FUNCTIONAL APPROACH TO STOCHASTIC INFLATION\textsuperscript{*}

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ABSTRACT

We propose functional approach to the stochastic inflationary universe dynamics. It is based on path integral representation of the solution to the differential equation for the scalar field probability distribution. In the saddle-point approximation scalar field probability distributions of various type are derived and the statistics of the inflationary-history-dependent functionals is developed.

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1. Introduction

There is an undecreasing interest in the stochastic dynamics of the inflationary universe (see [1] for the most recent developments). It is now well known that during inflation the inflaton scalar field is subject to quantum fluctuations and for this reason its behaviour under certain conditions resembles stochastic Brownian motion rather than regular evolution. As it has been discussed about ten years ago such a stochastic behaviour can be described by a probability distribution \( P(\phi, \tau) \) of the inflaton field values \( \phi \) in a Hubble-size domain at the inflationary stage. Here \( \tau \) denotes an arbitrary timelike evolution parameter which will be specified in a moment. More precisely, the function \( P(\phi, \tau) \) is the probability density for a hypothetical observer at the inflationary stage to find in its Hubble-size vicinity at the moment of “time” \( \tau \) the average value of the scalar field close to \( \phi \). This probability distribution evolves with “time” \( \tau \) and obeys the following differential equation [3]

\[
\frac{\partial P(\phi, \tau)}{\partial \tau} = \frac{\partial}{\partial \phi} \left( A(\phi)P(\phi, \tau) + \frac{1}{2}B^{1-\alpha}(\phi)\frac{\partial}{\partial \phi}B^{\alpha}(\phi)P(\phi, \tau) \right),
\]

where the constant parameter \( \alpha \) reflects some ambiguity in the order of the differential operations on the right-hand side (see e.g. [1]). The functions \( A(\phi) \) and \( B(\phi) \) depend on the specific choice of the parameter \( \tau \). In this paper we shall consider two such choices. Namely, if \( \tau \) is the cosmological time, \( \tau = t \), then

\[
A(\phi) = \frac{1}{3H(\phi)} \frac{dV(\phi)}{d\phi}, \quad B(\phi) = \frac{H^3(\phi)}{4\pi^2},
\]

(1.2)

If \( \tau \) is the expansion power, i.e. \( \tau = \log a \), where \( a \) is the scale factor of the universe, then

\[
A(\phi) = \frac{1}{3H^2(\phi)} \frac{dV(\phi)}{d\phi}, \quad B(\phi) = \frac{H^2(\phi)}{4\pi^2},
\]

(1.3)

In these expressions \( H(\phi) \) is the Hubble parameter which on the inflationary stage...
is given to a good precision by

\[ H^2(\phi) = \frac{8\pi}{3M_P^2} V(\phi), \]  

(1.4)

\( V(\phi) \) is the scalar field potential, and \( M_P \) is the Planck mass. In all the cases for simplicity we call the parameter \( \tau \) “time.”

The applicability limits of Eq. (1.1) were under consideration in [3] (see also [4, 1]). We shall discuss this point in the following chapter. Now it will be sufficient to note that for any reasonable inflaton potential there is a wide range of the scalar field values for which this equation is valid.

Investigation of Eq. (1.1) and search of its approximate solutions was the subject of many works (see [1] and references therein). The aim of this paper is to propose an approach based on functional calculus. As we will see, this approach is fairly transparent and sometimes turns out to be very convenient. Moreover, it represents a useful tool when studying statistics of the inflationary-history-dependent quantities. Such an observable value has been proposed in the work [5], in which also the approach which will be developed in this paper was partially used.

Our starting point is to write down the formal solution to (1.1) in terms of path integral

\[ Z(\phi_f, \phi_i, \tau) = N \int \exp(-I[\phi(\tau)]) \, d\phi. \]  

(1.5)

The function \( Z(\phi_f, \phi_i, \tau) \) represents the probability distribution of the transition from the initial scalar field value \( \phi_i \) to the final value \( \phi_f \) in a time \( \tau \), \( N \) here and below denotes an unspecified normalization constant,

\[ I[\phi(\tau)] = \frac{1}{2} \int \frac{(\dot{\phi} + A(\phi))^2}{B(\phi)} \, d\tau, \]  

(1.6)

is the “action”. The integral in (1.5) is taken over paths that lead from \( \phi_i \) to \( \phi_f \) in a time \( \tau \). Overdot denotes the derivative with respect to \( \tau \). The integration
measure \([d\phi]\) can be formally expressed as

\[
[d\phi] = \prod_{\tau} \frac{d\phi(\tau)}{\sqrt{B(\tau)}}.
\] (1.7)

We shall not be rigorous and shall not specify the meaning of the path integral (1.5) and of the measure (1.7). We only note that it depends on the specification of the operator ordering in (1.1).

