ON THE THERMODYNAMICS OF LAUGHLIN LIQUID FREEZING

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Abstract.

The competition between liquid and solid states of strongly correlated electron systems occurs in a novel way in a strong magnetic field. The fact that certain Landau level filling factors are especially favorable for the formation of strongly correlated liquid states, gives rise to the fractional quantum Hall effect. In this article we discuss some consequences of the existence of incompressible states with fractionally charged quasiparticle excitations for the thermodynamics of the liquid-solid transition.

1. INTRODUCTION

In the limit of strong magnetic fields electrons in two-dimensional (2D) systems are confined to the single-particle states of minimum quantized kinetic energy, i.e. to states in the lowest Landau level. In this limit all states of the system have the same kinetic energy so that the ground state, and for $k_B T \ll \hbar \omega_c$ also all thermodynamic properties, are determined entirely by electron-electron interactions. In this article we restrict our attention completely to the strong magnetic field limit where only the lowest spin-polarized Landau level is relevant.\textsuperscript{1} One single-particle state exists in the lowest Landau level for each magnetic flux quantum piercing the system and the electron density is usually measured in terms of the Landau level filling factor $\nu \equiv n\Phi_0/B = n\hbar c/eB$ where $n$ is the areal electron
density and $\Phi_0$ is the electron magnetic flux quantum. The lowest interaction energy state of 2D electrons is the Wigner\textsuperscript{2} crystal state in which electrons are localized at the sites of a triangular lattice. At zero field it is the kinetic-energy cost of localizing electrons which prohibits the formation of the crystal state except at extremely low densities. In a strong magnetic field, liquid and crystalline states of electrons compete in a different way; electrons cannot be localized to a length smaller than the cyclotron orbit radius corresponding to the quantized lowest kinetic energy, $\ell \equiv (\hbar c/eB)^{1/2}$. Only when $\ell$ is much smaller than the typical distance between electrons, i.e. only when $\nu \ll 1$ should the crystal state be expected to be the ground state. For weaker fields (larger filling factors) the ground state should be expected to be a liquid\textsuperscript{3,4}.

The discovery of the fractional quantum Hall effect\textsuperscript{5,6,7} and its subsequent explanation\textsuperscript{8} established the intricate and exotic nature of the electron liquid states which occur in the strong magnetic field limit. More recently data from transport experiments\textsuperscript{9}, and also from magneto-optical\textsuperscript{10} and frequency-dependent conductivity experiments\textsuperscript{11}, has accumulated\textsuperscript{12,13} which suggests the occurrence of the electron solid state. The experimental results are consistent with theoretical expectations that liquid states should tend to be favored near the filling factors where incompressible states occur and especially for $\nu$ near $1/m$ where $m$ is an odd integer. It is at these filling factors that the most stable incompressible liquid states, those first discovered by Laughlin\textsuperscript{8}, occur. Many aspects of the experiments which seem to indicate a Wigner crystal state remain poorly understood, as discussed elsewhere in this volume, and it seems likely that in real samples disorder must play an important role\textsuperscript{14} in determining the way in which Wigner crystallization is manifested. In this article we discuss only the ideal limit in which no disorder is present. We take the view\textsuperscript{15} that in this limit the transition between fractional Hall liquid states and the Wigner crystal state is strongly first order so that we can study the thermodynamics of the phase transition by comparing separate estimates of the filling factor and
temperature dependence of the free energies of liquid and solid states. In Section II we discuss the physics of fractional Hall liquid states for $\nu$ near $1/m$ and estimates of the temperature and filling factor dependence of the free energy which result. In Section III we briefly review what is known about the filling factor and temperature dependence of the free energy of the Wigner crystal state. Our considerations are based primarily on the harmonic approximation for the lattice state. The filling factor and temperature dependences of the free energy are very different for the two states and as we discuss in Section IV these differences suggest the occurrence of several unusual features in the phase boundary. In Section V we briefly summarize our findings.

