A criterion on vanishing cohomology

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Abstract: In Homotopy decomposition of classifying spaces via elementary Abelian subgroups, Stephan Jackowski and James McClure show, for functors admitting a Mackey complement over categories holding a direct product, a general result on vanishing cohomology. Here, we develop a framework leading to a general result on trivial homotopy which partially generalizes Jackowski and McClure's result in two different directions.

1. Introduction

1.1. In [1], Stephan Jackowski and James McClure state a quite general criterion on categories holding a direct product, which guarantees the annihilation of the positive cohomology groups for the contravariant \( \mathbb{A}_0 \)-valued functors admitting a Mackey complement. In [2, Theorem 6.26], we apply their argument to obtain a vanishing result on the context of a Frobenius \( P \)-category \( \mathcal{F} \) [2, 2.8].

1.2. An example of such categories is the additive cover — already introduced in [1] and recalled in section 4 below — of the full subcategory \( \tilde{\mathcal{F}}^\infty \) of the exterior quotient \( \tilde{\mathcal{F}} \) [2, 1.3], over the set of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \) [2, 4.8]; indeed, we show in [2, Proposition 6.14] that this category — noted \( \mathcal{ac}(\tilde{\mathcal{F}}^\infty) \) — admits a direct product. But, we have noticed a general kind of contravariant \( \text{mod}_\mathcal{O} \)-valued functors defined over \( \mathcal{ac}(\tilde{\mathcal{F}}^\infty) \) which need not admit a Mackey complement and nevertheless their standard differential complex (cf. 3.1 below) is homotopically trivial. On the other hand, the additive cover \( \mathcal{ac}(\tilde{\mathcal{F}}) \) of the exterior quotient \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) admits a family of commutative square diagrams (cf. 6.1 below) — which are not pull-backs — in such a way that, considering the stable cohomology groups [2, A3.17] together with the corresponding differential complex, the contravariant \( \text{mod}_\mathcal{O} \)-valued functors admitting a complement compatible with this family (cf. 6.7) supply again a homotopically trivial differential complex.

1.3. Our proofs of these two facts were similar enough to suggest the existence of a common argument; indeed, in section 2 we develop a quite abstract framework, leading to a general result on trivial homotopy in section 3. This result covers both situations mentioned above; in section 4 the first situation admits a more general discussion clarifying the relationship between the existence of a direct product in \( \mathcal{ac}(\tilde{\mathcal{F}}^\infty) \) and the fact that any \( \tilde{\mathcal{F}}^\infty \)-morphism is an epimorphism [2, Corollary 4.9]. From the first situation, in section 5 we get the triviality of some cohomology groups, leading to a key result for the existence and the uniqueness of the so-called perfect \( \mathcal{F} \)-locality \( \mathcal{L} \) [2, 17.13] associated with \( \mathcal{F} \). Finally, the second situation is discussed in section 6 and it plays a role in proving some functoriality of \( \mathcal{L} \), as we show in [3].
2. The general framework

2.1. We denote by $\mathfrak{Ab}$ the category of Abelian groups, by $\text{mod}_\mathcal{O}$ the category of finitely generated $\mathcal{O}$-modules where $\mathcal{O}$ is a complete discrete valuation ring with unequal characteristics, and by $\mathcal{CC}$ the category of small categories; all the other categories we consider are assumed to be small. Let $\mathcal{A}$ be a category; we call $\mathcal{A}$-category any category $\mathcal{B}$ containing $\mathcal{A}$ as a subcategory having the same objects, and fulfilling the following two conditions

2.1.1. Any $\mathcal{B}$-morphism $\varphi: R \to Q$ is the composition of a $\mathcal{B}$-isomorphism $\varphi_*: R \cong R_*$ and an $\mathcal{A}$-morphism $\iota: R_* \to Q$.

2.1.2. Any $\mathcal{B}$-isomorphism $\tau: R \cong R'$ such that $\iota' \circ \tau$ is an $\mathcal{A}$-morphism for some $\mathcal{A}$-morphism $\iota': R' \to Q'$, is an $\mathcal{A}$-isomorphism.

In some sense, these conditions generalize the divisibility condition [2, 2.3] in $\mathcal{F}$; we are interested in the cases where the category $\mathcal{B}$ is either a suitable extension $\tilde{\mathcal{F}}$ — defined in section 6 below — of the Frobenius $\mathcal{P}$-category $\mathcal{F}$, or the full subcategory $\mathcal{F}^\circ$ of $\mathcal{F}$; then $\mathcal{A}$ is either $\mathcal{T}_P \times \mathcal{P}$ (where $\mathcal{T}_P$ is the transporter category of $\mathcal{P}$ [2, 17.2] and see 6.3.1 below for this notation), or the “intersection” $\mathcal{F}^\circ_P$ with $\mathcal{F}^\circ$ of the “image” of $\mathcal{T}_P$ in $\mathcal{F}$.

2.2. For any $\mathcal{A}$-category $\mathcal{B}$, recall that an interior structure $\mathcal{I}$ of $\mathcal{B}$ is a correspondence sending any $\mathcal{B}$-object $Q$ to a subgroup $\mathcal{I}(Q)$ of $\mathcal{B}(Q)$ in such a way that we have $\varphi \circ \mathcal{I}(R) \subset \mathcal{I}(Q) \circ \varphi$ for any $\mathcal{B}$-morphism $\varphi: R \to Q$ [2, 1.3]; similarly, let us call co-interior structure $\mathcal{I}^\circ$ of $\mathcal{B}$ any interior structure of the opposite category $\mathcal{B}^\circ$; in this case, it follows from 2.1 that the correspondence sending any $\mathcal{A}$-object $Q$ to the intersection $\mathcal{I}(Q) \cap \mathcal{A}(Q)$ is a co-interior structure of $\mathcal{A}$. Then, a bi-interior structure $\mathcal{I}$, $\mathcal{I}^\circ$ of $\mathcal{B}$ is any pair formed by an interior and a co-interior structures $\mathcal{I}$ and $\mathcal{I}^\circ$ of $\mathcal{B}$ such that $\mathcal{I}(Q)$ and $\mathcal{I}^\circ(Q)$ centralize each other for any $\mathcal{B}$-object $Q$; we still denote by $\tilde{\mathcal{B}}$ the corresponding bi-exterior quotient; that is to say, for any pair of $\mathcal{B}$-objects $Q$ and $R$ we have

$$\tilde{\mathcal{B}}(Q, R) = \mathcal{I}(Q) \setminus \mathcal{B}(Q, R)/\mathcal{I}^\circ(R)$$

2.2.2.

Following [2, A2.2], a representation of $\mathcal{B}$ is nothing but a functor $\tau: \mathcal{B} \to \mathcal{CC}$ and then the semidirect product $\tau \times \mathcal{B}$ is a new category [2, A2.7] where an object is a pair $(a, Q)$ formed by a $\mathcal{B}$-object $Q$ and by a $\tau(Q)$-object $a$, and a morphism

$$(f, \varphi) : (b, R) \longrightarrow (a, Q)$$

2.2.3.

is a pair formed by a $\mathcal{B}$-morphism $\varphi: R \to Q$ and by an $\tau(Q)$-morphism

$$f : (\tau(\varphi))(b) \longrightarrow a$$

2.2.4.

Note that the category of sets is a subcategory of $\mathcal{CC}$ where any set is identified with the trivial category having only the identity morphisms.
2.3. Our criterion to get a \textit{trivial homotopy} is based on the existence of the following data. For an \(\mathcal{A}\)-category \(\mathcal{B}\), let us call \textit{homotopic system} \(\mathcal{H} = (\mathcal{I}, \mathcal{I}^\circ, \mathcal{G}, s, n, \nu)\) any quintuple formed by

\begin{enumerate}
  \item An \textit{bi-interior structure} \(\mathcal{I}, \mathcal{I}^\circ\) of \(\mathcal{B}\) such that \(\mathcal{I}(Q)\) is contained in \(\mathcal{A}(Q)\) for any \(\mathcal{B}\)-object \(Q\), and that the corresponding \textit{bi-exterior quotient} \(\tilde{\mathcal{A}}\) admits a final object \(P\).
  \item A \textit{subcategory} \(\mathcal{G}\) of \(\mathcal{B}\) with the same objects and only with \(\mathcal{G}\)-isomorphisms, in such a way that \(\mathcal{G}(Q)\) contains \(\mathcal{I}(Q)\) and \(\mathcal{I}^\circ(Q)\) for any \(\mathcal{B}\)-object \(Q\).
  \item A representation \(s: \mathcal{B} \to \mathcal{CC}\) of \(\mathcal{B}\) mapping any \(\mathcal{B}\)-object \(Q\) on a finite set \(s_Q\) and any \(\mathcal{B}\)-morphism \(\varphi: R \to Q\) on a map \(s_\varphi: s_R \to s_Q\).
  \item A \textit{natural map} \(\nu: n \to p\).
\end{enumerate}

We set \(\tilde{\mathcal{B}} = s \times \mathcal{B}\) and denote by \(\tilde{\mathcal{G}}\) the subcategory of \(\tilde{\mathcal{B}}\) determined by \(\mathcal{G}\), by \(\tilde{\mathcal{B}}\) the \(\mathcal{I}, \mathcal{I}^\circ\)-\textit{bi-exterior quotient} of \(\mathcal{B}\), by \(\tilde{\mathcal{G}}\) the image of \(\mathcal{G}\) in \(\tilde{\mathcal{B}}\) and by \(p: \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}\) the canonical functor; thus, a \(\mathcal{B}\)-object is a pair \((s, Q)\) where \(Q\) is an \(\mathcal{A}\)-object and \(s\) an element of \(s_Q\), and a \(\tilde{\mathcal{B}}\)-morphism \((t, R) \to (s, Q)\) is a \(\mathcal{B}\)-morphism \(\varphi: R \to Q\) such that \(s_\varphi(t) = s\); in order to avoid confusion, we often denote this \(\tilde{\mathcal{B}}\)-morphism by the pair \((t, \varphi)\).

2.3.4 A functor \(n: \mathcal{B} \to \tilde{\mathcal{A}} \subset \tilde{\mathcal{B}}\) sending \(\tilde{\mathcal{G}}\)-isomorphisms to \(\tilde{\mathcal{G}}\)-isomorphisms.

2.3.5 A natural map \(\nu: n \to p\).

Note that if \(\mathcal{G}\) contains all the \(\mathcal{B}\)-isomorphisms then condition 2.3.2 holds; on the contrary, if \(\mathcal{A}\) has a final object \(P\) and both \(\mathcal{I}\) and \(\mathcal{I}^\circ\) are \textit{trivial} then we can choose as \(\mathcal{G}\) the \textit{trivial subcategory} just containing the \textit{identity} \(\mathcal{B}\)-morphisms. Moreover, for any \(\mathcal{B}\)-object \(Q\), any \(s \in s_Q\) and any element \(\xi \in \mathcal{I}(Q)\), the \(\mathcal{A}\)-morphism

\[ n(s, \xi): n(s, Q) \cong n(s_\xi(s), Q) \]

is a \(\tilde{\mathcal{G}}\)-isomorphism.

2.4. In this situation, any \(\mathcal{B}\)-\textit{chain} \(q: \Delta_n \to \mathcal{B}\) \([2, \text{A2.2 and A2.8}]\) can be clearly lifted to a unique \(\tilde{\mathcal{B}}\)-\textit{chain} \(\tilde{q}_n: \Delta_n \to \tilde{\mathcal{B}}\) fulfilling \(\tilde{q}_n(0) = (s, q(0))\) for any choice of \(s \in s_q(0)\). Moreover, for any \(\tilde{\mathcal{B}}\)-\textit{chain} \(\tilde{q}: \Delta_n \to \tilde{\mathcal{B}}\) we have the \(\tilde{\mathcal{A}}\)-\textit{chain} \(n \circ \tilde{q}\) and, considering \(n \circ \tilde{q}\) as a \(\tilde{\mathcal{B}}\)-\textit{chain}, the natural map

\[ \nu \ast \tilde{q}: n \circ \tilde{q} \longrightarrow p \circ \tilde{q} \]

then, for any \(\ell \in \Delta_n\), as in \([2, \text{Lemma A4.2}]\) we denote by

\[ h^\vartriangleleft_\nu(\nu \ast \tilde{q})_\ell: \Delta_{n+1} \longrightarrow \tilde{\mathcal{B}} \]

the functor which coincides with \(n \circ \tilde{q}\) over \(\Delta_\ell\), maps \(i \in \Delta_{n+1} - \Delta_\ell\) on \((p \circ \tilde{q})(i - 1)\), maps \(i \cdot i + 1\) on \((p \circ \tilde{q})(i - 1 \cdot i)\) if \(i \leq n\), and maps \(\ell \cdot \ell + 1\) on

\[ (\nu \ast \tilde{q})_\ell: (n \circ \tilde{q})(\ell) \longrightarrow (p \circ \tilde{q})(\ell) \]
moreover, denote by
\[ h_{n+1}^n(\nu \circ \tilde{q}) : \Delta_{n+1} \to \tilde{A} \subset \tilde{B} \]
the $\tilde{A}$-chain $(n \circ \tilde{q})^P$ which extends $n \circ \tilde{q}$ to $\Delta_{n+1}$ sending $n + 1$ to $P$ and $n \cdot n + 1$ to the unique $\tilde{A}$-morphism from $(n \circ \tilde{q})(n)$ to $P$.

2.5. It is clear that $\mathcal{G}$ determines a subcategory $\tilde{\mathcal{G}}$ of $\tilde{B}$; then, for any $\tilde{\mathcal{G}}$-isomorphic pair of $\tilde{B}$-chains $\tilde{q}, \tilde{q}' \in \tilde{\mathcal{G}}(\Delta_n, \tilde{B})$ and any $\ell \in \Delta_{n+1}$ we claim that the $\tilde{B}$-chains $h^n_{\ell}(\nu \circ \tilde{q})$ and $h^n_{\ell}(\nu \circ \tilde{q}')$ are $\tilde{\mathcal{G}}$-isomorphic; more explicitly, for any natural $\tilde{\mathcal{G}}$-isomorphism $\tilde{\chi} : \tilde{q} \cong \tilde{q}'$ we claim that we have the natural $\tilde{\mathcal{G}}$-isomorphism
\[ h^n_{\ell}(\nu \circ \tilde{q}) : h^n_{\ell}(\nu \circ \tilde{q}) \cong h^n_{\ell}(\nu \circ \tilde{q}') \]
which coincides with $n \circ \tilde{\chi}$ over $\Delta_i$ if $\ell \leq n$, and either maps $i \in \Delta_{n+1} - \Delta_{\ell}$ on $(p \circ \tilde{\chi})_{i-1}$ or maps $\ell = n + 1$ on the identity $\tilde{B}$-morphism of $P$; indeed, for any $0 \leq i \leq \ell \leq n$, we have (cf. condition 2.3.4)
\[ h^n_{\ell}(\nu \circ \tilde{q})(i) = n(\tilde{q}(i)) \overset{n(\tilde{\chi}(i))}{=} n(\tilde{q}'(i)) = h^n_{\ell}(\nu \circ \tilde{q}')(i) \]
morover, for any $\ell + 1 \leq i \leq n + 1$, we also have
\[ h^n_{\ell}(\nu \circ \tilde{q})(i) = p(\tilde{q}(i - 1)) \overset{p(\tilde{\chi}(i - 1))}{=} p(\tilde{q}'(i - 1)) = h^n_{\ell}(\nu \circ \tilde{q}')(i) \]
finally, if $\ell \leq n$ then we get the commutative diagram (cf. condition 2.3.5)
\[ h^n_{\ell}(\nu \circ \tilde{q})(\ell + 1) = p(\tilde{q}(\ell)) \overset{p(\tilde{\chi}(\ell))}{=} p(\tilde{q}'(\ell)) = h^n_{\ell}(\nu \circ \tilde{q}')(\ell + 1) \]

2.6. On the other hand, any contravariant functor $a : \tilde{B} \to \mathfrak{Ab}$ determines a new contravariant functor from $\mathcal{B}$ to $\mathfrak{Ab}$ sending any $\mathcal{B}$-object $Q$ to the product $\prod_{s \in S_Q} (a \circ n)(s, Q)$ and any $\mathcal{B}$-morphism $\varphi : R \to Q$ to the group homomorphism
\[ \prod_{s \in S_Q} (a \circ n)(s, Q) \to \prod_{t \in S_R} (a \circ n)(t, R) \]
mapping the element $a = \sum_{s \in S_Q} a_s$, where $a_s$ belongs to $(a \circ n)(s, Q)$, on
\[ \sum_{t \in S_R} ((a \circ n)(t, \varphi))(a_{S_t}(t)) \in \prod_{t \in S_R} (a \circ n)(t, R) \]
in particular, $\mathcal{I}(Q)$ acts on $\prod_{s \in S_Q} (a \circ n)(s, Q)$ and we get a contravariant subfunctor $\mathcal{H}(a) : \mathcal{B} \to \mathfrak{Ab}$ sending any $\mathcal{B}$-object $Q$ to
\[ (\mathcal{H}(a))(Q) = (\prod_{s \in S_Q} (a \circ n)(s, Q))^{\mathcal{I}(Q)} \]
indeed, for any $\rho \in \mathcal{I}(R)$ we have $\varphi \circ \rho = \chi \circ \varphi$ for a suitable $\chi \in \mathcal{I}(Q)$ and therefore, if $a = \sum_{s \in s_Q} a_s$ belongs to $(\mathcal{H}(a))(Q)$ then for any $s \in s_Q$ we have

$$((a \circ n)(s, \chi))((a_{a_s}(s)) = a_s$$

so that we get

$$\left( (\mathcal{H}(a))(\rho) \right) \left( \sum_{t \in s_R} ((a \circ n)(t, \varphi))(a_{a_{\rho}(t)}) \right)$$

$$= \sum_{t \in s_R} ((a \circ n)(t, \rho)) \left( ((a \circ n)(s_{\rho}(t), \varphi))((a_{a_{\rho}(t)}) \right)$$

$$= \sum_{t \in s_R} ((a \circ n)(t, \varphi \circ \rho))((a_{a_{\rho}(t)})$$

$$= \sum_{t \in s_R} ((a \circ n)(t, \chi \circ \varphi))((a_{a_{\rho}(t)})$$

$$= \sum_{t \in s_R} ((a \circ n)(t, \varphi) \circ (a \circ n)(s_{\rho}(t), \chi))((a_{a_{\rho}(t)})$$

$$= \sum_{t \in s_R} ((a \circ n)(t, \varphi))((a_{a_{\rho}(t)})$$

thus, $a$ belongs to $(\mathcal{H}(a))(R)$; moreover, the above correspondence sending $a$ to $\mathcal{H}(a)$ is clearly functorial.

