LINEAR APPROXIMATION OF NONLINEAR SCHRÖDINGER EQUATIONS DRIVEN BY CYLINDRICAL WIENER PROCESSES

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Abstract. In this paper the existence and uniqueness of the solution for a stochastic nonlinear Schrödinger equation, which is perturbed by a cylindrical Wiener process is investigated. The existence of the variational solution and of the generalized weak solution are proved by using sequences of successive approximations, which are the solutions of certain linear problems.

1. Introduction. Linear or nonlinear Schrödinger equations describe how the quantum state of a physical system changes in time. The Schrödinger equation takes several different forms, depending on the physical situation. Using stochastic processes in Schrödinger equations one can model spontaneous emissions or thermic fluctuations or general random disturbances. Many authors investigated stochastic equations of Schrödinger type. Here we investigate a stochastic evolution equation of Schrödinger type over a triplet of complex rigged Hilbert spaces \((V,H,V^*)\):

\[
dX(t) = -iAX(t)dt + f(t,X(t))dt + g(t,X(t))dW(t), \quad X(0) = x_0,
\]

where \(A\) is a certain linear and continuous operator from \(V\) to \(V^*\), the nonlinear terms \(f\) and \(g\) are Lipschitz continuous. Usually, in the applications \(A\) is the Laplacian operator, while typical Lipschitz continuous nonlinearities are the exponential law

\[
f(t,z) = \frac{1}{\lambda}(1 - e^{-\lambda|z|^2})z
\]

and the saturating law

\[
f(t,z) = \frac{\lambda|z|^2}{1 + \lambda|z|^2}z,
\]

where \(\lambda > 0\) and \(z \in \mathbb{C}\).

The saturating law describes the variation of the dielectric constant of gas vapors through which a laser beam propagates [2]. The exponential nonlinearity serves as

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a useful model in homogenous, unmagnetized plasmas and laser-produced plasmas [5].

Stochastic partial differential equations can be interpreted as stochastic evolution equations and the solutions are defined in a generalized sense (see, e.g., [8], [16]). For example, the mild solution of a stochastic Schrödinger type equation is discussed in [3], [4], [7], see also the literature therein. The variational solution and generalized weak solution of (1) are considered in [10], where the solution process is approximated by the Galerkin method, but no error estimates were given.

Often it is possible to approximate the solutions of partial differential equations by linearization, see, e.g., [13]. This idea was also considered in the stochastic case (see, e.g., [12], [15]). Such approximations are useful for further investigations of optimal control problems. The linearization method discussed in detail in this paper enables us to give error estimates, i.e. to derive expressions for the bound of the difference between the solution of the considered equation and its approximations.

In our paper we approximate the variational solution and generalized weak solution of (1) by the sequence of solutions of the linearized problem: for 

\[ dX_{n+1}(t) = -iAX_{n+1}(t)dt + f(t, X_n(t))dt + g(t, X_n(t))dW(t), \quad X_0(t) = x_0. \] 

This paper has the following structure: In Section 2 we introduce a list of assumptions and give the definitions for the variational solution and generalized weak solution and present the method to approximate them. In Section 3 we investigate an auxiliary linear problem, which helps us to construct the sequence of successive approximations (2). This sequence is proved to converge to the variational solution in Section 4 and error estimates are derived. Section 5 is devoted to the approximation of the generalized weak solution. In Section 6 we give concrete examples for Schrödinger type problems for which the solution can be approximated by our method.

2. Assumptions and formulation of the problem. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a filtered complete probability space. Consider \((V, (\cdot, \cdot)_{V})\) and \((H, (\cdot, \cdot))\) be complex Hilbert spaces, such that \((V, H, V^*)\) forms a triplet of rigged Hilbert spaces. Let \(K\) be a separable real Hilbert space and \((W(t))_{t \geq 0}\) to be a \(K\)-valued cylindrical Wiener process adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\).

We indicate a list of assumptions and for each specific solution type and properties of it, we mention which assumptions we use:

- \([\mathbb{I}]\) \(x_0\) is \(\mathcal{F}_0\)-measurable;
- \(x_0 \in L^2(\Omega; V)\);
- \(x_0 \in L^2(\Omega; H)\).

- \([A]\) \(A : V \to V^*\) has the following properties:
  - \(A\) is linear and continuous \(\|Au\|_{V^*} \leq c_A \|u\|_V\);
  - \(\langle A(u), v \rangle = \langle A(v), u \rangle\) for all \(u, v \in V\);
  - there exists constants \(\alpha_1 \in \mathbb{R}\) and \(\alpha_2 > 0\), such that for all \(v \in V\) it holds \(\langle A(v), v \rangle \geq \alpha_1 \|v\|^2_H + \alpha_2 \|v\|^2_V\);

- Let \((h_n)_n \subset H\) be the eigenvectors of the operator \(A\), for which we assume that \(Ah_n \in H\) for all \(n \in \mathbb{N}^*\) and \((h_n)_n\) is a complete orthonormal system in \(H\). We denote by \((\lambda_n)_n\) the sequence of the corresponding eigenvalues of \(A\) and suppose
that $\lambda_n > 0$ for all $n \in \mathbb{N}^*$.

[Definition 2.1.] A process $X \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V)$ is called variational solution of (1), if for all $t \in [0, T], \omega \in \Omega$ it holds
\begin{equation}
(X(t), v) = (x_0, v) - i \int_0^t \langle AX(s), v \rangle ds + \int_0^t (f(s, X(s)), v) ds \text{ (3)}
+ \int_0^t g(s, X(s)) dW(s), v.
\end{equation}
The main result of our paper is that, if \([I-(1)], [A], [f-(1)(2)], [g-(1)(2)]\) are satisfied, then (3) admits a unique variational solution \(X \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)\), which can be approximated by a sequence of successive approximations \((X_n)_n\) in the following sense
\[
X_n \to X \text{ in } L^2(\Omega; C([0,T];H))
\]
and
\[
X_n \to X \text{ in } L^2(\Omega \times [0,T]; V),
\]
where \(X_n \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)\) is the solution of the following linearized problem
\[
(X_{n+1}(t), v) = (x_0, v) - i \int_0^t (AX_{n+1}(s), v)\, ds
+ \int_0^t (f(s, X_n(s)), v)\, ds + (\int_0^t g(s, X_n(s))\, dW(s), v)
\] (4)
for all \(t \in [0,T], v \in V, n \geq 0, \text{ a.e. } \omega \in \Omega, \text{ and } X_0 = x_0.\)

**Definition 2.2.** A process \(X \in L^2(\Omega; C([0,T];H))\) is called generalized weak solution of (1), if for all \(t \in [0,T], v \in D(A) = \{v \in V : Av \in H\}\) and \(\text{a.e. } \omega \in \Omega\) it holds
\[
(X(t), v) = (x_0, v) - i \int_0^t (X(s), Av)\, ds
+ \int_0^t (f(s, X(s)), v)\, ds + (\int_0^t g(s, X(s))\, dW(s), v).
\] (5)

The second result of our paper is that, if \([I-(2)], [A], [f-(1)(3)], [g-(1)(3)]\) are satisfied, then equation (5) admits a unique generalized weak solution \(X \in L^2(\Omega; C([0,T];H))\), which can be approximated by a sequence of successive approximations \((X_n)_n\) in the following sense
\[
X_n \to X \text{ in } L^2(\Omega; C([0,T];H)),
\]
where \(X_n \in L^2(\Omega; C([0,T];H))\) is the solution of the following linearized problem
\[
(X_{n+1}(t), v) = (x_0, v) - i \int_0^t (X_{n+1}(s), Av)\, ds
+ \int_0^t (f(s, X_n(s)), v)\, ds + (\int_0^t g(s, X_n(s))\, dW(s), v)
\] (6)
for all \(t \in [0,T], v \in D(A) = \{v \in V : Av \in H\}, n \geq 0, \text{ a.e. } \omega \in \Omega\) and \(X_0 = x_0.\)

Note, that the methods to prove the existence of the variational solution of (3) and of the generalized weak solution of (5) are different than those given in \([10]\). In our paper we obtain the solution as the limit of a sequence of successive approximations, which enables us to give error estimates, i.e. to derive expressions for the bound of the difference between the solution of the considered equation and its approximations.

