UNIFORM CONVERGENCE AND KNOT EQUIVALENCE

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Dedicated to my grandparents, Albert and Elizabeth Kobayashi.

Abstract. Given a uniformly convergent sequence of ambient isotopies \((\mathcal{H}_n)_{n \in \mathbb{N}}\), bijectivity of the limit function \(\mathcal{H}_\infty\) together with a minor compactness condition guarantees that \(\mathcal{H}_\infty\) is also an ambient isotopy. By offloading the uniform convergence hypothesis to a more diagrammatic condition, we obtain sufficient conditions for performing countably-many Reidemeister moves. We use this to construct examples of tame knots with countably-many crossings and discuss what distinguishes these from similar-looking wild curves.

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1. Introduction

This work is based on results from the author’s undergraduate thesis [5].

The goals of this document are to understand when one can apply countable sequences of Reidemeister moves and preserve ambient isotopy in the limit. The motivating examples are the following two curves.

![Two scintillating curves](image)

(a) A wild-looking unknot.  (b) An unknotted-looking wild knot.

**Figure 1.** Two scintillating curves.

**Tameness of Fig. 1a.** As indicated by the caption, the curve in Fig. 1a is tame. In fact, it is the unknot. An explicit ambient isotopy taking it to the unknot can be constructed as follows. From time $t = 0$ to $t = \frac{1}{2}$, use a Reidemeister I move to remove loop $L_1$. Then, from time $t = \frac{1}{2}$ to $t = \frac{3}{4}$, use a Reidemeister I move to remove loop $L_2$. So on and so forth, removing loop $L_n$ between time $t = 1 - \frac{1}{2^n}$ and $t = 1 - \frac{1}{2^n}$.

If one is careful about exactly how the Reidemeister I moves are performed, then the result will be an ambient isotopy. We provide the details in Proposition 4.2.

**Wildness of Fig. 1b.** By contrast, the curve in Fig. 1b is wild, a result established by Ralph Fox in [3]. His argument uses techniques co-developed with Emil Artin in [4], namely, a sort of “invariant” for tameness of arcs. We summarize the relevant result in Appendix A.

**The Problem.** At first, the wildness of the curve in Fig. 1b can appear very counterintuitive. As with the loops in Fig. 1a, any finite number of the “stitches” in Fig. 1b can be safely removed using Reidemeister II moves. The procedure is as follows. First, use a Reidemeister II move to move $L_{-1}$ into $L_1$. Next, use another Reidemeister II move to remove both $L_{-1}$ and $L_2$. 
Note, this leaves us with a shrunk copy of the same diagram we started with. Hence we can repeat the process any finite number of times. In general, to perform step \( n \) we use a Reidemeister II move to slide loop \( L_{2n-3} \) into loop \( L_{2n-1} \), then we remove \( L_{2n-3} \) and \( L_{2n} \) by using another Reidemeister II move.

What’s to stop us from using the same approach as in Fig. 1a, where we just performed step \( n \) between \( t = 1 - \frac{1}{2^n} \) and \( t = 1 - \frac{1}{2^{n+1}} \)? Indeed, if we choose our Reidemeister II moves carefully, it’s possible to guarantee continuity of the resulting limit function.

The issue is with bijectivity on the ambient space. As we show in Section 5, there is no way to choose our sequence of Reidemeister II moves without accidentally dragging points from the ambient space down to the wild point in the limit. The proof is a simple geometric argument but it can be easy to overlook.

**The Upshot.** To clarify the situation, in Theorems 3.3 and 3.8 we generalize the strategy we described for Fig. 1a and make it rigorous. Theorem 3.3 uses the language of ambient homeomorphisms while Theorem 3.8 uses the language of ambient isotopies; other than that, they are equivalent.

The layout of the rest of the paper is as follows:

- **(§2)** In Section 2, we go over the definitions of knots, ambient homeomorphisms, ambient isotopies, PL-ness, and tameness/wildness. It might be helpful to review these concepts since we will be working directly with equivalence in this paper instead of by proxy through Reidemeister’s Theorem. We also give the definition of uniform convergence and recall two elementary results from first courses in Analysis and Topology, respectively; the main theorems will follow from these.

- **(§3)** In Section 3, we give the definition of uniform convergence and use it to build up our main results (Theorems 3.3 and 3.8). In most cases Theorem 3.3 tends to be more ergonomic than Theorem 3.8, but we will generally employ Theorem 3.8 because it seems the language of ambient isotopy is more ubiquitous than that of ambient homeomorphism.

- **(§4)** In Section 4, we apply Theorem 3.8 to various curves with countably-many crossings.
In Section 5, we show what goes wrong if we try to apply Theorem 3.8 to two noteworthy examples; the second is the wild curve from Fig. 1b. Finally, some notation.

| Symbol | Interpretation |
|--------|----------------|
| $f$    | Embedding and/or knot (usually). |
| $h$    | Homeomorphism. |
| $h$    | PL Homeomorphism. |
| $\mathcal{R}$ | Homeomorphism constructed by composing other homeomorphisms. |
| $H$    | Ambient isotopy. |
| $H$    | PL Ambient isotopy. |
| $\mathcal{R}$ | Ambient isotopy constructed by iteratively gluing other ambient isotopies. |
| $(X, d_X)$ | Metric space $X$ with metric $d_X$. |
| $(X, T_X)$ | Topological space $X$ with topology $T_X$. |
| $\bigcirc_{k=1}^{n} f_k$ | The composite function $f_n \circ \cdots \circ f_2 \circ f_1$. $n$ generally reserved for a “particular” index (e.g., $n \in \{1, \ldots, n\}$ above). |
| $f_k \to f$ | The $f_k$ converge to $f$ pointwise. |
| $f_k \uarr f$ | The $f_k$ converge to $f$ uniformly. Note, some authors use $\Rightarrow$ instead of $\uarr$. |
| $\mathcal{A}^\circ$ | Interior of $A$. |
| $\overline{\mathcal{A}}$ | Closure of $A$. |
| $A_1 \setminus A_2$ | Set difference of $A_1$ and $A_2$. |
| $A_1 \sqcup A_2$ | Disjoint union of $A_1$ and $A_2$. |
| $[..]$ | Used to indicate parsing order in grammatically-ambiguous sentences. |
| $\checkmark$ | Indicates the completion of a case in a proof by casework. |

2. Background

Today we will be working with knot equivalence directly instead of making appeals to Reidemeister’s theorem. This is because we’re interested in knots that might be wild (Definition 2.4), but Reidemeister’s theorem assumes tameness (also Definition 2.4). We begin with a reminder of some fundamental definitions.

2.1. Fundamental Definitions. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. An embedding of $X$ into $Y$ is a function $f : X \to Y$ such that restricting the codomain of $f$ to $f(X)$ gives us a homeomorphism $\tilde{f} : X \to f(X)$.

![Figure 3. Example of embedding an $X$ into $\mathbb{R}^2$.](image-url)
Since embeddings must be injective, some authors choose to denote them by \( f: X \hookrightarrow Y \). Here we call \( Y \) the \textit{ambient space} and refer to \( f(X) \) as \( X \) \textit{embedded by} \( f \) \textit{in} \( Y \).

Two embeddings \( f_1, f_2: X \rightarrow Y \) are said to be \textit{equivalent} (denoted \( f_1 \cong f_2 \)) if there exists a homeomorphism \( h: Y \rightarrow Y \) such that for all \( x \in X \),

\[
(h \circ f_1)(x) = f_2(x).
\]

Since \( h \) is a homeomorphism on the \textit{ambient} space, we refer to it as an \textit{ambient homeomorphism}.

\textit{Remark 2.1.} This definition requires pointwise equality for \((h \circ f_1), f_2\). In general, this is stronger than requiring \( h(f_1(X)) = f_2(X) \) as sets.\(^1\) The interested reader should see [1, 6] for more.

