Jordan algebras and 3-transposition groups

Tom De Medts    Felix Rehren

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Abstract

An idempotent in a Jordan algebra induces a Peirce decomposition of the algebra into subspaces whose pairwise multiplication satisfies certain fusion rules $\Phi(1/2)$. On the other hand, 3-transposition groups $(G, D)$ can be algebraically characterised as Matsuo algebras $M_\alpha(G, D)$ with idempotents satisfying the fusion rules $\Phi(\alpha)$ for some $\alpha$. We classify the Jordan algebras $J$ which are isomorphic to a Matsuo algebra $M_{1/2}(G, D)$, in which case $(G, D)$ is a subgroup of the (algebraic) automorphism group of $J$; the only possibilities are $G = \text{Sym}(n)$ and $G = 3^2 : 2$. Along the way, we also obtain results about Jordan algebras associated to root systems.

The celebrated theory of 3-transposition groups, developed by B. Fischer in the 1960s, captures the symmetric groups, certain finite groups of Lie type over small characteristic, and the sporadic Fischer groups, and had a profound impact on 20th century group theory. The inherent combinatorial data has a formulation in terms of certain graphs, called Fischer spaces, captured by F. Buekenhout’s well-known geometrical characterisation of 3-transposition groups. Recently an algebraic characterisation of 3-transposition groups became available [HRS14]—in terms of a special class of nonassociative algebras.

These algebras have their origin in the Griess algebra, the 196884-dimensional nonassociative algebra whose automorphism group is the Monster, the largest sporadic simple group and itself a 6-transposition group. Developments in vertex operator algebras provided a wealth of new examples of Griess-like algebras. Via foundational work of M. Miyamoto and A. A. Ivanov et al., they were axiomatised as $\Phi$-axial algebras: algebras generated by idempotents whose eigenvectors multiply according to some fusion rules $\Phi$. Remarkably, the simplest case $\Phi(\alpha)$, in Table 1, of $\mathbb{Z}/2$-graded fusion rules exactly characterises 3-transposition groups.

We apply this new point of view to study a classical subject in nonassociative algebras: Jordan algebras. It is a well-known fact going back to the beginnings of the subject, that the eigenvalues of any idempotent in a Jordan algebra are $\{1, 0, 1/2\}$ and the eigenspaces satisfy the fusion rules $\Phi(1/2)$. This is the Peirce decomposition. It turns out that $\alpha = 1/2$ leads to very distinguished behaviour for $\Phi(\alpha)$-axial algebras; for all other values of $\alpha$, the classification of
\[ \begin{array}{c|ccc} * & 1 & 0 & \alpha \\ \hline 1 & \{1\} & \emptyset & \{\alpha\} \\ 0 & \{0\} & \{\alpha\} \\ \alpha & \{1,0\} \\ \end{array} \]

Table 1: Jordan fusion rules \( \Phi(\alpha) \)

\( \Phi(\alpha) \)-axial algebras leads to Matsuo algebras of 3-transposition groups. The wilder situation for \( \Phi(1/2) \) accommodates Jordan and Matsuo algebras, but also admits further possibilities.

In this note, we answer the question: which Jordan algebras are Matsuo algebras? Equivalently, we classify the Jordan algebras \( J \) containing a generating set of idempotents which are well-behaved in the sense that they induce automorphisms of \( J \) which form a 3-transposition group inside \( \text{Aut}(J) \).

**Main Theorem** (Theorems 3.1, 4.1, 4.7, 5.1). Let \( F \) be a field, \( \text{char}(F) \neq 2 \), and let \( J \) be a Jordan algebra over \( F \) generated by idempotents whose associated involutions generate a 3-transposition group \((G, D)\). Then \( J \) is a direct sum of Matsuo algebras \( J_i = M_{1/2}(G_i, D_i) \), where

i. either \( G_i = \text{Sym}(n) \), and \( J_i \) is the Jordan algebra of \( n \times n \) symmetric matrices over \( F \) with zero row sums;

ii. or \( G_i \cong 3^2 : 3 \), and

(a) either \( \text{char}(F) \neq 3 \) and \( J_i \) is the Jordan algebra of hermitian \( 3 \times 3 \) matrices over the quadratic étale extension \( E = F[x]/(x^2 + 3) \),

(b) or \( \text{char}(F) = 3 \) and \( J_i \) is a certain 9-dimensional degenerate Jordan algebra with an 8-dimensional radical.

The paper is organised as follows. Section 1 recalls elementary facts on Fischer spaces, root systems and 3-transposition groups. Section 2 gives definitions and basic results for Jordan and Matsuo algebras. Section 3 proves that the Matsuo algebra for \( \text{Sym}(n) \) is the Jordan algebra of zero-sum \( n \times n \) symmetric matrices, as well as giving details of a construction of a Jordan algebra of projection matrices coming from a root system. Section 4 proves ii. of the Main Theorem; in particular (a) involves recovering a Peirce decomposition in the Matsuo algebra, and for (b) we give a full description of a chain of ideals in the degenerate algebra. Section 5 finally shows that these are the only Matsuo algebras which are also Jordan algebras.

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1 Combinatorial preliminaries

A linear 3-graph is a pair $(\mathcal{G}, \mathcal{L})$, where $\mathcal{G}$ is a set of points and $\mathcal{L} \subseteq 2^\mathcal{G}$ is a set of lines such that any $\ell \in \mathcal{L}$ has size 3 and any two distinct lines intersect in at most one point. Often $\mathcal{G}$ alone refers to $(\mathcal{G}, \mathcal{L})$. In a linear 3-graph $\mathcal{G}$, for any two collinear points $x, y \in \mathcal{G}$ there exists a unique line $\ell$ connecting $x$ and $y$, and a unique element denoted $x \wedge y \in \mathcal{G}$ such that $\ell = \{x, y, x \wedge y\}$. Linear 3-graphs are also known as partial (Steiner) triple systems. A subset $S$ of $\mathcal{G}$ is called a subspace if it is closed under the operation $\wedge$, and if $T$ is any subset of $\mathcal{G}$, then we write $\langle T \rangle$ for the subspace generated by $T$, i.e., the smallest subspace of $\mathcal{G}$ containing $T$.

In such a $\mathcal{G}$, for two distinct points $x, y \in \mathcal{G}$ we write that $x \sim y$ if $x$ and $y$ are collinear, that is, if there exists a line containing $x$ and $y$, and $x \not\sim y$ otherwise. We can partition $\mathcal{G}$ with respect to $x$ as $\{x\} \cup x^\sim \cup x^\not\sim$, where $x^\sim = \{y \in \mathcal{G} \mid x \sim y\}$ and $x^\not\sim = \{y \in \mathcal{G} \mid x \not\sim y\}$.

The dual $\mathcal{G}^\vee$ of a hypergraph $(\mathcal{G}, \mathcal{L})$ is the graph with point set $\mathcal{L}$ and line set $\{\{\ell \in \mathcal{L} \mid x \in \ell\} \mid x \in \mathcal{G}\}$. The affine plane $\mathcal{P}_n$ of order $n$ (where $n$ is a prime power) is the graph with point set $\mathbb{F}_n^2$ and lines $\{U + v \mid U \leq \mathbb{F}_n^2, \dim U = 1, v \in \mathbb{F}_n^2\}$. Figure 1 illustrates $\mathcal{P}_2^\vee$ and $\mathcal{P}_3$.

![Figure 1: The dual affine plane $\mathcal{P}_2^\vee$ and the affine plane $\mathcal{P}_3$.](image)

**Definition 1.1** ([Asc97]). A Fischer space is a linear 3-graph for which, if $\ell_1, \ell_2$ are any two distinct intersecting lines, the subspace $\langle \ell_1 \cup \ell_2 \rangle$ is isomorphic to the dual affine plane $\mathcal{P}_2^\vee$ of order 2 or to the affine plane $\mathcal{P}_3$ of order 3. A Fischer space is said to be of symplectic type if $\mathcal{P}_3$ does not occur, i.e., if the subspace $\langle \ell_1 \cup \ell_2 \rangle$ is always isomorphic to $\mathcal{P}_2^\vee$.

