Entropy of Irregular Points for Some Dynamical Systems

Katrin Gelfert · Maria José Pacifico · Diego Sanhueza

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Abstract
We derive sufficient conditions for a dynamical systems to have a set of irregular points with full topological entropy. Such conditions are verified for nonuniformly hyperbolic systems such as positive entropy surface diffeomorphisms and rational functions on the Riemann sphere.

Keywords
Irregular points · Topological entropy · Specification property · Lyapunov exponents

Mathematics Subject Classification
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1 Introduction

Given a continuous map $f : X \to X$ on a compact metric space $X$ and a continuous observable $\varphi : X \to \mathbb{R}$, its Birkhoff average along an orbit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

(for $x \in X$ for which this limit exists) plays an important role because of its intimate relation with convergence in the weak* topology. Recall that the set of $\varphi$-irregular points for which
the averages do not converge,

\[ \hat{X}(f, \varphi) \overset{\text{def}}{=} \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \text{ does not exist} \right\}, \]

has zero measure for every \( f \)-invariant measure. Ruelle [24] coined the term “points with historical behaviour” because they trace the history of the system, whereas points whose averages converge only exhibit mean behaviour. Denote the set of irregular points of \( f \) by

\[ \hat{X}(f) \overset{\text{def}}{=} \bigcup \left\{ \hat{X}(f, \varphi) : \varphi : X \to \mathbb{R} \text{ continuous} \right\}. \tag{1} \]

Recall also that there exist several so-called “non-statistical” differentiable dynamical systems for which \( \hat{X}(f) \) has positive Lebesgue measure. One of the first, and classical, examples is the so-called Bowen’s eye (see [27] and also [29] and references therein). Although neither \( \hat{X}(f) \) nor \( \hat{X}(f, \varphi) \) is detected by any \( f \)-invariant measure, it can be “large” from another point of view such as, for example, fractal dimension, entropy, or general topology. This type of question is typical in multifractal analysis. Irregular points form an essential part of the multifractal decomposition of \( X \) (relative to \( \varphi \)),

\[ X = \bigcup_{\alpha \in \mathbb{R}} X_\alpha(f, \varphi) \cup \hat{X}(f, \varphi), \]

where \( X_\alpha(f, \varphi) \) denotes the set of \( \varphi \)-regular points with average \( \alpha \),

\[ X_\alpha(f, \varphi) \overset{\text{def}}{=} \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}. \]

Our focus will be on entropy. Even though any of the above defined sets is \( f \)-invariant, in general it is noncompact. Hence we rely on the concept of topological entropy introduced in [5] (we briefly recall its definition in Sect. 2). We denote by \( h(f, A) \) the topological entropy of \( f \) on \( A \subset X \). The following estimates hold true in general

\[ 0 \leq h(f, \hat{X}(f, \varphi)) \leq h(f, \hat{X}(f)) \leq h(f, X). \]

Each inequality can be strict. Recall, for instance, the example of a minimal dynamical system (hence satisfying \( \hat{X}(f) = \emptyset \)) with positive entropy in [13]. To our best knowledge, there is no nontrivial (that is, say, topologically transitive) example for which \( 0 < h(f, \hat{X}(f)) < h(f, X) \).

Previous approaches to analyze regular and irregular points commonly require certain “orbit-gluing properties”. For example, [10, 20] consider the case of a subshift of finite type. Just assuming that \( f \) has the so-called specification property, [28] establishes the restricted variational principle

\[ h(f, X_\alpha(f, \varphi)) = \sup \left\{ h_\mu(f) : \mu \text{ f-invariant, } \int \varphi \, d\mu = \alpha \right\}. \]

The specification property roughly says that given any number of arbitrarily long orbit segments, there exists an orbit which stays \( \varepsilon \)-close to each segment and between segments there are only a bounded number of iterations whose cardinality only depends on \( \varepsilon \) (we refer to [26, 30] for the full definition). For example, for a basic set of an Axiom A diffeomorphism, the existence of a Markov partition enables a symbolic description of orbits and shadowing permits to “connect” orbit segments which were symbolically coded previously. As such
partitions can be chosen with arbitrarily small diameter, this guarantees the existence of “arbitrarily specified orbits”.

