SOME RESULTS ON THE TOPOLOGY OF REAL BOTT TOWERS

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ABSTRACT. The main aim of this article is to study the topology of real Bott towers as special and interesting examples of real toric varieties. We first give a presentation of the fundamental group of a real Bott tower and show that the fundamental group is abelian if and only if the real Bott tower is a product of circles. We further prove that the fundamental group of a real Bott tower is always solvable and it is nilpotent if and only if it is abelian. We then describe the cohomology ring of a real Bott tower and also give recursive formulae for the Steifel Whitney classes. We derive combinatorial characterization for orientability of these manifolds and further give a combinatorial formula for the $(n-1)$th Steifel Whitney class. In particular, we show that if a Bott tower is orientable then the $(n-1)$th Steifel Whitney class must also vanish. Moreover, by deriving a combinatorial formula for the second Steifel-Whitney class we give a necessary and sufficient condition for the Bott tower to admit a spin structure. We finally prove the vanishing of all the Steifel-Whitney numbers and hence establish that these manifolds are null-cobordant.

1. INTRODUCTION

Bott towers are iterated fibre bundles with fibre at each stage being $\mathbb{P}^1_C$. In particular they are smooth projective complex toric varieties. They were constructed in [13] by M.Grossberg and Y.Karshon who show that a Bott-Samelson variety can be deformed to a Bott tower. Bott-Samelson manifolds were first constructed in [3] to study cohomology of generalized flag varieties. M.Demazure and D.Hansen used it to obtain desingularizations of Schubert varieties in generalized flag varieties. Moreover, the underlying differentiable structure is preserved under the deformation. Because of their relation with Bott Samelson manifolds which in turn are related to the Schubert varieties, along with their amenable structure as iterated $\mathbb{P}^1_C$-bundles, the Bott towers have been important and interesting objects of study.

Topological invariants for these manifolds have been studied using their iterated sphere bundle structure. The equivariant cohomology of Bott Samelson manifolds have been studied with applications to the cohomology of Schubert varieties in [7]. Also see [24] for equivariant $K$-theory of Bott towers with application to the equivariant $K$-theory of flag manifolds.

Indeed, from the viewpoint of toric topology, Bott-towers can be seen to also have the structure of a quasi-toric manifold [6] with the quotient polytope being the $n$-dimensional cube $I^n$ where $n$ is the complex dimension of the Bott tower. The second named author and P.Sankaran described the topological $K$-ring of the quasi toric manifolds in [24], where the topological $K$-ring of Bott towers and Bott

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Samelson manifolds have been described as a special example. Also see [3], [4] and [1] for results on complex $K$-theory, $KO$-theory as well as the complex cobordism ring of Bott towers and quasi-toric manifolds.

Recently there has been extensive work on the topology and geometry of Bott towers viewed as a quasi-toric manifold (see for example [16], [10]). These works are especially related to the problem of cohomological rigidity of Bott manifolds or more generally of quasi-toric manifolds.

There has also been a parallel study on the topology of real Bott towers. These manifolds are constructed as iterated $\mathbb{P}^1 = S^1$-bundles and can be viewed as a special example of a small cover defined by Davis and Januszkiewicz (see for example [15], [20], [9]).

The cohomology of a small cover have been described in [6]. Indeed in [6, Theorem 5.12] Davis and Januszkiewicz give a presentation for the cohomology ring with $\mathbb{Z}_2$-coefficients as a quotient of the Stanley-Reisner ring of the simple convex polytope by certain canonical linear relations. A similar description for the cohomology ring with $\mathbb{Z}_2$-coefficients of the real part of a smooth projective complex toric variety is due to Jurkiewicz (see [18, Theorem 4.3.1]).

In [6, Section 6], there is also a description of the tangent bundle of a small cover and a formula for its Steifel-Whitney class in terms of the generators of the cohomology ring. These generators are in turn the first Steifel-Whitney classes of certain canonical line bundles on the small cover.

In [22], the second named author of the current article described the fundamental group of a real toric variety associated to any smooth fan $\Delta$ in $\mathbb{Z}^n$. She in particular gave a presentation of the fundamental group, a combinatorial criterion on $\Delta$ for the fundamental group to be abelian along with a criterion for the real toric variety to be aspherical. The fundamental group of a small cover has also earlier been described in [6, Corollary 4.5] as the kernel of a natural map from the right-angled Coxeter group associated to the simple convex polytope $P$ to $\mathbb{Z}_2^n$. The natural map is the composition of the characteristic map $\lambda$ of the small cover with the abelianization map of the Coxeter group.

The authors in the current article were interested in the study of real Bott towers as a nice class of small covers or real toric varieties, which are characterized by the upper triangular real Bott matrix $C = (c_{i,j})$ with entries in $\mathbb{Z}_2$ (see definition below).

Our main motivation here is to give a precise and elegant description of all the above mentioned topological invariants namely, the fundamental group, cohomology ring and the Stiefel Whitney classes, in terms of the Bott numbers $\{c_{i,j}\}$. We exploit the inductive definition of this class of manifolds and derive precise topological information about them wherever possible. We further classify Bott towers satisfying a specific topological property like orientability or admission of spin structure, by means of certain algebraic identities on the $c_{i,j}$’s.

We now develop some notations before outlining our main results in the next section.

1.1. Notations and Conventions. In this section we recall the definition of a Bott tower and fix some notations (see [13]).

A Bott tower is a smooth complete complex toric variety which is constructed iteratively as follows:

Let $Y_1 = \mathbb{C}P^1$. Let $L_2$ be a holomorphic complex line bundle on $\mathbb{C}P^1$. We then let $Y_2 = \mathbb{P}(1 \oplus L_2)$ where $1$ is the trivial line bundle on $\mathbb{C}P^1$. Then $Y_2$ is a $\mathbb{C}P^1$
bundle over $\mathbb{CP}^1$ which is a Hirzebruch surface. We can iterate this process for $2 \leq j \leq n$, where at each step, $L_j$ is a complex line bundle over $Y_{j-1}$, and the variety $Y_j = \mathbb{P}(1 \oplus L_j)$ is a $\mathbb{CP}^1$ bundle over $Y_{j-1}$. The variety $Y_n$ thus obtained after $n$-steps is called an $n$-step Bott tower.

**Definition 1.1.** In fact an $n$-step Bott tower is a smooth complete toric variety of dimension $n$ whose fan $\Delta$ can be described as follows:

We take a collection of integers $\{c_{i,j}\}$, $1 \leq i < j \leq n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. Let $v_j = e_j$ for $1 \leq j \leq n$,

$$v_{n+j} = -e_j + \sum_{k=j+1}^{n} c_{j,k} \cdot e_k$$

for $1 \leq j \leq n-1$ and $v_{2n} = -e_n$. We define the fan $\Delta$ in $\mathbb{R}^n$ consisting of cones generated by the set of vectors in any subcollection of $\{v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ which does not contain both $v_i$ and $v_{n+i}$ for $1 \leq i \leq n$.

**Definition 1.2.** We can also view a Bott tower as a quasi-toric manifold (see [6]) over the $n$-cube $I^n$ which is a simple convex polytope of dimension $n$. If we index the $2n$ facets of $I^n$ by $F_1, F_2, \ldots, F_n, F_{n+1}, \ldots, F_{2n}$, then the characteristic function is defined on the collection of facets to $\mathbb{Z}^n$ as follows: $\lambda(F_j) = e_j$ for $1 \leq j \leq n$,

$$\lambda(F_{n+j}) = -e_j + \sum_{k=j+1}^{n} c_{j,k} \cdot e_{j+k}$$

for $1 \leq j \leq n-1$ and $\lambda(F_{2n}) = -e_n$.

1.1.1. **Real Bott tower.** We shall call the real part of the $n$-step complex Bott tower as the real $n$-step Bott tower.

In particular, $(Y_2)_\mathbb{R}$ is an $\mathbb{RP}^1$ bundle over $(Y_1)_\mathbb{R} = \mathbb{RP}^1$. Iteratively we construct $(Y_j)_\mathbb{R}$ as an $\mathbb{RP}^1$ bundle over $(Y_{j-1})_\mathbb{R}$ for $2 \leq j \leq n$. The real $n$-step Bott tower $(Y_n)_\mathbb{R}$ is indeed the real toric variety associated to the fan $\Delta$ described above (see [18], Section 2.4 and [22]).

