Dynamical Systems Analysis in Post-Friedmann Parametrizations of Modified Theories of Gravity

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Abstract. Provided that tests of the expansion history would not suffice to discriminate the best model of gravity at cosmological scales, upcoming high precision measurements of the structure of galaxies at very large scales will be a key piece of evidence that will allow us to break such a degeneracy. In order to test the predicted structure formation that comes from cosmological perturbation theory, the Parametrized Post-Friedmann (PPF) formalism provides an excellent theoretical framework to carry out such analysis as it parametrizes the cosmological perturbation theory of a large collection of Modify Gravity (MG) models. In this work, we carry out a dynamical analysis of the PPF parametrization for the first time in order to shed some light on the fate and evolution of the structure formation for different models of MG by writing the modified field equations using properly normalized variables. A Universe filled with only dark matter and dark energy is considered, represented by an effective equation of state, $\omega_{\text{eff}}$, and a negligible anisotropic stress. We consider different cases, from the simplest cold dark matter (CDM) model, passing through the $\Lambda$CDM model (with cosmological constant), finally arriving at the scale-independent and -dependent parametrizations of MG theories, in particular two models of $f(R)$ metric theories and Chameleon-like theories are considered. Given the employed formalism, for last two cases we found that the critical points and stability features of the dynamical systems are the same as those found in the fiducial $\Lambda$CDM model. However the behavior of the dynamics around them suffers important modifications. We find that signatures of these MG arise either in measurements of the modes of the lensing potential and velocity perturbations while the density contrast itself turns out to be less sensitive to the parametrization taken into consideration. We also provide a percentage estimation of the extent of the modification of perturbations in the MG models considered in comparison to those arising the fiducial model along the expansion history and for a number of scale-lengths.
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## 1 Introduction

Different measurements suggest that our Universe is currently experiencing a phase of accelerated expansion [1–3]. For more than a decade, cosmological data of the Cosmic Microwave Background (CMB) anisotropies [4] and supernovae (SN) Ia [5], plus the mapping of galaxies [6], have established that the Universe is prevailed by a new form of energy, called *dark energy* (DE), that is responsible of such expansion [7–9], making around 68% of the total matter-energy density [10]. Together with the other ingredients (mainly, dark matter and baryons), DE can fit fairly well the current observational data. The presence of a cosmological constant \( \Lambda \) in the Einstein field equations, which is based on the standard model of cosmology \( \Lambda \)CDM, is the simplest way to explain such accelerated expansion. However, it still faces severe objections, such as the coincidence and cosmological constant problems [8], and even though it looks as the most straightforward explanation, the presence of such a small, but different from zero constant, is one of the most exciting questions in today’s physics.

Such a challenging subject shows an imbalance in the Friedmann equations, and the physics community has addressed such problem either by proposing new sources of matter.
or by modifying (or extending) Einstein’s general relativistic description of gravity. In the framework of standard cosmology, the first one is addressed through DE, while the second one suggests the introduction of some modifications in the gravity part of the equations, known as MG theories [11, 12].

Einstein’s General Relativity (GR) is a well founded theory and though there have been different theoretical reasons to expand it, the case is that GR has passed all tests of experimental gravitation, from very small to Solar System scales [see for example 13, 14]. However, cosmological tests of GR are still an open challenge. Currently, the dynamics of the cosmological background and the properties of the perturbed Universe demand to be tested in consistency to the measurements of the Hubble (\(H(z)\)) [15], Baryon Acoustic Oscillations (BAO) distances [16], the Redshift Space Distortions (RSD) [17] expected from the gravitational attraction of cosmological structures and the Cosmic Microwave Background (CMB) [10], among others.

The assumption of DE comes once the Einstein’s field equations are established on the Friedmann-Lemaître-Robertson-Walker (FLRW) metric that gives rise to the Friedmann equations. These equations imply that a stress-energy-momentum component with negative pressure is required to describe cosmic acceleration. This substance is usually interpreted as the vacuum energy of the Universe (i.e., \(\Lambda\) in the \(\Lambda CDM\) model) or a scalar field (SF) [11].

From the theoretical point of view, dynamical DE models that come from a minimally coupled, canonical SF give an equation of state of the form, 

\[
P_{DE} = w(z) \rho_{DE},
\]

where \(P_{DE}\) is the pressure and \(\rho_{DE}\) is the energy density of the dark energy component. These models require the equation of state to be close to -1 at low redshifts (\(z \leq 1\)) and, in addition, assumptions are made for its perturbed behavior (i.e. sound velocity). Alternatively, in MG models a large-scale modification to GR is achieved by introducing new degrees of freedom to the geometric sector and therefore the cosmic expansion is due to the dynamics of these new modes. Diverse models can be found, such as scalar-tensor theories [19–21], \(f(R)\) metric theories [22–24], Hordenski models [25–27], among others, which on one hand pass local gravitation experiments in accordance with GR [28] by means of screening mechanisms, and on the other, they alter the laws of gravity at large scales.

Observations of the cosmic expansion history cannot differentiate DE from MG [29–31]. Forthcoming and ongoing redshift and weak lensing surveys, such as the Dark Energy Survey (DES) [32], Dark Energy Spectroscopic Instrument (DESI) [33], Euclid [34] and the Large Synoptic Survey Telescope (LSST) [35], together with Planck measurements [10] and other cosmological studies, will trace the growth of cosmic structures precisely through different epochs. They will give the opportunity to analyze GR by checking the relationship between the matter distribution, the gravitational potential and the lensing potential on various scales. Such experiments may give clues to the physics generating cosmic acceleration or increase the range of scales over which Einstein’s gravity has been proven up to now, and may also give an opportunity to discriminate between DE and MG models.

In this direction, the Parametrized Post-Friedmann (PPF) approach [36, 37] consists of formulating a parametrized system that can be used to investigate cosmological linear perturbations in a very general, model-independent way for a broad range of MG models, assuming that their background evolution is equivalent that of \(\Lambda CDM\) \(^1\). Unlike the Parametrized Post-Newtonian (PPN) approach, the PPF framework holds for arbitrary background metrics such as the FLRW metric, given that perturbations to the curvature scalar remain small. Being this approach linear, it is applicable to big length-scales on which matter perturbations have

\(^1\)It is well known that the expansion history is degenerated between models and a given MG parametrization can always be mapped to the \(\Lambda CDM\) background.
not yet passed the non-linear cutoff \((\delta_{nl}(k) \sim 1)\) and which lay inside the horizon. The PPF frame is thus useful to test gravity on cosmological scales and also to discriminate MG from DE. This type of parametric approach is not new; there has been a big amount of effort to understand the physics along these lines before \([36, 38, 39]\).

Departures from GR could be crucial not only in the linear regime of cosmological perturbations but also in the nonlinear regime (scales larger than the Solar System). Nonlinear effects may admit MG to be distinguished from exotic DE assuming that DE fluctuations are small. So, although the linear growth of structure gives a strong analysis of GR versus MG, it must be connected to other investigations. It is probable that a mixture of galaxy clustering, peculiar velocities, and weak lensing observations will be necessary to get robust constraints on scale-dependent MG theories.

In the present paper we pursuit to study MG models using the numerical phase space techniques known as \textit{dynamical systems} (DS). Because the field equations are complex, DS methods provide a nice and powerful way of analyzing the physical nature of perturbations in such theories \([40, 41]\), as it gives a rather straightforward way for achieving a qualitative picture of the dynamics of these models. In addition, in the context of MG, it facilitates to qualitatively estimate the sensitivity of the dynamics of perturbations to modifications to the geometric sector. These DS approaches have already been proved to a large range of space-times (FLRW, Bianchi models, etc.) \([42]\), for which the evolution equations for the expansion-normalized phase variables can be reduced to a system of autonomous, ordinary differential equations (ODE) that describe a self-consistent phase space \([43, 46, 47]\). Therefore, such tool allows for a preliminary analysis of these theories, suggesting what kind of models deserve further attention or which could be used to identify possible sources of observational signatures. Such techniques has been used to perform analysis in some DE/MG models in various recent investigations, see for example \([48]\) and references therein, and all of them consider only background dynamics. In the present work we extend the formalism to first order perturbation theory.

The main part of this paper is dedicated to contrast the parametrized models to the standard cosmological \(\Lambda\)CDM model, either numerically and analytically through the DS tool. Prosperous MG models are known to satisfy the desired cosmological dynamics, but its perturbations are more challenging to treat. In this sense, we give the critical points and eigenvalues for the different models which will describe many interesting results from the phase space analysis of the linear perturbations. Although the method is used for some specific cases in order to study the physics behind the models, the exposed tool is general and can be used to implement dynamical analysis of models of DE/MG (and even dark matter). Our main findings are the following: i) Critical points of either CDM, \(\Lambda\)CDM and MG parametrizations are essentially the same and they have the same stability features. This is not surprising since these parametrizations are constructed under the assumption that \(\Lambda\)CDM is a fiducial model from which modifications arise in specific scale (spatial and temporal) regimes depending on the type of underling theory; ii) The dynamical stream-field in the phase space of scale-independent PPF parametrizations is practically identical to that of \(\Lambda\)CDM; iii) By comparing the phase-stream-field nearby the critical points of scale-dependent PPF parametrizations against the \(\Lambda\)CDM fiducial model, we analyze the scale dependence of perturbations dynamics introduced by different MG parametrizations. We estimate the percentage of modification of each scale-dependent variable \((x_1, x_2, x_3, \text{ and } x_4)\) in comparison to the fiducial model, showing how the amplitude of such deviations depend on the scale \(k\) of the perturbation and the scale factor \(a\) for the MG models.
This work is organized as follows: In Section 2, the basic equations of the present paper are given. The cosmological perturbations inside the PPF formalism are studied, focusing primarily on the terms and equations that are of significance for this analysis. In Section 3, we briefly revise the DS theory. In Section 4 the evolution equations of the perturbed PPF system are transformed into an autonomous system by a convenient definition of the dynamical variables. In Section 5 we present the dynamical analysis of the standard CDM and $\Lambda$CDM cosmologies. The dynamical analysis is also carried out for the MG models considered, investigating the different trajectories that show up (vector field diagrams). We mainly focus on the information that the density contrast, velocity field and the scalar metric perturbations can give us about the physical behavior of the systems. The critical points are listed and the phase space analysis is given. We also show the numerical solutions and stability of the system in the vicinity of the critical points found, finally showing the percentage of deviation in the amplitude of the perturbations of the MG model compared to those in the fiducial $\Lambda$CDM model. Finally, Section 6 is dedicated to discussion and conclusions.

