More on the restricted almost unbiased Liu-estimator in Logistic regression

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Abstract

To address the problem of multicollinearity in the logistic regression model, in this paper we propose a new estimator called Stochastic restricted almost unbiased logistic Liu-estimator (SRAULLE) when the prior information is available in the form of stochastic linear restrictions. A Monte Carlo simulation study was carried out to compare the performance of the proposed estimator with some existing estimators in the scalar mean squared error (SMSE) sense. Finally, a real data example was given to appraise the performance of the estimators.

Keywords: Logistic Regression; Multicollinearity; Stochastic linear restrictions; Almost unbiased logistic Liu estimator; Stochastic restricted almost unbiased logistic Liu estimator.

1 Introduction

The maximum likelihood estimation technique is commonly used method to estimate the parameters of the logistic regression model. The multicollinearity severely affects the variance of the estimates of parameters in the logistic regression model. As a result model produces inefficient estimates. To overcome this issue, several alternative estimators have been proposed in the literature. These estimators were introduced mainly based on two types. The first type of estimators are based only on the sample information and the second type of estimators are based on the sample and priori available information which may be in the form of exact or stochastic linear restrictions. Logistic Ridge Estimator (LRE) by Schaefer et al. (1984), the Principal Component Logistic Estimator (PCLE) by Aguilera et al. (2006), the Modified Logistic Ridge Estimator (MLRE) by Nja et al. (2013), the Logistic Liu Estimator (LLE) by Mansson et al. (2012), the Liu-Type Logistic Estimator (LTLE) by Inan and Erdogan (2013), the Almost Unbiased Ridge Logistic Estimator (AURLE) by Wu and Asar (2016), the Almost Unbiased Liu Logistic Estimator (AULLE) by Xinfeng (2015), and the Optimal Generalized Logistic Estimator
(OGLE) by Varathan and Wijekoon (2017) are some of the first type of estimators proposed in the literature. Under the second type of estimators, the Restricted Maximum Likelihood Estimator (RMLE) by Duffy and Santner (1989), the Restricted Logistic Liu Estimator (RLLE) by Siray et al. (2015), the Modified Restricted Liu Estimator by Wu (2015), the Restricted Logistic Ridge Estimator (RLRE) by Asar et al. (2016a), the Restricted Liu-Type Logistic Estimator (RLTLE) by Asar et al. (2016b), and the Restricted Almost Unbiased Ridge logistic Estimator (RAURLE) by Varathan and Wijekoon (2016a) were introduced to improve the performance of the logistic model when the exact linear restrictions are available in addition to sample model. When the restrictions on the parameters are stochastic, the Stochastic Restricted Maximum Likelihood Estimator (SRMLE) (Nagarajah and Wijekoon, 2015), the Stochastic Restricted Ridge Maximum Likelihood Estimator (SRRMLE) (Varathan and Wijekoon, 2016b), and the Stochastic Restricted Liu Maximum Likelihood Estimator (SRLMLE) (Varathan and Wijekoon, 2016c) were proposed in the literature. In this research, following Xinfei (2015), a new estimator namely, Stochastic restricted almost unbiased logistic Liu Estimator (SRAULLE) is proposed for the logistic regression model with the presence of stochastic linear restrictions as prior information. The organization of the paper is as follows. The model specification and estimation are given in Section 2. Proposed estimators and their asymptotic properties are discussed in Section 3. In Section 4, the conditions for superiority of SRAULLE over some existing estimators are derived with respect to mean square error (MSE) criterion. A Monte Carlo simulation study is conducted to investigate the performance of the proposed estimator in the scalar mean squared error (SMSE) sense in Section 5. A numerical example is discussed in Section 6. Finally, some conclusive remarks are given in Section 7.

2 Model Specification and estimation

Consider the logistic regression model

\[ y_i = \pi_i + \varepsilon_i, \quad i = 1, \ldots, n \]  

(2.1)

which follows Binary distribution with parameter \( \pi_i \) as

\[ \pi_i = \frac{\exp(x_i' \beta)}{1 + \exp(x_i' \beta)}, \]

(2.2)

where \( x_i \) is the \( i^{th} \) row of \( X \), which is an \( n \times p \) data matrix with \( p \) explanatory variables and \( \beta \) is a \( p \times 1 \) vector of coefficients, \( \varepsilon_i \) are independent with mean zero and variance \( \pi_i (1 - \pi_i) \) of the response \( y_i \). The Maximum likelihood estimate (MLE) of \( \beta \) can be obtained as follows:

\[ \hat{\beta}_{MLE} = C^{-1} X' \hat{W} Z, \]

(2.3)

where \( C = X' \hat{W} X; \ Z \) is the column vector with \( i^{th} \) element equals \( \text{logit}(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i (1 - \hat{\pi}_i)} \) and \( \hat{W} = \text{diag} [\hat{\pi}_i (1 - \hat{\pi}_i)] \). Note that \( \hat{\beta}_{MLE} \) is an unbiased estimate of \( \beta \) and its covariance matrix is

\[ \text{Cov}(\hat{\beta}_{MLE}) = \{X' \hat{W} X\}^{-1}. \]  

(2.4)
The MSE and SMSE of $\hat{\beta}_{MLE}$ are

$$
MSE[\hat{\beta}_{MLE}] = Cov[\hat{\beta}_{MLE}] + B[\hat{\beta}_{MLE}]B'[\hat{\beta}_{MLE}] \\
= \{X'WX\}^{-1} \\
= C^{-1}
$$

and

$$
SMSE[\hat{\beta}_{MLE}] = tr[C^{-1}]
$$

When the multicollinearity presents in the logistic regression model (2.1), many alternative estimators have been proposed in the literature. Among those, in this research we consider the Logistic Liu estimator (LLE) by Mansson et al., (2012) and the Almost Unbiased Logistic Liu Estimator (AULLE) by Xinfeng (2015) under the first type of estimators.