On the other hand, “chaotic dynamics” commonly gives rise to a set of irregular points which is dense and has full entropy. For a full shift of two symbols \( \sigma : \Sigma_2 \to \Sigma_2 \), by [20, Lemma 6] it holds \( h(\sigma, \hat{\Sigma}_2(\sigma)) = h(\sigma, \Sigma_2) \). If \( f \) satisfies the specification property, then it is a consequence of [26, Theorem 4] (see also [6, Theorem 3]) that \( \hat{X}(f) \) is nonempty and has full entropy [9].

In general, specification, or any of its weaker versions, does not hold or is rather difficult to verify. We illustrate that it is also not necessary in this strong sense in order to deduce “maximal historic behavior”. Here the key observation is that such strong hypotheses are not required globally, but only for invariant subsystems whose entropies are sufficiently large and that entropy in some sense is “a local quantifier”. What is relevant indeed is detailed knowledge of the topological dynamics which gives rise to historic behavior and its interplay with ergodic properties.

We start by considering the set of irregular points, \textit{a priori} not fixing any observable, and assuming hyperbolicity.

**Theorem A** Every Axiom A \( C^1 \)-diffeomorphism \( f : X \to X \) on an \( n \)-dimensional closed manifold \( X \), \( n \geq 2 \), satisfies

\[
h(f, \hat{X}(f)) = h(f, X).
\]

In a “nonuniformly hyperbolic context”, the proof of the next result takes into consideration approximations in the weak* topology and in entropy of positive-entropy ergodic measures by horseshoes.

**Theorem B** Every \( C^{1+\alpha} \)-diffeomorphism \( f : X \to X \) on a closed surface \( X \) satisfies

\[
h(f, \hat{X}(f)) = h(f, X).
\]

A version of Theorem B was obtained independently in [3, Corollary 2.4].

To guarantee that \( h(f, \hat{X}(f, \varphi)) \) is positive (or even large), some hypotheses on the observable are absolutely necessary. Indeed, if \( \varphi \) is cohomologous to a constant, then \( \hat{X}(f, \varphi) \) is empty (and hence has zero entropy). We will recall some facts about cohomologous functions in Sect. 2. In general terms, the following dichotomy was shown in several contexts:

- either \( \hat{X}(f, \varphi) = \emptyset \)
- or \( \hat{X}(f, \varphi) \) has full topological entropy.

Indeed, this dichotomy about the set of \( \varphi \)-irregular points holds true for a topologically mixing subshift of finite type (or any topologically conjugate system) [2] and for dynamical systems satisfying the almost specification property ([30], see also [19] for the context of flows) or the gluing orbit property (see [17] and further references therein). Assuming the shadowing property, [7, Theorem 1.5] states an analogous dichotomy for the set of irregular points. Moreover, assuming the asymptotic average shadowing property (AASP for short) and a certain condition on the measure center, the set \( \hat{X}(f, \varphi) \) is either residual or empty (see [8]). It is not known if the AASP also implies the above dichotomy.

The following result provides a sufficient condition on \( \varphi \) to “detect” historic behavior and forces \( h(f, \hat{X}(f, \varphi)) \) to be large (or even full). We state it in a fairly general version in order to allow also the consideration of observables which are continuous on subsets but \textit{a priori} can be discontinuous on the full ambient space \( X \). Let us denote by \( \mathcal{M}(f) \) the set of \( f \)-invariant Borel probability measures.
Theorem C Let \( f : X \rightarrow X \) be a continuous map on a compact metric space and let \( \varphi : X \rightarrow \mathbb{R} \) be a function such that there is a sequence \((\Gamma_n)_n\) of compact \( f \)-invariant subsets of \( X \) and numbers \( \ell_n \in \mathbb{N} \) having the following properties:

(i) \( \varphi \) is continuous on \( \Gamma_n \),
(ii) \( f^{\ell_n} |_{\Gamma_n} \) has the almost specification property, and
(iii) it holds

\[
\inf_{\mu \in \mathcal{M}(f|_{\Gamma_n})} \int \varphi \, d\mu < \sup_{\mu \in \mathcal{M}(f|_{\Gamma_n})} \int \varphi \, d\mu. \tag{2}
\]

