MULTIPLICATIVE EXCELLENT FAMILIES OF ELLIPTIC SURFACES OF TYPE E\textsubscript{7} OR E\textsubscript{8}

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Abstract. We describe explicit multiplicative excellent families of rational elliptic surfaces with Galois group isomorphic to the Weyl group of the root lattices E\textsubscript{7} or E\textsubscript{8}. The Weierstrass coefficients of each family are related by an invertible polynomial transformation to the generators of the multiplicative invariant ring of the associated Weyl group, given by the fundamental characters of the corresponding Lie group. As an application, we give examples of elliptic surfaces with multiplicative reduction and all sections defined over \(\mathbb{Q}\) for most of the entries of fiber configurations and Mordell-Weil lattices in [OS], as well as examples of explicit polynomials with Galois group \(W(E_7)\) or \(W(E_8)\).

1. Introduction

Given an elliptic curve \(E\) over a field \(K\), the determination of its Mordell-Weil group is a fundamental question in algebraic geometry and number theory. When \(K = k(t)\) is a rational function field in one variable, this question becomes a geometrical question of understanding sections of an elliptic surface with section. Lattice theoretic methods to attack this problem were described in [Sh1]. In particular, when \(E \to \mathbb{P}^1\) is a rational elliptic surface given as a minimal proper model of

\[
y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)
\]

with \(a_i(t) \in k[t]\) of degree at most \(i\), the possible configurations (types) of bad fibers and Mordell-Weil groups were analyzed by Oguiso and Shioda [OS].

In [Sh2], the second author studied sections for some families of elliptic surfaces with an additive fiber, by means of the specialization map, and obtained a relation between the coefficients of the Weierstrass equation and the fundamental invariants of the corresponding Weyl groups. This was expanded in [SU], which studied families with a bad fiber of additive reduction more exhaustively. The formal notion of an excellent family was defined (see the next section), and the authors found excellent families for many of the “admissible” types.

The analysis of rational elliptic surfaces of high Mordell-Weil rank, but with a fiber of multiplicative reduction, is much more challenging. However, understanding this situation is arguably more fundamental, since if we write down a “random” elliptic surface, then with probability close to 1 it will have Mordell-Weil lattice \(E_8\) and twelve nodal fibers (i.e. of multiplicative reduction). To be more precise, if we choose Weierstrass coefficients \(a_i(t)\) of degree \(i\), with coefficients chosen uniformly at

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random from among rational numbers (say) of height at most \( N \), then as \( N \to \infty \) the surface will satisfy the above condition with probability approaching 1. One can make a similar statement for rational elliptic surfaces chosen to have Mordell-Weil lattice \( E_7^* \), \( E_8^* \), etc.

In a recent work [Sh3], this study was carried out for elliptic surfaces with a fiber of type I_3 and Mordell-Weil lattice isometric to \( E_8^* \), through a “multiplicative excellent family” of type \( E_6 \). We will describe this case briefly in Section 3. The main result of this paper shows that two explicitly described families of rational elliptic surfaces with Mordell-Weil lattices \( E_7^* \) or \( E_8^* \) are multiplicative excellent. The proof involves a surprising connection with representation theory of the corresponding Lie groups, and in particular, their fundamental characters. In particular, we deduce that the Weierstrass coefficients give another natural set of generators for the multiplicative invariants of the respective Weyl groups, as a polynomial ring. We note that similar formulas were derived by Eguchi and Sakai [ES] using calculations from string theory and mirror symmetry.

The idea of an excellent family is quite useful and important in number theory. An excellent family of algebraic varieties leads to a Galois extension \( F(\mu)/F(\lambda) \) of two purely transcendental extensions of a number field \( F \) (say \( \mathbb{Q} \)), with Galois group a desired finite group \( G \). This setup has an immediate number theoretic application, since one may specialize the parameters \( \lambda \) and apply Hilbert’s irreducibility theorem to obtain Galois extensions over \( \mathbb{Q} \) with the same Galois group. Furthermore, we can make the construction effective if appropriate properties of the group \( G \) are known (see examples 8 and 19 for the case \( G = W(E_7) \) or \( W(E_8) \)). At the same time, an excellent family will give rise to a split situation very easily, by specializing the parameters \( \mu \) instead. For examples, in the situation considered in our paper, we obtain elliptic curves over \( \mathbb{Q}(t) \) with Mordell-Weil rank 7 or 8 together with explicit generators for the Mordell-Weil group (see examples 7 and 18). There are also applications to geometric specialization or degeneration of the family. Therefore, it is desirable (but quite nontrivial) to construct explicit excellent families of algebraic varieties. Such a situation is quite rare in general: theoretically, any finite reflection group is a candidate, but it is not generally neatly related to an algebraic geometric family. Hilbert treated the case of the symmetric group \( S_n \), corresponding to families of zero-dimensional varieties. Not many examples were known before the (additive) excellent families for the Weyl groups of the exceptional Lie groups \( E_6 \), \( E_7 \) and \( E_8 \) were given in [Sh2], using the theory of Mordell-Weil lattices. Here, we finish the story for the multiplicative excellent families for these Weyl groups.

2. Mordell-Weil lattices and excellent families

Let \( X \to \mathbb{P}^1 \) be an elliptic surface with section \( \sigma : \mathbb{P}^1 \to X \), i.e. a proper relatively minimal model of its generic fiber, which is an elliptic curve. We denote the image of \( \sigma \) by \( O \), which we take to be the zero section of the Néron model. We let \( F \) be the class of a fiber in \( \text{Pic}(X) \cong \text{NS}(X) \), and let the reducible fibers of \( \pi \) lie over \( \nu_1, \ldots, \nu_k \in \mathbb{P}^1 \). The non-identity components of \( \pi^{-1}(\nu_i) \) give rise to a sublattice \( T_i \) of \( \text{NS}(X) \), which is (the negative of) a root lattice (see [Ko] [11]). The trivial lattice \( T \) is \( \mathbb{Z}O \oplus \mathbb{Z}F \oplus (\oplus T_i) \), and we have the isomorphism \( \text{MW}(X/\mathbb{P}^1) \cong \text{NS}(X)/T \), which describes the Mordell-Weil group. In fact, one can induce a positive definite pairing on the Mordell-Weil group modulo torsion, by inducing it from the negative of the intersection pairing on \( \text{NS}(X) \). We refer the reader to [Sh1] for more details. In this paper, we will call \( \oplus T_i \) the fibral lattice.

Next, we recall the notion of an excellent family with Galois group \( G \) from [SU]. Suppose \( X \to \mathbb{A}^n \) is a family of algebraic varieties, varying with respect to \( n \) parameters \( \lambda_1, \ldots, \lambda_n \). The generic member of this family \( X_\lambda \) is a variety over the rational function field \( k_0 = \mathbb{Q}(\lambda) \). Let \( k = k_0 \) be the algebraic closure, and suppose that \( \mathcal{C}(X_\lambda) \) is a group of algebraic cycles on \( X_\lambda \) over the field \( k \) (in
We now define the notion of a multiplicative excellent family of algebraic varieties, varying with respect to an excellent family with Galois group $G$. As an example, for $G = \mathbb{A}^n$ over $\mathbb{Q}$, we let $k_0 = \mathbb{Q}(\lambda)$, where $\lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30})$. It is shown in [Sh2] that this is an excellent family with Galois group $G$. The $p_i$ are related to the fundamental invariants of the Weyl group of $E_8$, as is suggested by their degrees (subscripts).

We now define the notion of a multiplicative excellent family for a group $G$. As before, $X \rightarrow \mathbb{A}^n$ is a family of algebraic cycles on $\mathbb{A}^n$, isomorphic (via a fixed isomorphism) to a fixed abelian group $M$. The fields $k_0$ and $k$ are as before, and we have a Galois representation

$$\rho_{\lambda} : Gal(k/k_0) \rightarrow Aut(\mathcal{C}(X_{\lambda})) \rightarrow Aut(M).$$

Suppose that $G$ is a group acting on $M$.

**Definition 1.** We say $\{X_{\lambda}\}$ is an excellent family with Galois group $G$ if the following conditions hold:

1. The image of $\rho_{\lambda}$ is equal to $G$.
2. There is a $Gal(k/k_0)$-equivariant evaluation map $s : \mathcal{C}(X_{\lambda}) \rightarrow k$.
3. There exists a basis $\{Z_1, \ldots, Z_n\}$ of $\mathcal{C}(X_{\lambda})$ such that if we set $u_i = s(Z_i)$, then $u_1, \ldots, u_n$ are algebraically independent over $\mathbb{Q}$.
4. $\mathbb{Q}[\{u_1, \ldots, u_n\}]^G = \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$.

As an example, for $G = W(E_8)$, consider the following family of rational elliptic surfaces

$$y^2 = x^3 + x \left( \sum_{i=0}^{3} p_{20-6i} t^i \right) + \left( \sum_{j=0}^{3} p_{30-6j} t^j + t^5 \right)$$

over $k_0 = \mathbb{Q}(\lambda)$, with $\lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30})$. It is shown in [Sh2] that this is an excellent family with Galois group $G$. The $p_i$ are related to the fundamental invariants of the Weyl group of $E_8$, as is suggested by their degrees (subscripts).

We now define the notion of a multiplicative excellent family with Galois group $G$. As before, $X \rightarrow \mathbb{A}^n$ is a family of algebraic cycles on $\mathbb{A}^n$, isomorphic (via a fixed isomorphism) to a fixed abelian group $M$. The fields $k_0$ and $k$ are as before, and we have a Galois representation

$$\rho_{\lambda} : Gal(k/k_0) \rightarrow Aut(\mathcal{C}(X_{\lambda})) \rightarrow Aut(M).$$

Suppose that $G$ is a group acting on $M$.

**Definition 2.** We say $\{X_{\lambda}\}$ is a multiplicative excellent family with Galois group $G$ if the following conditions hold:

1. The image of $\rho_{\lambda}$ is equal to $G$.
2. There is a $Gal(k/k_0)$-equivariant evaluation map $s : \mathcal{C}(X_{\lambda}) \rightarrow k^*$.
3. There exists a basis $\{Z_1, \ldots, Z_n\}$ of $\mathcal{C}(X_{\lambda})$ such that if we set $u_i = s(Z_i)$, then $u_1, \ldots, u_n$ are algebraically independent over $\mathbb{Q}$.
4. $\mathbb{Q}[\{u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}\}]^G = \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$.

