ON KERNEL OF THE REGULATOR MAP

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Abstract. By using the infinitesimal methods due to Bloch, Green and Griffiths in [1, 4], we construct an infinitesimal form of the regulator map and verify that its kernel is $\Omega^1_{\mathbb{C}/\mathbb{Q}}$, which suggests that Question 1.1 seems reasonable at the infinitesimal level.

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1. Background and question

Let $X$ be a smooth projective curve over the complex number field $\mathbb{C}$. In 1970s, Bloch constructed the regulator map $R: K_2(X) \to H^1(X, \mathbb{C}^*)$ in several ways. Later, Deligne found a different construction by considering $H^1(X, \mathbb{C}^*)$ as the group of line bundles with connections. We recall his construction very briefly as follows.

For $x$ a point on $X$, we use $i_x$ to denote the inclusion $x \to X$. The flasque BGQ resolution of $K_2(O_X)$

$$0 \to K_2(O_X) \to K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} i_x^* K_1(\mathbb{C}(x)) \to 0,$$

shows that $H^0(K_2(O_X))$ can be computed as $\text{Ker}\{ K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)) \}.$

So we have the exact sequence of groups

$$0 \to H^0(K_2(O_X)) \to K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)).$$

It’s known that there exists the following Gysin exact sequence in topology,

$$0 \to H^1(X, \mathbb{C}^*) \to H^1(\mathbb{C}(X), \mathbb{C}^*) \to \bigoplus_{x \in X^{(1)}} \mathbb{C}^*.$$
where $H^1(C(X), \mathbb{C}^*) = \lim_{\to} H^1(X - S, \mathbb{C}^*)$ and $S$ is finite points on $X$.

The main ingredient to construct the regulator map $R: H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ is the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H^0(K_2(O_X)) \\
\downarrow & & \downarrow R \\
0 & \longrightarrow & H^1(X, \mathbb{C}^*) \\
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & K_2(C(X)) \\
\downarrow & \downarrow R & \downarrow \cong \\
\longrightarrow & \longrightarrow & \bigoplus_{x \in X^{(1)}} K_1(C(x)) \\
\end{array}
\]

\[
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \bigoplus_{x \in X^{(1)}} \mathbb{C} \\
\downarrow & \downarrow \sim & \downarrow \\
\longrightarrow & \longrightarrow & \bigoplus_{x \in X^{(1)}} \mathbb{C}. \\
\end{array}
\]

That is, one constructs a map $R: K_2(C(X)) \to H^1(C(X), \mathbb{C}^*)$ and use it to deduce the regulator map $R: H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$. We refer the readers to [1] and Section 6 in [5] for more details.

This regulator map has nice motivic features and is related with a general program of Bloch-Beilinson conjecture. In this short note, we focus on the following question, see Section 2 in [3] for related discussion. To fix notations, for any abelian group $M$, $M_{\mathbb{Q}}$ denotes the image of $M$ in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ in the following.

**Question 1.1** (Conjecture 2.4 in [3]). Let $R: H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ be the regulator map, then $\text{Ker}(R)_{\mathbb{Q}} = K_2(C)_{\mathbb{Q}}$.

This question is very difficult to approach, though it has very simple form. For $X = \mathbb{P}^1$, this conjecture has been verified by Kerr [6].

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## 2. Main results

In this section, we shall define an infinitesimal form of the regulator map $R: H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ and verify its kernel is $\Omega^1_{C/\mathbb{Q}}$. Our approach is inspired by the following result due to Green and Griffiths:

**Theorem 2.1** (Page 74 and page 125 in [4]). Let $X$ be a smooth projective curve over $\mathbb{C}$, the Cousin flasque resolution of $\Omega^1_{X/\mathbb{Q}}$

\[
0 \rightarrow \Omega^1_{X/\mathbb{Q}} \rightarrow \Omega^1_{C(X)/\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} i_{x,*}H^1_x(\Omega^1_{X/\mathbb{Q}}) \rightarrow 0,
\]
is the tangent sequence to $BGQ$ flasque resolution of the sheaf $K_2(O_X)$

$$0 \to K_2(O_X) \to K_2(C(X)) \to \bigoplus_{x \in X^{(1)}} i_x_* K_1(C(x)) \to 0,$$

where the map $\rho$ is known to take principal parts.

