Univariate tight wavelet frames of minimal support

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Abstract
This work characterizes (dyadic homogeneous) wavelet frames for $L^2(\mathbb{R})$ by means of spectral techniques. These techniques use decomposability properties of the frame operator in spectral representations associated with the dilation operator. The approach is closely related to usual Fourier domain fiberization techniques, dual Gramian analysis, and extension principles. Spectral formulas are used to determine all the tight wavelet frames for $L^2(\mathbb{R})$ with a fixed finite number of generators of minimal support. The method associates wavelet frames of this type with certain inner operator-valued functions in Hardy spaces. The cases with one and two generators are completely solved.

Keywords Wavelet frames \cdot Spectral techniques \cdot Hardy spaces \cdot Inner functions

Mathematics Subject Classification 42C15 \cdot 47B15 \cdot 30J05

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with norm $| | \cdot | |_{\mathcal{H}}$ and scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ (linear in the first component and conjugate-linear in the second). A countable subset $X$ of $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$, such that the following inequalities hold:

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\[
A \|f\|^2 \leq \sum_{x \in X} |\langle f, x \rangle_{\mathcal{H}}|^2 \leq B \|f\|^2, \quad (f \in \mathcal{H}).
\]

(1)

It is well known (see, e.g., [7, Chapter 5]) that \(X\) is a frame for \(\mathcal{H}\) if and only if the corresponding synthesis operator:

\[
T_X : \ell^2(X) \to \mathcal{H}, \quad T_X \{c_x\}_{x \in X} := \sum_{x \in X} c_x x
\]

(2)
is well defined and bounded from \(\ell^2(X)\) onto \(\mathcal{H}\). In such a case, the analysis operator is the adjoint operator of \(T_X\), given by \(T_X^* : \mathcal{H} \to \ell^2(X), T_X^* f = \{\langle f, x \rangle_{\mathcal{H}}\}_{x \in X}\), and the frame operator \(S = T_X T_X^*\):

\[
S : \mathcal{H} \to \mathcal{H}, \quad Sf = T_X T_X^* f = \sum_{x \in X} \langle f, x \rangle_{\mathcal{H}} x,
\]

(3)
is bounded, positive, and invertible. Furthermore, \(S^{-1} X = \{S^{-1} x : x \in X\}\) is also a frame, called the canonical dual frame of \(X\), and the “perfect reconstruction formula”:

\[
f = \sum_{x \in X} \langle f, S^{-1} x \rangle_{\mathcal{H}} x, \quad (f \in \mathcal{H}),
\]

(4)
is satisfied. In (3) and (4), the series converges unconditionally for all \(f \in \mathcal{H}\), i.e., for every permutation of the summands the resulting series is convergent. For a frame \(X\), the sharpest possible constants \(A, B\) in (1) are \(A = ||S^{-1}||^{-1}\) and \(B = ||S|| = ||T_X||^2 = ||T_X^*||^2\) and are usually referred to as the frame bounds. A frame \(X\) is called a tight frame if its frame bounds coincide.

In this work, we focus attention on \(\mathcal{H} = L^2(\mathbb{R})\) and a special type of subsets, and the (dyadic homogeneous) wavelet systems of \(L^2(\mathbb{R})\) of the form:

\[
X = X_{\Psi} := \{\psi_{k,j} := D^k T^j \psi : \psi \in \Psi, k, j \in \mathbb{Z}\},
\]

(5)
where \(\Psi\) is a finite or countable family of \(L^2(\mathbb{R})\), and \(T\) and \(D\) are the translation and (dyadic) dilation operators on \(L^2(\mathbb{R})\) defined by:

\[
[Tf](x) := f(x - 1), \quad [Df](x) := 2^{1/2} f(2x), \quad (f \in L^2(\mathbb{R}), x \in \mathbb{R}).
\]

(6)
Wavelet systems of the form (5) which are frames for \(L^2(\mathbb{R})\) are called (dyadic homogeneous) wavelet frames for \(L^2(\mathbb{R})\) with generator set \(\Psi\).

Wavelet frames for \(L^2(\mathbb{R}^d)\) have been extensively studied in the last 3 decades. One of the main lines of study involves fiberization techniques [12, 20, 35–40], translation (shift) invariant subspaces [6, 35], and refinable functions [2, 5], leading to the unitary extension principle (UEP) [3, 9, 36, 37], subsequently extended in the form of oblique extension principle (OEP) [1, 10, 12, 21] and duality principle [13, 14]. Since the introduction of the extension principles, the main part of the literature devoted to the construction of wavelet frames uses them looking for the corresponding framelet filter banks and paying attention to properties like vanishing moments, symmetry, number of generators, support, etc. In
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the univariate case, see, e.g., [8–12, 23–25, 29, 37, 45]. The multivariate case in \( L^2(\mathbb{R}^d) \) is discussed in, e.g., [14, 22, 28, 31, 34, 44]. See the cited references and the recent book [26] for an exhaustive overview.

Here, we present an alternative approach to study wavelet frames for \( L^2(\mathbb{R}) \), an approach already introduced in [15–18] for the analysis of orthonormal wavelets. The adjective “spectral” for the techniques we develop comes from the use of suitable spectral representations for the translation and dilation operators \( T \) and \( D \). Such spectral representations are given in Sect. 2.

Section 3 contains some results derived from the spectral techniques for wavelet frames. Theorem 5 characterizes the wavelet systems \( X \) of the form (5) that are frames and tight frames for \( L^2(\mathbb{R}) \). The result is based on the fact that the frame operator \( S = T_X^* T_X \) is decomposable (diagonal for tight frames) in the direct integral associated with the spectral representation of the dilation operator \( D \). The matrix elements and fibers of the decomposable expression of \( S \) are given in Theorem 8. Theorems 5 and 8 lead to an affordable description of tight wavelet frames in Corollary 9. Let us note that the usual fiberization techniques work on the Fourier domain, i.e., on a spectral representation of the translation operator \( T \), in spite of wavelet systems of the form (5) are dilation invariant but not translation (shift) invariant; see Remark 6 for details.

Section 4 shows how by means of Corollary 9, it is possible to determine all the tight wavelet frames for \( L^2(\mathbb{R}) \) with a fixed number (say \( r \)) of generators of minimal support. We solve explicitly the cases \( r = 1 \) and \( r = 2 \). To our knowledge, existing literature has not deal with this problem. In the context of extension principles, the tight framelet filter banks with \( r = 1 \) are characterized in [25, Theorem 7]. For \( r = 2 \), Theorem 4.2 in [23] gives the tight framelet filter banks with complex symmetry and other partial results can be found in, e.g., [9, 10, 12, 37, 45]. In particular, the case with B-splines as refinable functions has been extensively studied; see, e.g., [14, Section 4.4]. See also [26] and references therein.

Like in [15], Hardy classes [19, 41, 42] play a central role here. In particular, operator-valued functions called rigid Taylor operator functions by Halmos [19] and \( M^+ \)-inner functions by Rosenblum and Rovnyak [41]. See Section 4.1 below for details. Roughly speaking, Lemma 12 (Halmos), Lemma 13 (Rovnyak), and Proposition 14 imply the following result:

Let \( X_\Psi \) be a wavelet system of the form (5), with cardinal of \( \Psi \) finite, say \( r \), and such that the support of each \( \psi \in \Psi \) is included in the interval \([0, 1]\). Then, \( X_\Psi \) is a tight wavelet frame for \( L^2(\mathbb{R}) \) if and only if \( \Psi \) is associated with an \( M^+\)-inner \((r \times r)\)-matrix function \( A^+ (\omega) \) satisfying certain properties.

This result comes from a particular choice of orthonormal bases \( \{L_i^{(n)}\} \) and \( \{K_{x,j}^{(m)}\} \) in the spectral method, the Haar orthonormal bases given in Appendix B, and the corresponding distribution of indices in the set of equations (19)—see Proposition 11, in particular, Table 1.

We discuss the cases \( r = 1 \) and \( r = 2 \) in Sects. 4.2 and 4.3, respectively. For \( r = 1 \) the solution is given in Corollary 15:
The only function \( \psi \in L^2[0, 1] \) such that the wavelet system \( X_\psi \) of the form \((5)\) generated by \( \Psi = \{ \psi \} \) is a tight frame for \( L^2(\mathbb{R}) \), with frame bound \( B \), is proportional to the Haar wavelet:

\[
\psi = \beta [\chi_{[0,1/2)} - \chi_{(1/2,1]}],
\]

where \( \beta \in \mathbb{C} \) and \( |\beta|^2 = B \). It is associated with the constant \( M^+ \)-inner scalar function

\[
A^+ : \partial \mathbb{D} \to \mathbb{C} : \omega \mapsto \beta |\beta|.
\]

This result invalidates Theorem 5 in [15] (see Remark 16 for details).

For \( r = 2 \), i.e., \( \Psi = \{ \psi_1, \psi_2 \} \subset L^2[0, 1] \), the \( M^+ \)-inner \((2 \times 2)\)-matrix functions \( A^+(\omega) \) of interest appear in Proposition 19 and the final solution is given in Propositions 20 and 21:

There are five types of families of \( M^+ \)-inner \((2 \times 2)\)-matrix functions

\[
A^+(\omega) = \begin{pmatrix} a^{(0)}(\omega) & a^{(1)}(\omega) \\ a^{(0)}_2(\omega) & a^{(1)}_2(\omega) \end{pmatrix} = \begin{pmatrix} a^{(0)}(\omega) & a^{(1)}(\omega) \end{pmatrix}
\]

leading to tight wavelet frames \( X_\psi \) of the form \((5)\) generated by \( \Psi \).

To obtain \( \psi_1 \) and \( \psi_2 \) from the function \( A^+(\omega) \), one must proceed in the following way. Since \( \text{supp} \psi_j \subset [0, 1], \) \( (j = 1, 2) \), their expansions in the Haar orthonormal basis \( \{ L_i^{(a)} \} \) — see Eq. \((10)\) — read:

\[
\psi_j = \sum_{i \in \mathbb{N} \cup \{0\}} [\hat{\psi}_j]_i^{(a)} L_i^{(a)}, \quad (j = 1, 2).
\]

Let us write:

\[
\Psi_i := \begin{pmatrix} [\hat{\psi}_1]_i^{(a)} \\ [\hat{\psi}_2]_i^{(a)} \end{pmatrix} \in \mathbb{C}^2, \quad (i \in \mathbb{N} \cup \{0\}).
\]
Then, for a frame bound $B > 0$:

$$\sum_{k=0}^{\infty} \omega^k \Psi_{2^k} = B_0 a^{(0)}(\omega),$$

$$\sum_{k=0}^{\infty} \omega^k \Psi_{2^{p+k+1}+l} = C_l a^{(1)}(\omega), \quad (p \geq 0, 2^p \leq l < 2^{p+1}),$$

where $B_0 \in \mathbb{C}$ is such that $|B_0|^2 = B$ and the sequence $\{C_l\}_{l \in \mathbb{N}} \subset \mathbb{C}$ is given by (35)–(38). Corollary 22 gives an explicit description of the coefficients $\{\Psi_i\}_{i \geq 0}$ for the five families of wavelet frames $\{\psi_1, \psi_2\}$ with support in $[0, 1]$. For type 1 functions $A^+(\omega)$, both functions $\psi_1, \psi_2$ are proportional to the Haar wavelet. For types 2–5 functions $A^+(\omega)$, some examples of real functions $\psi_1, \psi_2$ are shown in Appendix A. Corollary 22 permits us to study properties of $\psi_1$ and $\psi_2$ such as vanishing moments, symmetry, etc. Due to the restrictive condition of minimal support, we cannot expect good properties (see Remark 23 for details). Really, such properties are not relevant here. The objective of this work is to investigate connections between wavelet frames theory and Halmos–Helson theory of invariant and wandering subspaces [30], and Hardy classes and operator theory [19, 41].

Another important wavelet systems are nonhomogeneous wavelet systems given by:

$$X_{\phi,\Psi} := \{T^j \phi : \phi \in \Phi, j \in \mathbb{Z}\} \cup \{D^k T^j \psi : \psi \in \Psi, k, j \in \mathbb{Z}\},$$

where $\Phi, \Psi$ are finite or countable families of $L^2(\mathbb{R})$. Nonhomogeneous wavelet systems are closely related to refinable structures and filter banks, and have been systematically studied in [22]. The problem of linking homogeneous wavelet systems to nonhomogeneous wavelet systems has been considered in [27]. In Remark 24, we tackle the problem of linking the homogeneous tight wavelet frames $X_{\Psi}$ with $\Psi = \{\psi_1, \psi_2\}$ of minimal support to nonhomogeneous ones. Proposition 25 shows that each homogeneous tight wavelet frame $X_{\Psi}$ with $\Psi = \{\psi_1, \psi_2\}$ of minimal support can be linked to a nonhomogeneous tight wavelet frame $X_{\phi,\Psi}$ and a refinable structure. Moreover, $\Phi$ is described explicitly. This work is only a first step in understanding connections between $M^+$-inner matrix functions and filter banks.

## 2 Spectral representations for $T$ and $D$

We begin by introducing spectral representations for the translation and dilation operators, $T$ and $D$, defined on $L^2(\mathbb{R})$ by (6). These representations have already been considered in [15–18]. They live in spaces of the form $L^2(\partial \mathbb{D}; \mathcal{H})$ we next define.

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $\partial \mathbb{D}$ its boundary:

$$\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \quad \partial \mathbb{D} := \{\omega \in \mathbb{C} : |\omega| = 1\}.$$
In \( \partial \mathbb{D} \), interpret measurability in the sense of Borel and consider the normalized Lebesgue measure \( d\omega/(2\pi) \). Given a separable Hilbert space \( \mathcal{H} \), let \( L^2(\partial \mathbb{D}; \mathcal{H}) \), denote the set of all measurable functions \( v : \partial \mathbb{D} \rightarrow \mathcal{H} \), such that:

\[
\int_{\partial \mathbb{D}} ||v(\omega)||^2 \frac{d\omega}{2\pi} < \infty
\]

(modulo sets of measure zero); measurability here can be interpreted either strongly or weakly, which amounts to the same due to the separability of \( \mathcal{H} \). The functions in \( L^2(\partial \mathbb{D}; \mathcal{H}) \) constitute a Hilbert space with pointwise definition of linear operations and inner product given by:

\[
\langle u, v \rangle_{L^2(\partial \mathbb{D}; \mathcal{H})} := \int_{\partial \mathbb{D}} \langle u(\omega), v(\omega) \rangle_{\mathcal{H}} \frac{d\omega}{2\pi}, \quad (u, v \in L^2(\partial \mathbb{D}; \mathcal{H})).
\]

The space \( L^2(\partial \mathbb{D}; \mathcal{H}) \) is a particular case of direct integral of Hilbert spaces; see [33, Chapter 14] for details.

A bounded operator \( S : L^2(\partial \mathbb{D}; \mathcal{H}) \rightarrow L^2(\partial \mathbb{D}; \mathcal{H}) \) is said to be decomposable when there is a function \( \omega \mapsto S(\omega) \) on \( \partial \mathbb{D} \), such that \( S(\omega) : \mathcal{H} \rightarrow \mathcal{H} \) is a bounded operator and for each \( u \in L^2(\partial \mathbb{D}; \mathcal{H}) \), \( S(\omega)u(\omega) = [Su](\omega) \) for almost every (shortly, a.e.) \( \omega \in \partial \mathbb{D} \). For a decomposable operator \( S \), we shall write:

\[
S = S(\omega).
\]

If, in addition, \( S(\omega) = s(\omega)I_{\mathcal{H}} \), where \( I_{\mathcal{H}} \) is the identity operator on \( \mathcal{H} \) and \( s : \partial \mathbb{D} \rightarrow \mathbb{C} \) is a measurable function, we say that \( S \) is diagonalizable and write \( S = s(\omega)I_{\mathcal{H}} \).

For the sake of completeness, the following Proposition includes two well-known results on operator theory. Definitions and terminology can be found in the references cited in the proof.

**Proposition 1** Let \( S : L^2(\partial \mathbb{D}; \mathcal{H}) \rightarrow L^2(\partial \mathbb{D}; \mathcal{H}) \) be a bounded operator.

(i) A bounded operator \( S : L^2(\partial \mathbb{D}; \mathcal{H}) \rightarrow L^2(\partial \mathbb{D}; \mathcal{H}) \) commutes with every diagonalizable operator if and only if \( S \) commutes with the diagonalizable operator \( \omega I_{\mathcal{H}} \).

(ii) The set of decomposable operators in \( L^2(\partial \mathbb{D}; \mathcal{H}) \) is a von Neumann algebra with abelian commutant coinciding with the family of diagonalizable operators.

**Proof** (i) is a consequence of the spectral theory for unitary operators (of constant multiplicity): see, for example, Theorems 5.4.8, 6.2.4 and 7.2.1 in [4]. (ii) is a particular case of [33, Th.14.1.10]. \( \square \)

The spectral representations of \( T \) and \( D \) we consider are given in Propositions 3 and 2, respectively, where \( T \) and \( D \) are transformed into diagonalizable operators of the form \( \omega I_{\mathcal{H}} \) on suitable spaces \( L^2(\partial \mathbb{D}; \mathcal{H}) \). Proofs and more details can be found in [16]. We begin by considering an orthonormal basis (shortly,
ONB) \( \{L_i^{(0)}(x)\}_{i \in \mathbb{I}} \) of \( L^2[0, 1] \) and ONBs \( \{K_{s,j}^{(0)}(x)\}_{j \in \mathbb{J}} \) of \( L^2[\pm 1, \pm 2] \), where \( \mathbb{I}, \mathbb{J} \) are denumerable sets of indices (usually, \( \mathbb{N}, \mathbb{N} \cup \{0\} \) or \( \mathbb{Z} \))—here \( L^2[-1, -2] \) means \( L^2(-2, -1] \). Obviously, the families:

\[
\{L_i^{(n)}(x) := [T^n L_i^{(0)}](x) = L_i^{(0)}(x - n)\}_{i \in \mathbb{I}, n \in \mathbb{Z}},
\]

\[
\{K_{s,j}^{(m)}(x) := [D^m K_{s,j}^{(0)}](x) = 2^{m/2} K_{s,j}^{(0)}(2^m x)\}_{j \in \mathbb{J}, m, n \in \mathbb{Z}, s = \pm}
\]

are ONBs of \( L^2(\mathbb{R}) \) and, for each \( f \in L^2(\mathbb{R}) \), one has (in \( L^2 \)-sense):

\[
f = \sum_{i,n} \hat{f}_i^{(n)} L_i^{(n)}, \quad \text{with} \quad \hat{f}_i^{(n)} := \langle f, L_i^{(n)} \rangle_{L^2(\mathbb{R})},
\]

\[
f = \sum_{s,j,m} \hat{f}_{s,j}^{(m)} K_{s,j}^{(m)}, \quad \text{with} \quad \hat{f}_{s,j}^{(m)} := \langle f, K_{s,j}^{(m)} \rangle_{L^2(\mathbb{R})}.
\]

In what follows, fixed ONBs \( \{L_i^{(n)}(x)\}_{i \in \mathbb{I}, n \in \mathbb{Z}} \) and \( \{K_{s,j}^{(m)}(x)\}_{j \in \mathbb{J}, m \in \mathbb{Z}, s = \pm} \) of \( L^2(\mathbb{R}) \) as above, for each \( f \in L^2(\mathbb{R}) \), we shall write:

\[
f = \{\hat{f}_i^{(n)}\} = \{\hat{f}_{s,j}^{(n)}\}.
\]

A spectral representation for the dilation operator \( D \) on \( L^2(\mathbb{R}) \) is given in the next result. Here, \( l^2(\mathbb{J}) \) denotes the Hilbert space of sequences of complex numbers \( (c_j)_{j \in \mathbb{J}} \), such that \( \sum_{j \in \mathbb{J}} |c_j|^2 < \infty \), \( \{u_{s,j}\}_{j \in \mathbb{J}, s = \pm} \) is a fixed ONB of \( l^2(\mathbb{J}) \oplus l^2(\mathbb{J}) \) and \( \oplus \) denotes orthogonal sum.

