COMPACT VECTORIAL TOEPLITZ OPERATORS ON THE SEGAL-BARGMANN SPACE

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Abstract. We provide a sufficient condition for the compactness of a Toeplitz operator acting on the Segal-Bargmann space of vector-valued functions written in terms of an associated operator-valued kernel.

1. Introduction

A well-known result due to Stroethoff states that a Toeplitz operator \( T_{\phi} \), with \( \phi \in L^\infty(\mathbb{C}^n) \), acting on the Segal-Bargmann space \( B \) is compact if and only if \( \|P(\phi \circ \tau_\lambda)\| \to 0 \) as \( |\lambda| \to \infty \), where \( \tau_\lambda \) is the translation on \( \mathbb{C}^n \) by \( \lambda \) and \( P \) is the orthogonal projection from \( L^2(\mu_G) \) onto \( B \) (see [23, Theorem 5]). This result is related to a characterization of compact multiplication operators in terms of Berezin symbols due to Berger and Coburn (see [6, Theorem C]). The paper aims at generalizing Stroethoff’s result to the context of Toeplitz operators acting on the Segal-Bargmann space of vector-valued functions.

Toeplitz operators on Segal-Bargmann spaces arise in quantization of classical mechanics and are related to pseudodifferential operators (see [1, 13, 14, 16, 20, 21, 22]). They have been studied since the work of Berezin (see [4, 5]) in the classical context of complex-valued entire functions (see, e.g., [2, 3, 11, 12, 17, 18, 19]) and also in the more general setting of vector-valued functions (see, e.g., [8, 9, 10]). The literature concerning these operators is broad and still growing.

The main result of the paper states that a vectorial Toeplitz operator \( T_{\Phi} \) with \( L^\infty \) symbol \( \Phi \) is compact whenever its “oscillation at infinity” is small (see Theorem 3.6). The condition for that, namely condition (3.1), is a generalization of the Stroethoff’s one and employs the Hilbert-Schmidt norm. The reason for using the Hilbert-Schmidt norm lies in the fact that the compactness of \( T_{\Phi} \) is shown by approximating its adjoint, proved to be a vectorial integral operator with kernel related to translations (see Proposition 2.3), with a sequence of Hilbert-Schmidt vectorial integral operators. To this end we adapt a method used in the classical context (see Lemma 2.1, Proposition 3.3, and Corollary 3.5). It remains an open question whether condition (3.1) is necessary for the compactness of \( T_{\Phi} \). A necessary condition we obtained (see Proposition 3.7) turns out to be essentially weaker (see Examples 3.8, 3.9, and 3.10).

2. Preliminaries

In all what follows \( \mathbb{R} \) and \( \mathbb{C} \) stand for the sets of real and complex numbers, respectively, \( \mathbb{N} \) denotes the set of all natural numbers \( \{1, 2, 3, \ldots\} \), \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( \mathbb{R}_+ = [0, \infty) \), and \( \mathbb{R}_+^\infty = \mathbb{R}_+ \cup \{\infty\} \). For \( R \in (0, \infty) \), \( B_R \) stands for the ball \( \{z \in \mathbb{C}^n : |z| < R\} \), where \( |\cdot| \) is the standard Euclidean norm in \( \mathbb{C}^n \). The following standard multiindex notation is used: \( z^j = z_1^{j_1} \cdots z_n^{j_n} \), \( j! = j_1! \cdots j_n! \), and \( |j| = j_1 + \cdots + j_n \) for all \( j = (j_1, \ldots, j_n) \in \mathbb{Z}_+^n \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), \( n \in \mathbb{N} \).

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Let $\mathcal{H}$ be a (complex) Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let $A$ be a (linear) operator in $\mathcal{H}$. Then $\mathcal{D}(A)$ stands for the domain of $A$ and $A^*$ stands for the adjoint of $A$, whenever it exists. $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on $\mathcal{H}$ with the standard operator norm $\| \cdot \|_{\mathcal{B}(\mathcal{H})}$. If this leads to no confusion we write $\| \cdot \|$ instead of $\| \cdot \|_{\mathcal{B}(\mathcal{H})}$. The Banach space of all the Hilbert-Schmidt operators equipped with Hilbert-Schmidt norm $\| \cdot \|_{HS}$ is denoted by $\mathcal{HS}(\mathcal{H})$. Throughout the paper we adhere to the convention that if $A$ is not a Hilbert-Schmidt operator, then $\|A\|_{HS} = \infty$.

Let $\nu$ be a positive measure on $\mathscr{B}(\mathbb{C}^n)$, the $\sigma$-algebra of Borel sets in $\mathbb{C}^n$. Then as usual $L^2(\nu)$ stands for the space of all square-summable (with respect to $\nu$) complex-valued Borel functions on $\mathbb{C}^n$. By $\nu \otimes \nu$ we denote the product measure on $\mathscr{B}(\mathbb{C}^n \times \mathbb{C}^n) = \mathscr{B}(\mathbb{C}^{2n})$.

Let $\mu_{G}$ be the $n$-dimensional Gaussian measure on $\mathbb{C}^n$, i.e.,

$$\mu_{G}(\sigma) = \frac{1}{\pi^n} \int_{\sigma} \exp \left( - |z|^2 \right) dV(z), \quad \sigma \in \mathscr{B}(\mathbb{C}^n),$$

where $V$ denotes the Lebesgue measure on $\mathscr{B}(\mathbb{C}^n)$. The Segal-Bargmann space $\mathcal{B}$ (a.k.a. the Fock space) is a closed subspace of $L^2(\mu_{G})$ consisting of all analytic functions belonging to $L^2(\mu_{G})$. It is well-known that $\mathcal{B}$ is a RKHS with the kernel $k(z, w) = \exp \langle z, w \rangle$, $z, w \in \mathbb{C}^n$. Given $\lambda \in \mathbb{C}^n$, we define $k_\lambda(z) = \exp \langle z, \lambda \rangle$, $z \in \mathbb{C}^n$.

