Simple proofs of uniformization theorems

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Abstract

The measurable Riemann mapping theorem proved by Morrey and in some particular cases by Ahlfors, Lavrentiev and Vekua, says that any measurable almost complex structure on \( \mathbb{R}^2 (S^2) \) with bounded dilatation is integrable: there is a quasiconformal homeomorphism of \( \mathbb{R}^2 (S^2) \) onto \( \mathbb{C} (\overline{\mathbb{C}}) \) transforming the given almost complex structure to the standard one. We give an elementary proof of this theorem that is done as follows. Firstly we prove its double-periodic version: each \( C^\infty \) almost complex structures on the two-torus can be transformed by a diffeomorphism to the standard complex structure on appropriate complex torus. The proof is based on the homotopy method for the Beltrami equation on \( \mathbb{T}^2 \) with parameter. (As a by-product, we present a simple proof of the Poincaré–Koebe theorem saying that each simply-connected Riemann surface is conformally equivalent to either \( \mathbb{C} \), or \( \mathbb{C} \), or the unit disc.) Afterwards the general case is treated by \( C^\infty \) double-periodic approximation and simple normality arguments (involving Grötzsch inequality) following the classical scheme.

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1 Introduction, the plan of the paper and history

1.1 Uniformization theorems. The plan of the paper

A linear complex structure on $\mathbb{R}^2$ is a structure of a linear space over $\mathbb{C}$ (we fix an orientation and consider it to be compatible with the complex structure). The \textit{(almost) complex structure} on a real two-dimensional surface is a family of linear complex structures on the tangent planes at its points. A linear complex structure on $\mathbb{R}^2$ defines an ellipse in $\mathbb{R}^2$ centered at 0, which is an orbit under the $S^1$-action by multiplication by complex numbers with unit module. (The ellipse corresponding to the standard complex structure on $\mathbb{C}$ is a circle.) The \textit{dilatation} of a nonstandard linear complex structure on $\mathbb{C}$ (with respect to the standard complex structure) is the excentricity of the corresponding ellipse (i.e., the ratio of the largest radius over the smallest one). An almost complex structure defines an ellipse field in tangent planes, and vice versa: the ellipse field determines the almost complex structure in a unique way.

If our surface is a Riemann surface (with a fixed complex structure), then any (nonstandard) almost complex structure has a well-defined dilatation at each point of the surface. In this case an almost complex structure is said to be \textit{bounded}, if its dilatation is bounded. The \textit{(total) dilatation} of a bounded almost complex structure is the supremum of its dilatations (more precisely, the minimal supremum of dilatations after possible correction of the almost complex structure over a measure zero set).

Each real linear isomorphism $\mathbb{C} \to \mathbb{C}$ acts on the space of the ellipses centered at 0, and hence, on the space of linear complex structures. Its \textit{dilatation} is defined to be the dilatation of the image of the standard complex structure (which is equal to the excentricity of the image of a circle centered at 0). The action of a differentiable homeomorphism of domains in $\mathbb{C}$ on the almost complex structures and its dilatation (at a point) are defined to be those of its derivative. Its \textit{(total) dilatation} is the supremum of the dilatations through all the points.

It appears that any $C^\infty$ (and even measurable) bounded almost complex structure is integrable, that is, can be transformed to a true complex structure by a $C^\infty$ (respectively, quasiconformal) homeomorphism, see the following Definition and Theorem.

1.1 Definition (see, e.g., [Ah2]). Let $K > 0$. A homeomorphism of domains in $\mathbb{C}$ is said to be $K$-\textit{quasiconformal} (or $K$-homeomorphism), if it has local $L^2$ derivatives and its dilatation (at the differentiability points with nonzero derivative) is no greater than $K$. A homeomorphism is said to be quasiconformal if it is $K$-quasiconformal for some $K > 0$.

1.2 Remark The dilatations of a differentiable homeomorphism and its inverse are equal. In particular, the inverse to a $K$-diffeomorphism is also a $K$-diffeomorphism. The composition of two $K$-diffeomorphisms is a $K^2$-diffeomorphism. This follows from definition.

1.3 Proposition (see [Ah2]) The quasiconformal homeomorphisms of a Riemann surface form a group.
1.4 Proposition The image of a zero measure set under a quasiconformal homeomorphism has also zero measure.

1.5 Corollary For any quasiconformal homeomorphism the set of its differentiability points with zero derivative has measure zero.

Proof The image of the set from the Corollary has zero measure by definition. Therefore, the set itself has zero measure (Proposition 1.4 applied to the inverse mapping, which is quasiconformal by Proposition 1.3).

Both Propositions are proved in Subsection 3.4 and neither them, nor the Corollary will be used in the paper.

1.6 Definition A homeomorphism $\mathbb{C} \to \mathbb{C}$ is said to be normalized, if it fixes 0 and 1.

1.7 Theorem ([AhB], [M]). For any measurable bounded almost complex structure $\sigma$ on $\mathbb{C}$ there exists a unique normalized quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$ that transforms $\sigma$ to the standard complex structure (at the differentiability points with nonzero derivative). If $\sigma$ is $C^\infty$ in some domain, then the homeomorphism is a $C^\infty$ diffeomorphism while restricted to this domain.

Addendum [AhB]. If a bounded almost complex structure on $\mathbb{C}$ varies analytically in a complex parameter, then so does the corresponding homeomorphism from Theorem 1.7.

1.8 Remark A quasiconformal homeomorphism of a once punctured domain extends quasiconformally to the puncture (in particular, the homeomorphism from Theorem 1.7 is quasiconformal at infinity). This follows easily from the local uniqueness of the quasiconformal homeomorphism up to composition with conformal mapping (Proposition 3.10, see Subsection 3.4) and the theorem on erasing isolated singularities of bounded holomorphic functions.

In the present paper we give proofs of Theorem 1.7 (Sections 2, 3) and the Addendum (Subsection 3.5) that seem to be simpler than the known proofs and easier to explain. A historical overview will be given in Subsection 1.4.

1.9 Remark The proof of the local integrability of an analytic almost complex structure is elementary: it is done immediately by analyzing the complexification of the corresponding $\mathbb{C}$-linear 1-form (this proof is due to Gauss). But it is already nontrivial in the $C^\infty$ case.

The measurable versions of the Theorem and the Addendum have many very important applications in various domains of mathematics, especially in holomorphic dynamics and the Kleinian group theory (quasiconformal surgery, where one deals with invariant almost complex structures that are discontinuous...), see, e.g., [CG].

For the proof of Theorem 1.7 we firstly prove (in Section 2) its version for $C^\infty$ almost complex structures on the two-torus: the proof uses only elementary Fourier analysis.

1.10 Theorem ([Ab]) For any $C^\infty$ bounded almost complex structure $\sigma$ on $\mathbb{T}^2$ there exists a $C^\infty$ diffeomorphism of $\mathbb{T}^2$ onto appropriate complex torus (the latter torus depends on $\sigma$) that transforms $\sigma$ to the standard complex structure.
Then in Section 3 we deduce Theorem 1.7 from Theorem 1.10 by using double-periodic approximations of a given almost complex structure on \( \mathbb{C} \) and simple normality arguments involving a Grötzsch inequality for annuli diffeomorphisms. This deduction follows the classical scheme [Ah2].

The proof of Theorem 1.10 presented below is implicitly contained in the previous paper [Gl] by the author, where the same method was used to prove a foliated version of Theorem 1.7. We prove the existence of a global nowhere vanishing \( \sigma \)-holomorphic differential. To do this, we use the homotopy method for the Beltrami equation with parameter, which reduces the proof to solving a linear ordinary differential equation in \( L^2(T^2) \). We prove regularity of its solution by showing that the equation is bounded in any Sobolev space \( H^s(T^2) \).

In Subsection 1.3 we give a proof of the classical Poincaré-Köbe uniformization theorem using Theorem 1.10:

1.11 Theorem [Ko1], [Ko2], [P]. Each simply-connected Riemann surface is conformally equivalent to either unit disc, or \( \mathbb{C} \), or the Riemann sphere.