Geometrically, we think of \( h \) as “deforming” the ambient space to take \( f_1(X) \) to \( f_2(X) \).

\[\text{Figure 4. An example } h \text{ taking } f_1(X) \text{ to a distorted version representing } f_2(X).\]

A \textit{knot} is an embedding \( f: S^1 \hookrightarrow \mathbb{R}^3 \). One should note that some authors take the codomain to be \( S^3 \) instead of \( \mathbb{R}^3 \) because \( S^3 \) is compact. Our proofs today only require that \( Y \) be a metric space, hence we are free to choose either option. We could even choose a thickened orientable surface in order to work with virtual knots. However, we will do neither of these things and instead choose \( Y = \mathbb{R}^3 \) because it is easier to represent graphically.

For embeddings in \( \mathbb{R}^3 \), defining equivalence through ambient homeomorphisms is equivalent to defining equivalence with \textit{ambient isotopy} (defined below). We refer to this fact as the \textit{equivalence of equivalences}. For further discussion (as well as a list of references about the correspondence in each of the PL, \( C^\infty \), and \textit{Topological} categories), see [5], particularly \( \S 6.3 \).

\textbf{Definition 2.2 (Ambient Isotopy).} Let \((X, \mathcal{T}_X), (Y, \mathcal{T}_Y)\) be topological spaces. Let \( f_1, f_2: X \hookrightarrow Y \) be embeddings. Then a function \( H: [0, 1] \times Y \rightarrow Y \) is called an \textit{ambient isotopy} iff

\[
1\text{The correspondence holds for tame knots and certain everywhere-wild knots. An example of a knot for which it fails is Fig. 1b. The idea is that the ambient homeomorphism can only send wild points to other wild points, and this makes it impossible to pull the strand “through” the wild point.}
(1) $H$ is continuous,

(2) $H(0, \cdot)$ is the identity on $Y$,

(3) For all $t \in [0, 1]$, the function $H(t, \cdot): Y \to Y$ is a homeomorphism, and

(4) For all $x \in X$, we have

$$(H(1, \cdot) \circ f_1)(x) = f_2(x).$$

We often refer to $H$ as an ambient isotopy from $f_1$ to $f_2$.

Note that $H(1, \cdot)$ is an ambient homeomorphism from $f_1$ to $f_2$. We can think of the $t$ variable as describing a “time” parameter in a movie connecting $H(0, \cdot)$ to $H(1, \cdot)$.

We will prove our results today for both ambient homeomorphisms and ambient isotopies. Although this is not strictly necessary in light of the equivalence of equivalences, we have chosen to include both arguments to illustrate the additional step required for working with ambient isotopy.

### 2.2. Tameness and Wildness

Oftentimes, knot theory is restricted to the study of tame knots, which we define in a moment. We think of tame knots as being well-behaved because they belong to equivalence classes of knots that have representative elements that can described with finite information, namely polygonal or PL knots. This is the loose intuition underpinning Reidemeister’s theorem.

**Definition 2.3** (Polygonal Knot). A polygonal knot is a knot that is comprised of a finite union of straight line segments.

We can think of the $t$ variable as describing a “time” parameter in a movie connecting $H(0, \cdot)$ to $H(1, \cdot)$.

![Figure 5. 5 freeze-frames from an ambient isotopy where $H(1, \cdot)$ is the $h$ in Fig. 4](image)

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![Figure 6. Examples of some polygonal knots](image)
**Definition 2.4** (Tame & Wild Knots). A *tame knot* is a knot that is ambient isotopic to a polygonal knot. A *wild knot* is a knot that is not tame.

![Example of a tame knot](image)

**Figure 7.** Example of a tame knot

**Remark 2.5.** There are many other common definitions for tameness and wildness. A discussion of these definitions (and some of the equivalences) can be found in [5].

2.3. **Uniform Convergence.** Finally, we recall the definition of *uniform convergence*.

**Definition 2.6** (Uniform Convergence). Let \((X, d_X), (Y, d_Y)\) be metric spaces, and consider a sequence of functions \(f_n : X \to Y\). Suppose that the \(f_n\) converge pointwise to some \(f : X \to Y\). Then we say the \(f_n\) *converge to* \(f\) *uniformly* iff for all \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(x \in X\), \(n > n_0\) implies

\[
d_Y(f_n(x), f(x)) < \varepsilon.
\]

We typically denote uniform convergence by \(f_n \xrightarrow{u} f\). Geometrically, we think of this in terms of pictures like the following:

![Example of some \(f_n : \mathbb{R} \to \mathbb{R}\) satisfying \(\|f_n - f\|_\infty < \varepsilon\). Dashed lines indicate \(f(x) \pm \varepsilon\).](image)

**Figure 8.** Example of some \(f_n : \mathbb{R} \to \mathbb{R}\) satisfying \(\|f_n - f\|_\infty < \varepsilon\). Dashed lines indicate \(f(x) \pm \varepsilon\).

**Remark 2.7.** It’s worth noting that, as with the definition of ambient homeomorphism, this definition is generally stronger than requiring convergence of the images as sets. In fact, we can construct simple examples of \(f_n : [0, 1] \to \mathbb{R}^3\) such that for all \(n, m \in \mathbb{N}\), \(f_n([0, 1]) = f_m([0, 1])\), but the \(f_n\) do not even converge pointwise. One way to achieve this is to alternate between two different parameterizations of the same curve.
In the figures above, the colors correspond to value of $t \in [0, 1]$ that yields the given point under $f_n$. In order for $f_n \xrightarrow{u} f$, not only do the curves have to take on the same “shape,” but also every point of a given color in $f_n((0, 1])$ has to be less than $\varepsilon$ away from the corresponding point in $f((0, 1])$.

We recall two results from first courses in Analysis and Topology, respectively.

**Proposition 2.8.** Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. For each $k \in \mathbb{N}$, let $f_k : X \to Y$ be continuous. Suppose that there exists $f : X \to Y$ such that $f_k \xrightarrow{u} f$. Then $f$ is continuous.

**Proposition 2.9.** Let $(X, T_X)$, $(Y, T_Y)$ be topological spaces. Suppose that $X$ is compact and $Y$ is Hausdorff. Now suppose that $f : X \to Y$ is bijective and continuous. Then $f$ is a homeomorphism.

We are now ready to begin our discussion of these results in the context of knots.

### 3. Uniform Convergence & Knots

These two propositions give us the following simple result.

**Corollary 3.1.** Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces, and suppose that $X$ is compact. For each $k \in \mathbb{N}$, let $f_k : X \hookrightarrow Y$ be an embedding. Suppose that the $f_k$ converge uniformly to some $f : X \to Y$. Then if $f$ is injective, it follows that $f$ is an embedding.

**Proof.** By hypothesis, for all $k \in \mathbb{N}$ we have $f_k$ is an embedding and hence continuous. Since we also have $f_k \xrightarrow{u} f$, Proposition 2.8 implies $f$ is continuous. $(Y, d_Y)$ is a metric space and thus Hausdorff. So $f(X)$ with the subspace topology is Hausdorff. Now, $f$ is injective and thus $f$ is a bijection between $X$ and $f(X)$. By Proposition 2.9 it follows that $f$ is a homeomorphism between $X$ and $f(X)$. Thus $f$ is an embedding. 

Among other things, Corollary 3.1 can be used to construct fractal-like knot diagrams.
Example 3.2. Consider the knot constructed by the following iterative procedure, loosely inspired by the Koch Snowflake:

One can show that with the proper choice of parameterizations for the $f_k$, Corollary 3.1 guarantees the limit function $f_\infty = \lim_{k \to \infty} f_k$ is an embedding. The main challenge is explicitly proving injectivity; this can be done as long as the “shrink” factor for the twists is sufficiently small.

