GLOBAL EXISTENCE OF SOLUTIONS TO THE 2D SUBCRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION AND PERSISTENCY OF THE INITIAL REGULARITY

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Abstract. In this paper, we prove that if the initial data \( \theta_0 \) and its Riesz transforms \((R_1(\theta_0)\) and \(R_2(\theta_0)\)) belong to the space \( S(\mathbb{R}^2))^{B_{1-2\alpha}^1,\infty}_\infty \), where \( \alpha \in ]\frac{1}{2}, 1[ \), then the 2D Quasi-Geostrophic equation with dissipation \( \alpha \) has a unique global in time solution \( \theta \). Moreover, we show that if in addition \( \theta_0 \in X \) for some functional space \( X \) such as Lebesgue, Sobolev and Besov’s spaces then the solution \( \theta \) belongs to the space \( C([0, +\infty[ , X) \).

1. Introduction and main results

In this paper, we are concerned with the initial Value-Problem for the two-dimensional quasi-geostrophic equation with sub-critical dissipation

\[
\begin{aligned}
\partial_t \theta + (-\Delta)^\alpha \theta + \nabla \cdot (\theta u) &= 0 \text{ on } \mathbb{R}^+ \times \mathbb{R}^2 \\
\theta(0, x) &= \theta_0(x), \ x \in \mathbb{R}^2
\end{aligned}
\]

where \( \alpha \in ]\frac{1}{2}, 1[ \) is a fixed parameter and \( \nabla \) denotes the divergence operator with respect to the space variable \( x \in \mathbb{R}^2 \). The scalar function \( \theta \) represents the potential temperature. The velocity \( u = (u_1, u_2) \) is divergence free and determined from \( \theta \) through the Riesz transforms

\[
u = R^\perp(\theta) \equiv (-R_2(\theta), R_1(\theta)).
\]

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The non local operator \((-\Delta)^\alpha\) is defined through the Fourier transform
\[ \mathcal{F}((-\Delta)^\alpha f)(\xi) = |\xi|^{2\alpha} \mathcal{F}(f)(\xi) \]
where \(\mathcal{F}(f)\) is the Fourier transform of \(f\) defined by:
\[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-i(x,\xi)}dx. \]

To study the existence of the solutions of the equations \((QG_\alpha)\) we will follow the Fujita-Kato method. Thus we convert the equations \((QG_\alpha)\) into the fixed point problem
\[ (1.1) \quad \theta(t) = e^{-t(-\Delta)^\alpha} \theta_0 + \mathcal{B}_\alpha [\theta, \theta](t). \]
Here \((e^{-t(-\Delta)^\alpha})_{t>0}\) is the semi-group defined by:
\[ \mathcal{F}(e^{-t(-\Delta)^\alpha} f)(\xi) = e^{-t|\xi|^{2\alpha}} \mathcal{F}(f)(\xi) \]
and \(\mathcal{B}_\alpha\) is the bi-linear operator given by:
\[ (1.2) \quad \mathcal{B}_\alpha [\theta_1, \theta_2](t) = -\mathcal{L}_\alpha (\theta_1 \mathcal{R}^\perp(\theta_2)) \]
where, for \(v = (v_1, v_2)\),
\[ (1.3) \quad \mathcal{L}_\alpha(v)(t) = \int_0^t e^{-(t-s)(-\Delta)^\alpha} \nabla.v ds. \]

In the sequel, we mean by a mild solution on \([0, T[\) to the equations \((QG_\alpha)\) with data \(\theta_0\) a function \(\theta\) belonging to the space \(L^{2}_{loc}([0, T[, F_2)\) and satisfying in \(\mathcal{D}'([0, T[ \times \mathbb{R}^2)\) the equation \((1.1)\) where \(F_2\) is the completion of \(S(\mathbb{R}^2)\) with respect to the norm
\[ \|f\|_{F_2} \equiv \sup_{x_0 \in \mathbb{R}^2} \left( \|1_{B(x_0, 1)}f\|_2 + \|1_{B(x_0, 1)}\mathcal{R}^\perp(f)\|_2 \right). \]

One of the main property of the equations \((QG_\alpha)\) is the following scaling invariance property: if \(\theta\) is a solution of \((QG_\alpha)\) with data \(\theta_0\) then, for any \(\lambda > 0\), the function \(\theta_\lambda(t, x) \equiv \lambda^{2\alpha-1}\theta(\lambda^{2\alpha}t, \lambda x)\) is a solution of \((QG_\alpha)\) with data \(\theta_{0, \lambda}(x) \equiv \lambda^{2\alpha-1}\theta_0(\lambda x)\). This leads us to introduce the following notion of super-critical space: A Banach space \(X\) will be called super-critical space if \(S(\mathbb{R}^2) \hookrightarrow X \hookrightarrow S(\mathbb{R}^2)\) and there exists a constant \(C_X \geq 0\) such that
\[ \forall f \in X, \sup_{0<\lambda\leq1} \lambda^{2\alpha-1} \|f(\lambda \cdot)\|_X \leq C_X \|f\|_X. \]

For instance, the Lebesgue space \(L^p(\mathbb{R}^2)\) (respectively, the Sobolev space \(H^s(\mathbb{R}^2)\)) is super-critical space if \(p \geq p_c \equiv \frac{2}{2\alpha-1}\) (respectively, \(s \geq s_c \equiv 2 - 2\alpha\)). Moreover, one can easily prove that the Besov space \(B^{1-2\alpha, \infty}_\infty(\mathbb{R}^2)\) is the greatest super-critical space. The first purpose of this paper, is to prove the global existence of smooth solutions of the equations...
GLOBAL EXISTENCE AND PERSISTENCY OF THE INITIAL REGULARITY

(QG\(\alpha\)) for initial data in a sub-critical space \(\tilde{B}^\alpha\) closed to the space \(B^{1-2\alpha,\infty}_\infty(\mathbb{R}^2)\). Our space \(\tilde{B}^\alpha\) is the completion of \(S(\mathbb{R}^2)\) with respect to the norm

\[
\|f\|_{\tilde{B}^\alpha} \equiv \|f\|_{B^{1-2\alpha,\infty}_\infty} + \|\mathcal{R}^\perp(f)\|_{B^{1-2\alpha,\infty}_\infty}.
\]

Before setting precisely our global existence result, let us recall some known results in this direction: In [19], J. Wu proved that for any initial data \(\theta_0\) in the space \(L^p(\mathbb{R}^2)\) with \(p > p_c\) the equations \((QG^a)\) has a unique and global solution \(\theta\) belonging to the space \(L^\infty([0, +\infty[)\). Similarly, P. Constantin and J. Wu [4] showed the global existence and uniqueness for arbitrary initial data in the Sobolev space \(H^s(\mathbb{R}^2)\) where \(s > s_c\). Notice that these results don’t cover the limit cases \(p = p_c\) and \(s = s_c\).

Our global existence result reads as follows.

**Theorem 1.1.** Let \(\nu = 1 - \frac{1}{2\alpha}\). For any initial data \(\theta_0 \in \tilde{B}^\alpha\) the equation \((QG^\alpha)\) has a unique global solution \(\theta\) belonging to the space \(C^\infty([0, T] \times \mathbb{R}^2)\) with respect to the norm

\[
\|v\|_{E_T} \equiv \sup_{0 \leq t \leq T} \nu^t (\|v(t)\|_\infty + \|\mathcal{R}^\perp(v)(t)\|_\infty).
\]

Moreover,

\[C([0, +\infty[, \tilde{B}^\alpha).\]

Our second main result is a persistency theorem that states that the solution \(\theta\) given by the previous theorem keeps its initial regularity. Precisely, our theorem states as follows.

**Theorem 1.2.** Let \(X\) be one of the following Banach spaces:

- \(X = L^p(\mathbb{R}^2)\) with \(1 \leq p \leq \infty\).
- \(X = B_s^{p,q}(\mathbb{R}^2)\) with \(s > -1\) and \(1 \leq p, q \leq \infty\).
- \(X = \dot{B}_s^{p,q}(\mathbb{R}^2)\) with \(s > -1\) and \(1 \leq p, q \leq \infty\).

Assume \(\theta_0 \in \tilde{B}^\alpha \cap X\). Then the mild solution \(\theta\) of the equation \((QG^\alpha)\) given by Theorem 1.1 belongs to the space \(L^\infty_{loc}([0, +\infty[, X)\). Moreover, if \(\theta_0 \in \tilde{B}^\alpha \cap S(\mathbb{R}^2)^X\) then \(\theta\) belongs to \(C([0, +\infty[, S(\mathbb{R}^2)^X)\).

