Modelling of oscillations in slow/fast models of population dynamics

E Shchepakina

1Samara National Research University, Moskovskoe shosse 34, Samara, Russia, 443086
e-mail: shchepakina@yahoo.com

Abstract. This paper deals with the modelling of oscillations in a class of population dynamics models with slow and fast variables. The approach discussed is based on the geometric method of invariant manifolds, and the techniques of canards and black swans.

1. Introduction
The goal of the paper is to describe the technique for modelling various oscillations for a class of singularly perturbed systems. The approach is based on the geometric method of invariant manifolds, and the techniques of canards and black swans. The main feature of the systems under consideration is the presence of an exact slow invariant manifold of variable stability (or black swan). It should be noted that this feature is a characteristic of many models of population dynamics.

Among the various oscillatory processes, we distinguish canard-type oscillations. The distinctive features of these oscillations are predominantly slow dynamics and/or small amplitudes compared with the amplitudes of relaxation oscillations.

The term “canard” had been originally given by French mathematicians to the intermediate periodic trajectories of the van der Pol equation between the small and the large orbits due to their special shapes [1, 2]. Later the canards were investigated for other types of the singularly perturbed systems including the ones of higher dimensions, i.e. when the phase variables are vectors (see, for example, [3–13] and the references therein).

According to the geometrical theory of singular perturbations, a canard may be considered as a result of gluing stable and unstable slow invariant manifolds at one point of the breakdown surface. This is possible due to the availability of an additional scalar parameter in the differential system. This approach was first proposed in [14,15] and was then applied in [6,16–22].

Other examples of the stable/unstable trajectories may be a canard cascade [23, 24] and a canard doublet [25]. Both of these objects are the results of gluing stable (attractive) and unstable (repulsive) slow invariant manifolds at several points of the breakdown surface. To make this gluing possible, several additional parameters are required.

If we glue the stable and unstable slow invariant manifolds at all points of the breakdown surface at the same time, we obtain a black swan [10,18–20]. In many cases, such continuous stable/unstable invariant surfaces consist entirely of canards.
In this paper, we use black swans for modelling of a wide variety of oscillations. To demonstrate the approach discussed we consider a prey – predator – superpredator food chain model.

2. Invariant manifolds with a change of stability
Let us consider an autonomous singularly perturbed system
\[
\frac{dx}{dt} = f(x, y, \alpha, \varepsilon),
\]
\[
\varepsilon \frac{dy}{dt} = g(x, y, \alpha, \varepsilon),
\]
where \(x\) and \(y\) are vectors in Euclidean spaces \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively, \(\varepsilon\) is a small positive parameter, \(\alpha\) is an additional parameter, vector–functions \(f\) and \(g\) are sufficiently smooth and their values are comparable to one as \(\varepsilon \to 0\). The slow and fast subsystems are represented by (1) and (2), respectively.

**Definition 1** A smooth surface \(S_\varepsilon\) is called an invariant manifold of the system (1), (2) if any trajectory of the system that has at least one point in common with \(S_\varepsilon\) lies entirely on \(S_\varepsilon\).

Among the invariant manifolds we distinguish the invariant surfaces of slow motions whose dimension is equal to that of the slow subsystem, the so-called slow invariant manifolds. The stability or instability of the slow invariant manifold is defined by the stability or instability of its zero-order approximation \((\varepsilon = 0)\), the so-called slow surface [10, 24, 26].

**Definition 2** The surface \(S\) described by the equation
\[
g(x, y, \alpha, 0) = 0
\]
is called a slow surface. When the dimension of this surface is equal to one, it is called a slow curve.

Let a vector-function \(y = \varphi(x, \alpha)\) is an isolated root of the equation (3).

**Definition 3** The subset of \(S\) is stable (or attractive) if the spectrum of the Jacobian matrix
\[
J = \frac{\partial g}{\partial y}(x, \varphi(x, \alpha), \alpha, 0)
\]
is located in the left open complex half-plane. If there is at least one eigenvalue of the Jacobian matrix (4) with a positive real part then the subset of the slow surface is unstable (or repulsive).

The stable and unstable parts of the slow surface are separated by a breakdown surface (curve or points) at which \(\det J = 0\).

