LATTICES AND CORRECTION TERMS

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Abstract. Let $L$ be a nonunimodular definite lattice, $L^*$ its dual lattice, and $\lambda$ the discriminant form on $L^*/L$. Using a theorem of Elkies we show that whether $L$ embeds in the standard definite lattice of the same rank is completely determined by a collection of lattice correction terms, one for each metabolizing subgroup of $(L^*/L, \lambda)$. As a topological application this gives a rephrasing of the obstruction for a rational homology 3–sphere to bound a rational homology 4–ball coming from Donaldson’s theorem on definite intersection forms of 4–manifolds. Furthermore, from this perspective it is easy to see that if the obstruction to bounding a rational homology ball coming from Heegaard Floer correction terms vanishes, then (under some mild hypotheses) the obstruction from Donaldson’s theorem vanishes too.

1. Introduction

The purpose of this note is to give an explicit treatment of certain ideas, some of which are known to varying degrees, but which have not been presented fully in the form appearing here. In [Elk95] Elkies showed that every unimodular positive definite lattice $L$ of rank $n$ contains characteristic vectors with square less than or equal to $n$, and if there are no characteristic vectors with square strictly less than $n$ then $L$ is isomorphic to the standard lattice $(\mathbb{Z}^n, I)$. One can define a lattice correction term

$$d_L = \min \left\{ \frac{\chi^2 - n}{4} \right\},$$

where the minimum is over all characteristic vectors $\chi \in L$ (see [Gre13]). This is well-defined for all positive definite unimodular lattices, and Elkies’ result translates to the statement that $d_L \leq 0$ and $d_L = 0$ if and only if $L$ is isomorphic to the standard lattice. Our first goal is to generalize this to the case where $L$ is not unimodular. In this setting one can ask whether a definite lattice embeds in the standard lattice of the same rank.

Recall that we have a sequence $0 \to L \to L^* \xrightarrow{\pi} L^*/L \to 0$, and if $L$ is not unimodular then the discriminant group $L^*/L$ is non-trivial. There is a one-to-one correspondence between metabolizers $M < L^*/L$ and unimodular lattices $U$ with $L \subset U \subset L^*$, given by $U := \pi^{-1}(M)$ (Proposition 4). (Recall that a metabolizer is a subgroup $M$ with $|L^*/L| = |M|^2$ and such that the discriminant form $\lambda$ is identically zero on $M$.) For a metabolizer $M$ we denote the corresponding unimodular lattice $U(M)$. Then $U(M)$ will necessarily be positive definite of rank $n$, and hence we have a lattice correction term $d_{U(M)}$. We derive the following as a corollary of Elkies’ theorem.
Theorem 1. For $L$ a positive definite lattice of rank $n$, consider the set $D := \{d_{U(M_i)}\}$ of lattice correction terms, where we range over all metabolizers $M_i < L^*/L$. Then $L$ embeds in the standard lattice of rank $n$ if and only if $D$ contains $0$.

Note that $D$ is a finite set since $L^*/L$ is a finite group, and $D$ is empty (and hence $L$ does not embed in the standard lattice) if there do not exist any metabolizers. Our main interest in this result is in application to the following question in low-dimensional topology:

Question ([Kir78], Problem 4.5). When does a rational homology 3–sphere bound a rational homology 4–ball?

We are interested in the relationship between two obstructions to a rational homology 3–sphere $Y$ smoothly bounding a rational homology 4–ball. Suppose $Y$ bounds a smooth positive definite 4–manifold $X$ with $H_1(X) = 0$. If $Y$ bounds a smooth rational homology ball $W$ as well, we can form a smooth, closed, definite 4–manifold $Z = X \cup Y - W$. By Donaldson’s theorem [Don83, Don87] on definite intersection forms of smooth, closed 4–manifolds, the lattice $(H_2(Z), Q_Z)$ must be isomorphic to the standard lattice. It then follows that the lattice $(H_2(X), Q_X)$ must embed in the standard lattice of the same rank. This is what we call the obstruction to $Y$ bounding a rational homology ball coming from Donaldson’s theorem.

