On the fourth moment of a random determinant

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Abstract

In this paper, we generalise the formula for the fourth moment of a random determinant to account for entries with asymmetric distribution. We also derive the second moment of a random Gram determinant.

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A Matrix symbols

1 Introduction

Let $X_{ij}$’s be i.i.d. (independent and identically distributed) random variables and $A = (X_{ij})_{n \times n}$ a random square matrix having these variables as its entries. We are interested in expressing the moments of the determinant $|A|$, that is

$$f_k(n) = E|A|^k,$$  \hspace{1cm} (1)
as a function of moments \( m_r = \text{EX}_{ij}^r \). It is easy to see that \( f_k(n) \) is a polynomial in \( m_1, m_2, \ldots, m_k \). Moreover, when \( k \) is odd (and \( n > 1 \)), then \( f_k(n) \) is automatically zero due to the anti-symmetric property of a determinant. The only nontrivial cases are hence when \( k \) is even. An equivalent formulation of the problem is to find the generating function

\[
F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_k(n),
\]

from which one could deduce \( f_k(n) \) by its Taylor expansion. However, this definition of the generating function makes sense only for \( k \leq 5 \), otherwise it does not in general define an analytic function of \( t \) on an interval containing zero. Although, treated formally, it satisfies

\[
\frac{\partial F_k(t)}{\partial \mu_k} = t F_k(t).
\]

Sometimes, we restrict the distribution of \( X_{ij} \)’s:

- We say \( X_{ij} \)’s follow a **symmetric** distribution, if the odd moments are equal to zero up to the order \( k \) (that is, \( m_{2l+1} = 0 \) for \( 2l + 1 \leq k \)). We denote \( f_{sym}^k(n) \) and \( F_{sym}^k(t) \) the corresponding \( k \)-th moment of the random determinant formed by those random variables, and its generating function, respectively.

- We say \( X_{ij} \)’s follow a **centered** distribution (or equivalently, we say \( X_{ij} \)’s are centered random variables) if \( m_1 = 0 \). For those variables, we consider \( f_{cen}^k(n) \) and \( F_{cen}^k(t) \) in the same way.

The problem of finding the moments of a random determinant was studied extensively in a series of papers published in the 1950s \[4\][3][7][13]. For \( k = 2 \), there is a well known general formula

\[
f_2(n) = n! (m_2 + m_2^2 (n-1))(m_2 - m_1^2)^{n-1}
\]

attributed originally to Fortet \[4\] as a special case of a more general setting, although the formula itself could be derived in a much more elementary way \[11\].

However, no such formula was available for higher moments given \( X_{ij} \)’s being generally distributed, although there are three notable special cases:

1. Nyquist, Rice and Riordan \[7\] derived

\[
F_{sym}^4(t) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^3}.
\]

from which they obtained

\[
f_{sym}^4(n) = (n!)^2 m_2^{2n} \sum_{j=0}^{n} \frac{1}{j!} \left( \frac{m_4}{m_2} - 3 \right)^j \left( \frac{n-j+2}{2} \right).
\]

In fact, this formula holds even if \( X_{ij} \)’s follow just a centered distribution. That is, \( f_{cen}^4(n) = f_{sym}^4(n) \). This is due to the fact that \( m_3 \) appears always as a product \( m_1 m_3 \) in the \( f_4(n) \) polynomial.

2. In the same paper, they also derived that if \( X_{ij} \)’s follow a standard normal distribution, then \( f_k(n) \) could be expressed for any even \( k = 2m \) as

\[
f_{2m}(n) = (n!)^m \prod_{r=0}^{m-1} \left( \frac{n+2r}{2r} \right).
\]

A more elementary derivation of this result was later given by Prékopa \[8\].
3. Just recently, Lv and Potechin \([3]\) also obtained an explicit formula for \(f_{6}^{\text{sym}}(n)\). After some simplifications, their result is equivalent to

\[
f_{6}^{\text{sym}}(n) = (n!)^2 m_2^{3n} \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(n-j)!} \left(\frac{14+j+2i}{j-i}\right) \left(\frac{m_6}{m_2^3} - 15 \frac{m_4}{m_2^2} + 30\right)^{n-j} \left(\frac{m_4}{m_2^2} - 3\right)^{j-i}. \tag{8}\]

However, due to nontrivial \(m_3^2\) terms, \(f_{6}^{\text{sym}}(n)\) and \(f_{6}^{\text{cen}}(n)\) do not generally coincide. Luckily, using the same methods as in their paper, it can be easily derived that

\[
f_{6}^{\text{cen}}(n) = (n!)^2 m_2^{3n} \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(n-j)!} \left(\frac{10}{k}\right) \left(\frac{14+j+2i}{j-i}\right) n_6^{n-j-k} n_4^{j-i} n_3^{k}. \tag{9}\]

where

\[
n_6 = \frac{m_6}{m_2^3} - 10 \frac{m_3^2}{m_2^2} - 15 \frac{m_4}{m_2^2} + 30, \quad n_4 = \frac{m_4}{m_2^2} - 3, \quad n_3 = \frac{m_3^2}{m_2^2}. \tag{10}\]

More generally, denote \(U = (X_{ij})_{n \times p}\) a rectangular matrix with i.i.d. random variable entries \(X_{ij}\). This time, we are interested in expressing the moments of the determinant \(|U^T U|\) as a polynomial of the moments \(m_r = \mathbb{E} X_{ij}^r\) of the entries. For even \(k\), we denote

\[
f_k(n, p) = \mathbb{E}|U^T U|^{k/2} \quad \text{and} \quad F_k(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^n \omega^n - pf_k(n, p) \tag{11}\]

with \(f_k(n, 0) = 1\) by definition (we put \(|U^T U| = 1\) when \(p = 0\)). Notice that, when \(n = p\), we get by the multiplicative property of determinant

\[
f_k(n, n) = f_k(n) \quad \text{and thus} \quad F_k(t, 0) = F_k(t). \tag{12}\]

Again, restricting the distribution of \(X_{ij}\)'s, we write

- **(centered distribution)** \(f_k(n, p) = f_k^{\text{cen}}(n, p)\) and \(F_k(t, \omega) = F_k^{\text{cen}}(t, \omega)\) if \(m_1 = 0\); and similarly
- **(symmetrical distribution)** \(f_k(n, p) = f_k^{\text{sym}}(n, p)\) and \(F_k(t, \omega) = F_k^{\text{sym}}(t, \omega)\) if \(m_1 = m_3 = m_5 = \ldots = 0\).

The fact that \(f_k(n, p)\) is a polynomial in \(m_p\) leads to the important equality

\[
f_k^{\text{cen}}(n, p) = f_k^{\text{sym}}(n, p) \quad \text{valid for} \quad k = 2, 4. \tag{13}\]

When \(k \geq 6\), \(f_k^{\text{cen}}(n, p)\) contains extra products of even powers of odd moments \((m_2^2, \ldots)\). Let us have a quick overview of some special cases. As a simple consequence of Cauchy-Binet formula \([10]\), we have

\[
f_2(n, p) = p! \binom{n}{p} (m_2 + m_2^2 (p-1))(m_2^2 - m_2^2)^{p-1} \tag{14}\]

It turns out, this was the only known general formula. Although, as a special case, Dembo \([1]\) showed

1. For any even \(k\), formally,

\[
\frac{\partial F_k(t, \omega)}{\partial \mu} = t F_k(t, \omega). \tag{15}\]

2. If \(X_{ij}\)'s follow the standard normal distribution, then \((k = 2m \text{ even})\)

\[
f_{2m}(n, p) = p! m^{m-1} \prod_{r=0}^{m-1} \binom{n+2r}{n-p+2r}. \tag{16}\]

However, this result is older and there had even been a generalization of it based on known properties of the non-central Wishart distribution: Let \(X_{ij} \sim \mathcal{N}(\mu, \sigma^2)\), then (see Theorem 10.3.7 in \([6]\))

\[
f_{2m}(n, p) = p! m^{2m} \sigma^{2m} \prod_{r=0}^{m-1} \binom{n+2r}{n-p+2r} \sum_{s=0}^{m} \binom{m}{s} \binom{n-2}{n+2s-2}! \left(\frac{np\mu^2}{\sigma^2}\right)^s. \tag{17}\]
3. For symmetrical distribution of \(X_{ij}\)’s,

\[
    F_{4}^{\text{sym}}(t, \omega) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^2 (1 - \omega - m_2^2 t)}, \quad f_{4}^{\text{sym}}(n, p) = p^2 \binom{n}{p} m_2^p \sum_{j=0}^{p} \frac{1}{j!} \left( \frac{m_4}{m_2^2} - 3 \right)^j \binom{n-j+2}{2}.
\]  

(18)

Note that, letting \(\omega = 0\) (or \(p = n\)), we recover the formulae of Nyquist, Rice and Riordan \([7]\).

\[
    F_{4}^{\text{sym}}(t) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^3}, \quad f_{4}^{\text{sym}}(n) = (nt)^2 m_2^n \sum_{j=0}^{n} \frac{1}{j!} \left( \frac{m_4}{m_2^2} - 3 \right)^j \binom{n-j+2}{2}.
\]  

(19)

**Main results**

The aim of this paper is to generalize the result of Nyquist, Rice and Riordan \([7]\) to express the full \(f_4(n)\). That is, with \(X_{ij}\)’s being generally distributed. Furthermore, we aim to generalize the result of Dembo \([1]\) to express the full \(f_4(n, p)\). We present the following theorems and their corollaries:

**Theorem 1.**

\[
    F_4(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1 - \mu_2^2 t)^3} \left( 1 + \sum_{k=1}^{6} p_k t^k \right),
\]  

(20)

where

\[
    p_1 = m_4^4 + 6m_2^2m_4^2 - 2m_2^4 + 4m_1m_3, \quad p_2 = 7m_4^3 - 6m_2^2m_4^3 + \mu_2^4 + 12m_1m_2m_3 - 8m_1m_2^2m_3 + 6m_1^2m_3^2, \\
    p_3 = 2m_1(2m_4^2 - 6m_2^2m_4 + 2m_2^4m_3 + 3m_1m_2^2m_3^2 - 6m_1m_2^2m_3^2 + 2m_1^2m_3^3), \\
    p_4 = m_2^2(3m_1^2m_3^2 - 6m_1m_2^2m_3 + 6m_1^2m_3^2), \quad p_5 = 2m_1m_2m_3^2(2m_2 - m_1m_3), \quad p_6 = m_1^4m_3^4.
\]

and (central moments)

\[
    \mu_2 = m_2 - m_1^2, \quad \mu_3 = -3m_1m_2 + 2m_3^3, \quad \mu_4 = -4m_1m_3 + 6m_1^2m_2 - 3m_4.
\]

**Corollary 1.1.** Defining \(\mu_j\) as above, we have, by Taylor expansion,

\[
    f_4(n) = (nt)^2 \mu_2^n \sum_{j=0}^{n} \frac{1}{j!} \left( \frac{\mu_4}{\mu_2^2} - 3 \right)^j \sum_{i=2}^{4} q_i \left( \frac{n-j+i}{i} \right).
\]  

(21)

where

\[
    q_{-2} = \frac{m_4^4}{\mu_2^8}, \quad q_{-1} = -4m_2^2m_4^3 \frac{(\mu_2^2 + m_1m_3)}{\mu_2^2}, \quad q_0 = 6m_1^2m_3^2 \frac{(\mu_2^4 + 2m_1m_2m_3 + m_1^2m_3^2 - m_2^2m_3^2)}{\mu_2^2}, \\
    q_1 = 2m_1 \left( 6m_1^2m_3^2m_5 - 2m_1m_2^2m_3^2 - 2m_1^2m_2m_3^2 + 3m_1^2m_3^2m_3 - 6m_1m_2m_3^3 - 6m_1^2m_2m_3^3 - 2m_1^3m_5 \right), \\
    q_2 = 1 + m_1 \left( 19m_1^3m_5 - 6m_1m_2^2 - 24m_1^2m_2m_3^2 + 4m_1^2m_2m_3^2 - 18m_1m_2m_3^3 + 6m_1m_2m_3^3 + 4m_1^2m_2m_3^3 + m_1^3m_5 \right), \\
    q_3 = \frac{3m_1^2(2m_2^3 - 9m_1m_2m_3^2 + 4m_1m_2m_3^2 + 2m_1^2m_3^2)}{\mu_2^5}, \quad q_4 = \frac{12m_1^4}{\mu_2^2}.
\]

**Remark 1.** By definition we put \(\binom{-2}{2} = \binom{-1}{1} = 1\), \(\binom{-1}{-2} = -1\) and \(\binom{j}{-2} = \binom{-1}{j} = 0\), \(j \geq 0\).