**Definition 1.3.** As in the complex case we can also view $(Y_n)_\mathbb{R}$ as a small cover over the simple convex polytope $I^n$, where the characteristic map $\lambda$ is defined on the collection of facets $\mathcal{F}$ to $\mathbb{Z}^n_2$ as follows: $\lambda(F_j) = e_j$ for $1 \leq j \leq n$,

$$\lambda(F_{n+j}) = e_j + \sum_{k=j+1}^{n} c_{j,k} \cdot e_k$$

for $1 \leq j \leq n-1$ and $\lambda(F_{2n}) = e_n$. Here $c_{i,j} \in \mathbb{Z}_2$ for $1 \leq i < j \leq n$. Thus $(Y_n)_\mathbb{R}$ is homeomorphic to the identification space $\mathbb{Z}^n_2 \times I^n / \sim$ where $(t, p) \sim (t', p')$ if and only if $p = p'$ and $t \cdot (t')^{-1} \in G_{F(p)}$. Here $F(p) = F_1 \cap \cdots \cap F_l$ is the unique face of $I$ which contains $p$ in its relative interior and $G_{F(p)}$ is the rank-$l$ subgroup of $\mathbb{Z}^n_2$ determined by the span of $\lambda(F_1), \ldots, \lambda(F_l)$. Now, let $\pi : (Y_n)_\mathbb{R} \to I^n$ denote the second projection which maps $[t, p] \mapsto p$, and let $M_i := \pi^{-1}(F_i)$ denote the characteristic submanifold for $1 \leq i \leq 2n$. (See Section 1 of [6]).

The topological structure of an $n$-step real Bott tower is completely determined by the simple convex polytope $I^n$ and the data encoded by the matrix

$$C = (c_{i,j}) \in M_n(\mathbb{Z}_2)$$

(1.1)
where $c_{i,i} = 1$ and $c_{i,j} = 0$ for $i > j$. Note that the $i$th row of $C$ is $\lambda(F_{n+1}) \in \mathbb{Z}_2^n$ for $1 \leq i \leq n$. We call $C$ the Bott matrix. Thus $Y_n = Y(C)$ the real Bott tower associated to $C$.

The 2-step real Bott tower is the torus or the Klein bottle depending on whether $c_{1,2} = 0$ or $c_{1,2} = 1$. The 3-step real Bott tower is an $\mathbb{R}P^1$ bundle over the torus or the Klein bottle whose topological structure depends on $c_{1,2}, c_{1,3}$ and $c_{2,3}$.

**Notation 1.4.** In this article, since we are mainly interested in the study of the real Bott tower, for notational simplicity we shall henceforth denote $(Y_n)_{\mathbb{R}}$ by $Y_n$.

### 1.2. Overview of the main results.
In Section 2 we give a description of the fundamental group of the real Bott tower. In particular, we give a presentation of the fundamental group in terms of generators and relations in Theorem 2.1 and in Corollary 2.2 we prove that $Y_n$ has abelian fundamental group if and only if $Y_n$ is a product of circles. Further, in Proposition 2.6 we show that the commutator subgroup of $\pi_1(Y_n)$ is always abelian, so that $\pi_1(Y_n)$ is solvable. In Proposition 2.9 we prove that the fundamental group is nilpotent if and only if it is abelian. By abelianizing the fundamental group we further determine the first homology with integer coefficients $H_1(Y_n; \mathbb{Z})$ explicitly. We also conclude by induction that $Y_n$ is always aspherical. The main tool used in this section is the presentation of the fundamental group of the real part of any smooth toric variety in [22], where combinatorial characterizations are also given for the fundamental group to be abelian and the manifold to be aspherical. However, as mentioned earlier, in the case of the real Bott tower the fundamental group gets a neater and simpler presentation in terms of the entries of the characterizing Bott matrix. Moreover, the presentation also enables us to make further conclusions about the group theoretic properties of the fundamental group of these special class of real toric varieties. Further, proofs of all results in this section except that of Theorem 2.1 which uses [22, Lemma 3.2], are made self-contained and specific to the case of the real Bott tower.

In Section 3 we study the cohomology ring of these manifolds with $\mathbb{Z}_2$-coefficients. The tool here is to apply the description of the cohomology ring of small covers in [6] or else that of a real toric variety in [18, Section 2.4] to these class of manifolds as a special case. We describe the cohomology ring in terms of generators and relations in Theorem 3.17, where again the presentation given in terms of the entries of the Bott matrix becomes simple in this case due to the iterative structure of these manifolds.

In Section 4, using the presentation of the cohomology ring and applying the more general results of [6] we give explicit description of the Steifel-Whitney classes of $Y_n$ which are smooth connected compact manifolds. The description is again in terms of the entries of the Bott matrix. More explicitly, in Theorem 4.1 we give an inductive formula for total Steifel-Whitney class as well as the $k$th Steifel-Whitney class of $Y_n$ in terms of those of $Y_{n-1}$.

These results are applied to give a combinatorial characterisation for orientability of $Y_n$ in Theorem 4.5 criterion for $Y_n$ to admit a spin structure in Theorem 4.10 and a nice formula for $w_{n-1}(Y_n)$ in Corollary 4.9. We further show in Corollary 4.10 that if $Y_n$ is orientable then $w_{n-1}(Y_n)$ also vanishes. We finally prove the vanishing of all Steifel-Whitney numbers of $Y_n$, which enables us to conclude in Theorem 4.24 that the real Bott towers are null cobordant.
We aim to continue these methods to prove results, for instance, about the immersion and embedding dimensions and parallelizability of these manifolds in future work.

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## 2. THE FUNDAMENTAL GROUP

In this section we shall get a presentation for the fundamental group of the real Bott tower $Y(\Delta) = Y_n$ where $\Delta$ is as in Definition 1.1.

For obtaining a presentation for $\pi_1(Y_n)$ we apply [22, Proposition 3.1]. We shall first fix some notations which are slightly modified from [22] suitable to our setting. Recall that we have an exact sequence

$$0 \to \pi_1(Y_n) \to W(\Delta) \to \mathbb{Z}_2^n \to 0 \tag{2.2}$$

where $W(\Delta)$ is the right angled Coxeter group associated to $\Delta$ with the following presentation:

$$W(\Delta) = \langle s_j \mid s_j^2, 1 \leq j \leq 2n \rangle \text{ and } (s_is_j)^2 \text{ for all } 1 \leq i < j \leq 2n \text{ with } j \neq i + n \rangle.$$

The last arrow in the above exact sequence is obtained by composing the natural abelianization map from $W(\Delta)$ to $\mathbb{Z}^{2n}$ with the characteristic map $\lambda$ from $\mathbb{Z}^{2n}$ to $\mathbb{Z}^n$.

Let $\alpha_j := s_j s_{j-n} c_{j-n}^{j-n+1} \cdots c_{j-n-k}^{j-n+k} \cdots c_{j-n-n}^{j-n}$ for all $n + 1 \leq j \leq 2n - 1$ and $1 \leq k \leq 2n - j$ and $\alpha_{2n} := s_{2n} s_{2n-n}$. Let $b_j^i := c_{j-n,i}$ for $j - n + 1 \leq i \leq n$, $b_{j-n}^j = 1$ and $b_k^j = 0$ for $k < j - n$, for every $n + 1 \leq j \leq 2n$. Thus

$$b_j^i = (b_j^i)_{i=1,\ldots,n} \tag{2.3}$$

denotes the $(j-n)$th row vector of the Bott matrix $C$.

For $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$ where $\epsilon_i \in \{0, 1\}$, let

$$t_\epsilon := s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \tag{2.4}$$
in $W(\Delta)$.