Then, it holds

\[
\limsup_{n \to \infty} h(f, \Gamma_n) \leq \limsup_{n \to \infty} h(f, \hat{\Gamma}_n(f|_{\Gamma_n}, \varphi|_{\Gamma_n})) \leq h(f, \hat{X}(f)) = h(f, X).
\]

Remark 1.1 In many cases (such as, for example, under the hypotheses of Theorem D below), the sets \( \Gamma_n \) to verify the hypotheses of Theorem C can be chosen to be nested, that is, \( \Gamma_n \subseteq \Gamma_{n+1} \), and so it suffices to show that (2) holds for some \( n \) (and hence for all \( n' \geq n \)).

Fixing some observable \( \varphi : X \rightarrow \mathbb{R} \), the value \( h(f, \hat{X}(f, \varphi)) \) may \( \text{a priori} \) be much smaller than \( h(f, X) \). To exemplify this, recall that a \( C^1 \) diffeomorphism \( f \) is Axiom A if its nonwandering set \( \Omega(f) \) is hyperbolic and the periodic points of \( f \) are dense in \( \Omega(f) \). Given a \( C^1 \) Axiom A diffeomorphism \( f \) with nontrivial spectral decomposition \( \Omega(f) = \Omega_1 \cup \cdots \cup \Omega_\ell \) such that \( h(f, \Omega_i) < h(f, \Omega_j) \) for some index pair \( i \neq j \), then for \( \varphi : M \rightarrow \mathbb{R} \) such that \( \varphi|_{\Omega_i} \) is not cohomologous to a constant and \( \varphi|_{\Omega_j} = 0 \) it holds

\[
\hat{X}(f, \varphi) \neq \emptyset \quad \text{and} \quad h(f, \hat{X}(f, \varphi)) < \max_{j \neq i} h(f, \Omega_j) = h(f, X).
\]

Finally, we apply Theorem C to rational functions \( f \) on the Riemann sphere which can be considered as “nonuniformly hyperbolic” maps. It extends the above mentioned dichotomy to the set of \( \varphi^u \)-irregular points of the so-called geometric observable

\[
\varphi^u \overset{\text{def}}{=} \log |f'|,
\]

and moreover forces \( f \) to be of a certain type.

Theorem D Suppose that \( f \) is a rational function of degree \( d \geq 2 \) on the Riemann sphere and \( J(f) \) its Julia set. Then

\[
h(f, \hat{J}(f)) = h(f, J(f)) = \log d.
\]

Moreover, for the set of \( \varphi^u \)-irregular points

\[
\hat{J}(f, \varphi^u) \overset{\text{def}}{=} \left\{ z \in J : \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(z)| \text{ does not exist} \right\}
\]

the following dichotomy holds:

- either \( \hat{J}(f, \varphi^u) = \emptyset \) and hence (up to a conjugacy by a Möbius transformation) it holds \( f(z) = z^{\pm d} \),
- or \( h(f, \hat{J}(f, \varphi^u)) = h(f, J(f)) = \log d \).

Note that the setting in Theorem D is in general very far from being hyperbolic, in particular in the case when there exists a critical point inside the Julia set. A classical example is the Ulam-von Neumann map \( z \mapsto z^2 - 2 \) whose Julia set is the closed interval \([-2, 2]\) which
contains the critical point $z = 0$. The Hausdorff dimension of irregular points was studied in [14] (see also Sect. 4.1). See also [29] for an investigation of non-statistical behavior.

The article is organized as follows. In Sect. 2 we recall some preliminaries. In Sect. 3 we prove Theorems A, B, and C. In Sect. 4 we study rational maps and prove Theorem D.

## 2 Preliminaries

Note that $\hat{X}(f, \varphi)$ is empty whenever $\varphi$ is “essentially constant”. More precisely, consider the space $C(X)$ of continuous observables $\varphi : X \to \mathbb{R}$. Two functions $\psi, \varphi \in C(X)$ are cohomologous in $C(X)$ (with respect to $f$) if there exists $u \in C(X)$ such that $\psi = \varphi + u - u \circ f$. A function which is cohomologous to a constant is a coboundary. The following facts are immediate.