We will also make use of root systems, for which we refer the reader to any book on the subject, such as [Hum78, Chapter III]. We will write $R$ for a root system, and we write $R = \ldots$
$R_+ \cup R_-$ for some partition of $R$ such that $R_- = -R_+$. (Typically, $R_+$ would be the set of positive roots of $R$, but this is not always necessary.)

**Lemma 1.2.** Suppose $R$ is a simply-laced root system and $G$ is the graph with point set $R_+$ and lines $\{r, s, t\}$ for distinct roots $r, s, t$ spanning a root system of type $A_2$. Then $G$ is a Fischer space of symplectic type.

**Proof.** Suppose that $R$ is a simply-laced root system spanning $V$. Evidently $G$ as defined is a 3-graph. If two lines $\ell_1, \ell_2$ intersect in two points $r, s$, then $r, s$ span a root system of type $A_2$ in a subspace $U \subseteq V$ of dimension 2. Then $U \cap R_+$ has size 3, so the third point in both $\ell_1$ and $\ell_2$ is uniquely determined, so that $\ell_1 = \ell_2$. Thus $G$ is a linear 3-graph.

Now suppose that $\ell_1, \ell_2$ are two distinct intersecting lines, say $\ell_1 = \{r, s, t\}$ and $\ell_2 = \{r, u, v\}$. Therefore $U = \langle \ell_1 \cup \ell_2 \rangle$ in $V$ is 3-dimensional and, as $A_3$ is the only simply-laced root system of rank 3, $\ell_1 \cup \ell_2$ span a root system of type $A_3$, which has 6 positive roots and 4 distinct subspaces $A_2$. It is thus easy to check that $\langle \ell_1 \cup \ell_2 \rangle$ in $G$ is isomorphic to $P_2^\perp$ with 6 points and 4 lines. \qed

By abuse of notation, if $R$ is a simply-laced root system we also use $R$ to denote the Fischer space associated to $R$ by the above lemma.

Finally, we recall from [Asc97]:

**Definition 1.3.** A 3-transposition group is a pair $(G, D)$ where $G$ is a group generated by $D \subseteq G$ a subset of involutions closed under conjugation such that, for any $c, d \in D$, $|cd| \leq 3$.

The 3-transposition group $(G, D)$ of a Fischer space $G$ has $G$ generated by $D = \{\tau(x) \mid x \in G\}$ subject to the relations, for all $x, y \in G$, that $\tau(x)^2 = 1$ and $\tau(x)\tau(y) = \tau(x \wedge y)$. Any center-free 3-transposition group $(G, D)$ can be realised in this way from a unique Fischer space $G$.

We note that the 3-transposition group $(G, D)$ of $P_3$ is $G = 3^2 : 2$, an elementary abelian group of order 9 together with an inverting involution, where $D$ is the conjugacy class of this involution (which has size 9). The 3-transposition group of the Fischer space $A_n$ (of the root system of type $A_n$) is $(\text{Sym}(n + 1), (1, 2)^{\text{Sym}(n + 1)})$.

## 2 Algebraic preliminaries

An $\mathbb{F}$-algebra is a vector space over $\mathbb{F}$ equipped with an $\mathbb{F}$-bilinear multiplication. We do not require our algebras to be associative or unital, but all algebras in this text are commutative.

From now on we only consider fields $\mathbb{F}$ of characteristic $\text{char}(\mathbb{F})$ not 2.

We will be interested in two kinds of algebras: Jordan algebras and Matsuo algebras.
Definition 2.1. A **Jordan algebra** over $\mathbb{F}$ is a commutative algebra $J$ over $\mathbb{F}$ such that $(ab)(aa) = a(b(aa))$ for all $a, b \in J$.

**Example 2.2.** Associative algebras are an important source of Jordan algebras:

i. If $A$ is an associative $\mathbb{F}$-algebra, then $A^+$ with the same underlying vector space and **Jordan product** $x \bullet y = \frac{1}{2}(xy + yx)$ is a Jordan algebra [Alb47]. If $A$ is unital, then $A^+$ is also unital, with the same unit.

ii. If $A$ is an associative $\mathbb{F}$-algebra with involution $*$, then the subspace $\mathcal{H}(A, *) := \{ x \in A \mid x^* = x \}$ forms a Jordan subalgebra; it is called the Jordan algebra of **hermitian elements** of $A$ (with respect to $*$).

iii. The **Jordan algebra of symmetric zero-sum** $n \times n$ matrices over $\mathbb{F}$ is $\mathcal{H}(A, t)$ for $A$ the associative algebra of $n \times n$ matrices whose rows and columns all sum to 0, and $t$ the usual matrix transposition as involution. This algebra is unital if and only if $n \neq 0$ in $\mathbb{F}$, in which case the identity element is the matrix with each diagonal entry equal to $(n - 1)/n$ and each non-diagonal entry equal to $-1/n$.

**Definition 2.3** ([Mat03]). Let $\alpha \in \mathbb{F}$ and $\mathcal{G}$ a linear 3-graph. The **Matsuo algebra** $M_\alpha(\mathcal{G})$ is the $\mathbb{F}$-algebra with basis $\mathcal{G}$, where the multiplication of two basis elements $x, y \in \mathcal{G}$ is given by

$$xy = \begin{cases} x & \text{if } x = y \\ 0 & \text{if } x \not\sim y \\ \frac{\alpha}{2}(x + y - x \wedge y) & \text{if } x \sim y. \end{cases}$$

We will view $\mathcal{G}$ as embedded in $M_\alpha(\mathcal{G})$. Hence any $x \in \mathcal{G}$ is an idempotent, that is, $xx = x$.

To avoid degeneracy, from now on we assume $\alpha \neq 1, 0$.

**Lemma 2.4.** The eigenspaces of $x \in \mathcal{G}$ in $M_\alpha(\mathcal{G})$ are

$$\langle x \rangle$$, its 1-eigenspace, \hspace{1cm} (2)

$$\langle y + x \wedge y - \alpha x \mid y \sim x \rangle \oplus \langle y \mid y \not\sim x \rangle$$, its 0-eigenspace, and \hspace{1cm} (3)

$$\langle y - x \wedge y \mid y \sim x \rangle$$, its $\alpha$-eigenspace. \hspace{1cm} (4)

The algebra $M_\alpha(\mathcal{G})$ decomposes as a direct sum of these eigenspaces for any $x \in \mathcal{G}$.

**Proof.** The points of $\mathcal{G}$ form a basis for $A = M_\alpha(\mathcal{G})$. Evidently $x$ is a 1-eigenvector, and $x \not\sim$ is a set of 0-eigenvectors. Now partition $x\sim$ into sets $\{y, x \wedge y\}$ for $y \sim x$. Then the subspace $\langle x, y, x \wedge y \rangle$ of $A$ is spanned by $x, y - x \wedge y$ and $y + x \wedge y - \alpha x$, and these are 1, $\alpha$, 0-eigenvectors of $x$ respectively: $xx = x$, and

$$x(y - x \wedge y) = \frac{\alpha}{2}(x + y - x \wedge y - x \wedge y + y) = \alpha(y - x \wedge y),$$

$$x(y + x \wedge y - \alpha x) = \frac{\alpha}{2}(x + y - x \wedge y + x \wedge y - y) - \alpha x = \alpha x - \alpha x = 0.$$
Thus every pair \( \{y, x \wedge y\} \) gives a pair of 0, \( \alpha \)-eigenvectors of \( x \), and so we have a bijection between a basis of \( A \) and a collection of linearly independent 1, \( \alpha \)-eigenvectors of \( x \). \( \square \)

