Nonlinear Spencer operators on differentiable groupoids

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Abstract

We construct the first, second and sophisticated non-linear and linear Spencer complexes for a differentiable Lie groupoid $G$. To do this, we extend the diagonal calculus, as applied by Malgrange to the groupoid $M \times M$, to the context of $I \times G$, where $I$ is the manifold of identities of $G$.

1 Introduction

In [S], D. C. Spencer studying deformation of structures defined by transitive pseudo groups, introduced a nonlinear complex associated to the pseudogroup. There was work in the following years to simplify the introduction of this complex, as in Van Quê [N], Kumpera and Spencer [KS] and B. Malgrange [Ma1], [Ma2]. The same ideas in the context of G-structures were developed by Guillemin and Sternberg [GuS]. Malgrange obtained a construction for the nonlinear Spencer complex through the diagonal calculus of Grothendieck for the groupoid $M \times M$, where $M$ is a differentiable manifold. An open problem is to define the nonlinear Spencer complex for the groupoid $G_k$ of k-jets of bisections of a differentiable groupoid $G$ (see the Appendix of [KS]). Our goal in this paper is to define this complex and introduce its properties. For this, we explore the relationship between vector fields on $I \times G$ and the actions of bisections of $G_{k+1}$ on sections of the algebroid $J^k g$ associated to the groupoid $G_k$. Here $I$ is the submanifold of identities of $G$ and $g$ is the algebroid associated to $G$.

B. Malgrange in [Ma1], [Ma2] considered the groupoid $M \times M$ and $Q^k(M)$ the groupoid of k-jets of bisections of $M \times M$, and the algebroid $J^k TM$ associated to $Q^k(M)$. A section of $J^k TM$ is identified to the quotient of a $\pi_1$ vertical vector field on $M \times M$ module the vector fields that are null up to order $k$ on the diagonal of $M \times M$. This identification is possible because $M \times M$ is a transitive groupoid, and a right invariant vector field on the $s$-fiber, $s$ the source of $G$, extends uniquely as a right invariant vector field all over the groupoid $M \times M$. When the groupoid is not transitive, the knowing of a right invariant vector field on the s-fiber is insufficient to extend it to all the groupoid $G$. To utilize the technique devised by Malgrange in the intransitive case we must consider right invariant vector fields on $I \times G$ by the right action of $G$ on $I \times G$ given by $(x, X).Y = (x, X.Y)$. The diagonal of $M \times M$ is replaced by the diagonal $\Delta \subset I \times I \subset I \times G$. Let be $\mathcal{R}$ the sheaf of germs of vector fields on $I \times G$ that are $\rho_1$ projectables and right invariants ($\rho_1$ and $\rho_2$ are the projections of $I \times G$ on $I$ and $G$ respectively). A section of $J^k g$ identifies with a $\rho_1$ vertical field of $\mathcal{R}$ module the vector fields in $\mathcal{R}$ that are null up to order $k$ on $\Delta$. The $\rho_2$ vertical vector fields in $\mathcal{R}$ identifies with $\mathcal{T}$ the sheaf of sections of the tangent space $T = TI$ of $I$. Thus


\( J^k g = T \oplus J^k g \) identifies to the sheaf \( R \) module the sub sheaf of \( R \) that are null up to order \( k \) on \( \Delta \). In \cite{Ma1} \( J^k g \) is obtained as sum of \( J^k g \) and \( J^k g_1 \). Here \( J^k g \) is the quotient of vector fields in \( R \) tangents to the submanifold \( \{(t(X), X) \in I \times G | X \in G\} \), where \( t \) is the target of \( G \). The action of bisections of \( G_{k+1} \) on \( T \) was obtained through the actions on \( J^k g \) and \( J^k g_1 \). When \( G \) is intransitive, \( J^k g \) is not this sum. It is necessary to do a direct calculus to obtain the action of bisections of \( G_{k+1} \) on \( T \) which is done in Proposition 3.5.

For another approach to Lie groupoids and algebroids, and Spencer operators, see \cite{CSS}. A more recent version of Malgrange construction is in \cite{V}.

We resume briefly the content of each section. In section 2 we introduce the basic definitions of groupoids and algebroids, and the actions of jets of admissible sections on the jets of sections of tangent spaces. Section 3 is the core section, where we introduce the diagonal calculus and the Lie algebra sheaf \( \wedge (J^\infty g)^* \otimes (J^\infty g) \). In section 4 we introduce the first linear and non-linear Spencer complexes, through the introduction of sub-sheaf \( \wedge T^* \otimes J^\infty g \), and give the basic properties of these complexes. In a similar way, in section 5 we introduce the second linear and non-linear Spencer complexes, through the introduction of sub-sheaf \( \wedge \tilde{T}^* \otimes \tilde{J}^\infty g \). Finally, in section 6 we introduce the sophisticated linear and non-linear Spencer complexes with their properties.

This paper is dedicated to the memory of Alexandre Martins Rodrigues.

\section{Preliminaires}

\subsection{Groupoids and algebroids}

\textbf{Definition 2.1} A differentiable groupoid \( G \) is a differentiable manifold \( G \) with a regular submanifold \( I \), two submersions \( s, t : G \to I \) with \( s^2 = s, t^2 = t \), and the following operations on \( G \):

1. There exists a differentiable operation called \textit{composition} in \( G \),

\[
\begin{align*}
(s \times t)^{-1}(\Delta) & \to G, \\
(Y, X) & \to YX
\end{align*}
\]

where \( \Delta = \{(x, x) : x \in I\} \) with the following properties:

(a) if \( (Y, X), (Z, Y) \in (s \times t)^{-1}(\Delta) \) then \( (Z, YX), (ZY, X) \in (s \times t)^{-1}(\Delta) \) and \( Z(YX) = (ZY)X \);

(b) \( Xs(X) = t(X)X = X \).

2. There exists a diffeomorphism \( \iota \) of \( G \) called \textit{inversion}

\[
\begin{align*}
\iota : G & \to G, \\
X & \to X^{-1}
\end{align*}
\]

such that \( (X, X^{-1}), (X^{-1}, X) \in (s \times t)^{-1}(\Delta) \), and \( X^{-1}X = s(X), XX^{-1} = t(X) \).

The projections \( s \) and \( t \) are called of source and target respectively.
EXAMPLE. If $M$ is a differentiable manifold, $M \times M$ is a differentiable groupoid with $s = (\pi_2, \pi_2)$, $t = (\pi_1, \pi_1)$ and operations $(z, y)(y, x) = (z, x)$ and $(y, x)^{-1} = (x, y)$.

EXAMPLE. Let $(E, M, \pi)$ be a differentiable vector bundle and $P$ the set of linear isomorphisms between the fibers of $E$. This means that $X \in P$ if $X : E_a \to E_b$ is a linear isomorphism. The composition and inversion in $P$ are the composition and inversion of linear transformations. The identities are the identities $I_a$ in each $E_a$. Therefore the manifold $I$ of identities is diffeomorphic to $M$. The source map $s$ is defined by $s(X) = I_a$ and the target map $t$ by $t(X) = I_b$ for $X : E_a \to E_b$.

We denote by $G(x) = s^{-1}(x)$ the $s$-fiber of $G$ on $x \in I$; by $G(\cdot, y) = t^{-1}(y)$ the $t$-fiber of $G$ on $y \in M$ and $G(x, y) = G(x) \cap G(\cdot, y)$. The set $G(x, y)$ is a group, the so called isotropy group of $G$ at point $x$. If $U, V$ are open sets of $I$, we introduce the notations $G(U) = \cup_{x \in U} G(x)$, $G(\cdot, V) = \cup_{y \in V} G(\cdot, y)$, and $G(U, V) = G(U) \cap G(\cdot, V)$.

A (differentiable) section $F$ of $G$ defined on an open set $U$ of $M$ is a differentiable map $F : U \to G$ such that $s(F(x)) = x$. If $t(F(U)) = V$ and $f = t \circ F : U \to V$ is a diffeomorphism, we say that the section $F$ is a bisection. We write $U = s(F)$, $V = t(F)$ and $t \circ F = tF$.

We denote by $\mathcal{G}$ the set of bisections of $G$. Naturally $\mathcal{G}$ has a structure of groupoid. If $F, H \in \mathcal{G}$ with $t(F) = s(H)$, then $HF(x) = H(f(x))F(x)$ and $F^{-1}(y) = F(f^{-1}(y))^{-1}$, $y \in t(F)$ where $f = tF$.

2.2 Actions on $TG$

A bisection $F$, with $s(F) = U$, $t(F) = V$, defines a diffeomorphism

$$\tilde{F} : G(\cdot, U) \to G(\cdot, V) \quad \text{with} \quad X \mapsto F(t(X))X.$$ 

The differential $\tilde{F}_*: TG(\cdot, U) \to TG(\cdot, V)$ depends, for each $X \in G(\cdot, V)$, only of $j^1_{t(X)}F$. This defines an action

$$j^1_{t(X)}F : TXG \to T_F(t(X))XG \quad v \mapsto j^1_{t(X)}F \cdot v = (\tilde{F}_*)_X(v).$$ (1)

The application $\mathbb{1}$ defines a left action of the set $G_1$ of 1-jets of bisections of $G$ on $TG$

$$G_1 \times TG \to TG \quad (j^1_{t(X)}F, v \in TXG) \mapsto j^1_{t(X)}F \cdot v \in T_F(t(X))XG.$$ (2)

If $V_t \subset TG$ denotes the sub vector bundle of $t_*$ vertical vectors, then the action $\mathbb{2}$ depends only on $F(t(X))$:

$$X \times V_t \to V_t \quad (F(t(X)), v \in (V_t)_X) \mapsto F(t(X)) \cdot v \in (V_t)_{F(t(X))X}.$$ 

In a similar way, $F$ defines a right action which is a diffeomorphism

$$\overline{F} : G(V) \to G(U) \quad X \mapsto XF(f^{-1}s(X)).$$ (3)

The differential $\overline{F}_*$ of $\overline{F}$ induces the right action

$$TG \times G_1 \to TG \quad (v \in TXQ^*_F, j^1_{f^{-1}s(X)}F) \mapsto v \cdot j^1_{f^{-1}s(X)}F = (\overline{F}_*)_X(v).$$ (4)

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As $t(YX) = t(Y)$, it follows that

$$t_*(v \cdot j_{f^{-1}(s(X))} F) = t_*(v),$$

where $t_* : TG \to TI$ is the differential of $t : G \to I$. We deduce from (3) that the function $F$ restricted to the $s$-fiber $G(y)$ depends only on the value of $F$ in $f^{-1}(y)$. If $VG = \ker s_*$, then the right action (4) depends only on the value of $F$ at each point, and by restriction gives the action

$$VG \times G \to VG$$

$$(v \in V_X G, Y \in G(\cdot, s(X))) \mapsto v \cdot Y \in V_{XY} G.$$  \hspace{1cm} (5)

A vector field $\xi$ on $G$ with values in $VG$ is said right invariant if $\xi(YX) = \xi(X) \cdot Y$. The vector field $\xi$ is determined by its restriction $\xi$ to $I$.

If $\xi$ is a section of $V$ on $U \subset I$, let be

$$\xi(X) = \xi(t(X)) \cdot X,$$

the right invariant vector field on $G(\cdot, U)$. Then $\xi$ has $F_u$, $-\epsilon < u < \epsilon$, as one parameter group of diffeomorphisms induced by bisections $F_u$ of $G$ such that

$$\frac{d}{du} F_u|_{u=0} = \xi.$$

Therefore, $F_0 = I$ and

$$\frac{d}{du} F_u(x)|_{u=0} = \xi(x).$$

**Definition 2.2** The vector bundle $g = VG|_I$ on $I$ is the (differentiable) algebroid associated to the groupoid $G$.

Given sections $\xi, \eta$ of $s : g \to I$ defined on an open set $U$ of $I$, it is well defined the Lie bracket $[\cdot, \cdot]$ on local sections of $g$, given by

$$[\xi, \eta] = [\xi, \eta]|_I.$$  \hspace{1cm} (6)

**Proposition 2.1** If $f$ is a real function on $U$, $\xi, \eta$ sections of $g$ on $U$, then

$$[f \xi, \eta] = f[\xi, \eta] - (t_* \eta)(f) \xi.$$

**Proof.** As $f \xi = (f \circ t) \xi$, it follows

$$[f \xi, \eta] = [(f \circ t) \xi, \eta]|_I = ((f \circ t) \xi, \eta]|_I = (f \xi, \eta) - (t_* \eta)(f) \xi.$$  \hspace{1cm} \square

If $\Gamma(g)$ denotes the sheaf of germs of local sections of $g$, then $\Gamma(g)$ is a Lie algebra sheaf, with the Lie bracket $[\cdot, \cdot]$.

**Proposition 2.2** If $t$ denotes the inverse in $G$ and if $\xi \in g_x$, then

$$t_*(\xi) = t_*(\xi) - \xi.$$
Proof. Consider $X_\epsilon$ a curve in $G_x$ such that $\frac{d}{d\epsilon}X_\epsilon|_{\epsilon=0} = \xi$. Denote by $f(\epsilon) = t(X_\epsilon)$. The composition $X_\epsilon.(X_\epsilon)^{-1} = I_{f(\epsilon)}$ so

$$\frac{d}{d\epsilon}(X_\epsilon.X_\epsilon^{-1})|_{\epsilon=0} = \frac{d}{d\epsilon}I_{f(\epsilon)}|_{\epsilon=0}.$$ 

Observe that $\xi$ is $s$-vertical and $\iota_\ast(\xi)$ is $t$-vertical. Therefore

$$\frac{d}{d\epsilon}X_\epsilon|_{\epsilon=0}.I_x + I_x.\frac{d}{d\epsilon}X_\epsilon^{-1}|_{\epsilon=0} = \frac{d}{d\epsilon}I_{f(\epsilon)}|_{\epsilon=0}$$

and we get

$$\xi + \iota_\ast(\xi) = t_\ast(\xi).$$

\[\square\]

### 2.3 Right invariant diffeomorphisms of $G$

**Definition 2.3** A diffeomorphism $F : G \to G$ is right invariant if $F(XY) = F(X)Y$ for every $(X,Y) \in (s \times t)^{-1} \Delta$.

Every right invariant diffeomorphism is defined by a bisection $\sigma : I \to G$ defined as $\sigma(x) = F(I_x)$. In fact

$$F(X) = F(I_{t(X)}X) = F(I_{t(X)})X = \sigma(t(X))X.$$ 

From this formula it follows that $F$ takes $G(.,t(X))$ on $G(.,f(t(X)))$ where $f = tF$. As $F$ is a diffeomorphism, the function $f$ must be a diffeomorphism and $\sigma$ must be a bisection.