**Example 1** (General Gaussian distribution). If \(X_{ij} \sim N(\mu, \sigma^2)\), we have \(m_1 = \mu\), \(m_2 = \mu_2\), \(m_3 = \mu_3\), \(m_4 = \mu_4\), \((q_{-2}, q_{-1}, q_0, q_1, q_2, q_3, q_4) = (0, 0, 0, -4\mu^4, 19\mu^4 - 6\mu^2\sigma^2 + \sigma^4, 6\mu^2\sigma^2 - 27\mu^4, 12\mu^4) / \sigma^4\), from which we get

\[
    f_4(n) = \frac{1}{2}(nt)^2(1 + n)^{\sigma^4(k-1)}(n^3\mu^4 + (2 + n)\sigma^2(2n\mu^2 + \sigma^2)).
\]  

(22)
Example 2. \(((0, 2)\) matrices). Let \(X_{ij} = 0.2\) with equal probability, thus \((m_1, m_2, m_3, m_4) = (1, 2, 4, 8)\) and \((\mu_2, \mu_3, \mu_4) = (1, 0, 1)\). As pointed out by Terence Tao [12], the determinant of a random \(n \times n\) \((-1, +1)\) matrix is equal to the determinant of a random \(n - 1 \times n - 1\) \((0, 2)\) matrix for which \((m_1, m_2, m_3, m_4) = (0, 1, 0, 1)\). In terms of generating functions, that means

\[
F_4(t) = \frac{\partial}{\partial t} \left( t \frac{e^{sym}(t)}{4} \right) = \frac{\partial}{\partial t} \left( t \frac{e^{-2t}}{(1 - t)^3} \right) = \frac{e^{-2t} \left( 1 + 5t + 2t^2 + 4t^3 \right)}{(1 - t)^5},
\]

where in \(e^{sym}(t)\) we put \((m_1, m_2, m_3, m_4) = (0, 1, 0, 1)\). This result coincides exactly with our general formula for \(F_4(t)\) with \((m_1, m_2, m_3, m_4) = (1, 2, 4, 8)\).

Example 3 (Exponential distribution). If \(X_{ij} \sim \text{Exp}(1)\), that is if \(m_j = j!\), we have \((\mu_2, \mu_3, \mu_4) = (1, 2, 9)\) and \((q_1, q_2, q_0, q_1, q_2, q_3, q_4) = (16, -96, 192, -124, -26, 27, 12)\). Using Mathematica, we get an asymptotic behaviour for large \(n\),

\[
f_4(n) \approx \frac{1}{2} e^{6(n!)^2} (450 + 141n - 27n^2 - 5n^3 + n^4).
\]

(24)

The first ten exact moments are shown in Table 1 below.

\[
\begin{array}{cccccccc}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
f_4(n) & 24 & 960 & 51840 & 3511872 & 287953920 & 27988001280 & 27988001280 \\
\hline
n & 8 & 9 & 10 \\
\hline
f_4(n) & 418846663065600 & 6339954982846460 & 10964925305310412800 \\
\hline
\end{array}
\]

Table 1: Fourth moment of a random determinant with entries exponentially distributed.

These tabulated numbers are of particular interest in the field of random geometry. Let us have a \(d\)-dimensional simplex with unit \(d\)-volume, from which we select \(d + 1\) points uniformly and independently. A convex hull of those random points forms a smaller simplex with \(d\)-volume \(V_d\), which is now a random variable. As shown by Reed [9], the even moments of \(V_d\) are given by

\[
EV_d^{2l} = \left( \frac{dt}{(d + 2)!} \right)^{d+1} f_{2l}(d + 1),
\]

(25)

our result applied on \(X_{ij} \sim \text{Exp}(1)\) thus implies an explicit formula for the fourth moment of \(V_d\).

Theorem 2. Defining \(p_k\) and \(\mu_i\) as above, we have

\[
F_4(t, \omega) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1 - \mu_2^2)^4} \left( 1 + \frac{\omega}{1 - \mu_2^2} \sum_{k=1}^{6} \mu_k t^k + \frac{\omega m_1^2}{1 - \omega - \mu_2^2} \sum_{k=1}^{4} \bar{\mu}_k t^k + \frac{2 \omega^2 m_1^4 \mu_2^2 t^2}{(1 - \omega - \mu_2^2)^2} \right).
\]

(26)

where

\[
\bar{\mu}_1 = m_1^2 + 2\mu_2, \quad \bar{\mu}_2 = 5m_1^2 \mu_2^2 + 4m_1 \mu_2 \mu_3 - 2\mu_2^2, \quad \bar{\mu}_3 = 2m_1^2 \mu_2^4 - 4m_1 \mu_2 \mu_3^2 + 2m_1^2 \mu_2 \mu_3 ^2, \quad \bar{\mu}_4 = -2m_1^2 \mu_2^3.
\]

Remark 2. Letting \(\omega = 0\), we recover \(F_4(t)\). On the other hand, letting \(m_1 = 0\), we get \(F_4^{\text{sym}}(t, \omega)\).

Corollary 2.1. Defining \(q_i\) and \(\mu_i\) as above, we get, by Taylor expansion,

\[
f_4(n, p) = p^{2} \mu_2^2 \sum_{i=0}^{p} \frac{1}{i!} \left( \frac{\mu_4}{\mu_2^2} - 3 \right)^i \sum_{j=-2}^{4} \binom{n}{p} \frac{q_i (n-p) + \bar{q}_i (n-p)(n-p+7)}{(n-p+i+1)}.
\]

(27)

where

\[
q_0 = \frac{2 m_1^4}{\mu_2^2}, \quad q_1 = \frac{2 m_1^2 (2 \mu_2^2 \mu_3 + 3 m_1 \mu_2^2 - m_1 \mu_3^2)}{\mu_2^2}, \quad q_2 = \frac{m_1^2 (3 m_1^2 \mu_3^2 - 2 \mu_4 - 8 m_1 \mu_2 \mu_3 - 6 m_1^2 \mu_3^2)}{\mu_2^2},
\]

\[
q_3 = \frac{2 m_1^4}{\mu_2^2}, \quad q_4 = \frac{m_1^4}{\mu_2^2}
\]

and \(q_i, \bar{q}_i\) otherwise zero.
Example 4 (General Gaussian distribution). If $X_{ij} \sim N(\mu, \sigma^2)$, we have $m_1 = \mu$, $(\mu_2, \mu_3, \mu_4) = (\sigma^2, 0, 3\sigma^4)$, which gives, after series of simplifications,

$$f_4(n, p) = \frac{n!(n+1)!\sigma^4(p-1)}{(n-p)!(n-p+2)!} \left( np^2\mu^4 + (n+2) (2p\mu^2\sigma^2 + \sigma^4) \right).$$  \hspace{1cm} (28)

This formula agrees with the general case given by Equation 17.

Example 5 (Exponential distribution). If $X_{ij} \sim \text{Exp}(1)$, that is if $m_j = j!$, we have $(\mu_2, \mu_3, \mu_4) = (1, 2, 9)$ and $(q-2, q-1, q_0, q_1, q_2, q_3, q_4, q_0, q_1, q_2, q_3, q_4, q_5, q_6) = (16, -96, 192, -124, -26, 27, 12, -8, 30, -39, 17, 1, -2, 1)$. The exact moments $f_4(n, p)$ for low $n$ and $p$ are shown in Table 2 below.

| $f_4(n, p)$ | 1   | 2   | 3   | 4   | 5   | 6   |
|-------------|-----|-----|-----|-----|-----|-----|
| $n-p$       |     |     |     |     |     |     |
| 0           | 24  | 960 | 51840 | 3511872 | 287953920 | 27988001280 |
| 1           | 56  | 3744 | 297216 | 27708480 | 3004024320 | 375698373120 |
| 2           | 96  | 9432 | 1022400 | 124675200 | 17182609920 | 2675406827520 |
| 3           | 144 | 19320 | 2724480 | 419207040 | 71341240320 | 13491506810880 |
| 4           | 200 | 34920 | 6189120 | 1169602560 | 240336875520 | 54144163584000 |
| 5           | 264 | 57960 | 12579840 | 2858913792 | 696776048640 | 18409928343360 |
| 6           | 336 | 90384 | 23538816 | 6325119360 | 1801876285440 | 551197391754240 |
| 7           | 416 | 134352 | 41299200 | 12939696000 | 4256462960640 | 1491202996208640 |

Table 2: Second moment of a random Gram determinant with entries exponentially distributed
2 Proof of Theorem

2.1 NRR’s generating function

We briefly discuss what we believe is a simpler derivation of \( F_{\text{sym}}^4(t) \) of Nyquist, Rice and Riordan [7]. We were inspired by the paper of Lv and Potechin [5].

**Lemma 3.** Let \( S_n \) be the set of all permutations of order \( n \) and \( D_n \) the set of all derangements of the same order (that is, \( D_n \) is a subset of those permutations in \( S_n \) which have no fixed points). Denote \( C(\pi) \) the number of cycles in a permutation \( \pi \), then

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\pi \in D_n} u^{C(\pi)} = \frac{e^{-ux}}{(1-x)^u}, \tag{29}
\]

**Proof.** See [2], chapter on Bivariate generating function. ■

**Proposition 4.**

\[
f_4(n) = E |A|^2 = E \sum_{\pi_1, \pi_2, \pi_3, \pi_4 \in S_n} \prod_{r=1}^{4} \left( sgn(\pi_r) \prod_{i=1}^{n} X_{i\pi_r(i)} \right). \tag{30}
\]

**Proof.** Follows from the definition of determinant. ■

The summation above is carried over all permutation fours \( (\pi_r)_{r=1}^{4} \), which can be viewed, the same way as it is in the original article of Nyquist, Rice and Riordan [7], as a sum over all possible permutation tables. Since we assume \( X_{ij} \) follow symmetric distribution, many of the terms vanish.

**Definition 1** (Permutation tables). We say \( t \) is a permutation four-table of length \( n \) if its rows are exactly the permutations \( \pi_r \) of length \( n \). A table \( t \) is called symmetric, if its columns fall into the admissible categories below. Furthermore, we assign weight to each column. The weight \( w(t) \) of the table \( t \) is then simply a product of weights of its columns. Similarly we define sign of a table as a product of sign of the permutations in each row. The admissible columns in symmetric four-tables are:

- 4-columns: four copies of a single number (weight \( m_4 \))
- 2-columns: two pairs of distinct numbers (weight \( m_2^2 \))

We denote \( T_{4,n}^{\text{sym}} \) the set of all symmetric four-tables of length \( n \).

**Remark 3.** To distinguish between tables, we sometimes write \( t_r \) instead of \( \pi_r \) for the rows of \( t \).

**Proposition 5.**

\[
f_4^{\text{sym}}(n) = \sum_{t \in T_{4,n}^{\text{sym}}} w(t) \text{sgn}(t). \tag{31}
\]

**Proof.** Follows from the definitions above. ■

We group the summands according to number of 2-columns in \( t \). Those columns form a subtable \( s \) and the rest of the columns form another, a complementary subtable \( t' \). The signs of those tables are related as

\[
\text{sgn}(t) = \text{sgn}(s) \text{sgn}(t'). \tag{32}
\]

Denote \( [n] = \{1, 2, 3, \ldots, n\} \). For a given \( J \subset [n] \), we define \( T_{4,j}^{\text{sym}} \) a set of all symmetric four-tables of length \( j = |J| \) composed with numbers in \( J \). The set \( T_{4,n}^{\text{sym}} \) coincide then with \( T_{4,[n]}^{\text{sym}} \). Denote \( D_{4,J} \) the set of all four-tables composed only from 2-columns of numbers in \( J \). We can write our sum, since the selection \( J \) does not depend on position in table \( t \), as

\[
f_4^{\text{sym}}(n) = \sum_{J \subset [n]} \binom{n}{j} \sum_{t' \in T_{4,[n]}^{\text{sym}} / J} w(t) \text{sgn}(t) \sum_{s \in Q_{4,J}} w(s) \text{sgn}(s). \tag{33}
\]
No matter which numbers \( j \) are selected, as long as we select the same amount of them, the contribution is the same. Hence,

\[
f_4^{\text{sym}}(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{t' \in T_{4,n-j}} w(t) \text{sgn}(t) \sum_{s \in D_{4,j}} w(s) \text{sgn}(s),
\]

(34)

where \( Q_{4,j} = Q_{4,|j|} \). For the first inner sum, notice that table \( t' \) is composed of only four-columns, so \( w(t') = m_{4-j}^4 \) and \( \text{sgn}(t') = (\pm 1)^4 = 1 \). Also note that \( |T_{4,n-j}| = (n-j)! \). For the second inner sum, by symmetry, we can fix the first permutation in \( s \) to be identity, giving us the factor of \( j! \). Upon noticing also that \( w(s) = m_2^{2j} \) and \( \text{sgn}(s) = (\pm 1)^2 = 1 \), we get,

\[
f_4^{\text{sym}}(n) = \sum_{j=0}^{n} \binom{n}{j}^2 (n-j)!m_{4-j}^4 m_2^{2j} \sum_{s \in D_{4,j}\atop s_1 = \text{id}} 1.
\]

(35)

We group the summands according to the following permutation structure: Let \( b \) be a number in the first row of a given column of table \( s \). Since it is a 2-column, we denote the other number in the column as \( b' \). We construct a permutation \( \pi(s) \) to a given table \( s \) as composed from all those pairs \( b \to b' \). Note that since \( b \) and \( b' \) are allways different, the set off all \( \pi(s) \) corresponds to the set \( D_1 \) of all derangements. Since there are 3 possibilities how to arrange the leftover 3 numbers in the 2-columns corresponding to a given cycle of \( \pi(s) \), we get

\[
\sum_{s \in D_{4,j}\atop s_1 = \text{id}} 1 = \sum_{\pi \in D_1} 3^{C(\pi)}.
\]

(36)

Hence, in terms of generating functions,

\[
F_4^{\text{sym}}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_4^{\text{sym}}(n) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j}^2 \frac{(m_4 t)^{n-j} (m_2 t)^j}{(n-j)! j!} \sum_{\pi \in D_1} 3^{C(\pi)} = \sum_{j=0}^{\infty} \frac{(m_4 t)^j}{j!} \sum_{\pi \in D_1} 3^{C(\pi)} = \sum_{j=0}^{\infty} \frac{(m_4 t)^j}{j!} \sum_{\pi \in D_1} 3^{C(\pi)} = e^{m_4 t} \sum_{j=0}^{\infty} \frac{(m_2 t)^j}{j!} e^{-3m_2^2 t} = e^{m_4 t} (1 - m_2^2 t)^{-3}.
\]

(37)

The final equality is a special case of Lemma 6.