Let $\mathcal{H}$ be a separable Hilbert space. The space of all analytic functions $F: \mathbb{C}^n \to \mathcal{H}$ such that $\int_{\mathbb{C}^n} \|F(z)\|^2 d\mu_{G}(z) < \infty$ is to be identified with $\mathcal{B} \otimes \mathcal{H}$, the Hilbert tensor product of $\mathcal{B}$ and $\mathcal{H}$. We call it a (vectorial) Segal-Bargmann space. For $f \in \mathcal{B}$ and $h \in \mathcal{H}$, $f \otimes h$ stands for the function defined as $(f \otimes h)(z) = f(z)h$, $z \in \mathbb{C}^n$. Then $\mathcal{F} \otimes \mathcal{H}$ stands for the linear span of $\{k_\lambda \otimes g : \lambda \in \mathbb{C}^n, g \in \mathcal{H}\}$. The Segal-Bargmann space has the following reproducing property:

$$(2.1) \quad \langle F, k_z \otimes h \rangle = \langle F(z), h \rangle, \quad z \in \mathbb{C}^n, h \in \mathcal{H}, F \in \mathcal{B} \otimes \mathcal{H}.$$ 

Now, given a Borel function $\Phi: \mathbb{C}^n \to \mathcal{B}(\mathcal{H})$, we define the (vectorial) Toeplitz operator

$$T_\Phi: \mathcal{B} \otimes \mathcal{H} \supseteq \mathcal{D}(T_\Phi) \to \mathcal{B} \otimes \mathcal{H}$$

by the formula

$$\mathcal{D}(T_\Phi) = \{ F \in \mathcal{B} \otimes \mathcal{H} : \Phi F \in L^2(\mu_{G}) \otimes \mathcal{H} \},$$

$$T_\Phi F = P(\Phi F), \quad F \in \mathcal{D}(T_\Phi),$$

where $P$ denotes the orthogonal projection from $L^2(\mu_{G}) \otimes \mathcal{H}$ onto $\mathcal{B} \otimes \mathcal{H}$. Clearly, due to the reproducing property (2.1) we have

$$(2.2) \quad \langle (T_\Phi F)(z), h \rangle = \int_{\mathbb{C}^n} \langle \Phi(w)F(w), h \rangle k_z(w) d\mu_{G}(w), \quad z \in \mathbb{C}^n, F \in \mathcal{D}(T_\Phi), h \in \mathcal{H}. $$

Given a $\mathcal{B}(\mathcal{H})$-valued kernel on $\mathbb{C}^n$, i.e., a Borel function $\Theta: \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{B}(\mathcal{H})$, we define the (vectorial) integral operator

$$J_\Theta: \mathcal{B} \otimes \mathcal{H} \supseteq \mathcal{D}(J_\Theta) \to \mathcal{B} \otimes \mathcal{H}$$

with the following formula

$$\mathcal{D}(J_\Theta) = \{ F \in \mathcal{B} \otimes \mathcal{H} : J_\Theta F \in L^2(\mu_{G}) \otimes \mathcal{H} \},$$

$$J_\Theta F = P(J_\Theta F), \quad F \in \mathcal{D}(J_\Theta),$$

where

$$\langle (J_\Theta F)(z), h \rangle = \int_{\mathbb{C}^n} \Theta(z, w)F(w) d\mu_{G}(w), \quad z \in \mathbb{C}^n, F \in \mathcal{B} \otimes \mathcal{H}.$$
is understood in the weak sense. The kernel $\Theta$ is called Hilbert-Schmidt if it satisfies the condition
\begin{equation}
\int_{C^n} \int_{C^n} \|\Theta(z, w)\|_{HS}^2 d\mu_G(z) d\mu_G(w) < \infty.
\end{equation}

Let us recall a vectorial counterpart of the well-known classical result considering integral operators (see [7, p. 253]). For completeness and the reader convenience we supply a proof (different one than that in [7]).

**Lemma 2.1.** Let $\Theta: C^n \times C^n \to B(H)$ be a Hilbert-Schmidt kernel. Then $J_\Theta$ is a Hilbert-Schmidt operator on $B \otimes H$.

**Proof.** Since $\|A\| \leq \|A\|_{HS}$ for every Hilbert-Schmidt operator $A$, the Cauchy-Schwarz inequality and (2.3) imply that the operator $J_\Theta$ belongs to $B(B \otimes H)$.

Due to (2.3) and the fact that $HS(H)$ is separable, the kernel $\Theta$ is belongs to the Bochner-Lebesgue space $L^2(\mu_G \otimes \mu_G; HS(H))$, thus there exists a sequence of simple kernels $\{\Theta_j\}_{j=1}^\infty$ converging to $\Theta$ in $L^2(\mu_G \otimes \mu_G; HS(H))$ as $n \to \infty$. Each of the kernels $\Theta_j$, $j \in \mathbb{N}$, has the form $\Theta_j = \sum_{i=1}^{N_j} \chi_{\Omega_i} S_i$ with $\{\Omega_i\}_{i=1}^{N_j} \subseteq B(C^n \times C^n)$ and $\{S_i\}_{i=1}^{N_j} \subseteq HS(H)$. Since for all $\Omega \subseteq B(C^n \times C^n)$ and $S \in HS(H)$ we have

\[ J_\Omega F(z) = \int_{C^n} \Xi(z, w) F(w) d\mu_G(w) = S \int_{C^n} \chi_\Omega(z, w) F(w) d\mu_G(w), \quad F \in B \otimes H, \]