The composition of right invariant diffeomorphisms is a right invariant diffeomorphism. In fact, if $\tilde{F}$ is a right invariant diffeomorphism such that $\tilde{F}(Y) = \tilde{\sigma}(t(Y))Y$, then

$$\tilde{F} \circ F(X) = \tilde{F}(F(X)) = \tilde{\sigma}(t(F(X)))F(X) = \tilde{\sigma}(t(\sigma(t(X))))\sigma(t(X))X = \tilde{\sigma} \circ \sigma(t(X))X.$$ 

**Proposition 2.3** If $v$ is a vector field tangent to $I$, and $\sigma \in G$ is such that $t\sigma = f$, then

$$j^1\sigma. v. j^1\sigma^{-1} = f_\ast(v).$$

Proof. Posing $v = \frac{d}{du}x_u|_{u=0}$, we obtain

$$j^1\sigma. v. j^1\sigma^{-1} = \frac{d}{du}(\sigma(x_u).\sigma^{-1}(f(x_u)))|_{u=0} = \frac{d}{du}f(x_u)|_{u=0} = f_\ast v,$$

\[\square\]
2.4 Groupoids and algebroids of jets

Let be $G$ a differentiable groupoid and $G_k$ the manifold of $k$-jets of local bisections of $G$. This manifold has a natural structure of Lie groupoid given by composition of jets

$$j^k_{f(x)} H j^k_x F = j^k_x (HF),$$

and inversion

$$(j^k_x F)^{-1} = j^k_{f(x)} F^{-1},$$

where $F : U \to G$, $H : V \to G$ are local bisections of $G$, $f = tF$, $V = f(U)$ and $x \in U$. The groupoid $G_k$ has a natural submanifold of identities $I_k = j^k I$, where $I$ is the identity section of $G$. We have a natural identification of $I$ with $I_k$, given by $I(x) \mapsto I_k(x)$. Therefore we can think of $I$ as a submanifold of $G_k$. There are two submersions $s, t : G_k \to I$, the canonical projections source, $s(j^k_x F) = x$, and target, $t(j^k_x F) = f(x)$. We also consider $s$ and $t$ with values in $I$, by the above identification of $I_k$ with $I$.

There are natural projections $\pi^k_l = \pi_t : G_k \to G_l$, for $l \geq 0$, defined by $\pi_l(j^k_x F) = j^l_x F$. Observe that $G_0 = G$. The projections $\pi^k_l$ commute with the operations of composition and inversion in $G_k$.

A bisection $F_k$ of $G_k$ is holonomic if there exist a bisection $F$ of $G$ such that $F_k = j^k F$. Therefore, if $F_k$ is holonomic, we have $F_k = j^k(\pi_0 F_k)$.

**Definition 2.4** The vector bundle $\mathfrak{g}_k = VG_k|_I$ on $I$ is the (differentiable) algebroid associated to the groupoid $G_k$.

The vector bundle $J^k \mathfrak{g}$ is a vector bundle on $I$, and we also denote by $\pi : J^k \mathfrak{g} \to I$ the map $\pi(j^k_x \theta) = x$, where $\theta : U \subset I \to \mathfrak{g}$ is a local section. If $f_t$ is the 1-parameter group of local diffeomorphisms of $G$ such that $\frac{d}{dt}f_t|_{t=0} = \theta$, then we get, for $x \in U$,

$$\frac{d}{dt} j^k_x f_t|_{t=0} = j^k_x \theta.$$

This means we have a natural identification

$$\mathfrak{g}_k = J^k \mathfrak{g}.$$

**Proposition 2.4** The bracket $[ , ]_k$ on $\Gamma(\mathfrak{g}_k)$ is determined by:

(i) $[j^k_x \xi, j^k_x \eta]_k = j^k_x [\xi, \eta]$, $\xi, \eta \in \Gamma(\mathfrak{g})$

(ii) $[\xi_k, f \eta_k]_k = f[\xi_k, \eta_k]_k + (t*\xi_k)(f)\eta_k$,

where $\xi_k, \eta_k \in \Gamma(\mathfrak{g}_k)$, $f$ a real function on $I$.

Let be $T = TI$ the tangent bundle of $I$, and $\mathcal{T}$ the sheaf of germs of local sections of $T$. Therefore, as $TG_k|_I = TI \oplus VG_k|_I$,

$$TG_k|_I \cong T \oplus \mathfrak{g}_k,$$

and if we denote by

$$\tilde{J}^k \mathfrak{g} = T \oplus J^k \mathfrak{g},$$

$$\mathfrak{g}_k = (\tilde{J}^k \mathfrak{g})|_I.$$
then $TQ^k|_I \cong J^k\mathfrak{g}$. Observe that $J^k\mathfrak{g}$ is a vector bundle on $I$. The restriction of $t_* : TG_k \to T$ to $TG_k|_I$, and the isomorphism $TG_k|_I \cong J^k\mathfrak{g}$ defines the map, that we denote again by $t_*$,

$$t_* : J^kT \to T \quad v + j^k_x\theta \in (J^kT)_x \quad \mapsto \quad v + t_*\theta(x) .$$

We denote by the same symbols as above the projections $\pi^k_l = \pi_l : J^k\mathfrak{g} \to J^l\mathfrak{g}$, $l \geq 0$, defined by $\pi_l(j^k\theta) = j^l\theta$. If $\xi_k$ is a point or a section of $J^k\mathfrak{g}$, we denote by $\xi_l = \pi^k_l(\xi_k)$. The vector bundle $J^0\mathfrak{g}$ is isomorphic to $\mathfrak{g}$.

We have the canonical inclusions

$$\lambda^l : J^{k+l}\mathfrak{g} \to J^l J^k \mathfrak{g}$$

$$j^{k+l}_x \theta \mapsto j^l_x(\pi^k_l(\theta)) .$$

for $\theta \in \Gamma(\mathfrak{g})$.

Analogously to definition of bisections of $G_k$, a section $\xi_k$ of $J^k\mathfrak{g}$ is holonomic if there exist $\xi \in \Gamma(\mathfrak{g})$ such that $\xi_k = j^k\xi$. Therefore, if $\xi_k$ is holonomic, we have $\xi_k = j^k(\pi_0\xi_k)$.

We denote by $G_{k,l}$ the groupoid

$$G_{k,l} = \{ j^l_x F : x \in M, F \text{ a bisection of } G_k \}$$

and by $\pi_{k',l'}^{k,l} = \pi_{k',l'} : G_{k,l} \to G_{k',l'}$ the natural projections $\pi_{k',l'}^{k,l}(j^l_x F) = j^{l'}_{x}(\pi_{k',l'} F)$.

The Lie algebroid associated to $G_{k,l}$ is $J^l(J^k\mathfrak{g})$ which we denote also by $J^{k,l}\mathfrak{g}$.

**2.5 The affine structures on $J^1G$ and $J^1\mathfrak{g}$**

Given two sections $\sigma, \eta$ of $\mathfrak{g}$ such that $\sigma(x) = \eta(x)$ and $v \in T_x$ we have $\sigma_* v - \eta_* v \in V_{\eta(x)} \mathfrak{g}$. As $\mathfrak{g}$ is a vector bundle, $V\mathfrak{g} \cong \mathfrak{g}$, therefore $\sigma_* v - \eta_* v \in \mathfrak{g}_x$. This means $j^1_x\sigma - j^1_x\eta \in T^* \otimes \mathfrak{g}$. Inversely, given $j^1_x\sigma$ and $u \in T^* \otimes \mathfrak{g}$ there exists $\eta$ section of $\mathfrak{g}$ such that $j^1_x\sigma - j^1_x\eta = u$. As $J^1\mathfrak{g}$ has the canonical 0-section we get the exact sequence

$$0 \to T^* \otimes \mathfrak{g} \to J^1\mathfrak{g} \to \mathfrak{g} \to 0 .$$

(9)

The same argument applies to $J^1G$. In this case, if $\sigma, \eta$ are sections of $G$ such that $\sigma(x) = \eta(x)$ and $v \in T_x$ we have $\sigma_* v - \eta_* v \in V_{\eta(x)} G$, where

$$VG = \{ v \in TG : t_* v = 0 \} .$$

This means $j^1_x\sigma - j^1_x\eta \in T^*_x \otimes V_{\eta(x)} G$. Inversely, given $j^1_x\sigma$ and $u \in T^*_x \otimes V_x G$ there exists $\eta$ section of $G$ such that $j^1_x\sigma - j^1_x\eta = u$. A special case occurs when $\eta$ is the identity section $I$. As $VG|_I = \mathfrak{g}$, in this case we get $j^1_x\sigma - j^1_xI \in T^*_x \otimes \mathfrak{g}_x$. Inversely, given $u \in T^*_x \otimes \mathfrak{g}_x$ there exists $\sigma$ section of $G$ such that $j^1_x\sigma - j^1_xI = u$.

It is important to characterize the set of $u \in T^*_x \otimes \mathfrak{g}_x$ such that $j^1_x\sigma - j^1_xI = u \in T^*_x \otimes \mathfrak{g}_x$ is such that $\sigma$ is a bisection of $G$:

**Proposition 2.5** The set of $u \in T^*_x \otimes \mathfrak{g}_x$ such that $j^1_x\sigma = j^1_xI + u$ is the jet of a bisection $\sigma$ of $G$ is characterized by the application

$$v \in T_x \mapsto v + t_*(i(v)u) \in T_x$$

to be invertible.
As $J^k g \subset J^1 J^{k-1} g$ we obtain as a particular case of (9) the exact sequence
$$0 \to S^k T^* \otimes g \to J^k g \to J^{k-1} g \to 0.$$For a proof see [N].

2.6 The linear Spencer operator

If $\theta : U \subset I \to J^k g$ is a section, let be $\xi = j^1_x \theta \in J^1_x J^k g$, $x \in U$. Then $\xi$ can be identified to a linear application
$$\xi : T_x I \to (T_{\theta(x)} J^k g) \, v \mapsto \theta_*(v).$$If $\eta \in J^1_x J^k g$ is given by $\eta = j^1_x \mu$, with $\mu(x) = \theta(x)$, then $(\pi)_*(\eta_* - \xi_*) v = 0$, where we remember that $\pi : J^k g \to I$ is defined by $\pi(j^k \theta) = x$. So $\eta_* - \xi_* \in T^*_x I \otimes V_{\pi_\xi} J^k g$, where $V J^k g = \ker \pi_*$. But $J^k g$ is a vector bundle, then $V_{\pi_\xi} J^k g \equiv J^k g_\xi$, so $\eta_* - \xi_* \in T^*_x I \otimes J^k g_\xi$. The sequence
$$0 \to T^*_x I \otimes J^k g \to J^1_x J^k g \xrightarrow{\pi_\xi^1} J^k g \to 0$$obtained in this way is exact.

The linear operator $D$ defined by
$$D : \Gamma(J^k g) \to T^* \otimes \Gamma(J^{k-1} g) \, \xi_k \mapsto D\xi_k = j^1_k \xi_{k-1} - \lambda^1(\xi_k),$$is the linear Spencer operator. We remember that $\xi_{k-1} = \pi^k_{k-1} \xi_k$ and
$$\lambda^1 : J^k g \to J^1 J^{k-1} g \, j^k_\xi \mapsto j^1_k (j^{k-1} \xi),$$The difference in (11) is in $J^1 J^{k-1} g$ and is in $T^* \otimes J^{k-1} g$, by (10).

The operator $D$ is null on a section $\xi_k$ if and only if it is holonomic, i.e., $D\xi_k = 0$ if and only if there exists $\theta \in \Gamma(g)$ such that $\xi_k = j^k \theta$.

**Proposition 2.6** The operator $D$ is characterized by

(i) $D \circ j^k = 0$

(ii) $D(f \xi_k) = df \otimes \xi_{k-1} + f D\xi_k$,

with $\xi_k \in \Gamma(J^k g)$, $\xi_{k-1} = \pi_{k-1} \xi_k$ and $f$ a real function on $I$.

For a proof, see [KS].

The operator $D$ extends to
$$D : \wedge^l T^* \otimes \Gamma(J^k g) \to \wedge^{l+1} T^* \otimes \Gamma(J^{k-1} g) \, \omega \otimes \xi_k \mapsto D(\omega \otimes \xi_k) = d\omega \otimes \xi_{k-1} + (-1)^l \omega \wedge D\xi_k.$$
3 The calculus on the diagonal

Following [Ma1], [Ma2], [KS] and [V], we will relate \( \tilde{J}^k g \) with vector fields along the diagonal of \( I \times G \) and actions of bisections in \( G_k \) with diffeomorphisms of \( I \times G \) which are right invariants, \( \rho_1 \) projectables and preserve \( B(G) = \{(tX, X) : X \in G \} \).

We denote the diagonal of \( I \times G \) by \( \Delta = \{(x, x) \in I \times G | x \in I \} \), and by \( \rho_1 : I \times G \to I \) and \( \rho_2 : I \times G \to G \) the first and second projections, respectively. The restrictions \( \rho_1|\Delta, \rho_2|\Delta \) and \( t \circ \rho_2|\Delta \) are diffeomorphisms of \( \Delta \) on \( I \). A sheaf on \( I \) will be identified to its inverse image by \( \rho_1|\Delta \).

For example, if \( O_I \) denotes the sheaf of germs of real functions on \( I \), then we will write \( \tilde{O}_I \mid \Delta \) instead of \( (\rho_1|\Delta)^{-1}O_I \). Therefore, a \( f \in \tilde{O}_I \) will be considered in \( \tilde{O}_\Delta \) or in \( \tilde{O}_{I \times G} \) through the map \( f \to f \circ \rho_1 \).

The transposition in \( I \times I \) is denoted by \( \epsilon : I \times I \to I \times I \)

\[
(x, y) \to (y, x).
\]

(12)

The right action of \( G \) on \( G \) extends to \( I \times G \) by

\[
(I \times G) \times G \to I \times G \quad ((a, Y), X) \to (a, YX)
\]

where \( (X, Y) \in (s \times t)^{-1}(\Delta) \). A vector field \( \xi \) on \( I \times G \) is right invariant if is tangent to the submanifolds \( I \times s^{-1}(x) \) for every \( x \in I \) and \( \xi(a, XY) = \xi(a, X)Y \). A right invariant vector field on \( I \times G \) is defined by its restriction to \( I \times I \) since that \( \xi(a, X) = \xi(a, t(X))X \).

3.1 Brackets in \( \tilde{J}^k g \)

We denote by \( T(I \times G) \) the sheaf of germs of local sections of \( T(I \times G) \to I \times G \); by \( \mathcal{R} \) the sub sheaf in Lie algebras of \( T(I \times G) \) of right invariant vector fields whose elements are \( \rho_1 \)-projectables; by \( \mathcal{H}_\mathcal{R} \) the sub sheaf in Lie algebras of \( \mathcal{R} \) that projects on 0 by \( \rho_2 \), i.e. \( \mathcal{H}_\mathcal{R} = (\rho_2)^{-1}_*(0) \cap \mathcal{R} \); and by \( \mathcal{V}_\mathcal{R} \) the sub sheaf in Lie algebras defined by \( \mathcal{V}_\mathcal{R} = (\rho_1)^{-1}_*(0) \cap \mathcal{R} \). Clearly,

\[
\mathcal{R} = \mathcal{H}_\mathcal{R} \oplus \mathcal{V}_\mathcal{R},
\]

and

\[
[\mathcal{H}_\mathcal{R}, \mathcal{V}_\mathcal{R}] \subset \mathcal{V}_\mathcal{R}.
\]

Then

\[
(\rho_1)_* : \mathcal{H}_\mathcal{R} \to \mathcal{T}
\]

is an isomorphism, so we identify \( \mathcal{H}_\mathcal{R} \) naturally with \( \mathcal{T} \) by this isomorphism, and utilize both notations indistinctly.