2 Theoretical Background

Despite that the Universe may look practically homogeneous at the so-called homogeneity scales (over $150-300$ Mpc) the Universe is definitely inhomogeneous at smaller scales in which structure formation has taken place and hence the use of perturbation theory is needed. Given that MG theories give rise to the same predictions at the level of the cosmological background, data from large scale structure (LSS) is able to break such degeneracy between models and new signatures are to be identified only by means of perturbations upon a FLRW Universe. The dynamics of subhorizon perturbations is sufficient to describe the LSS formation and therefore the PPF formalism provides a convenient framework to encompass a large set of theories. In this work we shall use the PPF formalism, which implements a universal set of predicted parameters which are constructed through observational constraints [49]. The starting idea behind the PPF approach is to parametrize the right-hand-side of the $\Lambda$CDM equations for subhorizon perturbations of the metric by using free functions of scale-size and scale factor. Any modification of gravity at the level of the background is ignored and only modifications at the level of the perturbations are introduced by means of the PPF free functions defining the parametrization. The formalism is then implemented, for a fixed $\Lambda$CDM background history, and from it, we focus on the dynamics of the perturbations. This approach is supported by the fact that the standard cosmological model is in agreement with present constraints of the expansion history, thus, modifications to GR or exotic fluids/DE must be about the same at the unperturbed level. Therefore models are discriminated at the level of perturbations [36–38].

2.1 General Parametrized Post-Friedmann Formalism

The metric tensor in the Newtonian gauge with only scalar perturbations is given by

$$ds^2 = a^2(\eta) \left[ -(1 + 2\psi)d\eta^2 + (1 + 2\phi)\delta_{ij} dx^i dx^j \right], \quad (2.1)$$

where $a$ is the scale factor and $\eta$ is the conformal time (related to the cosmological time $t$ by $d/d\eta = a \, d/dt$). The perturbations that rule the growth of structure are described by $\phi$ (the Newtonian potential) and $\psi$ (the spatial curvature potential); $\phi$ and $\psi$ alter both the energy of photons (Integrated Sachs-Wolfe [ISW] effect) and their direction of propagation (gravitational lensing).
Einstein’s field equations to first order in perturbations dictate the evolution of the scalar potentials. Besides, the conservation of the energy-momentum tensor governs the density contrast and velocities of the different fluids that conform the Universe. We can then inquire if there is a general way to modify the cosmological perturbation equations such that they may incorporate many of the different MG theories. In order to mimic some MG behavior, the relationship between the left-hand and right-hand sides of Einstein’s equations has to be changed, but neither, Bianchi’s identities nor the energy-momentum conservation tensor should be modified (continuity and Euler equations).

The observables that characterize the large-scale structure are determined using cosmological perturbation theory in Fourier space. As mentioned before, the important variables are the two scalar potentials, $\phi$ and $\psi$, along with the matter density contrast $\delta$ and the matter velocity perturbation $v$. Thus, one needs four equations to describe the evolution of these four variables. Two equations are given by the covariant conservation of the matter energy-momentum tensor. The other two equations are assumed by a theory of gravity that defines how the metric responds to the matter stress-energy tensor.

In the case of the $\mu - \gamma$ parametrization the changes appear by writing the functions $\mu(a,k)$ and $\gamma(a,k)$ (with wavenumber $k$ dependence) in the Poisson and shear equations. A way to pose this in the Newtonian metric, in an epoch where radiation can be neglected, is by writing the adiabatic fluid equations in the following way, where the modifications are present in Eqs. (2.4) and (2.5):

$$
\dot{\delta} = -(1 + w)(\theta + 3\dot{\phi}) - 3H(c_s^2 - w)\delta, \tag{2.2}
$$

$$
\dot{\theta} = -\left[H(2 - 3w) + \frac{\dot{w}}{1 + w}\right] \theta + \frac{k^2}{a^2} \left(\frac{c_s^2}{1 + w}\delta + \psi - \sigma\right), \tag{2.3}
$$

$$
k^2\phi = 4\pi G a^2 \rho_t \left[\delta_t + 3a^2 H (w_{\text{eff}} + 1) \frac{\theta_t}{k^2} + 3(1 + w_{\text{eff}})\sigma\right] \mu, \tag{2.4}
$$

$$
k^2(\phi + \gamma\psi) = 12\pi G a^2 \rho_t \mu (1 + w_{\text{eff}})\sigma, \tag{2.5}
$$

where

$$
\rho_t = \rho_m + \rho_\Lambda, \tag{2.6}
$$

$$
w_{\text{eff}} = \Omega_m w_m + \Omega_\Lambda w_\Lambda, \tag{2.7}
$$

$$
\theta_t = \frac{(1 + w_m)\Omega_m \theta_m + (1 + w_\Lambda)\Omega_\Lambda \theta_\Lambda}{1 + w_{\text{eff}}}, \tag{2.8}
$$

$$
\delta_t = \Omega_m \delta_m + \Omega_\Lambda \delta_\Lambda. \tag{2.9}
$$

This is a closed system for $\delta$, $\theta$ (the velocity potential defined as $\theta \equiv \nabla_i v^i = \nabla_i dx^i/dt$), $\phi$ and $\psi$; $\sigma$ is the anisotropic stress. In Eqs. (2.2)-(2.3), the dots mean derivative with respect to the cosmological time $t$; the subindex ‘eff’ in the equations refers to effective quantities if there is more than one component (in our case, only DM and DE with the effective total equation of state $\omega_{\text{eff}}$); the subindex t means total quantities and $\rho_m$ denotes the DM density component. When $\gamma(a,k) = \mu(a,k) = 1$, the case of GR is recovered. We define $G_{\text{eff}} \equiv \mu(a,k)G$ (with $G$ the Newton’s gravitational constant), which in Eqs. (2.4) and (2.5) introduces a gravitational modification at cosmological scales. For simplicity, we will only focus in flat cosmologies ($\kappa = 0$) omitting entropy perturbations. These definitions infer that anisotropic stress of matter can be ignored at the epoch of interest, although it can be included, if needed, as it was done in [50, 51].
2.2 Scale-independent Parametrizations

During the radiation-dominated era, the Jeans length of the matter-radiation system is comparable to the Hubble length of the Universe. In this case, GR and MG operate similarly. However, at late times, the Jeans length has fallen to a few Mpc or less, and modifications of gravity come into play for most of the sub-horizon modes of the perturbations. Thus, the time and space dependence of perturbations must factorize for wavelengths longer than the Jeans length [38]:

\[ \phi(k, a) = D(a)\zeta(k), \]  
(2.10)

where \(D(a)\) is a function of the scale factor only and \(\zeta(k)\) is the curvature perturbation, which at the same time corresponds to the transfer function of the potentials. This factorization implies that the factor of both potentials depends only on the scale factor:

\[ \phi(k, a) = \gamma(a)\psi(k, a) + \mathcal{O}(k^2\zeta). \]  
(2.11)

In MG theories, scalar perturbations with very long wavelengths are mainly affected by \(\gamma\), and \(\mu\) is identified to the analysis of the physical gravitational constant and with experiments of the weak equivalence principle. By assuming that the \(\mathcal{O}(k^2\zeta)\) terms are negligible on sub-horizon scales larger than the Jeans length, a class of theories is defined: the scale-independent MG models [38]. Under the previous hypothesis, modifications of gravity are entirely described by the dynamics of the perturbations at large scales, and \(\gamma\) and \(\mu\) are given by

\[ \gamma(a) = \mu(a) = 1 + \beta a^s, \]  
(2.12)

where \(\beta\) and \(s\) are constants that can be obtained from observational constraints. The simplest case of this kind of theories (2.12) is the standard Brans-Dicke theory (BDT), that is basically the same as GR since \(\gamma \simeq 1 \simeq \mu\). \(^2\)

2.3 Scale-dependent Parametrizations

In scale-dependent MG theories, the parameters are functions not just of the length-scale but also of the scale factor, and there is no straightforward relationship between them. Thus, more parameters are required to describe such theories [52]. In the quasi-static approximation (QSA), the Poisson and trace shear equations can again be parametrized in terms of both the time- and scale-dependent functions, \(\mu(a, k)\) and \(\gamma(a, k)\) as shown in [22]. Under quite general conditions, and the QSA \((k/aH \gg 1)\), \(\mu\) and \(\gamma\) should always acquire a form of ratio of polynomials in \(k\). The coefficients inside the polynomials are functions of the background quantities and can be expected to be gradually changing functions. Technically, the number of these time-dependent coefficients is infinite if one allows for an arbitrary modification of GR. Quantitative differences in the predictions for these coefficients are important to discriminate between MG theories. This form of parametrization has been considered in a large number of works, e.g. [36–39], using various functional forms of the scale-dependence of \((\mu, \gamma)\), corresponding to different theories.

