Fuzzy Lie Groups

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Abstract

In this paper, we have tried to apply the concepts of fuzzy sets to the Lie groups and its relative concepts. By considering the definition of $C^1$-fuzzy manifolds, we define $C^1$-fuzzy submanifolds. In the main section, we defined the fuzzy Lie groups, fuzzy transformation groups and fuzzy $G$-invariants. Finally, our aim is to construct fuzzy differential invariants.

Keywords. fuzzy Lie group, fuzzy invariant, fuzzy $G$-invariant, fuzzy transformation group.

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1 Introduction

The notion of a fuzzy Lie group is depend on the basic concepts in fuzzy topology [6], $C^1$-fuzzy manifold and fuzzy differentiable function between two $C^1$-fuzzy manifolds [7].

In this paper, we introduce the fuzzy sets with two equivalent notations.

First, let $X$ be a non-empty set of points and $I = [0, 1]$; then $I^X$ will be denoted the set of all functions $\mu : X \rightarrow I$. A member of $I^X$ is called a fuzzy subset of $X$. If $x \in X$ and $p \in [0, 1]$, by the fuzzy point $x_p$, we mean the fuzzy subset of $X$ which takes the value $p$ at the point $x$ and 0 elsewhere. Let $\mu$ be a fuzzy subset of $X$ and $t \in [0, 1]$,
the set \( \{ x \in X : \mu(x) \geq t \} \) is called a level subset of \( \mu \) and is symbolized by \( \mu_t \). An element \( a \in X \) is called a normal element of \( \lambda \) with respect to \( \mu \), if \( \lambda(a) \geq \mu(y) \) for all \( y \in X \), [13].

Second, a fuzzy set \( A \) in \( X \) is characterized by a membership (characteristic) function \( \mu_A : X \rightarrow I \), which associates with each points in \( X \), a real number in interval \([0, 1]\), with the value \( \mu_A(x) \) at \( x \) representing the grade of membership of \( x \) in \( A \). We have:

i) if \( A \subseteq B \), then \( \mu_A \leq \mu_B \);

ii) if \( C = A \cup B \), then \( \mu_C(x) = \max\{ \mu_A(x), \mu_B(x) : x \in X \} \); and

iii) if \( D = A \cap B \), then \( \mu_D(x) = \min\{ \mu_A(x), \mu_B(x) : x \in X \} \).

more generally, for a family of fuzzy sets, \( A = \{ A_i : i \in J \} \), the union, \( C = \cup_{i \in J} A_i \), and intersection, \( D = \cap_{i \in J} A_i \), are defied by

\[
\mu_C(x) = \sup\{ \mu_{A_i}(x) : i \in J, x \in X \},
\]
\[
\mu_D(x) = \inf\{ \mu_{A_i}(x) : i \in J, x \in X \},
\]

Convention. We will consider \( I = [0, 1] \subset \mathbb{R} \) and \( J \) an indexing set.

2 Preliminaries

In this section, we define some fuzzy concepts.

**Definition 2.1.** If \( \lambda \in I^X \), \( \mu \in I^Y \) then \( \lambda \times \mu \in I^{X \times Y} \) is defined by

\[
(\lambda \times \mu)(x, y) := \min\{ \lambda(x), \mu(y) \}, \quad \forall (x, y) \in X \times Y.
\]

Let \( F : \lambda \rightarrow \mu \) be a fuzzy function. If \( A \leq \lambda \), \( B \leq \mu \), then \( F(A), F^{-1}(B) \) are defined by
(F(A))(y) := \sup \left\{ \min\{F(x,y), A(x)\} : x \in X \right\} \text{ for all } y \in Y, \text{ and}

(F^{-1}(B))(x) := \sup \left\{ \min\{F(x,y), B(x)\} : y \in Y \right\} \text{ for all } x \in X.