**Proposition 2** [16, Proposition 1] The operator \( \mathcal{G} \) defined by:

\[
\mathcal{G} : L^2(\mathbb{R}) \rightarrow L^2(\partial \mathbb{D}; l^2(\mathbb{J}) \oplus l^2(\mathbb{J}))
\]

\[
f \mapsto \tilde{f} := \bigoplus_{s = \pm} \bigoplus_{j \in \mathbb{J}} \sum_{m \in \mathbb{Z}} \omega^m \hat{f}_{s,j}^{(m)} u_{s,j}
\]

determines a spectral model for the dilation operator \( D \), i.e., \( \mathcal{G} \) is unitary and:

\[
\mathcal{G} D \mathcal{G}^{-1} = \omega I_{l^2(\mathbb{J}) \oplus l^2(\mathbb{J})}.
\]

Now, for the translation operator \( T \) on \( L^2(\mathbb{R}) \), if \( \{u_i\}_{i \in \mathbb{I}} \) is a fixed ONB of \( l^2(\mathbb{I}) \), one has:

**Proposition 3** [16, Proposition 3] The operator \( \mathcal{F} \) given by:

\[
\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\partial \mathbb{D}; l^2(\mathbb{I}))
\]

\[
f \mapsto \tilde{f} := \bigoplus_{i \in \mathbb{I}} \left[ \sum_{n \in \mathbb{Z}} \omega^n \hat{f}_i^{(n)} \right] u_i
\]
determines a spectral model for the translation operator $T$, i.e., $\mathcal{F}$ is unitary and:

$$\mathcal{F}T\mathcal{F}^{-1} = \omega I_{\ell(3)}.$$  

In the sequel, we shall write:

$$\hat{f}_i(\omega) := \sum_{n \in \mathbb{Z}} \omega^n \hat{f}_i^{(n)}, \quad (f \in L^2(\mathbb{R}), i \in \mathbb{I}),$$

$$\hat{f}_{s,j}(\omega) := \sum_{m \in \mathbb{Z}} \omega^m \hat{f}_{s,j}^{(m)}, \quad (f \in L^2(\mathbb{R}), s = \pm, j \in \mathbb{J}).$$

The change of representation between both expansions (10) and (11) is governed by a matrix $(\alpha_{i,n}^{s,j,m})$, where:

$$\alpha_{i,n}^{s,j,m} := \langle L_i^{(n)}, K_{s,j}^{(m)} \rangle_{L^2(\mathbb{R})}. \quad (12)$$

Useful identities shall be:

$$\begin{align*}
\left[ L_i^{(n)} \right]_{s,j}(\omega) &= \sum_m \omega^m \left[ L_i^{(n)} \right]_{s,j}^{(m)} \\
&= \sum_m \omega^m \langle L_i^{(n)}, K_{s,j}^{(m)} \rangle_{L^2(\mathbb{R})} = \sum_m \omega^m \alpha_{i,n}^{s,j,m}. \quad (13)
\end{align*}$$

### 3 Spectral techniques for wavelet frames

Although the results of this section can be given in the more general setting of affine systems in $L^2(\mathbb{R}^d)$, we restrict attention on the Hilbert space $L^2(\mathbb{R})$ and wavelet systems $X$ of the form (5). Theorems 5 and 8 and Corollary 9 below work on the dilation representation of Proposition 2 and not on the usual Fourier domain of fiberization techniques (see, e.g., [35, Th.3.3.5] and [37, Th.3.1]). Some comments comparing spectral with fiberization techniques are included in Remark 6. First, we include a technical result necessary to prove Theorem 5:

**Lemma 4** Let $X = X_\Psi$ be a wavelet system in $L^2(\mathbb{R})$ of the form (5), where $\Psi$ is a finite or countable family of $L^2(\mathbb{R})$, and such that:

$$\sup_{\psi \in \Psi} ||\psi||_{L^2(\mathbb{R})} = M < \infty. \quad (14)$$

---

1 Lemma 4 is just the equivalence between items (8) and (11) of Proposition 4.2.1 in [26] (courtesy of reviewers). We maintain here an alternative proof.
Then, the corresponding operator $T_X$, given by (2), is a well-defined bounded operator from $\ell^2(X)$ into $L^2(\mathbb{R})$ if and only if the corresponding operator $S$, given by (3), is a well-defined bounded operator on $L^2(\mathbb{R})$.

**Proof** If $T_X$ is a well-defined bounded operator from $\ell^2(X)$ into $L^2(\mathbb{R})$, then the adjoint $T_X^*$ and $S = T_X T_X^*$ are also well-defined bounded operators. For the opposite implication, it is obvious that the synthesis operator $T_X$, given by (2), is well defined at least on the dense subspace $\ell^0_0(X)$ of $\ell^2(X)$ formed by the sequences $\{c_n\} \in \ell^2(X)$ with a finite number of non-zero components. In principle, consider $T_X$ defined on $\ell^0_0(X)$. Such operator $T_X$ is preclosed if and only if (14) is satisfied. To see this, recall that $T_X$ is preclosed if and only if for every sequence $\{c_n\} \subset \ell^0_0(X)$, such that $\lim_{n \to \infty} c_n = 0$, one has $\lim_{n \to \infty} T_X c_n = 0$ (see, for example, [32, page 155]). If (14) is satisfied and $\lim_{n \to \infty} c_n = 0$, then $\lim_{n \to \infty} ||T_X c_n||_{L^2(\mathbb{R})} \leq M \lim_{n \to \infty} ||c_n||_{\ell^2(X)} = 0$. Conversely, if (14) is not satisfied, consider a sequence $\{\psi_n\} \subset \Psi$, such that $\lim_{n \to \infty} ||\psi_n||_{L^2(\mathbb{R})} = \infty$ and a sequence $\{c_n\} \subset \ell^0_0(X)$, such that the only non-zero element of $c_n$ is the $n$-component with modulus equal to $||\psi_n||_{L^2(\mathbb{R})}^{-1}$ for $n \geq n_0$. Moreover, if $T_X$ is preclosed with closure $\overline{T_X}$, the adjoint $T_X^*$ is defined in a dense domain $D(T_X^*)$ of $L^2(\mathbb{R})$ and $T_X^*$ is a closed operator, $T_X^* = [T_X]^*$, $T_X^{**} = \overline{T_X}$ and the domain of $T_X T_X^*$, $D(T_X T_X^*)$, is a core for $T_X^{**}$; see Remark 2.7.7 and Theorem 2.7.8 in [32]. Assume that $T_X T_X^*$ is bounded on $D(T_X T_X^*)$ and, then, $\overline{T_X} T_X^*$ admits a well-defined bounded extension $S$ on $L^2(\mathbb{R})$. Since, for $h \in D(T_X T_X^*)$:

$$||\langle S h, h \rangle_{L^2(\mathbb{R})}|| = ||\langle T_X T_X^* h, h \rangle_{L^2(\mathbb{R})}|| = ||T_X T_X^* h, \overline{T_X T_X^* h}_{L^2(\mathbb{R})}||$$

$$= ||T_X T_X^* h||_{\ell^2(X)}^2 \leq ||S|| ||h||_{L^2(\mathbb{R})}^2,$$

$T_X^*$ is also bounded on $D(T_X T_X^*)$, so that $T_X^*$ admits a well-defined bounded extension on $L^2(\mathbb{R})$ whose (well-defined bounded) adjoint extends $T_X$ and $\overline{T_X}$. Thus, $T_X$, defined in principle on $\ell^0_0(X)$, admits a well-defined bounded extension on $\ell^2(X)$.

Recall that a countable subset $X$ of a Hilbert space $\mathcal{H}$ is called a Bessel system if the second inequality in (1) is satisfied for some constant $B > 0$. In such a case, every number $B$ satisfying (1) is called a Bessel bound for $X$.

**Theorem 5** Let $X$ be a wavelet system in $L^2(\mathbb{R})$ of the form (5) and such that (14) is satisfied. Then:

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1 The first part of item 1 of Theorem 5 is just the equivalence between items (2) and (11) of Proposition 4.2.1 in [26] (courtesy of reviewers). In item 2 of Theorem 5, that $A_1 L^2(\mathbb{R}) \leq S \leq B L^2(\mathbb{R})$ implies $B^{-1} L^2(\mathbb{R}) \leq S^{-1} \leq A^{-1} L^2(\mathbb{R})$, can be deduced from Proposition 4.2.6 of [26] (courtesy of reviewers). Here, besides these facts, we must deal with the decomposability of $S$ and $S^{-1}$ with respect to the dilation representation given in Proposition 2 and the properties of the components $S(\omega)$ and $S^{-1}(\omega)$.
1. \( X \) is a Bessel system if and only if the corresponding operator \( S \), given by (3), is a well-defined bounded operator on \( L^2(\mathbb{R}) \). In such a case, \( S \) commutes with \( D \) and, then, going to the dilation representation given in Proposition 2, \( GS^{-1} \) is a decomposable operator on \( L^2(\mathbb{R}) \), \( G \) is the dilation generator.

\[
G S^{-1} = S(\omega).
\]

Moreover, \( S \) is positive, \( S(\omega) \) is positive for a.e. \( \omega \in \mathbb{D} \) and:

\[
||S|| = \text{ess sup}_{\omega \in \mathbb{D}} ||S(\omega)|| = \text{ess sup}_{\omega \in \mathbb{D}} \sup_{||u||_{L^2(\mathbb{R})}^2=1} ||S(\omega)u||_{F(\mathbb{R})} < \infty.
\]

2. \( X \) is a frame for \( L^2(\mathbb{R}) \) if and only if \( S \) is a well-defined bounded operator on \( L^2(\mathbb{R}) \) with bounded two-sided inverse \( S^{-1} \). In such a case, \( S^{-1} \) also commutes with \( D \) and

\[
G S^{-1} G^{-1} = S(\omega)^{-1}.
\]

Equivalently, \( X \) is a frame for \( L^2(\mathbb{R}) \) if and only if:

\[
\alpha := \text{ess sup}_{\omega \in \mathbb{D}} \sup_{||u||_{L^2(\mathbb{R})}^2=1} ||S(\omega)u||_{F(\mathbb{R})} < \infty
\]

and

\[
\beta := \text{ess inf}_{\omega \in \mathbb{D}} \inf_{||u||_{L^2(\mathbb{R})}^2=1} ||S(\omega)u||_{F(\mathbb{R})} > 0;
\]

in such a case, \( ||S|| = \alpha \) and \( ||S^{-1}|| = \beta^{-1} \).

3. \( X \) is a tight frame for \( L^2(\mathbb{R}) \), with frame bound \( B \), if and only if \( S = BI_{L^2(\mathbb{R})} \) or, equivalently:

\[
G S^{-1} = B I_{F(\mathbb{R})}.
\]

**Proof** (1) It is well known [7, Th.3.2.3] that a countable subset \( X \) of a Hilbert space \( \mathcal{H} \) is a Bessel system with Bessel bound \( B \) if and only if the corresponding synthesis operator \( T_X \), given by (2), is a well-defined bounded operator and \( ||T_X|| \leq B^{1/2} \). By Lemma 4, \( T_X \) is a well-defined bounded operator if and only if the corresponding operator \( S \), given by (3), is a well-defined bounded operator on \( L^2(\mathbb{R}) \). In such a case, \( ||T_X^*|| = ||T_X|| \), and \( S \) is positive, since \( \langle Sf,f \rangle_{L^2(\mathbb{R})} = \langle T_X^*f,T_X^*f \rangle_{F(\mathbb{R})} \geq 0 \). Moreover, \( S \) commutes with the dilation operator \( D \):

\[
SDf = \sum_{k \in \mathbb{Z}} \langle Df, D^k T^i \psi \rangle D^k T^i \psi
\]

\[
= \sum_{k \in \mathbb{Z}} \langle f, D^{k-1} T^i \psi \rangle D^k T^i \psi = DSf, \quad (f \in L^2(\mathbb{R})).
\]
(We use that $D$ is unitary, $D^* = D^{-1}$, and that the series defining $S$ converges unconditionally for all $f \in L^2(\mathbb{R})$; see [7, Corollary 3.2.5] and Lemma 7 below.) That $S$ and $D$ commute implies that $GSG^{-1}$ is a decomposable operator on $L^2(\partial \mathbb{D}, L^2(\mathbb{J}) \oplus L^2(\mathbb{J}))$, the representation space for $D$ in Proposition 1. Since $S$ is positive, $S(\omega)$ is positive a.e. (see [33, prop.14.1.8-9]). Being $G$ unitary, $||S|| = ||GSG^{-1}||$ and, by [33, Prop.14.1.9]:

$$||S|| = ||GSG^{-1}|| = \text{ess sup}_{\omega \in \partial \mathbb{D}} ||S(\omega)||.$$ 

(2) In terms of $S$, the inequalities in (1) read:

$$A ||f||^2_{L^2(\mathbb{R})} \leq \langle Sf, f \rangle \leq B ||f||^2_{L^2(\mathbb{R})}, \quad (f \in L^2(\mathbb{R})), $$

i.e., $A L^2(\mathbb{R}) \leq S \leq BL^2(\mathbb{R})$. In particular, the first inequality implies $A ||f||^2_{L^2(\mathbb{R})} \leq ||Sf||^2_{L^2(\mathbb{R})}$, for every $f \in L^2(\mathbb{R})$. Since $S$ is positive, this fact is equivalent to the existence of the two-sided bounded inverse $S^{-1}$ of $S$ (see [43, Th.12.12.c]). That $S^{-1}$ exists implies that $\text{Range}(S) = \text{Range}(T_X) = L^2(\mathbb{R})$. And a Bessel system $X$ is a frame if and only if this last condition is satisfied (see [7, Th.5.5.1]). Furthermore, $AL^2(\mathbb{R}) \leq S \leq BL^2(\mathbb{R})$ implies that $0 \leq I_{L^2(\mathbb{R})} - B^{-1}S \leq \frac{B-A}{B}I_{L^2(\mathbb{R})}$ and consequently:

$$||I_{L^2(\mathbb{R})} - B^{-1}S|| = \sup_{||f||=1} \left| \langle (I_{L^2(\mathbb{R})} - B^{-1}S)f, f \rangle \right| \leq \frac{B-A}{B} < 1,$$

so that:

$$S^{-1} = B^{-1} \sum_{k=0}^{\infty} (I_{L^2(\mathbb{R})} - B^{-1}S)^k,$$

where the last series converges in norm (uniformly). Since the set of decomposable operators is a $C^*$-algebra (moreover, a von Neumann algebra, see Proposition 1), $S^{-1}$ is also a decomposable operator in the dilation representation given in Proposition 2 and (see [33, prop.14.1.8])

$$G S^{-1} G^{-1} = S(\omega)^{-1}.$$ 

(Nota: $S(\omega)$ and $S(\omega)^{-1}$ are defined for a.e. $\omega \in \partial \mathbb{D}$.) Now, recall that for a bounded normal operator $S$ on a Hilbert space $\mathcal{H}$, $S$ has a bounded two-sided inverse if and only if $0 < \beta := \inf \{||Sx||_{\mathcal{H}} : x \in \mathcal{H}, ||x||_{\mathcal{H}} = 1\}$ and, then, $||S^{-1}|| = \beta^{-1}$; see [32, Lemma 2.4.8] and [43, Th.12.12.c]. Thus, being $S$, $GSG^{-1}$ and $S(\omega)$ positive, the bounded two-sided inverse $S^{-1}$ exists if and only if:
\[ \inf \frac{1}{\|u\|_{L^2(\mathbb{D},F(\mathcal{J}))}} \inf \|S(\omega)u\|_{F(\mathcal{J})} \leq \inf \|S(\omega)u\|_{F(\mathcal{J})} \leq \sup \|S(\omega)u\|_{F(\mathcal{J})} \leq \sup \|S(\omega)^{-1}u\|_{L^2(\mathbb{D},F(\mathcal{J}))} \]

see [33, prop.14.1.8–9] for details.

(3) If \( X \) is a frame, in terms of pseudo-inverses:

\[ S^{-1} = T_X^*-T_X^{-1} = (T_X^{-1})^*T_X^{-1}, \]

and the frame bounds are \( B = \|T_X\|^2 = \|T_X^*\|^2 \) and \( A = 1/\|T_X^{-1}\|^2 = 1/\|T_X^*\|^{-1} \); see [7, Lemmas 5.5.4 and A.7.2] for details. Thus, \( X \) is a tight frame if and only if \( X \) is a frame and \( \|T_X^{-1}\| \cdot \|T_X^{-1}\| = 1 \). In such a case, for \( f \in \mathcal{H}T_X \), \( \|f\| = \|T_X^{-1}T_Xf\| \leq \|T_X^{-1}\| \|T_Xf\| \), so that:

\[ \|T_X\| \|f\| = \frac{1}{\|T_X^{-1}\|} \|f\| \leq \|T_Xf\| \leq \|T_X\| \|f\| \]

and, then, \( \|T_X\| \|f\| = \|T_Xf\| \). That is, \( T_X/\|T_X\| \) is a partial isometry with the initial space \( \mathcal{H}T_X \) and final space \( L^2(\mathbb{R}) \). The orthogonal projection over the final space is

\[ I_{L^2(\mathbb{R})} = \frac{T_XT_X^*}{\|T_X\|^2} = \frac{S}{\|S\|} = B^{-1}S. \]

The converse follows from the fact that (1) is equivalent to \( AI \leq S \leq BI \). Finally, that \( GUG^{-1} \) is the constant diagonalizable operator \( I_{F(\mathcal{J})} \) if and only if \( U = I_{L^2(\mathbb{R})} \) is just [33, prop.14.1.8.iv].

**Remark 6** In Theorem 5, the frame operator \( S = T_XT_X^* \) has a decomposable image \( GUG^{-1} \) on the dilation representation given in Proposition 2 thanks to the fact that \( S \) and \( D \) commute (see Proposition 1). What about the commutation relations between \( S \) and the translation (shift) operator \( T \) in order that \( S \) have decomposable images \( FS_F^{-1} \) and \( F^*_aSF^*_a^{-1} \) on the translation representation given in Proposition 3? This

---

3 Let \( \mathcal{H}, \mathcal{H}' \) be Hilbert spaces and suppose that \( U : \mathcal{H} \to \mathcal{H}' \) is a bounded operator with closed range \( \mathcal{R}_U \) and kernel \( \mathcal{N}_U \). The pseudo-inverse of \( U \) is the unique operator \( U^{-1} : \mathcal{H}' \to \mathcal{H} \) satisfying \( \mathcal{N}_{U^{-1}} = \mathcal{R}_U^\perp \), \( \mathcal{R}_{U^{-1}} = \mathcal{R}_U^\perp \) and \( UU^{-1}f = f \) for \( f \in \mathcal{R}_U \) and \( U^{-1}Uf = f \) for \( f \in \mathcal{R}_U^\perp \), where \( \perp \) denotes orthogonal complement.
question is the cornerstone to develop fiberization techniques for wavelet systems in $L^2(\mathbb{R})$ of the form (5) on the Fourier domain.

For the translation and dilation operators, $T$ and $D$, defined on $L^2(\mathbb{R})$ by (6), one has $TD = DT^2$. Taking adjoints, $D^{-1}T^{-1} = T^{-2}D^{-1}$. Also, $D = T^{-1}DT^2$ or $DT^{-2} = T^{-1}D$ and $D^{-1} = T^{-2}D^{-1}T$ or $T^2D^{-1} = D^{-1}T$. Thus, in general:

$$T^jD^k = D^kT^{2j}, \quad \text{if } k > 0 \text{ and } j \in \mathbb{Z},$$

$$T^{j2^s}D^k = D^kT^j, \quad \text{if } k < 0 \text{ and } j \in \mathbb{Z}.$$ (15)

Moreover, the functional calculus for the unitary operators $T$ and $D$ permits us to extend the above formulas at least to $j, k \in \mathbb{Q}$. Along this work, we will only use $T^{j/k}f(\cdot) = f(\cdot - \frac{1}{k})$, for $k \in \mathbb{N}$ and $f \in L^2(\mathbb{R})$.

Given a Bessel wavelet system $X$ in $L^2(\mathbb{R})$ of the form (5) and the corresponding operator $S = T_XT_X^*$, for each $f \in L^2(\mathbb{R})$:

$$STf = \sum_{k \in \mathbb{Z}} \sum_{\psi \in \Psi} \langle f, T^{-1}D^kT^j\psi \rangle D^kT^j\psi$$

$$= \sum_{k \geq 0} \sum_{\psi \in \Psi} \langle f, T^{-1}D^kT^j\psi \rangle D^kT^j\psi + \sum_{k < 0} \sum_{\psi \in \Psi} \langle f, T^{-1}D^kT^j\psi \rangle D^kT^j\psi$$

$$= \sum_{k \geq 0} \sum_{\psi \in \Psi} \langle f, D^kT^{j-2k}\psi \rangle D^kT^j\psi + \sum_{k < 0} \sum_{\psi \in \Psi} \langle f, T^{j2^s}D^k\psi \rangle T^{j2^s}D^k\psi$$

and

$$TSf = \sum_{k \in \mathbb{Z}} \sum_{\psi \in \Psi} \langle f, D^kT^j\psi \rangle TD^kT^j\psi$$

$$= \sum_{k \geq 0} \sum_{\psi \in \Psi} \langle f, D^kT^j\psi \rangle TD^kT^j\psi + \sum_{k < 0} \sum_{\psi \in \Psi} \langle f, D^kT^j\psi \rangle TD^kT^j\psi$$

$$= \sum_{k \geq 0} \sum_{\psi \in \Psi} \langle f, D^kT^j\psi \rangle D^kT^{j+2k}\psi + \sum_{k < 0} \sum_{\psi \in \Psi} \langle f, T^{j2^s}D^k\psi \rangle T^{j+2^s}D^k\psi.$$ (16)

The sums for $k \geq 0$ coincide, but not the sums for $k < 0$. If for each $k < 0$, we add to the affine system $X$ the functions:

$$\psi_{k,j}^l = T^{l2^s}D^k\psi = T^lD^kT^j\psi = 2^{k/2}\psi(2^{k-l} - j), \quad l = 1, 2, \ldots, 2^{-k} - 1,$$

one obtains a system $\tilde{X}^q$ associated with $X$, such that the corresponding frame operator $S = T_XT_X^*$ commutes with the translation operator $T$. Thus, such $S$ shall be a decomposable operator on any spectral representation of $T$. Moreover, we have

$$\tilde{X}^q = \tilde{X}^q_+ \cup \tilde{X}^q_-, \text{ where}:$$
\[
\tilde{X}_+^q = \left\{ \psi_{k,j} := D^k T^j \psi : \psi \in \Psi, \ k \geq 0, j \in \mathbb{Z} \right\} \\
= \left\{ T^a D^k T^b \psi : \psi \in \Psi, \ k \geq 0, a \in \mathbb{Z}, 0 \leq b < 2^k \right\}, \\
\tilde{X}_-^q = \left\{ \psi_{k,j}^l := T^l D^k T^j \psi : \psi \in \Psi, \ k < 0, j \in \mathbb{Z}, 0 \leq l < 2^{-k} \right\} \\
= \left\{ T^a D^k \psi : \psi \in \Psi, \ k < 0, a \in \mathbb{Z} \right\}.
\]

A variant of \( \tilde{X}^q \) is what Ron and Shen [37, Section 5] call the quasi-affine system \( X^q \) associated with \( X \). \( X^q = X^q_+ \cup X^q_- \), where \( X^q_+ = \tilde{X}^q_+ \), the truncated affine system \( X_0 \) according to Ron and Shen [37, Section 4], and:

\[
X^q_+ = \left\{ 2^{k/2} \psi^l_{k,j} := 2^{k/2} T^l D^k T^j \psi : \psi \in \Psi, \ k < 0, j \in \mathbb{Z}, 0 \leq l < 2^{-k} \right\} \\
= \left\{ 2^{k/2} T^a D^k \psi : \psi \in \Psi, \ k < 0, a \in \mathbb{Z} \right\}.
\]

Working in the Fourier domain, one is forced to consider the translation invariant system \( X^q \) or \( \tilde{X}^q \). Ron and Shen [37, Theorem 5.5] prove a variant of the following result: the wavelet system \( X \) is a frame if and only if its quasi-affine counterparts \( X^q \) or \( \tilde{X}^q \) are a frame. In particular, the frame \( X \) is tight if and only if the quasi-affine system \( X^q \) or \( \tilde{X}^q \) is tight. Furthermore, the two systems \( X \) and \( X^q \) have identical frame bounds. The choice of the dilation representation of Proposition 2 (or any other spectral representation for \( D \)) avoids this inconvenience, since a wavelet system (in general, any affine system) is dilation invariant.

