COMPONENTS AND SINGULARITIES OF QUOT SCHEMES AND VARIETIES OF COMMUTING MATRICES

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Abstract. We investigate the variety of commuting matrices. We classify its components for any number of matrices of size at most 7. We prove that starting from quadruples of of $8 \times 8$ matrices, this scheme has generically nonreduced components, while up to degree 7 it is generically reduced. Our approach is to recast the problem as deformations of modules and generalize an array of methods: apolarity, duality and Białynicki-Birula decompositions to this setup. We include a thorough review of our methods to make the paper self-contained and accessible to both algebraic and linear-algebraic communities. Our results give the corresponding statements for the Quot schemes of points, in particular we classify the components of $\text{Quot}_{d}(\mathcal{O}_{\mathbb{A}^r}^{\oplus n})$ for $d \leq 7$ and all $r, n$.

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Appendix A. Functorial approach to comparison between $C_n(M_d)$ and Quot$^d$

1. INTRODUCTION

The variety $C_n(M_d)$ of $n$-tuples of commuting $d \times d$ matrices is an object of great interest for linear algebraists, in representation theory [CBS02] and in deformation theory. It has applications in complexity theory, see below. However, its geometry is complicated and surprisingly almost nothing about its components is present in the literature, especially when $n \geq 4$. The aim of the current work is to exhibit new components and clarify the general picture by building a robust toolbox.

The variety $C_n(M_d)$ is a cone over the zero tuple, so it is connected. It has a distinguished principal component that is the closure of the locus of tuples of diagonalizable matrices. Asking whether $C_n(M_d)$ is irreducible can be rephrased as asking whether every tuple is a limit of diagonalizable ones. The variety $C_2(M_d)$ is irreducible for each $d$ [MT55]. In contrast, for $n \geq 4$ the variety $C_n(M_d)$ is irreducible if and only if $d \leq 3$ [Ger61, Gur92], and the question of irreducibility of $C_3(M_d)$ is not solved completely yet: it is reducible for $d \geq 29$, see [HO01], [NŠ14, p. 238], and irreducible for $d \leq 10$, see [Šiv12]. In fact, it is known to be irreducible also for $d = 11$, although this result has not been published yet, see [HO15, p. 271]. These irreducibility results hold in characteristic zero.

To our knowledge literally no components of $C_n(M_d)$ other than the principal one are described in the literature. In this article we describe all components of $C_n(M_d)$ for arbitrary $n$ and for small $d$.

**Theorem A.** Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. The number of irreducible components of $C_n(M_d)$ for $d \leq 7$ is as shown in Table 1. All components have smooth points.

| $d \leq 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ | $d \gg 0$ |
|-----------|---------|---------|---------|---------|---------|----------|
| $n \leq 2$ | 1       | 1       | 1       | 1       | 1       | 1        |
| $n = 3$    | 1       | 1       | 1       | 1       | 1       | $\gg 0$  |
| $n = 4$    | 1       | 1       | 2       | 2       | 2       | $\gg 0$  |
| $n = 5$    | 1       | 1       | 2       | 4       | 8       | $\gg 0$  |
| $n = 6$    | 1       | 1       | 2       | 4       | 7       | $\gg 0$  |
| $n \geq 7$ | 1       | 1       | 2       | 4       | 7       | 13       |

**Table 1.** Number of components of $C_n(M_d)$

The components themselves are also described: the elementary components are presented in Section 6.1 and others are obtained using Proposition 4.27.

The study of $C_n(M_d)$ is a classical subject on its own, but our work is also inspired by complexity theory, namely by deciding whether a concise ternary tensor has minimal border rank. Let $A, B, C$ be $d$-dimensional $\mathbb{k}$-vector spaces and let $T \in A \otimes B \otimes C$. Assume that $T$ is **concise**, which means that the inclusion $T(B^\vee \otimes C^\vee) \subset A$ is an equality and the same holds for two other contractions. Assume moreover that $T$ is $1_A$-**generic**, which means that there exists a functional $\alpha \in A^\vee$ such that $T(\alpha) \in B \otimes C$ has full rank. Using $T(\alpha)$ we see that $T$ is isomorphic
to a tensor in $A \otimes B \otimes B^\vee$ so it can be identified with a tuple $\mathcal{E}(T)$ of matrices parameterized by $A$, one of them equal to the identity matrix [LM17, §2.1]. The tensor $T$ satisfies Strassen’s equations for the minimal border rank if and only if the matrices in $\mathcal{E}(T)$ commute [LM17, Lemma 2.6]. The fundamental observation [LM17, Proposition 2.8] is that $T$ has minimal border rank if and only if $\mathcal{E}(T) \in C_d(\mathbb{M}_d)$ lies in the principal component. The identity matrix may be discarded from the tuple to obtain $\mathcal{E}'(T) \in C_{d-1}(\mathbb{M}_d)$ and $\mathcal{E}'(T)$ lies in the principal component if and only if $\mathcal{E}(T)$ lies in the principal component. From this point of view, the nonprincipal components of $C_{d-1}(\mathbb{M}_d)$ yield classes of tensors satisfying Strassen’s equations yet having higher border rank. This is useful for investigation of equations to higher secant varieties of the Segre variety, a major open problem [Lan12].

We shift focus to the Quot scheme $\text{Quot}^d$ of zero-dimensional, degree $d$ quotient modules of $S^{\oplus r}$, where $S = \mathbb{k}[y_1, \ldots, y_n]$; see Section 3.2 for details. A classification of such modules is possible only in very small degrees [MR18]. By the classical ADHM construction [Nak99, Chapter 2] the geometry of $\text{Quot}^d$ is equivalent to the geometry of an open subset of $C_n(\mathbb{M}_d)$; this subset is the whole $C_n(\mathbb{M}_d)$ for $r \geq d$. An analysis of components from Theorem A shows that we have the following number of components for $\text{Quot}^d = \text{Quot}^d(\mathbb{A}^n)$.

| $d \leq 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ | $d = 7$ | $d \gg 0$ |
|-----------|---------|---------|---------|---------|---------|---------|
| $n \leq 2$ | 1,1,... | 1,... | 1,... | 1,... | 1,... | 1,... |
| $n = 3$   | 1,... | 1,... | 1,... | 1,... | 1,... | $\gg 0$ |
| $n = 4$   | 1,... | 1,... | 1,2,... | 1,2,... | 1,2,... | $\gg 0$ |
| $n = 5$   | 1,... | 1,... | 1,2,... | 1,3,4,... | 1,3,4,... | 1,4,7,8,... | $\gg 0$ |
| $n = 6$   | 1,... | 1,... | 1,2,... | 1,3,4,... | 1,4,6,7,... | 1,5,9,11,... | $\gg 0$ |
| $n \geq 7$ | 1,... | 1,... | 1,2,... | 1,3,4,... | 1,4,6,7,... | 1,6,10,12,13,... | $\gg 0$ |

Table 2. Number of components of $\text{Quot}^d$. In each entry, consecutive numbers correspond to the number of components for $r = 1,2,...$ and “...” means that the numbers stabilize at the value of the last entry. In particular, we see that for $r \geq 5$ we already have all the components (for $d \leq 7$).

We review the basics of ADHM constructions in Section 3.2. This connection is frequently used to analyse $\text{Quot}^d$, for example [HJ18] deduced irreducibility of $\text{Hilb}_{10}(\mathbb{A}^3)$ from the aforementioned irreducibility of $C_3(\mathbb{M}_{10})$; see also [HG21]. In this paper we use it backwards: we take advantage of the sophisticated commutative algebra tools such as duality for finite free resolutions (see Section 3.6) and Green’s Linear Syzygy Theorem (see proof of Theorem 6.14) to understand $\text{Quot}^d$ and then use this knowledge to understand $C_n(\mathbb{M}_d)$.

The Quot scheme $\text{Quot}^d$ is a natural generalization of the Hilbert scheme of $d$ points on $\mathbb{A}^n$, the latter just corresponds to $r = 1$. There are two tools of great importance in the analysis of this Hilbert scheme, namely

- Macaulay’s inverse systems, also known as apolarity, which are used to classify or produce explicit examples [IE78, IK99, CEVV09, Lan12],
- Białynicki-Birula decomposition, which is used to find components without the need of describing them [Jel19] and understand singularities [Jel20].

We generalize both tools to the setting of $\text{Quot}^d$: we introduce apolarity for modules (Section 4.1) and study the Białynicki-Birula decomposition (Section 5) for $\text{Quot}^d$. The Białynicki-Birula decomposition is quite technical, so in this introduction we discuss only apolarity.

There are at least three ways of explicitly writing down our objects of interest:
(1) Commuting matrices. In this form the degree \( d \) is explicitly given as the size. The relations are implicit in the commutativity relations. The presentation takes much space. Describing explicit deformations is easy, but proving irreducibility of loci is tedious.

(2) Modules given by generators and relations. Here the relations are explicit and numerous, while the degree is nontrivial to compute. Describing explicit deformations or proving irreducibility of loci are both tedious.

(3) Modules given by inverse systems, i.e., using apolarity. Here the relations and \( d \) are both implicit but relatively easy to compute. This presentation is compact. Producing explicit deformations is neither easy nor hard, but proving irreducibility is easy.

Let us contrast the three ways on an explicit example. Consider a quadruple of \( 4 \times 4 \) matrices

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

The matrices pairwise commute. We equip \( k^4 \) with a \( k[y_1, \ldots, y_4] \)-module structure, where \( y_i \cdot v = x_i(v) \) for every \( i = 1, 2, 3, 4 \) and \( v \in k^4 \). Denote the resulting module by \( M \). It is generated by elements \( e_3 := (0, 0, 1, 0)^T, e_4 := (0, 0, 0, 1)^T \in k^4 \) so we can write it as a quotient of \( F := Se_3 \oplus Se_4 \). In fact, we have

\[
M \simeq \frac{F}{(y_1e_4, y_2e_3, y_2e_4 - y_1e_3, y_3e_4, y_4e_3, y_4e_4 - y_3e_3)S}.
\]

This is the presentation by generators and relations. The inverse system is obtained as follows. Consider the graded dual module \( F^* := \bigoplus_i \text{Hom}(F, k) \). The monomial basis of \( F \) gives a natural dual basis of \( F^* \); we denote by \( z_1z_2^2e_3^* \) the element of \( F^* \) dual to \( y_1y_2^2e_3 \). One can verify that \( (y_1e_4, y_2e_3, y_2e_4 - y_1e_3, y_3e_4, y_4e_3, y_4e_4 - y_3e_3)S \subset F \) is equal to the set \( (N \cdot S)^\perp \), where \( N \subset F^* \) is a linear span of \( z_1e_4^*, z_2e_3^*, z_3e_3^* + z_4e_4^* \) and \( (-)^\perp \) denotes the orthogonal space is the above duality. We write this as

\[
M \simeq \frac{F}{((z_1e_3^* + z_2e_4^* + z_3e_3^* + z_4e_4^*)S)^\perp}.
\]

We refer the reader to Section 4.1 for further details.

Having discussed apolarity, we return to considering components. The usual method of producing components of \( C_n(M_d) \), which we apply successfully in Theorem A, is to produce an irreducible subvariety \( Z \subset C_n(M_d) \) of dimension \( D \) and find a point \( x \in Z \) such that \( \dim T_xC_n(M_d) = D \). This method requires the resulting component \( Z \) to have a smooth point \( x \). Below, we show that \( C_4(M_8) \) has a generically nonreduced component which thus violates this condition. While it follows from earlier results that \( C_n(M_d) \) is very singular when \( n, d \gg 0 \), see Remark 3.8, the importance of Theorem B lies in that the nonreducedness appears for small \( d \) and \( n \) and that we obtain a generically nonreduced component.

**Theorem B.** Let \( k \) be an algebraically closed field of characteristic zero. Consider the locus \( \mathcal{L} \) of 4-tuples of \( 8 \times 8 \) matrices with nonzero entries only in the top right \( 4 \times 4 \) corner. Then \( (kI_4)^4 + \mathcal{L} \) is a component of \( C_4(M_8) \) that is generically nonreduced (so it has no smooth points). Consequently, the variety \( C_n(M_d) \) has generically nonreduced components for all \( n \geq 4 \) and \( d \geq 8 \). In contrast, the variety \( C_n(M_d) \) is generically reduced for \( d \leq 7 \) and all \( n \). Similarly, the Quot scheme \( \text{Quot}_r^d \) has a generically nonreduced component for \( n, r \geq 4 \) and \( d \geq 8 \) while it is generically reduced for \( d \leq 7 \) and all \( r, n \).
Exhibiting a generically nonreduced component of a moduli space is subtle, as seen in Mumford’s celebrated example [Har10, §13] see also [Vak06, Kas15, Ric20]. In our setup, we use the machinery of differential graded Lie algebras (DGLA) to obtain an explicit description of the primary obstruction for Quot$^d_r$. This allows us to obtain the quadratic part of equations of the complete local ring at a point of $\mathcal{L}$. The locus $\mathcal{L}$ is a sink for an appropriate torus action, so this ring is a completion of a graded ring, hence the obtained part accounts for all quadratic relations. We prove that they cut out the ring of dimension equal to the dimension of $(\mathbb{k}I)^4 + \mathfrak{gl}_8 \cdot \mathcal{L}$, which concludes the proof for Quot$^8_3$. The result for $C_4(\mathbb{M}_8)$ follows from ADHM construction. The whole approach seems to be new and useful also in other cases.

The geometry of commuting matrices is still largely unexplored. One open question is to find the smallest $d$ such that $C_3(\mathbb{M}_d)$ is reducible. Another question, inspired by Table 1 is: there is a natural “add zero matrices” map from the set of components of $C_n(\mathbb{M}_d)$ to the set of components of $C_{n+1}(\mathbb{M}_d)$. Is this map a bijection for all $n \geq d$? It is possible to see that this map is a bijection for all $n$ larger than the maximal dimension of a commutative subalgebra of $\mathbb{M}_d$, in particular for $n \geq \lfloor (d/2)^2 \rfloor + 1$, see [Sch05]. For related ideas, see [LNŠ21, Introduction]. Finally, despite the challenges rising from Theorem B it is likely that one can classify components of $C_n(\mathbb{M}_d)$ for $d = 8$ and perhaps further. The general tools summarized and developed in this article would be very useful in investigating these questions.

Let us briefly summarize the contents of this article. In Section 3 we summarize the basics of $C_n(\mathbb{M}_d)$, Quot$^d_r$ and their connection. Most of these results are folklore but frequently references are missing or are available only in special cases or in language inaccessible to the linear-algebra community. In Sections 4-5 we present advanced general tools, which are new: apolarity, primary obstructions for Quot, concatenation maps, Białyńnicki-Birula decompositions etc. This is the core part in terms of general theory. Finally, in Section 6 we apply these results to obtain our main theorems. An appendix contains a brief introduction to functors of points.

2. Notation

| Symbol | Explanation |
|--------|-------------|
| $d$    | degree of the module = size of matrices |
| $n$    | number of matrices = number variables in polynomial ring $S$ |
| $r$    | number of generators of the module |
| $x_1, \ldots, x_n$ | commuting matrices |
| $y_1, \ldots, y_n$ | variables of polynomial ring $S$ |
| $V$    | a fixed $d$-dimensional vector space over $\mathbb{k}$ |

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3. Preliminaries

Let $\mathbb{k}$ be an algebraically closed field. So far we do not put any restrictions on its characteristic (these will be explicitly included in Sections 5-6), although in examples we assume characteristic zero. Let $S = \mathbb{k}[y_1, \ldots, y_n]$. We will now define several spaces that parameterize $S$-modules. In
the table below, \emph{modules} means zero-dimensional \(S\)-modules of degree \(d\). Here and elsewhere the \emph{degree} of a module \(M\) is just its dimension as a \(k\)-linear space. The spaces are linked by the forgetful functors and form the diagram, see Table 3. For more details on the maps, see Section 3.2.

| space          | objects                                |
|----------------|----------------------------------------|
| \(\text{Mod}^d(\mathbb{A}^n)\) | modules                                |
| \(\text{Quot}^d_r\) | modules with fixed \(r\) generators    |
| \(C_n(\mathbb{M}_d)\) | modules with fixed basis               |
| \(\mathcal{U}^\text{st}\) | modules with fixed basis and fixed sequence of \(r\) generators |

\[
\begin{array}{ccc}
\mathcal{U}^\text{st} & \xrightarrow{\text{smooth fib.dim.}} & r d \\
\downarrow / \text{GL}_d & & \downarrow / \text{GL}_d \\
\text{Quot}^d_r & \xrightarrow{} & \text{Mod}^d(\mathbb{A}^n) \\
\end{array}
\]

\textbf{Table 3.} Moduli spaces

3.1. **Commuting matrices.** Let \(\mathbb{M}_d\) be the affine space \(\text{Hom}(V,V)\) for a fixed \(d\)-dimensional vector space \(V\). By \(I_d\) we denote the \(d \times d\) identity matrix. Let

\[
C_n(\mathbb{M}_d) = \{(x_1, \ldots, x_n) \in \mathbb{M}_d^n ; \forall i,j : x_ix_j = x_jx_i\}
\]

be the \emph{variety of \(n\)-tuples of \(d \times d\) commuting matrices}. More precisely, we define \(C_n(\mathbb{M}_d)\) as the subscheme cut out by the quadratic equations coming from \(x_ix_j - x_jx_i = 0\). We show that the word \emph{variety} even though accepted in the literature is a slight abuse: it is known that \(C_n(\mathbb{M}_d)\) is in general reducible and in Section 6.5 we prove that it has generically nonreduced components.

\textbf{Lemma 3.1.} The tangent space to the scheme \(C_n(\mathbb{M}_d)\) defined above is

\[
T_{(x_1,\ldots,x_n)}C_n(\mathbb{M}_d) = \{(z_1, \ldots, z_n) \in \mathbb{M}_d^n ; \forall i,j : [x_i, z_j] + [z_i, x_j] = 0\}.
\]

For the variety \((C_n(\mathbb{M}_d))\text{red}\) the tangent space to it at \((x_1, \ldots, x_n)\) is contained in the space above.

\textbf{Proof.} We use the description of the tangent space via maps from \(\text{Spec}(k[\varepsilon]/\varepsilon^2)\), see for example [EH00, VI.1.3]. The tangent space to \(C_n(\mathbb{M}_d)\) at the point \((x_1, \ldots, x_n)\) is the vector space of tuples \((z_1, \ldots, z_n) \in \mathbb{M}_d^n\) such that \((X_1, \ldots, X_n) = (x_1 + \varepsilon z_1, x_2 + \varepsilon z_2, \ldots, x_n + \varepsilon z_n)\) satisfies the equations \(X_iX_j = X_jX_i\) modulo \(\varepsilon^2\). But

\[
X_iX_j - X_jX_i = (x_i + \varepsilon z_i)(x_j + \varepsilon z_j) - (x_j + \varepsilon z_j)(x_i + \varepsilon z_i) \equiv
\]

\[
\equiv x_i x_j - x_j x_i + e(z_i x_j + x_i z_j - x_j z_i - z_j x_i) = \varepsilon([x_i, z_j] + [z_i, x_j]) \pmod{\varepsilon^2},
\]

so the equations are satisfied precisely when \([x_i, z_j] + [z_i, x_j] = 0\). Finally, the tangent space to the underlying variety is always contained in the tangent space to the scheme. \(\square\)

We now show that \(C_n(\mathbb{M}_d)\) indeed corresponds to \(S\)-modules with a basis.

\textbf{Lemma 3.2.} The points of \(C_n(\mathbb{M}_d)\) are in bijection with \(S\)-modules with a fixed \(k\)-linear basis.

\textbf{Proof.} Let \(V = \bigoplus e_1 \oplus \ldots \oplus \bigoplus e_d\). Having a point \((x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)\), we define an \(S\)-module structure on \(V\) by \(y_i \cdot v = x_i(v)\). The equations \(x_i x_j = x_j x_i\) imply that \(y_i \cdot (y_j \cdot v) = y_j \cdot (y_i \cdot v)\)
for all \(i,j\), so indeed we get an \(S\)-module. Conversely, for an \(S\)-module \(M\) with a basis \(\mathcal{B}\) and \(i = 1,2,\ldots,n\) consider the multiplication by \(y_i\) as an endomorphism of \(M\) and let \(x_i\) be its matrix written in basis \(\mathcal{B}\). Since \(S\) is commutative, the matrices \(x_1,\ldots,x_n\) commute. \(\square\)

### 3.2. Quot and commuting matrices

Let \(S = \mathbb{k}[y_1,\ldots,y_n]\) be a polynomial ring. An \(S\)-module \(M\) has **finite degree** if the \(k\)-vector space \(M\) has finite dimension. We say that \(M\) has **degree** \(d\) if the \(k\)-vector space \(M\) has dimension \(d\). Fix a free \(S\)-module \(F = S e_1 \oplus S e_2 \oplus \cdots \oplus S e_r\) of rank \(r\). The **Quot scheme** \(\text{Quot}^d_r\) of points on \(\mathbb{A}^n = \text{Spec}(S)\) parameterizes degree \(d\) quotient modules of the \(S\)-module \(F\). In other words, a \(k\)-point of \(\text{Quot}^d_r\) is a quotient \(F/K\) of \(S\)-modules that is a \(d\)-dimensional vector space. For a quotient \(F/K\) we denote by \([F/K]\) the corresponding point in \(\text{Quot}^d_r\). To give such a quotient is the same as to give an \(S\)-module together with a fixed set of \(r\) generators.

To give the Quot scheme the topological space and scheme structures we need to define it using functorial language, see appendix for details. However, the willing reader can take this for granted.

**Example 3.3.** When \(r = 1\), we are speaking about quotients of \(S^{\oplus 1}\), so about modules of the form \(S/I\). Therefore \(\text{Quot}^1_1 = \text{Hilb}_d(\mathbb{A}^n)\).

The schemes \(C_n(M_d)\) and \(\text{Quot}^d_r\) are tightly connected by the ADHM construction \([\text{Nak99, HG21, Bar00}]\), named after \([\text{ADHM78}]\), which we now recall. In terms of Table 3, we just take a module, fix its basis and \(r\) generators and consider the two projections: forgetting about the generating set and forgetting about the basis. We now explain this more carefully. Throughout the article, a key fact to keep in mind is that the spaces in Table 3 are very singular, but, as we prove below, the maps are smooth, so intuitively these objects are “singular in the same way”.

Fix a number \(r\) and consider the product

\[
U := C_n(M_d) \times V^r.
\]

For every tuple \(T = (x_1,\ldots,x_n,v_1,\ldots,v_r)\) in this product we may take the smallest subspace \(W \subset V\) containing \(v_1,\ldots,v_r\) and preserved by operators \(x_1,\ldots,x_n\). We say that \(T\) is **stable** if \(W = V\). In other words, \(T\) is stable iff \(v_1,\ldots,v_r\) generate \(V\) as a \(\mathbb{k}[x_1,\ldots,x_n]\)-module. For every \(i\) the power \(x_i^d\) is a linear combination of smaller powers of \(x_i\), so \(T\) is stable if and only if the elements

\[
\{x_1^{a_1} \circ x_2^{a_2} \circ \cdots \circ x_n^{a_n}(v_j) \mid a_i < d, \ j = 1,\ldots,r\}
\]

span \(V\). This is an open condition, so the locus \(U^\text{st}\) of stable tuples is open. To a stable tuple \(T\) we associate a point of \(\text{Quot}^d_r\) as follows.

1. the tuple \((x_1,\ldots,x_n)\) induces an \(S\)-module structure on \(V\), where \(y_i(v) = x_i(v)\) for all \(v \in V\); here we use commutativity of \(x_i\) and \(x_j\). We denote the obtained \(S\)-module by \(M\) and call it the module corresponding to \((x_1,\ldots,x_n)\).
2. the stability of a tuple \((x_1,\ldots,x_n,v_1,\ldots,v_r)\) means that the \(S\)-module \(M\) is generated by \(v_1,\ldots,v_r\). There is a unique epimorphism of \(S\)-modules \(\pi_M : F \to M\) that sends \(e_i\) to \(v_i\) for \(i = 1,\ldots,r\). The map \(\pi_M\) restricts to an isomorphism of \(S\)-modules \(\overline{\pi_M} : F/\text{ker} \pi_M \to M\).

**Lemma 3.4.** The map \((x_1,\ldots,x_n,v_1,\ldots,v_r) \mapsto (\overline{\pi_M},\varphi)\) is a bijection between the \(\mathbb{k}\)-points of \(U^\text{st}\) and the set

\[
\left\{ \left( \frac{F}{K},\varphi \right) \mid [F/K] \in \text{Quot}^d_r, \varphi : F/K \to V \text{ is a } \mathbb{k}\text{-linear isomorphism} \right\}.
\]
Proof. It remains to construct an inverse. The multiplication by $y_i$ is a linear map $\mu_i\colon F/K \to F/K$, so $\varphi \circ \mu_i \circ \varphi^{-1} : V \to V$ is a $k$-linear endomorphism of $V$, so an element of $\mathbb{M}_d$ which we denote by $x_i$. Let $v_j = \varphi(\overline{e_j})$ for $j = 1, 2, \ldots, r$. Since the module $F/K$ is generated by images of $e_1, \ldots, e_r$, the tuple $(x_1, \ldots, x_n, v_1, \ldots, v_r)$ is stable. \hfill \Box

Lemma 3.4 gives us a map $U^\mathrm{st} \to \text{Quot}^d$ that first transforms $(x_1, \ldots, x_n, v_1, \ldots, v_r)$ to $(F/K, \varphi)$ and then forgets about the isomorphism $\varphi$. The fiber of $U^\mathrm{st} \to \text{Quot}^d$ consists of all possible $\varphi$, whence it is in bijection with $GL_d$.

Example 3.5. Consider the case $d = n = r = 1$. The closed points of $\text{Quot}^d$ are modules of the form $k[y_1]/(y_1 - \alpha)$ for $\alpha \in k$. The space $V$ is one-dimensional. The closed points of $U$ are pairs $(x_1, v_1)$, where $v_1 \in V$ and $x_1 \in \mathbb{M}_1 \simeq k$. The pair $(x_1, v_1)$ is stable if and only if $v_1 \neq 0$. In this case $x_1 v_1 = \alpha v_1$ for some $\alpha$. The projection $\pi_M:\mathbb{M}_1 \to M$ sends $y_1 - \alpha$ to zero and so

$$\ker \pi_M = (y_1 - \alpha)$$

as expected. For every $\lambda \in k^*$, under the identification from Lemma 3.4 the pairs $(x_1, v_1)$ and $(x_1, \lambda v_1)$ are sent to the same module $\mathbb{M}[y_1]/(y_1 - \alpha)$, though with different isomorphisms $\varphi$. The fiber of the above map $U^\mathrm{st} \to \text{Quot}^d$ over $\mathbb{M}[y_1]/(y_1 - \alpha)$ is equal to $\{(x_1, \lambda v_1) \mid \lambda \in k^*\}$, so it is isomorphic to $k^*$.

We have a natural action of $GL(V)$ on $C_n(\mathbb{M}_d)$, $U$ and $U^\mathrm{st}$. Namely, $GL(V)$ acts on $\mathbb{M}_d$ by conjugation, so it also acts on $C_n(\mathbb{M}_d)$ by simultaneous conjugation:

$$g \circ (x_1, \ldots, x_n) = (gx_1 g^{-1}, \ldots, gx_n g^{-1})$$

for $g \in GL(V)$. Similarly, $GL(V)$ acts on $V^{\times r}$ by $g(v_1, \ldots, v_r) = (gv_1, \ldots, gv_r)$, so we obtain an action on $U$ by

$$g \circ (x_1, \ldots, x_n, v_1, \ldots, v_r) = (gx_1g^{-1}, \ldots, gx_ng^{-1}, gv_1, \ldots, gv_r)$$

for $g \in GL(V)$. If the tuple $T = (x_1, \ldots, x_n, v_1, \ldots, v_r)$ was stable, then also the tuple $g \circ T$ is stable, so the action on $U$ restricts to an action on $U^\mathrm{st}$.