**LLE** : $\hat{\beta}_{LLE} = Z_d\hat{\beta}_{MLE}$; where $Z_d = (C + I)^{-1}(C + dI), 0 < d < 1$

**AULLE** : $\hat{\beta}_{AULLE} = W_d\hat{\beta}_{MLE}$; where $W_d = [I - (1 - d)^2(C + I)^{-2}], 0 < d < 1$

The asymptotic properties of LLE:

$$
E[\hat{\beta}_{LLE}] = E[Z_d\hat{\beta}_{MLE}] = Z_d\beta,
$$

$$
D[\hat{\beta}_{LLE}] = Cov[\hat{\beta}_{LLE}] = Cov[Z_d\hat{\beta}_{MLE}] = Z_dC^{-1}Z_d',
$$

Consequently, the bias vector and the mean square error matrix of LLE are obtained as

$$
B[\hat{\beta}_{LLE}] = E[\hat{\beta}_{LLE}] - \beta = [Z_d - I]\beta = \delta_1, (say)
$$

and

$$
MSE[\hat{\beta}_{LLE}] = D[\hat{\beta}_{LLE}] + B[\hat{\beta}_{LLE}]B'[\hat{\beta}_{LLE}] = Z_dC^{-1}Z_d' + \delta_1\delta_1'
$$
respectively.

The asymptotic properties of AULLE:

\[
E[\hat{\beta}_{AULLE}] = E[W_d\hat{\beta}_{MLE}]
= W_d\beta,
\]

\[
D[\hat{\beta}_{AULLE}] = Cov[\hat{\beta}_{AULLE}]
= Cov[W_d\hat{\beta}_{MLE}]
= W_dC^{-1}W_d,
\]

Then, the bias vector and the mean square error matrix of AULLE are obtained as

\[
B[\hat{\beta}_{AULLE}] = E[\hat{\beta}_{AULLE}] - \beta
= [W_d - I]\beta
= \delta_2,
\]

and

\[
MSE[\hat{\beta}_{AULLE}] = D[\hat{\beta}_{AULLE}] + B[\hat{\beta}_{AULLE}]B'[\hat{\beta}_{AULLE}]
= W_dC^{-1}W_d + \delta_2\delta_2'
\]

respectively. As an alternative technique to stabilize the variance of the estimator due to multicollinearity, one can use prior information, if available, in addition to the sample model (2.1) either as exact linear restrictions or stochastic linear restrictions.

Suppose that the following stochastic linear prior information is given in addition to the general logistic regression model (2.1).

\[
h = H\beta + v; \quad E(v) = 0, \quad Cov(v) = \Psi.
\]

where \( h \) is an \((q \times 1)\) stochastic known vector, \( H \) is a \((q \times p)\) of full rank \( q \leq p \) known elements and \( v \) is an \((q \times 1)\) random vector of disturbances with mean 0 and dispersion matrix \( \Psi \), which is assumed to be known \((q \times q)\) positive definite matrix. Further, it is assumed that \( v \) is stochastically independent of \( \varepsilon \), i.e. \( E(\varepsilon v') = 0 \).

In the presence of stochastic linear restrictions (2.17) in addition to the logistic regression model (2.1), Nagarajah and Wijekoon (2015) introduced the Stochastic Restricted Maximum Likelihood Estimator (SRMLE).

\[
\hat{\beta}_{SRMLE} = \hat{\beta}_{MLE} + C^{-1}H'(\Psi + HC^{-1}H')^{-1}(h - H\hat{\beta}_{MLE})
\]
The asymptotic properties of SRMLE:

\[
E(\hat{\beta}_{SRMLE}) = \beta, \tag{2.19}
\]

\[
Cov(\hat{\beta}_{SRMLE}) = C^{-1} - C^{-1}H'(\Psi + HC^{-1}H')^{-1}HC^{-1}
\]

\[
= (C + H'\Psi^{-1}H)^{-1},
\]

\[
= R \text{(say)}
\]

and

\[
Bias[\hat{\beta}_{SRMLE}] = E[\hat{\beta}_{SRMLE}] - \beta = 0. \tag{2.21}
\]

The MSE of SRMLE is

\[
MSE[\hat{\beta}_{SRMLE}] = Cov(\hat{\beta}_{SRMLE}) + B[\hat{\beta}_{SRMLE}]B'[\hat{\beta}_{SRMLE}] \tag{2.22}
\]

\[
= (C + H'\Psi^{-1}H)^{-1}
\]

\[
= R
\]

The Proposed Estimator

In this section, by replacing \(\hat{\beta}_{MLE}\) by \(\hat{\beta}_{SRMLE}\) in (2.8), we propose a new estimator which is called as the Stochastic restricted almost unbiased logistic Liu Estimator (SRAULLE) and defined as

\[
\hat{\beta}_{SRAULLE} = W_d\hat{\beta}_{SRMLE} \tag{3.1}
\]

where \(W_d = [I - (1 - d)^2(C + I)^{-2}], 0 < d < 1\).

