Choiceless Löwenheim-Skolem property and uniform definability of grounds

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Abstract. In this paper, without the axiom of choice, we show that if a certain downward Löwenheim-Skolem property holds then all grounds are uniformly definable. We also prove that the axiom of choice is forceable if and only if the universe is a small extension of some transitive model of ZFC.

Keywords: Axiom of Choice, Forcing method, Set-theoretic geology

1 Introduction

Set-theoretic geology, which was initiated by Fuchs-Hamkins-Reitz [2], is a study of the structure of all ground models of the universe. In standard set-theoretic geology, the universe is assumed to be a model of ZFC, and all ground models are also supposed to satisfy ZFC. On the other hand, it is possible that the universe is a generic extension of some choiceless model, moreover in modern set theory, the forcing method over choiceless models has become a common tool, e.g., Woodin’s P-max forcing over L(\mathbb{R}). So it is natural to consider set-theoretic geology without the Axiom of Choice (AC). The base theory in this paper is ZF unless otherwise specified. Let us say that a transitive model W of ZF is a ground of V if there is a poset P \in W and a (W, P)-generic G with V = W[G]. V is a trivial ground of V. Again, we do not assume that a ground satisfies AC, unless otherwise specified.

A first problem in developing set-theoretic geology without AC is the uniform definability of all grounds. Laver [12], and independently Woodin, proved that, in ZFC, the universe V is definable in its forcing extension V[G] by a first-order formula with parameters. Fuchs-Hamkins-Reitz [2] refined their result and showed that, in ZFC, all grounds are uniformly definable by a first-order formula:

Theorem 1 (Fuchs-Hamkins-Reitz [2], Reitz [14], in ZFC). There is a first-order formula \varphi(x, y) of set-theory such that:

1. For every set r, the class W_r = \{x | \varphi(x, r)\} is a ground of V with r \in W_r, and satisfies AC.
2. For every ground W of V, if W satisfies AC then W = W_r for some r.
First-order definability is an important property, which allows us to treat all grounds within the first-order theory ZFC. However, their proofs heavily rely on AC, and it is still open if all grounds are uniformly definable without AC. Gitman-Johnstone [5] obtained a partial result under a fragment of AC. For instance, they showed that if DCδ holds and a poset P has cardinality ≤ δ (P is assumed to be well-ordeable), then the universe V is definable in its forcing extension via P. For this problem, we give another partial answer. We prove that if a certain downward Löwenheim-Skolem property holds, then all grounds are uniformly definable as in Theorem 1. We also show that such a downward Löwenheim-Skolem property holds in many natural models of ZF, or if V has many large cardinals. So we can start studying set-theoretic geology in many choiceless models.

We introduce the following notion, which corresponds to the Löwenheim-Skolem theorem in the context of ZFC:

Definition 1. An uncountable cardinal κ is a Löwenheim-Skolem cardinal (LS cardinal, for short) if for every γ < κ ≤ α and x ∈ Vα, there is β > α and an elementary submodel X ⊆ Vβ such that:
1. Vγ ⊆ X.
2. x ∈ X.
3. The transitive collapse of X belongs to Vκ.
4. Vγ(X ∩ Vα) ⊆ X.

Clearly a limit of LS cardinals is also an LS cardinal, hence a singular LS cardinal can exist. In ZFC, a cardinal κ is LS if and only if κ = ℶκ, so there are proper class many LS cardinals.

In ZF, every supercompact cardinal (see Definition 3 below) is an LS cardinal. We show that if there are proper class many LS cardinals, e.g., there are proper class many supercompact cardinals, then all grounds are uniformly definable.

Theorem 2. Suppose there are proper class many LS cardinals. Then all grounds are uniformly definable, that is, there is a first-order formula φ(x, y) of set-theory such that:
1. For every r, Wr = {x | φ(x, r)} is a ground of V with r ∈ Wr.
2. For every ground W of V, there is r with Wr = W.

We also prove that the statement “there are proper class many LS cardinals” is absolute between V and its forcing extensions. Hence under the assumption, in any grounds and generic extensions of V, we can define all its grounds uniformly.

In ZFC, there are proper class many LS cardinals, which is a consequence of the Löwenheim-Skolem theorem. This means that if there is a poset which forces AC, then we can conclude that V has proper class many LS cardinals. Let us say that AC is forceable if there is a poset which forces AC. This result lead us to the problem of when AC is forceable. Blass [1] already considered a necessary and sufficient condition for it. The principle SVC, Small Violation of Choice, is the assertion that there is a set X such that for every set Y, there is an ordinal α and a surjection f : X × α → Y.
Theorem 3 (Blass [1]). The following are equivalent:

1. AC is forceable.
2. SVC holds.

Blass also showed that SVC holds in many choiceless models, such as symmetric models. So such models have proper class many LS cardinals, and all grounds are uniformly definable. We give another characterization, which tells us that AC is forceable if and only if $V$ is a small extension of a model of ZFC. For a transitive model $W$ of ZF and a set $X$, let $W(X)$ be the minimal transitive model of ZF with $W \subseteq W(X)$ and $X \in W(X)$ (see Definition 2 below).

Theorem 4. The following are equivalent:

1. AC is forceable.
2. There is a transitive model $W$ of ZFC and a set $X$ such that $W$ is definable in $V$ with parameters from $W$ and $V = W(X)$.
3. There is a transitive model $W$ of ZFC and a set $X$ such that $W$ is definable in $V$ with parameters from $W$, $V = W(X)$, and $W$ is a ground of some generic extension of $V$.

This characterization clarifies the structure of all grounds of $V$ under AC.

Theorem 5. Suppose $V$ satisfies AC. Then for every transitive model $M$ of ZF, $M$ is a ground of $V$ if and only if there is a ground $W$ of $V$ and a set $X$ such that $W$ satisfies AC and $M = W(X)$. In particular the collection of all grounds satisfying AC is dense in all grounds, with respect to $\subseteq$.

2 Preliminaries

In this paper, we say that a collection $M$ of sets is a class of $V$ if $(V; \in, M)$ satisfies the collection scheme, that is, for every formula $\varphi$ in the language $\{\in, M\}$ (where we identify $M$ as a unary predicate) and all sets $a, v_0, \ldots, v_n$, if the sentence $\forall b \in a \exists c \varphi(b, c, v_0, \ldots, v_n)$ holds in $V$, then there is a set $d$ such that the sentence $\forall b \in a \exists c \in d \varphi(b, c, \ldots, v_n)$ holds. A class needs not be definable in $V$, but every definable collection of sets is a class in our sense. Note also that, by the forcing theorem, if $W$ is a ground of $V$, then $W$ is a class of $V$.

The following fact is well-known. See e.g. Jech [7] for the definitions and the proof.

Theorem 6. Let $M$ be a transitive class containing all ordinals. Then $M$ is a model of ZF if and only if $M$ is closed under the Gödel operations and $M$ is almost universal, that is, for every set $x \subseteq M$, there is $y \in M$ with $x \subseteq y$.

For a transitive model $M$ of ZF and an ordinal $\alpha$, let $M_\alpha$ be the set of all $x \in M$ with rank $< \alpha$.

We can develop a standard theory of the forcing method without AC. See e.g. Grigorieff [4] for the following facts:

...
Theorem 7. Let $V[G]$ be a forcing extension of $V$ via a poset $P \in V$, and $V[G][H]$ of $V[G]$ via a poset $Q \in V[G]$. Then there is a poset $R \in V$ and a $(V, R)$-generic $G'$ such that $V[G][H] = V[G']$.

This fact shows that if $M$ is a ground of $V$ and $W$ is of $M$, then $W$ is a ground of $V$ as well.

A poset $P$ is weakly homogeneous if for every $p, q \in P$, there is an automorphism $f : P \to P$ such that $f(p)$ is compatible with $q$.

Theorem 8. Suppose $P$ is a weakly homogeneous poset. For every $x_0, \ldots, x_n \in V$ and formula $\varphi$, either $\models_P \varphi(x_0, \ldots, x_n)$ or $\models_P \neg \varphi(x_0, \ldots, x_n)$ in $V$.

For a set $S$, let $\text{Col}(S)$ be the poset consisting of all finite partial functions $\mathcal{F}$ from $\omega$ to $S$ ordered by reverse inclusion. $\text{Col}(S)$ is weakly homogeneous, and if $S$ is an ordinal definable set then so is $\text{Col}(S)$.

Theorem 9. Let $P$ be a poset, and $G$ be $(V, P)$-generic. Let $\alpha$ be a limit ordinal with $\alpha > \text{rank}(P) \cdot \omega$. Let $H$ be $(V[G], \text{Col}(V[G]_\alpha))$-generic. Then there is a $(V, \text{Col}(V_\alpha))$-generic $H'$ with $V[G][H] = V[H']$.

