FINITE F-INVERSE COVERS DO EXIST

JULIAN BITTERLICH
TECHNISCHE UNIVERSITÄT DARMSTADT

Abstract. We show that every finite inverse monoid has an idempotent-
separating cover by a finite F-inverse monoid. This provides a positive answer
to a conjecture of Henckell and Rhodes [4].

1. Introduction

An important theorem in the area of inverse monoids is McAlister’s Covering
Theorem [5, 6].

Theorem 1. For every inverse monoid $M$ there is an $E$-unitary inverse monoid
$N$ and an idempotent-separating cover $\theta : N \to M$.

Typically, the statement of McAlister’s Covering Theorem is just given as: “every
inverse monoid has an $E$-unitary cover.”

The proof of McAlister’s Covering Theorem can be adapted to finite inverse
monoids as well. This finite version reads in short: “every finite inverse monoid has
a finite $E$-unitary cover.”

Theorem 2. For every finite inverse monoid $M$ there is a finite, $E$-unitary inverse
monoid $N$ and an idempotent-separating cover $\theta : N \to M$.

Actually, in [5] the classical version and the finite version of McAlister’s Covering
Theorem are proved as a single statement.

A strengthening of McAlister’s Covering Theorem is Lawson’s Covering Theorem
[5]. It shows the existence of F-inverse covers instead of ‘just’ $E$-unitary covers.

Theorem 3. For every inverse monoid $M$ there is an $F$-inverse monoid $N$ and an
idempotent-separating cover $\theta : N \to M$.

In the sequel we show that the finite version of Lawson’s Covering Theorem is
also valid.

Theorem 4. For every finite inverse monoid $M$ there is a finite $F$-inverse monoid
$N$ and an idempotent-separating cover $\theta : N \to M$.

Theorem 4 was first conjectured by Henckell and Rhodes [4] as a possible route
to an affirmative answer for the pointlike conjecture, an important conjecture in the
theory of monoids (cf. [3]). The ‘pointlike conjecture’ was proved to be true by Ash
[1] but the validity of the finite version of Lawson’s Covering Theorem remained an
open problem. Some conditional results on the conjecture of Henckell and Rhodes
are given in [2, 9, 10].

Date: August 16, 2018.
The proof of Theorem 4 is divided in two parts. In Section 3 we introduce a compatibility relation between groups and inverse monoids and show how these compatible groups can be used to obtain F-inverse covers (Lemma 2 (ii)). In Section 5, which constitutes the technical part of this work, we then show the existence of finite such compatible groups (Lemma 5). While developing these techniques we also give proofs of McAlister’s and Lawson’s covering theorems.

In Section 4 we discuss these covering theorems for inverse semigroups and how they are implied by the inverse monoid versions.

The presentation of the results here is self-contained. We provide all definitions about E-unitary and F-inverse monoids in the next section. For further information about this topic, the reader may consult the monograph of Lawson [5].

2. Inverse monoids

Basic definitions. An inverse monoid is a monoid $M$ that has for each element $x \in M$ a unique element $x^{-1} \in M$ such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. The partial bijections on a set $X$ form the symmetric inverse monoid on $X$; denoted by $I(X)$. The Wagner–Preston Representation Theorem tells us that every inverse monoid is a submonoid of a symmetric inverse monoid. Consequently, we can visualise many definitions and results about inverse monoids in an illustrative way using partial bijections.

The set of idempotents of an inverse monoid $M$ is denoted by $E(M)$. In $I(X)$ the idempotents are just the restrictions of the identity. So clearly, the idempotents of $I(X)$ commute and thus, by the Wagner–Preston Theorem, the idempotents of any inverse monoid do commute.

The natural partial order on $M$ is defined by

$$x \leq y :\iff x = ey \text{ for some } e \in E(M).$$

In $I(X)$ the natural partial order $x \leq y$ just amounts to ‘$x$ is a restriction of $y$’. From this we can deduce, using Wagner–Preston, that the natural partial order on an inverse submonoid $M$ of an inverse monoid $N$ is induced by the natural partial order on $N$. Also note that the idempotents $E(M)$ are those elements that lie beneath 1 w.r.t. the natural partial order.

