Introduction

The Kerr-Newman (KN) solution has been found to be very special in many aspects. It represents the unique family of stationary asymptotically flat black holes with non-degenerate event horizon of 4D Einstein-Maxwell (EM) theory. In the general case, it represents an isolated charged rotating black hole and it comprises the Kerr (uncharged), Reissner-Nordström (static), and Schwarzschild (uncharged and static) solutions as limits.

The generalization of these black hole solutions to $D > 4$ dimensions was pioneered by T'angherlini [1] for static black holes, and by Myers and Perry (MP) [2] for rotating vacuum black holes. The corresponding $D > 4$ charged rotating black holes of EM theory could not yet be obtained in closed form [2,3], although a subset of solutions has been found numerically in odd dimensions [4].

Based on the strong interest in higher dimensional black holes in recent years, we here take the first step towards obtaining analytical expressions for the higher dimensional generalizations of the KN solutions, by studying the charged 5D MP hole solutions perturbatively, solving up to 4th order in the perturbative parameter, the electric charge.

Intuitively, lowest order perturbation theory gives for the gyromagnetic ratio the result $g = D - 2$ [3], which seems a natural higher dimensional generalization of the gyromagnetic ratio in $D = 4$ dimensions: $g = 2$. However, numerical calculations revealed that in higher dimensions, the gyromagnetic ratio should not be constant, but deviate from $D - 2$ for finite values of the charge. Here we show, that in higher order perturbation theory the gyromagnetic ratio indeed differs from $D - 2$.

EM black holes

We consider the 5D Einstein-Maxwell action with Lagrangian

$$L = \frac{1}{16 \pi G} \sqrt{-g} (R - F_{\mu \nu} F^{\mu \nu}) ,$$

with curvature scalar $R$, 5-dimensional Newton constant $G$, and field strength tensor $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu$ denotes the gauge potential.

Generic stationary EM black hole solutions with spherical horizon topology possess two independent angular momenta associated with two orthogonal planes of rotation [2]. In the case that the two angular momenta have equal magnitude, the isometry group enlarges and the system of coupled Einstein and matter field equations reduces to a system of ordinary differential equations [3].

Perturbations

We consider perturbations around the MP solutions, when both angular momenta have equal magnitude, we employ the following parametrization for the metric

$$ds^2 = g_{tt} dt^2 + \frac{dr^2}{W} + r^2 (d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2) + N(\varepsilon_1 \sin^2 \theta d\phi_1 + \varepsilon_2 \cos^2 \theta d\phi_2)^2 + 2B(\varepsilon_1 \sin^2 \theta d\phi_1 + \varepsilon_2 \cos^2 \theta d\phi_2) dt ,$$

and the gauge potential

$$A_\mu dx^\mu = a_t dt + a_\phi(\varepsilon_1 \sin^2 \theta d\phi_1 + \varepsilon_2 \cos^2 \theta d\phi_2) .$$

The metric functions $g$, $W$, $N$, and $B$, and the functions $a_t$, $a_\phi$ for the gauge potential then depend on $r$ only. Here $\varepsilon_k = \pm 1, k = 1, 2$ denote the sense of rotation in the $k$-th orthogonal plane of rotation.

Perturbations

We consider perturbations around the MP solutions, with the electric charge as the perturbative parameter. Taking into account the symmetry with respect to charge reversal, the perturbations take the form

$$g_{tt} = -1 + \frac{2M}{r^2} + q^2 g_{tt}^{(2)} + q^4 g_{tt}^{(4)} + O(q^6) ,$$

$$W = 1 - \frac{2M}{r^2} + \frac{2J^2}{Mr^4} + q^2 W^{(2)} + q^4 W^{(4)} + O(q^6) ,$$

$$N = \frac{2J^2}{Mr^4} + q^2 N^{(2)} + q^4 N^{(4)} + O(q^6) ,$$

$$B = \frac{2J}{r^2} + q^2 B^{(2)} + q^4 B^{(4)} + O(q^6) ,$$

$$a_t = qa_t^{(1)} + q^3 a_t^{(3)} + O(q^5) ,$$

$$a_\phi = qa_\phi^{(1)} + q^3 a_\phi^{(3)} + O(q^5) ,$$

(4)
q being the perturbative parameter associated with the electric charge (see Eq. (1) below).

When Eqs. (1) are substituted in the system of ODE’s, obtained with the ansatz Eqs. (2)-(3), from the field equations, this results in a perturbative sequence of systems of ODE’s, which have to be solved order by order.