**Proposition 3.1** The Lie bracket in \( \mathcal{R} \) satisfies:

\[
[v + \xi, f(w + \eta)] = v(f)(w + \eta) + f(v + \xi, w + \eta),
\]

with \( f \in \mathcal{O}_I \), \( v, w \in \mathcal{H}_\mathcal{R} \), \( \xi, \eta \in \mathcal{V}_\mathcal{R} \). In particular, the Lie bracket in \( \mathcal{V}_\mathcal{R} \) is \( \mathcal{O}_I \)-bilinear.
Proof. Let be \( f \in \mathcal{O}_I, \xi, \eta \in \mathcal{V}_R \). Then
\[
[v + \xi, (f \circ \rho_1)(w + \eta)] = (v + \xi)(f \circ \rho_1)(w + \eta) + (f \circ \rho_1)[v + \xi, w + \eta].
\]
As \( f \circ \rho_1 \) is constant on the sub manifolds \( \{x\} \times G \) and \( \xi \) is tangent to them, we obtain \( \xi(f \circ \rho_1) = 0 \), and the proposition follows.

We know that a right invariant vector field on \( G \) is defined by its restriction to \( I \), therefore identifies to an element of \( \Gamma(g) \). A vector field in \( \mathcal{V}_R \) is given by a family of sections of \( \Gamma(g) \) parameterized by an open set of \( I \). Therefore there exists a surjective morphism
\[
\Upsilon_k : \mathcal{R} \to \mathcal{T} \oplus \Gamma(J^k g)
\]
where \( v \in \mathcal{H}_R, \xi \in \mathcal{V}_R \), and
\[
\xi_k(x) = j^k_{(x,x)}(\xi|_{\{x\} \times I}).
\]
The kernel of this morphism is the sub sheaf \( \mathcal{V}_R^{k+1} \) of \( \mathcal{V}_R \) constituted by vector fields that are null on \( \Delta \) up to order \( k \). Therefore \( \mathcal{R}/\mathcal{V}_R^{k+1} \) is null outside of \( \Delta \), and can be considered as the sections of a vector bundle on \( \Delta \), and this vector bundle is isomorphic to the vector bundle on \( I, \mathcal{T} \oplus J^k g \).

Observe that the sections considered in the quotient are sections on open sets of \( I \). As
\[
[\mathcal{R}, \mathcal{V}_R^{k+1}] \subset \mathcal{V}_R^k,
\]
the bracket on \( \mathcal{R} \) induces a bilinear antisymmetric map which we call the first bracket of order \( k \),
\[
[\cdot, \cdot]_k = (\mathcal{T} \oplus \Gamma(J^k g)) \times (\mathcal{T} \oplus \Gamma(J^k g)) \to \mathcal{T} \oplus \Gamma(J^{k-1} g)
\]
defined by
\[
[v + \xi, w + \eta]_k = \Upsilon_{k-1}([v + \xi, w + \eta]),
\]
where \( \Upsilon_k(\xi) = \xi_k \) and \( \Upsilon_k(\eta) = \eta_k \).

It follows from proposition 3.1 that \( [\cdot, \cdot]_k \) satisfies:
\[
[v + \xi_k, f(w + \eta_k)]_k = v(f)(w + \eta_{k-1}) + \frac{f[v + \xi_k, w + \eta_k]}{k},
\]
(13)

\[
[[v + \xi_k, w + \eta_k], z + \theta_{k-1}]_{k-1} + [[[w + \eta_k, z + \theta_k], v + \xi_{k-1}]_{k-1} + [[[z + \theta_k, v + \xi_k], w + \eta_{k-1}]_{k-1} = 0,
\]

for \( v, w, z \in \mathcal{T}, \xi_k, \eta_k, \theta_k \in \Gamma(J^k g) \), \( f \in \mathcal{O}_I \). In particular, the first bracket is \( \mathcal{O}_I \)-bilinear on \( \Gamma(J^k g) \). Also
\[
[\Gamma(J^0 g), \Gamma(J^0 g)]_0 = 0.
\]

The following proposition relates \( [\cdot, \cdot]_k \) with the bracket in \( \mathcal{T} \) and the linear Spencer operator \( D \) in \( \Gamma(J^k g) \).

Proposition 3.2 Let be \( v, w \in \mathcal{T}, \theta, \mu \in \Gamma(g), \xi_k, \eta_k \in \Gamma(J^k g) \) and \( f \in \mathcal{O}_I \). Then:

(i) \( [v, w]_k = [v, w] \), where the bracket at right is the bracket in \( \mathcal{T} \);
(ii) \( [v, \xi_k]_k = i(v)D\xi_k \);

(iii) \([j^k\theta, j^k\mu]_k = j^{k-1}[\theta, \mu] \), where the bracket at right is the bracket in \( \Gamma(\mathfrak{g}) \).

PROOF. (i) This follows from the identification of \( T \) with \( \mathcal{H}_\mathcal{R} \).

(ii) First of all, if \( \theta \in \Gamma(\mathfrak{g}) \), let be \( \Theta \in \mathcal{V}_\mathcal{R} \) defined by \( \Theta(x, Y) = \theta(t(Y))Y \). Then \( Y_k(\Theta) = j^k\theta \).

If \( v \in \mathcal{H}_\mathcal{R} \), then \( v \) and \( \Theta \) are both \( \rho_1 \) and \( \rho_2 \) projectables, \( (\rho_1)_*(\Theta) = 0 \) and \( (\rho_2)_*(v) = 0 \), so we get \([v, \Theta] = 0 \). Consequently
\[
[v, j^k\theta]_k = Y_{k-1}([v, \Theta]) = 0. \tag{15}
\]

Also by \((14)\) we have
\[
[v, f\xi]_k = v(f)\xi_{k-1} + f[v, \xi]_k. \tag{16}
\]

As \((15)\) and \((16)\) determine \( D \) (cf. proposition 2.6), we get (ii).

(iii) Given \( \theta, \mu \in \Gamma(\mathfrak{g}) \), we define \( \Theta, H \in \mathcal{V}_\mathcal{R} \) as in (ii), \( \Theta(x, Y) = \theta(t(Y))Y \) and \( H(x, Y) = \mu(t(Y))Y \). Therefore
\[
[j^k\theta, j^k\mu]_k = [Y_k\Theta, Y_kH]_k = Y_{k-1}([\Theta, H]) = j^{k-1}[\theta, \mu].
\]

\( \square \)

Let be \( \mathcal{V}_\mathcal{R} \) the subsheaf in Lie algebras of \( \mathcal{R} \) such that \( \dot{\xi} \in \mathcal{V}_\mathcal{R} \) if and only if \( \dot{\xi} \) is tangent to the submanifold \( B(G) \) of \( I \times G \) image of \( G \) by the injective function
\[
B : G \rightarrow I \times G, \\
X \mapsto (t(X), X),
\]
i.e. \( B(G) = \{(t(X), X) \in I \times G | X \in G \} \). If \( \dot{\xi} = \xi_H + \xi \in \mathcal{V}_\mathcal{R}, \xi_H \in \mathcal{H}_\mathcal{R}, \xi \in \mathcal{V}_\mathcal{R} \), then
\[
\xi_H(t(X), X) + \xi(t(X), X) = \dot{\xi}(t(X), X).
\]

If \( X(u), u \in (-a, a) \), is a curve such that \( \frac{d}{du}X(u)|_{u=0} = \xi \), then
\[
\dot{\xi} = \frac{d}{du}(t(X(u)), X(u))|_{u=0} = \epsilon_*(t_*\xi) + \xi.
\]

Therefore \( \xi_H = \epsilon_*(t_*\xi) \). Remember that \( \epsilon \) is the transposition \((12)\).

Consequently, if \( \xi_k = Y_k(\xi) \), then \( \xi_H = \epsilon_*t_*(\xi_k) \), where we remember that \( t_* : J^k\mathfrak{g} \rightarrow T \) is defined in \((5)\). From now on, \( \xi_H \) denotes the horizontal component of \( \xi \in \mathcal{V}_\mathcal{R} \), so \( \dot{\xi} = \xi_H + \xi \), with \( \xi_H \in \mathcal{H} \) and \( \xi \in \mathcal{V} \).

We denote by \( \Gamma(J^k\mathfrak{g}) \) the subsheaf of \( \mathcal{T} \oplus \Gamma(J^k\mathfrak{g}) \), whose elements are
\[
\dot{\xi}_k = \xi_H + \xi_k,
\]
where \( \xi_H = \epsilon_*t_*(\xi_k) \). If \( \eta \) is a vector field in \( \mathcal{V}_\mathcal{R}^{k-1} \) and if \( (x, x) \in \Delta \), then \( \eta(x) = 0 \) up to order \( k \). It follows from \((5)\) that if \( t(X) = x \) then \( \eta_X = \eta_x.X \) is null of order \( k \) at \( X \). Therefore \( \eta \) is null of order \( k \) on \( B(G) \) and \( \mathcal{V}_\mathcal{R}^{k+1} \subset \mathcal{V}_\mathcal{R} \). Furthermore \( \Gamma(J^k\mathfrak{g}) \) identifies with \( \mathcal{V}_\mathcal{R}/\mathcal{V}_\mathcal{R}^{k+1} \) since that
\[
[\mathcal{V}_\mathcal{R}, \mathcal{V}_\mathcal{R}^{k+1}] \subset \mathcal{V}_\mathcal{R}^{k+1}, \tag{17}
\]
then:

\[ \frac{\partial}{\partial t} \cdot \frac{\partial}{\partial x} : \Gamma(J^k g) \times \Gamma(J^k g) \to \Gamma(J^k g) \]

\[ (\xi_H + \xi_k, \eta_H + \eta_k) \mapsto \Upsilon_k([\xi, \eta]), \]

where \( \Upsilon_k(\xi) = \xi_H + \xi_k \) and \( \Upsilon_k(\eta) = \eta_H + \eta_k \). Unlike the first bracket, we do not lose one order doing the bracket in \( J^k g \). The second bracket \( [\cdot, \cdot]_k \) is a Lie bracket on \( J^k g \). The proposition below relates it with the bracket \([\cdot, \cdot]_k\), defined in (6).

The projection

\[ \nu : H_R \oplus V_R \to V_R \]

\[ v + \xi \mapsto \xi \]

quotients to

\[ \nu_k : T \oplus \Gamma(J^k g) \to \Gamma(J^k g) \]

\[ v + \xi_k \mapsto \xi_k. \]

and the restriction \( \nu_k : J^k g \to J^k g \) is an isomorphism of vector bundles.

**Proposition 3.3** If \( \xi_k, \eta_k \in \Gamma(J^k g) \) then

\[ [\xi_k, \eta_k]_k = \nu_k([\tilde{\xi}_k, \tilde{\eta}_k]_k), \]

where \( \xi_k = \nu_k(\tilde{\xi}_k), \eta_k = \nu_k(\tilde{\eta}_k) \).

**Proof.** We will verify properties (i) and (ii) of Proposition 3.2. If \( \theta, \mu \in g \), let be \( \Theta, H \in V_R \) as in the proof of proposition 3.2. Then:

\[ (i) \quad \nu_k([\xi, \theta + j^k \theta, \epsilon_s t_s \mu + j^k \mu]_k) = \nu_k(\Upsilon_k(\{\xi, \theta + \epsilon_s t_s \mu + H\})) \]

\[ = \Upsilon_k(\nu(\epsilon_s t_s [\theta, \mu] + [\Theta, H])) = j^k([\theta, \mu]) = [j^k \theta, j^k \mu]_k. \]

\[ (ii) \quad \nu_k([\tilde{\xi}_k, f \tilde{\eta}_k]_k) = \nu_k(f[\tilde{\xi}_k, \tilde{\eta}_k]_k + \xi_H(f)\tilde{\eta}_k) \]

\[ = f\nu_k([\tilde{\xi}_k, \tilde{\eta}_k]_k) + (\epsilon_s t_s \xi_k)(f)\eta_k. \]

\[ \square \]

**Corollary 3.1** If \( \xi_k, \eta_k \in \Gamma(J^k g) \), then

\[ \nu_k([\xi_k, \eta_k]_k) = \{\xi_k, \eta_k\} = i(\xi_H)D\eta_{k+1} - i(\eta_H)D\xi_{k+1} + [\xi_{k+1}, \eta_{k+1}]_{k+1}, \]

where \( \xi_H = \epsilon_s t_s \xi_k, \eta_H = \epsilon_s t_s \eta_k \in T \) and \( \xi_{k+1}, \eta_{k+1} \in J^{k+1} g \) projects on \( \xi_k, \eta_k \) respectively.

**Proof.** It follows from Propositions 3.2 and 3.3.

As a consequence of proposition 3.3 we obtain that

\[ \nu_k : J^k g \to J^k g \]

\[ \xi_k \mapsto \xi_k. \]

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is an isomorphism of Lie algebras sheaves, where the bracket in $J^{k+1}\mathfrak{g}$ is the second bracket $[,]_k$ as defined in (3.11), and the bracket in $J^k\mathfrak{g}$ is the bracket $[,]_k$ as defined in (3).

In a similar way, we obtain from (17) that we can define the *third bracket* as

$$\[,[,]_k : \Gamma(J^{k+1}\mathfrak{g}) \times \Gamma(J^k\mathfrak{g}) \to \Gamma(J^k\mathfrak{g})$$

$$(\xi_H + \xi_{k+1}, v + \eta_k) \mapsto \eta_k([\xi, v + \eta]).$$

where $\xi, v + \eta \in \mathcal{R}$.

**Proposition 3.4** The third bracket has the following properties:

(i) $$\[,[,]_k : \Gamma(J^{k+1}\mathfrak{g}) \times \Gamma(J^k\mathfrak{g}) \to \Gamma(J^k\mathfrak{g})$$

$$(\xi_H + \xi_{k+1}, v + \eta_k) \mapsto \eta_k([\xi, v + \eta]).$$

(ii) $$\[,[,]_k : \Gamma(J^{k+1}\mathfrak{g}) \times \Gamma(J^k\mathfrak{g}) \to \Gamma(J^k\mathfrak{g})$$

$$(\xi_H + \xi_{k+1}, v + \eta_k) \mapsto \eta_k([\xi, v + \eta]).$$

(iii) $$\[,[,]_k : \Gamma(J^{k+1}\mathfrak{g}) \times \Gamma(J^k\mathfrak{g}) \to \Gamma(J^k\mathfrak{g})$$

$$(\xi_H + \xi_{k+1}, v + \eta_k) \mapsto \eta_k([\xi, v + \eta]).$$

where $\xi_{k+1} = \xi_H + \xi_{k+1} \in \Gamma(J^{k+1}\mathfrak{g})$, $\eta_k \in \Gamma(J^k\mathfrak{g})$, $\eta_{k+1} = v + \eta_{k+1} \in \Gamma(J^{k+1}\mathfrak{g})$, $\xi_k = \pi_k(\xi_{k+1})$, $\eta_k = \pi_k(\eta_{k+1})$.

