IRREGULAR SAMPLING AND THE RADON TRANSFORM

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ABSTRACT. In image reconstruction there are techniques that use analytical formulae for the Radon transform to recover an image from a continuum of data. In practice, however, one has only discrete data available. Thus one often resorts to sampling and interpolation methods. This article presents an approach to the inversion of the Radon transform that uses a discrete set of samples which need not be completely regular.

1. Introduction

The Radon transform of a function \( \varphi(x) \) in the plane \( \mathbb{R}^d \) is defined by

\[
R\varphi(\theta, s) = \int_{x \cdot \theta = s} \varphi(x) \, dx,
\]

whenever the integral makes sense. Here \( \theta \) is a unit direction vector and \( s \) is a scalar translation parameter. A principal problem in image reconstruction is the recovery of the values of \( \varphi(x) \) from the data \( \{R\varphi(\theta, s)\} \) for all \( \theta \) and all \( s \). The algorithm that is commonly called Fourier reconstruction [7] is a discretization of the Fourier-slice or projection-slice formula:

\[
\hat{\varphi}(\theta \tau) = (2\pi)^{(1-d)/2} \hat{(R\varphi)}(\theta, \tau),
\]

where on the right one has the Fourier transform in the second variable of the Radon transform \( R\varphi \).

We consider a finite set of directions \( \theta_j \in S^{d-1}, j = 1, 2, ..., p \). Using a set of equally spaced samples of the Radon transform \( (R\varphi)(\theta_j, s_\gamma), \gamma = 1, 2, ..., q \), we can reconstruct \( (R\varphi)(\theta_j, s) \) as a function of one variable \( s \). The common way of reconstruction is by applying the Shannon-Whittaker formula. Taking the Fourier transform in the single variable \( s \) we obtain functions \( \hat{(R\varphi)}(\theta_j, \tau) \).

Given functions \( \hat{(R\varphi)}(\theta_j, \tau) \) along all rays \( \theta_j \in S^{d-1}, j = 1, 2, ..., p \), we can estimate their values on a certain polar grid and then reconstruct the function \( \hat{\varphi}(\theta \tau) \). The usual way of reconstruction is again through the Shannon-Whittaker sampling theorem.

It is well known [2], [7] that one of the main problems with the Fourier reconstruction algorithm is that the Shannon-Whittaker sampling theorem can be used only in the case of lattice points (regular sampling). But in many situations there is no way to construct such a uniform cartesian grid using the naturally available polar grid. A number of different ways to avoid this obstacle can be found in the book of Natterer [7]. But, in any case, this difficulty reduces the accuracy of the Fourier reconstruction algorithm. Our idea is to use a sampling theorem which

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does not require the uniformity property of the sample points. The considerations in the present paper are purely qualitative. The paper [3] gives, in the case \(d = 2\), a different approach to the reconstruction of the image using an irregular set of samples.

2. AN IRREGULAR SAMPLING THEOREM

In what follows we use the notations below.

\(B_\sigma(\mathbb{R})\):

This denotes the set of band limited functions \(B_\sigma(\mathbb{R}^d)\) is the set of all \(f \in L^2(\mathbb{R}^d)\) such that the Fourier transform

\[
\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx
\]

has support in the ball \(B(0, \sigma)\) of radius \(\sigma\) centered at 0.

\(\|f\|\):

The symbol \(\|f\|\) denotes the \(L^2(\mathbb{R}^d)\)-norm of \(f\).

\(X(\lambda)\):

Let \(\lambda\) be a positive number. Let \(X(\lambda)\) denote a countable set of points \(\{x_\gamma\}\) in \(\mathbb{R}^d\) with the following property.

- Each \(\{x_\gamma\}\) contains exactly one point among the collection \(\{x_\gamma\}\).
- The closure \(\overline{Q(x_\gamma, \lambda)}\) is diffeomorphic, as a manifold with boundary, to a closed ball.
- Each \(Q(x_\gamma, \lambda)\) is of diameter \(\leq \lambda\).
- The sets \(\{Q(x_\gamma, \lambda)\}\) are pairwise disjoint.
- The closures \(\overline{Q(x_\gamma, \lambda)}\) cover \(\mathbb{R}^d\).

We will often call \(X(\lambda)\) the knot set.

The classical Shannon-Whittaker sampling theorem says that if \(f \in L^2(\mathbb{R})\) and its Fourier transform \(\hat{f}\) has support in \([-\omega, \omega]\), then \(f\) is completely determined by its values at points \(n\Omega\), where \(\Omega = \pi/\omega\) and, in the \(L^2\)-sense,

\[
f(t) = \sum f(n\Omega) \frac{\sin(\pi(t-n\Omega))}{\pi(t-n\Omega)}.
\]

The functions \(f \in L^2(\mathbb{R})\) with the property \(\text{supp} \hat{f} \subset [-\omega, \omega]\) form the Paley-Wiener class \(PW_\omega\). The Paley-Wiener theorem states that \(f\) is in \(PW_\omega\) if and only if \(f\) is an entire function of exponential type \(\omega\).