The representation of the solution \(Z(\phi_f, \phi_i, \tau)\) in the form (1.5) makes it possible to interpret the expression \(\exp(-I[\phi(\tau)])\) as the probability density in the space of paths. Such an interpretation becomes more evident if one considers the problem of calculating mean values of functionals \(F[\phi(\tau)]\) under the condition that the scalar field evolution started at \(\phi_i\), ended at \(\phi_f\) and lasted for the time \(\tau\). The mean value of the functional \(F[\phi(\tau)]\) will be given by the path integral similar to (1.5)

\[
< F[\phi(\tau)] > = \mathcal{N} \int F[\phi(\tau)] \exp (-I[\phi(\tau)]) \ [d\phi].
\] (1.8)

Path integration method can be used in deriving the statistical properties of such functionals. In Chapter 5 we will see how it works.

Of course in most cases it is impossible to calculate the path integral (1.5) or (1.8) explicitly as well as it is impossible to explicitly solve the differential equation (1.1). Therefore we must look for approximate solutions. In this paper we shall consider the case when the saddle-point approximation is good so that instead of (1.5) we can write approximately

\[
Z(\phi_f, \phi_i, \tau) \approx \mathcal{N} \exp \left( -I(\phi_f, \phi_i, \tau) \right),
\] (1.9)

where \(I(\phi_f, \phi_i, \tau)\) is the action on the extremal path. The applicability limits of such an approximation will be studied in the following chapter. We will see that
these limits are not at all restrictive, and, in fact, reduce to
\[ V(\phi) \ll M_P^4 \] (1.10)
for all the values of $\phi$ along the extremal path. The condition (1.10) just meets the condition of validity of Eq. (1.1) itself (see [3, 4, 1]). Thus we shall conclude that to the extent to which Eq. (1.1) is valid its approximate solution (1.9) is also valid. This conclusion will be justified also in Chapter 4 in which we consider a special case of quartic potential $V(\phi)$ which allows for analytic solution. This circumstance makes the saddle-point approximation representative enough.

In Chapter 3 we shall consider the scalar field probability distributions of various types. We will notice dramatic dependence of the results on the specific choice of the parameter $\tau$ and on the type of the distribution we consider. In Chapter 4, as we already mentioned, we consider a special case of quartic potential $V(\phi)$ with parameter $\tau$ being cosmological time, which can be solved exactly. In Chapter 5 we shall develop a general method of investigation of the path functionals statistics in the saddle-point approximation. In the final Chapter 6 we shall briefly discuss possible applications of the tools developed.

2. Validity limits of the saddle-point approximation

First of all it will be convenient to rewrite the action (1.6) as follows
\[ I[\phi(\tau)] = S(\phi_f, \phi_i) + I_0[\phi(\tau)] , \] (2.1)
where
\[ S(\phi_f, \phi_i) = \int_{\phi_i}^{\phi_f} \frac{A(\phi)}{B(\phi)} d\phi = \frac{3M_P^4}{16} \left( \frac{1}{V(\phi_i)} - \frac{1}{V(\phi_f)} \right) , \] (2.2)

1) This and similar expressions resemble the wave function of the universe in the de Sitter minisuperspace model. The authors of [4] and [1] regard this fact rather seriously and in its view try to relate minisuperspace quantum cosmology to the stochastic approach to inflation. To us it seems that this coincidence is but occasional.
and
\[ I_0[\phi(\tau)] = \frac{1}{2} \int \frac{\dot{\phi}^2 + A^2(\phi)}{B(\phi)} d\tau. \] (2.3)

From the action (2.3) one finds the generalized momentum
\[ p = \frac{\dot{\phi}}{B(\phi)}. \] (2.4)

Our experience with quantum mechanics tells us that the saddle-point (semiclassical) approximation is applicable when the functions \( A(\phi) \) and \( B(\phi) \) do not change significantly at one “wavelength” distance \( l \sim |p|^{-1} \) determined by the momentum \( p \) of the extremal trajectory. As both \( A(\phi) \) and \( B(\phi) \) depend only on the potential \( V(\phi) \) this condition reduces to
\[ l \frac{d \log V(\phi)}{d\phi} \ll 1. \] (2.5)

Assuming semiclassical approximation to be valid we note that in this case the peak of the distribution (1.5) at the moment \( \tau \) will lie close to the value \( \phi(\tau) \) of the extremal trajectory which obeys \( \dot{\phi} = -A(\phi) \). This follows from the observation that the action (1.6) achieves its minimum on this trajectory. Hence the condition (2.5) is to be checked on this trajectory. Using the expressions (1.2) and (1.3) for \( A(\phi) \) and \( B(\phi) \) we obtain
\[ H^4(\phi) \ll V(\phi), \] (2.6)
which is equivalent to (1.10).

