A CLASS OF HESSIAN QUOTIENT EQUATIONS IN THE WARPED PRODUCT MANIFOLD

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Abstract. In this paper, we consider a class of Hessian quotient equations in the warped product manifold $M = I \times \lambda M$. Under some sufficient conditions, we obtain an existence result for the star-shaped compact hypersurface $\Sigma$ in $M$ using standard degree theory based on a priori estimates for solutions to the Hessian quotient equations.

1. Introduction

In this paper, we consider the problem of prescribed Weingarten curvatures for closed, star-shaped hypersurfaces in the warped product manifold. Let $(M, g')$ be a compact Riemannian manifold and $I$ be an open interval in $\mathbb{R}$. The warped product manifold $M = I \times \lambda M$ is endowed with the metric

$$\overline{g}^2 = dr^2 + \lambda^2(r)g', \quad (1.1)$$

where $\lambda: I \to \mathbb{R}^+$ is a positive $C^2$ differentiable function. Let $\Sigma$ be a compact star-shaped hypersurface in $\overline{M}$, thus $\Sigma$ can be parametrized as a radial graph over $M$. Specifically speaking, there exists a differentiable function $r: M \to I$ such that the graph of $\Sigma$ can be represented by

$$\Sigma = \{(r(u), u) \mid u \in M\}.$$ 

We consider the following prescribed Weingarten curvature equation

$$\frac{\sigma_k}{\sigma_l}(\mu(\eta)) = f(V, \nu(V)), \quad \forall \ V \in \overline{M}, \quad (1.2)$$

where $2 \leq k \leq n, 0 \leq l \leq k-2$, $V = \lambda \frac{\partial}{\partial r}$ is the position vector field of hypersurface $\Sigma$ in $\overline{M}$, $\sigma_k$ is the $k$-th elementary symmetric function, $\mu(\eta)$ is the eigenvalue of $g^{-1}\eta$, $f$ is a

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given smooth function and $\nu(V)$ is the unit outer normal vector at $V$. The $(0, 2)$-tensor $\eta$ on $\Sigma$ is defined by

$$\eta_{ij} = H g_{ij} - h_{ij},$$

where $g_{ij}$ and $h_{ij}$ are the first and second fundamental forms of $\Sigma$ respectively, $H(V)$ is the mean curvature at $V \in \Sigma$. In fact, $\eta$ is the first Newton transformation of $h$ with respect to $g$. Given $r_1, r_2$ with $r_1 < r_2$, we define the annulus domain $\{(r, u) \in \overline{M} | r_1 \leq r \leq r_2\}$. The main theorem is as follows.

**Theorem 1.1.** Let $M$ be a compact Riemannian manifold, $\overline{M}$ be the warped product manifold with the metric (1.1) and $\Gamma$ be an open neighborhood of unit normal bundle of $M$ in $\overline{M} \times S^n$. Assume that $\lambda$ is a positive $C^2$ differentiable function and $\lambda' > 0$. Suppose that $f$ satisfies

\begin{align}
(1.3) & \quad f(V, \nu) > \frac{C_r^k}{C_l^n}((n - 1)\zeta(r))^{k-l}, \quad \forall \ r \leq r_1, \\
(1.4) & \quad f(V, \nu) < \frac{C_r^k}{C_l^n}((n - 1)\zeta(r))^{k-l}, \quad \forall \ r \geq r_2 \\
\text{and} & \quad \frac{\partial}{\partial \overline{r}}(\lambda^{k-l}f(V, \nu)) \leq 0, \quad \forall \ r_1 < \overline{r} < r_2,
\end{align}

where $V = \lambda \frac{\partial}{\partial \overline{r}}$ and $\zeta(r) = \lambda'(r)/\lambda(r)$. Then there exists a $C^{4, \alpha}$, $(\eta, k)$-convex, star-shaped and closed hypersurface $\Sigma$ in $\{(r, u) \in \overline{M} | r_1 \leq r \leq r_2\}$ satisfies the equation (1.2) for any $\alpha \in (0, 1)$.

**Remark 1.2.** The key to prove Theorem 1.1 is to obtain the curvature estimate for the Hessian quotient equation (1.2) in the warped product manifold, which is established in Theorem 3.4.

This kind of Hessian quotient equation is stimulated by many important geometric problems. When $k = n$, $l = 0$ and $\lambda(r) = r$, the equation (1.2) becomes the following equation for an $(\eta, n)$-convex hypersurface

$$\det(\eta(V)) = f(V, \nu),$$

which is studied intensively by Sha [26, 27], Wu [29] and Harvey-Lawson [16]. When the left hand of (1.6) is replaced by $\sigma_k(\eta(V))$, Chu-Jiao established curvature estimates for
this kind of equation in [6]. Inspired by above results, the authors in [8] considered the corresponding Hessian quotient type prescribed curvature equations in Euclidean space. In this paper, we generalize the existence results in [8] to the warped product manifold for the prescribed curvature problem. The remarkable fact is that Theorem 1.1 recovers the existence results in [6, 8].

When \( \mu(\eta) \) is replaced by \( \kappa(V) \) and \( l = 0 \), the equation (1.2) becomes this kind of prescribed curvature equation

\[
(1.7) \quad \sigma_k(\kappa(V)) = f(V, \nu),
\]

which has been widely studied in the past two decades. The key to this prescribed curvature equation is the curvature estimate. In Euclidean space, Caffarelli-Nirenberg-Spruck established the curvature estimates for \( k = n \) in [3]. Guan-Ren-Wang proved the \( C^2 \) estimates of the equation (1.7) for \( k = 2 \) in [14]. Spruck-Xiao extended the 2-convex case to space forms and gave a simple proof for the Euclidean case in [28]. Ren-Wang proved the \( C^2 \) estimates for \( k = n - 1 \) and \( n - 2 \) in [24, 25]. When \( 2 < k < n \), the \( C^2 \) estimates for the equation of prescribing curvature measures were also proved in [12, 13], where \( f(V, \nu) = \langle V, \nu \rangle \tilde{f}(V) \). Ivochkina considered the Dirichlet problem of the equation (1.7) on domains in \( \mathbb{R}^n \) and obtained the \( C^2 \) estimates according to the dependence of \( f \) on \( \nu \) under some extra conditions in [17, 18]. Caffarelli-Nirenberg-Spruck [4] and Guan-Guan [10] proved the \( C^2 \) estimates when \( f \) was independent of \( \nu \) and depended only on \( \nu \) respectively. Moreover, some results have been derived by Li-Oliker [23] on unit sphere, Barbosa-de Lira-Oliker [2] on space forms, Jin-Li [19] on hyperbolic space and Andrade-Barbosa-de Lira [11] on the warped product manifold. In particular, Chen-Li-Wang [17] generalized the results in [14] and Ren-Wang [24] extended to the \( (n - 2) \)-convex hypersurface in the warped product manifold.

The organization of the paper is as follows. In Sect. 2 we start with some preliminaries. The \( C^0, C^1 \) and \( C^2 \) estimates are given in Sect. 3. In Sect. 4 we finish the proof of Theorem 1.1.

2. Preliminaries

2.1. Star-shaped hypersurfaces in the warped product manifold. Let \( M \) be a compact Riemannian manifold with the metric \( g' \) and \( I \) be an open interval in \( \mathbb{R} \). Assume
that \( \lambda : I \to \mathbb{R}^+ \) is a positive differential function and \( \lambda' > 0 \). Clearly,

\[
\lambda(r) = \begin{cases} 
  r & \text{on } [0, \infty) \\
  \sin r & \text{on } [0, \frac{\pi}{2}) \\
  \sinh r & \text{on } [0, \infty) 
\end{cases}
\Rightarrow \mathcal{M} = \begin{cases} 
  \mathbb{R}^{n+1} \\
  S^{n+1} \\
  \mathbb{H}^{n+1}
\end{cases}
\]

The manifold \( \mathcal{M} = I \times_\lambda M \) is called the warped product if it is endowed with the metric

\[
\bar{g}^2 = dr^2 + \lambda^2(r)g'.
\]

The metric in \( \mathcal{M} \) is denoted by \( \langle \cdot, \cdot \rangle \). The corresponding Riemannian connection in \( \mathcal{M} \) will be denoted by \( \nabla \). The usual connection in \( M \) will be denoted by \( \nabla' \). The curvature tensors in \( M \) and \( \mathcal{M} \) will be denoted by \( R \) and \( \mathcal{R} \) respectively.

