THE HIGHEST WEIGHT STRUCTURE FOR STRICT POLYNOMIAL FUNCTORS

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Dedicated to the memory of Professor Sandy Green.

Abstract. We explain the highest weight structure for categories of strict polynomial functors, using the theory of Schur functors. A consequence is the well-known fact that Schur algebras are quasi-hereditary.

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1. Introduction

Polynomial representations of general linear groups are equivalent to modules over Schur algebras [10], and it is well-known that these algebras are quasi-hereditary [4, 6]. These notes present an alternative approach to this subject. We explain the highest weight structure [5] for categories of strict polynomial functors [8], working over an arbitrary commutative ring and using some of the principal results from the theory of Schur functors [11].

The essential ingredients of the highest weight structure are:
- The Schur functors are precisely the costandard objects (Corollary 4.8).
- The Cauchy decomposition provides a filtration of any injective object whose associated graded object is a direct sum of Schur functors (Corollary 5.2).

The material in these notes is elementary, based to a large extent on classical facts from multilinear algebra. In particular, properties of divided powers are used, for which we refer to [2, IV.5]. The language of strict polynomial functors is employed because of its flexibility. Evaluating strict polynomial functors at a free module of finite rank makes it easy to transfer this work to the representation theory of Schur algebras; for an explicit discussion we recommend Hashimoto’s notes [12].
2. Divided powers and strict polynomial functors

Strict polynomial functors were introduced by Friedlander and Suslin \[8\]. In this section we recall the definition and some basic properties, using an equivalent description in terms of representations of divided powers. For details and further references, see \[14, 15, 16, 18\].

**Finitely generated modules.** Throughout we fix a commutative ring \(k\). Let \(\mathcal{P}_k\) denote the category of finitely generated projective \(k\)-modules. Given \(V, W\) in \(\mathcal{P}_k\), we write \(V \otimes W\) for their tensor product over \(k\) and \(\text{Hom}(V, W)\) for the group of \(k\)-linear maps \(V \rightarrow W\). This provides two bifunctors

\[
\begin{array}{ccc}
- \otimes - & : & \mathcal{P}_k \times \mathcal{P}_k \rightarrow \mathcal{P}_k \\
\text{Hom}(-, -) & : & (\mathcal{P}_k)^{op} \times \mathcal{P}_k \rightarrow \mathcal{P}_k
\end{array}
\]

and the functor sending \(V\) to \(V^* = \text{Hom}(V, k)\) yields a duality

\[(\mathcal{P}_k)^{op} \sim \mathcal{P}_k.
\]

**Divided and symmetric powers.** Fix a positive integer \(d\) and denote by \(\mathcal{S}_d\) the symmetric group permuting \(d\) elements. For each \(V \in \mathcal{P}_k\), the group \(\mathcal{S}_d\) acts on \(V \otimes^d\) by permuting the factors of the tensor product. Denote by \(\Gamma^d V\) the submodule \((V \otimes^d)^{\mathcal{S}_d}\) of \(V \otimes^d\) consisting of the elements which are invariant under the action of \(\mathcal{S}_d\); it is called the module of divided powers (more correctly: symmetric tensors) of degree \(d\). The maximal quotient of \(V \otimes^d\) on which \(\mathcal{S}_d\) acts trivially is denoted by \(S^d V\) and is called the module of symmetric powers of degree \(d\). Set \(\Gamma^0 V = k\) and \(S^0 V = k\).

From the definition, it follows that \((\Gamma^d V)^* \cong S^d(V^*)\). Note that \(S^d V\) is a free \(k\)-module provided that \(V\) is free. Thus \(\Gamma^d V\) and \(S^d V\) belong to \(\mathcal{P}_k\) for all \(V \in \mathcal{P}_k\), and we obtain functors \(\Gamma^d, S^d : \mathcal{P}_k \rightarrow \mathcal{P}_k\).

**The category of divided powers.** We consider the category \(\Gamma^d \mathcal{P}_k\) which is defined as follows. The objects are the finitely generated projective \(k\)-modules and for two objects \(V, W\) set

\[\text{Hom}_{\Gamma^d \mathcal{P}_k}(V, W) = \Gamma^d \text{Hom}(V, W).
\]

This identifies with \(\text{Hom}(V \otimes^d, W \otimes^d)^{\mathcal{S}_d}\), where \(\mathcal{S}_d\) acts on \(\text{Hom}(V \otimes^d, W \otimes^d)\) via \((\sigma f)(v) = \sigma^{-1} f(\sigma v)\) for \(f : V \otimes^d \rightarrow W \otimes^d\) and \(\sigma \in \mathcal{S}_d\). Using this identification one defines the composition of morphisms in \(\Gamma^d \mathcal{P}_k\). The duality for \(\mathcal{P}_k\) induces a duality

\[(\Gamma^d \mathcal{P}_k)^{op} \sim \Gamma^d \mathcal{P}_k.
\]

**Example 2.1.** Let \(n\) be a positive integer and set \(V = k^n\). Then \(\text{End}_{\Gamma^d \mathcal{P}_k}(V)\) is isomorphic to the Schur algebra \(S_k(n, d)\) as defined by Green \[10\], Theorem 2.6c.

Following \[15\], this example suggests for \(\Gamma^d \mathcal{P}_k\) the term Schur category.

**Strict polynomial functors.** Let \(M_k\) denote the category of \(k\)-modules. We study the category of \(k\)-linear representations of \(\Gamma^d \mathcal{P}_k\). This is by definition the category of \(k\)-linear functors \(\Gamma^d \mathcal{P}_k \rightarrow M_k\) and we write by slight abuse of notation

\[\text{Rep} \Gamma^d_k = \text{Fun}_k(\Gamma^d \mathcal{P}_k, M_k).
\]

For objects \(X, Y\) in \(\text{Rep} \Gamma^d_k\) the set of morphisms is denoted by \(\text{Hom}_{\text{Rep} \Gamma^d_k}(X, Y)\).

The representations of \(\Gamma^d \mathcal{P}_k\) form an abelian category, where (co)kernels and (co)products are computed pointwise over \(k\). 
The Yoneda embedding. The Yoneda embedding
\[(\Gamma^d P_k)^{op} \to \text{Rep} \Gamma_k^d, \quad V \mapsto \text{Hom}_{\Gamma^d P_k}(V, -)\]
identifies \(\Gamma^d P_k\) with the full subcategory consisting of the representable functors. For \(V \in \Gamma^d P_k\) we write
\[\Gamma^d V = \text{Hom}_{\Gamma^d P_k}(V, -).\]
For \(X \in \text{Rep} \Gamma_k^d\) there is the Yoneda isomorphism
\[\text{Hom}_{\Gamma^d}(\Gamma^d V, X) \cong X(V)\]
and it follows that \(\Gamma^d V\) is a projective object in \(\text{Rep} \Gamma_k^d\).

