On $q$-deformed Farey sum and a homological interpretation of $q$-deformed real quadratic irrational numbers

Xin Ren

Abstract

The left and right $q$-deformed rational numbers were introduced by Bapat, Becker and Licata via regular continued fractions, and they gave a homological interpretation for left and right $q$-deformed rational numbers. In the present paper, we focus on negative continued fractions and defined left $q$-deformed negative continued fractions. We give a formula for computing the $q$-deformed Farey sum of the left $q$-deformed rational numbers based on it. We use this formula to give a combinatorial proof of the relationship between the left $q$-deformed rational number and the Jones polynomial of the corresponding rational knot which was proved by Bapat, Becker and Licata using a homological technique. Finally, we combine their work and the $q$-deformed Farey sum, and give a homological interpretation of the $q$-deformed Farey sum. We also give an approach to finding a relationship between real quadratic irrational numbers and homological algebra.

keywords: continued fractions, $q$-deformed Farey sum, Jones polynomial, real quadratic irrational numbers, 2-Calabi–Yau category

1 Introduction

The notion of $q$-deformed rational numbers [15] was introduced by Morier-Genoud and Ovsienko based on some combinatorial properties of rational numbers. They further extended this notion to arbitrary real numbers [16] by some number-theoretic properties of irrational numbers. These works are related to many directions including Jones polynomial of rational knots [9, 11, 18, 15], Teichmüller spaces [5], the Markov-Hurwitz approximation theory [8, 10, 13, 22], the modular group and the Picard group [12, 21], combinatorics of posets [14, 19, 20] and triangulated category.

For a formal parameter $q$ and an irreducible fraction $\frac{r}{s}$, as an enhancement of $q$-deformed rational numbers, Bapat, Becker and Licata defined left $q$-deformed rational number $\left[\frac{r}{s}\right]_{q}^{\flat}$ and right $q$-deformed rational number $\left[\frac{r}{s}\right]_{q}^{\sharp}$ via the regular continued fractions of $\frac{r}{s}$, and the right

MSC Classification(2020): 11A55, 05A30, 57K14, 18G80
$q$-deformed rational number \( \frac{r}{s} \) is exactly \( q \)-deformed rational number \( \frac{r}{s} \) considered by Morier-Genoud and Ovsienko. Following [15] and [2], the right \( q \)-deformed rational numbers can be expressed by the right \( q \)-deformed regular or negative continued fraction expansions, and the left \( q \)-deformed rational numbers can be expressed by the left \( q \)-deformed regular continued fraction expansions. These \( q \)-deformations of the fractions are rational expressions in the variable \( q \) with integer coefficients. Such as [15] [12], and so on, it may be more convenient from the perspective of the negative continued fraction expansion when we consider some properties of left and right \( q \)-deformed rational numbers and their applications. In particular, the formula for the right \( q \)-deformed Farey sum based on the negative continued fraction is more concise [15, Section 2]. This induces us to consider the \( q \)-deformed Farey sum of the left \( q \)-deformed rational numbers.

In the present paper, we define the left \( q \)-deformed negative continued fraction expansion. Then we give a formula for computing the \( q \)-deformed Farey sum of the left \( q \)-deformed rational numbers based on negative continued fraction (see Theorem 3.3).

As an application of the right \( q \)-deformed rational numbers, given a rational number \( \frac{r}{s} \), we can use the numerator and denominator of \( \left[ \frac{r}{s} \right] \) to represent the Jones polynomial of the rational knot to which \( \frac{r}{s} \) corresponds [15, Proposition A.1]. On the other hand, Bapat, Becker and Licata prove that the Jones polynomial for the rational knot corresponding to \( \frac{r}{s} \) can be represented by just the numerator of \( \left[ \frac{r}{s} \right] \) [2, Theorem A.3] by considering a homological interpretation of \( \left[ \frac{r}{s} \right] \) and \( \left[ \frac{r}{s} \right] \). Considering the zigzag algebra on the \( A_2 \) quiver, we can obtain a triangulated category \( C_2 \) called 2-Calabi–Yau category associated to the \( A_2 \) quiver [2, Section 4]. For spherical objects on \( C_2 \), Bapat, Becker and Licata defined two functions, denoted as \( \text{occ}_q \) and \( \text{hom}_q \), and they proved that \( \left[ \frac{r}{s} \right] \) and \( \left[ \frac{r}{s} \right] \) can be expressed in terms of \( \text{occ}_q \) and \( \text{hom}_q \), respectively [2, Theorems 3.7 and 3.8]. There are two questions worth considering. Can we give a combinatorial proof of [2, Theorem A.2] without homology techniques? Can we give a homological interpretation of the \( q \)-deformed irrational numbers defined in [16]? In the present paper, we apply Theorem 3.3 to give a combinatorial proof of [2, Theorem A.3] without using homology techniques (see Theorem 4.2). Then, we combine the homological interpretation of the left and right \( q \)-deformed rational numbers and the \( q \)-deformed Farey sum, and give a homological interpretation of the \( q \)-deformed Farey sum (see Propositions 5.13 and 5.14). We also apply the results in [2] Theorems 4.7 and 4.8 to real quadratic irrational numbers with periodic type (see Theorem 5.15).

This paper is organized into the following sections. In Section 2, we first recall some definitions related to the left and right \( q \)-deformed rational numbers, including the (right) \( q \)-deformed Euler continuants, which were introduced by Morier-Genoud and Ovsienko [15]. Similarly, we define the left \( q \)-deformed negative continued fractions and left \( q \)-deformed Euler continuants.

We prove that the left \( q \)-deformed negative continued fractions and the left \( q \)-deformed regular continued fractions are equal. In Section 3, we give \( q \)-deformed Farey sum of left \( q \)-rational num-
bers based on negative continued fraction, and derive a weighted triangulation and \( q \)-deformed Farey tessellation corresponding to left \( q \)-deformed rational numbers. In Section 4, we give a new proof of [2, Theorem A.2] as an application of \( q \)-deformed Farey sum of left \( q \)-deformed rational numbers by induction through the length of the negative continued fraction expansion. In Section 5, we first combine the results of [2, Theorems 4.7 and 4.8] with the \( q \)-deformed Farey sum to give a homological interpretation of the \( q \)-deformed Farey sum. Then we consider a real quadratic irrational number with periodic type, its \( q \)-deformation can also be expressed in a special form that is related to the \( q \)-deformation rational number that approximates it \([12]\). Based on the results of these \( q \)-deformations, we give the relations between real quadratic irrational numbers and homological algebra.

2 \( q \)-deformed continued fractions and \( q \)-deformed Euler continuants

In this section, we first briefly review some definitions related to left and right \( q \)-deformed rational numbers (see [2] and [15] for details). We define the left \( q \)-deformed negative continued fraction expansion and introduce the left \( q \)-deformed Eulerian continuants. We simply check that the left \( q \)-deformed negative continued fraction expansion is indeed consistent with the left \( q \)-deformed regular continued fraction expansion.

2.1 Left and right \( q \)-integers and \( q \)-deformed rational numbers

It is well-known that an irreducible fraction \( \frac{r}{s} \in \mathbb{Q} \cup \{\infty\} \) has unique regular and negative continued fraction expansions as follows:

\[
\frac{r}{s} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2m}}}} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_k}}}
\]

with \( a_1 \in \mathbb{Z}, a_i \in \mathbb{Z} \setminus \{0\} (i \geq 2), \) and \( c_1 \in \mathbb{Z} \setminus \{0\} (i \geq 2) \) and \( c_j \in \mathbb{Z} \setminus \{-1, 0, 1\} (j \geq 2) \). If \( \frac{r}{s} \) is negative, then \( a_1, \ldots, a_{2m} \) and \( c_1, \ldots, c_k \) are negative, and if \( \frac{r}{s} \) is positive, then \( a_1, \ldots, a_{2m} \) and \( c_1, \ldots, c_k \) are positive. We denote this expansion by \([a_1, \ldots, a_{2m}]\) and \([c_1, \ldots, c_k]\), respectively. As special cases, the regular and negative continued fraction expansions of 0 and \( \infty \) (\( \infty := \frac{1}{0} \)) are \([-1, 1], [[1, 1]]\) and empty expansion \([\ ], [[[\ ]]]\), respectively.

We consider the following three matrices. \( \sigma_1 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \)

We know that the modular group \( \text{PSL}_2(\mathbb{Z}) \) can be generated by \( \{\sigma_1, \sigma_2\} \) or \( \{\sigma_1, S\} \). The modular
group \( \text{PSL}_2(\mathbb{Z}) \) acts on \( \mathbb{Q} \cup \{\infty\} \) by the fractional linear transformation:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x) = \frac{ax + b}{cx + d},
\]

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}), \ x \in \mathbb{Q} \cup \{\infty\} \). Then a rational number \( \frac{r}{s} = [a_1, \ldots, a_{2m}] = \left[ [c_1, \ldots, c_k] \right] \) can be expressed by the following formulas:

\[
\frac{r}{s} = \sigma_1^{-a_1} \sigma_2^{-a_2} \sigma_1^{-a_3} \sigma_2^{-a_4} \cdots \sigma_1^{-a_{2m-1}} \sigma_2^{-a_{2m}}(\infty), \quad (2.1)
\]

\[
\frac{r}{s} = \sigma_1^{-c_1} S \sigma_2^{-c_2} S \cdots \sigma_1^{-c_k} S(\infty). \quad (2.2)
\]

**Definition 2.1** \((\mathbb{Z})\). Let \( q \) be a formal parameter. For a rational number \( \frac{r}{s} = [a_1, \ldots, a_{2m}] \), we denote by \( \text{PSL}_{2,q}(\mathbb{Z}) \) the subgroup of group \( \text{GL}_2(\mathbb{Z}[q^{\pm 1}]) \) generated by the following two elements:

\[
\sigma_{1,q} = \begin{pmatrix} q^{-1} & -q^{-1} \\ 0 & 1 \end{pmatrix}, \quad \sigma_{2,q} = \begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix}.
\]

Then the right \( q \)-deformed rational number is

\[
\left[ \frac{r}{s} \right]^q = \sigma_{1,q}^{-a_1} \sigma_{2,q}^{-a_2} \sigma_{1,q}^{-a_3} \sigma_{2,q}^{-a_4} \cdots \sigma_{1,q}^{-a_{2m-1}} \sigma_{2,q}^{-a_{2m}}(\infty),
\]

and the left \( q \)-deformed rational number is

\[
\left[ \frac{r}{s} \right]^q = \sigma_{1,q}^{-a_1} \sigma_{2,q}^{-a_2} \sigma_{1,q}^{-a_3} \sigma_{2,q}^{-a_4} \cdots \sigma_{1,q}^{-a_{2m-1}} \sigma_{2,q}^{-a_{2m}} \left( \frac{1}{1 - q} \right).
\]

**2.2 Left \( q \)-deformed negative continued fractions**

**Definition 2.2** \((\mathbb{Z})\). Let \( q \) be a formal parameter. We consider an integer \( n \), the following two rational forms \( [n]^q \) and \( [n]^q \) in \( q \) are called the right \( q \)-integer of \( n \) and the left \( q \)-integer of \( n \), respectively.

\[
[n]^q := \frac{1 - q^n}{1 - q}, \quad [n]^q := \frac{1 - q^{n-1} + q^n - q^{n+1}}{1 - q}.
\]

**Remark 2.3.** Suppose that \( m, n \in \mathbb{Z} \). It can be easy to check that the right \( q \)-integers and left \( q \)-integers satisfy the following properties.