In the proofs of the previously mentioned Theorems we use the well-known notations (recalled in the next Subsection) concerning almost complex structures.

1.2 Complex structures and uniformizing differentials. Basic notations

To a (nonstandard) almost complex structure (denoted \( \sigma \)) on a subset \( D \subset \mathbb{C} \) we put into correspondence a \( \mathbb{C} \)-valued 1-form that is \( \mathbb{C} \)-linear with respect to \( \sigma \). The latter form can be normalized to have the type

\[
\omega_\mu = dz + \mu(z)d\bar{z}, \quad |\mu| < 1. \tag{1.1}
\]

The function \( \mu \) is uniquely defined by \( \sigma \). Vice versa, for arbitrary complex-valued function \( \mu, \ |\mu| < 1 \), the 1-form (1.1) defines the unique complex structure for which it is \( \mathbb{C} \)-linear. We denote by \( \sigma_\mu \) the almost complex structure thus defined (whenever the contrary is not specified). Then \( \sigma_\mu \) is bounded, if and only if \( \sup |\mu| < 1 \).

1.12 Remark The ellipse associated to \( \sigma_\mu \) on the tangent plane at a point \( z \) is given by the equation \( |dz + \mu(z)\bar{d}z| = 1 \); the dilatation (excentricity) is equal to \( \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \).

We will be looking for a differentiable homeomorphism \( \Phi(z) \) that is holomorphic, i.e., that transforms \( \sigma_\mu \) to the standard complex structure. This is equivalent to say that the differential of \( \Phi \) (which is a closed form) is a \( \mathbb{C} \)-linear form, i.e., has the type \( f(z)(dz + \mu d\bar{z}) \):

\[
\frac{\partial \Phi}{\partial \bar{z}} = \mu \frac{\partial \Phi}{\partial z}.
\]

1.13 Remark Conversely, let \( \mu \) be \( C^\infty \), \( |\mu| < 1 \). Then any \( C^\infty \) closed 1-form \( f(z)(dz + \mu d\bar{z}) \) is \( \sigma_\mu \)-holomorphic, i.e., is a differential of a complex-valued \( C^\infty \) function \( \Phi \) transforming \( \sigma_\mu \) to the standard complex structure. A form \( f(z)(dz + \mu d\bar{z}) \) is closed if and only if

\[
\partial_\bar{z} f = \partial_\bar{z}(\mu f). \tag{1.2}
\]
1.14 Definition A Riemann surface is said to be parabolic, if its universal covering is conformally equivalent to \( \mathbb{C} \) (i.e., the surface is either \( \mathbb{C} \), or \( \mathbb{C}^* \), or a complex torus).

1.15 Definition The uniformizing differential on \( \mathbb{C} \) (or on a complex torus) with the affine coordinate \( z \) is the 1-form \( dz \) or its nonzero constant multiple. More generally, a holomorphic 1-form on a parabolic Riemann surface is said to be a uniformizing differential, if the primitive of its lifting to the universal cover is a conformal isomorphism onto \( \mathbb{C} \).

1.16 Remark The uniformizing differential is well-defined up to multiplication by constant. It coincides with the unique (up to constant) nowhere vanishing holomorphic differential whose squared module is a complete metric.

1.17 Proposition Let \( \mu : \mathbb{T}^2 \to \mathbb{C} \) be a \( C^\infty \) function, \( |\mu| < 1 \). Suppose there is a \( C^\infty \) nowhere vanishing function \( f : \mathbb{T}^2 \to \mathbb{C} \setminus 0 \) satisfying (1.2). Then the corresponding almost complex structure \( \sigma_\mu \) is integrable and the form \( f\omega_\mu \) is a uniformizing differential of \( (\mathbb{T}^2, \sigma_\mu) \).

The Proposition follows from compactness and the two previous Remarks.

1.18 Corollary For any bounded \( C^\infty \) almost complex structure \( \sigma \) on the closed unit disc \( \overline{D} \) there exists a \( C^\infty \) diffeomorphism of the open disc \( D \) onto itself transforming \( \sigma \) to the standard complex structure.

Proof Let us extend \( \sigma \) to \( \mathbb{R}^2 \) up to a double-periodic bounded \( C^\infty \) almost complex structure (say, with periods 4 and 4\( i \)) and consider the quotient torus equipped with the induced almost complex structure. Then the corresponding tori diffeomorphism from Theorem 1.14 transforms the latter structure to the standard one. Its lifting to the universal covers transforms \( D \) to a simply-connected domain in \( \mathbb{C} \) and sends \( \sigma \) to the standard complex structure. Now applying the Riemann mapping theorem to the image of \( D \) proves the Corollary.

We assume that the Riemann surface \( S \) is contractible, hence, admits a \( C^\infty \) 1-to-1 parametrization by \( \mathbb{R}^2 \). Its complex structure induces a \( C^\infty \) almost complex structure (denote it \( \sigma \)) on \( \mathbb{R}^2 \) (not necessarily bounded). Take a growing sequence of discs \( S_1 \subset S_2 \subset \ldots \subset S \) exhausting \( S \) centered at 0. On each \( S_n \) the almost complex structure \( \sigma \) is bounded. By the Corollary, for any \( n \) there is a diffeomorphism \( \phi_n : S_n \to D \) conformal with respect to the complex structure of \( S \), \( \phi_n(0) = 0 \). Let \( w \) be a local holomorphic chart on \( S \) near 0, \( w(0) = 0 \). Let us change \( \phi_n \) to its constant multiple \( \Phi_n = \lambda_n \phi_n \) having unit derivative in \( w \) at 0. The
family $\Phi_n$ is normal: each subsequence contains a subsequence converging uniformly on compact sets in $S$. Indeed, fix a $k \in \mathbb{N}$ and consider the $C^\infty$ injections $\Phi_n \circ \phi_n^{-1} : D \to \Phi_n(S_n)$, $n \geq k$. By construction, the latters are holomorphic and univalent, they send 0 to 0 and have one and the same derivative at 0. Therefore, they form a normal family, see [CG], hence, so do the $\Phi_n$’s. By construction, the limit of a converging subsequence of the $\Phi_n$’s is a conformal diffeomorphism of $S$ onto either a disc, or $\mathbb{C}$. Theorem 1.11 is proved.

1.4 Historical overview

The local integrability of a $C^\infty$ (and even Hölder) almost complex structure was proved by Korn [Korn] and Lichtenstein [Licht]; a simpler proof was obtained by Chern [Chern] and Bers [Be]. The local integrability together with the Poincaré-Köbe uniformization Theorem imply the global integrability statement of Theorem 1.11. Lavrentiev [La] gave a direct proof of Theorem 1.11 for continuous almost complex structures. Later Ahlfors [Ah1] and Vekua [Vek] gave another direct proofs under the previous (stronger) Hölder condition.

In the general measurable case Theorem 1.11 was proved by Morrey [M]. Later new proofs were obtained by Ahlfors and Bers [AhB], Bers and Nirenberg [BeN] and Boyarskii [Bo]. (In fact, Lavrentiev and Morrey stated their theorems for almost complex structures on a disc, but their versions on $\mathbb{R}^2$ follow immediately, e.g., by the arguments from the previous Subsection.) A new simpler proof of Theorem 1.11 using $L_2$ analysis and Fourier transformation on $\mathbb{R}^2$ was recently obtained by A.Douady and X.Buff [DB].

2 Smooth complex structures on $\mathbb{T}^2$. Proof of Theorem 1.10

2.1 Homotopy method. The sketch of the proof of Theorem 1.10

Let $\mu : \mathbb{T}^2 \to \mathbb{C}$ be a $C^\infty$ complex-valued function, $|\mu| < 1$, $\sigma_\mu$ be the corresponding almost complex structure, see (1.1). Theorem 1.10 says that there exists a diffeomorphism transforming $(\mathbb{T}^2, \sigma_\mu)$ into a complex torus equipped with the standard complex structure. To prove this statement, it suffices to construct a uniformizing differential, more precisely, a $C^\infty$ nowhere vanishing function $f : \mathbb{T}^2 \to \mathbb{C} \setminus 0$ such that the form $f \omega_\mu$ is closed (see Proposition 1.17), i.e., to solve partial differential equation (1.2) in a $C^\infty$ nowhere vanishing function $f$.