The asymptotic properties of \(\hat{\beta}_{SRAULLE}\) are

\[
E[\hat{\beta}_{SRAULLE}] = E[W_d\hat{\beta}_{SRMLE}]
\]

\[
= W_d\beta, \tag{3.2}
\]

\[
D(Cov(\hat{\beta}_{SRAULLE})) = Cov(\hat{\beta}_{SRAULLE})
\]

\[
= Cov(W_d\hat{\beta}_{SRMLE})
\]

\[
= W_dCov(\hat{\beta}_{SRMLE})W_d'
\]

\[
= W_dRW_d'. \tag{3.3}
\]
and
\[ \text{Bias}(\hat{\beta}_{\text{SRAULLE}}) = E[\hat{\beta}_{\text{SRAULLE}}] - \beta \]  \quad (3.4)
\[ = [W_d - I] \beta \]
\[ = \delta_2. \]

Consequently, the mean square error can be obtained as,
\[ \text{MSE}(\hat{\beta}_{\text{SRAULLE}}) = D(\hat{\beta}_{\text{SRAULLE}}) + \text{Bias}(\hat{\beta}_{\text{SRAULLE}})\text{Bias}(\hat{\beta}_{\text{SRAULLE}})' \]  \quad (3.5)
\[ = W_d R W_d' + \delta_2 \delta_2'. \]

4 Mean square error comparisons

When different estimators are available for the same parameter vector \( \beta \) in the regression model one must solve the problem of their comparison. Usually, as a general measure, the mean square error matrix is used, and is defined by
\[ \text{MSE}(\hat{\beta}, \beta) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \]  \quad (4.1)
\[ = D(\hat{\beta}) + B(\hat{\beta})B'(\hat{\beta}) \]
where \( D(\hat{\beta}) \) is the dispersion matrix, and \( B(\hat{\beta}) = E(\hat{\beta}) - \beta \) denotes the bias vector.

The Scalar Mean Squared Error (SMSE) of the estimator \( \hat{\beta} \) can be defined as
\[ \text{SMSE}(\hat{\beta}, \beta) = \text{trace}[\text{MSE}(\hat{\beta}, \beta)] \]  \quad (4.2)

For two given estimators \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \), the estimator \( \hat{\beta}_2 \) is said to be superior to \( \hat{\beta}_1 \) under the MSE criterion if and only if
\[ M(\hat{\beta}_1, \hat{\beta}_2) = \text{MSE}(\hat{\beta}_1, \beta) - \text{MSE}(\hat{\beta}_2, \beta) \geq 0. \]  \quad (4.3)

4.1 Comparison of SRAULLE with MLE

To compare the estimators \( \hat{\beta}_{\text{MLE}} \) and \( \hat{\beta}_{\text{SRAULLE}} \), we consider their MSE differences as below:
\[ \text{MSE}(\hat{\beta}_{\text{MLE}}) - \text{MSE}(\hat{\beta}_{\text{SRAULLE}}) = \{D(\hat{\beta}_{\text{MLE}}) + B(\hat{\beta}_{\text{MLE}})B'(\hat{\beta}_{\text{MLE}})\} \]
\[ - \{D(\hat{\beta}_{\text{SRAULLE}}) + B(\hat{\beta}_{\text{SRAULLE}})B'(\hat{\beta}_{\text{SRAULLE}})\} \]
\[ = C^{-1} - \{W_d R W_d' + \delta_2 \delta_2'\} \]
\[ = U_1 - V_1 \]
where \( U_1 = C^{-1} \) and \( V_1 = W_d R W_d' + \delta_2 \delta_2' \). One can obviously say that \( W_d R W_d' \) and \( U_1 \) are positive definite matrices and \( \delta_2 \delta_2' \) is non-negative definite matrix. Further by Lemma 1 (see Appendix A), it is clear that \( V_1 \) is positive definite matrix. By Lemma 2 (see Appendix A), if \( \lambda_{\max}(V_1 U_1^{-1}) < 1 \), then \( U_1 - V_1 \) is a positive definite matrix, where \( \lambda_{\max}(V_1 U_1^{-1}) \) is the largest eigen value of \( V_1 U_1^{-1} \). Based on the above arguments, it can be concluded that the estimator SRAULLE is superior to MLE if and only if \( \lambda_{\max}(V_1 U_1^{-1}) < 1 \).
4.2 Comparison of SRAULLE with LLE

Consider the MSE differences of \( \hat{\beta}_{\text{LLE}} \) and \( \hat{\beta}_{\text{SRAULLE}} \)

\[
MSE(\hat{\beta}_{\text{LLE}}) - MSE(\hat{\beta}_{\text{SRAULLE}}) = \{D(\hat{\beta}_{\text{LLE}}) + B(\hat{\beta}_{\text{LLE}})B'(\hat{\beta}_{\text{LLE}})\} - \{D(\hat{\beta}_{\text{SRAULLE}}) + B(\hat{\beta}_{\text{SRAULLE}})B'(\hat{\beta}_{\text{SRAULLE}})\} \\
= \{Z_dC^{-1}Z_d' + \delta_1\delta_1'\} - \{W_dRW_d' + \delta_2\delta_2'\}
\]

where \( U_2 = Z_dC^{-1}Z_d' + \delta_1\delta_1' \) and \( V_2 = W_dRW_d' + \delta_2\delta_2' \). One can easily say that \( W_dRW_d' \) and \( Z_dC^{-1}Z_d' \) are positive definite matrices and \( \delta_1\delta_1' \) and \( \delta_2\delta_2' \) are non-negative definite matrices. Further by Lemma 1, it is clear that \( U_2 \) and \( V_2 \) are positive definite matrices. By Lemma 2, if \( \lambda_{\max}(V_2U_2^{-1}) < 1 \), then \( U_2 - V_2 \) is a positive definite matrix, where \( \lambda_{\max}(V_2U_2^{-1}) \) is the largest eigen value of \( V_2U_2^{-1} \). Based on the above results, it can be said that the estimator SRAULLE is superior to LLE if and only if \( \lambda_{\max}(V_2U_2^{-1}) < 1 \).