(i) \( [n]^q = [n]^q + q^{n-1} - q^n \);

(ii) \( m + n [n]^q = [m]^q + q^m [n]^q = [n]^q + q^n [m]^q = [n]^q + q^n [m]^q = [n]^q + q^n [m]^q = [n]^q + q^n [m]^q \).
(iii) $[-n]_q^\sharp = -q^{-1} [n]_{q-1}^\sharp, \quad [-n]_q^\flat = -q^{-1} [n]_{q-1}^\flat$

(iv) $q^n [n]_{q-1}^\sharp = q [n]_q^\sharp, \quad q^n ([n]_{q-1}^\flat - [0]_{q-1})^\flat = q ([n]_q^\flat - [0]_q^\flat)$

Suppose that $\frac{r}{s} = [a_1, \ldots, a_{2m}] = [\lfloor c_1, \ldots, c_k \rfloor]$. From [15] and [2], the right $q$-deformed rational number $\frac{r}{s}_q^\sharp$ has both the following $q$-deformed positive and negative continued fraction expansions.

\[
\frac{r}{s}_q^\sharp = [a_1, a_2, \ldots, a_{2m}] : = [a_1]_q^\sharp + \frac{q^{a_1}}{[a_2]_q^\sharp - q^{a_2}} + \frac{q^{a_3}}{[a_3]_q^\sharp - q^{a_4}} + \cdots + \frac{q^{a_{2m-1}}}{[a_{2m-1}]_q^\sharp - q^{a_{2m}}}.
\]

\[
\frac{r}{s}_q^\flat = [\lfloor c_1, c_2, \ldots, c_k \rfloor] : = [c_1]_q^\flat - \frac{q^{c_1-1}}{[c_2]_q^\flat - q^{c_2-1}} + \frac{q^{c_3-1}}{[c_3]_q^\flat - q^{c_4-1}} + \cdots + \frac{q^{c_{k-1}-1}}{[c_{k-1}]_q^\flat - q^{c_k-1}}.
\]

For the left $q$-deformed rational number $\frac{r}{s}_q^\flat$, Bapat, Becker and Licata proved that the right $q$-deformed rational number $\frac{r}{s}_q^\flat$ has a $q$-deformed positive continued fraction expansion [2] as follows:
Similarly to the formula (2.4), we can also define the left \( q \)-deformed negative continued fraction expansion as follows:

**Definition 2.4** (left \( q \)-deformation of negative continued fraction expansion).

\[
[c_1, c_2, \ldots, c_k]_q^\flat := [c_1]_q^\flat - \frac{q^{c_1-1}}{[c_2]_q^\flat - \frac{q^{c_2-1}}{[c_3]_q^\flat - \frac{q^{c_3-1}}{[c_4]_q^\flat - \ldots \frac{q^{c_k-1}}{[c_{k-1}]_q^\flat - [c_k]_q^\flat}}}})
\]

Note that it differs from the right \( q \)-deformed negative continuous fraction expansion only in the last term.

As in the case of right \( q \)-deformation, we have the following conclusion for the case of left \( q \)-deformation.

**Theorem 2.5.** If a rational number \( \frac{r}{s} \) is given in the form \( \frac{r}{s} = [a_1, \ldots, a_{2m}] = [[c_1, \ldots, c_k]] \), then

\[
[a_1, \ldots, a_{2m}]_q^\flat = [[c_1, \ldots, c_k]]_q^\flat.
\]

We will prove this formula in Section 2.4.

By [15] and [2], the left and right \( q \)-rational can be expressed by the following quotients of two polynomial in \( q \) with integer coefficients as follow:

\[
[r]_q^\sharp = \frac{R^\sharp(q)}{S^\sharp(q)}, \quad [r]_q^\flat = \frac{R^\flat(q)}{S^\flat(q)}.
\]
where $\mathcal{R}^q(q)$ and $\mathcal{S}^q(q)$ are coprime, and $\mathcal{R}^q(q), \mathcal{S}^q(q) \in \mathbb{Z}[q]$ are monic polynomials ($\dagger \in \{\sharp, \flat\}$).

In particular, we have

$$[0]^\sharp_1 = 0, \quad [0]^\flat_1 = 1; \quad [\infty]^\sharp_q = \frac{1}{0}, \quad [\infty]^\flat_q = \frac{1}{1-q}.$$ 

\subsection*{2.3 The left and right $q$-deformed Euler continuants}

\textbf{Definition 2.6 (right $q$-deformed Euler continuants)}.

$$E^\sharp_k(c_1, \ldots, c_k)_q := \begin{vmatrix} [c_1]^\sharp_q & q^{c_1-1} & 1 \\ 1 & [c_2]^\sharp_q & q^{c_2-1} \\ \vdots & \ddots & \ddots \\ 1 & [c_{k-1}]^\sharp_q & q^{c_{k-1}-1} \\ 1 & [c_k]^\sharp_q \end{vmatrix}$$

where $c_i$'s are integers, and for convenience, we set $E^\sharp_0() = 1$ and $E^\sharp_{-1}() = 0$.

For the numerators and denominators of the right $q$-deformed rational numbers, we have the following conclusion.

\textbf{Proposition 2.7 ([15 Proposition 5.3])}. For a rational number $\frac{r}{s} = [c_1, \ldots, c_k]$, we have

$$\mathcal{R}^q(q) = E^\sharp_k(c_1, \ldots, c_k)_q, \quad \mathcal{S}^q(q) = E^\sharp_{k-1}(c_2, \ldots, c_k)_q.$$ 

\textbf{Definition 2.8 (left $q$-deformed Euler continuants)}.

$$E^\flat_k(c_1, \ldots, c_k)_q := \begin{vmatrix} [c_1]^\flat_q & q^{c_1-1} & 1 \\ 1 & [c_2]^\flat_q & q^{c_2-1} \\ \vdots & \ddots & \ddots \\ 1 & [c_{k-1}]^\flat_q & q^{c_{k-1}-1} \\ 1 & [c_k]^\flat_q \end{vmatrix}$$

where $c_i$'s are integers, and for convenience, we set $E^\flat_0() = 1$ and $E^\flat_{-1}() = 1-q$.

By the definition of $E^\flat_k(c_1, \ldots, c_k)_q$, we know that for $c_l > 2$, we have

$$q(E^\sharp_{l+1}(c_1, \ldots, c_l, 2)_q - E^\sharp_l(c_1, \ldots, c_l)_q) = E^\sharp_{l+2}(c_1, \ldots, c_l, 2, 2)_q - E^\sharp_{l+1}(c_1, \ldots, c_l, 2)_q.$$ 

For a non-negative integer $h$, by induction, it can be checked that

$$q^h(E^\sharp_{l+1}(c_1, \ldots, c_l, 2)_q - E^\sharp_l(c_1, \ldots, c_l)_q) = E^\sharp_{l+h+1}(c_1, \ldots, c_l, 2^{(h+1)})_q - E^\sharp_{l+h}(c_1, \ldots, c_l, 2^{(h)})_q.$$ 

(2.10)
Moreover, by the definition of $E_k^q(c_1, \ldots, c_k)_q$ and $E_k^q(c_1, \ldots, c_k)_q$, we can know that

\[
E_k^q(c_1, \ldots, c_k)_q = [c_k]_q E_{k-1}^q(c_1, \ldots, c_{k-1})_q - q^{c_k-1}E_{k-2}^q(c_1, \ldots, c_{k-2})_q = E_k^q(c_1, \ldots, c_k)_q - q^{c_k-1}(1-q)E_{k-1}^q(c_1, \ldots, c_{k-1})_q. \tag{2.11}
\]

### 2.4 Proof of Theorem 2.5

Before we prove Theorem 2.5, let us prove the following proposition.