**Proof.** The proof follows the same lines as the proof of proposition 3.3. \qed

### 3.2 Action of bisectors on $J^k\mathfrak{g}$

Let’s now verify the relationship that exists between the action of right invariant diffeomorphisms of $I \times G$, that are $\rho_1$-projectable and preserve $B(G)$ on $\mathcal{R}$ and actions (2) and (3) of $G_1G_k$ on $TG_k$. Let be $\sigma$ a (local) right invariant diffeomorphism of $I \times G$ that is $\rho_1$-projectable. Then

$$\sigma(x, Y) = (f(x), \Phi(x, Y)),$$

where $f \in \text{Diff} (I)$, $\Phi(x, Y) = \phi_x(t(Y))Y$ and $\phi_x : I \to G$ is a bisection for all $x \in I$.

It follows from $\sigma(B(G)) \subset B(G)$ that

$$\sigma(t(X), X) = (f(t(X)), \Phi(t(X), X)) = (f(t(X)), \Phi(t(X), t(X))X)$$

therefore $f(t(X)) = t(\Phi(t(X), t(X))X) = t\Phi(t(X), t(X))$. It follows that

$$f(x) = t\Phi(x, x) = t\phi_x(x).$$

The inverse is given by

$$\sigma^{-1}(y, Y) = (f^{-1}(y), \phi_{f^{-1}(y)}^{-1}(tY)Y).$$

As a special case

$$t(\phi_{f^{-1}(x)})^{-1}(x) = f^{-1}(x).$$
Let’s denote by $\mathcal{J}$ the set of (local) right invariant diffeomorphisms of $I \times G$ that are $\rho_1$-projectable and preserve $B(G)$. We have naturally the application

$$
\mathcal{J} \to \Gamma(G_k), \\
\sigma \mapsto \sigma_k,
$$

where $\sigma_k(x) = j^k_x \phi_x$, $x \in I$. If $\sigma' \in \mathcal{J}$, with $\sigma' = (f', \Phi')$, then

$$
(\sigma'.\sigma)(x, y) = (f'(f(x)), \Phi'(x))(y) = ((f' \circ f)(x), (\phi'_{f(x)} \phi_x)(y)),
$$

and from this it follows

$$
(\sigma' \circ \sigma)_k(x) = j^k_x(\phi'_{f(x)} \phi_x) = j^k_{f(x)}(\phi'_{f(x)} \phi_x) = \sigma'_k(f(x)).\sigma_k(x) = (\sigma'_k \sigma_k)(x),
$$

for each $x \in I$. So (18) is a surjective morphism of groupoids. If $\phi$ is a rigth invariant diffeomorphism of $G$, let be $\tilde{\phi} \in \mathcal{J}$ given by

$$
\tilde{\phi}(x, Y) = (t \phi(Y), \phi(tY)Y).
$$

It is clear that

$$
(\tilde{\phi})_k = j^k \phi.
$$

It follows from definitions of $\mathcal{J}$ and $\mathcal{R}$, that is well defined the action

$$
\mathcal{J} \times \mathcal{R} \to \mathcal{R}, \\
(\sigma, v + \xi) \mapsto \sigma_*(v + \xi).
$$

Then $\mathcal{J}$ acts on $\mathcal{V}_R$. Also, as $\tilde{\mathcal{V}}_R$ is tangent to $B(G)$ and $B(G)$ is invariant by elements of $\mathcal{J}$, $\tilde{\mathcal{V}}_R$ is invariant by $\mathcal{J}$.

**Proposition 3.5** Let be $\sigma \in \mathcal{J}$, $v \in \mathcal{H}_R$, $\xi \in \mathcal{V}_R$. We have:

(i) $(\sigma_*v)_k = f_*(v) + (j^1 \sigma_k.v.l^1 \sigma_{k+1}^{-1} - l^1 \sigma_{k+1}.v.l^1 \sigma_{k+1}^{-1})$;

(ii) $(\sigma * \xi)_k = l^1 \sigma_{k+1}.\xi_k.\sigma_k^{-1}$;

(iii) $(\sigma * \tilde{\xi})_k = f_*(\xi_H) + j^1 \sigma_k.\xi_k.\sigma_k^{-1}$.

**Proof.** If

$$
\sigma(x, Y) = (f(x), \phi_x(tY)Y),
$$

then

$$
\sigma^{-1}(x, Y) = (f^{-1}(x), (\phi^{-1})_x(tY)Y),
$$

where

$$
(\phi^{-1})_x = (\phi f^{-1}(x))^{-1}.
$$

(i) Let be $v = \frac{d}{du}H_u|_{u=0}$ where $H_u(x, Y) = (h_u(x), Y)$. Therefore

$$
(\sigma_*v) = \frac{d}{du}(\sigma \circ H_u \circ \sigma^{-1})|_{u=0},
$$

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or
\[(\sigma_* v)(x, Y) = \frac{d}{du} ((f \circ h_u \circ f^{-1})(x), (\phi_{(h_u \circ f^{-1})(x)})(\phi^{-1}(x))(tY) |_{u=0}).\]

Let be \(\sigma \circ H_u \circ \sigma^{-1} = S_u \circ R_u\), where
\[R_u(x, X) = (r_u(x), X) = ((t\phi_{f^{-1}(x)} \circ h_u \circ f^{-1})(x), X)\]
and
\[S_u(y, X) = ((f \circ h_u \circ f^{-1} \circ r_u^{-1})(y), (\phi_{(h_u \circ f^{-1} \circ r_u)^{-1}}(y))(\phi_{(f^{-1} \circ r_u)^{-1}}(y))^{-1})(tX)X).\]

Therefore
\[\frac{d}{du} S_u|_{u=0}(y, X) = \frac{d}{du} (f \circ h_u \circ f^{-1})|_{u=0}(y) - \frac{d}{du} r_u|_{u=0}(y) + \frac{d}{du} (\phi_{(h_u \circ f^{-1} \circ r_u)^{-1}}(y))(\phi_{(f^{-1} \circ r_u)}(y))^{-1})|_{u=0}(tX)X\]
and
\[\Upsilon_k(\frac{d}{du} S_u|_{u=0})(y) = (f_* v)(y) - \frac{d}{du} r_u|_{u=0}(y) + j^k_y \left[\frac{d}{du} (\phi_{(h_u \circ f^{-1} \circ r_u)^{-1}}(y))(\phi_{(f^{-1} \circ r_u)}(y))^{-1})\right]|_{u=0}.\]

Let be \(u \to x_u\) the family of trajectories in \(I\) defined by \(x_u = r_u^{-1}(y)\). We can write
\[j^k_y \left(\phi_{(h_u \circ f^{-1} \circ r_u)^{-1}}(y)\right)(\phi_{(f^{-1} \circ r_u)}(y))^{-1}) = j^k_y(\phi_{f^{-1}(x_u) \circ h_u \circ f^{-1}}(x_u))(\phi_{(h_u \circ f^{-1}(x_u))}(\phi_{f^{-1}(x_u)}))^{-1}) - j^k_y(\phi_{(h_u \circ f^{-1}(x_u))}(\phi_{(h_u \circ f^{-1}(x_u))})^{-1}) + j^k_y(\phi_{(f^{-1}(x_u))}(\phi_{(f^{-1}(x_u))})^{-1})\]
From this equality we get
\[\Upsilon_k(\frac{d}{du} S_u|_{u=0})(y) = (f_* v)(y) - \frac{d}{du} r_u|_{u=0}(y)
+ \frac{d}{du} \sigma_k((h_u \circ f^{-1})(x_u)) j^k_y(\phi_{f^{-1}(x_u)})(\phi_{f^{-1}(x_u)})^{-1})|_{u=0} + \frac{d}{du} x_u|_{u=0}
= (f_* v)(y) - \frac{d}{du} r_u|_{u=0}(y)
+ \frac{d}{du} \sigma_k((h_u \circ f^{-1})(y)) j^k_y(\phi_{f^{-1}(y)})(\phi_{f^{-1}(y)})^{-1})|_{u=0} + \frac{d}{du} x_u|_{u=0}
= (f_* v)(y) - \frac{d}{du} r_u|_{u=0}(y)
+ j^k_y(\phi_{f^{-1}(y)})(\sigma_k^{-1}(y))\frac{d}{du} r_u|_{u=0}(y) + \frac{d}{du} x_u|_{u=0},\]

since that \((\sigma_k^{-1})^{-1}(y) = (\sigma_{k+1}(f^{-1}(y))^{-1})^{-1}\).

It follows from \(r_u(x_u) = y\) that
\[\frac{d}{du} r_u|_{u=0}(y) + \frac{d}{du} x_u|_{u=0} = 0.\]

As \(r(u) = (t\phi_{f^{-1}(x)} \circ h_u \circ f^{-1})(x) = (t\phi_{f^{-1}(x)} \circ h_u)(t\phi_{f^{-1}(x)})^{-1})(x) = t j^k \phi_{f^{-1}(x)} \circ h_u \circ t(j^k \phi_{f^{-1}(x)})^{-1}(x),\)
we obtain from proposition \(2.3\) that
\[\frac{d}{du} r_u|_{u=0} = [\sigma_{k+1}, v, (\sigma_{k+1})^{-1}](x).\]
We conclude

\[
\begin{align*}
(\sigma_* v)_k(x) &= \left. \frac{d}{du}(S_u + H_u) \right|_{u=0}(x) \\
&= \left. (f_* v)(x) + j^1_{f^{-1}(x)} \sigma_k.(\sigma_{k+1})^{-1}(x).[\sigma_{k+1}.v.(\sigma_{k+1})^{-1}](x) - (\sigma_{k+1}.v.(\sigma_{k+1})^{-1})(x). \right.
\end{align*}
\]

We proved where \( \tilde{g} \) and \( \sigma \).

Then

\[
\begin{align*}
(\sigma_* v)_k = f_* v + j^1 \sigma_k.(\sigma_{k+1})^{-1} - \sigma_{k+1}.v.(\sigma_{k+1})^{-1}
\end{align*}
\]

(ii) Let be

\[
\xi = \left. \frac{d}{du} V_u \right|_{u=0},
\]

where \( V_u(x, Y) = (x, \eta_x^u(tY)Y) \), with \( \eta_x^u \in \mathcal{G} \) for each \( u \), and \( g_u(x) = t \eta_x^u(x) \). Then

\[
(\sigma \circ V_u \circ \sigma^{-1})(x, Y) = (x, (\phi_{f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1})(tY)Y),
\]

and

\[
(\sigma_* \xi)_k(x, Y) = \left. \frac{d}{du}(\phi_{f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right|_{u=0}(tY)Y)
\]

Consequently

\[
\begin{align*}
\Upsilon_k(\sigma_* \xi)(x) &= j^k_x (\left. \frac{d}{du}(\phi_{f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right|_{u=0}) \\
&= \left. \frac{d}{du}(j^k_x (g_u \circ f^{-1})(x) \cdot \phi_{f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right|_{u=0} \\
&= \frac{d}{du} \left( (\tilde{\phi}_{f^{-1}(x)})_k (g_u \circ f^{-1})(x) \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right) \bigg|_{u=0} \\
&= \frac{d}{du} \left( j^k_{f^{-1}(x)} (\tilde{\phi}_{f^{-1}(x)})_k \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right) \bigg|_{u=0} \\
&= \lambda^1 (\sigma_{k+1}(f^{-1}(x))) \cdot \xi_k (f^{-1}(x)) \cdot \sigma^{-1}_k(x),
\end{align*}
\]

since that

\[
\begin{align*}
j^1_{f^{-1}(x)} (\tilde{\phi}_{f^{-1}(x)})_k = j^1_{f^{-1}(x)} j^k \phi_{f^{-1}(x)} = \lambda^1 (j^k_{f^{-1}(x)} \phi_{f^{-1}(x)}) = \lambda^1 (\sigma_{k+1}(f^{-1}(x))).
\end{align*}
\]

We proved

\[
(\sigma \ast \xi)_k = \lambda^1 \sigma_{k+1}.\xi_k.\sigma^{-1}_k.
\]

(iii) Let be, as in (ii), \( \xi = \left. \frac{d}{du} V_u \right|_{u=0} \), where \( V_u(x, Y) = (x, \eta_x^u(tY)Y) \), with \( \eta_x^u \in \mathcal{G} \) for each \( u \), and \( g_u(x) = t \eta_x^u(x) \). Then

\[
\tilde{\xi} = \left. \frac{d}{du} \tilde{V}_u \right|_{u=0},
\]

where \( \tilde{V}_u(x, Y) = (g_u(x), \eta_x^u(tY)Y) \). Therefore

\[
(\sigma \circ \tilde{V}_u \circ \sigma^{-1})(x, Y) = (f \circ g_u \circ f^{-1}(x), (\phi_{g_u \circ f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1})(tY)Y),
\]

and

\[
(\sigma_* \tilde{\xi})(x, Y) = f_* \xi_H (x) + \left. \frac{d}{du}(\phi_{g_u \circ f^{-1}(x)} \cdot \eta_x^u \cdot (\phi_{f^{-1}(x)})^{-1}) \right|_{u=0}(tY)Y
\]
Observe this formula depends only of \( \sigma_k \).

We can give another proof combining (i) and (ii):

\[
(\sigma * \xi)_k = (\sigma * \xi_H)_k + (\sigma \xi)_k
\]
\[
(\xi)_k = (f_* \xi_H + j^1 \sigma_k \cdot \xi_H \cdot (\xi_{k+1}))^{-1} + (\lambda^1 \sigma_{k+1} \cdot \xi_H \cdot (\sigma_{k+1})^{-1}) + (\lambda^1 \sigma_{k+1} \cdot \xi_k \cdot \sigma_k^{-1})
\]

As \( t_*(\xi_k - \xi_H) = 0 \), we get \( j^1 \sigma_k \cdot (\xi_k - \xi_H) = \sigma_{k+1} \cdot (\xi_k - \xi_H) \) so

\[
(\sigma * \xi)_k = f_* \xi_H + j^1 \sigma_k \cdot \xi_H \cdot (\xi_{k+1})^{-1} + j^1 \sigma_k \cdot (\xi_k - \xi_H) \cdot (\sigma_{k+1})^{-1}
\]

\[
= f_* \xi_H + j^1 \sigma_k \cdot \xi_k \cdot (\sigma_{k+1})^{-1}.
\]

\[\square\]

It follows from Proposition 3.3 that action (19) projects on an action ( ):

\[
G^{k+1} \times (T \oplus \Gamma(J^k \mathfrak{g})) \rightarrow T \oplus \Gamma(J^k \mathfrak{g})
\]

where

\[
(\sigma_{k+1})_*(v + \xi_k) = f_* v + (j^1 \sigma_k \cdot v \cdot \lambda^1 \sigma_{k+1}^{-1} - \lambda^1 \sigma_{k+1} \cdot v \cdot \lambda^1 \sigma_{k+1}^{-1}) + (\lambda^1 \sigma_{k+1} \cdot \xi_k \cdot \sigma_k^{-1}).
\]

This action verifies

\[
[(\sigma_{k+1})_*(v + \xi_k), (\sigma_{k+1})_*(w + \eta_k)]_k = (\sigma_k)_*[v + \xi_k, w + \eta_k]_k.
\]

(21)

It follows from proposition 3.3 and (20) that \( (\sigma_{k+1})_*(\xi_k)(x) \) depends only on the value of \( \sigma_{k+1}(x) \) at the point \( x \) where \( \xi \) is defined, and \( (\sigma_{k+1})_*(v)(x) \) depends on the value of \( \sigma_{k+1} \) on a curve tangent to \( v(x) \).

