Relaxation and dephasing in a many-fermion generalization of the Caldeira-Leggett model

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We analyze a model system of fermions in a harmonic oscillator potential under the influence of a fluctuating force generated by a bath of harmonic oscillators. This represents an extension of the well-known Caldeira-Leggett model to the case of many fermions. Using the method of bosonization, we calculate Green’s functions and discuss relaxation and dephasing of a single extra particle added above the Fermi sea. We also extend our analysis to a more generic coupling between system and bath, that results in complete thermalization of the system.

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The interaction between a system and its environment is an important fundamental issue in quantum mechanics. It is at the basis of relaxation phenomena (like spontaneous emission), is essential for the measurement process, and leads to the destruction of interference effects (“decoherence” or “dephasing”). In the theory of quantum-dissipative systems, there are only few exactly solvable models, most notably the Caldeira-Leggett model of a single particle coupled to a bath of harmonic oscillators. This is the simplest possible model in which friction and fluctuations appear. If the particle is free, then this model can be used to study the quantum analogue of Brownian motion. The model remains exactly solvable if the particle moves in a parabolic potential (the damped quantum harmonic oscillator).

However, in many solid state applications, we actually consider dephasing of an electron inside a Fermi sea. It is difficult to apply the insights gained from single-particle calculations in such cases, since the Pauli principle may play an important role in relaxation processes. There have been comparatively few detailed studies of quantum-dissipative many-particle systems. Among them we mention a general discussion of dephasing in a Luttinger liquid, a study of fermions coupled to independent baths, and a formally exact extension of the Feynman-Vernon influence functional to fermions. In other cases, the Pauli principle has been introduced “by hand”, by keeping only the thermal part of the bath spectrum.

In this Letter, we study a natural extension of the Caldeira-Leggett model to a many-fermion case. The model consists of a sea of fermions populating the lower energy levels of a harmonic oscillator. We are interested in the effects that arise when a bath is coupled to this system via a fluctuating spatially homogeneous force. In contrast to an analogous system of free fermions, the bath leads to transitions between levels, with strong effects of the Pauli principle. This model might also prove relevant to the discussion of cold fermionic atoms in a 1d harmonic trap under the influence of fluctuations of the trapping potential.

We rewrite and solve the Hamiltonian using the method of bosonization, for the case of large particle numbers. This enables us to evaluate Green’s functions and to describe relaxation and dephasing of an extra particle added above the Fermi sea. Finally, we will extend our model to a more generic type of coupling.

The model - We consider a system of \(N\) identical fermions (non-interacting and spinless) confined in a one-dimensional harmonic oscillator potential (see Fig. 1). A fluctuating force \(\hat{F}\) leads to a coupling of the form \(\hat{F} \sum_j \hat{x}_j\), yielding, in second quantization:

\[
\hat{H} = \omega_0 \sum_{n=0}^{\infty} n \hat{c}^\dagger_n \hat{c}_n + \hat{H}_B + \frac{\hat{F}}{\sqrt{2m\omega_0}} \sum_{n=0}^{\infty} \sqrt{n+1} (\hat{c}^\dagger_{n+1} \hat{c}_n + h.c.) \tag{1}
\]

The operators \(\hat{c}_n\) annihilate fermions in the oscillator levels \(n\). The bath Hamiltonian \(\hat{H}_B\) describes an infinite number of harmonic oscillators, and the force \(\hat{F}\) is a sum...
over the bath normal coordinates $\hat{Q}_j$. It is characterized
fully by its power spectrum $\langle \hat{F} \hat{F} \rangle_\omega$. The special case
of an Ohmic bath, used for Quantum Brownian motion
[2], has $\langle \hat{F} \hat{F} \rangle_\omega = (\eta \omega / \pi) \theta (\omega - \omega_c) \theta (\omega)$ at $T = 0$, where
$\eta = m \gamma$ is the friction coefficient, $\gamma$ the damping rate,
and $\omega_c$ the cutoff. As the form of the coupling [1] is
not translationally invariant, the frequency $\omega_0$ contains a
stabilizing counterterm [1][2][11].

