We point out that the Neyman–Pearson lemma applies to Bayes factors if we consider expected type-1 and type-2 error rates. That is, the Bayes factor is the test statistic that maximizes the expected power for a fixed expected type-1 error rate. For Bayes factors involving a simple null hypothesis, the expected type-1 error rate is just the completely frequentist type-1 error rate. Lastly we remark on connections between the Karlin–Rubin theorem and uniformly most powerful tests, and Bayes factors. This provides frequentist motivations for computing the Bayes factor and could help reconcile Bayesians and frequentists.

1. Introduction

Testing models and hypotheses against experimental data is a fundamental part of science yet the statistical approaches for doing it remain contentious. In many fields null hypothesis significance testing via $p$-values is popular. In this approach, one tests a particular null hypothesis by computing the probability of obtaining data as or more extreme than that observed, assuming that the null hypothesis were true. This probability, the $p$-value, may be used as either a measure of evidence against the null hypothesis, or as a means to controlling a type-1 error rate (Neyman 1977). This error rate is the rate at which we would reject the null hypothesis when it was true. If we reject the null hypothesis only when $p < \alpha$, the type-1 error rate would be $\alpha$.

The use of $p$-values has been contentious for decades (Edwards, Lindman, and Savage 1963), and in recent years the situation appeared to reach a critical point (Benjamin et al. 2018; Lakens et al. 2018), after it became apparent that many effects discovered through null hypothesis significance testing were spurious and could not be reproduced by other researchers (Ioannidis 2005). All kinds of remedies have been proposed, from abandoning null hypothesis significance testing altogether (McShane et al. 2019) through to Bayesian methods, including Bayes factors (Jeffreys 1939; Kass and Raftery 1995).

In the Bayesian approach, we compute directly the relative probability that two models, $H_0$ and $H_1$, predict the observed data, $D$. 
This so-called Bayes factor updates the relative plausibility of two models. The numerator and denominator in the Bayes factor are Bayesian evidence. In general, for composite models they may be written

\[ p(D|H) = \int p(D|H, \theta) p(\theta|H) \, d\theta, \quad (1.2) \]

that is, as the marginal likelihood, where \( \theta \) are the model’s \( n \) unknown parameters with prior density \( p(\theta|H) \).

In the null hypothesis significance test, we are left to decide what possible data would be more extreme than that observed. If we consider a fixed type-1 error rate, this amounts to choosing a rejection region, \( \mathcal{R} \), in the sample space of the experiment of size \( \alpha \). Although other points of view exist, the dominant approach is choosing a rejection region that maximizes the chance of rejecting the null hypothesis when it is false, i.e., maximizing the power, \( 1 - \beta \). In other words, we chose \( \mathcal{R} \) such that for fixed

\[ \alpha = P(\mathcal{R}|H_0) \quad (1.3) \]

we maximize

\[ 1 - \beta = P(\mathcal{R}|H_1). \quad (1.4) \]

Equivalently, we minimize the type-2 error rate \( \beta \), the probability of not rejecting the null hypothesis when it is false. The Neyman–Pearson lemma (Neyman, Pearson, and Pearson 1933) and the Karlin–Rubin theorem (Casella and Berger 2002) indicate the most powerful tests are often built upon likelihood ratios, and likelihood ratios are ubiquitous in significance tests.

We could, on the other hand, relinquish control of the type-1 error rate, but insist on a rejection region that minimizes a weighted sum of type-1 and type-2 error rates (Lehmann 1958; Lindley 1953; Mudge et al. 2012; Savage 1961),

\[ W \equiv w_1 \alpha + w_2 \beta, \quad (1.5) \]

with weights \( w_1 \) and \( w_2 \). For the purposes of the calculations and arguments in this work, this is equivalent to the conventional approach discussed above. To see that, suppose that the region that minimizes Equation (1.5) has size \( \alpha' \). Since this region minimizes \( W \), among all possible regions of size \( \alpha' \), it must be the one that maximizes the power, \( 1 - \beta \).