The expression (1.10) is very notable: it is the condition under which the universe can be treated as being classical. It is of a pleasant surprise that in determining the validity limits of the saddle-point (semiclassical) approximation to the scalar field dynamics we ultimately came to this condition. Furthermore, the condition (1.10) is the same as the validity condition of the equation (1.1) itself (see [3, 4, 1]). All this enables us to think that the saddle-point approximation considered in this paper is representative enough and that the methods developed in its context can be applied to a wide class of problems.
3. Scalar field probability distributions

3.1. Direct (in time) transition probability

The transition probability distribution is given by Eq. (1.5), and, in the saddlepoint approximation, by Eq. (1.9). From Eq. (1.6) it is clear that the peak of the distribution (1.9) is at $\phi_f = \phi_d(\tau)$, where $\phi_d(\tau)$ is the direct (in time) extremal path with the lowest possible action, that is, the solution to the equation

$$\frac{d\phi}{d\tau} = -A(\phi), \quad (3.1)$$

with the initial condition $\phi(0) = \phi_i$.

We can develop the function $I(\phi_f, \phi_i, \tau)$ in powers of $\phi_f - \phi_d(\tau)$ as follows

$$I(\phi_f, \phi_i, \tau) = \frac{(\phi_f - \phi_d(\tau))^2}{2! \Delta_d(\tau)} + \frac{(\phi_f - \phi_d(\tau))^3}{3! \Gamma_d(\tau)} + \ldots, \quad (3.2)$$

and restrict ourselves to terms quadratic in $\phi_f - \phi_d(\tau)$. This assumes the distribution to be close to Gaussian. The expression for the value $\Delta_d(\tau)$ is easily obtained by making use of the Hamilton-Jacobi equation for the function $I(\phi_f, \phi_i, \tau)$. The result is

$$\Delta_d = A^2(\phi_d) \left| \int_{\phi_i}^{\phi_d} \frac{B(\phi)}{A^3(\phi)} d\phi \right|. \quad (3.3)$$

In the work [4] the same expression for the variance $\Delta_d$ of the scalar field distribution in Gaussian approximation was obtained by a different method.

As an example, consider the case of the power law potential

$$V(\phi) = \frac{\lambda \phi^n}{nM_{\text{Pl}}^{n-4}} \quad (n \text{ is even}). \quad (3.4)$$
If $\tau$ is the cosmological time then taking into account (1.2) we will have

$$\Delta d = \frac{4}{3} \frac{\phi_i^4 - \phi_d^4}{n^2 M^2_{\text{Pl}}} \lambda \phi_d^{n-2}. \quad (3.5)$$

If $\tau$ is the expansion power, $\tau = \log a$, we obtain using (1.3)

$$\Delta d = \frac{16}{3} \frac{\phi_i^{n+4} - \phi_d^{n+4}}{n^2 (n + 4) M^2_{\text{Pl}}} \lambda \phi_d^{-2}. \quad (3.6)$$

The dramatic difference between the variances (3.5) and (3.6) can be seen in their behaviour at sufficiently small $\phi_d$. For $n > 2$ if $\tau$ is the cosmological time then the variance, which is given by (3.5), is decreasing with time when $\phi_d$ is sufficiently small. The fluctuations thereby always remain sharply peaked around their maximum. Yet if $\tau$ is the expansion power the variance is increasing rather than decreasing. Surprisingly enough, the approaches to the scalar field statistics that seem to be just slightly different bring so different results.

Natural question is: when is it correct to treat the fluctuations as Gaussian. The answer is that it is possible to do so if the condition (1.10) is valid (see the work [6] and Chapter 4 of the present paper where this is shown for the exactly solvable case of $\lambda \phi^4$ potential). We remember (this has been shown in Chapter 2) that under this condition also the saddle-point approximation is valid. The condition (1.10) thus turns out to be sufficient for all the assumptions made in this paper.