Item (iii) of proposition 3.3 says that restriction to \( J^k \mathfrak{g} \) of action (20) is well defined:

\[
G^k \times \Gamma(J^k \mathfrak{g}) \rightarrow \Gamma(J^k \mathfrak{g})
\]

where

\[
(\sigma_k)_*(\xi_k) = f_* \xi_H + j^1 \sigma_k \cdot \xi_k \cdot \sigma_k^{-1}.
\]

In this case we get

\[
[(\sigma_k)_*(\xi_k), (\sigma_k)_*(\eta_k)]_k = (\sigma_k)_*[\xi_k, \eta_k]_k = \nu_k(\sigma_k)_*\nu_k^{-1}[\xi_k, \eta_k]_k
\]

and each \( \nu_k(\sigma_k)_*\nu_k^{-1} \) acts as an automorphism of the Lie algebra sheaf \( \Gamma(J^k \mathfrak{g}) \):

\[\nu_k(\sigma_k)_*\nu_k^{-1}[\xi_k, \eta_k]_k = \nu_k(\sigma_k)_*\nu_k^{-1}[\xi_k, \eta_k]_k\]
3.3 The Lie algebra sheaf $\wedge(\mathcal{J}^\infty g)^\ast \otimes (\mathcal{J}^\infty g)$

We denote by $\mathcal{J}^\infty g$ the projective limit of $\mathcal{J}^k g$, say, $\mathcal{J}^\infty g = \lim \text{proj} \mathcal{J}^k g$, and

$$\mathcal{J}^\infty g = T \oplus \mathcal{J}^\infty g.$$ 

As $T \oplus J^k g \cong TG^k|_I$, we have the identification of $\mathcal{J}^\infty T$ with $\lim \text{proj} \Gamma(TG^k|_I)$, where $\Gamma(TG^k|_I)$ denotes the sheaf of germs of local sections of the vector bundle $TG^k|_I \to I$. From the fact that $T \oplus J^k g$ is a $O_I$-module, we get $\mathcal{J}^\infty g$ is a $O_I$-module. In the following we use the notation

$$\xi = v + \lim \text{proj} \xi_k, \quad \eta = w + \lim \text{proj} \eta_k \in T \oplus \mathcal{J}^\infty g.$$ 

We define the first bracket in $\mathcal{J}^\infty g$ as:

$$\llbracket \xi, \eta \rrbracket_\infty = \lim \text{proj} [v + \xi_k, w + \eta_k]_k$$

With the bracket defined by (3.3), $\mathcal{J}^\infty g$ is a Lie algebra sheaf. Furthermore

$$\llbracket \xi, f \eta \rrbracket_\infty = v(f)\eta + f \llbracket \xi, \eta \rrbracket_\infty.$$ 

We extend now, as in [Ma1], [Ma2] or [KS], the bracket on $\mathcal{J}^\infty g$ to a Nijenhuis bracket (see [FN]) on $\wedge(\mathcal{J}^\infty g)^\ast \otimes (\mathcal{J}^\infty g)^\ast$, where

$$(\mathcal{J}^\infty g)^\ast = \lim \text{ind} (\mathcal{J}^k g)^\ast.$$ 

We introduce the exterior differential $d$ on $\wedge(\mathcal{J}^\infty g)^\ast$, by:

(i) if $f \in O_I$, then $df \in (\mathcal{J}^\infty g)^\ast$ is defined by

$$< df, \xi > = v(f).$$

(ii) if $\omega \in (\mathcal{J}^\infty g)^\ast$, then $d\omega \in \wedge^2(\mathcal{J}^\infty g)^\ast$ is defined by

$$< d\omega, \xi \wedge \eta > = \mathcal{L}(\xi) < \omega, \eta > - \mathcal{L}(\eta) < \omega, \xi > - < \omega, [\xi, \eta]_\infty >,$$

where $\mathcal{L}(\xi)f = < df, \xi >$.

We extend this operator to forms of any degree as a derivation of degree $+1$

$$d : \wedge^r(\mathcal{J}^\infty g)^\ast \to \wedge^{r+1}(\mathcal{J}^\infty g)^\ast.$$ 

The exterior differential $d$ is linear,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^r \omega \wedge d\tau,$$

for $\omega \in \wedge^r(\mathcal{J}^\infty g)^\ast$, and $d^2 = 0$.

Remember that $(\rho_1)_* : T \oplus \mathcal{J}^\infty g \to T$ is the projection given by the decomposition in direct sum of $\mathcal{J}^\infty g = T \oplus \mathcal{J}^\infty g$. (We could use, instead of $(\rho_1)_*$, the natural map $s_* : T \oplus J^k g \to T$, given by $s_* : TG^k|_I \to T$, and the identification (7)). Then $(\rho_1)^* : T^* \to (\mathcal{J}^\infty g)^\ast$ and this map extends to $(\rho_1)^* : \wedge^* \to \wedge^*(\mathcal{J}^\infty g)^\ast$. If $\omega \in \wedge^r(\mathcal{J}^\infty T)^\ast$, then

$$< (\rho_1)^* \omega, \tilde{\xi}_1 \wedge \cdots \wedge \tilde{\xi}_r > = < \omega, v_1 \wedge \cdots \wedge v_r >,$$

where $\tilde{\xi}_j = v_j + \xi_j, j = 1, \cdots, r$. It follows that $d((\rho_1)^* \omega) = (\rho_1)^*(d\omega)$. We identify $\wedge^* T^\ast$ with its image in $\wedge(\mathcal{J}^\infty g)^\ast$ by $(\rho_1)^*$, and we write simply $\omega$ instead of $(\rho_1)^* \omega$. 

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Let be \( u = \omega \otimes \xi \in (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g), \tau \in \wedge (\mathcal{J}^\infty g)^* \), with \( \deg \omega = r \) and \( \deg \tau = s \). We also define \( \deg u = r \). Then we define the derivation of degree \((r - 1)\)

\[
i(u) : \wedge^s (\mathcal{J}^\infty g)^* \to \wedge^{s+r-1}(\mathcal{J}^\infty g)^*
\]

by

\[
i(u)\tau = i(\omega \otimes \xi)\tau = \omega \wedge i(\xi)\tau
\]

and the Lie derivative

\[
\mathcal{L}(u) : \wedge^r (\mathcal{J}^\infty g)^* \to \wedge^{r+s}(\mathcal{J}^\infty g)^*
\]

by

\[
\mathcal{L}(u)\tau = i(u)d\tau + (-1)^r d(i(u)\tau),
\]

which is a derivation of degree \( r \). If \( v = \tau \otimes \eta \) we define

\[
[u, v] = [\omega \otimes \xi, \tau \otimes \eta] = \omega \wedge \tau \otimes [\xi, \eta]_\infty + \mathcal{L}(\omega \otimes \xi)\tau \otimes \eta - (-1)^r \mathcal{L}(\tau \otimes \eta)\omega \otimes \xi.
\]

A straightforward calculation shows that:

\[
[u, \tau \otimes \eta] = \mathcal{L}(u)\tau \otimes \eta + (-1)^r \tau \wedge [u, \eta] - (-1)^{r+s} d\tau \wedge i(\eta)u,
\]

where \( i(\eta)u = i(\eta)(\omega \otimes \xi) = i(\eta)\omega \otimes \xi \). On verify that

\[
[u, v] = -(-1)^r[v, u]
\]

and

\[
[u, [v, w]] = [[u, v], w] + (-1)^r[v, [u, w]],
\]

where \( \deg u = r \), \( \deg v = s \).

With this bracket, \( \wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g) \) is a Lie algebra sheaf. Furthermore, if

\[
[\mathcal{L}(u), \mathcal{L}(v)] = \mathcal{L}(u)\mathcal{L}(v) - (-1)^r \mathcal{L}(v)\mathcal{L}(u)
\]

then

\[
[\mathcal{L}(u), \mathcal{L}(v)] = \mathcal{L}([u, v]).
\]

In particular, we have the following formulas:

**Proposition 3.6** If \( u, v \in (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g), \omega \in (\mathcal{J}^\infty g)^*, \xi, \eta \in \mathcal{J}^\infty g \), then:

(i) \quad \langle \mathcal{L}(u)\omega, \xi \wedge \eta \rangle = \mathcal{L}(i(\xi)u)\langle \omega, \eta \rangle - \mathcal{L}(i(\eta)u)\langle \omega, \xi \rangle - \langle \omega, [i(\xi)u, \eta]_\infty + [\xi, i(\eta)u]_\infty - i([\xi, \eta]_\infty)u \rangle

(ii) \quad i(\xi)[u, \eta] = [i(\xi)u, \eta]_\infty - i([\xi, \eta]_\infty)u,

(iii) \quad [u, v, \xi \wedge \eta] = [i(\xi)u, i(\eta)v]_\infty - [i(\xi)u, \eta]_\infty - i([i(\xi)u, \eta]_\infty)u - [i(\eta)v, \xi]_\infty - i([i(\eta)v, \xi]_\infty)u - [i(\xi)v, \xi]_\infty - i([i(\xi)v, \xi]_\infty)u - [i(\eta)v, \xi]_\infty - i([i(\eta)v, \xi]_\infty)u.
Proof. It is a straightforward calculus applying the definitions.

If we define the groupoid

\[ G_\infty = \lim \mathrm{proj} \ G_k, \]

then for \( \sigma = \lim \mathrm{proj} \sigma_k \in G_\infty \), we obtain, from (20), the action

\[ \sigma : \mathcal{J}^\infty g \to \mathcal{J}^\infty g, \]
\[ \xi = v + \lim \mathrm{proj} \xi_k \ \implies \ \sigma_* \xi = \lim \mathrm{proj} (\sigma_{k+1})_* (v + \xi_k), \]

so it is well defined

\[ G_\infty \times \mathcal{J}^\infty g \to \mathcal{J}^\infty g, \]
\[ (\sigma, \xi) \mapsto \sigma_* \xi. \]

It follows from (21) that \( \sigma_* : \mathcal{J}^\infty g \to \mathcal{J}^\infty g \) is an automorphism of Lie algebra sheaf.

Given \( \sigma \in G_\infty \), \( \sigma \) acts on \( \wedge (\mathcal{J}^\infty g)^* \):

\[ \sigma^* : \wedge (\mathcal{J}^\infty g)^* \to \wedge (\mathcal{J}^\infty g)^*, \]
\[ \omega \mapsto \sigma^* \omega, \]

where, if \( \omega \) is a \( r \)-form,

\[ < \sigma^* \omega, \xi_1 \wedge \cdots \wedge \xi_r > = < \omega, \sigma_*^{-1}(\xi_1) \wedge \cdots \wedge \sigma_*^{-1}(\xi_r) >. \] (26)

Consequently, \( G_\infty \) acts on \( \wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g) \):

\[ G_\infty \times (\wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g)) \to \wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g), \]
\[ (\sigma, u) \mapsto \sigma_* u, \]

where

\[ \sigma_* u = \sigma_* (\omega \otimes \xi) = \sigma^* (\omega) \otimes \sigma_* (\xi). \] (27)

The action of \( \sigma_* \) is an automorphism of the Lie algebra sheaf \( \wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g) \), i.e.,

\[ [\sigma_* u, \sigma_* v] = \sigma_* [u, v]. \]

4 The first linear and non-linear Spencer complex

In this section we will study the sub sheaf \( \wedge T^* \otimes \mathcal{J}^\infty g \) and introduce linear and non-linear Spencer complexes. Principal references are [Ma1], [Ma2] and [KS].

**Proposition 4.1** The sheaf \( \wedge T^* \otimes \mathcal{J}^\infty g \) is a Lie algebra sub sheaf of \( \wedge (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g) \), and

\[ [\omega \otimes \xi, \tau \otimes \eta] = \omega \wedge \tau \otimes [\xi, \eta]_\infty, \]

where \( \omega, \tau \in \wedge T^*, \xi, \eta \in \mathcal{J}^\infty g \).

**Proof.** Let be \( u = \omega \otimes \xi \in \wedge T^* \otimes \mathcal{J}^\infty g \). For any \( \tau \in \wedge T^* \), \( i(\xi) \tau = 0 \), then, applying (22) we obtain \( i(u) \tau = 0 \). So (21) implies \( [\omega \otimes \xi, \tau \otimes \eta] = \omega \wedge \tau \otimes [\xi, \eta]_\infty. \) □

Let be the fundamental form

\[ \chi \in (\mathcal{J}^\infty g)^* \otimes (\mathcal{J}^\infty g) \]
Then it is well defined the first linear Spencer complex
therefore, \( D \)
that
As
\[ v \in \bigwedge T \oplus \mathcal{J} \infty \mathfrak{g}. \]
In another words, \( \chi \) is the projection of \( \mathcal{J} \infty \mathfrak{g} \) on \( T \), parallel to \( \mathcal{J} \infty \mathfrak{g} \).
If \( \mathbf{u} = \lim \mathbf{u}_k \), we define \( \mathbf{D} \mathbf{u} = \lim \mathbf{D} \mathbf{u}_k \).

**Proposition 4.2** If \( \omega \in \bigwedge T^* \), and \( \mathbf{u} \in \bigwedge T^* \otimes \mathcal{J} \infty \mathfrak{g} \), then:

(i) \( \mathcal{L}(\chi) \omega = d\omega \);

(ii) \( [\chi, \chi] = 0 \);

(iii) \( [\chi, \mathbf{u}] = \mathbf{D} \mathbf{u} \).

**Proof.** Let be \( \tilde{\xi} = v + \xi, \tilde{\eta} = w + \eta \in T \oplus \mathcal{J} \infty \mathfrak{g} \).

(i) As \( \mathcal{L}(\chi) \) is a derivation of degree 1, it is enough to prove (i) for 0-forms \( f \) and 1-forms \( \omega \in (\mathcal{J} \infty \mathfrak{g})^* \).