Duality. Given a representation \(X \in \text{Rep} \Gamma_k^d\), its dual \(X^\circ\) is defined by
\[X^\circ(V) = X(V^*)^\ast.\]
We have for all \(X, Y \in \text{Rep} \Gamma_k^d\) a natural isomorphism
\[\text{Hom}_{\Gamma^d}(X, Y^\circ) \cong \text{Hom}_{\Gamma^d}(Y, X^\circ).\]
The evaluation morphism \(X \to X^{\circ\circ}\) is an isomorphism when \(X\) takes values in \(P_k\).

Example 2.2. The divided power functor \(\Gamma^d\) and the symmetric power functor \(S^d\) belong to \(\text{Rep} \Gamma_k^d\). In fact
\[\Gamma^d = \text{Hom}_{\Gamma^d P_k}(k, -) \quad \text{and} \quad S^d \cong (\Gamma^d)^\circ.\]

The algebra of divided powers. Given \(V \in P_k\), we set \(\Gamma V = \bigoplus_{d \geq 0} \Gamma^d V\). For non-negative integers \(d, e\) the inclusion \(\mathcal{S}_d \times \mathcal{S}_e \subseteq \mathcal{S}_{d+e}\) induces natural maps
\[(\Gamma^d + e V) \to \Gamma^d V \otimes \Gamma^e V \quad \text{and} \quad \Gamma^d V \otimes \Gamma^e V \to \Gamma^{d+e} V.\]
The first map is given by
\[(V \otimes \delta_{d+e})_{\mathcal{S}_{d+e}} \subseteq (V \otimes \delta_{d+e})_{\mathcal{S}_d \times \mathcal{S}_e} \cong (V \otimes \delta_{d})_{\mathcal{S}_d} \otimes (V \otimes \delta_{e})_{\mathcal{S}_e}.\]
The second map sends \(x \otimes y \in \Gamma^d V \otimes \Gamma^e V\) to
\[xy = \sum_{g \in \mathcal{S}_{d+e}/\mathcal{S}_d \times \mathcal{S}_e} g(x \otimes y)\]
where \(g(x \otimes y) = \sigma(x \otimes y)\) for a coset \(g = \sigma(\mathcal{S}_d \times \mathcal{S}_e)\). This multiplication gives \(\Gamma V\) the structure of a commutative \(k\)-algebra.

Now suppose that \(V\) is a free \(k\)-module with basis \(\{v_1, \ldots, v_n\}\). Let \(\Lambda(n, d)\) denote the set of sequences \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) of non-negative integers such that \(\sum \lambda_i = d\). Then the elements
\[v_\lambda = \prod_{i=1}^n v_i^{\lambda_i} \quad \text{for} \quad \lambda \in \Lambda(n, d)\]
form a \(k\)-basis of \(\Gamma^d V\).

Let \(\{v_1^\ast, \ldots, v_n^\ast\}\) denote the dual basis of \(V^\ast\). We identify the symmetric algebra \(S(V^\ast) = \bigoplus_{d \geq 0} S^d(V^\ast)\) with the polynomial algebra \(k[v_1^\ast, \ldots, v_n^\ast]\). Let \(\{v_\lambda^\ast\}_{\lambda \in \Lambda(n, d)}\) be the basis of \((\Gamma^d V)^\ast\) dual to \(\{v_\lambda\}_{\lambda \in \Lambda(n, d)}\). Then the canonical isomorphism \((\Gamma^d V)^\ast \cong S^d(V^\ast)\) maps each \(v_\lambda^\ast\) to \(\prod_{i=1}^n (v_i^\ast)^{\lambda_i}\).
Tensor products. For non-negative integers \(d, e\) there is a tensor product
\[- \otimes - : \text{Rep} \Gamma_d^k \times \text{Rep} \Gamma_e^k \rightarrow \text{Rep} \Gamma_{d+e}^k.\]
Let \(X \in \text{Rep} \Gamma_d^k\) and \(Y \in \text{Rep} \Gamma_e^k\). The functor \(X \otimes Y\) acts on objects via
\[(X \otimes Y)(V) = X(V) \otimes Y(V)\]
and on morphisms via the map
\[
\Gamma_{d+e}^d \text{Hom}(V, W) \rightarrow \Gamma_d^d \text{Hom}(V, W) \otimes \Gamma_e^e \text{Hom}(V, W)
\]
given by (2.1). Note that
\[(X \otimes Y)^{(i)} \cong X^{(i)} \otimes Y^{(i)}\]
when \(X\) and \(Y\) take values in \(P^k\).

Graded representations. It is sometimes convenient to consider the category
\[
\prod_{d \geq 0} \text{Rep} \Gamma_d^k
\]
consisting of graded representations \(X = (X^0, X^1, X^2, \ldots)\). An example is for each \(V \in P^k\) the representation
\[
\Gamma V = (\Gamma^0 V, \Gamma^1 V, \Gamma^2 V, \ldots).
\]
The tensor product \(X \otimes Y\) of graded representations \(X, Y\) is defined in degree \(d\) by
\[
(X \otimes Y)^{(d)} = \bigoplus_{i+j=d} X^i \otimes Y^j.
\]
Decomposing divided powers. The assignment which takes \(V \in P^k\) to the symmetric algebra \(SV = \bigoplus_{d \geq 0} S^d V\) gives a functor from \(P^k\) to the category of commutative \(k\)-algebras which preserves coproducts. Thus
\[
SV \otimes SW \cong S(V \oplus W)
\]
and therefore by duality
\[
\Gamma V \otimes \Gamma W \cong \Gamma(V \oplus W).
\]
This yields an isomorphism of graded representations
\[
\Gamma V \otimes \Gamma W \cong \Gamma V \otimes \Gamma W.
\]
Thus for each positive integer \(n\), one obtains in degree \(d\) a decomposition
\[
\Gamma_{d,k}^d = \bigoplus_{i=0}^d \left( \Gamma^{d-i,k} \otimes \Gamma^i \right)
\]
and using induction a canonical decomposition
\[
\Gamma_{d,k}^d = \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^\lambda.
\]

The decomposition of divided powers implies that the finitely generated projective objects in \(\text{Rep} \Gamma_d^k\) are precisely the direct summands of finite direct sums of functors \(\Gamma^\lambda\), where \(\lambda = (\lambda_1, \ldots, \lambda_n)\) is any sequence of non-negative integers satisfying \(\sum \lambda_i = d\) and \(n\) is any positive integer.
Exterior powers. Given $V \in P_k$, let $\Lambda V = \bigoplus_{d \geq 0} \Lambda^d V$ denote the exterior algebra, which is obtained from the tensor algebra $TV = \bigoplus_{d \geq 0} V^\otimes d$ by taking the quotient with respect to the ideal generated by the elements $v \otimes v$, $v \in V$.