The minimum group congruence on $M$ is defined by

$$x \sigma y :\iff z \leq x, y \text{ for some } z \in M.$$

As the name suggest, $\sigma$ is the smallest congruence $\rho$ on $M$ for which $M/\rho$ is a group. Note that all idempotents lie in the same $\sigma$-class and that 1 is a maximal element in this class w.r.t. the natural partial order.

E-unitary inverse monoids and F-inverse monoids. A subset $X$ of an inverse monoid $M$ is called unitary if for any elements $x \in X$ and $z \in M$, $xz \in X$ or $zx \in X$ implies $z \in X$.

An inverse monoid is E-unitary if $E(M)$ is unitary. It is easy to see that an inverse monoid is E-unitary if, and only if,

$$e \leq x \text{ for some } e \in E(M) \implies x \in E(M)$$

for all $x \in M$. 

An $F$-inverse monoid is an inverse monoid in which each $\sigma$-class has a maximum w.r.t. the natural partial order. A inverse monoid is an F-inverse monoid if, and only if,

$$z \leq x, y \text{ for some } z \implies x, y \leq z \text{ for some } z$$

for all $x, y \in M$. F-inverse monoids are also $E$-unitary: if $e \leq x$ with $e \in E(M)$, then $e \leq x, 1$. So there is a $z$ with $x, 1 \leq z$. Clearly, $z = 1$ and so $x \leq 1$. Thus $x \in E(M)$.

Covers. A homomorphism between inverse monoids is a semigroup homomorphism. A homomorphisms $\theta: N \to M$ maps idempotents to idempotents and is order-preserving w.r.t. the natural partial order, i.e.,

$$x \leq y \implies \theta(x) \leq \theta(y).$$

A homomorphism is a cover if it is surjective; it is idempotent-separating if it is injective on the idempotents.

3. Constructing covers using compatible groups

Inverse monoids with generators. It is beneficial for us to use inverse monoids with a fixed set of generators. We let $P$ always stand for a set of symbols with an associated involution $(\cdot)^{-1}$. A $P$-generated inverse monoid is an inverse monoid $M$ with a set of generators $\{p^M \mid p \in P\}$, s.t. $(p^M)^{-1} = (p^{-1})^M$.

For $u = p_1 \ldots p_n \in P^*$ we set $u^M = p_1^M \ldots p_n^M$. Note that $(\cdot)^M$ constitutes a monoid homomorphism and that $(\cdot)^M$ is also compatible with inverses, i.e., $(u^{-1})^M = (u^M)^{-1}$ where $u^{-1} := p_n^{-1} \ldots p_1^{-1}$.

Clearly any inverse monoid can be cast as a $P$-generated inverse monoid for a suitable choice of $P$, and for finite inverse monoids we can choose this $P$ to be finite as well. We always assume that $P$ is finite if we talk about finite $P$-generated inverse monoids.

The product construction. We describe a product of $P$-generated inverse monoids and $P$-generated groups that induces idempotent-separating covers.

For a $P$-generated monoid $M$ and a $P$-generated group $G$ let $M \times_P G$ be the $P$-generated inverse monoid given as a submonoid of $M \times G$ with generators $\{(p^M, p^G) \mid p \in P\}$.

The natural partial order on $M \times_P G$ is given by

$$(m_1, g_1) \leq (m_2, g_2) \iff m_1 \leq m_2 \text{ and } g_1 = g_2,$$

and the idempotents are given by

$$(m, g) \in E(M \times_P G) \iff m \in E(M) \text{ and } g = 1.$$

Lemma 1. Let $\pi: M \times_P G \to M$ be the projection to the first component. Then $\pi: M \times_P G \to M$ is an idempotent-separating, surjective homomorphism.

Proof. Clearly $\pi$ is a homomorphism and surjective. $\pi$ is injective on $E(M \times_P G)$ since idempotents in $M \times_P G$ are purely characterised by their first component. \(\square\)
Compatible groups. We introduce two compatibility notions between $P$-generated inverse monoids and $P$-generated groups that ensure that the product $M \times_P G$ is (i) $E$-unitary or (ii) an $F$-inverse monoid.