**Lemma 2.5** ([HRS14, Theorem 5.3, Proposition 5.4]). Let \( G \) be a Fischer space without isolated points. For each \( x \in G \), let \( \tau(x) \) be the automorphism of \( M_\alpha(G) \) acting on the eigenspaces of \( x \) by

\[
y^{\tau(x)} = \begin{cases} y & \text{if } xy = 0 \text{ or } xy = y, \\ -y & \text{if } xy = \alpha y. \end{cases}
\]

(7)

Then each \( \tau(x) \) is an involution, \(|\tau(x)\tau(y)| \leq 3\) for any \( x, y \in G \), and the map

\[
G \to \text{Aut}(M_\alpha(G)), \quad x \mapsto \tau(x)
\]

is an injection. \( \square \)

The automorphism \( \tau(x) \) is known as a *Miyamoto involution*; restricted to the points \( G \subseteq M_\alpha(G) \), these \( \tau(x) \) are the same as those following Definition 1.3. If \( D_G = \{\tau(x) \mid x \in G\} \) and \( G_G = \langle D_G \rangle \), then \( (G_G, D_G) \) is a 3-transposition group. By this lemma, it is sufficient to find the Matsuo subalgebra \( M_\alpha(G) \) in an algebra \( A \) to realise the 3-transposition \( (G_G, D_G) \) group as a subgroup \( G_G \subseteq \text{Aut}(A) \) of the automorphism group of \( A \).

**Definition 2.6.** *Fusion rules* are a pair \((\Phi, \star)\), consisting of a set \( \Phi \subseteq F \) of *eigenvalues* lying in a field \( F \) and a mapping \( \star : \Phi \times \Phi \to 2^\Phi \). We also use \( \Phi \) to refer to \((\Phi, \star)\).

For example, \( \Phi(\alpha) \) are the *Jordan fusion rules*\(^1\) with eigenvalues \( \{1, 0, \alpha\} \subseteq F \) for \( \alpha \neq 1, 0 \) and \( \star \) symmetric as given by Table 1.

**Definition 2.7.**

i. For \( x \in A \), we call the eigenvalues, eigenvectors and eigenspaces of the adjoint map \( \text{ad}(x) \in \text{End}(A) \) the eigenvalues, eigenvectors and eigenspaces of \( x \), respectively. The \( \alpha \)-eigenspace of \( x \) in \( A \) is denoted \( A^x_\alpha = \{a \in A \mid xa = \alpha a\} \). By extension, if \( \Psi \subseteq F \) is a set, we write \( A^x_\Phi = \bigoplus_{\phi \in \Phi} A^x_\phi \), and \( A^x_\emptyset = 0 \).

ii. An idempotent \( e \) in an algebra \( A \) is a \( \Phi \)-axis if \( \text{ad}(e) \) is diagonalisable, takes all its eigenvalues in \( \Phi \), so that \( A \) decomposes into the direct sum of \( \text{ad}(e) \)-eigenspaces

\[
A = \bigoplus_{\phi \in \Phi} A^e_\phi,
\]

(8)

and the multiplication of eigenvectors satisfies the fusion rules \( \Phi \):

\[
A^e_\phi A^e_\psi \subseteq A^e_{\phi \star \psi}.
\]

(9)

In other words, \( xy \in A^e_{\phi \star \psi} = \bigoplus_{\chi \in \phi \star \psi} A^e_\chi \) for all \( x \in A^e_\phi, y \in A^e_\psi \).

**Lemma 2.8** ([Jac68, Chapter III, Lemma 1.1, p. 119]). Let \( J \) be a Jordan algebra over \( F \), and let \( e \in J \) be an idempotent. Then \( e \) is a \( \Phi(\nicefrac{1}{2}) \)-axis.

\(^1\) This name comes from the fact that the eigenspaces of an idempotent in a Jordan algebra multiply according to these fusion rules for \( \alpha = \nicefrac{1}{2} \); see Lemma 2.8.
The following results, together with Lemma 2.8, give a largely satisfactory answer to the question of which Jordan algebras are axial, that is, generated by \(\Phi(1/2)\)-axes. Recall that \(x \in A\) is nilpotent if there exists some integer \(n\) such that \(x^n = 0\), and that an ideal \(I\) is solvable if there is an integer \(k \geq 0\) such that \(I^{2k} = 0\), where \(I^{2k}\) is defined inductively by \(I^{20} = I\) and \(I^{2k} = (I^{2k-1})^2\).

**Theorem 2.9.** Suppose that \(J\) is a finite-dimensional Jordan algebra over \(F\).

i. ([Alb47, Lemma 4]) If \(a \in J\) is not nilpotent, then \(F[a] \subseteq J\) contains a nonzero idempotent.

ii. ([Alb47, Theorem 5]) There is a unique largest solvable ideal \(\text{Rad}(J)\) of \(J\), called its radical. All elements of \(\text{Rad}(J)\) are nilpotent, and \(J/\text{Rad}(J)\) is a semisimple Jordan algebra.

iii. ([Jac68, Chapter VIII, Section 3, Lemma 2]) If \(J\) is semisimple and \(F\) is algebraically closed, then \(J\) is spanned by idempotents. \(\Box\)

## 3 The Matsuo algebra of the symmetric groups

**Theorem 3.1.** For each integer \(n \geq 1\), the Matsuo algebra \(M_{1/2}(A_n)\) over \(F\) is isomorphic to the Jordan algebra of symmetric zero-sum \((n + 1) \times (n + 1)\) matrices over \(F\) from Example 2.2 iii.

Before the proof, we first present a construction of Jordan algebras arising from root systems. Suppose that \(R\) is a root system of rank \(n\); recall that this means \(R\) spans \(R^n\) with Euclidean form \((\cdot, \cdot)\). Based on the integral lattices of the root systems, following [Car05], we can consider the root systems inside \(V = F^{n+1}\); by assumption \(\text{char}(F) \neq 2\), so \(-1 \neq 1\). We also have to exclude \(\text{char}(F) = 3\) when \(R\) contains \(G_2\) as one of its irreducible components.

For \(v \in V\) with \(v v^t \neq 0\), write \(m_v\) for the projection matrix of the 1-dimensional subspace \(\langle v \rangle \subseteq V\), i.e., \(m_v = \frac{1}{v v^t} v v^t\). Let \(J(R)\) be the subspace spanned by the projection matrices \(\{m_a \mid a \in R\}\). Notice that \(J(R)\) is not closed under matrix multiplication in general. Also note that \(m_a = m_{-a}\), so it suffices to consider the projection matrices for a set \(R_+\) of positive roots.

**Example 3.2.** Let \(V = F^{n+1}\) with standard ordered basis \(v_1, \ldots, v_{n+1}\). There is an embedding \((A_n)_+ = \{a_{ij} = v_i - v_j \mid 1 \leq j < i \leq n + 1\} \subseteq V\), (10) and its projection matrices are

\[
m_{a_{ij}} = \frac{1}{2}(e_{ii} - e_{ij} - e_{ji} + e_{jj}),
\]

(11)

where \(e_{ij}\) is the \((n + 1) \times (n + 1)\) matrix whose entries are 0 everywhere except in position \((i, j)\) where it has entry 1.

**Lemma 3.3.** Let \(R\) be an irreducible root system. Suppose that \(a, b \in R\) are two roots, choose \(k \in \{1, 2, 3\}\) to satisfy \(\|b\| = \sqrt{k} \|a\|\), and assume that \(\text{char}(F) \neq 3\) when \(k = 3\) (i.e., when
R = G_2). Let \((a, b)\) denote the root system generated by \(a\) and \(b\). Then

\[
m_a \cdot m_b = \begin{cases} m_a & \text{if } a = \pm b, \text{ i.e., } \langle a, b \rangle \cong A_1, \\ 0 & \text{if } (a, b) = 0; \langle a, b \rangle \cong A_1 \times A_1, \\ \frac{1}{4}(m_a + k \cdot m_b - m_c) & \text{otherwise: } c \in \langle a, b \rangle \cap \{a \pm b\}; \langle a, b \rangle \cong A_2, B_2 \text{ or } G_2. \end{cases}
\]  

(12)

In particular, \(J(R) = \langle m_a \mid a \in R \rangle\) with the Jordan product \(\bullet\) is a Jordan algebra.