2. THERMODYNAMICS OF THE LIQUID STATES

For pairs of electrons restricted to the lowest Landau level the possible relative motion states may be labelled by angular momentum $m = 0, 1, \cdots$. There is only one relative motion state of the pair for each value of $m$ and for the fully spin-polarized states of the electron gas assumed here electrons can be found only in states of odd relative angular momentum. The Hamiltonian of the system can be completely specified in terms of the interaction energies of pair of electrons with relative angular momentum $m$, $V_m$. For $N_\Phi \equiv AB/\Phi_0 = m(N - 1) + 1$ and any odd integer $m$ it is possible to show\textsuperscript{7} that there is only one state in the many-particle Hilbert space for which the amplitude for finding any pair of electrons in a state of relative angular momentum less than $m$ is zero. (Here $N$ is the number of electrons and $A$ is the area of the 2D system.) That state is the strongly-correlated electron liquid state discovered by Laughlin,\textsuperscript{8,16} $|\Psi_L\rangle$. Since electrons are closer together when they are in a state of lower relative angular momentum it follows, for sufficiently short-range repulsive interactions, that at this particular magnetic field the Laughlin state will be separated from all other states in the Hilbert space by an energy
gap $\Delta \sim V_{m-2}$. Similarly the chemical potential at zero temperature will jump from a value $\sim V_m$ to a value $\sim V_{m-2}$ when the filling factor increases beyond $\nu = 1/m$. The existence of the energy gap and the associated chemical potential discontinuity leads to the fractional quantum Hall effect. The chemical potential discontinuity must also lead to anomalies in the thermodynamic properties of the electron system at low temperatures for $\nu$ near $1/m$.

For $N_\phi = m(N-1) + 1 + N_{qh}$ the many-body states in which relative angular momenta less than $m$ can be completely avoided can be mapped to the many-body states for $N_{qh}$ fermion holes in a Landau level with a degeneracy equal$^{17}$ to $\tilde{N}_\phi = N + N_{qh} - 1$. The quasiholes must have fractional charge $e/m$ since $m$ of them are created when one electron is removed from the system at fixed magnetic fields. When the fractional quantum Hall effect occurs states in this class should be separated from higher energy states by a gap $\sim \Delta$. Similarly, it can be convincingly argued$^{18,19,20}$ from several different points of view that the low-energy states for $N_\phi = m(N-1) + 1 - N_{qe}$ can be mapped to the many-body states of $N_{qe}$ fractionally charged fermion particles in a Landau level with degeneracy $\tilde{N}_\phi = N + 1 - N_{qe}$. We refer to these excitations with fractional negative charge as quasielectrons and use the word quasiparticles to refer to the fractionally charged excitations generically. Note that when they are considered as fermions the quasiholes see both electrons and other quasiparticles as sources of effective magnetic flux with the flux from quasiparticles directed in opposition to that from the other sources. In the following we assume that this picture can be extended to apply to low-lying excitations as well as to the ground state.

We estimate the thermodynamic properties of the electron system by assuming that its elementary excitations are quasiparticle-quasihole pairs. Since we are interested in filling factors near $\nu = 1/m$ and low-temperatures we will assume that the quasiparticles are sufficiently dilute that we can neglect their mutual interactions. This approximation is adequate for most purposes$^{21}$ and as we shall see below some interesting non-trivial effects
exist in the distribution functions for these ideal gases even when quasiparticle interactions are neglected.

