THE DIRICHLET PROBLEM IN A CLASS OF GENERALIZED WEIGHTED SPACES

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Abstract. We show continuity in generalized weighted Morrey spaces $M_{p,\varphi}(w)$ of sub-linear integral operators generated by some classical integral operators and commutators. The obtained estimates are used to study global regularity of the solution of the Dirichlet problem for linear uniformly elliptic operators with discontinuous data.

1. Introduction

In the present work we study the global regularity in generalized weighted Morrey spaces $M_{p,\varphi}(w)$ of the solutions of a class of elliptic partial differential equations (PDEs). Recall that the classical Morrey spaces $L_{p,\lambda}$ were introduced by Morrey in [34] in order to study the local Hölder regularity of the solutions of elliptic systems. In [5] Chiarenza and Frasca show boundedness in $L_{p,\lambda}(\mathbb{R}^n)$ of the Hardy-Littlewood maximal operator $\mathcal{M}$ and the Calderón-Zygmund operator $\mathcal{K}$

$$\mathcal{M} f(x) = \sup_{B(x)} \int_{B(x)} |f(y)| \, dy, \quad \mathcal{K} f(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n} \, dy.$$
Integral operators of that kind appear in the representation formulae of the solutions of various PDEs. Thus the continuity of the Calderón-Zygmund integral in certain functional space permit to study the regularity of the solutions of boundary value problems for linear PDEs in the corresponding space.

In [33] Mizuhara extended the definition of $L^p,\lambda$ taking a non-negative measurable function $\phi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ instead of the Morrey weight $r^\lambda$ in the definition of $L^p,\lambda$. Precisely, $f \in L^p,\phi(\mathbb{R}^n)$ if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $p \in [1, \infty)$ and
\[
\|f\|_{p,\phi} = \sup_{B_r(x)} \left( \frac{1}{\phi(x,r)} \int_{B_r(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty
\]
and the supremo is taken over all balls in $\mathbb{R}^n$.

Later Nakai extended the results of Chiarenza and Frasca to the case of $L^p,\phi$. Imposing the next integral and doubling conditions on $\phi$ (see [35])
\[
k_1^{-1} \leq \frac{\phi(x_0,t)}{\phi(x_0,r)} \leq k_1, \quad r \leq t \leq 2r,
\]
\[
\int_r^\infty \frac{\phi(x_0,t)}{t^{n+1}} \, dt \leq k_2 \frac{\phi(x_0,r)}{r^n}
\]
he proved boundedness of $M$ and $K$
\[
\|Mf\|_{p,\phi} \leq C\|f\|_{p,\phi}, \quad \|Kf\|_{p,\phi} \leq C\|f\|_{p,\phi}
\]
for all $f \in L^p,\phi(\mathbb{R}^n)$, $p \geq 1$.

The next extension of the Morrey spaces is given by the first author. He defined generalized Morrey spaces $M_{p,\varphi}$ with normalized norm under more general condition on the weight $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and considered continuity of various classical integral operators from one space $M_{p,\varphi_1}$ to another $M_{p,\varphi_2}$ under suitable condition on the pair $(\varphi_1, \varphi_2)$. In [11] (see also [12, 13]) it is shown that if
\[
\int_r^\infty \varphi_1(x,t) \frac{dt}{t} \leq C \varphi_2(x,r)
\]
then the operator $K$ is bounded from $M_{p,\varphi_1}$ to $M_{p,\varphi_2}$ for $p > 1$ and from $M_{1,\varphi_1}$ to the weak space $WM_{1,\varphi_2}$. In [2, 19], Guliyev et al. introduced a weaker condition on the pair $(\varphi_1, \varphi_2)$ under which boundedness of the classical integral operators from $M_{p,\varphi_1}$ to $M_{p,\varphi_2}$ is proved. Precisely, if

$$\int_r^\infty \frac{\operatorname{ess \ inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^\frac{n}{p}} \frac{dt}{t} \leq C \varphi_2(x, r),$$

then $K$ is bounded from $M_{p,\varphi_1}$ to another $M_{p,\varphi_2}$ for $p > 1$ and from $M_{1,\varphi_1}$ to the weak space $WM_{1,\varphi_2}$. Let us note that the condition (1.1) describes wider class of weight functions than (1.2) (see [16]).

For more recent results on boundedness and continuity of singular integral operators in generalized Morrey and new functional spaces and their application in the theory of the differential equations see [2, 13, 17, 18, 22, 23, 36, 39, 40] and the references therein.

Consider now the weighted $L^p$-spaces $L^p_w$ consisting of measurable functions $f$ for which

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(y)|^p w(y) \, dy\right)^{\frac{1}{p}}.$$

In [30] Muckenhoupt showed that the well known maximal inequality holds in $L^p_w$ if and only if the weight $w$ satisfies certain integral condition called $A_p$-condition. Later, Coifman and Fefferman [8] studied the continuity of some classical singular integrals in the Muckenhoupt spaces (see also [31, 32]).

Recently, Komori and Shirai [28] defined the weighted Morrey spaces $L^p_{\kappa}(w)$ endowed by the norm

$$\|f\|_{p_w,\kappa} = \sup_{B} \left(\frac{1}{w(B)^\kappa} \int_B |f(y)|^p w(y) \, dy\right)^{\frac{1}{p}}.$$ 

They studied the boundedness of the Calderón-Zygmund operator $K$ in these spaces. A natural extension of their results are the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ with $w \in A_p$ and $\varphi$ satisfying (1.1).
In [16] (see also [20, 21]) it is proved boundedness in $M_{p,\varphi}(w)$ of sub-linear operators generated by classical operators as $\mathcal{M}$, $\mathcal{K}$, the Riesz potential and others, covering such way the results obtained in [35] and [28]. Our goal here is to obtain a priori estimate for the solution of the Dirichlet problem for linear elliptic equations in these spaces.

The paper is organized as follows. We begin introducing the functional spaces that we are going to use. In Sections 3 and 4 we study continuity in the spaces $M_{p,\varphi}(w)$ of certain sub-linear integrals and their commutators with functions with bounded mean oscillation. These results permit to obtain continuity of the Calderón-Zygmund operator, with bounded functions and some nonsingular integrals which is done in Section 6. The last section is dedicated to the Dirichlet problem for linear elliptic equations with discontinuous coefficients. This problem is firstly studied by Chiarenza, Frasca and Longo. In their pioneer works [6, 7] they prove unique strong solvability of

$$\begin{cases} 
L u \equiv a^{ij}(x)D_{ij}u = f(x) & \text{a.a. } x \in \Omega, \\
u \in W^2_p(\Omega) \cap W^{1,1}_p(\Omega), & p \in (1, \infty), \ a^{ij} \in VMO \end{cases}$$

extending this way the classical theory of operators with continuous coefficients to those with discontinuous coefficients. Later their results have been extended in the Sobolev-Morrey spaces $W^2_{p,\lambda}(\Omega) \cap W^1_{p,\lambda}(\Omega)$, $\lambda \in (1, n)$ (see [9]) and the generalized Sobolev-Morrey spaces $W^2_{p,\phi}(\Omega) \cap W^1_{p,\phi}(\Omega)$ (see [40]) with $\phi$ as in [35]. In [22] we have studied the regularity of the solution of (1.3) in generalized Sobolev-Morrey spaces $W^2_{p,\varphi}(\Omega)$ where the weight function $\varphi$ satisfies a certain supremal condition derived from (1.2). We show that $Lu \in M_{p,\varphi}(\Omega)$ implies $D_{ij}u \in M_{p,\varphi}(\Omega)$ satisfying the estimate

$$\|D^2 u\|_{p,\varphi;\Omega} \leq C(\|Lu\|_{p,\varphi;\Omega} + \|u\|_{p,\varphi;\Omega}).$$