As a consequence of the previous theorems, we have the following theorem that generalizes the existence results of J. Wu [19] and P. Constantin and J. Wu [4] recalled above.

**Theorem 1.3.** Let \(X\) be the Lebesgue space \(L^p(\mathbb{R}^2)\) with \(p \geq p_c = \frac{2}{2\alpha - 1}\) or the Sobolev space \(H^s(\mathbb{R}^2)\) with \(s \geq s_c = 2 - 2\alpha\). Assume \(\theta_0 \in X\). Then the equation \((QG^\alpha)\) with initial data \(\theta_0\) has a unique global mild solution \(\theta\) belonging to the space \(C([0, +\infty[, X)\).
The remainder of this paper is as follows: in section 2 we recall some definitions and we give some useful Lemmas that will be used in this paper. In section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2 and in section 4, we will prove Theorem 1.3.

2. Preliminaries

2.1. Notations. In this subsection, we introduce some notations that will be used frequently in this paper:

1. Let $X$ be a Banach space such that $S(\mathbb{R}^2) \hookrightarrow X \hookrightarrow S'(\mathbb{R}^2)$. We denote by $X_R$ the space $X_R = \{ f \in X; \mathcal{R}^1(f) \in X^2 \}$ endowed with the norm $\| f \|_{X_R} = \| f \|_X + \| \mathcal{R}^1(f) \|_X$.

We recall that $\mathcal{R}^1(f) = (-\mathcal{R}_2f, \mathcal{R}_1f)$ where $\mathcal{R}_1$ and $\mathcal{R}_2$ are Riesz transforms.

2. Let $T > 0$, $r \in [1, \infty]$ and $X$ be a Banach space. $L^r_T.X$ denotes the space $L^r([0,T],X)$. In particular, $L^r_T.L^p$ will denote the space $L^r([0,T],L^p(\mathbb{R}^2))$.

3. Let $X$ be a Banach space, $T > 0$ and $\mu \in \mathbb{R}^+$. We denote by $L^\infty_\mu([0,T],X)$ the space of functions $f : [0,T] \to X$ such that $\| f \|_{L^\infty_\mu([0,T],X)} = \sup_{0 < t \leq T} t^\mu \| f(t) \|_X < \infty$ and $\lim_{t \to 0} t^\mu \| f(t) \|_X = 0$.

The sub-space $C^0_\mu([0,T],X)$ of $L^\infty_\mu([0,T],X)$ is defined by $C^0_\mu([0,T],X) \equiv L^\infty_\mu([0,T],X) \cap C([0,T],X)$.

4. Let $A$ and $B$ be two reals functions. The notation $A \lesssim B$ means that there exists a constant $C$, independent of the effective parameters of $A$ and $B$, such that $A \leq CB$.

2.2. Besov spaces. The standard definition of Besov spaces passes through the Littlewood-Paley dyadic decomposition [1], [7], and [10]. To this end, we take an arbitrary function $\psi \in S(\mathbb{R}^2)$ whose Fourier transform $\hat{\psi}$ is such that $\text{supp}(\hat{\psi}) \subset \{ \xi, \frac{1}{2} \leq |\xi| \leq 2 \}$, and for $\xi \neq 0$, $\sum_{j \in \mathbb{Z}} \hat{\psi}(\frac{\xi}{2^j}) = 1$, and define $\varphi \in S(\mathbb{R}^2)$ by $\hat{\varphi}(\xi) = 1 - \sum_{j \geq 0} \hat{\psi}(\frac{\xi}{2^j})$. For $j \in \mathbb{Z}$, we write $\varphi_j(x) = 2^{2j}\varphi(2^j x)$ and $\psi_j(x) = 2^{2j}\psi(2^j x)$ and we denote the convolution operators $S_j$ and $\Delta_j$, respectively, the convolution operators by $\varphi_j$ and $\psi_j$. 
Definition 2.1.
Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$.
1. A tempered distribution $f$ belongs to the (inhomogeneous) Besov space $B^{s,q}_p$ if and only if
\[
\|f\|_{B^{s,q}_p} \equiv \|S_0 f\|_p + \left( \sum_{j>0} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}} < \infty.
\]
2. The homogeneous Besov space $\dot{B}^{s,q}_p$ is the space of $f \in S'(\mathbb{R}^2)/\mathbb{R}[X]$ such that
\[
\|f\|_{\dot{B}^{s,q}_p} \equiv \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}} < \infty,
\]
Where $\mathbb{R}[X]$ is the space of polynomials [14].

An equivalent definition more adapted to the Quasi-geostrophic equations involves the semigroup $(e^{-t(-\Delta)^\alpha})_{t>0}$.

Proposition 2.1. If $s < 0$ and $q = \infty$. Then
\[
(2.1) \quad f \in \dot{B}^{s,\infty}_p \iff \sup_{t>0} t^{\frac{1}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_p < \infty,
\]
\[
(2.2) \quad f \in B^{s,\infty}_p \iff \forall T > 0, \sup_{0<t<T} t^{\frac{1}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_p \leq C_T.
\]

Proof: This proposition can be easily proved by following the same lines as in the proof of Theorem 5.3 in [10] in the case of the heat Kernel. One can see also the proof of Proposition 2.1 in [11].

2.3. Intermediate results. We shall frequently use the following estimates on the operator $e^{-t(-\Delta)^\alpha}$.

Proposition 2.2. For $t > 0$, we set $K_t$ the kernel of $e^{-t(-\Delta)^\alpha}$. Then for all $r \in [1, \infty]$ we have,
\[
(2.3) \quad \|K_t\|_r = C_1 r t^{\sigma_r},
\]
\[
(2.4) \quad \|\nabla K_t\|_r = C_2 r t^{\sigma_r - \frac{1}{2\alpha}},
\]
\[
(2.5) \quad \|\mathcal{R}_j \nabla K_t\|_r = C_3 r t^{\sigma_r - \frac{1}{2\alpha}},
\]
where $\sigma_r = \frac{1}{\alpha} (\frac{1}{r} - 1)$ and $C_1, C_2$, and $C_3$ are constants independent of $t$. 
Proof: For the proof of (2.3) see [11]. The estimate (2.5) can be obtained by following the same argument of the proof of Proposition 11.1 in [10].

Following the work of P.G. Lemarié-Rieusset, we introduce the notion of shift invariant functional space:

**Definition 2.2.** A Banach space $X$ is called shift invariant functional space if

- $S(\mathbb{R}^2) \hookrightarrow X \hookrightarrow S'(\mathbb{R}^2)$,
- $\forall \varphi \in S(\mathbb{R}^2)$ and $f \in X$, $\|\varphi * f\|_X \leq C_X \|\varphi\|_1 \|f\|_X$.

**Remark 2.1.** The Lebesgue spaces and Besov spaces are shift invariant functional spaces.

The proof of Theorem 1.1 requires the following lemmas.

**Lemma 2.1.** Let $X$ be a shift invariant functional space. If $f \in X$ then

$$
(2.6) \quad \sup_{t>0} \|e^{-t(-\Delta)^{\alpha}} f\|_X \leq C_X \|f\|_X.
$$

Moreover, if $f \in \overline{S(\mathbb{R}^2)}^X$ then: $e^{-t(-\Delta)^{\alpha}} f \in C([0, \infty[ \), $\overline{S(\mathbb{R}^2)}^X$) and $e^{-t(-\Delta)^{\alpha}} f \to f$ in $X$ as $t \to 0^+$.