**Remark.** Though we use similar notation, the specialization map $s$ and the $u_i$ in the multiplicative case are quite different from the ones in the additive case. Intuitively, one may think of these as exponentiated versions of the corresponding objects in the additive case. However, we want the specialization map to be an algebraic morphism, and so in general (additive) excellent families specified by Definition 1 will be very different from multiplicative excellent families specified by Definition 2.
In our examples, $G$ will be a finite reflection group acting on a lattice in Euclidean space, which will be our choice for $M$. However, what is relevant here is not the ring of (additive) invariants of $G$ on the vector space spanned by $M$. Instead, note that the action of $G$ on $M$ gives rise to a “multiplicative” or “monomial” action of $G$ on the group algebra $\mathbb{Q}[M]$, and we will be interested in the polynomials on this space which are invariant under $G$. This is the subject of multiplicative invariant theory (see, for example, [L]). In the case when $G$ is the automorphism group of a root lattice or root system, multiplicative invariants were classically studied by using the terminology of “exponentiated” roots $e^a$ (for instance, see [B] Section VI.3).

3. The $E_6$ case

We now briefly describe the construction of multiplicative excellent family in [Sh4]. Consider the family of rational elliptic surfaces $S_\lambda$ with Weierstrass equation 

$$y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2) x + q_0 + q_1 t + q_2 t^2 + t^3$$

with parameter $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$. The surface $S_\lambda$ generically only has one reducible fiber at $t = \infty$, of type $I_3$. Therefore, the Mordell-Weil lattice $M_\lambda$ of $S_\lambda$ is isomorphic to $E_6^*$. There are 54 minimal sections of height $4/3$, and exactly half of them have the property that $x$ and $y$ are linear in $t$. If we have

$$x = at + b$$
$$y = ct + d,$$

then substituting these back in to the Weierstrass equation, we get a system of equations, and we may easily eliminate $b, c, d$ from the system to get a monic equation of degree 27 (subject to a genericity assumption), which we write as $\Phi_\lambda(t)$. Let the specializations be $s_i = -1/a_i$ for $1 \leq i \leq 27$. We have

$$\Phi_\lambda(X) = \prod_{i=1}^{27} (X - a_i)$$

$$= \prod_{i=1}^{27} (X + 1/s_i)$$

$$= X^{27} + \epsilon_{-1} X^{26} + \epsilon_{-2} X^{25} + \cdots + \epsilon_4 X^4 + \epsilon_3 X^3 + \epsilon_2 X^2 + \epsilon_1 X + 1.$$

Here $\epsilon_i$ is the $i$'th elementary symmetric polynomial of the $s_i$ and $\epsilon_{-i}$ that of the $1/s_i$. The coefficients of $\Phi_\lambda(X)$ are polynomials in the coordinates of $\lambda$, and we may use the equations for $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_{-1}$ and $\epsilon_{-2}$ to solve for $p_0, \ldots, q_3$. However, the resulting solution has $\epsilon_{-2}$ in the denominator. We may remedy this situation as follows: consider the construction of $E_6^*$ as described in [Sh5]: let $v_1, \ldots, v_6$ be vectors in $\mathbb{R}^6$ with $\langle v_i, v_j \rangle = \delta_{ij} + 1/3$, and let $u = (\sum v_i)/3$. The Z-span of $v_1, \ldots, v_6, u$ is a lattice $L$ isometric to $E_6^*$. It is clear that $v_1, \ldots, v_5, u$ form a basis of $L$. Here, we choose an isometry between the Mordell-Weil lattice and the lattice $L$, and let the specializations of $v_1, \ldots, v_6, u$ be $s_1, \ldots, s_6, r$ respectively. These satisfy $s_1 s_2 \ldots s_6 = r^3$. The fifty-four nonzero minimal vectors of $E_6^*$ split up into two cosets (modulo $E_6$) of twenty-seven each, of which we have chosen one. The specializations of these twenty-seven special sections are given by

$$\{s_1, \ldots, s_{27}\} := \{s_i : 1 \leq i \leq 6\} \bigcup \{s_i/r : 1 \leq i \leq 6\} \bigcup \{r/(s_i s_j) : 1 \leq i < j \leq 6\}.$$
Let
\[ \delta_1 = r + \frac{1}{r} + \sum_{i \neq j} s_i \frac{s_j}{s_i s_j s_k} + \sum_{i < j < k} \left( \frac{r}{s_i s_j s_k} + \frac{s_i s_j s_k}{r} \right). \]

Then \( \delta_1 \) belongs to the \( G = W(E_6) \)-invariants of \( \mathbb{Q}[s_1, \ldots, s_5, r, s_i^{-1}, s_j^{-1}, r^{-1}] \), and it is shown by explicit computation in [Sh4] that
\[ \mathbb{Q}[s_1, \ldots, s_5, r, s_i^{-1}, s_j^{-1}, r^{-1}]^G = \mathbb{Q}[\delta_1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}] = \mathbb{Q}[p_0, p_1, q_0, q_1, q_2]. \]

The explicit relation showing the second equality is as follows.
\begin{align*}
\delta_1 &= -2p_1 \\
\epsilon_1 &= 6p_2 \\
\epsilon_{-1} &= p_2^2 - q_2 \\
\epsilon_2 &= 13p_2^2 + p_0 - q_2 \\
\epsilon_{-2} &= -2p_1p_2 + 6p_2 + q_1 \\
\epsilon_3 &= 8p_3^2 + 2p_0p_2 + p_1^2 - 6p_1 - q_0 + 9.
\end{align*}

We make the following additional observation. The six fundamental representations of the Lie algebra \( E_6 \) correspond to the fundamental weights in the following diagram, which displays the standard labeling of these representations.

```
1 3 4 5 6
```

The dimensions of these representations \( V_1, \ldots, V_6 \) are 27, 78, 351, 2925, 351, 27 respectively, and their characters are related to \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}, \delta_1 \) by the following nice transformation.
\begin{align*}
\chi_1 &= \epsilon_1 \\
\chi_4 &= \epsilon_3 \\
\chi_2 &= \delta_1 + 6 \\
\chi_5 &= \epsilon_{-2} \\
\chi_3 &= \epsilon_2 \\
\chi_6 &= \epsilon_{-1}.
\end{align*}

This explains the reason for bringing in \( \delta_1 \) into the picture, and also why there is a denominator when solving for \( p_0, \ldots, q_2 \) in terms of \( \epsilon_1, \ldots, \epsilon_4, \epsilon_{-1}, \epsilon_{-2} \), as remarked in [Sh4]. The coefficients \( \epsilon_j \) are essentially the characters of \( \Lambda^j V \), where \( V = V_1 \) is the first fundamental representation, while \( \epsilon_{-j} \) are those of \( \Lambda^j V^* \), where \( V_6 = V^* \). Note that \( \Lambda^3 V \cong \Lambda^3 V^* \). Therefore, from the expressions for \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2} \), we may obtain \( p_2, q_2, p_0, q_1, q_0 \), in terms of the remaining variable \( p_1 \), without introducing any denominators. However, representation \( V_2 \) cannot be obtained as a direct summand with multiplicity 1 from a tensor product of \( \Lambda^j V \) (for \( 1 \leq j \leq 3 \)) and \( \Lambda^k V^* \) (for \( 1 \leq k \leq 2 \)). On the other hand, we do have the following isomorphism:
\[(V_2 \otimes V_5) \oplus V_5 \oplus V_1 \cong \Lambda^4 V_1 \oplus (V_3 \otimes V_6) \oplus (V_6 \otimes V_6).\]

Therefore, we are able to solve for \( p_1 \) if we introduce a denominator of \( \epsilon_{-2} \), which is the character of \( V_5 \).
4. The $E_7$ case

4.1. Results. Next, we exhibit a multiplicative excellent family for the Weyl group of $E_7$. It is given by the Weierstrass equation

$$y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)x + q_0 + q_1 t + q_2 t^2 + q_3 t^3 - t^4.$$ 

For generic $\lambda = (p_0, \ldots, p_2, q_0, \ldots, q_3)$, this rational elliptic surface $X_\lambda$ has a fiber of type $I_2$ at $t = \infty$, and no other reducible fibers. Hence, the Mordell-Weil group $M_\lambda$ is $E_7^\tau$. We note that any elliptic surface with a fiber of type $I_2$ can be put into this Weierstrass form (in general over a small degree algebraic extension of the ground field), after a fractional linear transformation of the parameter $t$, and Weierstrass transformations of the variables $x$ and $y$.

Lemma 4. The smooth part of the special fiber is isomorphic to the group scheme $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$. The identity component is the non-singular part of the curve $y^2 + xy = x^3$. A section of height 2 has $x$- and $y$-coordinates polynomials of degrees 2 and 3 respectively, and its specialization at $t = \infty$ is $t = \infty \in k^* \times \{0, 1\}$. A section of height 3/2 has $x$ and $y$ coordinates of the form

$$x = at + b$$

$$y = ct^2 + dt + e.$$ 

and specializes at $t = \infty$ to $(c, 1)$.

Proof. First, observe that to get a local chart for the elliptic surface near $t = \infty$, we set $x = \bar{x}/u^2$, $y = \bar{y}/u^3$ and $t = 1/u$, and look for $u$ near 0. Therefore, the special fiber (before blow-up) is given by $\bar{y}^2 + \bar{x}\bar{y} = \bar{x}^3$, where $\bar{x} = \bar{x}|_{u=0}$ and $\bar{y} = \bar{y}|_{u=0}$ are the reductions of the coordinates at $u = 0$ respectively. It is an easy exercise to parametrize the smooth locus of this curve: it is given, for instance, by $\bar{x} = s/(s-1)^2$, $\bar{y} = s/(s-1)^3$. We then check that $s = (\bar{y} + \bar{x})/\bar{y}$ and the map from the smooth locus to $\mathbb{G}_m$ which takes the point $(\bar{x}, \bar{y})$ to $s$ is a homomorphism from the secant group law to multiplication in $k^*$. This proves the first half of the lemma. Note that we could just as well have taken $1/s$ to be the parameter on $\mathbb{G}_m$; our choice is a matter of convention. To prove the specialization law for sections of height 3/2, we may, for instance, take the sum of such a section $Q$ with a section $P$ of height 2 with specialization $(s, 0)$. A direct calculation shows that the $y$-coordinate of the sum has top (quadratic) coefficient $cs$. Therefore the specialization of $Q$ must have the form $\kappa c$, where $\kappa$ is a constant not depending on $Q$. Finally, the sum of two sections $Q_1$ and $Q_2$ of height 3/2 and having coefficients $c_1$ and $c_2$ for the $t^2$ term of their $y$-coordinates can be checked to specialize to $(c_1 c_2, 0)$. It follows that $\kappa = \pm 1$, and we take the plus sign as a convention. (It is easy to see that both choices of sign are legitimate, since the sections of height 2 generate a copy of $E_7$, whereas the sections of height 3/2 lie in the nontrivial coset of $E_7$ in $E_7^\tau$).