**Proof.** Projections are idempotents, so that \(m_a \cdot m_{-a} = m_a \cdot m_a = m_a^2 = m_a\) for all \(a \in R\).

Suppose that \((a, b) = 0\), so that \(a\) and \(b\) are orthogonal with respect to the Euclidean form. Then \(m_a\) and \(m_b\) are mutually orthogonal projections in \(V\), and hence \(m_a m_b = m_b m_a = 0\), so \(m_a \cdot m_b = 0\).

Suppose now (by replacing \(b\) with its negative if necessary) that the angle between \(a\) and \(b\) in \(R\) lies strictly between \(\pi/2\) and \(\pi\). Then an inspection of the possible root systems of rank 2 shows that \(a\) and \(b\) are the fundamental roots of a root system of type \(A_2, B_2\) or \(G_2\), so in particular \(a + b \in R\). We now deal with these three possible cases separately.

Assume first that \(a\) and \(b\) generate a root system of type \(A_2\). Then without loss of generality, we may assume that \(R = \mathbb{F}^3\) according to Example 3.2, and

\[
a = (1, -1, 0), \quad b = (0, 1, -1), \quad a + b = (1, 0, -1),
\]

(13)

\[
m_a = \frac{1}{2} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_b = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{a+b} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.
\]

(14)

Indeed \(m_a \cdot m_b = \frac{1}{4}(m_a + m_b - m_c)\) in this representation, and hence in general.

Assume next that \(a\) and \(b\) generate a root system of type \(B_2\). Then without loss of generality, we may assume that \(B_2\) in \(\mathbb{F}^2\), and

\[
a = (1, 0), \quad b = (-1, 1), \quad a + b = (0, 1),
\]

(15)

\[
m_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_b = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \quad m_{a+b} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(16)

Here we see that \(m_a \cdot m_b = \frac{1}{4}(m_a + 2m_b - m_c)\).

Assume finally that \(a\) and \(b\) generate a root system of type \(G_2\), and that \(\text{char}(\mathbb{F}) \neq 3\). In the standard construction of \(G_2\) in \(\mathbb{F}^3\),

\[
a = (1, -1, 0), \quad b = (-1, 2, -1), \quad a + b = (0, 1, -1),
\]

(17)

\[
m_a = \frac{1}{2} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad m_b = \frac{1}{2} \begin{pmatrix} 0 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_{a+b} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(18)

Similarly, \(m_a \cdot m_b = \frac{1}{4}(m_a + 3m_b - m_c)\).

The elements \(m_a \in J(R)\), for \(a \in R\), are linearly independent when \(R = A_n\) (see Example 3.2), but this is not true in general. More specifically, we have the following result.

**Lemma 3.4.** Let \(R\) be an irreducible root system of rank \(n\), and assume \(\text{char}(\mathbb{F}) \neq 3\) if \(R = G_2\). Then \(J(R)\) has dimension \(\frac{1}{2}n(n+1)\).
Proof. Let \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) be a set of fundamental roots for \( R \), and view the roots as elements of the row space \( F^{n+1} \). We claim that the set

\[
B = \{\alpha_i^j\alpha_j + \alpha_j^i\alpha_i \mid 1 \leq i \leq j \leq n\}
\]

consisting of \( \frac{1}{2}n(n + 1) \) matrices forms a basis for \( J(R) \).

As any projection matrix \( m_a \) is a scalar multiple of \( a^ja \), \( J(R) \) is spanned by the set \( C = \{a^ja \mid a \in R_+\} \). Since any \( a \in R_+ \) is an integral linear combination \( \sum_{i=1}^n \lambda_i\alpha_i \) of the fundamental roots, \( J(R) \) is contained in \( \text{span}(B) \): as \( \text{char}(F) \neq 2 \) we have \( \alpha_i^j\alpha_i \in \text{span}(B) \), and

\[
a^ja = \sum_{i=1}^n \lambda_i^2\alpha_i^j\alpha_i + \sum_{1 \leq i < j \leq n} \lambda_i\lambda_j(\alpha_i^j\alpha_j + \alpha_j^i\alpha_i). \tag{19}
\]

To prove the converse, that \( B \) is contained in \( \text{span}(C) \), we use induction on the distance \( d = d(i, j) \) between the nodes \( i \) and \( j \), for an element \( \alpha_i^j\alpha_j + \alpha_j^i\alpha_i \in B \), in the Dynkin diagram of \( R \) formed by \( \Pi \). The distance is well-defined because \( R \) is irreducible. The claim is obvious for \( d = 0 \), that is, \( i = j \). Assume now that each element \( \alpha_i^j\alpha_j + \alpha_j^i\alpha_i \) where \( d(i, j) < d \) is contained in \( \text{span}(C) \), and consider an element \( \alpha_i^j\alpha_j + \alpha_j^i\alpha_i \) with \( d(i, j) = d \). Let \( a \in R_+ \) be an arbitrary positive root that is an integral linear combination of the roots on the unique path from \( i \) to \( j \) in the Dynkin diagram, having a non-zero coefficient for both \( i \) and \( j \). Write \( a = \lambda_i\alpha_i + \cdots + \lambda_j\alpha_j \), so \( \lambda_i\lambda_j \neq 0 \). It now suffices to expand the expression for \( a^ja \in C \) to see that \( a^ja \) is the sum of \( \lambda_i\lambda_j(\alpha_i^j\alpha_j + \alpha_j^i\alpha_i) \) and terms that are in \( \text{span}(C) \) by the induction hypothesis, and we conclude that \( (\alpha_i^j\alpha_j + \alpha_j^i\alpha_i) \in \text{span}(C) \) as well.

This shows that \( \text{span}(B) = \text{span}(C) = J(R) \), and it remains to show that the elements of \( B \) are linearly independent. This is clear, however, because the set \( \Pi \) extends to a basis of the vector space \( F^{n+1} \), and with respect to this basis, distinct elements of \( B \) have nonzero entries in distinct positions. \( \square \)

Proof of Theorem 3.1. Let \( A_n \) be embedded in \( V \cong F^{n+1} \) as in (10). By the previous Lemmas 3.3, 3.4, \( J(A_n) \) is a Jordan algebra of dimension \( \frac{1}{2}n(n + 1) \) which, since it satisfies the same multiplication, is a quotient of the Matsuo algebra \( M_{I/2}(A_n) \). But \( M_{I/2}(A_n) \) has dimension \( |(A_n)_+| \), and \( |A_n| = n(n + 1) \), so \( \dim M_{I/2}(A_n) = \frac{1}{2}n(n + 1) = \dim J(A_n) \) and therefore \( J(A_n) \) is isomorphic to \( M_{I/2}(A_n) \). From (11), the elements of \( J(A_n) \) are symmetric zero-sum matrices, and hence \( J(A_n) \) is a subalgebra of the Jordan algebra of symmetric zero-sum \( (n + 1) \times (n + 1) \) matrices. As the latter algebra also has dimension \( \frac{1}{2}n(n + 1) \), the result follows. \( \square \)

4 The Matsuo algebra of the group \( 3^2 : 2 \)

For the Matsuo algebra of \( P_3 \), it will turn out that the situation completely degenerates when the characteristic of the underlying field \( F \) is equal to 3. Therefore, we distinguish two cases;
see Theorems 4.1 and 4.7.

For \( \text{char}(F) \neq 3 \), we will need the following definition. Let \( E \) be the quadratic étale extension \( E = F[x]/(x^2 + 3) \) of \( F \), and let \( \sigma \in \text{Gal}(E/F) \) be its non-trivial Galois automorphism. This \( E \) may or may not be a field, depending on whether \( -3 \) is a square in \( F \). We write \( E = \mathbb{F}[\zeta] \) with \( \zeta^2 = -3 \), so in particular \( \zeta^3 = -\zeta \).

