \left(1 - 2c_j^\dagger c_j\right) c_i$$

By identifying

$$\vec{\sigma} = \vec{S}$$

it is easy to see that

$$[\sigma_i^+, \sigma_j^-] = \delta_{ij} \sigma_i^\pm$$
$$[\sigma_i^z, \sigma_j^\pm] = \pm 2 \delta_{ij} \sigma_i^\mp$$

and ordinary fermionic commutation relations. The J-W transform works such that the string is cooked up so that it changes sign from $+1$ to $-1$ depending on whether the number of fermions to the left of site $i$ is even or odd.

J-W transformation is vital to numerical Monte Carlo simulation of 1-D systems because of the absence of "fermion sign problem". Quantum Monte Carlo methods calculates an integral for a quantity like partition function or correlation function by random sampling. The problem with fermions is that because of the change of sign under exchange of any two wavefunctions, series expansion must generate terms of both signs. This causes difficulty since partial results fluctuate wildly when the actual quantity to be calculated is much smaller in magnitude (as often the case). On the other hand if the resulting values calculated by this sampling process are of the same sign (so that the overall magnitude of the answer is necesarily much larger than of each individual term) then the truncation errors are much less severe.

\[^{12}\]Just as $S_i^z$ initiate the Holstein-Primakoff transformation.
Other numerical methods like the density-matrix renomalization group (DMRG), which is an extension of Wilson iterative numerical RG approach to 1-D chains, are also quite successful for low energy states. This DMRG method has been able to calculate the Haldane gap in spin-1 chain to many decimal places.

B.3.2 J-W transformation to solve the $XX$ chain

We now use the above J-W transformation to solve the so-called $XX$ chain. This is like the Heisenberg model with no $z$ coupling:

$$H_{XX} = \frac{J}{4} \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+)$$

Performing the J-W transformation, we have

$$H_{XX} = \frac{J}{4} \sum_i \left( \left( \prod_{j<i} \left( 1 - 2c_j^c c_j \right) \right) \left( \prod_{j<i+1} \left( 1 - 2c_j^c c_j \right) \right) \right)$$

B.4 The Bogoliubov-Valatin transformation for superfluidity

Consider the Bogoliubov transformation. We have,

$$\Gamma = U A_k$$

$$U = \begin{pmatrix} u & -\nu \\ \nu & u \end{pmatrix}$$

where

$$|u|^2 + |\nu|^2 = 1$$

where

$$u^* \nu - \nu^* u = 0$$

by orthogonality condition.

The unitary operator of transformation $S(r_k)$ can be recognized to be the squeezing operator,

$$S(r) = \exp \left[ \sum_{k \neq 0} \frac{r_k}{2} \left( a_k^+ a_{-k}^+ + a_{-k} a_k \right) \right]$$

where $r = \{r_k\}$. Note: $H = \sum_k \left[ E_k a_k^+ a_k + \frac{a_k^+}{2} (a_k a_{-k}^+ + h.c.) \right]$ and

$$\gamma^{k,\uparrow} = u a_{k,\uparrow} - v a_{-k,\downarrow}$$

$$\gamma^{k,\downarrow} = u a_{-k,\downarrow}^+ + v a_{k,\uparrow}$$
Consider the nonunitary transformation,
\[ e^{\alpha}a_k e^{-\beta} = e^{\alpha-\beta}a = \tilde{a}_k \]
We take \( e^{\alpha-\beta} = u \). Then we can have
\[
\alpha - \beta = \ln u (\cosh^2 x - \sinh^2 x) \\
= \ln u \\
\alpha = \ln u \cosh^2 x \\
\beta = \ln u \sinh^2 x
\]
Let the first non-unitary transformation be
\[
\tilde{a}_{k,\uparrow} = ua_{k,\uparrow} \\
\tilde{a}^\dagger_{-k,\downarrow} = ua^\dagger_{-k,\downarrow}
\]
after which we apply the second transformation: \( S\tilde{a}_k S^{-1} = b_k \). This can either be made unitary by affixing \( i \) or making \( r_k \) a pure imaginary in the exponent or non-unitary by just using real exponent but canonical transformation. Even with this two options, the overall transformation is non-unitary because of the first transformation to 'tilde' operators.

**Proof.**
\[
S\tilde{a}_k S^{-1} = b_k \\
= \exp \left[ \sum_{k \neq 0} \frac{r_k}{2} a_{k,\uparrow} a_{-k,\downarrow} \right] \tilde{a}_k \exp \left[ -\sum_{k \neq 0} \frac{r_k}{2} a_{k,\uparrow} a_{-k,\downarrow} + a_{-k} a_k \right]
\]
To prove, we make use of the following well-known relations,
\[
\exp (\lambda A) B \exp (-\lambda A) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \{ A^n, B \} = [A, ..., [A, [A, B]]], \ n \text{ commutations}
\]
with special relation for \( n = 0 \)
\[
\{ A^0, B \} = B.
\]
If\[
[A, B] = c I
\]
then\[
\exp (\lambda A) B \exp (-\lambda A) = B + \lambda [A, B]
\]
and higher-order commutators are zero.
\[
[a, a^\dagger] = 1, \ [a, a] = [a^\dagger, a^\dagger] = 0,
\]
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Now consider the commutator

\[
\left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) \tilde{a}_{k'} = \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) \tilde{a}_{k'} - \tilde{a}_{k'} \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right)
\]

We have

\[
\left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) \tilde{a}_{k'} = \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) \tilde{a}_{k'} - \tilde{a}_{k'} \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right)
\]

Thus, with \( r_k = r_{-k} \)

\[
\left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) a_{k'} = a_{k'} \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right) - \left( \sum_{k \neq 0} \frac{r_k}{2} a_{-k} a_{-k} + a_{-k} a_{-k} \right)
\]

So

\[
\left[ \left( \sum_{k \neq 0} \frac{r_k}{2} a_k^\dagger a_{-k} + a_{-k} a_k \right), a_{k'} \right] = \left( -\frac{r_{k'}}{2} \left( a_{-k'}^\dagger + a_{-k'}^\dagger \right) \right)
\]

and second commutator is

\[
\left[ \left( \frac{r_{k'}}{2} \left( a_{-k'}^\dagger + a_{-k'}^\dagger \right) \right), a_{k'} \right] = \left( \frac{r_{k'}}{2} \left( a_{-k'} a_{k'} + a_{-k'} a_{k'} \right) \right) - \left( \frac{r_{k'}}{2} \left( a_{-k'} a_{k'} + a_{-k'} a_{k'} \right) \right)
\]
\[
\left( \frac{r_{k'}}{2} \right) \left( a_{-k'}^\dagger a_{k'} + a_{-k'} a_{k'}^\dagger \right) = \left( \frac{r_{k'}}{2} \right) \left( \left\{ a_{-k'}^\dagger a_{-k'} \right\} + a_{k'} a_{-k'}^\dagger \right)
\]

which yields
\[
\left[ \left( \frac{r_{k'}}{2} \left( a_{-k'}^\dagger + a_{-k'} \right) \right), a_{k'} \right] = 0
\]

Therefore
\[
S \tilde{a}_k S^{-1} = b_k
\]
\[
= \exp \left[ \sum_{k \neq 0} \frac{r_k}{2} a_{-k}^\dagger a_{k} - a_{-k} a_k \right] a_k \exp \left[ - \sum_{k \neq 0} \frac{r_k}{2} a_{-k} a_{k}^\dagger - a_{-k} a_k \right]
\]
\[
b_k = a_k - r_{k'} a_{-k}^\dagger
\]

