Interaction of the Electromagnetic p-Wave with Thin Metal Film in the Field of Resonant Frequencies

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Abstract

It is shown that for thin metallic films thickness of which does not exceed thickness of skin layer, the problem allows analytical solution. In the field of resonant frequencies the analysis of dependence of coefficients of transmission, reflection and absorbtion on an electromagnetic wave is carried out. Dependence on pitch angle, thickness of the layer and coefficient of specular reflection and on effective electron collision frequency is carried out. The formula for contactless determination (calculation) of a thickness of a film by observable resonant frequencies is deduced.

Key words: degenerate plasma, electromagnetic p-wave, thin metallic film, coefficients of transmission, reflection and absorption.

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1. Introduction

The problem of interaction of an electromagnetic wave with the metal film attracts attention to itself for the long time already \[1\] – \[5\]. It is connected as with theoretical interest to this problem, and with numerous practical appendices as well \textsuperscript{6} and \textsuperscript{7}.

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Researches of interaction of an electromagnetic wave with metallic films were carried out basically for a case of specular dissipation of electrons on a film surface. It is connected with the fact that for more general boundary conditions the problem becomes essentially complicated and does not suppose the analytical solution generally.

In the present work it is shown that for thin films, a thickness of which does not exceed a thickness of a skin – layer, the problem allows the analytical solution. In previous work [8] the case when the frequency of electromagnetic wave is less than plasma (Langmuir) frequency was considered.

Let us note, that the most part of reasonings carried out below would be true for the more general case of conducting (in particular, semi-conductor) film.

2. Problem Statement

We consider the thin slab of conducting medium on which the electromagnetic wave falls. We denote the pitch angle by θ. We will assume, that the vector of magnetic field of electromagnetic wave is parallel to the surface of the slab. Such wave is called $p$–wave (see [3] or [1]).

We take Cartesian coordinate system with origin of coordinates on one of the surfaces of a slab, with axis $x$, directed deep into the slab. We direct the axis $y$ parallel with the vector of magnetic field of electromagnetic wave. Under such choice of the system of coordinates the electric field vector and magnetic field vector have the following structure

$$
\mathbf{E} = \{E_x(x, y, z), 0, E_z(x, y, z)\} \quad \text{and} \quad \mathbf{H} = \{0, H_y(x, y, z), 0\}.
$$

Further components of electric and magnetic field vectors we search in the form

$$
E_x(x, z, t) = E_x(x)e^{-i\omega t+ik\sin \theta z}, \quad E_z(x, z, t) = E_z(x)e^{-i\omega t+ik\sin \theta z},
$$
and
\[ H_y(x, z, t) = H_y(x)e^{-i\omega t + ik\sin \theta z}. \]

Now behaviour of electric and magnetic fields of the wave in the slab is described by the following system of differential equations [3]
\[
\begin{align*}
\frac{dE_z}{dx} &- ik \sin \theta E_x + ikH_y = 0 \\
\frac{dH_y}{dx} + ikE_z &= \frac{4\pi}{c}j_z.
\end{align*}
\]
(1)

Here \( c \) is the velocity of light, \( j \) is the current density, \( k \) is the wave number.

We denote the thickness of the slab by \( d \).

The coefficients of transmission \( T \), reflection \( R \) and absorption \( A \) of the electromagnetic wave by the slab are described by the following expressions [1], [9]
\[
T = \frac{1}{4} \left| P^{(1)} - P^{(2)} \right|^2, \quad (2a)
\]
\[
R = \frac{1}{4} \left| P^{(1)} + P^{(2)} \right|^2, \quad (2b)
\]
and
\[
A = 1 - T - R. \quad (2c)
\]

The quantities \( P^{(j)} \) \((j = 1, 2)\) are defined by the following expressions
\[
P^{(j)} = \frac{\cos \theta + Z^{(j)}}{\cos \theta - Z^{(j)}}, \quad j = 1, 2. \quad (3)
\]

The quantity \( Z^{(1)} \) corresponds to impedance on the lower surface of slab under symmetrical configuration of the external magnetic field. This is the case 1 when
\[
H_y(0) = H_y(d), \quad E_x(0) = E_x(d), \quad E_z(0) = -E_z(d)
\]
The quantity $Z^{(2)}$ corresponds to the impedance on the lower surface of the slab under antysymmetrical configuration of the external magnetic field. This is the case 2 when

$$H_y(0) = -H_y(d), \quad E_x(0) = -E_x(d), \quad E_z(0) = E_z(d).$$

The impedance is thus defined as follows

$$Z^{(j)} = \frac{E_z(-0)}{H_y(-0)}, \quad j = 1, 2.$$  \hspace{1cm} (4)

We will consider the case when the width of the slab $d$ is less than the depth of the skin – layer $\delta$. Let’s note, that depth of the skin – layer depends essentially on frequency of radiation, monotonously decreasing in process of growth of the last. The value $\delta$ possesses the minimum value in so-called infrared case \[10\]

$$\delta_0 = \frac{c}{\omega_p},$$

where $\omega_p$ is the plasma frequency.

