Extended Formulations for Polytopes of Regular Matroids

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Abstract

We present a simple proof of the fact that the base (and independence) polytope of a rank \(n\) regular matroid over \(m\) elements has an extension complexity \(O(mn)\).

1 Introduction

Consider a matroid \(M = (E, \mathcal{I})\) where \(E\) is a finite set of cardinality \(m\) and \(\mathcal{I} \subseteq 2^E\). Let \(\mathcal{B} \subseteq \mathcal{I}\) be the set of bases of \(E\), i.e., inclusionwise maximal sets in \(\mathcal{I}\). It is known that all the sets in \(\mathcal{B}\) have the same cardinality, which is called the rank of the matroid and we denote it by \(n\). For every \(I \in \mathcal{I}\), we can associate a vector \(1_I \in \{0,1\}^E\) that is its indicator vector. Let \(P(I) \subset \mathbb{R}^m\) denote the independence polytope of \(I\) that is obtained by taking the convex hull of the vectors \(1_I\) for all \(I \in \mathcal{I}\). Similarly, let \(P(\mathcal{B}) \subset \mathbb{R}^m\) denote the base polytope of \(I\) that is obtained by taking the convex hull of the vectors \(1_I\) for all \(I \in \mathcal{B}\). While the number of facets required to describe such polytopes is typically exponential, a question that has received a great deal of attention is for what matroids the extension complexity of \(P(I)\) is small. An extension of a polytope \(P\) is another polytope \(Q\) so that \(P\) is the image of \(Q\) under a linear map and the extension complexity of \(P\), denoted \(xc(P)\), is the smallest size extension of \(P\) where size is measured by the number of facets in the extension. [Rot13] gave an existential argument that there are matroids whose independence polytope has an exponential extension complexity. On the other hand, [KLWW16] proved that the independence polytopes of regular matroids have a polynomial extension complexity. In this paper we give a simple proof of this latter result. In particular we prove the following theorem.

Theorem 1.1 Let \(M = (E, \mathcal{I})\) be a regular matroid with \(|E| = m\) and rank \(n\). Then there is a polytope \(Q_M \subset \mathbb{R}^m \times \mathbb{R}^{2mn}\) such that

1. the number of facets in the description of \(Q_M\) is at most \(2mn + m\),
2. the number of equality constraints in the description of \(Q_M\) is \(n^2 + 1\), and
3. the projection of \(Q_M\) on to the first \(m\) coordinates is exactly \(P(\mathcal{B})\).

In other words, \(xc(P(\mathcal{B})) \leq 2mn + m\).

By a standard reduction, our result implies that the extension complexity of the independence polytope of a regular matroid is also \(O(mn)\) (see, for example, [KLWW16, Lemma 2.1]). Our proof is inspired by the construction of Wong [Won80] for the spanning tree polytope (presented in [Wol11]) and does not rely on the non-trivial decomposition theorem of Seymour [Sey80].


2 Proof of Theorem 1.1

We start with a well-known characterization of regular matroids: a matroid \( M = (E, I) \) is regular if and only if it can be realized by the columns of a totally unimodular matrix \( A_M \in \mathbb{R}^{n \times m} \). Recall that a totally unimodular matrix has all its minors 0, 1 or \(-1\). In particular this means that all the entries of \( A_M \) are from \{-1, 0, 1\} and the inverse of any of its (invertible) submatrices has entries from \{-1, 0, 1\}. For simplicity of notation, let \( A \) denote \( A_M \) and let \( E = \{1, 2, \ldots, m\} \). The collection of base sets \( \mathcal{B} \) is

\[
\mathcal{B} := \{ S \subseteq E \mid |S| = n \text{ and } \text{rank}(A_S) = n \},
\]

where \( A_S \) be the \( n \times n \) submatrix of \( A \) with columns indexed by \( S \).

We work with \( 2mn \) variables \( \{y_{i,j}\} \) and \( \{z_{i,j}\} \) for \( 1 \leq i \leq m, 1 \leq j \leq n \). Let \( Y = (y_{i,j}) \) and \( Z = (z_{i,j}) \) each be an \( m \times n \) matrix and \( I_n \) denote the \( n \times n \) identity matrix. Consider the polytope \( P \) defined by

\[
ya_{i,j}, z_{i,j} \geq 0 \quad \forall i, j \tag{1}
\]

\[
(A \ A) \begin{pmatrix} Y \\ -Z \end{pmatrix} = I_n. \tag{2}
\]

Claim 2.1 The polytope \( P \) has integral vertices.