As noted above the slow surface can be considered as a zero-order approximation of the slow invariant manifold, hence, in an \(\varepsilon\)–neighborhood of a stable (unstable) subset of the slow surface there exists a stable (unstable) slow invariant manifold. A slow invariant manifold can change its stability in some specific cases. We will consider the case when only one eigenvalue of the Jacobian matrix \(J\) can change its sign.

In the case of the scalar variables \(x\) and \(y\), the stable and unstable subsets of the slow curve \(S\) for which \(g_y < 0\) and \(g_y > 0\), respectively, are separated by the breakdown point(s) at which \(g_y = 0\).

The presence of the additional scalar parameter \(\alpha\) provides the possibility of gluing the stable and unstable invariant manifolds at the breakdown point to form a single trajectory, a canard [10, 24, 26].
**Definition 4** Trajectories which at first move along the stable slow invariant manifold and then continue for a while along the unstable slow invariant manifold are called canards or duck-trajectories.

**Definition 5** Trajectories which at first move along the unstable slow invariant manifold and then continue for a while along the stable slow invariant manifold are called false canard trajectories.

According to the theory of invariant manifolds, this gluing procedure means the following. The canard and the corresponding parameter value \( \alpha = \alpha^*(\varepsilon) \) allow for asymptotic expansions in powers of the small parameter \( \varepsilon \):

\[
y = h(x, \varepsilon) = h_0(x) + \varepsilon h_1(x) + \ldots + \varepsilon^k h_k(x) + \ldots, \tag{5}
\]

\[
\alpha^*(\varepsilon) = \alpha_0 + \varepsilon \alpha_1 + \ldots + \varepsilon^k \alpha_k + \ldots. \tag{6}
\]

We can calculate the functions \( h_0, h_1, h_2, \) etc. from the invariance equation:

\[
\varepsilon \frac{\partial h}{\partial x} f(x, h(x, \varepsilon), \alpha^*(\varepsilon), \varepsilon) = g(x, h(x, \varepsilon), \alpha^*(\varepsilon), \varepsilon),
\]

which follows from the system (1), (2) and the asymptotic expansions (5) and (6). However, all functions in (5) have a discontinuity at the breakdown point. A proper choice of \( \alpha_0, \alpha_1, \) etc. provides to avoid this discontinuity. This choice of the coefficients in (6) means that we glue the stable and unstable slow invariant manifolds at the breakdown point step by step in their zero-order approximation, first-order approximation, etc. The outcome of this procedure is a canard. The reader can find the algorithm and justifications of this procedure in [14–16,26].

It should be noted that, in the case of the scalar variables \( x \) and \( y \), the canards are exponentially close to each other near the slow curve and have the same asymptotic expansion (5) in powers of \( \varepsilon \). An analogous assertion is true for corresponding parameter values (6). Namely, any two values of the parameter \( \alpha \) for which canards exist have the same asymptotic expansions, and the difference between them is given by \( \exp(-1/c\varepsilon) \), where \( c \) is some positive number. But in the case \( \dim x \geq 2 \) the situation is essentially another: if the differential system has a canard then it has one-parameter family of canards at once, and a choice of a value of an additional parameter \( \alpha \) means a selection of a point on the breakdown surface at which the stable and unstable invariant manifolds are glued. The mathematical justification of this fact for the case \( \dim x \geq 2, \dim y = 1 \) one can find [6,10], and for the case \( \dim x = n, \dim y = m \), see, for example, [18–20].

Note that in many problem it is more convenient to find a canard in a parametric form [24,27]. Moreover, for the constructing the asymptotic expansions (5) it is assumed that the degenerate equation (3) allows one to find the slow surface explicitly. In many problems this is not possible due to the fact that the degenerate equation is either a high degree polynomial or transcendental. However, in many problems the slow surface can be described in parametric form

\[
x = \chi_0(v), \quad y = \varphi_0(v),
\]

where the parameter \( v \in R^n \). In this case a canard may also be found in parametric form as asymptotic expansions:

\[
x = \chi(v, \varepsilon) = \chi_0(v) + \varepsilon \chi_1(v) + \ldots + \varepsilon^k \chi_k(v) + \ldots,
\]

\[
y = \psi(v, \varepsilon) = \psi_0(v) + \varepsilon \psi_1(v) + \ldots + \varepsilon^k \psi_k(v) + \ldots.
\]
If this is not possible, it is necessary to use an implicit form for asymptotic representations [24,27].