To solve (1.2), we use the homotopy method. Namely, we include $\sigma_\mu$ into the one-parametric family of complex structures (denoted by $\sigma_\nu$) defined by their $\mathbb{C}$- linear 1-forms

$$\omega_\nu = dz + \nu(z,t)d\bar{z}, \ \nu(z,t) = t\mu(z), \ t \in [0,1].$$

The complex structure corresponding to the parameter value $t = 0$ is the standard one, the given structure $\sigma_\mu$ corresponds to $t = 1$. We will find a $C^\infty$ family $f(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C} \setminus 0$ of complex-valued nowhere vanishing $C^\infty$ functions on $\mathbb{T}^2$ depending on the same parameter $t$, $f(z,0) \equiv 1$, such that the differential forms $f(z,t)\omega_\nu$ are closed, i.e.,

$$\partial_\bar{z} f = \partial_\bar{z} (f \nu). \quad (2.1)$$

Then the function $f = f(z,1)$ is the one we are looking for.

To construct the previous family of functions, we will find firstly a family $f(z,t)$ of nonidentically-vanishing (not necessarily nowhere vanishing) functions satisfying (2.1):
2.1 Lemma Let $\nu(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ be a $C^\infty$ family of $C^\infty$ functions on $\mathbb{T}^2$, $|\nu| < 1$, $\nu(z,0) \equiv 0$, $z$ be the complex coordinate on $\mathbb{T}^2$. There exists a $C^\infty$ family $f(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ of $C^\infty$ functions on $\mathbb{T}^2$ that are solutions of (2.1) with the initial condition $f(z,0) \equiv 1$ such that for any fixed $t \in [0,1]$ $f(z,t) \not\equiv 0$ in $z$.

The Lemma will be proved in the next Subsection.

Below we show that in fact, the functions $f(z,t)$ from the Lemma vanish nowhere. To do this (and only in this place) we use the local integrability of a $C^\infty$ complex structure:

2.2 Proposition ([Korn], [Licht], [La], [Chern], [Be]). Let $D \subset \mathbb{C}$ be a disc centered at 0, $\mu : D \to \mathbb{C}$, $\mu \in C^\infty$, $|\mu| < 1$, $\sigma_\mu$ be the corresponding almost complex structure, see (1.11). There exists a local $\sigma_\mu$- holomorphic univalent coordinate near 0.

The Proposition will be proved in Subsection 2.3.

Proof of Theorem 1.10 modulo Lemma 2.1 and Proposition 2.2. Let $f(z,t)$ be a family of functions from the previous Lemma. By the previous discussion, it suffices to show that $f(z,t) \not\equiv 0$. This inequality holds for $t = 0$, where $f = 1$.

Let us prove that $f(z,t) \not\equiv 0$ by contradiction. Suppose the contrary. Then the set of the parameter values $t$ corresponding to the functions $f(z,t)$ having zeroes is nonempty (denote this set by $M$). Its complement $[0,1] \setminus M$ is open by definition. Let us show that the set $M$ is open as well. This will imply that the parameter segment is a union of two disjoint open sets, which will bring us to contradiction. It suffices to show that the (local) presence of a zero of a function $f$ persists under perturbation.

Suppose $f(z_0,t) = 0$ for some $z_0$, $t$ (let us fix them). It suffices to show that for $t'$ close to $t$ the function $f(z,t')$ has a zero near $z_0$. Let $w$ be the local holomorphic coordinate on $\mathbb{T}^2$ near $z_0$ from the previous Proposition corresponding to $\nu = \nu(z,t)$, $w(z_0) = 0$. Suppose that the function $f(z,t)$ does not vanish identically on $\mathbb{T}^2$ locally near $z_0$: one can achieve this by changing $z_0$, since $f$ does not vanish identically. Recall that $f\omega_\nu$ is a closed $\mathbb{C}$- linear 1-form with respect to the variable complex structure $\sigma_\nu$, hence, it is holomorphic in the coordinate $w$. Therefore, $f\omega_\nu = (w^k + \text{higher terms})dw$, $k \geq 1$. Now by the index argument, the local presence of zero of $f$ on $\mathbb{T}^2$ persists under perturbation. This together with the previous discussion proves the inequality $f(z,t) \not\equiv 0$ and Theorem 1.10.

2.2 Variable holomorphic differential: proof of Lemma 2.1

Differentiating (2.1) in $t$ yields (we denote $\dot{f}$ the partial derivative in $t$ of a function $f$)

$$\partial_z \dot{f} - (\partial_z \circ \nu) \dot{f} = (\partial_z \circ \nu) f.$$  \hfill (2.2)

where $\partial_z \circ \nu$ ($\partial_z \circ \dot{\nu}$) is the composition of the operator of the multiplication by the function $\nu$ (respectively, $\dot{\nu}$) and the operator $\partial_z$. Any solution $f$ of equation (2.2) with the initial condition $f(z,0) \equiv 1$ that vanishes identically on the torus for no value of $t$ is a one we are looking for. Let us show that (2.2) is implied by a bounded linear differential equation in $L_2(\mathbb{T}^2)$. To do this, we use the following properties of the operators $\partial_z$ and $\partial_{\bar{z}}$.

2.3 Remark Denote $z = x_1 + ix_2$, $x = (x_1,x_2) \in \mathbb{R}^2$. The operators $\partial_z$, $\partial_{\bar{z}}$ on $\mathbb{T}^2$ have common eigenfunctions $e_n(x) = e^{in(x)}$, $n = (n_1,n_2) \in \mathbb{Z}^2$. The corresponding eigenvalues
(denote them \( \lambda_n \) and \( \lambda'_n \) respectively) have equal modules, more precisely,
\[
\lambda'_n = -\overline{\lambda_n}.
\]
This is implied by the fact that the operator \( \partial_z \) is conjugated to \( -\partial_z \) in the \( L_2 \) scalar product, which follows from definition. In fact,
\[
\lambda_n = \frac{i}{2}(n_1 - in_2), \quad \lambda'_n = \frac{i}{2}(n_1 + in_2).
\]

2.4 Corollary There exists a unique unitary operator \( U : L_2(\mathbb{T}^2) \to L_2(\mathbb{T}^2) \) preserving averages such that "\( U = \partial_z^{-1} \circ \partial_z \)" (more precisely, \( U \circ \partial_z = \partial_z \circ U = \partial_z \)). The operator \( U \) commutes with partial differentiations and extends up to a unitary operator to any Hilbert Sobolev space of functions on \( \mathbb{T}^2 \). In particular, it preserves the space of \( C^\infty \) functions.

Proof The operator \( U \) from the Corollary is defined to have the previous eigenfunctions \( e_n \) with the eigenvalues \( \lambda_n = \frac{n_1 - in_2}{n_1 + in_2} \). Its uniqueness follows immediately from the previous operator equation on \( U \) applied to the functions \( e_n \). The rest of the statements of the Corollary follow immediately from definition and Sobolev embedding theorem (see [Ch], p.411).

