JOHN-NIRENBERG-Q SPACES VIA CONGRUENT CUBES

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Abstract To shed some light on the John-Nirenberg space, the authors of this article introduce the John-Nirenberg-$Q$ space via congruent cubes, $JNQ_{p,q}^\alpha(\mathbb{R}^n)$, which, when $p = \infty$ and $q = 2$, coincides with the space $Q_\alpha(\mathbb{R}^n)$ introduced by Essén, Janson, Peng and Xiao in [Indiana Univ Math J, 2000, 49(2): 575–615]. Moreover, the authors show that, for some particular indices, $JNQ_{p,q}^\alpha(\mathbb{R}^n)$ coincides with the congruent John-Nirenberg space, or that the (fractional) Sobolev space is continuously embedded into $JNQ_{p,q}^\alpha(\mathbb{R}^n)$. Furthermore, the authors characterize $JNQ_{p,q}^\alpha(\mathbb{R}^n)$ via mean oscillations, and then use this characterization to study the dyadic counterparts. Also, the authors obtain some properties of composition operators on such spaces. The main novelties of this article are twofold: establishing a general equivalence principle for a kind of ‘almost increasing’ set function that is here introduced, and using the fine geometrical properties of dyadic cubes to properly classify any collection of cubes with pairwise disjoint interiors and equal edge length.

Key words John-Nirenberg space; congruent cube; $Q$ space; (fractional) Sobolev space; mean oscillation; dyadic cube; composition operator

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1 Introduction

Throughout this article, a measurable function $f$ means that $f$ is Lebesgue measurable and $|f| < \infty$ almost everywhere; a cube $Q$ means that it has finite edge length and that all of its edges are parallel to the coordinate axes, but that $Q$ is not necessary to be open or closed. Recall that the Lebesgue space $L^q(\mathbb{R}^n)$ with $q \in [1, \infty]$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$
\|f\|_{L^q(\mathbb{R}^n)} := \begin{cases} 
\left( \int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{\frac{1}{q}}, & q \in (1, \infty), \\
\text{ess sup}_{x \in \mathbb{R}^n} |f(x)|, & q = \infty
\end{cases}
$$

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is finite. In what follows, we use $1_E$ to denote the characteristic function of any set $E \subset \mathbb{R}^n$, and $L^q_{\text{loc}}(\mathbb{R}^n)$ to denote the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $f1_E \in L^q(\mathbb{R}^n)$ for any bounded measurable set $E \subset \mathbb{R}^n$.

The John-Nirenberg space via congruent cubes (for short, the congruent John-Nirenberg space) was introduced by Jia, Tao, Yang, Yuan and Zhang [20] to shed some light on the mysterious space $JN_p$, and the space $B$, which were introduced, respectively, by John and Nirenberg [24] and Bourgain, Brezis and Mironescu [7]. In what follows, for any $\ell \in (0, \infty)$, let $\Pi_\ell$ denote the class of all collections $\{Q_j\}_j$ of subcubes of $\mathbb{R}^n$ with pairwise disjoint interiors and equal edge length $\ell$. Moreover, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and any bounded measurable set $E \subset \mathbb{R}^n$ with positive measure, let

$$f_E := \int_E f(y) dy := \frac{1}{|E|} \int_E f(y) dy.$$

**Definition 1.1** Let $p$, $q \in [1, \infty]$. The congruent John-Nirenberg space $JN^\text{con}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{JN^\text{con}_{p,q}(\mathbb{R}^n)} := \sup_{\ell \in (0,\infty)} \left\{ \sum_{j} |Q_j| [\text{MO}_{f,q}(Q_j)]^p \right\}^{\frac{1}{p}} < \infty$$

with the mean oscillation

$$\text{MO}_{f,q}(Q_j) := \left[ \int_{Q_j} |f(x) - f_{Q_j}|^q dx \right]^\frac{1}{q}$$

(1.1)

for any $j$. Moreover, let $JN^\text{con}_p(\mathbb{R}^n) := JN^\text{con}_{p,1}(\mathbb{R}^n)$.

Very recently, the authors of [21–23] further showed that several important operators (such as the Hardy-Littlewood maximal operator, Calderón-Zygmund operators, fractional integrals, and Littlewood-Paley operators) are bounded on congruent John-Nirenberg spaces. Thus, it is meaningful to study and reveal more properties of the congruent John-Nirenberg space.

Another well-known space that appeared in John and Nirenberg [24] is $\text{BMO}(\mathbb{R}^n)$, the space containing functions of bounded mean oscillation, which can be regarded as the limit space of $JN^\text{con}_{p,q}(\mathbb{R}^n)$ as $p \to \infty$; see [20, Proposition 2.21] and also [33, Proposition 2.6]. The space $\text{BMO}(\mathbb{R}^n)$ has wide applications in harmonic analysis and partial differential equations; see, for instance, [2, 11, 12, 15–18, 26, 30, 37]. In particular, we refer the reader to [31] for a systematic survey on function spaces of John-Nirenberg type. Later, Essén, Janson, Peng and Xiao [19] introduced $Q$ spaces which generalize the space $\text{BMO}(\mathbb{R}^n)$. To be more precise, the space $Q_\alpha(\mathbb{R}^n)$ with $\alpha \in \mathbb{R}$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{Q_\alpha(\mathbb{R}^n)} := \sup_{\text{cube } Q} \left[ |Q|^{\frac{\alpha}{n-1}} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dy dx \right]^\frac{1}{2}$$

is finite, where the supremum is taken over all cubes $Q$ of $\mathbb{R}^n$. Then Essén, Janson, Peng and Xiao [19, Theorem 2.3(iii)] showed that, for any $\alpha \in (-\infty, 0)$, $Q_\alpha(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$ with equivalent norms. Later, based on the open problems posed in [19, Section 8], such $Q$ spaces attracted a lot of attention; see, for instance, [14, 25, 27, 35, 36, 38, 39]. We also refer the reader to the recent monograph [34] by Xiao and the references therein for the elaborate development of $Q$ spaces. Thus, it is natural to consider the corresponding $Q$ spaces of congruent John-Nirenberg spaces; this is the main motivation of the present article.
In this article, we introduce the John-Nirenberg-\(Q\) space via congruent cubes (for short, the \(JNQ\) space), \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\), which, when \(p = \infty\) and \(q = 2\), coincides with the space \(Q_\alpha(\mathbb{R}^n)\). Moreover, we show that, for some particular indices, the \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\) coincides with the congruent John-Nirenberg space, or that the (fractional) Sobolev space is continuously embedded into \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\). Furthermore, we characterize \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\) via mean oscillations, and then use this characterization to study the dyadic counterparts of \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\). Also, we obtain some properties of the composition operators on such spaces. The main novelties of this article are twofold: establishing a general equivalence principle for a kind of ‘almost increasing’ set function that is here introduced, and using the fine geometrical properties of dyadic cubes to properly classify any collection of cubes with pairwise disjoint interiors and equal edge length.

**Definition 1.2** Let \(p, q \in [1, \infty)\) and \(\alpha \in \mathbb{R}\). The space \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\) is defined as the set of all measurable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} := \sup_{\ell \in \mathbb{N}_0} \left\{ \sum_j |Q_j| \left[ \Phi_{f,q,\alpha}(Q_j)^p\right]^{\frac{1}{p}} \right\}^{\frac{1}{p}} < \infty,
\]
and here and thereafter, for any cube \(Q\) of \(\mathbb{R}^n\),
\[
\Phi_{f,q,\alpha}(Q) := \left[ |Q|^{\frac{1}{n(\ell+1)}} \int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x-y|^{n+q\alpha}} dydx\right]^{\frac{1}{q}}.
\]  

\(\text{Remark 1.3} \) (i) Since \(\|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} = 0\) if and only if \(f\) is a constant almost everywhere, we regard \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\) as a function space of modulo constants. Thus, throughout this article, we simply write \{a.e. constant\} by \(\{0\}\).

(ii) Let \(q \in [1, \infty)\) and \(\alpha \in \mathbb{R}\). The space \(JNQ_{p,q}^{\infty,\alpha}(\mathbb{R}^n)\) can be automatically defined as the set of all measurable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{JNQ_{p,q}^{\infty,\alpha}(\mathbb{R}^n)} := \sup_{\text{cube } Q} \Phi_{f,q,\alpha}(Q) < \infty,
\]
where \(\Phi_{f,q,\alpha}(Q)\) is as in (1.2), and the supremum is taken over all cubes \(Q\) of \(\mathbb{R}^n\). Then \(JNQ_{p,q}^{\infty,\alpha}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)\) with equal norms, and hence \(Q_\alpha(\mathbb{R}^n)\) can be regarded as the limit space of \(JNQ_{p,q}^{\alpha}(\mathbb{R}^n)\) as \(p \to \infty\); see also Proposition 2.10 below.

The remainder of this article is organized as follows: Section 2 is devoted to revealing the relations between \(JNQ\) spaces and some other function spaces including congruent John-Nirenberg spaces, \(Q\) spaces, and (fractional) Sobolev spaces. To be more precise, we show that \(JNQ_{p,q}^\alpha(\mathbb{R}^n) = JNQ_{p,q}^{\infty,\alpha}(\mathbb{R}^n)\) for any \(\alpha \in (-\infty, 0)\) in Theorem 2.7. A primary tool used in this proof is the equivalent integral-type norm (see Corollary 2.6). Indeed, we obtain a more general equivalence principle in Proposition 2.5 via introducing the ‘almost increasing’ set function \(\Phi\) (see Definition 2.3). It should be pointed out that Proposition 2.5 may be of independent interest because, as a special case of this proposition with \(\Phi\) replaced by the mean oscillation as in (1.1), [20, Proposition 2.2] proves extremely useful when studying the boundedness of the Hardy-Littlewood maximal operator, Calderón-Zygmund operators, fractional integrals, and Littlewood-Paley operators on congruent John-Nirenberg spaces; see [21–23] for more details. Moreover, we show that \(Q_\alpha(\mathbb{R}^n)\) can be regarded as the limit space of \(JNQ_{p,q}^\alpha(\mathbb{R}^n)\) as \(p \to \infty\).
in Proposition 2.10, and also obtain an extension result over cubes (with finite edge length) in Proposition 2.11. Furthermore, for some non-negative \( \alpha \), we prove that the (fractional) Sobolev space is continuously embedded into the \( J N Q \) space in Propositions 2.15 and 2.16. At the end of this section, we sum up these relations in Theorem 2.17, which is a complete classification over \( p, q \in [1, \infty) \) and \( \alpha \in \mathbb{R} \).

In Section 3, we first characterize \( J N Q \) spaces via mean oscillations in Theorem 3.1. Then we use this characterization to study the dyadic counterparts of \( J N Q \) spaces which prove to be the intersection of \( J N Q \) spaces and congruent John-Nirenberg spaces; see Theorem 3.8. To handle a collection of cubes in the supremum of \( \| \cdot \|_{JNQ_{p,q}(\mathbb{R}^n)} \) rather than a single cube in the supremum of \( \| \cdot \|_{Q_{\alpha}(\mathbb{R}^n)} \), we use some fine geometrical properties of dyadic cubes to properly classify any collection of cubes with pairwise disjoint interiors and equal edge length; see Lemma 3.10.

Section 4 is devoted to investigating the left and the right composition operators on \( J N Q \) spaces. As an application of Proposition 2.16 (namely, the Sobolev space is continuously embedded into the \( J N Q \) space), we show that the left composition operator is bounded on the \( J N Q \) space if and only if the corresponding function belongs to the Lipschitz space; see Theorem 4.1. Moreover, we give a brief discussion about the right composition operator on the \( J N Q \) space; see Proposition 4.3 and Remark 4.4.

Finally, we establish some conventions on notation. Let \( \mathbb{N} := \{1, 2, \cdots \} \), \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \), \( \mathbb{Z}_+^n := (\mathbb{Z}_+)^n \), 0 denote the origin of the Euclidean space, and let \( \nabla b \) denote the gradient of \( b \). For any \( \gamma := (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}_+^n \), let \( D^\gamma := (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n} \). For any \( s \in \mathbb{N} \), \( C^s(\mathbb{R}^n) \) denotes the set of all functions \( f \) on \( \mathbb{R}^n \) whose derivatives \( \{D^\gamma f\}_{\gamma=0}^{s} \) are continuous. In addition, we use \( C(\mathbb{R}^n) \) to denote the set of all continuous functions on \( \mathbb{R}^n \), and \( C^\infty(\mathbb{R}^n) \) to denote the set of all infinitely differentiable functions on \( \mathbb{R}^n \). For any \( p \in [1, \infty] \), let \( p' \) be its conjugate index; that is, \( p' \) satisfies \( 1/p + 1/p' = 1 \). We use \( 1_E \) to denote the characteristic function of a set \( E \subset \mathbb{R}^n \), and let \( |E| \) denote the Lebesgue measure when \( E \subset \mathbb{R}^n \) is measurable. Also, we use \( \sharp A \) to denote the cardinality of the set \( A \). Moreover, for any \( z \in \mathbb{R}^n \) and \( \ell \in (0, \infty) \), we use \( Q(z, \ell) \) to denote the cube centered at \( z \) with the length \( \ell \); for any cube \( Q \) of \( \mathbb{R}^n \) and any \( \lambda \in (0, \infty) \), we use \( \lambda Q \) to denote the cube with edge length \( \lambda \ell(Q) \) and the same center as \( Q \).

2 Relations with Other Function Spaces

In this section, we show the relations between \( J N Q \) spaces with congruent John-Nirenberg spaces, \( Q \) spaces, and fractional Sobolev spaces.

2.1 Relations with Congruent John-Nirenberg Spaces

We first observe \( JNQ_{p,q}^{n/2}(\mathbb{R}^n) = JNQ_{p,q}^{n} \) via the following lemma:

**Lemma 2.1** Let \( q \in [1, \infty) \), \( E \) be a bounded measurable set in \( \mathbb{R}^n \) with positive measure, and \( f \) be a measurable function on \( \mathbb{R}^n \). Then,

(i) if \( f 1_E \in L^1(\mathbb{R}^n) \),

\[
\left[ \int_E |f(x) - f_E|^q \, dx \right]^\frac{1}{q} \leq \left[ \int_E \int_E |f(x) - f(y)|^q \, dy \, dx \right]^\frac{1}{q} \leq 2 \left[ \int_E |f(x) - f_E|^q \, dx \right]^\frac{1}{q}
\]

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and, in particular,
\[ 2 \int_E |f(x) - f_E|^2 \, dx = \int_E \int_E |f(x) - f(y)|^2 \, dy \, dx; \]

(ii) if \( f1_E \notin L^1(\mathbb{R}^n) \),
\[ \int_E \int_E |f(x) - f(y)|^q \, dy \, dx = \infty. \]

**Proof** Let \( q, E, \) and \( f \) be as in the present lemma. We first prove (i). In this case, we have that \( f_E < \infty \). By this, Hölder’s inequality, and Minkowski’s inequality, we have that
\[
\left[ \int_E |f(x) - f_E|^q \, dx \right]^{\frac{1}{q}} \leq \left\{ \int_E \left[ \int_E |f(x) - f(y)|^q \, dy \right]^{\frac{1}{q}} \right\}^{\frac{1}{q}} \\
\leq \left[ \int_E \int_E |f(x) - f(y)|^q \, dy \, dx \right]^{\frac{1}{q}} \\
\leq \left[ \int_E \int_E |f(x) - f_E|^q \, dy \, dx \right]^{\frac{1}{q}} + \left[ \int_E \int_E |f_E - f(y)|^q \, dy \, dx \right]^{\frac{1}{q}} \\
= 2 \left[ \int_E \frac{|f(x) - f_E|^q}{x} \, dx \right]^{\frac{1}{q}}.
\]

Moreover, for the case \( q = 2 \), we have that
\[
\int_E |f(x) - f_E|^2 \, dx = \int_E \int_E |f(x) - f_E||f(x) - f_E| \, dy \, dx = \int_E f(x) f(x) \, dx - f_E f_E
\]
and
\[
\int_E \int_E |f(x) - f(y)|^2 \, dy \, dx = \int_E \int_E |f(x) - f(y)||f(x) - f(y)| \, dy \, dx \\
= 2 \left[ \int_E f(x) f(x) \, dx - f_E f_E \right].
\]

Thus,
\[
2 \int_E |f(x) - f_E|^2 \, dx = \int_E \int_E |f(x) - f(y)|^2 \, dy \, dx,
\]
which completes the proof of (i).