We consider the system at a constant magnetic field. Dropping terms \( \sim 1/N \) we define \( \tilde{N} \equiv N - N_0 \) where \( N_0 = N_\phi/m \) so that by total charge conservation \( N_{qe} = N_{qh} + m\tilde{N} \). The energies of the states of the system at a given \( N \) and hence at a given \( \tilde{N} \) may be expressed in terms of the number of quasielectrons and quasiholes:

\[
E(N_{qe}, N_{qh}) = \epsilon_0 N_0 + \epsilon_{qe} N_{qe} + \epsilon_{qh} N_{qh}. \tag{1}
\]

Here \( \epsilon_0 \) is the energy per electron of the incompressible state while \( \epsilon_{qe} \) and \( \epsilon_{qh} \) are the energies to make quasiholes and quasiparticles at fixed magnetic field. For the physically interesting case of Coulomb interactions between the electrons estimates\(^{22}\) exist for these three parameters: For \( m = 5 \) and in \( e^2/\ell \) units\(^{23}\) \( \epsilon_{0L} \approx -0.3277, \epsilon_{qh} \approx 0.1072, \) and \( \epsilon_{qe} \approx -0.076. \) (We use \( e^2/\ell \) units for energies throughout this paper.) Notice that the quasielectron-quasihole pair creation energy \( \Delta \equiv \epsilon_{qh} + \epsilon_{qp} \approx 0.031. \) This means for \( N_\phi = m(N-1) + 1 \) the liquid ground state is separated from excited states in this approximation by an excitation gap equal\(^{24}\) to \( \Delta. \) At zero temperature the upward discontinuity in the chemical potential is \( m\Delta \) at \( N_\phi = m(N-1) + 1. \)

We evaluate thermodynamic properties at finite temperature in the grand canonical ensemble. The grand partition function is

\[
Z_G = \exp(-\beta(\epsilon_0 - \mu)N_0) \sum_{\tilde{N}, N_{qe}, N_{qh}} \delta_{N_{qe} - N_{qh}, m\tilde{N}} \exp[\beta \mu N_{qe}/m - \beta F^{(0)}(N_{qe}, \tilde{N}_\phi, \epsilon_{qe})] \exp[-\beta \mu N_{qh}/m - \beta F^{(0)}(N_{qh}, \tilde{N}_\phi, \epsilon_{qh})], \tag{2}
\]

where \( \tilde{N}_\phi = N + N_{qh} - N_{qe} \) is the degeneracy of the quasiparticle and quasihole Landau levels. We see below that the dependence of the quasi-Landau-level degeneracies on the number of quasiholes and quasiparticles which are present on the chemical potential unusual. In Eq. (2) \( F^{(0)}(N, N_\phi, \epsilon) \) is the finite temperature free energy for \( N \) non-interacting
Fermi particles in a system with a single energy level of energy $\epsilon$ and degeneracy $N_\phi$, which is known exactly\textsuperscript{25}:

$$F^{(0)}(N, N_\phi, \epsilon) = N_\phi [\epsilon \nu + k_B T (\nu \ln(\nu) + ((1 - \nu) \ln(1 - \nu))] \equiv N_\phi f^{(0)}(\nu, \epsilon)$$  \hspace{1cm} (3)

where $\nu \equiv N/N_\phi$. To evaluate Eq. (2) we use the $\delta$ function to eliminate the sum over $\tilde{N}$. At a given $T$ and $\mu$ the most probable values of $N_{qe}$ and $N_{qh}$ can be determined by finding the extrema of the summand in Eq. (2). Setting the derivative with respect to $N_{qe}$ to zero gives

$$\mu = m\mu_{qe} - (m - 1)k_B T [\ln(1 - \nu_{qe}) + \ln(1 - \nu_{qh})]$$  \hspace{1cm} (4)

while setting the derivative with respect to $N_{qh}$ to zero gives

$$\mu = -m\mu_{qh} - (m - 1)k_B T [\ln(1 - \nu_{qe}) + \ln(1 - \nu_{qh})].$$  \hspace{1cm} (5)

Here $\nu_{qp} \equiv N_{qp}/\tilde{N}_\phi$ is the quasiparticle Landau level filling factor. The second terms on the right hand side of Eq. (4) and Eq. (5) come from the derivatives of the quasiparticle Landau level degeneracies with respect to the quasiparticle and quasihole numbers. In these equations we have defined

$$\mu_{qp} \equiv \frac{df^{(0)}(\nu_{qp}, \epsilon_{qp})}{d\nu_{qp}} = \epsilon_{qp} + k_B T \ln[\nu_{qp}/(1 - \nu_{qp})].$$  \hspace{1cm} (6)

so that the quasiparticle filling factors are related to their chemical potentials by the usual Fermi distribution function. Subtracting Eq. (4) and Eq. (5) we see that $\mu_{qe} = -\mu_{qh}$ so that, given $\mu$, $\mu_{qe}$ can be determined by requiring Eq. (4) to be satisfied when $\mu_{qh} = -\mu_{qe}$.

