Theory of Membrane in Heegaard Diagram Expansion

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Abstract

The vacuum amplitude of the closed membrane theory is investigated using the fact that any three-dimensional manifold has the corresponding Heegaard diagram (splitting) although it is not unique. We concentrate on the topological aspect with the geometry considered only perturbatively. In the simplest case where the action describes the free fields we find that the genus one amplitudes (lens space) are obtained from the $S^3$ amplitude by merely renormalizing the membrane tension. The amplitudes corresponding to the Heegaard diagram of genus two or higher can be calculated as the Coulomb amplitudes with arbitrary charge distributed on a knot or a link which corresponds to the set of branch points of a given regular or an irregular covering space. We also discuss the case of membrane instanton.

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1 Introduction

Although the M theory seems to be occupying one corner of the moduli space of all the string theories [1], it still lacks a formulation which is as intuitively comprehensible as the other string theories. TypeIIB theory is known to have both the string and the matrix formulations [2] and the M theory has so far only the matrix formulation [3]. The latter is known to have some connection to the membrane theory [4] but its legitimacy as the fundamental theory is in doubt [5].

Whether the theory is fundamental or just effective it may happen that the vacuum amplitude is dominated by the process of all possible closed membranes being created and annihilated. If it is true, then, the scattering amplitude of closed membranes can be calculated just by inserting some vertex operators before the path integration is performed for the vacuum amplitude.

To locate a vertex operator in some finite position in the Euclidean space, the conformal invariance was essential in the case of string theory. We implicitly assume that the same should be true also in the case of membrane theory, although the essential part of our discussions does not depend on the form of the action. In the case of closed string (fig. 1 (a)), the vacuum amplitude is obtained by summing all the amplitudes corresponding to fig. 1 (b), namely all the amplitudes for the closed two-dimensional surfaces. In the case of closed membrane theory we must consider the processes which involve all types of closed surfaces shown in fig. 1 (b). These surfaces should be created and annihilated in the vacuum process. The surfaces with a higher genus may be regarded as a bound state of lower genus surfaces but instead we treat them on an equal footing in the approach described here. The process of creation and the subsequent propagation, or the splitting etc. and finally the annihilation of a closed string is described by an amplitude on a closed surface. This should be extended also to the case of closed surface. This means that the vacuum amplitude of the closed membrane theory must be the sum of amplitudes of all possible three-dimensional closed manifolds.

It is well known that the torii with genus number $g = 0, 1, 2, \ldots, \infty$ exhaust all possible two-dimensional connected orientable surfaces. The three-dimensional case is more complicated. Here we rely on the fact that any closed, orientable three-dimensional manifold has at least one Heegaard diagram (or Heegaard splitting): let $H_1$ and $H_2$ be solid torii of the same genus $g$, and let $h : \partial H_2 \rightarrow \partial H_1$ be a homeomorphism. Then any closed connected orientable three-dimensional manifold $M^3$ can be identified with $H_1 \bigcup_h H_2$;

$$M^3 = H_1 \bigcup_h H_2.$$  

There is no uniqueness of the diagram as is shown in fig. 2 where $S^3$ is shown to have a Heegaard diagram $(H_1, H_2, h)$ with $H_1$ and $H_2$ of any number of genus $g$. This causes some interesting complications which can be solved if the Poincaré conjecture is valid for the three-dimensional manifolds as will be explained in the subsequent sections.

The application of membrane theory to physics may not come from the fundamental theory (even if it exists) but rather from the solitonic membrane. The process of membranes appearing and disappearing in the Euclidean time is nothing but the membrane instanton [6]. In some models, this is the only source of four-dimensional superpotential and, thus, this may be the reason for the smallness
of masses of quarks and leptons \[7\]. The issue will be briefly discussed in the subsequent sections although, admittedly, a lot will remain in the future investigation.

Section 2 will be devoted to the general discussions and mathematical preliminaries. Section 3 will deal with the case of \( g = 1 \) which corresponds to the lens spaces. Section 4 treats the case of \( g = 2 \) and higher. Section 5 contains summary and discussions.