Entire functions of finite exponential type are also uniquely determined by and can be recovered from their values on specific irregular sets of points \(x_\gamma\). As was shown by Paley and Wiener it is enough to assume that the functions \(\exp ix_\gamma t, n \in \mathbb{Z}\) form a Riesz basis for \(L^2([-\pi, \pi])\).
One can consider even more general assumptions about the sequence \( \{x_n\} \). New and old results in the case when the functions \( \exp ix_n t \) form different kinds of frames in \( L^2([-\omega, \omega]) \) were summarized in [1] and [4].

Our goal is to show that every band limited function can be reconstructed from an appropriate irregular set of points using translations of the fundamental solution of any operator of the form \( \Delta + \varepsilon \), \( \varepsilon \geq 0 \), where \( \Delta \) is the Laplacian in Euclidean space. A similar result for the operator \( \Delta \) was considered in [8]. We consider the operator \( D = D_{\varepsilon} = \Delta + \varepsilon \), \( \varepsilon \geq 0 \). The fundamental solution \( E^k = E^k_{\varepsilon} \) of the operator \( D^k \) is the inverse Fourier transform of the function \((|\xi|^2 + \varepsilon^2)^{-k}\). In the case when \( k > d/2 \) and \( \varepsilon > 0 \) this function is smooth in \( L_2(\mathbb{R}^d) \) and has fast decay at infinity (see below). The last property illustrates an important difference between the cases \( \varepsilon = 0 \) and \( \varepsilon > 0 \).

If \( k > d/2 \) and \( \varepsilon > 0 \) then \((|\xi|^2 + \varepsilon)^{-k}\) is an integrable function and, because it is radial, its Fourier transform can be expressed in terms of the Bessel functions \( J_{d/2-1} \):

\[
E^k_\varepsilon(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ix\xi}}{(|\xi|^2 + \varepsilon)^k} d\xi = \frac{|x|^{1-d/2}}{(2\pi)^{d/2}} \int_0^{\infty} \frac{t^{d/2}}{(t^2 + \varepsilon)^k} J_{d/2-1}(t|x|) dt =
\frac{2^{1-k}}{(2\pi)^{d/2} \Gamma(k)} (\varepsilon^{-1/2}|x|)^{k-d/2} K_{k-d/2}(\varepsilon^{1/2}|x|).
\]

The last function is a locally integrable since \( K_\nu(t) \) grows as \( t^{-\nu} \ln t \) for \( t \to 0 \). It is also a rapidly decreasing \( C^\infty \) function outside the origin. We show in the Lemmas 2.2 and 2.3 that for every \( k \geq d/2 \) there are infinite linear combinations \( L^k_{x_\nu} \in L_2(\mathbb{R}^d) \) of translates of the fundamental solution \( E^k \) of the operator \( D^k \) for which \( L^k_{x_\nu}(x_\gamma) = \delta_{\gamma,\nu} \), \( x_\gamma \in X(\lambda) \).

In general one does not know an explicit formula for \( L^k_{x_\nu} \), so one has to find approximations to \( L^k_{x_\nu} \) using finite sets of knots. It is possible to do so because the fundamental solutions \( E^k_{\varepsilon} \), \( k > d/2 \), \( \varepsilon > 0 \) have fast decay at infinity. The value of \( L^k_{x_\nu} \) at a point depends essentially on a finite number of points from the knot set \( X(\lambda) \). In what follows we will use the notation \( L^k_{\nu} \) for \( L^k_{x_\nu} \).

We prove the following.

**Theorem 2.1.** There exists a constant \( c = c(d, \varepsilon) \) that depends only on the dimension \( d \) and the parameter \( \varepsilon \) such that for any \( \sigma \geq 0 \) every knot set \( X(\lambda) \) with \( \lambda < ((\sigma + \varepsilon)c(d, \varepsilon))^{-1} \) and every integer \( r \geq [d/2] + 1 \),

\[
f = \lim_{l \to \infty} \sum_{x_\nu \in X(\lambda)} f(x_\nu) L^k_{\nu} x_{\nu}, \quad l \in \mathbb{N},
\]

for all \( f \in B_\sigma(\mathbb{R}^d) \).

Moreover, an error estimate for this approximation is

\[
\|f - \sum_{\nu} f(x_\nu) L^k_{\nu}\| \leq 2(c(d, \varepsilon)\lambda(\sigma + \varepsilon))^{d+1} r \|f\|.
\]

The proof of the Theorem will follow from some preliminary results below.
Given a sample set $X(\lambda)$ and a sequence $\{s_\gamma\} \in l_2$ we will be interested in finding a function $s_k \in H^{2k}$, for $k$ large enough, such that

a) $s_k(x_\gamma) = s_\gamma$, for $x_\gamma \in X(\lambda)$.

b) The function $s_k$ minimizes the functional $u \rightarrow \|D^ku\|$.

