THE GROWTH OF THE DISCRIMINANT OF THE ENDOMORPHISM RING OF
THE REDUCTION OF A RANK 2 GENERIC DRINFELD MODULE

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Dedicated to Professor Ernst-Ulrich Gekeler

Abstract. For \( q \) an odd prime power, \( A = \mathbb{F}_q[T] \), and \( F = \mathbb{F}_q(T) \), let \( \psi : A \rightarrow F\{\tau}\) be a Drinfeld \( A \)-module over \( F \) of rank 2 and without complex multiplication, and let \( p = pA \) be a prime of \( A \) of good reduction for \( \psi \), with residue field \( \mathbb{F}_p \). We study the growth of the absolute value \( |\Delta_p| \) of the discriminant of the \( \mathbb{F}_p \)-endomorphism ring of the reduction of \( \psi \) modulo \( p \) and prove that, for all \( p \), \( |\Delta_p| \) grows with \( |p| \). Moreover, we prove that, for a density 1 of primes \( p \), \( |\Delta_p| \) is as close as possible to its upper bound \( |a_2^2 - 4\mu_pp| \), where \( X^2 + a_2X + \mu_pp \in A[X] \) is the characteristic polynomial of \( \tau_{\text{deg}p} \).

1. Introduction

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements, \( A := \mathbb{F}_q[T] \) be the ring of polynomials in \( T \) over \( \mathbb{F}_q \), \( F := \mathbb{F}_q(T) \) be the field of fractions of \( A \), and \( \mathbb{F}_p^\text{alg} \) be a fixed algebraic closure of \( F \). We call a nonzero prime ideal of \( A \) simply a prime of \( A \). The main results of this paper concern the reductions modulo primes of \( A \) of a fixed Drinfeld module over \( F \). To state these results, we first recall some basic concepts from the theory of Drinfeld modules.

An \( A \)-field is a field \( L \) equipped with a homomorphism \( \gamma : A \rightarrow L \). Two \( A \)-fields of particular prominence in this paper are \( F \) and \( \mathbb{F}_p := A/p \), where \( p \triangleleft A \) is a prime. When \( L \) is an extension of \( F \), we implicitly assume that \( \gamma : A \hookrightarrow L \) is obtained from the natural embedding of \( A \) into its field of fractions \( A \hookrightarrow F \hookrightarrow L \); when \( L \) is a finite extension of \( \mathbb{F}_p \), we implicitly assume that \( \gamma : A \rightarrow A/p \hookrightarrow L \) is obtained from the natural quotient map.

For an \( A \)-field \( L \), denote by \( L\{\tau\} \) the noncommutative ring of polynomials in \( \tau \) with coefficients in \( L \) and subject to the commutation rule \( \tau c = c^q\tau \), \( c \in L \). A Drinfeld \( A \)-module of rank \( r \geq 1 \) defined over \( L \) is a

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ring homomorphism \( \psi : A \to L\{\tau\} \), \( a \mapsto \psi_a \), uniquely determined by the image of \( T \):

\[
\psi_T = \gamma(T) + \sum_{1 \leq i \leq r} g_i(T)\tau^i, \quad g_r(T) \neq 0.
\]

The endomorphism ring of \( \psi \) is the centralizer of the image \( \psi(A) \) of \( A \) in \( L\{\tau\} \):

\[
\text{End}_L(\psi) := \{ u \in L\{\tau\} : u\psi_a = \psi_a u \text{ for all } a \in A \}
\]

\[
= \{ u \in L\{\tau\} : u\psi_T = \psi_T u \}.
\]

As \( \text{End}_L(\psi) \) contains \( \psi(A) \cong A \) in its center, it is an \( A \)-algebra. It can be shown that \( \text{End}_L(\psi) \) is a free \( A \)-module of rank \( \leq r^2 \); see [Dr74, Sec. 2].

Now let \( \psi : A \to F\{\tau\} \) be a Drinfeld \( A \)-module of rank \( r \) over \( F \) defined by

\[
\psi_T = T + g_1\tau + \ldots + g_r\tau^r.
\]

We say that a prime \( \mathfrak{p} \triangleleft A \) is a prime of good reduction for \( \psi \) if \( \text{ord}_\mathfrak{p}(g_i) \geq 0 \) for all \( 1 \leq i \leq r-1 \) and if \( \text{ord}_\mathfrak{p}(g_r) = 0 \). If that is the case, we view \( g \in A_{\mathfrak{p}} \) as elements of the completion \( A_\mathfrak{p} \) of \( A \) at \( \mathfrak{p} \) and define the reduction of \( \psi \) at \( \mathfrak{p} \) as the Drinfeld \( A \)-module \( \psi \otimes F_\mathfrak{p} : A \to F_\mathfrak{p}\{\tau\} \) given by

\[
(\psi \otimes F_\mathfrak{p})_{\mathfrak{p}} = \mathfrak{T} + \overline{g_1}\tau + \cdots + \overline{g_r}\tau^r,
\]

where \( \overline{g} \) is the image of \( g \in A_\mathfrak{p} \) under the canonical homomorphism \( A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p} \). Note that \( \psi \otimes F_\mathfrak{p} \) has rank \( r \) since \( \overline{g_r} \neq 0 \). It is clear that all but finitely many primes of \( A \) are primes of good reduction for \( \psi \); we denote the set of these primes by \( \mathcal{P}(\psi) \).

Let \( \mathfrak{p} = pA \) be a prime of good reduction of \( \psi \), where \( p \) denotes the monic generator of \( \mathfrak{p} \). Denote by \( \mathcal{E}_{\psi,\mathfrak{p}} = \text{End}_{F_\mathfrak{p}}(\psi \otimes F_\mathfrak{p}) \) the endomorphism ring of \( \psi \otimes F_\mathfrak{p} \). It is easy to see that \( \pi_\mathfrak{p} := \tau^{\deg p} \) is in the center of \( F_\mathfrak{p}\{\tau\} \), hence \( \pi_\mathfrak{p} \in \mathcal{E}_{\psi,\mathfrak{p}} \). Using the theory of Drinfeld modules over finite fields, it is easy to show that \( A[\pi_\mathfrak{p}] \) and \( \mathcal{E}_{\psi,\mathfrak{p}} \) are \( A \)-orders in the imaginary field extension \( F(\pi_\mathfrak{p}) \) of \( F \) of degree \( r \) ("imaginary" means that there is a unique place of \( F(\pi_\mathfrak{p}) \) over the place \( \infty := 1/T \) of \( F \)); see [GaPa19, Prop. 2.1]. Here we remind the reader that \( A[\pi_\mathfrak{p}] = \psi(A)[\pi_\mathfrak{p}] \), i.e. that \( A \cong \psi(A) \) also denotes the image of \( A \) under \( \psi \). Then, denoting by \( \mathcal{O}_{F(\pi_\mathfrak{p})} \) the integral closure of \( A \) in \( F(\pi_\mathfrak{p}) \), we obtain a natural inclusion of \( A \)-orders

\[
A[\pi_\mathfrak{p}] \subseteq \mathcal{E}_{\psi,\mathfrak{p}} \subseteq \mathcal{O}_{F(\pi_\mathfrak{p})}.
\]

It is an interesting problem, with important applications to the arithmetic of \( F \), to compare the above orders as \( \mathfrak{p} \) varies. For example, it is proved in [CoPa15, Thm. 1] (for \( r = 2 \)) and in [GaPa19, Thm. 1.2], [GaPa20, Thm. 1.1] (for \( r \geq 2 \)) that the quotient \( \mathcal{E}_{\psi,\mathfrak{p}}/A[\pi_\mathfrak{p}] \) captures the splitting behavior of \( \mathfrak{p} \) in the division fields of \( \psi \); this result then leads to non-abelian reciprocity laws in function field arithmetic. Also, in [GaPa19, Thm. 1.1], it is shown that the quotients \( \mathcal{E}_{\psi,\mathfrak{p}}/A[\pi_\mathfrak{p}] \) and \( \mathcal{O}_{F(\pi_\mathfrak{p})}/\mathcal{E}_{\psi,\mathfrak{p}} \) can be arbitrarily large as \( \mathfrak{p} \) varies, whereas in [CoPa15, Thm. 6] an explicit formula is given for the density of primes for which \( A[\pi_\mathfrak{p}] = \mathcal{E}_{\psi,\mathfrak{p}} \).
In this paper, we are interested in the growth of the discriminant of \( E_{\psi,p} \) as \( p \) varies, and in applications of this growth to the arithmetic of \( \psi \). Our results assume that \( q \) is odd and \( r = 2 \). These assumptions are to be kept from here on without further notice.

