SOME RESULTS ON VARIABLE GAUSSIAN BESOV-LIPSCHITZ AND VARIABLE GAUSSIAN TRIEBEL-LIZORKIN SPACES.

EBNER PINEDA, LUZ RODRIGUEZ, AND WILFREDO O. URBINA

ABSTRACT. In a previous paper [6] two of the authors introduced and study Gaussian Besov-Lipschitz spaces $B^{\alpha,p}_{\gamma_d}(\mathbb{R}^d)$ and Gaussian Triebel-Lizorkin spaces $F^{\alpha,p}_{\gamma_d}(\mathbb{R}^d)$. Now, in this paper we introduce the variable Gaussian Besov-Lipschitz spaces $B^{\alpha,p}_{\cdot \gamma_d}(\mathbb{R}^d)$ and the variable Gaussian Triebel-Lizorkin spaces $F^{\alpha,p}_{\cdot \gamma_d}(\mathbb{R}^d)$, that is to say, Gaussian Besov-Lipschitz and Triebel-Lizorkin spaces with variable exponents, following [7] and [6] under certain additional regularity conditions on the exponents $p(\cdot)$ and $q(\cdot)$ introduced by Dalmasso and Scotto in [7]. Trivially, they include the Gaussian Besov-Lipschitz spaces $B^{\alpha,p}_{\gamma_d}(\mathbb{R}^d)$ and Gaussian Triebel-Lizorkin spaces $F^{\alpha,p}_{\gamma_d}(\mathbb{R}^d)$. We consider some inclusion relations of those spaces and finally we also prove some interpolation results for them.

1. INTRODUCTION AND PRELIMINARIES

Let us consider the Gaussian measure

$$\gamma_d(dx) = e^{-\frac{1}{2}\|x\|^2} dx, \ x \in \mathbb{R}^d$$

on $\mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle.$$ 

Let $\nu = (\nu_1, \ldots, \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \ldots, d$, let $\nu! = \prod_{i=1}^d \nu_i!$, $|\nu| = \sum_{i=1}^d \nu_i$, $\partial_\nu = \frac{\partial}{\partial x}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_{\nu_1} \cdots \partial_{\nu_d}$. Consider the normalized Hermite polynomials of order $\nu$ in $d$ variables,

$$h_\nu(x) = \frac{1}{(2^{\nu!} \nu!)^{1/2}} \prod_{i=1}^d (-1)^{\nu_i} e^{x_i^2} \partial_{\nu_i}^\nu (e^{-x_i^2}).$$

The Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$ is defined by

$$T_t f(x) = \frac{1}{1 - e^{-2t}} \int_{\mathbb{R}^d} e^{-2(x-y)^2} f(y) \gamma_d(dy).$$

Using the Bochner subordination formula

$$e^{-t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} e^{-\frac{1}{4}u^2} du,$$

2010 Mathematics Subject Classification. Primary 42B25, 42B35; Secondary 46E30, 47G10.

Key words and phrases. Hermite expansions, variable exponent, Besov-Lipschitz, Triebel-Lizorkin, Gaussian measure.
we introduce the Poisson-Hermite semigroup by

\[
P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u} u^{\frac{-1}{2}} T_{\sqrt{4u}} f(x) du.
\]

Now, taking the change of variables \( s = \frac{t^2}{4u} \), \( P_t f(x) \) can be written as

\[
P_t f(x) = \mu_t^{(1/2)}(ds),
\]

where

\[
\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds,
\]

is the one-sided stable measure on \((0, \infty)\) of order \(1/2\).

It is easy to see that \( \mu_t^{(1/2)} \) is a probability measure on \((0, \infty)\).

It is well known, that the Hermite polynomials are eigenfunctions of the operator \( L \),

\[
L h_\nu(x) = -|\nu| h_\nu(x).
\]

In consequence

\[
T_t h_\nu(x) = e^{-|\nu|} h_\nu(x),
\]

and

\[
P_t h_\nu(x) = e^{-t\sqrt{\nu}} h_\nu(x),
\]

i.e. the Hermite polynomials are also eigenfunctions of \( T_t \) and \( P_t \) for any \( t \geq 0 \), for more details, see [10].

Next, we present some technical results for the measure \( \mu_t^{(1/2)} \) needed in what follows. First, as \( \mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds \), for any \( k \in \mathbb{N} \), we use the notation \( \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \) for

\[
\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) := \frac{\partial^k g(t, s)}{\partial t^k} ds.
\]

Lemma 1.1. Given \( k \in \mathbb{N} \)

\[
\frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) = \left( \sum_{i,j} a_{i,j} \frac{t^i}{s^j} \right) \mu_t^{(1/2)}(ds)
\]

where \( \{a_{i,j}\} \) is a finite set of constants and the indexes \( i \in \mathbb{Z}, j \in \mathbb{N} \) verifies the equation \( 2j - i = k \).

Lemma 1.2. Given \( k \in \mathbb{N} \) and \( t > 0 \)

\[
\int_0^{+\infty} \frac{1}{s^k} \mu_t^{(1/2)}(ds) = \frac{C_k}{t^{2k}},
\]

where \( C_k = \frac{2^{2k} \Gamma(k+\frac{1}{2})}{\pi^k} \).
Corollary 1.1. Given $k \in \mathbb{N}$ and $t > 0$

(1.12) \[ \int_{0}^{+\infty} \left| \frac{\partial^{k} \mu_{t}^{(1/2)}}{\partial t^{k}} \right| \, (ds) \leq \frac{C_{k}}{t^{k}}. \]

On the other hand, by considering the maximal function of the Ornstein-Uhlenbeck semigroup

\[ T^{*} f(x) = \sup_{t > 0} |T_{t} f(x)|, \]

we obtain the inequality:

Lemma 1.3. Let $f \in L^{1}(\gamma_{d}), x \in \mathbb{R}^{d}$ and $k \in \mathbb{N}$

(1.13) \[ \left| \frac{\partial^{k} P_{t} f(x)}{\partial t^{k}} \right| \leq C_{k} T^{*} f(x)t^{-k}, \quad \forall t > 0. \]

For the proofs of these technical results, see [6] or [10].

Now, for completeness, we need some background on variable Lebesgue spaces with respect to a Borel measure $\mu$.

A $\mu$-measurable function $p(\cdot) : \Omega \subset \mathbb{R}^{d} \to [1, \infty]$ is said to be an exponent function; the set of all the exponent functions will be denoted by $\mathcal{P}(\Omega, \mu)$. For $E \subset \Omega$ we set

\[ p_{-}(E) = \text{ess inf}_{x \in E} p(x) \quad \text{and} \quad p_{+}(E) = \text{ess sup}_{x \in E} p(x). \]

$\Omega_{\infty} = \{ x \in \Omega : p(x) = \infty \}$.

We use the abbreviations $p_{+} = p_{+}(\Omega)$ and $p_{-} = p_{-}(\Omega)$.

Definition 1.1. Let $E \subset \mathbb{R}^{d}$, we say that $p(\cdot) : E \to \mathbb{R}$ is locally log-Hölder continuous, denote by $p(\cdot) \in LH_{0}(E)$, if there exists a constant $C_{1} > 0$ such that

\[ |p(x) - p(y)| \leq \frac{C_{1}}{\log(e + \frac{1}{|x - y|})} \]

for all $x, y \in E$.

We say that $p(\cdot)$ is log-Hölder continuous at infinity with base point at $x_{0} \in \mathbb{R}^{d}$, and denote this by $p(\cdot) \in LH_{\infty}(E)$, if there exist constants $p_{\infty} \in \mathbb{R}$ and $C_{2} > 0$ such that

\[ |p(x) - p_{\infty}| \leq \frac{C_{2}}{\log(e + |x - x_{0}|)} \]

for all $x \in E$.

We say that $p(\cdot)$ is log-Hölder continuous, and denote this by $p(\cdot) \in LH(E)$ if both conditions are satisfied. The maximum, $\max\{C_{1}, C_{2}\}$ is called the log-Hölder constant of $p$.

