REPRESENTATIONS OF $\mathbb{U}_q\mathfrak{sl}(2|1)$ AT EVEN ROOTS OF UNITY

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Abstract. We construct all projective modules of the restricted quantum group $\mathbb{U}_q\mathfrak{sl}(2|1)$ at an even, $2p$th, root of unity. This $64p^4$-dimensional Hopf algebra is a common double bosonization, $\mathcal{B}(X^*) \otimes \mathcal{B}(X) \otimes H$, of two rank-2 Nichols algebras $\mathcal{B}(X)$ with fermionic generator(s), with $H = \mathbb{Z}_{2p} \otimes \mathbb{Z}_{2p}$. The category of $\mathbb{U}_q\mathfrak{sl}(2|1)$-modules is equivalent to the category of Yetter–Drinfeld $\mathcal{B}(X)$-modules in $\mathcal{C}_\rho = H_{YD}$, where coaction is defined by a universal $R$-matrix $\rho$. As an application of the projective module construction, we find the associative algebra structure and the dimension, $5p^2 - p + 4$, of the $\mathbb{U}_q\mathfrak{sl}(2|1)$ center.

1. Introduction

We study the representation theory of a particular version of the $\mathfrak{sl}(2|1)$ quantum group at even roots of unity. For an integer $p \geq 2$, our $64p^4$-dimensional $\mathbb{U}_q\mathfrak{sl}(2|1)$ at $q = e^{i\pi/p}$ is a double bosonization of any of the rank-2 Nichols algebras defined by the braiding matrices

(1.1) $\begin{pmatrix} -1 & q^{-1} \\ q^{-1} & q^2 \end{pmatrix} \quad$ and $\quad \begin{pmatrix} -1 & -q \\ -q & -1 \end{pmatrix}, \quad q = e^{i\pi/p}.$

The interest in these Nichols algebras is motivated by the fact that they centralize extended chiral algebras (vertex-operator algebras) of logarithmic models of two-dimensional conformal field theory with $\hat{\mathfrak{sl}}(2)_k$ symmetry, at the level $k = \frac{1}{p} - 2$. We recall that Nichols algebras are graded braided Hopf algebras “universally” associated with a braided vector space $X$ (see [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] and the (co)references therein). The “centralizer” relation between vertex-operator algebras and (braided or nonbraided) Hopf algebras is a key ingredient underlying the Kazhdan–Lusztig correspondence [20] [21] and the quest for nice relations between suitable categories defined in

This paper was written as “Logarithmic $\hat{\mathfrak{sl}}(2)$ CFT models from Nichols algebras.” But because it has no direct bearing on logarithmic models (which is left for the future) and in the hope that the algebraic structures discussed here may be interesting in their own right, we do not mention logarithmic CFT in the title.

We refer to [1] [2] [3] [4] for the origin of logarithmic conformal field theory (LCFT) and to [5] for the idea to define LCFT models as centralizers of screening operators. The screenings themselves turn out to generate a Nichols algebra [6], hence the importance of Nichols algebras in this context.
terms of Hopf algebras and representation categories of extended symmetry algebras of the corresponding CFT models.

A “logarithmic” Kazhdan–Lusztig correspondence can be particularly nice (an equivalence of categories) \([22, 23]\) (also see \([24]\) and, among more recent papers, \([25]\)). However, the case studied in \([22, 23]\) is nearly trivial in the language of Nichols algebras, the case of rank one \([26]\); true, the braided world can be quite rich, and already the corresponding nonbraided Hopf algebra is \(U_q\mathfrak{sl}_2\), a not altogether trivial quantum \(\mathfrak{sl}_2\) at an even \((2p)\)th root of unity. It is this \(U_q\mathfrak{sl}_2\) that features in the categorial equivalence with the \((1, p)\) LCFT models \([27, 22]\); the correspondence between Hopf-algebraic and LCFT realms, moreover, extends to modular group representations: those on the quantum group center and on the torus amplitudes of the logarithmic model turn out to coincide.

The ideas of the logarithmic Kazhdan–Lusztig correspondence need to be extended to higher-rank Nichols algebras; this would show, among other things, how much of what we know in the rank-1 case is “accidental,” and which features are indeed generic. Moving to higher-rank Nichols algebras was initiated in \([32]\) and, with precisely the two Nichols algebras defined by braiding matrices \((1.1)\), in \([33]\).

First and foremost, with the structural theory of finite-dimensional Nichols algebras with diagonal braiding completed \([15, 18, 19]\), knowledge about their appropriate representation categories (of Yetter–Drinfeld modules) is highly desirable.\(^3\) In this paper, we address the representation theory of the common double bosonization of the two chosen Nichols algebras, which is the \(64p^4\)-dimensional \(U_q\mathfrak{sl}_2(2|1)\). We first show that the category of \(U_q\mathfrak{sl}_2(2|1)\) modules is equivalent to the category \(\mathfrak{B}(X)\mathcal{YD}_{\varphi}\) of Yetter–Drinfeld \(\mathcal{B}(X)\) modules in \(\mathcal{C}_\varphi = H_\varphi \mathcal{YD}\), where \(\mathfrak{B}(X)\) is the corresponding Nichols algebra and \(H = \mathbb{Z}_{2p} \otimes \mathbb{Z}_{2p}\) is the Cartan subalgebra of \(U_q\mathfrak{sl}_2(2|1)\), with its coaction on \(H_\varphi \mathcal{YD}\) defined by the universal \(R\)-matrix \(\varphi \in H \otimes H\). Our main result is then the construction of projective \(U_q\mathfrak{sl}_2(2|1)\) modules. The ensuing picture is rather involved, and may be a good illustration of the intricacies occurring at roots of unity.

An essential part of the structure of projective modules can be conveniently expressed in terms of directed graphs whose vertices are simple subquotients and the edges are associated with elements of \(\text{Ext}^1\) groups, weighted with some coefficients (finding which is a major part of the existence proof for a given projective module). The paper therefore contains a number of pictures showing graphs of projective modules.

\(^2\)A feature that remarkably survives in the case where the representation category on the LCFT side is not that nice \([28, 29, 30, 31]\).

\(^3\)Once again on the subject of LCFT models, we note that their fusion has a good chance to be described just by the ring structure of a suitable category of finite-dimensional modules, finding which, difficult though it may be, is “infinitely” easier than deriving the fusion algebra directly. The examples where the fusion algebra known or reasonably conjectured in other approaches coincides with the one taken over from the Hopf-algebra side are quite encouraging (see, e.g., \([5, 27, 34, 35]\) and the references therein).
As an application of using the data presented in these graphs, we find the associative algebra structure of the center of $U_q\mathfrak{sl}(2|1)$:

$$Z = Z_{at} \oplus \bigoplus_{j=1}^{(p-1)(p-2)} Z_j \oplus \bigoplus_{j=1}^{4(p-1)} Z_{st}^j$$

where $Z_{at}$, $Z_j^i$, and $Z_{st}^j$ are commutative associative unital algebras; all $Z_{st}^j$ are 1-dimensional, each $Z_j^i$ is 5-dimensional (contains 4 nilpotents in addition to the unit), and $Z_{at}$ is 10$p - 2$ dimensional (10$p - 3$ nilpotents in addition to the unit).

In Sec. 2, we define our Hopf algebra $U(X) \equiv \overline{U_q\mathfrak{sl}(2|1)}$ and discuss some of its simple properties (including Casimir elements and the universal $R$-matrix). In Sec. 3, we prove that the category of its modules is equivalent to $U(X)$ $\overline{\mathbb{D}}$. In Sec. 4, we classify its simple modules, describing them quite explicitly. We continue in Sec. 4 with listing the Ext$^1$ spaces for the simple modules. We then construct the projective $U(X)$ modules in Sec. 6.

As an application of our treatment of projective modules, we find the center of $U(X)$ in Sec. 7. In an attempt to improve the readability of this inevitably technical paper, we isolate some computational details in the appendices.

We use $q$-integers and factorials defined as

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [1] \ldots [n],$$

all of which are assumed specialized to $q = q$ in (1.1).

All (co)modules in this paper are finite-dimensional.

2. The Hopf Algebra $U(X)$

2.1. The notation $U(X)$ for our 64$p^4$-dimensional quantum group $U_q\mathfrak{sl}(2|1)$ is a legacy of the Nichols-algebra setting, where $X$ in $\mathcal{B}(X)$ is a two-dimensional braided vector space—in our case, specifically, the one with diagonal braiding defined by any of the two braiding matrices in (1.1). The Hopf algebra $U(X)$ was derived from each of these two Nichols algebras in [33].

2.1.1. The Hopf algebra $U(X) \equiv \overline{U_q\mathfrak{sl}(2|1)}$. Our $U(X)$ is the algebra on $B, F, K, C, E$ with the relations listed in (2.1)–(2.3). We first identify an important Hopf subalgebra in $U(X)$, the restricted quantum group $U_q\mathfrak{sl}(2)$. It is generated by $E, K$, and $F$, with the relations

$$(2.1) \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad KE = q^2EK,$$

$$F^p = 0, \quad E^p = 0, \quad K^{2p} = 1.$$ 

Next, $U_q\mathfrak{sl}(2)$ and $k$ generate a larger algebra $\overline{U_q\mathfrak{sl}(2)}$ with further relations

$$(2.2) \quad kF = qFK, \quad kE = q^{-1}Ek, \quad k^{2p} = 1, \quad kK = Kk.$$
The other relations in $U(X)$ — those involving fermions $B$ and $C$ — are

\begin{equation}
KB = qBK, \quad kB = -Bk, \quad KC = q^{-1}CK, \quad kC = -Ck,
\end{equation}

\begin{align*}
B^2 &= 0, \quad BC - CB = \frac{k-q^{-1}}{q-q^{-1}}, \quad C^2 = 0, \\
FC - CF &= 0, \quad BE - EB = 0, \\
FFB - [2]FBF + BFF &= 0, \quad EEC - [2]ECE + CEE = 0.
\end{align*}

The Hopf-algebra structure of $U(X)$ is furnished by the coproduct, antipode, and counit given by

\begin{align*}
\Delta(F) &= F \otimes 1 + K^{-1} \otimes F, \quad \Delta(E) = E \otimes K + 1 \otimes E, \\
\Delta(B) &= B \otimes 1 + k^{-1} \otimes B, \quad \Delta(C) = C \otimes k + 1 \otimes C, \\
S(B) &= -kB, \quad S(F) = -KF, \quad S(C) = -Ck^{-1}, \quad S(E) = -EK^{-1}, \\
\varepsilon(B) &= 0, \quad \varepsilon(F) = 0, \quad \varepsilon(C) = 0, \quad \varepsilon(E) = 0,
\end{align*}

with $k$ and $K$ group-like.

By the $U(X)$ generators in what follows, we always mean $B, F, C, E, k, K$.

2.1.2. The second Hopf algebra structure [33]. The Hopf algebra $U(X)$ admits a non-trivial “twist”—an invertible normalized 2-cocycle (see B.3)

$$\Phi = 1 \otimes 1 + (q^{-1})Bk \otimes Ck^{-1} \in U(X) \otimes U(X)$$

 twisting by which gives rise to the second coalgebra structure

$$\tilde{\Delta}(x) = \Phi^{-1} \Delta(x) \Phi.$$

(Two coalgebra structures naturally come from the two underlying Nichols algebras.)

For the $U(X)$ generators chosen above, the coproducts are

\begin{align*}
\tilde{\Delta}(F) &= F \otimes 1 + K^{-1} \otimes F + (q-q^{-1})FBk \otimes Ck^{-1} + (1-q^2)BFk \otimes Ck^{-1}, \\
\tilde{\Delta}(B) &= B \otimes k^{-2} + k^{-1} \otimes B, \\
\tilde{\Delta}(E) &= 1 \otimes E + E \otimes K + (q-q^{-1})Bk \otimes ECk^{-1} + (1-q^2)Bk \otimes CEk^{-1}, \\
\tilde{\Delta}(C) &= C \otimes k + k^2 \otimes C.
\end{align*}

We note that the new coproduct has the following skew-primitive elements:

\begin{align*}
\tilde{\Delta}(Bk) &= 1 \otimes Bk + Bk \otimes k^{-1}, \\
\tilde{\Delta}(FB - qBF) &= (FB - qBF) \otimes 1 + K^{-1}k^{-1} \otimes (FB - qBF), \\
\tilde{\Delta}(Ck^{-1}) &= Ck^{-1} \otimes 1 + k \otimes Ck^{-1}, \\
\tilde{\Delta}(EC - qCE) &= 1 \otimes (EC - qCE) + (EC - qCE) \otimes Kk.
\end{align*}
2.2. PBW basis in $U(X)$. We have a linear-space isomorphism

$$U_\succ \otimes U_\prec \rightarrow U(X),$$

where $U_\succ$ is the subalgebra in $U(X)$ generated by $E$ and $C$, and $U_\prec$ is the subalgebra generated by $F$, $B$, $K$, and $k$. A PBW basis in $U_\prec$ (which we refer to as the PBW basis in $U_\prec$) can be chosen as $F^nK^ik^j, F^nBF^ik^j, F^nBFK^ik^j$, where $0 \leq n \leq p - 1$ and $0 \leq i, j \leq 2p - 1$; the PBW basis in $U_\succ$ can be chosen similarly, as $E^n$, $E^{n+1}C$, $E^nCE$, $E^nCEC$, where $0 \leq n \leq p - 1$. The PBW basis in $U(X)$ is then given by the product of these two PBW bases.

2.3. Casimir elements. There are two elements of a particular form in the center of $U(X)$, which we call Casimir elements:

$$C_1 = EFFK^2 + CBK^2k^3 - q(EC - qCE)(FB - q^{-1}BF)Kk^3 + \frac{q}{(q - q^{-1})^2}k^2 + \frac{q^{-1}}{(q - q^{-1})^2}K^2k^2 - \frac{q^{-1}}{(q - q^{-1})^2}K^2k^4,$$

$$C_2 = EFFK^{-1}k^{-2} + CBK^{-2}k^{-3} - q^{-1}(EC - q^{-1}CE)(FB - qBF)K^{-1}k^{-3} + \frac{q^{-1}}{(q - q^{-1})^2}k^{-2} + \frac{q}{(q - q^{-1})^2}K^{-2}k^{-2} - \frac{q}{(q - q^{-1})^2}K^{-2}k^{-4}.$$

That they are central is verified directly.

2.3.1. Proposition. Each of the Casimir elements satisfies a minimal polynomial relation of degree $p^2 - 2p + 4$:

$$\left( C_1 - \frac{q}{(q - q^{-1})^2} \right)^3 \prod_{s=1}^{p-1} \prod_{r=1}^{s-1} \left( C_1 - \frac{q^{1-4r}(q^{2r} - q^{2s} + q^{2r+2s})}{(q - q^{-1})^2} \right) \prod_{r=1}^{p-1} \left( C_1 - \frac{q^{1-4r}(2q^{2r} - 1)}{(q - q^{-1})^2} \right) = 0,$$

and the minimal relation for $C_2$ is obtained from here by replacing $q \rightarrow q^{-1}$.

This is an immediate corollary of our construction of projective $U(X)$-modules in Sec.6. We simply take the product of $(C_i - \lambda)$ factors over all different eigenvalues $\lambda$ of a given $C_i$ on the (linkage classes of) projective modules, with each factor raised to the power given by the corresponding Jordan cell size (three for the atypical linkage class, two for each of the typical linkage classes, and one for each Steinberg module/class; see 5.3).

A somewhat more involved derivation shows that the two Casimir elements satisfy a degree-$p$ polynomial relation,

$$\sum_{i=1}^{p} \left( \frac{-1}{i} \right)^i \frac{(p - 1)^i}{i - 1} (q - q^{-1})^{2i} (q^{-i}C_1^i - q^iC_2^i) = 0.$$

The full list of (“mixed”) relations satisfied by $C_1$ and $C_2$ is currently unknown.
2.4. Quasitriangular and related structures.

2.4.1. Theorem. The algebra $U(X)$ is quasitriangular, with the universal $R$-matrix given by

$$R = \rho \tilde{R},$$

(2.5)

$$\rho = \frac{1}{(2p)^2} \sum_{i=0}^{2p-1} \sum_{j=0}^{2p-1} \sum_{m=0}^{2p-1} \sum_{n=0}^{2p-1} (-1)^{jn} q^{-2im+jm+in} K^i k^j \otimes K^m k^n,$$

(2.6)

$$\tilde{R} = \sum_{a=0}^{p-1} \frac{q^{\frac{1}{2}} (q^{-1} a)!}{[a]!} E^a \otimes F^a \left( 1 \otimes 1 - (q - q^{-1}) C \otimes B \right)$$

$$\times \left( 1 \otimes 1 + q(q - q^{-1}) C \otimes \tilde{B} \right) \left( 1 \otimes 1 + (q - q^{-1})^3 \tilde{C} \otimes \tilde{B} \right),$$

where

$$\tilde{C} = E C - q^{-1} C E, \quad \tilde{B} = F B - q^{-1} F B,$$

$$\tilde{C} = C \tilde{C} = CEC, \quad \tilde{B} = B \tilde{B} = BF B.$$

This must be possible to extract from [36]; we give the proof by direct calculation in [B.1].

The category of $U(X)$ modules is therefore braided, with the braiding $Y \otimes Z \rightarrow Z \otimes Y$ given by $y \otimes z \mapsto R^{(2)} z \otimes R^{(1)} y$ (where we standardly write $R = R^{(1)} \otimes R^{(2)}$).

We let $U(X)\text{-}\text{MOD}$ denote the braided monoidal category of $U(X)$-modules.

2.4.2. We recall the general properties of the universal $R$-matrix for further reference: $R \Delta(x) = \Delta^{op}(x) R$, i.e., $R^{(1)} x^i \otimes R^{(2)} x^j = x^i \otimes R^{(1)} x^j$ for all $x$ in the algebra, and

$$R^{(1)'} \otimes R^{(1)''} \otimes R^{(2)} = R^{(1)} \otimes P^{(1)} \otimes R^{(2)},$$

$$R^{(1)} \otimes R^{(2)'} \otimes R^{(2)''} = R^{(1)} P^{(1)} \otimes P^{(2)} \otimes R^{(2)},$$

(with $R = R^{(1)} \otimes R^{(2)} = P^{(1)} \otimes P^{(2)}$, and so on).

2.4.3. Drinfeld map. With the above $R$-matrix, we introduce the so-called M[onodromy] “matrix”

$$M = R_{21} R \in U(X) \otimes U(X).$$

It is known to give rise to the Drinfeld map from the space $\mathcal{C}h$ of $q$-characters to the center $Z$ of $U(X)$:

$$\chi : \mathcal{C}h \rightarrow Z : \alpha \mapsto \alpha \otimes \text{id}(M) = \alpha(M_1)M_2,$$

(2.7)

where we standardly write $M = M_1 \otimes M_2$. We recall that $\mathcal{C}h$ is the space of elements of $U(X)^*$ that are invariant under the left coadjoint action of $U(X)$. 
2.4.4. Proposition. For the R-matrix in 2.4.1 the M-matrix can be written as

\[ M = \tilde{M} \tilde{\rho} \tilde{R}, \]

where

\[
\tilde{M} = \sum_{a=0}^{\frac{p-1}{2}} q^{-\frac{1}{2}a^2 - \frac{1}{2}a(q-q^{-1})a} \frac{F^a K^a \otimes E^a K^{-a}}{[a]!} (1 \otimes 1 + (q - q^{-1})Bk \otimes Ck^{-1}) \]
\[
\times (1 \otimes 1 - (q^2 - 1) \tilde{B}k \otimes \tilde{C}k^{-1}k^{-1}) (1 \otimes 1 + q^2 (q - q^{-1})^2 \tilde{B}k^2 \otimes \tilde{C}k^{-1}k^{-2})
\]
\[
\tilde{\rho} = \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{i'=0}^{p-1} \sum_{j'=0}^{p-1} q^{2i' + j + 2i' j' - 4i' j'} K^{2i'j} k^2 \otimes K^{2j'k} k^2.
\]

We show this in B.2

2.4.5. The quasitriangular structure corresponding to \( \tilde{\Delta} \) in 2.1.2 is given by

\[ \tilde{R} = \Phi_2^{-1} R \Phi, \]

as readily follows from \( R \Delta(x) = \Delta^\text{op}(x) R \). Hence, the corresponding M-matrix is

\[ \tilde{M} = \Phi^{-1} M \Phi. \]

We want to compare the two Drinfeld maps \( \chi : \mathfrak{ch} \rightarrow \mathfrak{g} \) and \( \tilde{\chi} : \mathfrak{ch} \rightarrow \mathfrak{g} \), where \( \mathfrak{ch} \) is the space of elements that are invariant under the “tilded” coadjoint action.