### 2.2 Matrix determinant lemma

The proof of Theorem 1 relies on the fact that \( f_4^{\text{even}}(n) = f_4^{\text{sym}}(n) \) combined with the following key lemma:

**Lemma 6.** Let \( C = (c_{ij})_{n \times n} \) be any real matrix, \( u = (u_i)_{n \times 1} \), \( v = (v_i)_{n \times 1} \) real vectors and \( \lambda \in \mathbb{R} \), then

\[
|C + \lambda uv^T| = |C| + \lambda v^T C^\text{adj}u,
\]

(38)

where \( (C^\text{adj})_{ij} = (-1)^{i+j} |C|_{ij} \) is called the adjugate matrix of \( C \) and \( C_{ij} \) denotes a matrix formed from \( C \) by deleting its \( j \)-th row and \( i \)-th column, as usual.

**Proof.** In fact, the lemma is a special case of the **Weinsteins–Aronszajn identity**. To see this, consider

\[
|C + \lambda uv^T| = |C| |I + \lambda C^{-1} uv^T| = |C| |I + \lambda v^T C^{-1} u| = |C| (1 + \lambda v^T C^{-1} u) = |C| + \lambda v^T C^\text{adj}u.
\]

(39)

By continuity, we conclude that the lemma holds even for \( C \) being noninvertible. \( \blacksquare \)

**Definition 2** \((Y_{ij}, \mu_r)\). We denote \( Y_{ij} = X_{ij} - m_1 \) and \( \mu_r = EY_{ij}^T \).

**Remark 4.** Clearly, \( Y_{ij}'s \) are centered i.i.d. random variables with moments depending on \( m_j \) as such

\[
\mu_1 = 0, \quad \mu_2 = m_2 - m_1^2, \quad \mu_3 = m_3 - 3m_1 m_2 + 2m_1^3, \quad \mu_4 = m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4,
\]

(40)

and so on.
Definition 3 (B, g_k(n), G_k(t)). Given Y_i j’s, we form a matrix \( B = (Y_{ij})_{n \times n} \) and denote \( g_k(n) = E |B|^k \) and

\[
G_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} g_k(n). \tag{41}
\]

Remark 5. Since the moments of a random determinant are dependent only on moments of its random entries, we get that \( g_k(n) \) is equal to \( f_k^{\text{cen}}(n) \) in which we replace \( m_i \) by \( \mu_i \). So, for \( k = 4 \),

\[
G_4(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1 - \mu_2^2 t)^3}. \tag{42}
\]

Proposition 7.

\[
|A| = |B| + m_1 S, \quad \text{where} \quad S = \sum_{ij} (-1)^{i+j} |B_{ij}|. \tag{43}
\]

Proof. By definition of \( Y_{ij} \)’s and \( B \), we can write

\[
A = B + m_1 u u^T, \tag{44}
\]

where \( u \) is a column vector with \( n \) rows having all components equal to one. Hence, by Lemma 6,

\[
|A| = |B + m_1 u u^T| = |B| + m_1 u^T B \text{adj} u = |B| + m_1 \sum_{ij} u_i (-1)^{i+j} |B_{ij}| u_j = |B| + m_1 S. \tag{45}
\]

\[\blacksquare\]

Corollary 7.1. We thus get an expression for \( f_4(n) \) in terms of the following summands

\[
f_4(n) = E |A|^4 = E (|B| + m_1 S)^4 = E |B|^4 + 4 m_1 E |B|^3 S + 6m_1^2 E |B|^2 S^2 + 4 m_1^3 E |B| S^3 + m_1^4 E S^4. \tag{46}
\]

Remark 6. The first summand is trivial, since we already know that \( E |B|^4 = g_4(n) \). The goal of the rest of our paper is to express the other summands in terms of \( g_4(n) \) as well. This was be possible due to the crucial fact that \( B \) now has only centered random entries \( Y_{ij} \)’s. The main tool to obtain such relations is using the Laplace expansion of determinants via their rows (or columns) repeatedly.

Definition 4 (\( B_{ij,k,l} \), matrix symbols). We denote \( B_{ij,k,l} \) a matrix \( B \) from which the rows \( i, j \) and columns \( k, l \) were deleted. To improve readability, we adopt a graphical notation (matrix symbols) for determinants \( |B_{ij}| \) and \( |B_{ij,k,l}| \). We write, for example,

\[
\begin{bmatrix}
\end{bmatrix} = |B_{22}|, \quad \begin{bmatrix}
\end{bmatrix} = |B_{23,24}|, \quad \begin{bmatrix}
\end{bmatrix} = \begin{bmatrix}
\end{bmatrix} = |B_{12}|.
\]

A row painted in black (or a column as in the example above) shows where the Laplace expansion is being performed in the next step.

Remark 7. Table 6 in the appendix shows all matrix symbols used in this paper.

2.3 Second summand

Proposition 8.

\[
E |B|^3 S = n^2 \mu_3 g_4(n - 1). \tag{47}
\]

Proof. By symmetry, \( E |B|^3 S = n^2 E |B|^3 |B_{11}| \), that is\[
E |B|^3 S = n^2 E \left( \begin{bmatrix}
\end{bmatrix} \right)^3 \begin{bmatrix}
\end{bmatrix} = n^2 \mu_3 E \left( \begin{bmatrix}
\end{bmatrix} \right)^4 - n^2 (n - 1) \mu_3 E \left( \begin{bmatrix}
\end{bmatrix} \right)^3 \begin{bmatrix}
\end{bmatrix} = n^2 \mu_3 g_4(n - 1). \tag{48}\]

\[\blacksquare\]
2.4 Third summand

Proposition 9.

\[ E|B|^2 S^2 = n^2 h_0(n) + n^2(n - 1)^2 \mu_3^2 g_4(n - 2), \]

where \( h_0(n) \) satisfies the recurrence relation

\[ h_0(n) = \mu_2 g_4(n - 1) + (n - 1)^2 \mu_2 h_0(n - 1). \]

Proof. By definition of \( S \), we have

\[ E|B|^2 S^2 = \sum_{ijkl} (-1)^{i+j+r+t} E|B|^2 |B_{ij}||B_{kl}|. \]

The terms \( E|B|^2 |B_{ij}||B_{kl}| \) in the sum above form equivalence classes in which each member has the same contribution (up to a sign). Each class is characterised by having the same relative arrangement of pairs of indices \((ij)\) and \((kl)\) in the \( n \times n \) matrix grid. Representants drawn from each class together with their signs and values denoted \( h_i(n) \) are shown in Table 3 below (the diagrams represent the relative arrangement of indices for a given representant). The table also shows the total number of terms in the same equivalence class (size of a class).

| sign | + | - | + |
|------|---|---|---|
| class | [ ] | [ ] | [ ] |
| size | \( n^2 \) | \( 2n^2(n - 1) \) | \( n^2(n - 1)^2 \) |
| value | \( h_0(n) \) | \( h_1(n) \) | \( h_2(n) \) |

Table 3: Classes of equivalent terms in the third summand

Thus, \( E|B|^2 S^2 = n^2 h_0(n) - 2n^2(n - 1)h_1(n) + n^2(n - 1)^2h_2(n) \) (52)

with \( h_0(n) = E|B|^2 |B_{11}|^2, \quad h_1(n) = E|B|^2 |B_{11}| |B_{12}|, \quad h_2(n) = E|B|^2 |B_{11}| |B_{12}|. \) (53)

We shall now perform the Laplace expansion on those terms until we get a recurrence relation.

\[ h_0(n) = E|B|^2 |B_{11}|^2 = E \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array} \right)^2 \left( \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{array} \right)^2 = \mu_2 E|B_{11}|^4 + (n - 1)\mu_2 E \left( \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{array} \right)^2 \left( \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{array} \right)^2 = \]

\[ = \mu_2 g_4(n - 1) + (n - 1)^2 \mu_2 E \left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{array} \right)^2 \left( \begin{array}{c} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{array} \right)^2 = \mu_2 g_4(n - 1) + (n - 1)^2 \mu_2 h_0(n - 1). \]

\[ h_1(n) = E|B|^2 |B_{11}| |B_{12}| = E \left( \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{array} \right)^2 \left( \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \right)^2 = \]

\[ = \mu_2 E \left( \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \end{array} \right)^3 \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{array} \right) + \mu_2 E \left( \begin{array}{c} \mu_1 \\ \mu_2 \\ \mu_3 \end{array} \right)^3 \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \right) + (n - 2)\mu_2 E \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \right)^2 \left( \begin{array}{c} \pi_1 \\ \pi_2 \\ \pi_3 \end{array} \right)^2 = \]

\[ = (n - 1)(n - 2)\mu_2 \mu_3 E \left( \begin{array}{c} \rho_1 \\ \rho_2 \\ \rho_3 \end{array} \right)^2 \left( \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array} \right)^2 = 0. \]

\[ h_2(n) = E|B|^2 |B_{11}| |B_{12}| = E \left( \begin{array}{c} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{array} \right)^2 \left( \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array} \right)^2 = \]

\[ = \mu_3 E \left( \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{array} \right)^3 \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right) - (n - 2)\mu_3 E \left( \begin{array}{c} \nu_1 \\ \nu_2 \end{array} \right)^2 \left( \begin{array}{c} \pi_1 \\ \pi_2 \end{array} \right)^2 = \mu_3 \left[ E|B|^3 |B_{11}| \right]_{n \rightarrow n - 1} = \mu_3 g_4(n - 2). \]

\[ \blacksquare \]
2.5 Fourth summand

Proposition 10.

\[ \mathbb{E}|B|^3 = 3n^2(n-1)^2 \mu_3 h_0(n-1) + n^2(n-1)^2(n-2)^2 \mu_3 g_4(n-3). \]  

(57)

Proof.

\[ \mathbb{E}|B|^3 = \sum_{ijklrs} (-1)^{i+j+k+l+r+s} |E|B_{ij}|B_{kl}|B_{rs}|. \]  

(58)

The following Table 4 summarizes all the possible classes of terms according to the arrangement of \((ij),(kl),(rs)\) indices. Note that there are some representants whose value is trivially zero (they contain a row or a column such that the expansion in which gives zero). Thus

\[ \mathbb{E}|B|^3 = 3n^2(n-1)^2 h_3(n) + 6n^2(n-1)^2 h_4(n) + 6n^2(n-1)^2(n-2) h_5(n) + n^2(n-1)^2(n-2)^2 h_6(n). \]  

(59)

| sign | class | size | value |
|------|-------|------|-------|
| +    |      | \(n^2\) | 0     |
| -    |      | \(6n^2(n-1)\) | 0     |
| +    |      | \(3n^2(n-1)^2\) | \(h_3(n)\) |
| +    |      | \(6n^2(n-1)^2\) | \(h_4(n)\) |
| +    |      | \(6n^2(n-1)^2(n-2)\) | \(h_5(n)\) |
| +    |      | \(n^2(n-1)^2(n-2)^2\) | \(h_6(n)\) |
| -    |      | \(2n^2(n-1)(n-2)\) | 0     |

Table 4: Classes of equivalent terms in the fourth summand

We now proceed to expand the values of the nontrivial representants until we get recurrence relations.

\[ h_3(n) = \mathbb{E}|B||B_{11}|^2|B_{22}| = \mathbb{E} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \mu_3 \mathbb{E} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \mu_3 \left[ \mathbb{E}|B|^2|B_{11}|^2 \right]_{n\to n-1} = \mu_3 h_0(n-1). \]  

(60)

\[ h_4(n) = \mathbb{E}|B||B_{12}||B_{21}||B_{22}| = \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = (n-2)\mu_3 \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0, \]  

(61)

\[ h_5(n) = \mathbb{E}|B||B_{11}||B_{13}||B_{22}| = \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = (n-2)\mu_3 \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 0, \]  

(62)

\[ h_6(n) = \mathbb{E}|B||B_{11}||B_{22}||B_{33}| = \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \mu_3 \mathbb{E} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \mu_3 h_2(n-1) = \mu_3 g_4(n-3). \]  

(63)

2.6 Fifth summand

Proposition 11.

\[ \mathbb{E}S^4 = n^2 g_4(n-1) + 6n^2(n-1)^2 \mu_2 h_0(n-1) + 3n^2(n-1)^2 h_0(n) + 6n^2(n-1)^2 h_10(n) + + 6n^2(n-1)^2(n-2)^2 \mu_2^2 h_0(n-2) + n^2(n-1)^2(n-2)^2(n-3)^2 \mu_3^3 g_4(n-4), \]  

(64)

where \(h_9(n)\) and \(h_{10}(n)\) satisfy the recurrence relations

\[ h_9(n) = \mu_2 h_0(n-1) + (n-2)^2 \mu_2^2 h_0(n-2) + (n-2)^2 \mu_2^2 h_9(n-1), \]  