Now, from the relations in $W(\Delta)$ it follows that:

$$t_\epsilon \alpha_j t_\epsilon = \begin{cases} 
\alpha_j & \text{if } \epsilon_{j-n} = 0 \\
\alpha_j^{-1} & \text{if } \epsilon_{j-n} = 1
\end{cases} \tag{2.4}$$

Since $b_{j-n}^j = 1$ and $b_k^j = 0$ for $k - n < j - n$ it follows that:

$$t_{b_j^j} \alpha_j t_{b_j^j} = \alpha_j^{-1} \tag{2.5}$$

for all $n + 1 \leq j \leq 2n$ and

$$t_{b_j^j} \alpha_k t_{b_j^j} = \alpha_k \tag{2.6}$$

for $n + 1 \leq k < j \leq 2n$. Moreover, since $b_k^j = c_{j-n,k-n}$ we have:

$$t_{b_j^j} \alpha_k t_{b_j^j} = \begin{cases} 
\alpha_k & \text{if } c_{j-n,k-n} = 0 \\
\alpha_k^{-1} & \text{if } c_{j-n,k-n} = 1
\end{cases} \tag{2.7}$$

for all $n + 1 \leq j < k \leq 2n$. 
Theorem 2.1. We have a presentation for \( \pi_1(Y_n) = \langle S \mid R \rangle \) where
\[
S = \{ \alpha_j : n + 1 \leq j \leq 2n \}
\]
and
\[
R = \{ x_{p,q} : n + 1 \leq p < q \leq 2n \}
\]
where
\[
x_{p,q} = \begin{cases} 
\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} & \text{if } c_{p-n,q-n} = 0 \\
\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} & \text{if } c_{p-n,q-n} = 1
\end{cases}
\]

Proof: Since \( v_1 = e_1, \ldots, v_n = e_n \) form a basis of \( N \otimes \mathbb{Z}_2 \) and also pairwise form cones in \( \Delta \), we can apply [22, Lemma 3.2]. We can relate the above notations with those in the proof of [22, Lemma 3.2] as follows:
\[
y_{j,\epsilon} = t_\epsilon \cdot \alpha_j \cdot t_\epsilon
\]
for \( n + 1 \leq j \leq 2n \) and \( \epsilon \in \mathbb{Z}_2^n \). Thus \( y_{j,\epsilon} = \alpha_j^\pm 1 \) by (2.3). Let \( B^p := \epsilon + b^p \) and \( B^q := \epsilon + b^q \) in \( \mathbb{Z}_2^n \), where \( b^p \) and \( b^q \) are as defined in (2.3). Then it can be seen that
\[
y_{p,q} \cdot y_{q,B^p} \cdot y_{p,B^q} \cdot y_{q,B^q} = t_\epsilon \cdot \alpha_p \cdot t_{b^q} \cdot \alpha_q \cdot t_{b^q} \cdot t_{b^p} \cdot \alpha_p \cdot t_{b^p} \cdot t_\epsilon
\]
whenever \( n + 1 \leq p, q \leq 2n \). Moreover, by (2.4), (2.5), (2.6) and (2.7) we can further see that if \( p < q \) and \( c_{p-n,q-n} = 0 \) then
\[
\alpha_p \cdot t_{b^q} \cdot \alpha_q \cdot t_{b^q} \cdot t_{b^p} \cdot \alpha_p \cdot t_{b^p} = (\alpha_q \cdot t_{b^q} \cdot \alpha_p \cdot t_{b^q} \cdot t_{b^p} \cdot \alpha_p \cdot t_{b^p})^{-1} = \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1}
\]
and if \( p < q \) and \( c_{p-n,q-n} = 1 \) then
\[
\alpha_p \cdot t_{b^q} \cdot \alpha_q \cdot t_{b^q} \cdot t_{b^p} \cdot \alpha_p \cdot t_{b^p} = (\alpha_q \cdot t_{b^q} \cdot \alpha_p \cdot t_{b^q} \cdot t_{b^p} \cdot \alpha_p \cdot t_{b^p})^{-1} = \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1}.
\]
Also it can be seen that when \( p = q \) we simply get a trivial relation. Thus by [22, Lemma 3.2] it follows that \( \pi_1(Y_n) \) has a presentation with generators
\[
S = \{ \alpha_j : n + 1 \leq j \leq 2n \}
\]
and the relations
\[
R' = \{ x_{p,q}^\epsilon : n + 1 \leq p < q \leq 2n \}
\]
where
\[
x_{p,q}^\epsilon = \begin{cases} 
t_\epsilon \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon & \text{if } c_{p-n,q-n} = 0 \\
t_\epsilon \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon & \text{if } c_{p-n,q-n} = 1
\end{cases}
\]
for \( \epsilon \in \mathbb{Z}_2^n \). Moreover, it can also be seen that each \( x_{p,q}^\epsilon \) is conjugate to either \( x_{p,q} \) or \( x_{p,q}^{-1} \) by an element of the free group on \( S \). For instance, in the case when \( \epsilon_{p-n} = 0 \) and \( \epsilon_{n,q-n} = 1 \) we have:
\[
x_{p,q}^\epsilon = t_\epsilon \cdot \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon
\]
\[
= t_\epsilon \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon \cdot t_\epsilon \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon
\]
\[
= \alpha_p \alpha_q \alpha_p^{-1} \alpha_q
\]
\[
= (\alpha_q^{-1}) \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot (\alpha_q)
\]
\[
= \alpha_q \cdot x_{p,q}^{-1} \cdot \alpha_q.
\]
The other cases can be proved similarly (see Appendix). Thus it follows that \( R \) given by (2.9) is a complete set of relations for \( \pi_1(Y_n) \). Hence the theorem. \( \Box \)
Corollary 2.2. The group $\pi_1(Y_n)$ is abelian if and only if $Y_n$ is a product of $n$-copies of $\mathbb{P}^1$.

**Proof:** From the above theorem we see that $\pi_1(Y_n)$ is abelian if and only if $c_{p-n,q-n} = 0$ for all $n+1 \leq p < q \leq 2n$. Hence the corollary. □

Lemma 2.3. $Y_n$ is an aspherical manifold.

**Proof:** Note that $Y_n$ is an $S^1$-bundle over $Y_{n-1}$ so by the long exact homotopy sequence we get:

$$\ldots \rightarrow \pi_2(Y_{n-1}) \rightarrow \pi_1(S^1) \rightarrow \pi_1(Y_n) \rightarrow \pi_1(Y_{n-1}) \rightarrow 0 \ldots$$

Since $\pi_k(Y_1) = 0$ for $k \geq 2$, the proof follows by induction on $n$. □

Corollary 2.4. The group $\pi_1(Y_n)$ is always torsion free.

**Proof:** This follows readily since $Y_n$ is an aspherical manifold. □

Remark 2.5. Lemma 2.3 alternately follows from [22, Theorem 1.2], since it can easily be seen that the fan $\Delta$ associated to $\Delta$ is flag like. More precisely, we cannot find three edge vectors $\{v_i, v_j, v_k\}$ which pairwise form a cone in $\Delta$ but together do not form a cone in $\Delta$. Also, Corollary 2.3 can alternately be derived from the combinatorial criterion on $\Delta$ for the fundamental group to be abelian in [22, Theorem 5.1].

2.1. Further group theoretic properties of $\pi_1(Y_n)$.

Proposition 2.6. The commutator subgroup $[\pi_1(Y_n), \pi_1(Y_n)]$ is abelian. In particular, $\pi_1(Y_n)$ is a solvable group.

**Proof:** For every $v_i$, $1 \leq i \leq n$ (resp. $n+1 \leq i \leq 2n$) there exists a unique $v_{i+n}$ (resp. $v_{i-n}$) such that $v_i, v_{i+n}$ (resp. $v_i, v_{i-n}$) do not form a cone in $\Delta$. Thus by [22, Lemma 4.1], $[W, W]$ is abelian. Further, since $\pi_1(Y_n)$ is a subgroup of $W$ (see 2.2) it follows that $[\pi_1(Y_n), \pi_1(Y_n)]$ is a subgroup of $[W, W]$ and is hence abelian. Now, $1 \leq [\pi_1, \pi_1] \leq \pi_1(Y_n)$ gives an abelian tower for $\pi_1(Y_n)$ so that $\pi_1(Y_n)$ is solvable. □

Let $\overline{\alpha_j}$ denote the image of $\alpha_j$ under the canonical abelianization homomorphism

$$\pi_1(Y_n) \rightarrow H_1(Y_n; \mathbb{Z}) \simeq \pi_1(Y_n)/[\pi_1(Y_n), \pi_1(Y_n)].$$

We then have the following description of $H_1(Y_n; \mathbb{Z})$:

**Proposition 2.7.** The group $H_1(Y_n; \mathbb{Z})$ has a presentation with generators

$$\langle \overline{\alpha_j} : n+1 \leq j \leq 2n \rangle$$

and relations

$$\alpha_p \cdot \alpha_q \cdot \alpha_p^{-1} \cdot \alpha_q^{-1}$$

for $n+1 \leq p, q \leq 2n$ and

$$\overline{\alpha_q^2}$$

for those $n+1 \leq q \leq 2n$ for which there exists a $p < q$ such that $c_{p-n, q-n} = 1$. Thus additively we have an isomorphism $H_1(Y_n; \mathbb{Z}) \simeq \mathbb{Z}^{n-r} \bigoplus \mathbb{Z}_2^r$ where $r$ is the number of $n+1 \leq q \leq 2n$ for which there exists a $p < q$ with $c_{p-n, q-n} = 1$. \\

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Corollary 2.8. The commutator subgroup \([\pi_1(Y_n), \pi_1(Y_n)]\) is a free abelian group with generators \(\alpha_q^2\) where \(n + 1 \leq q \leq 2n\) is such that there exists \(p < q\) with \(c_{p-n,q-n} = 1\).

