Donoho–Stark’s Uncertainty Principles in Real Clifford Algebras

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Abstract. The Clifford Fourier transform (CFT) has been shown to be a powerful tool in the Clifford analysis. In this work, several uncertainty inequalities are established in the real Clifford algebra $\text{Cl}_{(p,q)}$, including the Hausdorf–Young inequality, and three qualitative uncertainty principles of Donoho–Stark.

Keywords. Clifford algebras, Clifford-Fourier transform, Uncertainty principle, Donoho–Stark’s uncertainty principle.

1. Introduction

It is well known that the uncertainty principles (UPs) give information about a function and its Fourier transform. Their importance is due to their applications in different areas, e.g. quantum physics and signal processing. In quantum physics, they tell us that the position and the momentum of a particle cannot both be measured with precision.

The qualitative UP is a kind of UPs, which tells us how a signal $f$ and its Fourier transform $\hat{f}$, behave under certain conditions. One such example can be Donoho–Stark’s UP [4], which expresses the limitations on the simultaneous concentration of $f$, and $\hat{f}$.

The aim of this work is to generalize Donoho–Stark’s UP in Clifford’s analysis, using the basic properties of Clifford’s algebras and its Fourier transform.

For more details on Clifford Fourier’s transformations, their historical development and applications, we refer to [1,2,5,6].

In [3] Thm. 5.1, and [7] Thm. 8, the authors establish, in different ways, the UP of Donoho–Stark in quaternion algebra which is isomorphic at $\text{Cl}_{(0,2)}$.

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The first inequality we deal with is a generalization of the Hausdorff–Young inequality by means of the kernel of the CFT introduced by Hitzer [5].

Based on this inequality, and following the Donoho–Stark’UP proof techniques for the Dunkel-transform [11], we investigate three inequalities in terms of “ε-concentration” in the Clifford algebra $\mathbb{C}l_{(p,q)}$.

This paper is organized as follows. Section 2 is devoted to a reminder of the basics of Clifford algebras. In Sect. 3, we introduce the CFT and review its important properties, and prove the Hausdorff–Young inequality. In Sect. 4, we define the concept of “ε-concentration” in CFT-domain, and establish UPs of concentration type, then prove Donoho–Stark bandlimited UP for the CFT. Finally, we give a conclusion in Sect. 5.

2. Preliminaries

Let \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal basis of the real Euclidean vector space \( \mathbb{R}^{(p,q)} \), with \( p + q = n \).

The Clifford geometric algebra (see [10]) over \( \mathbb{R}^{(p,q)} \) denoted by \( \mathbb{C}l_{(p,q)} \), is defined as an associative, non commutative algebra which has the graded \( 2^n \)-dimensional basis

\[
\{ e_A = e_{\alpha_1}e_{\alpha_2}\ldots e_{\alpha_k}, : A \subseteq \{1,2,\ldots,n\}, 1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq n \} \tag{2.1}
\]

with \( e_\emptyset = 1 \) is the unit element (grade 0), \( e_1, e_2, \ldots, e_n \in \mathbb{R}^n \) are vectors (grade 1), and \( e_k e_l, 1 \leq k \leq l \leq n \) are bivectors (grade 2), and so on, and \( e_1e_2\ldots e_n \) (grade \( n \)), is the highest grade blade element in \( \mathbb{C}l_{(p,q)} \).

The multiplication of the basis vectors satisfy the rules

\[
e_k e_l + e_l e_k = 2\epsilon_k \delta_{k,l}, \quad \text{for} \quad 1 \leq k, l \leq n,
\]

With \( \delta_{k,l} \) is the Kronecker symbol, and \( \epsilon_k = +1, \) for \( k = 1, \ldots, p \), and \( \epsilon_k = -1, \) for \( k = p + 1, \ldots, n \).

Every element \( f \) of Clifford algebra \( \mathbb{C}l_{(p,q)} \), is called multivector, and can be expressed in the form

\[
f = \sum_A f_A e_A = f_\emptyset + \sum_{k \in \{1,2,\ldots,n\}} f_k e_k + \sum_{1 \leq k < l \leq n} f_{kl} e_k e_l + \cdots + f_{123\ldots n} e_1 e_2 e_3 \ldots e_n, \tag{2.2}
\]

where \( f_A \), are real-valued.