(65)

\[ h_{10}(n) = (n-2)^2 \mu_2^3 h_0(n-2) + (n-2)^2 \mu_2^3 h_{10}(n-1). \]
Proof.

\[ \mathbb{E} S^4 = \sum_{ijklrstuv} (-1)^{i+j+k+l+r+s+u+v} \mathbb{E} |B_{ij}||B_{kl}||B_{rs}||B_{uv}|. \] (66)

As in the previous cases, we have summarised the representatives of all classes in Table 5. Again note that the values of some of them are trivially zero.

| sign class | size value |
|------------|------------|
| +          | \( n^2 \)  |
| +          | \( 6n^2(n-1) \) |
| +          | \( 3n^2(n-1)^2 \) |
| +          | \( 6n^2(n-1)^2 \) |
| -          | \( 12n^2(n-1)(n-2) \) |
| -          | \( 12n^2(n-1)^2(n-2) \) |
| +          | \( 6n^2(n-1)^2(n-2)^2 \) |
| +          | \( 24n^2(n-1)^2(n-2)^2 \) |

| sign class | size value |
|------------|------------|
| -          | \( 24n^2(n-1)^2(n-2) \) |
| +          | \( 6n^2(n-1)^2(n-2)^2 \) |
| +          | \( 6n^2(n-1)^2(n-2)(n-3) \) |
| -          | \( 12n^2(n-1)^2(n-2)(n-3) \) |
| +          | \( 2n^2(n-1)(n-2)(n-3) \) |
| +          | \( n^2(n-1)^2(n-2)^2(n-3)^2 \) |
| +          | \( 24n^2(n-1)^2(n-2)(n-3)^2 \) |
| +          | \( 8n^2(n-1)^2(n-2)(n-3) \) |

| sign class | size value |
|------------|------------|
| -          | \( 0 \)  |
| +          | \( 0 \)  |
| +          | \( 0 \)  |
| -          | \( 0 \)  |
| +          | \( 0 \)  |
| +          | \( 0 \)  |
| +          | \( 0 \)  |
| -          | \( 0 \)  |

Table 5: Classes of equivalent terms in the fifth summand

Employing the Laplace expansion on the nontrivial terms, we obtain

\[ h_7(n) = \mathbb{E} |B_{11}|^4 \mathbb{E} |B_{12}|^4 \mathbb{E} |B_{22}|^4 \mathbb{E} |B_{11}|^2 |B_{12}|^2 |B_{22}|^2 = 94(n-1). \] (67)

\[ h_8(n) = \mathbb{E} |B_{11}|^2 |B_{12}|^2 \mathbb{E} = (n-1)^2 |B_{11}|^2 |B_{12}|^2 |B_{22}|^2 = (n-1)^2 |B_{11}|^2 |B_{12}|^2 |B_{22}|^2. \] (68)

\[ h_9(n) = \mathbb{E} |B_{11}|^2 |B_{22}|^2 = \mu_2 \mathbb{E} |B_{11}|^2 |B_{22}|^2 = \mu_2 |B_{11}|^2 |B_{22}|^2 = \mu_2 |B_{11}|^2 |B_{22}|^2. \] (69)

\[ h_{10}(n) = \mathbb{E} |B_{11}| |B_{12}| |B_{22}| = (n-2)^2 |B_{11}| |B_{12}| |B_{22}| = (n-2)^2 |B_{11}| |B_{12}| |B_{22}|. \] (70)

\[ h_{11}(n) = \mathbb{E} |B_{11}|^2 |B_{12}| |B_{13}| = -(n-1) |B_{11}|^2 |B_{12}| |B_{13}| = -(n-1) |B_{11}|^2 |B_{12}| |B_{13}| = 0. \] (71)

\[ h_{12}(n) = \mathbb{E} |B_{11}|^2 |B_{22}| |B_{23}| = (n-2) |B_{11}|^2 |B_{22}| |B_{23}| = (n-2) |B_{11}|^2 |B_{22}| |B_{23}| = 0. \] (72)
\[
\begin{align*}
    h_{13}(n) &= E|B_{11}|^2|B_{22}|B_{33}| = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 \\
    &= \mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-3)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = (n-3)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{14}(n) &= E|B_{11}|^2|B_{22}|B_{33}| = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-3)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{15}(n) &= E|B_{11}|^2|B_{22}|B_{13} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-2)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{16}(n) &= E|B_{11}|^2|B_{22}|B_{13} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-3)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{17}(n) &= E|B_{11}|^2|B_{23} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-2)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{18}(n) &= E|B_{11}|^2|B_{23}|B_{14} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-3)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{19}(n) &= E|B_{11}|^2|B_{24} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-2)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0, \\
    h_{20}(n) &= E|B_{11}|^2|B_{23}|B_{14} = E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 - (n-4)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = (n-4)\mu_3 E \left( \begin{array}{c}
        H_1 \\
        H_2 \\
        H_3
    \end{array} \right)^2 = 0.
\end{align*}
\]

\subsection*{2.7 Conclusion}

\textbf{Definition 5} \((H_0(t), H_9(t), H_{10}(t))\). Given \(h_0(n), h_9(n), h_{10}(n)\) as before, we define auxiliary generating functions

\[
H_0(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} h_0(n), \quad H_9(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} (n-1)^2 h_9(n), \quad H_{10}(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} (n-1)^2 h_{10}(n).
\]

\begin{proposition}

\begin{align*}
    H_0(t) &= \frac{\mu_2 tG_4(t)}{1 - \mu_2^2 t}, \\
    H_9(t) &= \mu_2^2 t^3 \frac{(1 + \mu_2^2 t)G_4(t)}{(1 - \mu_2^2)^2}, \\
    H_{10}(t) &= \frac{\mu_2^3 t^4 G_4(t)}{(1 - \mu_2^2)^2}.
\end{align*}
\end{proposition}

\begin{proof}

By summing up the recurrence relations for \(h_0(n), h_9(n), h_{10}(n)\) in Propositions 9 and 11, we get

\[
\begin{align*}
    H_0(t) &= \mu_2 tG(t) + \mu_2^2 tH_0(t), \\
    H_9(t) &= \mu_2 tH_0(t) + \mu_2^3 t^2 H(t) + \mu_2^3 tH(t), \\
    H_{10}(t) &= \mu_2^3 t^2 H_0(t) + \mu_2^3 tH_10(t).
\end{align*}
\]

By using simple algebraic manipulations, we get the desired statement.
\end{proof}

\textbf{Corollary 12.1.} \textit{By using Corollary 7.1 and Propositions 8, 9, 10, 11 and 12, we get, by summation,}

\[
F_4(t) = G_4(t) + 4m_1 \mu_3 G_4(t) + 6m_1^2 (H_0(t) + \mu_3^2 tG_4(t)) + 4m_1^3 (3\mu_3 tH_0(t) + \mu_3^3 t^2 G_4(t)) + m_1^4 (tG_4(t) + 6\mu_2 tH_0(t) + 3H_0(t) + 6H_1(t) + 6\mu_3 t^2 H_0(t) + \mu_3^3 t^4 G_4(t)),
\]

\textit{from which Theorem 7 follows immediately.}
3 Proof of Theorem 2

3.1 Generating function expansion

We can expand \( F_4(t, \omega) \) in a form of a Taylor-like series in \( t \) and \( \omega \),

\[
F_4(t, \omega) = \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \Phi_j(t). \tag{85}
\]

We claim this is without loss of generality. To see this, perform the Taylor expansion of the bracket, write \( \Phi_j(t) \) as a series in \( t \) and then compare the \( t \) and \( \omega \) coefficients with \( f_4(n, p) \) in (11). This particular choice of expansion was made to make the binomial transform in \( \omega \) behave nicely, we have

\[
\mathbb{E} [F_4(t, \omega)] = \frac{1}{1 - \omega} F_4\left(t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) = \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \mu_2^j t^j \Phi_j(t). \tag{86}
\]

The proof of Theorem 2 relies on a crucial fact that \( \Phi_j(t) = 0 \) for \( j \geq 3 \). \tag{87}

That is,

\[
F_4(t, \omega) = \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \left( \Phi_0(t) + \omega \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \Phi_1(t) + \omega^2 \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^2 \Phi_2(t) \right). \tag{88}
\]

The remaining functions \( \Phi_0(t), \Phi_1(t), \Phi_2(t) \) can be then found just by the methods used proving 1. Specially,

\[
\Phi_0(t) = F_4(t, 0) = F_4(t) = \frac{e^{t (\mu_4 - 3 \mu_2^2)}}{(1 - \mu_2^2 t)^3} \left( 1 + \sum_{k=1}^{6} p_k t^k \right). \tag{89}
\]

To show (87), we use Cauchy-Binet formula.

3.2 Cauchy-Binet formula

**Proposition 13** (Cauchy-Binet formula). Let \( C = (c_{ij})_{n \times p} \) and \( D = (d_{ij})_{n \times p} \) be real matrices and \( C_{(i_1, i_2, \ldots, i_p)} \) and \( D_{(i_1, i_2, \ldots, i_p)} \) be square matrices formed from those by selecting the rows \( i_1, i_2, \ldots, i_p \), then

\[
|C^T D| = \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq n} |C_{(i_1, i_2, \ldots, i_p)}||D_{(i_1, i_2, \ldots, i_p)}|. \tag{90}
\]

*Note that there is an equivalent formulation using deleting rows instead of selecting.* Namely, denoting \( C_{(i_1, i_2, \ldots, i_p)} \) a matrix formed from \( C \) by deleting its rows \( i_1, \ldots, i_p \), we have then

\[
|C^T D| = \sum_{1 \leq i_1 < i_2 < \ldots < i_p < n} |C_{(i_1, i_2, \ldots, i_{n-p})}||D_{(i_1, i_2, \ldots, i_{n-p})}|. \tag{91}
\]

**Remark 8.** If \( p > n \), \( |C^T D| = 0 \) automatically. That means also that \( f_k(n, p) = 0 \) whenever \( p > n \). As stated earlier, the formula offers a simple derivation of \( f_2(n, p) \). This is done by choice \( C = D = U \), so

\[
|U^T U| = \sum_{1 \leq i_1 < i_2 < \ldots < i_p < n} |U_{(i_1, i_2, \ldots, i_p)}|^2. \tag{92}
\]

Then, taking the expectation and by linearity, we get \( \binom{n}{p} \) identical terms, each attending the value \( 1 \),

\[
\mathbb{E} |U_{(1, 2, \ldots, p)}|^2 = f_2(p) = p!(m_2 + m_1^2(p - 1))(m_2 - m_1^2)^{p-1}. \tag{93}
\]

hence

\[
\mathbb{E} |U^T U| = p!(\binom{n}{p})(m_2 + m_1^2(p - 1))(m_2 - m_1^2)^{p-1}. \tag{94}
\]

Somewhat similarly, to derive \( \mathbb{E} |U^T U|^2 \), we just square (92) and take the expectation repeatedly.
3.3 Dembo’s generating function

To illustrate the squaring technique, we rederive Dembo’s formula for $\Phi^\text{sym}_4(t, \omega)$. First, squaring (92),

$$|U^T U|^2 = \sum_{\scriptsize \begin{array}{c} 1 \leq i_1 < i_2 < \ldots < i_q \leq n \\ 1 \leq i'_1 < i'_2 < \ldots < i'_p \leq n \end{array}} |U_{(i_1, i_2, \ldots, i_q)}|^2 |U_{(i'_1, i'_2, \ldots, i'_p)}|^2.$$  

(95)

**Definition 6** ($c_{p,q}$). Given two identical copies of the set $\{1, 2, 3, \ldots, n\}$, we denote $c_{p,q}$ the number of ways how we can select $p$ numbers from the first copy and other $p$ numbers from the second copy, provided that exactly $q$ numbers in both selections were chosen simultaneously. Using standard combinatorics,

$$c_{p,q} = \binom{n}{q} \binom{n-q}{p-q} \binom{n-p}{p-q} = \frac{n!}{q!(p-q)!(n-2p+q)!}.$$  

(96)

**Definition 7** ($\mathcal{U}^{[q]}$, $\check{\mathcal{U}}^{[q]}$). Denote $\mathcal{U}^{[q]} = (X_{ij})_{p \times p}$ and $\check{\mathcal{U}}^{[q]} = (\check{X}_{ij})_{p \times p}$ a random pair of $p$ by $p$ square matrices being identical in the first $q$ columns, that is $X_{ij} = \check{X}_{ij}$ for $j \leq q$ and all $i$. Otherwise, in columns $j > q$, we assume $\check{X}_{ij}$ are independent from each other and from all $X_{ij}$’s, following the same distribution.

**Definition 8** ($\mathbb{E}'$). We denote $\mathbb{E}'$ the conditional expectation taken with respect only to the entries in the $j > q$ columns of a random matrix pair. By properties of conditional expectations, $\mathbb{E} = \mathbb{E} \mathbb{E}'$.