By Theorem 1, the obstruction coming from Donaldson’s theorem is completely determined by a collection of lattice correction terms, one for each metabolizing subgroup of $(H_1(Y), \lambda)$ (in this context $\lambda$ is known as the linking form). Then work of Ozsváth and Szabó [OS03] shows that the Heegaard Floer corrections terms of $Y$ put bounds on the values of these lattice correction terms (indeed, the definition [I] of lattice correction terms is motivated by properties of Heegaard Floer correction terms, see Section 3). We use these bounds to show that the vanishing of the Heegaard Floer correction terms $d(Y, t)$ on a metabolizer $M$ as above is strictly positive, then $Y$ cannot bound a positive definite 4–manifold $X$ with $H_1(X) = 0$ (Proposition 8). Recall that if $Y$ bounds a rational homology ball, then $d(Y, t) = 0$ for all spin$^c$ structures $t$ that extend over the rational homology ball. This can be interpreted in a convenient way if we further assume that $Y$ is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere (so $|H_1(Y)|$ is odd). In particular, if such a $Y$ bounds a rational homology ball then there exists a metabolizer $M$ such that $d(Y, t) = 0$ for all spin$^c$ structures $t$ with $\text{PD}(c_1(t)) \in M$ (see, for example, [HLR12]). This is what we call the obstruction to a $\mathbb{Z}/2\mathbb{Z}$-homology sphere $Y$ bounding a rational homology ball coming from correction terms, and hence Theorem 2 shows that (in this context) the obstruction to bounding a rational homology ball coming from Heegaard Floer...
Correction terms is always at least as strong as that coming from Donaldson’s theorem. Note that if $d(Y, t) = 0$ for all spin$^c$ structures $t$ with $\text{PD}(c_1(t)) \in M$, then $-Y$ also satisfies the conditions of Theorem 2 (since $d(-Y, t) = -d(Y, t)$), and so by reversing orientation we get a statement that also applies to negative definite fillings of $Y$.

**Corollary 3.** If $Y$ is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere on which the correction term obstruction to bounding a rational homology ball vanishes, then for any smooth (positive or negative) definite 4–manifold $X$ with $H_1(X) = 0$ and $\partial X = Y$, the lattice $(H_2(X), Q_X)$ must embed in the standard lattice of the same rank.

When $Y$ is not a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, the first Chern class mapping is no longer a bijection, and Theorem 2 is less useful. For example, the lens space $L(4, 1)$ (which does bound a rational homology ball) does not satisfy the hypotheses of Theorem 2 since $\text{PD}(c_1(t))$ belongs to the unique metabolizer for each spin$^c$ structure on $L(4, 1)$, and there exists a spin$^c$ structure whose corresponding correction term is negative.

That there is a close relationship between Donaldson’s theorem and the correction terms of Heegaard Floer homology was already established in the original paper defining correction terms [OS03]. Indeed, using the theorem of Elkies mentioned above, Ozsváth and Szabó gave a new proof of Donaldson’s theorem using properties of the unique correction term $d(N)$ for an integral homology sphere $N$ (this mirrored another proof in Seiberg-Witten Floer theory [Frø96]). Furthermore, they showed that the obstruction to an integral homology 3–sphere bounding an integral homology 4–ball coming from correction terms is at least as strong as that coming from Donaldson’s theorem. More precisely, suppose $N$ bounds a positive definite 4–manifold $X$. Then if $d(N) = 0$ (which must be the case if $N$ bounds an integral homology ball), $Q_X$ must be isomorphic to the standard form [OS03] (Corollary 3 is a generalization of this statement).

If we are dealing with a rational rather than integral homology sphere, the two obstructions are slightly more complicated, as we described above. The extra complication in the case of Donaldson’s theorem is because we have to consider embeddings, rather than isomorphisms, of lattices; in the case of correction terms it is because there is no longer a unique correction term, but rather a collection of correction terms corresponding to the set of spin$^c$ structures on the 3–manifold.

Nonetheless, relations between these two obstructions for rational homology spheres have appeared previously in the literature, usually in more specific contexts. In [GJ11] Greene and Jabuka show that in the application of these obstructions to showing that certain types of knots (e.g. alternating knots) are not slice, one can view the correction term obstruction as a second-order obstruction after the vanishing of the obstruction coming from Donaldson’s theorem (see [GJ11] Theorem 3.6 and the preceding exposition). More recently, Greene [Gre17] showed that in certain special cases these two obstructions can be used to achieve the same purpose. For example, he showed that either obstruction is sufficient to classify which lens spaces $L(p, q)$ with odd $p$ bound rational homology balls (which had been carried out by Lisca [Lis07] using the obstruction from Donaldson’s theorem, including those with even $p$). Indeed, the proof of Theorem 2 is very similar to the ideas presented in [Gre17, Proposition 2.1]. In particular, one direction of Greene’s argument gives Corollary 3.
when \(H_1(Y)\) is cyclic. Hence the present note can be thought of as a companion to that paper, where here we take a more general and elementary perspective.