The two Drinfeld maps turn out to coincide in the sense that the diagram

\[ \begin{array}{ccc}
\mathfrak{ch} & \xrightarrow{\chi} & \mathfrak{ch} \\
\downarrow & & \downarrow \\
\mathfrak{g} & \xrightarrow{\tilde{\chi}} & \mathfrak{g}
\end{array} \]

is commutative, where the horizontal arrow is a linear isomorphism \( \beta \mapsto (\beta \mapsto \xi) = \beta(\xi) \), where \( \xi = S(U^{-1}) U \) and \( U = S(\Phi_1 \Phi_2) \). We prove this in the general case in B.3.1

With the above \( \Phi \), it readily follows that

\[ U = 1 - (q - q^{-1}) Bk^{-1} \quad \text{and} \quad \xi = k^{-2} \]

3. \( \mathbf{U}(X) \) modules and Yetter–Drinfeld \( \mathcal{B}(X) \) modules

We show that the category of \( \mathbf{U}(X) \) modules is equivalent to a category of Yetter–Drinfeld modules of the corresponding Nichols algebra. The exact statement is in 3.2 below. We begin with briefly recalling the relation between \( \mathbf{U}(X) \) and Nichols algebras.
3.1. **$\mathbf{U}(X)$ as a double bosonization.** The Hopf algebra $\mathbf{U}(X)$ is a double bosonization \[^{33}\] (also see \[^{37}^{17}^{38}\]) of the Nichols algebra $\mathfrak{B}(X)$ of a two-dimensional braided vector space $X$ with a basis $(B,F)$ such that the corresponding braiding matrix is $Q_a$ in (1.1) \[^{4}\] This $\mathfrak{B}(X)$ is the quotient \[^{18}\]

\[(3.1)\]

$$\mathfrak{B}(X) = T(X)/(BF^2 - [2]FBBF + F^2, B^2, F^p)$$

for $p \geq 3$ (and $\mathfrak{B}(X) = T(X)/(B^2, BF^2 - FBFB, F^2)$ if $p = 2$), with $\dim \mathfrak{B}(X) = 4p$. Constructing its double bosonization requires choosing a cocommutative ordinary Hopf algebra $H$ such that $\mathfrak{B}(X) \in H \text{YD}$. Specifically, we take $H = \mathbb{Z}_{2p} \otimes \mathbb{Z}_{2p}$, with the generators $g_1 = k$ and $g_2 = K$ acting on the $\mathfrak{B}(X)$ generators $F_1 = B$ and $F_2 = F$ as $g_i \cdot F_j = q_{i,j} F_j$, where $q_{i,j}$ are the entries of the braiding matrix; $H$ coacts as $F_i \mapsto g_i \otimes F_i$. The double bosonization

\[(3.2)\]

$$\mathbf{U}(X) = \mathfrak{B}(X^*) \#' \mathfrak{B}(X) \# H$$

(where the right-hand side is a tensor product in $H \text{YD}$) then contains the Hopf subalgebra $\mathfrak{B}(X) \# H$ given by the standard, “single,” bosonization \[^{39}\], and similarly for $\mathfrak{B}(X^*) \#' H$, but with the $H$ action and coaction changed by composing each with the antipode (hence the prime); the cross relations are $\hat{F}_i F_j - q_{i,j} F_j \hat{F}_i = \delta_{i,j}(1 - (g_{i,j})^2)$ in terms of the dual basis $\hat{F}_i$ in $X^*$. After a suitable change of basis, this yields our $\mathbf{U}(X)$ \[^{33}\].

Naturally, $H$ is identified with the Hopf subalgebra in $\mathbf{U}(X)$ generated by $k$ and $K$. Moreover, we refine (2.4) by introducing the linear space isomorphism

\[(3.3)\]

$$U_< \otimes H \otimes U_> \rightarrow \mathbf{U}(X),$$

where $U_<$ is the subalgebra in $\mathbf{U}(X)$ generated by $F$ and $B$. Each of the algebras $U_<, H,$ and $U_>$ is a Hopf algebra.

3.1.1. **Definition.** The category $\mathfrak{C}_\rho$ is the category $H \text{YD}$ of Yetter–Drinfeld $H$-modules with the coaction

\[(3.4)\]

$$\delta : z \mapsto z_{(-1)} \otimes z_{(0)} = \rho^{(2)} \otimes \rho^{(1)} \cdot z,$$

where $\rho$ is given by (2.5).

Evidently, the braiding in $\mathfrak{C}_\rho$ is given by

\[(3.5)\]

$$u \otimes v \mapsto u_{(-1)} \cdot v \otimes u_{(0)} = \rho^{(2)} \cdot v \otimes \rho^{(1)} \cdot u.$$

3.2. **Theorem.** The category $\mathbf{U}(X)$-MOD is equivalent as a braided monoidal category to the category $\mathfrak{B}(X) \text{YD} \mathfrak{C}_\rho$ of Yetter–Drinfeld $\mathfrak{B}(X)$-modules in $\mathfrak{C}_\rho$.

\[^{4}\]We select the first braiding matrix in (1.1) because we mainly describe $\mathbf{U}(X)$ with the generators chosen as $B, F, C, E, k, K$, not those that are skew-primitive with respect to the second coproduct in 2.1.2.
We prove this in the rest of this section, in several steps. We first construct a transmutation functor \( \mathcal{F}_T : \mathcal{U}(X)\text{-MOD} \to \mathcal{Y}_D\mathcal{C}_\rho \) (cf. [40, 41]). Its inverse, a double bosonization functor \( \mathcal{F}_{DB} : \mathcal{Y}_D\mathcal{C}_\rho \to \mathcal{U}(X)\text{-MOD} \), is then constructed as a composition of the bosonization functor \( \mathcal{F}_B \) and the functor \( \mathcal{F}_{DD} \) sending Yetter–Drinfeld modules to modules of the Drinfeld double. Both \( \mathcal{F}_T \) and \( \mathcal{F}_{DB} \) are vector-space-preserving functors. We then verify that their compositions are indeed equivalent to the corresponding identity functors.

3.3. Proposition. There is a vector-space-preserving braided monoidal functor \( \mathcal{F}_T \) sending each \( \mathcal{U}(X) \)-module into a Yetter–Drinfeld \( \mathcal{B}(X) \)-module in \( \mathcal{C}_\rho \).

We prove this in \[3.3.1-3.3.7\] Whenever possible (and this is most often possible), we do not assume the “ground” Hopf algebra \( H \) to be cocommutative (nor is it assumed commutative). This in fact clarifies the calculations, highlighting the “true” reasons why certain identities hold. We frequently use the universal \( R \)-matrix properties in [2.4.2] for \( \rho \in H \otimes H \) in particular.

3.3.1. \( U_< \) is an algebra in \( \mathcal{C}_\rho \). We make \( U_< \) into an algebra object in \( \mathcal{H}_D \mathcal{Y} \) by defining the left (adjoint) action and left coaction as \(\begin{align*}
h \triangleright x &= h'xS(h''), \\
\delta : x &\mapsto x_{(-1)} \otimes x_{(0)} = \rho^{(2)} \otimes \rho^{(1)} \triangleright x.
\end{align*}\)

The Yetter–Drinfeld axiom \((h \triangleright x)_{(-1)} \otimes (h \triangleright x)_{(0)} = h'x_{(-1)}S(h'') \otimes h'' \triangleright x_{(0)}\) is then immediate to verify. That the product \( U_< \otimes U_< \to U_< \) is a \( \mathcal{C}_\rho \)-morphism is also evident.

3.3.2. \( U_< \) is a Hopf algebra in \( \mathcal{C}_\rho \). We next define the coaction
\begin{equation}
\Delta : U_< \to U_< \otimes U_< : x \mapsto x \hat{\otimes} \hat{x} = x' S(\rho^{(2)}) \otimes \rho^{(1)} \triangleright x''.
\end{equation}

A priori, the right-hand side is an element not of \( U_< \otimes U_< \) but of \( U_< \otimes U_< \). However, it is immediately verified for the generators \( A = F, B \) that
\[\Delta(A) = A \otimes 1 + 1 \otimes A \in U_< \otimes U_<,\]
and we then conclude that \( \Delta(U_<) \subset U_< \otimes U_< \) from the main axiom of braided Hopf algebras,
\begin{equation}
\Delta(xy) = x_\hat{\otimes}(x_{(-1)} \triangleright y_{(2)}) \otimes x_{(0)} \hat{\otimes} y_\hat{\otimes}.
\end{equation}

This last identity is a relatively standard statement [40, 41], but we prove it here for completeness:
\[\text{r.h.s. of } (3.7) = x' S(\rho^{(2)})(\sigma^{(2)} \triangleright y' S(\tau^{(2)})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'')(\tau^{(1)} \triangleright y'')\]

We do not indicate the embedding \( \iota : H \to \mathcal{U}(X) \) explicitly, and write \( h'xS(h'') \) for what should be \( \iota(h) \triangleright x \), etc.; accordingly, we do not use a special symbol for the antipode in \( H \), and so on.
\[= x' S(\rho^{(2)'})(\rho^{(2)''} \triangleright y' S(\tau^{(2)})) \otimes (\rho^{(1)} \triangleright x'')(\tau^{(1)} \triangleright y'') \]
\[= x' y' S(\rho^{(2)'} \tau^{(2)}) \otimes (\rho^{(1)} \triangleright x'')(\tau^{(1)} \triangleright y'') \]
\[= x' y' S(\rho^{(2)}) \otimes (\rho^{(1)} \triangleright x'')(\rho^{(1)''} \triangleright y'') \]
\[= x' y' S(\rho^{(2)}) \otimes (\rho^{(1)} \triangleright x'y'') = \text{l.h.s. of (3.7)}. \]

Also, $\Delta$ is an $H$-module morphism. For the $H$-coaction, we have to prove that
\[x_{(0)} \otimes x_{(0)} \overset{\Delta}{=} x_{(0)} x_{(0)} \]
where we calculate
\[\text{r.h.s.} = \tau^{(2)} \sigma^{(2)} \otimes (\sigma^{(1)} \triangleright x' S(\rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'') \]
\[= \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'' \rho^{(2)'} \triangleright x') \]
\[= \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'' \rho^{(2)'} \triangleright x') \]
\[= \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\rho^{(2)'} \tau^{(1)''} \rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'' \rho^{(2)'} \triangleright x'), \]
and
\[\text{l.h.s.} = \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x'' \rho^{(2)'} \triangleright x') \]
\[= \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})) \]
\[= \tau^{(2)} \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})) \otimes (\sigma^{(1)} \rho^{(1)} \triangleright x' S(\tau^{(1)''} \rho^{(2)'})), \]

which is the same. For the $H$ action, we readily see that $h \triangleright \Delta(x) = \Delta(h \triangleright x)$ using the fact that $H$ is cocommutative. We thus conclude that $U_{\prec}$ is a Hopf algebra object in $\mathcal{E}_{\rho}$. 3.3.3. Remark. As in [12], we can characterize $U_{\prec}$ inside $U_{\leq}$ as
\[U_{\prec} = \{ x \in U_{\leq} \mid x' \otimes \pi(x'') = x \otimes 1 \} \]
with a projection (Hopf-algebra map) $\pi : U_{\leq} \rightarrow H$. Then
\[\pi(x') \otimes x'' = \rho^{(2)} \otimes \rho^{(1)} \triangleright x. \]

3.3.4. $U_{\prec} = \mathfrak{B}(X)$. In $\mathcal{E}_{\rho}$, the braiding matrix for the generators $B$ and $F$ is exactly $Q_a$ in (1.1). Hence, $U_{\prec}$ endowed with the coproduct $\Delta$ (and with the antipode $x \mapsto \rho^{(2)} S(\rho^{(1)} \triangleright x)$, which we do not discuss in any further detail for brevity) is the Nichols algebra $\mathfrak{B}(X)$ of the two-dimensional braided linear space $X$ with the braiding matrix $Q_a$.

We write $\mathfrak{B}(X)$ instead of $U_{\prec}$ in what follows.

3.3.5. $U(X)$ modules are objects of $\mathcal{E}_{\rho}$. Let $\mathcal{Z} \in U(X)\text{-MOD}$, with a $U(X)$ action $x \otimes z \mapsto x \cdot z$. By restriction, $\mathcal{Z}$ is an $H$-module. We make $\mathcal{Z}$ into an object in $\mathcal{E}_{\rho}$ by endowing it with the $H$-coaction (3.4).
3.3.6. U(X) modules are objects of \( \mathfrak{H}_q \mathfrak{h}_\mathfrak{d} \). On each \( U(X) \)-module \( Z \), which is a \( \mathfrak{B}(X) \)-module by restriction, we also define the \( \mathfrak{B}(X) \)-coaction

\[
\delta : Z \rightarrow \mathfrak{B}(X) \otimes Z
\]

(3.8)

\[
z \mapsto \, z \otimes \tilde{z} = R^{(2)} S(\rho^{(2)}) \otimes \rho^{(1)} R^{(1)}. \, z = \rho^{(2)} \circ \tilde{R}^{(2)} \otimes \rho^{(1)} \tilde{R}^{(1)}. \, z,
\]

where \( \tilde{R} \) is defined in (2.6); the second form shows that \( \delta \) indeed maps to \( \mathfrak{B}(X) \otimes Z \). The coaction property

\[
z \otimes \tilde{z} \otimes \tilde{z} = \tilde{z} \otimes \tilde{z} \otimes \tilde{z} = \tilde{z} \otimes \tilde{z} \otimes \tilde{z}
\]

is established by direct calculation. We must also verify that \( \delta \) is an \( \mathfrak{H}_q \mathfrak{d} \) morphism. We calculate, writing \( \rho = \rho^{(1)} \otimes \rho^{(2)} = \sigma^{(1)} \otimes \sigma^{(2)} = \tau^{(1)} \otimes \tau^{(2)} = \ldots \) (and assuming neither cocommutativity nor commutativity of \( H \)):

\[
h(\delta z) = (h' \circ R^{(2)} S(\rho^{(2)})) \otimes h'' \rho^{(1)} R^{(1)}. \, z
\]

\[
= h'R^{(2)} S(h'' \rho^{(2)}) \otimes \rho^{(1)} h'' R^{(1)}. \, z
\]

\[
= h'R^{(2)} S(h'' \rho^{(2)}) \otimes \rho^{(1)} R^{(1)}. \, z
\]

\[
= R^{(2)} h'' S(\rho^{(2)} h'' \rho^{(1)} h'). \, z
\]

\[
= R^{(2)} \otimes \rho^{(1)} R^{(1)}. \, z = \delta(h \cdot z),
\]

and

\[
z_{-1(1)} \tilde{z}_{-1(0)} \otimes z_{-1(0)} \tilde{z}_{0(0)} = \rho^{(2)} \sigma^{(2)} \otimes (\rho^{(1)} \circ \tilde{R}^{(2)} S(\tau^{(2)})) \otimes \sigma^{(1)} \tau^{(1)} R^{(1)}. \, z
\]

\[
= \rho^{(2)} \otimes (\rho^{(1)} \circ \tilde{R}^{(2)} S(\tau^{(2)})) \otimes \rho^{(1)} R^{(1)}. \, z
\]

\[
= \rho^{(2)} \otimes (\rho^{(1)} \circ \tilde{R}^{(2)} S(\tau^{(2)})) \otimes \rho^{(1)} R^{(1)}. \, z
\]

\[
= \rho^{(2)} \otimes \rho^{(1)} R^{(2)} S(\tau^{(2)} \rho^{(1)} R^{(1)}. \, z
\]

\[
= \rho^{(2)} \otimes R^{(2)} S(\tau^{(2)} \rho^{(1)} R^{(1)}. \, z
\]

\[
= \rho^{(2)} \otimes \rho^{(1)} R^{(1)}. \, z = z_{-1} \otimes z_{0(0)} \tilde{z}_{0(0)}.
\]

We next verify the Yetter–Drinfeld axiom relating the \( \mathfrak{B}(X) \) action and coaction, which in the standard graphic notation (see, e.g., [42]) is expressed as

\[
\begin{array}{c}
\mathfrak{B}(X) Z \\
\downarrow \Delta \\
\mathfrak{B}(X) \mathfrak{B}(X) Z
\end{array}
\]

\[
\begin{array}{c}
\mathfrak{B}(X) Z \\
\downarrow \delta \\
\mathfrak{B}(X) \mathfrak{B}(X) Z
\end{array}
\]

With the above \( \Delta, \delta \), and braiding, this is equivalent to

\[
R^{(2)} S(\sigma^{(2)})(\pi^{(2)} \tau^{(1)} \rho^{(1)} \circ x'' \otimes \pi^{(1)} \sigma^{(1)} R^{(1)} x' S(\rho^{(2)}) \tau^{(2)}. \, z
\]

\[
= x' S(\rho^{(2)})(\sigma^{(2)} \circ R^{(2)} S(\tau^{(2)})) \otimes (\sigma^{(1)} \rho^{(1)} \circ x'' \tau^{(1)} R^{(1)}. \, z
\]
for $x \in \mathcal{B}(X)$ and $z \in \mathcal{Z}$. We prove the last identity:

$$
\text{l.h.s.} = R^{(2)} S(\sigma^{(2)})(\pi^{(2)} \rho^{(1)} \triangleright x'') \otimes \pi^{(1)} \sigma^{(1)} R^{(1)} x' S(\rho^{(2)}') \rho^{(2)''} \cdot z
$$

$$
= R^{(2)} S(\sigma^{(2)})(\pi^{(2)} \triangleright x'') \otimes \pi^{(1)} \sigma^{(1)} R^{(1)} x' \cdot z
$$

$$
= R^{(2)} S((\sigma^{(2)})'((\sigma^{(2)})'' \triangleright x'') \otimes \sigma^{(1)} R^{(1)} x' \cdot z
$$

$$
= R^{(2)} x' S(\sigma^{(2)}) \otimes \sigma^{(1)} R^{(1)} x' \cdot z
$$

$$
= x' R^{(2)} S(\sigma^{(2)}) \otimes \sigma^{(1)} R^{(1)} x' \cdot z,
$$

but

$$
\text{r.h.s.} = x' S(\rho^{(2)}') (\rho^{(2)''} \triangleright R^{(2)} S(\tau^{(2)})) \otimes (\rho^{(1)} \triangleright x'') \tau^{(1)} R^{(1)} \cdot z
$$

$$
= x' R^{(2)} S(\rho^{(2)} \tau^{(2)}) \otimes (\rho^{(1)} \triangleright x'') \tau^{(1)} R^{(1)} \cdot z
$$

$$
= x' R^{(2)} S(\rho^{(2)}) \otimes (\rho^{(1)'} \triangleright x'') \rho^{(1)''} R^{(1)} \cdot z,
$$

which is the same.

3.3.7. The functor $\mathcal{F}_T$ is monoidal. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two $\mathcal{U}(X)$ modules. When viewed as $\mathcal{B}(X)$ modules, their tensor product carries the $\mathcal{B}(X)$ action that actually evaluates as the $\mathcal{U}(X)$ action on a tensor product of its modules:

$$
x \cdot (y \otimes z) = (x \cdot \rho^{(2)} \cdot y) \otimes (\rho^{(1)} \triangleright x') \cdot z
$$

$$
= (x' S(\tau^{(2)}) \rho^{(2)} \cdot y) \otimes (\rho^{(1)} \tau^{(1)} \triangleright x'') \cdot z
$$

$$
= (x' S(\rho^{(2)}') \rho^{(2)''} \cdot y) \otimes (\rho^{(1)} \triangleright x'') \cdot z
$$

$$
= x' \cdot y \otimes x'' \cdot z.
$$

Essentially the same calculation shows that the functor is braided: braiding $c_{\mathcal{Y}, \mathcal{Z}}$ in the category $\mathcal{B}(X)\mathcal{Y}\mathcal{D}_{\rho}$ evaluates as the braiding in $\mathcal{U}(X)\text{-MOD}$:

$$
(y \cdot \rho^{(2)} \cdot z) \otimes \rho^{(1)'} \cdot y \cdot z = R^{(2)} \cdot z \otimes R^{(1)} \cdot y.
$$

3.4. The functor $\mathcal{F}_B$. To establish a functor inverse to $\mathcal{F}_T$, we first construct a functor $\mathcal{F}_B : \mathcal{B}(X)\mathcal{Y}\mathcal{D}_{\rho} \to \mathcal{B}(X)\mathcal{Y}\mathcal{D}_{\rho}$, which essentially amounts to “a Radford formula for comodules.”

In this subsection, we do not need to assume the existence of a universal $R$-matrix $\rho \in H \otimes H$; therefore, $H$ is an arbitrary Hopf algebra (with bijective antipode).

3.4.1. Reminder: Radford’s biproduct. If $H$ is an ordinary Hopf algebra and $\mathcal{R}$ is an algebra object in $\mathcal{H}\mathcal{Y}\mathcal{D}$ (a braided Hopf algebra), then Radford’s biproduct $[39]$ (bosonization $[43]$) defines the structure of an ordinary Hopf algebra on the smash product $\mathcal{R} \# H$. Its multiplication, comultiplication, and antipode are

$$
(r \# h)(s \# g) = r(h' \triangleright s) \# h'' g,
$$
(3.10) \[ \Delta : r \not\in H \mapsto (r \not\in H)^{(1)} \otimes (r \not\in H)^{(2)} = \left(r^1 \# r^2_{\# H'} \right) \otimes \left(r^2_{\# H'} \right) \]

(3.11) \[ S(r \not\in H) = (1 \# s(r_{\# H})) \left(S(r_{\# H}) \right) \]

(where \( s \) is the antipode of \( H \) and \( S \) is the antipode of \( R \)).

3.4.2. Our aim is to extend bosonization to Yetter–Drinfeld modules. In the same setting as above, let \( Y \) be a Yetter–Drinfeld \( R \)-module in \( ^R_H YD \). We write \( r \mapsto y \) and \( h, y \) for the \( R \) and \( H \) actions, and \( y \mapsto y_{\not\in H} \otimes y_{\# H} \in R \otimes Y \) and \( y \mapsto y_{\not\in H} \otimes y_{\# H} \in H \otimes Y \) for the coactions.

3.4.3. Proposition. Let \( R \) be a Hopf algebra in \( C = \mathbb{H}_H YD \) and \( Y \in ^R_H YD_C \). Then \( Y \) is a Yetter–Drinfeld \( R \# H \)-module with the \( R \# H \) action

(3.12) \[ (r \not\in H) \mapsto y = r \mapsto (h \cdot y), \]

and coaction

(3.13) \[ \delta : y \mapsto y_{\not\in H} \otimes y_{\# H} = \left(y_{\not\in H} \# y_{\# H}(1)\right) \otimes y_{\# H}(0) \in \mathbb{R} \# H \otimes Y. \]

This defines a vector-space-preserving monoidal braided functor

\[ \mathcal{F}_B : ^R_H YD_C \to ^{R \# H}_R \mathbb{H}_H YD \]

to the category of left–left Yetter–Drinfeld \( R \# H \)-modules.