Lemma 3.6. In the description from Lemma 3.4, the $GL(V)$-action on $U^\mathrm{st}$ is given by

$$g \circ \left( \frac{F}{K}, \varphi \right) = \left( \frac{F}{K}, g \circ \varphi \right).$$

Proof. Let $T = (x_1, \ldots, x_n, v_1, \ldots, v_r)$ be a stable tuple and $M$ be the associated $S$-module and epimorphism $\pi_M : F \to M$ so that $\pi_M(e_j) = v_j$. Then $y_i \cdot m = x_i(m)$ for every $m \in M$. Let $T' = g \circ T$ and $M'$ be the associated $S$-module with epimorphism $\pi_{M'} : F \to M'$. Then $y_i \circ m' = gx_i g^{-1}(m')$ and $\pi_{M'}(e_j) = g v_j$, so the following diagram is commutative

$$\begin{array}{ccccc}
F & \xrightarrow{\pi_M} & M & \xrightarrow{y_i} & M \\
\downarrow{\pi_{M'}} & \downarrow{g(-)} & \downarrow{g(-)} \\
M' & \xrightarrow{y_i} & M'.
\end{array}$$

Since $g \cdot (-)$ is an isomorphism, we have $\ker \pi_M = \ker \pi_{M'}$, which exactly means that the action of $g$ on the quotients of $F$ is trivial. Moreover, the isomorphisms $\overline{\pi_M}$ and $\overline{\pi_{M'}}$ satisfy $\overline{\pi_{M'}} = g \circ \overline{\pi_M}$. \hfill \Box

The set-theoretic map defined after Lemma 3.4 is a shade of a richer structure: a morphism of schemes. To construct it is a matter of introducing the language of functors; we defer this to the appendix but state the result here.
Proposition 3.7. There exists a map of schemes \( p : U^s \to \text{Quot}^d_r \) such that
\[
p(x_1, \ldots, x_n, v_1, \ldots, v_r) = [F/K]
\]
in the notation above. The \( GL(V) \)-action defined on points of \( U^s \) in Lemma 3.6 gives rise to a morphism of schemes \( GL(V) \times U^s \to U^s \), i.e., to an algebraic \( GL(V) \)-action. This \( GL(V) \)-action and \( p \) make \( U^s \) a principal \( GL(V) \)-bundle over \( \text{Quot}^d_r \). Explicitly, this means that there exists an open cover \( U_i \) of \( \text{Quot}^d_r \) and \( GL(V) \)-equivariant isomorphisms of schemes \( p^{-1}(U_i) \cong GL(V) \times U_i \) over \( U_i \).

Proof. See Corollary A.6. \( \square \)

Remark 3.8. The smooth maps \( \text{Quot}^d_r \leftarrow U^s \to C_n(M_d) \) can be used to show a Murphy’s Law type statement for \( C_n(M_d) \). Namely by [Jel20] the Hilbert scheme of points has almost arbitrary pathologies. The Hilbert scheme is just \( \text{Quot}^d_r \) for \( r = 1 \) and so using the smooth maps one can deduce (similarly as in [Jel20, §5]) that \( C_n(M_d) \) has bad singularities, for example it is nonreduced. However, these singularities appear for \( n \geq 16 \) and very high \( d \).

Lemma 3.9. Let \( F/K \) be a quotient module. The tangent space to \( \text{Quot}^d_r \) at \( [F/K] \) is isomorphic to \( \text{Hom}_S(K, F/K) \). Let \( (x_1, \ldots, x_n) \in C_n(M_d) \) be a corresponding element. Then we have
\[
\dim T_{[F/K]} \text{Quot}^d_r = r^2 - d^2 + \dim T_{(x_1, \ldots, x_n)} C_n(M_d).
\]

Proof. The bijection \( T_{[F/K]} \text{Quot}^d_r \cong \text{Hom}(K, F/K) \) is classical, see [Ser06, Proposition 4.4.4] or [FGI+05, Theorem 6.4.5(b)] or adapt the proof of [Str96, Theorem 10.1]. To compute the tangent space dimension, consider the maps from \( U^s \) to \( \text{Quot}^d_r, C_n(M_d) \) as in Table 3. Take any point \( u = (x_1, \ldots, x_n; v_1, \ldots, v_r) \) mapping to \( [F/K] \in \text{Quot}^d_r \). By construction the map \( U^s \to C_n(M_d) \) is a composition of the open embedding of \( U^s \) into \( U = C_n(M_d) \times V^r \) with the projection of \( C_n(M_d) \times V^r \) onto the first coordinate. By Proposition 3.7, locally near \( u \) the map \( U^s \to \text{Quot}^d_r \) is a projection \( GL_d \times U \to U \). The equality (3.10) follows from comparing the dimensions of \( T_u U^s, T_{[F/K]} \text{Quot}^d_r \) and \( T_{(x_1, \ldots, x_n)} C_n(M_d) \). \( \square \)

3.3. Support and eigenvalues. For a finite degree module \( M \), we define its support as the set of maximal ideals \( \mathfrak{m} \subset S \) such that \( M/\mathfrak{m}M \neq 0 \). The support is equal to the set of maximal ideals containing the ideal \( \text{Ann}(M) = \{s \in S \mid sM = 0\} \), as we explain below (this also shows that our definition of support agrees with the usual one). On the one hand, if \( \mathfrak{m} \) is in the support then the ideal \( \text{Ann}(M) + \mathfrak{m} \) annihilates a nonzero module \( M/\mathfrak{m}M \) and so \( \text{Ann}(M) + \mathfrak{m} \neq (1) \), so \( \text{Ann}(M) \subset \mathfrak{m} \). On the other hand, if \( \mathfrak{m} \) is not in the support then \( M = \mathfrak{m}M \) so by Nakayama’s lemma [sta17, Tag 00DV] the intersection \( (1 - \mathfrak{m}) \cap \text{Ann}(M) \) is nonempty, so \( \text{Ann}(M) \not\subset \mathfrak{m} \).

Lemma 3.11. Let \( (x_1, \ldots, x_n) \in C_n(M_d) \) and let \( M \) be the associated \( S \)-module. Suppose that \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \) are such that \( (y_1 - \lambda_1, \ldots, y_n - \lambda_n) \) is in the support of \( M \). Then for every \( i \) the element \( \lambda_i \) is an eigenvalue of the matrix \( x_i \).

Proof. If for some \( i \) the element \( \lambda_i \) is not an eigenvalue of \( x_i \), then \( x_i - \lambda_i I \) is invertible. This translates to the multiplication by \( y_i - \lambda_i \) on \( M \) being an isomorphism. It follows that \( M = (y_i - \lambda_i)M / (y_i - \lambda_i)M = 0 \) and consequently \( M/(y_1 - \lambda_1, \ldots, y_n - \lambda_n)M = 0 \) so \( (y_i - \lambda_i)_{i=1,2,\ldots,n} \) is not in the support.

Remark 3.12. The converse of Lemma 3.11 does not hold. For example consider a pair of diagonal \( 2 \times 2 \) matrices \( (x_1, x_2) = (\text{diag}(0,1), \text{diag}(0,1)) \). The corresponding module is \( M = k[y_1, y_2]/(y_1, y_2) \oplus k[y_1, y_2]/(y_1 - 1, y_2 - 1) \) so \( (y_1 - 1, y_2) \) is not in the support of \( M \) even though 0 and 1 are eigenvalues of both matrices.
Despite Remark 3.12, we do know that the two extremal situations: where the module is supported at single point or when the matrices are (up to adding multiple of identity) nilpotent, actually coincide.

**Lemma 3.13.** Let \((x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)\) and let \(M\) be the associated \(S\)-module. Then the support of \(M\) consists of one point precisely when every \(x_i\) has only one eigenvalue.

**Proof.** Fix \(i\). The support of \(M\) consists of maximal ideals in \(S\). By Nullstellensatz such ideals have the form \((y_1 - \lambda_1, y_2 - \lambda_2, \ldots, y_n - \lambda_n)\) for a point \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{k}^n\). We identify each such ideal with the corresponding point, so the support of \(M\) becomes a subset of \(\mathbb{k}^n\). For any \(\lambda_i \in \mathbb{k}\) the following are equivalent:

1. \(\lambda_i\) is an eigenvalue of \(x_i\),
2. \(x_i - \lambda_iI\) is not invertible,
3. \((y_i - \lambda_i)M \neq M\),
4. \(M/(y_i - \lambda_i)M \neq 0\),
5. there exists a maximal ideal \(\mathfrak{m}\) containing \(y_i - \lambda_i\) and such that \(M/\mathfrak{m}M \neq 0\). Indeed \(\mathfrak{m}\) can be chosen as any element of the support of the nonzero module \(M/(y_i - \lambda_i)M\),
6. the support of \(M\) intersects the hyperplane \(y_i = \lambda_i\).

The support of \(M\) is a point iff for every \(i\) the last condition is satisfied by at most one \(\lambda_i\). This happens iff for every \(i\) the matrix \(x_i\) has only one eigenvalue. \(\square\)

**Remark 3.14.** In the arguments below we use Hilbert functions of modules and Jordan block types for matrices. While those structures are connected, it is not a straightforward connection as it requires Lefschetz-type theorems, see for example [HMM+13, Prop. 3.5, Prop. 3.64], [HMMNW03] or [IMM20, Prop. 2.10, Lemma 2.11].

3.4. Components. We now discuss that components of \(C_n(\mathbb{M}_d)\) and \(\text{Quot}_d^r\) are in bijection for \(r \geq d\). We begin with the following general lemma.

**Lemma 3.15.** Let \(p: U \to X\) be a surjective morphism of finite type schemes. Assume that for an irreducible component \(W \subset X\) the preimage \(p^{-1}(W)\) is irreducible. Then the maps \(W \mapsto p^{-1}(W)\) and \(Z \mapsto \overline{p(Z)}\) give a bijection between irreducible components of \(U\) and \(X\).

**Proof.** Let \(Z_1, \ldots, Z_k\) be the irreducible components of \(U\). Their images in \(X\) are irreducible, so each of them lies in some irreducible component of \(X\). Let \(W\) be any irreducible component of \(X\) and let \(I_W \subset \{1, \ldots, k\}\) consists of those \(i\) for which \(\overline{p(Z_i)} \subset W\). Since \(p\) is surjective, the closed subsets \(p(Z_i)\) cover \(X\) so there exists an \(i_0 \in I_W\) such that \(\overline{p(Z_{i_0})} = W\). If \(I_W \neq \{i_0\}\) then \(p^{-1}(W)\) contains at least two components of \(U\) contradicting the assumption. Therefore \(I_W = \{i_0\}\) and this proves the claim. \(\square\)

**Lemma 3.16.** Both maps \(p_1: U^\text{st} \to p_1(U^\text{st}) \subset C_n(\mathbb{M}_d)\) and \(p_2: U^\text{st} \to \text{Quot}_d^r\) in Table 3 satisfy the assumptions of Lemma 3.15. These maps also map open sets to open sets.

**Proof.** The map \(p_1\) is an open embedding into a product \(C_n(\mathbb{M}_d) \times \mathbb{A}^d\) composed with projection, so it is flat, therefore it maps open subsets to open subsets [sta17, Tag 039K]. In particular \(p_1(U^\text{st})\) is open. Let \(W \subset p_1(U^\text{st})\) be irreducible. Then \(p_1^{-1}(W)\) is a nonempty open subset of the irreducible scheme \(W \times \mathbb{A}^d\), so it is irreducible as well. This concludes the proof for \(p_1\).

Proposition 3.7 implies that there is an open cover \(\{V_i\}\) of \(\text{Quot}_d^r\) such that for all \(i\) the map \(p_2^{-1}(V_i) \to V_i\) is isomorphic to the second projection \(\text{GL}_d \times V_i \to V_i\). In particular \(p_2\) is flat, so it maps open subsets to open subsets. Let \(W \subset \text{Quot}_d^r\) be irreducible and suppose
$p_2^{-1}(W)$ is reducible. Then there are two disjoint open subsets $U_1, U_2 \subset p_2^{-1}(W)$. Pick any $V_1, V_2$ from the above open cover (if necessary, refining it) such that $V_i \subset p_2(U_i)$ for $i = 1, 2$. The set $W$ is irreducible and $V_1, V_2 \subset W$ so $V_1 \cap V_2$ is nonempty and irreducible. It follows that $P := p_2^{-1}(V_1 \cap V_2)$ is nonempty, isomorphic to $\text{GL}_d \times (V_1 \cap V_2)$ so irreducible, and $U_1$ and $U_2$ intersect $P$, so $U_1 \cap P, U_2 \cap P$ are nonempty open inside irreducible $P$, hence intersect, so $U_1$ and $U_2$ intersect, a contradiction. □

Lemma 3.15 and Lemma 3.16 imply that there are bijections between irreducible components of $\text{Quot}_d^r$, of $\mathcal{U}^r$, and of $p_1(\mathcal{U}^r) \subset C_n(M_d)$. The components of the open subset $p_1(\mathcal{U}^r)$ are a subset of components of $C_n(M_d)$ so we obtain the following

\{ irreducible component of $\text{Quot}_d^r$ \} \cong \{ irreducible component of $\mathcal{U}^r$ \} \cong \{ irreducible component of $p_1(\mathcal{U}^r)$ \} 

For $r \geq d$ the map $\mathcal{U}^r \to C_n(M_d)$ is surjective; just pick $v_1, \ldots, v_r$ containing a basis of $V$. In this case we obtain a bijection between components of $\text{Quot}_d^r$ and $C_n(M_d)$.

**Example 3.17.** The assumption $r \geq d$ is sharp. Consider the zero tuple $(x_1, \ldots, x_n) = (0, 0, \ldots, 0)$. For $v_1, \ldots, v_r \in V$ the tuple $(x_1, \ldots, x_n, v_1, \ldots, v_r)$ is stable if and only if $v_1, \ldots, v_r$ span $V$. In particular this forces $r \geq d$. It follows that $\mathcal{U}^r \to C_n(M_d)$ is surjective if and only if $r \geq d$. If $r < d$ then $C_n(M_d)$ may have more components than $\text{Quot}_d^r$. For $(n, d, r) = (4, 4, 1)$ the Quot scheme is the Hilbert scheme (Example 3.3) which is irreducible [CEVV09]. In contrast, the variety of commuting matrices is reducible, see Example 6.1.

For a component $Z_{\text{Quot}} \subset \text{Quot}_d^r$ the dimension of the corresponding component $Z^C \subset C_n(M_d)$ satisfies $\dim Z_{\text{Quot}} = rd - d^2 + \dim Z^C$ because we take preimages and images under smooth maps whose fibres have dimension $rd$ and $d^2$ respectively (the details of this argument are for example in the proof of Lemma 3.9). Lemma 3.9 gives the same equality for tangent spaces, so we conclude that a point $[F/K] \in Z_{\text{Quot}}$ is smooth if and only if any corresponding point $(x_1, \ldots, x_n) \in C_n(M_d)$ is smooth.

The **principal component** of $C_n(M_d)$ is the closure of the locus of $n$-tuples of diagonalizable matrices. It contains an open subset consisting of $n$-tuples where the first matrix has $d$ distinct eigenvalues. A matrix that commutes with such a matrix $x$, can be uniquely written in the form $p(x)$ where $p$ is a polynomial of degree at most $d - 1$, see [HJ85, Theorem 3.2.4.2]. The principal component of $C_n(M_d)$ is therefore a closure of a set that is parameterized by the set of all matrices with $d$ distinct eigenvalues, which is open in $M_d$, and $n - 1$ copies of the space of uni-variate polynomials of degree at most $d - 1$, so the principal component has dimension $d^2 + (n - 1)d$ [GS00, Proposition 6].

A tuple of matrices is diagonalizable if and only if it corresponds to an $S$-module abstractly isomorphic to $\bigoplus_{i=1}^d S/m_i$ for some maximal ideals $m_i \subset S$ with $S/m_i = k$. The **principal component** of $\text{Quot}_d^r$ is the closure of the locus of quotients $F/K$ that are abstractly isomorphic to such modules. In terms of Table 3, the principal component of $\text{Quot}_d^r$ is the image of the preimage in $\mathcal{U}^r$ of the principal component of $C_n(M_d)$. It follows that the dimension of the principal component of $\text{Quot}_d^r$ is $(n + r - 1)d$.

Antipodal to the principal components are the elementary components. A component of $C_n(M_d)$ is **elementary** if its points correspond to tuples where each matrix has precisely one eigenvalue. A component of $\text{Quot}_d^r$ is **elementary** if it parameterizes modules $M$ supported at a single point. By Lemma 3.13 these two notions agree in the sense that the injection above restricts to a bijection between elementary components of $\text{Quot}_d^r$ and elementary components of $C_n(M_d)$ that intersect the image of $\mathcal{U}^r$. 
3.5. Transposes and duality. The commuting matrices have a natural involution: the transposition of each matrix in the tuple. The corresponding notion for modules is taking the dual module. We now review how the basic invariants behave under this operation.

Recall that an $S$-module has finite degree if it is finite dimensional as a $k$-vector space. Throughout this section we fix a finite degree $S$-module $M$. The vector space $M^\vee = \text{Hom}_k(M, k)$ has a natural structure of an $S$-module by precomposition: for $s \in S$ and $\varphi \in \text{Hom}_k(M, k)$ we define $(s \cdot \varphi)(m) = \varphi(sm)$ for every $m \in M$. This $S$-action is called contraction.

**Definition 3.18.** The dual module of $M$ is the space $M^\vee$ equipped with an $S$-action by contraction. We denote it by $M^\vee$ if no confusion is likely to occur.

The double dual map $M \to (M^\vee)^\vee$ is an isomorphism of $S$-modules and the annihilators of $M$ and $M^\vee$ in $S$ are equal. For an ideal $I \subset S$, let $(0 : I)_M = \{ m \in M \mid Im = \{0\} \}$ be the annihilator of $I$ in $M$.

**Lemma 3.19.** For every ideal $I$ the vector spaces $M/IM$ and $(0 : I)_M^\vee$ have the same dimension. Also the vector spaces $(0 : I)_M$ and $M^\vee/IM^\vee$ have the same dimension.

**Proof.** The tautological perfect pairing $M \times M^\vee \to k$ descends to a perfect pairing between $M/IM$ and the perpendicular to $IM$ inside $M^\vee$. A functional $\varphi \in M^\vee$ is perpendicular to $IM$ iff it vanishes on $IM$ iff $I\varphi = \{0\}$. Hence the subspace $(IM)^\perp \subset M^\vee$ is equal to $(0 : I)_M^\vee$ and the perfect pairing shows that it has dimension equal to $M/IM$. The second claim is proven by interchanging $M$ and $M^\vee$ using that the double dual is an isomorphism. \hfill \Box

Let $d = \dim_k M$. Let $m = (y_1, \ldots, y_n) \subset S$. Fix a basis on $M$ and the dual basis on $M^\vee$. Assume that $M$ is supported only at zero. In particular, this implies that the annihilator of $M$ contains some power of $m$, say $m^D$, so that $M$ is a module over the local ring $S/m^D$. Let $x = (x_1, \ldots, x_n) \in C_n(M_d)$ be the commuting tuple associated to $M$, then $x^T = (x_1^T, \ldots, x_n^T)$ is the commuting tuple associated to $M^\vee$. Thus $M$ lies in the principal component if and only if $M^\vee$ lies in the principal component. Similarly, $M$ lies in a non-elementary component if and only if $M^\vee$ lies in a non-elementary component. For later use we summarize the following data.

\[
\begin{align*}
\text{min. no of gen. of } M &= \dim_k (M/mM) = \dim_k (0 : m)_M^\vee = \dim_\mathbb{k} \text{ of common kernel of } x^T \\
\text{min. no of gen. of } M^\vee &= \dim_k (M^\vee/m^2M^\vee) = \dim_k (0 : m^2)_M = \dim_\mathbb{k} \text{ of common kernel of } x \\
\dim_k (M^\vee/m^2M^\vee) &= \dim_k (0 : m^2)_M = \dim_k \bigcap_{i,j=1}^{n} \ker(x_i x_j) \\
d - \dim_k ((0 : m)_M^\vee \cap mM) &= \dim_k (mM + (0 : m)_M) = \dim_k \left( \sum_{i=1}^{n} \text{im } x_i + \bigcap_{i=1}^{n} \ker(x_i) \right)
\end{align*}
\]

3.6. Duality and minimal graded resolutions. The duality above is tightly connected with resolutions. Let us recall the graded case. Let $S = \mathbb{k}[y_1, \ldots, y_n]$ be a polynomial ring in $n$ variables graded by $\deg(y_i) = 1$. For a graded $S$-module $N$ and $j \in \mathbb{Z}$ we denote by $N(j)$ the module $N$ with grading shifted down by $j$. Explicitly, this means that $N(j)_i = N_{i+j}$ for every $i \in \mathbb{Z}$. For example, $S(j)$ is a free module generated by an element of degree $-j$. Let $M$ be a finite degree graded $S$-module. By Hilbert’s Syzygy Theorem and Auslander-Buchsbaum formula [Eis95, Chapter 19] the module $M$ has a unique up to isomorphism minimal graded free resolution which has length $n$:

\[
0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \leftarrow 0
\]

Each $F_i$ is a finitely generated free $S$-module which is graded. We write $F_i = \bigoplus_{j \in \mathbb{Z}} S^{\beta_{ij}}(-j)$ which explicitly means that $F_i$ has $\beta_{ij}$ generators of degree $j$. A subset of elements of $F_i$ is
minimal if its image in $F_i/(y_1, \ldots, y_n)F_i$ is linearly independent. In particular, for $F_0$ we speak of minimal generators of $M$, for $F_1$ about minimal relations and for $F_2$ about minimal syzygies of $M$.

**Example 3.20.** The module $F = S e_1 \oplus S e_2 \oplus S e_3$ graded by setting $\text{deg}(e_1) = 1$, $\text{deg}(e_2) = \text{deg}(e_3) = 3$ is denoted by $F = S(-1) \oplus S^{\oplus 2}(-3)$.

**Example 3.21.** For $n = 2$ the module $M = S/(y_1^2, y_1 y_2, y_2^2)$ has a minimal graded resolution that has length exactly two:

\[
\begin{align*}
0 & \leftarrow M \leftarrow S^{\oplus 1} \left[ \begin{array}{ccc} y_2 & 0 \\ -y_1 & y_2 \\ 0 & -y_1 \end{array} \right] S^{\oplus 3}(-2) \leftarrow S^{\oplus 2}(-3) \leftarrow 0.
\end{align*}
\]

A beautiful feature is that the resolution of the dual module $M^\vee$ is dual to the resolution of $M$. This is known to experts, but we provide a proof that does not employ the full machinery.

**Lemma 3.22** ([BH93, Ex 4.4.20]). Let $R$ be a graded ring and $0 \neq y \in R$ a nonzerodivisor which is homogeneous of degree $r$. Let $M$ and $N$ be graded $R$-modules such that the multiplication by $y$ is injective on $N$ and $yM = 0$. Then $\text{Hom}_R(M, N) = 0$ and for every $i \geq 0$ we have an isomorphism of graded $R$-modules

\[\text{Ext}^{i+1}_R(M, N)(−r) \simeq \text{Ext}^i_{R/(y)}(M, N/yN).\]

**Sketch of proof.** For every $\varphi \in \text{Hom}_R(M, N)$ we have $0 = \varphi(yM) = y\varphi(M)$ so $\varphi(M) = 0$ and $\varphi = 0$. For the Ext-part, repeat the proof of [BH93, Lemma 3.1.16] in the graded setting. □

**Theorem 3.23** (duality). Let $M$ be a graded $S$-module of finite degree and $M^\vee$ be a dual module.

Let $F_\bullet$ be minimal graded free resolution of $M$. Then $F_\bullet$ has length exactly $n = \text{dim}(S)$. The complex $\text{Hom}(F_\bullet, S)$ is a minimal graded free resolution of $M^\vee(n)$.

This is well known to experts, but we provide a proof that does not employ the full machinery of canonical modules.

**Proof.** Since $M$ is graded $S$-module of finite degree, there exists $N$ large enough such that $y_1^i M = 0$ for $i = 1, \ldots, n$. By Lemma 3.22 for every $0 \leq i \leq n$ we have $\text{Ext}^i_S(M, S)(−iN) = \text{Hom}_S/(y_1^N, \ldots, y_n^N)(M, S/(y_1^N, \ldots, y_n^N))$. For $i < n$ the same Lemma applied to $y_i^N$ gives that the right hand side is zero. The above Ext groups are homology of the complex $\text{Hom}(F_\bullet, S)$, so this complex is exact except for the $n$-th position: it is a resolution of $\text{Ext}^n_S(M, S) \simeq \text{Hom}_S/(y_1^N, \ldots, y_n^N)(M, S/(y_1^N, \ldots, y_n^N)(nN))$. Let $R = S/(y_1^N, y_n^N)$, let $d = n(N - 1)$ and let $\pi: R \rightarrow k(-d)$ be a $k$-linear projection onto the top degree part: it maps the monomial $\prod_{i=1}^n y_i^{-N}$ to 1 and all other monomials to zero. For any finitely generated graded $R$-module $P$ we obtain a graded $k$-linear map $\Phi_P: \text{Hom}_R(P, R) \rightarrow \text{Hom}_k(P, k(-d)) = P^\vee(-d)$ defined by $\Phi_P(\varphi) = \pi \circ \varphi$. The map $\Phi_P$ is a homomorphism of $R$-modules, because for every $\varphi: P \rightarrow R$ and $r \in R, p \in P$ we have

\[(\Phi_P(r \varphi))(p) = \pi(r \varphi(p)) = \pi(\varphi(rp)) = (\pi \circ \varphi)(rp) = (r(\pi \circ \varphi))(p) = (r \Phi_P(\varphi))(p).\]

We claim that $\Phi_P$ is an isomorphism for every $P$ as above. Taking the claim for granted, we take $P = M$ and obtain an isomorphism of graded $R$-modules

\[\text{Hom}_R(M, R)(nN) \rightarrow M^\vee(−d + nN) = M^\vee(n).\]
Together with the isomorphism $\Ext^n_S(M, S) \simeq \Hom_R(M, R)(nN)$ this concludes the proof of theorem; it remains to prove the claim. Fix any $P$ and its graded presentation $G_2 \to G_1 \to P \to 0$ where $G_i$ are finitely generated graded free $R$-modules. We obtain a commutative diagram of graded $R$-modules

$$
\begin{array}{cccc}
0 & \to & \Hom_R(P, R) & \to \Hom_R(G_1, R) & \to \Hom_R(G_2, R) \\
\downarrow \Phi_P & & \downarrow \Phi_{G_1} & & \downarrow \Phi_{G_2} \\
0 & \to & P^\vee(-d) & \to G_1^\vee(-d) & \to G_2^\vee(-d),
\end{array}
$$

where $\Phi_P$ is isomorphism whenever $\Phi_{G_1}$ and $\Phi_{G_2}$ are. It is thus enough to prove that $\Phi_P$ is an isomorphism for $P$ free. By splitting direct sums, we reduce to the case $P = R(-e)$. By shifting degrees we reduce to the case $P = R$. This case asks whether $\Phi_R: \Hom_R(R, R) \to R^\vee(-d)$ is an isomorphism. Both sides are vector spaces of the same dimension, so it is enough to prove that $\Phi_R$ is injective. Take a nonzero homomorphism $\varphi$. Then $\ker(\varphi)$ is a nonzero homogeneous ideal of $R$. We see directly that every such ideal contains the top degree part $R_d$ of $R$, so $\varphi$ maps some element of $R$ into $R_d$ hence its composition with $\pi$ is nonzero. This concludes the proof of the claim.

\begin{example}
In the setting of Example 3.21 the dual of the resolution of $M$ is

$$
0 \leftarrow M^\vee(2) \leftarrow S^\oplus 3 \leftarrow S^\oplus 2(3) \leftarrow S^\oplus 2(2) \leftarrow S^\oplus 1 \leftarrow 0.
$$

The surjection $S^\oplus 2(3) \to M^\vee(2)$ says that $M^\vee = \Hom_k(k[y_1, y_2]/(y_1y_2)^2, k)$ is generated by two elements of degree minus one: the projections onto $k y_i$ for $i = 1, 2$.
\end{example}

4. Structural results on $C_n(M_d)$ and $\Quot^d_d$

In this section we present general tools for investigation of $C_n(M_d)$. These will be heavily employed in the proofs of our main results, but are of general interest. In contact with Section 3 the results presented below are new (to our best knowledge).

4.1. Apolarity for modules. Writing down equations for a finite degree module $F/K$ can be cumbersome, since $K$ typically has many generators. This can be resolved by introducing apolarity for modules. Apolarity for algebras is already an extremely important tool in the classification of algebras [IK99, Jel17], apolarity for modules generalizes it in a natural way. Recall that an $S$-module has finite degree if it is finite dimensional as a $k$-vector space. We say that an $S$-submodule $K \subset F$ has cofinite degree (shortly: is cofinite) if the module $F/K$ has finite degree.

Let $F$ be a free $S$-module and $\Hom_k(F, k)$ be the space of functionals. To a cofinite submodule $K \subset F$ we associate a subspace $K^\perp \subset \Hom_k(F, k)$ defined by

$$
K^\perp := \{ \varphi: F \to k \mid \varphi(K) = \{0\} \}.
$$

This subspace is isomorphic to $\Hom_k(F/K, k)$ and thus finite dimensional. The aim of this subsection is to explore the relation between $K \subset F$ and $K^\perp$. Recall from Section 3.5 that $\Hom_k(F, k)$ is an $S$-module via contraction action: for $s \in S$ and $\varphi \in \Hom_k(F, k)$ we have $(s \cdot \varphi)(f) = \varphi(sf)$ for every $f \in F$. The following observation is fundamental:
Lemma 4.1. For every $K \subset F$ the subspace $K^\perp$ is an $S$-submodule.