Effectively, the force acts only on the center-of-mass
(c.m.) motion of the particles which, for the harmonic
oscillator, is independent of the relative motion. Thus,
in principle our model reduces to a single damped har-
monic oscillator, analyzed in Ref. [2]. However, we are
interested in single-fermion properties, and not in the col-
clective c.m. motion itself. Although the problem can be
solved exactly via normal modes of the complete set of
oscillators and antisymmetrizing with respect to fermion
coordinates, this procedure gets extremely cumbersome.
Instead, we employ an approximation for large fermion
numbers $N$, which also allows an extension to a more
generic coupling between system and bath.

Bosonization - For sufficiently large $N$ the lowest lev-
els are always occupied (at the given interaction strength
and temperatures), i.e. excitations are confined to the
region near the Fermi level. Then we may employ the
method of bosonization, rewriting the energy of fermions
as a sum over boson modes [9]. This is possible since
the energies of the oscillator levels increase linearly with
quantum number, just as the kinetic energy in the Lut-
tinger model of interacting electrons in one dimension
(for recent reviews see [11]).

We introduce (approximate) boson operators $\hat{b}_q =
\frac{1}{\sqrt{q}} \sum_{n=0}^{\infty} \hat{c}_{n+q} \hat{c}_n$ ($q \geq 1$), which destroy particle-hole ex-
citations. Then, the Hamiltonian given above becomes
approximately

$$\hat{H} \approx \omega_0 \sum_{q=1}^{\infty} q \hat{b}_q \hat{b}_q + \frac{N}{2 \Omega \omega_0} \hat{F} (\hat{b}_1 + \hat{b}_1^\dagger) + \hat{H}_B + E_N , \quad (2)$$

which will form the basis of our analysis. Here $E_N =
\omega_0 N (N - 1)/2$ is the total energy of the $N$-fermion non-
interacting ground state. Eq. [2] reveals that $\hat{F}$ only
couples to the lowest boson mode ($q = 1$), corresponding
to the c.m. motion. The damped motion of the c.m. os-
cillator can be solved exactly, along the lines of Ref. [2]
or [12], providing us with correlators such as $\langle \hat{b}_1 (t) \hat{b}_1^\dagger (0) \rangle$.

Derivation of Green’s functions - In order to find the
Green’s functions, we have to go back from the boson op-
erators $\hat{b}_q$ to the fermion operators $\hat{c}_n$, by employing well-
known finite-size bosonization identities. In our case, we
first have to introduce auxiliary fermion operators $\psi (x)$:

$$\hat{\psi} (x) = \frac{1}{\sqrt{2\pi}} \sum_n e^{inx} \hat{c}_n , \quad \hat{c}_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} \hat{\psi} (x) \, dx \quad (3)$$

The coordinate $x$ does not refer to the motion in the
oscillator. Rather, we have effectively mapped our prob-
lem to a chiral Luttinger liquid on a ring with a coupling
$\alpha \hat{F} \cos (x)$ ($x \in [0, 2\pi]$), see Fig. 11 (right). Thus, the
following results also describe relaxation of momentum
states in that model. Although a generic discussion of
dissipative Luttinger liquids has been provided in [2], the
particular questions we are going to study have not been
analyzed before.

The operators $\hat{\psi} (x)$ may be expressed as [11]:

$$\hat{\psi} (x) = \hat{K} \hat{\lambda} (x) e^{i\hat{\phi}^\dagger (x)} e^{i\hat{\phi} (x)} = \hat{K} \lambda e^{i\hat{\phi} r} , \quad (4)$$

with

$$\hat{\phi} = \hat{\phi} + \hat{\phi}^\dagger , \quad \phi (x) = - i \sum_{q=1}^{\infty} \frac{1}{\sqrt{q}} e^{iqx} \hat{b}_q . \quad (5)$$

The “Klein factor” $\hat{K}$ annihilates a particle, with
$[\hat{K}, \hat{b}_q ^\dagger] = 0$ and $\hat{K} (t) = \hat{K} \exp (-i \Omega_0 (N - 1) t)$. We have
$$\hat{\lambda} (x) = \exp (i (N - 1) x)/\sqrt{2\pi} \quad \text{and} \quad r = \exp (-[\hat{\phi}^\dagger, \hat{\phi}]/2).$$
(The exponent in $r$ diverges, so a formal cutoff at high $q$
should be introduced, which will drop out in the end
result.)