Then, (2.2) can be written as

\[
f = \sum_{k=0}^{k=n} < f >_k = < f >_0 + < f >_1 + < f >_2 + \cdots + < f >_n.
\]
where \( <f>_k = \sum_{|A|=k} f_A e_A \), denote the \( k \)-vector part of \( f \). As examples, 
\( <f>_0 \) denotes the scalar part, \( <f>_1 \) the vector part, \( <f>_2 \) the bivector 
part and \( <f>_n \) the pseudoscalar part.

Moreover, the principal reverse \( \tilde{f} \) of \( f \) is given by
\[
\tilde{f} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} <f>_k, 
\]

Where \( \tilde{f} \) means to change in the basis decomposition of \( f \) the sign of every vector of negative square 
\( e_A = \epsilon_{\alpha_1} e_{\alpha_1} \ldots \epsilon_{\alpha_k} e_{\alpha_k} \), \( 1 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k \leq n \).

It is straightforward to verify that the principal reverse is linear, involutive 
and anti-automorphic, this means
\[
\tilde{\tilde{f}} = f, \quad \tilde{f} + g = \tilde{f} + \tilde{g}, \quad \tilde{fg} = \tilde{f} \tilde{g}, \quad f, g \in Cl_{(p,q)}. \tag{2.3}
\]

For \( f, \tilde{g} \in Cl_{(p,q)} \), the scalar product \( f \ast \tilde{g} \), is defined by 
\[
f \ast \tilde{g} = <f \tilde{g}>_0 = \sum_A f_A g_A.
\]

In particular, if \( f = g \), then we obtain the modulus of a multivector 
\( f \in Cl_{(p,q)} \), defined as
\[
|f| = \sqrt{<f \tilde{f}>_0} = \sqrt{\sum_A f_A^2}. \tag{2.4}
\]

For \( 1 \leq a < \infty \), The linear spaces \( L^a(\mathbb{R}^{(p,q)}, Cl_{(p,q)}) \) are introduced as:
\[
L^a(\mathbb{R}^{(p,q)}, Cl_{(p,q)}) = \left\{ \mathbb{R}^{(p,q)} \rightarrow Cl_{(p,q)} : \|f\|_a = \left( \int_{\mathbb{R}^{(p,q)}} |f(t)|^a dt \right)^{\frac{1}{a}} < \infty \right\}.
\]

For \( a = \infty \), the \( L^\infty \)- norm is defined by 
\[
\|f\|_\infty = \text{ess sup}_{t \in \mathbb{R}^{(p,q)}} |f(t)|.
\]

In the following, we will prove two essential inequalities: the first is the Clifford 
triangle inequality, the second is related to the norm of the product of 
two multivectors.

**Lemma 2.1.** For \( \lambda, \rho \in Cl_{(p,q)} \), the following properties hold

(i) \( |\lambda + \rho| \leq |\lambda| + |\rho| \).

(ii) \( |\lambda \rho| \leq 2^n |\lambda| |\rho| \).

**Proof.** (i) 
\[
|\lambda + \rho|^2 \overset{(2.4)}{=} <(\lambda + \rho)(\lambda + \rho)>_0 \leq <\lambda \tilde{\lambda}>_0 + <\rho \tilde{\rho}>_0 + <\lambda \rho >_0 + <\rho \lambda>_0.
\]

In the second equality we used (2.3), applied the linearity of \(<.>_0 \) in the 
third.

Applying the property \(<v>_0 = <\tilde{v}>_0, \forall v \in Cl_{(p,q)}\),
we find that \( < \rho \tilde{\lambda} >_0 = < \rho \tilde{\lambda} >_{0(2.3)} < \lambda \tilde{\rho} >_0 \) then, we get
\[
|\lambda + \rho|^2 = |\lambda|^2 + |ho|^2 + 2 < \lambda \tilde{\rho} >_0.
\]

Now, using the Cauchy–Schwarz inequality
\[
< \lambda \tilde{\rho} >_0 \leq |\lambda| |\rho|.
\]
(For the proof see [9, Appendix]) we finish the proof. \( \square \)

(ii) If \( \Sigma_A \rho_A e_A \) is the decomposition of \( \rho \) in the basis (2.1) then
\[
\lambda \rho = \lambda \Sigma_A \rho_A e_A = \Sigma_A \lambda \rho_A e_A.
\]
Hence by the property (i) we have
\[
|\lambda \rho| \leq \Sigma_A |\lambda \rho_A e_A| = \Sigma_A |\rho_A| |\lambda e_A|,
\]
where we used the homogeneity property of the Clifford norm i.e. \(|av| = |a||v| \\ \forall v \in Cl_{(p,q)}, a \in \mathbb{R} \).

