A LIPSCHITZ REFINEMENT OF THE BEBUTOV–KAKUTANI DYNAMICAL EMBEDDING THEOREM

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ABSTRACT. We prove that an $\mathbb{R}$-action on a compact metric space embeds equivariantly in the space of one-Lipschitz functions $\mathbb{R} \to [0, 1]$ if its fixed point set can be topologically embedded in the unit interval. This is a refinement of the classical Bebutov–Kakutani theorem (1968).

1. INTRODUCTION

The purpose of this short paper is to refine a classical theorem of Bebutov [Beb40] and Kakutani [Kak68] on dynamical systems. We call $(X, T)$ a flow if $X$ is a compact metric space and

$$T : \mathbb{R} \times X \to X, \quad (t, x) \mapsto T_t x$$

is a continuous action of $\mathbb{R}$. We define $\text{Fix}(X, T)$ (sometimes abbreviated to $\text{Fix}(X)$) as the set of $x \in X$ satisfying $T_t x = x$ for all $t \in \mathbb{R}$. We define $C(\mathbb{R})$ as the space of continuous maps $\varphi : \mathbb{R} \to [0, 1]$. It is endowed with the topology of uniform convergence over compact subsets of $\mathbb{R}$, namely the topology given by the distance

$$(1.1) \quad \sum_{n=1}^{\infty} 2^{-n} \max_{|t| \leq n} |\varphi(t) - \psi(t)|, \quad (\varphi, \psi \in C(\mathbb{R})).$$

The group $\mathbb{R}$ continuously acts on it by the translation:

$$(1.2) \quad \mathbb{R} \times C(\mathbb{R}) \to C(\mathbb{R}), \quad (s, \varphi(t)) \mapsto \varphi(t + s).$$

A continuous map $f : X \to C(\mathbb{R})$ is called an embedding of a flow $(X, T)$ if $f$ is an $\mathbb{R}$-equivariant topological embedding. Bebutov [Beb40] and Kakutani [Kak68] found that the $\mathbb{R}$-action on $C(\mathbb{R})$ has the following remarkable “universality”:

**Theorem 1.1** (Bebutov–Kakutani). A flow $(X, T)$ can be equivariantly embedded in $C(\mathbb{R})$ if and only if $\text{Fix}(X, T)$ can be topologically embedded in the unit interval $[0, 1]$.

The “only if” part is trivial because the set of fixed points of $C(\mathbb{R})$ is homeomorphic to $[0, 1]$. So the main statement is the “if” part.
Although the Bebutov–Kakutani theorem is clearly a nice theorem, it has one drawback: The space $C(\mathbb{R})$ is not compact (nor locally compact). So it is not a “flow” in the above definition. This poses the following problem:

**Problem 1.2.** Is there a compact invariant subset of $C(\mathbb{R})$ satisfying the same universality?

The purpose of this paper is to solve this problem affirmatively. Let $L(\mathbb{R})$ be the set of maps $\varphi : \mathbb{R} \to [0, 1]$ satisfying the one-Lipschitz condition:

\begin{equation}
\forall s, t \in \mathbb{R} : \ |\varphi(s) - \varphi(t)| \leq |s - t|.
\end{equation}

$L(\mathbb{R})$ is a subset of $C(\mathbb{R})$. It is compact with respect to the distance (1.1) by Ascoli–Arzela’s theorem. The $\mathbb{R}$-action (1.2) preserves $L(\mathbb{R})$. So it becomes a flow. Our main result is the following. This solves [GJ16, Question 4.1].

**Theorem 1.3.** A flow $(X, T)$ can be equivariantly embedded in $L(\mathbb{R})$ if and only if $\text{Fix}(X, T)$ can be topologically embedded in the unit interval $[0, 1]$.

As in the case of the Bebutov–Kakutani theorem, the “only if” part is trivial because the fixed point set $\text{Fix}(L(\mathbb{R}))$ is homeomorphic to $[0, 1]$. Since $L(\mathbb{R})$ is compact, it is a more reasonable choice of such a “universal flow”.

The proof of Theorem 1.3 is based on the techniques originally used in the proof of the Bebutov–Kakutani theorem (in particular, the idea of local section). A main new ingredient is the topological argument given in Section 2, which has some combinatorial flavor.