To glue stable and unstable slow invariant manifolds at several breakdown points, we need several additional parameters and as a result we obtain a one-dimensional slow invariant manifold with multiple change of stability. A canard doublet [24–26] and a canard cascade [23] are the examples of such slow invariant manifold.

**Definition 6** Trajectories which at first pass along a stable part of a slow curve, then continue for a while along an unstable part of the slow curve and after that jump in the direction of another stable part of the slow curve, pass along this attractive part of the slow curve, then continue for a while along another unstable part of the slow curve are called canard doublets.

**Definition 7** The continuous one-dimensional slow invariant manifold of (1), (2) which contains at least two canards or false canards is called a canard cascade.

In the case of dim x ≥ 2 the gluing procedure may create slow invariant surfaces of variable stability (or black swans) [10,18–20,24,26]. Such surfaces may be considered as a multidimensional analogue of the notion of a canard. An additional parameter is used to glue together the stable and unstable parts of a canard, and we need an additional function to glue invariant manifolds whose dimension is greater than one. The argument of this function is a vector variable parameterizing the breakdown surface.

The presence of an exact black swan in a differential system can significantly simplify canard chase in multidimensional spaces. Moreover, the existence of an exact stable/unstable slow invariant manifolds underlies an effective approach to modelling various oscillatory processes in the differential system [28]. In the next section, this approach is illustrated via the Rosenzweig-MacArthur food chain model.

### 3. Model

Consider the Rosenzweig-MacArthur food chain model [29]:

\begin{align}
\dot{x} &= x \left( r - \frac{rx}{K} - \frac{p_1y}{H_1 + x} \right), \\
\dot{y} &= y \left( \frac{c_1x}{H_1 + x} - d_1 - \frac{p_2z}{H_2 + y} \right), \\
\dot{z} &= z \left( \frac{c_2y}{H_2 + y} - d_2 \right).
\end{align}

Here x is the population density of a logistic prey; y is the population density of a Holling type II predator; z is the population density of a Holling type II superpredator; r is the maximum per–capita growth rate for the prey and K its carrying capacity; p_1 is the maximum per–capita predation rate and H_1 is the semisaturation constant at which the per-capita predation rate is half of its maximum (p_1/2); c_1 is the maximum per–capita growth rate of the predator; d_1 is the per–capita natural death rate for the predator; p_2 and H_2 have similar meanings as p_1 and H_1, except that the predator y is the prey for the super-predator z. Similar explanations also apply to c_2 and d_2 [30–34].

With the following changes of variables and parameters [35–37],

\begin{align}
t &\to c_1t, \quad x \to \frac{x}{K}, \quad y \to \frac{p_1y}{rK}, \quad z \to \frac{p_1p_2z}{c_1rK}, \\
\zeta &= \frac{c_1}{r}, \quad \varepsilon = \frac{c_2}{c_1}, \quad \beta_1 = \frac{H_1}{K}, \quad \beta_2 = \frac{p_1H_2}{rK}, \quad \delta_1 = \frac{d_1}{c_1}, \quad \delta_2 = \frac{d_2}{c_2}.
\end{align}
the system (7)–(9) are transformed to the following form:

\[
\dot{\zeta} = x \left(1 - x - \frac{y}{\beta_1 + x}\right), \quad (10)
\]
\[
\dot{y} = y \left(\frac{x}{\beta_1 + x} - \delta_1 - \frac{z}{\beta_2 + y}\right), \quad (11)
\]
\[
\dot{z} = \varepsilon z \left(\frac{y}{\beta_2 + y} - \delta_2\right). \quad (12)
\]

Under the drastic trophic time diversification hypothesis that the maximum per–capita growth rate decreases from bottom to top along the food chain, namely

\[r \gg c_1 \gg c_2 > 0,\]

from what follows that

\[0 < \zeta \ll 1, \quad 0 < \varepsilon \ll 1.\]

Thus, (10)–(12) is the singularly perturbed system with three time scales. This fact allows us to use the geometric theory of singular perturbation for its analysis.