Let \( \{e_1, \ldots, e_{n-1}\} \) be an orthonormal frame field in \( M \) and let \( \{\theta_1, \ldots, \theta_{n-1}\} \) be the associated dual frame. The connection forms \( \theta_{ij} \) and curvature forms \( \Theta_{ij} \) in \( M \) satisfy the structural equations

\[
d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} = -\theta_{ji},
\]

\[
d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l.
\]

An orthonormal frame in \( \mathcal{M} \) may be defined by \( \overrightarrow{e}_i = \frac{1}{\lambda} e_i, 1 \leq i \leq n-1 \) and \( \overrightarrow{e}_0 = \frac{\partial}{\partial r} \). The associated dual frame is that \( \overrightarrow{\theta}_i = \lambda \theta_i, 1 \leq i \leq n-1 \) and \( \overrightarrow{\theta}_0 = dr \). Then, we have the following lemma (See [15]).

**Lemma 2.1.** Given a differentiable function \( r : M \to I \), its graph is defined by the hypersurface

\[
\Sigma = \{(r(u), u) : u \in M\}.
\]

Then, the tangential vector takes the form

\[
V_i = \lambda \overrightarrow{e}_i + r_i \overrightarrow{e}_0,
\]

where \( r_i \) are the components of the differential \( dr = r_i \theta^i \). The induced metric on \( \Sigma \) has

\[
g_{ij} = \lambda^2(r) \delta_{ij} + r_i r_j,
\]

and its inverse is given by

\[
g^{ij} = \frac{1}{\lambda^2} (\delta_{ij} - \frac{r_i r_j}{v^2}).
\]
We also have the outward unit normal vector of $\Sigma$
\[ \nu = \frac{1}{v} \left( \lambda \vec{e}_0 - r^i \vec{e}_i \right), \]
where $v = \sqrt{\lambda^2 + |\nabla'r|^2}$ with $\nabla'r = r^i \vec{e}_i$. Let $h_{ij}$ be the second fundamental form of $\Sigma$ in term of the tangential vector fields $\{X_1, \ldots, X_n\}$. Then,
\[ h_{ij} = -\langle \nabla X_j X_i, \nu \rangle = \frac{1}{v} \left( -\lambda r_{ij} + 2\lambda' r_i r_j + \lambda^2 \lambda' \delta_{ij} \right) \]
and
\[ h^i_j = \frac{1}{\lambda^2 v^2} \left( \delta_{ik} - \frac{r^i r^k}{v^2} \right) \left( -\lambda r_{kj} + 2\lambda' r_k r_j + \lambda^2 \lambda' \delta_{kj} \right), \]
where $r_{ij}$ are the components of the Hessian $\nabla^2 r = \nabla'dr$ of $r$ in $M$.

Let $\Gamma_k$ be the connected component of $\{\kappa \in \mathbb{R}^n \mid \sigma_m > 0, m = 1, \ldots, k\}$, the operator $\sigma_k(\kappa)$ for $\kappa = (\kappa_1, \ldots, \kappa_n) \in \Gamma_k$ is defined by
\[ \sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k}. \]
A smooth hypersurface $M \subset \mathbb{R}^{n+1}$ is called $(\eta, k)$-convex if $\mu(\eta) \in \Gamma_k$ for any $V \in M$, where $\Gamma_k$ is the Garding cone
\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\mu) > 0, \forall 1 \leq j \leq k \}. \]
For convenience, we introduce the following notations:
\[ G(\eta) := \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{\frac{1}{k-l}}, \quad G^{ij} := \frac{\partial G}{\partial \eta_{ij}}, \quad G^{ij,rs} := \frac{\partial^2 G}{\partial \eta_{ij} \partial \eta_{rs}}, \quad F^{ii} := \sum_{k \neq i} G^{kk}. \]
Thus,
\[ G^{ii} = \frac{1}{k - l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{\frac{1}{k-l} - 1} \left( \sigma_{k-1}(\eta|i) \sigma_l(\eta) - \sigma_k(\eta) \sigma_{l-1}(\eta|i) \right) \sigma_l^2(\eta). \]
If $\eta = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, then we have
\[ G^{11} \geq G^{22} \geq \cdots \geq G^{nn}, \quad F^{11} \leq F^{22} \leq \cdots \leq F^{nn}. \]
Note that $\sum_i F^{ii} = (n-1) \sum_i G^{ii} \geq (n-1) \left( \frac{C^k_n}{C_l^k} \right)^{\frac{1}{k-l}}$ and
\[ F^{ii} \geq F^{22} \geq \frac{1}{n(n-1)} \sum_i F^{ii}. \]

To handle the ellipticity of the equation (1.2), we need the following important propositions and their proof are the same as Proposition 2.2.3 in [5].
Proposition 2.2. Let \( \eta \) be a diagonal matrix with \( \mu(\eta) \in \Gamma_k \), \( 0 \leq l \leq k - 2 \) and \( k \geq 3 \). Then

\[-G_{ii}^{11}(\eta) = \frac{G_{11}^{11} - G_{ii}^{ii}}{\eta_{ii} - \eta_{11}}, \ \forall \ i \geq 2.\]

Proposition 2.3. Let \( M \) be a smooth \((\eta, k)\)-convex closed hypersurface in \( \mathbb{R}^{n+1} \) and \( 0 \leq l < k - 1 \). Then the operator

\[G(\eta_{ij}(V)) = \left( \frac{\sigma_k(\mu(\eta))}{\sigma_l(\mu(\eta))} \right)^{\frac{1}{k-l}}\]

is elliptic and concave with respect to \( \eta_{ij}(V) \). Moreover we have

\[\sum G_{ii}^{ii} \geq \left( \frac{C^n_k}{C^n_l} \right)^{\frac{1}{k-l}}.\]

The Codazzi equation is a commutation formula for the first order derivative of \( h_{ij} \) given by

\[h_{ij} - h_{kj} = R_{0ij},\]

and the Ricci identity is a commutation formula for the second order derivative of \( h_{ij} \) given by [7, Lemma 2.2], the following lemma can be derived.

Lemma 2.4. Let \( \overline{X} \) be a point of \( \Sigma \) and \( \{E_0 = \nu, E_1, \cdots, E_n\} \) be an adapted frame field such that each \( E_i \) is a principal direction and \( \omega_i^k = 0 \) at \( \overline{X} \). Let \( (h_{ij}) \) be the second quadratic form of \( \Sigma \). Then, at the point \( \overline{X} \), we have

\[h_{ii} - h_{i11} = h_{ii}h_{11}^2 - h_{11}h_{ii}^2 + 2(h_{ii} - h_{11})R_{1i11} + h_{ii}R_{1100} - h_{i1}R_{1101} + \overline{R}_{1i1i} - \overline{R}_{1i1i}.\]

Consider the function

\[\tau = \langle V, \nu \rangle, \quad \Lambda(r) = \int_0^r \lambda(s) ds\]

with the position vector field

\[V = \lambda(r) \frac{\partial}{\partial r}.\]

Then, we need the following lemma for \( \tau \) and \( \Lambda \).