Definition 2. For a transitive model $M$ of ZF containing all ordinals and a set $X$, let $M(X) = \bigcup_{\alpha \in \text{ON}} L(M_\alpha \cup \{X\})$. If $M$ is a class of $V$, then $M(X)$ is the minimal transitive class model of ZF with $M \subseteq M(X)$ and $X \in M(X)$.

The following useful fact will be applied frequently:

Theorem 10 (Theorem B in Grigorieff [4]). Let $W \subseteq V$ be a ground of $V$. Let $M$ be a transitive model of ZF and suppose $W \subseteq M \subseteq V$. Then the following are equivalent:

1. $V$ is a generic extension of $M$.
2. $M$ is of the form $W(X)$ for some $X \in M$.

We also use the following fact due to Solovay.

Theorem 11 (Solovay, see Fuchs-Hamkins-Rietz [2]). Let $P$, $Q$ be posets, and $G \times H$ be $(V, P \times Q)$-generic. Then $V[G] \cap V[H] = V$.

Lemma 1 (Folklore). Let $P$ be a poset, and $\alpha > \omega$ a limit ordinal with $P \in V_\alpha$. Let $G$ be $(V, P)$-generic. For a set $Y \in V$, let $Y[G] = \{\dot{a}_G \mid \dot{a} \in Y$ is a $P$-name$, \}$, where $\dot{a}_G$ is the interpretation of $\dot{a}$ by $G$. Then $V[G]_\alpha = V_\alpha[G]$.

Proof (Sketch of proof). One can check that for every $P$-name $\dot{a}$, we have $\text{rank}(\dot{a}_G) \leq \text{rank}(\dot{a})$, hence $V_\alpha[G] \subseteq V[G]_\alpha$. For the converse, by induction on $\beta < \alpha$ with $P \in V_\beta$, we can take a $P$-name $\dot{\sigma}$ such that rank($\dot{\sigma}$) $< \beta + \omega$ $\leq \alpha$ and $\models_P \dot{\sigma} = V[G]_\beta$ (we do not need AC). Hence if $P \in V_\alpha$ then $V[G]_\alpha \subseteq V_\alpha[G]$. □

1 In [4], our $M(X)$ is referred to $M[X]$. 
3 L"owenheim-Skolem cardinals

In this section we shall observe some basic properties of LS cardinals, but results in this section are not required to prove the main theorems.

First we prove that in ZF, every supercompact cardinal is an LS-cardinal.

**Definition 3** (Woodin, Definition 220 in [16]). An uncountable cardinal \( \kappa \) is supercompact if for every \( \alpha \geq \kappa \), there is \( \beta \geq \alpha \), a transitive set \( N \), and an elementary embedding \( j : V_\beta \to N \) such that the critical point of \( j \) is \( \kappa \), \( \alpha < j(\kappa) \), and \( V_\alpha \subseteq N \).

If \( \kappa \) is supercompact, then \( \kappa \) is regular and \( V_\kappa \) is a model of ZF.

**Lemma 2.** Every supercompact cardinal is an LS cardinal, and a limit of LS cardinals.

This lemma is an immediate consequence of the following result of Woodin. For an ordinal \( \gamma \), \( V_\gamma \prec \Sigma_1 \) means that \( V_\gamma \prec \Sigma_1 \) and for all \( \alpha < \gamma \), \( a \in V_\gamma \), and for all \( \Sigma_0 \)-formula \( \varphi(x,y) \), if there is \( b \in V \) such that \( \varphi(a,b) \) holds and \( V_\alpha b \subseteq b \) then there is \( b \in V_\gamma \) such that \( \varphi(a,b) \) holds and \( V_\alpha b \subseteq b \).

**Theorem 12** (Woodin, Lemma 222 in [16]). For an uncountable cardinal \( \kappa \), the following are equivalent:

1. \( \kappa \) is supercompact.
2. For all \( \gamma > \kappa \) such that \( V_\gamma \prec \Sigma_1 \) and for all \( a \in V_\gamma \), there exists \( \pi < \kappa, \pi \in V_{\gamma^+} \), and an elementary embedding \( j : V_{\gamma^+} \to V_{\gamma^+} \) with critical point \( \pi < \kappa \) such that \( j(\pi) = \kappa, j(\pi) = a \), and such that \( V_{\pi^+} \prec \Sigma_1 \).

In ZFC, the existence of proper class many LS cardinals is provable, and LS cardinal is not a large cardinal. However we will see that the existence of an LS cardinal is not provable from ZF.

**Definition 4.** An uncountable cardinal \( \kappa \) is weakly L"owenheim-Skolem (weakly LS, for short) if for every \( \gamma < \kappa \leq \alpha \) and \( x \in V_\alpha \), there is \( X \prec V_\alpha \) such that:

1. \( V_\gamma \subseteq X \).
2. \( x \in X \).
3. The transitive collapse of \( X \) belongs to \( V_\kappa \).

Clearly every LS cardinal is weakly LS.

**Lemma 3.** Let \( \kappa \) be a weakly LS cardinal.

1. For every \( x \in V_\kappa \), there is no surjection from \( x \) onto \( \kappa \).
2. For every cardinal \( \lambda \geq \kappa \) and \( x \in V_\kappa \), there is no cofinal map from \( x \) into \( \lambda^+ \). In particular \( \text{cf}(\lambda^+) \geq \kappa \).
Proof. (1). Suppose not. Then there is \( \gamma < \kappa \) and a surjection \( f \) from \( V_\gamma \) onto \( \kappa \).

Take a large \( \alpha \geq \kappa \) and \( X \prec V_\alpha \) such that \( V_\gamma \subseteq X \), \( \gamma, f \in X \), and the transitive collapse of \( X \) is in \( V_\kappa \). Clearly \( |X \cap \kappa| < \kappa \), otherwise the transitive collapse of \( X \) cannot be in \( V_\kappa \). However, since \( V_\gamma \subseteq X \) and \( f \in X \), we have \( \kappa = f^*V_\gamma \subseteq X \), this is a contradiction.

(2). Suppose to the contrary that there is a set \( x \in V_\kappa \) and a cofinal map \( f \) from \( x \) into \( \lambda^+ \). Fix \( \gamma < \kappa \) with \( x \in V_\gamma \). Take a large \( \alpha > \lambda^+ \) and \( X \prec V_\alpha \) such that:

1. \( V_\gamma \subseteq X \).
2. The transitive collapse of \( X \) is in \( V_\kappa \).
3. \( X \) contains all relevant objects.

Note that \( x \subseteq X \), hence \( f^*x \subseteq X \) and \( \lambda^+ = \sup(f^*x) = \sup(X \cap \lambda^+) \).

Let \( Y \) be the transitive collapse of \( X \), and \( \pi : X \rightarrow Y \) be the collapsing map. Now define \( g : Y \times \lambda \rightarrow \lambda^+ \) as follows: For \( (a, \beta) \in Y \times \lambda \), if \( \pi^{-1}(a) \) is a surjection from \( \lambda \) onto some ordinal \( < \lambda^+ \), then \( g(a, \beta) = \pi^{-1}(a)(\beta) \), and \( g(a, \beta) = 0 \) otherwise. Since \( \sup(X \cap \lambda^+) = \lambda^+ \), \( g \) is a surjection from \( Y \times \lambda \) onto \( \lambda^+ \). For \( \beta < \lambda \), let \( R_\beta = \{ g(a, \beta) \mid a \in Y \} \subseteq \lambda^+ \). We know \( \lambda^+ = \bigcup_{\beta < \lambda} R_\beta \).

If \( \operatorname{ot}(R_\beta) < \kappa \) for every \( \beta < \lambda \), we can take a canonical surjection from \( \lambda \times \kappa \) onto \( \lambda^+ \). However we can prove \( |\lambda \times \kappa| = \lambda \) in \( \text{ZF} \), hence \( |\lambda^+| = |\lambda \times \kappa| = \lambda \), so this is impossible. Thus there is \( \beta < \lambda \) with \( \operatorname{ot}(R_\beta) \geq \kappa \). This means that there is a surjection from \( Y \) onto \( \kappa \) via \( R_\beta \), contradicting (1).

Consider the model of \( \text{ZF} \) constructed by Gitik [3], which has no regular uncountable cardinals. By Lemma 3 there are no (weakly) LS cardinals in this model.

**Corollary 1.** An uncountable cardinal \( \kappa \) is an LS-cardinal if and only if for every set \( x \) and \( \gamma < \kappa \), there is \( \alpha \geq \kappa \) and \( X \prec V_\alpha \) such that \( x \in X \), \( V_\gamma \subseteq X \), \( V_\alpha \setminus X \subseteq X \), and the transitive collapse of \( X \) belongs to \( V_\kappa \).