2 General discussions

Much of our discussions are independent of the action we adopt for the membrane. Still, the concrete results seem to depend on it rather crucially because of the geometrical nature of our results. Let us start from the following expression for the generic vacuum amplitude:

\[
F = \sum_{\Sigma_i} \int e^{-S} \prod_{\alpha} d\psi_\alpha ,
\]

where \( \Sigma_i \) stands for all possible closed connected orientable three manifolds, and the action \( S \) is of the form:

\[
S = \int_{\Sigma_i} L_m d^3\sigma ,
\]

with an appropriate membrane Lagrangian \( L_m \). \( \psi_\alpha \) stands for the generic field which comes into the Lagrangian \( L_m \). We can adopt as \( L_m \) the super symmetric Lagrangian given in ref. \[7\] or, for simplicity, the following Weyl invariant bosonic membrane Lagrangian as an effective Lagrangian,

\[
L_m = \frac{T_m}{2} \sqrt{\text{det} g_{\alpha\beta}} \phi \left( g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - \phi^2 \right) ,
\]

where \( T_m \) is the membrane tension, \( g_{\alpha\beta} \ (\alpha, \beta = 1, 2, 3) \) is the metric tensor, \( X^\mu(\sigma) \ (\mu = 1, \ldots, D) \) is the membrane coordinate in the \( D \)-dimensional target space and \( \phi \) is the compensating field. Both the target space and the world volume space are taken to be Euclidean. Whether the Lagrangian \( \[3\] \) or its supersymmetric extension can be free of Weyl anomaly for certain value of \( D \) is itself an interesting problem but we will not touch on this issue here. Our main concern in this work is the sum over \( \Sigma_i \).

As is explained in the introduction each \( \Sigma_i \) has at least one Heegaard diagram \( (H_1, H_2, h) \). The fig. 3 (a) illustrates the case of \( g = 1 \) and the fig. 3 (b) corresponds to the case of \( g = 2 \). Each case can be transformed to a more physically intuitive fig. 4 (a) and fig. 4 (b), respectively. In fig. 4 (a), a thin torus surface is created at the Euclidean time \( t = t_1 \) and propagates within \( H_1 \) until \( t = 0 \). At \( t = 0 \), all sorts of \( h \) transform the torus surface into \( H_2 \). The surface propagates within \( H_2 \) until it disappears at \( t = t_3 \). Similar interpretation can be given to fig. 4 (b) with \( g = 2 \) torus. The \( g = 0 \) case is omitted here as the simplest exercise for the reader.

The whole problem, therefore, comes down to the issue of classification of \( h \) in each genus case. This, of course, is basically solved in the well established theorem due to Lickorish \[8\] which states that any \( h \) for genus \( g \) Heegaard diagram is isotopic to the composition of the Dehn twists along \( 3g-1 \) closed curves. The curves for \( g = 1 \) and \( g = 2 \) cases are already given in fig. 3 (a) and fig. 3 (b) as \( C_1, C_2 \) and \( C_1, \ldots, C_5 \) respectively and the general case in fig. 5. Although much is known for this problem \[9\], it still looks formidable to perform the summation in eq. \[1\] for all possible \( h \). Fortunately, there seem to
be ways out as will be explained in the following sections for each value of \( g \). The method is different in each case of \( g \) so that the discussion will be left to the sections which deal with individual cases.

In each case we cannot perform the integration for \( g_{\alpha\beta} \). We concentrate on the topological aspect and we pick a particular value for \( g_{\alpha\beta} \) and try to perturb \( g_{\alpha\beta} \) around this value. Instanton case is slightly different in that it is independent to \( g_{\alpha\beta} \) from the beginning and we must consider effectively the following case \[6\],

\[
L_m = \int_{\Sigma} C
\]

where \( C \) is a certain three form defined on \( \Sigma \) which is a three-dimensional manifold in a certain seven-dimensional compact space in case of M theory. \( C \) can be written as

\[
C = \sum_{i=1}^{b^{(3)}} a_i \omega_i
\]

where \( \omega_i \) is a component of a basis of three forms and \( b^{(3)} \) is the third Betti number of \( \Sigma \). These can be calculated in principle for a given Heegaard diagram \( (H_1, H_2, h) \). We touch on this case in the following separate sections for \( g = 1 \) and for \( g \geq 2 \).