Lemma 2.5. Let \( \tau, \Lambda \) be functions as above, then we have

\[\nabla_{E_i} \Lambda = \lambda(\overline{v}_0, E_i) E_i,\]

\[\nabla_{E_i} \tau = \sum_j (\nabla_{E_j} \Lambda) h_{ij},\]
\[ \nabla^2_{E_i,E_j} \Lambda = \lambda' g_{ij} - \tau h_{ij} \]

and

\[ \nabla^2_{E_i,E_j} \tau = -\tau \sum_k h_{ik} h_{kj} + \lambda' h_{ij} + \sum_k (h_{ijk} - \mathcal{R}_{0ijk}) \nabla_{E_k} \Lambda. \]

**Proof.** See Lemma 2.2, Lemma 2.6 and Lemma 2.3 in [7], [11] or [19] for the details. \(\square\)

### 3. A priori estimates

In order to prove Theorem 1.1, we use the degree theory for the nonlinear elliptic equation developed in [22] and the proof here is similar to those in [1, 19, 21, 23]. First, we consider the family of equations for \(0 \leq t \leq 1\)

\[ \frac{\sigma_k(\mu(\eta))}{\sigma_1(\mu(\eta))} = f^t(V, \nu(V)), \]

where \(f^t = tf(r,u,\nu) + (1-t)\varphi(r)\frac{C_k}{C_n}((n-1)\zeta(r))^{k-l}, \zeta(r) = \frac{\lambda'}{\lambda} \) and \(\varphi\) is a positive function which satisfies the following conditions:

(a) \(\varphi(r) > 0,\)

(b) \(\varphi(r) \geq 1 \text{ for } r \leq r_1,\)

(c) \(\varphi(r) \leq 1 \text{ for } r \geq r_2,\)

(d) \(\varphi'(r) < 0.\)

#### 3.1. \(C^0\) Estimates

Now, we can prove the following proposition which asserts that the solution of the equation (1.2) has uniform \(C^0\) bounds.

**Proposition 3.1.** Under the assumptions (1.3) and (1.4), if the \((\eta,k)\)-convex hypersurface \(\Sigma = \{(r(u),u) \mid u \in M\} \subset M\) satisfies the equation (3.1) for a given \(t \in (0,1],\)

then

\[ r_1 < r(u) < r_2, \quad \forall \ u \in M. \]

**Proof.** Assume \(r(u)\) attains its maximum at \(u_0 \in M\) and \(r(u_0) \geq r_2,\) then recall

\[ h^i_j = \frac{1}{\lambda^2 v}(\delta_{ik} - \frac{r^i r^k}{v^2})\left(-\lambda r_{kj} + 2\lambda'r_k r_j + \lambda^2 \lambda' \delta_{kj}\right), \]

which implies together with the fact that the matrix \(r_{ij}\) is non-positive definite at \(u_0\)

\[ h^i_j(u_0) = \frac{1}{\lambda^2}( -\lambda r_{ij} + \lambda^2 \lambda' \delta_{ij}) \geq \frac{\lambda'}{\lambda} \delta_{ij}. \]
Then
\[ \eta^i_j(u_0) = H\delta^i_j - h^i_j \geq \frac{(n-1)\lambda'}{\lambda} \delta_{ij}. \]

Note that \( \sigma_k(\mu(\eta)) \) for \( 0 \leq l \leq k - 2 \) is concave in \( \Gamma_k \). Thus
\[ \frac{\sigma_k(\mu(\eta))}{\sigma_l(\mu(\eta))} \geq \frac{\sigma_k((n-1)\lambda')\delta_{ij}}{\sigma_l((n-1)\lambda')\delta_{ij}} = \frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l}. \]

So, we arrive at
\[ tf(r, u, \nu) + (1-t)\varphi(r)\frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l} \geq \frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l}. \]

Thus, we obtain at \( u_0 \)
\[ f(r, u, \nu) \geq \frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l}, \]
which is in contradiction to (1.4). Thus, we have \( r(u) < r_2 \) for \( u \in M \). Similarly, we can obtain \( r(u) > r_1 \) for \( u \in M \). \( \square \)

Now, we prove the following uniqueness result.

**Proposition 3.2.** There exists an unique \((\eta, k)\)-convex solution to the equation (3.1) with \( t = 0 \), namely \( \Sigma_0 = \{(r(u), u) \in M \mid r(u) = r_0\} \), where \( r_0 \) satisfies \( \varphi(r_0) = 1 \).

**Proof.** Let \( \Sigma_0 \) be a solution of (3.1) for \( t = 0 \), then
\[ \frac{\sigma_k(\mu(\eta))}{\sigma_l(\mu(\eta))} - \varphi(r)\frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l} = 0. \]

Assume \( r(u) \) attains its maximum \( r_{\text{max}} \) at \( u_0 \in M \), then we have at \( u_0 \)
\[ h^i_j = \frac{1}{\lambda^i} \left( -\lambda r_{ij} + \lambda^2 \lambda' \delta_{ij} \right), \]
which implies together with the fact that the matrix \( r_{ij} \) is non-positive definite at \( u_0 \)
\[ \frac{\sigma_k(\mu(\eta))}{\sigma_l(\mu(\eta))} \geq \frac{C_n^k}{C_n^l}((n-1)\lambda')^{k-l}. \]

By the equation (3.1)
\[ \varphi(r_{\text{max}}) \geq 1. \]

Similarly,
\[ \varphi(r_{\text{min}}) \leq 1. \]
Thus, since $\varphi$ is a decreasing function, we obtain
$$
\varphi(r_{\min}) = \varphi(r_{\max}) = 1.
$$

We conclude
$$
r(x) = r_0, \quad \forall (r(u), u) \in \overline{M},
$$
where $r_0$ is the unique solution of $\varphi(r_0) = 1$.

3.2. $C^1$ Estimates. In this section, we establish the gradient estimates for the equation (3.1). The treatment of this section follows from [7, Lemma 3.1].

We recall that a star-shaped hypersurface $\Sigma$ in $M$ can be represented by
$$
\Sigma = \{ V(u) = (r(u), u) \mid u \in M \},
$$
where $V$ is the position vector field of hypersurface $\Sigma$ in $M$. We define a function $\tau = \langle V, \nu \rangle$. It is clear that
$$
\tau = \frac{r^2}{\sqrt{r^2 + |Dr|^2}}.
$$

**Theorem 3.3.** Under the assumption (1.5), if the closed star-shaped $(\eta, k)$-convex hypersurface $\Sigma = \{(r(u), u) \in \overline{M} \mid u \in M\}$ satisfies the curvature equation (3.1) and $r$ has positive upper and lower bound, then there exists a constant $C$ depending only on $n, k, l, \|\lambda\|_{C^1}, \inf_\Sigma r, \sup_\Sigma r, \inf_\Sigma f$ and $\|f\|_{C^1}$ such that
$$
|Dr| \leq C.
$$

**Proof.** It is sufficient to obtain a positive lower bound of $\tau$. We consider the function
$$
\Phi = -\ln \tau + \gamma(\Lambda),
$$
where $\gamma(\Lambda)$ is a function which will be chosen later. Assume that $\Phi$ attains its maximum value at point $u_0$. If $V$ is parallel to the normal direction $\nu$ at $u_0$, we have $\langle V, \nu \rangle = |V|$. Thus our result holds. So we assume $V$ is not parallel to the normal direction $\nu$ at $u_0$.