Using Eq. (3), we find for the hole-propagator:

$$\langle \hat{c}_n (t) \hat{c}_n \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{i(n' x' - nx)} \langle \hat{\psi}^\dagger (x', t) \hat{\psi} (x, 0) \rangle \, dx \, dx'. \quad (6)$$

The $\psi$-Green’s function is given directly in terms of the
$\phi$-correlator, using Eq. (4):

$$\langle \hat{\psi} (x', t) \hat{\psi} (x, 0) \rangle \equiv \frac{r^2}{2\pi} e^{in_F (x' - x) + \omega_0 t} \langle \hat{\phi} (x', t) \hat{\phi} (x, 0) \rangle \quad (7)$$

(with $n_F = N - 1$). The expectation value on the
right-hand side may be evaluated exactly [11], since the
system-bath coupling is bilinear. This yields $\exp (E)$
with:

$$E = - \frac{1}{2} \left( \langle \hat{\phi} (x', t)^2 \rangle + \langle \hat{\phi} (x, 0)^2 \rangle \right) + \langle \hat{\phi} (x', t) \hat{\phi} (x, 0) \rangle \quad (8)$$

The correlator of $\hat{\phi}$ is a polynomial in $X \equiv \exp (ix)$ and
$X' \equiv \exp (ix')$ (see Eq. 15). Now the double Fourier
integral in Eq. (15) may be evaluated by expanding $\exp (E)$
as a series in $X$ and $X'$. We find that $\langle \hat{c}_n (t) \hat{c}_n \rangle$ is the
coefficient of $X^n X'^m$ in the expansion of
\[ e^{\delta E(X,X',t)} \sum_{k \leq n_F} e^{i\omega k t} (X/X')^k, \tag{9} \]

where the noninteracting exponent has been subtracted in \( \delta E = E - E(0) \), which thus contains only the correlator of the damped c.m. mode \( q = 1 \). Detailed plots of the Green’s function will be published elsewhere \[10\]. Here we provide the result in the weak-coupling approximation, where we neglect the bath-induced smearing of the equilibrium Fermi level and use an exponential decay for the c.m. motion. Both assumptions are summarized in \( n \rho \sum_{k \leq n_F} \) above the Fermi sea, creating the many-particle state over placing an electron in a superposition of levels.

\[ \text{relaxation of level populations and dephasing.} \]

\[ \text{Con-} \]

\[ \text{rized in} \]

\[ \text{\Gamma is the decay rate, with} \]

\[ \Gamma = N \gamma \]

\[ \text{for the c.m. motion.} \]

\[ \text{Both assumptions are summa-} \]

\[ \text{tion, where we neglect the bath-induced smearing of} \]

\[ \text{Here we provide the result in the weak-coupling approx-} \]

\[ \text{imation, for which the following results have} \]

\[ \text{been evaluated. The reduced single-particle density ma-} \]

\[ \text{trix evolves according to:} \]

\[ \text{\Gamma t/2} \]

\[ \text{(Eq. (3)), leading to a four-fold Fourier integral,} \]

\[ \text{analogous to Eq. (4). Using Eq. (4), this may be eval-} \]

\[ \text{uated by a series expansion in four exponentials} \exp(\pm i\omega t), \]

\[ \text{exp}(\pm i\nu t), \text{similar to Eq. (5).} \]

\[ \text{We omit the lengthy general formula} \[10\], but discuss a limiting case below.} \]

\[ \langle \hat{c}_n (t) \hat{c}_n \rangle \approx e^{i\omega n t} \sum_{m=0}^{n_F-n} \frac{\nu(t)^m}{m!}, \tag{10} \]

where \( \nu(t) = \exp(-i(\omega_0 - \omega_n)t - \Gamma t/2) - 1 \).

We find that the hole (particle) propagator does not decay to zero in the limit \( t \to \infty \), for any \( n < n_F - 1 \) \((n > n_F + 2)\), since \( \nu(t) \to -1 \). This is in contrast to the naive single-particle picture of complete decay for any level \( n \neq n_F, n_F + 1 \) not directly at the Fermi level (i.e. the result suggested by the leading order self-energy). Physically, adding a hole (particle) creates an excited many-particle state which also contains contributions where the c.m. mode is not excited, and these will not decay, because only the c.m. mode is damped. A more generic coupling, leading to ergodicity, will be discussed at the end of this Letter.

