Shape Optimization in High Temperature Processes

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High temperature processes have a wide field of applications in industrial processes, e.g., optimal design in glass cooling or shape optimization for melting furnaces. We consider here the \( SP_1 \) approximation of the Radiative Heat Equation which includes reflecting boundary effects. This leads to a PDE-constrained shape optimization problem, where the shape of the domain is the design variable.

\section{1 Introduction}

In order to increase the efficiency of a flame driven furnace, we are investigating its optimal domain with respect to a given cost functional. A sophisticated model in this context is the stationary Radiative Heat Transfer Equation (abbr. RHTE) of gray matter in a homogeneous medium inside domain \( \Omega \):

\[ -\nabla \cdot (k \nabla T(x)) = -\int_{S^2} \kappa (aT(x)^4 - I(x, \omega))d\omega \quad \text{in} \ \Omega, \]

\[ \forall \omega \in S^2 : \omega \cdot \nabla_x I(x, \omega) = \kappa(aT(x)^4 - I(x, \omega)) \quad \text{in} \ \Omega, \]

with radiative intensity \( I(x, \omega) \) at position \( x \) in the domain \( \Omega \) and travelling directions \( \omega \) on the unit sphere, \( T \) the material temperature, \( \kappa \) the absorption coefficient, \( k \) the heat conduction coefficient and \( a \) the Stefan-Boltzmann constant. As boundary conditions we use a Robin type condition for the temperature and semi-transparent boundary conditions \[1\] for the intensity. Due to its five dimensional phase space, we use the approximate \( SP_1 \) model, which is able to describe boundary layer effects.

\section{2 Modelling approach}

As governing equations of the shape optimization problem we use the \( SP_1 \)-system, like in the optimal control problem for radiative heat transfer in [2] which reads in scaled form:

\[ -k \Delta T = \frac{1}{\epsilon^2} (\kappa \phi - \kappa 4\pi aT^4 + f_2) + f_1 \quad \text{in} \ \Omega, \]

\[ -\frac{\epsilon^2}{3\kappa} \Delta \phi = \kappa 4\pi aT^4 - \kappa \phi + f_2 \quad \text{in} \ \Omega, \]

with boundary conditions

\[ \epsilon k \nabla_n T = h(T_B - T) \quad \text{on} \ \partial \Omega, \]

\[ \epsilon \frac{1}{3\kappa} \nabla_n \phi = \chi (4\pi aT_B^4 - \phi) \quad \text{on} \ \partial \Omega, \]

where \( \chi = \frac{1 - 2\pi a}{2\pi \kappa} \), and the radiative flux is given by \( \phi(x) = \int_{S^2} I(x, \omega)d\omega \). The reflectivity constants \( r_1, r_2 \) are calculated according to \[1\]. The \( SP_1 \) system can be derived by an asymptotic expansion of the RHTE using a formal Neumann-Series approach and an introduction of the scaling parameter \( \epsilon \) for the optical thickness, see \[3\]. Gaussian source terms \( f_1 \) for the temperature and \( f_2 \) for the radiative intensity, which model the flame inside the furnace, are introduced. The modelling of the sources can be formulated as an inverse problem in radiative heat transfer, compare \[4\].

\section{3 Shape Optimization}

Let \((T_\Omega, \phi_\Omega)\) be the solution of system (1). We want to minimize the \( L_2 \) difference between the temperature profile \( T_\Omega \) at the bottom of the furnace \( \Gamma_B \subset \partial \Omega \) and a given temperature profile \( T_d \). The constrained shape optimization problem reads:

\[ \min_{\Omega} J(\Omega) = \min_{\Omega} \bar{J}(\Omega, T_\Omega) = \int_{T_B} ((T_\Omega - T_d)^2)ds \]

\[ \text{s.t.} \ (T_\Omega, \phi_\Omega) \quad \text{fulfills} \ (1). \]

We use a gradient descent method for the optimization based on the shape derivative. Let \( v \) and \( w \) be the adjoint variables of the temperature \( T \) and the radiative flux \( \phi \). We denote \( v_\Omega \) and \( w_\Omega \) as the solutions of the \( SP_1 \) adjoint system respectively.

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We consider the Lagrangian on the perturbed domain \( \Omega_t = T_t(\Omega) = (I + tV)(\Omega) \) with a family of diffeomorphisms \( T_t \). Let \( \Psi^t(\cdot) = (\cdot) \circ T_t^{-1} \) be the push forward operator and \( \xi_1(T, v) \) and \( \xi_2(\phi, w) \) be the weak forms of the \( SP_1 \)-System (1):

\[
\mathcal{L} \left( t, (T \phi)^T, (v w)^T \right) = \tilde{J} \left( \Omega_t, \Psi^t(T) \right) - \xi_1 \left( \Psi^t(T), \Psi^t(v) \right) - \xi_2 \left( \Psi^t(\phi), \Psi^t(w) \right).
\]

Following the approach of Sturm [5] we compute the shape derivative \( dJ(\Omega)[V] \) as follows:

\[
dJ(\Omega)[V] = \partial_t \mathcal{L} \left( 0, (T \phi_\Omega)^T, (v w_\Omega)^T \right).
\]

4 Numerical Results in Two Dimensions

We consider an initial domain \( \Omega = [0, 1] \times [0, 0.5] \) and use the following physical parameters: \( \varepsilon = 2, a = 5.67 \times 8 \left[ \frac{W}{m^2 K^4} \right], \)
\( k = 20 \left[ \frac{W}{m K} \right], h = 1.4 \left[ \frac{W}{m^2 K} \right], r_1 = 0.48, r_2 = 0.31, \kappa = 3 \left[ \frac{1}{m^2} \right], T_b = 300[K] \) and \( T_d = 800[K] \) at the bottom. We compute the shape gradient \( u \) as \( H^1(\Omega) \times H^1(\Omega) \) Riesz representative of the shape derivative using the variational form

\[
\int_{\Omega} \nabla \mathcal{L} \left( \phi, \nabla \phi \right) d\Omega = -dJ(\Omega)[V] \text{ for all } V \in H^1(\Omega) \times H^1(\Omega)
\]

with suitable boundary conditions depending on the respective application. Than, we perform a gradient descent method with Armijo step size control. For industrial purposes we project the shape derivative in a way such that the bottom \( \Gamma_B \) stays steady. The implementations were done in Python 3.6 with FEniCS 2017.2 as the finite elements framework.

![Fig. 1: Temperature distribution in the initial geometry](image1)

![Fig. 2: Temperature distribution in the optimized geometry](image2)

![Fig. 3: Temperature profiles at the bottom of the geometries](image3)

From Fig. 1 and Fig. 2 we see that the furnace admits a higher temperature after shrinking its size and smoothing its corners. This leads to a final temperature profile at the bottom which is very similar to the the desired temperature profile, see Fig. 3.

5 Conclusion

Shape optimization of high temperature processes modelled by the \( SP_1 \) system results in realistic optimal designs of the furnace. Further work on radiation dominated shape optimization in optically thinner regimes is in progress. For this purpose we need to investigate shape sensitivity on the kinetic level (RHTE).

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