Codimension One Regular Foliations on Rationally Connected Threefolds

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Abstract
In his work on birational classification of foliations on projective surfaces, Brunella showed that every regular foliation on a rational surface is algebraically integrable with rational leaves. This led Touzet to conjecture that every regular foliation on a rationally connected manifold is algebraically integrable with rationally connected leaves. Druel proved this conjecture for the case of weak Fano manifolds. In this paper, we extend this result showing that Touzet’s conjecture is true for codimension one foliations on threefolds with nef anti-canonical bundle.

Keywords Regular foliation · Holomorphic foliation · Foliated MMP · Rationally connected manifolds

Mathematics Subject Classification Primary 14M22; Secondary 37F75

1 Introduction
A singular holomorphic foliation on a normal complex variety $X$ is defined by a coherent subsheaf $\mathcal{F} \subset T_X$ which is closed under the Lie bracket and such that $T_X/\mathcal{F}$ is torsion free. There exists an open subset $X_0$ of the regular locus of $X$ over which $\mathcal{F}$ is a subbundle of $T_X$. The rank of $\mathcal{F}$ is defined as the rank of $\mathcal{F}|_{X_0}$. We say that $\mathcal{F}$ is a regular foliation if $X_0 = X$.

By the classical Frobenius integrability theorem (Theorem 2.2), $\mathcal{F}$ defines a decomposition of $X_0$ into a disjoint union of complex immersed submanifolds, having dimension equal to the rank of $\mathcal{F}$, such that locally analytically around each point $x \in X_0$, these submanifolds are the fibers of a submersion $f : U \to V$, where $\dim(V) = \dim(X) - r$. These immersed submanifolds are called the leaves of the foliation.

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foliation $\mathcal{F}$. We will say that $\mathcal{F}$ is algebraically integrable if for every leaf $F \subset X$ of $\mathcal{F}$, we have $\dim(F) = \dim(\overline{F}_{\text{Zar}})$, where $\overline{F}_{\text{Zar}}$ is the Zariski closure of $F$ in $X$.

As an example of a foliation, consider a vector field $v \in H^0(X, T_X)$, vanishing in at most a finite set of points. Then the sheaf generated by $v$ defines a foliation of rank one on $X$. The leaves are obtained by solving the ordinary differential equation $\gamma'(t) = v(\gamma(t))$. Thus, to determine whether a foliation of rank one is algebraically integrable is equivalent to determine whether the solutions of an ordinary differential equation are algebraic.

Naturally associated to a foliation $\mathcal{F}$ there is a canonical sheaf $K_{\mathcal{F}} = \det(\mathcal{F}^*)$. Conjecturally, the numerical properties of the canonical sheaf of a foliation $\mathcal{F}$ governs the geometry of $\mathcal{F}$, just like the canonical sheaf of a projective variety $X$ governs the geometry of $X$. In particular, there is a conjectural MMP for foliations (using $K_{\mathcal{F}}$ in the place of $K_X$), currently established if $\dim(X) \leq 3$ (Brunella 1999; McQuillan 2008; Cascini and Spicer 2018, 2020; Spicer and Svaldi 2019).

This foliated Minimal Model Program is significantly simplified for regular foliations of codimension 1 (Spicer 2020). If $\dim(X) = 2$ and $\mathcal{F}$ is a regular foliation of rank 1 on $X$, then either $K_{\mathcal{F}}$ is nef or $\mathcal{F}$ is induced by a rational fibration (hence algebraically integrable). If $\dim(X) = 3$ and $\mathcal{F}$ a regular foliation of codimension 1 on $X$, then after a sequence of smooth blow-downs we obtain a pair $(Y, G)$ where $Y$ and $G$ are regular, and either $K_G$ is nef, or $Y$ is a Mori Fiber Space with $K_G$-negative fibers tangent to $G$.

In this paper we use this MMP to classify regular foliations of codimension 1 on 3-folds. To illustrate this method, let $X$ be a rational surface. If $\mathcal{F}$ is not algebraically integrable, then $K_{\mathcal{F}}$ is nef. By Bott’s vanishing theorem (see Theorem 2.8), we have $c_1(T_X/\mathcal{F})^2 = c_2(T_X/\mathcal{F}) = 0$, and this implies that $c_1(T_X)^2 - c_2(T_X) \geq 0$. Thus $X$ has to be a Hirzebruch surface. However, Brunella shows in Brunella (1997) that every regular foliation on a Hirzebruch surface is algebraically integrable, and thus he concludes the following:

**Theorem 1.1** (Brunella 1997) Let $X$ be a smooth projective rational surface and let $\mathcal{F}$ be a regular foliation of rank 1 on $X$. Then $\mathcal{F}$ is induced by a smooth morphism with rational fibers.

It is then natural to wonder if a similar result holds for manifolds of higher dimension. This led Touzet to make the following conjecture (see Druel 2017):

**Conjecture 1.2** (Touzet) Let $X$ be a projective rationally connected manifold and let $\mathcal{F}$ be a regular foliation on $X$. Then $\mathcal{F}$ is induced by a smooth morphism with rationally connected fibers.

This conjecture is open for $\dim(X) \geq 3$. By a well known result of Campana and Kollar–Miyaoka–Mori, every weak Fano manifold is rationally connected. Druel showed the following theorem, which confirms Touzet’s conjecture in the case of weak Fanos:

**Theorem 1.3** (Druel 2017) Let $X$ be weak Fano manifold, i.e. $-K_X$ is nef and big. Let $\mathcal{F}$ be a regular foliation on $X$. Then $\mathcal{F}$ is induced by a smooth morphism with rationally connected fibers.
In this paper we address Touzet’s conjecture in the case when \( \dim(X) = 3 \) and \( F \) has codimension 1. Our main result is the following, which generalizes Druel’s result in the case of codimension one foliations on threefolds:

**Theorem 1.4** Let \( X \) be a smooth rationally connected threefold with \( -K_X \) nef. Let \( F \) be a regular foliation of codimension 1 on \( X \). Then \( F \) is induced by a smooth morphism with rational fibers.