3.1. Iteratively Constructing (Ambient) Homeomorphisms. For the rest of this document we will be primarily interested in a special case of Corollary 3.1. Namely, when $X = Y$, the $f_k$’s become homeomorphisms from $X$ to itself. When $X$ is a compact subset of $\mathbb{R}^3$ this will give us a way to construct ambient homeomorphisms (and later, ambient isotopies) by composing countably-many Reidemeister moves.

To that end we repack Corollary 3.1 into a form that is more ergonomic when working with this special case. In particular, we’ll write the limit function in terms of a composition of homeomorphisms, which is more in line with our intuition of applying multiple Reidemeister moves in succession. This will also allow us to offload the uniform convergence requirement to one of acting on a shrinking collection of neighborhoods, which is easier to interpret in terms of knot diagrams. Note, the following theorem is true for homeomorphisms in general, but we are only interested in applying it to ambient homeomorphisms.
Theorem 3.3. \((Y,d)\) be a metric space. For all \(k \in \mathbb{N}\), let \(h_k: Y \to Y\) be a homeomorphism, and for all \(n \in \mathbb{N}\), define
\[
h_n = \bigcap_{k=1}^{n} h_k = (h_n \circ h_{n-1} \circ \cdots \circ h_2 \circ h_1).
\]
For each \(k\) let \(V_k \subseteq Y\) such that \(h_k\) is identity on \(V_k^c\). Then provided

1. The \(V_k\) satisfy
   \[
   \lim_{n \to \infty} \diam\left(\bigcup_{k=n}^{\infty} V_k\right) = 0,
   \]
2. There exists a compact \(A \subseteq Y\) such that
   \[
   \left(\bigcup_{k=1}^{\infty} V_k\right) \subseteq A^o,
   \]
and
3. \(h_\infty\) defined by
   \[
   h_\infty = \lim_{n \to \infty} h_n
   \]
exists and is bijective,
then \(h_\infty\) is a homeomorphism.

Before continuing to the proof, we make some remarks about the statement.

Remark 3.4. The hypothesis that the limit \(h_\infty\) exists is superfluous as it is implied by conditions (1) and (2).

Remark 3.5. One can replace the somewhat-technical conditions on the \(V_k\) with simpler ones. E.g., conditions (1) and (2) could be substituted with

1. Requiring
   \[
   \cdots \subseteq V_{k+1} \subseteq V_k \subseteq \cdots \subseteq V_1 \subseteq A^o
   \]
and
2. \(\lim_{k \to \infty} \diam(V_k) = 0\).

Another option would be to replace condition (1) with something like “\(\diam(\lim \sup V_n) = 0\).” In any case, we avoided simplifications like these in an effort to make correspondence with the hypotheses of Corollary 3.1 more direct.

Remark 3.6. We require \(\lim_{n \to \infty} \diam\left(\bigcup_{k=n}^{\infty} V_k\right) = 0\) instead of \(\lim_{n \to \infty} \diam(V_k) = 0\) to avoid situations where the \(V_k\) have diameters like \(\frac{1}{k}\). If we were to allow cases like these the divergence of the harmonic series would cause problems.

Remark 3.7. Though perhaps tempting, it is not sufficient to do away with the conditions on the \(V_k\) by requiring something like “\(h_k \xrightarrow{u} 1_Y\).” As a counterexample, consider \(Y = [0,1]\) with the standard metric on \(\mathbb{R}\). For all \(k \in \mathbb{N}\), define
\[
h_k = x^{(k+1)/k}.
\]
Then $h_k \xrightarrow{u} 1_{[0,1]}$. But note, $h_k = x^k$, and thus

$$h_\infty(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

which is not a homeomorphism.

**Proof.** We will employ the gluing lemma. To that end, we need to partition $Y$ into two closed sets and show that $h_\infty$ is a homeomorphism on both. A natural choice is to consider $A^c$ and $A$. Note, the compactness of the latter will allow us to appeal to Corollary 3.1.

We examine these two sets separately.

1. (On $A^c$): By construction, each $h_k$ is identity on $V_k^c$. Since each $V_k \subseteq A^c$, it follows $h_\infty$ is identity (and hence a homeomorphism) on $(A^c)^c = A$.

2. (On $A$): Now, we show that $h_\infty$ is a homeomorphism on $A$. By Corollary 3.1, because $h_\infty$ was assumed to be bijective it suffices to show that the restrictions $h_k|_A$ converge uniformly to $h_\infty|_A$. We will suppress writing the $|_A$ for now because it clutters the notation too much.

Let $\varepsilon > 0$ be given. Recall that by hypothesis, we have

$$\lim_{n \to \infty} \text{diam} \left( \bigcup_{k=n}^{\infty} V_k \right) = 0,$$

hence there exists $n_0 \in \mathbb{N}$ such that

$$\text{diam} \left( \bigcup_{k>n_0}^{\infty} V_k \right) < \varepsilon.$$

We have the following claim.
Claim: For all \( n > n_0 \), for all \( y \in A \), we have
\[
d(h_n(y), h_\infty(y)) < \varepsilon.
\]

Proof of Claim: Fix an \( n > n_0 \) and let \( y \in A \) be arbitrarily chosen. We have two subcases.

(a) First, suppose \( h_{n_0}(y) \notin \bigcup_{k>n_0} V_k \).

Recall that we defined the \( h_k \) such that each \( h_k \) is identity outside \( V_k \).
It follows that for all \( k > n_0 \) we have \( h_k(y) = y \). Hence
\[
h_n(y) = h_\infty(y),
\]
so
\[
d(h_n(y), h_\infty(y)) = 0 < \varepsilon,
\]
as desired. \( \checkmark \)

![Diagram](image)

**Figure 12.** An example of this case with \( n_0 = 4 \). The shaded portions represent \( \bigcup_{k>n_0} V_k \). Here, \( y \) starts in \( V_1 \), is mapped into \( V_2 \) by \( h_1 \), into \( V_4 \) by \( h_2 \), skipped by \( h_3 \), then finally mapped to another point of \( V_4 \) by \( h_4 \), before remaining fixed for all \( k > 4 \).

(b) Now, suppose \( h_{n_0}(y) \in \bigcup_{k>n_0} V_k \). Note, since (for all \( k > n_0 \), \( h_k \) is bijective and is identity outside \( \bigcup_{k>n_0} V_k \)), it follows that for all \( n > n_0 \),
\[
h_n(y) \in \bigcup_{k>n_0} V_k
\]
and hence
\[
h_\infty(y) \in \bigcup_{k>n_0} V_k.
\]
Note, the set in the second is just the closure of the set in the first; thus they have the same diameter. By definition of \( n_0 \), we have \( \text{diam} \left( \bigcup_{k>n_0} V_k \right) < \varepsilon \), hence
\[
d(h_n(y), h_\infty(y)) < \varepsilon
\]
as desired.

In either case, we get that \( d(h_n(y), h_\infty(y)) < \varepsilon \). Now (writing the restrictions explicitly again) it follows that \( h_n \mid_A \) is a sequence of homeomorphisms with \( (h_n \mid_A) \overset{n}{\rightarrow} (h_\infty \mid_A) \).

Finally, recall that by hypothesis, \( h_\infty \) is a bijection. This implies \( h_\infty \mid_A \) is too, and since \( A \) is compact, Corollary 3.1 now guarantees \( h_\infty \mid_A \) is a homeomorphism on \( A \).

Now, applying the gluing lemma to \( (h_\infty \mid_A) \) and \( (h_\infty \mid_{A^c}) \) we conclude that \( h_\infty \) is continuous. An identical argument shows \( h_\infty^{-1} \) is continuous. It follows that \( h_\infty \) is a homeomorphism, as desired.

