Schrödinger functional formalism with domain-wall fermion

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Abstract: Finite volume renormalization scheme is one of the most fascinating scheme for non-perturbative renormalization on lattice. By using the step scaling function one can follow running of renormalized quantities with reasonable cost. It has been established the Schrödinger functional is very convenient to define a field theory in a finite volume for the renormalization scheme. The Schrödinger functional, which is characterized by a Dirichlet boundary condition in temporal direction, is well defined and works well for the Yang-Mills theory and QCD with the Wilson fermion. However one easily runs into difficulties if one sets the same sort of the Dirichlet boundary condition for the overlap Dirac operator or the domain-wall fermion. In this paper we propose an orbifolding projection procedure to impose the Schrödinger functional Dirichlet boundary condition on the domain-wall fermion.
1. Introduction

Perturbative renormalization factor is a source of systematic errors in numerical investigation of lattice QCD. There has been progress in numerical simulation with dynamical fermions nowadays and sources of systematic error is decreasing. Evaluation of renormalization factors in non-perturbative method is also required. Finite volume renormalization scheme is one of the most fascinating procedure to define non-perturbative renormalization scheme on lattice. By using the step scaling function one can follow running of renormalized quantities from low energy region to perturbative region with reasonable cost for recent computers. It has been established that the Schrödinger functional is very convenient to define a field theory in a finite volume for renormalization scheme.

The Schrödinger functional (SF) is defined as a transition amplitude between two boundary states with finite time separation [1, 2, 3, 4]

$$Z = \langle C'; x_0 = T | C; x_0 = 0 \rangle = \int D\Phi e^{-S[\Phi]}$$

and is written in a path integral representation of the field theory with some boundary condition. For renormalization of a finite volume theory defined through the SF the renormalization scale is introduced by a finite volume $T \times L^3 \sim L^4$ of the system. The formulation is already accomplished for the non-linear $\sigma$-model [5], the non-Abelian gauge theory [6] and the QCD with the Wilson fermion [7, 8] including $O(a)$ improvement procedure [9, 10]. (See Ref. [11] for review.)

Several renormalization quantities like running gauge coupling [12, 13, 14, 15, 16, 17, 18], $Z$-factors and $O(a)$ improvement factors [19, 20, 21, 22, 23, 24] are extracted conveniently by using a Dirichlet boundary conditions for spatial component of the gauge field

$$A_k(x)|_{x_0=0} = C_k(\vec{x}), \quad A_k(x)|_{x_0=T} = C_k'(\vec{x})$$

and for the quark fields

$$P_+ \psi(x)|_{x_0=0} = \rho(\vec{x}), \quad P_- \psi(x)|_{x_0=T} = \rho'(\vec{x}),$$

$$\overline{\psi}(x)P_-|_{x_0=0} = \overline{\rho}(\vec{x}), \quad \overline{\psi}(x)P_+|_{x_0=T} = \overline{\rho}'(\vec{x}),$$

$$P_\pm = \frac{1 \pm \gamma_0}{2}.$$  

One of advantage of this Dirichlet boundary condition is that the system acquire a mass gap proportional to $1/T$ and there is no infra-red divergence. The finite volume plays a role of an infra-red cut-off. Field theory with Dirichlet boundary condition is shown to be renormalizable for the pure gauge theory [6] and QCD with the Wilson fermion [8].

Although it is essential to adopt Dirichlet boundary condition for a mass gap and renormalizability, it has a potential problem of zero mode in fermion system. For instance starting from a free Lagrangian

$$\mathcal{L} = \overline{\psi} (\gamma_\mu \partial_\mu + m) \psi$$
with positive mass $m > 0$ and the Dirichlet boundary condition

$$P_- \psi |_{x_0=0} = 0, \quad P_+ \psi |_{x_0=T} = 0$$

(1.7)

the zero eigenvalue equation $(\gamma_0 \partial_0 + m) \psi = 0$ in temporal direction allows a solution

$$\psi = P_+ e^{-mx_0} + P_- e^{-m(T-x_0)}$$

(1.8)

in $T \to \infty$ limit and a similar solution remains even for finite $T$ with an exponentially small eigenvalue $\propto e^{-mT}$. In the SF formalism this solution is forbidden by adopting an “opposite” Dirichlet boundary condition (1.3) and the system has a finite gap even for $m = 0$ [7].

For the Wilson fermion [7] on lattice the Dirichlet boundary condition is automatically chosen among

$$P_- \psi |_{x_0=0} = 0, \quad P_+ \psi |_{x_0=T} = 0$$

(1.9)

depending on signature of the Wilson term. For example if we adopt a typical signature of the Wilson term

$$D_W = \gamma_\mu \frac{1}{2} (\nabla^*_\mu + \nabla_\mu) - \frac{a}{2} \nabla^*_\mu \nabla_\mu + M$$

(1.10)

the allowed Dirichlet boundary condition is the same as (1.3). In this case the zero mode solution is forbidden by choosing a proper signature for the mass term; the mass should be kept positive $M \geq 0$ to eliminate the zero mode [7].

However as was discussed in the previous paper [25, 26] this zero mode problem may become fatal in the overlap Dirac operator [27, 28] and the domain-wall fermion [29, 30, 31]. Both the overlap Dirac operator and the domain-wall fermion is defined through the four dimensional Wilson Dirac operator (1.10) but with an opposite signature for the Wilson fermion mass parameter $M$ (domain-wall height) to the Wilson parameter $r$. An opposite signature is necessary to impose heavy masses on the doublers and a single massless mode to survive. A requirement to the four dimensional Wilson Dirac operator is that $D_W$ should not have a continuous zero mode. If this is not the case the chiral Ward-Takahashi identity is broken dynamically for the domain-wall fermion that the explicit breaking term does not vanish [31]. For the overlap Dirac operator a continuous zero mode may break locality of the Dirac operator [32].

If the Dirichlet boundary condition (1.3) (1.4) is imposed to all fermion fields of the overlap Dirac operator or the domain-wall fermion exponentially small eigenvalues are allowed in the kernel $D_W$ because of an opposite signature of the Wilson parameter and the domain-wall height. Since these small eigenvalues are continuous in spatial momentum they may be a lethal problem in large $T$ limit to break essential properties of the chiral Dirac operator.

One may wonder that the small eigenvalues are boundary effect and should be localized near the temporal boundary. If one considers physics apart from the boundary there should be no harm. However this is not the case for our purpose to define renormalization scheme.
In finite volume scheme the renormalization scale is given by a size of the box, which is
realized by considering a correlation function of operators to be separated by an order of
box size. At least one of operators cannot be away from the boundary. Furthermore it is
convenient for the SF scheme to set one of the operator at the boundary.

In order to solve this problem an orbifolding projection procedure was proposed for
the overlap Dirac operator in Ref. [25]. In this formulation we start from a theory on
\(S^1 \times \mathbb{R}^3\) and impose orbifolding projection \(S^1/Z_2\) on temporal direction. Since we have
set anti-periodic boundary condition in temporal direction \(S^1\) before projection we have
a mass gap proportional to \(1/T\), which is not broken by the orbifolding. Because of this
mass gap we can avoid the zero mode problem of Dirichlet boundary condition.

In this paper the orbifolding formulation of the SF boundary condition is applied to
the domain-wall fermion. In section 2 the domain-wall fermion on \(S^1 \times T^3\) is introduced.
Formulation of domain-wall fermion in finite volume with the SF boundary condition is
discussed in section 3. Application of orbifolding procedure to fermionic part is almost
straightforward as was discussed in Ref. [26]. We can use the same kind of symmetry
argument as in the previous paper [25]. Difficulty is in a treatment of the Pauli-Villars
field. We adopted effective Dirac operator for this purpose. The proper Dirichlet boundary
condition (1.3) (1.4) may not be the unique choice to define a finite volume renormalization
scheme. In section 4 a chirally twisted boundary condition is discussed to define a finite
volume field theory keeping a good property of the SF boundary condition. Section 5 is
devoted for conclusion.

2. Domain-wall fermion action

The purpose of this paper is to introduce the domain-wall fermion system, with which
we can define a finite volume renormalization scheme (Schrödinger functional scheme).
The formulation for the pure Yang-Mills theory has been established in Ref. [6] by using
a transition amplitude between two boundary states (Schrödinger functional). In this
formulation the gauge field (link variable) lives in a finite box \(N_T \times N^3_L\) with a periodic
boundary condition in spatial direction and the SF Dirichlet boundary condition at the
temporal boundary

\[
U_k(\vec{x},0) = W_k(\vec{x}), \quad U_k(\vec{x},N_T) = W'_k(\vec{x}). \tag{2.1}
\]

We shall adopt this procedure for the gauge part and treat the gauge field as an external
field in this paper.

The transition amplitude of the fermion field has been introduced for the Wilson
fermion using the transfer matrix in Ref. [7]. The fermion field resides in the same finite
box for the path integral formalism with periodic or twisted boundary condition [9] in
spatial direction and the SF Dirichlet boundary condition (1.3) and (1.4) in temporal
direction. This fermion system is renormalizable including a shift in the boundary field \(\rho\)

\[\text{\footnotesize \cite{33}}\]
and \( \mathcal{P} \) \[8\]. Another specific property is that this system has a mass gap proportional to the temporal length \( 1/T \) and the finite box serves as an infra-red regulator.

We shall construct the domain-wall fermion system in a finite box keeping the same sort of properties as the Wilson fermion; (i) the theory has a mass gap proportional to \( 1/T \), (ii) there are boundary fields \( \rho \) and \( \mathcal{P} \) in temporal direction and the theory is renormalizable including a shift in these fields. If one naively impose the boundary condition (1.3) and (1.4) to all the fifth dimensional field \( \psi(x,s) \) then the chiral symmetry is broken “dynamically” as explained in the introduction. In order to avoid this problem we adopt an orbifolding procedure, where we start from doubled time length \( 2N_T \) and fermion fields in the finite box of length \( N_T \) with the Dirichlet boundary condition is realized by an orbifolding projection. For this purpose we copy gauge configuration with the SF boundary condition (2.1) into negative region and produce a time reflection symmetric configuration, which satisfies

\[
U_k(\vec{x}, x_0) = U_k(\vec{x}, -x_0), \quad U_0(\vec{x}, x_0) = U_0^T(\vec{x}, -x_0 - 1) \tag{2.2}
\]

as in the previous formulation of overlap Dirac operator \[25\]. The periodic boundary condition is set with length \( 2N_T \)

\[
U_\mu(\vec{x}, x_0 + 2N_T) = U_\mu(\vec{x}, x_0). \tag{2.3}
\]

In this paper we adopt the Shamir’s domain-wall fermion \[30, 31\] on a lattice \( 2N_T \times N^3_L \times N_5 \)

\[
S = \sum_{\vec{x}, \vec{y}, x_0, y_0 = -N_T +1}^{N_T} \sum_{s,t=1}^{N_5} \overline{\psi}(x,s)D_{\text{dwf}}(x,y; s,t)\psi(y,t). \tag{2.4}
\]

\( x_0 \) and \( y_0 \) represent the temporal coordinate which runs \(-N_T +1 \leq x_0 \leq N_T \). \( s \) and \( t \) are used for the fifth dimensional coordinate which runs \( 1 \leq s \leq N_5 \). For later use of orbifolding we set the anti-periodic boundary condition in temporal direction

\[
\psi(\vec{x}, x_0 + 2N_T, s) = -\psi(\vec{x}, x_0, s), \quad \overline{\psi}(\vec{x}, x_0 + 2N_T, s) = -\overline{\psi}(\vec{x}, x_0, s). \tag{2.5}
\]

The Dirac operator is given as a five dimensional Wilson’s one with conventional Wilson parameter \( r = 1 \) and negative mass parameter (domain-wall height) \(-M\) with \( 0 < M < 2 \)

\[
D_{\text{dwf}}(x,y; s,t) = \gamma_M D_M - \frac{1}{2} D^2 - M
\]

\[
= \left( -\frac{1 + \gamma_0}{2} U_0(0)W^+_{x_0,y_0} + \frac{1 - \gamma_0}{2} U_0^T(0)W^-_{x_0,y_0} \right) \delta_{x+y, \delta_{s,t}}
\]

\[
+ \left( -\frac{1 + \gamma_i}{2} U_i(x)\delta_{y, x+1} + \frac{1 - \gamma_i}{2} U_i^T(y)\delta_{y, x-1} \right) \delta_{x, \delta_{s,t}}
\]

\[
+ \left( -\frac{1 + \gamma_5^\pm(m_f)}{2} \Omega^+(m_f)_{s,t} + \frac{1 - \gamma_5}{2} \Omega^-(m_f)_{s,t} \right) \delta_{x,y}
\]

\[
+(5-M)\delta_{x,y} \delta_{s,t}. \tag{2.6}
\]
where $W^\pm$ are hopping operator in temporal direction with anti-periodic boundary condition, whose explicit form for $2N_T = 6$ is written as

$$W_{x_0,y_0}^+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad W^- = (W^+)^\dagger. \quad (2.7)$$

$\Omega^\pm$ are hopping operator in fifth direction with Dirichlet boundary condition (for massless case), whose matrix form for $N_5 = 6$ is given by

$$\Omega^{+,\pm}(m_f)_{s,t} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -m_f & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega^{-}(m_f) = (\Omega^{+,\pm}(m_f))^\dagger. \quad (2.8)$$

Here $m_f$ is a physical quark mass.