To prove this result, we apply the foliated MMP to \( F \). When \( K_F \) is not pseudo-effective, we end up with a pair \((Y, G)\) where \( Y \) and \( G \) are regular and \( Y \) is a Mori Fiber Space with fibers tangent to \( G \). By using Mori’s classification of smooth Mori Fiber Spaces of dimension 3, we show directly that the foliation \( G \) is induced by a smooth morphism with rational leaves.

When \( K_F \) is pseudo-effective, we end up with a pair \((Y, G)\) where \( Y \) and \( G \) are regular and \( K_G \) is nef. We then show that \( -K_Y \) is nef, and apply results of Bauer-Peternell, which classify the nef reduction map of \( -K_Y \) in the case of rationally connected threefolds. We are able to show that this map is actually a fibration by \( K3 \)-surfaces, and induces \( G \). We will show that this is a contradiction. Hence \( K_F \) cannot be pseudo-effective, and \( F \) is induced by a smooth morphism with rational fibers.

# 2 Preliminaries

In this section we define foliations and discuss some of their properties. In particular, we state the foliated Minimal Model Program for regular foliations of codimension one on threefolds, proved by Spicer. We also collect some technical results on nef reduction maps, isotrivial families and Mori contractions in the presence of foliations, which will be later used in the proof of our main result.

**Holomorphic Foliations**

We begin with the basic definitions and results which we will use concerning holomorphic foliations.

**Definition 2.1** A holomorphic foliation of rank \( r \) (or codimension \( n - r \)) on a normal complex variety \( X \) of dimension \( n \) is defined by a coherent subsheaf \( \mathcal{F} \subset T_X \), with generic rank \( r \), such that \([\mathcal{F}, \mathcal{F}] \subset \mathcal{F}\) and \( T_X / \mathcal{F} \) is torsion free. The canonical sheaf of \( \mathcal{F} \) is defined as \( K_\mathcal{F} = \det(\mathcal{F}^*) \). The singular locus of \((X, \mathcal{F})\) is defined as \( \text{sing}(\mathcal{F}) = X \setminus X_0 \), where \( X_0 \) is the open subset of the regular locus of \( X \) where \( \mathcal{F} \) is a subvector-bundle of \( T_X \). We say \( \mathcal{F} \) is a regular foliation if \( X \) is regular and \( \mathcal{F} \) is a subvector-bundle of \( T_X \) (i.e. \( \text{sing}(\mathcal{F}) = \emptyset \)).

The following theorem implies that \( \mathcal{F} \) decomposes \( X_0 \) into a union of immersed submanifolds of dimension \( r = \text{rank}(\mathcal{F}) \) which will be called the leaves of \( \mathcal{F} \).

**Theorem 2.2** (Clebsch 1866) Let \( X \) be a complex manifold of dimension \( n \) and \( T \subset T_X \) a subbundle of rank \( r \). Suppose that for any two sections \( v, w \) of \( T \), we have \([v, w] \) a
section of $T$. Then, for any point $p \in X$, there exists an open neighborhood $U$ of $p$ in $X$, such that $U \cong \mathbb{D}^r \times V$, where $V$ is a manifold of dimension $n - r$ and $\mathbb{D} \subset \mathbb{C}$ is the open unitary disk, such that $T|_U \cong \ker(d\pi)$, where $\pi : \mathbb{D}^r \times V \to V$ is the second projection.

**Definition 2.3** Let $X$ be a normal complex variety and let $\mathcal{F}$ be a foliation of rank $r$ on $X$. Let $X_0$ be the open subset of the regular locus of $X$ such that $\mathcal{F}$ is a subvector-bundle of $T_X$ on $X_0$. We say that an immersed submanifold $F \subset X_0$ of dimension $r$ is a leaf of $\mathcal{F}$ if $T_F \to (T_X)_{|F}$ factors through $\mathcal{F}_{|F} \to (T_X)_{|F}$.

**Remark 2.4** We might also define foliations by twisted reflexive differential forms. Indeed, let $X_0$ be as above. On $X_0$ we have an exact sequence of vector-bundles:

$$0 \to \mathcal{F}_{|X_0} \to (T_X)_{|X_0} \to (T_X/F)_{|X_0} \to 0.$$ Dualizing $(T_X)_{|X_0} \to (T_X/F)_{|X_0} \to 0$, we get an injective morphism

$$0 \to (T_X/F)^*_{|X_0} \to (\Omega^1_X)_{|X_0},$$

and thus a global section

$$0 \to \mathcal{O}_{X_0} \to (\Omega^1_X)_{|X_0} \otimes (T_X/F)_{|X_0},$$

i.e. a non-zero section $\omega \in H^0(X_0, (\Omega^1_X)_{|X_0} \otimes (T_X/F)_{|X_0})$. The sheaf $T_X/F$ is called the normal sheaf of $\mathcal{F}$, and we will denote it by $N_F$.

The previous remark allows us to define the pullback of foliations by dominant maps:

**Remark 2.5** (Druel 2021, 3.2) Let $\pi : Y \dashrightarrow X$ be a dominant rational map between normal varieties. Let $\mathcal{F}$ be a codimension $q$ foliation on $X$. Then there exists a codimension $q$ foliation $\pi^{-1}(\mathcal{F})$ on $Y$, called the pullback of $\mathcal{F}$ by $\pi$, such that the leaves of $\pi^{-1}(\mathcal{F})$ are pre-images of leaves of $\mathcal{F}$.

We see that if we pullback a foliation by a morphism $\pi$, then the general fiber of $\pi$ is tangent to this pullback foliation. The following lemma shows that the converse is also true, i.e., that if there is a fibration $\pi$ with general fiber tangent to $\mathcal{F}$, then $\mathcal{F}$ is the pullback of a foliation by $\pi$.