where $\Xi(z, w) = \chi_\Omega(z, w) S$, we see that for every $j \in \mathbb{N}$, the operator $J_{\Theta_j}$ is Hilbert-Schmidt and $\|J_{\Theta_j}\|_{HS(H)} = \sum_{i=1}^{N_j} \|S_i\|_{HS(H)}(\mu_G \otimes \mu_G)(\Omega_i) = \|\Theta_j\|_{L^2(\mu_G \otimes \mu_G; HS(H))}^2$. This implies convergence of $\{J_{\Theta_j}\}_{j=1}^\infty$ in $HS(B \otimes H)$. On the other hand, for every $F \in B \otimes H$ we have

\[ \|J_{\Theta_j} - J_\Theta\| F^2 \leq \int_{C^n} \int_{C^n} (\Theta_j(z, w) - \Theta_j(z, w))^2 F(w) d\mu_G(w) \leq \|F\|^2 \int_{C^n} \int_{C^n} (\Theta_j(z, w) - \Theta_j(z, w))^2 d\mu_G(z) \]

\[ \leq \|F\|^2 \int_{C^n} \int_{C^n} \|\Theta_j(z, w) - \Theta_j(z, w)\|_{HS}^2 d\mu_G(z) \]

\[ = \|F\|^2 \{\Theta_j - \Theta\}_{L^2(\mu_G \otimes \mu_G; HS(H))}. \]

Hence we deduce that $\{J_{\Theta_j}\}_{j=1}^\infty$ converges to $J_\Theta$ in $HS(H)$, which completes the proof. \qed

As shown below, the adjoint of a Toeplitz operator can be recovered from an integral operator with a kernel that is associated to the symbol of the Toeplitz operator via translations. For that we recall a translation-related formula. Given $\lambda \in C^n$, the translation $\tau_\lambda: C^n \to C^n$ is given by $\tau_\lambda(w) = w + \tau$, $w \in C^n$. Using the change-of-variable formula, we get

\begin{equation}
\int_{C^n} (f \circ \tau_\lambda)(w) d\mu_G(w) = \int_{C^n} f(w) \tilde{k}_\lambda(w) d\mu_G
\end{equation}

with

\[ \tilde{k}_\lambda(w) = \frac{|k_\lambda(w)|^2}{k_\lambda(\lambda)}, \quad w \in C^n, \]

holding for every Borel function $f: C^n \to \mathbb{R}_+$ or every $f \in L^2(\tilde{k} d\mu_G)$.

Here, and later on, given $\Phi: C \to B(H)$ and $g \in H$, we denote by $\Phi \otimes g$ the function $C^n \to H$ defined by $(\Phi \otimes g)(z) = \Phi(z) g$.

Toepplitz operators act on functions $k_\lambda \otimes g$, with $\lambda \in C^n$ and $g \in H$, in accordance to the following formula involving translations (cf. [24 Proposition 1]):
Lemma 2.2. Let $\lambda \in \mathbb{C}^n$ and $g \in \mathcal{H}$ be such that $k_\lambda \otimes g \in \mathcal{D}(T_\Phi)$. Then
\[(T_\Phi k_\lambda \otimes g)(z) = k_\lambda(z) \left( \mathcal{P}(\Phi \otimes g) \right)(\tau_{-\lambda}(z)) \quad z \in \mathbb{C}^n,
\]
where $\Phi = \Phi \circ \tau_\lambda$.

Proof. Fix $z \in \mathbb{C}^n$, $h \in \mathcal{H}$. Since $k_\lambda \otimes g \in \mathcal{D}(T_\Phi)$, we see that (2.4) implies
\[
\int_{\mathbb{C}^n} \|\Phi(u)g\|^2 d\mu_G(u) = \int_{\mathbb{C}^n} \|\Phi(u)\|^2 |k_\lambda(u)|^2 d\mu_G(u)
= \frac{1}{k_\lambda(\lambda)} \int_{\mathbb{C}^n} \|\Phi(u)\|^2 |k_\lambda(u)|^2 d\mu_G(u)
= \frac{1}{k_\lambda(\lambda)} \int_{\mathbb{C}^n} \|\Phi(u)k_\lambda(u)g\|^2 d\mu_G(u)
= \frac{1}{k_\lambda(\lambda)} \int_{\mathbb{C}^n} \|\Phi(u)(k_\lambda \otimes g)(u)\|^2 d\mu_G(u) < \infty,
\]
which means that $u \mapsto \Phi(u)g$ belongs to $L^2(\mu_G) \otimes \mathcal{H}$. Now, it is easily seen that (cf. [23])
\[
k_\lambda(u)\overline{k_\lambda(u)} = k_\lambda(\lambda)|k_{\tau_{-\lambda}}(\lambda)|^2 (\tau_{-\lambda}(u)), \quad u \in \mathbb{C}^n.
\]
This together with (2.2), (2.4), and (2.1) gives
\[
\langle (T_\Phi k_\lambda \otimes g)(z), h \rangle = \int_{\mathbb{C}^n} \langle \Phi(u)g, h \rangle k_\lambda(u)\overline{k_\lambda(u)} d\mu_G(u)
= \int_{\mathbb{C}^n} \langle \Phi(u)g, h \rangle k_\lambda(\lambda)|k_{\tau_{-\lambda}}(\lambda)|^2 (\tau_{-\lambda}(u)) d\mu_G(u)
= k_\lambda(\lambda) \int_{\mathbb{C}^n} \langle \Phi(u)g, h \rangle \overline{k_{\tau_{-\lambda}}(\lambda)}(\tau_{-\lambda}(u)) d\mu_G(u)
= k_\lambda(\lambda) \int_{\mathbb{C}^n} \langle \Phi(u)g, (k_{\tau_{-\lambda}} \otimes h)(u) \rangle d\mu_G(u)
= k_\lambda(\lambda) \langle (\mathcal{P}(\Phi \otimes g))(\tau_{-\lambda}(z)), h \rangle,
\]
which completes the proof. \(\blacksquare\)

Having the above result, we can show that the adjoint of a vectorial Toeplitz operator is a suboperator of a vectorial integral operator with translation-related kernel. For a general studies of the adjoints of vectorial Toeplitz operators we refer the reader to [9, 10].