4.3 Comparison of SRAULLE with AULLE

Consider the MSE differences of \( \hat{\beta}_{\text{AULLE}} \) and \( \hat{\beta}_{\text{SRAULLE}} \)

\[
MSE(\hat{\beta}_{\text{AULLE}}) - MSE(\hat{\beta}_{\text{SRAULLE}}) = \{D(\hat{\beta}_{\text{AULLE}}) + B(\hat{\beta}_{\text{AULLE}})B'(\hat{\beta}_{\text{AULLE}})\} - \{D(\hat{\beta}_{\text{SRAULLE}}) + B(\hat{\beta}_{\text{SRAULLE}})B'(\hat{\beta}_{\text{SRAULLE}})\} \\
= \{W_dC^{-1}W_d' + \delta_2\delta_2'\} - \{W_dRW_d' + \delta_2\delta_2'\} \\
= W_d(C^{-1} - R)W_d' \\
= C^{-1}H'(\Psi + HC^{-1}H')^{-1}HC^{-1} \\
> 0
\]

Since the above mean square error difference is positive definite, it can be concluded that SRAULLE is always superior than AULLE.

4.4 Comparison of SRAULLE with SRMLE

Consider the MSE differences of \( \hat{\beta}_{\text{SRMLE}} \) and \( \hat{\beta}_{\text{SRAULLE}} \)

\[
MSE(\hat{\beta}_{\text{SRMLE}}) - MSE(\hat{\beta}_{\text{SRAULLE}}) = \{D(\hat{\beta}_{\text{SRMLE}}) + B(\hat{\beta}_{\text{SRMLE}})B'(\hat{\beta}_{\text{SRMLE}})\} - \{D(\hat{\beta}_{\text{SRAULLE}}) + B(\hat{\beta}_{\text{SRAULLE}})B'(\hat{\beta}_{\text{SRAULLE}})\} \\
= R - \{W_dRW_d' + \delta_2\delta_2'\} \\
= U_3 - V_3
\]

where \( U_3 = R \) and \( V_3 = W_dRW_d' + \delta_2\delta_2' \). It can be easily seen that \( W_dRW_d' \) and \( R \) are positive definite matrices and \( \delta_2\delta_2' \) is non-negative definite matrix. Further by Lemma 1, it is clear that \( V_3 \) is positive definite matrix. By Lemma 2, if \( \lambda_{\max}(V_3U_3^{-1}) < 1 \), then \( U_3 - V_3 \) is a positive definite matrix, where \( \lambda_{\max}(V_3U_3^{-1}) \) is the largest eigen value of \( V_3U_3^{-1} \). Based on the above results,
it can be said that the estimator SRAULLE is superior to SRMLE if and only if \( \lambda_{\text{max}}(V_3U_3^{-1}) < 1 \).

According to the results obtained from above mean square error comparisons it can be concluded that the proposed estimator SRAULLE is always superior than AULLE. However, under certain conditions SRAULLE performs well over MLE, LLE, and SRMLE with respect to the mean square error sense.

5 A Simulation study

To examine the performance of the proposed estimator; SRAULLE with the existing estimators: MLE, LLE, AULLE and SRMLE in this section, we conduct the Monte Carlo simulation study. The simulations are based on different levels of multicollinearity; \( \rho = 0.7, 0.8, 0.9 \) and \( 0.99 \) and different sample sizes; \( n = 25, 50, 75 \) and 100. The Scalar Mean Square Error (SMSE) is considered for the comparison. Following McDonald and Galarneau (1975) and Kibria (2003), the explanatory variables are generated as follows:

\[
x_{ij} = (1 - \rho^2)^{1/2}z_{ij} + \rho z_{i,p+1}, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., p
\]  

(5.1)

where \( z_{ij} \) are independent standard normal pseudo-random numbers and \( \rho \) is specified so that the theoretical correlation between any two explanatory variables is given by \( \rho^2 \). Four explanatory variables are generated using (5.1). The dependent variable \( y_i \) in (2.1) is obtained from the Bernoulli(\( \pi_i \)) distribution where \( \pi_i = \frac{\exp(x'_i\beta)}{1+\exp(x'_i\beta)} \). The parameter values of \( \beta_1, \beta_2, ..., \beta_p \) are chosen so that \( \sum_{j=1}^{p} \beta_j^2 = 1 \) and \( \beta_1 = \beta_2 = ... = \beta_p \). Following Asar et al. (2016b), Wu and Asar (2015) and Mansson et al. (2012), the optimum value of the biasing parameter \( d \) can be obtained by minimizing SMSE value with respect to \( d \). However, for simplicity in this paper we consider some selected values of \( d \) in the range \( 0 < d < 1 \). Moreover, we consider the following restrictions.

\[
H = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

(5.2)

The simulation is repeated 1000 times by generating new pseudo-random numbers and the simulated SMSE values of the estimators are obtained using the following equation.