**3.2. Iteratively Constructing (Ambient) Isotopies.** We now state the analogous result for isotopies. As with Theorem 3.3, the theorem below is valid for isotopies in general but we are only interested in applying it to ambient isotopies. It might be easy to get bogged down by the additional details so we summarize the main ideas.

Given a sequence of isotopies \( H_k \), if the associated homeomorphisms \( h_k(\cdot) := H_k(1, \cdot) \) satisfy the hypotheses of Theorem 3.3, then we can stitch the \( H_k \)'s together into an isotopy \( \mathcal{H}_\infty \) as follows. First, define \( t_0 = 0 \) and let \( (t_k)_{k=1}^\infty \) be a strictly increasing sequence in \((0, 1)\). Define \( \mathcal{H}_\infty \) to apply the effects of \( H_1 \) over the compressed time interval \([t_0, t_1]\). Then, do the same to apply \( H_2 \) over \([t_1, t_2]\). Continue this process, in general applying \( H_k \) over the interval \([t_{k-1}, t_k]\).

Stopping the construction above after \( n \) steps will give us an isotopy \( \mathcal{H}_n \). Taking \( n \rightarrow \infty \) we will get a function \( \mathcal{H}_\infty \) with \( \mathcal{H}_\infty(1, \cdot) = h_\infty \). By Theorem 3.3, \( h_\infty \) will be a homeomorphism. And since the \( \mathcal{H}_n \) are all isotopies, we’ll see that \( \mathcal{H}_\infty(t, \cdot) \) will be a homeomorphism for all \( t \in [0, 1) \). Applying a uniform convergence argument to the \( \mathcal{H}_n \) will then show \( \mathcal{H}_\infty \) is continuous overall and thus an isotopy!

**Theorem 3.8.** Let \((Y, d_Y)\) be a metric space. For all \( k \in \mathbb{N} \), let \( H_k : [0, 1] \times Y \rightarrow Y \) be an isotopy, and let \( V_k \subseteq Y \) such that \( H_k \) is identity on \([0, 1] \times (V_k^c)\). For each \( k \) define \( h_k : Y \rightarrow Y \) by \( h_k(y) = H_k(1, y) \); note that by definition of an isotopy, \( h_k \) is a homeomorphism.

Suppose that the \( h_k \)'s and \( V_k \)'s satisfy the hypotheses of Theorem 3.3, and for all \( n \in \mathbb{N} \) define \( h_n = \bigcap_{k=1}^n h_k \). Let \( t_0 = 0 \) and let \( (t_k)_{k=1}^\infty \) be a strictly increasing sequence in \((0, 1)\). Then, for all \( t \in [0, 1) \) define an isotopy \( H(t, \cdot) \) as follows:

\[
H(t, y) = \begin{cases} 
H_k(t, y) & \text{if } t \in [t_{k-1}, t_k) \\
H_k(t_k, y) & \text{if } t = t_k
\end{cases}
\]

where \( H_k(1, y) = h_k(y) \) for all \( y \in Y \). Then, \( H \) will be an isotopy on \( Y \) as desired.
sequence in $(0, 1)$ converging to $1$. Then \( \mathcal{H}_\infty: [0, 1] \times Y \to Y \) defined by

\[
\mathcal{H}_\infty(t, y) = \begin{cases} 
H_1 \left( \frac{t-t_0}{t_1-t_0}, y \right) & \text{if } t \in [t_0, t_1] \\
H_2 \left( \frac{t-t_1}{t_2-t_1}, h_1(y) \right) & \text{if } t \in (t_1, t_2] \\
H_3 \left( \frac{t-t_2}{t_3-t_2}, h_2(y) \right) & \text{if } t \in (t_2, t_3] \\
\vdots & \\
H_k \left( \frac{t-t_k-1}{t_{k+1}-t_k-1}, h_{k-1}(y) \right) & \text{if } t \in (t_k-1, t_k] \\
\vdots & \\
h_\infty(y) & \text{if } t = 1,
\end{cases}
\]

is an isotopy.

**Proof.** To show \( \mathcal{H}_\infty \) is an isotopy we must show

1. \( \mathcal{H}_\infty(0, \cdot) \) is identity,
2. For each \( t \in [0, 1] \), \( \mathcal{H}_\infty(t, \cdot) \) is a homeomorphism, and
3. \( \mathcal{H}_\infty \) is continuous.

We prove these in the order above.

1. For all \( y \in Y \), \( \mathcal{H}_\infty(0, \cdot) = H_1(0, \cdot) \). Since \( H_1 \) is an isotopy, \( H_1(0, \cdot) \) is identity (by definition) and this proves (1).

2. To prove (2) we break things up into three subcases.
   
   (a) Suppose \( t = 0 \). Then \( \mathcal{H}_\infty(t, \cdot) = \mathcal{H}_\infty(0, \cdot) \) which is the identity and thus a homeomorphism.

   (b) Suppose \( t \in (0, 1) \). Then there exists \( k \in \mathbb{N} \) such that \( t \in (t_{k-1}, t_k] \).

   Recall that by construction,

   \[
   \mathcal{H}_\infty(t, \cdot) = H_k \left( \frac{t-t_{k-1}}{t_k-t_{k-1}}, h_{k-1}(\cdot) \right).
   \]  

   Define \( g(\cdot) := H_k \left( \frac{t-t_{k-1}}{t_k-t_{k-1}}, \cdot \right) \). By definition of an isotopy, \( g \) is a homeomorphism. Also, \( h_{k-1} \) is a finite composition of homeomorphisms and thus a homeomorphism. Rewriting Eq. (3.1) gives us that

   \[
   \mathcal{H}_\infty(t, \cdot) = (g \circ h_{k-1})(\cdot)
   \]

   which is a finite composition of homeomorphisms and hence itself a homeomorphism, as desired.

   (c) Now suppose \( t = 1 \). Then \( \mathcal{H}_\infty(t, \cdot) = h_\infty(\cdot) \). Since the \( h_k \)'s, \( V_k \)'s were assumed to satisfy the hypotheses of Theorem 3.3 we get that \( h_\infty \) is a homeomorphism as desired.

In either case, we see \( \mathcal{H}_\infty(t, \cdot) \) is a homeomorphism.

3. Finally, it remains to show \( \mathcal{H}_\infty \) is continuous. We employ uniform convergence.
Define a sequence of isotopies \( \mathcal{H}_n \) as follows: For each \( n \in \mathbb{N} \), let \( \mathcal{H}_n : [0, 1] \times Y \to Y \) be given by

\[
\mathcal{H}_n(t, y) = \begin{cases} 
H_1 \left( \frac{t-t_0}{t_1-t_0}, y \right) & \text{if } t \in [t_0, t_1] \\
H_2 \left( \frac{t-t_1}{t_2-t_1}, \mathcal{h}_1(y) \right) & \text{if } t \in (t_1, t_2] \\
\vdots \\
H_n \left( \frac{t-t_{n-1}}{t_n-t_{n-1}}, \mathcal{h}_{n-1}(y) \right) & \text{if } t \in (t_{n-1}, t_n] \\
\mathcal{h}_n(y) & \text{if } t \in (t_n, 1].
\end{cases}
\]

That is, we follow \( \mathcal{H}_\infty(t, y) \) until we reach \( t = t_n \) and then we freeze. One can verify that the \( \mathcal{H}_n(t, y) \) are indeed isotopies; of particular note, they are continuous. We now show \( \mathcal{H}_n \xrightarrow{\mathcal{u}} \mathcal{H}_\infty \).

Let \( \varepsilon > 0 \) be given. Then by the hypotheses on the \( V_k \) there exists \( n_0 \in \mathbb{N} \) such that

\[
\text{diam} \left( \bigcup_{k>n_0} V_k \right) < \varepsilon.
\]

Let \( n > n_0 \) be arbitrarily chosen, and similarly let \( (t, y) \in [0, 1] \times Y \). We show \( d(\mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y)) < \varepsilon \). We have two subcases.