There are 56 sections of height 3/2, with $x$ and $y$ coordinates in the form above. Substituting the above formulas for $x$ and $y$ into the Weierstrass equation, we get the following system of equations.

$$c^2 + ac + 1 = 0$$

$$q_3 + ap_2 + a^3 = (2c + a)d + bc$$

$$q_2 + bp_2 + 3a^2b = (2c + a)e + (b + d)d$$

$$q_1 + bp_1 + ap_0 + 3ab^2 = (2d + b)e$$

$$q_0 + bp_0 + b^3 = c^2.$$
We solve for $a$ and $b$ from the first and second equations, and then $e$ from the third, assuming $c \neq 1$. Substituting these values back into the last two equations, we get two equations in the variables $c$ and $d$. Taking the resultant of these two equations with respect to $d$, and dividing by $c^{30}(c^2-1)^4$, we obtain an equation of degree 56 in $c$, which is monic, reciprocal and has coefficients in $\mathbb{Z}[\lambda] = \mathbb{Z}[p_0, \ldots, q_3]$. We denote this polynomial by

$$
\Phi_\lambda(X) = \prod_{i=1}^{56} (X - s(P)) = X^{56} + \epsilon_1 X^{55} + \epsilon_2 X^{54} + \cdots + \epsilon_1 X + \epsilon_0,
$$

where $P$ ranges over the 56 minimal sections of height $3/2$. It is clear that $a, b, d, e$ are rational functions of $c$ with coefficients in $k_0$.

We have a Galois representation on the Mordell-Weil lattice

$$\rho_\lambda : \text{Gal}(k/k_0) \to \text{Aut}(M_\lambda) \cong \text{Aut}(E_7^\ast).$$

Here $\text{Aut}(E_7^\ast) \cong \text{Aut}(E_7) \cong W(E_7)$, the Weyl group of type $E_7$. The splitting field of $M_\lambda$ is the fixed field $k_\lambda$ of $\text{Ker}(\rho_\lambda)$. By definition, $\text{Gal}(k_\lambda/k_0) \cong \text{Im}(\rho_\lambda)$. The splitting field $k_\lambda$ is equal to the splitting field of the polynomial $\Phi_\lambda(X)$ over $k_0$, since the Mordell-Weil group is generated by the 56 sections of smallest height $P_i = (a_i t + b_i, c_i t^2 + d_i t + e_i)$. We also have that

$$k_\lambda = k_0(P_1, \ldots, P_{56}) = k_0(c_1, \ldots, c_{56}).$$

We shall sometimes write $e^\alpha$, (for $\alpha \in E_7^\ast$ a minimal vector) to refer to the specializations of these sections $c(P_i)$, for convenience.

**Theorem 5.** Assume that $\lambda$ is generic over $\mathbb{Q}$, i.e the coordinates $p_0, \ldots, q_3$ are algebraically independent over $\mathbb{Q}$. Then

1. $\rho_\lambda$ induces an isomorphism $\text{Gal}(k_\lambda/k_0) \cong W(E_7)$.
2. The splitting field $k_\lambda$ is a purely transcendental extension of $\mathbb{Q}$, isomorphic to the function field $\mathbb{Q}(Y)$ of the toric hypersurface $Y \subset G_7'$ defined by $s_1 \cdots s_7 = r^3$. There is an action of $W(E_7)$ on $Y$ such that $\mathbb{Q}(Y)^{W(E_7)} = k_{\lambda}^{W(E_7)} = k_0$.
3. The ring of $W(E_7)$-invariants in the affine coordinate ring

$$\mathbb{Q}[Y] = \frac{\mathbb{Q}[s_1, r, \frac{1}{s_1}, \frac{1}{r}]}{(s_1 \cdots s_7 - r^3)} \cong \mathbb{Q}[s_1, \ldots, s_6, r, s_1^{-1}, \ldots, s_6^{-1}, r^{-1}]$$

is equal to the polynomial ring $\mathbb{Q}[\lambda]$: $\mathbb{Q}[Y]^{W(E_7)} = \mathbb{Q}[[\lambda]] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3]$.

In fact, we shall prove an explicit, invertible polynomial relation between the Weierstrass coefficients $\lambda$ and the fundamental characters for $E_7$. Let $V_1, \ldots, V_7$ be the fundamental representations of $E_7$, and $\chi_1, \ldots, \chi_7$ their characters (on a maximal torus), as labelled below. For a description of the fundamental modules for the exceptional Lie groups see [C, Section 13.8].

Note that since the weight lattice $E_7^\ast$ has been equipped with a nice set of generators $(v_1, \ldots, v_7, u)$ with $\sum v_i = 3u$ (as in [Si5]), the characters $\chi_1, \ldots, \chi_7$ lie in the ring of Laurent polynomials...
\[ Q[s_i, r, 1/s_i, 1/r] \] where \( s_i \) corresponds to \( e^{v_i} \) and \( r \) to \( e^u \), and are obviously invariant under the (multiplicative) action of the Weyl group on this ring of Laurent polynomials. Explicit formulae for the \( \chi_i \) are given in the auxiliary files.

We also note that the roots of \( \Phi_\lambda \) are given by

\[ s_i, \frac{1}{s_i} \text{ for } 1 \leq i \leq 7 \text{ and } \frac{r}{s_i s_j} \text{ for } 1 \leq i < j \leq 7. \]

**Theorem 6.** For generic \( \lambda \) over \( Q \), we have

\[ Q[\chi_1, \ldots, \chi_7] = Q[p_0, p_1, q_0, q_1, q_2, q_3]. \]

The transformation between these sets of generators is

\[
\begin{align*}
\chi_1 &= 6p_2 + 25 \\
\chi_2 &= 6q_3 - 2p_1 \\
\chi_3 &= -q_2 + 13p_2^2 + 108p_2 + p_0 + 221 \\
\chi_4 &= 9q_3^2 - 6p_1q_3 - q_2 - q_0 + 8p_2^3 + 85p_2^2 + (2p_0 + 300)p_2 + p_1^2 + 10p_0 + 350 \\
\chi_5 &= (6p_2 + 26)q_3 + q_1 - 2p_1p_2 - 10p_1 \\
\chi_6 &= -q_2 + p_2^2 + 12p_2 + 27 \\
\chi_7 &= q_3
\end{align*}
\]

with inverse

\[
\begin{align*}
p_2 &= (\chi_1 - 25)/6 \\
p_1 &= (6\chi_7 - \chi_2)/2 \\
p_0 &= -(3\chi_6 - 3\chi_3 + \chi_1^2 - 2\chi_1 + 7)/3 \\
q_3 &= \chi_7 \\
q_2 &= -(36\chi_6 - \chi_1^2 - 22\chi_1 + 203)/36 \\
q_1 &= (24\chi_7 + 6\chi_5 + (-\chi_1 - 5)\chi_2)/6 \\
q_0 &= (27\chi_1^2 - 8\chi_1^3 - 84\chi_1^2 + 120\chi_1 - 136)/108 \\
&- (\chi_1 + 2)\chi_6/3 - \chi_4 + (\chi_1 + 5)\chi_3/3.
\end{align*}
\]

We note that our formulas agree with those of Eguchi and Sakai [ES], who seem to derive these by using an ansatz.

Next, we describe two examples through specialization, one of “small Galois” (in which all sections are defined over \( Q[t] \)) and one with “big Galois” (which has Galois group the full Weyl group).

**Example 7.** The values

\[
\begin{align*}
p_0 &= 244655370905444111/(3\mu^2), \quad p_1 = -4788369529481641525125/(16\mu^2) \\
q_3 &= 184185687325/(4\mu), \quad p_2 = 199937106590279644475038924955076599/(12\mu^4) \\
q_2 &= 57918534120411335989995011407800421/(9\mu^3) \\
q_1 &= -179880916617213624948875556502808560625/(4\mu^4) \\
q_0 &= 3531614375491975115952802080469762936626890880469201091/(1728\mu^6),
\end{align*}
\]
where \( \mu = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 102102 \), give rise to an elliptic surface for which we have \( r = 2 \), \( s_1 = 3 \), \( s_2 = 5 \), \( s_3 = 7 \), \( s_4 = 11 \), \( s_5 = 13 \), \( s_6 = 17 \), the simplest choice of multiplicatively independent elements. The Mordell-Weil group has a basis of sections for which we write down their \( x \)-coordinates below:

\[
\begin{align*}
x(P_1) &= -(10/3)t - 707606695171055129/1563722760600 \\
x(P_2) &= -(26/5)t - 611410735289928023/1563722760600 \\
x(P_3) &= -(50/7)t - 513728975686763429/1563722760600 \\
x(P_4) &= -(122/11)t - 316023939417997169/1563722760600 \\
x(P_5) &= -(170/13)t - 216677827127591279/1563722760600 \\
x(P_6) &= -(290/17)t - 17562556436754779/1563722760600 \\
x(P_7) &= -(229/30)t - 140574879644393807/390930690150.
\end{align*}
\]

In the auxiliary files the \( x \)-and \( y \)-coordinates are listed, and it is verified that they satisfy the Weierstrass equation.

**Example 8.** The value \( \lambda = \lambda_0 := (1,1,1,1,1,1,1) \) gives rise to an explicit polynomial \( f(X) = \Phi_{\lambda_0}(X) \), given by

\[
f(X) = X^{56} - X^{55} + 40X^{54} - 22X^{53} + 797X^{52} - 190X^{51} + 9878X^{50} - 1513X^{49} \\
+ 82195X^{48} - 17689X^{47} + 496844X^{46} - 175584X^{45} + 2336237X^{44} \\
- 1196652X^{43} + 8957717X^{42} - 5726683X^{41} + 28574146X^{40} \\
- 20119954X^{39} + 75465618X^{38} - 53541106X^{37} + 163074206X^{36} \\
- 110505921X^{35} + 287854250X^{34} - 181247607X^{33} + 420186200X^{32} \\
- 243591901X^{31} + 518626022X^{30} - 278343633X^{29} + 554315411X^{28} \\
- 278343633X^{27} + 518626022X^{26} - 243591901X^{25} + 420186200X^{24} \\
- 181247607X^{23} + 287854250X^{22} - 110505921X^{21} + 163074206X^{20} \\
- 53541106X^{19} + 75465618X^{18} - 20119954X^{17} + 28574146X^{16} \\
- 5726683X^{15} + 8957717X^{14} - 1196652X^{13} + 2336237X^{12} \\
- 175584X^{11} + 496844X^{10} - 17689X^{9} + 82195X^{8} - 1513X^{7} \\
+ 9878X^{6} - 190X^{5} + 797X^{4} - 22X^{3} + 40X^{2} - X + 1,
\]

for which we can show that the Galois group is the full group \( W(E_7) \), as follows. The reduction of \( f(X) \) modulo 7 shows that \( \text{Frob}_7 \) has cycle decomposition of type \( (7)^8 \), and similarly, \( \text{Frob}_{101} \) has cycle decomposition of type \( (3)^2(5)^4(15)^2 \). This implies, as in ([Sh3], Example 7.6), that the Galois group is the entire Weyl group.