For typical metals \[10\] $\delta_0 \sim 100$ nm.

Hence for the films thickness of which $d$ is less than $\delta_0$ our assumption is true for any frequencies.

3. Problem solution

The quantities $H_y$ and $E_z$ change a little on distances smaller than the depth of a skin – layer.

Therefore under fulfilment of the given assumption $d < \delta$ these electrical and magnetical fields will change a little in the slab.

In case 1 when $H_y(0) = H_y(d)$ it is possible to accept that the value $H_y$ is constant within the slab.
Variation of the quantity of \( y \)-projection of electric field on thickness of slab can be defined from the first equation of the system (1)

\[
E_y(d) - E_y(0) = -ikdH_y + ik \sin \theta \int_0^d E_x dx.
\]  

(5)

From the second equation of the system (1) it follows that on the boundary of the film the following relationship is satisfied

\[
E_x(0) = E_x(d) = H_y \sin \theta.
\]  

(6)

The integral from the relation (5) is proportional to the value of the quantity of normal to the surface component of electrical field on the surface and therefore according to the relation (6) it’s proportional to the quantity \( H_y \).

We define the coefficient of proportionality as

\[
G = \frac{1}{E_x(0)d} \int_0^d E_x(x) dx = \frac{1}{H_y d \sin \theta} \int_0^d E_x(x) dx.
\]  

(7)

With the help of (7) we rewrite the relation (5) in the following form

\[
E_z(d) - E_z(0) = (-ikd + ikGd \sin^2 \theta)H_y.
\]

Considering the antisymmetric character of the projection of electric field \( E_y \) in this case we receive

\[
E_y(0) = ikd(1 - G \sin^2 \theta)H_y.
\]  

(8)

Therefore for the impedance we have

\[
Z^{(1)} = \frac{ikd}{2}(1 - G \sin^2 \theta).
\]  

(9)

For the case 2 when \( E_z(0) = E_z(d) \), it is possible to assume that \( z \) – projection of electric field \( E_z \) is constant in the slab.
Then the magnetic field change on the width of a slab can be determined from the third equation of the system (1)

$$H_y(d) - H_y(0) = -ikdE_z + \frac{4\pi}{c} \int_0^d j_z(x)dx.$$  

(10)

Thus

$$j_z(x) = \sigma(x)E_z,$$

where $\sigma(x)$ is the conductance that in general case depends on coordinate $x$.

Let’s introduce the longitudinal conductivity averaged by thickness of the slab,

$$\sigma_d = \frac{1}{E_zd} \int_0^d j_z(x)dx = \frac{1}{d} \int_0^d \sigma(x)dx.$$  

(11)

Then the relation (10) can be rewritten with help (11) in the following form

$$H_y(d) - H_y(0) = -ikdE_z + \frac{4\pi d\sigma_d}{c}E_z.$$  

Considering symmetry of the magnetic field, from here we have

$$H_y(0) = \frac{1}{2}ikdE_z - \frac{2\pi d\sigma_d}{c}E_z.$$  

For the impedance (4) we have

$$Z^{(2)} = \frac{2c}{ickd - 4\pi \sigma_d d}.$$  

(12)

4. Coefficients of transmission, reflection and absorption

From here according to (3) we receive expressions for the quantities $P^{(j)} (j = 1, 2)$
\[
P^{(1)} = \frac{2\cos \theta + ikd(1 - G\sin^2 \theta)}{2\cos \theta - ikd(1 - G\sin^2 \theta)}, \quad (13a)
\]
\[
P^{(2)} = \frac{(4\pi \sigma_d - ikc)d\cos \theta - 2c}{(4\pi \sigma_d - ikc)d\cos \theta + 2c}. \quad (13b)
\]