The electron filling factor $\nu \equiv N/N_\phi$ can be related to the quasiparticle filling factors:

$$\nu = \frac{1 + \nu_{qe} - \nu_{qh}}{m + (m - 1)(\nu_{qe} - \nu_{qh})}.$$  \hspace{1cm} (7)

We will be interested in comparing the Helmholtz free energy of the liquid at a particular filling factor with that of the solid. Noting that fluctuations become negligible in the thermodynamic limit we find from Eq. (2) that

$$F \equiv -k_B T \ln Z_G + \mu N = \epsilon_0 N_0 + \tilde{N}_\phi [f^{(0)}(\epsilon_{qe}, \nu_{qe}) + f^{(0)}(\epsilon_{qh}, \nu_{qh})]$$  \hspace{1cm} (8)
so that the free energy per electron is

$$f \equiv \frac{\epsilon_0}{m\nu} + (\nu^{-1} - (m - 1))[f^{(0)}(\epsilon_{qe}, \nu_{qe}) + f^{(0)}(\epsilon_{qh}, \nu_{qh})]. \tag{9}$$

To determine $\nu_{qe}$ and $\nu_{qh}$ at a given filling factor and temperature we express the right hand side of Eq. (7) in terms of the quasiparticle energies and $z_{qe} \equiv \exp[\mu_{qe}/(k_B T)] = \exp[-\mu_{qh}/(k_B T)]$ and solve for its value. Given $z_{qe}$ we can easily calculate $\nu_{qe}$, $\nu_{qh}$ and the free energy. Some typical results are shown in Fig.[1]. The most noteworthy feature in this figure is related to the entropy,

$$S = -\frac{\partial F}{\partial T}. \tag{10}$$

At $\nu = 1/m$ the ground state is non-degenerate and separated from excited states by the gap $\Delta$. It follows that the entropy vanishes like $\exp[-\Delta/(2k_B T)]$ at low temperatures so that the free energy, which must decrease monotonically with temperature, is nearly constant. For $\nu \neq 1/m$ the ground state is degenerate as discussed near the beginning of this section, and the entropy approaches a constant. For $\nu < 1/m$, the degeneracy of the many-body ground state is

$$g = \binom{N + N_{qh}}{N_{qh}} \tag{11}$$

so that the entropy at zero temperature in the thermodynamic limit is

$$\frac{S}{Nk_B} = (1 + \nu^{-1} - m) \ln[1 + \nu^{-1} - m] - (\nu^{-1} - m) \ln[\nu^{-1} - m]. \tag{12}$$

Similarly the zero temperature entropy for $\nu > 1/m$ is

$$\frac{S}{Nk_B} = (m - \nu^{-1}) \ln[(m - \nu^{-1})^{-1} - 1] - (1 - 2(m - \nu^{-1})) \ln[(1 - 2(m - \nu^{-1}))/((1 - (m - \nu^{-1}))]. \tag{13}$$

The finite zero-temperature free energy gives rise to the linear decrease in free energy with increasing temperatures which is evident at low temperatures in Fig.[1]. In a more realistic model the entropy would drop to zero on a temperature scale reflecting quasiparticle
interactions\textsuperscript{26}; however these interactions would not greatly alter the comparisons between liquid and solid free energies which we make below.