\[
\hat{\text{SMSE}}(\hat{\beta}^*) = \frac{1}{1000} \sum_{r=1}^{1000} (\hat{\beta}_r - \beta)'(\hat{\beta}_r - \beta)
\]  

(5.3)

where \( \hat{\beta}_r \) is any estimator considered in the \( r^{th} \) simulation. The simulation results are displayed in Tables 5.1 - 5.3 (Appendix). As we observed from the theoretical results, the proposed estimator SRAULLE is superior to AULLE in the mean square error sense with respect to all the sample sizes \( n = 25, 50, 75 \), and 100 and all the \( \rho = 0.7, 0.8, 0.9, \) and 0.99. From the Tables 5.1- 5.3, it is further noted that if the multicollinearity is very high (for example \( \rho \geq 0.9 \)) the proposed estimator SRAULLE is a very good alternative to MLE, LLE, AULLE and SRMLE regardless
of the values of $n$ and $d$. However, the performance of LLE is considerably good for very small $d$ values and moderate $\rho$ values. Moreover, as we expected, MLE has the worst performance in all of the cases (having the largest SMSE values).

6 Numerical example

In order to check the performance of the new estimator SRAULLE, in this section, we used a real data set, which is taken from the Statistics Sweden website (http://www.scb.se/). The data consists the information about 100 municipalities of Sweden. The explanatory variables considered in this study are Population ($x_1$), Number unemployed people ($x_2$), Number of newly constructed buildings ($x_3$), and Number of bankrupt firms ($x_4$). The variable Net population change ($y$) is considered as response variable, which is defined as

$$y = \begin{cases} 1 & \text{if there is an increase in the population;} \\ 0 & \text{o/w.} \end{cases}$$

The correlation matrix of the explanatory variables $x_1$, $x_2$, $x_3$, and $x_4$ is displayed in Table 6.1. It can be noticed from the Table 6.1 that, all the pair wise correlations are very high (greater than 0.95). Hence a clear high multicollinearity exists in this data set. Further, the condition number being a measure of multicollinearity is obtained as 188 showing that there exists severe multicollinearity with this data set. Moreover, we use the same restrictions as in (5.2) for the prior information.

The SMSE values of MLE, LLE, AULLE, SRMLE, and SRAULLE for some selected values of biasing parameter $d$ in the range $0 < d < 1$ are given in the Table 6.2. It can be clearly noticed from the Table 6.2 that the proposed estimator SRAULLE outperforms the estimators MLE, LLE, AULLE, and SRMLE in the SMSE sense, with respect to all the selected values of biasing parameter $d$ in the range $0 < d < 1$ except $d = 0.01$. Further, SRAULLE is having better performance compared to AULLE for all the values of $d$.

Table 6.1: The correlation matrix of the explanatory variables

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|------|------|------|------|
| $x_1$ | 1.000 | 0.998 | 0.971 | 0.970 |
| $x_2$ | 0.998 | 1.000 | 0.960 | 0.958 |
| $x_3$ | 0.971 | 0.960 | 1.000 | 0.987 |
| $x_4$ | 0.970 | 0.958 | 0.987 | 1.000 |

7 Concluding Remarks

In this research, we proposed the Stochastic restricted almost unbiased logistic Liu estimator (SRAULLE) for logistic regression model in the presence of linear stochastic restriction when the multicollinearity problem exists. The conditions for superiority of the proposed estimator over some existing estimators were derived with respect to MSE criterion. Further, a numerical example and a Monte Carlo simulation study were done to illustrate the theoretical findings. Results
reveal that the proposed estimator is always superior to AULLE in the mean square error sense and it can be a better alternative to the other existing estimators under certain conditions.

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Appendix

Lemma 1: Let $A : n \times n$ and $B : n \times n$ such that $A > 0$ and $B \geq 0$. Then $A + B > 0$. (Rao and Toutenburg, 1995)

Lemma 2: Let the two $n \times n$ matrices $M > 0, N \geq 0$, then $M > N$ if and only if $\lambda_{\text{max}}(NM^{-1}) < 1$. (Rao et al., 2008)

Table 5.1: The estimated MSE values for different $d$ when $n = 25$

| $\rho$ | $d$ | MLE | LLE | AULLE | SRMLE | SRAULLE |
|---|---|---|---|---|---|---|
| $0.70$ | $0.01$ | 1.4798 | 0.8052 | 1.2383 | 1.0138 | 0.8536 |
| | $0.1$ | 1.4798 | 0.8600 | 1.2780 | 1.0138 | 0.8798 |
| | $0.2$ | 1.4798 | 0.9233 | 1.3186 | 1.0138 | 0.9067 |
| | $0.3$ | 1.4798 | 0.9893 | 1.3552 | 1.0138 | 0.9309 |
| | $0.4$ | 1.4798 | 1.0579 | 1.3875 | 1.0138 | 0.9524 |
| | $0.5$ | 1.4798 | 1.1291 | 1.4153 | 1.0138 | 0.9708 |
| | $0.6$ | 1.4798 | 1.2030 | 1.4383 | 1.0138 | 0.9861 |
| | $0.7$ | 1.4798 | 1.2794 | 1.4564 | 1.0138 | 0.9981 |
| | $0.8$ | 1.4798 | 1.3585 | 1.4694 | 1.0138 | 1.0068 |
| | $0.9$ | 1.4798 | 1.4402 | 1.4772 | 1.0138 | 1.0120 |
| | $0.99$ | 1.4798 | 1.5160 | 1.4798 | 1.0138 | 1.0137 |