We can choose the local orthonormal frame $\{E_1, \cdots, E_n\}$ on $\Sigma$ satisfying
$$
\langle V, E_1 \rangle \neq 0, \quad \text{and} \quad \langle V, E_i \rangle = 0, \quad \forall \ i \geq 2.
$$

Obviously, $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$. Then, we arrive at $u_0$
$$
0 = \Phi_i = -\frac{\nabla E_i \tau}{\tau} + \gamma \nabla E_i \Lambda,
$$

(3.2)

$$
0 \geq \Phi_{ii} = -\frac{\nabla^2 E_i E_i \tau}{\tau} + \frac{|\nabla E_i \tau|^2}{\tau^2} + \gamma' \nabla^2 E_i E_i \Lambda + \gamma'' |\nabla E_i \Lambda|^2.
$$

(3.3)
From Lemma 2.5, (3.2) and (3.3), we have

$$0 \geq -\frac{1}{\tau}(-\tau h_{ii}h_{ii} + \lambda'h_{ii} + (h_{ii} - \overline{R}_{0ii})\Lambda_l) + (\gamma'' + (\gamma')^2)\Lambda_l^2 + \gamma'(\lambda'g_{ii} - \tau h_{ii}).$$

By (2.4) and (3.2), we obtain

$$h_{11} = \tau \gamma', \quad h_{i1} = 0, \quad \forall \ i \geq 2.$$ 

Therefore, it is possible to rotate the coordinate system such that \{\(E_1, \ldots, E_n\)\} are the principal curvature directions of the second fundamental form \((h_{ij})\), i.e., \(h_{ij} = h_{ii}\delta_{ij}\).

Thus, from (2.4)-(2.6), (3.3) and (3.4), we get

$$0 \geq F_{ii}h_{ii}^2 - \frac{1}{\tau} F_{ii}h_{ii} + \frac{1}{\tau} F_{ii}(h_{ii} - \overline{R}_{0ii})\Lambda_l + (\gamma'' + (\gamma')^2)\Lambda_l^2 + \gamma'(\lambda'g_{ii} - \tau h_{ii}).$$

Since \(\eta_{ii} = \sum_{j \neq i} h_{jj}\), then

$$\sum_i F_{ii}h_{ii} = \sum_i \left( \sum_k G^{kk} - G^{ii} \right) \left( \frac{1}{n-1} \sum_l \eta_{ii} - \eta_{ii} \right)$$

$$= \sum_i G^{ii} \eta_{ii}$$

$$= \frac{1}{k-l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{k-l-1} \sum_i \eta_{ii} \sigma_{k-1}(\eta|i) \sigma_l(\eta) - \sigma_k(\eta) \sum_i \eta_{ii} \sigma_{l-1}(\eta|i)$$

$$= \tilde{f},$$

where \(\tilde{f} = f^{\frac{1}{k-l}}\). Note that the curvature equation (1.2) can be written as

$$G(\eta) = \tilde{f}.$$ 

Differentiating (3.7) with respect to \(E_1\), we obtain

$$G_{ii} \eta_{i1} = d_V \tilde{f}(\nabla E_1 V) + h_{11}d_\nu \tilde{f}(E_1).$$
In fact
\[ F^{ii}h_{ii} = \sum_i \left( \sum_j (G^{jj} - G^{ii}) \right) h_{ii} = \sum_i G^{ii} \eta_{ii} \]

(3.8)
\[ = d_V \tilde{f}(\nabla_{E_1} V) + h_{11} d_{\nu} \tilde{f}(E_1). \]

Putting (2.3), (3.4), (3.6) and (3.8) into (3.5), we derive
\[ (\gamma'' + (\gamma')^2)F^{11} \Lambda^2 + \gamma'F^{ii}g_{ii} - \frac{\lambda'}{\tau} \tilde{f} - \gamma' \tau \tilde{f}. \]

(3.9)

Choosing the function \( \gamma(r) = \frac{\alpha}{r} \) for a positive constant \( \alpha \), we get
\[ \gamma'(r) = -\frac{\alpha}{r^2}, \quad \gamma''(r) = \frac{2\alpha}{r^3}. \]
By (3.4) and the choice of function $\gamma(r)$, we have $h_{11} < 0$ at $u_0$. From $H > 0$ we know that

$$(3.11) \quad F^{11} = \sum_{j \neq 1} G^{jj} \geq \frac{1}{2} \sum_i G^{ii} = \frac{1}{2(n-1)} \sum_i F^{ii} \geq \frac{1}{2} \left( \frac{C_n^k}{C_n^l} \right)^{1/k}. $$

Putting (3.10) into (3.9), we have

$$0 \geq F^{ii} h_{ii} - d_V \tilde{f} (\nabla \nu V) + \frac{\alpha}{r^2} d_V \tilde{f} (E_1) \langle V, E_1 \rangle - \frac{1}{\tau} F^{ii} \mathcal{T}_{0ii} \langle V, E_1 \rangle$$

$$+ \frac{2\alpha}{r^4} F^{11} \langle V, E_1 \rangle^2 - \frac{\alpha}{r^2} \lambda' F^{ii} g_{ii} + \frac{\alpha}{r^2} \tau \tilde{f}$$

$$\geq \tau^2 \frac{\alpha^2}{r^4} F^{11} + \frac{2\alpha}{r^4} F^{11} \langle V, E_1 \rangle^2 + \frac{\alpha}{r^2} d_V \tilde{f} (E_1) \langle V, E_1 \rangle$$

$$- \frac{1}{\tau} F^{ii} \mathcal{T}_{0ii} \langle V, E_1 \rangle + d_V \tilde{f} (\nabla \nu V) - \frac{\alpha}{r^2} \lambda' F^{ii} g_{ii} + \frac{\alpha}{r^2} \tau \tilde{f}$$

$$= \tau^2 \frac{\alpha^2}{r^4} F^{11} |V|^2 + \frac{2\alpha}{r^4} F^{11} \langle V, E_1 \rangle^2 + \frac{\alpha}{r^2} d_V \tilde{f} (E_1) \langle V, E_1 \rangle$$

$$- \frac{1}{\tau} F^{ii} \mathcal{T}_{0ii} \langle V, E_1 \rangle + d_V \tilde{f} (\nabla \nu V) - \frac{\alpha}{r^2} \lambda' \sum_i F^{ii} + \frac{\alpha}{r^2} \tau \tilde{f},$$

the third equality comes from $|V|^2 = \langle V, E_1 \rangle^2 + \langle V, \nu \rangle^2$.

Since $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$, we can find that $V \perp \text{Span}\{E_2, \ldots, E_n\}$. On the other hand, $E_1, \nu \perp \text{Span}\{E_2, \ldots, E_n\}$. It is possible to choose coordinate systems such that $\nu \perp \text{Span}\{E_2, \ldots, E_n\}$, which implies that the pair $\{V, \nu\}$ and $\{\nu, E_1\}$ lie in the same plane and

$$\text{Span}\{E_2, \ldots, E_n\} = \text{Span}\{\nu, \nu\}. $$

Therefore, we can choose $E_2 = \nu, \ldots, E_n = \nu$. The vector $\nu$ and $E_1$ can be decomposed into

$$\nu = \langle \nu, \nu \rangle \nu_0 + \langle \nu, \nu \rangle \nu_1 = \frac{\tau}{\lambda} \nu_0 + \langle \nu, \nu \rangle \nu_1, $$

$$E_1 = \langle E_1, \nu \rangle \nu_0 + \langle E_1, \nu \rangle \nu_1.$$
By (3.13) and (3.14), we obtain
\[
\overline{R}_{0i1} = \frac{\tau}{\lambda} \langle E_1, \nu \rangle \overline{R}(\nu, \nu) + \langle \nu, \nu \rangle \overline{R}(\nu, \nu)
\]
(3.15)
\[
= \frac{\tau}{\lambda} \langle E_1, \nu \rangle \overline{R}(\nu, \nu) - \frac{\tau}{\lambda} \langle \nu, \nu \rangle \overline{R}(\nu, \nu),
\]
the second equality comes from \(0 = \overline{R}_{ijkl0}\) (see [7, Lemma 2.1]), the third equality comes from \(0 = \langle V, \nu \rangle \).