In what follows we will use the fact that, in the case $\varepsilon > 0$, the functional $u \rightarrow \|D^ku\|$ is equivalent to the Sobolev norm.

For the given sequence $\{s_\gamma\} \in l_2$ consider a function $f$ from $H^{2k}$ such that $f(x_\gamma) = s_\gamma$. Let $Pf$ denote the orthogonal projection of this function $f$ (in the Hilbert space $H^{2k}$ with the natural inner product) on the subspace

$$U^{2k}(X(\lambda)) = \{f \in H^{2k} | f(x_\gamma) = 0\}$$

with $H^{2k}$-norm. Then the function $g = f - Pf$ will be the unique solution of the above minimization problem for the functional $u \rightarrow \|D^ku\|$, $k > d/2$.

Given a function $f \in H^k$, where $k > d/2$, the function $s_k(f)$ will denote the solution to the above optimization problem with $s_\gamma = f(x_\gamma)$.

We will denote by $S^{2k}(X(\lambda))$ the set of all $L_2$- solutions of the equation

$$D^{2k}u = \sum_{x_\gamma \in X(\lambda)} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure and $\{\alpha_\gamma\} \in l_2$. Our next goal is to show that every $s_k(f)$ belongs to $S^{2k}(X(\lambda))$.

Indeed, suppose that $s_k \in H^{2k}$ is a solution to the minimization problem and $h \in U^{2k}(X(\lambda))$. Then

$$\|D^k(s_k + \lambda h)\|^2 = \|D^ks_k\|^2 + 2Re\lambda \int D^ks_kD^kh dx + |\lambda|^2 \|D^kh\|^2.$$

The function $s_k$ can be a minimizer only if for any $h \in U^{2k}(X(\lambda))$

$$\int D^ks_kD^kh dx = 0.$$

So, the function $g = D^ks_k \in L_2$ is orthogonal to $D^kU^{2k}(X(\lambda))$. Let $\varphi_\gamma \in C_0^\infty$ have disjoint supports and $\varphi_\gamma(x_\gamma) = 1$ and $h \in C_0^\infty$. Then the function $h - \sum h(x_\gamma)\varphi_\gamma$ belongs to the space $U^{2k}(X(\lambda)) \cap C_0^\infty$. Thus,

$$0 = \int gD^k(h - \sum h\varphi_\gamma) dx = \int gD^kh dx - \sum h(x_\gamma) \int gD^k\varphi_\gamma dx.$$

In other words

$$D^k g = \sum_{x_\gamma \in X(\lambda)} \alpha_\gamma \delta(x_\gamma),$$

or

$$D^{2k} s_k = \sum_{x_\gamma \in X(\lambda)} \alpha_\gamma \delta(x_\gamma),$$

where $\delta(x)$ is the Dirac measure.

Moreover, for any integer $r > 0$
\[
\sum_{\gamma=1}^{r} |\alpha_\gamma|^2 \leq \sum_{\gamma=1}^{\infty} \alpha_\gamma \delta(x_{\gamma}), \quad \sum_{\gamma=1}^{r} \alpha_\gamma \varphi_\gamma \leq \sum_{\gamma=1}^{\infty} \alpha_\gamma \delta(x_{\gamma}) \|_{H^{-2k}} (\sum_{\gamma=1}^{r} |\alpha_\gamma|^2)^{1/2},
\]

where \( C \) is independent of \( r \). This shows that the sequence \( \{\alpha_\gamma\} \) belongs to \( l_2 \).

Now suppose that \( f \in H^\infty \) and

\[
D^{2k} f = \sum_{x_{\gamma} \in X(\lambda)} \alpha_\gamma \delta(x_{\gamma}),
\]

where \( \{\alpha_\gamma\} \in l_2 \).

It was shown in [8] the norm of the Sobolev space \( H^k \) is equivalent to the norm

\[
\|D^{k/2} f\| + (\sum |f(x_{\gamma})|^2)^{1/2}.
\]

So for any \( \mu > 0 \) we have

\[
| < D^{2k} f, g > | = | < \sum \alpha_\gamma \delta(x_{\gamma}), g > | \leq \left( \sum |\alpha_\gamma|^2 \right)^{1/2} \left( \sum |g(x_{\gamma})|^2 \right)^{1/2} \leq C \left( \sum |\alpha_\gamma|^2 \right)^{1/2} \|g\|_{H^{d/2+\mu}}.
\]

This shows that the distribution \( \sum_{\gamma=1}^{\infty} \alpha_\gamma \delta(x_{\gamma}) = D^{2k} f \) belongs to \( H^{-d/2-\mu} \). Since the operator \( D^{2k} \) is \( C^\infty \)-uniformly elliptic of order \( 4k \) we can use a corresponding regularity result which gives that \( f \) belongs to \( H^{-d/2-\mu+4k} \), which is included in \( H^{2k} \) for all \( k > d \). The assertion that the orthogonal complement of \( D^k U^{2k}(X(\lambda)) \) is a subset of \( S^{2k}(X(\lambda)) \) is proved.