**Remark 2.1** The stochastic integral considered in (3) and in (5) is defined as follows: For a \(K\)-valued cylindrical Wiener process \(\left(W(t)\right)_{t \geq 0}\), for \(g : \Omega \times [0,T] \times H \to L_2(K, H)\) satisfying the assumptions \([g-(1)(3)]\) (note, that \([g-(1)(2)]\) implies...
[g-(1)(3)] and for \( v \in L^2(\Omega \times [0, T]; H) \) the stochastic integral \( \int_0^T g(s, v) dW(s) \) is defined as an \( H \)-valued Gaussian random variable with zero mean and given by

\[
\int_0^T g(s, v) dW(s) := \sum_{j=1}^{\infty} \int_0^T g(s, v(s)) e_j dw_j(s),
\]

where the series above converges in \( L^2(\Omega; H) \), \( \left( (w_n(t))_{t \geq 0} \right)_n \) is a sequence of mutually independent real-valued Wiener processes and \( (e_n)_n \) is an orthonormal basis in \( K \). One can prove that

\[
E \left\| \int_0^T g(s, v) dW(s) \right\|^2 = E \int_0^T \| g(s, v(s)) \|^2_{L^2(K, H)} ds.
\]

For more details, see [8].

2) Note, that any variational solution is also a generalized weak solution.

3) The operator \( iA \) with \( D(A) = \{ v \in V : Av \in H \} \) generates a \( C_0 \) group \( (\mathcal{U}_t)_{t \in \mathbb{R}} \) of unitary operators in \( H \). This fact follows from Stone’s Theorem (see Theorem 3.1.4 in [1]). We can introduce the mild solution by

\[
X(t) = \mathcal{U}_t x_0 + i \int_0^t \mathcal{U}_{t-s} f(s, X(s)) ds + i \int_0^t \mathcal{U}_{t-s} g(s, X(s)) dW(s)
\]

for all \( t \in [0, T] \) and for a.e. \( \omega \in \Omega \). The equivalence of generalized weak and mild solutions is well known for the case of \( C_0 \) semigroups, see for example Theorem 1.1 of Chapter 2 in [11]. Consequently, the generalized weak solution definition (5) is equivalent to the mild solution definition (7).

3. An auxiliary linear problem. Consider a stochastic process \( \xi \in L^2(\Omega; C([0, T]; V)) \cap L^2(\Omega \times [0, T]; V) \) and the following stochastic linear equation

\[
(Y^{\xi}(t), v) = (x_0, v) - i \int_0^t \langle AY^{\xi}(s), v \rangle ds + i \int_0^t \langle f(s, \xi(s)), v \rangle ds + i \int_0^t \langle g(s, \xi(s)) dW(s), v \rangle
\]

for all \( t \in [0, T], v \in V \) and a.e. \( \omega \in \Omega \).

Since we will use the Galerkin method to prove the existence of the solution of (8), for each \( n \geq 1 \) we introduce the finite dimensional spaces \( H_n := \text{sp}\{h_1, h_2, \ldots, h_n\} \) (equipped with the norm induced from \( H \) and \( K_n := \text{sp}\{e_1, e_2, \ldots, e_n\} \) (equipped with the norm induced from \( K \)). We define \( \pi_n : H \rightarrow H_n \) the orthogonal projection of \( H \) on \( H_n \) by \( \pi_n h := \sum_{i=1}^{n} \langle h, h_i \rangle h_i \). Let \( A_n : H_n \rightarrow H_n, f_n : \Omega \times [0, T] \rightarrow H_n, g_n : \Omega \times [0, T] \times H_n \rightarrow L(K_n, H_n) \) be defined respectively by

\[
A_n u = \sum_{i=1}^{n} \langle A u, h_i \rangle h_i, \quad f_n(t, u) = \pi_n f(t, u), \quad g_n(t, u) v = \pi_n g(t, u) v \quad \text{for} \quad v \in K_n
\]

and we denote \( x_{0n} = \pi_n x_0 \) and \( W_n(t) = \sum_{j=1}^{n} e_j w_j(t) \in K_n \).
Consider the finite dimensional Galerkin approximations corresponding to (8): for all $t \in [0, T]$, all $j \in \{1, \ldots, n\}$ and a.e. $\omega \in \Omega$
\[
(Y_n^\xi(t), h_j) = (x_{0n}, h_j) - i \int_0^t (AY_n^\xi(s), h_j)ds + \int_0^t (f_n(s, \xi(s)), h_j)ds \tag{9}
+ \left( \int_0^t g_n(s, \xi(s))dW_n(s), h_j \right).
\]

**Theorem 3.1.** Let $\xi \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V)$ and assume that [I-(1)], [A], [f-(1)(2)], [g-(1)(2)] are satisfied. Equation (9) admits a unique variational solution $Y_n^\xi \in L^2(\Omega; C([0, T]; H_n))$. There exists a constant $C_1 > 0$ such that following estimate holds
\[
\sup_{s \in [0, t]} E\|Y_n^\xi(s)\|_V^2 \leq C_1 (1 + E\|x_0\|_V^2 + E \int_0^t \|\xi(s)\|_V^2 ds) \text{ for all } n \in \mathbb{N} \text{ and all } t \in [0, T].
\]

**Proof.** Consider the complex valued processes $(\beta_{j,n}(t))_{t \in [0, T]}$, $j \in \{1, \ldots, n\}$, to be the unique solution of the following finite dimensional system (which is linear in $\beta_{j,n}$)
\[
\beta_{j,n}(t) = (x_{0n}, h_j) - i \lambda_j \int_0^t \beta_{j,n}(s)ds + \int_0^t (f_n(s, \xi(s)), h_j)ds \\
+ \left( \int_0^t g_n(s, \xi(s))dW_n(s), h_j \right)
\]
for all $t \in [0, T]$, all $j \in \{1, \ldots, n\}$ and a.e. $\omega \in \Omega$. Define the $H_n$-valued process
\[
Y_n^\xi(t) = \sum_{j=1}^n \beta_{j,n}(t) h_j \in H_n \text{ for each } t \in [0, T] \tag{10}
\]
which obviously verifies
\[
(AY_n^\xi, h_j) = \lambda_j (Y_n^\xi, h_j) = \lambda_j \beta_{j,n}
\]
and is the solution of (9). In what follows we derive some estimates for $Y_n^\xi$. Using the finite dimensional Itô formula we have
\[
|\langle Y_n^\xi(t), h_j \rangle|^2 = |\langle x_{0n}, h_j \rangle|^2 + 2i \Re \int_0^t \lambda_j |\langle Y_n^\xi(s), h_j \rangle|^2 ds \tag{11}
+ 2 \Re \int_0^t \langle f_n(s, \xi(s)), (Y_n^\xi(s), h_j) \rangle ds
+ 2 \Re \sum_{k=1}^n \int_0^t \langle g_n(s, \xi(s))e_k, (Y_n^\xi(s), h_j) \rangle dw_k(s) + \sum_{k=1}^n \int_0^t |\langle g_n(s, \xi(s))e_k, h_j \rangle|^2 ds.
\]
Summing up for $j = 1$ to $n$
\[
\|Y_n^\xi(t)\|^2 = \|x_{0n}\|^2 + 2 \Re \int_0^t \langle f_n(s, \xi(s)), Y_n^\xi(s) \rangle ds \\
+ 2 \Re \int_0^t \langle g_n(s, \xi(s))dW_n(s), Y_n^\xi(s) \rangle + \sum_{j=1}^n \sum_{k=1}^n \int_0^t |\langle g_n(s, \xi(s))e_k, h_j \rangle|^2 ds
\]
and taking expectation, we get
\[
E \sup_{s \in [0, t]} \|Y_n^\xi(s)\|^2 \leq E\|x_{0n}\|^2 + 2E \int_0^t |\langle f_n(s, \xi(s)), Y_n^\xi(s) \rangle| ds
\]
\[ + 2E \sup_{s \in [0,t]} \left| \int_0^s (g_n(r, \xi(r))dW_n(r), Y_n^\xi(r)) \right| \]
\[ + \sum_{j=1}^n \sum_{k=1}^n E \int_0^t \|(g_n(s, \xi(s))e_k, h_j)^2 ds. \]