2.7. Then, denoting by $\hat{a}: \mathcal{B} \rightarrow \mathbb{Ab}$ the composition of $a$ with the structural functor $\mathcal{B} \rightarrow \hat{B}$, we claim that there is a natural map

$$\Delta_{\mathcal{H}(a)}: \hat{a} \longrightarrow \mathcal{H}(a)$$

sending any $\mathcal{B}$-object $Q$ to the group homomorphism

$$\Delta_{\mathcal{H}(a)}(Q): a(Q) \longrightarrow \left( \prod_{s \in s_Q} ((a \circ n)(s, Q))^{\mathcal{I}(Q)} \right)$$

mapping $a \in a(Q)$ on

$$\Delta_{\mathcal{H}(a)}(Q)(a) = \sum_{s \in s_Q} ((a \ast \nu)(s, Q)(a)$$

indeed, this makes sense since $p(s, Q)$ coincides with $Q$; moreover, for any $\mathcal{B}$-morphism $\varphi: R \rightarrow Q$, we have (cf. condition 2.3.5)

$$\left( (\mathcal{H}(a))(\varphi) \circ \Delta_{\mathcal{H}(a)}(Q) \right)(a) = \sum_{t \in s_R} ((a \circ n)(t, \varphi)) \left( ((a \ast \nu)(s_{\varphi}(t), Q)(a) \right)$$

$$= \sum_{t \in s_R} ((a \ast \nu)(t, R) \left( ((a(\varphi))(a) \right) = (\Delta_{\mathcal{H}(a)}(R) \circ a(\hat{\varphi}))(a)$$

2.6.4,
We are ready to state our criterion on the \textit{contravariant} functor; we say that a \textit{contravariant} functor \(a: \tilde{B} \to \mathfrak{Ab}\) is \(\mathcal{H}\)-split if the \textit{natural map} \(\Delta_{\mathcal{H}}(a)\) admits a \textit{natural section}
\[\theta: \mathcal{H}(a) \to \tilde{a}\] 2.7.5.

3. The trivial homotopy

3.1. Let \(\mathcal{B}\) be an \(\mathcal{A}\)-category, \(\mathcal{H} = (I, I^\circ, G, s, n, \nu)\) an \textit{homotopic system} for \(\mathcal{B}\) and \(a: \tilde{B} \to \mathfrak{Ab}\) a \textit{contravariant} \(\mathcal{H}\)-split functor; we keep the notation above. For any \(n \in \mathbb{N}\), denoting by \(\mathfrak{Ft}(\Delta_n, B)\) the set of functors from \(\Delta_n\) to \(\tilde{B}\), we set
\[C^n(\tilde{B}, a) = \prod_{\tilde{q} \in \mathfrak{Ft}(\Delta_n, \tilde{B})} a(\tilde{q}(0))\] 3.1.1;

in this situation, we say that an element \(a = (a_\tilde{q})_{\tilde{q} \in \mathfrak{Ft}(\Delta_n, \tilde{B})}\) in \(C^n(\tilde{B}, a)\) is \(\tilde{G}\)-\textit{stable} \([2, A3.17]\) if for any \(\tilde{G}\)-isomorphic pair of \(\tilde{B}\text{-chains}\) \(\tilde{q}\) and \(\tilde{q}'\) in \(\mathfrak{Ft}(\Delta_n, \tilde{B})\), and any \textit{natural} \(\tilde{G}\)-\textit{isomorphism} \(\tilde{\chi}: \tilde{q} \cong \tilde{q}'\), we have
\[a_\tilde{q} = (a(\tilde{\chi}_0))(a_{\tilde{q}'})\] 3.1.2.

Then, denoting by \(C_\tilde{G}^n(\tilde{B}, a)\) the subgroup of \(\tilde{G}\)-\textit{stable} elements of \(C^n(\tilde{B}, a)\), in this section we prove that the \textit{differential subcomplex} with \(n\)-term \(C_\tilde{G}^n(\tilde{B}, a)\) and with the usual \textit{differential map} sending \(a\) to the \(C_\tilde{G}^{n+1}(\tilde{B}, a)\)-element \(d_n(a) = (d_n(a)_\tilde{q})_{\tilde{q} \in \mathfrak{Ft}(\Delta_{n+1}, \tilde{B})}\) defined by
\[d_n(a)_\tilde{q} = (a(t(0\cdot 1)))(a_{\tilde{q}0}) + \sum_{i=1}^{n+1} (-1)^i a_{\tilde{q}i0}\] 3.1.3

is \textit{homotopically trivial}. Note that, denoting by \(\tilde{a}: \mathcal{B} \to \mathcal{A}\) the composition of \(a\) with the \textit{structural functor} \(\mathcal{B} \to \tilde{B}\), the group homomorphism sending any element \((a_\tilde{q})_{\tilde{q} \in \mathfrak{Ft}(\Delta_n, \tilde{B})}\) of \(C_\tilde{G}^n(\tilde{B}, a)\) to the element \((\tilde{a}_\tilde{q})_{\tilde{q} \in \mathfrak{Ft}(\Delta_n, \tilde{B})}\) of \(C_\tilde{G}^n(\tilde{B}, \tilde{a})\) defined by \(\tilde{a}_\tilde{q} = a_\tilde{q}\), where \(\tilde{q}\) is the composition of \(\tilde{q}: \Delta_n \to \tilde{B}\) with the \textit{structural functor} \(\mathcal{B} \to \tilde{B}\), determines a group isomorphism
\[C_\tilde{G}^n(\tilde{B}, a) \cong C_\tilde{G}^n(\mathcal{B}, \tilde{a})\] 3.1.4

and we identify to each other both members of this isomorphism.

3.2. We are ready to define the \textit{homotopy map}; for any \(n \in \mathbb{N}\), we consider the group homomorphism
\[h^n: C_\tilde{G}^{n+1}(\tilde{B}, a) \to C_\tilde{G}^n(\tilde{B}, a)\] 3.2.1
sending any element \( a = (a_t)_{t \in \mathfrak{gct}(\Delta_n, \mathcal{B})} \) of \( C^{n+1}_G(\mathcal{B}, a) \) to the element

\[
h^n(a) = (h^n(a)_q)_{q \in \mathfrak{gct}(\Delta_n, \mathcal{B})} \in C^n_G(\mathcal{B}, \mathfrak{g}) = C^n_G(\mathcal{B}, a)
\]

defined by

\[
h^n(a)_q = \sum_{\ell=0}^{n+1} (-1)^\ell \theta_{q(0)} \left( \sum_{s \in \mathfrak{s}_q(0)} a_{b_\ell^n(\nu \ast \hat{q}_s)} \right)
\]

3.2.2.

This makes sense since for any \( \xi_0 \in I(q(0)) \) we can construct a natural \( \mathcal{B} \)-automorphism \( \xi : q \cong q \) which starts by \( \xi_0 : q(0) \cong q(0) \) and, for any \( i \in \Delta_n \), \( \xi_i \) belongs to \( I(q(i)) \); in particular, \( \xi \) is a natural \( \mathcal{G} \)-automorphism (cf. condition 2.3.2) and therefore it follows from 2.4 above that, for any \( s \in \mathfrak{s}_q(0) \), we get a natural \( \mathcal{G} \)-isomorphism

\[
\hat{\xi}_s : \hat{q}_s \cong \hat{q}_{\xi_0(s)}
\]

3.2.3.

3.3. Then, from 2.5 above, for any \( s \in \mathfrak{s}_q(0) \) and any \( \ell \in \Delta_{n+1} \) we still get a natural \( \mathcal{G} \)-isomorphism

\[
b_\ell^n(\nu \circ \hat{\xi}_s) : b_\ell^n(\nu \ast \hat{q}_s) \cong b_\ell^n(\nu \ast \hat{q}_{\xi_0(s)})
\]

3.3.1.

and therefore the \( \mathcal{G} \)-stability of \( a = (a_t)_{t \in \mathfrak{gct}(\Delta_n, \mathcal{B})} \in C^{n+1}_G(\mathcal{B}, a) \) forces

\[
a_{b_\ell^n(\nu \ast \hat{q}_s)} = \left( (a \circ n)(a_{\xi_0}(s), \xi_0) \right) \left( a_{b_\ell^n(\nu \ast \hat{q}_{\xi_0(s)})} \right)
\]

3.3.2.

at this point, setting

\[
b_\ell = \sum_{s \in \mathfrak{s}_q(0)} a_{b_\ell^n(\nu \ast \hat{q}_s)} \in \left( \mathcal{H}(a) \right)(q(0))
\]

3.3.3.

we have (cf. 3.3.2)

\[
\xi_0(b_\ell) = \sum_{s \in \mathfrak{s}_q(0)} (a \circ n)(s, \xi_0) \left( a_{b_\ell^n(\nu \ast \hat{q}_s)} \right)
\]

3.3.4.

Since this is true for any \( \xi_0 \in I(q(0)) \), \( b_\ell \) belongs to \( \left( \mathcal{H}(a) \right)(q(0)) \).

3.4. Similarly, for any \( \mathcal{B} \)-chain \( q' \in \mathfrak{gct}(\Delta_n, \mathcal{B}) \) \( \mathcal{G} \)-isomorphic to \( q \) and any natural \( \mathcal{G} \)-isomorphism \( \chi : q \cong q' \), it follows from 2.4 above that for any \( s \in \mathfrak{s}_q(0) \) we get a natural \( \mathcal{G} \)-isomorphism

\[
\hat{\chi}_s : \hat{q}_s \cong \hat{q}_{\chi_0(s)}
\]

3.4.1.

then, from 2.5 above, we still get the \( \mathcal{G} \)-isomorphism

\[
b_\ell^n(\nu \circ \hat{\chi}_s) : b_\ell^n(\nu \ast \hat{q}_s) \cong b_\ell^n(\nu \ast \hat{q}_{\chi_0(s)})
\]

3.4.2.

and therefore the \( \mathcal{G} \)-stability of \( a \) forces

\[
a_{b_\ell^n(\nu \ast \hat{q}_s)} = \left( (a \circ n)(a_{\chi_0}(s), \chi_0) \right) \left( a_{b_\ell^n(\nu \ast \hat{q}_{\chi_0(s)})} \right)
\]

3.4.3.
At this point, setting
\[ b'_\ell = \sum_{s' \in \Sigma_{q'}(0)} \alpha_{b'_{\ell}}(\nu \ast \tilde{t}_{s'}) \in (H(a))(q'(0)) \]  
for any \( \ell \in \Delta_{n+1} \), we have (cf. 3.4.3)
\[ b'_\ell = \sum_{s \in \Sigma_q(0)} ((a \circ n)(s\chi_0(s), \chi_0))^{-1}(a_{b'_{\ell}}(\nu \ast \tilde{t}_s)) = (H(a))(\chi_0)^{-1}(b_\ell) \]  
3.4.5.

Consequently, we obtain
\[ h^n(a)_{q'} = \sum_{\ell=0}^{n+1} (-1)^{\ell} \theta_{q'(0)}(b'_\ell) = \sum_{\ell=0}^{n+1} (-1)^{\ell} \left( \theta_{q'(0)} \circ (H(a))(\chi_0)^{-1} \right)(b_\ell) \]  
\[ = \sum_{\ell=0}^{n+1} (-1)^{\ell} (a(\tilde{\chi}_0)^{-1} \circ \theta_{q(0)}) (b_\ell) = a(\tilde{\chi}_0)^{-1}(h^n(a)_{q'}) \]  
3.4.6,
so that \( h^n(a) \) is \( \mathcal{G} \)-stable and can be identified to an element of \( \mathbb{C}_\mathcal{G}^n(\overline{B}, a) \).

**Theorem 3.5.** With the notation above, let \( \mathcal{H} = (\mathcal{I}, \mathcal{I}^0, \mathcal{G}, s, n, \nu) \) be a homotopic system for \( \mathcal{B} \) and \( a : \overline{B} \to \mathfrak{Ab} \) a contravariant \( \mathcal{H} \)-split functor. Then for any \( n \in \mathbb{N} \) we have
\[ d_n \circ h^n + h^{n+1} \circ d_{n+1} = \text{id}_{\mathbb{C}_\mathcal{G}^{n+1}(\overline{B}, a)} \]  
3.5.1.

In particular, \( \mathbb{H}_n^\mathcal{G}(\overline{B}, a) = \{0\} \) for any \( n \geq 1 \).

**Proof:** For any \( n \in \mathbb{N} \), any \( a = (a_T)_{T \in \mathcal{I} \in \mathcal{G} \cap (\Delta_{n+1}, B)} \) in \( \mathbb{C}^{n+1}_\mathcal{G}(\overline{B}, a) \) and any \( \tilde{r} \in \mathfrak{G} \cap (\Delta_{n+1}, \overline{B}) \), it suffices to prove that we have
\[ d_n(h^n(a))_{\tilde{r}} + h^{n+1}(d_{n+1}(a))_{\tilde{r}} = a_{\tilde{r}} \]  
3.5.2.

Up to the identification in 3.1.4 above and according to the definition of the differential map, for any \( \mathcal{B} \)-chain \( \tau : \Delta_{n+1} \to \mathcal{B} \) we have
\[ d_n(h^n(a))_{\tau} = (a(\tilde{\tau}(0 \bullet 1)))(h^n(a)_{\tau \circ \delta^n_i}) + \sum_{i=1}^{n+1} (-1)^i h^n(a)_{\tau \circ \delta^n_i} \]  
3.5.3;
similarly, according to the very definition of \( h^{n+1} \), we have
\[ h^{n+1}(d_{n+1}(a))_{\tau} = \sum_{\ell=0}^{n+2} (-1)^{\ell} \theta_{\tau(0)}(\sum_{s \in \Sigma_{q(0)}} d_{n+1}(a)_{b'_{\ell}}(\nu \ast \tilde{t}_s)) \]  
3.5.4.
But, for any \( s \in \mathcal{A}_r(0) \) and any \( \ell \in \Delta_{n+2} \) we know that
\[
d_{n+1}(a)_{h_{n+1}^\nu} = \left( a(h_{n+1}^\nu((\nu \circ \hat{t}_s)(0 \cdot 1))) \right)_{(\nu \circ \hat{t}_s \circ \delta_{i+1}^n)} + \sum_{i=1}^{n+2} (-1)^i a_{h_{n+1}^\nu} \circ \delta_{i+1}^n
\]
3.55;
thus, since \( h_{n+1}^\nu((\nu \circ \hat{t}_s)(0 \cdot 1)) = \nu_{(s,r(0))} \), from [2, Lemma A4.2] it follows that for \( \ell = 0 \) we get
\[
d_{n+1}(a)_{h_{n+1}^\nu} = (a(\nu_{(s,r(0)))})(a \nu_i) + \sum_{i=1}^{n+2} (-1)^i a_{h_{n+1}^\nu} \circ \delta_{i+1}^n
\]
3.56.
Moreover, for any \( 1 \leq \ell \leq n+2 \) we have \( h_{n+1}^\nu((\nu \circ \hat{t}_s)(0 \cdot 1)) = (n \circ \hat{t}_s)(0 \cdot 1) \); but, according to 2.3 above, the \( \hat{B} \)-morphism
\[\hat{t}_s(0 \cdot 1) : (s, r(0)) \rightarrow (\mathcal{A}_r(0 \cdot 1), r(1))\]
is denoted by the pair \((s, r(0 \cdot 1))\), and note that for any \( i \in \Delta_{n+1} \) we have
\[
h_{n+1}^\nu((\nu \circ \hat{t}_s) \circ \delta_{i+1}^n) = (n \circ \hat{t}_s) \circ \delta_{i+1}^n = (n \circ \hat{t}_s \circ \delta_{i+1}^n)
\]
3.58;
hence, from [2, Lemma A4.2] again and this equality, for any \( 1 \leq \ell \leq n+2 \) we obtain
\[
d_{n+1}(a)_{h_{n+1}^\nu} = ((a \circ n)(s, r(0 \cdot 1))))(a_{h_{n+1}^\nu} \circ (\nu \circ \hat{t}_s) \circ \delta_{i+1}^n)
\]
3.59.
Similarly, it follows from [2, Lemma A4.2] that for any \( 1 \leq \ell \leq n+2 \) we still get
\[
\sum_{i=1}^{n+2} (-1)^i a_{h_{n+1}^\nu} \circ \delta_{i+1}^n = \sum_{i=1}^{\ell-1} (-1)^i a_{h_{n+1}^\nu} \circ \delta_{i+1}^n
\]
3.59.10,
and from equality 3.5.8 for \( \ell = n + 2 \) we still obtain

\[
\sum_{i=1}^{n+2} (-1)^i a_{h_{\ell+1}^{n+1}(\nu \ast \tau_\gamma)} \circ \delta_r^n = \sum_{i=1}^{n+1} (-1)^i a_{h_{\ell+1}^{n+1}(\nu \ast \tau_\gamma)} + (-1)^n a_{h_{\ell+2}^{n+1}(\nu \ast \tau_\gamma)} \circ \delta_r^{n+1}
\]

3.5.11.

