Continuous two-photon source using a single quantum dot in a photonic crystal cavity

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Abstract

We propose methods for the realization of a continuous two-photon source using a coherently driven quantum dot embedded inside a photonic crystal cavity. We analyze the steady state population in quantum dot energy levels and field inside the cavity mode. We find conditions for population inversion in coherently driven and incoherently driven quantum dots. We discuss the effect of phonon coupling using the recently developed polaron transformed master equation at low temperatures. We show that on increasing temperature, two-photon gain decreases and single photon gain increases, due to phonon-assisted cavity mode feeding. We also propose a scheme for generating a squeezed state of field using four wave mixing.

Keywords: two-photon laser, quantum dot laser, nonclassical light source

(Some figures may appear in colour only in the online journal)

1. Introduction

In the quest for scalable on-chip quantum technology, semiconductor quantum dots (QDs) have emerged as a potential candidate [1–3]. With advances in lithography techniques, it is now possible to grow a quantum dot at a desired location inside a photonic crystal microcavity [4, 5]. As a result, new solid state on-chip cavity quantum electrodynamics (cavity-QED) systems have been developed. The strong coupling regime, where the dipole coupling strength between a single photon and a single QD becomes greater than the damping rates in the system, has been realized [6]. Various other highly regarded phenomena, in microwave and optical cavity-QED systems using trapped or Rydberg atoms, such as the appearance of higher rungs of the Jaynes–Cummings ladder [7], photon blockade [8], and Mollow triplets [9] have also been observed. Furthermore, remarkable success has been achieved in generating sources of nonclassical light, such as sources of entangled photons [10] and single photon sources possessing high efficiency and indistinguishability [11]. However, being solid state devices, interactions with longitudinal acoustic phonons are unique in these semiconductor cavity-QED systems. Interactions between phonons and excitons lead to dephasing [12] in the coupled dynamics of exciton–photon interaction, as well as off-resonant cavity mode feeding [13]. Various new phenomena, such as the high fidelity generation of exciton and biexciton states [14, 15], phonon assisted population inversion in two level systems [16], and the appearance of new features in Mollow triplets [17] have been observed due to phonon interactions. Interactions between phonons and excitons play a particularly significant role in off resonant exciton–photon interactions [18].

Generating coherent light by placing a large number of emitters inside a cavity has been a subject of some fascination for researchers, and various types of laser have been developed [19]. In conventional lasers, a large number of emitters are used as a gain medium to overcome photon losses from the cavity. However, microlasers, where a single emitter acts as the gain medium inside high quality cavity, have also been realized [20]. These systems are particularly useful for applications in the field of quantum information processing. The application of QDs in conventional lasers as a gain medium has had only limited success, due to variations in their sizes and exciton resonance frequencies [21]. However, single emitter micro-lasers
using single QDs embedded inside high quality photonic crystal cavities [22, 23], and QD coupled with coplanar microwave cavity [24, 25] have recently been realized. Furthermore, it has been observed that phonon interactions significantly alter the dynamics in both types of microlaser, i.e., a single QD embedded in a photonic crystal cavity [23], or a single QD coupled with coplanar cavity [25], and a greatly enhanced output power has been achieved due to phonon assisted resonant transitions.

Similarly to single photon lasers, where lasing occurs due to stimulated emission of single photons, it has also been predicted that a similar coherent generation of light is also possible through stimulated two-photon processes [26]. In addition, the output of a two-photon laser would be a squeezed coherent state, exhibiting quantum properties [27]. It should be noted that the realization of the two-photon laser had not achieved a great deal of success prior to the realization of the two-photon mazer [28]. The two-photon laser has been demonstrated by Gauthier et al. [29], using strongly driven two level atoms as gain media, and probing by a weak field with a frequency resonant to one of the side bands. The absence of squeezing in the output has been ascribed to the enhancement of noise in the spontaneous generation of photons. However, it has also been predicted that a two-photon correlated emission laser can show squeezing for certain parameters when the noise in one photon emission cancels the other [30]. We note that two photon lasing in a single QD using incoherent pumping has recently been proposed [31] which does not consider exciton–phonon coupling. We further note that sources of N-photons [32] and N-phonons [33] via QD cavity-QED have also been proposed. Here, we propose a scheme for the realization of a two-photon laser using a single QD embedded in a photonic crystal cavity. We discuss the effect of temperature on both single photon and two-photon gains. The two-photon emission through cavity mode is dominant in the case where single photon transitions are far off-resonant, and two-photon resonance conditions are satisfied [34]. Since the photon pair is generated through biexciton decay into the far off-resonant cavity mode, phonon interaction can play a significant role. We use recently developed master equation techniques to incorporate such interactions [35]. In this work, we investigate the effects of the coherent nature of a pump in order to achieve correlated two-photon emission in a single QD two-photon laser, with the aim of predicting nonclassical features in the laser output.

