BYZSHIELD: AN EFFICIENT AND ROBUST SYSTEM FOR DISTRIBUTED TRAINING

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ABSTRACT

Training of large scale models on distributed clusters is a critical component of the machine learning pipeline. However, this training can easily be made to fail if some workers behave in an adversarial (Byzantine) fashion whereby they return arbitrary results to the parameter server (PS). A plethora of existing papers consider a variety of attack models and propose robust aggregation and/or computational redundancy to alleviate the effects of these attacks. In this work we consider an omniscient attack model where the adversary has full knowledge about the gradient computation assignments of the workers and can choose to attack (up to) any q out of n worker nodes to induce maximal damage. Our redundancy-based method ByzShield leverages the properties of bipartite expander graphs for the assignment of tasks to workers; this helps to effectively mitigate the effect of the Byzantine behavior. Specifically, we demonstrate an upper bound on the worst case fraction of corrupted gradients based on the eigenvalues of our constructions which are based on mutually orthogonal Latin squares and Ramanujan graphs. Our numerical experiments indicate over a 36% reduction on average in the fraction of corrupted gradients compared to the state of the art. Likewise, our experiments on training followed by image classification on the CIFAR-10 dataset show that ByzShield has on average a 20% advantage in accuracy under the most sophisticated attacks. ByzShield also tolerates a much larger fraction of adversarial nodes compared to prior work.

1 INTRODUCTION

The explosive growth of machine learning applications have necessitated the routine training of large scale ML models in a variety of domains. Owing to computational and space constraints this training is typically run on distributed clusters. A common architecture of distributed training consists of a single central server (parameter server or PS) which maintains a global copy of the model and several worker machines. The workers compute gradients of the loss function with respect to the parameters being optimized. The results are returned to the PS which aggregates them and updates the model. A new copy of the model is broadcasted from the PS to the workers and the above procedure is repeated until convergence. TensorFlow (Abadi et al., 2016), MXNet (Chen et al., 2015), CNTK (Seide & Agarwal, 2016) and PyTorch (Paszke et al., 2019) are all examples of frameworks that utilize this architecture.

A major challenge with such setups as they scale is the fact that the system is vulnerable to adversarial attacks by the computing nodes (i.e., Byzantine behavior), or by failures/crashes thereof. 1 As a result, the corresponding gradients returned to the PS are potentially unreliable for use in training and model updates. Achieving robustness in the face of Byzantine node behavior (which can cause gradient-based methods to converge slowly or converge to a bad solution) is of crucial importance and has attracted significant attention in the community (Gunawi et al., 2018; Dean et al., 2012; Kotla et al., 2007).

One class of methods investigates techniques for suitably aggregating the results from the workers. (Blanchard et al., 2017) establishes that no linear aggregation method (such as averaging) can be robust even to a single faulty worker. Robust aggregation has been proposed as an alternative. Majority voting, geometric median and squared-distance-based techniques often coupled with averaging all fall into this category (Damaskinos et al., 2019; Yin et al., 2019; 2018; Xie et al., 2018; Blanchard et al., 2017; Chen et al., 2017). The benefit of such approaches is the robustness guarantees they provide. However, they have serious weaknesses. First, they are very limited in terms of the fraction of Byzantine nodes they can withstand. Moreover, their complexity often scales quadratically with the number of workers (Xie et al., 2018). Finally, their theoretical guarantees are insufficient to establish convergence and require strict assumptions such as convexity of the loss function.

1In this work, we assume that the PS is reliable.
Another category of defenses is based on redundancy and seeks resilience to Byzantine workers by replicating the gradient computation tasks such that each of them is computed by more than one machine (Rajput et al., 2019; Chen et al., 2018; Data et al., 2018; Yu et al., 2018). Even though this approach fundamentally requires more computation load, it comes with stronger guarantees of correcting the erroneous gradients. Existing techniques are sometimes combined with robust aggregation (Rajput et al., 2019). The main drawback of recent work in this category is that the model training can be easily disrupted by a powerful omniscient adversary who has full control of a subset of the nodes and can mount judicious attacks.

1.1 Contributions
In this work, we propose our redundancy-based method ByzShield that provides robustness in the face of a significantly stronger attack model than has been considered in prior work. In particular, we consider an omniscient adversary who has full knowledge of the tasks assigned to the workers. The adversary is allowed to pick any subset of q worker nodes to attack at each iteration of the algorithm. Our model allows for collusion among the Byzantine workers to inflict maximum damage on the training process.

Our work demonstrates an interesting link between achieving robustness and spectral graph theory. Specifically, we demonstrate that using optimal hypergraphs (constructed from mutually orthogonal Latin squares and Ramanujan graphs) allows us to assign redundant tasks to workers in such a way that the adversary’s ability to corrupt the gradients is significantly reduced as compared to the prior state of the art.

Our theoretical analysis shows that our method achieves a much lower fraction of corrupted gradients compared to other methods (Rajput et al., 2019); this is supported by numerical simulations which indicate over a 36% reduction on average in the fraction of corrupted gradients. Our experimental results on large scale training for image classification (CIFAR-10) shows that ByzShield has on average a 20% advantage in accuracy under the most sophisticated attacks.

Finally, we demonstrate the fragility of existing schemes (Rajput et al., 2019) to simple attacks under our considered attack model when the number of Byzantine workers is large. In these cases, prior state-of-the-art methods either diverge or are not applicable (Damaskinos et al., 2019; Rajput et al., 2019; El Mhamdi et al., 2018), whereas ByzShield can converge to accurate models.

1.2 Related Work
Prior work in this area considers adversaries with a wide range of powers. For instance, DETOX and DRACO consider adversaries that are only able to attack a random set of workers. In our case, an adversary possesses full knowledge of the task assignment and attacked workers can collude. Prior work which considers the same adversarial model as ours includes (Baruch et al., 2019; Xie et al., 2018; Yin et al., 2018; El Mhamdi et al., 2018; Shen et al., 2016).

One of the most popular robust aggregation techniques is known as mean-around-median or trimmed mean (Xie et al., 2018; Yin et al., 2018; El Mhamdi et al., 2018). It handles each dimension of the gradient separately and returns the average of a subset of the values that are closest to the median. Auror introduced in (Shen et al., 2016) is a variant of trimmed median which partitions the values of each dimension into two clusters using k-means and discards the smaller cluster if the distance between the two exceeds a threshold; the values of the larger cluster are then averaged. signSGD in (Bernstein et al., 2019) transmits only the sign of the gradient vectors from the workers to the PS and exploits majority voting to decide the overall update; this practice improves communication time and denies any individual worker too much effect on the update.

Krum in (Blanchard et al., 2017) chooses a single honest worker for the next model update. The chosen gradient is the one closest to its k ∈ N nearest neighbors. The authors recognized in later work (El Mhamdi et al., 2018) that Krum may converge to an ineffectual model in the landscape of non-convex high dimensional problems. They showed that a large adversarial change to a single parameter with a minor impact on the Lp norm can make the model ineffective. They present an alternative defense called Bulyan to oppose such attacks. Nevertheless, if K machines are used, Bulyan is designed to defend only up to (K − 3)/4 fraction of corrupted workers. Similarly, the method of (Chen et al., 2017) is based on the geometric median of means.

In the area of redundancy-based methods, DRACO in (Chen et al., 2018) uses a simple Fractional Repetition Code (FRC) (that operates by grouping workers) and the cyclic repetition code introduced in (Raviv et al., 2018; Tandon et al., 2017) to ensure robustness; majority vote and Fourier decoders try to alleviate the adversarial effects. That work ensures exact recovery (as if the system had no adversaries) with q Byzantine nodes and each task needs to be replicated r ≥ 2q + 1 times; the bound is information-theoretic minimum and DRACO is not applicable if it is violated. Nonetheless, this requirement is very restrictive for the typical assumption that up to half of the workers can be Byzantine.