**Time-evolution of density matrix -** We now turn to the two-particle Green’s function in order to learn about relaxation of level populations and dephasing. Consider placing an electron in a superposition of levels above the Fermi sea, creating the many-particle state \( \sum_{n_0} \Psi_{n_0} \hat{c}_{n_0} |FS\rangle \) at time 0. We assume the levels \( n_0 \) to be unoccupied. This will hold for \( n_0 > n_F \) in the weak-coupling limit, for which the following results have been evaluated. The reduced single-particle density matrix evolves according to:

\[ \rho_{nn'}(t) = \sum_{n_0, n_0'} \Psi_{n_0} \Psi_{n_0'}^* \langle \hat{c}_{n_0} \hat{c}_{n_0'}^*(t) \hat{c}_{n_0'}(t) \hat{c}_{n_0} \rangle. \tag{11} \]

We may rewrite the Green’s function in Eq. (11) in terms of \( \psi(x) \) (Eq. (3)), leading to a four-fold Fourier integral, analogous to Eq. (4). Using Eq. (4), this may be evaluated by a series expansion in four exponentials \( \exp(\pm i\omega t), \exp(\pm i\nu t), \text{similar to Eq. (5).} \)

\[ \text{We omit the lengthy general formula} \[10\], but discuss a limiting case below.}
Golden Rule behaviour is recovered (both for relaxation, \(n_0 = n'_0\), and dephasing, \(n_0 \neq n'_0\)). In the long-time limit we get a stationary distribution, \(\rho_{\text{decay}}(m,t) \rightarrow (2^{m}/m!)e^{-\frac{\Gamma t}{2}}\).

“Heating” around the Fermi level is encoded in (see Fig. 3)

\[
\rho_{\text{heat}}(n,n',t) = \nu(t)^{n-n'} \sum_{m_1,m_2} \frac{|\nu(t)|^{2(m_1+m_2)}}{m_1!m_2!^{\frac{1}{2}}} \left((-1)^{m_2+n_1}e^{-\Gamma t}\right),
\]

where the triple sum runs over \(m_1 = \max(0,n'+1)\ldots \infty, m_2 = 0 \ldots \infty,\) \(m_1 = \max(0,m_2+n+1)\ldots n-n'+m_1+m_2,\) and we have \(m_2 = m_1 + m_2 - m_1 + n - n'.\) In the short-time limit, \(\rho_{\text{heat}}(n,n,t)\) approximates to 1 for \(n < -1,\) to \(1 - |\nu(t)|^2\) for \(n = -1,\) to \(|\nu(t)|^2\) for \(n = 0,\) and 0 for \(n > 0\) (to \(\mathcal{O}(|\nu|^2)\)), describing the unperturbed Fermi sea and the onset of heating. Comparing to the full results (Fig. 2), we find that the limiting case (12) is a very good approximation even for small excitation energies.

Generic coupling - Up to now, we have considered a coupling where the particle coordinates enter linearly, and consequently only the c.m. mode is damped.

We now extend our analysis to a more generic situation, replacing the interaction in Eq. (2) by:

\[
\sqrt{\frac{N}{2m\omega_0}}\sum_{q=1}^{\infty} f_q q \sqrt{\tilde{b}_q + \tilde{b}_q^\dagger}.
\]

Now the bath induces transitions between levels \(n+q\) and \(n,\) with an arbitrary (real-valued) amplitude \(\propto f_q\) (which, however, must not depend on \(n\)). For \(f_1 = 1, f_q = 0(q > 1)\) we recover the original model. For \(f_q \neq 0\) all the boson modes are damped and couple to each other via the bath. Formally, the correlators \(\langle \tilde{b}_q(t)\tilde{b}_q^\dagger \rangle\) can be written in terms of the resolvent of the classical problem of boson oscillators coupled to bath oscillators (10), compare (12).