Proof. The “if” part is clear. For the converse, take a set \( x \) and \( \gamma < \kappa \). By Lemma 3 we can find \( \alpha > \kappa \) such that \( x \in V_\alpha \), but there is no \( y \in V_\kappa \) for which there is a cofinal map \( f : y \rightarrow \alpha \). Since \( \kappa \) is LS, we can find \( \beta > \alpha \) and \( X' \prec V_\beta \) such that \( x, \alpha \in X', V_\gamma \subseteq X', V_\alpha \setminus X' \subseteq X' \), and the transitive collapse of \( X' \) is in \( V_\kappa \). Let \( X = X' \cap V_\alpha \). We have that \( X \prec V_\alpha \), \( x \in X \), and its transitive collapse belongs to \( V_\kappa \). Next take \( f : V_\gamma \rightarrow X \). We know \( f \in X' \). By the choice of \( \alpha \), the set \( \{ \text{rank}(f(z)) \mid z \in V_\gamma \} \) is bounded in \( \alpha \). Hence \( f \in V_\alpha \), and \( f \in X' \cap V_\alpha = X \).

Next we prove that if \( \kappa \) is weakly LS, then the club filter over \( \lambda^+ \) for \( \lambda \geq \kappa \) is \( \kappa \)-complete.

**Lemma 4.** Let \( \kappa \) be a weakly LS cardinal. Let \( \lambda \geq \kappa \) be a cardinal and \( x \in V_\kappa \). Let \( f \) be a function from \( x \) into the club filter over \( \lambda^+ \). Then \( \bigcap f^*x \) contains a club in \( \lambda^+ \). In particular, the club filter over \( \lambda^+ \) is \( \kappa \)-complete.
Proof. Take $\gamma < \kappa$ with $x \in V_\gamma$ and sufficiently large $\alpha > \lambda^+$. Take $X \prec V_\alpha$ such that $V_\gamma \subseteq X$, $f, \lambda^+ \in X$, and the transitive collapse of $X$ is in $V_\kappa$. Put $C = \{C \in X \mid C$ is a club in $\lambda^+\}$. We know that for every $a \in x$ there is a club $C \in C$ with $C \subseteq f(a)$. Let $D = \bigcap C \subseteq \bigcap f^* x$. It is enough to see that $D$ is a club in $\lambda^+$. Closure is clear, so we check that $D$ is unbounded in $\lambda^+$. Take $\xi < \lambda^+$. Fix $\delta < \kappa$ such that the transitive collapse of $X$ is in $V_\delta$. Again, take another large $\beta > \alpha$ and $Y \prec V_\beta$ such that $V_\delta \subseteq Y$, $X, \xi, D \in Y$, the transitive collapse of $Y$ is in $V_\kappa$. There is a surjection from $V_\delta$ onto $X$, hence we have $X \subseteq Y$, and $C \subseteq Y$. Note that $\sup(Y \cap \lambda^+) < \lambda^+$, otherwise we can take a cofinal map from the transitive collapse of $Y$ into $\lambda^+$, which contradicts to Lemma 8.

By Lemma 8 $\xi < \sup(Y \cap \lambda^+) < \lambda^+$, so it is sufficient to check that $\sup(Y \cap \lambda^+) \in D = \bigcap C$. For each $C \in C$, we have $C \subseteq Y$. Since $C$ is a club in $\lambda^+$ and $\sup(Y \cap \lambda^+) < \lambda^+$, we have $\sup(Y \cap \lambda^+) \in C$. □

We also show a variant of Fodor’s lemma for weakly LS cardinals.

Lemma 5. Let $\kappa$ be a weakly LS cardinal. Let $\lambda \geq \kappa$ be a cardinal, and $f : \lambda^+ \setminus \{0\} \rightarrow \lambda^+$ be a regressive function. Then there is $\gamma < \lambda^+$ such that the set $\{\xi \in \lambda^+ \mid f(\xi) \leq \gamma\}$ is stationary in $\lambda^+$.

Proof. Let $f : \lambda^+ \setminus \{0\} \rightarrow \lambda^+$. Take a large $\alpha > \lambda^+$ and $X \prec V_\alpha$ such that $\lambda^+, f \in X$ and the transitive collapse of $X$ is in $V_\kappa$. We know $\eta = \sup(X \cap \lambda^+) < \lambda^+$. Pick $\gamma \in X \cap \lambda^+$ with $f(\eta) \leq \gamma$. Then $S = \{\xi \in \lambda^+ \mid f(\xi) \leq \gamma\} \in X$. If $S$ is non-stationary in $\lambda^+$, then there is a club $C \in X$ with $C \cap S = \emptyset$. But $\eta = \sup(X \cap \lambda^+) \in C$, so $\eta \in C \cap S$. This is a contradiction. □

That there are no LS cardinals holds in any model in which there are no regular uncountable cardinals, but this later statement is known to have large cardinals strength. So the following natural question arises: What is the consistency strength of “no (weakly) LS cardinals”? For this question, Asaf Karagila pointed out the following:

Theorem 13 (Karagila [8]). Suppose $V$ satisfies $\mathsf{AC}$ and $\mathsf{GCH}$. Then there is an extension of $V$ with some cofinalities as $V$, such that Fodor’s lemma fails and the club filter is not $\sigma$-complete on every regular uncountable cardinal.

In his model, there are no (weakly) LS cardinals by Lemmas 4 and 5.

Woodin (Theorem 227 in [16]) proved that if $\lambda$ is a singular cardinal and a limit of supercompact cardinals, then $\lambda^+$ is regular, and the club filter over $\lambda^+$ is $\lambda^+$-complete. Now we can replace supercompact cardinals in Woodin’s result by weakly LS cardinals:

Corollary 2. Let $\lambda$ be a singular weakly LS cardinal (e.g., a singular limit of weakly LS cardinals).

1. There is no cofinal map from $V_\lambda$ into $\lambda^+$. In particular $\lambda^+$ is regular.
2. Let $f$ be a function from $V_\lambda$ into the club filter over $\lambda^+$. Then $\bigcap f^* V_\lambda$ contains a club in $\lambda^+$. In particular the club filter over $\lambda^+$ is $\lambda^+$-complete.
3. For every regressive function $f : \lambda^+ \setminus \{0\} \to \lambda^+$, there is $\gamma < \lambda^+$ such that the set $\{\xi < \lambda^+ | f(\xi) = \gamma\}$ is stationary in $\lambda^+$.

Proof. Fix an increasing sequence $(\lambda_i | i < \text{cf}(\lambda))$ with limit $\lambda$.

(1). Let $g : V_\lambda \to \lambda^+$. For $i < \text{cf}(\lambda)$, let $\alpha_i = \sup(g^{\lambda+})$. We have $\alpha_i < \lambda^+$ by Lemma 3. Again, since $\text{cf}(\lambda) < \lambda$, we have $\sup(g^{\lambda+}) = \sup\{\alpha_i | i < \text{cf}(\lambda)\} < \lambda^+$ by Lemma 4.

(2). For a given $f$, we have that for every $i < \text{cf}(\lambda)$, $\cap f^{\lambda+}$ contains a club in $\lambda^+$ by Lemma 4. Then $\cap f^{\lambda+} = \cap_{i < \text{cf}(\lambda)} f^{\lambda+}$ contains a club by Lemma 4 again.

For (3), by Lemma 5 there is the minimal $\gamma < \lambda^+$ such that $\{\xi < \lambda^+ | f(\xi) \leq \gamma\}$ is stationary in $\lambda^+$. We will show that the set $\{\xi < \lambda^+ | f(\xi) = \gamma\}$ is stationary. Since $\lambda$ is singular, we have $\text{cf}(\gamma) < \lambda$. Take a sequence $(\gamma_i | i < \text{cf}(\gamma))$ with limit $\gamma$, and let $S_i = \{\xi < \lambda^+ | f(\xi) \leq \gamma_i\}$. By the minimality of $\gamma$, each $S_i$ is non-stationary. Then $\bigcup_{i < \text{cf}(\gamma)} S_i$ is non-stationary by (2). This means that the set $\{\xi < \lambda^+ | f(\xi) = \gamma\}$ must be stationary. \qed

**Question 1.**
1. Is the statement $\text{ZF}^{+}$“there is a weakly LS cardinal which is not LS” consistent?
2. Suppose there is a supercompact cardinal (or an extendible cardinal). Then are there proper class many (weakly) LS cardinals?
3. Suppose $\lambda$ is a singular weakly LS cardinal. Is the club filter over $\lambda^+$ normal?