Proof: One obtain easily (2.6) from (2.3). Let us prove the last statement. For $t > 0$, we denote by $K_t$ the kernel of the operator $e^{-t(-\Delta)^{\alpha}}$. Then $K_t(.) = t^{-\frac{1}{\alpha}} K(t^{-\frac{1}{\alpha}} \cdot)$ where $K = K_{t=1}$. Since $K \in L^1(\mathbb{R}^2)$ and $\int K(x) dx = 1$, there exists a sequence $(K_n)_{n} \in (C_c^\infty(\mathbb{R}^2))^N$ such that for all $n$, $\int K_n(x) dx = 1$ and $(K_n)_{n} \to K$ in $L^1(\mathbb{R}^2)$. Let $(f_n)_n$ be a sequence in $C_c^\infty(\mathbb{R}^2)$ satisfying $(f_n)_n \to f$ in $X$. Now we consider the functions $(u_n)_n$ and $u$ defined on $\mathbb{R}^+ \times \mathbb{R}^2$ by

$$
u(t, x) = K_t * f \quad \text{and} \quad u_n(t, x) = K_n(t) \ast f_n$$

where $K_n(t)(\cdot) = t^{-\frac{1}{\alpha}} K(t^{-\frac{1}{\alpha}} \cdot)$. One can easily verify that for all $n$, the function $\hat{u}_n(t, \xi) = \hat{K}_n(t^{-\frac{1}{\alpha}} \xi) \hat{f}_n(\xi)$ belongs to the space $C(\mathbb{R}^+, S(\mathbb{R}^2))$ and satisfies $\hat{u}_n(t, \cdot) \to \hat{f}_n$ in $S(\mathbb{R}^2)$ as $t$ goes to $0^+$. This implies that for all $n$, $u_n$ can be extended to a function in $C(\mathbb{R}^+, S(\mathbb{R}^2))$ with $f_n$ as value at $t = 0$. Consequently, to conclude the proof of the Lemma, we just need to show that the sequence $(u_n)_n$ converges to $u$ in the space $L^\infty(\mathbb{R}^+, X)$. To do this, we notice that for any $t > 0$ and any $n \in \mathbb{N}$ we have,

$$
u_n(t) - u(t) = K_n(t) \ast (f_n - f) + (K_n(t) - K_t) \ast f.$$
Hence,
\[ \| u_n(t) - u(t) \|_X \leq \| K_{(n),t} \|_1 \| f_n - f \|_X + \| K_{(n),t} - K_t \|_1 \| f \|_X \]
\[ \leq C \| f_n - f \|_X + \| K_{(n)} - K \|_1 \| f \|_X, \]
which leads to the desired result.

The next lemma will be useful in the sequel.

**Lemma 2.2.** Let \( X \) be a shift invariant functional space, \( T > 0 \) and \( \mu \geq 0 \). Then, for all \( f \in L^\infty_\mu([0,T],X) \), the function \( L_\alpha(f) \) belongs to \( L^\infty_{\mu'}([0,T],X_R) \) and satisfies
\[ \| L_\alpha(f) \|_{L^\infty_{\mu'}([0,T],X_R)} \leq C \| f \|_{L^\infty_{\mu}([0,T],X)} \]
where \( \mu' = \mu - 1 + \frac{1}{2\alpha} \) and \( C \) is a constant depending only on \( \alpha \) and \( X \). Moreover, if \( f \) belongs to \( L^\infty_\mu([0,T],S(\mathbb{R}^2)X) \) then \( L_\alpha(f) \) belongs to the space \( C^0_\mu([0,T],(S(\mathbb{R}^2)^X)_R) \).

**Proof :** The first assertion is an immediate consequence of estimates (2.4)-(2.5). The last assertion can be easily proved by using the previous lemma and the Lebesgue’s dominated convergence theorem, we left details to the reader.

**Lemma 2.3.** Let \( T > 0 \).
The following assertions hold true:

1. The linear operator \( e^{-t(-\Delta)^\alpha} \) is continuous from \( \tilde{B}^\alpha \) to \( E^\mu_T \).
2. The bilinear operator \( B_\alpha \) is continuous from \( E^\mu_T \times E^\mu_T \to E^\mu_T \) and its norm is independent of \( T \).

**Proof :** The first assertion follows from the characterization of Besov spaces by the kernel \( e^{-t(-\Delta)^\alpha} \) and the definition of \( \tilde{B}^\alpha \). The second assertion, is a direct consequence of the previous lemma and the fact that \( E^\mu_T = C^0_\mu([0,T],(C_0(\mathbb{R}^2))_R) \)

The following Lemma, which is a direct consequence of the preceding one will be useful in the proof of Theorem 1.2.

**Lemma 2.4.** Let \( \theta_0 \in \tilde{B}^\alpha \). The sequence \( \phi_n(\theta_0) \) defined by
\[ \phi_0(\theta_0) = e^{-t(-\Delta)^\alpha} \theta_0, \]
\[ \phi_{n+1}(\theta_0) = e^{-t(-\Delta)^\alpha} \theta_0 + B_\alpha[\phi_n(\theta_0),\phi_n(\theta_0)], \]
belongs to \( \bigcap_{T>0} E^\mu_T \). Moreover, there exists a constant \( \mu_0 > 0 \) (depending only on \( \alpha \)) such that if for some \( T > 0 \) we have \( \| \phi_0(f) \|_{E^\mu_T} \leq \mu_0 \) then \( \forall n \in \mathbb{N}^* \),

\[ \| \phi_n(\theta_0) \|_{E^\mu_T} \leq 2\| \phi_0(\theta_0) \|_{E^\mu_T}, \]
\[ \| \phi_{n+1}(\theta_0) - \phi_n(\theta_0) \|_{E^\mu_T} \leq \frac{1}{2^n}. \]
In particular, the sequence \((\phi_n(\theta_0))_n\) converges in the space \(E_{T}^{\nu}\) and its limit \(\theta\) is a mild solution to the equation \((QG_\alpha)\) with initial data \(\theta_0\).

The following elementary lemma will play a crucial role in this paper.

**Lemma 2.5. (Gronwall type Lemma)** Let \(T > 0, c_1, c_2 \geq 0, \kappa \in ]0,1[\) and \(f \in L^\infty(0,T)\) such that for all \(t \in [0,T]\)

\[
(2.9) \quad f(t) \leq c_1 + c_2 \int_0^t \frac{f(s)}{(t-s)\kappa}. \tag{2.9}
\]

Then

\[
(2.10) \quad \forall t \in [0,T], \quad f(t) \leq 2c_1e^{\nu t}, \tag{2.10}
\]

where \(\nu = \nu_{c_1,c_2} > 0\).

**Proof:** Let \(\nu > 0\) to be precise in the sequel and consider the function \(g\) defined on \([0,T]\) by

\[
g(t) = \sup_{0 < s < t} e^{-\nu s} f(s).
\]

Clearly, we have

\[
g(t) \leq c_1 + c_2 \int_0^t e^{-\nu(t-s)} g(s) ds, \\
\leq c_1 + c_2 \gamma_{\kappa} \nu^{\kappa-1} g(t),
\]

where \(\gamma_{\kappa} = \int_0^\infty \frac{e^{-t}}{t^\kappa}\). Thus, if we choose \(\nu > 0\) such that \(c_2 \gamma_{\kappa} \nu^{\kappa-1} = \frac{1}{2}\), we get the estimate \((2.10)\).

**Lemma 2.6. (Maximum Principal)**

Let \(\theta\) be a mild solution to the equation \((1.1)\) belonging to the space \(C([0,T],(C_0(\mathbb{R}^2))_{\mathbb{R}})\). Then \(\forall t \in [0,T]\), we have

\[
(2.11) \quad ||\theta(t)||_{\infty} \leq ||\theta_0||_{\infty}, \tag{2.11}
\]

\[
(2.12) \quad ||\mathcal{R}^\perp(\theta)(t)||_{\infty} \leq 2||\mathcal{R}^\perp(\theta_0)||_{\infty} e^{\eta t}, \tag{2.12}
\]

where \(\eta = \eta_{\alpha,||\theta_0||_{\infty}} > 0\).

**Proof:** The inequality \((2.11)\) is proved in \([15], [5]\) and \([19]\), for sufficiently smooth solution \(\theta\). To prove it in our case, we will proceed by linearization of the equations and regularization of the initial data. We consider a sequence of linear system \((QGL_n)_n:\)

\[
(QGL_n) \quad \left\{ \begin{array}{l}
\partial_t v - (-\Delta)^{\alpha} v + \nabla.(u_n v) = 0 \\
v(0,.) = \theta_n(.) \end{array} \right.
\]
where \((\theta_n)_n\) is a given sequence in \(C^\infty_c(\mathbb{R}^2)\) converging to \(\theta(0)\) in the space \(L^\infty(\mathbb{R}^2)\) and \(u_n = \omega_n \ast R^\perp(\theta)\) with \(\omega_n(\cdot) = n^2 \omega(n)\) where \(\omega \in C^\infty_c(\mathbb{R}^2)\) and \(\int \omega dx = 1\).