Similarly, we have
\[
\left[ \left( \sum_{k \neq 0} \frac{r_k}{2} a_{-k}^\dagger a_{-k} + a_{-k} a_k \right), \tilde{a}_{-k'} \right]
\]
\[
= \left( u \sum_{k \neq 0} \frac{r_k}{2} a_{-k}^\dagger a_{-k} + a_{-k} a_k \right) a_{-k'} - a_{-k'} \left( \sum_{k \neq 0} \frac{r_k}{2} a_{-k} a_{-k}^\dagger + a_{-k} a_k \right)
\]

Consider
\[
\left( \sum_{k \neq 0} \frac{r_k}{2} \left( a_{-k}^\dagger a_{-k} a_{k'}^\dagger + a_{-k} a_k a_{-k'}^\dagger \right) \right)
\]
\[
= \left( \sum_{k \neq 0} \frac{r_k}{2} \left( a_{k} a_{-k}^\dagger a_{-k'}^\dagger + a_{k} a_{-k} a_{-k'}^\dagger \right) \right)
\]
\[
= \left( \sum_{k \neq 0} \left( a_{k} a_{-k}^\dagger - \frac{r_k}{2} \right) a_{-k} \left[ a_{-k}^\dagger a_{-k} + \delta_{-k} a_{-k'}^\dagger \right] \right)
\]
\[
= \left( \sum_{k \neq 0} \left( a_{k} a_{-k}^\dagger - \frac{r_k}{2} \right) a_{-k} \left[ a_{-k}^\dagger a_{-k} + \delta_{-k} a_{-k'}^\dagger \right] \right)
\]
\[
= \sum_{k \neq 0} \left( a_{k} a_{-k}^\dagger - \frac{r_k}{2} \right) a_{-k} \left[ a_{-k}^\dagger a_{-k} + \delta_{-k} a_{-k'}^\dagger \right]
\]
\[
= \sum_{k \neq 0} \left( a_{k} a_{-k}^\dagger - \frac{r_k}{2} \right) a_{-k} \left[ a_{-k}^\dagger a_{-k} + \delta_{-k} a_{-k'}^\dagger \right]
\]
\[
= a_{-k'} \sum_{k \neq 0} \left( a_{k} a_{-k}^\dagger - \frac{r_k}{2} \right) a_{-k} + \frac{r_k}{2} \left( a_{k'} a_{k'}^\dagger \right)
\]

Therefore
\[
\left[ \left( \sum_{k \neq 0} \frac{r_k}{2} a_{-k} a_{-k}^\dagger - a_{-k} a_k \right), a_{-k'} \right] = u a_{-k'} + u r_{k'} a_{k'}^\dagger
\]
\[
= b_{-k'}^\dagger
\]

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Summarizing, we have

\[ b_k = (u a_k - u r_k a_{-k}^\dagger) \]
\[ b_{-k}^\dagger = (u a_{-k}^\dagger + u r_k a_k) \]

or in matrix form

\[
\begin{pmatrix}
  b_k \\
  b_{-k}^\dagger
\end{pmatrix} =
\begin{pmatrix}
  u & -u r_k \\
  u r_k & u
\end{pmatrix}
\begin{pmatrix}
  a_k \\
  a_{-k}^\dagger
\end{pmatrix}
\]

Putting \( r_k \) a pure imaginary and \( u \) a \( c \)-number, i.e., \( \{r_k, u\} \in \mathbb{C} \), we put

\[ u r_k = u \frac{v}{u} = v \text{ is a pure imaginary} \]

We can also put \( \{r_k, u\} \in \mathbb{R} \) and the transformation can still be canonical. Therefore the whole transformation idea is not unique and can at most only be partly unitary. We take \( v \) as pure imaginary

\[
\begin{pmatrix}
  b_k \\
  b_{-k}^\dagger
\end{pmatrix} =
\begin{pmatrix}
  u & -v \\
  -v^* & u^*
\end{pmatrix}
\begin{pmatrix}
  a_k \\
  a_{-k}^\dagger
\end{pmatrix}
\]

Taking the inverse, we have [diagonal of product = determinant of original matrix]

\[
\begin{pmatrix}
  u & -v \\
  v & u
\end{pmatrix}
\begin{pmatrix}
  u & v \\
  -v & u
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  |u|^2 + |v|^2 & (uv - vu) \\
  (vu - uv) & |u|^2 + |v|^2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}
\]

The make the whole transformation non-unitary. The condition

\[ u^2 + v^2 = 1 \]

makes the transformation canonical. Whereas, the condition \( uv - vu = 0 \) occurs by virtue of orthogonality of the eigenfunctions. Thus,

\[
\begin{pmatrix}
  a_k \\
  a_{-k}^\dagger
\end{pmatrix} =
\begin{pmatrix}
  u & v \\
  -v & u
\end{pmatrix}
\begin{pmatrix}
  b_k \\
  b_{-k}^\dagger
\end{pmatrix}
\]
B.4.1 Check for microcausality

\[
\begin{pmatrix}
  b_k \\
  b_k^\dagger
\end{pmatrix} =
\begin{pmatrix}
  u & -v \\
  v & u^\ast
\end{pmatrix}
\begin{pmatrix}
  a_k \\
  a_k^\dagger
\end{pmatrix}
\]

\[
[b_k, b_k^\dagger] = 1
\]

\[
\begin{align*}
\left[a_k - r_k a_{-k}^\dagger, \left(a_k^\dagger + r_k a_{-k}\right)\right] & = \left(a_k - r_k a_{-k}^\dagger\right) \left(a_k^\dagger + r_k a_{-k}\right) - \left(a_k^\dagger + r_k a_{-k}\right) \left(a_k - r_k a_{-k}^\dagger\right) \\
& = \left(a_k - r_k a_{-k}^\dagger\right) \left(a_k^\dagger - r_k a_{-k} a_{-k}^\dagger\right) - \left(a_k^\dagger + r_k a_{-k}\right) \left(a_k - r_k a_{-k}^\dagger\right) \\
& = u^2 \left[a_k, a_k^\dagger\right] + u^2 r_k^2 \left[a_{-k}, a_{-k}^\dagger\right] = u^2 + u^2 r_k^2 \\
& = u_k^2 + u^2 r_k^2, \text{let } r_k^2 = \frac{v_k^2}{u_k^2}
\end{align*}
\]

Hyperbolic Functions

\[
\begin{align*}
\cosh(2r_k) & = \frac{g_k}{E_k} \\
\sinh(2r_k) & = \frac{g_k}{E_k}
\end{align*}
\]

\[
E_k \cosh(2r_k) - g_k \sinh(2r_k) = 0
\]

Note that the low-energy excitation modes in the superfluid are found to be proportional to the \(k\) exhibiting the property of sound waves. In this case Bogoliubov transformation is not defined for \(k = 0\). This infrared divergence is identified by \(r_o \sim O(\ln N)\) so that it is divergent actually only in the thermodynamic limit.

