TRAVELING WAVES IN A NONLOCAL DISPERAL PREDATOR-PREY MODEL

Yu-Xia Hao, Wan-Tong Li* and Fei-Ying Yang

School of Mathematics and Statistics, Lanzhou University
Gansu, Lanzhou 730000, People’s Republic of China

Abstract. This paper is concerned with the traveling wave solutions for a class of predator-prey model with nonlocal dispersal. By adopting the truncation method, we use Schauder’s fixed-point theorem to obtain the existence of traveling waves connecting the semi-trivial equilibrium to non-trivial leftover concentrations for \( c > c^* \), in which \( c^* \) is the minimal wave speed. Meanwhile, through the limiting approach and the delicate analysis, we establish the existence of traveling wave solutions with the critical speed. Finally, we show the nonexistence of traveling waves for \( 0 < c < c^* \) by the characteristic equation corresponding to the linearization of original system at the semi-trivial equilibrium. Throughout the whole paper, we need to overcome the difficulties brought by the nonlocal dispersal and the non-preserving of system itself.

1. Introduction. In population dynamics, different predator-prey systems have been proposed to model different processes of biological interactions since Lotka [32] and Volterra [36]. The dynamic relationship between predators and preys has been one of the dominant themes in ecology due to its important application and universal existence. Based on different settings, there are many different kinds of predator-prey models described by differential systems in the literature, see [10, 12, 14, 18, 19, 21, 23, 26, 33, 31, 40] and so on.

Traveling wave solutions in population dynamic play an important role in understanding the long time asymptotic property of systems and which of reaction-diffusion equations have been studied since 1937 ([17, 24]). For the study of traveling wave solutions of predator-prey systems, there are many works nowadays. Through the invariant manifold theory, the shooting method, Hopf bifurcation analysis and LaSalle’s invariance principle, Dunbar [12, 13, 14] established the existence of several kinds of traveling waves for diffusive predator-prey system with Holling’s type I, II functional responses. Huang et al. [23] extended the work in [14] to \( \mathbb{R}^4 \) by using Dunbar’s method in [13]. Li and Wu [26] proved the existence of traveling waves in a diffusive predator-prey system with a simplified Holling’s type III functional response by Dunbar’s method. In [31], Lin et al. considered the following system
with sigmoidal response function

\[
\begin{align*}
\frac{\partial N_1}{\partial t} &= D_1 \frac{\partial^2 N_1}{\partial x^2} + r N_1 (1 - \frac{N_1}{k}) - \frac{N_1^2 N_2}{a_1 + b_1 N_1 + N_1^2}, \\
\frac{\partial N_2}{\partial t} &= D_2 \frac{\partial^2 N_2}{\partial x^2} + N_2 \left( \frac{\alpha N_1^2}{a_1 + b_1 N_1 + N_1^2} - e \right),
\end{align*}
\]

in which they studied traveling wave solutions of system (1) as \(D_1 = 0\). This response function in some applications is more realistic than the Holling’s type I, II functional responses, and more general than a simplified form of the Holling’s type III functional response considered before. By using the original Wazewski’s theorem, they obtained the existence of traveling wave solutions. Recently, using a shooting method, Huang [23] showed the existence of traveling wave fronts for a class of diffusive predator-prey systems. It should be pointed out that his approach is a significant improvement of techniques introduced by Dunbar and provides a more efficient way to study the existence of traveling wave solutions for general predator-prey systems. For more information about traveling waves of predator-prey systems, one can see [2, 15, 20, 30, 22, 42] and the references therein.

As we know, the nonlocal dispersal as a long range process is sometimes better to describe some natural phenomena rather than the local one in many situations, such as in population ecology, material science, infectious diseases and so on ([1, 33]). Here, the nonlocal dispersal is characterized by some integral operators, for instance, the following involution operator

\[
L[u](x) := J * u(x, t) - u(x, t) = \int_{\mathbb{R}} J(x - y) u(y, t) dy - u(x, t).
\]

As mentioned in [1], if \(u(x, t)\) is thought of as a density at a point \(x\) at time \(t\), and \(J(x - y)\) is thought of as the probability distribution of jumping from location \(y\) to location \(x\), then \(\int_{\mathbb{R}} J(x - y) u(y, t) dy\) is the rate at which individuals are arriving at position \(x\) from all other places and \(\int_{\mathbb{R}} J(x - y) u(x, t) dy\) \((\int_{\mathbb{R}} J(x) dx = 1)\) is the rate at which they are leaving location \(x\) to travel to all other sites. In recent years, the study of traveling wave solutions of nonlocal dispersal problems has attracted much attention, one can see [4, 8, 11, 5, 25, 3, 41] for traveling waves and entire solutions with monostable, bistable and ignition nonlinearities respectively. For systems of integral differential equations, there are some works for Lotka-Volterra competition and cooperative systems for traveling waves, such as [28, 34, 6, 16, 7] and so on. However, owing to the fact that the effect of nonlocal dispersal and the non-monotonic of predator-prey systems themselves, the study of traveling waves is very limited ([35, 41]). We only refer to the readers [27, 29, 37, 39, 38] for some nonlocal dispersal epidemic models with predator-prey property.

In this paper, we are concerned with the nonlocal counterpart of (1). That is, we consider the following nonlocal dispersal predator-prey system

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= d_1 \left( J * u(x, t) - u(x, t) \right) + ru(x, t) \left( 1 - \frac{u(x, t)}{k} \right) - \frac{u^2(x, t)v(x, t)}{a + bu(x, t) + u^2(x, t)}, \\
\frac{\partial v(x, t)}{\partial t} &= d_2 \left( J * v(x, t) - v(x, t) \right) + v(x, t) \left( \frac{\alpha u^2(x, t)}{a + bu(x, t) + u^2(x, t)} - z \right),
\end{align*}
\]

where \(u(x, t)\) and \(v(x, t)\) are the population densities of the prey and predator respectively, \(r\) is the prey growth rate, \(z\) is the predator natural death rate, \(d_1, d_2 > 0\) are respectively the diffusion coefficients of prey and predator species, and \(a, b, \alpha\) are all positive parameters. To write system (2) in a non-dimensional form, we rescale
the variables
\[ A = \frac{a}{k^2}, \quad B = \frac{b}{k}, \quad \phi = \frac{u}{k}, \quad \psi = \frac{v}{rk}, \quad t' = rt. \]
By dropping the prime on \( t \) for notational convenience, system (2) becomes
\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= d_1 r (J * (\phi - \phi) + \phi (1 - \phi) - \frac{\phi^2 \psi}{A + B \phi + \phi^2} + A(\beta - 1) - \frac{\beta^2}{2}), \\
\frac{\partial \psi}{\partial t} &= d_2 r (J * (\psi - \psi) + \frac{\psi}{r} (\alpha \phi^2 A + B \phi + \phi^2 - z) + z). 
\end{align*}
\]
(3)
where \( r, z, \alpha > 0, A, B \geq 0 \). Let \( \beta = \frac{\alpha z}{A} \) and assume \( \beta > 1 + A + B \). Then, system (3) has three spatially constant equilibria given by
\[ E_0(0, 0), \quad E_1(1, 0) \quad \text{and} \quad E_2(u^*, v^*), \]
where
\[ u^* = B + \sqrt{B^2 + 4A(\beta - 1)} \quad \text{and} \quad v^* = \frac{(1 - u^*)[A + Bu^* + (u^*)^2]}{u^*}. \]
(4)
We will establish the existence of traveling wave solutions of system (3) connecting the equilibria \( E_1 \) and \( E_2 \). Throughout this paper, we need the following assumptions of the kernel function \( J \):

(1) \( J \in C^1(\mathbb{R}), \quad J(x) = J(-x) \geq 0, \quad \int_{\mathbb{R}} J(y) dy = 1 \) and \( J \) is compactly supported.

In the present paper, we focus on the existence and nonexistence of traveling wave solutions of system (3) with \( d_1 > 0, \quad d_2 > 0 \). In [31], the authors considered the traveling wave solutions of system (1) with \( d_1 = 0, \quad d_2 > 0 \) by the original Wazewski’s theorem. For system (3), due to the appearance of nonlocal dispersal, the method used in [31] can not be applied. Meanwhile, the Dunbar’s method and the monotone dynamical theory are all invalid for system (3). However, inspired by some works in [9, 27, 37, 38, 39], we will get the existence of traveling waves through constructing a truncate problem combining the supper and lower solutions method with Schauder’s fixed-point theorem when \( c > c^* \) (\( c^* \) is the minimal wave speed). It is noticed that it is very difficult to determine the final state about traveling waves for system (3). The reason is that the system itself is not monotone and the nonlocal dispersal leads us not to find a suitable Lyapunov function to ensure that traveling waves converge \( E_2 \) at \( +\infty \), which is a common method. Despite this, in view of the point of biological, the invasion of predators is successful if traveling waves are persistent at the end. Thus, it is enough to obtain the weak traveling wave solutions if we only want to know whether the invasion is successful and what the invasion speed is (\textsc{[42]}). Thus, by some delicate analysis, we can obtain the traveling waves connecting \( E_1 \) to non-trivial leftover concentrations, which implies that the traveling waves are persistent at last. At the same time, we can assert that both predator and prey coexist behind the front and the coexistence state \( E_2 \) is the only possible constant leftover state. For \( c = c^* \), the existence of traveling waves is obtained by the limiting approach and the analytical methods. In general, we need to overcome the difficulties brought by the non-compact of nonlocal dispersal operator and the lack of regularity of solutions through the whole proof. Finally, through the characteristic equation coming from the linearization of the corresponding wave equation for system (3) at \( E_1 \), we can get the nonexistence of traveling waves when \( 0 < c < c^* \).

The remaining part of this paper is organized as follows. In Section 2, we consider the existence of traveling waves as \( c \geq c^* \). Then the nonexistence of traveling waves with \( 0 < c < c^* \) is proved in Section 3.
2. Existence of traveling waves. In this section, we shall consider the existence of the non-negative non-trivial solutions for the system (3) with the form \( u(x, t) = u(x + ct) \) and \( v(x, t) = v(x + ct) \). Substituting this solution into (3), we have the wave system

\[
\begin{align*}
&cu' = \frac{d_1}{r} \left( \int_R J(y) u(\xi - y) dy - u(\xi) \right) + u(\xi) \left( 1 - u(\xi) \right) - \frac{u^2(\xi)v(\xi)}{A + Bu(\xi) + u^2(\xi)}, \\
&cv' = \frac{d_2}{r} \left( \int_R J(y) v(\xi - y) dy - v(\xi) \right) + \frac{v(\xi)}{r} \left( \frac{\alpha u^2(\xi)}{A + Bu(\xi) + u^2(\xi)} - z \right).
\end{align*}
\]

We want to find positive solutions of (5) satisfying

\[
(u, v)(-\infty) = (1, 0)
\]

and

\[
0 < \liminf_{\xi \to +\infty} u(\xi) \leq u^* \leq \limsup_{\xi \to -\infty} u(\xi) < 1, \quad 0 < \liminf_{\xi \to +\infty} v(\xi) \leq v^* \leq \limsup_{\xi \to -\infty} v(\xi) < +\infty.
\]

Linearizing the second equation of system (5) at \((u, v) = (1, 0)\), we have

\[
c\varphi'(\xi) = \frac{d_2}{r} \left( \int_R J(y) \varphi(\xi - y) dy - \varphi(\xi) \right) + \frac{\alpha \varphi(\xi)}{r(A + B + 1)} - \frac{z}{r} \varphi(\xi).
\]

Letting \( \varphi(\xi) = e^{\lambda \xi} \) yields the characteristic equation

\[
f(\lambda, c) := \frac{d_2}{r} \left( \int_R J(y)e^{-\lambda y} dy - 1 \right) - c\lambda + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - z \right) = 0.
\]

By a direct calculation, we have

\[
f(0, c) = \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - z \right) > 0,
\]

\[
\frac{\partial f(0, c)}{\partial \lambda} = \left. \frac{d_2}{r} \int_R J(y)(-y)e^{-\lambda y} dy - c \right|_{\lambda=0} = -c < 0, \quad \forall c > 0,
\]

\[
\frac{\partial f(\lambda, c)}{\partial c} = -\lambda < 0, \quad \forall \lambda > 0,
\]

\[
\frac{\partial^2 f(\lambda, c)}{\partial \lambda^2} = \frac{d_2}{r} \int_R y^2 J(y)e^{-\lambda y} dy > 0, \quad \forall \lambda > 0.
\]

Furthermore, we can get the following lemma easily.

**Lemma 2.1.** There exists a positive pair of \((\lambda^*, c_*)\) such that

\[f(\lambda^*, c_*) = 0\] and \(\frac{\partial f(\lambda^*, c_*)}{\partial \lambda} = 0\).

Furthermore,

(i) if \(c > c_*\), then \(f(\lambda, c) = 0\) has two different real roots \(\lambda_1 = \lambda_1(c), \lambda_2 = \lambda_2(c)\) with \(0 < \lambda_1 < \lambda^* < \lambda_2 < \lambda_0\) for some \(\lambda_0 \in (0, +\infty)\) and

\[
f(\lambda, c) \begin{cases} < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2), \\ > 0 & \text{for } \lambda \in (0, \lambda_1) \cup (\lambda_2, \lambda_0); \end{cases}
\]

(ii) if \(0 < c < c_*\), then \(f(\lambda, c) > 0\) for all \(\lambda \in (0, \lambda_0]\).
2.1. Non-critical waves. In this section, we fix \( c > c_* \). Define some continuous functions as follows

\[
\begin{align*}
\mathfrak{v}(\xi) &= 1, \\
\mathfrak{w}(\xi) &= e^{\lambda \xi}, \\
\mathfrak{u}(\xi) &= \begin{cases} 1 - \sigma e^{\epsilon \xi}, & \xi < \xi_1, \\
0, & \xi \geq \xi_1, \end{cases}
\end{align*}
\]

where \( \epsilon, \sigma, \eta, M \) are all positive constants determined later. Also, we define

\[
\xi_1 = \frac{1}{c} \ln \frac{1}{\sigma}, \quad \xi_2 = \frac{1}{\eta} \ln \frac{1}{M},
\]

and one can choose \( \sigma > 0 \) and \( M > 0 \) large enough such that \( \xi_2 < \xi_1 \).