Proof. Take $\varphi \in K^\perp$, so that $\varphi(K) = \{0\}$. For every $s \in S$ we have

$$(s \cdot \varphi)(K) = \varphi(s \cdot K) \subset \varphi(K) = \{0\}.$$  

By Lemma 4.1, the map $K \mapsto K^\perp$ sends cofinite $S$-submodules of $F$ to finite degree $S$-submodules of $\text{Hom}_k(F, k)$. Now we construct a map in the opposite direction. To a finite degree $S$-submodule $M \subset \text{Hom}_k(F, k)$, we associate an $S$-submodule

$$M^\perp := \{f \in F \mid \forall \varphi \in M : \varphi(f) = 0 \} \subset F.$$ 

Applying $\text{Hom}_k(-, k)$ to the inclusion of $M$ into $\text{Hom}_k(F, k)$ we get a surjection

$$\text{Hom}_k(\text{Hom}_k(F, k), k) \to \text{Hom}_k(M, k).$$

Since $M$ has finite degree, the composed map $F \to \text{Hom}_k(M, k)$ is also surjective. The module $M^\perp$ is by definition its kernel, so we get a bijective linear map $F/M^\perp \to \text{Hom}_k(M, k)$.

Lemma 4.2. The map $F/M^\perp \to \text{Hom}_k(M, k)$ is an isomorphism of $S$-modules, where $S$ acts on $\text{Hom}_k(M, k)$ by contraction.

Proof. We already know that the map is a bijection, it remains to check that it is $S$-linear. Unraveling definitions we see that the image of $f \in F$ in $\text{Hom}_k(M, k)$ is an element $\mu_f$ such that $\mu_f(m) = m(f)$ for all $m \in M \subset \text{Hom}_k(F, k)$. Therefore

$$\mu_{sf}(m) = m(sf) = (s \cdot m)(f) = \mu_f(s \cdot m) = (s \cdot \mu_f)(m),$$

so the map $f \mapsto \mu_f$ is $S$-linear as claimed.

Considering both maps $K \mapsto K^\perp$ and $M \mapsto M^\perp$ we obtain the following bijection which is an embedded form of the double dual map.

Proposition 4.3 (Apolarity for modules). The maps $K \mapsto K^\perp, M \mapsto M^\perp$ give a bijection between cofinite submodules of $F$ and finite degree submodules of $\text{Hom}_k(F, k)$.

Proof. It remains to check that the natural maps $M \to (M^\perp)^\perp$ and $K \to (K^\perp)^\perp$ are identities and we leave it to the reader. 

The above may rightfully look like too abstract to apply, since the space $\text{Hom}_k(F, k)$ is huge, of uncountable dimension over $k$. We shrink it a bit now and give a down-to-earth presentation. Let $F_i \subset F$ be the linear subspace of elements of degree $i$ where $F$ is generated by elements of degree zero. Define

$$F^* := \bigoplus_i \text{Hom}_k(F_i, k) \subset \text{Hom}_k(F, k).$$

The space $F^*$ can be thought of as a restricted dual of $F$. Note that in the current article we use $(-)^\vee$ to denote the dual space. Alternatively, we can view $F^*$ as the space of all those functionals on $F$ that vanish on $(y_1, \ldots, y_n)^D F$ for some $D$. Fix an $S$-basis $e_1, \ldots, e_r$ of $F$, so that $F = \bigoplus_{j=1}^r \mathbb{k}[y_1, \ldots, y_n]e_j$. Each module $F_i$ has a “monomial” basis consisting of elements $y_1^{\alpha_1} \cdots y_n^{\alpha_n} e_j$ with $\sum \alpha_i = i$ and the dual basis on $F^*$ identifies it with the linear space

$$\bigoplus_{j=1}^r \mathbb{k}[z_1, \ldots, z_n]e_j^*. $$
The $S$-module structure coming from contraction becomes “coefficientless derivation”: the element $y_i$ acts of $F^*$ by
\[
y_i \circ (z_1^{a_1} \ldots z_n^{a_n})e_j^* = \begin{cases} 
0 & \text{if } a_i = 0 \\
(z_1^{a_1} \ldots z_i^{a_i-1} \cdot z_i^{a_i} \cdot z_{i+1}^{a_{i+1}} \ldots z_n^{a_n})e_j^* & \text{otherwise.}
\end{cases}
\]

For example, we have $y_2 \circ (z_1z_2e_1^* + z_3z_1e_2^* + z_2^2e_3^*) = z_1e_1^* + z_2e_3^*$. The grading on $F$ naturally induces a grading on $F^*$, where $\deg(z_i) = -1$. By slight abuse of notation we forget about the minus and consider $F^*$ as positively graded, i.e., with $\deg(1) = 1$. For example, $\deg(z_1^2 e_1 + z_2 z_3 e_2) = 2$. In this setup we obtain a restricted version of inverse systems, more resembling the classical one.

**Proposition 4.4** (Apolarity for modules, local). The maps $K \mapsto K^\perp, M \mapsto M^\perp$ give a bijection between cofinite $S$-submodules $K \subset F$ such that $F/K$ is supported only at the origin and finite degree $S$-submodules of $F^*$.

**Proof.** We will prove that the maps from Proposition 4.3 restrict to the given classes of modules. For any element $f \in F^*$ there is a $D$ such that $(y_1, \ldots, y_n)^D$ annihilates $f$. Therefore, the same holds for finitely many elements of $F^*$ and this shows that for every finitely generated submodule $M \subset F^*$, the module $F/M^\perp$ is annihilated by $(y_1, \ldots, y_n)^D$ so it is supported only at the origin. Conversely, take a cofinite module $K \subset F$ such that $F/K$ is supported only at the origin. The support of $F/K$ is equal to the set of prime ideals containing $\Ann(F/K)$. Since there is only one such, the radical of $\Ann(F/K)$ is equal to $(y_1, \ldots, y_n)$. Since $S$ is Noetherian, some power of $(y_1, \ldots, y_n)$ is contained in $\Ann(F/K)$, so we have $(y_1, \ldots, y_n)^DF \subset K$ for large enough $D$ so that $K^\perp$ is contained in $F^*_{\leq D-1}$.

**Definition 4.5.** For elements $\sigma_1, \ldots, \sigma_r \in F^*$ generating an $S$-submodule $M \subset F^*$ the apolar module of $\sigma_1, \ldots, \sigma_r$ is $F/M^\perp$. Conversely, for an $S$-module $F/K$ supported only at the origin any set of generators of the $S$-module $K^\perp \subset F^*$ is called a set of dual generators of $F/K$.

We now give some examples for later use. We will freely use the linear algebra from §3.5 coupled with Lemma 4.2 to compute some invariants.

**Example 4.6.** Let $S$ be any polynomial ring, $F = Se_1$ be a rank one free module and $F^* = Se_1^*$. Take any elements $\sigma_1, \ldots, \sigma_r \in F^*$ and the module $M \subset F^*$ generated by them. Then $F/M^\perp$ has the form $Se_1/Ie_1$ for an ideal $I \subset S$ and in fact $I = \Ann(\sigma_1, \ldots, \sigma_r)$ is the usual apolar ideal of $\sigma_1, \ldots, \sigma_r$ as in [IK99] for the contraction action. This shows how the apolarity for modules generalizes the one for algebras.

**Example 4.7.** Fix any $a, b$ such that $a \leq \binom{b+1}{2}$ and any $n \geq b$. Let $S = k[y_1, \ldots, y_n]$ with $m = (y_1, \ldots, y_n)$ and let $F = \bigoplus S e_i$ be a free module of rank at least one. Let $Q_1 = (\sum_{i=1}^b z_i^2)e_1^*$ and $Q_2, \ldots, Q_a \in k[z_1, \ldots, z_b]e_1^* \subset F^*$ be any quadrics such that $Q_1, \ldots, Q_a$ are linearly independent. The $S$-submodule $M \subset F^*$ generated by quadrics $Q_1, \ldots, Q_a$ is an $(a+b+1)$-dimensional vector space with basis $\{Q_i\}_{i=1, \ldots, a} \cup \{z_j e_1^*\}_{1 \leq j \leq b} \cup \{e_1^*\}$ and $(0 : m)_M$ is one-dimensional. The module $F/M^\perp$ is generated by $e_1$ and isomorphic to $S^{\oplus 1}/I$, where $S/I$ is a graded algebra with Hilbert function $(1, b, a)$.

**Example 4.8.** To give a very concrete example of a non-cyclic module, let $S = k[y_1, \ldots, y_5]$ with $m = (y_1, \ldots, y_5)$, let $F = Se_1 \oplus Se_2$ and $Q = (z_1^2 + z_2^2 + z_3^2)e_1^* + z_4 e_2^* \in F^*$. The $S$-submodule $M \subset F^*$ generated by $Q$ is a 6-dimensional vector space spanned by $Q, y_1 \circ Q = z_1 e_1^*$,
$y_2 \circ Q = z_2 e_1^*, y_3 \circ Q = z_3 e_1^*, y_4 \circ Q = e_2^*$ and $y_2^2 \circ Q = y_2^2 \circ Q = y_2^3 \circ Q = e_1^*$. We see that $(y_1, \ldots, y_5)^3$ annihilates $M$, while $(y_1, \ldots, y_5)^2$ does not. The module $M^\perp \subset F$ is minimally generated by

$$(y_1 e_1)_{j=4,5}, (y_i e_2)_{i=1,2,3,5}, (y_i y_j e_1)_{1 \leq i < j \leq 3}, (y_i y_j (y_j - y_j^2)e_1)_{1 \leq i < j \leq 3}, y_1 e_1 - y_4 e_2.$$  

Using the table from Section 3.5, we compute that $(F/M^\perp)/m(F/M^\perp)$ has dimension $\dim_k (0 : m)_M = \dim_k (e_1^*, e_2^*) = 2$, so the module $F/M^\perp$ is not cyclic.

The following two examples will be useful in Section 6. The exact assumptions are tailored to the needs of that section, so are by no means “natural”.

**Example 4.9.** Fix any $c$ and $a \leq \left(\frac{c+1}{2}\right)$ and additionally fix any $b \geq c$. Let $S$ be a polynomial ring of dimension $n \geq b$ and let $m = (y_1, \ldots, y_n)$. Let $F$ be a free $S$-module of rank $r = \max(a, b - c)$ with basis $e_1, e_2, \ldots, e_r$. Pick a linearly independent quadrics $Q_1, \ldots, Q_a$ in $k[z_1, \ldots, z_c]$ with $Q_i = \sum_{i=1}^{c} z_i^2$ and consider the element $Q := \sum_{i=1}^{a} Q_i e_i^* \in F^*$. Finally consider the submodule $M$ in $F^*$ generated by $Q$ and $\{z_j e_1^*\}_{j=c+1}^b$. Since the quadrics $Q_i$ are linearly independent, we have $(y_1, \ldots, y_c)^2 M = \langle e_1^*, \ldots, e_a^* \rangle = (0 : m)_M$. For further reference, we note that $mM/m^2M$ has dimension $c$ with a basis $\{y_i \circ Q\}_{1 \leq i \leq c}$ while $(0 : m^2)_M/m^2M$ has dimension $b$ with a basis consisting of the $c$ elements above and $\{z_j e_1^*\}_{j=c+1}^b$. Finally, $M$ itself is $(a + b + 1)$-dimensional. Note that in this example we consider $M$ and not $F/M^\perp$.

**Example 4.10.** Let $S = k[y_1, \ldots, y_n]$ be a polynomial ring, $F$ be a free $S$-module and $Q_1, Q_2 \in F^*$ be two homogeneous quadrics such that the submodule $M$ in $F^*$ generated by them has dim $M_1 \leq 3$, where $S_1, M_1$ are the linear parts of $S$ and $M$ respectively. We claim that up to coordinate change we have $y_i M = 0$ for $i \geq 4$. Fix $L(Q_i) = \{\ell \in S_1 | \ell Q_i = 0\} \subset S_1$. The space $L(Q_i)$ is the kernel of a map $S_1 \rightarrow M_1$, so $\dim L(Q_i) \geq n - 3$. Assume $\dim L(Q_1) \leq \dim L(Q_2)$ and consider two cases:

1. $\dim L(Q_1) = n - 3$. Up to coordinate change, we may assume $y_i Q_1 = 0$ for $i \geq 4$ and $y_1 Q_1, y_2 Q_1, y_3 Q_1$ span $M_1$. In particular, the space $M_1$ is annihilated by $y_i$ for every $i \geq 4$. Pick any $i \geq 4$. For every $j$ the element $y_j Q_2$ lies in $M_1$, so is annihilated by $y_i$. In other words, we have $y_j(y_j Q_2) = y_j Q_2 = 0$ for every $j$. But this means that every linear form annihilates $y_j Q_2$ which can happen only if $y_j Q_2 = 0$. This proves that $L(Q_2)$ contains $L(Q_1)$ and concludes this case.

2. $\dim L(Q_1) \geq n - 2$, so that $\dim L(Q_2) \geq n - 2$. Then we have $\dim (L(Q_1) \cap L(Q_2)) \geq n - 4$. If strict equality happens, we are done. If not, up to change of basis we may assume $L(Q_1) = \langle y_3, y_4, \ldots \rangle$ and $L(Q_2) = \langle y_1, y_2, y_5, y_6, \ldots \rangle$. The space $M_1$ is at most three-dimensional and contains $y_1 Q_1, y_2 Q_1, y_3 Q_2, y_4 Q_2$, so there is a linear dependence: some nonzero linear form $\ell$ lies both in the submodule generated by $Q_1$ and by $Q_2$. But then the annihilator of $\ell$ contains $L(Q_1) + L(Q_2) = S_1$, so actually $\ell = 0$, a contradiction.

4.2. **Obstruction theories.** In the following we will use obstruction theories, so we give a brief outline of them (see [FM98, Har10, Ser06] for more complete account). Roughly speaking, obstructions are a useful black box for proving that a given morphism $X \rightarrow Y$ is smooth or étale at a given $k$-point $x$.

Let $(A, m)$ be a noetherian complete local $k$-algebra with residue field $k$, such as $\hat{O}_{X,x}$. By Cohen’s structure theorem we can write $A = \hat{S}/I$ where $\hat{S} = k[[y_1, \ldots, y_d]]$ is a power series ring with maximal ideal $m_\hat{S} = (y_1, \ldots, y_d)$ and $I \subset m_\hat{S}^2$. The prototypical obstruction space for $A$ is $Ob := (I/m_\hat{S}I)^\vee$. We now construct the prototypical obstruction map for $Ob$. Suppose we have
an Artin local \( k \)-algebra \((B, n)\) with residue field \( k \) and a surjection \( B \to B_0 = B/J \) such that \( J \cdot n = 0 \).

**Example 4.11.** We could take \( B = k[\varepsilon]/\varepsilon^3 \) and \( B_0 = k[\varepsilon]/\varepsilon^2 \) or more generally \( B = k[\varepsilon]/\varepsilon^{r+1} \) and \( B_0 = k[\varepsilon]/\varepsilon^r \) for some \( r \geq 2 \).

Suppose further that we have a \( k \)-algebra homomorphism \( \varphi_0 : A \to B_0 \). We want to answer the question:

When does \( \varphi_0 \) lift to \( \varphi : A \to B \)?

We can always lift (not uniquely) the morphism \( k[[y_1, \ldots, y_d]] \to B_0 \) to \( B \): we first lift it to a homomorphism \( k[[y_1, \ldots, y_d]] \to B \) and this homomorphism then extends to \( k[[y_1, \ldots, y_d]] \to B \), because some power of the maximal ideal \( n \) of the Artin local \( k \)-algebra \( B \) is zero. So we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \longrightarrow & \hat{S} & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow \psi | I & & \downarrow \psi & & & \downarrow \varphi_0 & & \downarrow 0 \\
0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow & 0.
\end{array}
\]

Clearly, if \( \psi|_I = 0 \) then we obtain a map \( A \to B \). Moreover, \( \psi|_I(\mathfrak{m}_S I) \subset n \cdot J = 0 \), so actually \( \psi|_I = 0 \) if and only if the induced map \( I/\mathfrak{m}_S I \to J \) is zero. Hence, \( \psi|_I = 0 \) if and only if the induced \( k \)-linear map \( \text{ob}_{B \to B_0, \varphi_0} : J' \to \text{Ob} \) is zero. The map \( \text{ob}_{B \to B_0, \varphi_0} \) is called the obstruction map. It does not depend on the choice of lifting \( \psi \), thanks to the fact that \( J \cdot n = 0 \) and \( I \subset \mathfrak{m}_S^2 \).

An obstruction theory is an abstracted version of \( \text{Ob} \).

**Definition 4.12.** An obstruction theory for a complete local \( k \)-algebra \((A, \mathfrak{m})\) with residue field \( k \) is a \( k \)-vector space \( O \) and for every exact sequence \( 0 \to J \to B \to B_0 \to 0 \) as above and a \( k \)-algebra homomorphism \( \varphi_0 : A \to B_0 \), a \( k \)-linear map \( \text{ob}_{B \to B_0, \varphi_0} : J' \to O \) such that \( \text{ob}_{B \to B_0, \varphi_0} \) is zero if and only if \( \varphi_0 \) lifts to \( \varphi : A \to B \). Moreover, the maps \( \text{ob} \) are required to be appropriately functorial: if \( 0 \to J' \to B' \to B_0' \to 0 \) is another sequence as above and \( \rho, \rho_0 \) are \( k \)-algebra homomorphisms which fit into a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B_0 & \longrightarrow & 0 \\
\downarrow \rho |_J & & \downarrow \rho & & & \downarrow \rho_0 & & \downarrow 0 \\
0 & \longrightarrow & J' & \longrightarrow & B' & \longrightarrow & B_0' & \longrightarrow & 0.
\end{array}
\]

then \( \text{ob}_{B' \to B_0', \rho_0 \circ \varphi_0} = (\text{ob}_{B \to B_0, \varphi_0}) \circ (\rho | J)' \) as maps \( (J')' \to O \), see [FM98, Def 1.3, Def 3.1]. The space \( O \) is called the obstruction space. If \( X \) is a scheme, then an obstruction theory of a point \( x \in X \) is an obstruction theory for the complete local ring \( \hat{O}_{X,x} \).

Let \((A', \mathfrak{m}')\) be another complete local \( k \)-algebra with residue field \( k \) and with a \( k \)-algebra homomorphism \( f : A' \to A \) and let \( O_{A, A'} \) be obstruction spaces for some obstruction theories for \( A \) and \( A' \). A map (or morphism) of obstruction theories is a linear map \( O_f : O_A \to O_{A'} \) such that for every exact sequence \( 0 \to J \to B \to B_0 \to 0 \) as above and \( \varphi_0 : A \to B_0 \) the obstruction map for lifting \( \varphi_0 \circ f : A' \to B_0 \) to \( A' \to B \) is equal to \( O_f \circ \text{ob}_{B \to B_0, \varphi_0} \).

We stress that for a given \((A, \mathfrak{m})\) many obstruction theories with different obstruction spaces exist. For example, given one such theory with obstruction space \( O \), we can choose any space \( O' \) with subspace \( O \hookrightarrow O' \) and obtain a new theory with obstruction space \( O' \). Geometrically, if \( A = \hat{O}_{X,x} \) for a \( k \)-point \( x \) of a scheme \( X \), then a morphism \( \varphi_0 \) corresponds exactly to \( \text{Spec}(B_0) \to X \) and \( \varphi \) to \( \text{Spec}(B) \to X \). So the question of lifting becomes the question of lifting a given map.
Example 4.13. In the example above, we lift a map from \( \text{Spec}(k[\varepsilon]/\varepsilon^2) \to X \) to \( \text{Spec}(k[\varepsilon]/\varepsilon^3) \to X \). More generally, we could try to lift this map to \( \text{Spec}(k[\varepsilon]/\varepsilon^n) \) for \( n = 3, 4, \ldots \). If all those lifts exist, they glue to a map \( \text{Spec}(k[[t]]) \to X \). The closure of the image of this map is either just \( x \) or a curve passing through \( x \).

Example 4.14. If \( A \) has an obstruction theory with obstruction space zero, then the lifting automatically exists for all homomorphisms. In this case \( A \) is actually isomorphic to \( \check{S} \), hence, in the geometric sense, the point \( x \in X \) is smooth. Indeed, if \( \check{S} \to A \) is not an isomorphism then \( \check{S}/m^r \to A/m^r \) is not an isomorphism for some \( r \); take the smallest such \( r \). Taking \( B_0 = A/m^r \), \( \varphi_0 : A \to A/m^r \) the canonical map and \( B = \check{S}/m^r \) we obtain a lifting \( \varphi : A \to B \). From the definition of lifting it follows that \( \varphi \) maps no nonzero linear form from \( m \) to \( m^r / m^r \), and consequently it maps no nonzero form of degree \( k \) from \( m^k \) to \( m^{k+1} / m^r \) for \( k < r \). However, this is a contradiction with \( I \not\subseteq m^r \).

We will be interested in the following facts about obstruction theories:

1. By [FGI+05, Theorem 6.4.9] a point \([F/K] \in \text{Quot}^d\) has an obstruction theory with obstruction space \( \text{Ext}^1(K, F/K) \). For comparison, note that the tangent space at \([F/K] \) is \( \text{Hom}(K, F/K) \), by Lemma 3.9.
2. Suppose that \( x \in X \) has obstruction theory \( Ob_x \), \( y \in Y \) has obstruction theory \( Ob_y \) and \( f : X \to Y \) is a map of schemes with \( f(x) = y \) that induces a map of obstruction theories \( Ob_x \to Ob_y \). If this map is injective and the tangent map \( df : T_{X,x} \to T_{Y,y} \) is surjective then \( f \) is smooth at \( x \). This is called the Fundamental Theorem of obstruction calculus, see [FM98, Lemma 6.1]. If moreover \( df \) is bijective, then \( f \) is étale at \( x \).

This generalizes Example 4.14; indeed this example corresponds to the case \( Y = \text{Spec}(k), Ob_x = 0 \).

Finally, we discuss primary obstructions, which will be used in the proof of generic nonreducedness. For technical reasons we assume \( \text{char} k \neq 2 \). Let \( A \) and \( \check{S} \) be as before. Since \( I \) is contained in \( m^2 \), the surjection \( \check{S}/m^2 \to A/m^2 \) is an isomorphism and in particular it gives a homomorphism \( A/m^2 \to \check{S}/m^2 \). Let \((O, ob)\) be any obstruction theory for \((A, m)\). Taking \( B = \check{S}/m^2, B_0 = \check{S}/m^2, J = m^2 / \check{S} \) and \( \varphi_0 : A \to A/m^2 \simeq B_0 \) the canonical projection in the definition of obstruction theory above, we obtain a map \( ob_0 : (m^2 / m^3) \to O \). Since \( \check{S} \) is a power series ring, the domain of this map is dual to \( \text{Sym}_2(m^2 / m^3) \) and we obtain the primary obstruction map

\[
ob_0 : (\text{Sym}_2(m^2 / m^3))^\vee \to O.
\]

Dualizing, we get a map \( ob_0^\vee : O^\vee \to \text{Sym}_2(m^2 / m^3) = m^3 / m^2 \). Consider now a slightly more general situation. Fix a linear subspace \( K \subset m^2 \) and take \( B_K = \check{S} / (m^2 + K), B_0 = \check{S}/m^2 \).

Again using the definition, we get an obstruction map

\[
ob_K : \left( \frac{m^2}{m^2 + K} \right)^\vee \to O.
\]

Moreover, by the functoriality from the Definition 4.12, the map \( B \to B_K \) induces a factorization \( ob_K = ob_0 \circ f \) where \( f \) is the dual to the canonical surjection \( m^2 / m^3 \to m^2 / (m^2 + K) \). Dualizing similarly as above we get a map \( ob_K^\vee : O^\vee \to m^2 / (m^2 + K) \) which is the composition of \( ob_0^\vee \) with the canonical projection. Now we are ready to observe that the following are equivalent:

1. the obstruction \( ob_K \) is zero,
We now use this observation to compute the image of dual to the primary obstruction map. We have $A/m^3 \simeq \hat{S}/(m_S^3 + I)$. We show that the canonical projection $\varphi_0: A \to B_0$ lifts to a $k$-algebra homomorphism $\varphi_K: A \to B_K$ if and only if $I + m^3_S \subset K + m^3_S$. In one direction this is clear: if $I + m^3_S \subset K + m^3_S$, we may take for $\varphi_K$ the composition of canonical projections $A \to A/m^3 \to B_K$.

Conversely, assume that $\varphi_K: A \to B_K$ is a $k$-algebra homomorphism which is a lifting of $\varphi_0$. Then $\varphi_K(m^3)$ is equal to zero, so $\varphi_K$ factors through $\overline{\varphi_K}: A/m^3 \simeq \hat{S}/(m_S^3 + I) \to B_K = \hat{S}/(m_S^3 + K)$. If $\ell \in \hat{S}$ is any linear form, then it is clear that $\overline{\varphi_K}(\ell)$ is of the form $\ell + q$ for some $q \in m^3_S$. Since $\overline{\varphi_K}$ is a $k$-algebra homomorphism, it follows that $I + m^3_S \subset K + m^3_S$. Finally, the map $ob_K$ is zero if and only if the map $\varphi_0$ lifts to a $k$-algebra homomorphism $A \to B_K$. The above observations imply that $im ob_0 \subset (K + m^3_S)/m^3_S$ if and only if $I + m^3_S \subset K + m^3_S$, so $im ob_0 = (I + m^3_S)/m^3_S$. Intuitively, the above shows that $im ob_0$ recovers the quadratic part of $I$.

To make this useful, we need to know how to find the primary obstruction map explicitly. Recall that $m_S^3/m_S^2 \simeq m/m^2$ is called the cotangent space of $(A,m)$ and its dual $(m_S/m^2_S)^{\vee} \simeq (m/m^2)^{\vee}$ is called the tangent space of $(A,m)$. In the case $A = \hat{O}_X$ these spaces are the cotangent and tangent space to $X$ at $x$, respectively. Assume the characteristic of $k$ is not two.

There is a $GL(m_S^3/m^2_S)$-equivariant isomorphism

\[ (4.15) \quad \Sym_2((m_S/m_S^2)^{\vee}) \simeq (\Sym_2(m_S/m_S^2))^{\vee} \]

which is unique up to multiplication by a scalar. We fix one such isomorphism: it maps the product $(y_i^2) \cdot (y_j^2)$ of the elements of the dual basis (which is an element of the left hand side) to the functional $(y_i y_j)^{*}$ on the right hand side for $i \neq j$ and it maps $(y_i^2)^2$ to $2(y_i^2)$. The reader might wonder why the constant 2 is necessary; the reason is essentially the same as for $\frac{1}{2}$ constants that appear while writing a quadratic form as a symmetric matrix. Using (4.15) we view $ob_0$ as a map from $\Sym_2((m_S/m_S^2)^{\vee})$. Consider $\varphi_1, \varphi_2 \in (m_S/m_S^2)^{\vee}$. For $a \in A$ let $\bar{a} \in A$ be the image of $a$ under the composition $A \to A/m = k \hookrightarrow A$. The elements $\varphi_i$, $i = 1, 2$ of the tangent space induce $k$-algebra homomorphisms $f_i: A/m^2 \to k[\varepsilon_i, \varepsilon_i^{-1}]$ given by $f_i(a) = \bar{a} + \varepsilon_i \varphi_i(a - \bar{a})$ which jointly give a $k$-algebra homomorphism $f_{12}: A/m^2 \to k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1, \varepsilon_2)$ defined by $f_{12}(a) = \bar{a} + \sum_{i=1}^2 \varepsilon_i \varphi_i(a - \bar{a})$. We have a commutative diagram

\[ (4.16) \]

A diagram chase shows that the image of $(\varepsilon_1 \varepsilon_2)^{*}$ in $(m_S/m_S^2)^{\vee}$ is equal, under the isomorphism (4.15) to $\varphi_1 \varphi_2$, this is proven most conveniently by changing coordinates on $\hat{S}$ so that $\varphi_i = y_i^2$ for $i = 1, 2$ or $\varphi_1 = \varphi_2 = y_1^2$. Using the functoriality from Definition 4.12 we derive that the image of $(\varepsilon_1 \varepsilon_2)^{*}$ under the composition $(\varepsilon_1 \varepsilon_2)^{\vee} \to (m_S/m_S^2)^{\vee} \to O$ is the value of the primary obstruction on $\varphi_1 \varphi_2 \in \Sym_2((m_S/m_S^2)^{\vee})$.