Proof: For \( 1 \leq j \leq n \), consider the polytope \( P_j \) in variables \( \{y_{1,j}, \ldots, y_{m,j}\} \) and \( \{z_{1,j}, \ldots, z_{m,j}\} \) defined by

\[
ya_{i,j}, z_{i,j} \geq 0 \quad \forall i, \tag{3}
\]

\[
(A \ A) \begin{pmatrix} Y_j \\ -Z_j \end{pmatrix} = e_j, \tag{4}
\]

where \( Y_j \) and \( Z_j \) are the \( j \)-th columns of \( Y \) and \( Z \), respectively, and \( e_j \) is the \( j \)-th elementary unit vector. We first claim that polytope \( P_j \) is integral. Observe that \((A \ A)\) is totally unimodular. To see this, consider any square submatrix of \((A \ A)\) – either the submatrix takes two copies of the same column from \( A \), in which case its determinant is zero. Or it takes at most one copy of each column, in which case it is a submatrix of \( A \) and has determinant 0, 1 or \(-1\).

Since \( P_j \subset \mathbb{R}^{2m} \), any vertex of \( P_j \) is obtained by intersecting \( 2m \) bounding hyperplanes. That is, every vertex is obtained as a solution of the following: for some sets \( S, T \subseteq E \) with \( |S| + |T| = n \),

\[
(A_S \ A_T) \begin{pmatrix} Y_{S,j} \\ -Z_{T,j} \end{pmatrix} = e_j
\]

and, \( y_{i,j} = 0 \) for \( i \not\in S \) and \( z_{i,j} = 0 \) for \( i \not\in T \). The vector \( \begin{pmatrix} Y_{S,j} \\ -Z_{T,j} \end{pmatrix} \) is the \( j \)-th column of \((A_S \ A_T)^{-1}\), whose entries are from \{-1, 0, 1\}, since \((A \ A)\) is totally unimodular. Since \( P_j \) satisfies \( y_{i,j}, z_{i,j} \geq 0 \), together we obtain that all vertices of \( P_j \) come from \( \{0, 1\}^{2m} \).

Now, note that

\[
P = \{(y, z) \mid (y_j, z_j) \in P_j \text{ for all } 1 \leq j \leq n \}.
\]

We claim that any vertex of \( P \) projects to a vertex of \( P_j \). Let us say \((y, z)\) is a point in \( P \) with its projection \((y_j, z_j)\) not being a vertex of \( P_j \). Then \((y_j, z_j)\) can be written as a non-trivial convex
combination of two points \((a, b)\) and \((c, d)\) in \(P_{j}\). Define \((y', z')\) and \((y'', z'')\) which are same as \((y, z)\) except \((y'_j, z'_j) = (a, b)\) and \((y''_j, z''_j) = (c, d)\). Clearly, \((y, z)\) is a non-trivial convex combination of \((y'_j, z'_j)\) and \((y''_j, z''_j)\) which are in \(P\). Thus, \((y, z)\) cannot be a vertex of \(P\). Hence, vertices of \(P\) come from \(\{0, 1\}^{2mn}\).

Now, we introduce new variables \({x_i}_{i=1}^{m}\) and add the following constraints for \(1 \leq i \leq m, 1 \leq j \leq n\),

\[
x_i \geq y_{i,j} + z_{i,j}.
\]

(3)

Let the new polytope described by (1), (2) and (3) be \(Q\).

**Claim 2.2** Any vertex of \(Q\) must project to a vertex of \(P\).

**Proof:** The only constraints where \(x_i\) appears are (3). Let \((x, y, z)\) be a vertex of \(Q\). Then it must be the case that \(x_i = y_{i,j} + z_{i,j}\) for some \(j\), for each \(i - \) if not then one can find a nonzero vector \(\varepsilon\) such that both \((x + \varepsilon, y, z)\) and \((x - \varepsilon, y, z)\) are in \(Q\), implying that \((x, y, z)\) is not a vertex. Now, suppose \((y, z)\) is a vector in \(P\) is not a vertex. Then it can be written as a non-trivial convex combination of two points in \(P\), say \((y', z')\) and \((y'', z'')\). Define \(x'\) and \(x''\) as \(x'_i = z'_{i,j} + y'_{i,j}\) and \(x''_i = z''_{i,j} + y''_{i,j}\) for each \(i\). Then \((x, y, z)\) is a non-trivial convex combination of \((x', y', z')\) and \((x'', y'', z'')\) and thus, is not a vertex.