Next, we prove (ii). In this case, we have that \( \int_E |f(x)| \, dx = \infty \), and hence there exists a measurable subset \( F \subset E \) such that \( |f(x)| \geq 1 \) for any \( x \in F \). By this, along with Hölder’s inequality, we conclude that
\[
\left[ \int_E \int_E |f(x) - f(y)|^q \, dy \, dx \right]^{\frac{1}{q}} \geq \int_E \int_E |f(x) - f(y)| \, dy \, dx \\
\geq |E|^{-2} \int_F \int_{E \setminus F} |f(x)| - |f(y)| \, dy \, dx \\
\geq |E|^{-2} \int_F \int_{E \setminus F} |f(x)| - 1 \, dy \, dx \\
= |E|^{-2} |E \setminus F| \left[ \int_F |f(x)| \, dx - |F| \right] = \infty,
\]
where, in the last inequality, we used the observations \( |E| > 0, |F| \leq |E| < \infty \), and
\[
\int_F |f(x)| \, dx = \int_E |f(x)| \, dx - \int_{E \setminus F} |f(x)| \, dx \geq \int_E |f(x)| \, dx - |E| = \infty.
\]
This finishes the proof of (ii) and hence of Lemma 2.1.

Remark 2.2 Let \( \| f \|_{J^p_{n,q} (\mathbb{R}^n)} := \infty \) if \( f \mathbf{1}_E \notin L^1 (\mathbb{R}^n) \). Then we can extend the definition of \( JN^p_{n,q} (\mathbb{R}^n) \) to all measurable functions \( f \) on \( \mathbb{R}^n \) such that \( \| f \|_{JN^p_{n,q} (\mathbb{R}^n)} < \infty \). Therefore, as an immediate consequence of Lemma 2.1, we obtain that \( JN_{p,q}^{-n/q} (\mathbb{R}^n) = JN_{p,q}^{\infty} (\mathbb{R}^n) \) with equivalent norms.

Motivated by [20, Proposition 2.2] and [19, Lemma 5.7], we establish an equivalence principle (namely, Proposition 2.5) which shows that, in the congruent setting (namely, all cubes \( \{Q_j\} \) have equal edge length), the summation is equivalent to the integral so long as the integrand \( \Phi \) is ‘almost increasing’.

Definition 2.3 A non-negative set function \( \Phi \) on all cubes (or balls) \( Q \) is said to be almost increasing if there exists a positive constant \( C \) such that, for any cubes (or balls) \( Q_1, Q_2 \) satisfying \( Q_1 \subset Q_2 \) and \( |Q_1| \leq 2^{-n}|Q_2| \), it holds that

\[
\Phi(Q_1) \leq C \Phi(Q_2). \tag{2.1}
\]

Remark 2.4 We present two examples of almost increasing set functions:

(i) For any cube \( Q \),

\[
\text{MO}_{f,q}(Q) = \left[ \int_Q |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}},
\]

as in (1.1) with \( f \in L^q_{\text{loc}} (\mathbb{R}^n) \) and \( q \in [1, \infty) \); see [20, Lemma 2.1].

(ii) For any cube \( Q \),

\[
\Psi_{f,2,\alpha}(Q) := \sum_{k=0}^{\infty} \sum_{I \in \mathcal{D}_k(Q)} 2^{(2\alpha - n)k} \text{MO}_{f,2}(I),
\]

with \( f \in L^2_{\text{loc}} (\mathbb{R}^n) \) and \( \alpha \in (-\infty, \frac{1}{2}) \); see [19, Lemma 5.7] and also Lemma 3.11. Here and hereafter, for any \( k \in \mathbb{Z}_+ := \{0, 1, \cdots \} \) and any cube \( Q \) of \( \mathbb{R}^n \), \( \mathcal{D}_k(Q) \) denotes the dyadic cubes contained in \( Q \) of level \( k \).

In what follows, for any \( z \in \mathbb{R}^n \) and \( \ell \in (0, \infty) \), we use \( Q(z, \ell) \) to denote the cube centered at \( z \) with the edge length \( \ell \); for any cube \( Q \) of \( \mathbb{R}^n \) and any \( \lambda \in (0, \infty) \), we use \( \lambda Q \) to denote the cube with edge length \( \lambda \ell(Q) \) and the same center as \( Q \); moreover, we use \( A \) to denote the cardinality of the set \( A \).

Proposition 2.5 Let \( p \in [1, \infty) \) and \( \Phi \) be an almost increasing set function defined on cubes (or balls) of \( \mathbb{R}^n \) as in Definition 2.3. Then

\[
\sup_{\ell \in (0, \infty)} \left\{ \sum_{j} |Q_j| [\Phi(Q_j)]^p \right\}^{\frac{1}{p}} \sim \sup_{\ell \in (0, \infty)} \left\{ \int_{\mathbb{R}^n} [\Phi(Q(z, \ell))]^p \, dz \right\}^{\frac{1}{p}}
\]

\[
\sim \sup_{\ell \in (0, \infty)} \left\{ \sum_{j} |Q_j| [\Phi(Q_j)]^p \right\}^{\frac{1}{p}}, \tag{2.2}
\]

with the positive equivalence constants depending only on \( n \) and \( \Phi \). Moreover, (2.2) also holds with \( Q(z, \ell) \) replaced by \( B(z, \ell) := \{ x \in \mathbb{R}^n : |x - z| < \ell \} \).
Proof. Let \( p \) and \( \Phi \) be as in the present proposition. First, we show that

\[
\sup_{\ell \in (0,\infty)} \left\{ \int_{\mathbb{R}^n} |\Phi(Q(z, \ell))|^p \, dz \right\}^{\frac{1}{p}} \lesssim \sup_{\ell \in (0,\infty)} \left\{ \sum_{j \in \Pi_k} |Q_j| \left| \Phi(Q_j) \right|^p \right\}^{\frac{1}{p}}.
\]  

(2.3)

For any \( \ell \in (0,\infty) \), let \( m_\ell \in \mathbb{Z} \) satisfy \( \ell \in (2^{-m_\ell-1}, 2^{-m_\ell}) \), and, for any \( j \in \mathbb{Z}^n \), let \( Q_{\ell,j} := 2^{-m_\ell} j + [0,2^{-m_\ell})^n \). Then, for any \( \ell \in (0,\infty) \) and \( j \in \mathbb{Z}^n \), by some geometrical observations, we conclude that, for any \( z \in Q_{\ell,j} \),

\[ Q(z, \ell) \subset 2Q_{\ell,j} \quad \text{and} \quad |Q(z, \ell)| \leq 2^{-n}|2Q_{\ell,j}|, \]

which, together with (2.1), further implies that

\[ \Phi(Q(z, \ell)) \lesssim \Phi(2Q_{\ell,j}). \]  

(2.4)

Let \( V := \{ i = (i_1, \ldots, i_n) : i_1, \ldots, i_n \in \{0,1\} \} \) and \( 2\mathbb{Z}^n := \{ 2j : j \in \mathbb{Z}^n \} \). Then, for any \( i \in V \) and \( \ell \in (0,\infty) \), using some geometrical observations again, we find that

\[ \{2Q_{\ell,(j+i)}\}_{j+2\mathbb{Z}^n} \subset \Pi_{2^{-m_\ell+1}} \]  

(2.5)

and

\[ \mathbb{R}^n = \bigcup_{i \in V} \bigcup_{j \in 2\mathbb{Z}^n} Q_{\ell,(j+i)}. \]  

(2.6)

From (2.6), (2.4), (2.5), and the fact that \( \sharp V = 2^n \), it follows that, for any \( \ell \in (0,\infty) \),

\[
\begin{align*}
\int_{\mathbb{R}^n} |\Phi(Q(z, \ell))|^p \, dz &= \sum_{i \in V} \sum_{j \in 2\mathbb{Z}^n} \int_{Q_{\ell,(j+i)}} |\Phi(Q(z, \ell))|^p \, dz \\
&\lesssim \sum_{i \in V} \sum_{j \in 2\mathbb{Z}^n} \int_{2Q_{\ell,(j+i)}} |\Phi(2Q_{\ell,(j+i)})|^p \, dz \\
&\sim \sum_{i \in V} \sum_{j \in 2\mathbb{Z}^n} |2Q_{\ell,(j+i)}| \left| \Phi(2Q_{\ell,(j+i)}) \right|^p \\
&\lesssim \sharp V \sup_{j \in 2\mathbb{Z}^n} \left\{ \sum_{j} |Q_j| \left| \Phi(Q_j) \right|^p \right\} \\
&\sim \sup_{(Q_j), j \in \Pi_k} \left\{ \sum_{j} |Q_j| \left| \Phi(Q_j) \right|^p \right\}.
\end{align*}
\]

with the implicit positive constants independent of \( \ell \). Taking the supremum over \( \ell \in (0,\infty) \), we find that (2.3) holds.

Next, we prove that

\[
\sup_{\ell \in (0,\infty)} \left\{ \sum_{j} |Q_j| \left| \Phi(Q_j) \right|^p \right\}^{\frac{1}{p}} \lesssim \sup_{\ell \in (0,\infty)} \left\{ \int_{\mathbb{R}^n} |\Phi(Q(z, \ell))|^p \, dz \right\}^{\frac{1}{p}}.
\]  

(2.7)

Let \( \{Q_j\} \in \Pi_\ell \) with \( \ell \in (0,\infty) \). Observe that, for any \( z \in Q_j \),

\[ Q_j \subset Q(z, 2\ell) \quad \text{and} \quad |Q_j| = 2^{-n}|Q(z, 2\ell)|, \]

which, combined with (2.1), implies that

\[ \Phi(Q_j) \lesssim \Phi(Q(z, 2\ell)). \]

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and hence,
\[
\sum_j |Q_j| \left| \Phi(Q_j) \right|^p = \sum_j \int_{Q_j} \left| \Phi(Q_j) \right|^p \, dz \lesssim \sum_j \int_{Q_j} \left| \Phi(Q(z, 2\ell)) \right|^p \, dz
\]
\[
= \int_{\mathbb{R}^n} \left| \Phi(Q(z, 2\ell)) \right|^p \, dz.
\]
This implies (2.7), and hence (2.2) holds.

Finally, since balls and cubes are mutually comparable, if \( Q(z, \ell) \) in (2.2) is replaced by \( B_\ell(z) \), then the conclusion still holds true by repeating the above argument; we omit the details here. This finishes the proof of Proposition 2.5.

**Corollary 2.6** Let \( p, q \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). Then \( f \in JNQ_{p,q}^\alpha(\mathbb{R}^n) \) if and only if \( f \) is measurable on \( \mathbb{R}^n \) and
\[
\| f \|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} := \sup_{r \in (0, \infty)} \left\{ \int_{\mathbb{R}^n} r^{q\alpha-n} \int_{B(z,r)} \int_{B(z,r)} \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha q}} \, dx \, dy \, dz \right\}^{1/\gamma} < \infty.
\]
Moreover, \( \| \cdot \|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} \sim \| \cdot \|_{\tilde{JNQ}_{p,q}^\alpha(\mathbb{R}^n)} \).

**Proof** Let \( p, q, \alpha, \) and \( f \) be as in the present corollary. For any cube (or ball) \( Q \) of \( \mathbb{R}^n \), let \( \Phi_{f,q,\alpha}(Q) \) be the same as in (1.2). We claim that \( \Phi_{f,q,\alpha} \) is almost increasing as in Definition 2.3. Indeed, for any cubes (or balls) \( Q_1, Q_2 \) satisfying \( Q_1 \subset Q_2 \) with \( |Q_1| = C_1 |Q_2| \), by the properties of Lebesgue integral, we have that
\[
\Phi_{f,q,\alpha}(Q_1) = |Q_1|^{\frac{q}{n}} \int_{Q_1} \int_{Q_1} \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha q}} \, dx \, dy \leq C_1^{\frac{q}{n}} |Q_2|^{\frac{q}{n}} \int_{Q_2} \int_{Q_2} \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha q}} \, dx \, dy = C_1^{\frac{q}{n}} \Phi_{f,q,\alpha}(Q_2),
\]
and hence \( \Phi_{f,q,\alpha} \) is almost increasing, as in Definition 2.3. From this claim and Proposition 2.5, it follows that \( \| \cdot \|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} \sim \| \cdot \|_{\tilde{JNQ}_{p,q}^\alpha(\mathbb{R}^n)} \), which completes the proof of Corollary 2.6.

The basic properties of \( JNQ \) spaces are presented as follows (these are fine counterparts of the corresponding properties of \( Q \) spaces in [19, Theorem 2.3]):

**Theorem 2.7** Let \( p, q \in [1, \infty) \). Then we have the following:
(i) (Decreasing in \( \alpha \)) If \( -\infty < \alpha_1 \leq \alpha_2 < \infty \), then \( JNQ_{p,q}^{\alpha_1}(\mathbb{R}^n) \supset JNQ_{p,q}^{\alpha_2}(\mathbb{R}^n) \);
(ii) (Triviality for large \( \alpha \)) If \( \alpha \in \mathbb{R} \) satisfies \( \alpha > n(\frac{1}{q} - \frac{1}{p}) \) or \( \alpha \geq 1 \), then \( JNQ_{p,q}^{\alpha}(\mathbb{R}^n) = \{0\} \);
(iii) (Triviality for negative \( \alpha \)) If \( \alpha \in (-\infty, 0) \), then \( JNQ_{p,q}^{\alpha}(\mathbb{R}^n) = JNQ_{p,q}^{\alpha}(\mathbb{R}^n) \) with equivalent norms.

**Proof** Let \( p, q \in [1, \infty) \). We first show (i). Let \( \alpha_1, \alpha_2 \in \mathbb{R} \) with \( \alpha_1 \leq \alpha_2 \). Then, for any \( f \in JNQ_{p,q}^{\alpha_2}(\mathbb{R}^n) \) and any ball \( B \) of \( \mathbb{R}^n \), we have that
\[
|B|^{\frac{q}{n} - 1} \int_B \int_B \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha_1 q}} \, dx \, dy = |B|^{\frac{q}{n} - 1} \int_B \int_B \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha_1 q}} \, dx \, dy \lesssim |B|^{\frac{q}{n} + \frac{\alpha_2 - \alpha_1}{n}} \int_B \int_B \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha_2 q}} \, dx \, dy \sim |B|^{\frac{q}{n} - 1} \int_B \int_B \frac{|f(x) - f(y)|^q}{|x - y|^{n+\alpha_2 q}} \, dx \, dy.
\]
On the other hand, if consider the following two cases on prove (iii), it suffices to show that
This finishes the proof of (ii).

We next show (ii). Let \( f \in JNQ_{p,q}^\alpha(\mathbb{R}^n) \). By Definition 1.2, we conclude that, for any given cube \( Q \) of \( \mathbb{R}^n \) with edge length \( \ell(Q) \in (0,\infty) \),

\[
\left[ \int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} \, dx \, dy \right]^{\frac{1}{q}} \leq |Q|^\frac{1}{q} + \frac{\alpha}{n} \|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)}, \tag{2.8}
\]

Letting \( |Q| \to \infty \) in (2.8), it follows that \( f \) is a constant almost everywhere on \( \mathbb{R}^n \) if \( \alpha > n(\frac{1}{q} - \frac{1}{p}) \).