On the other hand
\[
|\lambda e_A|^2 \overset{(2.4)}{=} < \lambda e_A \tilde{\lambda} e_A >_0 \\
= < \lambda e_A \tilde{\lambda} >_0 \\
= < \lambda \tilde{\lambda} >_0 \\
= |\lambda|^2,
\]
where we applied (2.3) in the second equality, and the property \( e_A \tilde{e_A} = 1 \) in the third one.

And therefore,
\[
|\lambda \rho| \leq \Sigma_A |\rho_A| |\lambda|.
\]
By remarking that \( |\rho_A| \leq |\rho| \), and the cardinality of the basis (2.1) is \( 2^n \), we conclude the proof. \( \square \)

Lemma 2.2. Let \( \theta \in \mathbb{R} \), and \( \mu \in Cl_{(p,q)} \), with \( \mu^2 = -1 \), we have a natural generalization of Euler’s formula for Clifford algebra, as follows
\[
e^{\theta \mu} = \cos (\theta) + \mu \sin (\theta).
\]
Proof. As \( \mu^2 = -1 \), we have for any real \( \theta \)
\[
e^{\mu \theta} = \sum_{k=0}^{\infty} \frac{(\mu \theta)^k}{k!} \\
= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + \mu \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k + 1)!} \\
= \cos (\theta) + \mu \sin (\theta).
\]
\( \square \)
3. Clifford-Fourier transform

In this section, we introduce the Clifford Fourier transform (CFT) recall its properties, add one result related to the kernel of the CFT, and we prove the Hausdorff–Young inequality associated with the CFT.

**Definition 3.1.** Let $\mu \in Cl_{(p,q)}$ be a square root of $-1$, i.e. $\mu^2 = -1$.

The general Clifford Fourier transform (CFT) (see [5]) of $f \in L^1(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, with respect to $\mu$ is

$$\mathcal{F}_\mu \{f\} (\xi) = \int_{\mathbb{R}^{(p,q)}} f(t) e^{-\mu u(t,\xi)} dt,$$

where $dt = dt_1 \ldots dt_n$, $t, \xi \in \mathbb{R}^{(p,q)}$, and $u : \mathbb{R}^{(p,q)} \times \mathbb{R}^{(p,q)} \to \mathbb{R}$.

We assume, in the rest of this work, that $u(t, \xi) = l = \sum_{l=1}^{n} t_l \xi_l$.

3.1. Properties of the CFT

In the following, we give some important properties of the CFT. For more detailed discussions of the properties of the CFT and their proofs, see e.g. [1,5,6]

- **Left linearity**
  For $f_1, f_2 \in L^1(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, and constants $\alpha, \beta \in Cl_{(p,q)}$,
  $$\mathcal{F}_\mu \{\alpha f_1 + \beta f_2\} = \alpha \mathcal{F}_\mu \{f_1\} + \beta \mathcal{F}_\mu \{f_2\}.$$

- **Inversion formula**
  For $f, \mathcal{F}_\mu \{f\} \in L^1(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, we have
  $$f(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{(p,q)}} \mathcal{F}_\mu \{f\} (\xi) e^{\mu u(t,\xi)} d\xi,$$
  where $d\xi = d\xi_1 \ldots d\xi_n$, $t, \xi \in \mathbb{R}^{(p,q)}$.

- For the function $f \in L^2(\mathbb{R}^{(p,q)}, Cl_{(p,q)})$, one has the Parseval identity
  $$\| \mathcal{F}_\mu \{f\} \|_2 = (2\pi)^{\frac{n}{2}} \| f \|_2.$$

