Resolving disagreement for $\eta/s$ in a CFT plasma at finite coupling

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**Abstract**

The ratio of shear viscosity to entropy density in a strongly coupled CFT plasma can be computed using the AdS/CFT correspondence either from equilibrium correlation functions or from the Janik-Peschanski dual of the boost invariant plasma expansion. We point out that the previously found disagreement for $\eta/s$ at finite 't Hooft coupling is resolved once the incoming-wave boundary condition for metric fluctuations at the horizon of the dual geometry is properly imposed.

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1 Introduction

Gauge theory/string theory correspondence of Maldacena [1, 2] has been useful in analysis of the transport properties of the strongly coupled gauge theory plasma [3]. In particular, it was proven that the ratio of shear viscosity to the entropy density $\frac{\eta}{s}$ at infinite ’t Hooft coupling is universal in all gauge theory plasmas which allow for a holographically dual string theory description [4–7]. At finite ’t Hooft coupling (but still in the planar limit), this ratio receives leading contribution from $O(\alpha'^3)$ string theory corrections to the dual type IIB supergravity background. In [8] it was argued that such corrections are universal as well, as long as the dual gauge theory plasma is conformal.

The correction to the ratio $\frac{\eta}{s}$ can be computed either from equilibrium correlation functions (as in [9, 10]) or by imposing a non-singularity condition of the $O(\alpha'^3)$ corrected Janik-Peschanski [11] dual of the boost invariant CFT plasma expansion (as in [12]). In the former case it was found that [9, 10]

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + \frac{135}{8} \zeta(3) l^{-3/2} + \cdots \right), \quad (1.1)$$

while in the latter [12]

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + \frac{120}{8} \zeta(3) l^{-3/2} + \cdots \right), \quad (1.2)$$

where $l$ is the $\mathcal{N} = 4$ supersymmetric Yang-Mills ’t Hooft coupling.

In this paper we resolve the discrepancy between (1.1) and (1.2). It turns out that the incoming-wave boundary condition on metric fluctuations used to obtain (1.1) were imposed at the supergravity level, rather than at $O(\alpha'^3)$ string theory corrected background. In what follows we show that once the boundary conditions are properly imposed, the shear viscosity to the entropy ratio obtained from equilibrium correlation functions agrees with (1.2).

2 Incoming wave boundary condition

We consider here the shear quasinormal mode in $O(\alpha'^3)$ near-extremal D3 brane geometry. Discussion extends to both the sound quasinormal mode and the scalar quasinormal mode.
Equations of motion to the shear quasinormal mode in $\mathcal{O}(\alpha'^3)$ near-extremal D3 brane geometry were derived in [10]. These equations can be expanded perturbatively in $\gamma \equiv \frac{1}{8} \zeta(3) (\alpha')^3$, provided we introduce

$$Z_{\text{shear}} = Z_{\text{shear},0} + \gamma Z_{\text{shear},1} + \mathcal{O}(\gamma^2).$$

(2.1)

We find

$$0 = Z''_{\text{shear},0} + \frac{x^2 q^2 + w^2}{x(w^2 - x^2 q^2)} Z'_{\text{shear},0} + \frac{w^2 - x^2 q^2}{x^2(1 - x^2)^{3/2}} Z_{\text{shear},0},$$

$$0 = Z''_{\text{shear},1} + \frac{x^2 q^2 + w^2}{x(w^2 - x^2 q^2)} Z'_{\text{shear},1} + \frac{w^2 - x^2 q^2}{x^2(1 - x^2)^{3/2}} Z_{\text{shear},1} + J_{\text{shear,0}} [Z_{\text{shear},0}],$$

(2.2)

where the source $J_{\text{shear,0}}$ is a functional of the zero’s order shear mode $Z_{\text{shear},0}$

$$J_{\text{shear,0}} [Z_{\text{shear},0}] = C^{(4)}_{\text{shear}} \frac{d^4 Z_{\text{shear},0}}{dx^4} + C^{(3)}_{\text{shear}} \frac{d^3 Z_{\text{shear},0}}{dx^3} + C^{(2)}_{\text{shear}} \frac{d^2 Z_{\text{shear},0}}{dx^2} + C^{(1)}_{\text{shear}} \frac{dZ_{\text{shear},0}}{dx} + C^{(0)}_{\text{shear}} Z_{\text{shear},0}.$$}