It follows that $H^0(\Omega^1_{X/Q})$ can be computed as $\text{Ker}\{\Omega^1_{C(X)/Q} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H^1_x(\Omega^1_{X/Q})\}$. So we have the exact sequence of groups

$$0 \to H^0(\Omega^1_{X/Q}) \to \Omega^1_{C(X)/Q} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H^1_x(\Omega^1_{X/Q}).$$

**Definition 2.2** (page 71 and page 125 in [4]). For $X$ a smooth projective curve over $\mathbb{C}$ and $x$ a point on $X$, there exists a residue map

$$\text{Res} : H^1_x(\Omega^1_{X/Q}) \to \mathbb{C},$$

which is defined as follows:

Using $\Omega^1_{O_{X,x}/Q}(nx)$ to denote the absolute 1-forms with poles of order at most $n$ at $x$, we define $\text{Res}_x$ as the following composition:

$$\Omega^1_{O_{X,x}/Q}(nx) \xrightarrow{\Omega^1_{O_{X,x}/C}(nx)} \Omega^1_{O_{X,x}/Q} \xrightarrow{\text{Res}} \mathbb{C}.$$

If $\xi$ is the local uniformizer centered at $x$, an element of $H^1_x(\Omega^1_{X/Q})$ is represented by the following diagram

$$\begin{cases}
O_{X,x} \xrightarrow{\xi} O_{X,x} & \quad \text{with } O_{X,x}/(\xi^k) \longrightarrow 0 \\
O_{X,x} \xrightarrow{\psi} \Omega^1_{O_{X,x}/Q} & \quad \text{such that } \psi(\frac{\xi^k}{\xi^k}) \in \mathbb{C}.
\end{cases}$$

For such an element, we define $\text{Res}_x(\psi(\frac{\xi^k}{\xi^k})) \in \mathbb{C}$.

It is known that the tangent space to $\mathbb{C}^*$, which is defined to be the kernel of the natural projection:

$$\mathbb{C}[\varepsilon]^* \xrightarrow{\varepsilon=0} \mathbb{C}^*,$$

can be identified with $\mathbb{C}$ and the tangent map $\text{tan} : \mathbb{C}[\varepsilon]^* \to \mathbb{C}$ is given by $z_0 + z_1 \varepsilon \rightarrow \frac{z_1}{z_0}$. This tangent map further induces a map between cohomology groups $\text{tan} : H^1(X, \mathbb{C}[\varepsilon]^*) \to H^1(X, \mathbb{C})$. With this interpretation, one can consider $H^1(X, \mathbb{C})$ as the tangent space to $H^1(X, \mathbb{C}^*)$ (this is used in [1]).

There exists the following Gysin exact sequence in topology:

$$0 \to H^1(X, \mathbb{C}) \to H^1(C(X), \mathbb{C}) \to \bigoplus_{x \in X^{(1)}} \mathbb{C},$$
The boundary map \( H^1(C(X), \mathbb{C}) \to \bigoplus_{x \in X^{(1)}} \mathbb{C} \) can be described via Hodge theory as follows. Let \( D = \{p_1, \ldots, p_n\} \) be finite points on \( X \) and let \( U \) be the open complement, \( U = X - D \). Let \( i_D : D \to X \) denote the inclusion, the residue map Res: \( \Omega^1_X(\log D) \to i_D^* \Omega^1_D^{-1} \) induces Res: \( H^1(\Omega^1_X(\log D)) \to H^0(\Omega^1_D) \). This gives the map Res: \( H^1(U, \mathbb{C}) \to \bigoplus_{i=1}^{n} \mathbb{C} \), by using the identifications \( H^1(\Omega^1_X(\log D)) \cong H^1(U, \mathbb{C}) \) and \( H^0(\Omega^1_D) = H^0(D, \mathbb{C}) \cong \bigoplus_{i=1}^{n} \mathbb{C} \).

The following theorem is an infinitesimal form of diagram (1.1):

**Theorem 2.3.** There exists the following commutative diagram

\[
\begin{array}{cccc}
0 & \to & H^0(\Omega^1_{X/Q}) & \to & \Omega^1_{C(X)/Q} & \to & \bigoplus_{x \in X^{(1)}} H^1(\Omega^1_{X/Q}) \\
\downarrow R' & & \downarrow R' & & \downarrow \text{Res} & \downarrow \text{Res} & \\
0 & \to & H^1(X, \mathbb{C}) & \to & H^1(C(X), \mathbb{C}) & \to & \bigoplus_{x \in X^{(1)}} \mathbb{C},
\end{array}
\]

where the map R’s are the natural maps sending \( d/Qf \) to \( d/Cf \).

**Proof.** The map \( R': \Omega^1_{C(X)/Q} \to H^1(C(X), \mathbb{C}) \) can be described as follows. Let \( U \) be open affine in \( X \), \( H^1(U, \mathbb{C}) \) can be computed as \( \Gamma(U, \Omega^1/U)/d/\mathbb{C}\Gamma(U, O_U) \). Given any element \( \alpha \in \Omega^1_{U/Q} \), its image \([\alpha] \) in \( \Omega^1_{U/C} \) defines an element in \( H^1(U, \mathbb{C}) \).

To check the commutativity of the right square, working locally in a Zariski open affine neighborhood \( U \), we can write an element \( \beta \in \Omega^1_{C(X)/Q} \) as

\[
\beta = \frac{h}{f_1^{l_1} \cdots f_k^{l_k}},
\]

where \( f_1, \ldots, f_k, h \in \Gamma(U, O_U) \) are relatively prime and \( f_i \)'s are irreducible.

