GAUGE INVARIANT QUANTIZATION OF DISSIPATIVE SYSTEMS OF CHARGED PARTICLES IN EXTENDED PHASE SPACE

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Abstract—Recently, it is shown that the extended phase space formulation of quantum mechanics is a suitable technique for studying the quantum dissipative systems. Here, as a further application of this formalism, we consider a dissipative system of charged particles interacting with an external time dependent electric field. Such a system has been investigated by Buch and Denman, and two distinct solutions with completely different structure have been obtained for Schrödinger’s equation in two different gauges. However, by generalizing the gauge transformations to the phase space and using the extended phase space technique to study the same system, we demonstrate how both gauges lead to the same conductivity, suggesting the recovery of gauge invariance for this physical quantity within the extended phase space approach.

Keywords—Extended Phase Space, Electrodynamics, Gauge Transformation, Conductivity, Quantum Dissipative System, Gauge Invariant.
1. INTRODUCTION

Although the gauge invariance of electrodynamics is essentially satisfied in theory, however, in practice, some authors have shown that the different gauges do not yield the same physical results. For example, Kobe and Wen [1] have investigated an oscillating charged particle in a time dependent electric field, in two different gauges. They have shown that the transition probability amplitude as an important physical quantity in quantum mechanics, is gauge dependent. The equivalence or non equivalence of minimally coupled and multipolar hamiltonians, which are related to each other by a gauge transformation, is not well understood as noted by Golshan and Kobe [2], and Cohen-Tannoudji, et. al. [3]. Noteworthy challenges about Maxwell’s equations using multipolar and minimally coupled hamiltonians are done by Mandel [4], Healy [5], Ackerhalt [6] and others. However, their results are not consistent with each other in all respects. Buch and Denman [7] have considered the quantum mechanical evolution of a system of charged particles and have obtained different structure for solutions of Schrödinger equation in two different gauges.

Another point in the problem existing with the quantization of dissipative systems. Inspite of the vast efforts done, these systems have not a well understood quantum mechanics. The most well-known approaches to this problem are:

a) The Kaldirolla - Kanai method [8,9], which assumes a hamiltonian for a system of damped harmonic oscillators giving the correct classical equations of motions. Although, the Kaldirolla and Kanai hamiltonian is correct classically, it has its own problem in quantum mechanics and it violates the Heisenberg uncertainty principle.

b) The Bateman method [10], which considers a dual hamiltonian for a given dissipative system. The classical equations of motion of the system consider the evolution of the system and its mirror image, simultaneously. The energy dissipated by the original system is completely absorbed at the same rate by the mirror image system. Therefore, the dual hamiltonian becomes a constant of the motion. The difficulty existing with this
method is that the Hamiltonian is not equal to the total energy of the system and its mirror image. This presents its own hindrance to the problem.

c) The Schrödinger-Languvan method [11], in which a non-linear Hamiltonian is employed in the Schrödinger’s equation. The outcome of this nonlinearity is, in turn, the violation of the superposition principle.

d) The Dodonov-Mankov method [12], in which the loss energy states with complex eigenvalues are introduced.

In a review article including considerable number of references, Dekker [13] concludes that ”...although completeness is certainly not claimed, it is left that the present text covers a substantial portion of the relevant work done during the last half century. All models agree on the classical dynamics. ... closer inspection of the models shows that none of them, ... are completely satisfactory in all respects”.

Recently, we have employed the extended phase space (EPS) formulation of quantum mechanics [14,15,16] to investigate the evolution of dissipative systems [17,18,19]. We have shown that the extension of the phase space allows the presence of a mirror image system that absorbs the energy with the same rate that the given system dissipates. The whole system behaves as a conservative system which evolves together in the course of time and, therefore, the problems due to nonconservative nature of the dissipative systems are removed. Here, as a further application of the EPS technique, we treat the dynamics of a system of charged particles interacting with a time dependent electric field. To do this, one needs to generalize the conventional gauge transformations for the EPS. They are called the extended gauge transformations and are shown to be canonical transformations in EPS. Using the Kanai Hamiltonian it is shown that, in contrast to the Buch and Denman treatment [7], the solutions of the evolution equation have the same structure in two different gauges and are related to each other by a unitary transformation. Using these solutions the conductivity is shown to be a gauge invariant quantity.

The layout of this article is as follows: In section 2 a brief review of the EPS formulation
is introduced. In section 3 the gauge transformations are generalized for EPS and in section 4, by using the proposed technique, the quantum state functions of the system is obtained and are used to calculate the conductivity in two different gauges. Section 5 is devoted to conclusions.