3.2. Reverse (in time) transition probability

Besides the direct (in time) transition probability we can consider the probability that the scalar field initially had the value $\phi_i$ if after the time $\tau$ it has the value $\phi_f$. The equation for this probability is obtained from (1.1) by replacing the
differential operator in the right-hand side by its conjugate [1]

\[ \frac{\partial P(\phi, \tau)}{\partial \tau} = -A(\phi) \frac{\partial P(\phi, \tau)}{\partial \phi} + \frac{1}{2} B^\alpha(\phi) \frac{\partial}{\partial \phi} B^{1-\alpha}(\phi) \frac{\partial P(\phi, \tau)}{\partial \phi}. \] (3.7)

It can be shown that the solution to this equation which describes the reverse (in time) transition from \( \phi_i \) to \( \phi_f \) is given by the same function \( Z(\phi_f, \phi_i, \tau) \) represented by the path integral (1.5). In the saddle-point approximation this probability is given by the equation (1.9). The peak of the distribution (1.9), regarded as function of \( \phi_i \), is at \( \phi_i = \phi_r(\tau) \), where \( \phi_r(\tau) \) is the reverse extremal trajectory with minimal possible action, or solution to the equation

\[ \frac{d\phi}{d\tau} = A(\phi), \] (3.8)

with the initial condition \( \phi(0) = \phi_f \).

The extremal action in (1.9) can be developed in powers of \( \phi_i - \phi_r(\tau) \) (similar to its development (3.2) in powers of \( \phi_f - \phi_d \))

\[ I(\phi_f, \phi_i, \tau) = \frac{(\phi_i - \phi_r(\tau))^2}{2! \Delta_r(\tau)} + \frac{(\phi_i - \phi_r(\tau))^3}{3! \Gamma_r(\tau)} + \ldots, \] (3.9)

and the expression for \( \Delta_r(\tau) \) is easily found to be

\[ \Delta_r = A^2(\phi_r) \left| \int_{\phi_f}^{\phi_r} \frac{B(\phi)}{A^3(\phi)} d\phi \right|. \] (3.10)

For the scalar field potential (3.4) in the case when \( \tau \) is the cosmological time we will have

\[ \Delta_r = \frac{4}{3} \left| \phi_r^{n+4} - \phi_f^{n+4} \right| \lambda \phi_r^{n-2}. \] (3.11)

If \( \tau \) is the expansion power, \( \tau = \log a \), we obtain

\[ \Delta_r = \frac{16}{3} \left| \phi_r^{n+4} - \phi_f^{n+4} \right| \lambda \phi_r^{-2}. \] (3.12)
We see that in both cases as $\phi_r \gg \phi_f$ the behaviour of the variance is

$$\Delta_r \sim \frac{V(\phi_r)}{M_p^4} \phi_r^2. \quad (3.13)$$

As $\phi_r^2$ grows with time, the variance $\Delta_r$ is also growing, but as long as $V(\phi_r) \ll M_p^4$ the distribution is sharply peaked around its maximum.

### 3.3. Conditional probability distribution

In most of the works on the topic under consideration only simple direct or reverse transition probability distributions were considered. However it seems even more interesting to inquire about the conditional distributions. Specifically, the distribution $Z(\phi_f, \phi_i, \tau_*; \phi, \tau)$ for the values of the scalar field $\phi$ at the moment of time $\tau < \tau_*$ under the condition that the scalar field passes from $\phi_i$ to $\phi_f$ in a time $\tau_*$ is given by

$$Z(\phi_f, \phi_i, \tau_*; \phi, \tau) = \frac{Z(\phi_f, \phi, \tau_* - \tau) \cdot Z(\phi, \phi_i, \tau)}{Z(\phi_f, \phi_i, \tau_*)}. \quad (3.14)$$

In Gaussian approximation this expression will look as follows

$$Z(\phi_f, \phi_i, \tau_*; \phi, \tau) \approx \mathcal{N} \exp \left( -\frac{(\phi - \phi_m)^2}{2\Delta_m} \right), \quad (3.15)$$