Choosing a basis \( E_{\psi,p} = A\alpha_1 + A\alpha_2 \) of \( E_{\psi,p} \) as a free \( A \)-module of rank 2, the discriminant \( \Delta_p := \Delta_{\psi,p} \) of \( E_{\psi,p} \) is \( \det(\text{Tr}_{F(\tau)/F}(\alpha_i\alpha_j))_{1 \leq i,j \leq 2} \), which is well-defined up to a multiple by a square in \( \mathbb{F}_q^\times \). If we write
\[
\Delta_p = c_p^2 \cdot \Delta_{\psi,p},
\]
where \( c_p, \Delta_{\psi,p} \in A \) with \( \Delta_{\psi,p} \) square-free, then \( E_{\psi,p} = A + c_p \mathcal{O}_{F(\tau)}, \Delta_{\psi,p} \) is the discriminant of \( \mathcal{O}_{F(\tau)}, \) and \( \mathcal{O}_{F(\tau)}/E_{\psi,p} \cong A/c_p A \) as \( A \)-modules. Note that, similarly to \( \Delta_p \), the polynomial \( c_p \) also depends on \( \psi \), although this will not be explicitly indicated in our notation.

Denote by \( |\cdot| = |\cdot|_\infty \) the absolute value on \( F \) corresponding to \( 1/T \), normalized so that \( |a| = q^{\deg a} \) for \( a \in A \), with \( \deg a \) denoting the degree of \( a \) as a polynomial in \( T \) and subject to the convention that \( \deg 0 = -\infty \). The first main result of this paper is the following:

**Theorem 1.** Assume \( \text{End}_{\mathcal{O}_{F(\tau)}}(\psi) = A \). Then
\[
|\Delta_p| \gg \psi \frac{\log |p|}{(\log \log |p|)^2},
\]
where the implied \( \gg \)-constant depends on \( q \) and on the coefficients of the polynomial \( \psi_T \in F[\tau] \).

By considering \( \tau \) as the Frobenius automorphism of \( F_p \) relative to \( F_q \), that is, as the map \( \alpha \mapsto \alpha^q \), we can also consider \( F_p \) as an \( A \)-module via \( \psi \otimes F_p \) (hence \( T \) acts on \( F_p \) as \( (\psi \otimes F_p)_T \)). This module will be denoted \( \psi F_p \). The properties of the torsion elements of this module lead to an \( A \)-module isomorphism
\[
\psi F_p \cong A/d_{1,p} A \times A/d_{2,p} A
\]
for uniquely determined nonzero monic polynomials \( d_{1,p}, d_{2,p} \in A \) such that \( d_{1,p} \mid d_{2,p} \); as above, the polynomials \( d_{1,p}, d_{2,p} \) also depend on \( \psi \), although this will not be explicitly indicated in our notation.

Note that \( d_{2,p} \) may be regarded as the exponent of the \( A \)-module \( \psi F_p \), and that \( |d_{1,p} \cdot d_{2,p}| = |p| \); thus we have the trivial lower bound \( |d_{2,p}| \geq |p|^\frac{1}{2} \). Theorem 1 allows us to deduce a stronger lower bound on \( |d_{2,p}| \).

**Theorem 2.** Assume \( \text{End}_{\mathcal{O}_{F(\tau)}}(\psi) = A \). Then
\[
|d_{2,p}| \gg \psi \frac{|p|^\frac{1}{2} \log |p|}{\log \log |p|},
\]
where the implied \( \gg \)-constant depends on \( q \) and on the coefficients of the polynomial \( \psi_T \in F[\tau] \).

Theorems 1 and 2 are the Drinfeld module analogues of results by Schoof for elliptic curves over \( \mathbb{Q} \); see the statement and proof of the main result of [Se91]. Our proof of Theorem 1 is inspired by Schoof’s arguments. It relies on Drinfeld’s now-classical function field analogue of the analytic theory of elliptic curves [Dr74], on the growth properties of the function field counterpart of the \( j \)-function, proved by Gekeler [Ge99], and on the more recent Drinfeld module analogue of Deuring’s lifting lemma, proved in an earlier paper by the present authors [CoPa15].
RemarK 3. According to Theorem 1, $|\Delta_p|$ grows with $\deg p$. In relation to the growth of $|\Delta_p|$, Theorem 1.1 of [GaPa20] implies that, for any fixed number $\kappa > 0$, we can find $p$ such that $|c_p| > \kappa$; therefore, for such $p$, $|\Delta_p| = |c_p|^2 \cdot |\Delta_{F(\pi_p)}| > \kappa$. However, Theorem 1.1 of [GaPa20] does not imply that $|\Delta_p|$ has to grow with $\deg p$. In fact, computationally, Garai and the second author have found that it happens that $c_p = 1$; in this case, their aforementioned theorem does not give any lower bound on $|\Delta_p|$. On the other hand, Theorem 1 of the present paper does not imply that we can find any $p$ such that $|c_p| > \kappa$. Thus these two results are complementary to each other.

RemarK 4. If $\End_{F_{alb}}(\psi) \neq A$, then $\End_{F_{alb}}(\psi) = O$ is an order in an imaginary quadratic extension $K$ of $F$, in which case the growth of $|\Delta_p|$ is vastly different from that shown in Theorem 1. On one hand, if $p \in \mathcal{P}(\psi)$ splits in $K$, then $O \subseteq \End_{F_{alb}}(\psi) \otimes \mathbb{F}_p \subseteq O_K$ (see [Ge83, Lem. 3.3]), which implies that $|\Delta_p| \leq |\Delta_O|$, where $\Delta_O$ is the discriminant of $O$. Hence $|\Delta_p|$ remains bounded as $p$ varies over the primes that split in $K$. In particular, Theorem 1 is false without its assumption. On the other hand, if $p \in \mathcal{P}(\psi)$ is inert in $K$, then $\psi \otimes \mathbb{F}_p$ is supersingular, which implies that $A[\pi_p] = A[\sqrt{mp}]$ for some $\alpha \in \mathbb{F}_q^\times$, and that $A[\pi_p] = E_p = O_{F(\pi_p)}$ (see Lemma 5.2 and Theorem 5.3 of [Ge83]). Hence $|\Delta_p| = |p|$, a much larger growth than that shown in Theorem 1. One can also prove that, in the supersingular case, $d_{1,p} = 1$ and $d_{2,p} = p - \beta$ for some $\beta \in \mathbb{F}_q^\times$ (see [CoPa15, Cor. 3]). Hence $|d_{2,p}| = |p|$, which is as large as possible.

RemarK 5. From the theory of Drinfeld modules over finite fields, one can deduce that the discriminant of $A[\pi_p]$ has degree $\leq \deg p$. More precisely, the characteristic polynomial of $\pi_p$ is of the form $X^2 + a_pX + \mu_pp \in A[X]$, where $\mu_p \in \mathbb{F}_q^\times$ and $\deg a_p \leq \frac{\deg p}{2}$. This implies that $|\Delta_p| \leq |a_p^2 - 4\mu_pp| \leq |p|$. Note that the coefficients $a_p$ and $\mu_p$ also depend on $\psi$, although this will not be explicitly indicated in our notation.

The lower bound on $|\Delta_p|$ in Theorem 1 holds for all primes $p = pA \triangleleft A$, with finitely many exceptions. The next theorem gives a stronger lower bound, almost as close as the upper bound of Remark 5, which holds for a set of primes $p = pA \triangleleft A$ of Dirichlet density 1:

Theorem 6. Assume $\End_{F_{alb}}(\psi) = A$. For any positive valued function $f : \mathbb{N} \to (0, \infty)$ with $\lim_{x \to \infty} f(x) = \infty$, we have that, as $x \to \infty$,

$$\# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |\Delta_p| > \frac{|a_p^2 - 4\mu_pp|}{q^{\deg p}} \right\} \sim \pi_F(x),$$

where $\pi_F(x) := \# \{ p \triangleleft A : \deg p = x \}$. Moreover, the Dirichlet density of the set

$$\left\{ p \in \mathcal{P}(\psi) : |\Delta_p| > \frac{|a_p^2 - 4\mu_pp|}{q^{\deg p}} \right\}$$

exists and equals 1.