## 4 Uniform definability of grounds

In this section, we prove that if there are proper class many LS cardinals, then all grounds are uniformly definable. For this purpose, we introduce a very rough measure on sets, which will be used instead of cardinality.

**Definition 5.** For a set $x$, the norm of $x$, $\|x\|$, is the least ordinal $\alpha$ such that there is a surjection from $V_\alpha$ onto $x$.

The following is easy to check:

**Lemma 6.**
1. $\|x\| \leq \text{rank}(x)$.
2. If $x \subseteq y$ then $\|x\| \leq \|y\|$.
3. If $M \subseteq V$ is a transitive model of (a sufficiently large fragment of) $\text{ZF}$ and $x \in M$, then $\|x\|^{M} \geq \|x\|$.
4. If $X$ is an extensional set (that is, for every $x, y \in X$, $x = y \iff \forall z \in X(z \in x \leftrightarrow z \in y)$) and its transitive collapse belongs to $V_\alpha$ for some $\alpha$, then $\|X\| < \alpha$.

**Definition 6.** Let $\mathbb{Z}^*$ be the theory $\mathbb{Z}$, $\text{ZF} - \text{Replacement Scheme}$, with the conjunction of the following statements:

1. Every set $x$ has transitive closure $\text{trcl}(x)$.
2. Every set $x$ has rank, that is, there is a surjection from $\text{trcl}(x)$ onto some ordinal $\alpha$ such that $f(y) = \sup\{f(z) + 1 | z \in y\}$ for every $y \in \text{trcl}(x)$. Such an $\alpha$ is the rank of $x$. 
3. For every ordinal \( \alpha \), the collection of sets with rank \( \alpha \) forms a set.
4. Every extensional set has a (unique) transitive collapse and a collapsing map, that is, for every set \( X \), if \( \forall x, y \in X (x = y \iff \forall z \in X (z \in x \iff z \in y)) \), then there is a transitive set \( Y \) and an \( \in \)-isomorphism from \( X \) onto \( Y \).

For a transitive model \( M \) of \( Z^* \) and \( \alpha \in M \cap ON \), let \( M_\alpha = M \cap V_\alpha \). We know \( M_\alpha \in M \).

Note 1. Let \( M \) be a transitive model of \( Z^* \).
1. For \( \alpha \in M \cap ON \), \( M_{\alpha + 1} = \mathcal{P}(M_\alpha)^M = \mathcal{P}(M_\alpha) \cap M \), and \( M_\alpha = \bigcup_{\beta < \alpha} M_\beta \) if \( \alpha \) is limit.
2. \( M = \bigcup_{\alpha \in M \cap ON} M_\alpha \).
3. For \( \gamma \in M \cap ON \), if \( V_\gamma \) is a model of \( Z^* \) then \( M_\gamma \) is also a model of \( Z^* \).

For models of \( Z^* \), we define variants of the covering and approximation properties in Hamkins [6].

**Definition 7.** Let \( M \subseteq V \) be a transitive model of \( Z^* \). Let \( \alpha \in M \) be an ordinal.
1. We say that \( M \) satisfies the \( \alpha \)-norm covering property (for \( V \)) if for every \( \beta \in M \cap ON \) and set \( x \subseteq M_\beta \), if \( \|x\| < \alpha \) then there is \( y \in M \) such that \( x \subseteq y \) and \( \|y\|^M < \alpha \).
2. We say that \( M \) satisfies the \( \alpha \)-norm approximation property (for \( V \)) if for every \( \beta \in M \cap ON \) and set \( x \subseteq M_\beta \), if \( x \cap a \in M \) for every \( a \in M \) with \( \|a\|^M < \alpha \), then \( x \in M \).

Note 2. 1. \( M \) satisfies the \( \alpha \)-norm covering property if and only if for every \( \beta \in M \cap ON \) and set \( x \subseteq M_\beta \), if \( \|x\| < \alpha \) then there is \( y \in M_{\beta + 1} \) such that \( x \subseteq y \) and \( \|y\|^M < \alpha \).
2. \( M \) satisfies the \( \alpha \)-norm approximation property if and only if for every \( \beta \in M \cap ON \) and set \( x \subseteq M_\beta \), if \( x \cap a \in M \) for every \( a \in M_{\beta + 1} \) with \( \|a\|^M < \alpha \), then \( x \in M \).

**Lemma 7.** Let \( M \subseteq V \) be a transitive model of \( Z^* \). Let \( \gamma \in M \cap ON \) be an ordinal, and suppose \( M \) satisfies the \( \gamma \)-norm covering and the \( \gamma \)-norm approximation properties for \( V \). Fix \( \alpha > \gamma \) with \( \alpha \in M \), and let \( \beta > \alpha \) and \( X \prec V_\beta \) be such that:
1. \( V_\gamma \subseteq X \).
2. \( M_{\alpha + 1}, \gamma \in X \).
3. \( V_\gamma (X \cap V_\alpha) \subseteq X \).

Then \( X \cap M_\alpha \in M \).

**Proof.** By the \( \gamma \)-norm approximation property of \( M \), it is enough to see that for every \( \alpha \in M_{\alpha + 1} \), if \( \|a\|^M < \gamma \) then \( a \cap X \cap M_\alpha \in M \). Fix \( a \in M_{\alpha + 1} \) with \( \|a\|^M < \gamma \). Note that \( \|a\| < \gamma \). Since \( a \cap X \cap M_\alpha \subseteq X \) and \( \|a \cap X \cap M_\alpha \| \leq \|a\| < \gamma \), there is a surjection from \( V_\gamma \) onto \( a \cap X \cap M_\alpha \). \( V_\gamma (X \cap V_\alpha) \subseteq X \), hence we have
a \cap X \cap M_\alpha \in X. Because M satisfies the \gamma\text{-norm covering property for } V, there is some \( x \in M_{\alpha+1} \) such that \( \|x\|^M < \gamma \) and \( a \cap X \cap M_\alpha \subseteq x \). By the elementarity of \( X \), we may assume that \( x \in X \). Note that \( x \subseteq X \) since \( \|x\| \leq \|x\|^M < \gamma \) and \( V_\gamma \subseteq X \). Then we have \( a \cap X \cap M_\alpha = a \cap x \in M \).

**Lemma 8.** Let \( M, N \subseteq V \) be transitive models of \( \text{Z}^* \) with \( M \cap ON = N \cap ON \). Let \( \kappa \) be an LS cardinal with \( \kappa \in M \cap N \), and suppose there is \( \gamma < \kappa \) such that \( M \) and \( N \) satisfy the \( \gamma \text{-norm covering} \) and the \( \gamma \text{-norm approximation properties} \) for \( V \). If \( M_\kappa = N_\kappa \), then \( M = N \).

**Proof.** We show \( M_\kappa = N_\kappa \) by induction on \( \alpha \in M \cap ON \). The cases that \( \alpha \leq \kappa \) and \( \alpha \) is limit are clear. So suppose \( \alpha = \bar{\alpha} + 1 \) for some \( \bar{\alpha} \geq \kappa \) and \( M_{\bar{\alpha}} = N_{\bar{\alpha}} \).

First, we show that for every \( x \in M_{\alpha} \), if \( \|x\| < \gamma \) then \( x \in N \). Since \( \kappa \) is LS, we can find a large \( \beta > \alpha \) and \( X \prec V_\beta \) such that:

1. \( V_\gamma \subseteq X \),
2. \( V_\gamma '(X \cap V_\alpha) \subseteq X \),
3. the transitive collapse of \( X \) is in \( V_\kappa \), and
4. \( X \) contains all relevant objects.

Then, by Lemma 7, we have that \( X \cap M_\alpha \in M \) and \( X \cap N_\alpha \in N \). In particular \( X \cap M_{\bar{\alpha}} \in M \) and \( X \cap N_{\bar{\alpha}} \in N \). On the other hand, \( M_{\bar{\alpha}} = N_{\bar{\alpha}} \) by the induction hypothesis. Hence we have \( X \cap M_{\bar{\alpha}} = X \cap N_{\bar{\alpha}} \in M \cap N \). Since \( \|x\| < \gamma \), we have \( x \subseteq X \). Thus we have \( x \subseteq X \cap M_{\bar{\alpha}} = X \cap N_{\bar{\alpha}} \). \( X \cap M_{\bar{\alpha}} \) is extensional, so we can take the transitive collapse \( Y \) of \( X \cap M_{\bar{\alpha}} \) and the collapsing map \( \pi : X \cap M_{\bar{\alpha}} \rightarrow Y \). Note that \( Y \) and \( \pi \) are in \( M \cap N \) because \( M \) and \( N \) are models of \( \text{Z}^* \). The transitive collapse of \( X \) is in \( V_\kappa \), hence \( Y \) is also in \( V_\kappa \) and thus \( Y \in M_\kappa = N_\kappa \). Put \( y = \pi^{-1}x \subseteq Y \in M_\kappa \). \( y \) is in \( M_\kappa \), and hence is also in \( N_\kappa \). Now we have \( x = \pi^{-1}y \in N \).