By Lemma 3.9 for a point $[F/K]$ on the Quot scheme, the tangent space is given by $\Hom(K, F/K)$. Fix a free resolution $F_\bullet$ of $F/K$, beginning with $F_0 = F$. For every $\varphi: K \to F/K$ in this space
we can lift it to a chain complex map

\[
\begin{array}{cccccc}
0 & \to & K & \to & F_1 & \to & F_2 & \to & \cdots \\
& & d_0 & & d_1 & & d_2 & & \\
\varphi & & s_1(\varphi) & & s_2(\varphi) & & s_3(\varphi) & & \\
0 & \to & F/K & \to & F_0 & \to & F_1 & \to & \cdots \\
& & \pi & & d_0 & & d_1 & & 
\end{array}
\]

(4.17)

**Theorem 4.18.** Consider a point \([F/K]\) on the Quot scheme and the associated obstruction theory with obstruction space \(\text{Ext}^1(K,F/K)\). Then for \(\varphi_1, \varphi_2 \in \text{Hom}(K,F/K)\) the primary obstruction map sends \(\varphi_1 \varphi_2 \in \text{Sym}_2(\text{Hom}(K,F/K))\) to

\[
\pi \circ (s_1(\varphi_1) \circ s_2(\varphi_2) + s_1(\varphi_2) \circ s_2(\varphi_1))
\]

in \(\text{Ext}^1(K,F/K)\).

We remark that

\[
\pi \circ s_1(\varphi_1) \circ s_2(\varphi_2) \circ d_2 = \varphi_1 \circ d_0 \circ d_1 \circ s_3(\varphi_2) = \varphi_1 \circ 0 \circ s_3(\varphi_2) = 0,
\]

so indeed the expression (4.19) is a cycle in \(\text{Hom}(F_\bullet, F/K)\), hence it makes sense to take its homology class which is by definition the \(\text{Ext}^1(K,F/K)\) group. Below, we give two proofs. The first one is completely abstract and in essence tells that the result is known as a consequence of much deeper insights. The second one is down to earth, but certain details are left to the reader. Regrettfully, we do not know a reference which is both complete and accessible.

**First proof of Theorem 4.18.** Let \(M = F/K\) and let \((A,m)\) be the complete local ring of \([F/K]\) in the Quot scheme. Take \(\varphi_1, \varphi_2 \in \text{Hom}(K,M)\). As discussed above the image of \(\varphi_1 \varphi_2\) in the primary obstruction is the image of \((\varepsilon_1 \varepsilon_2)^*\) by the obstruction map associated to the bottom row of Diagram 4.16. Since \(F\) is a free \(S\)-module, the map \(\text{Ext}^1(K,M) \to \text{Ext}^2(M,M)\) is an isomorphism, thus we can compute the obstruction after forgetting that \(M\) is a quotient of \(F\). By the main result of [FIM12] the infinitesimal deformation functor of \(M\) is isomorphic to the Maurer-Cartan functor for the differential graded Lie algebra \(\text{End}(F_\bullet)\). By [Man11, Section 5] or [Man99, Example 2.16] the primary obstruction for a differential graded Lie algebra \(L\) is equal to the Lie bracket in its cohomology algebra. The cohomology of \(\text{End}(F_\bullet)\) is by definition \(\text{Ext}(M,M)\) and the bracket is just the bracket in the Yoneda pairing, i.e., for classes \(\varphi_1, \varphi_2 \in \text{Ext}^1(M,M)\) it returns the sum of Yoneda products \(\varphi_1 \varphi_2 + \varphi_2 \varphi_1\) in \(\text{Ext}^2(M,M)\), where the plus sign is due to the fact that we take brackets of odd degree elements, see [Man11, Definition 1.1]. Now, the expression in the theorem is exactly this sum under the isomorphism \(\text{Ext}^2(M,M) \simeq \text{Ext}^1(K,M)\). \(\Box\)

**Sketch of second proof of Theorem 4.18.** Let \(q \in \text{Sym}_2(\text{Hom}(K,F/K))\) be written as \(q = \sum_{i \leq j} \lambda_{ij} \varphi_i \varphi_j\) where \(\varphi_1, \ldots, \varphi_d\) is a basis of \(\text{Hom}(K,F/K)\). Let \(\varepsilon_1, \ldots, \varepsilon_d \in \hat{S}/m_S^2\) be a dual basis. We lift it to a generating set of \(m_S^3\). Using (4.15) we can interpret \(q\) as a functional \(\breve{q} : m_S^2/m_S^3 \to \mathbb{k}\) and build the algebra

\[
B = \frac{\hat{S}}{m_S^3 + \ker(\breve{q})}.
\]

Let \(B_0 = \hat{S}/m_S^2 = A/m^2\) and let \(q^* \in m_S^2/m_S^3\) satisfy \(\breve{q}(q^*) = 1\). Using the explicit formula below (4.15), we check that in the algebra \(\hat{B}\) we have \(\varepsilon_i \varepsilon_j = \lambda_{ij} q^*\) for \(i \neq j\) while \(\varepsilon_i^2 = 2 \lambda_{ii} q^*\). As discussed above, there is a natural morphism \(f_{\text{tan}} : \text{Spec}(B_0) \to \text{Quot}^d\) which by the universal
property of Quot, see Example A.3, corresponds to a module $M := (F \otimes_k B_0)/K$. In this version of the proof we will only show that if the morphism $f_{\text{tan}}$ extends to a morphism $\text{Spec}(B) \to \text{Quot}^d$, then the element of $\text{Ext}^1(K, M)$ defined by the formula $\sum_{i \leq j} \lambda_{ij} \pi \circ (s_1(\varphi_i) \circ s_2(\varphi_j) + s_1(\varphi_j) \circ s_2(\varphi_i))$ from the Theorem is zero. (This is enough for the application in the current article.)

For better clarity, we write subscript $B$ instead of $\otimes_k B$; similarly for $B_0$. Recall that $F_1$ is the free module with a canonical surjection $d_0: F_1 \to K$. By Nakayama lemma any lift of $d_0$ to a $S_{B_0}$-module homomorphism $(F_1)_{B_0} \to K$ is surjective. By the description of the tangent space to Quot scheme (see references in Lemma 3.9) one such lift $d_0' : (F_1)_{B_0} \to K$ is

$$d_0'(m) = d_0(m) + \sum_{i=1}^d \varepsilon_i s_1(\varphi_i)(m).$$

Suppose that $f_{\text{tan}}$ extends to a map $\text{Spec}(B) \to \text{Quot}^d$. By the definition of Quot, Example A.3, it follows that there exists an $S$-linear map $h: F_1 \to F$ such that if we define a map $\tilde{d}_0: (F_1)_B \to F_B$ by $\tilde{d}_0(m) = d_0(m) + q^* h(m)$, then $F_B/\text{im}(\tilde{d}_0)$ is flat over $B$. Flatness implies that $\ker(\tilde{d}_0)$ reduces to $\ker(d_0)$ modulo $(\varepsilon_1, \ldots, \varepsilon_d)$. Let $d_1: (F_2)_B \to \ker(\tilde{d}_0)$ be any $S_B$-module homomorphism that reduces to $d_1$ modulo $(\varepsilon_1, \ldots, \varepsilon_d)$. Nakayama lemma implies that $d_1$ is surjective. We obtain an exact sequence of $S_B$-modules

$$0 \leftarrow F_B/\text{im}(\tilde{d}_0) \leftarrow \frac{\pi_0}{F_B} \leftarrow d_0 \leftarrow (F_1)_B \leftarrow d_1 : (F_2)_B$$

which $(\mod (\varepsilon_1, \ldots, \varepsilon_d))$ gives the beginning of the resolution of $F/K$. Write $\tilde{d}_1|_{F_2} = d_1 + \sum_{i=1}^d s_2 i \varepsilon_i + q^* j$ where $s_2, j: F_2 \to F_1$. The condition $d_0 \circ d_1 = 0$ decomposes in the basis $\varepsilon_1, \ldots, \varepsilon_d, q^*$ of $B$ into equalities

$$d_0 s_{2i} + s_1(\varphi_i) d_1 = 0 \quad \text{for } i = 1, 2, \ldots, d$$

$$d_0 j + \sum_{i < j} \lambda_{ij} (s_1(\varphi_i)s_{2j} + s_1(\varphi_j)s_{2i}) + \sum_i 2 \lambda_{ii} s_1(\varphi_i)s_{2i} + \pi h_1 = 0.$$  

The first one implies that the maps $-s_{2i}: F_2 \to F_1$ are lifts of $s_1(\varphi_i)$, for $i = 1, 2, \ldots, d$, as in (4.17). Since the class in $\text{Ext}^1(K, F/K)$ does not depend on the choice of such lifts, we may take $s_2(\varphi_i) = -s_{2i}$. Then the third equation becomes

$$d_0 j - \sum_{i \leq j} \lambda_{ij} (s_1(\varphi_i)s_{2j} + s_1(\varphi_j)s_{2i}) + \pi h_1 = 0.$$ 

After composition with $\pi$ we obtain

$$-\pi \circ \left( \sum_{i \leq j} \lambda_{ij} (s_1(\varphi_i)s_{2j} + s_1(\varphi_j)s_{2i}) \right) + \pi h_1 = 0,$$

which shows that our class is equal to the boundary $(\pi h) \circ d_1$ and it is zero in $\text{Ext}^1(K, F/K)$. □

4.3. Natural endomorphisms of $C_n(M_d)$. The following lemma is obvious, but we state it as we will frequently use it to assume some open conditions on the matrices.

\textbf{Lemma 4.20.} If $V$ is an irreducible subvariety of $C_n(M_d)$ and $U$ is a nonempty Zariski-open subset of $V$, then $U$ and $V$ belong to the same irreducible component of $C_n(M_d)$.

\textbf{Lemma 4.21.} Let $\varphi: \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map, and let the map $C_n(M_d) \to C_n(M_d)$ be defined by applying $\varphi$ to the $n$-tuples of matrices. By a slight abuse of notation we denote it again by $\varphi$. If $C$ is any irreducible component of $C_n(M_d)$, then $\varphi(C) \subseteq C$. □
Proof. Let $D \geq 1$ be greater than the degree of any coordinate of $\varphi$. The set of all polynomial maps $\mathbb{A}^n \to \mathbb{A}^n$ of degree at most $D$ in each of the variables forms an affine space, say $\mathbb{A}^N$. Since $\mathcal{C}$ and $\mathbb{A}^N$ are irreducible, the image of the map $\mathbb{A}^N \times \mathcal{C} \to C_n(\mathcal{M}_d)$ defined by $(\psi, x_1, \ldots, x_n) \mapsto \psi(x_1, \ldots, x_n)$ is irreducible. This image clearly contains $\mathcal{C}$ because $id$ is among $\psi$. Since $\mathcal{C}$ is a component, the image is the whole $\mathcal{C}$. The lemma now follows. \hfill $\square$

**Corollary 4.22.** Two tuples of commuting matrices that generate the same unital $k$-algebra lie in the same components of $C_n(\mathcal{M}_d)$.

Proof. Fix any component $\mathcal{C}$ of $C_n(\mathcal{M}_d)$ containing $(x_1, \ldots, x_n)$ and let $(y_1, \ldots, y_n) \in C_n(\mathcal{M}_d)$ be an $n$-tuple that generates the same algebra as $(x_1, \ldots, x_n)$. Then there exist polynomial maps $\varphi, \psi : \mathbb{A}^n \to \mathbb{A}^n$ with $(y_1, \ldots, y_n) = \varphi(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n) = \psi(y_1, \ldots, y_n)$. Lemma 4.21 implies that $(y_1, \ldots, y_n) \in \varphi(\mathcal{C}) \subseteq \mathcal{C}$. Symmetrically, if $(y_1, \ldots, y_n)$ belongs to some component $\mathcal{C}'$, then $(x_1, \ldots, x_n) \in \psi(\mathcal{C}') \subseteq \mathcal{C}'$, again by Lemma 4.21. \hfill $\square$

**Corollary 4.23.** Suppose a tuple $(x_1, \ldots, x_n) \in C_n(\mathcal{M}_d)$ lies on a non-elementary component. Then also its sub-tuple $(x_1, x_2, \ldots, x_{n-1}) \in C_{n-1}(\mathcal{M}_d)$ lies on a non-elementary component.

Proof. Let the non-elementary component containing $(x_1, \ldots, x_n)$ be $\mathcal{Z}$ and let $\pi : C_n(\mathcal{M}_d) \to C_{n-1}(\mathcal{M}_d)$ forget the last matrix. Then $\pi(\mathcal{Z})$ is an irreducible locus containing $(x_1, \ldots, x_{n-1})$. By Corollary 4.22 or simply by $\text{GL}_n$-action, for a general tuple in $\mathcal{Z}$ the first matrix has at least two eigenvalues. The same is then true for $\pi(\mathcal{Z})$ which shows that this locus is contained in a non-elementary component. \hfill $\square$

Important classes of maps $\varphi$ above are translations and linear coordinate changes. More precisely, for a tuple $\alpha_\bullet = (\alpha_1, \ldots, \alpha_n) \in k^n$, we have a map $\mathbb{A}^n \to \mathbb{A}^n$ that translates by $\alpha_\bullet$, so we also have translation map shift$\alpha : C_n(\mathcal{M}_d) \to C_n(\mathcal{M}_d)$:

\begin{equation}
\text{shift}_\alpha(x_1, \ldots, x_n) = (x_1 + \alpha_1 I_d, \ldots, x_n + \alpha_n I_d).
\end{equation}

In fact, the shift$\circ$ is an action of $(\mathbb{A}^n, +)$ on $C_n(\mathcal{M}_d)$. Similarly, for a linear coordinate change $\mathbb{A}^n \to \mathbb{A}^n$ we have an induced $\text{GL}_n$-action on $C_n(\mathcal{M}_d)$, where $A = [a_{ij}] \in \text{GL}_n$ acts by

$$A \circ (x_1, \ldots, x_n) = A \cdot (x_1, \ldots, x_n)^T := \left( \sum_j a_{ij} x_j \right)_i.$$

The actions of $\text{GL}_n$ and $\mathbb{A}^n$ do not commute, rather they form a semidirect product; the group of affine coordinate changes. Both $\text{GL}_n$ and $\mathbb{A}^n$ commute with the $\text{GL}(V)$-action.

Let $x_1$ be a nilpotent matrix in the Jordan form and let $a_1 \leq a_2 \leq \cdots \leq a_m$ be the sizes of Jordan blocks of $x_1$. Consider a block matrix $y$ in the form

$$y = \begin{bmatrix}
y_{a_m, a_m} & y_{a_m, a_m-1} & \cdots & y_{a_m, a_1} 
y_{a_m-1, a_m} & y_{a_m-1, a_m-1} & \cdots & y_{a_m-1, a_1} 
\vdots & \vdots & \ddots & \vdots 
y_{a_1, a_m} & y_{a_1, a_m-1} & \cdots & y_{a_1, a_1}
\end{bmatrix}.$$
where \( y_{k,l} \) are \( k \times l \) matrices. We say that \( y \) is an upper-triangular Toeplitz matrix if each \( y_{k,l} \) is an upper-triangular Toeplitz matrix, i.e., it has the form

\[
\begin{bmatrix}
0 & \ldots & 0 & z_0 & z_1 & z_2 & \ldots & z_{k-1} \\
0 & \ldots & 0 & z_0 & z_1 & \ldots & z_{k-2} \\
0 & \ldots & 0 & 0 & z_0 & \ldots & z_{k-3} \\
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & 0 & z_0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
z_0 & z_1 & z_2 & \ldots & z_{l-1} \\
n_0 & z_1 & \ldots & z_{l-2} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & z_0 \\
0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 \\
0
\end{bmatrix}
\]

**Lemma 4.25.** Suppose that \((x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)\) where \(x_1\) is nilpotent and in the Jordan canonical form. Then each \(x_i, i \geq 2\) is an upper-triangular Toeplitz matrix.

**Proof.** The proof is well-known, we give it to introduce some notation and concepts used later. Recall that \(x_i\) are linear operators on \(V\). Introduce a \(k[t]\)-module structure on \(V\) by \(t \cdot v = x_1(v)\). Since all \(x_i\) commute with \(x_1\), they are endomorphisms of the resulting \(k[t]\)-module. Since \(x_1\) is in Jordan form, this module is isomorphic to

\[
\bigoplus_{i=1}^m \frac{k[t]}{(t^{a_i})}
\]

where \(a_1, \ldots, a_m\) are sizes of Jordan blocks of \(x_1\). Let \(M_i := k[t]/(t^{a_i})\). Then \(V = \bigoplus_i M_i\), so every endomorphism of \(V\) has the form

\[
\begin{bmatrix}
y_{a_m,a_m} & y_{a_m,a_m-1} & \cdots & y_{a_m,a_1} \\
y_{a_m-1,a_m} & y_{a_m-1,a_m-1} & \cdots & y_{a_m-1,a_1} \\
\vdots & \vdots & \ddots & \vdots \\
y_{a_1,a_m} & y_{a_1,a_m-1} & \cdots & y_{a_1,a_1}
\end{bmatrix}
\]

for some \(y_{i,i'} \in k[t]\). Moreover if \(a_i > a_{i'}\) then \(y_{i,i'}\) is divisible by \(t^{a_i-a_{i'}}\). It remains to go back to matrices and see that \(y_{i,j} = z_0 + z_1 t + z_2 t^2 + \ldots\) with \(z_i \in k\) corresponds to the upper-triangular Toeplitz matrix above. \(\square\)

**Remark 4.26.** By Corollary 4.22, we may add a polynomial in \(x_1\) to each \(x_i, i \geq 2\) so that a chosen diagonal block \(y_{k,k}\) is zero. We will frequently use this to make \(y_{a_m,a_m}\) a zero block.

### 4.4. Concatenation and components

For any \(d, d'\) the naive concatenation map \(C_n(\mathbb{M}_d) \times C_n(\mathbb{M}_{d'}) \to C_n(\mathbb{M}_{d+d'})\) maps tuples \(x = (x_i), x' = (x'_i)\) to the tuple

\[
\left( \begin{bmatrix} x_i \\ 0 \\ x'_i \end{bmatrix} \mid i = 1, 2, \ldots, n \right)
\]

This map is nowhere dominant since we can conjugate the target tuple by an element of \(\text{GL}_{d+d'}\). To remedy this, we consider the concatenation map \(\text{concat} : \text{GL}_{d+d'} \times C_n(\mathbb{M}_d) \times C_n(\mathbb{M}_{d'}) \to C_n(\mathbb{M}_{d+d'})\) that sends a triple \((g'', x, x')\) to the tuple

\[
\left( g'' \cdot \begin{bmatrix} x_i \\ 0 \\ x'_i \end{bmatrix} \cdot (g'')^{-1} \mid i = 1, 2, \ldots, n \right)
\]

In the theorem below, we show that concat map is usually dominant.
We say that tuples $\mathbf{x}, \mathbf{x}'$ have intersecting supports if for every $i$ the matrices $x_i$ and $x'_i$ have a common eigenvalue. We say that tuples have disjoint supports if they do not have intersecting supports. If the tuples have disjoint supports then the supports of the modules associated to $\mathbf{x}$ and $\mathbf{x}'$ are disjoint, but not vice versa, see §3.3.

**Proposition 4.27** (Concatenation of components).

1. Let $\mathbf{x} \in C_n(\mathbb{M}_d)$ and $\mathbf{x}' \in C_n(\mathbb{M}_{d'})$ be tuples with disjoint supports. Let $g \in \text{GL}_{d+d'}$. Then concat is a smooth map near the point $(g, \mathbf{x}, \mathbf{x}')$. The fiber $\text{concat}^{-1}(g, \mathbf{x}, \mathbf{x}')$ has dimension $d^2 + (d')^2$ at $(g, \mathbf{x}, \mathbf{x}')$.

2. Let $\mathcal{C}$ be an irreducible component of $C_n(\mathbb{M}_d)$ and let $\mathcal{C}'$ be an irreducible component of $C_n(\mathbb{M}_{d'})$. Then the closure of $\text{GL}_{d+d'} \times \mathcal{C} \times \mathcal{C}'$ is an irreducible component of $C_n(\mathbb{M}_{d+d'})$ which has dimension $2dd' + \dim \mathcal{C} + \dim \mathcal{C}'$. We call this component the concatenation of $\mathcal{C}$ and $\mathcal{C}'$. If $\mathcal{C}, \mathcal{C}'$ have smooth points then their concatenation also has a smooth point.

**Proof.** We first discuss how (2) follows from (1). Choose general points $\mathbf{x}$ and $\mathbf{x}'$ of $\mathcal{C}, \mathcal{C}'$. Thanks to the translation maps (4.24), we assume that for every $i$ the matrices $x_i$ and $x'_i$ have disjoint eigenvalues. Then there exists an open neighbourhood $W$ of $(\mathbf{x}, \mathbf{x}')$ in $\mathcal{C} \times \mathcal{C}'$ where this condition holds. By (1), the map $\text{concat}_{\text{GL}_{d+d'} \times W} : \text{GL}_{d+d'} \times W \to C_n(\mathbb{M}_{d+d'})$ is smooth and its fibers have dimension $d^2 + (d')^2$ at every point. Therefore the image of this map is open and has dimension $(d+d')^2 + \dim \mathcal{C} + \dim \mathcal{C}' - d^2 - (d')^2$. Since $W$ is open in $\mathcal{C} \times \mathcal{C}'$ the scheme $\text{GL}_{d+d'} \times W$ is irreducible hence its image is an open irreducible subset of required dimension.

Now we prove (1). Let $\mathbf{x}'' = \text{concat}(g, \mathbf{x}, \mathbf{x}')$. Let $M, M'$ be the modules associated to $\mathbf{x}$, $\mathbf{x}'$. By Lemma 3.2 they come with canonical $k$-linear bases $b, b'$. The module associated to point $\mathbf{x}''$ is $M \oplus M'$ by Lemma 3.6 and comes with a canonical basis $b'' = g \cdot (b, b')$. The point $\mathbf{x}'' \in C_n(\mathbb{M}_{d+d'})$ has an obstruction theory with obstruction group $\text{Ext}^2(M \oplus M', M \oplus M')$, see Corollary A.9. Similarly, the point $(g, \mathbf{x}, \mathbf{x}') \in \text{GL}_{d+d'} \times C_n(\mathbb{M}_d) \times C_n(\mathbb{M}_{d'})$ has an obstruction theory with obstruction group $\text{Ext}^2(M, M') \oplus \text{Ext}^2(M', M')$. By an examination of the construction of these obstruction theories [FGI+05, Theorem 6.4.9], the map concat induces a map of those theories which is injective onto the direct summand $\text{Ext}^2(M, M) \oplus \text{Ext}^2(M', M')$ of $\text{Ext}^2(M \oplus M', M \oplus M')$. If we prove that

$$d \text{concat} : T_{(g, \mathbf{x}, \mathbf{x}')} \text{GL}_{d+d'} \times C_n(\mathbb{M}_d) \times C_n(\mathbb{M}_{d'}) \to T_{\mathbf{x}''} C_n(\mathbb{M}_{d+d'})$$

is surjective, then by the Fundamental Theorem of obstruction calculus, see §4.2, the map concat is smooth at $(g, \mathbf{x}, \mathbf{x}')$. Then the dimension of the fiber of concat at $(g, \mathbf{x}, \mathbf{x}')$ is computed as $\dim_k \ker(d \text{concat})$.

The vector space $T_{\mathbf{x}''} C_n(\mathbb{M}_{d+d'})$ is described in Lemma 3.1 as tuples of commuting matrices in $\mathbb{M}_{d+d'}(k[\varepsilon]/\varepsilon^2)$ which reduce to $\mathbf{x}''$ modulo $\varepsilon$. We fix such a tuple $\mathbf{x}''$. Arguing as in Lemma 3.2, we obtain an associated module $\mathcal{M}$ over $S[\varepsilon]/\varepsilon^2$ together with a fixed $k[\varepsilon]/\varepsilon^2$-linear basis $\mathbf{b}''$. Since $\mathbf{x}''$ reduces to $\mathbf{x}''$ modulo $\varepsilon$, we have $\mathcal{M}/\varepsilon \mathcal{M} \simeq M \oplus M'$ and the basis $\mathbf{b}''$ reduces to $\mathbf{b}''$ modulo $\varepsilon$.

We claim that the direct sum decomposition $\mathcal{M}/\varepsilon \mathcal{M} \simeq M \oplus M'$ lifts to a direct sum decomposition of $\mathcal{M}''$. As in Section 3.3, for $I = \text{Ann}(M)$ and $I' = \text{Ann}(M')$ we have that the set of maximal ideals containing $I$ (respectively, $I'$) is the support of $M$ (respectively, of $M'$). Since these supports are disjoint, no maximal ideal contains $I + I'$, so $I + I' = (1)$. Since $I'M' = 0$ and $IM = 0$, we have

$$I(M \oplus M') = IM' = IM' + I'M' = (I + I')M' = M'$$
and similarly $I'(M \oplus M') = M$, so we obtain $II'(M \oplus M') = 0$. Since $M''/\varepsilon M'' = M \oplus M'$, we have that $II'M'' \subset \varepsilon M''$. The multiplication by $\varepsilon$ gives a surjection $M''/\varepsilon M'' \to \varepsilon M''$. The subset $II'\varepsilon M''$ is the image of $II'(M''/\varepsilon M'') = II'(M \oplus M') = 0$, so that $(II')^2 M'' \subset II'\varepsilon M'' = 0$. Since $I + I' = (1)$ also $I^2 + (I')^2 = (1)$. Let $J, J' \subset S[\varepsilon]/\varepsilon^2$ be the ideals generated by $I, I'$ respectively. They satisfy $J^2 + J'^2 = (1)$ and $J^2 J^2 M'' = (J^2 + J'^2) \cdot (J^2 M'' \cap J^2 M'') \subset J^2 J^2 M'' = 0$. By Chinese Remainder Theorem, we obtain

$$M'' = \frac{M''}{J^2 M'' \cap J^2 M''} \cong \frac{M''}{J^2 M''} \oplus \frac{M''}{J^2 M''}.$$

Let $\mathcal{M} := M''/J^2 M''$ and $\mathcal{M}' := M''/J^3 M''$. We have

$$\mathcal{M} \cong \frac{M''}{\varepsilon \mathcal{M}} \cong \frac{M''}{\varepsilon M''/\varepsilon^2 M''} \cong \frac{M''}{I^2(M \oplus M')} = \frac{M \oplus M'}{M'} = M,$$

so the submodule $\mathcal{M}$ reduces to $M$ modulo $\varepsilon$ and the same goes for $\mathcal{M}'$. Via the direct sum $M'' = M \oplus M'$ we view $\mathcal{M}, \mathcal{M}'$ as submodules of $M''$. Since $J, J'$ are coprime, the submodule $\mathcal{M}$ is exactly the set of elements of $M''$ annihilated by $J^2$, and similarly for $\mathcal{M}'$.

The $S[\varepsilon]/\varepsilon^2$-modules $\mathcal{M}, \mathcal{M}'$ are free $k[\varepsilon]/\varepsilon^2$-modules as direct summands of the free $k[\varepsilon]/\varepsilon^2$-module $M''$. Fix any $k[\varepsilon]/\varepsilon^2$-linear bases $\mathbf{b}, \mathbf{b}'$ of $\mathcal{M}, \mathcal{M}'$ which reduce to bases $\mathbf{b}, \mathbf{b}'$ respectively. By the argument of Lemma 3.2 from the pair $(\mathcal{M}, \mathbf{b})$ we obtain a tuple $\tilde{x}$ of commuting matrices with entries in $k[\varepsilon]/\varepsilon^2$ which reduces to $x$ modulo $\varepsilon$. By Lemma 3.1 this gives an element of $T_x C_n(M_d)$ which we denote also by $\tilde{x}$. From $(\mathcal{M}, \mathbf{b}')$ we analogously get an element $\tilde{x}' \in T_{\tilde{x}} C_n(M_{d'})$. The bases $\mathbf{b}''$ and $g \cdot (\mathbf{b}, \mathbf{b}')$ are two bases that reduce to $b'' = g \cdot (b, b')$ modulo $\varepsilon$, so there exists a unique element $\tilde{g} \in T_y GL_d$ such that $\tilde{g} \cdot (\mathbf{b}, \mathbf{b}') = \mathbf{b}''$. If we view elements of tangent space as morphisms from Spec($k[\varepsilon]/\varepsilon^2$), then the tangent map $d$ concatenates is given by composing them with concat. Therefore, the triple $(\tilde{g}, \tilde{x}, \tilde{x}')$ in the source of $d$ concatenates maps to $\tilde{x}''$. It remains to compute the kernel of $d$ concatenates. So we assume $\tilde{x}'' = \tilde{x}''$ and $\mathcal{M}'' = (M \oplus M')[\varepsilon]/\varepsilon^2$. If an element $(\tilde{g}, \tilde{x}, \tilde{x}')$ maps to $\tilde{x}''$, then it induces a decomposition of $\mathcal{M}'' = \mathcal{N} \oplus \mathcal{N}'$. Since $\tilde{x}$ reduces to $x$ modulo $\varepsilon$, we have $\mathcal{N}/\varepsilon \mathcal{N} = M$. Since $IM = 0$, we have $J\mathcal{N} \subset \varepsilon \mathcal{N}$ and so $J^2 \mathcal{N} = 0$. Similarly $J^2 \mathcal{N}' = 0$, so by uniqueness above, we have $\mathcal{N} = \mathcal{M}$ and $\mathcal{N}' = \mathcal{M}'$ and hence the triple $(\tilde{g}, \tilde{x}, \tilde{x}')$ is uniquely determined by a choice of bases $\mathbf{b}, \mathbf{b}'$: for these we have respectively a $d^2$ and $(d')^2$ dimensional space of choices.