GR and MG theories have in common that the curvature perturbation is a conserved quantity at some range of scales. Therefore, the scale-dependence of the transfer functions

\(^2\)According to Solar System constraints.
only appear below the Jeans length or at scales of strong-field sector, for MG theories where the factorization of cosmological perturbations no longer holds. Theories of this kind are called *scale-dependent* MG models. Examples of this class of models are the $f(R)$ metric theories, where the wavelength of the scalar degree of freedom of the modification characterizes the theory. Eventually, measuring this scale-dependence (at several redshifts) can constrain scale-dependent MG theories.

A suitable scheme to encompass all theories having a modification of gravity with second order equations of motion are included in the Hordenski class [55], for which the ratio of even polynomials is as follows [53]:

\[
\mu(a,k) = 1 + \frac{p_3(a)k^2}{p_4(a) + p_5(a)k^2}, \tag{2.13}
\]

\[
\gamma(a,k) = \frac{p_1(a) + p_2(a)k^2}{1 + p_3(a)k^2}, \tag{2.14}
\]

where the $p_i(a)$’s are, in general, free functions of the scale factor.

Even though Eqs. (2.13)-(2.14) come from general arguments, the subspace of possible models to which we are restraining corresponds to the set of models contained in the Hordenski class, which includes many of the possible theories of DE and MG [54, 55]. It is important to note that although this ansatz was obtained using the QSA, it allows for near and super-horizon modifications of gravity: $\gamma(a,k \to 0) = p_1(a) \neq 1$.

Specific forms of the $p_i$ functions can be obtained from the particular MG theories. Having five free functions gives an unnecessary level of indetermination in this analysis. Then, it is good to demand physical arguments in order to assemble an improved parametrization of this type of theories. It is common to accommodate the transition scale, which separates the different regimes in which gravity acts in different ways. This transition can be studied by the following parametrization [56]:

\[
\mu(a,k) = 1 + \frac{\beta_1 \lambda_1^2 k^2 a^s}{1 + \lambda_1^2 k^2 a^s}, \tag{2.15}
\]

\[
\gamma(a,k) = 1 + \frac{\beta_2 \lambda_2^2 k^2 a^s}{1 + \lambda_2^2 k^2 a^s}, \tag{2.16}
\]

where the parameters $\lambda_i$ have dimensions of length, while the $\beta_i$’s serve as dimensionless couplings. According to [56], the above parametrization is suitable to scalar-tensor theories which include a host of cosmologically relevant models. Table 1 shows well-known examples found in the literature [see e.g. 56].

| Model               | $\beta_1$ | $\beta_2$ | $\lambda_1^2$ | $\lambda_2^2$ | $a^s$ |
|---------------------|-----------|-----------|----------------|----------------|-------|
| $f(R)$ I            | 4/3       | 1/2       | $\beta_1^{-1}10^4$ Mpc$^2$ | $10^9$ Mpc$^2$ | 4     |
| $f(R)$ II           | 4/3       | 1/2       | $\beta_1^{-1}10^4$ Mpc$^2$ | $10^8$ Mpc$^2$ | 4     |
| Chameleon-like I    | 9/8       | 7/9       | $\beta_1^{-1}10^3$ Mpc$^2$ | $10^9$ Mpc$^2$ | 2     |
| Chameleon-like II   | 9/8       | 7/9       | $\beta_1^{-1}10^3$ Mpc$^2$ | $10^8$ Mpc$^2$ | 2     |

**Table 1**: MG scale-dependent model parameters.

### 3 A brief revision of Dynamical Systems

The qualitative analysis of different cosmological models whose evolution is ruled by a finite-dimensional ODE, autonomous system has become a nice tool which plays a big role [17]. The
system is said to be autonomous if for \( \dot{x}_i = f_i(x_1, ..., x_n, t) \), the functions \( f_i \) do not include explicit time-dependent terms. The DS approach provides the opportunity of studying the stability of the solutions of the system in a simple way by investigating the linearization of the system around its critical points. For example, equilibrium points of the reduced system can correspond to dynamically evolving cosmological models [18].

Since our intention is to provide a qualitative picture of the perturbations inside the PPF formalism, a DS approach is undertaken. Commonly, a dimensionless (logarithmic) time variable, \( N \) (e-folding), is introduced so that the evolution is valid for all times (i.e., \( N \) accepts all real values), and a normalized set of variables is build for a number of reasons: i) This usually leads to a bounded system, ii) the variables are well behaved and often have a straightforward clear physical explanation, and iii) due to a symmetry in the equations, one of the equations decouple from the others (in GR the expansion rate is used to normalize the variables), and the arising simplified reduced system is studied. The goal of our qualitative analysis is to understand the nature of typical solutions of this kind of systems.

In this direction, the dynamics of the system (2.2)-(2.5) can be treated by inspecting its evolution around fixed/critical points, i.e. points \( P_i \) fulfilling the stability condition \( dP_i / d \ln a = 0 \). After obtaining the fixed points, one proceeds to get the eigenvalues \( \lambda_i \) of the Jacobian matrix of the system, in order to linearize it around each critical point. This determines the stability nature of a particular point, in other words, it controls how the system behaves when approaching to such critical point.

Now, in more than three dimensions it turns out difficult to label all possible critical points based on their eigenvalues. Nevertheless, for an \( n \)-dimensional system, if one has \( n \) eigenvalues for each point, the stability depending on the nature of these eigenvalues can be roughly interpreted given the following classification:

- All \( \lambda_i \) are real and have the same sign:
  - Negative eigenvalues: Stable node/Attractor.
  - Positive eigenvalues: Unstable node.

- All \( \lambda_i \) are real and at least one is positive and one negative: Saddle points.

- At least one eigenvalue is real and there are pairs of complex eigenvalues:
  - All eigenvalues have negative real parts: Stable Focus-Node.
  - All eigenvalues have positive real parts: Unstable Focus-Node.
  - At least one positive real part and one negative: Saddle Focus.

For a thorough description of the technique, and for some applications to other cosmological models, please refer to [43–45].

4 PPF Dynamical System

In this section we give the ODE that are to be solved in the present problem, and the convenient normalization variables are introduced in order to obtain the corresponding DS that will be carefully analyzed in the following sections. Linear perturbation theory is assumed to be valid throughout the paper.
4.1 The differential system

The system of cosmological ODE that describes the evolution of the matter perturbations in the PPF formalism are given by

\[
\begin{align*}
\dot{\delta} &= - (1 + w)(\theta + 3\phi) - 3H(c_s^2 - w)\delta, \\
\dot{\theta} &= - \left[ H(2 - 3w) + \frac{\bar{w}}{1 + w} \right] \theta + \frac{k^2}{a^2} \left( \frac{c_s^2}{1 + w} \delta + \psi - \sigma \right), \\
\dot{\phi} &= 4\pi G \frac{a^2}{k^2} \rho_t \mu \left[ \delta_t + 3\frac{a^2}{k^2} H\theta_{\text{eff}} \dot{\theta} + H(1 + w_{\text{eff}})\theta_t + 2H^2(1 + w_{\text{eff}})\theta_t \right] \\
&+ 4\pi G \frac{a^2}{k^2} \mu \left[ \delta_t + 3a^2 H \frac{k^2}{k^2} \rho_t (1 + w_{\text{eff}}) \theta_t + 6a^2 H^2 \frac{k^2}{k^2} \rho_t (1 + w_{\text{eff}}) \theta_t \right] \\
&+ 12\pi G \frac{a^2}{k^2} \rho_t \mu [\sigma \dot{w}_{\text{eff}} + \sigma(1 + w_{\text{eff}})] \\
&+ 12\pi G \frac{a^2}{k^2} \mu [\sigma(1 + w_{\text{eff}})(\dot{\phi} + 2H\rho_t)], \\
\dot{\psi} &= 12\pi G \frac{a^2}{k^2} \frac{\sigma}{\gamma}(1 + w_{\text{eff}})(\dot{\phi} + 2H\rho_t + \rho_t \dot{\mu}) + 12\pi G \frac{a^2}{k^2} \frac{\mu}{\gamma} \rho_t \left[ (1 + w_{\text{eff}}) \left( \dot{\sigma} - \sigma \frac{\dot{\gamma}}{\gamma} \right) + \sigma \dot{w}_{\text{eff}} \right] \\
&- \frac{1}{\gamma} \left( \dot{\phi} - \sigma \frac{\dot{\gamma}}{\gamma} \right).
\end{align*}
\]

The units of the variables involved are

\[
[a] = [\delta] = [\phi] = [\psi] = [\mu] = [\gamma] = \text{[dimensionless]},
\]

\[
[H] = [\theta] = [k] = \left[ \frac{1}{t} \right] = \text{[eV]},
\]

where we are assuming \( c = \hbar = 1 \).