**Proposition 2.1** For every $\varphi \in C(X)$, $c_1, c_2 \in \mathbb{R}, c_1 \neq 0$, and $k \in \mathbb{N}$, $\hat{X}(f, \varphi) = \hat{X}(f, c_1 \varphi + c_2) = \hat{X}(f, \varphi \circ f^k)$. If $\varphi, \psi \in C(X)$ are cohomologous, then $\hat{X}(f, \varphi) = \hat{X}(f, \psi)$. For every coboundary $\varphi$, it holds $\hat{X}(f, \varphi) = \emptyset$.

**Remark 2.2** Equip $C(X)$ with the usual sup-norm $\|\varphi\| \overset{\text{def}}{=} \sup |\varphi|$. For $\varphi \in C(X)$ the following facts are equivalent:

1. $\inf_{\mu \in \mathcal{M}(f)} \int \varphi \, d\mu < \sup_{\mu \in \mathcal{M}(f)} \int \varphi \, d\mu$.
2. $\varphi$ is not in the closure of the subset of coboundaries,
3. $\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j$ does not converge pointwise to a constant.

See, for example [30, Lemma 2.1], where further properties are stated. It is clear that if $\hat{X}(f, \varphi) \neq \emptyset$, then property (3) (and hence any other) holds true. Assuming the almost specification property, by [30, Theorem 4.1] property 1) (and hence any other) implies $\hat{X}(f, \varphi) \neq \emptyset$.

In general, $f$ can have a complicated dynamics and it can be more convenient to analyze certain subsystems. For example, $\hat{X}(f, \varphi) \neq \emptyset$ if property 1) in Remark 2.2 holds for a subsystem which has the specification property. To this end, we state the following lemma which is straightforward to show.

**Lemma 2.3** For each $\ell \in \mathbb{N}$ and $\varphi \in C(X)$, it holds

$$\hat{X}(f, \varphi) = \bigcup_{j=0}^{\ell-1} \hat{X}(f^j, \varphi \circ f^j).$$

In particular, $\hat{X}(f^\ell, \varphi) \subseteq \hat{X}(f, \varphi)$ and $\hat{X}(f) = \hat{X}(f^\ell)$.

Observe that $\hat{X}(f^\ell, \varphi)$ can be a proper subset of $\hat{X}(f, \varphi)$.

Let us finally briefly recall the definition of entropy on a set $Y \subseteq X$ (with respect to a continuous map $f : X \to X$ of a metric space $X$) [5]. Given a finite open cover $\mathcal{A}$ of $X$, let $n_\mathcal{A}(Y)$ be the smallest nonnegative integer $n$ such that $f^n(Y)$ is not contained in an element of $\mathcal{A}$; if $f^n(Y)$ is contained in an element of $\mathcal{A}$ for all integers $n \geq 0$, then let $n_\mathcal{A}(Y) \overset{\text{def}}{=} \infty$. For any countable collection of open sets $\mathcal{U}$ and $t \geq 0$ define

$$m(\mathcal{U}, t, \mathcal{A}) \overset{\text{def}}{=} \sum_{U \in \mathcal{U}} \exp(-t \cdot n_\mathcal{A}(U)).$$
Denote by $C_{\alpha} (Y, \varepsilon)$ the collection of all countable open covers $\mathcal{U}$ of $Y$ satisfying $\exp(-n_{\alpha} (U)) < \varepsilon$ for each $U \in \mathcal{U}$. Let

$$m_{\alpha} (Y, t) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \inf \left\{ m(\mathcal{U}, t, \alpha): \mathcal{U} \in C_{\alpha} (Y, \varepsilon) \right\}.$$ 

The topological entropy of $f$ on $Y$ is defined by

$$h(f, Y) \overset{\text{def}}{=} \sup_{\alpha} \inf \{ t \geq 0: m_{\alpha} (Y, t) = 0 \}.$$ 