DETOX in (Rajput et al., 2019) extends the work of (Chen et al., 2018) and uses a simple grouping strategy to assign the gradients. It performs multiple stages of aggregation to gradually filter the adversarial values. The first stage involves majority voting while the following stages perform robust aggregation, as discussed before. Unlike DRACO,
the authors do not seek exact recovery hence the minimum requirement in \( r \) is small. However, the theoretical resilience guarantees that DETOX provides depend heavily on a “random assignment” of tasks to workers and on “random choice” of the adversarial workers. Furthermore, their theoretical results hold when the fraction of Byzanines is less than \( 1/40 \). Under these assumptions, they show that on average a small percentage of the gradient computations will be distorted.

## 2 Parallel Training Formulation

The formulation we discuss is standard in distributed deep learning. Assume that the loss function of the \( i \)-th sample is \( l_i(w) \), where \( w \in \mathbb{R}^d \) is the parameter set of the model. We seek to minimize the empirical risk of the dataset, i.e.,

\[
\min_w L(w) = \min_w \frac{1}{n} \sum_{i=1}^{n} l_i(w)
\]

where \( n \) is the size of the dataset. This is typically solved iteratively via mini-batch Stochastic Gradient Descent (SGD) over distributed clusters. Initially, the parameters \( w \) are randomly set to \( w_0 \) (in general, we will denote the state of the model at the end of iteration \( t \) with \( w_t \)). Following this, a randomly chosen batch \( B_t \) of \( b \) samples is used to perform the update in the \( t \)-th iteration. Thus,

\[
w_{t+1} = w_t - \eta_t \frac{1}{|B_t|} \sum_{i \in B_t} \nabla l_i(w_t)
\]

where \( \eta_t \) denotes the learning rate of the \( t \)-th iteration. In this work, we focus on synchronous SGD in which the PS waits for all workers to return before performing an update.

Within the cluster of \( K+1 \) servers the PS updates the global model after receiving the computations from the \( K \) workers. It also stores the dataset and the model and coordinates the training protocol. The remaining servers, denoted \( U_0, \ldots, U_{K-1} \), will be the workers computing gradients on subsets of the batch.

**Worker Assignment:** A given batch \( B_t \) is split into \( f \) disjoint sets (or files) denoted \( B_{t,i} \), for \( i = 0, \ldots, f-1 \). These files are then assigned to the workers according to a bipartite graph \( G = (U \cup F_t, E) \), where \( U = \{U_0, U_1, \ldots, U_{K-1}\} \) and \( F_t = \{B_{t,i} : i = 0, 1, \ldots, f-1\} \) denote the workers and the files respectively. An edge exists between \( u \in U \) and \( v \in F_t \) if worker \( u \) is assigned file \( v \). In general, a given file is assigned to \( r > 1 \) workers \( (r \) is called the replication factor\); this allows for protection against Byzantine behavior. It follows that each worker is responsible for \( l = fr/K \) files; \( l \) is called the computational load of the system. We let \( N(S) \) denote the set of neighbors of a subset \( S \) of the nodes. Thus, \( N(B_{t,i}) \) denotes the set of files assigned to worker \( U_i \) and \( N(B_{t,i}) \) denotes the set of workers that are assigned file \( B_{t,i} \).

**Byzantine Attack Model:** We consider a model where \( q \) workers operate in a Byzantine fashion, i.e., they can return arbitrary values to the PS. Our attack setup is omniscient where a worker, Byzantine or honest, knows the data assignment of all other participants and the model at every iteration; the choice of the Byzantine workers can also change across iterations. Furthermore, the Byzantine machines can collude to induce maximal damage. We will suppose that the fraction of Byzantine workers \( \epsilon = q/K \) is less than \( 1/2 \).

We emphasize that this is a stronger adversary model than those considered in other redundancy-based work (Rajput et al., 2019; Chen et al., 2018). Let \( g_{t,i}^{(j)} \) be the value returned from the \( j \)-th worker to the PS for assigned file \( B_{t,i} \).

Then,

\[
g_{t,i}^{(j)} = \begin{cases} g_{t,i} & \text{if } U_j \text{ is honest}, \\ x & \text{otherwise} \end{cases}
\]

where \( g_{t,i} \) is the sum of the loss gradients on all samples in file \( B_{t,i} \), i.e.,

\[
g_{t,i} = \sum_{j \in B_{t,i}} \nabla l_j(w_t)
\]

and \( x \) is any arbitrary vector in \( \mathbb{R}^d \).

**Training Protocol:** The training protocol followed in our work is similar to the one introduced in (Rajput et al., 2019). Following the assignment of files, each worker \( U_j \) computes the gradients of the samples in \( N(U_j) \) and sends them to the PS. Subsequently, the PS will perform a majority vote across the computations of each file. Recall that each file has been processed by \( r \) workers. For each such file \( B_{t,i} \), the PS decides a majority value \( m_i \).

\[
m_i \overset{\text{def}}{=} \text{majority} \left\{ g_{t,i}^{(j)} : U_j \in N(B_{t,i}) \right\}.
\]

In our implementation, the majority function picks out the gradient that appears the maximum number of times. We ensure that there are no floating point precision issues with this method, i.e., all honest workers assigned to \( B_{t,i} \) will return the exact same gradient. However, potential precision issues can easily be handled by deciding the majority by defining appropriate clusters amongst the returned gradients.

Assume that \( r \) is odd (needed in order to avoid breaking ties) and let \( r' = \frac{r+1}{2} \). Under the majority rule in Eq.
We only consider biregular graphs with good expansion are one where any small enough set of nodes is guaranteed to have a large number of neighbors. This implies that the Byzantine nodes together process a large number of files which in turn means that their ability to corrupt the majority on many of them is limited.

For a bipartite graph $G = (V_1 \cup V_2, E)$ with $|V_1| = K_1, |V_2| = f$, its bi-adjacency matrix $H$ is defined as

$$H_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

We only consider biregular $G$ where the left and right degrees of all nodes are the same; these are denoted $d_L$ and $d_R$ respectively. Let $A = \frac{1}{\sqrt{d_L d_R}} H$ denote the normalized bi-adjacency matrix. It is known that the eigenvalues of $AA^T$ (or $A^T A$) are $1 = \mu_0 > \mu_1 \geq \cdots \geq \mu_{K-1} \geq \mu_K = \cdots = \mu_f = 0$ if $K < f$ (Chung, 1997).

For a subset of the vertices $S$, let $\text{vol}(S)$ denote the number of edges that connect a vertex in $S$ with another vertex not in $S$. We will use the following lemma (Zhu & Chugg, 2007).

**Lemma 1.** For any bipartite graph $G = (U \cup \mathcal{F}, E)$ and a subset $S$ of $U$ (or $\mathcal{F}$)

$$\frac{\text{vol}(N(S))}{\text{vol}(S)} \geq \frac{1}{\mu_1 + (1 - \mu_1) \frac{\text{vol}(S)}{|S|}}.$$ 

Now suppose that the bipartite graph $G$ chosen for the worker-file assignment has a second-eigenvalue $= \mu_1$. Based on Lemma 1, if $S$ is a set of $q$ Byzantine workers, then $\text{vol}(S) = q f$ since $G$ is bipartite. By substitution, we com-
We begin with the definition of a Latin square. We refer to the sets $R, C$ and $S$ as rows, columns and symbols of the Latin square, respectively. The representation of a LS is an $l \times l$ array $L$ in which the cell $L_{ij}$ contains the symbol $L(i,j)$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$L_1$ & $L_2$ & $L_3$ \\
\hline
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 \\
4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 \\
\hline
\end{tabular}
\caption{A set of three MOLS of degree 5.}
\end{table}

Claim 1. Let $\beta(q)$ be the maximum number of files one can distort with $q$ Byzantines (since there is a bijection between gradients and files distorting a file is equivalent to distorting the gradient of the samples it contains hence the two terms will be used interchangeably). Then,

$$\beta(q) \leq \frac{q\lceil S \rceil - \beta}{r' - 1} = \frac{q\lceil S \rceil - \beta}{(r'-1)/2} \leq \gamma.$$ (5)

The proof is in the Appendix.

4 TASK ASSIGNMENT

In this Section, we propose two different techniques of constructing the graph $G$ which determines the allocation of gradient tasks to workers in ByzShield. Our graphs have $|U| \leq |F|$ (i.e., fewer workers than files) and possess optimal expansion properties in terms of their spectrum.