4.2 The autonomous system

The system of Eqs. (4.1)-(4.4) will be written as an autonomous system. The first step in the implementation is the definition of the variables. We introduce the general dimensionless variables:

\[
\begin{align*}
x_1 &= \delta, \quad x_2 = \frac{\theta}{H}, \quad x_3 = \phi, \\
x_4 &= \psi, \quad x_5 = \frac{H^2}{k^2}, \quad x_6 = \Omega_m \quad x_7 = a^2,
\end{align*}
\]

where \( x_1 \) is the density contrast; \( x_2 \) is a re-definition of the velocity divergence \( \theta \) over the Hubble parameter; \( x_3 \) and \( x_4 \) are the scalar potentials of the metric; \( x_5 \) is the squared Hubble rate in units of the perturbation mode \( k \); \( x_6 \) is the matter density parameter and \( x_7 \) is the scale factor squared.
For $x_2$ one has that the velocity divergence decreases at early times when $H$ is large, but it increases at late times. $x_5$ conveniently provides a good comparison parameter of the size of a mode at a given time in the expansion history.

The physical bounds of the involved variables are: $-1 \ll x_1 \ll 1$ (linear approximation), $-1 \ll x_2 \ll 1$ (linear approximation), $0 \leq x_3, x_4 < 1$ (first order perturbation), $0 \leq x_5 < \infty$, $0 \leq x_6 \leq 1$ and $0 \leq x_7 \leq 1$.

Physically, all variables $x_i$ are bounded with the exception of $x_5$, which can tend towards infinity since $H$ has no upper bound. It is important then to build a bounded variable for $x_5$ to carry out a reliable search of the critical points in our system. To that aim the following change of variable is made:

$$y_5 \equiv \frac{1}{1 + x_5}.$$  \hspace{1cm} (4.9)

Once relation (4.9) is introduced, the new variable $y_5$ will be bounded between $[0, 1]$, which corresponds to $x_5 \to \infty$ and $x_5 = 0$, respectively. In general, the search for fixed points at infinity is done by using Poincaré compactification of the phase space. Still, from a phenomenological point of view, in this work, apart from $x_5$ we shall only determine the dynamics in the neighborhood of finite fixed points which are already physically bounded.

For a flat FLRW Universe the following constraint is realized for the effective equation of state

$$w_{\text{eff}} \equiv -1 - \frac{2}{3} \frac{H'}{H} = w_\Lambda (1 - \Omega_m) + w_m \Omega_m = \begin{cases} 0, & \text{CDM} \\ -\Omega_\Lambda = \Omega_m - 1, & \Lambda \text{CDM} \end{cases}$$  \hspace{1cm} (4.10)

where $'$ means derivative with respect to the e-folding: $d/(Hdt) = d/dN$, with $N = \ln a$.

Using Eqs. (4.1)-(4.4) and the definition of the bounded variable $y_5$ (Eq. (4.9)), our dimensionless DS reads as follows:

$$x'_1 = -(1 + \omega)(x_2 + 3x_3) - 3(c_s^2 - \omega)x_1,$$ \hspace{1cm} (4.11)

$$x'_2 = -x_2 \left[ \frac{1}{2} - 3\omega + \frac{\omega'}{1 + \omega} + \frac{3}{2}(1 - x_6) \right] + \frac{1}{x_5 x_7} \left( \frac{c_s^2}{1 + \omega} x_1 + x_4 - \sigma \right),$$ \hspace{1cm} (4.12)

$$x'_3 = \frac{3x_5 x_6 x_7 \mu}{2 + 9(1 + \omega) x_5 x_6 x_7 \mu} \left[ -(1 + \omega) x_2 - 3(c_s^2 - \omega)x_1 - 3(1 - x_6)(x_1 + 3x_2 x_5 x_7) 
- 3x_2 x_5 x_7 \left( \frac{1}{2} - 3\omega + \frac{\omega'}{1 + \omega} + \frac{3}{2}(1 - x_6) \right) + 3 \left( \frac{c_s^2}{1 + \omega} x_1 + x_4 - \sigma \right) 
- (3x_6 - 2)(x_1 + 6x_2 x_5 x_7) + \frac{\mu'}{\mu} (x_1 + 3x_2 x_5 x_7 + 3\sigma) + 3\sigma \left( \frac{\sigma'}{\sigma} - 1 \right) \right],$$ \hspace{1cm} (4.13)

$$x'_4 = \frac{9}{2} x_5 x_6 x_7 \frac{\mu \sigma}{\gamma} \left[ \frac{\mu'}{\mu} + \frac{\sigma'}{\sigma} - \frac{\gamma'}{\gamma} - 1 \right] - \frac{1}{\gamma} \left( x'_3 - x_3 \frac{\gamma'}{\gamma} \right),$$ \hspace{1cm} (4.14)

$$y'_5 = 3y_5^2 x_5 x_6,$$ \hspace{1cm} (4.15)

$$x'_6 = -3x_6(1 - x_6),$$ \hspace{1cm} (4.16)

$$x'_7 = 2x_7.$$ \hspace{1cm} (4.17)

The generalization to the multi-fluid case is trivial: one has just to add a new variable $\Omega_i$ for each new type of fluid. As a result, the number of dynamical equations increases and then also the dimension of the phase space.
Let us make an important point: the application of the \(\mu - \gamma\) parametrization to the equations of motions in more general cases does not always assume the QSA. In the \(\Lambda\)CDM limit, when \(\mu = \gamma = 1\), the exact equations of GR are recovered, while the parametrization admits for departures from \(\mu = \gamma = 1\) at all scales. Relativistic effects may be important for some MG models [57–59], but our present approach does not consider them.

Lastly, observe that the number of equations is always \(n \geq 3\) and this suggests that the model can permit chaotic behavior. Although this is not unexpected, it makes the understanding of the non-local properties of the phase space a very challenging exercise.

5 Results

In what follows, the critical points and eigenvalues of system (4.11)-(4.17) are obtained for different models and their cosmological viability is examined. Since we are interested in perturbations at late times, radiation has decoupled and can be safely ignored. The overall features of the dynamics will be given in the form of vector phase portraits, which reflect the phase space solutions of the problem. Also, when studying MG models we shall integrate out the system with initial conditions close to the critical points in order to analyze their behavior and compare them to the \(\Lambda\)CDM fiducial model. For this purpose, we carry out the corresponding MG-DS analysis of the models considered (\(f(R)\) and Chameleons), finally arriving to the conclusion that their critical points are exactly the same to those found in \(\Lambda\)CDM (see (5.8), (5.9), (5.19), (5.20)). The reason for this is that the MG PPF parametrization introduces multiplicative factors to equations (4.13), (4.14) so that the \(\Lambda\)CDM critical points are also a solution of the MG system. Other non-trivial critical points that could mathematically be initially be derived from e.g. Eq. (4.14) are not however a solution of the whole system. Given this though, the steam flows of the phase space in both models are quite different. Also, we point out that the DS-approach used here turns to be a powerful tool when it comes to quantify the extent of modification with respect to \(\Lambda\)CDM, by introducing MG parametrizations. In this direction, we extract useful information about how much \(\Lambda\)CDM differs from MG solutions, either with the scale \(k\) and/or the initial conditions around the critical points. This information can be useful when studying the phenomenology of these models, as we will see below, and provides valuable criteria to identify which data result convenient to use in order to constrain MG models.

5.1 CDM system

For the standard CDM case (no MG), a Universe filled with only this matter element is studied. For the variables of our dynamical system, Eqs.(4.11)-(4.17), this implies \(x_6 = \Omega_m = 1\) (\(\Omega_\Lambda = 0\)) and \(\mu = \gamma = 1\); one also has that \(\omega = 0\), \(c_s = 0\), \(\sigma = 0\). The DS is then
given by

\begin{align}
x_1' &= -(x_2 + 3x_3'), \\
x_2' &= \frac{x_2 - x_3}{2 - x_5 x_7}, \\
x_3' &= -\frac{3x_5 x_7}{2 + 9x_5 x_7} \left[ x_1 + 3x_3 + x_2 \left( 1 + \frac{15}{2} x_5 x_7 \right) \right], \\
x_4' &= -x_3', \\
2x_5' &= 3x_5^2 x_5, \\
x_6' &= 0, \\
x_7' &= 2x_7.
\end{align}

And the system has the following critical points:

\begin{align}
P_1 &= (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, y_5 = 0, x_6 = 1, x_7 = 0), \\
P_2 &= (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, y_5 = 1, x_6 = 1, x_7 = 0).
\end{align}

In this case, we are using the fact that in a Universe without anisotropic stress, the Einstein field equations yield \( \psi = -\phi \) (the stress-energy tensor is invariant under spatial rotations, or the three principal pressures are identical), which translates into \( x_4 = -x_3 \), and the scalar potentials result the same during the epoch of structure formation. However, this will no longer be true in MG theories, and the potentials need not to be the same; this case will be treated in the next sections. In general, the solution for the matter density satisfies \( x_6 = \text{constant} \) (free parameter), but as we are working on a flat FLRW background Universe with only dark matter, this implies \( x_6 = 1 \).