Theorem 6 is a Drinfeld module unconditional analogue of a recent result of the first author and Fitzpatrick for elliptic curves over $\mathbb{Q}$; see [CoFo20]. It is inspired by the results of [CoSh15] and relies on some of the main results of [CoPa15].

4
In Sections 2, 3, and 4 we review and prove several results about orders, quadratic forms, and \( j \)-invariants of Drinfeld modules needed in the proofs of the main theorems. In Sections 5 and 6 we present the proofs of Theorems 1, 2, and 6.

**Notation.** Throughout the paper, we use the standard \( \sim, o, O, \ll, \gg \) notation, which we now recall: given suitably defined real functions \( h_1, h_2 \), we say that \( h_1 \sim h_2 \) if \( \lim_{x \to \infty} h_1(x)/h_2(x) = 1 \); we say that \( h_1 = o(h_2) \) if \( \lim_{x \to \infty} h_1(x)/h_2(x) = 0 \); we say that \( h_1 = O(h_2) \) or \( h_1 \ll h_2 \) or \( h_2 \gg h_1 \) if \( h_2 \) is positive valued and there exists a positive constant \( C \) such that \( |h_1(x)| \leq C h_2(x) \) for all \( x \) in the domain of \( h_1 \); we say that \( h_1 = O_D(h_2) \) or \( h_1 \ll_D h_2 \) or \( h_2 \gg_D h_1 \) if \( h_1 \ll h_2 \) and the implied \( O \)-constant \( C \) depends on priorly given data \( D \). We make the convention that any implied \( O \)-constant may depend on \( q \) without any explicit specification.

### 2. \( A \)-orders

Let \( K/F \) be a quadratic imaginary extension. Let \( B \) be the integral closure of \( A \) in \( K \). An \( A \)-order in \( K \) is an \( A \)-subalgebra \( \mathcal{O} \) of \( B \) with the same unity element and such that \( B/\mathcal{O} \) has finite cardinality. Note that an \( A \)-order \( \mathcal{O} \) is a free \( A \)-module of rank \( 2 \) and that there is an \( A \)-module isomorphism \( B/\mathcal{O} \cong A/cA \) for a unique nonzero monic polynomial \( c \in A \), called the conductor of \( \mathcal{O} \) in \( B \). It is easy to show that \( \mathcal{O} = A + cB \).

Let \( \{ \alpha_1, \alpha_2 \} \) be a basis of \( \mathcal{O} \) as a free \( A \)-module, and \( \sigma \in \text{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z} \) be the generator of \( \text{Gal}(K/F) \). The discriminant of \( \{ \alpha_1, \alpha_2 \} \) is

\[
\text{disc}(\alpha_1, \alpha_2) = \det \begin{pmatrix}
\alpha_1 & \sigma(\alpha_1) \\
\alpha_2 & \sigma(\alpha_2)
\end{pmatrix}^2.
\]

If \( \{ \beta_1, \beta_2 \} \) is another \( A \)-basis of \( \mathcal{O} \), then \( \text{disc}(\beta_1, \beta_2) = \kappa^2 \cdot \text{disc}(\alpha_1, \alpha_2) \) for some \( \kappa \in \mathbb{F}_q^\times \). The discriminant \( \Delta_{\mathcal{O}} \) of \( \mathcal{O} \) is defined to be \( \text{disc}(\alpha_1, \alpha_2) \), up to an \( (\mathbb{F}_q^\times)^2 \)-multiple. It is elementary to show that \( \mathcal{O} = A[\sqrt{\Delta_{\mathcal{O}}}] \) and \( (\Delta_{\mathcal{O}}) = c^2 \cdot (\Delta_B) \), where \( (a) \) denotes the ideal generated by an element \( a \in A \). Note that \( \Delta_B \) is square-free.

**Remark 7.** The splitting behavior of \( \infty = 1/T \) in any quadratic extension \( K/F \) can be described using the discriminant \( \Delta_B \). Namely, \( \infty \) ramifies in \( K/F \iff \deg \Delta_B \) is odd; \( \infty \) splits in \( K/F \iff \deg \Delta_B \) is even and the leading coefficient of \( \Delta_B \) is a square in \( \mathbb{F}_q^\times \); \( \infty \) is inert in \( K/F \iff \deg \Delta_B \) is even and the leading coefficient of \( \Delta_B \) is not a square in \( \mathbb{F}_q^\times \).

There are two important groups associated to an \( A \)-order \( \mathcal{O} \): the unit group \( \mathcal{O}^\times \) of invertible elements of \( \mathcal{O} \) and the ideal class group \( \text{Cl}(\mathcal{O}) \) of classes of proper (invertible) fractional ideals of \( \mathcal{O} \); see [Cox89, p. 136].

The ideal class group is finite and its cardinality \( h(\mathcal{O}) \) is called the class number of \( \mathcal{O} \). The class numbers of \( \mathcal{O} \) and \( B \) are related by the following well-known formula (see [Cox89, Thm. 7.24, p. 146] or [Yu95a, pp. 323–324]):

\[
 h(\mathcal{O}) = h(B) \frac{|\mathcal{C}|}{[B^\times : \mathcal{O}^\times]} \prod_{\ell \mid \mathcal{C}} \left( 1 - \left( \frac{K}{\ell} \right) \frac{1}{[\ell]} \right),
\]
where
\[
\left( \frac{K}{\ell} \right) := \begin{cases} 
1 & \text{if } (\ell) \text{ splits in } K, \\
-1 & \text{if } (\ell) \text{ is inert in } K, \\
0 & \text{if } (\ell) \text{ ramifies in } K.
\end{cases}
\]

Hence
\[
h(\mathcal{O}) \leq h(B) \cdot |c| \prod_{\ell \mid c, \ell \text{ monic irreducible}} \left( 1 + \frac{1}{|\ell|} \right). \tag{2}
\]

The product on the right-hand side of (2) may be bounded from above as follows:

**Lemma 8.**
\[
\prod_{\ell \mid c, \ell \text{ monic irreducible}} \left( 1 + \frac{1}{|\ell|} \right) \ll \log \log |c|. \tag{3}
\]

**Proof.** To simplify the notation, we assume that \( \ell \) is always monic and irreducible in this proof. Note that
\[
\prod_{\ell \mid c} \left( 1 + \frac{1}{|\ell|} \right) < \prod_{\ell \mid c} \frac{|\ell|}{|\ell| - 1} = \frac{|c|}{\varphi_A(c)},
\]
where \( \varphi_A(c) = |c| \prod_{\ell \mid c} (1 - 1/|\ell|) \) is the analogue of the classical Euler function. On the other hand, the well-known bound
\[
|c|/\varphi_A(c) \ll \log \log |c|
\]
for the classical Euler function is valid also for its function field analogue (see [Br10, Lem. 2.2]). By combining these two observations, the lemma follows. \( \square \)

The class number \( h(B) \) of the maximal order may be estimated from above as follows:

**Lemma 9.**
\[
h(B) \leq \begin{cases} 
\frac{\sqrt{|\Delta_B|} \deg \Delta_B}{\sqrt{q}}, & \text{if } \deg \Delta_B \text{ is odd}, \\
2\frac{\sqrt{|\Delta_B|} \deg \Delta_B}{q+1}, & \text{otherwise}.
\end{cases}
\]

**Proof.** The quadratic symbol \( \left( \frac{K}{\ell} \right) \) gives rise to a quadratic character \( \chi_{\Delta_B}(\cdot) \) and to an associated \( L \)-function \( L(s, \chi_{\Delta_B}); \) see [Ro02, p. 316].

On one hand, by the function field analogue of the classical class number formula (see [Ro02, Thm. 17.8A]),
\[
L(1, \chi_{\Delta_B}) = \begin{cases} 
\frac{\sqrt{q}}{\sqrt{|\Delta_B|}} h(B), & \text{if } \deg \Delta_B \text{ is odd}, \\
\frac{q+1}{2\sqrt{|\Delta_B|}} h(B), & \text{otherwise}.
\end{cases}
\]

Note that, above, we made use of our assumption that \( K/F \) is imaginary.
On the other hand, by [Ro02, Lem. 17.10],

$$L(1, \chi_{\Delta_B}) = \sum_{0 \neq m \in A \atop m \text{ monic} \atop \deg m < \deg \Delta_B} \frac{\chi_{\Delta_B}(m)}{|m|}.$$ 

Hence

$$|L(1, \chi_{\Delta_B})| \leq \sum_{0 \neq m \in A \atop m \text{ monic} \atop \deg m < \deg \Delta_B} \frac{1}{|m|} = \sum_{0 \leq d \leq \deg(\Delta_B) - 1} q^{-d} \sum_{0 \neq m \in A \atop m \text{ monic} \atop \deg m = d} 1 = \deg \Delta_B,$$

where $|L(1, \chi_{\Delta_B})|$ denotes the usual absolute value on $\mathbb{C}$.