4.1 Task Assignment Based on Latin Squares

We extensively use combinatorial structures known as Latin squares (Van Lint & Wilson, 2001) in short, LS) in this protocol. The basic definitions and known results about them that we will need in our exposition are presented below.

4.1.1 Basic Definitions and Properties

We begin with the definition of a Latin square.

Definition 1. A Latin square of degree $l$ is a quadruple $(R, C, S; L)$ where $R, C$ and $S$ are sets with $l$ elements each and $L$ is a mapping $L : R \times C \rightarrow S$ such that for any $i \in R$ and $x \in S$, there is a unique $j \in C$ such that $L(i,j) = x$. Also, any two of $i \in R$, $j \in C$ and $x \in S$ uniquely determine the third so that $L(i,j) = x$.

We refer to the sets $R, C$ and $S$ as rows, columns and symbols of the Latin square, respectively. The representation of a LS is an $l \times l$ array $L$ in which the cell $L_{ij}$ contains the symbol $L(i,j)$.

Algorithm 2: Proposed MOLS-based assignment.

Input: Batch size $b$, computation load $l$, redundancy $r \leq l - 1$, empty graph $G$ with worker vertices $U$ and file vertices $F_i$.

1. The PS partitions $B_t$ into $l^2$ disjoint files of $b/l^2$ samples each $B_t = \{B_{t,i} : i = 0, 1, \ldots, l^2 - 1\} = F_t$.
2. The PS constructs a set of $r$ MOLS $L_1, L_2, \ldots, L_r$ of degree $l$ with symbols $S = \{0, 1, \ldots, l - 1\}$.
3. for $k = 0$ to $r - 1$ do
5. for $s = 0$ to $l - 1$ do
7. end end

Definition 2. Two Latin squares $L_1 : R \times C \rightarrow S$ and $L_2 : R \times C \rightarrow S$ (with the same row and column sets) are said to be orthogonal when for each ordered pair $(s, t) \in S \times T$, there is a unique cell $(i, j) \in R \times C$ so that $L_1(i,j) = s$ and $L_2(i,j) = t$.

A set of $k$ Latin squares of degree $l$ which are pairwise orthogonal are called mutually orthogonal or MOLS. It can be shown (Van Lint & Wilson, 2001) that the maximal size of an orthogonal set of Latin squares of degree $l$ is $l - 1$.

We use a standard construction which yields $l - 1$ MOLS of degree $l$. Initially, pick a prime power $l$. (i) All row, column and symbol sets are to be the elements of $\mathbb{F}_l$ (the finite field of size $l$), i.e., $R = C = S = \mathbb{F}_l = \{0, 1, \ldots, l - 1\}$. (ii) Then, for each nonzero element $\alpha$ in $\mathbb{F}_l$, define $L_\alpha(i,j) = \alpha i + j$ (addition is over $\mathbb{F}_l$). The $l - 1$ squares created in this manner are MOLS since linear equations of the form $\alpha i + bj = s, c\alpha + dj = t$ have unique solutions $(i, j)$ provided that $\alpha d - bc \neq 0$ which is the case here.

Table 1 shows an example of the above procedure that yields four MOLS of degree 5 (only the first three are shown and will be used in the sequel) for $R = C = S = \mathbb{F}_5 = \{0, 1, \ldots, 4\}$ and $L_\alpha(i,j) = \alpha i + j$ (mod 5), $\alpha = 1, 2, 3$.

4.1.2 Redundant Task Allocation

To allocate the batch of an iteration, $B_t$, to workers, first, we will partition $B_t$ into $f = l^2$ disjoint files $B_{t,0}, B_{t,1}, \ldots, B_{t,l^2-1}$ of $b/l^2$ samples each where $l$ is the degree of a set of MOLS constructed as in Sec. 4.1.1. Following this we construct a bipartite graph specifying the placement according to the MOLS (see Algorithm 2).

The following example showcases the proposed protocol of Algorithm 2 using the MOLS from Table 1.
Table 2: File allocation for $l = 5$, $r = 3$ based on MOLS.  

| Node | Stores   | Node | Stores   | Node | Stores   |
|------|----------|------|----------|------|----------|
| $U_0$ | 0.9, 13, 17, 21 | $U_0$ | 0.8, 11, 19, 22 | $U_0$ | 0.7, 14, 16, 23 |
| $U_1$ | 1.5, 14, 18, 22 | $U_1$ | 1.9, 12, 15, 24 | $U_1$ | 1.8, 10, 17, 24 |
| $U_2$ | 2.6, 10, 19, 23 | $U_2$ | 2.5, 13, 16, 24 | $U_2$ | 2.9, 11, 18, 20 |
| $U_3$ | 3.7, 11, 15, 24 | $U_3$ | 3.6, 14, 17, 20 | $U_3$ | 3.5, 12, 19, 21 |
| $U_4$ | 4.8, 12, 16, 20 | $U_4$ | 4.7, 10, 18, 21 | $U_4$ | 4.6, 13, 15, 22 |

2(a) 1st replica. 2(b) 2nd replica. 2(c) 3rd replica.

Example 1. Consider $K = 15$ workers $U_0, \ldots, U_{14}$, $l = 5$ and $r = 3$. Based on our protocol the $l^2 = 25$ files of each batch $B_t$ would be arranged on a grid $L$, as defined above. Since $r = 3$, the construction (see line 2 of the algorithm) involves the use of 3 MOLS of degree $l = 5$, shown in Table 1. For the LS $L_1$ ($k = 0$ in step 3 of the algorithm) choose, e.g., symbol $s = 0$. The locations of $s = 0$ in $L_1$ are (0, 0), (1, 4), (2, 3), (3, 2) and (4, 1) and the files in those cells in $L$ are $B_{1,0}, B_{4,9}, B_{13,13}, B_{17,17}$ and $B_{21,21}$. Hence, $U_0$ will receive those files from the PS. The complete file assignment for the cluster is shown in Table 2 (the index $i$ of file $B_{t, i}$ has been used instead for brevity).

It is evident that each worker stores $l$ files. Another observation that follows directly from the placement policy (cf. Algorithm 2) and Definition 1 is that any two workers populated on the same Latin square do not share any files while Definition 2 implies that a pair of workers populated on two different orthogonal Latin squares should have exactly one file in common.

4.2 Task Assignment Based on Direct Graph Constructions

We next focus on direct construction of bipartite graphs for the task assignment. In this regard, we consider Ramanujan bigraphs which provably achieve the smallest possible value of the second largest eigenvalue of $AA^T$, as defined in Section 3 (see also (Feng & Li, 1996)).

4.2.1 Redundant Task Allocation

There are many ways to construct a $(d_L, d_R)$-biregular bipartite graph with the Ramanujan property (Hoholdt & Janwa, 2012). We use the one presented in (Burnwal et al., 2019). This method yields a $(s, m)$ or an $(m, s)$ bipartite graph (depending on the relation between the integers $m, s$) for $m \geq 2$ and a prime $s$. It is based on LDPC “array code” matrices (Fan, 2001) and proceeds as follows.

- **Step 1:** Pick integer $m \geq 2$ and prime $s$.

- **Step 2:** Define a cyclic shift permutation matrix $P_{s \times s}$

$$P_{s \times s} = \begin{cases} 1 & \text{if } j \equiv i - 1 \pmod{s}, \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$.

- **Step 3:** Construct matrix $B$ as

$$B = \begin{bmatrix} I_s & I_s & I_s & \cdots & I_s \\ I_s & P & P^2 & \cdots & P^{m-1} \\ I_s & P^2 & P^4 & \cdots & P^{2(m-1)} \\ I_s & P^3 & P^6 & \cdots & P^{3(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_s & P^{s-1} & P^{2(s-1)} & \cdots & P^{(m-1)(s-1)} \end{bmatrix}$$

$B$ is a $s^2 \times ms$ matrix constructed as a $s \times m$ block matrix. If $m \geq s$, then $H = B$ is chosen to be the biadjacency matrix of the bipartite graph, otherwise $H = B^T$ is chosen. In both cases, $H$ represents a Ramanujan bigraph. Note that $H$ is a zero-one biadjacency matrix since $P$ and hence all of its powers are permutation matrices.