4. Slow invariant manifold

The slow surface of the system is described by the degenerate equation

\[x \left(1 - x - \frac{y}{\beta_1 + x}\right) = 0.\]

Note that \(x \equiv 0\) is the exact slow invariant manifold. Moreover, the plane \(x \equiv 0\) is the exact black swan of the system (10)–(12), since it consists of the stable part

\[S_s^1 = \{(0, y, z) : y > \beta_1\}\]

and the unstable part

\[S_s^1 = \{(0, y, z) : 0 < y < \beta_1\},\]

separated by the breakdown line

\[S_{br}^1 = \{(0, y, z) : y = \beta_1\}.\]

The second leaf of the slow surface

\[S_2 = \{(x, y, z) : 1 - x - \frac{y}{\beta_1 + x} = 0\}\]

is divided by the breakdown curve

\[S_{br}^2 = \{(x, y, z) : x = \frac{1 - \beta_1}{2}, \ y = \frac{(1 + \beta_1)^2}{4}\}\]

into its stable \((S_2^s)\) and unstable \((S_2^u)\) subsets, see Figure 1.

In an \(\zeta\)-neighborhood of the stable (unstable) subsets \(S_1^s\) and \(S_2^s\) \((S_1^u\) and \(S_2^u)\) of the slow surface there exist the stable (unstable) slow invariant manifolds \(S_{1,\varepsilon}^s\) and \(S_{2,\varepsilon}^s\) \((S_{1,\varepsilon}^u\) and \(S_{2,\varepsilon}^u)\), respectively. We can glue together \(S_{2,\varepsilon}^s\) and \(S_{2,\varepsilon}^u\) at some point on the breakdown line \(S_{br}^2\), using the standard procedure discussed above, to get a canard. Moreover, it is possible to construct a slow invariant manifold with changing stability (black swan) consisting entirely of canards. For this goal we need an additional function, say \(\delta_1 = \delta_1(\varepsilon, \zeta)\), to glue invariant manifolds at all points of the breakdown line \(S_{br}^2\) at the same time. The argument \(\zeta\) of this function is the variable parameterizing the breakdown line.
5. Black swan

For (10)–(12), it is convenient to find the black swan in the parametric form [24]:

\[ y = y(x, z, \zeta) = y_0(x, z) + \zeta y_1(x, z) + \zeta^2 y_2(x, z) + O(\zeta^3), \]  

(13)

where \( x \) and \( z \) are treated as parameters, while the corresponding gluing function \( \delta_1 = \delta_1(z, \zeta) \) has the following asymptotic expansions

\[ \delta_1 = \delta_1(z, \zeta) = b_0 + \zeta b_1 + \zeta^2 b_2 + O(\zeta^3). \]  

(14)

From (10)–(12) and (13), (14) using the invariance equation

\[ \varepsilon \zeta \frac{\partial y}{\partial x} \frac{dx}{dt} + \varepsilon \zeta \frac{\partial y}{\partial z} \frac{dz}{dt} = \varepsilon \zeta \frac{dy}{dt}, \]

we have

\[
\left( \frac{\partial y_0}{\partial x} + \zeta \frac{\partial y_1}{\partial x} + \zeta^2 \frac{\partial y_2}{\partial x} + O(\zeta^3) \right) x \left( 1 - x - \frac{y_0 + \zeta y_1 + \zeta^2 y_2}{\beta_1 + x} + O(\zeta^3) \right) \\
+ \zeta \left( \frac{\partial y_0}{\partial z} + \zeta \frac{\partial y_1}{\partial z} + \zeta^2 \frac{\partial y_2}{\partial z} + O(\zeta^3) \right) z \left( \frac{y_0 + \zeta y_1 + \zeta^2 y_2}{\beta_2 + y_0 + \zeta y_1 + \zeta^2 y_2} - \delta_2 + O(\zeta^3) \right) \\
= \zeta \left( y_0 + \zeta y_1 + \zeta^2 y_2 + O(\zeta^3) \right) \left( \frac{x}{\beta_1 + x} - b_0 - \zeta b_1 - \zeta^2 b_2 - \frac{z}{\beta_2 + y_0 + \zeta y_1 + \zeta^2 y_2} + O(\zeta^3) \right). 
\]