Let us write down equation (2.2) in terms of the new operator \( U \). Applying the "operator" \( \partial_z^{-1} \) to (2.2) and substituting \( U = \partial_z^{-1} \circ \partial_z \) yields
\[
(Id - U \circ \nu) \dot{f} = (U \circ \dot{\nu}) f.
\]
This equation implies (2.2). For any \( t \in [0, 1] \) the operator \( Id - U \circ \nu \) in the left-hand side is invertible in \( L_2(\mathbb{T}^2) \) and the norm of the inverse operator is bounded uniformly in \( t \), since \( U \) is unitary and the module \( |\nu| \) is less than 1 and bounded away from 1 by compactness. Thus, the last equation can be rewritten as
\[
\dot{f} = (Id - U \circ \nu)^{-1}(U \circ \dot{\nu}) f,
\]
which is an ordinary differential equation in \( f \in L_2(\mathbb{T}^2) \) with a uniformly \( L_2 \)-bounded operator in the right-hand side. As it is shown below (in Proposition 2.5), the inverse \((Id - U \circ \nu)^{-1}\) is also uniformly bounded in each Hilbert Sobolev space \( H^j(\mathbb{T}^2) \). Therefore, equation (2.2) written in arbitrary Hilbert Sobolev space has a unique solution with a given initial condition, in particular, with \( f(z, 0) \equiv 1 \) (the theorem on existence and uniqueness of solution of ordinary differential equation in Banach space with the right-hand side having uniformly bounded derivative [Ch]). For any \( t \in [0, 1] \) this solution does not vanish identically on \( \mathbb{T}^2 \) (uniqueness of solution) and belongs to all the spaces \( H^j(\mathbb{T}^2) \); hence, it is \( C^\infty(\mathbb{T}^2) \) by Sobolev embedding theorem (see [Ch], p.411). Thus, Lemma 2.1 is implied by the following

2.5 Proposition Let \( x = (x_1, x_2) \) be affine coordinates on \( \mathbb{R}^2 \), \( \mathbb{T}^2 = \mathbb{R}^2 / 2\pi \mathbb{Z}^2 \). Let \( s \geq 0 \), \( s \in \mathbb{Z} \), \( U \) be a linear operator in the space of \( C^\infty \) functions on \( \mathbb{T}^2 \) that commutes with the operators \( \frac{\partial}{\partial x_i} \), \( i = 1, 2 \), and extends to any Sobolev space \( H^j = H^j(\mathbb{T}^2) \), \( 0 \leq j \leq s \), up to a unitary operator. Let \( 0 < \delta < 1 \), \( \nu \in C^\delta(\mathbb{T}^2) \) be a complex-valued function, \( |\nu| \leq \delta \). The operator \( Id - U \circ \nu \) is invertible and the inverse operator is bounded in all the spaces \( H^j \), \( 0 \leq j \leq s \). For any \( 0 < \delta < 1 \), \( j \leq s \), there exists a constant \( C > 0 \) (depending only on \( \delta \) and \( s \)) such that for any complex-valued function \( \nu \in C^\delta(\mathbb{T}^2) \) with \( |\nu| \leq \delta \)
\[
||(Id - U \circ \nu)^{-1}||_{H^j} \leq C(1 + \sum_{k \leq j} \max \left| \frac{\partial^k \nu}{\partial x_{i_1}, \ldots, \partial x_{i_k}} \right|^2).
\]
Proof Let us prove the Proposition for $s = 1$. For higher $s$ its proof is analogous.

By definition, $\|U \circ \nu\|_{L^2} \leq \delta < 1$. (2.5)

Hence, the operator $Id - U \circ \nu$ is invertible in $L^2 = H^0$ and

$$(Id - U \circ \nu)^{-1} = Id + \sum_{k=1}^{\infty} (U \circ \nu)^k :$$ (2.6)

the sum of the $L^2$ operator norms of the sum entries in (2.6) is finite by (2.5). Let us show that the operator in the right-hand side of (2.6) is well-defined and bounded in $H^1$. To do this, it suffices to show that the sum of the operator $H^1$-norms of the same entries is finite.

Let $f \in H^1(T^2)$. Let us estimate $\|(U \circ \nu)^{k}f\|_{H^1}$. We show that for any $k \in \mathbb{N}$

$$\left\| \frac{\partial}{\partial x_r} ((U \circ \nu)^k f) \right\|_{L^2} < c k \delta^{k-1} \|f\|_{H^1}, \quad c = \delta + \max \left| \frac{\partial \nu}{\partial x_r} \right|, \quad r = 1, 2. \quad (2.7)$$

This will imply the finiteness of the operator $H^1$-norm of the sum in the right-hand side of (2.6) and Proposition 2.5 (with $C = 4 \sum_{k \in \mathbb{N}} k \delta^{k-1} = \frac{4}{(1-\delta)^2}$).

Let us prove (2.7), e.g., for $r = 1$. The derivative in the left-hand side of (2.7) equals

$$(U \circ \nu)^{k} \frac{\partial f}{\partial x_1} + \sum_{i=1}^{k} (U \circ \nu)^{k-i} \circ (U \circ \frac{\partial \nu}{\partial x_1}) \circ (U \circ \nu)^{i-1} f$$

(since $U$ commutes with the partial differentiation by the condition of Proposition 2.5). The $L^2$-norm of the first term in the previous formula is no greater than $\delta^k \|f\|_{H^1}$ by (2.5). Each term in its sum has $L^2$-norm no greater than $\delta^{k-1} \max \left| \frac{\partial \nu}{\partial x_r} \right| \|f\|_{L^2}$ by (2.5). This proves (2.7). The Proposition is proved. Lemma 2.1 is proved. 

2.6 Remark The solution of equation (2.4) with the initial condition $f|_{t=0} = 1$ admits the following formula:

$$f(x, t) = (Id - U \circ \nu)^{-1}(1) = 1 + U(\nu) + (U \circ \nu \circ U)(\dot{\nu}) + \ldots \quad (2.8)$$

Indeed, its right-hand side is a well defined $C^\infty$ family of $C^\infty$ functions on $T^2$, which follows from the uniform boundedness of the operators $(Id - U \circ \nu)^{-1}$ in any given Hilbert Sobolev space. By definition, it satisfies the unit initial condition. Differentiating (2.8) in $t$ yields

$$(Id - U \circ \nu)^{-1} \circ (U \circ \dot{\nu}) \circ (Id - U \circ \nu)^{-1}(1) = (Id - U \circ \nu)^{-1} \circ (U \circ \dot{\nu}) f(x, t).$$

Hence, the function (2.8) satisfies (2.4).

2.3 Zero of holomorphic differential. Proof of Proposition 2.2

Let us prove the existence of local holomorphic coordinate. Without loss of generality we assume that $\mu(0) = 0$ (applying a linear change of variables). One can achieve also that $\mu$ is arbitrarily small with derivatives of orders up to 3 applying a homothety and taking the restriction to a smaller disc centered at 0. We consider that the disc where $\mu$ is defined is
embedded into $\mathbb{T}^2$ and extend the function $\mu$ smoothly to $\mathbb{T}^2$. We assume that the extended function satisfies the inequality $|\mu|_{C^3(\mathbb{T}^2)} < 1$; one can make $\delta$ arbitrarily small.

Let $\nu(x, t) = t\mu$, $f(x, t)$ be the corresponding function family from Lemma 2.1 constructed as the solution of differential equation (2.4) with unit initial condition, $f(x) = f(x, 1)$. We show in the next paragraph that $f(0) \neq 0$, if the previous constant $\delta$ is small enough. Then the local coordinate we are looking for is the function

$$w(z) = \int_0^z f(dz + \mu d\bar{z}).$$

Indeed, it is well-defined and holomorphic by definition. Its local univalence follows from the nondegeneracy of its differential $f(0)(dz + \mu d\bar{z})$ at 0 (the inequalities $|\mu| < 1$, $f(0) \neq 0$).

Recall that by (2.8),

$$f(x, t) = (Id - tU \circ \mu)^{-1}(1), \text{ where } U = (\partial_z)^{-1}\partial_z.$$

The functions $f(x, t)$ are equal to 1, if $\mu = 0$. Let us show that they are $C^0$- close to 1 (and hence, $f(0, 1) \neq 0$), whenever $\mu$ is small enough with derivatives up to order 3. Consider the operator functional $A(\mu) = (Id - tU \circ \mu)^{-1}$: its value being an operator acting in $H^3(\mathbb{T}^2)$ (it is well-defined, see Proposition 2.5). As it will be shown in the next paragraph, it depends continuously on small functional parameter $\mu \in C^3(\mathbb{T}^2)$, $max|\mu| < 1$, in the $H^3(\mathbb{T}^2)$ operator norm, and moreover, it has a bounded derivative in $\mu$. Therefore, if $||\mu||_{C^3}$ is small enough, then each function $f(x, t)$ is close to 1 in $H^3$ (thus, in $C^0$, by Sobolev embedding theorem).