**Remark 1.4.** Problem 1.2 asks us to find a universal flow smaller than $C(\mathbb{R})$. If we look for a universal flow larger than $C(\mathbb{R})$, then it is much easier to find an example. Let $L^\infty(\mathbb{R})$ be the set of $L^\infty$-functions $\varphi : \mathbb{R} \to [0, 1]$. (We identify two functions which are equal to each other almost everywhere.) We consider the weak* topology on it. Namely a sequence $\{\varphi_n\}$ in $L^\infty(\mathbb{R})$ converges to $\varphi \in L^\infty(\mathbb{R})$ if for every $L^1$-function $\psi : \mathbb{R} \to \mathbb{R}$

\[ \lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n(t)\psi(t) \, dt = \int_{\mathbb{R}} \varphi(t)\psi(t) \, dt. \]

Then $L^\infty(\mathbb{R})$ is compact and metrizable by Banach–Alaoglu’s theorem and the separability of the space of $L^1$-functions, respectively. The group $\mathbb{R}$ acts continuously on it by translation. So it becomes a flow. Note that $\text{Fix}(L^\infty(\mathbb{R}))$ is homeomorphic to $[0, 1]$ and that the natural inclusion map $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is an equivariant continuous injection. Then the Bebutov–Kakutani theorem implies the universality of $L^\infty(\mathbb{R})$: A flow $(X, T)$ can be equivariantly embedded in $L^\infty(\mathbb{R})$ if and only if $\text{Fix}(X, T)$ can be topologically embedded in $[0, 1]$. 


Lemma 2.1. Let \( f \) be a continuous map. Then for any \( \delta > 0 \), we can choose a point \( s \) such that
\[
|f(s) - f(t)| < \frac{\delta}{2} 
\]
for all \( \delta > \frac{\delta}{2} \).

Proof. We take \( 0 < \delta < 1 \) satisfying
\[
\frac{\delta}{2} > \frac{\delta}{4}.
\]
Then for any \( \delta > 0 \) there exists \( g \in C(X,L[0,a]) \) satisfying
\[
\max_{x \in X} \|f(x) - g(x)\|_\infty < \delta.
\]
This implies that for all \( \delta > \frac{\delta}{4} \), the space of maps \( g : X \to [0,1] \) is endowed with the distance
\[
|g(s) - g(t)| < \frac{\delta}{4}.
\]
We define \( F_L[0,a] \subset L[0,a] \) as the space of constant functions \( \varphi : [0,a] \to [0,1] \), which is homeomorphic to \( [0,1] \).

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2. Topological preparations

Let \( a \) be a positive number. We define \( L[0,a] \) as the space of maps \( \varphi : [0,a] \to [0,1] \) satisfying
\[
\forall s,t \in [0,a] : |\varphi(s) - \varphi(t)| \leq |s - t|.
\]
\( L[0,a] \) is endowed with the distance \( |\varphi - \psi|_\infty = \max_{0 \leq t \leq a} |\varphi(t) - \psi(t)| \). We define \( F_L[0,a] \subset L[0,a] \) as the space of constant functions \( \varphi : [0,a] \to [0,1] \), which is homeomorphic to \( [0,1] \).

Let \( (X,d) \) be a compact metric space. We define \( C(X,L[0,a]) \) as the space of continuous maps \( f : X \to L[0,a] \), which is endowed with the distance
\[
\max_{x \in X} \|f(x) - g(x)\|_\infty.
\]

Lemma 2.1. Let \( f \in C(X,L[0,a]) \) and suppose there exists \( 0 < \tau < 1 \) satisfying
\[
\forall x \in X, \forall s,t \in [0,a] : |f(x)(s) - f(x)(t)| \leq \tau |s - t|.
\]
Then for any \( \delta > 0 \) there exists \( g \in C(X,L[0,a]) \) satisfying
1. \( \max_{x \in X} \|f(x) - g(x)\|_\infty < \delta \).
2. \( g(x)(0) = f(x)(0) \) and \( g(x)(a) = f(x)(a) \) for all \( x \in X \).
3. \( g(X) \cap F_L[0,a] = \emptyset \).

Proof. We take \( 0 < b < c < a \) satisfying \( b = a - c < \delta/4 \). We take an open covering \( \{U_1, \ldots, U_M\} \) of \( X \) satisfying
\[
\forall 1 \leq m \leq M : \text{diam} f(U_m) < \min \left( \frac{\delta}{4}, \frac{(1 - \tau)b}{2} \right).
\]
We take a point \( p_m \in U_m \) for each \( m \). We choose a natural number \( N \) satisfying
\[
N > M, \quad \Delta \overset{\text{def}}{=} \frac{c - b}{N - 1} < \frac{\delta}{4}.
\]
We divide the interval \([b,c]\) into \((N - 1)\) intervals of length \( \Delta \):
\[
b = a_1 < a_2 < \cdots < a_N = c, \quad a_{n+1} - a_n = \Delta \quad (\forall 1 \leq n \leq N - 1).
\]
Set \( A = \{a_1, \ldots, a_N\} \) and define a vector \( e \in \mathbb{R}^A \) by \( e = (1,1,\ldots,1) \). Notice that \( f(p_m)|_A \) is an element of \([0,1]^{A} \). Since \( N > M \) we can choose \( u_1, \ldots, u_M \in [0,1]^A \) satisfying
1. \( |f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1 - \tau)b/2) \) for all \( 1 \leq m \leq M \) and \( 1 \leq n \leq N \).
Claim 2.2. For proving $0 \leq x \leq 1$, this is trivial. For $1 \leq n \leq N - 1$, this is a direct consequence of the property (2) of $u_m$. So we consider the case of $n = 0$. (The case of $n = N$ is the same).