As mentioned earlier, we will assume $K \leq f$. The rows of $H$ will therefore correspond to workers and the columns will correspond to files in both of the above cases. Since all blocks of $H$ are permutation matrices the support of each row (size of storage of worker) and column (replication of file) can be easily derived. This yields two cases when this construction is applied to our setup

$$(K, f, l, r) = \begin{cases} (ms, s^2, s, m) & \text{if } m < s, \\ (s^2, ms, m, s) & \text{otherwise}. \end{cases} \quad (6)$$

We will refer to the case $m < s$ as Ramanujan Case 1 and to the other case as Ramanujan Case 2. We can have setups with identical parameters between the MOLS formulation and Case 1; specifically, one can choose the same computation load $l$ (for prime $l$) and replication factor $r$ such that $r < l$ which in both schemes will make use of $K = rl$ workers and $f = l^2$ files. The need for a prime $l$ makes the Ramanujan scheme less flexible in terms of choosing the computation load, though. Ramanujan Case 2 requires $r \geq l$ hence we examine this method separately.

5 Distortion Fraction Analysis

In this section, we perform a worst-case analysis of the fraction of distorted files incurred by various aggregation methods, including that of ByzShield. This fraction will be defined as $\hat{\epsilon} = c_{\text{max}}/f$.

A few reasons motivate this analysis. Principally, our deep learning experiments (upcoming Section 6.2) show that $\hat{\epsilon}$ is a very useful indicator of the convergence of a model’s training. Also, this comparison shows the superiority of ByzShield in terms of theoretical robustness guarantees since we achieve a much smaller $\hat{\epsilon}$ for the same $q$. Finally, we demonstrate via numerical experiments that the quantity
We denote an eigenvalue \( x \).

We next compute the spectrum of \( AA \) with \( (\text{wal et al., 2019}) \) [Theorem 6] have computed the set of singular values of \( H \) of Ramanujan Cases 1 and 2. In order to have a direct comparison between the spectral properties of the MOLS scheme and these Ramanujan graphs, we have normalized \( H \) and computed the eigenvalues of the corresponding \( AA \) instead. For the MOLS scheme, the normalized bi-adjacency matrix \( A \) has \( d_L = l \) and \( d_R = r \). We denote an eigenvalue \( x \) with algebraic multiplicity \( y \) with \( (x, y) \). All proofs for this Section are in the Appendix.

**Lemma 2.**

- **MOLS and Ramanujan Case 1:** \( (AA^T)^{MOLS} \) and \( (AA^T)^{Ram,1} \) require \( 2 < r < l \) and have spectrum

\[
\{(1,1), (1/r, r(l-1)), (0, r-1)\}.
\]

- **Ramanujan Case 2:** \( (AA^T)^{Ram,2} \) requires \( r \leq l \) and has spectrum

\[
\{(1,1), (1/r, r(r-1)), (0, r-1)\}.
\]

The eigenvalues are sorted in decreasing order of magnitude. Interestingly, \( (AA^T)^{Ram,1} \) has exactly the same spectrum as if the graph was constructed from a set of MOLS. Overall, the parameters \( l, r, q \) number of workers \( K = rl \), number of files \( f = l^2 \) and the second largest eigenvalue of \( AA^T \), \( \mu_1 = 1/r \), are all identical across the two schemes.

**5.1 Upper Bounds of \( \hat{\epsilon}^{ByzShield} \) Based on Graph Expansion**

**5.1.1 MOLS-based Graphs**

Based on Lemma 2, \( \mu_1 = 1/r \) is the second largest eigenvalue of \( (AA^T)^{MOLS} \). Substituting in the expression of Eq. (5), Claim 1 gives us the exact value for \( \gamma \). It is a tight upper bound on \( c^{(q)}_{\text{max}} \) for the MOLS protocol and by normalizing it we conclude that an optimal attack when a set of \( q \) Byzantines is chosen, distorts at most a fraction

\[
\hat{\epsilon}^{MOLS} \leq \frac{\gamma}{f} = \frac{2q^2}{r^2} \frac{1}{1 + (r-1) \frac{2q^2}{r^2}}
\]

of the files.

**5.1.2 Ramanujan Graphs**

By Lemma 2, the underlying graph of the MOLS assignment is a Ramanujan graph and all results of Section 5.1.1 carry over verbatim to Case 1.

Case 2 needs to be treated separately since it leads to a different cluster setup with \( K = s^2 = r^2 \) workers and \( f = rl \) files with an underlying graph with \( \mu_1 = 1/s = 1/r \). A simple computation yields

\[
\hat{\epsilon}^{Ram,2} \leq \frac{\gamma}{f} = \frac{2q^2}{r^2} \frac{1}{r + (r-1) \frac{2q^2}{r^2}}.
\]

**5.2 Exact Value of \( \hat{\epsilon}^{ByzShield} \) in the Regime \( q \leq r \)**

When \( q \leq r \), i.e., in the regime where there are only a few Byzantine workers, we can provide exact values of \( \hat{\epsilon} \) for our constructions. In particular we pick the \( q \) workers to be Byzantine such that the multiset of the files stored collectively across them (also referred to as multiset sum) has the maximal possible number of files repeated at least \( r' \) times. In this way, the majority of the copies of those files is distorted and the aggregation produces erroneous gradients.

**Claim 2.** The maximum distortion fraction incurred by an optimal attack for the regime \( q \leq r \) is characterized as follows.

- If \( r = 3 \), then

\[
\hat{\epsilon}^{ByzShield} = \begin{cases} 
0 & \text{if } q < 2, \\
1/f & \text{if } q = 2, \\
3/f & \text{if } q = 3.
\end{cases}
\]

- If \( r > 3 \), then

\[
\hat{\epsilon}^{ByzShield} = \begin{cases} 
0 & \text{if } q < r', \\
1/f & \text{if } r' \leq q < r, \\
2/f & \text{if } q = r.
\end{cases}
\]

**5.3 Comparisons With Prior Work**

We compare with the values of \( \hat{\epsilon} \) achieved by baseline and other redundancy-based methods. Baseline approaches do not involve redundancy or majority voting; their aggregation is applied directly to the \( K \) gradients returned by the workers and hence \( f = K \) and \( c^{(q)}_{\text{max}} = q \) which yields a distortion fraction \( \hat{\epsilon} = q/K \). We also compare the actual values of \( c^{(q)}_{\text{max}} \) for ByzShield against the theoretical upper bounds based on the expansion analysis in Section 5.1.

**5.3.1 Achievable \( \hat{\epsilon} \) of Prior Work Under Worst-case Attacks**

We now present a direct comparison between our work and state-of-the-art schemes that are redundancy-based. DETOX in (Rajput et al., 2019) and DRACO in (Chen et al., 2018) both use the same Fractional Repetition Code (FRC) for the assignment and both use majority vote aggregation, hence we will treat them in a unified manner. Our goal is to establish that redundancy is not enough to achieve resilience
We pinpoint that at the regime $K/r$ (however the actual task assignment is in general different). An interesting observation is that the simulations of the ac-
tual value of $c_{\text{max}}^{q}$ were also identical across the two; thus, both have been summarized in the same Table 3. More sim-
ulations including Ramanujan Case 2 are shown in Section A.5 of the Appendix. From these tables we deduce that $\gamma$ is
a very accurate worst-case approximation of $c_{\text{max}}^{q}$.
Comparing with the achieved $\hat{\epsilon}^{\text{FRC}}$ computed as in Section 5.3.1, note that ByzShield can tolerate a higher number of Byzantines while consistently maintaining a small fraction $\hat{\epsilon}$. In contrast, a worst-case attack on FRC raises this error very quickly with respect to $q$. On average, $\hat{\epsilon}^{\text{ByzShield}} = 0.64\hat{\epsilon}^{\text{FRC}}$ in Table 3. Finally, ByzShield has a benefit over baseline methods for up to $q = 5$ with respect to $\hat{\epsilon}$.