**Definition 2.2.** A fuzzy function \( F : \lambda \rightarrow \mu \) is said to be

i) **injective** if \( F(x_1) = F(x_2) \) then \( x_1 = x_2 \); 

ii) **surjective** if for all \( y \in Y \) with \( \mu(y) \neq 0 \), there exists \( x \in X \) such that \( F(x,y) = \lambda(x) \); 

iii) **bijective** if \( F \) is both injective and surjective.

**Definition 2.3. (Chakrabarty and Ahsanullah [2])** Let \( \mu \) be a fuzzy subset of \( X \). A collection \( \tau \) of fuzzy subsets of \( \mu \) satisfying:

i) \( t \cap \mu \in \tau \) for all \( t \in I \),

ii) if \( \mu_i \in \tau \) for all \( i \in J \) then \( \bigcup \{\mu_i : i \in J\} \in \tau \),

iii) if \( \mu, \nu \in \tau \) then \( \mu \cap \nu \in \tau \).

\( \tau \) is called a fuzzy topology on \( \mu \). The pair \((\mu, \tau)\) is called a fuzzy topological space. Members of \( \tau \) will be called fuzzy open sets and their complements with respect to \( \mu \) are called fuzzy closed sets of \((\mu, \tau)\). \( \mathcal{B} \subseteq \tau \) is called an open base of \( \tau \) if every member of \( \tau \) can be expressed as union of some members of \( \mathcal{B} \).

Let \((\lambda, \tau)\) and \((\mu, \tau')\) are two fuzzy topological spaces, the collection

\[ \mathcal{B} = \{\gamma \times \eta : \gamma \in \tau, \eta \in \tau'\} \]

form an open base of a fuzzy topology in \( \lambda \times \mu \). The fuzzy topology in \( \lambda \times \mu \), induced by \( \mathcal{B} \) is called the product fuzzy topology of \( \tau \) and \( \tau' \) and is denote by \( \tau \times \tau' \). The fuzzy topological space \((\lambda \times \mu, \tau \times \tau')\) is called the product of the fuzzy topological spaces \((\lambda, \tau)\) and \((\mu, \tau')\).

A fuzzy topological space is called a fuzzy \( T_1 \)-space, if every fuzzy point is a closed fuzzy set. ([2] and [7])
Definition 2.4. \((\lambda, \tau)\) is said to be a fuzzy Hausdorff space if \(x_p, y_p \in \lambda\ (x \neq y)\), there exist \(\mu, v \in \tau\) such that \(x_p \in \mu, y_q \in v\) and \(\mu \cap v = 0\).

Definition 2.5. A fuzzy proper function \(F : (\lambda, \tau) \rightarrow (\mu, \tau')\) is said to be

i) fuzzy continuous if \(F^{-1}(\nu) \in \tau\) for all \(\nu \in \tau'\),

ii) fuzzy open if \(F(\delta) \in \tau'\) for all \(\delta \in \tau\),

iii) fuzzy homomorphism if \(F\) be bijective, fuzzy continuous and open.

Definition 2.6. Let \(G\) be a group. \(\mu \in I^G\) is said to be a fuzzy subgroup of \(G\) if \(\forall x, y \in G\)

i) \(\mu(xy) \geq \min\{\mu(x), \mu(y)\}\),

ii) \(\mu(x^{-1}) = \mu(x)\).

Definition 2.7. (Das [5]). A fuzzy topology \(\tau\) on a group \(G\) is said to be compatible if the mappings

\[
m : (G \times G, \tau \times \tau) \rightarrow (G, \tau) \quad (x, y) \mapsto xy
\]

\[
i : (G, \tau) \rightarrow (G, \tau) \quad x \mapsto x^{-1}
\]

are fuzzy continuous. A group \(G\) equipped with a compatible fuzzy topology \(\tau\) on \(G\) is called a fuzzy topological group.