By using the assumptions on \( f \) we have
\[ 2ReE \int_0^t (f_n(s, \xi(s)), Y_n^\xi(s))ds \leq 2c_f E \int_0^t \|\xi(s)\|^2 ds + 2c_{HV} k_f + E \int_0^t \|Y_n^\xi(s)\|^2 ds, \]
and by the assumptions on \( g \) it holds
\[ \sum_{j=1}^n \sum_{k=1}^n E \int_0^t \|(g_n(s, \xi(s))e_k, h_j)^2 ds \leq 2c_g E \int_0^t \|\xi(s)\|^2 ds + 2c_{HV} k_g, \]
as well as by the Burkholder-Davis-Gundy inequality (see [19, p. 44, Theorem 7])
\[ 2E \sup_{s \in [0,t]} \left| \int_0^s (g_n(r, \xi(r))dW_n(r), Y_n^\xi(r)) \right| \leq 6E \left( \sup_{s \in [0,t]} \|Y_n^\xi(s)\|^2 \right)^{\frac{1}{2}} \left( \int_0^t \|g(s, \xi(s))\|^2_{L^2(K,H)} ds \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{2} E \sup_{s \in [0,t]} \|Y_n^\xi(s)\|^2 + 18E \int_0^t \|g(s, \xi(s))\|^2_{L^2(K,H)} ds \]
\[ \leq \frac{1}{2} E \sup_{s \in [0,t]} \|Y_n^\xi(s)\|^2 + 36c_g E \int_0^t \|\xi(s)\|^2 ds + 36c_{HV} k_g. \]

Then,
\[ \frac{1}{2} E \sup_{s \in [0,t]} \|Y_n^\xi(s)\|^2 \leq E\|x_{0,n}\|^2 + 2(c_f + 19c_g) E \int_0^t \|\xi(s)\|^2 ds \]
\[ + 2c_{HV} (k_f + 19k_g) + E \int_0^t \|Y_n^\xi(s)\|^2 ds. \]

Gronwall’s lemma yields
\[ E \sup_{t \in [0,T]} \|Y_n^\xi(t)\|^2 \leq C_0 (1 + E\|x_0\|^2 + E \int_0^T \|\xi(s)\|^2 ds), \quad (12) \]
where \( C_0 > 0 \) is a constant independent of \( n \), but depending on \( c_f, c_g, c_{HV}, k_f, k_g, T \).
This implies that \( Y_n^\xi \in L^2(\Omega; C([0,T]; H_n)). \)

Now we derive estimates in the \( V \) norm. Multiplying both sides of the equality (11) with \( \lambda_j \) and then summing up for \( j = 1 \) to \( n \), using \([A], [\xi-(2)], [g-(2)]\) and taking expectation we get
\[ E\langle Y_n^\xi(t), Y_n^\xi(t) \rangle = E\langle Ax_{0,n}, x_{0,n} \rangle + 2Re \sum_{j=1}^n \lambda_j E \int_0^t (f_n(s, \xi(s)), (Y_n^\xi(s), h_j)h_j)ds \]
\[ + \sum_{j=1}^n \sum_{k=1}^n \lambda_j E \int_0^t \|(g_n(s, \xi(s))e_k, h_j)^2 ds \]
\[ \leq E\langle Ax_0, x_0 \rangle + Tc_A(k_f + k_g) + c_A(k_f + k_g) E \int_0^t \|\xi(s)\|_V^2 ds + c_A E \int_0^t \|Y_n^\xi(s)\|_V^2 ds. \]
Moreover, by \([A]\) and \([I-(1)]\) it follows
\[
\alpha_2 E\|Y_n^\xi(t)\|^2 + \alpha_1 E\|Y_n^\xi(t)\|^2 \leq E(Ax_0, x_0) + Tc_A(k_f + k_g) + c_A(k_f + k_g) \int_0^t \|\xi(t)\|^2 dt + c_A E \int_0^t \|Y_n^\xi(t)\|^2 dt.
\]
By (12) and by Gronwall’s lemma it follows that there exists a constant \(C_1 > 0\) (independent of \(n\) but depending on \(T, c_A, c_f, k_f, k_g, C_1, C_H, \alpha_1, \alpha_2\)) such that
\[
\sup_{s \in [0,t]} E\|Y_n^\xi(s)\|^2 \leq C_1 (1 + E\|x_0\|^2 + E \int_0^t \|\xi(s)\|^2 ds) \text{ for all } n \in \mathbb{N} \text{ and all } t \in [0,T].
\]
\[\square\]

**Theorem 3.2.** Let \(\xi \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)\) and assume that \([I-(1)], [A], [I-(1)(2)], [g-(1)(2)]\) are satisfied. Equation (8) admits a unique variational solution
\[
Y^\xi \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V).
\]
The following estimate holds
\[
E\int_0^t \|Y^\xi(s)\|^2 ds \leq C_1 \int_0^t \left(1 + E\|x_0\|^2 + E \int_0^s \|\xi(r)\|^2 dr\right) ds \text{ for each } t \in [0,T].
\]
(13)

**Proof.** The almost sure uniqueness of the solution of (8), follows by assuming that there exist two variational solutions \(Y\) and \(\hat{Y}\) of (8) and computing
\[
\|Y(t) - \hat{Y}(t)\|^2 = 2Im \int_0^t (Y(s) - \hat{Y}(s), Y(s) - \hat{Y}(s)) ds
\]
for all \(t \in [0,T]\) and a.e. \(\omega \in \Omega\). By the properties of \(A\) we obtain \(Y(t) = \hat{Y}(t)\) for all \(t \in [0,T]\) and a.e. \(\omega \in \Omega\).