Now for the proof of Proposition 2.2 it suffices to prove the boundedness of the previous derivative $A'(\mu)$. For any $0 < \delta' < 1$ the $A(\mu)$ is uniformly bounded in all $\mu$ with $||\mu||_{C^3} < \delta'$ (Proposition 2.3), so, we can apply the usual formula for the derivative of the inverse operator: the derivative of $A(\mu)$ along a vector $h \in C^3(\mathbb{T}^2)$ is equal to

$$\nabla_h A(\mu) = A(\mu) \circ U \circ h \circ A(\mu).$$

To prove the boundedness of the derivative, we have to show that the $H^3$- norm of the operator in the right-hand side of the previous formula is no greater than some constant (depending on $\mu$) times $||h||_{C^3}$. Indeed, the previous $H^3$ operator norm is no greater than $||A(\mu)||_{H^3}^2$ times the $H^3$- norm of the operator of multiplication by the function $h$, the latter is no greater than $||h||_{C^3}$ times some universal constant. This proves the boundedness of the derivative. Proposition 2.2 is proved. The proof of Theorem 1.7 is completed.

3 Quasiconformal mappings. Proof of Theorem 1.7

3.1 The plan of the proof of Theorem 1.7

We have already proved the statement of Theorem 1.7 for a $C^\infty$ double-periodic almost complex structure on $\mathbb{C}$ (i.e., a lifting to the universal cover $\mathbb{C}$ of a $C^\infty$ complex structure on $\mathbb{T}^2$). In this case the diffeomorphism $\mathbb{C} \rightarrow \mathbb{C}$ from the Theorem is the lifting to the universal covers of the diffeomorphism of the tori given by Theorem 1.10. To prove Theorem 1.7 in the general case (let $\sigma$ be a given (may be measurable) bounded complex structure on $\mathbb{C}$) we consider a sequence $\sigma_n$ of $C^\infty$ double-periodic complex structures on $\mathbb{C}$ with growing periods and uniformly bounded dilatations (say less than a fixed $K > 0$) that converge to $\sigma$ almost
everywhere. For each $\sigma_n$ there is a normalized quasiconformal diffeomorphism $\Phi_n : \mathbb{C} \to \mathbb{C}$ transforming $\sigma_n$ to the standard complex structure. We show that the diffeomorphisms $\Phi_n$ converge (uniformly on $\mathbb{C}$) to a homeomorphism (denoted $\Phi$). We will prove that $\Phi$ is a quasiconformal homeomorphism sending $\sigma$ to the standard complex structure (see the end of the Subsection). The uniqueness of a latter homeomorphism and its diffeomorphic property on a smoothness domain of $\sigma$ will be proved in 3.4. Its analytic dependence on parameter (the Addendum to Theorem 1.7) will be proved in 3.5.

We prove the convergence of $\Phi_n$ by equicontinuity of the normalized $K$-homeomorphisms:

**3.1 Lemma [Ah2].** For any $K > 0$ the normalized $K$-homeomorphisms $\mathbb{C} \to \mathbb{C}$ (see Definition 1.1) are equicontinuous with their inverses as mappings of the Riemann sphere.

Lemma 3.1 (proved in 3.2) together with Arzela-Ascoli theorem imply the following

**3.2 Corollary** For any $K > 0$ each sequence of normalized $K$-homeomorphisms $\mathbb{C} \to \mathbb{C}$ contains a subsequence converging to a homeomorphism $\mathbb{C} \to \mathbb{C}$ uniformly on $\mathbb{C}$.

**3.3 Lemma [Ah2].** Let $K > 0$, $U \subset \mathbb{C}$ be a domain (that may be the whole $\mathbb{C}$) $\Phi_n : U \to \Phi_n(U) \subset \mathbb{C}$ be a sequence of $K$-homeomorphisms converging uniformly on compact subsets to a homeomorphism (denote $\Phi$ the limit). Let $\sigma_n$ be the almost complex structures sent to the standard one by $\Phi_n$. Let $\sigma_n$ converge almost everywhere (denote $\sigma$ their limit). Then $\Phi$ is a $K$-homeomorphism sending $\sigma$ to the standard complex structure.

Lemma 3.3 will be proved in Subsection 3.3 (using Lemma 3.1 and Corollary 3.2).

**Proof of existence in Theorem 1.7 modulo Lemmas 3.1 and 3.3.** Let $\sigma_n$, $\sigma$, $K$, $\Phi_n$ be as at the beginning of the Section. Then $\Phi_n$ are $K$-diffeomorphisms. Passing to a subsequence, one can achieve that $\Phi_n$ converge to a homeomorphism (Corollary 3.2 denote $\Phi$ the limit homeomorphism). By Lemma 3.3 $\Phi$ is a $K$-homeomorphism transforming $\sigma$ to the standard complex structure. Theorem 1.7 is proved.

**3.4 Remark** In the proof of the existence in Theorem 1.7 we had used only the statements of the previous Lemmas for $C^\infty$ diffeomorphisms. Their statements for general quasiconformal homeomorphisms will be used in the proof of the uniqueness in Theorem 1.7 (Subsection 3.4).

**3.2 Normality. Proof of Lemma 3.1.**

The proof of Lemma 3.1 is based on the Grötzsch inequality (the next Lemma) comparing moduli of $K$-homeomorphic complex annuli. To state it, let us firstly recall the following

**3.5 Definition** see [Ah2]. The modulus of an annulus $A = \{r < |z| < 1\}$ is $m(A) = -\frac{1}{2\pi} \ln r$.

**3.6 Remark** Consider the cylinder $\mathbb{R} \times S^1$ with the coordinates $(x, \phi)$, $S^1 = \mathbb{R}/2\pi \mathbb{Z}$, and the standard complex structure, which is induced by the Euclidean metric $dx^2 + d\phi^2$.

For any $R > 0$ put $A(R) = \{0 < x < R\}$; then $m(A(R)) = \frac{R}{2\pi}$. (3.1)

The modulus of an annulus is invariant under conformal mappings [Ah2].
3.7 Lemma (Grötzsch, see [Ah2]). Let $K > 0$, $f : A_1 \to A_2$ be a $K$- homeomorphism of complex annuli. Then

$$m(A_2) \geq K^{-1}m(A_1).$$

(3.2)

Proof For completeness of presentation, we give the classical proof of the Lemma. Firstly we prove the Lemma for a $K$- diffeomorphism; the general case is treated analogously (see the end of the proof). Let us consider that the annuli are drawn on the previous cylinder, say, $A_1 = A(R_1)$, $A_2 = A(R_2)$, then $m(A_i) = \frac{R_i}{2\pi}$, $i = 1, 2$, see (3.1). Thus, it suffices to show that $R_2 \geq K^{-1}R_1$. To do this, consider the pullback (denoted $g$) to $A_1$ under $f$ of the Euclidean metric of $A_2$ (denote $| \cdot |_g$ (Area) the corresponding norm of vector fields on $A_1$ (respectively, the area), Area being the Euclidean area). One has Area$(A_i) = 2\pi R_i$, Area$(A_2) = Area_g(A_1)$. We show that

$$Area_g(A_1) \geq K^{-1}Area(A_1).$$

(3.3)

This together with the previous formulas for the areas will prove the Lemma. For the proof of (3.3) we consider the family $A(r) = \{0 < x < r \} \subset A_1$ of subannuli in $A_1$, $r \leq R_1$, and prove a lower bound of the derivative $(Area_g(A(r)))'$. To do this, consider the vector field $\frac{\partial}{\partial x}$ as the sum of its component tangent to the circles $x = const$ and the $g$- orthogonal component

![](image1)

(denote the latter component normal to the circles by $n$, see Fig.1). The vector field $n$ has the same projection to the $x$-axis, as $\frac{\partial}{\partial x}$ and its flow leaves invariant the fibration by circles $x = const$: its time $t$ flow map transforms $A(r)$ to $A(r + t)$. Therefore,

$$\left(\frac{\partial}{\partial x}\right)^t = \int_{x=r, \phi \in [0,2\pi]} \frac{\partial}{\partial \phi} |_g |n|_g d\phi.$$  

(3.4)