We will assume that length of a wave of incident radiation surpasses essentially the thickness of the slab, i.e. \(kd \ll 1\). Then expressions (9) and (12) for impedances and expression (13) for quantity \(P^{(j)}\) \((j = 1, 2)\) become a little simpler

\[
Z_0^{(1)} = -\frac{1}{2}ikGd\sin^2 \theta, \quad Z_0^{(2)} = -\frac{c}{2\pi \sigma_d d}. \quad (14)
\]

Substituting (14) into (3), we have:

\[
P_0^{(1)} = \frac{2\cos \theta - ikGd\sin^2 \theta}{2\cos \theta + ikGd\sin^2 \theta}, \quad P_0^{(2)} = \frac{2\pi \sigma_d d\cos \theta - c}{2\pi \sigma_d d\cos \theta + c}. \quad (15)
\]

Thus quantities \(R, T, A\) can be found according to the formulas (2).

In a limiting case of non-conducting slab, when \(\sigma_d \to 0, G \to 1\) from these expressions we have

\[
P^{(1)} = -P^{(2)} = \frac{2 + ikd\cos \theta}{2 - ikd\cos \theta},
\]

from which

\[
T = 1, \quad R = 0, \quad A = 0.
\]

Under almost tangent incidence when \(\theta \to \pi/2\) we receive \(P^{(1)} \to -1, P^{(2)} \to -1\). Thus, we obtain that \(T \to 0, R \to 1, A \to 0\).

Let the relation \(kl \ll 1\) be true. Then in a low-frequency case, when \(\omega \to 0\), the quantity \(\sigma_d\) for a metal film can be presented in the following form [11]

\[
\sigma_d = \frac{w}{\Phi(w)} \sigma_0, \quad w = \frac{d}{l}, \quad (16)
\]

where

\[
\frac{1}{\Phi(w)} = \frac{1}{w} - \frac{3}{2w^2}(1 - p) \int_{1}^{\infty} \left(\frac{1}{t^3} - \frac{1}{t^5}\right) \frac{1 - e^{-wt}}{1 - pe^{-wt}} dt.
\]
Here $l$ is the mean free path of electrons, $p$ is the coefficient of specular reflection, $\sigma_0 = \omega_p^2 \tau / (4\pi)$ is the static conductivity of a volume pattern, $\tau = l/v_F$ is the time of mean free path of electrons, $v_F$ is the Fermi velocity.

In a low-frequency case when the formula (16) is applicable, the coefficients $T, R, A$ do not depend on frequency of the incoming radiation according to the formulas (2).

For arbitrary frequencies these expressions are true under condition, that it is necessary to use the following expression $l \to \frac{v_F \tau}{1 - i\omega \tau}$, as a quantity $l$ and the expression $\sigma_0 \to \frac{\sigma_0}{1 - i\omega \tau}$ instead of $\sigma_0$.

For the case $kl \ll 1$ the quantity $G$ can be calculated from the problem of behaviour of a plasma slab in variable electric field, which is perpendicular to the surface of slab [13].

Let’s calculate the coefficients of transmission and reflection in the case when $kd \ll 1$. We will substitute expressions $P^{(1)}$ and $P^{(2)}$, defined according to (15), into the formulas (2). We receive

$$T = \cos^2 \theta \left| \frac{1 - ik \frac{d}{2} G \sin^2 \theta \frac{2\pi d \sigma_d}{c}}{(\cos \theta + ik \frac{d}{2} G \sin^2 \theta)(1 + \frac{2\pi d \sigma_d}{c} \cos \theta)} \right|^2, \quad (17)$$

$$R = \left| \frac{ik \frac{d}{2} G \sin^2 \theta - \frac{2\pi d \sigma_d}{c} \cos^2 \theta}{(\cos \theta + ik \frac{d}{2} G \sin^2 \theta)(1 + \frac{2\pi d \sigma_d}{c} \cos \theta)} \right|^2, \quad (18)$$

where the quantity $G$ may be found from the solution of the problem of plasma oscillations [12] and [13],

$$G = \frac{1}{d} \int_0^d e(x) dx, \quad (19)$$

and $e(x)$ is the electric field.

5. Coefficients of transmission, reflection and absorption in the case of specular reflection of electrons
In the case of mirror reflection of electrons according to (16) we have \( \sigma_d = \sigma_0 \), and the quantity \( G \) under the formula (19) can be calculated precisely, using the electric field in plasma layer constructed in [13].