For each $d \geq 0$, the $k$-module $\Lambda^d V$ is free provided that $V$ is free. Thus $\Lambda^d V$ belongs to $P_k$ for all $V \in P_k$, and this gives a functor $\Gamma^d P_k \to P_k$, since the ideal generated by the elements $v \otimes v$ is invariant under the action of $S_d$ on $V^\otimes d$. There is a natural isomorphism

$$\Lambda^d (V^*) \cong (\Lambda^d V)^*$$

induced by $(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det(f_i(v_j))$, and therefore $(\Lambda^d)^* \cong \Lambda^d$.

Representations of Schur algebras. Strict polynomial functors and modules over Schur algebras are closely related, since for any $X \in \text{Rep} \Gamma^d_k$ the Schur algebra $S_k(n,d)$ acts on $X(k^n)$; cf. Example 2.1.

Let $\text{Mod} S_k(n,d)$ denote the category of modules over $S_k(n,d)$. Then the functor

$$(2.3) \quad \text{Rep} \Gamma^d_k \to \text{Mod} S_k(n,d), \quad X \mapsto X(k^n)$$

is for $n \geq d$ an equivalence $[1]$.

Base change. Let $k \to k'$ be a homomorphism of commutative rings. The functor $- \otimes_k k'$: $P_k \to P_{k'}$ induces for each positive integer $d$ functors

$$\Gamma^d P_k \to \Gamma^d P_{k'}$$ and $\text{Rep} \Gamma^d_k \to \text{Rep} \Gamma^d_{k'}$$

which we denote again by $- \otimes_k k'$. For example, $\Gamma^\lambda \otimes_k k' = \Gamma^{\lambda'}$ for each $\lambda \in \Lambda(n,d)$.

We note that most results in this work are invariant under base change.

3. Schur and Weyl functors

Schur and Weyl functors were introduced by Akin, Buchsbaum, and Weyman [1]. We give the definition and refer to the next section for a description in terms of (co)standard objects.

Partitions and Young diagrams. Fix a positive integer $d$. A partition of weight $d$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers satisfying $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\sum \lambda_i = d$. Its conjugate $\lambda'$ is the partition where $\lambda'_i$ equals the number of terms of $\lambda$ that are greater or equal than $i$.

Fix a partition $\lambda$ of weight $d$. Each integer $r \in \{1, \ldots, d\}$ can be written uniquely as sum $r = \lambda_1 + \ldots + \lambda_{i-1} + j$ with $1 \leq j \leq \lambda_i$. The pair $(i,j)$ describes the position (ith row and jth column) of $r$ in the Young diagram corresponding to $\lambda$. The partition $\lambda$ determines a permutation $\sigma_\lambda \in S_d$ by $\sigma_\lambda(r) = \lambda'_1 + \ldots + \lambda'_{j-1} + i$, where $1 \leq i \leq \lambda_j$. Note that $\sigma_\lambda = \sigma_\lambda^{-1}$. Here is an example.

$$\lambda = (3,2) \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \quad \lambda' = (2,2,1) \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & \end{array} \quad \sigma_\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$$

1Our preference is to work with the functor category $\text{Rep} \Gamma^d_k$ because the Schur category $\Gamma^d P_k$ carries useful structure (e.g. $\oplus$ or $\otimes$) which ‘disappears’ when one evaluates at a single object $k^n$. 
Schur and Weyl modules. Fix a partition $\lambda$ of weight $d$, and assume that $\lambda_1 + \cdots + \lambda_n = d = \lambda'_1 + \cdots + \lambda'_m$. For $V \in P_k$ one defines the Schur module $S_\lambda V$ as image of the map

$$\Lambda^\lambda V \otimes \cdots \otimes \Lambda^{\lambda_m} V \xrightarrow{\Delta \otimes \cdots \otimes \Delta} V \otimes d \xrightarrow{\Delta \otimes \cdots \otimes \Delta} S_\lambda V \otimes \cdots \otimes S_\lambda V.$$  

Here, we denote for an integer $r$ by $\Delta : \Lambda^r V \to V \otimes r$ the comultiplication given by

$$\Delta(v_1 \wedge \cdots \wedge v_r) = \sum_{\sigma \in S_r} \text{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)},$$

$\nabla : V \otimes r \to S^r V$ is the multiplication, and $s_\lambda : V \otimes d \to V \otimes d$ is given by

$$s_\lambda(v_1 \otimes \cdots \otimes v_d) = v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_d(d)}.$$

The corresponding Weyl module $W_\lambda V$ is by definition the image of the analogous map

$$\Gamma^\lambda V \otimes \cdots \otimes \Gamma^{\lambda_m} V \xrightarrow{\Delta \otimes \cdots \otimes \Delta} V \otimes d \xrightarrow{\Delta \otimes \cdots \otimes \Delta} \Lambda^\lambda V \otimes \cdots \otimes \Lambda^{\lambda_m} V.$$  

Note that $(W_\lambda V)^* \cong S_\lambda(V^*)$.

Young tableaux. Suppose that $V$ is a free $k$-module with basis $\{v_1,\ldots, v_r\}$. We fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and describe an explicit basis for $S_\lambda V$ and $W_\lambda V$.

A filling of a Young diagram is a map which assigns to each box a positive integer. A Young tableau is a filling that is weakly increasing along each row and strictly increasing down each column.