The following diagram is commutative:

\[
\begin{array}{ccc}
\frac{h}{f_1^{l_1} \cdots f_k^{l_k}} & \xrightarrow{\rho} & \sum_i \frac{h}{f_1^{l_1} \cdots \hat{f_i} \cdots f_k^{l_k}} \\
\downarrow \text{R'} & & \downarrow \text{Res} \\
\frac{h}{f_1^{l_1} \cdots f_k^{l_k}} & \xrightarrow{\text{Res}} & \sum_i \text{Res}_{x_i}(\frac{h}{f_1^{l_1} \cdots \hat{f_i} \cdots f_k^{l_k}}),
\end{array}
\]

where \( x_i = \{f_i = 0\} \) and \( \hat{f_i} \) means to omit the \( i^{th} \) term.

The map \( R': \Omega^1_{C(X)/Q} \to H^1(C(X), \mathbb{C}) \) induces \( R': H^0(\Omega^1_{X/Q}) \to H^1(X, \mathbb{C}) \). \( \square \)
Let \( \{f_0, g_0\} \in H^0(K_2(O_X)) \) and let \((N, \nabla)\) denote the bundle with connection \(\nabla\), as recalled on page 4 in [1]. There exists the following commutative diagram:

\[
\begin{array}{cccc}
\{f_0, g_0\} & \overset{\varepsilon = 0}{\leftarrow} & \{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\} & \overset{\text{tan}}{\rightarrow} \\
\downarrow R & & \downarrow & \downarrow R' \\
\{f_0, g_0\}^*(N, \nabla) & \overset{\varepsilon = 0}{\leftarrow} & \{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\}^*(N, \nabla) & \overset{\text{tan}}{\rightarrow} \\
\end{array}
\]

The commutativity of left square is trivial. To check the right one, since \(\{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\} = \{f_0, g_0\}\{f_0, 1 + \varepsilon \frac{g_1}{f_0}\}\{1 + \varepsilon \frac{f_1}{f_0}, 1 + \varepsilon \frac{g_1}{f_0}\}\), we reduce to considering \(\{1 + \varepsilon f_1, g_0\}\) which is obvious:

\[
\begin{array}{cc}
\{1 + \varepsilon f_1, g_0\} & \overset{\text{tan}}{\rightarrow} \\
\downarrow & \\
\{1 + \varepsilon f_1, g_0\}^*(N, \nabla) & \overset{\text{tan}}{\rightarrow} \\
\end{array}
\]

where the up tan map is well-known and the down tan map is the formula (2.12) on page 14 in [1].

In this sense, we consider the map \(R' : H^0(\Omega^1_{X/Q}) \to H^1(X, \mathbb{C})\) as the infinitesimal form of the regulator map \(R : H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)\) and compute the kernel of \(R'\).

Since \(H^1(X, \mathbb{C})\) has Hodge decomposition \(H^1(X, \mathbb{C}) \cong H^0(\Omega^1_{X/Q}) \oplus H^1(O_X)\) and the map \(R' : H^0(\Omega^1_{X/Q}) \to H^1(X, \mathbb{C})\) naturally maps \(d_{/Q} f\) to \(d_{/C} f\), so \(R'\) is the composition \(H^0(\Omega^1_{X/Q}) \to H^0(\Omega^1_{X/C}) \hookrightarrow H^1(X, \mathbb{C})\).

Hence \(\text{Ker}(R') = \text{Ker}\{H^0(\Omega^1_{X/Q}) \to H^0(\Omega^1_{X/C})\}\).

**Theorem 2.4.** \(\text{Ker}(R') = \Omega^1_{C/Q}\).

**Proof.** There is a natural short exact sequence of sheaves

\[
0 \to \Omega^1_{C/Q} \otimes_c O_X \to \Omega^1_{X/Q} \to \Omega^1_{X/C} \to 0.
\]

The associated long exact sequence is of the form

\[
0 \to H^0(\Omega^1_{C/Q} \otimes_c O_X) \to H^0(\Omega^1_{X/Q}) \to H^0(\Omega^1_{X/C}) \to H^1(\Omega^1_{C/Q} \otimes_c O_X) \to \cdots.
\]

So the kernel of \(H^0(\Omega^1_{X/Q}) \to H^0(\Omega^1_{X/C})\) can be identified with \(H^0(\Omega^1_{C/Q} \otimes_c O_X) \cong H^0(O_X) \otimes \Omega^1_{C/Q} \cong \mathbb{C} \otimes \Omega^1_{C/Q} \cong \Omega^1_{C/Q}\). 

\[\Box\]
Since the tangent space to $K_2(C)$ is $\Omega^1_{C/Q}$, this theorem suggests that Question 1.1 seems reasonable at the infinitesimal level.

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