The physical quark field is defined by the fifth dimensional boundary field with chiral projection

$$q(x) = (P_L\delta_{s,1} + P_R\delta_{s,N_5}) \psi(x,s), \quad (2.9)$$

$$\bar{q}(x) = \overline{\psi}(x,s) (\delta_{s,N_5}P_L + \delta_{s,1}P_R), \quad (2.10)$$

$$P_{R/L} = \frac{1 \pm \gamma_5}{2}. \quad (2.11)$$

The physical quark mass term is given as an ordinary form $\mathcal{L}_{\text{mass}} = m_f\bar{q}q$ with this quark field.

3. Schrödinger functional with conventional boundary condition

In this section we shall construct the domain-wall fermion system in finite box, in which the conventional SF Dirichlet boundary condition (1.3) and (1.4) is satisfied by the physical quark field. This formulation will be done by making use of an orbifolding in temporal direction.

3.1 Orbifolding construction of SF boundary condition

Since we adopted (anti) periodic boundary condition in temporal direction with period $2N_T$ fields live on $S^1$. The orbifolding $S^1/\mathbb{Z}_2$ is to identify the negative time coordinate with the positive one $x_0 = -x_0$. Identification of fields on $S^1$ is performed according to the symmetry of the theory including the time reflection. A homogeneous Dirichlet boundary condition will appear at fixed points.
The time reversal symmetry of the domain-wall fermion is given by
\[
\psi(\vec{x}, x_0, s) \rightarrow \sum_{x_0, y_0, s, t} \psi(\vec{x}, y_0, t), \quad \overline{\psi}(\vec{x}, x_0, s) \rightarrow \overline{\psi}(\vec{x}, y_0, t) \sum_{y_0, t, s},
\]
(3.1)
\[
\sum_{x_0, y_0, s, t} = i \gamma_5 \gamma_0 R_{x_0, y_0} P_{s, t},
\]
(3.2)
where \( P \) is a parity transformation in fifth direction
\[
P_{s, t} \psi(\vec{x}, x_0, s) = \psi(\vec{x}, x_0, N_5 - s + 1),
\]
whose matrix representation is
\[
P_{s, t} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (N_5 = 6)
\]
(3.3)
and \( R \) is a time reflection operator acting on the temporal direction
\[
R_{x_0, y_0} \psi(\vec{x}, y_0, s) = \psi(\vec{x}, -x_0, s),
\]
whose matrix form is given by
\[
R_{x_0, y_0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad (2N_T = 6)
\]
(3.4)
to satisfy anti-periodicity in \( 2N_T \). We notice that \( R \) has a symmetric fixed point \( x_0 = 0 \) and an anti-symmetric fixed point \( x_0 = N_T \)
\[
R\psi(\vec{x}, 0, s) = \psi(\vec{x}, 0, s), \quad R\psi(\vec{x}, N_T, s) = -\psi(\vec{x}, N_T, s).
\]
(3.5)
The domain-wall fermion Dirac operator is invariant under the time reflection
\[
[\Sigma, D_{\text{dwf}}] = 0
\]
(3.6)
since the reflection invariant gauge configuration (2.2) is adopted.

In order to realize the SF boundary condition at the fixed points we need to combine the chiral transformation with the time reflection [25]. The chiral transformation is given by a vector like rotation of fermion field but with a different charge for two boundaries in fifth direction [31]
\[
\psi(x, s) \rightarrow i Q_{s, t} \psi(x, t), \quad \overline{\psi}(x, s) \rightarrow -\overline{\psi}(x, t)i Q_{t, s},
\]
(3.7)
where \( Q \) is the vector charge matrix which flips sign in the middle of the fifth direction
\[
Q_{s, t} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}, \quad (N_5 = 6).
\]
(3.8)
We consider massless $m_f = 0$ theory in this sub-section.

Here we should notice that this chiral rotation is not an exact symmetry of the domain-wall fermion Dirac operator but we have an explicit breaking term

$$Q D_{dwx} Q - D_{dwx} = 2X,$$

(3.9)

where $X$ is a contribution from the middle layer, which picks up a charge difference there

$$X = \left( P_L \delta_{s, \frac{N_5}{2}} \delta_{t, \frac{N_5}{2} + 1} + P_R \delta_{s, \frac{N_5}{2} + 1} \delta_{t, \frac{N_5}{2}} \right) \delta_{x,y}.$$  

(3.10)

However it was discussed in Ref. [31] that if we consider correlation functions between the bilinear $\bar{\psi}X\psi$ and the physical quark operators contribution is suppressed exponentially in $N_5$ under the condition that the transfer matrix in fifth direction has a gap from unity. Furthermore the domain-wall fermion Dirac operator with explicit time reflection invariance (3.6) does not have index [25], since the contribution to the index [34]

$$\lim_{N_5 \to \infty} a^4 \sum_x \langle \bar{\psi}(x,s) \gamma_5 X_{s,t} \psi(x,t) \rangle = -\lim_{N_5 \to \infty} \text{tr} \left( \gamma_5 X \frac{1}{D_{dwx}} \right)$$

(3.11)

can be shown to vanish by using anti-commutativity $\{\gamma_5 X, \Sigma\} = 0$. We expect that $X$ has no effect on anomaly. We shall ignore this term in the following by constraining that we treat the physical quark Green’s functions only.

Another way to avoid the explicit breaking term is to include it into the Dirac operator. By using an anti-commutative nature $\{Q, X\} = 0$ we can define a chiral symmetric Dirac operator by

$$D_{dwx}^{\text{sym}} = D_{dwx} + X,$$

(3.12)

which commutes with $Q$ exactly even at finite $N_5$. The orbifolding projection in the following can be defined in an exact sense. A compensation of the exact chiral symmetry at finite $N_5$ is a non-locality in the effective Dirac operator, which however is suppressed exponentially in $N_5$. Detailed property of this Dirac operator is deferred in appendix A.

Combining the time reversal transformation (3.1) and the chiral transformation (3.7) we define the orbifolding transformation

$$\psi(\vec{x}, x_0, s) \to A_{x_0,y_0;s,t} \psi(\vec{x}, y_0, t), \quad \bar{\psi}(\vec{x}, x_0, s) \to \bar{\psi}(\vec{x}, y_0, t) A_{y_0,x_0;t,s},$$

(3.13)

$$A_{x_0,y_0;s,t} = \gamma_0 \gamma_5 (PQ)_{s,t} R_{x_0,y_0}.$$  

(3.14)

The domain-wall fermion Dirac operator has time reversal symmetry (3.6) and we assume that the chiral transformation is an exact symmetry of the Dirac operator

$$[Q, D_{dwx}] = 0$$

(3.15)

by ignoring effect of the explicit breaking term $X$ or by adopting the symmetric Dirac operator. The orbifolding transformation becomes symmetry of the Dirac operator

$$[A, D_{dwx}] = 0.$$  

(3.16)
In order to show this we may use a relation \( \{ P, Q \} = 0 \).

The operator \( A \) satisfies a property \( A^2 = 1 \) and can be used to define a projection operator. The orbifolding identification of the fermion field is given by projecting out the following symmetric sub-space

\[
\Pi_\pm \psi(x, s) = 0, \quad (\bar{\psi} \Pi_\pm)(x, s) = 0, \quad \Pi_\pm = \frac{1 \pm A}{2}.
\]  

(3.17)

This projection relates fields in negative region to those in the positive region \( \psi(\vec{x}, -x_0, s) = 0 \), which means fields in the negative is not independent. As will be discussed in appendix B if we consider non-negative region \( 0 \leq x_0 \leq T \), fields in the bulk \( 0 < x_0 < T \) is not constrained. Only the boundary fields obey a projection condition

\[
\begin{align*}
\overline{\Pi}_- \psi(\vec{x}, 0, s) &= 0, \quad \overline{\Pi}_+ \psi(\vec{x}, N_T, s) = 0 \\
(\overline{\psi} \overline{\Pi}_-)(\vec{x}, 0, s) &= 0, \quad (\overline{\psi} \overline{\Pi}_+)(\vec{x}, N_T, s) = 0
\end{align*}
\]

(3.18)  

(3.19)

with projection operator

\[
\Pi_\pm = \frac{1 \pm \Gamma}{2}, \quad \Gamma = \gamma_0 \gamma_5 P Q.
\]  

(3.20)

The orbifolding projection for the physical quark field is given by picking up the boundary components from the projected fermion field

\[
\begin{align*}
(P_L \delta_{s, 1} + P_R \delta_{s, N_T}) (\Pi_-)_{s, t} \psi(x, t) &= \Pi_+ q(x) = 0, \\
(\overline{\psi}(x, t) (\Pi_-)_{t, s} (\delta_{s, N_5} P_L + \delta_{s, 1} P_R) = \overline{\psi}(x) \Pi_- = 0, \\
\Pi_\pm &= \frac{1 \pm \Gamma}{2}, \quad \Gamma = \gamma_0 \gamma_5 P R,
\end{align*}
\]

(3.21)  

(3.22)  

(3.23)

which turns out to be the same condition for the continuum theory in Ref. [25]. The proper homogeneous SF Dirichlet boundary condition is provided at fixed points \( x_0 = 0, N_T \) for the physical quark fields

\[
\begin{align*}
P_+ q(x)|_{x_0 = 0} &= 0, \quad P_- q(x)|_{x_0 = N_T} = 0, \\
\overline{\psi}(x) P_-|_{x_0 = 0} &= 0, \quad \overline{\psi}(x) P_+|_{x_0 = N_T} = 0.
\end{align*}
\]  

(3.24)  

(3.25)

The massless orbifolded action is given by projection

\[
S_{SF} = \sum \frac{1}{2} \overline{\psi} D_{dwl}^{SF} \psi, \quad D_{dwl}^{SF} = \Pi_+ D_{dwl} \Pi_+.
\]  

(3.26)

We notice the massless SF Dirac operator \( D_{dwl}^{SF} \) breaks “chiral symmetry” (3.7) explicitly by the projection \( \Pi_+ \). However the symmetry breaking effect comes from the projection (3.18) (3.19) at the boundary. Ordinary chiral Ward-Takahashi identity [31] is satisfied in the bulk \( 0 < x_0 < N_T \) where fields are not constrained.

Since our orbifolded action is given by projecting onto a symmetric sub-space of the theory and the orbifolding symmetry is not broken by anomaly † the renormalizability is kept trivially ‡.

† According to a similar discussion to that for chiral index (3.11) we can easily show that orbifolding matrix \( A \) does not have index.

‡ Boundary source fields are introduced in later sub-section.
Our original theory on $S^1$ has a gap because of the anti-periodic boundary condition. This gap is kept intact after orbifolding, which can be confirmed at tree level. We have a Hermiticity relation for the SF Dirac operator

$$ (D_{\text{dwf}}^{SF})^\dagger = \gamma_5 P D_{\text{dwf}}^{SF} \gamma_5 P $$

and this Dirac operator connects the same Hilbert sub-space

$$ D_{\text{dwf}}^{SF} : \mathcal{H}_- \rightarrow \mathcal{H}_- , \quad \mathcal{H}_- = \{ \psi | \Pi_- \psi = 0 \} . $$

It is straightforward to solve the eigenvalue problem numerically at tree level. Here we omit the detail but we can easily see that the lowest eigenvalue (a gap) converge to $\pi/2T$ in the continuum limit, which agrees with that of continuum massless theory [7].

As will be discussed in appendix B the bulk part of this projected Dirac operator is exactly the same as that of the ordinary domain-wall fermion. The physical quark fields satisfies the proper boundary condition. Together with the renormalizability and existence of the mass gap this orbifolded system is a strong candidate of QCD with the SF boundary condition to define a finite volume scheme.