One has $|n|_g \geq K^{-1} |\frac{\partial}{\partial \phi}|_g$. Indeed, the $g$- norm $| \cdot |_g$ of a vector tangent to $A_1$ is equal to the standard Euclidean norm $| \cdot |$ of its image under $f$: $|n|_g = |f_* n|_g$. $|\frac{\partial}{\partial \phi}|_g = |f_* \frac{\partial}{\partial \phi}|$. By definition, $|\frac{\partial}{\partial x}| = 1$, $|n| \geq |\frac{\partial}{\partial x}| = 1 = |\frac{\partial}{\partial \phi}|$. Therefore, by the $K$- quasiconformality of $f$ (see, Definition 1.1), $|n|_g = |f_* n| \geq K^{-1} |f_* \frac{\partial}{\partial \phi}| = K^{-1} |\frac{\partial}{\partial \phi}|_g$. Hence, the previous derivative is no less than

$$K^{-1} \int_{x=r, \phi \in [0,2\pi]} \frac{\partial}{\partial \phi}|^2 d\phi \geq K^{-1}(2\pi)^{-1} \int_{\phi \in [0,2\pi]} |\frac{\partial}{\partial \phi}|^2 d\phi.$$  

(Cauchy-Bouniakovskii-Schwarz inequality). The latter integral is no less than $2\pi$. Indeed, it is equal to the length in the metric $g$ of the circle $x = r$, or in other terms, the Euclidean length of its image under $f$, which is a closed curve in $A_2$ isotopic to a circle $x = const$. Therefore, $(Area_g(A(r)))' \geq 2\pi K^{-1}$, thus, Area$g(A_1) \geq 2\pi K^{-1} R_1 = K^{-1} \cdot Area(A_1)$. This proves (3.3) and the Lemma for a $K$- diffeomorphism $f$. In the case, when $f$ is a $K$- homeomorphism, thus just having local $L_2$ derivatives, the previous discussion remains valid: the previous integrals are well-defined for almost all $r$, since the subintegral expression $|\frac{\partial}{\partial \phi}|_g |n|_g$ in (3.4) is bounded from above by $||df(r, \phi)||^2$ times a constant depending on $K$. This follows from definition and the uniform boundedness of the Euclidean norm $|n|$: by definition, $n$ is projected to the vector field $\frac{\partial}{\partial x}$ with unit norm; the angle between $n$ and a circle $x = const$ is bounded from below by a constant depending on $K$ (quasiconformality). Lemma 3.7 is proved. \qed
To prove Lemma 3.1, we need to show that close points cannot be mapped to distant points under a normalized K-homeomorphism or its inverse. This is proved by comparing moduli of appropriate annuli with those of their images (using Lemma 3.7).

For the proof of Lemma 3.1 we recall the notion of the Poincaré metric [CG]. The Poincaré metric of the unit disc $|z| < 1$ is $\frac{4|dz|^2}{(1-|z|^2)^2}$ (it is invariant under its conformal automorphisms). A Riemann surface is hyperbolic, if its universal covering is conformally equivalent to the unit disc (see Theorem 1.11), e.g., any domain in $\mathbb{C}$ whose complement contains more than one point. The Poincaré metric of a hyperbolic Riemann surface is the pushforward of the Poincaré metric of the unit disc under the universal covering.

3.8 Remark (see [CG]). The Poincaré metric is well-defined, complete and decreasing: the Poincaré metric of a subdomain of a hyperbolic Riemann surface is greater than that of the ambient surface. The Poincaré metric of $\mathbb{C}\setminus\{0,1\}$ is greater than its standard spherical metric times a constant.

In the proof of Lemma 3.1 we use the following relation of modulus of an annulus and its Poincaré metric, whose proof is a straightforward calculation.

3.9 Proposition see, e.g., [DH]. The modulus of an annulus is equal to $\pi$ times the inverse of the length of its closed geodesic.

Let us prove the equicontinuity of normalized $K$-homeomorphisms by contradiction. Suppose the contrary, i.e., there exist an $\varepsilon > 0$, a sequence of normalized $K$-homeomorphisms $\Phi_n : \mathbb{C} \to \mathbb{C}$ and a sequence of pairs $x_n, y_n \in \mathbb{C}$, $|x_n - y_n| \to 0$, $|\Phi_n(x_n) - \Phi_n(y_n)| > \varepsilon$ (in the spherical metric of $\mathbb{C}$). Without loss of generality we assume that the sequence $x_n$ (and hence, $y_n$) converges (one can achieve this by passing to a subsequence, denote $x$ the limit). Then there is a sequence $A_n$ of annuli in $\mathbb{C}\setminus\{0,1,\infty\}$ bounded by circles centered at $x$ and surrounding the pairs $x_n, y_n$: one of the circles is fixed, the other one contracts to $x$, as $n \to \infty$ see Fig.2a. By definition, the annuli $A_n$ tend to once punctured disc, hence, $m(A_n) \to \infty$. The point $x$ may coincide with some of the three points 0, 1, $\infty$. Let us take two of the latters that are distinct from $x$ (say, let them be 0, 1). Then each annulus $A_n$ separates the pairs $(x_n, y_n)$ and $(0,1)$. By Lemma 3.7 $m(\Phi_n(A_n)) \to \infty$ as well. Hence, by Proposition 3.9 the lengths of the geodesics (denoted by $\gamma_n$) of the annuli $\Phi_n(A_n)$ in their Poincaré metrics tend to zero. But the latter lengths are greater than the lengths of $\gamma_n$ taken in the Poincaré metric of $\mathbb{C}\setminus\{0,1\}$, and hence, also greater than their lengths in the spherical metric times a constant independent from $n$ (by the previous Remark). Thus, each $\gamma_n$ separates the pairs $(\Phi_n(x_n), \Phi_n(y_n))$ and $(0,1)$ and is a closed curve with spherical length tending to 0. Hence, the spherical distance between $\Phi_n(x_n)$ and $\Phi_n(y_n)$ tends to 0 - a contradiction.
Now let us prove that the inverses to the normalized $K$-homeomorphisms are also equicontinuous by contradiction, analogously to the previous discussion. Suppose the contrary: there exist an $\varepsilon > 0$, a sequence of normalized $K$-homeomorphisms $\Phi_n : \mathbb{C} \to \mathbb{C}$ and a sequence of pairs $x_n, y_n \in \mathbb{C}$, $|x_n - y_n| \to 0$, $|\Phi_n^{-1}(x_n) - \Phi_n^{-1}(y_n)| > \varepsilon$ (in the spherical metric of $\mathbb{C}$). Without loss of generality we assume that all the sequences $x_n, y_n, \Phi_n^{-1}(x_n), \Phi_n^{-1}(y_n)$ converge (one can achieve this by passing to a subsequence); denote their limits by $x, y, \tilde{x}, \tilde{y}$ respectively. By definition, $x = y, \tilde{x} \neq \tilde{y}$. Firstly consider the case, when $x \neq 0, 1, \infty$. Then $\tilde{x}, \tilde{y} \neq 0, 1, \infty$ as well: otherwise $\Phi_n^{-1}(x_n), \Phi_n^{-1}(y_n)$ would accumulate to $\{0, 1, \infty\}$, while their $\Phi_n$-images $x_n, y_n$ would not - a contradiction to the equicontinuity of the $\Phi_n$'s (already proved). Fix an annulus $A$ separating the pair $0, \tilde{x}$ and the triple $1, \tilde{y}, \infty$ (we assume that its closure is disjoint from $\Phi_n^{-1}(x_n), \Phi_n^{-1}(y_n)$ for any $n$). Its images $\Phi_n(A)$ are disjoint from $0, 1, x_n, y_n$ and have moduli bounded away from zero (Lemma 3.7), and hence, closed geodesics (denoted $\gamma_n$, see Fig.2b) of uniformly bounded lengths. Thus, the lengths of $\gamma_n$ in the Poincaré metric of $\mathbb{C} \setminus \{0, 1, x_n, y_n\}$ are also uniformly bounded. On the other hand, $\gamma_n$ separates the pair $(0, x_n)$ and the triple $(1, y_n, \infty)$ for any $n$, see Fig.2b. The points $0, x_n$ in the first pair are distant ($x = \lim x_n \neq 0$), thus, the spherical length of $\gamma_n$ is bounded from below. The points $x_n, y_n$, which are separated by $\gamma_n$, collide towards $x$, so, $\gamma_n$ comes arbitrarily close to $x_n$, as $n \to \infty$. This implies that $\gamma_n$ has length tending to infinity in the Poincaré metric of $\mathbb{C} \setminus \{0, x_n\}$, and hence, in the Poincaré metric of $\mathbb{C} \setminus \{0, 1, x_n, y_n\}$. If $x_n$ does not move while $n$ changes, this follows from the completeness of the Poincaré metric (Remark 3.3). The case, when $x_n \neq \text{const}$, is reduced to the previous one by applying the variable change $w = \frac{z}{x_n}$. This contradicts to the previous statement saying that the latter Poincaré length of $\gamma_n$ is uniformly bounded.