**Lemma 2.2.** The function \( \mathfrak{v}(\xi) \) satisfies

\[
\epsilon \mathfrak{v}'(\xi) \geq \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) \mathfrak{v}(\xi - y) dy - \mathfrak{v}(\xi) \right) + \mathfrak{v}(\xi) \left( 1 - \mathfrak{u}(\xi) - \frac{\mathfrak{w}^2(\xi) \mathfrak{u}(\xi)}{A + B \mathfrak{v}(\xi) + \mathfrak{w}(\xi)} \right),
\]

(9)

The proof is trivial and omitted here.

Below, denote

\[
\Delta(\lambda, c) = \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1 \right) - c \lambda.
\]

The direct calculation yields that

\[
\left. \frac{\partial \Delta}{\partial \lambda} \right|_{\lambda=0} = \left. \left( \frac{d_1}{r} \int_{\mathbb{R}} (\lambda y) J(y) e^{-\lambda y} dy - c \right) \right|_{\lambda=0} = -c < 0.
\]

Due to the fact that \( \Delta(0, c) = 0 \), thus there exists some \( \tilde{\lambda} > 0 \) such that \( \Delta(\tilde{\lambda}, c) < 0 \). Then, we have the following result.

**Lemma 2.3.** Assume \( 0 < \epsilon < \min\{\tilde{\lambda}, \lambda_1\} \) and \( \sigma > \max\{1, \frac{1}{-\Delta(\epsilon, c)/(A + \eta \lambda_1)}\} \) large enough. Then, \( \mathfrak{u}(\xi) \) satisfies the following inequality

\[
\epsilon \mathfrak{u}'(\xi) \leq \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) \mathfrak{u}(\xi - y) dy - \mathfrak{u}(\xi) \right) + \mathfrak{w}(\xi) \left( 1 - \mathfrak{u}(\xi) - \frac{\mathfrak{w}^2(\xi) \mathfrak{u}(\xi)}{A + B \mathfrak{w}(\xi) + \mathfrak{w}(\xi)} \right)
\]

(10)

for any \( \xi \neq \xi_1 \).

**Proof.** If \( \xi \geq \xi_1 \), then \( \mathfrak{u}(\xi) = 0 \), thus it is easy to get (10).

On the other hand, if \( \xi < \xi_1 \), then \( \mathfrak{u}(\xi) = 1 - \sigma e^{\epsilon \xi} > 0 \), \( \mathfrak{v}(\xi) = e^{\lambda \xi} < 1 \). By the direct computation, there is

\[
\begin{align*}
\epsilon \mathfrak{u}'(\xi) &- \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) \mathfrak{u}(\xi - y) dy - \mathfrak{u}(\xi) \right) - \mathfrak{w}(\xi) \left( 1 - \mathfrak{u}(\xi) - \frac{\mathfrak{w}^2(\xi) \mathfrak{u}(\xi)}{A + B \mathfrak{w}(\xi) + \mathfrak{w}(\xi)} \right) \\
&= -\epsilon \sigma e^{\epsilon \xi} - \frac{d_1}{r} \left[ \int_{\mathbb{R}} J(y) \left( 1 - \sigma e^{\epsilon \xi - w} \right) dy - (1 - \sigma e^{\epsilon \xi}) \right] - (1 - \sigma e^{\epsilon \xi}) \sigma e^{\epsilon \xi} \\
&+ \frac{(1 - \sigma e^{\epsilon \xi})^2 \epsilon \mathfrak{v}(\xi)}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} \\
&= \epsilon e^{\epsilon \xi} \left[ -\epsilon c + \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) e^{-\epsilon y} dy - 1 \right) \right] + \frac{(1 - \sigma e^{\epsilon \xi})^2 \epsilon \mathfrak{v}(\xi)}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} \\
&\leq \epsilon e^{\epsilon \xi} \left[ -\epsilon c + \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) e^{-\epsilon y} dy - 1 \right) \right] - (1 - \sigma e^{\epsilon \xi}) \sigma e^{\epsilon \xi} + \frac{e^{\lambda \xi}}{A + B + 1}.
\end{align*}
\]
This completes the proof.

**Lemma 2.4.** The function \( \bar{\nu}(\xi) \) satisfies

\[
\frac{\nu'(\xi)}{r} \geq \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \bar{\nu}_{\xi}(\xi - y) dy - \nu(\xi) \right) + \frac{\nu(\xi)}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B \nu(\xi) + \nu^2(\xi)} - z \right). 
\]

(11)

**Proof.** Since \( \bar{\nu}(\xi) = e^{\lambda_1 \xi} \), \( \bar{\nu}(\xi) = 1 \), we have

\[
\frac{\nu'(\xi)}{r} - \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) e^{\lambda_1 \xi - y} dy - \nu(\xi) \right) = \frac{\nu(\xi)}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B \nu(\xi) + \nu^2(\xi)} - z \right) = 0.
\]

This completes the proof.

**Lemma 2.5.** Suppose \( 0 < \eta < \min\left\{ \frac{\rho - \lambda_1}{2}, \epsilon \right\} \) and let

\[
M > \max\{1, -\frac{2\alpha \sigma}{f(\lambda_1 + \eta, c)rA}\}
\]

large enough. Then, the function \( \nu(\xi) \) satisfies

\[
\frac{\nu'(\xi)}{r} \leq \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \nu_{\xi}(\xi - y) dy - \nu(\xi) \right) + \frac{\nu(\xi)}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B \nu(\xi) + \nu^2(\xi)} - z \right)
\]

(12)

for any \( \xi \neq \xi_2 \).

**Proof.** If \( \xi > \xi_2 \), then \( \nu(\xi) = 0 \), thus (12) holds.

If \( \xi < \xi_2 \), then \( \nu(\xi) = 1 - \sigma e^{\epsilon \xi} \), \( \nu(\xi) = e^{\lambda_1 \xi}(1 - Me^{\eta \xi}) \). By Lemma 2.1(ii) one has \( f(\lambda_1 + \eta, c) < 0 \), the direct computation gives that

\[
\frac{\nu'(\xi)}{r} - \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) e^{\lambda_1 \xi - y} dy - \nu(\xi) \right) = \frac{\nu(\xi)}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B \nu(\xi) + \nu^2(\xi)} - z \right)
\]

\[
eq \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) e^{\lambda_1 \xi - y} dy - \nu(\xi) \right) + \frac{1}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} - z \right)
\]

\[
= \frac{d_2}{r} \int_{\mathbb{R}} J(y) e^{\lambda_1 \xi - y} dy - \frac{1}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} + z \right)
\]

\[
+ Me^{\lambda_1 \xi}(1 - Me^{\eta \xi}) \left( \frac{\alpha (1 - \sigma e^{\epsilon \xi})^2}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} - z \right)
\]

\[
= \frac{d_2}{r} \int_{\mathbb{R}} J(y) e^{\lambda_1 \xi - y} dy + \frac{1}{r} \left( \frac{\alpha \nu^2(\xi)}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} + z \right)
\]

\[
+ Me^{\lambda_1 \xi}(1 - Me^{\eta \xi}) \left( \frac{\alpha (1 - \sigma e^{\epsilon \xi})^2}{A + B(1 - \sigma e^{\epsilon \xi}) + (1 - \sigma e^{\epsilon \xi})^2} - z \right)
\]

This completes the proof.
The proof is completed. \(\square\)

Let \(N > -\xi_2\) and we define the following set

\[
\Gamma_N = \left\{ (\phi(\cdot), \varphi(\cdot)) \in C([-N,N], \mathbb{R}^2) \left| \begin{array}{l}
\frac{\partial u}{\partial t} - \Delta u + f(u) \varphi(N) + g(u) \phi(N) \leq 0, \\
\frac{\partial v}{\partial t} - \Delta v + f(v) \varphi(N) + g(v) \phi(N) \leq 0,
\end{array} \right. \right\}
\]

It is easy to obtain that \(\Gamma_N\) is a closed, convex subset of \(C([-N,N], \mathbb{R}^2)\). For any \((\phi(\cdot), \varphi(\cdot)) \in \Gamma_N\), define

\[
\tilde{\phi}(\xi) = \begin{cases} 
\phi(N), & \xi > N, \\
\phi(\xi), & |\xi| \leq N, \\
\psi(\xi), & \xi < -N,
\end{cases}
\]

\[
\tilde{\varphi}(\xi) = \begin{cases} 
\varphi(N), & \xi > N, \\
\varphi(\xi), & |\xi| \leq N, \\
\psi(\xi), & \xi < -N.
\end{cases}
\]

Then, we consider the following initial value problem

\[
\begin{cases}
\frac{d}{dt} (\bar{u}(\cdot) e^{-(\lambda_1 + \eta)\theta} u(y)) - \int_{\mathbb{R}} J(y) e^{-(\lambda_1 + \eta)\theta} du(y) - \frac{\lambda_1}{\lambda_1 + \lambda_2} (\bar{u}(\cdot) e^{-(\lambda_1 + \eta)\theta} u(y)) + \frac{1}{r} \left( \frac{\alpha (1 - \sigma e^{\xi})^2}{A + B(1 - \sigma e^{\xi}) + (1 - \sigma)} \right) du(y) = 0, \\
\frac{d}{dt} (\bar{v}(\cdot) e^{-(\lambda_1 + \eta)\theta} v(y)) - \int_{\mathbb{R}} J(y) e^{-(\lambda_1 + \eta)\theta} dv(y) - \frac{\lambda_1}{\lambda_1 + \lambda_2} (\bar{v}(\cdot) e^{-(\lambda_1 + \eta)\theta} v(y)) + \frac{1}{r} \left( \frac{\alpha (1 - \sigma e^{\xi})^2}{A + B(1 - \sigma e^{\xi}) + (1 - \sigma)} \right) dv(y) = 0, \\
u(-N) = u(-N), \quad v(-N) = v(-N),
\end{cases}
\]

(13)
By the ODE theory, we have that (13) admits a unique solution \((u_N(\xi), v_N(\xi))\) satisfying \(u_N(\cdot), v_N(\cdot) \in C^1([-N, N])\). Define an operator \(F = (F_1, F_2) : \Gamma_N \to C([-N, N])\) as

\[
F_1[\phi, \varphi](\xi) = u_N(\xi), \quad F_2[\phi, \varphi](\xi) = v_N(\xi) \quad \text{for} \quad \xi \in (-N, N).
\]

**Lemma 2.6.** The operator \(F : \Gamma_N \to \Gamma_N\) is completely continuous.

**Proof.** We first show that \(F : \Gamma_N \to \Gamma_N\). For any \((\phi(\cdot), \varphi(\cdot)) \in \Gamma_N\), we should prove that

\[
F_1[\phi, \varphi](-N) = u(-N), \quad F_2[\phi, \varphi](-N) = v(-N)
\]

and

\[
u(\xi) \leq F_1[\phi, \varphi](\xi) \leq 1, \quad v(\xi) \leq F_2[\phi, \varphi](\xi) \leq \tau(\xi) \quad \text{for any} \quad \xi \in (-N, N).
\]

By the definition of the operator \(F\), it is obvious to see

\[
F_1[\phi, \varphi](-N) = u_N(-N) = u(-N), \quad F_2[\phi, \varphi](-N) = v_N(-N) = v(-N).
\]

For the other case, we only consider \(F_1[\phi, \varphi](\xi), F_2[\phi, \varphi](\xi)\) is the same. To get this goal, it is sufficient to show \(u(\xi) \leq u_N(\xi) \leq 1\). According to the definition of \(\tilde{\phi}(\xi)\), we know that

\[
\frac{d_1}{r} \int_{\mathbb{R}} J(y) \left( \tilde{\phi}(\xi - y) - 1 \right) dy - \frac{\phi(\xi) \varphi(\xi)}{A + B\phi(\xi) + \phi^2(\xi)} \leq 0,
\]

which implies that 1 is a super-solution of the first equation of (13), thus one can get that \(u_N(\xi) \leq 1\) for \(\xi \in (-N, N)\). Additionally, applying Lemma 2.3 yields that

\[
cu(\xi) - \frac{d_1}{r} \int_{\mathbb{R}} J(y) \left( \tilde{\phi}(\xi - y) - u(\xi) \right) dy - \phi(\xi) (1 - u(\xi)) + \frac{\phi(\xi) \varphi(\xi) u(\xi)}{A + B\phi(\xi) + \phi^2(\xi)} \\
\leq cu(\xi) - \frac{d_1}{r} \int_{\mathbb{R}} J(y) \left( u(\xi - y) - u(\xi) \right) dy - u(\xi) (1 - u(\xi)) + \frac{\tau(\xi)}{A + B + 1} \\
\leq 0
\]

for any \(\xi \in (-N, N)\). Since \(u_N(-N) = u(-N)\), the comparison principle implies that \(u(\xi) \leq u_N(\xi)\) for \(\xi \in [-N, N]\). So we get that \(u(\xi) \leq u_N(\xi) \leq 1\) for all \(\xi \in [-N, N]\). Hence, we get \(F : \Gamma_N \to \Gamma_N\).