We have a comment on mass term. We dropped quark mass term since it breaks the chiral symmetry. However as was discussed in Ref. [25] it is possible to introduce a mass term which is consistent with the orbifolding symmetry (3.13). One of candidates is

$$ S_{\text{mass}} = \sum_x m_f \bar{q}(x) \eta(x_0) q(x), $$

where $\eta$ is an anti-symmetric step function

$$ \eta(-x_0) = -\eta(x_0), \quad \eta(x_0 + 2T) = \eta(x_0), $$

$$ \eta(x_0) = 1 \quad \text{for} \quad 0 < x_0 < N_T . $$

### 3.2 Free propagator

In order to check that the orbifolded domain-wall fermion system describes the QCD with the SF boundary condition properly we consider the physical quark propagator at tree level. The massless fermion propagator is given as an inverse of the projected Dirac operator

$$ G_{\text{dwf}}^{SF}(x, y; s, t) = 2 \left( D_{\text{dwf}}^{SF} \right)^{-1}_{x, y; s, t} = 2 \left( \Pi_+ \frac{1}{D_{\text{dwf}}} \Pi_+ \right)_{x, y; s, t} . $$

where inverse is defined in the sub-space $\mathcal{H}_-$

$$ D_{\text{dwf}}^{SF} \left( D_{\text{dwf}}^{SF} \right)^{-1} = (D_{\text{dwf}})^{-1} D_{\text{dwf}}^{SF} = \Pi_+ . $$

At tree level this propagator can be written in a simple form as

$$ G_{\text{dwf}}^{SF}(x, y; s, t) = \frac{1}{N^3 L} \sum_{\vec{p}} e^{i \vec{p} \cdot (\vec{x} - \vec{y})} G_{\text{dwf}}^{SF}(\vec{p}; x_0, y_0; s, t) , $$

$$ G_{\text{dwf}}^{SF}(\vec{p}; x_0, y_0; s, t) = \frac{1}{2a N_T} \sum_{n=-N_T}^{N_T} \left( \frac{1}{D_{\text{dwf}}(\vec{p})} \right)_{s, t'} \left\{ (e^{ip_0(x_0-y_0)} + e^{ip_0(x_0+y_0)}) (\vec{P}_+)_{t', t} + (e^{ip_0(x_0-y_0)} - e^{ip_0(x_0+y_0)}) (\vec{P}_-)_{t', t'} \right\} , $$

$$ (3.34) $$
where the projection operator $\mathcal{P}_\pm$ is defined in (3.20). The temporal momentum $p_0$ satisfies the quantization condition

$$p_0 = \frac{2n - 1}{2N_T} \pi, \quad -N_T + 1 \leq n \leq N_T$$

(3.35)

for anti-periodicity in $2N_T$. $D_{dwf}(p)$ is the domain-wall fermion Dirac operator in momentum space without orbifolding projection

$$aD_{dwf}(p) = i\gamma_\mu \sin p_\mu + W(p) - P_L \Omega^+ - P_R \Omega^-,$$

(3.36)

$$W = 1 - M + \sum_\mu (1 - \cos a p_\mu).$$

(3.37)

The explicit form of its inverse can be derived according to Ref. [30], which we defer to appendix C.

The physical quark propagator is given by selecting the contribution from the boundary fields in fifth direction

$$G_{\text{quark}}(x,y) = \left( P_L \delta_{s,1} + P_R \delta_{s,N_5} \right) G_{\text{dwf}}(x,y; s,t) \left( \delta_{t,N_5} P_L + \delta_{t,1} P_R \right)$$

(3.38)

$$= 2 \left( \Pi_- G_{\text{quark}} \Pi_+ \right)_{x,y},$$

where

$$G_{\text{quark}}(x,y) = \left( P_L \delta_{s,1} + P_R \delta_{s,N_5} \right) \left( \frac{1}{D_{dwf}} \right)_{x,y; s,t} \left( \delta_{t,N_5} P_L + \delta_{t,1} P_R \right)$$

(3.39)

is the physical quark propagator in $2N_T \times N_3^3$ space-time without any projection. The proper Dirichlet boundary conditions [10]

$$P_+ G_{\text{quark}}(x,y) \big|_{x_0=0} = 0, \quad P_- G_{\text{quark}}(x,y) \big|_{x_0=N_T} = 0,$$

(3.40)

$$G_{\text{quark}}(x,y) \big|_{y_0=0} P_+ = 0, \quad G_{\text{quark}}(x,y) \big|_{y_0=N_T} P_- = 0$$

(3.41)

are satisfied for this quark propagator because of the projection $\Pi_{\pm}$. By ignoring subleading terms in $e^{-N_5}$ the propagator takes the following form at tree level

$$\alpha^3 \sum_{\vec{x}} e^{-i\vec{p}(\vec{x}-\vec{y})} G_{\text{quark}}^{\text{SF}}(x,y) = \frac{1}{2aN_T} \sum_{n=-N_T+1}^{N_T} \left( \frac{i\gamma_\mu \sin p_\mu}{1 - e^{\alpha W(p)}} \right) e^{ip_0 x_0}$$

$$\times \left\{ (e^{-ip_0 y_0} + e^{ip_0 y_0}) P_+ + (e^{-ip_0 y_0} - e^{ip_0 y_0}) P_- \right\},$$

(3.42)

which can be shown to approach to the continuum SF propagator of Ref. [10] without any $O(a)$ term. This system has no extra zero mode we encountered in naive formulation and we conclude that this is equivalent to the QCD with SF boundary condition.

### 3.3 Surface term

In the orbifolding construction of the SF formalism only the homogeneous boundary condition (3.24) (3.25) can be introduced. However in general SF formalism the Dirichlet boundary condition is inhomogeneous as (1.3) and (1.4). The boundary values $\rho, \cdots, \bar{\rho}$
are regarded as external source fields coupled to the dynamical fields and the correlation functions involving the boundary fields

$$\zeta(\vec{x}) = \frac{\delta}{\delta \rho(\vec{x})}, \quad \zeta^\prime(\vec{x}) = -\frac{\delta}{\delta \rho'(\vec{x})},$$  \hspace{1cm} (3.43)

$$\zeta(\vec{x}) = \delta \rho(\vec{x}), \quad \zeta^\prime(\vec{x}) = -\delta \rho'(\vec{x}) \hspace{1cm} (3.44)$$

are used conveniently to extract the renormalization factors.

Coupling between the boundary source fields and the dynamical fields are not introduced automatically in our formulation since the boundary value vanishes by projection. In other words our formulation is protected by an orbifolding symmetry. The action (3.26) is invariant under a transformation

$$\delta \left( \Pi_+ \psi \right) (x, s) = \alpha \left( \Pi_+ \psi \right) (x, s), \hspace{1cm} (3.45)$$

$$\delta \left( \psi \Pi_+ \right) (x, s) = -\alpha \left( \psi \Pi_+ \right) (x, s), \hspace{1cm} (3.46)$$

where remaining degrees of freedom $\Pi_- \psi$ and $\psi \Pi_-$ are kept intact. The boundary source fields are elements of $\Pi_- \psi$ and $\psi \Pi_-$. Since the surface term is given to connect the boundary source fields and the dynamical fields $\Pi_+ \psi$, $\psi \Pi_+$ it is not consistent with the orbifolding symmetry.

In this paper we define a surface term as an orbifolding symmetry breaking term, which is consistent with other symmetries of the orbifolded domain-wall fermion. The symmetries are; parity

$$\psi(x, s) \rightarrow \gamma_0 P_{st} \psi(-\vec{x}, x_0, t), \quad \bar{\psi}(x, s) \rightarrow \bar{\psi}(-\vec{x}, x_0, t) \gamma_0 P_{ts},$$  \hspace{1cm} (3.47)

charge conjugation

$$\psi(x, s) \rightarrow CP_{st} \psi^T(x, t), \quad \bar{\psi}(x, s) \rightarrow \bar{\psi}^T(x, t) P_{ts} (-C^{-1}), \quad C = \gamma_2 \gamma_0 \hspace{1cm} (3.48)$$

and chiral symmetry (3.7) in the bulk $0 < x_0 < N_T$, where chiral Ward-Takahashi identity of Ref. [31] is satisfied.

Using the orbifolding projection (3.17) it is easy to show that the orbifolding transformation (3.45) (3.46) is a “chiral” transformation only at the boundary

$$\delta \left( \mathcal{P}_+ \psi \right) (\vec{x}, 0, s) = \alpha \left( \mathcal{P}_+ \psi \right) (\vec{x}, 0, s), \hspace{1cm} (3.49)$$

$$\delta \left( \mathcal{P}_- \psi \right) (\vec{x}, 0, s) = -\alpha \left( \mathcal{P}_- \psi \right) (\vec{x}, 0, s), \hspace{1cm} (3.50)$$

$$\delta \left( \mathcal{P}_- \psi \right) (\vec{x}, N_T, s) = \alpha \left( \mathcal{P}_- \psi \right) (\vec{x}, N_T, s), \hspace{1cm} (3.51)$$

$$\delta \left( \mathcal{P}_- \psi \right) (\vec{x}, N_T, s) = -\alpha \left( \mathcal{P}_- \psi \right) (\vec{x}, N_T, s) \hspace{1cm} (3.52)$$

and is a vector $U(1)$ transformation in the bulk $0 < x_0 < N_T$

$$\delta \psi(\vec{x}, x_0, s) = \alpha \psi(\vec{x}, x_0, s), \quad \delta \bar{\psi}(\vec{x}, x_0, s) = -\alpha \bar{\psi}(\vec{x}, x_0, s).$$  \hspace{1cm} (3.53)
boundary can be broken by a physical quark mass term \( m \bar{q} q \) keeping the bulk vector symmetry. But this is forbidden by the chiral Ward-Takahashi identity in the bulk. The symmetry should be broken only at the boundary.

We introduce boundary source fields as a component of projected out degrees of freedom in (3.18) and (3.19)

\[
\begin{align*}
\lambda(\vec{x}, s) &= \mathcal{T}_- \psi(\vec{x}, 0, s), \\
\lambda(\vec{x}, s) &= \mathcal{T}_+ \psi(\vec{x}, N_T, s), \\
\bar{\lambda}(\vec{x}, s) &= (\bar{\psi} \mathcal{T}_-)(\vec{x}, 0, s), \\
\bar{\lambda}(\vec{x}, s) &= (\bar{\psi} \mathcal{T}_+)(\vec{x}, N_T, s)
\end{align*}
\]

(3.54)

(3.55)

The orbifolding symmetry breaking term takes the form

\[
S_{\text{breaking}} = \bar{\lambda}(\vec{x}, s) \hat{O}_{st} \mathcal{T}_+ \psi(\vec{x}, 0, t) + (\bar{\psi} \mathcal{T}_+)(\vec{x}, 0, s) \hat{O}_{st} \lambda(\vec{x}, t)
\]

\[
+ \bar{\lambda}(\vec{x}, s) \hat{O}_{st} \mathcal{T}_- \psi(\vec{x}, N_T, t) + (\bar{\psi} \mathcal{T}_-)(\vec{x}, N_T, s) \hat{O}_{st} \lambda(\vec{x}, t),
\]

(3.56)

where \( \hat{O} \) is a local operator which anti-commute with \( \mathcal{T} = \gamma_0 \gamma_5 PQ \). Candidates of \( \hat{O} \) are \( \gamma_0, \gamma_5, Q, P \) and

\[
K(u)_{st} = (P_L \delta_{s,N_5+1-u} + P_R \delta_{s,u})(P_L \delta_{t,u} + P_R \delta_{t,N_5+1-u}),
\]

(3.57)

\[
\bar{K}(u)_{st} = (P_R \delta_{s,N_5+1-u} + P_L \delta_{s,u})(P_R \delta_{t,u} + P_L \delta_{t,N_5+1-u}),
\]

(3.58)

where \( K(1) \) produces a physical quark mass term \( \bar{q} K(1) \psi = \bar{q} q \).