Note how the value corresponding to the density perturbations is \( x_1 = \delta = 0 \) at the critical points. Since the value corresponding to the potentials \( \phi = -\psi = x_3 = 0 \) at this point, there are no gravitational wells to form density perturbations. Also, the divergence of the velocity perturbation divided by the Hubble parameter \( x_2 = \theta/H = 0 \) at the critical points. Given that \( \theta = \mathbf{v} \cdot \mathbf{v} \), in Fourier space the condition \( x_2 \) is equivalent to \( \mathbf{k} \cdot \mathbf{v} = 0 \), that is, the velocity projected onto the direction of the wavevector vanishes independently of the magnitude of \( k \). Therefore the condition \( k \cdot \mathbf{v} \ll 1 \) has two possibilities: i) the velocity perturbation is always transverse to the wavevector, or ii) it vanishes. The first possibility may not necessarily be true, hence we adopt the second possibility and then the Universe has no peculiar velocities at these critical points.

The value corresponding to the ratio of the wavenumber to the Hubble parameter, \( x_5 = H/k \), can take two values at the critical points; \( P_1 \) corresponds to \( x_5 \to \infty \) which either means \( H \) very large (this case corresponds to the Big Bang singularity) or mathematically \( k \to 0 \) corresponding to large scales (This latter option is not valid when QSA is applied and shall not be considered in such limit). \( P_2 \) corresponding to \( x_5 = 0 \) implies \( H = 0 \) (static Universe) or \( k \gg 1 \) (corresponding to small structures), which cannot be since we are working inside the linear regime. Finally, \( a \) turns out to be very small in both cases.

Furthermore, the system seems to diverge for some values of the critical points (\( x_5 \) and \( x_7 \)). This happens because in the operations involved in the derivation of the autonomous system, terms like \( 1/x_5 x_7 \) come out in the denominators. Nevertheless, this is not a real problem for the method but a result of the fact that for these values of the parameter the cosmological equations take an special form.
The eigenvalues $\lambda_i$ corresponding to the Jacobian matrix of CDM evaluated at the critical points are given by:

| Critical point | Eigenvalues $\lambda$ |
|----------------|------------------------|
| $P_1$          | $\{3.0, 3.0, 2.0, -1.840, 1.219, 0.0, 0.0\}$ |
| $P_2$          | $\{3.0, -3.0, 2.0, -1.798, 0.649 + 0.642i, 0.649 - 0.642i, 0.0\}$ |

Table 2: Eigenvalues for the two critical points in the CDM case.

For $P_1$ all eigenvalues are real, with two of them being equal to zero and one negative. For $P_2$, we have five real eigenvalues and a pair of complex conjugates. From the Hartman-Grobman theorem [46], it is known that if all eigenvalues of the Jacobian matrix satisfy $\text{Re}(\lambda) \neq 0$, then the point is hyperbolic. If at least one eigenvalue is zero, then the point is said to be non-hyperbolic, and no further can be said apart from that. In the case of $P_1$ and $P_2$, both critical points have at least one vanishing eigenvalue, so they are non-hyperbolic in nature. The usual study of the critical points in this case is then unsuccessful in providing a trustworthy idea about the behavior of the flux around them. Still, as it will be seen in the next subsections, the phase portraits corresponding and numerical solutions near $P_1$ and $P_2$ can give us a good understanding of the nature of these points.

### 5.2 $\Lambda$CDM system

A $\Lambda$CDM expansion history is defined by $1 - \Omega_\Lambda = \Omega_m$. Also, $\dot{H} = 0 (H' = 0)$ corresponds to either a static or a de Sitter solution ($\Omega_m = 0$ and $w_{\text{eff}} = -1$). In addition, we have the energy constraint condition

$$\Omega_m + \Omega_\Lambda = 1 \rightarrow \Omega_\Lambda = 1 - \Omega_m = 1 - x_6,$$  \hspace{1cm} (5.10)

which can be used to shorten the dimensions of the system. For the system of equations that describe the $\Lambda$CDM case we have: $w_{\text{eff}} = -\Omega_\Lambda = \Omega_m - 1$, and the Hubble evolution equation can then be written as

$$\frac{H'}{H} = -\frac{3}{2}x_6.$$  \hspace{1cm} (5.11)

The dynamical system is then given by

$$x'_1 = -(x_2 + 3x'_3),$$  \hspace{1cm} (5.12)

$$x'_2 = -\frac{x_2}{2} (4 - 3x_6) - \frac{x_3}{x_5 x_7},$$  \hspace{1cm} (5.13)

$$x'_3 = -\frac{3x_5 x_6 x_7}{2 + 9x_5 x_6 x_7} \left\{ x_1 (2x_6 - 1) + 3x_3 + x_2 \left[ 1 + 3x_5 x_7 \left( 1 + \frac{3}{2}x_6 \right) \right] \right\},$$  \hspace{1cm} (5.14)

$$x'_4 = -x'_3,$$  \hspace{1cm} (5.15)

$$y'_5 = 3x_3 y_5^2 3x_6,$$  \hspace{1cm} (5.16)

$$x'_6 = -3x_6 (1 - x_6)$$  \hspace{1cm} (5.17)

$$x'_7 = 2x_7.$$  \hspace{1cm} (5.18)

The $\Lambda$CDM system has four critical points: two of them corresponding to $P_1$ and $P_2$ from the CDM model (Table 2); i. e., they correspond to a submanifold of the $\Lambda$CDM case.
and their physical interpretation is the same as that given in the previous case. There are also two new critical points, $P_3$ and $P_4$, given by

$$P_3 = (x_1 = p, x_2 = 0, x_3 = 0, x_4 = 0, y_5 = 0, x_6 = 0, x_7 = 0), \quad (5.19)$$
$$P_4 = (x_1 = q, x_2 = 0, x_3 = 0, x_4 = 0, y_5 = 1, x_6 = 0, x_7 = 0), \quad (5.20)$$

where $p$ and $q$ are arbitrary constants; we take $p = q = 0$ for the sake of self-consistency, since there are no gravitational contributions. $P_3$ and $P_4$ are the new critical points which correspond to a DE-only dominated Universe, an empty ($\rho_m = 0$) de Sitter Universe where a cosmological constant pervades.

Now, we obtain the eigenvalues of the autonomous system (5.12)-(5.18) in order to determine their stability.

| Critical point | Eigenvalues $\lambda$ |
|---------------|------------------------|
| $P_1$         | $\{3.0, 3.0, 2.0, -1.840, 1.219, 0.0, 0.0\}$ |
| $P_2$         | $\{3.0, -3.0, 2.0, -1.798, 0.649 + 0.642i, 0.649 - 0.642i, 0.0\}$ |
| $P_3$         | $\{-3, -2, 2, 0, 0, 0, 0\}$ |
| $P_4$         | $\{-3, -2, 2, 0, 0, 0, 0\}$ |

Table 3: Eigenvalues for the four critical points in the $\Lambda$CDM case.

There are several important facts to consider from the critical points and their eigenvalues. Again, from $P_1$ and $P_2$ it can be seen that if there are no gravitational potentials there will be no source of matter; i.e., if $x_3 = 0$ ($x_4 = 0$) this implies $x_1 = 0$. As in the CDM case, $P_1$ corresponds to the point of Big Bang singularity. In the case of $P_2$, we have again $y_5 = 1$ which means $x_5 = H^2/k^2 \to 0$ ($H = 0$ is a stationary solution), and both points correspond to the previous results obtained in Sec. 5.1. $P_3$ and $P_4$ correspond to an empty de Sitter Universe filled up with only DE, infinitely expanding for the case $y_5 = 0$ ($x_5 \to \infty$) and stationary in the case $y_5 = 1$ ($x_5 = 0$). In all the critical points we have $x_2 = \theta/H = 0$.

Figs. 1-4 show the critical points of the stream fields $x_1' - x_6'$, $x_4' - x_7'$, $x_5' - x_6'$, and $x_6' - x_7'$, respectively, for the $\Lambda$CDM model, projected onto the corresponding dynamical variables. We have particularly chosen these stream fields and projections as they can clearly show the behavior of the corresponding perturbations with respect the content of matter density ($x_6$) and the evolution of the scale factor ($x_7$).
Figure 1: Phase portraits for the matter perturbations vs. matter density in the $\Lambda$CDM model. Projection of the stream fields $x'_1 - x'_6$ onto the plane $x_6 - x_1$. The evolution from $y_5 = 0$ to $y_5 = 1$ is shown (from left to right). The phase portraits show the behavior of the perturbations from $x_6 = 0$ (DE dominated Universe) to $x_6 = 1$ (DM dominated Universe). Case $y_5 = 0$ (Big Bang): it can be seen how an only DE dominated Universe is a minimum for matter perturbations, i. e., an stable homoclinic orbit (point $(0, 0)$). Meanwhile, the maximum for the perturbations corresponds to the point $(1, 0)$ (DM only Universe). Case $y_5 = 1$ (stationary Universe): In all cases the evolution is stationary.
As mentioned above, a critical point of the system is a saddle point when it is neither a sink (all $\Re(\lambda) < 0$) nor a source ($\Re(\lambda) > 0$), as it is the case of the perturbations in DE dominated solutions as shown in Figs. 3 and 4, where it can be seen that these points attract trajectories in some directions but repel them along others, being always unstable.

Physically, this illustrates the behavior for the perturbations from a DM to a DE dominated Universe. Once the point $x_6 = 0$ is reached, the solution is completely DE dominated, acting as a cosmological constant. The numerical projection plots of the system show that this set cannot be stable. Weinberg [60] was the first to point out that once the cosmological constant comes to prevail the dynamics of the Universe, then structure formation stops because density perturbations stop growing.