3. THERMODYNAMICS OF THE HARMONIC ELECTRON SOLID

In this section we briefly review known results for the thermodynamics of the harmonic 2D Wigner crystal in the strong-magnetic-field limit. In using the harmonic approximation for the electron crystal we are ignoring the possibility of large departures of the electrons from their lattice sites. The use of this approximation is consistent with our assumption that the phase-transition is strongly first order since the harmonic approximation can then still be valid when the solid melts. We remark that the harmonic approximation is still useful for estimating thermodynamic properties at low temperatures even though true long-range-order in the Wigner crystal cannot exist at finite temperatures.

Adding the Lorentz force to the classical equations of motion it is easy to demonstrate that the harmonic phonon energies in the strong magnetic field limit are related to the zero-field energies\textsuperscript{27} by

$$
ε_+(\vec{q}) = \hbar \omega_c + \sum_{λ} \epsilon^2_λ(\vec{q})/2\hbar \omega_c
$$

(14)

and

$$
ε_-(\vec{q}) = \prod_{λ} \epsilon_λ(\vec{q})/\hbar \omega_c.
$$

(15)

(See for example Ref.[28]. Here $ε_λ$ is a zero-field phonon energy.) At long wavelengths the zero-field resonance frequencies are a purely longitudinal plasmon mode, $ε^2_{pl}(q) = 2π\hbar^2 ne^2q/m^*$ and a linearly dispersing transverse mode $ε_t = c_t q$. It follows that $ε_+(q) = \hbar \omega_c + ε^2_{pl}/2\hbar \omega_c$ while $ε_-(q) = c_t qε_{pl}(q)/\hbar \omega_c \sim q^{3/2}$.

At general wavelengths the zero-field phonon energies can be expressed in the form\textsuperscript{29}

$$
ε_{λ,\vec{q}} = \left(\frac{\hbar^2 e^2(2πn)^{3/2}}{m^*}\right)^{1/2} \tilde{ε}(\vec{q})
$$

(16)
where \( \epsilon(\tilde{q}) \) is a readily calculable pure number, \( \tilde{q} \equiv n^{1/2}q \), and \( n \) is the areal density of the electron system. It follows that at strong fields

\[
\epsilon_+(\tilde{q}) = \hbar \omega_c + \frac{1}{2}(e^2/\ell)\nu^{3/2} \sum_{\lambda} \tilde{\epsilon}_\lambda^2(\tilde{q}) \equiv \hbar \omega_c + (e^2/\ell)\nu^{3/2}\tilde{\epsilon}_+(\tilde{q})
\]

(17)

and

\[
\epsilon_-(\tilde{q}) = (e^2/\ell)\nu^{3/2} \prod_{\lambda} \tilde{\epsilon}_\lambda(\tilde{q}) \equiv (e^2/\ell)\nu^{3/2}\tilde{\epsilon}_-(\tilde{q})
\]

(18)

Note that the low energy modes are independent of the electron mass and therefore must correspond to intra-Landau-level excitations of the system. For \( k_B T \ll \hbar \omega_c \) the mean harmonic-oscillator quantum number is zero for the high-frequency cyclotron mode so that the internal energy is given by 

\[
E = N\hbar \omega_c/2 + E_{mad} + N(e^2/\ell)\nu^{3/2}e(t)
\]

where

\[
e(t) = \frac{1}{2N} \sum \tilde{q} [\tilde{\epsilon}_+(\tilde{q}) + (2n(\tilde{\epsilon}_-(\tilde{q})/t) + 1)\tilde{\epsilon}_-(\tilde{q})],
\]

(19)

\( E_{mad} \) is the classical point lattice energy, \( n(x) = (\exp(x) + 1)^{-1} \) and \( t \equiv k_B T/(e^2/\ell)\nu^{3/2} \).