Our paper is organized as follows. In section 2, we present a model for a two-photon laser using an incoherent pump. We discuss the effects of phonon interaction at low temperatures in terms of steady state population dynamics, single photon gain, two-photon gain, and the Wigner function of the cavity field. In section 3, we discuss two-photon lasing using coherently driven QD. Continuous generation of squeezed state of field using four-wave mixing is discussed in section 4. Finally, conclusions are presented in section 5.

2. Two-photon lasing using an incoherent pump

We consider a single quantum dot embedded in a single mode photonic crystal microcavity. The quantum dot consists of four energy levels: ground state $|g\rangle$, two exciton states $|x\rangle$, $|y\rangle$ and biexciton state $|u\rangle$. The transitions $|g\rangle \leftrightarrow |x\rangle$ and $|x\rangle \leftrightarrow |u\rangle$ are driven by an $x$-polarized pump field, and the transitions $|u\rangle \leftrightarrow |y\rangle$ and $|y\rangle \leftrightarrow |g\rangle$ are coupled with a $y$-polarized cavity mode. In the QD, excitation state $|x\rangle$, and biexciton state $|u\rangle$ are created via an incoherent $x$-polarized pump field. The directions of polarization of the cavity mode and the pumping fields are selected to be perpendicular to each other. A similar scheme has been proposed by Elena del Valle et al. [29], without considering the effect of acoustic phonons. We investigate the effect of exciton–phonon coupling on two-photon lasing, in terms of both lasing conditions and the properties of the laser output. These effects are particularly significant because the transitions used for two-photon generation in cavity mode via biexciton–exciton cascade decay are far off-resonant. The Hamiltonian of the system in rotating frame with cavity mode frequency is given by

$$H = h(\delta_1 + \delta_x)\sigma_{xx} + h(\Delta_1 + \Delta_x - \Delta_{xx})\sigma_{uu} + h\delta_1\sigma_{yy} + h(\sigma_{xg}a + g\sigma_{yg}A + \text{H.c.}) + H_{ph},$$

(1)

where $\delta_1 = \omega_1 - \omega_c$ denotes detuning between exciton resonance frequency $\omega_1$ and cavity mode frequency of frequency $\omega_c$, $g_1$ and $g_2$ are the dipole coupling constants for transitions $|y\rangle \leftrightarrow |g\rangle$ and $|u\rangle \leftrightarrow |y\rangle$ with the cavity mode, respectively, $\sigma_{ij} = |i\rangle \langle j|$, and $|i\rangle$ are QD operators, and $a$ is the annihilation operator for photons in cavity mode. The longitudinal acoustic photphon bath and exciton–phonon–photon interactions are included in $H_{ph} = h\sum_k \omega_kb_kb_k + h\sum_{i,j} \lambda_{ik}\sigma_i(b_k + b_k^\dagger)$, where $\lambda_{ik}$ are exciton phonon coupling constants, and $b_k$ & $b_k^\dagger$ are the annihilation and creation operators for the $i$th phonon mode of frequency $\omega_k$. In order to maintain all orders of exciton–phonon coupling, we use a polaron transformed Hamiltonian. The transformed Hamiltonian $H' = e^PHe^{-P}$, with $P = \sum_{i,j} \delta_{ij}\sigma_i(b_i - b_i^\dagger)$ can be written as the sum of the terms corresponding to cavity-QD system, phonon bath and system-bath interactions as $H' = H_s + H_b + H_{ph}$, where

$$H_s = h(\Delta_1 + \delta_1)\sigma_{xx} + h(2\Delta_1 + \delta_x - \Delta_{xx})\sigma_{uu} + h\Delta_1\sigma_{yy} + (B)X_g,$$

(2)

$$H_b = h\sum_k \omega_kb_kb_k^\dagger,$$

(3)

$$H_{ph} = \xi_bX_b + \xi_dX_d.$$

(4)

The polaron shifts $\sum_{k} \lambda_{ik}^2/\omega_k$ are included in the effective detunings $\Delta_1$ and $\Delta_2$. The system operators are given by $X_g = (g_1\sigma_{xg}a + g_2\sigma_{yg}a + \text{H.c.})$, $X_b = i(b_1\sigma_{yg}a + g_2\sigma_{yg}a + \text{H.c.})$, and the bath fluctuation operators are $\xi_b = \frac{\delta}{2}(B_+ + B_- - 2(B))$, $\xi_d = \frac{\delta}{2}(B_+ - B_-)$. The phonon displacement operators are $B_k = \exp[\pm \sum_l \lambda_{ik}^2(b_l - b_l^\dagger)]$ with an expectation value of $\langle B_k \rangle = (B_k)$. Due to interaction with phonons, excitons and biexcitons form dressed states called polaron states. The polaron transform changes the basis from bare QD states to polaron states, where the Hamiltonian (1), in the absence of cavity interaction, becomes diagonal. The physical reason behind
this is that the equilibrium positions for phonon oscillations are different for ground state and exciton states, and the polaron transform, which is equivalent to the state dependent displacement operator, accounts for the displacement of equilibrium positions [36]. The polaron transform has been widely used in relation to charge transfer, tunneling phenomena, and quantum phase transitions [37]. Transforming to a polaron frame, we renormalize the exciton–cavity coupling and minimize the QD–phonon processes are given by