These studies are extended on divergence form elliptic/parabolic equations in [3, 24].
In this paper we use the following notions:

\[ D_i u = \partial u / \partial x_i, \quad Du = (D_1 u, \ldots, D_n u) \]
means the gradient of \( u \),

\[ D_{ij} u = \partial^2 u / \partial x_i \partial x_j, \quad D^2 u = \{ D_{ij} u \}_{ij=1}^n \]
means the Hessian matrix of \( u \),

\[ B_r (x_0) = \{ x \in \mathbb{R}^n : | x - x_0 | < r \} \]
is a ball centered at a fixed point \( x_0 \in \mathbb{R}^n \),

\[ B_r (x) \equiv B_r \equiv B \]
is a ball centered at any point \( x \in \mathbb{R}^n \), \( |B_r| = C r^n \),

\[ B_r^c = \mathbb{R}^n \setminus B_r, \quad 2B_r = B_{2r} \]

\[ \mathbb{S}^{n-1} = \{ y \in \mathbb{R}^n : | y - x | = 1 \} \]
is a unit sphere at \( \mathbb{R}^n \) centered in \( x \in \mathbb{R}^n \),

\[ \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n > 0 \} \]

For any measurable set \( A \) and \( f \in L_p (A), 1 < p < \infty \) we write

\[ \| f \|_{L_p (A)} = \| f \|_{p; A} = \left( \int_A | f (y) |^p \, dy \right)^{1/p}, \quad \| \cdot \|_{p; \mathbb{R}^n} \equiv \| \cdot \|_p . \]

The standard summation convention on repeated upper and lower indices is adopted. The letter \( C \) is used for various positive constants and may change from one occurrence to another.

### 2. Weighted spaces

We start with the definitions of some function spaces that we are going to use.

**Definition 2.1.** (see [26, 37]) Let \( a \in L^1_{\text{loc}} (\mathbb{R}^n) \) and \( a_{B_r} = \frac{1}{|B_r|} \int_{B_r} a (x) \, dx \).

Define

\[ \gamma_a (R) = \sup_{r \leq R} \frac{1}{|B_r|} \int_{B_r} | a (y) - a_{B_r} | \, dy \quad \forall R > 0. \]

We say that \( a \in \text{BMO} \) (bounded mean oscillation) if

\[ \| a \|_* = \sup_{R > 0} \gamma_a (R) < +\infty. \]

The quantity \( \| a \|_* \) is a norm in \( \text{BMO} \) modulo constant functions under which \( \text{BMO} \) is a Banach space. If

\[ \lim_{R \to 0} \gamma_a (R) = 0 \]
then $a \in VMO$ (vanishing mean oscillation) and we call $\gamma_a(R)$ a VMO-modulus of $a$.

For any bounded domain $\Omega \subset \mathbb{R}^n$ we define $BMO(\Omega)$ and $VMO(\Omega)$ taking $a \in L_1(\Omega)$ and integrating over $\Omega_r = \Omega \cap B_r$.

According to [1], having a function $a \in BMO(\Omega)$ or $VMO(\Omega)$ it is possible to extend it in the whole space preserving its $BMO$-norm or $VMO$-modulus, respectively. In the following we use this extension without explicit references.

**Lemma 2.1.** (John-Nirenberg lemma, [26]) Let $a \in BMO$ and $p \in (1, \infty)$. Then for any ball $B$ holds

$$\left(\frac{1}{|B|} \int_B |a(y) - a_B|^p dy\right)^{\frac{1}{p}} \leq C(p)\|a\|_s.$$  

As an immediate consequence of Lemma 2.1 we get the next property.

**Corollary 2.1.** Let $a \in BMO$ then for all $0 < 2r < t$ holds

$$|a_{B_r} - a_{B_t}| \leq C\|a\|_s \ln \frac{t}{r}$$

where the constant is independent of $a, x, t$ and $r$.

We call weight a non-negative locally integrable function on $\mathbb{R}^n$. Given a weight $w$ and a measurable set $E$ we denote the $w$-measure of $E$ by

$$w(E) = \int_E w(x) \, dx.$$  

Denote by $L_{p,w}(\mathbb{R}^n)$ or $L_{p,w}$ the weighted $L_p$ spaces. It turns out that the strong type $(p, p)$ inequality

$$\left(\int_{\mathbb{R}^n} (Mf(x))^p w(x) \, dx\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx\right)^{\frac{1}{p}}$$

holds for all $f \in L_{p,w}$ if and only if the weight function satisfies the Muckenhoupt $A_p$-condition

$$[w]_{A_p} := \sup_B \left\{ \frac{1}{|B|} \int_B w(x) \, dx \right\} \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx\right)^{p-1} < \infty.$$
The expression \([w]_{A_p}\) is called \textit{characteristic constant} of \(w\). The function \(w\) is \(A_1\) weight if \(\mathcal{M}w(x) \leq C_1w(x)\) for almost all \(x \in \mathbb{R}^n\). The minimal constant \(C_1\) for which the inequality holds is the \(A_1\) \textit{characteristic constant} of \(w\).

We summarize some basic properties of the \(A_p\) weights in the next lemma (see [10, 30] for more details).

**Lemma 2.2.** (1) Let \(w \in A_p\) for \(1 \leq p < \infty\). Then for each \(B\)
\[
1 \leq [w]_{A_p(B)}^{\frac{1}{p}} = |B|^{-1} \|w\|_{L_1(B)}^{\frac{1}{p}} \|w^{-\frac{1}{p}}\|_{L_{p'}(B)} \leq [w]_{A_p}^{\frac{1}{p}}.
\]
(2) The function \(w^{-\frac{1}{p-1}}\) is in \(A_{p'}\) where \(\frac{1}{p} + \frac{1}{p'} = 1, 1 < p < \infty\) with characteristic constant
\[
[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.
\]
(3) The classes \(A_p\) are increasing as \(p\) increases and
\[
[w]_{A_q} \leq [w]_{A_p}, \quad 1 \leq q < p < \infty.
\]
(4) The measure \(w(x)dx\) is doubling, precisely, for all \(\lambda > 1\)
\[
w(\lambda B) \leq \lambda^{np}[w]_{A_p}w(B).
\]
(5) If \(w \in A_p\) for some \(1 \leq p \leq \infty\), then there exist \(C > 0\) and \(\delta > 0\) such that for any ball \(B\) and a measurable set \(E \subset B\),
\[
\frac{1}{|w|_{A_p}} \left(\frac{|E|}{|B|}\right) \leq \frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|}\right)^\delta.
\]
(6) For each \(1 \leq p < \infty\) we have
\[
\bigcup_{1 \leq p < \infty} A_p = A_\infty \quad \text{and} \quad [w]_{A_\infty} \leq [w]_{A_p}.
\]
(7) For each \(a \in BMO, 1 \leq p < \infty\) and \(w \in A_\infty\) we have
\[
\|a\|_* = C \sup_B \left(\frac{1}{w(B)} \int_B |a(y) - a_B|^p w(y) dy\right)^{\frac{1}{p}}.
\]