At the proof of decomposition of the decision of an initial boundary problem in [13] the electric field in a metal layer has actually been constructed

\[
e(x) = \frac{\lambda_1}{\lambda_\infty} + \frac{2 \lambda_1 \eta_0 \cosh(z_0 x/\eta_0)}{(ac - \eta_0^2)\lambda'(-\eta_0)\cosh(z_0/\eta_0)} + \frac{\lambda_1}{2} \int_{-1}^{1} \frac{\eta_0^2 \cosh(z_0 x/\eta)}{\lambda^+(\eta)\lambda^-(\eta)\cosh(z_0/\eta)} d\eta.
\]

In this formula \( \lambda(z) \) is the dispersion function from the problem of plasma oscillations,

\[
\lambda(z) = c^2 + \frac{z^2}{2} \int_{-1}^{1} \frac{\eta_1^2 - \tau^2}{\tau^2 - z^2} d\tau, \quad \eta_1^2 = ac, \quad a = \frac{a_0 \nu}{v_F \kappa},
\]

\[
\kappa = \frac{9 a_0^2}{r_D^2}, \quad \eta_1^2 = \frac{3 v_F^2}{\omega_p^2}, \quad c = \frac{a_0 (\nu - i\omega)}{\kappa v_F}, \quad z_0 = c \kappa,
\]

\( r_D \) is the Debye radius, \( \lambda_1 = \lambda(\eta_1) = c^2 - ac, \eta_k^* \) are the zeroes of the functions \( \cosh(z_0/\eta) \),

\[
\eta_k^* = -\frac{2 z_0 i}{\pi (2k + 1)} = -\frac{2 a_0 (\omega + i\nu)}{\pi (2k + 1) v_F}, \quad k = 0, \pm 1, \pm 2, \cdots
\]

In the layer \( 0 \leq x \leq d \) the electric field has the following form

\[
e(x) = \frac{\cosh[z_0(2x - d)/\eta_1 d]}{\cosh(z_0/\eta_1)} - \frac{\lambda_1}{z_0} \sum_{k=-\infty}^{k=+\infty} \frac{\eta_k^*^3 \cosh[z_0(2x - d)/\eta_k^* d]}{\lambda(\eta_k^*) (\eta_k^* - \eta_1^2) \sinh(z_0/\eta_k^*)}. \tag{20}
\]

The quantity \( G \) can be found easily by means of the equalities (19) and (20) (see also [13]) and has the following form:

\[
G = \frac{\lambda_1}{\lambda_\infty} + \frac{2 \lambda_1 \eta_0^2 \tanh(z_0/\eta_0)}{z_0 (ac - \eta_0^2)\lambda'(-\eta_0)} + \frac{\lambda_1}{2 z_0} \int_{-1}^{1} \frac{\tanh(z_0/\eta)\eta^3 d\eta}{\lambda^+(\eta)\lambda^-(\eta)}. \tag{21}
\]
Here
\[ \lambda^\pm(\mu) = \lambda(\mu) \pm i\frac{\pi}{2} \mu(\eta_1^2 - \mu^2). \]

Integral from (21) we will calculate by means of methods of contour integration. Let us take advantage further of obvious equality
\[ \frac{1}{\lambda^+(\mu)\lambda^-(\mu)} = \frac{1}{i\pi\mu(\mu^2 - \eta_1^2)} \left[ \frac{1}{\lambda^+(\mu)} - \frac{1}{\lambda^-(\mu)} \right]. \]

Therefore the integral from (21) is equal to
\[ \frac{1}{2} \int_{-1}^{1} \frac{\tanh(z_0/\tau)\tau^3}{\lambda^+(\tau)\lambda^-(\tau)} d\tau = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{\lambda^+(\tau) - \lambda^-(\tau)} \frac{\tanh(z_0/\tau)\tau^2}{\tau^2 - \eta_1^2} d\tau. \]

We take now a circle \( \gamma_R (\gamma_R : |z| = R) \) with so big radius \( R \), that all finite singular points of function
\[ f(z) = \frac{\tanh(z_0/z)z^2}{(z^2 - \eta_1^2)\lambda(z)} \]
lay inside \( \gamma_R \). Such points are points \( z = \pm \eta_1 \), zeroes of the dispersion function \( \lambda(z) \) are points \( z = \pm \eta_0 \) (if \( (\gamma, \varepsilon) \in D^+ \) (see [13]), and also polar singularity of the function \( \tanh(z_0/z) \). The last points are points \( z = \eta_k^\pm, k = 0, \pm 1, \pm 2, \ldots \).