Definition 1.2. Let $E \subset \mathbb{R}^{d}$, we say that $p(\cdot) \in \mathcal{P}_{d}^{\log}(E)$, if $\frac{1}{p(\cdot)}$ is log-Hölder continuous and denote by $C_{\log}(p)$ or $C_{\log}$ the log-Hölder constant of $\frac{1}{p(\cdot)}$. 
Definition 1.3. Let $\Omega \subset \mathbb{R}^d$ and $p(\cdot) \in \mathcal{P}(\Omega, \mu)$. For a $\mu$-measurable function $f : \Omega \to \mathbb{R}$, we define the modular
\begin{equation}
\rho_{p(\cdot), \mu}(f) = \int_{\Omega(\Omega, \mu)} |f(x)|^{p(x)} \mu(dx) + \|f\|_{L^{p(\cdot)}(\Omega, \mu)},
\end{equation}
and the norm
\begin{equation}
\|f\|_{L^{p(\cdot)}(\Omega, \mu)} = \inf \{\lambda > 0 : \rho_{p(\cdot), \mu}(f / \lambda) \leq 1\}.
\end{equation}

Definition 1.4. The variable exponent Lebesgue space on $\Omega \subset \mathbb{R}^d$, $L^{p(\cdot)}(\Omega, \mu)$ consists on those $\mu$-measurable functions $f$ for which there exists $\lambda > 0$ such that $\rho_{p(\cdot), \mu}(f / \lambda) < \infty$, i.e.
\begin{equation*}
L^{p(\cdot)}(\Omega, \mu) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \rho_{p(\cdot), \mu}(f / \lambda) < \infty, \text{ for some } \lambda > 0 \right\}.
\end{equation*}

Observation 1.1. When $\mu$ is the Lebesgue measure, we write $p(\cdot)$ and $\|f\|_{p(\cdot)}$ instead of $\rho_{p(\cdot), \mu}$ and $\|f\|_{p(\cdot), \mu}$.

Theorem 1.1. (Norm conjugate formula) Let $\nu$ a complete, $\sigma$-finite measure on $\Omega$. $p(\cdot) \in \mathcal{P}(\Omega, \nu)$, then

\begin{equation}
\frac{1}{2} \|f\|_{p(\cdot), \nu} \leq \|f\|_{\nu} \leq 2 \|f\|_{p(\cdot), \nu},
\end{equation}
for all $f$ $\nu$-measurable on $\Omega$.

donde $\|f\|_{p(\cdot), \nu} = \sup \left\{ \int_\Omega |f| \nu d\mu : g \in L^{p'(\cdot)}(\Omega, \nu), \|g\|_{p'(\cdot), \nu} \leq 1 \right\}$.

Proof. See Corollary 3.2.14 in [3]

Theorem 1.2. (Hölder’s inequality) Let $\nu$ a complete, $\sigma$-finite measure on $\Omega$. $r(\cdot), q(\cdot) \in \mathcal{P}(\Omega, \nu)$, define $p(\cdot) \in \mathcal{P}(\Omega, \nu)$ by
\begin{equation}
\frac{1}{p(x)} = \frac{1}{r(x)} + \frac{1}{q(x)} \text{ a.e. } x \in \Omega.
\end{equation}
Then for all $f \in L^{p(\cdot)}(\Omega, \nu)$ and $g \in L^{q(\cdot)}(\Omega, \nu)$, $f g \in L^{p(\cdot)}(\Omega, \nu)$ and
\begin{equation}
\|fg\|_{p(\cdot), \nu} \leq 2 \|f\|_{q(\cdot), \nu} \|g\|_{r(\cdot), \nu}.
\end{equation}

Proof. See Lemma 3.2.20 in [3]

Theorem 1.3. (Minkowski’s integral inequality for variable Lebesgue spaces) Given $\mu$ and $\nu$ complete $\sigma$-finite measures on $X$ and $Y$ respectively, $p \in \mathcal{P}(X, \mu)$. Let $f : X \times Y \to \mathbb{R}$ measurable with respect to the product measure on $X \times Y$, such that for almost every $y \in Y$, $f(\cdot, y) \in L^{p(\cdot)}(X, \mu)$. Then
\begin{equation}
\left\| \int_Y f(\cdot, y) \nu(dy) \right\|_{p(\cdot), \mu} \leq \int_Y \left\| f(\cdot, y) \right\|_{p(\cdot), \mu} \nu(dy)
\end{equation}

Proof. It is completely analogous to the proof of Corollary 2.38 in [1] by interchanging the Lebesgue measure for complete $\sigma$-finite measures $\mu$ and $\nu$ on $X$ and $Y$ respectively, and by using (1.17), Fubini’s theorem and then (1.16).
In what follows $\mu$ represents the measure $\mu(dt) = \frac{dt}{t}$ on $\mathbb{R}^+$.

**Observation 1.2.** For a $\mu$-measurable function $f : \mathbb{R}^+ \to \overline{\mathbb{R}}$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^+, \mu)$, and any $\lambda > 0$

$$\rho_{q(\cdot), \mu}(f) = \int_0^\infty \frac{|f(t)|^{q(t)}\mu(dt)}{\lambda} = \int_0^\infty \left| \frac{\gamma^{1/q(t)} f(t)}{\lambda} \right|^{q(t)} dt$$

Thus,

(1.19) \[ \|f\|_{q(\cdot), \mu} \leq \|\gamma^{1/q(\cdot)} f\|_{q(\cdot)} \]

Next, we present an useful technical results for the measure $\mu$.

**Lemma 1.4.** For $q(\cdot) \in \mathcal{P}(\mathbb{R}^+, \mu)$

i) For any $\alpha, \beta > 0$ and $q_+ < \infty$, $\|t^\alpha e^{-\beta t}\|_{q(\cdot), \mu} < \infty$.

ii) For any $\alpha > 0$, $\|t^\alpha \chi_{(0,1]}\|_{q(\cdot), \mu} < \infty$.

iii) For any $\alpha > 0$, $\|t^{-\alpha} \chi_{(1,\infty)}\|_{q(\cdot), \mu} < \infty$.

iv) For any $t_0 > 0$, $(\ln 2)^{\frac{1}{q(\cdot)}} \leq \|\chi_{(t_0/2, t_0]}\|_{q(\cdot), \mu} \leq 1$.

**Proof.** Let us prove i). Set $f = t^\alpha e^{-\beta t}$

$$\rho_{q(\cdot), \mu}(f) = \int_0^\infty \frac{|f(t)|^{q(t)}\mu(dt)}{\lambda} = \int_0^1 \frac{t^\alpha e^{-\beta t} t^{q(t)} dt}{t} + \int_1^\infty \frac{t^\alpha e^{-\beta t} t^{q(t)} dt}{t}$$

Now,

$$\int_0^1 \frac{t^\alpha e^{-\beta t} t^{q(t)} dt}{t} = \int_0^1 \frac{t^{\alpha q(t)-1} e^{-\beta q(t)} dt}{t} \leq \int_0^1 t^{\alpha-1} dt < \infty,$$

since $\alpha, \beta > 0$ and $0 \leq t \leq 1$. On the other hand, by making the change of variables $u = t\beta q_-$

$$\int_1^\infty \frac{t^\alpha e^{-\beta t} t^{q(t)} dt}{t} = \int_1^\infty \frac{t^{\alpha q(t)} e^{-\beta q(t)} dt}{t} \leq \int_0^\infty \frac{u^{\alpha q_-} e^{-u} du}{u} = \frac{1}{(\beta q_-)^{\alpha q_-}} \Gamma(\alpha q_-) < \infty,$$

since $\alpha, \beta > 0$ and $q_+ < \infty$. Thus, $\rho_{q(\cdot), \mu}(f) < \infty$, and therefore

$$\|t^\alpha e^{-\beta t}\|_{q(\cdot), \mu} < \infty.$$
The proof of ii) and iii) are immediate. Now, in order to prove iv), set \( g = x_{[t_0/2, t]} \)

\[
\rho_{q(\cdot),\mu}(g) = \int_0^\infty |g(t)|^{q(t)} \mu(dt) = \int_{t_0/2}^{t_0} \frac{dt}{t} = \ln 2 < 1.
\]

Then, \( \lambda \geq 1 \) implies \( \rho_{q(\cdot),\mu}(\frac{g}{\lambda}) \leq \rho_{q(\cdot),\mu}(g) \leq 1 \). Thus, \( \|g\|_{q(\cdot),\mu} \leq 1 \).