$$|g(x)(b) - f(x)(0)| \leq \sum_{m=1}^{M} h_m(x)|u_m(b) - f(p_m)(b)| + \sum_{m=1}^{M} h_m(x)|f(p_m)(b) - f(x)(b)|$$

We apply to each term of the right-hand side the property (1) of $u_m$, $\text{diam}f(U_m) < (1 - \tau)b/2$ in (2.2) and $|f(x)(b) - f(x)(0)| \leq \tau b$ in (2.1) respectively. Then this is bounded by

$$\frac{(1 - \tau)b}{2} + \frac{(1 - \tau)b}{2} + \tau b = b.$$ 

This proves $g(x) \in L[0, a]$.

Next we show $|g(x)(a_n) - f(x)(a_n)| < \delta/2$ for all $0 \leq n \leq N + 1$. For $n = 0, N + 1$, this is trivial. For $1 \leq n \leq N$, we can bound $|g(x)(a_n) - f(x)(a_n)|$ from above by

$$\sum_{m=1}^{M} h_m(x)|u_m(a_n) - f(p_m)(a_n)| + \sum_{m=1}^{M} h_m(x)|f(p_m)(a_n) - f(x)(a_n)|$$

$$< \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} \quad \text{(by the property (1) of $u_m$ and $\text{diam}f(U_m) < \delta/4$ in (2.2)).}$$

Finally, let $a_n < t < a_{n+1}$. We can bound $|g(x)(t) - f(x)(t)|$ by

$$|g(x)(t) - g(x)(a_n)| + |g(x)(a_n) - f(x)(a_n)| + |f(x)(a_n) - f(x)(t)|$$

$$< 2(a_{n+1} - a_n) + \frac{\delta}{2} \quad \text{(by $f(x), g(x) \in L[0, a]$)}$$

$$< \delta \quad \text{(by $a_{n+1} - a_n \leq \max(b, \Delta) < \frac{\delta}{4}$).}$$

□
For every \( x \in X \), the function \( g(x) : [0, a] \to [0, 1] \) is a non-constant function because
\[
g(x)|_\Lambda = \sum_{m=1}^{M} h_m(x)u_m \not\in \mathbb{R}e \quad \text{(by the property (3) of } u_m).\]

This proves the statement. \( \square \)

We need two lemmas on linear algebra. For \( u = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \) we set
\[
Du = (x_2 - x_1, x_3 - x_2, \ldots, x_{n+1} - x_n) \in \mathbb{R}^n.
\]

**Lemma 2.3.** Let \( l \geq m + 1 \) and set \( e = (1, 1, \ldots, 1) \in \mathbb{R}^l \). The set of \( (u_1, \ldots, u_m) \in \mathbb{R}^{l+1} \times \cdots \times \mathbb{R}^{l+1} = (\mathbb{R}^{l+1})^m \) such that
\[
(2.3) \quad \text{the vectors } e, Du_1, Du_2, \ldots, Du_m \text{ are linearly independent}
\]
is open and dense in \( (\mathbb{R}^{l+1})^m \).

**Proof.** The condition \( (2.3) \) defines a Zariski open set in \( (\mathbb{R}^{l+1})^m \). So it is enough to show that the set is non-empty because a non-empty Zariski open set is always dense in the Euclidean topology. We set
\[
u_i = (-1, \ldots, -1, 0, \ldots, 0), \quad (1 \leq i \leq m).
\]
Then
\[
Du_i = (0, \ldots, 0, 1, 0, \ldots, 0).
\]
The vectors \( e, Du_1, \ldots, Du_m \) are linearly independent. \( \square \)