From \( \mathcal{L}(\chi) \) we have \( \mathcal{L}(\chi) f = i(\chi) df = df \). It follows from proposition 3.6 (i) that

\[
\langle \mathcal{L}(\chi) \omega, \tilde{\xi} \wedge \tilde{\eta} \rangle = \mathcal{L}(\chi)(v \wedge \omega) - \mathcal{L}(\omega) (v \wedge \tilde{\xi}) = - \langle \omega, [v, \tilde{\xi}] \rangle_{\mathfrak{g}} + \langle \tilde{\xi}, [v, \omega] \rangle_{\mathfrak{g}} = \mathcal{L}(\chi)(v \wedge \omega) - \mathcal{L}(\omega) (v \wedge \tilde{\xi}) = \langle \omega, [v, \tilde{\xi}] \rangle_{\mathfrak{g}} - \langle \tilde{\xi}, [v, \omega] \rangle_{\mathfrak{g}}.
\]

(ii) Applying proposition 3.6 (iii), we obtain

\[
\langle \frac{1}{2}[\chi, \chi], \tilde{\xi} \wedge \tilde{\eta} \rangle = [v, w] - i([v, \tilde{\eta}] - [w, \tilde{\xi}] - \rho_1 [\tilde{\xi}, \tilde{\eta}]_{\mathfrak{g}}) \chi = [v, w] - ([v, w] - [v, v]) = 0.
\]

(iii) It follows from \( \mathcal{L}(\chi) \) that, for \( \mathbf{u} = \omega \otimes \xi \),

\[
[\chi, \mathbf{u}] = \mathcal{L}(\chi) \omega \otimes \xi + (-1)^{r} \omega \wedge [\chi, \xi] - (-1)^{2r} d\omega \wedge i(\xi) \chi = d\omega \otimes \xi + (-1)^{r} \omega \wedge [\chi, \xi].
\]

As \( D \) is characterized by proposition 2.6, it is enough to prove \( [\chi, \xi] = D \xi \). It follows from propositions 3.2 (ii) and 3.6 (ii) that

\[
i(\tilde{\eta})[\chi, \xi] = [i(\tilde{\eta}) \chi, \xi]_{\mathfrak{g}} - i([\tilde{\eta}, \xi]_{\mathfrak{g}}) \chi = [\omega, \xi]_{\mathfrak{g}} = i(\tilde{\eta}) D \xi.
\]

\( \Box \)

If \( \mathbf{u}, \mathbf{v} \in \bigwedge T^* \otimes \mathcal{J} \infty \mathfrak{g} \), with \( \deg \mathbf{u} = r \), \( \deg \mathbf{v} = s \), then we get from \( \mathcal{L}(\chi) \) and proposition 4.2 (iii) that

\[
D[\mathbf{u}, \mathbf{v}] = [D \mathbf{u}, \mathbf{v}] + (-1)^{r} [\mathbf{u}, D \mathbf{v}],
\]

and

\[
[\chi, [\chi, \mathbf{u}]] = [[\chi, \chi], \mathbf{u}] - [\chi, [\chi, \mathbf{u}]] = -[\chi, [\chi, \mathbf{u}]],
\]

therefore, \( D^2 \mathbf{u} = 0 \), or

\[
D^2 = 0.
\]

Then it is well defined the first linear Spencer complex,

\[
0 \to \Gamma(\mathfrak{g}) \to \mathcal{J} \infty \mathfrak{g} \xrightarrow{D} T^* \otimes \mathcal{J} \infty \mathfrak{g} \xrightarrow{D} \to
\]

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\[ \wedge^2 T^* \otimes \mathcal{J}^\infty g \xrightarrow{D} \cdots \xrightarrow{D} \wedge^m T^* \otimes \mathcal{J}^\infty g \to 0, \]

where \( \dim T = m \). This complex projects on

\[ 0 \to \Gamma(g) \xrightarrow{j^k} \mathcal{J}^k g \xrightarrow{D} \mathcal{T}^* \otimes \mathcal{J}^{k-1} g \xrightarrow{D} \wedge^2 T^* \otimes \mathcal{J}^{k-2} g \xrightarrow{D} \cdots \xrightarrow{D} \wedge^m T^* \otimes \mathcal{J}^{k-m} g \to 0. \]

This complex is exact (see [Ma1], [Ma2], [KS]).

Let be \( \gamma^k \) the kernel of \( \pi_k : J^k g \to J^{k-1} g \). Denote by \( \delta \) the restriction of \( D \) to \( \gamma^k \). It follows from proposition (2.6ii) that \( \delta \) is \( \mathcal{O}_T \)-linear and \( \delta : \gamma^k \to T^* \otimes \gamma^{k-1} \). This map is injective, in fact, if \( \xi \in \gamma^k \), then by [10], \( \delta \xi = -\lambda^1(\xi) \) is injective. As

\[ i(v)D(i(w)D\xi) - i(w)D(i(v)D\xi) - i([v, w])D\pi_{k-1} \xi = 0, \]

for \( v, w \in \mathcal{T} \), \( \xi \in \gamma^k \subset \mathcal{J}^k g \), we obtain that \( \delta \) is symmetric, \( i(v)\delta(i(w)\delta\xi) = i(w)\delta(i(v)\delta\xi) \). Observe that we get the map

\[ \iota : \gamma^k \to S^2 T^* \otimes \gamma^{k-2} \]

defined by \( i(v, w)\iota(\xi) = i(v)\delta(i(w)\delta\xi) \), and if we go on, we obtain the isomorphism

\[ \gamma^k \cong S^k T^* \otimes J^0 g, \]

where, given basis \( e_1, \ldots, e_m \in T^\ast, f_1, \ldots, f_r \in g \) with the dual basis \( e_1^\ast, \ldots, e_m^\ast \in T^* \), we obtain the basis

\[ f^{k_1, k_2, \ldots, k_m}_l = \frac{1}{k_1!k_2!\cdots k_m!}(e^1)^{k_1}(e^2)^{k_2}\cdots(e^m)^{k_m} \otimes J^0 f_l \]

of \( S^k T^* \otimes J^0 g \), where \( k_1 + k_2 + \cdots + k_m = k, k_1, \ldots, k_m \geq 0 \) and \( l = 1, \ldots, r \). In this basis

\[ \delta(f^{k_1, k_2, \ldots, k_m}_l) = -\sum_{i=1}^m e^i \otimes f^{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_m}_l. \]

From the linear Spencer complex, we obtain the exact sequence of morphisms of vector bundles

\[ 0 \to \gamma^k \xrightarrow{\delta} T^* \otimes \gamma^{k-1} \xrightarrow{\delta} \wedge^2 T^* \otimes \gamma^{k-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \wedge^m T^* \otimes \gamma^{k-m} \to 0. \]

Let’s now introduce the first nonlinear Spencer operator \( \mathcal{D} \). The “finite” form \( \mathcal{D} \) of the linear Spencer operator \( D \) is defined by

\[ \mathcal{D}\sigma = \chi - \sigma^{-1}_\ast(\chi), \]

where \( \sigma \in \mathcal{G}_\infty \).

**Proposition 4.3** The operator \( \mathcal{D} \) take values in \( T^* \otimes \mathcal{J}^\infty g \), so

\[ \mathcal{D} : \mathcal{G}_\infty \to T^* \otimes \mathcal{J}^\infty g, \]

and

\[ i(v)(\mathcal{D}\sigma)_k = \lambda^1 \sigma^{-1}_{k+1} \cdot J^1 \sigma_k v - v, \quad (28) \]

where \( \sigma = \lim \text{proj} \sigma_k \in \mathcal{G}_\infty \).
PROOF. Applying (26) and (27), it follows for $\xi \in \mathcal{J}^\infty \mathfrak{g}$,
\[
i(\xi)D\sigma = i(\xi)\chi - \sigma_*^{-1}(i(\sigma_*(\xi))\chi) = 0,
\]
and for $v \in \mathcal{T}$,
\[
i(v)D\sigma = i(v)\chi - \sigma_*^{-1}(i(\sigma_*(v))\chi) = v - \sigma_*^{-1}(f_*v),
\]
where $f = t \circ \sigma$. By proposition 3.5 (i),
\[
i(v)(D\sigma)_k = v - (f_*^{-1}(f_*v) + j^1_\sigma v, \lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k).
\]
Posing $v = \frac{d}{du}x_u|_{u=0}$, we obtain
\[
j^1_\sigma v, j^1_\sigma = \frac{d}{du}(\sigma_k(x_u), \sigma_k^{-1}(f(x_u)))|_{u=0} = \frac{d}{du}f(x_u)|_{u=0} = f_*v, \quad (29)
\]
and replacing this above, we get
\[
i(v)(D\sigma)_k = -j^1_\sigma v, j^1_\sigma = (\lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k).\lambda^1_\sigma v^k + \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k = (\lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k).
\]
since that $\lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k$ is s-vertical (cf. (5)).

**Corollary 4.1** We have $D\sigma = 0$ if and only if $\sigma = j^\infty(\pi_0 \circ \sigma)$, where $\pi_0 : \mathcal{G}_\infty \to \mathcal{G}$.

**Corollary 4.2** If $\sigma_{k+1} \in \mathcal{G}_{k+1}$, then
\[
(\sigma_{k+1})_*(v) = f_*v + (\sigma_{k+1})_*(i(v)(D\sigma)_{k+1}),
\]
for $v \in \mathcal{T}$.

**Proof.** It follows from (20) and proposition 4.3 that
\[
(\sigma_{k+1})_*(i(v)(D\sigma)_{k+1}) = \lambda^1_\sigma v^k. \lambda^1_\sigma v^k = \lambda^1_\sigma v^k. (\lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k) = j^1_\sigma v, \lambda^1_\sigma v^k - \lambda^1_\sigma v^k f_*v, \lambda^1_\sigma v^k = (\sigma_{k+1})_*(v) - f_*v.
\]

**Proposition 4.4** The operator $D$ has the following properties:

(i) If $\sigma, \sigma' \in \mathcal{G}_\infty$,
\[
D(\sigma' \circ \sigma) = D\sigma + \sigma_*^{-1}(D\sigma').
\]
In particular
\[
D\sigma^{-1} = -\sigma_*(D\sigma).
\]
(ii) If $\sigma \in \mathcal{G}_\infty$, $u \in \wedge^T \otimes \mathcal{J}_\infty^\infty$, $\mathcal{D}((\sigma^{-1}_* u) = \sigma^{-1}_* (Du) + [D\sigma, \sigma^{-1}_* u].$

(iii) If $\xi = \frac{d}{du} \sigma_u|_{u=0}$, with $\xi \in \mathcal{J}_\infty^\infty$, and $\sigma_u \in \mathcal{G}_\infty$ is the 1-parameter group associated to $\xi$, then

$$D\xi = \frac{d}{du} D\sigma_u|_{u=0}.$$ 

**Proof.**

(i)

$$\mathcal{D}(\sigma' \circ \sigma) = \chi - \sigma^{-1}_*(\chi) + \sigma^{-1}_*(\chi - (\sigma')^{-1}_*(\chi)) = \mathcal{D}\sigma + \sigma^{-1}_*(\mathcal{D}\sigma').$$

(ii)

$$\mathcal{D}(\sigma^{-1}_* u) = \sigma^{-1}_*[\sigma_\chi, u] = \sigma^{-1}_* [\chi - \mathcal{D}\sigma^{-1}, u]$$

$$= \sigma^{-1}_*(Du) - [\sigma^{-1}_*(\mathcal{D}\sigma^{-1}), \sigma^{-1}_* u] = \sigma^{-1}_*(Du) + [\mathcal{D}\sigma, \sigma^{-1}_* u].$$

(iii)

$$\frac{d}{du} D\sigma_u|_{u=0} = -\frac{d}{du} (\sigma^{-1}_u)_*(\chi) = -[\xi, \chi] = D\xi.$$

Proposition 4.3 says that $\mathcal{D}$ is projectable:

$$\mathcal{D}_k \mathcal{G}_{k+1} \rightarrow \mathcal{J}^k_\infty \mathcal{J}_k^\infty,$$

where

$$i(v) \mathcal{D}\sigma_{k+1} = \lambda_1^1 \sigma_{k+1}^{-1} \sigma_k^j v - v.$$ 

It follows from $[\chi, \chi] = 0$ that

$$0 = \sigma^{-1}_*[\chi, \chi] = [\sigma^{-1}_*(\chi), \sigma^{-1}_*(\chi)] = [\chi - \mathcal{D}\sigma, \chi - \mathcal{D}\sigma] = [\mathcal{D}\sigma, \mathcal{D}\sigma] - 2\mathcal{D}(\mathcal{D}\sigma),$$

therefore

$$\mathcal{D}(\mathcal{D}\sigma) - \frac{1}{2}[\mathcal{D}\sigma, \mathcal{D}\sigma] = 0. \quad (30)$$

If we define the non linear operator

$$\mathcal{D}_1 : \mathcal{T}^* \otimes J^\infty \mathcal{g} \rightarrow \wedge^2 \mathcal{T}^* \otimes J^\infty \mathcal{g},$$

then we can write (30) as

$$\mathcal{D}_1 \mathcal{D} = 0.$$ 

The operator $\mathcal{D}_1$ projects in order $k$ to

$$\mathcal{D}_1 : \mathcal{T}^* \otimes \mathcal{J}^k \mathcal{g} \rightarrow \wedge^2 \mathcal{T}^* \otimes \mathcal{J}^{k-1} \mathcal{g},$$

where

$$\mathcal{D}_1 u = Du - \frac{1}{2}[u, u]_k.$$ 

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We define the first non-linear Spencer complex by

\[
1 \to \mathcal{G}^j \to \mathcal{G}_{k+1}^j \xrightarrow{\mathcal{D}} T^* \otimes J^k \mathfrak{g} \xrightarrow{\mathcal{D}_1} \wedge^2 T^* \otimes J^{k-1} \mathfrak{g},
\]

which is exact in \( \mathcal{G}_{k+1}^j \).

Let be \( p^1(\mathcal{D}) : J^1 \mathcal{G}_{k+1} \to T^* \otimes J^k \mathfrak{g} \) the morphism associated to the differential operator \( \mathcal{D} \). It follows from proposition 4.3 that

\[
i(v)p^1(\mathcal{D})(j^1 \sigma_{k+1}) = \lambda^1 \sigma_{k+1}^{-1}(x) j^1_x \sigma_{v} - v
\]

where \( \sigma_{k+1} \) is an admissible section of \( \mathcal{G}_{k+1} \) and \( \sigma_k = \pi_k \sigma_{k+1} \).

**Proposition 4.5** The image of \( \mathcal{G}_{k,1} \) by \( p^1(\mathcal{D}) \) is the set

\[
B_{k,1} = \{ X \in T^* \otimes J^k \mathfrak{g} : v \in T \to t_{\ast}(X(v) + v) \in T \text{ is invertible} \}.
\]

**Proof.** It follows from (28) and (29) that \( t_{\ast}(i(v)D \sigma_{k+1}) = \sigma_{k+1}^{-1} f_{\ast} v. \sigma_{k+1} - v \), so \( t_{\ast}(i(v)D \sigma_{k+1} + v) = \sigma_{k+1}^{-1} f_{\ast} v. \sigma_{k+1} \) therefore \( D \sigma_{k+1} \in B_{k,1} \).