6 LARGE-SCALE DEEP LEARNING EXPERIMENTS

6.1 Experiment Setup

We have evaluated our method in classification tasks on large datasets on Amazon EC2 clusters. This project is written in PyTorch (Paszke et al., 2019) and uses the MPICH library for communication among the devices. The implementation builds on DETOX and has been provided.\footnote{https://github.com/hwang595/DETOX} The principal metric used in our evaluation is the top-1 test accuracy.

The classification tasks were performed using CIFAR-10 with ResNet-18 (He et al., 2016). Experimenting with hyperparameter tuning we have chosen the values for batch size, learning rate scheduling, and momentum, among others. We used clusters of $K = 25$ and $K = 15$ workers of type c5.4xlarge on Amazon EC2 for various values of $q$. The PS was an i3.16xlarge instance in both cases. More details are in the Appendix.

The cluster of $K = 25$ workers utilizes the Ramanujan Case 2 construction presented in Section 4.2.1 with $r = l = 5$ resulting in partitioning each batch into $f = rl = 25$ files. For $K = 15$, we used the MOLS scheme (cf. Section 4.1) for $r = 3$ and $l = 5$ such that $f = l^{2} = 25$. The corresponding simulated values of $c_{\text{max}}^{q}$ and the fraction of distorted files, $\hat{\epsilon}$, under the attack on DETOX proposed in Section 5.3 are in Tables 4 and 3, respectively.

The attack models we tried are the following

- **ALIE** (Baruch et al., 2019): This attack involves communication among the Byzantines in which they jointly estimate the mean $\mu_{i}$ and standard deviation $\sigma_{i}$ of the batch’s gradient for each dimension $i$. They subsequently send a function of those moments to the PS such that it will distort the median of the results.

- **Constant**: Adversarial workers send a constant matrix with all elements equal to a fixed value; the matrix has the same dimensions as the true gradient.

---

Table 3: Distortion fraction evaluation for MOLS-based assignment for $(K, f, l, r) = (15, 25, 5, 3)$ and comparison.

| $q$ | $c_{\text{max}}^{q}$ | $\hat{\epsilon}^{\text{ByzShield}}$ | $\hat{\epsilon}^{\text{Baseline}}$ | $\hat{\epsilon}^{\text{FRC}}$ | $\gamma$ |
|-----|----------------------|-----------------------------|-------------------------------|-----------------------------|---------|
| 2   | 1                    | 0.04                        | 0.13                          | 0.2                         | 2.11    |
| 3   | 3                    | 0.12                        | 0.2                           | 0.2                         | 4.29    |
| 4   | 5                    | 0.2                         | 0.2                           | 0.4                         | 6.96    |
| 5   | 8                    | 0.32                        | 0.33                          | 0.4                         | 10      |
| 6   | 12                   | 0.48                        | 0.4                           | 0.6                         | 13.33   |
| 7   | 14                   | 0.56                        | 0.47                          | 0.6                         | 16.9    |

\footnote{https://github.com/kkonstantinidis/ByzShield}
The ALIE algorithm is, to the best of our knowledge, the most sophisticated attack in literature for centralized setups. Our experiments showcase this fact. Constant attack often alters the gradients towards wrong directions in order to disrupt the model’s convergence and is also a powerful attack. Reversed gradient is the weakest among the three, as we will discuss in the Section 6.2.

In all of our experiments, we chose the q Byzantines such that \( \tilde{c} \) is maximized; this is equivalent to a worst-case omniscient adversary which can control all devices.

The defense mechanism ByzShield applied on the outputs of the majority votes is coordinate-wise median (Yin et al., 2019) across the \( f \) gradients computed as in Algorithm 1 (lines 14-16). We decided to pair our method with median-based defenses (CIFAR-10).

The most popular baseline techniques we compare against are median-of-means (Minsker, 2015), coordinate-wise majority voting with \textit{signSGD} (Bernstein et al., 2019), \textit{Multi-Krum} (Damaskinos et al., 2019) and \textit{Bulyan} (El Mhamdi et al., 2018). The latter two are robust to only a small fraction of erroneous results and their implementations rely on this assumption. Specifically, if \( c_{\max}^{(q)} \) is the number of adversarial operands then \textit{Multi-Krum} requires at least \( 2c_{\max}^{(q)} + 3 \) total number of operands while the same number for \textit{Bulyan} is \( 4c_{\max}^{(q)} + 3 \). These constraints are very restrictive and the methods are inapplicable for larger values of \( q \) for which our method is Byzantine-robust. DETOX (Rajput et al., 2019) uses the aforementioned baseline methods for aggregating
the “winning” gradients after the majority voting.

6.2 Results

In order to effectively interpret the results, the fraction of distorted gradients $\hat{\epsilon}$ needs to be taken into account; we will refer to the values in Appendix Table 4 throughout this exposition. Due to the nature of gradient-descent methods, $\hat{\epsilon}$ may not be always proportional to how well a method works but it is a good starting point. We will restrict our interpretation of the results to the case of $K = 25$. For $K = 15$, we performed a smaller range of experiments demonstrating similar effects (cf. Figures 9, 10, 11 in Appendix).

Let us focus on the results under the median-based defenses. In Figure 2, we compare ByzShield, the baseline implementation of coordinate-wise median ($\hat{\epsilon} = 0.12, 0.2$ for $q = 3, 5$, respectively) and DETOX with median-of-means ($\hat{\epsilon} = 0.2$) under the ALIE attack. ByzShield achieves at least a 20% average accuracy overhead for both values of $q$ ($\hat{\epsilon} = 0.04, 0.08$ for $q = 3, 5$, respectively). In Figure 6 (reversed gradient), we observe similar trends. However, for $q = 9$ in DETOX, a fraction of $\hat{\epsilon} = 0.6$ majorities is distorted and even though this attack is considered weak, the method breaks (constant 10% accuracy).

Similarly, ByzShield significantly outperforms Bulyan and Multi-Krum-based schemes on ALIE (Figures 3, 4); Bulyan cannot be paired with DETOX for $q \geq 1$ for our setup since the requirement $f \geq 4c_{\text{max}}^{(q)}/3$, as discussed in Section 6.1, cannot be satisfied. The maximum allowed $q$ for DETOX with Multi-Krum (Figures 4 and 8) is $q = 5$.

In signSGD, only the sign information of the gradient is retained and sent to the PS by the workers. The PS will output the majority of the signs for each dimension. The method’s authors (Bernstein et al., 2019) suggest that gradient noise distributions are in principle unimodal and symmetric due to the Central Limit Theorem. Following the advice of (Rajput et al., 2019), we pair this defense with the stronger constant attack as sign flips (e.g., reversed gradient) are unlikely to affect the gradient’s distribution. ByzShield with median still enjoys an accuracy improvement of about 20% for $q = 3$ and a smaller one for $q = 5$. The results are in Figure 5.

All schemes defend well under the reversed gradient attack for small values of $q$ (Figures 6, 7, 8). Nevertheless, in Figure 7, by testing ByzShield on large number of Byzantine workers ($q = 9$) for which a fraction of $\hat{\epsilon} = 0.36$ files are distorted, we realize that it still converges to a relatively high accuracy; Bulyan cannot be applied in this case. Multi-Krum has an advantage over our method for this attack for $q = 9$ (Figure 8). DETOX cannot be paired with Multi-Krum in this case as it needs at least $2c_{\text{max}}^{(q)} + 3 = 7$ groups.

Note that in our adversarial scenario, DETOX may perform worse than its baseline counterpart depending on the fraction of corrupted gradients. For example, in Figure 6, for $q = 3$, $\hat{\epsilon}_{\text{DOETO}} = 0.2$ while $\hat{\epsilon}_{\text{Median}} = 3/25 = 0.12$ and this reflects on better accuracy values for the baseline median.

In summary, there are two takeaways from these results

- The distortion fraction $\hat{\epsilon}$ is a good predictor of an aggregation scheme’s convergence.
- ByzShield supersedes prior schemes by an average of 20% in terms of top-1 accuracy for large values of $q$ under various powerful attacks.