Although the systems may be solved for generic values of $M$ and $J$, the expressions for the metric and gauge potential perturbations are very involved. Since the main features are shared by the extremal case we present here most expressions only for the extremal solutions, while the general case will be presented elsewhere [6].

In order to perform the perturbative scheme, it is convenient to fix several quantities from the beginning. In the extremal case, we have fixed the angular momentum for any perturbative order, and we have imposed the extremality condition for all orders. This choice fixes all integration constants, and it has the advantage to allow us to compare the perturbative analytical solutions with the non-perturbative numerical solutions obtained previously [4].

Introducing the parameter $\nu$ for the extremal MP solutions by $M = 2\nu^2$, $J = 2\nu^3$, the perturbations up to 4th order read

$$g_{tt} = -1 + 4\nu^2 \frac{r^2 - 4\nu^2}{r^2} q^2 + \frac{1}{36\nu^6} \left[ 11r^4 - 32\nu^2 r^2 + 16\nu^4 \right] q^4 + O(q^6) ,$$

$$W = 1 - 4\nu^2 \frac{r^2 - 2\nu^2}{r^2} q^2 + \frac{1}{36\nu^6} \left[ 24r^6 - 12\nu^2 r^4 + 18\nu^4 r^2 - 64\nu^6 \right] q^4 + O(q^6) ,$$

$$N = \frac{4\nu^4}{r^2} - \frac{2(r^2 + 2\nu^2)}{3\nu^4} q^2 - \frac{8r^8 - 8\nu^2 r^6 - 7\nu^4 r^4 + 16\nu^6 r^2 - 16\nu^8}{36\nu^6}$$

$$+ \frac{(r^2 - 2\nu^2)(3\nu^6 + 8\nu^6)}{27\nu^8 r^4} \ln \left( 1 - \frac{2\nu^2}{r^2} \right) q^4 + O(q^6) ,$$

$$B = \frac{4\nu^3}{r^2} - \frac{4\nu^4}{3\nu^4} q^2 - \frac{(r^2 - 2\nu^2)(3\nu^4 - 6\nu^2 r^2 + 16\nu^4)}{54\nu^6 r^4} \ln \left( 1 - \frac{2\nu^2}{r^2} \right) q^4 + O(q^6) ,$$

$$+ a_\ell \frac{1}{r^2} q + \frac{2(r^2 - \nu^2)}{9\nu^4 r^4} \ln \left( 1 - \frac{2\nu^2}{r^2} \right) q^4 + O(q^5) ,$$

$$a_r = -\frac{\nu}{r^2} q - \left[ \frac{2r^4 + \nu^2 r^2 - 4\nu^4}{18\nu^6 r^4} \right] q^4 + \frac{r^4 - 4\nu^4}{18\nu^6 r^2} \ln \left( 1 - \frac{2\nu^2}{r^2} \right) q^3 + O(q^5) .$$

We observe that apart from the usual $1/r$ polynomial expressions, logarithms are present. When going to higher order, more complicated structures appear [6].

Physical quantities

From the analytical perturbative solutions, Eq. (5), one can extract the perturbative expressions for the physical quantities of these charged rotating black holes. Employing the same conventions as in [4], the mass $M$, the equal magnitude angular momenta $J$, and the charge $Q$ can be shown to be

$$M = \frac{3}{2} \pi \nu^2 + \frac{\pi}{8\nu^2} q^2 + \frac{\pi}{288\nu^6} q^4 + O(q^6) ,$$

$$J = \pi \nu^3 \text{ (for any order)} , \quad Q = \pi q ,$$

while the magnetic moment $\mu_{\text{mag}}$ is given by

$$\mu_{\text{mag}} = \pi \nu q - \frac{\pi}{18\nu^3} q^3 + O(q^5) .$$

These perturbative extremal black holes possess an event horizon located at $r = r_H$, where

$$r_H = \sqrt{2\nu} + \frac{\sqrt{2}}{24\nu^3} q^2 + \frac{11\sqrt{2}}{1152\nu^6} q^4 + O(q^6) ,$$

which rotates with a horizon angular velocity

$$\Omega = \frac{1}{2\nu} - \frac{1}{24\nu^2 q^2} - \frac{1}{288\nu^6} q^4 + O(q^6) .$$

Introducing further the area of the horizon $A_H$ and the electrostatic potential at the horizon $\Phi_H$

$$A_H = 8\pi^2 \nu^3 + O(q^6) ,$$

$$\Phi_H = \frac{1}{4\nu^2 q} \left( \frac{3}{128\nu^6} q^3 + O(q^5) \right) ,$$

one can easily see that the Smarr formula [7, 8]

$$M = \frac{3}{2} \kappa_{\text{sg}} A_H + \frac{3}{2} 2\Omega J + \Phi_H Q ,$$

is satisfied up to 4th order (note, that the surface gravity $\kappa_{\text{sg}}$ vanishes for extremal solutions).