Each filling $T$ with entries in $\{1, \ldots, r\}$ yields two elements

$$v_T \in \Gamma^\lambda V \otimes \cdots \otimes \Gamma^{\lambda_m} V \quad \text{and} \quad \hat{v}_T \in \Lambda^\lambda V \otimes \cdots \otimes \Lambda^{\lambda_m} V$$

by replacing any $i$ in a box by $v_i$. Here is an example of a Young tableau

$$\lambda = (5, 3, 3, 2)$$

and here are the corresponding elements.

$$v_T = (v_1 \otimes v_2) (v_3 \otimes v_4) (v_5) \otimes (v_6) \otimes (v_7)$$

$$\hat{v}_T = (v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_6 \otimes v_7) \otimes (v_8 \otimes v_9 \otimes v_{10}) \otimes (v_{11} \otimes v_{12} \otimes v_{13}) \otimes (v_{14} \otimes v_{15} \otimes v_{16}) \otimes (v_{17} \otimes v_{18} \otimes v_{19}) \otimes (v_{20} \otimes v_{21} \otimes v_{22})$$

More precisely, let $T(i, j)$ denote the entry of the box $(i, j)$ and define $\alpha_i = \lambda_i$ by setting $\alpha'_i = \text{card}\{t \mid T(i, t) = j\}$. Then $v_T = v_{\alpha'_1} \otimes \cdots \otimes v_{\alpha'_n}$. Note that the elements $v_T$ form a $k$-basis of $\Gamma^\lambda V$ as $T$ runs through all fillings (weakly increasing along each row).

Proposition 3.1 ([1] Theorems II.2.16 and II.3.16]). Let $\lambda$ be a partition and $V$ a free $k$-module of rank $r$.

1. The canonical map $\Lambda^\lambda V \to S_\lambda V$ sends the elements $\hat{v}_T$ with $T$ a Young tableau on $\lambda$ with entries in $\{1, \ldots, r\}$ to a $k$-basis of $S_\lambda V$.
2. The canonical map $\Gamma^\lambda V \to W_\lambda V$ sends the elements $v_T$ with $T$ a Young tableau on $\lambda$ with entries in $\{1, \ldots, r\}$ to a $k$-basis of $W_\lambda V$. \hfill $\square$

For expositions on Schur and Weyl modules, see [1] §8.1] or [19] §2.1. There one finds proofs of Proposition 3.1 and presentations of these modules, which are relevant for the proof of Theorem 4.7.
**Schur and Weyl functors.** The definition of Schur and Weyl modules gives rise to functors $S_\lambda$ and $W_\lambda$ in $\text{Rep} \Gamma_k^d$ for each partition $\lambda$ of weight $d$. Note that $S_\lambda \cong W_\lambda$ and $W_\lambda \cong S_\lambda$.

**Example 3.2.** We have $S_1 = \Lambda^d$ and $S(d) = S^d$.

### 4. Weight spaces and (co)standard objects

#### Weight space decompositions.
Fix a free $k$-module $V$ with basis $\{v_1, \ldots, v_n\}$.

For any $X \in \text{Rep} \Gamma_k^d$ we describe a decomposition of $X(V)$ into weight spaces; see also [8, Corollary 2.12] for this decomposition and a different argument.

The canonical decomposition

$$\Gamma^d.V = \bigoplus_{\mu \in \Lambda(n,d)} \Gamma^\mu.$$ 

induces via the Yoneda isomorphism $\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, X) \cong X(V)$ a decomposition

$$X(V) = \bigoplus_{\mu \in \Lambda(n,d)} X(V)^\mu \quad \text{with} \quad \text{Hom}_{\Gamma_k^d}(\Gamma^\mu, X) \cong X(V)^\mu.$$ 

For each $\mu \in \Lambda(n,d)$ this isomorphism can be written as composition of

$$\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, X) \cong \text{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)), \quad \phi \mapsto \phi_V$$

and

$$\text{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)) \cong X(V)^\mu, \quad \psi \mapsto \psi(v_1^{\otimes \mu_1} \otimes \cdots \otimes v_n^{\otimes \mu_n}).$$

Here, we identify $\text{End}_{\Gamma_k^d}(V) = S_k(n,d)$ and note that $v_1^{\otimes \mu_1} \otimes \cdots \otimes v_n^{\otimes \mu_n}$ generates $\Gamma^\mu(V)$ as $S_k(n,d)$-module.

The following lemma summarises this discussion.

**Lemma 4.1.** Let $\mu \in \Lambda(n,d)$ and set $V = k^n$. For $X \in \text{Rep} \Gamma_k^d$ there are natural isomorphisms

$$\text{Hom}_{\Gamma_k^d}(\Gamma^\mu, X) \cong \text{Hom}_{S_k(n,d)}(\Gamma^\mu(V), X(V)) \cong X(V)^\mu.$$ 

We observe that the duality preserves the weight space decomposition.

**Lemma 4.2.** Let $\mu \in \Lambda(n,d)$ and set $V = k^n$. For $X \in \text{Rep} \Gamma_k^d$ there is a natural isomorphism

$$X^\circ(V)^\mu \cong (X(V)^\mu)^\ast.$$ 

**Proof.** We have

$$\text{Hom}_{\Gamma_k^d}(\Gamma^{d,V}, X^\circ) \cong X^\circ(V) = X(V)^\ast \cong \text{Hom}_{\Gamma_k^d}(\Gamma^{d,V^\ast}, X)^\ast.$$ 

Now use Lemma [4.1] and the canonical decomposition

$$\Gamma^{d,V} \cong \bigoplus_{\mu \in \Lambda(n,d)} \Gamma^\mu \cong \Gamma^{d,V^\ast}. \quad \square$$
Standard morphisms. We compute the weight spaces for $\Gamma^\lambda$ and $S^\lambda$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be sequences of non-negative integers satisfying $\sum \lambda_i = d = \sum \mu_j$. Given a matrix $A = (a_{ij})$ of non-negative integers with $\lambda_i = \sum a_{ij}$ and $\mu_j = \sum a_{ij}$ for all $i, j$, there is a standard morphism

$$\gamma_A: \Gamma^\mu = \bigotimes_j \Gamma^\mu_j \longrightarrow \bigotimes_j \left( \bigotimes_i \Gamma^a_{ij} \right) = \bigotimes_j \left( \bigotimes_i \Gamma^a_{ij} \right) \longrightarrow \bigotimes_i \Gamma^{\lambda_i} = \Gamma^\lambda$$

where the first morphism is the tensor product of the natural inclusions $\Gamma^\mu_j \rightarrow \bigotimes_i \Gamma^a_{ij}$ and the second morphism is the tensor product of the natural product maps $\bigotimes_j \Gamma^a_{ij} \rightarrow \Gamma^{\lambda_i}$, as given by (2.1). Analogously, there is a morphism

$$\sigma_A: \Gamma^\mu = \bigotimes_j \Gamma^\mu_j \longrightarrow \bigotimes_j \left( \bigotimes_i T^{a_{ij}} \right) = \bigotimes_j \left( \bigotimes_i T^{a_{ij}} \right) \longrightarrow \bigotimes_i S^{\lambda_i} = S^\lambda$$

where $T^r = \otimes^r$ for any non-negative integer $r$, the first morphism is the tensor product of the natural inclusions $\Gamma^\mu_j \rightarrow \bigotimes_i T^{a_{ij}}$, and the second morphism is the tensor product of the natural product maps $\bigotimes_j T^{a_{ij}} \rightarrow S^{\lambda_i}$.