From (3.11) and (3.15), (3.12) becomes
\[
0 \geq C_1 F^{11} \alpha^2 r_2 - C_2 F^{11} \frac{\alpha}{r_2} - C_3 \frac{\alpha}{r_1} |d_{\nu} \overline{f}(E_1)| - C_4 F^{11} - |d_{V} \overline{f}(\nabla_{\nu} V)| - C_5
\]
\[
\geq C \alpha^2 F^{11} - C_2 \alpha F^{11} - C_3 \alpha |d_{\nu} \overline{f}(E_1)| - C F^{11} - |d_{V} \overline{f}(\nabla_{\nu} V)| - C,
\]
where \(r_1 = \inf_{\Sigma} r, r_2 = \sup_{\Sigma} r\), \(C_1, C_2, C_3, C_4, C_5\) depend on \(n, r_1, r_2, \inf_{\Sigma} f\), the \(C^1\) bounds of \(\lambda\) and curvature \(\overline{R}\). Thus, we have a contradiction when \(\alpha\) is large enough. Hence, \(V\) is parallel to the normal \(\nu\) which implies the lower bound of \(\tau\).

3.3. \(C^2\) Estimates. Under the assumption (1.3)-(1.5), from Theorem 3.1 and Theorem 3.3 we know that there exists a positive constant \(C\) depending on \(\inf_{\Sigma} r\) and \(\|r\|_{C^1}\) such that
\[
\frac{1}{C} \leq \inf_{\Sigma} \tau \leq \sup_{\Sigma} \tau \leq C.
\]

**Theorem 3.4.** Let \(\Sigma\) be a closed star-shaped \((\eta, k)\)-convex hypersurface satisfying the curvature equation (3.1) and the assumption of Theorem 1.1. Then, there exists a constant \(C\) depending only on \(n, k, l, \inf_{\Sigma} \lambda, \inf_{\Sigma} r, \inf_{\Sigma} f, \|r\|_{C^2}, \|f\|_{C^2}\) and the curvature \(\overline{R}\) such that for \(1 \leq i \leq n\)
\[
|\kappa_i(u)| \leq C, \quad \forall u \in M.
\]

**Proof.** Since \(\eta \in \Gamma_k \subset \Gamma_1\), we see that the mean curvature is positive. It suffices to prove that the largest curvature \(\kappa_{\text{max}}\) is uniformly bounded from above. Take the auxiliary function
\[
P = \ln \kappa_{\text{max}} - \ln (\tau - a) + \Lambda,
\]
where \( a = \frac{1}{2} \inf_{\Sigma} (\tau) \) and \( A > 1 \) is a constant to be determined later. Assume that \( P \) attains its maximum value at point \( u_0 \). We can choose a local orthonormal frame \( \{ E_1, E_2, \ldots, E_n \} \) near \( u_0 \) such that
\[
h_{ii} = \delta_{ij} h_{ij}, \quad h_{11} \geq h_{22} \geq \cdots \geq h_{nn}
\]
at \( u_0 \). Recalling that \( \eta_{ii} = \sum_{k \neq i} h_{kk} \), we have
\[
\eta_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn}.
\]
It follows that
\[
G^{11} \geq G^{22} \geq \cdots \geq G^{nn}, \quad F^{11} \leq F^{22} \leq \cdots \leq F^{nn}.
\]
We define a new function \( Q \) by
\[
Q = \ln h_{11} - \ln (\tau - a) + A\Lambda.
\]
Since \( h_{11}(u_0) = \kappa_{\max}(u_0) \) and \( h_{11} \leq \kappa_{\max} \) near \( u_0 \), \( Q \) achieves a maximum at \( u_0 \).

Hence
\[
0 = Q_i = \frac{h_{11i}}{h_{11}} - \frac{\tau_i}{\tau - a} + A\Lambda_i,
\]
\[
0 \geq F^{ii} Q_{ii} = F^{ii}(\ln h_{11})_{ii} - F^{ii}(\ln(\tau - a))_{ii} + A F^{ii} \Lambda_{ii}.
\]

We divide our proof into four steps.

**Step 1:** We claim that
\[
0 \geq \frac{2}{h_{11}} \sum_{i \geq 2} G^{i,i,i} h_{i1}^2 h_{1i}^2 \frac{-F^{ii} h_{11i}^2}{h_{11}^2} + \frac{a F^{ii} h_{11i}^2}{\tau - a} + \frac{F^{ii} \tau_i^2}{(\tau - a)^2} + (A\Lambda' - C_0) \sum_i F^{ii} - C_0 h_{11} - \frac{C_0(1 + \sum_i F^{ii})}{h_{11}} - AC_0,
\]
where \( C_0 \) depends on \( \inf_{\Sigma} r, \inf_{\Sigma} f, \|r\|_{C^2}, \|f\|_{C^2} \) and the curvature \( \tilde{\kappa} \).

Using the similar argument in (3.8), we obtain
\[
F^{ii} h_{iij} = d_V \tilde{f}(\nabla_{E_j} V) + h_{jj} d_{\nu} \tilde{f}(E_j).
\]
By Gauss formula and Weingarten formula,
\[
\tau_i = h_{ii}(V, E_i), \quad \tau_{ii} = \sum_j h_{iji}(V, E_j) - \tau h_{ii}^2 + h_{ii}.
\]
Combined with (3.19), (3.20) and Codazzi formula, we have

$$-F^{ii}(\ln(\tau - a))_{ii} = -F^{ii}(\frac{\tau_{ii}}{\tau - a} - \frac{\tau_i^2}{(\tau - a)^2})$$

$$= -\frac{1}{\tau - a} \sum \eta_{jj}(d_{\nu}\tilde{f})(E_j)(V, E_j) - \frac{1}{\tau - a} \sum d_{\nu}\tilde{f}(\nabla E_j V)(V, E_j)$$

$$- \frac{1}{\tau - a} \sum R_{0ji}F^{ii}(V, E_j) + \frac{\tau F^{ii}h_{ii}^2}{\tau - a} - \frac{1}{\tau - a} \tilde{f} + F^{ii}\frac{\tau_i^2}{(\tau - a)^2}$$

(3.21) \geq -\frac{1}{\tau - a} \sum \eta_{jj}(d_{\nu}\tilde{f})(E_j)(V, E_j) + \frac{\tau F^{ii}h_{ii}^2}{\tau - a} + F^{ii}\frac{\tau_i^2}{(\tau - a)^2} - C_1 \sum_i F^{ii} - C_1,$$

where $C_1$ depends on $\inf_{\Sigma} r$, $\|r\|_{C^1}$, $\|f\|_{C^1}$ and the curvature $\overline{\nabla}$.

Differentiating (3.7) with respect to $E_1$ twice, we obtain

$$G^{ij}\eta_{j1} = d_{\nu}\tilde{f}(\nabla E_1 V) + h_{1k}d_{\nu}\tilde{f}(E_k)$$

and

$$G^{ij,rs}\eta_{j1}\eta_{rs1} + G^{ij}\eta_{j111} = d_{\nu}\tilde{f}(\nabla E_1 V, \nabla E_1 V) + d_{\nu}\tilde{f}(\nabla^2 E_1, E_1 V)$$

$$+ 2d_{\nu}d_{\nu}\tilde{f}(\nabla E_1 V, \nabla E_1 \nu) + d_{\nu}\tilde{f}(\nabla^2 E_1 \nu, \nabla E_1 \nu) + d_{\nu}\tilde{f}(\nabla^2 E_1, E_1 \nu)$$

$$\geq \sum_i h_{1i1}(d_{\nu}\tilde{f})(E_i) - C_2 h_{11}^2 - C_2 h_{11} - C_2.$$

Applying the concavity of $G$, we derive

$$-G^{ij,rs}\eta_{j1}\eta_{rs1} \geq -2 \sum_{i \geq 2} G^{1i,1i}\eta_{i11}^2 = -2 \sum_{i \geq 2} G^{1i,1i} h_{1i1}^2.$$

It follows that

(3.22) $F^{ii}h_{ii11} = G^{ii}\eta_{ii11} \geq -2 \sum_{i \geq 2} G^{1i,1i} h_{1i1}^2 + \sum_i h_{1i1}(d_{\nu}\tilde{f})(E_i) - C_2 h_{11}^2 - C_2 h_{11} - C_2,$$

where $C_2$ depends on $\|f\|_{C^2}$, $\|r\|_{C^2}$. 