The Jordan algebra \( \mathcal{H}_3(E,\ast) \) consists of \( 3 \times 3 \) matrices over \( E \) fixed by \( \ast \), where \( \ast \) is the involution on \( \text{Mat}_3(E) \) given by conjugate transposition, i.e., \( (x_{ij})^\ast = (x_{ji}^\sigma) \); see Example 2.2 ii.

**Theorem 4.1.** Assume that \( \text{char}(F) \neq 2,3 \). The Matsuo algebra \( M_{1/2}(P_3) \) over \( F \) is isomorphic to the Jordan algebra \( \mathcal{H}_3(E,\ast) \).

Recall the affine plane \( P_3 \) from Figure 1 and let \( A = M_{1/2}(P_3) \). For each \( i \in \{1, \ldots, 9\} \), we let \( p_i \) be the generator of the Matsuo algebra corresponding to the point \( i \) in Figure 1.

**Lemma 4.2.** The algebra \( A \) is unital, with \( \text{id} = \frac{1}{9} \sum_{i=1}^{9} p_i \).

**Proof.** Let \( z = \sum_{i=1}^{9} p_i \). By symmetry and linearity, it suffices to verify that \( zp_1 = 3p_1 \). Indeed,

\[
zp_1 = p_1 + \frac{1}{9} \sum_{i=2}^{9} (p_1 + p_j - p_1 \wedge p_j) = 3p_1 + \frac{1}{9} \sum_{i=2}^{9} (p_j - p_1 \wedge p_j) = 3p_1
\]

since each of the 8 elements \( p_2, \ldots, p_9 \) occurs once with each sign in the sum. \( \square \)

We will require idempotents associated with lines of \( P_3 \). Let \( L \) be any of the 12 lines of \( P_3 \). Then we define

\[
e_L = -\frac{1}{3} \sum_{i \in L} p_i + \frac{1}{3} \sum_{i \notin L} p_i \quad \text{and} \quad f_L = \text{id} - e_L = \frac{2}{3} \sum_{i \in L} p_i.
\]

**Lemma 4.3.** For each line \( L \) of \( P_3 \), \( e_L \) and \( f_L \) are idempotents in \( A \). Furthermore, if \( L \) and \( M \) are two parallel lines in \( P_3 \), then \( e_L \) and \( e_M \) are orthogonal, i.e., \( e_L e_M = 0 \).

**Proof.** Without loss of generality, we may assume that \( L = \{1,2,3\} \). Firstly, \( f_L \) is idempotent:

\[
f_L^2 = \frac{4}{9}(p_1 + p_2 + p_3)^2 = \frac{4}{9}(p_1 + p_2 + p_3) + 2 \cdot \frac{4}{9} \cdot \frac{1}{3} \sum_{1 \leq i < j \leq 3} (p_i + p_j - p_i \wedge p_j)
\]

\[
= \frac{4}{9}(p_1 + p_2 + p_3) + \frac{2}{9}(p_1 + p_2 + p_3) = \frac{2}{3}(p_1 + p_2 + p_3) = f_L.
\]

It follows that \( e_L \) is idempotent, as

\[
e_L e_L = (\text{id} - f_L)(\text{id} - f_L) = \text{id} - 2\text{id} f_L + f_L = \text{id} - f_L = e_L.
\]

Now notice that \( e_L e_M = 0 \) if and only if \( f_L f_M = f_L + f_M - \text{id} \). Without loss of generality, for \( L \) and \( M \) parallel, we may assume that \( L = \{1,2,3\} \) and \( M = \{4,5,6\} \). Then, as claimed,

\[
f_L f_M = \frac{4}{9} \sum_{i=1}^{3} \sum_{j=4}^{6} p_i p_j = \frac{1}{9} \sum_{i=1}^{3} \sum_{j=4}^{6} (p_i + p_j - p_i \wedge p_j)
\]
\[ \frac{1}{3} \left( 3 \sum_{i=1}^{6} p_i - 3 \sum_{i=7}^{9} p_i \right) = \frac{1}{3} \sum_{i=1}^{6} p_i - \frac{1}{3} \sum_{i=7}^{9} p_i = f_L + f_M - \id. \]

**Lemma 4.4.** The eigenvalues of \( e_L \) are 1, 0 and \( \frac{1}{2} \). Denote the \( \alpha \)-eigenspace of \( e_L \) by \( A^L_\alpha \). Then
\[
\begin{align*}
A^L_1 &= \{ \lambda e_L \mid \lambda \in \mathbb{F} \}, \\
A^L_0 &= \{ \sum_{i \in L} \lambda_i p_i + \lambda \left( \sum_{i \in M} p_i - \sum_{i \in N} p_i \right) \mid \lambda_i, \lambda \in \mathbb{F} \}, \\
A^L_{1/2} &= \left\{ \left( \sum_{i \in M} \lambda_i p_i + \sum_{j \in N} \mu_j p_j \right) \mid \sum_i \lambda_i = 0, \sum_j \mu_j = 0 \right\}.
\end{align*}
\]

**Proof.** To prove this, without loss of generality, we assume that \( L = \{1, 2, 3\} \), \( M = \{4, 5, 6\} \) and \( N = \{7, 8, 9\} \). Let \( x = \sum_{i=1}^{9} \lambda_i p_i \) be an arbitrary element of \( A \). Then
\[
x e_L = -\frac{1}{6} (\lambda_1 + \cdots + \lambda_9)(p_1 + p_2 + p_3) + \frac{1}{2} (\lambda_4 p_4 + \cdots + \lambda_9 p_9) \\
+ \frac{1}{6} (\lambda_4 + \lambda_5 + \lambda_6)(p_7 + p_8 + p_9) + \frac{1}{6} (\lambda_7 + \lambda_8 + \lambda_9)(p_4 + p_5 + p_6).
\]
It is now straightforward to verify that the elements occurring in the statement of the lemma are indeed eigenvectors for \( e_L \); since the dimensions of these three subspaces are 4, 4 and 1 respectively, they together span all of \( A \), and hence we have found all eigenvectors. \( \square \)

As a consequence of the previous result, we get a “Peirce decomposition” for \( A \), although in fact we have not yet established whether or not \( A \) is a Jordan algebra.

**Corollary 4.5.** Let \( \{L_1, L_2, L_3\} \) be a set of parallel lines in \( P_3 \), and denote the corresponding idempotents by \( e_1, e_2 \) and \( e_3 \), respectively. Let \( A_{ii} = A^L_{ii} = \langle e_i \rangle \) for each \( i \), and let \( A_{ij} = A^L_{ij} \cap A^L_{ji} \) for \( i \neq j \). Then for any choice of \( \{i, j, k\} = \{1, 2, 3\} \), we have
\[
A_{ij} = \left\{ \sum_{\ell \in L_k} \lambda_\ell p_\ell \mid \sum_\ell \lambda_\ell = 0 \right\},
\]
so \( \dim A_{ij} = 2 \), and
\[
A = A_{11} \oplus A_{22} \oplus A_{33} \oplus A_{12} \oplus A_{13} \oplus A_{23}. \quad \square
\]

We now establish some notation for \( \mathcal{H}_3(E, \ast) \). Let \( e_{ij} \) be the usual matrix units in \( \text{Mat}_3(E) \). Following the notation in [Jac68, p. 125], we define
\[
x[ij] = xe_{ij} + x^\sigma e_{ji} \in J
\]
for all \( i, j \); in particular, \( x[ii] = (x + x^\sigma) e_{ii} \) for all \( i \), and \( x[ji] = x^\sigma[ij] \) for all \( i, j \). Recall from [Jac68, p. 126] that the multiplication in \( J \) is completely determined by the multiplication rules
\[
2x[ij] \cdot y[jk] = xy[ik] \quad \text{for all } i, j, k \text{ distinct},
\]
\[
2x[ii] \cdot y[ij] = (x + x^\sigma) y[ij] \quad \text{for all } i \neq j,
\]
\[
2x[ij] \cdot y[ij] = xy[ii] + x^\sigma[ij] \quad \text{for all } i \neq j,
\]

\[
= \frac{1}{3} \left( 3 \sum_{i=1}^{6} p_i - 3 \sum_{i=7}^{9} p_i \right) = \frac{1}{3} \sum_{i=1}^{6} p_i - \frac{1}{3} \sum_{i=7}^{9} p_i = f_L + f_M - \id. \]
\[ 2x[ii] \cdot y[ii] = (x + x')(y + y')[ii] \quad \text{for all } i, \quad (28) \]
\[ x[ij] \cdot y[k\ell] = 0 \quad \text{if } \{i, j\} \cap \{k, \ell\} = \emptyset. \quad (29) \]

Finally, let \( J_{ij} = \{x[ij] \mid x \in E\} \) for all \( 1 \leq i \leq j \leq 3 \), so in particular
\[ J = J_{11} \oplus J_{22} \oplus J_{33} \oplus J_{12} \oplus J_{13} \oplus J_{23}. \quad (30) \]

The final step in our proof is to directly establish the isomorphism.