where $\phi_m$ is the value of the scalar field $\phi$ at the peak of the distribution, and $\Delta_m$ is the variance analogous to $\Delta_d$ and $\Delta_r$ of the previous sections. If the value of $\tau_*$ is sufficiently close to the amount of time it takes to proceed from $\phi_i$ to $\phi_f$ along the trajectory defined by (3.1), then the peaks of both multipliers in the numerator of (3.14) at the moment $\tau < \tau_*$ will be around the same value of $\phi \approx \phi_1(\tau) \approx \phi_2(\tau)$. The values $\phi_1(\tau)$ and $\phi_2(\tau)$ are defined as follows: $\phi_1(\tau)$ is the starting point of the extremal path determined by (3.1) which ends at $\phi_f$ and which lasts for the time $\tau_* - \tau$, and $\phi_2(\tau)$ is the end point of the extremal path determined by (3.8).
which starts at $\phi_i$ and which lasts for the time $\tau$. Then writing for both multipliers in the numerator of Eq. (3.14) the expressions similar to (1.9) and developing the extremal actions in the way similar to (3.2) and (3.9) we can obtain the following approximate expression for (3.14)

$$Z(\phi_f, \phi_i, \tau^*; \phi, \tau) \approx N \exp \left( -\frac{(\phi - \phi_1)^2}{2\Delta_1} - \frac{(\phi - \phi_2)^2}{2\Delta_2} \right). \quad (3.16)$$

The variances $\Delta_1$ and $\Delta_2$ are given by the expressions similar to (3.3) and (3.10). From (3.16) one can see that the values $\phi_m$ and $\Delta_m$ in (3.15) are given by

$$\phi_m = \frac{\phi_1 \Delta_2 + \phi_2 \Delta_1}{\Delta_1 + \Delta_2}, \quad (3.17)$$

$$\Delta_m = \frac{\Delta_1 \Delta_2}{\Delta_1 + \Delta_2}. \quad (3.18)$$

We can see that the peak $\phi_m$ of the distribution (3.14) is between $\phi_1$ and $\phi_2$, and the variance $\Delta_m$ is smaller than in the case of the unconditioned distribution (1.9). We must note, however, that if the value of $\tau^*$ strongly deviates from the value of time it takes to proceed from $\phi_i$ to $\phi_f$ along the trajectory defined by (3.1), then the approximation (3.16) will be not good. In this case higher order terms in the developments (3.2) and (3.9) of the extremal actions will contribute significantly to the values of $\phi_m$ and $\Delta_m$. In Chapter 5 the variance of the conditional probability distribution will be calculated more precisely using path functional statistics.
4. Exact analytic solution

In the paper [6] it was demonstrated that the stochastic equations can be solved analytically for the theory with quartic potential

\[ V(\phi) = \frac{\lambda}{4} \phi^4, \]  

(4.1)

and in the case when \( \tau \) is proportional to the cosmological time \( t \). Derivation in [6] was based on the Langevin stochastic equation from which the Fokker-Planck equation (1.1) is usually obtained (see the Appendix). In this chapter we show that the case under consideration in our formalism is described by a solvable path integral\(^1\) (see also Ref. 7). If we return to the expressions (1.5)-(1.7), make a transformation from \( \phi \) to a new variable

\[ x = \sqrt{\frac{3}{2\lambda}} \left( \frac{M_P}{\phi} \right)^2, \]

(4.2)

and (for convenience) rescale the time as follows

\[ \tau = \sqrt{\frac{2\lambda}{3\pi}} M_P t, \]

(4.3)

then the expression (1.6) for the action \( I \) will acquire a simple form

\[ I[x(\tau)] = \frac{1}{2} \int (\dot{x} - x)^2 \, d\tau, \]

(4.4)

and the expression similar to (1.5) will become

\[ Z(x_f, x_i, \tau) = \mathcal{N} \int \exp (-I[x(\tau)]) \, [dx], \]

(4.5)

with the standard measure

\[ [dx] = \prod_\tau dx(\tau). \]

(4.6)

Path integral (4.5) with the quadratic action (4.4) can be calculated exactly

\(^1\) Such a possibility was communicated to us by V. Mukhanov.
and one obtains the distribution for \( x_f \) with fixed initial \( x_i \) (direct transition probability) in the form

\[
Z(x_f, x_i, \tau) = \mathcal{N} \exp \left( -\frac{(x_f - x_d(\tau))^2}{(x_d(\tau)/x_i)^2 - 1} \right) . \tag{4.7}
\]

The same expression can be rewritten to yield the distribution for \( x_i \) with fixed final \( x_f \) (reverse transition probability) as follows

\[
Z(x_f, x_i, \tau) = \mathcal{N} \exp \left( -\frac{(x_i - x_r(\tau))^2}{1 - (x_r(\tau)/x_f)^2} \right) . \tag{4.8}
\]

The definition of the functions \( x_d(\tau) \) and \( x_r(\tau) \) is quite analogous to that of \( \phi_d(\tau) \) and \( \phi_r(\tau) \) of the previous section: \( x_d(\tau) \) is the solution to the equation of motion that stems from the action (4.4):

\[
\dot{x} = x , \tag{4.9}
\]

with the initial condition \( x(0) = x_i \), and \( x_r(\tau) \) is the solution to the reverse (in time) equation

\[
\dot{x} = -x , \tag{4.10}
\]

with the initial condition \( x(0) = x_f \).