For a Bessel wavelet system \( X \) in \( L^2(\mathbb{R}) \) of the form (5), the operator \( S = T_X T_X^* \) in the dilation representation of Proposition 2, \( GSG^{-1} \), is given by:

\[
L^2(\partial \mathbb{D}; \mathcal{F}(\mathbb{J}) \oplus \mathcal{F}(\mathbb{J})) \xrightarrow{G^{-1}} L^2(\mathbb{R}) \xrightarrow{S} L^2(\mathbb{R}) \xrightarrow{G} L^2(\partial \mathbb{D}; \mathcal{F}(\mathbb{J}) \oplus \mathcal{F}(\mathbb{J}))
\]

\[
Gf = \tilde{f} = \left\{ \tilde{f}_{k,j}^{(m)} \right\} \xrightarrow{G^{-1}} f \xrightarrow{S} f \xrightarrow{G} \sum_{k,j}^u \left\langle f, \psi_{k,j} \right\rangle_{L^2(\mathbb{R})} \psi_{k,j}
\]

The superindex “\( u \)” added to the sum symbol \( \sum \) in the last expressions reflects that the series defining \( S \) converges unconditionally for all \( f \in L^2(\mathbb{R}) \); see [7, Corollary 3.2.5].

**Lemma 7** [7, Lemma 2.1.1] Let \( \{ y_k \}_{k=1}^\infty \) be a sequence in a Banach space \( Y \), and let \( y \in Y \). Then, the following are equivalent:

(i) \( \sum_{k=1}^\infty y_k \) converges unconditionally to \( y \) in \( Y \).
(ii) For every \( \epsilon > 0 \), there exists a finite set \( F \), such that \( ||y - \sum_{k \in I} y_k|| \leq \epsilon \) for all finite sets \( I \subset \mathbb{N} \) containing \( F \).
According to Lemma 7, \( \left[ \sum_{k \in \mathbb{Z}} \psi_{\psi}^{k} \right]^{u} \) means that, for each \( f \in L^{2}(\mathbb{R}) \), one must take the limit of sums over suitable finite sets of triplets \((k, j, \psi) \in \mathbb{Z} \times \mathbb{Z} \times \Psi \). This is the correct way to interpret the sums and avoids any possible “infinity” in partial calculations dealing with the expressions we will encounter in what follows.

To take advantage of the dilation representation of Proposition 2, the matrix \( \left( \alpha_{l,n}^{s,j,m} \right) \), defined by (12), must appear on stage. Next result gives an expression for the matrix elements and fibers of the decomposable operator \( G_{SG}^{-1} \) associated with the Bessel wavelet system \( X \). They are written in terms of the \( \alpha_{l,n}^{s,j,m} \)'s and the components \( \left\{ \hat{\psi}_{s,j}^{(n)} \right\} ^{u} \) of each \( \psi \in \Psi \) (and not in terms of the components \( \left\{ \hat{\psi}_{s,j} \right\} ^{u} \) !). The result is given for the ONB \( \{ u_{s,l} \}_{l \in J, s = \pm} \) of \( \hat{I}^{2}(\mathbb{J}) \bigoplus \hat{I}^{2}(\mathbb{J}) \) fixed in Proposition 2.

**Theorem 8** For a Bessel wavelet system \( X \) in \( L^{2}(\mathbb{R}) \) of the form (5), the operator \( S = T_{X}^{*} T_{X} \) in the dilation representation of Proposition 2, \( G_{SG}^{-1} \), has matrix elements \( \left[ G_{SG}^{-1} \right]_{s,l}^{s',l'} \) given by:

\[
\left[ G_{SG}^{-1} \right]_{s,l}^{s',l'} : L^{2}(\partial \mathbb{D}, \mathbb{C}) \rightarrow L^{2}(\partial \mathbb{D}, \mathbb{C})
\]

\[
h(\omega) \mapsto h(\omega) \sum_{\sigma} \alpha_{l,n}^{s,l} \sum_{k,j \in \mathbb{Z}} \left( \sum_{l, n \in \mathbb{Z}} \alpha_{l,n}^{s,j,m} \alpha_{l',n'}^{s',j,m} \right) \left( \sum_{\psi \in \Psi} u^{(n)} \hat{\psi}_{s,j}^{(n')} \right),
\]

where \( l, l' \in \mathbb{J}, s, s' = \pm \). Thus, the fibers of \( G_{SG}^{-1} \) are:

\[
S(\omega) : \hat{I}^{2}(\mathbb{J}) \bigoplus \hat{I}^{2}(\mathbb{J}) \rightarrow \hat{I}^{2}(\mathbb{J}) \bigoplus \hat{I}^{2}(\mathbb{J})
\]

\[
u_{s,l} \mapsto \bigoplus_{s',l'} \nu_{s',l'} \sum_{\sigma} \alpha_{l,n}^{s,l} \sum_{k,j \in \mathbb{Z}} \left( \sum_{l, n \in \mathbb{Z}} \alpha_{l,n}^{s,j,m} \alpha_{l',n'}^{s',j,m} \right) \left( \sum_{\psi \in \Psi} u^{(n)} \hat{\psi}_{s,j}^{(n')} \right),
\]

for a.e. \( \omega \in \partial \mathbb{D} \).

**Proof** The following identities are direct consequences of (10), (11), and (12):

\[
\sum_{r,p,q} \overline{\psi}_{r,p}^{(q)} \alpha_{l,n-j}^{r,p,q} = \sum_{r,p,q} \overline{\psi}_{r,p}^{(q)} \langle L_{i}^{(n-j)}, K_{r,p}^{(q)} \rangle_{L^{2}(\mathbb{R})}^{L^{2}(\mathbb{R})}
\]

\[
= \langle L_{i}^{(n-j)}, \sum_{r,p,q} \overline{\psi}_{r,p}^{(q)} K_{r,p}^{(q)} \rangle_{L^{2}(\mathbb{R})}^{L^{2}(\mathbb{R})} = \langle L_{i}^{(n-j)}, \psi \rangle_{L^{2}(\mathbb{R})} \overline{\psi}_{i}^{(n-j)}.
\]

For \( f = K_{s,l}^{(m)} \), one has:
\[
\omega^m u_{s,l} \mapsto K_{s,l}^{(m)} \mapsto S K_{s,l}^{(m)} = \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \langle K_{s,l}^{(m)}, \psi_{k,j} \rangle_{L^2(\mathbb{R})} \psi_{k,j}
\]

\[
\therefore \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \langle K_{s,l}^{(m)}, \psi_{k,j} \rangle_{L^2(\mathbb{R})} \mathcal{G} \psi_{k,j}.
\]

Using (11) and the definition of \( \mathcal{G} \) in Proposition 2:

\[
\sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \langle K_{s,l}^{(m)}, \psi_{k,j} \rangle_{L^2(\mathbb{R})} \mathcal{G} \psi_{k,j}
\]

\[
= \sum_{\tau \in \mathbb{P}} \left( \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \alpha^m \left[ D^k T j \psi \right]_{s,l}^{(m)} \left( \sum_{m', l'} \alpha^{m'} \left[ D^k T j \psi \right]_{s', l'}^{(m')}ight) u_{s', l'}
\]

\[
= \sum_{\tau \in \mathbb{P}} \left( \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \alpha^m \left[ D^k T j \psi \right]_{s,l}^{(m)} \left[ T j \psi \right]_{s', l'}^{(m-k)} \right) u_{s', l'}
\]

By [16, Lemma 5], the last expression coincides with:

\[
\sum_{\tau \in \mathbb{P}} \left( \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \alpha^m \left[ D^k T j \psi \right]_{s,l}^{(m)} \left[ T j \psi \right]_{s', l'}^{(m-k)} \right) u_{s', l'}
\]

and, by (16), this is equal to:

\[
\sum_{\tau \in \mathbb{P}} \left( \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \alpha^m \left[ D^k T j \psi \right]_{s,l}^{(m)} \left[ T j \psi \right]_{s', l'}^{(m-k)} \right) u_{s', l'}
\]

Thus, for \( m \in \mathbb{Z} \), the matrix element \([\mathcal{G} S G^{-1}]_{s,l} \) satisfies:

\[
[\mathcal{G} S G^{-1}]_{s,l} (\omega^m) = \omega^m \sum_{\rho \in \mathbb{P}} \sum_{\psi \in \mathcal{P}} \left( \sum_{k \in \mathbb{Z}} \sum_{\psi \in \mathcal{P}} \alpha^m \left[ D^k T j \psi \right]_{s,l}^{(m)} \left[ T j \psi \right]_{s', l'}^{(m-k)} \right) u_{s', l'}.
\]

Since \( \{ \omega^m \}_{m \in \mathbb{Z}} \) is an ONB of \( L^2(\partial \mathbb{D}, \mathbb{C}) \), we get the result. \( \square \)
In particular, for tight wavelet frames:

**Corollary 9** Let $X$ be a wavelet system in $L^2(\mathbb{R})$ of the form (5) and such that (14) is satisfied. Then, $X$ is a tight frame for $L^2(\mathbb{R})$, with frame bound $B$, if and only if:

$$
\sum_{i,n,k} \left( \sum_{i',n',j} \alpha_{i,k}^{x,i',j} \alpha_{i',j,k}^{y,i,j} \right) \left( \sum_{\psi \in \mathcal{P}} \left( \sum_{i,n,k} \alpha_{i,n,k}^{y,i,n,k} \hat{\psi}_i^{(n)} \hat{\psi}_i^{(n')} \right) \right) = B \delta_{s,s'} \delta_{l,l'} \delta_{\sigma,\sigma'},
$$

(17)

where $s, s' = \pm, l, l' \in \mathcal{J}$, $\sigma \in \mathcal{Z}$, and $\delta$ denotes the Dirac $\delta$-function.

**Proof** By Theorem 5, $X$ is a tight frame with frame bound $B$ if and only if $S = BI_{L^2(\mathbb{R})}$ or, equivalently, $GS^{-1} = BI_{P(\mathcal{J}) \oplus P(\mathcal{J})}$, i.e., $S(\omega) = B I_{P(\mathcal{J}) \oplus P(\mathcal{J})}$, for a.e. $\omega \in \partial \mathbb{D}$. Now, consider the expression of the fibers $S(\omega)$ given in Theorem 8.

4 Tight wavelet frames of minimal support

In this section, Corollary 9 is used to determine all the tight wavelet frames for $L^2(\mathbb{R})$ of the form (5), with cardinal of $\Psi$ finite and such that the support of each $\psi \in \Psi$, $\text{supp} \ \psi$, is included in the interval $[0, 1]$. Note that, since the cardinal of $\Psi$ is finite, condition (14) is trivially satisfied.

Due to the structure of the ONB $\{L_i^{(j)}\}$, defined by (8), the fact that $\text{supp} \ \psi \subseteq [0, 1]$, ($\psi \in \Psi$), implies that their expansions (10) read:

$$
\psi = \sum_{i \in \mathbb{I}} \hat{\psi}_i^{(0)} L_i^{(0)}, \quad (\psi \in \Psi),
$$

since $\hat{\psi}_i^{(n)} = 0$ for $n \neq 0$. Thus, for non-zero summands in the left-hand side of (17), it must be $n = n' = 0$, so that:

$$
\sum_{i,n,k} \left( \sum_{i',n',j} \alpha_{i,k}^{x,i',j} \alpha_{i',j,k}^{y,i,j} \right) \left( \sum_{\psi \in \mathcal{P}} \left( \sum_{i,n,k} \alpha_{i,n,k}^{y,i,n,k} \hat{\psi}_i^{(n)} \hat{\psi}_i^{(n')} \right) \right) = \sum_{i,n,k} \left( \sum_{i',n',j} \alpha_{i',j,k}^{x,i',j} \alpha_{i,j,k}^{y,i,j} \right) \left( \sum_{\psi \in \mathcal{P}} \left( \sum_{i,n,k} \alpha_{i,n,k}^{y,i,n,k} \hat{\psi}_i^{(n)} \hat{\psi}_i^{(n')} \right) \right).
$$

(18)

Now, being finite the cardinal of $\Psi$, according to Lemma 7 and the comments that follow it, the unconditional sums in (18) may be calculated, for example, in the following way:

$$
\sum_{i,n,k} \left( \sum_{i',n',j} \alpha_{i,k}^{x,i',j} \alpha_{i',j,k}^{y,i,j} \right) \left( \sum_{\psi \in \mathcal{P}} \left( \sum_{i,n,k} \alpha_{i,n,k}^{y,i,n,k} \hat{\psi}_i^{(n)} \hat{\psi}_i^{(n')} \right) \right) = \lim_{a \to \infty} \sum_{i,n,k} \left[ \sum_{i',n',j} \alpha_{i',n',j}^{x,i',j} \alpha_{i,j,n,k}^{y,i,j} \right] \left[ \sum_{\psi} \left( \sum_{i,n,k} \alpha_{i,n,k}^{y,i,n,k} \hat{\psi}_i^{(n)} \hat{\psi}_i^{(n')} \right) \right].
$$
Then, in this particular case, Corollary 9 can be rewritten as follows:

**Proposition 10** Let $X$ be a wavelet system in $L^2(\mathbb{R})$ of the form (5), where $\Psi$ has finite cardinal and $\text{supp} \psi \subseteq [0, 1]$ for every $\psi \in \Psi$. Then, $X$ is a tight frame for $L^2(\mathbb{R})$, with frame bound $B$, if and only if:

$$
\lim_{a \to \infty} \sum_{i,j,l} \left( \sum_{k=-a}^{a} \alpha_{i,j}^{a,k} \alpha_{i,j}^{a,k+\sigma} \right) \left[ \sum_{\psi \in \Psi} \tilde{\psi}_i(0) \tilde{\psi}_j(0) \right] = B \delta_{s,s'} \delta_{l,l'} \delta_{\sigma}, \quad (s,s' = \pm, l,l' \in \mathbb{J}, \sigma \in \mathbb{Z}).
$$

(19)

From now on, we consider the Haar orthonormal bases $\{L_i^{(0)}\}$ and $\{K_{s,l}^{(k)}\}$ and the corresponding matrix $\left( a_{i,j}^{s,k} \right)$ given in Appendix B. This choice leads to the following result, which we write in vectorial form:

**Proposition 11** Let $\{L_i^{(0)}\}_{i \in \mathbb{N} \cup \{0\}}$ be the Haar orthonormal basis of $L^2[0,1]$ given in (99). Let $r \in \mathbb{N}$ and:

$$\Psi = \{\psi_1, \psi_2, \ldots, \psi_r\} \subset L^2[0, 1],$$

where $\psi_j = \sum_i [\tilde{\psi}_i]_j^{(0)} L_i^{(0)}$, $(j = 1, \ldots, r)$. Let us put:

$$\Psi_i := \left[ \begin{array}{c} [\tilde{\psi}_1]_i^{(0)} \\
[\tilde{\psi}_2]_i^{(0)} \\
\vdots \\
[\tilde{\psi}_r]_i^{(0)} \end{array} \right] \in \mathbb{C}^r, \quad (i \in \mathbb{N} \cup \{0\}).$$

Then, the wavelet system $X$ of the form (5) generated by $\Psi$ is a tight frame for $L^2(\mathbb{R})$, with frame bound $B$, if and only if the following conditions are satisfied:

1. $\Psi_0 = 0 \in \mathbb{C}^r$.
2. One has:

$$\sum_{k=\sup\{1,1-\sigma\}}^{\infty} \langle \Psi_{2k+l+1}, \Psi_{2k+l} \rangle_{\mathbb{C}^r} = B \delta_{\sigma}, \quad (\sigma \in \mathbb{Z}).$$

(20)

3. For $l \geq 1$, $l = 2^p + \sum_{l=0}^{p-1} l_2^l$, $(p \geq 0)$:

$$\sum_{k=-r}^{0} ||\Psi_{2^{p+k}+\sum_{l=0}^{p-1} l_2^l}||_{\mathbb{C}^r}^2 + \sum_{k=1}^\infty ||\Psi_{2^{p+k}+l}||_{\mathbb{C}^r}^2 = B,$$

(21)

$$\sum_{k=\sup\{1,1-\sigma\}}^{\infty} \langle \Psi_{2^{p+k}+l}, \Psi_{2^{p+k+1}-l} \rangle_{\mathbb{C}^r} = 0, \quad (\sigma \in \mathbb{Z} \setminus \{0\}).$$

(22)
\[
\sum_{k=\sup\{1,1-\sigma\}}^{\infty} \langle \Psi_{2^p+i+k+\sigma}, \Psi_{2^p-1} \rangle_{C^r} = 0, \quad (\sigma \in \mathbb{Z}).
\] (23)

4. For \( l, l' \geq 1, l \neq l', l = 2^p + \sum_{t=0}^{p-1} l_t 2^t, l' = 2^{p'} + \sum_{t=0}^{p'-1} l'_t 2^t, (p, p' \geq 0):

\[
\sum_{k=\sup\{-p,-p'\}}^{0} \delta(\sum_{t=0}^{k} l_t 2^t - (\sum_{t=0}^{k} l'_t 2^t)) \times \langle \Psi_{2^p+i+k}, \Psi_{2^{p'}+i+k} \rangle_{C^r} + \sum_{k=1}^{\infty} \langle \Psi_{2^p+i+k}, \Psi_{2^{p'}+i+k} \rangle_{C^r} = 0,
\] (24)

\[
\sum_{k=\sup\{1,1-\sigma\}}^{\infty} \langle \Psi_{2^p+i+k+\sigma}, \Psi_{2^p+i+k+1} \rangle_{C^r} = 0, \quad (\sigma \in \mathbb{Z}\setminus\{0\}).
\] (25)

**Proof** To avoid additional indices along the proof, we work with generic \( \psi \in \Psi \) and not with \( \psi_1, \ldots, \psi_r \). Consider the Haar orthonormal bases \( \{ L^{(i)} \} \) and \( \{ K^{(k)} \} \) and the corresponding matrix \( \alpha_{s,t}^{y,k} \) given in Appendix B. For \( s = s' = +, l = l' = 0, \) and \( \sigma = 0 \) in (19), one obtains:

\[
\lim_{a \to \infty} \sum_{l,l'} \left[ \sum_{k=-2^a}^{2^a} \alpha_{l,j}^{y,k} \alpha_{l',j}^{y,k} \right] \left[ \sum_{\psi} \hat{\psi}_0^{(0)} \hat{\psi}_0^{(0)} \right] = \lim_{a \to \infty} \left( a + 1 + \sum_{k=1}^{a} 2^{-k} \right) \left[ \sum_{\psi} |\hat{\psi}_0^{(0)}|^2 \right] + \sum_{r=0}^{\infty} \left[ \sum_{\psi} |\hat{\psi}_r^{(0)}|^2 \right].
\]

The last expression can be equal to \( B \) if and only if \( \hat{\psi}_0^{(0)} = \int \psi = 0 \) for every \( \psi \in \Psi \), which is condition 1 in the statement, and \( \sum_{t=0}^{\infty} |\sum_{\psi} |\hat{\psi}_0^{(0)}|^2| = B \), the condition (20) for \( \sigma = 0 \) in the statement. Assuming then that \( \hat{\psi}_0^{(0)} = 0 \) for every \( \psi \in \Psi \), straightforward calculations with the \( \alpha_{s,t}^{y,k} \)'s lead to the fact that the conditions in (19), for \( s = s' = +, \) are related with Table 1.