(2.3)

The coefficients $C^{(i)}_{\text{shear}}$ are given explicitly in appendix A of [10]. In (2.2) we introduced

$$w = \frac{\omega}{2\pi T_0}, \quad q = \frac{q}{2\pi T_0}.$$}

(2.4)

where $T_0$ is a near-extremal D3 brane temperature in the supergravity approximation.

At the supergravity level, i.e., for $Z_{\text{shear,0}}$, the incoming-wave boundary condition at the horizon implies that in the hydrodynamic approximation

$$Z_{\text{shear,0}} = x^{-iw} \left( z^{(0)}_{\text{shear,0}} + i q z^{(1)}_{\text{shear,0}} + \mathcal{O}(q^2) \right),$$

(2.5)

where $z^{(i)}_{\text{shear,0}}$ are regular at the horizon. While it is possible to use ansatz

$$Z_{\text{shear,1}} = x^{-iw} \left( z^{(0)}_{\text{shear,1}} + i q z^{(1)}_{\text{shear,1}} + \mathcal{O}(q^2) \right),$$

(2.6)

with regular $z^{(i)}_{\text{shear,1}}$ at the horizon (as was done in [9, 10]) to order $\mathcal{O}(q)$, it is straightforward to verify that $\mathcal{O}(q^2)$ term in (2.6) is always singular. The reason for this is that the asymptotic $\propto x^{-iw}$ is an incoming-wave boundary condition only at the supergravity level, but is modified at $\mathcal{O}(\gamma)$.

To determine the correct incoming-wave boundary condition we have to go back to the equation of motion for $Z_{\text{shear}}$ (2.1):

$$0 = Z''_{\text{shear}} + \frac{x^2 q^2 + w^2}{x(w^2 - x^2 q^2)} Z'_{\text{shear}} + \frac{w^2 - x^2 q^2}{x^2(1 - x^2)^{3/2}} Z_{\text{shear}} + J_{\text{shear,0}} [Z_{\text{shear}}] + \mathcal{O}(\gamma^2).$$

(2.7)
We look for solution to (2.7) to order $\mathcal{O}(\gamma)$ within the ansatz

$$Z_{\text{shear}} = x^\beta \left( z_{\text{shear}}^{(0)} + iq z_{\text{shear}}^{(1)} + \mathcal{O}(q^2) \right) \times (1 + \mathcal{O}(\gamma^2)) ,$$

(2.8)

with regular $z_{\text{shear}}^{(i)}$ at the horizon. Substituting (2.8) into (2.7) we find (to order $\mathcal{O}(\gamma)$):

$$0 = x^\beta \left( \frac{1}{x^2} \left( \beta^2 + w^2 (1 - 30 \gamma) \right) + \mathcal{O}(x^0) \right) .$$

(2.9)

From (2.9) we see that the incoming wave boundary condition is

$$\beta = -i w (1 - 15 \gamma) + \mathcal{O}(\gamma^2) .$$

(2.10)

Eq. (2.10) is an expected modification. Indeed, computations of the spectrum of shear and sound quasinormal modes in more complicated supergravity backgrounds [13, 14], as well as the general arguments for the scalar quasinormal mode in [6], show that the incoming-wave boundary condition at the horizon always takes the form $\propto x^{-i \omega T}$. Thus, a $(1 - 15 \gamma)$ rescaling of the supergravity boundary conditions is simply a well-known rescaling of the near-extremal D3 brane temperature due to string theory higher derivative corrections [15]:

$$T = T_0 (1 + 15 \gamma + \mathcal{O}(\gamma^2)) .$$

(2.11)

We emphasize again that the boundary condition (2.10) is required not only to obtain correct physical results, but it is mandatory if one attempts to extend computation of the spectrum of quasinormal modes beyond the first order in the hydrodynamic approximation.