In the process of adding the contributions from \( g = 0, 1, 2, \ldots \) we encounter the problem of double counting due to the non-uniqueness of the Heegaard diagram although it is meaningless until we know exactly how to determine the coefficient in front of each contribution. In principle, this can be solved assuming the validity of the Poincaré conjecture which states that the homology group together with the fundamental group classifies the topology of the manifolds. For example, \( g = 0 \) is \( S^3 \) and we know out of \( g = 1 \) Heegaard diagrams there exists only one with the trivial fundamental group. We, therefore, subtract this contribution from the \( g = 1 \) contribution. This kind of renormalization or subtraction procedure can in principle be applied to higher genus cases although Poincaré conjecture in three dimensional case is yet to be proven beyond \( g \geq 3 \).

3 \quad g = 1 \text{ case}

The Heegaard diagram \( (H_1, H_2, h) \) for \( g = 1 \) is well understood. Each \( h \) corresponds to the following \( SL_2(Z) \) operation

\[
\begin{pmatrix} i_2 \\ j_2 \end{pmatrix} = \begin{pmatrix} q' & p \\ p' & q \end{pmatrix} \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \quad qq' - pp' = 1 .
\]

(6)

Here \( (i_1, j_1) \) or \( (i_2, j_2) \) corresponds to the \( i_1 \) \( (i_2) \)-th meridian or to the \( j_1 \) \( (j_2) \)-th longitude respectively. The reducible case is \( q + q' = \pm 2 \).

The products of Dehn twists correspond to \( (i_1, j_1) = (1, 0) \) in eq. \[5\]:

\[
\begin{pmatrix} i_2 \\ j_2 \end{pmatrix} = \begin{pmatrix} q & p' \\ p & q' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} .
\]

(7)

\[1\]We are not certain whether we should include irreducible ones in our summation.
Since \((H_1, H_2, h^{-1}) = (H_1, H_2, h)\), we should calculate what sort of product of Dehn twists corresponds to \(h^{-1}\). We easily see that
\[
\left( \begin{array}{cc} q & p' \\ p & q' \end{array} \right)^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} q' \\ -p \end{array} \right) \simeq \left( \begin{array}{c} q' \\ p \end{array} \right),
\]
where \(\simeq\) means the equivalence as a Heegaard diagram or the topological equivalence. Together with eq. (6) this implies
\[
L(p, q) = L(p, q') \quad \text{when} \quad qq' = 1 \pmod{p},
\]
where \(L(p, q)\) signifies the Heegaard diagram corresponding to the product of Dehn twists given in eq. (7). Brody’s theorem \([10]\) states that the converse is true: The only topologically equivalent case for \(L(p, q)\) is given by eq. (9) except for the following trivial cases,
\[
L(p, q) \cong L(p, -q) \cong L(-p, -q) \cong L(p, q + kp).
\]

\(L(p, q)\) is known to be a lens space. The correspondence between the lens space and the Dehn twist is illustrated in fig. 6 (a) \(\sim\) fig. 6 (d) where we treat the case of \(L(5, 2)\) \([11]\).

We now work with the unit ball depicted in fig. 6 (a) where a surface point on the northern hemisphere after an eastward rotation of an angle equal to \(2\pi q/p\) is identified with the southern point with the same latitude. The \(q = 1\) case corresponds to \(S^3/Z_p\) and was used by E. Witten \([7]\) to construct a model for the doublet-triplet splitting without the twist the sector problem which accompanies the orbifold cases \([12]\). The fundamental group of \(L(p, q)\) is \(Z_p\) independently of \(q\) and, therefore, can be used instead of \(S^3/Z_p\) for the purpose of doublet-triplet splitting although the physical implication is not clear.