Training Time: In ByzShield, within an iteration, a worker performs $l$ forward/backward propagations (one for each file) and transmits $l$ gradients back to the PS. The rest of the approaches would have each worker return a single gradient. Hence, ByzShield spends more time on communication owing to the $l$ transmissions. In both DETOX and ByzShield a worker processes $r$ times more samples compared to a baseline technique. Thus, we expect redundancy based methods to consume more time on computation at the worker level than their baseline counterparts. Aggregation time varies a lot by the method and the number of files to be processed by the PS for the model update. As an example, baseline median, ByzShield and DETOX median-of-means took 3.14, 10.81 and 4 hours, respectively, for the full training under the ALIE attack with $q = 3$ (Figure 2). The corresponding per-iteration time estimate (average across iterations) is in Figure 12 of the Appendix.

We emphasize though that the central finding of our work is that careful assignment of tasks to workers can significantly improve the overall robustness of the training and our focus was not on optimizing the implementation. There is ample scope for improving the implementation to reduce the overall time consumption. This can be done, e.g., by using GPUs or workers with higher network bandwidth.

7 Conclusions and Future Work

We chose to utilize median for our method as it was performing well in all of our experiments. However, ByzShield can also be used with non-trivial aggregation schemes such as Bulyan and Multi-Krum and potentially yield even better results. Our scheme is significantly more robust than the most sophisticated defenses in literature at the expense of increased computation and communication time. We believe that implementation-related improvements such as adding support for GPUs can alleviate the computation overhead. Algorithmic improvements to make it more communication-efficient are also interesting directions for future work.
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A APPENDIX

A.1 Asymptotic Complexity

If the gradient computation has linear complexity and since each worker is assigned to $l$ files of $b/f$ samples each, the gradient computation cost at the worker level is $O((lb/f)d)$ ($K$ such computations in parallel). In our schemes, however, $b$ is a constant multiple of $f$ and $l \leq r$; hence, the complexity becomes $O(rd)$ which is identical to other redundancy-based schemes in (Rajput et al., 2019; Chen et al., 2018). The complexity of aggregation varies significantly depending on the operator. For example, majority voting can be done in time which scales linearly with the number of votes using MJRTY proposed in (Boyer & Moore, 1991). In our case, this is $O(Kd)$ as the PS needs to use the $d$-dimensional input from all $K$ machines. Krum (Blanchard et al., 2017), Multi-Krum (Damaskinos et al., 2019) and Bulyan (El Mhamdi et al., 2018) are applied to all $K$ workers by default and require $O(K^2(d + \log K))$.

A.2 Proof of Claim 1

Proof. As all nodes in $S$ have degree $l$, the number of outgoing edges from $S$ is $l|S|$. For a file to be distorted it needs at least $r'$ of its copies to be processed by Byzantine nodes. Furthermore, we know that collectively the nodes process at least $\beta$ files. Using these observations, we have

$$
\begin{align*}
\lvert S \rvert &\geq c^{(q)}_{\max} r' + (\lvert N(S) \rvert - c^{(q)}_{\max}) \\
\Rightarrow \lvert S \rvert &\geq c^{(q)}_{\max} r' + (\beta - c^{(q)}_{\max}) \\
\Rightarrow \lvert S \rvert &\geq c^{(q)}_{\max} (r' - 1) + \beta \\
\Rightarrow -\beta &\geq c^{(q)}_{\max} (r' - 1) \\
\Rightarrow c^{(q)}_{\max} &\leq \frac{\lvert S \rvert - \beta}{r' - 1} = \frac{ql - \beta}{(r-1)/2} \overset{\gamma}{=} \\
\end{align*}
$$

where in Eq. (7) we have substituted the result from Eq. (5).

A.3 Proof of Lemma 2

Proof. From (Burnwal et al., 2019) [Theorem 6], the singular values of the zero-one (unnormalized) biadjacency matrix $H$ can be characterized in three distinct cases; we utilize the first two of them. They are presented below with the singular values reported in decreasing order of magnitude.

- **Ramanujan Case 1:** If $2 \leq m < s$, then $H = B^T$ has singular values and corresponding multiplicities

  $$
  \{(\sqrt{sm}, 1), (\sqrt{s, m(s-1)}), (0, m-1)\}.
  $$

- **Ramanujan Case 2:** If $m \geq s$ and $s|m$, then $H = B$ has singular values and corresponding multiplicities

  $$
  \{(\sqrt{sm}, 1), (\sqrt{m, (s-1)s}), (0, s-1)\}.
  $$

In order to perform a direct comparison between the MOLS and Ramanujan construction we want to examine the eigenvalues of the normalized biadjacency matrix $A = \frac{1}{\sqrt{d_Ld_R}}H$ (from Section 3) multiplied by its transpose, i.e., we are interested in the spectrum of $(AA^T)_{Ran.1}$ and $(AA^T)_{Ran.2}$. In all cases, observe that $d_Ld_R = sm$ and the spectrum of $AA^T = \frac{1}{sm}HH^T$ in Lemma 2 of the main text is easily deduced; these eigenvalues will be the squares of the singular values of $H$ multiplied by $\frac{1}{sm}$.

We continue by computing the spectrum of $(AA^T)_{MOLS}$. Suppose that the workers are partitioned into $r$ parallel classes depending on the LS used to populate them. The classes for Latin squares $L_1, \ldots, L_r$ will be denoted with $P_1, \ldots, P_r$, respectively. The underlying principle is that a parallel class includes exactly one replica of each file. Let

$$
X = l \left( \frac{1}{\sqrt{lr}} \right)^2 I_l = \frac{1}{r} I_l = \begin{pmatrix}
1/r & 0 \\
0 & 1/r
\end{pmatrix}
$$

and

$$
Y = 1 \left( \frac{1}{\sqrt{lr}} \right)^2 J_l = \frac{1}{r} J_l
$$

where $J_n$ is the all-ones $n \times n$ matrix.

Based on the placement scheme introduced in Section 4.1.2 the matrix $(AA^T)_{MOLS}$ takes the following $K \times K$ block form.

$$
(AA^T)_{MOLS} = 
\begin{bmatrix}
X & Y & Y & \cdots & Y \\
Y & X & Y & \cdots & Y \\
Y & Y & X & \cdots & Y \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
Y & \cdots & Y & \cdots & X \\
Y & \cdots & Y & \cdots & X
\end{bmatrix}
$$

Hence, the blocks of $(AA^T)_{MOLS}$ take one of the forms $X, Y$. This is shown by observing that blocks $X$ concern multiplication of rows of $A$ with columns of $A^T$ which correspond to workers within a parallel class (if the index of the row of $A$ is the same as the index of the column of $A^T$ then $l$ entries $(\frac{1}{\sqrt{lr}})^2$ are added, otherwise there is no common file and the product is zero). Similarly, blocks that have the form of $Y$ concern pairs of workers from different parallel classes which share exactly one file.
We observe that

\[(AA^T)_{\text{MOLS}} = \frac{1}{I_r} \left( \begin{array}{cccc} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{array} \right) \otimes J_l + \frac{1}{r} I_{r}J_r. \quad (8)\]

The characteristic polynomial of \( J_l \) is \((\lambda - l)\lambda^{l-1}\) and its spectrum (with corresponding algebraic multiplicities) is \( \sigma(J_l) = \{(l, 1), (0, l - 1)\} \) (Horn & Johnson, 2012) which coincides with the spectrum of the adjacency matrix of a complete graph with self-edges.

**Note 1.** If \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( x \), then \( \lambda + c \) is an eigenvalue of \( A + cI \) with the same eigenvector \( x \).

For \( C \), note that \( \sigma(C) \) becomes \((r - 1, 1), (-1, r - 1)\) since it is a rank-1 update of \( J_r \) (cf. Note 1).

Let \( D := C \otimes J_l \) and spectra \( \sigma(C) = \{\lambda_1, \ldots, \lambda_r\} \) and \( \sigma(AA^T) = \{\mu_1, \ldots, \mu_l\} \) then

\[\sigma(D) = \{\lambda_i \mu_j, i = 1, \ldots, r, j = 1, \ldots, l\} = \{(l(r - 1), 1), (-l, r - 1), (0, (l - 1)r)\} \]

**Note 2.** If \( \lambda \) is an eigenvalue of \( A \) with eigenvector \( x \), then \( \alpha \lambda \) is an eigenvalue of \( \alpha A \) with the same eigenvector \( x \).