Conversely, if \( f, h \) belong to \( S^{2k}(X(\lambda)) \) and \( U^{2k}(X(\lambda)) \cap C^\infty \) respectively, then, since \( f \in H^{2k} \) and \( h \in H^{2k} \) and the pairing \( < \ldots > \) is an extension of the scalar product in \( L^2 \),

\[
\int D^k \overline{f} \, dx = < D^k f, \overline{h} > = \sum \alpha_\gamma \overline{h(x_{\gamma})} = 0.
\]

Thus we proved the following.

**Lemma 2.2.** A function \( f \in L_2 \) belongs to \( S^{2k}(X(\lambda)) \), i.e. satisfies the equation

\[
D^{2k} f = \sum_{x_{\gamma} \in X(\lambda)} \alpha_\gamma \delta(x_{\gamma}),
\]

where \( \{\alpha_\gamma\} \in l_2 \) if and only if \( f \) is a solution to the minimization problem for the functional \( u \to \|D^k u\| \).

In particular, every solution to the minimization problem is a linear combination of the translates of the fundamental solution \( E^k_\varepsilon \) of the operator \( D^k_\varepsilon = (\Delta + \varepsilon)^k, \varepsilon > 0 \).

In particular, for any \( x_{\gamma} \in X(\lambda) \) there exists a unique \( L^{2k}_x(X(\lambda)) \in S^{2k}(X(\lambda)) \) that takes the value 1 at the point \( x_{\gamma} \) and 0 at all other points in \( X(\lambda) \). These functions form a Riesz basis in \( S^{2k}(X(\lambda)) \).

Recall that the last assertion means that for any \( g \in S^{2k}(X(\lambda)) \) in \( L_2 \) we have

\[
g = \sum_{\gamma} g(x_{\gamma}) L^{2k}_x
\]
and there are constants $C_1, C_2 > 0$ such that
\[
\|g\|_2 \leq C_1 \left( \sum |g(x_\gamma)|^2 \right)^{1/2} \leq C_2\|g\|_2, \quad (g \in S^{2k}(X(\lambda))).
\]

This statement is a consequence of the next result.

**Lemma 2.3.** Every function from $S^{2k}(X(\lambda))$, $k = 2^d$, is uniquely determined by its values at points $x_\gamma \in X(\lambda)$. Moreover, for any $f \in S^{2k}(X(\lambda))$ the norm $(\sum |f(x_\gamma)|^2)^{1/2}$ is equivalent to the $L_2$-norm and to the Sobolev norm.

**Proof.** Since $S^{2k}(X(\lambda))$ is closed in the $L_2$-norm and $S^{2k}(X(\lambda)) \subset H^{2k}$ the $L_2$-norm and $H^{2k}$ norm are equivalent on $S^{2k}(X(\lambda))$. Moreover, one can show that on the space $S^{2k}(X(\lambda))$, $k = 2^d$, the norm $H^{2k}$ is equivalent to the norm $(\sum |f(x_\gamma)|^2)^{1/2}$. This statement is a consequence of the next result.

Indeed, if the functions $\varphi_\gamma \in C^\infty$ have disjoint supports in $B(x_\gamma, \lambda/4)$ and $\varphi_\gamma(x_\mu) = \delta_{\gamma \mu}, |\varphi_\gamma| \leq 1$, then the function $F = \sum_{\gamma \in N} f(x_\gamma)\varphi_\gamma$ is in $H^{2k}$ and $f(x_\gamma) = F(x_\gamma), k > d/2$. Because of the minimization property, we have
\[
\|D^k f\| \leq \|D^k F\| \leq C \left( \sum_\gamma |f(x_\gamma)|^2 \right)^{1/2}.
\]
Since for $k = 2^d$ the $H^{2k}$ norm on $S^{2k}$ is equivalent to the norm $\|D^k f\|$, this implies its equivalence to the norm $(\sum_\gamma |f(x_\gamma)|^2)^{1/2}$.

Now we can prove the following approximation property.

**Theorem 2.4.** For any integer $r \geq [d/2] + 1$ and any $f \in H^{2d+1+1}(R^d)$,
\[
f(x) = \lim_{l \to \infty} s_{2^l r}(f) = \lim_{l \to \infty} \sum_{x_\nu \in X(\lambda)} f(x_\nu)\gamma_{2^l r}(x).
\]
Moreover, there exists a constant $C(d, \varepsilon)$ such that the following error estimate is valid:
\[
\|f - \sum_\nu f(x_\nu)\gamma_{2^l r}\| \leq 2(c(d, \varepsilon)\lambda)^{2^{d+1} r} \|D^r f\|, l = 0, 1, \ldots
\]

**Proof.** If
\[
f \in H^{2k}, k = 2^d, \quad l = 0, 1, \ldots
\]
and
\[
s_k(f) = \sum_\nu f(x_\nu)\gamma_{2^k r}
\]
then
\[
f - s_k(f) \in U^{2k}(X(\lambda))
\]
and as it was shown in [8] we have
\[
\|f - s_k(f)\| \leq (C(d, \varepsilon)\lambda)^k \|D^{k/2} f - s_k(f)\|, k = 2^d, (l = 0, 1, \ldots).
\]
Using the minimization property of $s_k(f)$ we obtain
\[
\|f - s_k(f)\| \leq (c(d, \varepsilon)\lambda)^k \|D^{k/2} f\|, c(d, \varepsilon) = 2C(d, \varepsilon), k = 2^d, \quad (l = 0, 1, \ldots).
\]
The approximation theorem is proved. 

