Circle domains

**Definition**

A domain \( \Omega \subset \hat{\mathbb{C}} \) is a **circle domain** if \( \partial \Omega \) consists of points and circles.

The boundary of a circle domain contains at most **countably many circles**.
Koebe’s Conjecture

Conjecture (Kreisnormierungsproblem, Koebe 1908)

Every domain $\Omega$ is conformally equivalent to a circle domain.

- Simply connected domains: Riemann 1851
- Finitely connected domains: Koebe 1920
- Countably connected domains: He-Schramm 1993
- Uncountably connected domains: Open
Uniqueness

- Finitely connected domains: Koebe 1920
- Countably connected domains: He-Schramm 1993
- Uncountably connected domains: Fails in general!

Observation

If $E$ is a totally disconnected compact set with $\text{Area}(E) > 0$, then there exists a non-Möbius homeomorphism $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ that is conformal on $\hat{\mathbb{C}} \setminus E$.

Set $\mu = \frac{1}{2} \chi_E$

Solve Beltrami equation: $f_z = \mu f_z$ \Rightarrow f
Rigidity

Definition
A circle domain $\Omega$ is **conformally rigid** if every conformal map from $\Omega$ onto another circle domain is the restriction of a Möbius transformation.

Rigid: Finitely connected, Countably connected
Non-rigid: $\partial \Omega$ is totally disconnected and $\text{Area}(\partial \Omega) > 0$
Removability

Definition

Let $K \subset \hat{\mathbb{C}}$ be a compact set. $K$ is **conformally removable** if every homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is conformal in $\hat{\mathbb{C}} \setminus K$ is conformal in $\hat{\mathbb{C}}$. 

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Rigidity theorems for circle domains
Examples of removable sets

- Sets of $\sigma$-finite length (e.g. smooth curves) (Besicovitch 1931)
- Quasicircles
- Boundaries of John/Hölder domains (quasihyperbolic condition by Jones-Smirnov 2000)
- NED sets (Negligible for Extremal Distance), e.g., $C$, $C \times C$ (Ahlfors-Beurling 1950)

Figure: von Koch snowflake
Examples of non-removable sets

- Sets of positive area
- \( C \times [0, 1] \) and some product sets \( C \times E \), where \( C, E \) are Cantor sets
- Bishop’s flexible curves, with Hausdorff dimension 1 (1994)
- Sierpiński carpets (N. 2019)
- Sierpiński gasket (N. 2019)
The Rigidity Conjecture

Conjecture (He-Schramm 1994)

Let $\Omega$ be a circle domain. The following are equivalent:

(i) $\Omega$ is conformally rigid
(ii) $\partial \Omega$ is conformally removable

If $\partial \Omega$ is totally disconnected, then (i) $\Rightarrow$ (ii).
Known results

|                          | $\partial \Omega$ removable? | $\Omega$ rigid?          |
|--------------------------|-------------------------------|--------------------------|
| Area $> 0$               | N                             | N (Sibner 1968)          |
| NED                      | Y (Ahlfors-Beurling 1950)     | Y (Ahlfors-Beurling 1950)|
| finite                   | Y                             | Y (Koebe 1920)           |
| countable                | Y (Besicovitch 1931)          | Y (He-Schramm 1993)      |
| $\sigma$-finite          | Y (Besicovitch 1931)          | Y (He-Schramm 1994)      |
| John/Hölder              | Y (Jones-Smirnov 2000)        | Y (N.-Younsi 2019)       |
| quasi-hyperbolic         | Y (Jones-Smirnov 2000)        | Y (N.-Younsi 2019)       |
The quasihyperbolic condition

\[ D \subsetneq \mathbb{C}, \quad \delta_D(x) = \text{dist}(x, \partial D) \]

\[ k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{1}{\delta_D} ds \]

Facts:

- \( k_D \simeq h_D \) if \( D \) is simply connected
- \( k_D(x_1, x_2) \simeq \#\{\text{Whitney cubes needed to connect } x_1 \text{ to } x_2\} \)

Definition

A domain \( D \) satisfies the **quasihyperbolic condition** if

\[ \int_D k_D(x, x_0)^2 dx < \infty \text{ for some } x_0 \in D. \]

Examples: Quasidisks, John domains, Hölder domains
Theorem (N.-Younsi, Invent. Math. 2019)

Let \( \Omega \) be a circle domain with \( \infty \in \Omega \) and consider a ball \( B(0, R) \supset \partial \Omega \). If \( D = B(0, R) \cap \Omega \) satisfies the quasihyperbolic condition, then \( \Omega \) is rigid.
Proof