Using Note 2, the spectrum of the term \( \frac{1}{r}D \) in Eq. (8) becomes

\[\left\{ \left( \frac{l(r - 1)}{r} \right), (1, r - 1), \left( \frac{l}{r} (0, (l - 1)r) \right) \right\} \]

\[= \left\{ (1, l), (-1/r, r - 1), (0, (l - 1)r) \right\} \]

which by Note 2 yields

\[\sigma((AA^T)_{\text{MOLS}}) = \left\{ \left( \frac{1}{r} (l(r - 1) + 1/r, 1) \right), \cdots \right. \]

\[\left. (-1/r + 1/r, r - 1), \cdots \right\}

\[= \{(1, 1), (0, r - 1), (1/r, (l - 1)r)\} \]

\[\square \]

A.4 Proof of Claim 2

**Proof.** We begin by proving the result for the MOLS-based scheme. We note that there is a bijection between the files and the grid’s cells \((i, j)\) (cf. Section 4.1); we will refer to the files using their cell’s coordinates and vice versa. A worker \( U_k \) which belongs to \( L_\alpha \) is such that its files are completely characterized by the solutions to an equation of the form \( \alpha i + j = z \).

These facts immediately imply that workers belong to a LS \( L_\alpha \) do not have any files in common. As the equations corresponding to \( L_\alpha \) and \( L_\beta \) for \( \alpha \neq \beta \) are linearly independent, we also have that any two workers from different Latin squares share exactly one file.

The respective proofs for the two cases of the claim are given below.

- **Case 1:** \( r = 3 \).

  For \( q = 2 \) and based on the above observations, it is evident that attacking \( q = 2 \) workers cannot distort more than one file, their common file. This yields \( c_{max}^q = 1 \) and \( \hat{e} = 1/3 \).

  For \( q = 3 \), we want to show that we can distort up to \( c_{max}^q = 3 \) files. Suppose that the three Latin squares in which the workers lie are denoted \( L_{\alpha_i}, i = 1, 2, 3 \) where the \( \alpha_i \)’s are distinct. We can choose any \( z_1 \) and \( z_2 \) such that the workers \( U_1 \) and \( U_2 \) specified by \( \alpha_1 i + j_1 = z_1 \) and \( \alpha_2 j + j_2 = z_2 \) respectively. Suppose that the common file in these workers is denoted by \((i_1, j_1)\). For \( U_3 \) specified by \( \alpha_3 i + j = z_3 \) we only need to ensure that we pick \( z_3 \neq \alpha_3 j_1 + j_1 \). This can always be done since there are \( l - 1 \) other choices for \( z_3 \).

  With these we can see that any two workers within the set \( \{U_1, U_2, U_3\} \) share a file. As \( r = 3 \), this implies that \( c_{max}^q = 3 \) for \( q = 3 \) and \( \hat{e} = 3/3 \).

- **Case 2:** \( r > 3 \).

  The argument we make for this case follows along the same lines as that for \( q = 3 \) above. Note that to distort a given file we have to pick \( r' \) workers \( U_{k_1}, \ldots, U_{k_r} \) from distinct LS \( L_{\alpha_1}, \ldots, L_{\alpha_r} \) which correspond to a system of equations of the form

\[\alpha_1 i + j = z_{k_1} \]

\[\alpha_2 i + j = z_{k_2} \]

\[\vdots \]

\[\alpha_r i + j = z_{k_r} \]

This system of equations has to have a common solution denoted by \((i_1, j_1)\). We can pick at most \( r' \) more workers \( U_{k_{r+1}}, \ldots, U_{k_r} \) since \( q \leq r \). If these workers have to distort a different file, then they need to certainly have a common solution denoted \((i_2, j_2)\) along with one worker from \( \{U_{k_1}, \ldots, U_{k_r}\} \). Note that we cannot pick two workers from \( \{U_{k_1}, \ldots, U_{k_r}\} \) since any two such workers already share \((i_1, j_1)\).

Finally, we note that among these workers we cannot find another subset of \( r'' \) workers that share a third
The structure of the bi-adjacency matrix $B$

We want to derive the equations for the remaining parallel case. We continue by proving the claim for Ramanujan Case 2. Since Ramanujan Case 2 uses $B^2$ instead of $B$, that proof is very similar and will be omitted.

Let us start by substituting the parameters for Case 1 which are $m = r$ and $s = l$ so that $B$ becomes an $l \times r$ block matrix with dimensions $l^2 \times rl$ (tall matrix)

$$B = \begin{bmatrix} I_l & I_l & I_l & \ldots & I_l \\ I_l & P & P^2 & \ldots & P^{r-1} \\ I_l & P & P^4 & \ldots & P^{2(r-1)} \\ I_l & P^3 & P^6 & \ldots & P^{3(r-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_l & P^{r-1} & P^{2(l-1)} & \ldots & P^{(r-1)(l-1)} \end{bmatrix}. \quad (10)$$

As in the case of MOLS, suppose that the workers are partitioned into $r$ parallel classes $\mathcal{P}_1, \ldots, \mathcal{P}_r$ depending on which block-column of $B$ was used to populate them. In this case, the $i^{th}$ block-column of $B$ is mapped to the $i^{th}$ parallel class of workers, $\mathcal{P}_i$.

Also, define a bijection $\phi$ between the indices $0, \ldots, l^2 - 1$ of the files (rows of $B$) and tuples of the form $(x, y)$ where $x, y \in \{0, \ldots, l-1\}$ such that the mapping for a file index $i$ is

$$\phi: i \rightarrow (x, y) = ([i/l], i \pmod{l}).$$

In the remaining part, we will be referring to the files as tuples $(x, y)$.

Our next objective is to describe each worker $U_k$ with a linear equation which encodes the relationship between $x$ and $y$ of the files $(x, y)$ assigned to $U_k$.

Due to its special structure, let us focus on the first block column which corresponds to $\mathcal{P}_1$. Note that the files stored by each of the first $l$ workers $U_0, \ldots, U_{l-1}$ respectively satisfy the following equations

$$y = 0 \pmod{l}$$

$$y = 1 \pmod{l}$$

$$\vdots$$

$$y = l-1 \pmod{l}. \quad (11)$$

We want to derive the equations for the remaining parallel classes $\mathcal{P}_2, \ldots, \mathcal{P}_r$. For a class $\mathcal{P}_c \in \{\mathcal{P}_2, \ldots, \mathcal{P}_r\}$ and letting $U_{k_1}, \ldots, U_{k_{l-1}}$ be the workers of $\mathcal{P}_c$, their files respectively satisfy

$$(i-1)x - y = 0 \pmod{l}$$

$$(i-1)x - y = 1 \pmod{l}$$

$$\vdots$$

$$(i-1)x - y = l-1 \pmod{l}. \quad (12)$$

for all values of $i \in \{2, \ldots, r\}$. Simple observations of the structure of powers of the permutation matrix $P$ of Step 2 in Section 4.2.1 and that of matrix $B$ in Eq. (10) should convince the reader of the above fact.

Let us proceed to the optimal distortion analysis. The ensuing discussion is almost identical to that for MOLS presented earlier in the current section (refer to that for more details).

- **Case 1: $r = 3$.**

For $q = 2$, it is straightforward to see that any pair of linearly independent equations (from two adversarial workers of different classes) yields their unique common file $(x, y)$ which is distorted; hence $c_{\max}^{(q)} = 1$ and $\hat{\epsilon} = 1/f$.

For $q = 3$, we can either pick one equation (worker) from Eq. (11) and the remaining two from Eq. (12) or pick all three from Eq. (12). In any case, an appropriate choice of the three equations can guarantee that any subset of two of them are linearly independent so that $c_{\max}^{(q)} = 3$ gradients are distorted and $\hat{\epsilon} = 3/f$.