Proposition 2.3. Suppose $\mathcal{K} \otimes \mathcal{H} \subseteq \mathcal{D}(T_\Phi)$. Then $\mathcal{D}(T_\Phi^*) \subseteq \mathcal{D}(J_{\Theta_\Phi})$ and
\[T_\Phi^* F = J_{\Theta_\Phi} F, \quad F \in \mathcal{D}(T_\Phi^*),\]
where $\Theta_\Phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{B}(\mathcal{H})$ is the kernel given by
\[\Theta_\Phi(z, w)^* g = k_z(w) \left( \mathcal{P}(\Phi_z \otimes g) \right)(\tau_{-z}(w)), \quad g \in \mathcal{H}, \quad z, w \in \mathbb{C}^n.
\]

Proof. Let $F \in \mathcal{D}(T_\Phi^*)$. Then, by (2.2) and Lemma 2.2 we have
\[
\int_{\mathbb{C}^n} \langle \Theta_\Phi(z, w) F(w), g \rangle d\mu_G(w) = \int_{\mathbb{C}^n} \langle F(w), \Theta_\Phi(z, w)^* g \rangle d\mu_G(w)
= \int_{\mathbb{C}^n} \langle F(w), (T_\Phi k_z \otimes g)(w) \rangle d\mu_G(w)
= \langle (T_\Phi^* F)(z), g \rangle, \quad g \in \mathcal{H}, \quad z \in \mathbb{C}^n,
\]
which means that $J_{\Theta_\Phi} F : \mathbb{C}^n \to \mathcal{H}$ is well-defined (the integral exists in the weak sense). Also, by the above computations, we have
\[
\langle (J_{\Theta_\Phi} F)(z), g \rangle = \langle (T_\Phi^* F)(z), g \rangle, \quad g \in \mathcal{H}, \quad z \in \mathbb{C}^n.
\]
This implies that \((J_{\varphi_2} F)(z) = (T_\varphi F)(z)\) for every \(z \in \mathbb{C}^n\). As a consequence, \(J_{\varphi_2} F \in L^2(\mu_G) \otimes \mathcal{H}\) and \(J_{\varphi_2} F = T_\varphi F\), which completes the proof.

3. Main results

Generalizing Stroethoff’s result we use various kernels of related to the formula for the adjoint of a Toeplitz operator that was presented in Proposition 2.3. The following lemma provides a sufficient condition for one of those kernels to be \(\text{HS}(\mathcal{H})\)-valued.

**Lemma 3.1.** Suppose \(\mathcal{K} \otimes \mathcal{H} \subseteq \mathcal{D}(T_\varphi)\) and \(z, w \in \mathbb{C}^n\). Let \(\Xi_\varphi(z, w) \in \mathcal{B}(\mathcal{H})\) be given by

\[
\Xi_\varphi(z, w) = \int_{\mathbb{C}^n} \Phi_z(u)g \, k_w(u) \, d\mu_G(u) = P(\Phi_z \otimes g)(w), \quad g \in \mathcal{H}.
\]

Then the following conditions are satisfied:

(i) \(\|\Xi_\varphi(z, w)\|_{\text{HS}}^2 \leq k_w(w) \int_{\mathbb{C}^n} \|\Phi_z(u)\|_{\text{HS}}^2 d\mu_G(u)\),

(ii) if \(\Phi\) is analytic, then \(\|\Xi_\varphi(z, w)\|_{\text{HS}} = \|\Phi_z(w)\|_{\text{HS}}\).

**Proof.** Let \(\{g_i\}_{i=1}^\infty\) an orthonormal basis for \(\mathcal{H}\).

(i) Assume that \(\int_{\mathbb{C}^n} \|\Phi_z(u)\|_{\text{HS}}^2 d\mu_G(u) < \infty\). Then for \(\mu_G\)-a.e. \(u \in \mathbb{C}^n\), \(\Phi_z(u)\) is Hilbert-Schmidt. Then we get

\[
\sum_{i,j=1}^\infty |\langle \Xi_\varphi(z, w)g_i, g_j \rangle|^2 = \sum_{i,j=1}^\infty \left| \int_{\mathbb{C}^n} \langle \Phi_z(u)g_i, g_j \rangle k_w(u) \, d\mu_G(u) \right|^2 \\
\leq \sum_{i,j=1}^\infty \left( \int_{\mathbb{C}^n} |\langle \Phi_z(u)g_i, g_j \rangle|^2 \, d\mu_G(u) \right) \left( \int_{\mathbb{C}^n} |k_w(u)|^2 \, d\mu_G(u) \right) \\
= \|k_w\|^2 \int_{\mathbb{C}^n} \left( \sum_{i,j=1}^\infty |\langle \Phi_z(u)g_i, g_j \rangle|^2 \right) d\mu_G(u) \\
= k_w(w) \int_{\mathbb{C}^n} \|\Phi_z(u)\|_{\text{HS}}^2 d\mu_G(u),
\]

which gives (i).