3.4.4. Proof of 3.4.3. First, the coaction property is readily verified for (3.13) using the fact that the \( R \)-coaction is an \( H \)-comodule morphism. Next, we must show the Yetter–Drinfeld condition in \( (R \# H) \otimes Y \),

(3.14) \[ \left( (r \not\in H)^{(1)} \mapsto y_{\not\in H} \right) \otimes \left( (r \not\in H)^{(2)} \mapsto y_{\not\in H} \right) = (r \not\in H)^{(1)} \otimes (r \not\in H)^{(2)} \mapsto y_{\not\in H}, \]

where the right-hand side is simple:

\[ = \left(r^1 \# r^2_{\# H'} \right) \left(y_{\not\in H} \# y_{\# H}(1)\right) \otimes \left(r^2_{\# H'} \right) \cdot y_{\# H}(0). \]

As regards the left-hand side, we first note that for \( z \in \mathbb{H} \), \( s \in R \), and \( g \in H \),

\[ z_{\not\in H}(s \# g) \otimes z_{\not\in H} = \left(z_{\not\in H} \# z_{\# H}(1)\right) \left(s \# g\right) \otimes z_{\# H}(0) \]

\[ = \left(z_{\not\in H} \left(z_{\# H}(1) \cdot s\right) \# z_{\# H}(0)\right) = \left(z_{\not\in H} \left(z_{\# H}(1) \cdot s\right) \# z_{\# H}(0)\right) \otimes z_{\# H}(0). \]

Using this, we readily see directly from the definitions that

l.h.s. of (3.14) =

\[ \left( (r^1 \# r^2_{\# H'} \mapsto y_{\not\in H}) \right) \otimes \left( (r^1 \# r^2_{\# H'} \mapsto y_{\not\in H}) \right) \otimes \left( (r^1 \# r^2_{\# H'} \mapsto y_{\not\in H}) \right) \otimes r^2_{\# H'}(0) \]

\[ \otimes \left(r^2_{\# H'} \right) \cdot y_{\# H}(0). \]
\[
\begin{align*}
(r \rightarrow (r_{(-1)}^2, y)) \triangleleft (r \rightarrow (r_{(-1)}^2, y))_{(0)} \triangleright r_{(0)}^2 \otimes (r \rightarrow (r_{(-1)}^2, y))_{(0)} \\
= r \rightarrow (r_{(-1)}^2, y) \triangleright y_{(-1)} \otimes (r_{(0)}^2 \rightarrow y_{(0)}),
\end{align*}
\]
whence
\[
\text{l.h.s. of (3.14)} = r \rightarrow (r_{(-1)}^2, y) \triangleright (h', y)_{(-1)} \neq (r_{(0)}^2 \rightarrow (h', y))_{(-1)} h'' \otimes (r_{(0)}^2 \rightarrow (h', y))_{(0)}.
\]
We continue by using the fact that the \( R \)-action \( \rightarrow \) is a morphism of \( H \)-comodules,
\[
= r \rightarrow (r_{(-1)}^2, y) \triangleright (h', y)_{(0)} \neq r_{(0)}^2 \rightarrow (h', y)_{(0)}
\]
and that the \( R \) coaction is a morphism of \( H \)-modules,
\[
= r \rightarrow (r_{(-1)}^2, h' \triangleright y_{(-1)}) \neq r_{(0)}^2 \rightarrow (h'' \rightarrow y)_{(0)}
\]
after which the \( H \)-Yetter–Drinfeld condition for \( \mathcal{Y} \) yields
\[
= r \rightarrow (r_{(-1)}^2, h' \rightarrow y_{(-1)}) \neq r_{(0)}^2 \rightarrow (h'' \rightarrow y)_{(0)}
\]
= r.h.s. of (3.14).

This shows that we have a functor.

The functor is monoidal: for \( \mathcal{Y}, \mathcal{Z} \in \mathcal{YD} \), the action of \( R \) on a tensor product \( \mathcal{Y} \otimes \mathcal{Z} \) is given by the throughout map in
\[
r \otimes (y \otimes z) \mapsto r \otimes r \otimes y \otimes z \mapsto r \otimes r_{(-1)} \cdot y \otimes r_{(0)}^2 \otimes z \mapsto (r \rightarrow (r_{(-1)}^2, y)) \otimes r_{(0)}^2 \rightarrow z.
\]
But with \( \mathcal{Y}, \mathcal{Z} \in \mathcal{YD} \), the \( R \# H \) action on the tensor product is
\[
(r \# h) \otimes (y \otimes z) \mapsto ((r \# h)^{[1]} \rightarrow y) \otimes ((r \# h)^{[2]} \rightarrow z)
\]
\[
= (r \# r_{(-1)}^2, h') \rightarrow y \otimes ((r_{(0)}^2 \# h'') \rightarrow z)
\]
\[
= (r \rightarrow (r_{(-1)}^2, h' \rightarrow y)) \otimes (r_{(0)}^2 \rightarrow (h'' \rightarrow z)),
\]
and hence \( r \neq 1 \) acts the same as \( r \) in the preceding formula.

The functor is braided: for \( y \in \mathcal{Y} \) and \( z \in \mathcal{Z} \), with \( \mathcal{Y}, \mathcal{Z} \in \mathcal{YD} \), the braiding is
\[
c_{\mathcal{Y}, \mathcal{Z}}^{\mathcal{YD}^\mathcal{R}} : y \otimes z \mapsto (y_{(-1)} \rightarrow (y_{(-1)} \otimes z)) \otimes y_{(0)}.
\]
On the other hand, when \( \mathcal{Y} \) and \( \mathcal{Z} \) are viewed as objects in \( \mathcal{YD}^{\mathcal{R} \# H} \), the braiding is
\[
c_{\mathcal{Y}, \mathcal{Z}}^{\mathcal{YD}^{\mathcal{R} \# H}} : y \otimes z \mapsto (y_{[-1]} \rightarrow z) \otimes y_{[0]} = (y_{[-1]} \neq y_{[-1]} \rightarrow z \otimes y_{(0)})
\]
\[
= (y_{(-1)} \rightarrow (y_{(-1)} \otimes z)) \otimes y_{(0)},
\]
which is the same.

For \( \mathcal{Y} \otimes \mathcal{Z} \in \mathcal{YD} \), the coaction is the throughout map in
\[
y \otimes z \mapsto y_{(-1)} \otimes y_{(0)} \otimes z_{(-1)} \otimes z_{(0)} \mapsto y_{(-1)} \rightarrow (y_{(-1)} \rightarrow z_{(-1)} \otimes y_{(0)} \otimes z_{(0)} \rightarrow y_{(-1)} \rightarrow (y_{(-1)} \rightarrow z_{(-1)}) \otimes y_{(0)} \otimes z_{(0)},
\]
which lies in \( R \otimes Y \otimes \mathbb{Z} \). Now for \( Y \otimes Z \in \mathcal{R}^H \otimes YD \), the coaction is simply
\[
y_{[-1]} \otimes y_{[0]} \otimes z_{[0]} = (y_{[-1]} + y_{[-1]}') \otimes y_{[0]} \otimes z_{[0]}
\]
which is in \( \mathcal{R} \oplus Y \otimes Z \). Applying \( \text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id} \) restores the previous formula.

3.4.5. Remark. A particular case of 3.4.3 is well known: for \( Y = (R, \triangleright) \) with the left adjoint action, we write (3.12) as (with \( r, p \in R \))
\[
(r \triangleright p) = r \triangleright (r \triangleright p) \triangleright p = r \triangleright (r \triangleright p) \triangleright p = r \triangleright (r \triangleright p) \triangleright p
\]
(see (3.11) for the antipode), which is a restriction of the ("nonbraided") left adjoint action of \( \mathcal{R} \oplus H \) on itself.

3.5. The functor \( F_{DB} \). To construct a "double bosonization" functor \( F_{DB} \) inverse to \( F_T \), we compose \( F_B \) with the standard functor establishing the equivalence \[44\] of Yetter–Drinfeld modules with modules of the Drinfeld double \( D(B(X) \oplus H) = D(U_\infty) \), \( F_{DB} : \mathbb{B}(X) \otimes H \rightarrow D(U_\infty) \)-\text{MOD}.

Applied to any \( Y \in \mathbb{B}(X) \otimes \mathcal{D}Y \), \( F_{DB} = F_{DD} F_B \) then produces a \( U(X) \)-module because the action of the generators of \( H^* \in D(U_\infty) \) is completely determined by the action of the generators of \( H \in D(U_\infty) \), which means that the module is actually a module of \( U(X) \simeq D(U_\infty)/(H^* \simeq H) \) (the quotient by relations expressing \( H^* \) through \( H \)). Indeed, for \( \mu \in H^* \), its action on \( Y \) viewed as a Yetter–Drinfeld \( U_\infty \)-module is standardly given by
\[
\mu \downarrow y = \langle \mu, S^{-1}(y_{[-1]}) \rangle \otimes y_{[0]}
\]
It follows that the generators of \( H^* \) (see [B.1.2]) act as
\[
L \downarrow z = K \cdot z, \quad \ell \downarrow z = k \cdot z,
\]
and hence we have a functor
\[
F_{DB} : \mathbb{B}(X) \otimes \mathcal{D}Y \rightarrow U(X) \text{-MOD}.
\]

3.6. Composing the functors. To verify that \( F_{DB} F_T \sim 1_{U(X)-\text{MOD}} \), we first calculate \( F_B F_T \). Applied to \( U_\infty \subset U(X) \), \( F_T \) gives a braided Hopf algebra with coproduct (3.6), \( x \otimes x = x' S(\rho(2)) \otimes x'' (x \in U_\infty) \); applying \( F_B \)—using Radford’s formula (3.10)—we obtain the coproduct \( \Delta : U_\infty \rightarrow U_\infty \otimes U_\infty \) that evaluates as
\[
\Delta(x \oplus h) = x' h' \otimes x'' h'' ,
\]
which is the original coproduct on $U_{\prec}$. For $\mathcal{F}_T$ produces coproduct $3.8$, $y_{-1} \otimes y_0 = R^{(2)} S(\rho^{(2)}) \otimes \rho^{(1)} R^{(1)} \cdot y$; to further apply $\mathcal{F}_B$, we substitute the last formula in the “Radford formula for comodules,” Eq. (3.13), to obtain the $U_{\prec}$ coaction

$$\delta(y) = R^{(2)} \otimes R^{(1)} \cdot y \in U_{\prec} \otimes \mathcal{Y}.$$  

The resulting $U_{\prec}$ action, obviously, is the restriction of the $U(X)$ action. To the Yetter–Drinfeld $U_{\prec}$-module $\mathcal{Y}$ thus obtained, we now apply $\mathcal{F}_{DB}$, making it into a module of the Drinfeld double $\mathcal{D}(U_{\prec})$. Then $U_{\prec}^\ast$ (a Hopf subalgebra in $\mathcal{D}(U_{\prec})$) acts as

$$\mu \triangleright y = \langle \mu, S^{-1}(y_{[1]}) \rangle y_{[0]} = \langle \mu, S^{-1}(R^{(2)}) \rangle R^{(1)} \cdot y = \langle \mu, R^{(2)} S(R^{(1)}) \rangle \cdot y.$$  

With the $R$-matrix in 2.4.1 and with the duality worked out in B.1.1 we find, along with (3.15), that

$$E \triangleright y = E \cdot y, \quad C \triangleright y = C \cdot y.$$  

Comparing with the formulas in B.1.2 shows that the resultant $\mathcal{D}(U_{\prec})$-module is in fact a $U(X)$-module, naturally isomorphic to the original $\mathcal{Y}$.

We show similarly that $\mathcal{F}_T \mathcal{F}_{DB} \sim l_{\mathcal{B}(X) YD e_\rho}$. Starting from $Z \in \mathcal{B}(X) YD e_\rho$ (where we write the action of both $H$ and $U_{\prec}$ as $x \otimes z \mapsto x \cdot z$ for simplicity, and the coaction as $z \mapsto z_{-1} \otimes z_0$) and applying $\mathcal{F}_T \mathcal{F}_{DB}$, we arrive at the Yetter–Drinfeld $U_{\prec}$-module with the coaction

$$z \mapsto \rho^{(2)} \triangleright R^{(2)} \otimes \langle \tilde{R}^{(1)}, S^{-1}(z_{-1} \# \tau^{(2)}) \rangle \rho^{(1)} \cdot z_0$$

$$= \rho^{(2)} \triangleright R^{(2)} \otimes \langle \tilde{R}^{(1)}, S^{-1}(z_{-1} \# \rho^{(2)\prime}) \rangle \rho^{(1)} \cdot z_0$$

$$= \rho^{(2)\prime} \triangleright R^{(2)} \otimes \langle \tilde{R}^{(1)}, S^{-1}(z_{-1} \# \rho^{(2)\prime}) \rangle \rho^{(1)} \cdot z_0$$

$$= \rho^{(2)\prime} \triangleright R^{(2)} \otimes \langle \rho^{(2)} \triangleright R^{(1)}, S^{-1}(z_{-1}) \rangle \rho^{(1)} \cdot z_0$$

$$= R^{(2)} \otimes \langle \tilde{R}^{(1)}, S^{-1}(z_{-1}) \rangle z_0 = z_{-1} \otimes z_0.$$  

Hence, we have constructed an inverse functor $\mathcal{F}_{DB}$ to the functor $\mathcal{F}_T$ in 3.3. The categories $U(X)$-mod and $\mathcal{B}(X) YD e_\rho$ are equivalent. In the rest of the paper, we study the modules of $U(X)$.

4. Simple modules of $U(X)$

The following theorem is of course contained in [17] [38]; we spell out the details here in order to fix our notation and conventions.

4.1. Theorem. The algebra $U(X)$ has $4p^2$ simple modules, labeled as

$$z_{s, r}^{\alpha, \beta}, \quad \alpha, \beta = \pm, \quad s = 1, \ldots, p, \quad r = 0, \ldots, p - 1,$$
with the dimensions

\[
\dim_{\mathbb{A}_s, p}^{\alpha, \beta} = \begin{cases} 
2s - 1, & 1 \leq s \leq p, \quad r = 0, \\
2s + 1, & 1 \leq s \leq p - 1, \quad r = s, \\
4s, & 1 \leq s \leq p - 1, \quad r \neq 0, s, \\
4p, & s = p, \quad 1 \leq r \leq p - 1.
\end{cases}
\]

(4.1)

The modules in the first two lines are atypical, and the others typical.

The cases occurring in the theorem can be illustrated by the diagram in Fig. 4.1.

Figure 4.1. The \( p^2 \) simple \( U(X) \) modules, labeled by \( s \) and \( r \).

modules are the “bulk” of the diagram and the greatest part of the right column. In what follows, we refer to these last modules (the fourth case in (4.1)) as Steinberg modules. The notation in the figure distinguishes more cases than just those in (4.1) for our later purposes.

4.2. Constructing simple \( U(X) \)-modules. We construct the simple \( U(X) \) modules explicitly, by “gluing together” some modules of the algebra \( \overline{U}_q \mathfrak{sl}(2) \) defined in (2.1) and (2.2). We let simple \( \overline{U}_q \mathfrak{sl}(2) \) modules be denoted by \( \chi_{s, r}^{\alpha, \beta} \), where \( \alpha, \beta = \pm, s = 1, \ldots, p, \) and \( r = 0, \ldots, p - 1 \). As a \( \overline{U}_q \mathfrak{sl}(2) \)-module, an \( \chi_{s, r}^{\alpha, \beta} \) with any \( \beta \) and \( r \) is isomorphic to \( \chi_s^\alpha \) (see A.1). In particular, there is a highest-weight vector \( |\alpha, s, \beta, r\rangle_0 = 0 \) such that

\[
E |\alpha, s, \beta, r\rangle_0 = 0 \quad \text{and} \quad K |\alpha, s, \beta, r\rangle_0 = \alpha q^{s-1} |\alpha, s, \beta, r\rangle_0,
\]

and the second Cartan generator of \( U(X) \) acts on this vector as

\[
k |\alpha, s, \beta, r\rangle_0 = \beta q^{-r} |\alpha, s, \beta, r\rangle_0.
\]

It follows that \( \dim \chi_{s, r}^{\alpha, \beta} = s \).
4.2.1. $\overline{U}_q^{\ast} sl(2)$ decompositions and bases of simple modules. For each of the four cases in (4.1), we now list the $\overline{U}_q^{\ast} sl(2)$ decompositions of simple $U(X)$ modules, specify the corresponding choice of basis, and describe how the $\overline{U}_q^{\ast} sl(2)$ constituents are glued together by the fermionic $U(X)$ generators $B$ and $C$. There are two sorts of basis vectors $| \gamma_m^{-}\rangle$ and $| \gamma_m^{+}\rangle$ for atypical modules and two more, $| \gamma_m^{\uparrow}\rangle$ and $| \gamma_m^{\downarrow}\rangle$, for typical modules; in all cases,

$$C | \gamma_m^{\pm}\rangle = 0 \quad \text{and} \quad B | \gamma_m^{\pm}\rangle = 0$$

in any simple $U(X)$ module.

Atypical modules, $1 \leq s \leq p, r = 0$: as $\overline{U}_q^{\ast} sl(2)$ modules, these simple $U(X)$ modules decompose as

$$\mathcal{Z}_{s,0}^{\alpha,\beta} = \mathcal{X}_{s,0}^{\alpha,\beta} \oplus \mathcal{X}_{s-1,p-1}^{\alpha,\beta}, \quad 2 \leq s \leq p,$$

which is illustrated in Fig. 4.2 left, and as $\mathcal{Z}_{1,0}^{\alpha,\beta} = \mathcal{X}_{1,0}^{\alpha,\beta}$ (one-dimensional modules). We choose a basis in $\mathcal{Z}_{s,0}^{\alpha,\beta}$ in accordance with this decomposition, as

$$\left( | \alpha, s, \beta, 0 \rangle_{n}^{-\rangle} \in \mathcal{X}_{s,0}^{\alpha,\beta} \right)_{0 \leq n \leq s-1}, \quad \left( | \alpha, s, \beta, 0 \rangle_{m}^{+\rangle} \in \mathcal{X}_{s-1,p-1}^{\alpha,\beta} \right)_{0 \leq m \leq s-2}. $$
The arrows in the notation for basis vectors refer to the visualization of $\mathcal{Z}_{s,0}^{\alpha,\beta}$ as in Fig. 4.2, left. The fermionic generators relate the two types of vectors as

$$B|\alpha, s, \beta, 0\rangle_n^\rightarrow = -[m]|\alpha, s, \beta, 0\rangle_{n-1}, \quad C|\alpha, s, \beta, 0\rangle_m^\rightarrow = \beta|\alpha, s, \beta, 0\rangle_{m+1}.$$

Here and hereafter, we set $|\alpha, s, \beta, r\rangle_m^\ast = 0$ whenever $m$ is outside the range indicated for vectors of a given type.

**Atypical modules, $1 \leq s \leq p - 1, r = s$:** the module decomposes as

$$\mathcal{Z}_{s,s}^{\alpha,\beta} = \mathcal{X}_{s,s}^{\alpha,\beta} \oplus \mathcal{X}_{s+1,s}^{\alpha,-\beta},$$

as is illustrated in Fig. 4.2, middle. We choose a basis in $\mathcal{Z}_{s,s}^{\alpha,\beta}$ accordingly:

$$\left(\langle \alpha, s, \beta, s \rangle_n^\rightarrow \in \mathcal{X}_{s,s}^{\alpha,\beta}\right)_{0 \leq n \leq s}, \quad \left(\langle \alpha, s, \beta, s \rangle_m^\rightarrow \in \mathcal{X}_{s+1,s}^{\alpha,-\beta}\right)_{0 \leq m \leq s}.$$

The fermions act as

$$B|\alpha, s, \beta, s\rangle_n^\rightarrow = [s-n]|\alpha, s, \beta, s\rangle_n^\rightarrow, \quad C|\alpha, s, \beta, s\rangle_m^\rightarrow = \beta|\alpha, s, \beta, s\rangle_m^\rightarrow.$$

**Typical modules, $1 \leq s \leq p - 1, r \neq 0, s$:** the modules decompose as

$$\mathcal{Z}_{s,r}^{\alpha,\beta} = \mathcal{X}_{s,r}^{\alpha,\beta} \oplus \mathcal{X}_{s+1,r}^{\alpha,-\beta} \oplus \mathcal{X}_{s-1,r,1}^{\alpha,-\beta} \oplus \mathcal{X}_{s,r,1}^{\alpha,\beta}, \quad 2 \leq s \leq p - 1,$$

which is illustrated in Fig. 4.2, right, and as $\mathcal{Z}_{1,r}^{\alpha,\beta} = \mathcal{X}_{1,r}^{\alpha,\beta} \oplus \mathcal{X}_{2,r}^{\alpha,-\beta} \oplus \mathcal{X}_{1,r-1}^{\alpha,\beta}$. We choose a basis in $\mathcal{Z}_{s,r}^{\alpha,\beta}$ accordingly, such that the bases in respective $\mathcal{U}_q^\ast \mathfrak{sl}(2)$ submodules are

$$\left(\langle \alpha, s, \beta, r \rangle_j^\ast \right)_{0 \leq j \leq s-1}, \quad \left(\langle \alpha, s, \beta, r \rangle_j \right)_{0 \leq m \leq s}, \quad \left(\langle \alpha, s, \beta, r \rangle_j \right)_{0 \leq j \leq s-1}.$$

The fermions glue the $\mathcal{U}_q^\ast \mathfrak{sl}(2)$ modules together as

$$B|\alpha, s, \beta, r\rangle_j^\rightarrow = \left[\frac{m}{s}\right]|\alpha, s, \beta, r\rangle_j^\rightarrow + \beta\left[\frac{s-n}{s}\right]|\alpha, s, \beta, r\rangle_j^\ast,$$

$$B|\alpha, s, \beta, r\rangle_m^\ast = \beta|\alpha, s, \beta, r\rangle_m^\ast, \quad C|\alpha, s, \beta, r\rangle_m^\rightarrow = |\alpha, s, \beta, r\rangle_m^\rightarrow,$$

$$C|\alpha, s, \beta, r\rangle_j^\ast = \left[\frac{s-n}{s}\right]|\alpha, s, \beta, r\rangle_j^\ast + \beta|\alpha, s, \beta, r\rangle_j^\rightarrow.$$