The same argument shows that for every \( x \in N_{\alpha} \), if \( \|x\| < \gamma \) then \( x \in M \). Finally, by the \( \gamma \text{-norm approximation property} \) of \( M \) and \( N \), we have \( \mathcal{P}(M_{\bar{\alpha}}) \cap M = \mathcal{P}(M_{\bar{\alpha}}) \cap N \), hence \( M_\alpha = N_\alpha \).

**Corollary 3.** Suppose there are proper class many LS cardinals. Then there is a first-order formula \( \varphi'(x, y) \) of set-theory such that:

1. \( \mathcal{W}'_{\alpha} = \{ x \mid \varphi'(x, r) \} \) is a transitive class model of \( \text{ZF} \) containing all ordinals such that \( r \in \mathcal{W}'_{\alpha} \), and \( \mathcal{W}'_{\alpha} \) satisfies the \( \alpha \text{-norm covering and approximation properties} \) for \( V \) for some \( \alpha \).
2. For every transitive class model \( W \subseteq V \) of \( \text{ZF} \) containing all ordinals, if \( W \) satisfies the \( \alpha \text{-norm covering and approximation properties} \) for \( V \) for some \( \alpha \), then there is \( r \) with \( \mathcal{W}'_{\alpha} = W \).

**Proof.** For a set \( r \), we define \( \mathcal{W}'_{\alpha} \) if \( r \) satisfies the following conditions:

1. \( r = (X, \kappa, \alpha) \) where \( \kappa \) is an LS cardinal, \( \alpha < \kappa \), and \( X \) is a transitive set with \( X \cap ON = \kappa \).
2. For each cardinal $\lambda > \kappa$, if $V_\lambda$ is a model of $Z^*$, then there is a unique transitive model $X^{r,\lambda}$ of $Z^*$ such that $X^{r,\lambda} \cap \text{ON} = \lambda$, $(X^{r,\lambda})_\kappa = X$, and $X^{r,\lambda}$ satisfies the $\alpha$-norm covering and the $\alpha$-norm approximation properties for $V$.

In this case, let $W'_r = \bigcup \{X^{r,\lambda} \mid \lambda > \kappa$ is a cardinal$\}$. Otherwise, let $W'_r = V$. It is clear that the collection $\{W'_r \mid r \in V\}$ is a uniformly definable collection of classes. We see that $\{W'_r \mid r \in V\}$ is as required.

First we check condition $(1)$. If $W'_r = V$, then it is clear. Suppose not, then $r$ is of the form $(X, \kappa, \alpha)$. It is clear that $W'_r$ is transitive and contains all ordinals.

Now fix cardinals $\lambda_0 > \lambda_1 > \kappa$ such that $V_{\lambda_0}$ and $V_{\lambda_1}$ are models of $Z^*$, and let $X^{r,\lambda_0}, X^{r,\lambda_1}$ be transitive models of $Z^*$. It is routine to check that $(X^{r,\lambda_0})_{\lambda_1}$ is a model of $Z^*$, and satisfies the $\alpha$-norm covering and approximation properties. Because $X = (X^{r,\lambda_0})_\kappa = (X^{r,\lambda_1})_\kappa$, we have $(X^{r,\lambda_0})_{\lambda_1} = X^{r,\lambda_1}$ by Lemma 8. This means that $(W'_r)_\lambda = X^{r,\lambda}$ for every cardinal $\lambda > \kappa$ with $V_\lambda$ a model of $Z^*$, and that $W'_r$ is almost universal and closed under the Gödel operations. Thus we have that $W'_r$ is a model of $ZF$ by Theorem 6. Moreover, by the definition of the norm covering and approximation properties, it is also easy to check that $W'_r$ satisfies the $\alpha$-norm covering and approximation properties. Finally, we have $X = (W'_r)_\kappa \in W'_r$, and $r \in W'_r$.

For $(2)$, suppose $W$ is a transitive class model of $ZF$ and $W$ satisfies the $\alpha$-norm covering and approximation properties for $V$ for some $\alpha$. Fix an LS cardinal $\kappa > \alpha$, and let $X = M_\kappa$ and $r = (X, \kappa, \alpha)$. For each cardinal $\lambda > \kappa$, if $V_\lambda$ is a model of $Z^*$ then $W_\lambda$ is a transitive model of $Z^*$, satisfies the $\alpha$-norm covering and approximation properties for $V$, and, by Lemma 8, $W_\lambda$ is a unique transitive model $M$ of $Z^*$ satisfying the $\alpha$-norm covering and approximation properties for $V$ and $M_\kappa = X$. Then we have $W = W'_r$ by the definition of $W'_r$.

Note 3. In item $(2)$ of the previous corollary, $W$ need not to be a class of $V$; In fact, it is sufficient that $W$ is a transitive model of $ZF$ satisfying the norm covering and approximation properties.

**Lemma 9.** Let $\kappa$ be a weakly LS cardinal. Let $M \subseteq V$ be a ground of $V$, and suppose there is a poset $P \in M_\kappa$ and an $(M, P)$-generic $G$ with $V = M[G]$. Then $M$ satisfies the $\kappa$-norm covering and the $\kappa$-norm approximation properties for $V$.

**Proof.** First we show that $M$ satisfies the $\kappa$-norm covering property for $V$. Take $\alpha$ and $x \subseteq M_\alpha$ with $||x|| < \kappa$. Fix a limit $\gamma < \kappa$ and a surjection $f : V_\gamma \rightarrow x$. We may assume that $P \in V_\gamma$. We know $V_\gamma = \{\dot{y}_G \mid \dot{y} \in M_\gamma \text{ is a } P\text{-name}\}$ (see Lemma 3), where $\dot{y}_G$ is the interpretation of $\dot{y}$ by $G$. Hence we have a canonical surjection $\dot{y} \mapsto \dot{y}_G$ from all $P$-names in $M_\gamma$ onto $V_\gamma$, and so we can take a surjection $g$ from $M_\gamma$ onto $x$. Let $\dot{y}$ and $\dot{x}$ be $P$-names for $g$ and $x$ respectively. We work in $M$. Fix $p_0 \in G$ such that $p_0 \Vdash \forall^{\bar{\gamma}} \dot{y}$ is a surjection from $M_\gamma$ onto $\dot{x} \subseteq M_\alpha$ in $M$. For $a \in M_\gamma$ and $p \in P$ with $p \leq p_0$, take a unique $x_{a,p} \in M_a$ with $p \Vdash \forall^{\bar{\gamma}} \dot{y}(a) = x_{a,p}$ if it exists. If there is no such $x_{a,p}$, let $x_{a,p} = \emptyset$. Let $x' = \{x_{a,p} \mid a \in M_\gamma, p \in P\} \subseteq M$. Clearly $x \subseteq x'$. Moreover we can easily take a surjection from $M_\gamma \times P$ onto $x'$, hence $||x'||^M \leq \gamma + \omega < \kappa$. 


For the $\kappa$-norm approximation property of $M$, take $\alpha \in M$, $A \subseteq M_\alpha$, and suppose $A \cap a \in M$ for every $a \in M_{\alpha+1}$ with $\|a\|^M < \kappa$. Take a $\mathbb{P}$-name $\dot{A} \in M$ for $A$. Take $p_0 \in G$ with $p_0 \Vdash^{\mathbb{P}} "A \subseteq M_\alpha", \text{and } A \cap a \in M$ for every $a \in M_{\alpha+1}$ with $\|a\|^M < \kappa" \in M$. For $p \in \mathbb{P}$ with $p \leq p_0$, let $A_p = \{ a \in M_\alpha \mid p \Vdash^{\mathbb{P}} "a \in \dot{A}" \} \in M$. We claim that there is $p \in \mathbb{P}$ with $A_p = A$, which completes our proof.

Suppose to the contrary that there is no $p \in \mathbb{P}$ with $A_p = A$. Take $\gamma < \kappa$ with $\mathbb{P} \in V_\gamma$. Since $\kappa$ is a weakly LS cardinal, we can find $\beta > \alpha$ and $X \prec V_\beta$ such that:

1. $V_\gamma \subseteq X$.
2. The transitive collapse of $X$ is in $V_\kappa$.
3. $X$ contains all relevant objects.