In this work, we use the Bayes factor as a test statistic in a frequentist setting, argue that it is an optimal choice and that this may help reconcile advocates of Bayesian and frequentist procedures for hypothesis testing. The idea of using the Bayes factor in this way and its potential for harmonizing Bayesian and frequentist procedures (Bayarri et al. 2016; Berger 2003) dates as far back as the 1950s (Good 1957, 1961, 1992), though Berger, Boukai, and Wang (1997) notes that it failed to gain traction among practitioners. In Section 2 we briefly review the Neyman–Pearson lemma and the Karlin–Rubin theorem, and in Sections 3 and 4, we show they may be applied to Bayes factors, such that the Bayes factor is the most powerful choice of test statistic. This was previously discussed formally in Zhang (2017). Besides drawing attention to the relevant
results and presenting them in an original, pedagogical manner, we combine the ideas of Good (1957, 1961, 1992) and results of Zhang (2017). We thus argue in Section 5 that the Neyman–Pearson lemma and Karlin–Rubin theorem in fact put Bayes factors at the center of frequentist hypothesis tests as they are an optimal choice of test statistic. This may help finally reconcile Bayesian and frequentist approaches to testing.

2. The Neyman–Pearson lemma

For simple hypotheses with no unknown parameters, the Neyman–Pearson lemma (Neyman, Pearson, and Pearson 1933) tell us that the likelihood ratio,

$$\lambda(D) = \frac{p(D|H_1)}{p(D|H_0)},$$

(2.1)

is an optimal test statistic. The rejection region of size $\alpha$ that maximizes the power must be a contour of the likelihood ratio, $R = \{D : \lambda(D) \geq \lambda_0\}$, where the value at the contour, $\lambda_0$, depends on the desired type-1 error rate and would be found from Equation (1.3). The Neyman–Pearson lemma doesn’t extend to composite hypotheses that depend on unknown parameters, as the optimal choice of test statistic would in general depend on the assumed values for the unknown parameters. In some cases, a uniformly most powerful test (UMPT) exists, which maximizes the power for any values of the unknown parameters in the alternative hypothesis. The Karlin–Rubin theorem demonstrates cases in which a UMPT exists; see Section 4.

3. The Neyman–Pearson lemma for Bayes factors

In the Bayesian formalism, for any composite model, we may find a simple model by marginalizing the $n$ unknown parameters,

$$p(D|H) = \int p(D|H, \theta)p(\theta|H)\ d^n\theta.$$  

(3.1)

This is the prior predictive for the data and when evaluated at the observed data, the Bayesian evidence for the model. We may thus use our Bayes factor in place of the likelihood ratio in the Neyman–Pearson lemma. In this case, though, the error rates are subtly re-interpreted, and to distinguish them we denote them with a bar,

$$\bar{\alpha} \equiv P(R|H_0) = \int P(R|H_0, \theta_0)\ p(\theta_0|H_0)\ d^n\theta_0$$  

(3.2)

$$1 - \bar{\beta} \equiv P(R|H_1) = \int P(R|H_1, \theta_1)\ p(\theta_1|H_1)\ d^n\theta_1$$  

(3.3)

That is, here they are the expected error rates, averaging over the possible values for the $n$ unknown model parameters $\theta_0$ in the null hypothesis and for the $m$ unknown model parameters $\theta_1$ in the alternative hypothesis. They do not in general correspond to any observable long-run error rates.

This requires, of course, choices of prior density, $p(\theta_0|H_0)$ and $p(\theta_1|H_1)$; see e.g., Kass and Wasserman (1996); Consonni et al. (2018) for discussion of rules for choosing
priors in a Bayesian setting. In a frequentist setting, the priors needn’t be interpreted in the same manner as in subjective or objective Bayesian approaches. The prior for any parameters in the alternative hypothesis, $p(\theta_1 | H_1)$, may be thought of as a weight function that indicates which choices of parameters we want our test to be most powerful for (Bayarri et al. 2016; Good and Crook 1974). Similarly, the prior for any parameters in a composite null hypothesis, $p(\theta_0 | H_0)$, weights which choices of parameters we most want to control the type-1 error rate for.