C Decoupling degrees of freedom in CMP and QFT

Another important transformations which has to do with separating degrees of freedom are the so-called decoupling transformations. We will only present
here the basic principle which goes back to Foldy and Woutheysen \[78,80\],
decoupling positive and negative energy states of the relativistic Dirac electrons.
The technique is an essentially iterative procedure to all orders in decoupling by
adding to the transformed Hamiltonian at each order of the calculation a suitable
decoupling perturbation. Similar technique has been employed in the calculation
of the magnetic susceptibility of interacting Bloch fermions in solids \[15\]. More
recently, this technique appears in somewhat similar form in the literature which
address suppressing decoherence in quantum information systems \[81\].

C.1 Decoupling transformation for Dirac energy bands

The decoupling of energy bands of $4 \times 4$ Hamiltonian of the relativistic Dirac
equation was initiated by Foldy-Woutheysen transformation. This work perhaps
marks the birth of dressing transformation in quantum physics, i.e., diagonal-
ization or removal of off-diagonal terms to arbitrary order.

The Dirac equation is given by,

$$i\frac{\partial \Psi}{\partial t} = (\beta m + \vec{\alpha} \cdot \vec{p}) \Psi \quad (208)$$

where $p$ is the momentum operator, $\alpha$ and $\beta$ are the well-known Dirac matrices
(in the usual representation with $\beta$ diagonal) and units in which $\hbar = c = 1$.
Restoring dimensional units of $c$ and $\hbar$ we have

$$i\hbar\frac{\partial \Psi}{\partial t} = (\beta mc^2 + c\vec{\alpha} \cdot \vec{p}) \Psi$$

where

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \gamma^0, \text{ 'even' since diagonal only}$$

$$\alpha_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix}, \text{ 'odd' since diagonal is zero}$$

Notice:

$$\frac{1 + \beta}{2} = \frac{1}{2} \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\frac{1 - \beta}{2} = \frac{1}{2} \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Therefore

$$\frac{1 + \beta}{2} \Psi + \frac{1 + \beta}{2} \Psi = \Psi$$
Therefore

\[ i \frac{\partial}{\partial t} \Psi = H' \Psi \]

\[ i \frac{\partial}{\partial t} \left( \Phi' X' \right) = \sqrt{p^2 + m^2} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} \Phi' \\ X' \end{pmatrix} \]

\[ = \begin{pmatrix} \sqrt{p^2 + m^2} \Phi' \\ -\sqrt{p^2 + m^2} X' \end{pmatrix}, \text{ this is a decoupled bands!} \]

or

\[ i \frac{\partial}{\partial t} \Phi' = \sqrt{p^2 + m^2} \Phi' \]

\[ i \frac{\partial}{\partial t} X' = -\sqrt{p^2 + m^2} X' \]

\[
\begin{align*}
\beta \alpha_{\mu} &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_{\mu} \\ -\sigma_{\mu} & 0 \end{pmatrix} \\
\alpha_{\mu} \beta &= \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_{\mu} \\ -\sigma_{\mu} & 0 \end{pmatrix} \\
\beta \alpha_{\mu} \beta &= \begin{pmatrix} 0 & \sigma_{\mu} \\ -\sigma_{\mu} & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_{\mu} \\ -\sigma_{\mu} & 0 \end{pmatrix} = -\alpha_{\mu} \\
\{\alpha_{\mu}, \beta\} &= 0 \\
\beta^2 &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = I \\
\vec{\alpha} \cdot \vec{\alpha} &= \sum_{\mu} \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_{\mu} \\ \sigma_{\mu} & 0 \end{pmatrix} = \sum_{\mu} \begin{pmatrix} \sigma_{\mu}^2 & 0 \\ 0 & \sigma_{\mu}^2 \end{pmatrix} = \sum_{\mu=1}^{3} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \\
\sigma_x^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_y^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_z^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_x \sigma_y &= i \sigma_z \\
\sigma_y \sigma_z &= i \sigma_x \\
\sigma_z \sigma_x &= -i \sigma_y \\
\end{align*}

Thus

\[ H_{\text{Dirac}} = \begin{pmatrix} mc^2 & c \vec{p} \cdot \vec{\sigma} \\ c \vec{p} \cdot \vec{\sigma} & -mc^2 \end{pmatrix} \]
The eigenfunction obeys

\[ i \frac{\partial \psi}{\partial t} = (\beta m + \vec{\alpha} \cdot \vec{p}) \psi \]

\[ = E \psi \]

The eigenfunctions are of the form of Bloch function: \( u(p) \) \( e^{-\vec{p} \cdot \vec{x}} \). For each value of \( \vec{p} \) there are 4 linearly independent spinors \( u(p) \)

\[ u(p) = \begin{pmatrix} u_+ & u_- \\ u_+ & u_- \end{pmatrix} \]

corresponding to the two energy eigenvalues \( \pm \sqrt{m^2 + \vec{p}^2} \) and two eigenvalues \( \pm 1 \) for the \( z \)-component related to spin. Equation (208) contains odd operators, specifically the components of \( \alpha \). It is possible to perform canonical transformation which remove all odd operators. This is the Foldy-Wouthuesen transformation.

If \( S \) is an Hermitian operator, then

\[ \Psi' = e^{iS} \Psi \]

\[ H' = e^{iS} H e^{-iS} - e^{iS} \left( i \frac{\partial}{\partial t} \right) e^{-iS} \]

leaves

\[ \left( i \frac{\partial}{\partial t} \right) e^{-iS} e^{iS} \Psi = H e^{-iS} e^{iS} \Psi \]

\[ \left( i \frac{\partial}{\partial t} \right) e^{-iS} e^{iS} \Psi = H e^{-iS} e^{iS} \Psi \]

\[ \left[ e^{iS} \left( i \frac{\partial}{\partial t} \right) e^{-iS} \right] e^{iS} \Psi + \left( i \frac{\partial}{\partial t} \right) e^{iS} \Psi = \left( e^{iS} H e^{-iS} \right) e^{iS} \Psi \]

\[ \left( i \frac{\partial}{\partial t} \right) e^{iS} \Psi = \left[ (e^{iS} H e^{-iS}) - e^{iS} \left( i \frac{\partial}{\partial t} \right) e^{-iS} \right] e^{iS} \Psi \]

\[ \left( i \frac{\partial}{\partial t} \right) \Psi' = H' \Psi' \]

Let \( S \) be a non-explicity time-dependent operator \( [(\beta m + \vec{\alpha} \cdot \vec{p})]: i \frac{\partial \Psi}{\partial t} = (\beta m + \vec{\alpha} \cdot \vec{p}) \Psi \)

\[ S = -i \left( \frac{1}{2m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \]

where the function \( \omega \left( \frac{\vec{p}}{m} \right) \) is to be determined such that \( H' \) is free of odd operators.

\[ T (\vec{p}) \Psi (x) = \frac{1}{(2\pi\hbar)^2} \int d\vec{p} d\vec{x} T (\vec{p}') \exp \left[ \frac{i}{\hbar} \vec{p}' \cdot (x - x') \right] \Psi (x') \]
If \( \{H, S\} = 0 \), then

\[
H' = e^{iS} H e^{-iS} = e^{2iS} H = e^{2iS} \cos (2S + i \sin 2S)
\]