(a) First, suppose \( t \in [0, t_n] \). Then \( \mathcal{H}_n(t, y) = \mathcal{H}_\infty(t, y) \) and so we have \( d(\mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y)) = 0 \) and the bound holds.

(b) Now, suppose \( t \in (t_n, 1] \). If \( y \notin \left( \bigcup_{k>n_0} V_k \right)^C \), then \( \mathcal{H}_n(t, y) = \mathcal{H}_\infty(t, y) \), so we have \( d(\mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y)) = 0 \) and the bound holds. Else, note that both of \( \mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y) \in \bigcup_{k>n_0} V_k \), hence

\[
d(\mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y)) < \varepsilon
\]

as desired.

In either case, we have \( d(\mathcal{H}_n(t, y), \mathcal{H}_\infty(t, y)) < \varepsilon \). As \( (t, y) \) were arbitrarily chosen, this implies \( \mathcal{H}_n \xrightarrow{\mathcal{u}} \mathcal{H}_\infty \). By Proposition 2.8, \( \mathcal{H}_\infty \) is continuous.

It follows that \( \mathcal{H}_\infty \) is an isotopy as desired.

In the next section, we apply this result to a variety of curves, beginning with the example from Fig. 1.

4. Various Applications of Theorem 3.8

The first few examples will all make use of the following lemma, which allows us to remove the bijectivity hypothesis from Theorem 3.8.

**Lemma 4.1.** Let all variables be quantified as in Theorem 3.8. Then if the \( V_k \)'s are all disjoint, \( \mathcal{H}_\infty(1, \cdot) \) is guaranteed to be a bijection.

**Proof.** If the \( V_k \) are all disjoint, then defining \( U = \bigcup_{k=1}^\infty V_k \) we can write \( Y \) as the disjoint union

\[
Y = (U^C) \sqcup \left( \bigcup_{k=1}^\infty V_k \right).
\]
Note, $H_\infty(1, \cdot)$ is identity on $U^c$ and hence a bijection. Since the $V_k$ are all disjoint, $H_\infty(1, \cdot)|_{V_k} = H_k(1, \cdot)|_{V_k}$, the latter of which is a homeomorphism and thus bijective, so $H_\infty(1, \cdot)$ is a bijection on each of the $V_k$.

Thus $H_\infty(1, \cdot)$ is a bijection overall.

Proposition 4.2. The curve shown in Fig. 13 below is tame. In particular, it is an unknot.

Proof. Let $f_0 : S^1 \hookrightarrow \mathbb{R}^3$ be the standard unknot, and let $f_1 : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding yielding a diagram like Fig. 13. We apply Theorem 3.8 to construct an ambient isotopy $H_\infty : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ taking $f_0$ to $f_1$.

Consider the sequence of $V_k$’s chosen as follows.

---

3Strictly speaking we have not defined tameness for curves, only for knots. A tame curve is a curve that’s ambient homeomorphic (equivalently, ambient isotopic) to a polygonal curve.

4A parametrization can be found in [5], although it is given in the context of a “theorem” about tameness and parametrizations that turns out to be incorrect.
One can verify that \( \lim_{n \to \infty} \text{diam} \left( \bigcup_{k=n}^{\infty} V_k \right) = 0 \) and that there exists a compact set \( A \subseteq \mathbb{R}^3 \) such that \( \bigcup_{k=1}^{\infty} V_k \subseteq A \).

For all \( k \in \mathbb{N} \), let \( H_k : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3 \) be an ambient isotopy inserting a Reidemeister I into the arc bound in \( V_k \). Use these \( H_k \) to define \( H_\infty \) as in Theorem 3.8. By Lemma 4.1, \( H_\infty(1, \cdot) \) is a bijection. Thus by Theorem 3.8, \( H_\infty \) is an ambient isotopy from \( f_0 \) to \( f_1 \).

**Proposition 4.3.** The following curve is tame.

\[ \text{Proof (Sketch).} \] We apply Theorem 3.8 twice. This two-step method is not strictly necessary, but the diagram is a bit less cluttered this way. Consider the following starting curve:

Apply Reidemeister II moves within the dotted regions below:

Since these \( V_k \) are disjoint, we can again apply Lemma 4.1 to obtain an ambient isotopy. The result looks something like the following:
Now, perform Reidemeister II moves in the following regions:

Again, the $V_k$ here are all disjoint, hence one can apply Lemma 4.1 to show that this is an ambient isotopy. The end result is

which is the desired diagram.

We now consider a similar curve, this time constructed using Reidemeister I moves. This will be the most technical argument of the paper. We advise the reader to read through Example 5.2 in the next section before continuing. This is because Example 5.2 shows how we can lose bijectivity in the limit, and the bulk of the challenge in Proposition 4.4 is addressing similar concerns. We have to address bijectivity in a manner like this whenever we have points that are moved by infinitely many of the $H_k$’s (whereas in Proposition 4.3, each point is moved by only finitely many $H_k$).

**Proposition 4.4.** Let $f_0: [0, 1] \hookrightarrow \mathbb{R}^3$ be
and let $f_1 : [0, 1] \hookrightarrow \mathbb{R}^3$ be

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig15}
\caption{Our starting arc.}
\end{figure}

Then $f_0 \cong f_1$.

Proof (Sketch). We will construct the ambient isotopy from $f_0$ to $f_1$ by a recursive process. We will repeatedly insert Reidemeister I moves like the following:

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig16}
\caption{A different countable sequence of Reidemeister I moves.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig17}
\caption{The general procedure.}
\end{figure}

We must do so in such a way that we can still argue bijectivity of $\mathcal{H}_\infty(1, \cdot)$. The key idea is to choose our $H_k$’s so that only one point (denoted $y_\infty$) gets moved infinitely-many times.\footnote{$y_\infty$ will be the point that gets sent to the limit of the twists in Fig. 16. In our construction, $y_\infty$ will be the vertex in Fig. 15, but one can create other constructions where it is a different point.} We explicitly guarantee this by constructing our $H_k$’s so that for all $y \in \mathbb{R}^3 \setminus \{y_\infty\}$, there exists $n_0$ such that for $n > n_0$, $h_{n_0}(y)$ is unmoved by $H_n(t, \cdot)$.\footnote{\textit{y}_\infty$ will be the point that gets sent to the limit of the twists in Fig. 16. In our construction, $y_\infty$ will be the vertex in Fig. 15, but one can create other constructions where it is a different point.}
To that end, define $\ell$ as shown in Fig. 18, and let $\varepsilon > 0$ with $\varepsilon \ll 1$. Define $V_1$ to be a closed rectangular prism of dimensions $(6 + \varepsilon)\ell \times (2 + \varepsilon)\ell \times (2 + \varepsilon)\ell$, and let $H_1$ be a PL ambient isotopy inserting the first loop such that $H_1$ is identity off $V_1$. Note, even though we define $V_1$ to be closed, we’ll draw it with dotted lines in the below.

![Figure 18. The same figure, now showing $V_1$.](image)

Now, we describe the general strategy for inserting the $k + 1$st loop given the first $k$ loops. We want $V_{k+1}, H_{k+1}$ to be half-scale versions of $V_k, H_k$. The figure below shows this for $k = 1$.

![Figure 19. $V_1$ and $V_2$, with application of $H_2$ shown.](image)

But, to make things work, it will be important to first apply some intermediate ambient isotopy $H_{k+5}$ in between $H_k$ and $H_{k+1}$ such that $H_{k+5}$ does not change the diagram, but does help ensure that points in the ambient space aren’t lost in the limit.