Lemma 4.3 ([17] p. 8). Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be sequences of non-negative integers with $\sum \lambda_i = d = \sum \mu_j$.

(1) The morphisms $\gamma_A$ form a $k$-basis of $\text{Hom}_{\Gamma_k^e}(\Gamma^\mu, \Gamma^\lambda)$.

(2) The morphisms $\sigma_A$ form a $k$-basis of $\text{Hom}_{\Gamma_k^e}(\Gamma^\mu, S^\lambda)$.

Proof. We may assume that $\lambda, \mu \in \Lambda(n, d)$ and apply Lemma 4.1. Fix a free $k$-module $V$ with basis $\{v_1, \ldots, v_n\}$. Then we have an isomorphism

$$\text{Hom}_{\Gamma_k^e}(\Gamma^\mu, \Gamma^\lambda) \cong \text{Hom}_{\Lambda(n, d)}(\Gamma^\mu V, \Gamma^\lambda V) \cong (\Gamma^\lambda V)_\mu.$$  

A standard morphism $\gamma_A$ evaluated at $V$ takes the element $v_{1}^{\otimes \mu_1} \otimes \cdots \otimes v_{n}^{\otimes \mu_n}$ to $v_A = v_{a_1} \otimes \cdots \otimes v_{a_n}$ with $a^i \in \Lambda(n, \lambda_i)$ and $a^i = a_{ij}$. Now the assertion of part (1) follows from the fact that the elements $v_A$ form a basis of $\Gamma^\lambda V$ as $\mu$ runs through $\Lambda(n, d)$; cf. Example 4.3

The proof of part (2) is analogous. \qed

For example, let $\lambda = (5, 3, 3, 2)$ and $\mu = (1, 3, 3, 2, 2, 2)$. For

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

the morphism $\gamma_A$ evaluated at $V = k^6$ takes $v_{1}^{\otimes \mu_1} \otimes \cdots \otimes v_{n}^{\otimes \mu_n}$ to the element

$$(v_1(v_2 \otimes v_3)(v_3 \otimes v_5)) \otimes (v_2v_3v_5) \otimes ((v_4 \otimes v_4)v_6) \otimes (v_5v_6).$$

The special case $\lambda = (1, \ldots, 1) = \mu$ yields the isomorphism

$$\text{End}_{\Gamma_k^e}(\Gamma^{(1, \ldots, 1)}) \cong k\mathfrak{S}_d.$$

Let $\lambda$ be a partition and $T$ a filling of the corresponding Young diagram. The content of $T$ is by definition the sequence $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i$ equals the number of times the integer $i$ occurs in $T$.

Example 4.4. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and set $V = k^n$. For a filling $T$ of the corresponding Young diagram with entries in $\{1, \ldots, n\}$, the element $v_T$ belongs to $(\Gamma^\lambda V)_\mu$ where $\mu$ equals the content of $T$. The standard morphism $\gamma_A: \Gamma^\mu \rightarrow \Gamma^\lambda$ given by $a_{ij} = \text{card}\{t \mid T(t) = j\}$ and evaluated at $V$ sends $v_{1}^{\otimes \mu_1} \otimes \cdots \otimes v_{n}^{\otimes \mu_n}$ to

\[2\text{This yields a basis of the Schur algebra } S_k(n, d) \cong \bigoplus_{\lambda, \mu \in \Lambda(n, d)} \text{Hom}_{\Gamma_k^e}(\Gamma^\mu, \Gamma^\lambda).\]
The dominance order. We consider the dominance order on the set of partitions of weight $d$. Thus $\mu \leq \lambda$ if $\sum_{i=1}^{r} \mu_i \leq \sum_{i=1}^{r} \lambda_i$ for all $r$.

The following simple lemma explains the relevance of Young tableaux.

**Lemma 4.5.** Let $\lambda$ and $\mu$ be partitions. Then there exists a Young tableau of shape $\lambda$ with content $\mu$ if and only if $\mu \leq \lambda$.

The next proposition describes the weight spaces for Schur and Weyl functors.

**Proposition 4.6.** Let $\lambda$ and $\mu$ be partitions of weight $d$.

1. $\text{Hom}_{\mathcal{F}}(\Gamma^n, W_\lambda) \neq 0$ if and only if $\mu \leq \lambda$. Moreover, $\text{Hom}_{\mathcal{F}}(\Gamma^\lambda, W_\lambda) \cong k$.

2. $\text{Hom}_{\mathcal{F}}(\Gamma^n, S_\lambda) \neq 0$ if and only if $\mu \leq \lambda$. Moreover, $\text{Hom}_{\mathcal{F}}(\Gamma^\lambda, S_\lambda) \cong k$.

**Proof.** We apply Lemma 4.4. The assertion for $W_\lambda$ then follows from the computation in Example 4.4 and Lemma 4.5 using the basis of a Weyl module from Proposition 3.1. For $S_\lambda$ the assertion follows from the first part since $S_\lambda \cong W_\lambda^\ast$, using Lemma 4.2.

**Standard objects.** Let $\lambda$ be a partition of weight $d$. For $X \in \text{Rep} \Gamma^d_\mathcal{F}$ and any partition $\mu$ of weight $d$ we define the trace

$$\text{tr}_\mu X = \sum_{\phi : \Gamma^n \to X} \text{Im} \phi.$$ 

The standard object corresponding to $\lambda$ is by definition

$$\Delta(\lambda) = \Gamma^\lambda / (\sum_{\mu \not\leq \lambda} \text{tr}_\mu \Gamma^\lambda)$$

where $\mu$ runs through all partitions of weight $d$.