On the other hand, if \( \alpha \geq 1 \), then

\[
\int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} \, dx \, dy \gtrsim \int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n+q} |f(Q)|^q (\alpha-1)} \, dx \, dy,
\]

which, combined with (2.8), further implies that

\[
\int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n+q}} \, dx \, dy \lesssim |Q|^{1-\frac{n}{q} - \frac{\alpha}{n}} \|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)},
\]

and hence

\[
\inf_{(x,y)\in Q \times Q} \left[ \frac{|f(x) - f(y)|}{|x - y|} \right]^q \int_Q \int_Q \frac{\, dx \, dy}{|x - y|^n} \leq \int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n+q}} \, dx \, dy \lesssim |Q|^{1-\frac{n}{q} - \frac{\alpha}{n}} \|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} < \infty.
\]

From this and the observation that

\[
\int_Q \int_Q \frac{\, dx \, dy}{|x - y|^n} = \infty,
\]

we deduce that

\[
\inf_{(x,y)\in Q \times Q} \left[ \frac{|f(x) - f(y)|}{|x - y|} \right]^q = 0
\]

for any cube \( Q \) of \( \mathbb{R}^n \), which further implies that \( f \) is a constant almost everywhere on \( \mathbb{R}^n \).

This finishes the proof of (ii).

Now, we show (iii). Recall that \( JNQ_{p,q}^\alpha(\mathbb{R}^n) = JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \); see Remark 2.2. Thus, to prove (iii), it suffices to show that \( JNQ_{p,q}^\alpha(\mathbb{R}^n) = JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \) for any \( \alpha \in (-\infty,0) \), and we consider the following two cases on \( \alpha \):

Case 1) \( \alpha \in (-\infty,-\frac{2}{q}] \). In this case, by (i), we conclude that

\[
JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \subset JNQ_{p,q}^\alpha(\mathbb{R}^n).
\]

Conversely, let \( f \in JNQ_{p,q}^\alpha(\mathbb{R}^n) \) and let \( B \) be a ball of \( \mathbb{R}^n \) with radius \( r_B \in (0,\infty) \). Observe that, for any \( x, y \in B \),

\[
\left| \{ z \in B : \min\{|x - z|, |y - z|\} > 2^{-1} r_B \} \right| = \left| B \setminus \left[ B(x, 2^{-1} r_B) \cup B(y, 2^{-1} r_B) \right] \right| \geq \frac{1}{2} |B|.
\]

From this and \( -q\alpha - n \geq 0 \), we deduce that

\[
\int_B \min\{|x - z|^{-q\alpha - n}, |y - z|^{-q\alpha - n}\} \, dz
\]

\[
= \int_B \min\{|x - z|, |y - z|\}^{-q\alpha - n} \, dz
\]

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\[ \geq \int_{\{z \in B : \min \{|x-z|, |y-z|\} > 2^{-1} r_B\}} [\min \{|x-z|, |y-z|\}]^{-q_0 \cdot n} \, dz \]

and hence,

\[
\int_B \int_B |f(x) - f(y)|^q \, dx \, dy 
\leq |B|^{\frac{p}{q} - 1} \int_B \int_B \min \{|x-z|^{-q_0 \cdot n}, |y-z|^{-q_0 \cdot n}\} \, |f(x) - f(y)|^q \, dx \, dy 
\leq |B|^{\frac{p}{q} - 1} \int_B \int_B \min \{|x-z|^{-q_0 \cdot n}, |y-z|^{-q_0 \cdot n}\} \times |\min \{|x-z|^{-q_0 \cdot n}, |y-z|^{-q_0 \cdot n}\}| \, |f(x) - f(y)|^q \, dx \, dy 
\leq |B|^{\frac{p}{q} - 1} \int_B \int_B \frac{|f(x) - f(z)|^q}{|x-z|^{n+q_0}} \, dx \, dy + |B|^{\frac{p}{q} - 1} \int_B \int_B \frac{|f(y) - f(z)|^q}{|y-z|^{n+q_0}} \, dy \, dx 
\sim |B|^{\frac{p}{q} - 1} \int_B \int_B \frac{|f(x) - f(y)|^q}{|x-y|^{n+q_0}} \, dx \, dy. \tag{2.9}
\]

Then (2.9) implies that

\[ \|f\|_{JNQ_{p,q}^{-n/q}(\mathbb{R}^n)} \lesssim \|f\|_{JNQ_{p,q}^n(\mathbb{R}^n)}, \]

and hence \( JNQ_{p,q}^\alpha(\mathbb{R}^n) \subseteq JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \). This shows that \( JNQ_{p,q}^\alpha(\mathbb{R}^n) = JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \) when \( \alpha \in (-\frac{n}{q}, 0) \).

Case 2) \( \alpha \in [-\frac{n}{q}, 0) \). In this case, by (i), we conclude that

\[ JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \supset JNQ_{p,q}^\alpha(\mathbb{R}^n). \]

Conversely, let \( z \in \mathbb{R}^n \), \( r \in (0, \infty) \), and let \( f \in JNQ_{p,q}^{-n/q}(\mathbb{R}^n) \). By the Tonelli theorem, we have that

\[
\int_{B(z,r)} \int_{B(z,r)} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q_0}} \, dx \, dy = \int_{B(z,r)} \int_{B(z-r, y)} \frac{|f(x+y) - f(y)|^q}{|x-y|^{n+q_0}} \, dx \, dy 
\leq \int_{B(z,r)} \int_{B(z-r, y)} \frac{|f(x-y)|^q}{|x-y|^{n+q_0}} \, dx \, dy 
= \int_{B(0,2r)} \int_{B(z,r)} \frac{|f(x+y) - f(y)|^q}{|x-y|^{n+q_0}} \, dy \, dx 
\leq \int_{B(0,2r)} \int_{B(z,r)} \frac{1}{|x|^{n+q_0}} \int_{B(z, r)} |f(x+y) - f_B(z, r)|^q \, dy \, dx 
\leq \int_{B(0,2r)} \int_{B(z,r)} \frac{1}{|x|^{n+q_0}} \int_{B(z,3r)} |f_B(z, r) - f(y)|^q \, dy \, dx 
\leq \int_{B(0,2r)} \int_{B(z,r)} \frac{1}{|x|^{n+q_0}} |f(y) - f_B(z, r)|^q \, dy \, dx 
\sim r^{-q_0} \int_{B(z,3r)} |f(y) - f_B(z, r)|^q \, dy,
\]

where the implicit positive constants are independent of \( r \). This, together with Corollary 2.6, shows that

\[
\|f\|_{JNQ_{p,q}^\alpha(\mathbb{R}^n)} \sim \sup_{r \in (0, \infty)} \left[ \int_{\mathbb{R}^n} \left( \int_{B(z,r)} \int_{B(z,r)} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q_0}} \, dx \, dy \right)^{\frac{p}{q}} \, dz \right]^{\frac{q}{p}}.
\]
Then, by the arbitrariness of \[Q\] and since (ii) and (iii) of Theorem 2.7, we conclude that the observation that \(\{p, q\} \in (1, \infty)\) as a generalization of the John-Nirenberg space shows that \(J N Q_{p, q}^n(\mathbb{R}^n)\) as well, which completes the proof of Corollary 2.8.

**Remark 2.9** Corollary 2.8 partially answers the open question posed in [33, Remark 4.2(ii)] (see also [31, Question 1(ii)]) for the case \(q > p\), but it is still unclear for the case \(q = p\).

### 2.2 Relations with \(Q\) Spaces

The \(J N Q\) space is closely connected with the \(Q\) space. Indeed, the following proposition shows that \(Q_\alpha(\mathbb{R}^n)\) serves as a limit space of \(J N Q_{p, 2}^n(\mathbb{R}^n)\) when \(p \to \infty\), and hence it is reasonable to define \(J N Q_{\infty, 2}^n(\mathbb{R}^n) := Q_\alpha(\mathbb{R}^n)\) in Remark 1.3(ii).

**Proposition 2.10** Let \(p \in [1, \infty)\) and \(\alpha \in \mathbb{R}\). Then, for any \(f \in \bigcup_{r \in [1, \infty)} \bigcap_{p \in [r, \infty)} J N Q_{p, 2}^n(\mathbb{R}^n)\), it holds that

\[
\lim_{p \to \infty} \|f\|_{J N Q_{p, 2}^n(\mathbb{R}^n)} = \|f\|_{Q_\alpha(\mathbb{R}^n)}.
\]

**Proof** Let \(p \in [1, \infty)\), \(\alpha \in \mathbb{R}\), and \(f\) be a measurable function on \(\mathbb{R}^n\). Then, for any given cube \(Q\) of \(\mathbb{R}^n\), by Definition 1.2, we have that

\[
\|f\|_{J N Q_{p, 2}^n(\mathbb{R}^n)} \geq |Q|^\frac{1}{p} \left[ |Q|^{\frac{2\alpha}{n} - 1} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n + 2\alpha}} dy dx \right]^\frac{1}{p}.
\]

Since \(|Q|\frac{1}{p} \to 1\) as \(p \to \infty\), it follows that

\[
\lim_{p \to \infty} \|f\|_{J N Q_{p, 2}^n(\mathbb{R}^n)} \geq \left[ |Q|^{\frac{2\alpha}{n} - 1} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x - y|^{n + 2\alpha}} dy dx \right]^\frac{1}{p}.
\]

Then, by the arbitrariness of \(Q\), we find that

\[
\lim_{p \to \infty} \|f\|_{J N Q_{p, 2}^n(\mathbb{R}^n)} \geq \|f\|_{Q_\alpha(\mathbb{R}^n)}.
\]
On the other hand, let \( f \in \bigcup_{r \in [1, \infty)} \bigcap_{p \in [r, \infty)} JNQ_{p,2}^\alpha(\mathbb{R}^n) \). Then there exists an \( r_0 \in [1, \infty) \) such that \( f \in JNQ_{p,2}^\alpha(\mathbb{R}^n) \) for any \( p \in [r_0, \infty) \). We claim that

\[
\limsup_{p \to \infty} \|f\|_{JNQ_{p,2}^\alpha(\mathbb{R}^n)} \leq \|f\|_{Q_\alpha(\mathbb{R}^n)}.
\] (2.10)

Indeed, if \( \|f\|_{Q_\alpha(\mathbb{R}^n)} = \infty \), then (2.10) trivially holds. If \( \|f\|_{Q_\alpha(\mathbb{R}^n)} < \infty \), then we may assume, without loss of generality, that \( \|f\|_{Q_\alpha(\mathbb{R}^n)} = 1 \). Thus, for any cube \( Q \subseteq \mathbb{R}^n \),

\[
|Q|^{2\alpha-1} \int_Q \int_Q \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}}dydx \leq \|f\|_{Q_\alpha(\mathbb{R}^n)} = 1,
\]

and hence, for any \( p \in [r_0, \infty) \), we have that

\[
\|f\|_{JNQ_{p,2}^\alpha(\mathbb{R}^n)} \leq \sup_{Q_{j} \in N} \left\{ \sum_j |Q_j| \left[ |Q_j|^{2\alpha-1} \int_{Q_j} \int_{Q_j} |f(x) - f(y)|^2 \frac{dydx}{|x-y|^{n+2\alpha}} \right]^{\frac{p}{p-2\alpha}} \right\}^{\frac{p-2\alpha}{p}}.
\]

Letting \( p \to \infty \), we obtain that

\[
\limsup_{p \to \infty} \|f\|_{JNQ_{p,2}^\alpha(\mathbb{R}^n)} \leq 1 = \|f\|_{Q_\alpha(\mathbb{R}^n)}.
\]

This finishes the proof of (2.10), and hence of Proposition 2.10.

Next, we show some extension properties of \( JNQ \) spaces. Recall that we can extend a measurable function \( f \) via a fundamental invariance principle: for any \((x, t) \in \mathbb{R}^{n+1}\),

\[
F(x, t) := f(x).
\] (2.11)

It is easy to show that \( \|F\|_{L^\infty(\mathbb{R}^{n+1})} = \|f\|_{L^\infty(\mathbb{R}^n)} \). Moreover, by [19, Theorem 2.6], we also have that

\[
\|F\|_{Q_\alpha(\mathbb{R}^{n+1})} \sim \|f\|_{Q_\alpha(\mathbb{R}^n)}
\]

for any \( \alpha \in \mathbb{R} \), where the implicit equivalence constants are independent of \( f \) and \( F \). Correspondingly, as in [13, Proposition 4.1] and [20, Lemma 2.18], we next show that such an extension also holds for the \( JNQ \) space on cubes. In what follows, for any \( p, q \in [1, \infty) \), any \( \alpha \in \mathbb{R} \), any cube \( Q_0 \subset \mathbb{R}^n \), and any measurable function \( f \) on \( Q_0 \), let

\[
\|f\|_{JNQ_{p,q}^\alpha(Q_0)} := \sup \left\{ \sum_j |Q_j| \left[ |Q_j| \Phi_{f,q,\alpha}(Q_j)^q \right] \right\}^{\frac{1}{q}},
\]

where \( \Phi_{f,q,\alpha}(Q_j) \) is as in (1.2) with \( Q \) replaced by \( Q_j \), and where the supremum is taken over all collections \( \{Q_j\}_j \) of subcubes of \( Q_0 \) with pairwise disjoint interiors. Moreover, the space \( JNQ_{p,q}^\alpha(Q_0) \) is defined by setting

\[
JNQ_{p,q}^\alpha(Q_0) := \left\{ f \text{ is measurable on } Q_0 : \|f\|_{JNQ_{p,q}^\alpha(Q_0)} < \infty \right\}.
\]

**Proposition 2.11** Let \( p, q \in [1, \infty) \), \( \alpha \in \mathbb{R} \), \( Q_0 \) be a cube of \( \mathbb{R}^n \) with edge length \( \ell_0 \), and let \( \tilde{Q}_0 := Q_0 \times [t_0, t_0 + \ell_0] \) for any given \( t_0 \in \mathbb{R} \). Let \( f \) be a measurable function on \( Q_0 \), and let \( F(x, t) := f(x) \) for any \((x, t) \in \tilde{Q}_0\). Then \( F \) is a measurable function on \( \tilde{Q}_0 \), and

\[
F \in JNQ_{p,q}^\alpha(\tilde{Q}_0) \iff f \in JNQ_{p,q}^\alpha(Q_0).
\] (2.12)
Moreover,

\[ \|F\|_{J N Q_p,q(\mathbb{R}^n)} \sim \ell_0^{1/p} \|f\|_{J N Q_p,q(\mathbb{R}^n)}, \]

with the positive equivalence constants independent of \( f, F, \) and \( Q_0. \)

**Proof** Let all symbols be as in the present proposition. We claim that, if \( \alpha > -\frac{n}{q}, \) then, for any \( x, y \in \mathbb{R}^n \) with \( |x - y| \leq \ell \) for some \( \ell \in (0, \infty), \)

\[ \int_{t_0}^{t_0+\ell} \int_{t_0}^{t_0+\ell} \frac{dtdu}{|(x - y, t - u)|^{n+1+q\alpha}} \sim \ell|x-y|^{-n-q\alpha}. \tag{2.13} \]

Indeed, on one hand, by \( \alpha > -\frac{n}{q}, \) we have that

\[ \int_{t_0}^{t_0+\ell} \int_{t_0}^{t_0+\ell} \frac{dtdu}{|x - y, t - u|^{n+1+q\alpha}} \sim \ell \int_0^\infty ds \int_{|x-y|}^{\infty} ds \frac{1}{s^{n+1+q\alpha}} \sim \ell|x-y|^{-n-q\alpha}. \tag{2.14} \]

On the other hand, from \( |x - y| \leq \ell, \) it follows that

\[ \int_{t_0}^{t_0+\ell} \int_{t_0}^{t_0+\ell} \frac{dtdu}{|x - y, t - u|^{n+1+q\alpha}} \sim \int_{t_0}^{t_0+\ell} \int_{t_0}^{t_0+\ell} \frac{dtdu}{|x - y|^{n+1+q\alpha} + |t - u|^{n+1+q\alpha}} \sim \int_{t_0}^{t_0+\ell} \int_{t_0}^{t_0+\ell} \frac{dtdu}{|x - y|^{n+1+q\alpha} + |t - u|^{n+1+q\alpha}} \sim \ell|x-y|^{-n-q\alpha}. \tag{2.15} \]

Combining (2.14) and (2.15), we obtain (2.13), and hence the above claim holds.