Combined with (2.2), (3.22) and Codazzi formula, we have

\[
F_{ii}(\ln h_{1i})_{ii} = \frac{F_{ii}h_{11i}}{h_{11}} - \frac{F_{ii}h_{11i}^2}{h_{11}^2}
\]

\[
\geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i} h_{1i}^2 + \frac{1}{h_{11}} \sum_i h_{11i} (d_\nu \tilde{f})(E_i) + \frac{1}{h_{11}} \sum_i R_{0i11}(d_\nu \tilde{f})(E_i)
\]

\[
+ h_{11} \tilde{f} - F_{ii} h_{1i}^2 + \frac{F_{ii}}{h_{11}} (R_{1i0;1} - \tilde{R}_{0i1i0}) + 2 F_{ii} \tilde{R}_{1i1i} - F_{ii} R_{0000}
\]

\[
-C_2 \sum_i F_{ii} + \frac{\tilde{f}}{h_{11}} \frac{F_{ii} h_{11i}^2}{h_{11}^2} - C_2 h_{11} - C_2 - C_2
\]

\[
(3.23)
\]

\[
\geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i} h_{1i}^2 + \frac{1}{h_{11}} \sum_i h_{11i} (d_\nu \tilde{f})(E_i) - F_{ii} h_{1i}^2
\]

\[
- \frac{F_{ii} h_{11i}^2}{h_{11}} - C_3 h_{11} - C_3 \sum_i F_{ii} - C_3 C_3 - C_3
\]

where $C_3$ depends on $\inf \Sigma f$, $\|f\|C_2$, $\|r\|C_2$ and the curvature $\tilde{R}$.

By (2.5), we derive

\[
(3.24) \quad AF_{ii} \Lambda_{ii} = A\lambda' F_{ii} g_{ii} - A \tau F_{ii} h_{1i} = A\lambda' \sum_i F_{ii} - A \tau \tilde{f}.
\]

Taking (3.21), (3.23) and (3.24) into (3.17), we get

\[
0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i} h_{1i}^2 + \frac{1}{h_{11}} \sum_i h_{11i} (d_\nu \tilde{f})(E_i)
\]

\[
- \frac{1}{\tau - a} \sum_j h_{jjj} (d_\nu \tilde{f})(E_j) \langle V, E_j \rangle + \frac{a F_{ii} h_{11i}^2}{\tau - a} + F_{ii} \frac{\tau_i^2}{(\tau - a)^2}
\]

\[
+ (A\lambda' - C_4) \sum_i F_{ii} - C_4 h_{11} - C_4 (1 + \sum_i F_{ii}) - AC_4.
\]

By (3.16), (3.20),

\[
\frac{1}{h_{11}} \sum_i h_{11i} (d_\nu \tilde{f})(E_i) - \frac{1}{\tau - a} \sum_j h_{jjj} (d_\nu \tilde{f})(E_j) \langle V, E_j \rangle
\]

\[
= \sum_i \left( \frac{h_{11i}}{h_{11}} - \frac{\tau_i}{\tau - a} \right) (d_\nu \tilde{f})(E_i)
\]

\[
= -A \sum_i (d_\nu \tilde{f})(E_i) \langle V, E_i \rangle
\]

\[
\geq -AC_4,
\]
which implies the inequality (3.18).

**Step 2:** There exists a positive constant \( \delta < \frac{1}{n-2} \) such that

\[
\frac{C_{n-1}^{k-1}[1 - (n - 2)\delta]^{k-1} - (n - 1)\delta C_{n-1}^{k-2}[1 + (n - 2)\delta]^{k-2}}{C_n[1 + (n - 2)\delta]^l} > \frac{C_{n-1}^{k-1}}{2C_n^l}.
\]

We claim that there exists a constant \( B > 1 \) depending on \( n, k, l, \delta, \inf \Sigma r, \inf \Sigma f, \|r\|_{C^2}, \|f\|_{C^2} \) and the curvature \( \overline{R} \), such that

\[
\frac{a F_{ii} h_{ii}^2}{2(\tau - a)} + \frac{A\lambda - C_0}{2} \sum_i F_{ii} \geq C_0 h_{11},
\]

if \( h_{11} \geq B, A = \left( 4\|f\|_{C^0} \frac{1}{k-1} \frac{KC_n^l}{(n-k+1)C_{n-1}^{k-1} \inf \lambda'} + \frac{27}{\inf \lambda'} \right) C_0.\)

Case 1: \( |h_{ii}| \leq \delta h_{11} \) for all \( i \geq 2. \)

In this case, we have

\[
|\eta_{11}| \leq (n - 1)\delta h_{11}, \quad [1 - (n - 2)\delta] h_{11} \leq \eta_{22} \leq \cdots \leq \eta_{nn} \leq [1 + (n - 2)\delta] h_{11}.
\]

By the definitions of \( G_{ii} \) and \( F_{ii} \), we obtain

\[
\sum_i F_{ii} = (n - 1) \sum_i G_{ii} = \frac{n - 1}{k - l} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{k-1} \frac{(n - k + 1)\sigma_{k-1}(\eta)\sigma_l(\eta) - (n - l + 1)\sigma_k(\eta)\sigma_{l-1}(\eta)}{\sigma_l^2(\eta)}
\]

\[
\geq \frac{C_n^k}{C_n^{k-1}} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{k-1} \left( \frac{\sigma_{k-1}(\eta)}{\sigma_l(\eta)} \right)
\]

\[
= \frac{C_n^k}{C_n^{k-1}} \left( \frac{\sigma_k(\eta)}{\sigma_l(\eta)} \right)^{k-1} \left( \frac{\sigma_{k-1}(\eta) + \eta_{11}\sigma_{k-2}(\eta)\eta_{11}}{\sigma_l(\eta)} \right)
\]

\[
\geq \frac{n - k + 1}{k} \frac{1}{k-1} \frac{C_n^{k-1}[1 - (n - 2)\delta]^{k-1} - (n - 1)\delta C_n^{k-2}[1 + (n - 2)\delta]^{k-2}}{C_n[1 + (n - 2)\delta]^l} \frac{h_{11}^{k-1-l}}{h_{11}},
\]

which implies that

\[
C_0 h_{11} \leq \frac{A\lambda - 27C_0}{2} \sum_i F_{ii}.
\]
Case 2: $h_{22} > \delta h_{11}$ or $h_{nn} < -\delta h_{11}$.

In this case, we have

$$\frac{a F^{ii} h_{ii}^2}{2(\tau - a)} \geq \frac{a}{2(\sup \tau - a)} (F^{22} h_{22}^2 + F^{nn} h_{nn}^2)$$

$$\geq \frac{a \delta^2}{2(\sup \tau - a)} F^{22} h_{11}^2$$

$$\geq \frac{a \delta^2}{2n(\sup \tau - a)} \sum_i G^{ii} h_{11}^2$$

$$\geq \left(\frac{C_n^k}{C_n^l} \right)^{\frac{1}{2l}} \frac{a \delta^2 h_{11}}{2(\sup \tau - a)} h_{11}.$$

Then, we conclude that

$$\frac{a F^{ii} h_{ii}^2}{2(\tau - a)} \geq C_0 h_{11},$$

if

$$h_{11} \geq \left(\left(\frac{C_n^k}{C_n^l} \right)^{\frac{1}{2l}} \frac{a \delta^2}{2(\sup \tau - a)} \right)^{-1} C_0.$$

Step 3: We claim that

$$|h_{ii}| \leq C_5 A, \quad \forall \ i \geq 2,$$

if $h_{11} \geq B > 1$, where $C_5$ is a constant depending on $n, k, l, \inf r, \inf f, \|r\|_{C^2}, \|f\|_{C^2}$ and the curvature $R$.