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## 2. Lattices

First we develop the necessary terminology about lattices (cf. [Gre13, Section 2]). In this paper a lattice \((L, Q)\) is a finite rank free abelian group \(L\) together with a symmetric, bilinear form \(Q: L \times L \to \mathbb{Q}\). We will assume that \(Q\) is nondegenerate, i.e., for every non-zero \(x \in L\) there exists some \(y \in L\) such that \(Q(x, y) \neq 0\). Usually the form will be understood and we will just say \(L\) is a lattice. If the image of the form lies in \(\mathbb{Z}\), then the lattice will be called integral. We will always use \(L\) to denote an integral lattice. An isomorphism of lattices is an isomorphism of the free abelian groups that preserves the forms, and an embedding of lattices is a monomorphism that preserves the forms.

We say \(L\) is positive definite if the rank of \(Q\) equals its signature, and negative definite if the rank of \(Q\) equals \(-1\) times the signature. The standard positive definite lattice, or more simply, the standard lattice (of rank \(n\)), is \((\mathbb{Z}^n, I)\). This means that in a chosen basis the form is represented by the identity matrix.

The form \(Q\) extends to a rational valued form on \(L \otimes \mathbb{Q}\), and the dual lattice \(L^*\) is defined as the subset \(\{x \in L \otimes \mathbb{Q} \mid Q(x, y) \in \mathbb{Z}, \forall y \in L\}\). The quotient \(L^*/L\) is called the discriminant group, and its order is the discriminant of \(L\), denoted \(\text{disc}(L)\). If \(\text{disc}(L) = 1\), then we say \(L\) is unimodular. Note that we have a sequence \(0 \to L \to L^* \xrightarrow{\pi} L^*/L \to 0\). We can define a symmetric bilinear form \(\lambda: (L^*/L) \times (L^*/L) \to \mathbb{Q}/\mathbb{Z}\), called the discriminant form, as follows. For any \(x, y \in L^*/L\), take lifts \(\bar{x}, \bar{y} \in L^*\) (so \(\pi(\bar{x}) = x\) and \(\pi(\bar{y}) = y\)), and define \(\lambda(x, y) = -Q(\bar{x}, \bar{y}) \pmod{1}\) (mod 1).

As mentioned in the introduction, a subgroup \(M < L^*/L\) satisfying \(\text{disc}(L) = |M|^2\) and \(\lambda_{M \times M} = 0\) is called a metabolizer. The following proposition is well-known in various forms (see [Tab12, Lemma 2.5]), and is central to our argument.

**Proposition 4.** There is a one-to-one correspondence between metabolizers of \((L^*/L, \lambda)\) and unimodular integral lattices \(U\) with \(L \subset U \subset L^*\), given by the assignment \(U(M) := \pi^{-1}(M)\), for each metabolizer \(M\).

**Proof.** Let \(M\) be a metabolizer of \((L^*/L, \lambda)\), and \(U(M) = \pi^{-1}(M)\) as above. Then \(U(M)\) is a subgroup of \(L^*\) and the rational valued form \(Q\) on \(L^*\) restricts to an integral form on \(U(M)\). To see this, observe that for \(\bar{x}, \bar{y} \in U(M)\), \(x = \pi(\bar{x})\) and \(y = \pi(\bar{y})\) are in \(M\), and so \(Q(\bar{x}, \bar{y}) \equiv -\lambda(x, y) = 0 \pmod{1}\).

\(U(M)\) is unimodular if and only if \(U(M) = U(M)^*\), or equivalently, if \([U(M)^* : U(M)] = 1\). Now \([U(M) : L] = |M| = \sqrt{\text{disc}(L)}\), and \([L^* : U(M)] = [L^*/L : M] = \sqrt{\text{disc}(L)}\) as well. Since

\[
\text{disc}(L) = [L^* : L] = [L^* : U(M)^*][U(M)^* : U(M)][U(M) : L],
\]

it suffices to show that \([U(M) : L] = [L^* : U(M)^*]\). This follows from Lemma 5 below.
So far we have shown that $U(M)$ is a unimodular integral lattice. Now let $U$ be an arbitrary unimodular integral lattice with $L \subset U \subset L^*$. Similar to an argument above, the fact the $U$ is integral implies that $\lambda$ vanishes on $\pi(U)$. So to show that $\pi(U)$ is a metabolizer we only need to show that $|L^*/L| = |\pi(U)|^2$. This holds because

$$|\pi(U)| = |U : L| = |L^* : U^*| = |L^* : U| = |L^*/L|/|\pi(U)|,$$

where the second equality uses Lemma 4 again. Hence $\pi(U)$ is a metabolizer, and $U = \pi^{-1}(\pi(U))$ since $U \subset \pi^{-1}(\pi(U)) \subset U^* \subset U$. This shows that the assignment $M \mapsto U(M)$ is a bijection. \hfill \Box