Taking expectation of (95), transposing each matrix and collecting identical terms, we get

$$f_4(n, p) = \mathbb{E} |U^T U|^2 = \sum_{q=0}^{p} c_{p,q} \mathbb{E} |\mathcal{U}^{[q]}|^2 |\check{\mathcal{U}}^{[q]}|^2.$$  

(97)

Now, we use the key assumption that $X_{ij}$’s follow a symmetrical distribution, that is $m_1 = m_3 = 0$ and $f_4(n, p) = \Phi^\text{sym}_4(n, p)$. Expanding the independent columns of $\mathcal{U}^{[q]}$ and $\check{\mathcal{U}}^{[q]}$, respectively and taking $\mathbb{E}'$,

$$\mathbb{E}' |\mathcal{U}^{[q]}|^2 = \mathbb{E}' |\check{\mathcal{U}}^{[q]}|^2 = (p-q)! m_2^{p-q} \sum_{1 \leq i_1 < i_2 < \ldots < i_q \leq p} |\mathcal{U}^{[q]}_{(i_1, i_2, \ldots, i_q)}|^2 = (p-q)! m_2^{p-q} |\mathcal{U}^T \mathcal{U}|.$$  

(98)

where in the last step we used Cauchy-Binet formula again and denoted $\mathcal{U}'$ a $p \times q$ matrix formed from $\mathcal{U}^{[q]}$ by selecting its first $q$ columns. Therefore

$$\mathbb{E} |\mathcal{U}^{[q]}|^2 |\check{\mathcal{U}}^{[q]}|^2 = \mathbb{E} \left[ \mathbb{E}' |\mathcal{U}^{[q]}|^2 |\mathbb{E}' |\check{\mathcal{U}}^{[q]}|^2 \right] = (p-q)!^2 m_2^{2(p-q)} \mathbb{E} |\mathcal{U}^T \mathcal{U}'|^2 = (p-q)!^2 m_2^{2(p-q)} f_4(p, q).$$  

(99)

Inserting the result into (97), we get the recurrence relation

$$\Phi^\text{sym}_4(n, p) = \sum_{q=0}^{p} \frac{n! m_2^{2(p-q)} f_4^{\text{sym}}(p, q)}{q!(n-2p+q)!}. $$  

(100)

This is, inserting to (11) and by straightforward manipulations, equivalent to

$$\Phi^\text{sym}_4(t, \omega) = \frac{1}{1-\omega} \Phi^\text{sym}_4 \left( t, \frac{\omega m_2^2 t}{1-\omega} \right) = \mathbb{E} \left[ \Phi^\text{sym}_4 \right](t, \omega).$$  

(101)

Using our ansatz for generating functions, namely

$$\Phi^\text{sym}_4(t, \omega) = \frac{1}{1-\omega-m_2^2 t} \Phi^\text{sym}_4(t, 0) = \frac{1}{1-\omega-m_2^2 t} \Phi^\text{sym}_4(t, 0) = \frac{1}{1-\omega-m_2^2 t} F_4^\text{sym}(t, \omega) = \frac{e^{t(m_4-3m_2^2)}}{(1-m_2^2 t)^2(1-\omega-m_2^2 t)}.$$  

(105)
3.4 Matrix resolvents

**Definition 9** (\(V, V'\)). Similarly as for \(U\) and \(U'\), which are given as

\[
U = \begin{pmatrix}
X_{11} & \cdots & X_{1p} \\
\vdots & \ddots & \vdots \\
X_{n1} & \cdots & X_{np}
\end{pmatrix}
\quad \text{and} \quad
U' = \begin{pmatrix}
X_{11} & \cdots & X_{1q} \\
\vdots & \ddots & \vdots \\
X_{p1} & \cdots & X_{pq}
\end{pmatrix},
\]

we denote

\[
V = \begin{pmatrix}
Y_{11} & \cdots & Y_{1p} \\
\vdots & \ddots & \vdots \\
Y_{n1} & \cdots & Y_{np}
\end{pmatrix}
\quad \text{and} \quad
V' = \begin{pmatrix}
Y_{11} & \cdots & Y_{1q} \\
\vdots & \ddots & \vdots \\
Y_{p1} & \cdots & Y_{pq}
\end{pmatrix}.
\]

**Definition 10** (\(\UP, \UPT, \V, \V'\)). Denote \(\UP\) an \(n \times (p + 1)\) matrix formed from \(U\) by attaching to it \((p+1)\)-th column filled with 1's. Similarly, \(\UPT\) be a \(p \times (q+1)\) matrix formed from \(U'\) by the same way. Symbolically,

\[
\UP = \begin{pmatrix}
X_{11} & \cdots & X_{1p} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
X_{n1} & \cdots & X_{np} & 1
\end{pmatrix}
\quad \text{and} \quad
\UPT = \begin{pmatrix}
X_{11} & \cdots & X_{1q} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
X_{p1} & \cdots & X_{pq} & 1
\end{pmatrix}.
\]

Similarly, we denote

\[
\V = \begin{pmatrix}
Y_{11} & \cdots & Y_{1p} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{n1} & \cdots & Y_{np} & 1
\end{pmatrix}
\quad \text{and} \quad
\V' = \begin{pmatrix}
Y_{11} & \cdots & Y_{1q} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{p1} & \cdots & Y_{pq} & 1
\end{pmatrix}.
\]

And finally, as a special case when \(p = 0\), we write

\[
\UP = \V = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T \quad \text{and} \quad \UPT = \V' = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}^T q.
\]

**Lemma 14.**

\[
\UP^T \UP = \V^T \V
\]

**Proof.** Using Cauchy-Binet formula,

\[
\|\UP^T \UP\| = \sum_{1 \leq i_1 < i_2 < \ldots < i_{p+1} \leq n} |\UP(i_{i_1}, i_{i_2}, \ldots, i_{i_{p+1}})|^2.
\]

By properties of determinant, note that, assuming \(m_i \neq 0, \)

\[
|\UP(i_{i_1}, i_{i_2}, \ldots, i_{i_{p+1}})| = \frac{1}{m_1} \begin{vmatrix}
X_{i_{i_1}} & \cdots & X_{i_{i_p}} & 1 \\
\vdots & \ddots & \vdots & \vdots \\
X_{i_{i_{p+1}}} & \cdots & X_{i_{i_{p+1}} & 1
\end{vmatrix}.
\]

By Lemma \(\square\), choosing \(\lambda = m_1, u = (1, \ldots, 1)^T\) and \(v = (1, \ldots, 1)^T\),

\[
\begin{vmatrix}
X_{i_1} & \cdots & X_{i_p} & m_1 \\
\vdots & \ddots & \vdots & \vdots \\
X_{i_{p+1}} & \cdots & X_{i_{p+1}} & m_1
\end{vmatrix} = \begin{vmatrix}
Y_{i_1} & \cdots & Y_{i_p} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{i_{p+1}} & \cdots & Y_{i_{p+1}} & 0
\end{vmatrix} + m_1 \sum_{ij} (-1)^{i+j} \det \begin{vmatrix}
Y_{i_1} & \cdots & Y_{i_p} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{i_{p+1}} & \cdots & Y_{i_{p+1}} & 0
\end{vmatrix}.
\]
The first $Y$ determinant is automatically zero due to the last column filled with zeroes. Similarly, the only nonzero terms in the sum are those for $j = p + 1$, thus

$$|\mathbb{M}_{(i_1, i_2, \ldots, i_{p+1})}| = \sum_i (-1)^{i+p+1} \det \left( \begin{array}{ccc} Y_{i_1 1} & \cdots & Y_{i_1 p} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ Y_{i_{p+1} 1} & \cdots & Y_{i_{p+1} p} & 0 \\ \end{array} \right) = |\mathbb{M}_{(i_1, i_2, \ldots, i_{p+1})}|,$$  \hspace{1cm} (115)

which we have identified as the expansion of $|\mathbb{M}_{(i_1, i_2, \ldots, i_{p+1})}|$ in the last column. The proof of the proposition is finished by again employing the Cauchy-Binet formula. By continuity, the lemma holds even for $m_1 = 0$. \hspace{1cm} \blacksquare

**Definition 11** ($O^{[q]}, V^{[q]}$). Denote

$$O^{[q]} = \begin{pmatrix} X_{11} & \cdots & X_{1q} & Y_{1,q+1} & \cdots & Y_{1,p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{p1} & \cdots & X_{pq} & Y_{p,q+1} & \cdots & Y_{p,p} \end{pmatrix} \quad \text{and} \quad V^{[q]} = \begin{pmatrix} Y_{11} & \cdots & Y_{1q} & Y_{1,q+1} & \cdots & Y_{1,p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{p1} & \cdots & Y_{pq} & Y_{p,q+1} & \cdots & Y_{p,p} \end{pmatrix}.$$  

**Proposition 15.**

$$E'[O^{[q]}]^2 = (p-q)! \mu_2^{p-q} |U^{T}U'|, \quad E'[V^{[q]}]^2 = (p-q)! \mu_2^{p-q} |V^{T}V'|.$$  

**Proof.** Thanks to $Y_{ij}$'s being central, the formulae are a direct consequence of (98). \hspace{1cm} \blacksquare

**Proposition 16.**

$$|U^{[q]}| = |O^{[q]}| + m_1 \sum_{i \in \{ 1, \ldots, p \}} (-1)^{i+1} |O^{[q]}_i|.$$  

**Proof.** Use Lemma 6 with $\lambda = m_1$, $C = O^{[q]}$, $u = (1, \ldots, 1)^T$ and $v = (0, \ldots, 0, 1, \ldots, 1)^T$. \hspace{1cm} \blacksquare

**Proposition 17.** When $X_{ij}$'s follow general distribution, we have, in contrast to (98),

$$E'[U^{[q]}]^2 = (p-q)! \mu_2^{p-q} \left( |U^{T}U'| + \frac{m_1^2}{\mu_2} |\mathbb{M}^{T}\mathbb{M}'| \right)$$  

**Proof.** Squaring the Proposition 16

$$|U^{[q]}|^2 = |O^{[q]}|^2 + 2m_1 \sum_{i \in \{ 1, \ldots, p \}} (-1)^{i+1} |O^{[q]}_i||O^{[q]}_i| + m_1^2 \sum_{l,t \in \{ 1, \ldots, p \}} \sum_{j,s \in \{ q+1, \ldots, p \}} (-1)^{i+j+s+t} |O^{[q]}_i||O^{[q]}_j||O^{[q]}_s||O^{[q]}_t|.$$  \hspace{1cm} (116)

We now take the $E'$ expectation. Expanding $|O^{[q]}_i|$ in the $j$-th column, where $j > q$, we notice that

$$E'[O^{[q]}_i||O^{[q]}_i] = E' \sum_{k=1}^p (-1)^{k+1} Y_{kj} |O^{[q]}_k||O^{[q]}_i| = 0$$  \hspace{1cm} (117)

since $E'Y_{kj} = 0$. Similarly, by expanding $|O^{[q]}_i|$ in the $s$-th column, where $s > q$, we get

$$E'[O^{[q]}_i||O^{[q]}_s] = 0 \quad \text{when} \quad s \neq j.$$  \hspace{1cm} (118)

Only terms with $j = s$ survive. Therefore, by symmetry,

$$E'[U^{[q]}]^2 = E'[O^{[q]}]^2 + m_1^2 (p-q) E' \left( \sum_{i=1}^n (-1)^{i+j} |O^{[q]}_i| \right)^2.$$  \hspace{1cm} (119)
Notice that in (119), we can interpret the sum in the bracket as another expansion, namely
\[ \sum_{i=1}^{\p} (-1)^{i+j} |O_{i,q+1}^{[q]}| = \begin{bmatrix} X_{11} & \ldots & X_{1q} & 1 & Y_{1,q+2} & \ldots & Y_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{p1} & \ldots & X_{pq} & 1 & Y_{1,q+2} & \ldots & Y_{1p} \end{bmatrix}. \]  
(120)

Thus, taking \( E' \) (that is, taking the expectation with respect to \( Y_{ij} \)'s), we get
\[ E' \left( \sum_{i=1}^{\p} (-1)^{i+j} |O_{i,q+1}^{[q]}| \right)^2 = (p - q - 1)! \mu_2^{p-q-1}|\Upsilon'^T \Upsilon'|. \]  
(121)

Together with Lemma 14, we have \( |\Upsilon'^T \Upsilon'| = |\Upsilon'^T \Upsilon'| \), which concludes the proof of the proposition.  

**Definition 12** (\( \Upsilon^{[q]}, \Upsilon^{[q]}', E' \)). Denote
\[ \Upsilon^{[q]} = \begin{pmatrix} Y_{11} & \ldots & Y_{1q} & Y_{1,q+1} & \ldots & Y_{1,p-1} & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{p1} & \ldots & Y_{pq} & Y_{p,q+1} & \ldots & Y_{p,p-1} & 1 \end{pmatrix}, \]  
\[ \Upsilon^{[q]'} = \begin{pmatrix} Y_{11} & \ldots & Y_{1q} & Y_{1,q+1} & \ldots & Y_{1,p-1} & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{p1} & \ldots & Y_{pq} & Y_{p,q+1} & \ldots & Y_{p,p-1} & 1 \end{pmatrix} \]  
(122)
a random pair of \( p \) by \( p \) square matrices being identical in the first \( q \) columns, that is \( Y_{ij} = \tilde{Y}_{ij} \) for \( j \leq q \) and all \( i \). Otherwise, in columns \( j > q \), we assume \( \tilde{Y}_{ij} \) are independent from each other and from all \( \tilde{Y}_{ij} \)'s, following the same distribution. For a reminder, the \( E' \) expectation is constructed such it acts only on those \( j > q \) columns.