Now, we show (2.12). By Theorem 2.7(iii), we only need to prove (2.12) with \( \alpha \in (-\frac{n}{q}, \infty). \)

Let \( \{\tilde{Q}_i\}_{i=1} \) be any given collection of subcubes of \( Q_0 \) with pairwise disjoint interiors and equal edge length \( \ell_i. \) Then, from the Tonelli Theorem and (2.13), it follows that

\[ \sum_{i} |\tilde{Q}_i| \left[ \ell_1^{\alpha_n-(n+1)} \int_{Q_i} \int_{Q_i} \frac{|F(x, t) - F(y, u)|^q}{|x - y|^{n+q\alpha}} dtdy \right]^p, \]

\[ \sum_{i} |\tilde{Q}_i| \left[ \ell_1^{\alpha_n-(n+1)} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} dtdy \right]^p, \]

\[ \sum_{i} |Q_i| \ell_1^{\alpha-n} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} dtdy, \]

\[ \sum_{i} |Q_i| \int_{0}^{t_0} 1_{I_i}(t) dt \left[ \ell_1^{\alpha_n-n} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} dtdy \right]^p. \tag{2.16} \]

Notice that, for any given \( t \in (0, t_0), \) \( \{Q_i : I_i \ni t\} \) is a collection of subcubes of \( Q_0 \) with pairwise disjoint interiors and equal edge length \( \ell_i, \) which implies that

\[ \sum_{i : I_i \ni t} |Q_i| \left[ \ell_1^{\alpha-n} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^q}{|x - y|^{n+q\alpha}} dtdy \right]^p \leq \|f\|_{J N Q_p,q(\mathbb{R}^n)}^p. \]

\[ \frac{1}{\|f\|_{J N Q_p,q(\mathbb{R}^n)}^p} \]
By this and (2.16), we obtain that
\[
\sum_{i} |Q_i| \left[ \ell_1^{(n+1)-q} \int_{Q_i} \int_{Q_i} \frac{|F(x,t) - F(y,u)|^q}{|x-y|^{n+q}} \, dx \, dy \, dt \, du \right]^p \leq \int_0^{\ell_0} \|f\|_{JNQ^p_{p,q}(Q_0)}^p \, dt \sim \ell_0^p \|f\|_{JNQ^p_{p,q}(Q_0)}^p.
\]
From this and the arbitrariness of \(\{Q_i\}_i\), it follows that
\[
\|F\|_{JNQ^p_{p,q}(Q_0)}^p \leq \ell_0^{1/p} \|f\|_{JNQ^p_{p,q}(Q_0)}^p, \tag{2.17}
\]
Conversely, let \(\{Q_i\}_i\) be any given collection of subcubes of \(Q_0\) with pairwise disjoint interiors and equal edge length \(\ell_1\). Also, for any \(j \in \{1, \cdots, J\}\) with \(J\) being the largest integer not greater than \(\ell_0/\ell_1\), let
\[
Q_{i,j} := Q_j \times [t_0 + (j-1)\ell_1, t_0 + j\ell_1].
\]
Then it is obvious that \(\{Q_{i,j}\}_{i,j}\) is a collection of subcubes of \(Q_0\) with pairwise disjoint interiors and equal edge length \(\ell_1\), and that \(J\ell_1 \geq \ell_0/2\). By this and (2.13), we have that
\[
\|F\|_{JNQ^p_{p,q}(Q_0)}^p \geq \sum_{i,j} |Q_{i,j}| \left[ \ell_1^{q(n+1)} \int_{Q_{i,j}} \int_{Q_{i,j}} \frac{|F(x,t) - F(y,u)|^q}{|x-y|^{n+q}} \, dx \, dy \, dt \, du \right]^p 
\sim \sum_{j=1}^{J} \sum_{i} |Q_{i,j}| \ell_1 \left[ \ell_1^{q(n+1)} \int_{Q_{i,j}} \int_{Q_{i,j}} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q}} \, dx \, dy \right]^p 
\geq \frac{\ell_0}{2} \sum_{i} |Q_i| \left[ \ell_1^{q(n+1)} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q}} \, dx \, dy \right]^p.
\]
From this and the arbitrariness of \(\{Q_i\}_i\), it follows that
\[
\|F\|_{JNQ^p_{p,q}(Q_0)}^p \geq \ell_0^{1/p} \|f\|_{JNQ^p_{p,q}(Q_0)},
\]
which, combined with (2.17), further shows that
\[
\|F\|_{JNQ^p_{p,q}(Q_0)} \sim \ell_0^{1/p} \|f\|_{JNQ^p_{p,q}(Q_0)}.
\]
This finishes the proof of Proposition 2.11. \(\square\)

**Remark 2.12** The factor \(\ell_0^{1/p}\) in Proposition 2.11 indicates that the invariance principle (2.11) is no longer feasible (unless \(p = \infty\)) for the extension from \(JNQ^\alpha_{p,q}(\mathbb{R}^n)\) to \(JNQ^\alpha_{p,q}(\mathbb{R}^{n+1})\).

### 2.3 Relations with (Fractional) Sobolev Spaces

Now, we show the relations between \(JNQ\) spaces and (fractional) Sobolev spaces.

Recall that the fractional Sobolev space \(W^{s,p}(\mathbb{R}^n)\) with \(p \in [1, \infty)\) and \(s \in \mathbb{R}\) is defined to be the set of all measurable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{W^{s,p}(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right]^{1/p} < \infty.
\]
When \(s \in (0, 1)\), \(\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}\) is usually called the Gagliardo semi-norm. It is well known that, for any \(s \in [1, \infty)\), \(W^{s,p}(\mathbb{R}^n)\) contains only the functions that are almost everywhere constant; see, for instance, [8, 9] and the references therein. Also, the same triviality holds for \(s \in (-\infty, 0]\).

**Proposition 2.13** Let \(p \in [1, \infty)\) and \(s \in (-\infty, 0]\). Then \(W^{s,p}(\mathbb{R}^n) = \{0\}\).
Proposition 2.13 might be well known, however, we did not find a complete proof in the existing literature. For the convenience of the reader, we present the details here. To prove Proposition 2.13, we establish the next lemma. In what follows, for any \( x, h \in \mathbb{R}^n \) and \( t \in (0, \infty) \), let

\[
B_t^+ := \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| \leq t \text{ and } x_j \geq 0 \text{ for any } j \in \{1, \cdots, n\} \},
\]

\[
\Delta_h f(x) := f(x + h) - f(x), \quad \omega(f, t) := \sup_{h \in B_t^+} \|\Delta_h f\|_{L^p(\mathbb{R}^n)},
\]

and

\[
\omega^*(f, t) := \frac{1}{t^n} \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_h f\|_{L^p(\mathbb{R}^n)}^p dh.
\]

Lemma 2.14 Let \( p \in [1, \infty) \) and \( f \) be a measurable function on \( \mathbb{R}^n \). Then there exists a positive constant \( C \), depending only on \( n \) and \( p \), such that, for any \( t \in (0, \infty) \),

\[
\omega(f, t) \leq C \omega^*(f, t).
\]

Proof Let \( p \) and \( f \) be as in the present lemma. Notice that, for any \( h, x, s \in \mathbb{R}^n \),

\[
\Delta_h f(x) = f(x + h) - f(x)
\]

\[
\quad = [f(x + h + s) - f(x)] - [f(x + h + s) - f(x + h)]
\]

\[
\quad = \Delta_{h+s} f(x) - \Delta_s f(x + h),
\]

which implies that

\[
|\Delta_h f(x)| \leq |\Delta_{h+s} f(x)| + |\Delta_s f(x)|.
\]

Thus, for any \( h \in B_{t/2}^+ \) and \( s \in \mathbb{R}^n \), we have that

\[
\|\Delta_h f\|_{L^p(\mathbb{R}^n)}^p \leq \|\Delta_{h+s} f\|_{L^p(\mathbb{R}^n)}^p + \|\Delta_s f\|_{L^p(\mathbb{R}^n)}^p,
\]

which implies that

\[
\int_{B_{3t}^+ \setminus B_t^+} \|\Delta_h f\|_{L^p(\mathbb{R}^n)}^p ds \leq \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_{h+s} f\|_{L^p(\mathbb{R}^n)}^p ds + \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_s f\|_{L^p(\mathbb{R}^n)}^p ds,
\]

and hence,

\[
\|\Delta_h f\|_{L^p(\mathbb{R}^n)}^p \leq t^{-n} \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_{h+s} f\|_{L^p(\mathbb{R}^n)}^p ds + t^{-n} \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_s f\|_{L^p(\mathbb{R}^n)}^p ds
\]

\[
\leq t^{-n} \int_{B_{3t}^+ \setminus B_t^+} \|\Delta_s f\|_{L^p(\mathbb{R}^n)}^p ds \sim \omega^*(f, t). \tag{2.18}
\]

Moreover, observe that, for any \( t \in (0, \infty) \),

\[
\omega(f, t) = \sup_{h \in B_t^+} \int_{\mathbb{R}^n} |f(x + h) - f(x)|^p dx
\]

\[
\leq \sup_{h \in B_t^+} \left[ \int_{\mathbb{R}^n} \left| f(x + h) - f \left( x + \frac{h}{2} \right) \right|^p dx + \int_{\mathbb{R}^n} \left| f \left( x + \frac{h}{2} \right) - f(x) \right|^p dx \right]
\]

\[
\leq \sup_{h \in B_{t/2}^+} \int_{\mathbb{R}^n} \left| f(x + \tilde{h}) - f(x) \right|^p dx \sim \omega(f, \frac{t}{2}).
\]
By this and (2.18), we conclude that, for any $t \in (0, \infty)$,
\[
\omega(f, t) \lesssim \omega(f, \frac{t}{2}) \sim \sup_{h \in B_{t/2}^+} \|\Delta_h f\|_{L^p(R^n)}^p \lesssim \omega^*(f, t),
\]
which completes the proof of Lemma 2.14.

Next, we prove Proposition 2.13 via Lemma 2.14.

**Proof of Proposition 2.13** Let $p \in [1, \infty)$, $s \in (-\infty, 0]$, and $f \in W^{s, p}(R^n)$. Then
\[
\omega(f, t) \lesssim \omega(f, \frac{t}{2}) \sim \sup_{h \in B_{t/2}^+} \|\Delta_h f\|_{L^p(R^n)}^p \lesssim \omega^*(f, t),
\]
which implies that, for any $h \in B_{\infty}^+ := \{x = (x_1, \cdots, x_n) \in R^n : x_j \geq 0 \text{ for any } j \in \{1, \cdots, n\}\},
\[
0 = \|\Delta_h f\|_{L^p(R^n)}^p = \int_{R^n} |f(x + h) - f(x)|^p dx.
\]
Therefore,
\[
0 = \int_{B_{\infty}^+} \int_{R^n} |f(x + h) - f(x)|^p dx dh. \tag{2.19}
\]

Next, we consider the case $h \in R^n \setminus B_{\infty}^+$. For any $j \in \{1, \cdots, n\}$, let
\[
\tau_j : \begin{cases} \{x \in R^n \} & \rightarrow R^n, \\
(x_1, \cdots, x_j, \cdots, x_n) & \rightarrow (x_1, \cdots, -x_j, \cdots, x_n).
\end{cases}
\]
Notice that each octant $\tilde{B}_{\infty}^+$ of $R^n$ can be mapped onto $B_{\infty}^+$ via a composition of reflections; namely, there exists a $\tau_0 := \tau_{j_1} \circ \cdots \circ \tau_{j_k}$ with $\{j_1, \cdots, j_k\} \subset \{1, \cdots, n\}$ such that $\tau_0(B_{\infty}^+) = B_{\infty}^+$, which implies that
\[
\int_{B_{\infty}^+} \int_{R^n} |f(x + h) - f(x)|^p dx dh = \int_{B_{\infty}^+} \int_{R^n} |f(x + h) - f(x)|^p dx dh.
\]
Combining this with (2.19), we obtain that
\[
0 = \int_{R^n} \int_{R^n} |f(x + h) - f(x)|^p dx dh = \int_{R^n} \int_{R^n} |f(x) - f(y)|^p dx dy.
\]
This shows that $f$ is equal to some constant almost everywhere on $R^n$, which completes the proof of Proposition 2.13. \qed
Proposition 2.15  Let \( p, q \in [1, \infty) \) and \( \alpha_0 := n\left(\frac{1}{q} - \frac{1}{p}\right) \). Then,

(i) if \( \alpha_0 \in (0, \infty) \), \( JNQ^\alpha\gamma_p q(R^n) \supset W^{\alpha_0 \cdot q}(R^n) \) and \( \| \cdot \| JNQ^\alpha\gamma_{p q}(R^n) \leq \| \cdot \| W^{\alpha_0 \cdot q}(R^n) \);

(ii) if \( \alpha_0 = 0 \), \( JNQ^\alpha\gamma_p q(R^n) = W^{\alpha_0 \cdot q}(R^n) = \{0\} \).

Proof  We first prove (i). If \( \alpha_0 \in (0, \infty) \), then \( \frac{1}{q} - \frac{1}{p} > 0 \), and hence \( \frac{p}{q} > 1 \). Therefore, for any measurable function \( f \) on \( R^n \),

\[
\| f \| JNQ^\alpha\gamma_{p q}(R^n) = \sup_\ell \left\{ \sum_j \left[ \int_{Q_j} \int_{Q_j} \frac{|f(x) - f(y)|^q}{|x - y|^{n + \alpha_0}} dy dx \right]^{\frac{p}{q}} \right\}^{\frac{1}{p}} \\
\leq \sup_\ell \left\{ \sum_j \int_{Q_j} \int_{Q_j} \frac{|f(x) - f(y)|^q}{|x - y|^{n + \alpha_0}} dy dx \right\}^{\frac{1}{p}} \\
= \left[ \int_{R^n} \int_{R^n} \frac{|f(x) - f(y)|^q}{|x - y|^{n + \alpha_0}} dy dx \right]^{\frac{1}{p}} = \| f \| W^{\alpha_0 \cdot q}(R^n).
\]

This shows that \( W^{\alpha_0 \cdot q}(R^n) \subset JNQ^\alpha\gamma_{p q}(R^n) \) and \( \| \cdot \| JNQ^\alpha\gamma_{p q}(R^n) \leq \| \cdot \| W^{\alpha_0 \cdot q}(R^n) \).

Next, we prove (ii). If \( \alpha_0 = 0 \), then \( p = q \), and hence, for any measurable function \( f \) on \( R^n \),

\[
\| f \| JNQ^\alpha\gamma_{p q}(R^n) = \sup_\ell \left\{ \sum_j \int_{Q_j} \int_{Q_j} \frac{|f(x) - f(y)|^p}{|x - y|^n} dy dx \right\}^{\frac{1}{p}} \\
= \left[ \int_{R^n} \int_{R^n} \frac{|f(x) - f(y)|^p}{|x - y|^n} dy dx \right]^{\frac{1}{p}} = \| f \| W^{\alpha_0 \cdot q}(R^n).
\]

This shows that \( JNQ^\alpha\gamma_{p q}(R^n) = W^{\alpha_0 \cdot q}(R^n) \) which, together with Proposition 2.13, further implies that \( JNQ^\alpha\gamma_{p q}(R^n) = \{0\} \). This finishes the proof of Proposition 2.15.

Also, recall that the Sobolev space \( W^{1, \gamma}(R^n) \) with \( \gamma \in (0, \infty) \) is defined as the set of all weakly differentiable functions \( f \) on \( R^n \) such that

\[
\| f \| W^{1, \gamma}(R^n) := \left[ \int_{R^n} |\nabla f(x)|^{\gamma} dx \right]^{\frac{1}{\gamma}} < \infty.
\]

Borrowing some ideas from [34, p. 8, Theorem 1.4(ii)], we establish the following continuously embedding from \( W^{1, \gamma}(R^n) \) to \( JNQ^\alpha_{p q}(R^n) \).