**Proposition 2.9.** Consider the element $S_q := \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$ in $\text{PSL}_2(q(\mathbb{Z}))$, then

\[
\sigma_{1,q}^{-c_1} S_q \sigma_{1,q}^{-c_2} S_q \cdots \sigma_{1,q}^{-c_k} S_q \left( \frac{1}{1-q} \right) = [[c_1, \ldots, c_k]]_q^p. \tag{2.12}
\]

**Proof.** By Proposition 4.3 of [15] and Proposition 2.7, one has

\[
\sigma_{1,q}^{-c_1} S_q \sigma_{1,q}^{-c_2} S_q \cdots \sigma_{1,q}^{-c_k} S_q = \begin{pmatrix} E_k^q(c_1, \ldots, c_k)_q & -q^{c_k-1}E_{k-1}^q(c_1, \ldots, c_{k-1})_q \\ E_{k-1}^q(c_2, \ldots, c_k)_q & -q^{c_k-1}E_{k-2}^q(c_2, \ldots, c_{k-1})_q \end{pmatrix}.
\]

We view the left $q$-rational $[\infty]_q^p = \frac{1}{1-q}$ as a vector in the projective space. Note that

\[
E_k^q(c_1, \ldots, c_k)_q = [c_k]_q E_{k-1}^q(c_1, \ldots, c_{k-1})_q - q^{c_k-1}E_{k-2}^q(c_1, \ldots, c_{k-2})_q,
\]

and by Remark 2.3 we have

\[
\sigma_{1,q}^{-c_1} S_q \sigma_{1,q}^{-c_2} S_q \cdots \sigma_{1,q}^{-c_k} S_q \left( \frac{1}{1-q} \right)
= \begin{pmatrix} E_k^q(c_1, \ldots, c_k)_q - q^{c_k-1}E_{k-1}^q(c_1, \ldots, c_{k-1})_q + q^{c_k}E_{k-1}^q(c_1, \ldots, c_{k-1})_q \\ E_{k-1}^q(c_2, \ldots, c_k)_q - q^{c_k-1}E_{k-2}^q(c_2, \ldots, c_{k-1})_q + q^{c_k}E_{k-2}^q(c_2, \ldots, c_{k-1})_q \end{pmatrix}
= \begin{pmatrix} [c_k]_q E_{k-1}^q(c_1, \ldots, c_{k-1})_q - q^{c_k-1}E_{k-2}^q(c_1, \ldots, c_{k-2})_q \\ [c_k]_q E_{k-2}^q(c_2, \ldots, c_{k-1})_q - q^{c_k-1}E_{k-3}^q(c_2, \ldots, c_{k-2})_q \end{pmatrix}
= \begin{pmatrix} E_k^q(c_1, \ldots, c_k)_q \\ E_{k-1}^q(c_2, \ldots, c_k)_q \end{pmatrix}.
\]
Thus, by expanding the determinant (2.9), we can infer that

$$\frac{E_k^q(c_1, \ldots, c_k)q}{E_{k-1}^q(c_2, \ldots, c_k)q} = [c_1]_q^2 - \frac{q^{c_1-1}}{E_{k-2}^q(c_3, \ldots, c_k)q} = [c_1]_q^2 - \frac{q^{c_2-1}}{E_{k-3}^q(c_4, \ldots, c_k)q} = \cdots = [[c_1, \ldots, c_k]]_q^b.$$  

\[ \square \]

**Proof of Theorem 2.5**

By Proposition 4.9 in [15], it follows that

$$q^{a_2+a_4+\cdots+a_{2m}}\sigma_{1,q}^{-a_1}\sigma_{2,q}^{a_2}\sigma_{1,q}^{-a_3}\sigma_{2,q}^{a_4}\cdots\sigma_{1,q}^{-a_{2m-1}}\sigma_{2,q}^{a_{2m}} = \sigma_{1,q}^{-c_i}S_q\sigma_{1,q}^{-c_j}S_q\cdots\sigma_{1,q}^{-c_k}S_q\sigma_{1,q}^{-1}$$

and hence

$$[a_1, \ldots, a_{2m}]_q^b = \sigma_{1,q}^{-a_1}\sigma_{2,q}^{a_2}\sigma_{1,q}^{-a_3}\sigma_{2,q}^{a_4}\cdots\sigma_{1,q}^{-a_{2m-1}}\sigma_{2,q}^{a_{2m}} \left( \frac{1}{1-q} \right)$$

$$= \sigma_{1,q}^{-c_i}S_q\sigma_{1,q}^{-c_j}S_q\cdots\sigma_{1,q}^{-c_k}S_q\sigma_{1,q}^{-1} \left( \frac{1}{1-q} \right)$$

$$= \sigma_{1,q}^{-c_i}S_q\sigma_{1,q}^{-c_j}S_q\cdots\sigma_{1,q}^{-c_k}S_q \left( \frac{1}{1-q} \right)$$

$$= [[c_1, \ldots, c_k]]_q^b.$$  

Through the above arguments, we have

$$[[T]_q^{-b}]_q = \sigma_{1,q}^{-c_i}S_q\sigma_{1,q}^{-c_j}S_q\cdots\sigma_{1,q}^{-c_k}S_q\sigma_{1,q}^{-1} \left( \frac{1}{1-q} \right).$$

### 2.5 Basic properties of the numerator and denominator of left $q$-rational numbers

Morier-Genoud and Ovsienko give the basic properties of the numerator and denominator of right $q$-deformed rationals as follows [15]:

For $i = 1, 2, \ldots, k$, we have

$$\mathcal{R}_i^q(q) = \mathcal{R}_i^q(q), \quad \mathcal{R}_{i+1}^q(q) = [c_{i+1}]_q^b \mathcal{R}_i^q(q) - q^{c_i-1} \mathcal{R}_{i-1}^q(q).$$
\[ S^\sharp_k(q) = S^\sharp(q), \quad S^\sharp_{i+1}(q) = [c_{i+1}]_q S^\sharp_i(q) - q^{c_{i+1} - 1} S^\sharp_{i-1}(q), \]

where the initial data
\[ R^\sharp_0(q) = 1, \quad R^\sharp_1(q) = [c_1]_q, \quad S^\sharp_0(q) = 0, \quad S^\sharp_1(q) = 1, \]
them it follows that
\[ \frac{R^\sharp_i(q)}{S^\sharp_i(q)} = [c_1, \ldots, c_i]_q. \]

Similarly, we have the corresponding property for the left \( q \)-rationals as follows. For \( i = 1, 2, \ldots, k \), we have
\[ R^\flat_k(q) = R^\flat(q), \quad R^\flat_{i+1}(q) = [c_{i+1}]_q R^\flat_i(q) - q^{c_{i+1} - 1} R^\flat_{i-1}(q), \]
\[ S^\flat_k(q) = S^\flat(q), \quad S^\flat_{i+1}(q) = [c_{i+1}]_q S^\flat_i(q) - q^{c_{i+1} - 1} S^\flat_{i-1}(q), \]
where the initial data
\[ R^\flat_0(q) = 1, \quad R^\flat_1(q) = [c_1]_q, \quad S^\flat_0(q) = 1 - q, \quad S^\flat_1(q) = 1, \]
then it follows that
\[ \frac{R^\flat_i(q)}{S^\flat_i(q)} = [c_1, \ldots, c_i]_q. \]

### 3 \( q \)-deformed Farey sum and \( q \)-deformed Farey triangles

In this section, we give formulas corresponding to the \( q \)-deformed Farey sum of the left \( q \)-deformed rational numbers. We use this formula to obtain a \( q \)-deformed Farey tessellation and weighted triangulation on the left \( q \)-deformed rational numbers. For convenience, from this section onwards, we always assume that the rational numbers are non-negative. The case of negative rational numbers can be considered by symmetry.

#### 3.1 \( q \)-deformed Farey sum of left and right \( q \)-rational numbers

We consider two non-negative irreducible fractions \( \frac{r}{s} \) and \( \frac{r'}{s'} \) (we always assume that \( \frac{1}{0} \) is an irreducible fraction), then we say \( \frac{r}{s}, \frac{r'}{s'} \) are Farey neighbors if \( |sr' - rs'| = 1 \). Different from the ordinary sum of fractions, we denote the Farey sum of \( \frac{r}{s} \) and \( \frac{r'}{s'} \) by
\[ \frac{r}{s} \# \frac{r'}{s'} := \frac{r + r'}{s + s'}. \] (3.1)

The \( q \)-deformed Farey sum of right \( q \)-deformed rational numbers has been introduced in \[15\].
Theorem 3.1 (Morier-Genoud and Ovsienko [15]). For a positive rational number $\alpha = [c_1, \ldots, c_k]$ which is the Farey sum of

$$\beta = \begin{cases} \lfloor c_1, \ldots, c_{l-1} \rfloor & \text{for } c_k = c_{k-1} = \cdots = c_{l+1} = 2, \ c_l > 2, \ 1 \leq l \leq k \\ \lfloor 1 \rfloor & \text{for } k = 1, \ c_k = 2 \\ \lfloor 1, 1 \rfloor & \text{for } c_k = c_{k-1} = \ldots = c_2 = 2, \ c_1 = 1 \end{cases}$$

and

$$\gamma = \begin{cases} \lfloor c_1, \ldots, c_{k-1} \rfloor & \text{for } k \geq 2 \\ \lfloor \rfloor & \text{for } k = 1, \end{cases}$$

if we assume that

$$[\alpha]^q = \frac{R_\alpha^q(q)}{S_\alpha^q(q)}, \quad [\beta]^q = \frac{R_\beta^q(q)}{S_\beta^q(q)}, \quad [\gamma]^q = \frac{R_\gamma^q(q)}{S_\gamma^q(q)}.$$

then

$$\frac{R_\alpha^q(q)}{S_\alpha^q(q)} = \frac{R_\beta^q(q) + q^{c_k-1}R_\gamma^q(q)}{S_\beta^q(q) + q^{c_k-1}S_\gamma^q(q)}.$$  \hspace{1cm} (3.4)

Hence, we define the $q$-defomed Farey sum $\#_q^z$ of $[\beta]^q$ and $[\gamma]^q$ by

$$[\beta]^q \#_q^z [\gamma]^q = \frac{R_\beta^q(q) + q^{c_k-1}R_\gamma^q(q)}{S_\beta^q(q) + q^{c_k-1}S_\gamma^q(q)}.$$

Example 3.2.

$$\frac{12}{5} = \lfloor 3, 2, 3 \rfloor, \quad \frac{7}{3} = \lfloor 3, 2, 2 \rfloor, \quad \frac{5}{2} = \lfloor 3, 2 \rfloor,$$ then

$$\left[ \frac{12}{5} \right]^q = \frac{1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5}{1 + q + 2q^2 + q^3},$$

$$\left[ \frac{7}{3} \right]^q = \frac{1 + 2q + 2q^2 + q^3 + q^4}{1 + q + q^2}, \quad \left[ \frac{5}{2} \right]^q = \frac{1 + 2q + q^2 + q^3}{1 + q}.$$

$$\left[ \frac{7}{3} \right]^q \#_q^z \left[ \frac{5}{2} \right]^q = \frac{(1 + 2q + 2q^2 + q^3 + q^4) + q^2(1 + 2q + q^2 + q^3)}{(1 + q + q^2) + q^2(1 + q)} = \left[ \frac{12}{5} \right]^q.$$
Similarly, we consider a left \( q \)-deformed rational number for a left \( q \)-deformed Farey sum. The following theorem gives the formula for the \( q \)-deformed Farey sum of a left \( q \)-deformed rational number. It is interesting to note that formula (3.5) forms a formal symmetry with the formula (3.4).