Let \(n \in \mathbb{N}\). By converting the system \((QGL_n)\) into the integral equation
\[
(IQGL_n) \quad v(t) = e^{-t(-\Delta)^\alpha} \theta_n - \int_0^t \nabla \cdot e^{-(t-s)(-\Delta)^\alpha} (u_n v) ds
\]
and by following a standard method, one can easily prove that the system \((QGL_n)\) has a unique global solution \(v_n \in \cap_{k \in \mathbb{N}} C^\infty([0, T], H^k(\mathbb{R}^2))\). Hence we are allowed to make the following computations: Let \(p \in [2, \infty]\). For any \(t \in [0, T]\) we have
\[
\frac{1}{p} \frac{d}{dt} \|v_n(t)\|^p = - \int ((-\Delta)^\alpha v) v |v|^{p-2} dx - \int \nabla (u_n v) v |v|^{p-2} dx
\]
\[
\equiv I_1(t) + I_2(t).
\]
Firstly, a simple integration by parts implies that \(I_2(t) = -I_2(t)\) and so
\[
I_2(t) = 0.
\]
Secondly, by the positivity Lemma (see \([15]\) and \([6]\)), we have
\[
I_1(t) \leq 0.
\]
Therefore,
\[
\sup_{t \in [0, T]} \|v_n(t)\|_p \leq \|\theta_n\|_p.
\]
Letting \(p \to +\infty\), yields
\[
\sup_{t \in [0, T]} \|v_n(t)\|_\infty \leq \|\theta_n\|_\infty.
\]
Consequently, to obtain the inequality (2.11) we just need to show that the sequence \((v_n)_n\) converges to the function \(\theta\) in the space \(L^\infty([0, T], L^\infty(\mathbb{R}^2))\). To do this, we consider the sequence \((w_n)_n = (v_n - \theta)_n\). Let \(t \in [0, T]\) and \(n \in \mathbb{N}\). We have
\[
w_n(t) = e^{-t(-\Delta)^\alpha} (w_n(0)) - \int_0^t \nabla \cdot e^{-(t-s)(-\Delta)^\alpha} \left((u_n - R^\perp(\theta)) v_n\right) ds
\]
\[
- \int_0^t \nabla \cdot e^{-(t-s)(-\Delta)^\alpha} \left(R^\perp(\theta) w_n\right) ds.
\]
Thus, by using the Young inequality and Proposition 2.2 we easily get
\[
\|w_n(t)\|_\infty \leq \|\theta_n - \theta(0)\|_\infty + C_\alpha T^\nu A_n B_n + C_\alpha M_\theta \int_0^t \|\frac{w_n(s)}{(t-s)^{1/2\alpha}}\|_\infty ds
\]
where \(C_\alpha\) is a constant depending only on \(\alpha\),
\[
A_n = \sup_{0 \leq t \leq T} \|u_n(t) - R^\perp(\theta)(t)\|_\infty.
\]
\[ B_n = \sup_{0 \leq t \leq T} \| v_n(t) \|_\infty \]

and

\[ M_\theta = \sup_{0 \leq t \leq T} \| R^\perp(\theta)(t) \|_\infty. \]

Applying Lemma 2.5, we get

\[ \sup_{0 \leq t \leq T} \| w_n(t) \|_\infty \leq C \left[ \| \theta_n - \theta(0) \|_\infty + C_\alpha T^\nu A_n B_n \right] \]

where \( C \) is a constant depending on \( \alpha, T \) and \( \theta \) only.

Therefore, to obtain the desired conclusion, we just have to notice that the sequence \((B_n)_n\) is bounded and that \( A_n \to 0 \) as \( n \to \infty \) thanks to the uniform continuity of the function \( R^\perp(\theta) \) on \([0, T] \times \mathbb{R}^2\), which is a consequence of the fact \( R^\perp(\theta) \in C([0, T], C_0(\mathbb{R}^2)) \).

Now, let us establish the inequality (2.12). For any \( t \in [0, T] \), we have

\[ R^\perp(\theta)(t) = e^{-t(-\Delta)^\alpha} \left( R^\perp(\theta)(0) \right) - \int_0^t R^\perp \nabla e^{-(t-s)(-\Delta)^\alpha} (R^\perp(\theta) \theta) ds. \]

Applying the Young inequality and (2.5), we get

\[ \| R^\perp(\theta)(t) \|_\infty \leq \| R^\perp(\theta)(0) \|_\infty + C \| \theta(0) \|_\infty \int_0^t \frac{\| R^\perp(\theta)(s) \|_\infty ds}{(t-s)^{1/2\alpha}} \]

where the constant \( C \) depends only on \( \alpha \). Hence, Lemma 2.5 leads the desired inequality.

3. Proof of Theorem 1.1

According to Lemma 2.3, there exists \( T > 0 \) such that \( \| e^{-t(-\Delta)^\alpha} \theta_0 \|_{E_T^\nu} \leq \mu_0 \) where \( \mu_0 \) is the real defined by Lemma 2.4. Therefore, the same lemma ensures that the equation \((QG_\alpha)\) with initial data \( \theta_0 \) has a mild solution \( \theta \) belonging to the space \( E_T^\nu \). Following a standard arguments (see for example the proof of the [10 Lemma]), the uniqueness of the solution \( \theta \) can be easily deduced from the continuity of the operator \( B_\alpha \) on the space \( E_T^\nu \).

Hence, there exists a unique maximal solution,

\[ \theta \in \bigcap_{0 < T < T^\ast} E_T^\nu, \]

where \( T^\ast \) is the maximal time existence. Let us show that,

\[ \theta \in C([0, T^\ast), \tilde{B}^\alpha). \]
GLOBAL EXISTENCE AND PERSISTENCY OF THE INITIAL REGULARITY

Thanks to the embedding,

\[(C_0(\mathbb{R}^2))_{\mathcal{R}} \subset \tilde{\mathbb{B}}^\alpha,\]

and Lemma 2.1, we just need to prove the continuity of,

\[N(\theta)(t) = B_{\alpha}[\theta, \theta](t),\]

at \(t = 0^+\) in the space \(\tilde{\mathbb{B}}^\alpha\). Even more, we show that

\[\lim_{t \to 0^+} N(\theta)(t) = 0, \quad \text{in} \quad \tilde{\mathbb{B}}^\alpha.\]

For that, we use Proposition 2.2, the Young inequality and estimates (2.4) – (2.5), to get

\[||N(\theta)(t)||_{\tilde{\mathbb{B}}^\alpha} \lesssim \sup_{0 < t' < 1} t'^{\nu} \int_0^t (t + t' - \tau)^{-\frac{\alpha}{2} - 2\nu} d\tau \ ||\theta||_{E_{\nu}}^2,\]

and hence we obtain,

\[(3.1) \quad ||N(\theta)(t)||_{\tilde{\mathbb{B}}^\alpha} \lesssim ||\theta||_{E_{\nu}}^2.\]

Since the right hand side of (3.1) goes to 0 as \(t\) goes \(0^+\) we obtain the desired result.

It remains to show that the solution \(\theta\) is global, that is \(T^* = \infty\). We argue by contradiction. If \(T^* < \infty\) then, from Lemma 2.4, we must have,

\[\forall 0 < t_0 < T^*, \quad \|e^{t(\Delta)^\alpha}\theta(t_0)\|_{\mathcal{E}_{\nu}^{t_0}} \geq \mu_0,\]

which yields by the Young inequality

\[(3.2) \quad ||\theta(t_0)||_{\infty} + ||\mathcal{R}(\theta)(t_0)||_{\infty} \geq \frac{c}{(T^* - t_0)^{\nu}},\]

where \(c > 0\) is a universal constant. Which contradicts the maximum principal (Lemma 2.5).

4. Proof of Theorem 1.2

Along this section, we consider \(\theta_0\) a given initial data belonging to the space \(\tilde{\mathbb{B}}^\alpha\) and we denote by \(\theta\) the solution to the equation (QG\(_\alpha\)) given by Theorem 1.1. We will establish the persistency of the regularity of the initial data. That is, if moreover \(\theta_0 \in X\) for a suitable Banach spaces \(X\) then the solution \(\theta \in C([0, \infty), X)\).
4.1. Propagation of the $L^p$ regularity. In this subsection we will prove the propagation of the initial $L^p$ regularity. Precisely, we prove the following proposition.