Consider $M_\alpha \cap X$. Since the transitive collapse of $X$ is in $V_\kappa$, we have $\|X\| < \kappa$, and $\|M_\alpha \cap X\| \leq \kappa$ as well. We know that $M$ satisfies the $\kappa$-norm covering property, thus we can find $x \in X$ such that $\|x\|^M < \kappa$ and $M_\alpha \cap X \subseteq x$. We may assume that $x \in M_{\alpha+1}$. By the assumption, we have $A' = A \cap x \in M$. Thus there is $p \in G$ such that $p \leq p_0$ and $p \Vdash^{\mathbb{P}} "A \cap x = A"$, which means that $A \cap x = A_p \cap x$. Since $\mathbb{P} \in X$ and $\mathbb{P} \in V_\gamma$, we have $\mathbb{P} \subseteq X$; hence $p \in X$, and $A_p \in X$ as well. Since $A_p \neq A$, there is $a \in A \triangle A_p$. Because $A_p, A \in X$, we may assume $a \in X$, so $a \in X \cap M_\alpha \subseteq x$. Then $a \in (A \triangle A_p) \cap x = (A \cap x) \triangle (A_p \cap x)$, this is a contradiction. \hfill \Box

Now the uniform definability of grounds is immediate from Corollary 3 and Lemma 9.

**Corollary 4.** Suppose there are proper class many LS cardinals. Then there is a formula $\varphi(x, y)$ of set-theory such that:

1. $W_r = \{ x \mid \varphi(x, r) \}$ is a ground of $V$ with $r \in W_r$.
2. For every ground $W$ of $V$, there is $r$ with $W_r = W$.

**Proof.** Let $\{ W'_r \mid r \in V \}$ be the collection defined in Corollary 3. Then define $\{ W_r \mid r \in V \}$ as follows: For a set $r$, if there are some poset $\mathbb{P} \in W'_r$ and an $(W'_r, \mathbb{P})$-generic $G$ with $W'_r[G] = V$, then let $W_r = W'_r$. If otherwise, put $W_r = V$. By Corollary 3 and Lemma 9 the collection $\{ W_r \mid r \in V \}$ is all grounds of $V$. \hfill \Box

**Question 2.** Suppose there is one supercompact cardinal (or one extendible cardinal). Are all grounds uniformly definable as in Theorem 2?

Finally we shall prove that the statement “there are proper class many LS cardinals” is absolute between $V$ and its forcing extensions.

**Lemma 10.** Let $\mathbb{P}$ be a poset, and $\kappa < \lambda$ cardinals with $\mathbb{P} \in V_\kappa$. If $\Vdash^\mathbb{P} "\kappa \text{ and } \lambda \text{ are LS}"$, then $\lambda$ is LS in $V$.

**Proof.** Take a set-forcing extension $V[G]$ of $V$ via $\mathbb{P}$. In $V[G]$, since $\kappa$ is an LS cardinal and $\mathbb{P} \in V_\kappa$, $V$ satisfies the $\kappa$-norm covering and approximation properties for $V[G]$ by Lemma 9. We shall see that $\lambda$ is LS in $V$. Take $\gamma <$
Claim. Let $\lambda \leq \alpha$ and $x \in V_\alpha$. Take a large $\beta$ and $X \prec V[G]_\beta$ such that $V[G]_\beta \subseteq X$. $X$ contains all relevant objects, the transitive collapse of $X$ is in $V[G]_\lambda$, and $V[G]_\gamma (X \cap V[G]_\alpha) \subseteq X$. Let $Y = X \cap V_{\alpha+1}$. We know $Y \prec V_{\alpha+1}$. Moreover $Y \in V$ by Lemma 7. Since the transitive collapse of $X$ is in $V[G]_\lambda$, we have that $Y$ is in $V_\lambda$. Since $V_\gamma \subseteq V[G]_\gamma \subseteq X$, we have $V_\gamma \subseteq X \cap V_{\alpha+1} = Y$. Finally we check that $(V_\gamma(Y \cap V_\alpha))V \subseteq Y$. Take $f: V_\gamma \to Y \cap V_\alpha$ with $f \in V$. We know $V_\gamma \subseteq V[G]_\gamma, V_\alpha \subseteq V[G]_\alpha$, and $V[G]_\gamma(X \cap V[G]_\alpha) \subseteq X$, so $f \in X \cap V_{\alpha+1} = Y$. □

Lemma 11 (Folklore). Let $P$ be a poset, and $\alpha > \omega$ a limit ordinal with $P \in V_\alpha$. Suppose also that $V_\alpha$ satisfies the $\Sigma_1$-collection scheme. For every $X \prec V_\alpha$ with $P \in X$ and $P \subseteq X$, we have $X[G] \prec V[G]_\alpha$.

Proof (Sketch of proof). If $M$ is a transitive model of $\mathsf{ZFC}$, then we can define the forcing relation and prove the forcing theorem for every poset $P \in M$, formula $\varphi$, $P$-names $\dot{a}_0, \ldots, \dot{a}_n \in M$, and $(M,P)$-generic $G$, we have that $M[G] \models \varphi(\dot{a}_0, \ldots, \dot{a}_n)$ if and only if $P \Vdash \varphi(\dot{a}_0, \ldots, \dot{a}_n)$ holds in $M$ for some $p \in G$.

Now suppose $V_\alpha$ satisfies the $\Sigma_1$-collection scheme. Note that $V[G]_\alpha = V[\alpha]$ by Lemma 8. To see that $X[G] \prec V[G]_\alpha = V[\alpha][G]$, by Tarski-Vaught criterion, it is enough to see that for every formula $\varphi$ and $\dot{a}_0, \ldots, \dot{a}_n \in X[G]$, if $V[G]_\alpha \models \exists \sigma \varphi(\dot{a}_0, \ldots, \dot{a}_n, x)$ then there is $\dot{b} \in X[G]_\alpha$ with $V[G]_\alpha \models \varphi(\dot{a}_0, \ldots, \dot{a}_n, \dot{b})$. Let $\dot{a}_0, \ldots, \dot{a}_n \in X$ be $P$-names for $a_0, \ldots, a_n$ respectively. By the forcing theorem, there is some $p \in G$ and $P$-name $\dot{\sigma} \in V_\alpha$ such that $P \Vdash \varphi(\dot{a}_0, \ldots, \dot{a}_n, \dot{\sigma})$ in $V_\alpha$. Hence the statement $\exists \sigma (\dot{\sigma} \in X[G]$ and $V[G]_\alpha \models \varphi(\dot{a}_0, \ldots, \dot{a}_n, \dot{\sigma})$ holds in $V_\alpha$. Because $X \prec V_\alpha$, we can find a witness $\dot{\tau} \in X$. Then $(\dot{\sigma})_G \in X[G]$ and $V[G]_\alpha \models \varphi((\dot{a}_0)_G, \ldots, (\dot{a}_n)_G, (\dot{\tau})_G)$, as required. □

Lemma 12. Let $\kappa$ be a cardinal limit of $\mathsf{LS}$ cardinals (hence $\kappa$ itself is an $\mathsf{LS}$ cardinal) and $P \in V_\kappa$ be a poset. Let $G$ be $(V,P)$-generic. Then $\kappa$ is $\mathsf{LS}$ in $V[G]$.

Proof. In $V[G]$, fix an ordinal $\alpha > \kappa$, $x \in V[G]_\alpha$, and $\gamma < \kappa$. We will find some $\beta > \alpha$ and $X \prec V[G]_\beta$ such that $V[G]_\beta \subseteq X$, $x \in X$, the transitive collapse of $X$ is in $V[G]_\alpha$, and $V[G]_\gamma (X \cap V[G]_\alpha) \subseteq X$. Let $\dot{x}$ be a name for $x$.

In $V$, take a limit $\beta > \alpha$ such that $V_\beta$ satisfies the $\Sigma_1$-collection scheme. Take an $\mathsf{LS}$ cardinal $\delta < \kappa$ with $\gamma < \delta$, and a submodel $Y \prec V_\beta$ such that $V_\delta \subseteq Y$, $\dot{x} \in Y$, the transitive collapse of $Y$ is in $V_\kappa$, and $V_\delta (Y \cap V_\kappa) \subseteq Y$. We may assume that $P \subseteq Y$. We will show that $Y[G]$ is as required.

We have $x \in Y[G] \prec V_\beta[G] = V[G]_\beta$ and $V[G]_\gamma \subseteq V[G]_\delta = V_\delta[G] \subseteq Y[G]$. To show that $V[G]_\gamma (Y[G] \cap V[G]_\alpha) \subseteq Y[G]$, take $f: V[G]_\gamma \to Y[G] \cap V[G]_\alpha$. We will find $y \in Y[G]$ with $\text{range}(f) \subseteq y$ and $\|y\|^{V[G]} < \delta$. Then we will have $f \in Y[G]$ since $\mathfrak{P}(y) \subseteq Y[G]$.