Similarly the entropy has contributions only from the intra-Landau-level phonon modes and depends on temperature only through the dimensionless parameter \( t \); 
\( S = Nk_B s(t) \) where \( s(t) \) is readily evaluated numerically. In Fig.[2], Fig.[3], and Fig.[4] we plot results for the specific heat \( (C_V = Nk_B e'(t)) \), the entropy and the Helmholtz free energy \( (F = N\hbar \omega_c/2 + N(e^2/\ell)(-0.78213\nu^{1/2} + \nu^{3/2}f(t)) \) respectively. Here \( f(t) = e(t) - ts(t) \) and the second term in the Free energy is the Madulung energy. All quantities are evaluated for the triangular electron lattice which has the lowest free energy over the temperature range of interest. In the next section we use this free energy to construct an estimate of the shape of the liquid-solid phase boundary near an incompressible filling factor. The free-energy of the solid, unlike that of the liquid, has a smooth dependence on filling factor. At low temperatures the vibrational contribution to the free-energy per electron is \( f(t) \sim -9(e^2/\ell)\nu^{3/2}t^{7/3} \); this power law follows from the \( q^{3/2} \) dispersion of the intra-Landau-level mode and should be compared to the linear temperature dependence which
occurs except at \( \nu = 1/m \) for the liquid. Thus finite temperature effects should generally be expected to favor the liquid over the solid, as is commonly the case. An exception occurs for \( \nu \) very close to \( 1/m \) where there are few low energy excitations for the liquid.

When leading anharmonic corrections are included the ground state energy of the Wigner crystal is given by

\[
\epsilon_S = N\hbar\omega_c/2 + N(e^2/\ell)[-0.78213\nu^{1/2} + 0.2410\nu^{3/2} + 0.087\nu^{5/2}]
\] (20)

In Eq. (20) the first term in square brackets is the energy of the classical Wigner lattice, the second term is the harmonic zero point energy, and the last term comes from anharmonic corrections. (The second term should be compared with the zero-temperature limit of the vibrational free energy shown in Fig.[4]). We see that for \( \nu \sim 1/5 \), the region of filling factor we focus on below, anharmonic corrections are \( \sim 10^{-3}(e^2/\ell) \) per particle. A similar uncertainty exists in estimates of the energy per particle of the incompressible state. Below we take the attitude that the difference between the ground state energies of liquid and solid states at \( \nu = 1/m \) cannot be determined with sufficient accuracy and may in practice be altered somewhat by changes in the quantum confinement width of the two-dimensional electron gas. We will treat this difference as a phenomenological parameter.

4. THE LIQUID-SOLID PHASE BOUNDARY

In Fig.[5] we plot the filling factor dependences of the free-energies for liquid and solid states obtained using the approximations discussed in the previous two sections. These curves were obtained assuming that the temperature dependence of the small anharmonic contribution to the energy of the solid can be neglected. The ground state energy of the liquid state has the cusp responsible for the fractional quantum Hall effect at \( \nu = 1/m \). The ground state energy of the liquid is lower than that of the solid for a finite range of filling.
factors around $\nu = 1/m$. Well away from $\nu = 1/m$ the degeneracy of the liquid ground state causes the free-energy to decrease more rapidly for the liquid than for the solid. We have constructed the phase diagram for our model of the liquid-solid phase transition by mapping out the filling factors at which the free-energy differences cross zero. The results are shown in Fig.[6]. There is qualitative agreement between these phase boundaries and those mapped out by various experimental probes$^{13}$ of the two-dimensional electron gas system in a strong magnetic field in this regime of filling factors. We emphasize that this level of agreement is achieved without any adjustable parameters.

The temperature scale in Fig.[6] should be compared with the classical melting temperature of the Wigner solid which is $T_M \sim 6 \times 10^{-2} \nu_{1/2} (e^2/\ell k_B)$. Thus, in our approximation, the solid phase persists to temperatures well above the classical melting temperature. This behaviour seems to be improbable, although experimental fingerprints of the solid phase do also seem to persist beyond the classical melting temperature. The temperature scale in our figures is strongly sensitive to the quasiparticle-quasihole creation energy and could be adjusted downward by adjusting $\Delta$. Our results are also strongly sensitive to the ground state energy difference between the liquid and solid states at $\nu \equiv 1/m$. This difference is obtained by subtracting two quantities which have a relative difference of only several parts per thousand and must be regarded as highly uncertain. It may even be altered importantly by changes in the effective interaction between electrons due to differences in the electronic width of a heterojunction or quantum well between different samples.