Next, we illustrate that the operator \(F\) is completely continuous. According to (13), by direct computation, there is

\[
u_N(\xi) = v(-N) \exp \left\{ -\frac{1}{c} \int_{-N}^{\xi} \left( \frac{d_2}{r} + \phi(s) + \frac{\varphi(s) \phi(s)}{A + B\phi(s) + \phi^2(s)} \right) ds \right\} \\
+ \frac{1}{c} \int_{-N}^{\xi} \exp \left\{ \frac{1}{c} \int_{\xi}^{\eta} \left( \frac{d_2}{r} + \phi(s) + \frac{\varphi(s) \phi(s)}{A + B\phi(s) + \phi^2(s)} \right) ds \right\} \left( \phi(\eta) + \frac{d_1}{r} f(\eta) \right) d\eta
\]

and

\[
u_N(\xi) = v(-N) \exp \left\{ -\frac{1}{cr} \int_{-N}^{\xi} \left( d_2 + z - \frac{\alpha \phi^2(s)}{A + B\phi(s) + \phi^2(s)} \right) ds \right\} \\
+ \frac{d_2}{cr} \int_{-N}^{\xi} \exp \left\{ \frac{1}{cr} \int_{\xi}^{\eta} \left( d_2 + z - \frac{\alpha \phi^2(s)}{A + B\phi(s) + \phi^2(s)} \right) ds \right\} g(\eta) d\eta,
\]

(14)
Lemma 2.7. \( f_\phi(\eta) = \int_{-N}^{N} J(\eta - y)\psi(-N)dy + \int_{-N}^{N} J(\eta - y)\phi(y)dy + \int_{N}^{+\infty} J(\eta - y)\phi(N)dy, \)

\( g_\varphi(\eta) = \int_{-N}^{N} J(\eta - y)\psi(-N)dy + \int_{-N}^{N} J(\eta - y)\varphi(y)dy + \int_{N}^{+\infty} J(\eta - y)\varphi(N)dy. \)

Note that for any \( \xi \in [-N, N] \)

\[
|f_{\phi_1}(\eta) - f_{\phi_2}(\eta)| = \left| \int_{-N}^{N} J(\eta - y) (\phi_1(y) - \phi_2(y)) dy + \int_{N}^{+\infty} J(\eta - y) (\phi_1(N) - \phi_2(N)) dy \right|
\leq \left| \int_{-N}^{N} J(\eta - y) (\phi_1(y) - \phi_2(y)) dy \right| + \left| \int_{N}^{+\infty} J(\eta - y) (\phi_1(N) - \phi_2(N)) dy \right|
\leq 2 \max_{y \in [-N, N]} |\phi_1(y) - \phi_2(y)|,
\]

and the same computation can give

\[
|g_{\varphi_1}(\eta) - g_{\varphi_2}(\eta)| \leq 2 \max_{y \in [-N, N]} |\varphi_1(y) - \varphi_2(y)|.
\]

Then, by the definition of the operator \( F \), one can get the continuity of \( F \) from (14) and (15).

Finally, we show that \( F \) is compact. Since \( u_N \) and \( v_N \) are all of class \( C^1([-N, N]) \), according to (13) for any \( (u_N(\xi), v_N(\xi)) \in \Gamma_N \), we know \( u_N' \) and \( v_N' \) are all bounded for any \( \xi \in [-N, N] \). Thus, it is obtained that \( F \) is compact. Then the proof is completed.

Furthermore, owing to \( \Gamma_N \) is closed and convex, thus the Schauder’s fixed-point theorem implies that there exists \( (u_N(\cdot), v_N(\cdot)) \in \Gamma_N \) such that

\[
(u_N(\xi), v_N(\xi)) = F[u_N, v_N](\xi), \quad \forall \xi \in (-N, N).
\]

To obtain the existence of solutions for system (5), we need some estimates about \( (u_N(\cdot), v_N(\cdot)) \).

Set

\[
C^{1,1}([-N, N]) = \{ u \in C^1([-N, N]) | u' \text{ is Lipschitz continuous} \}
\]

and

\[
||u(x)||_{C^{1,1}([-N, N])} = \max_{x \in [-N, N]} |u(x)| + \max_{x \in [-N, N]} |u'(x)| + \max_{x, y \in [-N, N], x \neq y} \frac{|u'(x) - u'(y)|}{|x - y|}.
\]

Lemma 2.7. There exists some constant \( C(N) > 0 \) such that

\[
||u_N(\cdot)||_{C^{1,1}([-N, N])} \leq C(N), \quad ||v_N(\cdot)||_{C^{1,1}([-N, N])} \leq C(N)
\]

for any \( N > -\xi_2 \).

Proof. By the above discussion, we know that \( (u_N(\cdot), v_N(\cdot)) \) satisfies

\[
cu_N'(\xi) = \frac{d1}{r} \int_{R} J(y) [\bar{u}_N(\xi - y) - u_N(\xi)] dy + u_N(\xi) (1 - u_N(\xi)) - \frac{u_N^2(\xi) v_N(\xi)}{A + B u_N(\xi) + u_N^2(\xi)} \]

(16)
and
\[
c v'_{N}(\xi) = \frac{d_2}{r} \int_{R} J(y) \left[ \hat{v}_{N}(\xi - y) - v_{N}(\xi) \right] dy + \frac{v_{N}(\xi)}{r} \left( \frac{\alpha u_{N}^2(\xi)}{A + Bu_{N}(\xi) + u_{N}^2(\xi)} - z \right),
\]
where
\[
\begin{align*}
\hat{u}_{N}(\xi) &= \begin{cases} u_{N}(N), & \xi > N, \\
u_{N}(\xi), & |\xi| \leq N, \\
u_{N}(\xi), & \xi < -N,
\end{cases} \\
\hat{v}_{N}(\xi) &= \begin{cases} v_{N}(N), & \xi > N, \\
v_{N}(\xi), & |\xi| \leq N, \\
u_{N}(\xi), & \xi < -N.
\end{cases}
\end{align*}
\]

Following that \(0 \leq u_{N}(\xi) \leq 1\) and \(0 \leq v_{N}(\xi) \leq e^{\lambda_{1}N}\) for \(\xi \in [-N, N]\). Letting \(N_{0}(N) := \max\{1, e^{\lambda_{1}N}\}\), then there hold
\[
\begin{align*}
|u'_{N}(\xi)| &= \frac{1}{c} \left| \frac{d_1}{r} \int_{R} J(y) \left[ \hat{u}_{N}(\xi - y) - u_{N}(\xi) \right] dy + u_{N}(\xi) \left( 1 - u_{N}(\xi) \right) - \frac{u_{N}^2(\xi)v_{N}(\xi)}{A + Bu_{N}(\xi) + u_{N}^2(\xi)} \right| \\
&\leq \frac{d_1}{cr} \left| \int_{R} J(y) \left[ \hat{u}_{N}(\xi - y) - u_{N}(\xi) \right] dy \right| + \frac{1}{c} \left| u_{N}(\xi) \left( 1 - u_{N}(\xi) \right) \right| \\
&\quad + \frac{1}{c} \left| \frac{u_{N}^2(\xi)}{A + Bu_{N}(\xi) + u_{N}^2(\xi)} \right| |v_{N}(\xi)| \\
&\leq \frac{d_1}{cr} \left| \int_{R} J(y) \hat{u}_{N}(\xi - y) dy \right| + \frac{d_1}{cr} |u_{N}(\xi)| + \frac{1}{4c} + \frac{1}{c} \left| \frac{A}{w_{N}(\xi)} + \frac{B}{w_{N}(\xi)} + 1 \right| |v_{N}(\xi)| \\
&\leq \frac{d_1}{cr} + \frac{d_1}{cr} + \frac{1}{4c} + \frac{N_{0}(N)}{c} \\
&= \frac{8d_1 + r + 4rN_{0}(N)}{4cr}.
\end{align*}
\]

and
\[
\begin{align*}
|v'_{N}(\xi)| &= \left| \frac{d_2}{cr} \int_{R} J(y) \left[ \hat{v}_{N}(\xi - y) - v_{N}(\xi) \right] dy + \frac{v_{N}(\xi)}{cr} \left( \frac{\alpha u_{N}^2(\xi)}{A + Bu_{N}(\xi) + u_{N}^2(\xi)} - z \right) \right| \\
&\leq \frac{d_2}{cr} \left| \int_{R} J(y) \hat{v}_{N}(\xi - y) dy \right| + \frac{d_2}{cr} |v_{N}(\xi)| + \frac{1}{cr} |v_{N}(\xi)| \left| \frac{\alpha u_{N}^2(\xi)}{A + Bu_{N}(\xi) + u_{N}^2(\xi)} - z \right| \\
&\leq \frac{d_2N_{0}(N)}{cr} \left| \int_{R} J(y)e^{-\lambda_{1}v_{N}dy} + \frac{d_2N_{0}(N)}{cr} + \frac{(\alpha + z)d_2N_{0}(N)}{cr} \right| \left| \frac{(\alpha + z + d_2)N_{0}(N)}{cr} \right| \\
&= \frac{d_2N_{0}(N)}{cr} \left| \int_{R} J(y)e^{-\lambda_{1}v_{N}dy} + \frac{(\alpha + z + d_2)N_{0}(N)}{cr} \right| .
\end{align*}
\]

Thus, there exists some constant \(C_{1}(N) > 0\) such that
\[
\|u_{N}\|_{C^{1}([{-N,N})]} < C_{1}(N) \quad \text{and} \quad \|v_{N}\|_{C^{1}([{-N,N})]} < C_{1}(N).
\]

It is obvious to obtain that
\[
|u_{N}(\xi) - u_{N}(\eta)| < C_{1}(N)|\xi - \eta| \quad \text{and} \quad |v_{N}(\xi) - v_{N}(\eta)| < C_{1}(N)|\xi - \eta| 
\]
for any $\xi, \eta \in [-N, N]$. In view of (16), we have

$$c |u_N' (\xi) - u_N' (\eta)|$$

$$\leq \frac{d_1}{r} \left| \int_{-R}^{+R} J(y) [\hat{u}_N(\xi - y) - \hat{u}_N(\eta - y)] dy + |u_N(\xi)(1 - u_N(\xi)) - u_N(\eta)(1 - u_N(\eta))| \right.$$  

$$+ \left| \frac{u_N^2(\xi)v_N(\xi)}{A + Bu_N(\xi) + u_N^2(\xi)} - \frac{u_N^2(\eta)v_N(\eta)}{A + Bu_N(\eta) + u_N^2(\eta)} \right| + \frac{d_1}{r} |u_N(\xi) - u_N(\eta)|$$

$$:= \frac{d_1}{r} u_1 + u_2 + u_3 + \frac{d_1}{r} u_4. \quad (19)$$

By the condition $(J)$, we can assume that $L$ is its Lipschitz constant and $R$ is the radius of $\text{supp} J$. Then, there are

$$u_1 = \left| \int_{-R}^{+R} J(y) \hat{u}_N(\xi - y) dy - \int_{-R}^{+R} J(y) \hat{u}_N(\eta - y) dy \right|$$

$$= \left| \int_{-R}^{\xi + R} J(\xi - y) \hat{u}_N(y) dy - \int_{-R}^{\eta + R} J(\eta - y) \hat{u}_N(y) dy \right|$$

$$= \left| \int_{-R}^{\eta + R} J(\xi - y) \hat{u}_N(y) dy - \int_{-R}^{\xi + R} J(\eta - y) \hat{u}_N(y) dy \right.$$  

$$+ \int_{\eta + R}^{\xi + R} [J(\xi - y) - J(\eta - y)] \hat{u}_N(y) dy \left| \right.$$  

$$\leq 2 \|J\|_{L^\infty} |\xi - \eta| + \left| \int_{\eta + R}^{\xi + R} [J(\xi - y) - J(\eta - y)] \hat{u}_N(y) dy \right|$$

$$+ \left| \int_{\eta + R}^{\xi + R} [J(\xi - y) - J(\eta - y)] \hat{u}_N(y) dy \right|$$

$$\leq (4 \|J\|_{L^\infty} + 2RL) |\xi - \eta|,$$

$$u_2 = |u_N(\xi)(1 - u_N(\xi)) - u_N(\eta)(1 - u_N(\eta))|$$

$$= |u_N(\xi) - u_N^2(\xi) - u_N(\eta) + u_N^2(\eta)|$$

$$\leq |u_N(\xi) - u_N(\eta)| + |u_N^2(\xi) - u_N^2(\eta)|$$

$$\leq |u_N(\xi) - u_N(\eta)| + |u_N(\xi) - u_N(\eta)| \cdot |u_N(\xi) + u_N(\eta)|$$

$$\leq 3 |u_N(\xi) - u_N(\eta)|,$$

and there are constants $L_0 \geq A + B + 1$ and $\tilde{L} \geq (2AN_0(N) + BN_0(N))$ such that

$$u_3 = \left| \frac{u_N^2(\xi)v_N(\xi)}{A + Bu_N(\xi) + u_N^2(\xi)} - \frac{u_N^2(\eta)v_N(\eta)}{A + Bu_N(\eta) + u_N^2(\eta)} \right|$$

$$\leq 2 \left| \frac{u_N^2(\cdot)}{A + Bu_N(\cdot) + u_N^2(\cdot)} \right|_{L^\infty [-N, N]} |v_N(\xi) - v_N(\eta)|$$

$$+ (Au_N^2(\xi) + Bu_N(\eta)u_N^2(\xi) + u_N^2(\xi)u_N^2(\eta)) |v_N(\xi) - v_N(\eta)|$$

$$+ (Au_N(\xi)(u_N(\xi) + u_N(\eta)) + Bu_N(\xi)u_N(\eta)v_N(\xi)) |v_N(\xi) - v_N(\eta)|.$$
for any $\xi, \eta$ equations (16) and (17). According to the estimates in Lemma 2.7, for the sequence $L$

Thus, there is a constant $L_1(N) > 0$ such that

Then applying (17), we also have

Then applying (17), we also have

Then applying (17), we also have

Then applying (17), we also have

Then applying (17), we also have

Then applying (17), we also have
Lemma 2.8. The functions $u$ and $v$ are non-trivial, in the sense that

$$0 < u < 1, \quad v > 0 \text{ in } \mathbb{R}.$$ 

Proof. Firstly, owing to the definition of $u$, we have $v > 0$ in $\xi \in (\xi_2, \xi)$. Thus, we only need to show that $v > 0$ in $\xi \in [\xi_2, \infty)$. Contrarily, we assume that there exists a real number $\xi_0 \in [\xi_2, \infty)$, such that $v(\xi_0) = 0$. Since $v(\xi) \geq 0$ in $\mathbb{R}$, hence $v'(\xi_0) = 0$. From the second equation of (5), we get that $v(\xi_0 - y) = v(\xi_0) = 0$ for any $y \in \mathbb{R}$, one contradiction to the fact that $v > 0$ for all $\xi \leq \xi_2$.