Among these candidates \( \gamma_5 \) and \( Q \) are forbidden by the parity symmetry. \( \gamma_0 \) is not consistent with the charge conjugation. \( P, K(u) \) and \( \bar{K}(u) \) break chiral symmetry, which however is broken at the boundary. Since \( P, K(u) \) and \( \bar{K}(u) \) are consistent with parity and charge conjugation they are proper candidates of orbifolding symmetry breaking term. However we adopt only \( K(1) \) in this paper to reproduce the same surface term as the continuum theory. We define the surface term as

\[
S_{\text{surface}} = -a^3 \sum_{\vec{x}} \left( \bar{\lambda}(\vec{x}, s) K(1)_{st} \mathcal{T}_+ \psi(\vec{x}, 0, t) + (\bar{\psi} \mathcal{T}_+)(\vec{x}, 0, s) K(1)_{st} \lambda(\vec{x}, t) \right)
\]

\[
+ \bar{\lambda}(\vec{x}, s) K(1)_{st} \mathcal{T}_- \psi(\vec{x}, N_T, t) + (\bar{\psi} \mathcal{T}_-)(\vec{x}, N_T, s) K(1)_{st} \lambda(\vec{x}, t) \right)
\]

\[
= a^3 \sum_{\vec{x}} \left( -\bar{\rho}(\vec{x}) P_- q(x)|_{x_0=0} - \bar{q}(x) P_+ \rho(\vec{x})|_{x_0=0} \right)
\]

\[
- \bar{\rho}'(\vec{x}) P_+ q(x)|_{x_0=N_T} - \bar{q}'(x) P_- \rho'(\vec{x})|_{x_0=N_T},
\]

(3.59)

where \( q \) and \( \bar{q} \) are active dynamical fields at the temporal boundary. \( \rho \) and \( \bar{\rho} \) are boundary source fields for the physical quark fields

\[
\begin{align*}
P_+ q(x)|_{x_0=0} &= \rho(\vec{x}), & P_- q(x)|_{x_0=N_T} &= \rho'(\vec{x}),
\end{align*}
\]

(3.60)

\[
\begin{align*}
\bar{q}(x) P_- |_{x_0=0} &= \bar{\rho}(\vec{x}), & \bar{q}(x) P_+ |_{x_0=N_T} &= \bar{\rho}'(\vec{x}).
\end{align*}
\]

(3.61)

Since this surface term is not a general orbifolding symmetry breaking term, general anticipation is that we will need all sort of breaking terms to renormalize quantum corrections. However if we consider Green functions constructed with physical quark operators
only we may expect that quantum corrections which appear in these Green functions can be renormalized into a shift of physical operators and physical quark source fields $\rho$, $\overline{\rho}$, $\rho'$ and $\overline{\rho}'$. The situation is similar to the physical quark mass term $m_f \overline{q} q$. This term is not a general chiral symmetry breaking term. However if we consider a Green function of the physical quark fields only, quantum corrections which appear with chiral symmetry breaking can be renormalized into the physical operators and mass term. We did not need all the breaking terms for renormalization if we consider the physical quark Green function. Explicit calculation is necessary to confirm this expectation for source fields.

We check validity of this surface term at tree level. According to Ref. [10] we introduce the generating functional

$$Z_F [\overline{\psi}', \rho'; \overline{\psi}, \rho, \overline{\eta}, \eta; U] = \int D\psi D\overline{\psi} \exp \left\{ -S_F [U, \overline{\psi}, \psi; \overline{\psi}', \rho', \rho, \overline{\eta}, \eta] + a^4 \sum_{x, s} (\overline{\psi}(x, s) \eta(x, s) + \overline{\eta}(x, s) \psi(x, s)) \right\},$$

(3.62)

where $\eta(x)$ and $\overline{\eta}(x)$ are source fields for the fermion fields and the total action $S_F$ is given as a sum of the bulk action (3.26) and the surface term (3.59). We notice that the fermion fields $\psi$ and $\overline{\psi}$ obey the orbifolding condition (3.17). The correlation functions between the boundary fields are derived with the same procedure as Ref. [10].

$$\langle \psi(x, s) \overline{\psi}(y, t) \rangle = G^\text{SF}_{\text{dwf}}(x, y; s, t),$$

(3.63)

$$\langle q(x) \overline{q}(y) \rangle = G^\text{SF}_{\text{quark}}(x, y),$$

(3.64)

$$\langle q(x) \overline{\zeta}(\overline{y}) \rangle = G^\text{SF}_{\text{quark}}(x, y) P_+ |_{y_0=0},$$

(3.65)

$$\langle q(x) \overline{\zeta}'(\overline{y}) \rangle = G^\text{SF}_{\text{quark}}(x, y) P_- |_{y_0=N_T},$$

(3.66)

$$\langle \zeta(\overline{x}) \overline{q}(y) \rangle = P_- G^\text{SF}_{\text{quark}}(x, y) |_{x_0=0},$$

(3.67)

$$\langle \zeta'(\overline{x}) \overline{q}(y) \rangle = P_+ G^\text{SF}_{\text{quark}}(x, y) |_{x_0=N_T},$$

(3.68)

$$\langle \zeta(\overline{x}) \overline{\zeta}(\overline{y}) \rangle = P_- G^\text{SF}_{\text{quark}}(x, y) P_+ |_{x_0=0, y_0=0},$$

(3.69)

$$\langle \zeta'(\overline{x}) \overline{\zeta}(\overline{y}) \rangle = P_+ G^\text{SF}_{\text{quark}}(x, y) P_- |_{x_0=0, y_0=N_T},$$

(3.70)

$$\langle \zeta'(\overline{x}) \overline{\zeta}'(\overline{y}) \rangle = P_+ G^\text{SF}_{\text{quark}}(x, y) P_- |_{x_0=N_T, y_0=0},$$

(3.71)

$$\langle \zeta'(\overline{x}) \overline{\zeta}'(\overline{y}) \rangle = P_+ G^\text{SF}_{\text{quark}}(x, y) P_- |_{x_0=N_T, y_0=N_T}. $$

(3.72)

The propagator $G^\text{SF}_{\text{dwf}}$ and $G^\text{SF}_{\text{quark}}$ are given in (3.31) and (3.38). We notice that the above propagators between the boundary fields and physical quark fields approach to the continuum SF boundary propagator without any $O(a)$ term at tree level.

### 3.4 Effective action of the domain-wall fermion

In order to perform numerical simulation with dynamical fermion we need to introduce the Pauli-Villars field to cancel bulk contribution in fifth direction. The Pauli-Villars field is a
four component complex scalar and its action is given by

\[ S_{PV} = \sum_{\vec{x}, \vec{y}} \sum_{x_0, y_0 = -N_T + 1}^{N_T} \sum_{x, y = 1}^{N_5} \bar{\phi}(x, s) D_{PV}(x, y; s, t) \phi(y, t), \] (3.73)

where Dirac operator for the Pauli-Villars field is given in the same form as the domain-wall fermion Dirac operator (2.6) with \( m_f = 1 \)

\[ D_{PV} = D_{dwt}(m_f = 1). \] (3.74)

This Dirac operator does not commute with the orbifolding operator \( A = \gamma_0 \gamma_5 PQR \) because of the mass term. It is not straightforward to introduce the Pauli-Villars field by orbifolding. In this paper we propose to implement it by the effective Dirac operator \([34, 35]\).

The effective Dirac operator appears in an effective action of the physical quark field (2.9) (2.10) and “physical” Pauli-Villars field

\[ Q(x) = (P_L \delta_{s,1} + P_R \delta_{s,N_5}) \phi(x, s), \] (3.75)

\[ \overline{Q}(x) = \bar{\phi}(x, s) \left( \delta_{s,N_5} P_L + \delta_{s,1} P_R \right). \] (3.76)

The effective action is given by integrating out all the bulk fields other than physical fields at the fifth dimensional boundary \([34]\)

\[ S_{\text{eff}} = \bar{q}(x) (D_{\text{eff}})_{xy} q(y) + \overline{Q}(x) (D_{\text{eff}} + 1)_{xy} Q(y). \] (3.77)

In its derivation the effective Dirac operator \( D_{\text{eff}} \) is given as an inverse of the full physical quark propagator

\[ D_{\text{eff}} = \frac{1}{\langle q \bar{q} \rangle}, \] (3.78)

whose explicit form is

\[ D_{\text{eff}} = \frac{1 + \gamma_5 S}{1 - \gamma_5 S}, \quad S = \frac{1 - T N_5}{1 + T N_5}, \quad T = \frac{1 - H'}{1 + H'}, \quad H' = \gamma_5 D_W \frac{1}{2 + D_W}. \] (3.79)

Here \( D_W \) is a four dimensional Wilson Dirac operator with negative mass \(-2 < -M < 0\). In \( N_5 \to \infty \) limit the Dirac operator becomes

\[ D_{\text{eff}} = \frac{1 + \gamma_5 \epsilon \left( \tilde{H} \right)}{1 - \gamma_5 \epsilon \left( \tilde{H} \right)}, \quad T = e^{-\tilde{H}}, \] (3.80)

where \( \epsilon(x) \) is a sign function

\[ \epsilon(x) = \frac{x}{\sqrt{x^2}}. \] (3.81)

We can easily check that this Dirac operator is exactly chiral symmetric

\[ \{ \gamma_5, D_{\text{eff}} \} = 0 \] (3.82)
in $N_5 \to \infty$ limit and should be non-local to satisfy the Nielsen-Ninomiya’s no-go theorem.

The effective Dirac operator is related to the original domain-wall fermion and the Pauli-Villars Dirac operator through determinant

$$\det \frac{1}{D_{PV}} D_{dwf} = \det \frac{D_{eff}}{D_{eff} + 1} = \det D_{N_5}, \quad (3.83)$$

where $D_{N_5}$ is a truncated overlap Dirac operator [34, 35]. Hereafter we take $N_5 \to \infty$ limit implicitly and write $D_{N_5 \to \infty} = D_{OD}$. In terms of the domain-wall fermion the overlap Dirac operator is defined as

$$D_{OD} = \frac{D_{eff}}{D_{eff} + 1}. \quad (3.84)$$

$D_{OD}$ satisfies the Ginsparg-Wilson relation [36]

$$\{ \gamma_5, D_{OD} \} = 2D_{OD} \gamma_5 D_{OD}. \quad (3.85)$$

If we introduce physical quark mass term we have a massive overlap Dirac operator through determinant

$$D_{OD}(m_f) = \frac{D_{eff} + m_f}{D_{eff} + 1} = D_{OD} + m_f (1 - D_{OD}). \quad (3.86)$$

The effective Dirac operator of the orbifolded domain-wall fermion system is defined in a similar way. Since the four dimensional Wilson Dirac operator $D_W$ commute with the four dimensional time reflection operator $\Sigma = i \gamma_5 \gamma_0 R$ we have following anti-commutation relations

$$\{ \Sigma, H' \} = 0, \quad \{ \Sigma, \bar{H} \} = 0. \quad (3.87)$$

By using these relations we can easily show that the effective Dirac operator (3.80) anti-commute with the four dimensional orbifolding operator $\Gamma$ defined in (3.23)

$$\{ \Gamma, D_{eff} \} = 0. \quad (3.88)$$

The massless overlap Dirac operator defined in the above satisfy “Ginsparg-Wilson relation” for the orbifolding transformation [25]

$$\{ \Gamma, D_{OD} \} = 2D_{OD} \Gamma D_{OD}. \quad (3.89)$$

We define the Schrödinger functional effective Dirac operator as an inverse of the orbifolded full quark propagator (3.38)

$$D_{SF}^{\text{eff}} = \Pi_+ \frac{1}{\langle q\bar{q} \rangle} \Pi_- = \Pi_+ D_{eff} \Pi_-, \quad (3.90)$$

where inverse means that in a sub-space

$$D_{eff}^S G_{\text{quark}}^{SF} = 2\Pi_+, \quad G_{\text{quark}}^{SF} D_{eff} = 2\Pi_. \quad (3.91)$$
Contribution from the Pauli-Villars field is introduced to reproduce the Schrödinger functional overlap Dirac operator defined in Ref. [25] §

\[
D_{\text{OD}}^{\text{SF}} = \Pi_+ D_{\text{eff}} \Pi_- \frac{1}{D_{\text{eff}} + 1} = \Pi_+ D_{\text{OD}} \hat{\Pi}_-, \tag{3.92}
\]

where

\[
\hat{\Pi}_\pm = \frac{1 \pm \hat{\Gamma}}{2}, \quad \hat{\Gamma} = \Gamma (1 - 2D_{\text{OD}}). \tag{3.93}
\]

This is not a unique definition of the SF overlap Dirac operator but we can define another Dirac operator as

\[
\overline{D}_{\text{OD}}^{\text{SF}} = \frac{1}{D_{\text{eff}} + 1} \Pi_+ D_{\text{eff}} \Pi_. \tag{3.94}
\]

These two Dirac operators are related by unitary operators

\[
u = \frac{1 + \Sigma}{2} (1 - 2D_{\text{OD}}) + \frac{1 - \Sigma}{2}, \quad u' = \gamma_5 u \gamma_5 \tag{3.95}
\]

by

\[
u D_{\text{OD}}^{\text{SF}} u^\dagger = \overline{D}_{\text{OD}}^{\text{SF}}, \quad u'^\dagger D_{\text{OD}}^{\text{SF}} u' = \overline{D}_{\text{OD}}^{\text{SF}}. \tag{3.96}
\]

Here we used a fact that the effective and the overlap Dirac operators commute with the four dimensional time reflection operator \(\Sigma\).