**Proof:** In \([W, W]\), \(w \cdot [s_i, s_j] \cdot w^{-1} = [s_j, s_i]\) if the reduced word \(w\) contains either \(s_i\) or \(s_j\) but not both, \(w \cdot [s_i, s_j] \cdot w^{-1} = [s_i, s_j]\) otherwise. But we know that

\([s_i, s_j] = (s_i, s_j)^2 = 1\) when \(j \neq i + n\)

So, \([W, W] = \langle w \cdot [s_i, s_j] \cdot w^{-1} \mid 1 \leq i, j \leq d, w \in W \rangle = \langle (s_i \cdot s_{i+n})^2 \mid 1 \leq i \leq n \rangle\).

Observe that \(\alpha^2_j = (s_j \cdot s_{j-n-1} \cdots s_{j-n})^2 = (s_j \cdot s_{j-n})^2\) for \(j = n + 1 \leq j \leq 2n\).

Thus \([W, W]\) is generated by \(\langle \alpha_j^2 \mid n + 1 \leq j \leq 2n \rangle\) as a subgroup of \(\pi_1(Y_n)\).

Moreover, by [22, Lemma 4.2], the torsion elements of \(W\) and hence in \([W, W]\) are only of order 2. Hence it follows that \([W, W]\) is a free abelian group generated by \(\alpha_j^2, n + 1 \leq j \leq 2n\). Since \([\pi_1(Y_n), \pi_1(Y_n)]\) is a subgroup of \([W, W]\), by Proposition 2.7 the corollary follows. \(\square\)

Proposition 2.9. The group \(\pi_1(Y_n)\) is nilpotent if and only if it is abelian.

**Proof:** Let \(\pi_1^n, n + 1 \leq q \leq 2n\) is such that there exists a \(p < q\) such that \(c_{p-n,q-n} = 1\). Then we see that \(\alpha_q^2 \alpha_p \alpha_q^{-2} \alpha_p^{-1} = (s_q s_{q-n})^4 = \alpha_q^4\), which belongs to \([\pi_1(Y_n), [\pi_1(Y_n), \pi_1(Y_n)]]\). Proceeding similarly by induction we get that

\[\alpha_q^{2(k-1)} \alpha_p \alpha_q^{-2(k-1)} \alpha_p^{-1} = \alpha_q^{2k}\]

belongs to \(\pi_1(Y_n)^{(k)}\). Here \(\pi_1(Y_n)^{(1)} := [\pi_1(Y_n), \pi_1(Y_n)]\) and \(\pi_1(Y_n)^{(k)} := [\pi_1(Y_n), \pi_1(Y_n)^{(k-1)}]\). Since \(\alpha_q\) is of infinite order in \(W\), the proposition follows. \(\square\)

Remark 2.10. In particular, by Corollary 2.8, \(\pi_1(Y_n)\) is nilpotent if and only if \(Y_n\) is a product of \(n\)-copies of \(\mathbb{P}^1\)'s.

Remark 2.11. Here we mention that the fundamental group of a real toric variety is not in general solvable. For example, the non-orientable surfaces of genus \(g\) are real toric varieties (see for example [18, Section 4.5, Remark 4.5.2] and [22, Remark 3.3]), whose fundamental groups contain free subgroups of rank \(g - 1\) (see for example [2, p. 62, Section 4]). Thus whenever \(g \geq 3\), these groups are not solvable. Hence the property proved in Proposition 2.6 is specific to the real Bott tower.

3. Cohomology ring with \(\mathbb{Z}_2\)-coefficients

In this section we shall describe cohomology ring \(H^*(Y_n; \mathbb{Z}_2)\) in terms of generators and relations by applying [6, Theorem 5.12], viewing \(Y_n\) as a small cover. More precisely, we state the following theorem.

**Theorem 3.1.** Let \(\mathcal{R} := \mathbb{Z}_2[x_1, x_2, \ldots, x_{2n}]\) and let \(\mathcal{I}\) denote the ideal in \(\mathcal{R}\) generated by the following set of elements

\[\{x_i x_{n+i}, x_i - x_{n+i} + \sum_{j=1}^{i-1} c_{i,j} \cdot x_{n+j} \mid 1 \leq i \leq n}\].

As a graded \(\mathbb{Z}_2\)-algebra \(H^*(Y_n; \mathbb{Z}_2)\) is isomorphic to \(\mathcal{R}/\mathcal{I}\).

**Proof:** The proof immediately follows by applying [6, Theorem 4.13] viewing \(Y_n\) as a small cover over an \(n\)-cube with the characteristic map given by means of the Bott matrix. Thus we get the following isomorphism of graded \(\mathbb{Z}_2\)-algebras:

\[\psi : \mathcal{R}/\mathcal{I} \simeq H^*(Y_n; \mathbb{Z}_2)\]
where \( \psi \) maps \( x_i + I \) to the fundamental class of the characteristic submanifold \( [M_i] \in H^1(Y_n; \mathbb{Z}_2) \) for \( 1 \leq i \leq 2n \). Hence the theorem. \( \square \)

**Corollary 3.2.** Let \( \tilde{R} := \mathbb{Z}_2[y_1, y_2, \ldots, y_n] \) and let \( \tilde{I} \) denote the ideal in \( \tilde{R} \) generated by the following set of elements

\[
\{ y_i^2 - \sum_{j=1}^{i-1} c_{j,i} \cdot y_i y_j \mid 1 \leq i \leq n \}.
\]

We have the following isomorphism of graded \( \mathbb{Z}_2 \)-algebras:

\[
\tilde{\psi} : \tilde{R}/\tilde{I} \simeq H^*(Y_n; \mathbb{Z}_2)
\]

where \( \tilde{\psi} \) maps \( y_i + I \) to the fundamental class of the characteristic submanifold \( [M_{i+n}] \in H^1(Y_n; \mathbb{Z}_2) \) for \( 1 \leq i \leq n \).

**Proof:** In (3.16), by substituting the second relation in the first, we can simplify the presentation of \( R/I \) by reducing the generators to \( x_{n+1}, \ldots, x_{2n} \) and the relations to

\[
x_{n+i} \cdot [x_{n+i} - \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+j}] \forall 1 \leq i \leq n.
\]

Now, by sending \( y_i \) to \( x_{i+n} \) we get the required isomorphism from \( R/I \) to \( \tilde{R}/\tilde{I} \).

**Remark 3.3.** We can also apply results in [18, Section 4.3] to get Theorem 3.1 by viewing \( Y_n \) as a real toric variety.

**Remark 3.4.** Note that, in fact there exists a canonical line bundle \( [L_i] \) on \( Y_n \) such that \( [M_i] = w_1(L_i) \) for every \( 1 \leq i \leq 2n \), where \( w_1(L_i) \) denotes the first Stiefel Whitney class of \( L_i \). Thus the cohomology ring is generated by \( w_1(L_i) \) for \( 1 \leq i \leq 2n \) (see [6, Section 6.1]).

### 4. Stiefel Whitney Classes of \( Y_n \)

Let \( w_k(Y_n) \) denote the \( k \)th Stiefel-Whitney class of \( Y_n \) for \( 0 \leq k \leq n \) with the understanding that \( w_0(Y_n) = 1 \). Then \( w(Y_n) = 1 + w_1(Y_n) + \cdots + w_n(Y_n) \) is the total Stiefel-Whitney class of \( Y_n \).