According to the theorem of the full sum of residues we receive
\[ \left[ \text{Res}_{z=\infty} + \sum_{k=-\infty}^{k=+\infty} \text{Res}_{z=\eta_k^\pm} \right] \frac{\tanh(z_0/z)z^2}{(z^2 - \eta_1^2)\lambda(z)} = \]
\[ = \frac{1}{2\pi i} \int_{-1}^{1} \frac{1}{\lambda^+(\tau)} - \frac{1}{\lambda^-(\tau)} \frac{\tanh(z_0/\tau)\tau^2}{\tau^2 - \eta_1^2} d\tau. \]

We note that
\[ \text{Res}_{z=\infty} \frac{\tanh(z_0/z)z^2}{(z^2 - \eta_1^2)\lambda(z)} = -\frac{z_0}{\lambda_\infty}, \]
\[ \text{Res}_{z=\eta_k^\pm} \frac{\tanh(z_0/z)z^2}{(z^2 - \eta_1^2)\lambda(z)} = -\frac{2\eta_k^4}{z_0 \lambda(\eta_k^\pm)(\eta_k^\pm)^2 - \eta_1^2}. \]
Thus, the integral from (21) is equal to

\[
\frac{1}{2} \int_{-1}^{1} \frac{\tanh(z_0/\tau)\tau^3 d\tau}{\lambda^+(\tau)\lambda^-(\tau)} = -\frac{z_0}{\lambda_\infty} + \frac{\eta_1 \tanh(z_0/\eta_1)}{2\lambda(\eta_1)} +
\]

\[+ \frac{2\eta_0^2 \tanh(z_0/\eta_0)}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} - 2 \sum_{n=-\infty}^{+\infty} \frac{\eta_n^4}{\lambda(\eta_n^2)(\eta_n^2 - \eta_1^2)}.\]

Hence, the quantity \(G\) is equal to

\[
G = \frac{\eta_1}{z_0} \tanh \frac{z_0}{\eta_1} - \frac{\lambda_1}{z_0^2} \sum_{k=0}^{+\infty} \frac{\eta_k^4}{\lambda(\eta_k^2)(\eta_k^2 - \eta_1^2)}. \tag{22}
\]

Let us note, that according to the evenness of the expression standing under a sign of the sum in (22), it is possible to simplify this sum and present the expression (22) in the following form

\[
G = \frac{\eta_1}{z_0} \tanh \frac{z_0}{\eta_1} - \frac{2\lambda_1}{z_0^2} \sum_{k=0}^{+\infty} \frac{\eta_k^4}{\lambda(\eta_k^2)(\eta_k^2 - \eta_1^2)}. \tag{23}
\]

Let us note, that the sum of the series from (23) is well approximated by the first member, i.e. instead of (23) it is possible to take

\[
G_1 = \frac{\eta_1}{z_0} \tanh \frac{z_0}{\eta_1} - \frac{2\lambda_1\eta_0^4}{z_0^2 \lambda(\eta_0^2)(\eta_0^2 - \eta_1^2)}. \]

Let us carry out numerical calculations. We will enter the relative error

\[
O_1(\Omega, \varepsilon, d) = \left| \frac{G - G_1}{G} \right| \cdot 100\%, \quad \Omega = \frac{\omega}{\omega_p}, \quad \varepsilon = \frac{\nu}{\omega_p}. \]

Then for a film of sodium (\(\omega_p = 6.5 \cdot 10^{15} \text{ sec}^{-1}, \nu_F = 8.52 \cdot 10^7 \text{ cm/ sec}\)) of the thickness of 1 nanometer, 5 nanometers and 10 nanometers under \(\omega = \omega_p\) and \(\nu = 10^{-3}\omega_p \text{ sec}^{-1}\) accordingly we have: \(O_1 = 1.42\%, 1.38\%\) and
1.98%. For $G$ we have replaced an infinite series with the finite sum for $N = 10^6$ members.

The quantity $G$ is approximated by first two members of the decomposition (21) even more effectively, i.e. when we replace $G$ with the quantity

$$G_2 = \frac{\lambda_1}{\lambda_\infty} + \frac{2\lambda_1\eta_0^2 \tanh(z_0/\eta_0)}{z_0(ac - \eta_0^2)\lambda'(\eta_0)}.$$ (24)

The formula (24) means, that we have replaced the electric field (20) by two first components of Drude and Debaye, corresponding to the discrete spectrum.