**Lemma 3.2.** For $t, \xi \in \mathbb{R}^{(p,q)}$, and $\mu \in Cl_{(p,q)}$, with $\mu^2 = -1$, the following inequality holds:

$$|e^{-\mu u(t,\xi)}| \leq (1 + |\mu|^2)^{\frac{1}{2}}.$$

**Proof.** By means of Lemma 2.2 and the definition (2.4) of the Clifford norm, we obtain

$$|e^{-\mu u(t,\xi)}|^2 = \cos^2 (u(t,\xi)) + \sum_A \sin^2 (u(t,\xi)) \mu_A^2$$

$$\leq 1 + \sum_A \mu_A^2 = 1 + |\mu|^2.$$
Therefore,
\[ |e^{-\mu u(t,\xi)}| \leq (1+|\mu|^2)^{\frac{1}{2}}. \]

However, by combining Lemmas 3.2 and 2.1, we do have the following result:

**Lemma 3.3.** Let \( \lambda \in \text{Cl}_{(p,q)} \), and \( \mu \in \text{Cl}_{(p,q)} \) be a square root of \(-1\). Then,
\[ |\lambda e^{-\mu u(x,y)}| \leq 2^n |\lambda| (1 + |\mu|^2)^{\frac{1}{2}}. \]  

(3.3)

**Proposition 3.4.** Riesz–Thorin theorem ([8], Thm. 2.1)

Let \((X,A,\mu)\) be a measurable space (i.e, \(X\) is a set, \(A\) denotes \(\sigma\)-algebra of measurables subsets of \(X\), and \(\mu\) is a measure defined on \(A\)), likewise let \((Y,B,\nu)\) be a measurable space, and Let \(T\) be a bounded linear operator from \(L^{a_0}(X,A,\mu)\) to \(L^{b_0}(Y,B,\nu)\) with norm \(M_0\) and from \(L^{a_1}(X,A,\mu)\) to \(L^{b_1}(Y,B,\nu)\) with norm \(M_1\). Then \(T\) is bounded from \(L^{a}(X,A,\mu)\) to \(L^{b}(Y,B,\nu)\) with norm \(M_\theta\)
\[ M_\theta \leq M_0^{1-\theta} M_1^{\theta}, \]
with
\[ \frac{1}{a} = \frac{1-\theta}{a_0} + \frac{\theta}{a_1}, \quad \frac{1}{b} = \frac{1-\theta}{b_0} + \frac{\theta}{b_1}, \quad \theta \in (0,1). \]  

(3.4)

**Theorem 3.5.** Hausdorff–Young inequality associated with CFT.

Let \( f \in L^{a}(\mathbb{R}^n,\text{Cl}_{(p,q)}) \), \( 1 \leq a \leq 2 \), then \( \mathcal{F}^\mu \{f\} \in L^{b}(\mathbb{R}^n,\text{Cl}_{(p,q)}) \) with \( \frac{1}{a} + \frac{1}{b} = 1 \), and we have
\[ \|\mathcal{F}^\mu \{f\}\|_b \leq C_h \|f\|_a, \]  

(3.5)

where \( C_h \) is the constant of Hausdorff–Young inequality given by \( C_h = (2^n C_\mu)^{1-\frac{a}{2}} (2\pi)^{\frac{a}{2}} \) with \( C_\mu = (1 + |\mu|^2)^{\frac{1}{2}} \).

**Proof.** We have
\[ |\mathcal{F}^\mu \{f\}(\xi)| = \left| \int_{\mathbb{R}^n} f(t) e^{-\mu u(t,\xi)} dt \right| \]
\[ \leq \int_{\mathbb{R}^n} |f(t) e^{-\mu u(t,\xi)}| dt \]
\[ \leq 2^n (1 + |\mu|^2)^{\frac{1}{2}} \int_{\mathbb{R}^n} |f(t)| dt. \]

Where we used (3.3).

Thus,
\[ \|\mathcal{F}^\mu \{f\}\|_\infty \leq 2^n C_\mu \|f\|_1. \]  

(3.6)

Hence, \( \mathcal{F}^\mu \) is a bounded linear operator of type \((1,\infty)\) with norm \(2^n C_\mu\).