(15)

Setting \( \zeta = 0 \) in (15) we obtain

\[ y_0(x, z) = (1 - x)(\beta_1 + x). \]  

(16)

We now equate terms in \( \zeta^1 \) in (15):

\[
\frac{\partial y_1}{\partial x} x \left( 1 - x - \frac{y_0}{\beta_1 + x} \right) - \frac{xy_1}{\beta_1 + x} \frac{\partial y_0}{\partial x} + \varepsilon \frac{\partial y_0}{\partial z} z \left( \frac{y_0}{\beta_2 + y_0} - \delta_2 \right) \\
- \frac{xy_1}{\beta_1 + x} \frac{\partial y_0}{\partial z} z \left( \frac{y_0}{\beta_2 + y_0} - \delta_2 \right) \\
= \zeta \left( y_0 + \zeta y_1 + \zeta^2 y_2 + O(\zeta^3) \right) \left( \frac{x}{\beta_1 + x} - b_0 - \zeta b_1 - \zeta^2 b_2 - \frac{z}{\beta_2 + y_0 + \zeta y_1 + \zeta^2 y_2} + O(\zeta^3) \right). 
\]
\[ y = y_0 \left( \frac{x}{\beta_1 + x} - b_0 - \frac{z}{\beta_2 + y_0} \right) \]

or, taking into account (16),

\[ \frac{-xy_1}{\beta_1 + x} \frac{\partial y_0}{\partial x} = y_0 \left( \frac{x}{\beta_1 + x} - b_0 - \frac{z}{\beta_2 + y_0} \right) . \]  \hspace{1cm} (17) 

On the breakdown surface \( S_{br}^2 \) we have

\[ \frac{\partial y_0}{\partial x} = 1 - \beta_1 - 2x \equiv 0. \]

By continuity of the function \( y_0 = y_0(x, z) \) we thus require the following condition from (17):

\[ b_0(x) = \left. \frac{x}{\beta_1 + x} - \frac{z}{\beta_2 + y_0} \right|_{x=1-\beta_1, y_0=(1+\beta_1)^2} = \frac{1-\beta_1}{1+\beta_1} - \frac{4z}{4\beta_2 + (1 + \beta_1)^2} . \]  \hspace{1cm} (18) 

Then from (17) we obtain:

\[ y_1(x, z) = \frac{(1-x)(\beta_1 + x)^2}{x(1-\beta_1 - 2x)} \times \left( \frac{z}{\beta_2 (1-x)(\beta_1 + x)} + \frac{1-\beta_1}{1+\beta_1} - \frac{x}{\beta_1 + x} - \frac{4z}{4\beta_2 + (1 + \beta_1)^2} \right) . \]  \hspace{1cm} (19) 

Similarly, equating terms in \( \zeta^2 \) in (15) we can calculate \( b_1 \) and \( y_2(x, z) \), etc. The outcome of this gluing process is the continuous stable/unstable slow invariant manifold or black swan, represented on Figure 2. The black swan consists entirely of canards. Each corresponds to the specified initial data and passes through a definite point on the breakdown line \( S_{br}^2 \), see Figure 3.

6. Modelling of oscillations

Recall that \( x \equiv 0 \) is the exact black swan of the system (10)–(12). In this special case, the trajectories of the system, starting in the basin of attraction of \( S_1^s \), will continue their movement for a while along \( S_1^u \). Therefore, we can transform a single canard, passing through a point \( z_c \) on the breakdown line \( S_{br}^2 \), to a shape of a canard doublet [23, 25] by vary the value of \( \delta_1 = \delta_1(z_c, \zeta) \) on a value of order \( O(\exp(-1/c\zeta)) \), where \( c \) is some positive number. Figures 4 (a)-(e) show the results of such transformation for a canard with a starting point \( (x(0) = 1, y(0) = 0.15, z(0) = 0.01) \). It should be noted that all these trajectories correspond to the function (14) and also lie on the black swan, see Figure 5.