Thus, \(Q\) also has integral vertices. Now, we argue that the vertices of \(Q\) come from full-rank sets.

**Claim 2.3** Let \(S \subseteq E\) be a set with rank\((A_S) < n\) and \(x \in \mathbb{R}^m\) be a vector supported on \(S\). Then for any \((y, z) \in \mathbb{R}^{2mn}\), \((x, y, z) \notin Q\).

**Proof:** Let us say \((x, y, z) \in Q\). For any \(i \in \overline{S}\), \(x_i = 0\), which implies \(z_{i,j} = 0\) and \(y_{i,j} = 0\) for all \(j\) (from (3)). This means that in (2), only the columns of \(A_S\) can contribute. As rank\((A_S) < n\), we know rank\((A_S A_S)\) cannot be equal to the identity matrix.

**Claim 2.4** Let \(T \subseteq E\) be a set with rank\((A_T) = n\) and \(|T| = n\). Let \(x \in \{0, 1\}^m\) be its indicator vector. Then there exists \((y, z) \in \mathbb{R}^{2mn}\) such that \((x, y, z) \in Q\).

**Proof:** Consider the matrix \(A_T^{-1}\), which has entries from \([-1, 0, 1]\), since \(A\) is totally unimodular. The rows of \(A_T^{-1}\) will be indexed by the elements in \(T\) and columns will be indexed by \(1 \leq j \leq n\). For each \(i \notin T\), assign \(y_{i,j} = 0\) and \(z_{i,j} = 0\) for all \(j\). For each \(i \in T\) and \(1 \leq j \leq n\), assign

\[
\begin{align*}
\text{if } A_T^{-1}(i, j) = 0 & \text{ then } y_{i,j} = 0, z_{i,j} = 0, \\
\text{if } A_T^{-1}(i, j) = 1 & \text{ then } y_{i,j} = 1, z_{i,j} = 0, \\
\text{if } A_T^{-1}(i, j) = -1 & \text{ then } y_{i,j} = 0, z_{i,j} = 1.
\end{align*}
\]

Now, let us verify that \((y, z)\) satisfies (2). As \(y_{i,j} = z_{i,j} = 0\) for \(i \notin T\). We can write the l.h.s. of (2) as

\[
(A_T A_T) \begin{pmatrix} Y_T \\ -Z_T \end{pmatrix} = A_T(Y_T - Z_T).
\]

But, note that \(A_T^{-1}(i, j) = Y_T(i, j) - Z_T(i, j)\). In other words, \(A_T^{-1} = Y_T - Z_T\) and thus,

\[
A_T(Y_T - Z_T) = I_n.
\]
Now, we verify that the vector $x$ satisfies (3). Observe that $y_{i,j} + z_{i,j} \leq 1$ when $i \in T$ and $y_{i,j} + z_{i,j} = 0$ otherwise. As $x$ is the indicator vector of $T$ we can see that for each $i, j$

$$x_i \geq y_{i,j} + z_{i,j}.$$  

Finally, we show that the base polytope $P(\mathcal{B})$ comes from a face of $Q$.

**Claim 2.5** For any point $(x, y, z) \in Q$, $\sum_{i=1}^{m} x_i \geq n$.

**Proof:** It suffices to prove the claim for the vertices of $Q$. Let $(x, y, z)$ be a vertex of $Q$, which we have seen is integral. From Claim 2.3 $x$ has support at least $n$. Since, $x$ is integral and $x \geq 0$, we get that $\sum_{i=1}^{m} x_i \geq n$. \qed 

Claim 2.4 implies that there exist points $(x, y, z) \in Q$ with $\sum_{i=1}^{m} x_i = n$, namely when $x$ is the indicator vector of a base set $T$. Thus from Claim 2.5 adding the following equation with $Q$ gives us a face of $Q$.

$$\sum_{i=1}^{m} x_i = n. \quad (4)$$

To conclude the proof of Theorem 1.1 note that (1), (2), (3) and (4) describe the extension polytope, whose projection on to the $x$ coordinates is the base polytope $P(\mathcal{B})$.

**References**

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