(ii) Due to the reproducing property we have

\[
\langle P(\Phi_z \otimes g_i)(w), g_j \rangle = \int_{\mathbb{C}^n} \langle \Phi_z(u)g_i, g_j \rangle k_w(u) \, d\mu_G(u) = \langle \Phi_z(w)g_i, g_j \rangle, \quad i, j \in \mathbb{N}.
\]

Therefore we get

\[
\sum_{i,j=1}^\infty |\langle \Xi_\varphi(z, w)g_i, g_j \rangle|^2 = \sum_{i,j=1}^\infty |\langle \Phi_z(w)g_i, g_j \rangle|^2,
\]

which proves (ii) and completes the proof. □

Employing Lemma 2.4 we get.

**Corollary 3.2.** Suppose \(\mathcal{K} \otimes \mathcal{H} \subseteq \mathcal{D}(T_\varphi)\). Suppose that \(\Phi\) is analytic and kernel \((z, w) \mapsto \Phi(z+w)\) is Hilbert-Schmidt. Then the integral operator \(J_{\Xi_\varphi}\) with \(\Xi_\varphi\) as in Lemma 3.1 is Hilbert-Schmidt on \(\mathcal{B} \otimes \mathcal{H}\).

As shown below, the \(\mathcal{B}(\mathcal{H})\)-valued kernel of the adjoint of a Toeplitz operator multiplied by appropriate complex function induces a Hilbert-Schmidt integral operator.

**Proposition 3.3.** Let \(\mathcal{K} \otimes \mathcal{H} \subseteq \mathcal{D}(T_\varphi)\) and \(\Xi_\varphi : \mathbb{C}^n \times \mathbb{C}^n \to \mathcal{B}(\mathcal{H})\) be as in Lemma 3.1. Suppose that \(\gamma : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}\) is Borel and the function \(\alpha : \mathbb{C}^n \to \mathbb{R}_+\) given by

\[
\alpha(z) = \int_{\mathbb{C}^n} |\gamma(z, w)|^2 \|\Xi_\varphi(z, w)\|_{\text{HS}}^2 d\mu_G(w), \quad z \in \mathbb{C}^n,
\]


Proof. According to Lemma 2.1 the proof amounts to showing that the kernel \( \Theta_{\gamma, \phi} \) is Hilbert-Schmidt. Since, by the Fubini’s theorem and (2.1), we have

\[
\int_{C^n} \sum_{i,j} |\gamma(z, \tau_z(w))| \langle \Xi_{\phi}(z, w) \rangle^2 \, d\mu_G(z, w) < \infty,
\]

the claim follows.

\[\square\]

Corollary 3.4. Let \( \Phi \in L^2(\mu_G; B(H)) \). Let \( \Delta \subseteq C^n \) be a bounded Borel set. If

\[
\int_\Delta \int_{C^n} \|\Phi_z(u)\|_{\text{HS}}^2 \, d\mu_G(u) \, dV(z) < \infty,
\]

then the integral operator \( J_{\Theta_{\chi, \Delta, \Delta, *}} \), given as in Proposition 3.3, is Hilbert-Schmidt on \( B \otimes H \).

Proof. In view of Proposition 3.3, \( J_{\Theta_{\chi, \Delta, \Delta, *}} \) is Hilbert-Schmidt whenever

\[
\int_{C^n} \int_{C^n} \chi_{\Delta \times \Delta}(z, w) \|\Xi_{\phi}(z, w)\|_{\text{HS}}^2 \, d\mu_G(w) \, dV(z) < \infty.
\]

Since, by Lemma 3.1(i), we have

\[
\int_{C^n} \int_{C^n} \chi_{\Delta \times \Delta}(z, w) \|\Xi_{\phi}(z, w)\|_{\text{HS}}^2 \, d\mu_G(w) \, dV(z) \leq V(\Delta) \int_{C^n} \|\Phi_z(u)\|_{\text{HS}}^2 \, d\mu_G(u) \, dV(z),
\]

inequality (3.1) follows from the assumptions.

\[\square\]

Corollary 3.5. Let \( \Phi \in L^\infty(\mu_G; HS(H)) \). Let \( \Delta \subseteq C^n \) be a bounded Borel set. Then the integral operator \( J_{\Theta_{\gamma, \Delta, \Delta, *}} \) with \( \gamma_{\Delta}: C^n \times C^n \to \mathbb{R} \) given by \( \gamma_{\Delta} := \chi_{\Delta \times C^n} \) is Hilbert-Schmidt on \( B \otimes H \).

Proof. First we show that

\[
\|\Xi_{\phi}(z, w)\|_{\text{HS}}^2 \leq C \exp \left( -\frac{|w|^2}{2} \right),
\]

where \( C > 0 \) is a constant such that \( \|\Phi(u)\|_{\text{HS}} \leq C \) for \( \mu_G \)-a.e. \( u \in C^n \). Indeed, let \( \{g_i\}_{i=1}^\infty \) be an orthonormal basis for \( H \). Then, applying Jensen’s inequality with a probabilistic measure \( d\nu_{z, w}(t) = |k_w(t - z)| \exp \left( -|t - z|^2 - \frac{|w|^2}{4} \right) dV(t) \), we get

\[
\sum_{i,j=1}^\infty |\langle \Xi_{\phi}(z, w) g_i, g_j \rangle|^2 = \sum_{i,j=1}^\infty \left| \int_{C^n} \langle \Phi_z(u) g_i, g_j \rangle k_w(u) \, d\mu_G(u) \right|^2
\]

\[
\leq \sum_{i,j=1}^\infty \left( \int_{C^n} |\langle \Phi_z(u) g_i, g_j \rangle|^2 k_w(u) \, d\mu_G(u) \right)^2
\]

\[
= \sum_{i,j=1}^\infty \left( \int_{C^n} |\langle \Phi(t) g_i, g_j \rangle|^2 k_w(t - z) \exp \left( -|t - z|^2 \right) dV(t) \right)^2
\]