By Theorem 3.1 it follows that \((Y_n^\xi)_{n \geq 1}\) is bounded in the space \(L^2(\Omega \times [0,T]; V)\). Then by \([21, p. 258, Proposition 21.23 (i)]\) there exist a subsequence of \((Y_n^\xi)_{n \geq 1}\), for which we use the same notation, and \(Y \in L^2(\Omega \times [0,T]; V)\) such that
\[
Y_n^\xi \rightharpoonup Y \text{ in } L^2(\Omega \times [0,T]; H) \text{ and in } L^2(\Omega \times [0,T]; V).
\]

Taking limit in (9) and using that
\[
\int_0^t f_n(s, \xi(s)) ds \rightarrow \int_0^t f(s, \xi(s)) ds \text{ in } L^2(\Omega \times [0,T]; H),
\]
\[
\int_0^t g_n(s, \xi(s)) dW_n(s) \rightarrow \int_0^t g(s, \xi(s)) dW(s) \text{ in } L^2(\Omega \times [0,T]; H),
\]
we get for each \(j \geq 1\) and a.e. \((\omega, t) \in \Omega \times [0,T]\)
\[
(Y(t), h_j) = (x_0, h_j) - i \int_0^t \langle AY(s), h_j \rangle ds + \int_0^t (f(s, \xi(s)), h_j) ds + (\int_0^t g(s, \xi(s)) dW(s), h_j).
\]
There exists an \(\mathcal{F}_t\)-adapted \(H\)-valued process which is equal to \(Y(t)\) for a.e. \((\omega, t) \in \Omega \times [0,T]\) and equal to the right-hand side of the above equality for all \(t \in [0,T]\) and
a.e. \( \omega \in \Omega \). We also denote this process by \((Y(t))_{t \in [0,T]}\). Hence, for each \( t \in [0,T] \), each \( j \geq 1 \) and a.e. \( \omega \in \Omega \) it holds

\[
(Y(t), h_j) = (x_0, h_j) - i \int_0^t \langle AY(s), h_j \rangle ds + \int_0^t (f(s, \xi(s)), h_j) ds
+ (\int_0^t g(s, \xi(s))dW(s), h_j).
\]

The process \((Y(t))_{t \in [0,T]}\) has in \( H \) almost surely continuous trajectories (see [19, p. 73, Theorem 2]). In fact, \( Y^\xi := Y \) is the unique solution of (8). We know that a subsequence of \((Y^\xi_n)\) converges to \( Y^\xi \) strongly in \( L^2(\Omega \times [0,T]; H) \) and weakly in \( L^2(\Omega \times [0,T]; V) \). In fact, the whole sequence has these properties because of [20, p. 480, Proposition 10.13 (1) and (2)] and since (8) possesses a unique solution. By using Theorem 3.1 and the above mentioned weak convergence, the following estimate holds

\[
E \int_0^t \|Y^\xi(s)\|^2_T ds \leq \liminf_{n \to \infty} E \int_0^t \|Y^\xi_n(s)\|^2_T ds
\leq C_1 \int_0^t \left(1 + E\|x_0\|^2_V + E\int_0^s \|\xi(r)\|_V^2 dr \right) ds
\]

for each \( t \in [0,T] \).

4. Approximation of the variational solution. We consider successively \( \xi := X_n \) and denote \( X_{n+1} := Y^{X_n} \) for each \( n \geq 0 \) with \( X_0 = x_0 \), where \( x_0 \in L^2(\Omega; V) \) is the initial condition. Then by (8) and Theorem 3.2 we have that there exists \( X_n \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V) \) such that (4) holds for each \( n \geq 0 \).

Note, that the usual fixed point argument cannot be applied since \( f(t, \cdot) \) and \( g(t, \cdot) \) are not Lipschitz continuous from \( V \) to \( V \) and therefore the sequence \((X_n)\) of successive approximations does not converge in the space \( L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V) \).

**Theorem 4.1.** If [I-(1)], [A], [F-(1) (2)], [g-(1) (2)] are satisfied, then equation (3) admits a unique solution \( X \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V) \) such that the sequence \((X_n)\) of successive approximations, i.e. the solutions of (4), converges to this solution in the following sense

\[
X_n \to X \text{ in } L^2(\Omega; C([0,T]; H))
\]

and

\[
X_n \to X \text{ in } L^2(\Omega \times [0,T]; V).
\]

**Proof.** Suppose that \( X, \hat{X} \in L^2(\Omega \times [0,T]; V) \cap L^2(\Omega; C([0,T]; H)) \) are two solutions of (3). By the stochastic energy equality we have

\[
\|X(t) - \hat{X}(t)\|^2 = 2i\text{m} \int_0^t \langle A(X(s) - \hat{X}(s)), X(s) - \hat{X}(s) \rangle ds
+ 2Re \int_0^t (f(s, X(s)) - f(s, \hat{X}(s)), X(s) - \hat{X}(s)) ds
+ 2Re \sum_{j=1}^{\infty} \int_0^t (g(s, X(s))e_j - g(s, \hat{X}(s))e_j, X(s) - \hat{X}(s)) dw_j(s)
+ \int_0^t \|g(s, X(s)) - g(s, \hat{X}(s))\|^2_{L^2(K,H)} ds.
\]
By the Burkholder-Davis-Gundy inequality, by the Schwarz inequality and by \([f-(1)]\) and \([g-(1)]\), we have for all \(t \in [0,T]\)

\[
\frac{1}{2} E \sup_{s \in [0,t]} \|X(s) - \hat{X}(s)\|^2 \leq (2\sqrt{c_f} + 19c_g) E \int_0^t \|X(s) - \hat{X}(s)\|^2 ds.
\]

Gronwall’s lemma implies that for all \(t \in [0,T]\) and a.e. \(\omega \in \Omega\) it holds \(X(t) = \hat{X}(t)\).

By using (8) we write

\[
\|Y^\xi(t) - Y^\eta(t)\|^2 = 2Im \int_0^t \langle AY^\xi(s) - AY^\eta(s), Y^\xi(s) - Y^\eta(s) \rangle ds
\]

\[
+ 2Re \int_0^t (f(s, \xi(s)) - f(s, \eta(s)), Y^\xi(s) - Y^\eta(s)) ds
\]

\[
+ 2Re \sum_{k=1}^\infty \int_0^t (g(s, \xi(s))e_k - g(s, \eta(s))e_k, Y^\xi(s) - Y^\eta(s)) dw_k(s)
\]

\[
+ \sum_{k=1}^\infty \int_0^t \|g(s, \xi(s))e_k - g(s, \eta(s))e_k\|^2 ds.
\]

Then

\[
E \sup_{s \in [0,t]} \|Y^\xi(s) - Y^\eta(s)\|^2 \leq 2E \int_0^t \|(f(s, \xi(s)) - f(s, \eta(s)))\| \|Y^\xi(s) - Y^\eta(s)\| ds
\]

\[
+ 2E \sup_{s \in [0,t]} \sum_{k=1}^\infty \int_0^s \|g(r, \xi(r))e_k - g(r, \eta(r))e_k, Y^\xi(r) - Y^\eta(r)\| dw_k(r)
\]

\[
+ E \int_0^t \|g(s, \xi(s)) - g(s, \eta(s))\|^2_{L_2(K,H)} ds.
\]

By the Burkholder-Davis-Gundy inequality and \([g-(1)]\) we get

\[
2E \sup_{s \in [0,t]} \sum_{k=1}^\infty \int_0^s \|g(r, \xi(r))e_k - g(r, \eta(r))e_k, Y^\xi(r) - Y^\eta(r)\| dw_k(r)
\]

\[
\leq 6E \left( \sup_{s \in [0,t]} \|Y^\xi(s) - Y^\eta(s)\|^2 \int_0^t \|g(s, \xi(s)) - g(s, \eta(s))\|^2_{L_2(K,H)} ds \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} E \sup_{s \in [0,t]} \|Y^\xi(s) - Y^\eta(s)\|^2 + 18c_g E \int_0^t \|\xi(s) - \eta(s)\|^2 ds.
\]

By the above estimates we obtain

\[
\frac{1}{2} E \sup_{s \in [0,t]} \|Y^\xi(s) - Y^\eta(s)\|^2 \leq (c_f + 19c_g) E \int_0^t \|\xi(s) - \eta(s)\|^2 ds
\]

\[
+ E \int_0^t \|Y^\xi(s) - Y^\eta(s)\|^2 ds.
\]