Since

\[
2iS = \left( \frac{1}{m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) = \Omega \, \vec{\alpha} \cdot \vec{p}
\]

\[
H' = \exp \left( \left( \frac{1}{m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right) H
\]

\[
= \left\{ 1 + \Omega \beta \vec{\alpha} \cdot \vec{p} - \Omega^2 (\vec{\alpha} \cdot \vec{p})^2 - \frac{\Omega^3}{e!} (\vec{\alpha} \cdot \vec{p})^4 \right. \left( \beta \vec{\alpha} \cdot \vec{p} \right) + \frac{\Omega^4}{4!} (\vec{\alpha} \cdot \vec{p})^4 + \left. \frac{\Omega^5}{5!} (\vec{\alpha} \cdot \vec{p})^4 \right) (\beta \vec{\alpha} \cdot \vec{p}) - \ldots \right\} H
\]

\[
= \left[ 1 - \Omega^2 (\vec{\alpha} \cdot \vec{p})^2 + \frac{\Omega^4 (\vec{\alpha} \cdot \vec{p})^4}{4!} - \frac{\Omega^6 (\vec{\alpha} \cdot \vec{p})^6}{6!} + \ldots \right] H
\]

\[
= \left\{ \text{Cos} (p \Omega) + \sin (p \Omega) \frac{\beta \vec{\alpha} \cdot \vec{p}}{p} \right\} H
\]

\[
= \left\{ \text{Cos} (p \Omega) \beta m + \sin (p \Omega) \frac{\beta \vec{\alpha} \cdot \vec{p}}{p} \beta m \right\} = \left\{ \text{Cos} (p \Omega) \beta m - \sin (p \Omega) \frac{\beta \vec{\alpha} \cdot \vec{p}}{p} \beta m \right\}
\]

\[
= \left\{ \text{Cos} (p \Omega) \frac{\alpha \cdot \vec{p} + \sin (p \Omega) \beta \vec{p}}{p} \right\} = \left\{ p \text{Cos} (p \Omega) \frac{\alpha \cdot \vec{p}}{p} + \sin (p \Omega) \beta \vec{p} \right\}
\]

\[
\text{Grouping, we obtained by separating the odd terms, } \vec{\alpha} \cdot \vec{p}.
\]

\[
H' = \beta \left\{ m \text{Cos} (p \Omega) + p \sin (p \Omega) + \frac{\vec{\alpha} \cdot \vec{p}}{p} \left[ p \text{Cos} (p \Omega) - m \sin (p \Omega) \right] \right\}
\]

Therefore free of odd terms if

\[
[p \cos (p \Omega) - m \sin (p \Omega)] = 0
\]

\[
\frac{p}{m} = \frac{\sin (p \Omega)}{\cos (p \Omega)} = \tan^{-1} (p \Omega)
\]

\[
= \tan^{-1} \left( \frac{p}{m} \omega \left( \frac{\vec{p}}{m} \right) \right)
\]
\[ H' = \beta \left\{ m \cos(p\Omega) + p \sin(p\Omega) \right\} \]
\[ = \beta \left\{ m \frac{m}{\sqrt{p^2 + m^2}} + p \frac{p}{\sqrt{p^2 + m^2}} \right\} \]
\[ = \beta \sqrt{p^2 + m^2} = \beta E_p \]
\[ \cos(p\Omega) = \frac{m}{\sqrt{p^2 + m^2}} \]

Appendix

Now the powers of

\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^2 \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^3 \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^4 \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^5 \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^6 \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^{2n} \)
\( (\beta \mathbf{\alpha} \cdot \mathbf{p})^{2n+1} \)

We have to show that indeed, \( \{H, S\} = 0 \).

\[ \{H, S\} = HS + SH \]
\[ = (\beta m + \vec{\alpha} \cdot \vec{p}) \left[ -i \left( \frac{1}{2m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] \]
\[ + \left[ -i \left( \frac{1}{2m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] (\beta m + \vec{\alpha} \cdot \vec{p}) \]
\[ = \left\{ \left[ -i \left( \frac{1}{2m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] \right\} \]
\[ + \left\{ \left[ -i \left( \frac{1}{2m} \right) \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] \right\} \]
\[ + \left\{ \left[ -i \left( \frac{1}{2m} \right) \beta \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] \right\} \]
\[ + \left\{ \left[ -i \left( \frac{1}{2m} \right) \vec{\alpha} \cdot \vec{p} \omega \left( \frac{\vec{p}}{m} \right) \right] \right\} \]
\[ = 0, \text{ since } \{\beta \vec{\alpha} \cdot \vec{p} + \vec{\alpha} \cdot \vec{p} \beta\} = 0 \]

C.2 Effective band theory of magnetic susceptibility of relativistic Dirac fermions

In this section, we formulate the magnetic susceptibility of relativistic Dirac fermions analogous to energy-band dynamics of crystalline solids. The Hamiltonian of free relativistic Dirac fermions is of the form

\[ \mathcal{H} = \beta \Delta + c\vec{\alpha} \cdot \vec{P}. \] (209)
We designate quantum operators in capital letters and their corresponding eigenvalues in small letters. The equation for the eigenfunctions and eigenvalues is

\[ Hb_\lambda(x, p) = E_\lambda(x, p)b_\lambda(x, p), \]  

where \( E_\lambda(p) = \pm E(p) \), and \( E_\lambda(q' - q) = \frac{1}{(2\pi\hbar)^2} \int d\bar{p} e^{i\bar{p}(\bar{q}' - \bar{q})} E_\lambda(p) \), \( \lambda \) labels the band index: \( \pm \) spin band for positive energy states and \( \pm \) spin band for negative energy states.

\[ E_\lambda(p) = \pm \sqrt{(cp)^2 + (mc^2)^2}. \]

The doubly degenerate bands is reminiscent of the Kramer conjugates in bismuth and Bi-Sb alloys. The localized function \( a_\lambda(\bar{x} - \bar{q}) \) is the ‘Wannier function’ for relativistic Dirac fermions, defined below.

In the absence of magnetic field we may define the Wannier function and Bloch function of a relativistic Dirac fermions as

\[ b_\lambda(x, p) = \frac{1}{(2\pi\hbar)^2} e^{i(\frac{\bar{p}}{\hbar})\bar{x}} u_\lambda(\bar{p}), \]

\[ a_\lambda(\bar{x} - \bar{q}) = \frac{1}{(2\pi\hbar)^2} \int d\bar{p} e^{i(\frac{\bar{p}}{\hbar})\bar{x}} b_\lambda(x, p), \]

where \( b_\lambda(x, p) \) is the Bloch function, and \( a_\lambda(\bar{x} - \bar{q}) \) the corresponding Wannier function. \( u_\lambda(\bar{p}) \) is a four-component function. The \( u_\lambda(\bar{p})'s \) are related to the \( u_\lambda(0)'s \) by a unitary transformation, \( S \), which also transforms the Dirac Hamiltonian into an even form, i.e., no longer have interband terms or the negative and positive energy states are decoupled. This is equivalent to the transformation from Kohn-Luttinger basis to Bloch functions in \( \bar{k} \cdot \bar{p} \) theory. We have

\[ S = \frac{E + \beta \mathcal{H}}{\sqrt{2E(E + \Delta)}}. \]

which can be written in matrix form as

\[ S = \left( \begin{array}{cc} \sqrt{\frac{(E + \Delta)^2}{2E}} & e^{i\bar{p}} \sqrt{\frac{1}{2E}} \\ e^{-i\bar{p}} \sqrt{\frac{1}{2E}} & \sqrt{\frac{(E + \Delta)^2}{2E}} \end{array} \right), \]

where the entries are \( 2 \times 2 \) matrices, \( \Delta = mc^2 \), and all matrix elements may be viewed as matrix elements of \( S \) between the \( u_\lambda(0)'s \), which are the spin functions in the Pauli representation. The transformed Hamiltonian is

\[ \mathcal{H} = S\mathcal{H}S^\dagger = \beta E(\bar{p}). \]  