**Theorem 3.3.** For a rational number \( \alpha = [c_1, \ldots, c_k] \) which is the Farey sum of \( \beta \) and \( \gamma \) defined by (3.2) and (3.3), if we assume that

\[
[\alpha]_q = \frac{R^\flat_\alpha(q)}{S^\flat_\alpha(q)}, \quad [\beta]_q = \frac{R^\flat_\beta(q)}{S^\flat_\beta(q)}, \quad [\gamma]_q = \frac{R^\flat_\gamma(q)}{S^\flat_\gamma(q)},
\]

then

\[
\frac{R^\flat_\alpha(q)}{S^\flat_\alpha(q)} = \frac{q^{k-l+1}R^\flat_\beta(q) + R^\flat_\gamma(q)}{q^{k-l+1}S^\flat_\beta(q) + S^\flat_\gamma(q)},
\]

(3.5)

where \( c_k = c_{k-1} = \cdots = c_{l+1} = 2, \ c_l > 2, \ 1 \leq l \leq k. \) In particular, for \( k = 1 \), we have

\[
\frac{R^\flat_\alpha(q)}{S^\flat_\alpha(q)} = \frac{qR^\flat_\beta(q) + R^\flat_\gamma(q)}{qS^\flat_\beta(q) + S^\flat_\gamma(q)}.
\]

Hence, we define the \( q \)-deformed Farey sum \( \#^\flat_q \) of \( [\beta]_q \) and \( [\gamma]_q \) by

\[
[\beta]_q \#^\flat_q [\gamma]_q = \frac{q^{k-l+1}R^\flat_\beta(q) + R^\flat_\gamma(q)}{q^{k-l+1}S^\flat_\beta(q) + S^\flat_\gamma(q)}.
\]

**Proof.** Suppose that \( \alpha = [c_1, \ldots, c_l, 2^{(k-l)}] \), where \( 2^{(k-l)} \) stands for \( k-l \) copies of \( 2, \ \beta = [c_1, \ldots, c_l - 1], \ \gamma = [c_1, \ldots, c_l, 2^{(k-l-1)}] \), then

\[
R^\flat_\alpha(q) = E^\flat_k(c_1, \ldots, c_l, 2^{(k-l)})_q
= [2]_q E^\flat_{k-1}(c_1, \ldots, c_l, 2^{(k-l-1)})_q - qE^\flat_{k-2}(c_1, \ldots, c_l, 2^{(k-l-2)})_q
= (1 + q^2 + q^3)E^\flat_{k-2}(c_1, \ldots, c_l, 2^{(k-l-2)})_q - (q + q^3)E^\flat_{k-3}(c_1, \ldots, c_l, 2^{(k-l-3)})_q,
\]

and

\[
R^\flat_\gamma(q) = E^\flat_k(c_1, \ldots, c_l, 2^{(k-l)})_q
= [2]_q E^\flat_{k-2}(c_1, \ldots, c_l, 2^{(k-l-2)})_q - qE^\flat_{k-3}(c_1, \ldots, c_l, 2^{(k-l-3)})_q
= (1 + q^2)E^\flat_{k-2}(c_1, \ldots, c_l, 2^{(k-l-2)})_q - qE^\flat_{k-3}(c_1, \ldots, c_l, 2^{(k-l-3)})_q.
\]
Thus, by (2.10),
\[ R_\alpha^b(q) - R_\gamma^c(q) = E_{k+1}^b(c_1, \ldots, c_l, 2^{(k-l+1)})_q - E_k^b(c_1, \ldots, c_l, 2^{(k-l)})_q. \]

On the other hand,
\[ R_\beta^b(q) = E_i^b(c_1, \ldots, c_l - 1)_q \]
\[ = [c_l - 1]_q E_{i-1}^b(c_1, \ldots, c_{l-1})_q - q^{c_{l-1}-1}E_{i-2}^b(c_1, \ldots, c_{l-2})_q \]
\[ = ([c_l]_q - q^{c_{l-2}})E_{i-1}^b(c_1, \ldots, c_{l-1})_q - q^{c_{l-2}}E_{i-2}^b(c_1, \ldots, c_{l-2})_q \]
\[ = E_i^b(c_1, \ldots, c_l)_q - q^{c_{l-2}}E_{i-1}^b(c_1, \ldots, c_{l-1})_q. \]

Again, by (2.10), we have
\[ q^{k-l+1}R_\beta^b(q) = q^{k-l}(E_{i+1}^b(c_1, \ldots, c_l)_q - q^{c_{l-2}}E_{i-1}^b(c_1, \ldots, c_{l-1})_q) \]
\[ = q^{k-l}(E_{i+1}^b(c_1, \ldots, c_l, 2)_q - E_i^b(c_1, \ldots, c_l)_q) \]
\[ = E_{i+1}^b(c_1, \ldots, c_l, 2^{(k-l+1)})_q - E_i^b(c_1, \ldots, c_l, 2^{(k-l)})_q. \]

Hence, we proved that,
\[ R_\alpha^b(q) = q^{k-l+1}R_\beta^b(q) + R_\gamma^c(q). \]

The proof of \( S_\alpha^b(q) = q^{k-l+1}S_\beta^b(q) + S_\gamma^c(q) \) is similar. \[\square\]

**Example 3.4.**

(1) Since \( \frac{12}{5} = [3, 2, 3] \), \( \frac{7}{3} = [3, 2, 2] \), \( \frac{5}{2} = [3, 2] \), then by Theorem 3.3, it follows that
\[ \left[ \frac{12}{5} \right]^b_q = \frac{1 + 2q + 2q^2 + 2q^3 + 3q^4 + q^5 + q^6}{1 + q + q^2 + q^3 + q^4}, \]
\[ \left[ \frac{7}{3} \right]^b_q = \frac{1 + q + q^2 + 2q^3 + q^4 + q^5}{1 + q^2 + q^3}, \]
\[ \left[ \frac{5}{2} \right]^b_q = \frac{1 + q + q^2 + q^3 + q^4}{1 + q^2}, \]
\[ \left[ \frac{7}{3} \right]^b_q \#^b_q \left[ \frac{5}{2} \right]^b_q = \frac{q(1 + q + q^2 + 2q^3 + q^4 + q^5) + (1 + q + q^2 + q^3 + q^4)}{q(1 + q^2 + q^3) + (1 + q^2)} = \left[ \frac{12}{5} \right]^b_q. \]

(2) Since \( \frac{7}{2} = [4, 2, 1] \), \( \frac{3}{1} = [3] \), \( \frac{4}{1} = [4] \), then by Theorem 3.3, it follows that
\[ \left[ \frac{7}{2} \right]^b_q = \frac{1 + q + 2q^2 + q^3 + q^4 + q^5}{1 + q^2}, \]
\[ \left[ \frac{3}{1} \right]^b_q = \frac{1 + q + q^3}{1}, \]
\[ \left[ \frac{4}{1} \right]^b_q = \frac{1 + q + q^2 + q^4}{1}. \]
In this section, following [15, 2], we discuss a relationship between left $q$-deformed rational numbers and Farey tessellation (see [6] for more details). We assume that all rational numbers are represented as irreducible fractions. We order the elements of $\mathbb{Q}_{>0} \cup \{\infty\}$ by horizontal segment drawn in the plane, then Farey tessellation consists of all triangles whose forms are as in the left of Figure 1 (a rational number $\alpha$ which is the Farey sum of $\beta$ and $\gamma$ defined by (3.2) and (3.3)), and each vertex corresponds to a rational number, and any two vertices that are Farey neighbors are connected by a semicircle. We call these triangles Farey triangles, and the initial Farey triangle is on the right of Figure 1.

3.2 $q$-deformed Farey tessellation about left $q$-deformed rational numbers

In this section, following [15, 2], we discuss a relationship between left $q$-deformed rational numbers and Farey tessellation (see [6] for more details). We assume that all rational numbers are represented as irreducible fractions. We order the elements of $\mathbb{Q}_{>0} \cup \{\infty\}$ by horizontal segment drawn in the plane, then Farey tessellation consists of all triangles whose forms are as in the left of Figure 1 (a rational number $\alpha$ which is the Farey sum of $\beta$ and $\gamma$ defined by (3.2) and (3.3)), and each vertex corresponds to a rational number, and any two vertices that are Farey neighbors are connected by a semicircle. We call these triangles Farey triangles, and the initial Farey triangle is on the right of Figure 1.

The $q$-deformed Farey tessellation considered in [15] and [2] is composed of $q$-deformed Farey triangles which are obtained by basing on the original Farey triangle and each vertex is a right $q$-deformed rational number and each edge is weighted. Every $q$-deformed Farey triangle can be obtained according to the laws of Theorem 3.1. Now we replace Theorem 3.1 with Theorem 3.3 and setting the initial $q$-deformed Farey triangle as the left of Figure 2, then we can obtain a new Farey tessellation consisting of $q$-deformed Farey triangles as in the right of Figure 2. Each
vertex of a $q$-deformed Farey triangle corresponds to a left $q$-deformed rational number (as a simple example, see Figure 3).

Following [2, Section 2.2], we choose two infinitely close Farey triangle sequences from the left and right sides near the rational number $\alpha$. Considering the Farey tessellation according to the laws of Theorem 3.3, then we find that the Farey triangle sequence on the left side of $\alpha$ converges to exactly one point. However, when $q$ is not equal to 1, the one on the right side of $\alpha$ cannot converge to a point. Thus we obtain a figure with mirror symmetry to [2, Figure 5].

### 3.3 Weighted triangulation about left $q$-deformed rational numbers

Consider a positive rational number $\alpha = [a_1, \ldots, a_{2m}] = [[c_1, \ldots, c_k]]$ which is the Farey sum of $\beta$ and $\gamma$ defined by (3.2) and (3.3). According to [17], it follows that $\alpha$ corresponds to a triangulation as in Figure 1. If we give the initial values as in Figure 5, then the remaining vertices can be computed according to the Farey sum.

![Figure 4: Triangulation of $\alpha$.](image)
Figure 3: A part of the Farey tessellation with weights carried by the edges and left $q$-deformed rational numbers labeling the vertices.
Example 3.5. The triangulations of $\frac{8}{11} = [0, 1, 2, 1, 1, 1]$ and $\frac{11}{8} = [1, 2, 2, 1]$ can be expressed as Figure 6.