![Phase portrait](image.png)

**Figure 2**: Phase portrait for the matter perturbations vs scale factor in the $\Lambda$CDM model. Projection of the stream fields $x'_1 - x'_7$ onto the plane $x_7 - x_1$ for $y_5 = 0$ (the case $y_5 = 1$ is exactly the same as before). The phase portrait shows the behavior of the perturbations from $x_7 = 0$ (initial singularity) to $x_7 = 1$ (today). Case $y_5 = 0$ (Big Bang), it can be seen how the initial condition $a = 0$ is a maximum for the system, it corresponds to the point $(0,0)$. Case $y_5 = 1$ (stationary Universe); In all cases the evolution is stationary. In comparison to the phase portraits of Fig.1, in this case there is only one critical point corresponding to the Big Bang. There are no critical points corresponding to the evolution of the perturbations in the future. The phase portrait correspond to $x_6 = 1$ (DM dominated). For the case $x_6 = 0$ (DE dominated), the evolution is always stationary.
Figure 3: Phase portraits for the metric perturbations vs matter density in the ΛCDM model. Projection of the stream fields $x_3' - x_6'$ onto the plane $x_6 - x_3$ for $y_5 = 0$ (the case $y_5 = 1$ is exactly the same as before). The phase portrait shows the behavior of the gravitational perturbations ($x_3 = -x_4$) from $x_6 = 0$ (DE dominated Universe) to $x_6 = 1$ (DM dominated Universe). Case $y_5 = 0$ (Big Bang), it can be seen how an only DE dominated Universe is a minimum for gravitational perturbations, i.e., an stable homoclinic orbit (point $(0,0)$). Meanwhile, the maximum for the perturbations corresponds to the point $(1,0)$ (DM only Universe). Case $y_5 = 1$ (stationary Universe); In all cases the evolution is stationary.
Figure 4: Phase portrait for the matter density vs scale factor in the ΛCDM model. Projection of the stream fields $x_6' - x_7'$ onto the planes $x_6 - x_7$ for all $y_5$. The phase portraits show the behavior of the matter density parameter from $x_7 = 0$ (initial singularity) to $x_7 = 1$ (today). It can be seen how the initial condition $a = 0$ is a minimum for DM when there is only a DE contribution, it corresponds to the point $(0,0)$. Finally, the point $(0,1)$ corresponds to a maximum for the DM density.

5.3 Modified gravity models

In this section we study some of the most significant linear MG models in cosmology, making use of the full DS given by Eqs. (4.11)-(4.17). In principle, cosmological observations of cosmic structure on linear scales can be used to measure $\mu$ and $\gamma$, and to constrain specific models. It is overall required that GR holds at early times, meaning that $s > 0$.

5.3.1 Scale-Independent Parametrizations

In the scale-independent case, the modifications of gravity are completely described by the dynamics of the perturbations at large scales, and the dynamical variable connected to $\gamma = \mu$ is given by

$$\gamma = \mu = 1 + \beta(\sqrt{x_7})^s,$$

with $\beta$ and $s$ constants. This parametrization is purely phenomenological and it is motivated by the arguments given in Section 2.2. Its corresponding differential equation in terms of the
e-folding ($N = \ln a$) becomes

$$\gamma' = \mu' = \beta s (\sqrt{\gamma})^s. \quad (5.22)$$

From Eqs. (4.11)-(4.17) it can be seen that $x_7 = 0$ is a critical point for the system, and we note that the scale-independent parametrization depends only on this variable. From Eq. (5.21), $\gamma = \mu = 1$, both at the critical point. With these values and comparing this system with the DS of the $\Lambda$CDM case (Eqs. (5.12)-(5.18)), we can see that the physics of the scale-independent DS compared to that of $\Lambda$CDM is the same. The phase portraits for this case are numerically very similar to those obtained for the $\Lambda$CDM model at the level of the perturbations, so we do not show them here again. In conclusion, from the DS point of view, there is not a big difference between $\Lambda$CDM and the scale-independent MG theories.

### 5.3.2 General scale-dependent parametrizations

From the point of view of model testing, scale-dependent MG introduces a richer phenomenology compared to the scale-independent case treated in the previous subsection. These models have two extra parameters plus the exponent giving the time dependence. Most importantly, the natural coherence of $\Lambda$CDM perturbations is lost in these models since a characteristic scale is introduced, below which modifications to gravity show up. This key feature brings up interesting phenomenological features of LSS formation in the context of MG theories.

Now, as mentioned in Sec. 2.3, an appropriate way to encompass all theories having a modification of gravity is through the Hordenski class, for which the dimensionless PPF parametrizations are given by the following equations, according to Eq. (2.15) and Eq. (2.16):

$$\mu(x_7) = \frac{1 + \beta_1 l_1 x_7^{s/2}}{1 + l_1 x_7^{s/2}}, \quad (5.23)$$

$$\gamma(x_7) = \frac{1 + \beta_2 l_2 x_7^{s/2}}{1 + l_2 x_7^{s/2}}, \quad (5.24)$$

where $\beta_1$ and $\beta_2$ are fixed dimensionless coupling constants and $s$ is a dimensionless constant that should be constrained by observations. In this parametrization, for notational convenience, we have defined the new auxiliary and dimensionless parameter

$$l_i \equiv \lambda_i^2 k^2 \quad \text{for } i = 1, 2. \quad (5.25)$$

For simple PPF models, the parameters $\lambda_i$ define a characteristic length-scale for a given modified model: $\lambda_1$ corresponds to the scale below which the gravitational potential $x_3$ turns out to be modified. Similarly, $\lambda_2$ is the upper bound for scales of modified modes of the scalar curvature perturbation $x_4$. Notice that the $l_i$’s are the only free dimensionless parameters which depend on the wavenumber (scale of the perturbation), and correspond to the ratio of the wavelength of the perturbation to the characteristic scales of the model.

Now, if we want to take into account such type of parametrizations into our DS (Eqs. (4.11)-
The corresponding dimensionless differential equations for $\mu$ and $\gamma$ read:

$$
\mu'(x_6, x_7) = \frac{1 + \beta_1 l_1 x_7^{s/2}}{1 + l_1 x_7^{s/2}} \left( 3x_6 + 3l_1 x_6 x_7^{s/2} - sl_1 x_7^{s/2} \right) + \frac{\beta_1 l_1 x_7^{s/2}(s - 3x_6) - 3x_6}{1 + l_1 x_7^{s/2}},
$$

$$
\gamma'(x_6, x_7) = \frac{1 + \beta_2 l_2 x_7^{s/2}}{1 + l_2 x_7^{s/2}} \left( 3x_6 + 3l_2 x_6 x_7^{s/2} - sl_2 x_7^{s/2} \right) + \frac{\beta_2 l_2 x_7^{s/2}(s - 3x_6) - 3x_6}{1 + l_2 x_7^{s/2}}.
$$

As it can be seen from Eqs. (5.26) and (5.27) models with higher $s$ produce bigger changes in the DS portraits because they lead to larger time derivatives, i.e., larger values of $s$ produce larger changes in the late time behavior of the different models.

### 5.3.3 Parametrization in $f(R)$ and Chameleon models

In this subsection we make use of the general DS given by Eqs. (4.11)-(4.17) to find the special characteristics for the I and II-$f(R)$ and Chameleon models shown in Table 1. To this end, we have written $\lambda_2$ in terms of $\lambda_1$ as: $\lambda_2 = \sqrt{\lambda_1^2 \beta_1}$, so the number of parameters reduces to four ($\lambda_1$, $\beta_1$, $\beta_2$ and $s$) with only one free parameter given by the wavenumber $k$.

In what follows, the phase portraits shown will be only those of the variables that seem to be most affected by the modifications to GR. The rest behave in the same manner as in the $\Lambda$CDM model.

The phase portraits plotted in Fig. 5 show the behavior of the stream field $x_1' - x_7'$, from $x_7 = 0$ (Big Bang singularity) to $x_7 = 1$ (today). The phase portraits correspond to $x_6 = 1$ (DM dominated). For the case $x_6 = 0$ (DE dominated), the evolution is exactly the same as in the $\Lambda$CDM model. There is only one critical point corresponding to the Big Bang and the case $y_5 = 0$ corresponds to such singularity. The initial condition $a = 0$ ($x_7 = 0$) is a source point interpreted as the time at which perturbations can start growing. Strictly speaking the solution should be valid since last scattering, $z_{ls} = 1080$ that yields $x_7 = (1/1081)^2 \approx 10^{-6}$. The case $y_5 = 1$ corresponds to a stationary Universe. There are no critical points corresponding to the evolution of the perturbations in the future. It can be noticed how, as the wavenumber increases (heading to the non-linear regime), the transition to the final stationary state is faster. Also, as $k$ increases, the differences between $f(R)$ and Chameleon models are more remarkable for the perturbations into the future. As in the previous cases, note how the curves arise from the past attractor (Big bang), expand monotonously, and evolve into the future.
Figure 5: Phase portraits for the matter perturbations vs scale factor in MG models. Projection of the stream fields $x'_1 - x'_7$ onto the plane $x_1 - x_7$ for different wavenumbers: $k = 0.012 \text{ Mpc}^{-1}$ (upper panel) and $k = 0.12 \text{ Mpc}^{-1}$ (bottom panel). Red arrows represent the stream fields in $f(R)$ models and the blue arrows represent the stream fields for Chameleon models.
Figure 6: Phase portraits for the velocity perturbations vs scale factor in MG models. Projection of the stream fields $x_1' - x_7'$ onto the plane $x_2 - x_7$. The panel represents the MG models, $f(R)$ (red) and Chameleon (blue), both for $k = 0.012$ Mpc$^{-1}$. The phase portraits show the behavior of the stream field of the velocity perturbation from $x_7 = 0$ (Big Bang) to $x_7 = 1$ (today).