Gronwall’s lemma implies

\[
E \sup_{t \in [0,T]} \|Y^\xi(t) - Y^\eta(t)\|^2 \leq C_2 E \int_0^T \|\xi(s) - \eta(s)\|^2 ds \tag{14}
\]
where \( C_2 = 2(c_f + 19c_n)e^{2T} > 0 \). Successively applying inequality (14) we obtain for \( n \geq 0 \)

\[
E \sup_{t \in [0,T]} \|X_{n+1}(t) - X_n(t)\|^2 \leq C_2 E \int_0^T \|X_n(s) - X_{n-1}(s)\|^2 ds \leq \ldots 
\]

\[
\leq \frac{(C_2 T)^n}{n!} E \sup_{t \in [0,T]} \|X_1(t) - X_0(t)\|^2.
\]

For simplicity denote \( \mathcal{H} := L^2(\Omega; C([0,T]; H)) \). By (15) we get

\[
\|X_{n+1} - X_n\|_{\mathcal{H}} \leq \sqrt{\frac{(C_2 T)^n}{n!}} \|X_1 - X_0\|_{\mathcal{H}}.
\]

We denote \( a_n = \sqrt{\frac{(C_2 T)^n}{n!}} \). Then

\[
\|X_{n+k} - X_n\|_{\mathcal{H}} \leq \|X_1 - X_0\|_{\mathcal{H}} \cdot \sum_{j=n}^{n+k-1} a_j.
\]

Moreover,

\[
\frac{a_{n+1}}{a_n} = \sqrt{\frac{C_2 T}{n+1}} \rightarrow 0.
\]

Then by D’Alembert’s criterion for convergent series, it follows that the series \( \sum_{j=1}^{\infty} a_j \)

is convergent, hence

\[
\sum_{j=n}^{n+k-1} a_j \rightarrow 0 \text{ as } n,k \rightarrow \infty,
\]

and \( (X_n)_n \) is a Cauchy sequence in the Banach space \( \mathcal{H} = L^2(\Omega; C([0,T]; H)) \). Therefore, there exists \( \hat{X} \in L^2(\Omega; C([0,T]; H)) \) such that

\[
E \sup_{t \in [0,T]} \|X_n(t) - \hat{X}(t)\|^2 \rightarrow 0.
\]

We successively apply the inequality (13) from Theorem 3.2, where we assume without loss of generality that \( C_1 > 1 \), to get

\[
E \int_0^T \|X_n(s)\|^2_V ds \leq C_1 \int_0^T \left( 1 + E\|x_0\|^2_V + E \int_0^s \|X_{n-1}(r)\|^2_V dr \right) ds \leq \ldots
\]

\[
\leq (1 + E\|x_0\|^2_V) \sum_{k=1}^{n+1} \frac{(C_1 T)^k}{k!} \leq (1 + E\|x_0\|^2_V) e^{C_1 T}.
\]

Hence \( (X_n)_n \) is a bounded sequence in \( L^2(\Omega \times [0,T]; V) \) and there exist a subsequence \( (X_{n_k})_k \) of \( (X_n)_n \) and \( X \in L^2(\Omega \times [0,T]; V) \) such that

\[
X_{n_k} \rightarrow X \text{ in } L^2(\Omega \times [0,T]; H) \text{ and in } L^2(\Omega \times [0,T]; V).
\]

By (17) we must have

\[
E \int_0^T \|X(s) - \hat{X}(s)\|^2 ds = 0.
\]
We pass to the limit in (4) as $k \to \infty$, then it follows that
\[
(X(t), v) = (x_0, v) - i \int_0^t \langle AX(s), v \rangle ds + \int_0^t (f(s, X(s)), v) ds + \int_0^t g(s, X(s)) dW(s, v)
\]
for a.e. $(\omega, t) \in \Omega \times [0, T]$ and all $v \in V$. There exists an $\mathcal{F}_t$-adapted $H$-valued process which is equal to $X(t)$ for a.e. $(\omega, t) \in \Omega \times [0, T]$ and equal to the right-hand side of the above equality for all $t \in [0, T]$ and a.e. $\omega \in \Omega$. We also denote this process by $(X(t))_{t \in [0, T]}$. Then
\[
(X(t), v) = (x_0, v) - i \int_0^t \langle AX(s), v \rangle ds + \int_0^t (f(s, X(s)), v) ds + \int_0^t g(s, X(s)) dW(s, v)
\]
for each $t \in [0, T]$, each $v \in V$ and a.e. $\omega \in \Omega$. The process $(X(t))_{t \in [0, T]}$ has in $H$ almost surely continuous trajectories (see [19, p. 73, Theorem 2]).

We know that a subsequence of $(X_n)_n$ converges to $X$ strongly in $L^2(\Omega \times [0, T]; H)$ and weakly in $L^2(\Omega \times [0, T]; V)$. In fact, the whole sequence has these properties because of [20, p. 480, Proposition 10.13 (1) and (2)] and since (3) possesses a unique solution.

In the following theorem we give some error estimates. We denote by
\[
C := C_2 T = 2(c_f + 19c_g)e^{2T},
\]
where $C_2$ was the constant determined during the proof of Theorem 4.1.

**Theorem 4.2.** If [A-(1)], [A], [F-(1)(2)], [g-(1)(2)] are satisfied and $n \in \mathbb{N}$ such that $n > C - 1$, then the following error estimates hold:

1. $E \sup_{t \in [0, T]} \|X(t) - X_n(t)\|^2 < E \sup_{t \in [0, T]} \|X_1(t) - X_0\|^2 \frac{(n + 1)C^n}{n!(\sqrt{n + 1} - \sqrt{C})^2}$.

2. $E \sup_{t \in [0, T]} \|X(t) - X_{n+1}(t)\|^2 \leq E \sup_{t \in [0, T]} \|X_{n+1}(t) - X_n(t)\|^2 \left( \sum_{j=1}^{\infty} \frac{C_j}{j!} \right)^2$.

3. $E \sup_{t \in [0, T]} \|X(t) - X_{n+1}(t)\|^2 \leq \frac{C}{T} \int_0^T E \sup_{s \in [0, t]} \|X(s) - X_n(s)\|^2 dt$.

**Proof.** 1) By using the notation $a_n = \sqrt{\frac{C}{n^3}}$ from Theorem 4.1 (where $C = C_2 T$) we compute for each $k > 1$
\[
\frac{a_{n+k-1}}{a_n} = \sqrt{\frac{C}{n+k-1}} \cdots \sqrt{\frac{C}{n+1}} \leq \left( \frac{C}{n+1} \right)^{\frac{k-1}{2}}.
\]
For $n$ sufficiently large such that $C < n + 1$ we have
\[
\sum_{j=n}^{n+k-1} a_j \leq a_n \sum_{j=1}^{k} \left( \frac{C}{n+1} \right)^{\frac{j-1}{2}} \leq \frac{a_n}{1 - \frac{C}{n+1}}.
\]
Hence, by taking \( k \to \infty \) in the inequality (16) which was derived in the proof of Theorem 4.1 we obtain
\[
\|X - X_n\|_H \leq \|X_1 - X_0\|_H \sqrt{\frac{C^n}{n!} \cdot \frac{\sqrt{n} + 1}{\sqrt{n} + 1 - \sqrt{C}}}
\]
2) Similar to (15) we derive for \( j > 1 \)
\[
\|X_{n+j} - X_{n+j}||_H \leq a_j \|X_{n+1} - X_n\|_H.
\]
Then,
\[
\|X_{n+k+1} - X_{n+1}\|_H \leq \|X_{n+1} - X_n\|_H \cdot \sum_{j=1}^{k} a_j
\]
and by \( k \to \infty \) we obtain
\[
\|X - X_{n+1}\|_H \leq \|X_{n+1} - X_n\|_H \cdot \sum_{j=1}^{\infty} a_j.
\]
3) As in (15) we have
\[
E \sup_{t \in [0, T]} \|X_{n+k+1}(t) - X_{n+1}(t)\|^2 \leq \frac{C}{T} \int_0^T \sup_{s \in [0, t]} \|X_{n+k}(s) - X_n(s)\|^2 dt.
\]
Taking \( k \to \infty \) and using the strong convergence result from Theorem 4.1 we get
\[
E \sup_{t \in [0, T]} \|X(t) - X_{n+1}(t)\|^2 \leq \frac{C}{T} \int_0^T \sup_{s \in [0, t]} \|X(s) - X_n(s)\|^2 dt.
\]