\[
= \sum_{i,j=1}^\infty \left( \int_{C^n} |\langle \Phi(t) g_i, g_j \rangle| \exp \left( -\frac{|w|^2}{4} \right) d\nu_{z, w}(t) \right)^2
\]

belongs to \( L^1(V) \). Then the integral operator \( J_{\Theta_{\gamma, \phi}} \) with \( \Theta_{\gamma, \phi}: C^n \times C^n \to B(H) \) given by

\[
\Theta_{\gamma, \phi}(x, y)^* g = \gamma(x, \tau_x(y)) k_x(y) (P \Phi_x \otimes g)(\tau_x(y)), \quad g \in H, \ x, y \in C^n,
\]
is Hilbert-Schmidt on \( B \otimes H \).
which yields (3.2). Consequently, we deduce that

\[
\int_{C^n} |\gamma_\Delta(z, w)|^2 \|\Xi(z, w)\|_{H^s}^2 \, d\mu_G(w) \leq C \int_{C^n} \exp \left( \frac{|w|^2}{2} \right) \, d\nu_{z, w}(t),
\]

which yields (3.2). Consequently, we deduce that

\[
\int_{C^n} k_z(z) \left( \int_{C^n} |\gamma_\Delta(z, w)|^2 \|\Xi(z, w)\|_{H^s}^2 \, d\mu_G(w) \right) \, d\mu_G(z) = \int_{\Delta} k_z(z) \left( \int_{C^n} \|\Xi(z, w)\|_{H^s}^2 \, d\mu_G(w) \right) \, d\mu_G(z)
\]

\[
\leq C \int_{\Delta} k_z(z) \int_{C^n} \exp \left( \frac{|w|^2}{2} \right) \, d\mu_G(w) \, d\mu_G(z) < \infty.
\]

This implies that $J_{\Theta_\gamma, \phi}$ is Hilbert-Schmidt (see the proof of Proposition 3.3). \qed

Having the above result, we are able now to provide a sufficient condition for the compactness of $J_\phi$ à la Stroethoff.

**Theorem 3.6.** Let $\Phi \in L^\infty(\mu_G; HS(H))$. Suppose that:

\[
\lim_{|z| \to \infty} \int_{C^n} \|\Xi(z, w)\|_{H^s}^2 \, d\mu_G(w) = 0.
\]

Then $T_\Phi$ is compact.

**Proof.** For a given $r > 0$ we define a function $\gamma_r: C^n \times C^n \to \mathbb{R}$ by $\gamma_r(z, w) = \chi_{\{|z| < r\}}(z, w)$, where as usual $\{|z| < r\} := \{z \in C^n : |z| < r\}$. Then, by Corollary 3.3, the integral operator $J_{\Theta_\gamma, \phi}$ is Hilbert-Schmidt for every $r > 0$. Hence, showing that $J_{\Theta_\gamma, \phi} \to T_\Phi$ as $r \to \infty$ yields the claim.

Observe that by Proposition 2.3 we have

\[
T_\Phi - J_{\Theta_\gamma, \phi} = J_{\Theta_\gamma} - J_{\Theta_\gamma, \phi} = J_{\Theta^{(r)}},
\]

where

\[
\Theta^{(r)}(z, w) = \Theta_{\gamma_r, \phi}(z, w), \quad z, w \in C^n,
\]

with $\gamma_r = \chi_{\{|z| > r\}} \times C^n$. We will evaluate the norm of $J_{\Theta^{(r)}}$ using the Schur test. First, we note that there exists a constant $\alpha > 0$ such that

\[
\int_{C^n} \eta_r(z)|k_z(w)||\Xi(z, \tau_z(w))||H^s_k(z)\frac{1}{2} \, d\mu_G(z) \leq \alpha k_w(w)\frac{1}{2}, \quad w \in C^n.
\]

Indeed, by (3.2), we have

\[
\int_{C^n} \eta_r(z)|k_z(w)||\Xi(z, \tau_z(w))||H^s_k(z)\frac{1}{2} \, d\mu_G(z) \leq C \int_{C^n} |k_z(w)| \exp \left( \frac{\tau_z(w)}{4} \right) k_z(z)\frac{1}{2} \, d\mu_G(z)
\]

\[
= C k_w(w)\frac{1}{2} \int_{C^n} |k_z(w)|\frac{1}{2} \, d\mu_G(z).
\]

Since the last integral equals $2^n(2\pi)^n k_w(w)^\frac{1}{2}$, we get (3.5). On the other hand, there exists a function $\beta: \mathbb{R} \to \mathbb{R}$ such that

\[
\lim_{r \to \infty} \beta(r) = 0
\]