Recall some basic properties:
- (Monotonicity) $Y \subset Z \subset X$ implies $h(f, Y) \leq h(f, Z)$,
- $h(f^m, Y) = m \cdot h(f, Y)$ for all $m \in \mathbb{N}$,
- (Countable stability) $h(f, \bigcup_{n \in \mathbb{N}} Y_n) = \sup_{n \in \mathbb{N}} h(f, Y_n)$

### 3 Proof of Theorems A, B, and C

**Proof of Theorem C** For each $n \geq 1$, [30, Theorem 4.1] implies

$$h(f^\ell_n, \Gamma_n) = h(f^\ell_n|_{\Gamma_n}, \dot{X}(f^\ell_n|_{\Gamma_n}, \varphi|_{\Gamma_n}))$$

and hence Lemma 2.3 together with $\dot{X}(f|_{\Gamma_n}, \varphi|_{\Gamma_n}) \subseteq \dot{X}(f, \varphi)$ and monotonicity of entropy gives

$$h(f, \Gamma_n) = h(f|_{\Gamma_n}, \dot{X}(f^\ell_n|_{\Gamma_n}, \varphi|_{\Gamma_n})) \leq h(f, \dot{X}(f, \varphi)) \leq h(f, \dot{X}(f)) \leq h(f, X).$$

Letting $n \to \infty$, this implies the assertion. \hfill \Box

**Proof of Theorem B** If $h(f, X) = 0$, the result is immediate.

If $h(f, X) > 0$, then by the variational principle for entropy, there is a sequence of ergodic measures $(\mu_n)_n$ such that $h_{\mu_n}(f) \to h(f, X)$. By Ruelle’s inequality, any $\mu_n$ with positive entropy is hyperbolic. Let us hence consider any hyperbolic ergodic measure $\mu$ with entropy arbitrarily close to $h(f, X)$. By Katok’s horseshoe construction [15, TheoremS.5.9], for every $\varepsilon > 0$, there exists a basic set $\Gamma \subset M$ and $m \in \mathbb{N}$ such that $f^m|_{\Gamma}$ is topologically mixing and hence has the specification property, and $h_{\mu}(f) - \varepsilon < h(f, \Gamma)$. Hence,

$$h(f, \Gamma) = \frac{1}{m} h(f^m, \Gamma) = \frac{1}{m} h(f^m, \dot{X}(f^m|_{\Gamma}))$$

$$\leq \frac{1}{m} h(f^m, \dot{X}(f|_{\Gamma})) = h(f|_{\Gamma}, \dot{X}(f|_{\Gamma})) \leq h(f, \dot{X}(f)).$$

Choosing $\mu$ with entropy arbitrarily close to $h(f, X)$ implies the claim. \hfill \Box

In higher dimensions there may not exist any hyperbolic ergodic measure and hence the strategy of the proof of Theorem B does not work. In an *a priori* hyperbolic context, we invoke [30, Theorem 4.1] (or [2]) directly.

**Proof of Theorem A** By the spectral decomposition theorem, the set of nonwandering points splits as

$$\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_m.$$

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Moreover, for every $i = 1, \ldots, m$ there are $\ell_i \in \mathbb{N}$ and a decomposition $\Omega_i = X_{1,i} \cup \ldots \cup X_{\ell_i,i}$ such that $f^{\ell_i}|_{X_{\ell_i,i}}$ is topologically mixing, for each $1 \leq k \leq \ell_i$, and hence has the specification property. The basic properties of entropy imply the following estimates

$$h(f, X) = \max_i h(f, \Omega_i) = \max_i \frac{1}{\ell_i} h(f^{\ell_i}, X_{\ell_i,i}) \leq h(f, \hat{X}(f)) \leq h(f, X),$$

and hence equality of all terms. This proves the assertion. □

## 4 Irregular Points in Julia Sets

In this section, let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of degree $d \overset{\text{def}}{=} \deg(f) \geq 2$ and consider its Julia set $J = J(f)$. Recall that $J$ is nonempty, compact, and coincides with the closure of the set of repelling periodic points. Moreover, $f|_J$ is topologically exact and its entropy equals $h(f, J) = \log d$ (see, for example, [4, 18]).