**Remark 3.** Observe that the smallest \( n \in \mathbb{N} \) such that \( n > C - 1 \) is \( n = [C] \). Then a possible bound for the convergent series mentioned in the above theorem is
\[
\sum_{j=1}^{[C]-1} a_j + \sum_{j=[C]}^{\infty} a_j < \sum_{j=1}^{[C]-1} \sqrt{\frac{C}{j!}} + \sqrt{\frac{C^{[C]}}{[C]!}} \cdot \frac{\sqrt{[C] + 1}}{\sqrt{[C] + 1 - \sqrt{C}}} \quad \text{if} \ [C] \geq 2,
\]
and
\[
\sum_{j=1}^{\infty} a_j < \frac{\sqrt{([C] + 1)C}}{\sqrt{[C] + 1 - \sqrt{C}}} \quad \text{if} \ [C] \in \{0, 1\},
\]
where \([C]\) is the integer part of the positive constant \( C \).

5. Approximation of the generalized weak solution.

**Theorem 5.1.** If [I-(2)], [A], [I-(1)(3)], [g-(1)(3)] are satisfied, then equation (5) admits a unique generalized weak solution \( X \in L^2(\Omega; C([0, T]; H)) \), which can be approximated by a sequence of successive approximations \((X_n)\) in the following sense
\[
X_n \to X \text{ in } L^2(\Omega; C([0, T]; H)),
\]
where \( X_n \) is the solution of (6).
Moreover, for \( n \in \mathbb{N} \) such that \( n > C - 1 \) (with \( C \) given in (20)), the following error estimates hold:

1) \( E \sup_{t \in [0, T]} \|X(t) - X_n(t)\|^2 < E \sup_{t \in [0, T]} \|X_1(t) - X_0\|^2 \frac{(n + 1)C^n}{n!(\sqrt{n} + 1 - \sqrt{C})^2}. \)
2) \[ E \sup_{t \in [0, T]} \| X(t) - X_{n+1}(t) \|^2 \leq E \sup_{t \in [0, T]} \| X_{n+1}(t) - X_n(t) \|^2 \left( \sum_{j=1}^{\infty} \sqrt{\frac{C_j}{j!}} \right)^2; \]

3) \[ E \sup_{t \in [0, T]} \| X(t) - X_{n+1}(t) \|^2 \leq \frac{C}{T} \int_0^T E \sup_{s \in [0, t]} \| X(s) - X_n(s) \|^2 dt. \]

Proof. The uniqueness follows with an indirect proof: assume that

\[ U, V \in L^2(\Omega; C([0, T]; H)) \]

are two solutions of (5). We have no Itô formula for computing \( \| U(t) - V(t) \|^2 \), so we will take into consideration the finite dimensional approximations, i.e. we denote \( U_n = \pi_n U \) and \( V_n = \pi_n V \) for each \( n \geq 1 \) and \( t \in [0, T] \) and use (5) to write

\[
\| U_n(t) - V_n(t) \|^2 = 2Re \int_0^t (f(s, U(s)) - f(s, V(s)), U_n(s) - V_n(s)) ds \\
+ 2Re \sum_{k=1}^n \int_0^t (g(s, U(s)) e_k - g(s, V(s)) e_k, U_n(s) - V_n(s)) dw_k(s) \\
+ \sum_{k=1}^n \int_0^t \| g(s, U(s)) e_k - g(s, V(s)) e_k \|^2 ds.
\]

By the Burkholder-Davis-Gundy inequality and some estimates similar to those given in the proof of Theorem 4.1 we get

\[
E \sup_{t \in [0, T]} \| U_n(t) - V_n(t) \|^2 \leq C_2 E \int_0^T \| U(s) - V(s) \|^2 ds,
\tag{21}
\]

where \( C_2 \) is the same constant as in (14). But \( U_n = \pi_n U \) and \( V_n = \pi_n V \) and then

\[
E \| U_n(t) - V_n(t) \|^2 \to E \| U(t) - V(t) \|^2 \text{ for each } t \in [0, T].
\]

By (21) we obtain

\[
E \| U(t) - V(t) \|^2 \leq C_2 E \int_0^T \| U(s) - V(s) \|^2 ds \text{ for each } t \in [0, T].
\]

Gronwall’s lemma implies \( E \| U(t) - V(t) \|^2 = 0 \) for each \( t \in [0, T] \). But \( U, V \in L^2(\Omega; C([0, T]; H)) \), hence it holds \( \| U(t) - V(t) \|^2 = 0 \) for each \( t \in [0, T] \) and for a.e. \( \omega \in \Omega \).

For \( \xi \in L^2(\Omega; C([0, T]; H)) \) we consider the finite dimensional equations (9) which can be rewritten as

\[
(Y_n^{\xi}(t), h_j) = (x_{0n}, h_j) - i \int_0^t (Y_n^{\xi}(s), Ah_j) ds + \int_0^t (f_n(s, \xi(s)), h_j) ds \\
+ (\int_0^t g_n(s, \xi(s)) dW_n(s), h_j).
\]

for all \( t \in [0, T] \), all \( j \in \{1, \ldots, n\} \) and a.e. \( \omega \in \Omega \). By Theorem 3.1 it follows that \( Y_n^{\xi} \in L^2(\Omega; C([0, T]; H_n)). \)

We prove that \( (Y_n^{\xi})_n \) is a Cauchy sequence in \( L^2(\Omega; C([0, T]; H)) \): By using (22) we write for \( n > m \)

\[
\| Y_n^{\xi}(t) - Y_m^{\xi}(t) \|^2 = E \| x_{0n} - x_{0m} \|^2 \\
+ 2Re \int_0^t (f_n(s, \xi(s)) - f_m(s, \xi(s)), Y_n^{\xi}(s) - Y_m^{\xi}(s)) ds.
\]
+ 2Re \sum_{k=1}^{n} \int_{0}^{t} (g_n(s, \xi(s))e_k - g_m(s, \xi(s))e_k, Y^\xi_m(s) - Y^\xi_m(s))dw_k(s)

+ \sum_{k=m+1}^{n} \int_{0}^{t} \|g_n(s, \xi(s))e_k\|^2 ds.