**Theorem 4.1.** (i) Under the isomorphism (3.17) of \( H^*(Y_n; \mathbb{Z}_2) \) with \( R/I \) we have the identification

\[
w(Y_n) = \prod_{i=1}^{2n} (1 + x_i)
\]

where \( x_i \) for \( 1 \leq i \leq 2n \) satisfy (3.16).

(ii) We further have the following recursive formula

\[
w(Y_n) = w(Y_{n-1}) \cdot (1 + x_n)(1 + x_{2n}),
\]

where

\[
x_n \cdot x_{2n} = 0, x_n = x_{2n} - \sum_{i=1}^{n-1} c_{i,n}x_{n+i}.
\]
Proof: The proof of (i) follows readily by applying [6] Corollary 6.8 for $Y_n$.

Now, we shall prove (ii).

Note that the defining Bott matrix for $Y_{n-1}$ is the $n-1 \times n-1$ submatrix of $C$ obtained by deleting the $n$th row and the $n$th column of $C$.

Moreover, let $\pi_n : Y_n \to Y_{n-1}$ denote the canonical projection of the $\mathbb{RP}^1$-bundle. Then via pullback along $\pi_n^*$, $H^*(Y_{n-1}; \mathbb{Z}_2)$ can be identified with the subring $\mathcal{R}'/\mathcal{I}'$ of $\mathcal{R}/\mathcal{I}$ where $\mathcal{R}' = \mathbb{Z}_2[x_1,x_2,\ldots,x_{n-1},x_{n+1},\ldots,x_{2n-1}]$ and $\mathcal{I}'$ is the ideal generated by the relations

\begin{equation}
(4.24) \quad \{ x_{i}x_{n+i}, x_i - x_{n+i} + \sum_{j=1}^{i-1} c_{j,i}x_{n+j} \text{ for } 1 \leq i \leq n-1 \}.
\end{equation}

Since $Y_n$ is an $\mathbb{RP}^1$-bundle over $Y_{n-1}$, we further have the following presentation of $H^*(Y_n; \mathbb{Z}_2)$ as an algebra over $H^*(Y_{n-1}; \mathbb{Z}_2)$:

\begin{equation}
(4.25) \quad H^*(Y_n; \mathbb{Z}_2) \simeq H^*(Y_{n-1}; \mathbb{Z}_2)[x_n,x_{2n}]/J
\end{equation}

where $J$ is the ideal generated by the relations

\begin{equation}
(4.26) \quad x_n \cdot x_{2n}, x_n - x_{2n} + \sum_{i=1}^{n-1} c_{i,n}x_{n+i}.
\end{equation}

Furthermore, via $\pi_n^*$ we can identify $w(Y_{n-1})$ with the expression

\begin{equation}
(4.27) \quad w(Y_{n-1}) = \prod_{i=1}^{n-1} (1 + x_i) \cdot \prod_{i=n+1}^{2n-1} (1 + x_i)
\end{equation}

in $\bar{R}$ where $x_i$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq 2n-1$ satisfy the relations (4.24).

Now by (4.21) and (4.27), (ii) follows. □

Corollary 4.2.

(i) The following hold in the $\mathbb{Z}_2$-algebra $\mathcal{R}/\mathcal{I}$:

\begin{equation}
(4.28) \quad w(Y_n) = w(Y_{n-1})(1 + \sum_{i=1}^{n-1} c_{i,n}x_{n+i}),
\end{equation}

\begin{equation}
(4.29) \quad w_k(Y_n) = w_k(Y_{n-1}) + w_{k-1}(Y_{n-1}) \cdot (\sum_{i=1}^{n-1} c_{i,n}x_{n+i})
\end{equation}

for $n \geq 2$ and $1 \leq k \leq n$.

(ii) For every $1 \leq k \leq n$, $w_k(Y_n)$ is a $\mathbb{Z}_2$-linear combination of square free monomials of degree $k$ in the variables $x_{n+1}, \ldots, x_{2n-1}$ modulo $\mathcal{I}$.

Proof: The equation (4.21) reduces to (4.28) by applying (4.23). Note that under the isomorphism $\psi$ of graded algebras $H^*(Y_n; \mathbb{Z}_2)$ and $\mathcal{R}/\mathcal{I}$, $w_k(Y_n) \in H^k(Y_n; \mathbb{Z}_2)$ corresponds to a polynomial of degree $k$ in $x_i, 1 \leq i \leq 2n$ modulo $\mathcal{I}$ for $1 \leq k \leq n$. Thus we get (4.29) by comparing the degree $k$-terms on either side of (4.28) and (i) follows.

Observe that by applying (4.24), in $\mathcal{R}'/\mathcal{I}'$ and hence in $\mathcal{R}/\mathcal{I}$, we can substitute for $x_i$ in terms of $x_{n+1}, \ldots, x_{n+i}$ modulo $\mathcal{I}$ using the equality

\begin{equation}
(4.30) \quad x_i = x_{n+i} + \sum_{j=1}^{i-1} c_{j,i}x_{n+j}.
\end{equation}
In particular, $w_k(Y_{n-1})$ (resp. $w_{k-1}(Y_{n-1})$) can be written as a polynomial of degree $k$ (resp. $k-1$) in $x_{n+i}$, $1 \leq i \leq n-1$. Furthermore, multiplying either side of (4.30) with $x_{n+i}$, along with the equality $x_i \cdot x_{n+i} = 0$ gives

$$x_{n+i}^2 = \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+i} \cdot x_{n+j}$$

(4.31)

for $1 \leq i \leq n-1$. It follows that $w_k(Y_{n-1})$ (resp. $w_{k-1}(Y_{n-1})$) can be expressed as square free monomials of degree $k$ (resp. $k-1$) in $x_{n+i}$, $1 \leq i \leq n-1$ in the algebra $\mathcal{R}/\mathcal{I}$. Now, assertion (ii) follows readily by applying (4.31) again in (4.29).

**Corollary 4.3.** We have the following elegant formula for $w_{n-1}(Y_n)$ in $H^*(Y_n; \mathbb{Z}_2)$ in terms of the Bott numbers $c_{i,j}$:

$$w_{n-1}(Y_n) = c_{1,2} \cdot c_{2,3} \cdots c_{n-1,n} \cdot x_{n+1} \cdot x_{n+2} \cdots x_{2n-1}.$$  

**Proof:** The proof follows by induction on $n$ and (4.29) using the fact that in $H^*(Y_n; \mathbb{Z}_2)$ the following relations hold:

$$x_{n+1}^2 = 0; x_{n+1} \cdot x_{n+2}^2 = 0; x_{n+1} \cdot x_{n+2} \cdot x_{n+3}^2 = 0 \cdots x_{n+1} \cdot x_{n+2} \cdots x_{2n-2}^2 = 0.$$  

4.1. Orientability of the real Bott tower. In this section we give a necessary and sufficient condition for $Y_n$ to be orientable.

**Lemma 4.4.** We have the following expression for the $w_1(Y_n)$ in $\mathcal{R}/\mathcal{I}$:

$$w_1(Y_n) = \sum_{i=1}^{n-1} (\sum_{j=i+1}^{n} c_{i,j}) \cdot x_{n+i}.$$  

**Proof:** The lemma follows by putting $k = 1$ in (4.29) followed by induction on $n$. □

**Theorem 4.5.** The space $Y_n$ is orientable if and only if

$$\sum_{j=i+1}^{n} c_{i,j} \equiv 0 \pmod{\mathbb{Z}_2} \text{ for every } 1 \leq i \leq n-1,$$

where $c_{i,j}$ are the entries of the defining Bott matrix $C$ (see [1,1]).

**Proof:** Note that via the isomorphism (3.17) of graded algebras $\mathcal{R}/\mathcal{I}$ and $H^*(Y_n; \mathbb{Z}_2)$, $\{x_{n+1}, x_{n+2}, \ldots, x_{2n}\}$ corresponds to a basis over $\mathbb{Z}_2$ of $H^1(Y_n; \mathbb{Z}_2)$. Thus by (4.33), it follows that $w_1(Y_n) = 0$ if and only if

$$\sum_{j=i+1}^{n} c_{i,j} \equiv 0 \pmod{\mathbb{Z}_2} \text{ for every } 1 \leq i \leq n-1.$$  

(4.35)

Furthermore, since a necessary and sufficient condition for a compact connected differentiable manifold $M$ to be orientable is $w_1(M) = 0$, the theorem follows. □

**Corollary 4.6.** Let $Y_n$ be an oriented real Bott tower $Y_n$. Then $w_{n-1}(Y_n) = 0$.

**Proof:** By (4.33) it follows that $c_{n-1,n} = 0$ if $Y_n$ is orientable. Now by (4.34) the corollary follows. □
Remark 4.7. The assertion of Corollary 4.6 is true for any even dimensional manifold but not in general true when the dimension is odd (see [14, Theorem II and examples on p. 94]). Our assertion although specific to the case of a real Bott tower, holds in all dimensions.