Definition 2.8. A fuzzy topological vector space, is a vector space \(E\) over the field \(F\) of real or complex numbers, if \(E\) is equipped with a fuzzy topology \(\tau\) and \(F\) equipped with the usual topology \(\mathcal{F}\), such that following two mappings are fuzzy continuous:

\[
(E, \tau) \times (E, \tau) \rightarrow (E, \tau) \quad (x, y) \mapsto x + y
\]

\[
(F, \mathcal{F}) \times (E, \tau) \rightarrow (E, \tau) \quad (\alpha, x) \mapsto \alpha x.
\]
Definition 2.9. Let $E, F$ be two fuzzy topological vector space, the mapping $\phi : E \rightarrow F$ is said to be tangent to $0$ if given a neighborhood $W$ of $0_\delta$, $0 < \delta \leq 1$, in $F$ there exists a neighborhood $V$ of $0$, $0 < \epsilon < \delta$, in $E$ such that

$$\phi[tV] \subset o(t)W,$$

for some function $o(t)$.

Definition 2.10. (Ferraro and Foster [7]). Let $E, F$ be two fuzzy topological vector space, each endowed with a $T_1$-fuzzy topology. Let $f : E \rightarrow F$ be a fuzzy continuous mapping. The $f$ is said to be fuzzy differentiable at a point $x \in E$ if there exists a linear fuzzy continuous mapping $u : E \rightarrow F$ (It is denoted by $u \in \mathcal{L}(E, F)$.) such that

$$f(x + y) = f(x) + u(y) + \phi(y), \quad y \in E,$$

where $\phi$ is tangent $0$. The mapping $u$ is called the fuzzy derivative of $f$ in $x$ that is denoted by $f'(x); f'(x) \in \mathcal{L}(E, F)$. The mapping $f$ is fuzzy differentiable if it is fuzzy differentiable at every point of $E$.

Definition 2.11. A fuzzy proper function $F : \lambda \rightarrow \mu$ is said to be a fuzzy homomorphism if $F(x, y) = \lambda(x), F(z, w) = \lambda(z)$ imply

$$F(xz, yw) = \lambda(xz) \quad \text{for all } x, y, z, w \in G.$$

Definition 2.12. Let $E, F$ be fuzzy topological vector spaces. A bijection $f : E \rightarrow F$ is said to be a $C^1$-fuzzy diffeomorphism if it and its inverse $f^{-1}$ are fuzzy differentiable, and $f'$ and $(f^{-1})'$ are fuzzy continuous.

Definition 2.13. (Kim and Lee [8]) Let $X$ be a vector space over a field $F$ with Lie bracket $X \times X \ni (x, y) \mapsto [x, y] \in X$. $X$ is called a Lie algebra over $F$ if

i) The Lie bracket is bilinear,
ii) \([x, y] = -[y, x]\) for all \(x, y \in X\),

iii) (Jacobi identity) \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\) for all \(x, y, z \in X\).

Now \(\mu \in I^X\) is called a fuzzy Lie subalgebra of \(X\) if, for all \(\alpha \in F, x, y \in X\), the following requirements are met

i) \(\mu(x + y) \geq \min\{\mu(x), \mu(y)\}\),

ii) \(\mu(\alpha x) \geq \mu(x)\),

iii) \(\mu([x, y]) \geq \min\{\mu(x), \mu(y)\}\).

If the condition (iii) is replaced by

\[\mu([x, y]) \geq \max(\mu(x), \mu(y))\]

then \(\mu\) is called a fuzzy Lie ideal of \(X\).

**Example 2.14.** Let \(X = \mathbb{R}^3\) and \([x, y] = x \times y\), where \(\times\) is cross product, for all \(x, y \in X\). Then \(X\) is a Lie algebra over a field \(\mathbb{R}\). Define \(\mu : \mathbb{R}^3 \to I\) by

\[
\mu(x) = \begin{cases} 
1 & \text{if } x = y = z = 0, \\
\frac{1}{4} & \text{if } x = y = 0 \text{ and } z \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

then \(\mu\) is a fuzzy subalgebra of \(X\). But \(\mu\) is not a fuzzy Lie ideal of \(X\); because

\[
\mu(([0, 0, 1], (1, 1, 1))) = \mu((0, 0, 1) \times (1, 1, 1)) = \mu(-1, 1, 0) = 0,
\]

while

\[
\max\{\mu(0, 0, 1), \mu(1, 1, 1)\} = \max\left\{\frac{1}{4}, 0\right\} = \frac{1}{4}.
\]