By combining this bound with the class number formula, we obtain the stated bound for $h(B)$. $\square$

Putting together (2), Lemma 8, and Lemma 9, we obtain an upper bound for the class number $h(O)$ of the arbitrary order $O$:

$$h(O) \ll \sqrt{|\Delta_O|} \cdot (\deg \Delta_O)^2. \quad (4)$$

3. Quadratic Forms

Let $f(x, y) = ax^2 + bxy + cy^2 \in A[x, y]$ be a quadratic form. The discriminant of $f(x, y)$ is $b^2 - 4ac$. The quadratic form $f(x, y)$ is primitive if $\gcd(a, b, c) = 1$.

The group $\text{GL}_2(A)$ acts on the set of primitive quadratic forms as follows: if \[
\begin{pmatrix}
    a & b/2 \\
    b/2 & c
\end{pmatrix}
\] is the matrix of $f$, that is, $f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b/2 \\
    b/2 & c \end{pmatrix} \begin{pmatrix} x \\
    y \end{pmatrix}$, and if $M \in \text{GL}_2(A)$, then $M^t \begin{pmatrix} a & b/2 \\
    b/2 & c \end{pmatrix} M$ is the matrix of $Mf$.

Two primitive quadratic forms $f$ and $g$ are properly equivalent if $g = Mf$ for some $M \in \text{SL}_2(A)$.

In the proof of Theorem 1 we will need the following analogue of a well-known classical result:

**Theorem 10.** Let $O$ be an imaginary quadratic $A$-order, of discriminant $\Delta_O$. If $ax^2 + bxy + cy^2 \in A[x, y]$ is a primitive quadratic form of discriminant $\Delta_O$, then $A + \frac{-b + \sqrt{\Delta_O}}{2a} A$ is a proper fractional ideal of $O$. The map

$$ax^2 + bxy + cy^2 \mapsto A + \frac{-b + \sqrt{\Delta_O}}{2a} A$$

induces a bijection between the proper equivalence classes of primitive quadratic forms of discriminant $\Delta_O$ and $\text{Cl}(O)$.

**Proof.** The proof of Theorem 7.7 in [Cox89, pp. 137–140] works also in this context; see [Be09] for the details. $\square$

**Lemma 11.** Every primitive quadratic form over $A$ is properly equivalent to a quadratic form $ax^2 + bxy + cy^2$ such that

$$\deg b < \deg a \leq \deg c. \quad (5)$$
Proof. Among all forms properly equivalent to the given one, pick \( f(x,y) = ax^2 + bxy + cy^2 \) so that \( \deg b \) is as small as possible (note that \( b \) can be zero, in which case, by our convention, \( \deg 0 = -\infty \)).

If \( \deg a \leq \deg b \), then \( f \) is properly equivalent to

\[
g(x,y) = f(x+my,y) = ax^2 + (2am+b)xy + c'y^2
\]

for some \( m \in A \). Using the division algorithm in \( A \), we can choose \( m \) so that \( \deg(2am+b) < \deg a \), which contradicts our choice of \( f(x,y) \). Thus \( \deg b < \deg a \). The inequality \( \deg b < \deg c \) follows similarly.

If \( \deg a > \deg c \), then we interchange the outer coefficients via the proper equivalence \((x,y) \mapsto (-y,x)\).

The resulting form satisfies \( \deg b < \deg a \leq \deg c \). \( \square \)

Let \( F_\infty \) be the completion of \( F \) with respect to the absolute value \( |\cdot| \), and \( C_\infty \) be the completion of the algebraic closure \( F_\infty^{\text{alg}} \). We use the same notation for the unique extension of \( |\cdot| \) to \( C_\infty \). The imaginary part of \( z \in C_\infty \) is

\[
|z|_i := \min_{x \in F_\infty} |z-x|.
\]

Obviously, \( |z|_i \leq |z| \).

Lemma 12. Let \( f(x,y) = ax^2 + bxy + cy^2 \) be a primitive quadratic form over \( A \) with discriminant \( \Delta \). Assume \( F(\sqrt{\Delta}) \) is imaginary and \( f(x,y) \) satisfies (5). Then

\[
1 \leq \left| \frac{-b + \sqrt{\Delta}}{2a} \right|_i = \left| \frac{-b + \sqrt{\Delta}}{2a} \right| \leq |\sqrt{\Delta}|.
\]

Proof. Since \( \Delta = b^2 - 4ac \) and \( \deg b < \deg a \leq \deg c \), we obtain that

\[
\deg \Delta = \deg(b^2 - 4ac) = \deg(ac) \geq 2 \deg a.
\]

Hence

\[
|\sqrt{\Delta}| \geq |a| > |b|.
\]

Then, using the strong triangle inequality, we obtain that

\[
\left| \frac{-b + \sqrt{\Delta}}{2a} \right| = \frac{|-b + \sqrt{\Delta}|}{|a|} = \frac{|\sqrt{\Delta}|}{|a|}.
\]

Since \( |a| \geq 1 \), we deduce that

\[
1 \leq \frac{|\sqrt{\Delta}|}{|a|} \leq |\sqrt{\Delta}|.
\]

Thus the two outer inequalities of (6) hold, that is,

\[
1 \leq \left| \frac{-b + \sqrt{\Delta}}{2a} \right| \leq |\sqrt{\Delta}|.
\]

To prove the middle equality of (6), we proceed in two different ways, according to the parity of \( \deg \Delta \).
First, suppose that $\deg \Delta$ is odd. Observe that (7) gives

$$
\log_q \left| \frac{-b + \sqrt{\Delta}}{2a} \right| = \log_q \left| \frac{\sqrt{\Delta}}{a} \right| = \frac{\deg \Delta}{2} - \deg a,
$$

Given our assumption on $\deg \Delta$, the above implies that $\log_q \left| \frac{-b + \sqrt{\Delta}}{2a} \right| \not\in \mathbb{Z}$. In this case, the desired middle equality of (6) follows from the more general fact that, if $z \in C_\infty$ is such that $\log_q |z| \not\in \mathbb{Z}$, then $|z_i| = |z|$, which we now explain. On one hand, if $\log_q |z| \not\in \mathbb{Z}$, then for any $x \in F_\infty$ we have $|z| \neq |x|$. As such, the strong triangle inequality implies $|z - x| = \max \{|z|, |x|\} \geq |z|$, showing that $|z_i| \geq |z|$. On the other hand, $|z_i| \leq |z|$. Thus we must have $|z_i| = |z|$.

Next, suppose that $u := \deg \Delta$ is even. Since $F(\sqrt{\Delta})$ is assumed to be imaginary, the leading coefficient of $\Delta$ is not a square in $F_q^\times$ (see Remark 7) and $F_\infty(\sqrt{\Delta}) = F_q^2 F_\infty$. Choosing $1/T$ as the uniformizer of $F_q^2 F_\infty$, we can expand $\sqrt{\Delta}$ as

$$
\sqrt{\Delta} = \alpha \left( \frac{1}{T} \right)^{-u/2} + \text{higher degree terms in } \frac{1}{T},
$$

where $\alpha \in F_q^\times \setminus F_q$. Since $u/2 \geq \deg a > \deg b$, the $1/T$-expansion of $(-b + \sqrt{\Delta})/2a$ is

$$
\frac{-b + \sqrt{\Delta}}{2a} = \beta \left( \frac{1}{T} \right)^{-v} + \text{higher degree terms in } \frac{1}{T},
$$

where $v := \frac{u}{2} - \deg a \geq 0$ and $\beta \in F_q^2 \setminus F_q$.

If $\left| \frac{-b + \sqrt{\Delta}}{2a} - x \right| < \left| \frac{-b + \sqrt{\Delta}}{2a} \right|$ for some $x \in F_\infty$, then the $1/T$-expansion of $x$ must have the form

$$
\beta \left( \frac{1}{T} \right)^{-v} + \text{higher degree terms in } \frac{1}{T},
$$

But this is not possible, since $\beta \not\in F_q$. Therefore, $\left| \frac{-b + \sqrt{\Delta}}{2a} - x \right| \geq \left| \frac{-b + \sqrt{\Delta}}{2a} \right|$ for all $x \in F_\infty$, which implies that $\left| \frac{-b + \sqrt{\Delta}}{2a} \right| = \left| \frac{-b + \sqrt{\Delta}}{2a} \right|_i$.