Now let $x \in \{0, 1, \infty\}$, say, $x = 1$. Then $\tilde{x}, \tilde{y} \neq 0, \infty$, as before, and at least one of $\tilde{x} \neq \tilde{y}$ (say, $\tilde{x}$) is distinct from 1. In these notations we repeat the previous argument. Lemma 3.1 is proved.

### 3.3 Quasiconformality and weak convergence. Proof of Lemma 3.3

Let $\Phi_n, \sigma_n, \Phi, \sigma$ be as in Lemma 3.3. Recall that the dilatations of the $\sigma_n$'s are no greater than $K$, as are those of the $\Phi_n$'s, hence, the same is true for $\sigma$. Let us show that $\Phi$ is quasiconformal, more precisely: 1) has local $L_2$ derivatives that are weak $L_2$ limits of those of $\Phi_n$; 2) transforms $\sigma$ to the standard complex structure (and hence, is $K$-quasiconformal). This will prove Lemma 3.3.
For the proof of statement 1) we use the fact that the norms of the differentials $d\Phi_n$ (in the spherical metric of $\mathbb{C}$) are uniformly bounded in each space $L_2(D), D \subset \mathbb{C}$. Indeed, on each disc $D \subset \mathbb{C}$ $||d\Phi_n||_{L_2(D)}^2 \leq K(Area(\Phi_n(D)))$, which follows from definition and $K$-quasiconformality (the areas are taken in the spherical metric). The latter areas converge to $Area(\Phi(D))$, hence, they are uniformly bounded, and so are the previous $L_2$-norms.

Thus, the derivatives are locally $L_2$-bounded, hence, passing to a subsequence one can achieve that they converge $L_2$-weakly. On the other hand, they converge to the derivative of $\Phi$ in sense of distributions. Therefore, the latter is also $L_2$ locally and the convergence is $L_2$-weak. Statement 1) is proved.

Let $\mu_n, \mu$ be the functions from $\{1\}$ defining the complex structures $\sigma_n$ and $\sigma$ respectively, thus, $d\Phi_n = f_n(dz + \mu_n dz)$. By assumption, $|\mu_n| < 1$, $\mu_n \to \mu$ almost everywhere. We have to show that $\frac{\partial \Phi}{\partial z} = \mu \frac{\partial \Phi}{\partial z}$. Indeed, $f_n \to f = \frac{\partial \Phi}{\partial z}$; $f_n \mu_n \to \frac{\partial \Phi}{\partial z}$ (both $L_2$ weakly), as $n \to \infty$. Since, $f_n$ are uniformly bounded in a local space $L_2$ and weakly converge, $\mu_n$ are uniformly bounded and converge almost everywhere, the weak limit of their product is the product $f \mu$ of their limits. This proves the previous partial differential equation on $\Phi$ together with statement 2) and Lemma $\{1\}$.

3.4 Uniqueness, smoothness and group property

Here we prove the uniqueness of the normalized quasiconformal homeomorphism from Theorem $\{1\}$ and the group and measure properties of quasiconformal mappings (Propositions $\{1\}$ and $\{1\}$). The uniqueness follows from the local uniqueness up to composition with a conformal mapping and from normalizedness. The local uniqueness (together with the diffeomorphic property on a smoothness domain of the complex structure) are implied by the following

3.10 Proposition Let $D \subset \mathbb{C}$ be a simply-connected domain, $\sigma$ be a bounded measurable almost complex structure on $D$, $\Phi : D \to \Phi(D) \subset \mathbb{C}$ be a quasiconformal homeomorphism transforming $\sigma$ to the standard complex structure. Then $\Phi$ is unique up to left composition with a conformal mapping. It is a $C^\infty$ diffeomorphism, if $\sigma$ is $C^\infty$.

Proof Let $\mu : D \to \mathbb{C}$ be the function defining the almost complex structure $\sigma$.

Case $\mu \equiv 0$. Then $\frac{\partial \Phi}{\partial z} = 0$ and $\Phi$ has local $L_2$ derivatives. Let us show that $\Phi$ is conformal. Fix a $z_0 \in D$ and put $U(z) = \int_{z_0}^z \Phi(\zeta) d\zeta$. We show that the function $U(z)$ is well-defined (independent on the choice of path connecting $z_0$ to $z$). Then it is holomorphic by definition, hence, so is $\Phi(z) = \frac{dU}{dz}$. It suffices to show that the integral of the form $\Phi dz$ along any Jordan curve is zero. Since the derivatives of $\Phi$ are locally $L_2$, we can apply the Stokes formula: the previous integral is equal to the integral of the differential $d(\Phi dz)$ over the domain bounded by the curve. But $d(\Phi dz) = \frac{\partial \Phi}{\partial \bar{z}} \bar{z} dz = 0$, so, it is zero.

Case $\mu \in C^\infty$. There exists at least one $C^\infty$ quasiconformal diffeomorphism $\Psi$ transforming $\sigma$ to the standard complex structure (Theorem $\{1\}$, see also the discussion in Section 1.2). The composition $\Phi \circ \Psi^{-1}$ preserves the standard complex structure by definition and is quasiconformal: it has local $L_2$ derivatives, since so does $\Phi$ and $\Psi^{-1}$ is $C^\infty$. Therefore, it is conformal, as is proved above, hence, $\Phi$ is a $C^\infty$ diffeomorphism.

Case $\mu$ is measurable. Let $0 < \delta < 1, |\mu| < \delta$, $\mu_n \to \mu$ almost everywhere (we extend $\mu_n, \mu$ to $\mathbb{C}$ with the latter inequality and convergence). Consider the corresponding almost complex structures $\sigma_{\mu_n}$, see $\{1\}$, and
the quasiconformal diffeomorphisms (denoted \( \Phi_n \)) from Theorem 1.7 (the latters exist as is proved above). Passing to subsequence, one can assume that they converge uniformly on \( \overline{\mathbb{C}} \) (by Lemma 3.3). Denote \( \Psi \) their limit, which is a quasiconformal homeomorphism transforming the extended complex structure \( \sigma \) to the standard one (Lemma 3.3). It suffices to show that \( \Phi \circ \Psi^{-1} : \Psi(D) \to \Phi(D) \) is a conformal homeomorphism. It is a homeomorphism, since so are \( \Phi \) and \( \Psi \), and preserves the standard complex structure, thus, if we show that it is quasiconformal, this will imply conformality (as is proved in the previous case \( \mu \equiv 0 \)). To do this, consider the homeomorphisms \( h_n = \Phi \circ \Phi_n^{-1} : \Phi_n(D) \to \Phi(D) \). They are quasiconformal homeomorphisms with uniformly bounded dilatations, as in the previous paragraph. They converge to \( \Phi \circ \Psi^{-1} \) uniformly on compact subsets of \( \Psi(D) \). The corresponding pullbacks of the standard complex structure converge to the latter almost everywhere, which follows from definition and convergence \( \mu_n \to \mu \). Hence, by Lemma 3.3, the limit is quasiconformal. Proposition 3.10 is proved. Theorem 1.7 is proved. \( \square \)