On the other hand, taking \( 0 < \lambda < 1 \)

\[
\rho_{q(\cdot),\mu}(\frac{g}{\lambda}) = \int_{t_0/2}^{t_0} \lambda^{-\frac{q(t)}{t}} \mu(dt) \geq \int_{t_0/2}^{t_0} \lambda^{-\frac{q}{t}} \mu(dt) = \lambda^{-\frac{q}{t}} \ln 2
\]

then \( \lambda < (\ln 2)^{1/q-} \) implies \( \rho_{q(\cdot),\mu}(\frac{g}{\lambda}) > 1 \). Therefore, \( \rho_{q(\cdot),\mu}(\frac{g}{\lambda}) \leq 1 \) implies \( \lambda \geq (\ln 2)^{1/q-} \) and then

\[
\|g\|_{q(\cdot),\mu} \geq (\ln 2)^{1/q-}.
\]

In the case \( \Omega = \mathbb{R}^+ \), we denote \( M_{0,\infty} \) the set of all measurable functions \( p(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) which satisfy the following conditions:

i) \( 0 \leq p_- \leq p_+ < \infty \),

ii) there exists \( p(0) = \lim_{x \to 0} p(x) \) and \( |p(x) - p(0)| \leq \frac{A}{\ln(1/x)} \), \( 0 < x < 1/2 \)

ii) there exists \( p(\infty) = \lim_{x \to \infty} p(x) \) and \( |p(x) - p(\infty)| \leq \frac{A}{\ln(x)} \), \( x > 2 \).

we denote \( P_{0,\infty} \) the subset of functions \( p(\cdot) \) such that \( p_- \geq 1 \).

Let \( \alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+) \), bounded with

\[
\alpha(0) < \frac{1}{p'(0)}, \alpha(\infty) < \frac{1}{p'(\infty)}
\]

and

\[
\beta(0) > -\frac{1}{p(0)}, \beta(\infty) > -\frac{1}{p(\infty)}
\]

**Theorem 1.4.** Let \( p(\cdot) \in P_{0,\infty} \), \( \alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+) \), bounded. Then the Hardy-type inequalities

\[
\left\| x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)}} dy \right\|_{p(\cdot)} \leq C_{\alpha(\cdot),p(\cdot)} \|f\|_{p(\cdot)}
\]

\[
\left\| x^{\beta(y)} \int_x^\infty \frac{f(y)}{y^{\beta(y)-1}} dy \right\|_{p(\cdot)} \leq C_{\beta(\cdot),p(\cdot)} \|f\|_{p(\cdot)}
\]

are valid, if and only if, \( \alpha(\cdot), \beta(\cdot) \) satisfy conditions (1.20) and (1.21)

**Proof.** For the proof see Theorem 3.1 and Remark 3.2 in [4].

As a consequence, we obtain the Hardy’s inequalities associated to the exponent \( q(\cdot) \in P_{0,\infty} \) and the measure \( \mu \).
Corollary 1.2. Let \( q(\cdot) \in \mathcal{P}_{0,\infty} \) and \( r > 0 \), then
\[
(1.24) \quad \left\| \int_{-t}^{t} \int_{0}^{\infty} g(y)dy \right\|_{q(\cdot)\mu} \leq C_{r,q(\cdot)} \left\| y^{r+1} g \right\|_{q(\cdot)\mu}
\]
and
\[
(1.25) \quad \left\| \int_{t}^{\infty} \int_{0}^{\infty} g(y)dy \right\|_{q(\cdot)\mu} \leq C_{r,q(\cdot)} \left\| y^{r+1} g \right\|_{q(\cdot)\mu}
\]
Proof. Let \( \alpha(t) = -r + \frac{1}{q(\cdot)} = -r + 1 - \frac{1}{q(\cdot)} \), for any \( t \in \mathbb{R}^+ \), \( f(y) = y^{\alpha(t)} g(y) \), for any \( y \in \mathbb{R}^+ \) then \( \alpha(\cdot) \in LH(\mathbb{R}^+) \) and bounded, \( \alpha(0) = -r + \frac{1}{q(\cdot)} < \frac{1}{q(\cdot)} \) and \( \alpha(\infty) = -r + \frac{1}{q(\cdot)} < \frac{1}{q(\cdot)} \). Then, using (1.19) and (1.22)
\[
\left\| \int_{-t}^{t} \int_{0}^{\infty} g(y)dy \right\|_{q(\cdot)\mu} = \left\| \int_{-t}^{t-\frac{1}{q(\cdot)}} \int_{0}^{t} g(y)dy \right\|_{q(\cdot)} = \left\| \int_{0}^{t} \int_{0}^{\infty} g(y)dy \right\|_{q(\cdot)} \leq C_{r,q(\cdot)} \left\| y^{\alpha(t)} g \right\|_{q(\cdot)} = C_{r,q(\cdot)} \left\| y^{r+1} g \right\|_{q(\cdot)\mu}.
\]
On the other hand, by taking \( \beta(t) = r - \frac{1}{q(\cdot)}, \forall t \in \mathbb{R}^+ \), \( f(y) = y^{\beta(t)+1} g(y) \), \( \forall y \in \mathbb{R}^+ \) then \( \beta(\cdot) \in LH(\mathbb{R}^+) \) and the proof of (1.25) is completely analogous.

In what follows we will consider only Lebesgue variable spaces with respect to the Gaussian measure \( \gamma_d \), \( L^{p(\cdot)}(\mathbb{R}^d, \gamma_d) \). The next condition was introduced by E. Dalmasso and R. Scotto in [2] and it is crucial to deal with the Gaussian measure.

Definition 1.5. Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d) \), we say that \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \) if there exist constants \( C_{\gamma_d} > 0 \) and \( p_{\infty} \geq 1 \) such that
\[
(1.26) \quad |p(x) - p_{\infty}| \leq \frac{C_{\gamma_d}}{|x|^2},
\]
for \( x \in \mathbb{R}^d \setminus \{(0,0,\ldots,0)\} \).

Observation 1.3. It can be proved that if \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \), then \( p(\cdot) \in LH_{\infty}(\mathbb{R}^d) \).

2. The main results

In this section we are going to define the variable Gaussian Besov-Lipschitz spaces and the variable Gaussian Triebel-Lizorkin spaces, which are the main goal of the paper.

The following two technical results are needed for defining variable Gaussian Besov-Lipschitz spaces.

Lemma 2.1. Let \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_{0}(\mathbb{R}^d) \) and \( f \in L^{p(\cdot)}(\gamma_d), \alpha \geq 0, y, k, l \) integers greater than \( \alpha \), then
\[
\left\| \frac{\partial^k u(\cdot,t)}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq A_k t^{-k+\alpha} \text{ if and only if } \left\| \frac{\partial^l u(\cdot,t)}{\partial t^l} \right\|_{p(\cdot),\gamma_d} \leq A_l t^{-l+\alpha}.
\]
Moreover, if $A_k(f), A_l(f)$ are the smallest constants in the inequalities above then there exist constants $A_{k, l, \alpha, P(\cdot)}$ and $D_{k, l, \alpha}$ such that

$$A_{k, l, \alpha, P(\cdot)} A_k(f) \leq A_l(f) \leq D_{k, l, \alpha} A_k(f),$$

for all $f \in L^{p(\cdot)}(\gamma_d)$.

**Proof.** Let us suppose without loss of generality that $k \geq l$. We start by proving the direct implication. For this we use the representation of the Poisson-Hermite semigroup (1.6), this is,

$$P_t f(x) = \int_0^{+\infty} T_s f(x) \mu_t^{1/2}(ds).$$

Then, by differentiating $k$-times with respect to $t$ and by using the dominated convergence theorem, we get

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k \mu_t^{1/2}}{\partial t^k}(ds).$$

By using Lemma 1.3, it’s easy to prove that for all $m \in \mathbb{N}$

$$\lim_{t \to +\infty} \frac{\partial^m P_t f(x)}{\partial t^m} = 0.$$ 

Now, given $n \in \mathbb{N}, n > \alpha$

$$- \int_t^{+\infty} \frac{\partial^{n+1} P_s f(x)}{\partial s^{n+1}} ds = - \lim_{s \to +\infty} \frac{\partial^n P_s f(x)}{\partial s^n} + \frac{\partial^n P_t f(x)}{\partial t^n}.$$

Thus, for Minkowski’s integral inequality (1.18)

$$\left\| \frac{\partial^n u(\cdot, t)}{\partial t^n} \right\|_{p(\cdot), \gamma_d} \leq 4 \int_t^{+\infty} \left\| \frac{\partial^{n+1} u(\cdot, s)}{\partial s^{n+1}} \right\|_{p(\cdot), \gamma_d} ds \leq 4 \int_t^{+\infty} A_{n+1}(f) s^{-(n+1)+\alpha} ds = 4 \frac{A_{n+1}(f)}{n-\alpha} t^{-(n+1)-\alpha}.$$ 