**Definition 1.** A $P$-generated group $G$ is

(i) **compatible** with $M$ if for all $u \in P^*$

$$u^G = 1 \implies u^M \leq 1.$$  

(ii) **strongly compatible** with $M$ if for all $u, w \in P^*$

$$u^G = w^G \implies v^G = u^G = w^G \text{ and } u^M, w^M \leq v^M \text{ for some } v \in P^*.$$  

**Lemma 2.** Let $M$ be a $P$-generated inverse monoid and $G$ a $P$-generated group. Then $M \times_P G$ is

(i) $E$-unitary if $G$ is compatible with $M$.

(ii) an $F$-inverse monoid if $G$ is strongly compatible with $M$.

**Proof.** Let $N = M \times_P G$. We use the fact that we can denote the elements in $N$ by $u^N = (u^M, u^G)$ for $u \in P^*$.

(i) Let $u^N \leq w^N$ and $u^N \in E(N)$. Then $u^G = w^G$ and $u^G = 1$. Hence $w^G = 1$. Since $G$ is compatible with $M$, this gives $w^M \leq 1$, i.e., $w^M = e \in E(M)$. So $w^N = (w^M, w^G) = (e, 1) \in E(N)$.

(ii) Let $w^N \leq u^N_1, u^N_2$. Then $u^N_1 = w^G = u^G_2$. Since $G$ is strongly compatible with $M$, we obtain a $v \in P^*$ s.t. $v^G = u^G_1 = u^G_2$ and $u^M_1, u^M_2 \leq v^M$. So $u^N_1 = (u^M_1, u^G_1) \leq (v^M, v^G) = v^N$ and $u^N_2 = (u^M_2, u^G_2) \leq (v^M, v^G) = v^N$.

$\square$

Note that $\text{FG}(P)$, the free group over $P$, is strongly compatible with any $P$-generated inverse monoid. With this we can prove the covering theorems of McAlister and Lawson.

**Proof of Theorem 4 and Theorem 5.** It suffices to show Theorem 4 since $F$-inverse monoids are also $E$-unitary.

Let $M$ be an inverse monoid. We can see $M$ as a $P$-generated inverses monoid by a suitable choice of $P$ and generators of $M$. Then $M \times_P \text{FG}(P)$ is an $F$-inverse monoid by Lemma 2. Furthermore, $\pi : M \times_P \text{FG}(P) \to M$ is an idempotent-separating cover by Lemma 11.

$\square$

We now want to prove the finite versions of these covering theorems. For that we have to provide finite (strongly) compatible groups.

**Lemma 3.** For every finite $P$-generated inverse monoid there is a finite compatible $P$-generated group.

**Proof.** By the Wagner–Preston Theorem we can think of the $P$-generated inverse monoid $M$ as a submonoid of $I(X)$ for some finite set $X$. Then every $p^M$ is a partial bijection on $X$. We can extend each of the $p^M$ to a bijection $p^G$. Then the subgroup of the symmetric group of $X$ generated by the $\{ p^G \mid p \in P \}$ is compatible with $M$.

$\square$

We can now finish the proof of the finite version of McAlister’s Covering Theorem.
**Proof of Theorem 2.** Let $M$ be a finite inverse monoid. We can see $M$ as a finite $P$-generated inverse monoid by a suitable choice of $P$ and generators of $M$. Lemma 3 guarantees the existence of a finite $P$-generated group $G$ compatible with $M$. The product $M \times_P G$ is a finite, E-unitary inverse monoid by Lemma 2. Furthermore, $\pi : M \times_P G \to M$ is an idempotent-separating cover by Lemma 1. □

Similarly, with the help of the next lemma we can obtain the finite version of Lawson’s Covering Theorem.

**Lemma 4.** For every finite $P$-generated inverse monoid there is a finite strongly compatible $P$-generated group.

**Proof.** The proof is the subject of Section 5. There we describe a construction of a finite $P$-generated group $G$ for a given finite $P$-generated inverse monoid $M$ that is compatible with $M$ by Lemma 5. □

**Proof of Theorem 4.** Let $M$ be a finite inverse monoid. We can see $M$ as a finite $P$-generated inverse monoid by a suitable choice of $P$ and generators of $M$. Lemma 3 guarantees the existence of a finite $P$-generated group $G$ strongly compatible with $M$. The product $M \times_P G$ is a finite, F-inverse monoid by Lemma 2. Furthermore, $\pi : M \times_P G \to M$ is an idempotent-separating cover by Lemma 1. □

4. Covering theorems for inverse semigroups

The covering theorems of McAlister and Lawson are originally stated for for inverse semigroups. In this section we want to argue that these distinctions in the covering theorems do not matter as we can deduce the inverse semigroup versions from the inverse monoid versions and vice versa.