\(\Omega, \Omega^* \subset \hat{\mathbb{C}}\) circle domains, containing \(\infty\)
\(f: \Omega \to \Omega^*\) conformal

Step 1: \(f\) extends to a homeomorphism \(\overline{\Omega} \to \overline{\Omega}^*\)

Step 2: \(f\) extends to a homeomorphism \(\hat{\mathbb{C}} \to \hat{\mathbb{C}}\)

Step 3: \(f\) is \(K\)-quasiconformal on \(\hat{\mathbb{C}}\), \(K = K(\Omega)\)

Step 4: \(K = 1\) so \(f\) is Möbius
Step 1: Extension to $\partial \Omega$

- $f$ does not map boundary circles (continua) to points
- $f$ does not map boundary points to circles (continua)

$$W_r = f^{-1}(B(0, r))$$
\[ W_r = f^{-1}(B(0, r)) \]

Finitely many circles:

\[ r \leq \int_{\gamma_\theta \cap W_r} |f'| ds + \sum_{C \cap \gamma_\theta \neq \emptyset} \text{diam}(f(C)) \]
\[
\begin{align*}
r &\leq \int_{\gamma_0 \cap W_r} |f'| \, ds + \sum_{C \cap \gamma_0 \neq \emptyset} \text{diam}(f(C)) + \sum_{i=1}^{N} \int_{\gamma_i} |f'| \, ds
\end{align*}
\]
The detours $\gamma_i$

- $\gamma_i$ is a concatenation of quasihyperbolic geodesics from the basepoint $x_0$
- $\gamma_i$ lies in $N_\varepsilon(\partial \Omega)$: $\varepsilon$-neighborhood of $\partial \Omega$
- $\gamma_i$ and $\gamma_j$ intersect distinct Whitney cubes for $i \neq j$

\[
\int_0^{2\pi} \sum_{i=1}^N \int_{\gamma_i} |f'| ds \, d\theta \lesssim C(f) \cdot \left( \int_{N_\delta(\partial \Omega)} k(x, x_0)^2 \, dx \right)^{1/2} \xrightarrow{\varepsilon \to 0} 0
\]
Step 2: Extension to $\hat{\mathbb{C}}$

The extension conjugates the **Schottky groups** of $\Omega$ and $\Omega^*$ and is unique:

\[
R = \text{reflection along } C \\
R^* = \text{reflection along } f(C)
\]

\[
R^* = f \circ R \circ f^{-1}
\]
Step 3: Quasiconformality of $f$

It suffices to show that $\text{mod}(A) \geq 1$ implies $\text{mod}(f(A)) \geq C$.

- Reflect finitely many times and get $\Omega_k = \text{union of countably many copies of } \Omega$
- $\text{diam}(A) \gg \text{diam}(C)$, for all circles $C$ in $\partial \Omega_k$
We need estimates of the form

\[
\text{diam}(f(\gamma)) \leq \int_{\gamma \cap \Omega_k} |f'| ds + \sum_{C \cap \gamma \neq \emptyset} \text{diam}(f(C)) + \text{Error(detours)}
\]

- Each reflected copy \( T(\Omega) \) of \( \Omega \) satisfies quasihyperbolic condition (bi-Lipschitz invariant)

- Problem: need **infinitely many** detours!

- Solution: Quasihyperbolic condition in \( T(\Omega) \) + continuity
  \( \Rightarrow \) \( f \) is **ACL up to** \( \partial T(\Omega) \)
Step 4: Conformality of $f$

- $f: \Omega \to \Omega^*$ extends uniquely to a $K$-quasiconformal map of $\hat{\mathbb{C}}$ that conjugates the Schottky groups of $\Omega$ and $\Omega^*$. ($K = K(\Omega)$)
- If $K > 1$, set $\nu = c\mu_f$, $c > 1$. Solve Beltrami equation
  
  \[ h_z = \nu h_z \]

  and get $h: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is not $K$-quasiconformal.
- $\mu_h = \nu$ is invariant under the Schottky group of $\Omega$, so $h(\Omega)$ is a circle domain and $h$ conjugates the Schottky groups of $\Omega$ and $h(\Omega)$.
- $h|_\Omega$ is conformal ($\nu = 0$) so there exists a unique extension $\tilde{h}$ that is $K$-quasiconformal and conjugates the Schottky groups of $\Omega$ and $h(\Omega)$.
- Uniqueness: $\tilde{h} = h$. Contradiction!
Thank you!