The behavior of the stream-field of $x_2 = \theta/H$ seems to be independent of the wavenumber and of the MG model, but it is completely different from the $\Lambda$CDM model in which all cases is static. In the MG models the case $y_5 = 0$ seems to be vertically stationary for the velocity field. There is only one critical point corresponding to the Big Bang. There are no critical points corresponding to the evolution of the perturbations in the future. The phase portraits correspond to $x_6 = 1$ (DM dominated). We can see how the point $(0,0)$ is a saddle point for $x_2$, in this case there is an homoclinic orbit (minimum in energy) at this point when $y_5 = 1$ (which corresponds to $H = 0$). For the $\Lambda$CDM model the evolution form $y_5 = 0$ to $y_5 = 1$ is always stationary (there is no change in the direction of the fields). The smallness-to-the-eye of the scale-dependency of the projected stream field of $x_2$ shown by this plot is only apparent. Solutions with initial conditions nearby the critical points also show this “small-to-the-eye” effect (see Fig. 11 later on). However, as it will be seen in the next pages, Fig. 12 will show the real extent of the deviation from the $\Lambda$CDM fiducial solutions which turns out to be quite significant.

In Fig. 7 the behavior of the field seems to be independent of $y_5$ (we have the same evolution for any $y_5$). This evolution is dependent on the wavenumber. As the wavenumber increases (nonlinear regime) a critical point seems to appear at $(0,0)$ (panel for $k = 0.12$ Mpc$^{-1}$). This critical point corresponds to a homoclinic orbit. The changes appear to be more significant for f(R) models than for Chameleon models. The phase portraits correspond to $x_6 = 1$ (DM dominated). For the $\Lambda$CDM model the evolution is always stationary.
Figure 7: Phase portraits for the gravitational potential vs scale factor in MG models. Projection of the stream fields $x_7' - x_7$ onto the planes $x_7 - x_3$ for all $y_5$. The red vectors represent f(R) and the blue vectors represent the Chameleon models. In this case the evolution is shown for different wavenumbers ($k = 0.012$ Mpc$^{-1}$, $k = 0.037$ Mpc$^{-1}$ and $k = 0.12$ Mpc$^{-1}$, respectively). The phase portraits show the behavior of the gravitational potential $x_3$ from $x_7 = 0$ (Big Bang) to $x_7 = 1$ (today).
Figure 8: Phase portraits for the curvature potential $\psi$ vs scale factor in MG models. Projection of the stream fields $x_4' - x_7'$ onto the plane $x_4 - x_7$. The red vector field represents $f(R)$ models and the blue vector field represents Chameleon models, both for $k = 0.012$ Mpc$^{-1}$. The phase portraits show the behavior of the curvature potential from $x_7 = 0$ (Big Bang) to $x_7 = 1$ (today).

In this case, the behavior of the field seems to be independent of the wavenumber and of the MG model, but it is completely different from $\Lambda$CDM which in all cases is static. For the MG models, in the case $y_5 = 0$ it can be seen that the point $(0,0)$ is a saddle point for $x_4$, in this case there is an homoclinic orbit. There is only one critical point corresponding to the Big Bang. There are no critical points corresponding to the evolution of the potential in the future. The phase portraits correspond to $x_6 = 1$ (DM dominated). At $y_5 = 1$ (which corresponds to $H = 0$) the evolution is stationary and the same as the $\Lambda$CDM model.

5.4 Modified Gravity Solutions with Initial Conditions in a Vicinity of the Critical Points

In this section we analyze the evolution of a set of solutions for different initial conditions close to each critical point.

Specifically, the following figures show solutions of the perturbations with initial conditions close to the different critical points obtained previously for different scales above and below the characteristic scale $\lambda_1$ of the corresponding model. Each set of lines with a given color correspond to solutions with a fixed initial condition laying in a subspace $S \equiv \{(x_1, ..., x_4) | x_1 = x_2 = x_3 = x_4 = P_0\}$. In order to figure out whether any scale dependency is affected by initial conditions, we overlap different wavelengths indistinguishably for each $P_0$ considered. Later we shall get back to study the scale-dependency of particular cases in more detail. Notice that the solutions for a given color (initial condition) show scale-dependency as different lines split apart from each other.
Table 4: Initial condition for each critical point. The initial conditions were selected near the critical points, we use the parameters $P_0 = 10^{-2}$, $\epsilon = 10^{-3}$, the integration starts at $\log a = -6$, i.e. $x_{7}^{ini} = 10^{-12}$.

An important remark that we must make at this point is that the solutions shown in this work are not necessarily those picked by observations, rather we aim to have a glance into the plethora of possible solutions laying nearby the critical points of the dynamical system describing the dynamics of linear perturbations for parametrized models of gravity within the context of PPF in order to be able to compare them to the fiducial $\Lambda$CDM model and understand the nature of the modifications to gravity.

As Fig. 9 shows, solutions of the perturbations $x_1, x_2,$ and $x_3$ are rather sensitive to initial conditions in the subspace $S$ in the vicinity of $P_1$. Besides, modes of perturbations $x_1, x_2,$ and $x_3$ evolve almost independently of the scale-size either in the large and small scale regimes of MG. It is interesting to notice that in all cases, at small redshifts, solutions of $x_2$ and $x_3$ have an attractor behavior towards a constant value independently of their wavelength and initial conditions. In addition, modes of $x_3$ suddenly drop at early times and the depth of such drop has a large dependence upon initial conditions. Particularly, $x_3$ solutions nearby $P_1$ and $P_2$ have a deeper fall as the initial condition $P_0$ is smaller in contrast to what happens for solutions close to $P_3$ and $P_4$. Besides, smaller values of $\epsilon$ lead to a quicker transition from its minimum to a stationary value at late times. This process is carried out independently of the wavelength. On the other hand, interestingly modes of $x_4$ undergo through a transition from a stationary state to another with a higher value which is independent of of the scale-size and it only determines when such transition happens. Notice that this does not occur in $\Lambda$CDM where $x_3 = x_4$. Such difference at late times can be interpreted as a “gravitational slip” due to modifications to the metric which observationally would lead to lensing effects. It is worth of noticing that the evolution of $x_4$ is totally independent on local initial conditions within $\mathcal{P}$ associated to $\Lambda$CDM critical points $(P_{1,2})$ in contrast to what happens around CDM ones $(P_{3,4})$ where $x_4$ is slightly sensitive to initial conditions. An interesting feature of $x_4$ corresponding to the CDM critical points $(P_{3,4})$ in contrast to the ones in $\Lambda$CDM $(P_{1,2})$ is that at early times they suffer a small drop produced by matter. Let us say a quick word about the matter contrast $x_1$, its behavior w.r.t. variations of initial conditions is roughly the same for all critical points except for $P_3$ where larger $\epsilon$ leads to smaller perturbations.
Figure 9: Solutions for different modes of the four perturbations are plotted for various values of initial conditions (see Table 4) inside a vicinity to each critical point. Dynamical variables associated to perturbations are plotted in every panel corresponding to each critical point. Different initial conditions are labeled with colors and each color contains a number of indistinguishable lines for different scales. As lines of the same color split apart of each other we can recognize dependencies on the length-scale.

5.5 Signatures of Modifications of Gravity onto the Solution of the Perturbations

An important feature of $\Lambda$CDM perturbations at the linear regime –and therefore within the PPF context– is that they are coherent. Such a quality is clearly manifested in the dynamical system, as the evolution of any mode is equivalent to any other. However, in typical modified theories of gravity this scale invariance is broken by construction when introducing the characteristic scale $\lambda_1$, which establishes a spatial turnover from which the modifications to the space-time geometry arise and become important. Therefore, linear perturbations in these models turn out to lose their coherence at some point regarding to the sort of phenomena
they are intended to explain and this decoherence is manifested in their evolution by becoming scale dependent.

In this subsection we aim to study, firstly in subsection 5.5.1 the scale-dependence of some modes for our considered parametrized \( f(R) \) and Chameleon models close to the critical points and compare the outcomes from both models. Secondly, in subsection 5.5.2 we quantify the percentage level at which the solutions deviate from the fiducial ΛCDM model close to each critical point.

### 5.5.1 Scale-Dependence of perturbations

As expected, either in parametrized Chameleon or in \( f(R) \) models, the main modification arises in the perturbations of the metric and it is scale-dependent. Naively, one might think that this fact is not surprising since these models are intended to alter only the geometric sector. However, since all perturbations are coupled, indirect decoherence might be inherited by matter perturbations as well. Figs. 10 and 11 illustrate the previous situation, where the main modification happens in the potential \( x_4 \) affecting its dynamics dramatically while a sub-dominant scale-dependence manifests in the velocity perturbation at late times. Apparently to the eye the scale-dependence of \( x_2 \) looks very small, however a comparison between these solutions within MG and GR will reveal that this variation is actually very important as we shall see later.