For that reason we plot in Fig.[7a-e] a series of phase diagrams obtained by arbitrarily adjusting the ground state energy of the liquid at $\nu = 1/m$ so that it passes through our approximate value of the ground state energy of the solid at this filling factor. Members of this sequence of phase diagrams should also apply qualitatively, with suitable downward adjustments to the temperature scale, to the shape of the phase boundary for $\nu$ near $1/m$ at larger values of $m$ since the ground state energy difference of the two states is expected
to cross zero\textsuperscript{4,30} with increasing \(m\). As the ground state energy of the liquid is increased the width of the filling factor interval where the liquid is stable at \(T = 0\) decreases. At the same time the \(T = 0\) entropy of the liquid at the position of the phase boundary decreases so that the effect of increasing temperature in expanding the width of the liquid stability interval is diminished. This is seen in Fig.[7a] and Fig.[7b] where the phase boundary lines become more vertical. When the ground state energy is raised still further the entropy of the solid along the phase boundary can increase above that of the liquid as temperature increases so that, in accordance with the Clausius-Clapeyron equation\textsuperscript{31}, the width of the liquid stability interval decreases with temperature. As we see in Fig.[7c] and Fig.[7d] the stability interval can decrease to zero so that the solid becomes stable at \(\nu \equiv 1/m\) over some finite temperature interval before melting again. Finally when the ground state at \(\nu = 1/m\) is raised above that of the solid, as in Fig.[7e], the solid will melt at a temperature which is minimized at \(\nu = 1/m\). It seems likely that this is typically the situation for \(\nu = 1/9\) and may often be the case for \(\nu = 1/7\). The behavior at a particular filling factor in a particular system may depend on the degree of Landau level mixing \textit{i.e.}, on the electron density\textsuperscript{30}, as well as on geometric details of particular systems which influence effective electron-electron interactions.

5. SUMMARY

In this article we have taken the view that the transition between Wigner crystal and fluid states in two-dimensions in a strong magnetic field is strongly first order. The thermodynamics of the phase-transition is then dominated by the same anomaly which is responsible for the fractional quantum Hall effect; namely the existence of incompressible states. When interactions between fractionally charged quasiparticles are neglected the entropy of the liquid has a finite contribution at zero-temperature which vanishes as the
incompressible filling factor is approached. As a result the shape of the liquid-solid phase boundary changes qualitatively as the filling factors along the phase boundary approach the incompressible filling factor. When the ground state energies of the Wigner solid and the incompressible liquid are nearly identical the liquid can freeze and remelt with increasing temperature.
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FIGURES

Fig.[1] Free energy versus temperature in a fractional Hall liquid for $\nu$ near $1/m$ with $m = 5$.

Fig.[2] Temperature dependence of the specific heat of the harmonic Wigner lattice in the strong magnetic field limit.

Fig.[3] Temperature dependence of the entropy of harmonic Wigner lattice in the strong magnetic field limit.

Fig.[4] Temperature dependence of the free energy of the harmonic Wigner lattice in the strong magnetic field limit. The dashed line shows the temperature dependence of the energy.

Fig.[5] Free energy difference between liquid and solid states as a function of filling factor for a series of temperatures. The temperatures listed are in units of $e^2/\ell k_B$.

Fig.[6] Phase boundary between Wigner solid and fractional Hall liquid states of the two-dimensional electron gas at strong magnetic fields near $\nu = 1/5$.

Fig.[7] Phase boundaries between Wigner solid and fractional Hall liquid states of the two-dimensional electron for arbitrarily adjusted incompressible ground state energies.