A compact $f$-invariant set $R \subset J$ is a **uniformly expanding repeller** if there is $n \in \mathbb{N}$ so that

$$\inf_{z \in R} |(f^n)'(z)| > 1,$$

$f|_R$ is topologically transitive, and $R$ is isolated, that is, there exists an open set $U \supset R$ such that $f^n(z) \in U$ for all $n \geq 0$ implies $z \in R$. Note that if, moreover, $f|_R$ is topologically mixing, then $f|_R$ has the specification property. Recall that $f|_J$ is **expansive** if there is $\delta > 0$ such that for every $z, w \in J, z \neq w$, there is $n \in \mathbb{N}$ such that $d(f^n(z), f^n(w)) \geq \delta$.

A rational function $f$ is expansive if and only if $J$ does not contain critical points of $f$. For every rational function $f$ without critical points in $J$, $J$ is either a uniformly expanding repeller or admits a **parabolic point**, that is, a periodic point $z = f^p(z) \in J$ so that $(f^p(z))'(z)$ is a root of unity. See [31] for details.

### 4.1 Irregular Points

Let us first provide some more information on the rich set of irregular points $\hat{J}(f)$ as defined in (1) (that is, with respect to continuous observables). By [14, Theorem 16], its **Hausdorff dimension** satisfies

$$\dim_H(\hat{J}(f)) \geq t(f) \overset{\text{def}}{=} \min \{ t \in \mathbb{R} : P(f, -t \varphi^u) = 0 \},$$

where $P(f, -t \varphi^u) \overset{\text{def}}{=} \sup_{\mu \in \mathcal{M}(f)} (h_\mu(f) - t \int \varphi^u d\mu).$

Here $t(f)$ is the smallest zero of the so-called **variational pressure function** $t \mapsto P(f, -t \varphi^u)$ associated to the geometric observable $\varphi^u = \log |f'|$. It encodes several further characteristic quantities. For example,

$$t(f) = hD(f, J) \overset{\text{def}}{=} \sup_Y \dim_H(Y) \overset{(4)}{=} DD(f, J) \overset{\text{def}}{=} \sup_{\mu} \dim_H(\mu) \leq \dim_H(J), \quad \overset{(5)}{\text{where}}$$

where in (4) the supremum is taken over all uniformly expanding repellers $Y \subset J$. In (5), $\dim_H \mu$ denotes the **Hausdorff dimension** of $\mu$ and the supremum is taken over all ergodic states.
measures $\mu$ with positive entropy. See [23, Chapter 12.3] and [31] for details. If $f|_J$ is expansive then $t(f) = \dim_H(J)$.1 Note that there exist rational functions for which $J$ has Lebesgue measure zero and which satisfy $t(f) = 2$ (such maps are constructed in [25]). To the best of our knowledge, it is not known if there are maps satisfying

$$\dim_H(J(f)) < \dim_H(J).$$

Finally note that for certain rational functions, the set of accumulation points of a sequence of empirical measures of Lebesgue-almost every $z \in \mathbb{C}$ is shown to be the largest possible set, that is, the set of all $f$-invariant measures $\mathcal{M}(f)$. In particular, any such $z$ is an irregular point for $f : \mathbb{C} \to \mathbb{C}$ (see [29] for details).

### 4.2 $\varphi^u$-Irregular Points

Let us now consider the set of $\varphi^u$-irregular points. As already indicated above, this observable, along with its scaled version $t\varphi^u$, $t \in \mathbb{R}$, plays an important role in the thermodynamics formalism (see also [23, 31] and references therein). If $f$ has no critical points in $R \subset J$ then $\varphi^u$ is continuous on $R$. For every $\mu \in \mathcal{M}_{\text{erg}}(f)$, it holds $\varphi^u \in L^p(\mu)$ for any $p > 0$. Let

$$\mathcal{X}(\mu) \overset{\text{def}}{=} \int \log |f'| \, d\mu = \int \varphi^u d\mu$$