To obtain the conditions in (19), for \( s = s' = +, \) Table 1 is used as follows: choose the values of \( l \) and \( l' \), and consider the corresponding files in the table. Choose the value of \( \sigma \). In each entry of the table, the indices \( k \) and \( i \) are associated with \( l \), and the indices \( k + \sigma \) and \( i' \) are associated with \( l' \). One must pair the columns with the same \( k \) and \( j \) in both files, multiply \( \hat{\psi}_j^{(0)} \) by \( \hat{\psi}_i^{(0)} \) for the indices \( i, i' \) selected in the paired columns, sum over \( \psi \in \Psi \) and, finally, sum over all the paired columns. For example, for \( l = l' = 0 \) and \( \sigma = 0 \) we arrive to the already known equation:

\[
\sum_{k=1}^{\infty} \left[ \sum_{\psi} |\hat{\psi}_2^{(0)}|^2 \right] = B,
\]
the condition (20) for $\sigma = 0$ in the statement. For $l = l' = 0$ and $\sigma > 0$, we get (note that in this case $j = 0$ in every column):

$$\sum_{k=1}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2k-1}^{(l)} \tilde{\psi}_{2k-1+\sigma}^{(l)} \right] = 0. \quad (26)$$

For $l = l' = 0$ and $\sigma < 0$, the resultant condition:

$$\sum_{k=1-\sigma}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2k-1}^{(l)} \tilde{\psi}_{2k-1+\sigma}^{(l)} \right] = 0$$

coincides with (26) for $-\sigma$. Both of them correspond to the condition (20) for $\sigma \neq 0$ in the statement. For $l = l' = 2^p + \sum_{t=0}^{p-1} l_t 2^t$, $(p \geq 0)$, and $\sigma = 0$:

$$\sum_{k=-p}^{-1} \left[ \sum_{\psi} |\tilde{\psi}_{2p+k}^{(l)}| \right]^2 + \left[ \sum_{\psi} |\tilde{\psi}_l^{(l)}| \right]^2 + \sum_{k=1}^{\infty} \left[ \sum_{\psi} |\tilde{\psi}_{2p+k+l}^{(l)}| \right]^2 = B,$$

which is the condition (21) in the statement. For $l = l' = 2^p + \sum_{t=0}^{p-1} l_t 2^t$, $(p \geq 0)$, and $\sigma > 0$ (note that in this case there are different $j$'s in the first columns of Table 1):

$$\sum_{k=1}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2p+k+l}^{(l)} \tilde{\psi}_{2p+k+l+\sigma}^{(l)} \right] = 0. \quad (27)$$

For $l = l' = 2^p + \sum_{t=0}^{p-1} l_t 2^t$, $(p \geq 0)$, and $\sigma < 0$, condition:

$$\sum_{k=1-\sigma}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2p+k+l}^{(l)} \tilde{\psi}_{2p+k+l+\sigma}^{(l)} \right] = 0$$

coincides with (27). Both of them correspond to condition (22) in the statement. For $l = 0, l' = 2^p + \sum_{t=0}^{p-1} l_t 2^t$, $(p \geq 0)$, and $\sigma \in \mathbb{Z}$:

$$\sum_{k=\text{sup}{1,1-\sigma}}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2k-1}^{(l)} \tilde{\psi}_{2k-1+\sigma}^{(l)} \right] = 0,$$

the condition (23) in the statement. For $l = 2^p + \sum_{t=0}^{p-1} l_t 2^t, \ l' = 2^{p'} + \sum_{t=0}^{p'-1} l'_t 2^t, \ (p, p' \geq 0)$, and: $\sigma = 0$,

$$\sum_{k=\text{sup}{1,1-\sigma}}^{-1} \delta((\sum_{t=0}^{\frac{p}{2}} l_t 2^t) - (\sum_{t=0}^{\frac{p'}{2}} l'_t 2^t)) \times \left[ \sum_{\psi} \tilde{\psi}_{2p+k+l}^{(l)} \tilde{\psi}_{2p+k+l+\sigma}^{(l)} \right] = 0,$$

$$\sum_{k=1}^{\infty} \left[ \sum_{\psi} \tilde{\psi}_{2p+k+l}^{(l)} \tilde{\psi}_{2p+k+l+\sigma}^{(l)} \right] = 0.$$
which is condition (24) in the statement. For \( l = 2^p + \sum_{t=0}^{p-1} l_t 2^t, l' = 2^{p'} + \sum_{t=0}^{p'-1} l'_t 2^t, \) \((p, p' \geq 0),\) and \( \sigma \in \mathbb{Z} \setminus \{0\}:\)

\[
\sum_{k=\sup\{1,1-\sigma\}}^{\infty} \left[ \sum_{\psi} \overline{\psi^{(0)}_{2^{p+l}+l}} \psi^{(0)}_{2^{p'+s}+p} \right] = 0,
\]

the condition (25) in the statement. For \( s = s' = -, \) conditions derived from the set of Eq. (19) are equivalent to the former ones for \( s = s' = +. \) For \( s = + \) and \( s' = -, \) or \( s = - \) and \( s' = +, \) the set of equations (19) leads to trivial conditions. \( \square \)

### 4.1 Hardy functions

It is not easy to handle the set of conditions for the vectors \( \Psi_i \in C' \) in Proposition 11. A better way to tackle this set of conditions consists in writing them in terms of Hardy functions in \( H^+(\partial \mathbb{D}, C') \) we next define. In such approach, inner matrix functions and results by Halmos (Lemma 12 below) and Rovnyak (Lemma 13) play a central role.

Recall that \( \mathbb{D} \) denotes the open unit disc of the complex plane \( \mathbb{C} \) and \( \partial \mathbb{D} \) its boundary, the unit circle. Let \( \mathcal{H} \) be a separable Hilbert space and denote by \( \mathcal{L}(\mathcal{H}) \) the space of bounded operators on \( \mathcal{H} \) (in the sequel, we shall only need to consider Hilbert spaces of finite dimension, \( \mathcal{H} = C' \), for which \( \mathcal{L}(\mathcal{H}) \) can be identified with the space of complex \((r \times r)\)-matrices). We denote by \( H^2(\mathbb{D}; \mathcal{H}) \) the Hardy class of functions:

\[
\tilde{h}(\lambda) = \sum_{k=0}^{\infty} \lambda^k h_k, \quad (\lambda \in \mathbb{D}, h_k \in \mathcal{H}),
\]

with values in \( \mathcal{H}, \) such that \( \sum |h_k|_{\mathcal{H}}^2 < \infty. \) For each function \( \tilde{h} \in H^2(\mathbb{D}; \mathcal{H}), \) the non-tangential limit in strong sense:

\[
s-lim_{\lambda \to \omega} \tilde{h}(\lambda) = \sum_{k=0}^{\infty} \omega^k h_k =: h(\omega)
\]

exist for almost all \( \omega \in \partial \mathbb{D}. \) The functions \( \tilde{h}(\lambda) \) and \( h(\omega) \) determine each other (they are connected by Poisson formula), so that we can identify \( H^2(\mathbb{D}; \mathcal{H}) \) with a subspace of \( L^2(\partial \mathbb{D}; \mathcal{H}), \) say \( H^+(\partial \mathbb{D}; \mathcal{H}), \) thus providing \( H^2(\mathbb{D}; \mathcal{H}) \) with the Hilbert space structure of \( H^+(\partial \mathbb{D}; \mathcal{H}) \) and embedding it in \( L^2(\partial \mathbb{D}; \mathcal{H}) \) as a subspace (the space \( L^2(\partial \mathbb{D}; \mathcal{H}) \) has been defined in Sect. 2).

From now on, the operator “multiplication by \( \omega \)” on \( H^+(\partial \mathbb{D}; \mathcal{H}) \) shall be denoted by \( M^+, \) that is:

\[
[M^+ h](\omega) := \omega \cdot h(\omega), \quad (h \in H^+(\partial \mathbb{D}; \mathcal{H}), \omega \in \partial \mathbb{D}). \tag{28}
\]

The operator \( M^+ \) is an isometry from \( H^+(\partial \mathbb{D}; \mathcal{H}) \) into \( H^+(\partial \mathbb{D}; \mathcal{H}). \) A subspace \( \mathfrak{M} \subseteq H^+(\partial \mathbb{D}; \mathcal{H}) \) is called a wandering subspace for \( M^+ \) if \( [M^+]^m \mathfrak{M} \perp [M^+]^n \mathfrak{M} \) whenever \( m \) and \( n \) are distinct non-negative integers.
Consider the subspace $\mathcal{G}$ of $H^+(\partial \mathbb{D}; \mathcal{H})$ consisting of all constant functions, i.e., the functions $f : \partial \mathbb{D} \to \mathcal{H}$, such that there exists a vector $h \in \mathcal{H}$ with $f(\omega) = h$ for a.e. $\omega \in \partial \mathbb{D}$. A weakly measurable\(^4\) operator-valued function:

$$A^+ : \partial \mathbb{D} \to \mathcal{L}(\mathcal{H}) : \omega \mapsto A(\omega)$$

is called\(^5\) a $M^+$-inner function or rigid Taylor operator function if $A^+$ maps $\mathcal{G}$ into $H^+(\partial \mathbb{D}; \mathcal{H})$ and $A^+(\omega)$ is for a.e. $\omega \in \partial \mathbb{D}$ a partial isometry\(^6\) on $\mathcal{H}$ with the same initial space.

According to Halmos [19, Lemma 5], wandering subspaces for $M^+$ and $M^+$-inner functions (or rigid Taylor operator functions) are related as follows:

**Lemma 12** (Halmos) A subspace $\mathcal{M}$ of $H^+(\partial \mathbb{D}; \mathcal{H})$ is a wandering subspace for $M^+$ if and only if there exists a $M^+$-inner function $A^+$, such that $\mathcal{M} = A^+ \mathcal{G}$. The subspace $\mathcal{M}$ uniquely determines $A^+$ to within a constant partially isometric factor on the right.

Another fundamental result in what follows is due to Rovnyak [42, Lemma 5]:

**Lemma 13** (Rovnyak) If $\mathcal{H}$ has finite dimension $r$, there is no orthonormal set $h_1, \ldots, h_{r+1}$ in $H^+(\partial \mathbb{D}; \mathcal{H})$ containing $r + 1$ elements and such that $\omega^m h_i(\omega)$ is orthogonal to $\omega^n h_j(\omega)$ whenever $m \neq n$.

In terms of Hardy functions, Proposition 11 reads as follows:

**Proposition 14** Under the conditions of Proposition 11, for $l \geq 0$, consider the Hardy function $h_l \in H^+(\partial \mathbb{D}, \mathbb{C}^r)$ defined by:

$$h_0(\omega) := \sum_{k=0}^{\infty} \omega^k \Psi_{2^k},$$

$$h_l(\omega) := \sum_{k=0}^{\infty} \omega^k \Psi_{2^p k + l + 2^p}, \quad (2^p \leq l < 2^{p+1}, \ p \geq 0).$$

Then, the wavelet system $X$ of the form (5) generated by $\Psi$ is a tight frame, with frame bound $B$, if and only if the following conditions are satisfied:

$$\Psi_0 = 0 \in \mathbb{C}^r,$$

\(^4\) $A^+$ weakly measurable means the scalar product $(A^+(\omega) h, g)_{\mathcal{H}}$ is a Borel measurable scalar function on $\partial \mathbb{D}$ for each $h, g \in \mathcal{H}$.

\(^5\) The name rigid Taylor operator function is introduced by Halmos [19]. The name $M^+$-inner function is used by Rosenblum and Rovnyak [41] and co-workers.

\(^6\) An operator $B \in \mathcal{L}(\mathcal{H})$ is a partial isometry when there is a (closed) subspace $\mathcal{M}$ of $\mathcal{H}$, such that $||Bu|| = ||u||$ for $u \in \mathcal{M}$ and $Bv = 0$ for $v \in \mathcal{M}^\perp$. In such a case, $\mathcal{M}$ is called the initial space of $B$. 

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\[ \langle \omega^m \tilde{h}_m, \omega^n \tilde{h}_n \rangle \rangle_{H^+(\partial \mathbb{D}, \mathbb{C})} = \delta_{m-n} \beta_{l, l'}, \quad (l, l', m, n \in \mathbb{N} \cup \{0\}) \]  

where:

\[ \beta_{l, l'} := \begin{cases} 
B, & \text{if } l = l' = 0, \\
0, & \text{if } l = 0, l' \geq 1, \\
B - \sum_{k=-p}^{0} ||\mathcal{P}_{2^{p+k+1}l}||_{C^l}^2, & \text{if } l = l' \geq 1, l = 2^p + \sum_{i=0}^{p-1} l_i 2^i, \\
-\sum_{k=\sup\{\sup_-p, -p'\}}^{0} \delta_{(\sum_{m=0}^{-p}, l, 2^i)} \mathcal{P}_{2^{p+k+1}l} \mathcal{P}_{2^{p+k+1}l}^\top \mathcal{P}_{2^{p+k+1}l} \mathcal{P}_{2^{p+k+1}l}^\top, & \text{if } l, l' \geq 1, l \neq l', l = 2^p + \sum_{i=0}^{p-1} l_i 2^i, l' = 2^{p'} + \sum_{i=0}^{p'-1} l_i 2^i. 
\end{cases} \]

**Proof** Condition 1 of Proposition 11 coincides with (29), and conditions 2–4 in Proposition 11, i.e., Eqs. (20)–(25), are equivalent to condition (30) of the statement. In detail, conditions (20), (22), (23), and (25), all of them for \( \sigma \in \mathbb{Z} \setminus \{0\} \), are in correspondence with condition (30) for \( m \neq n \); condition (20) for \( \sigma = 0 \) corresponds to the first line of the definition of \( \beta_{l, l'} \) in condition (30); condition (23) for \( \sigma = 0 \) corresponds to the second line of the definition of \( \beta_{l, l'} \) in condition (30); condition (21) corresponds to the third line of the definition of \( \beta_{l, l'} \) in condition (30); finally, condition (24) corresponds to the fourth line of the definition of \( \beta_{l, l'} \) in condition (30). \( \square \)

### 4.2 Case \( r = 1; \Psi = \{ \psi \} \subseteq L^2[0, 1] \)

For \( r = 1 \), Proposition 14 leads to the following:

**Corollary 15** The only function \( \psi \in L^2[0, 1] \), such that the wavelet system \( \mathcal{X} \) of the form (5) generated by \( \Psi = \{ \psi \} \) is a tight frame for \( L^2(\mathbb{R}) \), with frame bound \( B \), is proportional to the Haar wavelet:

\[ \psi = \beta [\chi_{(0,1/2)} - \chi_{(1/2,1)}], \]

where \( \beta \in \mathbb{C} \) and \( |\beta|^2 = B \).

**Proof** Condition (30) in Proposition 14, with \( l = l' \geq 0 \) and \( m \neq n \), implies that each \( \tilde{h}_l \) is a scalar \((M^+)-(l, l', 2^i)\)-inner function in \( H^+(\partial \mathbb{D}, \mathbb{C}) \), unless \( \beta(l, l) = 0 \). Since \( \beta(0, 0) = B > 0 \), \( \tilde{h}_0 \) is a scalar inner function in \( H^+(\partial \mathbb{D}, \mathbb{C}) \). Condition (30) again, now with \( m = n = 0 \), \( l = 0 \) and \( l' \geq 1 \), assures that \( \tilde{h}_0 \) is orthogonal to every \( \tilde{h}_l \), \( l' \geq 1 \). Then, according to Rovniak’s Lemma 13, one has \( \tilde{h}_{l'} = 0 \) for every \( l' \geq 1 \). From condition (30) once more, with \( m = n = 0 \) and \( l = l' = 1 \), one gets \( ||\mathcal{P}_{1}||^2 = ||\tilde{\psi}_{1}||^2 = B \), so that \( ||\mathcal{P}_{l}||^2 = ||\tilde{\psi}_{l}||^2 = 0 \) for \( l \neq 1 \). \( \square \)
Remark 16 Corollary 15 implies, in particular, that the only orthonormal wavelet \( \psi \in L^2(\mathbb{R}) \) with \( \text{supp} \psi \subseteq [0, 1] \) is the Haar wavelet. This fact contradicts Theorem 5 in [15], which asserts that it is possible to obtain orthonormal wavelets \( \psi \in L^2(\mathbb{R}) \) of minimal support, different from the Haar wavelet, using certain families of Hardy functions. The main problem in [15] is to check the completeness condition (ii) of Corollary 3 there. The sufficient condition given in item (2) of Proposition 4 there, \( \tilde{\psi}_{+1} \neq 0 \), fails to be right. According to Corollary 15 here, the completeness condition (ii) of Corollary 3 in [15] is satisfied if and only if \( \tilde{\psi}_{+1} = 1 \). Thus, for the functions \( \psi \) given in Theorem 5 of [15], save the Haar wavelet, the family \( \{ \psi_{m,n} := D^m T^n \psi : m, n \in \mathbb{Z} \} \) is an orthonormal system of \( L^2(\mathbb{R}) \), but it is not complete.

4.3 Case \( r = 2: \mathcal{U} = \{ \psi_1, \psi_2 \} \subset L^2[0, 1] \)

For \( r = 2 \), condition (30) in Proposition 14 implies, among other things, that the closed subspace generated by any subfamily of the set \( \{ \mathfrak{b}_l \}_{l \in \mathbb{N} \cup \{0\}} \) is a wandering subspace for \( M^+ \) in \( H^+_C \) = \( H^+(\partial \mathbb{D}, \mathbb{C}^2) \). According to Halmos’s Lemma 12, such a wandering subspace has at most dimension 2 and is of the form \( A^+ \mathfrak{C} \), for some rigid Taylor \( (M^+-\text{inner}) \) operator-valued function \( A^+ \), where \( \mathfrak{C} \) denotes the subspace of constant functions. Thus, \( A^+(\omega) : \mathbb{C}^2 \to \mathbb{C}^2 \) is for a.e. \( \omega \in \partial \mathbb{D} \) a partial isometry with the same initial subspace. For non-zero \( A^+ \), such initial subspace, say \( C \subset \mathbb{C}^2 \), can have dimension 1 or 2.

Consider, in particular, the closed wandering subspace for \( M^+ \) generated by \( \{ \mathfrak{b}_0, \mathfrak{b}_1 \} \) in \( H^+_C \) and the corresponding rigid Taylor \( (M^+-\text{inner}) \) operator-valued function \( A^+ \) with initial subspace \( C \subset \mathbb{C}^2 \). Condition (30) implies, in particular, that:

\[
||\mathfrak{b}_0||^2_{H^+_C} = B \neq 0; \quad \mathfrak{b}_0 \perp \mathfrak{b}_l, \quad (l \geq 1); \tag{31}
\]

\[
||\mathfrak{b}_1||^2_{H^+_C} = B - ||\psi_1||^2_{C^2}; \quad (\mathfrak{b}_l, \mathfrak{b}_l)_{H^+_C} = -(\psi_l, \psi_1)_{C^2}, \quad (l \geq 2). \tag{32}
\]

Then, if \( \dim C = 1 \), the only option is that \( \mathfrak{b}_0 \neq 0 \) and \( \mathfrak{b}_1 = 0 \), so that \( B = ||\psi_1||^2_{C^2} \); in such a case, conditions (29) and (30) are satisfied if and only if \( \psi_1 \neq 0 \) and \( \psi_l = 0 \) for every \( l \neq 1 \) in \( \mathbb{N} \cup \{0\} \). That is, if \( \dim C = 1 \), in \( \mathcal{U} = \{ \psi_1, \psi_2 \} \) both functions \( \psi_1 \) and \( \psi_2 \) are proportional to the Haar wavelet. Thus, the only possible non-trivial situation requires that \( \dim C = 2 \). For \( \dim C = 2 \), the rigid Taylor \( (M^+-\text{inner}) \) operator-valued function \( A^+ \) can be written as a \( (2 \times 2) \) matrix inner function:

\[
A^+(\omega) = \begin{pmatrix}
\mathfrak{a}_1(0)(\omega) & \mathfrak{a}_1(1)(\omega) \\
\mathfrak{a}_2(0)(\omega) & \mathfrak{a}_2(1)(\omega)
\end{pmatrix},
\]

whose entries are functions belonging to the scalar Hardy space \( H^+_C = H^+(\partial \mathbb{D}, \mathbb{C}) \) and such that \( A^+(\omega) \) is unitary for a.e. \( \omega \in \partial \mathbb{D} \). In other words, the columns of \( A^+ \), say:
\[ a^{(0)}(\omega) := \begin{pmatrix} a_1^{(0)}(\omega) \\ a_2^{(0)}(\omega) \end{pmatrix}, \quad a^{(1)}(\omega) := \begin{pmatrix} a_1^{(1)}(\omega) \\ a_2^{(1)}(\omega) \end{pmatrix}, \]

are elements of \( H^+_{c^2} = H^+(\partial \mathbb{D}, \mathbb{C}^2) \) satisfying:

\[
\langle a^{(i)}, \omega^m a^{(j)} \rangle_{H^+_{c^2}} = \delta_m \delta_{i-j}, \quad (m \in \mathbb{Z}), (i,j = 0, 1). \tag{33}
\]