2. A REVIEW OF THE EPS FORMULATION

A direct approach to quantum statistical mechanics is proposed by Sobouti and Nasiri [14], by extending the conventional phase space and by applying the canonical quantization procedure to the extended quantities in this space. Assuming the phase space coordinates $p$ and $q$ to be independent variables on the virtual trajectories, allows one to define momonanta $\pi_p$ and $\pi_q$, conjugate to $p$ and $q$, respectively. This is done by introducing the extended lagrangian

$$\mathcal{L}(p, q, \dot{p}, \dot{q}) = -\dot{p}q - \dot{q}p + \mathcal{L}^p(p, \dot{p}) + \mathcal{L}^q(q, \dot{q}),$$

(1)

where $\mathcal{L}^p$ and $\mathcal{L}^q$ are the $p$ and $q$ space lagrangians of the given system. Using Eq. (1) one may define the momenta, conjugate to $p$ and $q$, respectively, as follow

$$\pi_p = \frac{\partial \mathcal{L}}{\partial \dot{p}} = \frac{\partial \mathcal{L}^p}{\partial \dot{p}} - q,$$

(2a)

$$\pi_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}^q}{\partial \dot{q}} - p.$$  

(2b)

In the EPS defined by the set of variables $\{p, q, \pi_p, \pi_q\}$, one may define the extended hamiltonian

$$\mathcal{H}(\pi_p, \pi_q, p, q) = \dot{p}\pi_p + \dot{q}\pi_q - \mathcal{L} = H(p + \pi_q, q) - H(p, q + \pi_p)$$

$$= \sum \frac{1}{n!} \left\{ \frac{\partial^n H}{\partial p^n} \pi_p^n - \frac{\partial^n H}{\partial q^n} \pi_q^n \right\},$$

(3)

where $H(p, q)$ is the hamiltonian of the system. Using the canonical quantization rule, the following postulates are outlined:
a) Let \( p, q, \pi_p \) and \( \pi_q \) be operators in a Hilbert space, \( X \), of all square integrable complex functions, satisfying the following commutation relations

\[
\begin{align*}
[\pi_q, q] &= -i\hbar, & \pi_q &= -i\hbar \frac{\partial}{\partial q}, \\
[\pi_p, p] &= -i\hbar, & \pi_p &= -i\hbar \frac{\partial}{\partial p}, \\
[p, q] &= [\pi_p, \pi_q] = 0.
\end{align*}
\] (4a, 4b, 4c)

By virtue of Eqs. (4), the extended hamiltonian, \( \mathcal{H} \), will also be an operator in \( X \).

b) A state function \( \chi(p, q, t) \in X \) is assumed to satisfy the following dynamical equation

\[
\begin{align*}
\frac{i\hbar}{\partial t} \chi &= \mathcal{H} \chi = [H(p - i\hbar \frac{\partial}{\partial q}, q) - H(p, q - i\hbar \frac{\partial}{\partial p})] \chi \\
&= \sum_{n} \frac{(-i\hbar)^n}{n!} \left\{ \frac{\partial^n H}{\partial p^n} \frac{\partial^n}{\partial q^n} - \frac{\partial^n H}{\partial q^n} \frac{\partial^n}{\partial p^n} \right\} \chi.
\end{align*}
\] (5)

c) The averaging rule for an observable \( O(p, q) \), a c-number operator in this formalism, is given as

\[
< O(p, q) > = \int O(p, q) \chi^\ast(p, q, t) dp dq.
\] (6)

For details of selection procedure of the admissible state functions, see Sobouti and Nasiri [14].

3. GENERALIZATION OF GAUGE TRANSFORMATIONS FOR EPS

Interaction of a charged particle with an external electromagnetic field is described by the following hamiltonian:

\[
H = \frac{1}{2m} \left[ p - \frac{e}{c} A \right]^2 + e\phi,
\] (7)
where \( A(q,t) \) and \( \phi(q,t) \) are electromagnetic vector and scalar potentials and \( e \) is the electric charge of the particle. The conventional gauge transformations are unitary transformations as follows

\[
F = \exp \left( \frac{-ie}{\hbar c} f(q,t) \right),
\]

(8)

where \( f(q,t) \) is an arbitrary gauge function in \( q \)-space \([20]\). Shrödinger’s equation in \( q \)-representation would be form invariant under the above transformation. This requires that \( A \) and \( \phi \) transform as follows \([20]\)

\[
A' = A + \nabla_q f(q,t),
\]

(9a)

and

\[
\phi' = \phi - \frac{1}{c} \frac{\partial f(q,t)}{\partial t}.
\]

(9b)

Alternatively, this gauge invariance may be obtained by assuming \( H \rightarrow H - \frac{e}{c} \frac{\partial f(q,t)}{\partial t} \), and \( p \rightarrow p - \frac{e}{c} \nabla f(q,t) \). This gives, \( p - \frac{e}{c} A \rightarrow (p - \frac{e}{c} \nabla f) - \frac{e}{c} A = p - \frac{e}{c} (A + \nabla f) \).