Therefore

$$A_n(f) \leq 4 \frac{A_{n+1}(f)}{n-\alpha},$$

and, since $n > \alpha$ is arbitrary, then, by using the above result $k - l$ times, we obtain

$$A_l(f) \leq 4 \frac{A_{l+1}(f)}{l-\alpha} \leq 4 \frac{A_{l+2}(f)}{(l-\alpha)(l+1-\alpha)} \leq \ldots \leq 4^{k-l} \frac{A_k(f)}{(l-\alpha)(l+1-\alpha) \ldots (k-1-\alpha)} = D_{k, l, \alpha} A_k(f).$$

To prove the converse, we use again the representation (1.6) and we obtain that

$$u(x, t_1 + t_2) = P_{t_1}(P_{t_2} f)(x) = \int_0^{+\infty} T_s (P_{t_2} f)(x) \mu_t^{1/2}(ds).$$
Thus, taking $t = t_1 + t_2$ and differentiating $l$ times with respect to $t_2$ and $k - l$ times with respect to $t_1$, we get

$$\frac{\partial^k u(x, t)}{\partial t^k} = \int_0^{+\infty} T_s \left( \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right) \frac{\partial^{k-l} \mu_t}{\partial t_1^{k-l}}(ds).$$

(2.1)

Then, by Corollary 1.1, Minkowski’s integral inequality (1.18) and the $L^{p(\cdot)}$-boundedness of the Ornstein-Uhlenbeck semigroup (see [5]), we get

$$\left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq 4 \int_0^{+\infty} \left\| \frac{\partial^l P_{t_2} f}{\partial t_2^l} \right\|_{p(\cdot), \gamma_d} \left\| \frac{\partial^{k-l} \mu_t}{\partial t_1^{k-l}}(ds) \right\| \leq 4C_{p(\cdot)} \left\| \frac{\partial^l P_{t_2} f}{\partial t_2^l} \right\|_{p(\cdot), \gamma_d} \int_0^{+\infty} \left\| \frac{\partial^{k-l} \mu_t}{\partial t_1^{k-l}}(ds) \right\| \leq 4C_{p(\cdot)} \left\| \frac{\partial^l P_{t_2} f}{\partial t_2^l} \right\|_{p(\cdot), \gamma_d} C_{k-l}^{l-k} \leq 4C_{p(\cdot)} A_l(f) C_{k-l}^{l-\alpha} t_1^{-k}.$$ 

Therefore, taking $t_1 = t_2 = \frac{l}{2}$,

$$\left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq 4C_{p(\cdot)} A_l(f) C_{k-l}^{l-\alpha} \frac{t_1^{l}}{2}^{-k}.$$ 

Thus

$$A_k(f) \leq 4C_{p(\cdot)} \frac{C_{k-l}}{2^{-k+\alpha}} A_l(f).$$

\[\square\]

**Lemma 2.2.** Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0, \infty}$. Let $\alpha \geq 0$ and $k, l$ integers greater than $\alpha$. Then

$$\left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d}^{l-\alpha} \left\| \frac{\partial^l u(\cdot, t)}{\partial t^l} \right\|_{p(\cdot), \gamma_d} < \infty$$

if and only if

$$\left\| \frac{\partial^l u(\cdot, t)}{\partial t^l} \right\|_{p(\cdot), \gamma_d}^{l-\alpha} \left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} < \infty.$$ 

Moreover, there exist constants $A_{k,l,\alpha, p(\cdot)}$ and $D_{k,l,\alpha, q(\cdot)}$ such that

$$D_{k,l,\alpha, q(\cdot)} \left\| \frac{\partial^l u(\cdot, t)}{\partial t^l} \right\|_{p(\cdot), \gamma_d}^{l-\alpha} \left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq A_{k,l,\alpha, p(\cdot)} \left\| \frac{\partial^l u(\cdot, t)}{\partial t^l} \right\|_{p(\cdot), \gamma_d}^{l-\alpha} \left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot), \gamma_d},$$

for all $f \in L^{p(\cdot)}(\gamma_d)$. 

Proof. Suppose without loss of generality that \( k \geq l \). We prove first the converse implication; by proceeding as in lemma 2.1, taking \( t_1 = t_2 = \frac{t}{2} \), we have

\[
\left\| \frac{\partial^k u(t, \cdot)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq 4 C_{p(\cdot)} \left\| \frac{\partial^l p_{\gamma_d}}{\partial t^l_2} \right\|_{p(\cdot), \gamma_d} C_{k-l} \left( \frac{t}{2} \right)^{l-k},
\]

\[
= 4 C_{p(\cdot)} \left\| \frac{\partial^l p_{\gamma_d}}{\partial (\frac{t}{2})^l} \right\|_{p(\cdot), \gamma_d} C_{k-l} \left( \frac{t}{2} \right)^{l-k}.
\]

Thus

\[
\left\| \frac{\partial^k u(t, \cdot)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq 4 C_{p(\cdot)} \frac{C_{k-l}}{2^{l-k}} \left\| \frac{\partial^l u(t, \cdot)}{\partial (\frac{t}{2})^l} \right\|_{p(\cdot), \gamma_d} \left\| \frac{\partial^l s(t, \cdot)}{\partial s^l} \right\|_{p(\cdot), \gamma_d} \mu.
\]

with \( A_{k,l,\alpha,p(\cdot)} = 4 C_{p(\cdot)} C_{k-l} 2^{k-\alpha} \).

For the direct implication, given \( n \in \mathbb{N}, n > \alpha \), again, as in the above lemma

\[
\left\| \frac{\partial^n u(t, \cdot)}{\partial t^n} \right\|_{p(\cdot), \gamma_d} \leq 4 \int_t^{+\infty} \left\| \frac{\partial^{n+1} u(s, \cdot)}{\partial s^{n+1}} \right\|_{p(\cdot), \gamma_d} ds.
\]

Therefore, by the Hardy’s inequality (1.25)

\[
\left\| \frac{\partial^n u(t, \cdot)}{\partial t^n} \right\|_{p(\cdot), \gamma_d} \leq 4 \left\| \frac{\partial^{n+1} u(s, \cdot)}{\partial s^{n+1}} \right\|_{p(\cdot), \gamma_d} \mu.
\]

Now, since \( n > \alpha \) is arbitrary, by using the previous result \( k-l \) times, we obtain

\[
\left\| \frac{\partial^k u(t, \cdot)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \leq 4 C_{l,\alpha,q(\cdot)} \left\| \frac{\partial^{l+1} u(t, \cdot)}{\partial t^{l+1}} \right\|_{p(\cdot), \gamma_d} \mu.
\]

\[
\leq 4^2 C_{l,\alpha,q(\cdot)} C_{l+1,\alpha,q(\cdot)} \left\| \frac{\partial^{l+2} u(t, \cdot)}{\partial t^{l+2}} \right\|_{p(\cdot), \gamma_d} \mu.
\]

\[
\vdots
\]

\[
\leq D_{k,l,\alpha,q(\cdot)} \left\| \frac{\partial^k u(t, \cdot)}{\partial t^k} \right\|_{p(\cdot), \gamma_d} \mu.
\]

where \( D_{k,l,\alpha,q(\cdot)} = 4^{k-l} C_{l,\alpha,q(\cdot)} \cdots C_{k-l,\alpha,q(\cdot)} \). □

The next technical result will be the key to define the variable Gaussian Triebel-Lizorkin spaces.
Lemma 2.3. Let $p(\cdot) \in \mathcal{P}_d^{\infty}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,d}$. Let $\alpha \geq 0$ and $k, l$ integers greater than $\alpha$. Then

$$\left\| t^{-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{q(\cdot)\mu} \right\|_{p(\cdot),\gamma_d} < \infty$$

if and only if

$$\left\| t^{-\alpha} \left\| \frac{\partial^l}{\partial t^l} P_t f \right\|_{q(\cdot)\mu} \right\|_{p(\cdot),\gamma_d} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha,q(\cdot)}, D_{k,l,\alpha,q(\cdot)}$ such that

$$D_{k,l,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^l}{\partial t^l} P_t f \right\|_{q(\cdot)\mu} \right\|_{p(\cdot),\gamma_d} \leq A_{k,l,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{q(\cdot)\mu} \right\|_{p(\cdot),\gamma_d},$$

for all $f \in L^{\infty}(\gamma_d)$.