The distribution (4.7) was first obtained in Ref. 6 by solving the stochastic Langevin equation which corresponds to the Fokker-Planck equation (1.1). In [6] this expression was also analysed. Here we shall perform similar brief analysis of the distributions (4.7) and (4.8) to demonstrate excellent agreement of the saddle-point approximation of the previous chapter with these exact distributions. Using (4.1) and (4.2) we can express the distributions (4.7) and (4.8) in terms of the
scalar field variable $\phi$ as follows

$$Z(\phi_f, \phi_i, \tau) = \mathcal{N} \exp \left(-\frac{3M_P^4 (\phi_f - \phi_d(\tau))^2 (\phi_f + \phi_d(\tau))^2}{8 (V_i - V_d(\tau)) \phi_f^4}\right)$$

(4.11)

$$= \mathcal{N} \exp \left(-\frac{3M_P^4 (\phi_i - \phi_r(\tau))^2 (\phi_i + \phi_r(\tau))^2}{8 (V_r(\tau) - V_f) \phi_i^4}\right),$$

where we made notations $V_i = V(\phi_i), V_f = V(\phi_f), V_d(\tau) = V(\phi_d(\tau)), V_r(\tau) = V(\phi_r(\tau))$.

It is straightforward to see that as long as the conditions (which are essentially (1.10))

$$\frac{V_i - V_d(\tau)}{M_P^4} \ll 1,$$

(4.12)

and

$$\frac{V_r(\tau) - V_f}{M_P^4} \ll 1,$$

(4.13)

are valid, the distributions given by (4.11) are to a great precision Gaussian with peak and variance in accord with the saddle-point approximation of the previous chapter. Thus, validity of our approach under the condition (1.10) is justified also by comparison with an exact analytic solution.

5. Path functional statistics

The aim of this chapter is to develop methods of investigation the statistics of path functionals. That is, we shall learn to calculate mean values $\langle F \rangle$ of path functionals $F[\phi(\tau)]$. In this paper we shall deal with the conditional statistics only, the condition being that the scalar field passes from $\phi_i$ to $\phi_f$ in a time $\tau_*$.

If it is known, statistics of any other type in principle can be easily determined.
Like in the previous chapters we restrict ourselves to the saddle-point approximation. In Chapter 2 we have seen that this approximation is good enough for the scalar field probability distributions. For it to be also good for the functional statistics, the functionals are to vary sufficiently slowly (not exponentially) in the space of paths $\phi(\tau)$, the condition which we shall assume valid.

Consider a set of path functionals $\{F_i\}, i = 1, 2, \ldots$. Then knowing their statistics is equivalent to knowing the generating function $W$ of the set of real variables $\{\mu_i\}$ which is defined as follows

\[
\exp (-W(\mu)) = \mathcal{N} \int \exp \left( -I[\phi(\tau)] - \sum_k \mu_k F_k[\phi(\tau)] \right) \, d[\phi],
\]

(5.1)

where by $\mu$ we shortly denoted the whole set $\{\mu_i\}$. Functional integral in the last expression must be evaluated under an appropriate condition; as it is clear from what has been emphasized above, in our case this is the condition of the scalar field passing from $\phi_i$ to $\phi_f$ in a time $\tau_*$. Normalization constant $\mathcal{N}$ in (5.1) is supposed to be chosen in such a way that $W(0) = 0$. We shall use the following properties of the function $W(\mu)$:

\[
\frac{\partial W(0)}{\partial \mu_i} = < F_i >,
\]

(5.2)

\[
\frac{\partial^2 W(0)}{\partial \mu_i \partial \mu_j} = - < \delta F_i \delta F_j >,
\]

(5.3)

where $\delta F_i = F_i - < F_i >$.