**Lemma 2.4.** Let \( n > l \geq 2m \). The set of \( (u_1, \ldots, u_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = (\mathbb{R}^n)^m \) such that, for any integer \( \alpha \) with \( 2 \leq \alpha \leq n - l + 1 \),
\[
(2.4) \quad u_{1|1}^\alpha, u_{1|1}^{\alpha + l - 1}, u_{2|1}^{\alpha + l - 1}, \ldots, u_{m|1}^{\alpha + l - 1} \text{ are linearly independent in } \mathbb{R}^l
\]
is open and dense in \( (\mathbb{R}^n)^m \). Here for \( u_i = (x_{i1}, \ldots, x_{in}) \)
\[
u_i|1^\alpha = (x_{i1}, \ldots, x_{il}), \quad u_i|\alpha^{l} = (x_{i,\alpha}, \ldots, x_{i,\alpha+l-1}).
\]

**Proof.** The condition \( (2.4) \) defines a Zariski open set in \( (\mathbb{R}^n)^m \). Hence it is enough to show that for each \( 2 \leq \alpha \leq n - l + 1 \) we can choose \( (u_1, \ldots, u_m) \in (\mathbb{R}^n)^m \) satisfying \( (2.4) \).

We define \( u_i = (x_{i1}, \ldots, x_{in}) \) \((1 \leq i \leq m)\) by
\[
x_{ij} = 1 \quad (j = i, \alpha + l - i), \quad x_{ij} = 0 \quad \text{(otherwise)}.
\]
Then it is direct to check that these \( u_i \) satisfy \( (2.4) \). One can also use a proof from \[\text{[Lin99, Lemma 5.5]}\]. \( \square \)

**Lemma 2.5.** Let \( f \in C(X, L[0, a]) \) and suppose there exists \( 0 < \tau < 1 \) satisfying \( (2.1) \). Then for any \( \delta > 0 \) there exists \( g \in C(X, L[0, a]) \) satisfying
(1) \( \max_{x \in X} \| f(x) - g(x) \|_\infty < \delta. \)

(2) \( g(x)(0) = f(x)(0) \) and \( g(x)(a) = f(x)(a) \) for all \( x \in X. \)

(3) If \( x, y \in X \) and \( 0 \leq \varepsilon \leq a/2 \) satisfy

\[
\forall t \in [0, a - \varepsilon] : g(x)(t + \varepsilon) = g(y)(t)
\]

then \( \varepsilon = 0 \) and \( d(x, y) < \delta. \)

**Proof.** Except for the use of the above two lemmas on linear algebra, the proof is close to Lemma 2.1. We take \( 0 < b < c < a \) with \( b = a - c < \min(\delta/4, a/4) \). We take an open covering \( \{U_1, \ldots, U_M\} \) satisfying \( \text{diam} \, U_m < \delta \) and \( \text{diam} \, f(U_m) < \min(\delta/4, (1 - \tau)b/2) \) for all \( 1 \leq m \leq M \). Take \( p_m \in U_m \) for each \( m \). Let \( N \geq 2 \) be a natural number and set \( \Delta = (c - b)/(N - 1) \). We introduce a partition \( b = a_1 < a_2 < \cdots < a_N = c \) by \( a_n = b + (n - 1)\Delta \). We set \( A = \{a_1, \ldots, a_N\} \) and \( \Lambda = A \cap [b, a/4] = \{a_1, \ldots, a_L\} \). We also set \( e = (1, 1, \ldots, 1) \in \mathbb{R}^L \). We choose \( N \) sufficiently large so that

\[
\Delta < \frac{\delta}{4}, \quad N > L \geq 2M.
\]

Since \( L \geq 2M \geq M + 1 \), by using Lemmas 2.3 and 2.4, we can choose \( u_1, \ldots, u_M \in [0, 1]^A \) satisfying

(1) \( |f(p_m)(a_n) - u_m(a_n)| < \min(\delta/4, (1 - \tau)b/2) \) for all \( 1 \leq m \leq M \) and \( 1 \leq n \leq N \).

(2) \( |u_m(a_{n+1}) - u_m(a_n)| < \Delta \) for all \( 1 \leq m \leq M \) and \( 1 \leq n \leq N - 1 \).

(3) Define \( D_L u_m = (u_m(a_2) - u_m(a_1), \ldots, u_m(a_{L+1}) - u_m(a_L)) \in \mathbb{R}^L \). Then the \((M + 1)\) vectors \( e, D_L u_1, \ldots, D_L u_M \) in \( \mathbb{R}^L \) are linearly independent.