As an example, let us now try to compute \(F\) given in eq. (1) for the case of \(\Sigma_i = L(p, q)\) with the action given in eq. (3). We adopt the \(\phi = 1\) gauge and assume that the \(g_{\alpha\beta}\) can be perturbed around the metric corresponding to \(L(p, q)\) of fig. 6 (a). \(F\) can be written in the following form:
\[
F = \sum_{p, q} \int e^{-\frac{\beta^2}{2}T_m} d^3\sigma \partial_\alpha X \partial_\alpha X \prod_\sigma dx(\sigma),
\]
where the boundary condition for \(X(\sigma) = X(r, \theta, \varphi)\) with the usual spherical coordinate \((r, \theta, \varphi)\) is given by
\[
X(1, \theta, \varphi) = X(1, \pi - \theta, \varphi + \frac{2\pi q}{p}).
\]
It is easy to see that the expression given in eq. (11) is independent of \(q\). Assuming that we do not need any \(q\)-dependent arbitrary constant before summing over \(q\), we have
\[
F = \sum_p n_p e^{-\frac{\beta^2}{2}T_m} \int d^3\sigma \partial_\alpha X \partial_\alpha X \prod_\sigma dx(\sigma),
\]
where \(n_p\) is the number of \(q ( < p)\) with \(q\) and \(p\) having no common divisor. The boundary condition \([12]\) turns into a restriction on the coefficients in the expansion
\[
X(r, \theta, \varphi) = \sum_{n = 1}^{\infty} \sum_{m = -n}^{n} a_{n, m}(r) Y_n^m(\theta, \varphi),
\]
where \( Y^m_n(\theta, \varphi) \) is the usual spherical harmonics. We find

\[
n + \left(1 + \frac{2p}{p}\right) m = 2N, \tag{15}
\]

where \( N \) is an arbitrary integer. This leads to the following restrictions:

For odd \( p \)
- if \( kp \leq n < (k + 1)p \) then,
  - for even \( n \), \( m = 0, \pm 2p, \ldots, \pm (k)p, \)
  - for odd \( n \), \( m = \pm p, \pm 3p, \ldots, \pm [k]p, \)

where \( (k) \) is the largest even number smaller than or equal to \( k \) and \([k]\) is the largest odd number smaller than or equal to \( k \). Similar restriction can be obtained for even \( p \). The integration in (13) can be done in an elementary way and we obtain,

\[
F = \sum_{p, q} F \left( L(p, q), T_m \right) = \sum_p n_p F(S^3 / Z_p, T_m) = \sum_p n_p F(S^3, T_m^p), \tag{16}
\]

where \( F(A, T_m) \) is the contribution to \( F \) in eq. (13) from a given manifold \( A \) with the membrane tension \( T_m \). We have the expression

\[
T_m^p = T_m \frac{\sum_k (k(k + 1) + 1)}{\sum_k (kp(kp + 1) + 1)} \approx p^{-2}T_m. \tag{17}
\]

We may have a \( p \)-dependent constant \( \alpha_p \) multiplied to each contribution in equation (16). The number \( n_p \), therefore, may not mean anything but we list first several numbers in Table 1 anyway.

| \( p \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | \ldots |
|\ --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( n_p \) | 1 | 1 | 2 | 2 | 3 | 2 | 5 | 4 | 5 | 5 | 9 | 4 | 11 | 6 | \ldots |

Table 1: Values of first several \( n_p \)

Equation (16) together with (17) shows that the contribution from \( L(p, q) \) can be calculated from that of \( S^3 \) simply by renormalizing the membrane tension. Since \( p = 1 \) corresponds to \( S^3 \) either we subtract this from eq. (16) or we omit the contribution from \( g = 0 \) which also corresponds to \( S^3 \). These discussions are relevant only when we know how to compute \( \alpha_p \) on the basis of the unitarity or some other physical principles. We may take a limit of thin membrane and relate this to the string theory, where unitarity determines \( \alpha_p \). The case of membrane instanton (eq. 4) is much easier for \( L(p, q) \). We take the transverse coordinate to be vanishing in eq. (4). Then we have

\[
L_m = \int_{L(p, q)} Pd^3\sigma \tag{18}
\]

where \( P \) is a pseudoscalar density. This leads to

\[
F(L(p, q)) = F(S^3), \tag{19}
\]

for all values of \( p \) and \( q \).
4  \( g \geq 2 \) case

We start with the case of \( g = 2 \) Heegaard diagram \((H_1, H_2, h)\) where \( h \) corresponds to a product of any number of matrices each factor corresponding to one of five Dehn twists along the curves \( C_1, \ldots, C_5 \) in fig. 3 (b). Each Dehn twist can be expressed as a \( 4 \times 4 \) matrix since only four out of five curves \( C_1, \ldots, C_5 \) constitute a basis. What is remarkable in the \( g = 2 \) case is that we can choose \( C_1, \ldots, C_5 \) and, therefore, all the Dehn twists to be invariant under the \( 180^\circ \) rotation around the \( X \) axis shown in fig. 3 (b).