As was discussed in Ref. [25] the SF overlap Dirac operator does not have \(\gamma_5\) Hermiticity relation. Instead we have

\[(D_{\text{OD}}^{\text{SF}})^\dagger = \gamma_5 \overline{D}_{\text{OD}}^{\text{SF}} \gamma_5. \tag{3.97}\]

In order to define real fermion determinant we may need even numbers of flavours and different Dirac operators for each flavours. An example for two flavours case is

\[
D_{\text{SF}}^{(2)} = \begin{pmatrix} \overline{D}_{\text{OD}}^{\text{SF}} & \overline{D}_{\text{OD}}^{\text{SF}} \\ \overline{D}_{\text{OD}}^{\text{SF}} & \overline{D}_{\text{OD}}^{\text{SF}} \end{pmatrix}. \tag{3.98}\]

We notice that \(U(2)\) vector flavour symmetry is broken to \(U(1) \times U(1)\). Determinant of this Dirac operator is

\[
\det D_{\text{SF}}^{(2)} = \det D_{\text{SF}} \gamma_5 = \det \begin{pmatrix} \Pi_+ D_{\text{eff}} & \frac{1}{D_{\text{eff}} + 1} \\ \Pi_+ D_{\text{eff}} & \frac{1}{D_{\text{eff}} + 1} \end{pmatrix}, \tag{3.99}\]

which is re-written in terms of pseudo-fermion field \(\chi\)

\[
\det D_{\text{SF}}^{(2)} = \int \mathcal{D} \left( \Pi_+ \chi \right) \mathcal{D} \left( \Pi_+ \chi \right) \exp \left( - \chi^\dagger \Pi_+ \left( \frac{1}{D_{\text{eff}} + 1} \right) \left( \frac{1}{D_{\text{eff}} + 1} \right) \Pi_+ \chi \right). \tag{3.100}\]

\[\text{§The Ginsparg-Wilson relation (3.89) in this paper is different from that in Ref. [25] by factor two and so is the definition of } \hat{\Gamma}.\]
The determinant is defined in a sub-space $\mathcal{H}_- = \{ \psi | \Pi_- \psi = 0 \}$ of eigenfunctions. In evaluation of the fermion force we need to calculate

$$\left( \frac{1}{D_{\text{eff}}} + 1 \right)^{-1} = (\langle q\overline{q} \rangle + 1)^{-1},$$

which corresponds to inverse of the overlap Dirac operator.

The orbifolded effective Dirac operator is modified as follows when we introduce the mass term (3.29)

$$D_{\text{eff}}^{\text{SF}}(m_f) = \frac{1}{2} \Pi_+ (D_{\text{eff}} + m_f \eta) \Pi_-.$$  

(3.102)

Taking into account a contribution from the Pauli-Villars Dirac operator the massive SF overlap Dirac operator is defined as

$$D_{\text{OD}}^{\text{SF}}(m_f) = \frac{1}{2} \Pi_+ (D_{\text{eff}} + m_f \eta) \Pi_+ \frac{1}{D_{\text{eff}} + 1} = \frac{1}{2} \Pi_+ (D_{\text{OD}} + m_f \eta (1 - D_{\text{OD}})) \hat{\Pi}_-,$$

(3.103)

$$\overline{D}_{\text{OD}}^{\text{SF}}(m_f) = \frac{1}{2} D_{\text{eff}} + 1 \Pi_+ (D_{\text{eff}} + m_f \eta) \Pi_-.$$  

(3.104)

Although we do not have a unitary transformation to relate $D_{\text{OD}}^{\text{SF}}(m_f)$ and $\overline{D}_{\text{OD}}^{\text{SF}}(m_f)$ we have a Hermiticity relation

$$\left( D_{\text{OD}}^{\text{SF}}(m_f) \right)^\dagger = \gamma_5 \overline{D}_{\text{OD}}^{\text{SF}}(m_f) \gamma_5.$$  

(3.105)

We also need even numbers of flavours to define a real fermion determinant.

4. Schrödinger functional with twisted boundary condition

In the previous section we presented an orbifolding formulation of domain-wall fermion in finite box, in which the homogeneous proper boundary condition (3.24) (3.25) is satisfied. This is a solution of our purpose to define a finite volume renormalization scheme. However this may not be the unique solution of our requirement that the theory has a mass gap and is kept to be renormalizable in a finite box. In this section we propose another orbifolding formulation to adopt chirally twisted boundary condition [25, 37, 38]. As was discussed in Ref. [25] the chirally twisted boundary condition has advantages that the fermion determinant becomes real and the mass term is introduced easier. For domain-wall fermion the Pauli-Villars field can be treated in a straightforward way by orbifolding.

4.1 Orbifolding construction of chirally twisted boundary condition

As will be discussed later the fermion determinant becomes real for even numbers of flavours. In this section we adopt two flavours case for instance. We start from the massless orbifolded action (3.26) and introduce the twisted orbifolding by chirally rotating the fermion field

$$\psi = e^{i \frac{2}{3} Q^a \tau^3} \psi', \quad \overline{\psi} = \overline{\psi'} e^{-i \frac{2}{3} Q^a \tau^3},$$

(4.1)
where $\tau^3$ is the Pauli matrix to act on flavour space and $Q$ is the vector charge (3.8) for chiral transformation. In terms of the rotated field the orbifolded action is given by

$$ S_{SF} = \sum \frac{1}{2} \bar{\psi}' \tilde{D}_{dwf}^S \psi', \quad \tilde{D}_{dwf}^S = \bar{\Pi}_- D_{dwf} \Pi_-, $$

(4.2)

where

$$ \bar{\Pi}_\pm = \frac{1 \pm \Sigma \tau^3}{2} $$

(4.3)

is a twisted orbifolding projection with time reflection operator $\Sigma$ defined in (3.1).

As was discussed in sub-section 3.1 the Dirac operator has no index and the chiral transformation is not anomalous even for Abelian case. This formulation with twisted orbifolding projection is equivalent to the original one for massless theory. Here we notice that the twisted orbifolding operator $\Sigma \tau^3$ commute with the massive domain-wall fermion Dirac operator

$$ \left[ \Sigma \tau^3, D_{dwf}(m_f) \right] = 0 $$

(4.4)

since we adopted time reflection invariant gauge configuration. We can extend this twisted formulation to massive theory

$$ S_{\text{twist}}_{dwf} = \sum \frac{1}{2} \bar{\psi}^T \tilde{D}_{dwf}^S (m_f) \psi, \quad \tilde{D}_{dwf}^S = \bar{\Pi}_- D_{dwf} (m_f) \Pi_- $$

(4.5)

It is straightforward to introduce the Pauli-Villars field through orbifolding

$$ S_{\text{twist}}_{PV} = \sum \frac{1}{2} \bar{\phi} \tilde{D}_{PV}^S \phi, \quad \tilde{D}_{PV}^S = \bar{\Pi}_- D_{PV} \Pi_- $$

(4.6)

since $D_{PV} = D_{dwf}(m_f = 1)$ and is commutable with the orbifolding operator.

The fermion fields satisfy the twisted orbifolding projection condition

$$ \bar{\Pi}_+ \psi = 0, \quad \bar{\psi} \Pi_+ = 0, $$

(4.7)

which brings the following boundary conditions

$$ \bar{\Pi}_+ \psi(\vec{x}, 0, s) = 0, \quad \bar{\Pi}_- \psi(\vec{x}, N_T, s) = 0 $$

$$ \left( \bar{\psi} \bar{P}_+ \right) (\vec{x}, 0, s) = 0, \quad \left( \bar{\psi} \bar{P}_- \right) (\vec{x}, N_T, s) = 0 $$

(4.8)

(4.9)

with projection operator

$$ \bar{\Pi}_\pm = \frac{1 \pm i \gamma_5 \gamma_0 \tau^3}{2}. $$

(4.10)

In terms of the physical quark field the projection condition becomes

$$ \left( P_L \delta_{s,1} + P_R \delta_{s, N_5} \right) \left( \bar{\Pi}_+ \right)_{s,t} \psi(x, t) = \bar{\Pi}_+ q(x) = 0, $$

(4.11)

$$ \left( \bar{\psi} \bar{P}_+ \right)_{t,s} \left( \delta_{s, N_5} P_L + \delta_{s,1} P_R \right) \eta(x) \bar{\Pi}_+ = 0, $$

(4.12)

$$ \bar{\Pi}_\pm = \frac{1 \pm \Sigma \tau^3}{2}, \quad \Sigma = i \gamma_5 \gamma_0 R, $$

(4.13)
where $\Sigma$ is the time reflection operator in four dimensions. The boundary condition for the physical quark field is

\[ \overline{P}_- q(x)|_{x_0=0} = 0, \quad \overline{P}_+ q(x)|_{x_0=N_T} = 0, \quad (4.14) \]
\[ \overline{q}(x) \overline{P}_- |_{x_0=0} = 0, \quad \overline{q}(x) \overline{P}_+ |_{x_0=N_T} = 0, \quad (4.15) \]
\[ \overline{P}_\pm = \frac{1 \pm i\gamma_5\gamma_0\tau_3}{2}. \quad (4.16) \]

We have two comments. The orbifolded Dirac operator with twisted projection has a following Hermiticity relation

\[ \overline{D}_{\text{SF}}(m)^\dagger = \gamma_5\tau_1 \overline{D}_{\text{SF}}(m) \gamma_5\tau_1, \quad (4.17) \]

which is also the same for the orbifolded Pauli-Villars Dirac operator. The $SU(2)$ flavour symmetry is broken to $U(1)_V \times U(1)_3$ as in the chirally twisted mass QCD.

### 4.2 Free propagator

The original theory before orbifolding has a mass gap proportional to $1/T$ because of anti-periodicity in temporal direction. This property is robust against orbifolding process and survive in the twisted orbifolding formulation. We will check this property at tree level by using propagator.

The fermion propagator is defined as an inverse of the orbifolded Dirac operator in a sub-space

\[ \overline{G}_{\text{dwf}}^{\text{SF}}(x, y; s, t) = 2 \left( \overline{D}_{\text{dwf}}^{\text{SF}} \right)^{-1} x, y; s, t = 2 \left( \overline{\Pi} - \frac{1}{\overline{D}_{\text{dwf}}^{\text{SF}}} \overline{\Pi} \right)_{x, y; s, t}, \quad (4.18) \]
\[ \overline{D}_{\text{dwf}}^{\text{SF}} \left( \overline{D}_{\text{dwf}}^{\text{SF}} \right)^{-1} = \left( \overline{D}_{\text{dwf}}^{\text{SF}} \right)^{-1} \overline{D}_{\text{dwf}}^{\text{SF}} = \overline{\Pi}. \quad (4.19) \]

At tree level this propagator can be written in a simple form as

\[ \overline{G}_{\text{dwf}}^{\text{SF}}(x, y; s, t) = \frac{1}{N_L^3} \sum_{\vec{p}} e^{i\vec{p}(x-y)} G_{\text{dwf}}^{\text{SF}}(\vec{p}, x_0, y_0; s, t), \quad (4.20) \]
\[ \overline{G}_{\text{dwf}}^{\text{SF}}(\vec{p}, x_0, y_0; s, t) = \frac{1}{2aN_T} \sum_{n=-N_T+1}^{N_T} (D_{\text{dwf}}(p))_{s, t} \left\{ e^{ip_0(x_0-y_0)} - e^{ip_0(x_0+y_0)} \right\} (\overline{P}_+{t'}) \left\{ e^{ip_0(x_0-y_0)} + e^{ip_0(x_0+y_0)} \right\} (\overline{P}_-{t'}), \quad (4.21) \]

$D_{\text{dwf}}(p)$ is the domain-wall fermion Dirac operator in momentum space without orbifolding projection, whose inverse is given in appendix C. We notice that the temporal momentum $p_0$ satisfies the quantization condition (3.35) and there is no extra fermion zero mode.