**Proof of Theorem 4.1.** Consider the decomposition of \( A \) of Corollary 4.5. Let \( \eta \) be the \( \mathbb{F} \)-vector space isomorphism from \( A \) to \( J \) given on each of the six Peirce subspaces by, for \( \lambda, \mu \in \mathbb{F} \),
\[ e_i \mapsto e_{ii} = \frac{1}{2}[ii] \quad \text{for all } i, \]
\[ \lambda p_1 + \mu p_2 - (\lambda + \mu)p_3 \mapsto \left( \frac{3}{4}(\lambda + \mu) + \frac{1}{4}(\lambda - \mu)\right) \zeta \quad (23), \]
\[ \lambda p_4 + \mu p_5 - (\lambda + \mu)p_6 \mapsto \left( \frac{3}{4}(\lambda + \mu) + \frac{1}{4}(\mu - \lambda)\right) \zeta \quad (13), \]
\[ \lambda p_7 + \mu p_8 - (\lambda + \mu)p_9 \mapsto \left( \frac{3}{4}(\lambda + \mu) + \frac{1}{4}(\lambda - \mu)\right) \zeta \quad (12). \]

We will verify that \( \eta \) is an isomorphism of Jordan algebras by going through each of the cases occurring in the multiplication rules (25) through (29).

For case (25), assume that \( i = 1, j = 2 \) and \( k = 3 \); the other possibilities for \( i, j, k \) are completely similar. So let \( x_{12} = \lambda p_7 + \mu p_8 - (\lambda + \mu)p_9 \in A_{12} \) and \( y_{23} = \lambda' p_1 + \mu' p_2 - (\lambda' + \mu')p_3 \in A_{23} \) be arbitrary. Then
\[
2x_{12}y_{23} = \frac{1}{2}(-\lambda'p_4 - \mu'p_6 + \lambda'(\lambda' + \mu')p_5 - \mu'\lambda'p_6 - \mu'\mu_5 + \mu(\lambda' + \mu)p_4) \\
+ (\lambda + \mu)\lambda'p_5 + (\lambda + \mu)\mu'p_4 - (\lambda + \mu)(\lambda' + \mu)p_6) \\
= \frac{1}{2}(-\lambda' + 2\mu' + \lambda' + \mu)\lambda'p_4 + \frac{1}{2}(2\lambda' - \mu' + \lambda' + \mu)p_5) \\
+ \frac{1}{2}(-\lambda' - \mu' - 2\mu' - 2\mu)p_6, \\
\]
so
\[ \eta(2x_{12}y_{23}) = \left( \frac{3}{8}(\lambda' + \mu') + 2\mu' + 2\lambda') + \frac{3}{8}(\lambda' - \mu')\zeta \right) \quad (29). \]

On the other hand,
\[
\left( \frac{3}{8}(\lambda + \mu) + \frac{1}{4}(\lambda - \mu)\zeta \right) \cdot \left( \frac{3}{4}(\lambda' + \mu') + \frac{1}{4}(\lambda' - \mu')\zeta \right) \\
= \left( \frac{9}{32}(\lambda + \mu)(\lambda' + \mu') - \frac{9}{32}(\lambda - \mu)(\lambda' - \mu') \right) + \frac{3}{32}(\lambda + \mu)(\lambda' - \mu') + (\lambda - \mu)(\lambda' + \mu') \zeta \zeta \zeta \\
= \left( \frac{3}{8}(\lambda' + \mu') + 2\mu' + 2\lambda') + \frac{3}{8}(\lambda' - \mu')\zeta \right); \\
\]
we conclude that \( \eta(2x_{12}y_{23}) = 2\eta(x_{12})\eta(y_{23}). \)

The multiplication rule (26) is equivalent to the statement that \( y[ij] \) is a \( \frac{1}{2} \)-eigenvector for \( e_{ii} \). Since \( A_{ij} \) is contained in the \( \frac{1}{2} \)-eigenspace of \( e_i \), it follows that \( \eta(e_iy_{ij}) = \eta(e_i)\eta(y_{ij}) \) for all \( i \neq j \) and all \( y_{ij} \in A_{ij} \).

We now check (27), and again we assume that \( i = 1 \) and \( j = 2 \) since the other cases are completely similar. So let \( x_{12} = \lambda p_7 + \mu p_8 - (\lambda + \mu)p_9 \in A_{12} \) and \( y_{12} = \lambda' p_7 + \mu' p_8 - (\lambda' + \mu')p_9 \in A_{23} \).
be arbitrary. Then
\[ 2x_{12}y_{12} = \left( (\lambda\lambda' + \mu\mu') + \frac{1}{2}(\lambda\mu' + \mu\lambda') \right) (p_7 + p_8 + p_9) = \left( \frac{3}{2}(\lambda\lambda' + \mu\mu') + \frac{3}{4}(\lambda\mu' + \mu\lambda') \right) (e_1 + e_2). \]

On the other hand,
\[
\begin{align*}
&\left( \frac{3}{2}(\lambda + \mu) + \frac{1}{2}(\lambda - \mu)\zeta \right) \cdot \left( \frac{3}{2}(\lambda' + \mu') + \frac{1}{2}(\lambda' - \mu')\zeta \right) \\
&\quad = \left( \frac{3}{2}(\lambda + \mu) + \frac{1}{2}(\lambda - \mu)\zeta \right) \cdot \left( \frac{3}{2}(\lambda' + \mu') - \frac{1}{2}(\lambda' - \mu')\zeta \right) \\
&\quad = \left( \frac{3}{2}(\lambda\lambda' + \mu\mu') + \frac{3}{2}(\lambda\mu' + \mu\lambda') \right) + \frac{3}{4}(\lambda\mu' - \mu\lambda')\zeta,
\end{align*}
\]
and hence
\[
\left( \frac{3}{2}(\lambda + \mu) + \frac{1}{2}(\lambda - \mu)\zeta \right) \left( \frac{3}{2}(\lambda' + \mu') + \frac{1}{2}(\lambda' - \mu')\zeta \right)^{\sigma}[ii] = \left( \frac{3}{2}(\lambda\lambda' + \mu\mu') + \frac{3}{4}(\lambda\mu' + \mu\lambda') \right) e_{ii}.
\]

We conclude that \( \eta(2x_{12}y_{12}) = 2\eta(x_{12})\eta(y_{12}) \).

Case (28) is a consequence of the definition of \( x[ii] = (x + x^\sigma)e_{ii} \) combined with the fact that \( e_i \), by Lemma 4.3, and \( e_{ii} \) are idempotents.