**Steinberg modules, $s = p, 1 \leq r \leq p - 1$:** the decomposition is

$$\mathcal{Z}_{p,r}^{\alpha,\beta} = \mathcal{X}_{p,r}^{\alpha,\beta} \oplus \mathcal{Y}_{p-1,r,1}^{\alpha,-\beta} \oplus \mathcal{X}_{p,r,1}^{\alpha,\beta},$$

where $\mathcal{Y}_{p-1,r,1}^{\alpha,-\beta}$ is a projective $\mathcal{U}_q^\ast \mathfrak{sl}(2)$-module. This can be illustrated with much the same diagram as in Fig. 4.2 right, but with two differences in the middle columns: first, there are exactly $p$ $\textbullet$ and $p$ $\text{	extdagger}$ vectors and, second, they are not a
We note that if \( \text{modules} \) where \( \text{tr} \) (4.7), then the (“quantum”) trace operation on short exact sequences 20 A. M. SEMIKHATOV AND I. YU. TIPUNIN.

\[ \text{Drinfeld map into the Casimir elements:} \]

\[
\begin{align*}
\alpha, p, \beta \rightarrow r_n^{\alpha, \beta}, & \quad \alpha, p, \beta \rightarrow r_n^{\alpha, \beta}, & \quad \alpha, p, \beta \rightarrow r_n^{\alpha, \beta}, & \quad \alpha, p, \beta \rightarrow r_n^{\alpha, \beta}, & \quad \alpha, p, \beta \rightarrow r_n^{\alpha, \beta}, \\
\text{where } n = 0, \ldots, p - 1. & \quad \text{The fermions act as} \\
& \quad B|\alpha, p, \beta, r_n^{\alpha, \beta}| = [n - 1]|\alpha, p, \beta, r_n^{\alpha, \beta}| + |n|\alpha, p, \beta, r_n^{\alpha, \beta}, & \quad C|\alpha, p, \beta, r_n^{\alpha, \beta}| = -\beta[r]|\alpha, p, \beta, r_n^{\alpha, \beta}, \\
& \quad B|\alpha, p, \beta, r_n^{\alpha, \beta}| = [n]|\alpha, p, \beta, r_n^{\alpha, \beta}|, & \quad C|\alpha, p, \beta, r_n^{\alpha, \beta}| = \beta[r]|\alpha, p, \beta, r_n^{\alpha, \beta}. \\
& \quad C|\alpha, p, \beta, r_n^{\alpha, \beta}| = \beta[r - 1]|\alpha, p, \beta, r_n^{\alpha, \beta}| + \beta[r]|\alpha, p, \beta, r_n^{\alpha, \beta}. \\
& \quad \text{The remaining formulas for the } U(X) \text{ action are collected in } A.3 \]

4.3. Casimirs from simple modules. We recall the Drinfeld map (2.7). If \( \mathcal{Z} \) is a \( U(X) \)-module, then the (“quantum”) trace operation

\[
\text{Tr}_{\mathcal{Z}} : A \mapsto \text{tr}_{\mathcal{Z}}(AK^p k^2) : U(X) \rightarrow \mathbb{C},
\]

where \( \text{tr}_{\mathcal{Z}} \) evaluates the ordinary trace in any chosen basis in \( \mathcal{Z} \), defines an element of \( \mathfrak{C} \mathfrak{H} \). We note that if \( \gamma = \text{Tr}_{\mathcal{Z}} \), then it follows from (2.10) that \( \mathfrak{C} \mathfrak{H} \supseteq \tilde{\gamma} : A \mapsto \text{tr}_{\mathcal{Z}}(AK^p) \). Calculating with (2.8), we see that traces over three-dimensional representations are mapped by the Drinfeld map into the Casimir elements:

\[
\chi : \text{Tr}_{\mathcal{Z}_{1,1}} \alpha^{p-1} q^{-1} (q - q^{-1})^2 C_1, \\
\chi : \text{Tr}_{\mathcal{Z}_{2,0}} \alpha^p q (q - q^{-1})^2 C_2.
\]

5. \( \text{Ext}^1 \) spaces for simple \( U(X) \)-modules

We now describe the \( \text{Ext}^1 \) groups for simple \( U(X) \)-modules. We recall that for two modules \( \mathcal{Z}_2 \) and \( \mathcal{Z}_1 \), \( \text{Ext}^1(\mathcal{Z}_1, \mathcal{Z}_2) \) is a linear space with the basis identified with nontrivial short exact sequences

\[
0 \rightarrow \mathcal{Z}_2 \rightarrow \mathcal{Z}_1 \oplus \mathcal{Z}_2 \rightarrow \mathcal{Z}_1 \rightarrow 0
\]
modulo a certain equivalence relation \cite{45}.

The action of any $U(X)$ generator $A$ on $\mathbb{Z}_1 \oplus \mathbb{Z}_2$ is given by

$$\rho_A = \rho_A^{(0)} + \xi_A$$

where $\rho_A^{(0)}$ is the direct sum of the actions of $U(X)$ generators on simple modules and $\xi_A = \xi_A^{\mathbb{Z}_1, \mathbb{Z}_2} : \mathbb{Z}_1 \to \mathbb{Z}_2$ are linear maps (also linear in $A$). We list the $\xi_A$ for each extension in what follows.

5.1. $\text{Ext}^1$ spaces for typical simple modules. The nontrivial $\text{Ext}^1(\mathbb{Z}_1, \mathbb{Z}_2)$ spaces for the typical simple $U(X)$-modules are one-dimensional. These are $\text{Ext}^1(\mathbb{Z}_s, \mathbb{Z}_{p-s, p+r-s})$ and $\text{Ext}^1(\mathbb{Z}_s, \mathbb{Z}_{p+s, p-r-s})$ for each pair $(s, r)$ such that

$$1 \leq s \leq p - 1, \quad 1 \leq r \leq p - 1, \quad s \neq r.$$ 

To avoid notational complications, we here adopt the convention that $\mathbb{Z}_{s,p+r} = \mathbb{Z}_{s,r}$ for $1 \leq r \leq p - 1$.

We fix a basis element in each of the $\text{Ext}^1$ spaces, writing

$$\text{Ext}^1(\mathbb{Z}_s, \mathbb{Z}_{p-s, p+r-s}) = \{f_{p-s, p+r-s}\},$$

and

$$\text{Ext}^1(\mathbb{Z}_s, \mathbb{Z}_{p-s, p-r-s}) = \{e_{p-s, p+r-s}\}.$$

Here and hereafter, $\alpha$ and $\beta$ are omitted in the right-hand sides in order not to overburden the notation; they are in all cases easily reconstructed from the context. The maps $\mathbb{Z}_{s,r} \to \mathbb{Z}_{p-s, p+r-s}$ by the $U(X)$ generators associated with (5.1) are

$$\xi_F : [\alpha, s, \beta, r]_m \mapsto \delta_{m,s} - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1,$$

$$\xi_C : [\alpha, s, \beta, r]_m \mapsto -\beta [r] \delta_{m,s} - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1,$$

$$\xi_F : [\alpha, s, \beta, r]_m \mapsto [s - r] [r] \delta_{m,s-2} - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1,$$

$$\xi_F : [\alpha, s, \beta, r]_m \mapsto -\beta \delta_{m,s-1} [r] - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1,$$

$$\xi_F : [\alpha, s, \beta, r]_m \mapsto [s - r] \delta_{m,s-1} [r] - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1,$$

$$\xi_F : [\alpha, s, \beta, r]_m \mapsto \delta_{m,s-1} [s] - \alpha, p - s, -\beta, p + r - s \delta_{0,1}^1.$$ 

(And zero otherwise). The maps $\mathbb{Z}_{s,r} \to \mathbb{Z}_{p-s, p-r-s}$ by the $U(X)$ generators associated with (5.2) are

$$\xi_E : [\alpha, s, \beta, r]_m \mapsto \frac{[s + 1]}{[r - s][r]} \delta_{m,0} - \alpha, p - s, \beta, p + r - s \delta_{p-s-2,1},$$

$$\xi_E : [\alpha, s, \beta, r]_m \mapsto \beta \frac{[s]}{[r-s]} \delta_{m,0} - \alpha, p - s, \beta, p + r - s \delta_{p-s-1,1},$$

$$\xi_E : [\alpha, s, \beta, r]_m \mapsto [s - 1] \delta_{m,0} - \alpha, p - s, \beta, p + r - s \delta_{p-s,1},$$

$$\xi_E : [\alpha, s, \beta, r]_m \mapsto \delta_{m,0} - \alpha, p - s, \beta, p + r - s \delta_{p-s,1}.$$
5.2. Ext spaces for atypical simple U(X)-modules. The Ext$^1(Z_1, Z_2)$ groups for atypical simple U(X)-modules are either 1- or 0-dimensional. The nonzero Ext$^1$ spaces can be arranged into several series, in addition to which there are a few “exceptional” cases, where one of the modules involved is $Z_1^{1,0}$. These cases can be conveniently absorbed into the series by adopting the convention that $Z_0^{1,0} = Z_1^{1,0}$ and setting $|\alpha, 0, \beta, 0\rangle_0 = |\alpha, 1, -\beta, 0\rangle_0$ and $|\alpha, 0, \beta, 0\rangle_m = 0, m \neq 0$ for the basis vectors. Then the extensions are defined by the following formulas, where we choose a basis vector in each space, placing it in curly brackets, and specify how the U(X) generators map from $Z_1$ to $Z_2$ (again omitting the $\alpha$ and $\beta$ indices from the notation)

\[
\begin{align*}
\text{Ext}^1(Z_{s,0}^{1}, Z_{s+1,0}^{1}) = \{b_{s+1}\}, & \xi_B : |\alpha, s, \beta, 0\rangle_m \mapsto -[s-m]|\alpha, s+1, -\beta, 0\rangle_m, \\
(1 \leq s \leq p-1) & \xi_B : |\alpha, s, \beta, 0\rangle_m \mapsto [s-m]|\alpha, s+1, -\beta, 0\rangle_m, \\
\text{Ext}^1(Z_{s,0}^{1}, Z_{s-1,0}^{1}) = \{c_{s-1}\}, & \xi_C : |\alpha, s, \beta, 0\rangle_m \mapsto |\alpha, s-1, -\beta, 0\rangle_m, \\
(2 \leq s \leq p) & \xi_C : |\alpha, s, \beta, 0\rangle_m \mapsto |\alpha, s-1, -\beta, 0\rangle_m, \\
\text{Ext}^1(Z_{s,s}^{1}, Z_{p-s,0}^{1}) = \{f_{p-s}\}, & \xi_F : |\alpha, s, \beta, 0\rangle_{s-1} \mapsto |\alpha, p-s, -\beta, 0\rangle_0, \\
(0 \leq s \leq p-1) & \xi_F : |\alpha, s, \beta, 0\rangle_{s-1} \mapsto |\alpha, p-s, -\beta, 0\rangle_0, \\
\text{Ext}^1(Z_{s,s}^{1}, Z_{p-s,0}^{1}) = \{e_{p-s}\}, & \xi_E : |\alpha, s, \beta, 0\rangle_{s} \mapsto |s| - (s+1)|\alpha, p-s, -\beta, 0\rangle_{p-s-2}, \\
(0 \leq s \leq p-1) & \xi_E : |\alpha, s, \beta, 0\rangle_{s} \mapsto |s+1| - (s+1)|\alpha, p-s, -\beta, 0\rangle_{p-s-2}, \\
\text{Ext}^1(Z_{s,s}^{1}, Z_{1,s-1}^{1}) = \{b_{s-1}\}, & \xi_B : |\alpha, s, \beta, 0\rangle_{m} \mapsto -[m]|\alpha, s-1, -\beta, 0\rangle_{m-1}, \\
(1 \leq s \leq p-1) & \xi_B : |\alpha, s, \beta, 0\rangle_{m} \mapsto [m]|\alpha, s-1, -\beta, 0\rangle_{m-1}, \\
\text{Ext}^1(Z_{s,s}^{1}, Z_{s+1,0}^{1}) = \{c_{s+1}\}, & \xi_C : |\alpha, s, \beta, 0\rangle_{m} \mapsto |\alpha, s+1, -\beta, 1\rangle_{m+1}, \\
(0 \leq s \leq p-2) & \xi_C : |\alpha, s, \beta, 0\rangle_{m} \mapsto |\alpha, s+1, -\beta, 1\rangle_{m+1}, \\
\text{Ext}^1(Z_{s,0}^{1}, Z_{p-s,p-s}^{1}) = \{f_{p-s}\}, & \xi_F : |\alpha, s, \beta, 0\rangle_{s-1} \mapsto -|\alpha, p-s, -\beta, 0\rangle_0, \\
(1 \leq s \leq p) & \xi_F : |\alpha, s, \beta, 0\rangle_{s-1} \mapsto -|\alpha, p-s, -\beta, 0\rangle_0,
\end{align*}
\]
5.3. Linkage classes. It follows that the simple $U(X)$ modules are divided into linkage classes as follows.

1. There are $4(p-1)$ Steinberg linkage classes, labeled by $\alpha = \pm, \beta = \pm$, and $1 \leq r \leq p-1$; each class contains a single module $\mathcal{Z}_{\alpha, \beta}^{p, r}$.
2. There are $(p-1)(p-2)$ typical linkage classes, labeled by $(\alpha, s, r)$ with $\alpha = \pm$ and $1 \leq r < s \leq p-1$; each such class contains four simple modules $\mathcal{Z}_{\alpha, s, r}^{\pm, \pm}$, $\mathcal{Z}_{\alpha, s, r}^{\pm, \mp}$, $\mathcal{Z}_{\alpha, s, r}^{-\pm, \pm}$, and $\mathcal{Z}_{\alpha, s, r}^{-\pm, \mp}$.
3. There is one atypical linkage class containing $4(2p-1)$ simple modules: $\mathcal{Z}_{s, s}^{\pm, \pm}$ with $1 \leq s \leq p-1$ and $\mathcal{Z}_{s, 0}^{\pm, \pm}$ with $1 \leq s \leq p$ (with uncorrelated signs in either case).

6. Projective $U(X)$ modules

6.1. Theorem. Projective modules of $U(X)$ are exhausted by the following list (with $\alpha, \beta = \pm$ in all cases).

- Steinberg modules $\mathcal{Q}_{\alpha, \beta}^{p, r}$ with $1 \leq r \leq p-1$, which are the simple $4p$-dimensional modules described in \[4.2.1\]
- Typical modules $\mathcal{Q}_{\alpha, \beta}^{\pm, \pm}$ with $1 \leq s \leq p-1$, $1 \leq r \leq p-1$ and $r \neq s$, described in \[6.3.2\] each has four simple subquotients and dimension $8p$.
- Atypical modules:
  - The $12p$-dimensional $\mathcal{Q}_{\alpha, \beta}^{p, 0}$ modules described in \[6.3.3\] with 12 simple subquotients each.
  - The $12p$-dimensional modules $\mathcal{Q}_{\alpha, \beta}^{p, p-1}$ described in \[6.3.4\] with 12 simple subquotients.
  - The $16p$-dimensional $\mathcal{Q}_{\alpha, \beta}^{s, 0}$ modules with $1 \leq s \leq p-1$ described in \[6.3.5\] with 16 simple subquotients.
  - The $16p$-dimensional modules $\mathcal{Q}_{\alpha, \beta}^{s, p-2}$ described in \[6.3.6\] with 16 simple subquotients.
  - The $24p$-dimensional modules $\mathcal{Q}_{\alpha, \beta}^{1, 0}$ described in \[6.3.7\] with 24 simple subquotients.

We construct each projective module $\mathcal{Q}$ explicitly by choosing a basis and defining the $U(X)$ action in that basis. The set of basis vectors is the union of the bases of all simple
subquotients. This means that for each projective \( Q \), we choose and fix a linear space isomorphism

\[
\mu : Q \to \tilde{Q} = \bigoplus_i \mathbb{Z}_i,
\]

with the direct sum of all simple subquotients of \( Q \). In accordance with the direct sum decomposition, we also write \( \mu = \sum_i \mu_i \).

6.2. The \( U(X) \) action and graphs. We explain how we construct the action of \( U(X) \) generators on projective modules. For a generator \( A \), its action on \( v \in Q \) is given by

\[
\rho_A(v) = \rho_A^{(0)}(v) + \rho_A^{(1)}(v) + \rho_A^{(2)}(v),
\]

with the following ingredients.

1. To define \( \rho_A^{(1)} \), we recall that in [5.1] and [5.2], we chose a collection of maps \( \xi_{\mathbb{Z}} \) between simple \( U(X) \) modules, which we now write using a more detailed notation, as \( \xi_{\mathbb{Z}} \). Then

\[
\rho_A^{(1)} = \mu^{-1} \circ \bar{\rho}_A \circ \mu,
\]

where \( c_{\mathbb{Z}} \) are some coefficients, depending on a pair of simple subquotients in the projective module in question.

For a projective module \( Q \), the \( \rho_A^{(1)} \) maps can be represented as a directed graph with the set of vertices given by the \( \mathbb{Z}_i \) in (6.1) and the set of edges corresponding to the nonzero products \( c_{\mathbb{Z}} \) (nonzero for at least one \( A \)). The edge is directed from \( \mathbb{Z}_k \) to \( \mathbb{Z}_i \). We construct such graphs in what follows, decorating the edges with \( c_{\mathbb{Z}} \).

2. Finally,

\[
\rho_A^{(2)} = \mu^{-1} \circ \eta_A \circ \mu,
\]

where \( \eta_A \) are linear maps (also linear in \( A \)) and \( \mathbb{Z}_i \) is a descendant but not a child of \( \mathbb{Z}_k \) in the graph defined by the \( c_{\mathbb{Z}} \). These maps are needed for (6.2) to be a \( U(X) \) action.

Proving the existence of the projective cover \( Q \) of a simple module \( \mathbb{Z}_x \) amounts to finding the coefficients \( c_{\mathbb{Z}} \) and the maps \( \eta_A \) such that the graph has a root vertex given by \( \mathbb{Z}_x \) and Eq. (6.2) is a \( U(X) \) action (and the resultant module is maximal indecomposable). We solve for such \( c_{\mathbb{Z}} \) and \( \eta_A \) in what follows. The solution is not unique due to the freedom of taking linear combinations of (the respective basis vectors in) isomorphic

\footnote{Not on pairs of isomorphism classes of simple \( U(X) \) modules.}
subquotients, but the existence of a solution proves the existence of the corresponding module; that it is maximal indecomposable is then verified by inspection in each particular case.

6.3. Constructing the projective modules. We proceed with projective modules starting with the simple ones in [6.3.1] and ending with those having 24 simple subquotients in [6.3.7].

6.3.0. We need the obvious notion of level in a directed graph with a root vertex *. That vertex is assigned level 1, every child of * has level 2, and so on (in the graphs we are dealing with, this defines the level of each vertex uniquely).

6.3.1. Simple projective modules: \( s = p, \ 1 \leq r \leq p - 1 \). These are the simple (“Steinberg”) modules discussed in and after Eq. (4.6). Each of them is also projective.

6.3.2. Projective covers of typical simple modules with \( 1 \leq s \leq p - 1, \ r \neq 0, s \). The “bulk” of the diagram in Fig. 4.1 yields \( 4(p - 1)(p - 2) \) modules \( Q_{\alpha,\beta}^{s,r} \), each of dimension \( 8p \), labeled by

\[
1 \leq s \leq p - 1, \ 1 \leq r \leq p - 1, \ r \neq s, \ \alpha, \beta = \pm.
\]

Their graphs (see [6.2]) are very simple:

\[
(6.5)
\]

Here, \( (\alpha, \beta)^{s,r} = Z^{\alpha,\beta}_{s,r} \) are the simple subquotients\(^7\) and the arrows are the (basis elements in) the corresponding \( \text{Ext}^1 \) and the units on the arrows are the coefficients \( c^{X,1} \) standing in front of these elements (see (6.3)).

We consider the arrow with a checkmark as an example. From the subquotients that it connects, we see that the relevant extension is \( f_{p-s, p+r-s} \) in (5.1); the corresponding \( \xi_A \) piece of the action of \( U(X) \) generators is therefore given by (5.3) times the coefficient 1. For the opposite arrow in the diagram, the same maps (5.3) should be used with \( \alpha \to -\alpha, \ s \to p - s, \) and \( r \to p + r - s, \) and similarly for the other arrows.

\(^7\)We omit the uninformative \( Z \) for conformity with similar, but much more complicated graphs in what follows, where extra symbols would complicate the picture even more.
It remains to specify the $\eta_A$ piece of the action such that Eqs. (6.2)–(6.4) define a $U(X)$ action. The choice of the $\eta_A$ maps in a given basis is not unique (the different solutions being mapped into one another by basis changes), and we choose $\eta_A$ to be nonzero only for $A = B, E$, and only acting on the top subquotient. Its basis vectors (4.5), now denoted as

$$\begin{align*}
|\alpha, s, \beta, r\rangle_j^{-\Delta}, & \quad |\alpha, s, \beta, r\rangle^1_j, & \quad |\alpha, s, \beta, r\rangle^1_n, & \quad |\alpha, s, \beta, r\rangle_n^{-\Delta}, \\
|\alpha, s, \beta, r\rangle_n^{-\Delta}, & \quad |\alpha, s, \beta, r\rangle^1_n, & \quad |\alpha, s, \beta, r\rangle^1_m, & \quad |\alpha, s, \beta, r\rangle_m^{-\Delta},
\end{align*}$$

are than mapped into the respective vectors in the bottom subquotient, denoted as

$$\begin{align*}
|\alpha, s, \beta, r\rangle_j^{-\Delta}, & \quad |\alpha, s, \beta, r\rangle^1_j, & \quad |\alpha, s, \beta, r\rangle^1_n, & \quad |\alpha, s, \beta, r\rangle_n^{-\Delta},
\end{align*}$$

The nonzero maps $\eta_B$ and $\eta_E$ are given by

$$\begin{align*}
\eta_B|\alpha, s, \beta, r\rangle_j^{-\Delta} &= \alpha \Gamma_{s,r,j}|\alpha, s, \beta, r\rangle^1_j, \\
\eta_B|\alpha, s, \beta, r\rangle_m^{-\Delta} &= \alpha \Gamma_{s,r,m}|\alpha, s, \beta, r\rangle^1_m, \\
\eta_B|\alpha, s, \beta, r\rangle_n^{-\Delta} &= \alpha \Gamma_{s,r,n+1}|\alpha, s, \beta, r\rangle^1_n, \\
\eta_E|\alpha, s, \beta, r\rangle_j^{-\Delta} &= [s] \alpha |\alpha, s, \beta, r\rangle_{j-1}^{-\Delta}, \\
\eta_E|\alpha, s, \beta, r\rangle_m^{-\Delta} &= [s] \alpha |\alpha, s, \beta, r\rangle_{m-1}^{-\Delta}, \\
\eta_E|\alpha, s, \beta, r\rangle_n^{-\Delta} &= [s] \alpha |\alpha, s, \beta, r\rangle_{n-1}^{-\Delta},
\end{align*}$$

where

$$\Gamma_{s,r,m} = -\frac{[m-1]}{[s]} + \frac{1}{2} \left(-\left(q^r + q^{-r}\right)\frac{[s-1]}{[s]} + \left[r\right] \frac{[2s-1]-3}{[s]^2}\right) \left[\frac{m}{r-s}\right].$$

Direct calculation shows that with the coefficients 1 in (6.5), all relations for the $U(X)$ generators are satisfied.