In the saddle-point approximation the expression for $W(\mu)$ is given by

\[
W(\mu) \approx I(\mu) - I(0) + \sum_k \mu_k F_k(\mu),
\]

(5.4)

where $I(\mu)$ and $F_k(\mu)$ are the corresponding functionals’ values on the extremal
path of the extended “action”

\[ I_{\text{ext}}[\phi(\tau)] \equiv I[\phi(\tau)] + \sum_k \mu_k F_k[\phi(\tau)]. \quad (5.5) \]

All the expressions below will refer to the saddle-point approximation.

From the above definitions it is clear that

\[ < F_i >_{\mu \neq 0} = \frac{\partial W(\mu)}{\partial \mu_i} = F_i(\mu), \quad (5.6) \]

\[ < \delta F_i \delta F_j > = - \frac{\partial F_i(0)}{\partial \mu_j} = - \frac{\partial F_j(0)}{\partial \mu_i}. \quad (5.7) \]

Rewriting the last expression in somewhat other terms we obtain

\[ < \delta F_i \delta F_j > = - \int_0^\tau d\tau \frac{\delta F_i[\phi_c]}{\delta \phi(\tau)} \frac{\partial \phi_c(\tau, 0)}{\partial \mu_j}, \quad (5.8) \]

where \( \phi_c(\tau, \mu) \) is the extremal path of the extended action (5.5). All that remains
to do now is to calculate the value \( \partial \phi_c(\tau, 0)/\partial \mu_j \). Let us denote this value by \( \beta_j(\tau) \):

\[ \beta_j(\tau) \equiv \frac{\partial \phi_c(\tau, 0)}{\partial \mu_j}. \quad (5.9) \]

Differential equation for \( \beta_j(\tau) \) can be obtained from the equation of motion for \( \phi(\tau, \mu) \). It can be put in the following form

\[ \frac{d}{d\tau} \left( \frac{\beta_j}{\phi_c} \right) = \frac{B(\phi_c)}{\phi_c^2} \left( \int_0^{\tau} d\sigma \frac{\delta F_j[\phi_c]}{\delta \phi(\sigma)} \phi_c(\sigma) + C_j \right), \quad (5.10) \]

where \( C_j = \text{const} \). In this expression \( \mu \) everywhere should be put to zero. The
solution to the last equation is

$$\beta_j(\tau) = \dot{\phi}_c(\tau) \int_0^\tau d\sigma \frac{B(\phi_c(\sigma))}{\dot{\phi}_c^2(\sigma)} \left( \int_0^\sigma d\rho \frac{\delta F_j[\phi_c]}{\delta \phi(\rho)} \dot{\phi}_c(\rho) + C_j \right). \quad (5.11)$$

The constants $C_j$ are determined from the condition

$$\beta_j(\tau_*) = 0, \quad (5.12)$$

which stems from our boundary condition $\phi(\tau_*, \mu) \equiv \phi_f$.

Now inserting the expression for $\beta_j(\tau)$ given by (5.11) into (5.8) and making use of the condition (5.12) we obtain our final result in the form

$$< \delta F_i \delta F_j > = \int_0^{\tau_*} d\tau \frac{B(\phi_c(\tau))}{\dot{\phi}_c^2(\tau)} f_i(\tau) F_j(\tau) - \int_0^{\tau_*} d\tau \frac{B(\phi_c(\tau))}{\dot{\phi}_c(\tau)} f_i(\tau) \cdot \int_0^{\tau_*} d\tau \frac{B(\phi_c(\tau))}{\dot{\phi}_c(\tau)} f_j(\tau), \quad (5.13)$$

where

$$f_k(\tau) = \int_0^\tau d\rho \frac{\delta F_k[\phi_c]}{\delta \phi(\rho)} \dot{\phi}_c(\rho). \quad (5.14)$$

Let us consider a simple example. Let $F[\phi(\tau)] = \phi(\xi)$. In this case the formula (5.13) yields

$$< \delta \phi(\xi) \delta \phi(\eta) > = \frac{\dot{\phi}_c(\xi) \dot{\phi}_c(\eta) \int_0^\eta d\rho \frac{B(\phi_c(\rho))}{\dot{\phi}_c(\rho)} \cdot \int_0^{\tau_*} d\sigma \frac{B(\phi_c(\sigma))}{\dot{\phi}_c(\sigma)} \cdot \int_0^{\tau_*} d\tau \frac{B(\phi_c(\tau))}{\dot{\phi}_c(\tau)}}{\int_0^{\tau_*} d\tau \frac{B(\phi_c(\tau))}{\dot{\phi}_c(\tau)}}, \quad (5.15)$$

for $\eta \leq \xi$. It is easy to show that at $\eta = \xi$, and if $\tau_*$ is the time in which the extremal trajectory defined by (3.1) proceeds from $\phi_i$ to $\phi_f$, this expression coincides with
the variance given by the equation (3.18). This, of course, is of no surprise since in this case we calculate one and the same value. If the time $\tau_*$ deviates strongly from the time in which the extremal trajectory (3.1) passes from $\phi_i$ to $\phi_f$ then the expression (3.18) for the variance ceases to be valid, and one has to use more exact expression (5.15).