For the calculation of the quantity $G_2$ explicit expression of zero of dispersion function $\eta_0 = \eta_0(\Omega, \varepsilon)$ is required. We will write the factorization formula of dispersion function (see [13]) without the proof

$$\lambda(z) = \lambda_\infty(\eta_0^2 - z^2)X(z)X(-z).$$ (25)

In (25) we introduce the following notations

$$\lambda_\infty = \lambda(\infty) = \frac{1}{3} + ac - c^2 = \frac{1}{3}(1 - \Omega^2 - i\varepsilon\Omega),$$

$$X(z) = \frac{1}{z} \exp V(z), \quad V(z) = \frac{1}{2\pi i} \int_0^1 \frac{\ln G(\tau) - 2\pi i}{\tau - z} d\tau,$$

$$G(\tau) = \frac{\lambda^+ (\tau)}{\lambda^- (\tau)}, \quad \lambda^\pm (\tau) = c^2 - ac - (\tau^2 - ac)\lambda_0^\pm (\tau),$$

$$\lambda_0^\pm (\tau) = \lambda_0 (\tau) \pm \frac{\pi}{2} i \tau, \quad \lambda_0 (\tau) = 1 + \frac{\tau}{2} \int_{-1}^1 \frac{d\tau'}{\tau' - \tau} = 1 + \frac{\tau}{2} \ln \frac{1 - \tau}{1 + \tau}.$$

If we calculate values of the left and right parts of the equation (25) in the point $z = i$, then for square of zero of dispersion function after some transformations we receive following expression

$$\eta_0^2 = -1 + \frac{\lambda(i)}{\lambda_\infty X(i)X(-i)} = -1 + \frac{\lambda(i)}{\lambda_\infty} \exp \left[- V(i) - V(-i)\right].$$
Considering, that $\lambda_0(i) = 1 - \frac{\pi}{4}$, we have

$$\lambda(i) = c^2 - ac + \left(1 - \frac{\pi}{4}\right)(1 + ac) = -\frac{1}{3}(\Omega^2 + i\varepsilon\Omega) + \left(1 - \frac{\pi}{4}\right)\left[1 + \frac{1}{3}(\varepsilon^2 - i\varepsilon\Omega)\right].$$

It is possible to present the function $X(z) = \frac{1}{z}e^{V(z)}$ in the following form

$$X(z) = \frac{1}{z - 1}e^{V_0(z)}, \quad V_0(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln G(\tau) d\tau}{\tau - z}.$$

Let us find the sum

$$V_0(i) + V_0(-i) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln G(\tau) d\tau}{\tau - i} + \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln G(\tau) d\tau}{\tau + i} =$$

$$= \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln G(\tau) d\tau}{\tau - i} = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\tau \ln G(\tau) d\tau}{\tau^2 + 1}.$$

By means of these formulas we will transform the formula for the square of zero of the dispersion functions to the form

$$\eta_0^2 = -1 + \frac{2\lambda(i)}{\lambda_\infty} \exp \left[ -\frac{1}{2\pi i} \int_{-1}^{1} \frac{\tau G_1(\tau) d\tau}{\tau^2 + 1} \right],$$

or

$$\eta_0^2 = -1 + \frac{2\lambda(i)}{\lambda_\infty} \exp \left[ \frac{i}{\pi} \int_{0}^{1} \frac{\tau G_1(\tau) d\tau}{\tau^2 + 1} \right], \quad (26)$$

where

$$G_1(\tau) = \ln \frac{(3\tau^2 - \varepsilon^2 + i\varepsilon\Omega)(\lambda_0(\tau) + \frac{\pi}{2}\tau i) + \Omega^2 + i\varepsilon\Omega}{(3\tau^2 - \varepsilon^2 + i\varepsilon\Omega)(\lambda_0(\tau) - \frac{\pi}{2}\tau i) + \Omega^2 + i\varepsilon\Omega}.$$

Now a relative error

$$O_2(\Omega, \varepsilon, d) = \left| \frac{G - G_2}{G} \right| \cdot 100\%$$
for sodium films of the thickness of 1 nanometer, 5 nanometers and 10 nanometers at $\omega = \omega_p$ and $\nu = 10^{-3}\omega_p$ accordingly equals: $O_2 = 0.575\%, 0.003\%$ and $0.0004\%$.

The graph of the relative error $O_2(\Omega, \varepsilon, d)$ as a function of the variable $\Omega$ for a film with the thickness of 10 nm at $\nu = 0.001\omega_p$ sec$^{-1}$ is represented on the Fig. 1.

![Graph](image-url)

Figure 1: Influence of continuous spectrum, $d = 10$ nm, $\nu = 0.001\omega_p$, $\theta = 75^\circ$.