**Proposition 18.**
\[ E' |\Upsilon^{[q]}|^2 = (p - q - 1)! \mu_2^{p-q-1}|\Upsilon'^T \Upsilon'|. \]

**Proof.** Denoting
\[ \Upsilon' = \begin{pmatrix} Y_{11} & \ldots & Y_{1q} & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ Y_{p1} & \ldots & Y_{pq} & 1 & 1 \end{pmatrix} \]  
(123)
we have, shifting the 1's column to the left and by using Proposition 17
\[ E' |\Upsilon^{[q]}|^2 = \left| \begin{array}{cccccc} Y_{11} & \ldots & Y_{1q} & 1 & Y_{1,q+1} & \ldots & Y_{1,p-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ Y_{p1} & \ldots & Y_{pq} & 1 & Y_{p,q+1} & \ldots & Y_{p,p-1} \end{array} \right|^2 = (p - q - 1)! \mu_2^{p-q-1} \left( \Upsilon'^T \Upsilon' \right) + \frac{m_2^2}{\mu_2} |\Upsilon'^T \Upsilon'|. \]  
(124)
However, \( |\Upsilon'^T \Upsilon'| = 0 \) trivially (parallellepiped with two spanning vectors being identical has zero volume).

**Definition 13** (\( W, W', \sigma, \sigma' \)). Let us define another pair of random matrices
\[ W = \begin{pmatrix} Y_{11} & \ldots & Y_{1p} \\ \vdots & \ddots & \vdots \\ Y_{n1} & \ldots & Y_{np} \\ 1 & \ldots & 1 \end{pmatrix} \]  
(125)
and
\[ W' = \begin{pmatrix} Y_{11} & \ldots & Y_{1q} \\ \vdots & \ddots & \vdots \\ Y_{p1} & \ldots & Y_{pq} \\ 1 & \ldots & 1 \end{pmatrix}, \]
from which we construct two sums \( \sigma \) and \( \sigma' \) as
\[ \sigma = \sum_{1 \leq i_1 < i_2 < \ldots < i_{p-1} \leq n} |W_{(i_1,i_2,\ldots,i_{p-1})}|^2 \]  
and
\[ \sigma' = \sum_{1 \leq i_1 < i_2 < \ldots < i_{q-1} \leq n} |W'_{(i_1,i_2,\ldots,i_{q-1})}|^2. \]  
(126)
By definition, we put \( \sigma = 0 \) when \( p = 0 \).
**Definition 14** \( W^{[q]}, \tilde{W}^{[q]} \). Denote

\[
W^{[q]} = \begin{pmatrix}
Y_{11} & \cdots & Y_{1q} & Y_{1,q+1} & \cdots & Y_{1,p+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
Y_{p1} & \cdots & Y_{pq} & Y_{p,q+1} & \cdots & Y_{p,p+1}
\end{pmatrix}, \quad \tilde{W}^{[q]} = \begin{pmatrix}
Y_{11} & \cdots & Y_{1q} & \tilde{Y}_{1,q+1} & \cdots & \tilde{Y}_{1,p+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
Y_{p1} & \cdots & Y_{pq} & \tilde{Y}_{p,q+1} & \cdots & \tilde{Y}_{p,p+1}
\end{pmatrix}.
\]

**Proposition 19.**

\[
E' |W^{[q]}|^2 = (p - q + 1)! \mu_2^{-q} (\mu_2 \sigma' + |V'TV'|)
\]

**Proof.** Denote

\[
Z^{[q]} = \begin{pmatrix}
Y_{11} & \cdots & Y_{1q} & Y_{1,q+1} & \cdots & Y_{1,p+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
Y_{p1} & \cdots & Y_{pq} & Y_{p,q+1} & \cdots & Y_{p,p+1}
\end{pmatrix}.
\]

By Lemma 5 (or by expansion in the last row), we get

\[
|W^{[q]}| = |Z^{[q]}| + \sum_{j=q+1}^{p+1} (-1)^{p+1+j} |Z^{[q]}_{p+1,j}|.
\]

Squaring,

\[
|W^{[q]}|^2 = |Z^{[q]}|^2 + 2 \sum_{j=q+1}^{p+1} (-1)^{p+1+j} |Z^{[q]}||Z^{[q]}_{p+1,j}| + \sum_{j=q+1}^{p+1} \sum_{s=q+1}^{p+1} (-1)^{j+s} |Z^{[q]}_{p+1,j}||Z^{[q]}_{p+1,s}|.
\]

We now take the \( E' \) expectation. Expanding \( |Z^{[q]}| \) in the \( j \)-th column, where \( j > q \), we notice that

\[
E' |Z^{[q]}||Z^{[q]}_{p+1,j}| = E' \sum_{k=1}^{p} (-1)^{k+j} Y_{kj} |Z^{[q]}||Z^{[q]}_{p+1,j}| = 0
\]

since \( E' Y_{kj} = 0 \). Similarly, expanding \( |Z^{[q]}_{p+1,j}| \) in the \( s \)-th column, where \( s > q \), we get

\[
E' |Z^{[q]}_{p+1,j}||Z^{[q]}_{p+1,s}| = 0 \quad \text{when} \quad s \neq j.
\]

Only terms with \( j = s \) survive. Therefore, by symmetry,

\[
E' |W^{[q]}|^2 = E' |Z^{[q]}|^2 + (p - q + 1) E' |Z^{[q]}_{p+1,p+1}|^2.
\]

But \( Z^{[q]}_{p+1,q+1} = V^{[q]} \). So, using Proposition 15

\[
E' |Z^{[q]}_{p+1,q+1}|^2 = (p - q)! \mu_2^{-q} |V'TV'|.
\]

On the other hand, expanding \( |Z^{[q]}| \) in all columns with \( j > q \) and collecting terms with the same value,

\[
E' |Z^{[q]}|^2 = (p - q + 1)! \mu_2^{-q+1} \sigma'.
\]

\( \blacksquare \)
3.5 Auxiliary moments and their recurrences

Definition 15 (B, g₁(n), g₂(n, p), G₄(t), G₄(t, ω), Ψ(ς)). Given Yᵢ’s, we form a matrix B = (Yᵢ)ᵢ×ᵢ and denote g₄(n) = E|B|⁴ and

\[ G₄(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)²} g₄(n). \]  

(135)

Similarly, denote g₄(n, p) = E|VᵀV|². For its generating function, we write

\[ G₄(t, ω) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p ω^{n-p} g₄(n, p). \]  

(136)

Remark 9. Since the moments of a random determinant are dependent only on moments of its random entries, we get that g₄(n) and g₄(n, p) are equal to f₄(n) and f₄(n, p), respectively, in which we replace mᵢ by μᵢ when j > 1. So, by Dembo’s formula and by the ansatz expansion

\[ G₄(t, ω) = \sum_{j=0}^{∞} \omega^j \left( \frac{1 - m₂ t}{1 - ω - m₂ t} \right)^{j+1} \Psi_j(t) = \frac{e^{t(μ₄ - 3μ₂)}}{(1 - μ₂ t)^2(1 - ω - μ₂ t)}. \]  

(137)

where

\[ ψ₀(ς) = G₄(ς, 0) = G₄(ς) = \frac{e^{t(μ₄ - 3μ₂)}}{(1 - μ₂ t)^3} \text{ and } ψ_j(ς) = 0 \text{ for } j ≥ 1. \]  

(138)

Definition 16 (α(n, p), β(n, p), γ(n, p), δ(n, p), ε(n, p), ρ(n, p), η(n, p), κ(n, p)). We define the following

\[ α(n, p) = E(|U|⁴|V|⁴), \quad β(n, p) = |V|⁴|U|⁴, \quad γ(n, p) = E(|U|⁴|V|⁴), \quad δ(n, p) = E(|U|⁴|V|⁴), \quad ε(n, p) = E|U|⁴|V|⁴, \]

\[ ρ(n, p) = E|V|⁴|U|⁴, \quad η(n, p) = E|V|⁴|U|⁴, \quad κ(n, p) = E⁴. \]

Remark 10. By definition, we have γ(n, 0) = ε(n, 0) = ρ(n, 0) = η(n, 0) = 0 and δ(n, 0) = 1, α(n, 0) = η(n, 0) = n and β(n, 0) = n². And due to vanishment for large p, we also have α(n, p) = β(n, p) = ε(n, p) = η(n, p) = 0 for p ≥ n and γ(n, p) = δ(n, p) = ρ(n, p) = 0 for p ≥ n + 1. Less straightforwardly, it can be shown that σ = 0 when p ≥ n + 2, so κ(n, p) = 0 for p ≥ n + 2 (but we don’t need it).

Remark 11. When m₁ = 0, note that f₄(n, p) = δ(n, p) = g₄(n, p), γ(n, p) = ρ(n, p) and α(n, p) = η(n, p).

Proposition 20.

\[ f₄(n, p) = \sum_{q=0}^{p} \frac{n!μ₂(q-p)}{(n-2p+q)!} \left( f₄(p, q) + \frac{2m₂}{μ₂} α(p, q) + \frac{m₁}{μ₂} β(p, q) \right). \]

Proof. By Cauchy-Binet formula, taking expectation, transposing and collecting identical terms, we get

\[ f₄(n, p) = E|U|⁴|U|⁴ = E \sum_{1≤i₁<i₂<...<iₚ≤n} |U(i₁, i₂,...,iₚ)|² |U(i₁, i₂,...,iₚ)|² = \sum_{q=0}^{p} c_{p,q} E|U|^q |\bar{U}|^q. \]  

(139)

By Proposition 47 with the fact that E = E E’,

\[ E|U|^q |\bar{U}|^q = E \left( (p-q)! μ₂²-q \left( |U|⁴ |U’| + \frac{m₂}{μ₂} |\bar{U}|⁴ |\bar{U’}| \right) \right)^2 \]

\[ = (p-q)!² μ₂²(p-q) \left( f₄(p, q) + \frac{2m₂}{μ₂} α(p, q) + \frac{m₁}{μ₂} β(p, q) \right), \]  

(140)

inserting this result into (139), we get the desired recurrence relation for f₄(n, p).
**Definition 17** \((d_{p, q})\). Given two identical copies of the set \([1, 2, 3, \ldots, n]\), we denote \(d_{p, q}\) the number of ways how we can select \(p\) numbers from the first copy and \(p + 1\) numbers from the second copy, provided that exactly \(q\) numbers in both selections were chosen simultaneously. Using standard combinatorics,

\[
d_{p, q} = \binom{n}{q} \binom{n-q}{p-q} \binom{n-p}{p+1-q} = \frac{n!}{q!(p-q)!(p-q+1)!(n-2p+q-1)!}. \tag{141}
\]

**Proposition 21.**

\[
\alpha(n, p) = \sum_{q=0}^{p} \frac{n! \mu_2^{2(p-q)}}{q!(n-2p+q-1)!} \left( \mu_2 \gamma(p, q) + \delta(p, q) + m_2^2 \epsilon(p, q) + \frac{m_2^2}{\mu_2} \eta(p, q) \right).
\]

*Proof.* By Cauchy-Binet formula and by taking expectation, transposing and collecting identical terms,

\[
\alpha(n, p) = \mathbb{E}[\|U\|^2|\|V\|^2] = \mathbb{E} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq n} |U(i_1, i_2, \ldots, i_p)|^2 |V(i'_1, i'_2, \ldots, i'_{p+1})|^2 = \sum_{q=0}^{p} d_{p, q} \mathbb{E}[\|U|^q]^2 |\tilde{W}^q|^2. \tag{142}
\]

Combining Propositions [17] and [19] with the fact that \(\mathbb{E} = \mathbb{E}'\),

\[
\mathbb{E}[\|U|^q]^2 |\tilde{W}^q|^2 = \mathbb{E} (p-q)! \mu_2^{2(p-q)} \left( |U^\top U'| + \frac{m_2^2}{\mu_2} |\tilde{V}^\top V'| \right) (p-q+1)! \mu_2^{2-p-q} \left( \mu_2 \sigma + |V'\tilde{V}'| \right)
\]

\[
= (p-q)! (p-q+1)! \mu_2^{2(p-q)} \left( \mu_2 \gamma(p, q) + \delta(p, q) + m_2^2 \epsilon(p, q) + \frac{m_2^2}{\mu_2} \eta(p, q) \right). \tag{143}
\]

Inserting this result into (142), we get the desired relation for \(\alpha(n, p)\). \(\blacksquare\)

**Proposition 22.**

\[
\eta(n, p) = \sum_{q=0}^{p} \frac{n! \mu_2^{2(p-q)}}{q!(n-2p+q-1)!} \left( \mu_2 \rho(p, q) + \eta_4(p, q) \right).
\]

*Proof.* In Proposition 21, put \(m_1 = 0\) and use Remark 11. \(\blacksquare\)

**Proposition 23.**

\[
\gamma(n, p) = \sum_{q=0}^{p-1} \frac{n! \mu_2^{2p-2q-1}}{q!(n-2p+q+1)!} \left( \alpha(p, q) + \frac{m_2^2}{\mu_2} \beta(p, q) \right).
\]

*Proof.* By Cauchy-Binet formula and the definition of \(\sigma\), taking expectation, transposing and collecting identical terms,

\[
\gamma(n, p) = \mathbb{E}[\|U\|^p|\|\tilde{V}\|^q] = \mathbb{E} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq n} |U(i_1, i_2, \ldots, i_p)|^2 |\tilde{W}(i'_1, i'_2, \ldots, i'_{p+1})|^2 = \sum_{q=0}^{p-1} d_{p-1, q} \mathbb{E}[\|U|^q]^2 |\tilde{W}^q|^2. \tag{144}
\]