Combining Eqs. (10), we define the gyromagnetic ratio $g$, 

$$g = \frac{2M \mu_{\text{mag}}}{Q J} = 3 + \frac{1}{12\nu^4} q^2 + O(q^4) .$$

In the non-extremal case (for fixed $r_H$), the gyromagnetic ratio $g$ obtained after lengthy calculation is given by [6]

$$g = 3 \left( \frac{M^2 - \eta(M^2 + 3\eta)}{M^2(M^2 + \eta)^2} \right) q^2 + O(q^4) ,$$

where $\eta$ is the electric charge (see Eq. (6) below).
where \( \eta = \sqrt{\dot{M}(\dot{M}^3 - 2J^2)} \). Note, that Eq. (13) reduces to Eq. (12) for extremal solutions.

Quality of the perturbative solutions

To obtain an assessment of the quality of the perturbative solutions we compare them with the corresponding numerical solutions [1].

Let us first address the black hole properties extracted from the metric, which are obtained perturbatively up to 4th order in the charge. The mass \( M \) is obtained with high accuracy (\(< .04\%\)) up to \( Q/M \approx 0.7 \), and it is still rather good (\(< .3\%\)) up to \( Q/M \approx 1 \), independent of the angular momentum \( J \).

This is demonstrated in Fig. 1, where we exhibit the domain of existence of these EM black holes. Here the scaled angular momentum \( |J|/M^{3/2} \) of the extremal EM black holes is shown versus the scaled charge \( Q/M \) for the exact numerical solutions and for the perturbative solutions in 2nd and 4th order.

The horizon properties of the black holes are reproduced with as good accuracy as the global mass. We demonstrate this for the horizon angular velocity in Fig. 2.

The quality of the metric functions themselves is demonstrated exemplarily in Fig. 3, where we show the metric coefficient \( g_{tt} \) as a function of the compactified radial coordinate \( 1 - r_H/r \) for an extremal black hole with \( J = 5 \) and \( Q = 3 \) and compare with the perturbative. One can clearly see that as the order of the perturbations increases, the agreement with the non-perturbative numerical solution improves.

The gyromagnetic ratio \( g \), defined in Eq. (12), is very sensitive to the accuracy of the calculations. Although the perturbative result in lowest order leads to the constant value \( g = 3 \) [3], the higher-order perturbative calculations, presented here, Eqs. (12,13), reveal a non-constant value for the gyromagnetic ratio. We exhibit the gyromagnetic ratio in Fig. 4 for extremal solutions, comparing the 2nd order results with the corresponding numerical values. (Note, that 4th order perturbations for the gyromagnetic ratio require 5th order perturbations for the magnetic moment.)

As expected, for small charges the gyromagnetic ratio agrees very well with the numerical results. But the accuracy holds only up to \( Q/M \approx 0.2 \), since \( g \) is obtained only in 2nd order, when the metric and the gauge potential are obtained in 4th order. Assuming a tentative \( \nu^{-8} \) dependence for the 4th order correction seems suited to
We have presented perturbative analytical solutions for charged rotating 5D EM black holes with spherical horizon topology, using the electric charge as the perturbative parameter. Contrary to the case of 4D KN black holes solutions, the perturbative series cannot be truncated in a consistent way to produce an exact analytical solution to the equations. Moreover, the 4th order perturbations contain logarithms and in higher order more complicated structures appear [6], in contrast to the $1/r$ polynomial expressions of 4D KN solutions.

For the quality of the approximate perturbative solutions we find, that the 4th order approximation is accurate up to $Q/M \approx 0.7$ and rather good up to $Q/M \approx 1$ (recall, that $Q/M \leq \sqrt{3}/2$). Since the 4th order approximation of the metric and gauge potential functions gives rise to a 2nd order approximation of the gyromagnetic ratio, this quantity is less accurate for larger values of $Q/M$. However, the new perturbative results clearly show, that $g \neq 3$ in general.

Although the results presented here were mainly for extremal solutions, they are easily extended to non-extremal solutions [6].

We anticipate that this perturbative approach may be applied to other theories where so far only numerical solutions are available, leading to further insight into such phenomena as non-uniqueness, instability, counter-rotation, or negative horizon mass, as encountered for instance in Einstein-Maxwell-Chern-Simons black holes [8, 9].

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