An inverse semigroup is a semigroup that has unique inverses as defined for inverse monoids. All notions introduced so far translate to inverse semigroups as well. (F-inverse semigroups are an exception; we discuss this after the following lemma.)

It is easy to see that the covering theorems for inverse semigroups directly imply their counterparts for inverse monoids by virtue of the following lemma.

**Lemma.** Let $\theta : S \to M$ be a surjective, idempotent-separating homomorphism of inverse semigroups. Then $S$ is a monoid if $M$ is a monoid.

**Proof.** Let $f \in S$ with $\theta(f) = 1$. W.l.o.g. $f \in E(S)$, otherwise we continue with $ff^{-1}$. For $e \in E(S)$, $\theta(fe) = \theta(e)$ and so $fe = e$ as $\theta$ is injective on $E(S)$. Thus $fe = e$ for all $e \in E(S)$. Now, for $x \in S$, $fx = fxe^{-1}x = (fxe^{-1})xe = xe^{-1}x = x$.

Similarly $xf = x$. Thus $f$ is a neutral element in $S$. □

For F-inverse covers the situation is actually a bit more complicated. If we transfer the definition of F-inverse monoids to inverse semigroups we get that each such ‘F-inverse semigroup’ is automatically a monoid. However, there is a definition of F-inverse semigroups that extends the notion of F-inverse monoids but does not force a semigroup to be a monoid (see [5, Chapter 7.4] for the definition and properties of F-inverse semigroups). Lawson’s Covering Theorem is given with this definition of F-inverse semigroups in mind. Nevertheless, the notions of F-inverse semigroups and F-inverse monoids agree on the class of monoids and so Lawson’s Covering
Theorem for inverse semigroups directly implies Lawson’s Covering Theorem for inverse monoids.

Now we describe how to obtain the covering theorems for inverse semigroups using the covering theorems for inverse monoids. Let $S$ be an inverse semigroup. Then $S^1$ is constructed from $S$ by adding a neutral element. Note that $S^1$ does not have any non-trivial units. The idea is now that we can apply the covering theorems to $S^1$ and subsequently remove the 1 again.

The next lemma shows that we can get McAlister’s Covering Theorem for inverse semigroups from the corresponding theorem for inverse monoids.

**Lemma.** Let $N$ be an $E$-unitary inverse monoid, $S$ an inverse semigroup and $\theta: N \to S^1$ an idempotent-separating cover. Then $N \setminus \ker(\theta)$ is an $E$-unitary inverse semigroup and $\theta|_{N \setminus \ker(\theta)}: N \setminus \ker(\theta) \to S$ is an idempotent-separating cover.

**Proof.** Clearly, $N \setminus \ker(\theta)$ is closed under inverses. It is also closed under products since $S^1$ does not have any non-trivial units. So $N \setminus \ker(\theta)$ is an inverse subsemigroup of $N$. Being $E$-unitary is a universal statement and hence it is preserved under passage to inverse subsemigroups. So $N \setminus \ker(\theta)$ is $E$-unitary. It is clear that $\theta|_{N \setminus \ker(\theta)}: N \setminus \ker(\theta) \to S$ is an idempotent separating cover. □

We want to prove a similar statement for Lawson’s Covering Theorem. For that we need the following result [5, Chapter 7.4 Lemma 8]

**Lemma.** Let $S$ be an inverse subsemigroup of an $F$-inverse semigroup $T$ s.t.

1. $E(S)$ is an order ideal of $E(T)$, i.e.,
   $$f \leq e \quad \Rightarrow \quad f \in E(S)$$
   for $e \in E(S)$ and $f \in E(T)$,

2. for $t \in T$,
   $$t^{-1}tt^{-1} \in S \quad \Rightarrow \quad t \in S.$$

Then $S$ is also an $F$-inverse semigroup.