The correlator \(\langle \hat{\phi}(x',t)\hat{\phi}(x,0) \rangle\) now contains contributions for all pairs \(q,q'.\) The evaluation of the Green’s functions proceeds as before. Unfortunately, one fails to deal with far more terms. However, interesting behaviour is already found in the weak-coupling limit, which here implies neglecting the effective coupling between boson modes that has been induced by the bath, and describing the correlator of each boson mode separately as a damped oscillation.

For the case of constant \(f_q = 1\) (up to some cutoff) and an Ohmic bath spectrum, the boson correlator decay rate \(\propto q^2\Gamma/2\). This fits the expectation about Pauli blocking: The decay of a particle from state \(n_F + \delta n + 1\) is due to transitions by 1 to \(\delta n\) levels, and adding up their rates (which grow linearly) leads to a total rate \(\propto \delta n^2\), consistent with the decay rate of the highest boson mode \(q = \delta n\) that is excited by adding this particle. The actual evolution of the Green’s function is a superposition of decays, with rates up to this value.

An example for the resulting time-evolution is shown in Fig. 4. Starting from a state where a single extra particle has been added in level \(n_0\) above the Fermi sea, one can observe the evolution of the populations \(\rho_{nn}(t)\) (Ohmic bath, \(T = 0\)). At intermediate times, heating around the Fermi level takes place (barely visible, in contrast to Fig. 2). In contrast to the previous case, the relaxation towards the \(N+1\)-particle ground state is complete, the system is ergodic.

Conclusions - We have analyzed a many-fermion generalization of the single particle in a damped harmonic oscillator, illustrating relaxation and dephasing in a dissipative many-particle system. Using the method of bosonization (in the limit of large particle number), we have derived exact expressions for the Green’s functions and discussed them in limiting cases. We have analyzed the decay of an excited state created by adding one par-
ticle above the Fermi level, where one can observe the “heating” around the Fermi level (due to the effective interaction between particles), as well as the incomplete decay of the excited particle. Finally, we have extended our analysis to a more generic type of coupling between system and bath, where the system becomes fully ergodic.

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[1] U. Weiss: *Quantum Dissipative Systems*, World Scientific, Singapore (2000).
[2] A. O. Caldeira and A. J. Leggett, Physica 121A, 587 (1983); Phys. Rev. A 31, 1059 (1985).
[3] A. H. C. Neto, C. D. Chamon, C. Nayak, Phys. Rev. Lett. 79, 4629 (1997).
[4] J. M. Wheatley, Phys. Rev. Lett. 67, 1181 (1991). P. C. Howell and A. J. Schofield, cond-mat/0103191
[5] D. S. Golubev and A. D. Zaikin, Phys. Rev. B 59, 9195 (1999).
[6] B. L. Altshuler, A. G. Aronov, and D. E. Khmelnitsky, J. Phys. C Solid State 15, 7367 (1982); S. Chakravarty and A. Schmid, Phys. Rep. 140, 195 (1986). A. Stern, Y. Aharonov, and Y. Imry, Phys. Rev. A 41, 3436 (1990); D. Cohen and Y. Imry, Phys. Rev. B 59, 11143 (1999).
[7] F. Marquardt and C. Bruder, Phys. Rev. B 65, 125315 (2002).
[8] F. Schreck et al., Phys. Rev. Lett. 87, 080403 (2001); S. R. Granade et al., ibid. 88, 120405 (2002); T. Loftus et al., ibid. 88, 173201 (2002); A. Recati et al., ibid. 90, 020401 (2003).
[9] W. Wonneberger, Phys. Rev. A 63, 063607 (2001); G. Xianlong and W. Wonneberger, Phys. Rev. A 65, 033610 (2002); G. Xianlong, F. Gleisberg, F. Lochmann, and W. Wonneberger, Phys. Rev. A 67, 023610 (2003); G. Xianlong and W. Wonneberger, J. Phys. B 37, 2363 (2004).
[10] F. Marquardt and D. S. Golubev, cond-mat/0409401
[11] J. v. Delft and H. Schoeller, Annalen der Physik, Vol. 4, 225 (1998). H. Grabert, in "Exotic States in Quantum Nanostructures" ed. by S. Sarkar, Kluwer (2001).
[12] V. Hakim and V. Ambegaokar, Phys. Rev. A 32, 423 (1985).