On the other hand, by (3.2) one sees that \( \mathcal{F}^\mu \) is of type \((2,2)\) with norm \((2\pi)^{\frac{1}{2}}\).
Proposition 3.4 reveals that $\mathcal{F}_\mu$ is of type $(a, b)$ i.e is bounded from $L^a(\mathbb{R}^{(p,q)}, Cl(p,q))$ to $L^b(\mathbb{R}^{(p,q)}, Cl(p,q))$, with norm $M_\theta$, such that

$$M_\theta \leq (2^n C_\mu)^{1-\theta} ((2\pi)^{\frac{n}{2}})^\theta.$$  

(3.4) implies

$$\frac{1}{a} = 1 - \frac{\theta}{2} = 1 - \frac{\theta}{\infty}, \quad \frac{1}{b} = \frac{\theta}{2}, \quad \theta \in (0, 1).$$

Then

$$\frac{1}{a} = 1 - \frac{1}{b}, \quad 1 \leq a \leq 2,$$

and

$$M_\theta \leq (2^n C_\mu)^{1-\frac{\theta}{2}} (2\pi)^{\frac{\theta}{2}}.$$  

This completes the proof. \(\square\)

4. Donoho–Stark Uncertainty Principles in Clifford algebra $\mathcal{C}l(p,q)$

The Donoho–Stark UPs involve the concept of $\epsilon$-concentration. Before we provide the main results of this section we introduce two localization operators with characteristic functions as symbols. Their importance is due to the fact that Donoho–Stark hypothesis of $\epsilon$-concentration can be interpreted in terms of the action of these two operators[4]. Let $T$ and $\Omega$ be measurable subsets of $\mathbb{R}^n$ and $f$ be a Clifford algebra-valued function of $\mathbb{R}^n$, and $\mathcal{F}_\mu\{f\}$ be (if exists) its Clifford Fourier transform.

The first operator is the time-limiting operator $P_T$ given by

$$(P_T f)(t) \overset{\text{def}}{=} (\chi_T f)(t) = \begin{cases} f(t), & \text{if } t \in T, \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

This operator delete the part of $f$ outside $T$.

The second operator is the frequency-limiting operator $Q_\Omega$ defined by

$$\mathcal{F}_\mu\{Q_\Omega f\}(\xi) \overset{\text{def}}{=} (\chi_\Omega \mathcal{F}_\mu \{f\})(\xi) = \int_{\Omega} f(t) e^{-\mu u(t, \xi)} dt. \quad (4.2)$$

Which means $Q_\Omega f$ is a partial reconstruction of $f$ using only frequency information from frequencies in $\Omega$, and $\mathcal{F}_\mu\{Q_\Omega f\}$ vanishes outside $\Omega$.

**Definition 4.1.** A function $f$ is $\epsilon_T$ concentrated in the $L^a$-norm on $T$ if

$$\left(\int_{\mathbb{R}^n \setminus T} |f(t)|^a dt\right)^{\frac{1}{a}} \leq \epsilon_T \|f\|_a.$$  

**Remark 4.2.** If $\epsilon_T = 0$, then $T$ is the exact support of $f$. 

Lemma 4.3. If $1 < a \leq 2$, $\frac{1}{a} + \frac{1}{b} = 1$, and $f \in L^a(\mathbb{R}^n, Cl_{(p,q)})$, then
\[
\|F^\mu \{Q\Omega Pf\}\|_b \leq 2^n C_\mu |T|^\frac{1}{b} |\Omega|^\frac{1}{\frac{1}{b}} \|f\|_a.
\] (4.3)

Where $|T|$ is denoted as the Lebesgue measure of $T$.

Proof. Without loss of generality, we assume that $|T| < \infty$ and $|\Omega| < \infty$, we have
\[
F^\mu \{Q\Omega Pf\} = \chi_\Omega F^\mu \{Pf\},
\]
Thus,
\[
\|F^\mu \{Q\Omega Pf\}\|_b = \left( \int_\Omega |F^\mu \{Pf\}(\xi)|^b d\xi \right)^{\frac{1}{b}}.
\] (4.4)

In view of
\[
F^\mu \{Pf\}(\xi) = \int_T f(t)e^{-\mu u(t,\xi)} dt,
\]
we have by Hölder inequality, and (3.3))
\[
|F^\mu \{Pf\}(\xi)| \leq 2^n \left( \int_T |f(t)|^a dt \right)^{\frac{1}{a}} \left( \int_T |e^{-\mu u(t,\xi)}|^b dt \right)^{\frac{1}{b}} \leq \|f\|_a 2^n C_\mu |T|^\frac{1}{b}.
\]
Consequently, (4.4) yields
\[
\|F^\mu \{Q\Omega Pf\}\|_b \leq 2^n C_\mu |T|^\frac{1}{b} |\Omega|^\frac{1}{\frac{1}{b}} \|f\|_a.
\]

□

We are now in the position to establish the first UP of concentration type for $L^a$ theory.