However, if we change the value of the function \( \delta_1 = \delta_1(z_c, \zeta) \) significantly, we thereby destroy the glueing of the stable integral manifold \( S_2^u \) and the unstable integral manifold \( S_2^u \). As a result, the trajectories of the system will describe relaxation oscillations, see Figure 4 (f).

The presence of the exact trivial black swan, attracting to itself the trajectory of the system (10)–(12), can explain the so-called effect of apparent disappearance [38]: the system exhibits the dynamics with long periods of low abundance, when one of the populations, or all populations, remain at very low and probably undetectable levels (the apparent disappearance), alternating with comparatively short periods of a high abundance. In the system (10)–(12), the longest periods of low abundance of the population of prey one can observe in the case of the relaxation oscillations.
Figure 2. The slow surface (grey) and the black swan (green) of the system (10)–(12).

Figure 3. The canards of the system (10)–(12).

7. Conclusions
In this paper, we have demonstrated a new approach to the modelling of oscillations in a class of population dynamics models. The approach discussed is based on the technique of slow invariant manifolds with changing stability. For the Rosenzweig–MacArthur food chain model,
Figure 4. The projections of intermediate trajectories of the system (10)–(12) during the transformation from a single canard (a, b) to a canard doublet (c, d, e), and a relaxation cycle (f); $\zeta = 0.02$, $\varepsilon = 0.3$, $\beta_1 = 0.21$, $\beta_2 = 0.7$, $\delta_2 = 0.35$, and (a) $\delta_1 = 0.5944473$, (b) $\delta_1 = 0.5944469$, (c) $\delta_1 = 0.5944468$, (d) $\delta_1 = 0.5944467$, (e) $\delta_1 = 0.59442$, (f) $\delta_1 = 0.59338$. 
Figure 5. The slow surface (gray), the black swan (green), and the canard doublets (blue) of the system (10)–(12).

we demonstrated how to model canard-type oscillations via a black swan construction. We pointed out the link between the existence of the exact trivial black swan and canards with the phenomenon of apparent disappearance in the prey population.

From the standpoint of chaos generating mechanisms in the Rosenzweig–MacArthur food chain model, canards were considered in [39].

8. References
[1] Diener M 1979 Nessie et Les Canards (Strasbourg: Publication IRMA)
[2] Benoit E, Callot J L, Diener F and Diener M 1981-1982 Chasse au canard Collect. Math. 31-32 37-119
[3] Benoit E and and Lobry C 1982 Les canards de R3 C.R. Acad. Sc. Paris 294 483-488
[4] Benoit E 1983 Systemes lents-rapides dans R3 et leurs canards Societe Mathematique de France, Asterisque 109-110 159-191
[5] Mishchenko E F, Kolesov Yu S, Kolesov A Yu and Rozov N Kh 1995 Asymptotic Methods in Singularly Perturbed Systems (New York: Plenum Press)
[6] Sobolev V A and Shchepakina E A 1996 Duck trajectories in a problem of combustion theory Differential Equations 32 1177-1186
[7] Szmolyan P and Wechselberger M 2001 Canards in R3 J. Diff. Eq. 177 419-453
[8] Wechselberger M 2005 Existence and bifurcation of canards in R3 in the case of a folded node J. Appl. Dyn. Syst 4(1) 101-139
[9] Xie F, Han M and Zhang W 2005 Canard Phenomena in Oscillations of a Surface Oxidation Reaction J Nonlinear Sci 15 363-386
[10] Shchepakina E and Sobolev V 2005 Black Swans and Canards in Laser and Combustion Models (Singular Perturbation and Hysteresis) (Philadelphia: SIAM) 207-255
[11] Xie F, Han M and Zhang W 2006 The persistence of canards in 3-D singularly perturbed systems with two fast variables Asymp. Anal. 47(1) 95-106
[12] Marino F, Marin F, Balle S and Piro O 2007 Chaotically spiking canards in an excitable system with 2D inertial fast manifolds Phys. Rev. Lett. 98 074104
[13] Desroches M, Krauskopf B and Osinga H M 2010 Numerical continuation of canard orbits in slow-fast dynamical systems *Nonlinearity* **23** 739-765