Conversely, let be \( Y \in B_{k,1} \) and consider \( \tilde{Y} \in B_{k,1} \) such that \( \text{id}^\ast \otimes \pi_{k}(\tilde{Y}) = Y \). Let be the map \( \Sigma : T_{x} \to T_{I_{k+1}(x)} \mathcal{G}_k \) defined by \( \Sigma(v) = i(v)\tilde{Y} + v \). There exists a local section \( \sigma \in \mathcal{G}_k \) such that \( \sigma(v) = I_{k+1}(x) \) and \( \sigma(v) = \Sigma(v) \). Then

\[
i(v)p^1(\mathcal{D})(j^1 \sigma) = \sigma^{-1}(x) j^1_x (v) - v
\]

\[
= t_{\ast}(i(v)Y + v) - v = i(v)Y.
\]

\( \square \)

5 The second linear and non-linear Spencer complex

Consider the projection \( \rho : \tilde{J}^\infty \mathfrak{g} \to T \) defined by \( \rho(v + \xi) = v - t_{\ast}(\xi) \). The kernel of \( \rho \) is \( \tilde{J}^\infty \mathfrak{g} \). We can include \( \wedge T^* \) in \( \tilde{J}^\infty \mathfrak{g} \) by the pullback for \( \rho^\ast \). Therefore we denote by \( \wedge T^* = \rho^\ast(\wedge T^*) \).

**Proposition 5.1** The sheaf \( \wedge T^* \otimes \tilde{J}^\infty \mathfrak{g} \) is a Lie algebra sub sheaf of \( \wedge(\tilde{J}^\infty \mathfrak{g})^* \otimes (\tilde{J}^\infty \mathfrak{g}) \). and

\[
[\tilde{\omega} \otimes \tilde{\xi}, \tilde{\tau} \otimes \tilde{\eta}] = [\tilde{\omega} \wedge \tilde{\tau} \otimes [\tilde{\xi}, \tilde{\eta}]_{\infty},
\]

where \( \tilde{\omega}, \tilde{\tau} \in \wedge T^*, \tilde{\xi}, \tilde{\eta} \in \tilde{J}^\infty \mathfrak{g} \).

**Proof.** Let be \( u = \tilde{\omega} \otimes \tilde{\xi} \in \wedge T^* \otimes \tilde{J}^\infty \mathfrak{g} \). For any \( \tilde{\tau} \in \wedge T^* \), \( i(\tilde{\xi})\tilde{\tau} = 0 \), then, applying (22) we obtain \( i(u)\tilde{\tau} = 0 \), and by (23) \( \Sigma(u)\tilde{\tau} = 0 \). So (24) implies \( [\tilde{\omega} \otimes \tilde{\xi}, \tilde{\tau} \otimes \tilde{\eta}] = [\tilde{\omega} \wedge \tilde{\tau} \otimes [\tilde{\xi}, \tilde{\eta}]_{\infty} \).

\( \square \)

**Corollary 5.1** \( \wedge T^* \otimes \tilde{J}^k \mathfrak{g} \) is a sheaf in Lie algebras for \( k \geq 1 \).
Proof. The bracket $[\mathbf{1}]_\infty$ projects to $[\mathbf{1}]_k$ in $\check{J}^k\mathfrak{g}$. Therefore $\check{\mathcal{T}}^* \otimes \check{J}^k\mathfrak{g}$ is well defined as a sheaf.

Let be the fundamental form
$$\chi \in (\check{J}^\infty\mathfrak{g})^* \otimes (\check{J}^\infty\mathfrak{g})$$
defined by
$$i(\hat{\xi})\chi = (\rho_1 + t_*)(\hat{\xi}) = v - t_* (\xi),$$
where $\hat{\xi} = v + \xi \in \mathcal{T} \oplus \check{J}^\infty\mathfrak{g}$. In another words, $\chi$ is the projection of $\check{J}^\infty\mathfrak{g}$ on $\mathcal{T}$, parallel to $\check{J}^\infty\mathfrak{g}$.

We define the second linear Spencer operator $\overline{\mathcal{D}}$ by
$$\overline{\mathcal{D}} : \check{\mathcal{T}}^* \otimes \check{J}^\infty\mathfrak{g} \to \check{\mathcal{T}}^* \otimes \check{J}^\infty\mathfrak{g},$$
where
$$\nu : \check{\mathcal{T}}^* \otimes \check{J}^\infty\mathfrak{g} \to \check{\mathcal{T}}^* \otimes \check{J}^\infty\mathfrak{g},$$
$$\hat{\omega} \otimes \hat{\xi} \to \omega \otimes \xi.$$  
We project this isomorphism $\nu$ in order $k$ to
$$\nu_k : \check{\mathcal{T}}^* \otimes \check{J}^k\mathfrak{g} \to \check{\mathcal{T}}^* \otimes \check{J}^k\mathfrak{g},$$
$$\hat{\omega} \otimes \hat{\xi}_k \to \omega \otimes \xi_k.$$

Proposition 5.2 If $\omega \in \check{\mathcal{T}}^*$, and $u \in \check{\mathcal{T}}^* \otimes \check{J}^\infty\mathfrak{g}$, then:

(i) $\mathcal{L}(\chi)\hat{\omega} = d\hat{\omega};$

(ii) $[\chi, \chi] = 0;$

(iii) $[\chi, u] = \overline{\mathcal{D}}u.$

Proof. Let be $\hat{\xi} = v + \xi$, $\hat{\eta} = w + \eta \in \mathcal{T} \oplus \check{J}^\infty\mathfrak{g}.$

(i) As $\mathcal{L}(\chi)$ is a derivation of degree 1, it is enough to prove (i) for 0-forms $f$ and 1-forms $\hat{\omega} \in (\check{J}^\infty\mathfrak{g})^*$. From (23) we have $\mathcal{L}(\chi) f = i(\chi) df = df$. It follows from proposition 3.6(i) that

$$< \mathcal{L}(\chi)\hat{\omega}, \hat{\xi} \wedge \hat{\eta} > = \mathcal{L}(v - \xi_H) < \hat{\omega}, \hat{\eta} > - \mathcal{L}(w - \eta_H) < \hat{\omega}, \hat{\xi} >$$
$$= - \omega, [v - \xi_H, \hat{\eta}] + [\hat{\xi}, w - \eta_H] - \mathcal{L}(v - \xi_H) < \hat{\omega}, \hat{\xi} >$$
$$= - \omega, (v - \xi_H, w) - t_* (i(v - \xi_H) D\eta) + [v, w - \eta_H] + t_* (i(w - \eta_H) D\xi) >$$
$$+ \omega, [v, w] - t_* (i(v) D\eta) - i(w) D\xi + \mathcal{L}(\hat{\xi}, \hat{\eta}) >$$
$$= \mathcal{L}(v - \xi_H) < \omega, w - \eta_H > - \mathcal{L}(w - \eta_H) < \omega, v - \xi_H >$$
$$- \omega, [v, w] - \xi_H, w] - [v, \eta_H] + t_* (i(\xi_H) D\eta - i(\eta_H) D\xi + [\xi, \eta]) >$$
$$= \mathcal{L}(v - \xi_H) < \omega, w - \eta_H > - \mathcal{L}(w - \eta_H) < \omega, v - \xi_H >$$
$$- \omega, [v, w] - \xi_H, w] - [v, \eta_H] + [\xi, \eta_H] >$$

$$= \mathcal{L}(v - \xi_H) < \omega, w - \eta_H > - \mathcal{L}(w - \eta_H) < \omega, v - \xi_H >$$
$$- \omega, [v, w] - \xi_H, w] - [v, \eta_H] >$$

$$= \mathcal{L}(\hat{\omega}) = \mathcal{L}(\chi) f = df.$$
(ii) Applying proposition \[3.6\] (iii), we obtain
\[
\langle \frac{1}{2}[\chi, \chi], \xi \wedge \eta \rangle = [i(\hat{\xi})\chi, i(\hat{\eta})\chi] - i((i(\hat{\xi})\chi, \hat{\eta}) - [i(\hat{\eta})\chi, \hat{\xi}] - i((\hat{\xi}, \hat{\eta})\chi) \chi \\
= [v - \xi_H, w - \eta_H] - ([v - \xi_H, w] - t_*(i(v - \xi_H)D\eta)) \\
+ (w - \eta_H, v) - t_*(i(w - \eta_H)J\xi) \\
+ (v, w) - t_*(i(v)\eta - i(w)\xi - [\xi, \eta]_\infty)) \\
= \left[\xi_H, \eta_H\right] - t_*(i(\xi_H)D\eta - i(\eta_H)D\xi + [\xi, \eta]_\infty) \\
= \left[\xi_H, \eta_H\right] - t_*(\xi, \eta) = 0.
\]

(iii) It follows from \[24\] that, for \(u = \omega \otimes \xi\),
\[
[\chi, u] = \mathcal{L}(\chi)\omega \otimes \xi + (-1)^r\omega \wedge [\chi, \xi] - (-1)^{2r}d\omega \wedge i(\xi)\chi \\
= d\omega \otimes \xi + (-1)^r\omega \wedge [\chi, \xi].
\]

As \(D\) is characterized by proposition \[2.6\], it is enough to prove \([\chi, \xi] = D\xi\). It follows from propositions \[3.2\] (ii) and \[3.6\] (ii) that
\[
i(\eta)[\chi, \xi] = [i(\eta)\chi, \xi]_\infty - i(\eta, \xi)_\infty)\chi \\
= [w - \eta_H, \xi]_\infty - [w - \eta_H, \xi]_\infty)\chi \\
= [w - \eta_H, \xi]_\infty - [w - \eta_H, \xi]_\infty)\chi \\
= [w - \eta_H, \xi]_\infty + i(w - \eta_H)D\xi - (w - \eta_H, \xi) - t_*(i(w - \eta_H)D\xi) \\
= \rho_*(i(w - \eta_H)D\xi) \\
= i(\eta)\nu^{-1}D\xi = i(\eta)D\xi.
\]

\[\square\]

If \(u, v \in \wedge T^* \otimes \tilde{J}^\infty g\), with \(\deg u = r\), \(\deg v = s\), then we get from \[25\] and proposition \[4.2\] (iii) that
\[
D[u, v] = [D u, v] + (-1)^r[u, D v],
\]
and
\[
[\chi, [\chi, u]] = [[\chi, \chi], u] - [\chi, [\chi, u]] = -[\chi, [\chi, u]],
\]
therefore, \(D^2 u = 0\), or
\[
D^2 = 0.
\]

Then it is well defined the second linear Spencer complex,
\[
0 \rightarrow \Gamma(g) \xrightarrow{j_k} \tilde{J}^k g \xrightarrow{\nabla} \tilde{T}^* \otimes \tilde{J}^\infty g \xrightarrow{\nabla} \wedge^2 \tilde{T}^* \otimes \tilde{J}^\infty g \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \wedge^m \tilde{T}^* \otimes \tilde{J}^\infty g \rightarrow 0,
\]
where \(\dim T = m\). This complex projects on
\[
0 \rightarrow \Gamma(g) \xrightarrow{j_k} \tilde{J}^k g \xrightarrow{\nabla} \tilde{T}^* \otimes \tilde{J}^{k-1} g \xrightarrow{\nabla} \wedge^2 \tilde{T}^* \otimes \tilde{J}^{k-1} g \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \wedge^m \tilde{T}^* \otimes \tilde{J}^{k-m} g \rightarrow 0.
\]

This complex is exact (see \[Ma1\], \[Ma2\], \[KS\]).

Let’s now introduce the second nonlinear Spencer operator \(\overline{D}\). The “finite” form \(\overline{D}\) of the linear Spencer operator \(\overline{D}\) is defined by
\[
\overline{D}\sigma = \chi - \sigma^{-1}(\chi),
\]
where \(\sigma \in \mathcal{G}_\infty\).
Proposition 5.3 The operator $\mathcal{D}$ take values in $\mathcal{J}^* \otimes \mathcal{J}^\infty_1 g$, so
\[
\mathcal{D} : G_\infty \rightarrow \mathcal{J}^* \otimes \mathcal{J}^\infty_1 g,
\]
and
\[
i(v) (\mathcal{D}\sigma)_k = v - f_s^{-1}(\sigma_{k+1}.v.\sigma_{k+1}^{-1}) + (v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v),
\]
where $\sigma = \lim \text{proj} \sigma_k \in G_\infty$.

Proof. Applying (26) and (27), it follows for $\xi \in \mathcal{J}^\infty_1 g$,
\[
i(\xi) \mathcal{D}\sigma = i(\xi) \overline{\chi} - \sigma_s^{-1}(i(\sigma_s(\xi)) \overline{\chi}) = 0,
\]
and for $v \in \mathcal{T}$,
\[
i(v) \mathcal{D}\sigma = i(v) \overline{\chi} - \sigma_s^{-1}(i(\sigma_s(v)) \overline{\chi}),
\]
therefore from proposition 2.3
\[
i(v) (\mathcal{D}\sigma)_k = i(v) \overline{\chi} - (\sigma_s^{-1})_k \overline{\chi} + (i(f_s(v) + j^1 \sigma_k.v.\lambda^1 \sigma_k^{-1} - \lambda^1 \sigma_{k+1}.v.\lambda^1 \sigma_{k+1}^{-1}) \overline{\chi})
\]
\[
= v - (\sigma_s^{-1})_k \overline{\chi} + (f_s v - t_s(j^1 \sigma_k.v.\sigma_{k+1}^{-1} - \lambda^1 \sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi})
\]
\[
= v - (\sigma_s^{-1})_k \overline{\chi} + (f_s v - (j^1 \sigma_k.v.\lambda^1 \sigma_k^{-1} - \lambda^1 \sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi})
\]
\[
= v - (\sigma_s^{-1})_k \overline{\chi} + (f_s v - f_s v + \sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi}
\]
\[
= v - (\sigma_s^{-1})_k \overline{\chi} + (f_s v - f_s v - \sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi}
\]
\[
= v - (\sigma_s^{-1})_k \overline{\chi} + (f_s v - f_s v - \sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi}
\]
\[
= v - j^1 \sigma_k^{-1}.(\sigma_{k+1}.v.\sigma_{k+1}^{-1}) \overline{\chi} + (v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v)
\]
\[
= v - f_s^{-1}(\sigma_{k+1}.v.\sigma_{k+1}^{-1}) + (v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v).
\]
Since that
\[
t_s(v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v) = t_s(v - j^1 \sigma_k^{-1}.(\sigma_{k+1}.v.\sigma_{k+1}^{-1}).j^1 \sigma_k) = v - f_s^{-1}(\sigma_{k+1}.v.\sigma_{k+1}^{-1})
\]
we get
\[
\nu(i(v)(\mathcal{D}\sigma)_k) = v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v.
\]
\[
\Box
\]

Corollary 5.2 We have $\mathcal{D}\sigma = 0$ if and only if $\sigma = j^\infty_1(\pi_0 \circ \sigma)$, where $\pi_0 : G_\infty \rightarrow G$.