Thus, one may consider the term \( \frac{e}{c} \nabla f \) either with \( p \) or with \( A \). So the ordinary gauge transformations may be looked either as a gauged transformation on potentials or a canonical coordinate transformation on the phase space coordinates \( p \) and \( q \) with fixed \( q \). The later interpretation will be important in the definition of gauge transformations in the extended phase space.

Using Eq. (3), one may obtain the extended hamiltonian for a charged particle interacting with an external electromagnetic field, as follows

\[
\mathcal{H} = \frac{1}{2m} [\pi_q - \frac{e}{c} A(q,t)]^2 + \frac{1}{2m} [\pi_p - \frac{e}{c} A(q,t)] + e\phi(q,t)
- \frac{1}{2mc} [\frac{e}{c} A(q + \pi_p, t)]^2 - \frac{1}{m} [\frac{e}{c} p. A(q + \pi_p, t)] - e\phi(q + \pi_p, t).
\]

(10)

We assume that the gauge transformation in EPS may be described by a unitary operator as follows

\[
\Gamma = \exp \left( \frac{-ie}{\hbar c} \gamma(q,p,t) \right),
\]

(11)
where $\gamma(q, p, t)$ is now an arbitrary gauge function of the phase space coordinates. In order that Eq. (5) to be form invariant under transformation Eq. (11), one requires

$$
\chi' = \Gamma \chi, \\
(12a)
$$

$$
\mathcal{H}' = \Gamma \mathcal{H} \Gamma^\dagger - i\hbar \Gamma \left( \frac{\partial \Gamma^\dagger}{\partial t} \right), \\
(12b)
$$

where $\chi'$ and $\mathcal{H}'$ are gauge transformed state function and extended hamiltonian, respectively. To be consistent with conventional gauge transformations in $q$-representation, one requires the following form for $\Gamma$:

$$
\Gamma = FG^\dagger, \\
(13a)
$$

where

$$
G = \exp \left( \frac{-ie}{\hbar c} g(p, t) \right). \\
(13b)
$$

In Eq. (13b), $g(p, t)$ is an arbitrary gauge function in $p$-space. In fact, the unitary transformation $G$ leaves Schrödinger’s equation form invariant in $p$-representation in parallel with the $F$ which does the same job for $q$-representation. Eventually, using Eq. (12b), the gauge transformed extended hamiltonian will have the following form:

$$
\mathcal{H}' = \left( FH(p + \pi_q, q)F^\dagger - i\hbar F \frac{\partial F^\dagger}{\partial t} \right) - \left( G^\dagger H(p, q + \pi_p)G - i\hbar G^\dagger \frac{\partial G}{\partial t} \right) \\
= \frac{1}{2m}[\pi_q - \frac{e}{c} \nabla_q f(q, t) - \frac{e}{c} A(q, t)]^2 + \frac{1}{m} p.[\pi_q - \frac{e}{c} \nabla_q f(q, t) - \frac{e}{c} A(q, t)] + e\phi(q, t) \\
- \frac{1}{2m} [\frac{e}{c} A(q + \pi_p + \frac{e}{c} \nabla_p g(p, t))]^2 - \frac{e}{mc} p.A(q + \pi_p + \frac{e}{c} \nabla_p g(p, t)) \\
-e\phi(q + \pi_p + \frac{e}{c} \nabla_p g(p, t), t). \\
(14)
$$

Note that, the arguments of $A$ and $\phi$ in the last three terms of $\mathcal{H}'$ in Eq. (14), does not allow to express the extended gauge transformation in their standard form, that is, in terms of vector and scalar potentials as in Eq.(9). In fact the additive property of $A$ and $\phi$
with derivative of gauge functions is lost. However, another possibility emerges. One may consider the extended gauge transformations as the following coordinate transformations

\[ \pi'_p = \pi_p + \frac{e}{c} \nabla_p g(p,t), \quad p' = p, \] (15a)

\[ \pi'_q = \pi_q - \frac{e}{c} \nabla_q f(q,t), \quad q' = q, \] (15b)

where \( \nabla_p \) denotes the derivative with respect to \( p \). It is simply verified that the Eqs. (15) in classical level, are canonical coordinate transformations which in quantum level correspondes to unitary transformations. Note that the interpretation of gauge transformations as canonical coordinate transformations is valid in both of the ordinary phase space as well as extended phase space. While the usual form of the gauge transformations on the electromagnetic potentials are valid only in ordinary phase space as noted earlier.