Now, we show that $u > 0$ over $\mathbb{R}$. Indeed, if $u(\hat{\xi}) = 0$ for some real number $\hat{\xi}$, then

$$0 = -cu'(\hat{\xi}) + \frac{d_1}{r} \int_{\mathbb{R}} J(y) \left[ u(\hat{\xi} - y) - u(\hat{\xi}) \right] dy + u(\hat{\xi}) \left( 1 - u(\hat{\xi}) \right) - \frac{u^2(\hat{\xi})v(\hat{\xi})}{A + Bu(\hat{\xi}) + u^2(\hat{\xi})},$$

where $u'(\hat{\xi}) = 0$, whence $u(\hat{\xi} - y) = 0$ for any $y \in \mathbb{R}$, which implies $u \equiv 0$ in $\mathbb{R}$. This contradicts to the fact that $u \geq u$.

Similarly, we claim that $u < 1$ in $\mathbb{R}$ by a contradiction argument. If there exists a real number $\xi_*$ such that $u(\xi_*) = 1$, then

$$0 = -cu'(\xi_*) + \frac{d_1}{r} \int_{\mathbb{R}} J(y) \left[ u(\xi_* - y) - u(\xi_*) \right] dy + u(\xi_*) \left( 1 - u(\xi_*) \right) - \frac{u^2(\xi_*)v(\xi_*)}{A + Bu(\xi_*) + u^2(\xi_*)} \leq - \frac{v(\xi_*)}{A + B + 1} < 0.$$ 

This contradiction leads to the inequality $u < 1$ and the proof is finished. \hfill \Box

Lemma 2.9. There exists some positive constant $C_0$, such that

$$\int_{\mathbb{R}} J(y) \left| \frac{v(\xi - y)}{v(\xi)} \right| dy < C_0, \quad \left| \frac{v'(\xi)}{v(\xi)} \right| < C_0.$$ 

Proof. Let $\omega(\xi) = \frac{v'(\xi)}{v(\xi)}$ and $\mu = \frac{2 + \alpha}{cr}$. Since $v(\xi)$ satisfies

$$cv'(\xi) = \frac{d_2}{r} \int_{\mathbb{R}} J(y) \left[ v(\xi - y) - v(\xi) \right] dy + \frac{v(\xi)}{r} \left( \frac{\alpha u^2(\xi)}{A + Bu(\xi) + u^2(\xi)} - \frac{z}{cr} \right),$$

then

$$\omega(\xi) = \frac{d_2}{er} \int_{\mathbb{R}} J(y) \frac{v(\xi - y)}{v(\xi)} dy - \frac{d_2}{cr} + \frac{\alpha}{cr} \frac{u^2(\xi)}{A + Bu(\xi) + u^2(\xi)} - \frac{z}{cr}$$

$$= \frac{d_2}{cr} \int_{\mathbb{R}} J(y) e^{\frac{r}{\alpha} \omega(s)ds} dy - \mu + \frac{\alpha}{cr} \frac{u^2(\xi)}{A + Bu(\xi) + u^2(\xi)}$$

$$\geq \frac{d_2}{cr} \int_{\mathbb{R}} J(y) e^{\frac{r}{\alpha} \omega(s)ds} dy - \mu.$$
Set \( \rho = \frac{d_L}{d_R} \) and \( L(\xi) = \exp \left\{ \mu \xi + \int_0^\xi \omega(s) ds \right\} \). Thus, a direct computation gives
\[
I'(\xi) = (\mu + \omega(\xi)) L(\xi) \geq \rho \int_R J(y) e^{\int_\xi^y \omega(s) ds} dy L(\xi),
\]
(21)
which implies that \( L(\xi) \) is non-decreasing. Meanwhile, \( \lim_{\xi \to -\infty} L(\xi) = 0 \). Denote the radius of \( \text{supp} J \) by \( r_0 \). Following from (J), we know \( r_0 > 0 \). Thus, we can take some \( r_* > 0 \) with \( 2r_* < r_0 \). Now, integrating both sides of (21) from \( -\infty \) to \( \xi \), we get
\[
L(\xi) \geq \rho \int_{-\infty}^{\xi} \int_R J(y) e^{\int_x^y \omega(s) ds} dy L(x) dx
= \rho \int_R J(y) e^{\mu y} \int_{-\infty}^{\xi} L(x-y) dx dy
\geq \rho \int_R J(y) e^{\mu y} \int_{\xi-r_*}^{\xi} L(x-y) dx dy
\geq \rho r_* \int_R J(y) e^{\mu y} L(\xi - r_- y) dy,
\]
and so
\[
\int_{-\infty}^0 J(y) e^{\mu y} \frac{L(\xi - r_+ y)}{L(\xi)} dy \leq (\rho r_*)^{-1}.
\] (22)
Moreover, integrating two sides of (21) from \( \xi - r_* \) to \( \xi \) yields
\[
L(\xi) - L(\xi - r_*) \geq \rho \int_{\xi-r_*}^{\xi} \int_R J(y) e^{\int_x^y \omega(s) ds} dy L(x) dx
= \rho \int_R J(y) e^{\mu y} \int_{\xi-r_*}^{\xi} L(x-y) dx dy
\geq \rho r_* \int_{-\infty}^{-2r_*} J(y) e^{\mu y} L(\xi - r_- y) dy
\geq \rho r_* \int_{-\infty}^{-2r_*} J(y) e^{\mu y} dy L(\xi + r_*).
\]
Since \( 2r_* < r_0 \), \( \int_{-\infty}^{-2r_*} J(y) e^{\mu y} dy > 0 \). Let
\[
\delta_0 := \frac{1}{\rho r_* \int_{-\infty}^{-2r_*} J(y) e^{\mu y} dy}.
\]
Thus, there holds
\[
L(\xi + r_+) \leq \delta_0 L(\xi) \text{ for all } \xi \in \mathbb{R}.
\] (23)
Note that
\[
J(y) \frac{v(\xi - y)}{v(\xi)} dy = \int_R J(y) e^{\mu y} \frac{L(\xi - y)}{L(\xi)} dy,
\]
then, it follows from (22) and (23) that
\[
\int_R J(y) \frac{v(\xi - y)}{v(\xi)} dy = \int_{-\infty}^0 J(y) e^{\mu y} \frac{L(\xi - y)}{L(\xi)} dy + \int_{0}^{+\infty} J(y) e^{\mu y} \frac{L(\xi - y)}{L(\xi)} dy
\leq \int_{-\infty}^0 J(y) e^{\mu y} \frac{L(\xi - y)}{L(\xi)} dy + \int_{0}^{+\infty} J(y) e^{\mu y} dy.
\[
\begin{aligned}
& \leq \delta_0 \int_{-\infty}^{0} J(y) e^{\mu y} \frac{L(\xi - r_\ast - y)}{L(\xi)} dy + \int_{0}^{+\infty} J(y) e^{\mu y} dy \\
& \leq \frac{\delta_0}{\rho v^*_\ast} + \int_{0}^{+\infty} J(y) e^{\mu y} dy.
\end{aligned}
\]

Furthermore, noting
\[
|\omega(\xi)| \leq \rho \int_{\mathbb{R}} J(y) \frac{v(\xi - y)}{v(\xi)} dy + \frac{d_2 + z}{cr} + \frac{\alpha}{cr A + B + 1},
\]
we complete the proof. \hfill \Box

**Lemma 2.10.** Choose \(c_0 \in [c_1, c_2]\) with \(0 < c_1 \leq c_2\) and let \(\{c_k, u_k, v_k\}\) be a sequence of traveling waves of system (5) with speeds \(\{c_k\}\). If there is a sequence \(\{\xi_k\}\) such that \(v_k(\xi_k) \to +\infty\) as \(k \to +\infty\), then \(u_k(\xi_k) \to 0\) as \(k \to +\infty\).

**Proof.** Assume that there exists a subsequence of \(\{\xi_k\}_{k \in \mathbb{N}}\), also denoted by \(\xi_k\), such that \(v_k(\xi_k) \to +\infty\) as \(k \to +\infty\) and \(u_k(\xi_k) \geq \epsilon\) in \(\mathbb{R}\) for all \(k \in \mathbb{N}\) with some positive constant \(\epsilon\). Since \(0 < u_k < 1\) and \(v_k > 0\) in \(\mathbb{R}\), then the first equation of (5) yields that
\[
u_k'(\xi) \leq \frac{r + d_1}{c_1 r} \in \mathbb{R}.
\]
Thus, there exists a constant \(\tau\) such that
\[
u_k(\xi) \geq \frac{\epsilon}{2}, \forall \xi \in [\xi_k - \tau, \xi_k]
\]
for all \(k \in \mathbb{N}\), where \(\tau = \frac{c_0 + \tau}{2(r + d_1)}\). Additionally, Lemma 2.9 gives that \(\left|\frac{\nu_k'(\xi)}{\nu_k(\xi)}\right| \leq M_0\) in \(\mathbb{R}\) for some \(M_0 > 0\) independently of \(k\). Therefore, there is
\[
u_k(\xi) = \exp \left\{ \int_{\xi}^{\xi_k} \frac{\nu_k'(s)}{\nu_k(s)} ds \right\} \leq e^{M_0 \tau}, \forall \xi \in [\xi_k - \tau, \xi_k]
\]
for all \(k \in \mathbb{N}\). Obviously, following from \(v_k(\xi_k) \to +\infty\) as \(k \to +\infty\) that
\[
\min_{\xi \in [\xi_k - \tau, \xi_k]} v_k(\xi) \geq e^{-M_0 \tau} v_k(\xi_k) \to +\infty \text{ as } k \to +\infty.
\]
Furthermore, by the first equation of (5), one has
\[
\max_{\xi \in [\xi_k - \tau, \xi_k]} \frac{1}{c_1} \left( \frac{d_1}{r} + \frac{1}{4} \right) - \frac{c^2}{c_2 (A + B \tau^2 + \frac{1}{4})} \leq \min_{\xi \in [\xi_k - \tau, \xi_k]} v_k(\xi) \to -\infty \text{ as } k \to +\infty.
\]
Take \(M = \frac{2}{\tau}\), some \(k_0 > 0\) exists such that
\[
u_k(\xi) \leq -M, \forall k \geq k_0 \text{ and } \forall \xi \in [\xi_k - \tau, \xi_k].
\]
Since \(u_k < 1\) in \(\mathbb{R}\) for each \(k \in \mathbb{N}\), the above inequality gives that \(u_k(\xi_k) \leq -1\) for all \(k \geq k_0\). This contradicts to the fact that \(u_k(\xi) > 0\) in \(\mathbb{R}\) and this ends the proof. \hfill \Box

**Lemma 2.11.** If \(\limsup_{k \to +\infty} v(\xi) = +\infty\), then \(\lim_{\xi \to +\infty} v(\xi) = +\infty\).

**Proof.** Conversely, assume \(v_- = \liminf_{\xi \to +\infty} v(\xi) < +\infty\). Then, there is some sequence \(\{\xi_k\}\) such that \(\lim_{k \to +\infty} v(\xi_k) = v_-\) with \(\xi_k \to +\infty\) as \(k \to +\infty\). Without loss of generality, we can assume \(v(\xi_k) \leq v_- + 1\) for all \(k \in \mathbb{N}\).
Lemma 2.12. Since $\lim_{k \to +\infty} v(\xi) = +\infty$, then $\lim_{k \to +\infty} v(\eta_k) = +\infty$. Hence, setting $m_1 = \sup_{\xi \in \mathbb{R}} \left| \frac{v(\xi)}{v(\eta_k)} \right|$, one can assume $v(\eta_k) \geq (v_- + 1)e^{m_1r_0}$ without loss of generality, in which $r_0$ is the radius of $\text{supp}J$. Note that

$$v(\eta_k) = e^{\int_{\xi_k}^{\eta_k} \frac{v(s)}{v(\eta_k)} ds} \leq e^{m_1|\xi - \eta_k|} \leq e^{m_1r_0}, \text{ if } |\xi - \eta_k| \leq r_0.$$ 

Which gives $v(\xi) \geq v_- + 1$ for all $\xi \in [\eta_k - r_0, \eta_k + r_0]$. Thus, $[\eta_k - r_0, \eta_k + r_0] \cap (\xi_k, \xi_{k+1})$. The second equation of (5) yields that

$$0 = cv'(\eta_k) = \frac{d^2}{r} \int_{\mathbb{R}} J(y) [v(\eta_k - y) - v(\eta_k)] dy + \frac{\alpha u^2(\eta_k)}{r} \left( \frac{\alpha u^2(\eta_k)}{A + Bu(\eta_k) + u^2(\eta_k)} - z \right)$$

$$\leq \frac{v(\eta_k)}{r} \left( \frac{\alpha u^2(\eta_k)}{A + Bu(\eta_k) + u^2(\eta_k)} - z \right).$$

(24)

Further, since $\lim_{k \to +\infty} v(\eta_k) = +\infty$, Lemma 2.10 implies that $u(\eta_k) \to 0$ as $k \to +\infty$. Therefore, there exists some $k_0$ so that

$$\frac{v(\eta_k)}{r} \left( \frac{\alpha u^2(\eta_k)}{A + Bu(\eta_k) + u^2(\eta_k)} - z \right) < 0$$

for all $k \geq k_0$, which contradicts to inequality (24). This completes the proof. 