At this point, for any \( s \in s_{\tau(0)} \) and any \( 1 \leq \ell \leq n + 2 \), let us set

\[
\alpha_{s,0} = (a(\nu(\tau(s,0))))(a_{\bar{\tau}})
\]

\[
\alpha_{s,\ell} = (a \circ n)(s, \tau(0 \cdot 1))(a_{h_{\ell-1}^{n}(\nu \ast \tau_\gamma)} \circ \delta_r^n)
\]

3.5.12;

moreover, for any \( 0 \leq \ell \leq n + 1 \) we still set [2, Lemma A4.2]

\[
\beta_{s,\ell} = (-1)^\ell a_{h_{\ell+1}^{n+1}(\nu \ast \tau_\gamma)} \circ \delta_r^{n+1} = (-1)^\ell a_{h_{\ell+1}^{n+1}(\nu \ast \tau_\gamma)} \circ \delta_r^{n+1}
\]

\[
0 = \beta_{s,-1} = \beta_{s,n+2} = \gamma_{s,-1} = \gamma_{s,n+2}
\]

3.5.13.

\[
\gamma_{s,\ell} = \sum_{i=1}^{n+1} (-1)^i a_{h_{\ell}^{n}(\nu \ast \tau_\gamma)} \circ \delta_r^n \quad \text{and} \quad \gamma'_{s,\ell} = \sum_{i=\ell+1}^{n+1} (-1)^i a_{h_{\ell}^{n}(\nu \ast \tau_\gamma)} \circ \delta_r^n
\]

With all this notation, from equalities 3.5.6 and 3.5.9 for any \( s \in s_{\tau(0)} \) we get

\[
\sum_{\ell=0}^{n+2} (-1)^\ell d_{n+1}(a)_{h_{\ell}^{n+1}(\nu \ast \tau_\gamma)} = \sum_{\ell=0}^{n+1} (-1)^\ell (\alpha_{s,\ell} + \beta_{s,\ell-1} - \beta_{s,\ell} + \gamma_{s,\ell-1} - \gamma'_{s,\ell})
\]

3.5.14.

\[
= \sum_{\ell=0}^{n+2} (-1)^\ell \alpha_{s,\ell} - \sum_{\ell=0}^{n+1} \sum_{i=1}^{n+1} (-1)^{\ell+i} a_{h_{\ell}^{n}(\nu \ast \tau_\gamma)} \circ \delta_r^n
\]

More precisely, for the first term of the bottom sum, since \( a \) is \( \mathcal{H} \)-split, we have (cf. 3.5.12)

\[
\theta_{t(0)} \left( \sum_{s \in \Sigma_{t(0)}} \alpha_{s,0} \right) = \theta_{t(0)} \left( \sum_{s \in \Sigma_{t(0)}} (a(\nu(\tau(s,0))))(a_{\bar{\tau}}) \right) = a_{\bar{\tau}}
\]

3.5.15.

In conclusion, for any \( \ell \in \Delta_{n+1} \) and any \( i \in \Delta_n \) setting

\[
\alpha_{s,\ell} = \theta_{t(0)} \left( \sum_{s \in \Sigma_{t(0)}} \alpha_{s,\ell+1} \right)
\]

\[
\bar{\varepsilon}_{t,i} = \theta_{t(0)} \left( \sum_{s \in \Sigma_{t(0)}} a_{h_{\ell}^{n}(\nu \ast \tau_\gamma)} \circ \delta_r^{n+1} \right)
\]

3.5.16,
from 3.5.4, 3.5.14 and 3.5.15 we get

$$h^{n+1}(d_{n+1}(a))_\ell = a_\ell - \sum_{\ell=0}^{n+1} (-1)^\ell (a_\ell + \sum_{i=1}^{n+1} (-1)^i \varepsilon_{\ell,i-1})$$  \[3.5.17\]

On the other hand, setting

$$\xi_\ell = \sum_{s \in \mathcal{s}_r(n)} a_{h_n}^r(\nu s, \circ \delta^n_0)$$  \[3.5.18\]

from the *naturality* of $\theta$ for the first term in the right-hand member of equality 3.5.3 we get

$$\left( a(\bar{r}(0 \bullet 1)) \right) (h^n(a)_\circ \delta^n_0) = \left( a(\bar{r}(0 \bullet 1)) \right) \left( \sum_{\ell=0}^{n+1} (-1)^\ell \theta(\nu(1)) \xi_\ell \right)$$  \[3.5.19\]

$$= \sum_{\ell=0}^{n+1} (-1)^\ell \left( \theta(0) \circ (\mathcal{H}(a))(\nu(0 \bullet 1)) \right) \xi_\ell$$

but, by the very definition of $\mathcal{H}(a)$ we still get (cf. 2.6.2 and 3.5.12)

$$\left( \left( \mathcal{H}(a) \right)(\nu(0 \bullet 1)) \right) \left( \mathcal{s}_r(0) \right) = \sum_{s \in \mathcal{s}_r(0)} (a \circ (s, \nu(0 \bullet 1)) \left( a_{h_n}^r(\nu s, \circ \delta^n_0) \right)$$  \[3.5.20\]

hence, we obtain (cf. 3.5.16)

$$\left( a(\bar{r}(0 \bullet 1)) \right) (h^n(a)_\circ \delta^n_0) = \sum_{\ell=0}^{n+1} (-1)^\ell a_\ell$$  \[3.5.21\]

Moreover, if $i \neq 0$, for the $i$-th term in the right-hand member of equality 3.5.3 and for any $s \in \mathcal{s}_r(0)$, it follows from 2.4 that $(\bar{r} \circ \delta^n_s)_s = \bar{r}_s \circ \delta^n_s$ and therefore we get

$$h^n(a)_\circ \delta^n_s = \sum_{\ell=0}^{n+1} (-1)^\ell \theta(\nu(1)) \left( \sum_{s \in \mathcal{s}_r(n+1)} a_{h_n}^r(\nu s, \circ \delta^n_s) \right)$$  \[3.5.22\]

Finally, equality 3.5.2 follows from the suitable alternating sum of equalities 3.5.17, 3.5.19, and 3.5.22. We are done.
3.6. A serious advantage in getting a trivial homotopy of a differential complex — rather than the simple annulation of its cohomology groups — is that this triviality is inherited by the subcomplexes which are compatible with the homotopy maps. More generally, for our application in [3] we have to consider the following sophisticated framework that we apply in section 5.

3.7. With the notation above, let $\mathcal{H} = (\mathcal{I}, \mathcal{I}^o, \mathcal{G}, s, n, \nu)$ be a homotopic system for $\mathcal{B}$ where $\mathcal{I}$, $\mathcal{I}^o$ and $\mathcal{G}$ are trivial, and $a : \mathcal{B} \to \mathfrak{Ab}$ is a contravariant $\mathcal{H}$-split functor. Consider a subcategory $\mathcal{C}$ of the category of chains $\text{ch}(\mathcal{B})$ [2, A2.8] such that any $\text{ch}^\ast(\mathcal{B})$-morphism $q \to r$ from a $\mathcal{C}$-object $q$ is a $\mathcal{C}$-morphism, and denote by $\mathcal{C}_A$ the "intersection" of $\mathcal{C}$ with the subcategory $\text{ch}(\mathcal{A})$ of $\text{ch}(\mathcal{B})$, and by $s^\mathcal{C} : \mathcal{C} \to \mathcal{C}\mathcal{E}$ the restriction to $\mathcal{C}$ of the composition $s \circ v_B$ of $s$ with the evaluation functor $v_B : \text{ch}(\mathcal{B}) \to \mathcal{B}$ [2, A3.7.3]; actually, the subcategory $\mathcal{C}_A$ of $\text{ch}(\mathcal{A})$ inherits the property above. From 2.4 we get a functor from $(s \circ v_B) \times \text{ch}(\mathcal{B})$ to $\text{ch}(s \times \mathcal{B})$ and therefore $n : s \times \mathcal{B} \to \mathcal{A} \subset \mathcal{B}$ and $\nu : n \to p$ respectively induce a functor and a natural map

$$n^{\nu} : (s \circ v_B) \times \text{ch}(\mathcal{B}) \to \text{ch}(\mathcal{A}) \subset \text{ch}(\mathcal{B}) \quad \text{and} \quad \nu^{\nu} : n^{\nu} \to p^{\nu} \quad 3.7.1$$

where $p^{\nu} : (s \circ v_B) \times \text{ch}(\mathcal{B}) \to \text{ch}(\mathcal{B})$ denotes the structural functor.

3.8. Then assume that, by restriction, they induce a functor and a natural map

$$n^\mathcal{C} : s^\mathcal{C} \times \mathcal{C} \to \mathcal{C}_A \subset \mathcal{E} \quad \text{and} \quad \nu^\mathcal{C} : n^\mathcal{C} \to p^\mathcal{C} \quad 3.8.1$$

where $\mathcal{E}$ and $\mathcal{C}_A$ denote the respective "images" in $\text{ch}(\mathcal{B})$ of $\mathcal{C}$ and $\mathcal{C}_A$, and $p^\mathcal{C} : s^\mathcal{C} \times \mathcal{C} \to \mathcal{E}$ denotes the structural functor; thus, for any $\mathcal{C}$-object $q$ and any element $s$ in $s_{q(0)}$ we get $\mathcal{E}_A$- and $\mathcal{E}$-objects $n^\mathcal{C}(s, q)$ and $p^\mathcal{C}(s, q)$, which are actually functors from $\Delta_n$ to $\mathcal{B}$ for some $n \in \mathbb{N}$, and get a $\mathcal{C}$-morphism

$$\nu^\mathcal{C}_{(s,q)} : n^\mathcal{C}(s, q) \to p^\mathcal{C}(s, q) = \bar{q} \quad 3.8.2$$

which is actually a natural map between these functors; moreover, let us assume that for any $\ell \in \Delta_n$ the functor $b^\nu_{\ell}(\nu^\mathcal{C}_{(s,q)}) : \Delta_{n+1} \to \mathcal{B}$ is also a $\mathcal{C}$-object.

3.9. In this situation, denoting by $a^\mathcal{C} : \mathcal{E} \to \mathfrak{Ab}$ the restriction to $\mathcal{E}$ of the composition $a \circ v_B : \text{ch}(\mathcal{B}) \to \mathfrak{Ab}$, we can consider the new differential complex holding the $n$-term

$$\mathcal{C}^n(\mathcal{E}, a^\mathcal{C}) = \prod_{q} a(q(0)) \quad 3.9.1$$

where $q : \Delta_n \to \mathcal{B}$ runs over the set of functors which are $\mathcal{E}$-objects; indeed, for any $n \in \mathbb{N}$ the differential map defined in 3.1.3 induces a differential map

$$d^\mathcal{C}_n : \mathcal{C}^n(\mathcal{E}, a^\mathcal{C}) \to \mathcal{C}^{n+1}(\mathcal{E}, a^\mathcal{C}) \quad 3.9.2$$
since, for any functor \( \tau : \Delta_{n+1} \to \mathcal{B} \) which is a \( \mathcal{C} \)-object, the functors \( \tau \circ \delta^n \) are also \( \mathcal{C} \)-objects for any \( i \in \Delta_{n+1} \); let us denote by \( \mathbb{H}^n(\mathcal{C}, \mathcal{A}^\xi) \) the corresponding cohomology group. Moreover, it is clear that the restriction to \( \mathcal{C} \) induces a surjective group homomorphism

\[
 r^n_\mathcal{C} : \mathbb{C}^n(\mathcal{B}, \mathcal{A}) \to \mathbb{C}^n(\mathcal{C}, \mathcal{A}^\xi)
\]

for any \( n \in \mathbb{N} \), and that these group homomorphisms are compatible with both differential maps \( d_n \) and \( d_{n+1}^\mathcal{C} \); the point is that the cohomology of the new differential complex only depends on the restriction to \( \mathcal{C}_A \); precisely, denoting by \( i : \mathcal{A} \to \mathcal{B} \) and \( i^\xi : \mathcal{C}_A \to \mathcal{C} \) the obvious inclusion functors, and by

\[
 r^n_A : \mathbb{C}^n(\mathcal{B}, \mathcal{A}) \to \mathbb{C}^n(\mathcal{A}, \mathcal{A}^\xi) \quad \text{and} \quad r^n_{A, \mathcal{C}} : \mathbb{C}^n(\mathcal{B}, \mathcal{A}) \to \mathbb{C}^n(\mathcal{C}, \mathcal{A}^\xi \circ i^\xi)
\]

the group homomorphisms induced by restriction for any \( n \in \mathbb{N} \), we have the following consequence of Theorem 3.5.

**Corollary 3.10.** With the notation and the hypothesis above, the restriction induces an injective group homomorphism from \( \mathbb{H}^n(\mathcal{C}, \mathcal{A}^\xi) \) to \( \mathbb{H}^n(\mathcal{C}_A, \mathcal{A}^\xi \circ i^\xi) \) for any \( n \geq 1 \).

**Proof:** Let \( a = (a_q)_{q \in \mathcal{B} \cap (\Delta_n \times \mathcal{B})} \) be an element of \( \mathbb{C}^n(\mathcal{B}, \mathcal{A}) \) such that \( r^n_\mathcal{C}(a) \) is a \( n \)-cocycle in \( \mathbb{C}^n(\mathcal{C}, \mathcal{A}^\xi) \); then, it is clear that \( r^n_A(a) \) is also a \( n \)-cocycle in \( \mathbb{C}^n(\mathcal{C}_A, \mathcal{A}^\xi \circ i^\xi) \); thus, it suffices to prove that if \( r^n_A(a) \) has a trivial image in \( \mathbb{H}^n(\mathcal{C}_A, \mathcal{A}^\xi \circ i^\xi) \) then \( r^n_\mathcal{C}(a) \) also has a trivial image in \( \mathbb{H}^n(\mathcal{C}, \mathcal{A}^\xi) \).

Assume that \( r^n_A(a) \) has a trivial image in \( \mathbb{H}^n(\mathcal{C}_A, \mathcal{A}^\xi \circ i^\xi) \); then, we may choose \( b \) in \( \mathbb{C}^{n-1}(\mathcal{C}, \mathcal{A}^\xi) \) such that we have \( r^n_A(a)_q = d_{n-1}^\mathcal{C}(b)_q \) for any \( \mathcal{A}_A \)-object \( q : \Delta_n \to \mathcal{A} \subset \mathcal{B} \); moreover, we may choose \( c \) in \( \mathbb{C}^n(\mathcal{B}, \mathcal{A}) \) lifting \( d_{n-1}^\mathcal{C}(b) \) and fulfilling \( c_\mathcal{A} = a_q \) for any \( \mathcal{A} \)-chain \( q : \Delta_n \to \mathcal{A} \subset \mathcal{B} \), so that we have \( r^n_A(a - c) = 0 \); set \( e = a - c \).

In any case, it follows from 3.5.1 that

\[
a = d_{n-1}(h^{n-1}(a)) + h^n(d_n(a)) \quad \text{3.10.1}
\]

and note that \( r^n_\mathcal{C}(d_{n-1}(h^{n-1}(a))) = d_{n-1}^\mathcal{C}(r^n_\mathcal{C}(h^{n-1}(a))) \); since

\[
r^{n+1}_\mathcal{C}(d_n(a)) = d_{n-1}^\mathcal{C}(r^n_\mathcal{C}(a)) = 0
\]

we also have

\[
r^{n+1}_\mathcal{C}(d_n(e)) = 0
\]

thus, it follows from Lemma 3.11 below that \( h^n(d_n(e)) \) belongs to \( \text{Ker}(r^n_\mathcal{C}) \) and, in particular, it has a trivial image in \( \mathbb{H}^n(\mathcal{C}, \mathcal{A}^\xi) \); finally, it suffices to prove that the \( n \)-cocycle \( r^n_\mathcal{C}(h^n(d_n(e))) \) has a trivial image in \( \mathbb{H}^n(\mathcal{C}, \mathcal{A}^\xi) \);
but, according to 3.5.1, we have
\[ r^\ell_n(h^n(d_n(c))) = r^\ell_n\left(c - d_{n-1}(h^{n-1}(c))\right) = d^\ell_{n-1}(b) - d^\ell_{n-1}\left(r^\ell_{n-1}(h^{n-1}(c))\right) \]

3.10.4.

We are done.

**Lemma 3.11.** With the notation and the hypothesis above, we have
\[ h^n(Ker(r_A^{n+1}) \cap Ker(r_C^{n+1})) \subset Ker(r_A^n) \]

Moreover, for any stable element \( a \) in Ker\((r_A^{n+1})\), \( h^n(a) \) belongs to Ker\((r_A^n)\).

**Proof:** If \( a = (a_\ell)_{\ell \in S} \in Ker(r_A^{n+1}) \) belongs to Ker\((r_A^{n+1}) \cap Ker(r_C^{n+1})\) then, for any \( C \)-object \( q : \Delta_n \rightarrow B \), any \( s \in s_q(0) \) and any \( 0 \leq \ell \leq n + 1 \), in 3.2.2 either \( h^n_{\ell}(\nu * q_s) \) is also a \( C \)-object or we have \( \ell = n + 1 \) and, according to 2.4.4, \( h^n_{n+1}(\nu * q_s) \) is an \( \tilde{A} \)-chain; in both cases we have \( a_{h^n_{\ell}(\nu * q_s)} = 0 \) and therefore \( h^n(a)_q \) is also zero.

Moreover, for any stable element \( a = (a_\ell)_{\ell \in S} \in Ker(r_A^{n+1}) \), any \( \tilde{A} \)-chain \( q : \Delta_n \rightarrow \tilde{A} \), any \( s \in s_q(0) \) and any \( i \in \Delta_n \), it follows from condition 2.1.1 that the \( B \)-morphism (cf. 2.4.1)
\[ (\nu * q_s)_i : (n \circ q_s)(i) \rightarrow \tilde{q}(i) \]

3.11.2

can be factorized
\[ (n \circ q_s)(i) \xrightarrow{\mu_i} \tilde{q}(i) \xrightarrow{\iota_i} \tilde{q}(i) \]

3.11.3

where \( \iota_i \) is an \( \tilde{A} \)-morphism and \( \mu_i \) a \( B \)-isomorphism; then, setting
\[ \tilde{q}'(j * i) = \mu_i \circ (n \circ q_s)(j * i) \circ \mu_j^{-1} \]

3.11.4

for any \( 0 \leq j \leq i \), we get an obvious \( \tilde{B} \)-chain \( \tilde{q}'_s : \Delta_n \rightarrow \tilde{B} \), a natural isomorphism \( \mu : n \circ q_s \cong \tilde{q}' \) and a natural map \( \iota : \tilde{q}' \rightarrow \tilde{q} \).