The graph on Fig. 1 shows, that in the area $\omega \geq \omega_p$ the contribution to the electric field, corresponding to the continuous spectrum, is insignificant, and can be neglected. Thus, function $G$ is approximated by two composed Drude and Debaye, corresponding to discrete spectrum, according to (24).

Using the formulas (17) and (18), with the use of the expression (24) for functions $G$, we will carry out graphic research of the coefficients of transmission, reflecion and absorption.

6. Discussion of the results

Let us consider the case of thin sodium film. We will construct graphics of the dependences of transmission, reflecion and absorption on quantity $\Omega = \omega/\omega_p$ at pitch angle $\theta = 75^\circ$ (Figs. 2 – 10).
We will note, that near to a plasma resonance ($\omega \sim \omega_p$) the coefficient of transmission has a minimum, and reflection and absorption coefficients have minimum values. Under a thickness of the film of 1.5 nanometers and at $\nu = 0.05\omega_p$ in the areas of resonant frequencies ($\omega > \omega_p$) all coefficients have one maximum more. Under increase of the thickness of the film from 1.5 to 10 nanometers the second maximum vanishes.

For a film with the thickness of 5 nanometers and under $\nu = 0.02\omega_p$ in the field of resonant frequencies behaviour of all coefficients has so-called ”edge” character (”paling”). Under the further increase of the thickness of the film frequency of ”combs” increases also, and we can see growth of quantity of its ”teeths” (Figs. 5 – 10). If for a film with the thickness of 5 nanometers the quantity $\varepsilon = \nu/\omega_p$ decreases, the quantity of ”teeths” of combs grows sharply.
Figure 2: Transmittance, $\theta = 75^\circ$. Curves 1, 2, 3 correspond to values of parameters $d = 2$ nm, $\nu = 0.05\omega_p$; $d = 5$ nm, $\nu = 0.03\omega_p$; $d = 10$ nm, $\nu = 0.05\omega_p$.

Figure 3: Reflectance, $\theta = 75^\circ$. Curves 1, 2, 3 correspond to values of parameters $d = 2$ nm, $\nu = 0.05\omega_p$; $d = 5$ nm, $\nu = 0.03\omega_p$; $d = 10$ nm, $\nu = 0.05\omega_p$.

Figure 4: Absorptance, $\theta = 75^\circ$. Curves 1, 2, 3 correspond to values of parameters $d = 2$ nm, $\nu = 0.05\omega_p$ sec$^{-1}$; $d = 5$ nm, $\nu = 0.03\omega_p$; $d = 10$ nm, $\nu = 0.05\omega_p$. 
Figure 5: Transmittance, $d = 5 \text{ nm}, \nu = 0.001\omega_p, \theta = 75^\circ$.

Figure 6: Reflectance, $d = 5 \text{ nm}, \nu = 0.001\omega_p, \theta = 75^\circ$.

Figure 7: Absorptance, $d = 5 \text{ nm}, \nu = 0.001\omega_p, \theta = 75^\circ$. 
Figure 8: Transmittance, $d = 10 \text{ nm}$, $\nu = 0.001\omega_p$, $\theta = 75^\circ$.

Figure 9: Reflectance, $d = 10 \text{ nm}$, $\nu = 0.001\omega_p$, $\theta = 75^\circ$.

Figure 10: Absorptance, $d = 10 \text{ nm}$, $\nu = 0.001\omega_p$, $\theta = 75^\circ$. 
At the further increase of the thickness of a film frequency of teeths of the comb (number of links of paling) increases. Let us note, that Fig. 10 actually coincides with Fig. 3 of work [1], and Fig. 9 coincides with Fig. 2 of [1].

On Figs. 11 and 12 dependences of coefficients of transmission, reflection and absorption from the pitch angle $\theta$ of the electromagnetic waves on a film are presented. These graphics show monotonously decreasing character of the transmission coefficient as in area before resonant frequencies ($\omega < \omega_p$), and in the field of resonant frequencies as well. The coefficient of reflection in the field of resonant frequencies has one minimum for thin films (for example, for $d = 1$ nm and $\nu = 0.001\omega_p$ sec$^{-1}$).

On Fig. 13 we will show dependence of coefficients of transmission, reflection and absorption on quantity of a thickness of a film $d$, $1$ nm $\leq d \leq 100$ nm, by $\nu = 0.001\omega_p$ sec$^{-1}$, $\theta = 75^\circ$. Transmittance has one minimum in a considered range of thickness, and the absorptance has one maximum.