By using Propositions [17] and [18] and \(\mathbb{E} = \mathbb{E}'\),

\[
\mathbb{E}[\|U|^q]^2 |\tilde{W}^q|^2 = \mathbb{E} (p-q)! \mu_2^{2(p-q)} \left( |U^\top U'| + \frac{m_2^2}{\mu_2} |\tilde{V}^\top V'| \right) (p-q-1)! \mu_2^{2-p-q} |\tilde{V}'\tilde{W}'|
\]

\[
= (p-q)! (p-q-1)! \mu_2^{2(p-q-1)} \left( \alpha(p, q) + \frac{m_2^2}{\mu_2} \beta(p, q) \right). \tag{145}
\]

Inserting this result into (144), we get the desired relation for \(\gamma(n, p)\). \(\blacksquare\)

**Proposition 24.**

\[
\rho(n, p) = \sum_{q=0}^{p-1} \frac{n! \mu_2^{2p-2q-1} \eta(p, q)}{q!(n-2p+q+1)!}.
\]
Proposition 25. 
\[ \beta(n, p) = \sum_{q=0}^{p+1} \frac{n!\mu_2^{2(p-q)}}{q!(n-2p+q-2)!} \left( \mu_2^2 \kappa(p, q) + 2\mu_2 \rho(p, q) + g_4(p, q) \right). \]

Proof. By Cauchy-Binet formula and by taking expectation, transposing and collecting identical terms,
\[ \beta(n, p) = E [\mathbf{Y}^T \mathbf{Y}]^2 = E \sum_{1 \leq i_1 < i_2 < \ldots < i_{p+1} \leq n} |\mathbf{Y}_{(i_1, i_2, \ldots, i_{p+1})}|^2 |\mathbf{Y}_{(i'_1, i'_2, \ldots, i'_{p+1})}|^2 = \sum_{q=0}^{p+1} c_{p+1, q} E [W^{[q]}]^2 |\tilde{W}^{[q]}|^2. \] (146)

By using Proposition 19 and \( E = E' \),
\[ E |W^{[q]}|^2 |\tilde{W}^{[q]}|^2 = E \left( (p-q+1)! \mu_2^{p-q} (\mu_2 \sigma' + |V^T V'|)^2 - (p-q+1)! \right) (p-q+1)! \mu_2^{2(p-q)} (\mu_2^2 \kappa(p, q) + 2\mu_2 \rho(p, q) + g_4(p, q)) \] (147)
and inserting this result into (146), we get the desired relation for \( \beta(n, p) \).

Proposition 26. 
\[ \kappa(n, p) = \sum_{q=0}^{p-1} \frac{n!\mu_2^{2(p-q-1)}}{q!(n-2p+q+2)!} \beta(p, q). \]

Proof. By the definition of \( \sigma \), taking expectation, transposing and collecting identical terms,
\[ \kappa(n, p) = E \sigma^2 = E \sum_{1 \leq i_1 < i_2 < \ldots < i_{p-1} \leq n} |\mathbf{W}_{(i_1, i_2, \ldots, i_{p-1})}|^2 |\mathbf{W}_{(i'_1, i'_2, \ldots, i'_{p-1})}|^2 = \sum_{q=0}^{p-1} c_{p-1, q} E [\mathbf{Y}^{[q]}]^2 |\tilde{\mathbf{Y}}^{[q]}|^2. \] (148)

By using Proposition 18 and \( E = E' \),
\[ E |\mathbf{Y}^{[q]}|^2 |\tilde{\mathbf{Y}}^{[q]}|^2 = E \left( (p-q-1)! \mu_2^{p-q-1} \mathbf{Y}^T \mathbf{Y} \right)^2 = (p-q-1)! \mu_2^{2(p-q-1)} \beta(p, q), \] (149)
inserting this result into (148), we get the desired relation for \( \kappa(n, p) \).

Proposition 27. 
\[ \delta(n, p) = \sum_{q=0}^{p} \frac{n!\mu_2^{2(p-q)}}{q!(n-2p+q)!} \left( \delta(p, q) + \frac{m_2^2}{\mu_2} \eta(p, q) \right). \]

Proof. By Cauchy-Binet formula and by taking expectation, transposing and collecting identical terms,
\[ \delta(n, p) = E [\mathbf{U}^T \mathbf{U}] |\mathbf{V}^T \mathbf{V}| = E \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq n} |\mathbf{U}_{(i_1, i_2, \ldots, i_p)}|^2 |\mathbf{V}_{(i'_1, i'_2, \ldots, i'_p)}|^2 = \sum_{q=0}^{p} c_{p, q} E [U^{[q]}]^2 |V^{[q]}|^2. \] (150)

By using Propositions 15 and 17 and \( E = E' \),
\[ E |U^{[q]}|^2 |V^{[q]}|^2 = E (p-q)! \mu_2^{p-q} \left( |U^T U'| + \frac{m_2^2}{\mu_2} |V^T V'| \right) (p-q)! \mu_2^{p-q} |V^T V'| \]
\[ = (p-q)! \mu_2^{2(p-q)} \left( \delta(p, q) + \frac{m_2^2}{\mu_2} \eta(p, q) \right), \] (151)
inserting this result into (150), we get the desired relation for \( \delta(n, p) \).
**Definition 18** ($e_{p,q}$). Given two identical copies of the set $\{1, 2, 3, \ldots, n\}$, we denote $e_{p,q}$ the number of ways how we can select $p + 1$ numbers from the first copy and $p - 1$ numbers from the second copy, provided that exactly $q$ numbers in both selections were chosen simultaneously. Using standard combinatorics,

$$e_{p,q} = \binom{n}{q} \binom{n-q}{p+1-q} \binom{n-(p+1)}{p-1-q} = \frac{n!}{q!(p-q+1)!(p-q-1)!(n-2p+q)!}. \quad (152)$$

**Proposition 28.**

$$e(n, p) = \sum_{q=0}^{p-1} \frac{n!2^{p-2q-1}}{q!(n-2p+q)!} (\mu_2 e(p, q) + \eta(p, q)).$$

**Proof.** By Cauchy-Binet formula and the definition of $\sigma$, taking expectation, transposing and collecting identical terms,

$$e(n, p) = E[\mathcal{V}^T \mathcal{V} \sigma] = E \sum_{1 \leq i_1 < i_2 < \cdots < i_{p+1} \leq n} |\mathcal{V}_{i_1, i_2, \ldots, i_{p+1}}|^2 |\mathcal{V}_{i_1', i_2', \ldots, i_{p+1}'}|^2 = \sum_{q=0}^{p-1} e_{p,q} E[|\mathcal{V}^T|^2 |\mathcal{V}^T|^2]. \quad (153)$$

By using Propositions 18 and 19 and $E = E' E'$,

$$E[|\mathcal{V}^T|^2 |\mathcal{V}^T|^2]^2 = E(p-q+1)! \mu_2^{p-q} (\mu_2 \sigma' + |\mathcal{V}^T \mathcal{V}'|) (p-q-1)! \mu_2^{p-q-1} |\mathcal{V}^T \mathcal{V}'|$$

$$= (p-q+1)! (p-q-1)! \mu_2^{2p-2q-1} (\mu_2 e(p, q) + \eta(p, q)). \quad (154)$$

inserting this result into (153), we get the desired relation for $e(n, p)$. 

**Remark 12.** Dependencies of auxiliary moments on themselves are shown graphically in Figure 1.

![Graph of dependencies in recurrence relations](image)

**Figure 1:** Graph of dependencies in recurrence relations

### 3.6 Auxiliary generating functions

**Definition 19** ($\tilde{\alpha}(t, \omega)$, $\tilde{\beta}(t, \omega)$, $\tilde{\gamma}(t, \omega)$, $\tilde{\delta}(t, \omega)$, $\tilde{\epsilon}(t, \omega)$, $\tilde{\rho}(t, \omega)$, $\tilde{\eta}(t, \omega)$, $\tilde{\kappa}(t, \omega)$). We define the following generating functions

$$\tilde{\alpha}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \alpha(n, p),$$

$$\tilde{\beta}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \beta(n, p),$$

$$\tilde{\gamma}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p+1)!}{n!p!} t^p \omega^{n-p+1} \gamma(n, p),$$

$$\tilde{\delta}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p+1)!}{n!p!} t^p \omega^{n-p+1} \delta(n, p),$$

$$\tilde{\epsilon}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \epsilon(n, p),$$

$$\tilde{\rho}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \rho(n, p),$$

$$\tilde{\eta}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \eta(n, p),$$

$$\tilde{\kappa}(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} \kappa(n, p).$$
\( \dot{\epsilon}(t, \omega) = \sum_{n=0}^{\infty} \frac{(n-p+1)!}{n!p!} t^n \omega^{n-p+1} \epsilon(n, p), \)
\( \dot{\eta}(t, \omega) = \sum_{n=0}^{\infty} \frac{(n-p+1)!}{n!p!} t^n \omega^{n-p+1} \eta(n, p), \)
\( \dot{\rho}(t, \omega) = \sum_{n=0}^{\infty} \frac{(n-p+2)!}{n!p!} t^n \omega^{n-p+2} \rho(n, p), \)
\( \dot{\kappa}(t, \omega) = \sum_{n=0}^{\infty} \frac{(n-p+2)!}{n!p!} t^n \omega^{n-p+2} \kappa(n, p). \)

**Definition 20** \((\alpha_j(t), \beta_j(t), \gamma_j(t), \delta_j(t), \epsilon_j(t), \rho_j(t), \eta_j(t), \kappa_j(t)).\) Also, we define the ansatz coefficients via expansions

\[
\begin{align*}
\dot{\alpha}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \alpha_j(t), \\
\dot{\beta}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \beta_j(t), \\
\dot{\gamma}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \gamma_j(t), \\
\dot{\delta}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \delta_j(t), \\
\dot{\epsilon}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \epsilon_j(t), \\
\dot{\eta}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \eta_j(t), \\
\dot{\rho}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \rho_j(t), \\
\dot{\kappa}(t, \omega) &= \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \kappa_j(t).
\end{align*}
\]

**Definition 21** \((G^4_1, \Psi^*_j).\) On top of that, we define the following extra generating function

\[
G^4_1(t, \omega) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(n-p+2)!}{n!p!} t^n \omega^{n-p} g_4(n, p) = \sum_{j=0}^{\infty} \omega^j \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+1} \Psi^*_j(t).
\]

**Proposition 29.**

\[
G^4_1(t, \omega) = 2\omega^2 \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^3 \Psi^*_j(t).
\]

That is \( \Psi^*_2(t) = 2G^4_1(t) \) and \( \Psi^*_j(t) = 0 \) otherwise.

**Proof.** Note that

\[
G^4_1(t, \omega) = \omega \frac{\partial}{\partial \omega} \left( \omega \frac{\partial}{\partial \omega} (\omega G^4_1(t, \omega)) \right) = \sum_{j=0}^{\infty} \omega^{j+2} \left( \frac{1 - \mu_2^2 t}{1 - \omega - \mu_2^2 t} \right)^{j+3} (j+2)(j+1) \Psi^*_j(t),
\]

so in other words

\[
\Psi^*_j(t) = j(j+1) \Psi^*_{j-2}(t).
\]

Since \( \Psi^*_0(t) = G^4_1(t) \) and otherwise \( \Psi^*_j(t) \) is zero for \( j \geq 1 \), this finishes the proof.