4. The $j$-invariant of a rank 2 Drinfeld module

Let $\gamma : A \rightarrow L$ be an $A$-field and let $\psi : A \rightarrow L\{\tau\}$ be a Drinfeld $A$-module over $L$ of rank 2, defined by $\psi_T = \gamma(T) + g_1 \tau + g_2 \tau^2$ for some $g_1, g_2 \in L$ with $g_2 \neq 0$. The quantity

$$
\begin{align*}
\frac{g_1^{q+1}}{g_2} & \in L
\end{align*}
$$

is called the $j$-invariant of $\psi$.

In general, two Drinfeld $A$-modules $\psi$ and $\phi$ are said to be isomorphic over an extension $L'$ of $L$ if $\psi_T = c^{-1} \phi_T c$ for some $c \in L'$. It is easy to show that two Drinfeld $A$-modules $\phi$ and $\psi$ of rank 2 are isomorphic over $L^{\text{alg}}$ if and only if $j(\phi) = j(\psi)$.
Now assume that $L = \mathbb{C}_\infty$ and let $\Omega := \mathbb{C}_\infty - F_\infty$ be the Drinfeld half-plane. The group $GL_2(A)$ acts on $\Omega$ by linear fractional transformations,
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d},
\]
and the set
\[
\mathcal{F} := \{ z \in \Omega : |z| = |z|_i \geq 1 \}
\]
is as close as possible to a “fundamental domain” for the action of $GL_2(A)$ on $\Omega$; see [Ge99, Prop. 6.5]. In particular, every element of $\Omega$ is $GL_2(A)$-equivalent to some element of $\mathcal{F}$.

To each $z \in \Omega$, we associate the lattice $A + Az \subset \mathbb{C}_\infty$. By the analytic theory of Drinfeld modules, the lattice $A + Az$ corresponds to a Drinfeld $A$-module $\psi^z$ of rank 2 defined over $\mathbb{C}_\infty$. Moreover, it can be shown that $\psi^z \cong \psi^{z'}$ (over $\mathbb{C}_\infty$) if and only if $z = \gamma z'$ for some $\gamma \in GL_2(A)$; see [Dr74, Sec. 6]. Therefore, the map $z \mapsto \psi^z$ induces a bijection between the orbits $GL_2(A) \setminus \Omega$ and the isomorphism classes of rank 2 Drinfeld $A$-modules over $\mathbb{C}_\infty$.

Thanks to the above properties, there exists a $GL_2(A)$-invariant function $j : \Omega \rightarrow \mathbb{C}_\infty$, $j(z) := j(\psi^z)$. (10)

**Theorem 13.** For $z \in \mathcal{F}$, we have $\log_q |j(z)| < q^2|z|$.

**Proof.** See Theorem 6.6 in [Ge99]. □

A Drinfeld $A$-module $\psi$ of rank 2 over $\mathbb{C}_\infty$ is said to have **complex multiplication** if $\text{End}_{\mathbb{C}_\infty}(\psi) \neq A$.

**Theorem 14.** Suppose that the Drinfeld $A$-module $\psi := \psi^z$ defined by some $z \in \Omega$ has complex multiplication. Then the following properties hold.

(i) $K := F(z)$ is an imaginary quadratic extension of $F$.

(ii) $\mathcal{O} := \text{End}_{\mathbb{C}_\infty}(\psi)$ is an $A$-order in $K$.

(iii) $K(j(\psi))/K$ is a finite abelian extension.

(iv) $j(\psi)$ is integral over $A$.

(v) $\text{Gal}(K(j(\psi))/K) \cong \text{Cl}(\mathcal{O})$.

(vi) $\{ \sigma(j(\psi)) \mid \sigma \in \text{Gal}(K(j(\psi))/K) \} = \left\{ j \left( \frac{-b + \sqrt{\Delta}}{2a} \right) \mid [ax^2 + bxy + cy^2]_{\Delta, \mathcal{O}}/\text{SL}_2(A) \right\}$, where $[ax^2 + bxy + cy^2]_{\Delta, \mathcal{O}}/\text{SL}_2(A)$ denotes the proper equivalence class of the primitive quadratic form $ax^2 + bxy + cy^2$ of discriminant $\Delta$.

**Proof.** For (i)-(v), see Section 4 in [Ge83]. For (vi), proceed as follows. On one hand, by [Ge83, Cor. 4.5], the set of Galois conjugates of $j(\psi)$ is equal to the set $\{ j(z') \}$, where $A + Az'$ runs over the equivalence classes of proper fractional ideals of $\mathcal{O}$. On the other hand, Theorem 10 gives explicit expressions for representatives of these ideal classes in terms of the equivalence classes of quadratic forms. This completes the proof. □
Let \( \psi \) be a Drinfeld \( A \)-module of rank 2 over \( F \) and let \( p \triangleleft A \) be a fixed prime where \( \psi \) has good reduction. Let \( \psi \otimes \mathbb{F}_p \) be the reduction of \( \psi \) at \( p \). As we mentioned in the introduction, \( \mathcal{E}_{\psi, p} := \text{End}_{\mathbb{F}_p}(\psi \otimes \mathbb{F}_p) \) is an \( A \)-order in the imaginary quadratic extension \( F(\pi_p) \) of \( F \). Since \( p \) remains fixed in this section, for simplicity of notation, in the proofs below we will write

\[
\mathcal{E} := \mathcal{E}_{\psi, p}, \quad \Delta := \Delta_p, \quad \text{and} \quad K := F(\pi_p).
\]

### 5.1. Proof of Theorem 1.

**Proposition 15.** There exists a Drinfeld \( A \)-module \( \Psi \) of rank 2 over \( \mathbb{C}_\infty \) for which the following properties hold.

\( (i) \) \ \text{End}_{\mathbb{C}_\infty}(\Psi) = \mathcal{E}. \)

\( (ii) \) There exists a prime \( \mathfrak{P} \) of \( K(j(\Psi)) \) lying over \( p \) such that \( j(\Psi) \equiv j(\psi) \mod \mathfrak{P} \).

**Proof.** Note that in \( (ii) \) we implicitly use Theorem 14: assuming \( (i) \), by Theorem 14, \( j(\Psi) \) is algebraic over \( A \), so \( K(j(\psi)) \) is a finite algebraic extension of \( F \) and \( j(\psi) \) mod \( \mathfrak{P} \) makes sense.

Since the rank of \( \psi \otimes \mathbb{F}_p \) is 2, by [CoPa15, Prop. 24], the field \( K \) is “good” for \( \psi \otimes \mathbb{F}_p \) in the sense therein. Then by Theorem 22 and its proof in [CoPa15], there exists a discrete valuation ring \( R \) with maximal ideal \( \mathcal{M} \), equipped with an injective homomorphism \( \gamma : A \to R \), and having the properties:

\( (a) \) \ \text{\( \mathcal{M} \cap A = p \) and \( R/\mathcal{M} \cong A/p \);} \n
\( (b) \) there exists a Drinfeld \( A \)-module \( \Psi : A \to R\{\tau\} \) of rank 2 such that \( \mathcal{E} \subseteq \text{End}_R(\Psi) \);

\( (c) \) \ \text{\( \Psi \otimes \mathbb{F}_p := \Psi \mod \mathcal{M} \) is isomorphic to \( \psi \otimes \mathbb{F}_p \) over \( \mathbb{F}_p \).}

It is not hard to deduce from [Ge83, Lem. 3.3] that, under reduction modulo \( \mathcal{M} \), we get an injection \( \text{End}_L(\Psi) \hookrightarrow \text{End}_{\mathbb{F}_p}(\psi \otimes \mathbb{F}_p) \), where \( L \) is the fraction field of \( R \). Hence \( \mathcal{E} \subseteq \text{End}_L(\Psi) \subseteq \text{End}_{\mathbb{F}_p}(\psi \otimes \mathbb{F}_p) = \mathcal{E} \), which implies that \( \text{End}_L(\Psi) = \mathcal{E} \). By considering the action of \( \mathcal{E} \) on the tangent space of \( \Psi \), one deduces that \( K \) is a subfield of \( L \). Thus \( K(j(\Psi)) \) is a subfield of \( L \).