Since the closed subspace generated by \( \{ \vartheta_0, \vartheta_1 \} \) coincides with \( A^+ \mathbb{C} \), the functions \( \vartheta_0 \) and \( \vartheta_1 \) can be taken to be proportional to these vectors: \( \vartheta_0 = B_0 a^{(0)} \) and \( \vartheta_1 = C_1 a^{(1)} \) for certain non-null constants \( B_0, C_1 \in \mathbb{C} \). Moreover, according to (31), (32) and Rovniak’s Lemma 13:

\[
\vartheta_0(\omega) = B_0 a^{(0)}(\omega); \quad \vartheta_1(\omega) = C_1 a^{(1)}(\omega), \quad (l \geq 1),
\]

where the constants \( B_0, C_1 \in \mathbb{C} \) must satisfy:

\[
|B_0|^2 = B \neq 0, \quad |C_1|^2 = B - ||\vartheta_1||^2 \neq 0,
\]

\[
C_1 C_1^* = \langle \vartheta_0, \vartheta_1 \rangle_{H^+_{c^2}} = -\langle \vartheta_1, \vartheta_1 \rangle_{c^2}, \quad (l \geq 2). \tag{34}
\]

Consider the Taylor–Fourier series:

\[
a^{(0)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(0)}, \quad a^{(1)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(1)},
\]

where \( a_k^{(0)}, a_k^{(1)} \in \mathbb{C}^2, (k \in \mathbb{N} \cup \{0\}) \), and:

\[
\vartheta_0(\omega) = \sum_{k=0}^{\infty} \omega^k \vartheta_2^k = \sum_{k=0}^{\infty} \omega^k \begin{pmatrix} [\vartheta_1]^{(0)}_{2^k} \\ [\vartheta_2]^{(0)}_{2^k} \end{pmatrix} = B_0 \sum_{k=0}^{\infty} \omega^k a_k^{(0)},
\]

\[
\vartheta_1(\omega) = \sum_{k=0}^{\infty} \omega^k \vartheta_2^{k+1} = \sum_{k=0}^{\infty} \omega^k \begin{pmatrix} [\vartheta_1]^{(0)}_{2^k+1}^{(0)} \\ [\vartheta_2]^{(0)}_{2^k+1}^{(0)} \end{pmatrix} = C_1 \sum_{k=0}^{\infty} \omega^k a_k^{(1)}.
\]

In these terms:

\[
|C_1|^2 = B - ||\vartheta_1||^2 \neq 0, \tag{35}
\]

\[
C_{2^k} = -\frac{1}{C_1} \langle \vartheta_2^{k}, \vartheta_1 \rangle_{c^2} = -\frac{B}{C_1} \langle a_k^{(0)}, a_k^{(0)} \rangle_{c^2}, \quad (k \in \mathbb{N}), \tag{36}
\]

\[
C_{2^k+1} = -\frac{1}{C_1} \langle \vartheta_2^{k+1}, \vartheta_1 \rangle_{c^2} = -\frac{C_1 B_0}{C_1} \langle a_{k-1}^{(1)}, a_k^{(0)} \rangle_{c^2}, \quad (k \in \mathbb{N}). \tag{37}
\]

Any other \( C_l, (l \geq 2) \), can be expressed in terms of the \( C_{2^k} \) and \( C_{2^k+1} \):
Lemma 17 For $l \geq 2$, $l \not= 2^k$, $2^k + 1$, with $l = 2^p + 2^{p_1} + 2^{p_2} + \cdots + 2^{p_s}$, where $p > p_1 > p_2 > \cdots > p_s \geq 0$, one has:

$$C_l = \begin{cases} C_1^{-(l-1)} C_{2^{p-p_1+1}} \cdots C_{2^{p_{s-1}-p_{s-1}+1}} C_{2^{p_{s-1}+1}}, & \text{if } p_s = 0, \\ C_1 C_{2^{p-p_1+1}} \cdots C_{2^{p_{s-1}-p_{s-1}+1}} C_{2^{p_s}}, & \text{if } p_s > 0. \end{cases}$$

(38)

Proof For $l \geq 2$, $l \not= 2^k$, $2^k + 1$, so that $l = 2^p + l_1$ with $l_1 \not= 0, 1$, and $l_1 = 2^{p_1} + l_2$, since $\mathbf{b}_l(\alpha) := \sum_{k=0}^\infty \alpha^k \mathbf{c}_{2^k + 1 + l}$, the vector $\mathbf{c}_l = \mathbf{c}_{2^p + l_1}$ is the $k$-coefficient in the Taylor series of $\mathbf{b}_l$, where $k = p - p_1 - 1$. Thus, being $\mathbf{b}_l = C_l \mathbf{a}^{(1)}$:

$$C_l = -\frac{1}{C_1} \langle \mathbf{c}_l, \mathbf{a}^{(1)} \rangle C_2^2 = -\frac{1}{C_1} \langle C_l \mathbf{a}^{(1)}_{p-p_1-1}, B_0 \mathbf{a}^{(0)} \rangle C_2^2 = \frac{C_{2^{p_p+1}}}{C_1} C_l.$$

(39)

The same argument for $l_1, l_2, \ldots, l_{p_s}$ leads to:

$$C_l = \frac{C_1^{-(l-1)} C_{2^{p-p_1+1}} C_{2^{p_{s-1}-p_{s-1}+1}}}{C_{2^{p_s+1}}^{l-1}} C_{2^{p+1}}^{l_s+2}.$$

In the final step, if $p_s = 0$, then $C_{2^{p_s+1}}^{l_s+2} = C_{2^{p+1}}^{l_s+1}$; on the other hand, if $p_s > 0$, as before:

$$C_{2^{p+1}}^{l_s+2} = \frac{C_{2^{p+1}-p_s+1}}{C_1} C_{2^{p_s}}.$$

In general, since $\mathbf{b}_l = C_l \mathbf{a}^{(1)}$ for $l \geq 1$:

$$||\mathbf{b}_l||_{H^+_{C_2}} = ||C_l||^2 ||\mathbf{a}^{(1)}||_{H^+_{C_2}} = ||C_l||^2.$$

On the other hand, Eq. (30) with $m = n$ and $l = l' \geq 1, l = 2^p + \sum_{i=0}^{p-1} l_i 2^i$, implies:

$$||\mathbf{b}_l||_{H^+_{C_2}} = B - \sum_{k=-p}^{0} ||\mathbf{c}_{2^{p+k+1}} \sum_{i=0}^{p+k+1} l_i 2^i ||^2_{C_2}.$$ 

In particular, for $l = 2^p$ with $p \geq 0$:

$$|C_{2^p}|^2 = ||\mathbf{b}_{2^p}||_{H^+_{C_2}} = B - \sum_{k=0}^{p} ||\mathbf{c}_{2^k}||^2_{C_2} = B \left(1 - \sum_{k=0}^{p} ||\mathbf{a}^{(0)}_k||^2_{C_2}\right), \quad (p \geq 0).$$

(40)

And, for $l = 2^p + 1$ with $p > 0$:  

\[ \text{Birkhäuser} \]
\[ |C_{2^p+1}|^2 = ||h_{2^p+1}||_{H^+_{C^2}} = B - ||\Psi_1||_{C^2}^2 - \sum_{k=1}^{p} ||\Psi_{2^k+1}||_{C^2}^2 \]

\[ = B \left( 1 - ||a_0^{(0)}||_{C^2}^2 \right) \left( 1 - \sum_{k=0}^{p-1} ||a_k^{(1)}||_{C^2}^2 \right), \quad (p > 0). \tag{41} \]

In a similar way, from Eq. (30), now with \( m = n \) and \( l \neq l', l = 2^p \) or \( l = 2^p + 1, \)
\( l' = 2^{p'}, \) or \( l' = 2^{p'} + 1, \) one gets:

\[ C_{2^{p'}} \overline{C_{2^p}} = \langle h_{2^p}, h_{2^{p'}} \rangle_{H^+_{C^2}} = - \sum_{r=0}^{p} \langle \Psi_{2^{p'}-r}, \Psi_{2^p-r} \rangle_{C^2} \]

\[ = - B \sum_{r=0}^{p} \langle a_0^{(0)} - r, a_0^{(0)} - r \rangle_{C^2}, \quad (0 \leq p < p'), \tag{42} \]

\[ C_{2^{p'+1}} \overline{C_{2^{p+1}}} = \langle h_{2^{p'+1}}, h_{2^{p+1}} \rangle_{H^+_{C^2}} = - \sum_{r=0}^{p-1} \langle \Psi_{2^{p'-1}+r}, \Psi_{2^{p+1}+r} \rangle_{C^2} \]

\[ = - B \left( 1 - ||a_0^{(0)}||_{C^2}^2 \right) \sum_{r=0}^{p-1} \langle a_1^{(1)} - r, a_1^{(1)} - r \rangle_{C^2}, \quad (0 < p < p'). \tag{43} \]

\[ C_{2^{p'+1}} \overline{C_{2^p}} = \langle h_{2^{p'+1}}, h_{2^p} \rangle_{H^+_{C^2}} \]

\[ = \begin{cases} 
- \sum_{r=0}^{p} \langle \Psi_{2^{p'-1}+r}, \Psi_{2^{p+1}+r} \rangle_{C^2} = -C_1 B_0 \sum_{r=0}^{p} \langle a_1^{(1)} - r, a_0^{(0)} - r \rangle_{C^2}, & \text{if } 0 \leq p < p', \\
- \sum_{r=0}^{p-1} \langle \Psi_{2^{p'-1}+r}, \Psi_{2^{p+1}+r} \rangle_{C^2} = -C_1 B_0 \sum_{r=0}^{p-1} \langle a_1^{(1)} - r, a_0^{(0)} - r \rangle_{C^2}, & \text{if } 0 < p' \leq p. \end{cases} \tag{44} \]

Let us note that (35) coincides with (40) for \( p' = 0, \) (36) is (42) for \( p = 0 \) and \( p' = k \in \mathbb{N}, \) and (37) is (44) for \( p = 0 \) and \( p' = k \in \mathbb{N}. \)

Conditions (40)–(44) are part of the conditions (30) in Proposition 14, but they are sufficient for the family of Hardy functions \( \{ h_i \}_{i \in \mathbb{N} \cup \{ 0 \}} \subset H^+_{C^2} \) to satisfy the complete set of conditions (30):

**Proposition 18** Let:

\[ a^{(0)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(0)}, \quad a^{(1)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(1)} \]

be a pair of functions in \( H^+_{C^2} = H^+(\partial \mathbb{D}, \mathbb{C}^2) \) satisfying (33) and \( ||a_0^{(0)}||_{C^2} < 1. \) Let \( B_0 \in \mathbb{C}, \) such that:

\[ |B_0|^2 = B, \tag{45} \]
and let \( \{ C_l \}_{l \in \mathbb{N}} \subset \mathbb{C} \) be a sequence verifying (40)–(44). For \( l \in \mathbb{N} \cup \{ 0 \} \), define the Hardy function \( h_l \in H^+_{\mathbb{C}^2} \) by:

\[
h_0(\omega) = \sum_{k=0}^{\infty} \omega^k \Psi_{2^l} := B_0 \alpha^{(0)}(\omega) = B_0 \sum_{k=0}^{\infty} \omega^k a_k^{(0)},
\]

\[
h_l(\omega) = \sum_{k=0}^{\infty} \omega^k \Psi_{2^{p+1}+l+1} := C_l \alpha^{(1)}(\omega) = C_l \sum_{k=0}^{\infty} \omega^k a_k^{(1)}, \quad (2^p \leq l < 2^{p+1}, \ p \geq 0).
\]

Then, the family \( \{ h_l \}_{l \in \mathbb{N} \cup \{ 0 \}} \subset H^+_{\mathbb{C}^2} \) satisfies the complete set of conditions (30).

**Proof** (33) for \( m \neq 0 \) implies (30) for \( m \neq n \). (33) for \( m = 0 \) and \( i = j = 0 \), together with (45), coincide with (30) for \( m = n \) and \( l = l' = 0 \). (33) for \( m = 0 \), \( i = 0 \), and \( j = 1 \) leads to (30) for \( m = n \), \( l = 0 \), and \( l' \geq 1 \). (40)–(44) are just (30) for \( m = n \) and \( l \neq 0 \) or \( l = 2^p + 1 \).

For \( m = n \) and \( l = l' = 2^p + \sum_{t=0}^{p-1} \lambda_t 2^t \), \( l \neq 2^k \), \( 2^k + 1 \), so that \( l = 2^p + l_1 \) with \( l_1 \neq 0 \), 1, condition (30) is:

\[
|C_l|^2 = \langle h_l, h_l \rangle_{H^+_{\mathbb{C}^2}} = B - \sum_{k=-p}^{0} ||\Psi_{2^{p+1}+l} \sum_{t=0}^{p-1} \lambda_t 2^t ||^2_{C^r}
\]

\[
= |C_l|^2 - \sum_{k=-(p-1)}^{0} |||\Psi_{2^{p+1}+l}||^2_{C^r}
\]

\[
= |C_l|^2 \left( 1 - \sum_{t=0}^{p-1} |||a^{(1)}||^2_{C^r} \right) = |C_l|^2 \frac{|C_{2^p+l}||^2}{B \left( 1 - |||a^{(0)}||^2_{C^r} \right)},
\]

which is true, due to (39) together with (40) for \( p = 1 \).

For \( m = n \) and \( l, l' \geq 2, l \neq l', l, l' \neq 2^k \), \( 2^k + 1 \), so that \( l = 2^p + l_1, l' = 2^p' + l'_1 \) with \( l_1, l'_1 \neq 0, 1 \), and \( l_1 = 2^{h_1} + l_2, l'_1 = 2^{h'_1} + l'_2 \), condition (30) reads:

\[
C_{l_1} \overline{C_{l_1}} = \langle h_{l_1}, h_{l_1} \rangle_{H^+_{\mathbb{C}^2}}
\]

\[
= \delta_{(p-p_1)-(p'-p'_1)} \langle h_{l_1}, h_{l_1} \rangle_{H^+_{\mathbb{C}^2}}
\]

\[
- \sum_{k=\sup\{p_1-p+1, p'-1\}}^{0} \langle \Psi_{2^{p+1}+l_1}, \Psi_{2^{p'+1}+l'_1} \rangle_{C^2}
\]

\[
= C_{l_1} \overline{C_{l_1}} \left( \delta_{(p-p_1)-(p'-p'_1)} \right)
\]

\[
- \sum_{t=0}^{p-1} \langle a^{(1)}_{p'-r-p'_1-1}, a^{(1)}_{p-r-p_1-1} \rangle_{C^2},
\]
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which, by (39), coincides with (41) when \( p - p_1 = p' - p'_1 \) and coincides with (43) when \( p - p_1 \neq p' - p'_1 \).

And in a similar way for the two remaining cases: for \( m = n, l = 2^p \) or \( l = 2^p + 1 \), and \( l' \neq 2^k, 2^k + 1 \).

Proposition 18 says that everything can be written in terms of the \( M^+ \)-inner matrix \( A^+ \) or, equivalently, in terms of the pair of functions \( \mathbf{a}^{(0)} \) and \( \mathbf{a}^{(1)} \) of \( H^+_{\mathbb{C}^2} = H^+(\partial \mathbb{D}, \mathbb{C}^2) \):

Proposition 19 Let:

\[
\mathbf{a}^{(0)}(\omega) = \sum_{k=0}^{\infty} \omega^k \mathbf{a}^{(0)}_k \quad \text{and} \quad \mathbf{a}^{(1)}(\omega) = \sum_{k=0}^{\infty} \omega^k \mathbf{a}^{(1)}_k
\]

be a pair of functions in \( H^+_{\mathbb{C}^2} \) satisfying (33), \( ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2} < 1 \), and such that:

\[
\frac{|\langle \mathbf{a}^{(0)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2}|^2}{1 - ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2}^2} = 1 - \sum_{k=0}^{p} ||\mathbf{a}^{(0)}_k||_{\mathbb{C}^2}, \quad (p > 0),
\]

(46)

\[
\frac{|\langle \mathbf{a}^{(1)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2}|^2}{1 - ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2}^2} = 1 - \sum_{k=0}^{p} ||\mathbf{a}^{(1)}_k||_{\mathbb{C}^2}, \quad (p \geq 0),
\]

(47)

\[
\frac{\langle \mathbf{a}^{(1)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2} \langle \mathbf{a}^{(0)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2}}{1 - ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2}^2} = - \sum_{r=0}^{p} \langle \mathbf{a}^{(0)}_{p'-r}, \mathbf{a}^{(0)}_{p-r} \rangle_{\mathbb{C}^2}, \quad (0 < p < p'),
\]

(48)

\[
\frac{\langle \mathbf{a}^{(1)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2} \langle \mathbf{a}^{(0)}_p, \mathbf{a}^{(1)}_0 \rangle_{\mathbb{C}^2}}{1 - ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2}^2} = - \sum_{r=0}^{p} \langle \mathbf{a}^{(1)}_{p'-r}, \mathbf{a}^{(1)}_{p-r} \rangle_{\mathbb{C}^2}, \quad (0 \leq p < p'),
\]

(49)

\[
\frac{\langle \mathbf{a}^{(1)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2} \langle \mathbf{a}^{(0)}_p, \mathbf{a}^{(0)}_0 \rangle_{\mathbb{C}^2}}{1 - ||\mathbf{a}^{(0)}_0||_{\mathbb{C}^2}^2} = \begin{cases} 
- \sum_{r=0}^{p} \langle \mathbf{a}^{(1)}_{p'-r}, \mathbf{a}^{(0)}_{p-r} \rangle_{\mathbb{C}^2}, & \text{if } 0 < p \leq p', \\
- \sum_{r=0}^{p'} \langle \mathbf{a}^{(1)}_{p'-r}, \mathbf{a}^{(0)}_{p-r} \rangle_{\mathbb{C}^2}, & \text{if } 0 \leq p' < p.
\end{cases}
\]

(50)

Let \( B_0 \in \mathbb{C} \), such that:

\[
|B_0|^2 = B,
\]

(51)

and let \( \{C_l\}_{l \in \mathbb{N}} \subset \mathbb{C} \) be a sequence of scalars, where:
\[ |C_1|^2 = B(1 - ||a_0^{(0)}||^2_{C^2}) \neq 0, \quad (52) \]

\[ C_{2^k} = - \frac{B}{C_1} \langle a_k^{(0)}, a_0^{(0)} \rangle_{C^2}, \quad (k \in \mathbb{N}), \quad (53) \]

\[ C_{2^{k+1}} = - \frac{C_1B_0}{C_1} \langle a_k^{(1)}, a_0^{(0)} \rangle_{C^2}, \quad (k \in \mathbb{N}), \quad (54) \]

and the rest of \( C_i \)'s are given by (38). For \( l \in \mathbb{N} \cup \{0\} \), define the Hardy function \( \mathfrak{h}_l \in H^+_{C^2} \) by:

\[
\mathfrak{h}_0(\omega) = \sum_{k=0}^{\infty} \omega^k \Psi_{2^k} := B_0 \ a_0^{(0)}(\omega) = B_0 \sum_{k=0}^{\infty} \omega^k a_k^{(0)},
\]

\[
\mathfrak{h}_l(\omega) = \sum_{k=0}^{\infty} \omega^k \Psi_{2^{k+l+1}} := C_l \ a_l^{(1)}(\omega) = C_l \sum_{k=0}^{\infty} \omega^k a_k^{(1)}, \quad (2^p \leq l < 2^{p+1}, \ p \geq 0).
\]

Then, the family \( \{\mathfrak{h}_l\}_{l \in \mathbb{N} \cup \{0\}} \subset H^+_{C^2} \) satisfies the complete set of conditions (30).

**Proof** Conditions (46)–(50) are just conditions (40)–(44) where the \( C_i \)'s are eliminated using (35)–(37), except condition (40) for \( p = 0 \), condition (42) for \( p = 0 \) and condition (44) for \( p = 0 \). These three excluded conditions of Proposition 18 are the added relations (52)–(54) in Proposition 19 to define the sequence \( \{C_l\}_{l \in \mathbb{N}} \) (which coincide with (35)–(37)). \[ \square \]

Taking into account that \( 1 = ||a^{(0)}||_{H^+_{C^2}} = \sum_{k=0}^{\infty} ||a_k^{(0)}||^2_{C^2}, \quad (j = 0, 1), \) (see (33)), from (40) and (41), one deduces that, given \( p \geq 1 \):

\[
C_{2^p} = 0 \Leftrightarrow \ C_{2^k} = 0 \text{ for all } k \geq p \Leftrightarrow a_k^{(0)} = 0 \text{ for all } k > p, \quad (55)
\]

\[
C_{2^{p+1}} = 0 \Leftrightarrow \ C_{2^{k+1}} = 0 \text{ for all } k \geq p \Leftrightarrow a_k^{(1)} = 0 \text{ for all } k \geq p. \quad (56)
\]

Now, we are ready to obtain all the non-trivial families \( \{\mathfrak{h}_l\}_{l \in \mathbb{N} \cup \{0\}} \subset H^+_{C^2} \) that satisfy the complete set of conditions (30) or, equivalently, all the \( M^+ \)-inner \((2 \times 2)\)-matrix functions \( A^+(\omega) \) verifying the conditions (46)–(50). We shall collect the results in Propositions 20 and 21.