| $0.80$ | $0.01$ | 1.9817 | 0.8781 | 1.4912 | 1.1793 | 0.9039 |
| | $0.1$ | 1.9817 | 0.9602 | 1.5699 | 1.1793 | 0.9479 |
| | $0.2$ | 1.9817 | 1.0566 | 1.6513 | 1.1793 | 0.9935 |
| | $0.3$ | 1.9817 | 1.1584 | 1.7253 | 1.1793 | 1.0350 |
| | $0.4$ | 1.9817 | 1.2658 | 1.7912 | 1.1793 | 1.0720 |
| | $0.5$ | 1.9817 | 1.3786 | 1.8481 | 1.1793 | 1.1041 |
| | $0.6$ | 1.9817 | 1.4968 | 1.8955 | 1.1793 | 1.1308 |
| | $0.7$ | 1.9817 | 1.6205 | 1.9329 | 1.1793 | 1.1518 |
| | $0.8$ | 1.9817 | 1.7497 | 1.9599 | 1.1793 | 1.1670 |
| | $0.9$ | 1.9817 | 1.8843 | 1.9763 | 1.1793 | 1.1762 |
| | $0.99$ | 1.9817 | 2.0101 | 1.9817 | 1.1793 | 1.1793 |

| $0.90$ | $0.01$ | 3.5707 | 0.9334 | 1.9075 | 1.5271 | 0.9039 |
| | $0.1$ | 3.5707 | 1.0954 | 2.1522 | 1.5271 | 0.9479 |
| | $0.2$ | 3.5707 | 1.2945 | 2.4151 | 1.5271 | 0.9935 |
| | $0.3$ | 3.5707 | 1.5137 | 2.6625 | 1.5271 | 1.0350 |
| | $0.4$ | 3.5707 | 1.7530 | 2.8885 | 1.5271 | 1.0720 |
| | $0.5$ | 3.5707 | 2.0124 | 3.0881 | 1.5271 | 1.1041 |
| | $0.6$ | 3.5707 | 2.2919 | 3.2573 | 1.5271 | 1.1308 |
| | $0.7$ | 3.5707 | 2.5915 | 3.3924 | 1.5271 | 1.1518 |
| | $0.8$ | 3.5707 | 2.9112 | 3.4908 | 1.5271 | 1.1670 |
| | $0.9$ | 3.5707 | 3.2510 | 3.5506 | 1.5271 | 1.1762 |
| | $0.99$ | 3.5707 | 3.5740 | 3.5705 | 1.1793 | 1.1793 |

| $0.99$ | $0.01$ | 33.1595 | 0.4893 | 1.2984 | 2.4804 | 0.2878 |
| | $0.1$ | 33.1595 | 1.2132 | 3.5907 | 2.4804 | 0.4482 |
| | $0.2$ | 33.1595 | 2.5413 | 7.2401 | 2.4804 | 0.7008 |
| | $0.3$ | 33.1595 | 4.4324 | 11.5879 | 2.4804 | 1.0004 |
| | $0.4$ | 33.1595 | 6.8751 | 16.2114 | 2.4804 | 1.3183 |
| | $0.5$ | 33.1595 | 9.8731 | 20.7438 | 2.4804 | 1.6294 |
| | $0.6$ | 33.1595 | 13.4266 | 24.8751 | 2.4804 | 1.9128 |
| | $0.7$ | 33.1595 | 17.5353 | 28.3515 | 2.4804 | 2.1510 |
| | $0.8$ | 33.1595 | 22.1995 | 32.8571 | 2.4804 | 2.3308 |
| | $0.9$ | 33.1595 | 27.4190 | 32.6065 | 2.4804 | 2.4425 |
| | $0.99$ | 33.1595 | 32.5914 | 33.1540 | 2.4804 | 2.4800 |
Table 5.2: The estimated MSE values for different $d$ when $n = 50$