Let \( \mathfrak{P} := \mathcal{M} \cap K(j(\Psi)) \). Since \( A/\mathfrak{P} \subseteq \mathcal{O}_{K(j(\Psi))}/\mathfrak{P} \subseteq R/\mathcal{M} \), \( \mathfrak{P} \) is a maximal ideal of the integral closure \( \mathcal{O}_{K(j(\Psi))} \) of \( A \) in \( K(j(\Psi)) \), with residue field \( \mathbb{F}_p \). From the construction, it is clear that \( \Psi \mod \mathfrak{P} \) is isomorphic to \( \psi \otimes \mathbb{F}_p \) over \( \mathbb{F}_p \). In particular, \( j(\Psi) \equiv j(\psi) \mod \mathfrak{P} \).

Finally, we embed \( L \) into \( \mathbb{C}_\infty \) and consider \( \Psi \) as a Drinfeld \( A \)-module over \( \mathbb{C}_\infty \). Since \( \text{End}_{\mathbb{C}_\infty}(\Psi)/\text{End}_L(\Psi) \) is a free \( A \)-module and rank\( A \) \( \text{End}_{\mathbb{C}_\infty}(\Psi) \leq 2 \), we conclude that \( \text{End}_{\mathbb{C}_\infty}(\Psi) = \mathcal{E} \). \[ \square \]

Now assume that \( \text{End}_{\mathcal{F}_{\text{alg}}}(\psi) = A \). Let \( \Psi \) be a Drinfeld \( A \)-module over \( \mathbb{C}_\infty \) as in Proposition 15. We have \( j(\psi) \neq j(\Psi) \) (as elements of \( \mathbb{C}_\infty \)), since, otherwise, \( \psi \cong \Psi \) over \( \mathbb{C}_\infty \), which implies \( \text{End}_{\mathcal{F}_{\text{alg}}}(\psi) = \text{End}_{\mathbb{C}_\infty}(\psi) = \mathcal{E} \), a contradiction. Write \( j(\psi) = n/m \) with relatively prime \( n, m \in A \).

Let \( \mathfrak{P} \) be the prime of \( K(j(\Psi)) \) from Proposition 15. Then \( j(\Psi) \equiv \frac{n}{m} \mod \mathfrak{P} \) and \( j(\Psi) \neq \frac{n}{m} \) imply \( 0 \neq n - m \cdot j(\Psi) \in \mathfrak{P} \).
By Theorem 14, \( K(j(\Psi))/K \) is an abelian extension and

\[
\text{Nr}_{K(j(\Psi))/K}(n - m \cdot j(\Psi)) = \prod_{\sigma \in \text{Gal}(K(j(\Psi))/K)} (n - m \cdot \sigma(j(\Psi)))
\]

\[
= \prod_{[ax^2 + bxy + cy^2]_{\Delta}/\text{SL}_2(A)} \left( n - m \cdot j\left( \frac{-b + \sqrt{\Delta}}{2a} \right) \right).
\]

On one hand, since \( n - m \cdot j(\Psi) \in \mathfrak{P} \), we have \( \alpha := \text{Nr}_{K(j(\Psi))/K}(n - m \cdot j(\Psi)) \in \mathfrak{p}' \), where \( \mathfrak{p}' \) is the prime of \( K \) lying under \( \mathfrak{P} \). Letting \( \alpha' \) be the conjugate of \( \alpha \) over \( F \), we obtain that \( \alpha \alpha' \in \mathfrak{p} \), which gives \( |\alpha| \geq |\mathfrak{p}|^{1/2} \).

On the other hand, by the strong triangle inequality,

\[
|\alpha| = \prod_{[ax^2 + bxy + cy^2]_{\Delta}/\text{SL}_2(A)} \max \left\{ |n|, |m| \cdot j\left( \frac{-b + \sqrt{\Delta}}{2a} \right) \right\}.
\]

Remark that, by Lemma 11, we can assume that the triples \((a, b, c)\) above satisfy (5). Under this assumption, Lemma 12 implies that \( \frac{-b + \sqrt{\Delta}}{2a} \in \mathcal{F} \), with \( \mathcal{F} \) as defined in (9). Then Theorem 13 and Lemma 12 imply

\[
\log_q \left| j\left( \frac{-b + \sqrt{\Delta}}{2a} \right) \right| \leq q^2 \left| \frac{-b + \sqrt{\Delta}}{2a} \right| \leq q^2 |\sqrt{\Delta}|.
\]

Combining our estimates, we get

\[
|\mathfrak{p}|^{\frac{1}{2}} \leq \prod_{[ax^2 + bxy + cy^2]_{\Delta}/\text{SL}_2(A)} \max \left\{ |n|, |m| \cdot q^2 |\sqrt{\Delta}| \right\} = \max \left\{ |n|, |m| \cdot q^2 |\sqrt{\Delta}| \right\}^{h(\mathcal{E})}.
\]

Since \( n \) and \( m \) are determined by \( \psi \), we deduce that

\[
\deg p \ll_{\psi} h(\mathcal{E}) \cdot |\sqrt{\Delta}|.
\]

Furthermore, since, by (4), \( h(\mathcal{E}) \ll \sqrt{|\Delta| \cdot (\deg \Delta)^2} \), we deduce that

\[
\deg p \ll_{\psi} |\Delta| \cdot (\deg \Delta)^2. \quad (11)
\]

We claim that (11) implies

\[
|\Delta| \gg_{\psi} \frac{\log |\mathfrak{p}|}{(\log \log |\mathfrak{p}|)^2}. \quad (12)
\]

Indeed, if \( |\Delta| \geq \log |\mathfrak{p}| \), then obviously \( |\Delta| > \log |\mathfrak{p}|/(\log \log |\mathfrak{p}|)^2 \). On the other hand, if \( |\Delta| \leq \log |\mathfrak{p}| \), then \( \log |\Delta| \leq \log \log |\mathfrak{p}| \), and so from (11) we get

\[
|\Delta| \gg_{\psi} \frac{\log |\mathfrak{p}|}{(\log \Delta)^2} = \frac{\log |\mathfrak{p}|}{(\log |\Delta|)^2} (\log q)^2 \gg_{\psi} \frac{\log |\mathfrak{p}|}{(\log \log |\mathfrak{p}|)^2}.
\]

This completes the proof of Theorem 1.
5.2. Proof of Theorem 2. We start by recalling a few more general facts. Let \( \gamma : A \to L \) be an \( A \)-field and let \( \phi \) be a Drinfeld \( A \)-module of rank \( r \) over \( L \). Then \( \phi \) endows any field extension \( L' \) of \( L \) with an \( A \)-module structure, where \( m \in A \) acts by \( \phi_m \). More precisely, if \( \phi_m = \gamma(m) + \sum_{1 \leq i \leq r \deg m} g_i(m) \tau^i \), then put \( \phi_m(x) = \gamma(m)x + \sum_{1 \leq i \leq r \deg m} g_i(m)x^i \in L[x] \) and, for \( \lambda \in L' \), define \( m \circ \lambda := \phi_m(\lambda) \). The \( m \)-torsion \( \phi[m] \subset L^{ab} \) of \( \phi \) is the set of zeros of the polynomial \( \phi_m(x) \). It is clear that \( \phi[m] \) has a natural structure of an \( A \)-module and it is not hard to show that, as \( A \)-modules, \( \phi[m] \subseteq (A/mA)^{\otimes r} \), with an equality if and only if \( m \) is relatively prime to \( \ker(\gamma) \); see [Go96, Ch. 4].