6.3.3. $\blacksquare$: Projective cover of $\mathcal{Z}_{p,0}^{\alpha,\beta}$. The projective cover $\mathcal{Q}_{p,0}^{\alpha,\beta}$ of $\mathcal{Z}_{p,0}^{\alpha,\beta}$ (denoted by $\blacksquare$ in Fig. 4.1) has 12 simple subquotients and is $12p$-dimensional. The corresponding graph is shown in Fig. 6.1. Each simple subquotient $\mathcal{Z}_{s,r}^{\alpha,\beta}$ is identified by its parameters $(\alpha, \beta)$ and is in addition labeled by $\ell_n$, where $\ell$ is the level and $n$ consecutively labels subquotients within each level.

As previously, each link $\mathcal{Z}_1 \to \mathcal{Z}_2$ corresponds to the basis element in $\text{Ext}^1(\mathcal{Z}_1, \mathcal{Z}_2)$ (which, we recall, is one-dimensional) times the coefficient placed on the link. For example, consider the level-2-to-level-3 link $\left(\begin{array}{c}
\alpha \\
-p -1
\end{array}\right)_{2_1} \rightarrow \left(\begin{array}{c}
\alpha \\
1
\end{array}\right)_{3_3}$, which of course stands for $\mathcal{Z}_{p-1,0}^{\alpha,\beta} \rightarrow \mathcal{Z}_{1,1}^{\alpha,\beta}$. It corresponds to the element $\tilde{e}_1$ in (5.2) (the last in the list) in accordance with the somewhat truncated notation used here, which ignores the upper indices. The coefficient $-\alpha$ on the link means that $E$ and $B$ map from $\mathcal{Z}_{p-1,0}^{\alpha,\beta}$ to $\mathcal{Z}_{1,1}^{\alpha,\beta}$ as

$$\xi_E : |\alpha, p - 1, -\beta, 0\rangle_0^- \mapsto -\alpha \cdot |\alpha, 1, -\beta, 1\rangle_0^-,$$
\[ \xi_E : [\alpha, p - 1, -\beta, 0] \to -\alpha \cdot [ -2 ] [ -\alpha, 1, -\beta, 1] \]
\[ \xi_B : [\alpha, p - 1, -\beta, 0] \to -\alpha \cdot (-\alpha) [ -\alpha, 1, -\beta, 1] \]

and these are the only maps by the U(X) generators between the simple subquotients \( Z_{p-1,0} \) and \( Z_{1,1} \) of \( Q_{p,0} \).

The same \( Z_{p-1,0} \) is also linked in the graph to \( Z_{1,1} \), with a link decorated by \(-\alpha\); this is \( f_1 \) from the list in [5.2] and the nonzero maps by U(X) generators are therefore given by
\[ \xi_F : [\alpha, p - 1, -\beta, 0]_{p-2} \to -\alpha \cdot [ -\alpha, 1, \beta, 1]_{p-2} \]
\[ \xi_F : [\alpha, p - 1, -\beta, 0]_{p-3} \to -\alpha \cdot ( - ) [ -\alpha, 1, \beta, 0]_{p-3} \]

In addition to the graph, we have to specify the \( \eta_A \) maps such that Eqs. (6.2)–(6.4) define a U(X) action. This part of the action of U(X) generators can be chosen as follows:
\[ \eta_E [\alpha, p, \beta, 0]_{n,10} = -\alpha [\alpha, p, \beta, 0]_{n-1,3} - \beta [\alpha, p-1, \beta, p-1]_{n-1,3} \]
\[ \eta_B [\alpha, p, \beta, 0]_{n,10} = -\beta [n] [\alpha, p, \beta, 0]_{n-1,5} + [n+1] [\alpha, p, \beta, 0]_{n-1,5} \]
\[ + \alpha \beta [n+1] [\alpha, p-1, \beta, p-1]_{n-1,3} \]
\[ \eta_B|\alpha, p, \beta, 0\rangle_{n, 1} = \alpha \beta [n + 2]|\alpha, p - 1, \beta, p - 1\rangle_{n, 32}, \]
\[ \eta_C|\alpha, p, \beta, 0\rangle_{n, 10} = -\alpha \delta_{n, p - 1} - \alpha, 1, \beta, 1\rangle_{0, 31}, \]
\[ \eta_C|\alpha, p, \beta, 0\rangle_{n, 1} = \alpha \delta_{n, p - 2} - \alpha, 1, \beta, 1\rangle_{0, 31}, \]
\[ \eta_E|\alpha, p - 1, -\beta, 0\rangle_{n, 2} = -\alpha|\alpha, p - 1, -\beta, 0\rangle_{n - 1, 41}, \]
\[ \eta_E|\alpha, p - 1, -\beta, 0\rangle_{n, 2} = -\alpha[2]|\alpha, p - 1, -\beta, 0\rangle_{n - 1, 41} \]
\[ \eta_B|\alpha, p - 1, -\beta, 0\rangle_{n, 2} = [n + 1]|\alpha, p - 1, -\beta, 0\rangle_{n - 1, 41} - [n]|\alpha, p, \beta, 0\rangle_{n, 50}, \]
\[ \eta_B|\alpha, p - 1, -\beta, 0\rangle_{n, 2} = [n + 1]|\alpha, p, \beta, 0\rangle_{n, 50}, \]
\[ \eta_E|\alpha, p, \beta, 0\rangle_{n, 3} = -\alpha|\alpha, p, \beta, 0\rangle_{n - 1, 50}, \]
\[ \eta_B|\alpha, p, \beta, 0\rangle_{n, 3} = \beta[n + 1]|\alpha, p, \beta, 0\rangle_{n - 1, 50}, \]
\[ \eta_B|\alpha, 1, \beta, 1\rangle_{n, 3} = \alpha \delta_{n, n}|\alpha, p, \beta, 0\rangle_{p - 1, 50}, \]
\[ \eta_B|\alpha, 1, \beta, 1\rangle_{n, 3} = \alpha \delta_{n, n}|\alpha, p, \beta, 0\rangle_{p - 2, 50}, \]
\[ \eta_E|\alpha, p - 1, \beta, p - 1\rangle_{n, 3} = |\alpha, p, \beta, 0\rangle_{n - 1, 50}. \]

With these \( \eta_A \) and with the \( c_{i, k} \) read off from the graph, Eqs. (6.2)–(6.4) define an \( U(X) \) action, as can be verified directly. Inspection shows that the resulting module is indecomposable and maximal.

**6.3.4. \( \blacksquare \): Projective cover of \( \mathcal{Z}_{p - 1, p - 1}^{\alpha, \beta} \).** The projective cover \( \Omega_{p - 1, p - 1}^{\alpha, \beta} \) (denoted by \( \blacksquare \) in Fig. 4.1) also has 12 subquotients and is 12\( p \)-dimensional. Its graph is shown in Fig. 6.2 with the same notation and conventions as for the preceding projective module.

The \( \eta \) piece of the action by the \( U(X) \) generators needed in (6.2) is as follows. On the basis vectors of the top subquotient, we have

\[ \eta_E|\alpha, p - 1, \beta, p - 1\rangle_{n, 1} = -2|\delta_{n, p - 2} - \alpha, 1, \beta, 0\rangle_{0, 41}, \]
\[ \eta_E|\alpha, p - 1, \beta, p - 1\rangle_{n, 1} = -2|\alpha, p, \beta, 0\rangle_{n - 1, 3} - [2]|\alpha, p - 1, \beta, p - 1\rangle_{n - 1, 50} \]
\[ + |\alpha, p - 1, \beta, p - 1\rangle_{n - 1, 3} - [2]|\delta_{n, n} - \alpha, 1, -\beta, 0\rangle_{0, 42}, \]
\[ \eta_B|\alpha, p - 1, \beta, p - 1\rangle_{n, 1} = -\alpha \beta [n + 1]|\alpha, p - 2, -\beta, p - 2\rangle_{n - 1, 40} \]
\[ + \alpha[n]|\alpha, p - 1, \beta, p - 1\rangle_{n - 1, 3} - \alpha \beta \delta_{n, n} - \alpha, 2, \beta, 0\rangle_{0, 1}, \]
\[ \eta_B|\alpha, p - 1, \beta, p - 1\rangle_{n, 1} = \alpha \beta [n + 1]|\alpha, p - 2, -\beta, p - 2\rangle_{n - 1, 40} - \alpha \beta \delta_{n, n} - \alpha, 2, \beta, 0\rangle_{0, 1}, \]
\[ \eta_C|\alpha, p - 1, \beta, p - 1\rangle_{n, 1} = \alpha \beta |\alpha, p - 1, \beta, p - 1\rangle_{n, 50} - \alpha \beta \delta_{n, p - 1} - \alpha, 1, \beta, 0\rangle_{0, 1}. \]

The \( \eta_A \) maps from level-2 vectors are

\[ \eta_E|\alpha, p - 2, -\beta, p - 2\rangle_{n, 2} = [2]|\alpha, p - 2, -\beta, p - 2\rangle_{n - 1, 40}, \]
\[ \eta_E|\alpha, p - 2, -\beta, p - 2\rangle_{n, 2} = |\alpha, p - 2, -\beta, p - 2\rangle_{n - 1, 40}, \]
\[ \eta_B|\alpha, p - 2, -\beta, p - 2\rangle_{n, 2} = \alpha[n + 1]|\alpha, p - 2, -\beta, p - 2\rangle_{n, 40}. \]
Direct calculation shows that the above $Z_{p-1,p-1}$, (any of the $\alpha_{\beta}$),

$$\eta_{\beta} - \alpha,1,\beta,0 \rangle_{n,21} = \alpha[2] \delta_{0,n} |\alpha, p-1, \beta, p-1 \rangle_{p-1,50},$$

and those from level 3,

$$\eta_{\alpha} |\alpha, p, \beta, 0 \rangle_{n,33} = |\alpha, p-1, \beta, p-1 \rangle_{n-1,50},$$

$$\eta_{\beta} |\alpha, p, \beta, 0 \rangle_{n,33} = -\alpha[n+1] |\alpha, p-1, \beta, p-1 \rangle_{n-1,50},$$

$$\eta_{\alpha} |\alpha, p, \beta, 0 \rangle_{n,33} = -\alpha[n+2] |\alpha, p-1, \beta, p-1 \rangle_{n,50},$$

$$\eta_{\alpha} |\alpha, p-1, \beta, p-1 \rangle_{n,33} = \alpha |\alpha, p-1, \beta, p-1 \rangle_{n-1,50},$$

$$\eta_{\beta} |\alpha, p-1, \beta, p-1 \rangle_{n,33} = [n] |\alpha, p-1, \beta, p-1 \rangle_{n,50},$$

$$\eta_{\alpha} |\alpha, 2, \beta, 0 \rangle_{n,33} = \delta_{1,n} |\alpha, p-1, \beta, p-1 \rangle_{0,50},$$

$$\eta_{\alpha} |\alpha, 2, \beta, 0 \rangle_{n,33} = -\delta_{0,n} |\alpha, p-1, \beta, p-1 \rangle_{0,50},$$

Direct calculation shows that the above $\eta_4$ and the $c_{l,n}$ read off from the graph ensure that Eqs. (6.2)–(6.4) define a projective $U(X)$ module.

6.3.5. $\mathbb{Z}$: Projective cover of $Z_{s,0}^{\alpha,\beta}$ for $2 \leq s \leq p-1$. The projective cover $Q_{s,0}^{\alpha,\beta}$ of $Z_{s,0}^{\alpha,\beta}$ (any of the $\mathbb{Z}$ in Fig. 4.1) with $2 \leq s \leq p-1$ has 16 simple subquotients and is $16p$-dimensional. Its graph is shown in Fig. 6.3. A feature not encountered in the previous graphs is the occurrence of links from a given subquotient leading to isomorphic
Figure 6.3. Subquotients of the projective module $\Omega_{s,0}^{\alpha,\beta}$ for $2 \leq s \leq p - 1$. 
subquotients on the next level. Two (or more) such links

\[ \begin{align*}
(\alpha \beta)_{\ell_n} & \quad \xrightarrow{c_1} \quad (\alpha' \beta')_{(\ell+1)n_1} \\
& \quad \xrightarrow{c_2} \quad (\alpha' \beta')_{(\ell+1)n_2}
\end{align*} \]

mean that the \( U(X) \) generators map via \( c_1 \xi_1^A + c_2 \xi_2^A \), where \( \xi_i^A : (\alpha \beta)_{\ell_n} \rightarrow (\alpha' \beta')_{(\ell+1)n_i} \) are the maps in \( \mathbb{A}_2 \) associated with each link. In other words, the corresponding basis vectors in the two isomorphic subquotients occur in linear combinations with \( c_1 \) and \( c_2 \) as the coefficients.

The \( \eta_A \) maps from the top-level subquotient are

\[ \begin{align*}
\eta_E(\alpha, s, \beta, 0)_{n,1_0} & = -[s-1][s] \alpha, s, \beta, 0_{n-1,3_0} \\
\eta_E(\alpha, s, \beta, 0)_{n,1_0} & = -[s-1][s] \alpha, s, \beta, 0_{n-1,3_0} \\
\eta_B(\alpha, s, \beta, 0)_{n,1_0} & = \beta [n][s] \alpha, s, \beta, 0_{n-1,3_5} + \alpha [s-1][s] \alpha, s, \beta, 0_{n-1,3_0} \\
& \quad + \alpha \beta [s-1][s] \delta_{0,n} - \alpha, p-s, \beta, p-s_{p-s,4_3} \\
& \quad - \alpha [n][s-1][s] \alpha, s+1, -\beta, 0_{n,4_1} \\
& \quad + \alpha \beta [s][s-1][s-n] - [n][2s-1] \alpha, s, \beta, 0_{n-1,5_0}, \\
\eta_B(\alpha, s, \beta, 0)_{n,1_0} & = \alpha [n+1][s-1][s] \alpha, s+1, -\beta, 0_{n,4_1}, \\
\eta_C(\alpha, s, \beta, 0)_{n,1_0} & = [1-s][s] \delta_{n,s-1} - \alpha, p-s+1, \beta, p-s_{p-s,4_3} \\
& \quad - \alpha [s][s] \alpha, s-1, -\beta, 0_{n,4_0}, \\
\eta_C(\alpha, s, \beta, 0)_{n,1_0} & = [s-1][s] \delta_{n,s-2} - \alpha, p-s+1, \beta, p-s_{p-s,4_3} \\
& \quad - \alpha [s][s] \alpha, s-1, -\beta, 0_{n,4_0} + \alpha [s][s] \alpha, s, \beta, 0_{n+1,5_0}
\end{align*} \]

and those from level 2,

\[ \begin{align*}
\eta_B(-\alpha, p-s, -\beta, p-s)_{n,2_0} & = -\alpha [s-1] \delta_{0,n} - \alpha[s-1] \delta_{0,n} - \alpha, s+1, -\beta, 0_{s,4_1} \\
& \quad - \beta [n+s] - \alpha, p-s, -\beta, p-s_{n,4_2}, \\
\eta_B(-\alpha, p-s, -\beta, p-s)_{n,2_0} & = -\alpha [s-1] \delta_{0,n} - \alpha[s-1] \delta_{0,n} - \alpha, s+1, -\beta, 0_{s-1,4_1}, \\
\eta_B(-\alpha, p-s, -\beta, p-s)_{n,2_1} & = \beta [s-1][n+s] - \alpha, p-s, \beta, p-s_{n,4_3}, \\
\eta_E(-\alpha, s-1, -\beta, 0)_{n,2_2} & = [s-1][s] \alpha, s-1, -\beta, 0_{n-1,4_0},
\end{align*} \]

\(^8\)The isomorphic targets could of course be identified differently, so as to “split” any given triple, but this cannot be done for all such triples simultaneously. We did not attempt to “optimize” the possible linear combinations, in particular because there seems to be no well-defined optimum.
η_E|α, s−1, −β, 0⟩_{n, 2}^{−} = [s−2][s−1]|α, s−1, −β, 0⟩_{n−1, 4_0}^{−},
η_B|α, s−1, −β, 0⟩_{n, 2}^{−} = α[n−s]|α, s−1, −β, 0⟩_{n−1, 4_0}^{−} − α[n]|α, s, β, 0⟩_{n−5_0}^{−},
η_B|α, s−1, −β, 0⟩_{n, 2}^{−} = α[n+1]|α, s, β, 0⟩_{n−5_0}^{−},
η_E|α, s+1, −β, 0⟩_{n, 3}^{−} = [s−1][s+1]|α, s+1, −β, 0⟩_{n−1, 4_1}^{−},
η_E|α, s+1, −β, 0⟩_{n, 3}^{−} = [s−1][s]|α, s+1, −β, 0⟩_{n−1, 4_0}^{−},
η_B|α, s+1, −β, 0⟩_{n, 3}^{−} = −α[s−1][s−n]|α, s+1, −β, 0⟩_{n−1, 4_1}^{−},
η_C|α, s+1, −β, 0⟩_{n, 3}^{−} = [s−1][s]|δ_{n,s} − α, p−s, −β, p−s⟩_{0,4_2}^{−},
η_C|α, s+1, −β, 0⟩_{n, 3}^{−} = [s−1][s]|δ_{n,s−1} − α, p−s, −β, p−s⟩_{0,4_2}^{−},

and from level 3,

η_E|α, s, β, 0⟩_{n, 3}^{−} = −β[n]|α, s, β, 0⟩_{n−1, 5_0}^{−},
η_B|−α, p−s+1, β, p−s+1⟩_{n, 3}^{−} = αδ_{0,n}|α, s, β, 0⟩_{s−2, 5_0}^{−},
η_B|−α, p−s+1, β, p−s+1⟩_{n, 3}^{−} = αδ_{0,n}|α, s, β, 0⟩_{s−2, 5_0}^{−},
η_E|α, s, β, 0⟩_{n, 3}^{−} = [s−1][s]|α, s, β, 0⟩_{n−1, 5_0}^{−},
η_E|α, s, β, 0⟩_{n, 3}^{−} = [s−1][s]|α, s, β, 0⟩_{n−1, 5_0}^{−},
η_B|α, s, β, 0⟩_{n, 3}^{−} = −α[s−1−n]|α, s, β, 0⟩_{n−1, 5_0}^{−}.

6.3.6. ⊁: Projective cover of \( \mathcal{Z}_{s,s}^{α,β} \) for \( 1 \leq s \leq p−2 \). For each \( 1 \leq s \leq p−2 \), the projective module \( \mathcal{Q}_{s,s}^{α,β} \) (any of the ⊁ in Fig. 4.11) is 16p-dimensional and has 16 subquotients. Its graph is shown in Fig. 6.4, with all the previous conventions in force.