The \( a_\lambda(\bar{x} - \bar{q}) \) is not a \( \delta \)-function because of the dependence of \( u_\lambda(\bar{p}) \) on \( \bar{p} \); it is spread out over a region of the order of the Compton wavelength, \( \frac{\hbar}{m} \), of the electron and no smaller, as pointed out first by Newton and Wigner [82], Foldy and Wouthuysen [78], and by Blount. [83]
The Weyl correspondence for the momentum and coordinate operator giving the correct dynamics of quasiparticles is given by the prescription that the momentum operator $\vec{P}$ and coordinate operator $\vec{Q}$ be defined with the aid of the Wannier function and the Bloch function as

$$\vec{P}_b(\vec{x}, \vec{p}) = \vec{p}_b(\vec{x}, \vec{p}),$$
$$\vec{Q}_a(\vec{x} - \vec{q}) = \vec{q}_a(\vec{x} - \vec{q}),$$

and the uncertainty relation follows in the formalism,

$$[Q_i, P_j] = i\hbar \delta_{ij}.$$

From Eq. (210), we have

$$\frac{1}{(2\pi\hbar)^2} \int dq \ e^{(-\frac{i}{\hbar})\vec{p} \cdot \vec{q}} \mathcal{H}_a(\vec{x} - \vec{q}) = E_\lambda(p) \frac{1}{(2\pi\hbar)^2} \int dq \ e^{(-\frac{i}{\hbar})\vec{p} \cdot \vec{q}} a_\lambda(\vec{x} - \vec{q}),$$
$$\mathcal{H}_a(\vec{x} - \vec{q}') = \int dq \ E_\lambda(\vec{q}' - q) a_\lambda(\vec{x} - \vec{q}).$$

These relations allows us to transform the ‘bare’ Hamiltonian operator to an ‘effective Hamiltonian’ expressed in terms of the $\vec{P}$ operator and the $\vec{Q}$ operator. This is conveniently done by the use of the ‘lattice’ Weyl transform (‘lattice’ Weyl transform and Weyl transform will be used interchangeably for infinite translationally invariant system including crystalline solids). Thus, any operator $A(\vec{P}, \vec{Q})$ which is a function of $\vec{P}$ and $\vec{Q}$ can be obtained from the matrix elements of the ‘bare’ operator, $A_{\text{op}}$, between the Wannier functions or between the Bloch functions as,

$$A(\vec{P}, \vec{Q}) = \sum_{\lambda\lambda'} \int d\vec{u} \ d\vec{v} \ a_{\lambda \lambda'}(\vec{u}, \vec{v}) \exp \left[ -\frac{i}{\hbar} (\vec{Q} \cdot \vec{u} + \vec{P} \cdot \vec{v}) \right] \Omega_{\lambda \lambda'},$$
$$a_{\lambda \lambda'}(\vec{u}, \vec{v}) = \hbar^{-8} \int d\vec{p} \ d\vec{q} \ a_{\lambda \lambda'}(\vec{p}, \vec{q}) \exp \left[ -\frac{i}{\hbar} (\vec{Q} \cdot \vec{u} + \vec{P} \cdot \vec{v}) \right],$$
$$a_{\lambda \lambda'}(\vec{p}, \vec{q}) = \int d\vec{u} e^{\frac{i}{\hbar} \vec{p} \cdot \vec{u}} \langle \vec{p} - \frac{1}{2} \vec{v}, \lambda | A_{\text{op}} \ | \vec{q} + \frac{1}{2} \vec{v}, \lambda' \rangle$$
$$= \int d\vec{u} e^{\frac{i}{\hbar} \vec{q} \cdot \vec{u}} \langle \vec{p} + \frac{1}{2} \vec{u}, \lambda | A_{\text{op}} \ | \vec{q} - \frac{1}{2} \vec{u}, \lambda' \rangle,$$

where $|\vec{p}, \lambda \rangle$ and $|\vec{q}, \lambda \rangle$ are the state vectors representing the Bloch functions and Wannier functions, respectively, and

$$\Omega_{\lambda \lambda'} = \int d\vec{p} |\vec{p}, \lambda \rangle \langle \vec{p}, \lambda' |$$
$$= \int d\vec{q} |\vec{q}, \lambda \rangle \langle \vec{q}, \lambda' |.$$
C.3 Dressing of conjugate variables in energy-band quantum dynamics

A few more words about $\vec{Q}$ and $\vec{P}$. The use of $\vec{Q}$, conjugate to the operator $\vec{P}$ of the Hamiltonian in even form, is preferred in the band-dynamical formalism. The reason we now associate $\vec{Q}$ with the operator $\vec{P}$ of the Hamiltonian in even form is that this momentum operator now belongs to the respective bands (each of infinite width) of the decoupled Dirac Hamiltonian. This operator is now analogous to the crystal momentum operator in crystalline solids. For the original Dirac Hamiltonian $\dot{x} = c$ [from Eq. (209)] leading to a complex zitterbewegung motion in $x$-space, whereas for the Hamiltonian in even form $\dot{Q} = v$ [from Eq. (211)], $c$ is the speed of light and $v$ the velocity of a wave packet in the classical limit, and thus $Q$ is more closely related to the band dynamics of fermions than $x$. Moreover, on the cognizance that the continuum is the limit when the lattice constant of an array of lattice points goes to zero, there is a more compelling fundamental basis for using the lattice-position operator $Q$. Since quantum mechanics is the mathematics of measurement processes, the most probable measured values of the positions are the lattice-point coordinates. Indeed, these lattice points, or atomic sites, are where the electrons spend some time in crystalline solids. Therefore the lattice points and crystal momentum are clearly the observables of the theory and $q$ and $p$ constitute the eigenvalues of the lattice-point position operator $Q$ and crystal momentum operator $P$, respectively. Thus, $Q$ is considered here as the generalized position operator in quantum theory for describing energy-band quantum dynamics, canonical conjugate to ‘crystal’ momentum operator $\vec{P}$ of the Hamiltonian in even form. Although the ‘bare’ operator $x$ can still be used as position operator it only unnecessarily renders very complicated and almost intractable resulting expressions, since this does not directly reflect the appropriate observables in band dynamics as first enunciated by Newton and Wigner and by Wannier several decades ago. Thus, in understanding the dynamics of Dirac relativistic quantum mechanics succinctly, position space should be defined at discrete points $q$ which are eigenvalues of the operator $Q$.