Finally, to deal with case (29), we have to verify that \( A_{ij}A_{k\ell} = 0 \) as soon as \( \{i, j\} \cap \{k, \ell\} = \emptyset \). If \( i = j \) and \( k = \ell \), then this again is an immediate consequence of Lemma 4.3, that \( e_i \) and \( e_j \) are orthogonal idempotents. If \( i = j \) and \( k \neq \ell \), then \( A_{k\ell} \) is contained in the \( \frac{1}{2} \)-eigenspace of both \( e_k \) and \( e_\ell \), and hence in the 0-eigenspace of \( \operatorname{id} - e_k - e_\ell = e_i \); it follows that \( A_{ii}A_{k\ell} = 0 \). \( \square \)

If we knew in advance that \( M_{1/2}(P_3) \) was a Jordan algebra, the calculations in the proof of Theorem 4.1 could be replaced by an application of Jacobson’s “Strong Coordinatization Theorem”, [Jac68, Theorem 5, p. 133]. Indeed, the idempotents \( e_1, e_2 \) and \( e_3 \) are strongly connected, and the coordinatizing algebra, an algebra structure on \( A_{ij} \) for \( i \neq j \), is isomorphic to the \( \mathbb{F} \)-algebra \( E \), which can be obtained from [Jac68, Lemma 3, p. 135]. This is how we obtained the formulas for the isomorphism \( \eta \).

In order to describe the situation in the case of \( \text{char}(\mathbb{F}) = 3 \), we require some more definitions from the theory of Jordan algebras.

**Definition 4.6 ([Jac68]).** Let \( J \) be a Jordan algebra over \( \mathbb{F} \).

i. For every \( a, b \in J \), we define \( U_a(b) := 2a(ab) - a^2b \). This defines, for each \( a \), a linear map \( U_a : J \to J \), known as the \( U \)-operator of \( a \).

ii. An element \( a \in J \) is called an absolute zero divisor if \( U_a \) is the zero map.

iii. An element \( a \in J \) is called trivial if \( U_a \) is the zero map and moreover \( a^2 = 0 \).

**Theorem 4.7.** The Matsuo algebra \( M_{1/2}(P_3) \) over a field \( \mathbb{F} \) of characteristic 3 is isomorphic to a 9-dimensional non-unital Jordan algebra with an 8-dimensional radical \( R \). Furthermore, there is a chain of ideals of \( J \)
\[
0 < Z < T < R < J \quad \text{with dim} \ Z = 1, \ \text{dim} \ T = 6, \ \text{and} \ Z = T^2, \ T = R^2, \quad (32)
\]
such that the elements of $Z$ are trivial, the elements of $T$ are absolute zero divisors, and $J/R$ is a unital Jordan algebra isomorphic to $F$.

Proof. Let $A = M_{1/2}(P_3)$. As previously, we will use the affine plane $P_3$ from Figure 1, and for each $i ∈ \{1, \ldots, 9\}$, we let $p_i$ be the generator of the Matsuo algebra corresponding to the point $i$ in Figure 1. Since $\text{char}(F) = 3$, however, the element $z = \sum_{i=1}^{9} p_i$ is an annihilating element of the algebra $A$, i.e., $zx = 0$ for all $x ∈ A$. In particular, $A$ is non-unital.

It is a straightforward but lengthy calculation to verify that the linearised Jordan identity, see e.g., [McC04], Proposition 1.8.5 (1),

\[(xz)w + ((zw)y)x + ((wx)y)z = (xz)(yw) + (zw)(yx) + (wx)(yz),\]  

holds over $F_3$ and hence $A$ is a Jordan algebra over $F_3$ and over any field extension of $F_3$, that is, over any field of characteristic 3. We performed this check by computer.

We also leave open the straightforward verification that

\[Z = \langle p_1 + \cdots + p_9 \rangle, \quad T = \langle p_i + p_j + p_k | \{i, j, k\} \text{ a line} \rangle, \quad R = \left\langle \sum_{i=1}^{9} \lambda_i p_i \mid \sum_{i=1}^{9} \lambda_i = 0 \right\rangle,\]  

are a chain of ideals in $A$, with $\text{dim} \, Z = 1, \text{dim} \, T = 6, \text{dim} \, R = 8$, and that

\[R^2 = T, \quad T^2 = Z, \quad Z^2 = 0,\]  

and therefore $R$ is a solvable ideal. We already showed that $z$, spanning $Z$, is trivial. To show that $T$ consists of absolute zero divisors, let $t ∈ T$ be arbitrary. As $T^2 = Z$ we have $tt ∈ Z$ and hence $(tt)x = 0$ for all $x ∈ A$. To show that $t(tx) = 0$, by linearity of $x ∈ A$ we may take $x = p_i$ a point. If $ℓ$ is a line containing $p_i$, then $\left(\sum_{p ∈ ℓ} p\right)p_i = 0$; if $p_i ∉ ℓ$, then $p_i$ lies in one of the two lines $ℓ′, ℓ''$ parallel to $ℓ$, say $ℓ′$, and $\left(\sum_{p ∈ ℓ} p\right)p_i = \sum_{p ∈ ℓ} p - \sum_{p ∈ ℓ''} p$. Let $m$ be any line; then

\[\left(\sum_{q ∈ m} \sum_{p ∈ ℓ} p - \sum_{p ∈ ℓ''} p\right) = 0.\]  

As $t$ is a linear sum of terms of the form $\sum_{p ∈ ℓ} p$ for $ℓ$ a line, this shows that $t(tp_i) = 0$ for all $i$ and hence $t(tx) = 0$ for all $x ∈ A$.

The quotient $J/R$ is a 1-dimensional algebra spanned by the image $\bar{p}_1$ of $p_1 ∈ J$, which satisfies $\bar{p}_1 \cdot \bar{p}_1 = \bar{p}_1$, so $J/R$ has unit $\bar{p}_1$ and is isomorphic to the Jordan algebra of the field $F$. Thus $J$ has a nonsolvable quotient, and in particular is not itself solvable. As $R$ has codimension 1, $R$ is the maximal solvable ideal, i.e., the radical, of $J$.

\[\square\]

5 Classification

In the next proof, we will require two well-known definitions and facts. The noncommuting graph on a subset $D ⊆ G$ is the graph $D$ with points $D$ and lines $\{c, d\}$ for $c, d ∈ D$ with $[c, d] ≠ 1$. If $D$ is a generating set of involutions closed under conjugation, then the graph $D$ is
related to the Fischer space of \((G, D)\): it has the same point set, and two points are connected
in the noncommuting graph if and only if they are connected in the Fischer space, but lines are
sets of size 2 instead of size 3. Such noncommuting graphs can also be seen as (simply-laced)
Coxeter diagrams.

Given a graph \(G\) in which two points lie on at most a single edge, we define the Coxeter
group \(\text{Cox}(G)\) on \(G\) to be the pair \((G, D)\) defined as follows: let \(D'\) be a set of generators in
bijection with the points of \(G\), \(G\) be the group with presentation
\[
G = \langle d \in D' \mid d^2 = 1, \ |cd| = 3 \text{ for } c \in D' \text{ collinear, } |cd| = 2 \text{ otherwise}\rangle,
\]
and set \(D = D'^G\).

**Theorem 5.1.** Let \(J\) be a finite-dimensional Jordan algebra over \(\mathbb{F}\) which is also a Matsuo
algebra \(M_{1/2}(G)\). Then \(G\) is a disjoint union of Fischer spaces \(P_3\) and \(A_n, n \geq 1\). In particular,
\(J\) is a direct product of the Jordan algebras described in Theorems 3.1 and 4.1.

**Proof.** Suppose that \(G\) is a Fischer space and that \(G = G_0 \cup G_1\) is a partition into two mutually
disconnected nontrivial Fischer spaces. Then \(M_\alpha(G) = M_\alpha(G_0) \times M_\alpha(G_1)\) as \(\mathbb{F}\)-algebras. Hence
we may assume without loss of generality that \(G\) is connected, and we proceed to show that \(G\)
is either \(A_n\) for some \(n \geq 1\) or \(P_3\).