The next result follows from [16, Lemma 4.4].
Lemma 2.3. Let $w \in A_p$ with $1 < p < \infty$ and $a \in BMO$. Then
\[
\left( \frac{1}{w^{1-p'}(B)} \int_B |a(y) - a_B|^{p'} w(y)^{1-p'} \, dy \right)^{\frac{1}{p'}} \leq C [w]_{A_p} \|a\|_*,
\]
where $C$ is independent of $a$, $w$ and $B$.

Definition 2.2. Let $\varphi(x, r)$ be weight in $\mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $w \in A_p$, $p \in [1, \infty)$. The generalized weighted Morrey space $M_{p,\varphi}(\mathbb{R}^n, w)$ or $M_{p,\varphi}(w)$ consists of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that
\[
\|f\|_{p,\varphi,w} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \left( w(B_r(x))^{-1} \int_{B_r(x)} |f(y)|^{p} w(y) \, dy \right)^{\frac{1}{p}} < \infty.
\]
For any bounded domain $\Omega$ we define $M_{p,\varphi}(\Omega, w)$ taking $f \in L_{p,w}(\Omega)$ and integrating over $\Omega_r = \Omega \cap B_r(x), x \in \Omega$.

Generalized Sobolev-Morrey space $W^{2}_{p,\varphi}(\Omega, w)$ consists of all functions $u \in W^{2}_{p,w}(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega, w)$, $0 \leq |s| \leq 2$ endowed by the norm
\[
\|u\|_{W^{2}_{p,\varphi}(\Omega, w)} = \sum_{0 \leq |s| \leq 2} \|D^s f\|_{p,\varphi,w;\Omega}.
\]
The space $W^{2}_{p,\varphi}(\Omega, w) \cap \overset{\circ}{W}^1_{p,w}(\Omega)$ consists of all functions $u \in W^{2}_{p,w}(\Omega) \cap \overset{\circ}{W}^1_{p,w}(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega, w)$, $0 \leq |s| \leq 2$ and is endowed by the same norm. Recall that $\overset{\circ}{W}^1_{p,w}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^1_{p,w}(\Omega)$.

Remark 2.1. The density of the $C_0^\infty$ functions in the weighted Lebesgue space $L_{p,w}$ is proved in [38, Chapter 3, Theorem 3.11].

3. Sublinear operators generated by singular integrals in $M_{p,\varphi}(w)$

Let $T$ be a sub-linear operator. Suppose that $T$ satisfy
\[
|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^w} \, dy
\]
for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$. 
The next results generalize some estimates obtained in [11, 13, 19, 20, 21]. The proof is as in [19] and makes use of the boundedness of the weighted Hardy operator

\[ H^*_\psi g(r) := \int_r^\infty g(t)\psi(t) \, dt, \quad 0 < r < \infty. \]

**Theorem 3.1.** ([14, 15]) Suppose that \(v_1, v_2,\) and \(\psi\) are weights on \(\mathbb{R}_+\). Then the inequality

\[ \text{ess sup}_{r>0} v_2(r) H^*_\psi g(r) \leq C \text{ess sup}_{r>0} v_1(r) g(r) \]

holds with some \(C > 0\) for all non-negative and nondecreasing \(g\) on \(\mathbb{R}_+\) if and only if

\[ B := \text{ess sup}_{r>0} v_2(r) \int_r^\infty \frac{\psi(t)}{\text{ess sup}_{t<s<\infty} v_1(s)} \, dt < \infty \]

and \(C = B\) is the best constant in (3.2).

**Theorem 3.2.** Let \(1 < p < \infty, w \in A_p\) and the pair \((\varphi_1, \varphi_2)\) satisfy

\[ \int_r^\infty \text{ess inf}_{t<s<\infty} \frac{\varphi_1(x,s)w(B_s(x))^{\frac{1}{p}}}{w(B_t(x))^{\frac{1}{p}}} \, dt \leq C \varphi_2(x,r), \]

and \(T\) be a sub-linear operator satisfying (3.1). If \(T\) is bounded on \(L^p,w\) and \(\|Tf\|_{p,w} \leq C[w]^{\frac{1}{p}}_A \|f\|_{p,w}\), then \(T\) is bounded from \(M_{p,\varphi_1}(w)\) to \(M_{p,\varphi_2}(w)\) and

\[ \|Tf\|_{p,\varphi_2,w} \leq C[w]^{\frac{1}{p}}_A \|f\|_{p,\varphi_1,w} \]

with a constant independent of \(f\).

For any \(a \in BMO\) consider the commutator \(T_a f = aTf - T(af)\). Let \(T_a\) be a sub-linear operator satisfying

\[ |T_a f(x)| \leq C \int_{\mathbb{R}^n} |a(x) - a(y)||f(y)| \frac{1}{|x-y|^n} \, dy \]

for any \(f \in L_1(\mathbb{R}^n)\) with a compact support and \(x \not\in \text{supp}f\). Suppose in addition that \(T_a\) is bounded in \(L_{p,w}\) and satisfies \(\|T_a f\|_{p,w} \leq \|f\|_{p,w} \leq \}

\[ C \|a\|_{A_p}^{1/p} \|f\|_{p,w}. \] Then the next result is valid and the proof is as in [19], making use of Theorem 3.1.

**Theorem 3.3.** Let \( p \in (1, \infty), w \in A_p, a \in BMO \) and the pair \((\varphi_1, \varphi_2)\) satisfy

\[ (3.7) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t<s<\infty} \varphi_1(x, s) \, w(B_t(x))^{1/p}}{w(B_t(x))^{1/p}} \frac{dt}{t} \leq C \varphi_2(x, r) \]

with a constant independent on \( x \) and \( r \). Suppose that \( T_a \) is bounded in \( L_{p,w} \) and satisfies \( (3.6) \). Then \( T_a \) is bounded from \( M_{p,\varphi_1}(w) \) to \( M_{p,\varphi_2}(w) \) and

\[ (3.8) \quad \|T_a f\|_{p,\varphi_2,w} \leq C[w]_{A_p}^{1/p} \|a\|_{*} \|f\|_{p,\varphi_1,w}. \]

4. **Sublinear operators generated by nonsingular integrals in** \( M_{p,\varphi}(w) \)

For any \( x \in \mathbb{R}_+^n \) define \( \tilde{x} = (x_1, \ldots, x_{n-1}, -x_n) \). Let \( \tilde{T} \) be a sublinear operator with a nonsingular kernel. Suppose that \( \tilde{T} \) satisfy the condition

\[ (4.1) \quad |\tilde{T} f(x)| \leq C \int_{\mathbb{R}_+^n} \frac{|f(y)|}{|\tilde{x} - y|^n} \, dy \]

for any \( f \in L_1(\mathbb{R}_+^n) \) with a compact support.