In particular, we have the commutative diagram
\[ \begin{array}{ccc}
\tilde{q}'(i) & \xrightarrow{\iota_i} & \tilde{q}(i) \\
\tilde{q}'(j * i) & \xrightarrow{\iota_{j * i}} & \tilde{q}(j * i) \\
\tilde{q}'(j) & \xrightarrow{\iota_j} & \tilde{q}(j) \\
\end{array} \]

3.11.5

where all the maps \( \tilde{q}'(j * i) \) are \( \tilde{A} \)-morphisms; then, it easily follows from conditions 2.1.1 and 2.1.2 that \( \tilde{q}'(j * i) \) is also an \( \tilde{A} \)-morphism. At this point, it is easily checked that, for any \( \ell \in \Delta_n \), \( h^n_\ell(\nu * q_s) \) is isomorphic to \( h^n_\ell(\nu) \) and therefore the stability of \( a \) implies that
\[ a_{h^n_\ell(\nu * q_s)} = (a(\mu_0))(a_{h^n_\ell(\nu)}) \]

3.11.7

but, \( h^n_\ell(\nu) \) is clearly an \( \tilde{A} \)-chain, so that we have \( a_{h^n_\ell(\nu)} = 0 \); moreover, we already know that \( h^n_{n+1}(\nu * q_s) \) is an \( \tilde{A} \)-chain; hence, we get \( h^n(a)_q = 0 \).
4. The direct product case

4.1. Let \( \mathcal{A} \) be a category with a final object \( P \), and \( \mathcal{B} \) an \( \mathcal{A} \)-category (cf. 2.1); recall that the additive cover \( \text{ac}(\mathcal{B}) \) of \( \mathcal{B} \) — introduced in [1] — is the category where the objects are the finite sequences \( \bigoplus_{i \in I} Q_i \) of \( \mathcal{B} \)-objects \( Q_i \) and where the morphisms
\[
(f, \varphi) : \bigoplus_{j \in J} R_j \longrightarrow \bigoplus_{i \in I} Q_i
\]
are the pairs formed by a map \( f : J \rightarrow I \) and by a family \( \varphi = \{\varphi_j\}_{j \in J} \) of \( \mathcal{B} \)-morphisms
\[
\varphi_j : R_j \longrightarrow Q_{f(j)}
\]
where \( j \) runs over \( J \) [2, A2.7.3], the composition being induced by the composition of maps and the composition in \( \mathcal{B} \). This category admits an obvious direct sum and we denote by
\[
j : \mathcal{B} \longrightarrow \text{ac}(\mathcal{B})
\]
The canonical functor mapping any \( \mathcal{B} \)-object \( Q \) on the \( \text{ac}(\mathcal{B}) \)-object \( \bigoplus_{\{\emptyset\}} Q \) that we still denote by \( Q \).

4.2. Let us say that \( \mathcal{B} \) is a multiplicative \( \mathcal{A} \)-category whenever the additive cover \( \text{ac}(\mathcal{B}) \) is endowed with a direct product which is distributive with respect to the direct sum. Moreover, let us say that \( \mathcal{B} \) is ordered if it fulfills the following condition [2, A5.1.1]
\[
4.2.1 \quad \text{Any pair of } \mathcal{B} \text{-objects } Q \text{ and } R \text{ which admit } \mathcal{B} \text{-morphisms } R \rightarrow Q \text{ and } Q \rightarrow R \text{ are } \mathcal{B} \text{-isomorphic and then all the } \mathcal{B} \text{-morphisms } R \rightarrow Q \text{ are } \mathcal{B} \text{-isomorphims.}
\]
As we mention in the Introduction, the case we are interested in is the following. Let \( P \) be a finite \( p \)-group and \( \mathcal{F} \) a Frobenius \( P \)-category, and denote by \( \mathcal{F}^e \) the full subcategory of \( \mathcal{F} \) over the set of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \) [2, 4.8]. We show in [2, Proposition 6.14] that \( \text{ac}(\mathcal{F}^e) \) admits a direct product; consequently, denoting by \( \mathcal{F}_P \) the Frobenius category of the group \( P \) [2, 1.8] and by \( \mathcal{F}_P^e \) the full subcategory of \( \mathcal{F}_P \) over the same set of \( \mathcal{F} \)-selfcentralizing subgroups, \( P \) is a final \( \mathcal{F}_P^e \)-object and \( \mathcal{F}_P^e \) is clearly a \( \mathcal{F}_P \)-category which is ordered and multiplicative; thus, the main result of this section can be applied to \( \mathcal{F}_P^e \).

**Lemma 4.3.** In an ordered multiplicative \( \mathcal{A} \)-category \( \mathcal{B} \) any morphism is an epimorphism.

**Proof:** For any \( \mathcal{B} \)-object \( Q \) and any \( \mathcal{B} \)-automorphism \( \sigma \in \mathcal{B}(Q) \) we have a unique \( \text{ac}(\mathcal{B}) \)-morphism
\[
\delta_{\sigma} : Q \longrightarrow Q \times Q
\]
which composed with the structural ac(\mathcal{B})-morphisms of \( Q \times Q \) yields \( \text{id}_Q \) and \( \sigma \); since \( \mathcal{B} \) is an ordered category, up to suitable identifications, we have

\[ Q \times Q = \bigoplus_{\sigma \in \mathcal{B}(Q)} Q \oplus W \quad 4.3.2 \]

where \( W \) is an ac(\mathcal{B})-object such that there is no \( \mathcal{B} \)-morphism from \( Q \) to \( W \), and where the structural ac(\mathcal{B})-morphisms

\[ Q \leftarrow \bigoplus_{\sigma \in \mathcal{B}(Q)} Q \oplus W \rightarrow Q \quad 4.3.3 \]

respectively induce \( \text{id}_Q \) and \( \sigma \) over the term indexed by \( \sigma \in \mathcal{B}(Q) \).

Let \( \psi : T \rightarrow R \), \( \varphi : R \rightarrow Q \) and \( \varphi' : R \rightarrow Q \) be \( \mathcal{B} \)-morphisms such that

\[ \varphi' \circ \psi = \varphi \circ \psi \quad 4.3.4 \]

then, there is a unique ac(\mathcal{B})-morphism \( \theta : R \rightarrow Q \times Q \) which composed with the structural ac(\mathcal{B})-morphisms of \( Q \times Q \) yields

\[ \varphi : R \rightarrow Q \quad \text{and} \quad \varphi' : R \rightarrow Q \quad 4.3.5 \]

moreover, the composition \( \theta \circ \psi \) has to coincide with the unique ac(\mathcal{B})-morphism

\[ T \rightarrow Q \times Q = \bigoplus_{\sigma \in \mathcal{B}(Q)} Q \oplus W \quad 4.3.6 \]

which composed with the structural ac(\mathcal{B})-morphisms of \( Q \times Q \) yields \( \varphi \circ \psi \) and \( \varphi' \circ \psi \); but, since \( \varphi \circ \psi = \varphi' \circ \psi \), this ac(\mathcal{B})-morphism necessarily coincides with the ac(\mathcal{B})-morphism formed by the map sending to \( \text{id}_Q \), in the set of indices associated with \( \bigoplus_{\sigma \in \mathcal{B}(Q)} Q \oplus W \), the unique element \( \emptyset \) in the set associated with \( T \), and by the \( \mathcal{B} \)-morphism \( \varphi \circ \psi : T \rightarrow Q \) (cf. 4.1); consequently, the ac(\mathcal{B})-morphism \( \theta \) is also formed by the map sending \( \emptyset \) to \( \text{id}_Q \) and by a \( \mathcal{B} \)-morphism \( \varphi'' : R \rightarrow Q \) which composed with the identity map over \( Q \) yields \( \varphi \) and \( \varphi' \) which forces

\[ \varphi = \varphi'' = \varphi' \quad 4.3.7 \]

4.4. As a matter of fact, for any triple of \( \mathcal{B} \)-objects \( Q \), \( R \) and \( T \) in an ordered \( \mathcal{A} \)-category \( \mathcal{B} \) where any morphism is an epimorphism, any \( \mathcal{B} \)-morphism \( \alpha : Q \rightarrow R \) induces an injective map from \( \mathcal{B}(T, R) \) to \( \mathcal{B}(T, Q) \) and, as in [2, 6.4], we can consider the elements of \( \mathcal{B}(T, Q) \) which, even partially, cannot be extended \( \text{via} \ \alpha \); precisely, we set

\[ \mathcal{B}(T, Q)_\alpha = \mathcal{B}(T, Q) - \bigcup_{\theta'} \mathcal{B}(T, Q') \circ \theta' \quad 4.4.1 \]

where \( \theta' \) runs over the set of \( \mathcal{B} \)-nonisomorphisms \( \theta' : Q \rightarrow Q' \) from \( Q \) fulfilling \( \alpha' \circ \theta' = \alpha \) for some \( \alpha' \in \mathcal{B}(R, Q') \) which is then unique, and we simply say that \( \theta' \) \textit{divides} \( \alpha \) setting \( \alpha' = \alpha / \theta' \). Note that, a \( \mathcal{B} \)-morphism \( \beta : Q \rightarrow T \) belongs to \( \mathcal{B}(T, Q)_\alpha \) if and only if \( \alpha \) belongs to \( \mathcal{B}(R, Q)_\beta \) and, as in [2, 6.9],
let us call strict such a triple \((\alpha, Q, \beta)\) in \(B\); moreover, we say that two strict triples \((\alpha, Q, \beta)\) and \((\alpha', Q', \beta')\) in \(B\) are equivalent if there exists a \(B\)-isomorphism \(\theta : Q \cong Q'\) fulfilling

\[ \alpha' \circ \theta = \alpha \quad \text{and} \quad \beta' \circ \theta = \beta \]

4.4.2;

let us denote by \(\Sigma_{R,T}\) a set of representatives for the set of equivalence classes of strict triples to \(R\) and \(T\) in \(B\). Then the analogous of equality [2, 6.7.1] still holds in this general setting and, in some sense, it characterizes the multiplicity of \(B\) and the direct product in \(\text{ac}(B)\).

**Proposition 4.5.** Let \(B\) be an ordered \(A\)-category. Then, \(B\) is multiplicative if and only if any \(B\)-morphism is an epimorphism and, for any triple of \(B\)-objects \(Q\), \(R\) and \(T\), and any \(\alpha \in B(R, Q)\), we have

\[ B(T, Q) = \bigsqcup_{\theta'} B(T, Q')_{\alpha/\theta' \circ \theta'} \]

4.5.1

where \(\theta' : Q \to Q'\) runs over a set of representatives for the \(B\)-isomorphism classes of \(B\)-morphisms starting on \(Q\) and dividing \(\alpha\). In this case we have a unique \(\text{ac}(B)\)-isomorphism

\[ \bigoplus_{(\alpha', Q', \beta') \in \Sigma_{R,T}} Q' \cong R \times T \]

4.5.2

which composed with the structural \(\text{ac}(B)\)-morphisms from \(R \times T\) to \(R\) and \(T\) respectively yields \(\alpha'\) and \(\beta'\).

**Proof:** Assume first that \(B\) is multiplicative; then, it follows from Lemma 4.3 that any \(B\)-morphism is an epimorphism. On the other hand, once we fix \(\alpha\) in \(B(R, Q)\), we have a bijection between \(B(T, Q)\) and the set of \(\text{ac}(B)\)-morphisms from \(Q\) to the direct product

\[ R \times T = \bigoplus_{i \in I} Q_i \]

4.5.3

which composed with the structural \(\text{ac}(B)\)-morphism \(R \times T \to R\) coincide with \(\alpha\); moreover, the \(\text{ac}(B)\)-morphism corresponding to \(\beta \in B(T, Q)\) is given by some \(i \in I\) and by a suitable \(B\)-morphism \(\theta_i : Q \to Q_i\) dividing \(\alpha\) and \(\beta\).

In particular, if \(\beta\) belongs to \(B(T, Q)\), then \(\theta_i\) has to be a \(B\)-isomorphism. In any case, we have \(\alpha = \alpha_i \circ \theta_i\) and \(\beta = \beta_i \circ \theta_i\) for suitable \(\alpha_i \in B(R, Q_i)\) and \(\beta_i \in B(T, Q_i)\), and it is quite clear that \(\beta_i\) necessarily belongs to \(B(T, Q_i)\alpha_i\); now, equality 4.5.1 follows easily. Moreover, this argument applied to any \((\alpha', Q', \beta') \in \Sigma_{R,T}\) determines a unique \(i \in I\) together with a \(B\)-isomorphism \(Q' \cong Q_i\), which proves isomorphism 4.5.2.
Conversely, assume that any \( B \)-morphism is an epimorphism and that equalities 4.5.1 hold; it is easily checked that in order to define a *distributive direct product* in \( \text{ac}(B) \), it suffices to define the *direct product* of any pair of \( B \)-objects \( R \) and \( T \); thus, choosing a set of representatives \( \mathcal{X}_{R,T} \) for the set of equivalence classes of strict triples to \( R \) and \( T \) in \( B \), we simply set

\[
R \times T = \bigoplus_{(\alpha',Q',\beta') \in \mathcal{X}_{R,T}} Q' \quad 4.5.4.
\]

Indeed, for any \( B \)-object \( Q \) and any pair of \( B \)-morphisms

\[
\alpha : Q \rightarrow R \quad \text{and} \quad \beta : Q \rightarrow T \quad 4.5.5,
\]

we have the corresponding equality 4.5.1 determined by \( Q \), \( R \), \( T \) and \( \alpha \); then, the \( B \)-morphism \( \beta \) determines a unique term \( \theta'' : Q \rightarrow Q'' \) in the disjoint union, together with a unique \( \beta'' \in B(T,Q)_{\alpha/\theta''} \); that is to say, we get an \( \text{ac}(B) \)-morphism

\[
Q \rightarrow \bigoplus_{(\alpha',Q',\beta') \in \mathcal{X}_{R,T}} Q' \quad 4.5.6
\]

which composed with the structural homomorphisms yields \( \alpha \) and \( \beta \), and the uniqueness is clear. We are done.

4.6. From now on, let \( B \) be an *ordered multiplicative\( A \)-category and \( G \) a subcategory of \( B \) with the same objects and only with \( G \)-isomorphisms. We have the functor

\[
m_P : B \rightarrow \text{ac}(B) \quad 4.6.1
\]

sending any \( B \)-object \( Q \) to the direct product \( Q \times P \) and any \( B \)-morphism \( \varphi : R \rightarrow Q \) to the unique \( \text{ac}(B) \)-morphism

\[
\varphi \times \text{id}_P : R \times P \rightarrow Q \times P \quad 4.6.2
\]

determined by the pair of \( B \)-morphisms

\[
R \times P \rightarrow R \xrightarrow{\varphi} Q \quad \text{and} \quad R \times P \rightarrow P \xrightarrow{\text{id}_{P}} P \quad 4.6.3;
\]

it is clear that any *contravariant* functor \( a : B \rightarrow \text{Ab} \) determines an *additive contravariant* functor

\[
a^{\text{ac}} : \text{ac}(B) \rightarrow \text{Ab} \quad 4.6.4
\]

and, in this section, we will prove that, for suitable choices of \( G \), the *differential complex* with \( n \)-term \( C^n_0(B,a^{\text{ac}} \circ m_P) \) (cf. 3.1) and with the usual *differential map* is homotopically trivial. Note that we have a *natural map*

\[
\omega_P : m_P \rightarrow j \quad 4.6.5
\]

sending any \( B \)-object \( Q \) to the structural \( \text{ac}(B) \)-morphism \( Q \times P \rightarrow Q \).
4.7. Explicitly, for any $\mathcal{B}$-object $Q$, setting $I_Q = \Xi_{Q,p}$ we certainly may assume that

$$Q \times P = \bigoplus_{(\alpha', Q') \in I_Q} Q'$$  \hspace{1cm} \text{4.7.1;}

moreover, it follows from condition 2.1.1 that, up to $\alpha(\mathcal{B})$-isomorphisms, we still may assume that $\nu' = \nu_Q'$ and that the strict triple $(\id_Q, \zeta_Q, \nu_Q)$ belongs to $I_Q$. On the other hand, for any $\mathcal{B}$-morphism $\varphi: R \rightarrow Q$, the $\alpha(\mathcal{B})$-morphism $\varphi \times \id_P$ determines both a map $I_\varphi: I_R \rightarrow I_Q$ and, setting $I_\varphi(\beta', R', \iota_R) = (\alpha', Q', \nu_Q)$ for a triple $(\beta', R', \iota_R)$ in $I_R$, a $\mathcal{B}$-morphism $\varphi_{(\beta', R', \iota_R)}: R' \rightarrow Q'$ fulfilling (cf. 4.1)

$$\alpha' \circ (\beta', \nu_{R'}) = \varphi \circ \beta' \quad \text{and} \quad \nu_{Q'} \circ \varphi_{(\beta', R', \iota_R)} = \iota_{R'}$$  \hspace{1cm} \text{4.7.2;}

actually, according to conditions 2.1.1 and 2.1.2, the right-hand equality is equivalent to say that $\varphi_{(\beta', R', \iota_R)}$ is an $\mathcal{A}$-morphism and, in this case, this $\mathcal{A}$-morphism is uniquely determined by the left-hand equality; moreover, we assume that if $\varphi$ is a $\mathcal{G}$-isomorphism then $\varphi_{(\beta', R', \iota_R)}$ is also a $\mathcal{G}$-isomorphism.