**Lemma 2.6** (Araujo and Druel 2013, Lemma 6.7) Let $\pi : X \to Y$ be an equidimensional morphism with connected fibers between normal varieties. Let $\mathcal{F}$ be a foliation of rank $r$ on $X$. Suppose the general fiber of $\pi$ is tangent to $\mathcal{F}$. Then there exists a foliation $\mathcal{G}$ on $Y$, of rank $r - (\dim(X) - \dim(Y))$, such that the following sequence is exact:

$$0 \to T_{X/Y} \to \mathcal{F} \to (\pi^*\mathcal{G})^*.$$ In this case, $\mathcal{F} = \pi^{-1}(\mathcal{G})$. \[\square\] Springer
The importance of the normal bundle for regular foliations will come from the following two results. The first one (Remark 2.7) says that when a foliation $\mathcal{F}$ is regular and $Z \subset X$ is a submanifold which is tangent to $\mathcal{F}$, then all the Chern classes of $N_{\mathcal{F}|Z}$ vanish. The second one (Theorem 2.8) says that all Chern polynomials (see the definition of Chern polynomial below) on $N_{\mathcal{F}}$ of degree greater than the codimension of $\mathcal{F}$ vanish in case $\mathcal{F}$ is regular. They will be essential in the proof of our main result.

**Remark 2.7** Let $X$ be a manifold and let $\mathcal{F}$ be a regular foliation on $X$. Consider $N_{\mathcal{F}}$ the normal bundle of $\mathcal{F}$. Then there is a natural $\mathcal{F}$-connection $\nabla$ on $N_{\mathcal{F}}$. Indeed, let $U$ be a local section of $N_{\mathcal{F}}$ and let $V$ be a local section of $\mathcal{F}$. Then we may define $\nabla_V(U) = p([V,T])$, where $p: T_X \to N_{\mathcal{F}}$ is the projection and $T$ is a local section of $T_X$ such that $p(T) = U$. Since $\mathcal{F}$ is closed under the Lie bracket, this is well defined.

Now, let $Z \subset X$ be a submanifold which is tangent to $\mathcal{F}$, i.e. $T_Z \subset (\mathcal{F})|_Z$. Then, $\nabla|_Z$ is a holomorphic connection on $N_{\mathcal{F}|Z}$. In particular, all the Chern classes of $N_{\mathcal{F}|Z}$ vanish (see Atiyah 1957, Theorem 4).

Before stating the other result, let us define a Chern polynomial of degree $d$ of a vector bundle $E$ on a manifold $X$ as any element which is a linear combination of monomials $c_{i_1}(E) \cdot c_{i_2}(E) \cdots c_{i_s}(E) \in H^{2i_1+2i_2+\cdots+2i_s}(X, \mathbb{C})$, where $d = i_1 + \cdots + i_s$.

**Theorem 2.8** (Baum and Bott 1972) Let $X$ be a manifold and let $\mathcal{F}$ be a codimension $q$ foliation on $X$. Let $\varphi = \varphi(N_{\mathcal{F}}) \in H^{2l}(X, \mathbb{C})$ be a Chern polynomial on $N_{\mathcal{F}}$ of degree $l$, with $q < l \leq \dim(M)$. If $\mathcal{F}$ is regular, then $\varphi(N_{\mathcal{F}}) = 0$.

**Remark 2.9** Let $X$ be a manifold and $\mathcal{F}$ a regular codimension 1 foliation on $M$. Let $C \subset X$ be a smooth proper curve on $X$. By Remark 2.7, if $C$ is tangent to $\mathcal{F}$, then $c_1(N_{\mathcal{F}}) \cdot C = 0$. Now suppose $C$ is not tangent to $\mathcal{F}$. Let

$$\omega \in H^0(X, \Omega^1_X \otimes N_{\mathcal{F}})$$

define $\mathcal{F}$. Then, since $C$ is not tangent to $\mathcal{F}$,

$$\omega|_C \in H^0(C, \Omega^1_C \otimes N_{\mathcal{F}|C})$$

is non-zero. This implies that $\Omega^1_C \otimes N_{\mathcal{F}|C}$ is a vector bundle on $C$ with non negative degree. Since $\deg(\Omega^1_C) = 2g(C) - 2$, where $g(C)$ is the genus of $C$, we conclude that if $C$ is not tangent to $\mathcal{F}$, then $c_1(N_{\mathcal{F}}) \cdot C \geq 2 - 2g(C)$.

In particular, if $g(C) = 0$, then $c_1(N_{\mathcal{F}}) \cdot C \geq 0$ and equality holds if and only if $C$ is tangent to $\mathcal{F}$.

**Foliated Minimal Model Program**

We state the foliated Minimal Model Program for regular codimension one foliations on 3-folds.
Theorem 2.10 (Spicer 2020) Let $X$ be a projective manifold of dimension 3. Let $\mathcal{F}$ be a regular codimension one foliation on $X$. Then there is a sequence of smooth blow-ups centered at smooth curves:

$$X \to X_1 \to \cdots \to X_n,$$

such that if $\mathcal{F}_i$ is the foliation induced by $\mathcal{F}$ on $X_i$, then $\mathcal{F}_i$ is regular for each $i$, and the non-trivial fibers of $X_i \to X_{i+1}$ are tangent to $\mathcal{F}_i$. Moreover one the following holds:

(a) Either $K_{\mathcal{F}_n}$ is nef;
(b) Or there exists a structure of Mori Fiber Space $X_n \to Y$, whose fibers are tangent to $\mathcal{F}_n$.

In the case (b) of the theorem above, we have a classification by Mori of the possibilities for $X_n \to Y$:

Theorem 2.11 (Mori 1982, Theorem 3.3, Theorem 3.5) Let $X$ be a projective manifold of dimension 3. Let $R \subset N\!E(X)$ be a $K_X$-negative extremal ray, and consider $\varphi: X \to Y$ the contraction associated to $R$. Then either $\varphi$ contracts an irreducible divisor $D$ or $\varphi$ is a fibration by Fano varieties and $Y$ is smooth. Moreover, we have the following cases:

1. If $\varphi$ is divisorial, then one of the following holds:
   (a) $\varphi$ is the blow-up of a smooth curve $C$ on $Y$ and $Y$ is smooth;
   (b) $Q = \varphi(D)$ is a point, $Y$ is smooth, $D \cong \mathbb{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-1)$;
   (c) $Q = \varphi(D)$ is a point, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_D(D)$ is of degree $(-1, -1)$ and $s \times \mathbb{P}^1 \sim t \times \mathbb{P}^1$ (linear equivalence) on $X$ ($s, t \in \mathbb{P}^1$);
   (d) $Q = \varphi(D)$ is a point, $D$ is isomorphic to an irreducible reduced singular quadric in $\mathbb{P}^3$, $\mathcal{O}_D(D) \cong \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}}(-1)$; or
   (e) $Q = \varphi(D)$ is a point, $D \cong \mathbb{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}}(-2)$.