**Proposition 4.1.** Let $X = L^p$; with $p \in [1, \infty]$. If $\theta_0 \in X$ then $\theta \in \mathcal{C}(0, T, X)$. Moreover, if $\theta_0 \in \mathcal{S}(\mathbb{R}^2)^X$ then $\theta \in \mathcal{C}(0, \infty, \mathcal{S}(\mathbb{R}^2)^X)$.

**Proof:** assume $\theta_0 \in X$ and let $T > 0$. We consider the Banach spaces $\mathcal{Z}_1 = \mathcal{E}_T^{\nu}$ and $\mathcal{Z}_2 = L^\infty([0, T], X)$ endowed respectively with the norm
\[
\|v\|_{\mathcal{Z}_1} = \sup_{0 < t < T} e^{-\lambda t} t^\nu \|v(t)\|_\infty \quad \text{and} \quad \|v\|_{\mathcal{Z}_2} = \sup_{0 < t < T} e^{-\lambda t} \|v(t)\|_p,
\]
where $\lambda > 0$ to be fixed later. We consider the linear integral equation,
\[
(4.1) \quad v = \Psi_\theta(v) \equiv e^{t\lambda^{2\alpha}} \theta_0 + B_\alpha[\theta, v].
\]
Let $k \in \{1; 2\}$. According to Lemma 2.7, the affine functional $\Psi_\theta : \mathcal{Z}_k \to \mathcal{Z}_k$ is continuous. Let us estimate the norm of its linear part,
\[
K_\theta(v) = B_\alpha[\theta, v].
\]
Let $\varepsilon > 0$ to be chosen later. A direct computation using (2.3) gives,
\[
\|K_\theta\|_{\mathcal{L}(\mathcal{Z}_1)} = \sup_{\|v\|_{\mathcal{Z}_1}} \|K_\theta(v)\|_{\mathcal{Z}_1}
\leq C_1 \sup_{0 < t < T} t^\nu \int_0^t (t - \tau)^{-\frac{1}{2\alpha} - 2\nu} e^{-\lambda (t - \tau)} \|\theta\|_{\mathcal{E}_T^{\nu}} d\tau
\leq C_2 \left( \|\theta\|_{\mathcal{E}_T^{\nu}} \sup_{0 < t < \varepsilon} t^\nu \int_0^t (t - \tau)^{-\frac{1}{2\alpha} - 2\nu} d\tau + T^\nu \varepsilon^{-2\nu} \|\theta\|_{\mathcal{E}_T^{\nu}} \lambda^{-\nu} \Gamma(\nu) \right)
\leq C_3 \left( \|\theta\|_{\mathcal{E}_T^{\nu}} + T^\nu \varepsilon^{-2\nu} \lambda^{-\nu} \|\theta\|_{\mathcal{E}_T^{\nu}} \right),
\]
where the constants $C_1, C_2$ and $C_3$ depend only on $\alpha$. Similarly, we prove the estimate,
\[
\|K_\theta\|_{\mathcal{L}(\mathcal{Z}_2)} \leq C \left( \|\theta\|_{\mathcal{E}_T^{\nu}} + T^\nu \varepsilon^{-2\nu} \lambda^{-\nu} \|\theta\|_{\mathcal{E}_T^{\nu}} \right)
\]
where $C$ is a constant depending only on $\alpha$. Since, $\|\theta\|_{\mathcal{E}_T^{\nu}} \to 0$ as $\varepsilon \to 0^+$, one can choose, successively, $\varepsilon$ small enough and $\lambda$ large enough so that $\Psi_\theta$ becomes a contraction on $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and therefore on $\mathcal{Z}_1 \cap \mathcal{Z}_2$. Let $v_1$ and $v_{1,2}$ be the unique fixed point of $\Psi_\theta$ respectively in $\mathcal{Z}_1$ and $\mathcal{Z}_1 \cap \mathcal{Z}_2$. Now, since $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subset \mathcal{Z}_1$ then $v_1 = v_{1,2}$. Moreover, by construction $\theta$ is a fixed point of $\Psi_\theta$ in $\mathcal{Z}_1$ thus $\theta = v_1 = v_{1,2}$ and hence $\theta \in L^\infty([0, T], X)$.

The proof of the last statement of the proposition is identically similar, we have only to replace $\mathcal{Z}_2$ by $C([0, T], \mathcal{S}(\mathbb{R}^2)^X)$. \hfill \blacksquare
4.2. **Propagation of $\dot{B}^{s,q}_p$ regularity for $s > 0$.** In this section, we prove an abstract result, which implies in particular the persistence of the $\dot{B}^{s,q}_p$ regularity for $s > 0$. Our result states as follows:

**Proposition 4.2.** Let $X$ be a shift invariant functional space such that for a constant $C$

\[
\forall f, g \in X \cap L^\infty(\mathbb{R}^2), \quad \|fg\|_X \leq C (\|f\|_\infty \|g\|_X + \|g\|_\infty \|f\|_X).
\]

If the initial data $\theta_0$ belongs to $X$ then the solution $\theta$ belongs to $\bigcap_{T > 0} L^\infty([0,T], X)$. Moreover, if $\theta_0$ belongs to $(S(\mathbb{R}^2)^\alpha)_R$ then $\theta$ belongs to $C\left(\mathbb{R}^+, (S(\mathbb{R}^2)^\alpha)_R\right)$.

The proof of this proposition relies essentially on the two followings lemmas. The first one is an elementary compactness lemma:

**Lemma 4.1.** Let $\lambda > 0$ and $K$ a compact subset of $\tilde{B}^\alpha$. Then there exists $\delta = \delta(K, \lambda) > 0$ such that

\[
\forall f \in K, \quad \|e^{-t(-\Delta)^\alpha}f\|_{E_\delta^\alpha} \leq \lambda.
\]

**Proof:** For $n \in \mathbb{N}^*$, we set

\[
V_n = \left\{ f \in \tilde{B}^\alpha, \quad \|e^{-t(-\Delta)^\alpha}f\|_{E_t^\alpha/n} < \lambda \right\}.
\]

We claim that, $\forall n \in \mathbb{N}^*$, $V_n$ is an open subset of $\tilde{B}^\alpha$ and $\bigcup_n V_n = \tilde{B}^\alpha$. This follows easily from the continuity of the linear operator $e^{-t(-\Delta)^\alpha}$ from $B^\alpha$ into $E_T^\alpha$ for all $T > 0$ and the propriety

\[
\forall f \in \tilde{B}^\alpha, \quad \lim_{T \to 0} \|e^{-t(-\Delta)^\alpha}f\|_{E_T^\alpha} = 0.
\]

Thus, since $K$ is a compact subset of $\tilde{B}^\alpha$, there exists a finite subset $I \subset \mathbb{N}^*$ such that $K \subset \bigcup_I V_n = V_n$. where $n^* = \max(n \in I)$. Hence, we conclude that the choice $\delta = 1/n^*$ is suitable.

The second lemma establishes a local in time propagation of the $X$ regularity.

**Lemma 4.2.** Let $X$ be as in the Proposition 4.2. If $\theta_0$ belongs to $X$ (resp. $(S(\mathbb{R}^2)^\alpha)_R$) then there exists $\delta = \delta(X, \alpha) > 0$ such that the solution $\theta \in L^\infty([0,\delta], X)$ ( resp. $C\left([0,\delta], (S(\mathbb{R}^2)^\alpha)_R\right)$). Moreover, the time $\delta$ is bounded below by

\[
\sup \left\{ T > 0, \quad \|e^{-t(-\Delta)^\alpha}\theta_0\|_{E^\alpha_T} \leq \mu \right\},
\]

where $\mu$ is a non negative constant depending on $X$ and $\alpha$ only.
Proof: let us consider the case of \( \theta_0 \in X_R \). The proof in the other case is similar. Let \( \mu \in ]0, \mu_0[ \) to be chosen later and let \( T > 0 \) such that \( \|e^{-t(-\Delta)}\theta_0\|_{E_T^\mu} \leq \mu \). According to the Lemma 2.4, the sequence \( (\phi_n(\theta_0))_n \) converges in \( E_T^\mu \) to the solution \( \theta \) and satisfies the following estimates

\[
(4.3) \quad \sup_n \|\phi_n(\theta_0)\|_{E_T^\mu} \leq \mu
\]

\[
(4.4) \quad \forall n \in \mathbb{N}, \quad \|\phi_{n+1}(\theta_0) - \phi_n(\theta_0)\|_{E_T^\mu} \leq 2^{-n}.
\]