**Lemma 4.1.** Let \( w \in A_p, p \in (1, \infty), \) the operator \( \tilde{T} \) satisfy \( (4.1) \) and \( \tilde{T} \) is bounded on \( L_{p,w}(\mathbb{R}_+^n) \). Let also for any fixed \( x_0 \in \mathbb{R}_+^n \) and for any \( f \in L^\text{loc}_{p,w}(\mathbb{R}_+^n) \)

\[ (4.2) \quad \int_r^\infty w(B_t^+(x_0))^{-\frac{1}{p}} \|f\|_{p,w;B_t^+(x_0)} \frac{dt}{t} < \infty. \]

Then

\[ (4.3) \quad \|\tilde{T} f\|_{p,w;B_r^+(x_0)} \leq C[w]_{A_p}^{1/p} w(B_r^+(x_0))^{1/p} \int_{2r}^\infty w(B_t^+(x_0))^{-\frac{1}{p}} \|f\|_{p,w;B_t^+(x_0)} \frac{dt}{t} \]

with a constant independent of \( x_0, r, \) and \( f \).
Proof. Consider the decomposition $f = f_1 + f_2$ with $f_1 = f \chi_{2B^+}(x_0)$ and $f_2 = f \chi(2B^+)(x_0)^c$. Because of the boundedness of $\tilde{T}$ in $L_{p,w}(\mathbb{R}^n_+)$ we have as in [22]

$$
\|\tilde{T} f_1\|_{p,w;B^+_r(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,w;2B^+_r(x_0)},
$$

Since for any $\tilde{x} \in B^+_r(x_0)$ and $y \in (2B^+_r(x_0))^c$ it holds

(4.4) \quad \frac{1}{2}|x_0 - y| \leq |\tilde{x} - y| \leq \frac{3}{2}|x_0 - y|,

we get as in [22]

$$
|\tilde{T} f_2(x)| \leq C \int_{2r}^{\infty} \left( \int_{B^+_r(x_0)} |f(y)| dy \right) \frac{dt}{t^{n+1}}.
$$

Making use of the Hölder inequality and (2.3) we get

(4.5) \quad |\tilde{T} f_2(x)| \leq C \int_{2r}^{\infty} \|f\|_{p,w;B^+_r(x_0)} \|w^{-\frac{1}{p}}\|_{\ell,p;B^+_r(x_0)} \frac{dt}{t^{n+1}}

\leq C[w]_{A_p}^{\frac{1}{p}} \int_{2r}^{\infty} w(B^+_r(x_0))^{-\frac{1}{p}} \|f\|_{p,w;B^+_r(x_0)} \frac{dt}{t}.

Direct calculations give

(4.6) \quad \|\tilde{T} f_2\|_{p,w;B^+_r(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \frac{\|f\|_{p,w;B^+_r(x_0)}}{w(B^+_r(x_0))^{\frac{1}{p}}} \frac{dt}{t}

for all $f \in L_{p,w}(\mathbb{R}^n_+)$ satisfying (4.2). Thus,

(4.7) \quad \|\tilde{T} f\|_{p,w;B^+_r(x_0)} \leq \|\tilde{T} f_1\|_{p,w;B^+_r(x_0)} + \|\tilde{T} f_2\|_{p,w;B^+_r(x_0)}

\leq C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,w;2B^+_r(x_0)}

+ C[w]_{A_p}^{\frac{1}{p}} w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \frac{\|f\|_{p,w;B^+_r(x_0)}}{w(B^+_r(x_0))^{\frac{1}{p}}} \frac{dt}{t}.
On the other hand, by (2.3)
\[
\|f\|_{p,w;2B_t^+(x_0)} \leq C|\mathcal{B}_r^+(x_0)| \|f\|_{p,w;2B_t^+(x_0)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}
\]
\[
\leq C|\mathcal{B}_r^+(x_0)| \int_{2r}^{\infty} \|f\|_{p,w;2B_t^+(x_0)} \frac{dt}{t^{n+1}}
\]
\[
\leq C[w]_{A_p}^\frac{1}{p} w(B_r^+(x_0))^\frac{1}{p} \int_{2r}^{\infty} \|f\|_{p,w;2B_t^+(x_0)} \|w^{-\frac{1}{p}}\|_{p';B_t^+(x_0)} \frac{dt}{t^{n+1}}
\]
\[
\leq C[w]_{A_p}^\frac{1}{p} w(B_r^+(x_0))^\frac{1}{p} \int_{2r}^{\infty} [w]_{A_p}^\frac{1}{p} w(B_r^+(x_0))^{-\frac{1}{p}} \|f\|_{p,w;2B_t^+(x_0)} \frac{dt}{t}
\]
(4.8)
which unified with (4.7) gives (4.3). \hfill \square

**Theorem 4.1.** Suppose that \( w \in A_p, p \in (1, \infty) \), the pair \((\varphi_1, \varphi_2)\) satisfies the condition (3.4) for any \( x \in \mathbb{R}_+^n \) and (4.1) holds. Then if \( \mathcal{T} \) is bounded in \( L_{p,w}(\mathbb{R}_+^n) \), then it is bounded from \( M_{p,\varphi_1}(\mathbb{R}_+^n, w) \) in \( M_{p,\varphi_2}(\mathbb{R}_+^n, w) \) and
\[
\|\mathcal{T}f\|_{p,\varphi_2,w;\mathbb{R}_+^n} \leq C[w]_{A_p}^\frac{1}{p} \|f\|_{p,\varphi_1,w;\mathbb{R}_+^n}
\]
with a constant independent of \( f \).

**Proof.** By Lemma 4.1 we have
\[
\|\mathcal{T}f\|_{p,\varphi_2,w;\mathbb{R}_+^n} \leq C[w]_{A_p}^\frac{1}{p} \sup_{x \in \mathbb{R}_+^n, r > 0} \varphi_2(x, r)^{-1} \int_{r}^{\infty} w(B_r^+(x))^{-\frac{1}{p}} \|f\|_{p,w;2B_t^+(x)} \frac{dt}{t}.
\]

Applying the Theorem 3.1 with
\[
v_1(r) = \varphi_1(x, r)^{-1} w(B_r^+(x))^{-\frac{1}{p}}, \quad v_2(r) = \varphi_2(x, r)^{-1},
\]
\[
\psi(r) = w(B_r^+(x))^{-\frac{1}{p}} r^{-1}, \quad g(r) = \|f\|_{p,w;2B_t^+(x)}
\]
to the above integral, we get as in [22]
\[
\|\mathcal{T}f\|_{p,\varphi_2,w;\mathbb{R}_+^n} \leq C[w]_{A_p}^\frac{1}{p} \sup_{x \in \mathbb{R}_+^n, r > 0} \varphi_1(x, r)^{-1} w(B_r^+(x))^{-\frac{1}{p}} \|f\|_{p,w;2B_t^+(x)}
\]
\[
= C[w]_{A_p}^\frac{1}{p} \|f\|_{p,\varphi_1,w;\mathbb{R}_+^n}.
\]
\hfill \square
5. Commutators of sub-linear operators generated by nonsingular integrals in $M_{p,\varphi}(w)$