**Desired Properties of $H_{k+5}$**: We want $H_{k+5}$ to preemptively “unsquish” points that might be compressed together by $H_{k+1}$. To determine exactly how much unsquishing we have to do, we look at a sort of inverse Lipschitz condition.

Let $h_{k+1} = H_{k+1}(1, \cdot)$. Since $H_{k+1}$ is a PL ambient isotopy, $h_{k+1}$ is a PL ambient homeomorphism, and thus there exists $c \in (0, 1)$ such that for all $x_1, x_2 \in \mathbb{R}^3$,

$$c \cdot d(x_1, x_2) \leq d(h_{k+1}(x_1), h_{k+1}(x_2)).$$

(4.1)

---

6 Actually, $\varepsilon > 0$ can be arbitrarily chosen so long as for all $k \in \mathbb{N}$ we have $V_{k+1} \subseteq V_k^0$. We just choose $\varepsilon \ll 1$ because it makes for cleaner-looking diagrams.

7 We can assume PL-ness because the modifications can be realized by elementary moves.

8 The 6 in our prism dimensions comes from the fact $\ell$ is defined to be $1/3$rd of the length of the twist inserted in Fig. 18, and the moves halve in size at each iteration.

9 This essentially pops out of the finiteness condition on our simplicial complexes for PL maps.
Note that because the $H_k$ are all identical up to scaling, $c$ is independent of $k$.\footnote{One might ask why we can’t have $c \geq 1$. Note that $V_{k+1}$ being bounded precludes $c > 1$. For $c = 1$, note that $h_{k+1}$ is not a vector space isomorphism of $\mathbb{R}^3$, and hence not an isometry on $\mathbb{R}^3$, since $h_{k+1}$ is identity outside $V_{k+1}$, isometry must fail on $V_{k+1}$.}

We interpret Eq. (4.1) as giving us a bound on how much $h_{k+1}$ can “squish” points in the space together. Let $q_k$ be the tip of the twist before applying $H_{k+1}$:

![Figure 20](image)

**Figure 20.** $q_k$ labeled.

For all $p \in V_{k+1}$, we want $H_{k+0.5}$ to be constructed to guarantee that either

1. $h_{k+1}(h_{k+0.5}(p)) \in V_k \setminus V_{k+1}$ (i.e. $p$ gets moved to the outer box), or
2. $p$ gets “moved farther from $q_k$ than it can be squished in later”:

$$\frac{1}{c} \cdot d(p, q_k) \leq d(h_{k+0.5}(p), h_{k+0.5}(q_k)).$$  \hspace{1cm} (4.2)

**Constructing $H_{k+0.5}$:** Let $V_{k+0.5}$ be a slightly-scaled-up version of $V_{k+1}$ such that $V_{k+1} \subset V_{k+0.5} \subset V_k$. To make things easier, we will require that $V_{k+0.5}$ also only intersects with $(h_k \circ h_{k-1} \circ \cdots \circ h_1 \circ f_0)([0,1])$ in a wedge-shape and that $V_{k+0.5}$ and $V_{k+1}$ share the same center of mass and have all sides parallel (see Fig. 21).

![Figure 21](image)

**Figure 21.** $V_{k+0.5}$ and $V_{k+1}$

We can parameterize every point $p \in V_{k+0.5}$ in terms of a piecewise linear function as detailed below. The construction is a bit unergonomic to formalize explicitly, but it is meant to capture the ideas of Fig. 22.
We construct it in two parts and then scale them by half and glue them together.

(1) If \( p \in V_{k+1} \setminus V_{k+1} \), there exist unique points \( v_{k+\frac{1}{2}} \in \partial V_{k+1} \) and \( v_{k+1} \in \partial V_{k+1} \) such that \( v_{k+\frac{1}{2}} \) is the point in \( V_{k+1} \) “corresponding” to \( v_{k+1} \) in \( V_{k+1} \), and \( p \) is on the line segment \( v_{k+\frac{1}{2}}v_{k+1} \). Thus there exists a unique \( s \in [0,1] \) such that we can write \( p \) as a convex combination

\[
p = s \cdot v_{k+\frac{1}{2}} + (1-s) \cdot v_{k+1}.
\]

(2) If \( p \in V_{k+1} \), there exists a unique point \( v_{k+1} \in \partial V_{k+1} \) such that \( p \) is on the line segment \( q_k v_{k+1} \). Analogously to the above, there exists a unique \( s \in [0,1] \) such that we can write \( p \) as

\[
p = s \cdot v_{k+1} + (1-s) \cdot q_k.
\]

We re-scale \( s \) to glue these two parameterizations into a single function which we call \( H'_{k+\frac{1}{2}} \):

\[
H'_{k+\frac{1}{2}}(s, v_{k+\frac{1}{2}}, v_{k+1}) = \begin{cases} 
2s \cdot v_{k+1} + (1-2s) \cdot q_k & s \in [0, \frac{1}{2}] \\
(2s-1) \cdot v_{k+\frac{1}{2}} + (2-2s) \cdot v_{k+1} & s \in (\frac{1}{2}, 1].
\end{cases}
\]

We’ll now do something a bit bizarre and rewrite the parameters in \( H'_{k+\frac{1}{2}} \) as functions of \( p \). Note that with the re-scaling of \( s \), we now have \( s \) uniquely determined by \( p \). Also recall that by construction, \( v_{k+\frac{1}{2}} \) and \( v_{k+1} \) are each uniquely determined by \( p \). Hence, we can think of \( s, v_{k+\frac{1}{2}}, v_{k+1} \) as being functions of \( p \). One can show that these are all continuous. As such, we can indeed think of \( H'_{k+\frac{1}{2}} \) as just being a complicated way of writing the identity function on \( V_{k+1} \).

To turn \( H'_{k+\frac{1}{2}} \) into our ambient isotopy \( H_{k+\frac{1}{2}} \), we now introduce time dependence in \( s \). Define \( s_{c_0} = \frac{1}{2} \) and \( s_{c_1} = \frac{1}{2} \), and observe \( s_{c_0} < s_{c_1} \). Define \( s_c : [0,1] \to [s_{c_0}, s_{c_1}] \)

\[11\] By “corresponds,” we mean that given a linear function that scales up \( V_{k+1} \) to yield \( V_{k+\frac{1}{2}} \), \( v_{k+1} \) gets mapped to \( v_{k+\frac{1}{2}} \).

\[12\] The reason that \( c \) appears in this expression is because we’re trying to get Eq. (4.2) out in the end.
by

\[ s_c(t) = t \cdot s_{c_1} + (1 - t) \cdot s_{c_0}, \]

and use this to define

\[
s'(t, p) = \begin{cases} 
    \left( \frac{s(p)}{s_{c_0}} \right) \cdot s_c(t) & \text{if } s(p) \in [0, s_{c_0}] \\
    \left( \frac{s(p) - s_{c_0}}{1 - s_{c_0}} \right) \cdot 1 + \left( 1 - \left( \frac{s(p) - s_{c_0}}{1 - s_{c_0}} \right) \right) \cdot s_c(t) & \text{if } s(p) \in (s_{c_0}, 1]
\end{cases}
\]

This looks unpleasant but the idea is simple. First, recall that \( s(p) \) represents a parameter in \([0, 1]\) that tells us how to write \( p \) as a convex combination of other points. One can verify that when \( t = 0 \), \( s'(t, p) \) reduces to \( s(p) \). Then, as \( t \) increases to 1, \( s'(t, p) \) distorts the interval represented by \( s(p) \) until we end up with something like the following, in which \( s_{c_0} \) gets mapped to where \( s_{c_1} \) was initially:

\[ \begin{array}{ccc}
    s_{c_0} & & s_{c_1} \\
    \downarrow & & \downarrow \\
    \left[ \begin{array}{c}
    \hline
    \hline
    \hline
    \hline
    \end{array} \right] & & \left[ \begin{array}{c}
    \hline
    \hline
    \hline
    \hline
    \end{array} \right]
\end{array} \]

**Figure 23.** The interval \([0, 1]\) represented by \( s'(t, p) \) as \( t \) goes from 0 to 1. The light portion represents the values where \( s(p) \in [0, s_{c_0}] \) and the dark portion represents \( s(p) \in [s_{c_0}, 1] \).