\[ T = \sum_{i=1}^{c} \kappa \langle \sigma_{ii} \rangle \rho_{s} - \sum_{i=1}^{c} \gamma \delta \langle \sigma_{ii} \rangle \rho_{s} \]

where \( \kappa \) denotes the photon leakage rate from the cavity mode and \( \gamma \) and \( \gamma \) are the spontaneous decay rates. The dephasing rates for exciton states are given by \( \gamma_{2} \) and \( \gamma_{1} \). Each \( \gamma_{2} \) correspond to exciton and biexciton pumping rates. The Lindblad super operator corresponding to an operator \( \hat{a} \) is defined as \( \mathcal{L}(\hat{a}) = \hat{a} \rho_{s} - \rho_{s} \hat{a} \). The phonon induced processes are given by

\[ \mathcal{L}_{ph}(\rho_{s}) = \frac{1}{\hbar} \int \tau \sum_{j \neq a} G_{j}(\tau) \{ X_{j}(\tau), X_{j}(\tau) \rho_{s} \} + \text{H.c.} \]

where \( X_{j}(\tau) = e^{-i L_{j}(\tau) / \hbar} X_{j}(\tau) e^{i L_{j}(\tau) / \hbar} \) and \( G_{j}(\tau) = \langle B \rangle^{2} \{ \cosh [\phi(\tau)] - 1 \} \), and \( G_{j}(\tau) = \langle B \rangle^{2} \sinh[\phi(\tau)] \). The spectral density function for the phonon bath is given by \( J(\omega) = \sum_{k} |\alpha_{k}|^{2} \delta(\omega - \omega_{k}) \). Considering the spherical QD and Gaussian wave-functions for electron and hole, these simplify to \( J(\omega) = \alpha_{p} \omega_{p}^{3} \exp[-\omega^{2} / 2 \omega_{0}^{2}] \) [38], where the parameters \( \alpha_{p} \) and \( \omega_{h} \) are the electron–phonon coupling and cutoff frequency respectively. In our calculations we use \( \alpha_{p} = 1.42 \times 10^{-3} \) and \( \omega_{h} = 10 \), which gives \( \langle B \rangle = 1.0, 0.90, 0.84, \) and \( 0.73 \) for \( T = 0K, T = 5K, 10K, \) and \( 20K \), respectively. The system–phonon interactions are included in phonon correlation function \( \phi(\tau) \), given by

\[ \phi(\tau) = \int_{0}^{\infty} \frac{d \omega}{\omega^{2}} \frac{J(\omega)}{\omega^{2}} \left[ \frac{\omega_{h}}{2K_{b}T} \right] \cos[\phi(\tau)] + i \sin[\phi(\tau)] \]

where \( K_{b} \) and \( T \) are the Boltzmann constant and the temperature of the phonon bath, respectively. We solve master equation (5) numerically using the quantum optics toolbox [39].

In order to analyze the two-photon lasing, we plot steady state populations and cavity field statistics results in figures 1 and 2 using phonon bath temperatures \( T = 5K, T = 20K \), and typical values of parameters which are compatible with the experiments. In figure 1, we fix biexciton binding energy corresponding to \( \Delta_{cx} = 10 \), and by changing the cavity frequency, the detuning parameter \( \Delta_{1} \) is altered at a constant temperature [40]. The cavity assisted two-photon resonance occurs for \( \Delta_{1} = 2 \Delta_{cx} / 3 \). Single photon resonances can occur when either cavity is resonant to exciton transition, i.e. \( \Delta_{1} = 0 \), or the cavity is resonant to the biexciton to exciton transition, i.e. \( \Delta_{1} = \Delta_{cx} / 2 \). Since we have considered pumping to biexciton state \( |u \rangle \) only through transitions \( |g \rangle \rightarrow |x \rangle \), and \( |x \rangle \rightarrow |u \rangle \), the resonant photon emission corresponding to the single photon resonance at \( \Delta_{1} = 0 \) does not occur, as exciton state \( |y \rangle \), which is coupled through cavity mode, remains less populated because the biexciton to exciton transition \( |u \rangle \rightarrow |y \rangle \) becomes far off-resonant. When detuning \( \Delta_{1} \) is increased from zero, the steady-state population in the biexciton state \( \langle u \rho_{s} \langle u \rangle \rangle \), calculated after tracing over cavity states, is much larger than the populations in other states of QD. Therefore, one can easily achieve population inversion in single QD, when incoherent pumping is larger than other losses in the system. For \( \Delta_{1} = 4.3 \), i.e. when two-photon resonance occurs, a large population from the biexciton state

Figure 1. In (a) and (c), steady state populations in quantum dot energy states are shown as follows: \( |u \rangle \) (blue), \( |x \rangle \) (red), \( |y \rangle \) (black), \( |g \rangle \) (cyan), and average photon number \( \langle n \rangle \) (green) for \( \eta_{1} = \eta_{2} = 0.5 \). In (b) and (d), average photon numbers are shown as follows: \( \langle n \rangle \) (blue), \( g^{2}(0) \) (red), Fano factor \( F \) (black) for \( \Delta_{1} = 4.5 \), where \( \eta_{1} = \eta_{2} = \eta \). Here, temperature \( T = 5K \), cavity leakage \( \kappa = 0.2 \), cavity field couplings \( g_{1} = g_{2} \), spontaneous decay rates \( \gamma_{1} = \gamma_{2} = 0.01 \), biexciton binding energy \( \Delta_{cx} = 10 \), and anisotropic energy gap \( \delta_{1} = 1.0 \).