Let $\dot{f}$ be a $P$-name for $f$. In $V$, since $\delta$ is $\mathsf{LS}$, there is $Z \prec V_\beta$ such that $V_\delta \subseteq Z, Y, \dot{f}, \ldots \in Z$ and the transitive collapse of $Z$ is in $V_\delta$. Let $R = \{\dot{a} \in Z \cap Y \mid \exists p \in \mathfrak{P} \exists \dot{b} \in V, (p \Vdash^\mathfrak{P} \dot{f}(\dot{b}) = \dot{a})\}$. 

Claim. $\text{range}(f) \subseteq \{\dot{a}_G \mid \dot{a} \in R\}$.

2 Actually Kripke-Platek set-theory is sufficient.
Proof (Proof of Claim). Take $a \in \text{range}(f)$. Then there are $\mathbb{P}$-names $\dot{b} \in V_\gamma$ and $\dot{a} \in Y \cap V_\alpha$ such that $f(\dot{b}) = a = \dot{a}_G$. Take $p \in \mathbb{P}$ with $p \models "f(\dot{b}) = \dot{a}"$. Then the statement $\exists \dot{a}' \in Y (p \models "f(\dot{b}) = \dot{a}'")$ holds in $V_\beta$. Because $\mathbb{P} \subseteq V_\gamma \subseteq Z$ and $V_\gamma, Y \in Z$, we can find $\dot{a}' \in Z \cap Y$ such that $p \models "f(\dot{b}) = \dot{a}'"$, then $\dot{a}' \in R$ and $a = \dot{a}'_G$. □

Now $R \subseteq Y$ and $\|R\| \leq \|Z\| < \delta$. Since $V_\gamma(Y \cap V_\alpha) \subseteq Y$, we have $R \in Y$. Let $y = \{\dot{a}_G \mid \dot{a} \in R\} \in Y[G]$. We have $\text{range}(f) \subseteq y$, and, since $\|R\|^V < \delta$, we know $\|y\|^V[G] < \delta$ as well. □

Corollary 5. Let $V[G]$ be a generic extension of $V$. Then the statement that “there are proper class many LS cardinals” is absolute between $V$ and $V[G]$.

We say that all grounds are uniformly definable in the generic multiverse if there is a first-order formula $\varphi(x, y)$ of set-theory such that, in all grounds and generic extensions of $V$, all its grounds are uniformly definable by $\varphi$ as in Theorem 2.4.

By Corollary 5 and the proofs of Corollaries 3 and 4, we have:

Corollary 6. Suppose there are proper class many LS cardinals. Then all grounds are uniformly definable in the generic multiverse.

If AC is forceable, then there are proper class many LS cardinals in some generic extension of $V$, and hence also in $V$ by Corollary 6. Hence we also have:

Corollary 7. Suppose AC is forceable. Then there are proper class many LS cardinals, and all grounds are uniformly definable in the generic multiverse.

In the next section, we discuss when AC is forceable.

By Corollary 2 we can easily construct a model $V$ such that $V$ does not satisfy AC but $V$ has proper class many LS cardinals; For instance, AC is forceable over $L(\mathbb{R})$, so $L(\mathbb{R})$ has proper class many LS cardinals. On the other hand it is possible that $L(\mathbb{R})$ does not satisfy AC (e.g., see Theorem 3 below). However the author does not know if the converse direction of Corollary 7 fails:

Question 3. Is it consistent that there are proper class many LS cardinals but AC is never forceable?

This question might be connected with Woodin’s Axiom of Choice Conjecture:

Conjecture 14 (Axiom of Choice Conjecture, Woodin, Definition 231 in [16])
If $V$ has a large cardinal, e.g., extendible cardinal, then AC is forceable.

We know some notable models of ZF in which AC is never forceable, for instance:

1. A model of ZF which has no regular uncountable cardinals (Gitik [1]).

Of course this definition cannot be formalized within ZF, so we will use it informally.
2. A model of ZF which has proper class many infinite but Dedekind-finite sets (Monro [13]).
3. A model of ZF in which Fodor’s lemma fails everywhere and every club filter is not \( \sigma \)-complete (Karagila [8]).
4. The Bristol model \( M \), a transitive model of ZF which lies between \( L \) and the Cohen forcing extension \( L[c] \), definable in \( L[c] \) (Karagila [9]).

Daisuke Ikegami pointed out that Chang’s model \( L(\text{ON}^\omega) \) is also an example. Kunen [10] showed that, in ZFC, AC fails in \( L(\text{ON}^\omega) \) if there are uncountably many measurable cardinals.

5. Chang’s model \( L(\text{ON}^\omega) \) assuming that there are proper class many measurable cardinals.

If there are proper class many measurable cardinals, we can check that AC is not forceable over \( L(\text{ON}^\omega) \) by a similar argument used in [10].

**Question 4.** Do these models have proper class many LS cardinals?

As stated before, models (1) and (3) have no LS cardinals.

**Question 5.** What does the geology of these models looks like? For instance, are all grounds uniformly definable in these models?

We know few things about the geology of these models.

## 5 The mantle and the generic mantle

In this section we briefly discuss the mantle and the generic mantle of the universe.

**Definition 8.** Suppose all grounds are uniformly definable as in Theorem 2. The mantle \( M \) is the intersection of all grounds, that is, \( M = \bigcap_r W_r \).

The mantle is a parameter-free definable transitive class containing all ordinals. In ZFC, the intersection of all grounds satisfying AC is a model of ZFC ([2], [15]), so a natural and important question is:

**Question 6.** Is the mantle a model of ZF or ZFC?

Note that if all grounds of \( V \) are downward directed, that is, every two grounds of \( V \) have a common ground, then we can prove that the mantle is a model of ZF as in the context of ZFC (see [2]). In the ZFC-context, it is known that all grounds are downward directed (see Theorem 16 below). However, in the ZF-context, this downward directedness can fail. Now let us sketch the proof.

For sets \( X \) and \( Y \), let \( Fn(X,Y) \) be the poset consisting of all finite partial functions from \( X \) into \( Y \) with the reverse inclusion order. The following is known, e.g., see Exercise E in Chapter VII in Kunen [11]:

**Theorem 15 (Millar).** Suppose \( V = L \). Let \( G \) be \( (V,Fn(\omega_1,2))-\)generic. Then, in \( V[G] \), \( L(\mathbb{R}^{V[G]}) \) does not satisfy AC.
Now suppose \( V = L \). Let \( \mathbb{P} = \text{Fn}(\omega_1, 2) \). Take a \((V, \mathbb{P} \times \mathbb{P})\)-generic \( G \times H \), and work in \( V[G \times H] \). Let \( M_G = L(\mathbb{R}^{V[G]}) \), and \( M_H = L(\mathbb{R}^{V[H]}) \). By Theorem \[15\] \( M_G \) and \( M_H \) do not satisfy AC. Hence \( V = L \) is not a common ground of \( M_G \) and \( M_H \). \( M_G \) and \( M_H \) are grounds of \( V[G \times H] \) by Theorem \[11\] On the other hand, by Theorem \[11\] we have \( V[G] \cap V[H] = V = L \). Because \( M_G \subseteq V[G] \) and \( M_H \subseteq V[H] \), we have \( M_G \cap M_H = V = L \). This shows that \( M_G \) and \( M_H \) cannot have a common ground.

Again, suppose all grounds are uniformly definable in the generic multiverse. Then every forcing extension of \( V \) can define its mantle in the same way. The following is immediate from Theorem \[8\] and the weak homogeneity of \( \text{Col}(V_\alpha) \):

**Lemma 13.** Suppose all grounds are uniformly definable in the generic multiverse.

1. For every limit ordinal \( \alpha \) and \((V, \text{Col}(V_\alpha))\)-generic \( G_0, G_1 \), we have \( M^{V[G_0]}_V = M^{V[G_1]}_V \subseteq M^V \subseteq V \). Hence, for some/any \((V, \text{Col}(V_\alpha))\)-generic \( G \), \( M^{V[G]}_V \) can be denoted as \( M^{\text{Col}(V_\alpha)}_V \).

2. The collection \( \{M^{\text{Col}(V_\alpha)}_V \mid \alpha \text{ is a limit ordinal} \} \) is uniformly definable in \( V \).

3. Let \( V[G] \) be a forcing extension of \( V \). Then there is a limit ordinal \( \alpha \) such that \( M^{\text{Col}(V_\alpha)}_V \subseteq M^V[G] \).