4.8. At this point, we have a representation $I$ of $\mathcal{B}$ (cf. 2.2) mapping any $\mathcal{B}$-object $Q$ on the set $I_Q$ considered as a trivial category, and any $\mathcal{B}$-morphism $\varphi: R \rightarrow Q$ on the map $I_\varphi: I_R \rightarrow I_Q$ considered as a trivial functor; let us denote by $\widehat{\mathcal{B}} = I \times \mathcal{B}$ the corresponding semidirect product (cf. 2.2). That is to say, according to 2.2 above, we can identify the $\widehat{\mathcal{B}}$-objects to the triples $(Q', \alpha', Q)$ where $Q$ is a $\mathcal{B}$-object and $(\alpha', Q', \nu_{Q'})$ belongs to $I_Q$; similarly, if $(R', \beta', R)$ is another $\widehat{\mathcal{B}}$-object, we can identify the $\widehat{\mathcal{B}}$-morphisms from $(R', \beta', R)$ to $(Q', \alpha', Q)$ to the pairs

$$(\varphi', \varphi): (R', \beta', R) \rightarrow (Q', \alpha', Q)$$  \hspace{1cm} \text{4.8.1}

formed by an $\mathcal{A}$-morphism $\varphi': R' \rightarrow Q'$ and by a $\mathcal{B}$-morphism $\varphi: R \rightarrow Q$ fulfilling

$$\alpha' \circ \varphi' = \varphi \circ \beta'$$  \hspace{1cm} \text{4.8.2.}

Let us denote by $p: \widehat{\mathcal{B}} \rightarrow \mathcal{B}$ the structural functor; it is easily checked that we have a second functor and a natural map

$$n: \widehat{\mathcal{B}} \rightarrow \mathcal{A} \subset \mathcal{B} \quad \text{and} \quad \nu: n \rightarrow p$$  \hspace{1cm} \text{4.8.3}

respectively mapping the $\widehat{\mathcal{B}}$-object $(Q', \alpha', Q)$ on $Q'$ and on $\alpha'$, and the $\widehat{\mathcal{B}}$-morphism $(\varphi', \varphi)$ on $\varphi'$ [2, A2.18]; note that if $\varphi$ is an $\mathcal{G}$-isomorphism then, according to our hypothesis, $\varphi'$ is a $\mathcal{G}$-isomorphism too.

4.9. Similarly, the restriction $I_A$ of $I$ to $\mathcal{A}$ defines a representation of $\mathcal{A}$; we set $\widehat{\mathcal{A}} = I_A \times \mathcal{A}$ which is a subcategory of $\widehat{\mathcal{B}}$ having the same objects;
moreover, with the notation above, \((\phi', \phi)\) is an \(\hat{A}\)-morphism if and only if \(\phi\) is an \(A\)-morphism; then, it is clear that \((P, \text{id}_P, P)\) is a final object in \(\hat{A}\) and that \(\hat{B}\) is an \(\hat{A}\)-category. Now, the quintuple \(\hat{H}\) formed by

4.9.1 the trivial bi-interior structure over \(\hat{B}\),

4.9.2 the converse image \(\hat{G}\) in \(\hat{B}\) of the subcategory \(G\) of \(B\),

4.9.3 the trivial representation \(\hat{s}: \hat{B} \to \mathcal{C}\) mapping any \(\hat{B}\)-object on the set \(\{\emptyset\}\) and any \(\hat{B}\)-morphism on the corresponding identity map,

4.9.4 the functor \(\hat{n}: \hat{s} \times \hat{B} = \hat{B} \to \hat{A} \subset \hat{B}\) mapping any \(\hat{B}\)-object \((Q', \alpha', Q)\) on \((\hat{Q}', \text{id}_{Q'}, \hat{Q}')\) and any \(\hat{B}\)-morphism

\[
(\phi', \phi): (R', \beta', R) \to (Q', \alpha', Q)
\]

on the \(\hat{A}\)-morphism \(\phi', \phi\),

4.9.5 the natural map \(\hat{\nu}: \hat{n} \to \text{id}_\hat{B}\) sending any \(\hat{B}\)-object \((Q', \alpha', Q)\) to the \(\hat{B}\)-morphism

\[
(\text{id}_{Q'}, \alpha'): (Q', \text{id}_{Q'}, Q') \to (Q', \alpha', Q)
\]

is clearly a homotopic system of the \(\hat{A}\)-category \(\hat{B}\) (cf. 2.3). Note that

\[
n \circ \hat{n} = n \quad \text{and} \quad p \ast \hat{\nu} = \nu
\]

4.10. On the other hand, for any contravariant functor \(a: B \to \mathfrak{Ab}\) and any \(n \in \mathbb{N}\), it is clear that we have (cf. 3.1 and 4.7.1)

\[
\mathcal{C}^n(B, a^{\text{ac}} \circ m_P) = \prod_{q \in \hat{\text{ct}}(\Delta_n, B)} a^{\text{ac}}(q(0) \times P)
\]

\[
= \prod_{q \in \hat{\text{ct}}(\Delta_n, B)} \prod_{(\alpha', Q', \iota_{Q'}) \in I_q(0)} a(Q')
\]

\[
= \prod_{q \in \hat{\text{ct}}(\Delta_n, B)} \prod_{(\alpha', Q', \iota_{Q'}) \in I_q(0)} (a \circ n)(Q', \alpha', Q)
\]

and that there is a bijection

\[
\hat{\text{ct}}(\Delta_n, \hat{B}) \cong \bigsqcup_{q \in \hat{\text{ct}}(\Delta_n, B)} I_q(0)
\]

mapping any \(\hat{B}\)-chain \(\hat{q}: \Delta_n \to \hat{B}\) on the pair \((q, (\alpha', Q', \iota_{Q'}))\) formed by the \(B\)-chain \(q = p \circ \hat{q}\) and by the element \((\alpha', Q', \iota_{Q'})\) of \(I_q(0)\) fulfilling

\[
\hat{q}(0) = (Q', \alpha', Q)
\]
4.11. Indeed, for any \( q \in \mathcal{Fct}(\Delta_n, \mathcal{B}) \), any \((\alpha', Q', \iota_{Q'}) \in I_{q(0)}(0)\) and any \( i \in \Delta_n \), it follows from equality 4.5.1 that the \( \mathcal{B} \)-morphism

\[
q(0 \cdot i) \circ \alpha' : Q' \rightarrow q(i)
\]

4.11.1 determines a \( \mathcal{B} \)-object \( Q_i' \) and two \( \mathcal{B} \)-morphisms

\[
\theta_i : Q' \rightarrow Q_i' \quad \text{and} \quad \alpha'_i : Q_i' \rightarrow q(i)
\]

4.11.2

such that \((\alpha'_i, Q'_i, \iota_{Q'_i})\) belongs to \( I_{q(i)}(i) \) and that we have

\[
q(0 \cdot i) \circ \alpha' = \alpha'_i \circ \theta_i \quad \text{and} \quad \iota_{Q'} = \iota_{Q_i'} \circ \theta_i
\]

4.11.3;

then, we clearly can define a \( \hat{\mathcal{B}} \)-\textit{chain} \( \hat{q} : \Delta_n \rightarrow \hat{\mathcal{B}} \) fulfilling \( \hat{q}(i) = (Q'_i, \alpha'_i, q(i)) \) and the uniqueness of \( \hat{q} \) follows from Lemma 4.3. Consequently, it is easily checked that, setting \( \hat{a} = a \circ n \), we get a canonical isomorphism

\[
\mathbb{C}^n_G(\mathcal{B}, a^{\text{aff}} \circ m_P) \cong \mathbb{C}^n_{\hat{\mathcal{G}}}(\hat{\mathcal{B}}, \hat{a})
\]

4.11.4

which is clearly compatible with the corresponding \textit{differential maps}. Thus, our claim in 4.6 above follows from Theorem 3.5 and from the next result.

\textbf{Proposition 4.12.} With the notation and the hypothesis above, the contravariant functor \( a \circ n \) is \( \hat{H} \)-\textit{split}. In particular, for any \( n \geq 1 \) we have

\[
\mathbb{H}^n_G(\mathcal{B}, a^{\text{aff}} \circ m_P) = \{0\}
\]

4.12.1

\textbf{Proof:} Since \( n \circ n = n \) (cf. 4.9.6), we actually have (cf. 2.6 and 2.7)

\[
\hat{H}(\hat{a}) = \hat{a} \quad \text{and} \quad \Delta_{\hat{n}}(\hat{a}) = \text{id}_{\hat{a}}
\]

4.12.2

We are done.

5. A quotient in the direct product case

5.1. Consider \( A, B, P \) and \( a \) as in section 4; for our application in [3], we have to deal with a quotient of the functor \( a^{\text{aff}} \circ m_P \) above. More precisely, assume that \( B \) has \textit{minimal objects} and fix a minimal \( \mathcal{B} \)-object \( U \) and a \textit{contravariant} functor \( a_U : B \rightarrow A \) vanishing over all the \( \mathcal{B} \)-objects not \( \mathcal{B} \)-isomorphic to \( U \); then, the natural map \( \omega_P : m_P \rightarrow j \) (cf. 4.6.5) determines the following natural map

\[
a_U^{\text{aff}} \ast \omega_P : a_U \rightarrow a_U^{\text{aff}} \circ m_P
\]

5.1.1

and we have to deal with the quotient

\[
\bar{a}_U^{\text{aff}} \circ m_P = (a_U^{\text{aff}} \circ m_P) / \text{Im}(a_U^{\text{aff}} \ast \omega_P)
\]

5.1.2

in this section we state some conditions on \( \mathcal{B} \) guaranteeing the annulation of the \( \bar{a}_U^{\text{aff}} \circ m_P \)-valued \( n \)-cohomology group of \( \mathcal{B} \) for any \( n \geq 1 \).
5.2. Our argument starts from the trivial fact that, denoting by \( i: \mathcal{A} \to \mathcal{B} \) the \textit{inclusion} functor, since \( \mathcal{A} \) has a final object, for any \( n \geq 1 \) we have \([2, \text{Corollary A4.8}]\)
\[
\mathbb{H}^n(\mathcal{A}, \underline{\alpha}^n_U \circ \underline{m}^n_U \circ i) = \{0\}
\]
and it employs Corollary 3.10 in order to move from \( \mathcal{A} \) to \( \mathcal{B} \): more precisely, we will apply Corollary 3.10 to suitable subcategories \( \mathcal{C}^n \) of \( \mathcal{ch}(\mathcal{B}) \). First of all, we consider subsets
\[
\mathcal{N} \subset \bigsqcup_{Q,V} \mathcal{B}(Q,V)
\]
where \((Q,V)\) runs over the set of pairs of \( \mathcal{B} \)-objects with \( V \) \( \mathcal{B} \)-isomorphic to \( U \), fulfilling the following two conditions:

5.2.3 \textit{For any element} \( \theta: V \to Q \) of \( \mathcal{N} \) \textit{and any pair of} \( \mathcal{B} \)-\textit{isomorphisms} \( \alpha: Q \cong Q' \) \textit{and} \( \eta: V' \cong V \), \textit{the composition} \( \alpha \circ \theta \circ \eta \) \textit{belongs to} \( \mathcal{N} \).

5.2.4 \textit{Any} \( \mathcal{B} \)-\textit{morphism} \( \eta: V \to R \) \textit{dividing an element} \( \theta: V \to Q \) \textit{of} \( \mathcal{N} \) \textit{belongs to} \( \mathcal{N} \).

It follows from 4.10 that we can identify the \( \mathcal{ch}(\mathcal{B}) \)-objects with the triples \((Q', \alpha', q)\) where \( q: \Delta_n \to \mathcal{B} \) is a \( \mathcal{B} \)-\textit{chain} and \((\alpha', Q', \iota_{Q'})\) belongs to \( I_{q(0)} \); then, we denote by \( \mathcal{E}^\mathcal{N} \) the \textit{full} subcategory of \( \mathcal{ch}(\mathcal{B}) \) over the \( \mathcal{ch}(\mathcal{B}) \)-\textit{objects} \((Q', \alpha', q)\) such that either \( Q' \) is not \( \mathcal{B} \)-isomorphic to \( U \) or \( q(0 \bullet n) \circ \alpha' \) belongs to \( \mathcal{N} \), and by \( O_n(\mathcal{E}^\mathcal{N}) \) the set of \( \mathcal{E}^\mathcal{N} \)-\textit{objects} \((Q', \alpha', q)\) such that \( q \) is an \( n \)-\textit{chain}.

5.3. Note that, if \((R', \beta', r)\) is a second \( \mathcal{ch}(\mathcal{B}) \)-\textit{object} where \( r: \Delta_m \to \mathcal{B} \) is a second \( \mathcal{B} \)-\textit{chain}, a \( \mathcal{ch}(\mathcal{B}) \)-\textit{morphism} \((\mu', \mu, \delta)\) from \((Q', \alpha', q)\) to \((R', \beta', r)\) is given by \( \mathcal{ch}(\mathcal{B}) \)-\textit{morphism} \([2, \text{A2.8}]\)
\[
(\mu, \delta) : (q, \Delta_n) \longrightarrow (r, \Delta_m)
\]
and by an \( \mathcal{A} \)-\textit{morphism} \( \mu' : Q' \to R' \) \textit{fulfilling}
\[
\beta' \circ \mu' = \mu_0 \circ q(0 \bullet \delta(0)) \circ \alpha'
\]
In particular, if \((Q', \alpha', q)\) is a \( \mathcal{E}^\mathcal{N} \)-\textit{object} and \((\mu', \mu, \delta)\) is a \( \mathcal{ch}(\mathcal{B}) \)-\textit{morphism}, then either \( R' \) is not isomorphic to \( U \) and therefore \((R', \beta', r)\) is a \( \mathcal{E}^\mathcal{N} \)-\textit{object}, or \( R' \cong U \) which implies that we have \( Q' \cong U \) and that \( \mu' \) is an isomorphism (cf. 4.2.1); then \( q(0 \bullet n) \circ \alpha' \) belongs to \( \mathcal{N} \) and, since \( \mu : q \circ \delta \to r \) is a \textit{natural isomorphism} \([2, \text{A2.8}]\) and equality 5.3.2 implies that \( r(0 \bullet m) \circ \beta' \circ \mu' \) \textit{divides} \( q(0 \bullet n) \circ \alpha' \), \( r(0 \bullet m) \circ \beta' \) also belongs to \( \mathcal{N} \), so that \((R', \beta', r)\) is also a \( \mathcal{E}^\mathcal{N} \)-\textit{object}. Thus, \( \mathcal{E}^\mathcal{N} \) fulfills the condition in 3.7.

5.4. Now, in order to apply Corollary 3.10, we have to prove that the hypothesis in 4.8 holds. \textit{Mutatis mutandis}, denote by \( \mathcal{E}^\mathcal{N}_A \) the "intersection"
which actually corresponds to a functor from $\Delta^n$ which proves our claim. In conclusion, we have proved that the $q$-object

$$C \in \text{ch} \left( \hat{\mathcal{B}} \right)$$

is not isomorphic to $\mathcal{B}$ which fulfills $q'(0) = Q'$ and admits a natural map

$$\text{ch}(\nu)_{(Q',\alpha',q)} : q' \to q$$

sending $0 \in \Delta_n$ to $\alpha'$ and $i \in \Delta_n$ to a $\mathcal{B}$-morphism $\alpha'_i : q'(i) \to q(i)$ such that $(\alpha'_i, q'(i), t_{q'(i)})$ belongs to $I_{q(i)}$ (cf. 4.11). In particular, if $(Q', \alpha', q)$ is a $\mathcal{C}^n$-object, either $Q'$ is not isomorphic to $U$ and $(Q', \text{id}_{Q'}, q')$ is a $\mathcal{C}^n$-object, or $q(0 \cdot n) \circ \alpha'$ belongs to $\mathcal{N}$ and therefore, since $q'(0 \cdot n)$ divides $q(0 \cdot n) \circ \alpha'$, $q'(0 \cdot n)$ also belongs to $\mathcal{N}$, so that $(Q', \text{id}_{Q'}, q')$ is again a $\mathcal{C}^n$-object; hence, $(Q', \text{id}_{Q'}, q')$ is a $\mathcal{C}_A^n$-object and, by restriction, $\text{ch}(n)$ and $\text{ch}(\nu)$ induce a functor and a natural map

$$n^\mathcal{C} : \mathcal{C}^n \to \mathcal{C}_A^n \subset \mathcal{C}^n \text{ and } \nu^\mathcal{C} : n^\mathcal{C} \to \text{id}_{\mathcal{E}^n}$$