Lemma 2.12. [41] Assume $c > 0$ and $B(\cdot)$ is a continuous function with $B(\pm \infty) := \lim_{\xi \to \pm \infty} B(\xi)$. Let $z(\xi)$ be a measurable function satisfying

$$cz(\xi) = \int_{\mathbb{R}} J(y)e^{\xi - y} z(s) ds dy + B(\xi) \text{ in } \mathbb{R}.$$ 

Then, $z$ is uniformly continuous and bounded. Moreover, $\mu^\pm := \lim_{\xi \to \pm \infty} z(\xi)$ exist and are real roots of the characteristic equation

$$c\mu = \int_{\mathbb{R}} J(y)e^{-\mu y} dy + B(\pm \infty) \ (i = 1, 2).$$

Lemma 2.13. $v(\xi)$ is bounded in $\mathbb{R}$.

Proof. Owing to $v(\xi) > 0$ in $\mathbb{R}$, then we only need to prove $\lim_{\xi \to \pm \infty} v(\xi) < +\infty$. Assume that $\lim_{\xi \to \pm \infty} v(\xi) = +\infty$. Then, Lemma 2.11 gives that $\lim_{\xi \to +\infty} v(\xi) = +\infty$ and by Lemma 2.10, we have $\lim_{\xi \to +\infty} u(\xi) = 0$. Let $\omega(\xi) = \frac{v'(\xi)}{v(\xi)}$ and according to the second equation of (5), there is

$$c\omega(\xi) = \frac{d_2}{r} \int_{\mathbb{R}} J(y)e^{\xi - y} \omega(y) dy - \frac{d_2}{r} + \frac{1}{r} \left( \frac{\alpha u^2(\xi)}{A + Bu(\xi) + u^2(\xi)} - z \right).$$

(25)

Since $\lim_{\xi \to +\infty} u(\xi) = 0$ and $\lim_{\xi \to -\infty} u(\xi) = 1$, applying Lemma 2.12 can obtain that

$$\lim_{\xi \to +\infty} \omega(\xi) \text{ exists and satisfies the following equation}$$

$$\frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\lambda y} dy - 1 \right) - c\lambda - \frac{z}{r} = 0. \quad (26)$$
Letting
\[ g(\lambda, c) = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1 \right) - c\lambda - \frac{z}{r}, \]
it is easy to calculate that \( g(0, c) = -\frac{z}{r} < 0 \) and
\[ \frac{\partial g}{\partial \lambda}\big|_{\lambda=0} = -c < 0, \quad \frac{\partial^2 g}{\partial \lambda^2} = \frac{d_2}{r} \int_{\mathbb{R}} J(y)g^2e^{-\lambda y}dy > 0. \]
Then, there is a unique positive root \( \lambda_* \) of (26). Since \( v(\xi) > 0 \) and \( \lim_{\xi \to +\infty} v(\xi) = +\infty \), we have \( \lim_{\xi \to +\infty} \omega(\xi) = 0 \). Thus, \( \lim_{\xi \to +\infty} \omega(\xi) = \lambda_* \).
Since \( v(\xi) \leq e^{\lambda_*\xi} \) in \( \mathbb{R} \), where \( \lambda_1 \) is the smaller positive real root of
\[ \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\lambda_1 y}dy - 1 \right) - c\lambda_1 + \frac{1}{r} A + B + 1 - \frac{z}{r} = 0 \quad (27) \]
and \( \lambda_2 \) is the bigger real root of (27). Notice that
\[ \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\lambda_2 y}dy - 1 \right) - c\lambda_2 - \frac{z}{r} = -\frac{1}{r} A + B + 1 < 0, \]
we have \( \lambda_2 < \lambda_* \). It follows from \( \lim_{\xi \to +\infty} \omega(\xi) = \lambda_* \) that there exists some \( \xi_* \) large enough such that
\[ v(\xi) \geq C e^{\frac{\lambda_2+\lambda_*}{2} \xi} \quad \text{for all} \quad \xi \geq \xi_*, \]
in which \( C \) is some positive constant. This is a contradiction because of \( \lambda_1 < \frac{\lambda_2+\lambda_*}{2} \) and \( v(\xi) \leq \Omega(\xi) = e^{\lambda_*\xi} \). Thus, \( v(\xi) \) is bounded. The proof is ended. \( \square \)

**Lemma 2.14.** \( \inf_{\mathbb{R}} u > 0. \)

**Proof.** Notice that the \( C^\infty \) function \( u \) satisfies \( 0 < u < 1 \) in \( \mathbb{R} \) and \( u(-\infty) = 1 \). Contrarily, assume that \( \inf_{\mathbb{R}} u = 0 \). Then there exists a sequence \( \{\xi_k\} \) converging to \( +\infty \) such that \( u(\xi_k) \to 0 \) as \( k \to +\infty \). On the other hand, since both functions \( u(\xi) \) and \( v(\xi) \) are bounded, system (5) guarantees that the function \( u(\xi) \) and \( v(\xi) \) are \( C^\infty \) bounded. Therefore, by the Arzela-Ascoli theorem, the functions \( u_k(\xi) := u(\xi + \xi_k) \) and \( v_k(\xi) := v(\xi + \xi_k) \) converge in \( C^\infty_{\nu} (\mathbb{R}) \) as \( k \to +\infty \), up to extraction of a subsequence, to some nonnegative \( C^\infty \) functions \( u_\infty \) and \( v_\infty \). Furthermore,
\[ cu'_\infty(\xi) = \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y)u_\infty(\xi - y)dy - u_\infty(\xi) \right) + u_\infty(\xi) \left( 1 - u_\infty(\xi) \right) - \frac{u^2_\infty(\xi)}{A + Bu_\infty(\xi) + \frac{d_2}{r}} \quad (28) \]
in \( \mathbb{R} \) and \( u_\infty(0) = 0 \). Owing to 0 is a global minimum of \( u_\infty \), one has \( u'_\infty(0) = 0 \) and the above equality implies that \( u_\infty(-y) = 0 \) for all \( y \in \mathbb{R} \). By the second equation of (5), we have
\[ cv'_\infty(\xi) = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)v_\infty(\xi - y)dy - v_\infty(\xi) \right) - \frac{z}{r} v_\infty(\xi). \quad (29) \]
Then, integrating both sides of (29) from \(-j\) to \( j \) yields
\[ c (v_\infty(j) - v_\infty(-j)) = \frac{d_2}{r} \int_{-j}^{j} \left( \int_{\mathbb{R}} J(y)v_\infty(\xi - y)dy - v_\infty(\xi) \right) d\xi - \frac{z}{r} \int_{-j}^{j} v_\infty(\xi)d\xi. \]
Due to the fact that $v_{\infty}$ is bounded in $\mathbb{R}$, we can assume that there exists a constant $M > 0$ such that $v_{\infty} \leq M$, then

$$
\frac{z}{r} \int_{-j}^{j} v_{\infty}(\xi)d\xi = \frac{d_{2}}{r} \int_{-j}^{j} \left( \int_{\mathbb{R}} J(y)v_{\infty}(\xi - y)dy - v_{\infty}(\xi) \right)d\xi - c(v_{\infty}(j) - v_{\infty}(-j)) 
$$

$$
\leq 2M\frac{d_{2}}{r} \int_{\mathbb{R}} J(\xi)|d\xi| + cM.
$$

Thus, we can get $v_{\infty} \in L^{1}(\mathbb{R})$. Applying the Fourier transform to (29), there is

$$
-ct\xi \hat{v}_{\infty} = \frac{d_{2}}{r}(\hat{J} - 1)\hat{v}_{\infty} - \frac{z}{r} \hat{v}_{\infty}.
$$

Since $\hat{J}(0) = 1$, the above equality gives that $0 = -\frac{z}{r} \int_{\mathbb{R}} v_{\infty}(\xi)d\xi$ at 0, which implies $v_{\infty} = 0$.

Define $\Phi_{k}(\xi) = \frac{u(\xi + \xi_{k})}{u(\xi_{k})} := \frac{u_{k}(\xi)}{u(\xi_{k})}$, by the first equation of (5), we have

$$
c\Phi_{k}'(\xi) = \frac{d_{1}}{r}(J * \Phi_{k} - \Phi_{k}) + \Phi_{k}(1 - u_{k}) - \Phi_{k} \frac{u_{k}v_{k}}{A + Bu_{k} + u_{k}^{2}}.
$$

Lemma 2.13 gives that $v(\xi)$ is bounded in $\mathbb{R}$. Hence, as in the proof of Lemma 2.9, we can get $\frac{u'(\xi)}{u(\xi)}$ is locally uniformly bounded in $\mathbb{R}$ easily. Note that $\Phi_{k}(\xi) = \exp \left\{ \int_{\xi_{k}}^{\xi} \frac{u'(\tau)}{u(\tau)} d\tau \right\}$ is locally bounded in $\mathbb{R}$, which implies that $\Phi_{k}(\xi)$ is locally uniformly bounded in $\mathbb{R}$. Thus, up to extraction of a subsequence, some $\Phi_{\infty}(\xi)$ exists such that $\Phi_{k}(\xi) \to \Phi_{\infty}(\xi)$ in $C^{1}_{\text{loc}}(\mathbb{R})$ as $k \to \infty$ with

$$
c\Phi_{\infty}' = \frac{d_{1}}{r}(J * \Phi_{\infty} - \Phi_{\infty}) + \Phi_{\infty}.
$$

It is easy to know that $\Phi_{\infty}(\xi) > 0$ in $\mathbb{R}$. Setting $\Theta(\xi) = \frac{\Phi_{\infty}'(\xi)}{\Phi_{\infty}(\xi)}$, then there holds

$$
c\Theta = \frac{d_{1}}{r} \left( \int_{\mathbb{R}} J(y)e^{\xi - \Theta(y)}dy - 1 \right) + 1.
$$

Lemma 2.12 gives that $\Theta(\pm \infty)$ exist and are positive real roots of the characteristic equation

$$
c\lambda = \frac{d_{1}}{r} \left( \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1 \right) + 1.
$$

Hence, we have $\Theta(\pm \infty) > 0$. Meanwhile, according to the definition of $\Theta$, some $\xi_{0}$ exists so that $\Phi_{\infty}'(\xi) > 0$ for any $\xi \leq \xi_{0}$. But, since $0 < u_{k}(\xi) < 1$ in $\mathbb{R}$ and $u(-\infty) = 1$ for every given $k$, there is some $\xi_{1}$ exists such that $u'_{k}(\xi) \leq 0$ for any $\xi \leq \xi_{1}$. Hence, taking $\xi \leq \min\{\xi_{0}, \xi_{1}\}$, a contradiction happens. The proof is thereby completed.

\[Q.E.D.\]

**Lemma 2.15.** Given some positive real numbers $c_{1}, c_{2}$ with $c_{1} \leq c_{2}$. For any solution $(u, v)$ of system (5) with speed $c \in [c_{1}, c_{2}]$ satisfying $0 < u < 1$, $v > 0$, there is $\varepsilon > 0$ so that if $v(\xi) \leq \varepsilon$ for any $\xi \in \mathbb{R}$, then $v'(\xi) > 0$.

**Proof.** On the contrary, we assume that there is no such $\varepsilon$. Then there exists a sequence of real numbers $\{c_{k}\}$ in $[c_{1}, c_{2}]$, a sequence of solutions $\{(u_{k}, v_{k})\}$ of (5) with speed $c = c_{k}$ and $0 < u_{k} < 1$, $v_{k} > 0$ in $\mathbb{R}$, and a sequence of real numbers $\xi_{k}$ such that

$$
v_{k}(\xi_{k}) \to 0 \text{ as } k \to +\infty \text{ and } v'_{k}(\xi_{k}) \leq 0 \text{ for all } k \in \mathbb{N}.
$$
Up to a shift of the origin, without loss of generality, we suppose that \( \xi_k = 0 \) for all \( k \in \mathbb{N} \). Up to extraction of a subsequence, one can assume that \( c_k \to c_\infty \in [c_1, c_2] \) as \( k \to +\infty \).

Notice that Lemma 2.9 and system (5) satisfy by \( (u_k, v_k) \) with \( c_k \in [c_1, c_2] \subset (0, +\infty) \) imply that the sequence \( \left\{ \frac{v_k}{u_k} \right\} \) is bounded in \( L^\infty(\mathbb{R}) \), that is, there is \( C_1 > 0 \) such that \(|v_k'(\xi)| \leq C_1 v_k(\xi)\) for all \( k \in \mathbb{N} \) and \( \xi \in \mathbb{R} \). Since \( v_k(0) \to 0^+ \) as \( k \to +\infty \), it follows that

\[ v_k \to 0 \text{ locally uniformly in } \mathbb{R} \text{ as } k \to +\infty. \]

Consequently, it also holds that \( v_k' \to 0 \) locally uniformly in \( \mathbb{R} \) as \( k \to +\infty \).