Our Theorem 2.1 follows from the above approximation theorem and the Bernstein inequality satisfied by any function from $B_2$:

$$\|D^{k/2} f\| \leq (\sigma + c)^k \|f\|.$$ 

Theorem 2.1 is proved.

irregular set of knots and the operator $\Delta + \varepsilon$, for $\varepsilon > 0$, the case of equally spaced points for the operator $\Delta$ is of special interest because in this case explicit formulas for the Fourier transform of $L^k/(\Delta)$ are known; here $\bar{n}$ is the integer lattice. Indeed one can verify (see [6]) that in the case of the standard lattice $\bar{n}$ of $\mathbb{R}^d$ the function $\Lambda_0^k = L_0^k$ is

$$\Lambda_0^k(\xi) = (2\pi)^{-d/2}(|\xi|^{2k} \sum_{j \in \mathbb{Z}^d} |\xi - 2\pi j|^{-2k})^{-1}$$

and all other $L_0^k$ are translations of $L_0^k$.

These functions $L_0^k$ have very fast decay at infinity in the sense that for every $k$ there are $a = a(k) > 0, b = b(k) > 0$ such that $|L_0^k(x)| \leq ae^{-b|\bar{n} - x|}$.

We also want to make the following remark. Our sampling theorem requires in general some oversampling. This means that the distance between sampling points needs to be small enough compared to the size of the support of the Fourier transform of the given band limited function. We will show now that if one is going to consider lattice sampling points then the oversampling is not necessary. For example if the Fourier transform of a function is in the cube $[-\pi, \pi]^d$ then the natural lattice in $\mathbb{R}^d$ can be chosen as the sampling set. Such a rate of sampling is known to be the best possible and is called the Nyquist rate.

Indeed, we can rewrite the formula for the function $\Lambda_0^k$:

$$\Lambda_0^k(\xi) = (|\xi|^{-2k} |\xi - 2\pi j|^{2k}) L_0^{kj}(\xi).$$

This shows that $\lim_{k \to \infty} \Lambda_0^k(\xi)$ is zero for every $\xi$ outside the cube $[-\pi, \pi]^d$.

Next, the formula

$$\Lambda_0^k(\xi) = (2\pi)^{-d/2} (1 + |\xi|^{2k} \sum_{j \in \mathbb{Z}^d} |\xi - 2\pi j|^{-2k})^{-1},$$

where $\mathbb{Z}^d_+$ is the set of all non zero $d$-tuples, implies that the limit $\lim_{k \to \infty} \Lambda_0^k(\xi)$ is $(2\pi)^{-d/2}$ for all $\xi$ in the cube $[-\pi, \pi]^d$.

In other words

$$\lim_{k \to \infty} L_k^0(x) = \frac{\sin(\pi x_1)}{\pi x_1} \frac{\sin(\pi x_2)}{\pi x_2} \cdots \frac{\sin(\pi x_d)}{\pi x_d}.$$ 

Together with the classical Shannon-Whittaker sampling theorem,

$$\hat{\varphi}(t) = \sum_{\bar{n} \in \mathbb{Z}^d} \hat{\varphi}(\bar{n}) \frac{\sin(\pi(t_1 - n_1))}{\pi(t_1 - n_1)} \cdots \frac{\sin(\pi(t_d - n_d))}{\pi(t_d - n_d)},$$

where $\varphi$ has support in the cube $[-\pi, \pi]^d$, $\bar{n} = (n_1, \ldots, n_d), t = (t_1, \ldots, t_d)$, this proves the formula.
\[ \varphi(x) = \lim_{k \to \infty} \sum_{\bar{n} \in \mathbb{Z}^d} \hat{\varphi}(\bar{n}) \Lambda^k_{\bar{n}}(x), \quad x \in \mathbb{R}^d. \]

This interpolation formula seems to be new.

3. **Inversion of the Radon transform in \( \mathbb{R}^n \) using irregular sampling**

For a given function \( \varphi \) on \( \mathbb{R}^d \) the Radon transform \( R_\varphi \) is defined by

\[ R_\varphi(s) = \int_{\theta \perp} \varphi(s\theta + y)dy, \]

where \( \theta \) is a direction vector belonging to the unit sphere \( S^{d-1} \) and \( s \) is a real number. In other words the Radon transform \( R_\varphi(s) = R_\varphi(\theta, s) \) is the integral of \( \varphi \) over the hyperplane in \( \mathbb{R}^d \) defined by \( x : <x, \theta> = s \). The backprojection operator is defined by

\[ R^* g(x) = \int_{S^{d-1}} g(\theta, <x, \theta>)d\theta, \]

where \( x \in \mathbb{R}^d \), \( g \) is defined on the direct product of \( S^{d-1} \), and \( \mathbb{R} \), which can be identified with the set of hyperplanes in \( \mathbb{R}^{d-1} \).