Proposition 2.16  Let \( \alpha \in (-\infty, 1) \), \( p, q \in [1, \infty) \), and \( \gamma \in [q, \infty) \) with \( \frac{1}{p} = \frac{1}{q} + \frac{1}{\gamma} \). Then \( W^{1, \gamma}(R^n) \subset JNQ^\alpha_{p q}(R^n) \), and there exists a positive constant \( C \) such that, for any \( f \in W^{1, \gamma}(R^n) \),

\[
\| f \| JNQ^\alpha_{p q}(R^n) \leq C \| f \| W^{1, \gamma}(R^n).
\]

Proof  Let all of the symbols be as in the present proposition. We claim that, for any given cube \( Q \), with edge length \( \ell_0 \),

\[
\left[ \int_Q \int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{n + \alpha_0}} dxdy \right]^{\frac{1}{q}} \leq |Q|^{\frac{1 - \alpha_0}{\alpha}} \left[ \int_{(1 + 2^{\sqrt{\gamma}})Q} |\nabla f(w)|^\gamma dw \right]^{\frac{1}{\gamma}}.
\]  \hspace{1cm} (2.20)

Indeed, for any \( y, z \in R^n \), from the fundamental theorem of calculus, it follows that

\[
f(y + z) - f(y) = \int_0^1 z \cdot \nabla f(y + tz) dt,
\]

\(\square\) Springer
and hence,

\[ |f(y + z) - f(y)| \leq \int_0^1 |z| |\nabla f(y + tz)| \, dt. \]

By this, the Minkowski integral inequality, and \( \alpha \in (-\infty, 1) \), we conclude that

\[
\left[ \int_Q \int_{Q} \left| \frac{f(x) - f(y)}{|x - y|^{n+\alpha q}} \right|^q \, dx \, dy \right]^{\frac{1}{q}} \leq \left\{ \int_Q \int_{B(0, \sqrt{n} \bar{c}_o)} \left[ \left| \frac{f(x) - f(y)}{|x|^{\alpha q}} \right|^q \right] |z|^{q(1-\alpha) - n} \, dz \, dy \right\}^{\frac{1}{q}} \\
\leq \left\{ \int_Q \int_{B(0, \sqrt{n} \bar{c}_o)} \left[ \int_0^1 |\nabla f(y + tz)| \, dt \right]^q |z|^{q(1-\alpha) - n} \, dz \, dy \right\}^{\frac{1}{q}} \\
\leq \int_0^1 \left[ \int_Q \int_{B(0, \sqrt{n} \bar{c}_o)} |\nabla f(y + tz)|^q |z|^{q(1-\alpha) - n} \, dz \, dy \right]^{\frac{1}{q}} \, dt \\
\leq \left[ \int_{(1+2\sqrt{n})Q} |\nabla f(w)|^q \int_{B(0, \sqrt{n} \bar{c}_o)} |z|^{q(1-\alpha) - n} \, dz \, dw \right]^{\frac{1}{q}} \\
\sim |Q|^{\frac{q}{\alpha p + q}} \left[ \int_{(1+2\sqrt{n})Q} |\nabla f(w)|^q \, dw \right]^{\frac{1}{q}}.
\]

This shows that (2.20) holds, and hence we have that

\[
\left[ |Q|^{\frac{n-1}{p}} \int_Q \int \left| \frac{f(x) - f(y)}{|x - y|^{n+\alpha q}} \right|^q \, dx \, dy \right]^{\frac{1}{q}} \lesssim \left| \int_{(1+\sqrt{n})Q} |\nabla f(w)|^q \, dw \right|^{\frac{1}{q}}. \tag{2.21}
\]

Now, let \( \{Q_j\} \in \Pi_\ell \) with \( \ell \in (0, \infty) \). Then a geometrical observation shows that there exists a positive constant \( C_n \), depending only on \( n \), such that, for any \( x \in \mathbb{R}^n \),

\[
\sum_j 1_{(1+2\sqrt{n})Q_j}(x) \leq C_n. \tag{2.22}
\]

From (2.21), \( \frac{1}{p} + \frac{1}{n} = \frac{1}{q} \leq \frac{1}{\gamma} \), Hölder’s inequality, and (2.22), we deduce that

\[
\sum_j |Q_j| \left[ |Q_j|^{\frac{n-1}{p}} \int_Q \int_{Q_j} \left| \frac{f(x) - f(y)}{|x - y|^{n+\alpha q}} \right|^q \, dx \, dy \right]^{\frac{1}{q}} \\
\lesssim \sum_j \left[ \int_{(1+\sqrt{n})Q_j} |\nabla f(w)|^q \, dw \right]^{\frac{1}{q} + \frac{1}{p}} \\
\lesssim \sum_j \left[ \int_{(1+2\sqrt{n})Q_j} |\nabla f(w)|^{1/(1+\gamma)q} \, dw \right]^{p(\frac{1}{q} + \frac{1}{p})} \\
\lesssim \left[ \int_{\mathbb{R}^n} |\nabla f(w)|^{1/(1+\gamma)q} \, dw \right]^{p(\frac{1}{q} + \frac{1}{p})},
\]

where the implicit positive constants are independent of \( \ell \), \( \{Q_j\} \), and \( f \). This implies that

\[
\|f\|_{JNQ_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{W^{1,\gamma}(\mathbb{R}^n)},
\]

with \( \frac{1}{q} = \frac{1}{p} + \frac{1}{n} \), and hence finishes the proof of Proposition 2.16.

As a summary of the main results in this section, we have the following complete classification.

\[ \square \] Springer
Theorem 2.17 Let \( p, q \in [1, \infty), \alpha \in \mathbb{R}, \) and \( \alpha_0 := n(\frac{1}{q} - \frac{1}{p}) \). Then,

(i) if \( \alpha_0 < 0 \), \( JNQ^\alpha_{p,q}(\mathbb{R}^n) = \{0\} \);
(ii) if \( \alpha_0 = 0 \),
\[
JNQ^\alpha_{p,q}(\mathbb{R}^n) = \begin{cases} 
JN_{p,p}^{\infty}(\mathbb{R}^n), & \alpha \in (-\infty, 0), \\
\{0\}, & \alpha \in [0, \infty);
\end{cases}
\]
(iii) if \( \alpha_0 \in (0, 1) \),
\[
JNQ^\alpha_{p,q}(\mathbb{R}^n) = \begin{cases} 
JN_{p,q}^{\infty}(\mathbb{R}^n) \supset W^{\alpha_0,q}(\mathbb{R}^n), & \alpha \in [0, \alpha_0], \\
\{0\}, & \alpha \in (\alpha_0, \infty);
\end{cases}
\]
(iv) if \( \alpha_0 \in [1, \infty) \),
\[
JNQ^\alpha_{p,q}(\mathbb{R}^n) = \begin{cases} 
JN_{p,p}^{\infty}(\mathbb{R}^n), & \alpha \in (-\infty, 0), \\
W^{1,\frac{1}{p} + \frac{1}{n} - 1}(\mathbb{R}^n), & \alpha \in [0, 1), \\
\{0\}, & \alpha \in [1, \infty);\end{cases}
\]

Proof First, (i) immediately follows from Corollary 2.8 and Theorem 2.7(i).

Next, we prove (ii). Indeed, Theorem 2.7(iii) shows the case \( \alpha \in (-\infty, 0) \), Proposition 2.15 (ii) shows the case \( \alpha = 0 \), and Theorem 2.7(ii) shows the case \( \alpha \in (0, \infty) \). This finishes the proof of (ii).

Now we show (iii). Indeed, Theorem 2.7(iii) indicates the case \( \alpha < 0 \); by Theorem 2.7(i) and Proposition 2.15(i), we have that
\[
JN_{p,q}^{\alpha_0}(\mathbb{R}^n) \supset JN_{p,q}(\mathbb{R}^n) \supset W^{\alpha_0,q}(\mathbb{R}^n),
\]
when \( \alpha \in [0, \alpha_0] \); Theorem 2.7(ii) shows the case \( \alpha \in (\alpha_0, \infty) \), which completes the proof of (iii).

Finally, we prove (iv). The cases \( \alpha \in (-\infty, 0) \) and \( \alpha \in [1, \infty) \) have been shown, respectively, in (iii) and (ii) of Theorem 2.7. Moreover, notice that
\[
\alpha_0 \geq 1 \iff \frac{1}{p} + \frac{1}{n} \leq \frac{1}{q}.
\]
Using this and Proposition 2.16, we obtain the case \( \alpha \in [0, 1) \). This finishes the proof of (iv), and hence of Theorem 2.17.

Remark 2.18 It is still unclear whether or not the inclusions in (iii) and (iv) of Theorem 2.17 is proper.

3 Mean Oscillations and Dyadic Counterparts

In this section, we show that \( JNQ \) spaces can be equivalently characterized by mean oscillations in Subsection 3.1. Moreover, we use this characterization to study the dyadic counterpart of \( JNQ \) spaces in Subsection 3.2.
3.1 Characterization via Mean Oscillations

Now, we characterize JNQ spaces in terms of mean oscillations as in [19, Section 5]. By Corollary 2.17(i), we know that the space \( JNQ^p_{\alpha,q}(\mathbb{R}^n) \) is trivial when \( p < q \), and hence we only pay attention to the case \( p \geq q \). In what follows, for any \( k \in \mathbb{Z}_+ \) and any cube \( Q \) of \( \mathbb{R}^n \), we use \( \mathcal{D}_k(Q) \) to denote the dyadic cubes contained in \( Q \) of level \( k \); for any given \( q \in [1, \infty) \), \( \alpha \in \mathbb{R} \), measurable function \( f \), and any cube \( Q \) of \( \mathbb{R}^n \), let

\[
\Psi_{f,q,\alpha}(Q) := \left[ \sum_{k=0}^{\infty} 2^{(q\alpha-n)k} \sum_{l \in \mathcal{D}_k(Q)} \int_I |f(x) - f_I|^q dx \right]^{\frac{1}{q}},
\]

where we define \( \int_I |f(x) - f_I|^q dx = \infty \) if \( f \) is not \( \mathcal{L}^q(\mathbb{R}^n) \) as in Remark 2.2. Moreover, for any \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( |x|_\infty := \max_{i \in \{1, \ldots, n\}} |x_i| \).

**Theorem 3.1** Let \( 1 \leq q \leq p < \infty \) and \( \alpha \in \mathbb{R} \). Then \( f \in JNQ^p_{\alpha,q}(\mathbb{R}^n) \) if and only if \( f \) is measurable on \( \mathbb{R}^n \) and

\[
\|f\|_{JNQ^p_{\alpha,q}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{D}, \alpha \in \mathbb{R}} \left\{ \int_Q \left\{ \sum_j |Q_j| \Psi_{f,q,\alpha}(Q_j) \right\} \right\}^{\frac{1}{p}} < \infty,
\]

where \( \Psi_{f,q,\alpha}(Q_j) \) is as in (3.1) with \( Q \) replaced by \( Q_j \) for any \( j \). Moreover, \( \| \cdot \|_{JNQ^p_{\alpha,q}(\mathbb{R}^n)} \sim \| \cdot \|_{JNQ^p_{\alpha,q}(\mathbb{R}^n)} \).

**Proof** Let \( p, q, \alpha, \) and \( f \) be as in the present theorem. We claim that, for any cube \( Q \) of \( \mathbb{R}^n \) with edge length \( \ell(Q) \in (0, \infty) \),

\[
\left[ \Psi_{f,q,\alpha}(Q) \right]^q \lesssim 2^{q\alpha-n} \int_Q \int_Q |f(x) - f(y)|^q |x - y|^{n+q\alpha} dx dy \tag{3.3}
\]

and

\[
2^{q\alpha-n} \int_Q \int_Q |f(x) - f(y)|^q |x - y|^{n+q\alpha} dx dy \lesssim \left[ \Psi_{f,q,\alpha}(Q) \right]^q + \int_{E_Q} \left[ \Psi_{f,q,\alpha}(Q + z) \right]^q dz, \tag{3.4}
\]

where \( Q + z := \{ x + z : x \in Q \} \), \( E_Q := \{ z \in \mathbb{R}^n : |z|_\infty \leq \ell(Q) \} \) and the implicit positive constants are independent of \( Q \). Indeed, for the case \( q = 2 \), this claim was proven in [19, Lemmas 5.3 and 5.4] (see also [34, p.11, Theorem 1.5]). For the general case \( q \in [1, \infty) \), we present some details here for the sake of completeness.

We first show (3.3). From (3.1) and Lemma 2.1(i), it follows that

\[
\left[ \Psi_{f,q,\alpha}(Q) \right]^q = \sum_{k=0}^{\infty} 2^{(q\alpha-n)k} \sum_{l \in \mathcal{D}_k(Q)} \int_I |f(x) - f_I|^q dx
\]

\sim \sum_{k=0}^{\infty} 2^{(q\alpha-n)k} \sum_{l \in \mathcal{D}_k(Q)} \int_I \int_I |f(x) - f(y)|^q dx dy
\]

\sim \sum_{k=0}^{\infty} \sum_{l \in \mathcal{D}_k(Q)} \frac{2^{(q\alpha-n)k}}{(2^{-nk}(Q))^{q}} \int_I \int_I |f(x) - f(y)|^q dx dy
\]

\sim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k_Q(x, y)|f(x) - f(y)|^q dx dy, \tag{3.5}
\]

where

\[
k_Q(x, y) := \sum_{k=0}^{\infty} \sum_{l \in \mathcal{D}_k(Q)} 2^{(q\alpha+n)k} |Q|^{-2} |1_f(x) - 1_f(y)|. \tag{3.6}
\]
Now, to finish the proof of (3.3), we consider the following two cases on $\alpha$:

Case 1) $\alpha \in (-\frac{n}{q}, \infty)$. In this case, since

$$x, \; y \in I \in \mathcal{D}_k(Q) \implies |x - y| \leq \sqrt{n} \ell(I) = \sqrt{n} 2^{-k} \ell(Q)$$

$$\iff 2^k \leq \sqrt{n} \ell(Q)/|x - y|,$$

it follows that, for any $x, \; y \in Q$,

$$k_Q(x, y) = \sum_{k=0}^{\infty} \sum_{I \in \mathcal{D}_k(Q)} 2^{(q_n+n)k} |Q|^{-2} 1_I(x) 1_I(y)$$

$$\leq \sum_{\{k \in \mathbb{Z}_+: 2^k \leq \sqrt{n} \ell(Q)/|x - y|\}} 2^{(q_n+n)k} |Q|^{-2} \lesssim \left[ \frac{\ell(Q)}{|x - y|} \right]^{q_n+n} |Q|^{-2}.$$

Using this, the observation that $k_Q(x, y) = 0$ unless $x, \; y \in Q$, and (3.5), we obtain (3.3) when $\alpha \in (-\frac{n}{q}, \infty)$.

Case 2) $\alpha \in (-\infty, -\frac{n}{q}]$. We first claim that, if $\alpha \in (-\infty, 0)$, then

$$\Psi_{f,q,\alpha}(Q) \sim \MO_{f,q}(Q), \quad (3.7)$$

where $\MO_{f,q}(Q)$ is as in (1.1), and the positive equivalence constants are independent of $f$ and $Q$. Indeed, notice that, for any $I \in \mathcal{D}_1(Q)$, by Minkowski’s inequality and Hölder’s inequality, we have that

$$\left[ \int_I |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}} \leq \left[ \int_I |f(x) - f_I|^q \, dx \right]^{\frac{1}{q}} + |f_I - f_Q|$$

$$\leq \left[ \int_I |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}} + 2 |f_I - f_Q|$$

$$\leq \left[ \int_I |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}} + 2 \int_I |f(x) - f_Q| \, dx$$

$$\leq 3 \left[ \int_I |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}},$$

which implies that

$$\left[ \int_I |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}} \sim \MO_{f,q}(I) + |f_I - f_Q|,$$

and hence,

$$[\MO_{f,q}(Q)]^q = \frac{1}{|Q|} \sum_{I \in \mathcal{D}_1(Q)} \int_I |f(x) - f_Q|^q \, dx$$

$$\sim 2^{-n} \sum_{I \in \mathcal{D}_1(Q)} \{[\MO_{f,q}(I)]^q + |f_I - f_Q|^q\} \gtrsim 2^{-n} \sum_{I \in \mathcal{D}_1(Q)} [\MO_{f,q}(I)]^q.$$

From this, along with the mathematical induction, it follows that, for any $k \in \mathbb{N}$,

$$[\MO_{f,q}(Q)]^q \gtrsim 2^{-kn} \sum_{I \in \mathcal{D}_k(Q)} [\MO_{f,q}(I)]^q. \quad (3.8)$$

Moreover, notice that

$$\Psi_{f,q,\alpha}(Q) = [\MO_{f,q}(Q)]^q + \sum_{k=1}^{\infty} \sum_{I \in \mathcal{D}_k(Q)} 2^{(q_n-n)k} [\MO_{f,q}(I)]^q.$$
This, combined with (3.8) and \( \alpha \in (-\infty, 0) \), implies that
\[
[MO_{f,q}(Q)]^q \leq \Psi_{f,q,\alpha}(Q) \lesssim \sum_{k=0}^{\infty} 2^{q\alpha k} [MO_{f,q}(Q)]^q \sim [MO_{f,q}(Q)]^q,
\]
which shows that (3.7) holds. Using (3.7), Lemma 2.1(i), and (2.9), we obtain (3.3) when \( \alpha \in (-\infty, -\frac{2}{q}] \).