Moreover, the functions \( u_k' \) are locally bounded and the functions \( u_k \) are globally bounded. Therefore, up to extraction of a subsequence, the functions \( u_k \) converge in \( C^1_{\text{loc}}(\mathbb{R}) \) to a function \( 0 \leq u_\infty \leq 1 \). Letting \( k \to +\infty \), applying system (5), we have

\[
\begin{align*}
c_\infty u_k'(\xi) &= \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) u_\infty(\xi - y) dy - u_\infty(\xi) \right) + u_\infty(\xi)(1 - u_\infty(\xi)) \text{ in } \mathbb{R}. \tag{30}
\end{align*}
\]

Call \( \alpha = \inf_\mathbb{R} u_\infty \) and let \( \{\zeta_m\} \) be sequence of real numbers such that \( u_\infty(\zeta_m) \to \alpha \) as \( m \to +\infty \). Up to extraction of a subsequence, the functions \( u_\infty(\xi + \zeta_m) \) converge as \( m \to +\infty \) in \( C^1_{\text{loc}}(\mathbb{R}) \) to a function \( \phi_\infty \) solving

\[
\begin{align*}
c_\infty \phi_\infty'(\xi) &= \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) \phi_\infty(\xi - y) dy - \phi_\infty(\xi) \right) + \phi_\infty(\xi)(1 - \phi_\infty(\xi)) \text{ in } \mathbb{R},
\end{align*}
\]

where \( \alpha \leq \phi_\infty \leq 1 \) and \( \phi_\infty(0) = \alpha \). As a consequence, \( \phi_\infty'(0) = 0 \) and

\[
\frac{d_1}{r} \left( \int_{\mathbb{R}} J(y) \phi_\infty(-y) dy - \phi_\infty(0) \right) \geq 0.
\]

Thus, the first equation of (5) gives that \( \phi_\infty(0)(1 - \phi_\infty(0)) \leq 0 \). Whence, \( \alpha \geq 1 \).

However, since \( \alpha = \inf_\mathbb{R} u_\infty \) and \( u_\infty \leq 1 \) in \( \mathbb{R} \), one can infer that \( u_\infty(\xi) = 1 \) in \( \mathbb{R} \).

Now set

\[
\psi_k(\xi) = \frac{v_k(\xi)}{v_k(0)}
\]

for \( k \in \mathbb{N} \) and \( \xi \in \mathbb{R} \). Since the sequence \( \left\{ \frac{v_k}{u_k} \right\} \) is bounded in \( L^\infty(\mathbb{R}) \), the positive functions \( \psi_k \) are locally bounded. Therefore, the functions

\[
\psi_k'(\xi) = \frac{\psi_k'(\xi)}{\psi_k(0)} = \frac{\psi_k'(\xi)}{\psi_k(\xi)} \times \psi_k(\xi)
\]

are locally bounded too. Since each \( \psi_k(\xi) \) obeys

\[
\begin{align*}
c_k \psi_k'(\xi) &= \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \psi_k(\xi - y) dy - \psi_k'(\xi) \right) + \psi_k'(\xi) \left( \frac{\alpha u_k^2(\xi)}{A + B u_k(\xi) + u_k^2(\xi)} - z \right)
\end{align*}
\]

in \( \mathbb{R} \) and the sequence \( \{u_k\} \) is bounded in \( C^1_{\text{loc}}(\mathbb{R}) \), one concludes that the functions \( \psi_k'(\xi) \) are locally bounded too. By the Arzela-Ascoli theorem, up to extraction of a subsequence, the positive functions \( \psi_k(\xi) \) converge in \( C^1_{\text{loc}}(\mathbb{R}) \) to a nonnegative solution \( \psi_\infty(\xi) \) satisfying

\[
\begin{align*}
c_\infty \psi_\infty'(\xi) &= \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \psi_\infty(\xi - y) dy - \psi_\infty'(\xi) \right) + \psi_\infty'(\xi) \left( \frac{\alpha}{A + B + 1} - z \right), \tag{31}
\end{align*}
\]
where the fact that \( u_k(\xi) \to u_\infty(\xi) = 1 \) as \( k \to +\infty \) for all \( \xi \in \mathbb{R} \) has been used. Further, we claim that \( \psi_\infty(\xi) > 0 \) in \( \mathbb{R} \). Instead, there is \( \xi_0 \in \mathbb{R} \) such that \( \psi_\infty(\xi_0) = 0 \) and \( \psi_\infty'(\xi_0) = 0 \). Following from (31) that \( \psi_\infty(\xi_0 - y) = \psi_\infty(\xi_0) = 0 \) for all \( y \in \mathbb{R} \), which implies \( \psi_\infty \equiv 0 \) in \( \mathbb{R} \). This contradicts to the fact that \( \psi_\infty(0) = 1 \), hence \( \psi_\infty(\xi) > 0 \) for all \( \xi \in \mathbb{R} \).

Let \( \omega := \frac{\psi'}{\psi_\infty} \). Then, \( \omega \) satisfies

\[
c_\infty \omega(\xi) = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{\int_{\xi}^{\infty} \omega(s)ds}dy - 1 \right) + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - \frac{\alpha}{\lambda} \right).
\]

Thus, by Lemma 2.12, \( \omega(\xi) = \frac{\psi_\prime(\xi)}{\psi_\infty(\xi)} \) has finite limits as \( \xi \to \pm \infty \), which are roots of the characteristic equation

\[
c_\infty \mu_\pm = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\mu_\pm y}dy - 1 \right) + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - \frac{\alpha}{\lambda} \right).
\]

Since \( \frac{\alpha}{A + B + 1} > z \), then \( \mu_\pm > 0 \). Naturally, \( \psi \) is positive at \( \pm \infty \). Then, by differentiating (32), we have

\[
c_\infty \omega'(\xi) = \frac{d_2}{r} \int_{\mathbb{R}} J(y) \frac{\psi_\prime(\xi - y)}{\psi_\infty(\xi)} (\omega(\xi - y) - \omega(\xi)) dy.
\]

Thus, if \( \omega \) has a minimum at point \( \xi \) in \( \mathbb{R} \), then \( \omega'(\xi) = 0 \) and \( \omega(\xi - y) = \omega(\xi) \) for all \( y \in \mathbb{R} \). As a consequence,

\[
\inf_{\xi} \omega \geq \min\{\omega(-\infty), \omega(+\infty)\} > 0.
\]

Finally, due to \( \psi_\prime > 0 \) in \( \mathbb{R} \), therefore \( 0 < \psi_\prime (0) = \lim_{k \to +\infty} \psi_\prime (0) = \lim_{k \to +\infty} \frac{\psi'(0)}{\psi_\infty(0)} \) and \( v'_\prime (0) > 0 \) for all \( k \) large enough. This contradicts to the fact that \( v'_\prime (0) \leq 0 \) for all \( k \in \mathbb{N} \). The proof is completed.

Below, we give the main results in this subsection.

**Theorem 2.16.** Assume that (J) holds and let \( B \geq 1 - A \). Then for any \( c > c_* \), there exists a bounded classical solution \((\tilde{u}(\xi), \tilde{v}(\xi))\) of system (5) with \( 0 < \tilde{u} < 1 \), \( \tilde{v} > 0 \) in \( \mathbb{R} \) satisfying (6) and (7).

**Proof.** By Lemma 2.15 and the positivity of \( \tilde{v} \) in \( \mathbb{R} \), there holds

\[
\lim \inf_{\xi \to +\infty} \tilde{v}(\xi) > 0.
\]

We claim that

\[
\lim \sup_{\xi \to +\infty} \tilde{u}(\xi) < 1.
\]

Indeed, otherwise, there exists a sequence of real numbers \( \{ \xi_k \} \) converging to \( +\infty \) such that \( u(\xi_k) \to 1 \) as \( k \to +\infty \). As in the proof of Lemma 2.14, up to extraction of a subsequence, the functions \( u_k(\xi) := \tilde{u}(\xi + \xi_k) \) and \( v_k(\xi) := \tilde{v}(\xi + \xi_k) \) converge as \( k \to +\infty \) in \( C_b^\infty(\mathbb{R}) \) to some nonnegative \( C^\infty \) functions \( u_\infty \) and \( v_\infty \) solving (5). Furthermore, \( 0 \leq u_\infty \leq 1 \) and \( v_\infty > 0 \) in \( \mathbb{R} \) from Lemma 2.14 and (34). Since \( u_\infty(0) = 1 \), one has \( u'_\infty(0) = 0 \). By the first equation of (5), we have

\[
0 = cu'_\infty(0) = \frac{d_1}{r} \left( \int_{\mathbb{R}} J(y)u_\infty(-y)dy - u_\infty(0) \right) + u_\infty(0)(1 - u_\infty(0)) - \frac{u^2_\infty(0)v_\infty(0)}{A + Bu_\infty(0) + u^2_\infty(0)}.
\]
\[
\frac{d_1}{r} \left( \int_{\mathbb{R}} J(y)u_\infty(-y)dy - 1 \right) - \frac{v_\infty(0)}{A + B + 1} < 0,
\]

which leads to a contradiction. Thus, the claim (35) holds.

Next, we will show that

\[
\liminf_{\xi \to +\infty} \hat{u}(\xi) \leq u^* \leq \limsup_{\xi \to +\infty} \hat{u}(\xi), \quad \liminf_{\xi \to +\infty} \hat{v}(\xi) \leq v^* \leq \limsup_{\xi \to +\infty} \hat{v}(\xi). \tag{36}
\]

Denote \( u_- = \liminf_{\xi \to +\infty} \hat{u}(\xi) \), \( u_+ = \limsup_{\xi \to +\infty} \hat{u}(\xi) \), \( v_- = \liminf_{\xi \to +\infty} \hat{v}(\xi) \) and \( v_+ = \limsup_{\xi \to +\infty} \hat{v}(\xi) \).

One already knows from (34), (35), Lemmas 2.13 and 2.14 that

\[
0 < u_- \leq u_+ < 1 \quad \text{and} \quad 0 < v_- \leq v_+ < \infty.
\]

Consider now a sequence \( \{\xi_k\} \) converging to \(+\infty\) such that \( \hat{v}(\xi_k) \to v_+ \) as \( k \to +\infty \). Up to extraction of a subsequence, the functions \( u_k(\xi) := \hat{u}(\xi + \xi_k) \) and \( v_k(\xi) := \hat{v}(\xi + \xi_k) \) converge in \( C_{loc}^\infty(\mathbb{R}) \) to some bounded functions \( 0 < u_\infty < 1 \) and \( v_\infty > 0 \) satisfying (5). Furthermore, \( 0 < v_+ = v_\infty(0) = \max_{\mathbb{R}} v_\infty \), therefore, \( v_\infty(0) = 0 \) and \( \int_{\mathbb{R}} J(y)v_\infty(-y)dy - v_\infty(0) \leq 0 \). Then, the second equation of (5) gives

\[
\frac{v_\infty(0)}{r} \left( \frac{\alpha v_\infty^2(0)}{A + Bu_\infty(0) + u_\infty^2(0)} - z \right) \geq 0.
\]

That is

\[
\frac{u_\infty^2(0)}{A + Bu_\infty(0) + u_\infty^2(0)} \geq \frac{z}{\alpha} = \frac{1}{\beta}.
\]

Hence

\[
u_+ = \limsup_{\xi \to +\infty} \hat{u}(\xi) \geq \frac{B + \sqrt{B^2 + 4A(\beta - 1)}}{2(\beta - 1)} = u^*.
\]

Similarly, it follows that \( u_- = \liminf_{\xi \to +\infty} \hat{u}(\xi) \leq u^* \). Consider also a sequence \( \{\xi_k\} \) converging to \(+\infty\) such that \( \hat{u}(\xi_k) \to u_+ \) as \( k \to +\infty \). As above, up to extraction of a subsequence, the functions \( \phi_k(\xi) := \hat{u}(\xi + \xi_k) \) and \( \psi_k(\xi) := \hat{v}(\xi + \xi_k) \) converge in \( C_{loc}^\infty(\mathbb{R}) \) to some bounded functions \( 0 < \phi_\infty < 1 \) and \( \psi_\infty > 0 \) satisfying (5). Furthermore, \( 0 < u_+ = \phi_\infty(0) = \max_{\mathbb{R}} \phi_\infty \). Therefore, \( \phi_\infty(0) = 0 \) and \( \int_{\mathbb{R}} J(y)\phi_\infty(-y)dy - \phi_\infty(0) \leq 0 \). Then, the first equation of (5) implies that

\[
u_+(1 - u_+) - \frac{(u_+)^2\psi_\infty(0)}{A + Bu_+ + (u_+)^2} \geq 0.
\]

Since \( u_+ \geq u^* \) and \( B \geq 1 - A \), one can get that

\[
\psi_\infty(0) \leq \frac{(1 - u_+)[A + Bu_+ + (u_+)^2]}{u_+} \leq \frac{(1 - u^*)[A + Bu^* + (u^*)^2]}{u^*} = v^*,
\]

whence \( v_- = \liminf_{\xi \to +\infty} \hat{v}(\xi) \leq v^* \). Similarly, it follows that \( v_+ = \liminf_{\xi \to +\infty} \hat{v}(\xi) \geq v^* \).