For any $a \in BMO$ consider the commutator $\widetilde{T}_a f = aTf - \widetilde{T}(af)$ where $\widetilde{T}$ is the nonsingular operator satisfying (4.1) and $f \in L_1(\mathbb{R}^n_+)$ with a compact support. Suppose that for $x \notin \text{supp}f$

\begin{equation}
|\widetilde{T}_a f(x)| \leq C \int_{\mathbb{R}^n_+} |a(x) - a(y)| \frac{|f(y)|}{|x - y|^n} \, dy,
\end{equation}

where $C$ is independent of $f, a,$ and $x.$

Suppose in addition that $\widetilde{T}_a$ is bounded in $L_{p,w}(\mathbb{R}^n_+), w \in A_p, p \in (1, \infty)$ satisfying the estimate $\|\widetilde{T}_a f\|_{p,w;\mathbb{R}^n_+} \leq C [w]_{A_p}^{\frac{1}{p}} \|a\|_\varphi \|f\|_{p,w;\mathbb{R}^n_+}.$ Our aim is to show boundedness of $\widetilde{T}_a$ in $M_{p,\varphi}(\mathbb{R}^n_+, w).$

To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 4.1 (see [22] for details).

**Lemma 5.1.** Let $w \in A_p, p \in (1, \infty), a \in BMO$ and $\widetilde{T}_a$ be a bounded operator in $L_{p,w}(\mathbb{R}^n_+)$ satisfying (5.1) and the estimate $\|\widetilde{T}_a f\|_{p,w;\mathbb{R}^n_+} \leq C[w]_{A_p}^{\frac{1}{p}} \|a\|_\varphi \|f\|_{p,w;\mathbb{R}^n_+}.$ Suppose that for all $f \in L_{p,w}^{\text{loc}}(\mathbb{R}^n_+), x_0 \in \mathbb{R}^n_+$ and $r > 0$ applies the next condition

\begin{equation}
\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{p,w;B^+_r(x_0)}}{w(B^+_r(x_0))^{1/p}} \, dt < \infty.
\end{equation}

Then

\begin{equation}
\|\widetilde{T}_a f\|_{p,w;B^+_r(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} \|a\|_\varphi \|f\|_{p,w;B^+_r(x_0)} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{p,w;B^+_r(x_0)}}{w(B^+_r(x_0))^{1/p}} \, dt.
\end{equation}

**Proof.** The decomposition $f = f \chi_{2B^+_r(x_0)} + f \chi_{(2B^+_r(x_0))^c} = f_1 + f_2$ gives

\begin{equation}
\|\widetilde{T}_a f\|_{p,w;B^+_r(x_0)} \leq \|\widetilde{T}_a f_1\|_{p,w;B^+_r(x_0)} + \|\widetilde{T}_a f_2\|_{p,w;B^+_r(x_0)}.
\end{equation}

From the boundedness of $\widetilde{T}_a$ in $L_{p,w}(\mathbb{R}^n_+)$ it follows

\begin{equation}
\|\widetilde{T}_a f_1\|_{p,w;B^+_r(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} \|a\|_\varphi \|f\|_{p,w;2B^+_r(x_0)}.
\end{equation}
On the other hand, because of (4.4) we can write

\[ ||\tilde{T}_n f_2||_{p,w;B^+_t(x_0)} \]

\[ \leq C \left( \int_{2r} \int_{B^+_t(x_0)} \left( \frac{|a(y) - a_{B^+_t(x_0)}||f(y)|}{|x_0 - y|^p} \right) w(x) \, dx \, dy \right)^{\frac{1}{p}} \]

\[ + C \left( \int_{2r} \int_{B^+_t(x_0)} \left( \frac{|a(x) - a_{B^+_t(x_0)}||f(y)|}{|x_0 - y|^p} \right) w(x) \, dx \, dy \right)^{\frac{1}{p}} \]

\[ = I_1 + I_2. \]

Where, as in [22], we have

\[ I_1 \leq C w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \int_{B^+_t(x_0)} |a(y) - a_{B^+_t(x_0)}||f(y)| \, dy \, dt^{\frac{1}{p}}. \]

Applying Hölder’s inequality, Lemma 2.1, (2.1) and (2.5), we get

\[ I_1 \leq C w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \int_{B^+_t(x_0)} |a(y) - a_{B^+_t(x_0)}||f(y)| \, dy \, dt^{\frac{1}{p}} \]

\[ + C w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \int_{B^+_t(x_0)} |a_{B^+_t(x_0)} - a_{B^+_t(x_0)}||f(y)| \, dy \, dt^{\frac{1}{p}} \]

\[ \leq C w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \left( \int_{B^+_t(x_0)} |a(y) - a_{B^+_t(x_0)}|^{p'} w(y)^{1-p'} \, dy \right)^{\frac{1}{p'}} \]

\[ \times \left( \int_{2r} \int_{B^+_t(x_0)} |a_{B^+_t(x_0)} - a_{B^+_t(x_0)}| \, dy \, dt \right)^{\frac{1}{p}}. \]

By Lemma 2.1 and (4.5) we get

\[ I_2 \leq C[w]^{\frac{1}{p}}_{A_p} \|a\| w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \ln \frac{t}{r} \|f\|_{p,w;B^+_t(x_0)} w(B^+_t(x_0))^{-\frac{1}{p}} \, dt \]

\[ \leq C[w]^{\frac{1}{p}}_{A_p} \|a\| w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \ln \frac{t}{r} \|f\|_{p,w;B^+_t(x_0)} w(B^+_t(x_0))^{-\frac{1}{p}} \, dt \]

\[ \leq C[w]^{\frac{1}{p}}_{A_p} \|a\| w(B^+_r(x_0))^{\frac{1}{p}} \int_{2r} \left( 1 + \ln \frac{t}{r} \right) \|f\|_{p,w;B^+_t(x_0)} w(B^+_t(x_0))^{-\frac{1}{p}} \, dt. \]
Summing up $I_1$ and $I_2$ we get that for all $p \in (1, \infty)$

$$
(5.4) \quad \| \tilde{T}_a f \|_{p, w, B(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} \| a \|_* \left( w(B(x_0)) \right)^{\frac{1}{p}} \int_2^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\| f \|_{p, w, B(x_0)}^{\frac{1}{p}}}{w(B(x_0))^{\frac{1}{p}}} \frac{dt}{t}.
$$