When \( a_0^{(0)} = 0 \), relations (53) and (54) imply that \( C_{2^k} = C_{2^{k+1}} = 0 \) for every \( k \geq 1 \), so that, by (55) and (56), \( a_k^{(0)} = 0 \) for every \( k > 1 \), \( a_k^{(1)} = 0 \) for every \( k > 0 \), and, by (38), \( C_l = 0 \) for every \( l \geq 2 \). In this case, \( \{a_0^{(0)}, a_0^{(1)}\} \) is an orthonormal basis of \( C^2 \) (by (54) for \( k = 1 \), since \( C_3 = 0 \)) and \( |B_0|^2 = |C_1|^2 = B \). This leads to the family of
type 2 in Propositions 20 and 21 below, where \(u_0 = a_1^{(0)}\) and \(u_1 = a_0^{(1)}\). It is trivial to check that this type of families of Hardy functions satisfies conditions (29) and (30) of Proposition 14.

In what follows, we assume that \(0 < ||a_0^{(0)}|| < 1\).

If \(C_2 = 0\), from (40) with \(p = 1\):
\[
\mathbf{h}_0(\omega) = \Psi_1 + \omega \Psi_2 = B_0(a_0^{(0)} + \omega a_1^{(0)}), \quad ||a_1^{(0)}||^2_{C^2} = 1 - ||a_0^{(0)}||^2_{C^2}; \quad (57)
\]
by (36) with \(k = 1\) or (42) with \(p = 0, p' = 1\):
\[
\langle a_1^{(0)}, a_0^{(0)} \rangle_{C^2} = 0; \quad (58)
\]
by (44) with \(l = p = p'\),
\[
\langle a_0^{(1)}, a_1^{(0)} \rangle_{C^2} = 0; \quad (59)
\]
and by (44) with \(l = p < p'\):
\[
\langle a_{p'}^{(1)}, a_1^{(0)} \rangle_{C^2} = -\langle a_{p'-1}^{(1)}, a_0^{(0)} \rangle_{C^2}, \quad (p' > 0). \quad (60)
\]
If \(C_3 = 0\), from (41) with \(p = 1\):
\[
\mathbf{h}_1(\omega) = \Psi_3 = C_1 a_0^{(1)}, \quad ||a_0^{(1)}||^2_{C^2} = 1; \quad (61)
\]
and by (37) with \(k = 1\) and (44) with \(l = p' \leq p\):
\[
\langle a_0^{(1)}, a_p^{(0)} \rangle_{C^2} = 0 \quad (p \geq 0). \quad (62)
\]
Thus, if \(C_2 = C_3 = 0\), by (58) and (62), the three non-null vectors \(a_0^{(0)}, a_1^{(0)}, a_0^{(1)}\) should be orthogonal to each other in \(C^2\), which is not possible.

If \(C_3 \neq 0\), by (37) with \(k = 1\):
\[
\langle a_0^{(1)}, a_0^{(0)} \rangle_{C^2} \neq 0, \quad (63)
\]
and equating the expressions for \(C_{2_l+1} \overline{C_3} \) obtained using (37) and (43) with \(l = p < p'\), i.e., by (49) with \(l = p < p'\):
\[
\frac{\langle a_{p'}^{(1)}, a_0^{(0)} \rangle_{C^2} \langle a_0^{(0)}, a_0^{(1)} \rangle_{C^2}}{1 - ||a_0^{(0)}||^2_{C^2}} = -\langle a_{p'}^{(1)}, a_0^{(1)} \rangle_{C^2}, \quad (p' > 0). \quad (64)
\]
Due to (63), \(\frac{\langle a_{p'}^{(1)}, a_0^{(0)} \rangle_{C^2} a_0^{(0)} + a_0^{(1)} \neq 0, \) and (64) is equivalent to:
\[
da_{p'}^{(1)} \perp \frac{\langle a_0^{(1)}, a_0^{(0)} \rangle_{C^2}}{1 - ||a_0^{(0)}||^2_{C^2}} a_0^{(0)} + a_0^{(1)} \neq 0, \quad (p > 0). \quad (65)
\]
Now, if \(C_2 = 0\) and \(C_3 \neq 0\), by (58), (59), and: (63),
\[a_0^{(1)} = \lambda_0 a_0^{(0)}, \quad (0 \neq \lambda_0 \in \mathbb{C}),\]

\[
\frac{\langle a_0^{(1)}, a_0^{(0)} \rangle_{C^2}}{1 - ||a_0^{(0)}||^2_{C^2}} a_0^{(0)} + a_0^{(1)} = \frac{\lambda_0}{1 - ||a_0^{(0)}||^2_{C^2}} a_0^{(0)}, \tag{66}
\]

by (58)–(60), (65), and (66):

\[a_1^{(1)} = \lambda_1 a_1^{(0)}, \quad (0 \neq \lambda_1 \in \mathbb{C}),\]

\[
\langle a_1^{(1)}, a_1^{(0)} \rangle_{C^2} = -\langle a_1^{(1)}, a_1^{(0)} \rangle_{C^2}, \tag{67}
\]

\[a_p^{(1)} = 0, \quad (p > 1),\]

and by (40), (41), (66), and (67),

\[
\lambda_1 = -\frac{||a_0^{(0)}||^2_{C^2}}{1 - ||a_0^{(0)}||^2_{C^2}} \lambda_0, \quad ||\lambda_0||^2 = \frac{1 - ||a_0^{(0)}||^2_{C^2}}{||a_0^{(0)}||^2_{C^2}}. \tag{68}
\]

Here, being \(a_p^{(0)} = a_0^{(1)} = 0\) for \(p > 1\), by (36) and (37), \(C_{2k} = 0\) for \(k > 0\) and \(C_{2k+1} = 0\) for \(k > 1\). Thus, by (38), \(C_k \neq 0\) only if \(l = 2^p - 1, (p > 0)\), and:

\[
C_{2^p-1} = C_{2^p-2} = \frac{C_{2^p-1}}{C_{2^p-2}}\left(\frac{C_{2^p-1}B_0}{C_1}\langle a_0^{(1)}, a_0^{(0)} \rangle_{C^2}\right)^{p-1}
= C_1 \left(\frac{B_0 ||a_0^{(0)}||^2_{C^2} \lambda_0}{C_1}\right)^{p-1}, \quad (p > 1).
\]

This case leads to the family of type 3 in Propositions 20 and 21 below, with \(p = ||a_0^{(0)}||_{C^2}, a_0^{(0)} = \rho u_0, a_1^{(0)} = (1 - \rho^2)^{1/2} u_1\) and \(\theta = \arg(\lambda_0)\).

If \(C_2 \neq 0\), by (36) with \(k = 1\):

\[
\langle a_1^{(0)}, a_0^{(0)} \rangle_{C^2} \neq 0, \tag{69}
\]

and equating the expressions for \(C_{2^p} C_2\) obtained using (36) and (42) with \(1 = p < p'\), i.e., by (48) with \(1 = p < p'\):

\[
\frac{\langle a_p^{(0)}, a_0^{(0)} \rangle_{C^2} \langle a_0^{(0)}, a_1^{(0)} \rangle_{C^2}}{1 - ||a_0^{(0)}||^2_{C^2}} = -\langle a_p^{(0)}, a_1^{(0)} \rangle_{C^2} - \langle a_{p-1}^{(0)}, a_0^{(0)} \rangle_{C^2}, \quad (p' > 1). \tag{70}
\]

Then, when \(C_2 \neq 0\) and \(C_3 = 0\), by (62):

\[a_p^{(0)} = \gamma_p a_0^{(0)}, \quad \text{where } \gamma_p \in \mathbb{C}, \quad (p > 0), \tag{71}
\]

and by (69), (70), and (71):
\[ \gamma_1 \neq 0, \quad \gamma_p = \left( -\frac{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}}{\gamma_1} \right)^{p-1} \gamma_1, \quad (p > 0). \tag{72} \]

In this case, relations (33) for \( i = j = 0 \) and \( m \in \mathbb{Z} \), together with (72), lead to 
\[ |\gamma_1| = \frac{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}}{||a_0^{(0)}||^2_{\mathbb{C}^2}}. \] If \( \theta = \arg(\gamma_1) \):
\[ \gamma_p = \left( -||a_0^{(0)}||^2_{\mathbb{C}^2} e^{i\theta} \right)^{p-1} \frac{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}}{||a_0^{(0)}||^2_{\mathbb{C}^2}} e^{i\theta}, \quad (p > 0). \tag{73} \]

Since \( C_3 = 0 \), by (41) and Lemma 17, \( C_i \neq 0 \) only if \( l = 2^p, \ (p \geq 0) \). From (36) and (73):
\[ C_{2^p} = -\frac{B}{C_1} \langle a^{(0)}_{p'}, a^{(0)}_0 \rangle_{\mathbb{C}^2} = \frac{B}{C_1} \left( -||a_0^{(0)}||^2_{\mathbb{C}^2} e^{i\theta} \right)^p (1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}), \quad (p \in \mathbb{N}). \]

This case corresponds with the family of type 4 in Propositions 20 and 21 below, where \( \rho = ||a_0^{(0)}||_{\mathbb{C}^2}, a_0^{(0)} = \rho u_0, a_0^{(1)} = u_1 \) and \( \theta = \arg(\gamma_1) \).

Finally, when \( C_2 \neq 0 \) and \( C_3 \neq 0 \), equating the expressions for \( C_3 C_{2^p} \) obtained using (36), (37) and (44) with \( l = p' \leq p, \ i.e., \) by (50) with \( l = p' \leq p \): 
\[ a^{(0)}_{p'} \perp \frac{\langle a^{(0)}_1, a^{(0)}_0 \rangle_{\mathbb{C}^2}}{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}} a^{(0)}_0 + a^{(1)}_0 \neq 0, \quad (p > 0), \tag{74} \]

and equating the expressions for \( C_{2^{p'+1}} C_{2^p} \) obtained using (36), (37) and (44) with \( l = p \leq p', \ i.e., \) by (50) with \( l = p \leq p' \):
\[ a^{(1)}_0 \perp \frac{\langle a^{(0)}_1, a^{(0)}_0 \rangle_{\mathbb{C}^2}}{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}} a^{(0)}_0 + a^{(1)}_0 \neq 0, \tag{75} \]

\[ \frac{\langle a^{(1)}_{p'}, a^{(0)}_0 \rangle_{\mathbb{C}^2} \langle a^{(0)}_1, a^{(1)}_0 \rangle_{\mathbb{C}^2}}{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}} = -\langle a^{(1)}_{p'}, a^{(1)}_0 \rangle_{\mathbb{C}^2} = -\langle a^{(1)}_{p'-1}, a^{(0)}_0 \rangle_{\mathbb{C}^2}, \tag{76} \]

\((p' > 0)\).

Recall that \( \frac{\langle a^{(0)}_0, a^{(0)}_0 \rangle_{\mathbb{C}^2}}{1 - ||a_0^{(0)}||^2_{\mathbb{C}^2}} a^{(0)}_0 + a^{(1)}_0 \neq 0 \) due to (69). Then, by (65) and (74), there exist three unitary vectors \( u_0, u_1, v \in \mathbb{C}^2 \) and two sequences \( \{\rho_p\}_{p \geq 0} \) and \( \{\tau_p\}_{p \geq 0} \) of complex numbers, such that:
\[ a^{(0)}_0 = \rho_0 u_0, \quad a^{(1)}_0 = \tau_0 u_1, \quad a^{(0)}_{p'} = \rho_p v, \quad a^{(1)}_{p'} = \tau_p v, \quad (p > 1). \tag{77} \]

Moreover, \( 0 < |\rho_0| = ||a_0^{(0)}||^2_{\mathbb{C}^2} < 1, \rho_1 \neq 0 \) (because \( C_2 \neq 0 \)), \( \tau_0 \neq 0 \) (since \( C_3 \neq 0 \)). Also, by (63) and (69):
\[ \langle u_1, u_0 \rangle_{C^2} \neq 0, \quad \langle v, u_0 \rangle_{C^2} \neq 0; \tag{78} \]

by (64) or (75) or (76),

\[ \frac{|\rho_0|^2}{1 - |\rho_0|^2} \langle u_0, u_1 \rangle_{C^2} \langle v, u_0 \rangle_{C^2} + \langle v, u_1 \rangle_{C^2} = 0; \tag{79} \]

and by (78) and (79):

\[ \frac{|\rho_0|^2}{1 - |\rho_0|^2} = -\frac{\langle v, u_1 \rangle_{C^2}}{\langle u_0, u_1 \rangle_{C^2} \langle v, u_0 \rangle_{C^2}}, \quad \text{so that} \quad \langle v, u_1 \rangle_{C^2} \neq 0. \tag{80} \]

Taking into account that the sequences \( \{\rho_p\}_{p \geq 1} \) and \( \{\tau_p\}_{p \geq 0} \) satisfy the respective recurrence relations (70) and (76), and that both relations coincide, we have:

\[ \tau_1 = -\frac{\bar{\rho}_0}{\bar{\rho}_1} \frac{\langle u_1, u_0 \rangle_{C^2}}{1 + \frac{|\rho_0|^2}{1 - |\rho_0|^2} \langle v, u_0 \rangle_{C^2}^2} \tau_0; \tag{81} \]

\[ \rho_p = r^{p-1} \rho_1, \quad \tau_p = r^{p-1} \tau_1, \quad (p > 1); \]

\[ r = -\frac{\bar{\rho}_0}{\bar{\rho}_1} \frac{\langle v, u_0 \rangle_{C^2}}{1 + \frac{|\rho_0|^2}{1 - |\rho_0|^2} \langle v, u_0 \rangle_{C^2}^2}. \tag{82} \]

The conditions (33) lead to:

\[ 1 = |\rho_0|^2 + |\rho_1|^2 \frac{1}{1 - |r|^2}, \quad 1 = |\tau_0|^2 + |\tau_1|^2 \frac{1}{1 - |r|^2}, \tag{83} \]

\[ 0 = \rho_0 \langle u_0, v \rangle_{C^2} + \rho_1 \frac{\bar{r}}{1 - |r|^2}, \quad 0 = \tau_0 \langle u_1, v \rangle_{C^2} + \tau_1 \frac{\bar{r}}{1 - |r|^2}, \tag{84} \]

\[ 0 = \rho_0 \bar{\tau}_0 \langle u_0, u_1 \rangle_{C^2} + \rho_1 \bar{\tau}_1 \frac{1}{1 - |r|^2}. \]

From (81)–(84), we get:

\[ |\rho_1|^2 = \frac{1 - |\rho_0|^2}{1 + \frac{|\rho_0|^2}{1 - |\rho_0|^2} \langle v, u_0 \rangle_{C^2}^2}, \quad |\tau_0|^2 = \frac{1}{1 + \frac{|\rho_0|^2}{1 - |\rho_0|^2} \langle u_1, u_0 \rangle_{C^2}^2}, \tag{85} \]

\[ \tau_1 = \frac{\rho_1 \langle u_1, v \rangle_{C^2}}{\rho_0 \langle u_0, v \rangle_{C^2}}, \]

and the interesting additional relations:
\[ |r|^2 = \frac{|\rho_0|^2}{1 - |\rho_0|^2} \frac{|\langle v, u_0 \rangle_{C^2}|^2}{1 + |\rho_0|^2} \frac{|\langle v, u_0 \rangle_{C^2}|^2}{1 - |\rho_0|^2}, \]

\[ \frac{|\rho_0|^2}{1 - |\rho_0|^2} |\langle v, u_0 \rangle_{C^2}|^2 = \frac{|r|^2}{1 - |r|^2}, \quad \frac{|\tau_1|^2}{|\rho_1|^2} = \frac{1 - |\tau_0|^2}{1 - |\rho_0|^2}. \]

From (35)–(37) and (81)–(84), for \( p > 0 \):

\[ C_{2^p} = \frac{B}{C_1} (1 - |\rho_0|^2)^{p^p} = C_1^{p^p}, \]

\[ C_{2^{p+1}} = \frac{C_1 B_0}{C_1} \frac{\tau_1}{\rho_1} (1 - |\rho_0|^2)^{p^p-1} = \frac{C_1^2}{B_0} \frac{\tau_0}{\rho_0 (u_0, v)_{C^2}} r^{p-1}. \]

In this case, the family of \( M^+ \)-inner \((2 \times 2)\)-matrix functions \( A^+(\omega) \) and Hardy functions \( \{ h_i \}_{i \in \mathbb{N} \cup \{0\}} \) of \( H^+_C \) is that of type 5 in Propositions 20 and 21.

Let us remember that the condition \( ||a^{(0)}||_{C^2} < 1 \) in Proposition 19 restricts the attention to \( M^+ \)-inner matrices \( A^+ \) with initial subspace \( C \) of dimension two, i.e., such that \( A^+(\omega) \) is unitary for a.e. \( \omega \in \partial \mathbb{D} \). On the other hand, as we have seen at the beginning of this section, when dimension of \( C \) is one, the only feasible pair \( \{ h_0, h_1 \} \) leading to a tight wavelet frame, with frame bound \( B \), must satisfy \( h_0 = \Psi_1 = B a^{(0)} \) and \( h_1 = 0 \), with \( ||\Psi_1||_{C^2} = B \) or \( ||a^{(0)}||_{C^2} = 1 \), so that one gets a trivial Haar wavelet frame. The corresponding \( M^+ \)-inner matrix \( A^+ \) is of the form:

\[ A^+(\omega) = \left( \begin{array}{cc} a^{(0)}(\omega) & a^{(1)}(\omega) \\ a^{(0)}(\omega) & a^{(1)}(\omega) \end{array} \right) = \left( \begin{array}{c} a^{(0)}(\omega) \\ 0 \end{array} \right) . \]

Including this case as “Type 1”, we have proved the following:

**Proposition 20** There are five types of families of \( M^+ \)-inner \((2 \times 2)\)-matrix functions:

\[ A^+(\omega) = \left( \begin{array}{cc} a^{(0)}(\omega) & a^{(1)}(\omega) \\ a^{(0)}(\omega) & a^{(1)}(\omega) \end{array} \right) , \]

satisfying conditions (46)–(50). They are as follows:

- **Type 1**: Given \( u_0 \in C^2 \), such that \( ||u_0||_{C^2} = 1 \),

\[ \begin{cases} a^{(0)}(\omega) = u_0, \\ a^{(1)}(\omega) = 0. \end{cases} \]

- **Type 2**: Given an orthonormal basis \( \{ u_0, u_1 \} \) of \( C^2 \):

\[ \begin{cases} a^{(0)}(\omega) = \omega u_0, \\ a^{(1)}(\omega) = u_1. \end{cases} \]

- **Type 3**: Given an orthonormal basis \( \{ u_0, u_1 \} \) of \( C^2, 0 < \rho < 1 \) and \( \theta \in \mathbb{R} \):
\[
\begin{align*}
\{ a^{(0)}(\omega) &= \rho u_0 + \omega(1 - \rho^2)^{1/2}u_1, \\
\{ a^{(1)}(\omega) &= e^{i\theta}[(1 - \rho^2)^{1/2}u_0 - \omega \rho u_1].
\end{align*}
\]

- **Type 4:** Given an orthonormal basis \(\{u_0, u_1\}\) of \(\mathbb{C}^2\), \(0 < \rho < 1\) and \(\theta \in \mathbb{R}\):

\[
\begin{align*}
a^{(0)}(\omega) &= \left( \rho + (1 - \rho^2)e^{i\theta}\sum_{k=1}^{\infty} \omega^k \left(-\rho e^{i\theta}\right)^k \right)u_0, \\
a^{(1)}(\omega) &= u_1.
\end{align*}
\]

- **Type 5:** Given \(0 < |\rho_0| < 1\), choose three unitary vectors \(u_0, u_1\) and \(v\) in \(\mathbb{C}^2\), such that (79) is satisfied.\(^7\) Then, \(|\rho_1|\) and \(|\tau_0|\) are given by (85). Once the free arguments for \(\rho_0, \rho_1\), and \(\tau_0\) have been selected, say \(\theta_0, \theta_1\), and \(\tau_0\), the value of \(r\) is determined by (82) and the value of \(\tau_1\) is given by (85). Then:

\[
\begin{align*}
a^{(0)}(\omega) &= \rho_0 u_0 + \rho_1 v \sum_{k=1}^{\infty} \omega^k r^{k-1}, \\
a^{(1)}(\omega) &= \tau_0 u_1 + \tau_1 v \sum_{k=1}^{\infty} \omega^k r^{k-1}.
\end{align*}
\]

In terms of the families \(\{\mathbf{h}_l\}_{l \in \mathbb{N} \cup \{0\}} \subset H^+_c(\mathbb{C}^2)\), the result reads as follows:

**Proposition 21** There are five types of families \(\{\mathbf{h}_l\}_{l \in \mathbb{N} \cup \{0\}} \subset H^+_c(\mathbb{C}^2)\) satisfying the complete set of conditions (30). They are as follows:

- **Type 1:** Given \(u_0 \in \mathbb{C}^2\), such that \(||u_0||_{\mathbb{C}^2} = 1\), and \(B_0 \in \mathbb{C}\), with \(|B_0|^2 = B\):

\[
\begin{align*}
\mathbf{h}_0(\omega) &= \Psi_1 = B_0 u_0, \\
\mathbf{h}_l(\omega) &= 0, \text{ for } l \geq 1.
\end{align*}
\]

- **Type 2:** Given an orthonormal basis \(\{u_0, u_1\}\) of \(\mathbb{C}^2\) and a pair of constants \(B_0, C_1 \in \mathbb{C}\), with \(|B_0|^2 = |C_1|^2 = B\):

\[
\begin{align*}
\mathbf{h}_0(\omega) &= \omega \Psi_2 = \omega B_0 u_0, \\
\mathbf{h}_1(\omega) &= \Psi_3 = C_1 u_1, \\
\mathbf{h}_l(\omega) &= 0, \text{ for } l \geq 2.
\end{align*}
\]

- **Type 3:** Given an orthonormal basis \(\{u_0, u_1\}\) of \(\mathbb{C}^2\), \(0 < \rho < 1\), constants \(B_0, C_1 \in \mathbb{C}\), such that \(|B_0|^2 = B, |C_1|^2 = B(1 - \rho^2)\), and \(\theta \in \mathbb{R}\):

\[\text{\footnotesize\(7\)}\] When the coordinates of \(u_0, u_1\) and \(v\) are real, (79) is equivalent to \(\frac{1}{1-|\rho_0|^2} = \tan(\widehat{\nu_0})\tan(\widehat{\nu_0 \nu_1})\), where \(\widehat{\nu}\) denotes the angle from \(a\) to \(b\) as vectors in \(\mathbb{R}^2\).  