Then

\[ E \sup_{s \in [0,t]} \|Y^\xi_n(s) - Y^\xi_m(s)\|^2 \leq E\|x_{0n} - x_{0m}\|^2 \]

\[ + 2E \int_{0}^{t} \|f_n(s, \xi(s)) - f_m(s, \xi(s))\|\|Y^\xi_n(s) - Y^\xi_m(s)\| ds \]

\[ + 2E \sup_{s \in [0,t]} \sum_{k=1}^{n} \int_{0}^{s} (g_n(r, \xi(r))e_k - g_m(r, \xi(r))e_k, Y^\xi_n(r) - Y^\xi_m(r))dw_k(r)\]  

\[ + E \int_{0}^{t} \|g_n(s, \xi(s)) - g_m(s, \xi(s))\|_{L^2(K,H)}^2 ds. \]

By the Burkholder-Davis-Gundy inequality we have

\[ 2E \sup_{s \in [0,t]} \sum_{k=1}^{n} \int_{0}^{s} (g_n(r, \xi(r))e_k - g_m(r, \xi(r))e_k, Y^\xi_n(r) - Y^\xi_m(r))dw_k(r)\]  

\[ \leq \frac{1}{2} E \sup_{s \in [0,t]} \|Y^\xi_n(s) - Y^\xi_m(s)\|^2 + 18E \int_{0}^{t} \|g_n(s, \xi(s)) - g_m(s, \xi(s))\|_{L^2(K,H)}^2 ds. \]

Then by Gronwall’s lemma we obtain

\[ E \sup_{t \in [0,T]} \|Y^\xi_n(t) - Y^\xi_m(t)\|^2 \leq 2e^{2T} \left( E\|x_{0n} - x_{0m}\|^2 \right. \]

\[ + \left. E \int_{0}^{T} \|f_n(s, \xi(s)) - f_m(s, \xi(s))\|_{L^2(K,H)}^2 + 19\|g_n(s, \xi(s)) - g_m(s, \xi(s))\|_{L^2(K,H)}^2 ds \right). \]

By the properties of \( x_0 \), \( f \) and \( g \), it follows that \( (Y^\xi_n)_n \) is a Cauchy sequence in \( L^2(\Omega; C([0,T]; H)) \) and there exists \( Y^\xi \in L^2(\Omega; C([0,T]; H)) \) such that

\[ E \sup_{t \in [0,T]} \|Y^\xi(t) - Y^\xi_n(t)\|^2 \to 0 \quad (23) \]

and for each \( t \in [0,T] \), each \( j \geq 1 \) and a.e. \( \omega \in \Omega \) it holds

\[ (Y^\xi(t), h_j) = (x_0, h_j) - i \int_{0}^{t} (\overline{Y^\xi(s), Ah_j}) ds + \int_{0}^{t} (f(s, \xi(s)), h_j) ds \]

\[ + (\int_{0}^{t} g(s, \xi(s)) dW(s), h_j). \]

We consider successively \( \xi := X_n \) and denote \( X_{n+1} := Y^X_n \) for each \( n \geq 0 \) with \( X_0 = x_0 \), where \( x_0 \in L^2(\Omega; H) \) is the initial condition. By the above reasoning we have that there exists \( X_n \in L^2(\Omega; C([0,T]; H)) \) such that equation (6) holds for each \( n \geq 0 \).

Consider \( \eta \in L^2(\Omega; C([0,T]; H)) \). Since we have no Itô formula for computing \( \|Y^\xi(t) - Y^\eta(t)\|^2 \), we will take into consideration the finite dimensional approximations, i.e. we use (22) to get

\[ \|Y^\xi_n(t) - Y^\eta_n(t)\|^2 = 2Re \int_{0}^{t} (f_n(s, \xi(s)) - f_n(s, \eta(s)), Y^\xi_n(s) - Y^\eta_n(s)) ds \]
\[ + 2Re \sum_{k=1}^{n} \int_{0}^{t} (g_n(s, \xi(s))e_k - g_n(s, \eta(s))e_k, Y^\xi_n(s) - Y^n_\eta(s))dw_k(s) \]
\[ + \sum_{k=1}^{n} \int_{0}^{t} \|g_n(s, \xi(s))e_k - g_n(s, \eta(s))e_k\|^2 ds. \]

By the Burkholder-Davis-Gundy inequality and estimates similar to those computed in the proof of Theorem 4.1 we get
\[
E \sup_{t \in [0,T]} \|Y^\xi_n(t) - Y^n_\eta(t)\|^2 \leq C_2 E \int_{0}^{T} \|\xi(s) - \eta(s)\|^2 ds,
\]
where \(C_2\) is the same constant as in (14). But the convergence result from (23) implies
\[
E \sup_{t \in [0,T]} \|Y^\xi_n(t) - Y^n_\eta(t)\|^2 \leq C_2 E \int_{0}^{T} \|\xi(s) - \eta(s)\|^2 ds. \tag{24}
\]
Successively applying for \(\xi = X_n\) and \(\eta = X_{n-1}\) inequality (24) we obtain
\[
E \sup_{t \in [0,T]} \|X_{n+1}(t) - X_n(t)\|^2 \leq C_2 E \int_{0}^{T} \|X_n(s) - X_{n-1}(s)\|^2 ds \leq \ldots \tag{25}
\]
\[
\leq \frac{(C_2 T)^n}{n!} E \sup_{t \in [0,T]} \|X_1(t) - X_0(t)\|^2.
\]

Exactly as in the proof of Theorem 4.1 we get that \((X_n)\) is a Cauchy sequence in the Banach space \(L^2(\Omega; C([0,T]; H))\). Therefore, there exists \(X \in L^2(\Omega; C([0,T]; H))\) such that
\[
E \sup_{t \in [0,T]} \|X_n(t) - X(t)\|^2 \to 0.
\]

We pass to the limit in (6) as \(n \to \infty\), then it follows that
\[
(X(t), v) = (x_0, v) - i \int_{0}^{t} (X(s), Av) ds
\]
\[
+ \int_{0}^{t} (f(s, X(s)), v) ds + (\int_{0}^{t} g(s, X(s))dW(s), v)
\]
for each \(t \in [0,T]\), \(v \in D(A)\) and a.e. \(\omega \in \Omega\).

The error estimates are derived similarly to those given in Theorem 4.2. \(\square\)

6. Examples.

6.1. Saturable nonlinearities. We consider the following saturable nonlinearities (see [5, Subsection 2.1.1 and Chapter 7], [18], [6])
\[
f_1(z) = (1 - e^{-|z|^2})z, \quad z \in \mathbb{C}
\]
and
\[
f_2(z) = \frac{|z|^2}{1 + |z|^2} z, \quad z \in \mathbb{C}.
\]
The two mappings
\[
(x, y) \in \mathbb{R}^2 \to \left((1 - e^{-x^2-y^2})x, (1 - e^{-x^2-y^2})y\right) \in \mathbb{R}^2
\]

and

\[(x, y) \in \mathbb{R}^2 ightarrow \left(\frac{x^2 + y^2}{1 + x^2 + y^2}, \frac{x^2 + y^2}{1 + x^2 + y^2}\right) \in \mathbb{R}^2\]

have all partial derivatives of first order bounded, therefore each of the above functions is Lipschitz continuous from \(\mathbb{C}\) to \(\mathbb{C}\) and hence \([f-(1)]\) is verified. By computation, one can prove that also \([f-(2)]\) holds.