The component part of the previous theorem was proven by different means in [CBS02]. However, it seems that the smoothness of the map concatenates and the dimension of fibers, which are important for us, are not present in that paper.

4.5. Specific Jordan blocks.

**Lemma 4.28** (Very big largest Jordan block). Let $(x_1, \ldots, x_n) \in C_n(M_d)$ be an $n$-tuple of nilpotent matrices such that the sizes $a_m > a_{m-1} \geq \cdots \geq a_1$ of Jordan blocks of the Jordan form of some linear combination of $x_1, \ldots, x_n$ satisfy $a_m > 2a_{m-1}$. Then the $n$-tuple $(x_1, \ldots, x_n)$ belongs to a non-elementary component of $C_n(M_d)$.

**Proof.** The action of $GL_n \times GL(V)$ stabilizes the irreducible components of $C_n(M_d)$, therefore we may assume that $x_1$ is in the Jordan canonical form:

$$x_1 = \begin{bmatrix} J_{a_m} & & \\ & \ddots & \\ & & J_{a_1} \end{bmatrix}$$
where $J_{a_k}$ denotes the nilpotent Jordan block of size $a_k$ for each $k = 1, \ldots, m$. By Lemma 4.25 the matrices $x_2, \ldots, x_n$ are of the form

$$x_i = \begin{bmatrix} x_{a_m, a_m}^{(i)} & x_{a_m, a_{m-1}}^{(i)} & \cdots & x_{a_m, a_1}^{(i)} \\ x_{a_{m-1}, a_m}^{(i)} & x_{a_{m-1}, a_{m-1}}^{(i)} & \cdots & x_{a_{m-1}, a_1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a_1, a_m}^{(i)} & x_{a_1, a_{m-1}}^{(i)} & \cdots & x_{a_1, a_1}^{(i)} \end{bmatrix}$$

where each $x_{a_j, a_k}^{(i)}$ is an $a_j \times a_k$ upper triangular and Toeplitz matrix. By Remark 4.26 we assume that $x_{a_m, a_m}^{(i)} = 0$ for every $i \geq 2$. In particular, since $a_m > 2a_{m-1}$, the $(a_{m-1} + 1)$-th row and column of the matrix $x_i$ are zero for each $i \geq 2$. Consequently, the matrix $E_{a_{m-1}+1, a_{m-1}+1}$ that has 1 at the intersection of the $(a_{m-1} + 1)$-th row and column and zeros elsewhere commutes with $x_i$ for $i \geq 2$. For a nonzero $\lambda$, the matrix $x_1 + \lambda E_{a_{m-1}+1, a_{m-1}+1}$ is not nilpotent and in fact has more than one eigenvalue. Therefore the line $\{(x_1 + \lambda E_{a_{m-1}+1, a_{m-1}+1}, x_2, \ldots, x_n) : \lambda \in \mathbb{k}\}$ intersects in an open subset some component of $C_n(\mathbb{M}_d)$ that is non-elementary. But a component is closed, so it contains the whole line, in particular $(x_1, \ldots, x_n)$, and the lemma follows. \hfill \Box

**Lemma 4.29** (One Jordan block). Let $d$ be arbitrary and let $(x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)$ be an $n$-tuple of nilpotent matrices such that some linear combination of $x_1, \ldots, x_n$ has a one-dimensional kernel. Then the $n$-tuple $(x_1, \ldots, x_n)$ belongs to the principal component of $C_n(\mathbb{M}_d)$.

**Proof.** Using the actions of $\text{GL}(V)$ and $\text{GL}_n$ we may assume that $x_1 = J_d$ has a one-dimensional kernel, so by Lemma 4.25 every other matrix in the tuple is a polynomial in $x_1$ so the tuple generates the algebra $\mathbb{k}[x_1]$. By Corollary 4.22 it follows that this tuple lies in the principal component. \hfill \Box

**Proposition 4.30** (Two Jordan blocks structure). Let $(x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)$ be an $n$-tuple of nilpotent matrices such that some linear combination of $x_1, \ldots, x_n$ has a two-dimensional kernel. Pick such a combination and let $\Delta$ be the difference of the sizes of its two Jordan blocks. Then up to $\text{GL}(V)$-action and nonlinear change of generators as in Corollary 4.22, there exists a $j$ such that the $n$-tuple $(x_1, \ldots, x_n)$ is a limit of tuples of the form

$$(4.31) \quad x_1 = \begin{bmatrix} t \\ \end{bmatrix}, \quad x_2 = \begin{bmatrix} t^i & 0 \\ 0 & 0 \end{bmatrix}, \quad x_i = \begin{bmatrix} 0 & b_i \\ c_i & 0 \end{bmatrix} \quad \text{for } i \geq 3$$

where for every $i \geq 3$ we have $c_i \in t^i \mathbb{k}[t]$ and $b_i \in t^{i+\Delta} \mathbb{k}[t]$ and $t^j x_i = 0$. Here, we use the “polynomial” notation from Lemma 4.25.

**Proof.** After a coordinate change we may assume that $x_1$ is in the Jordan canonical form with two Jordan blocks of sizes $m \geq k$:

$$x_1 = \begin{bmatrix} J_m \\ \end{bmatrix}. \quad x_2 = \begin{bmatrix} J_k \end{bmatrix}.$$

Introduce a $\mathbb{k}[t]/(t^m)$-module structure on $V$ where $t \cdot v = x_1(v)$ for all $v \in V$. Then $V = \mathbb{k}[t]/(t^m) \oplus \mathbb{k}[t]/(t^k)$, see Lemma 4.25. Recall that the only ideals in $\mathbb{k}[t]/(t^m)$ are generated by powers of $t$. For an element $f \in \mathbb{k}[t]$ let the valuation of $f$, denoted $\nu(f)$ be the maximal power of $t$ that divides $f$. This valuation descends to a function on $\mathbb{k}[t]/(t^m)$. For any $g, h \in \mathbb{k}[t]/(t^m)$, we have $\nu(g) \leq \nu(h)$ if and only if the ideal $(g)$ contains the ideal $(h)$ in $\mathbb{k}[t]/(t^m)$. 


Below we repeatedly make use of Corollary 4.22. We have \( \Delta = m-k \geq 0 \). Write each \( x_i \) as a matrix

\[
(4.32) \quad x_i = \begin{bmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{bmatrix}
\]

for \( a^{(i)} \in k[t]/(t^m) \), \( b^{(i)} \in t^\Delta k[t]/(t^m) \) and \( c^{(i)}, d^{(i)} \in k[t]/(t^k) \). By definition, the matrix \( x_1 \) is diagonal with both diagonal entries equal to \( t \). Let the vanishing order \( \text{ord}(x_1) \) of \( x_1 \) be defined as \( \min \left( \nu(a^{(i)} - d^{(i)}), \nu(c^{(i)}), \nu(b^{(i)}) - \Delta \right) \). Let \( j = \min(\text{ord}(x_i) \mid i = 2, 3, \ldots) \). After a coordinate change, we may assume \( j = \text{ord}(x_2) \). Moreover, if \( \Delta = 0 \) we may use the \( \text{GL}_2 \)-action on the ring of the endomorphisms of the \( k[t]/(t^m) \)-module \( V \) to assume that the matrix of the coefficients of \( x_2 \) at \( t_j \) is lower triangular (for example, the transpose of a Jordan normal form), in particular \( b^{(2)} \in t^{j+1} k[t]/(t^m) \).

We subtract an appropriate polynomial in \( x_1 \) from each \( x_i, i \geq 2 \), and hence obtain \( d^{(i)} = 0 \) for all \( i \geq 2 \). Note that this does not change the vanishing orders of the matrices \( x_i \). Therefore we have

\[
(4.33) \quad [x_1, x_j] = x_1 x_j - x_j x_1 = \begin{bmatrix} b^{(i)} c^{(j)} - b^{(j)} c^{(i)} & a^{(i)} b^{(j)} - a^{(j)} b^{(i)} \\ c^{(i)} a^{(j)} - c^{(j)} a^{(i)} & c^{(i)} b^{(j)} - c^{(j)} b^{(i)} \end{bmatrix}.
\]

We will now divide \( x_3, \ldots, x_n \) by \( x_2 \) with remainder: we claim that each \( x_i \) for \( i \geq 3 \) can be written as \( x_i' + r_i \), where \( x_i' \in x_2 k[x_1] \) and

\[
 r_i = \begin{bmatrix} a'_i & b'_i \\ c'_i & 0 \end{bmatrix} \quad \text{with} \quad t^j r_i = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad a'_i, c'_i \in t^j k[t], \quad b'_i \in t^{j+\Delta} k[t].
\]

We have three cases.

1. \( j = \nu(a^{(2)}) \). Then \( a^{(2)} = t^j u \) with \( u \in k[t]/(t^m) \) invertible. For all \( i \geq 3 \) we have \( j \leq \text{ord}(x_i) \leq \nu(a^{(i)}) \), hence \( \nu(a^{(2)}) \leq \nu(a^{(i)}) \), so \( a^{(i)} \) is a multiple of \( a^{(2)} \) in \( k[t]/(t^m) \), say \( a^{(i)} = a^{(2)} \cdot q(t) \) where \( q(t) \in k[t] \). Let \( x_i' := x_2 \cdot q(x_1) \) and let \( r_i = x_i - x_i' \). Then \( r_i \) is a matrix

\[
\begin{bmatrix} 0 & b'_i \\ c'_i & 0 \end{bmatrix}.
\]

The commutativity \( [x_2, x_i] = 0 \) yields \( [x_2, r_i] = 0 \), so by (4.33) we get

\[
0 = \begin{bmatrix} b^{(2)} c'_i - b'_i c^{(2)} & a^{(2)} b'_i \\ -c'_i a^{(2)} & c^{(2)} b'_i - c'_i b^{(2)} \end{bmatrix}
\]

so indeed

\[
t^j \cdot r_i = t^j \cdot \begin{bmatrix} 0 & b'_i \\ c'_i & 0 \end{bmatrix} = \begin{bmatrix} 0 & t^j b'_i \\ t^j c'_i & 0 \end{bmatrix} = u^{-1} \begin{bmatrix} 0 & a^{(2)} b'_i \\ a^{(2)} c'_i & 0 \end{bmatrix} = 0
\]

as claimed.

2. \( j = \nu(c^{(2)}) \). Analogously as above, we divide every \( x_i \) to obtain a remainder with \( c'_i = 0 \) and then use (4.33) to conclude that \( c^{(2)} b'_i = 0 \) in \( k[t]/(t^m) \) hence also \( t^j b'_i = 0 \). 

3. \( j = \nu(b^{(2)}) - \Delta \). Then \( \nu(b^{(2)}) = j + \Delta \). Analogously as above, we divide every \( x_i \) to obtain a remainder \( r_i \) with \( b'_i = 0 \) and use (4.33) to conclude that \( b^{(2)} c'_i = 0 \) in \( k[t]/(t^m) \). This implies that \( c'_i \) is divisible by \( t^{m - \nu(b^{(2)})} = t^{m - j - \Delta} = t^{k-j} \), so \( t^j \cdot c'_i = 0 \) in \( k[t]/(t^k) \) and hence the only nonzero entry of \( t^j \cdot r_i \) is in the top left corner.

Moreover, \( a'_i, c'_i \in t^j k[t] \) since the corresponding entries of both \( x_i \) and \( x'_i \) lie there; same for \( b'_i \in t^{j+\Delta} k[t] \). We replace \( x_i \) by \( r_i \), by Corollary 4.22 this does not change the components
containing our tuple. We keep the notation that

\[ x_i = \begin{bmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{bmatrix}. \]

Consider the matrix \( A = \begin{bmatrix} t^j & 0 \\ 0 & 0 \end{bmatrix} \). Clearly, \( A \) is an endomorphism of the \( \mathbb{k}[t] \)-module \( V \), so the underlying matrix in \( M_d \) commutes with \( x_1 \). From the above considerations we see that

\[ A \cdot x_i = x_i \cdot A = \begin{bmatrix} t^j a^{(i)} & 0 \\ 0 & 0 \end{bmatrix} \text{ for } i \geq 3. \]

For each \( \lambda \in \mathbb{k} \) define \( X_2(\lambda) := \lambda A + x_2 \). For each \( \lambda \in \mathbb{k} \), this matrix commutes with \( x_1, x_3, x_4, \ldots, x_n \). For all but one choices of \( \lambda \in \mathbb{k} \) we have \( \nu(\lambda t^j + a^{(2)}) = j \). Since we want to prove that our starting tuple is a limit, we may replace \( x_2 \) by suitable \( X_2(\lambda) \) and hence we can assume that \( \nu(a^{(2)}) = j \) and that \( x_i \) is as in the case (1) above for \( i \geq 3 \).

Now we put \( x_2 \) in a normal form by using the automorphisms of the module \( V \). For every \( \alpha \in \mathbb{k}[t], \beta \in t^A \mathbb{k}[t] \) such that \( 1 - \alpha \beta \in \mathbb{k}[t]/(t^m) \) is an invertible element (which is always the case if \( A > 0 \)) we have an automorphism \( \Phi(\alpha, \beta) \) of the module \( V \) given by the matrix

\[
\frac{1}{1-\alpha \beta} \begin{bmatrix} 1 & \beta \\ -\alpha & 1 \end{bmatrix}.
\]

whose inverse is the matrix \( \frac{1}{1-\alpha \beta} \begin{bmatrix} 1 & -\beta \\ -\alpha & 1 \end{bmatrix} \). The coordinate change \( \Phi(\alpha, \beta) \) maps \( x_1 \) to itself and each \( x_i \), for \( i \geq 2 \) to

\[
\frac{1}{1-\alpha \beta} \begin{bmatrix} a^{(i)} - \alpha \beta c^{(i)} + \beta b^{(i)} & -\beta a^{(i)} + b^{(i)} - \beta^2 c^{(i)} \\ \alpha a^{(i)} - \alpha^2 b^{(i)} + c^{(i)} & -\alpha \beta a^{(i)} + \alpha b^{(i)} - \beta c^{(i)} \end{bmatrix}.
\]

Now, since \( \nu(a^{(2)}) = j \leq \nu(b^{(2)}) \), \( \nu(c^{(2)}) \) and \( b^{(2)} \in t^{j+1} \mathbb{k}[t] \) (also for \( \Delta = 0 \)), we can solve the equation \( \alpha a^{(2)} - \alpha^2 b^{(2)} + c^{(2)} = 0 \) in \( \alpha \in \mathbb{k}[t]/(t^m) \), hence after a transformation \( \Phi(\alpha, 0) \) we have \( c^{(2)} = 0 \). The change of coordinates might have not preserved the conditions \( a^{(i)} = 0 \) for \( i \geq 3 \) and \( d^{(i)} = 0 \) for \( i \geq 2 \), but it did preserve the conditions \( a^{(i)} \in t^{i} \mathbb{k}[t]/(t^m), b^{(i)} \in t^{i+1} \mathbb{k}[t]/(t^m), c^{(i)}, d^{(i)} \in t^{i} \mathbb{k}[t]/(t^k) \) and \( \nu(a^{(2)}) = j \). We now subtract an appropriate element of \( \mathbb{k}[x_1] \) from each \( x_i, i \geq 3 \), to guarantee \( a^{(i)} = 0 \) for \( i \geq 3 \) and \( d^{(i)} = 0 \) for \( i \geq 2 \). Now, \( \nu(b^{(2)}) - \Delta \geq \nu(a^{(2)}) \), so we may also solve the equation \( -\beta a^{(2)} + b^{(2)} = 0 \) in \( \beta \in t^A \mathbb{k}[t]/(t^m) \). Applying \( \Phi(0, \beta) \) we get \( b^{(2)} = 0 \). Finally we rescale \( x_2 \) by an invertible element \( u \in \mathbb{k}[t]/(t^m) \) such that \( ua^{(2)} = t^j \) so that

\[
x_2 = \begin{bmatrix} t^j & 0 \\ 0 & 0 \end{bmatrix}.
\]

Again, we subtract appropriate elements of \( \mathbb{k}[x_1, x_2] \) from each \( x_i \) to get \( a^{(i)} = d^{(i)} = 0 \) for each \( i \geq 3 \). This concludes the proof. \( \square \)

**Proposition 4.34 (Two Jordan blocks).** Let \( d > 4 \) be arbitrary and let \( (x_1, \ldots, x_n) \in C_n(\mathbb{M}_d) \) be an \( n \)-tuple of nilpotent matrices such that some linear combination of \( x_1, \ldots, x_n \) has a two-dimensional kernel. Then the \( n \)-tuple \( (x_1, \ldots, x_n) \) belongs to a non-elementary component of \( C_n(\mathbb{M}_d) \).

**Proof.** Using Proposition 4.30 and closedness of non-elementary locus, we assume that \( (x_1, \ldots, x_n) \) is in the form (4.31). We consider three cases:
(1) \( j \geq \frac{k+1}{2} \). By adding appropriate powers of \( x_t \) to \( x_2 \), we may assume that \( a^{(t)} = 0 \) for \( i \geq 2 \). (This operation makes \( d^{(2)} \) nonzero.) Since every block entry of every \( x_i \) with \( i \geq 2 \) is divisible by \( t^j \), the matrix \( x_i \) has zeros in the \( \left[ \frac{k}{2} \right] \)-th row and column. In this case, the matrix unit \( E_{\left[ \frac{k}{2} \right],\left[ \frac{k}{2} \right]} \) commutes with \( x_i \) for each \( i \geq 2 \). The line \( \{ (x_1 + \lambda E_{\left[ \frac{k}{2} \right],\left[ \frac{k}{2} \right]}, x_2, \ldots, x_n) : \lambda \in \mathbb{k} \} \) intersects in an open subset a component of \( C_n(M_d) \) that contains \( n \)-tuples of matrices with more than one eigenvalue, so this component must contain the whole line, and in particular it contains \( (x_1, \ldots, x_n) \).

(2) \( j < \frac{k}{2} \). Since \( t^j \) annihilates the left-lower and right-upper block entries of all \( x_i \) with \( i \geq 2 \), we see that each of these matrices has zeros in the \( (m + \left[ \frac{k}{2} \right]) \)-th row and column. In this case the matrix unit \( E_{m+\left[ \frac{k}{2} \right],m+\left[ \frac{k}{2} \right]} \) commutes with \( x_i \) for each \( i \geq 2 \) and as in the previous case, the tuple \( (x_1, \ldots, x_n) \) lies on a non-elementary component.

(3) \( j = \frac{k}{2}, k \) even. In this case every two matrices in the form \( (4.31) \) commute by equation \( (4.33) \). So we have an affine space of tuples of commuting matrices and it is enough to prove that a general element of this space lies on a non-elementary component. We thus assume that the space spanned by \( \mathbb{k}[x_1]x_2, \mathbb{k}[x_1]x_3, \mathbb{k}[x_1]x_4 \) is equal to the space spanned by

\[
\mathbb{k}[t] \cdot \begin{bmatrix} t^j & 0 \\ 0 & 0 \end{bmatrix}, \mathbb{k}[t] \cdot \begin{bmatrix} 0 & t^j+\Delta \\ 0 & 0 \end{bmatrix}, \mathbb{k}[t] \cdot \begin{bmatrix} 0 & 0 \\ t^j & 0 \end{bmatrix}.
\]

Then every other matrix \( x_i \) is an element of \( (x_2, x_3, x_4)k[x_1] \), so by Corollary 4.22 we assume \( x_i = 0 \) for \( i \geq 5 \). Finally, we replace \( x_2 \) by \( x_2^2 - x_2 \) and get

\[
x_2 = \begin{bmatrix} 0 & 0 \\ 0 & t^j \end{bmatrix}.
\]

If \( m > k = 2j \), then all matrices \( x_i \) for \( i \geq 2 \) have \( (j + 1) \)-th row and column zero, so \( E_{j+1,j+1} \) commutes with them and the tuple lies on a non-elementary component by the same argument as in previous cases. If \( m = k \), then the argument is slightly more complicated: the matrix \( E := E_{1,1} + E_{j+1,j+1} + E_{m+1,m+1} + E_{m+j+1,m+j+1} \) commutes with \( x_i \) for all \( i \geq 2 \). If \( d > 4 \), then \( m > 2 \) so \( E \) is not the identity matrix and we conclude as before.

\[ \square \]

Remark 4.35. If \( d = 4 \), then there are two possible Jordan structures for a matrix with 2-dimensional kernel: \((3, 1)\) and \((2, 2)\). In the first case the proposition still holds, by Lemma 4.28. On the other hand, in the second case the proposition is false, as will be shown in Example 6.1.

5. Białynicki-Birula decompositions and components of Quot

In this section, we assume \( \text{char} \mathbb{k} = 0 \) for technical reasons; see [Jel19] for details. Oversimplifying, the Białynicki-Birula on Quot works as follows: consider the locus

\[ \mathcal{E}e = \{ [F/K] \mid F/K \text{ is supported on a single point of } \mathbb{A}^n \} \subset \text{Quot}^d. \]

The Białynicki-Birula decomposition of Quot is a certain subdivision of \( \mathcal{E}e \) into loci \( \mathcal{E}e_{\epsilon_1}, \ldots, \mathcal{E}e_{\epsilon_s} \).

For a point \([F/K] \in \mathcal{E}e\) there exists a simple linear algebraic condition: having \( \text{trivial negative tangents} \) \((5.4)\), which implies that \( \mathcal{E}e_{\epsilon_1} \to \text{Quot}^d \) is an open immersion near \([F/K]\), see Proposition 5.5. In that case \([F/K]\) is a point of an elementary component of Quot. So the basic takeaway from this section could be that

\[ \text{To find an elementary component of Quot}^d \text{ it is enough to prove that a given point [F/K] satisfies the condition \((5.4)\).} \]
Example 5.2. In applications below use only the most natural action, where deg(\(x_i\)) = 1 for all \(i\) and deg \(e_j\) = 0 for all \(j\).

The above choice of degrees of generators of \(F\) determines an action of \(\mathbb{G}_m\) on \(F\), namely \(t \circ f := t^{-\deg f}f\) for a homogeneous element \(f \in F\) and \(t \in \mathbb{G}_m(k)\).

Example 5.2. If \(S = k[x_1, x_2]\) with \(\deg x_1 = \deg x_2 = 1\) and \(F = S(-1) \oplus S\), then its element \(e_1 + x_2e_2\) is homogeneous of degree one, hence \(t \circ (e_1 + x_2e_2) = t^{-1}(e_1 + x_2e_2)\).

The action on \(F\) induces an action on \(\text{Quot}^d\), we just have \(t \circ (F/K) = F/(t \circ K)\), where \(t \circ K = \{t \circ k \mid k \in K\}\). Let \(\mathbb{G}_m = \text{Spec } k[t^{-1}] = \mathbb{G}_m \cup \{\infty\}\). The (negative) Białynicki-Birula decomposition of \(\text{Quot}^d\) is uniquely determined by the \(\mathbb{G}_m\)-action. Specifically, for any \(k\)-algebra \(A\), the \(A\)-points of this decompositions are

\[
\text{Quot}^{d,+}_r(A) = \left\{ \varphi: \mathbb{G}_m \times \text{Spec}(A) \to \text{Quot}^d_r \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant} \right\}.
\]

The superscript “+” instead of the more natural “−” is introduced for consistency with [Jel19].

The above formula describes a functor \(\text{Quot}^{d,+}_r\). This functor is represented by a scheme by [JS19, Prop 4.5, 5.3] since \(\text{Quot}^d_r\) is covered by \(\mathbb{G}_m\)-invariant affine open subschemes given in Example A.4. Below we denote by \(\text{Quot}^{d,+}_r\) both the functor and the representing scheme. The \(k\)-points of \(\text{Quot}^{d,+}_r\) by definition correspond to \(\mathbb{G}_m\)-equivariant maps \(\varphi: \mathbb{G}_m \to \text{Quot}^d_r\). Each such map can be restricted to \(\varphi^0: \mathbb{G}_m \to \text{Quot}^d_r\). But a \(\mathbb{G}_m\)-equivariant map from \(\mathbb{G}_m\) is just the orbit map of \(\varphi^0(1) = \varphi(1)\) from this point of view, \(\varphi(\infty)\) is the limit of the orbit of \(\varphi(1)\) and informally speaking \(\text{Quot}^{d,+}_r\) parameterizes points of \(\text{Quot}^d_r\) together with the limits at infinity of their \(\mathbb{G}_m\)-orbits.

The mapping \(\varphi \mapsto \varphi(1, -)\) gives a natural forgetful morphism

\[
\theta_0: \text{Quot}^{d,+}_r \to \text{Quot}^d_r
\]

which is universally injective, or, in other words, injective on \(L\)-points for every field \(L\). We identify the \(k\)-points of \(\text{Quot}^{d,+}_r\) with their images in \(\text{Quot}^d_r\). Below we check which \(k\)-points of \(\text{Quot}^d_r\) are obtained in this way.

Lemma 5.3. The \(k\)-points of \(\text{Quot}^{d,+}_r\) are exactly the \(k\)-points of \(\text{Quot}^d_r\) which correspond to modules \(M\) supported only at the origin of \(k^n\).

Proof. The argument is very similar to [Jel19, Proposition 3.3].

Let \(M\) correspond to a point of \(\text{Quot}^{d,+}_r\). This means that the \(\mathbb{G}_m\)-orbit of \([M]\) extends to a morphism \(\varphi: \mathbb{G}_m \to \text{Quot}^d\). Pick \(i \in \{1, 2, \ldots, n\}\) and \(\alpha \in k\) and suppose that \(\text{Supp } M\) intersects the hyperplane \(V(x_i - \alpha)\). Then the support of \(t \circ M\) intersects the hyperplane \(V(t \circ (x_i - \alpha)) = V(x_i - m \circ x_i \alpha)\). If \(\alpha \neq 0\) this means that \(\text{Supp } (t \circ M)\) is divergent, a contradiction.
since it converges to the support of \( \varphi(\infty) \in \text{Quot}_d^d \). This shows that \( \text{Supp} M = \{0\} \). Conversely, suppose that \( M \) is supported at zero. We may view \([M]\) as a point on the Quot scheme of \( \mathcal{O}_{\mathbb{P}^n}^{\oplus r} \). Such Quot is projective, hence every \( \mathbb{G}_m \) orbit extends, moreover the limit is still supported at zero so it lies in \( \text{Quot}_d^d \).

Consider a module \( M = F/K \) supported at the origin. We say that \( M \) has trivial negative tangents if
\[
\dim_k \left( \text{Hom}(K, M) / \text{Hom}(K, M)_{\geq 0} \right) = n.
\]

Recall from §3.4 that an irreducible component of \( \text{Quot}_d^d \) is elementary if its geometric points correspond to modules supported only at one point. Let \( \theta : \mathbb{A}^n \times \text{Quot}_d^{d,+} \to \text{Quot}_d^d \) be the morphism defined on points by \( \theta(w, [F/K]) = [F/K] + w \). In other words, \( \theta(w, [F/K]) \) is the module \( F/K \) translated by vector \( w \).

**Proposition 5.5.** If \( M \) has trivial negative tangents, then \( \theta : \mathbb{A}^n \times \text{Quot}_d^{d,+} \to \text{Quot}_d^d \) is an open immersion near \([M]\). Conversely, if \( Z \subset \text{Quot}_d^d \) is a generically reduced elementary component, then a general point of \( Z \) has trivial negative tangents.

**Proof.** This follows similarly to [Jel19, Theorem 4.5, Theorem 4.9].

While we do not employ it significantly in the current article, Proposition 5.5 is very useful to describe new elementary components of Quot schemes, see the introduction of [Jel19] for the case of Hilbert schemes. Also, all examples in Section 6.1 below have trivial negative tangents.

With a bit of deformation theory, we can even check that a given point with trivial negative tangents is smooth.

**Lemma 5.6.** The tangent space to \( \text{Quot}_d^{d,+} \) at a point \([M = F/K]\) is equal to \( \text{Hom}_{\mathbb{G}_m}(K, M)_{\geq 0} \). Moreover, the point \([M = F/K]\) has an obstruction theory with obstruction space \( \text{Ext}^1_{\mathbb{G}_m}(K, M)_{\geq 0} \).

**Proof.** This follows exactly as in [Jel19, Theorem 4.2].

**Theorem 5.7.** Let \( M = F/K \) be a module of finite degree supported at the origin and such that \( \text{Ext}^1_{\mathbb{G}_m}(K, M)_{\geq 0} = 0 \). Then \([M] \in \text{Quot}_d^{d,+} \) is a smooth point. If moreover \( M \) has trivial negative tangents, then \([M] \in \text{Quot}_d^d \) is a smooth point on an elementary component.