Combining Cases 1) and 2), we find that (3.3) holds.

Next, we prove (3.4). By (3.5) and the Fubini theorem, we obtain that
\[
\int_{E_Q} [\Psi_{f,q,\alpha}(Q + z)]^q \, dz = \int_{\mathbb{R}^n} \int_{E_Q} k_{Q+z}(x,y) \, dz \, |f(x) - f(y)|^q \, dx \, dy.
\]
Thus, to prove (3.4), it suffices to show that, for any given \( x, y \in Q \),
\[
\frac{\ell(qa-n)}{|x-y|^{n+qa}} \lesssim k_{Q}(x,y) + \int_{E_Q} k_{Q+z}(x,y) \, dz. \tag{3.9}
\]
We first consider the case \( x, y \in Q \) with \( |x-y|_{\infty} \leq 2^{-1} \ell(Q) \). In this case, there exists an \( m \in \mathbb{Z}_+ \) such that
\[
2^{-m-2} \ell(Q) < |x-y|_{\infty} \leq 2^{-m-1} \ell(Q). \tag{3.10}
\]
Notice that, if \( |z|_{\infty} > \ell(Q) \), then \( x \notin Q + z \), and hence \( k_{Q+z}(x,y) = 0 \). From this, (3.6), and (3.10), we deduce that
\[
\int_{E_Q} k_{Q+z}(x,y) \, dz = \frac{1}{|E_Q|} \int_{\mathbb{R}^n} k_{Q+z}(x,y) \, dz \geq \frac{1}{|Q|} \int_{\mathbb{R}^n} \sum_{I \in \mathcal{D}_m(Q+z)} 2^{(qa+n)|I|} |Q|^{-2} 1_I(x) 1_I(y) \, dz \\
\geq |\ell(Q)|^{-3q} \left[ \frac{\ell(Q)}{|x-y|_{\infty}} \right]^{qa+n} \sum_{I \in \mathcal{D}_m(Q)} 1_{I+z}(x) 1_{I+z}(y) \, dz.
\]

Observe that, for any \( I \in \mathcal{D}_m(Q) \), and \( z, x, y \in \mathbb{R}^n \),
\[
1_{I+z}(x) 1_{I+z}(y) = 1_{I-x}(-z) 1_{I-y}(-z),
\]
and hence,
\[
\sum_{I \in \mathcal{D}_m(Q)} \int_{\mathbb{R}^n} 1_{I+z}(x) 1_{I+z}(y) \, dz = \sum_{I \in \mathcal{D}_m(Q)} |(I-x) \cap (I-y)| \geq \sum_{I \in \mathcal{D}_m(Q)} 2^{-n}|I| = 2^{-n}|Q|,
\]
where the inequality holds because (3.10) \( \implies \) \( \ell(I) - |x-y|_{\infty} \geq 2^{-1} \ell(I) \implies \) there exists some cube \( \tilde{I} \) with edge length \( \frac{1}{2} \ell(I) \) such that \( [(I-x) \cap (I-y)] \supset \tilde{I} \). By this and (3.11), we conclude that
\[
\int_{E_Q} k_{Q+z}(x,y) \, dz \gtrsim \frac{\ell(qa-n)}{|x-y|^{n+qa}}. \tag{3.12}
\]
We then consider the case \( x, y \in Q \) with \( |x-y|_{\infty} > 2^{-1} \ell(Q) \). In this case, we have \( |x-y| \sim \ell(Q) \), which, together with (3.6) with \( k = 0 \), further implies that
\[
k_{Q}(x,y) \geq |Q|^{-2} \sim \frac{\ell(qa-n)}{|x-y|^{n+qa}}.
\]
Combining this with (3.12), we obtain (3.9), and hence (3.4) holds. Altogether, we have completed the proofs of both (3.3) and (3.4), and hence the proof of the above claim.

Now, from (3.3), we deduce that, if \( f \in JNQ_{p,q}^\alpha(R^n) \), then

\[
\|f\|_{JNQ_{p,q}^\alpha(R^n)} = \sup_{j \in \mathbb{N}_k} \left\{ \sum_j |Q_j| |\Psi_{f,q,\alpha}(Q_j)|^p \right\}^{1/p} \\
\lesssim \sup_{j \in \mathbb{N}_k} \left\{ \sum_j |Q_j| \left[ (\frac{n}{n+q})^{-1} \int_{Q_j} \int_{Q_j} |f(x) - f(y)|^q |x-y|^{n+q\alpha} \, dx \, dy \right]^{1/p} \right\}^{1/p} \\
\sim \|f\|_{JNQ_{p,q}^\alpha(R^n)} < \infty,
\]

and hence (3.2) holds. Conversely, if the measurable function \( f \) satisfies (3.2), then, by (3.4), \( p/q \geq 1 \), and the Minkowski inequality, we conclude that

\[
\|f\|_{JNQ_{p,q}^\alpha(R^n)} = \sup_{j \in \mathbb{N}_k} \left\{ \sum_j |Q_j| |\Psi_{f,q,\alpha}(Q_j)|^p \right\}^{1/p} \\
\lesssim \sup_{j \in \mathbb{N}_k} \left\{ \sum_j |Q_j| \left[ (\frac{n}{n+q})^{-1} \int_{Q_j} \int_{Q_j} |f(x) - f(y)|^q |x-y|^{n+q\alpha} \, dx \, dy \right]^{1/p} \right\}^{1/p} \\
+ \sup_{j \in \mathbb{N}_k} \left\{ \int_{E_Q} \left\{ \sum_j |Q_j| \left[ (\frac{n}{n+q})^{-1} \int_{Q_j} \int_{Q_j} |f(x) - f(y)|^q |x-y|^{n+q\alpha} \, dx \, dy \right]^{1/p} \right\}^{1/p} \right\}^{1/p} \\
\lesssim \sup_{j \in \mathbb{N}_k} \left\{ \sum_j |Q_j| |\Psi_{f,q,\alpha}(Q_j)|^p \right\}^{1/p} \\
+ \sup_{j \in \mathbb{N}_k} \left\{ \int_{E_Q} \left\{ \sum_j |Q_j| \left[ (\frac{n}{n+q})^{-1} \int_{Q_j} \int_{Q_j} |f(x) - f(y)|^q |x-y|^{n+q\alpha} \, dx \, dy \right]^{1/p} \right\}^{1/p} \right\}^{1/p} \\
\sim \|f\|_{JNQ_{p,q}^\alpha(R^n)} < \infty.
\]

This finishes the proof of Theorem 3.1. □

3.2 Dyadic \( JNQ \) Spaces

Following [19, Section 7], we study the dyadic counterparts of \( JNQ \) spaces in this subsection. In what follows, for any \( k \in \mathbb{Z} \), \( \mathcal{D}_k(R^n) \) denotes the set of all dyadic cubes contained in \( \mathbb{R}^n \) of level \( k \), and \( \mathcal{D}(R^n) := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k(R^n) \). In addition, for any \( x, y \in \mathbb{R}^n \), the dyadic distance \( \delta(x,y) \) is defined by setting

\[
\delta(x,y) := \inf \{ \ell(I) : x, y \in I \in \mathcal{D}(R^n) \}.
\]

Notice that the dyadic distance is infinite between points in different octants and, for any \( x, y \in Q \in \mathcal{D}(R^n) \),

\[
|x - y| \leq \sqrt{n} \delta(x,y) \leq \sqrt{n} \ell(Q). \tag{3.13}
\]
Definition 3.2 Let \( p, q \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). The dyadic \( JNQ \) space \( JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that
\[
\| f \|_{JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n)} := \sup_{k \in \mathbb{Z}} \left\{ \sum_{Q \in \mathcal{D}_k(\mathbb{R}^n)} |Q| \left[ \frac{|Q|^{\alpha}}{|\delta(x,y)|^{n+\alpha}} \right] \right\}^{\frac{1}{\alpha}} < \infty.
\]

The following lemma is a slight variant (changing ‘2’ into ‘\( q \)’) of [19, Lemma 7.1] (we omit the details here):

Lemma 3.3 Let \( q \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). Then there exists a positive constant \( C \) such that, for any dyadic cube \( Q \in \mathcal{D}(\mathbb{R}^n) \) and any \( f \in L^q(Q) \),
\[
C^{-1} \Psi_{f,q,\alpha}(Q) \leq \left[ |Q|^{\frac{\alpha}{q^2}} \int_Q \frac{|f(x) - f(y)|^q}{|\delta(x,y)|^{n+\alpha}} \, dx \, dy \right]^{\frac{1}{q}} \leq C \Psi_{f,q,\alpha}(Q),
\]
where \( \Psi_{f,q,\alpha}(Q) \) is the same as in (3.1).

As a consequence of Lemma 3.3, we immediately have the following equivalent norm on the space \( JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n) \):

Proposition 3.4 Let \( p, q \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). Then \( f \in JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n) \) if and only if \( f \) is measurable on \( \mathbb{R}^n \) and
\[
\| f \|_{\overset{\text{JNQ}}{\alpha,\text{dyadic}}(\mathbb{R}^n)} := \sup_{k \in \mathbb{Z}} \left\{ \sum_{Q \in \mathcal{D}_k(\mathbb{R}^n)} |Q| \left[ \Psi_{f,q,\alpha}(Q) \right]^\frac{1}{p} \right\}^p < \infty,
\]
where \( \Psi_{f,q,\alpha}(Q) \) is the same as in (3.1). Moreover, \( \| \cdot \|_{\overset{\text{JNQ}}{\alpha,\text{dyadic}}(\mathbb{R}^n)} \sim \| \cdot \|_{JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n)} \).

Remark 3.5 The next Lemma 3.12 implies that, if \( p, q \in [1, \infty) \) and \( \alpha \in (-\infty, 1/q) \), then
\[
\| f \|_{\overset{\text{JNQ}}{\alpha,\text{dyadic}}(\mathbb{R}^n)} \sim \sup_{k \in \mathbb{Z}} \left\{ \sum_{Q \in \mathcal{D}_k(\mathbb{R}^n)} |Q| \left[ \Phi_{f,q,\alpha}(Q) \right]^\frac{1}{p} \right\}^p,
\]
with \( \Phi_{f,q,\alpha} \) the same as in (1.2), and the positive equivalence constants independent of \( f \).

Also, we have an analogue of Theorem 2.7 for dyadic \( JNQ \) spaces.

Theorem 3.6 Let \( p, q \in [1, \infty) \). Then dyadic \( JNQ \) spaces enjoy the following properties:
(i) (Decreasing in \( \alpha \)) If \( -\infty < \alpha_1 \leq \alpha_2 < \infty \), then \( JNQ^{\alpha_1,\text{dyadic}}(\mathbb{R}^n) \supset JNQ^{\alpha_2,\text{dyadic}}(\mathbb{R}^n) \).
(ii) (Triviality for large \( \alpha \)) If \( \alpha \in \left( \frac{n}{q} - 1, \infty \right) \), then \( JNQ^{\alpha,\text{dyadic}}(\mathbb{R}^n) \) contains only functions that are almost everywhere constant in each octant.
(iii) (Triviality for negative \( \alpha \)) If \( \alpha_1, \alpha_2 \in (-\infty, 0) \), then
\[
JNQ^{\alpha_1,\text{dyadic}}(\mathbb{R}^n) = JNQ^{\alpha_2,\text{dyadic}}(\mathbb{R}^n)
\]
with equivalent norms.

Proof First, using (3.13), we find that, for any \( \alpha_1, \alpha_2 \in \mathbb{R} \) with \( \alpha_1 \leq \alpha_2 \), any \( f \in JNQ^{\alpha_2,\text{dyadic}}(\mathbb{R}^n) \), and any dyadic cube \( Q \in \mathcal{D}(\mathbb{R}^n) \),
\[
\left| Q \right|^{\frac{\alpha_2}{n+\alpha_2}} \int_Q \frac{|f(x) - f(y)|^q}{|\delta(x,y)|^{n+\alpha_2}} \, dx \, dy
\]

\[\triangledown\] Springer
\[
\begin{align*}
\leq & \left[\ell(Q)^{\alpha_1-n}\right] Q \int_Q \int_Q \frac{|f(x) - f(y)|^q}{\delta(x,y)^{n+q\alpha_2}} |f(Q)|^{q(\alpha_2-\alpha_1)} \, dx \, dy \\
= & |Q|^{\alpha_2-1} \int_Q \int_Q |f(x) - f(y)|^q \delta(x,y)^{n+q\alpha_2} \, dx \, dy,
\end{align*}
\]
which implies that \( \|f\|_{JNQ_{p,q}^{\alpha_1,dyadic}(\mathbb{R}^n)} \leq \|f\|_{JNQ_{p,q}^{\alpha_2,dyadic}(\mathbb{R}^n)} \), and hence
\[
JNQ_{p,q}^{\alpha_1,dyadic}(\mathbb{R}^n) \supset JNQ_{p,q}^{\alpha_2,dyadic}(\mathbb{R}^n).
\]

This shows (i) of the present theorem.

Next, we prove (ii) of the present theorem. Let \( \alpha \in \left(n\left(\frac{1}{p} - \frac{1}{2}\right)\right) \), \( f \in JNQ_{p,q}^{\alpha,dyadic}(\mathbb{R}^n) \), and \( Q \in \mathcal{D}(\mathbb{R}^n) \). Suppose that \( Q_m \supseteq Q \) and \( \ell(Q_m) = 2^m \ell(Q) \).

From Proposition 3.4 and (3.1), it follows that
\[
\infty > \|f\|_{JNQ_{p,q}^{\alpha,dyadic}(\mathbb{R}^n)} \geq |Q_m|^{1/\alpha} \Psi_{f,q,\alpha}(Q_m) \geq 2^{[\alpha-n(\frac{1}{p} - \frac{1}{2})]} \left[ \int_Q |f(x) - f|_q \, dx \right]^{\frac{1}{q}}.
\]
Letting \( m \to \infty \), we obtain that \( \int_Q |f(x) - f|_q \, dx = 0 \), which implies that \( f \) is a constant almost everywhere in each octant, and hence that (ii) holds.

Finally, (iii) follows from Proposition 3.4 and (3.7), which completes the proof of Theorem 3.6. \( \square \)

**Remark 3.7** Let \( p, q \in [1, \infty) \) and \( \alpha_0 := n\left(\frac{1}{q} - \frac{1}{p}\right) \). Then, corresponding to Proposition 2.15, we claim that \( JNQ_{p,q}^{\alpha_0,dyadic}(\mathbb{R}^n) \supset W^{\alpha_0,q}(\mathbb{R}^n) \) if \( \alpha_0 \geq 0 \). Indeed, by (3.3) and the observation that \( \alpha_0 \geq 0 \iff p/q \geq 1 \), we obtain that
\[
\begin{align*}
\sup_{k \in \mathbb{Z}} \left\{ \sum_{Q \in \mathcal{G}_k(\mathbb{R}^n)} |Q| \left[ \Psi_{f,q,\alpha_0}(Q) \right]^p \right\}^{\frac{1}{p}} \\
\leq \sup_{k \in \mathbb{Z}} \left[ \sum_{Q \in \mathcal{G}_k(\mathbb{R}^n)} |Q| \left[ \ell(Q)^{q\alpha_0-n} \int_Q \int_Q |f(x) - f(y)|^q \, dx \, dy \right]^{\frac{1}{q}} \right]^{\frac{1}{p}} \\
\sim \sup_{k \in \mathbb{Z}} \left[ \sum_{Q \in \mathcal{G}_k(\mathbb{R}^n)} \int_Q \int_Q |f(x) - f(y)|^q \, dx \, dy \right]^{\frac{1}{q}} \\
\leq \sup_{k \in \mathbb{Z}} \left[ \sum_{Q \in \mathcal{G}_k(\mathbb{R}^n)} \int_Q \int_Q |f(x) - f(y)|^q \, dx \, dy \right]^{\frac{1}{q}},
\end{align*}
\]
which, together with Proposition 3.4, further implies that \( \|f\|_{JNQ_{p,q}^{\alpha_0,dyadic}(\mathbb{R}^n)} \lesssim \|f\|_{W^{\alpha_0,q}(\mathbb{R}^n)} \).