The formula (24) means, that periodic character of transmittance, reflectance and absorptance explaints under the presence function $\tanh(z_0/\eta_0)$ in the second member of this formula. This member is called Debye mode.

On Figs. 14 and 15 is shown, that transmittance, reflectance and absorptance have extrema in the same points $\Omega_n$, independently of quantity of the pitch angle of the electromagnetic wave. These reasons allow to find a thickness of a film on those points $\Omega_n = \omega_n/\omega_p$, in which coefficients $T$, $R$ and $A$ have an extremum.

Let us pass to a deducing of the formula for calculation of the thickness of a film in those points $\Omega_n$, in which coefficients of transmission, reflection and absorption have extrema. We consider coefficient of reflection. For this coefficient on Fig. 16 the first links of the comb represented earlier on Figs.
8 - 10 are considered. In this figure the dot curve corresponding to discrete and continuous spectrum, coincides with continuous curve, answering to discrete spectrum.

Points $\Omega_n$ in which the reflection coefficient has minimum, in accuracy coincide with points, in which function $\cos(\text{Re} \, iz_0/\eta_0)$ possesses the value of zero (see Fig. 16). From the equation $\cos(\text{Re} \, iz_0/\eta_0) = 0$ we find:

$$\text{Re} \left( \frac{i \cdot z_0(\Omega_n, \varepsilon, d)}{\eta_0(\Omega_n, \varepsilon)} \right) = \frac{\pi}{2} + \pi n, \quad n = 0, 1, 2, 3, \ldots,$$

or, in explicit form,

$$\text{Re} \left( \frac{i \cdot \omega_p \cdot 10^{-7} \cdot (\varepsilon - i \Omega_n)}{2v_F\eta_0(\Omega_n, \varepsilon)} d \right) = \frac{\pi}{2} + \pi n, \quad n = 0, 1, 2, 3, \ldots. \quad (27)$$

In the formula (27) the quantity $d$ (thickness of a film) is measured in nanometers. The exact formula for calculation of the thickness of a film on frequencies $\Omega_n$, $n = 0, 1, 2, \ldots$ is deduced from the formula (27). In the points $\Omega_n$, $n = 0, 1, 2, \ldots$ the reflection coefficient has local minima:

$$d = \frac{10^7\pi v_F(1 + 2n)}{\omega_p \text{Re} \left( \frac{\Omega_n + i\varepsilon}{\eta_0(\Omega_n, \varepsilon)} \right)}, \quad n = 0, 1, 2, 3, \ldots. \quad (28)$$

From Fig. 16 it is visible that at $n = 3$ $\Omega_3 = 1.025$. The formula (28) gives the thickness: $d = 9.968$ nm, i.e. an error in visual determination of the thickness of the film gives 0.3%.

7. Conclusion

In the present work for the thin films which thickness does not surpass thickness skin layer, the formulas for calculation of transmittance, reflectance and absorptance are received. The analysis of these coefficients is carried out. The formula for a finding of a thickness of a film by resonances in the field of resonant frequencies is deduced.
Figure 11: Transmittance, $d = 2$ nm, $\nu = 0.005\omega_p$. Curves 1, 2, 3 correspond to values of parameters $\omega = 0.9\omega_p$, $\omega = \omega_p$, $\omega = 1.01\omega_p$.

Figure 12: Reflectance, $d = 1$ nm, $\nu = 0.001\omega_p$. Curves 1, 2, 3, 4 correspond to values of parameters $\omega = 0.95\omega_p$, $\omega = \omega_p$, $\omega = 1.1\omega_p$, $\omega = 1.2\omega_p$.

Figure 13: Dependence of transmittance, reflectance and absorptance from film thickness $d$, $1$ nm $\leq d \leq 100$ nm, $\omega = \omega_p$, $\nu = 0.001\omega_p$. 
Figure 14: Transmittance, $d = 1$ nm, $\nu = 0.01\omega_p$. Curves 1, 2, 3 correspond to values of angle $\theta = 75^\circ, 60^\circ, 45^\circ$.

Figure 15: Absorptance, $d = 1$ nm, $\nu = 0.01\omega_p$. Curves 1, 2, 3 correspond to values of angle $\theta = 75^\circ, 60^\circ, 45^\circ$.

Figure 16: Reflectance, $d = 10$ nm, $\nu = 0.001\omega_p$. Periodic curve is set by equation $y = 0.8 + 0.25\cos(\text{Re}(i\omega_0/\eta_0))$. 
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