The physical quark propagator is given by selecting the contribution from the boundary fields in fifth direction

\[ \overline{G}_{\text{quark}}^{\text{SF}}(x, y) = (P_L \delta_{s,1} + P_R \delta_{s,N_5}) \overline{G}_{\text{dwf}}^{\text{SF}}(x, y; s, t) (\delta_{t,N_5} P_L + \delta_{t,1} P_R) = 2 \left( \overline{\Pi} G_{\text{quark}} \overline{\Pi} \right)_{x, y}, \quad (4.22) \]
where \( G_{\text{quark}}(x, y) \) is the physical quark propagator in \( 2N_T \times N^3_L \) space-time without any projection. Following Dirichlet boundary conditions

\[
\begin{align*}
\tilde{P}_+ G_{\text{quark}}^\text{SF}(x, y)|_{x_0=0} &= 0, & \tilde{P}_- G_{\text{quark}}^\text{SF}(x, y)|_{x_0=N_T} &= 0, \\
G_{\text{quark}}^\text{SF}(x, y)|_{y_0=0} \tilde{P}_+ &= 0, & G_{\text{quark}}^\text{SF}(x, y)|_{y_0=N_T} \tilde{P}_- &= 0
\end{align*}
\]  

are satisfied for this quark propagator. By ignoring sub-leading terms in \( e^{-N_5} \) the propagator takes the following form at tree level

\[
a^3 \sum_{\vec{x}} e^{-i\vec{p}(\vec{x}-\vec{y})} G_{\text{quark}}^\text{SF}(x, y) = \frac{1}{2aN_T} \sum_{n=-N_T+1}^{N_T} \left( \frac{i\gamma_\mu \sin p_\mu}{1-e^{iW(p)}} \right) e^{ip_0x_0} \\
\times \left\{ (e^{-ip_0y_0} - e^{ip_0y_0}) \tilde{P}_+ + (e^{-ip_0y_0} + e^{ip_0y_0}) \tilde{P}_- \right\}
\]  

We emphasize that the physical quark has a gap proportional to \( 1/T \) because of the anti-periodicity (3.35). This formulation satisfies one of the requirement.

### 4.3 Surface term

In this subsection we consider a twisted orbifolding symmetry and introduce a coupling to the boundary source field (surface term) as a symmetry breaking term. The orbifolded action (4.5) is invariant under the following twisted orbifolding transformation

\[
\delta \left( \bar{\Pi}_- \psi \right) (x, s) = \alpha \left( \bar{\Pi}_- \psi \right) (x, s), \quad \delta \left( \bar{\psi} \Pi_- \right) (x, s) = -\alpha \left( \bar{\psi} \Pi_- \right) (x, s), \tag{4.26}
\]

where remaining degrees of freedom \( \bar{\Pi}_+ \psi \) and \( \bar{\psi} \Pi_+ \) are kept intact. The boundary source fields are elements of \( \bar{\Pi}_+ \psi \) and \( \bar{\psi} \Pi_+ \).

We define a surface term as an orbifolding symmetry breaking term, which is consistent with parity

\[
\psi(x, s) \rightarrow \gamma_0 P_{st} \tau^{1,2} \psi(-\vec{x}, x_0, t), \quad \bar{\psi}(x, s) \rightarrow \bar{\psi}(-\vec{x}, x_0, t) \gamma_0 P_{ts} \tau^{1,2}, \tag{4.27}
\]

charge conjugation

\[
\psi(x, s) \rightarrow CP_{st} \tau^{1,2} \bar{\psi}^T(x, t), \quad \bar{\psi}(x, s) \rightarrow \bar{\psi}^T(x, t) (C^{-1}) P_{ts} \tau^{1,2}, \quad C = \gamma_2 \gamma_0 \tag{4.28}
\]

and

\[
\psi(x, s) \rightarrow C(PQ)_{st} \bar{\psi}^T(x, t), \quad \bar{\psi}(x, s) \rightarrow \bar{\psi}^T(x, t) (C^{-1})(PQ)_{ts}, \quad C = \gamma_2 \gamma_0 \tag{4.29}
\]

and vector \( U(1)_3 \) symmetry

\[
\delta \psi(x, s) = \beta \tau^3 \psi(x, s), \quad \delta \bar{\psi}(x, s) = -\beta \bar{\psi}(x, s) \tau^3. \tag{4.30}
\]

of the orbifolded domain-wall fermion. Here we modified the parity and the charge conjugation transformation to be consistent with the twisted orbifolding projection.
Using the orbifolding projection (4.7) the orbifolding transformation (4.26) is shown to be a “chiral” transformation at the boundary in which a half of degrees is rotated
\[
\delta \left( \bar{P}_- \psi \right) (\vec{x}, 0, s) = \alpha \left( \bar{P}_- \psi \right) (\vec{x}, 0, s), \quad (4.31)
\]
\[
\delta \left( \bar{\psi} P_- \right) (\vec{x}, 0, s) = -\alpha \left( \bar{\psi} P_- \right) (\vec{x}, 0, s), \quad (4.32)
\]
\[
\delta \left( \bar{P}_+ \psi \right) (\vec{x}, N_T, s) = \alpha \left( \bar{P}_+ \psi \right) (\vec{x}, N_T, s), \quad (4.33)
\]
\[
\delta \left( \bar{\psi} P_+ \right) (\vec{x}, N_T, s) = -\alpha \left( \bar{\psi} P_+ \right) (\vec{x}, N_T, s) \quad (4.34)
\]
and is a vector \( U(1) \) transformation in the bulk \( 0 < x_0 < N_T \)
\[
\delta \psi(\vec{x}, x_0, s) = \alpha \psi(\vec{x}, x_0, s), \quad \delta \bar{\psi}(\vec{x}, x_0, s) = -\alpha \bar{\psi}(\vec{x}, x_0, s). \quad (4.35)
\]
The symmetry should be broken at the boundary.

We introduce boundary source fields as a component of projected out degrees of freedom
\[
\lambda(\vec{x}, s) = \bar{P}_+ \psi(\vec{x}, 0, s), \quad \lambda'(\vec{x}, s) = \bar{P}_- \psi(\vec{x}, N_T, s), \quad (4.36)
\]
\[
\bar{\lambda}(\vec{x}, s) = \left( \bar{\psi} P_+ \right) (\vec{x}, 0, s), \quad \bar{\lambda}'(\vec{x}, s) = \left( \bar{\psi} P_- \right) (\vec{x}, N_T, s). \quad (4.37)
\]
The orbifolding symmetry breaking term takes the form
\[
S_{\text{breaking}} = \bar{\lambda}(\vec{x}, s) \tilde{O}_{st} \bar{P}_- \psi(\vec{x}, 0, t) + \left( \bar{\psi} P_- \right) (\vec{x}, 0, s) \tilde{O}_{st} \lambda(\vec{x}, t)
\]
\[
+ \bar{\lambda}'(\vec{x}, s) \tilde{O}_{st} \bar{P}_+ \psi(\vec{x}, N_T, t) + \left( \bar{\psi} P_+ \right) (\vec{x}, N_T, s) \tilde{O}_{st} \lambda'(\vec{x}, t), \quad (4.38)
\]
where \( \tilde{O} \) is a local operator which anti-commute with \( i\gamma_5\gamma_0 P\tau^3 \). Candidates of \( \tilde{O} \) which is consistent with the parity, charge conjugation and \( U(1)_3 \) symmetries are \( PQ\tau^3, K(u)Q\tau^3 \) and \( \tilde{K}(u)Q\tau^3 \), where \( K \) and \( \tilde{K} \) are defined in (3.57) (3.58).

As in the previous section we adopt \( K(1)Q \) for the surface term to couple only to the physical quark field
\[
S_{\text{surface}} = -\alpha^3 \sum_{\vec{x}} \left( \bar{\lambda}(\vec{x}, s)(K(1)Q)_{st} \gamma^3 \bar{P}_- \psi(\vec{x}, 0, t) + \left( \bar{\psi} P_- \right) (\vec{x}, 0, s)(K(1)Q)_{st} \gamma^3 \lambda(\vec{x}, t) \right.
\]
\[
+ \bar{\lambda}'(\vec{x}, s)(K(1)Q)_{st} \gamma^3 \bar{P}_+ \psi(\vec{x}, N_T, t) + \left( \bar{\psi} P_+ \right) (\vec{x}, N_T, s)(K(1)Q)_{st} \gamma^3 \lambda'(\vec{x}, t) \bigg)
\]
\[
= \alpha^3 \sum_{\vec{x}} \left( - \bar{\rho}(\vec{x}) \gamma^3 P_- \bar{q}(x) \bigg|_{x_0=0} - \bar{\varphi}(x) \tilde{P}_- \gamma^3 \rho(\vec{x}) \bigg|_{x_0=0} 
\]
\[
- \bar{\rho}'(\vec{x}) \gamma^3 P_+ \bar{q}(x) \bigg|_{x_0=N_T} - \bar{\varphi}(x) \tilde{P}_+ \gamma^3 \rho'(\vec{x}) \bigg|_{x_0=N_T} \right), \quad (4.39)
\]
\( \rho \) and \( \bar{\rho} \) are boundary source fields for the physical quark fields
\[
\bar{P}_+ q(x)|_{x_0=0} = \rho(\vec{x}), \quad \bar{P}_- q(x)|_{x_0=N_T} = \rho'(\vec{x}), \quad (4.40)
\]
\[
\bar{\rho}(x) \tilde{P}_+|_{x_0=0} = \bar{\rho}(\vec{x}), \quad \bar{\varphi}(x) \tilde{P}_-|_{x_0=N_T} = \bar{\varphi}(\vec{x}). \quad (4.41)
\]

Although this surface term is not a general symmetry breaking term, we also expect that quantum corrections can be renormalized into a shift of physical operators and physical quark source fields \( \rho, \bar{\rho}, \rho' \) and \( \bar{\varphi} \) if we consider Green functions constructed with physical quark operators only.
4.4 Effective action of the domain-wall fermion

For the twisted orbifolding formulation of finite volume field theory the Pauli-Villars field is introduced directly as in (4.6). Total contributions from fermion and Pauli-Villars field is

$$\det_{\tilde{\mathcal{H}}_+} \frac{1}{D_{PV}} D_{\text{dwf}}(m_f) \tilde{\Pi}_{-},$$

(4.42)

where the determinant is defined in a sub-space $\tilde{\mathcal{H}}_+ = \{ \psi | \tilde{\Pi}_{+} \psi = 0 \}$ of eigenfunctions. In this sub-section we will show that this determinant is equivalent to that of the overlap Dirac operator with twisted orbifolding [25]

$$\det_{\tilde{\mathcal{H}}_+} \frac{1}{D_{PV}} D_{\text{dwf}}(m_f) \Pi_{-} = \det_{\tilde{\mathcal{H}}_+} D_{OD}(m_f) \Pi_{-}. \quad (4.43)$$

For this purpose we adopt the Schur decomposition procedure for the effective Dirac operator [39, 40]. Statement of the Schur decomposition is that the overlap Dirac operator is given as a Schur complement of the domain-wall fermion Dirac operator divided by the Pauli-Villars Dirac operator

$$\frac{1}{D_{PV}} D_{\text{dwf}}(m_f) = \mathcal{P} U^{-1} (1) D_{OD}^{(5)}(m_f) U(m_f) \mathcal{P}^\dagger. \quad (4.44)$$

Here $\mathcal{P}$, $U(m_f)$ and $D_{OD}^{(5)}(m_f)$ are matrices in fifth dimension and their explicit forms for $N_5 = 6$ case are given by

$$\mathcal{P} = \begin{pmatrix} P_R & P_R & P_R & P_R & P_L \\ P_L & P_L & P_L & P_L & P_R \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_L & P_L & P_L & P_L & P_R \end{pmatrix} = P_R + \Omega^{-1}(1) P_L, \quad (4.45)$$

$$U(m_f) = \begin{pmatrix} 1 & -T^4 (P_L - m_f P_R) \\ 1 & -T^3 (P_L - m_f P_R) \\ 1 & -T^2 (P_L - m_f P_R) \\ 1 & -T (P_L - m_f P_R) \end{pmatrix}, \quad (4.46)$$

$$D_{OD}^{(5)}(m_f) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad (4.47)$$

where $D_{OD}(m_f)$ is a truncated four dimensional massive overlap Dirac operator

$$D_{OD}(m_f) = \frac{1}{2} (1 + \gamma_5 S) + m_f \left( 1 - \frac{1}{2} (1 + \gamma_5 S) \right) \quad (4.48)$$
with the same definition for \( S \) in (3.79). The truncated Dirac operator turns out to be the ordinary overlap Dirac operator (3.86) in \( N_5 \to \infty \) limit. \( \Omega^-(m_f) \) is a hopping operator in fifth direction (2.8). So we have
\[
\det \frac{1}{D_{PV}} D_{dwf}(m_f) = \det D_{OD}(m_f) \tag{4.49}
\]
for ordinary domain-wall fermion system.