C.4 The Even Form of Dirac Hamiltonian in a Uniform Magnetic Field

The Dirac Hamiltonian for an electron with anomalous magnetic moment in a magnetic field is

$$\mathcal{H}_{op} = \vec{\alpha} \cdot \vec{\Pi}_{op} + \beta mc^2 - \frac{1}{2}(g - 2)\mu_B \beta \vec{\sigma} \cdot \vec{B},$$

where

$$\vec{\Pi}_{op} = c\vec{P}_{op} - e\vec{A} \left( \vec{Q}_{op} \right),$$

$$\mu_B = \frac{e\hbar}{2mc}.$$
The transformed Hamiltonian in even form $\mathcal{H}'_B$ is given by Ericksen and Kolsrud [85]

$$\mathcal{H}'_B = \beta \left[ m^2 c^4 + \Pi^2 - \epsilon_e c(1 + \lambda') \vec{\sigma} \cdot \vec{B} + \beta \left( \frac{\lambda' \hbar}{2mc} \right) \sigma \cdot (B \times \Pi - \Pi \times B) \right]^{1/2}. \quad (212)$$

where $\lambda' = \frac{1}{2} (g - 2)$, and

$$\tilde{\Pi} = cP - eA(Q) - eA(r),$$
$$A(Q + r) = \frac{1}{2} B \times (Q + r),$$
$$r = \beta \left( \frac{\lambda' \hbar}{mc} \right) \sigma.$$

The above Hamiltonian can be written as

$$\mathcal{H}'_B = \beta \left[ m^2 c^4 + \Pi^2 - e\epsilon_e c(1 + \lambda') \vec{\sigma} \cdot \vec{B} - 2 \left( \frac{1}{2} B \times r \cdot \Pi \right) \right]^{1/2}$$
$$= \beta \left[ m^2 c^4 + \Pi^2 - e\epsilon_e c(1 + \lambda') \vec{\sigma} \cdot \vec{B} - 2 A(r) \cdot \Pi \right]^{1/2}$$
$$= \beta \left[ m^2 c^4 + \Pi^2 - e\epsilon_e c(1 + \lambda') \vec{\sigma} \cdot \vec{B} - A^2(r) \right]^{1/2}$$
$$= \beta \left[ m^2 c^4 + \Pi^2 - e\epsilon_e c(1 + \lambda') \vec{\sigma} \cdot \vec{B} - \left( \frac{\lambda' \hbar}{2mc} \right)^2 B^2 \right]^{1/2}. \quad (213)$$

C.5 Translation operator, $T_M(q)$, under uniform magnetic fields

In the presence of a uniform magnetic field, magnetic Wannier Functions, $A_\lambda(x - q)$, and magnetic Bloch functions, $B_\lambda(x, p)$, exist. This is proved by using symmetry arguments. In general, these two basis functions are complete and span all the eigensolutions of the magnetic Hamiltonian belonging to a band index $\lambda$. The magnetic Wannier Functions $A_\lambda(x - q)$ and magnetic Bloch functions $B_\lambda(x, p)$ are related by similar unitary transformation in the absence of magnetic field, namely,

$$B_\lambda(x, p) = \frac{1}{(2\pi \hbar)^{3/2}} e^{i(\frac{\pi}{\hbar})\vec{p} \cdot \vec{x}} u_\lambda(p),$$
$$A_{\lambda(x - q)} = \frac{1}{(2\pi \hbar)^{3/2}} \int d\vec{p} \ e^{i(\frac{\pi}{\hbar})\vec{p} \cdot \vec{q}} B_\lambda(x, p),$$

where $\vec{p}$ and $\vec{q}$ are quantum labels.
Under a uniform magnetic fields, we have for a translation operator, $T_M(q)$, obeying the relation,
\[
\nabla \cdot T_M(q) = [P, T_M(q)] = \frac{ie}{\hbar c} A(q) T_M(q).
\]
Therefore,
\[
T_M(q) = \exp \left( -\frac{ie}{\hbar c} A(r) \cdot q \right) C(q),
\]
where $C_0(q)$ is an operator which do not depend explicitly on $r$. Since $T_M(q)$ is a translation operator by amount $q$ leads us to write
\[
C_0(q) = \exp(-q \cdot \nabla r), \quad \text{a pure displacement operator by amount } -q.
\]
Equation (214) means that $[P, T_M(q)]$ is diagonal if $T_M(q)$ is diagonal, and therefore they have the same eigenfunctions and the same quantum label. Therefore displacement operator in a translationally symmetric system under a uniform magnetic field acquire the so-called ‘Peierls phase factor’.

Clearly, bringing the wavepacket or Wannier function around a closed loop, or around plaquette in the tight-binding limit, would acquire a phase equal to the magnetic flux through the area defined by the loop. This is the so-called Bohm-Aharonov effect or Berry phase. Thus, the concept of Berry phase has actually been floating around in the theory of band dynamics since the time of Peierls. Berry [86] has brilliantly generalized the concept to parameter-dependent Hamiltonians even in the absence of magnetic field through the so-called Berry connection, Berry curvature, and Berry flux.

The magnetic translation operator generates all magnetic Wannier functions belonging to band index $\lambda$ from a given magnetic Wannier function centered at the origin, $A_0^\lambda (r - 0)$, as
\[
A_\lambda (r - q) = T_M(q) A_0^\lambda (r - 0) = \exp \left( -\frac{ie}{\hbar c} A(r) \cdot q \right) A_0^\lambda (r - q).
\]
We also have the following relation,
\[
T_M(q) T_M(\rho) = \exp \left( \frac{ie}{\hbar c} A(q) \cdot \rho \right) T_M(q + \rho),
\]
\[
[T_M(q), T_M(\rho)] = \exp \left( \frac{ie}{\hbar c} A(q) \cdot \rho \right) T_M(q + \rho) - \exp \left( \frac{ie}{\hbar c} A(\rho) \cdot q \right) T_M(\rho + q) = 2i \sin \left( \frac{e}{\hbar c} A(q) \cdot \rho \right) T_M(q + \rho).
\]
Moreover, we have,

\[
\hat{H}B_\lambda(x, p) = E_\lambda \left( p - \frac{e}{c} A(q) \right) B_\lambda(x, p), \\
\hat{H}A_\lambda(x - \bar{q}, p) = \int dq \, e^{i\hat{p}(q - \bar{q})} E_\lambda(q - \bar{q}) A_\lambda(x - \bar{q}),
\]

and the lattice Weyl transform of any operator, \( A_{op} \), is

\[
a_{\lambda\lambda'}(p, q) = \int d\vec{v} \, e^{i\vec{p} \cdot \vec{v}} \left\langle A_\lambda \left( \vec{q} - \frac{1}{2} \vec{v} \right) \left| \hat{H}_{B} \right| A_{\lambda'} \left( \vec{q} + \frac{1}{2} \vec{v} \right) \right\rangle. \tag{216}
\]

The Weyl transform of the Hamiltonian operator is easily calculated using Eq. (215) and Eq. (216). The reader is referred to Ref. [3, 5] for details of the derivation. Applying Eq. (216) to the even form of the Dirac Hamiltonian, we have

\[
h_B'(\vec{p}, \vec{q})_{\lambda\lambda'} = \int d\vec{v} \, e^{i\vec{p} \cdot \vec{v}} \left\langle A_\lambda \left( \vec{q} - \frac{1}{2} \vec{v} \right) \left| \hat{H}_B \right| A_{\lambda'} \left( \vec{q} + \frac{1}{2} \vec{v} \right) \right\rangle = \int d\vec{v} \, \exp \left[ \frac{i}{\hbar} \left( p - \frac{e}{c} A(q) \right) \cdot \vec{v} \right] E_{\lambda}(\vec{v}; B) \delta_{\lambda\lambda'}.
\]