Finally,

$$
\| \tilde{T}_a f \|_{p, w, B(x_0)} \leq C[w]_{A_p}^{\frac{1}{p}} \| a \|_* \left( \| f \|_{p, w, 2B(x_0)} \right)^{\frac{1}{p}} + w(B(x_0))^{\frac{1}{p}} \int_2^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\| f \|_{p, w, B(x_0)}^{\frac{1}{p}}}{w(B(x_0))^{\frac{1}{p}}} \frac{dt}{t},
$$

and the statement follows by (4.8).

\[ \Box \]

\textbf{Theorem 5.1.} Let $w \in A_p$, $p \in (1, \infty)$, $a \in BMO$ and $(\varphi_1, \varphi_2)$ be such that

$$
(5.5) \quad \int_2^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\text{ess inf}_{t < s < \infty} \varphi_1(x, s) w(B(x))^{\frac{1}{p}}}{w(B(x))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r).
$$

Suppose $\tilde{T}_a$ is a sub-linear operator satisfying (5.1) and bounded on $L_{p, w}(\mathbb{R}_+^n)$. Then $\tilde{T}_a$ is bounded from $M_{p, \varphi_1}(\mathbb{R}_+^n, w)$ to $M_{p, \varphi_2}(\mathbb{R}_+^n, w)$ and

$$
(5.6) \quad \| \tilde{T}_a f \|_{p, \varphi_2, w, \mathbb{R}_+^n} \leq C[w]_{A_p}^{\frac{1}{p}} \| a \|_* \| f \|_{p, \varphi_1, w, \mathbb{R}_+^n}
$$

with a constant independent of $f$ and $a$.

The statement of the theorem follows by Lemma 5.1 and Theorem 3.1 in the same manner as the proof of Theorem 4.1.

\section{Calderón-Zygmund operators in $M_{p, \psi}(w)$}

In the present section we deal with Calderón-Zygmund type integrals and their commutators with BMO functions. We start with the definition of the corresponding kernel.

\textbf{Definition 6.1.} A measurable function $K(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

i) $K(x, \cdot)$ is a Calderón-Zygmund kernel for almost all $x \in \mathbb{R}^n$;
\(i_a\) \(K(x, \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})\),

\(i_b\) \(K(x, \mu \xi) = \mu^{-n} K(x, \xi) \quad \forall \mu > 0,\)

\(i_c\) \[\int_{\mathbb{S}^{n-1}} K(x, \xi) d\sigma_\xi = 0 \quad \int_{\mathbb{S}^{n-1}} \left|K(x, \xi)\right| d\sigma_\xi < +\infty,\]

\(iib\) \[\max_{|\beta| \leq 2n} \left\|D^\beta K\right\|_{\infty; \mathbb{R}^n \times \mathbb{S}^{n-1}} = M < \infty.\]

The singular integrals

\[\mathcal{K}f(x) := P.V. \int_{\mathbb{R}^n} K(x, x - y) f(y) dy\]

\[\mathcal{C}[a, f](x) := P.V. \int_{\mathbb{R}^n} K(x, x - y) [a(x) - a(y)] f(y) dy = a \mathcal{K}f(x) - \mathcal{K}(af)(x)\]

are bounded in \(L_{p,w}\) (see [21] for more references) and satisfy (3.1) and (5.1). Hence the next results hold as a simple application of the estimates from Sections 3 and 4 (see [22] for details).

**Theorem 6.1.** Let \(w \in A_p, p \in (1, \infty)\) and \(\varphi\) be weight such that for all \(x \in \mathbb{R}^n\) and \(r > 0\)

\[\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\text{ess inf}_{t < s < \infty} \varphi(x, s) w(B_s(x))^{\frac{1}{p}}}{w(B_t(x))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi(x, r).\]

Then for any \(f \in M_{p,\varphi}(\mathbb{R}^n, w)\) and \(a \in BMO\) there exist constants depending on \(n, p, \varphi, w,\) and the kernel such that

\[\left\|\mathcal{K}f\right\|_{p,\varphi,w} \leq C[w]_{A_p}^{\frac{1}{p}} \left\|f\right\|_{p,\varphi,w};\]

\[\left\|\mathcal{C}[a, f]\right\|_{p,\varphi,w} \leq C[w]_{A_p}^{\frac{1}{p}} \left\|a\right\|_* \left\|f\right\|_{p,\varphi,w}.\]

The assertion follows by (4.9) and (5.6).

**Corollary 6.1.** Let \(\Omega \subset \mathbb{R}^n, \partial \Omega \in C^{1,1}, K : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\) be as in Definition 6.1, \(a \in BMO(\Omega)\) and \(f \in M_{p,\varphi}(\Omega, w)\) with \(p, \varphi,\) and \(w\) as in Theorem 6.1. Then

\[\left\|\mathcal{K}f\right\|_{p,\varphi,w;\Omega} \leq C[w]_{A_p}^{\frac{1}{p}} \left\|f\right\|_{p,\varphi,w;\Omega};\]

\[\left\|\mathcal{C}[a, f]\right\|_{p,\varphi,w;\Omega} \leq C[w]_{A_p}^{\frac{1}{p}} \left\|a\right\|_* \left\|f\right\|_{p,\varphi,w;\Omega}.\]
with $C = C(n,p,\varphi,[w]_{A_p},|\Omega|,\mathcal{K})$.

**Corollary 6.2.** *(see [6, 22])* Let $p$, $\varphi$, and $w$ be as in Theorem 6.1 and $a \in VMO$ with a VMO-modulus $\gamma_a$. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon,\gamma_a)$ such that for any ball $B_r$ with a radius $r \in (0,\rho_0)$ and all $f \in M_{p, \varphi}(B_r, w)$

$$\|C[a,f]\|_{p, \varphi, w; B_r} \leq C\varepsilon\|f\|_{p, \varphi, w; B_r},$$

with $C$ independent of $\varepsilon$, $f$, and $r$.

For any $x, y \in \mathbb{R}_n^+$ define the generalized reflection $T(x; y)$

$$T(x; y) = x - 2x_n a_n(y) T(x) = T(x; x) : \mathbb{R}_+^n \to \mathbb{R}_n^n$$

where $a_n$ is the last row of the matrix $a = \{a_{ij}\}_{i,j=1}^n$. Then there exist positive constants $C_1, C_2$ dependent on $n$ and $\Lambda$, such that

$$C_1|x - y| \leq |T(x) - y| \leq C_2|x - y| \quad \forall \ x, y \in \mathbb{R}_+^n.$$

Then the nonsingular integrals

$$\tilde{K}f(x) := \int_{\mathbb{R}_+^n} K(x, T(x) - y)f(y) \, dy$$

$$\tilde{C}[a,f](x) := \int_{\mathbb{R}_+^n} K(x, T(x) - y)[a(x) - a(y)]f(y) \, dy$$

are sub-linear and according to the results in Sections 4 and 5 we have.