Figure 2. Steady state populations in quantum dot energy states and cavity field parameters for temperature \( T = 20K \) using the same parameters and color scheme as in figure 1.
is transferred to ground state |g⟩, thus indicating the generation of photons in pairs inside the cavity mode from |n⟩ → |g⟩ via |y⟩. The population in exciton state |y⟩ keeps increasing monotonically for larger positive values of Δ1 and becomes larger than the population in the biexciton state around the single photon resonance at Δ1 = Δx1 − κ. This is due to the fact that on increasing Δ1, the detuning between cavity mode and biexciton to exciton transition decreases. The mean number of photons in cavity mode also exhibits a sharp peak around the two-photon resonance, and a broader peak corresponding to phonon assisted single photon transition is observed at Δ1 = Δx1 − δ1. In Figures 1(b) and 2(b), mean photon number ⟨n⟩ = ⟨a†a⟩ and the second order photon correlation for zero time delay, g2(0) = ⟨a†a⟩²/⟨a†a⟩², which indicates two-photon coincidence detection, has been plotted. The Fano factor F = (⟨n²⟩ − ⟨n⟩²)/⟨n⟩, defined as the ratio of variance for the field inside the cavity mode to that for the coherent state, having same average number of photons, has also been plotted. In Figure 1(b), when detuning Δ1 increases from zero to the value (Δx1 − δ1)/2 corresponding to two-photon resonance, the average number of photons inside cavity mode ⟨n⟩ reach maximum at the same time as the photon correlation function g2(0) reaches minimum. It has been proved that the value of g2(0) for two photon coherent states could be larger than 1 [27]. The Fano factor F also acquires minimum value, indicating the suppression of noise and a simultaneous large enhancement in cavity field at two-photon resonance. Therefore, two-photon lasing in single QD using an incoherent pump is possible.

In subplot (c), and (d) of figures 1 and 2, steady state populations and cavity field statistics are shown for different values of pump strength, using η1 = η2 and Δ1, corresponding to two-photon resonance. An increase in pump intensity, prompts the growth of further population in the biexciton state, and beyond a certain threshold value, population inversion is achieved. Population inversion increases with further increases in pump power, and saturates after attaining maximum value. The average number of photons inside cavity mode ⟨n⟩ also increases at first on increasing the pump field, and reaches maximum at a certain pump strength, on increasing pump power further than the value of ⟨n⟩, which is a well-understood self quenching effect in single emitter lasers.

In Figure 2, the temperature of the phonon bath is selected as T = 20 K. Comparing figures 1 and 2, we notice that at higher temperatures, due to phonon assisted off resonant transitions, the level of emission into cavity mode is enhanced for all off-resonant values of detuning Δ1. However, the emission around two-photon resonance reduces slightly and enhances around single photon resonance at Δ1 = (Δx1 − δ1)/2. Furthermore, the maximum values of population inversion, and the average number of photons inside the cavity mode decreases. On increasing the temperature, the values of g2(0) and Fano factor F increase under two-photon lasing conditions showing an enhancement in noise.

For the purpose of comparing single-photon and two-photon gains, we calculate the evolution for the diagonal elements of density matrix ρn using the two-photon resonant condition, and a simplified master equation. See appendices A and B for the derivation of the simplified master equation, and an evaluation of diagonal density matrix ρn. Having traced over the QD states, the probability of having n photons in cavity field Pn is given by

$$P_n = -\alpha np_n + G_n^{(2)} P_{n-2} + G_{n-1}^{(1)} P_{n-1} + \alpha_n^{(1)} P_{n+1} + \alpha_n^{(2)} P_{n+2}.$$  (8)

where we have assumed cavity damping is small enough that (n + 1)nPn+1 − nnPn = 0. The two-photon stimulated emission rate and the one-photon stimulated emission rate are given by

$$G_n^{(2)} = \alpha_n^{(2)}, \quad G_n^{(1)} = \alpha_n^{(1)},$$

with respect to steady state, αn = G_n^{(2)} + G_n^{(1)} + α_n^{(2)} + α_n^{(1)}. In figure 3, we plot stimulated emission rates and absorption rates for single-photon and two-photon processes at different temperatures. We find that for small values of pump strength, stimulated emission rates are always higher than absorption rates, even where there is no population inversion in QD. Further, for small values of pump strength, two-photon gain is larger than one-photon gain. At a higher temperature T = 20 K, as shown in figure 3(b), two-photon gain decreases, and one-photon gain increases. For smaller values of pump strength and at lower temperatures, two-photon gain dominates.