Now, let us consider the $q$-deformation of the triangulation which is called weighted triangulation. For the vertices and edges of the two kinds of triangles in the triangulation, we will set them with the $q$-deformed Farey sum from Theorem 3.1 in Figure 7, and the initial setting is as Figure 8 (See [15] for details).
Figure 7: The triangle set by Theorem 3.1.

Figure 8: The initial settings of our’s weighted triangulation for the cases \( \alpha > 1 \)(left) and \( 0 < \alpha \leq 1 \)(right).

Similarly, for the vertices and edges of the two kinds of triangles in the triangulation, we will set them with the \( q \)-deformed Farey sum from Theorem 3.3 in Figure 9 and the initial setting is as Figure 10. Different from the above case, the weights of the edges with endpoints \([\alpha]_q\) and \([\gamma]_q\) become 1, and the weights of the edges with endpoints \([\alpha]_q\) and \([\beta]_q\) become some power of \( q \), which is symmetric to the case of the right \( q \)-deformed rational numbers. This phenomenon also corresponding to the fact that for the \( q \)-deformed Farey sum, there is a symmetric between the case of the right \( q \)-deformed rational numbers and the case of the left \( q \)-deformed rational numbers.
Figure 9: The triangle set by Theorem 3.3

Figure 10: The initial settings of our’s weighted triangulation for the cases $\alpha > 1$ (left) and $0 < \alpha \leq 1$ (right).

**Example 3.6.** The weighted triangulations of $\left[ \frac{8}{11} \right]_q^\sharp = [0, 1, 2, 1, 1, 1]_q^\sharp = [[1, 4, 3]]_q^\sharp$ and $\left[ \frac{11}{8} \right]_q^\sharp = [1, 2, 2, 1]_q^\sharp = [[2, 2, 4]]_q^\sharp$ can be expressed as Figures 11 and 12.
Figure 11: Weighted triangulations of $\left[\frac{8}{11}\right]^\sharp_q$.

Figure 12: Weighted triangulations of $\left[\frac{11}{8}\right]^\sharp_q$.

**Example 3.7.** The weighted triangulations of $\left[\frac{8}{11}\right]^b_q = [0, 1, 2, 1, 1]^\flat_q = [[1, 4, 3]]_q^b$ and $\left[\frac{11}{8}\right]^b_q = [1, 2, 2, 1]^\flat_q = [[2, 2, 4]]_q^b$ can be expressed as Figures 13 and 14.
4 Jones polynomial and left $q$-rational numbers

Following [11, Proposition 1.2 (b)], [15, Proposition A.1] and [2, Theorem A.3], we can obtain the relationship between left and right $q$-deformed rational numbers and Jones polynomials. In this section, we give a new proof of Theorem A.3 in [2] without the homological argument.
For a rational number \( \alpha = [c_1, \ldots, c_k] \in \mathbb{Q}_{>1} \), following [2], we suppose that \( V_\alpha(q) \) is the Jones polynomial associated with the rational knot parametrized by \( \alpha \), and \( |V_\alpha(q)| \) denote the polynomial obtained by making each coefficient positive. Following [15], let \( J_\alpha(q) \) be the normalized Jones polynomial associated with the rational knot parametrized by \( \alpha \). The next lemma can be checked by [15, Proposition A.1] and Theorem 3.1.

**Lemma 4.1.** For a rational number \( \alpha = [c_1, \ldots, c_k] \) which is the Farey sum of \( \beta \) and \( \gamma \) defined by (3.2) and (3.3). Then one has

\[
J_\alpha(q) = J_\beta(q) + q^{c_k-1}J_\gamma(q).
\]

Following [2], the sequence of coefficients of the normalized Jones polynomial \( J_\alpha(q) \) is just the reverse of the sequence of coefficients of the Jones polynomial \( |V_\alpha(q)| \), then the equation

\[
|V_\alpha(q)| = R_\alpha^b(q)
\]

will be proved by showing the next theorem.

**Theorem 4.2.** For a rational \( \alpha = [c_1, \ldots, c_k] \), the normalized Jones polynomial of \( \alpha \) satisfies the following formula:

\[
J_\alpha(q) = q^{m}R_\alpha^b(q^{-1}),
\]

where \( m = \deg(R_\alpha^b(q)) = \sum_{j=1}^{k} c_j - k + 1 \).

**Proof.** For the rational \( \alpha = [c_1, \ldots, c_k] \), it is easy to check the case of \( k = 1 \) and \( k = 2 \). We consider the following induction hypothesis:

\[
J_{[c_1, \ldots, c_i]}(q) = q^{m}R_{[c_1, \ldots, c_i]}^b(q^{-1}), \quad \text{for } 1 \leq i \leq k.
\]  

(4.1)

We prove that

\[
J_{[c_1, \ldots, c_i, c_{k+1}]}(q) = q^{\deg(R_{\alpha'}^b(q))}R_{\alpha'}^b(q^{-1}),
\]

where \( \alpha' = [c_1, \ldots, c_k, c_{k+1}] \).

For the case of \( c_k = 2 \), we have

\[
\alpha' = \beta' \# \alpha,
\]

where \( \beta = \left\{ \begin{array}{l} [c_1, \ldots, c_l - 1] \quad \text{for } c_k = c_{k-1} = \cdots = c_{l+1} = 2, c_l > 2, 1 \leq l \leq k, \\ [1] \quad \text{for } k = 1, c_k = 2. \end{array} \right. \)

Suppose that

\[
m_1 = \deg(R_{\beta'}^b(q)) = \left\{ \begin{array}{ll} \sum_{j=1}^{l} c_j - l & \text{for } c_{k+1} = c_k = \cdots = c_{l+1} = 2, c_l > 2, 1 \leq l \leq k + 1, \\ 1 & \text{for } k = 1, c_k = 2, \end{array} \right.
\]

22
then by the induction hypothesis \((4.1)\), Lemma \(4.1\) and Theorem \(3.3\) it follows that

\[
J_{\alpha'}(q) = J_{\beta'}(q) + qJ_{\alpha}(q) \\
= q^{m_1} R_{\beta'}^b(q^{-1}) + q^{m+1} R_{\alpha}^b(q^{-1}) \\
= q^{m+1}(q^{-(m-m_1+1)}R_{\beta'}^b + R_{\alpha}^b(q^{-1})) \\
= q^{m+1}R_{\alpha'}^b(q^{-1}),
\]

where \(m + 1 = \sum_{j=1}^{k} c_j - k + 2 = \deg(R_{\alpha'}^b(q))\).

Now we assume \(\alpha' = [c_1, \ldots, c_k, c]\) for some \(c \geq 2\). We set the following induction hypothesis:

\[
J_{\alpha'}(q) = q^{m'} R_{\alpha'}^b(q^{-1}), \tag{4.2}
\]

where \(m' = \deg(R_{\alpha'}^b(q)) = \sum_{j=1}^{k} c_j - k + c\).

Suppose that \(\alpha'' = [c_1, \ldots, c_k, c+1]\), then we have \(\alpha'' = \alpha' \# \alpha\). By the induction hypothesis \((4.1), (4.2), \text{Lemma } 4.1 \text{ and Theorem } 3.3\) it follows that

\[
J_{\alpha''}(q) = J_{\alpha'}(q) + q^{c} J_{\alpha}(q) \\
= q^{m'} R_{\alpha'}^b(q^{-1}) + q^{m'+c} R_{\alpha}^b(q^{-1}) \\
= q^{m+c}(q^{-1}R_{\alpha'}^b + R_{\alpha}^b(q^{-1})) \\
= q^{m'+c} R_{\alpha''}^b(q^{-1}),
\]

where \(m + c = \sum_{j=1}^{k} c_j - k + 1 + c = \deg(R_{\alpha''}^b(q))\).

\[ \square \]

**Example 4.3.** For \(9_4 = [3, 2, 2, 2]\), since

\[
\deg(R_{9_4}^b(q)) = 3 + 2 + 2 + 2 - 4 + 1 = 6,
\]

and

\[
\begin{bmatrix} 9 \end{bmatrix}_q^b = \frac{1 + q + q^2 + 2q^3 + 2q^4 + q^5 + q^6}{1 + q^2 + q^3 + q^4},
\]

then by Theorem \(4.2\) we have

\[
J_{9_4}(q) = q^6 R_{9_4}^b(q^{-1}) = 1 + q + 2q^2 + 2q^3 + q^4 + q^5 + q^6.
\]
5 Relationship to 2-Calabi–Yau category associated to the $A_2$ quiver

A relation between $q$-deformed rational numbers and homology algebra is given in [2]. In this section, we first briefly recall the relevant definitions and conclusions. We derive a homological interpretation of $q$- Farey sum of $q$-deformed rational numbers by combining Theorems 3.7 and 3.8 in [2] with Theorems 3.1 and 3.3 in Section 3. In addition, we consider any continued fraction expansion of a real quadratic irrational number of purely cyclic type and give its homological interpretation.

A relation between $q$-deformed rational numbers and homology algebra is given in [2]. In this chapter, we briefly recall the relevant definitions and conclusions (c.f. [1, 2, 4, 3, 23] for more details in this chapter).

5.1 2-Calabi–Yau category associated to the $A_2$ quiver and spherical objects

We fix a field $k$ which is algebraic closed and char($k$) = 0. Consider $DA_2$ the double of $A_2$ quiver as

$$
1 \xrightarrow{2} 2,
$$

where 1 and 2 are vertices.

**Definition 5.1** (Zigzag algebra of $A_2$). The zigzag algebra $Z_2$ of $A_2$ is the quotient of the path algebra $k(DA_2)$ by the two-sided ideal generated by all length three paths.

We regard $Z_2$ as a differential graded algebra by assuming that the grading is the path length and the differential is zero. We denote the homotopy category of differential graded modules over $Z_2$ by $\mathcal{H}_2$, and denote by $D_2$ the derived category of differential graded modules over $Z_2$ (by inverting quasi-isomorphisms in the $\mathcal{H}_2$). Note that $D_2$ is not the derived category of complexes of graded $Z_2$-module as an ordinary algebra (c.f. [23, Section 4]). For $i = 1, 2$, we denote the differential graded module $Z_2(i)$ by $P_i$, where $(i)$ is the trivial path. By an ambusing notion, we will denote $P_i$ as an object of $D_2$.