**Lemma.** Let $N$ be an $F$-inverse monoid, $S$ an inverse semigroup and $\theta: N \to S^1$ an idempotent-separating cover. Then $N \setminus \ker(\theta)$ is an $F$-inverse semigroup and $\theta|_{N \setminus \ker(\theta)}: N \setminus \ker(\theta) \to S$ is an idempotent separating cover.

**Proof.** The only non-trivial part is to check that $N \setminus \ker(\theta)$ satisfies the properties (i) and (ii) of the previous lemma.

For (i) let $e, f \in E(N)$ with $f \leq e$ and $\theta(e) \neq 1$. Then $\theta(f) \leq \theta(e) \neq 1$ and so $\theta(f) \neq 1$. The right-hand side of (ii) translates to ‘$\theta(t)$ is a unit’ but the only unity in $S^1$ is 1. So (ii) is satisfied as well. □

5. Constructing strongly compatible groups

This section is solely dedicated to the proof of Lemma [4] We describe a construction that for any given $P$-generated inverse monoid $M$ produces a $P$-generated group strongly compatible with $M$. A key part (or one could say black box) in this construction is a theorem by Otto (Theorem [5]).

Before we can describe and discuss the construction we need to introduce some notations regarding finitely generated groupoids, in order to state Otto’s Theorem.
Finitely generated groupoids and the Theorem of Otto. We see a groupoid $G$ as a generalisation of a group in which every element $g \in G$ has an associated source, $s(g)$, and a target, $t(g)$, which impose the usual restrictions on the multiplication operation. The sources and targets of $G$ constitute the objects of the groupoid. We denote the neutral element at an object $a$ by $\text{id}_a$.

Similarly to $P$-generated groups we want to work with groupoids with generator $s$. By their typed nature it is natural to use graphs to describe the generators.

A multidigraph $I = (V, E)$ is a two-sorted structure with vertices $V$ and edges $E$. Every edge $e \in E$ has a source, $s(e) \in V$, and target, $t(e) \in V$. We also assume that there is an involution $(\cdot)^{-1}$ on $E$ s.t. $s(e^{-1}) = t(e)$ and $t(e^{-1}) = s(e)$. A walk $u$ in $I$ is a sequence of edges $u = e_1 \ldots e_n$ s.t. $t(e_i) = s(e_{i+1})$. We denote all walks over $I$ by $I^*$. For $\alpha \subseteq E$ closed under $(\cdot)^{-1}$ we let $I(\alpha)$ be the subgraph $(V, \alpha)$. So $I(\alpha)^*$ denotes all walks in $I$ which only consist of edges in $\alpha$.

An $I$-groupoid is a groupoid $G$ with generators $\{ e^G \mid e \in E \}$ s.t.

$$s(e^G) = s(e), t(e^G) = t(e), \text{ and } (e^{-1})^G = (e^{-1})^G.$$

We let $u^G := e_1^G \ldots e_n^G$ for $u = e_1 \ldots e_n \in E^*$, and $G(\alpha) := \{ u^G \mid u \in I(\alpha)^* \}$ for $\alpha \subseteq E$ closed under $(\cdot)^{-1}$.

$G$ is 2-acyclic if for all $\alpha, \beta \subseteq E$ closed under $(\cdot)^{-1}$

$$G(\alpha) \cap G(\beta) = G(\alpha \cap \beta).$$

See Figure 1 for a sketch that depicts which forms of cyclic configurations are forbidden by 2-acyclicity.

A symmetry of $I$ is a two sorted map $\varphi = (\varphi_V, \varphi_E)$ s.t.

$$s(\varphi_E(e)) = \varphi_V(s(e)), t(\varphi_E(e)) = \varphi_V(t(e)), \text{ and } \varphi_E(e^{-1}) = \varphi_E(e)^{-1}.$$

An $I$-groupoid $G$ is symmetric if every symmetry $\varphi$ of $I$ induces an automorphism $\varphi_G$ of $G$ induced by $\varphi_G(e^G) = \varphi_E(e)^G$. The following theorem is due to Otto [7] (also see [8] for an extended version).