Then, if \( \varphi \in C_0^\infty(\mathbb{R}^d) \), the identity

\[ R^* T^{1-d} R\varphi = \varphi, \]

holds, where, for \( \alpha \) is real and \( T^\alpha \) is the Riesz potential operator, i.e., the Fourier transform of the function \( T^\alpha \varphi \) is defined as \( |\xi|^{-\alpha} \hat{\varphi}(\xi) \). For proofs see [7]. Our goal is to introduce a different reconstruction formula which only require s a discrete set of values of the Radon transform.

The analogous formula in Fourier analysis is the Poisson summation formula for the functions \( \varphi \) from \( L^2(\mathbb{R}) \) with support in \([-\pi, \pi] \):

\[ \varphi(t) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n)e^{int}. \quad (3.1) \]

The meaning of the last formula is that the Fourier coefficients of a function with compact support are regularly spaced samples of its Fourier transform. In this paper we will give an analog of the Poisson summation formula for the Radon transform. More precisely it will be shown that a compactly supported function can be reconstructed using even an irregular set of samples of its Radon transform.

Applying the Fourier transform to the formulas from the Theorem 2.1 we arrive at the following irregular version of the Poisson summation formula (3.1).

**Theorem 3.1.** If \( \varphi \in L^2(\mathbb{R}^d) \) has support in the ball \( B(\sigma, 0) \) and \( \Lambda^k_{\nu} \) is the inverse Fourier transform of the function \( L^k_{\nu} \) then

\[ \varphi = \lim_{l \to \infty} \sum_{\xi_\nu \in \Xi(\lambda)} \hat{\varphi}(\xi_\nu) \Lambda^l_{\nu}, \quad (3.2) \]
assuming \( l \in \mathbb{N} \), \( r \geq \lceil d/2 \rceil + 1 \), \( \lambda < (c(d, \varepsilon)(\sigma + \varepsilon))^{-1} \), and where the \( \Xi(\lambda) \) is an appropriate discrete set in the space of the dual variable \( \xi \).

An error estimate for this approximation is

\[
\| \varphi - \sum_{\xi_\nu \in \Xi(\lambda)} \hat{\varphi}(\xi_\nu) \Lambda_\nu^{2^d r} \| \leq 2(c(d, \varepsilon)\lambda(\sigma + \varepsilon))^{2^{d+1}r} \| \varphi \|.
\]

Note that \( \Lambda_\nu^{k} = \Lambda_{\nu, \varepsilon}^{k} \) is of the form

\[
\Lambda_{\nu, \varepsilon}^{k}(x) = (|x|^2 + \varepsilon)^{-k} \sum_{\xi_\mu \in \Xi(\lambda)} a_\mu(\nu, k) \exp(-i\xi_\mu x) \quad (3.3)
\]

and the coefficients \( a_\mu(\nu, k) \) can be determined using the conditions \( L_{\nu, \varepsilon}^{k}(x_\sigma) = \delta_{\nu, \sigma} \), where

\[
L_{\nu, \varepsilon}^{k}(x) = \sum_{\mu} a_\mu(\nu, k) E_{\varepsilon}^{k}(x - x_\mu).
\]

The following statement is an analog of the last theorem in the case of the Radon transform.

**Lemma 3.2.** Suppose that \( \varphi \in C^{\infty}_{0} \) has support in the ball \( B(\sigma, 0) \). If \( \Xi(\lambda) \) is a knot set in the space of the dual variable \( \xi \) with \( \lambda < (c(d, \varepsilon)(\sigma + \varepsilon))^{-1} \), where \( c(d, \varepsilon) \) is from the Theorem 2.1, then

\[
\varphi(x) = \lim_{l \to \infty} \frac{(2\pi)^{(1-d)/2}}{\sqrt{l}} \sum_{\xi_\nu \in \Xi(\lambda)} (\hat{R}\varphi)(\xi_\nu) \Lambda_\nu^{2^d r}(x) \quad (l \in \mathbb{N}, x \in \mathbb{R}^d),
\]

where \( \Lambda_\nu^{k} \) are from (3.3).

An error estimate is given by the inequality

\[
\| \varphi - (2\pi)^{(1-d)/2} \sum_{\nu} (\hat{R}\varphi)(\xi_\nu) \Lambda_\nu^{2^d r} \| \leq 2(c(d, \varepsilon)\lambda(\sigma + \varepsilon))^{2^{d+1}r} \| R\varphi \|.
\]

**Proof.** The Fourier slice theorem states that

\[
\hat{\varphi}(\theta \tau) = (2\pi)^{(1-d)/2}(\hat{R}\varphi)(\theta, \tau),
\]

where on the right we have the Fourier transform in the second variable.