Proof. Suppose without loss of generality that $k \geq l$. Let $n \in \mathbb{N}$ such that $n > \alpha$, we can prove that

$$\left\| t^{-\alpha} \left\| \frac{\partial^n}{\partial t^n} P_t f(x) \right\|_{q(\cdot)\mu} \right\| = \int_{-\infty}^{+\infty} \left\| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right\|_{q(\cdot)\mu} ds,$$

Then, by the Hardy’s inequality (1.25),

$$\left\| t^{-\alpha} \left\| \frac{\partial^n}{\partial t^n} P_t f(x) \right\|_{q(\cdot)\mu} \right\| \leq \left\| \left\| t^{-\alpha} \int_{-\infty}^{+\infty} \left\| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right\|_{q(\cdot)\mu} \right\| ds \right\| \leq C_{n,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \right\|_{q(\cdot)\mu} \right\|.$$

Now, since $n > \alpha$ is arbitrary, by iterating the previous argument $k - l$ times, we obtain

$$\left\| t^{-\alpha} \left\| \frac{\partial^l}{\partial t^l} P_t f(x) \right\|_{q(\cdot)\mu} \right\| \leq C_{l,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^{l+1}}{\partial t^{l+1}} P_t f(x) \right\|_{q(\cdot)\mu} \right\| \leq C_{l,\alpha,q(\cdot)} C_{l+1,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^{l+2}}{\partial t^{l+2}} P_t f(x) \right\|_{q(\cdot)\mu} \right\| \leq \cdots \leq C_{k,l,\alpha,q(\cdot)} \left\| t^{-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t f(x) \right\|_{q(\cdot)\mu} \right\|.$$
where \( D_{k,l,\alpha,q} = 1/C_{k,l,\alpha,q} \).

The other inequality is obtained from the case \( k = l + 1 \) by an inductive argument. Let \( t_1, t_2 > 0 \) and take \( t = t_1 + t_2 \), from (2.1) we get

\[
\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{t_1} T_s \left( \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right) \frac{\partial^{k-l}}{\partial t_1^{k-l}} \mu_{t_1}^{(1/2)}(ds),
\]

and since, \( \frac{\partial}{\partial t_1} \mu_{t_1}^{(1/2)}(ds) = (t_1^{-1} - \frac{t_1}{2s}) \mu_{t_1}^{(1/2)}(ds) \) we obtain

\[
\left| \frac{\partial^k u(x,t)}{\partial t^k} \right| \leq \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) (t_1^{-1} - \frac{t_1}{2s}) \mu_{t_1}^{(1/2)}(ds)
\]

\[
= t_1^{-1} \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \mu_{t_1}^{(1/2)}(ds) + \frac{t_1}{2} \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \frac{1}{s} \mu_{t_1}^{(1/2)}(ds).
\]

Therefore

\[
\left\| k^{-a} \frac{\partial^k u(x,t)}{\partial t^k} \right\|_{q(\cdot), \mu} \leq \left\| k^{-a} t_1^{-1} \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \mu_{t_1}^{(1/2)}(ds) \right\|_{q(\cdot), \mu}
\]

\[
+ \left\| k^{-a} \frac{t_1}{2} \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \frac{1}{s} \mu_{t_1}^{(1/2)}(ds) \right\|_{q(\cdot), \mu}.
\]

Now, by using Minkowski’s integral inequality twice (1.18) (since \( T_s \) is an integral transformation with positive kernel) and the fact that \( \mu_{t_1}^{(1/2)}(ds) \) is a probability measure, we get

\[
(I) = \left\| k^{-a} t_1^{-1} \int_0^{t_1} T_s \left( \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right) \mu_{t_1}^{(1/2)}(ds) \right\|_{q(\cdot), \mu}
\]

\[
\leq \int_0^{t_1} T_s \left( \left\| k^{-a} t_1^{-1} \int_0^{t_1} \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right\|_{q(\cdot), \mu} \right) \mu_{t_1}^{(1/2)}(ds)
\]

\[
\leq 16 \int_0^{t_1} T_s \left( \left\| k^{-a} t_1^{-1} \int_0^{t_1} \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right\|_{q(\cdot), \mu} \right) \mu_{t_1}^{(1/2)}(ds)
\]

\[
\leq 16 T^* \left( \left\| k^{-a} t_1^{-1} \int_0^{t_1} \left| \frac{\partial^l P_{t_2} f(x)}{\partial t_2^l} \right| \right\|_{q(\cdot), \mu} \right).
\]
For (II) we proceed in analogous way, and by using Lemma 1.2 we get

\[
(II) \leq \frac{16}{2} T^* \left( \left\| t_2^{k-\alpha} t_1 \left| \frac{\partial^t P_{t_2} f(x)}{\partial t_2^t} \right|_{q(t), \mu} \right\| \int_0^{+\infty} \frac{1}{s^t t_1^{1/2}}(ds) \right)
\]

\[
= 8T^* \left( \left\| t_2^{k-\alpha} t_1 \left| \frac{\partial^t P_{t_2} f(x)}{\partial t_2^t} \right|_{q(t), \mu} \right\| C_1 \frac{1}{t_1^t} \right).
\]

Now, since \( T^* \) is defined as a supremum, we get

\[
(II) \leq 8C_1 T^* \left( \left\| t_2^{k-\alpha} t_1^{-1} \left| \frac{\partial^t P_{t_2} f(x)}{\partial t_2^t} \right|_{q(t), \mu} \right\| \right).
\]

Then, taking \( t_1 = t_2 = \frac{1}{2} \) and the change of variable \( s = \frac{1}{2} t \), we have

\[
(I) \leq 16T^* \left( \left\| s^{l-\alpha} \left| \frac{\partial^t P_{s} f(x)}{\partial s^t} \right|_{q(t), \mu} \right\| \right)
\]

and

\[
(II) \leq 8C_1 T^* \left( \left\| s^{l-\alpha} \left| \frac{\partial^t P_{s} f(x)}{\partial s^t} \right|_{q(t), \mu} \right\| \right).
\]

Therefore, by the \( L^{p(\cdot)}(\gamma_d) \)-boundedness of \( T^* \) (see [5]),

\[
\left\| t_2^{k-\alpha} \left| \frac{\partial^k u(t, \cdot)}{\partial t^k} \right|_{q(t), \mu} \right\|_{p(\cdot), \gamma_d} \leq 2^{k-\alpha} 16 \left\| T^* \left( \left\| s^{l-\alpha} \left| \frac{\partial^t P_{s} f(x)}{\partial s^t} \right|_{q(t), \mu} \right\|_{p(\cdot), \gamma_d} \right) + 2^{k-\alpha} 8C_1 \left\| T^* \left( \left\| s^{l-\alpha} \left| \frac{\partial^t P_{s} f(x)}{\partial s^t} \right|_{q(t), \mu} \right\|_{p(\cdot), \gamma_d} \right) \right\|_{p(\cdot), \gamma_d} \right.
\]

\[
\leq 2^{k-\alpha} C_{p(\cdot)} (16 + 8C_1) \left\| s^{l-\alpha} \left| \frac{\partial^t P_{s} f(x)}{\partial s^t} \right|_{q(t), \mu} \right\|_{p(\cdot), \gamma_d}.
\]

Next, we need the following technical result for the \( L^{p(\cdot)}(\gamma_d) \)-norms of the derivatives of the Poisson-Hermite semigroup:

**Lemma 2.4.** Let \( p(\cdot) \in P_{\gamma_d}^\alpha(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \). Suppose that \( f \in L^{p(\cdot)}(\gamma_d) \), then for any integer \( k \),

\[
\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot), \gamma_d} \leq C_{p(\cdot)} \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p(\cdot), \gamma_d}, \text{ for whatever } 0 < s < t < +\infty.
\]

Moreover,

\[
(2.2) \left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot), \gamma_d} \leq C_{k, p(\cdot), \gamma_d} t, \quad t > 0.
\]

**Proof.** First, let us consider the case \( k = 0 \). Fixed \( t_1, t_2 > 0 \), by using the semigroup property of \( \{P_t\} \), we get

\[
P_{t_1+t_2} f(x) = P_{t_1}(P_{t_2} f(x))
\]

Thus, by the \( L^{p(\cdot)} \)-boundedness of \( \{P_t\} \) (see [5]),
\[ \|P_{s+t}f\|_{q,\gamma_d} \leq C_{p(\cdot),q(\cdot)}\|P_s f\|_{q(\cdot),\gamma_d}. \]