This shows the above claim, and hence, if \( \alpha_0 \in (0,1) \), then
\[
JNQ_{p,q}^{\alpha_0,dyadic}(\mathbb{R}^n) \neq \{ \text{a.e. constant in each octant} \}.
\]

Moreover, similarly to the proof of Theorem 2.17, we find that (3.14) is false for any \( \alpha_0 \in (-\infty,0] \). However, when \( \alpha_0 \in [1, \infty) \), it is interesting to ask whether or not (3.14) still holds, which is still unclear so far.

Recall that Essén, Janson, Peng and Xiao [19, Theorem 7.9] used \( \Psi_{f,q,\alpha}(Q) \) to establish the following relation between the space \( Q \) and its dyadic analogue \( Q^d_{\alpha}(\mathbb{R}^n) \): for any \( \alpha \in (-\infty,1/2) \),
\[
Q_{\alpha}(\mathbb{R}^n) = \left[ Q^d_{\alpha}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \right].
\]
Now, we establish a corresponding result for the space $JNQ$ as follows (notice that we only need to consider the case $p \geq q$, otherwise $p < q$ and hence both $JNQ_{p,q}^\alpha (\mathbb{R}^n)$ and $JN_{p,q}^{\text{con}} (\mathbb{R}^n)$ are trivial due to Theorem 2.17(i) and Corollary 2.8):

**Theorem 3.8** Let $1 \leq q \leq p < \infty$ and $\alpha \in (-\infty, 1/q)$. Then

$$JNQ_{p,q}^\alpha (\mathbb{R}^n) = [JNQ_{p,q}^{\text{dyadic}} (\mathbb{R}^n) \cap JN_{p,q}^{\text{con}} (\mathbb{R}^n)].$$

To prove Theorem 3.8, we need a geometrical lemma (Lemma 3.10) which is a refinement of both [32, Lemma 2.4] and [20, Lemma 2.5]. In what follows, for any set $A$ of $\mathbb{R}^n$, we use $\overline{A}$ to denote its closure; moreover, two sets $A$ and $B$ are said to be mutually adjacent if $A \cap B \neq \emptyset$.

**Definition 3.9** Let $k \in \mathbb{Z}$ and $Q$ be a cube of $\mathbb{R}^n$ with edge length $\ell \in (2^{-k-1}, 2^{-k}]$. Observe that there exist mutually adjacent dyadic cubes $\{Q_j\}_{j=1}^{2^n} \subset D_k (\mathbb{R}^n)$, with $Q^{(1)}$ being the left and lower one in $\{Q_j\}_{j=1}^{2^n}$ such that $Q \cap Q^{(1)} \neq \emptyset$ and $Q \subset \bigcup_{j=1}^{2^n} Q_j$; see the figure below for two examples of cubes when $n = 2$. Then the dominated cube of $Q$, denoted by $Q^\flat$, is defined by setting

$$Q^\flat := \bigcup_{j=1}^{2^n} Q_j.$$

Moreover, $Q^{(1)}$ is called the left and lower cube of $Q$.

![Diagram](image)

**Lemma 3.10** Let $k \in \mathbb{Z}$ and $\{Q_j\}_{j \in \mathbb{N}} \in \Pi_\ell$ with $\ell \in (2^{-k-1}, 2^{-k}]$. Then $\{Q_j\}_{j \in \mathbb{N}}$ can be divided into $6^n$ subfamilies $\{Q^b_m\}_{m=1}^{6^n}$ (may be empty for some $m$) such that

(i) $\{Q_j\}_{j \in \mathbb{N}} = \bigcup_{m=1}^{6^n} Q^b_m$;

(ii) for any $m \in \{1, \cdots, 6^n\}$, all cubes in $\{Q^b : Q \in Q^b_m\}$ are mutually disjoint, where $Q^b$ is as in Definition 3.9.

**Proof** Let all the symbols be as in the present lemma. Divide $\mathcal{D}_k (\mathbb{R}^n)$ into $3^n$ sparse subfamilies as follows:

$$\mathcal{I}_1 := \{(0, 2^{-k})^n + 3 \cdot 2^{-k} j : j \in \mathbb{Z}^n\},$$
$$\mathcal{I}_2 := \mathcal{I}_1 + (2^{-k}, 0, \cdots, 0),$$
$$\mathcal{I}_3 := \mathcal{I}_1 + (2 \cdot 2^{-k}, 0, \cdots, 0).$$
By some geometrical observations, we find that, if \( Q_1 \in \mathcal{J}_k \) and \( Q_2 \in \mathcal{J}_l \) with \( k \neq l \), then
\[
\overline{Q_1} \cap \overline{Q_2} = \emptyset;
\]
moreover, for any cubes \( Q_k, Q_l \in \{Q_j\}_{j \in \mathbb{N}} \),
\[
Q_k^j \cap Q_l^j = \emptyset \iff Q_k^{(1)}_{j} \cap Q_l^{(1)}_{j} = \emptyset
\]
(see, for instance, the cubes \( Q \) and \( \bar{Q} \) in the figure of Definition 3.9). Furthermore, from \( \{Q_j\}_{j \in \mathbb{N}} \in \Pi_\ell \) and \( \ell \in [2^{-k-1}, 2^{-k}] \), it follows that each dyadic cube in \( D_k(\mathbb{R}^n) \) may be the left and lower cube (see Definition 3.9) of at most \( 2^n \) cubes in \( \{Q_j\}_{j \in \mathbb{N}} \). Combining this, (3.15) and (3.16), we obtain \( 3^n \times 2^n = 6^n \) desired subfamilies, which completes the proof of Lemma 3.10.

The next two lemmas are, respectively, the variants of [19, Lemmas 5.7 and 5.8]. Indeed, Lemma 3.11 shows that \( \Psi_{f,q,\alpha} \) is almost increasing (see Definition 2.3), and Lemma 3.12 is the reverse of (3.3); we omit the proofs here because, as in the proof of (3.3) in Theorem 3.1, it suffices to change ‘2’ into ‘q’ and to make some corresponding modifications.

**Lemma 3.11** Let \( q \in [1, \infty) \), \( \alpha \in (-\infty, 1/q) \), and let \( f \) be a measurable function on \( \mathbb{R}^n \). Then there exists a positive constant \( C \), depending only on \( n, q, \) and \( \alpha \), such that, for any cubes \( Q_1, Q_2 \) with \( Q_1 \subset Q_2 \) and \( \ell(Q_1) = \frac{1}{2}\ell(Q_2) \),
\[
\Psi_{f,q,\alpha}(Q_1) \leq C\Psi_{f,q,\alpha}(Q_2),
\]
where \( \Psi_{f,q,\alpha} \) is the same as in (3.1).

**Lemma 3.12** Let \( q \in [1, \infty) \), \( \alpha \in (-\infty, 1/q) \), and let \( f \in L^q(Q) \). Let \( \Phi_{f,q,\alpha} \) and \( \Psi_{f,q,\alpha} \) be, respectively, as in (1.2) and (3.1). Then
\[
\Psi_{f,q,\alpha}(Q) \sim \Phi_{f,q,\alpha}(Q),
\]
with the positive equivalence constants depending only on \( n, q, \) and \( \alpha \).

Now, we prove Theorem 3.8.

**Proof of Theorem 3.8** Let \( 1 \leq q \leq p < \infty \) and \( \alpha \in \mathbb{R} \). It is obvious that
\[
\| \cdot \|_{JNQ_{p,q}^{\alpha, \text{dyadic}}(\mathbb{R}^n)} \leq \| \cdot \|_{JNQ_{p,q}^{\alpha}(\mathbb{R}^n)},
\]
and hence, that
\[
JNQ_{p,q}^{\alpha}(\mathbb{R}^n) \subset JNQ_{p,q}^{\alpha, \text{dyadic}}(\mathbb{R}^n),
\]
which, combined with Theorem 2.7, further implies that
\[
JNQ_{p,q}^{\alpha}(\mathbb{R}^n) \subset [JNQ_{p,q}^{\alpha, \text{dyadic}}(\mathbb{R}^n) \cap JN_{p,q}^{\text{con}}(\mathbb{R}^n)].
\]

Conversely, let \( \alpha \in (-\infty, 1/q) \), \( f \in JNQ_{p,q}^{\alpha, \text{dyadic}}(\mathbb{R}^n) \cap JN_{p,q}^{\text{con}}(\mathbb{R}^n) \), and \( \{Q_j\}_{j \in \mathbb{N}} \in \Pi_\ell \) with \( \ell \in [2^{-k-1}, 2^{-k}] \) for some given \( k \in \mathbb{Z} \). For any \( j \in \mathbb{N} \), let \( Q_j = \bigcup_{i=1}^{2^n} Q_j^{(i)} \) be the dominated cube of \( Q_j \) as in Definition 3.9. Also, let \( \{Q_m\}_{m=1}^{6^n} \) be as in Lemma 3.10. Then, by the observation...
with the supremum taken over all collections \( \{ Q_j \} \), we conclude that

\[
\sum_j |Q_j| \left[ |Q_j|^{\frac{n-1}{n-q}} \int_{Q_j} \int_{Q_j} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q\alpha}} \, dy \, dx \right]^\frac{1}{q}
\]

\[
\lesssim \sum_j |Q_j^p| \left( \left[ \int_{Q_j^p} \int_{Q_j^p} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q\alpha}} \, dy \, dx \right]^\frac{1}{q} \right)^p
\]

\[
\lesssim \sum_{m=1}^n \sum_{j: Q_j \subset Q_m} \left| Q_j \right| \left[ \left[ \int_{Q_j^p} \int_{Q_j^p} \frac{|f(x) - f(y)|^q}{|x-y|^{n+q\alpha}} \, dy \, dx \right]^\frac{1}{q} \right]^p
\]

\[
\lesssim \| f \|_{JNQ_p^q, dyadic(R^n)}^p + \| f \|_{JNQ_p^q, dyadic(R^n)}^p
\]

where the implicit positive constants depend only on \( n, p, q, \) and \( \alpha \). This implies that \( f \in JNQ_p^q(R^n) \) and \( \| f \|_{JNQ_p^q(R^n)} \lesssim \| f \|_{JNQ_p^q, dyadic(R^n)} \), and hence that

\[
JNQ_p^q(R^n) \supset JNQ_p^q, dyadic(R^n) \cap JNQ_p^q(R^n)
\]

which completes the proof of Theorem 3.8.

At the end of this section, we discuss some other dyadic \( JNQ \)-type norms. Letting \( \Phi_{f,q,\alpha} \) be the same as in (1.2),

\[
\| f \|_{JNQ_p^q,1,1(R^n)} := \sup \left\{ \sum_j |Q_j| \left[ \Phi_{f,q,\alpha}(Q_j) \right]^p \right\}^{\frac{1}{p}}
\]

with the supremum taken over all collections \( \{ Q_j \} \) of subcubes of \( R^n \) with pairwise disjoint interiors and same dyadic edge length,

\[
\| f \|_{JNQ_p^q,1,1,1(R^n)} := \sup_{k \in \mathbb{Z}} \left\{ \sum_{Q \in G_k(R^n)} |Q| \left[ \Phi_{f,q,\alpha}(Q) \right]^p \right\}^{\frac{1}{p}}
\]

and

\[
\| f \|_{JNQ_p^q,1,1,1(R^n)} := \sup \left\{ \sum_j |Q_j| \left[ \Phi_{f,q,\alpha}(Q_j) \right]^p \right\}^{\frac{1}{p}}
\]

with the supremum taken over all collections \( \{ Q_j \} \) of disjoint dyadic cubes of \( R^n \) (\( \{ Q_j \} \) may have different edge lengths).

**Remark 3.13** (i) From Proposition 2.5 and the proof of Corollary 2.6, we easily deduce that

\[
\| f \|_{JNQ_p^q,1,1,1(R^n)} \lesssim \| f \|_{JNQ_p^q,1,1,1(R^n)} \lesssim \| f \|_{JNQ_p^q,1,1,1(R^n)}.
\]
Moreover, we apparently have that
\[ \| \cdot \|_{J \mathcal{N} Q_{p,q}^{\alpha} (\mathbb{R}^n)} \geq \| \cdot \|_{J \mathcal{N} Q_{p,q}^{\alpha} II (\mathbb{R}^n)} \]
via their definitions, but the reverse inequality is obviously false (it suffices to consider functions which are almost everywhere constant in each octant, but not equal to some constant almost everywhere on \( \mathbb{R}^n \)).

(ii) From Lemmas 3.3 and 3.12, it follows that, for any \( \alpha \in (-\infty, \frac{1}{q}) \),
\[ \| \cdot \|_{J \mathcal{N} Q_{p,q}^{\alpha} dyadic (\mathbb{R}^n)} \sim \| \cdot \|_{J \mathcal{N} Q_{p,q}^{\alpha} II (\mathbb{R}^n)}, \]
but the case \( \alpha \in \left[ \frac{1}{q}, \infty \right) \) is still unclear so far. The main obstacle here is that we do not know whether or not Lemma 3.12 still holds for \( \alpha \in \left[ \frac{1}{q}, \infty \right) \); see also [34, p. 22, Section 1.5] for some similar problems related to \( Q \) spaces.

(iii) Recall that, in [1, Corollary 3.3], Berkovits, Kinnunen and Martell established the John-Nirenberg inequality of \( J \mathcal{N} Q^{\alpha} (Q_0) \) whose norm is taken over all collections of disjoint dyadic cubes of \( Q_0 \) (not necessary to have equal edge length). Moreover, Yue and Dafni [39] established the John-Nirenberg inequality of \( Q_{\alpha} (\mathbb{R}^n) \) with \( \alpha \in [0, 1) \); see also [34, p. 32, Theorem 2.3] and [38]. Thus, it is very interesting to find the John-Nirenberg inequality of \( J \mathcal{N} Q_{p,1}^{\alpha} (\mathbb{R}^n) \) with \( p \in [1, \infty) \) and \( \alpha \in \mathbb{R} \). This is a challenging problem which is still open so far.

4 Composition Operators on \( J N Q \) Spaces

This section is devoted to the left and the right composition operators on \( J N Q \) spaces. Let \( L : \mathbb{R} \rightarrow \mathbb{R} \) and \( R : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be suitable mappings. Then, for any measurable function \( f \) and any \( x \in \mathbb{R}^n \), the left composition operator \( \mathcal{C}_L \) is defined by setting
\[ \mathcal{C}_L(f)(x) = L(f(x)), \]
and the right composition operator \( \mathcal{C}_R \) is defined by setting
\[ \mathcal{C}_R(f)(x) = f(R(x)). \]
These concepts can be found in [34, 35], and we also recall that, for when \( L \) is a continuous self-map of \( \mathbb{R} \), Xiao [34, 35] proved that the left composition operator \( \mathcal{C}_L \) is bounded on \( Q \) spaces if and only if \( L \) is a Lipschitz function, while for when \( R \) is a quasiconformal mapping, the boundedness of the right composition operator \( \mathcal{C}_R \) on \( Q \) spaces was established in [25] (see also [36]).