C.5.1 The function \( E_\lambda(\vec{p} - \frac{e}{c} A(q); B) \delta_{\lambda\lambda'} \)

The function \( E_\lambda(\vec{p} - \frac{e}{c} A(q); B) \) is the Weyl transform of \( \beta [\mathcal{H}^2/2] \), where the matrix \( \beta \) served to designate the four bands. In order to calculate \( \chi \) we only need the knowledge of \( E_\lambda(\vec{p} - \frac{e}{c} A(q); B) \) as an expansion up to second order in the coupling constant \( e \) and after a change of variable [this is effected by setting \( A(q) = 0, p = \hbar \vec{k} \) in the expansion], we obtain the expression of \( E_\lambda(\vec{p} - \frac{e}{c} A(q); B)|_{A(q)=0} \), where the dependence in the field \( B \) is beyond the vector potential,

\[
E_\lambda(\vec{k}; B) = E_\lambda(\vec{k}; 0) + B E^{(1)}_\lambda(\vec{k}) + B^2 E^{(2)}_\lambda(\vec{k}) + \cdots
\]

The function \( E_\lambda(\vec{p} - \frac{e}{c} A(q); B)|_{A(q)=0} \) which includes the anomalous magnetic moment of the electron is obtained as

\[
E_\lambda(k; B) = \beta \left\{ E - \frac{e c}{2 E} \vec{L}_{c.m.} \cdot \vec{B} - \frac{(1 + \lambda')}{2E} \frac{e \hbar \vec{\sigma} \cdot \vec{B}}{8E^3} \left( e \hbar \vec{\sigma} \cdot \vec{B} \right)^2 + \frac{(e \hbar c)^2 c^2}{8 E^5} B^2 \left[ 1 + \left( \frac{\lambda E}{mc^2} \right)^2 \right] + O(e^3) \right\},
\]

\[83\]
where

\[ \vec{L}_{c.m.} = \beta \left( \frac{\lambda' \hbar}{mc} \right) \vec{\sigma} \times \vec{p}, \]

\[ \epsilon^2 = m^2 c^4 + c^2 \hbar^2 k_z^2, \]

\[ E \left( \vec{k} \right) = \sqrt{m^2 c^4 + c^2 \hbar^2 k_z^2}. \]

The term, $\vec{L}_{c.m.}$, is a magnetodynamic effect, i.e., due to hidden average angular momentum $\vec{L}_{c.m.}$ of a moving electron. Thus, the introduction of the Pauli anomalous term in $\mathcal{H}$ at the outset endows a rigid-body behavior to the electron, and its angular momentum about the origin $\vec{L}_0$ is

\[ \vec{L}_0 = \vec{L}_{MO} + \vec{L}_{c.m.}, \]

where $\vec{L}_{MO}$ is the angular momentum about the origin of the system of charge concentrated as a point at the center of mass and $\vec{L}_{c.m.}$ is the average angular momentum of the system, as a spread-out distribution of charge about the center of mass. Thus,

\[ \vec{L}_0 = \vec{q} \times \vec{p} + \left\langle \sum_i \vec{r}_i \times \vec{p}_i \right\rangle, \]

\[ \left\langle \sum_i \vec{r}_i \times \vec{p}_i \right\rangle = \beta \left( \frac{\lambda' \hbar}{mc} \right) \vec{\sigma} \times \vec{p}, \]

\[ M = - \left[ \frac{2E_\lambda^{(2)}(\vec{k})B}{sp} \right] \]

\[ = - \frac{(e\hbar c)^2}{4 \left[ E_\lambda(\vec{k}) \right]^5} \left[ 1 + \left( \frac{\lambda' E}{mc^2} \right)^2 \right] B. \]  

(217)

The induced magnetic moment due to a distribution of electric charge is

\[ M = - \frac{Be^2 \langle r^2 \rangle}{4mc^2}, \]  

(218)

where $\langle r^2 \rangle$ is the average of the square of the spatial spread of the distribution normal to the magnetic field. Equating Eqs. (217) with (218) we obtain

\[ \langle r^2 \rangle = \frac{mc^2(hc)^2 e^2}{E_\lambda(\vec{k})} \left[ 1 + \left( \frac{\lambda' E}{mc^2} \right)^2 \right]. \]  

(219)

For positive energy states $E_\lambda(k) = (c^2 \hbar^2 k_z^2 + m^2 c^4)^{\frac{1}{2}}$ and in the nonrelativistic limit, Eq. (219) reduces to

\[ \langle r^2 \rangle = (1 + \lambda^2) \left( \frac{\hbar}{mc} \right)^2, \]

and thus the effective spread of the electron at rest, and for $\lambda' = 0$, is precisely equal to the Compton wavelength.
The magnetic susceptibility is given by

\[
\chi = -\frac{1}{48\pi^3} \left( \frac{e}{\hbar c} \right)^2 \sum_{\lambda} \int d\mathbf{k} \left\{ \frac{\partial^2 E_\lambda(\mathbf{k};0)}{\partial k_x^2} \frac{\partial^2 E_\lambda(\mathbf{k};0)}{\partial k_y^2} - \left( \frac{\partial^2 E_\lambda(\mathbf{k};0)}{\partial k_x \partial k_y} \right)^2 \right\} \frac{\partial f(E_\lambda)}{\partial E_\lambda}
\]

\[
- \left( \frac{1}{2\pi} \right)^3 \sum_{\lambda} \int d\mathbf{k} \left| E^{(1)}_\lambda(k) \right|^2 \frac{\partial f(E_\lambda)}{\partial E_\lambda} - \left( \frac{1}{2\pi} \right)^3 \sum_{\lambda} \int d\mathbf{k} \left( E^{(2)}_\lambda(k) \right) f(E_\lambda).
\]

Using the following change of variable of integration,

\[
(hc)^3 \int d\mathbf{k} = \int_{-\infty}^{\infty} d\eta \int_0^{2\pi} d\phi \ E\left(\mathbf{k}\right) \ dE\left(\mathbf{k}\right),
\]

where

\[\eta = \hbar ck_z,\]

we obtain for the positive energy states the expression for \( \chi \) which can be divided into more physically meaningful terms as

\[
\chi = \chi_{LP} + \chi_P + \chi_{sp} + \chi_g + \chi_{MD},
\]

where

\[
\chi_{LP} = \frac{1}{24\pi^3} \left( \frac{e}{\hbar c} \right)^2 \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} \frac{\epsilon^2}{E^3} \frac{\partial f(E)}{\partial E} \ dE,
\]

(220)

\[
\chi_P = -\frac{1}{8\pi^2} \left( \frac{e}{\hbar c} \right)^2 \int_{-\infty}^{\infty} d\eta \int_{\epsilon}^{\infty} \frac{1}{E} \frac{\partial f(E)}{\partial E} \ dE,
\]

(221)

\[
\chi_{sp} = -\frac{1}{8\pi^2} \left( \frac{e^2}{\hbar c} \right) \int_{-\infty}^{\infty} d\eta \int_{\epsilon}^{\epsilon} \left[ 1 + \left( \frac{\lambda E}{mc^2} \right)^2 \right] f(E) \ dE,
\]

(222)

\[
\chi_g = \frac{1}{8\pi^2} \left( \frac{e^2}{\hbar c} \right) \int_{-\infty}^{\infty} d\eta \int_{\epsilon}^{\infty} \frac{f(E)}{E^2} \ dE,
\]