- **Case 2: $r > 3$.**

Pick the first $r$ adversaries $U_{k_1}, \ldots, U_{k_r}$ from distinct classes (the choice of classes is irrelevant). Jointly solving any two of them fully describes a gradient $(x, y)$ and substituting that solution to the remaining equations will guide us to the choice of the remaining Byzantines needed to distort the file $(x, y)$.

Then, for the other $r' - 1$ Byzantines we make sure that at least one of them, say $U_{k_d}$, does not belong to any of the classes which include $U_{k_1}, \ldots, U_{k_r}$ and that $U_{k_d}$ does not store $(x, y)$; this implies that by solving the system of the equations of $U_{k_d}$ and any other Byzantine chosen so far we compute a file $(x', y')$. To skew $(x', y')$ we need $r' - 2$ new adversaries $U_{k_{d+1}}, \ldots, U_{k_{r'}}$ which store it. It is clear that $(x', y')$ has to be a solution to their equations (none of these adversaries can be in $\{U_{k_1}, \ldots, U_{k_r}\}$). Then, $c_{\max}^{(q)} = 2$ and $\hat{\epsilon} = 2/f$.

A.5 **Extra Simulation Results on $c_{\max}^{(q)}$ and $\hat{\epsilon}$**

More simulations similar to those of Section 5.3 of the main paper are shown in Tables 4, 5 and 6. Note that we evaluate
Table 4: Distortion fraction evaluation for Ramanujan-based assignment (Case 2) for \((m, s) = (5, 5)\), i.e., \((K, f, l, r) = (25, 25, 5, 5)\) and comparison.

| \(q\) | \(c_{\text{max}}\) | \(\hat{\epsilon}_{\text{ByzShield}}\) | \(\hat{\epsilon}_{\text{Baseline}}\) | \(\hat{\epsilon}_{\text{FRC}}\) | \(\gamma\) |
|------|----------------|-----------------|-----------------|-----------------|-------|
| 3    | 1              | 0.04            | 0.12            | 0.2             | 2.43  |
| 4    | 1              | 0.04            | 0.16            | 0.2             | 3.9   |
| 5    | 2              | 0.08            | 0.2             | 0.2             | 5.56  |
| 6    | 4              | 0.16            | 0.24            | 0.4             | 7.35  |
| 7    | 5              | 0.2             | 0.28            | 0.4             | 9.25  |
| 8    | 7              | 0.28            | 0.32            | 0.4             | 11.23 |
| 9    | 9              | 0.36            | 0.36            | 0.6             | 13.28 |
| 10   | 12             | 0.48            | 0.4             | 0.6             | 15.38 |
| 11   | 14             | 0.56            | 0.44            | 0.6             | 17.54 |
| 12   | 17             | 0.68            | 0.48            | 0.8             | 19.73 |

only up to \(q = 13\) in Table 5 since the problem quickly becomes computationally intractable and takes too long to finish as the number of combinations scales exponentially.

A.6 Deep Learning Experiments Hyperparameters

The images have been normalized using standard values of mean and standard deviation for the dataset. We chose \(b = 750\) for both cases \((K = 25)\) and \((K = 15)\). The value used for momentum was set to 0.9 and we trained for 13 epochs in all experiments. In the following tables, a learning rate scheduling is denoted with \((x, y, z)\); this notation signifies the fact that we start with a rate equal to \(x\) and every \(z\) iterations we multiply the rate by \(y\). We will also index the schemes in order of appearance in the corresponding figure’s legend. Experiments of ByzShield which appear in multiple figures are not repeated here (we ran those training processes once). In order to pick the optimal hyperparameters for each scheme, we performed an extensive grid search involving different combinations of \((x, y, z)\). For each method, we ran 200 iterations for each such combination and chose the one which was giving the lowest value of average cross-entropy loss (principal criterion) and the highest value of top-1 accuracy (secondary criterion).

Table 5: Distortion fraction evaluation for MOLS-based assignment for \((K, f, l, r) = (35, 49, 7, 5)\) and comparison.

| \(q\) | \(c_{\text{max}}\) | \(\hat{\epsilon}_{\text{ByzShield}}\) | \(\hat{\epsilon}_{\text{Baseline}}\) | \(\hat{\epsilon}_{\text{FRC}}\) | \(\gamma\) |
|------|----------------|-----------------|-----------------|-----------------|-------|
| 3    | 1              | 0.02            | 0.12            | 0.14            | 2.68  |
| 4    | 1              | 0.02            | 0.16            | 0.14            | 4.39  |
| 5    | 2              | 0.04            | 0.2             | 0.14            | 6.36  |
| 6    | 4              | 0.08            | 0.24            | 0.29            | 8.54  |
| 7    | 5              | 0.1             | 0.28            | 0.29            | 10.89 |
| 8    | 8              | 0.16            | 0.32            | 0.29            | 13.37 |
| 9    | 10             | 0.2             | 0.36            | 0.43            | 15.97 |
| 10   | 11             | 0.22            | 0.4             | 0.43            | 18.67 |
| 11   | 14             | 0.29            | 0.44            | 0.43            | 21.44 |
| 12   | 16             | 0.33            | 0.48            | 0.57            | 24.29 |
| 13   | 20             | 0.41            | 0.52            | 0.57            | 27.2  |

Table 6: Distortion fraction evaluation for MOLS-based assignment for \((K, f, l, r) = (21, 49, 7, 3)\) and comparison.

| \(q\) | \(c_{\text{max}}\) | \(\hat{\epsilon}_{\text{ByzShield}}\) | \(\hat{\epsilon}_{\text{Baseline}}\) | \(\hat{\epsilon}_{\text{FRC}}\) | \(\gamma\) |
|------|----------------|-----------------|-----------------|-----------------|-------|
| 2    | 1              | 0.02            | 0.1             | 0.14            | 2.23  |
| 3    | 3              | 0.06            | 0.14            | 0.14            | 4.67  |
| 4    | 5              | 0.1             | 0.19            | 0.29            | 7.72  |
| 5    | 8              | 0.16            | 0.24            | 0.29            | 11.29 |
| 6    | 12             | 0.24            | 0.29            | 0.43            | 15.27 |
| 7    | 16             | 0.33            | 0.33            | 0.43            | 19.6  |
| 8    | 21             | 0.43            | 0.38            | 0.57            | 24.22 |
| 9    | 25             | 0.51            | 0.43            | 0.57            | 29.08 |
| 10   | 29             | 0.59            | 0.52            | 0.71            | 34.15 |
Table 7: Parameters used for training.

| Figure | Schemes | Learning rate schedule |
|--------|---------|------------------------|
| 2      | 1, 2    | (0.00625, 0.96, 15)    |
| 2      | 3       | (0.025, 0.96, 15)      |
| 2      | 4, 5, 6 | (0.01, 0.95, 20)       |
| 3      | 1, 2    | (0.003125, 0.96, 15)   |
| 4      | 1       | (0.00625, 0.96, 15)    |
| 4      | 2, 5, 6 | (0.01, 0.95, 20)       |
| 5      | 1, 2    | (0.0001, 0.99, 20)     |
| 5      | 3, 4    | (0.025, 0.96, 15)      |
| 5      | 5, 6    | (0.001, 0.5, 50)       |
| 6      | 1, 2, 4 | (0.05, 0.96, 15)       |
| 6      | 3       | (0.1, 0.95, 50)        |
| 6      | 5, 6    | (0.025, 0.96, 15)      |
| 7      | 1, 2    | (0.025, 0.96, 15)      |
| 7      | 4       | (0.05, 0.96, 15)       |
| 8      | 1, 2, 3 | (0.05, 0.96, 15)       |
| 8      | 7, 8    | (0.025, 0.96, 15)      |
| 9      | 1       | (0.003125, 0.96, 15)   |
| 9      | 2       | (0.01, 0.96, 15)       |
| 9      | 3       | (0.0125, 0.96, 15)     |
| 10     | 1       | (0.0015625, 0.96, 15)  |
| 11     | 1       | (0.003125, 0.96, 15)   |
| 11     | 3       | (0.0125, 0.96, 15)     |