The \( \eta_A \) piece of the action of \( U(X) \) generators on \( \mathcal{Q}_{s,s}^{α,β} \) is as follows. On the basis vectors of the top-level subquotient, we have

\[ \eta_E|α, s, β, s⟩_{n, 1_0}^{−} = −[s]^2[s+1]|α, s, β, s⟩_{n−1, 3_0}^{−} + 2β[s]^2[s+1]|α, s, β, s⟩_{n−1, 5_0}^{−}, \]
\[ \eta_E|α, s, β, s⟩_{n, 1_0}^{−} = −[s][s+1]|α, s, β, s⟩_{n−1, 3_0}^{−} + 3β[s][s+1]|α, s, β, s⟩_{n−1, 5_0}^{−}, \]
\[ \eta_B|α, s, β, s⟩_{n, 1_0}^{−} = −α[n][s+1]|α, s, β, s⟩_{n, 3_0}^{−} − α[s][s+1]|δ_{0,n} − α, p−s+1, β, 0⟩_{p−s, 3_0}^{−}, \]
\[ −β[s][s−n]|α, s, β, s⟩_{n−1, 4_0}^{−}, \]
\[ + αβ[s+1][n−s]|α, s, β, s⟩_{n−1, 5_0}^{−} , \]
\[ \eta_B|α, s, β, s⟩_{n, 1_0}^{−} = α[n+1][s][s+1]|α, s−1, −β, s−1⟩_{n−1, 4_0}^{−}, \]
\[ −α[s][s+1]|δ_{0,n} − α, p−s+1, β, 0⟩_{p−s, 3_0}^{−}, \]

On basis vectors of the level-2 modules, the \( \eta \) maps are

\[ \eta_E|−α, p−s, β, 0⟩_{n, 2_0}^{−} = 2β[s]|δ_{0,n} − α, s, β, s⟩_{s−1, 5_0}^{−}, \]
\[ \eta_E|−α, p−s, β, 0⟩_{n, 2_0}^{−} = 3β[s+1]|δ_{0,n} − α, s, β, s⟩_{s−1, 5_0}^{−}, \]
Figure 6.4. Graph of the projective module $Q_{s,s}^{\alpha,\beta}$ for $1 \leq s \leq p - 2$. 
\[ \eta_B|\alpha, p-s, \beta, 0 \rangle_{n, 2} = \beta|n\rangle - \alpha, p-s, \beta, 0 \rangle_{n-1, 4_2} + 2\alpha\beta \delta_{0, n}|\alpha, s, \beta, s \rangle_{n-1, 5_0}, \]
\[ \eta_E|\alpha, s-1, -\beta, s-1 \rangle_{n, 2} = [s-1][s][s+1]|\alpha, s-1, -\beta, s-1 \rangle_{n-1, 4_0}, \]
\[ \eta_E|\alpha, s-1, -\beta, s-1 \rangle_{n, 2} = [s][s+1]|\alpha, s-1, -\beta, s-1 \rangle_{n-1, 4_0}, \]
\[ \eta_B|\alpha, s-1, -\beta, s-1 \rangle_{n, 2} = \alpha[n+1][s+1]|\alpha, s-1, -\beta, s-1 \rangle_{n, 4_0}, \]
\[ \eta_C|\alpha, s-1, -\beta, s-1 \rangle_{n, 2} = \alpha|\alpha, s, \beta, s \rangle_{n+1, 5_0}, \]
\[ \eta_F|\alpha, p-s, -\beta, 0 \rangle_{n, 2} = -2\beta \delta_{n, p-s-1}|\alpha, s, \beta, s \rangle_{n, 5_0}, \]
\[ \eta_F|\alpha, p-s, -\beta, 0 \rangle_{n, 2} = 3\beta \delta_{n, p-s-2}|\alpha, s, \beta, s \rangle_{n, 5_0}, \]
\[ \eta_C|\alpha, p-s, -\beta, 0 \rangle_{n, 2} = -\beta|n\rangle - \alpha, p-s, -\beta, 0 \rangle_{n-1, 4_3}, \]
\[ \eta_C|\alpha, p-s, -\beta, 0 \rangle_{n, 2} = -\beta|n\rangle - \alpha, p-s, -\beta, 0 \rangle_{n-1, 4_3}, \]
\[ \eta_E|\alpha, s+1, -\beta, s+1 \rangle_{n, 2} = [s+1][s+2]|\alpha, s+1, -\beta, s+1 \rangle_{n-1, 4_1}, \]
\[ \eta_E|\alpha, s+1, -\beta, s+1 \rangle_{n, 2} = [s+1][s+1]|\alpha, s+1, -\beta, s+1 \rangle_{n-1, 4_1}, \]
\[ \eta_B|\alpha, s+1, -\beta, s+1 \rangle_{n, 2} = \alpha[n]|\alpha, s+1, -\beta, s+1 \rangle_{n, 4_1} - \alpha[n+1][s+1]|\alpha, s, \beta, s \rangle_{n-1, 5_0}, \]
\[ \quad - \alpha|s+1\delta_{0, n}-\alpha, p-s, -\beta, 0 \rangle_{p-s-1, 4_3}, \]
\[ \eta_B|\alpha, s+1, -\beta, s+1 \rangle_{n, 2} = \alpha[n+1][s+1]|\alpha, s, \beta, s \rangle_{n-1, 5_0}, \]
\[ \quad - \alpha|s+1\delta_{0, n}-\alpha, p-s, -\beta, 0 \rangle_{p-s-2, 4_3} \]

and on level-3 vectors,
\[ \eta_C|\alpha, s, \beta, s \rangle_{n, 3} = |\alpha, s, \beta, s \rangle_{n, 5_0}, \]
\[ \eta_E|\alpha, s, \beta, s \rangle_{n, 3} = [s][s+1]|\alpha, s, \beta, s \rangle_{n-1, 5_0}, \]
\[ \eta_E|\alpha, s, \beta, s \rangle_{n, 3} = [s][s+1]|\alpha, s, \beta, s \rangle_{n-1, 5_0}, \]
\[ \eta_B|\alpha, s, \beta, s \rangle_{n, 3} = \alpha[n+1]|\alpha, s, \beta, s \rangle_{n, 5_0}, \]
\[ \eta_C|\alpha, p-s+1, \beta, 0 \rangle_{n, 3} = \delta_{n, p-s} |\alpha, s, \beta, s \rangle_{n, 5_0}, \]
\[ \eta_C|\alpha, p-s+1, \beta, 0 \rangle_{n, 3} = -\delta_{n, p-s-1} |\alpha, s, \beta, s \rangle_{n, 5_0}, \]

6.3.7. \(*\): Projective cover of \(Z^\alpha_{1,0, \beta} \). We finally describe projective covers of the one-dimensional representations (\(\square\) in Fig. 4.1). The projective module \(\Omega^\alpha_{1,0, \beta} \) has dimension 24\(p\) and is built from 24 simple subquotients. Its rather involved graph is shown in Fig. 6.5. Maps into linear combinations of isomorphic subquotients, which already occurred in Figs. 6.3 and 6.4 here involve up to four modules.

The \(\eta\) piece of the action of \(U(X)\) generators is as follows. The maps from the top are
\[ \eta_F|\alpha, 1, \beta, 0 \rangle_{0, 1, 0} = 2\alpha[2]|\alpha, p-1, -\beta, p-1 \rangle_{0, 4_3} \]
FIGURE 6.5. Graph of the projective module $Q_{(α, β)}^{1,0}$
\[ \eta_E|\alpha, 1, \beta, 0\rangle_{n, 2} = 2\alpha[2] - \alpha \cdot p - 1, \beta, p - 1 \rangle_{p, 2, 4} \]

and those from level 2 are
\[ \eta_E|\alpha, 1, \beta, 1\rangle_{n, 2} = [2]\alpha, 1, \beta, 1 \rangle_{n-1, 4, 1} \]
\[ \eta_B|\alpha, 1, \beta, 1\rangle_{n, 2} = \alpha \delta_0, n - \alpha \cdot p - 1, \beta, p - 1 \rangle_{p-1, 4, 2} \]
\[ \eta_B|\alpha, 1, \beta, 1\rangle_{n, 2} = \alpha \delta_0, n - \alpha \cdot p - 1, \beta, p - 1 \rangle_{p-2, 4, 2} \]
\[ \eta_E|\alpha, 2, -\beta, 0\rangle_{n, 2} = [2]\alpha, 2, -\beta, 0 \rangle_{n-1, 4, 2} \]
\[ \eta_C|\alpha, 2, -\beta, 0\rangle_{n, 2} = \delta_1, n - \alpha \cdot p - 1, -\beta, p - 1 \rangle_{p-4, 3} \]
\[ \eta_C|\alpha, 2, -\beta, 0\rangle_{n, 2} = -\delta_0, n - \alpha \cdot p - 1, -\beta, p - 1 \rangle_{p-4, 4} \]
\[ \eta_E|-\alpha, p, -\beta, 0\rangle_{n, 2} = 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 2} - 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 3} \]
\[ \eta_B|-\alpha, p, -\beta, 0\rangle_{n, 2} = 2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 0} \]
\[ \eta_B|-\alpha, p, -\beta, 0\rangle_{n, 2} = -2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 3} \]
\[ \eta_E|-\alpha, p - 1, -\beta, p - 1 \rangle_{n, 2} = 2\alpha \cdot p - 1, -\beta, p - 1 \rangle_{n-1, 4, 0} \]
\[ \eta_B|-\alpha, p - 1, -\beta, p - 1 \rangle_{n, 2} = 2\alpha \cdot p - 1, -\beta, p - 1 \rangle_{n-1, 4, 3} \]
\[ \eta_E|-\alpha, p, -\beta, 0 \rangle_{n, 2} = 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 2} - 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 3} \]
\[ \eta_B|-\alpha, p, -\beta, 0 \rangle_{n, 2} = 2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 0} \]
\[ \eta_B|-\alpha, p, -\beta, 0 \rangle_{n, 2} = -2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 3} \]
\[ \eta_B|-\alpha, p, -\beta, 0 \rangle_{n, 2} = 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 2} - 2\alpha \cdot p, -\beta, 0 \rangle_{n-1, 4, 3} \]
\[ \eta_B|-\alpha, p, -\beta, 0 \rangle_{n, 2} = 2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 0} \]
\[ \eta_B|-\alpha, p, -\beta, 0 \rangle_{n, 2} = -2\alpha[n+1] - \alpha, p, -\beta, 0 \rangle_{n-1, 4, 3} \]

Together with the \(c^{X, 1}\) specified on the links, this defines a \(U(X)\) module, which is then seen to be maximal and indecomposable.

6.4. Completeness. We have constructed projective covers \(\Omega_i \to \mathbb{Z}_i\) for all simple \(U(X)\) modules. That these are all projective modules can also be verified by calculating the sum \(\sum \dim \Omega_i \cdot \dim \mathbb{Z}_i\):

\[ (6.6) \quad 4 \left( (p - 1) 4p + (p - 2) \sum_{s=1}^{p-1} 8p \cdot 4s + (12p) \cdot (2p - 1) + (12p) \cdot (2p - 1) \right) \]
(the overall 4 is for the values taken by $\alpha$ and $\beta$).

6.5. As an immediate corollary of the structure of projective $U(X)$-modules, we obtain the minimal polynomials for the Casimir elements in $2.3$ simply by finding the eigenvalues on modules from each linkage class and taking the Jordan-cell size into account (which is 3 for the atypical linkage class, 2 for each of the typical linkage classes, and 1 for each of the Steinberg classes; hence the multiplicities of the minimal-polynomial roots in $2.3$).

As a less trivial (although generally straightforward) corollary, with find the $U(X)$ center.

7. The $U(X)$ Center

7.1. Theorem. The algebra $U(X)$ has a $(5p^2 - p + 4)$-dimensional center. The center $Z$ decomposes into the direct sum

$$Z = \bigoplus_{j=1}^{p^2+p-1} \mathbb{C} \cdot e_j \bigoplus_{j=1}^{4p^2-2p+5} \mathbb{C} \cdot w_j$$

of linear subspaces generated by primitive central idempotents $e_j$ and central nilpotents $w_j$. The block decomposition of the center as an associative algebra is

$$Z = Z_{at} \bigoplus_{j=1}^{(p-1)(p-2)} Z^j_{tt} \bigoplus_{j=1}^{4(p-1)} Z^j_{st}$$

where $Z_{at}$ corresponds to atypical $Z^j_t$ to typical and $Z^j_{st}$ to Steinberg linkage class.

This theorem is an application of the construction of projective $U(X)$ modules. We calculate the $U(X)$ center as the center of the basic algebra (an approach also taken for $U_q sl(2)$ in [46]).

7.2. The basic algebra of $U(X)$. The basic algebra is the algebra of endomorphisms of the direct sum of projective modules taken with multiplicity 1 each. Basic algebra generators can be chosen as primitive idempotents and nilpotents: (i) each primitive idempotent $e_Q$ is the projector on a single projective module $Q$, and (ii) each nilpotent $w_{Q,n}$ is a morphism $Q \to Q'$ defined uniquely by the condition that it sends the top subquotient of $Q$ into an isomorphic subquotient on level two in a projective module $Q'$, is an isomorphism of these subquotients, and acts by zero on all projective modules other than $Q$. Hence, $n = 1, \ldots, N_Q$, where $N_Q$ is the number of level-2 subquotients in the linkage class that
are isomorphic to the top subquotient of \( \Omega \); it is in fact equal to the number of level-2 subquotients in \( \Omega \).

We describe this in more detail, invoking the structure of various projective modules:

(1) Each of the \( 4(p - 1) \) Steinberg modules (simple projective modules) contributes only an idempotent to the basic algebra generators. These idempotents are central.

(2) “Typical” projective modules contribute \( 4(p - 1)(p - 2) \) idempotents (projectors on each of typical projective modules) and \( 8(p - 1)(p - 2) \) nilpotents: for each typical projective module, these are two maps (distinguished by \( \pm \) in front of \( \beta \))

\[
Q_{s, r}^{\alpha, \beta} \rightarrow Q_{p-3, p-r-s}^{-\alpha, \pm \beta}, \quad r \neq 0, s
\]

sending the top subquotient into a level-2 subquotient.

(3) For the atypical projective modules, there are \( 4(p - 2) \) idempotents. As regards maps to level two in projective modules, we see from the graphs in Figs. 6.1–6.5 that the relevant number \( N = N_\Omega \) of such maps is as follows for the five species of atypical projective modules:

\[
\begin{align*}
Q_{p, 0}^{\alpha, \beta} \quad (\square) : & \quad N = 3, & Q_{p-1, p-1}^{\alpha, \beta} \quad (\square) : & \quad N = 3, \\
Q_{s, 0}^{\alpha, \beta} \quad (\square) : & \quad N = 4, & Q_{s, s}^{\alpha, \beta} \quad (\square) : & \quad N = 4, \\
Q_{1, 0}^{\alpha, \beta} \quad (\square) : & \quad N = 6.
\end{align*}
\]

There are respectively 4, 4, \( 4(p - 2) \), \( 4(p - 2) \), and 4 projective modules of each species, which gives the total of \( 4 \cdot 3 + 4 \cdot 3 + 4(p - 2) \cdot 4 + 4(p - 2) \cdot 4 + 4 \cdot 6 = 32p - 16 \) “atypical” generators \( \Omega_j \) of the basic algebra.

7.3. The dimension of the center. We now list the generators of the center. These are primitive central idempotents (which are enumerated immediately) and central nilpotents (finding which requires more work). Their total number (see 7.3.3 below) gives the dimension of the center of \( U(X) \).

7.3.1. Central idempotents and nilpotents. Each linkage class (see the list in 5.3) yields a primitive central idempotent, the projection onto that linkage class. In addition, the linkage classes except the Steinberg ones yield several central nilpotents each. The items describing them are listed below in the order of (rapidly) increasing complexity.

(1) Each of the \( 4(p - 1) \) Steinberg linkage classes produces a single central idempotent \( e_{r}^{\alpha, \beta} \) \( (r = 1, \ldots, p - 1) \).

(2) Each of the \( (p - 1)(p - 2) \) “typical” linkage classes yields one central idempotent \( e_{s, r}^{\alpha} \) and four central nilpotents \( w_{s, r}^{\alpha, \beta}(\gamma) \), where \( 1 \leq r < s \leq p - 1 \) and \( \alpha, \beta, \gamma = \pm \) (note that \( \alpha, s, \) and \( r \) enumerate a linkage class, while \( \beta \) and \( \gamma \) range over nilpotents inside a linkage class). These \( w_{s, r}^{\alpha, \beta}(\gamma) \) are the maps

\[
w_{s, r}^{\alpha, \beta}(+) : Q_{s, r}^{\alpha, \beta} \rightarrow Q_{s, r}^{\alpha, \beta}, \quad r \neq 0, s
\]
REPRESENTATIONS OF $\mathfrak{u}_q(s^\ell(2|1))$ AT EVEN ROOTS OF UNITY

and

$$w_{s,r}^{\alpha,\beta} : Q_{p-s,p+r-s}^\alpha \rightarrow Q_{p-s,p+r-s}^\alpha, \quad r \neq 0, s,$$

sending the top subquotient to the bottom subquotient in the same projective module.

(3) The “atypical” linkage class yields one central idempotent and central nilpotents that are of two groups: some follow immediately (item a) and the derivation of others is somewhat more involved (item b).

(a) There are $4(2p-1)$ central nilpotents $w_a^{\alpha,\beta}$, $a = 1, \ldots, 2p-1, \alpha, \beta = \pm$, one for each atypical projective module ($Q_{s,0}^{\alpha,\beta}$ with $1 \leq s \leq p$ and $Q_{s,s}^{\alpha,\beta}$ with $1 \leq s \leq p-1$). Each $w_a^{\alpha,\beta}$ maps the top subquotient into the isomorphic bottom subquotient in the same projective module (and is zero on all other projective modules). It then follows that $w_a^{\alpha,\beta} w_a^{\alpha',\beta'} = 0$ for all values of the indices.

(b) In addition, there are central nilpotents $w_b$ given by linear combinations of noncentral idempotents $W_B$ defined as follows. Each $W_B$ is a map from the top subquotient in one of the atypical projective modules into an isomorphic subquotient at level three in the same projective module. We can therefore write $W_B = W_{\Omega,m}$, where $\Omega$ is a projective module and $m$ labels its level-3 subquotients that are isomorphic to its top subquotient.

The graphs in Figs. 6.1–6.5 readily show that the number $M$ of such subquotients and hence the number of $W_B$ yielded by each species of projective modules is as follows:

- $Q_{p,0}^{\alpha,\beta} (\Box) : M = 4,$
- $Q_{s,0}^{\alpha,\beta}_{2 \leq s \leq p-1} (\square) : M = 4 \cdot 2(p-2),$  
- $Q_{p-1,p-1}^{\alpha,\beta} (\bigcirc) : M = 4,$  
- $Q_{s,s}^{\alpha,\beta}_{1 \leq s \leq p-2} (\bigotimes) : M = 4 \cdot 2(p-2),$  
- $Q_{1,0}^{\alpha,\beta} (\blacklozenge) : M = 4 \cdot 4.$

(a factor of 4 in each case is of course due to the four values taken by $(\alpha, \beta)$).

This gives a $(16p - 8)$-dimensional space of noncentral “level-3” nilpotents. Taking their linear combinations

$$w = \sum_{m} x_{\Omega,m} W_{\Omega,m} = \sum_{B=1}^{16p - 8} x_B W_B,$$

we require the commutativity with the basic algebra generators. This $w$ already commutes with all idempotents (which act either as identity or by zero) and, evidently, with the all “typical” nilpotents. It remains to require that it commute with the basic algebra generators described in item 3 in 7.2 (page 38):

(7.3)  

$$w_{\Omega_j} - \Omega_j w = 0, \quad j = 1, \ldots, 32p - 16,$$
Solving Eqs. (7.3) for the $x_{\Omega,m}$, we find a $(2p + 1)$-dimensional subspace of central “level-3” nilpotents.

7.3.2. Theorem. There are exactly $2p + 1$ linearly independent solutions $w_b$ of Eqs. (7.3).

We prove this in B.4 by deriving the explicit form of the equations and solving them.

7.3.3. As a corollary, we calculate the dimension of the center of $U(X)$ as follows, with “idem.” and “nilp.” referring to the idempotents and nilpotents described in items 1–3 above:

$$\dim Z = 4(p - 1)_{\text{idem., item 1}} + (p - 1)(p - 2)_{\text{idem., item 2}} + \frac{1}{p + 1}_{\text{idem., item 3}}$$

$$+ 4(p - 1)(p - 2)_{\text{nilp., item 2}} + 10p - 3_{\text{nilp., item 3}}$$

$$= 5p^2 - p + 4.$$

7.3.4. Remark. It follows immediately that the algebra of the $w_b$ (solutions of (7.3)) is

$$w_b w_c = \sum_{j, \alpha, \beta} f_{b,c}^{\alpha, \beta, j} w_j^{\alpha, \beta}$$

where all nonzero constants $f_{b,c}^{\alpha, \beta, j}$ can be chosen equal to 1 by rescaling the $w_j^{\alpha, \beta}$.

8. Conclusions

The full structure of projective modules is a powerful tool in studying an associative algebra. Various consequences of the construction in this paper are to be worked out with regard to the properties relevant for the LCFT counterpart of our $U_q(sl(2|1))$. The results can have a bearing on various facets of LCFT models, in the range from “spin chains,” potentially allowing physical applications (see [47, 48, 49] and the references therein), to no less exciting “categorial studies” (see [50, 51, 52] and the references therein). A spin chain that suggests itself in relation to our $U_q(sl(2|1))$ is the one composed of alternating “fundamental” ($s = 1, r = 1$) and “antifundamental” ($s = 2, r = 0$) 3-dimensional representations. A relevant tool in its study would be Lusztig’s divided-power version of the algebra. As regards the relation to “continuous” LCFT, it is sensitive to the Hopf algebra $H$ that defines the corresponding category of Yetter–Drinfeld modules [33]. As we see, $H$ comes with its universal $R$-matrix, which also is to play a role in CFT. Of primary interest is the modular group action on (a subalgebra in) the $U_q(sl(2|1))$ center, which is also linked to the study of the corresponding LCFT torus amplitudes and their modular and other properties. Tensor products of $U_q(sl(2|1))$ modules conjecturally correspond to fusion on the LCFT side.

We thank B. Feigin, A. Gainutdinov, and A. Kiselev for the useful discussions. This paper was supported in part by the RFBR grant 13-01-00386.
A.1. Simple $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$ modules [27]. The $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$ Hopf algebra is defined in Eqs. (2.1). Its simple modules can be labeled as $\mathcal{X}_s^\alpha$, where $\alpha = \pm$ and $s = 1, \ldots, p$ (with dim $\mathcal{X}_s^\alpha = s$). A basis in $\mathcal{X}_s^\alpha$ is denoted by $|\alpha, s\rangle_n$, $n = 0, \ldots, s - 1$, with $|\alpha, s\rangle_0$ being a highest-weight vector:

$$E|\alpha, s\rangle_0 = 0, \quad K|\alpha, s\rangle_0 = \alpha q^{s-1}|\alpha, s\rangle_0.$$ 

The $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$ action on all the $|\alpha, s\rangle_n$, in a “naturally isomorphic” notation, is given in [27].

A.2. Projective $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$ modules. The projective $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$-module $\mathcal{B}_{s,r}^{\alpha,\beta}$ that covers $\mathcal{X}_{s,r}^{\alpha,\beta}$ has a basis $|\alpha, s, \beta, r\rangle_n^a$, $|\alpha, s, \beta, r\rangle_n^b$ with $n = 0, \ldots, s - 1$ and $|\alpha, s, \beta, r\rangle_m^x$, $|\alpha, s, \beta, r\rangle_m^y$ with $m = 0, \ldots, p - s - 1$. The action of $\overline{\mathbb{U}}_q\mathfrak{sl}(2)$ in this basis is given in [27], and the action of $k$ is

$$k|\alpha, s, \beta, r\rangle_n^a = \beta q^{-r+n}|\alpha, s, \beta, r\rangle_n^a, \quad k|\alpha, s, \beta, r\rangle_n^b = \beta q^{-r+n}|\alpha, s, \beta, r\rangle_n^b,$$

$$k|\alpha, s, \beta, r\rangle_n^x = \beta q^{s-r-n-p}|\alpha, s, \beta, r\rangle_n^x, \quad k|\alpha, s, \beta, r\rangle_n^y = \beta q^{s-r-n}|\alpha, s, \beta, r\rangle_n^y.$$

A.3. The rest of the $\mathcal{U}(X)$ action on simple modules. For completeness, we give the formulas that fully define the $\mathcal{U}(X)$ action on simple modules in [4.2.1] (the action of fermionic generators on basis vectors was already described there).