This fact leads to a theorem by Birman and Hilden \[13\] which states that a \( g = 2 \) Heegaard diagram \((H_1, H_2, h)\) is topologically equivalent to a 2-fold branched coverings of \( S^3 \) with a knot or a link as the branch set. The proof is illustrated in fig. 7 (a) to fig. 7 (d) \[13\].

The generic expression for this case can be written as,

\[
F = \int e^{-s_1-s_2} \prod_{\tilde{\sigma} \in L} \delta(\psi^{(1)}(\tilde{\sigma}) - \psi^{(2)}(\tilde{\sigma})) \prod_{\sigma} d\psi^{(1)}(\sigma)d\psi^{(2)}(\sigma) ,
\]

where \( s_i = \int_{S^3_i} L(\psi^{(i)}(\sigma))d\sigma \) and \( L \) stands for a given link. \( \psi^{(i)}(\sigma) \) is a field which comes into the Lagrangian corresponding to the \( i \)-th sphere \( S^3_i \).

First, let us consider the action in eq. \[3\]. We take \( \phi = 1 \) gauge and treat \( g_{\alpha\beta} \) perturbatively as in the case of \( g = 1 \). We have,

\[
F = \int \exp \left[ -\frac{D-2}{2} T_m \sum_{i=1}^2 \int_{S^3_i} d^3 \sigma \partial_\alpha X^{(i)} \partial_\alpha X^{(i)} \right] \prod_{\tilde{\sigma} \in L} \delta(X^{(1)}(\tilde{\sigma}) - X^{(2)}(\tilde{\sigma})) \prod_{\sigma} dX^{(i)}(\sigma) . \]

The integration can be done in an elementary way with the result,

\[
F = \int \exp \left[ (D-2)T_m \int d^3 \sigma E_\alpha(\tilde{\sigma}) E_\alpha(\tilde{\sigma}) \right] \prod_s dk(s) , \]

where \( \partial_\alpha E_\alpha = \int_{\tilde{\sigma}} (s) E_\alpha(\tilde{\sigma}) - \tilde{\sigma}) K(s) \). \( s \) is a parameter along the link \( L \). This means that we are calculating nothing but a Coulomb amplitude with all possible charge distribution on the link. Eq. \[22\] can be rewritten as,

\[
F = \int \exp \left[ -\frac{(D-2)T_m}{4\pi} \int_{\tilde{\sigma}(s) \in L} ds \int_{\tilde{\sigma}(s') \in L} ds' \frac{k(s)k(s')}{|\tilde{\sigma}(s) - \tilde{\sigma}(s')|} \right] \prod_s dk(s)
\]

\[
= \sum_L \left[ \det \left( \frac{1}{|\tilde{\sigma}(s) - \tilde{\sigma}(s')|} \right) \right]^{-1/2} , \tag{23}
\]

where \( \tilde{\sigma}(s), \tilde{\sigma}(s') \in L \). The summation is over all possible links. We obtained the result for the flat metric.

It will be very interesting to see what we will get for the generic metric. Our approach is to use perturbation theory for \( g_{\alpha\beta} \). To avoid the double counting we need to subtract from eq. \[22\] a contribution from manifold which has the same homology and the fundamental group corresponding to \( L(p, q) \) treated in the previous section. We should be able to obtain these for each \( L \) in principle.

Let us now proceed to discuss the case of \( g \geq 3 \). Here we depend on the Alexander-Hilden-Montesinos theorem \[15\], the Hilden and Montesinos version of which states that every closed orientable
3-manifold $M$ is an irregular 3-fold branched covering of $S^3$ branched over a knot. The irregularity means the lack of symmetry which we have in the case of $g = 2$. This does not matter for our purposes. Moreover, since the theorem applies to any three-manifold, the calculation applies to any value of $g$.

The result obtained is similar to the case of $g = 2$ except that we have two kinds of charges distributed on the knots,

$$F = \int \exp \left[ -\frac{(D-2)T_m}{4\pi} \int_{\sigma \in L} \frac{k_1(s)k_1(s') - k_1(s)k_2(s') + k_2(s)k_2(s')}{|\tilde{\sigma}(s) - \tilde{\sigma}(s')|} dsds' \right] \prod_s dk(s) .$$

(24)

Although this formula can be applied to any value of $g$, this does not necessarily undermine the formulae for $g = 1$ and $g = 2$ which have been performed previously since they are much simpler. Again we note that eq. (24) was obtained using the flat metric.