Thus we can define the generic mantle \( g\mathbb{M} = \bigcap \{M^{\text{Col}(V_\alpha)}_V \mid \alpha \in \text{ON} \} \), which is the intersection of all mantles of all generic extensions. As in the context of ZFC (see \[2\]), we can check that \( g\mathbb{M} \) is a parameter-free definable transitive model of ZF containing all ordinals. Clearly \( g\mathbb{M} \subseteq \mathbb{M} \). In the ZFC-context, the mantle coincides with the generic mantle \((2, 15)\). How is it in ZF?

**Question 7.** Does \( \mathbb{M} = g\mathbb{M} \)?

### 6 When AC is forceable

In this section, we discuss when AC is forceable. For this purpose, we use the DDG, downward directedness of the grounds.

**Theorem 16 (Usaba \[15\], in ZFC).** Let \( \{W_r \mid r \in V \} \) be the uniformly definable collection of all grounds satisfying AC as in Theorem \[7\]. Let \( X \) be a set. Then there is a ground \( W \) of \( V \) such that \( W \) satisfies AC, and \( W \) is a ground of each \( W_r (r \in X) \).

**Proposition 1.** Suppose AC is forceable, and let \( V[G] \) be a generic extension of \( V \) such that \( V[G] \) satisfies AC. Then there is a ground \( W \) of \( V[G] \) and a set \( X \) such that \( V = W(X) \) and \( W \) satisfies AC.

**Proof.** Let \( \mathbb{P} \) be a poset, \( G \) be \((V, \mathbb{P})\)-generic, and suppose \( V[G] \) satisfies AC. Take a \((V[G], \mathbb{P})\)-generic \( H \). We may assume that \( V[H] \) satisfies AC. Then \( G \times H \) is \((V, \mathbb{P} \times \mathbb{P})\)-generic, and \( V[G \times H] \) is a common forcing extension of \( V[G] \) and \( V[H] \). Because both \( V[G] \) and \( V[H] \) satisfy AC, \( V[G \times H] \) also satisfies AC. Then,
by Theorem [10] there is a model $W$ of ZFC which is a common ground of $V[G]$ and $V[H]$. We know $V = V[G] \cap V[H]$ by Theorem [11] hence $W \subseteq V \subseteq V[G]$. $V[G]$ is a forcing extension of $W$, so $V$ must be of the form $W(X)$ for some $X \in V$ by Theorem [11].

Now we have the following characterization.

**Corollary 8.** The following are equivalent:

1. AC is forceable.
2. There is a transitive model $W$ of ZFC and a set $X$ such that $W$ is definable in $V$ with parameters from $W$ and $V = W(X)$.
3. There is a transitive model $W$ of ZFC and a set $X$ such that $W$ is definable in $V$ with parameters from $W$, $V = W(X)$, and $W$ is a ground of some generic extension of $V$.

**Proof.**

$(3) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (1)$. Suppose $V = W(X)$. Let $Y$ be the transitive closure of $X$, and $\mathbb{P} = \text{Col}(Y)$. Take a $(V, \mathbb{P})$-generic $G$. In $V[G]$, $Y$ is well-orderable. Because $W$ satisfies AC and $Y$ is well-orderable in $V[G]$, every element of $V = W(X)$ is well-orderable in $V[G]$. Then every element of $V[G]$ is well-orderable, because there is a canonical class surjection from $V$ onto $V[G]$.

$(1) \Rightarrow (3)$. By Proposition [1] we can find a set-forcing extension $V[G]$ of $V$ and a ground $W$ of $V[G]$ such that $V[G]$ satisfies AC, $W$ is a model of ZFC, and $V = W(X)$ for some $X \in V$. We have to check that $W$ is definable in $V$. Because $W$ is a ground of $V[G]$, $W$ satisfies the $\alpha$-norm covering and approximation properties for $V[G]$ for some large $\alpha$. Then it is easy to check that $W$ also satisfies the $\alpha$-norm covering and approximation properties for $V$. Since AC is forceable over $V$, $V$ has proper class many LS cardinals. Then, by Corollary [8] $W$ is of the form $W'_r$ for some $r \in W$, hence $W$ is definable in $V$. \hfill $\Box$

**Corollary 9.** Suppose AC is forceable. Then for every ground $W$ of $V$, there is a transitive model $M$ of ZFC and a set $X \in W$ such that $M$ is definable in $V$ with parameters from $M$ and $W = M(X)$.

**Proof.** If $W$ is a ground of $V$, then AC is forceable over $W$. Then the assertion follows from the previous corollary. \hfill $\Box$

**Corollary 10.** Suppose $V$ satisfies AC. Then for every transitive model $M$ of ZF, $M$ is a ground of $V$ if and only if there is a ground $W$ of $V$ and a set $X$ such that $W$ satisfies AC and $M = W(X)$.

**Proof.** If $M$ is a ground of $V$, then AC is forceable over $M$. We can find required $W$ and $X$ by Proposition [1] For the converse, suppose $M = W(X)$ for some ground $W$ of $V$ satisfying AC and a set $X$. Then $M$ is a ground by Theorem [10]. \hfill $\Box$

**Corollary 11.** Suppose AC is forceable. Then the generic mantle is a model of ZFC.
Proof. Since $AC$ is forceable, for every sufficiently large $\alpha$ and $(V, Col(V_\alpha))$-generic $G$, we have that $V[G]$ satisfies $AC$. By Corollary 10 if $M \subseteq V[G]$ is a ground of $V[G]$, then there is a ground $W$ of $V[G]$ such that $W \subseteq M$ and $W$ satisfies $AC$. Hence the mantle of $V[G]$ coincides with the intersection of all grounds of $V[G]$ satisfying $AC$. In $\text{ZFC}$, it is known that the intersection of all grounds satisfying $AC$ is a model of $\text{ZFC}$ (11). Therefore we have that for every large $\alpha$, $M^{\text{Col}(V_\alpha)}$ is a model of $\text{ZFC}$. Since the generic mantle is the intersection of the $M^{\text{Col}(V_\alpha)}$’s, we can check that the generic mantle is a model of $\text{ZFC}$. \qed

A symmetric model is a type of choiceless model constructed as a submodel of a generic extension. See Grigorieff [4] for the definition of symmetric models. We use the following characterization of symmetric models.

For a class $M$ of $V$, let $\text{OD}(M)$ be the collection of all sets $x$ such that $x$ is definable with parameters from $M$ and ordinals. $\text{HOD}(M)$ is the collection of all sets $x$ such that the transitive closure of $x$ is a subset of $\text{OD}(M)$. If a class $M$ is a transitive model of $\text{ZF}$ containing all ordinals, then $\text{HOD}(M)$ is a transitive model of $\text{ZF}$ with $M \subseteq \text{HOD}(M)$.

Theorem 17 (Theorem C in [4]). For transitive models $M$ and $N$ of $\text{ZF}$ and a generic extension $M[G]$ of $M$, suppose $M \subseteq N \subseteq M[G]$ and $N$ is a class of $M[G]$. Then $N$ is a symmetric submodel of $M[G]$ if and only if $N$ is of the form $\text{HOD}(M(X))^{M[G]}$ for some $X \in N$.

Proposition 2. Suppose $AC$ is forceable. Then there is a transitive model $M$ of $\text{ZFC}$ and a generic extension $M[G]$ of $M$ such that $M$ is definable in $V$ with parameters from $M$ and $V$ is a symmetric submodel of $M[G]$.

Proof. By Corollary 8 there is a definable transitive model $M$ of $\text{ZFC}$ such that $V = M(X)$ for some $X \in V$, and $M$ is a ground of some generic extension $V[H]$ of $V$. By appealing to Theorem 7 we may assume that $H$ is $(V, Col(V_\alpha))$-generic for some $\alpha$, and there is $\beta$ and $(M, Col(M_\beta))$-generic $G$ with $V[H] = M[G]$. Let us consider $\text{HOD}(M(X))^{M[G]}$, which is a symmetric submodel of $M[G]$. Since $Col(V_\alpha)$ is weakly homogeneous, ordinal definable, and $M \subseteq V$, we have $M(X) \subseteq \text{HOD}(M(X))^{M[G]} = \text{HOD}(M(X))^{V[H]} \subseteq \text{HOD}(M(X))^V \subseteq V$, hence $M(X) = \text{HOD}(M(X))^{M[G]} = V$. \qed

Corollary 12. The following are equivalent:

1. $AC$ is forceable.
2. There is a transitive model $M$ of $\text{ZFC}$ and a generic extension $M[G]$ of $M$ such that $M$ is definable in $V$ with parameters from $M$, and $V$ is a symmetric submodel of $M[G]$.

Proof. (1) $\Rightarrow$ (2) follows from the previous proposition, and (2) $\Rightarrow$ (1) follows from the result of Blass [1]. \qed
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