**Definition 5.2.** Let $\mathcal{C}_2$ be the full triangulated subcategory of $D_2$ which is the extension closure of \{ $P_1^{\leq m_1}[s_1] \bigoplus P_2^{\leq m_2}[s_2] : m_i \in \mathbb{Z}_{\geq 0}, s_i \in \mathbb{Z}, i = 1, 2$ \}.

**Theorem 5.3** ([1, 2, 4, 3, 23]). For the $\mathcal{C}_2$, we have the following facts.

(i) $\mathcal{C}_2$ is a finite type linear triangulated category;

(ii) $\mathcal{C}_2$ is a 2-Calabi–Yau category;
(iii) $P_1$ and $P_2$ are spherical objects in the finite type $C_2$, that is for $i, j \in 1, 2$ and any integer $m$, one has

$$\text{Hom}(P_i, P_j[m]) = \begin{cases} 
\mathbb{K} & \text{for } m = 0, 2 \text{ and } i = j, \\
\mathbb{K} & \text{for } m = 1 \text{ and } i \neq j, \\
0 & \text{for otherwise.}
\end{cases}$$

There is a unique morphism $\varphi_{12} : P_1[-1] \to P_2$, and also $\varphi_{21} : P_2[-1] \to P_1$, and we denote the cones of $\varphi_{12}$ and $\varphi_{21}$ by $P_{12}$ and $P_{21}$, respectively. We note that $P_1$, $P_2$, $P_{12}$ and $P_{21}$ are indecomposable spherical objects of $C_2$.

The extension closure of $P_1$ and $P_2$ is a heart of a bounded $t$-structure of $C_2$ (c.f. [7, Section 10.1], for associated definitions). We call this heart is a standard heart, and denote it by $\heartsuit_{\text{std}}$. Note that $P_1$ and $P_2$ are simple objects in $\heartsuit_{\text{std}}$, and $\heartsuit_{\text{std}}$ is a module category of the preprojective algebra of type $A_2$. We have a exact sequence

$$0 \to P_1 \to P_{21} \to P_2 \to 0.$$ 

5.2 Spherical twist functors on $C_2$

**Definition 5.4** (Spherical twist functors on $C_2$ [23]). Let $X$ is a spherical object in $C_2$. The spherical twist functor along $X$ is an autoequivalence on $C_2$ as follows.

$$\sigma_{P_i}(X) := \text{Cone} \left( \text{hom}^\bullet(P_i, X) \otimes_k P_i \overset{\text{ev}}{\to} X \right)$$

where $\text{hom}^\bullet(P_i, X)$ is a complex which for $k \in \mathbb{Z}$ the $k$-th term is defined by

$$\text{hom}^k(P_i, X) := \bigoplus_{j \in \mathbb{Z}} \text{Hom}(P_i[j], X[j - k]).$$

We also have a inverse of $\sigma_{P_i}$ which is defined as

$$\sigma_{P_i}^{-1}(X) := \text{Cone} \left( X \overset{\text{ev}}{\to} \text{Lin}^\bullet(\text{hom}^\bullet(P_i, X), P_i) \right)[-1].$$

Let $S_{\text{D}}$ be the set of spherical objects of $C_2$.

**Proposition 5.5** (Braid relation [23]). The group of spherical twist functor which generated by $\sigma_{P_1}$, $\sigma_{P_2}$ is isomorphic to $B_3$. Moreover for any $X \in S$ and for any spherical twist functors $\sigma$ on $C_2$, one has $\sigma(X) \in S_{\text{D}}$.

Henceforth, without causing confusion, we simply denote $\sigma_{P_1}$ and $\sigma_{P_2}$ as $\sigma_1$ and $\sigma_2$, respectively.

For an irreducible fraction $\alpha = [a_1, a_2, \ldots, a_{2m}]$, let the spherical object corresponding to $\alpha$ be

$$X_\alpha := \sigma_{a_1}^{-1} \sigma_{a_2} \cdots \sigma_{a_{2m-1}}^{-1} \sigma_{a_{2m}} P_1.$$  (5.1)

By Proposition 5.5, $X_\alpha$ belongs to $S_{\text{D}}$.  

25
5.3 Bridgeland stability conditions

Definition 5.6 ([1]). A stability condition on a triangulated category \( T \) is specified by two compatible structures \( \tau := (P, Z) \), where \( P \) is a slicing which consists of abelian subcategories \( P(\phi) \) of \( T \) for each real number \( \phi \), and \( Z \) is called the central charge which is a homomorphism of additive groups from the Grothendieck group \( K_0(T) \) to the complex numbers \( \mathbb{C} \). The slicing and the central charge satisfy the following conditions:

(i) \( P(\phi + 1) = P(\phi)[1] \);

(ii) For \( \phi > \psi \), if \( X \in \text{Ob} P(\phi) \), \( Y \in \text{Ob} P(\psi) \), then \( \text{Hom}(X, Y) = 0 \);

(iii) For any nonzero object \( X \) in \( T \), there is a Harder–Narasimhan (HN) filtration of \( X \), where each triangle in Figure 15 is exact and each \( Y_i \in P(\phi_i) \) for \( \phi_1 > \phi_2 > \cdots > \phi_n \);

\[
0 = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = X
\]

\[
Y_1 \quad Y_2 \quad \cdots \quad Y_{n-1} \quad Y_n
\]

Figure 15: The Harder–Narasimhan filtrations of \( X \)

(iv) For any \( X \in \text{Ob} P(\phi) \), there is some positive real number \( m \) such that

\[
Z([X]) = me^{i\pi \phi}.
\]

Definition 5.7 ([2]). A stability condition on \( \mathbb{C}^2 \) is standard if there is a positive real number \( \phi \), such that \( P([\phi, \phi + 1]) = \mathcal{O}_{\text{std}} \). More specifically, \( P([\phi, \phi + 1]) \) is formed by \( P_1 \) and \( P_2 \) on the spot.

A standard stability condition \( \tau \) gives rise to a special filtration of \( X_\alpha \) (see Figure 16),

\[
0 = (X_\alpha)_0 \rightarrow (X_\alpha)_1 \rightarrow (X_\alpha)_2 \rightarrow \cdots \rightarrow (X_\alpha)_{n-1} \rightarrow (X_\alpha)_n = X_\alpha
\]

\[
Y_1 \quad Y_2 \quad \cdots \quad Y_{n-1} \quad Y_n
\]

Figure 16: The Harder–Narasimhan filtrations of \( X_\alpha \)

where each subquotient (called \( \tau \)-HN-factor) \( Y_j \) is isomorphic to \( P_{\nu}[i] \) \( (j = 0, \ldots, n, i \in \mathbb{Z}, \nu \in \{1, 2, 12, 21\}) \).
5.4 Two kinds of functional $\text{occ}_q$ and $\overline{\text{hom}}_q$

Fix a standard stability condition $\tau$, let $[P_i, P_j]$ denote the set of all objects of $C_2$ whose $\tau$-Harder-Narasimhan filtration factors are shifts of either $P_i$ or $P_j$ where $i, j = 1, 2, 12, 21$ and $i \neq j$. By [2, Proposition 4.3], each spherical object $X \in S_0$ must belong to one of $[P_2, P_{21}], [P_{21}, P_1], [P_1, P_{12}], [P_{12}, P_2]$. Since we only consider the case $\alpha \in \mathbb{Q} \cap (0, \infty)$, then by [2, Figure 6 in Section 4.3], it must have $X \in [P_2, P_{21}] \cup [P_{21}, P_1]$.

**Definition 5.8** (Counting functional). The counting functional $\pi_\nu(X)$ of $\tau$-HN-factors is defined by an element of the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ as follows.

$$\pi_\nu(X) := \sum_{i \in \mathbb{Z}} \eta_\nu q^i,$$

where $\nu \in \{1, 2, 12, 21\}$ and $\eta_\nu$ is the number of $\tau$-HN-factors which are isomorphic to $P_\nu$.

**Definition 5.9** ([2, $\text{occ}_q$ and $\overline{\text{hom}}_q$]). For any $X, Y \in S_0$, there are two kinds of functionals $S_0 \times S_0 \to \mathbb{Z}[q^\pm]$ denoted by $\text{occ}_q$ and $\overline{\text{hom}}_q$, respectively, which are defined as follows.

$$\text{occ}_q(P_1, X) := \pi_2(X) + \pi_{12}(X) + \pi_{21}(X),$$

$$\text{occ}_q(P_2, X) := \pi_1(X) + \pi_{12}(X) + \pi_{21}(X),$$

$$\overline{\text{hom}}_q(X, Y) := \begin{cases} q^n(q^{-2} - q^{-1}) & \text{if } Y \cong X[n], \\ \sum_{n \in \mathbb{Z}} \dim_k \text{Hom}(X, Y[n])q^{-n} & \text{otherwise}. \end{cases}$$

Then we have the next theorem which gives a relationship between the $q$-deformed rational numbers and homological algebra.

**Theorem 5.10** (Bapat, Becker and Licata [2, A part of Theorems 4.7 and 4.8]). Consider a non-negative rational number $\alpha = [a_1, a_2, \ldots, a_{2m}]$. Suppose that

$$X_\alpha = \sigma_1^{-a_1} \sigma_2^{a_2} \cdots \sigma_1^{-a_{2m-1}} \sigma_2^{a_{2m}} P_1,$$

then we have

$$[\alpha]^+_q = \frac{\text{occ}_q(P_2, X_\alpha)}{\text{occ}_q(P_1, X_\alpha)},$$

and

$$[\alpha]^-_q = \frac{\overline{\text{hom}}_q(X_\alpha, P_2)}{q \overline{\text{hom}}_q(X_\alpha, P_1)}.$$
5.5 A homological interpretation of the \( q \)-deformed Farey sum