We can also describe degenerations of this family of rational elliptic surfaces \( X_\lambda \) by the method of “vanishing roots,” where we drop the genericity assumption, and consider the situation where the elliptic fibration might have additional reducible fibers. Let \( \psi : Y \rightarrow \mathbb{A}^7 \) be the finite surjective morphism associated to

\[
\mathbb{Q}[p_0, \ldots, q_3] \rightarrow \mathbb{Q}[y] \cong \mathbb{Q}[s_1, \ldots, s_6, r, s_1^{-1}, \ldots, s_6^{-1}, r^{-1}].
\]

For \( \xi = (s_1, \ldots, s_7, r) \in Y \), let the multiset \( \Pi_\xi \) consist of the 126 elements \( s_i/r \) and \( r/s_i \) (for \( 1 \leq i \leq 7 \)), \( s_i/s_j \) (for \( 1 \leq i \neq j \leq 7 \)) and \( s_is_jr/s_k \) and \( r/(s_is_js_k) \) for \( 1 \leq i < j < k \leq 7 \), corresponding to the 126 roots of \( E_7 \). Let \( 2\nu(\xi) \) be the number of times 1 appears in \( \Pi_\xi \), which is
also the multiplicity of 1 as a root of $\Psi_\lambda(X)$ (to be defined in Section 12), where $\lambda = \psi(\xi)$. We call the associated roots of $E_7$ the vanishing roots, in analogy with vanishing cycles in the deformation of singularities. By abuse of notation we label the rational elliptic surface $X_\lambda$ as $X_\xi$.

**Theorem 9.** The surface $X_\xi$ has new reducible fibers (necessarily at $t \neq \infty$) if and only if $\nu(\xi) > 0$. The number of roots in the root lattice $T_{\text{new}}$ is equal to $2\nu(\xi)$, where $T_{\text{new}} := \oplus_{v \neq \infty} T_v$ is the new part of the trivial lattice.

We may use this result to produce specializations with trivial lattice including $A_1$, corresponding to the entries in the table of [OS, Section 1]. Note that in earlier work [Sh2, SU], examples of rational elliptic surfaces were produced with a fiber of additive type, for instance, a fiber of type III (which contributes to the trivial lattice) or a fiber of type II. Using our excellent family, we can produce examples with the $A_1$ fiber being of multiplicative type $I_2$ and all other irreducible singular fibers being nodal ($I_1$). We list below those types which are not already covered by [Sh4]. To produce these examples, we use an embedding of the new part $T_{\text{new}}$ of the fibral lattice into $E_7$, which gives us any extra conditions satisfied by $s_1, \ldots, s_7, r$. The following multiplicative version of the labeling of simple roots of $E_7$ is useful (compare [Sh4]).

![Dynkin diagram](image)

For instance, to produce the example in line 18 of the table (i.e. with $T_{\text{new}} = D_4$), we may use the embedding into $E_7$ indicated by embedding the $D_4$ Dynkin diagram within the dashed lines in the figure above. Thus, we must force $s_2 = s_3 = s_4 = s_5$, and $r = s_1 s_2 s_3$, and a simple solution with no extra coincidences is given in the rightmost column (note that $s_7 = 18^3/(2 \cdot 3^4 \cdot 5) = 36/5$).

| Type in [OS] | Fibral lattice | MW group | $\{s_1, \ldots, s_6, r\}$ |
|--------------|----------------|----------|---------------------------|
| 2            | $A_1$          | $E_7^*$  | 3, 5, 7, 11, 13, 17, 2    |
| 4            | $A_1^2$        | $D_6^*$  | 3, 5, 7, 11, 13, 2        |
| 7            | $A_1^7$        | $D_1^* \oplus A_1^*$ | 3, 5, 7, 11, 2          |
| 10           | $A_1 \oplus A_3$ | $A_1^* \oplus A_3^*$ | 3, 3, 3, 5, 7, 2        |
| 13           | $A_1^4$        | $D_1^* \oplus \mathbb{Z}/2\mathbb{Z}$ | $-1, 2, 3, 5, 7, 49/30, 7$ |
| 14           | $A_1^2$        | $A_1^4$  | 3, 3, 3, 3, 5, 7, 2      |
| 17           | $A_1 \oplus A_4$ | $\frac{1}{10} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{pmatrix}$ | 3, 3, 3, 3, 5, 2        |
| 18           | $A_1 \oplus D_4$ | $A_1^3$  | 2, 3, 3, 3, 5, 18        |
| 21           | $A_1^2 \oplus A_3$ | $A_3^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 3, 5, 60, 30, 30, 30, 900 |
| 22           | $A_1^2 \oplus A_3$ | $A_1^* \oplus (1/4)$ | 3, 3, 5, 5, 5, 2         |
| 24           | $A_1^2$        | $A_1^3 \oplus \mathbb{Z}/2\mathbb{Z}$ | 15/4, 2, 2, 3, 3, 5, 15 |
| 28           | $A_1 \oplus A_5$ | $A_5^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 3, 6, 6, 6, 36       |
| 29           | $A_1 \oplus A_5$ | $A_1^* \oplus (1/6)$ | 2, 2, 2, 2, 2, 3         |
| 30           | $A_1 \oplus D_5$ | $A_1^* \oplus (1/4)$ | 2, 2, 2, 2, 3, 8         |
| 33           | $A_1^2 \oplus A_4$ | $\frac{1}{10} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ | 2, 2, 3, 3, 3, 12       |
| 34           | $A_1^2 \oplus D_4$ | $A_1^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 3, 3, 3, 6, 18       |
\[ V = 4.2. \]

This last example is the universal elliptic curve with type 74. For the examples in lines 58, 70 and 74 of the table, one can show that it is not possible to define a rational elliptic surface over \( \mathbb{Q} \) in the form we have assumed, such that all the specializations \( s_i, r \) are rational. However, there do exist examples with all sections defined over \( \mathbb{Q} \), not in the chosen Weierstrass form.

The surface with Weierstrass equation

\[ y^2 + xy + \left( \frac{c^2 - 1}{16} \right) y = x^3 + \frac{(c^2 - 1)(t^2 - 1)}{16} x^2 \]

has a 4-torsion section \((0, 0)\) and a non-torsion section \((c + 1)(t^2 - 1)/8, (c + 1)^2(t - 1)^2(t + 1)/32\) of height \(1/2\), as well as two reducible fibers of type \( I_1 \) and a fiber of type \( I_2 \). It is an example of type 58.

The surface with Weierstrass equation

\[ y^2 + xy + t^2 y = x^3 + t^2 x^2 \]

has a 4-torsion section \((0, 0)\), and reducible fibers of types \( I_8 \) and \( I_2 \). It is an example of type 70.

The surface with Weierstrass equation

\[ y^2 + xy - \left( t^2 - \frac{1}{16} \right) y = x^3 - \left( t^2 - \frac{1}{16} \right) x^2 \]

has two reducible fibers of type \( I_4 \) and two reducible fibers of type \( I_2 \). It also has a 4-torsion section \((0, 0)\) and a 2-torsion section \((4t - 1)/8, (4t - 1)^2/32\), which generate the Mordell-Weil group. It is an example of type 74. This last example is the universal elliptic curve with \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) torsion (compare [Ku]).

**Remark 10.** For the examples in lines 58, 70 and 74 of the table, one can show that it is not possible to define a rational elliptic surface over \( \mathbb{Q} \) in the form we have assumed, such that all the specializations \( s_i, r \) are rational. However, there do exist examples with all sections defined over \( \mathbb{Q} \), not in the chosen Weierstrass form.

**4.2. Proofs.** We start by considering the coefficients \( \epsilon_i \) of \( \Phi_{\lambda}(X) \); we know that \((-1)^i \epsilon_i \) is simply the \( i \)'th elementary symmetric polynomial in the 56 specializations \( s(P_i) \). One shows, either by explicit calculation with Laurent polynomials, or by calculating the decomposition of \( \Lambda V \) (where \( V = V_7 \) is the 56-dimensional representation of \( E_7 \)), and expressing its character as polynomials
in the fundamental characters, the following formulae. Some more details are in Section 4 and the auxiliary files.

\[
\begin{align*}
\epsilon_1 &= -\chi_7 \\
\epsilon_2 &= \chi_6 + 1 \\
\epsilon_3 &= - (\chi_7 + \chi_5) \\
\epsilon_4 &= \chi_6 + \chi_4 + 1 \\
\epsilon_5 &= -(\chi_6 + \chi_3 - \chi_1^2 + \chi_1 + 1)\chi_7 + (\chi_1 - 1)\chi_5 - \chi_2\chi_3 \\
\epsilon_6 &= \chi_1\chi_7^2 + (\chi_5 - (\chi_1 + 1)\chi_2)\chi_7 + \chi_6^2 + 2(\chi_3 - \chi_1^2 + \chi_1 + 1)\chi_6 \\
&\quad - \chi_2\chi_5 - (2\chi_1 + 1)\chi_4 + \chi_3^2 + 2(2\chi_1 + 1)\chi_3 \\
&\quad + \chi_1\chi_2^2 - 2\chi_1^3 + \chi_3^2 + 2\chi_1 + 1 \\
\epsilon_7 &= -(\chi_1 + 1)\chi_6 + 2\chi_4 - 2(\chi_1 + 1)\chi_3 + \chi_3^3 - 3\chi_1 - 1)\chi_7 \\
&\quad - 2(\chi_5 - \chi_1^2)\chi_6 - (\chi_3 - \chi_1^2 + \chi_1 + 2)\chi_5 + 3\chi_2\chi_4 \\
&\quad - (\chi_1 + 3)\chi_2\chi_5 - \chi_2^3 + (2\chi_1 - 1)\chi_1\chi_2.
\end{align*}
\]

On the other hand, we can explicitly calculate the first few coefficients \(\epsilon_i\) of \(\Phi_A(X)\) in terms of the Weierstrass coefficients, obtaining the following equations. Details for the method are in Section 8.