**Proposition 30.**

\[
\begin{align*}
F_4(t, \omega) &= \frac{1}{1 - \omega} \left( F_4 \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + \frac{2m_1^2}{\mu_2} \alpha \left( t, \frac{1 - \mu_2^2 t}{1 - \omega} \right) + \frac{m_4^2}{\mu_2^2} \beta \left( t, \frac{1 - \mu_2^2 t}{1 - \omega} \right) \right), \\
\dot{\alpha}(t, \omega) &= \frac{1}{\mu_2^2 (1 - \omega)} \left( \mu_2 \dot{\gamma} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + \dot{\delta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + m_2^2 \dot{\epsilon} \left( t, \frac{1 - \mu_2^2 t}{1 - \omega} \right) + m_4^2 \dot{\eta} \left( t, \frac{1 - \mu_2^2 t}{1 - \omega} \right) \right), \\
\dot{\beta}(t, \omega) &= \frac{1}{\mu_2^2 (1 - \omega)} \left( \mu_2 \dot{\kappa} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + 2m_2 \dot{\rho} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + G_4 \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) \right), \\
\dot{\gamma}(t, \omega) &= \frac{1}{\mu_2 (1 - \omega)} \left( \dot{\alpha} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + \frac{m_2^2}{\mu_2} \dot{\beta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) \right),
\end{align*}
\]
\[ \dot{\delta}(t, \omega) = \frac{1}{\mu_2^2 t(1 - \omega)} \left( \dot{\delta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + \frac{m_1^2}{\mu_2} \tilde{\eta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) \right). \]

\[ \dot{\epsilon}(t, \omega) = \frac{1}{\mu_2^2 t(1 - \omega)} \left( \mu_2 \dot{\epsilon} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + \tilde{\eta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) \right). \]

\[ \dot{\eta}(t, \omega) = \frac{1}{\mu_2^2 t^2(1 - \omega)} \left( \mu_2 \dot{\beta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) + G^4 \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right) \right). \]

\[ \dot{\beta}(t, \omega) = \frac{1}{\mu_2^2 t(1 - \omega)} \tilde{\eta} \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right). \]

\[ \dot{\kappa}(t, \omega) = \frac{1}{\mu_2^2 t^2(1 - \omega)} \beta \left( t, \frac{\omega \mu_2^2 t}{1 - \omega} \right). \]

**Proof.** Insert Propositions 20 – 28 into the definitions of the corresponding generating functions. \( \square \)

**Corollary 30.1.** In the terms of ansatz coefficients, comparing the terms of the expansions, this is equal to the linear system

\[ \Phi_j(t) = \mu_2^j t \left( \Phi_0(t) + \frac{2m_1^2}{\mu_2} \alpha(t) + \frac{m_1^2}{\mu_2} \beta_1(t) \right), \quad \alpha_1(t) = \mu_2^{j-1} t^{j-1} \left( \mu_2 \gamma_1(t) + \delta_1(t) + m_1^2 \epsilon_1(t) + \frac{m_1^2}{\mu_2} \eta_1(t) \right). \]

\[ \beta_j(t) = \mu_2^{j-4} t^{j-2} \left( \mu_2 \kappa_1(t) + 2 \mu_2 \rho_j(t) + \Psi_1(t) \right), \quad \gamma_1(t) = \mu_2^{j-1} t \left( \alpha_j(t) + \frac{m_1^2}{\mu_2} \beta_j(t) \right). \]

\[ \delta_j(t) = \mu_2^{j-2} t^{j-1} \left( \delta_1(t) + \frac{m_1^2}{\mu_2} \eta_1(t) \right), \quad \epsilon_j(t) = \mu_2^{j-3} t^{j-1} \left( \mu_2 \epsilon_j(t) + \eta_1(t) \right), \]

\[ \eta_1(t) = \mu_2^{j-4} t^{j-2} \left( \mu_2 \rho_1(t) + \Psi_1(t) \right), \quad \rho_j(t) = \mu_2^{j-3} t^{j-1} \eta_1(t), \quad \kappa_j(t) = \mu_2^{j-2} t^j \beta_j(t). \]

**Corollary 30.2.** Solving the linear system, we get

\[ \Phi_1(t) = m_1^2 t \frac{2 \mu_2 \alpha_1(t) + \frac{m_1^2}{\mu_2} \beta_1(t)}{1 - \mu_2^2 t}, \quad \Phi_2(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^4} G_4(t). \]

\[ \alpha_0(t) = 0, \quad \alpha_1(t) = \frac{\delta_1(t) + m_1 \epsilon_1(t) + \frac{m_1^2}{\mu_2} \beta_1(t)}{1 - \mu_2^2 t}, \quad \alpha_2(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^3} G_4(t). \]

\[ \beta_0(t) = 0, \quad \beta_1(t) = \frac{2}{1 - \mu_2^2 t} G_4(t), \quad \beta_2(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^2} G_4(t). \]

\[ \gamma_0(t) = 0, \quad \gamma_1(t) = t \frac{\mu_2 \delta_1(t) + m_1 \mu_2 \epsilon_1(t) + \frac{m_1^2}{\mu_2} \beta_1(t)}{1 - \mu_2^2 t}, \quad \gamma_2(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^3} G_4(t). \]

\[ \delta_0(t) = 0, \quad \delta_1(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^2} G_4(t), \quad \delta_2(t) = \frac{2 m_1^2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^2} G_4(t). \]

\[ \epsilon_0(t) = 0, \quad \epsilon_1(t) = \frac{2 \mu_2 t}{1 - \mu_2^2 t} G_4(t), \quad \epsilon_2(t) = \frac{2 \mu_2 t}{1 - \mu_2^2 t} G_4(t). \]

\[ \eta_0(t) = 0, \quad \eta_1(t) = 0, \quad \eta_2(t) = \frac{2}{1 - \mu_2^2 t} G_4(t). \]

\[ \rho_0(t) = 0, \quad \rho_1(t) = 0, \quad \rho_2(t) = \frac{2 \mu_2 t}{1 - \mu_2^2 t} G_4(t). \]

\[ \kappa_0(t) = 0, \quad \kappa_1(t) = t \beta_1(t), \quad \kappa_2(t) = \frac{2 \mu_2^2 t^2}{(1 - \mu_2^2 t)^2} G_4(t). \]

and

\[ \Phi_j(t) = \alpha_j(t) = \beta_j(t) = \gamma_j(t) = \delta_j(t) = \epsilon_j(t) = \eta_j(t) = \rho_j(t) = \kappa_j(t) = 0 \quad \text{for} \quad j \geq 3. \]
3.7 The remaining ansatz coefficients and the final conclusion

Proposition 31.

\[ \begin{align*}
\epsilon_1(t) &= \sum_{n=0}^{\infty} \frac{\epsilon(n, n)}{n!} t^n, \\
\delta_1(t) &= \sum_{n=0}^{\infty} \frac{\delta(n, n)}{n!} t^n, \\
\beta_1(t) &= \sum_{n=0}^{\infty} \frac{\beta(n + 1, n)}{(n + 1)!} t^n.
\end{align*} \]

Proof. In Definition 20, perform the Taylor expansion in \( \omega \) and then compare with Definition 19. \( \blacksquare \)

Proposition 32.

\[ \epsilon_1(t) = 0 \]

Proof. Trivial since \( \epsilon(n, n) = 0 \) (see Remark 10). \( \blacksquare \)

Proposition 33.

\[ \delta_1(t) = \frac{1 + (m_1^2 \mu_2 - \mu_2^2 + 2m_1 \mu_3) t + (m_1^2 \mu_2^2 - 2m_1 \mu_2^2 \mu_3) t^2 - m_1^2 \mu_2^2 \mu_3^2 t^3}{1 - \mu_2^2 t} G_4(t). \]

Proof. With \( U \) and \( V \) having dimensions \( n \times n \), we have similarly as in the previous proposition,

\[ \delta(n, n) = E|U^T U||V^T V| = E|A|^2|B|^2, \]

which can be now simplified using standard techniques as before. Recall the formula

\[ |A| = |B| + m_1 S, \quad \text{where} \quad S = \sum_{ij} (-1)^{i+j}|B_{ij}|, \]

then

\[ \delta(n, n) = E|B|^4 + 2m_1 E|B|^3 S + m_1^2 E|B|^2 S^2. \]

It turns out that the terms are either trivial or already expressed, namely

- \( E|B|^4 = g_4(n) \)
- \( E|B|^3 S = n^2 \mu_3 g_4(n - 1) \)
- \( E|B|^2 S^2 = n^2 h_0(n) + n^2(n - 1)^2 \mu_3^2 g_4(n - 2) \)

In total, multiplying (159) by \( t^n/n!^2 \) and summing up,

\[ \delta_0(t) = G_4(t) + 2m_1 \mu_3 t G_4(t) + m_1^2 \left( H_0(t) + \mu_3^2 t^2 G_4(t) \right), \]

from which the proposition follows since \( H_0(t) = \mu_2 t G_4(t)/(1 - \mu_2^2 t) \) (see Proposition 12). \( \blacksquare \)

Proposition 34.

\[ \beta_1(t) = \frac{1 + 2 \mu_2^2 t}{1 - \mu_2^2 t} G_4(t) \]

Proof. Let \( V \) has dimensions \( (n + 1) \times n \), thus \( V \) is a square \( (n + 1) \times (n + 1) \) matrix. By the multiplicative property of determinant,

\[ \beta(n + 1, n) = E|V^T V|^2 = E|V|^4. \]

By expansion in the last column,

\[ |V| = \sum_{i=1}^{n+1} (-1)^{i+n+1}|V_{i1}|. \]

Hence

\[ \beta(n + 1, n) = E|V|^4 = E \sum_{i,j,r,s \in \{1,\ldots,n+1\}} (-1)^{i+j+r+s}|V_{i1}||V_{ij}||V_{ir}||V_{rs}|. \]

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By symmetry in $ijrs$ and by omitting trivially vanishing terms, we obtain
\[ \beta(n+1, n) = (n+1)E|V_{i1}|^4 + 3(n+1)nE|V_{i1}|^2|V_{i2}|^2, \]  
which we can write as, recalling $h_8(n)$ and $B$ the $n \times n$ matrix of $Y_{ij}$'s,
\[ \frac{\beta(n+1, n)}{n+1} = E|B|^4 + 3n[E|B_{11}|^2|B_{12}|^2]_{n \rightarrow n+1} = g_4(n) + 3n^2\mu_2 h_0(n). \]  
In total, summing for all $n$ and by using the definitions of the generating functions,
\[ \beta(t) = G_4(t) + 3\mu_2 H_0(t) = \frac{1 + 2\mu_2^2 t}{1 - \mu_2^2} G_4(t). \]

**Proposition 35.** Combining Propositions 32 – 34 and by Corollary 30.2, we get
\[ \alpha_1(t) = 1 + \frac{1}{(1 - \mu_2^2)^2} (2m_1^2\mu_2 - \mu_2^2 + 2m_1\mu_3) t + \frac{1}{(1 - \mu_2^2)^3} (2m_1^2\mu_2^3 - 2m_1\mu_2\mu_3 + m_1^2\mu_3^2) t^2 - \frac{1}{(1 - \mu_2^2)^3} m_1^2\mu_2\mu_3^2 t^3 G_4(t). \]

**Proposition 36.** Combining Propositions 34 and 35 and by Corollary 30.2, we get, defining $\hat{p}_k$'s as before,
\[ \Phi_4(t) = \frac{m_2^2 G_4(t)}{(1 - \mu_2^2)^3} \sum_{k=1}^4 \hat{p}_k t^k. \]

**Corollary 36.1.** Together with the fact that $\Phi_0(t) = F_4(t)$ from [89], we recover Theorem 2.

### 4 Final remarks

We believe it might be still possible to derive the full $f_6(n)$ via the same treatment as presented in this paper (Lemma 6 and expansions in all classes). Similarly, by cubing Cauchy-Binet formula, one could obtain $f_6^{\text{sym}}(n, p)$ and possibly $f_6(n, p)$. But that task may be way harder.

### References

[1] Dembo A. 1989. On random determinants. *Quarterly of applied mathematics* 47.2:185–195.
[2] Flajolet P, Sedgewick R. 2009. *Analytic combinatorics*. Cambridge University press.
[3] Forsythe GE, Tukey JW. 1952. The extent of n-random unit vectors. *Bulletin of the American Mathematical Society* Vol. 58. 4:502–502.
[4] Fortet R. 1951. Random determinants. J. Research Nat. Bur. Standards 47:465–470.
[5] Lv Z, Potechin A. 2022. The Sixth Moment of Random Determinants. URL: [https://arxiv.org/abs/2206.11356](https://arxiv.org/abs/2206.11356)
[6] Muirhead RJ. 1982. *Aspects of multivariate statistical theory*. John Wiley & Sons.
[7] Nyquist H, Rice S, Riordan J. 1954. The distribution of random determinants. *Quarterly of Applied mathematics* 12.2:97–104.
[8] Prékopa A. 1967. On random determinants I. *Studia Sci. Math. Hungar* 2.1-2:125–132.
[9] Reed W. 1974. Random points in a simplex. *Pacific Journal of Mathematics* 54.2:183–198.
[10] Stanley RP. *Expected size of determinant of AA^T for non-square random A*. MathOverflow. (version: 2015-07-02). URL: [https://mathoverflow.net/q/210668](https://mathoverflow.net/q/210668)
[11] Stanley RP, Fomin S. 1999. *Enumerative Combinatorics*. Vol. 2. Cambridge Studies in Advanced Mathematics. Cambridge University Press:152.
[12] Tao T. *Expected determinant of a random NxN matrix*. MathOverflow. (version: 2010-06-09). URL: [https://mathoverflow.net/q/27616](https://mathoverflow.net/q/27616)
[13] Turán P. 1955. On a problem in the theory of determinants. *Acta Math. Sinica* 5.41:417–423.
### A Matrix symbols

| B   | B/1j | B/11 | B/12 | B/11/2j | B/11/3j | B/11/4j | B/11/i2 | B/11/i3 | B/11/i4 | B/11/i5 | B/12 | B/12/2j | B/12/3j | B/12/11 | B/12/i3 | B/13 | B/13/2j | B/13/i1 | B/13/i2 | B/14 | B/14/i1 | B/21 |
|-----|------|------|------|---------|---------|---------|---------|---------|---------|---------|------|---------|---------|---------|---------|------|---------|---------|---------|------|---------|------|
| B/23 | B/23/1j | B/23/2j | B/23/12 | B/23/13 | B/23/14 | B/23/i1 | B/23/i2 | B/24/i2 | B/31 | B/31/2j | B/32 | B/33/1j | B/33/i2 | B/34/1j | B/34/i1 | B/12.12 | B/12.13 | B/12.14 | B/12.23 | B/12.24 | B/12.25 |
| B/34 | B/44/1j | B/12.13 | B/12.14 | B/13.13 | B/13.23 | B/13.34 | B/13.35 | B/14.12 | B/14.13 | B/14.14 | B/14.45 | B/23.12 | B/23.14 | B/23.23 | B/23.24 | B/23.34 | B/24.13 | B/24.14 | B/24.15 |

| E|B_{11}|^2|B_{23.24}|^2|B_{22}|B_{21}| = E\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array}\right)^2 \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}\right)^2 

| Table 6: Table of all matrix symbols used |