**Proof of Proposition 1.3** The statement of the Proposition is local: it suffices to show that compositions (inverses) of local \( K \)-homeomorphisms are \( K^2 \)- (respectively, \( K \)-) quasiconformal. We prove this statement for composition (for inverse the proof is analogous): given domains \( U, V, W \subset \mathbb{C} \) and \( K \)-homeomorphisms \( \Psi : U \to V, \Phi : V \to W \), let us show that \( \Phi \circ \Psi \) is a \( K^2 \)-homeomorphism. By Remark 1.2 the previous statement holds for diffeomorphisms, and in the general case the dilatation of the composition is no greater than \( K^2 \), thus, to prove the quasiconformality means to show that the composition has local \( L_2 \) derivatives. To do this, consider the pullback \( \sigma(\Phi) \) of the standard complex structure under \( \Phi \). Let us extend it to \( \overline{\mathbb{C}} \) without increasing the dilatation and construct a sequence \( \sigma(\Phi_n) \) of \( C^\infty \) almost complex structures on \( \overline{\mathbb{C}} \) converging to \( \sigma(\Phi) \) almost everywhere with dilatations no greater than \( K \). Let \( \Phi_n : \mathbb{C} \to \mathbb{C} \) be the corresponding normalized quasiconformal homeomorphisms (which are \( K \)-homeomorphisms) from Theorem 1.7. They are \( C^\infty \) diffeomorphisms as is proved above. By Lemma 3.3 they converge uniformly on \( \overline{\mathbb{C}} \) to a \( K \)-homeomorphism \( \tilde{\Phi} : \mathbb{C} \to \mathbb{C} \) transforming \( \sigma(\Phi) \) to the standard complex structure. By the previous Proposition, \( \tilde{\Phi} = \Phi \) up to left composition with a conformal mapping. Now the compositions \( \Phi_n \circ \Psi \) are \( K^2 \)-homeomorphisms (since \( \Phi_n \) is \( C^\infty \)) converging to \( \tilde{\Phi} \circ \Psi \), and the corresponding pullbacks of the standard complex structure converge also. Hence, by Lemma 3.3 the limit is quasiconformal. Since the limit coincides with \( \Phi \circ \Psi \) up to composition with a conformal mapping, the latter is quasiconformal too. Proposition 1.3 is proved. \( \square \)

**Proof of Proposition 1.4** The statement of Proposition 1.4 is local and is reduced to the case of quasiconformal homeomorphisms \( \overline{\mathbb{C}} \to \overline{\mathbb{C}} \), as Proposition 1.3 proved above. We prove it by contradiction. Suppose the contrary: some quasiconformal homeomorphism \( h \) of the Riemann sphere sends a zero measure set \( S \) to a positive measure set \( h(S) \) (without loss of generality we assume that \( h \) fixes 0, 1 and \( \infty \)). Let \( \sigma \) be the pull-back under \( h \) of the standard complex structure (it is well-defined almost everywhere). Then \( h \) is the unique normalized quasiconformal homeomorphism transforming \( \sigma \) to the standard complex structure. Let us change the standard structure in the image as follows: on \( h(S) \) we change it to some constant nonstandard almost complex structure; on the rest we keep it standard. Denote \( \sigma' \) the almost complex structure thus obtained on the Riemann sphere in the image. By Theorem 1.7 there exists a unique normalized quasiconformal homeomorphism \( H \) transforming \( \sigma' \) to the standard complex structure. One has \( H \not= Id \), since the set \( h(S) \) has a positive measure. By definition and Proposition 1.3 the composition \( H \circ h \), which is different from
$h$, is a normalized quasiconformal homeomorphism transforming $\sigma$ to the standard complex structure. This contradicts the uniqueness of $h$. Proposition 1.4 is proved.

### 3.5 Analytic dependence on parameter. Proof of the Addendum

**Double-periodic case.** Consider a family of double-periodic $C^\infty$ almost complex structures $\sigma(t)$ on $\mathbb{C}$ depending holomorphically on a complex parameter $t$ (this means that the corresponding function $\mu = \mu(z, t)$ from (1.1) is holomorphic in $t$). We assume that the periods are fixed, thus, $\sigma(t)$ are the lifting to the universal cover $\mathbb{C}$ of an analytic family of almost complex structures on the two-torus. Then the corresponding quasiconformal diffeomorphisms (denoted $\Phi_t$) from Theorem 1.7 are holomorphic in $t$ as well. Indeed, their differentials are uniformizing differentials. Hence, for any $t$, $d\Phi_t = f_t(dz + \mu(z, t)dz)$ up to multiplication by complex constant depending on $t$, where $f_t$ is given by formula (2.8). The right-hand side of (2.8) is analytic in the functional parameter $\mu$, hence, $f_t$ is holomorphic in $t$ and $z \rightarrow \int_0^t f_t(dz + \mu(z, t)dz)$ is a holomorphic family of diffeomorphisms of $\mathbb{C}$. The family $\Phi_t$ is obtained from the latter by multiplication by a function in $t$ that makes the previous diffeomorphisms normalized (fixing 1), hence, the multiplier function (and thus, $\Phi_t$ as well) are also holomorphic in $t$. The Addendum is proved in the double-periodic case.

**General case.** Now consider arbitrary analytic family $\sigma(t)$ of bounded almost complex structures on $\mathbb{C}$ depending on a complex parameter $t$ (we suppose that $t$ runs through the unit disc $D$). Let $\mu(z, t)$ be the corresponding functions, see (1.1), which are holomorphic in $t$. Then there exists a $0 < \delta < 1$ such that $|\mu(z, 0)| < \delta$ for any $z$. The corresponding mapping $M_z : t \mapsto \mu(z, t)$ is a holomorphic mapping $D \rightarrow D$ depending on $z$ in a measurable way such that $|M_z(0)| < \delta$. (Recall that for a given $\delta < 1$ the space of holomorphic mappings $M : D \rightarrow D$ with $|M(0)| < \delta$ is compact, see [CG].) Vice versa, for any $0 < \delta < 1$ each measurable collection of holomorphic mappings $M_z : D \rightarrow D$ with $|M_z(0)| < \delta$ defines an analytic family of bounded almost complex structures; they are uniformly bounded when restricted to a smaller parameter disc $D_r = \{ |t| < r \}$, $r < 1$. Indeed, in the case, when $M_z(0) \equiv 0$, $|M_z|_{D_r} < \delta$ (Schwarz Lemma); the general case is easily reduced to the previous one.

Denote $\Phi_t$ the corresponding normalized quasiconformal homeomorphisms from Theorem 1.7. To prove the analyticity of $\Phi_t$ in $t$, we approximate $\sigma(t)$ (in the sense of convergence almost everywhere) by analytic families $\sigma_n(t)$ of $C^\infty$ double-periodic almost complex structures depending holomorphically on the same parameter $t$ with growing periods $2n, 2in, \sigma_n \rightarrow \sigma$. (For example, consider the restriction of $\sigma$ to the period square centered at 0 and take $\sigma_n$ to be its double-periodic extension. Then approximate the new double-periodic family $M_z$ by a $C^\infty$ family of holomorphic mappings $D \rightarrow D$.) One can do this in such a way that $\sigma_n(t)|_{t \in D_r}$ be uniformly bounded. Denote $\Phi_{n,t}$ the normalized quasiconformal homeomorphisms transforming $\sigma_n(t)$ to the standard complex structure. They depend analytically on $t$, as is proved above. By Lemma 3.3 for any $t, z$, $\Phi_{n,t}(z) \rightarrow \Phi_t(z), as n \rightarrow \infty$. Thus, $\Phi_t(z)$ is a function in $t$ that is a limit of pointwise converging sequence of holomorphic functions. Let us prove that for any fixed $z$ the functions $\Phi_{n,t}(z)$ in $t \in D_r$ are bounded uniformly in $n$: then their limit $\Phi_t(z)$ is holomorphic. Indeed, the almost complex structures $\sigma_n(t)|_{t \in D_r}$ are uniformly bounded. Therefore, the family $\Phi_{n,t}$ (depending on the two parameters $n$ and $t \in D_r$) together with their inverses is equicontinuous (Lemma 3.1). Hence, the previous functions are uniformly bounded, so, their limit is holomorphic. The Addendum is proved.
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