We start from the orbifolded domain-wall fermion Dirac operator divided by the Pauli-Villars Dirac operator
\[
D_{SF}^{(5)} = \bar{\Pi} \frac{1}{D_{PV}} D_{dwf}(m_f) \bar{\Pi} = \bar{\Pi} \mathcal{P} U^{-1}(1) D_{OD}^{(5)}(m_f) U(m_f) \mathcal{P}^\dagger \bar{\Pi}. \tag{4.50}
\]
We consider multiplication of the projection operator on unitary matrix \( \mathcal{P} \) and we have
\[
\bar{\Pi} \mathcal{P} = \bar{\Pi} \mathcal{P} \hat{\Pi}, \quad \hat{\Pi} = \frac{1 \pm P \Omega^{-(-1)} \Sigma \tau^3}{2}. \tag{4.51}
\]
We notice that a matrix in the projection \( \hat{\Pi} \) has a following form
\[
P \Omega^{-(-1)} = \Omega^+(1) P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.52}
\]
where \( P_{(N_5-1)} \) is a \((N_5-1) \times (N_5-1)\) matrix of the form
\[
P_{(N_5-1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (N_5 = 6). \tag{4.53}
\]
The projection operator \( \hat{\Pi} \) is written as a direct sum of two projections
\[
\hat{\Pi} = \begin{pmatrix} \Pi^{(N_5-1)} & \Pi_{(N_5-1)} \end{pmatrix}, \tag{4.54}
\]
where
\[
\Pi_{(N_5-1)} = \frac{1 \pm P_{(N_5-1)} \Sigma \tau^3}{2}. \tag{4.55}
\]
is a projection operator in \( N_5 - 1 \) sub-space.

Taking into account the explicit form of the matrix \( U(m_f) \) its determinant multiplied by the projection becomes
\[
\det_{(+\text{subspace})} U(m) \hat{\Pi}_- = \det_{(+\text{subspace})} \hat{\Pi}_- U(m) \hat{\Pi}_- = \det_{(+\text{subspace})} \begin{pmatrix} \Pi^{(N_5-1)} \Pi_{(N_5-1)} \end{pmatrix} = 1, \tag{4.56}
\]
\[
\det_{(+\text{subspace})} \hat{\Pi}_- U^{-1}(m) = \det_{(+\text{subspace})} \hat{\Pi}_- U^{-1}(m) \hat{\Pi}_- = \det_{(+\text{subspace})} \begin{pmatrix} \Pi^{(N_5-1)} \Pi_{(N_5-1)} \end{pmatrix} = 1. \tag{4.57}
\]
Substituting this relation determinant of the total Dirac operator is equivalent to that of the orbifolded overlap Dirac operator

\[
\det_{\mathcal{H}_+} D_{SF}^{(5)} = \det_{\mathcal{H}_+} \tilde{\Pi}_- P \tilde{\Pi}_- U^{-1}(1) \tilde{\Pi}_- D_{OD}^{(5)}(m_f) \tilde{\Pi}_- U(m_f) \tilde{\Pi}_- P^d \tilde{\Pi}_- \\
= \det_{(+\text{subspace})} \left( \tilde{\Pi}_- D_{OD}^{(5)}(m_f) \tilde{\Pi}_- \right) \\
= \det_{(+\text{subspace})} \left( \tilde{\Pi}_-^{(N_5-1)} \tilde{\Pi}_- \right) \left( \tilde{\Pi}_-^{1(N_5-1)} D_{OD}(m_f) \tilde{\Pi}_- \right) \\
= \det_{\mathcal{H}_+} \tilde{\Pi}_- D_{OD}(m_f) \tilde{\Pi}_- 
\]

and we get expected result.

At last we have a comment on Hermiticity. The five dimensional total Dirac operator \( D_{SF}^{(5)} \) has a following Hermiticity relation

\[
D_{SF}^{(5)\dagger} = \gamma_5 \tau^{1,2} D_{SF}^{(5)} \gamma_5 \tau^{1,2} 
\]

and its determinant is real. Since our domain-wall fermion Dirac operator does not have index the chiral rotation (4.1) is well defined even for single flavour case and we can define a single flavour orbifolded Dirac operator as

\[
D_{single}^{(5)} = 1 - \sum \frac{1}{2} D_{PV} \sum \frac{1}{2} D_{dwf}(m_f) 
\]

However we do not have a Hermiticity relation for this Dirac operator and the determinant is not shown to be real. We may need even numbers of flavours to avoid this problem.

5. Conclusion

In this paper the orbifolding formulation of the finite volume field theory is applied to the domain-wall fermion. In order to reproduce the proper SF Dirichlet boundary condition we need both the time reflection and the chiral symmetries. Application of this procedure to fermionic part is straightforward because of good chiral symmetry of the domain-wall fermion. Since there is no chiral symmetry for the Pauli-Villars field it is introduced by using the effective Dirac operator to reproduce the SF overlap Dirac operator. The surface term is given as an external source field to break the orbifolding symmetry.

The SF Dirichlet boundary condition may not be the unique choice to define a finite volume field theory suitable for renormalization scheme. A finite volume field theory with chirally twisted boundary condition is also proposed. Time reflection symmetry is enough to reproduce the twisted boundary condition by orbifolding. We can treat the fermionic part and the Pauli-Villars field in an equal footing. We have a \( \gamma_5 \) Hermiticity relation for the orbifolded Dirac operator and the total determinant is real. This formulation is applicable to two flavours dynamical simulation.
A. Effective action of chiral symmetric Dirac operator

In this appendix we derive the effective Dirac operator of the physical quark field for an action with the chiral symmetric Dirac operator (3.12). Four dimensional part of the symmetric Dirac operator is the same as the ordinary Dirac operator (2.6). Hopping term of this Dirac operator into the fifth direction takes the form

$$P_L \Omega^+(m_f = 0) + P_R \Omega^-(m_f = 0) = \begin{pmatrix} 0 & P_L & 0 & 0 & 0 & 0 \\ P_R & 0 & P_L & 0 & 0 & 0 \\ 0 & P_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_L & 0 & 0 \\ 0 & 0 & 0 & P_R & 0 & P_L \\ 0 & 0 & 0 & 0 & P_R & 0 \end{pmatrix}$$

(A.1)

for massless case. If there were no quark mass this Dirac operator is equivalent to two independent domain-wall fermion with half fifth dimensional size of \(N_5/2\). It is easily shown that there are two extra zero mode (doublers) at the middle boundary \(s = \frac{N_5}{2}\) and \(s = \frac{N_5}{2} + 1\) related to the exact chiral symmetry at finite \(N_5\).

The physical quark fields may be defined in the same manner as (2.9) and (2.10). We can integrate out the bulk field other than \(q\) and \(\overline{q}\) according to Ref. [34, 35, 39]. We start by writing the fermion field as a vector in fifth direction and chirality. For \(N_5 = 6\) we have

$$\Psi^T = (\psi_{1R} \psi_{1L} \psi_{2R} \psi_{2L} \psi_{3R} \psi_{3L} \psi_{4R} \psi_{4L} \psi_{5R} \psi_{5L} \psi_{6R} \psi_{6L}),$$

$$\Psi = (\overline{\psi}_{1L} \overline{\psi}_{1R} \overline{\psi}_{2L} \overline{\psi}_{2R} \overline{\psi}_{3L} \overline{\psi}_{3R} \overline{\psi}_{4L} \overline{\psi}_{4R} \overline{\psi}_{5L} \overline{\psi}_{5R} \overline{\psi}_{6L} \overline{\psi}_{6R}),$$

where

$$\psi_{R/L} = P_{R/L} \psi, \quad \overline{\psi}_{R/L} = \overline{\psi} P_{L/R}.$$  (A.2)

Then we change variable as

$$\Psi^T = (\psi_{1L} \psi_{1R} \psi_{2L} \psi_{2R} \psi_{3L} \psi_{3R} \psi_{4L} \psi_{4R} \psi_{5L} \psi_{5R} \psi_{6L} \psi_{6R}),$$

$$\Psi = (\overline{\psi}_{1R} \overline{\psi}_{2R} \overline{\psi}_{3R} \overline{\psi}_{1L} \overline{\psi}_{2L} \overline{\psi}_{3L} \overline{\psi}_{4R} \overline{\psi}_{4L} \overline{\psi}_{5R} \overline{\psi}_{5L} \overline{\psi}_{6R} \overline{\psi}_{6L}).$$

The Dirac operator is written as follows in terms of the primed field

$$D_{d/w} = \begin{pmatrix} \alpha & \beta & \alpha & \beta \\ \beta_0 & \alpha_0 & \alpha & \beta \\ \beta_0 & \alpha_0 & \alpha & \beta \\ -1 & \alpha_0 & \beta_0 & \alpha_0 \end{pmatrix},$$

(A.3)

where

$$\alpha = \begin{pmatrix} B & -C^\dagger \end{pmatrix}, \quad \alpha_0 = P_R \alpha.$$  (A.4)
\[ \beta = \begin{pmatrix} -1 & \phantom{1} \\ C & B \end{pmatrix}, \quad \beta_0 = P_L \beta, \quad \text{(A.5)} \]

\[ C_{xy} = \sigma_\mu \frac{1}{2} \left( \delta_{x+\mu, y} U_\mu(x) - \delta_{x-\mu, y} U^\dagger_\mu(y) \right), \quad \text{(A.6)} \]

\[ B_{xy} = (1 - M) \delta_{xy} - \frac{1}{2} \left( \delta_{x+\mu, y} U_\mu(x) + \delta_{x-\mu, y} U^\dagger_\mu(y) - 2 \delta_{xy} \right), \quad \text{(A.7)} \]

\[ \gamma_\mu = \begin{pmatrix} \sigma_\mu \\ \sigma_\mu^\dagger \end{pmatrix}. \quad \text{(A.8)} \]

We integrate out all the fields except for the physical quark field

\[ q(x) = P_L \psi(x, 1) + P_R \psi(x, N_5) = P_L \gamma_0 \psi'(x, 1) + P_R \gamma_0 \psi'(x, N_5), \quad \text{(A.9)} \]

\[ \overline{q}(x) = \overline{\psi}(x, 1) P_R + \overline{\psi}(x, N_5) P_L = \overline{\psi}(x, \frac{N_5}{2}) \gamma_0 P_R + \overline{\psi}(x, N_5) \gamma_0 P_L \quad \text{(A.10)} \]

according to Ref. [34]. Result is given as a full quark propagator

\[ \langle \overline{q}q \rangle = \frac{1}{2} \left( \frac{1}{D_{\text{eff}}} - \gamma_5 \frac{1}{D_{\text{eff}}} \gamma_5 \right) = \frac{1}{D_{\text{sym}}}. \quad \text{(A.11)} \]

Here \( D_{\text{eff}} \) is the truncated effective Dirac operator (3.79) with half size of fifth dimensional length

\[ D_{\text{eff}} = \frac{1 + \gamma_5 S}{1 - \gamma_5 S}, \quad S = \frac{1 - T_{\frac{N_5}{2}}}{1 + T_{\frac{N_5}{2}}}. \quad \text{(A.12)} \]

Transfer matrix is given by

\[ T = \gamma_5 \gamma_0 \left( -\alpha \beta^{-1} \right) \gamma_5 \gamma_5 = \frac{1 - H'}{1 + H'}. \quad \text{(A.13)} \]

The full quark propagator anti-commutes with \( \gamma_5 \) even at finite \( N_5 \). In \( N_5 \to \infty \) limit the effective Dirac operator \( D_{\text{eff}} \) anti-commutes with \( \gamma_5 \) exactly and the effective Dirac operator \( D_{\text{sym}} \) with symmetric construction becomes the same as that of the ordinary domain-wall fermion \( D_{\text{eff}} \).

We introduce the Pauli-Villars field in the same manner with the Dirac operator

\[ D_{\text{PV}}^\text{sym} = D_{\text{dwf}}(m_f = 1) + X. \quad \text{(A.14)} \]

The effective action of the physical quark field \( q, \overline{q} \) and the physical Pauli-Villars field \( Q, \overline{Q} \) is given by

\[ S_{\text{eff}} = \overline{q} D_{\text{eff}}^\text{sym} q + \overline{Q} \left( D_{\text{eff}}^\text{sym} + 1 \right) Q. \quad \text{(A.15)} \]

The overlap Dirac operator is given to reproduce the same determinant as the effective action

\[ D_{\text{OD}}^\text{sym} = \frac{D_{\text{eff}}^\text{sym}}{D_{\text{eff}}^\text{sym} + 1}. \quad \text{(A.16)} \]
Because of exact chiral symmetry of $D_{\text{sym}}$ the overlap Dirac operator $D_{\text{OD}}^{\text{sym}}$ satisfies the Ginsparg-Wilson relation even at finite $N_5$.