6.2. Stochastic equations of Schrödinger type. In all examples \((K, (\cdot, \cdot)_K)\) is a separable real Hilbert space and \((\epsilon_n)_n\) is an orthonormal basis in \(K\), and \(\left( (w_n(t))_{t \geq 0} \right)_n\) is a sequence of mutually independent real-valued Wiener processes. Let \((\nu_j)_j\) be a sequence of positive real numbers such that one of the following conditions is satisfied

\[
\sum_{j=1}^{\infty} (\nu_j)^2 < \infty; \quad (26)
\]

\[
\sum_{j=1}^{\infty} \nu_j^2 < \infty. \quad (27)
\]

**Example 1.** We consider a linear Schrödinger equation on a bounded domain with an additive infinite dimensional random noise: Let \(V = H^1_0([0,1]; \mathbb{C})\) and \(H = L^2([0,1]; \mathbb{C})\) and \(h_j(x) = \sin(2j\pi x) + i \sin(2j\pi x), \lambda_j = (2j\pi)^2, j \in \mathbb{N}^*\). For \(j \in \mathbb{N}^*\) we consider the measurable function \(\mu_j : \Omega \times [0,T] \rightarrow \mathbb{R}\), which is \(\mathcal{F}_t\)-adapted for each \(t \in [0,T]\) such that \(|\mu_j(t)| \leq \nu_j\) for all \(t \in [0,T]\) and \((\nu_j)_j\) satisfies (26). Let \(g : [0,T] \rightarrow L_2(K, V)\) be defined by

\[
g(t)y = i \sum_{j=1}^{\infty} \mu_j(t)(y, \epsilon_j)_K h_j \quad (28)
\]

for \(t \in [0,T]\) and \(y \in K\). Obviously the conditions \([g-(1)(2)]\) are fulfilled. Then we obtain an additive \(V\)-valued noise defined by

\[
\int_0^t g(s)dW(s) = i \sum_{j=1}^{\infty} \int_0^t \mu_j(s)dw_j(s)h_j. \quad (29)
\]

Let \(\psi : [0,1] \rightarrow \mathbb{R}\) be a \(C^1\) function. Then \(f : H \rightarrow H\) defined by \(f(v)(x) = i\psi(x)v(x) x \in [0,1]\) verifies \([f-(1)(2)]\). For \(x_0 \in V\) we introduce

\[
\int_0^1 X(t,x)\pi(x)dx = \int_0^1 x_0(x)\pi(x)dx - i \int_0^t \int_0^1 \frac{\partial}{\partial x}X(s,x)\frac{\partial}{\partial x}\pi(x)dxds \quad (30)
\]

\[
+ i \int_0^t \int_0^1 \psi(x)X(s,x)\pi(x)dxds + i \sum_{j=1}^{\infty} \int_0^t \mu_j(s)\int_0^1 h_j(x)\pi(x)dx dw_j(s)
\]

for all \(t \in [0,T]\), \(v \in V\) and a.e. \(\omega \in \Omega\). Problem (30) admits a unique variational solution.

If we assume that (27) holds, then \(g : [0,T] \rightarrow L_2(K, H)\) and it follows that there exists a generalized weak solution for the following problem with initial condition
For all $t \in [0, T]$, $v \in V \cap H^2([0,1]; \mathbb{C})$ and a.e. $\omega \in \Omega$. Each of the problems (30) and (31) describes a particle in $[0,1]$ driven by a potential $\psi$ and an additive infinite dimensional noise. If $\psi(x) \equiv 0$, then we get the Schrödinger equation for a randomly disturbed free particle.

Example 2. We consider a linear Schrödinger equation on $\mathbb{R}^d$ with an additive infinite dimensional random noise: Let $\psi \in L^\infty_t(L^2(\mathbb{R}^d; \mathbb{R}^+))$ be such that $\lim_{\|x\|_d \to \infty} \psi(x) = +\infty$. Then $A = -\Delta + \psi$ defines a self-adjoint operator on $\{u \in L^2(\mathbb{R}^d; \mathbb{R}^d) : Au \in L^2(\mathbb{R}^d; \mathbb{R}^d)\}$ and admits a sequence of eigenfunctions $(v_j)_j$ in $H^2(\mathbb{R}^d; \mathbb{R}^d)$ and the functions $h_j = v_j + iv_j$, $j \in \mathbb{N}^*$, form a complete orthonormal system in $H := L^2(\mathbb{R}^d; \mathbb{C})$ (see [17], Theorem XIII. 67, p. 249). Assume that the functions $\mu_j : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $j \geq 1$, are measurable and $F_t$-adapted for each $t \in [0, T]$ with

$$\sum_{j=1}^\infty \int_0^T \left( \int_{\mathbb{R}^d} |\mu_j(t, x)|^2 |h_j(x)|^2 dx \right) dt < \infty.$$

For $x_0 \in H$ the following problem has generalized weak solution

$$\int_{\mathbb{R}^d} X(t, x)\psi(x)dx = \int_{\mathbb{R}^d} x_0(x)\psi(x)dx + i \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} X(s, x) \frac{\partial^2}{\partial x_k^2} v(x)dxds$$

$$+ i \int_0^t \int_{\mathbb{R}^d} \psi(x)X(s, x)\psi(x)dxds + i \sum_{j=1}^\infty \int_0^t \left( \int_{\mathbb{R}^d} \mu_j(s, x)h_j(x)\psi(x)dx \right) dw_j(s)$$

for all $t \in [0, T], v \in H^2(\mathbb{R}^d; \mathbb{C})$ and a.e. $\omega \in \Omega$. It is well known that in the deterministic case for $\psi(x) = \frac{1}{2}|x|^2$ problem (32) defines the linear harmonic oscillator (see [9]). In our case we have additionally the influence of random noise.

Example 3. We now introduce a nonlinear problem with a multiplicative noise. Let $\psi$ and $(h_j)_j$ $(j \in \mathbb{N}^*)$ be the functions introduced in Example 2 for $d = 1$ and $H = L^2(\mathbb{R}; \mathbb{C})$. We consider functions $\rho_j : \mathbb{C} \to \mathbb{C}$ so that we have for all $z_1, z_2 \in \mathbb{C}$

$$|\rho_j(z_1) - \rho_j(z_2)| \leq \nu_j |z_1 - z_2| , \rho_j(0) = 0,$$

where $|\rho_j(z)| < \nu_j$ for each $z \in \mathbb{C}$ and $(\nu_j)_j$ satisfies (26). For

$$x_0 \in V := \left\{ v : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} v(x) \right|^2 + \psi(x)|v(x)|^2dx < \infty \right\}$$
we consider
\[
\int_{\mathbb{R}} X(t, x) \psi(x) dx = \int_{\mathbb{R}} x_0(x) \psi(x) dx - i \int_{\mathbb{R}} \int_0^t \frac{\partial}{\partial x} X(s, x) \frac{\partial}{\partial x} \psi(x) dx ds
\]
\[
+ i \int_0^t \int_{\mathbb{R}} \psi(x) X(s, x) \psi(x) dx ds + i \int_0^t \int_{\mathbb{R}} (1 - \exp \{- |X(t, x)|^2 \}) X(t, x) \psi(x) dx ds
\]
\[
+ \int_{\mathbb{R}} \left( \sum_{j=1}^{\infty} \int_0^t \rho_j(X(s, x)) dw_j(s) \right) h_j(x) \psi(x) dx
\]
for all \( t \in [0, T], v \in V \) and a.e. \( \omega \in \Omega \). Problem (33) admits a unique variational solution, which can be approximated by a sequence of successive approximations \( (X_n)_n \), where \( X_n \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V) \) is the solution of the following linearized problem
\[
\int_{\mathbb{R}} X_{n+1}(t, x) \psi(x) dx = \int_{\mathbb{R}} x_0(x) \psi(x) dx - i \int_{\mathbb{R}} \int_0^t \frac{\partial}{\partial x} X_{n+1}(s, x) \frac{\partial}{\partial x} \psi(x) dx ds
\]
\[
+ i \int_0^t \int_{\mathbb{R}} \psi(x) X_{n+1}(s, x) \psi(x) dx ds
\]
\[
+ i \int_0^t \int_{\mathbb{R}} (1 - \exp \{- |X_n(t, x)|^2 \}) X_n(t, x) \psi(x) dx ds
\]
\[
+ \int_{\mathbb{R}} \left( \sum_{j=1}^{\infty} \int_0^t \rho_j(X_n(s, x)) dw_j(s) \right) h_j(x) \psi(x) dx
\]
for all \( t \in [0, T], v \in V, n \geq 0 \), a.e. \( \omega \in \Omega \) and \( X_0 = x_0 \).

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