and recall that $\mu$-almost every $z$ is $\varphi^u$-regular, that is, its Birkhoff average exists, satisfies

$$\mathcal{X}(z) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(z)| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi^u(f^k(z)) = \mathcal{X}(\mu),$$

and is also called its characteristic exponent. Note that $\mathcal{X}(\mu) \geq 0$ (see [21]). Let

$$\alpha^- = \inf \{ \mathcal{X}(\mu) : \mu \in \mathcal{M}_{\text{erg}}(f) \} \quad \text{and} \quad \alpha^+ = \sup \{ \mathcal{X}(\mu) : \mu \in \mathcal{M}_{\text{erg}}(f) \}$$

denote the minimal and maximal value of the “spectrum of possible exponents” of $\varphi^u$-regular points, respectively.

A natural upper bound of the characteristic exponent of any $z \in \mathbb{C}$ is given by the value

$$\alpha^+ \leq \mathcal{X}_{\text{max}} \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log K(f^n), \quad \text{where} \quad K(f) \overset{\text{def}}{=} \max_{z \in \mathbb{C}} |f'(z)|.$$  

Here the existence of the limit is a consequence of subadditivity. Note that

$$\alpha^+ = \mathcal{X}_{\text{max}} = \sup_{z \in \mathbb{C}} \lim_{n \to \infty} \sup_{m \to \infty} \frac{1}{n} \log |(f^n)'(z)| = \sup_{z \in \text{Per}(f)} \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(z)|, \quad (6)$$

where $\text{Per}(f) \subset J$ denotes the set of periodic points (see [12, 22]).

Recall that there exists a unique (hence ergodic) measure of maximal entropy $\mu_0$. Together with Ruelle’s inequality, it holds

$$0 < \log d = h(f, J) = h_{\mu_0}(f) \leq \mathcal{X}(\mu_0) \leq \alpha^+.$$  

Note that there exists the following classification of rational functions in terms of the above defined quantifiers.

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1 Only recently it was shown that there indeed exist rational functions $f$ satisfying $\mathcal{H}(f|_J) < \dim_H(J)$ (by [1] so-called Feigenbaum maps with periodic combinatorics whose Julia set has positive Lebesgue measure have this property).
Remark 4.1 [Classification of rational functions] Let

$$\alpha(f) \overset{\text{def}}{=} \dim_H(\mu_0) = \frac{\log d}{\chi(\mu_0)}.$$ 

The only rational functions of degree $\geq 2$ for which the observable $\psi \overset{\text{def}}{=} \alpha(f)\varphi^u - \log d$ is cohomologous to zero in $L^2(\mu_0)$ (that is, there exists $u \in L^2(\mu_0)$ satisfying $\psi = u - f \circ u$) must (up to conjugacy by a Möbius transformation) be of one of the following types:

(i) either $f(z) = z^{\pm d}$, and hence

$$\alpha(f) = 1 \quad \text{and} \quad \log d = \chi(\mu_0) = \chi_{\text{max}},$$

(ii) or $f(z) = \pm$ Chebyshev polynomial of degree $d$, and hence

$$\alpha(f) = 1 \quad \text{and} \quad \log d = \chi(\mu_0) < \chi_{\text{max}} = 2 \log d,$$

(iii) or $f$ is a Lattès map, and hence

$$\alpha(f) = 2 \quad \text{and} \quad \frac{1}{2} \log d = \chi(\mu_0) < \chi_{\text{max}} = \log d.$$

For any other rational map, $\alpha(f)\varphi^u - \log d$ is not cohomologous to zero in $L^2(\mu_0)$. Moreover,

$$\chi(\mu_0) < \chi_{\text{max}}.$$

See [32] and references therein.

We are now ready to prove Theorem D.

Proof of Theorem D We first consider the set $\hat{J}(f)$. By (7), it holds $\chi(\mu_0) > 0$ and hence, by [23, Theorem 11.6.1], there exists a sequence $(\Gamma_n)_n \subset J$ of uniformly expanding repellers satisfying

$$h(f, J) - \frac{1}{n} \leq h(f, \Gamma_n) \quad \text{and} \quad \left| \chi(\mu_0) - \chi(\mu) \right| \leq \frac{1}{n} \quad \text{for every} \ \mu \in \mathcal{M}(f|\Gamma_n)$$

and numbers $m_n \in \mathbb{N}$ such that $f^{m_n}|\Gamma_n$ is mixing.