**Lemma 5.** Let $L'$ be an integral lattice with $L \subset L' \subset L^*$. Then $[L' : L] = [L^* : (L')^*]$.

**Proof.** Let $H = \pi(L')$, so $H \cong L'/L$. Furthermore let $H^\circ$ denote its annihilator, i.e., the subgroup of $L^*/L$ consisting of all elements that pair trivially with every element of $H$ under $\lambda$. Then observe that $H^\circ = \pi((L')^*)$. We claim that $(L^*/L)/H^\circ \cong H$. To see this, note that the map $\psi : L^*/L \rightarrow \text{Hom}(L^*/L, \mathbb{Q}/\mathbb{Z}) \cong L^*/L$ given by $\psi(x) = \lambda(x, \cdot)$ is an isomorphism since $\lambda$ is nondegenerate (c.f. [Wal63, Section 1]). Indeed, for each $x \in L^*/L$ there exists some $y \in L^*/L$ such that $\lambda(x, y) = 1/n \in \mathbb{Q}/\mathbb{Z}$, where $n$ is the order of $x$. We get a map $\psi' : L^*/L \rightarrow \text{Hom}(H, \mathbb{Q}/\mathbb{Z}) \cong H$ by restricting the domain of each $\psi(x)$ to $H$. Then $\psi'$ has image isomorphic to $H$ and kernel $H^\circ$, giving $(L^*/L)/H^\circ \cong H$. It then follows that

$$[L' : L] = [L'/L] = |H| = |(L^*/L)/H^\circ| = [L^* : (L')^*],$$

completing the proof. \hfill \Box

We introduce some additional terminology. A characteristic covector $\chi \in L^*$ is an element such that $Q(\chi, y) \equiv Q(y, y) \pmod{2}$ for all $y \in L$. Let $\text{Char}(L)$ denote the set of characteristic covectors. If a characteristic covector $\chi$ actually lies in $L$ (as will always be the case when $L$ is unimodular), we can simply call $\chi$ a characteristic vector. As in the introduction, if $L$ is unimodular and positive definite, we have a well-defined lattice correction term

$$d_L = \min_{\chi \in \text{Char}(L)} \left\{ \frac{\chi^2 - \text{rk}(L)}{4} \right\}.$$ 

([Gre13] contains an extended discussion of this invariant.) In this language we can state the result of Elkies as follows.

**Theorem 6 ([Elk95]).** For $L$ a unimodular positive definite lattice, $d_L \leq 0$ and $d_L = 0$ if and only if $L$ is isomorphic to the standard lattice.

Hence the lattice correction term completely determines when a unimodular positive definite lattice is isomorphic to the standard lattice. We can combine Theorem 6 and Proposition 4 to characterize which nonunimodular positive definite lattices embed in the standard lattice. Let $L$ be such a lattice, and $M < L^*/L$ be a metabolizer. Since $L$ is positive definite, $U(M)$ is positive definite as well, since $L \subset U(M)$ and both lattices have the same rank. Hence we can define a set $D := \{d_{U(M)}\}$ of lattice correction terms, where we range over all metabolizers $M_i < L^*/L$. Recall that Theorem 1 from the introduction states the $L$ embeds in the standard lattice of the same rank if and only if $D$ contain 0. We prove this theorem now.
Proof of Theorem 1. If \( D \) contains 0, then some \( U(M) \) satisfies \( d_{U(M)} = 0 \). By Theorem 6, \( U(M) \) is isomorphic to the standard lattice. Since \( L \subset U(M) \), one direction of the proof is finished.