Compensation of the exact chirality at finite $N_5$ is a non-locality in the overlap Dirac operator, which comes from the extra zero mode in the middle of fifth direction. However we can show that the non-locality is exponentially small in $N_5$ and disappears in $N_5 \to \infty$ limit. In order to extract the non-locality we define explicit breaking term of the chiral symmetry of the ordinary effective Dirac operator (3.79) or the truncated overlap Dirac operator at finite $N_5$

$$
\delta_{N_5} = \gamma_5 \frac{1}{D_{\text{eff}}(N_5)} + \frac{1}{D_{\text{eff}}(N_5)} \gamma_5 = \gamma_5 \frac{1}{D_{\text{OD}}(N_5)} + \frac{1}{D_{\text{OD}}(N_5)} \gamma_5 - 2\gamma_5. \quad (A.17)
$$

The chiral symmetric effective Dirac operator is re-written as

$$
\frac{1}{D_{\text{eff}}^{\text{sym}}} = \frac{1}{D_{\text{eff}}} - \frac{1}{2} \gamma_5 \delta_{N_5} \gamma_5. \quad (A.18)
$$

where we used a fact that the breaking term commutes with $\gamma_5$

$$
[\delta_{N_5}, \gamma_5] = 0. \quad (A.19)
$$

The chiral symmetric overlap Dirac operator is given in a following form

$$
D_{\text{OD}}^{\text{sym}} = \frac{1}{1 - \frac{1}{2} D_{\text{OD}} \gamma_5 \delta_{N_5}}. \quad (A.20)
$$

$D_{\text{OD}}$ in denominator may bring a non-local factor into the overlap Dirac operator. However as was shown in Ref. [34] $\delta_{N_5}$ is exponentially small in $N_5$. The physical part of the chiral symmetric Dirac operator $D_{\text{dof}}^{\text{sym}}$ coincides with that of the ordinary Dirac operator in $N_5 \to \infty$ limit.

B. Folding of temporal direction

In our formulation with the orbifolding (3.17) fermion fields in negative time $-N_T < x_0 < 0$ can be written in term of those in the positive region

$$
\psi(\vec{x}, -x_0, s) = (\Gamma)_{s,t} \psi(\vec{x}, x_0, t), \quad \Gamma = \gamma_0 \gamma_5 PQ. \quad (B.1)
$$

Half of the field degrees of freedom can be eliminated explicitly by folding the temporal axis into the non-negative range $0 \leq x_0 \leq N_T$ together with the boundary condition (3.18) (3.19).

For this purpose we introduce four projection operators in temporal direction

$$
\begin{align*}
T_- & \quad \text{for} \quad -N_T + 1 \leq x_0 \leq -1, \\
T_0 & \quad \text{for} \quad x_0 = 0, \\
T_+ & \quad \text{for} \quad 1 \leq x_0 \leq N_T - 1, \\
T_T & \quad \text{for} \quad x_0 = N_T,
\end{align*}
$$
which pick up the fermion fields on the corresponding region. For instance

\[
(T_+)_{x_0,y_0} \psi(y_0) = \begin{cases} 
\psi(x_0) & \text{for } 1 \leq x_0 \leq N_T - 1 \\
0 & \text{otherwise}
\end{cases}.
\] (B.2)

Summing up four projection operators we have a unity

\[
1 = T_- + T_0 + T_+ + T_T
\] (B.3)

and \(T_\alpha\)’s have a projection property

\[
T_\alpha T_\beta = T_\alpha \delta_{\alpha,\beta}.
\] (B.4)

These projection operators satisfy the following relation with the time reflection operator \(R\)

\[
RT_+ = T_- R, \quad RT_- = T_+ R, \quad RT_0 = T_0 R = T_0, \quad RT_T = T_T R = -T_T.
\] (B.5)

By using these properties we have an identity relation

\[
\Pi_+ = \Pi_+ (T_+ + T_- + T_0 + T_T) = \Pi_+ (T_+ + T_0 + T_T) (2T_+ + T_0 + T_T) \Pi_+.
\] (B.6)

and the orbifolded action (3.26) can be re-written in terms of the fermion fields depending on the non-negative region only

\[
S = \sum_{\vec{x},\vec{y},x_0,y_0=0}^{N_T} \sum_{s,t} \sum \overline{\psi''}(x,s) D_{\text{folded}} \psi''(y,t),
\] (B.7)

where \(\psi''\) and \(\overline{\psi''}\) are defined as

\[
\psi''(\vec{x},x_0,s) = ((T_+ + T_0 + T_T) \Pi_+)_{x_0,y_0;s,t} \psi(\vec{x},y_0,t),
\] (B.8)

\[
\overline{\psi''}(\vec{x},x_0,s) = \overline{\psi}(\vec{x},y_0,t) (\Pi_+ (T_+ + T_0 + T_T))_{y_0,x_0;t,s},
\] (B.9)

which have no dependence on negative time. These fields can further be written as

\[
\psi''(\vec{x},x_0,s) = (T_+ + T_0 \overline{\Pi}_+ + T_T \overline{\Pi}_-)_{x_0,y_0;s,t} \psi(\vec{x},y_0,t),
\] (B.10)

\[
\overline{\psi''}(\vec{x},x_0,s) = \overline{\psi}(\vec{x},y_0,t) (T_+ + \overline{T}_0 + \overline{T}_T)_{y_0,x_0;t,s},
\] (B.11)

by using (B.5) and identification (B.1). There is no constraint on positive bulk fields.

The folded Dirac operator \(\overline{D}_{\text{dwlf}}^{SF}\) is given formally as

\[
D_{\text{dwlf}}^{\text{folded}} = \frac{1}{2} (2T_+ + T_0 + T_T) \Pi_+ D_{\text{dwlf}} \Pi_+ (2T_+ + T_0 + T_T).
\] (B.12)
This Dirac operator can be written in more explicit form by using the orbifolding symmetry (3.16) and the ultra local property of the domain-wall fermion Dirac operator, with which we eliminate the term like \( T_+ D_{\text{dwf}} A T_+ = T_+ D_{\text{dwf}} T_- A \)

\[
D_{\text{dwf}}^{\text{folded}} = \frac{1}{2} T_0 \bar{P}_+ D_{\text{dwf}} \bar{P}_+ T_0 + T_0 \bar{P}_+ D_{\text{dwf}} T_+ + T_+ D_{\text{dwf}} \bar{P}_+ T_0 + T_+ D_{\text{dwf}} T_+ \\
+ T_T \bar{P}_- D_{\text{dwf}} T_+ + T_+ D_{\text{dwf}} \bar{P}_- T_T + \frac{1}{2} T_T \bar{P}_- D_{\text{dwf}} \bar{P}_- T_T 
\quad (B.13)
\]

\[
= \begin{pmatrix}
-\bar{P}_+ D_0^{(3+1)} \bar{P}_+ & -\bar{P}_+ U_0(0) \\
-\bar{P}_+ U_0^T(1) & -\bar{P}_- U_0(1) \\
-\bar{P}_+ U_0^T(2) & -\bar{P}_- U_0(2) \\
-\bar{P}_+ U_0^T(3) & \bar{P}_- D_0^{(3+1)} \bar{P}_-
\end{pmatrix}
\quad (B.14)
\]

where the matrix represents the Dirac operator in temporal direction for \( N_T = 4 \). \( D_0^{(3+1)} \) is the Dirac operator in spatial direction and the fifth direction

\[
D_{3+1}^{(3+1)}(x,y;s,t) = \left( \frac{-1 + \gamma^5 U_i(x) \delta_{y_i,x_i+1} + \frac{1}{2} - \gamma^5 U_i^T(y) \delta_{y_i,x_i-1}}{2} \right) \delta_{x_0,y_0} \delta_{s,t} \\
+ \left( \frac{-1 + \gamma^5 \Omega_{s,t}^+ + \frac{1}{2} - \gamma^5 \Omega_{s,t}^-}{2} \right) \delta_{x,y} + (5-M) \delta_{x,y} \delta_{s,t}.
\quad (B.15)
\]

There is no constraint for the bulk region \( 1 < x_0, y_0 < N_T - 1 \), which is nothing but ordinary domain-wall fermion Dirac operator.

We notice that the projection operator \( \bar{P}_\pm \) at the boundary does not commute with the \( \gamma_0 \) chiral projection \( P_\pm \). If we consider an eigenvalue equation of this Dirac operator a zero mode dumping solution

\[
\psi = P_-(1-M)^{x_0} + P_+(1-M)^{(N_T-x_0)}
\quad (B.16)
\]

in temporal direction, which have broken the chiral symmetry “dynamically” in a naive formulation, is forbidden by this boundary term.

The fermion propagator is given as an inverse of the folded Dirac operator

\[
G_{\text{dwf}}^{\text{folded}} = 2 (T_+ + T_0 + T_T) \bar{P}_+ D_{\text{dwf}}^{-1} \bar{P}_+ (T_+ + T_0 + T_T),
\quad (B.17)
\]

where the inverse is defined in the ordinary meaning for the positive bulk region \( 0 < x_0 < N_T \) and in terms of the projected sub-space at the boundary

\[
D_{\text{dwf}}^{\text{folded}} G_{\text{dwf}}^{\text{folded}} = T_+ + \bar{P}_+ T_0 + \bar{P}_- T_T.
\quad (B.18)
\]

**C. Free fermion propagator**

Inverse of the massless domain-wall fermion Dirac operator in momentum space is derived according to the procedure of Ref. [30]. In this appendix we omit derivation and give the result:

\[
\frac{1}{D_{\text{dwf}}(p)} = (-i \gamma_\mu \sin p_\mu + W - \Omega^-) G_R P_L + (-i \gamma_\mu \sin p_\mu + W - \Omega^+) G_L P_R,
\quad (C.1)
\]
where $\Omega$ and $W$ are defined in (2.8) and (3.37). $G_R$ and $G_L$ are defined as

\begin{align}
G_R(s, t) &= G^0(s - t) + A_{++} e^{\alpha(s+t)} + A_{+-} e^{\alpha(s-t)} + A_{--} e^{-\alpha(s+t)} + A_{-+} e^{-\alpha(s-t)}, \quad (C.2) \\
G_L(s, t) &= G^0(s - t) + B_{++} e^{\alpha(s+t)} + B_{+-} e^{\alpha(s-t)} + B_{--} e^{-\alpha(s+t)} + B_{-+} e^{-\alpha(s-t)}, \quad (C.3) \\
G^0(s - t) &= C \left( e^{\alpha(N_5 - |s-t|)} + e^{-\alpha(N_5 - |s-t|)} \right) \quad (C.4)
\end{align}

with exponent and coefficients given by

\begin{align}
\cosh \alpha &= \frac{1 + W^2 + \sin^2 p_\mu}{2|W|}, \quad (C.5) \\
C &= \frac{1}{4W \sinh \alpha \sin(\alpha N_5)}, \quad (C.6) \\
A_{++} &= F(1 - W e^{-\alpha})(e^{-2\alpha N_5} - 1), \quad A_{-+} = F(1 - W e^{\alpha})(1 - e^{2\alpha N_5}), \quad (C.7) \\
B_{++} &= e^{-2\alpha(N_5+1)} A_{-+}, \quad B_{-+} = e^{2\alpha(N_5+1)} A_{++}, \quad (C.8) \\
A_{--} &= A_{++} = B_{-+} = B_{++} = FW(e^\alpha - e^{-\alpha}), \quad (C.9) \\
F &= \frac{C}{e^{\alpha N_5}(1 - W e^\alpha) - e^{-\alpha N_5}(1 - W e^{-\alpha})}. \quad (C.10)
\end{align}

This notation is valid for positive $W$ and for negative case we define

\begin{align}
e^{\pm \alpha} &= \cosh \alpha \pm \sqrt{\cosh^2 \alpha - 1} \quad (C.11)
\end{align}

and flip their sign $e^{\pm \alpha} \to -e^{\pm \alpha}$ according to $\text{sgn}(W)$.

The physical quark propagator in momentum space is defined by picking up the boundary components

\begin{align}
\langle q(p)\bar{q}(-p) \rangle &= (P_L \delta_{s,1} + P_R \delta_{s,N_5}) \left( \frac{1}{D_{\text{dwt}}(p)} \right)_{s,t} (\delta_{t,N_5} P_L + \delta_{t,1} P_R) \\
&= -i\gamma_\mu \sin p_\mu G_R(N_5, N_5) + WG_R(1, N_5). \quad (C.12)
\end{align}

Ignoring the next to leading term in $N_5$ the quark propagator has a simple form

\begin{align}
\langle q(p)\bar{q}(-p) \rangle &= \frac{i\gamma_\mu \sin p_\mu}{1 - W e^\alpha}. \quad (C.13)
\end{align}

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