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\[ \begin{align*}
\mathfrak{h}_0(\omega) &= \Psi_1 + \omega \Psi_2 = B_0(\rho u_0 + \omega (1 - \rho^2)^{1/2} u_1), \\
\mathfrak{h}_{2^{p-1}}(\omega) &= B_0(1 - \rho^2)^{1/2} C_1 e^{i\theta} [(1 - \rho^2)^{1/2} u_0 - \omega \rho u_1], \quad (p \geq 1), \\
\mathfrak{h}_j(\omega) &= 0, \quad \text{else.}
\end{align*} \]

- Type 4: Given an orthonormal basis \{u_0, u_1\} of \( \mathbb{C}^2 \), \( 0 < \rho < 1 \), constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - \rho^2) \), and \( \theta \in \mathbb{R} \):

\[ \begin{align*}
\mathfrak{h}_0(\omega) &= \sum_{k=0}^{\infty} \omega^k \Psi_{2^k} = B_0 \left( \rho + (1 - \rho^2)e^{i\theta} \sum_{k=1}^{\infty} \omega^k \left( - \rho e^{i\theta} \right)^{k-1} \right) u_0, \\
\mathfrak{h}_{2^p}(\omega) &= \Psi_{2^{p+1}+2^{p-1}} = \left( - \rho e^{i\theta} \right)^p C_1 u_1, \quad (p \geq 0), \\
\mathfrak{h}_j(\omega) &= 0, \quad \text{else.}
\end{align*} \]

- Type 5: Given \( 0 < |\rho_0| < 1 \), choose three unitary vectors \( u_0, u_1 \) and \( v \) in \( \mathbb{C}^2 \), such that (79) is satisfied. Then, \( |\rho_1| \) and \( |\tau_0| \) are given by (85). Once the free arguments for \( \rho_0, \rho_1, \\text{and} \tau_0 \) have been selected, say \( \theta_{\rho_0}, \theta_{\rho_1} \) and \( \theta_{\tau_0} \), the value of \( r \) is determined by (82) and the value of \( \tau_1 \) is given by (85). Choose constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - |\rho_0|^2) \), and select their free arguments \( \theta_{B_0} \) and \( \theta_{C_1} \). Then:

\[ \begin{align*}
\mathfrak{h}_0(\omega) &= \sum_{k=0}^{\infty} \omega^k \Psi_{2^k} = B_0 \left( \rho_0 u_0 + \rho_1 v \sum_{k=1}^{\infty} \omega^k r^{k-1} \right), \\
\mathfrak{h}_1(\omega) &= \sum_{k=0}^{\infty} \omega^k \Psi_{2^{k+1}+1} = C_1 \left( \tau_0 u_1 + \tau_1 v \sum_{k=1}^{\infty} \omega^k i^{k-1} \right), \\
\mathfrak{h}_j(\omega) &= \sum_{k=0}^{\infty} \omega^k \Psi_{2^{p+1}+l} = \frac{C_l}{C_1} \mathfrak{h}_1(\omega), \quad (2^p \leq l < 2^{p+1}, \ p \geq 0),
\end{align*} \]

where \( C_{2^p} \) and \( C_{2^{p+1}} \), \( (p > 0) \), are given by (86), and the other \( C_i \)'s are calculated using (38).

Recall that the functions of the wavelet frame \{\psi_1, \psi_2\}, with support in \([0, 1]\), expand in the Haar orthonormal basis \( \{L_i^{(n)}\} \) as:

\[ \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = \sum_{p=0}^{\infty} \sum_{q=0}^{2^p-1} \Psi_{2^p+q} L_i^{(0)} = \sum_{p=0}^{\infty} \sum_{q=0}^{2^p-1} \Psi_{2^p+q} \psi^H_{p,q} = \sum_{p=0}^{\infty} \sum_{q=0}^{2^p-1} \Psi_{2^p+q} D^p T^q \psi^H, \]

where \( \psi^H := \chi_{(0, 1/2)} - \chi_{(1/2, 1)} \) is the Haar wavelet—see Eq. (10) and Appendix B—since \( \Psi_0 = 0 \) by (29). For the sake of clarity, we include an explicit description of
the coefficients \( \{ \Psi_i \}_{i \geq 0} \) for the five families of wavelet frames \( \{ \psi_1, \psi_2 \} \) with support in \([0, 1]\):

**Corollary 22.** There are five types of families of wavelet frames \( \{ \psi_1, \psi_2 \} \) with support in \([0, 1]\). Their coefficients \( \{ \Psi_i \}_{i \geq 0} \) are as follows:

- **Type 1:** Given \( u_0 \in \mathbb{C}^2 \), such that \( ||u_0||_{\ell^2} = 1 \), and \( B_0 \in \mathbb{C} \), with \( |B_0|^2 = B \),
  \[
  \begin{aligned}
  \Psi_1 &= B_0 u_0, \\
  \Psi_l &= 0, \text{ for } l \neq 1.
  \end{aligned} \tag{88}
  \]

- **Type 2:** Given an orthonormal basis \( \{ u_0, u_1 \} \) of \( \mathbb{C}^2 \) and a pair of constants \( B_0, C_1 \in \mathbb{C} \), with \( |B_0|^2 = |C_1|^2 = B \),
  \[
  \begin{aligned}
  \Psi_1 &= B_0 u_0, \\
  \Psi_2 &= C_1 u_1, \\
  \Psi_l &= 0, \text{ for } l \neq 2, 3.
  \end{aligned} \tag{89}
  \]

- **Type 3:** Given an orthonormal basis \( \{ u_0, u_1 \} \) of \( \mathbb{C}^2 \), \( 0 < \rho < 1 \), constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - \rho^2) \), and \( \theta \in \mathbb{R} \):
  \[
  \begin{aligned}
  \Psi_1 &= B_0 \rho u_0, \\
  \Psi_2 &= B_0(1 - \rho^2)^{1/2} u_1, \\
  \Psi_{2^p+2^{p-1}} &= \left( - \frac{C_1 \rho e^{i\theta}}{B_0(1 - \rho^2)^{1/2}} \right)^{p-1} C_1 e^{i\theta} (1 - \rho^2)^{1/2} u_0, \quad (p > 0), \\
  \Psi_{2^{p+1}+2^{p-1}} &= -\left( - \frac{C_1 \rho e^{i\theta}}{B_0(1 - \rho^2)^{1/2}} \right)^{p-1} C_1 e^{i\theta} \rho u_1, \quad (p > 0), \\
  \Psi_l &= 0, \quad \text{else}.
  \end{aligned} \tag{90}
  \]

- **Type 4:** Given an orthonormal basis \( \{ u_0, u_1 \} \) of \( \mathbb{C}^2 \), \( 0 < \rho < 1 \), constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - \rho^2) \), and \( \theta \in \mathbb{R} \):
  \[
  \begin{aligned}
  \Psi_1 &= B_0 \rho u_0, \\
  \Psi_2^p &= (1 - \rho e^{i\theta})^{p-1} B_0 u_0, \quad (p > 0), \\
  \Psi_{2^{p+1}+2^{p}} &= (1 - \rho e^{i\theta})^p C_1 u_1, \quad (p \geq 0), \\
  \Psi_l &= 0, \quad \text{else}.
  \end{aligned} \tag{91}
  \]

- **Type 5:** Given \( 0 < |\rho_0| < 1 \), choose three unitary vectors \( u_0, u_1 \) and \( v \) in \( \mathbb{C}^2 \), such that \( (79) \) is satisfied. Then, \( |\rho_1| \) and \( |\tau_0| \) are given by \( (85) \). Once the free arguments for \( \rho_0 \), \( \rho_1 \) and \( \tau_0 \) have been selected, say \( \theta_{\rho_0}, \theta_{\rho_1} \) and \( \theta_{\tau_0} \), the value of \( r \) is determined by \( (82) \) and the value of \( \tau_1 \) is given by \( (85) \). Choose constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - |\rho_0|^2) \), and select their free arguments \( \theta_{B_0} \) and \( \theta_{C_1} \). Then:
by (34) and (81), we can avoid the use of the auxiliary parameter \( \tau_1 \) and sequence \( \{ C_l \}_{l=1}^{\infty} \) to describe the Type 5 family. For it, we introduce two new complex constants:

\[
\alpha := \frac{C_1 \tau_0}{B_0 \rho_0} \frac{\langle v, u_1 \rangle_{C^2}}{\langle v, u_0 \rangle_{C^2} \langle u_0, u_1 \rangle_{C^2}}, \quad \beta := \frac{C_1 \tau_0 \rho_1}{B_0 \rho_0^2} \frac{|\langle u_1, v \rangle_{C^2}|^2}{\langle u_0, v \rangle_{C^2}^2 \langle u_0, u_1 \rangle_{C^2}},
\]

so that \( C_l \tau_l = \alpha \langle \Psi_l, u_0 \rangle_{C^2} \) and \( C_l \tau_l = \beta \langle \Psi_l, u_0 \rangle_{C^2} \) for \( l \geq 2 \).

- Type 5 (Second version). Given \( 0 < |\rho_0| < 1 \), choose three unitary vectors \( u_0, u_1 \), and \( v \) in \( C^2 \), such that (79) is satisfied. Then, \( |\rho_1| \) and \( |\tau_0| \) are given by (85). Once the free arguments for \( \rho_0, \rho_1, \) and \( \tau_0 \) have been selected, say \( \theta_{\rho_0}, \theta_{\rho_1}, \) and \( \theta_{\tau_0} \), the value of \( r \) is determined by (82). Choose constants \( B_0, C_1 \in \mathbb{C} \), such that \( |B_0|^2 = B, |C_1|^2 = B(1 - |\rho_0|^2) \), and select their free arguments \( \theta_{B_0} \) and \( \theta_{C_1} \). Then:

\[
\begin{align*}
\Psi_0 &= 0, \\
\Psi_1 &= B_0 \rho_0 \ u_0, \\
\Psi_{2p} &= r^{p-1} B_0 \rho_1 \ v, \quad (p > 0), \\
\Psi_{2p+1} &= C_1 \tau_0 \ u_1, \\
\Psi_{2p+1} &= r^{p-1} B_0 \rho_0 \ |r| B_0 \rho_1 \ v, \quad (p > 0), \\
\Psi_{2p+1} &= \alpha \langle \Psi_l, u_0 \rangle_{C^2} \ u_1, \quad (p > 0, \ 2^p \leq l < 2^{p+1}), \\
\Psi_{2p+1} &= r^{p-1} \beta \langle \Psi_l, u_0 \rangle_{C^2} \ u_1, \quad (k > 0, \ p \geq 0, \ 2^p \leq l < 2^{p+1}).
\end{align*}
\]

For Type 1 functions \( A^+(\omega) \), both functions \( \psi_1 \) and \( \psi_2 \) are proportional to the Haar wavelet. Type 3 functions \( A^+(\omega) \) lead to a reflected version of functions \( \psi_1 \) and \( \psi_2 \) obtained from Type 4 functions \( A^+(\omega) \) with same parameters. For Types 2–5 functions \( A^+(\omega) \), some examples of real functions \( \psi_1 \) and \( \psi_2 \) are shown in Appendix A.

**Remark 23** The expansion (87) for wavelet frames \( \{ \psi_1, \psi_2 \} \) and the explicit description of coefficients \( \{ \Psi_l \}_{l \geq 0} \) in Corollary 22 permit us to study properties of \( \psi_1 \) and \( \psi_2 \) such as vanishing moments, symmetry, etc. Due to the very restrictive condition of minimal support, we cannot expect good properties. For example, being \( \Psi_0 = 0, \psi_1 \) and \( \psi_2 \) have null 0-moments in any case. To calculate 1-moments, we only need to know the expansion (87) for \( f(x) = x, \ x \in [0, 1] \). It is:
Easy calculations show that there are no wavelet frames of Types 2–4, such that both functions \( \psi_1, \psi_2 \) have null 1-moments (because \( u_0 \) and \( u_1 \) must be orthogonal). Type 5 wavelet frames require to handle the recursive description of coefficients \( \{ \Psi^i \}_{i \geq 0} \) in (92) or (93). We do not enter into details, because these properties are not relevant here. The objective of this work is to investigate connections between wavelet frames theory and Halmos–Helson theory of invariant and wandering subspaces [30], Hardy classes, and operator theory [19, 41]. Without using extension principles, Fourier transform, and multiresolution analysis, spectral techniques characterize here tight wavelet frames of minimal support. Increasing the number of generators leads to inner matrix functions of bigger dimension and additional families of interconnected parameters. Increasing support requires several inner functions and the analysis of their interrelations.

**Remark 24** As commented in the Introduction, nonhomogeneous wavelet systems of the form (7) are closely related to filter banks and multiresolution analysis. Thus, it is of interest to tackle the problem of linking the homogeneous tight wavelet frames \( \Psi = \{ \psi_1, \psi_2 \} \) with minimal support studied here to nonhomogeneous ones.

It is well known that every nonhomogeneous tight wavelet frame leads to a homogeneous tight wavelet frame [22, Prop.5]. Conversely, the relations linking homogeneous wavelet systems to nonhomogeneous wavelets systems have been studied in [27]. The key issue to link a homogeneous tight wavelet frame \( X_\Psi \) to a nonhomogeneous tight wavelet frame \( X_{\Phi,\Psi} \) is whether

\[
\text{len}(S(H_\Psi)) < \infty,
\]

where \( H_\Psi := \{ D^{\pm j} \psi : \psi \in \Psi, j \in \mathbb{N} \} \), \( S(H_\Psi) \) is the integer translation (shift) invariant subspace generated by \( H_\Psi \), and \( \text{len}(S(H_\Psi)) \) is the minimal number of generators; see [27, Th.3.6].

For Type 1 wavelet frames \( \{ \psi_1, \psi_2 \} \), (88) says that both functions \( \psi_1 \) and \( \psi_2 \) are proportional to Haar wavelet \( \psi^H \), and it is clear that \( S(H_\Psi) \) is generated by the Haar scale function \( \Psi^H = \chi_{(0,1)} \)—see Appendix B—and \( \text{len}(S(H_\Psi)) = 1 \).

For Type 2 wavelet frames \( \{ \psi_1, \psi_2 \} \), according to (89), expansion (87) is given by:

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Psi_2 D \psi^H + \Psi_3 D T \psi^H.
\]

Thus, using relations (15), we get:

\[ f = \sum_{p=0}^{\infty} \sum_{q=0}^{2^p-1} \frac{-1}{2^{2(p+1)}} D^p T^q \psi^H. \]
\[ D^{-j} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_2 D^{1-j} \psi^H + \psi_3 T^{2j-1} D^{1-j} \psi^H, \quad (j > 0). \]

In this case, a minimal set of generators for \( S(H_f) \) is given, for example, by \( \{ D\varphi^H, DT\varphi^H \} \), so that \( \text{len}(S(H_f)) = 2 \).

For Type 4 wavelet frames \( \{ \psi_1, \psi_2 \} \), according to (91), expansion (87) reads:

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1 \psi^H + \sum_{p=1}^{\infty} \psi_2 D^p \psi^H + \sum_{p=0}^{\infty} \psi_2^{p+1+2p} D^{p+1} T^{2p} \psi^H.
\]

Let us consider the following two pairs of functions:

\[
\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} := \sum_{p=1}^{\infty} \psi_2 D^p \psi^H, \quad (94)
\]

\[
\begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix} := \sum_{p=0}^{\infty} \psi_2^{p+1+2p} D^{p+1} T^{2p} \psi^H.
\]

Relations (15) and the explicit expression of \( \{ \Psi_j \}_{j \geq 0} \) in (91) lead in this case to:

\[
D^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1 D^{-1} \psi^H + \psi_2 \psi^H + \left( -\rho e^{i\theta} \right) \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} +
\]

\[
+ \psi_3 T \psi^H + \left( -\rho e^{i\theta} \right) T^{1/2} \begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix}.
\]

Iteration of this formula for \( D^{-j} \), \( j \geq 2 \), implies that a set of generators for \( S(H_f) \) is given by \( \{ D\varphi^H, DT\varphi^H, \gamma_1, \gamma_2, \gamma_3, T^{1/2} \gamma_3, \gamma_4, T^{1/2} \gamma_4 \} \). Taking a glance at (91), we see that \( \gamma_1, \gamma_2, T^{-1/2} \gamma_3 \) and \( T^{-1/2} \gamma_4 \) are proportional. Thus, a minimal set of generators for \( S(H_f) \) is given, for example, by:

\[
\{ D\varphi^H, DT\varphi^H, \gamma_1, T^{1/2} \gamma_1 \}
\]

(if \( \gamma_1 = 0 \), we take any other nonzero \( \gamma_j, j = 2, 3, 4 \), so that \( \text{len}(S(H_f)) = 4 \).

For Type 3 wavelet frames \( \{ \psi_1, \psi_2 \} \), like arguments for Type 4, here with coefficients \( \{ \Psi_j \} \) given by (90) and functions:

\[
\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} := \sum_{p=1}^{\infty} \psi_2^{p+2p-1} D^p T^{2p-1} \psi^H, \quad (95)
\]

\[
\begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix} := \sum_{p=1}^{\infty} \psi_2^{p+1+2p-1} D^{p+1} T^{2p-1} \psi^H.
\]

lead to similar results: if \( \gamma_j \neq 0 \), then \( \{ D\varphi^H, DT\varphi^H, \gamma_j, T^{1/2} \gamma_j \} \) is a minimal set of generators for \( S(H_f) \) and \( \text{len}(S(H_f)) = 4 \).

For Type 5 wavelet frames \( \{ \psi_1, \psi_2 \} \), according to (92) or (93), expansion (87) is of the form:
\[
\begin{align*}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} &= \psi_1^H + \sum_{p=1}^{\infty} \sum_{l=2^p}^{2^{p+1}-1} \psi_{2l} D^p \psi^H + \sum_{p=0}^{\infty} \sum_{l=2^p}^{2^{p+1}-1} \psi_{2l+1} D^{p+1} T^l \psi^H \\
&\quad + \sum_{k=1}^{\infty} \sum_{p=0}^{k+1} \sum_{l=2^p}^{2^{p+1}-1} \psi_{2k+p+1} D^{k+p+1} T^l \psi^H.
\end{align*}
\]

In this case, let us consider the four pairs of functions:
\[
\begin{align*}
\begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} &= \sum_{p=1}^{\infty} \psi_{2p} D^p \psi^H, \\
\begin{pmatrix}
\gamma_3 \\
\gamma_4
\end{pmatrix} &= \sum_{p=0}^{\infty} \sum_{l=2^p}^{2^{p+1}-1} \psi_{2p+1} D^{p+1} T^l \psi^H, \\
\begin{pmatrix}
\gamma_5 \\
\gamma_6
\end{pmatrix} &= \sum_{k=1}^{\infty} \sum_{p=0}^{k} \sum_{l=2^p}^{2^{p+1}-1} \psi_{2k+p+1} D^{k+p+1} T^l \psi^H, \\
\begin{pmatrix}
\gamma_7 \\
\gamma_8
\end{pmatrix} &= \sum_{p=1}^{\infty} \sum_{q=0}^{p-1} \sum_{l'=2^q}^{2^{q+1}-1} \psi_{2p+1} D^{p+1} T^{l'} \psi^H.
\end{align*}
\] (96)

In the sequel, we shall use the following notation: given two constant vectors
\(v = (v_1, v_2)\) and \(w = (w_1, w_2)\) of \(\mathbb{C}^2\) and a pair of functions \(\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{C}\):
\[
\frac{v}{w} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} := \begin{pmatrix}
\frac{v_1}{w_1} \gamma_1 \\
\frac{v_2}{w_2} \gamma_2
\end{pmatrix},
\]
where \(\frac{v}{w} := 0\) if \(w_j = 0\), \(j = 1, 2\). From the recursive description of the coefficients \(\{\psi_i\}_{i \geq 0}\) given in (93), we obtain the following identities:
\[
\begin{align*}
D^{-1} \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} &= \psi_2 \psi^H + r \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}, \\
D^{-1} \begin{pmatrix}
\gamma_3 \\
\gamma_4
\end{pmatrix} &= T \psi_3 \psi^H + \alpha \langle v, u_0 \rangle_{\mathbb{C}^2} \frac{u_1}{v} T \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix} \\
&\quad + \alpha \langle u_1, u_0 \rangle_{\mathbb{C}^2} T \begin{pmatrix}
\gamma_3 \\
\gamma_4
\end{pmatrix} + \alpha \langle v, u_0 \rangle_{\mathbb{C}^2} \frac{u_1}{v} T \begin{pmatrix}
\gamma_7 \\
\gamma_8
\end{pmatrix}, \\
D^{-1} \begin{pmatrix}
\gamma_5 \\
\gamma_6
\end{pmatrix} &= \beta \frac{v}{u_1} \begin{pmatrix}
\gamma_3 \\
\gamma_4
\end{pmatrix} + r \begin{pmatrix}
\gamma_5 \\
\gamma_6
\end{pmatrix}, \\
D^{-1} \begin{pmatrix}
\gamma_7 \\
\gamma_8
\end{pmatrix} &= \beta \frac{v}{u_1} \begin{pmatrix}
\gamma_3 \\
\gamma_4
\end{pmatrix} + r \begin{pmatrix}
\gamma_7 \\
\gamma_8
\end{pmatrix}.
\end{align*}
\]

Since:
we deduce that a set of generators for \( S(H_\Psi) \) is given by \( \{ D\varphi^H, DT\varphi^H \} \cup \{ \gamma_j : 1 \leq j \leq 8 \} \). Depending on the value of parameters \( B, B_0, C_1, \rho_0, \rho_1, \tau_0, u_0, u_1, \) and \( v \), this set of generators may be linearly dependent (all the functions in the set have support in \([0, 1]\)), so that translations do not matter). Thus, \( \text{len}(S(H_\Psi)) \leq 10 \).