**Proof.** Follows by combining Lemma 5.6, Example 4.14, and Proposition 5.5.

Later we will also need the positive Białynicki-Birula decomposition that we describe below. Let \( \mathcal{T}_m = \text{Spec} \mathbb{k}[t] = \mathbb{G}_m \cup \{0\} \). For any \( k \)-algebra \( A \), the \( A \)-points of this decomposition are
\[
\text{Quot}_{d-}^d(A) = \left\{ \varphi : \mathcal{T}_m \times \text{Spec}(A) \to \text{Quot}_d^d \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant} \right\}.
\]

This decomposition too is represented by a scheme, which we denote \( \text{Quot}_{d-}^d \) and its point \( M = F/K \) has tangent space \( \text{Hom}(K, M)_{\leq 0} \) and an obstruction theory with obstruction space \( \text{Ext}^1(K, M)_{\leq 0} \).

6. Results specific for degree at most eight

Throughout this section we assume \( \text{char } \mathbb{k} = 0 \). Some of the arguments do not use this assumption, most others can be made for large enough characteristics. However, characteristic
zero is indispensable for proofs of surjectivity of tangent maps and for all arguments performed with the help of Macaulay2.

6.1. Examples of elementary components. In this subsection we gather examples of elementary components in $C_n(\mathbb{M}_d)$. For compactness we use the tensor notation: a tuple $(x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)$ is written as $\sum x_i \cdot e_i$ where $e_1, \ldots, e_n$ are formal coordinates. For example, the triple

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is presented as a single matrix $\begin{bmatrix} e_1 + e_2 & e_2 + e_3 \\ 0 & e_1 + e_2 \end{bmatrix}$.

For any $n$, $d$ and $m \in \{1, \ldots, d-1\}$ define

$$Z_{n,d-m}^m = \left\{ g \cdot \left( \begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & A_n \\ 0 & 0 \end{bmatrix} \right) \right\} \cdot g^{-1}; A_1, \ldots, A_n \in \mathbb{M}_{m \times (d-m)}, g \in \text{GL}_d \right\} \subset C_n(\mathbb{M}_d).$$

For a general tuple in $Z_{n,d-m}^m$, the set of $g \in \text{GL}_d$ that conjugate this tuple into a block upper triangular tuple has codimension $m(d - m)$ in $\text{GL}_d$. Therefore, $\dim Z_{n,d-m}^m = (n + 1)m(d - m)$.

Adding some multiple of identity matrices to each matrix, we get a locus of dimension $(n + 1)m(d - m) + n$, that we call the $m$-th square-zero locus of $C_n(\mathbb{M}_d)$. Below we say that a tuple $x$ witnesses that the square-zero locus is a component if the tangent space $T_x C_n(\mathbb{M}_d)$ has dimension $(n + 1)m(d - m) + n$; equal to the dimension of the square-zero locus.

**Example 6.1** $(4 \times 4$ square-zero quadruple, $m = 2$). The quadruple

$$\begin{bmatrix} 0 & 0 & e_1 & e_2 \\ 0 & 0 & e_3 & e_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

witnesses that the square-zero locus is a component for $(n, d, m) = (4, 4, 2)$. This tuple corresponds to a module $M$ with Hilbert function $(2, 2)$ that is generated by two elements and has trivial negative tangents. It was already known in [Gur92, p. 72] that this tuple does not lie on the principal component, while seemingly it was unknown which other components it lies on.

**Example 6.2** $(5 \times 5$ square-zero quintuple, $m = 2$). The tuple

$$\begin{bmatrix} 0 & 0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_4 & e_1 + e_5 & e_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

witnesses that the square-zero locus is a component for $(n, d, m) = (5, 5, 2)$. This tuple corresponds to a module with Hilbert function $(3, 2)$ that has trivial negative tangents and is a smooth point of an elementary 31-dimensional component of Quot.

**Example 6.3** $(6 \times 6$ square-zero sextuple, $m = 3$). The tuple $x_1 = E_{14} + E_{25} + E_{36}, x_2 = E_{15} + E_{26}, x_3 = E_{16}, x_4 = E_{24} + E_{35}, x_5 = E_{25} + E_{36}, x_6 = E_{34}$ witnesses that the square-zero locus is a component for $(n, d, m) = (6, 6, 3)$. This tuple corresponds to a module with Hilbert function $(3, 3)$ that has trivial negative tangents and is a smooth point of an elementary 51-dimensional component of Quot.

**Example 6.4** $(6 \times 6$ square-zero sextuple II, $m = 2$). The tuple $x_1 = E_{13} + E_{24}, x_2 = E_{14} + E_{25}, x_3 = E_{15} + E_{26}, x_4 = E_{16}, x_5 = E_{23}, x_6 = E_{24}$ witnesses that the square-zero locus is a
component for \((n, d, m) = (6, 6, 2)\). This tuple corresponds to a module with Hilbert function (4, 2) that has trivial negative tangents and is a smooth point of an elementary 50-dimensional component of Quot.

Example 6.5 \((7 \times 7 \text{ square zero quintuple, } m = 3)\). The tuple \(x_1 = E_{14} + E_{25} + E_{36}, x_2 = E_{15} + E_{26} + E_{37}, x_3 = E_{16} + E_{27}, x_4 = E_{24} + E_{35}, x_5 = E_{26} + E_{37}\) witnesses that the square-zero locus is a component for \((n, d, m) = (5, 7, 3)\). This tuple corresponds to a module with Hilbert function (4, 3) that has trivial negative tangents and is a smooth point of an elementary 56-dimensional component of Quot.

Example 6.6 \((7 \times 7 \text{ square zero septuple, } m = 2)\). The tuple \(x_1 = E_{13} + E_{24}, x_2 = E_{14} + E_{25}, x_3 = E_{15} + E_{26}, x_4 = E_{16} + E_{27}, x_5 = E_{17}, x_6 = E_{23}, x_7 = E_{24}\) witnesses that the square-zero locus is a component for \((n, d, m) = (7, 7, 2)\). This tuple corresponds to a module with Hilbert function (5, 2) that has trivial negative tangents and is a smooth point of an elementary 73-dimensional component of Quot.

Example 6.7 \((7 \times 7 \text{ non-square zero quintuples})\). This is the only example that does not parameterize square-zero matrices. Consider the locus of quintuples of \(7 \times 7\) matrices that have the form

\[
(6.8) \quad x_i = \begin{bmatrix}
\mu_i & 0 & 0 & \lambda_1;u_{11} + \lambda_2;u_{21} & \lambda_1;u_{21} + \lambda_2;u_{31} & * & * \\
0 & \mu_i & 0 & \lambda_1;u_{12} + \lambda_2;u_{22} & \lambda_1;u_{22} + \lambda_2;u_{32} & * & * \\
0 & 0 & \mu_i & \lambda_1;u_{13} + \lambda_2;u_{23} & \lambda_1;u_{23} + \lambda_2;u_{33} & * & * \\
0 & 0 & 0 & \mu_i & 0 & \lambda_1i & 0 \\
0 & 0 & 0 & 0 & \mu_i & 0 & \lambda_2i \\
0 & 0 & 0 & 0 & 0 & \mu_i & 0 \\
\end{bmatrix}
\]

where we take arbitrary \(\mu_i, \lambda_{1i}, \lambda_{2i}, u_{jk} \in \mathbb{k}\) for \(i = 1, 2, \ldots, 5\) and \(j, k = 1, 2, 3\) and stars denote arbitrary entries. The matrices in each quintuple commute. There are \(5 \cdot (3 + 6) + 9 = 54\) parameters, so we obtain a morphism \(\mathbb{A}^{54} \to C_5(M_7)\). For \(\lambda_{1i}; \lambda_{2i};^{2}_{i=1}\) linearly independent we can recover the \(u_{jk}\) from the matrices, so this map is generically one-to-one and so its image is a rational locus \(\mathcal{L}\) of dimension 54. Consider the locus \(\text{GL}_7 \cdot \mathcal{L}\). To obtain its dimension we pick a general point \(x = (x_1, \ldots, x_5) \in \mathcal{L}\) and compute the dimension of \(G := \{g \in \text{GL}_7 | gxg^{-1} \in \mathcal{L}\}\). This subgroup does not change when we subtract identity matrices, so we assume \(\mu_i = 0\) for \(i = 1, 2, \ldots, 5\). We have \(\bigcap_i \ker(x_i) = (*, *, *, 0, 0, 0, 0), \bigcap_i \ker(x_i) + \sum \ker(x_i) = (*, *, *, *, 0, 0, 0), \bigcap_{i,j} \ker(x_i \cdot x_j) = (*, *, *, *, *, 0, 0)\) so any element of \(G\) stabilizes those spaces. Therefore, \(G \subset \text{GL}_7\) has codimension at least 17 and so \(\dim(\text{GL}_7 \cdot \mathcal{L}) \geq 71\). The tangent space to \(C_n(M_d)\) at the quintuple

\[
\begin{bmatrix}
0 & 0 & 0 & e_1 & 0 & e_3 & 0 \\
0 & 0 & 0 & e_2 & e_1 & e_4 & 0 \\
0 & 0 & 0 & 0 & e_2 & e_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e_1 \\
0 & 0 & 0 & 0 & 0 & 0 & e_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

is 71-dimensional, so indeed we obtain a component. The tuple above corresponds to an \(k[y_1, \ldots, y_5]\)-module \(M\) with Hilbert function (2, 2, 3) that is generated by two elements. In the language of Section 6.3 below, this component is a part of \(\mathcal{W}_{a,b,d,a-b-c}^m\) for \((a, b, c) = (3, 3, 2)\).
Moreover, if \( \mu_i = 0 \) then \( x_i^2 \) has rank at most one, which implies that the associated module does not have the strong Lefschetz property. Note that this happens for any graded module in the open locus of this component.

**Remark 6.9.** All of the above components give rise to elementary components of \( C_n(\mathbb{M}_d) \) for \( n \) greater than in the examples. More precisely, consider a witness point \( x \in C_n(\mathbb{M}_d) \) as in each of the examples above; this point is smooth. Consider the point \( x' \in C_{n+e}(\mathbb{M}_d) \) obtained by padding the tuple \( x \) with \( e \) zero matrices. We claim that \( x' \) is smooth as well. By Lemma 3.1 the difference \( \dim T_xC_{n+e}(\mathbb{M}_d) - \dim T_xC_n(\mathbb{M}_d) \) is equal to \( e \cdot \dim C \), where \( C \subset \mathbb{M}_d \) is the space of matrices commuting with every matrix in \( x \). Now, in each case a direct check shows that this commutator is as small as possible: in the square-zero cases it is given by square-zero matrices (and scalar matrices) while in the case of Example 6.7 it is given by matrices of the shape (6.8) with the same \( (u_{ij}) \) as in \( x \). Knowing this, we see that the tangent space dimension is equal to the dimension of the locus defined analogously as in the examples, thus \( x' \) is smooth.

### 6.2. Cube nonzero cases.

**Proposition 6.10.** Let \( d \leq 7 \) and let \( (x_1, \ldots, x_n) \in C_n(\mathbb{M}_d) \) be a tuple of nilpotent matrices such that some linear combination of them has nonzero cube. Then the \( n \)-tuple belongs to a non-elementary component of \( C_n(\mathbb{M}_d) \).

**Proof.** Using the action of \( \text{GL}_n \) we assume that \( x_1^3 \neq 0 \). We put \( x_1 \) in a Jordan form. Let \( a \geq a' \) be the sizes of the two largest Jordan blocks of \( x_1 \), so \( a \geq 4 \). If the kernel of \( x_1 \) is at most two-dimensional, we conclude using Proposition 4.34 or Lemma 4.29. Otherwise, \( x_1 \) has at least three Jordan blocks, so \( a + a' \leq 6 \). If \( a \geq 2a' + 1 \), then we conclude using Lemma 4.28. If \( a \leq 2a' \) then \( a = 4 \), \( a' = 2 \), so the Jordan type of \( x_1 \) is \((4,2,1)\). Using Lemma 4.25 and Remark 4.26 we put the matrices \( x_i \) for \( i \geq 2 \) in the form

\[
 x_i = \begin{bmatrix}
 0 & 0 & 0 & b_0^{(i)} & b_1^{(i)} & c^{(i)} \\
 0 & 0 & 0 & 0 & b_0^{(i)} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & d_0^{(i)} & d_1^{(i)} & e_0^{(i)} & e_1^{(i)} & f^{(i)} \\
 0 & 0 & 0 & d_0^{(i)} & 0 & e_0^{(i)} & 0 \\
 0 & 0 & 0 & 0 & h^{(i)} & k^{(i)} \\
 \end{bmatrix}.
\]

Since \( x_i \) are nilpotent, we have \( k^{(i)} = 0 = e_0^{(i)} \) for every \( i \geq 2 \). The \((1,3)\)-entry of the commutator of \( x_i \) and \( x_j \) is \( b_0^{(i)}d_0^{(j)} - b_0^{(j)}d_0^{(i)} \), so any two pairs \((b_0^{(i)},d_0^{(i)})\) and \((b_0^{(j)},d_0^{(j)})\) with \( 2 \leq i < j \) are linearly dependent. We assume by Corollary 4.22 that \( b_0^{(i)} = 0 \) and \( d_0^{(i)} = 0 \) for each \( i \geq 3 \). Let \( y \) be the matrix with the \((3,6)\)-th entry equal to \(-b_0^{(2)}\) and all other entries zero. The tuple \((x_1 + \lambda E_{22}, x_2 + \lambda y, x_3, \ldots, x_n)\) commutes for each \( \lambda \in k \). For \( \lambda \neq 0 \) the matrix \( x_1 + \lambda E_{22} \) has two distinct eigenvalues, so such an \( n \)-tuple belongs to a non-elementary component of \( C_n(\mathbb{M}_d) \). Since components are closed, also \((x_1, \ldots, x_n)\) belongs to this component.

### 6.3. Cube zero, square nonzero cases.

**Proposition 6.10** takes care of the case when \( x_ix_jx_k \neq 0 \) for some \( i, j, k \). In this subsection we consider the next case: where \( x_ix_jx_k = 0 \) for all \( i, j, k \) but there exist some \( i, j \) such that \( x_ix_j \neq 0 \).
Now we decompose the cube-zero locus into subloci. For a tuple \( \mathbf{x} = (x_1, \ldots, x_n) \) in \( C_n(\mathbb{M}_d) \) let \( K_1(\mathbf{x}) := \bigcap_{i=1}^n \ker x_i \) and \( K_2(\mathbf{x}) := \bigcap_{i,j=1}^n \ker x_ix_j \) and \( \text{im}(\mathbf{x}) = \sum_{i=1}^n \text{im} x_i \). From cube-zero we get \( \text{im}(\mathbf{x}) \subset K_2(\mathbf{x}) \). Consider the locus

\[
W_{a,b,d-a-b}^n = \{ \mathbf{x} \in C_n(\mathbb{M}_d) \mid \dim K_1(\mathbf{x}) = a, \dim K_2(\mathbf{x}) = a + b, \forall i, j, k : x_ix_jx_k = 0 \}.
\]

and its subloci, for \( 1 \leq c \leq b \), defined by

\[
W_{a,b,d-a-bc}^n = \{ (x_1, \ldots, x_n) \in W_{a,b,d-a-b}^n \mid \dim(\text{im}(\mathbf{x}) + K_1(\mathbf{x})) = a + c \}.
\]

Observe that for \( c > b \) or \( c > n(d - a - b) \) the locus \( W_{a,b,d-a-bc}^n \) is empty. For any tuple \( \mathbf{x} \in W_{a,b,d-a-b}^n \), we have \( K_1(\mathbf{x}) \subset K_2(\mathbf{x}) \) and if we choose bases of these spaces compatibly, then the matrices \( x_i \) can be written as

\[
x_i = \begin{bmatrix} 0 & A_i & B_i \\ 0 & 0 & C_i \\ 0 & 0 & 0 \end{bmatrix}
\]

for some matrices \( A_i \in \mathbb{M}_{a \times b}, B_i \in \mathbb{M}_{d-a-b} \) and \( C_i \in \mathbb{M}_{b \times (d-a-b)} \). By construction, the common kernel of \( (A_i)_i \) is zero and the common kernel of \( (C_i)_i \) is zero. The introduction of parameters \( a, b, c \) is arbitrary, but it decomposes the cube zero locus into more approachable subloci. Below we prove that some of them are irreducible.

**Lemma 6.12.** Let \( a, b, c, d \) be positive integers, \( d - a - b = 1 \) and \( c \leq b, n \). Then the locally closed locus \( W_{a,b,d-a-bc}^n \) is irreducible of dimension \( n(a(b - c + 1) + c) + \frac{ac(c+1)}{2} + ab + bc - c^2 + a + b \).

*Proof.* In essence, this is the same parameter count as in Example 6.7. By definition, a tuple \( \mathbf{x} \in W_{a,b,d-a-bc}^n \) determines a flag \( K_1(\mathbf{x}) \subset K_2(\mathbf{x}) + \text{im}(\mathbf{x}) \subset K_2(\mathbf{x}) \) so the locus \( W_{a,b,d-a-bc}^n \) is fibered over the flag variety of subspaces \( V_1 \subset V_2 \subset V_3 \subset V \cong k^d \) where \( (\dim V_1, \dim V_2, \dim V_3) = (a, a+c, a+b) \). This flag variety has dimension \( ab + bc - c^2 + a + b \). It remains to compute the fiber, so we assume that \( K_1(\mathbf{x}) = \langle e_1, \ldots, e_a \rangle, K_1(\mathbf{x}) + \text{im}(\mathbf{x}) = \langle e_1, \ldots, e_a+c \rangle, K_2(\mathbf{x}) = \langle e_1, \ldots, e_a+b \rangle \). In this case, the matrices \( x_i \) have the form (6.11) with \( C_i \) having nonzero entries only in the first \( c \) rows. Up to a linear change of coordinates, which amounts to \( nc \) parameters, we assume \( C_i = e_i \) for \( i \leq c \) and \( C_i = 0 \) for \( i > c \). The commutativity condition then reduces to saying that for \( 1 \leq i < j \leq c \) the \( i \)-th column of \( A_j \) is the \( j \)-th column of \( A_i \) and additionally, the first \( c \) columns of \( A_i \) are zero for \( i > c \). Therefore we have \( na(b - c + 1) + ac^2 \) linear and independent conditions, so be obtain an affine space, in particular the whole fiber is irreducible. The flag variety is homogeneous under the action of \( \text{GL}(V) \), the whole locus \( W_{a,b,d-a-bc}^n \) is an image of the product of the fiber and \( \text{GL}(V) \) so this locus is irreducible. The dimension count follows. \( \Box \)

The following lemma will be frequently used in the proof of the main theorem of this section. It is a generalization of the case \( a_m = 3 \) of Lemma 4.28.

**Lemma 6.13.** Let \( \mathbf{x} \in C_n(\mathbb{M}_d) \) be a cube-zero tuple such that \( \dim \sum_{i,j} \text{im}(x_ix_j) = 1 \) and \( \dim K_2(\mathbf{x}) = d - 1 \). Then \( \mathbf{x} \) belongs to a non-elementary component of \( C_n(\mathbb{M}_d) \).

*Proof.* Let \( f \in V \setminus K_2(\mathbf{x}) \), then \( V = K_2(\mathbf{x}) + kf \). Let \( e \) span \( \sum_{i,j} \text{im}(x_ix_j) \). Then every \( x_ix_j \) is a multiple of the unique rank one matrix that sends \( f \) to \( e \). Replacing matrices with linear combinations, we may assume \( x_1^2 \neq 0 \) and then \( x_1x_i = 0 \) for all \( i \geq 2 \). Rearrange our fixed basis
of $V$ such that the first basis element is $e$. In this basis, our tuple becomes

\[
\begin{pmatrix}
0 & 0 & A'_1 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & C'_1 \\
\vdots & & & & \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & A'_n & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & C'_n \\
\vdots & & & & \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where $*$ denotes some irrelevant yet possibly nonzero parts. Consider the matrix $y$ with the middle block $C'_iA'_1$ and all other blocks zero. The relations $x_i x_i = x_i x_1 = 0$ for all $i \geq 2$ imply that $A'_i C'_i = 0$ and $A'_i C'_i = 0$ for these $i$ and this proves that the matrix $y$ commutes with $x_i$ for each $i \geq 2$. The trace of $y$ is nonzero, so for $\lambda \neq 0$ the matrix $x_1 + \lambda y$ has nonzero trace so it has a nonzero eigenvalue, so the family $(x_1 + \lambda y, x_2, \ldots, x_n)$ proves that $x$ lies on a non-elementary component.

\[\square\]

**Theorem 6.14.** Let $d \leq 7$ and let $a$, $b$, $c$, $n$ be positive integers with $c \leq b$, $c \leq n(d - a - b)$ and $a + b < d$ and let $\mathcal{V} = \mathcal{W}^n_{a,b,d-a-b;c} := (\mathbb{C})^n + \mathcal{W}^n_{a,b,d-a-b;c}$. Then the following holds:

1. For $(a,b,c,d) = (3,3,2,7)$ and $n \geq 5$ the variety $\mathcal{V}$ is an irreducible component of $\mathcal{C}_n(\mathbb{M}_d)$.

2. For $(a,b,c,d) = (2,2,2,7)$ and $n \geq 5$ the variety $\mathcal{V}$ has a component $\mathcal{Z}$ which is the transpose of the locus from case (1) and $\mathcal{Z}$ is a component of $\mathcal{C}_n(\mathbb{M}_d)$, while all other components of $\mathcal{V}$ belong to non-elementary components of $\mathcal{C}_n(\mathbb{M}_d)$.

3. In all other cases the variety $\mathcal{V}$ belongs to the union of non-elementary components of $\mathcal{C}_n(\mathbb{M}_d)$.

**Proof.** Fix a tuple $x \in \mathcal{W}^n_{a,b,d-a-b;c}$. First we discard several easy cases. If $n \leq 2$ then it is classically known that $\mathcal{C}_n(\mathbb{M}_d)$ is irreducible. If $n = 3$ then it is also known that $\mathcal{C}_n(\mathbb{M}_d)$ is irreducible for $d \leq 10$, see [Han05, Šiv12]. If $a = 1$ and $d - a - b = 1$ then dim $\sum_{i,j} \text{im}(x_i x_j) = 1$ so the tuple lies on a non-elementary component by Lemma 6.13. So, after possibly transposing the matrices, we assume $a \geq 2$.

Assume now $c = 1$. We will check that dim $\mathcal{K}_2(x) = d - 1$ and that dim $\sum_{i,j} \text{im}(x_i x_j) = 1$. Since $\text{im}(x) \not\subset \ker(x)$, up to linear coordinate change we assume $\text{im}(x) \not\subset \ker(x)$. Let $w \in V$ be such that $x_1(w) \not\in \ker(x)$ and denote this element by $v$. Since $x_1(w) \in \text{im}(x) \subset kv + \ker(x)$, we have $x_1(w) = \alpha v + k_1$ where $\alpha \in k$ and $k_1 \in \ker(x)$. Then $x_i(v) = x_i(x_1(w)) = x_i(x_1(w)) = x_1(\alpha_i v + k_i)$ so after linear change of coordinates we assume that $x_i(v) = 0$ for $i = 2,3,\ldots,n$. In particular, dim $\sum_{i,j} \text{im}(x_i x_j) = 1$. Since $x_2, \ldots, x_n$ annihilate $\text{im}(x)$ we have that $(x_2, \ldots, x_n) \cdot (x_1, \ldots, x_n)$ annihilates $V$ and so $\bigcap \ker(x_i x_j) = \ker(x_1^2)$ is of codimension one, as claimed. By Lemma 6.13, also this tuple lies on a non-elementary component. In the following we assume $c \geq 2$. Then $b \leq 4$. We will subdivide the remaining cases with respect to $d - a - b$.

**Case 1, $d - a - b = 1$.** Example 6.7 proves the part (1) for $n = 5$. To obtain examples with bigger $n$ simply add zero matrices, see Remark 6.9. It remains to prove that the remaining cases are non-elementary. The locus $\mathcal{W}^n_{a,b,d-a-b;c}$ is irreducible by Lemma 6.12.

Suppose first $b = c$. We claim that $\mathcal{W}^n_{a,b,d-a-b;c}$ contains cyclic tuples: the tuples $x = (y_1, \ldots, y_n)$ for which there exists a vector $v \in V$ such that $V = k[x_1, \ldots, x_n] \cdot v$. If $a \leq \left(\frac{b+1}{2}\right)$ then the claim follows from Example 4.7. If $a > \left(\frac{b+1}{2}\right)$ then let $\delta = a - \left(\frac{b+1}{2}\right)$ and consider the tuple corresponding to the algebra

\[
\frac{k[y_1, \ldots, y_b; y'_1, \ldots, y'_\delta]}{(y_1, \ldots, y_b)^3 + (y_1, \ldots, y_b; y'_1, \ldots, y'_\delta)}
\]
in its monomial basis. In our case \( d \leq 7 \) so the only option is \( a = 4, b = 2 \), then \( b + \delta = 3 \leq n \) so indeed the above algebra gives rise to a tuple in \( \mathcal{W}_{a,b,d-a-b,c} \); if \( n > b + \delta \) we pad the tuple with zero matrices. The set of cyclic tuples is open and corresponds to the locus of algebras of degree \( 1 + a + b \) in the ADHM construction, see §3.2. The Hilbert scheme of up to 7 points is irreducible [CEVV09, Theorem 1.1], so these tuples belong to the closure of the principal component.

In the following we can thus assume \( b > c \). We already have \( c \geq 2 \), so \( b \geq 3 \) and so \( a \leq 3 \). Since \( a \leq 3 \) and \( c \geq 2 \), we have \( a \leq \binom{c+1}{2} \). Recall that \( n \geq 4 \geq b \). Using Example 4.9 and irreducibility of \( \mathcal{W}_{a,b,d-a-b,c} \) we assume that the linear span of \((A_iC_j)_{1\leq i,j\leq n}\) is \( a \)-dimensional. We now make a series of reductions to relate the present case to the previous one. By a version of [Han05, Lemma 2.7] we may and will assume that \( B_i = 0 \). Using the \( GL_n \)-action, we assume that \( C_i = 0 \) for \( i > c \). Consider the subtuple \((x_1,\ldots,x_c)\). The matrices \( C_1,\ldots,C_c \) have nonzero common cokernel, so up to linear transformation some of the rows are identically zero in each \( x_i \), similarly for \( A_1,\ldots,A_c \) if those have nonzero common cokernel. Erase those rows and the corresponding columns. We obtain a commuting tuple which falls in the cube zero, square nonzero case; in fact it belongs to the case \( b = c \) just considered and is even cyclic, thus corresponds to an algebra.

By slight abuse of notation, we refer to it as \((x_1,\ldots,x_c)\). We know already that it lies on a non-elementary component, however, we need a bit more, so we make a provisional definition. We say that the tuple \((x_1,\ldots,x_c)\) is \textit{deformable in the middle} if there exists a tuple

\[
\zeta_i = \begin{bmatrix}
0 & 0 & 0 \\
0 & D_i & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

such that \((x_i + \lambda\zeta_i)_{i=1}^c\) is a commuting tuple for every \( \lambda \in k \) and moreover at least one \( \zeta_i \) has a nonzero eigenvalue. We now show that for \( a = 2 \) the tuple is deformable in the middle:

- If \( c \geq 3 \) then the tuple corresponds to an algebra with Hilbert function \((1,c,2)\). The deformation from [CEVV09, Proposition 4.10] gives a deformation in the middle for the monomial basis \( y_{c-1}^2, y_{c}^2, y_{c}, \ldots, y_{1} \). (The referenced result requires a generality assumption on the tuple, but we may impose arbitrary such assumptions.) Specifically, it gives a deformation

\[
(y_{i}y_{j} | i \neq j) + (y_1^2 - \lambda y_1 - a_1 y_{c-1}^2 - b_1 y_1^2) + (y_i^2 - a_i y_{c-1}^2 - b_i y_{c}^2 | i \geq 2),
\]

parameterized by \( \lambda \) where \( a_i, b_i \in k \) are constants.

- In the special case \( c = 2 \), the tuple corresponds to an algebra with Hilbert function \((1,2,2)\). This algebra is the quotient of a polynomial ring by an ideal generated by a single quadric — which we may assume by genericity has full rank — and all cubics, so it is isomorphic to \((y_1y_2, y_1^3, y_2^3)\). For \( \lambda \in k \) the deformation

\[
\frac{k[y_1,y_2]}{(y_1y_2, y_1^3, y_2^3 - \lambda y_2^2)}
\]

in the basis \( y_2^2 - \lambda y_2, y_1^2, y_2, y_1, 1 \) gives a deformation in the middle.