Hence, (36) is proved and this ends the proof of Theorem 2.16. \( \square \)
2.2. Critical waves. This section is devoted to showing the existence of a traveling wave \((u, v)\) of system (5) with speed \(c = c_*\). In order to achieve this goal, we choose a strictly decreasing sequence \(\{c_k\}\) of real numbers such that \(c_k \in (c_*, c_* + 1]\) for each \(k \in \mathbb{N}\), and
\[
c_k \to c_* \text{ as } k \to +\infty.
\]
In subsection 2.1, we have obtained the existence of traveling waves \((u_k, v_k)\) of system (5) with speed \(c_k\). We will pass to the limit as \(k \to +\infty\) to get the existence of a traveling wave with the limiting speed \(c_*\). To do this, we need some a priori bounds for the functions \(v_k\) in order to get a non-trivial solution at the limit. Here, since the lower solutions depend on \(c_k\) are degenerated as \(k \to +\infty\), we have to prove the convergence of \((u_k(\xi), v_k(\xi))\) as \(\xi \to -\infty\) with speed \(c = c_*\).

**Lemma 2.17.** For any \(c \geq c_*\), let \((u, v)\) be a bounded solution of system (5) with \(0 < u < 1\), \(v > 0\) in \(\mathbb{R}\) satisfying (6) and (7). Furthermore, if \(u(+\infty)\) or \(v(+\infty)\) exists, then they both exist and \((u(+\infty), v(+\infty)) = (u^*, v^*)\).

**Proof.** First, we assume that \(a = \lim_{\xi \to +\infty} u(\xi)\) exists. Then, by (35) and (36), we know \(0 < a = u^* < 1\). For any sequence \(\{\xi_k\}\) converges to \(+\infty\), the functions \(u_k(\xi) := u(\xi + \xi_k)\) and \(v_k(\xi) := v(\xi + \xi_k)\) converge in \(C^0_{\text{loc}}(\mathbb{R})\), up to extraction of a subsequence, to some functions \(u_\infty = a = u^*\) and \(v_\infty\) such that
\[
u^*(1-u^*) - \frac{(u^*)^2 v_\infty}{A + Bu^* + (u^*)^2} = 0 \text{ for all } \xi \in \mathbb{R}.
\]
Thus,
\[
v_\infty = \frac{(1-u^*)(A + Bu^* + (u^*)^2)}{u^*} = v^*.
\]
Since the limit does not depend on the sequence \(\{\xi_k\}\), one gets that \(\lim_{\xi \to +\infty} v(\xi) = v^*\).

On the other hand, if \(a_1 = \lim_{\xi \to +\infty} v(\xi)\) exists. From (34) and (36), one has \(0 < a_1 = v^*\). Considering any sequence \(\{\xi_k\}\) converging to \(+\infty\). Up to extraction of a subsequence, the functions \(u_k(\xi) := u(\xi + \xi_k)\) and \(v_k(\xi) := v(\xi + \xi_k)\) converge in \(C^0_{\text{loc}}(\mathbb{R})\) to some functions \(u_\infty\) and \(v_\infty = a_1 = v^*\) such that
\[
v^* \left( \frac{v}{r} \frac{\alpha(u_\infty)^2}{A + Bu_\infty + (u_\infty)^2} - z \right) = 0 \text{ for all } \xi \in \mathbb{R}.
\]
Hence,
\[
u_\infty = \frac{B + \sqrt{B^2 + 4A(\beta - 1)}}{2(\beta - 1)} = u^*.
\]
Owing to the fact that the limit does not depend on the sequence \(\{\xi_k\}\), one can conclude that \(\lim_{\xi \to +\infty} u(\xi) = u^*\).

Thus, if the limit \(a = \lim_{\xi \to +\infty} u(\xi)\) or the limit \(a_1 = \lim_{\xi \to +\infty} v(\xi)\) exists, then they both exist and there holds \((u(+\infty), v(+\infty)) = (u^*, v^*)\).

**Lemma 2.18.** \(\liminf_{k \to +\infty} \|v_k\|_{L^\infty(\mathbb{R})} > 0\).

**Proof.** Assume that the above conclusion does not hold. Then, up to extraction of a subsequence, one has \(\|v_k\|_{L^\infty(\mathbb{R})} \to 0\) as \(k \to +\infty\). Since \(c_k \in (c_*, c_* + 1]\) \(\subseteq (0, +\infty)\) for each \(k \in \mathbb{N}\), Lemma 2.15 gives that \(v_k^* > 0\) in \(\mathbb{R}\) for all \(k\) large enough. Owing to the fact that \(v_k\) is bounded on \(\mathbb{R}\), so the limit \(v_k(+\infty)\) exists in \(\mathbb{R}\) for all \(k\) large
enough. Due to the fact that \((u_k, v_k)\) satisfy the assumptions of Lemma 2.17, hence for any \(k\) large enough, one can get

\[
v_k(\pm \infty) = v^* = \frac{(1 - u^*) (A + Bu^* + (u^*)^2)}{u^*} > 0.
\]

This contradicts that \(\lim_{k \to \pm \infty} \|v_k\|_{L^\infty(\mathbb{R})} = 0\) and the proof is completed. \(\square\)

**Lemma 2.19.** \(\limsup_{k \to \pm \infty} \|v_k\|_{L^\infty(\mathbb{R})} < +\infty.\)

**Proof.** Similarly, we will prove this Lemma by the way of contradiction. Assume that the conclusion does not hold. Then, up to extraction of a subsequence, one can assume that \(\|v_k\|_{L^\infty(\mathbb{R})} \to +\infty\) as \(k \to +\infty\). For each \(k \in \mathbb{N}\), since \(v_k\) is positive and bounded in \(\mathbb{R}\), then there exists \(\xi_k \in \mathbb{R}\) such that

\[
v_k(\xi_k) \geq \left(1 - \frac{1}{k + 1}\right) \|v_k\|_{L^\infty(\mathbb{R})}.
\]

(37)

In particular, \(v_k(\xi_k) \to +\infty\) as \(k \to +\infty\). Consequently,

\[
\lim_{k \to +\infty} v_k(\xi + \xi_k) \to +\infty \text{ locally uniformly in } \xi \in \mathbb{R}.
\]

By Lemma 2.10, we have

\[
\phi_k(\xi) := u_k(\xi + \xi_k) \to 0
\]

as \(k \to +\infty\) locally uniformly in \(\xi \in \mathbb{R}\). Furthermore, for all \(k \in \mathbb{N}\), there is

\[
v_k'(\xi) \geq \frac{v_k(\xi)}{c_k} \int_\mathbb{R} J(y) v_k(\xi - y) dy - \frac{d_2 + z}{c_k r} v_k(\xi) \in \mathbb{R}.
\]

Since each \(v_k\) is positive, Lemma 2.9 implies that the functions \(\int_\mathbb{R} J(y) \frac{v_k(\xi - y)}{v_k(\xi)} dy\) are globally bounded in \(\mathbb{R}\) independently of \(k \in \mathbb{N}\), and the second equation of (5) gives that there exists some constant \(C\) such that \(|v_k'(\xi)| \leq C v_k(\xi)\).

Moreover, according to the boundedness of the sequence \(\left\{ \frac{v_k'}{v_k} \right\}\) in \(L^\infty(\mathbb{R})\), we can also obtain that the functions

\[
\psi_k(\xi) = \frac{v_k(\xi + \xi_k)}{v_k(\xi_k)}
\]

are locally bounded independently of \(k\). Each function \(\psi_k(\xi)\) satisfies

\[
c_k \psi_k' = \frac{d_2}{r} \left( \int_\mathbb{R} J(y) \psi_k(\xi - y) dy - \psi_k(\xi) \right) + \frac{\psi_k(\xi)}{r} \left( \frac{\alpha \phi_k^2}{A + B \phi_k + \phi_k^2} - z \right) \quad \text{in } \mathbb{R}.
\]

Hence, the functions \(\psi_k'\) are locally bounded too. By the Arzela-Ascoli theorem, up to extraction of a subsequence, the positive functions \(\psi_k\) converge locally uniformly in \(\mathbb{R}\) to a nonnegative continuous function \(\psi_\infty\). At the same time, from the above equation and the fact that \(\phi_k \to 0\) as \(k \to +\infty\) locally uniformly in \(\mathbb{R}\), we know the functions \(\psi_k'\) converge locally uniformly in \(\mathbb{R}\) too. Thus, the functions \(\psi_k\) converge in \(C^1_{loc}(\mathbb{R})\) to \(\psi_\infty\) and which satisfies

\[
c_\psi \psi_\infty' = \frac{d_2}{r} \left( \int_\mathbb{R} J(y) \psi_\infty(\xi - y) dy - \psi_\infty(\xi) \right) - \frac{\psi_\infty(\xi)}{r} z.
\]

(38)

Remember that the function \(\psi_\infty\) is thus automatically of class \(C^\infty(\mathbb{R})\). Therefore, \(\psi_\infty\) is nonnegative and \(\psi_\infty(0) = \lim_{k \to +\infty} \psi_k(0) = 1\). Furthermore, we can get that \(\psi_\infty\) is positive in \(\mathbb{R}\) as in the proof of Lemma 2.15 for the solution of (31).
At last, for every \( \xi \in \mathbb{R} \), \( v_k(\xi + \xi_k) \leq \|v_k\|_{L^\infty(\mathbb{R})} \leq (1 + \frac{1}{r}) v_k(\xi_k) \) holds from (37). That is, \( \psi_k(\xi) \leq 1 + \frac{1}{r} \) for every \( \xi \in \mathbb{R} \) and \( 1 \leq k \in \mathbb{N} \). Thus, \( \psi_\infty(\xi) \leq 1 \) for every \( \xi \in \mathbb{R} \). Due to \( \psi_\infty(0) = 1 \), then the function \( \psi_\infty \) get a global maximum at 0 and \( \psi_\infty'(0) = 0 \). The equation (38) gives that
\[
0 = \frac{d_2}{r} \left( \int_\mathbb{R} J(y) \psi_\infty(-y) dy - 1 \right) - \frac{\varepsilon}{r} < 0.
\]
This is a contradiction. Thus, the proof of Lemma 2.19 is finished. \( \square \)

**Theorem 2.20.** Suppose (J) holds and let \( B \geq 1 - A \). For \( c = c_\ast \), there is a solution \((u(\xi), v(\xi))\) of system (5) with \( 0 < u(\xi) < 1, \ v(\xi) > 0 \) satisfying (6) and (7).

**Proof.** From Lemma 2.15, we know that for every solution \((\phi, \psi)\) of system (5) with speed \( c_k \in [c_\ast, c_\ast + 1] \), there exists \( \varepsilon > 0 \) such that \( \psi'(\xi) > 0 \) as \( \psi(\xi) \leq \varepsilon \) for any \( \xi \in \mathbb{R} \). Without loss of generality, one can assume that
\[
0 < \varepsilon \leq v^* = \frac{(1 - u^*)(A + Bu^* + (u^*)^2)}{u^*}.
\]
According to Lemma 2.18 and the positivity of each \( v_k \), we can assume
\[
0 < \varepsilon < \inf \left\{ \|v_k\|_{L^\infty(\mathbb{R})} \right\}.
\]
Thus, for each \( k \in \mathbb{N} \), due to \( v_k(-\infty) = 0 \) and \( v_k > 0 \). Then, there exists \( \xi_k \in \mathbb{R} \) such that \( v_k(\xi_k) = \varepsilon \). Shift the origin at \( \xi_k \) and denote

\[
\tilde{u}_k(\xi) := u_k(\xi + \xi_k) \quad \text{and} \quad \tilde{v}_k(\xi) := v_k(\xi + \xi_k).
\]

Lemma 2.19 gives that the sequence \{\( \tilde{v}_k \)\} is bounded in \( L^\infty(\mathbb{R}) \). Since \( 0 < \tilde{u}_k < 1 \) in \( \mathbb{R} \) and \( c_k \to c_\ast > 0 \) as \( k \to +\infty \), the functions \( \tilde{u}_k \) and \( \tilde{v}_k \) converge in \( C^\infty_{\text{loc}}(\mathbb{R}) \), up to extraction of a subsequence, to some bounded \( C^\infty(\mathbb{R}) \) functions \( u \) and \( v \) solving system (5) with speed \( c_\ast \). Furthermore, \( 0 \leq u \leq 1 \) and \( v \geq 0 \) in \( \mathbb{R} \), while \( v(0) = \varepsilon > 0 \).

Next, we shall show that the pair \((u, v)\) with speed \( c = c_\ast \) is non-trivial and satisfies the desired limiting conditions at \( \pm \infty \). Firstly, let us show that \( v > 0 \) in \( \mathbb{R} \). Indeed, if there is \( \xi^* \in \mathbb{R} \) such that \( v(\xi^*) = 0 \), then \( v'(\xi^*) = 0 \) and the second equation of system (5) yields \( v(\xi^* - y) = 0 \) for any \( y \in \mathbb{R} \). Hence, the function \( v \equiv 0 \) in \( \mathbb{R} \), it is impossible because of \( v(0) = \varepsilon > 0 \). Therefore, \( v > 0 \) in \( \mathbb{R} \). By the positivity of \( v \) and the proof of Lemma 2.8, we get \( 0 < u < 1 \) in \( \mathbb{R} \).

Then, let us prove that the pair \((u, v)\) satisfies the limiting conditions at \( -\infty \). From the choice of \( \varepsilon \) above and \( v(0) = \varepsilon \), we have \( v' > 0 \) in \((\infty, 0)\). In particular, the limit \( \alpha_1 = \lim_{\xi \to -\infty} v(\xi) \) exists and \( \alpha_1 \in [0, \varepsilon] \). If \( \alpha_1 > 0 \), by the same arguments as in the proof of Lemma 2.17, we have \( u(-\infty) \) exists and \( u(-\infty) = -\frac{B + \sqrt{B^2 - 4A(\beta - 1)}}{2(\beta - 1)} \in (0, 1) \). The same arguments also yield \( \alpha_1 = v(-\infty) = (1 - u^*)(A + Bu^* + (u^*)^2) = v^* \). Therefore, \( v^* = \alpha_1 < \varepsilon \), contradicting (39). Hence,
\[
a_1 = v(-\infty) = 0.
\]
Furthermore, Considering any sequence \{\( \xi_k \)\} converging to \(-\infty \). Up to extraction of a sequence, the functions \( u_k(\xi) := u(\xi + \xi_k) \) and \( v_k(\xi) := v(\xi + \xi_k) \) converge in \( C^\infty_{\text{loc}}(\mathbb{R}) \) to a pair \((u_\infty, 0)\), for some function \( 0 \leq u_\infty \leq 1 \) solving (30) with speed \( c_\infty = c_\ast \). It follows from the proof of Lemma 2.15 that \( u_\infty = 1 \) in \( \mathbb{R} \). Due to the
limit does not depend on the choice of the sequence \( \{ \xi_k \} \), one can obtain the limit \( \lim_{\xi \to -\infty} u(\xi) \) exists, and \( u(-\infty) = 1 \).