6. Discussion

In this concluding chapter we shall briefly discuss possible applications of our approach. To our mind the biggest advantage of the functional approach is in the possibility of dealing with the statistics of the inflationary-history-dependent values. Of course, this will be of essential interest only for observable (directly or indirectly) values of such type. It is a general opinion that in the context of the chaotic inflation no present observable value is sensitive to the inflationary history in the very remote past [2]. The reason for this belief is that the spatial scales which correspond to the observable structure are much smaller than the Planck scale at most part of the inflationary history. They become larger than the Planck scale only at the (relative) end of inflation and it is assumed that after this moment there is no “memory” on the spatial scales of interest about the past evolution of the universe. However, this suggestion is not so evident and might turn out to be not quite correct. Thus in our work [5] we followed another assumption, namely, that the quantum state of the universe at the end of inflation might contain information about the whole past evolution of the universe on the inflationary stage. We assumed that the quantum state of the universe can be calculated using the standard quantum field theory methods even on spatial scales much below the Planck scale. Our result was that under these assumptions stochastic inflation can lead to anisotropic spectrum of the primordial density fluctuations. The degree of this anisotropy is a functional of the type discussed in the present paper. In our work [5] the functional method developed in the present paper was partially used. Although the value discovered in [5] is the only known example of interest which
depends substantially on the past inflationary history, other possibilities may well appear in the future. It is for such inflationary-history dependent functionals that the methods developed in this paper will be of use.

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APPENDIX

In this appendix we demonstrate how a path-integral solution to the diffusion equation (1.1) with $\alpha = 1/2$ (Stratonovich form) can be derived also from the underlying Langevin equation using functional methods.

Consider the following Langevin equation

$$\dot{x} = -A(x) + C(x)\eta,$$  
(A.1)

with random gaussian function $\eta(t)$ representing white noise

$$\langle \eta(t)\eta(u) \rangle = \delta(t - u).$$  
(A.2)

The formal probability distribution in the space of functions $\eta(t)$ is given by

$$\exp\left(-\frac{1}{2} \int \eta^2(t) \, dt\right).$$  
(A.3)
Given a functional $F[x(t)]$ we can calculate its mean value:

$$< F[x(t)] > = \mathcal{N} \int [d\eta] F[x_\eta(t)] \exp \left( -\frac{1}{2} \int \eta^2(t) \, dt \right),$$

(A.4)

where $x_\eta(t)$ is the solution to (A.1), and the formal path integration measure in (A.4) is

$$[d\eta] = \prod_t d\eta(t).$$

(A.5)

If the initial condition is $x(0) = x_0$ and $F[x(t)] = \delta(x - x(t))$, we obtain the “Green’s function”

$$Z(x, x_0, t) = < \delta(x - x(t)) > .$$

(A.6)

It can easily be shown [8] that this function obeys the following Fokker-Planck equation

$$\frac{\partial Z(x, x_0, t)}{\partial t} = \frac{\partial}{\partial x} \left( A(x) Z(x, x_0, t) + \frac{1}{2} C(x) \frac{\partial}{\partial x} C(x) Z(x, x_0, t) \right).$$

(A.7)

This equation is just the equation (1.1) with $B = C^2$, $\alpha = 1/2$, written in terms of the variable $x$ instead of $\phi$.

It is natural to proceed in (A.4) from integration over $\eta(t)$ to integration over $x(t)$. We will have then

$$< F[x(t)] > = \mathcal{N} \int [dx] F[x(t)] \exp \left( -I[x(t)] \right).$$

(A.8)

The “action” $I[x(t)]$ in this expression is given by (compare with (2.1))

$$I[x(t)] = \frac{1}{2} \int dt \frac{(\dot{x} + A(x))^2}{C^2(x)},$$

(A.9)

and the formal measure for path integration is found to be

$$[dx] = \prod_t \frac{dx(t)}{C(x(t))}.$$

(A.10)

We see that (A.8), (A.9) and (A.10) coincide respectively with (1.8), (1.6) and
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