\[
\begin{align*}
\epsilon_1 &= -q_3 \\
\epsilon_2 &= p_2^2 + 12p_2 - q_2 + 28 \\
\epsilon_3 &= -3(2p_2 + 9)q_3 - q_1 + 2p_1(p_2 + 5) \\
\epsilon_4 &= 9q_4^3 - 6p_1q_3 - 2q_2 - q_0 + 8p_2^3 + 86p_2^2 + 2(p_0 + 156)p_2 + p_1^2 + 10p_0 + 378 \\
\epsilon_5 &= (8q_2 - 20p_2^2 - 174p_2 - 7p_0 - 351)q_3 - 2p_1q_2 + 6(p_2 + 4)q_1 \\
&\quad + 14p_1p_2^2 + 108p_1p_2 + 2(p_0 + 101)p_1 \\
\epsilon_6 &= 12(4p_2 + 15)q_3^2 - (5q_1 + 38p_1p_2 + 140p_1)q_3 + 4q_2^2 \\
&\quad + (16p_2^2 + 96p_2 - 4p_0 + 155)q_2 + 2p_1q_1 + 3(4p_2 + 17)q_0 + 28p_2^2 + 360p_2 \\
&\quad + (4p_0 + 1765)p_2^2 + 2(4p_1^2 + 21p_0 + 1950)p_2 + 29p_1^2 + p_0^2 + 88p_0 + 3276 \\
\epsilon_7 &= -36q_3^3 + 42p_1q_3^2 + (4q_2 - 20q_0 - 56p_2^2 - 628p_2^2 - 14(p_0 + 168)p_2 - 16p_1^2 \\
&\quad - 46p_0 - 2925)q_3 + (3q_1 + 6p_1p_2 + 20p_1)q_2 + (21p_2^2 + 162p_2 - p_0 + 323)q_1 \\
&\quad + 6q_0 + 42p_1p_2^2 + 448p_1p_2^2 + 2(p_0 + 799)p_1p_2 + 2p_1^3 + 6(p_0 + 316)p_1.
\end{align*}
\]

Equating the two expressions we have obtained for each \(\epsilon_i\), we get a system of seven equations, the first being

\[-\chi_7 = -q_3.\]

We label these equations (1), \ldots, (7). The last few of these polynomial equations are somewhat complicated, and so to obtain a few simpler ones, we may consider the 126 sections of height 2, which we analyze as follows. Substituting

\[
\begin{align*}
x &= at^2 + bt + c \\
y &= dt^3 + et^2 + ft + g
\end{align*}
\]
in to the Weierstrass equation, we get another system of equations:

\[
\begin{align*}
a^3 &= d^2 + ad \\
3a^2b &= (2d + a)e + bd \\
a(p_2 + 3ac + 3b^2) &= (2d + a)f + c^2 + be + cd + 1 \\
q_3 + bp_2 + ap_1 + 6abc + b^3 &= (2d + a)g + (2e + b)f + ce \\
q_2 + cp_2 + bp_1 + ap_0 + 3ac^2 + 3b^2c &= (2e + b)g + f^2 + cf \\
q_1 + cp_1 + bp_0 + 3bc^2 &= (2f + c)g \\
q_0 + cp_0 + c^3 &= g^2.
\end{align*}
\]

The specialization of such a section at \( t = \infty \) is \( 1 + a/d \). Setting \( d = ar \), we may as before eliminate other variables to obtain an equation of degree 126 for \( r \). Substituting \( r = 1/(u - 1) \), we get a monic polynomial \( \Psi_\lambda(X) = 0 \) of degree 126 for \( u \). Note that the roots are given by elements of the form

\[
\frac{s_i}{s_i} \quad \text{for} \quad 1 \leq i \leq 7, \quad \frac{s_i s_j}{s_j} \quad \text{for} \quad 1 \leq i \neq j \leq 7 \quad \text{and} \quad \frac{s_i s_j s_k}{s_j s_k} \quad \text{for} \quad 1 \leq i < j < k \leq 7.
\]

As before, we can write the first few coefficients \( \eta_i \) of \( \Psi_\lambda \) in terms of \( \lambda = (p_0, \ldots, q_3) \), as well as in terms of the characters \( \chi_j \), obtaining some more relations. We will only need the first two:

\[
\begin{align*}
-\chi_1 + 7 &= \eta_1 = -18 - 6p_2 \\
-6\chi_1 + \chi_3 + 28 &= \eta_2 = p_0 + 72p_2 + 13p_2^2 - q_2 + 99
\end{align*}
\]

which we call (1') and (2') respectively.

Now we consider the system of six equations (1) through (4), (1') and (2'). These may be solved for \((p_2, p_0, q_3, q_2, q_1, q_0)\) in terms of the \( \chi_j \) and \( p_1 \). Substituting this solution into the other three relations (5), (6) and (7), we obtain three equations for \( p_1 \), of degrees 1, 2 and 3 respectively. These have a single common factor, linear in \( p_1 \), which we then solve. This gives us the proof of Theorem 0.

The proof of Theorem 5 is now straightforward. Part (1) asserts that the image of \( \rho_\lambda \) is surjective on to \( W(E_7) \): this follows from a standard Galois theoretic argument as follows. Let \( F \) be the fixed field of \( W(E_7) \) acting on \( \lambda = Q(\lambda)(s_1, \ldots, s_6, r) = Q(s_1, \ldots, s_6, r) \), where the last equality follows from the explicit expression of \( \lambda = (p_0, \ldots, q_3) \) in terms of the \( \chi_i \), which are in \( Q(s_1, \ldots, s_6, r) \). Then we have that \( k_0 \subset F \) since \( p_0, \ldots, q_3 \) are polynomials in the \( \chi_i \) with rational coefficients, and the \( \chi_i \) are manifestly invariant under the Weyl group. Therefore \( [k_\lambda : k_0] \geq [k_\lambda : F] = |W(E_7)| \), where the latter equality is from Galois theory. Finally, \( [k_\lambda : k_0] \leq |\text{Gal}(k_\lambda/k_0)| \leq |W(E_7)| \), since \( \text{Gal}(k_\lambda/k_0) \hookrightarrow W(E_7) \). Therefore, equality is forced.

Another way to see that the Galois group is the full Weyl group is to demonstrate it for a specialization, such as Example 8 and use [9, Section 9.2, Prop 2].

Next, let \( Y \) be the toric hypersurface given by \( s_1 \ldots s_7 = r^3 \). Its function field is the splitting field of \( \Phi_\lambda(X) \), as we remarked above. We have seen that \( Q(Y)^{W(E_7)} = k_0 = Q(\lambda) \). Since \( \Phi_\lambda(X) \) is a monic polynomial with coefficients in \( Q[\lambda] \), we have that \( Q(Y)^{W(E_7)} \) is integral over \( Q[\lambda] \). Therefore \( Q(Y)^{W(E_7)} \) is also integral over \( Q[\lambda] \), and contained in \( Q(Y)^{W(E_7)} = k_0 = Q(\lambda) \). Since \( Q[\lambda] \) is a polynomial ring, it is integrally closed in its ring of fractions. Therefore \( Q(Y)^{W(E_7)} \subset Q[\lambda] \).
We also know $\mathbb{Q}[\chi] = \mathbb{Q}[\chi_1, \ldots, \chi_7] \subset \mathbb{Q}[Y]^{W(E_7)}$, since the $\chi_j$ are invariant under the Weyl group. Therefore, we have

$$\mathbb{Q}[\chi] \subset \mathbb{Q}[Y]^{W(E_7)} \subset \mathbb{Q}[\lambda]$$

and Theorem 6 which says $\mathbb{Q}[\chi] = \mathbb{Q}[\lambda]$, implies that all these three rings are equal. This completes the proof of Theorem 5.

Remark 11. Note that this argument gives an independent proof of the fact that the ring of multiplicative invariants for $W(E_7)$ is a polynomial ring over $\chi_1, \ldots, \chi_7$. See [B, Théorème VI.3.1 and Exemple 1] or [L, Theorem 3.6.1] for the classical proof that the Weyl-orbit sums of a set of fundamental weights are a set of algebraically independent generators of the multiplicative invariant ring; from there to the fundamental characters is an easy exercise.

Remark 12. Now that we have found the explicit relation between the Weierstrass coefficients and the fundamental characters, we may go back and explore the “genericity condition” for this family to have Mordell-Weil lattice $E_7^*$. To do this we compute the discriminant of the cubic in $x$, after completing the square in $y$, and take the discriminant with respect to $t$ of the resulting polynomial of degree 10. A computation shows that this discriminant factors as the cube of a polynomial $A(\lambda)$, whose zero locus corresponds to the occurrence of a reducible multiplicative fiber. In fact, we calculate (for instance, by evaluating the split case), that $B(\lambda)$ is the product of $(e^\alpha - 1)$, where $\alpha$ runs over 126 roots of $E_7$. We deduce by further analyzing the type II case that the condition to have Mordell-Weil lattice $E_7^*$ is that

$$\prod (e^\alpha - 1) = \Psi_{\lambda}(1) \neq 0.$$

Note that this is essentially the expression which occurs in Weyl’s denominator formula. In addition, the condition for having only multiplicative fibers is that $\Psi_{\lambda}(1)$ and $A(\lambda)$ both be non-zero.

Finally, the proof of Theorem 9 follows immediately from the discussion in [Sh6, Sh7] (compare [Sh7, Section 2.3] for the additive reduction case).

5. The $E_8$ Case

5.1. Results. Finally, we show a multiplicative excellent family for the Weyl group of $E_8$. It is given by the Weierstrass equation

$$y^2 = x^3 + tx^2 + (p_0 + p_1 t + p_2 t^2) x + (q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4 + t^5).$$

For generic $\lambda = (p_0, \ldots, p_2, q_0, \ldots, q_4)$, this rational elliptic surface $X_\lambda$ has no reducible fibers, only nodal $I_1$ fibers at twelve points, including $t = \infty$. We will use the specialization map at $\infty$. The Mordell-Weil lattice $M_\lambda$ is isomorphic to the lattice $E_8$. Any rational elliptic surface with a multiplicative fiber of type $I_1$ may be put in the above form (over a small degree algebraic extension of the base field), after a fractional linear transformation of $t$ and Weierstrass transformations of $x, y$.

Lemma 13. The smooth part of the special fiber is isomorphic to the group scheme $\mathbb{G}_m$. The identity component is the non-singular part of the curve $y^2 = x^3 + x^2$. A section of height 2 has $x$- and $y$-coordinates polynomials of degrees 2 and 3 respectively, and its specialization at $t = \infty$ may be taken as $\lim_{t \to \infty} (y + tx)/(y - tx) \in k^*$. 
The proof of the lemma is similar to that in the $E_7$ case (and simpler!), and we omit it.