2. If $\varphi$ is a Mori fibration, then $Y$ is smooth and one of the following holds:
   (a) $\dim(Y) = 2$ and for every $y \in Y$, $\varphi^{-1}(y)$ is a conic in $\mathbb{P}^2$;
   (b) $\dim(Y) = 1$ and $-K_X$ is relatively ample; or
   (c) $\dim(Y) = 0$ and $X$ is Fano.

Moreover in case (a), the discriminant $\Delta = \{ y \in Y \mid \varphi^{-1}(y) \text{ is singular} \}$ of $\varphi$ has only ordinary double points as singularities, and $\varphi^{-1}(y)$ is a double line if and only if $y$ is a singular point of $\Delta$. In case (b), we have $1 \leq (K_{\varphi^{-1}(y)})^2 \leq 6$ or $(K_{\varphi^{-1}(y)})^2 = 8, 9$; if $(K_{\varphi^{-1}(y)})^2 = 9$, then $\varphi$ is a $\mathbb{P}^2$-bundle; if $(K_{\varphi^{-1}(y)})^2 = 8$, then $X$ is embedded in a $\mathbb{P}^3$-bundle $P$ over $Y$ such that, for all $y \in Y$, $\varphi^{-1}(y)$ is an irreducible reduced quadric of $\mathbb{P}^3$.
Nef Reduction for Anticanonical Bundles

Another tool we will use in our proof is the classification of the nef reduction map of the anticanonical bundle of rationally connected threefolds with nef anticanonical bundle, given by Bauer and Peternell.

**Theorem 2.12** (Bauer and Peternell 2004, Theorem 2.1) Let $X$ be a smooth projective threefold with $-K_X$ nef. Then there exists a morphism $f : X \rightarrow B$ to a normal projective variety $B$ such that

1. $-K_X$ is numerically trivial on all the fibers of $f$;
2. for every $x \in X$ general and every irreducible curve $C$ passing through $x$ such that $\dim f(C) > 0$, we have $-K_X \cdot C > 0$.

This result is more precise if $X$ is rationally connected with $K_X^2 \equiv 0$. We will show the condition $K_X^2 \equiv 0$ is always verified if $X$ admits a regular foliation of codimension one with nef canonical bundle. Moreover, we will only be concerned with the cases (1) and (2a) of Theorem 2.11. Under these conditions, we have the following theorem:

**Theorem 2.13** (Bauer and Peternell 2004, Corollary 2.4) Let $X$ be a smooth rationally connected projective threefold with $-K_X$ nef. Suppose that $K_X^2 \equiv 0$ and that $X$ is as in cases (1) or (2a) of Theorem 2.11. Then $-K_X$ induces a $K3$-fibration $f : X \rightarrow \mathbb{P}^1$ and $-K_X \cong f^* (\mathcal{O}_{\mathbb{P}^1}(1))$.

Isotrivial Families

In the proof of our main result, we will need to show that, under certain conditions, the $K3$ fibration obtained in Theorem 2.13 is isotrivial. We begin with the definition of isotrivial fibration:

**Definition 2.14** Let $f : X \rightarrow Y$ be a surjective morphism between normal projective varieties with connected general fiber $F$. We say $f$ is birationally isotrivial if $X \times_Y \text{Spec} \mathbb{C}(Y)$ is birational to $F \times \text{Spec} \mathbb{C}(Y)$, where $\mathbb{C}(Y)$ is the field of rational functions on $Y$.

The first result shows that under some conditions, a smooth family over $\mathbb{P}^1$ is isotrivial:

**Theorem 2.15** (Viehweg and Zuo 2001, Theorem 0.1) Suppose $f : X \rightarrow \mathbb{P}^1$ is a surjective morphism with connected general fiber $F$, where $X$ is a projective manifold. Suppose that $f$ is not birationally isotrivial, and that $F$ has a minimal model $F'$ with $K_{F'}$ semi-ample. Then $f$ has at least three singular fibers.

The second result shows that under some conditions, an isotrivial family of surfaces over a curve is trivial after an étale change of base:

**Lemma 2.16** (Oguiso and Viehweg 2001, Lemma 1.6) Let $f : X \rightarrow C$ be a smooth projective family of minimal surfaces of non-negative Kodaira dimension. Then $f$ is birationally isotrivial if, and only if, there exists a finite étale cover $C' \rightarrow C$ and a surface $F$ with $X \times_C C' \cong F \times C'$.
Mori Contractions in the Presence of a Codimension One Regular Foliation

Finally, we will need to see how certain Mori contractions (as in Theorem 2.11) behave in the presence of a regular foliation of codimension one. The following lemma shows that if we have a divisorial contraction to a point on $X$, then there is not regular foliation of codimension 1 on $X$.

**Lemma 2.17** Let $X$ be a smooth threefold and $\pi: X \to Y$ one of the divisorial contractions to a point in Theorem 2.11. If $\mathcal{F}$ is a codimension 1 foliation on $X$, then $\text{sing}(\mathcal{F}) \neq \emptyset$.

**Proof** Denote by $E$ the $\pi$-exceptional divisor. If $\ell \subset E$ is a rational curve contracted by $\pi$, then $N_{E/X} \cdot \ell < 0$. By Remark 2.7, if we had $\text{sing}(\mathcal{F}) = \emptyset$ and $E$ tangent to $\mathcal{F}$, then $N_{E/X} \cdot \ell = \deg(N_{\mathcal{F}|E}) = 0$. Thus, if $E$ is tangent to $\mathcal{F}$, then $\text{sing}(\mathcal{F}) \neq \emptyset$. We may then suppose that $E$ is not tangent to $\mathcal{F}$.