We first show the boundedness of \( \mathcal{C}_L \) in Theorem 4.1, which is an application of Proposition 2.16. Recall that the Lipschitz space \( \text{Lip}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) on \( \mathbb{R}^n \) such that
\[ \| f \|_{\text{Lip}(\mathbb{R}^n)} := \sup \left\{ \left| \frac{f(x) - f(y)}{|x - y|} \right| : x, y \in \mathbb{R}^n \text{ and } x \neq y \right\} < \infty, \]
and a well-known result of Campanato [10] indicates that, for any \( q \in [1, \infty) \),
\[ \| f \|_{\text{Lip}(\mathbb{R}^n)} \sim \sup_{\text{cube } Q \subset \mathbb{R}^n} |Q|^{-\frac{1}{q}} \left[ \int_Q |f(x) - f_Q|^q \, dx \right]^{\frac{1}{q}}, \tag{4.1} \]
with the positive equivalence constants independent of \( f \).
Theorem 4.1 Let \( \alpha \in (-\infty, 1) \), \( p, q \in [1, \infty) \), \( \gamma \in [q, \infty) \) with \( \frac{1}{\gamma} = \frac{1}{p} + \frac{1}{n} \), and let 
\( L : \mathbb{R} \to \mathbb{R} \) be a continuous mapping. Then
(i) \( \mathcal{E}_L \) is bounded on \( JNQ^\alpha_{p,q}(\mathbb{R}^n) \) if and only if \( L \in \text{Lip}(\mathbb{R}) \);
(ii) \( f \in JNQ^\alpha_{p,q}(\mathbb{R}^n) \) if and only if there exists a \( g \in JNQ^\alpha_{p,q}(\mathbb{R}^n) \) such that
\[ \frac{1}{g} \in JNQ^\alpha_{p,q}(\mathbb{R}^n) \quad \text{and} \quad f = g - \frac{1}{g}. \]

Proof Let all the symbols be as in the present proposition. We first show (i). If \( L \in \text{Lip}(\mathbb{R}) \), then, for any \( x, y \in \mathbb{R}^n \), we have that
\[ |L(f(x)) - L(f(y))| \leq \|L\|_{\text{Lip}(\mathbb{R})} |f(x) - f(y)|, \]
and hence,
\[ \|\mathcal{E}_L(f)\|_{JNQ^\alpha_{p,q}(\mathbb{R}^n)} \leq \|L\|_{\text{Lip}(\mathbb{R})} \|f\|_{JNQ^\alpha_{p,q}(\mathbb{R}^n)}. \]
Conversely, let \( \mathcal{E}_L \) be bounded on \( JNQ^\alpha_{p,q}(\mathbb{R}^n) \). To obtain \( L \in \text{Lip}(\mathbb{R}) \), via (4.1), it suffices to show that, for any given interval \( I \subset \mathbb{R} \) with finite length \( \ell \),
\[ |I|^{-1} \left( \int_I |L(t) - L_i|^q \, dt \right)^{\frac{1}{q}} \lesssim 1. \quad (4.2) \]
To this end, let \( Q := I^n := I \times \cdots \times I \) be a cube of \( \mathbb{R}^n \), and let
\[ f_1(x) := \begin{cases} x_1, & x = (x_1, \ldots, x_n) \in Q, \\ 0, & x \in \mathbb{R}^n \setminus Q. \end{cases} \]
Then, from Proposition 2.16, it follows that
\[ \|f_1\|_{JNQ^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \|f_1\|_{W^{1,\gamma}(\mathbb{R}^n)} \sim \left[ \int_Q |(1,0,\cdots,0)|^\gamma \, dx \right]^{\frac{1}{\gamma}} \sim |Q|^{\frac{1}{\gamma} + \frac{1}{n}}. \quad (4.3) \]
Notice that
\[ \mathcal{E}_L(f_1)(x) = L(f_1(x)) = \begin{cases} L(x_1), & x = (x_1, \ldots, x_n) \in Q, \\ 0, & x \in \mathbb{R}^n \setminus Q \end{cases} \]
and hence,
\[ \int_I |L(t) - L_i|^q \, dt = \int_Q |\mathcal{E}_L(f_1)(x) - [\mathcal{E}_L(f_1)]_Q|^q \, dx. \quad (4.4) \]
Moreover, by Definition 1.1, Theorem 2.7, the boundedness of \( \mathcal{E}_L \), and (4.3), we conclude that
\[ \left\{ \left( \int_Q |\mathcal{E}_L(f_1)(x) - [\mathcal{E}_L(f_1)]_Q|^q \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{n}} \lesssim \|\mathcal{E}_L(f_1)\|_{JNQ^\alpha_{p,q}(\mathbb{R}^n)} \lesssim \|f_1\|_{JNQ^\alpha_{p,q}(\mathbb{R}^n)} \lesssim |Q|^{\frac{1}{\gamma} + \frac{1}{n}}, \]
which, together with (4.4) and the fact that \( |Q| = |I|^n \), further implies that
\[ |I|^{-1} \left( \int_I |L(x) - L_i|^q \, dx \right)^{\frac{1}{q}} \lesssim |I|^{-1} |Q|^{\frac{1}{n}} \sim 1. \]
Hence (4.2) holds. This shows that \( \phi \in \text{Lip}(\mathbb{R}^n) \), and hence finishes the proof of (i).
We then show (ii). If there exists a $g \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$ such that $\frac{1}{g} \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$ and $f = g - \frac{1}{g}$, then we apparently have that $f \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$. Conversely, let $f \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, let
\[
g(x) := \frac{\sqrt{x^2 + 2^2} + x}{2}.
\]
Then, for any $x \in \mathbb{R}^n$,
\[
\frac{1}{g(x)} = \frac{\sqrt{x^2 + 2^2} - x}{2},
\]
and hence,
\[
x = g(x) - \frac{1}{g(x)}.
\]
Notice that $g, \frac{1}{g} \in \text{Lip}(\mathbb{R}^n)$. Then, from (i) and $f \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$, it follows that $g(f), \frac{1}{g(f)} \in JNQ_{p,q}^{\alpha}(\mathbb{R}^n)$, which, combined with (4.5), further implies that
\[
f = g(f) - \frac{1}{g(f)}.
\]
This finishes the proof of (ii), and hence of Theorem 4.1. \(\square\)

**Remark 4.2** We have the following observations:

(i) For more studies on the left composition operators on Lebesgue spaces, Sobolev spaces, and function spaces of Besov-Triebel-Lizorkin type, we refer the reader to [4–6, 29, 35].

(ii) For more studies on the left composition operators on BMO and $Q$ spaces, we refer the reader to [3, 34, 35].

Next, we consider the right composition operator. Recall that, for when $R$ is a quasiconformal mapping with some particular geometrical assumptions, Reimann [28, Theorem 2] showed that $\mathcal{C}_R$ is bounded on BMO $(\mathbb{R}^n)$, and Koskela, Xiao, Zhang and Zhou [25, Theorem 1.3] showed that $\mathcal{C}_R$ is also bounded on $Q_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, 1)$ (see also [36]). However, the following proposition indicates that the quasiconformal mapping may be an unsuitable object to study the boundedness of right composition operators on $JNQ$ spaces (in what follows, for any $r \in (0, \infty)$ and any cube $Q$ of $\mathbb{R}^n$, let $r \times Q := \{rx : x \in Q\}$):

**Proposition 4.3** Let $p \in [1, \infty)$, $r \in (0, \infty)$, and
\[
\mathcal{C}_R := \begin{cases}
\mathbb{R}^n & \text{to} \quad \mathbb{R}^n, \\
x = (x_1, \ldots, x_n) & \mapsto rx = (rx_1, \ldots, rx_n).
\end{cases}
\]
Then
\[
\|\mathcal{C}_R(f)\|_{JNQ_{p}^{\infty}(\mathbb{R}^n)} = r^{-n/p}\|f\|_{JNQ_{p}^{\infty}(\mathbb{R}^n)}.
\]

**Proof** Let all the symbols be as in the present proposition. Also, let $\{Q_j\} \in \Pi_\ell$ with $\ell \in (0, \infty)$. Then, for any $j$, we have that
\[
\int_{Q_j} \mathcal{C}_R(f)(y)dy = \int_{Q_j} f(ry)dy = \int_{r \times Q_j} f(z)dz,
\]
and hence, $(\mathcal{C}_R(f))_{Q_j} = f_{r \times Q_j}$. By this, we have that
\[
\sum_j |Q_j| \left[\int_{Q_j} \left|\mathcal{C}_R(f)(x) - (\mathcal{C}_R(f))_{Q_j}\right|^p dx\right]^\frac{1}{p}
\]
which, together with the observation that \( \{ r \times Q_j \}_j \in \Pi_{rt} \), further implies that (4.6) holds. This finishes the proof of Proposition 4.3. □

**Remark 4.4** Notice that \( \mathcal{J}_N^{\text{con}}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \); see Remark 1.3(ii) or [20, Proposition 2.11]. Also, observe that \( \mathcal{C}_R \) in Proposition 4.3 is the most elementary quasiconformal mapping. Letting \( r \to 0^+ \) in (4.6), we find that \( p = \infty \) is the only possible \( p \) such that right composition operators, generated by quasiconformal mappings, are bounded on \( \mathcal{J}_N^{\text{con}}(\mathbb{R}^n) \). Therefore, it is interesting to find suitable conditions of \( R \) such that \( \mathcal{C}_R \) is bounded on \( \mathcal{J}_N Q \) spaces, which is still unclear so far.

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**References**

[1] Berkovits L, Kinnunen J, Martell J M. Oscillation estimates, self-improving results and good-\( \lambda \) inequalities. J Funct Anal, 2016, **270**(9): 3559–3590

[2] Betancor J J, Duong X T, Li J, et al. Product Hardy, BMO spaces and iterated commutators associated with Bessel Schrödinger operators. Indiana Univ Math J, 2019, **68**(1): 247–289

[3] Bourdaud G, Lanza de Cristoforis M, Sickel W. Functional calculus on BMO and related spaces. J Funct Anal, 2002, **189**(2): 515–538

[4] Bourdaud G, Moussai M, Sickel W. A necessary condition for composition in Besov spaces. Complex Var Elliptic Equa, 2020, **65**(1): 22–39

[5] Bourdaud G, Moussai M, Sickel W. Composition operators acting on Besov spaces on the real line. Ann Mat Pura Appl, 2014, **193**(5): 1519–1554

[6] Bourdaud G, Moussai M, Sickel W. Composition operators on Lizorkin-Triebel spaces. J Funct Anal, 2010, **259**(5): 1098–1128

[7] Bourgain J, Brezis H, Mironescu P. A new function space and applications. J Eur Math Soc, 2015, **17**(9): 2083–2101

[8] Brezis H. How to recognize constant functions. A connection with Sobolev spaces. Uspekhi Mat Nauk, 2002, **57**(4): 59–74; Translation in Russian Math Surveys, 2002, **57**(4): 693–708

[9] Brezis H, Van Schaftingen J, Yung P L. A surprising formula for Sobolev norms. Proc Natl Acad Sci USA, 2021, **118**(8): e2025254118

[10] Campanato S. Proprietà di una famiglia di spazi funzionali. Ann Scuola Norm Sup Pisa Cl Sci, 1964, **18**(3): 137–160

[11] Chen P, Duong X T, Li J, et al. BMO spaces associated to operators with generalised Poisson bounds on non-doubling manifolds with ends. J Differential Equations, 2021, **270**: 114–184

[12] Chen P, Duong X T, Song L, Yan L. Carleson measures, BMO spaces and balayages associated to Schrödinger operators. Sci China Math, 2017, **60**(11): 2077–2092

[13] Dafni G, Hytönen T, Korte R, Yue H. The space \( \mathcal{J}_N^{p} \): Nontriviality and duality. J Funct Anal, 2018, **275**(3): 577–603

[14] Dafni G, Xiao J. Some new tent spaces and duality theorems for fractional Carleson measures and \( Q_a(\mathbb{R}^n) \). J Funct Anal, 2004, **208**(2): 377–422

[15] Duong X T, Li H, Li J, Wick B D. Lower bound of Riesz transform kernels and commutator theorems on stratified nilpotent Lie groups. J Math Pures Appl, 2019, **124**: 273–299
[16] Duong X T, Li J, Sawyer E, et al. A two weight inequality for Calderón-Zygmund operators on spaces of homogeneous type with applications. J Funct Anal, 2021, 281(9): 109190
[17] Duong X T, Li J, Wick B D, Yang D. Characterizations of product Hardy spaces in Bessel setting. J Fourier Anal Appl, 2021, 27(2): Art 24
[18] Duong X T, Yan L. Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. J Amer Math Soc, 2015, 18(4): 943–973
[19] Essén M, Janson S, Peng L, Xiao J. Q spaces of several real variables. Indiana Univ Math J, 2000, 49(2): 575–615
[20] Jia H, Tao J, Yang D, et al. Special John-Nirenberg-Campanato spaces via congruent cubes. Sci China Math, 2022, 65(2): 359–420
[21] Jia H, Tao J, Yang D, et al. Boundedness of Calderón-Zygmund operators on special John-Nirenberg-Campanato and Hardy-type spaces via congruent cubes. Anal Math Phys, 2022, 12(1): Art 15
[22] Jia H, Tao J, Yang D, et al. Boundedness of fractional integrals on special John-Nirenberg-Campanato and Hardy-type spaces via congruent cubes. Fract Calc Appl Anal, 2022, 25(6): 2446–2487
[23] Jia H, Yang D, Yuan W, Zhang Y. Estimates for Littlewood-Paley operators on ball Campanato-type function spaces. Results Math, 2023, 78(1): Art 37
[24] John F, Nirenberg L. On functions of bounded mean oscillation. Comm Pure Appl Math, 1961, 14: 415–426
[25] Koskela P, Xiao J, Zhang Y, Zhou Y. A quasiconformal composition problem for the Q-spaces. J Eur Math Soc, 2017, 19(4): 1159–1187
[26] Li J, Wick B D. Characterizations of $H^1_{\Delta N}(\mathbb{R}^n)$ and $\text{BMO}_{\Delta N}(\mathbb{R}^n)$ via weak factorizations and commutators. J Funct Anal, 2017, 272(12): 5384–5416
[27] Peng L Z, Yang Q X. Predual spaces for $Q$ spaces. Acta Math Sci, 2009, 29B(2): 243–250
[28] Reimann H M. Functions of bounded mean oscillation and quasiconformal mappings. Comment Math Helv, 1974, 49: 260–276
[29] Runst T, Sickel W. Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations. Berlin: Walter de Gruyter, 1996
[30] Tao J, Xue Q, Yang D, Yuan W. XMO and weighted compact bilinear commutators. J Fourier Anal Appl, 2021, 27(3): Art 60
[31] Tao J, Yang D, Yuan W. A survey on function spaces of John-Nirenberg type. Mathematics, 2021, 9(18): 2264
[32] Tao J, Yang D, Yuan W. Vanishing John-Nirenberg spaces. Adv Calc Var, 2022, 15(4): 831–861
[33] Tao J, Yang D, Yuan W. John-Nirenberg-Campanato spaces. Nonlinear Anal, 2019, 189: 111584
[34] Xiao J. Qα Analysis on Euclidean Spaces. Berlin: De Gruyter, 2019
[35] Xiao J. The transport equation in the scaling invariant Besov or Essén-Janson-Peng-Xiao space. J Differential Equations, 2019, 266(11): 7124–7151
[36] Xiao J, Zhou Y. A reverse quasiconformal composition problem for $Q_\alpha(\mathbb{R}^n)$. Ark Mat, 2019, 57(2): 451–469
[37] Yang S, Chang D C, Yang D, Yuan W. Weighted gradient estimates for elliptic problems with Neumann boundary conditions in Lipschitz and (semi-)convex domains. J Differential Equations, 2020, 268(6): 2510–2550
[38] Yue H. A fractal function related to the John-Nirenberg inequality for $Q_\alpha(\mathbb{R}^n)$. Canad J Math, 2010, 62(5): 1182–1200
[39] Yue H, Dafni G. A John-Nirenberg type inequality for $Q_\alpha(\mathbb{R}^n)$. J Math Anal Appl, 2009, 351(1): 428–439

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