Remark 4.8. In particular, a 3-step oriented real Bott tower \( Y_3 \) satisfies \( w_2(Y_3) = 0 \), and hence admits spin structure. This is a special case of the well known more general result of Steenrod that an oriented threefold is parallelizable.

Remark 4.9. Trivially product of \( n \) copies of \( \mathbb{P}^1 \)'s is a parallelizable Bott tower for any \( n \). Converse is not true as can be seen by the 3-step Bott tower associated with Bott numbers \( c_{1,2} = 1, c_{1,3} = 1 \) and \( c_{2,3} = 0 \), which is parallelizable but not a product of \( \mathbb{P}^1 \)'s.

Now we give a combinatorial characterization for \( Y_n \) to admit a spin structure.

**Theorem 4.10.** The orientable Bott tower \( Y_n \) admits a spin structure if and only if in addition to \((4.34)\) the following identities hold for \( 1 \leq j < k \leq n - 2 \):

\[
(4.36) \quad \sum_{r=j+1}^{n} \sum_{s=k+1}^{n} c_{j,r} \cdot c_{k,s} + c_{j,k} \sum_{r,s=k+1}^{n} c_{k,r} \cdot c_{k,s} \equiv 0 \pmod{2}
\]

where \( c_{i,j} \) are as defined in \((4.11)\).

**Proof:** By Theorem 3.1 and by equation \((4.21)\), \( w(Y_n) \) can be identified with the class in \( \mathcal{R}/\mathcal{I} \) of the following term

\[
(4.37) \quad \prod_{j=1}^{n} (1 + x_j + x_{n+j} + x_j \cdot x_{n+j})
\]

Further, using the relations \((3.16)\) in \( \mathcal{I} \) we can rewrite \((4.37)\) as

\[
(4.38) \quad \prod_{j=2}^{n} \left( 1 + \sum_{i=1}^{j-1} c_{i,j} \cdot x_{n+i} \right)
\]

Furthermore, since Theorem 3.1 gives an isomorphism of graded \( \mathbb{Z}_2 \)-algebras, the degree 2 term of \( w(Y_n) \), namely \( w_2(Y_n) \), can be identified with the degree 2 term of expression \((4.38)\) which is the class of the following term in \( \mathcal{R}/\mathcal{I} \):

\[
(4.39) \quad \sum_{1 \leq j < k \leq n-1} \left( \sum_{r=j+1}^{n} \sum_{s=k+1}^{n} c_{j,r} c_{k,s} \right) x_{n+j} x_{n+k} + \sum_{k=1}^{n-2} \left( \sum_{r,s=k+1}^{n} c_{k,r} c_{k,s} \right) x_{n+k}^2
\]

By substituting \( x_{n+k}^2 = \sum_{j=1}^{k-1} c_{j,k} \cdot x_{n+j} \cdot x_{n+k} \) from \((4.31)\) and \( c_{n-1,n} = 0 \) from \((4.34)\) in \((4.39)\), we get that \( w_2(Y_n) \) can be identified with the class of the following term in \( \mathcal{R}/\mathcal{I} \):

\[
(4.40) \quad \sum_{1 \leq j < k \leq n-2} \left( \sum_{r=j+1}^{n} \sum_{s=k+1}^{n} c_{j,r} c_{k,s} + c_{j,k} \cdot \sum_{r,s=k+1}^{n} c_{k,r} c_{k,s} \right) x_{n+j} x_{n+k}
\]

Further, as a graded \( \mathbb{Z}_2 \)-vector space \( H^2(Y_n; \mathbb{Z}_2) \) is isomorphic to the subspace of \( \mathcal{R}/\mathcal{I} \) freely generated over \( \mathbb{Z}_2 \) by the classes of \( x_{n+j} x_{n+k}, 1 \leq j < k \leq n \). Moreover,
the necessary and sufficient condition for an orientable manifold \( M \) to admit a spin structure is \( w_2(M) = 0 \). Thus the theorem follows from (4.40). □

**Example 4.11.** The 4-step Bott towers admitting spin structure are classified by the following list of associated Bott matrices.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

**Remark 4.12.** Note that the above list of Bott matrices exhausts all orientable 4-step Bott towers. Thus it follows that every orientable 4-step Bott tower is also spin. Moreover, it is known that a 4-manifold \( M \) is parallelizable if and only if it admits a spin structure (i.e. \( w_1(M) = w_2(M) = 0 \)) and has vanishing Euler characteristic and signature \( (\chi(M) = \sigma(M) = 0) \) (see [19, p. 699]). Moreover, by Hirzebruch signature formula, \( \sigma(M) = \frac{1}{4} p_1(M)|M| \), where \( p_1(M) \) is the first Pontrjagin class and \([M]\) the fundamental class of \( M \). Now, a real Bott tower has vanishing Euler characteristic (see Remark 4.22 and vanishing Pontrjagin classes by ([6] Corollary 6.8 (i))). Thus it follows that a 4-step Bott tower is orientable if and only if it is parallelizable. Further, it corresponds to one of the eight Bott matrices in the above list.

The following example shows that this is not the case in dimensions 5 and higher. Indeed there are \( n \)-step Bott towers which are orientable but not spin when \( n \geq 5 \).

**Example 4.13.** Let \( Y_n \) be the \( n \)-step Bott tower, \( n \geq 5 \), associated to the Bott numbers \( c_{1,2} = 1, c_{1,n-2} = 1; c_{n-2,n-1} = 1, c_{n-2,n} = 1; c_{i,j} = 0 \) for \( i \neq 1, n-2 \) and \( c_{1,j} = 0 \) for \( j \neq 2, n-2 \). These numbers clearly satisfy (4.34) but not (4.36). Indeed in this case, when \( j = 1 \) and \( k = n-2 \), the left hand side of (4.36) is \( c_{1,2}(c_{n-2,n-1} + c_{n-2,n}) + c_{1,n-2}(c_{n-2,n-1} + c_{n-2,n}) + c_{1,n-2}c_{n-2,n-1}c_{n-2,n} \equiv 1 \mod \mathbb{Z}_2 \).

**Definition 4.14.** We call the Bott matrix \( C \) spin if and only if the associated Bott tower \( Y_n = Y(C) \) is spin.

Let \( R_i \) denote the \( i \)-th row vector \((0, \ldots, 0, 1 = c_{i,i}, c_{i,i+1}, \ldots, c_{i,n})\) of \( C \). For every \( 1 \leq j < k \leq n \), we define an \( n \times n \) Bott matrix \( C_{j,k} \) with \( R_i \) as the \( j \)-th row and \( R_k \) as the \( k \)-th row. Further, for \( i 
eq j, k \), we let the \( i \)-th row of \( C_{j,k} \) have 1 as the \((i,i)\)-th entry and all other entries as 0.

**Corollary 4.15.** The Bott matrix \( C \) is spin if and only if \( C_{j,k} \) is spin for every \( 1 \leq j < k \leq n-2 \).

**Proof:** From Theorem 4.10 a necessary and sufficient condition for \( C \) to be spin is that the entries \( c_{i,j} \), \( i+1 \leq j \leq n \) on the row \( R_i \) for every \( 1 \leq i \leq n \) satisfy (4.34) and further, the entries \( c_{j,r}, j+1 \leq r \leq n \) of \( R_j \) and \( c_{k,s}, k+1 \leq s \leq n \) of \( R_k \) for every \( 1 \leq j < k \leq n \) satisfy (4.36).
Remark 4.16. See [12] for results on criterion for nonzero rows of the matrix $A$. Theorem 1.2 follows immediately from Corollary 4.15 above. This can be seen because the entries above the main diagonal on the $i$th row of $C_{jk}$ where $i \neq j, k$, trivially satisfy (4.34). Moreover, if either $i \neq j, k$ or $l \neq j, k$ and $1 \leq i < l \leq n$, the entries of $C_{jk}$ above the main diagonal on the $i$th and the $l$th row trivially satisfy (4.36). Hence the corollary.