| $\rho$  | $d = 0.01$ | $d = 0.1$ | $d = 0.2$ | $d = 0.3$ | $d = 0.4$ | $d = 0.5$ | $d = 0.6$ | $d = 0.7$ | $d = 0.8$ | $d = 0.9$ | $d = 0.99$ |
|---------|------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.70    | MLE 1.0662 | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    | 1.0662    |
|         | LLE 0.6249 | 0.6600    | 0.7004    | 0.7422    | 0.7854    | 0.8300    | 0.8761    | 0.9236    | 0.9726    | 1.0230    | 1.0696    |
|         | AULLE 0.9318| 0.9543    | 0.9770    | 0.9975    | 1.0154    | 1.0307    | 1.0434    | 1.0533    | 1.0604    | 1.0647    | 1.0662    |
|         | SRMILE 0.7173| 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    | 0.7173    |
|         | SRAULLE 0.6334| 0.6474    | 0.6617    | 0.6744    | 0.6856    | 0.6952    | 0.7031    | 0.7092    | 0.7137    | 0.7164    | 0.7173    |
| 0.80    | MLE 1.5025 | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    | 1.5025    |
|         | LLE 0.7156 | 0.7739    | 0.8419    | 0.9134    | 0.9883    | 1.0667    | 1.1485    | 1.2337    | 1.3224    | 1.4145    | 1.5003    |
|         | AULLE 1.1894| 1.2403    | 1.2926    | 1.3400    | 1.3819    | 1.4181    | 1.4481    | 1.4717    | 1.4887    | 1.4990    | 1.5024    |
|         | SRMILE 0.8807| 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    | 0.8807    |
|         | SRAULLE 0.7111| 0.7388    | 0.7671    | 0.7928    | 0.8155    | 0.8351    | 0.8513    | 0.8641    | 0.8733    | 0.8789    | 0.8807    |
| 0.90    | MLE 2.8448 | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    | 2.8448    |
|         | LLE 0.8150 | 0.9431    | 1.0989    | 1.2687    | 1.4526    | 1.6505    | 1.8626    | 2.0887    | 2.3288    | 2.5830    | 2.8239    |
|         | AULLE 1.6588| 1.8370    | 2.0267    | 2.2037    | 2.3625    | 2.5056    | 2.6248    | 2.7198    | 2.8407    | 2.8307    | 2.8446    |
|         | SRMILE 1.2238| 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    | 1.2238    |
|         | SRAULLE 0.7500| 0.8216    | 0.8976    | 0.9684    | 1.0325    | 1.0888    | 1.1363    | 1.1741    | 1.2015    | 1.2182    | 1.2237    |
| 0.99    | MLE 26.9632| 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   | 26.9632   |
|         | LLE 0.4165 | 1.0607    | 2.1916    | 3.7594    | 5.7641    | 8.2058    | 11.0845   | 14.4001   | 18.1527   | 22.3422   | 26.4863   |
|         | AULLE 1.2822| 3.2802    | 6.3089    | 9.8379    | 13.5461   | 17.1550   | 20.4289   | 23.1753   | 25.2440   | 26.5281   | 26.9589   |
|         | SRMILE 2.2638| 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    | 2.2638    |
|         | SRAULLE 0.1830| 0.3484    | 0.5958    | 0.8823    | 1.1824    | 1.4737    | 1.7377    | 1.9589    | 2.1255    | 2.2288    | 2.2635    |
Table 5.3: The estimated MSE values for different $d$ when $n = 75$

| $\rho$ | MLE | LLE | AULLE | SRMLE | SRAULLE | MLE | LLE | AULLE | SRMLE | SRAULLE | MLE | LLE | AULLE | SRMLE | SRAULLE | MLE | LLE | AULLE | SRMLE | SRAULLE |
|-------|-----|-----|-------|-------|---------|-----|-----|-------|-------|---------|-----|-----|-------|-------|---------|-----|-----|-------|-------|---------|
| 0.70  | 0.4775 | 0.3791 | 0.3940 | 0.3837 | 0.4647 | 0.4775 | 0.3791 | 0.3940 | 0.3837 | 0.4647 | 0.4775 | 0.6770 | 0.6770 | 0.6770 | 0.6770 | 0.6770 | 0.6770 | 0.6770 | 0.6770 |
| 0.80  | 0.4802 | 0.6404 | 0.5120 | 0.4850 | 0.6467 | 0.4968 | 0.5156 | 0.5349 | 0.5545 | 0.5443 | 0.4984 | 0.4984 | 0.4984 | 0.4984 | 0.4984 | 0.4984 | 0.4984 | 0.4984 |
| 0.90  | 1.1005 | 1.1005 | 1.0830 | 1.1005 | 1.0830 | 1.1005 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 | 1.0830 |
| 0.99  | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 | 13.1308 |
Table 5.4: The estimated MSE values for different $d$ when $n = 100$

| $\rho = 0.70$ | MLE | LLE | AULLE | SRMLE | SRAULLE |
|----------------|-----|-----|-------|--------|---------|
| $d = 0.01$     | 0.4172 | 0.3380 | 0.4078 | 0.3423 | 0.3350 |
| $d = 0.1$      | 0.4172 | 0.3450 | 0.4094 | 0.3423 | 0.3363 |
| $d = 0.2$      | 0.4172 | 0.3528 | 0.4111 | 0.3423 | 0.3375 |
| $d = 0.3$      | 0.4172 | 0.3608 | 0.4125 | 0.3423 | 0.3387 |
| $d = 0.4$      | 0.4172 | 0.3689 | 0.4137 | 0.3423 | 0.3396 |
| $d = 0.5$      | 0.4172 | 0.3771 | 0.4148 | 0.3423 | 0.3404 |
| $d = 0.6$      | 0.4172 | 0.3853 | 0.4157 | 0.3423 | 0.3411 |
| $d = 0.7$      | 0.4172 | 0.3937 | 0.4163 | 0.3423 | 0.3416 |
| $d = 0.8$      | 0.4172 | 0.4021 | 0.4168 | 0.3423 | 0.3420 |
| $d = 0.9$      | 0.4172 | 0.4107 | 0.4171 | 0.3423 | 0.3422 |
| $d = 0.99$     | 0.4172 | 0.4185 | 0.4172 | 0.3423 | 0.3423 |