(4) For any \( \varepsilon > 0 \) with \( \varepsilon + \Lambda \subset A \),

\[
u_1|\Lambda, u_1|\varepsilon + \Lambda, u_2|\Lambda, u_2|\varepsilon + \Lambda, \ldots, u_m|\Lambda, u_m|\varepsilon + \Lambda \text{ are linearly independent in } \mathbb{R}^A\]

For \( x \in X \) we define \( g(x) : [0, a] \to [0, 1] \) in the same way as in the proof of Lemma 2.1. Namely, we set \( g(x)(0) = f(x)(0), g(x)(a) = f(x)(a) \) and \( g(x)(a_n) = \sum_{m=1}^M h_m(x)u_m(a_n) \) for \( 1 \leq n \leq N \), where \( \{h_m\} \) is a partition of unity satisfying \( \text{supp} \, h_m \subset U_m \). We extend \( g(x) \) to \([0, a] \) by linearity. It follows that \( g(x) \in L[a, a] \) and \( \|g(x) - f(x)\|_{\infty} < \delta \) as before. We need to check the property (3) of the statement. Suppose there exist \( x, y \in X \) and \( 0 \leq \varepsilon \leq a/2 \) satisfying \( g(x)(t + \varepsilon) = g(y)(t) \) for all \( 0 \leq t \leq a - \varepsilon \).

First we show \( \varepsilon + \Lambda \subset A \). Otherwise, \( (\varepsilon + \Lambda) \cap A = \emptyset \). Then it follows from the piecewise linearity that the function \( g(y)(t) \) becomes differentiable at every \( t \in \Lambda \), which implies

\[
g(y)(a_{n+1}) - g(y)(a_n) = g(y)(a_{n+2}) - g(y)(a_{n+1}) \quad (1 \leq n \leq L - 1),
\]

and hence

\[
\sum_{m=1}^M h_m(y)(u_m(a_{n+1}) - u_m(a_n)) = \sum_{m=1}^M h_m(y)(u_m(a_{n+2}) - u_m(a_{n+1})) \quad (1 \leq n \leq L - 1).
\]
This means that \( \sum_{m=1}^{M} h_m(y)D_L u_m \in \mathbb{R}e \), which contradicts the property (3) of \( u_m \). So we must have \( \varepsilon + \Lambda \subset A \).

The equation \( g(x)(t + \varepsilon) = g(y)(t) (0 \leq t \leq a - \varepsilon) \) implies

\[
\sum_{m=1}^{M} h_m(x)u_m|_{\varepsilon+\Lambda} = \sum_{m=1}^{M} h_m(y)u_m|_{\Lambda}.
\]

It follows from the property (4) of \( u_m \) that \( \varepsilon = 0 \) and \( h_m(x) = h_m(y) \) for all \( 1 \leq m \leq M \). Then \( x, y \in U_m \) for some \( m \) and hence \( d(x,y) \leq \text{diam} U_m < \delta \). □

3. Proof of Theorem 1.3

Let \( (X,T) \) be a flow. Set \( F = \text{Fix}(X,T) \). We define \( F_L = \text{Fix}(L(\mathbb{R})) \). Namely \( F_L \) is the space of constant maps \( \varphi : \mathbb{R} \to [0,1] \), which is homeomorphic to \([0,1]\). Suppose there exists a topological embedding \( h : F \to F_L \). We would like to show that there exists an equivariant embedding \( f : X \to L(\mathbb{R}) \) with \( f|_F = h \). We define \( C_{T,h}(X,L(\mathbb{R})) \) as the space of equivariant continuous maps \( f : X \to L(\mathbb{R}) \) satisfying \( f|_F = h \), which is endowed with the compact-open topology. For \( f \in C_{T,h}(X,L(\mathbb{R})) \) we define \( \text{Lip}(f) \) as the supremum of

\[
\frac{|f(x)(t) - f(x)(s)|}{|s-t|}
\]

over all \( x \in X \) and \( s,t \in \mathbb{R} \) with \( s \neq t \).

**Lemma 3.1.** The space \( C_{T,h}(X,L(\mathbb{R})) \) is not empty. Moreover for any \( \delta > 0 \) there exists \( f \in C_{T,h}(X,L(\mathbb{R})) \) satisfying \( \text{Lip}(f) \leq \delta \).