Theorem 4.4. If a non-zero function $f \in L^1 \cap L^a(\mathbb{R}^n, Cl_{(p,q)})$, is $\varepsilon_T$ concentrated on $T$, in $L^a$-norm, and $F^\mu$ is $\varepsilon_\Omega$-concentrated on $\Omega$, in $L^b$-norm, $\frac{1}{a} + \frac{1}{b} = 1$.

Then,
\[
\|F^\mu \{f\}\|_b \leq \frac{C_\mu 2^n |T|^\frac{1}{b} |\Omega|^\frac{1}{\frac{1}{b}} + C_h \varepsilon_T}{1 - \varepsilon_\Omega} \|f\|_a.
\] (4.5)

Proof. Without loss of generality, we may assume that $T$ and $\Omega$ have finite measure.

Then we have
\[
\|f - Pf\|_a = \left( \int_{\mathbb{R}^n \setminus T} |f(t)|^a dt \right)^{\frac{1}{a}} \leq \varepsilon_T \|f\|_a.
\]
Since $F^\mu \{f\}$ is $\varepsilon_\Omega$-concentrated on $\Omega$, in $L^b$-norm, we obtain that
\[
\|F^\mu \{f\} - F^\mu \{Q\Omega f\}\|_b \leq \varepsilon_\Omega \|F^\mu \{f\}\|_b.
\] (4.6)
On the other hand,
\[ \| \mathcal{F}^\mu \{ Q_{\Omega} f \} - \mathcal{F}^\mu \{ Q_{\Omega} P_T f \} \|_b = \| \chi_{\Omega} \mathcal{F}^\mu \{ f \} - \chi_{\Omega} \mathcal{F}^\mu \{ P_T f \} \|_b \leq \| \mathcal{F}^\mu \{ f - P_T f \} \|_b \leq C_h \| f - P_T f \|_a. \]

Where we used the linearity of \( \mathcal{F}^\mu \) in the second inequality, and (3.5) in the last.

And consequently, by the triangle inequality
\[ \| \mathcal{F}^\mu \{ f \} - \mathcal{F}^\mu \{ Q_{\Omega} P_T f \} \|_b \leq \| \mathcal{F}^\mu \{ f \} - \mathcal{F}^\mu \{ Q_{\Omega} f \} \|_b + \| \mathcal{F}^\mu \{ Q_{\Omega} P_T f \} \|_b \leq \varepsilon_{\Omega} \| \mathcal{F}^\mu \{ f \} \|_b + C_h \| f - P_T f \|_a \leq \varepsilon_{\Omega} \| \mathcal{F}^\mu \{ f \} \|_b + C_h \varepsilon_T \| f \|_a. \quad (4.7) \]

Moreover, again using the triangle inequality, (4.3), and (4.7), implies that
\[ \| \mathcal{F}^\mu \{ f \} \|_b \leq \| \mathcal{F}^\mu \{ f \} - \mathcal{F}^\mu \{ Q_{\Omega} P_T f \} \|_b + \| \mathcal{F}^\mu \{ Q_{\Omega} P_T f \} \|_b \leq \varepsilon_{\Omega} \| \mathcal{F}^\mu \{ f \} \|_b + C_h \varepsilon_T \| f \|_a + 2^n C_{\mu} |T|^\frac{1}{2} |\Omega|^\frac{1}{2} \| f \|_a. \]

Hence,
\[ (1 - \varepsilon_{\Omega}) \| \mathcal{F}^\mu \{ f \} \|_b \leq (C_{\mu} 2^n |T|^\frac{1}{2} |\Omega|^\frac{1}{2} + C_h \varepsilon_T) \| f \|_a. \]

\[ \square \]

In view of the Parseval identity (3.2), and (4.5) and by taking \( a = b = 2 \) in Theorem 4.4 and remarking that \( C_h = (2\pi)^\frac{n}{2} \) in that case, we get

**Corollary 4.5.** Suppose that \( f \in L^2(\mathbb{R}^n, Cl_{(p,q)}) \), with \( f \neq 0 \), is \( \varepsilon_T \)-concentrated on \( T \), in \( L^2 \)-norm, and \( \mathcal{F}^\mu \) is \( \varepsilon_{\Omega} \)-concentrated on \( \Omega \), in \( L^2 \)-norm.