There are 240 sections of minimal height 2, with $x$ and $y$ coordinates of the form
\[ x = gt^2 + at + b, \]
\[ y = ht^3 + ct^2 + dt + e. \]

Under the identification with $G_m$ of the special fiber of the Néron model, this section goes to $(h + g)/(h - g)$. Substituting the above formulas for $x$ and $y$ into the Weierstrass equation, we get the following system of equations.

\[ h^2 = g^3 + g^2 \]
\[ 2ch = 3ag^2 + 2ag + 1 \]
\[ 2dh + c^2 = q_4 + gp_2 + 3bg^2 + (2b + 3a^2)g + a^3 \]
\[ 2eh + 2cd = q_3 + ap_2 + gp_1 + 6abg + 2ab + a^2 \]
\[ 2ce + d^2 = q_2 + bp_2 + ap_1 + gp_0 + 3b^2g + b^2 + 3a^2b \]
\[ 2de = q_1 + bp_1 + ap_0 + 3ab^2 \]
\[ c^2 = q_0 + bp_0 + b^3. \]

Setting $h = gu$, we eliminate other variables to obtain an equation of degree 240 for $u$. Finally, substituting in $u = (v + 1)/(v - 1)$, we get a monic reciprocal equation $\Phi_\lambda(X) = 0$ satisfied by $v$, with coefficients in $\mathbb{Z}[\lambda] = \mathbb{Z}[p_0, \ldots, p_2, q_0, \ldots, q_4]$. We have

\[ \Phi_\lambda(X) = \prod_{i=1}^{240} (X - s(P)) = X^{240} + \epsilon_1 X^{239} + \cdots + \epsilon_1 X + \epsilon_0, \]

where $P$ ranges over the 240 minimal sections of height 2. It is clear that $a, \ldots, h$ are rational functions of $v$, with coefficients in $k_0$.

We have a Galois representation on the Mordell-Weil lattice
\[ \rho_\lambda : \text{Gal}(k/k_0) \to \text{Aut}(M_\lambda) \cong \text{Aut}(E_8). \]

Here $\text{Aut}(E_8) \cong W(E_8)$, the Weyl group of type $E_8$. The splitting field of $M_\lambda$ is the fixed field $k_\lambda$ of $\text{Ker}(\rho_\lambda)$. By definition, $\text{Gal}(k_\lambda/k_0) \cong \text{Im}(\rho_\lambda)$. The splitting field $k_\lambda$ is equal to the splitting field of the polynomial $\Phi_\lambda(X)$ over $k_0$, since the Mordell-Weil group is generated by the 240 sections of smallest height $P_i = (g_it^2 + a_it + b_i, h_it^3 + c_it^2 + d_it + e_i)$. We also have that

\[ k_\lambda = k_0(P_1, \ldots, P_{240}) = k_0(v_1, \ldots, v_{240}). \]

**Theorem 14.** Assume that $\lambda$ is generic over $\mathbb{Q}$, i.e the coordinates $p_0, \ldots, q_4$ are algebraically independent over $\mathbb{Q}$. Then

1. $\rho_\lambda$ induces an isomorphism $\text{Gal}(k_\lambda/k_0) \cong W(E_8)$.
2. The splitting field $k_\lambda$ is a purely transcendental extension of $\mathbb{Q}$, isomorphic to the function field $\mathbb{Q}(Y)$ of the toric hypersurface $Y \subset \mathbb{P}^8_{\mathbb{Q}}$ defined by $s_1 \ldots s_8 = r^3$. There is an action of $W(E_8)$ on $Y$ such that $\mathbb{Q}(Y)^{W(E_8)} = k_\lambda^{W(E_8)} = k_0$.
3. The ring of $W(E_8)$-invariants in the affine coordinate ring
\[ \mathbb{Q}[Y] = \mathbb{Q}[s_i, r, 1/s_i, 1/r]/(s_1 \ldots s_8 - r^3) \cong \mathbb{Q}[s_1, \ldots, s_7, r, s_1^{-1}, \ldots, s_7^{-1}, r^{-1}] \]
is equal to the polynomial ring $\mathbb{Q}[\lambda]$: 

$$\mathbb{Q}[Y]^{W(E_8)} = \mathbb{Q}[\lambda] = \mathbb{Q}[p_0, p_1, q_0, q_1, q_2, q_3, q_4].$$

As in the $E_7$ case, we prove an explicit, invertible polynomial relation between the Weierstrass coefficients $\lambda$ and the fundamental characters for $E_8$. Let $V_1, \ldots, V_8$ be the fundamental representations of $E_8$, and $\chi_1, \ldots, \chi_8$ their characters as labelled below.

![Diagram of the Dynkin diagram for $E_8$]

Again, for the set of generators of $E_8$, we choose (as in [Sh5]) vectors $v_1, \ldots, v_8, u$ with $\sum v_i = 3u$ and let $s_i$ correspond to $v_i$ and $r$ to $u$, so that $\prod s_i = r^3$. The 240 roots of $\Phi_\lambda(X)$ are given by

$$s_i, \frac{1}{s_i} \text{ for } 1 \leq i \leq 8, \quad s_i s_j \text{ for } 1 \leq i \neq j \leq 8, \quad \frac{s_i s_j}{r}, \frac{r}{s_i s_j} \text{ for } 1 \leq i < j \leq 8 \quad \text{and} \quad \frac{s_i s_j s_k}{r}, \frac{r}{s_i s_j s_k} \text{ for } 1 \leq i < j < k \leq 8.$$

The characters $\chi_1, \ldots, \chi_7$ lie in the ring of Laurent polynomials $\mathbb{Q}[s_i, r, 1/s_i, 1/r]$, and are invariant under the multiplicative action of the Weyl group on this ring of Laurent polynomials. The $\chi_8$ may be explicitly computed using the software LiE, as indicated in Section [7] and the auxiliary files.

**Theorem 15.** For generic $\lambda$ over $\mathbb{Q}$, we have

$$\mathbb{Q}[\chi_1, \ldots, \chi_8] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3, q_4].$$

The transformation between these sets of generators is

$$\begin{align*}
\chi_1 &= -1600 q_4 + 1536 p_2 + 3875 \\
\chi_2 &= 2(-45600 q_4 + 12288 q_3 + 40704 p_2 - 16384 p_1 + 73625) \\
\chi_3 &= 64(14144 q_4^2 - 72(384 p_2 + 1225) q_4 + 11200 q_3 - 4096 q_2 + 13312 p_2^2 \\
&\quad + 87072 p_2 - 17920 p_1 + 16384 p_0 + 104625) \\
\chi_4 &= -9175040 q_4^3 + 12288(25600 p_2 + 222101) q_4^2 - 256(4530176 q_3 - 65536 q_2 \\
&\quad + 1392640 p_2^2 + 21778944 p_2 - 8159232 p_1 + 2621440 p_0 + 34773585) q_4 \\
&\quad + 32(4718592 q_3^2 + 384(80896 p_2 - 32768 p_1 + 225379) q_3 - 29589504 q_2 \\
&\quad + 30408704 q_1 - 33554432 q_0 + 4194304 p_2^3 + 88129536 p_2^2 \\
&\quad - 64(876544 p_1 - 2621440 p_0 - 4399923) p_2 + 8388608 p_2^2 - 133996544 p_1 \\
&\quad + 65175552 p_0 + 215596227).\
\end{align*}
\[ \chi_5 = 24760320q_4^2 - 64(106496q_3 + 738816p_2 - 163840p_1 + 2360085)q_4 \]
+ 12288(512p_2 + 4797)q_3 - 30670848q_2 + 16777216q_1 + 20250624p_2^2
\- 512(16384p_1 - 235911)p_2 - 45154304p_1 + 13631488r_0 + 146325270
\]
\[ \chi_6 = 110592q_4^2 - 1536(128p_2 + 1235)q_4 + 724992q_3 - 262144q_2 + 65536p_2^2 \]
\+ 1062912p_2 - 229376p_1 + 2450240
\]
\[ \chi_7 = -4(3920q_4 - 1024q_3 - 1152p_2 - 7595) \]
\[ \chi_8 = -8(8q_4 - 31). \]

**Remark 16.** We omit the inverse for conciseness here; it is easily computed in a computer algebra system and is available in the auxiliary files.

**Remark 17.** As before, our explicit formulas are compatible with those in [ES]. Also, the proof of Theorem 14 gives another proof of the fact that the multiplicative invariants for \( W(E_8) \) are freely generated by the fundamental characters (or by the orbit sums of the fundamental weights).

**Example 18.** Let \( \mu = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19) = 9699690. \)

\[ q_1 = -2243374456559366834339/(2^3 \cdot \mu^2) \]
\[ q_3 = 4308003431294033846226518246078567/(2^{11} \cdot \mu^3) \]
\[ q_2 = 725510194764901112739173303498415842573146409905769/(2^{22} \cdot 3^2 \cdot \mu^4) \]
\[ q_1 = (-1288109930551729138207432378468368491584063772556981164919245) \]
\[ 30489)/(2^{29} \cdot 3 \cdot \mu^5) \]
\[ q_0 = (882717679332361992427303384148545901911918837196838709750423283 \]
\[ 4433603579926502483)//(2^{42} \cdot 3 \cdot \mu^6) \]
\[ p_2 = 146156773903879871001810589/(2^9 \cdot 3 \cdot \mu^2) \]
\[ p_1 = -24909805041567866985469379779685360019313/(2^{20} \cdot \mu^3) \]
\[ p_0 = 14921071761102637668643191215765039801471771138867387/(2^{23} \cdot 3 \cdot \mu^4) \]

These values give an elliptic surface for which we have \( r = 2, s_1 = 3, s_2 = 5, s_3 = 7, s_4 = 11, s_5 = 13, s_6 = 17, s_7 = 19, \) the simplest choice of multiplicatively independent elements. Here, the specializations of a basis are given by \( v \in \{ 3, 5, 7, 11, 13, 17, 19, 15/2 \} \). Once again, we list the \( x \)-coordinates of the corresponding sections, and leave the remainder of the verification to the auxiliary files.

\[ x(P_1) = 3t^2 - (99950606190359/620780160)t \]
\[ + 43253275576474881209649813/2642523476911718400 \]
\[ x(P_2) = (5/4)t^2 - (153332163637781/1655413760)t \]
\[ + 5414114237697608646836821/5138596941004800 \]
\[ x(P_3) = (7/9)t^2 - (203120672689603/2793510720)t \]
\[ + 69431643485691306367886386397927570430735155200 \]
\[ x(P_4) = (11/25)t^2 - (8564057914757/147804800)t \]
\[ + 115126372233675800396600989/155442557465395200. \]
Theorem 20. The surface $T$ has number of roots in the root lattice $E_{sh2, sh4}$ or our examples for the entries of $OS$, and a nodal fiber. Below, we list those types which are not already covered by We may use this result to produce specializations with trivial lattice corresponding to most of the part of the trivial lattice.