| $\rho = 0.80$ | MLE | LLE | AULLE | SRMLE | SRAULLE |
|----------------|-----|-----|-------|--------|---------|
| $d = 0.01$     | 0.5932 | 0.4341 | 0.5662 | 0.4469 | 0.4278 |
| $d = 0.1$      | 0.5932 | 0.4477 | 0.5709 | 0.4469 | 0.4311 |
| $d = 0.2$      | 0.5932 | 0.4629 | 0.5755 | 0.4469 | 0.4344 |
| $d = 0.3$      | 0.5932 | 0.4785 | 0.5796 | 0.4469 | 0.4373 |
| $d = 0.4$      | 0.5932 | 0.4943 | 0.5832 | 0.4469 | 0.4398 |
| $d = 0.5$      | 0.5932 | 0.5104 | 0.5863 | 0.4469 | 0.4420 |
| $d = 0.6$      | 0.5932 | 0.5268 | 0.5888 | 0.4469 | 0.4437 |
| $d = 0.7$      | 0.5932 | 0.5435 | 0.5907 | 0.4469 | 0.4451 |
| $d = 0.8$      | 0.5932 | 0.5605 | 0.5921 | 0.4469 | 0.4467 |
| $d = 0.9$      | 0.5932 | 0.5777 | 0.5930 | 0.4469 | 0.4469 |
| $d = 0.99$     | 0.5932 | 0.5935 | 0.5932 | 0.4469 | 0.4469 |

| $\rho = 0.90$ | MLE | LLE | AULLE | SRMLE | SRAULLE |
|----------------|-----|-----|-------|--------|---------|
| $d = 0.01$     | 1.1311 | 0.6264 | 0.9785 | 0.6958 | 0.6086 |
| $d = 0.1$      | 1.1311 | 0.6659 | 1.0041 | 0.6958 | 0.6232 |
| $d = 0.2$      | 1.1311 | 0.7113 | 1.0300 | 0.6958 | 0.6381 |
| $d = 0.3$      | 1.1311 | 0.7584 | 1.0532 | 0.6958 | 0.6513 |
| $d = 0.4$      | 1.1311 | 0.8070 | 1.0736 | 0.6958 | 0.6630 |
| $d = 0.5$      | 1.1311 | 0.8573 | 1.0910 | 0.6958 | 0.6729 |
| $d = 0.6$      | 1.1311 | 0.9092 | 1.1053 | 0.6958 | 0.6811 |
| $d = 0.7$      | 1.1311 | 0.9627 | 1.1166 | 0.6958 | 0.6875 |
| $d = 0.8$      | 1.1311 | 1.0179 | 1.1246 | 0.6958 | 0.6921 |
| $d = 0.9$      | 1.1311 | 1.0746 | 1.1295 | 0.6958 | 0.6949 |
| $d = 0.99$     | 1.1311 | 1.1271 | 1.1311 | 0.6958 | 0.6958 |

| $\rho = 0.99$ | MLE | LLE | AULLE | SRMLE | SRAULLE |
|----------------|-----|-----|-------|--------|---------|
| $d = 0.01$     | 10.8045 | 5.9082 | 1.7797 | 1.8820 | 0.3585 |
| $d = 0.1$      | 10.8045 | 1.5492 | 2.7752 | 1.8820 | 0.5315 |
| $d = 0.2$      | 10.8045 | 2.4352 | 5.0155 | 1.8820 | 0.7427 |
| $d = 0.3$      | 10.8045 | 3.0700 | 6.5735 | 1.8820 | 0.9609 |
| $d = 0.4$      | 10.8045 | 4.0288 | 7.5399 | 1.8820 | 1.1742 |
| $d = 0.5$      | 10.8045 | 5.1198 | 8.7932 | 1.8820 | 1.3722 |
| $d = 0.6$      | 10.8045 | 6.3430 | 9.6474 | 1.8820 | 1.5462 |
| $d = 0.7$      | 10.8045 | 7.6985 | 10.2821 | 1.8820 | 1.6889 |
| $d = 0.8$      | 10.8045 | 9.1863 | 10.6972 | 1.8820 | 1.7948 |
| $d = 0.9$      | 10.8045 | 10.6884 | 10.8032 | 1.8820 | 1.8600 |
| $d = 0.99$     | 10.8045 | 10.8045 | 10.8032 | 1.8820 | 1.8817 |
Table 6.2: The SMSE values of estimators for the Numerical example

|     | d = 0.01 | d = 0.1 | d = 0.2 | d = 0.3 | d = 0.4 | d = 0.5 |
|-----|----------|---------|---------|---------|---------|---------|
| MLE | 0.0009457555 | 0.0009457555 | 0.0009457555 | 0.0009457555 | 0.0009457555 | 0.0009457555 |
| LLE | 0.0009441630 | 0.0009443098 | 0.0009444729 | 0.0009446361 | 0.0009447993 | 0.0009449624 |
| AULLE | 0.0009457541 | 0.0009457543 | 0.0009457546 | 0.0009457548 | 0.0009457550 | 0.0009457551 |
| SRMLE | 0.000945487 | 0.000945487 | 0.000945487 | 0.000945487 | 0.000945487 | 0.000945487 |
| SRAULLE | 0.000945472 | 0.000945475 | 0.000945477 | 0.000945480 | 0.000945481 | 0.000945483 |

|     | d = 0.6 | d = 0.7 | d = 0.8 | d = 0.9 | d = 0.99 |
|-----|---------|---------|---------|---------|---------|
| MLE | 0.0009457555 | 0.0009457555 | 0.0009457555 | 0.0009457555 | 0.0009457555 |
| LLE | 0.0009451256 | 0.0009452888 | 0.0009454521 | 0.0009456153 | 0.0009457622 |
| AULLE | 0.0009457553 | 0.0009457554 | 0.0009457554 | 0.0009457555 | 0.0009457555 |
| SRMLE | 0.000945487 | 0.000945487 | 0.000945487 | 0.000945487 | 0.000945487 |
| SRAULLE | 0.000945484 | 0.000945485 | 0.000945486 | 0.000945487 | 0.000945487 |