**Remark 4.16.** See [12] for results on criterion for $Y_n$ to admit spin structure using different methods. We however note here that the main result [12, Theorem 1.2] follows immediately from Corollary 4.15 above. This can be seen because the conditions (4.34) and (4.36) need to be checked only for the entries of pairs of nonzero rows of the matrix $A$ where $A := C - I$. In [12], $A$ is called the Bott matrix. Thus we need to check the spin condition only on those $A_{jk} := C_{jk} - I$ with $j$th and $k$th nonzero rows.

Moreover, our result is more general as we do not require the assumption in [12, Theorem 1.2] that the number of nonzero rows of the matrix $A$ is even (see [12, Remark 2.1]).

We illustrate Corollary 4.15 by the following examples.

**Example 4.17.**

\[(1)\]

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
C_{23} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
C_{24} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
C_{34} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Clearly $\sum_{i=1}^{n} c_{i,j} \equiv 0 \pmod{2}$ for all $1 \leq i \leq 5$.

When $j = 2$ and $k = 3$ the left hand side of (4.10) is $c_{2,3}(c_{3,4} + c_{3,5} + c_{3,6}) + c_{2,4}(c_{3,5} + c_{3,6}) + c_{2,5}(c_{3,4} + c_{3,6}) + c_{2,6}(c_{3,4} + c_{3,5}) + c_{2,8}(c_{3,4} + c_{3,5} + c_{3,6} + c_{3,5} c_{3,6}) \equiv 0 \pmod{2}$

When $j = 3$ and $k = 4$ the left hand side of (4.10) is $c_{3,4}(c_{4,5} + c_{4,6}) + c_{3,5} c_{4,6} + c_{3,6} c_{4,5} + c_{3,4} + c_{4,5} c_{4,6} \equiv 0 \pmod{2}$

When $j = 2$ and $k = 4$ the left hand side of (4.10) is $c_{2,3}(c_{4,5} + c_{4,6}) + c_{2,4}(c_{4,5} + c_{4,6}) + c_{2,5} c_{4,6} + c_{2,6} c_{4,5} + c_{2,4} c_{4,5} c_{4,6} \equiv 0 \pmod{2}$. 


Since $C_{23}, C_{24}, C_{34}$ are all spin by Corollary [1.15] $C$ is spin. We do not consider the matrices $C_{1l}$ for $2 \leq l \leq 4$, since the first row of $C$ has nonzero entry only on the main diagonal.

\[(2)\]

\[
C = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Clearly $\sum_{i=1}^{n} c_{1i} = 0$ for all $1 \leq i \leq 4$.

When $j = 1$ and $k = 2$ the left hand side of (1.10) is

$c_{1,2}(c_{2,3} + c_{2,4} + c_{2,5}) + c_{1,3}(c_{2,4} + c_{2,5}) + c_{1,4}(c_{2,3} + c_{2,5}) + c_{1,5}(c_{2,3} + c_{2,4}) + c_{1,2}(c_{2,3} + c_{2,4} + c_{2,5} + c_{2,4} + c_{2,5}) \equiv 0 (\text{mod } \mathbb{Z}_2).

When $j = 1$ and $k = 3$ the left hand side of (1.10) is

$c_{1,2}(c_{3,4} + c_{3,5}) + c_{1,3}(c_{3,4} + c_{2,5}) + c_{1,4} c_{3,5} + c_{1,5} c_{3,4} + c_{1,3} c_{3,4} c_{3,5} \equiv 1 (\text{mod } \mathbb{Z}_2).$ Thus $C$ is not spin since $C_{13}$ is not spin.

\[(3)\]

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Clearly $\sum_{i=1}^{n} c_{1i} = 0$ for all $1 \leq i \leq 6$.

When $j = 2$ and $k = 3$ the left hand side of (1.10) is

$c_{2,3}(c_{3,4} + c_{3,5} + c_{3,6} + c_{3,7}) + c_{2,4}(c_{3,5} + c_{3,6} + c_{3,7}) + c_{2,5}(c_{3,4} + c_{3,6} + c_{3,7}) + c_{2,6}(c_{3,4} + c_{3,5} + c_{3,7}) + c_{2,3}(c_{3,4} + c_{3,5} + c_{3,4} + c_{3,7}) + c_{3,6} c_{3,7} + c_{3,6} \equiv 1 (\text{mod } \mathbb{Z}_2).$

Thus $C$ is not spin since $C_{23}$ is not spin.
4.2. Existence of a nowhere vanishing tangent vector field.

**Proposition 4.18.** The $n$-step Bott tower is the total space of a fibre bundle over $S^1$ with fibre an $n-1$-step Bott tower corresponding to the Bott matrix $C^1$ of size $n-1 \times n - 1$, defined by deleting the first row and first column of $C$.

**Proof:** This can be seen as follows: Let $N'$ denote the lattice $\bigoplus_{i=2}^{n} \mathbb{Z}e_i$. Let $v'_1 = e_2, v'_2 = e_3, \ldots, v'_{n-1} = e_n$. Also let $v'_{n+1} = -e_2 + c_2 e_3 + \cdots + c_{2,n} e_n, \ldots, v'_{2n-2} = -e_{n-1} + c_{n-1,n} e_n$ and $v'_{2n-1} = -e_n$.

We define the fan $\Delta'$ in $N'$ consisting of cones generated by the set of vectors in any subcollection of $\{v'_1, v'_2, \ldots, v'_{n-1}, v'_{n+1}, \ldots, v'_{2n-1}\}$ which does not contain both $v'_i$ and $v'_{n+i}$ for $1 \leq i \leq n-1$.

Also let $N''$ denote the lattice $\mathbb{Z}e_1$ and $\Delta''$ denote the fan consisting of the one dimensional cones generated by the vectors $e_1$ and $-e_1$ and the cone $\{0\}$ which corresponds to the real toric variety $\mathbb{P}^1_k$.

By projecting to $\mathbb{Z}e_1$ we get an exact sequence of fans

\[(4.41) \quad 0 \to (\Delta', N') \to (\Delta, N) \to (\Delta'', N'') \to 0.\]

Moreover, the fan $\tilde{\Delta''}$ in $N$ consisting of the one dimensional cones generated by the vectors $e_1$ and $-e_1 + e_{1,2} e_2 + \cdots + e_{1,n} e_n$ and the cone $\{0\}$ is a lift of $\Delta''$. Moreover, it can be seen that every cone $\sigma$ of $(N, \Delta)$ is a sum $\sigma' + \sigma''$ of a cone in $(N', \Delta')$ and a cone in $(N, \tilde{\Delta''})$.

Thus we see that $Y_n = X(\Delta, N)$ is a toric fibre bundle over $S^1$ with fibre an $n-1$-step real Bott tower corresponding to the fan $(N', \Delta')$ (see [8, p. 41]). \[
\]

In this convention, we shall denote the $n$-step Bott tower by $Z_n$ and the fibre, which is the $n-1$-step real Bott tower associated to the matrix $C^1$ and hence the fan $(N', \Delta')$, by $Z_{n-1}$. We can iterate this process and view $Z_{n-1}$ again as a fibre bundle over $S^1$ with fibre $Z_{n-2}$ which is the $n-2$-step real Bott tower associated to the matrix $C^2$ of size $n-2 \times n - 2$, obtained by deleting the first and the second rows and columns of $C$. Continuing this process $n-2$ times we finally get that $Z_2$ is a two step Bott tower associated to the Bott matrix $C^{n-2}$, obtained by deleting the first $n-2$ rows and columns of $C$. Then $Z_2$ is a fibre bundle over $S^1$ with fibre $Z_1 \simeq S^1$.

Following the above convention, in this section we shall denote the $n$-step real Bott tower by $Z_n$. Let $p_n : Z_n \rightarrow S^1$ denote the projection of this fibre bundle.

**Theorem 4.19.** The $n$-step real Bott tower $Z_n$ has a nowhere vanishing continuous tangent vector field. Moreover, the tangent bundle splits into a direct sum of line bundles.

**Proof:** Note that the tangent bundle of $Z_n$ is the direct sum of the pull back of the tangent bundle of $S^1$ and the relative tangent bundle of $Z_{n-1}$.

Since the tangent bundle of $S^1$ is trivial its pull back to $Z_n$ is a trivial line bundle. Thus a nowhere vanishing section of the trivial line bundle gives a nowhere vanishing section for the tangent bundle of $Z_n$.