**Theorem 4.7.** Let $\lambda$ be a partition of weight $d$. The canonical morphism $\Gamma^\lambda \to \Delta(\lambda)$ induces isomorphisms

$$W_\lambda \xrightarrow{\sim} \Delta(\lambda) \quad \text{and} \quad \text{Hom}_{\mathcal{F}}(\Delta(\lambda), \Delta(\lambda)) \xrightarrow{\sim} \text{Hom}_{\mathcal{F}}(\Gamma^\lambda, \Delta(\lambda)) \xrightarrow{\sim} k.$$ 

**Proof.** The proof of [1] Theorem II.3.16] shows that the functor $W_\lambda$ admits a presentation

$$\bigoplus_{i \geq 1} \bigoplus_{t=1}^{\lambda_{i+1}} \Gamma^{\lambda(i,t)} \to \Gamma^\lambda \to W_\lambda \to 0$$

where

$$\lambda(i,t) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + t, \lambda_{i+1} - t, \lambda_{i+2}, \ldots)$$

and $\Gamma^{\lambda(i,t)}$ is the standard morphism $\gamma_\lambda$ given by the matrix

$$A = \text{diag}(\lambda_1, \lambda_2, \ldots) + tE_{i+1,i} - tE_{i+1,i+1}.$$ 

Thus the canonical morphism $\Gamma^\lambda \to \Delta(\lambda)$ induces a morphism $\phi : W_\lambda \to \Delta(\lambda)$, since $\lambda(i,t) \not\leq \lambda$ for all pairs $i, t$.

On the other hand, $\text{Hom}_{\mathcal{F}}(\Gamma^n, W_\lambda) = 0$ for all $\mu \not\leq \lambda$ by Proposition 4.6. Thus the canonical morphism $\Gamma^\lambda \to W_\lambda$ induces a morphism $\Delta(\lambda) \to W_\lambda$, which is an inverse for $\phi$.

For the other pair of isomorphisms apply $\text{Hom}_{\mathcal{F}}(-, \Delta(\lambda))$ to the exact sequence

$$0 \to \sum_{\mu \nleq \lambda} \text{tr}_\mu \Gamma^\lambda \to \Gamma^\lambda \to \Delta(\lambda) \to 0.$$
and use again Proposition 4.6. □

Costandard objects. The duality yields an analogue of Theorem 4.7 for Schur functors. The costandard object corresponding to a partition λ is by definition

\[ \nabla(\lambda) = \bigcap_{\mu \leq \lambda} \text{rej}_\mu S^\lambda \quad \text{with} \quad \text{rej}_\mu X = \bigcap_{\phi : X \to S^\mu} \ker \phi. \]

Corollary 4.8. Let λ be a partition of weight d. The canonical morphism \( \nabla(\lambda) \to S^\lambda \) induces isomorphisms

\[ \nabla(\lambda) \cong S_\lambda \quad \text{and} \quad \text{Hom}_{\Gamma_k}(\nabla(\lambda), \nabla(\lambda)) \cong \text{Hom}_{\Gamma_k}(\nabla(\lambda), S^\lambda) \cong k. \]

Moreover, the canonical morphism \( \Gamma^\lambda \to \Delta(\lambda) \) induces isomorphisms

\[ \text{Hom}_{\Gamma_k}(\Delta(\lambda), \nabla(\lambda)) \cong \text{Hom}_{\Gamma_k}(\Gamma^\lambda, \nabla(\lambda)) \cong k. \]

Proof. From the definition we have \( \nabla(\lambda) \cong \Delta(\lambda)^\circ \) since \( S^\mu \cong (\Gamma^\mu)^\circ \) for each partition \( \mu \). Thus the first set of isomorphisms follows directly from Theorem 4.7 by applying the duality.

For the last pair of isomorphisms apply \( \text{Hom}_{\Gamma_k}(-, \nabla(\lambda)) \) to the exact sequence 4.2 and use Proposition 4.6. □

Simple objects. We study the composition factors of each standard object \( \Delta(\lambda) \), assuming that \( k \) is a field. For a partition \( \lambda \) of weight \( d \) let \( U(\lambda) \) denote the maximal subobject of \( \Delta(\lambda) \) satisfying \( \text{tr}_\lambda U(\lambda) = 0 \) and set

\[ L(\lambda) = \Delta(\lambda)/U(\lambda). \]

Proposition 4.9. Suppose that \( k \) is a field and fix a partition \( \lambda \) of weight \( d \).

1. The functor \( U(\lambda) \) is the unique maximal subobject of \( \Delta(\lambda) \) in \( \text{Rep} \Gamma_k^d \).
2. The functor \( L(\lambda) \) is a simple object in \( \text{Rep} \Gamma_k^d \).
3. Each composition factor \( L(\mu) \) of \( U(\lambda) \) satisfies \( \mu \leq \lambda \).

Proof. (1) Let \( X \subseteq \Delta(\lambda) \) be a subobject. If \( \text{tr}_\lambda X \neq 0 \), then there is a nonzero morphism \( \Gamma^\lambda \to X \to \Delta(\lambda) \), which is an epimorphism by Theorem 4.7. Thus \( U(\lambda) \) is the unique maximal subobject of \( \Delta(\lambda) \).

(2) follows from (1).

(3) Let \( L(\mu) \) be a composition factor of \( U(\lambda) \). Then there is a nonzero morphism \( \Gamma^\mu \to U(\lambda) \). Thus \( \mu \neq \lambda \). We have \( \mu \leq \lambda \) by Proposition 4.6. □

The duality maps the unique simple quotient of \( \Delta(\lambda) \) to the unique simple subobject of \( \nabla(\lambda) \). Next we show that the socle of \( \nabla(\lambda) \) is isomorphic to \( L(\lambda) \).

Lemma 4.10. Let \( S \in \text{Rep} \Gamma_k^d \) be simple and \( \lambda = \max\{\mu \mid \text{Hom}_{\Gamma_k}(\Gamma^\mu, S) \neq 0\} \). Then \( S \cong L(\lambda) \).

Proof. Choose a nonzero morphism \( \Gamma^\lambda \to S \). This factors through the canonical morphism \( \Gamma^\lambda \to \Delta(\lambda) \). Thus \( S \cong L(\lambda) \) by Proposition 4.9. □

Proposition 4.11. Let \( \lambda \) be a partition. Then \( \Lambda(\lambda)^\circ \cong L(\lambda) \).