4. APPLICATION TO A SYSTEM OF DISSIPATIVE CHARGED PARTICLRES

Let us now apply this method, to study a dissipative system of non interacting charged particles in a time dependent uniform external electric field. The single particle hamiltonian for this medium with damping constant \( \alpha \) is as follows [9]

\[ H = \frac{1}{2m} e^{-\alpha t} [p - \frac{e}{c} A(q,t)]^2 + e^{\alpha t} e\phi(q,t). \] (16)

Extending of Eq. (16), one gets

\[ \mathcal{H}(p, q, \pi_p, \pi_q) = \frac{1}{2m} e^{-\alpha t} [p + \pi_q - \frac{e}{c} A(p,q)]^2 + e^{\alpha t} e\phi(q,t) \]
\[ - \frac{1}{2} e^{-\alpha t} [p - \frac{e}{c} A(q + \pi_p, t)]^2 - e^{\alpha t} e\phi(q + \pi_p, t). \] (17)

Using Eq. (17), we investigate the behavior of the system in two different gauges; one with zero scalar potential and the other with zero vector potential (hereafter called \( A \)-gauge and \( \phi \)-gauge, respectively). This is done in the following subsections.
4.1. A-GAUGE

A time dependent uniform electric field may be generated by setting

$$A(t) = -c \int^t e^{\alpha \lambda} E(\lambda) d\lambda, \quad \phi(q, t) = 0, \quad (18)$$

which is called A-gauge. Note that $A(t)$ depends only on time. In this gauge the extended Hamiltonian (17), assumes the following form

$$H_A = \frac{1}{2m} e^{-\alpha t} \left[ \pi_q^2 + 2\pi_q (p + e \int^t e^{\alpha \lambda} E(\lambda) d\lambda) \right]. \quad (19)$$

In obtaining Eq. (19), the gauge transformation was followed by the extension. A word of caution is in order. These operations do not, in general, commute with each other and have its own interesting consequences that will be presented elsewhere. Of course the electromagnetic field assumed here is a special case, hence, the ordering of operations concerning with extension and gauge transformation is not important. Using the above Hamiltonian, the evolution equation [15], becomes

$$i\hbar \frac{\partial \chi_A}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \chi_A}{\partial \xi^2} - i\hbar \frac{1}{m} \left( p + e \int^t e^{\alpha \lambda} E(\lambda) d\lambda \right) \frac{\partial \chi_A}{\partial q}. \quad (20)$$

Making the transformation

$$\xi = q - \frac{p}{m} \int^t e^{-\alpha \lambda} d\lambda - \frac{e}{m} \int^t e^{-\alpha \lambda} \int^\lambda e^{\alpha \nu} E(\nu) d\nu d\lambda, \quad (21a)$$

$$\eta = p, \quad (21b)$$

$$\tau = t, \quad (21c)$$

and using them in Eq. (20), one gets

$$i\hbar \frac{\partial \chi_A}{\partial \tau} = -\frac{\hbar^2}{2m} e^{-\alpha \tau} \frac{\partial^2 \chi_A}{\partial \xi^2}. \quad (22)$$
Equation (22) can be solved for $\chi_A$ by separation of variables and the result is

$$\chi_A = C_+ \exp \left[ +ik\xi + \frac{i\hbar k^2}{2m\alpha} e^{-\alpha t} \right] + C_- \exp \left[ -ik\xi + \frac{i\hbar k^2}{2m\alpha} e^{-\alpha t} \right], \quad (23)$$

where $C_\pm$ and $k$ are normalization and separation constants, respectively. One can use the state function $\chi_A$ of Eq. (23) to calculate the conductivity for the system of $N$ non-interacting charged particles given by

$$\sigma = \frac{Ne < \dot{q} >}{E(t)}, \quad (24)$$

where

$$\dot{q} = \frac{1}{m} pe^{-\alpha t} + \frac{e}{m} e^{-\alpha t} \int^t E(\lambda) e^{\alpha \lambda} d\lambda. \quad (25)$$