**Theorem 5.** For every finite $I = (V, E)$ there are finite, symmetric, 2-acyclic $I$-groupoids.
The main construction. We describe a construction of a finite $P$-generated group $G$ that is strongly compatible with a given finite, $P$-generated inverse monoid $M$. The construction proceeds in 4 steps:

Step 1: Let $F$ be a finite $P$-generated group compatible with $M$ (guaranteed to exist by Lemma \textup{[4]})

Step 2: Let $I$ be the Gaifman graph of $F$, i.e., the finite multidigraph $I = (F, E)$ where

\[
E = \{(f, p) \mid f \in F, p \in P\} \quad \text{and} \quad s((f, p)) = f, \quad t((f, p)) = fp^p, \quad (f, p)^{-1} = (fp^p, p^{-1}).
\]

Step 3: Let $\mathbb{H}$ be a finite, symmetric, 2-acyclic $I$-groupoid (guaranteed to exist by Otto’s Theorem (Theorem \textup{[6]})).

Step 4: Let $G$ be the $P$-generated group given as a subgroup of the symmetric group of $\mathbb{H}$ (here $\mathbb{H}$ seen as a plain set) generated by $\{p^G \mid p \in P\}$ where

\[
p^G(h) := h(t(h), p)\mathbb{H}.
\]

Clearly the resulting $P$-generated group $G$ is finite. It remains to show that $G$ is indeed strongly compatible with $M$.

In the following we reserve $a, b, c$ to denote elements in $P^*$ and $u, v, w$ to denote elements in $I^*$.

Note that we can think of $I$ as a cover of $P$, each edge $(f, p)$ in $I$ is canonically labelled by $p \in P$ and every vertex in $I$ is adjacent to exactly one edge with colour $p$. This enables us to pass from $I^*$ to $P^*$ by projecting walks $(f_1, p_1) \ldots (f_n, p_n)$ to words $p_1 \ldots p_n$; we write $\pi$ for this projection. On the other hand, for a fixed $f \in F$, we can uniquely lift a word $a$ over $P$ to a walk $u$ in $I$ starting at $f$, i.e., there is a unique $u \in I^*$ s.t. $s(u) = f$ and $\pi(u) = a$. So, lifts and projections give us means to translate between $P^*$ and $I^*$.

The following lemma shows that $G$ is strongly compatible with $M$. The lemma references some auxiliary statements that are proved subsequently. It might be instructive to read the proof of the lemma once to get an idea about the crucial steps, then read the proof of the auxiliary lemmas, and after that read this proof once more.

**Lemma 5.** Let $M$ be a finite $P$-generated inverse monoid and $G$ as above. Then $G$ is a finite $P$-generated group strongly compatible with $M$.

**Proof.** Clearly, $G$ is finite. We show that $G$ is strongly compatible with $M$. Let $a, b \in P^*$ with $a^G = b^G$. We set $u, v \in I^*$ to be the lifts of $a, b$ to $1$. Then, by Lemma \textup{[7] (i)}, $u^H = a^G(id_1)$ and $v^H = b^G(id_1)$. Thus $u^H = v^H$.

Set $\alpha = \{ e, e^{-1} \in E \mid e \text{ appears in } u \}$ and $\beta = \{ e, e^{-1} \in E \mid e \text{ appears in } w \}$. Then $u \in I(\alpha)$, $w \in I(\beta)$, and $u^H = w^H$. By 2-acyclicity of $\mathbb{H}$, there is a $v \in I(\alpha \cap \beta)$ s.t. $v^H = u^H = w^H$.

We show that $c := \pi(v)$ is as desired, i.e., $c^G = a^G = b^G$ and $a^M, b^M \leq c^M$. Note that $v$ is the lift of $e$ to $1$ and so, again by Lemma \textup{[7] (i)}, $v^H = c^G(id_1)$. As $a^G, b^G$ and $c^G$ agree for one argument, namely $id_1$, they have to be equal, according to Lemma \textup{[7] (ii)}.

By construction of $v$, every edge $e$ that appears in $v$ also appears in $u$ and in $w$ either directly as $e$ or as $e^{-1}$. So by Lemma \textup{[3]} we get that $\pi(u)^M, \pi(w)^M \leq \pi(v)^M$ and thus $a^M, b^M \leq c^M$. \hfill $\square$
We give now the proofs of the auxiliary lemmas.