3 Corrected shear and sound quasinormal modes to first order in the hydrodynamic approximation

In previous section we argued that the incoming-wave boundary condition for the near-extremal D3 brane quasinormal modes receives order $\mathcal{O}(\alpha'^3)$ correction. It turns out that this modification (2.10) is the source of the discrepancy between (1.1) and (1.2). To show the latter we have to recompute the spectrum of the shear and sound quasinormal modes. It is straightforward to do so following detailed discussion in [10].

For the shear quasinormal mode we find

$$z_{\text{shear},0}^{(0)} = 1 , \quad z_{\text{shear},0}^{(1)} = \frac{1}{2} \frac{q}{w} x^2 ,$$

(3.1)
\[
\begin{align*}
z^{(0)}_{\text{shear},1} &= \frac{25}{16} x^2 \left( x^4 - 4x^2 + 5 \right), \\
z^{(1)}_{\text{shear},1} &= -\frac{1}{32qw} x^2 \left( q^2 \left( -240 - 1565x^2 - 860x^4 + 695x^6 \right) \\
&\quad + 16w^2 \left( 594 - 264x^2 + 43x^4 \right) \right) + \delta z^{(1)}_{\text{shear},1},
\end{align*}
\]

and for the sound quasinormal mode we find
\[
\begin{align*}
z^{(0)}_{\text{sound},0} &= \frac{3w^2 + (x^2 - 2)q^2}{3w^2 - 2q^2}, \\
z^{(1)}_{\text{sound},0} &= \frac{2wq x^2}{3w^2 - 2q^2}, \tag{3.3}
\end{align*}
\]

\[
\begin{align*}
z^{(0)}_{\text{sound},1} &= \frac{5x^2}{16(3w^2 - 2q^2)^2} \left( q^4 \left( 2404 + 446x^2 - 4164x^4 + 2006x^6 \right) \\
&\quad - 3w^2 q^2 \left( 1588 + 183x^2 - 2072x^4 + 1003x^6 \right) + 45w^4 \left( 5 - 4x^2 + x^4 \right) \right), \\
z^{(1)}_{\text{sound},1} &= \frac{w x^2}{8q(3w^2 - 2q^2)^2} \left( q^4 \left( -13344 + 5846x^2 - 4520x^4 + 1734x^6 \right) \\
&\quad - 3w^2 q^2 \left( -9744 + 5035x^2 - 2604x^4 + 867x^6 \right) \\
&\quad - 36w^4 \left( 594 - 264x^2 + 43x^4 \right) \right) + \delta z^{(1)}_{\text{sound},1}, \tag{3.4}
\end{align*}
\]

where \(\delta z^{(1)}_{\text{shear},1}\) and \(\delta z^{(1)}_{\text{sound},1}\) are corrections due to the modified boundary condition (2.10):
\[
\delta z^{(1)}_{\text{shear},1} = -\frac{15q x^2}{2w}, \quad \delta z^{(1)}_{\text{sound},1} = \frac{30wq x^2}{2q^2 - 3w^2}. \tag{3.5}
\]

Imposing the Dirichlet condition on \(x^{\text{im}(1-15\gamma)}Z_{\text{shear},0}\) and \(x^{\text{im}(1-15\gamma)}Z_{\text{sound},0}\) at the boundary determines the lowest shear and sound quasinormal frequencies
\[
\begin{align*}
&\text{shear : } \ w = -iq^2 \left( \frac{1}{2} + \frac{105}{2} \gamma \right) + O(q^3, \gamma^2), \\
&\text{sound : } \ w = \frac{1}{\sqrt{3}}q - iq^2 \left( \frac{1}{3} + \frac{105}{3} \gamma \right) + O(q^3, \gamma^2). \tag{3.6}
\end{align*}
\]

Note that both channels lead to the same prediction for \(\frac{q}{s}\), namely, the one given by (1.2). Additionally, as expected, neither the speed of sound nor the bulk viscosity receives \(O(\gamma)\) corrections, which are forbidden by the conformal symmetry.

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