Next, we plot the Wigner distribution for the cavity field in figure 4. The Wigner distribution is calculated using the following definition [41]:

$$W(\alpha) = \frac{2}{\pi^2} e^{\frac{-|\alpha|^2}{2}} \int d^2 \beta \langle -\beta | \rho_c | \beta \rangle e^{-2(\beta^{*}a^{*} - \beta a)}$$  (9)

where, $\rho_c = \sum_{n,m} \rho_{nm} |n\rangle \langle m|$ denotes the density matrix for the cavity field at steady state, calculated after tracing over the QD states, and $|\beta\rangle$ is a coherent state. Using the density matrix $\rho_c$ in equation (9), the Wigner distribution takes the following form:

$$W(\alpha) = \frac{2}{\pi^2} e^{\frac{-|\alpha|^2}{2}} \sum_{n,m} \rho_{nm} \int d^2 \beta e^{-\frac{2(\beta^{*}p_{nm} - \beta p_{nm})}{\sqrt{n!m!}}} e^{-|\beta|^2} e^{-2(\beta^{*}a^{*} + \beta a)}.$$  (10)
Further, after evaluating the integration in equation (10), we get
\[ W(\alpha) = \frac{2}{\pi} e^{-2\langle \alpha^2 \rangle} \sum_{n,m} \frac{\rho_{nm}}{\sqrt{n!m!}} \frac{(-1)^{n+m}}{2^{n+m}} \frac{\partial^{n+m}}{\partial \alpha^n \partial \alpha^m} e^{-4\langle \alpha^2 \rangle}. \] (11)
In equation (11), the term \( \frac{\partial^{n+m}}{\partial \alpha^n \partial \alpha^m} e^{-4\langle \alpha^2 \rangle} \) is calculated using the Leibniz rule. Equation (11) is simplified to
\[ W(\alpha) = \frac{2}{\pi} e^{-2\langle \alpha^2 \rangle} \sum_{n,m} \rho_{nm} \sum_{i=0}^{n} \frac{(-1)^{n-i}}{i!} \frac{\partial^i}{\partial \alpha^i} \frac{\partial^{n-i}}{\partial \alpha^{n-i}} e^{-4\langle \alpha^2 \rangle}. \] (12)
In figure 4(a), we show the Wigner distribution (12) for parameters used in figure 1(a) and in figure 3(b) for parameters used in figure 2(a) with detuning \( \Delta_1 = 4.5g_1 \) corresponding to a two-photon resonance. Comparing figures 4(a) and (b), it becomes clear that on increasing temperature of the phonon bath the variance in cavity field increases. Moreover, squeezing in the cavity field, as predicted for two-photon coherent states, is absent. The absence of squeezing in two-photon lasers has been ascribed to the fact that the noise in the fields of each photon in the pair are added together, which negates squeezing in the cavity output.

3. Two-photon lasing using a coherent pump

For the purpose of pumping QDs in the biexciton state coherently, we consider an external laser applied between transitions \( |g\rangle \rightarrow |x\rangle \), and \( |x\rangle \rightarrow |u\rangle \). The polarization of the laser field is selected as orthogonal to the polarization of the cavity field. The Hamiltonian of the system in rotating frame, with frequency of pump laser \( \omega_p \) is given by
\[ H = \hbar \Delta_p \sigma_{xx} + \hbar (2\Delta_p - \delta_x - \Delta_{xx}) \sigma_{ax} + \hbar (\Delta_p - \delta_x) \sigma_{xy} + h(\Delta_p - \delta_x - \Delta_1) a^\dagger a + \langle B \rangle X_g, \] (14)
where \( \Delta_p = \omega_x - \omega_p \) and \( \Omega_1 \) is the detuning and dipole coupling strengths for the pump laser; all other parameters retain their previously defined meanings. Having followed the same procedure as above, we arrive at the same polaron transformed master equation (5), where system Hamiltonian and operators \( X_g \) and \( X_u \) are defined as follows:
\[ H_s = \hbar \Delta_p \sigma_{xx} + \hbar (2\Delta_p - \delta_x - \Delta_{xx}) \sigma_{ax} + \hbar (\Delta_p - \delta_x) \sigma_{xy} + h(\Delta_p - \delta_x - \Delta_1) a^\dagger a + \langle B \rangle X_g, \] (14)
\[ X_g = \hbar (\Omega_1 \sigma_{gx} + \Omega_2 \sigma_{ax} + g_1 \sigma_{xy} a + g_2 \sigma_{xy} a) + H.c. \] (15)
\[ X_u = i \hbar (\Omega_1 \sigma_{gx} + \Omega_2 \sigma_{ax} + g_1 \sigma_{xy} a + g_2 \sigma_{xy} a) + H.c. \] (16)
The polaron shifts are absorbed in the detunings.