For the atypical $\mathcal{Z}_{s,0}^{\alpha,\beta}$ modules in (4.2), we have

$$K|\alpha, s, \beta, 0\rangle_n^- = \alpha q^{s-2n-1}|\alpha, s, \beta, 0\rangle_n^-, \quad k|\alpha, s, \beta, 0\rangle_n^- = \beta q^n|\alpha, s, \beta, 0\rangle_n^-,$$

$$F|\alpha, s, \beta, 0\rangle_n^- = |\alpha, s, \beta, 0\rangle_{n+1}^-, \quad E|\alpha, s, \beta, 0\rangle_n^- = \alpha[n][s-n]|\alpha, s, \beta, 0\rangle_{n-1}^-,$$

$$K|\alpha, s, \beta, 0\rangle_m^+ = \alpha q^{s-2m-2}|\alpha, s, \beta, 0\rangle_m^+, \quad k|\alpha, s, \beta, 0\rangle_m^+ = -\beta q^{m+1}|\alpha, s, \beta, 0\rangle_m^+,$$

$$F|\alpha, s, \beta, 0\rangle_m^+ = |\alpha, s, \beta, 0\rangle_{m+1}^+, \quad E|\alpha, s, \beta, 0\rangle_m^+ = \alpha[m][s-m-1]|\alpha, s, \beta, 0\rangle_{m-1}^+,$$

where we set $|\alpha, s, \beta, 0\rangle_n^- = 0$ for $n < 0$ and $n > s - 1$, and $|\alpha, s, \beta, 0\rangle_m^+ = 0$ for $m < 0$ and $m > s - 2$. For $s = 1$, each module $\mathcal{Z}_{1,0}^{\alpha,\beta}$ with $\alpha, \beta = \pm$ is 1-dimensional. The basis consists of a single vector $|\alpha, 1, \beta, 0\rangle_0^-$ such that

$$K|\alpha, 1, \beta, 0\rangle_0^- = \alpha|\alpha, 1, \beta, 0\rangle_0^- - \beta|\alpha, 1, \beta, 0\rangle_0^-,$$

with the other generators acting trivially. For convenience in what follows (in the cases where $\mathcal{Z}_{1,0}^{\alpha,\beta}$ occur along with higher-dimensional modules), we set $|\alpha, 1, \beta, 0\rangle_m^+ = 0$ for all $m$.

For the atypical $\mathcal{Z}_{s,s}^{\alpha,\beta}$ modules in (4.3), we have

$$K|\alpha, s, \beta, s\rangle_n^- = \alpha q^{s-2n-1}|\alpha, s, \beta, s\rangle_n^-, \quad k|\alpha, s, \beta, s\rangle_n^- = \beta q^{-s+n}|\alpha, s, \beta, s\rangle_n^-,$$

$$F|\alpha, s, \beta, s\rangle_n^- = |\alpha, s, \beta, s\rangle_{n+1}^-, \quad E|\alpha, s, \beta, s\rangle_n^- = \alpha[n][s-n]|\alpha, s, \beta, s\rangle_{n-1}^-,$$

$$K|\alpha, s, \beta, s\rangle_m^+ = \alpha q^{s-2m}|\alpha, s, \beta, s\rangle_m^+, \quad k|\alpha, s, \beta, s\rangle_m^+ = -\beta q^{s+m}|\alpha, s, \beta, s\rangle_m^+,$$
where we set \( |\alpha, s, \beta, s_m^-\rangle = 0 \) for \( n < 0 \) and \( n > s - 1 \), and \( |\alpha, s, \beta, s_m^-\rangle = 0 \) for \( m < 0 \) and \( m > s \).

For the typical modules \( z_{s, s}^{\alpha, \beta} \) in (4.4),

\[
K |\alpha, s, \beta, r_j^-\rangle = \alpha q^{s-1-j} |\alpha, s, \beta, r_j^-\rangle, \quad k |\alpha, s, \beta, r_j^-\rangle = \beta q^{r+j} |\alpha, s, \beta, r_j^-\rangle,
\]

\[
F |\alpha, s, \beta, r_j^-\rangle = |\alpha, s, \beta, r_j^-\rangle + \alpha |\alpha, s, \beta, r_{j+1}^-\rangle, \quad E |\alpha, s, \beta, r_j^-\rangle = \alpha^2 |\alpha, s, \beta, r_{j-1}^-\rangle,
\]

\[
K |\alpha, s, \beta, r_m^\uparrow\rangle = \alpha q^{s-1} |\alpha, s, \beta, r_m^\uparrow\rangle, \quad k |\alpha, s, \beta, r_m^\uparrow\rangle = \beta q^{-r+m} |\alpha, s, \beta, r_m^\uparrow\rangle,
\]

\[
F |\alpha, s, \beta, r_m^\uparrow\rangle = |\alpha, s, \beta, r_m^\uparrow\rangle + \alpha |\alpha, s, \beta, r_{m+1}^\uparrow\rangle, \quad E |\alpha, s, \beta, r_m^\uparrow\rangle = \alpha^2 |\alpha, s, \beta, r_{m-1}^\uparrow\rangle,
\]

where we as usual assume that the vectors outside the ranges specified in (4.5) are equal to zero. If \( s = 1 \), the decomposition degenerates to \( z_{1, r}^{\alpha, \beta} = \chi_{1, r}^{\alpha, \beta} \oplus \chi_{2, r}^{\alpha, -\beta} \oplus \chi_{1, r-1}^{\alpha, \beta} \), with a basis \( |\alpha, 1, \beta, r_0^-\rangle, (|\alpha, 1, \beta, r_{m+1}^\downarrow\rangle)_{m=0,1}, |\alpha, 1, \beta, r_0^\uparrow\rangle \), and with the \( U(X) \) action easily deducible from the general case above.

For the Steinberg modules \( z_{s, s}^{\alpha, \beta} \) in (4.6),

\[
K |\alpha, p, \beta, r_{j}\rangle = \alpha q^{s-1} |\alpha, p, \beta, r_{j}\rangle, \quad k |\alpha, p, \beta, r_{j}\rangle = \beta q^{r-j} |\alpha, p, \beta, r_{j}\rangle,
\]

\[
F |\alpha, p, \beta, r_{j}\rangle = |\alpha, p, \beta, r_{j}\rangle + \alpha |\alpha, p, \beta, r_{j+1}\rangle, \quad E |\alpha, p, \beta, r_{j}\rangle = -\alpha |\alpha, p, \beta, r_{j-1}\rangle,
\]

\[
K |\alpha, p, \beta, r_{n}\rangle = \alpha q^{s-1} |\alpha, p, \beta, r_{n}\rangle, \quad k |\alpha, p, \beta, r_{n}\rangle = \beta q^{r-n} |\alpha, p, \beta, r_{n}\rangle,
\]

\[
F |\alpha, p, \beta, r_{n}\rangle = |\alpha, p, \beta, r_{n}\rangle + \alpha |\alpha, p, \beta, r_{n+1}\rangle, \quad E |\alpha, p, \beta, r_{n}\rangle = -\alpha |\alpha, p, \beta, r_{n-1}\rangle,
\]

APPENDIX B. PROOFS AND CALCULATION DETAILS

B.1. The universal \( R \)-matrix of \( U(X) \). We calculate the universal \( R \)-matrix for \( U(X) \) by relating \( U(X) \) to the Drinfeld double of \( U_{\leq} \) (see decomposition (2.4)).
B.1.1. The generators $E$ and $C$ can be viewed as functionals on the subalgebra $U_\leq$ such that
\[
\langle E, FK^i k_j \rangle := -\frac{1}{q-q^{-1}}, \quad \langle C, BK^i k_j \rangle := \frac{1}{q-q^{-1}},
\]
and all other evaluations of $E$ and $C$ on the PBW basis elements in $U_\leq$ vanish. We let $\tilde{U}_>$ temporarily denote the algebra generated by the functionals $E$ and $C$ with the product of any two functionals $\beta$ and $\gamma$ defined standardly as
\[
\langle \beta \gamma, x \rangle = \langle \beta, x' \rangle \langle \gamma, x'' \rangle,
\]
where $\Delta x = x' \otimes x''$ is the coproduct on $U_\leq$. It then readily follows that the relations $ECE - [2]ECE + CCE = 0$ and $CC = 0$ hold in $\tilde{U}_>$. Hence, polynomials in $E$ and $C$ can be expressed linearly in terms of $E^i, E^{i+1}C, E^iCE, E^iCEC, i \geq 0$. By induction on the power of the relevant generator, we then find that these tentative PBW basis elements have the following nonzero evaluations on the PBW basis elements in $U_\leq$:
\[
\begin{align*}
&\langle E^n, F^n K^i k_j \rangle = (-1)^n(q-q^{-1})^{-n}q^{\frac{1}{2}n(n-1)}[n]!, \\
&\langle E^{n+1}C, F^n BF K^i k_j \rangle = (-1)^{n+1}(q-q^{-1})^{-n-2}q^{\frac{1}{2}(n+2)(n-1)}[n+1]!, \\
&\langle E^n C, F^n BK^i k_j \rangle = (-1)^n(q-q^{-1})^{-n-1}q^{\frac{1}{2}n(n-1)}[n]!, \\
&\langle E^n CE, F^n BF K^i k_j \rangle = (-1)^{n+1}(q-q^{-1})^{-n-2}q^{\frac{1}{2}n(n-1)}(1 + q^{-n-1}[n])[n]!, \\
&\langle E^n CE, F^n+1 BK^i k_j \rangle = (-1)^{n+1}(q-q^{-1})^{-n-2}q^{\frac{1}{2}(n+2)(n-1)}[n+1]!, \\
&\langle E^n CEC, F^n BF BK^i k_j \rangle = (-1)^{n+1}(q-q^{-1})^{-n-2}q^{\frac{1}{2}(n+1)(n-2)}[n]!
\end{align*}
\]
(and all other evaluations vanish). It then follows that $E^p = 0$ in $\tilde{U}_>$, and $\tilde{U}_>$, with its PBW basis modeled on that in $U_>$, is isomorphic to $U_>$ as an algebra.

To diagonalize the “nondiagonal” part of the pairing $U_> \otimes U_\leq \to \mathbb{C}$, we define
\[
\begin{align*}
X_n &= [n]E^n - CE - (q^{2-n} + [n-1])E^n C, \\
Y_n &= E^n - CE - q^{-1}E^n C.
\end{align*}
\]
Then
\[
\begin{align*}
&\langle X_n, F^n BK^i k_j \rangle = (-1)^{n+1}q^{\frac{1}{2}(n-1)(n-2)}[n]!, \quad \langle X_n, F^n-1 BF K^i k_j \rangle = 0, \\
&\langle Y_n, F^n BK^i k_j \rangle = 0, \quad \langle Y_n, F^n-1 BF K^i k_j \rangle = (-1)^{n}q^{\frac{1}{2}n(n-3)}[n-1]!.
\end{align*}
\]

B.1.2. We next extend $\tilde{U}_>$ by generators $L, \ell \in H^*$, also functionals on $U_\leq$, that we require to commute with the generators of $U_\leq$ exactly as $K$ and $k$ do:
\[
\begin{align*}
FL &= q^2 LF, & BL &= q^{-1} LB, \\
F\ell &= q^{-1}\ell F, & B\ell &= -\ell B,
\end{align*}
\]
where for \(a \in U_\ll\) and a functional \(\beta\), we evaluate the product using the Drinfeld-double formula

\[
(a \bar{\beta} \rightarrow \beta 
\]

where \(\Delta a = a' \otimes a''\) is the coproduct on \(U_\ll\). It then follows that

\[
\langle L, K^n k^m \rangle = q^{n-2m}, \quad \langle \ell, K^n k^m \rangle = (-1)^n q^m
\]

(with the other evaluations vanishing). Using (B.1), we then establish further “cross-commutator” relations:

\[
FE - EF = \frac{L - K^{-1}}{q - q^{-1}}, \quad BC - CB = -\frac{\ell - k^{-1}}{q - q^{-1}},
\]

as well as \(FC - CF = 0\) and \(BE - EB = 0\) (and \(KL =LK\,\text{etc.}\)).

This essentially (modulo easily reconstructible details) shows that \(U(X)\) is the quotient of the Drinfeld double of \(U_\ll\) by the Hopf ideal generated by the relations \(L = K\) and \(\ell = k\). The (inverse) universal \(R\)-matrix is then inherited from the Drinfeld double in the form

\[
R^{-1} = \sum_{a=0}^{p-1} (-1)^a \frac{q^{-\frac{1}{2}a(a-1)(q - q^{-1})a}}{[a]!} \left( E^a \otimes F^a - q^{-1} X_a \otimes F^a B \right.
\]

\[
- q(q - q^{-1}) Y_{a+1} \otimes F^a B F - q(q - q^{-1})^2 E^a CEC \otimes F^a BFB \left) \rho^{-1},
\]

where

\[
\rho^{-1} = \frac{1}{(2p)^2} \sum_{j=0}^{2p-1} \sum_{m=0}^{2p-1} \sum_{n=0}^{2p-1} \sum_{i=0}^{2p-1} (-1)^i q^{2im- in - jm} K^i k^j \otimes K^m k^n.
\]

From here, the universal \(R\)-matrix \(R = (S^{-1} \otimes \text{id}) R^{-1}\) (where \(S\) is the antipode of \(U(X)\)) follows in the form \(R = \rho \bar{R}\), with \(\rho\) in (2.5) and

\[
\bar{R} = \sum_{a=0}^{p-1} \frac{q^{-\frac{1}{2}a(a-1)(q - q^{-1})a}}{[a]!} \left( E^a \otimes F^a \right.
\]

\[
- q^{-1} \left( q^{2-a}[a] E^{a-1} CE - (1 + q^{2-a}[a-1]) E^a C \right) \otimes F^a B
\]

\[
+ q^{-2} (q^2 - 1) \left( E^a CE - q E^{a+1} C \right) \otimes F^a B F - q^{-3} (q^2 - 1)^2 E^a CEC \otimes F^a BFB \left),
\]

which can be readily rewritten in the factored form as in (2.6).

**B.2. Proof of (2.4.4)** The \(M\)-matrix (see (2.4.3)) can obviously be written as \(M = (\rho_{21} \bar{R}_{21} \rho) \bar{R}\), where we calculate \(\rho_{21} \bar{R}_{21} \rho\) from the definition of \(\rho\) and \(\bar{R}\):

\[
\rho_{21} \bar{R}_{21} \rho = \frac{1}{(2p)^2} \sum_{i=0}^{2p-1} \sum_{j=0}^{2p-1} \sum_{m=0}^{2p-1} \sum_{n=0}^{2p-1} \sum_{i'=0}^{2p-1} \sum_{j'=0}^{2p-1} \sum_{m'=0}^{2p-1} \sum_{n'=0}^{2p-1} (-1)^{i+n+j'}
\]

\[
\times q^{-2i + jm + in} q^{-2i' m + j' m' + i' n'} (K^i k^j \otimes K^m k^n) \bar{R}_{21} (K^{i'} k^{j'} \otimes K^{m'} k^n).
Here, we next move all $K$ and $k$ factors to the right of $\tilde{R}_{21}$; after simple changes of summation variables, $i' \rightarrow i' - m$ and $j' \rightarrow j' - n$, we then see that the summations over $m$ and $n$ can give a nonzero result only for terms with particular values of $i$ and $j$, e.g., $i = m' - a$ and $j = n'$ or $i = m' - 1 - a$ and $j = n' - 1$, etc. (depending on the term taken in the expression for $\tilde{R}_{21}$). This reduces the $m'$ and $n'$ summations to the form

$$\sum_{m'=0}^{2p-1} \sum_{n'=0}^{2p-1} q^{m'+yn'} k^{2m'} k^{2n'},$$

where $x$ and $y$ are integer linear combinations of other summation indices. Next, splitting the range $[0, 2p - 1]$ of $m'$ into $[0, p - 1] \cup [p, 2p - 1]$ and shifting $m'$ by $p$ in the second half of the range, we obtain that the entire sum is proportional to $1 + (-1)^x$, which acts as a selection rule for the parity of one of the remaining summation indices involved in $x$. The same is repeated for the $n'$ sum, yielding the factor $1 + (-1)^y$ and another selection rule. The result of these straightforward manipulations is

$$\rho_{21} \tilde{R}_{21} \rho = \sum_{a=0}^{p-1} \frac{q^{1-a-\frac{3}{2}a^2} (q - q^{-1})^a}{[a]^t}$$

$$\times \left( F^a \otimes E^a + q^{2a-1} F^a B \otimes (q^{2-a}[a] E^{a-1} C E - (1 + q^{2-a}[a-1]) E^a C) (k \otimes k^{-1}) \right)$$

$$- q^{-2a-2} (q^2 - 1) F^a B F \otimes (E^a C E - q^{a+1} C) (K k \otimes K^{-1} k^{-1})$$

$$- q^{-1} (q^2 - 1)^2 (F^a B F \otimes E^a C E C) (K k^2 \otimes K^{-1} k^{-2})$$

$$\times \frac{1}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{i'=0}^{p-1} \sum_{j'=0}^{p-1} q^{2i'j + 2ij' - 4i'j} \left( K^{2i+a(k^2 \otimes K^{2j'} - a k^{2j'})} \right).$$

This is immediately verified to be equal to $\tilde{M} \rho$ in $\text{[2.4.4]}$.

**B.3. Coincidence of two Drinfeld maps.** Let $A$ be a Hopf algebra and $\Phi$ an invertible normalized two-cocycle, i.e., an invertible element $\Phi \in A \otimes A$ such that

$$\Phi'_1 F_1 \otimes \Phi'_2 F_2 \otimes \Phi_2 = \Phi_1 \otimes \Phi_2 F_1 \otimes \Phi'_2 F_2$$

and $\epsilon(\Phi_1) \Phi_2 = \Phi_1 \epsilon(\Phi_2) = 1,$

where $\Phi = \Phi_1 \otimes \Phi_2 = F_1 \otimes F_2$ (and $\epsilon$ is the counit). This standardly defines a new Hopf algebra structure—the one with the same product and counit, and with the coproduct and antipode given by

$$\widetilde{\Delta}(x) = \Phi^{-1} \Delta(x) \Phi, \quad \widetilde{S}(x) = U^{-1} S(x) U \quad \forall x \in A,$$

where

$$U = S(\Phi_1) \Phi_2.$$
We also note that $\tilde{S}^2(x) = \xi^{-1}S^2(x)\xi$, where
\begin{equation}
(\xi) = S(U^{-1})U.
\end{equation}

**B.3.1. Theorem.** Let $A$ be a quasitriangular Hopf algebra and $\Phi$ an invertible normalized 2-cocycle. Then diagram (2.9) is commutative, i.e.,
\begin{equation}
\tilde{\beta}(\tilde{M}_1)\tilde{M}_2 = \beta(M_1)M_2
\end{equation}
for any $\beta \in \mathfrak{h}$.

**Proof.** First, we have a linear space isomorphism $\mathfrak{h} \rightarrow \tilde{\mathfrak{h}}$ given by $\beta \mapsto (\beta - \xi)$. Indeed, if $\beta \in \mathfrak{h}$, which amounts to the condition that $\beta(xy) = \beta(S^2(y)x)$ for all $x, y \in A$, then the functional $\tilde{\beta} : x \mapsto \beta(\xi x)$ is invariant under the “tilded” coadjoint action, i.e., $\tilde{\beta}(xy) = \tilde{\beta}(S^2(y)x)$ for all $x, y \in A$.

We next note two simple consequences of the cocycle condition:
\begin{equation}
\phi_1 \otimes \Phi_1 \phi_2 \otimes \Phi_2 = \phi_1\Phi'_1 \otimes \phi'_2 \Phi''_1 \otimes \phi''_2 \Phi_2,
\end{equation}
\begin{equation}
F_1 \otimes F_2 \phi_1 \otimes \phi_2 = \phi'_1 F_1 \otimes \phi''_2 F_2 \otimes \phi'_2 \Phi_2''
\end{equation}
where $\Phi^{-1} = \phi_1 \otimes \phi_2$. From (B.4), applying the antipode and multiplying, we find the identity $\phi_1S(\Phi_1 \phi_2)\Phi_2 = 1$, whence, for $U$ in (B.2), it follows that $U^{-1} = \phi_1S(\phi_2)$. Hence, $\xi = S(U^{-1})U = S^2(f_2)S(f_1)S(F_1)F_2$, and we calculate
\begin{equation}
\tilde{\beta}(\tilde{M}_1)\tilde{M}_2 = \beta(\xi \phi_1 M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
\begin{equation}
= \beta(S(f_1)S(F_1)F_2 \phi_1 M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
where we next note that $S(F_1)F_2 \phi_1 \otimes \phi_2 = S(F_1)F_2 \otimes F_2''$ as a simple consequence of (B.5), and we can therefore continue
\begin{equation}
= \beta(S(f_1)S(F_1)F_2 M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
\begin{equation}
= \beta(S(f_1)M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
(by the property $M\Delta(x) = \Delta(x)M$ of the M-matrix), which after directly applying the cocycle condition becomes
\begin{equation}
= \beta(S(f_1)S(\Phi'_1 F_1)M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
\begin{equation}
= \beta(S(\Phi'_1 M_1 \phi_1') \phi_2 M_2 \Phi_2
\end{equation}
\begin{equation}
= \beta(M_1 \phi_1' S^{-1}(\Phi'_1) \phi_2 M_2 \Phi_2.
\end{equation}
This is the same as $\tilde{\beta}(M_1)M_2$. 
B.4. Proof of 7.3.2. We solve Eqs. (7.3). We recall that $\Omega_j$ are the $32p - 16$ nilpotent basic algebra generators defined in item 3 in 7.2 and the unknowns $x_{\Omega,m}$ (see (7.2)) are associated with the $16p - 8$ “level-3” nilpotent basic algebra elements $W_{\Omega,m}$ defined in item 3b in 7.3.1.