The net effect of \( H_{k+0.5} \) is to take a diagram like the following

\[ V_{k+0.5} \]

and turn it into
Finally, we have the following:

**Claim:** $H_{k+.5} : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by
\[
H_{k+.5}(t, p) = \begin{cases} 
H'_{k+.5}(s'(t, p), p) & p \in V_{k+.5} \\
p & p \notin V_{k+.5}
\end{cases}
\]
satisfies the desired properties of $H_{k+.5}$.

**Proof of Claim:** One can verify that $H_{k+.5}$ satisfies all the properties of an ambient isotopy.\footnote{Intuitively, all it is doing is sliding all the points in $V_{k+.5}$ along the lines in Fig. 22 until they are either in $V_{k+.5} \setminus V_{k+1}$ or $\frac{1}{e}$ times as far from $q_k$ as they were at the start.} It remains to show that $h_{k+.5} = H_{k+.5}(1, \cdot)$ satisfies the conditions stipulated near Eq. (4.2).

Let $p \in V_{k+.5}$ be arbitrary. We have two cases.

1. Suppose $s(p) \in (s_{c_0}, 1]$. Then $h_{k+.5}(p) \in V_{k+.5} \setminus V_{k+1}$, and hence $h_{k+1}(h_{k+.5}(p)) \in V_k \setminus V_{k+1}$.

2. Suppose $s(p) \in [0, s_{c_0}]$. One can verify that in this case, $H_{k+.5}(t, p)$ only slides $p$ along a ray segment originating from $q_k$, with the sliding dictated by $s'(t, p)$. Hence
\[
\frac{d(p, q_k)}{d(h_{k+.5}(p), h_{k+.5}(q_k))} = \frac{d(H_{k+.5}(0, p), H_{k+.5}(0, q_k))}{d(H_{k+.5}(1, p), H_{k+.5}(1, q_k))}
\]
\[
= \frac{s'(0, p)}{s'(1, p)}
\]
\[
= \frac{s(p)}{s(1, q_k)} \cdot \frac{s(p)}{s(1, q_k)}
\]
\[
= c.
\]

Simplifying gives us
\[
\frac{1}{c} \cdot d(p, q_k) = d(H_{k+.5}(1, p), H_{k+.5}(1, q_k)),
\]
as desired.
Guaranteeing Bijectivity: Observe that for all \( k \in \mathbb{N} \), for all \( p \in V_{k+1} \), we have
\[
d(h_{k+1}(h_{k+5}(p)), h_{k+1}(h_{k+5}(q_k))) \geq c \cdot d(h_{k+5}(p), h_{k+5}(q_k)) \\
\geq \frac{1}{c} \cdot d(p, q_k) \\
\geq d(p, q_k).
\] (4.3)

For each \( n \in \mathbb{N} \), let \( h_n \) denote the composition of all these homeomorphisms:
\[
h_n = \left( \bigcap_{k=1}^{n-1} (h_{k+1} \circ h_{k+5}) \right) \circ h_1 \\
= (h_n \circ h_{n-5} \circ \cdots \circ h_2 \circ h_{1,5} \circ h_1).
\]

Note that the sequence of points \( h_n^{-1}(q_n) \) is constant, hence the limit \( \lim_{n \to \infty} h_n^{-1}(q_n) \) exists; in particular, it is \( y_\infty \). For all \( y \in \mathbb{R}^3 \setminus \{y_\infty\} \), Eq. (4.3) shows that at each step, \( y \) is sent no closer to \( q_{k+1} \) than it was to \( q_k \). Since the boxes are shrinking it follows that each such \( y \) will eventually leave the boxes and thus remain fixed at subsequent steps. Explicitly: If \( n_0 \) satisfies
\[
\frac{(6 + 2\varepsilon)\ell}{2^{n_0}} < d(y, y_\infty),
\]
Then for all \( n > n_0 \) we have \( h_n(y) \notin V_n \), and hence
\[
h_n(y) = h_{n_0}(y).
\]

This implies \( h_\infty \) is a bijection between \( \mathbb{R}^3 \setminus \{y_\infty\} \) and \( \mathbb{R}^3 \setminus \{h_\infty(y_\infty)\} \). So Theorem 3.3 implies \( h_\infty \) is a homeomorphism between \( \mathbb{R}^3 \setminus \{y_\infty\} \) and \( \mathbb{R}^3 \setminus \{h_\infty(y_\infty)\} \). Thus \( h_\infty \) is bijective on \( \mathbb{R}^3 \), and Theorem 3.8 implies that \( \mathcal{H}_\infty(1, \cdot) \) is an ambient isotopy.

Finally, we have the following famous example.

**Proposition 4.5.** The following curve is a tame arc.
\[
\begin{center}
\includegraphics[width=0.5\textwidth]{figure25}
\end{center}
\]

**Sketch.** We apply Lemma 4.1. Consider a sequence of properly nested boxes \( V_1 \), \( V_2 \), \ldots, as follows:
\[
\begin{center}
\includegraphics[width=0.5\textwidth]{figure25}
\end{center}
\]

**Figure 25.** A countable connected sum of trefoils.
For all $k \in \mathbb{N}$, define $V'_k$ by $V'_k = V_k \setminus V_{k+1}$. Note the $V'_k$ are disjoint. We can define the ambient isotopies $H_k$ such that $H_k$ performs the following modification on $V'_k$: 
Taking the limit, we obtain an ambient isotopy unknotted the arc.

Having established the versatility of Theorem 3.8, we now discuss situations in which it cannot be applied. In a sense, we will see that each of the hypotheses of the theorem are sharp.

5. Cases Where Theorem 3.8 Does Not Apply

**Example 5.1.** If we extend the right hand side of Fig. 25 with a straight line segment, then we cannot apply Theorem 3.8. In fact, the curve is wild — see [2], Exercise 2.8.4.

What breaks here is that if we try to apply the same argument as we did with the non-extended version in Proposition 4.5, we can’t force \( \lim_{n \to \infty} \text{diam} \left( \bigcup_{k=n}^{\infty} V_k \right) = 0 \). In particular, \( \text{diam} \left( \bigcup_{k=n}^{\infty} V_k \right) \) is bounded below by the length of the straight line segment.

And now, as promised, we discuss the curve from Fig. 1b.

**Example 5.2.** One cannot apply Theorem 3.8 to the following curve:

![Figure 26. Fox’s “Remarkable Knotted Curve.”](image)

Here, the \( V_k \)'s are not the problem; rather it is bijectivity on the ambient space. Consider a “line” of points in the ambient space passing through the first loop:
Figure 27. The curve, now with an imaginary "line" of points from the ambient space.

As we remove the first loop, some points on the "line" get pulled through:
As we remove the second loop, a similar process occurs:
As \( n \to \infty \), the stitching process continues, with the number of lines doubling at each iteration. No matter what we try, in the limit, a countable subset of the original line gets mapped to the wild point. In fact, if we thicken the line in Fig. 27 to a cylinder, we see that *uncountably*-many points are lost in the limit!

This is reflected in the fact that if we were to try something like the approach taken in Proposition 4.4, we would not be able to define ambient isotopies that do the jobs of the \( H_{k+5} \)'s.