Corollary 5.3 If $\sigma_{k+1} \in G_{k+1}$, then
\[
(\sigma_{k+1})_s(v) = \sigma_{k+1}.v.\sigma_{k+1}^{-1} + (\sigma_{k+1})_s(i(v) \mathcal{D}\sigma_{k+1}),
\]
for $v \in \mathcal{T}$.

Proof. It follows from proposition 3.5 (iii) and proposition 5.3 that
\[
(\sigma_{k+1})_s(i(v) \mathcal{D}\sigma_{k+1}) = f_s (v - f_s^{-1}(\sigma_{k+1}.v.\sigma_{k+1}^{-1})) + j^1 \sigma_k. (v - j^1 \sigma_k^{-1}.\sigma_{k+1}.v). \sigma_k^{-1}
\]
\[
= f_s v - \sigma_{k+1}.v.\sigma_{k+1}^{-1} + j^1 \sigma_k.v.\sigma_{k+1}^{-1} - \sigma_{k+1}.v.\sigma_{k+1}^{-1}
\]
\[
= (\sigma_{k+1})_s(v) - \sigma_{k+1}.v.\sigma_{k+1}^{-1}.
\]
\[
\Box
\]
Proposition 5.4 The operator \( \overline{\mathcal{D}} \) has the following properties:

(i) If \( \sigma, \sigma' \in \mathcal{G}_\infty \),

\[
\overline{\mathcal{D}}(\sigma' \circ \sigma) = \overline{\mathcal{D}}\sigma + \sigma^{-1}_*(\overline{\mathcal{D}}\sigma').
\]

In particular

\[
\overline{\mathcal{D}}\sigma^{-1} = -\sigma_*(\overline{\mathcal{D}}\sigma).
\]

(ii) If \( \sigma \in \mathcal{G}_\infty \), \( u \in \wedge \tilde{T}^* \otimes \tilde{J}^\infty \mathfrak{g} \),

\[
\overline{\mathcal{D}}(\sigma^{-1}_*u) = \sigma^{-1}_*(\overline{\mathcal{D}}u) + [\overline{\mathcal{D}}\sigma, \sigma^{-1}_*u].
\]

(iii) If \( \xi = \frac{d}{du}\sigma_u |_{u=0} \), with \( \xi \in \mathcal{J}^\infty \mathfrak{g} \), and \( \sigma_t \in \mathcal{G}_\infty \) is the 1-parameter group associated to \( \xi \), then

\[
\overline{\mathcal{D}}\xi = \frac{d}{du}\overline{\mathcal{D}}\sigma_u |_{u=0}.
\]

PROOF. (i)

\[
\overline{\mathcal{D}}(\sigma' \circ \sigma) = \overline{\mathcal{D}}(\sigma' \circ \sigma) - \overline{\mathcal{D}}(\sigma' \circ \sigma) + \overline{\mathcal{D}}(\sigma' \circ \sigma) = \overline{\mathcal{D}}\sigma + \sigma^{-1}_*(\overline{\mathcal{D}}\sigma').
\]

(ii)

\[
\overline{\mathcal{D}}(\sigma^{-1}_*u) = \sigma^{-1}_*\overline{\mathcal{D}}u - \sigma^{-1}_*(\overline{\mathcal{D}}\sigma^{-1}_*u) = \sigma^{-1}_*(\overline{\mathcal{D}}u) + [\overline{\mathcal{D}}\sigma, \sigma^{-1}_*u].
\]

(iii)

\[
\frac{d}{du}\overline{\mathcal{D}}\sigma_u |_{u=0} = -\frac{d}{du}(\sigma_u^{-1})_*(\overline{\mathcal{D}}) = -[\xi, \overline{\mathcal{D}}\sigma] = -[\xi - \sigma H, \overline{\mathcal{D}}\sigma] = \overline{\mathcal{D}}\xi + [\xi H, \overline{\mathcal{D}}\sigma].
\]

It follows from Proposition 3.6(ii) that

\[
\overline{\mathcal{D}}\sigma u |_{u=0} = -\frac{d}{du}(\sigma_u^{-1})_*(\overline{\mathcal{D}}) = -[\xi, \overline{\mathcal{D}}\sigma] = -[\xi - \sigma H, \overline{\mathcal{D}}\sigma] = \overline{\mathcal{D}}\xi + [\xi H, \overline{\mathcal{D}}\sigma].
\]

where \( \tilde{\eta} = v + \eta, \eta H = t_*\eta, \tilde{\xi} = t_*\xi \). See that \( i([\tilde{\eta}, \xi H]_{\infty})(\overline{\mathcal{D}}\sigma u |_{u=0}) = 0. \)

Proposition 5.3 says that \( \overline{\mathcal{D}} \) is projectable:

\[
\overline{\mathcal{D}} : \mathcal{G}_{k+1} \rightarrow \tilde{T}^* \otimes \tilde{J}^k \mathfrak{g}
\]

where

\[
i(v)\overline{\mathcal{D}}\sigma_{k+1} = v - f_{\sigma_{k+1}}^{-1}(\sigma_{k+1}, v, \sigma_{k+1}^{-1}) + (v - j^1 \sigma_{k+1}^{-1} \sigma_{k+1}^{-1} v).
\]

(33)

It follows from \( [\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma] = 0 \) that

\[
0 = \sigma^{-1}_*(\sigma^{-1}_*([\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma]) = [\sigma^{-1}_*([\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma]) = [\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma] = [\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma] - 2\overline{\mathcal{D}}(\overline{\mathcal{D}}\sigma),
\]

therefore

\[
\overline{\mathcal{D}}(\overline{\mathcal{D}}\sigma) = -\frac{1}{2}[\overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma] = 0.
\]
If we define the non linear operator

\[ \overline{\mathcal{D}}_1 : \tilde{T}^* \otimes \mathfrak{j}^\infty \mathfrak{g} \to \wedge^2 \tilde{T}^* \otimes \mathfrak{j}^\infty \mathfrak{g}, \]

\[ u \mapsto \overline{\mathcal{D}}_1 u - \frac{1}{2} [u, u], \]

then we can write (34) as

\[ \overline{\mathcal{D}}_1 \mathcal{D} = 0. \]

The operator \( \overline{\mathcal{D}}_1 \) projects in order to

\[ \overline{\mathcal{D}}_1 : \tilde{T}^* \otimes \mathfrak{j}^k \mathfrak{g} \to \wedge^2 \tilde{T}^* \otimes \mathfrak{j}^{k-1} \mathfrak{g}, \]

where

\[ \overline{\mathcal{D}}_1 u = \mathcal{D} u - \frac{1}{2} [u, u]_k. \]

We define the second non-linear Spencer complex by

\[ 1 \to \mathcal{G} \to \mathcal{G}_{k+1} \xrightarrow{\overline{\mathcal{D}}} \tilde{T}^* \otimes \mathfrak{j}^k \mathfrak{g} \xrightarrow{\overline{\mathcal{D}}} \wedge^2 \tilde{T}^* \otimes \mathfrak{j}^{k-1} \mathfrak{g}, \]

which is exact in \( \mathcal{G}_{k+1} \).

Let be \( p^1(\overline{\mathcal{D}}) : \tilde{T}^* \otimes \mathfrak{j}^k \mathfrak{g} \) the morphism associated to the differential operator \( \overline{\mathcal{D}} \). It follows from (32) that

\[ p^1(\overline{\mathcal{D}})(j^1 \sigma_{k+1})(v) = t_*(v - j^1 \sigma_k^{-1} \sigma_{k+1}(x).v + v - j^1 \sigma_k^{-1} \sigma_{k+1}(x).v \]

where \( \sigma_{k+1} \) is an admissible section of \( \mathcal{G}_{k+1} \) and \( \sigma_k = \pi_0 \sigma_{k+1} \).

**Proposition 5.5** The image of \( \mathcal{G}_{k,1} \) by \( p^1(\overline{\mathcal{D}}) \) is the set

\[ \tilde{B}^{k,1} = \{ X \in \tilde{T}^* \otimes \mathfrak{j}^k \mathfrak{g} : v \in T \to v - \nu(X(v)) \in T \text{ is inversible} \}. \]

**Proof.** It follows from (33) that

\[ t_*(v - \nu(i(v)\overline{\mathcal{D}} \sigma_{k+1})) = v - f^{-1}_*(\sigma_{k+1}, v, \sigma_{k+1}^{-1}), \]

so \( t_*(v - \nu(i(v)\overline{\mathcal{D}} \sigma_{k+1})) = f^{-1}_*(\sigma_{k+1}, v, \sigma_{k+1}^{-1}) \) therefore \( \overline{\mathcal{D}} \sigma_{k+1} \in \tilde{B}^{k,1} \).

Conversely, let be \( Y \in \tilde{B}^{k,1}_2 \) and consider \( \tilde{Y} \in \tilde{B}^{k+1,1}_2 \) such that \( id^* \otimes \pi_0(\tilde{Y}) = Y \). Let be the map \( \Sigma : T_x \to T_{I_{k+1}(x)} \mathcal{G}_k \) defined by \( \Sigma(v) = v - \nu(i(v)\tilde{Y}) \). There exists a local section \( \sigma \in \mathcal{G}_{k+1} \) such that \( \sigma(x) = I_{k+1}(x) \) and \( \sigma^{-1} (v) = \Sigma(v) \). Then

\[ i(v)p^1(\overline{\mathcal{D}})(j^1 \sigma) = t_*(v - j^1(\pi_0 \sigma)^{-1} \sigma(x).v + v - j^1(\pi_0 \sigma)^{-1} \sigma(x).v \]

\[ = t_*(v - j^1(\pi_0 \sigma)^{-1}.v + v - j^1(\pi_0 \sigma)^{-1}.v \]

\[ = t_*(v - \pi_0 i(v) \Sigma) + v - \pi_0 i(v) \Sigma \]

\[ = t_*(v - (v - \nu(i(v)\tilde{Y})) + v - (v - \nu(i(v)\tilde{Y})) \]

\[ = i(v)\tilde{Y}. \]

\[ \square \]
6 The sophisticated Spencer complex

Let be \( \delta \) the restriction of \( -D \) to \( \wedge T \otimes \gamma^k \). Set

\[
B^{k,p} = \wedge^p \tilde{T}^* \otimes \tilde{J}_k^k / \delta \left( \wedge^{p-1} \tilde{T}^* \otimes \gamma^{k+1} \right)
\]

and

\[
B^k = \oplus B^{k,p}.
\]

Let \( B^k \) be the sheaf of sections of \( B^k \).

**Proposition 6.1** The subsheaf \( \Gamma \left( \delta \left( \wedge^p \tilde{T}^* \otimes \gamma^{k+1} \right) \right) \) is an ideal of \( \wedge \tilde{T}^* \otimes \tilde{J}_k^k \).

**Proof.** Let be \( u \in \wedge \tilde{T}^* \otimes \tilde{J}_k^k \), with \( \deg u = r \), \( v \in \Gamma \left( \wedge \tilde{T}^* \otimes \gamma^{k+1} \right) \) with \( \deg v = s - 1 \) and \( u_{k+1} \in \wedge \tilde{T}^* \otimes \tilde{J}_{k+1}^k \) such that \( \pi_k u_{k+1} = u_k \). Then we get from \( \pi_k [u_{k+1}, v]_{k+1} = [u, \pi_k v]_k = 0 \) that \( [u_{k+1}, v]_{k+1} \in \Gamma \left( \wedge \tilde{T}^* \otimes \gamma^{k+1} \right) \). Therefore we get from (31) that

\[
-\delta [u_{k+1}, v]_{k+1} = [D u_{k+1}, \pi_k v]_k + (-1)^r [u, -\delta v]_k = (-1)^r [u, -\delta v]_k
\]

and the proposition is proved.

\( \square \)

**Corollary 6.1** \( B^k \) is a Lie algebra sheaf.

Given \( u \in B^{k,p} \) let be \( u_{k+1} \in \wedge \tilde{T}^* \otimes \tilde{J}_{k+1}^k \) that projects on \( u \). We define the operator

\[
\hat{D} : B^{k,p} \to B^{k,p+1}
\]

by

\[
\hat{D} u = D u_{k+1} \mod \delta \left( \wedge^p \tilde{T}^* \otimes \gamma^{k+1} \right)
\]

The sophisticated Spencer complex is

\[
0 \to j^k \mathcal{G} \to B^k \xrightarrow{\hat{D}} B^{k,1} \xrightarrow{\hat{D}} B^{k,2} \to \ldots \xrightarrow{\hat{D}} B^{k,p} \xrightarrow{\hat{D}} B^{k,p+1} \xrightarrow{\hat{D}} \ldots \xrightarrow{\hat{D}} B^{k,n} \to 0 \quad (35)
\]

which is exact.

We introduce now the nonlinear version of (35). Let be

\[
G^{k+1}_k(x) = \{ X \in G_{k+1}(x) | T^{k+1}_k X = I_k(x) \}.
\]

We know that \( J^1 G_k \) has a affine structure on \( G_k \). The vector space for this affine structure is the fiber of

\[
v J^1 G_k = \{ w \in T J^1 G_k : \pi_0^1 w = 0 \}.
\]

Therefore we can identify

\[
\partial : G^{k+1}_k(x) \to T^*_x \gamma^k_x
\]

\[
X \quad \to \quad \partial X = X - I_{k+1}(x)
\]

It follows from (33) that for a section \( F \in G^{k+1}_k \) we get \( \overline{\delta} \overline{D} F = -\overline{\delta} (v_k^{-1} \partial F) \), where \( \partial F = F - (I_{k+1}) \).

Let be \( F \in G_k \) and \( F_1 \in G_{k+1} \) such that \( \pi_k F_1 = F \) and let be

\[
\hat{D} F = \overline{D} F_1 \mod \delta (\gamma^{k+1}) \in B^{k,1}
\]
Let’s prove that this class depends only of $F$. If $F_2$ is another section in $G_{k+1}$ such that $\pi_k F_2 = F$, then $F_2 = F_1 G$ with $G \in G_{k+1}^k$. Then

$$\overline{DF}_2 = \overline{D}(F_1 G) = \overline{D}G + G^{-1}(\overline{D}F_1).$$

It follows from proposition 3.5 (iii) that $G^{-1}$ acts trivially on $\overline{D}F_1$. Therefore

$$\overline{DF}_2 = -\delta(\nu_k^{-1} \partial G + \overline{D}F_1),$$

what proves the class depends only of $F$.

Let’s define for $u \in B^{k,1}$

$$\hat{D}_1 u = \hat{D} u - \frac{1}{2}[u, u]$$

We obtain the nonlinear sophisticated complex

$$1 \rightarrow \mathcal{G}^1 \rightarrow \mathcal{G}_{k+1}^1 \rightarrow B^{k,1} \rightarrow B^{k,2},$$

which is exact in $G_{k+1}$.

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