Then, to conclude we just need to show that \( (\phi_n(\theta_0))_n \) is a Cauchy sequence in the Banach space \( z_R = L^\infty([0,T], X_R) \) endowed with its natural norm,

\[
\|v\|_{z_R} = \sup_{0 < t < \delta} (\|v(t)\|_X + \|\mathcal{R}^+(v)(t)\|_X).
\]

Firstly, using the Lemma 2.2 and the fact that \( (\phi_n(\theta_0))_n \in E_T^\mu \), we infer inductively that the sequence \( (\phi_n(\theta_0))_n \) belongs to the space \( Z_R \). Secondly, once again the Lemma 2.2 implies that the sequence \( (\omega_{n+1})_n \equiv (\phi_{n+1}(\theta_0) - \phi_n(\theta_0))_n \) satisfies the following inequality

\[
\|\omega_{n+1}\|_{Z_R} \leq C \left( \|\phi_n(\theta_0)\|_{Z_R} + \|\phi_{n-1}(\theta_0)\|_{Z_R} \right) \|\omega_n\|_{E_T^\mu} + C \left( \|\phi_n(\theta_0)\|_{E_T^\mu} + \|\phi_{n-1}(\theta_0)\|_{E_T^\mu} \right) \|\omega_n\|_{Z_R},
\]

where \( C = C(X, \alpha) > 0 \). This inequality combined with the estimates \((4.3) - (4.4)\) yields

\[
\|\omega_{n+1}\|_{Z_R} \leq C \left( \frac{1}{2} \right)^n \left( \|\phi_n(\theta_0)\|_{Z_R} + \|\phi_{n-1}(\theta_0)\|_{Z_R} \right) + 4C\mu \|\omega_n\|_{Z_R}.
\]

Finally, if we choose \( \mu > 0 \) such that \( 4C\mu < 1 \) one can conclude the proof by using the following Lemma which is inspired from [8]. 

**Lemma 4.3.** Let \( (x_n)_n \) be a sequence in a normed vector space \( (Z, \|\cdot\|) \). If there exist a constant \( \lambda \in [0,1[ \) and \( (\sigma_n)_n \in l^1(\mathbb{N}) \) such that:

\[
(4.5) \quad \forall n \in \mathbb{N}^*, \quad \|x_{n+1} - x_n\| \leq \sigma_n (\|x_n\| + \|x_{n-1}\|) + \lambda \|x_n - x_{n-1}\|,
\]

then the series \( \sum_n \|x_{n+1} - x_n\| \) converges. In particular, \( (x_n)_n \) is a Cauchy sequence in \( Z \).

Proof: let us define the sequence \( M_n = \sup_{k \leq n} \|x_k\| \). It follows inductively from \((4.5)\)

\[
\|x_{n+1} - x_n\| \leq 2 \sum_{k=0}^{n-1} \sigma_{n-k} M_{n-k} \lambda^k,
\]

\[
(4.6) \quad \leq C_n M_n,
\]

R. May. and E. Zahrouni.
where \( \varpi_n = 2 \sum_{k=0}^{n-1} \sigma_{n-k} \lambda^k \).

Noticing that since \((\varpi_n)_n\) is a convolution of two sequences in \(l^1(\mathbb{N})\) then \((\varpi_n)_n\) belongs to \(l^1(\mathbb{N})\). Therefore, we just need to show that the sequence \((M_n)_n\) is bounded. This is somehow obvious. In fact, using the triangular inequality \( \|x_{n+1}\| \leq \|x_n\| + \|x_{n+1} - x_n\| \), (4.6) yields

\[
M_{n+1} \leq (1 + \varpi_n)M_n.
\]

Which in turn implies

\[
M_n \leq \Pi_{k=0}^{n-1} (1 + \varpi_k) \leq e^{\sum_{k=0}^\infty \varpi_k}.
\]

The proof is then achieved.

Now let us see how the two previous lemmas allow to prove the Proposition 4.2.

**Proof:** as usual we consider only the case of \( \theta_0 \in X_\mathcal{R} \). Let \( T > 0 \). By the Theorem 1.1, the solution \( \theta \) is continuous from \( \mathbb{R}^+ \) into \( \tilde{\mathcal{B}}^\alpha \), then \( K \equiv \theta([0,T]) \) is a compact subset of \( \tilde{\mathcal{B}}^\alpha \). Therefore, in view of the Lemma 4.1, there exists \( \delta > 0 \) such that

\[
\forall \tau \in [0,T], \quad \|e^{-\Delta \tau}\theta(\tau)\|_{E^\alpha} \leq \mu_0,
\]

where \( \mu_0 \) is the real given by Lemma 4.2. Now, we consider a repartition \( 0 = t_0 < \cdots < t_{N+1} = T \) of the interval \([0,T]\) such that \( \sup_i t_{i+1} - t_i \leq \frac{\delta}{2} \). We will show inductively that

\[
\theta \in L^\infty([t_i, t_{i+1}], X_\mathcal{R}),
\]

which implies in turn the desired result \( \theta \in L^\infty([0,T], X_\mathcal{R}) \). First, by the Lemma 4.2, the claim (4.8) is true for \( i = 0 \). Assume that, it is also true for \( i \leq N \). Then there exists \( \tau_0 \) in \([t_i, t_{i+1}]\) such that \( \bar{\theta}_0 \equiv \theta(\tau_0) \in X \cap \tilde{\mathcal{B}}^\alpha \). We notice that \( \bar{\theta} \equiv \theta(. + \tau_0) \) is the unique solution given by Theorem 1.1 of the Quasi-geostrophic equation with initial data \( \bar{\theta}_0 \). Then according to Lemma 4.2 and (4.7), we get \( \theta \in L^\infty([\tau_0, \tau_0 + \delta], X_\mathcal{R}) \). Hence, we are ready to conclude since \([t_{i+1}, t_{i+2}] \subset [\tau_0, \tau_0 + \delta] \).

4.3. Propagation of \( B_p^{s,q} \) regularity for \( s < 0 \).

**Proposition 4.3.** Let \( X \) be \( B_p^{s,q} \) or \( \tilde{B}_p^{s,q} \) with \( s < 0 \) and \( 1 \leq p, q \leq \infty \). If \( \theta_0 \) belongs to \( X_\mathcal{R} \) then the solution

\[
\theta \in \bigcap_{T>0} L^\infty([0,T], X_\mathcal{R}).
\]

As in the case \( s > 0 \), by using the compactness lemma 4.1 we just need to prove the following local persistency result:
Lemma 4.4. If \( \theta_0 \in X_R \) then there exists \( \delta > 0 \) such that,
\[
\theta \in L^\infty ([0, \delta], X_R).
\]
Moreover, the time \( \delta \) is bounded below by,
\[
\sup \left\{ T > 0 / \| e^{-t(-\Delta)^\alpha} \theta_0 \|_{E_T^\nu} \leq \mu_0 \right\},
\]
where \( \mu_0 \) is given by Lemma 2.4.