Combined with Step 1 and Step 2, we obtain

$$0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i1} h_{11}^2 - \frac{F^{ii} h_{11i}^2}{h_{11}^2} + \frac{a F^{ii} h_{ii}^2}{2(\tau - a)} + \frac{F^{ii} \tau_{1i}^2}{(\tau - a)^2}$$

$$+ \frac{A \lambda' - C_0}{2} \sum_i F^{ii} - \frac{C_0 (1 + \sum_i F^{ii})}{h_{11}^2} - AC_0.$$

(3.25)

Use (3.16), the concavity of $G$ and Cauchy-Schwarz inequality,

$$0 \geq -\frac{1 + \frac{\epsilon}{(\tau - a)^2}}{F^{ii} \tau_{1i}^2} - (1 + \frac{1}{\epsilon}) A^2 F^{ii} \Lambda_i^2$$

$$+ \frac{a F^{ii} h_{ii}^2}{2(\tau - a)} + \frac{F^{ii} \tau_{1i}^2}{(\tau - a)^2} + \frac{A \lambda' - C_0}{2} \sum_i F^{ii} - \frac{C_0 (1 + \sum_i F^{ii})}{h_{11}^2} - AC_0$$

$$\geq \left(\frac{a}{2(\tau - a)} - \frac{C_0 \epsilon}{(\tau - a)^2}\right) F^{ii} h_{ii}^2 - \left((1 + \frac{1}{\epsilon}) A^2 C_0 - \frac{A \lambda' - C_0}{2}\right) \sum_i F^{ii} - \left(\sum_i F^{ii} + A + 1\right) C_0.$$
where \( \tau_i = h_{ii}(V, E_i) \) in the second inequality. Choose \( \epsilon = \frac{a(\tau - a)}{4C_0} \), (3.26)

\[
0 \geq \frac{a}{4(\tau - a)} F_{ii}^2 h_{ii}^2 - \left( (1 + \frac{4C_0}{a(\tau - a)}) A^2 C_0 - \frac{AX' - C_0}{2} \right) \sum_i F_{ii} - \left( \sum_i F_{ii} + A + 1 \right) C_0
\]

\[
\geq \frac{a}{4(\sup \tau - a)} F_{ii}^2 h_{ii}^2 - \left( (1 + \frac{4C_0}{a^2}) A^2 C_0 - \frac{AX' - C_0}{2} \right) \sum_i F_{ii} - \left( \sum_i F_{ii} + A + 1 \right) C_0.
\]

By (2.1) and (3.26), we have

\[
0 \geq \frac{a}{4(\sup \tau - a)n(n-1)} \left( \sum_{k \geq 2} h_{kk}^2 \right) \sum_i F_{ii}
\]

\[
- \left( (1 + \frac{4C_0}{a^2}) A^2 C_0 - \frac{AX' - C_0}{2} + C_0 + \frac{(A + 1)C_0}{n-1} \left( \frac{C_k}{C_n} \right)^{-\frac{1}{k-l}} \right) \sum_i F_{ii},
\]

which implies that

\[
\sum_{k \geq 2} h_{kk}^2 \leq C_5^2 A^2,
\]

where \( C_5 \) is a constant depending on \( n, k, l, \inf \Sigma r, \inf \Sigma f, \| r \|_{C^2}, \| f \|_{C^2} \) and the curvature \( \overline{R} \).

**Step 4:** We claim that there exists a constant \( C \) depending on \( n, k, l, \inf \Sigma \lambda', \inf \Sigma r, \inf \Sigma f, \| r \|_{C^2}, \| f \|_{C^2} \) and the curvature \( \overline{R} \) such that

\[
h_{11} \leq C.
\]

Without loss of generality, we assume that

(3.27) \[
h_{11} \geq \max \left\{ B, \left( \frac{32nC_0A^2(\sup \tau - a)}{\varepsilon a} \right)^{\frac{1}{2}}, \frac{C_5 A}{\beta} \right\},
\]

where \( \beta < \frac{1}{2} \) will be determined later. Recalling \( \tau_1 = h_{11}(V, E_1) \), by (3.16) and Cauchy-Schwarz inequality, we have

\[
\frac{F_{11} h_{11}^2}{h_{11}^2} \leq \frac{1 + \epsilon}{(\tau - a)^2} F_{11}^2 \tau_1^2 + (1 + \frac{1}{\epsilon}) A^2 F_{11}^1 \tau_1^1 (V, E_1)^2
\]

\[
\leq \frac{C_0 \varepsilon F_{11}^1 h_{11}^2}{(\tau - a)^2} + \frac{C_0 A^2 F_{11}^1}{(\tau - a)^2} + \frac{1 + \frac{\varepsilon}{\epsilon}}{\epsilon} C_0 A^2 F_{11}.
\]

Choose \( \varepsilon \leq \min\{ \frac{a(\tau - a)}{16C_0}, 1 \} \), such that

\[
\frac{F_{11} h_{11}^2}{h_{11}^2} \leq \frac{F_{11}^1 \tau_1^2}{(\tau - a)^2} + \frac{a F_{ii}^2 h_{ii}^2}{(\tau - a)^2} + \frac{2C_0 A^2 F_{11}^1}{\varepsilon}.
\]
Hence according to Step 3 and (3.27), we know that

\[(3.28) \quad \frac{F^{11} h_{11}^2}{h_{11}^2} \leq \frac{F^{11} \tau_1^2}{(\tau - a)^2} + \frac{a F^{ii} h_{ii}^2}{8(\tau - a)}\]

and

\[|h_{ii}| \leq \beta h_{11}, \quad \forall i \geq 2.\]

Thus

\[\frac{1 - \beta}{h_{11} - h_{ii}} \leq \frac{1}{h_{11}} \leq \frac{1 + \beta}{h_{11} - h_{ii}}.\]

Combined with Proposition 2.2, we obtain

\begin{align*}
\sum_{i \geq 2} \frac{F^{ii} h_{11}^2}{h_{11}^2} & = \sum_{i \geq 2} \frac{F^{ii} - F^{11} h_{ii}^2}{h_{11}^2} + \sum_{i \geq 2} \frac{F^{11} h_{11i}^2}{h_{11}^2} \\
& \leq \frac{1 + \beta}{h_{11}} \sum_{i \geq 2} \frac{F^{ii} - F^{11} h_{11i}^2}{h_{11} - h_{ii}} + \sum_{i \geq 2} \frac{F^{11} h_{11i}^2}{h_{11}^2} \\
& = \frac{1 + \beta}{h_{11}} \sum_{i \geq 2} \frac{G^{11} - G^{ii}}{h_{ii}^2} h_{11i}^2 + \sum_{i \geq 2} \frac{F^{11} h_{11i}^2}{h_{11}^2} \\
& = - \frac{1 + \beta}{h_{11}} \sum_{i \geq 2} \frac{G^{ii,i1} h_{11i}^2}{h_{11}^2} + \sum_{i \geq 2} \frac{F^{11} h_{11i}^2}{h_{11}^2}. \tag{3.29}
\end{align*}