5.5. Thus, for any $\mathcal{C}^n$-object $(Q', \alpha', q)$ we get a $\mathcal{C}_A^n$-object $(Q', \text{id}_{Q'}, q')$, which actually corresponds to a functor from $\Delta_n$ to $\mathcal{B}$, and a $\mathcal{C}^n$-morphism

$$\nu^\mathcal{C}_{(Q',\alpha',q)} : (Q', \text{id}_{Q'}, q') \to (Q', \alpha', q)$$

which is actually a natural map between the corresponding functors; then, we claim that for any $\ell \in \Delta_n$, the functor

$$\mathcal{B}^{\nu}_{\ell} (\nu^\mathcal{C}_{(Q',\alpha',q)}) : \Delta_{n+1} \to \hat{\mathcal{B}}$$

is also a $\mathcal{C}^n$-object. Indeed, this functor sends $0 \in \Delta_{n+1}$ to $(Q', \text{id}_{Q'}, Q')$; thus, if $Q'$ is not isomorphic to $U$ then $\mathcal{B}^{\nu}_{\ell} (\nu^\mathcal{C}_{(Q',\alpha',q)})$ is a $\mathcal{C}^n$-object; otherwise we already know that $q(0 \cdot n) \circ \alpha'$ belongs to $\mathcal{N}$ and, from the very definition of $\mathcal{B}^{\nu}_{\ell} (\nu^\mathcal{C}_{(Q',\alpha',q)})$, it is easily checked that this functor corresponds to some triple $(Q', \text{id}_{Q'}, t_\ell)$ where $t_\ell : \Delta_{n+1} \to \mathcal{B}$ is a functor fulfilling

$$t_\ell (0 \cdot n + 1) = q(0 \cdot n) \circ \alpha'$$

which proves our claim. In conclusion, we have proved that the homotopic system $\hat{\mathcal{H}}$ for the category $\hat{\mathcal{B}}$, and the subcategory $\mathcal{C}^n$ of $\text{ch} \left( \hat{\mathcal{B}} \right)$, fulfill all the hypotheses in 3.7 and 3.8 above.
5.6. As in 3.9 above, denoting by \( \hat{a}^\mathbb{N}_U : \mathcal{C}^\mathbb{N} \to \mathfrak{A} \mathfrak{b} \) the restriction to \( \mathcal{C}^\mathbb{N} \) of the composition
\[
\alpha_U \circ n \circ \varphi_B : \text{ch}(\hat{B}) \to \mathfrak{A} \mathfrak{b}
\]
we can consider the new differential complex holding the \( n \)-term
\[
\mathbb{C}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U) = \prod_{\hat{q}} (\alpha_U \circ n)(\hat{q}(0))
\]
where \( \hat{q} : \Delta_n \to \hat{B} \) runs over the set of functors which are \( \mathcal{C}^\mathbb{N} \)-objects, together with the complex homomorphism determined by the restriction
\[
r^\mathbb{N}_i : \mathbb{C}^n(\hat{B}, \hat{a}^\mathbb{N}_U) \to \mathbb{C}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U)
\]
indeed, for any \( n \in \mathbb{N} \) the differential map defined in 3.1.3 induces a differential map
\[
d^\mathbb{N}_n : \mathbb{C}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U) \to \mathbb{C}^{n+1}(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U)
\]

5.7. Explicitly, if \( a = (a(Q, \alpha, \rho))_{(Q, \alpha, \rho) \in O_n(\mathcal{C}^\mathbb{N})} \) belongs to \( \mathbb{C}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U) \) then \( d^\mathbb{N}_n(a) \) evaluated at \( (R', \beta', r) \in O_{n+1}(\mathcal{C}^\mathbb{N}) \) yields either zero if \( R' \) is not isomorphic to \( U \), or otherwise
\[
d^\mathbb{N}_n(a)_{(R', \beta', r)} = a' + \sum_{i=1}^{n+1} (-1)^i a_{(R', \beta', r, i \delta^\rho)}
\]
where either \( r(0 \cdot 1) \circ \beta' \) does not belong to \( \mathcal{B}(r(1), R') \) and we set \( a' = 0 \), or \( r(0 \cdot 1) \circ \beta' \in \mathcal{B}(r(1), R') \) and, assuming that \( I_{\tau(0 \cdot 1)} : I_{\tau(0)} \to I_{\tau(1)} \) maps \( (\beta', R', r_{R'}) \) on \( (\alpha', Q', i_{Q'}) \) and that \( \rho' : R' \to Q' \) is an \( \mathfrak{A} \)-morphism fulfilling
\[
r(0 \cdot 1) \circ \beta' = \alpha' \circ \rho'
\]
we set \( a' = (a_U(\rho'))(a(Q', \alpha', r_{Q'}) \hat{a}^\mathbb{N}_U) \). Let us denote by \( \mathbb{H}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U) \) the corresponding cohomology group; as above, the point is that the cohomology of the new differential complex only depends on the restriction to \( \mathcal{C}^\mathbb{N}_A \) precisely, denoting by \( \hat{\mathfrak{A}} : \mathfrak{A} \to \hat{B} \) and \( \hat{i}^\mathbb{N} : \mathcal{C}^\mathbb{N}_A \to \mathbb{C}^\mathbb{N} \) the obvious inclusion functor, Corollary 3.10 implies the following result.

**Corollary 5.8.** With the notation above, the restriction induces an injective group homomorphism from \( \mathbb{H}^n(\mathcal{C}^\mathbb{N}, \hat{a}^\mathbb{N}_U) \) to \( \mathbb{H}^n(\mathcal{C}^\mathbb{N}_A, \hat{a}^\mathbb{N}_U \circ \hat{i}^\mathbb{N}) \) for any \( n \geq 1 \).

5.9. In our argument, we also need some lifting procedure — provided by Proposition 5.12 below — from \( \hat{\mathfrak{a}}^\mathbb{N}_U \circ m^\mathfrak{P} \) to \( \hat{\mathfrak{a}}^\mathbb{N}_U \circ m^\mathfrak{P} \)-valued items. Let us denote by \( \mathfrak{K}^\mathbb{N}_ \mathfrak{m} \) the differential subcomplex of \( \mathbb{C}(\hat{B}, \hat{a}_U) \) with \( n \)-term \( \ker(r^\mathbb{N}_n) \).
(cf. 5.6.3); more explicitly, \( \mathbb{K}_{n}^{a} \) can be identified with the set of families 
\( a = (a_{(V, \gamma, q)})_{(V, \gamma, q)} \) where \( V \) is \( \mathcal{B} \)-isomorphic to \( U \), \( q : \Delta_{n} \rightarrow \mathcal{B} \) is a functor and 
\( \gamma : V \rightarrow q(0) \) is a \( \mathcal{B} \)-morphism such that the triple \( (\gamma, \nu, \nu_{V}) \) belongs to \( I_{q(0)} \), and where, for such a triple \( (\gamma, \nu, \nu_{V}) \), \( a_{(V, \gamma, q)} \) belongs to \( a_{U}(V) \) and it vanish if \( \gamma \) does not belong to \( \mathcal{B}(q(0), \nu_{V}) \) or \( q(0 \bullet n) \circ \gamma \) belongs to \( \mathcal{M} \).
That is to say, we have the exact sequence of differential complexes

\[
0 \rightarrow \mathbb{K}_{n}^{a} \rightarrow C(\mathcal{B}, \hat{a}_{U}) \rightarrow C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \rightarrow 0
\]

Moreover, we have the canonical isomorphism \( C(\mathcal{B}, \hat{a}_{U}) \cong C(\mathcal{B}, a_{U}^{\mathbb{M}} \circ m_{P}) \)
(cf. 4.11.4), and the natural map 5.1.1 determines the exact sequence of differential complexes

\[
0 \rightarrow C(\mathcal{B}, a_{U}) \rightarrow C(\mathcal{B}, a_{U}^{\mathbb{M}} \circ m_{P}) \rightarrow C(\mathcal{B}, a_{U}^{\mathbb{M}} \circ m_{P}) \rightarrow 0
\]

then, let us denote by \( Z, Z_{n} \) the respective differential subcomplexes of \( C(\mathcal{B}, \hat{a}_{U}) \) and \( C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \) determined by the images of \( C(\mathcal{B}, a_{U}) \). Also, we denote by \( \mathcal{K}_{\mathbb{M}}, \mathcal{Z}_{\mathbb{M}} \) and \( \mathcal{K}_{\mathbb{M}} \) the respective intersections of \( \mathcal{K}_{\mathbb{M}} \) with \( C(\partial, \hat{a}_{U} \circ \iota) \) and of \( \mathcal{Z}_{\mathbb{M}} \) with \( C(\mathcal{C}_{\mathbb{M}}, \hat{a}_{U}^{\mathbb{M}} \circ i_{\mathbb{M}}) \).

5.10. Assume that \( \mathcal{M} \) is not empty and that, in the (pre)order defined by the divisibility, \( \mathcal{M} \) has some maximal element which we may assume that it has the form \( i_{U}^{\mathbb{M}} : U \rightarrow S \); for induction purposes, consider the set

\[
\mathcal{M}_{\mathbb{M}} = \mathcal{M} - \{ \alpha \circ i_{U}^{\mathbb{M}} \circ \eta \mid \alpha : S \cong S' \text{ and } \eta : U' \cong U, \mathcal{F}-isomorphisms \}
\]

which still fulfills conditions 5.2.3 and 5.2.4; in this situation, it is clear that \( \mathcal{K}_{\mathbb{M}} \) is contained in \( \mathcal{K}_{\mathbb{M}} \) and that, setting \( \mathcal{K}_{\mathbb{M}} = \mathcal{K}_{\mathbb{M}} \), the restriction from \( \mathcal{C} \) to \( \mathcal{C} \) induces an exact sequence of differential complexes

\[
0 \rightarrow \mathcal{K}_{\mathbb{M}} \rightarrow C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \rightarrow C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \rightarrow 0
\]

more explicitly, for any \( n \in \mathbb{N} \), an element \( a = (a_{(V, \gamma, q)})_{(V, \gamma, q)} \in C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \) belongs to the image of \( \mathcal{K}_{\mathbb{M}} \) if and only if we have \( a_{(V, \gamma, q)} = 0 \) whenever \( q(0 \bullet n) \circ \gamma \) does not belong to \( \mathcal{M} - \mathcal{M} \).

5.11. Further, the restriction from \( \mathcal{C} \) to \( \mathcal{C} \) clearly maps \( \mathcal{Z}_{\mathbb{M}} \) onto \( \mathcal{Z}_{\mathbb{M}} \) and therefore the kernel \( \mathcal{Z}_{\mathbb{M}} \) of this complex homomorphism can be identified to a subcomplex of \( \mathcal{K}_{\mathbb{M}} \); explicitly, \( a \) belongs to the image of \( \mathcal{Z}_{\mathbb{M}} \) if and only if we have \( a_{(V, \gamma, q)} = 0 \) whenever \( q(0 \bullet n) \) does not belong to \( \mathcal{M} - \mathcal{M} \) whereas if \( q(0 \bullet n) \) belongs to \( \mathcal{M} - \mathcal{M} \) we have

\[
(a_{U}(\gamma)^{-1}(a_{(V, \gamma, q)}) = (a_{U}(\delta)^{-1}(a_{(W, \delta, q)})) \]

for any \( \gamma \in \mathcal{B}(q(0), V) \) and any \( \delta \in \mathcal{B}(q(0), W) \) such that \( (V, \gamma, q) \) and \( (W, \delta, q) \) belong to \( O_{n}(\mathcal{C}) \). As above, we denote by \( \mathcal{K}_{\mathbb{M}} \) and \( \mathcal{Z}_{\mathbb{M}} \) the respective converse images of \( C(\mathcal{C}, \hat{a}_{U}^{\mathbb{M}}) \) in \( \mathcal{K}_{\mathbb{M}} \) and \( \mathcal{Z}_{\mathbb{M}} \). On the other hand,
Proof: For any $\nabla$ that $\nabla U$, isomorphic to $—$ where we identify $\iota$ also divides $\iota I$ belongs to $\mathcal{C}$ such that $q$ or it follows from equalities 5.11.1 that with the notation above, assume that the set $\{e\}$ and any minimal $\theta: V \to Q$ dividing $\theta \circ q(0\cdot n) \circ \gamma$ also divides $\iota V$; thus, since $(\gamma, V, \iota V) \in I(0)$, $\mathcal{T}_q$ is empty whenever $q(0) \neq U$. 

**Proposition 5.12.** With the notation above, assume that the set $\mathcal{T}_q$ is finite and that the multiplication by $|q|$ over $a(U)$ is invertible for any $B$-chain $q$ such that $q(0) \cong U$. Then, the inclusion of complexes $Z_{n+1}(\mathcal{C}) \subset K_{n+1}(\mathcal{C})$ admits a section $\nabla: K_{n+1}(\mathcal{C}) \to Z_{n+1}(\mathcal{C})$ mapping $K_{n+1}(\mathcal{C})$ onto $Z_{n+1}(\mathcal{C})$.

**Proof:** For any $n \in \mathbb{N}$ and any element $a = (a_{(V, \gamma, q)}(V, \gamma, q) \in O_n(\mathcal{C}^n)$ of $K_{n+1}(\mathcal{C})$ — where we identify $K_{n+1}(\mathcal{C})$ with its image in $\mathcal{C}^n(\mathcal{D}, a_U)$ — let us define that $\nabla(a)$ evaluated at $(W, \delta, r) \in O_n(\mathcal{C}^n)$ either yields zero if $r(0)$ is not isomorphic to $U$, or if $r(0) \cong U$ then we have

$$\nabla(a)(W, \delta, r) = \sum_{(\alpha, V, \gamma) \in \mathcal{T}_r} \frac{1}{|\mathcal{T}_r|} (a_{U}(\gamma^{-1} \circ \delta))(a_{(V, \gamma, r)}) \quad 5.12.1;$$

this element $\nabla(a)$ belongs to $Z_{n+1}(\mathcal{C})$ since if $r(0)$ is isomorphic to $U$ then we clearly have

$$(a_U(\delta))^{-1}(\nabla(a)(W, \delta, r)) = (a_U(\delta'))^{-1}(\nabla(a)(W', \delta', r)) \quad 5.12.2$$

for a second evaluation at $(W', \delta', r) \in O_n(\mathcal{C}^n)$.

On the other hand, $a$ belongs to $K_{n+1}(\mathcal{C})$ if and only if $a_{(V, \gamma, q)} = 0$ whenever $q: \Delta_n \to \mathcal{B}$ does not factorize through $\mathcal{A}$; in this case, we also have $\nabla(a)(W, \delta, r) = 0$ whenever $r: \Delta_n \to \mathcal{B}$ does not factorize through $\mathcal{A}$, and therefore $\nabla(a)$ belongs to $Z_{n+1}(\mathcal{C})$. Moreover, if $a$ belongs to $Z_{n+1}(\mathcal{C})$ then either $r(0)$ is not isomorphic to $U$ and we get

$$a_{(W, \delta, r)} = 0 = \nabla(a)(W, \delta, r) \quad 5.12.3$$

or it follows from equalities 5.11.1 that

$$\nabla(a)(W, \delta, r) = \sum_{(\alpha, V, \gamma) \in \mathcal{T}_r} \frac{1}{|\mathcal{T}_r|} a_{(W, \delta, r)} = a_{(W, \delta, r)} \quad 5.12.4.$$ 

Let us evaluate $\nabla(a)$ and $\nabla(a)$ at $(W, \delta, r) \in O_{n+1}(\mathcal{C}^n)$; if $r(0)$ is not isomorphic to $U$ then we have (cf. 5.7.1)

$$\nabla(a)(W, \delta, r) = 0 = a_{(W, \delta, r)} \quad 5.12.5.$$
Otherwise, on the one hand it follows from 5.7.1 that we have
\[
\nabla^{n+1}_{\delta,\gamma}(d^n_{\gamma}(a))_{(W,\delta,\tau)} \\
= \sum_{(a, V, \gamma) \in T_n} \frac{1}{|T_n|} (a_U(\gamma^{-1} \circ \delta)) (d^n_{\gamma}(a))_{(V, \gamma, \tau)}
\]
\[\text{(5.12.6)}\]
\[
= \sum_{(a, V, \gamma) \in T_n} \frac{1}{|T_n|} (a_U(\gamma^{-1} \circ \delta)) (a'_{(a, V, \gamma)} + \sum_{i=1}^{n+1} (-1)^i a_{(V, \gamma, \tau \circ \delta^n_i)})
\]
where either \(r_0 \circ 1 \circ \gamma\) does not belong to \(B\langle r_0 \circ 1 \rangle_{V, \gamma}\) and we set \(a'_{(a, V, \gamma)} = 0\), or \(r_0 \circ 1 \circ \gamma \in B\langle r_0 \circ 1 \rangle_{V, \gamma}\) and, assuming that \(I_{\tau(0 \circ 1)} : I_{\tau(0)} \rightarrow I_{\tau(1)}\) maps \((\gamma, V, \nu_\gamma)\) on \((\gamma, T, \nu_T)\) and that \(\theta : V \rightarrow T\) is an \(A\)-morphism fulfilling
\[
r_0 \circ 1 \circ \gamma = \eta \circ \theta
\]
we set
\[
a'_{(a, V, \gamma)} = (a_U(\theta)_\gamma) (a_{(T, \eta, \nu_\tau \circ \delta^n_i)})
\]
\[\text{(5.12.7)}\]
\[\text{(5.12.8)}\]
On the other hand, it follows again from 5.7.1 that we have
\[
d^n_{\gamma}(\nabla^{n+1}_{\delta,\gamma}(a))_{(W,\delta,\tau)} = b' + \sum_{i=1}^{n+1} (-1)^i \nabla^{n+1}_{\delta,\gamma}(a)_{(W,\delta,\tau \circ \delta^n_i)}
\]
\[\text{(5.12.9)}\]
\[
= b' + \sum_{i=1}^{n+1} \sum_{(a, V, \gamma) \in T_{\tau \circ \delta^n_i}} \left( \frac{1}{|T_{\tau \circ \delta^n_i}|} (a_U(\gamma^{-1} \circ \delta)) (a_{(V, \gamma, \tau \circ \delta^n_i)}) \right)
\]
where either \(r_0 \circ 1 \circ \delta\) does not belong to \(B\langle r_0 \circ 1 \rangle_{\nu_\tau \circ \delta^n_i}\) and we set \(b' = 0\), or \(r_0 \circ 1 \circ \delta \in B\langle r_0 \circ 1 \rangle_{\nu_\tau \circ \delta^n_i}\) and, assuming that \(I_{\tau(0 \circ 1)} : I_{\tau(0)} \rightarrow I_{\tau(1)}\) maps \((\delta, W, \nu_W)\) on \((\eta, T, \nu_T)\) and that \(\theta : W \rightarrow T\) is an \(A\)-morphism fulfilling
\[
r_0 \circ 1 \circ \delta = \eta \circ \theta
\]
we set \(b' = (a_U(\theta)) (\nabla^{n+1}_{\delta,\gamma}(a)_{(T, \eta, \nu_\tau \circ \delta^n_i)})\) which vanish unless \(r_1\) is isomorphic to \(U\) and then we get
\[
b' = \sum_{(a, V, \gamma) \in T_{\tau \circ \delta^n_i}} \frac{1}{|T_{\tau \circ \delta^n_i}|} (a_U(\gamma^{-1} \circ \eta \circ \theta)) (a_{(V, \gamma, \tau \circ \delta^n_i)})
\]
\[\text{(5.12.11)}\]
For any \(1 \leq i \leq n\), since we have
\[
(r \circ \delta^n_i)(0) = r(0) \quad , \quad (r \circ \delta^n_i)(n) = r(n+1)
\]
and \((r \circ \delta^n_i)(0 \cdot n) = r(0 \cdot n + 1)\)
\[\text{(5.12.12)}\]
we easily get $T_{\text{cod}_n} = T_r$ and therefore the $i$-terms on the bottom sums in 5.12.6 and 5.12.9 coincide with each other. If $i = n + 1$ and $\tau(n \bullet n + 1)$ is not an isomorphism then, in the $n + 1$-terms on the bottom sums in 5.12.6 and 5.12.9, the $B$-morphisms

$$(\tau \circ \delta_n(0 \bullet n) \circ \gamma = \tau(0 \bullet n) \circ \gamma)$$ 5.12.13

do not belong to $\mathfrak{M} - \mathfrak{M}$ and therefore in both cases all the items $a_{\tau(n \bullet n + 1)}$ vanish, so that both $n + 1$-terms vanish; if $\tau(n \bullet n + 1)$ is an isomorphism then we have a bijection

$$\tau^{n+1} : T_r \cong T_{\text{cod}_{n+1}}$$ 5.12.14

mapping $(\alpha, V, \gamma) \in T_r$ on $(\alpha \circ \tau(n \bullet n + 1), V, \gamma)$ which belongs to $T_{\text{cod}_{n+1}}$ since $\alpha \circ \tau(n \bullet n + 1)$ belongs to $B(P, (\tau \circ \delta_n(0 \bullet n + 1))(n))$ and we have

$$\alpha \circ \tau(0 \bullet n + 1) \circ \gamma = (\alpha \circ \tau(n \bullet n + 1)) \circ (\tau \circ \delta_n(0 \bullet n)) \circ \gamma$$ 5.12.15;

hence, in this case the $n + 1$-terms on the bottom sums in 5.12.6 and 5.12.9 still coincide with each other.