Going back to our original tuple, we see that it lies on a non-elementary component whenever \((x_1,\ldots,x_c)\) is deformable in the middle; this is because the first \( c \) columns of \( A_i \) matrices, for \( i > c \), are zero. Therefore, we automatically get that the tuple lies on a non-elementary component whenever \( a = 2 \). We have already reduced to the case \( a \leq 3 \), so it remains to consider \( a = 3 \). Since \( b \geq 3 \) and \( a + b \leq 7 - 1 \), we get \( b = 3 \). Since \( b > c \geq 2 \), we get \( c = 2 \). In this case for \( n \geq 5 \)
we have a component, see Example 6.7, so we assume \( n \leq 4 \), hence \( n = 4 \). Consider the tuple corresponding to the module whose dual generators are \( Q = z_1^2e_1^* + z_2e_2^* + z_3^2e_3^* \), \( L = z_3e_1^* + z_4e_2^* \). In tensor notation, in the basis \( e_1^*, e_2^*, e_3^*, L, z_1^2e_1^*, z_2e_2^*, z_1e_1^* + z_2e_2^*, z_2e_3^* \), \( Q \) it is given by the following matrix with \( \lambda = 0 \):

\[
\begin{bmatrix}
0 & 0 & 0 & e_3 & e_1 & 0 & 0 \\
0 & 0 & 0 & e_4 & e_2 & e_1 & 0 \\
0 & 0 & \lambda e_1 & 0 & 0 & e_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e_1 & 0 \\
0 & 0 & 0 & 0 & \lambda e_1 & e_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda e_1 \\
\end{bmatrix}
\]

The above tuple commutes for arbitrary \( \lambda \). For \( \lambda \neq 0 \) we get a tuple \( (x_1, \ldots, x_4) \) with \( x_1 \) having eigenvalues 0 and \( \lambda \). The tuple is then a product of

1. a tuple of \( 3 \times 3 \) matrices acting on the \( 3 \)-dimensional generalized eigenspace of \( x_1 \) for eigenvalue \( \lambda \). This tuple lies in the principal component.
2. a tuple of \( 4 \times 4 \) matrices acting on the \( 4 \)-dimensional generalized eigenspace of \( x_1 \) for eigenvalue zero. This tuple is square-zero.

By Proposition 4.27 this shows that our original tuple lies on the concatenation of \( 4 \times 4 \) square-zero component and the \( 3 \times 3 \) principal component. Those components have dimensions 24 and 18 respectively by §6.1 and §3.4 so by Proposition 4.27 the concatenation has dimension \( 24 + 18 + 2 \cdot 3 \cdot 4 = 66 \). A direct check shows that the tangent space to \( C_4(M_r) \) at our tuple is \( 66 \)-dimensional, so our tuple is a smooth point of this component and so the whole locus \( \mathcal{W}^4_{3,3,1,2} \), being irreducible, is contained in this component.

**Case 2, \( d - a - b \geq 2 \).** The spaces \( V/\left( \sum_{i,j} \text{im}(x_ix_j) \right) \) and \( \cap \ker(x_j^T x_i^T) \) are dual and they have dimension \( d - \text{dim} \left( \sum_{i,j} \text{im}(A_iC_j) \right) \). If they have codimension one, then the tuple \( x^T \) falls into Case 1 (this concerns in particular the component \( Z \) from part (2)). Since we already considered Case 1, below we assume \( \text{dim} \sum_{i,j} \text{im}(A_iC_j) \geq 2 \). In particular, \( a \geq 2 \) so \( b \leq 3 \).

It also suffices to consider \( n \)-tuples \( (x_1, \ldots, x_n) \in \mathcal{W}_{a,b,d-a-b}^n \) with \( x_i \) of the form (6.11) where some linear combination of the matrices \( A_1, \ldots, A_n \) has rank at least 2. Indeed, if each linear combination of the matrices \( A_i \) has rank at most 1, then the matrices \( A_i \) have either a common kernel of codimension 1 or a common 1-dimensional image, see for example [AL81, Lemma 2]. The common kernel of the matrices \( A_i \) is trivial, as discussed below (6.11), and \( b \geq 2 \), so these matrices do not have a codimension one common kernel. If the matrices \( A_i \) have common 1-dimensional image, then \( \dim \sum_{i,j} A_iC_j = 1 \), a contradiction.

Assume first that \( b = 2 \). As above, consider an arbitrary \( n \)-tuple \( (x_1, \ldots, x_n) \in \mathcal{W}_{a,2,d-a-2}^n \) with \( x_i \) of the form (6.11). By the argument above, some linear combination of the matrices \( A_i \) has rank 2, and using the actions of \( \text{GL}_n \) and \( \text{GL}_d \) we may assume that \( A_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \). Let

\[
A_i = \begin{bmatrix} A_i^1 & A_i^2 \\ A_i^3 & A_i^4 \end{bmatrix}
\]

for \( i \geq 2 \). The commutativity then implies \( C_i = A_i'C_1 \) for each \( i \geq 2 \), and hence \( 0 = \cap_{i=1}^n \ker C_i = \ker C_1 \). It follows that \( C_1 \) is injective and consequently \( d - a - 2 = 2 \). We may assume that \( C_1 = I \), and then \( C_i = A_i' \) for \( i \geq 2 \). Commutativity then reduces to \( [A_i', A_j'] = 0 \) for \( i, j \geq 2 \) and \( A_i'' = 0 \) for each \( i \geq 2 \). The maximal dimension of a commutative vector space of \( 2 \times 2 \) matrices is 2, therefore we use \( \text{GL}_n \)-action to assume that \( A_i' = 0 \) for \( i \geq 3 \) and that \( A_2' \) is singular. Since \( A_2 \) is singular, there exist nonzero vectors \( u, v \in k^2 \) such that \( A_2'u = 0 \) and
The matrix
\[ y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
then commutes with \( x_i \) for each \( i \geq 2 \). If \( v^T u \neq 0 \), then \( y \) has two distinct eigenvalues, and if \( v^T u = 0 \), then for \( \lambda \neq 0 \) the matrix \( x_1 + \lambda y \) is nilpotent, but with nonzero cube, so the \( n \)-tuple \((x_1, \ldots, x_n)\) in both cases belongs to a non-elementary component, in the second case, by Proposition 6.10. This in particular concludes the proof of part (2). In the rest of the proof we assume that \( b = 3 \). This forces \( a = 2 \) and \( d - a - b = 2 \).

If \( c = 2 \), let \((x_1, \ldots, x_n) \in W^{n}_{2,3,2,2} \) be an \( n \)-tuple of commuting matrices of the form (6.11). As \( c = 2 \) and \( b = 3 \), there exists a nonzero vector \( u \in k^b \) such that \( u^T C_i = 0 \) for each \( i = 1, \ldots, n \). Consider the transposed tuple \((x_1^T, \ldots, x_n^T)\). The vector \( u \) forces it to have at least three-dimensional common kernel, so either this tuple falls into the Case 1 or into the case \( b = 2 \) above.

We thus assume that \( c = 3 \). In the remaining case our strategy is to find deformations in the middle. The matrices \( B_i \) are irrelevant here, so we assume \( B_i = 0 \), so the associated module is naturally graded. We begin by proving that the matrices \((x_1, \ldots, x_n)\) span an at most three dimensional space. We use apolarity §4.1. Our matrices in the standard basis correspond to a module \( M \), where \( M \subset F^* \) is generated by two (homogeneous) quadrics \( Q_1, Q_2 \). Using Example 4.10, we get our claim. So we reduce to the case \( n = 3 \) and \( S = k[y_1, y_2, y_3] \) and below we assume \( n = 3 \).

We will use downward induction on \( d \) and for that reason we will consider also \( d = 8, 9 \) and \( a = 2, 3 \) even though such large \( d \) fall outside the scope of the current theorem. For a tuple \( x \) we also introduce the number \( a^T = a^T (x) := \dim \ker (x^T) \). This number can be computed from the associated module and by abuse of notation we will view \( a^T \) as a function of a module. In this setting it is simply the minimal number of generators of this module, see §3.5.

We first tackle the case \((a, a^T, b, c, d) = (3, 3, 3, 3, 9)\). Since \( d - a - b = 3 = a^T \), we have \( \ker (x) \supset \ker (x) \). By assumptions, the matrices \( A_1 \) have zero common kernel and cokernel. Assume that no linear combination of those matrices has full rank. By [EH88, Theorem 1.1] this implies that either all \( A_i \) are skew-symmetric or they form a compression space, which means that, up to base change, they jointly have the shape

\[ \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}. \]

We first show that the compression space case is in fact impossible. Suppose that the matrices have the shape (6.15). The common kernel of \( A_1, A_2, A_3 \) is zero, so up to linear change of coordinates we have

\[ A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}. \]

The condition \( A_i C_j = A_j C_i \) for all \( i, j \) then forces all \( C_i \) matrices to have zero last row, which implies that \( \dim (\ker (x) + K_1 (x)) \leq 5 \) and thus contradicts \( a + c = 6 \). Therefore, we assume that \( A_i \) are skew-symmetric. Since they have zero common kernel, in fact the matrices \( A_i \) form
a basis of the space of skew-symmetric matrices. But then commutativity implies that $C_i = 0$ for all $i$, a contradiction. Summing up, there exists a linear combination of $A_i$ that has full rank. Repeating the argument for $C_i$ and performing linear operations, we assume $A_1 = C_1 = 1$. Commutativity with $x_i$ implies $A_i = C_i$ for all $i$ and then $A_i$, $A_j$ commute for any $i$, $j$. We have now an obvious deformation in the middle that adds a block $A_i$ in the middle of each $x_i$. Since $A_1$ has three nonzero eigenvalues, this deformation splits the module into a degree three and six modules. (This case did not use the assumption $n = 3$.)

Now we consider the case $(a, a^T, b, c, d) = (3, 2, 3, 3, 8)$ and additionally make the following assumption: if $Q_1, Q_2 \in F_2$ are the homogeneous dual generators for the module $M$ associated to our tuple, then no linear combination of $Q_1$, $Q_2$ is “rank one”, i.e., annihilated by a two-dimensional subspace of $S_1$. This is the most challenging part. Consider the dual module $M^\vee$.

Since $a = 3$, the dual module is minimally generated by three elements $3.5$ so we present it as a quotient of $S^{\oplus 3}$. By apolarity §4.1 our module $M$ is a submodule of $(S^{\oplus 3})^*$. Since $a^T = 2$, the module $M$ is generated by two elements corresponding to $e_7$, $e_8$ vectors. Since $B_i = 0$ for all $i$, the module $M$ is naturally graded and the two above generators are of degree two.

We will show that $M^\perp$, the annihilator of $M$ in $S^{\oplus 3}$, has a degree two element among minimal homogeneous generators. If it is so, then replacing this generator $g \in M^\perp$ by $S_{1g}$ we get an inclusion $N \subset M^\perp$ with $\dim_k M^\perp/N = 1$. Therefore, we have $M^\vee \simeq S^{\oplus 3}/M^\perp$ is a quotient of a module $S^{\oplus 3}/N$ of degree 9. Since $B_i$ are assumed to be zero, the modules $M$, $M^\vee$, $S^{\oplus 3}/N$ are all graded and moreover $H_{S^{\oplus 3}/N} = (3, 3, 3)$. This shows that the module $S^{\oplus 3}/N$ corresponds to a tuple of matrices with invariants $(a, a^T, b, c, d) = (3, 3, 3, 3, 9)$. In terms of matrices, this means that the transpose of our tuple arises from a $9 \times 9$ tuple by erasing the first row and column. The deformation in the middle in the case $9 \times 9$, constructed above, gives a deformation in the middle in our case.

To look for generators of $M^\perp$ we will look at the syzygies of $M$. The module $M$ is generated by $Q_1, Q_2$ so we have $M = S^{\oplus 2}/K$. Since $\dim M_1 = 3$, we have $\dim K_1 = 3$. Let $k_1, k_2, k_3 \in S^{\oplus 2}$ be a $k$-basis of $K_1$.

Assume first that there are no linear forms $l_1, l_2, l_3 \in S$ not all equal to zero such that $\sum k_i l_i = 0$. Write $k_i = k_{i1} e_1 + k_{i2} e_2$ so that the vector $(k_1, k_2, k_3)$ becomes a matrix

$$
\begin{pmatrix}
\begin{array}{ccc}
k_{11} & k_{12} & k_{13} \\
k_{21} & k_{22} & k_{23}
\end{array}
\end{pmatrix}.
$$

Let $\Delta_i$ be the minor of this matrix obtained by removing $i$-th column. By a general fact, the vector $(\Delta_1, -\Delta_2, \Delta_3)^T$ lies in the kernel of this matrix. Let us translate this to the resolution language. This vector then becomes a quadratic syzygy for the module $M$. Our assumption on the nonexistence of linear forms $l_1, \ldots, l_3$ such that $\sum k_i l_i = 0$ implies that there are no linear syzygies between $k_1, k_2, k_3$ so the above syzygy is minimal and the minimal resolution of $M$ looks like

$$
0 \leftarrow M \leftarrow S^{\oplus 2}(-2) \leftarrow S^{\oplus 3}(-3) \oplus G_1 \leftarrow S^{\oplus 1}(-5) \oplus G_2 \leftarrow G_3 \leftarrow 0
$$

for some graded free $S$-modules $G_1, G_2, G_3$. Now by Theorem 3.23 the resolution of $S^{\oplus 3}/M^\perp$ is dual to the resolution of $M$ twisted by $n = 3$ so the minimal syzygy $S(-5)$ above corresponds to a minimal generator of degree $5 - 3 = 2$ in $M^\perp$ and we win.

It remains to consider the case when there are linear forms $l_1, l_2, l_3 \in S$ not all equal to zero such that $\sum k_i l_i = 0$. In other words assume that $M$ has some linear syzygy. Green’s Linear
Syzygy Theorem [Eis05, Theorem 7.1] implies that there are some linear forms annihilating a generator of our module. Below we prove this directly for our case, without using the theorem.

If \( l_1, l_2, l_3 \) are linearly independent, then up to coordinate change \((l_1, l_2, l_3) = (y_2, y_1, 0)\). By a direct computation, the triples \((k'_1, k'_2, k'_3)\) of linear forms such that \( \sum k'_i y_i = 0 \) are linear combinations of the rows of

\[
B = \begin{bmatrix}
0 & -y_3 & y_2 \\
y_3 & 0 & -y_1 \\
-y_2 & y_1 & 0
\end{bmatrix}
\]

so \((k_{i1}, k_{i2}, k_{i3})\) are linear combinations of the rows of \( B \) for \( i = 1, 2 \) so

\[
\begin{bmatrix}
k_{11} & k_{12} & k_{13} \\
k_{21} & k_{22} & k_{23}
\end{bmatrix} = A \cdot B
\]

for some matrix \( A \in \mathbb{M}_{2 \times 3}(k) \) of rank two. The matrix \( ABA^T \) is nonzero and antisymmetric so it has the form

\[
\begin{bmatrix}
0 & \ell \\
-\ell & 0
\end{bmatrix}
\]

for a non-zero linear form \( \ell \). This matrix is equal to \([k_{ij}] \cdot A^T\), so \(-\ell e_2, \ell e_1 \in K_1\). After a coordinate change we assume \( \ell = y_3 \) and view \( S^{\oplus 3}/M^\perp \) as a module over \( k[y_1, y_2]\). Since \( M^\perp_1 \) is 3-dimensional, it generates at most a 6-dimensional subspace of the 7-dimensional space \( M^\perp_2 \).

So there is a minimal quadric generator and we conclude.

If \( l_1, l_2, l_3 \) are linearly dependent, then up to coordinate change we have \((l_1, l_2, l_3) = (y_2, y_1, 0)\), so \( k_1 = \lambda y_1 e_1 + \mu y_1 e_2 \) and \( k_2 = -\lambda y_2 e_1 - \mu y_2 e_2 \) so both \( y_2 \) and \( y_1 \) annihilate \( \lambda Q_1 + \mu Q_2 \), which is thus a rank one quadric which contradicts our assumption that no such quadric exists in \( \langle Q_1, Q_2 \rangle \). This concludes the case \((a, b, c, d) = (3, 3, 3, 8)\).

The last remaining case for \( b = 3 \) is \((a, b, c, d) = (2, 3, 3, 7)\). As explained at the beginning of the Case 2, we have a matrix \( A_i \) of rank two, in particular the joint image of the matrices \( A_i \) spans the first two coordinates. Using \( c = 3 \) we deduce \( a^T = 2 \).

From the point of view of modules, in this case we consider the module \( S^{\oplus 2}/M^\perp \) with Hilbert function \((2, 3, 2)\). We have \( \dim M^\perp_1 = 3 \), so that \( \dim S_1 M^\perp_1 \leq 9 \) while \( \dim M^\perp_2 = 10 \). Therefore \( M^\perp \) has a minimal generator of degree two and so arguing as in the \( d = 8 \) case, we get that the dual of \( M \) is a quotient of a graded module \( N \) with Hilbert function \((2, 3, 3)\). The module \( S^{\oplus 2}/M^\perp \) is dual to \( M \) and \( a(M^\vee) = a^T(M) = 2 \) and \( a^T(M^\vee) = 2 \). As in the previous case, we get \( a(N) = 3, a^T(N) = 2, b(N) = c(N) = 3 \). Consider the degree zero part of \( S^{\oplus 2}/M^\perp \).

Suppose first that no element of this part is annihilated by a two-dimensional subspace of \( S_1 \), then the same holds for \( N \). In this case we reduce to the case \( d = 8 \): the deformation in the middle for \( N \) has been constructed above.

It remains to consider the case where there is a degree zero element of \( S^{\oplus 2}/M^\perp \) annihilated by a two-dimensional subspace of \( S_1 \). Consider homogeneous dual generators for this module. Since \( a = 2 \) and \( S^{\oplus 2}/M^\perp \) has Hilbert function \((2, 3, 2)\), these are two quadrics. So we consider the locus \( \mathcal{L} \) of pairs of quadrics \( Q_1, Q_2 \in (S^{\oplus 2})_2^* \) such that two conditions hold:

1. some non-zero linear combination of \( Q_1, Q_2 \) has “rank one”, i.e., it is annihilated by a codimension one space of \( S_1 \),
2. \( \dim_k(S_1 Q_1 + S_1 Q_2) \leq 3 \).

Up to linear transformations we may assume \( Q_1 = z_1^2 e_1^* \) and either \( y_1 Q_1 = y_2 Q_2 \) or \( y_1 (Q_1 - Q_2) = 0 \). The locus with \( y_1 (Q_1 - Q_2) = 0 \) lies in the closure of the one where \( y_1 Q_1 = y_2 Q_2 \). The locus where \( y_1 Q_1 = y_2 Q_2 \) is just an affine space. This shows that \( \mathcal{L} \) is irreducible.
To prove that the modules obtained from $L$ lie on a non-elementary component we will consider a deformation that is not in the middle, so below we also take into account the $B_i$ matrices and so we consider all $n$, not necessarily $n = 3$. By Corollary 4.23 we may assume $n \geq 4$. The addition of arbitrary $B_i$ matrices does not break the irreducibility of our locus (on the level of dual generators it amounts to adding arbitrary linear forms to each $Q_i$). For every $\lambda \in k$ consider the commuting $n$-tuple written in tensor notation as

$$x(\lambda) := \begin{bmatrix}
\lambda e_1 & 0 & e_2 & e_1 & 0 & 0 & 0 \\
0 & 0 & e_3 & 0 & e_1 & 0 & e_4 \\
0 & 0 & 0 & \lambda e_1 & 0 & 0 & e_1 \\
0 & 0 & 0 & 0 & 0 & e_2 & e_1 \\
0 & 0 & \lambda e_3 & 0 & \lambda e_1 & e_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

so that the matrices $x_i$ are zero for $5 \leq i \leq n$. The tuple $x(0)$ corresponds to a module from $L$ with $B_i$ matrices added. For $\lambda \neq 0$ the tuple $x(\lambda)$ corresponds to a module in the concatenation of the $4 \times 4$ square zero component in $C_n(M_4)$ and the principal component in $C_n(M_3)$. By Proposition 4.27 and using the formulas for dimensions given in §3.4 and §6.1 this component has dimension $34 + 8n$. For $n = 4$ we directly check that $T_{x(0)}C_4(M_7)$ is 66-dimensional. We also directly compute that a matrix commuting with all elements of $x(0)$ is an element of $k[x(0)]$ and $\dim_k k[x(0)] = 8$. As in Remark 6.9 this shows that for $n \geq 4$ the tangent space $T_{x(0)}C_n(M_7)$ has dimension $66 + 8(n - 4) = 34 + 8n$. Thus $x(0)$ is a smooth point of the concatenated component. By irreducibility the whole locus lies in the corresponding non-elementary component of Quot. This concludes the whole proof.

6.4. Square-zero cases. In this section we consider commuting matrices such that the square of any linear combination of them is zero. Since $\text{char } k \neq 2$, we equivalently assume $x_i x_j = 0$ for all $i, j$. This implies that the image of each $x_i$ is contained in the common kernel. Extending a basis of $\sum \text{im } x_i$ to a basis of $V$ we get matrices that have nonzero entries only in the top-right $m \times (d - m)$ corner.

For any $d, m \in \{1, \ldots, d-1\}$ and any $n \geq 1$ define

$$Z_{m,d-m}^{n,\text{upper}} = \left\{ \left( \begin{bmatrix}
0 & A_1 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
\end{bmatrix}, \ldots, \left( \begin{bmatrix}
0 & A_n \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
\end{bmatrix} \right) \right) : A_1, \ldots, A_n \in M_{m \times (d-m)} \right\} \subset C_n(M_d).$$

and let $Z_{m,d-m}^n = \text{GL}_d \cdot Z_{m,d-m}^{n,\text{upper}}$. The discussion above shows that a square-zero tuple $(x_i)$ with $\dim \sum \text{im } x_i = m$ corresponds to a point in $Z_{m,d-m}^n$. The locus $Z_{m,d-m}^n$ is irreducible for all values of $n, m, d$ and its dimension is $(n + 1)m(d - m)$ by §6.1.

Let $V_{m,d-m}^n \subset C_n(M_d)$ consist of tuples of commuting matrices with lower left $(d - m) \times m$ corner equal to zero. We have an inclusion $Z_{m,d-m}^{n,\text{upper}} \subset V_{m,d-m}^n$. Moreover, for every $t \in k$ and a commuting tuple of matrices $\begin{bmatrix}
B_i \\
0 \\
C_i \end{bmatrix}$ in $V_{m,d-m}^n$, the tuple $\begin{bmatrix} tB_i & A_i \\
0 & tC_i \end{bmatrix}$ commutes as well. Letting $t$ go to zero, we get that $Z_{m,d-m}^{n,\text{upper}} \subset V_{m,d-m}^n$ is a retract. Let $W_{m,d-m}^n$ be the intersection of the principal component of $C_n(M_d)$ with $V_{m,d-m}^n$. The proof in §3.4 that the principal component is irreducible and of dimension $d^2 + (n - 1)d$, adapts immediately and shows that $W_{m,d-m}^n$ is irreducible and of dimension $d^2 - (d - m)m + (n - 1)d$, since a polynomial in a matrix with zero lower left $(d - m) \times m$ corner is a matrix of the same shape. The retraction $V_{m,d-m}^n \to Z_{m,d-m}^{n,\text{upper}}$
restricts to a retraction $\mathcal{W}^{m,m-d-m} \rightarrow \mathcal{Z}^{n,m-d-m}_{\text{upper}} \cap \mathcal{W}^{n,m-d-m}$. We define

$$\pi^{n}_{m,d-m} : \mathcal{W}^{n,m-d-m} \rightarrow \mathcal{Z}^{n,m-d-m}_{\text{upper}}$$

as the composition of the above retraction and the inclusion in $\mathcal{Z}^{n,m-d-m}_{\text{upper}}$. The image of $\pi^{n}_{m,d-m}$ is $\mathcal{Z}^{n,m-d-m}_{\text{upper}} \cap \mathcal{W}^{n,m-d-m}$ so it is closed, but not necessarily the whole $\mathcal{Z}^{n,m-d-m}_{\text{upper}}$.

In the following theorem we restrict to $n \geq 4$, since for $n \leq 3$ the variety $C_{n}(\mathcal{M}_{d})$ is irreducible for all $d \leq 10$ by [MT55, Šiv12].

**Theorem 6.16.** Let $n \geq 4$ and $d \leq 7$ be arbitrary and $1 \leq m \leq d-1$. Then the following holds:

(a) If $m = 1$ or $m = d - 1$, then $\mathcal{Z}^{n,m-d-m}_{\text{upper}}$ belongs to the principal component.

(b) If $m \in \{2, d - 2\}$ and $n < d$, then $\mathcal{Z}^{n,m-d-m}_{\text{upper}}$ belongs to the principal component.

(c) The variety $\mathcal{Z}^{n}_{3,3}_{\text{upper}}$ belongs to the principal component for $n < 6$.

(d) If $d = 7$ and $m \in \{3, 4\}$, then $\mathcal{Z}^{n}_{4,m-d-m}_{\text{upper}}$ belongs to the concatenation of $(kI)^{4} + \mathcal{Z}^{3}_{2,2}_{\text{upper}}$ and the principal component of $C_{4}(\mathcal{M}_{d})$.

(e) In all other cases the set $(kI)^{n} + \mathcal{Z}^{n,m-d-m}_{\text{upper}}$ is an irreducible component of $C_{n}(\mathcal{M}_{d})$.

**Proof.** Case (a). Up to transposition we may assume $m = 1$. Up to $\text{GL}_{d}$ and $\text{GL}_{n}$ action we have $x_{i} = E_{1,i+1}$ for $i = 1, 2, \ldots, s$ and $x_{i} = 0$ for $i > s$. Then our tuple is a limit of commuting tuples of the form $(E_{1,i+1} + \lambda E_{i+1,j+1} : i = 1, 2, \ldots, s)$ which belong to the principal component.

Case (b). Recall that $d > n \geq 4$, so $d \geq 5$. Up to transposition we may assume $m = 2$. Additionally, by the same argument as in Corollary 4.23 we assume $n = d - 1$. We will show that the image of the map $\pi = \pi^{d-1}_{2,d-2}$ contains an open subset of $\mathcal{Z}^{d-1}_{2,d-2}_{\text{upper}}$. We define $x_{j} = x^{j}$ for $j = 1, \ldots, n = d - 1$ where

$$x = \begin{bmatrix}
-1 & 1 & 1 & \ddots & 1 \\
1 & -1 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & 1 \\
1 & \ddots & \ddots & \ddots & 1 \\
1 & \ddots & \ddots & \ddots & 1 \\
\end{bmatrix}.$$ 

The matrix $x_{1}$ has all eigenvalues distinct. We directly verify that the point $(x_{1}, \ldots, x_{d-1}) \in \mathcal{W}^{n}_{m,d-m}$ is smooth. We directly compute that the tangent map $d\pi$ is surjective at $(x_{1}, \ldots, x_{d-1})$ so the image of $\pi$ contains an open neighbourhood of $\pi((x_{1}, \ldots, x_{d-1}))$ in $\mathcal{Z}^{d-1}_{2,d-2}_{\text{upper}}$. On the other hand, the image of the map $\pi$ is a closed subset, so $\pi$ is surjective.

Case (c). We follow the argument from Case (b). It is enough to make the proof for $n = 5$ and we take the tuple $(x, x^{2}, x^{3}, x^{4}, x^{5})$ where

$$x = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 \\
\end{bmatrix}.$$ 

Case (d). Let $\mathcal{C}$ denote the concatenated component. We employ an argument analogous to the previous cases, but consider a retraction from $\mathcal{C} \cap \mathcal{W}^{n}_{m,d-m}$ to $\mathcal{C} \cap \mathcal{Z}^{n}_{m,d-m}_{\text{upper}}$ and a map $\pi'$ analogous to $\pi$. Transposing if necessary, we can assume $m = 3$. Consider a tuple $x_{0}$ of four matrices,
written in tensor notation (see §6.1) and another invertible matrix \( g \) as below.

\[
x_0 = \begin{bmatrix}
    e_3 + e_4 & 0 & 0 & 0 & 0 & 0 \\
    0 & e_2 + e_4 & 0 & 0 & 0 & 0 \\
    0 & 0 & e_1 + e_4 & 0 & 0 & 0 \\
    0 & 0 & 0 & e_2 & e_1 & 0 \\
    0 & 0 & 0 & 0 & e_4 & e_3 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
g = \begin{bmatrix}
    1 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 1 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

We take the tuple \( x = (gx_ig^{-1})_{x_i \in x_0} \). Directly by construction of \( x_0 \), the module corresponding to both tuples lies in the concatenated component. By Proposition 4.27 the component \( C \) is 66-dimensional. By the construction from that proposition, the intersection of \( C \) with the space of tuples of matrices with zero lower 4 × 3 corner contains the locus obtained by naively concatenating the components and then acting with the parabolic subgroup of \( \text{GL}_4 \) consisting of matrices with zero lower 4 × 3 corner. In particular, this intersection contains a locus of dimension at least 66 − 12 and our point lies on that locus. We directly compute that the tangent space at \( x \) to \( V_{3,4} \) has dimension \( 54 = 66 - 12 \), so \( x \) is a smooth point of \( C \cap V_{3,4} \). We also directly compute that \( d\pi' \) is surjective at this point.