(223)

\[
\chi_{MD} = -\frac{\lambda^2}{8\pi^2} \left( \frac{e^2}{\hbar c} \right) \int_{-\infty}^{\infty} d\eta \int_{\epsilon}^{\infty} \frac{(E^2 - \epsilon^2)}{(mc^2)^2 E} \frac{\partial f(E)}{\partial E} \ dE,
\]

(224)

where

\[
\left( \frac{ec}{2E} \right)_{\text{c.m.}}^2 = \left( \frac{\lambda e^2}{2mc^2} \right) \left( \frac{E^2 - \epsilon^2}{E^2} \right),
\]

\[
\vec{B} = B \frac{\vec{z}}{|\vec{z}|}.
\]

The total susceptibility for the positive energy states is

\[
\chi = \frac{1}{(2\pi)^2} \left( \frac{e^2}{\hbar c} \right) \left[ 1 + \lambda^2 - \frac{1}{3} \right] \int_{0}^{\infty} d\eta \frac{f(\epsilon)}{\epsilon} \left( \frac{e^2}{\hbar c} \right) \left( \frac{\lambda^2}{mc^2} \right)^2 \int_{0}^{\infty} d\eta \ G(\epsilon - \mu),
\]

(225)
where

\[ G(\epsilon - \mu) = k_B T \ln \left( 1 + \exp \left[ - \frac{(\epsilon - \mu)}{k_B T} \right] \right) \]

\[ = \int_{-\infty}^{\infty} f(E) dE. \]

The contributions of the holes is obtained by replacement of \( f(\epsilon) \) and \( G(\epsilon - \mu) \) in Eq. (225) by \( (1 - f(-\epsilon)) \) and \( G(\epsilon + \mu) \), respectively.

The relative importance of terms that made up \( \chi \) at \( T = 0 \) of Dirac fermions, where \( n \) is the electron density, \( k_F = (3\pi^2 n)^{\frac{1}{3}}, \eta_F = \hbar c k_F \) and \( E_F = (\Delta^2 + \eta_F^2)^{\frac{1}{2}} \), is summarized below.

| Various Contributions to \( \chi_{\text{Dirac}} \) at \( T = 0 \) | Nonrelativistic, \( \frac{\lambda}{\pi} \ll 1 \) | Ultrarelativistic, \( \frac{\lambda}{\pi} \gg 1 \) |
|---------------------------------------------------|-----------------|-----------------|
| \( \chi_{LP} = -\frac{1}{12\pi^2} \left( \frac{\lambda}{\pi} \right)^2 \left( \frac{e}{m} \right) \frac{\eta_F}{E_F} \left( \frac{1}{12} + \Delta^2 \eta_F \right) \) | \( -\frac{1}{12\pi^2} \left( \frac{e}{m} \right) \frac{\eta_F}{E_F} \) \( k_F \) | \( -\frac{1}{12\pi^2} \left( \frac{e}{m} \right) \) \( \frac{1}{4} \) |
| \( \chi_p = \frac{1}{4\pi^2} (1 + \lambda')^2 \left( \frac{e}{m} \right) \frac{\eta_F}{E_F} \) | \( \frac{1}{4\pi^2} (1 + \lambda')^2 \left( \frac{e}{m} \right) \) \( k_F \) | \( \frac{1}{4\pi^2} (1 + \lambda')^2 \left( \frac{e}{m} \right) \) |
| \( \chi_{MD} = -\frac{\lambda^2}{4\pi^2} \left( \frac{e}{m} \right) \frac{\eta_F}{E_F} \left( \frac{1}{12} - \Delta^2 \eta_F \right) \) | \( \Rightarrow 0 \) | \( \frac{\lambda^2}{4\pi^2} \left( \frac{e}{m} \right) \frac{1}{4} \left( \frac{\eta_F}{\lambda} \right)^2 \) |
| \( \chi_{\text{spread}} = -\frac{1}{12\pi^2} \left( \frac{\lambda}{\pi} \right)^2 \left( \frac{e}{m} \right) \sinh^{-1} \left( \frac{\eta_F}{\lambda} \right) - \chi_{LP} \) | \( \Rightarrow 0 \) | \( \frac{1}{12\pi^2} \left( \frac{e}{m} \right) \ln \frac{2n_F}{\lambda} - \frac{1}{4} \) |
| \( \Rightarrow 0 \) | \( \frac{\lambda^2}{4\pi^2} \left( \frac{e}{m} \right) \frac{1}{4} \left( \frac{\eta_F}{\lambda} \right)^2 + \frac{1}{4} \sinh^{-1} \left( \frac{\eta_F}{\lambda} \right) \) |
| \( \chi_q = \frac{1}{4\pi^2} (1 + \lambda')^2 \left( \frac{e}{m} \right) \sinh^{-1} \left( \frac{\eta_F}{\lambda} \right) - \frac{\eta_F}{\lambda} \) | \( \Rightarrow 0 \) | \( \frac{1}{4\pi^2} (1 + \lambda')^2 \left( \frac{e}{m} \right) \ln \frac{2n_F}{\lambda} - 1 \) |

### C.7 Displacement Operator under Uniform High External Electric Fields

To complement Sec. C.3, we give the translation operator for uniform electric field case, \( \mathcal{H} = \mathcal{H} - e F \cdot \vec{x} \). We have for the displacement operator, \( T_E(q) \), obeying the relation,

\[ i\hbar T_E(q) = [T_E(q), \mathcal{H}], \]

\[ T_E(q) = \frac{i e}{\hbar} F \cdot q \ T_E(q). \]

Therefore

\[ T_E(q) = C_0(q, \tau) \exp \left( \frac{i e}{\hbar} F t \cdot q \right), \]

where \( C_0(q, \tau) \) is an operator which do not depend explicitly on time, \( t \). \( T_E(q) \), being a displacement operator in space and time lead us to write the operator

\[ C_0(q, \tau) = \exp \left( q \cdot \frac{\partial}{\partial r} + \tau \frac{\partial}{\partial t} \right). \]
$T_E(q)$ plays critical role similar to $T_M(q)$ for establishing the phase space quantum transport dynamics at very high electric fields, where we consider realistic transport problems as time-dependent many-body problems. For zero field case we are dealing with biorthogonal Wannier functions and Bloch functions because the Hamiltonian is no longer Hermetian due to the presence of energy variable, $z$, in the self-energy. This means that $[T_E(q), \mathcal{H}]$ is diagonal in the bilinear expansion if $T_E(q)$ is diagonal. The eigenfunction of the ‘lattice’ translation operator $T_E(q)$ must then be labeled by a wavenumber $\vec{k}$ which is varying in time as

$$\vec{k} = \vec{k}_0 + \frac{e\vec{F}}{\hbar}t,$$

and $\mathcal{H}$ is also diagonal in $\vec{k}$. Similarly, the energy variable, $z$, in the Hamiltonian must also vary as

$$z = z_0 + e\vec{F}\cdot\vec{q}.$$

Similar developments for translationally invariant many-body system subjected to a uniform electric field allows us to define the corresponding electric Bloch functions and electric Wannier functions, in a unifying manner for both magnetic and electric fields. This electric-field version allows us to derive the quantum transport equation of the particle density at very high electric fields. This will be discussed in another communication dealing with quantum transport in many-body systems. A more general displacement operator is recently given by Buot [87].

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