**Proof.** Consider the map

\[
F \ni x \to h(x)(0) \in [0,1].
\]

By the Tietze extension theorem, we can extend this function to a continuous map \( h_0 : X \to [0,1] \). Let \( \varphi : \mathbb{R} \to [0,1] \) be a smooth function satisfying

\[
\int_{-\infty}^{\infty} \varphi(t) \, dt = 1, \quad \int_{-\infty}^{\infty} \varphi'(t) \, dt \leq \min(1, \delta).
\]

For \( x \in X \) we define \( f(x) : \mathbb{R} \to [0,1] \) by

\[
f(x)(t) = \int_{-\infty}^{\infty} \varphi(t-s)h_0(T_s x) \, ds.
\]

Then \( |f(x)'(t)| \leq \min(1, \delta) \) and \( f = h \) on \( F \). Hence \( f \in C_{T,h}(X,L(\mathbb{R})) \) and \( \text{Lip}(f) \leq \delta \). □

We borrow the next lemma from Auslander [Aus88, p. 186, Corollary 6].
Lemma 3.2. Let \( p \in X \setminus F \). There exist \( a > 0 \) and a closed set \( S \subset X \) containing \( p \) such that the map

\[
(3.1) [-a, a] \times S \to X, \quad (t, x) \mapsto T_t x
\]

is a continuous injection whose image contains an open neighborhood of \( p \) in \( X \). We call \( (a, S) \) a local section around \( p \) and denote the image of \([X,A]\) by \([-a, a] \cdot S\).

**Proof.** We explain the proof for the convenience of readers. We can find \( 0 < c < 0 \) of

\[
\text{Lemma 3.3.}
\]

We choose a continuous function \( h : X \to [0, 1] \) satisfying \( T_c p \not\subset \text{supp} \, h \) and \( h = 1 \) on a neighborhood of \( p \). We define \( f : X \to \mathbb{R} \) by

\[
f(x) = \int_c^0 h(T_t x) dt.
\]

We choose \( 0 < a < |c| \) and a closed neighborhood \( A \) of \( p \) satisfying

\[
\bigcup_{|t| \leq a} T_t(A) \subset \{ h = 1 \}, \quad \bigcup_{|t| \leq a} T_{t+c}(A) \cap \text{supp} \, h = \emptyset.
\]

It follows that \( f(T_t x) = f(x) + t \) for any \( x \in A \) and \( |t| \leq a \). Set \( S = \{ x \in A \mid f(x) = f(p) \} \).

Then \((a, S)\) becomes a local section. Indeed if \( x, y \in S \) and \( s, t \in [-a, a] \) satisfy \( T_s x = T_t y \), then \( s + f(p) = f(T_s x) = f(T_t y) = t + f(p) \) and hence \( s = t \) and \( x = y \). Thus the map \((3.1)\) is injective. We take \( 0 < b < a \) and an open neighborhood \( U \) of \( p \) satisfying \( \bigcup_{|t| < b} T_t(U) \subset A \). Then the set \([-a, a] \cdot S\) contains

\[
(3.2) \quad \{ x \in U \mid -b < f(x) - f(p) < b \}
\]

because if \( x \in U \) satisfies \( t \overset{\text{def}}{=} f(x) - f(p) \in (-b, b) \) then \( f(T_{-t} x) = f(x) - t = f(p) \) (i.e. \( T_{-t} x \in S \)) and \( x = T_t(T_{-t} x) \in [-a, a] \cdot S \). The set \((3.2)\) is an open neighborhood of \( p \). \( \square \)

**Lemma 3.3.** For any point \( p \in X \setminus F \) there exists a closed neighborhood \( A \) of \( p \) in \( X \) such that the set

\[
(3.3) \quad G(A) = \{ f \in C_{T,h}(X, L(\mathbb{R})) \mid f(A) \cap F_L = \emptyset \}
\]

is open and dense in the space \( C_{T,h}(X, L(\mathbb{R})) \).

**Proof.** Take a local section \((a, S)\) around \( p \). For \( x \in X \) we define \( H(x) \subset \mathbb{R} \) (the set of hitting times) as the set of \( t \in \mathbb{R} \) satisfying \( T_t x \in S \). Any two distinct \( s, t \in H(x) \) satisfy \( |s - t| > a \). Notice that if \( x \in F \) then \( H(x) = \emptyset \). We denote by \( \text{Int} \,([-a, a] \cdot S) \) the interior of \([-a, a] \cdot S \). We choose a closed neighborhood \( A_0 \) of \( p \) in \( S \) satisfying \( A_0 \subset \text{Int} \,([-a, a] \cdot S) \). We define a closed neighborhood \( A \) of \( p \) in \( X \) by

\[
A = \bigcup_{|t| \leq a} T_t(A_0).
\]