We could instead consider a fixed maximum type-1 error rate, $\hat{x}$, for any values of the unknown parameters in the null hypothesis. If we assume that it occurs when $\theta_0 = \hat{\theta}_0$, we may write

$$\hat{x} = P(\mathcal{H}|H_0, \hat{\theta}_0).$$  \hspace{1cm} (3.4)

This is in fact equivalent to a sharp prior $p(\theta_0 | H_0) = \delta(\theta_0 - \hat{\theta}_0)$, i.e., specifying through our prior that we must control the type-1 error rate for the parameters that maximize the type-1 error rate. In this case, we would use the Bayes factor,

$$B(\mathcal{H}) = \frac{\int p(\mathcal{H}, \theta_1) \ p(\theta_1 | H_1) \ d\theta_1}{p(\mathcal{H}|H_0, \theta_0)}$$  \hspace{1cm} (3.5)

in place of the likelihood ratio in the Neyman–Pearson lemma. Under the null hypothesis the expected magnitude of the preference for the alternative model from the Bayes factor in Equation (1.1) must be smaller than that from the one used when we control the maximum type-1 error rate in Equation (3.5). This result follows from Gibbs’ inequality,

$$\int \left[ \log \left( \frac{p(\mathcal{H}|H_1)}{p(\mathcal{H}|H_0)} \right) - \log \left( \frac{p(\mathcal{H}|H_1)}{p(\mathcal{H}|H_0, \hat{\theta}_0)} \right) \right] \ p(\mathcal{H}|H_0) \ d\mathcal{H} \leq 0. \hspace{1cm} (3.6)$$

So whilst controlling the maximum error rate rather than the expected error may seem conservative, the computation involves the Bayes factor in Equation (3.5) that we expect to overstate the evidence against the null hypothesis.

If the null hypothesis is simple, as is often the case, $\bar{x} = \hat{x} = \bar{z}$, and so the type-1 error rates may be interpreted in the usual, completely frequentist manner. The power, on the other hand, remains the expected power, averaged across the unknown parameters in the alternative hypothesis.

### 4. Karlin–Rubin theorem for Bayes factors

It would be desirable to extend the Neyman–Pearson lemma to composite models. Unfortunately, UMPT do not always exist as the optimal test generally depends on the values of the unknown parameters in the alternative hypothesis. There are approaches that sidestep the issue such as a minimax treatment of type-1 error rates and power. The Karlin–Rubin theorem (Casella and Berger 2002), on the other hand, extends the Neyman–Pearson lemma to a UMPT in a composite case in special circumstances.

To apply the theorem, the null and alternative hypothesis must be disjoint regions of a one-dimensional parameter space separated by a boundary at $\theta_C$, that is,
Suppose that a sufficient test statistic, $T$, exists and that the likelihood ratio

$$
\lambda(T; \theta, \theta') = \frac{p(T|\theta)}{p(T|\theta')}
$$

is a monotonic non-decreasing function of $T$ for any $\theta > \theta'$. We call these conditions the monotone likelihood ratio (MLR) conditions. Under these conditions, the UMPT is a threshold on $T$, $R = \{T : T \geq T_0\}$. The threshold $T_0$ is determined by fixing the maximum type-1 error rate,

$$
\tilde{\alpha} = P(\mathcal{R}|H_0, \theta_C)
$$

and occurs when $\theta = \theta_C$. This ensures the size of the test is no larger than $\tilde{\alpha}$ for any choice of unknown parameter in the null hypothesis.

We can make a similar statement for our Bayes factor,

$$
B(T) = \frac{p(T|H_1)}{p(T|H_0)} = \int \frac{p(T|H_1, \theta_1) p(\theta_1|H_1) d\theta_1}{p(T|H_0, \theta_0) p(\theta_0|H_0) d\theta_0}.
$$

We assume that the prior $p(\theta_0|H_0)$ for the unknown parameters in the null hypothesis was chosen. We suppose that, with that choice, the Bayes factor is always a monotonic function of $T$ for any choice of prior for the $m$ unknown parameters in the alternative model, $p(\theta_1|H_1)$. This is satisfied by the MLR conditions of the Karlin–Rubin theorem for any choice of $p(\theta_0|H_0)$; see Appendix A. The Karlin–Rubin theorem considered a fixed maximum type-1 error rate, $\tilde{\alpha}$. In our Bayesian interpretation, we could fix this or the mean error rate $\alpha$. If we fix the former, we consider the Bayes factor in Equation (3.5) in the following argument. If we choose to fix the expected type-1 error rate, on the other hand, we instead consider the Bayes factor in Equation (4.4) in the following argument.