In order to prove the general case, \( k > 0 \), using the dominated convergence theorem and differentiating the identity \( u(x, t_1 + t_2) = P_{t_1}(u(x, t_2)) \) \( k \)-times with respect to \( t_2 \) we obtain
\[
\frac{\partial^k u(x, t_1 + t_2)}{\partial (t_1 + t_2)^k} = P_{t_1} \left( \frac{\partial^k u(x, t_2)}{\partial t_2^k} \right)
\]
and then we proceed as in the previous argument. In order to prove (2.4) we use again the representation (1.6) of the Poisson-Hermite semigroup and differentiating \( k \)-times with respect to \( t \) to obtain
\[
\frac{\partial^k}{\partial t^k} u(x, t) = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_{t, s}^{(1/2)}(ds).
\]
Thus, by the Minkowski’s integral inequality, the \( L^{p(\cdot)} \)-boundedness of the Ornstein-Uhlenbeck semigroup (see [5]) and the Corollary 1.1, for \( k > 0 \)
\[
\left\| \frac{\partial^k u(\cdot, t)}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d} \leq 4 \int_0^{+\infty} \left\| T_s f \right\|_{p(\cdot),q(\cdot),\gamma_d} \left\| \frac{\partial^k \mu_{t, s}^{(1/2)}}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d} (ds) \leq 4C_{p(\cdot),q(\cdot),\gamma_d} \int_0^{+\infty} \left\| \frac{\partial^k \mu_{t, s}^{(1/2)}}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d} (ds) \leq 4C_{p(\cdot),q(\cdot),\gamma_d} \int_0^{+\infty} \left\| \frac{\partial^k \mu_{t, s}^{(1/2)}}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d} (ds) \leq C_{k,p(\cdot),q(\cdot),\gamma_d} \left\| f \right\|_{p(\cdot),q(\cdot),\gamma_d}.
\]

The Lipschitz spaces can be generalized of the following way (see, for example [6], [7],[8],[9]), using the Poisson-Hermite semigroup.

We are ready to define the variable Gaussian Besov-Lipschitz spaces \( B^{p, q}_p(\gamma_d) \), also called Gaussian Besov-Lipschitz spaces with variable exponents or variable Besov-Lipschitz spaces for expansions in Hermite polynomials.

**Definition 2.1.** Let \( p(\cdot) \in \mathcal{P}_{0,\infty}^\infty(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) and \( q(\cdot) \in \mathcal{P}_{0,\infty} \). Let \( \alpha \geq 0, k \) the smallest integer greater than \( \alpha \). The variable Gaussian Besov-Lipschitz space \( B^{p, q}_p(\gamma_d) \) is defined as the set of functions \( f \in L^{p(\cdot)}(\gamma_d) \) such that
\[
\left( f \right)^{\alpha} = \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d} < \infty.
\]

The norm of \( f \in B^{p, q}_p(\gamma_d) \) is defined as
\[
\| f \|_{B^{p, q}_p(\gamma_d)} := \| f \|_{p(\cdot),q(\cdot),\gamma_d} + \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),q(\cdot),\gamma_d}.
\]
The variable Gaussian Besov-Lipschitz space $B^{\alpha,\frac{d}{p}}_{p(\cdot),\infty}(\gamma_d)$ is defined as the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ for which there exists a constant $A$ such that

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq A t^{-k+\alpha}$$

and then the norm of $f \in B^{\alpha,\frac{d}{p}}_{p(\cdot),\infty}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha,\frac{d}{p}}_{p(\cdot),\infty}} := \|f\|_{p(\cdot),\gamma_d} + A_k(f),$$

where $A_k(f)$ is the smallest constant $A$ in the above inequality.

Lemmas 2.1 and 2.2 show that we could have replaced $k$ with any other integer $l$ greater than $\alpha$ and the resulting norms are equivalents.

Now, let us study some inclusion relations between variable Gaussian Besov-Lipschitz spaces. The next result is analogous to Proposition 10, page 153 in [7] (see also [6] or Proposition 7.36 in [10]).

**Proposition 2.1.** Let $p(\cdot) \in \mathcal{P}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. The inclusion $B^{\alpha_{1,2}}_{p(\cdot),q(\cdot)}(\gamma_d) \subset B^{\alpha_{1,2}}_{p(\cdot),q(\cdot)}(\gamma_d)$ holds if:

i) $\alpha_1 > \alpha_2 > 0$ ( $q_1(\cdot)$ and $q_2(\cdot)$ not need to be related), or

ii) If $\alpha_1 = \alpha_2$ and $q_1(t) \leq q_2(t)$ a.e.

**Proof.** To prove part ii), let us take $\alpha$ the common value of $\alpha_1$ and $\alpha_2$.

Let $f \in B^{\alpha_{1,2}}_{p(\cdot),q(\cdot)}$ and set $A = \| t^{-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \| q_1(\cdot) \|_{\mu} \cdot \gamma_d$.

Fixed $t_0 > 0$

$$\left\| \chi_{[\frac{t_0}{2}, t_0]} f \right\|_{p(\cdot),\gamma_d} \leq A.$$ 

However, by Lemma 2.4,

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \quad t \in \left[ \frac{t_0}{2}, t_0 \right].$$

Thus, we obtain

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)} \left\| \chi_{[\frac{t_0}{2}, t_0]} f \right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)} A.$$ 

Therefore,

$$\left( \frac{t_0}{2} \right)^{\alpha-\gamma} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)} A.$$
and by Lemma 1.4
\[
\left( \frac{t_0}{2} \right)^{k-\alpha} \left\| \frac{\partial^k P_{t_0} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \geq \left( \frac{t_0}{2} \right)^{k-\alpha} \left\| \frac{\partial^k P_{t_0} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \geq C_{p(\cdot)} A
\]

Then,
\[
\left\| \frac{\partial^k P_{t_0} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq \frac{C_{p(\cdot)} 2^{k-\alpha}}{(\ln 2)^{1/q_1}} A t_0^{k+\alpha},
\]

and since \( t_0 \) is arbitrary
\[
\left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{k,\alpha,p(\cdot)q_1(\cdot)} A t^{-k+\alpha},
\]

for all \( t > 0 \). In other words, \( f \in B^a_{p(\cdot)q_1(\cdot)} \) implies that \( f \in B^a_{p(\cdot),\infty} \).

Now, let us take \( g(t) = t^{k-\alpha} \left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \), then \( \rho_{q_1(\cdot),\mu}(g) < \infty \), since, \( f \in B^a_{p(\cdot)q_1(\cdot)} \).

Thus, as \( q_2(t) \geq q_1(t) \) a.e.

\[
\rho_{q_2(\cdot),\mu}(g) = \int_0^{\infty} \left( k-\alpha \left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right)^{q_2(t)} \frac{dt}{t} \geq \int_0^{\infty} \left( k-\alpha \left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right)^{q_1(t)} \frac{dt}{t} \leq (C_{k,\alpha,p(\cdot)q_1(\cdot)} A)^{q_2(t)-q_1(t)} \int_0^{\infty} \left( k-\alpha \left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right)^{q_1(t)} \frac{dt}{t} = (C_{k,\alpha,p(\cdot)q_1(\cdot)} A)^{q_2(t)-q_1(t)} \rho_{q_1(\cdot),\mu}(g) < +\infty.
\]

Hence, \( f \in B^a_{p(\cdot),q_2(\cdot)} \). In order to prove part i), by Lemma 2.4, we obtain
\[
\left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{k,\alpha} t^{-k}, \quad t > 0.
\]

Now, given \( f \in B^a_{p(\cdot)q_1(\cdot)} \), again by setting
\[
A = \left\| t^{k-\alpha} \left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_1(\cdot),\mu},
\]

we obtain, as in part ii),
\[
\left\| \frac{\partial^k P_{t} f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{k,\alpha_1,p(\cdot)q_1(\cdot)} A t^{-k+\alpha_1},
\]
for all $t > 0$. Therefore,

$$
\left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} \leq \left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} + \left\| \chi^{(1,\infty)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} = (I) + (II).
$$

Now, again by Lemma 1.4 we get,

$$(I) = \left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} \leq \left\| \chi^{(0,1)} t^{k-a_2} C_{k,\alpha_1,p(\cdot)q_1(\cdot)A} t^{-k+\alpha_1} \right\|_{q_2(\cdot),\mu} < \infty,$n

and also by Lemma 1.4,

$$(II) = \left\| \chi^{(1,\infty)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} \leq \left\| \chi^{(1,\infty)} t^{k-a_2} C_{k,p(\cdot)} f^{-\xi} \right\|_{q_2(\cdot),\mu} < \infty.$$

Hence,

$$
\left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} < +\infty,
$$

and then $f \in B^{\gamma_2}_{p(\cdot),\gamma_d}$.