and

\[
\int_{C^n} \eta_r(z)|k_z(w)||\Xi(z, \tau_z(w))||H^s_k(w)\frac{1}{2} \, d\mu_G(w) \leq \beta(r) k_z(z)\frac{1}{2}, \quad z \in C^n.
\]
This can be proved as follows. By the change-of-variable theorem we have
\[
\int_{\mathbb{C}^n} \eta_r(z)|k_z(w)||\Xi(z, \tau_z(w))||_{HS} k_w(w)^{\frac{1}{2}} d\mu_G(w) \\
= \int_{\mathbb{C}^n} \eta_r(z)|k_z(t)||\Xi(z, t)||_{HS} k_{\tau_z(t)}(\tau_z(t))^{\frac{1}{2}} \exp \left( -|\tau_z(t)|^2 \right) dV(t) \\
= \eta_r(z)k_z(z)^{\frac{1}{2}} \int_{\mathbb{C}^n} \|\Xi(z, t)||_{HS} \exp \left( -\frac{|t|^2}{2} \right) dV(t) 
\]
Since \( \exp \left( -\frac{|t|^2}{2} \right) = \exp \left( -\frac{5|t|^2}{12} \right) \exp \left( -\frac{|t|^2}{12} \right) \), applying Hölder’s inequality with conjugate exponents \( \frac{1}{2} \) and later \( (3.2) \) we see that
\[
\int_{\mathbb{C}^n} \eta_r(z)|k_z(w)||\Xi(z, \tau_z(w))||_{HS} k_w(w)^{\frac{1}{2}} d\mu_G(w) \\
\leq \eta_r(z)k_z(z)^{\frac{1}{2}} \left( \int_{\mathbb{C}^n} \|\Xi(z, t)||_{HS}^2 d\mu_G(t) \right)^{\frac{1}{2}} \left( \int_{\mathbb{C}^n} \exp \left( -\frac{|t|^2}{8} \right) dV(t) \right)^{\frac{1}{2}} \\
\leq \eta_r(z)k_z(z)^{\frac{1}{2}} \left( 2(2\pi)^n \right)^{\frac{1}{2}} C^\frac{1}{2} \left( \int_{\mathbb{C}^n} \|\Xi(z, t)||_{HS}^2 d\mu_G(t) \right)^{\frac{1}{2}}.
\]
Hence, in view of \((3.3)\), we deduce \((3.6)\) and \((3.7)\) with
\[
\beta(r) = \left( 2(2\pi)^n \right)^{\frac{1}{2}} C^\frac{1}{2} \sup_{\{\|\Xi(z, t)||_{HS}^2 \leq 1\}} \left( \int_{\mathbb{C}^n} \|\Xi(z, t)||_{HS}^2 d\mu_G(t) \right)^{\frac{1}{2}}, \quad r \in \mathbb{R}_+
\]
Now, using the Schur test (see \([15] \) Theorem 5.2), \((3.5)\), and \((3.7)\) we get
\[
\|J_{\Delta(r)}\|^2 \leq \alpha \beta(r), \quad r \in \mathbb{R}_+.
\]
Therefore, by \((3.5)\) and \((3.7)\), \(T_\Phi\) is compact. \(\square\)

It remains an open problem as to whether condition \((3.3)\) is necessary for compactness of \(T_\Phi\) with \(HS(\mathcal{H})\)-valued symbols. Adapting the methods from the classical complex-valued case we get a weaker condition.

**Proposition 3.7.** Let \(T_\Phi \in \mathcal{B}(\mathcal{B} \otimes \mathcal{H})\) be compact. Then
\[
(3.8) \quad \lim_{|z| \to \infty} \int_{\mathbb{C}^n} \|\hat{P}(\Phi_z \otimes g)(w)||^2 d\mu_G(w) = 0, \quad g \in \mathcal{H}.
\]

**Proof.** Applying twice Lemma \((2.2)\) we get
\[
\|T_\Phi(k_z \otimes g)\|^2 = \int_{\mathbb{C}^n} |k_z(w)|^2 \|\hat{P}(\Phi_z \otimes g)(\tau_z(w))||^2 d\mu_G(w), \quad z \in \mathbb{C}^n, g \in \mathcal{H}.
\]
Combining this with \((2.3)\) and
\[
(3.9) \quad \|k_z \otimes g\| = \|g\| \exp |z|^2, \quad z \in \mathbb{C}^n, g \in \mathcal{H},
\]
yields
\[
\|T_\Phi \left( \frac{k_z \otimes g}{\|k_z \otimes g\|} \right)\|^2 = \frac{1}{\|g\|^2} \int_{\mathbb{C}^n} \|\hat{P}(\Phi_z \otimes g)(\tau_z(w))||^2 k_z(w) d\mu_G(w) \\
= \frac{1}{\|g\|^2} \int_{\mathbb{C}^n} \|\hat{P}(\Phi_z \otimes g)(w)||^2 d\mu_G(w), \quad z \in \mathbb{C}^n, g \in \mathcal{H}.
\]
The compactness of \(T_\Phi\) implies that \(\|T_\Phi x\|\) converges to 0 for any sequence \(\{x\}\) converging to 0 weakly. In particular, we deduce from \((3.9)\) and
\[
\langle p \otimes h, k_z \otimes g \rangle = |\langle p(z) h, g \rangle| \leq |p(z)||h||\|g\|, \quad z \in \mathbb{C}^n, p \in \mathcal{P}, h, g \in \mathcal{H},
\]
that
\[ \lim_{|z| \to \infty} \left\| T_\Phi \left( \frac{k_z \otimes g}{\|k_z \otimes g\|} \right) \right\|^2 = 0. \]

This, in view of (3.10), proves the claim. \( \square \)

It is worth noting that there is quite a gap between conditions (3.3) and (3.8) as shown in the following examples. The first of them shows that even in a very simplified situation the integrals in (3.3) and (3.8) differ essentially and hints the importance of using the Hilbert-Schmidt norm.