By Theorem 2.11, we have three possibilities: $E \cong \mathbb{P}^2$, or $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, or $E$ is isomorphic to a quadric cone in $\mathbb{P}^3$. Suppose that $\mathcal{F}$ is regular. Let $\omega \in H^0(X, \Omega^1_X \otimes N_{\mathcal{F}})$ define $\mathcal{F}$.

If $E \cong \mathbb{P}^2$ or a quadric cone, then we claim that $\omega|E \equiv 0$. Indeed, consider $(N_{\mathcal{F}})|_E \in \text{Pic}(E) \cong \mathbb{Z}$. Then, there is a line $\ell$ in $E$ such that $(N_{\mathcal{F}})|_E \sim_{\mathbb{Q}} a\ell$, for some $a \in \mathbb{Q}$. Since $\mathcal{F}$ is regular, Theorem 2.8 implies that $N_{\mathcal{F}}^2 \equiv 0$ and thus $a^2\ell^2 \sim 0$. This can only happen if $a = 0$. We conclude that $(N_{\mathcal{F}})|_E \cong \mathcal{O}_E$. Therefore $\omega|E \in H^0(E, \Omega^1_E)$. In both cases, we have $H^0(E, \Omega^1_E) = 0$. Thus $\omega|E \equiv 0$ and $E$ is invariant by $\mathcal{F}$, a contradiction.

If $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, then by Lemma 2.18 below, $\mathcal{F}|_E$ is given by one of the projections $p: E \to \mathbb{P}^1$. In particular $N_{\mathcal{F}} \cdot \ell = 0$ for every fiber $\ell$ of $p$. Now, if $f$ is a fiber of the other projection $q: E \to \mathbb{P}^1$, then $f \sim \ell$ in $X$, which implies that $N_{\mathcal{F}} \cdot f = 0$ for every fiber of $q$. This implies that every fiber of $q$ is tangent to $\mathcal{F}$ as well (see Remark 2.9). We conclude that $E$ is invariant by $\mathcal{F}$, a contradiction. \hfill \square

In the previous proof, we used the following lemma:

**Lemma 2.18** (Brunella 2015, Lemma 3.5.3.1) Let $X$ be a projective manifold and let $S \subset X$ be a Hirzebruch surface. Suppose $\mathcal{F}$ is a regular codimension 1 foliation on $X$. Then $\mathcal{F}|_S$ is induced by a $\mathbb{P}^1$-bundle.

**Proof** Let $F$ be a fiber of $S \to \mathbb{P}^1$ and $C$ a minimal section, so that $F$ and $C$ generate $\text{Pic}(S)$. Write $D$ for the class of $(N_{\mathcal{F}})|_S$. Since $\mathcal{F}$ is regular, Theorem 2.8 implies that $N_{\mathcal{F}}^2 \equiv 0$ and for all rational curves $\ell \subset X$, $N_{\mathcal{F}} \cdot \ell \geq 0$ with equality if and only if $\ell$ is tangent to $\mathcal{F}$ (see Remark 2.9).

Write $D = aF + bC$, for $a, b \in \mathbb{Z}$, and $n = -C^2$. Then $D^2 = 2ab - b^2n$, $D \cdot F = b$ and $D \cdot C = a - bn$. We thus see that $2ab - b^2n = 0$, $b \geq 0$ and $a - bn \geq 0$. Now, suppose that $F$ is not tangent to $\mathcal{F}$. Then $b > 0$, and we may simplify the first equation to $2a - bn = 0$. This together with $a - bn \geq 0$ entails $bn \leq 0$, which can only happen if $n = 0$. In particular $a = 0$ and $D \cdot C = 0$, implying that $C$ is tangent to $\mathcal{F}$. Since $S = \mathbb{P}^0$, every other fiber of the second projection, linearly equivalent to $C$, has zero intersection with $D$ as well, and thus is tangent to $\mathcal{F}$. Therefore, $\mathcal{F}$ is given by the
second projection $S \to \mathbb{P}^1$. If $F$ is tangent to $\mathcal{F}$, the same argument shows that $\mathcal{F}$ is given by the original projection $S \to \mathbb{P}^1$. This concludes the proof.

Finally, the next lemma implies that in the case (2b) of Theorem 2.11, a regular foliation of codimension one is always induced by the Mori Fiber Space.

**Lemma 2.19** Let $Y$ be a projective manifold of dimension 3 and let $C$ be a smooth curve. Suppose $f : Y \to C$ is a fibration with $\rho(Y/C) = 1$ and for the general fiber $F$ of $f$, we have $\text{Pic}(F)$ torsion free and irregularity $q(F) = 0$. Let $\mathcal{G}$ be a codimension 1 regular foliation on $Y$. Then $\mathcal{G}$ is induced by $f$.

**Proof** Let $F$ be a smooth fiber of $f$. Let $\omega \in H^0(Y, \Omega^1_Y \otimes N_{\mathcal{G}})$ define $\mathcal{G}$ and let $\omega|_F \in H^0(F, \Omega^1_F \otimes (N_{\mathcal{G}})|_F)$ be induced by $F \hookrightarrow Y$. Since $\rho(Y/C) = 1$, we have either $(N_{\mathcal{G}})|_F$ ample, or $(N_{\mathcal{G}})^*_F$ ample, or $(N_{\mathcal{G}})|_F \equiv 0$. By Theorem 2.8, we have $N_{\mathcal{G}}^2 \equiv 0$, and thus, since $\dim(F) = 2$ and $\rho(Y/C) = 1$, we can only have $(N_{\mathcal{G}})|_F \equiv 0$. This shows that $(N_{\mathcal{G}})|_F$ is a torsion element of Pic($F$). Since Pic($F$) is torsion-free, it follows that $(N_{\mathcal{G}})|_F \sim 0$. This implies that $\omega|_F \in H^0(F, \Omega^1_F)$, which is zero since $q(F) = 0$. We conclude that $\omega|_F \equiv 0$, showing that $F$ is invariant by $\mathcal{G}$. Therefore, the general fiber of $f$ is invariant by $\mathcal{G}$, and thus $\mathcal{G}$ is induced by $f$ by Lemma 2.6. $\square$

### 3 Proof of the Main Result

Using results of the previous section, in this section we will prove Theorem 1.4 (which will follow from Corollaries 3.2 and 3.8). Let $X$ be a rationally connected threefold with $-K_X$ nef, and let $\mathcal{F}$ be a regular foliation of codimension 1 on $X$. Theorem 1.4 states that $\mathcal{F}$ is induced by a smooth morphism with rational fibers.