Finally, we claim that

$$\sum_{(\alpha, V, \gamma) \in T_r} \frac{1}{|T_r|} (a_{U}((\gamma^{-1} \circ \delta)))(a'_{(\alpha, V, \gamma)}) = b'$$ 5.12.16;

since we are assuming that $\tau(0)$ is isomorphic to $U$, $\gamma$ and $\delta$ are isomorphisms and therefore $\tau(0 \bullet 1) \circ \delta$ belongs to $B(\tau(1), V)_{\mu V}$ if and only if $\tau(0 \bullet 1) \circ \gamma$ belongs to $B(\tau(1), V)_{\nu V}$; thus, we may assume that this happens since otherwise we have

$$\sum_{(\alpha, V, \gamma) \in T_r} \frac{1}{|T_r|} (a_{U}((\gamma^{-1} \circ \delta)))(a'_{(\alpha, V, \gamma)}) = 0 = b'$$ 5.12.17.

Then, we claim that $\tau(0 \bullet 1) \circ \gamma$ is an isomorphism; otherwise, since $\tau(0 \bullet 1) \circ \gamma$ clearly divides $\alpha \circ \tau(0 \bullet n + 1) \circ \gamma$, a minimal $B$-morphism from $V$ dividing $\tau(0 \bullet 1) \circ \gamma$ divides $\nu V$, contradicting the fact that $\tau(0 \bullet 1) \circ \gamma$ belongs to $B(\tau(1), V)_{\nu V}$; hence, if $\tau(1)$ is not isomorphic to $U$ then equality 5.12.17 still holds.

Consequently, we may assume that $\tau(0 \bullet 1)$ is an isomorphism and then, since in 5.12.7 above the map $I_{\tau(0 \bullet 1)}$ is a bijection, once again we have a bijection

$$\tau^0 : T_r \cong T_{\text{cod}_0}$$ 5.12.18

mapping $(\alpha, V, \gamma) \in T_r$ on $(\alpha, T_{\gamma}, \eta_{\gamma})$ which belongs to $T_{\text{cod}_0}$ since we have

$$\alpha \circ \tau(0 \bullet n + 1) \circ \gamma = \alpha \circ (\tau \circ \delta_n(0 \bullet n)) \circ (\tau(0 \bullet 1) \circ \gamma)$$

$$= \alpha \circ (\tau \circ \delta_n(0 \bullet n)) \circ \eta_{\gamma} \circ \theta_{\gamma}$$ 5.12.19.
and, in this situation, \( \theta \) is necessarily an isomorphism; that is to say, according to equalities 5.12.8, 5.12.9 and 5.12.11, we get

\[
b' = \sum_{(\alpha, V, \gamma) \in T} \frac{1}{|T_{ro0}|} (a_U(\eta^{-1} \circ \eta \circ \theta)) (a_{(T, \eta, \gamma \circ \delta_n)})
\]

\[
= \sum_{(\alpha, V, \gamma) \in T} \frac{1}{|T_{ro0}|} (a_U(\eta^{-1} \circ \tau(0 \cdot 1) \circ \delta)) (a_{(T, \eta, \gamma \circ \delta_n)}) \tag{5.12.20}
\]

\[
= \sum_{(\alpha, V, \gamma) \in T} \frac{1}{|T_{ro0}|} (a_U(\gamma^{-1} \circ \delta)) (a'_{(\alpha, V, \gamma)})
\]

since \( \eta^{-1} \circ \tau(0 \cdot 1) = \theta \circ \gamma^{-1} \) (cf. 5.12.7). We are done.

5.13. Now, in the exact sequence 5.9.2, we denote by \( \mathbb{L}_{\mathcal{M}} \) the converse image of \( \mathcal{K}_{\mathcal{M}} \) in \( \mathbb{C}(\mathcal{B}, a_U) \) and set

\[
\mathcal{K}_{\mathcal{M}} = \mathcal{K}_{\mathcal{M}} / \mathbb{L}_{\mathcal{M}} \quad \text{and} \quad \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} = \mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M}) / \mathbb{Z}_{\mathcal{M}} \tag{5.13.1},
\]

so that we get the commutative diagram of exact sequences

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \mathbb{Z}_{\mathcal{M}} & \rightarrow \mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M}) & \rightarrow \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \mathbb{Z} & \rightarrow \mathbb{C}(\mathcal{B}, a_U^\mathcal{M} \circ m) & \rightarrow \overline{\mathbb{C}(\mathcal{B}, a_U^\mathcal{M} \circ m)} & \rightarrow 0 \tag{5.13.2} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \mathbb{L}_{\mathcal{M}} & \rightarrow \mathcal{K}_{\mathcal{M}} & \rightarrow \overline{\mathcal{K}_{\mathcal{M}}} & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Moreover, with the notation in 5.10 above, \( \mathbb{L}_{\mathcal{M}} \) is contained in \( \mathbb{L}_{\mathcal{M}'} \) and we have \( \mathbb{L}_{\mathcal{M}} / \mathbb{L}_{\mathcal{M}} \cong \mathbb{Z}_{\mathcal{M}, \mathcal{M}'} \); similarly, \( \overline{\mathbb{K}_{\mathcal{M}}} \) is “contained” in \( \overline{\mathbb{K}_{\mathcal{M}'} } \) and, setting \( \overline{\mathbb{K}_{\mathcal{M}, \mathcal{M}'}} = \overline{\mathbb{K}_{\mathcal{M}}}/\overline{\mathbb{K}_{\mathcal{M}'} } \), we get the new exact commutative diagram of exact sequences

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \overline{\mathbb{K}_{\mathcal{M}, \mathcal{M}'}} & \rightarrow \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} & \rightarrow \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \overline{\mathbb{K}_{\mathcal{M}, \mathcal{M}'}} & \rightarrow \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} & \rightarrow \overline{\mathbb{C}(\mathcal{C}_{\mathcal{M}}, \hat{a}_U^\mathcal{M})} & \rightarrow 0 \tag{5.13.3}; \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 \rightarrow \mathbb{Z}_{\mathcal{M}, \mathcal{M}'} & \rightarrow \mathbb{Z}_{\mathcal{M}} & \rightarrow \mathbb{Z}_{\mathcal{M}} & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

then, with obvious notation, the corresponding restrictions to \( \mathcal{A} \) of the left-hand vertical sequence yield the exact sequence

\[
0 \rightarrow \mathbb{Z}_{\mathcal{M}, \mathcal{M}'} \rightarrow \overline{\mathbb{K}_{\mathcal{M}, \mathcal{M}'}} \rightarrow \overline{\mathbb{K}_{\mathcal{M}, \mathcal{M}'}} \rightarrow 0 \tag{5.13.4}.
\]
Proposition 5.14. With the notation above, assume that the category $B$ is finite and that the multiplication by $|T_q|$ over $a_U(U)$ is invertible for any $B$-chain $q$ such that $q(0) \cong U$. Then, for any $n \geq 1$ we have
\[ \mathcal{H}^n(B, a_U^{ac} \circ m_P) = \{0\} \] 5.14.1.

Proof: Let $\bar{a} \in \mathbb{C}^n(B, a_U^{ac} \circ m_P)$ be an $n$-cocycle lifting a cohomological class $\bar{a}^b$ and denote by $\bar{a}_A$ the image of $\bar{a}$ in $\mathbb{C}^n(A, a_U^{ac} \circ m_P)$; since $A$ has the final object $P$, as announced in 5.1 we have
\[ \mathcal{H}^n(A, a_U^{ac} \circ m_P \circ i) = \{0\} \] 5.14.2.

and therefore we get $\bar{a}_A = d_{n-1}(\bar{b}_A)$ for some $\bar{b}_A \in \mathbb{C}^{n-1}(A, a_U^{ac} \circ m_P)$; thus, lifting $\bar{b}_A$ to $\bar{b} \in \mathbb{C}^{n-1}(B, a_U^{ac} \circ m_P)$ and replacing $\bar{a}$ by $\bar{a} - d_{n-1}(\bar{b})$, we may assume that $\bar{a}_A = 0$.

More generally, we claim that for any set $\mathfrak{M}$ fulfilling conditions 5.2.3 and 5.2.4 we can choose $\bar{b} \in \mathbb{C}^{n-1}(B, a_U^{ac} \circ m_P)$ in such a way that the $n$-cocycle $\bar{a} - d_{n-1}(\bar{b})$ belongs to $\mathbb{K}^n_{\mathfrak{M}}$; since $\mathbb{K}_0 = \mathbb{C}^n(B, a_U^{ac} \circ m_P)$, arguing by induction on $|\mathfrak{M}|$ we may assume that $\mathfrak{M} \neq \emptyset$. Now, in the order defined by the divisibility, choose a maximal $B$-morphism $i_S^U : U \to S$ in $\mathfrak{M}$; then, the set
\[ \mathfrak{M} = \mathfrak{M} - \{\alpha \circ i_S^U \circ \eta \mid \alpha : S \cong S' \text{ and } \eta : U' \cong U \text{ } \mathcal{F}\text{-isomorphisms}\} \] 5.14.3

still fulfills conditions 5.2.3 and 5.2.4; consequently, we can choose $\bar{b}$ in $\mathbb{C}^{n-1}(B, a_U^{ac} \circ m_P)$ such that the $n$-cocycle $\bar{a} - d_{n-1}(\bar{b})$ belongs to $\mathbb{K}^n_{\mathfrak{M}}$; in particular, denoting by $\bar{a}^n$ and $\bar{b}^n$ their respective images in $\mathbb{C}^n(\mathfrak{C}, a_U^{ac})$ and $\mathbb{C}^{n-1}(\mathfrak{C}, a_U^{ac})$ (cf. diagram 5.13.2), we still have $\bar{d}_n^n(\bar{a}^n - d_{n-1}^n(\bar{b}^n)) = 0$.

Let $a$ and $b$ be elements in $\mathbb{C}^n(B, a_U^{ac} \circ m_P)$ and $\mathbb{C}^{n-1}(B, a_U^{ac} \circ m_P)$ respectively lifting $\bar{a}$ and $\bar{b}$ in such a way that $a - d_{n-1}(b)$ belongs to $\mathbb{K}^n_{\mathfrak{M}}$ and that the image $a_A$ of $a$ in $\mathbb{C}^n(A, a_U^{ac} \circ m_P)$ is equal to zero, and denote by $a^n$ and $b^n$ their respective images in $\mathbb{C}^n(\mathfrak{C}, a_U^{ac})$ and $\mathbb{C}^{n-1}(\mathfrak{C}, a_U^{ac})$; then, identifying $\mathbb{K}^n_{\mathfrak{M}}$ with the corresponding subcomplex of $\mathbb{C}^n(\mathfrak{C}, a_U^{ac})$, it is clear from the exact sequence 5.10.2 that $a^n - d_{n-1}^n(b^n)$ belongs to $\mathbb{K}^n_{\mathfrak{M}}$; but, it follows from Proposition 5.12 and from 5.13 that
\[ \mathbb{K}^n_{\mathfrak{M}} \cong \mathcal{Z}^n_{\mathfrak{M}} \oplus \mathcal{H}^n_{\mathfrak{M}} \text{ and } \mathbb{K}^n_{\mathfrak{M}}/\mathcal{H}^n_{\mathfrak{M}} \cong \mathcal{Z}^n_{\mathfrak{M}} \] 5.14.4;

hence, we actually can choose $a$ in such a way that $a^n - d_{n-1}^n(b^n)$ belongs to $\text{Ker}(\nabla^n_{\mathfrak{M}}) \cong \mathbb{K}^n_{\mathfrak{M}}$ and then, since $d_n^n(\bar{a}^n - d_{n-1}^n(\bar{b}^n)) = 0$, from the commutative diagram 5.13.3 above we still get $d_n^n(\bar{a}^n - d_{n-1}^n(\bar{b}^n)) = 0$. 


Thus, the image of $a^n - d_{n-1}^m(b^n)$ in $\mathbb{C}^m(A, \tilde{\alpha}^m_U)$ is also an $n$-cocycle and actually it coincides with $d_{n-1}^m(b^n)$, so that it is an $n$-coboundary; at this point, it follows from Corollary 5.8 above that the image of $a^n - d_{n-1}^m(b^n)$ in $\mathbb{H}^n(\mathbb{C}^m, \tilde{\alpha}^m_U)$ is equal to zero; that is to say, we have $a^n - d_{n-1}^m(b^n) = d_{n-1}^m(c^n)$ for some $c^n \in \mathbb{C}^{n-1}(\mathbb{C}^m, \tilde{\alpha}^m_U)$; finally, lifting $c^n$ to $c \in \mathbb{C}^{n-1}(B, \alpha_U^B \circ m_P)$, it is clear that the $n$-cocycle $a - d_{n-1}(b + c)$ belongs to $K^n_\mathcal{A}$. In particular, considering the case (cf. 5.2.2)

$$\mathcal{H} = \bigcup_{Q,V} \mathcal{B}(Q,V)$$

we have $\mathcal{K}^n_\mathcal{A} = \{0\}$ and therefore there is $\tilde{b} \in \mathbb{C}^{n-1}(B, \alpha_U^B \circ m_P)$ such that $\tilde{a} = d_{n-1}(\tilde{b})$. We are done.

6. The compatible complement case

6.1. Let $P$ be a finite $p$-group and $\mathcal{F}$ a Frobenius $P$-category; we are interested in a family of commutative square diagrams in the additive cover $\text{ac}(\tilde{\mathcal{F}})$ (cf. 4.1) of the exterior quotient $\tilde{\mathcal{F}}$ (cf. 2.2) — called special $\text{ac}(\tilde{\mathcal{F}})$-squares. Explicitly, for any triple of subgroups $Q$, $R$ and $T$ of $P$, and any pair of $\tilde{\mathcal{F}}$-morphisms $\tilde{\alpha} : R \to Q$ and $\tilde{\beta} : T \to Q$, we choose a pair of representatives $\alpha$ of $\tilde{\alpha}$ and $\beta$ of $\tilde{\beta}$, and a set of representatives $W \subset Q$ for the set of double classes $\alpha(R)\backslash Q/\beta(T)$; moreover, for any $w \in W$, we set $U_w = \alpha(R)^w \cap \beta(T)$ and respectively denote by

$$\alpha_w : U_w \to R \quad \text{and} \quad \beta_w : U_w \to T$$

the $\mathcal{F}$-morphisms mapping $u \in U_w$ on $\alpha^*(uw^{-1})$ and on $\beta^*(u)$, where $\alpha^* : \alpha(R) \cong R$ and $\beta^* : \beta(T) \cong T$ are the inverse maps of the respective isomorphisms induced by $\alpha$ and $\beta$.