By an application of the Neyman–Pearson lemma for Bayes factors, the most powerful test at a fixed size should be a threshold on $B$, $\mathcal{R} = \{T : B(T) \geq B_0\}$. As the Bayes factor is a monotonic function of $T$, this is equivalent to a threshold on $T$, $\mathcal{R} = \{T : T \geq T_0\}$. As $T_0$ can be found independently from the prior $p(\theta_1|H_1)$ by Equation (3.2) or (3.4), it must be the most powerful test for any choice of prior $p(\theta_1|H_1)$, including point masses at any particular values of the unknown parameters. Thus, it is the UMPT.

We thus find connections between the Bayes factor and the UMPT. By generalizing the Karlin–Rubin theorem, we find that a UMPT exists whenever the Bayes factor is a monotonic function of a sufficient statistic for any choice of prior for the unknown parameters in the alternative model. The Bayes factor corresponding the observed $T$ must, however, depend on the choices of prior.

5. Reflections

The Neyman–Pearson lemma leads to optimal test statistics for null hypothesis significance tests for simple hypotheses. Interpreting the Bayes factor as a likelihood ratio for two simple models leads to a Bayesian interpretation of the Neyman–Pearson lemma.
that goes beyond simple models. The Bayes factor maximizes the expected power for a test of a fixed expected size.

On the Bayesian side, this could provide further justification for using Bayes factors for objective Bayesians or more generally those who are concerned about the frequentist properties of Bayes factors. If we place a threshold on the Bayes factor, for whatever type-1 error rate it to which that threshold corresponds, the Bayes factor was the statistic that maximized the expected power. On the frequentist side, even if you want to carry on computing p-values, there is justification for doing so using the Bayes factor as a test statistic, especially in the case of simple null hypotheses but composite alternatives. In the case of simple null hypotheses, using the Bayes factor results in the best expected power for a fixed completely frequentist type-1 error rate. The only concession required is that to construct the test we must choose a weight function that marks where we want power and talk about expected power. The test itself would, however, remain strictly frequentist.

The Karlin–Rubin theorem extends the Neyman–Pearson lemma to particular composite models. We found that the conditions of the Karlin–Rubin theorem may be recast as the requirement that the Bayes factor is a monotonic function of a sufficient statistic for any choices of prior for unknown parameters. This leads to a slightly novel proof of a generalized Karlin–Rubin theorem and a connection between the properties of Bayes factors and the existence of uniformly most powerful tests. The Bayesian interpreted Karlin–Rubin theorem provides conditions under which a test of fixed size always maximizes the expected power, regardless which prior was chosen for the unknown parameters in the alternative hypothesis in the computation of the expected power.

These results could help synthesize frequentist and Bayesian procedures, as it shows that the Bayes factors could lie at the heart of each one, and proponents of either should be interested, in principle, in computing the Bayes factor. The outstanding difference would be that the Bayesian would consider the magnitude of the observed Bayes factor, in accordance with the likelihood principle (Berger and Wolpert 1984), whereas the frequentist would consider the probability of obtaining a Bayes factor more extreme than that observed. In practice, computing the Bayes factor and finding its distribution could be challenging, as popular asymptotic approaches such as Wilks’ theorem (Wilks 1938) need not apply (though see Severeni (2010)).

Funding

This work was supported by National Natural Science Foundation of China (Grant no. 1195041050).