To prove this, we run a foliated MMP according to Theorem 2.10. Let $(Y, \mathcal{G})$ be the end result. There are two possibilities: either $K_X$ is pseudo-effective, or it is not. If $K_X$ is not pseudo-effective, then Theorem 2.10 says that there is a fibration $\pi : Y \to B$ in one of the cases in (2) of Theorem 2.11, and its fibers are tangent to $\mathcal{G}$. In particular, these fibers have dimension one or two. If $\pi$ is a del Pezzo fibration, then it follows that $\mathcal{G}$ is induced by $\pi$.

The case of conic bundle will follow from the following lemma:

**Lemma 3.1** Let $\pi : Y \to S$ be a conic bundle over a surface as in Theorem 2.11. Let $\mathcal{H}$ be foliation of rank 1 on $S$, such that the pulled back foliation $\pi^{-1}(\mathcal{H})$ is regular. Then $\mathcal{H}$ is a regular foliation on $S$.

**Proof** Let $\Delta \subset S$ be the set of points over which $\pi$ is not smooth. Then by Theorem 2.11, outside the singular set of $\Delta$, the fiber of $\pi$ is always reduced. In particular, outside the set of non reduced fibers, the non-smooth locus of $\pi$ has codimension at least 2: it consists of the singular points of the singular fibers of $\pi$. Let $p \in S$ be any point. Then there exists an analytic neighborhood $U$ of $p$ in $S$, and a holomorphic 1-form $\omega$ on $U$, such that $\mathcal{H}$ is induced by $\omega$. Suppose $\omega$ vanishes only at $p$. Then $\pi^*(\omega)$ vanishes along $\pi^{-1}(p)$. Moreover, it can vanish at points contained in the locus where $\pi$ is not smooth. Since this locus has codimension at least 2, we conclude that $\pi^*(\omega)$
vanishes along a set of codimension 2. In particular, this 1-form defines \( \pi^{-1}(\mathcal{H}) \) in a neighborhood of every point in \( \pi^{-1}(p) \). But this would imply that \( \pi^{-1}(\mathcal{H}) \) is singular along \( \pi^{-1}(p) \), a contradiction. Thus \( \mathcal{H} \) has to be regular.

In our context, Lemma 2.6 implies that \( G = \pi^{-1}(\mathcal{H}) \) for some foliation \( H \) of rank one on \( S \). Since \( G \) is regular, we have by Lemma 3.1 that the foliation \( \mathcal{H} \) is regular. Finally, since \( S \) is a smooth rational surface (by Theorem 2.11 \( S \) is smooth; if \( C \) is a general rational curve in \( X \), then its image to \( S \) is a rational curve as well, and thus \( S \) is rational), Theorem 1.1 implies that \( \mathcal{H} \) is induced by a smooth morphism with rational fibers. This shows that \( G \), and hence \( F \), is induced by a morphism with rational general fiber. Thus, when \( K_F \) is not pseudo-effective we have the following:

**Corollary 3.2** Let \( X \) be a rationally connected threefold. Let \( F \) be a regular foliation of codimension 1 on \( X \), with \( K_F \) non pseudo-effective. Then \( F \) is induced by a smooth morphism with rational fibers.

**Proof** The previous paragraph shows that \( F \) is induced by a rational map \( \varphi : X \to C \) whose general fiber is rational. Since \( F \) is regular, we may assume \( \varphi \) is a morphism. We only need to show that the morphism \( \varphi : X \to C \) inducing \( F \) is smooth. Indeed, since \( X \) is rationally connected, we must have \( C \cong \mathbb{P}^1 \). Since the foliation induced by \( \varphi \) is smooth, if \( F \) is any fiber of \( \varphi \), then there exists a smooth surface \( F' \) such that \( F = mF' \), for some \( m > 0 \). By Graber et al. (2003), \( \varphi \) admits a section \( \ell \). In particular, \( \ell \cdot F = 1 \), and we conclude that \( m = 1 \), for any fiber \( F = mF' \). Thus \( \varphi \) is smooth.

For the rest of the paper, we will suppose that \( K_F \) is pseudo-effective (our goal is to get a contradiction). Keeping the notation, let \( (Y, G) \) be an end result of a foliated MMP. Then \( K_G \) is nef. Now we are going to use the hypothesis \( -K_X \) nef. The first step is to show that \( -K_Y \) is nef as well. The two following lemmas will ensure this.

**Lemma 3.3** Let \( X \) be a projective manifold of dimension 3. Let \( A \) be a nef divisor and \( B \) a pseudo-effective divisor on \( X \). Suppose \( H \cdot B^2 \geq 0 \) for some ample divisor \( H \) on \( X \). If \( (A + B)^2 = 0 \), then \( \alpha A = \lambda B \) for some \( \alpha, \lambda \in \mathbb{R} \), not both zero.

**Proof** Let \( H \) be the ample divisor in the statement. Then \( H \cdot (A + B)^2 = 0 \), i.e. \( H \cdot A^2 + H \cdot B^2 + 2H \cdot A \cdot B = 0 \). This is a sum of three non-negative numbers, and thus each one has to be zero. By the Lefschetz decomposition (Griffiths and Harris 1978, page 122), \( N^1(X) = \mathbb{R}[H] \oplus P^2(X) \), where \( P^2(X) \) consists of divisor classes satisfying \( C \cdot H^2 = 0 \) (we will denote the divisor class of a divisor \( D \) by \( D \) as well). By the Hodge-Riemann bilinear relations (Griffiths and Harris 1978, page 123), for every non-zero divisor \( C \) with class in \( P^2(X) \), \( H \cdot C^2 > 0 \). If \( W = \langle A, B \rangle \) has dimension 2, then there is a non-zero element \( C \) of \( P^2(X) \) in \( W \). On the other hand, for all \( C \in W \), we have \( H \cdot C^2 = 0 \), and we get a contradiction. Thus \( W \) has dimension \( \leq 1 \).