We choose a continuous function \( q : S \to [0, 1] \) satisfying \( q = 1 \) on \( A_0 \) and \( \text{supp} \, q \subset \text{Int} \,([-a, a] \cdot S) \).
The set $G(A)$ defined in (3.3) is obviously open. So it is enough to prove that it is dense. Take $f \in C_{T,h} (X, L(\mathbb{R}))$ and $0 < \delta < 1$. By Lemma 3.1 we can find $f_0 \in C_{T,h} (X, L(\mathbb{R}))$ satisfying $\text{Lip}(f_0) \leq 1/2$. We define $f_1 \in C_{T,h} (X, L(\mathbb{R}))$ by

$$f_1(x)(t) = (1 - \delta)f(x)(t) + \delta f_0(x)(t).$$

It follows $\text{Lip}(f_1) \leq 1 - \delta/2 < 1$. We apply Lemma 2.1 to the map

$$X \ni x \mapsto f_1(x)|_{[0,a]} \in L[0, a].$$

Then we find $g \in C(X, L[0, a])$ satisfying

1. $|g(x)(t) - f_1(x)(t)| < \delta$ for all $x \in X$ and $0 \leq t \leq a$.
2. $g(x)(0) = f_1(x)(0)$ and $g(x)(a) = f_1(x)(a)$ for all $x \in X$.
3. $g(X) \cap F_L[0, a] = \emptyset$.

We set $u(x)(t) = g(x)(t) - f_1(x)(t)$ for $x \in X$ and $0 \leq t \leq a$. We define $g_1 \in C_{T,h} (X, L(\mathbb{R}))$ as follows: Let $x \in X$.

- For each $s \in H(x)$, we set
  $$g_1(x)(t) = f_1(x)(t) + q(T_x x) \cdot u(T_x x)(t - s) \quad \text{for} \ t \in [s, s + a].$$

- For $t \in \mathbb{R} \setminus \bigcup_{s \in H(x)} [s, s + a]$, we set $g_1(x)(t) = f_1(x)(t)$.

This satisfies

$$|g_1(x)(t) - f(x)(t)| \leq |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \leq 3\delta$$

for all $x \in X$ and $t \in \mathbb{R}$. If $x \in A$ then there exists $s \in [-a, a]$ with $T_s x \in A_0$ and hence

$$g_1(x)(s + t) = g(T_s x)(t) \quad \text{for} \ t \in [0, a].$$

It follows from the property (3) of $g$ that the function $g_1(x)$ is not constant. Thus $g_1 \in G(A)$. Since $f$ and $\delta$ are arbitrary, this proves that $G(A)$ is dense in $C_{T,h} (X, L(\mathbb{R}))$. □

**Lemma 3.4.** For any two distinct points $p$ and $q$ in $X \setminus F$ there exist closed neighborhoods $B$ and $C$ of $p$ and $q$ in $X$ respectively such that the set

$$G(B, C) = \{ f \in C_{T,h} (X, L(\mathbb{R})) \mid f(B) \cap f(C) = \emptyset \}$$

is open and dense in $C_{T,h} (X, L(\mathbb{R}))$.

**Proof.** Take local sections $(a, S_1)$ and $(a, S_2)$ around $p$ and $q$ respectively. We can assume that $[-a, a] \cdot S_1$ and $[-a, a] \cdot S_2$ are disjoint with each other. For $x \in X$ we define $H(x)$ as the set of $t \in \mathbb{R}$ satisfying $T_t x \in S_1 \cup S_2$. We choose closed neighborhoods $B_0$ of $p$ in $S_1$ and $C_0$ of $q$ in $S_2$ respectively satisfying $B_0 \subset \text{Int} ([-a, a] \cdot S_1)$ and $C_0 \subset \text{Int} ([-a, a] \cdot S_2)$. We take a continuous function $\tilde{q} : X \to [0, 1]$ satisfying $\tilde{q} = 1$ on $B_0 \cup C_0$ and $\text{supp} \tilde{q} \subset \text{Int} ([-a, a] \cdot S_1) \cup \text{Int} ([-a, a] \cdot S_2)$. We define closed neighborhoods $B$ and $C$ of $p$ and $q$ respectively by

$$B = \bigcup_{|t| \leq a/4} T_t(B_0), \quad C = \bigcup_{|t| \leq a/4} T_t(C_0).$$
The set \( G(B,C) \) defined in (3.3) is open. We show that it is dense. Take \( f \in C_{T,h}(X,L(\mathbb{R})) \) and \( 0 < \delta < 1 \). We can assume that

\[
\delta < d(B_0, C_0) \overset{\text{def}}{=} \min_{x \in B_0, y \in C_0} d(x, y).
\]

We define \( f_1 \in C_{T,h}(X,L(\mathbb{R})) \) exactly in the same way as in the proof of Lemma 3.3. It satisfies \( \text{Lip}(f_1) \leq 1 - \delta/2 \) and \( |f(x)(t) - f_1(x)(t)| \leq 2\delta \) for all \( x \in X \) and \( t \in \mathbb{R} \).