**Lemma 6.** Let $u,v \in I^*$ with the same sources and targets s.t. every edge or its inverse that appears in $v$ also appears in $u$. Then $\pi(u)^M \leq \pi(v)^M$.

**Proof.** Note that $\pi(w)^M \in E(M)$ if $s(w) = t(w)$: it can be shown by induction that $\pi(w)^F = s(w)^{-1}t(w)$ for all $w \in I^*$ (keep in mind that the vertices of $I$ are the elements of $F$). If now $s(w) = t(w)$, then $\pi(w)^F = 1$ and thus $\pi(w)^M \in E(M)$ as $F$ is compatible with $M$.

We prove the statement of the lemma by induction over the length of $v$. If $|v| = 0$, then $s(u) = t(u)$ and thus $\pi(u)^M \leq 1 = \pi(v)^M$. For the induction step let $e$ be the last edge in $v$, i.e., $v = v'e$. Then $e$ or $e^{-1}$ also appears somewhere in $u$. We distinguish these two cases.

1. $e$ appears in $u$. Then $u$ can be decomposed into $u = u_1ev_2$ as in the following sketch

\[ u_1 \rightarrow e \rightarrow v' \rightarrow u_2 \]

We see that $s(u_2) = t(u_2)$ and so $\pi(u_2)^M \in E(M)$. Note that we can apply the induction hypothesis to $u_1ev_2u_2^{-1}e^{-1}$ and $v'$, i.e., $\pi(u_1ev_2u_2^{-1}e^{-1})^M \leq \pi(v')^M$. So we get

\[
\pi(u_1ev_2)^M = \pi(u_1)^M \pi(eu_2)^M = \pi(u_1)^M \pi(eu_2)^M (\pi(eu_2)^M)^{-1} = \pi(u_1)^M \pi(eu_2)^M \\
= \pi(v')^M \pi(eu_2)^M \leq \pi(v)^M \\
= \pi(v'^{-1}e)^M \pi(u_2)^M \leq \pi(v'e)^M
\]

Note that the last inequality is true just by the definition of $\leq$.

2. $e^{-1}$ appears in $u$. Then $u$ can be decomposed into $u = u_1e^{-1}u_2$. A sketch of how $u$ and $v$ decompose is given here:

\[ u_1 \rightarrow v' \rightarrow u_2 \rightarrow e \]

We see that $s(e^{-1}u_2) = t(e^{-1}u_2)$ and so $\pi(e^{-1}u_2)^M \in E(M)$. Also note that we can apply the induction hypothesis to $u_1e^{-1}u_2u_2^{-1}$ and $v'$, i.e., $\pi(u_1e^{-1}u_2u_2^{-1})^M \leq \pi(v')^M$. So we get

\[
\pi(u_1e^{-1}u_2)^M = \pi(u_1)^M \pi(e^{-1}u_2)^M = \pi(u_1)^M \pi(e^{-1}u_2)^M \pi(e^{-1}u_2)^M \\
= \pi(u_1)^M \pi(e^{-1}u_2)^M \pi(e^{-1}u_2)^M = \pi(u_1e^{-1}u_2u_2^{-1})^M \pi(e)^M \\
\leq \pi(v')^M \pi(e)^M = \pi(v'e)^M.
\]