In figure 5, we show the steady state populations and cavity field statistics at \( T = 5K \), using a coherent driving field. For subplots (a) & (b) we consider pump strength such that \( \Omega_1 =
At two photon resonance. In this case also, no squeezing occurs due to the far-off-resonant driving field between $|u\rangle$ and $|x\rangle$.

$\Omega_2 = 2g_1$, and for subplots (c) & (d) we use $\Omega_1 = \Omega_2 = 4g_1$. For typical values of parameters, we obtain population inversion for a good range of positive values of $\Delta_1$. However, when $\Delta_1$ becomes comparable to biexciton binding energy, single photon transitions from $|u\rangle \rightarrow |y\rangle$ are enhanced, and populations in $|y\rangle$ begin to predominate. The driving laser is applied resonantly between $|g\rangle \rightarrow |x\rangle$ transition, resulting in transition $|x\rangle \rightarrow |u\rangle$ being detuned by biexciton binding energy. In the presence of phonon interaction, such a method of pumping has been found to be very efficient for the deterministic generation of biexciton states [14, 15]. One should note that, phonon interaction is essential for achieving population inversion by this method. Due to the resonant application of the pump laser between $|g\rangle$ and $|x\rangle$, dressed states $|\pm\rangle = (|x\rangle \pm |g\rangle)/\sqrt{2}$ are formed. Therefore, the ground state effectively splits into a doublet, separated in frequency by pump laser strength $2\Omega_1$. Due to this splitting of the ground state into a doublet, two-photon resonant emission in cavity mode occurs corresponding to two values of $\Delta_1$ separated by $\Omega_1$. Since the influence of phonon coupling is more pronounced for larger detunings, the two-photon resonance peak at higher values of $\Delta_1$ has a larger width. Similarly to the case of two-photon lasing using incoherent pumping, we observe troughs in the plots of $g^2(0)$ and $F$ for the values of $\Delta_1$, corresponding to two-photon resonant emission. A sharp rise in average number of photons in cavity mode and the reduction of variance in photon distribution clearly indicate two-photon lasing at two different values of $\Delta_1$.

The pumping mechanism to the biexciton state can be understood using the dressed state picture shown in figure 6. Two dressed states $|\pm\rangle = (|x\rangle \pm |g\rangle)/\sqrt{2}$ are formed, due to resonant laser interaction between ground state $|g\rangle$ and exciton state $x$. The coupling between biexciton state $|u\rangle$ and dressed states $|\pm\rangle$ is equal to $\Omega_2/\sqrt{2}$. The coherent driving field is far-off-resonant with transitions $|u\rangle \rightarrow |\pm\rangle$. Therefore, phonon-assisted incoherent pumping primarily occurs from $|\pm\rangle$ to biexciton state $|u\rangle$, and coherent transition remains negligible. However, due to coherent field $\Omega_2$, the Stark energy shift is produced for the biexciton state. Although using a coherent driving field, biexcitons are generated incoherently, this can provide efficient control over pumping strength.

In figure 7, we plot Wigner function, calculated using equation (11), and corresponding to the parameters used in figure 4 at two photon resonance. In this case also, no squeezing is observed, as the procedure for two-photon generation in cavity mode is unchanged. Furthermore, we observe greater broadening in the Wigner function, corresponding to larger values of $\Delta_1$.

4. Continuous source of squeezed light through fourwave mixing

In this case, we consider coherent pumping using the same setup as discussed in the previous section. Here, however, the QD is pumped in biexciton state through two-photon resonant transition $|g\rangle \rightarrow |u\rangle$ via exciton state $x$. We have selected detuning of the pump $\Delta_p = (\delta_x + \delta_1)/2$ which satisfies the two-photon resonant condition. The couplings of pump laser $\Omega_1$ and $\Omega_2$ are considered as greater than QD-cavity couplings $g_1$ and $g_2$ for stronger pumping.

In figure 8, we plot steady state population and cavity field parameters. In subplots (a) & (b), we do not include phonon–exciton interactions, and in (c) & (d) we consider phonon interaction at $T = 5K$. In this pumping method we obtain almost equal populations in ground state $|g\rangle$ and biexciton state $|u\rangle$, which can be dominant. However, no significant population inversion is achieved. The average number of cavity photons displays a sharp two-photon resonance peak corresponding to $\Delta_1 = (\Delta_{xx} - \delta_1)/2$ and $g^2(0)$ and Fano factor exhibit a sharp trough. On including phonon–exciton interaction, the value of maximum $\langle n\rangle$ inside the cavity mode decreases, and the value of $g^2(0)$ and the Fano factor increase, indicating enhancement in the variance of photon distribution. In figure 9, we show the Wigner distribution of the cavity field, corresponding to two-photon resonance and temperatures of $\Delta_1$. For typical values of parameters, we obtain population inversion of biexciton states [14, 15].
when the temperature is increased, and exciton–phonon inter-
actions become significant, the coherence between emitted
photons diminishes, and these features slowly disappear. How-
ever, at lower temperature of up to 5K, squeezing in the cavity
field is visible, and one can realize a continuous source
of squeezed light using single QD.