**Theorem 6.2.** Let $a \in BMO(\mathbb{R}_+^n)$, $w \in A_p$, $p \in (1, \infty)$ and $\varphi$ be Morrey weight satisfying (6.1). Then $\tilde{K}f$ and $\tilde{C}[a,f]$ are continuous in $M_{p,\varphi}(\mathbb{R}_+^n,w)$ and for all $f \in M_{p,\varphi}(\mathbb{R}_+^n,w)$ holds

$$\|\tilde{K}f\|_{p, \varphi, w; \mathbb{R}_+^n} \leq C[w]_{A_p}^\frac{1}{p} \|f\|_{p, \varphi, w; \mathbb{R}_+^n} \quad \|\tilde{C}[a,f]\|_{p, \varphi, w; \mathbb{R}_+^n} \leq C[w]_{A_p}^\frac{1}{p} \|a\|_* \|f\|_{p, \varphi, w; \mathbb{R}_+^n}$$

with constants dependent on known quantities only.

**Corollary 6.3.** *(see [6, 22])* Let $p$, $\varphi$ and $w$ be as in Theorem 6.2 and $a \in VMO$ with a VMO-modulus $\gamma_a$. Then for any $\varepsilon > 0$ there exists a
positive number \( \rho_0 = \rho_0(\varepsilon, \gamma_a) \) such that for any ball \( B_r^+ \) with a radius \( r \in (0, \rho_0) \) and all \( f \in M_{p,\varphi}(B_r^+, \omega) \)

\[
\| \mathcal{C}[a, f] \|_{p,\varphi, u; B_r^+} \leq C \varepsilon \| f \|_{p,\varphi, u; B_r^+},
\]

where \( C \) is independent of \( \varepsilon, f \) and \( r \).

7. The Dirichlet problem

Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \) be a bounded \( C^{1,1} \)-domain. We consider the problem

\[
\begin{cases}
Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x) & \text{a.a. } x \in \Omega, \\
u \in W^2_p(\Omega, w) \cap W^1_p(\Omega, w), & p \in (1, \infty)
\end{cases}
\]

subject to the following conditions:

\( H_1) \) Strong ellipticity: there exists a constant \( \Lambda > 0 \), such that

\[
\begin{cases}
\Lambda^{-1} |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2 & \text{a.a. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n \\
a^{ij}(x) = a^{ji}(x) & 1 \leq i, j \leq n
\end{cases}
\]

Let \( a = \{a^{ij}\} \), then \( a \in L_\infty(\Omega) \) and \( \|a\|_{\infty, \Omega} = \sum_{ij=1}^n \|a^{ij}\|_{\infty, \Omega} \) by (7.2).

\( H_2) \) Regularity of the data: \( a \in VMO(\Omega) \) with \( VMO \)-modulus \( \gamma_a := \sum \gamma_{a^{ij}}, b^i, c \in L_\infty(\Omega) \), and \( f \in M_{p,\varphi}(\Omega, w) \) with \( w \in A_p, 1 < p < \infty \) and \( \varphi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) measurable.

Let \( \mathcal{L} = a^{ij}(x)D_{ij} \), then \( \mathcal{L}u = f(x) - b^i(x)D_iu(x) - c(x)u \). As it is well known (see [6, 22] and the references therein) for any \( x \in \text{supp} u \), a ball \( B_r \subset \Omega' \) and a function \( v \in C^\infty_0(B_r) \) we have the representation

\[
D_{ij}v(x) = \text{P.V.} \int_{B_r} \Gamma_{ij}(x, x - y) \left[ \mathcal{L}v(y) + (a^{hk}(x) - a^{hk}(y))D_{hk}v(y) \right] dy
\]

\[
= \mathcal{K}_{ij} \mathcal{L}v(x) + \mathcal{C}_{ij} [a^{hk}, D_{hk}v](x) + \mathcal{L}v(x) \int_{S_{n-1}} \Gamma_j(x; y)y_{i}d\sigma_y
\]

According to Remark 2.1 the formula (7.3) holds true also for functions \( v \in W^2_{p,\omega}(B_r) \). Here \( \Gamma_{ij}(x, \xi) = \partial^2 \Gamma(x, \xi)/\partial \xi_i \partial \xi_j \) and \( \Gamma_{ij} \) are variable
Calderón-Zygmund kernels as in Definition 6.1 for all \(1 \leq i, j \leq n\). Then the operators \(R_{ij}\) and \(C_{ij}\) are singular as \(R\) and \(C\). In view of the results obtained in Section 6 we get for \(r\) small enough

\[
\|D^2v\|_{p,\varphi,w,B_r} \leq C \left( \varepsilon \|D^2v\|_{p,\varphi,w,B_r} + \|Lv\|_{p,\varphi,w,B_r} \right).
\]

Choosing \(r\) such that \(C\varepsilon < 1\) we can move the norm of \(D^2v\) on the left-hand side and write

\[
(7.4) \quad \|D^2v\|_{p,\varphi,w,B_r} \leq C\|Lv\|_{p,\varphi,w,B_r}.
\]

Take a cut-off function \(\eta(x) \in C_0^\infty(B_r)\)

\[
\eta(x) = \begin{cases} 1 & x \in B_{\theta r} \\ 0 & x \notin B_{\theta' r} \end{cases}
\]

such that \(\theta' = \theta(3-\theta)/2 > \theta\) for \(\theta \in (0, 1)\) and \(|D^s\eta| \leq C[\theta(1-\theta)r]^{-s}\) for \(s = 0, 1, 2\). Apply (7.4) to \(v(x) = \eta(x)u(x) \in W^2_{p,w}(B_r)\) we get

\[
\|D^2u\|_{p,\varphi,w,B_{\theta' r}} \leq \|D^2v\|_{p,\varphi,w,B_{\theta' r}} \leq C\|Lv\|_{p,\varphi,w,B_{\theta' r}}
\]

\[
\leq C \left( \|Lu\|_{p,\varphi,w,B_{\theta' r}} + \frac{\|Du\|_{p,\varphi,w,B_{\theta' r}}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi,w,B_{\theta' r}}}{\theta(1-\theta)r^2} \right).
\]

Since \(1 < \frac{1}{\theta(1-\theta)r}\) for \(r < 4\) and

\[
(7.5) \quad \|Lu\|_{p,\varphi,w,B_{\theta' r}} \leq C\left(\|Lu\|_{p,\varphi,w,B_{\theta' r}} + \|Dv\|_{p,\varphi,w,B_{\theta' r}} + \|u\|_{p,\varphi,w,B_{\theta' r}}\right)
\]

we can write

\[
\|D^2u\|_{p,\varphi,w,B_{\theta' r}} \leq C \left( \|Lu\|_{p,\varphi,w,B_{\theta' r}} + \frac{\|Du\|_{p,\varphi,w,B_{\theta' r}}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi,w,B_{\theta' r}}}{\theta(1-\theta)r^2} \right).
\]

Consider now the weighted semi-norms

\[
\Theta_s = \sup_{0 < \theta < 1} \left[ \theta(1-\theta)r \right]^s \|D^s u\|_{p,\varphi,w,B_{\theta r}}, \quad s = 0, 1, 2.
\]