In both cases $\pi(u)^M \leq \pi(v)^M$. \qed

We now want to show that the elements in $G$ are completely determined by one pair of value and argument. This fact uses that $H$ is symmetric.
Every \( f \in F \) defines a symmetry \( \phi_f \) of \( I \) whose action on the vertices is given by \( \phi_f(f') = ff' \) and its action on the edges given by \( \phi_f((f',p)) = (ff',p) \) (notationally we do not explicitly distinguish between the vertex part and the edge part of \( \phi \)). Note that \( \phi_f((f',p)) = (\phi_f(f'),p) \). Since \( \mathbb{H} \) is symmetric, \( \phi_f \) can be extended to a symmetry \( \phi_{f,\mathbb{H}} \) of \( \mathbb{H} \), i.e., \( \phi_{f,\mathbb{H}}(u^\mathbb{H}) := (\phi_f(u))^\mathbb{H} \) is well-defined. We give a proof sketch of the following fact:
\[
\phi_{f,\mathbb{H}}(a^G(h)) = a^G(\phi_{f,\mathbb{H}}(h))
\]
for \( a \in P^* \) and \( h \in \mathbb{H} \). The proof is by induction on the length of \( a \), and at its core lies the statement that \( \phi_{f,\mathbb{H}}(p^G(u^\mathbb{H})) = a^G(\phi_{f,\mathbb{H}}(u^\mathbb{H})) \) for \( p \in P \) and \( u \in I \). We show this by proving that both sides are equal to \( \phi_f(u(t(u),p))^\mathbb{H} \). For the left-hand side we get
\[
\phi_{f,\mathbb{H}}(p^G(u^\mathbb{H})) = \phi_{f,\mathbb{H}}(u^\mathbb{H}(t(u),p)^\mathbb{H}) = \phi_{f,\mathbb{H}}(u^\mathbb{H}(t(u),p)^\mathbb{H}) = \phi_f(u(t(u),p))^\mathbb{H}
\]
and for the right-hand side we get the same.

Lemma 7. Let \( a, b \in P^* \). Then
\[(i) \ a^G(id_1) = u^\mathbb{H}, \text{ where } u \text{ is the lift of } a \text{ to } 1 \text{ in } I,
(ii) \ a^G = b^G \text{ if } a^G(h) = b^G(h) \text{ for some } h \in \mathbb{H}.
\]
Proof. (i) is a consequence of the stronger statement
\[u^\mathbb{H} = \pi(u)^G(id_{\pi(u)}),\]
for \( u \in I^* \). This stronger statement can be easily proved by induction.
To prove (ii) we note that in general
\[a^G(h') = h' \cdot h^{-1} \cdot a^G(h) \text{ for all } h, h' \text{ with } t(h) = t(h'),\]
which can be proved easily by induction. With this we can show that for \( h, h' \in \mathbb{H}, \ a \in P^* \) and \( \eta = \phi_{t(h')h^{-1},\mathbb{H}} \) we have that
\[
a^G(h') = h' \cdot \eta^{-1}(h') \cdot a^G(id_{t(h')})) = h' \cdot \eta(\eta^{-1}(h') \cdot a^G(h))
\]
\[
= h' \cdot \eta(h^{-1} \cdot a^G(h)).
\]
We can now finish the argument for (ii). If \( a^G(h) = b^G(h) \), then \( a^G(h') = h' \cdot h^{-1} \cdot a^G(h) = h' \cdot h^{-1} \cdot b^G(h) = b^G(h') \) for every \( h' \in \mathbb{G}. \)

References

[1] C. J. Ash. Inevitable graphs: A proof of the type II conjecture and some related decision procedures. International Journal of Algebra and Computation, 01(01):127–146, Mar. 1991.
[2] K. Auinger and M. Szendrei. On F-inverse covers of inverse monoids. Journal of Pure and Applied Algebra, 204(3):493 – 506, Mar. 2006.
[3] K. Henckell, S. Margolis, J.-E. Pin, and J. Rhodes. Ashs type II theorem, profinite topology and Malcev products. International Journal of Algebra and Computation, 1:411–436, Dec 1991.
[4] K. Henckell and J. Rhodes. The theorem of Knast, the PG = BG and type-II conjectures. In J. Rhodes, editor, Monoids and Semigroups with Applications, pages 453–463. World Scientific, 1991.
[5] M. V. Lawson. *Inverse semigroups*. World Scientific Publishing, 1998.
[6] D. B. McAlister. Groups, semilattices and inverse semigroups. *Trans. Amer. Math. Soc.*, 192(192):227–244, May 1974.
[7] M. Otto. Groupoids, hypergraphs, and symmetries in finite models. In *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 53–62, June 2013.
[8] M. Otto. Finite groupoids, finite coverings and symmetries in finite structures. *ArXiv e-prints*, Apr. 2014. arXiv:1404.4599.
[9] N. Szakács. On the graph condition regarding the F-inverse cover problem. *Semigroup Forum*, 92(3):551–558, June 2016.
[10] N. Szakács and M. B. Szendrei. On F-inverse covers of finite-above inverse monoids. *Journal of Algebra*, 452(Supplement C):42 – 65, Apr. 2016.