Proof: we consider only the case of \( X = B_{s,q}^p \). The proof in the other case is similar. Let \( T > 0 \) such that
\[
\| e^{-t(-\Delta)^\alpha} \theta_0 \|_{E_T^\nu} \leq \mu_0.
\]
According to the Lemma 2.4 the sequence \( (\phi_n(\theta_0))_n \) satisfies
\[
(4.9) \quad \| \phi_{n+1}(\theta_0) - \phi_n(\theta_0) \|_{E_T^\nu} \leq \frac{1}{2^n},
\]
and converges to the solution \( \theta \) in \( E_T^\nu \). Our first task is to prove that \( (\phi_n(\theta_0))_n \) is a Cauchy sequence in the space,
\[
X_T^{\sigma,p} = \{ v : (0, T] \to L^p; \| v \|_{X_T^{\sigma,p}} \equiv \sup_{0 < t < T} t^{\frac{\sigma}{p}} (\| v(t) \|_p + \| R(v)(t) \|_p) < \infty \},
\]
where \( \sigma = -s \).
Thanks to the Besov characterization (2.2) and Lemma 2.2, we can show inductively that \( (\phi_n(\theta_0))_n \) belongs to \( X_T^{\sigma,p} \) and satisfies,
\[
(4.10) \quad \| \phi_{n+1}(\theta_0) - \phi_n(\theta_0) \|_{X_T^{\sigma,p}} \leq C \| \phi_n(\theta_0) - \phi_{n-1}(\theta_0) \|_{E_T^\nu} \max (\| \phi_n(\theta_0) \|_{X_T^{\sigma,p}}, \| \phi_{n-1}(\theta_0) \|_{X_T^{\sigma,p}}).
\]
Thus, By (4.9) and Lemma 4.3 we deduce that \( (\phi_n(\theta_0))_n \) is a Cauchy sequence in \( X_T^{\sigma,p} \).
Therefore its limit \( \theta \in X_T^{\sigma,p} \). Now by a simple computation using the characterization (2.2) we deduce that \( \theta \in L^\infty ([0, T_0], (B_{s,\infty}^p)_R) \). Moreover, for \( \epsilon > 0 \) such that
\[
(4.11) \quad -1 < s \pm \epsilon < 0,
\]
one can show that the nonlinear part \( N(\theta)(t) = B_\alpha [\theta, \theta](t) \) satisfies
\[
(4.12) \quad \| N(\theta)(t) \|_{B_{p}^{s,\infty}} + \| R^\perp N(\theta)(t) \|_{B_{p}^{s,\infty}} \leq C_{s,\epsilon} t^{-\frac{s}{2p}} \| \theta \|_{E_T^\nu} || \theta ||_{X_T^{\sigma,p}},
\]
Indeed, we have \( \tau \in [0,1] \)
\[
\tau^{-\frac{2\alpha}{m}} \left\| e^{-\tau(-\Delta)^{\alpha}} N(\theta)(t) \right\|_p \leq C \int_0^t (t + \tau - r)^{-\frac{2\alpha}{m}} \tau^{-\frac{m}{2}} \tau^{-\nu_r} \frac{\tau}{r} dr \left\| \theta \right\|_{E^\infty_r} \left\| \theta \right\|_{X^s_{\sigma, p}},
\]
\[
\leq C \int_0^t \left( \frac{t}{t + \tau - r} \right)^{-\frac{2\alpha}{m}} \left( t + \tau - r \right)^{-\frac{m}{2}} \tau^{-\nu_r} \frac{\tau}{r} dr \left\| \theta \right\|_{E^\infty_r} \left\| \theta \right\|_{X^s_{\sigma, p}},
\]
(4.13)
\[
\leq C \int_0^t (t - r)^{-\frac{1}{\alpha+1}} \tau^{-\nu_r} \frac{\tau}{r} dr \left\| \theta \right\|_{E^\infty_r} \left\| \theta \right\|_{X^s_{\sigma, p}},
\]
(4.14)
\[
\leq C t^{-\frac{2\alpha}{m}} \left\| \theta \right\|_{E^\infty_r} \left\| \theta \right\|_{X^s_{\sigma, p}},
\]

Where to obtain (4.13), we have used the facts that \( 0 \leq \frac{\tau}{t+\tau-r} \leq 1 \), \( t + \tau - r \geq t - r \) and (4.11). Similarly, we have the same estimate (4.14) for \( R^\perp N(\theta)(t) \). Hence, by Proposition 2.1, we get (4.12). Thus, by interpolation we obtain \( N(\theta) \in L^\infty([0,T], (B^{s,q}_p)_{R^\perp}) \) which implies \( \theta \in L^\infty([0,T], (B^{s,q}_p)) \).

4.4. The case of null regularity \( s = 0 \). In this subsection we aim to prove the following result,

**Proposition 4.4.** Let \( X \) be \( B^{0,q}_p \) or \( B^{0,q}_p \) with \( 1 \leq p, q \leq \infty \). If \( \theta_0 \in X \) then the solution
\[
\theta \in \bigcap_{T>0} L^\infty([0,T], X).
\]

Thanks to the following imbedding,
\[
\dot{B}^{0,1}_p \subset \dot{B}^{0,q}_p \subset \dot{B}^{0,\infty}_p,
\]

and
\[
\dot{B}^{0,1}_p \subset \dot{B}^{0,q}_p \subset \dot{B}^{0,\infty}_p,
\]

the proof of the above proposition is an immediate consequence of the following lemma,

**Lemma 4.5.** If \( \theta_0 \in \dot{B}^{0,\infty}_p \) then \( N(\theta) = B_{\alpha} [\theta, \theta](t) \) belongs to \( \bigcap_{T>0} L^\infty([0,T], \dot{B}^{0,1}_p) \).

**Proof:** By using the Young inequality we deduce that
\[
\dot{B}^{0,\infty}_p \cap \dot{B}^{-(2\alpha-1),\infty}_\infty \subset \dot{B}^{2\frac{1}{2} - \alpha,\infty}_{2p}.
\]

Observe that \( s^* = \frac{1}{2} - \alpha < 0 \) and hence according to the proof of the proposition 4.3 and to the continuity of the Riesz transforms on homogeneous Besov spaces, we have
\[
\theta \in \bigcap_{T>0} X^T_{\sigma^*, 2p} \text{ where } \sigma^* = \alpha - \frac{1}{2}. \text{ Let } T > 0 \text{ and } 0 < \sigma < 2\alpha - 1. \text{ The basic estimate,}
\]
\[
\left\| \nabla^{\pm \sigma} \nabla e^{-t(-\Delta)^{\sigma}} f \right\|_p \leq C_\sigma t^{-\frac{\sigma+1}{2\alpha}} \left\| f \right\|_p.
\]
yields immediately
\[ \| (\sqrt{-\Delta})^\pm \sigma N(\theta)(t) \|_p \leq C t^{-\frac{\pm\sigma}{2p}} \| \theta \|_{X_{\sigma,2}^T}^2. \]
Now, we use the interpolation result (see Theorem 6.3 in [1])
\[ \left[ (\sqrt{-\Delta})^\sigma L^p, (\sqrt{-\Delta})^{-\sigma} L^p \right]_{\frac{1}{2},1} = \hat{B}^{0,1}_p, \]
to deduce,
\[ \forall 0 < t < T, \quad \| N(\theta)(t) \|_{\hat{B}^{0,1}_p} \leq C \| \theta \|_{X_{\sigma,2}^T}^2, \]
that implies,
\[ N(\theta) \in \hat{L}^\infty ([0,T], \hat{B}^{0,1}_p). \]

Remark 4.1. As in the context of the Navier-Stokes equations [3], we observe thanks to (4.10) and (4.15) that in the case $-1 < s \leq 0$, the fluctuation term $w(t)$ is more regular than the tendency $e^{-t(-\Delta)^s} \theta_0$.

5. Proof of Theorem 1.3

The existence part is a direct consequence of Theorem 1.1, Theorem 1.2 and the following embedding (consequence of Bernstein’s inequality and the boundedness of the Riesz transforms on Lebesgue’s and Sobolev’s spaces)
\[ L^p(\mathbb{R}^2) \subset \tilde{B}_\sigma \forall p \geq p_c, \]
\[ H^s(\mathbb{R}^2) = B^{s,2}_2(\mathbb{R}^2) \subset \tilde{B}_\sigma \forall s \geq s_c. \]

Let us establish the uniqueness part. First we notice that since for $s \geq s_c$ then
\[ H^s(\mathbb{R}^2) \hookrightarrow H^{sc}(\mathbb{R}^2) \hookrightarrow L^{pc}(\mathbb{R}^2), \]
therefore, we just need to prove the uniqueness in the spaces $(C([0,T], L^p(\mathbb{R}^2)))_{p \geq p_c}$. This will be deduced from the following continuity result of the bilinear operator $B_\sigma$.

Lemma 5.1. Let $p \in ]p_c, \infty[, q \in ]1, \infty[ and T > 0$. There exists a constant $C$ independent of $T$ such that:

- for any $u, v$ in $L^\infty_T L^p$,
\[ \| B_\sigma [u,v] \|_{L^\infty_T L^p} \leq C T^\sigma \| u \|_{L^\infty_T L^p} \| v \|_{L^\infty_T L^p}, \]
where $\sigma = \frac{1}{\alpha} \left( \frac{1}{p_c} - \frac{1}{p} \right)$. 
GLOBAL EXISTENCE AND PERSISTENCY OF THE INITIAL REGULARITY

- for any \( u, v \) in \( L^\infty_T L^p_R \),
  \[
  \| B_\alpha[u, v] \|_{L^q_T L^p_R} + \| B_\alpha[v, u] \|_{L^q_T L^p_R} \leq C \| u \|_{L^\infty_T L^p_R} \| v \|_{L^q_T L^p_R}.
  \]

- for any \( u \in L^\infty_T L^\infty_R \) and \( v \in L^q_T L^p_R \),
  \[
  \| B_\alpha[u, v] \|_{L^q_T L^p_R} + \| B_\alpha[v, u] \|_{L^q_T L^p_R} \leq C T^{1 - \frac{1}{2\alpha}} \| u \|_{L^\infty_T L^\infty_R} \| v \|_{L^q_T L^p_R}.
  \]

**Proof:** Estimate (5.1) follows easily from the continuity of the Riesz transforms on the Lebesgue spaces \( L^r(\mathbb{R}^2) \) with \( 1 < r < \infty \), the Young and the Hölder inequality and the estimate 2.4 on the \( L^r(\mathbb{R}^2) \) norm of the kernel of the operator \( \nabla e^{-(t-s)(-\Delta)^\alpha} \).