5. Conclusions

We have discussed the effect of exciton–phonon coupling on
two-photon lasing in a single quantum dot embedded inside a
photonic crystal cavity. We have analyzed schemes using inco-
erent and coherent pumps for achieving two-photon lasing. In
order to visualize squeezing in the cavity field we have plotted
the Wigner function for each. In the case of two-photon lasing,

we do not find squeezing in the cavity field. However, we dis-
cuss a method of four-wave mixing for generating a continuous
source of squeezed state using single QD.

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Appendix A Deriving a simplified master equation

We are interested in resonant two-photon transitions from
biexcitons; therefore, we work under conditions where sin-
gle photon transitions are suppressed, i.e. the coupling con-
stants of the cavity field where exciton and biexciton states
are much smaller than their detunings ($g_1, g_2 \ll |\Delta_1|$), and
the two-photon resonance condition $2\Delta_1 + \delta_x - \Delta_\Delta = 0$ is satis-
fied. Under such conditions, the master equation (5) can be
further simplified using $H_t = h\Delta_1|y\rangle\langle y| + h(\Delta_1 + \Delta_2)|u\rangle\langle u|$, with
$\Delta_2 = \Delta_1 + \delta_x - \Delta_\Delta$, and neglecting the terms propor-
tional to $g_1$ and $g_2$ in equation (6) in the expression of $X(t, \tau)$. This
simplified form of master equation contains Lindblad
terms corresponding to different processes involved in the
dynamics. Under such an approximation, and after algebraic
manipulation [42], the master equation (5) takes the form

$$\dot{\rho}_s = -\frac{i}{\hbar} [H_{\text{eff}}, \rho_s] - \sum_{i=x,y} \left( \frac{\gamma_i}{2} \mathcal{L} [\sigma_{gi}] + \frac{\eta_i}{2} \mathcal{L} [\sigma_{ui}] \right) \rho_s - \frac{\kappa}{2} \mathcal{L} [a] \rho_s - \frac{\gamma_d}{2} \mathcal{L} [\sigma_{ud}] \rho_s - \frac{1}{2} \left( \Gamma_{i+}^{\dagger} \mathcal{L} [\sigma_{yi}a^\dagger] + \Gamma_{i-}^{\dagger} \mathcal{L} [\sigma_{yi}a] - \Gamma_i \mathcal{L} [\sigma_{yi}a^\dagger] \right) \rho_s - \frac{\Gamma_{yy}}{2} \mathcal{L} [\sigma_{yda^\dagger} - 2\sigma_{yda^\dagger} \rho_s - 2\sigma_{yda^\dagger} \rho_s \sigma_{yda^\dagger} - \rho_s \sigma_{yda^\dagger}^2] - \frac{\Gamma_{yg}}{2} \mathcal{L} [\sigma_{yda} - 2\sigma_{yda} \rho_s \sigma_{yda^\dagger} - \rho_s \sigma_{yda}^2], \quad (A1)
where the first term corresponds to the effective dynamics of the system when QDs are far off-resonant. The effective Hamiltonian is given by

$$H_{\text{eff}} = H_0 + h \left( \delta_1^a |a\rangle \langle a| + \delta_1^b |b\rangle \langle b| + \delta_1^c |c\rangle \langle c| + \delta_1^d |d\rangle \langle d| \right) + \hbar \Omega \left( \sigma_{\text{ad}} a^2 + \sigma_{\text{bd}} b^2 \right);$$  \hspace{1cm} (A2)

where $\delta^+$ are Stark shifts, and the third term represents two-photon processes. The expressions for Stark shifts and two-photon coupling are given by

$$\delta_1^+ = g_2^2 B^2 \Im \left[ \int_0^\infty d\tau \left( e^{i\omega\tau} - 1 \right) e^{i\Delta_1\tau} \right]$$ \hspace{1cm} (A3)

$$\Omega = g_1 g_2 B^2 \Re \int_0^\infty d\tau \left( e^{-i\omega\tau} - 1 \right) e^{i\Delta_1\tau}. \hspace{1cm} (A4)$$

The phonon induced single photon and two-photon cavity mode feeding rates $\Gamma_\text{s}$, and the phonon induced two-photon transition rates $\Gamma_\text{ag}$ are given by

$$\Gamma_\text{s} = 2 g_1^2 B^2 \Omega \int_0^\infty d\tau \left( e^{i\omega\tau} - 1 \right) \left( e^{i\Delta_1\tau} \right)$$ \hspace{1cm} (A5)

$$\Gamma_\text{ag} = 2 g_1 g_2 B^2 \Omega \int_0^\infty d\tau \left( e^{-i\omega\tau} - 1 \right) \left( e^{i\Delta_1\tau} \right). \hspace{1cm} (A6)$$

The numerical results using simplified master equation (A1), and the results obtained after integration of master equation (5) for parameters used in figures 2(c) and (d) are plotted in figure A.1 which matches very well.