Proof. The assertion follows from Lemma 4.10 using Lemma 4.2. □

Corollary 4.12. Suppose that \( k \) is a field and let \( \Lambda \) denote the set of partitions of weight \( d \). Then \( \{L(\lambda)\}_{\lambda \in \Lambda} \) is a representative set of simple objects in \( \text{Rep} \Gamma_k^d \). □
5. The Cauchy decomposition

The Cauchy decomposition formula for Schur functors \[1, 7\] is the analogue of Cauchy’s formula for symmetric functions \[3\]. More precisely, the term ‘Cauchy decomposition’ refers to a filtration of symmetric powers whose associated graded object is a direct sum of Schur functors. One obtains the formula for symmetric functions by passing in characteristic zero from polynomial representations of general linear groups to their characters.

Fix \(V, W \in \mathbb{P}_k\). For any non-negative integer \(r\) there is a unique map

\[\psi^r : \Gamma^r V \otimes \Gamma^r W \rightarrow \Gamma^r (V \otimes W)\]

making the following square commutative.

\[
\begin{array}{ccc}
\Gamma^r V \otimes \Gamma^r W & \xrightarrow{\psi^r} & \Gamma^r (V \otimes W) \\
V^\otimes r \otimes W^\otimes r & \sim & (V \otimes W)^{\otimes r}
\end{array}
\]

Extend this map for a partition \(\lambda = (\lambda_1, \ldots, \lambda_n)\) of weight \(d\) to a map

\[\psi^\lambda : \Gamma^\lambda V \otimes \Gamma^\lambda W \rightarrow \Gamma^d (V \otimes W)\]

which is given as composite

\[
\Gamma^\lambda V \otimes \Gamma^\lambda W \xrightarrow{\sim} (\Gamma^{\lambda_1} V \otimes \Gamma^{\lambda_1} W) \otimes \cdots \otimes (\Gamma^{\lambda_n} V \otimes \Gamma^{\lambda_n} W) \xrightarrow{\psi^{\lambda_1} \otimes \cdots \otimes \psi^{\lambda_n}} \\
\Gamma^{\lambda_1} (V \otimes W) \otimes \cdots \otimes \Gamma^{\lambda_n} (V \otimes W) \rightarrow \Gamma^d (V \otimes W)
\]

with the last map given by multiplication.

We consider the lexicographic ordering on the set of partitions of weight \(d\). Thus \(\mu \leq \lambda\) if \(\mu_i \leq \lambda_i\) whenever \(\mu_i = \lambda_i\) for all \(i < r\). For a partition \(\lambda\) let \(\lambda^-\) denote its immediate predecessor and \(\lambda^+\) its immediate successor. Set \((1, \ldots, 1)^- = -\infty\) and \((d)^+ = +\infty\).

The Cauchy filtration is by definition the chain

\[
F_{+\infty} \subseteq F_{(d)} \subseteq F_{(d-1, 1)} \subseteq \cdots \subseteq F_{(1, \ldots, 1, 2)} \subseteq F_{(1, \ldots, 1)} = \Gamma^d (V \otimes W)
\]

where \(F_\lambda = \sum_{\mu \geq \lambda} \text{Im} \psi^\mu\).

The following result describes the factors of the Cauchy filtration; it is the analogue of [1 Theorem III.1.4] for the Cauchy filtration of \(S^d (V \otimes W)^3\).

**Theorem 5.1** ([13 Theorem III.2.9]). Let \(V, W \in \mathbb{P}_k\) and fix a partition \(\lambda\) of weight \(d\). Then the morphism \(\psi^\lambda : \Gamma^\lambda V \otimes \Gamma^\lambda W \rightarrow F_\lambda\) induces an isomorphism

\[\Delta(\lambda)V \otimes \Delta(\lambda)W \xrightarrow{\sim} F_\lambda/F_{\lambda^+}\]

which is functorial in \(V\) and \(W\) (with respect to morphisms in \(\Gamma^d \mathbb{P}_k\)).

**Proof.** From the presentation \([4, 11]\) of \(W_\lambda \cong \Delta(\lambda)\) we deduce that there is a morphism \(\tilde{\psi}^\lambda\) making the following square commutative.

\[
\begin{array}{ccc}
\Gamma^\lambda V \otimes \Gamma^\lambda W & \xrightarrow{\psi^\lambda} & F_\lambda \\
\Delta(\lambda)V \otimes \Delta(\lambda)W & \xrightarrow{\tilde{\psi}^\lambda} & F_\lambda/F_{\lambda^+}
\end{array}
\]

\(^3\)The Cauchy filtration of \(S^d (V \otimes W)\) is obtained from \([5, 1]\) by duality. The approach via maps \(\Lambda^\lambda V \otimes \Lambda^\lambda W \rightarrow S^d (V \otimes W)\) is more complicated.
Corollary 5.2. Let \( \rho : \Gamma^{\lambda(i,t)} \to \Gamma^\lambda \) arising in (4.1). The composition

\[
\Gamma^{\lambda(i,t)} \otimes \Gamma^{\lambda(i,t)} W \xrightarrow{\gamma_1 \otimes \gamma_2} \Gamma^\lambda \otimes \Gamma^\lambda W \xrightarrow{\psi^\lambda} \Gamma^d(V \otimes W)
\]
equals a multiple of \( \psi^{\lambda(i,t)} \), and we have \( \text{Im} \psi^{\lambda(i,t)} \subseteq F_{\lambda^+} \) since \( \lambda(i,t) > \lambda \). This yields \( \psi^\lambda \); it is an epimorphism by construction. A computation of ranks (as in the proof of [11] Theorem III.1.4]) shows that \( \psi^\lambda \) is an isomorphism. \[\Box\]

Corollary 5.2. Let \( V \) be a free \( k \)-module of finite rank. There are filtrations

\[0 = X_+ \subseteq X(d) \subseteq X(d-1,1) \subseteq \cdots \subseteq X(1,\ldots,1,2) \subseteq X(1,\ldots,1) = \Gamma^d \text{Hom}(V,-)\]

and

\[0 = Y_- \subseteq Y(1,\ldots,1) \subseteq Y(1,\ldots,1,2) \subseteq \cdots \subseteq Y(d-1,1) \subseteq Y(d) = S^d(V \otimes -)\]
in \( \text{Rep} \Gamma^d_k \) such that for each partition \( \lambda \) the quotient \( X_\lambda / X_{\lambda^+} \) is a direct sum of copies of \( \Delta(\lambda) \) and \( Y_\lambda / Y_{\lambda^-} \) is a direct sum of copies of \( \nabla(\lambda) \).