Consider a positive rational number \( \alpha = [a_1, a_2, \ldots, a_{2m}] = [[c_1, \ldots, c_k]] \). Since Theorems 3.1 and 3.3 are based on \([[[c_1, \ldots, c_k]]]\), we first make a simple formal transformation of (5.1). By direct computation, \( S \in \text{Section 2} \) is represented by \( S = \sigma_1 \sigma_2 \sigma_1 \) in \( \text{PSL}_2(\mathbb{Z}) \), and then we have

\[
\sigma_1^{-a_1} \sigma_2^{a_2} \cdots \sigma_1^{-a_{2m-1}} \sigma_2^{a_{2m}} = \sigma_1^{-c_1} \sigma_2^{-c_2} \cdots \sigma_1^{-c_k} \sigma_2^{-1} = \sigma_1^{-c_1+1} \sigma_2^{-c_2+2} \sigma_1^{-c_3+2} \sigma_2 \cdots \sigma_1^{-c_k+2} \sigma_2,
\]

in \( \text{PSL}_2(\mathbb{Z}) \). Thus, we also have

\[
\sigma_1^{-a_1} \sigma_2^{a_2} \cdots \sigma_1^{-a_{2m-1}} \sigma_2^{a_{2m}} = \sigma_1^{-c_1+1} \sigma_2^{-c_2+2} \sigma_1^{-c_3+2} \sigma_2 \cdots \sigma_1^{-c_k+2} \sigma_2.
\]

in \( B_3 \). Hence, we have

\[
X_{\alpha} = \sigma_1^{-c_1+1} \sigma_2^{-c_2+2} \sigma_1^{-c_3+2} \sigma_2 \cdots \sigma_1^{-c_k+2} \sigma_2 P_1.
\]

(5.2)

If \( \alpha \) is the Farey sum of \( \beta \) and \( \gamma \) defined by (3.2) and (3.3), then by applying the above transformation to \( \beta \) and \( \gamma \), it follows that

\[
X_{\beta} = \begin{cases} 
\sigma_1^{-c_1+1} \sigma_2 \sigma_1^{-c_2+2} \sigma_2 \sigma_1^{-c_3+2} \sigma_2 \cdots \sigma_1^{-c_k+2} \sigma_2 \sigma_1^{-c_l+3} \sigma_2 P_1 & \text{for } c_k = c_{k-1} = \cdots = c_{l+1} = 2, c_l > 2, 1 \leq l \leq k, \\
\sigma_2 P_1 & \text{for } k = 1, c_k = 2, \\
P_2 & \text{for } c_k = c_{k-1} = \cdots = c_2 = 2, c_1 = 1,
\end{cases}
\]

(5.3)

and

\[
X_{\gamma} = \begin{cases} 
\sigma_1^{-c_1+1} \sigma_2 \sigma_1^{-c_2+2} \sigma_2 \sigma_1^{-c_3+2} \sigma_2 \cdots \sigma_1^{-c_k+2} \sigma_2 P_1 & \text{for } k \geq 2, \\
P_1 & \text{for } k = 1.
\end{cases}
\]

(5.4)

Note that \( \text{occ}_q(P_2, X_{\alpha}) \) and \( \text{occ}_q(P_1, X_{\alpha}) \) are not the numerator and denominator of the right \( q \)-deformed rational number, respectively. In fact, they differ by some power of \( q \) (c.f. [2, The proof of Theorems 4.7 and 4.8]). Hence, we first determine the power of \( q \) as follows.

**Theorem 5.11.** For the right \( q \)-deformed rational number \([\alpha]^*_q = \frac{R^*_q(\alpha)}{S^*_q(\alpha)} \), we have

\[
R^*_q(\alpha) = q^{k-1} \text{occ}_q(P_2, X_{\alpha}), \quad S^*_q(\alpha) = q^{k-1} \text{occ}_q(P_1, X_{\alpha}).
\]

**Proof.** For a polynomial \( f(q) \in \mathbb{Z}[q^{-1}, q] \), let \( md(f(q)) := \min \left\{ i : f(q) = \sum \rho_i q^i, \rho_i \neq 0 \right\} \). One has \( md(R^*_q(\alpha)) = 0 \). We consider the \( \tau \)-HN-factor \( Y_j \) of \( X_{\alpha} \), and let

\[
\iota(X_{\alpha}) := \min \left\{ i : P_\nu[i] \cong Y_j, \nu = 1, 2, 12, 21 \right\}.
\]

28
Then by the definition of $\text{occ}_q$, we have $md(\text{occ}_q(P_2, X_\alpha)) = md(\text{occ}_q(P_1, X_\alpha)) = \iota(X_\alpha)$. It can be checked that for $n_1, n_2, l_1, l_2 \in \mathbb{Z}_{\geq 0}$,

\[
\iota(P_1) = 0, \quad \iota(\sigma_2 P_1) = 0, \quad \iota(\sigma_1 \sigma_2 P_1) = n_1 - 1, \quad \iota(\sigma_1^l \sigma_2 P_1) = n_1 - 1,
\]

\[
\iota(\sigma_2 \sigma_1^l \sigma_2 P_1) = n_1 + n_2 - 1, \quad \iota(\sigma_1 \sigma_2 \sigma_1^l \sigma_2 P_1) = n_1 + n_2 - 1,
\]

by direct computation. For (5.2), repete the above computation, we have $\iota(X_\alpha) = k - 1$. Hence we have

\[
\mathcal{R}_\alpha^k(q) = q^{k-1} \text{occ}_q(P_2, X_\alpha), \quad \mathcal{S}_\alpha^k(q) = q^{k-1} \text{occ}_q(P_1, X_\alpha).
\]

\[
\square
\]

Similarly, we note that $\overline{\text{hom}}_q(X_\alpha, P_2)$ and $\overline{\text{hom}}_q(X_\alpha, P_1)$ are not the numerator and denominator of the right $q$-deformed rational number, respectively. However, we have

**Theorem 5.12.** For the left $q$-deformed rational number $[\alpha]^\flat = \frac{\mathcal{R}_\alpha^k(q)}{\mathcal{S}_\alpha^k(q)}$, we have

\[
\mathcal{R}_\alpha^k(q) = q^{\sum_{j=1}^e (c_j-2)+1} \overline{\text{hom}}_q(X_\alpha, P_2), \quad \mathcal{S}_\alpha^k(q) = q^{\sum_{j=1}^e (c_j-2)+2} \overline{\text{hom}}_q(X_\alpha, P_1).
\]

**Proof.** It can be proved by a similar argument of Theorem 5.11 and consider

\[
\iota'(X_\alpha) := \max \{i : P_\nu[i] \cong Y_\nu, \nu = 1, 2, 12, 21\}
\]

and the definition of $\overline{\text{hom}}_q$.

\[
\square
\]

For a rational number $\alpha = [\lfloor c_1, \ldots, c_k \rfloor]$ which is the Farey sum of $\beta$ and $\gamma$ defined by (3.2) and (3.3), if we consider the spherical objects corresponding $\alpha, \beta, \gamma$, then by Theorems 3.1 and 3.3, we have the following facts.

**Proposition 5.13.** Considering the spherical objects $X_\alpha, X_\beta$ and $X_\gamma$ in $C_2$, for $i = 1, 2$, we have the following formula.

(i) If $\alpha \in \mathbb{Q}_{>1} \setminus \mathbb{Z}_{>1}$, then

\[
\text{occ}_q(P_i, X_\alpha) = q^{l-k} \text{occ}_q(P_i, X_\beta) + q^{c_k-2} \text{occ}_q(P_i, X_\gamma),
\]

\[
\overline{\text{hom}}_q(X_\alpha, P_i) = q^{2(k-l)-\sum_{j=t+1}^k c_j} \overline{\text{hom}}_q(X_\beta, P_i) + q^{-c_k+2} \overline{\text{hom}}_q(X_\gamma, P_i);
\]

(ii) If $\alpha \in \mathbb{Z}_{>1}$ (i.e. $l = k = 1$, and $X_\gamma = P_1$), then

\[
\text{occ}_q(P_i, X_\alpha) = \text{occ}_q(P_i, X_\beta) + q^{c_l-1} \text{occ}_q(P_i, P_1),
\]

\[
\overline{\text{hom}}_q(X_\alpha, P_i) = \overline{\text{hom}}_q(X_\beta, P_i) + q^{-\sum_{j=1}^k (c_j - 2) + 1} \overline{\text{hom}}_q(P_1, P_i);
\]

29
(iii) If \( \alpha \in (\mathbb{Q} \cap (0,1)) \setminus \mathbb{Z}_{>1} \) (where \( \mathbb{Z}_{>1}^{-1} := \{ \frac{1}{n} : n \in \mathbb{Z}_{>1} \} \), then

\[
\text{occ}_q(P_i, \alpha) = q^{l-k} \text{occ}_q(P_i, \alpha) + q^{\alpha - 2} \text{occ}_q(P_i, \gamma),
\]

\[
\text{occ}_q(X_\alpha, P_i) = q^{2(k-l+1)} \text{occ}_q(X_\beta, P_i) + q^{-\alpha + 2} \text{occ}_q(X_\gamma, P_i);
\]

(iv) If \( \alpha \in \mathbb{Z}_{>1}^{-1} \) (i.e. \( X_\beta = P_2 \) and \( c_k - 2 = 0 \)), then

\[
\text{occ}_q(P_i, \alpha) = q^{-(k-1)} \text{occ}_q(P_i, P_2) + \text{occ}_q(P_i, X_\gamma),
\]

\[
\text{occ}_q(X_\alpha, P_i) = q^{-\sum_{j=l+1}^k c_j + 3k} \text{occ}_q(P_2, P_i) + \text{occ}_q(X_\gamma, P_i).
\]

By Corollary 5.13, we can obtain a formal Farey sum of the spherical objects in \( \mathcal{C}_2 \).

**Proposition 5.14.** Considering the spherical objects \( X_\alpha, X_\beta \) and \( X_\gamma \) in \( \mathcal{C}_2 \), for \( i = 1, 2 \), we have the following formula.

(i) If \( \alpha \in \mathbb{Q}_{>1} \setminus \mathbb{Z}_{>1} \), then

\[
X_\alpha = X_\beta[l - k] \oplus X_\gamma[c_k - 2];
\]

(ii) If \( \alpha \in \mathbb{Z}_{>1} \) (i.e. \( l = k = 1 \), and \( X_\gamma = P_1 \)), then

\[
X_\alpha = X_\beta \oplus P_1[c_1 - 1];
\]

(iii) If \( \alpha \in (\mathbb{Q} \cap (0,1)) \setminus \mathbb{Z}_{>1}^{-1} \) (where \( \mathbb{Z}_{>1}^{-1} := \{ \frac{1}{n} : n \in \mathbb{Z}_{>1} \} \), then

\[
X_\alpha = X_\beta[l - k - 1] \oplus X_\gamma[c_k - 2];
\]

(iv) If \( \alpha \in \mathbb{Z}_{>1}^{-1} \) (i.e. \( X_\beta = P_2 \) and \( c_k - 2 = 0 \)), then

\[
X_\alpha = P_2[-(k - 1)] \oplus X_\gamma.
\]

### 5.6 Real quadratic irrationals with periodic type

Let \( x > 1 \) be a real quadratic irrational number. Since \( [x]_q \) can be written as \( [x]_q = \frac{\mathcal{R} + \sqrt{\mathcal{P}}}{\mathcal{S}} \) where \( \mathcal{R}, \mathcal{P}, \mathcal{S} \in \mathbb{Z}[q] \), then we have the following conclusions which give a homological interpretation of \( [x]_q \).