Finally, we shall prove that the non-triviality weak conditions at \( +\infty \). As in the proof of Lemma 2.14, there holds \( \inf_R u > 0 \) with \( c = c_\ast \). Meanwhile, according to Lemma 2.15 and (34), we have \( \lim_{\xi \to +\infty} v(\xi) > 0 \) and by (35), we can infer that \( \lim_{\xi \to +\infty} u(\xi) < 1 \) for \( c = c_\ast \). As in the proof of Theorem 2.16, there is

\[
\lim_{\xi \to +\infty} \inf v(\xi) \leq v^* \leq \lim_{\xi \to +\infty} u(\xi), \quad \lim_{\xi \to +\infty} v(\xi) \leq v^* \leq \lim_{\xi \to +\infty} v(\xi).
\]

Therefore, the proof of this theorem is completed. \( \square \)

3. Nonexistence of traveling waves. In this section, we shall establish the nonexistence of traveling waves of system (5) with \( 0 < c < c_\ast \).

**Theorem 3.1.** Suppose that \( (J) \) holds. Then for any speed \( 0 < c < c_\ast \), system (5) has no nontrivial bounded positive solution satisfying (6).

**Proof.** We will prove this theorem by the way of contradiction. Here, we assume that there exist traveling wave solutions \( (u(\xi), v(\xi)) \) of system (5) satisfying the limit behavior at infinity. From the second equation of (5) and the positivity of \( u(\xi) \), we have

\[
c \frac{v'(\xi)}{v(\xi)} \geq \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \frac{v(\xi - y)}{v(\xi)} dy - 1 \right) - \frac{z}{r} \quad \text{for any } \xi \in \mathbb{R}.
\]

Lemma 2.9 yields that \( \int_{\mathbb{R}} J(y) \frac{v(\xi - y)}{v(\xi)} dy \) is bounded, and so is \( \frac{v'(\xi)}{v(\xi)} \). Now, we consider the sequence \( \{ \xi_k \} \) converging to \( -\infty \) as \( k \to +\infty \). The functions

\[
\Phi_k(\xi) := \frac{v(\xi + \xi_k)}{v(\xi_k)}
\]

are locally bounded and they obey

\[
c \Phi'_k(\xi) = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \Phi_k(\xi - y) dy - \Phi_k(\xi) \right) + \frac{\Phi_k(\xi)}{r} \left( \frac{\alpha u^2(\xi + \xi_k)}{A + Bu(\xi + \xi_k) + u^2(\xi + \xi_k)} - z \right)
\]

for any \( \xi \in \mathbb{R} \). Remembering that \( u(-\infty) = 1 \), then one can infer the functions \( \Phi'_k(\xi) \) are locally bounded too. By the Arzela-Ascoli theorem, the functions \( \Phi_k \) converge in \( C^1_{loc}(\mathbb{R}) \), up to extraction of a subsequence, to a function \( \Phi_\infty \) satisfying

\[
c \Phi'_\infty = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) \Phi_\infty(\xi - y) dy - \Phi_\infty \right) + \frac{\Phi_\infty}{r} \left( \frac{\alpha}{A + B + 1} - z \right) \quad \text{in } \mathbb{R}.
\]

Furthermore, \( \Phi_\infty \geq 0 \) in \( \mathbb{R} \) and \( \Phi_\infty(0) = 1 \). As in the proof of Lemma 2.15, it is easy to obtain \( \Phi_\infty > 0 \) in \( \mathbb{R} \). Letting

\[
\Psi(\xi) = \frac{\Phi'_\infty(\xi)}{\Phi_\infty(\xi)}
\]

Then \( \Psi(\xi) \) obeys

\[
c \Psi(\xi) = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y) e^{f_\xi - y \Psi(\tau) d\tau} dy - 1 \right) + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - z \right) \quad \text{in } \mathbb{R}.
\]
Since $\frac{\alpha}{A + B + 1} - z > 0$, according to Lemma 2.12, one can get the limits $\Psi(\pm \infty) = \nu^\pm$ exist in $\mathbb{R}$ and are positive roots of the equation
\[
c\mu = \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\mu y} dy - 1 \right) + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - z \right).
\]
Since $0 < c < c^*$, Lemma 2.1 gives
\[
c\mu > \frac{d_2}{r} \left( \int_{\mathbb{R}} J(y)e^{-\mu y} dy - 1 \right) + \frac{1}{r} \left( \frac{\alpha}{A + B + 1} - z \right)
\]
for all $\mu > 0$. This is one contradiction and we complete our proof.

**Acknowledgments.** The authors are very grateful to the anonymous referee for his/her valuable comments helping us to improve the original manuscript. Research of W.T. Li was partially supported by NSF of China (11731005, 11671180) and Research of F.Y. Yang was partially supported by NSF of China (11601205).

**REFERENCES**

[1] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. J. Toledo-Melero, *Nonlocal Diffusion Problems*, Mathematical Surveys and Monographs, 165, Amer. Math. Soc., Providence, Rhode Island, 2010.

[2] S. B. Ai, Y. H. Du and R. Peng, *Traveling waves for a generalized Holling-Tanner predator-prey model*, *J. Differential Equations*, 263 (2017), 7782–7814.

[3] M. Alfaro, J. Coville and G. Raoul, *Bistable travelling waves for nonlocal reaction diffusion equations*, *Discrete Contin. Dyn. Syst.*, 34 (2014), 1775–1791.

[4] N. F. Britton, *Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model*, *SIAM J. Appl. Math.*, 50 (1990), 1663–1688.

[5] P. W. Bates, P. C. Fife, X. F. Ren and X. F. Wang, *Traveling waves in a convolution model for phase transitions*, *Arch. Rational Mech. Anal.*, 138 (1997), 105–136.

[6] X. X. Bao, W. T. Li and W. X. Shen, *Traveling wave solutions of Lotka-Volterra competition systems with nonlocal dispersal in periodic habitats*, *J. Differential Equations*, 260 (2016), 8590–8637.

[7] X. X. Bao, W.-T. Li and Z.-C. Wang, *Time periodic traveling curved fronts in the periodic Lotka-Volterra competition-diffusion system*, *J. Dynam. Differential Equations*, 29 (2017), 981–1016.

[8] X. F. Chen, *Existence, uniqueness and asymptotic stability of travelling waves in non-local evolution equations*, *Adv. Differential Equations*, 2 (1997), 125–160.

[9] Y.-Y. Chen, J. S. Guo and C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator-prey model*, *J. Math. Anal. Appl.*, 445 (2017), 212–239.

[10] Y.-Y. Chen, J. S. Guo and C.-H. Yao, *Traveling wave solutions for a continuous and discrete diffusive predator-prey model*, *J. Math. Anal. Appl.*, 445 (2017), 212–239.

[11] J. Coville, J. Dávila and S. Martínez, *Nonlocal anisotropic dispersal with monostable nonlinearity*, *J. Differential Equations*, 244 (2008), 3080–3118.

[12] S. R. Dunbar, *Traveling wave solutions of diffusive Lotka-Volterra equations*, *J. Math. Biol.*, 17 (1983), 11–32.

[13] S. R. Dunbar, *Traveling wave solutions of diffusive Lotka-Volterra equations: A heteroclinic connection in $\mathbb{R}^4$*, *Trans. Amer. Math. Soc.*, 286 (1984), 557–594.

[14] S. R. Dunbar, *Traveling waves in diffusive predator-prey equations: Periodic orbits and point-to-periodic heteroclinic orbits*, *SIAM J. Appl. Math.*, 46 (1986), 1057–1078.

[15] W. Ding and W. Z. Huang, *Traveling wave solutions for some classes of diffusive predator-prey models*, *J. Dynam. Differential Equations*, 28 (2016), 1293–1308.

[16] F. D. Dong, W. T. Li and J. B. Wang, *Asymptotic behavior of traveling waves for a three-component system with nonlocal dispersal and its application*, *Discrete Contin. Dyn. Syst.*, 37 (2017), 6291–6318.

[17] R. A. Fisher, *The wave of advance of advantageous genes*, *Ann. Eugenic.*, 7 (1937), 355–369.

[18] S.-C. Fu and J.-C. Tsai, *Wave propagation in predator-prey systems*, *Nonlinearity*, 28 (2015), 4389–4423.
[19] C.-H. Hsu, C.-R. Yang, T.-H. Yang and T.-S. Yang, Existence of traveling wave solutions for diffusive predator-prey type systems, *J. Differential Equations*, **252** (2012), 3040–3075.
[20] Y. L. Huang and G. Lin, Traveling wave solutions in a diffusive system with two preys and one predator, *J. Math. Anal. Appl.*, **418** (2014), 163–184.
[21] J. H. Huang, G. Lu and S. G. Ruan, Existence of traveling wave solutions in a diffusive predator-prey model, *J. Math. Biol.*, **46** (2003), 132–152.
[22] K. Hong and P. X. Weng, Stability and traveling waves of a stage-structured predator-prey model with Holling type-II functional response and harvesting, *Nonlinear Anal. Real World Appl.*, **14** (2013), 83–103.
[23] W. Z. Huang, Traveling wave solutions for a class of predator-prey systems, *J. Dynam. Differential Equations*, **24** (2012), 633–644.
[24] A. N. Kolmogorov, I. G. Petrovskii and N. S. Piskunov, Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem, *Byul. Mosk. Gos. Univ. Ser. A: Mat. Mekh.*, **1** (1937), 1–26.
[25] W.-T. Li, Y.-J. Sun and Z.-C. Wang, Entire solutions in the Fisher-KPP equation with nonlocal dispersal, *Nonlinear Anal. Real World Appl.*, **11** (2010), 2302–2313.
[26] W.-T. Li and S.-L. Wu, Traveling waves in a diffusive predator-prey model with Holling type-III functional response, *Chaos Solitons Fractals*, **37** (2008), 476–486.
[27] W.-T. Li and F.-Y. Yang, Traveling waves for a nonlocal dispersal SIR model with standard incidence, *J. Integral Equations Appl.*, **26** (2014), 243–273.
[28] X.-S. Li and G. Lin, Traveling wavefronts in nonlocal dispersal and cooperative Lotka-Volterra system with delays, *Appl. Math. Comput.*, **204** (2008), 738–744.
[29] Y. Li, W.-T. Li and F.-Y. Yang, Traveling waves for a nonlocal dispersal SIR model with delay and external supplies, *Appl. Math. Comput.*, **247** (2014), 723–740.
[30] G. Lin, Invasion traveling wave solutions of a predator-prey system, *Nonlinear Anal.*, **96** (2014), 47–58.
[31] X. B. Lin, P. X. Weng and C. F. Wu, Traveling wave solutions for a predator-prey system with sigmoidal response function, *J. Dynam. Differential Equations*, **23** (2011), 903–921.
[32] A. J. Lotka, *Elements of Physicals Biology*, Williams and Wilkins Company, Baltimore, 1925.
[33] J. D. Murray, *Mathematical Biology. I: An Introduction*, 3rd edition, Interdisciplinary Applied Mathematics, 17. Springer-Verlag, New York, 2002.
[34] S. X. Pan, W.-T. Li and G. Lin, Travelling wavefronts in nonlocal delayed reaction-diffusion systems and applications, *Z. Angew. Math. Phys.*, **60** (2009), 377–392.
[35] J. A. Sherratt, Invasion generates periodic traveling waves (wavetrains) in predator-prey models with nonlocal dispersal, *SIAM J. Appl. Math.*, **76** (2016), 293–313.
[36] V. Volterra, Fluctuations in the abundance of a species considered mathematically, *Nature*, **119** (1927), 12–13.
[37] F.-Y. Yang and W.-T. Li, Traveling waves in a nonlocal dispersal SIR model with critical wave speed, *J. Math. Anal. Appl.*, **458** (2018), 1131–1146.
[38] F.-Y. Yang, W.-T. Li and Z.-C. Wang, Traveling waves in a nonlocal dispersal SIR epidemic model, *Nonlinear Anal. Real World Appl.*, **23** (2015), 129–147.
[39] F.-Y. Yang, Y. Li, W.-T. Li and Z.-C. Wang, Traveling waves in a nonlocal dispersal Kermack-McKendrick epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 1969–1993.
[40] G.-B. Zhang, W.-T. Li and G. Lin, Traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure, *Math. Comput. Modelling*, **49** (2009), 1021–1029.
[41] G.-B. Zhang, W.-T. Li and Z.-C. Wang, Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity, *J. Differential Equations*, **252** (2012), 5096–5124.
[42] T. R. Zhang, W. D. Wang and K. F. Wang, Minimal wave speed for a class of non-cooperative diffusion-reaction system, *J. Differential Equations*, **260** (2016), 2763–2791.

Received May 2019; revised October 2019.

E-mail address: haoyx15@lzu.edu.cn
E-mail address: wtli@lzu.edu.cn
E-mail address: yangfy@lzu.edu.cn