Proof. The filtration of \( \Gamma^d \text{Hom}(V,-) \) is given by the filtration (5.1), replacing \( V \) by \( V^* \) and using its functoriality in \( W \). It follows from Theorem 5.1 that \( X_\lambda / X_{\lambda^+} \) is a direct sum of copies of \( \Delta(\lambda) \); the multiplicity is given by the rank of \( \Delta(\lambda)V \) over \( k \).

For the second filtration let \( Y_\lambda \) denote the kernel of the epimorphism

\[S^d(V \otimes -) \xrightarrow{\sim} \Gamma^d \text{Hom}(V,-)^{\circ} \to X_\lambda^{\circ} .\]

Then we have

\[Y_\lambda / Y_{\lambda^-} \cong (X_\lambda / X_{\lambda^+})^{\circ}\]

which is a direct sum of copies \( \Delta(\lambda)^{\circ} \cong \nabla(\lambda) \) by the first part. \[\Box\]

Remark 5.3. The filtration of \( \Gamma^d \text{Hom}(V,-) \) induces one for each direct summand of \( \Gamma^d \text{Hom}(V,-) \). This follows from the functoriality of the filtration (5.1) in \( V \).

The canonical isomorphism

\[\text{End}_{\Gamma^d_k}(V)^{\text{op}} \xrightarrow{\sim} \text{End}_{\Gamma^d_k}(\Gamma^d \text{Hom}(V,-))\]

then shows that each decomposition of \( \Gamma^d \text{Hom}(V,-) \) yields a decomposition of each factor \( X_\lambda / X_{\lambda^+} \). An analogous statement holds for \( S^d(V \otimes -) \).

6. Highest weight categories

Highest weight categories were introduced by Cline, Parshall, and Scott [5]. We recall the definition.

Let \( k \) be a field and \( \mathcal{C} \) a \( k \)-linear abelian category satisfying the following properties:

1. \( \mathcal{C} \) admits directed unions of subobjects.
2. For a subobject \( A \) and a directed family of subobjects \( B_i \) of an object in \( \mathcal{C} \),
   \[A \cap (\bigcup B_i) = \bigcup (A \cap B_i).\]
3. Every object in \( \mathcal{C} \) is the union of its finite length subobjects.
4. \( \mathcal{C} \) has enough injective objects.

A simple object \( S \) is a composition factor of an object \( X \) in \( \mathcal{C} \) if it is a composition factor of a finite length subobject of \( X \). Let \([X:S]\) denote the supremum of the multiplicities \([X':S]\) of \( S \) in a composition series of a finite length subobject \( X' \subseteq X \).

The category \( \mathcal{C} \) is a highest weight category if there exists a poset \( \Lambda \) such that each interval is finite and and the following properties hold:
The dual assertions of (6).

Theorem 6.1. Let $d$ partitions of weight $\Lambda$ of strict polynomial functors is a highest weight category with respect to the set of composition factors $L(\mu)$ of $\nabla(\lambda) / L(\lambda)$ satisfy $\mu < \lambda$. For $\lambda, \mu \in \Lambda$, the length of the $k$-module $\text{Hom}_C(\nabla(\lambda), \nabla(\mu))$ and $[\nabla(\lambda):L(\mu)]$ are finite.

(7) The injective envelope $I(\lambda)$ of $L(\lambda)$ admits a filtration

$$0 = F_0(\lambda) \subseteq F_1(\lambda) \subseteq F_2(\lambda) \subseteq \ldots$$

such that

(a) $F_1(\lambda) \cong \nabla(\lambda)$ and for each $n > 1$ there is some $\mu = \mu(n) > \lambda$ with $F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu)$.

(b) The set $\{ n \in \mathbb{N} \mid \mu(n) = \mu \}$ is finite for each $\mu \in \Lambda$.

(c) $I(\lambda) = \bigcup F_n(\lambda)$.

The following theorem is the principal result of this work.

**Theorem 6.1.** Let $k$ be a field and $d$ a positive integer. Then the category $\text{Rep}_{k} \Gamma^d$ of strict polynomial functors is a highest weight category with respect to the set of partitions of weight $d$ and the lexicographic order.

**Proof.** The properties (1)–(4) are clear. Let $\Lambda$ denote set of partitions of weight $d$ and the lexicographic order. Note that the lexicographic order refines the dominance order.

The subcategory of finite length objects in $\text{Rep}_{k} \Gamma^d$ admits a duality (sending $X$ to $X^\circ$). Thus it suffices to verify the dual assertions of (5)–(7), keeping in mind that the duality stabilises the simple objects by Proposition 4.11.

Ad (5): From Corollary 4.1 and the subsequent Remark 5.3 we obtain a filtration of $P(\lambda)$ which satisfies the dual assertions of (7). More precisely, let

$$0 = X_{-\infty}^\lambda \subseteq X_d^\lambda \subseteq X_{d-1,1}^\lambda \subseteq \ldots \subseteq X_{1,...,1,2}^\lambda \subseteq X_{1,...,1}^\lambda = P(\lambda)$$

denote the direct summand of the filtration of $\Gamma^d V$ corresponding to $P(\lambda)$. The maximal partition $\mu$ with $X_\mu^\lambda = P(\lambda)$ equals $\lambda$ since $P(\mu)/X_\mu^\lambda \cong X_\mu^\lambda / X_{\mu^+}^\lambda$ is a direct sum of copies of $\Delta(\mu)$. Note that the multiplicity of $\Delta(\mu)$ in $P(\mu)/X_{\mu^+}^\lambda$ equals one, since $P(\mu)$ has simple top. Thus all remaining factors of the filtration are isomorphic to direct sums of copies of $\Delta(\mu)$ with $\mu > \lambda$.

The module category of a finite dimensional $k$-algebra $A$ is a highest weight category if and only if $A$ is quasi-hereditary [3]. Thus the equivalence (2.10) between $\text{Rep}_{k} \Gamma^d$ and the category of modules over the Schur algebra $S_k(n,d)$ for $n \geq d$ yields the following (see [11, §7] for historical comments).

**Corollary 6.2.** The Schur algebra $S_k(n,d)$ is quasi-hereditary for all $n \geq d$. □

**Remark 6.3.** The arguments of this proof can be modified to show that the Schur algebra $S_k(n,d)$ is quasi-hereditary for all $n$.

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