In the other direction, suppose \( L \) embeds in the standard lattice \((\mathbb{Z}^n, I)\) of the same rank. Hence we can suppose \( L \subset \mathbb{Z}^n \), and tensoring with \( \mathbb{Q} \) shows that \((\mathbb{Z}^n, I) \supset \mathbb{L} \). By Proposition 4, \((\mathbb{Z}^n, I) = U(M) \) for some metabolizer \( M \), and Theorem 6 implies that \( d_{U(M)} = 0 \). This completes the other direction of the proof. \( \square \)

Let \( n \) denote the rank of \( L \) (and \( U(M) \)). Recall that \( d_{U(M)} \) is defined as

\[
(2) \quad d_{U(M)} = \min_{\chi \in \text{Char}(U(M))} \left\{ \frac{\chi^2 - n}{4} \right\}.
\]

Since \( L \subset U(M) \subset L^* \), \( \text{Char}(U(M)) \subset \text{Char}(L) \). Indeed, \( \text{Char}(U(M)) \) is a subset of those characteristic covectors of \( L^* \) that map to elements of \( M \) under the projection \( \pi \). Hence from (2) we obtain

\[
(3) \quad d_{U(M)} \geq \min_{\chi \in \text{Char}(L)} \left\{ \frac{\chi^2 - n}{4} \right\}.
\]

This will be useful in the next section. Note that it is possible to show we have equality in (3) if \( \text{disc}(L) \) is odd.

3. Rational homology spheres and correction terms

We now turn to the topological application discussed in the introduction. Let \( Y \) be a rational homology 3–sphere that bounds a smooth positive definite 4–manifold \( X \) with \( H_1(X) = 0 \). By the long exact sequence of the pair \((X, Y)\) we get the presentation

\[
(4) \quad 0 \rightarrow H_2(X) \rightarrow H_2(X, Y) \rightarrow H_1(Y) \rightarrow 0.
\]

Under suitable choices of bases, the map \( H_2(X) \rightarrow H_2(X, Y) \) is given by the matrix representing the intersection form \( Q_X \) (see, for example, [GS99, Exercise 5.3.13 (f)]). Furthermore, if we let \( L \) denote the lattice \((H_2(X), Q_X)\), the dual lattice \( L^* \) is identified with \((H_2(X, Y), Q_X^{-1})\), and (4) becomes

\[
(5) \quad 0 \rightarrow L \xrightarrow{Q_X} L^* \xrightarrow{\pi} L^*/L \rightarrow 0.
\]

In this context the discriminant form is called the linking pairing \( \lambda \) on \( H_1(Y) \equiv L^*/L \), and is defined by \( \lambda(x, y) = -(Q_X)^{-1}(\pi^{-1}(x), \pi^{-1}(y)) \) (mod 1).

As explained in the introduction, a consequence of Donaldson’s theorem is that if \( Y \) smoothly bounds a rational homology ball, then the lattice \( L = (H_2(X), Q_X) \) embeds in the standard lattice of the same rank. By Theorem 6 this condition is completely determined by the collection of lattice correction terms \( \{d_{U(M_i)}\} \), where we range over metabolizers of \((H_1(Y), \lambda)\). These lattice correction terms are in turn bounded by the Heegaard Floer correction terms of \( Y \), as we now describe. Recall that in Ozsváth and Szabó’s Heegaard Floer homology, correction terms are rational valued invariants of spin\(^c\) rational homology spheres that are preserved under spin\(^c\) rational homology cobordism. For \( Y \) with spin\(^c\) structure \( t \), the corresponding correction term
is denoted \(d(Y, t)\). In the present context (\(Y\) bounding a positive definite \(X\)), we have the following theorem.

**Theorem 7** ([OS03]). If \(s\) is a spin\(^c\) structure on \(X\) with \(s|_Y = t\), then

\[
\frac{1}{4}(c_1(s))^2 - \text{rk}(H_2(X)) \geq d(Y, t).
\]

Now we relate this to lattices. The first Chern class mapping and Poincaré duality provide a bijection between spin\(^c\) structures on \(X\) and characteristic covectors in \(H^2(X,Y)\) ([GS99, Proposition 2.4.16]). Under this bijection, a spin\(^c\) structure \(s\) on \(X\) that extends a spin\(^c\) structure \(t\) on \(Y\) corresponds to a characteristic covector \(\chi\) in \(H^2(X,Y) = L^*\), such that \(\pi(\chi) = \text{PD}(c_1(t))\). (See Figure 1.) Then (6) implies that \(\frac{1}{4}(\chi^2 - \text{rk}(H_2(X))) \geq d(Y, t)\) for each such \(\chi\) (other applications of these bounds are found in [OS06] and [GW13]). For a metabolizer \(M\), we can combine this with the inequality (3) to obtain

\[
0 \geq d_{U(M)} \geq \min_{\chi \in \text{Char}(L)} \left\{ \frac{\chi^2 - \text{rk}(H_2(X))}{4} \right\} \geq \min_{t \in \text{Spin}^c(Y) : \text{PD}(c_1(t)) \in M} \left\{ d(Y, t) \right\}.
\]