Then, one has
\[ (2\pi)^\frac{n}{2} (1 - \varepsilon_{\Omega} - \varepsilon_T) \leq C_{\mu} 2^n |T|^\frac{1}{2} |\Omega|^\frac{1}{2}. \]

Choose \( \varepsilon_{\Omega} = \varepsilon_T = 0 \), in Corollary 4.5, and use remark 4.2.

We do have the following result

**Corollary 4.6.** Suppose that \( f \in L^2(\mathbb{R}^n, Cl_{(p,q)}) \), with \( \text{Supp} \ f \subseteq T \), and \( \text{Supp} \ \mathcal{F}^\mu \subseteq \Omega \).

Then,
\[ \frac{(\frac{n}{2})^n}{1 + \| \mu \|^2} \leq |T||\Omega|. \]

The second concentration UP of Donoho–Stark associated with CFT for the \( L^1 \cap L^a \) theory, is given in the following theorem.

**Theorem 4.7.** Suppose that a non zero function \( f \in L^1 \cap L^a(\mathbb{R}^n, Cl_{(p,q)}) \), \( 1 < a \leq 2 \), is \( \varepsilon_T \)-concentrated on \( T \), in \( L^1 \)-norm, and \( \mathcal{F}^\mu \) is \( \varepsilon_{\Omega} \)-concentrated on \( \Omega \), in \( L^b \)-norm, \( \frac{1}{a} + \frac{1}{b} = 1 \).

Then,
\[ \| \mathcal{F}^\mu \{ f \} \|_b \leq \frac{2^n C_{\mu}|T|^\frac{1}{2} |\Omega|^\frac{1}{2}}{(1 - \varepsilon_{\Omega})(1 - \varepsilon_T)} \| f \|_a. \]
Proof. We assume that $T$ and $\Omega$ have finite measure, we have by triangle inequality and (4.6)
\[
\| \mathcal{F}^\mu \{f\}\|_b \leq \| \mathcal{F}^\mu \{f\} - \mathcal{F}^\mu \{Q_\Omega f\}\|_b + \| \mathcal{F}^\mu \{Q_\Omega f\}\|_b
\leq \varepsilon_\Omega \| \mathcal{F}^\mu \{f\}\|_b + \left( \int_{\Omega} |\mathcal{F}^\mu \{f\}(\xi)|^b d\xi \right)^{\frac{1}{b}}.
\]
Using
\[
\left( \int_{\Omega} |\mathcal{F}^\mu \{f\}(\xi)|^b d\xi \right)^{\frac{1}{b}} \leq \| \mathcal{F}^\mu \{f\}\|_\infty |\Omega|^{\frac{1}{b}}.
\]
We indeed obtain by (3.6)
\[
(1 - \varepsilon_\Omega) \| \mathcal{F}^\mu \{f\}\|_b \leq |\Omega|^{\frac{1}{b}} 2^n C_\mu \|f\|_1.
\] (4.8)
Furthermore, by assuming that $f$ is $\varepsilon_T$ concentrated on $T$, in $L^1$-norm, we obtain
\[
\|f\|_1 \leq \|f - P_T f\|_1 + \|P_T f\|_1
\leq \varepsilon_T \|f\|_1 + \left( \int_T |f(t)| dt \right)
\leq \varepsilon_T \|f\|_1 + |T|^{\frac{1}{b}} \|f\|_a.
\]
Where we used the Hölder inequality.
Thus,
\[
(1 - \varepsilon_T) \|f\|_1 \leq |T|^{\frac{1}{b}} \|f\|_a.
\] (4.9)
Combining the results of (4.8) and (4.9) yields the desired result. \(\square\)

4.1. Donoho–Stark bandlimited UP for the CFT

Let $B^a(\Omega)$, $1 \leq a \leq 2$, be the set of functions $g \in L^a(\mathbb{R}^n, Cl_{(p,q)})$, that are bandlimited on $\Omega$, i.e. $Q_\Omega g = g$.