The membrane instanton for the case $g \geq 2$ does not seem to allow us to proceed without the knowledge of explicit form for basis three forms $\omega^{(3)}_{ijk,\alpha}$ in the following formula,

$$F = \int dk(s) \exp \left[ i \int d^3\sigma \left\{ p^{(1)}(\sigma) \int ds\delta(\tilde{\sigma}(s) - \tilde{\sigma}(s))p^{(1)}(\sigma)k(s) 
+ p^{(2)}(\sigma) + \int ds\delta(\tilde{\sigma}(s) - \tilde{\sigma}(s))p^{(2)}(\sigma)k(s) \right\} \right],$$

(25)

where $p = \varepsilon_{ijk} \left( \sum_{\alpha=1}^{b_3(X)} a_\alpha \omega_{ijk,\alpha} \right)$. This is the formula for $g = 2$ and we have a similar formula for $g \geq 2$. These formulae are quite useless unless we know what $\omega$’s are ($a_\alpha$ can be integrated as a four dimensional field).

5 summary and discussions

The vacuum amplitude for the membrane theory has been discussed. We started with the non-unique classification of three-dimensional manifolds using the Heegaard diagram. This gives some intuitive understanding of what is going on in the vacuum amplitude. That is, a thin torus of genus $g$ is created and becomes fatter. At certain stage it gets twisted in a homeomorphic manner and propagates until it disappears. A Heegaard diagram $(H_1, H_2, h)$ for $g = 1$ with reducible $h$ is a lens space $L(p, q)$. It has been shown that the vacuum amplitude contribution is the same as $S^3$ with the membrane tension renormalized. We have restricted all our calculations to zeroth order in $h_{\alpha\beta}$ where $g_{\alpha\beta} = g^{0}_{\alpha\beta} + h_{\alpha\beta}$. The contribution from $g \geq 2$ is described as a Coulomb amplitude with one ($g = 2$) or two ($g \geq 3$) kinds of arbitrary charges distributed on the knot or the link which plays the role of branch set of covering spaces. We have the two-fold covering space of $S^3$ for the case of $g = 2$ and three-fold covering space of $S^3$ for $g \geq 3$. Eq. (23) indicates that our result is a geometric one. Fig. 8 (a) and fig. 8 (b) contribute more or less the same amount to $F$. Both give a determinant of a matrix in eq. (23) with large diagonal elements and large matrix elements where $s$ and $s'$ are both close to either of $A$, $B$ and $C$. This indicates that not only the local but also global nature of the knot may be important. The summation over $g_{\alpha\beta}$ should play an important role in understanding the situation. Eq. (24) is valid for any three dimensional manifold with no reference to Heegaard diagram. This means in particular that we can use eq. (24) for the cases of $g = 1$ or $g = 2$. But still the Heegaard diagram seems to provide
us a means to understand the situation in an intuitive manner. It may also happen that smaller genus contributions provide us a good approximation. What we have done in this article is quite preliminary and more investigation has to be done especially to consider the generic $g_{\alpha \beta}$.

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fig. 1a a closed string

fig. 1b closed surfaces

fig. 2 \( g=2 \) Heegaard diagram for \( S^3 \)
fig.3a  Heegaard diagram $(H_1, H_2, h)$ for $g=1$

fig.3b  Heegaard diagram $(H_1, H_2, h)$ for $g=2$
fig. 4a  (H₁, H₂, h) depicted on the complex plane Z(t) for g=1

fig. 4b  (H₁, H₂, h) for g=2. Top(bottom) part is just a two closed loops b and d touching each other on P
fig. 5   general case for curves to be used for Dehn twist
fig.6a  Lens space $L(p, q)$ for $p = 5$, $q = 2$

fig.6b

fig.6c

fig.6d
fig.7a

fig.7b

fig.7c

fig.7d

fig.8a  clover knot on a plane except for points A, B and C

fig.8b  $S^1$ on a plane except for points A, B and C