6.2. Consider the $\text{ac}(\tilde{\mathcal{F}})$-object $U = \bigoplus_{w \in W} U_w$ and the two $\text{ac}(\tilde{\mathcal{F}})$-morphisms

$$\tilde{\gamma} : U \to R \quad \text{and} \quad \tilde{\delta} : U \to T$$

respectively determined by the families $\{\tilde{\alpha}_w\}_{w \in W}$ and $\{\tilde{\beta}_w\}_{w \in W}$; then, it is quite clear that the following $\text{ac}(\tilde{\mathcal{F}})$-diagram is commutative

$$\begin{array}{ccc}
Q & \xrightarrow{\alpha} & R \\
\downarrow{\tilde{\gamma}} & \searrow{\tilde{\alpha}} & \downarrow{\tilde{\beta}} \\
U & \xrightarrow{\tilde{\delta}} & T \\
\end{array}$$

Let us call special $\text{ac}(\tilde{\mathcal{F}})$-square any direct sum in $\text{ac}(\tilde{\mathcal{F}})$ of such commutative $\text{ac}(\tilde{\mathcal{F}})$-diagrams. As we show in Proposition 6.6 below, the special $\text{ac}(\tilde{\mathcal{F}})$-squares are actually the image of the pull-backs of a suitable extension of the category $\text{ac}(\tilde{\mathcal{F}})$. 
6.3. As above, denoting by $\mathcal{F}_P$ the Frobenius category of the group $P$ [2, 1.8], it is easily checked that $\mathcal{F}$ is a $\mathcal{F}_P$-category; but, we are interested in suitable extensions $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and $\tilde{\mathcal{F}}_P$ of $\mathcal{F}_P$ such that $\tilde{\mathcal{F}}$ is a $\tilde{\mathcal{F}}_P$-category (cf. 2.1) admitting an homotopic system (cf. 2.3). Recall that any Frobenius $P$-category comes from a basic $P \times P$-set [2, Propositions 21.9 and 21.12] and let $\Omega$ be an $\mathcal{F}$-basic $P \times P$-set [2, 21.3]; as in [2, 22.2], let us denote by $G$ the group of permutations of $\text{Res}_{\{1\} \times P}(\Omega)$ and identify $P$ with the image of $P \times \{1\}$ in $G$; then, we define $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_P$ as the categories with the same objects as $\mathcal{F}$ and, for any pair of subgroups $Q$ and $R$ of $P$, with the sets of morphisms

$$\tilde{\mathcal{F}}(Q, R) = T_G(R, Q) \times P \quad \text{and} \quad \tilde{\mathcal{F}}_P(Q, R) = T_P(R, Q) \times P \quad 6.3.1,$$

where $T_G$ and $T_P$ respectively denote the transporters in $G$ and in $P$, the composition being defined by the products in $G$ and in $P$.

6.4. Thus, by the very definition of $\mathcal{F}$-basic $P \times P$-sets [2, 21.3], $\mathcal{F}$ and $\mathcal{F}_P$ are indeed respective quotients of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_P$; more precisely, $\tilde{\mathcal{F}}$ admits the co-interior structure which maps any subgroup $Q$ of $P$ on $C_G(Q) \times P$ and then $\tilde{\mathcal{F}}$ is just the corresponding exterior quotient. It is also clear that $\tilde{\mathcal{F}}$ is actually an $\tilde{\mathcal{F}}_P$-category and we define below a homotopic system $\mathcal{H} = (I, I^o, G, s, n, v)$ for $\tilde{\mathcal{F}}$. Recall that, for any subgroup $Q$ of $P$, the stabilizer $(Q \times P)_\omega$ of $\omega \in \Omega$ in $Q \times P$ has the form

$$\Delta_Q(\Omega, \omega) = \{ (t^Q_v, v) \}_{v \in Q_\omega} \quad 6.4.1$$

for some subgroup $Q_\omega$ of $P$ and some $t^Q_\omega \in T_G(Q_\omega, Q)$; note that, for any $\tilde{\mathcal{F}}$-morphism $(x, u) : R \to Q$ and any $\omega \in \Omega$, we have

$$^u R_\omega \subset Q_{x \cdot \omega \cdot u^{-1}} \quad \text{and} \quad x \cdot t^R_\omega = t^Q_{x \cdot \omega \cdot u^{-1}} \cdot u \mod C_G(R_\omega) \quad 6.4.2.$$
This quintuple is a homotopic system of $\tilde{F}$; indeed, the functoriality of $s$ comes from the very definition of $\tilde{F}$; the functoriality of $n$ and the naturality of $\nu$ follow from 6.4.2.

**Proposition 6.6.** Denoting by $\tilde{\tilde{F}}^I$ the exterior $I$-quotient of $\tilde{F}$ and by $\tilde{\epsilon}: \tilde{\tilde{F}}^I \to \tilde{F}$ the canonical functor, the additive cover $\text{ac}(\tilde{\tilde{F}}^I)$ admits pull-backs, and the special $\text{ac}(\tilde{F})$-squares are the image by $\tilde{\epsilon}$ of all the $\text{ac}(\tilde{\tilde{F}}^I)$-pull-backs.

**Proof:** In order to prove that $\text{ac}(\tilde{\tilde{F}}^I)$ admits pull-backs, it suffices to consider any triple of subgroups $Q$, $R$ and $T$ of $P$, and any pair of $\tilde{\tilde{F}}^I$-morphisms $\tilde{x}: R \to Q$ and $\tilde{y}: T \to Q$. In this case, choose a pair of representatives $x \in T_G(R,Q)$ of $\tilde{x}$ and $y \in T_G(T,Q)$ of $\tilde{y}$, and a set of representatives $W \subset Q$ for the set of double classes $xR\setminus Q/yT$; moreover, for any $w \in W$, set $U_w = xR \cap wy(T)$, respectively denote by $x_w: U_w \to R$ and $y_w: U_w \to T$ the $\tilde{\tilde{F}}^I$-morphisms mapping $u \in U_w$ on $u \tilde{x}$ and on $u \tilde{y}$, and consider the two $\text{ac}(\tilde{\tilde{F}}^I)$-morphisms

$$\bigoplus_{w \in W} x_w: \bigoplus_{w \in W} U_w \to R \quad \text{and} \quad \bigoplus_{w \in W} y_w: \bigoplus_{w \in W} U_w \to T$$

respectively determined by the families $\{x_w\}_{w \in W}$ and $\{y_w\}_{w \in W}$. Then, we claim that the following $\text{ac}(\tilde{\tilde{F}}^I)$-diagram is a pull-back

$$
\begin{array}{cccccc}
& & Q & & \\
& & \uparrow \tilde{x} & & \\
& & R & & \leftarrow \tilde{y} & \\
\bigoplus_{w \in W} x_w & \downarrow & \bigoplus_{w \in W} U_w & \leftarrow & \bigoplus_{w \in W} y_w & \\
& & & & \\
& & & & \\
\end{array}
$$

Indeed, if $S$ is a subgroup of $P$ and we have two $\tilde{\tilde{F}}^I$-morphisms $\tilde{a}: S \to R$ and $\tilde{b}: S \to T$ fulfilling $\tilde{x} \tilde{a} = \tilde{y} \tilde{b}$ then, choosing representatives $a$ in $\tilde{a}$ and $b$ in $\tilde{b}$, there exists $u \in Q$ such that $xa = uyb$ and it is easily checked that the double class of $u$ in $xR\setminus Q/yT$ does not depend on our choices of $a$ and $b$; moreover, denoting by $w \in W$ the representative of the double class of $u$ and choosing $r \in R$ and $t \in T$ fulfilling $u = (x^r)^{-1}w^yt$, we get

$$xra = wytb$$

and therefore $xraS$ is contained in $U_w$; thus, we obtain an $\tilde{\tilde{F}}^I$-morphism $\tilde{xra}: S \to U_w$ and therefore an $\text{ac}(\tilde{\tilde{F}}^I)$-morphism

$$S \to \bigoplus_{w \in W} U_w$$
determined by the element $w \in W$ and by the $\tilde{F}^w$-morphism $\tilde{x}ra : S \to U_w$ which clearly fulfills
\[
\tilde{x}_w \circ \tilde{x}ra = \tilde{a} \quad \text{and} \quad \tilde{y}_w \circ \tilde{x}ra = \tilde{b}
\]
6.6.6.

Conversely, for any $\ac(\tilde{F})$-morphism $S \to \bigoplus_{w' \in W} U_{w'}$ determined by a suitable $w' \in W$ and by a $\tilde{F}^w$-morphism $\tilde{c} : S \to U_{w'}$ fulfilling $\tilde{x}_w \circ \tilde{c} = \tilde{a}$ and $\tilde{y}_w \circ \tilde{c} = \tilde{b}$, it is easily checked that $w' = w$ and that $\tilde{c} = \tilde{x}ra$. Now, the last statement is clear.

6.7. Any contravariant functor $a : \tilde{F} \to O\text{-mod}$ determines an additive contravariant functor $a^\circ : \ac(\tilde{F}) \to O\text{-mod}$; let us say that a functor $a^\circ : \tilde{F} \to O\text{-mod}$ is a compatible complement of $a$ if it fulfills the following three conditions:

6.7.1 For any subgroup $Q$ of $P$ we have $a^\circ(Q) = a(Q)$.

6.7.2 For any $\tilde{F}$-morphism $\tilde{\varphi} : R \to Q$ we have

\[
a^\circ(\tilde{\varphi}) \circ a(\tilde{\varphi}) = \frac{|Q|}{|R|} \cdot \text{id}_{a(Q)}.
\]

6.7.3 For any special $\ac(\tilde{F})$-square $R \to T$ we have the commutative $O\text{-mod}$-diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\tilde{\varphi}} & Q \\
\downarrow{\tilde{\varphi}} & & \downarrow{\tilde{\varphi}} \\
U & \xrightarrow{\tilde{\gamma}} & \delta
\end{array}
\]

Note that the equality in 6.7.2 forces $(a^\circ)^{\ast \ast}(\tilde{\varphi}) = (a^\circ)^{\ast \ast} - 1$ for any $\ac(\tilde{F})$-isomorphism $\tilde{\varphi}$. Actually, these conditions amount to saying that, in terms of [1], the obvious functor

\[
(a^\circ \circ \tilde{\varphi})^{\ast \ast} : \ac(\tilde{F}^x) \to O\text{-mod}
\]

is a cohomological Mackey complement of the contravariant functor

\[
(a \circ \tilde{\varphi})^{\ast \ast} : \ac(\tilde{F}^x) \to O\text{-mod}
\]

6.7.4

Proposition 6.8. With all the notation above, any contravariant functor $a : \tilde{F} \to O\text{-mod}$ admitting a compatible complement $a^\circ$ is $H$-split. In particular, for any $n \geq 1$ we have $H^n_{\tilde{G}}(\tilde{F}, a) = \{0\}$. 


**Proof:** Denoting by $\tilde{a}: \tilde{F} \to \mathcal{O}\text{-mod}$ the composition of $a$ with the structural functor, it follows from 2.7 that there is a *natural map* $\Delta_H(a): \tilde{a} \to H(a)$ sending any subgroup $Q$ of $P$ to the group homomorphism

$$\Delta_H(a)_Q : a(Q) \longrightarrow \big( \prod_{\omega \in \Omega} a(Q_\omega) \big)^{Q \times P}$$  \hspace{1cm} 

mapping $a \in a(Q)$ on

$$(\Delta_H(a)_Q)(a) = \sum_{\omega \in \Omega} (a(i_\omega^Q))(a)$$  \hspace{1cm} 

First of all note that, for any subgroup $Q$ of $P$ and any $\omega \in \Omega$, we have (cf 6.7.1)

$$(a^\circ(i_\omega^Q))(\big(a(i_\omega^Q)\big)(a)) = \frac{|Q|}{|Q_\omega|} a = a$$  \hspace{1cm} 

Then, choosing a set of representatives $\Gamma_Q \subset \Omega$ for the set of $Q \times P$-orbits in $\Omega$, we consider the map

$$\theta_Q : \big( \prod_{\omega \in \Omega} a(Q_\omega) \big)^{Q \times P} \longrightarrow a(Q)$$  \hspace{1cm} 

sending any element $\sum_{\omega \in \Omega} a_\omega$ of $(H(a))(Q)$, where $a_\omega$ belongs to $a(Q_\omega)$ for any $\omega \in \Omega$, to

$$\theta_Q\left( \sum_{\omega \in \Omega} a_\omega \right) = \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_Q} \left( a^\circ(i_\omega^Q) \right)(a_\omega)$$  \hspace{1cm} 

in particular, for any $a \in a(Q)$ we get (cf. 6.8.2 and 6.8.3)

$$\theta_Q\left( \Delta_H(a)_Q(a) \right) = \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_Q} \left| \frac{Q_\omega}{Q_\omega} \right| a$$

$$= \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_Q} \frac{|Q_\omega| \cdot |P|}{|P|} a = a$$  \hspace{1cm} 

Moreover, it easily follows from 6.4.2 that the map $\theta_Q$ does not depend on the choice of $\Gamma_Q$.

Consequently, according to Theorem 3.5, it suffices to prove that the correspondence mapping any subgroup $Q$ of $P$ on $\theta_Q$ is a *natural map*. Let $(x, u): R \to Q$ be an $\tilde{F}$-morphism and denote by $\tilde{x}: R \to Q$ its image in $\tilde{F}(Q, R)$; we claim that the following diagram is commutative

$$
\begin{array}{c}
\prod_{\omega \in \Omega} a(Q_\omega) \quad \theta_Q \quad a(Q) \\
\downarrow (H(a))(x, u) \downarrow a(\tilde{x})
\end{array}
$$

\hspace{1cm} 

$$
\begin{array}{c}
\prod_{\omega \in \Omega} a(R_\omega) \quad \theta_R \quad a(R) \\
\downarrow (H(a))(x, u) \downarrow a(\tilde{x})
\end{array}
$$  \hspace{1cm} 

6.8.7;
indeed, it follows from definition 6.8.5 that

\[(a(\tilde{x}) \circ \theta_Q)(\sum_{\omega \in \Omega} a_\omega) = \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_Q} (a(\tilde{x}) \circ a^\circ(t^Q_\omega)) (a_\omega) \quad 6.8.8.\]

But, for any \(\omega \in \Gamma_Q\), choosing a set of representatives \(V_\omega \subset Q\) for the set of double classes \(t^Q_\omega Q_\omega \backslash Q / ^x R\), so that we have

\[Q = \bigcup_{\omega \in V_\omega} (^x R)^{-1}(t^Q_\omega Q_\omega) \quad 6.8.a,\]

and setting

\[T_{\omega, v} = ^x R \cap t^Q_\omega Q_\omega \quad \text{and} \quad x_{\omega, v} = vx \quad 6.8.9\]

for any \(v \in V_\omega\), the two \(a(\tilde{F})\)-morphisms

\[\tilde{\gamma}_{\omega} : \bigoplus_{v \in V_\omega} T_{\omega, v} \rightarrow R \quad \text{and} \quad \tilde{\delta}_{\omega} : \bigoplus_{v \in V_\omega} T_{\omega, v} \rightarrow Q_\omega \quad 6.8.10\]

determined by the families of \(\tilde{F}\)-morphisms

\[\tilde{\gamma}_{\omega, v} = \widetilde{x_{\omega, v}}^{-1} : T_{\omega, v} \rightarrow R \quad \text{and} \quad \tilde{\delta}_{\omega, v} = \widetilde{t^Q_\omega}^{-1} : T_{\omega, v} \rightarrow Q_\omega \quad 6.8.11\]

form a special \(a(\tilde{F})\)-square (cf. 6.1 and 6.2)

\[
\begin{array}{ccc}
& & Q \\
& t^Q_\omega & \leftarrow \ \\
Q_\omega & \leftarrow & R \\
\Downarrow & \ \leftarrow & \uparrow \ \leftarrow \quad \tilde{\gamma}_{\omega} \\
\Theta_{v \in V_\omega} T_{\omega, v} & & \\
\end{array}
\]

Consequently, we still have the commutative \(\mathcal{O}\)-\textit{mod}-diagram

\[
\begin{array}{ccc}
a(Q) & \leftarrow & a(\tilde{x}) \\
a^\circ(t^Q_\omega) & \leftarrow & \quad a(Q_\omega) \quad \quad a(R) \\
a^{\ast\ast}(\tilde{\delta}_\omega) & \leftarrow & \quad \prod_{v \in V_\omega} a(T_{\omega, v}) \quad (a^{\ast\ast}(\tilde{\gamma}_\omega)) \\
\end{array}
\]

and therefore equality 6.8.8 becomes

\[(a(\tilde{x}) \circ \theta_Q)(\sum_{\omega \in \Omega} a_\omega) = \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_Q} \sum_{v \in V_\omega} (a^\circ(x_{\omega, v}^{-1}) \circ a(\tilde{t^Q_\omega}^{-1}))(a_\omega) \quad 6.8.14.\]
On the other hand, choosing a set of representatives $\Gamma_R \subset \Omega$ for the set of $R \times P$-orbits in $\Omega$, it follows again from definition 6.8.5 and from 6.4.2 that

$$
(\theta_R \circ (\mathcal{H}(a))(x, u)) \left( \sum_{\omega \in \Omega} a_\omega \right) = \frac{|P|}{|\Omega|} \sum_{\omega \in \Gamma_R} (a^\circ (t^\sim_{\omega} u) \circ a(\tilde{u})) (a_{x^{-1} u^{-1}}) \quad 6.8.15.
$$

But, for any $\omega \in \Gamma_Q$, it is clear that we have

$$
Q \omega \cdot P = \bigsqcup_{v \in V_\omega} (^v R)^{-1} \omega \cdot P = \bigsqcup_{v \in V_\omega} x R x^{-1} v^{-1} \omega \cdot P \quad 6.8.16
$$

and, in particular, we can choose

$$
\Gamma_R = \bigsqcup_{\omega \in \Gamma_Q} x^{-1} (V_\omega)^{-1} \omega \cdot u \quad 6.8.17
$$

as a set of representatives; in this case, for any $\omega \in \Gamma_Q$ and any $v \in V_\omega$, the terms respectively corresponding to the pair $(\omega, v)$ and to $x^{-1} v^{-1} \omega \cdot u \in \Gamma_R$ in the right-hand members of equalities 6.8.14 and 6.8.15 coincide with each other, as it can be checked from the following commutative diagram

\[
\begin{array}{ccc}
\alpha(Q_\omega) & \xrightarrow{\alpha(t^\sim_{\omega} v^{-1})} & \alpha(T_{\omega, v}) \\
\parallel & & \alpha(t^\sim_{\omega} u) \downarrow & \alpha(t^\sim_{v^{-1} \omega} u) \downarrow & \alpha(R) \\
\alpha(Q_{v^{-1} \omega}) & \xrightarrow{\alpha(\tilde{u})} & \alpha(R_{x^{-1} v^{-1} \omega \cdot u}) & \xrightarrow{\alpha(t^\sim_{x^{-1} v^{-1} \omega \cdot u})} & \alpha(R)
\end{array}
\]

We are done.

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