Let us now define the variable Gaussian Triebel-Lizorkin spaces $F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$, which represent another way to measure regularity of functions, proceeding as in [6], [8] or [9].

**Definition 2.2.** Let $p(\cdot) \in \mathcal{P}_{\gamma_d}^{\infty}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$ and $k$ the smallest integer greater than $\alpha$. The variable Gaussian Triebel-Lizorkin space $F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ is the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ such that

$$
(2.6) \quad \left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu} < \infty,
$$

The norm of $f \in F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ is defined as

$$
(2.7) \quad \|f\|_{F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)} := \|f\|_{p(\cdot),\gamma_d} + \left\| \chi^{(0,1)} t^{k-a_2} \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d|q_2(\cdot),\mu}.
$$

By Lemma 2.3, the definition of $F^\alpha_{p(\cdot),q(\cdot)}$ is independent of the integer $k > \alpha$ chosen and the resulting norms are equivalents.
Observation 2.1. The variable Gaussian Besov-Lipschitz and variable Gaussian Triebel-Lizorkin spaces are, by construction, subspaces of $L^{p(\cdot)}(\mathbb{R}^d)$. Moreover, since trivially $\|f\|_{p(\cdot),\gamma_d} \leq \|f\|_{B^\infty_{p(\cdot),q(\cdot)}}$ and $\|f\|_{p(\cdot),\gamma_d} \leq \|f\|_{F^\infty_{p(\cdot),q(\cdot)}}$, the inclusions are continuous. On the other hand, from (1.9) it is clear that for all $t > 0$ and $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial t^k} P_t h_\beta(x) = (-1)^k |\beta|^{k/2} e^{-t \sqrt{|\beta|}} h_\beta(x),$$

and again by Lemma 1.4,

$$\left\| t^{-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t h_\beta \right\|_{p(\cdot),\gamma_d} \right\|_{p(\cdot),\gamma_d} = \left\| t^{-\alpha} \left\| (-|\beta|^{1/2})^{k} e^{-t \sqrt{|\beta|}} h_\beta \right\|_{p(\cdot),\gamma_d} \right\|_{p(\cdot),\gamma_d} = |\beta|^{k/2} \| h_\beta \|_{p(\cdot),\gamma_d} \left\| t^{-\alpha} e^{-t \sqrt{\theta}} \right\|_{p(\cdot),\gamma_d} = C_{k,\alpha,\beta,q(\cdot)} \| h_\beta \|_{p(\cdot),\gamma_d} < \infty.$$

Thus, $h_\beta \in B^\alpha_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and

$$\| h_\beta \|_{p(\cdot),q(\cdot)} = (1 + C_{k,\alpha,\beta,q(\cdot)}) \| h_\beta \|_{p(\cdot),\gamma_d}.$$

In a similar way, $h_\beta \in F^\alpha_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and

$$\| h_\beta \|_{F^\alpha_{p(\cdot),q(\cdot)}} = \| h_\beta \|_{p(\cdot),\gamma_d} + \left\| t^{-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t h_\beta \right\|_{q(\cdot),\mu} \right\|_{p(\cdot),\gamma_d} = \| h_\beta \|_{p(\cdot),\gamma_d} + |\beta|^{k/2} \left\| t^{-\alpha} e^{-t \sqrt{\theta}} \right\|_{q(\cdot),\mu} \| h_\beta \|_{p(\cdot),\gamma_d} = (1 + C_{k,\alpha,\beta,q(\cdot)}) \| h_\beta \|_{p(\cdot),q(\cdot)}.$$

Hence, the polynomials $P$ is contained in $B^\alpha_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$ and in $F^\alpha_{p(\cdot),q(\cdot)}(\mathbb{R}^d)$.

Also, we have an inclusion result for the variable Gaussian Triebel-Lizorkin spaces, which is analogous to Proposition 2.1, see also [6] or Proposition 7.40 in [10].

**Proposition 2.2.** Let $p(\cdot) \in \mathcal{P}_\gamma(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. The inclusion $F_{p(\cdot),q_1(\cdot)}(\mathbb{R}^d) \subset F_{p(\cdot),q_2(\cdot)}(\mathbb{R}^d)$ holds for $\sigma_1 > \sigma_2 > 0$ and $q_1(t) > q_2(t)$ a.e.

**Proof.** Let us consider $f \in F_{p(\cdot),q_1(\cdot)}$, then

$$\left\| t^{-\alpha_2} \left\| \frac{\partial^k}{\partial t^k} P_t f(x) \right\|_{q_2(\cdot),\mu} \right\|_{q_2(\cdot),\mu} \leq \left\| t^{-\alpha_2} \left\| \frac{\partial^k}{\partial t^k} P_t f(x) \right\|_{q_1(\cdot),\mu} \right\|_{q_1(\cdot),\mu} + \left\| t^{-\alpha_2} \left\| \frac{\partial^k}{\partial t^k} P_t f(x) \right\|_{q_1(\cdot),\mu} \right\|_{q_1(\cdot),\mu} = (I) + (II).$$
Now, since \( q_1(t) > q_2(t) \) a.e., by taking \( r(t) = \frac{q_1(t)q_2(t)}{q_1(t) - q_2(t)} \), we obtain that \( r(\cdot) \geq 1 \) and \( \frac{1}{r(\cdot)} + \frac{1}{q_1(\cdot)} = \frac{1}{q_2(\cdot)} \), thus, by Hölder’s inequality (1.17) and Lemma 1.4

\[
(I) = \left\| \alpha_1^{a_2} \chi_{(0,1]}^{k-a_1} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_2(\cdot)} \\
\leq 2 \left\| \alpha_1^{a_2} \chi_{(0,1]} \right\|_{r(\cdot)} \left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_1(\cdot)} \\
= C_{\alpha_1,a_2,q_1(\cdot)} \left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_1(\cdot)}.
\]

Now, for the second term (II), by using Lemmas 1.4 and 1.3, we get

\[
(II) = \left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \chi_{(1,\infty]} \right\|_{q_2(\cdot)} \\
\leq C_k T^* f(x) \left\| \alpha_1^{a_2} \right\|_{q_2(\cdot)} = C_{k,a_2,q_2(\cdot)} T^* f(x).
\]

Then, by using the \( L^{p(\cdot)}(\gamma_d) \) boundedness of \( T^* \) (see [5]),

\[
\left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_2(\cdot)} \\
\leq C_{\alpha_1,a_2,q_1(\cdot),q_2(\cdot)} \left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_1(\cdot)} + C_{k,a_2,q_2(\cdot)} \left\| \alpha_1^{a_2} \right\|_{p(\cdot),\gamma_d} \\
\leq C_{\alpha_1,a_2,q_1(\cdot),q_2(\cdot)} \left\| \alpha_1^{a_2} \left| \frac{\partial^k P_1 f(x)}{\partial x^k} \right| \right\|_{q_1(\cdot)} + C_{k,a_2,p(\cdot),q_2(\cdot)} \left\| \alpha_1^{a_2} \right\|_{p(\cdot),\gamma_d} < +\infty.
\]

Therefore, \( f \in F_{\alpha_2}^{a_2}(p(\cdot),\gamma_d) \).

\[\square\]

3. Interpolation Results

Finally, we are going to consider some interpolation results for the Gaussian variable Besov-Lipschitz and the variable Triebel-Lizorkin spaces.

We will use the following results for general variable Lebesgue spaces \( L^{p(\cdot)}(X,\nu) \).

**Lemma 3.1.** Let \( p(\cdot) \in \mathcal{P}(\Omega,\nu) \) and \( s > 0 \) such that \( sp^- \geq 1 \). Then

\[
\|f\|_{p(\cdot),\nu} = \|f\|_{s p(\cdot),\nu}^s.
\]

**Proof.** It is the same proof of Lemma 3.2.6 in [3]. \[\square\]

**Lemma 3.2.** Let \( \nu \) a complete \( \sigma \)-finite measure on \( X \). \( r_j(\cdot) \in \mathcal{P}(X,\nu), \) \( 1 < r_j^-, r_j^+ < \infty, j = 0, 1 \). For all \( 0 < \lambda < 1, \) if \( f \in L^{r(\cdot)}(X,\nu), \) \( j = 0, 1 \) then \( f \in L^{r(\cdot)}(X,\nu) \) where

\[
\frac{1}{r(y)} = \frac{1 - \lambda}{r_0(y)} + \frac{\lambda}{r_1(y)}, \text{ a.e. } y \in X \text{ and } \nu
\]

\[
\tag{3.1}
\|f\|_{r(\cdot),\nu} \leq 2\|f\|_{r_0(\cdot),\nu}^{1-\lambda}\|f\|_{r_1(\cdot),\nu}^\lambda.
\]
Proof. It is a consequence of Hölder’s inequality (1.17) and Lemma 3.1. □

Now, we present the interpolation result.