Remark 5.3. Another perspective one might consider here is that there is no obvious way to “tie” the curve in Fig. 26 if starting with just an unknotted line. Indeed, there’s no “first loop” to insert — to tie one, we require infinitely-many of the others to be present already!

6. Discussion

We conclude with a discussion of directions for future work.

Theorems 3.3 and 3.8 give us one direction to a loose “countable analogue” of Reidemeister’s theorem. The restrictions on the \( V_k \)'s have a nice diagrammatic interpretation — “the total region we’re going modify has to shrink in the limit” — but so far, a similarly-concise description of the bijectivity requirement has eluded the author.

Qualitatively, it seems that problems tend to occur when the set of points that get moved infinitely-many times is not topologically discrete; however, it’s been difficult to find the right language to distinguish between cases when this gives rise to *legitimate* problems versus ones where the issue is superficial. As an example
of the former, consider Example 5.2, and as an example of the latter, consider a sequence of homeomorphisms \( h_k : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) where each \( h_k \) is defined by

\[
h_k(y) = y + \begin{bmatrix} \frac{1}{k} \\ 0 \\ 0 \end{bmatrix}.
\]

Then

\[
h_\infty(y) = y + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

and so all points in \( \mathbb{R}^3 \) are moved infinitely-many times by \( h_\infty \), yet we have no problems.

Thus, we have the following question:

**Question 1.** Is there a simpler way to guarantee bijectivity of the limit function in Theorem 3.8? In particular, is there a purely diagrammatic condition?

One of the things that makes the problem in Example 5.2 tricky to spot at first is that the limiting process yields an isotopy, just not an ambient isotopy (this is reminiscent of Bachelor’s unknotting). We wonder whether a similar effect can be obtained using only Reidemeister I or Reidemeister III moves, as detailed in the following questions:

**Question 2.** Does there exist a knot \( f : S^1 \hookrightarrow \mathbb{R}^3 \) such that applying a countable sequence of Reidemeister I moves to \( f \) yields an isotopy, but not an ambient isotopy?

**Question 3.** Same as Question 2, but with Reidemeister III moves instead of Reidemeister I moves.

Of course, we want to avoid trivial examples like taking \( H_1 \) to be Bachelor’s unknotting and then taking the remaining \( H_k \)’s to be identity.

Now, returning to the question of a countable analogue for Reidemeister’s theorem:

**Question 4.** If we restrict ourselves to Reidemeister moves, when do the converses of Theorems 3.3 and 3.8 hold? That is, given an ambient isotopy between two embeddings \( f_0, f_1 : S^1 \hookrightarrow \mathbb{R}^3 \), when can we guarantee the existence of the desired \( V_k \)’s and \( h_k \)’s, where each of the \( h_k \)’s represent single Reidemeister moves?

We have a conjecture in this direction. In [5] the author employed Theorem 3.8 to argue the following result:

**Theorem 6.1.** Call a knot diagram a **discrete diagram** if it satisfies all the axioms of a regular diagram except perhaps having finitely-many crossings.\(^{13}\)

\(^{13}\)The “discrete” in “discrete diagram” comes from the fact that the set of crossing points only needs to be topologically discrete rather than finite.
Now, let \( f : S^1 \rightarrow \mathbb{R}^3 \) be an arbitrary knot. Then if \( f \) admits a discrete diagram, \( f \) is ambient isotopic to a representative comprised of countably-many line segments.

This gave rise to the following conjecture (we thank Kye Shi for the insight of adding the fourth move):

**Conjecture 1.** Define the extended Reidemeister moves to be the standard move set together with a fourth move

![Figure 30. Fourth move](image)

where in the above, \( A \) is a compact set whose interior remains fixed relative to its boundary. Note, \( A \) can contain wild points.

Let \( f_0, f_1 : S^1 \rightarrow \mathbb{R}^3 \) be knots that admit discrete diagrams. Suppose \( f_0 \cong f_1 \). Then there exists a countable sequence of extended Reidemeister moves satisfying the hypotheses of Theorem 3.8 and taking \( f_0 \) to \( f_1 \).

For more details on the proposed approach, see [5], §9.3.1. We have some partial results in this direction but there remain important gaps.

7. Acknowledgements

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Appendix A. The Fox-Artin Tameness Invariant

We will now describe the invariant for tameness established by Fox and Artin.

**Theorem A.1** (Fox-Artin). Let \( f : [0, 1] \to \mathbb{R}^3 \) be a tame arc. Let \( p = f(0) \), and for all \( k \in \mathbb{N} \), let \( V_k \subseteq \mathbb{R}^3 \) be a closed set such that \( p \in V_k^c \). Suppose that
\[
\cdots \subseteq V_k \subseteq \cdots \subseteq V_2 \subseteq V_1,
\]
and
\[
\{p\} = \bigcap_{n=1}^\infty V_n.
\]
Then there exists \( n_0 \in \mathbb{N} \) such that the inclusion map
\[
\iota : \pi(V_{n_0} \setminus f([0, 1])) \to \pi(V_1 \setminus f([0, 1]))
\]
is the trivial homomorphism.

**Proof.** \( f \) is tame implies that there exists an ambient homeomorphism \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( h \circ f \) is a straight line segment. Since homeomorphism preserves the fundamental group (as well as all the conditions on the \( V_n \)), it suffices to prove the claim for a straight line.

Hence, without loss of generality, suppose \( f \) is a line segment.

\[ p \]
\[ \begin{array}{c}
\text{Figure 31. An example of } f([0,1]), \text{ with the choice of } p \text{ labeled.}
\end{array} \]

\[
\begin{array}{c}
\text{Figure 32. An example of some possible first few } V_n \text{'s.}
\end{array} \]

Since \( p \in V_1^c \), there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(p) \subseteq V_1 \). Now, because \( \bigcap_{n=1}^\infty V_n = \{p\} \), we have \( \text{diam}(V_k) \to 0 \), and hence there exists \( n_0 \in \mathbb{N} \) such that \( V_{n_0} \subseteq B_\varepsilon(p) \).

This gives us the inclusions (of sets)
\[
\iota_0 : V_{n_0} \hookrightarrow B_\varepsilon(p) \quad \text{and} \quad \iota_1 : B_\varepsilon(p) \hookrightarrow V_1.
\]
Then the inclusion (of sets) \( \iota : V_{n_0} \hookrightarrow V_1 \) is given by
\[
\iota = \iota_1 \circ \iota_0.
\]
The result is identical if we replace \( V_{n_0}, B_\varepsilon(p), \) and \( V_1 \) with \( V_{n_0} \setminus f([0,1]), B_\varepsilon(p) \setminus f([0,1]), \) and \( V_1 \setminus f([0,1]) \), respectively.

Since the inclusion maps are all continuous, they induce homomorphisms on the associated fundamental groups. Thus, defining
\[
(\iota_0)_* : \pi(V_{n_0} \setminus f([0,1])) \hookrightarrow \pi(B_\varepsilon(p) \setminus f([0,1])) \\
(\iota_1)_* : \pi(B_\varepsilon(p) \setminus f([0,1])) \hookrightarrow \pi(V_1 \setminus f([0,1]))
\]
we see that \( \iota_* : \pi(V_{n_0} \setminus f([0,1])) \hookrightarrow \pi(V_1 \setminus f([0,1])) \) is given by
\[
\iota_* = (\iota_1)_* \circ (\iota_0)_*.
\]
\( B_\varepsilon(p) \setminus f([0,1]) \) is just a ball with a radius removed, hence \( \pi(B_\varepsilon(p) \setminus f([0,1])) \) is the trivial group. It follows that \( (\iota_1)_* \) is the trivial homomorphism, and thus \( \iota_* \) is the trivial homomorphism. ■

In the case of the curve in Fig. 1b, one can find a sequence of \( V_k \) such that no such \( n_0 \) exists. It follows that the curve is wild.