In particular, $f^{m_n}|\Gamma_n$ has the specification property. As $\Gamma_n$ has positive topological entropy, $\mathcal{M}(f|\Gamma_n)$ is not a singleton and, in particular, there exists an observable $\phi_n \in C(J)$ which distinguishes measures in $\mathcal{M}(f|\Gamma_n)$ and satisfies (2). Invoking [11, Lemma 2], $(\Gamma_n)_n$ can in fact be chosen to be increasing, that is, $\Gamma_n \subset \Gamma_{n+1} \subset \ldots$. Thus, there exists some common $\phi \in C(J)$ satisfying (2) for every $n \in \mathbb{N}$. Thus, by Theorem C, we obtain

$$h(f, \hat{J}(f, \phi)) = h(f, \hat{f}(f)) = h(f, J) = \log d,$$

which proves the first assertion of the theorem.

Let us now study the set of $\psi^u$-irregular points. If $f(z) = z^{\pm d}$ then $|f'(z)| = d$ for every $z \in J$ and hence $\hat{J}(f, \psi^u) = \emptyset$. It hence remains to assume that $f(z) \neq z^{\pm d}$, and to show that $\hat{J}(f, \psi^u)$ is nonempty and has full entropy.

Let us first consider the particular case when $J$ is a uniformly expanding repeller. Then $\psi^u$ is continuous on $J$. Moreover, $f|_J$ has the shadowing property and topological exactness implies that $f|_J$ is topologically mixing. Hence, $f|_J$ satisfies the specification property (see, for example, [16, Lemma 9]) and thus $h(f, \hat{J}(f, \psi^u)) = h(f, J)$ follows from [30, Theorem 4.1].
Let us now provide the proof in the general case. Recall that, by Remark 4.1, if \( f(z) \neq z^{\pm d} \), then it holds \( 0 < \lambda'(\mu_0) < \lambda_{\text{max}} \). As in the first part of this proof, there exists a sequence \((\Gamma_n)_n \subset J\) of uniformly expanding repellers so that (8) holds. By (6)

\[
0 \leq \alpha^- \leq \lambda'(\mu_0) < \alpha^+ = \sup_{z \in \text{Per}(f)} \lambda(z).
\]

Hence, for \( n \) large enough, there exists an \( f \)-invariant measure \( \nu \) supported on a periodic hyperbolic orbit \( \Gamma_\nu \subset J \) satisfying

\[
\lambda'(\mu_0) + \frac{1}{n} < \lambda'_{\text{max}} - \frac{1}{n} < \lambda'(\nu).
\]

In particular, by (8), the repellers \( \Gamma_n \) and \( \Gamma_\nu \) are disjoint. Therefore, we can invoke [11, Lemma2] and conclude that there exists a transitive uniformly expanding Cantor repeller \( \Lambda_n \supset \Gamma_n \cup \Gamma_\nu \). In particular, there is \( \ell_n \in \mathbb{N} \) such that \( f^{\ell_n}|_{\Lambda_n} \) satisfies the specification property, \( \varphi^u \) is continuous on \( \Lambda_n \), and

\[
h(f, \Lambda_n) \geq h(f, \Gamma_n) \geq h(f, J) - \frac{1}{n}.
\]

Hence, from Theorem C and monotonicity of topological entropy, we conclude

\[
h(f, \hat{J}(f, \varphi^u)) = \limsup_{n \to \infty} h(f, \hat{\Lambda}_n(f|_{\Lambda_n}, \varphi^u|_{\Lambda_n})) = h(f, J).
\]

Finally, note that any of the above arguments remain true when instead of a rational function \( f \) we consider \( g = h^{-1} \circ f \circ h \), where \( h: \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) is some Möbius transformation. This finishes the proof of the theorem. \( \square \)

**Data availability** All data generated or analysed during this study are included in this published article (and its supplementary information files).

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