Estimate (5.2) is a consequence of the continuity of the Riesz transforms on the space \( L^p_R(\mathbb{R}^2) \), the Hölder inequality, the Sobolev embedding and the following maximal regularity property of the operator \( (-\Delta)^\alpha \)

\[
\left\| \int_0^t (-\Delta)^\alpha e^{-(t-s)(-\Delta)^\alpha} v ds \right\|_{L^q_T L^p_R} \lesssim \| v \|_{L^q_T L^p_R},
\]

which can be proved by following the proof of Theorem 7.3 in [10]. Let us now prove estimate (5.3). For any \( t \in [0, T] \) we have

\[
\left\| B_\alpha[u, v](t) \right\|_{L^p_R} \lesssim \int_0^t \frac{1}{(t-s)^{1/2\alpha}} \| \mathcal{R}^\perp(u)(s) \|_\infty \| v(s) \|_{p_c} ds
\]

\[
\lesssim \left\| \mathcal{R}^\perp(u) \right\|_{L^\infty_T L^\infty_R} \left( 1_{[0,T]} s^{-\frac{1}{2\alpha}} \right) \ast \left( 1_{[0,T]} \| v(s) \|_{p_c} \right)(t)
\]

where the star denotes the convolution in \( \mathbb{R} \). Hence the Young inequality yields

\[
\left\| B_\alpha[u, v] \right\|_{L^q_T L^p_R} \lesssim \left\| \mathcal{R}^\perp(u) \right\|_{L^\infty_T L^\infty_R} T^{1 - \frac{1}{2\alpha}} \| v \|_{L^q_T L^p_R}.
\]

Similarly, we obtain

\[
\left\| B_\alpha[v, u] \right\|_{L^q_T L^p_R} \lesssim T^{1 - \frac{1}{2\alpha}} \| u \|_{L^\infty_T L^\infty_R} \left\| \mathcal{R}^\perp(v) \right\|_{L^2_T L^p_R}
\]

\[
\lesssim T^{1 - \frac{1}{2\alpha}} \| u \|_{L^\infty_T L^\infty_R} \| v \|_{L^q_T L^p_R}.
\]

Estimate (5.3) is then proved.

Now we are ready to finish the proof of the uniqueness. Let \( p \geq p_c \) and \( T > 0 \) be two reals number and let \( \theta_1 \) and \( \theta_2 \) be two mild solutions of the equation \((QG_\alpha)\) with the same
data \( \theta_0 \) such that \( \theta_1, \theta_2 \in C([0, T], L^p(\mathbb{R}^2)) \). We aim to show that \( \theta_1 = \theta_2 \) on \([0, T] \). For this, we will argue by contradiction. Then we suppose that \( t_* < T \) where

\[
t_* \equiv \sup\{t \in [0, T]; \forall s \in [0, t], \theta_1(s) = \theta_2(s)\}.
\]

To conclude, we need to prove that there exists \( \delta \in ]0, T - t_*] \) such that \( \tilde{\theta}_1 = \tilde{\theta}_2 \) on \([0, \delta] \), where \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are the functions defined on \([0, T - t_*] \) by

\[
\tilde{\theta}_1(t) = \theta_1(t + t_*), \quad \tilde{\theta}_2(t) = \theta_2(t + t_*).
\]

We deal separately with the sub-critical case and the critical case:

**The first case:** \( p > p_c \). Thanks to the continuity of \( \theta_1 \) and \( \theta_2 \) on \([0, T] \), we have \( \theta_1(\tau_*) = \theta_2(t_*) \). Hence, the functions \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) are two mild solutions on \([0, \delta_0 \equiv T - t_*] \) of the equation \((QG_\alpha)\) with the same data \( \theta_1(\tau_*) \). Therefore, the function \( \tilde{\theta} \equiv \tilde{\theta}_1 - \tilde{\theta}_2 \) satisfies the equation

\[
\tilde{\theta} = B_\alpha[\tilde{\theta}_1, \tilde{\theta}] - B_\alpha[\tilde{\theta}_1, \tilde{\theta}_2].
\]

Thus, according to (5.1) we have for any \( \delta \in ]0, \delta_0] \)

\[
\left\| \tilde{\theta} \right\|_{L^p_{\tilde{\theta}}} \leq C\delta^\alpha \left( \left\| \tilde{\theta}_1 \right\|_{L^p_{\tilde{\theta}}} + \left\| \tilde{\theta}_2 \right\|_{L^p_{\tilde{\theta}}} \right) \left\| \tilde{\theta} \right\|_{L^p_{\tilde{\theta}}}
\]

\[
\leq C\delta^\alpha \left( \left\| \theta_1 \right\|_{L^p_{\tilde{\theta}}} + \left\| \theta_2 \right\|_{L^p_{\tilde{\theta}}} \right) \left\| \tilde{\theta} \right\|_{L^p_{\tilde{\theta}}}
\]

where \( C > 0 \) is independent on \( \delta \).

Consequently, for \( \delta \) small enough, \( \tilde{\theta} = 0 \) on \([0, \delta] \) which ends the proof in the sub-critical case.

**The second case:** \( p = p_c \). Choose a fix real \( q > 1 \) and let \( \varepsilon > 0 \) to be chosen later. By density of smooth functions in the space \( C([0, T], L^{p_c}(\mathbb{R}^2)) \), one can decompose \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) into \( \tilde{\theta}_1 = u_1 + v_1 \) and \( \tilde{\theta}_2 = u_2 + v_2 \) with

\[
\left\| u_1 \right\|_{L^q_{\tilde{\theta}_1} L^{p_c}} + \left\| u_2 \right\|_{L^q_{\tilde{\theta}_2} L^{p_c}} \leq \varepsilon,
\]

\[
\left\| v_1 \right\|_{L^q_{\tilde{\theta}_1} L^{\infty}} + \left\| v_2 \right\|_{L^q_{\tilde{\theta}_2} L^{\infty}} \equiv \mathcal{M} < \infty.
\]

As in the previous case, the function \( \tilde{\theta} \equiv \tilde{\theta}_1 - \tilde{\theta}_2 \) satisfies the equations

\[
\tilde{\theta} = B_\alpha[\tilde{\theta}_1, \tilde{\theta}] + B_\alpha[\tilde{\theta}_1, \tilde{\theta}_2]
\]

\[
= B_\alpha[u_1, \tilde{\theta}] + B_\alpha[v_1, \tilde{\theta}] + B_\alpha[u_2, \tilde{\theta}] + B_\alpha[v_2, \tilde{\theta}].
\]

Now by applying (5.2)-(5.3) and using (5.5)-(5.6) we get, for any \( \delta \in ]0, \delta_0] \), the following estimate

\[
\left\| \tilde{\theta} \right\|_{L^p_{\tilde{\theta}}} \leq C \left( \varepsilon + \delta^{1 - \frac{1}{q} + \frac{1}{p_c}} \mathcal{M} \right) \left\| \tilde{\theta} \right\|_{L^p_{\tilde{\theta}}}
\]
where $C > 0$ is a constant depending only on $\alpha, p$ and $q$.

Thus, by choosing $\varepsilon$ small enough, we conclude that there exists $\delta \in ]0, \bar{\delta}_0]$ such that $\|\tilde{\theta}\|_{L^q_tL^p} = 0$, which implies that $\tilde{\theta}_1 = \tilde{\theta}_2$ on $[0, \delta]$. The proof is then achieved.

**Remark 5.1.** The idea of the proof of the uniqueness in the critical case is inspired from the paper [12] of S. Monniaux.

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