By abuse of notation we label the rational elliptic surface $E$ the finite surjective morphism associated to situation where the elliptic fibration might have additional reducible fibers. Let $\psi : Y \to A^8$ be the finite surjective morphism associated to

$$
\xi = (s_1, \ldots, s_8, r) \in Y,$
$$
let the multiset $\Pi_\xi$ consist of the 240 elements $s_i$ and $1/s_i$ (for $1 \leq i \leq 8$), $s_i/s_j$ (for $1 \leq i \neq j \leq 8$), $s_is_j/r$ and $r/(s_is_j)$ (for $1 \leq i < j \leq 8$) and $s_is_j/s_k$ and $r/(s_is_j)$ for $1 \leq i < j < k \leq 8$, corresponding to the 240 roots of $E_6$. Let $2\nu(\xi)$ be the number of times 1 appears in $\Pi_\xi$, which is also the multiplicity of 1 as a root of $\Phi_\lambda(X)$, with $\lambda = \psi(\xi)$. We call the associated roots of $E_8$ the vanishing roots, in analogy with vanishing cycles in the deformation of singularities. By abuse of notation we label the real elliptic surface $X_\lambda$ as $X_\xi$.

**Theorem 20.** The surface $X_\xi$ has new reducible fibers (necessarily at $t \neq \infty$) if and only if $\nu(\xi) > 0$. The number of roots in the root lattice $T_{new}$ is equal to $2\nu(\xi)$, where $T_{new} := \oplus_{v \neq \infty} T_v$ is the new part of the trivial lattice.

We may use this result to produce specializations with trivial lattice corresponding to most of the entries of $OS$, and a nodal fiber. Below, we list those types which are not already covered by our examples for the $E_7$ case, which have an $I_2$ fiber.

| Type in $OS$ | Fibral lattice | MW group | {s_1, \ldots, s_6, r} |
|---------------|----------------|----------|----------------------|
| 1             | 0   | $E_8$   | 3, 5, 7, 11, 13, 17, 19, 2 |
| 5             | $A_3$ | $D_5^2$ | 2, 2, 2, 5, 7, 11, 3 |
| 8             | $A_4$ | $A_1$^2 | 2, 2, 2, 2, 2, 5, 3 |
| 15            | $A_5$ | $A_1^2 \oplus A_1^2$ | 2, 2, 2, 2, 2, 2, 5, 3 |
| 16            | $D_5$ | $A_1^2$ | 2, 3, 3, 3, 3, 5, 18 |
| 25            | $A_6$ | $F\left(\begin{array}{cc} 4 & -1 \\ -1 & 2 \end{array}\right)$ | 2, 2, 2, 2, 2, 2, 2, 3 |
| 26            | $D_4$ | $A_1^2$ | 2, 3, 3, 3, 3, 3, 18 |
| 35            | $A_3^2$ | $A_1^2 \oplus Z/2Z$ | 2, 3, 3, 3, 3, 3, 1, 1, -3 |
all the specializations

As before, for the examples in lines 63, 67 and 72 of the table, one can show that it

Remark 21. The surface with Weierstrass equation

\[
\begin{align*}
0 & = y^2 + tx^3 + 3y \frac{t^2(t-1)}{16} \frac{x^3 + t(t-1)x^2}{16} \\
0 & = (t+1)xy + ty \frac{t^2(t-1)}{16} \frac{x^3 + t(t-1)x^2}{16}
\end{align*}
\]

\[
\begin{align*}
0 & = y^2 + xy + t^3 y = x^3
\end{align*}
\]

has a 3-torsion point \((0,0)\) and a fiber of type I_9. It is an example of type 63.

The surface with Weierstrass equation

\[
\begin{align*}
0 & = y^2 + (t+1)xy + ty = x^3 + tx^2
\end{align*}
\]

has a 5-torsion section \((0,0)\) and two fibers of type I_5. It is an example of type 67.

The surface with Weierstrass equation

\[
\begin{align*}
0 & = y^2 + tx^3 + 3y \frac{t^2(t-1)}{16} \frac{x^3 + t(t-1)x^2}{16}
\end{align*}
\]

has a 4-torsion section \((0,0)\), and two fibers of types I_4 and I_5. It is an example of type 72.

Remark 22. Our tables and the one in [Sh4] cover all the cases of [OS], except lines 9, 27 and 73 of the table, with trivial lattice \(D_4\), \(E_6\) and \(D_7^2\) respectively. Since these have fibers with additive reduction, examples for them may be directly constructed using the families in [Sh2]. For instance, the elliptic surface

\[
\begin{align*}
y^2 & = x^3 - xt^2
\end{align*}
\]

has two fibers of type I_6^0 and Mordell-Weil group \((\mathbb{Z}/2\mathbb{Z})^2\). This covers line 73 of the table. For the other two cases, we refer the reader to the original examples of additive reduction in Section 3 of [Sh2].

5.2. Proofs. The proof proceeds analogously to the \(E_7\) case: with two differences: we only have one polynomial \(\Phi_\lambda(X)\) to work with (as opposed to having \(\Phi_\lambda(X)\) and \(\Psi_\lambda(X)\)), and the equations are a good deal more complicated.

We first write down the relation between the coefficients \(\epsilon_i\), \(1 \leq i \leq 9\), and the fundamental invariants \(\chi_j\); as before, we postpone the proofs to the auxiliary files and outline the idea in Section

| Type in [OS] | Fibral lattice | MW group |
|-------------|----------------|----------|
| 36          | \(A_3^3\)      | \(1/4\)  |
|             | \(E_7\)        | \(A_1^6\) |
| 44          | \(A_7\)        | \(A_1^6 \oplus \mathbb{Z}/2\mathbb{Z}\) |
| 45          | \(A_7\)        | \(1/8\)  |
| 46          | \(D_7\)        | \(1/4\)  |
| 54          | \(A_3 \oplus D_4\) | \(1/4 \oplus \mathbb{Z}/2\mathbb{Z}\) |
| 55          | \(A_3 \oplus A_4\) | \(1/20\) |
| 62          | \(E_8\)        | 0        |
| 63          | \(A_8\)        | \(\mathbb{Z}/3\mathbb{Z}\) |
| 64          | \(D_8\)        | \(\mathbb{Z}/2\mathbb{Z}\) |
| 67          | \(A_2^3\)      | \(\mathbb{Z}/5\mathbb{Z}\) |
| 72          | \(A_3 \oplus D_5\) | \(\mathbb{Z}/4\mathbb{Z}\) |

Here \(\zeta_3, I\) and \(\zeta_5\) are primitive third, fourth and fifth roots of unity.
Second, we write down the coefficients $\epsilon_i$ in terms of $\lambda = (p_0, \ldots, p_5, q_0, \ldots, q_4)$; see Section 6 for an explanation of how this is carried out. In the interest of brevity, we do not write out either of these sets of equations, but relegate them to the auxiliary computer files. Finally, setting the corresponding expressions equal to each other, we obtain a set of equations (1) through (9).

To solve these equations, proceed as follows: first use (1) through (5) to solve for $q_0, \ldots, q_4$ in terms of $\chi_j$ and $p_0, p_1, p_2$. Substituting these in to the remaining equations, we obtain (6') through (9'). These have low degree in $p_0$, which we eliminate, obtaining equations of relatively small degrees in $p_1$ and $p_2$. Finally, we take resultants with respect to $p_1$, obtaining two equations for $p_2$, of which the only common root is the one listed above. Working back, we solve for all the other variables, obtaining the system above and completing the proof of Theorem 15. The deduction of Theorem 14 now proceeds exactly as in the case of $E_7$.

Remark 23. As in the $E_7$ case, once we find the explicit relation between the Weierstrass coefficients and the fundamental characters, we may go back and explore the “genericity condition” for this family to have Mordell-Weil lattice isomorphic to $E_8$. To do this we compute the discriminant of the cubic in $x$, after completing the square in $y$, and take the discriminant with respect to $t$ of the resulting polynomial of degree 11. A computation shows that this discriminant factors as the cube of a polynomial $A(\lambda)$ (which vanishes exactly when the family has a fiber of additive reduction, generically type II), and the product of $(e^\alpha - 1)$, where $\alpha$ runs over minimal vectors of $E_8$. Again, the genericity condition to have Mordell-Weil lattice exactly $E_8$ is just the nonvanishing of

$$\Phi_{\lambda}(1) = \prod (e^\alpha - 1),$$

the expression which occurs in the Weyl denominator formula. Furthermore, the condition to have only multiplicative fibers is that $\Phi_{\lambda}(1)A(\lambda) \neq 0$.

As before, the proof of Theorem 20 follows immediately from the results of [Sh6, Sh7], by degeneration from a flat family.

6. Resultants, Interpolation and Computations

We now explain a computational aid, used in obtaining the equations expressing the coefficients of $\Phi_{\lambda}$ (for $E_8$) or $\Psi_{\lambda}$ (for $E_7$) in terms of the Weierstrass coefficients of the associated family of rational elliptic surfaces. We illustrate this using the system of equations obtained for sections of the $E_8$ family:

$$h^2 = g^3 + g^2$$
$$2ch = 3ag^2 + 2ag + 1$$
$$c^2 + 2dh = q_4 + gp_2 + 3bg^2 + (2b + 3a^2)g + a^2$$
$$2eh + 2cd = q_3 + ap_2 + gp_1 + 6abg + 2ab + a^3$$
$$2ce + d^2 = q_2 + bp_2 + ap_1 + gp_0 + 3b^2g + b^2 + 3a^2b$$
$$2de = q_1 + bp_1 + ap_0 + 3ab^2$$
$$e^2 = q_0 + bp_0 + b^3.$$

Setting $h = gu$ and solving the first equation for $g$ we have $g = u^2 - 1$. We solve the next three equations for $c, d, e$ respectively. This leaves us with three equations $R_1(a, b, u) = R_2(a, b, u) =$
$R_3(a, b, u) = 0$. These have degrees 2, 2, 3 respectively in $b$. Taking the appropriate linear combination of $R_1$ and $R_2$ gives us an equation $S_1(a, b, u) = 0$ which is linear in $b$. Similarly, we may use $R_1$ and $R_3$ to obtain another equation $S_2(a, b, u) = 0$, linear in $b$. We write

$$S_1(a, b, u) = s_{11}(a, u)b + s_{10}(a, u)$$
$$S_2(a, b, u) = s_{21}(a, u)b + s_{20}(a, u)$$
$$R_1(a, b, u) = r_2(a, u)b^2 + r_1(a, u)b + r_0(a, u).$$

The resultant of the first two polynomials gives us an equation

$$T_1(a, u) = s_{11}s_{20} - s_{10}s_{21} = 0$$

while the resultant of the first and third gives us

$$T_2(a, u) = r_2s_{10}^2 - r_1s_{10}s_{11} + r_0s_{11}^2 = 0.$$ 

Finally, we substitute $u = (v + 1)/(v - 1)$ throughout, obtaining two equations $\tilde{T}_1(a, v) = 0$ and $\tilde{T}_2(a, v) = 0$.