**Appendix B. Deriving single photon and two-photon absorption and stimulated emission rates**

We use simplified master equation (A1) for deriving the equation of motion for the coupled QD-cavity system using the notation $|U\rangle \equiv |u, n\rangle$, $|Y\rangle \equiv |y, n + 1\rangle$, $|G\rangle \equiv |g, n + 2\rangle$, $|X\rangle \equiv |x, n\rangle$, and $\{\{\rho_{ij}\}\} = \rho_{ij}$. The diagonal elements of the QD-cavity density matrix are given by

$$\dot{\rho}_{UU} = ig_2 \sqrt{n + 1} [\rho_{UY} - \rho_{UY}] + i \Omega \sqrt{(n + 1)(n + 2)} [\rho_{UG} - \rho_{GU}] - \frac{\Gamma_{ag}}{2} \sqrt{(n + 1)(n + 2)} [\rho_{UG} + \rho_{GU}]$$

$$- \left( \Gamma_1^+ (n + 1) + 2 \gamma \right) \rho_{UU} + \Gamma_1^- (n + 1) \rho_{YY} + \eta_2 \rho_{XX},$$

$$\dot{\rho}_{YY} = ig_1 \sqrt{n + 2} [\rho_{YG} - \rho_{GY}] - ig_2 \sqrt{n + 1} [\rho_{UY} - \rho_{UY}] + \Gamma_{ag} \sqrt{(n + 1)(n + 2)} [\rho_{UG} + \rho_{GU}]$$

$$- \left( \Gamma_1^+ (n + 1) + \Gamma_1^+ (n + 2) + \gamma \right) \rho_{YY} + \Gamma_1^- (n + 2) \rho_{GG} + \Gamma_1^- (n + 1) \rho_{UU} + \gamma (u, n + 1) |\rho_{u}| u, n + 1),$$

$$\dot{\rho}_{GG} = -ig_1 \sqrt{n + 2} [\rho_{YG} - \rho_{GY}] - i \Omega \sqrt{(n + 1)(n + 2)} [\rho_{UG} - \rho_{GU}] - \frac{\Gamma_{ag}}{2} \sqrt{(n + 1)(n + 2)} [\rho_{UG} + \rho_{GU}]$$

$$+ \Gamma_1^- (n + 2) \rho_{YY} - \Gamma_1^- (n + 2) \rho_{GG} + \gamma (y, n + 2) |\rho_{y}| y, n + 2) - \eta \rho_{GG},$$

and the off-diagonal density matrix elements required to determine diagonal density matrix are given by

$$\dot{\rho}_{UF} = - \left[ i \Delta_2 + 2 \delta_2 (n + 1) - i \delta_2^+ (n + 1) - i \delta_2^- (n + 2) + \frac{\Gamma_1^+}{2} (n + 1) + \frac{\Gamma_1^-}{2} (n + 1) + \frac{\Gamma_1^+}{2} (n + 2) + \frac{3 \gamma + 3 \gamma_d}{2} \right] \rho_{UF}$$

$$+ ig_2 \sqrt{n + 1} [\rho_{UU} - \rho_{YY}] - \left( \frac{\Gamma_{ag}}{2} + i \Omega \right) \sqrt{(n + 1)(n + 2)} \rho_{UG} + i g_1 \sqrt{n + 2} \rho_{UG},$$

$$\dot{\rho}_{YG} = - \left[ i \Delta_1 + 2 \delta_1 (n + 1) + i \delta_1^+ (n + 2) - i \delta_1^- (n + 2) + \frac{\Gamma_1^-}{2} (n + 1) + \frac{\Gamma_1^+}{2} (n + 2) + \frac{\eta + \gamma + \gamma_d}{2} \right] \rho_{YG}$$

$$+ ig_1 \sqrt{n + 2} [\rho_{YG} - \rho_{GG}] - \left( \frac{\Gamma_{ag}}{2} + i \Omega \right) \sqrt{(n + 1)(n + 2)} \rho_{YY} - ig_2 \sqrt{n + 2} \rho_{UG},$$

$$\dot{\rho}_{UG} = - \left[ i \delta_1^+ (n + 1) - i \delta_1^- (n + 2) + \frac{\Gamma_1^-}{2} (n + 1) + \frac{\Gamma_1^+}{2} (n + 2) + \gamma + \gamma_d + \frac{\eta}{2} \right] \rho_{UG}$$

$$- ig_2 \sqrt{n + 1} \rho_{YG} + ig_1 \sqrt{n + 2} \rho_{UU} - \left( \frac{\Gamma_{ag}}{2} + i \Omega \right) \sqrt{(n + 1)(n + 2)} \rho_{UU} - \left( \frac{\Gamma_{ag}}{2} + i \Omega \right) \sqrt{(n + 1)(n + 2)} \rho_{GG},$$

(B1)
here, we have assumed that $\gamma_1 \approx \gamma_2 = \gamma$ and that cavity damping $\kappa$ is negligible. We solve off-diagonal elements in steady state condition, and substitute the solution in equation (B1). Having traced over the QD states we obtain the evolution of the cavity field of the form (8).

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