Recall that the rank of a connected Fischer space \(G\) is the size of a smallest collection of
points generating \(G\). We proceed case-by-case for Fischer spaces of rank at most 4. A connected
Fischer space of rank 1 is a single point, and the associated 1-dimensional algebra is obviously
Jordan; this is the case \(G = A_1\). A connected Fischer space of rank 2 is a line, generated by
two points, so the only Matsuo-Jordan algebra here is \(M_{1/2}(A_2)\). Positive answers in rank 3,
that is, for \(P_2' \cong A_3\) and \(P_3\), are given by Theorems 3.1, 4.1. Recall that these are the only
Fischer spaces of rank 3 by definition.

The rank 4 Fischer spaces are classified by [Hal93, Proposition 2.9]. They are the Fischer
space \(A_4\) and the Fischer spaces of quotients of the 3-transposition groups
\[
(W_2(\tilde{A}_3), D_2), \quad (W_3(\tilde{A}_3), D_3),
\]
\[
(G_4 = 2^{1+6} : SU_3(2)', D_4), \quad \text{and M. Hall’s (}G_5 = 3^{10} : 2, D_5).\]

To define \(W_k(\tilde{A}_n)\), let \(G'\) be the \(\mathbb{F}_k\)-linear permutation representation of \(\text{Sym}(n+1)\), that is, the
semidirect product of \(\text{Sym}(n+1)\) with the module \(\mathbb{F}_k^{n+1}\), where the action is by permutation
of the standard ordered basis \(\{v_1, \ldots, v_{n+1}\}\) of \(\mathbb{F}_k^{n+1}\), and let \(D'\) be the conjugacy class of the
image of \((1, 2) \in \text{Sym}(n+1)\) in the semidirect product \(G'\). Then \(W_k(\tilde{A}_n)\) is the quotient \((G, D)\)
of \((G', D')\) by the diagonal \(\langle v_1 + \cdots + v_{n+1} \rangle\).

The latter two groups are defined by presentations. Let \(C\) be the complete graph on
\(\{a, b, c, d\}\), and \(C'\) the graph obtained from \(C\) by deleting the edge \(\{b, c\}\). Then
\[
G_4 = \text{Cox}(C')/((a^bd)^3 = (a^c d)^3 = (a^{bc}d)^3 = 1),
\]
\[ G_5 = \text{Cox}(C)/(b^c d)^3 = (a^b c)^3 = (a^b d)^3 = (a^c d)^3 = (a^{bd} c)^3 = (a^{cd} b)^3 = (a^{dc} b)^3 = 1. \] (40)

Let \( D_i \) be the image of the Coxeter involutions closed under conjugation in the above quotient for \( i = 4, 5 \). Then \( D_i \) generates \( G_i \) and \((G_i, D_i) \) is a 3-transposition group.

It follows by Theorem 3.1 that the Matsuo algebra of \( A_4 \) is Jordan. This is the only one out of the five groups which gives a Jordan algebra. For the others, we will, in each case, choose 4 generating transpositions \( a, b, c, d \) of \( G \), where \( G \) is one of \( W_2(\tilde{A}_3), W_3(\tilde{A}_3), G_4 \) or \( G_5 \), such that for \( x = a + b + c \) in the algebra \( M_{1/2}(G) \), we find \((xx)(dx) \neq ((xx)d)x \), whence \( A \) is not Jordan. We show that \( A \) is not Jordan for \( G = W_k(\tilde{A}_3) \) and \( k = 2, 3 \) by explicit calculation: set \( a', b', c', d' \) equal to

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix},
\end{pmatrix}
\]

respectively; then \( \{a', b', c', d'\} \) generates \( G' = k^4 \colon \text{Sym}(4) \), and if \( n = \left( \begin{array}{cc} I_4 & 0 \\ 1 & 1 \end{array} \right) \), then \( G = G'/\langle n \rangle, a = a'/\langle n \rangle \) and likewise define \( b, c, d \). It is easy to check that \( a, b, c, d \) are conjugate, and act as transpositions on \( k^3 \subseteq W_k(\tilde{A}_n) \), so that \( a, b, c, d \in D \). For \( (G, D) = W_2(\tilde{A}_3) \), the coefficient of \( a \) in \((xx)d)x \) is \( \frac{3}{8} \), and the coefficient of \( a \) in \((xx)(dx) \) is \( \frac{7}{16} \). For \( (G, D) = W_3(\tilde{A}_3) \), the respective coefficients are \( \frac{12}{32} \) and \( \frac{7}{16} \). We see that \( \frac{3}{8}, \frac{12}{32} \neq \frac{7}{16} \) in any characteristic (not 2 by assumption). In each case this shows that the Jordan identity does not hold.

Abusively, let now \( a, b, c, d \) stand for the images of \( a, b, c, d \) under the quotient \( \text{Cox}(C') \to G_4 \) or \( \text{Cox}(C) \to G_5 \). Then \( x = a+b+c \) in the algebra again gives \((xx)(dx) \neq ((xx)d)x \). In both cases, the idempotent corresponding to \( a^{cd} b \) has a nonzero contribution, namely with coefficient \( \frac{1}{32} \), on only the lefthand side. Therefore the Matsuo algebras of the Fischer spaces of \( (G_4, D_4) \) and \( (G_5, D_5) \) are also not Jordan algebras.

This handles the cases when \( (G, D) \) is one of the 3-transposition groups in (38). These groups admit finitely many quotients with rank 4 Fischer spaces, for which the same method shows that the coefficients of some element in \( (xx)(dx) \) and \((xx)(dx) \) differ by \( \frac{1}{32} \) or \( \frac{1}{64} \), whence these quotients cannot give rise to Jordan algebras either.

Suppose that \( (G, D) \) is a transposition group whose Fischer space \( G \) has rank \( r \) at least 5, such that the Matsuo algebra \( A = M_{1/2}(G, D) \) is Jordan. If \( T \subseteq D \) is a generating set for \( G \) and \( T \) is the noncommuting graph on \( T \), then \( G \) is a quotient of the Coxeter group on \( T \). Suppose that the subspace spanned by \( T' = \{d_1, \ldots, d_4\} \) has rank 4 in \( G \). By the above, \( (T') \cong \text{Sym}(5) \) and \( T' \) is a line with 4 nodes, since the subalgebra of \( A \) generated by \( d_1, \ldots, d_4 \) must itself be Jordan. Therefore if \( T = \{d_1, \ldots, d_r\} \subseteq D \) is a set of generators for \( G \) (which is connected, since the noncommuting graph on \( D \) is connected), then no vertex has valency 3 in \( T \). Therefore \( T \)
is either a line or a loop, corresponding to $A_r$ or $\tilde{A}_{r-1}$. By Theorem 3.1, $M_{1/2}(A_r)$ is Jordan. Suppose $T$ is $\tilde{A}_{r-1}$. Then $G$ is a quotient of $W_k(\tilde{A}_{r-1})$ [Hal93, p. 272]. But $W_k(\tilde{A}_{r-1})$ admits an embedding of $W_k(\tilde{A}_3)$ for all $r \geq 5$: for

$$a' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{r-2}, \quad b' = (1) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{r-3}, \quad c' = I_2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{r-4},$$

$$d' = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{r-5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{r-4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{r-3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{r-2}
\end{pmatrix}, \quad n = \begin{pmatrix} I_{r-1} & 0 \\ 0 & 1 \end{pmatrix},$$

we have that $W_k(\tilde{A}_{r-1})$ is the quotient of $k^r : \text{Sym}(r)$ by $\langle n \rangle$, and $a, b, c, d$ the images of $a', b', c', d'$ in $W_k(\tilde{A}_{r-1})$ generate a subgroup $W_k(\tilde{A}_3)$. Therefore the Matsuo algebra of $W_k(\tilde{A}_3)$ is a subalgebra of $A$, which is not Jordan, so $A$ is not Jordan. Hence the only possibility in rank $r \geq 5$ is that $T$ is $A_r$. 

\[\square\]

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$^2$preferred over the published version in J. Math. Soc. Japan 57: 639–649, 2005.