**Lemma 3.4** Let \( X \) and \( Y \) be projective manifolds of dimension 3. Let \( \varphi : X \to Y \) be a smooth blowup of a curve \( C \). Then \( K_Y^2 = \varphi_*(K_X^2) + C \) in \( A_1(Y) \).

**Proof** Let \( E \) be the exceptional divisor of \( \varphi \). Then \( K_X = \varphi^*(K_Y) + E \) and thus \( K_Y^2 = \varphi_*(\varphi^*(K_Y) \cdot K_X) = \varphi_*(K_X - E) \cdot K_X) = \varphi_*(K_X^2 - E \cdot K_X) \). Write \( (K_X)_E = \)
af + b\tilde{C}, where $f$ is a fiber and $\tilde{C}$ a section of $\varphi|_E : E \to C$. Since $\varphi$ is a blow up, $(K_X)|_E : f = -1$, and thus $(K_X)|_E = af - \tilde{C}$. We conclude then that $\varphi_*(-E \cdot K_X) = (\varphi|_E)_*(-(K_X)|_E) = (\varphi|_E)_*(-af + \tilde{C}) = (\varphi|_E)_*(\tilde{C}) = C$, and thus $\varphi_*(-E \cdot K_X) = C$. This shows that $K_Y^2 = \varphi_*(K_X^2) + C$ in $A_1(Y)$. \hfill$\square$

Lemma 3.3 and 3.4 imply the following proposition:

**Proposition 3.5** Let $X$ be a smooth rationally connected projective threefold. Suppose $-K_X$ is nef. Let $\mathcal{F}$ be a regular codimension 1 foliation on $X$, with $K\mathcal{F}$ pseudo-effective. Let $(Y, \mathcal{G})$ be the end result of a foliated MMP. Then $-K_Y$ is nef and $K_Y^2 \equiv 0$. Moreover, there is an extremal contraction $\varphi : Y \to Z$ which is either divisorial or such that $\dim(Z) = 2$.

**Proof** By Theorem 2.10, $\varphi : X \to Y$ is a composition of smooth blow-ups centered at smooth curves. Let us first show that $H \cdot K_Y^2 \geq 0$, for every ample divisor $H$ on $Y$. To show this, we may assume that $\varphi$ is a single blow-up centered at a smooth curve $C$. The general case follows from this one by an inductive argument. Let $H$ be an ample divisor in $Y$. Then, by Lemma 3.4,

$$H \cdot K_Y^2 = H \cdot (\varphi_*(K_X^2) + C) = \varphi^*(H) \cdot K_X^2 + H \cdot C.$$ 

By hypothesis, $H' \cdot K_X^2 \geq 0$ for every ample divisor $H'$ on $X$. Thus, the same is true for every nef divisor. We conclude that $H \cdot K_Y^2 \geq 0$, for every ample divisor $H$ on $Y$.

Now, $N_{\mathcal{G}} = K_{\mathcal{G}} - K_Y$. Moreover, since $\mathcal{G}$ is regular, $N_{\mathcal{G}}^2 \equiv 0$ by Theorem 2.8. Thus, we have $K_{\mathcal{G}}$ nef, $-K_Y$ pseudo-effective and $H \cdot K_Y^2 \geq 0$ for every ample divisor $H$ on $Y$, and it follows from Lemma 3.3 that there exist $\alpha, \lambda \in \mathbb{R}$, non simultaneously zero, such that $\alpha K_{\mathcal{G}} \equiv \lambda K_Y$. If $\alpha = 0$, then $K_Y \equiv 0$, contradicting the fact that $Y$ is rationally connected. We may then suppose that $\alpha = 1$. Moreover, $K_Y$ is not nef, while $K_{\mathcal{G}}$ is; this implies that $\lambda \leq 0$. If $\lambda < 0$, then $-K_Y$ is nef. We also have $N_{\mathcal{G}} = (\lambda - 1)K_Y$, and since $N_{\mathcal{G}} \neq 0$, we have $K_Y^2 \equiv 0$. If $\lambda = 0$, then $K_{\mathcal{G}} \equiv 0$.

By Touzet (2008) and Druel (2018) (see Druel 2021 for a precise statement), we have three possibilities for $Y$:

- $Y$ is the product of a torus and a manifold with numerically trivial canonical class;
- $Y$ is a rational fibration over a manifold with numerically trivial canonical class;
- $Y$ is the product of a curve of genus at least 2 and a manifold with numerically trivial canonical class.

Since $Y$ is rationally connected, we easily see that none of these cases is possible. Thus we cannot have $\lambda = 0$.

Let us finish by showing that $Y$ is in cases (1) or (2a) of Theorem 2.11. First, $Y$ cannot be Fano, because $K_Y^2 \equiv 0$. In this case then, there exists an extremal negative contraction $\pi : Y \to Z$ (classified by Theorem 2.11). If $\dim(Z) = 1$, then $\pi$ is a del Pezzo fibration. By Lemma 2.19, $\mathcal{G}$ is induced by $\pi$. But in this case $(K_{\mathcal{G}})|_F \equiv K_F$, for a general fiber $F$ of $\pi$, a contradiction to the fact that $F$ is rational and $K_{\mathcal{G}}$ is nef.

Thus, we conclude that the only remaining possibilities for $Y$ in Theorem 2.11 are the ones in (1) and (2a). \hfill$\square$
Thus we may take the nef reduction map $f : Y \to \mathbb{P}^1$ associated to $-K_Y$ according to Theorem 2.12. By Proposition 3.5 and Theorem 2.13, we conclude that $f$ is a K3-fibration, and $-K_Y \cong f^*(\mathcal{O}_{\mathbb{P}^1}(1))$. We first show that in this case, $\mathcal{G}$ is necessarily induced by $f$.