For a projective module $Q = Q^{\alpha,\beta}_r$, we let $W_{Q^{\alpha,\beta}_r,m}$ be denoted as $W^{\alpha,\beta}_{r,s}(m)$. The corresponding unknowns, accordingly, are then written as $x^{\alpha,\beta}_{r,s}(m)$, and we moreover drop the uninformative “$x$” and distinguish the variables pertaining to the five species of projective modules in 6.3.3–6.3.7 by an individual letter each:

$$
(W.6) \quad x^{\alpha,\beta}_{r,s}(m) = \begin{cases} 
\alpha^{\alpha,\beta}_p(m), & s = p, r = 0 \quad (\square), \\
\alpha^{\alpha,\beta}_{p-1}(m), & s = r = p - 1 \quad (\square), \\
\alpha^{\alpha,\beta}_s(m), & 2 \leq s \leq p - 1, r = 0 \quad (\square), \\
\alpha^{\alpha,\beta}_s(m), & 1 \leq s = r \leq p - 2 \quad (\square), \\
\alpha^{\alpha,\beta}_1(m), & s = 1, r = 0 \quad (\square).
\end{cases}
$$

We recall that the argument $m$ labels level-3 subquotients in a given projective module that are isomorphic to the top subquotient. It is convenient, for uniformity, to let $m$ take not consecutive values (e.g., from 1 to 4 for the projective module $Q^{\alpha,\beta}_1$ in Fig. 6.5) but the values that the relevant subquotients are already assigned in the graphs (which are 2, 4, 5, and 7 in Fig. 6.5). The argument in $W^{\alpha,\beta}_{r,s}(m)$ is then of course understood in the same way.

Each $W^{\alpha,\beta}_{r,s}(m)$, as well as each $\Omega_j$ in (7.3), can be regarded as a linear operator on the vector space with basis consisting of all simple subquotients of all projective modules, and is therefore completely determined by the coefficients with which it sends each subquotient into (linear combinations of) others. By definition, the top subquotient is mapped (into a level-3 subquotient by each $W$ and a level-2 subquotient by each $\Omega_j$) with the coefficient 1, while all other coefficients are obtained from the graphs in Figs. 6.1–6.5 simply from the condition that the basic algebra elements be $U(X)$ intertwiners. This is illustrated in Fig. B.1.

The commutativity in (7.3) then becomes the commutativity condition for the corresponding matrices. We illustrate this with equations involving the six basic algebra generators $\Omega^{\alpha,\beta}_{1,0,n}$, $1 \leq n \leq 6$, sending the top one-dimensional subquotient of $Q^{\alpha,\beta}_{1,0}$ into isomorphic level-2 subquotients (see 7.2 item 3). The isomorphic subquotients occur in the projective modules $Q^{-\alpha,\beta}_{p,0}$, $Q^{-\alpha,\beta}_{p,0}$, $Q^{-\alpha,\beta}_{p-1,p-1}$, $Q^{-\alpha,\beta}_{p-1,p-1}$, $Q^{-\alpha,\beta}_{2,0}$, and $Q^{-\alpha,\beta}_{1,1}$; we select the first one in this (arbitrary) ordering. Because the corresponding basic algebra generator $\Omega^{\alpha,\beta}_{1,0,1}$ acts by zero on all projective modules except $Q^{\alpha,\beta}_{1,0}$, commutator equation (7.3) takes the form
The top subquotient of the projective module on the left is sent into an isomorphic level-2 subquotient in the projective module on the right. Isomorphic descendants are then mapped into one another with the coefficients given by ratios of the weights associated with edges of the graphs. Those children of \((\alpha \beta s r)\) on the left that have no isomorphic subquotients among the children of \((\alpha' \beta' s' r')\) on the right are mapped to zero. The procedure continues similarly to the lower-lying levels.

\[
\begin{align*}
\Omega_{Q_{1,0}^{\alpha,\beta}} & \left( \xi_1^{\alpha,\beta} (2) W_{1,0}^{\alpha,\beta} (2) + \xi_1^{\alpha,\beta} (4) W_{1,0}^{\alpha,\beta} (4) + \xi_1^{\alpha,\beta} (5) W_{1,0}^{\alpha,\beta} (5) + \\
& + \xi_1^{\alpha,\beta} (7) W_{1,0}^{\alpha,\beta} (7) \right) - \sigma_p^{-\alpha,\beta} (0) W_{p,0}^{-\alpha,\beta} (0) \Omega_{Q_{1,0}^{\alpha,\beta}} = 0.
\end{align*}
\]

This equation for the unknowns \(\xi_1^{\alpha,\beta} (2)\), \(\xi_1^{\alpha,\beta} (4)\), \(\xi_1^{\alpha,\beta} (5)\), \(\xi_1^{\alpha,\beta} (7)\), and \(\sigma_p^{-\alpha,\beta} (0)\) is still an “operator” equation in the sense that it is written in terms of maps. A “scalar” equation follows by applying (B.7) to the top subquotient of \(Q_{1,0}^{\alpha,\beta}\). Simple analysis as in Fig. B.1 readily shows that, with \(\Omega_{Q_{1,0}^{\alpha,\beta}}\) sending the top subquotient of \(Q_{1,0}^{\alpha,\beta}\) as \((\alpha \beta)_{2,0} \rightarrow \)

\[
\left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{1,0} \rightarrow \frac{1}{2} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{1,0}, \quad \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{3,4} \rightarrow \frac{1}{2} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{3,4}, \quad \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{3,5} \rightarrow 0, \quad \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{3,7} \rightarrow 0.
\]

Also, \(W_{p,0}^{-\alpha,\beta} (0)\) maps as

\[
\left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{2,0} \rightarrow \alpha \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{1,0}.
\]

This gives the equation

\[
\alpha \sigma_p^{-\alpha,\beta} (0) + \alpha \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{2,0} \rightarrow \alpha \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)_{1,0}.
\]
The full list of equations that follow from commuting with $\Omega_{Q_{1,0}^+}^n$ is

$$\alpha\sigma_p^{-\alpha,+}(0) + \frac{\zeta_1^{\alpha,+}(4)}{2} - \frac{\zeta_1^{\alpha,+}(2)}{2} = 0, \quad \alpha\sigma_p^{-\alpha,-}(0) - \frac{\zeta_1^{\alpha,+}(4)}{2} + \frac{\zeta_1^{\alpha,+}(2)}{2} = 0,$$

$$\pi_{p-1}^{-\alpha,+}(1) - \frac{\zeta_1^{\alpha,+}(4)}{2} + \frac{\zeta_1^{\alpha,+}(5)}{2} = 0, \quad \pi_{p-1}^{-\alpha,-}(1) + \frac{\zeta_1^{\alpha,+}(4)}{2} + \frac{\zeta_1^{\alpha,+}(5)}{2} = 0,$$

$$-\omega_2^{\alpha,-}(0) - \frac{\zeta_1^{\alpha,+}(7)}{2} - \alpha\omega_2^{\alpha,-}(5) = 0,$$

$$\sigma_1^{\alpha,+}(0) - \frac{\alpha}{2}[2]\zeta_1^{\alpha,+}(2) + \frac{1}{2}[2]\zeta_1^{\alpha,+}(5) + \frac{1}{2}[2]\zeta_1^{\alpha,+}(7) = 0$$

(note the four $\alpha$ occurring as coefficients).

Similarly, the $4(p-2)$ equations that follow from commuting with $\Omega_{Q_{s}^+,+}^s$, $1 \leq s \leq p-2$, are as follows:

$$\frac{\sigma_1^{s,+}(0)}{[2]} + \frac{\zeta_1^{s,+}(2)}{2} + \frac{\zeta_1^{s,+}(5)}{2} + \frac{\zeta_1^{s,+}(7)}{2} = 0,$$

$$\sigma_s^{s,+}(5) + \omega_{p-s}^{-,-}(5) = 0, \quad 1 \leq s \leq p-2,$$

$$\sigma_s^{s,+}(5) - \omega_{p-s}^{s,+}(5) = 0, \quad 1 \leq s \leq p-2,$$

$$\frac{\sigma_s^{s+1,+}(0)}{[s+2]} - \frac{\sigma_s^{s,-}(0)}{[s]} + \frac{\sigma_s^{s,-}(5)}{[s]^2} = 0, \quad 1 \leq s \leq p-3,$$

$$\sigma_s^{s+1,+}(0) + \frac{\sigma_s^{s,-}(0)}{[s]} + \frac{\sigma_s^{s,-}(5)}{[s]^2} = 0, \quad 1 \leq s \leq p-3,$$

$$\frac{\sigma_s^{p-2,+}(0)}{[2]} + \frac{\sigma_s^{p-2,+}(5)}{[2]^2} + \pi_{p-1}(1) = 0,$$

(the projective modules with a level-2 subquotient isomorphic to $2Q_{Q,1}^+$ (the top subquotient of $Q_{s}^+$) are $Q_{p-1,0}^-, Q_{1,0}^+, Q_{p-1,0}^+$, and $Q_{2,2}^+$ for $s = 1$ and $Q_{p-s,0}^-, Q_{p-s,0}^+, Q_{p-s,0}^+$, $Q_{s-1,s-1}^+$, and $Q_{s+1,s+1}^+$ for $2 \leq s \leq p-2$).

It is impossible to write the entire system of equations here because of its length. Most of the equations have two or three terms, but the system consists of numerous “blocks” in accordance with the parameterization $\Omega_{Q_{s}^+,+}^s$ of basic algebra generators. Part of the system can be written “uniformly,” with the equations labeled by $s$ and having the same functional form for any $s$. But there are also “boundary effects”: the two length-$(p-2)$ series of projective modules ($\boxtimes$ and $\boxplus$) are followed at the end by modules of a somewhat reduced structure ($\square$ and $\square$), and are also “joined” by the $\boxtimes$ projective module with an “enhanced” structure. Both these effects are well seen in the above formulas ($\zeta_1$ in the first and $\pi_{p-1}$ in the last equation).
But it is possible to give the full solution of the system, which can be written relatively compactly. The comparative complexity or simplicity of the explicit solution—and indeed of the procedure of solving—depends rather strongly on the choice of free variables in terms of which the others are to be expressed. In choosing the free variables, we were guided by the desire to avoid final formulas with the number of terms growing with \( p \); this turned out to be possible, and was actually a factor underlying the success in solving the system explicitly. Numerous variations of our choice are of course possible. A drawback of the specific choice that we make is that the formulas become slightly sensitive to the parity of \( p \); we therefore write the solution explicitly only for odd \( p \). Specifically, the \( 2p + 1 \) free variables are chosen as

\[
\begin{align*}
\xi_1^{+,+}(2), & \quad \xi_1^{+,-}(4), \quad \xi_1^{-,-}(4), \quad \xi_1^{+,+}(5), \quad \omega_2^{-,-}(0), \\
\omega_{2i+1}^{+,+}(0), & \quad i = 1, \ldots, \frac{p-1}{2} - 1, \quad \omega_{2i}^{+,+}(0), \quad i = 1, \ldots, \frac{p-1}{2}, \\
\sigma_{2i-1}^{+,+}(0), & \quad i = 1, \ldots, \frac{p-1}{2} - 1, \quad \sigma_{2i}^{+,+}(0), \quad i = 1, \ldots, \frac{p-1}{2} - 1, \\
\pi_{p-1}^{+,+}(1).
\end{align*}
\]

The other \( 14p - 9 \) variables are expressed in terms of these as follows. First, there are 22 lower-\( s \) relations, occurring because of the special structure of the \( \Omega_{1,0}^{\alpha,\beta} \) projective module:

\[
\begin{align*}
\xi_1^{+,-}(4) & = -\xi_1^{-,-}(4), \\
\xi_1^{+,+}(4) & = -\xi_1^{+,-}(4), \\
\xi_1^{-,-}(2) & = \xi_1^{-,-}(4) - \frac{2\omega_{p-1}^{+,+}(0)}{[2]}, \\
\xi_1^{+,-}(2) & = \xi_1^{+,-}(2), \\
\xi_1^{+,+}(5) & = -\xi_1^{+,-}(4) - 2\pi_{p-1}^{+,+}(1), \\
\xi_1^{+,+}(5) & = \xi_1^{+,+}(5), \\
\xi_1^{+,+}(7) & = -\xi_1^{+,-}(2) - \xi_1^{+,+}(5) - \frac{2\sigma_{p-1}^{+,+}(0)}{[2]} + 2\xi_1^{-,-}(4), \\
\xi_1^{+,-}(7) & = -\xi_1^{+,-}(2) - \xi_1^{+,+}(5) - \frac{2\sigma_{p-1}^{+,+}(0)}{[2]}, \\
\xi_1^{-,-}(7) & = -\beta \xi_1^{+,+}(4) + \xi_1^{+,+}(2) + 2\pi_{p-1}^{+,+}(1) + \frac{2\sigma_{p-1}^{+,+}(0)}{[2]}, \\
\omega_1^{-,-}(0) & = \frac{\beta}{2} [2] \xi_1^{+,+}(4) - \frac{1}{2} [2] \xi_1^{+,+}(2) + \omega_{p-1}^{+,+}(0) - \sigma_{p-1}^{+,+}(0), \\
\omega_2^{-,-}(5) & = \frac{\beta}{2} [2] \xi_1^{+,+}(4) - \frac{\beta}{2} [2] \xi_1^{+,+}(2) - \beta [2] \xi_1^{+,+}(5) - \beta [2] \xi_1^{+,+}(1) \\
& \quad - \beta [2] \sigma_1^{+,-}(0) - \beta [2] \omega_2^{+,-}(0), \\
\omega_2^{+,+}(5) & = -\frac{\beta}{2} [2] \xi_1^{+,+}(4) - \frac{\beta}{2} [2] \xi_1^{+,+}(2) - \beta [2] \xi_1^{+,+}(5) - \beta [2] \sigma_1^{+,-}(0) + \beta [2] \omega_2^{+,+}(0),
\end{align*}
\]
\[ \omega_2^{-+}(0) = \omega_2^{-,-}(0) - [2]^3 \xi_1^{++}(4), \]
\[ \omega_3^{-,-}(0) = -\frac{1}{2}[2][3]^3 \xi_1^{++}(4) - \frac{1}{2}[2][3]^3 \xi_1^{++}(2) - [3]^3 \sigma_1^{-,-}(0) + [3]^2 \sigma_{p-3}^{++}(0), \]
\[ \omega_3^{-,-}(5) = -\frac{\beta}{2}[2][3]^4 \xi_1^{++}(4) + \frac{\beta}{2}[2][3]^4 \xi_1^{++}(2) + \beta [3]^4 \sigma_1^{++}(0) \]
\[ - \beta [3]^3 \sigma_{p-3}^{++}(0) + \beta \frac{[3]^4 \omega_{p-5}^{--}(0)}{[2]^2}. \]

Here and hereafter, \( \beta = \pm \). Next, there are \( 14(p - 1) - 31 \) “serial” relations for “generic” values of \( s \), which in our solution are split into even and odd ones:

\[ \omega_1^{-+2i}(5) = \beta [2i] \omega_1^{++}(0) - \beta \frac{[2i]^4 \omega_1^{++}(0)}{[2i-2][2i-1]^2}, \quad 2 \leq i \leq \frac{p-1}{2}, \]
\[ \omega_{2i}^{++}(5) = \beta [2i+1] \omega_{2i+1}^{++}(0) - \beta \frac{[2i+1]^4 \omega_{2i+1}^{++}(0)}{[2i-1][2i]^2}, \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \omega_{2i}^{++}(5) = -\beta [2i]^3 \sigma_{p-2i}^{++,+}(0) + \beta \frac{[2i]^4 \sigma_{p-2i}^{++,+}(0)}{[2i-2]}, \quad 2 \leq i \leq \frac{p-1}{2}, \]
\[ \omega_{2i-1}^{-,-}(5) = -\beta [2i+1]^3 \sigma_{p-1-2i}^{+,+}(0) + \beta \frac{[2i+1]^4 \sigma_{p-1-2i}^{+,+}(0)}{[2i-1]}, \quad 2 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \omega_{2i}^{-,-}(0) = \omega_{2i}^{+,+}(0) - [2i-1][2i]^3 \xi_1^{-,-}(4), \quad 1 \leq i \leq \frac{p-1}{2}, \]
\[ \omega_{2i+1}^{+,+}(0) = \omega_{2i+1}^{-,-}(0) - [2i][2i+1]^3 \xi_1^{-,-}(4), \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \omega_{2i}^{-,-}(0) = -\frac{\beta}{2}[2i-1][2i][2i]^3 \xi_1^{++}(4) - \frac{1}{2}[2i-1][2i]^3 \xi_1^{++}(2) \]
\[ - \frac{[2i-1][2i][2i]^3 \sigma_1^{++}(0)}{[2]} + [2i]^2 \sigma_{p-2i}^{++}(0), \quad 2 \leq i \leq \frac{p-1}{2}, \]
\[ \omega_{2i+1}^{-,-}(0) = \frac{\beta}{2}[2i][2i+1]^3 \xi_1^{++}(4) - \frac{1}{2}[2i][2i+1]^3 \xi_1^{++}(2) \]
\[ - \frac{[2i][2i+1]^3 \sigma_1^{++}(0)}{[2]} + [2i+1]^2 \sigma_{p-1-2i}^{++,+}(0), \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \sigma_{2i}^{-,-}(0) = -\frac{\beta}{2}[2i][2i+1] \xi_1^{++,+}(4) - \frac{1}{2}[2i][2i+1] \xi_1^{++,+}(2) \]
\[ + \omega_{p-2i}^{++,+}(0) - \frac{[2i][2i+1] \sigma_{p-1-2i}^{++,+}(0)}{[2]}], \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \sigma_{2i+1}^{-,-}(0) = \frac{\beta}{2}[2i+1][2i+2] \xi_1^{++,+}(4) - \frac{1}{2}[2i+1][2i+2] \xi_1^{++,+}(2) \]
\[ + \omega_{p-1-2i}^{++,+}(0) - \frac{[2i+1][2i+2] \sigma_{p-1-2i}^{++,+}(0)}{[2i+1]^2}, \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \sigma_{2i}^{-,-}(0) = \sigma_{2i}^{+,+}(0) - [2i][2i+1] \xi_1^{-,-}(4), \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
\[ \sigma_{2i+1}^{+,+}(0) = \sigma_{2i+1}^{-,-}(0) - [2i+1][2i+2] \xi_1^{-,-}(4), \quad 0 \leq i \leq \frac{p-1}{2} - 2, \]
\[ \sigma_{2i}^{-,-}(5) = \frac{\beta}{2}[2i][2i+1]^3 \xi_1^{++,+}(4) - \frac{1}{2}[2i][2i+1]^3 \xi_1^{++,+}(2) \]
\[ + \omega_{p-2i}^{++,+}(0) - \frac{[2i][2i+1] \sigma_{p-1-2i}^{++,+}(0)}{[2i+1]^2}, \quad 1 \leq i \leq \frac{p-1}{2} - 1, \]
This completes the list of formulas expressing 14 dependent variables and thus solving linear system (7.3).

And finally, the 14 high-s relations, whose form is largely determined by the somewhat special structure of \( Q_{p,0}^\alpha \beta \) and \( Q_{p-1,p-1}^\alpha \beta \), are

\[
\sigma_{p-3}^{+,-}(5) = -\frac{1}{2} [2][3]^2 \xi_1^{+,-}(4) + \frac{1}{2} [2][3]^2 \xi_1^{+,-}(2) + \frac{3}{2} \omega_2^{-,-}(0) - [3] \sigma_{p-3}^{+,-}(0),
\]

\[
\sigma_{p-2}^{+,-}(5) = -\frac{1}{2} [2]^2 \xi_1^{+,-}(2) - \frac{1}{2} [2]^2 \xi_1^{+,-}(5) - \beta [2] \sigma_{p-2}^{+,-}(0) + \frac{\omega_2^{-,-}(0)}{[2]},
\]

\[
\sigma_{p-2}^{+,-}(0) = -\frac{1}{2} [2]^2 \xi_1^{+,-}(4) + \frac{1}{2} [2]^2 \xi_1^{+,-}(2) + \frac{\omega_2^{-,-}(0)}{[2]} + \sigma_{p-2}^{+,-}(0) - [2] \xi_1^{+,-}(4),
\]

\[
\sigma_{p-2}^{+,-}(0) = -\frac{1}{2} [2]^2 \xi_1^{+,-}(4) + \frac{1}{2} [2]^2 \xi_1^{+,-}(2) + \frac{\omega_2^{-,-}(0)}{[2]} + \sigma_{p-2}^{+,-}(0),
\]

\[
\pi_{p-1}^{+,-}(1) = \frac{1}{2} \xi_1^{+,-}(4) - \frac{1}{2} \xi_1^{+,-}(5),
\]

\[
\pi_{p-1}^{+,-}(1) = \xi_1^{+,-}(4) + \pi_{p-1}^{+,-}(1),
\]

\[
\sigma_p^{+,-}(0) = \frac{\omega_{p-1}^{+,-}(0)}{[2]} - \xi_1^{+,-}(4),
\]

\[
\sigma_p^{+,-}(0) = \frac{\omega_{p-1}^{+,-}(0)}{[2]}
\]

\[
\sigma_p^{+,-}(0) = \frac{\omega_{p-1}^{+,-}(0)}{[2]}
\]

\[
\sigma_p^{+,-}(0) = \frac{\omega_{p-1}^{+,-}(0)}{[2]}
\]

This completes the list of formulas expressing 14 \( p - 9 \) variables in terms of 2\( p + 1 \) independent variables and thus solving linear system (7.3).

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