Because of the choice of \(\theta'\) we have \(\theta(1-\theta) \leq 2\theta'(1-\theta')\). Thus, after standard transformations and taking the supremum with respect to \(\theta \in (0, 1)\) we get

\[
(7.6) \quad \Theta_2 \leq C \left( r^2 \|Lu\|_{p,\varphi,w,B_{\theta' r}} + \Theta_1 + \Theta_0 \right).
\]
Lemma 7.1 (Interpolation inequality). There exists a constant $C$ independent of $r$ such that
\[ \Theta_1 \leq \varepsilon \Theta_2 + \frac{C}{\varepsilon} \Theta_0 \quad \text{for any } \varepsilon \in (0, 2). \]

Proof. For functions $u \in W_{p,w}^2(B_r)$, $p \in (1, \infty)$ and $w \in A_p$ we dispose with the following interpolation inequality proved in [27]
\[ \|Du\|_{p,w;B_r} \leq C \left( \|u\|_{p,w;B_r} + \|u\|_{p,w;B_r} \|D^2u\|_{p,w;B_r}^{\frac{1}{2}} \right). \]
Then for any $\varepsilon > 0$ we have
\[ \|Du\|_{p,w;B_r} \leq C \left( \left(1 + \frac{1}{2\varepsilon}\right) \|u\|_{p,w;B_r} + \varepsilon \|D^2u\|_{p,w;B_r} \right). \]
Choosing $\varepsilon$ small enough, such that $\delta = \frac{C\varepsilon}{2} < 1$, dividing all terms of $\varphi(x,r)w(B_r)\frac{\delta}{2}$ and taking the supremum over $B_r$, we get the desired interpolation inequality in $M_{p,\varphi}(w)$
\[ (7.7) \quad \|Du\|_{p,\varphi,w;B_r} \leq \delta \|D^2u\|_{p,\varphi,w;B_r} + \frac{C}{\delta} \|u\|_{p,\varphi,w;B_r}. \]
We can always find some $\theta_0 \in (0, 1)$ such that
\[ \Theta_1 \leq 2[\theta_0(1 - \theta_0)r] \|Du\|_{p,\varphi,w;B_{\theta_0r}} \leq 2[\theta_0(1 - \theta_0)r] \left( \delta \|D^2u\|_{p,\varphi,w;B_{\theta_0r}} + \frac{C}{\delta} \|u\|_{p,\varphi,w;B_{\theta_0r}} \right). \]
The assertion follows choosing $\delta = \frac{\varepsilon}{2}[\theta_0(1 - \theta_0)r] < \theta_0r$ for any $\varepsilon \in (0, 2)$. \qed

Interpolating $\Theta_1$ in (7.6) and taking $\theta = \frac{1}{2}$ as in [22] we get the Caccioppoli-type estimate
\[ \|D^2u\|_{p,\varphi,w;B_r/2} \leq C \left( \|Lu\|_{p,\varphi,w;B_r} + \frac{1}{r^2} \|u\|_{p,\varphi,w;B_r} \right). \]
Further, proceeding as in [22] and making use of (7.5) and (7.7) we get the following interior a priori estimate.

Theorem 7.1 (Interior estimate). Let $u \in W_{p,w}^{2,\text{loc}}(\Omega)$ and $L$ be a linear elliptic operator verifying $H_1$) and $H_2$) such that $Lu \in M_{p,\varphi}^{\text{loc}}(\Omega, w)$ with
p ∈ (1, ∞), w ∈ A_p and ϕ satisfying (6.1). Then \( D_{ij}u \in L_{p,\varphi}(\Omega', w) \) for any \( \Omega' \subset \Omega'' \subset \Omega \) and

\[
\|D^2 u\|_{p,\varphi;w;\Omega'} \leq C (\|u\|_{p,\varphi;w;\Omega''} + \|Lu\|_{p,\varphi;w;\Omega''})
\]

where the constant depends on known quantities and dist (\( \Omega', \partial \Omega'' \)).

Let \( x^0 = (x', 0) \) and denote by \( C^{\gamma} \) the space of functions \( u \in C^{\infty}_0(\mathcal{B}_r(x^0)) \) with \( u = 0 \) for \( x_n \leq 0 \). The space \( W^{2,\gamma}_{p,w}(\mathcal{B}_r(x^0)) \) is the closure of \( C^{\gamma} \) with respect to the norm of \( W^{2,\gamma}_{p,w} \). Then for any \( v \in W^{2,\gamma}_{p,w}(\mathcal{B}_r^+(x^0)) \) the next representation formula holds (see [7])

\[
D_{ij}v(x) = \tilde{R}_{ij}\mathcal{L}v(x) + \mathcal{E}_{ij}[a^{hk}D_{hk}v](x) + \mathcal{L}v(x) \int_{S^{n-1}} \Gamma_j(x,y)y_id\sigma_y + I_{ij}(x) \quad \forall \ i, j = 1, \ldots, n,
\]

where we have set

\[
I_{ij}(x) = \tilde{R}_{ij}\mathcal{L}v(x) + \tilde{C}_{ij}[a^{hk}D_{hk}v](x), \quad \forall \ i, j = 1, \ldots, n-1,
\]

\[
I_{in}(x) = I_{ni}(x) = \tilde{R}_{il}(D_n\mathcal{T}(x))^l\mathcal{L}v(x) + \mathcal{E}_{il}[a^{hk}D_{hk}v](x)(D_n\mathcal{T}(x))^l
\]

\[
\forall \ i = 1, \ldots, n-1,
\]

\[
I_{nn}(x) = \tilde{R}_{ln}(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s\mathcal{L}v(x) + \tilde{C}_{ls}[a^{hk}D_{hk}v(x)](D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s
\]

where

\[
D_n\mathcal{T}(x) = ((D_n\mathcal{T}(x))^1, \ldots, (D_n\mathcal{T}(x))^n) = \mathcal{T}(e_n, x).
\]

Applying the estimates (6.8) and (6.9), the interpolation inequality (7.7) and taking into account the VMO properties of the coefficients \( a^{ij} \)'s, it is possible to choose \( r_0 \) small enough such that

\[
\|D_{ij}v\|_{p,\varphi;w;\mathcal{B}_r^+} \leq C (\|Lv\|_{p,\varphi;w;\mathcal{B}_r^+} + \|u\|_{p,\varphi;w;\mathcal{B}_r^+})
\]

for all \( r < r_0 \) (see [22] for details). By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to
that covering and applying the estimate (7.9) we get a boundary a
priori estimate that unified with (7.8) gives the next theorem.

**Theorem 7.2** (Main result). Let \( u \in W^2_{p,\varphi}(\Omega, w) \cap W^1_p(\Omega, w) \) be a
solution of (7.1) under the conditions \( H_1 \) and \( H_2 \). Then the next
estimate holds for any \( w \in A_p, \ p \in (1, \infty) \) and \( \varphi \) satisfying (6.1)
\[
\| D^2 u \|_{p,\varphi, w; \Omega} \leq C \left( \| u \|_{p,\varphi, w; \Omega} + \| f \|_{p,\varphi, w; \Omega} \right)
\]
and the constant \( C \) depends on known quantities only.

Let us note that the solution of (7.1) exists according to Remark 2.1.

The a priori estimate follows as in [6, 7] making use of (7.5) and the
interpolation inequality in weighted Lebesgue spaces [27].

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