**Theorem 5.15.** Let \( x = [[c_1, \ldots, c_k, c_1, \ldots, c_k, \ldots]] > 1 \) be a real quadratic irrational number which the continued fraction expansion is purely periodic type. Suppose that \( \alpha = [[c_1, \ldots, c_k]] \), and \( \gamma = [[c_1, \ldots, c_{k-1}]] \). Considering two spherical objects \( X_\alpha \) and \( X_\gamma \) in \( \mathcal{C}_2 \) which are given by (5.2) and (5.4). Then, we have

30
Proof. We only prove (1), and the (2) can be prove by the same argument. By Proposition 2.7, we have

\[ [x]_q = \frac{A_1 + A_2 + \sqrt{(A_1 - A_2)^2 - 4q^{3k+2}c_i}}{B}, \]  

(5.5)

\[ A_1 = \text{occ}_q(P_2, X_\alpha), \quad A_2 = q^{c_k} \text{occ}_q(P_1, X_\gamma), \quad B = 2 \text{occ}_q(P_1, X_\alpha), \]

and

\[ [x]_q = \frac{A'_1 + A'_2 + \sqrt{(A'_1 - A'_2)^2 - 4c(q)}}{B'}, \]  

(5.6)

where

\[ A'_1 = (q - 1)\text{hom}_q(P_1, X_\alpha) + q\text{hom}_q(P_2, X_\alpha), \]
\[ A'_2 = q^{c_k-1}(\text{hom}_q(P_1, X_\gamma) + (1 - q)\text{hom}_q(P_2, X_\gamma)), \]
\[ B' = 2q(\text{hom}_q(P_1, X_\alpha) + (1 - q)\text{hom}_q(P_2, X_\alpha)), \]
\[ c(q) = q^{\sum_{i=1}^{k}c_i-3k+4} - 2q^{\sum_{i=1}^{k}c_i-3k+3} + 3q^{\sum_{i=1}^{k}c_i-3k+2} - 2q^{\sum_{i=1}^{k}c_i-3k+1} + q^{\sum_{i=1}^{k}c_i-3k}; \]

(2) In particular, if \( x = \frac{c_1 + \sqrt{c_1^2 - 4}}{2} \) \((c \geq 3)\), then

\[ [x]_q = \frac{\text{occ}_q(P_2, X_\alpha) + \sqrt{(\text{occ}_q(P_2, X_\alpha)^2 - 4q^{c_1-1}}}{2 \text{occ}_q(P_1, X_\alpha)\]  

(5.7)

By [12] Proposition 4.3, since \([x]_q\) can be written as

\[ [x]_q = \frac{R + \sqrt{P}}{S}, \]  

with

\[ R = E_k^\pm(c_1, \ldots, c_k)_q, \quad S = E_{k-1}^\pm(c_2, \ldots, c_k)_q, \]
\[ P = (E_k^\pm(c_1, \ldots, c_k)_q - q^{c_k-1}E_{k-2}^\pm(c_2, \ldots, c_k)_q)^2 - 4q^{\sum_{i=1}^{k}(c_i-1)}, \]
\[ S = 2E_{k-1}^\pm(c_2, \ldots, c_k)_q, \]

then, by a simple substitution, (5.5) is proved.
On the other hand, by \cite[Lemma 4.13]{2}, we can know that the relationship between the occ\_q and hom\_q as follows.

\[
\text{hom}_q(P_1, X) = q^{-1}\text{occ}_q(P_1, X) + (1 - q^{-1})\text{occ}_q(P_2, X),
\]

\[
\text{hom}_q(P_2, X) = (q^{-2} - q^{-1})\text{occ}_q(P_1, X) + q^{-1}\text{occ}_q(P_2, X).
\]

By solving the above two equations on occ\_q, we can obtain the following two equations.

\[
\text{occ}_q(P_1, X) = \frac{q\text{hom}_q(P_1, X)}{q + q^{-1} - 1} + \frac{(q - q^2)\text{hom}_q(P_2, X)}{q + q^{-1} - 1},
\]

\[
\text{occ}_q(P_2, X) = \frac{(q - 1)\text{hom}_q(P_1, X)}{q + q^{-1} - 1} + \frac{q\text{hom}_q(P_2, X)}{q + q^{-1} - 1}.
\]

Finally, we substitute these two equations into (5.5) to obtain (5.6).

\[\square\]

\textbf{Remark 5.16.} For (2) of the above Theorem, if we take \(c_1 = 3\), then we have

\[
X_\alpha = P_{21} \oplus P_{11} \oplus P_{12}
\]

hence, we get the case of the golden number as follows.

\[
\left[1 + \frac{\sqrt{5}}{2}\right]_q = q^{-1}\left(\text{occ}_q(P_2, X_\alpha) + \sqrt{\text{occ}_q(P_2, X_\alpha)^2 - 4q^{c_1-1} - 2\text{occ}_q(P_1, X_\alpha)}\right) \frac{2\text{occ}_q(P_1, X_\alpha)}{2\text{occ}_q(P_1, X_\alpha)}.
\]

\[
(5.8)
\]

\textbf{Acknowledgments}

I am greatly indebted to Professor Asilata Bapat for many useful discussions. I also grateful to Professor Akishi Ikeda for providing a notes about Stability conditions on triangulated categories, which helped me learn some basic concepts about it. I am greatly indebted to Professor Michihisa Wakui for many useful disscussions and for the guidance. I thank Professors Takeyoshi Kogiso and Kengo Miyamoto for giving me the opportunity to start this work.

\textbf{References}

[1] Bridgeland, T.: Stability conditions on triangulated categories, Ann. of Math. (2) 166 (2007), no. 2, 317–345.

[2] Bapat, A., Becker, L., Licata, A. M.: \(q\)-deformed rational numbers and the 2-Calabi–Yau category of type \(A_2\), Forum Math. Sigma11(2023), Paper No. e47, 41 pp.

[3] Bapat, A., Deopurkar, A., L., Licata, A. M.: Spherical objects and stability conditions on 2-Calabi–Yau quiver categories, Mathematische Zeitschrift (2023) 303:13.
[4] Bapat, A., Deopurkar, A., L., Licata, A. M.: A Thurston compactification of the space of stability conditions, arXiv:2011.07908, 2022.

[5] Fok, V. V.; Chekhov, L. O.: Quantum Teichmüller spaces, Theoret. and Math. Phys. 120 (1999), 1245–1259.

[6] Hardy, G. H., and Wright, E. M.: An introduction to the theory of numbers, Sixth edition. Revised by D. R. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles. Oxford University Press, Oxford, 2008. xxii+621 pp.

[7] Kashiwara, M., Schapira, P.: Sheaves on Manifolds, Grundlehren der Mathematischen Wissenschaften, vol. 292. Springer-Verlag, Berlin (1990).

[8] Kogiso, T.: $q$-deformations and $t$-deformations of Markov triples, arXiv:2008.12913, 2022.

[9] Kogiso, T., Wakui, M.: A bridge between Conway-Coxeter friezes and rational tangles through the Kauffman bracket polynomials, J. Knot Theory Ramifications 28 (2019), no. 14, 1950083, 40 pp.

[10] Labbé, S., and Lapointe, M.: The $q$-analog of the Markoff injectivity conjecture over the language of a balanced sequence, Comb. Theory 2 (2022), no. 1, Paper No. 9, 25 pp.

[11] Lee, K., Schiffler, R.: Cluster algebras and Jones polynomials, Selecta Math. (N.S.) 25 (2019), no. 4, Paper No. 58, 41 pp.

[12] Leclerc, L., Morier-Genoud, S.: The $q$-deformations in the modular group and of the real quadratic irrational numbers, Adv. in Appl. Math. 130 (2021), Paper No. 102223, 28 pp.

[13] Leclerc, L., Morier-Genoud, S., Ovsienko, V., Veselov, A.: On radius of convergence of $q$-deformed real numbers, Mosc. Math. J.24(2024), no.1, 1–19.

[14] McConville, T., Sagan, B. E., and Smyth, C. On a rank-unimodality conjecture of Morier-Genoud and Ovsienko, Discrete Math. 344 (2021), no. 8, Paper No. 112483, 13 pp.

[15] Morier-Genoud, S., Ovsienko, V.: $q$-deformed rationals and $q$-continued fractions, Forum Math. Sigma 8 (2020), Paper No. e13, 55 pp.

[16] Morier-Genoud, S., Ovsienko, V.: On $q$-deformed real numbers, Exp. Math. 31 (2022), no. 2, 652–660.

[17] Morier-Genoud, S., Ovsienko, V.: Farey boat: continued fractions and triangulations, modular group and polygon dissections, Jahresber. Dtsch. Math.-Ver. 121 (2019), no. 2, 91–136.

[18] Nagai, W., Terashima, Y.: Cluster variables, ancestral triangles and Alexander polynomials, Adv. Math. 363 (2020), 106965, 37 pp.

[19] Oguz, E. K.: Oriented posets and rank matrices, arXiv:2206.05517, 2022.
[20] Oguz, E. K., Ravichandran, M.: Rank polynomials of fence posets are unimodal, Discrete Math. 346 (2023), no. 2, Paper No. 113218, 20 pp.

[21] Ovsienko, V.: Towards quantized complex numbers: $q$-deformed gaussian integers and the Picard group, Open Communications in Nonlinear Mathematical Physics Vol. 1 (2021) pp 73–93.

[22] Ren, X.: On radiiuses of convergence of $q$-metallic numbers and related $q$-rational numbers, Res. Number Theory 8 (2022), no. 3, Paper No. 37, 14 pp.

[23] Seidel, P., Thomas, R.: Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001), no. 1, 37–108.

Xin Ren: Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan.

E-mail address: (1) ren.xin.sci@osaka-u.ac.jp; (2) xinren1213@gmail.com.