Furthermore, if we assume by induction that the tangent bundle of an $n-1$-step Bott tower is a direct sum of line bundles, then the relative tangent bundle of $Z_{n-1}$ in the above fibration is also a direct sum of the associated line bundles. It follows that the tangent bundle of the $n$-step Bott tower is a direct sum of $n$-line bundles. Hence the proof.\[
\]


Corollary 4.20. Let $Z_n$ be an orientable real Bott tower. Then the Euler class of $Z_n$ vanishes.

Proof: The proof is an immediate consequence of the [17, Theorem 4.19 and p. 101].

Corollary 4.21. The $n$-step real Bott tower $Z_n$ is orientable (respectively spin) implies that the successive fibres $Z_{n-1}, Z_{n-2}, \ldots, Z_2$ in the above iterated construction are all orientable (respectively spin).

Proof: This follows from (4.34) and (4.36) since the Bott matrix corresponding to $Z_k$ is $C^{n-k}$ which is the matrix obtained from $C$ by deleting the first $k$ rows and $k$ columns.

Remark 4.22. Alternately the assertion of Corollary 4.20 follows more directly from the fact that the Euler characteristic is multiplicative for the total space of fibre bundles. Indeed $\chi(Z_n) = 0$ since $\chi(S^1) = 0$. In fact this also implies the statement on existence of nowhere vanishing vector field in Theorem 4.19 by Hopf’s theorem.

4.3. Real Bott-towers bound.

Definition 4.23. A smooth compact $n$-dimensional manifold $M$ without boundary is null cobordant if it is diffeomorphic to the boundary of some compact smooth $n+1$-dimensional manifold $W$ with boundary.

Let $w_k := w_k(Y_n)$ for $1 \leq k \leq n$. Also let $\mu_{Y_n}$ denote the fundamental class of $Y_n$ in $H_n(Y_n; \mathbb{Z}_2)$. Then

\begin{equation}
\langle w_1^{r_1} \cdots w_n^{r_n}, \mu_{Y_n} \rangle \in \mathbb{Z}_2
\end{equation}

such that $\sum_{i=1}^n i \cdot r_i = n$ are the Steifel-Whitney numbers of $Y_n$.

Theorem 4.24. The $n$-step Bott tower $Y_n$ is null-cobordant.

Proof: From Corollary 4.2 (ii), it follows that, any monomial $w_1^{r_1} \cdots w_n^{r_n}$ of total dimension $n$, under the isomorphism $\psi$ corresponds in $R/I$, to a $\mathbb{Z}_2$-linear combination of square free monomials of degree $n$ in $x_{n+1}, x_{n+2}, \ldots, x_{2n-1}$. But there are no square free monomials of degree $n$ in $x_{n+j}, 1 \leq j \leq n-1$. Thus the monomial $w_1^{r_1} \cdots w_n^{r_n} = 0$ in $H^n(Y_n; \mathbb{Z}_2)$ so that the associated Steifel-Whitney number is zero. Therefore by Thom’s theorem it follows that $Y_n$ is null-cobordant.

Definition 4.25. A smooth compact $n$-dimensional manifold $M'$ without boundary is orientedly null cobordant if it is diffeomorphic to the boundary of some compact smooth $n+1$-dimensional oriented manifold $W'$ with boundary.

Let $Y_n$ denote an oriented $n$-step real Bott tower. Let $p_i := p_i(Y_n)$ denote the $i$th Pontryagin class of $Y_n$ in $H^{4i}(Y_n; \mathbb{Z})$ and $\mu_{Y_n}$ denote the fundamental homology class in $H_n(Y_n; \mathbb{Z}_2)$. Then for each $I = i_1, \ldots, i_r$ a partition of $k$, the $I$th Pontrjagin number of $Y_n$ is given by

\begin{equation}
\langle p_{i_1} \cdots p_{i_r}, \mu_{Y_n} \rangle \in \mathbb{Z}
\end{equation}

when $n = 4k$. It is zero when $n$ is not divisible by 4 (see [17, p. 185]).

Corollary 4.26. Let $Y_n$ be an oriented $n$-step real Bott tower, then it is orientedly null-cobordant.
Proof: Note that [6, Corollary 6.8 (i)], implies that all the Pontrjagin numbers of $Y_n$ vanish. Moreover, we have shown above in the proof of Theorem 4.24 that all the Stiefel-Whitney numbers of $Y_n$ vanish. Thus the corollary follows by Wall’s theorem ([23, Section 8, Corollary 1]). □

Remark 4.27. There are examples of real toric varieties whose top Stiefel-Whitney class does not vanish. For example, the non-orientable surfaces of odd genus are real toric varieties (see [18, Section 4.5, Remark 4.5.2] [22, Remark 3.3]) having non-vanishing second Stiefel-Whitney class. Thus the properties we have proved in this and the preceding section are all specific to real Bott towers.

Remark 4.28. Another motivation for this work was to relate the topology of the real Bott tower with the topology of the real Bott Samelson manifolds using the degeneration results of Grossberg and Karshon (see [13]). This could further be used to obtain results on the topology of real flag manifolds. In particular, to study the fundamental group and cohomology ring of the real Bott Samelson manifolds and real flag variety. This shall again be taken up in future work.

Remark 4.29. During the process of this work the authors also came across the work of Kamishima and Masuda [20] where among other results they also compute the fundamental group and cohomology ring of real Bott towers but using different methods.

Appendix

Proof that $x_{p,q}^\epsilon$ is a conjugate of either $x_{p,q}$ or $x_{p,q}^{-1}$ by an element of the free group generated by $S$:

\begin{enumerate}
  \item $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 0$ and $\epsilon_{q-n} = 0$:
    \[ x_{p,q}^\epsilon = \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} \]
    \[ = x_{p,q} \]
  \item $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 0$ and $\epsilon_{q-n} = 1$:
    \[ x_{p,q}^\epsilon = \alpha_p \alpha_q^{-1} \alpha_p^{-1} \]
    \[ = \alpha_q^{-1} \cdot (\alpha_p \alpha_p^{-1} \alpha_q^{-1}) \cdot \alpha_q \]
    \[ = \alpha_q^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_q \]
    \[ = \alpha_q^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_q \]
  \item $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 0$:
    \[ x_{p,q}^\epsilon = \alpha_p^{-1} \alpha_q \alpha_p \alpha_q^{-1} \]
    \[ = \alpha_p^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1}) \cdot \alpha_p \]
    \[ = \alpha_p^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_p \]
    \[ = \alpha_p^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_p \]
  \item $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 1$:
    \[ x_{p,q}^\epsilon = \alpha_p^{-1} \alpha_q^{-1} \alpha_p \alpha_q \]
    \[ = (\alpha_p \alpha_q)^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1}) \cdot (\alpha_p \alpha_q) \]
    \[ = (\alpha_p \alpha_q)^{-1} \cdot x_{p,q} \cdot (\alpha_p \alpha_q) \]
\end{enumerate}
(5) \( c_{p-n,q-n} = 1, \epsilon_{p-n} = 0 \) and \( \epsilon_{q-n} = 0 \):
\[
x_{p,q}^\epsilon = \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} \\
= x_{p,q}
\]

(6) \( c_{p-n,q-n} = 1, \epsilon_{p-n} = 1 \) and \( \epsilon_{q-n} = 0 \):
\[
x_{p,q}^\epsilon = \alpha_p^{-1} \alpha_q^{-1} \alpha_p \alpha_q^{-1} \\
= (\alpha_p \alpha_q^{-1})^{-1} \cdot (\alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1}) \cdot (\alpha_p \alpha_q^{-1}) \\
= (\alpha_p \alpha_q^{-1})^{-1} \cdot x_{p,q} \cdot (\alpha_p \alpha_q^{-1})
\]

(7) \( c_{p-n,q-n} = 1, \epsilon_{p-n} = 1 \) and \( \epsilon_{q-n} = 1 \):
\[
x_{p,q}^\epsilon = \alpha_p^{-1} \alpha_q \alpha_p \alpha_q \\
= \alpha_p^{-1} \cdot (\alpha_q \alpha_p \alpha_q \alpha_p) \cdot \alpha_p \\
= \alpha_p^{-1} \cdot (\alpha_q \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_p \\
= \alpha_p^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_p
\]

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