It remains to prove that the other cases correspond to elementary components. For \( m = 2 \) and \( n \geq d \) it is enough (Remark 6.9) to take \( n = d \) and the smooth points with trivial negative tangents are exhibited in Examples 6.1, 6.2, 6.4, 6.6. The transposes of those give the cases for \( m = d - 2 \). For \((m, d) = (3, 6)\) see Example 6.3, while for \((m, d) = (3, 7)\) see Example 6.5. \( \square \)

**Remark 6.17.** In the case \((d)\), we have strong computational evidence that the locus does not lie in the principal component, however this remains open. Also, this locus contains no smooth points of \( C_4(M_7) \).

**Proof of Theorem A and correctness of Table 2.** First consider elementary components of \( C_n(M_d) \). By Proposition 6.10, Theorem 6.14 and Theorem 6.16 apart from the principal component of \( C_n(M_1) \), all other elementary components are the ones listed in §6.1 and their transposes (as well as components formed by increasing \( n \), see Remark 6.9, but we ignore these, counting them only at the very end). Explicitly, these are the square-zero loci for

\[(d, m, n) = (4, 2, 4), (5, 2, 5), (5, 3, 5), (6, 2, 6), (6, 3, 6), (6, 4, 6), (7, 2, 7), (7, 3, 5), (7, 4, 5), (7, 5, 7)\]

and two other components: the cube-zero, square non-zero component from Example 6.7 and its transpose (both with \( n = 5 \)). Clearly this last component could only coincide with its transpose. But this does not happen, as the general tuple from this component has (after adding multiples of the identity matrix to make the matrices nilpotent) a three-dimensional common kernel, while its transpose has a two-dimensional common kernel. The square zero loci mentioned above are also pairwise distinct since \( m \) can be recovered from a general element of the square-zero locus as the dimension of the common kernel. We have the following number of those elementary components:

| \(d=4 \) | \(d=5 \) | \(d=6 \) | \(d=7 \) |
|---|---|---|---|
| \(n=4\) | 1 | | |
| \(n=5\) | 2 | 4 | |
| \(n=6\) | 3 | | |
| \(n=7\) | | 2 | |
By increasing \( n \) as in Remark 6.9 and/or concatenating with the principal component of \( C_n(\mathbb{M}_1) \) to increase \( d \) we get the numbers of components as in Table 1. Arguing as in Proposition 6.18 below, we conclude that these are all the components. Finally, to obtain the number of components for \( \text{Quot}_r^d \) we compute the number of generators for the modules corresponding to general points of the elementary components in Section 6.1 and, for the non-elementary components, note that if modules \( M_1 \) and \( M_2 \) have disjoint supports, then a surjection from \( S^{\oplus r} \) onto \( M_1 \oplus M_2 \) exists if and only if such surjections exist for both \( M_1 \) and \( M_2 \).

Every component contains a smooth point as proven in Proposition 6.18. \( \square \)

6.5. A generically nonreduced component of \( \text{Quot}_4^d \).

**Proposition 6.18.** For \( d \leq 7 \) and any \( n \), \( r \) the schemes \( C_n(\mathbb{M}_d) \) and \( \text{Quot}_r^d \) are generically reduced (their singular loci are nowhere dense).

**Proof.** It is enough to prove this for \( \text{Quot}_r^d \), since for \( r \) large enough \( C_n(\mathbb{M}_d) \) is an image of a smooth map from a scheme \( U^{\text{st}} \) smooth over \( \text{Quot}_r^d \); see §3.4 and the target of a smooth map is reduced if and only if the source is [sta17, Tag 039Q, Tag 025O]. The classification of elementary components for degrees at most 7 shows that each of them has a smooth point, so is generically reduced. Now we argue that for \( d \leq 7 \) every component of \( \text{Quot}_n^d \) is generically reduced. In one sentence, this is because each non-elementary component locally in étale topology (over \( \mathbb{C} \) one can take analytic topology) is a symmetrized product of elementary components. To make this more precise, consider an arbitrary component \( Z \) and a general point \([M]\) on it. Suppose that \( M = \bigoplus M_i \), where each \( M_i \) is supported at a single point. Since \([M]\) is general, the number of summands is maximal, so that \( M_i \) is not a limit of reducible modules. By generality, we may assume \([M_i]\) lies on a single elementary component \( Z_i \) and is a smooth point there. It follows that \( Z \) is a concatenation of \( Z_i \). By Proposition 4.27 and since \([M_i]\) are smooth, the point \([M]\) is smooth as well. \( \square \)

The aim of this section is to show that Proposition 6.18 is no longer valid for \( d = 8 \) and \( n \geq 4 \). Apart from theoretical importance, this is relevant for the search for elementary components, since one method for such a search is that a locus is constructed and then certified to be a component by exhibiting a smooth point. The result below shows that for larger \( d \), \( n \) this method is in general insufficient.

We first discuss the general assumptions necessary for our example and then provide a concrete example satisfying them. For a graded module \( N \) its Hilbert series is the formal series \( \sum_i (\dim_k N_i) T^i \). Consider the locus \( \mathcal{L}_0 = Z_{4,4}^{\text{upper}} \subset C_4(\mathbb{M}_8) \) of \( 4 \times 8 \) matrices that have nonzero entries only in the top right quadrant. From the module point of view we consider \( S = k[y_1, \ldots, y_4] \), a free module \( F = S^{\oplus 4} \) and quotient modules of the form \( F/K \) where \( K \) is generated by all quadratic forms and \( 4 \cdot 4 - 4 = 12 \) linear forms, so that the quotient \( F/K \) is graded with Hilbert series \( 4 + 4T \). Denote their locus (with reduced scheme structure) by \( \mathcal{L}_0^{\text{Quot}} \). Denote by \( \mathcal{L}_0^{\text{Quot}} \) the locus consisting of modules in \( \mathcal{L}_0^{\text{Quot}} \) translated by an arbitrary vector in \( \mathbb{A}^4 \). The above construction gives morphisms \( \text{Gr}(12, 16) \to \mathcal{L}_0^{\text{Quot}} \) and \( \text{Gr}(12, 16) \times \mathbb{A}^4 \to \mathcal{L}_0^{\text{Quot}} \) that are bijective on closed points, in particular \( \mathcal{L}_0^{\text{Quot}} \) is 52-dimensional.

Consider a point \([M]\) corresponding to a quotient \( F/K \) for \( K \) as above. We will now put several constrains on \( K \) that allow us to deduce nonreducedness and finally we exhibit an example of \( K \) satisfying those assumptions.

**Assumption I.** Assume that \( K \) is generated by linear forms.
Under the above assumption, the multiplication map \( S_1 \otimes_k K_1 \to K_2 \) is onto, so its kernel is 8-dimensional and so the minimal graded free resolution of \( F/K \) has the shape
\[
(6.19) \quad 0 \leftarrow F/K \leftarrow F \leftarrow S^{12}(-1) \leftarrow S^{\oplus 8}(-2) \oplus F'_2 \ldots
\]
where the free module \( F'_2 \) is a direct sum of \( S(-j) \) for varying \( j \geq 3 \).

**Lemma 6.20.** The tangent space \( T \) to \( \text{Quot}^8_4 \) at \([F/K]\) as above is homogeneous and satisfies \( \dim T_{-1} \geq 16 \) as well as \( \dim T_0 = 48 \).

**Proof.** By Lemma 3.9, the tangent space \( T \) is isomorphic to \( \text{Hom}_S(K, F/K) \) so it is homogeneous. From (6.19) we see that the minimal graded free resolution of \( K \) begins with
\[
0 \leftarrow K \leftarrow S^{12}(-1) \leftarrow S^{\oplus 8}(-2) \oplus F'_2 \leftarrow \ldots
\]
Applying \( \text{Hom}_S(-, F/K) \) we get an exact sequence
\[
0 \longrightarrow T \longrightarrow (F/K)^{\oplus 12}(1) \longrightarrow (F/K)^{\oplus 8}(2) \oplus \text{Hom}_S(F'_2, F/K).
\]
The Hilbert series of \( \text{Hom}_S(F'_2, F/K) \) is concentrated in degrees \( \leq -2 \). The Hilbert series of \( (F/K)^{\oplus 12}(1) \) is \( 48T^{1-1} + 48 \) while the Hilbert series of \( (F/K)^{\oplus 8}(2) \) is \( 32T^{-2} + 32T^{-1} \). Thus the kernel has Hilbert series \( \alpha T^{-1} + 48 \), where \( \alpha \geq 16 \), which is exactly the claim. \( \square \)

One can observe that the 48 tangent directions in degree zero arise from varying the \( 4 \cdot 12 \) coefficients of the linear forms in the linear ring in \( K_1 \).

**Assumption II.** Assume that the tangent space at \([F/K] \in \text{Quot}^8_4\) is 64-dimensional.

Consider now the complete local ring \( \widehat{O}[M] \) of \([M] \in \text{Quot}^8_4 \). By Example A.4, there is an affine open neighbourhood \( \text{Spec}(A) \) of \([M]\) preserved by the usual \( \mathbb{G}_m \)-action, see Section 5 for details on the \( \mathbb{G}_m \)-action. This action gives a \( \mathbb{Z} \)-grading on the algebra \( A \). The point \([M]\) is \( \mathbb{G}_m \)-fixed since \( K \) is homogeneous, so that the corresponding maximal ideal \( \mathfrak{m} \subset A \) is also graded. Therefore also \( A/\mathfrak{m}^k \) for every \( k \) and in particular \( \mathfrak{m}/\mathfrak{m}^2 \) are graded. Let \( \hat{S} = k[[t_1, \ldots, t_{16}, u_1, \ldots, u_{48}]] \) where \( t_i \) and \( u_j \) forms a basis of the cotangent space, where \( \deg t_i = 1 \) and \( \deg u_j = 0 \); note that the degrees on the cotangent space are the negatives of the degrees on the tangent space. Let \( \mathfrak{m}_\hat{S} = (t_1, \ldots, t_{16}, u_1, \ldots, u_{48}) \). By Assumption II the local ring \( \hat{O}[M] \) has the form \( \hat{O}[M] = \hat{S}/I \), where \( I \subset \mathfrak{m}_\hat{S}^2 \).

We now come to the main part of the argument. Recall the primary obstruction map from Theorem 4.18. Its dual is a homogeneous map
\[
o: \text{Ext}^1(K, M)^\vee \to \text{Sym}_2(\mathfrak{m}_\hat{S}^2/\mathfrak{m}_\hat{S}^2).
\]
We restrict it to the degree 2 part of \( \text{Ext}^1(K, M)^\vee \) and obtain the map
\[
(6.21) \quad \bar{o}: (\text{Ext}^1(K, M)^\vee)_2 \to \text{Sym}_2(\langle t_1, \ldots, t_{16} \rangle).
\]
We now make our third and final restriction about the point.

**Assumption III.** The quotient ring of \( k[[t_1, \ldots, t_{16}]] \) by the ideal generated by \( \text{im} \bar{o} \) has dimension at most 4.

**Theorem 6.22.** Let \([M] = [F/K]\) be a point of \( L^\text{Quot}_3 \subset \text{Quot}^8_4 \) satisfying Assumptions I-III. Then the image of \( L^\text{Quot} \to \text{Quot}^8_4 \) is \( |\mathcal{Z}| \) for an irreducible component \( \mathcal{Z} \) of \( \text{Quot}^8_4 \). The component \( \mathcal{Z} \) is generically nonreduced.
Proof. For every $k \geq 3$ consider the natural map
\[
\varphi_k : k[t_1, \ldots, t_{16}] \to \frac{\mathcal{O}_{[M]}}{m^k + (u_1, \ldots, u_{48})}.
\]
This is a homomorphism of graded rings so its kernel is a homogeneous ideal. This map is also bijective on tangent spaces and the maximal ideal of the ring on the right is nilpotent, so the induced map
\[
k[t_1, \ldots, t_{16}]/(t_1, \ldots, t_{16})^k \to \frac{\mathcal{O}_{[M]}}{m^k + (u_1, \ldots, u_{48})}
\]
is surjective by the nilpotent Nakayama lemma and hence $\varphi_k$ is surjective. The canonical projection
\[
\frac{\mathcal{O}_{[M]}}{(u_1, \ldots, u_{48})} \to \frac{\hat{S}}{m^2 + (u_1, \ldots, u_{48})} = \frac{\mathcal{O}_{[M]}}{m^2 + (u_1, \ldots, u_{48})}
\]
can be clearly lifted to a local ring homomorphism
\[
\frac{\mathcal{O}_{[M]}}{(u_1, \ldots, u_{48})} \to \frac{\hat{S}}{m^2 + I + (u_1, \ldots, u_{48})} = \frac{\mathcal{O}_{[M]}}{m^2 + (u_1, \ldots, u_{48})},
\]
so the corresponding obstruction map $\text{ob}_1 : (\hat{S}^2 / (\hat{S}^3 + I))^\vee \to \text{Ext}^1(K, M)$ is zero. By the discussion before Theorem 4.18 we therefore get that the image of the map $\hat{o}$ from (6.21) lies in $(m^3 + I)/m^2$, so in the kernel of $\varphi_3$. The degree two parts of ker $\varphi_k$ for all $k \geq 3$ are equal. In particular, the intersection of all ker $\varphi_k$ contains $\text{im} \hat{o}$. Passing to the limit of the maps from $k[t_1, \ldots, t_{16}]/((t_1, \ldots, t_{16})^k + \text{im} \hat{o})$ induced by $\varphi_k$, we get a surjective ring homomorphism
\[
\varphi : k[[t_1, \ldots, t_{16}]]/(\text{im} \hat{o}) \to \frac{\mathcal{O}_{[M]}}{(u_1, \ldots, u_{48})}.
\]
By Assumption III the ring $k[t_1, \ldots, t_{16}]/(\text{im} \hat{o})$ localized at $(t_1, \ldots, t_{16})$ has dimension at most four. Completion of a Noetherian local ring preserves dimension, see [AM69, Corollary 11.19] so also the dimension of $k[[t_1, \ldots, t_{16}]]/(\text{im} \hat{o})$ is at most four. Since $\varphi$ is surjective, the dimension of $\mathcal{O}_{[M]}/(u_1, \ldots, u_{48})$ is at most four. Since dividing a local ring by an element of the maximal ideal lowers the dimension by at most one, see [AM69, Corollary 11.18], the dimension of $\mathcal{O}_{[M]}$ is at most $48 + 4 = 52$, which is the dimension of $\mathcal{L}^{\text{Quot}}$. Therefore $\mathcal{L}^{\text{Quot}}$ corresponds to a minimal prime of $\mathcal{O}_{[M]}$ and hence to a component $\mathcal{Z}$ passing through $[M]$. By Lemma 6.20 the tangent space at every $k$-point of this component is at least 64, so that $\mathcal{Z}$ is generically nonreduced. \qed

**Example 6.23.** Let $k$ be a field of characteristic zero and consider $K$ generated by the columns of the following matrix
\[
\begin{pmatrix}
y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4
0 & 0 & y_4 & y_1 & y_2 & 0 & 0 & 0 & 2y_1 + 3y_2 + y_3 + 2y_4 & y_2 + y_3 & 2y_1 + 3y_2 + 5y_3 + 5y_4 & 0 & 0
\end{pmatrix}
\]
Using Macaulay2 and formula (4.19) we verify Assumptions I-III and additionally we check that the obstruction group $\text{Ext}^1(K, M)$ is concentrated in degree $-2$. The code for computing explicitly the dimension in Assumption III is included as an ancillary file $\text{VerifyComputationsI-nArticle.m2}$ in [JS21].

**Corollary 6.24.** The component $(kI)^4 + Z_{4,4}^4$ of $C_4(M_8)$ is generically nonreduced.
Proof. This component corresponds to the generically nonreduced component $Z \subset \text{Quot}_r^d$ constructed in Theorem 6.22 using Example 6.23. Being generically nonreduced, the component $Z$ has no smooth points. By the discussion of Section 3.4 also $(kI)^4 + \mathbb{Z}^4_{d,4}$ has no smooth points, so is generically nonreduced as well.

Proof of Main Theorem B. It is enough to prove that the generically nonreduced irreducible component $Z$ from Corollary 6.24 induces a generically nonreduced irreducible component of $C_n(M_d)$ for $n \geq 4$ and $d \geq 8$. To pass from $n$ to $n + 1$, consider the map $i : C_n(M_d) \to C_{n+1}(M_d)$ that adds the zero matrix. Forgetting about the last matrix gives a map $s : C_{n+1}(M_d) \to C_n(M_d)$ such that $si = id$. Let $x$ be a point of $Z$ that does not lie on other components and let $Z'$ be any component containing $i(x)$. A deformation of $i(x)$ induces a deformation of $x$ by $s$ so the component $Z'$ consists of square-zero matrices of the same shape as in $Z$. In particular, for any $n \geq 4$ the locus $(kI)^n + \mathbb{Z}^n_{d,4}$ is a component. Now a tangent space argument shows that it is nonreduced. To pass from $d$ to $d + 1$ take the concatenation with the principal component of $C_n(M_1)$ and use Proposition 4.27.

To pass from $C_n(M_d)$ to Quot schemes, fix $r \geq 4$. The tuples in the above components of $C_n(M_d)$ correspond to modules that have a four-element generating set, hence the components correspond to components of $\text{Quot}_r^d$ which are generically nonreduced by the comparison of tangent spaces (or arguing using the smooth maps from $U_{\text{et}}$).

Remark 6.25. We decided to avoid mentioning explicitly the Bialynicki-Birula decomposition in our argument. However, it does underlie the whole construction and we discuss here the argument from its perspective. Assumption I together with the equivalent of Proposition 5.5 imply that injection of the Bialynicki-Birula cell of the positive decomposition (constructed at the end of §5) is an open immersion near $[F/K]$. Therefore, locally near this point, the Quot scheme retracts to its $G_m$-fixed locus. By Assumption II, the $G_m$-fixed locus locally near our point is just the Grassmannian $\text{Gr}(12,16)$ and is smooth. By Assumption II again, the fiber of the retraction has 16-dimensional tangent space, while by Assumption III the fiber itself is 4-dimensional, hence the contradiction with being reduced. The details of this line of argument will appear in [Sza21].

Appendix A. Functorial Approach to Comparison between $C_n(M_d)$ and $\text{Quot}_r^d$

The bijection on points obtained in Lemma 3.4 is not enough to compare singularities of $C_n(M_d)$ and $\text{Quot}_r^d$ or to prove that the map $U_{\text{et}} \to \text{Quot}_r^d$ is a morphism. However, the same idea can be extended to give such a comparison, using the language of the functor of points [EH00, Section VI.1].

For a $k$-algebra $A$, an $A$-point of a scheme $X$ is just a morphism $\text{Spec}(A) \to X$ of schemes over $\text{Spec}(k)$. In particular, when $k$ is algebraically closed and $X$ is reasonable (for example, locally of finite type), the $k$-points of $X$ and closed points of $X$ agree. The set of $A$-points is denoted by $X(A)$. For a morphism of affine schemes $\text{Spec}(A) \to \text{Spec}(B)$ we get a map of sets $X(B) \to X(A)$. In this way $X(-)$ becomes a functor from $k$-algebras to sets.

The idea of the functor of points is that for every $A$ the set of $X(A)$ resembles the set $X(k)$ of closed points of $X$, so it is easier to deal with sets of $A$-points for every $A$ than with $X$ viewed as a locally ringed topological space.

Example A.1. What is an $A$-point of $M_d$? Since $M_d = \text{Spec} k[z_{ij} \mid 1 \leq i,j \leq d]$, an $A$-point of $M_d$ is a homomorphism $\varphi : k[z_{ij}] \to A$, i.e., a matrix $[\varphi(z_{ij})]_{i,j}$ with entries in $A$. Conversely,
having such a matrix \([a_{ij}]\) we can uniquely define \(\varphi\) by \(\varphi(z_{ij}) = a_{ij}\). The conclusion is that an \(A\)-point of \(M_d\) is a \(d \times d\) matrix with entries in \(A\). Similarly, since \(GL_d = \text{Spec} k[z_{ij}, \Delta \mid 1 \leq i, j \leq d]/(\Delta \cdot \det[z_{ij}] = 1)\), an \(A\)-point of \(GL_d\) is a matrix with entries in \(A\) and with invertible determinant which is exactly an element of \(GL_d(A)\).

**Example A.2.** The scheme \(C_n(M_d)\) is closed in \(M_d^n = \text{Spec} k[z_{ij}^e \mid 1 \leq i, j \leq d, 1 \leq e \leq n]\) given by the quadratic equations as explained in §3.1. An \(A\)-point of \(C_n(M_d)\) is a homomorphism \(\varphi: k[z_{ij}^e \mid 1 \leq i, j \leq d, 1 \leq e \leq n]/I(C_n(M_d)) \rightarrow A\). This gives an \(n\)-tuple \([\varphi(z_{ij}^e)]_{1 \leq i, j \leq d}\) for \(e = 1, 2, \ldots, n\) of \(d \times d\) matrices and the fact that \(\varphi\) factors through the quotient by \(I(C_n(M_d))\) implies exactly that those matrices commute. This shows that an \(A\)-point of \(C_n(M_d)\) is just a commuting \(n\)-tuple of \(d \times d\) matrices with entries in \(A\).

For simplicity of notation, let \(S_A := S \otimes_k A\), \(F_A := F \otimes_k A\) and \(V_A := V \otimes_k A\) where \(F := S^{\otimes r}\) is a free module. The \(\text{Quot}\) scheme is defined as a functor of points.

**Example A.3** ([FGI+05, Chapter 5]). We define the scheme \(\text{Quot}^d_r\) by declaring that for every \(k\)-algebra \(A\) the set of \(A\)-points of \(\text{Quot}^d_r\) is

\[
\text{Quot}^d_r(A) := \left\{ \frac{F_A}{K} \mid K \subset F_A \text{ is an } S_A\text{-submodule, the } A\text{-module } \frac{F_A}{K} \text{ is locally free and} \right. \\
\text{for every maximal } m \triangleleft A \text{ the } (A/m)\text{-vector space } \frac{F \otimes_k (A/m)}{K} \text{ has dimension } d \right\}
\]

For example, \(\text{Quot}^d_r(k) = \{ F/K \mid K \subset F \text{ an } S\text{-submodule, } \dim_k(F/K) = d \}\). An important observation is that \(K\) is a kernel of a surjection of locally free \(A\)-modules, so it is a locally free \(A\)-module as well. Warning: the \(S_A\)-module \(F_A/K\) is not locally free: it is not even torsion-free.

For a map \(A \rightarrow A'\) of algebras we declare that the map \(\text{Quot}^d_r(A) \rightarrow \text{Quot}^d_r(A')\) sends \(\frac{F_A}{K}\) to \(\frac{F_A}{K} \otimes_A A' \simeq \frac{F_{A'}}{K'\otimes_A A'}\). This defines a functor \(\text{Quot}^d_r\). It is a nontrivial theorem that this functor is represented by a scheme, see [FGI+05, Chapter 5].

A technically less demanding way of proving that \(\text{Quot}^d_r\) is a scheme is to exhibit an open cover by affine schemes and use [EH00, Theorem VI.14]. We present this approach, without much detail, below.

**Example A.4.** Fix \(d\) elements of the module \(F\) that are “monomials”, i.e., that have the form \(y_1^{a_1} \cdots y_n^{a_n} e_j\) for some \(a_i \in \mathbb{Z}_{\geq 0}, j \in \{1, 2, \ldots, r\}\) and denote their set by \(\lambda\). Inside \(\text{Quot}^d_r\) we consider the locus \(U_{\lambda}\) of quotients \(F/K\) such that the image of \(\lambda\) is a \(k\)-linear basis of \(F/K\). More precisely, for each \(A\), we define \(U_{\lambda}(A)\) as the set of \(S_A\)-submodules \(K \subset F_A\) such that the images of \(\lambda\) form an \(A\)-linear base of \(F_A/K\). This is an open condition and the resulting open subscheme \(U_{\lambda} \subset \text{Quot}^d_r\) is affine by the argument repeating the one done for the Hilbert scheme in [MS05, Section 18.1]. The subschemes \(U_{\lambda}/\lambda\) form an open cover of the \(\text{Quot}\) scheme. To prove representability, apply [EH00, Theorem VI.14].

Having discussed the existence of \(\text{Quot}^d_r\), we discuss the analogues of the maps defined in §3.2.

**Lemma A.5.** Let \(A\) be a \(k\)-algebra. The map \((x_1, \ldots, x_n, v_1, \ldots, v_r) \mapsto (\frac{F_A}{K})_{\text{free } \pi M }^{\pi M} (\pi M) \) is a bijection between the \(A\)-points of \(U^\text{free}\) and the set

\[
\left\{ \left( \frac{F_A}{K}, \varphi \right) \mid [F_A/K] \text{ is an } A\text{-point of } \text{Quot}^d_r, \varphi: F_A/K \rightarrow V_A \text{ is an isomorphism of } A\text{-modules} \right\}.
\]

This bijection gives rise to an isomorphism of functors.
Corollary A.6. There is a morphism of schemes \( p: \mathcal{U}^\text{st} \to \text{Quot}^d \) defined on \( A \)-points by the formula \( (x_1, \ldots, x_n, v_1, \ldots, v_r) \mapsto \left[ \frac{F_A}{\ker \pi_M} \right] \). This map makes \( \mathcal{U}^\text{st} \) a principal \( \text{GL}(V) \)-bundle over \( \text{Quot}^d \).

**Proof.** By Lemma A.5, the map above is a map of functors, so by Yoneda’s Lemma [EH00, Lemma VI.1] it gives a morphism of schemes \( p: \mathcal{U}^\text{st} \to \text{Quot}^d \). To prove that \( p \) is a principal \( \text{GL}(V) \)-bundle, we can argue locally on \( \text{Quot}^d \). Choose a point of this scheme and its open neighbourhood \( Z = \text{Spec}(B) \). The corresponding submodule \( K \subset F_B \) has a quotient

\[
Q = \frac{F_B}{K},
\]

which is a locally free \( B \)-module of rank \( d \). Shrink \( Z \) so that \( Q \) becomes a free \( B \)-module and choose an isomorphism \( \varphi_0: Q \to V_B \) of \( B \)-modules.

The preimage \( p^{-1}(Z) \) is the fiber product \( Z \times_{\text{Quot}^d} \mathcal{U}^\text{st} \), so an \( A \)-point of this preimage is a morphism \( j: \text{Spec}(A) \to \text{Spec}(B) \) and an \( A \)-point of \( \mathcal{U}^\text{st} \). By Lemma A.5, this \( A \)-point gives an \( S_A \)-submodule \( K \subset F_A \) together with an isomorphism of \( A \)-modules \( \varphi: F_A/K \simeq V_A \). The product is fibered over \( \text{Quot}^d \) which means that the submodules \( K \) and \( K \otimes_B A \) of \( F_A \) are equal.

Summing up, an \( A \)-point of \( p^{-1}(Z) \) is an isomorphism of \( A \)-modules \( \varphi: F_A/(K \otimes_B A) \to V_A \). Let \( \overline{\varphi}_0: Q \otimes_B A \to V_B \) be obtained from \( \varphi_0 \), then we get an automorphism \( \varphi \circ \overline{\varphi}_0^{-1}: V_A \to V_A \) of the \( A \)-module \( V_A \), hence an \( A \)-point of \( \text{GL}(V) \). Conversely, such an \( A \)-point gives an isomorphism of \( Q \otimes_B A \) with \( V_A \). This shows that the functor of points of \( p^{-1}(Z) \) is isomorphic to the functor of points of \( \text{GL}(V) \times Z \), so by Yoneda’s lemma we get the claim. \( \square \)

We can repeat the argument of Lemma A.5 for \( C_n(\mathbb{M}_d) \).

Lemma A.7. Let \( A \) be a \( k \)-algebra. The map \((x_1, \ldots, x_n) \mapsto (M, \text{id})\) is a bijection between the \( A \)-points of \( C_n(\mathbb{M}_d) \) and the set

\[
\left\{ (M, \varphi) \mid M \text{ a locally free } A \text{-module, } \varphi: M \to V_A \text{ is an } A \text{-linear isomorphism} \right\} / \text{iso}.
\]

There is no scheme \( X \) whose \( A \)-points correspond to locally free \( A \)-modules. However, there is such an algebraic stack (see [Ols16] for introduction to stacks) and it is called \( \text{Mod}^d(\mathbb{A}^n) \).

Corollary A.8. The variety \( C_n(\mathbb{M}_d) \) is an \( \text{GL}(V) \)-bundle over \( \text{Mod}^d(\mathbb{A}^n) \).

**Proof.** This follows from Lemma A.7 along the same lines as Corollary A.6. \( \square \)

Corollary A.9. The variety \( C_n(\mathbb{M}_d) \) has an obstruction theory, where a point \((x_1, \ldots, x_n)\) with corresponding module \( M \) has obstruction group \( \text{Ext}^2(M, M) \).

**Proof.** The map \( C_n(\mathbb{M}_d) \to \text{Mod}^d(\mathbb{A}^n) \) is smooth by Corollary A.8, so the obstruction theory for \( \text{Mod}^d(\mathbb{A}^n) \), see [FG1+05, Proposition 6.5.1], lifts to an obstruction theory for \( C_n(\mathbb{M}_d) \). \( \square \)
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