[14] Gorelov G N and Sobolev V A 1991 Mathematical modelling of critical phenomena in thermal explosion theory *Combust. Flame* **87** 203-210

[15] Gorelov G N and Sobolev V A 1992 Duck-trajectories in a thermal explosion problem *Appl. Math. Lett.* **5** 3-6

[16] Sobolev V A and Shchepakina E A 1993 Self-ignition of laden medium *J. Combustion, Explosion and Shock Waves* **29** 378-381

[17] Gol'dshtein V, Zinoviev A, Sobolev V and Shchepakina E 1996 Criterion for thermal explosion with reactant consumption in a dusty gas *Proc. London Roy. Soc. Ser. A* **452** 2103-2119

[18] Shchepakina E and Sobolev V 2001 Integral manifolds, canards and black swans *Nonlinear Analysis A* **44** 897-908

[19] Shchepakina E 2002 Slow integral manifolds with stability change in the case of a fast vector variable *Differential Equations* **38** 1146-1452

[20] Shchepakina E 2003 Black swans and canards in self-ignition problem *Nonlinear Analysis: Real Word Applications* **4** 45-50

[21] Schneider K, Shchepakina E and Sobolev V 2003 A new type of travelling wave *Mathematical Methods in the Applied Sciences* **26** 1349-1361

[22] Gorelov G N, Sobolev V A and Shchepakina E A 2006 Canards and critical behaviour in autocatalytic combustion models *Journal of Engineering Mathematics* **56** 143-160

[23] Sobolev V 2013 *Canard Cascades Discr. and Cont. Dynam. Syst. B* **18** 513-521

[24] Shchepakina E, Sobolev V and Mortell M P 2014 Singular Perturbations. Introduction to system order reduction methods with applications *Lect. Notes in Math.* **2114**

[25] Pokrovskii A, Shchepakina E and Sobolev V 2008 Canard doublet in a Lotka-Volterra type model *Journal of Physics: Conference Series* **138** 012019

[26] Shchepakina E and Sobolev V 2016 Invariant surfaces of variable stability *Journal of Physics: Conference Series* **727** 012016

[27] Sobolev V 2005 *Geometry of Singular Perturbations: Critical Cases (Singular Perturbation and Hysteresis)* (Philadelphia: SIAM) 153-206

[28] Shchepakina E and Sobolev V 2018 Cascade of 3D canard doublets *Journal of Physics: Conference Series* **1096** 012053

[29] Rosenzweig M L and MacArthur R H 1963 Graphical representation and stability conditions of predator-prey interactions *American Naturalist* **97** 209-223

[30] Hastings A and Powell T 1991 Chaos in a three-species food chain *Ecology* **72** 896-903

[31] Rinaldi S and Muratori S 1992 Slow-fast limit cycles in predator-prey models *Ecol. Modell.* **61** 287-308

[32] Muratori S and Rinaldi S 1992 Low- and high-frequency oscillations in three-dimensional food chain system *J. Appl. Math.* **52** 1688-1706

[33] McCann K and Yodzis P 1995 Bifurcation structure of a three-species food chain model *Theor. Popul. Biol.* **48** 93-125

[34] Kuznetsov Yu A and Rinaldi S 1996 Remarks on food chain dynamics *Math. Biosci.* **133** 1-33

[35] Deng B 2001 Food chain chaos due to junction-fold point *Chaos* **11** 514-525

[36] Deng B and Hines G 2002 Food chain chaos due to Shilnikov orbit *Chaos* **12** 533-538

[37] Deng B and Hines G 2003 Food chain chaos due to transcritical point *Chaos* **13** 578-585

[38] Gavin C, Pokrovskii A, Prentice M and Sobolev V 2006 Dynamics of a Lotka-Volterra type model with applications to marine phage population dynamics *Journal of Physics: Conference Series* **55** 80-93

[39] Deng B 2001 Food chain chaos with canard explosion *Chaos* **14** 1083-1092

**Acknowledgment**

This work was supported by the Ministry of Science and Higher Education of the Russian Federation under the Competitiveness Enhancement Program of Samara University (2013-2020).