Let \( \varphi \in L_2(\mathbb{R}^d) \) be supported in the ball \( B(0, \sigma) \). Then the function \( f \) such that \( f = \hat{\varphi} \) belongs to \( B_\sigma(\mathbb{R}^d) \). According to our sampling theorem,

\[
f(\xi) = \lim_{l \to \infty} \sum_{\xi_\nu \in \Xi(\lambda)} f(\xi_\nu) L_{\nu}^{2^d r}(\xi),
\]

where the \( \Xi(\lambda) \) is an appropriate discrete set in the space of the dual variable \( \xi \). The error estimate is given by the inequality

\[
\| f - \sum_{\nu} f(\xi_\nu) L_{\nu}^{2^d r} \| \leq 2(c(d, \varepsilon)\lambda(\sigma + \varepsilon))^{2^{d+1}r} \| f \|.
\]

So we have
where convergence is understood in the $L_2^2(R^d)$ sense.

Taking inverse Fourier transform we obtain

$$\varphi(x) = \lim_{l \to \infty} (2\pi)^{(1-d)/2} \sum_{\nu} \hat{\varphi}(\xi_\nu) \Lambda^{2l}_l(x) \quad (l \in \mathbb{N}, x \in \mathbb{R}^d),$$

(3.5)

where $\Lambda^{2l}_l$ is the inverse Fourier transform of $L^k_\nu$. The error estimate is given by the inequality

$$\|\varphi - (2\pi)^{(1-d)/2} \sum_{\nu} \hat{\varphi}(\xi_\nu) \Lambda^{2l}_l\| \leq 2(c(d, \varepsilon) \lambda(\sigma + \varepsilon))^{2l+1} \|\varphi\|.$$  

□

Although the formula (3.4) is a natural analog of the formula (3.2) it involves not only the Radon transform but also the Fourier transform in the second variable.

We are going to show how one can approximate the values $\hat{\varphi}(\xi_\nu)$ using just samples of the Radon transform. One way to do this is by using equally spaced samples and the Shannon-Whittaker formula. This method has the advantage that the Fast Fourier Transform can be used [7]. But it is also available in the context of our sampling Theorem 2.1 using an irregular set of samples.

It is convenient to assume now that every point in $\Xi(\lambda)$ has polar coordinates $(\theta_\nu, \tau_\mu)$ where $\theta_\nu$ belongs to the unit sphere $S^{d-1}$ and $\tau_\mu$ is the distance from 0.

In the one dimensional case, $d = 1$, we will use the notation $U^k_\nu$ for the functions $L^k_\nu$ which were constructed in the second section. Note that $U^k_\nu$ is a piecewise polynomial spline of order 2

Lemma 3.3. If $Y(\mu_j)$ is a sequence of knots with $\mu_j \to 0$ and $U^m_{2,j}$ is the corresponding set of one-dimensional Lagrangian splines then

$$\hat{\varphi}(\theta, \tau) = \lim_{j \to \infty} \sum_{s_\gamma \in Y(\mu_j)} (\hat{\varphi})(\theta, s_\gamma) V^m_{2,j} \Lambda^{2l}_l(\tau)$$

(3.6)

where $V^m_{2,j}$ is the Fourier transform of $U^m_{2,j}$.

Proof. Let $\lambda_i \to 0$. Now $X_i = X(\lambda_i)$ is the corresponding knot set and $\Lambda^{2l}_l(X(\lambda_i))$ is the set of Lagrange functions that correspond to $X_i$.

On the Fourier transform side our approximation theorem gives

$$\hat{\varphi} = \lim_{\nu \to \infty} \sum_{x_\nu \in X(\lambda_i)} g(x_\nu) \Lambda^{2l}_l(X(\lambda_i)), g \in C^\infty_0$$

(3.7)

with an error estimate

$$\|\hat{\varphi} - \sum_{x_\nu \in X(\lambda_i)} g(x_\nu) \Lambda^{2l}_l(X(\lambda_i))\| \leq 2(c(\lambda_i))^{2l+1} \|g\|_{H^{d+1}}, \quad (l = 0, 1, \ldots).$$

Note that the sum (3.6) is finite because $g$ has compact support. To describe the corresponding approximation to $R\varphi(\theta, s)$ as the function of the variable $s$ we introduce the sequence of $\mu_j \to 0$ and corresponding knot sequences $Y_j = Y(\mu_j)$. Then, since we are considering the one-dimensional case we have
\[ (R\varphi)(\theta, s) = \lim_{j \to \infty} \sum_{s_\gamma \in Y(\mu_j)} (R\varphi)(\theta, s_\gamma) U^m_{\gamma,j}(s), \quad m > 1, \]

and taking the Fourier transform in \( s \) we obtain

\[ (\hat{R}\varphi)(\theta, \tau) = \lim_{j \to \infty} \sum_{s_\gamma \in Y(\mu_j)} (R\varphi)(\theta, s_\gamma) V^m_{\gamma,j}(\tau) \]

where \( V^m_{\gamma,j} \) is the Fourier transform of \( U^m_{\gamma,j} \).