We have proved the following result:

**Proposition 25** The five types of families of wavelet frames \( \Psi = \{ \psi_1, \psi_2 \} \) with support in \([0, 1]\) satisfy \( \text{len}(S(H_\Psi)) < \infty \). To be precise:

- For Type 1 wavelet frames, \( \text{len}(S(H_\Psi)) = 1 \) and a minimal set of generators of \( S(H_\Psi) \) is given by \( \{ \varphi^H \} \), where \( \varphi^H = \chi_{(0,1)} \) is the Haar scale function.
- For Type 2 wavelet frames, \( \text{len}(S(H_\Psi)) = 2 \) and a minimal set of generators of \( S(H_\Psi) \) is given by \( \{ D\varphi^H, DT\varphi^H \} \).
- For Type 3 wavelet frames, \( \text{len}(S(H_\Psi)) = 4 \) and a minimal set of generators of \( S(H_\Psi) \) is given by \( \{ D\varphi^H, DT\varphi^H, \gamma_j, T^{1/2}\gamma_j \} \), where \( \gamma_j \) is any of the nonzero functions defined in (95).
- For Type 4 wavelet frames, \( \text{len}(S(H_\Psi)) = 4 \) and a minimal set of generators of \( S(H_\Psi) \) is given by \( \{ D\varphi^H, DT\varphi^H, \gamma_j, T^{1/2}\gamma_j \} \), where \( \gamma_j \) is any of the nonzero functions defined in (94).
- For Type 5 wavelet frames, \( \text{len}(S(H_\Psi)) \leq 10 \) and a set of generators of \( S(H_\Psi) \) is given by \( \{ D\varphi^H, DT\varphi^H \} \cup \{ \gamma_j : 1 \leq j \leq 8 \} \), the functions \( \gamma_j \) defined in (96).

Proposition 25 above together with Theorems 3.6 and 3.7 in [27] (see also [26, Theorem 4.5.4]) show that each homogeneous tight wavelet frame \( \Psi = \{ \psi_1, \psi_2 \} \) with minimal support is intrinsically linked to a nonhomogeneous tight wavelet frame and a refinable structure. Therefore, homogeneous tight wavelet frames \( \Psi = \{ \psi_1, \psi_2 \} \) with minimal support can be constructed from filter banks through the refinable structure.

Finally, we analyze connections between Types of wavelet frames \( \Psi = \{ \psi_1, \psi_2 \} \) of minimal support by means of the associated \( M^+ \)-inner matrix functions. For it, recall that the \( M^+ \)-inner matrix function \( A^+(\omega) \) and the \( M^+ \)-wandering subspace generated by \( \{ b_0, b_1 \} \) in \( H^+_\mathbb{C} \) are connected by Lemma 12 (Halmos). There, the subspace generated by \( \{ b_0, b_1 \} \) uniquely determines \( A^+(\omega) \) to within a constant partially isometric factor on the right. In particular, in the non-trivial Types 2–5 of Propositions 20 and 21, to within a constant unitary factor on the right.

**Proposition 26** Let \( A^+(\omega) = \begin{pmatrix} a^{(0)}(\omega) & a^{(1)}(\omega) \\ \end{pmatrix} \) be an \( M^+ \)-inner \((2 \times 2)\)-matrix function, unitary for a.e. \( \omega \in \partial \mathbb{D} \), with \( a^{(0)}, a^{(1)} \in H^+_\mathbb{C} \) satisfying conditions (46)–(50). Given an arbitrary constant unitary \((2 \times 2)\)-matrix \( U \), consider the \((2 \times 2)\)-matrix function \( B^+(\omega) \) defined by:
\[ B^+(\omega) = \left( \begin{array}{cc} b^{(0)}(\omega) & b^{(1)}(\omega) \end{array} \right) := A^+(\omega) \cdot U. \]

Then, \( B^+(\omega) \) is also an \( M^+ \)-inner \((2 \times 2)\)-matrix function, unitary for a.e. \( \omega \in \partial \mathbb{D} \), and such that \( b^{(0)}(\omega), b^{(1)}(\omega) \) verify conditions (46)–(50).

**Proof** Conditions (46)–(50) for \( a^{(0)}, a^{(1)} \) are given in terms of their Fourier-Taylor coefficients:

\[ a^{(0)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(0)}, \quad a^{(1)}(\omega) = \sum_{k=0}^{\infty} \omega^k a_k^{(1)}. \]

On the other hand, the constant matrix \( U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \) is unitary if and only if:

\[ |u_{12}|^2 = |u_{21}|^2 = 1 - |u_{11}|^2 = 1 - |u_{22}|^2 \]
\[ \theta_{11} - \theta_{12} = \theta_{21} - \theta_{22} + \pi \mod(2\pi) \] (97)

with \( u_{jk} = |u_{jk}| e^{i\theta_{jk}}, j, k \in \{1, 2\} \). Using (97), direct calculations show that \( b^{(0)}(\omega), b^{(1)}(\omega) \) verify conditions (46)–(50) if and only if:

\[ \left( 1 - \|a_0^{(0)}\|_{\mathbb{C}^2}^2 \right) |u_{11} - u_{21}|^2 \left( \frac{\langle a_0^{(1)}, a_0^{(0)} \rangle}{1 - \|a_0^{(0)}\|_{\mathbb{C}^2}^2} \right)^2 = 1 - \|a_0^{(0)} u_{11} + a_0^{(1)} u_{21}\|_{\mathbb{C}^2}^2. \] (98)

Finally, it is easy to see that, given \( A^+(\omega) = \begin{pmatrix} a^{(0)}(\omega) & a^{(1)}(\omega) \end{pmatrix} \) in any of the Types 2–5 of Proposition 20 and an arbitrary constant unitary matrix \( U \), condition (98) is always satisfied. \( \square \)

In other words, Proposition 26 asserts that, for every constant unitary \((2 \times 2)\)-matrix \( U \), given an \( M^+ \)-inner \((2 \times 2)\)-matrix function \( A^+(\omega) \) in the non-trivial Types 2–5 of Proposition 20, \( B^+(\omega) = A^+(\omega) \cdot U \) is also an \( M^+ \)-inner \((2 \times 2)\)-matrix function in the non-trivial Types 2–5 of Proposition 20. These transformations connect \( M^+ \)-inner matrix functions in Types 2 and 3 on the one hand (those with a finite number of non-zero Fourier-Taylor coefficients) and \( M^+ \)-inner matrix functions in Types 4 and 5 on the other hand (those with an infinite number of non-zero Fourier–Taylor coefficients). To be precise:

(i) Starting from a Type 2 matrix function \( A^+(\omega) \), where:

\[
\begin{align*}
\begin{cases}
a^{(0)}(\omega) &= \omega u_0, \\
a^{(1)}(\omega) &= u_1,
\end{cases}
\end{align*}
\]

for \( B^+(\omega) = A^+(\omega) \cdot U \), one has:
1. If $U$ is a diagonal unitary matrix, i.e., $u_{12} = u_{21} = 0$ and $|u_{11}| = |u_{22}| = 1$, then $B^+(\omega)$ is also a Type 2 matrix function:

$$
\begin{align*}
\mathbf{b}^{(0)}(\omega) &= \omega u'_0, \\
\mathbf{b}^{(1)}(\omega) &= u'_1,
\end{align*}
$$

where $u'_0 = e^{i\theta_{11}} u_0$, $u'_1 = e^{i\theta_{22}} u_1$.

2. If $U$ is not diagonal, that is, $|u_{12}| = |u_{21}| > 0$, then $B^+(\omega)$ is a Type 3 matrix function:

$$
\begin{align*}
\mathbf{b}^{(0)}(\omega) &= \rho u'_0 + \omega(1 - \rho^2)^{1/2} u'_1, \\
\mathbf{b}^{(1)}(\omega) &= e^{i\theta}(1 - \rho^2)^{1/2} u'_0 - \omega e^{i\theta} \rho u'_1,
\end{align*}
$$

where $u'_0 = e^{i\theta_{21}} u_1$, $u'_1 = e^{i\theta_{11}} u_0$, $\rho = |u_{12}|$ and $\theta = \theta_{22} - \theta_{21} = \theta_{12} - \theta_{11} + \pi$.

(ii) Starting from a Type 4 matrix function $A^+(\omega)$, where:

$$
\begin{align*}
\mathbf{a}^{(0)}(\omega) &= \rho u_0 + \sum_{k=1}^{\infty} \omega^k e^{i\theta} (1 - \rho^2)^{1/2} (-\rho e^{i\theta})^{k-1} u_0, \\
\mathbf{a}^{(1)}(\omega) &= u_1,
\end{align*}
$$

for $B^+(\omega) = A^+(\omega) \cdot U$, one has:

1. If $U$ is a diagonal unitary matrix, then $B^+(\omega)$ is also a Type 4 matrix function:

$$
\begin{align*}
\mathbf{b}^{(0)}(\omega) &= \rho u'_0 + \sum_{k=1}^{\infty} \omega^k e^{i\theta} (1 - \rho^2)^{1/2} (-\rho e^{i\theta})^{k-1} u'_0, \\
\mathbf{b}^{(1)}(\omega) &= u'_1,
\end{align*}
$$

where $u'_0 = e^{i\theta_{11}} u_0$, $u'_1 = e^{i\theta_{22}} u_1$.

2. If $U$ is not diagonal, then $B^+(\omega)$ is a Type 5 matrix function:

$$
\begin{align*}
\mathbf{b}^{(0)}(\omega) &= \rho_0 u'_0 + \rho_1 v' \sum_{k=1}^{\infty} r^{k-1} \omega^k, \\
\mathbf{b}^{(1)}(\omega) &= \tau_0 u'_1 + \tau_1 v' \sum_{k=1}^{\infty} r^{k-1} \omega^k,
\end{align*}
$$

where:
These relationships exhaust the four family Types 2–5 of Proposition 20.

Some examples

Examples of real wavelet frames \{ψ₁, ψ₂\} of Type 2:

1. Parameters: \(u_0 = (1, 0), \ u_1 = (0, 1), \ B_0 = 1\) and \(\text{arg}(C_1) = 0\).

2. Parameters: \(u_0 = (1, 1)/\sqrt{2}, \ u_1 = (1, -1)/\sqrt{2}, \ B_0 = 1\) and \(\text{arg}(C_1) = 0\).
Examples of real wavelet frames \( \{ \psi_1, \psi_2 \} \) of Type 3:

1. Parameters: \( u_0 = (1, 0), \; u_1 = (0, 1), \; \rho = 1/2, \; B_0 = 1, \; \arg(C_1) = 0 \) and \( \theta = 0 \).

2. Parameters: \( u_0 = (1, 1)/\sqrt{2}, \; u_1 = (1, -1)/\sqrt{2}, \; \rho = 3/4, \; B_0 = 1, \; \arg(C_1) = 0 \) and \( \theta = \pi \).
Examples of real wavelet frames \( \{\psi_1, \psi_2\} \) of Type 4:

1. Parameters: \( u_0 = (1, 0) \), \( u_1 = (0, 1) \), \( \rho = 1/2 \), \( B_0 = 1 \), \( \arg(C_1) = 0 \) and \( \theta = 0 \).

2. Parameters: \( u_0 = (1, 1)/\sqrt{2} \), \( u_1 = (1, -1)/\sqrt{2} \), \( \rho = 3/4 \), \( B_0 = 1 \), \( \arg(C_1) = 0 \) and \( \theta = \pi \).
Examples of real wavelet frames \( \{\psi_1, \psi_2\} \) of Type 5:

1. Parameters: \( \rho_0 = 0.1, \ u_0 = (1, 0), \ v = (1, 1)/\sqrt{2}, \ arg(\rho_1) = arg(\tau_0) = 0, \ B_0 = 1 \) and \( arg(C_1) = 0 \).

2. Parameters: \( \rho_0 = 1/2, \ u_0 = (1, 0), \ v = (1, 1)/\sqrt{2}, \ arg(\rho_1) = arg(\tau_0) = 0, \ B_0 = 1 \) and \( arg(C_1) = 0 \).
3. Parameters: \( \rho_0 = 0.9, u_0 = (1, 0), v = (1, 1)/\sqrt{2}, \arg(\rho_1) = \arg(\tau_0) = 0, B_0 = 1 \) and \( \arg(C_1) = 0 \).

4. Parameters: \( \rho_0 = -1/2, u_0 = (1, 0), v = (1, 1)/\sqrt{2}, \arg(\rho_1) = \arg(\tau_0) = 0, B_0 = 1 \) and \( \arg(C_1) = 0 \).
5. Parameters: $\rho_0 = -0.6$, $u_0 = (1, 1)/\sqrt{2}$, $v = (1, 2)/\sqrt{5}$, $\arg(\rho_1) = \arg(r_0) = 0$, $B_0 = 1$ and $\arg(C_1) = 0$.

6. Parameters: $\rho_0 = 1/2$, $u_0 = (1, 1)/\sqrt{2}$, $v = (1, 2)/\sqrt{5}$, $\arg(\rho_1) = \arg(r_0) = 0$, $B_0 = 1$ and $\arg(C_1) = 0$.

(Graphics created with Matlab.)

**Haar bases**

Let $\varphi^H$ and $\psi^H$ be the *Haar scaling function and wavelet* given by:

$$
\varphi^H := \chi_{[0,1)}, \quad \psi^H := \chi_{(0,1/2)} - \chi_{(1/2,1)},
$$

where $\chi$ denotes the characteristic function:
and consider “Haar orthonormal bases” \( \{L_i(\cdot)(x)\}_{i \in \mathbb{N}_0} \) of \( L^2[0, 1) \) and \( \{K_{±,d}(\cdot)(x)\}_{d \in \mathbb{N}_0} \) of \( L^2[±1, ±2) \) defined by:

\[
\begin{align*}
L_0^{(0)} & := \phi^H_{0,0}, \\
L_{2^p + q}^{(0)} & := \psi^H_{p,q}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1; \\
K_{+,0}^{(0)} & := \phi^H_{m,1}, \\
K_{+,2^p + q}^{(0)} & := \psi^H_{p+m,2^p + q}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1; \\
K_{-,0}^{(0)} & := \phi^H_{m,-2}, \\
K_{-,2^p + q}^{(0)} & := \psi^H_{p+m,-2^p + q}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1.
\end{align*}
\]

In accordance with (8) and (9), for each \( n, m \in \mathbb{Z} \):

\[
\begin{align*}
L_0^{(n)} & := \phi^H_{0,n}, \\
L_{2^p + q}^{(n)} & := \psi^H_{p,q+2^p n}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1; \\
K_{+,0}^{(m)} & := \phi^H_{m,1}, \\
K_{+,2^p + q}^{(m)} & := \psi^H_{p+m,2^p + q}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1; \\
K_{-,0}^{(m)} & := \phi^H_{m,-2}, \\
K_{-,2^p + q}^{(m)} & := \psi^H_{p+m,-2^p + q}, \quad \text{for } p = 0, 1, \ldots; q = 0, 1, \ldots, 2^p - 1.
\end{align*}
\]

For these bases, the change of representation matrix \( (\alpha_{s,i,m}^{j,n}) \), defined by (12), is as follows:

For \( n = 0, i = 0 \):

\[
\begin{align*}
\alpha_{s,i,m}^{0,0} = \begin{cases} 2^{−m/2}, & \text{if } s = +, \ j = 0, \ m > 0, \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

For \( n = 0, r = 0, 1, 2, \ldots \):

\[
\begin{align*}
\alpha_{s,i,m}^{0,r} = \begin{cases} 2^{−1/2}, & \text{if } s = +, \ j = 0, \ m = r + 1, \\
2^{(r−m)/2}, & \text{if } s = +, \ j = 0, \ m > r + 1, \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

For \( n = 0, r = 0, 1, 2, \ldots \) and \( t = 2^p + q \) (with \( 0 \leq p < r \) and \( q = 0, 1, \ldots, 2^p − 1 \)):

\[
\begin{align*}
\alpha_{s,i,m}^{0,t} = \begin{cases} 1, & \text{if } s = +, \ j = t, \ m = r − p, \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

For \( n = 1 \) and every \( i \geq 0 \):
\[
\alpha^{s,j,m}_{t,1} = \begin{cases} 
1, & \text{if } s = +, \ j = i, \ m = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n > 1 \) and \( i > 0 \), with \( n = 2^u + v \) (\( u = 1, 2, \ldots, v = 0, 1, \ldots, 2^u - 1 \)) and \( i = 2^r + t \) (\( r = 0, 1, \ldots, t = 0, 1, \ldots, 2^r - 1 \)):

\[
\alpha^{s,j,m}_{2^u+i,2^u+v} = \begin{cases} 
1, & \text{if } s = +, \ j = 2^r(2^u + v) + t, \ m = -u, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n > 1 \) and \( i = 0 \), with \( n = 2^u + v \) (\( u = 1, 2, \ldots, v = 0, 1, \ldots, 2^u - 1 \)):

\[
\alpha^{s,j,m}_{0,2^u+v} = \begin{cases} 
2^{-u/2}, & \text{if } s = +, \ j = 0, \ m = -u, \\
(-1)^{w(u,v,p)} 2^{(p-u)/2}, & \text{if } s = +, \ j = 2^p + \lfloor v/2^{u-p} \rfloor \\
0, & \text{otherwise,}
\end{cases}
\]

where \([ \cdot ]\) denotes “entire part of” and:

\[
w(u, v, p) = \left\lfloor \frac{v - 2^{u-p}[v/2^{u-p}]}{2^{u-p-1}} \right\rfloor. \tag{101}
\]

For \( n = -1 \), \( i = 0 \):

\[
\alpha^{s,j,m}_{0,-1} = \begin{cases} 
2^{-m/2}, & \text{if } s = -, \ j = 0, \ m > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n = -1 \), \( r = 0, 1, 2, \ldots \):

\[
\alpha^{s,j,m}_{2^r+1-1,-1} = \begin{cases} 
2^{-1/2}, & \text{if } s = -, \ j = 0, \ m = r + 1, \\
-2^{(r-m)/2}, & \text{if } s = -, \ j = 0, \ m > r + 1, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n = -1 \), \( r = 1, 2, \ldots, 0 \leq p < r \), and \( q = 0, 1, \ldots, 2^p - 1 \):

\[
\alpha^{s,j,m}_{2^r+1-2^p+1+q,-1} = \begin{cases} 
1, & \text{if } s = -, \ j = 2^p + q, \ m = r - p, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n = -2 \) and every \( i \geq 0 \):

\[
\alpha^{s,j,m}_{i,-2} = \begin{cases} 
1, & \text{if } s = -, \ j = i, \ m = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

For \( n < -2 \) and \( i > 0 \), with \( n = -2^u + v \) (\( u = 1, 2, \ldots, v = 0, 1, \ldots, 2^u - 1 \)) and \( i = 2^r + t \) (\( r = 0, 1, \ldots, t = 0, 1, \ldots, 2^r - 1 \)):

\[\text{Taking into account the binary expression } v = \sum_{k=0}^{u-1} t_k 2^k, \text{ with } t_k = 0 \text{ or } 1, \text{ one has } w(u, v, p) = t_{u-p-1} \text{ and } \left\lfloor v/2^{u-p} \right\rfloor = \sum_{k=u-p}^{u-1} t_k 2^{u-(u-p)}\]
For $n < -2$ and $i = 0$, with $n = -2^u + v (u = 1, 2, \ldots, v = 0, 1, \ldots, 2^u - 1)$:

$$\alpha_{s,j,m}^{x,0,-2^u+1+v} = \begin{cases} 
2^{-i/2}, & \text{if } s = -j = 0, \ m = -u, \\
(-1)^w(u,v,p) \ 2^{(p-i)/2}, & \text{if } s = -j = 2^p + \lfloor \sqrt{2^u - p} \rfloor \\
0, & \text{otherwise},
\end{cases}$$

where $w(u, v, p)$ is given by (101).

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