Use (3.16), (3.27), Cauchy-Schwarz inequality and the fact \(\tau_i = h_{ii} \langle V, E_i \rangle\),

\begin{align*}
\sum_{i \geq 2} \frac{F^{11} h_{11i}^2}{h_{11}^2} & \leq 2 \sum_{i \geq 2} \frac{F^{11} \tau_i^2}{(\tau - a)^2} + 2 A^2 \sum_{i \geq 2} F^{11} \langle V, E_i \rangle^2 \\
& \leq 2 C_0 \frac{a^2}{a^2} \sum_{i \geq 2} \frac{a F^{11} h_{ii}^2}{\tau - a} + 2 n C_0 A^2 F^{11} \\
& \leq \beta^2 2 n C_0 a F^{11} \frac{h_{11}^2}{\tau - a} + \frac{a}{16(\tau - a)} F^{11} h_{11}^2. \tag{3.30}
\end{align*}

By Cauchy-Schwarz inequality and Codazzi formula, we have

\begin{align*}
\sum_{i \geq 2} G^{ii,i1} h_{11i}^2 &= - \frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i1} (h_{11i} + R_{01i1})^2 \\
& \geq - \frac{2}{h_{11}} \sum_{i \geq 2} G^{ii,i1} \left( \frac{3}{4} h_{11i}^2 - 3 R_{01i1}^2 \right). \tag{3.31}
\end{align*}
When we choose $\beta$ sufficiently small such that $\beta \leq \min\left\{ \sqrt{\frac{a^2}{2nC_0}}, \frac{1}{2} \right\}$, by (3.29), (3.30), (3.31) and Proposition 2.2, we have

$$
\sum_{i\geq 2} \frac{F_{ii} h_{11i}}{h_{11}^2} \leq -\frac{3}{2h_{11}} \sum_{i\geq 2} G_{ii,1} h_{11i}^2 + \frac{a F_{11} h_{11}^2}{8(\tau - a)} \leq -\frac{2}{h_{11}} \sum_{i\geq 2} G_{ii,1} h_{11i}^2 + \frac{6}{h_{11}} \sum_{i\geq 2} G_{ii,1} F_{ii} - \sum_{i\geq 2} F_{ii} h_{11i}^2 + \frac{a F_{11} h_{11}^2}{8(\tau - a)} \leq -\frac{2}{h_{11}} \sum_{i\geq 2} G_{ii,1} h_{11i}^2 + \frac{6C_0}{h_{11}} \sum_{i\geq 2} G_{ii,1} (G_{ii} - \beta) + \frac{a F_{11} h_{11}^2}{8(\tau - a)},
$$

(3.32)

if $h_{11} \geq B > 1$. The last inequality comes from $1 - \frac{\beta}{h_{11} - h_{ii}} \leq \frac{1}{h_{11}} < 1$. Note that

$$
\sum_{i\geq 2} (G_{ii} - G_{ii}) = nG_{ii} - \sum_i G_{ii}\leq (n - 1) \sum_i G_{ii} = \sum_i F_{ii}.
$$

Then (3.32) gives that

$$
-\frac{2}{h_{11}} \sum_{i\geq 2} G_{ii,1} h_{11i}^2 - \sum_{i\geq 2} F_{ii} h_{11i}^2 \geq -\frac{6C_0}{1 - \beta} \sum_{i\geq 2} (G_{ii} - \beta) - \frac{a F_{11} h_{11}^2}{8(\tau - a)} \geq -12C_0 \sum_{i} F_{ii} - \frac{a F_{11} h_{11}^2}{8(\tau - a)},
$$

(3.33)

Substitute (3.28) and (3.33) into (3.25), then

$$
0 \geq -\frac{F_{11} \tau_{1}^2}{(\tau - a)^2} - \frac{a F_{ii} h_{11i}^2}{8(\tau - a)} - 12C_0 \sum_{i} F_{ii} - \frac{a F_{11} h_{11}^2}{8(\tau - a)} + \frac{a F_{ii} h_{ii}^2}{2(\tau - a)} + \frac{A\lambda' - C_0}{2} \sum_{i} F_{ii} - C_0(1 + \sum_{i} F_{ii}) - AC_0 \geq \frac{A\lambda' - 27C_0}{2} \sum_{i} F_{ii} - C_0(A + 1) \geq \frac{C_0}{2} h_{11} - C_0(A + 1),
$$

which implies that

$$
h_{11} \leq 2(A + 1).
$$

□
4. The proof of Theorem 1.1

In this section, we use the degree theory for nonlinear elliptic equation developed in [22] to prove Theorem 1.1. The proof here is similar to those in [1, 10, 21]. So, only sketch will be given below.

Based on a priori estimates in Theorem 3.1, Theorem 3.3 and Theorem 3.4, we know that the equation (3.1) is uniformly elliptic. From [9], [20] and Schauder estimates, we have

\[ |r|_{C^{4,\alpha}(M)} \leq C \]  

for any \((\eta, k)\)-convex solution \(\Sigma\) to the equation (3.1), where the position vector of \(\Sigma\) is \(V = (r(u), u)\) for \(u \in M\). We define

\[ C^{4,\alpha}_0(M) = \{ r \in C^{4,\alpha}(M) : \Sigma \text{ is } (\eta, k) - \text{convex} \}. \]

Let us consider

\[ F(\cdot; t) : C^{4,\alpha}_0(M) \rightarrow C^{2,\alpha}(M), \]

which is defined by

\[ F(r, u; t) = \frac{\sigma_k(\mu(\eta))}{\sigma_l(\mu(\eta))} - tf(r, u, \nu) - (1-t)\varphi(r) \frac{C^k_n}{C^l_n}((n-1)\zeta(r))^{k-l}. \]

Let

\[ \mathcal{O}_R = \{ r \in C^{4,\alpha}_0(M) : |r|_{C^{4,\alpha}(M)} < R \}, \]

which clearly is an open set of \(C^{4,\alpha}_0(M)\). Moreover, if \(R\) is sufficiently large, \(F(r, u; t) = 0\) has no solution on \(\partial \mathcal{O}_R\) by a priori estimate established in (4.1). Therefore the degree \(\text{deg}(F(\cdot; t), \mathcal{O}_R, 0)\) is well-defined for \(0 \leq t \leq 1\). Using the homotopic invariance of the degree, we have

\[ \text{deg}(F(\cdot; 1), \mathcal{O}_R, 0) = \text{deg}(F(\cdot; 0), \mathcal{O}_R, 0). \]

Proposition 3.2 shows that \(r_0 = 1\) is the unique solution to the above equation for \(t = 0\). Direct calculation shows that

\[ F(sr_0; 0) = (1 - \varphi(sr_0)) \frac{C^k_n}{C^l_n}((n-1)\zeta(r))^{k-l}(sr_0). \]

Then

\[ \delta_{r_0}F(r_0, u; 0) = \frac{d}{ds}|_{s=1}F(sr_0, u; 0) = -\varphi'(r_0) \frac{C^k_n}{C^l_n}((n-1)\zeta(r))^{k-l}(r_0) > 0, \]
where $\delta F(r_0, u; 0)$ is the linearized operator of $F$ at $r_0$. Clearly, $\delta F(r_0, u; 0)$ takes the form

$$\delta_w F(r_0, u; 0) = -a^{ij} w_{ij} + b^i w_i - \varphi'(r_0) \frac{C_k}{C_n} ((n-1)\zeta(r))^{k-l}(r_0),$$

where $a^{ij}$ is a positive definite matrix. Since $-\varphi'(r_0) \frac{C_k}{C_n} ((n-1)\zeta(r))^{k-l}(r_0) > 0$, thus $\delta_{\nu_0} F(r_0, u; 0)$ is an invertible operator. Therefore,

$$\text{deg}(F(; 1), \mathcal{O}_R; 0) = \text{deg}(F(; 0), \mathcal{O}_R, 0) = \pm 1.$$ 

So, we obtain a solution at $t = 1$. This completes the proof of Theorem 1.1.

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