We return to the Drinfeld \( A \)-module \( \psi \) of rank 2 over \( F \) and, as in the introduction, we consider \( F_p \) as an \( A \)-module via the action of \( \psi \otimes F_p \) and denote it by \( \psi F_p \); then

\[
\psi F_p \cong A/d_1, A \times A/d_2, A
\]

for uniquely determined nonzero monic polynomials \( d_1, d_2 \in A \) such that \( d_1 \mid d_2 \). (Note that there are at most two terms because \( \psi F_p \) is a finite \( A \)-module, so for some \( d \in A \) we have \( \psi F_p \subseteq \psi (\otimes F_p)|d| \subseteq A/dA \times A/dA \).) Since \( p \) remains fixed in this section, for simplicity of notation, in this proof we will write

\[
d_1 := d_1, d_2 := d_2.
\]

Suppose \( d_1 \neq 1 \). Then \( (\psi \otimes F_p)|d_1 \) is rational over \( F_p \), i.e., all roots of \( \psi_{d_1}(x) \) are in \( F_p \), and \( p \mid d_1 \). Let \( \pi_p = \psi^{\deg(p)} \in E \) be the Frobenius endomorphism of \( \psi \otimes F_p \). The fact that \( (\psi \otimes F_p)|d_1 \) is rational over \( F_p \) implies that \( \pi_p = 1 + d_1 \alpha \) for some \( \alpha \in E \); see the proof of Theorem 1.2 in [GaPa20]. From the theory of Drinfeld modules over finite fields (see [Yu95b, Thm. 1], [Ge91, Section 3]), we know that the minimal polynomial of \( \pi_p \) over \( A \) has the form

\[
P_{\psi, p}(X) = X^2 + a_p X + \mu_p p,
\]

where \( \mu_p \in F_q^\times \), and that

\[
\deg a_p \leq \frac{\deg p}{2}.
\]

Therefore \( |\pi_p| = |\pi'_p| = |p|^{\frac{1}{2}} \), where \( \pi'_p \) denotes the conjugate of \( \pi_p \) over \( F \). In particular,

\[
|p| = |\pi_p| \cdot |\pi'_p| = |1 + d_1 \alpha| \cdot |1 + d_1 \alpha'| = |d_1 \alpha| \cdot |d_1 \alpha'| = |d_1|^2 \cdot |\alpha \alpha'|,
\]

where \( \alpha' \) denotes the conjugate of \( \alpha \) over \( F \).

We write \( \alpha = a_1 + a_2 \sqrt{\Delta} \) for some \( a_1, a_2 \in A \). Note that \( a_2 \neq 0 \) since \( \pi \notin A \). Then

\[
|\alpha \alpha'| = |a_1^2 - a_2^2 \Delta|.
\]

The leading terms of \( a_1^2 \) and \( a_2^2 \Delta \), as polynomials in \( T \), cannot cancel. Indeed, \( a_1^2 \) and \( a_2^2 \) have even degrees and their leading coefficients are squares in \( F_q^\times \), whereas \( \Delta \) either has odd degree or has a leading coefficient which is not a square in \( F_q^\times \) (see Remark 7). This implies that

\[
|a_1^2 - a_2^2 \Delta| \geq |\Delta|.
\]
Combining Theorem 1 with (15), (16), (17), we obtain

\[ |p| \geq |d_1|^2 \cdot |\Delta| \gg \psi |d_1|^2 \cdot \frac{\log |p|}{(\log \log |p|)^2}, \]

or, equivalently,

\[ |d_1| \ll \psi \frac{\sqrt{|p| \cdot \log \log |p|}}{\sqrt{\log |p|}}. \]

Recalling that \( |d_1d_2| = |p| \), we deduce that

\[ |d_2| \gg \psi \sqrt{|p| \cdot \log \log |p|}. \]

This completes the proof of Theorem 2.

6. Proof of Theorem 6

Let \( \psi \) be a Drinfeld \( A \)-module of rank 2 over \( F \) and let \( p \triangleright A \) be a fixed prime where \( \psi \) has good reduction. As before, let \( \psi \otimes F_p \) be the reduction of \( \psi \) at \( p \). As we mentioned earlier, the rings \( A[\pi_p] \subseteq E_{\psi,p} \subseteq O_{F(\pi_p)} \) are \( A \)-orders in the imaginary quadratic extension \( K_p := F(\pi_p) \) of \( F \). Since \( p \) varies in this section, we will now specify the dependence on \( p \) of \( \Delta_p \) and of all other relevant data.

Similarly to the \( A \)-module isomorphism \( O_{F(\pi_p)}/E_{\psi,p} \cong A/c_pA \) mentioned in the introduction, there is an \( A \)-module isomorphism \( E_{\psi,p}/A[\pi_p] \cong A/b_pA \), where \( b_p = b_{\psi,p} \in A \) is a monic polynomial.

Comparing the discriminants of \( E_{\psi,p} \) and \( A[\pi_p] \) and remarking that the discriminant of \( A[\pi_p] \) is the discriminant of the polynomial \( P_{\psi,p} \) of (13), we find a (not necessarily monic) generator \( \delta_p = \delta_{\psi,p} \in A \) of the ideal \( (\Delta_p) \) that is uniquely determined by the relation

\[ a_p^2 - 4\mu_p p = b_p^2 \delta_p. \]

Then, by taking norms, we obtain

\[ \frac{|\Delta_p|}{|a_p^2 - 4\mu_p p|} = \frac{|\delta_p|}{|a_p^2 - 4\mu_p p|} = \frac{1}{|b_p|^2}. \] (19)

Letting \( f \) be any positive valued function such that \( \lim_{x \to \infty} f(x) = \infty \), we deduce from (19) that proving that

\[ \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, \ |\Delta_p| \geq \frac{|a_p^2 - 4\mu_p p|}{q^{f(\deg p)}} \right\} \sim \pi_F(x) \]

is equivalent to proving that

\[ \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, \ |b_p| \geq q^{\frac{f(\deg p)}{2}} \right\} = o(\pi_F(x)). \] (20)

To prove (20), our strategy will be to partition the primes \( p \) according to the values \( m \) taken by \( b_p \), to relax each equality \( b_p = m \) to the divisibility \( m \mid b_p \), and to reinterpret this divisibility as a condition that may be studied via density theorems, as we explain below.

First, let us remark that the condition \( m \mid b_p \) for some \( p \) with \( \deg p = x \) implies that

\[ \deg m \leq \frac{\deg (a_p^2 - 4\mu_p p)}{2} \leq \frac{x}{2}, \] (21)
since \( \deg a_p \leq \frac{\deg p}{2} \), as we recalled in (14). Thus

\[
\# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |b_p| \geq q^{\frac{\deg p}{2}} \right\} = \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |b_p| \geq q^{\frac{f(x)}{2}} \right\}
= \sum_{0 \neq m \in A \text{ monic}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, b_p = m \right\}
\leq \sum_{\frac{m}{\mu(q)}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, m \mid b_p \right\}. \quad (22)
\]

To analyze the inner counting function, let \( m \in A \) be a monic polynomial and denote by \( F(\psi[m]) \) the field obtained by adjoining to \( F \) the elements of \( \psi[m] \). We obtain a finite Galois extension of \( F \), whose Galois group \( \text{Gal}(F(\psi[m])/F) \) may be viewed as a subgroup of \( \text{GL}_2(A/mA) \). We distinguish the subfield

\[ J_m := \{ z \in F(\psi[m]) : \sigma(z) = z \forall \sigma \in \text{Gal}(F(\psi[m])/F) \text{ a scalar element} \}, \]

which is a finite Galois extension of \( F \) and whose Galois group \( \text{Gal}(J_m/F) \) may be viewed as a subgroup of \( \text{PGL}_2(A/mA) \).

An important consequence of [CoPa15, Thm. 1] is that, for any \( p \in \mathcal{P}(\psi) \) with \( p \nmid m \),

\[ m \mid b_p \iff p \text{ splits completely in } J_m/F. \quad (23) \]

Thus the divisibility \( m \mid b_p \) may be studied via the Chebotarev Density Theorem. For this, we use the standard notation

\[ \Pi_1(x, J_m/F) := \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, p \text{ splits completely in } J_m/F \right\}, \]

and we recall that it was shown in [CoPa15, Thm. 15] that an effective version of the Chebotarev Density Theorem of [MuSc94] applied to the extension \( J_m/F \) gives

\[
\Pi_1(x, J_m/F) = \frac{c_m(x)}{[J_m : F]} \cdot \frac{q^x}{x} + O(x^{\frac{1}{2} \deg m}), \quad (24)
\]

where

\[ c_m := [J_m \cap F_q : F_q] \]

and

\[ c_m(x) := \begin{cases} c_m & \text{if } c_m \mid x, \\ 0 & \text{otherwise.} \end{cases} \quad (25) \]
Upon fixing a parameter \( f(x)/2 \leq y = y(x) \leq x/2 \), which will be chosen optimally later, by using (24) in (22) we obtain that

\[
\# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |b_p| \geq q^{(deg p)} \right\} \leq \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq x}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, m \mid b_p \right\} + \Pi_1(x, J_m/F)
\]

\[
= \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} c_m(x) [J_m : F] + O \left( \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} q^{\frac{x}{2} \deg m} \right)
\]