![Figure 10](image_url)

**Figure 10**: Solutions for different modes of \( x_4 \) in \( f(R) \) (solid) and Chameleon (dashed) models with initial conditions close to \( P_1 \). In contrast to what happens in ΛCDM, in MG models, this potential suffers a transition. The final stationary state reached by \( x_4 \) is totally independent of the scale and initial conditions, however, the scale determines at which redshift the transition happens. Such transition is takes place in a shorter time scale in \( f(R) \) than in Chameleon models.
5.5.2 Quantitative comparison of MG gravity models against the fiducial $\Lambda$CDM

In this section we show the deviations of the dimensionless perturbations $x_1$, $x_2$, $x_3$, and $x_4$ from the $\Lambda$CDM case (when $\mu = \gamma = 1$, i.e. $\lambda_1 = \lambda_2 = 0$) nearby the critical points of the dynamical system. We define the deviation as

$$\Delta_i = \left( \frac{x_i}{x_i^{(0)}} - 1 \right) 100\%, \quad (5.28)$$

where $x_i^{(0)}$ is the dimensionless perturbation of the fiducial $\Lambda$CDM. By the definition $l_1 = k\lambda_1$ and the relation $\lambda_1 = \lambda_2\sqrt{B_1}$ we know that the case $l_1 = 0$ boils down to the $\Lambda$CDM case. Taking into account this, we solve the differential equations of dynamical system numerically and plot the percentage deviations from the fiducial $\Lambda$CDM using initial conditions close to the four critical points previously calculated, for different values of $l_1$. In Table 4 the initial conditions for each critical point are shown.

As we are treating with linear perturbations, the values of $l_1$ must be in this regime. Choosing the models: $f(R)$ II and Chameleon-like II from Table 1, in which $\lambda_2 = 10$ Mpc. From the matter power spectrum we know that the linear regime is for $k = (0, \sim 0.1 \text{ Mpc}^{-1})$, consequently $l_1 = (0, 1)$. We choose $l_1 = \{0, 0.1, 1, 10\}$, which are in the range between large scales ($l_1 \sim 0$) to small scales ($l_1 \sim 1$), the limit case ($l_1 \sim 10$) where the linearity starts to be invalid, and the null case ($l_1 = 0$) that is the fiducial $\Lambda$CDM. For practical purposes we maintain fixed $P_0$ and $\epsilon$ in our analysis.

In Fig. 12 the percentage deviations of the dimensionless perturbations $x_1$, $x_2$, $x_3$, and $x_4$ from the $\Lambda$CDM are plotted.
We notice that the percentage deviations of the dimensionless perturbations are quite similar for the different critical points sharing the main physical features, therefore, instead of describing each one separately we summarize them in general:

- The percentage of deviation of modified models of gravity with respect to the fiducial model as function of the scale-factor for each perturbation is of the same order in all critical points.

- $f(R)$ models have dramatically larger deviations than Chameleons in velocity perturbation modes $x_2$ and the curvature potential $x_4$. The previous is clearly reflected in phase-portraits which clearly shows the change in geometry of these stream field for the different models.
For modes of the density contrast $x_1$ and the gravitational potential $x_3$ important effects of MG arise. The first present a deviation of less than 1.2% and the second, in some cases reach a 100% deviation. Note that the deviations for $x_3$ are the only negative ones. This has interesting consequences at phenomenological level since it suggests that measurements of the density contrast in the Universe are not very useful to constraint significantly these models.

Modifications in Chameleon models occur earlier than for $f(R)$ owing to the shift in values of $s$ between models that consequently shifts the starting stage of the modification.

It is worth noticing the increasingly large sensitivity of the velocity perturbations at late times. This suggests that possible signatures of modified gravity may appear in phenomena regarding to redshift space distortions produced by peculiar velocities.

The deviations in $x_1$ for early times start being negligible, but increase after some time until they stabilize to an almost constant value at late times. For smaller wavelengths the final value of the perturbation mode is larger while modes with large wavelength do not grow significantly. This can also be seen in the phase-portraits corresponding to $x_1$.

Roughly speaking, deviations in $x_2$ have a similar behavior than the ones for $x_1$, they are small at early times and they start to grow until they stabilize to a constant value. However, in the latter case the difference is larger (in some cases it reaches $\sim 40\%$). The growth process for $x_2$ finishes at later times than for $x_1$.

Deviations of $x_3$ are negative$^3$ (and in some cases they can be very small, as for example $P_1 \sim 10^{-5}\%$). This case is particularly interesting because, unlike the others, it starts with no deviations and later decreases, large scales start this process first and after a while they return to zero at late times. This process happens at earlier times for Chameleons than for $f(R)$ models. Notice that in any case all perturbations start approaching back to $\Lambda$CDM (and hence deviations approach to zero) at the same time.

The deviations in $x_4$ have characteristics very similar to those of $x_2$, they start almost null and after a while large-scale modes start growing and then approach to a constant value. This growing process occurs later in $f(R)$ than Chameleons and larger values are reached in the former case. Deviations for both models are very significant ($\sim 20\%$ to $100\%$). The stabilization to a same value independently of the scale shows up the attractor feature of the critical point for this perturbation. Notice that for each model the deviations reach the same final value in the future independently of the scale. However, large-scale modes of perturbations increase first and arrive to the final value before the small-scale ones. The final value in $f(R)$ is around five times larger than the corresponding to Chameleons.

This analysis is mainly qualitative looking for the main physical differences between the models, quantitatively the main purpose is to calculate the orders of magnitude of the deviations. The approach clearly shows that the DS-machinery provides a powerful tool to test the sensitivity of perturbations to variations of a given parametric model. At the same

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$^3$This is a typical behavior of perturbations arising from this sort of parametrizations [see 38].
time it is helpful to determine which kind of phenomena may possibly show up detectable signatures of a given model.

6 Conclusions

In this paper a dynamical analysis of the perturbed linear cosmology within the PPF formalism for a dark fluid composed only of DM and DE has been carried out.

A set of variables that allows us to transform the differential system into a finite-dimensionless system have been introduced. The most well known way to do this is by parameterizing the system through the scale factor or the Hubble parameter. Some particular cases were then studied, finally arriving to the more general models of MG within the PPF scheme.

The machinery set up in this paper is general and easily applicable to other cases. Critical points, their existence, and their corresponding eigenvalues are shown in Tables 2 and 3 for different models which include $f(R)$ and Chameleon models, and the fiducial $\Lambda$CDM with respect to which some comparisons are made. After finding the critical points, an stability analysis was carried out for each one of them and their stability and cosmological characteristics where driven.

In Section 5.5.2, the percentage deviations of the dimensionless perturbations from the $\Lambda$CDM case nearby the critical points of the dynamical system were shown. The differences in perturbation’s evolution between different critical points are subtle, showing that near the critical points are essentially the same. The differences appear when varying the scale $k$ and the model.

The deviations in $x_1$ evolve from zero in the past to a stationary value in the future that depends of the scale and model, large-scale perturbations increase and reach a stationary value, this value being larger than the ones found for small-scale modes. $f(R)$ perturbations start to increase first than those of Chameleons and its final value is around 10 times larger; in all cases the deviations from the fiducial $\Lambda$CDM are small, $\sim 0.001\%$, the extreme case is $\sim 0.1\%$.

On the other hand, the deviations in $x_2$ evolve from zero, then increase with the same rate for different scales, it is not yet clear if it reaches a stationary value in the future as $x_4$ does. Large-scale deviations departure first and reach higher values, Chameleons begin to grow earlier, but $f(R)$ reaches higher values, in both cases the deviations are important (around $\sim 10\%$).

Unlike the other perturbations, the $f(R)$ and Chameleons’ deviations in $x_3$ are negative, evolve from zero, then decrease and finally grow to be almost zero again; the smaller values are reached by large-scale perturbations and also do it earlier than small-scales. Chameleon models evolve in the opposite direction.

Finally, deviations in $x_4$ have very similar characteristics to $x_2$, the main differences are: i) In both models the deviations reach a stationary value for all scales (large scales departures), and ii) The magnitude of the final stage of the deviations are around $\sim 20\%$ for Chameleon models and of 100% for $f(R)$ models.

The parametrizations used here are true from horizon scales down to the scales at which nonlinearities become meaningful. This spans a wide range of observables aimed by the next generation of experiments [32, 33, 35], including weak lensing (of galaxies and CMB), redshift space distortions, peculiar velocities surveys, the ISW effect and associated cross-correlations, and probably galaxy clusters [61]. The $(\beta,s)$ parameters of scale-independent
MG given in Section 2.2, and the parameters \((\lambda_i, \beta_i)\) of the scale-dependent MG models presented in Section 2.3, are convenient for testing observational data. If measurements of galaxy clustering, peculiar velocities, and weak lensing are all consistent with \(\lambda_i = \beta_i = 0\), for example, then MG and exotic DE models can both be ruled out. If measurements need nonzero parameters, however, DE and MG continue to be possible solutions until extra hypothesis are made to differentiate between them, for other constraints see [62].

The presented qualitative analysis of the percentage deviations is a useful tool because it shows the scale and MG model dependence of the scalar perturbations without having to solve in detail the complete set of equations, giving valuable clues for subsequent researchs which pursue some detailed information about scalar perturbations in MG models. The aim is to show a first general approximation in this theme, details such as parameter constraints according to observations is outside of the objectives and scope of this formalism.

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