**Example 3.8.** Let \( \mathcal{H} \) be an infinite dimensional Hilbert space and \( A \in \mathcal{B}(\mathcal{H}) \) be an operator that is not Hilbert-Schmidt. Define \( \Phi: \mathbb{C}^n \to \mathcal{B}(\mathcal{H}) \) by \( \Phi(z) = A \). Clearly, \( \Phi \) is analytic,

\[ \int_{\mathbb{C}^n} \| P(\Phi_z \otimes g)(w) \|^2 d\mu_G(w) = \int_{\mathbb{C}^n} \| Ag \|^2 d\mu_G(w) < \infty, \quad z \in \mathbb{C}^n, g \in \mathcal{H}, \]

and

\[ \int_{\mathbb{C}^n} \| \Xi_\Phi(z,w) \|^2 d\mu_G(w) = \int_{\mathbb{C}^n} \| A \|^2 d\mu_G(w) < \infty, \quad z \in \mathbb{C}^n. \]

On the other hand, for an orthonormal basis \( \{g_i\}_{i=1}^\infty \) of \( \mathcal{H} \) we have

\[ \| \Xi_\Phi(z,w) \|^2_{\text{HS}} = \sum_{i,j=1}^\infty \left| \langle \Phi_z(w)g_i, g_j \rangle \right|^2 = \| A \|^2_{\text{HS}} = \infty, \quad z, w \in \mathbb{C}^n, \]

which implies that

\[ \int_{\mathbb{C}^n} \| \Xi_\Phi(z,w) \|^2_{\text{HS}} d\mu_G(w) = \infty, \quad z \in \mathbb{C}^n. \]

Note that \( T_\Phi \) is not compact.

The second example provides a non-compact vectorial Toeplitz operator which satisfies (3.8). Its symbol is not \( \text{HS}(\mathcal{H}) \)-valued and (3.3) does not hold.

**Example 3.9.** Let \( \mathcal{H} \) be an infinite dimensional Hilbert space. Let \( \phi: \mathbb{C}^n \to \mathbb{C} \) be a non-zero function such that the classical Toeplitz operator \( T_\phi \) on \( \mathcal{B} \) is compact. In view of [23, Theorem 5] we necessarily have

(3.11) \[ \lim_{|z| \to \infty} \int_{\mathbb{C}^n} |P(\phi \circ \tau_z)(w)|^2 d\mu_G(w) = 0, \]

where \( P \) denotes an orthogonal projection from \( L^2(\mu_G) \) onto \( \mathcal{B} \). This implies that for a function \( \Phi: \mathbb{C}^n \to \mathcal{B}(\mathcal{H}) \) given by \( \Phi(z) = \phi(z) I \), \( I \) being the identity operator on \( \mathcal{H} \), we have

\[ \int_{\mathbb{C}^n} \| P(\Phi_z \otimes g)(w) \|^2 d\mu_G(w) = \int_{\mathbb{C}^n} \| P(\phi_z \otimes g)(w) \|^2 d\mu_G(w) \]

\[ = \int_{\mathbb{C}^n} \| P(\phi \circ \tau_z)(w) \|^2 d\mu_G(w), \quad z \in \mathbb{C}^n, g \in \mathcal{H}, \]

and thus by (3.11) we have

\[ \lim_{|z| \to \infty} \int_{\mathbb{C}^n} \| P(\Phi_z \otimes g)(w) \|^2 d\mu_G(w) = 0, \quad g \in \mathcal{H}. \]

On the other hand, since the identity operator on an infinite dimensional Hilbert space is not compact, we have

\[ \int_{\mathbb{C}^n} \| \Xi_\Phi(z,w) \|^2_{\text{HS}} d\mu_G(w) = \int_{\mathbb{C}^n} \| P(\phi \circ \tau_z)(w)) \|^2_{\text{HS}} d\mu_G(w) = \infty, \quad z \in \mathbb{C}^n. \]

Clearly, \( T_\Phi \) is not compact as the image of \( \{ \chi_{C^n} \otimes e_i : i \in \mathbb{N} \} \) via \( T_\Phi \), where \( \{ e_i : i \in \mathbb{N} \} \) is an orthonormal basis of \( \mathcal{H} \), does not contain a convergent subsequence.
The next example, which was pointed to us by an anonymous referee, is showing that condition \( (3.3) \) may characterize the compactness of \( T_\Phi \) only in the case when \( \Phi \) is \( \text{HS}(\mathcal{H}) \)-valued. The latter is not necessary in general.

**Example 3.10.** We modify the symbol \( \Phi \) from Example 3.9 by replacing the identity operator by any compact operator \( A \) which is not Hilbert-Schmidt, i.e., \( \Phi(z) = \phi(z)A, z \in \mathbb{C}^n \), where \( \phi : \mathbb{C}^n \to \mathbb{C} \) is a non-zero function such that \( T_\phi \) on \( \mathcal{B} \) is compact. Then \( (3.3) \) is satisfied while \( (3.3) \) is not, just as in Example \( (3.9) \). Nonetheless, \( T_\Phi \) is compact. Indeed, since \( T_\phi \) and \( A \) are compact for any bounded sequences \( \{f_i\}_{i=1}^\infty \subseteq L^2(\mu_G) \) and \( \{g_j\}_{j=1}^\infty \subseteq \mathcal{H} \) there are subsequences \( \{f_{ik}\}_{k=1}^\infty \) and \( \{g_{jk}\}_{k=1}^\infty \) such that \( \{T_\phi f_{ik}\}_{k=1}^\infty \) and \( \{Ag_{jk}\}_{k=1}^\infty \) which are convergent. This implies that \( \{T_\Phi f_{ik} \otimes g_{jk}\}_{k=1}^\infty \) is convergent. Since the simple tensors \( \{f \otimes g : f \in L^2(\mu_G), g \in \mathcal{H}\} \) are dense in \( L^2(\mu_G) \otimes \mathcal{H} \), we deduce that \( T_\Phi \) is compact.

We conclude the paper with bringing up to the reader’s attention the questions that remains open. The first asks whether there exists a compact \( T_\Phi \) induced by a \( \text{HS}(\mathcal{H}) \)-valued symbol \( \Phi \) for which \( (3.3) \) is not satisfied. The second asks if there is a \( \text{HS}(\mathcal{H}) \)-valued symbol \( \Phi \) such that \( (3.8) \) holds but \( T_\Phi \) is not compact.

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