ORCID

Andrew Fowlie http://orcid.org/0000-0001-5457-6329

References

Bayarri, M., D. J. Benjamin, J. O. Berger, and T. M. Sellke. 2016. Rejection odds and rejection ratios: A proposal for statistical practice in testing hypotheses. *Journal of Mathematical Psychology* 72:90–103. doi:10.1016/j.jmp.2015.12.007.
Benjamin, D. J., J. O. Berger, M. Johannesson, B. A. Nosek, E.-J. Wagenmakers, R. Berk, K. A. Bollen, B. Brems, L. Brown, C. Camerer, et al. 2018. Redefine statistical significance. *Nature Human Behaviour* 2 (1):6–10. doi:10.1038/s41562-017-0189-z.

Berger, J. O. 2003. Could Fisher, Jeffreys and Neyman have agreed on testing? *Statistical Science* 18 (1):1–32. doi:10.1214/ss/1056397485.

Berger, J. O., B. Boukai, and Y. Wang. 1997. Unified frequentist and Bayesian testing of a precise hypothesis. *Statistical Science* 12 (3):133–60. doi:10.1214/ss/103037904.

Berger, J. O., and R. L. Wolpert. 1984. *The likelihood principle*, vol. 6. Hayward, CA: IMS, Institute of Mathematical Statistics.

Casella, G., and R. L. Berger. 2002. *Statistical inference*, vol. 2. Duxbury Pacific Grove, CA: Cengage Learning.

Consonni, G., D. Fouskakis, B. Liseo, and I. Ntzoufras. 2018. Prior distributions for objective Bayesian analysis. *Bayesian Analysis* 13 (2):627–79. doi:10.1214/18-BA1103.

Edwards, W., H. Lindman, and L. J. Savage. 1963. Bayesian statistical inference for psychological research. *Psychological Review* 70 (3):193–242. doi:10.1037/h0044139.

Good, I. J. 1957. Saddle-point methods for the multinomial distribution. *The Annals of Mathematical Statistics* 28 (4):861–81. doi:10.1214/aoms/1177706790.

Good, I. J. 1992. The Bayes/non-Bayes compromise: A brief review. *Journal of the American Statistical Association* 87 (419):597–606. doi:10.1080/01621459.1992.10475256.

Good, I. J., and J. F. Crook. 1974. The Bayes/non-Bayes compromise and the multinomial distribution. *Journal of the American Statistical Association* 69 (347):711–20. doi:10.1080/01621459.1974.10480193.

Good, I. J. 1961. Weight of evidence, causality and false-alarm probabilities. In Information Theory: Fourth London Symposium, Butterworth, London, 125–136.

Ioannidis, J. P. A. 2005. Why most published research findings are false. *PLoS Medicine* 2 (8):e124. doi:10.1371/journal.pmed.0020124.

Jeffreys, H. 1939. The theory of probability. Oxford classic texts in the physical sciences. United Kingdom: Oxford University Press.

Kass, R. E., and A. E. Raftery. 1995. Bayes factors. *Journal of the American Statistical Association* 90 (430):773–95. doi:10.1080/01621459.1995.10476572.

Kass, R. E., and L. Wasserman. 1996. The selection of prior distributions by formal rules. *Journal of the American Statistical Association* 91 (435):1343–70. doi:10.1080/01621459.1996.10477003.

Lakens, D., F. G. Adolfi, C. J. Albers, F. Anvari, M. A. Apps, S. E. Argamon, T. Baguley, R. B. Becker, S. D. Benning, D. E. Bradford, et al. 2018. Justify your alpha. *Nature Human Behaviour* 2 (3):168–71. doi:10.1038/s41562-018-0311-x.

Lehmann, E. L. 1958. Significance level and power. *The Annals of Mathematical Statistics* 29 (4):1167–76. doi:10.1214/aoms/1177706448.

Lindley, D. V. 1953. Statistical inference. *Journal of the Royal Statistical Society: Series B (Methodological)* 15 (1):30–76. doi:10.1111/j.2517-6161.1953.tb00123.x.

McShane, B. B., D. Gal, A. Gelman, C. Robert, and J. L. Tackett. 2019. Abandon statistical significance. *The American Statistician* 73 (sup1):235–45. doi:10.1080/00031305.2018.1527253.