Note that if \( \varphi \) has compact support then its Radon transform \( R\varphi(s) \) is also of compact support in the variable \( s \) and, because of this, the last two sums are finite.

Keeping the same notations we summarize the last two lemmas in the following theorem.

**Theorem 3.4.** Suppose that \( \varphi \in C^\infty_0 \) has support in the ball \( B(\sigma, 0) \). If \( \Xi(\lambda) \) is a knot set in the space of the dual variable \( \xi \) with \( \lambda < (c(d, \varepsilon)(\sigma + \varepsilon)^{-1} \) where \( c(d, \varepsilon) \) is from the Theorem 2.1 then

\[ \varphi(x) = \lim_{l \to \infty} (2\pi)^{(1-d)/2} \sum_{\nu,\mu} (\hat{R}\varphi)(\theta_{\nu}, \tau_{\mu}) \Lambda^{2l}_{\nu,\mu}(x), l \in \mathbb{N} \quad (x \in \mathbb{R}^d), \]

where \( \Lambda^{2l}_{\nu,\mu} \) is the inverse Fourier transform of \( L^{k}_{\nu,\mu} \) and of the form (3.3).

An error estimate is given by the inequality

\[ \|\varphi - (2\pi)^{(1-d)/2} \sum_{\nu,\mu} (\hat{R}\varphi)(\theta_{\nu}, \tau_{\mu}) \Lambda^{2l}_{\nu,\mu} \| \leq 2(c(d, \varepsilon)\lambda(\sigma + \varepsilon))^{2l+1}\|R\varphi\|. \]

The approximate values of \( \hat{R}\varphi(\theta_{\nu}, \tau_{\mu}) \) can be determined by the formula

\[ (\hat{R}\varphi)(\theta_{\nu}, \tau_{\mu}) = \lim_{j \to \infty} \sum_{s_\gamma \in Y(\mu_j)} (R\varphi)(\theta_{\nu}, s_\gamma) V^m_{\gamma,j}(\tau_{\mu}). \]

4. **A Computational algorithm**

Alas, the amount of work which is needed for numerical implementations of the above algorithm is too big. In what follows we improve the standard Fourier Algorithm by using our irregular sampling theorem in conjunction with the Fast Fourier Transform. It will make our modification as efficient as the original Fourier Algorithm is. We restrict ourselves to the case \( d = 2 \) and \( \varepsilon = 1 \). In this case the constant \( c(d, \varepsilon) \) is not greater than 3.

Let \( \varphi \) have compact support and let \( g = R\varphi \) be sampled at

\[ (\theta_j, s_l), \quad j = 1, \ldots, p, \quad l = -q, \ldots, q, \]

where

\[ \theta_j = (\cos \alpha_j, \sin \alpha_j), \quad \alpha_j = (j-1)\pi/p, s_l = hl, h = 1/q. \]
It is easy to see that the optimal relation between $p$ and $q$ is given by the approximate formula $p \approx \pi q$.

STEP 1. For $j = 1, \ldots, p$, compute approximations $\hat{g}_{jr}$ to $\hat{g}(\theta_j, r\pi)$ by

$$\hat{g}_{jr} = (2\pi)^{-1/2} e^{i\pi r / q} \sum_{l=-q}^{q-1} g(\theta_j, s_l), r = -q, \ldots, q - 1.$$  

This step provides an approximation to $\hat{\varphi}$ on the polar grid $G_{p,q} = \pi r \theta_j : r = -q, \ldots, q - 1, j = 1, \ldots, p$.

Because we perform $p$ discrete Fourier transforms of length $2q$ and $p = \pi q$ this step requires $0(q^2 \ln q)$ operations.

STEP 2. For each $k \in \mathbb{Z}^2, |k| < q$ use the formula

$$\hat{\varphi}_k = \sum \hat{g}_{jl} L_{jl}^m$$

where the summation is taken over some points from the polar grid $G_{pq}$ which surround the point $k \in \mathbb{Z}^2$. Moreover due to the fact that the function $L_{jl}^m$ is localized essentially around point $\pi l \theta_j$ it is enough to keep a constant number of terms in this summation. This observation is very important since it implies that the second step requires essentially $0(q^2)$ operations.

STEP 3. Compute an approximation $\varphi_N$ to $\varphi(hN), N \in \mathbb{Z}^2$ by

$$\varphi_N = (1/2\pi)^{d/2} \sum_{|k|<q} e^{i\pi Nk / q} \varphi_k, |N| < q.$$  

To perform this step one needs $0(q^2 \ln q)$ steps. Thus the amount of work for our modified algorithm is the same as for the standard Fourier algorithm.

The high stability in the step 2 is a consequence of the estimate from Theorem 2.4.

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