Note that we would have a contradiction if there exists a metabolizer \(M\) for \(Y\) with

\[
\min_{t \in \text{Spin}^c(Y) : \text{PD}(c_1(t)) \in M} \left\{ d(Y, t) \right\} > 0,
\]

and so such a \(Y\) cannot bound a smooth positive definite 4–manifold \(X\) with \(H_1(X) = 0\). We record this here as a proposition. Note that this generalizes a similar result for integral homology spheres [OS03, Corollary 9.8] (also compare to [OS12a, Proposition 5.2]).

**Proposition 8.** Suppose a rational homology sphere \(Y\) has a metabolizer \(M\) for \((H_1(Y), \lambda)\) for which \(d(Y, t) > 0\) for each spin\(^c\) structure \(t\) with \(\text{PD}(c_1(t)) \in M\). Then \(Y\) cannot bound a smooth positive definite 4–manifold \(X\) with \(H_1(X) = 0\).

We can now prove the second theorem from the introduction.

**Proof of Theorem 2.** Recall we are assuming that \(Y\) is a rational homology 3–sphere that bounds a positive definite 4–manifold \(X\) with \(H_1(X) = 0\), and that there exists a metabolizer \(M\) of \(H_1(Y)\) such that \(d(Y, t) \geq 0\) for all spin\(^c\) structures \(t\) with
By Proposition 8, there must be at least one such $t$ such that $d(Y, t) = 0$, and hence
$$\min_{t \in \text{Spin}^c(Y) \cap \text{PD}(c_1(t)) \in M} \{ d(Y, t) \} = 0.$$  

Then equation (7) implies that $d(U(M)) = 0$, and by Theorem 1 the lattice $(H_2(X), Q_X)$ must embed in the standard lattice of the same rank. \hfill $\square$

4. Examples

Finally we give a couple examples to illustrate these ideas. First we use Proposition 8 to show that the connected sum of $n$ copies of +3-surgery on the right-handed trefoil, $\#^n S^3_3(T_{2,3})$, for any $n > 0$, does not bound a negative definite 4-manifold with trivial first homology (this also follows from [OS12a Proposition 5.2]). To see this, suppose that $\#^n S^3_3(T_{2,3})$ does bound such a negative definite 4-manifold, for some $n > 0$. Then $Y := \#^4n(-S^3_3(T_{2,3}))$ bounds a positive definite 4-manifold with vanishing first homology. The correction terms of surgeries on torus knots can be computed readily by combining work of [NW15] and [BL14] (see, for example, [AG17]). In this way one can check that every correction term for $-S^3_3(T_{2,3})$ is positive. Hence on any metabolizer $M$ of $Y$ (which do exist, since the Witt group $W(Q/Z)$ is 4-torsion [MH73]) we have that $d(Y, t) > 0$ for each spin$^c$ structure $t$ with $\text{PD}(c_1(t)) \in M$. This contradicts Proposition 8.

Next we consider +9-surgery on the left-handed trefoil, $S^3_9(T_{-2,3})$. Again let $Y$ denote this 3-manifold. Then $Y$ has a unique metabolizer, and the corresponding correction terms are $\{2, 0, 0\}$. Hence the correction term obstruction shows that $S^3_9(T_{-2,3})$ does not bound a rational homology ball. On the other hand, Theorem 2 states that for every positive definite 4-manifold $X$ with $H_1(X) = 0$, the lattice $(H_2(X), Q_X)$ must embed in the standard lattice of the same rank. Indeed it easy to check this condition for the two obvious positive definite 4–manifolds bounded by $Y$: the 2-handlebody given by the trace of the surgery, and the canonical definite plumbing associated to $S^3_9(T_{-2,3})$ as a Seifert fibered space. However, $Y$ also bounds a negative definite 4–manifold with trivial first homology (see [OS12b]), with intersection form

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Since the corresponding lattice does not embed in the standard negative definite lattice of rank 9, we see that the obstruction coming from Donaldson’s theorem can also be used to show that $Y$ does not bound a rational homology ball. It is an interesting question whether there exists a rational homology sphere $Y$ for which the correction term obstruction does not vanish, but for any positive or negative definite 4–manifold bounded by $Y$, the associated lattice must embed in the standard lattice of the same rank.
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