**Theorem 3.1.** Let \( p_j(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d) \) and \( q_j \in \mathcal{P}(\mathbb{R}^+, \mu), j = 0, 1 \)

1. For \( 1 < p_j^+, q_j^- < +\infty \) and \( \alpha_j \geq 0 \), if \( f \in B_{p_j(\cdot),q_j(\cdot)}^{\alpha_j}(\gamma_d), j = 0, 1 \),
then for all \( 0 < \theta < 1 \), \( f \in B_{p(\cdot),q(\cdot)}^{\alpha}(\gamma_d) \), where
\[
\alpha = \alpha_0(1 - \theta) + \alpha_1 \theta
\]
and
\[
\begin{align*}
\frac{1}{p(x)} &= \frac{1 - \theta}{p_0(x)} + \frac{\theta}{p_1(x)}, \text{ a.e. } x \in \mathbb{R}^d, \\
\frac{1}{q(t)} &= \frac{1 - \theta}{q_0(t)} + \frac{\theta}{q_1(t)}, \text{ a.e. } t \in \mathbb{R}^+.
\end{align*}
\]

2. For \( 1 < p_j^+, q_j^- < +\infty \) and \( \alpha_j \geq 0 \), if \( f \in F_{p_j(\cdot),q_j(\cdot)}^{\alpha_j}(\gamma_d), j = 0, 1 \),
then for all \( 0 < \theta < 1 \), \( f \in F_{p(\cdot),q(\cdot)}^{\alpha}(\gamma_d) \), where
\[
\alpha = \alpha_0(1 - \theta) + \alpha_1 \theta
\]
and
\[
\begin{align*}
\frac{1}{p(x)} &= \frac{1 - \theta}{p_0(x)} + \frac{\theta}{p_1(x)}, \text{ a.e. } x \in \mathbb{R}^d, \\
\frac{1}{q(t)} &= \frac{1 - \theta}{q_0(t)} + \frac{\theta}{q_1(t)}, \text{ a.e. } t \in \mathbb{R}^+.
\end{align*}
\]

Proof. i) Let \( k \) be an integer greater than \( \alpha_0 \) and \( \alpha_1 \), by using Lemma 3.2, we obtain for \( \alpha = \alpha_0(1 - \theta) + \alpha_1 \theta \),
\[
\left\| k^{-\alpha} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \right\|_{p(\cdot),\gamma \mu}^{1-\theta} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{q(\cdot),\mu}^\theta
\]
\[
\leq \left\| k^{-(\alpha_0(1-\theta)+\alpha_1 \theta)} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \right\|_{p_0(\cdot),\gamma_d}^{1-\theta} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_1(\cdot),\gamma_d}^\theta
\]
\[
= 2 \left\| f^{(1-\theta)(k-\alpha_0)+\theta(k-\alpha_1)} \right\|_{p_0(\cdot),\gamma_d} \left( \frac{\partial^k P_t f}{\partial t^k} \right)^{1-\theta} \left( \frac{\partial^k P_t f}{\partial t^k} \right)^\theta
\]
\[
= 2 \left\| k^{-\alpha_0} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \left( \frac{\partial^k P_t f}{\partial t^k} \right)^{\alpha_1} \left( \frac{\partial^k P_t f}{\partial t^k} \right)^{\alpha_1} \left( \frac{\partial^k P_t f}{\partial t^k} \right)^\theta \right\|_{p_0(\cdot),\gamma_d} \left( \frac{\partial^k P_t f}{\partial t^k} \right)^\theta
\]
Thus, by Hölder’s inequality (1.17) and Lemma 3.1,
\[
\left\| k^{-\alpha} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \right\|_{p(\cdot),\gamma \mu}^{1-\theta} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{q(\cdot),\mu}^\theta
\]
\[
\leq 4 \left\| k^{-\alpha_0} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \right\|_{p_0(\cdot),\gamma_d}^{1-\theta} \left\| k^{-\alpha_1} \left( \frac{\partial^k P_t f}{\partial t^k} \right) \right\|_{p_1(\cdot),\gamma_d}^{\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{q(\cdot),\mu}^\theta < +\infty,
\]
that is, \( f \in B_{p(\cdot),q(\cdot)}^{\alpha}(\gamma_d) \).

ii) Analogously, by Hölder’s inequality (1.17) and Lemma 3.1, we obtain for \( \alpha = \alpha_0(1-\theta) + \alpha_1 \theta \),

\[
\left\| k^{\alpha} \frac{\partial^k f(x)}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq 2 \left\| k^{\alpha_0} \frac{\partial^k f(x)}{\partial t^k} \right\|_{p(\cdot),\gamma_d}^{1-\theta} \left\| k^{\alpha_1} \frac{\partial^k f(x)}{\partial t^k} \right\|_{q(\cdot),\gamma_d}^\theta,
\]
a.e. \( x \in \mathbb{R}^d \). Therefore

\[
\left\| k^{\alpha} \frac{\partial^k f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq 2 \left\| k^{\alpha_0} \frac{\partial^k f}{\partial t^k} \right\|_{p(\cdot),\gamma_d}^{1-\theta} \left\| k^{\alpha_1} \frac{\partial^k f}{\partial t^k} \right\|_{q(\cdot),\gamma_d}^\theta,
\]

and again by Hölder’s inequality and Lemma 3.1,

\[
\left\| k^{\alpha} \frac{\partial^k f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq 4 \left\| k^{\alpha_0} \frac{\partial^k f}{\partial t^k} \right\|_{p(\cdot),\gamma_d}^{1-\theta} \left\| k^{\alpha_1} \frac{\partial^k f}{\partial t^k} \right\|_{q(\cdot),\gamma_d}^\theta < +\infty.
\]

That is, \( f \in F_{p(\cdot),q(\cdot)}^{\alpha}(\gamma_d) \).

References

[1] Cruz-Uribe, D. & Fiorenza, A. *Variable Lebesgue Spaces Foundations and Harmonic Analysis*, Applied and Numerical Harmonic Analysis Birkhäuser-Springer, Basel, (2013).

[2] Dalmasso, E. & Scotto, R. (2017) *Riesz transforms on variable Lebesgue spaces with Gaussian measure*, Integral Transforms and Special Functions, 28:5, 403-420. DOI: 10.1080/10652469.2017.1296835

[3] Diening, L., Harjulehto, P., Hästö, P. and Růžička, M. *Lebesgue and Sobolev spaces with variable exponents*. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.

[4] Diening, L., Samko, S. *Hardy inequality in variable exponent Lebesgue spaces*. Fractional Calculus & Applied Analysis, Volume 10, Number 1 (2007).

[5] Moreno, J., Pineda, E., & Urbina, W. *Boundedness of the maximal function of the Ornstein-Uhlenbeck semigroup on variable Lebesgue spaces with respect to the Gaussian measure and consequences*. Revista Colombiana de Matemáticas (2021) (to Appear)

[6] Pineda, E., Urbina, W. *Some results on Gaussian Besov-Lipschitz and Gaussian Triebel-Lizorkin spaces*. J. Approx. Theor.161(2), 529-564 (2009)

[7] Stein E. *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press. Princeton, New Jersey. (1970).

[8] Triebel, H *Theory of function spaces*, Birkhäuser Verlag. Basel. (1983).

[9] Triebel, H *Theory of function spaces II*, Birkhäuser Verlag, Basel. (1992).

[10] Urbina W. *Gaussian Harmonic Analysis*, Springer Monographs in Math. Springer Verlag, Switzerland AG (2019).
EBNER PINEDA, LUZ RODRIGUEZ, AND WILFREDO O. URBINA

Escuela Superior Politécnica del Litoral. ESPOL, FCNM, Campus Gustavo Galindo Km. 30.5
Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, ECUADOR.
Email address: epineda@espol.edu.ec

Escuela Superior Politécnica del Litoral. ESPOL, FCNM, Campus Gustavo Galindo Km. 30.5
Vía Perimetral, P.O. Box 09-01-5863, Guayaquil, ECUADOR.
Email address: luzeurod@espol.edu.ec

Department of Mathematics, Actuarial Sciences and Economics, Roosevelt University, Chicago,
IL, 60605, USA.
Email address: wurbinaromero@roosevelt.edu