A function $f$ is said $\varepsilon$-bandlimited on $\Omega$ in $L^a$-norm, if there is $g \in B^a(\Omega)$, with
\[
\|f - g\|_a \leq \varepsilon \|f\|_a.
\]

Lemma 4.8. Let $g \in B^a(\Omega)$, $1 \leq a \leq 2$, then
\[
\|P_T g\|_a \leq \frac{C_\mu C_h}{\pi^n} |T|^\frac{a}{2} |\Omega|^\frac{a}{2} \|g\|_a.
\]

Proof. We may assume that $|T| < \infty$ and $|\Omega| < \infty$.

Using inversion formula (3.1), and the assumption that $g \in B^a(\Omega)$, we get
\[
g(t) = \frac{1}{(2\pi)^n} \int_{\Omega} \mathcal{F}^\mu \{g\}(\xi) e^{i\mu u(t,\xi)} d\xi.
\]
By (3.3) and Hölder inequality, one has
\[
|g(t)| \leq \frac{C_\mu}{\pi^n} |\Omega|^{\frac{a}{2}} \|\mathcal{F}^\mu \{g\}\|_b, \quad \frac{1}{a} + \frac{1}{b} = 1.
\]
\[
\leq \frac{C_\mu C_h}{\pi^n} |\Omega|^{\frac{a}{2}} \|g\|_a. \quad (\text{By (3.5)})
\]
Thus, we have
\[ \|P_T g\|_a = \left( \int_T |g(t)|^a dt \right)^{1/a} \leq \frac{C\mu Ch}{\pi^n} |T|^{1/2} |\Omega|^{1/2} \|g\|_a. \]

Theorem 4.9. Let \( f \in L^1(\mathbb{R}^n, Cl_{(p,q)}) \), be \( \varepsilon_T \)-concentrated on \( T \), in \( L^a \)-norm and \( \varepsilon \)-bandlimited on \( \Omega \), in \( L^a \)-norm, \( 1 \leq a \leq 2 \).

Then,
\[ \frac{1 - \varepsilon_{\Omega} - \varepsilon_T}{1 + \varepsilon_{\Omega}} \leq \frac{C\mu Ch}{\pi^n} |T|^{1/2} |\Omega|^{1/2}. \]

Proof. By definition, there exists \( g \in B^a(\Omega) \), such that \( \|f - g\|_a \leq \varepsilon_{\Omega} \|f\|_a \).

This leads to
\[ \|P_T f\|_a \leq \|P_T g\|_a + \|P_T (f - g)\|_a \leq \|P_T g\|_a + \varepsilon_{\Omega} \|f\|_a. \]

From Lemma 4.8, and the fact \( \|g\|_a \leq (1 + \varepsilon_{\Omega}) \|f\|_a \), we get
\[ \|P_T f\|_a \leq \frac{C\mu Ch}{\pi^n} |T|^{1/2} |\Omega|^{1/2} (1 + \varepsilon_{\Omega}) \|f\|_a + \varepsilon_{\Omega} \|f\|_a \]
\[ = \left[ \frac{C\mu Ch}{\pi^n} |T|^{1/2} |\Omega|^{1/2} (1 + \varepsilon_{\Omega}) + \varepsilon_{\Omega} \right] \|f\|_a. \] (4.10)

On the other hand, as \( f \) is \( \varepsilon_T \)-concentrated on \( T \), in \( L^a \)-norm, we have
\[ \|f\|_a \leq \|f - P_T f\|_a + \|P_T f\|_a \leq \varepsilon_T \|f\|_a + \|P_T f\|_a. \]

Then,
\[ \|f\|_a \leq \frac{1}{1 - \varepsilon_T} \|P_T f\|_a. \] (4.11)

By combining (4.10) and (4.11), we conclude the proof. \( \square \)

5. Conclusion

In this paper, we have proven several uncertainty inequalities for the CFT. The first one is the Hausdorff–Young inequality in the Clifford algebra \( Cl_{(p,q)} \), which we think will be an important tool in the future to prove other geometric inequalities for the CFT. The other three inequalities are the generalization of UPs of concentration type, they are \( L^a(\mathbb{R}^n, Cl_{(p,q)}) \) versions. Two are dependent on signal \( f \). However, the third is independent of the bandlimited signal \( f \).

Seeing the fact that Clifford Fourier’s transformations have proved to be very useful tools for applications in color image processing, quantum mechanics, electromagnetism, signal processing, optics, electrodynamics, etc. we expect that our generalization of the uncertainty principles of Donoho–Stark by means of the Clifford algebra will be relevant to applied mathematics.
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