We apply Lemma 2.3 to the map

\[
X \ni x \mapsto f_1(x)\big|_{[0,a]} \in L[0,a].
\]

Then we find \( g \in C(X,L[0,a]) \) satisfying

1. \( |g(x)(t) - f_1(x)(t)| < \delta \) for all \( x \in X \) and \( 0 \leq t \leq a \).
2. \( g(x)(0) = f_1(x)(0) \) and \( g(x)(a) = f_1(x)(a) \) for all \( x \in X \).
3. If \( x, y \in X \) and \( 0 \leq \varepsilon \leq a/2 \) satisfy

\[
\forall t \in [0, a - \varepsilon] : g(x)(t + \varepsilon) = g(y)(t)
\]

then \( d(x,y) < \delta \).

We set \( u(x)(t) = g(x)(t) - f_1(x)(t) \) for \( x \in X \) and \( 0 \leq t \leq a \). We define \( g_1 \in C_{T,h}(X,L(\mathbb{R})) \) as before. Namely, for \( x \in X \),

- For each \( s \in H(x) \), we set

\[
g_1(x)(t) = f_1(x)(t) + \tilde{q}(T_s x) \cdot u(T_s x)(t - s) \quad \text{for} \quad t \in [s, s + a].
\]

- For \( t \in \mathbb{R} \setminus \bigcup_{s \in H(x)} [s, s + a] \), we set \( g_1(x)(t) = f_1(x)(t) \).

This satisfies \( |g_1(x)(t) - f(x)(t)| \leq |g_1(x)(t) - f_1(x)(t)| + |f_1(x)(t) - f(x)(t)| \leq 3\delta \).

We would like to show \( g_1(B) \cap g_1(C) = \emptyset \). Suppose \( x \in B \) and \( y \in C \) satisfy \( g_1(x) = g_1(y) \). There exist \( |s_1| \leq a/4 \) and \( |s_2| \leq a/4 \) satisfying \( T_{s_1}x \in B_0 \) and \( T_{s_2}y \in C_0 \). We can assume \( s_1 \leq s_2 \) without loss of generality. Set \( \varepsilon = s_2 - s_1 \in [0,a/2] \). We have

\[
g_1(x)(s_1 + t) = g(T_{s_1} x)(t) \quad \text{and} \quad g_1(y)(s_2 + t) = g(T_{s_2} y)(t) \quad \text{for} \quad t \in [0,a].
\]

\( g_1(x) = g_1(y) \) implies that

\[
g(T_{s_1} x)(t + \varepsilon) = g(T_{s_2} y)(t) \quad \text{for} \quad t \in [0,a - \varepsilon].
\]

It follows from the property (3) of \( g \) that \( d(T_{s_1} x, T_{s_2} y) < \delta \). Since \( \delta < d(B_0, C_0) \leq d(T_{s_1} x, T_{s_2} y) \), this is a contradiction. Therefore \( g_1(B) \cap g_1(C) = \emptyset \). This proves the lemma. \( \square \)

Now we can prove Theorem 1.3. By Lemmas 3.3 and 3.4, there exist families of closed sets \( \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \) and \( \{C_n\}_{n=1}^{\infty} \) of \( X \setminus F \) such that

- \( X \setminus F = \bigcup_{n=1}^{\infty} A_n \) and \( (X \setminus F) \times (X \setminus F) \setminus \{(x,x) : x \in X\} = \bigcup_{n=1}^{\infty} B_n \times C_n \).
- \( G(A_n) \) are open and dense in the space \( C_{T,h}(X,L(\mathbb{R})) \) for all \( n \geq 1 \).
- \( G(B_n,C_n) \) are open and dense in the space \( C_{T,h}(X,L(\mathbb{R})) \) for all \( n \geq 1 \).
By the Baire category theorem, the set
\[
\bigcap_{n=1}^{\infty} G(A_n) \cap \bigcap_{n=1}^{\infty} G(B_n, C_n)
\]
is dense and $G_\delta$ in $C_{T,h}(X, L(\mathbb{R}))$. In particular it is not empty. Any element $f$ in this set gives an embedding of the flow $(X, T)$ in $L(\mathbb{R})$.

**Remark 3.5.** The proof of the Bebutov–Kakutani theorem in [Kak68, Aus88] used the idea of “constructing large derivative”. It is possible to prove Theorem 1.3 by adapting this idea to the setting of one-Lipschitz functions. But this approach seems a bit tricky and less flexible than the proof given above. The above proof possibly has a wider applicability to different situations (e.g. other function spaces).

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