(26)

\[
+ \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop y < \deg m \leq \frac{x}{2}}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, m \mid b_p \right\}.
\]

(27)

To estimate the first summation in (26), we recall that it was proven in [Ge19, Thm. 4.1] that the integers \( c_m \) are absolutely bounded, that is,

\[ c_m \ll 1. \]  

(28)

Thus

\[
\sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} c_m(x) [J_m : F] \ll \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} \frac{1}{[J_m : F]} \leq \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m}} \frac{1}{[J_m : F]}.
\]

(29)

Furthermore, under the assumption \( \text{End}_{F_{\text{alg}}} (\psi) = A \), it was proven in [CoJo20, Cor. 3] that

\[ |m|^3 \ll \psi [J_m : F] \leq |m|^3. \]  

(30)

(Note that only the lower bound requires the assumption on the endomorphism ring.) Using the above lower bound in (29), we deduce that

\[
\sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} c_m(x) [J_m : F] \ll \psi \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m}} \frac{1}{|m|^3} \ll \frac{1}{q^{f(x)}}.
\]

(31)

To estimate the second summation in (26), we observe that

\[
\sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop f(x) \leq \deg m \leq y}} \deg m \leq \sum_{\substack{0 \neq m \in A \setminus \{0\} \\ \text{monic} \atop \deg m \leq y}} \deg m \leq q^y.
\]

(32)
By putting together (26), (27), (31), and (32), we deduce that
\[
\# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |b_p| \geq q^{f(\deg p)} \right\} \ll_{\psi} \frac{q^{r-f(x)}}{x} + q^{2x+y}
\]
(33)

and recalling that \( \lim_{y \to 0} \frac{r+y}{y} = 0 \), which confirms (20).

Let us remark that, since we are seeking to prove that (33) is \( o(q^x/x) \), we will need to choose \( y \) such that \( y \leq x/2 \). Thus the summation in (34) is non-empty.

An upper bound for this remaining sum over \( m \in A \) with \( y < \deg m \leq \frac{x}{2} \) can be obtained as an application of the Square Sieve, which once again makes use of (24), (28), and (30), but in a much more elaborate way than what we explained above. This application is detailed in the proof of the non-trivial estimate [CoPa15, (41)] for the quantity \( B_2(\psi, x, y) \) defined therein, which differs from our own summation in (34) only by having \( m \) squarefree. Note that the squarefreeness of \( m \) is never used in the estimates leading to formula (41) of [CoPa15]; as such, that formula holds also when \( m \) has square factors. In summary, as shown in [CoPa15, (41), pp. 7811–7812], we have
\[
\sum_{\substack{0 \leq m \leq A \\text{monic} \\ \text{and} \ \ \ y < \deg m \leq \frac{x}{2}}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, m | b_p \right\} \leq \sum_{\substack{0 \leq m \leq \frac{x}{2} \\text{monic} \\ \ y < \deg m \leq \frac{x}{2}}} \# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, m^2 | (a_p^2 - 4\mu_p p) \right\}
\]
(34)

\[
\ll_{\psi, x} q^{16x - 2y + x(x - 3)},
\]
(35)

valid for any \( x > 0 \), an arbitrary value of which we now fix.

Choosing
\[
y = y(x) := \frac{(11 + \varepsilon)x}{24}
\]
(36)

and recalling that \( \lim_{x \to \infty} f(x) = \infty \), we deduce from (33), (34), and (35) that
\[
\# \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |b_p| \geq q^{f(\deg p)} \right\} \ll_{\psi, x} q^{x-f(x)} + q^{2x+y+\varepsilon x} = o\left(\frac{q^{x}}{x}\right),
\]
(37)

which confirms (20).

Now let us prove that the set \( \left\{ p \in \mathcal{P}(\psi) : |\Delta_p| > \frac{|a_p^2 - 4\mu_p p|}{q^{f(\deg p)}} \right\} \) has Dirichlet density 1. For this, let \( s > 1 \) and consider the sum
\[
\sum_{p \in \mathcal{P}(\psi) : |\Delta_p| \leq \frac{|a_p^2 - 4\mu_p p|}{q^{f(\deg p)}}} q^{-sx} \deg p = \sum_{x \geq 1} q^{-sx} \left\{ p \in \mathcal{P}(\psi) : \deg p = x, |\Delta_p| \leq \frac{|a_p^2 - 4\mu_p p|}{q^{f(\deg p)}} \right\}
\]
(38)

By (26), (27), (32), (35), and our earlier choice (36) of \( y \), we obtain that the above is
\[
\leq \sum_{x \geq 1} q^{(1-x)x} \frac{1}{x} \sum_{\substack{0 \leq m \leq A \\text{monic} \\ \ y < \deg m \leq \frac{x}{2}}} \frac{c_m(x)}{J_m : F} + O_{\psi, x} \left( \sum_{x \geq 1} q^{(1-x)s} \right).
\]
(38)
To estimate the first double sum in (38), we use (25) to rewrite $c_m(x)$:

$$T_1 := \sum_{x \geq 1} \frac{q(1-s)x}{x} \sum_{0 \neq m \in A \atop \text{monic}} c_m(x) \left[ J_m : F \right]$$

$$= \sum_{x \geq 1} \frac{q(1-s)x}{x} \sum_{0 \neq m \in A \atop \deg m \leq \deg x} c_m \left[ J_m : F \right]$$

$$= \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \sum_{j \geq 1} \frac{q(1-s)c_mj}{j} \left[ J_m : F \right] \tag{39}$$

Next we fix $M > 0$ and observe that, since $\lim_{x \to \infty} f(x) = \infty$, there exists $n(M) \in \mathbb{N}$ such that

$$f(n) > M \quad \forall n \geq n(M). \tag{40}$$

We split the inner sum over $j$ in our last reformulation (39) of $T_1$ according to whether $c_mj \geq n(M)$ or $c_mj < n(M)$. For the first range $c_mj \geq n(M)$, we obtain

$$T_{1,1} := \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \sum_{c_mj \geq n(M)} \frac{q(1-s)c_mj}{j} \left[ J_m : F \right]$$

$$\leq \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \frac{1}{\left[ J_m : F \right]} \sum_{j \geq 1} \frac{q(1-s)c_mj}{j}$$

$$\ll \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \frac{1}{\left[ J_m : F \right]} \sum_{j \geq 1} \frac{q(1-s)j}{j}$$

$$\ll (\psi) \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \frac{1}{\left| m \right|^3} \sum_{j \geq 1} \frac{q(1-s)j}{j}$$

$$\ll \frac{1}{qM} \left| \log \left( 1 - q^{1-s} \right) \right|,$$

where we used (40) to pass to the second line, (28) to pass to the third line, the lower bound in (30) to pass to the fourth line, and [CoSh15, Lem. 2.2], together $s > 1$, to pass to the fifth line. It follows that

$$\lim_{s \to 1^+} -\log (1 - q^{1-s}) \ll (\psi) \frac{1}{qM}. \tag{41}$$

For the second range $c_mj < n(M)$ in the inner sum of (39), we remark that

$$T_{1,2} := \sum_{0 \neq m \in A \atop \deg m \leq \deg x} \sum_{c_mj < n(M)} \frac{q(1-s)c_mj}{j} \left[ J_m : F \right]$$
is a finite sum since \( \deg m \leq \frac{cn_2}{2} \ll j \) (using (28) for the \( \ll \)-bound) and since \( j < n(M) \). Observing that
\[
\lim_{s \to 1^+} \frac{q^{(1-s)\alpha}}{\log (1 - q^{1-s})} = 0 \text{ for any } \alpha \geq 1,
\]
it follows that
\[
\lim_{s \to 1^+} \frac{T_{1,2}}{- \log (1 - q^{1-s})} = 0. \tag{42}
\]

To estimate the remaining \( O \)-term in (38), we consider the sum
\[
T_2 := \sum_{x \geq 1} q^{(\frac{23}{24} + \epsilon - s)x}
\]
and observe that
\[
\lim_{s \to 1^+} \frac{T_2}{- \log (1 - q^{1-s})} = - \lim_{s \to 1^+} \frac{q^{\frac{23}{24} + \epsilon - s}}{(1 - q^{\frac{23}{24} + \epsilon - s}) \log (1 - q^{1-s})} = 0. \tag{43}
\]

We now take \( M \to \infty \) in (40) and put together (41), (42), and (43), completing the proof of Theorem 6.

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