Next, we would like to compute the resultant of $\tilde{T}_1(a, v)$ and $\tilde{T}_2(a, v)$, which have degrees 8 and 9 with respect to $a$, to obtain a single equation satisfied by $v$. However, the polynomials $\tilde{T}_1$ and $\tilde{T}_2$ are already fairly large (they take a few hundred kilobytes of memory), and since their degree in $a$ is not too small, it is beyond the current reach of computer algebra systems such as gp/PARI or Magma to compute their resultant. It would take too long to compute their resultant, and another issue is that the resultant would take too much memory to store, certainly more than is available on the authors' computer systems (for instance, it would take more than 16GB of memory).

To circumvent this issue, what we shall do is to use several specializations of $\lambda$ in $\mathbb{Q}^8$. Once we specialize, the polynomials take much less space to store, and the computations of the resultants becomes tremendously easier. Since the resultant can be computed via the Sylvester determinant

$$\begin{vmatrix}
 a_8 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\
 0 & a_8 & a_2 & a_1 & a_0 & 0 & \ldots & 0 \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 0 & \ldots & 0 & a_8 & \ldots & a_2 & a_1 & a_0 \\
 0 & \ldots & 0 & 0 & a_8 & \ldots & a_2 & a_1 \\
 b_9 & b_8 & \ldots & b_2 & b_1 & b_0 & 0 & \ldots & 0 \\
 0 & b_9 & b_8 & \ldots & b_2 & b_1 & b_0 & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
 0 & \ldots & 0 & b_9 & b_8 & \ldots & b_2 & b_1 & b_0
\end{vmatrix}$$

where $\tilde{T}_1(a, v) = \sum a_i(v)a^i$ and $\tilde{T}_2(a, v) = \sum b_i(v)a^i$, we see that the resultant is a polynomial $Z(v) = \sum z_i v^i$ with coefficients $z_i$ being polynomials in the coefficients of the $a_i$ and the $b_i$, which happen to be elements of $\mathbb{Q}[\lambda]$ (recall that $\lambda = (p_0, \ldots, p_2, q_0, \ldots, q_1)$). Furthermore, we can bound the degrees $m_i(j)$ of $z_i(v)$ with respect to the $j$’th coordinate of $\lambda$, by using explicit bounds on the multidegrees of the $a_i$ and $b_i$. Therefore, by using Lagrange interpolation (with respect to the eight variables $\lambda_j$) we can reconstruct $z_i(v)$ from its specializations for various values of $\lambda$. The same method lets us show that $Z(v)$ is divisible by $v^{22}$ (for instance, by showing that $z_0$ through $z_{21}$ are zero), and also by $(v + 1)^8_0$ (by first shifting $v$ by 1 and then computing the Sylvester determinant, and proceeding as before), as well as by $(v^2 + v + 1)^8$ (this time, using cube roots of unity). Finally, it is clear that $Z(v)$ is divisible by the square of the resultant $G(v)$ of $s_{11}$ and $s_{10}$ with respect to
a. Removing these extraneous factors, we get a polynomial \( \Phi_\lambda(v) \) which is monic and reciprocal of degree 240. We compute its top few coefficients by this interpolation method.

Finally, we note the interpolation method above in fact is completely rigorous. Namely, let \( \epsilon_i(\lambda) \) be the coefficient of \( v^i \) in \( \Phi_\lambda(v) \), with bounds \((m_1, \ldots, m_8)\) for its multidegree, and \( \epsilon'_i(\lambda) \) is the putative polynomial we have computed using Lagrange interpolation on a set \( L_i = \{\ell_{i,0}, \ldots, \ell_{i,m_i}\} \) for \( 1 \leq i \leq 8 \) are sets of integers chosen generically enough to ensure that \( G(v) \) has the correct degree and that \( Z(v) \) is not divisible by any higher powers of \( v, v+1 \) or \( v^2 + v + 1 \) than in the generic case. Then since \( \epsilon'_j(\ell_{i_1,i_2}, \ldots, \ell_{s,i_s}) = \epsilon'_j(\ell_{i_1,i_2}, \ldots, \ell_{s,i_s}) \) for all choices of \( i_1, \ldots, i_8 \), we see that the difference of these polynomials must vanish.

7. Representation theory, and some identities in Laurent polynomials

Finally, we demonstrate how to deduce the identities relating the coefficients of \( \Phi_{E_7,\lambda}(X) \) or \( \Psi_{E_7,\lambda}(X) \) to the fundamental characters for \( E_7 \) (and similarly, the coefficients of \( \Phi_{E_8,\lambda}(X) \) to the fundamental characters of \( E_8 \)).

Conceptually, the simplest way to do this is to express the alternating powers of the 56-dimensional representation \( V_7 \) or the 133-dimensional representation \( V_1 \) in terms of the fundamental modules of \( E_7 \) and their tensor products. We know that the character \( \chi_1 \) of \( V_1 \) is \( 7 + \sum e^\alpha \), where the sum is over the 126 roots of \( E_7 \). Therefore we have \((-1)^{\eta_1} = \chi_1 - 7 \). For the next example, we consider \( \Lambda^2 V_1 = V_3 \oplus V_1 \). This gives rise to the equation

\[
\eta_2 + 7 \cdot (-1)^{\eta_1} + \left(\frac{7}{2}\right) = \chi_3 + \chi_1
\]

which gives the relation \( \eta_2 = \chi_3 - 6\chi_1 + 28 \).

A similar analysis can be carried out to obtain all the other identities used in our proofs, using the software LiE, available from http://www-math.univ-poitiers.fr/~maavl/LiE/.

A more explicit method is to compute the expressions for the \( \chi_i \) as Laurent polynomials in \( s_1, \ldots, s_6, r \) (note that \( s_7 = r^3/(s_1 \ldots s_6) \)), and then do the same for the \( \epsilon_i \) or \( \eta_i \). The latter calculation is simplified by computing the power sums \( \sum(e^\alpha)^i \) (for \( \alpha \) running over the smallest vectors of \( E_7^* \) or \( E_8^* \)), for \( 1 \leq i \leq 7 \) and then using Newton’s formulas to convert to the elementary symmetric polynomials, which are \((-1)^i\epsilon_i \) or \((-1)^i\eta_i \). Finally, we check the identities by direct computation in the Laurent polynomial ring (it may be helpful to clear out denominators). This method has the advantage that we obtain explicit expressions for the \( \chi_i \) (and then for \( \lambda \) by Theorem 8) in terms of \( s_1, \ldots, s_6, r \), which may then be used to generate examples such as Example 7.

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The auxiliary computer files for checking our calculations are available from the arXiv.org e-print archive, where it is file number arXiv.org:1204.1531. To access the auxiliary files, download the source file for the paper. This will produce not only the \( \LaTeX \) file for this paper, but also the computer code. The file README.txt gives an overview of the various computer files involved.
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REFERENCES

[B] N. Bourbaki, Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968.

[C] Carter, Lie algebras of finite and affine type, Cambridge Studies in Advanced Mathematics, 96, Cambridge University Press, Cambridge, 2005.

[ES] T. Eguchi and K. Sakai, Seiberg-Witten curve for $E$-string theory revisited, Adv. Theor. Math. Phys. 7 (2003), no. 3, 419–455.

[JKZ] F. Jouve, E. Kowalski and D. Zywina, An explicit integral polynomial whose splitting field has Galois group $W(E_8)$, J. Théor. Nombres Bordeaux 20 (2008), no. 3, 761–782.

[Ko] K. Kodaira, On compact analytic surfaces II-III, Ann. of Math. 77 (1963), 563–626; 78 (1963), 1–40; Collected Works, III, 1269–1372, Iwanami and Princeton Univ. Press (1975).

[Ku] D. S. Kubert, Universal bounds on the torsion of elliptic curves, Proc. London Math. Soc. (3) 33 (1976), no. 2, 193–237.

[L] M. Lorenz, Multiplicative invariant theory, Encyclopaedia of Mathematical Sciences, 135: Invariant Theory and Algebraic Transformation Groups, VI, Springer-Verlag, Berlin, 2005.

[OS] K. Oguiso and T. Shioda, The Mordell-Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Paul. 40 (1991), no. 1, 83–99.

[Se] J.-P. Serre, Lectures on the Mordell-Weil theorem. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, Aspects of Mathematics, E15, Friedr. Vieweg & Sohn, Braunschweig, 1989.

[Sh1] T. Shioda, On the Mordell-Weil lattices, Comment. Math. Univ. St. Paul. 39 (1990), no. 2, 211–240.

[Sh2] T. Shioda, Construction of elliptic curves with high rank via the invariants of the Weyl groups, J. Math. Soc. Japan 43 (1991), no. 4, 673–719.

[Sh3] T. Shioda, Theory of Mordell-Weil lattices, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 473–489, Math. Soc. Japan, Tokyo, 1991.

[Sh4] T. Shioda, Multiplicative excellent family of type $E_6$, Proc. Japan Acad. Ser. A 88 (2012), no. 3, 46–51.

[Sh5] T. Shioda, A uniform construction of the root lattices $E_6$, $E_7$, $E_8$ and their dual lattices, Proc. Japan Acad. Ser. A Math. Sci. 71 (1995), no. 7, 140–143.

[Sh6] T. Shioda, Gröbner basis, Mordell-Weil lattices and deformation of singularities. I., Proc. Japan Acad. Ser. A Math. Sci. 86 (2010), no. 2, 21–26.

[Sh7] T. Shioda, Gröbner basis, Mordell-Weil lattices and deformation of singularities. II., Proc. Japan Acad. Ser. A Math. Sci. 86 (2010), no. 2, 27–32.

[Sh8] T. Shioda, Some explicit integral polynomials with Galois group $W(E_8)$, Proc. Japan Acad. Ser. A Math. Sci. 85 (2009), no. 8, 118–121.

[SU] T. Shioda and H. Usui, Fundamental invariants of Weyl groups and excellent families of elliptic curves, Comment. Math. Univ. St. Paul. 41 (1992), no. 2, 169–217.

[T] J. Tate, Algorithm for determining the type of a singular fiber in an elliptic pencil, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 33–52. Lecture Notes in Math., Vol. 476, Springer, Berlin, 1975.

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