Mudge, J. F., L. F. Baker, C. B. Edge, and J. E. Houlanhan. 2012. Setting an optimal a that minimizes errors in null hypothesis significance tests. *PLoS One* 7 (2):e32734–7. doi:10.1371/journal.pone.0032734.

Neyman, J. 1977. Frequentist probability and frequentist statistics. *Synthese* 36 (1):97–131. doi:10.1007/BF00485695.

Neyman, J., E. S. Pearson, and K. Pearson. 1933. IX. On the problem of the most efficient tests of statistical hypotheses. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character* 231 (694–706):289–337.

Savage, L. J. 1961. The foundations of statistics reconsidered. Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Univ. California Press, Berkeley, Calif, vol. I, 575–586.

Severeni, T. A. 2010. Likelihood ratio statistics based on an integrated likelihood. *Biometrika* 97 (2):481–96.
Appendix A:

Karlin–Rubin conditions for Bayes factors

Let us check the implications of the MLR conditions in the Karlin–Rubin theorem on the Bayes factor. For the hypotheses under consideration in the Karlin–Rubin theorem in Equation (4.1), the Bayes factor could be written

\[ B(T) = \frac{\int_{\theta > \theta_C} p(T|\theta) \, p(\theta|H_1) \, d\theta}{\int_{\theta \leq \theta_C} p(T|\theta) \, p(\theta|H_0) \, d\theta} \quad (A.1) \]

for some suitably normalized choices of prior densities \( p(\theta|H_1) \) and \( p(\theta|H_0) \). Starting from Equation (4.2), for \( T' > T \) and \( \theta > \theta' \), we have

\[ \frac{p(T'\mid \theta)}{p(T'\mid \theta')} \geq \frac{p(T\mid \theta)}{p(T\mid \theta')} \geq \frac{p(\theta|H_1)}{p(\theta'|H_0)} \quad (A.2) \]

\[ \frac{p(T'\mid \theta)}{p(T'\mid \theta')} \geq \frac{p(T\mid \theta)}{p(T\mid \theta')} \geq \frac{p(\theta|H_1)}{p(\theta'|H_0)} \quad (A.3) \]

\[ [p(T'\mid \theta) \, p(\theta|H_1)] \, [p(T'\mid \theta') \, p(\theta'|H_0)] \geq [p(T\mid \theta) \, p(\theta|H_1)] \, [p(T\mid \theta') \, p(\theta'|H_0)] \quad (A.4) \]

where we multiplied each side by the ratio of prior densities that appear in the Bayes factor and rearranged terms. Integrating each side with respect to \( \theta \) and \( \theta' \) only over the regions guaranteeing \( \theta > \theta' \), we find

\[ \int_{\theta > \theta_C} p(T'\mid \theta) \, p(\theta|H_1) \, d\theta \int_{\theta \leq \theta_C} p(T\mid \theta) \, p(\theta|H_0) \, d\theta \geq \int_{\theta > \theta_C} p(T\mid \theta) \, p(\theta|H_1) \, d\theta \int_{\theta \leq \theta_C} p(T'\mid \theta') \, p(\theta'|H_0) \, d\theta. \quad (A.5) \]

Finally, we rearrange terms to find

\[ \frac{\int_{\theta > \theta_C} p(T'\mid \theta) \, p(\theta|H_1) \, d\theta}{\int_{\theta \leq \theta_C} p(T'\mid \theta) \, p(\theta|H_0) \, d\theta} \geq \frac{\int_{\theta > \theta_C} p(T\mid \theta) \, p(\theta|H_1) \, d\theta}{\int_{\theta \leq \theta_C} p(T\mid \theta) \, p(\theta|H_0) \, d\theta} \quad (A.6) \]

such that \( B(T') \geq B(T) \) for \( T' \geq T \). We did not make use of any properties of any particular choice of prior densities. In other words, the ordinary MLR conditions of the Karlin–Rubin theorem mean that the Bayes factor is a monotonic function of \( T \) for any choices of prior densities \( p(\theta|H_1) \) and \( p(\theta|H_0) \).