The expression for $\dot{q}$ in Eq. (25) can be obtained using extended Hamilton’s equations as $\pi_q \to 0$ [14]. If one sets $C_+ = C_-$ and calculates the expectation value of $< \dot{q} >$ using the averaging rule of Eq. (6), one gets

$$\sigma = \frac{Ne^2}{m(\alpha + i\omega)}, \quad (26)$$

as $< p >$ vanishes. A time dependency of the form $E(t) = E_0 e^{i\omega t}$ is assumed for the external electric field. Equation (26) is consistent with the result obtained by Buch and Denman [7] in the same gauge.

4.2. $\phi$-GAUGE

The aforementioned electric field may be obtained by assuming

$$A(q,t) = 0, \quad \phi(q,t) = -qE(t). \quad (27)$$

The corresponding extended hamiltonian will be

$$\mathcal{H}_\phi = e^{-\alpha t} \frac{1}{2m} \left( \pi_q^2 + 2p\pi_q \right) + e^{\alpha t} e E(t) \pi_p. \quad (28)$$

The evolution equation now becomes

$$i\hbar \frac{\partial \chi_\phi}{\partial t} = \mathcal{H}_\phi \chi_\phi = e^{-\alpha t} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \chi_\phi}{\partial q^2} - \frac{i\hbar p}{m} \frac{\partial \chi_\phi}{\partial q} \right) - i\hbar e^{\alpha t} E(t) \frac{\partial \chi_\phi}{\partial p}. \quad (29)$$
In parallel with Eqs. (21), one may consider the following transformation

\[ \xi' = q - \frac{p}{m} \int_t^t e^{-\alpha \lambda} d\lambda + \frac{e}{m} \int_t^t e^{\alpha \lambda} E(\lambda) \int^\lambda e^{-\alpha \nu} d\nu d\lambda, \] (30a)

\[ \eta' = p - e \int_t^t e^{\alpha \lambda} E(\lambda) d\lambda. \] (30b)

\[ \tau' = t. \] (30c)

Equation (29) under the above transformation becomes

\[ i\hbar \frac{\partial \chi_\phi}{\partial \tau'} = -\frac{\hbar^2}{2m} e^{-\alpha \tau'} \frac{\partial^2 \chi_\phi}{\partial \xi'^2}. \] (31)

The remarkable point is that, except for different functional forms of \( \xi \) and \( \xi' \), Eq. (31) has exactly the same form as Eq. (22). Therefore, we conclude that, in contrast to the result obtained by Buch and Denman using the conventional Schrödinger quantum mechanics, the solution \( \chi_\phi \) of Eq. (31) has the same structure as \( \chi_A \) of Eq. (22) and differs only by a phase factor. This result may be considered as an advantage of the EPS method. In fact, \( \chi_\phi \) and \( \chi_A \) are related to each other by a unitary transformation, and therefore, one can easily verify that the value obtained for electrical conductivity in \( A \)-gauge will be valid for \( \phi \)-gauge, too.

5. CONCLUSIONS

The gauge transformations in electrodynamics are generalized to the extended phase space and is shown to have their own advantages in studying the quantum mechanical evolution of charged particles. They turned out to be canonical coordinate transformations in this space. The interpretation of the gauge transformations as change of electromagnetic potentials is no longer valid in EPS approach. However, treating the gauge transformations as canonical coordinate transformations is valid for EPS method as well as for conventional approaches. In other words, it seems more convenient and reasonable to treat the gauge
transformations as canonical coordinate transformations. The formalism is applied to study the evolution of a system of charged particles in the presence of a uniform and time dependent external electric field. The problem is handled in two different gauges, first employed by Buch and Denman to study the same problem by using the Schrödinger quantum mechanics. They obtained two physically different solutions for Schrödinger’s equation using above gauges. However, our solutions, in contrast to those of Buch and Denman, have a unique structure in both gauges for any arbitrary boundary conditions. They are related by a unitary transformation which guarantee the gauge equivalence of the physical quantities. As an example, the conductivity is calculated for the system and is shown that it is a gauge independent quantity.

There is also a noteworthy point in obtaining Eqs. (19) and (28). In general, the operations of extension and gauge transformations do not commute with each other. It depends on the form of hamiltonian and the interacting electromagnetic fields. However, for the special case considered here, no matter how these operations might be ordered. The more general case is investigated and the results will be appeared elsewhere.

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