**Lemma 3.6** Let $f : Y \to \mathbb{P}^1$ a K3-fibration induced by $-mK_Y$, for some $m > 0$. Suppose $\mathcal{G}$ is a codimension one foliation on $Y$ such that $N_\mathcal{G} \equiv \alpha K_Y$, for some $\alpha$. Then $\mathcal{G}$ is induced by $f$.

**Proof** Let $\omega \in H^0(Y, \Omega^1_Y \otimes N_\mathcal{G})$ define $\mathcal{G}$. Let $F$ be a general fiber of $f$. Then $N_\mathcal{G} \cdot F = 0$, because $\omega$ is given by $| - mK_Y|$ (hence $-mK_Y \cdot F = 0$). Since $F$ is K3, we have $\pi_1(F) = 0$, and this implies that $(N_\mathcal{G})_F \sim 0$. Thus $\omega|_F \in H^0(F, \Omega^1_F)$, and this group is 0, again by $\pi_1(F) = 0$. We conclude that $\omega|_F \equiv 0$, implying that $F$ is tangent to $\mathcal{G}$, for the general $F$, fiber of $f$. This shows that $\mathcal{G}$ is induced by $f$. \qed

Finally, we get a contradiction by showing that in this case, since $\mathcal{G}$ is regular, $f$ has to be smooth and hence isotrivial.

**Lemma 3.7** Let $Y$ be a rationally connected manifold of dimension 3. Suppose there is a fibration $f : Y \to \mathbb{P}^1$, whose general fiber has numerically trivial canonical bundle, such that $-K_Y = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$. If $\mathcal{G}$ is the foliation induced by $f$, then $\text{sing}(\mathcal{G}) \neq \emptyset$.

**Proof** Since $K_Y$ is not nef, there is an extremal contraction $\varphi : Y \to B$. Suppose that $\text{sing}(\mathcal{G}) = \emptyset$. Let us consider all the cases for $\varphi$ in Theorem 2.11. First, since $-K_Y = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $Y$ cannot be Fano. If $\varphi$ is a fibration by del Pezzo surfaces, with relative Picard number equal to one, then by Lemma 2.19, any regular foliation of codimension one on $Y$ is induced by $\varphi$. If $\varphi$ is a conic bundle as in Theorem 2.11, with fibers tangent to $\mathcal{G}$, then by Lemma 3.1 and Theorem 1.1, any regular foliation of codimension one on $Y$ has rational leaves. Thus, in all these cases we would contradict the hypothesis that the leaves of $\mathcal{G}$ have numerically trivial canonical bundle, and hence they cannot happen.

It remains to treat the cases $\varphi$ a divisorial contraction and $\varphi$ a conic bundle whose fibers are generically transverse to $\mathcal{G}$. By Lemma 2.17, if $\varphi$ is a divisorial contraction as in Theorem 2.11, then the only possible case is that it is a smooth blow-up of a curve.

Since we are supposing that the foliation induced by $f$ is regular, if $F'$ is a singular fiber of $f$, then there exists a regular surface $F'' \subset Y$, and a positive integer $m > 0$, such that $F' = mF''$. Moreover, since $-K_Y = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$, we have $-K_Y \sim F'$. Thus $K_Y \cdot F' = 0$ and $F' \cdot F' = 0$ (since it is a fiber), and we conclude that $K_Y \cdot F'' = 0$ and $F'' \cdot F'' = 0$. By the adjunction formula, $K_{F''} = 0$.

If $\varphi$ is the blow-up of a smooth 3-fold along a curve, then taking $\ell$ a rational curve contracted by it, we have $-K_Y \cdot \ell = 1$. This implies that, for any singular fiber $F' = mF''$ of $f$, we have $mF'' \cdot \ell = 1$, and thus $m = 1$. We conclude, in this case, that $f$ is smooth.

If $\varphi$ is a conic bundle with fibers generically transverse to $\mathcal{F}$, then for any such fiber $\ell$, we have $-K_Y \cdot \ell = 2$. Thus, for any fiber $F' = mF''$, we have $mF'' \cdot \ell = 2$, and hence $m = 1$ or $m = 2$. If $m = 1$ for all fibers, then again $f$ is smooth. Suppose
then that $m = 2$ for some fiber $F'$. Thus, in this case, $F'' \cdot \ell = 1$. This implies that the restriction $\varphi_{|F''} : F'' \to B$ is an isomorphism. Since $B$ is rational, this contradicts the fact that $K_{F''} \equiv 0$.

We conclude that in both cases, $f$ has to be smooth. By Theorem 2.15, $f$ is birationally isotrivial. Thus, by Lemma 2.16, since $\mathbb{P}^1$ is simply connected, we conclude that $f$ is trivial, in other words, $Y \cong F \times \mathbb{P}^1$, for $F$ a fiber of $f$, and $f$ is the projection to $\mathbb{P}^1$. Since $Y$ is rationally connected, this implies that $F$ is rational, a contradiction to the fact that $K_F \equiv 0$. Thus, no case of Theorem 2.11 is possible when $\mathcal{G}$ is regular, which shows that $\text{sing}(\mathcal{G}) \neq \emptyset$. 

The contradiction to the fact that $\mathcal{G}$ is regular obtained from the last lemma follows from the assumption that $K_{\mathcal{G}}$ is pseudo-effective. We thus conclude the following, which is the final step to show Theorem 1.4.

**Corollary 3.8** Let $X$ be a rationally connected threefold with $-K_X$ nef. Let $\mathcal{F}$ be a regular foliation of codimension 1 on $X$. Then $K_{\mathcal{F}}$ is not pseudo-effective.

**Proof of Theorem 1.4** Let $X$ be a rationally connected threefold with $-K_X$ nef and let $\mathcal{F}$ be a codimension 1 regular foliation on $X$. Then, by Corollary 3.8, $K_{\mathcal{F}}$ is not pseudo-effective, which implies, by Corollary 3.2, that $\mathcal{F}$ is induced by a smooth morphism with rational fibers. 

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