### 3 \(C^1\)-Fuzzy Submanifold

We start this section by defining a \(C^1\)-fuzzy manifold and some examples:
Definition 3.1. (Ferraro and Foster [7]) Let \( X \) be a set. A \( C^1 \)-fuzzy atlas on \( X \) is a collection of pairs \( \{(A_j, \phi_j)\}_{j \in J} \), which satisfies the following conditions:

i) Each \( A_j \) is a fuzzy set in \( X \) and \( \sup_j \{\mu_{A_j}(x)\} = 1 \), for all \( x \in X \).

ii) Each \( \phi_j \) is a bijection, defined on the support of \( A_j \),

\[
\{x \in X : \mu_{A_j}(x) > 0\},
\]

which maps \( A_j \) onto an open fuzzy set \( \phi_j[A_j] \) in some fuzzy topological vector space \( E_j \), and, for each \( l \in J \), \( \phi_j[A_j \cap A_l] \) is an open fuzzy set in \( E_j \).

iii) The mapping \( \phi_l \circ \phi_j^{-1} \), which maps onto \( \phi_j[A_j \cap A_l] \) is a \( C^1 \)-fuzzy diffeomorphism for each pair of indices \( j,l \).

Each pair \( (A_j, \phi_j)_{j \in J} \) is called a fuzzy chart of the fuzzy atlas. If a point \( x \in X \) lies in the support of \( A_j \) then \( (A_j, \phi_j)_{j \in J} \) is said to be a fuzzy chart at \( x \).

Let \((X, \tau)\) be a fuzzy topological space. Suppose there exists an open fuzzy set \( A \) in \( X \) and a fuzzy continuous bijective mapping \( \phi \) defined on the support of \( A \) and mapping onto an open fuzzy set \( V \) in some fuzzy topological vector space \( E \). Then \((A, \phi)\) is said to be compatible with the \( C^1 \)-atlas \( \{(A_j, \phi_j)\}_{j \in J} \) if each mapping \( \phi_j \circ \phi^{-1} \) of \( \phi[A \cap A_j] \) onto \( \phi_j[A \cap A_l] \) is a \( C^1 \)-fuzzy diffeomorphism. Two \( C^1 \)-fuzzy atlases are compatible if each fuzzy chart of one atlas is compatible with each fuzzy chart of the other atlas. It may be verified immediately that the relation of compatibility between \( C^1 \)-fuzzy atlases is an equivalence relation. An equivalence class of \( C^1 \) fuzzy atlases on \( X \) is said to define a \( C^1 \)-fuzzy manifold on \( X \).

Proposition 3.3. Let \( X,Y \) be the fuzzy manifolds; then the product \( X \times Y \) is a fuzzy manifold.

Definition 3.4. Let \( X,Y \) be the fuzzy manifolds and let \( f \) be a mapping of \( X \) into \( Y \). Then \( f \) is said to be fuzzy differentiable at a point \( x \in X \) if there is a fuzzy chart
(U, φ) at x ∈ X and a fuzzy chart (V, φ) at f(x) ∈ Y such that the mapping φ o f o φ⁻¹, which maps φ[U ∩ f⁻¹[V]] into φ[V] is fuzzy differentiable at φ(x). The mapping f is fuzzy differentiable if it is fuzzy differentiable at every point of X; it is a C¹-fuzzy diffeomorphism if φ o f o φ⁻¹ is a C¹-fuzzy diffeomorphism.

Example 3.5. Suppose that X = S¹ is the set of points of the unit circle in R². If U is the fuzzy set of S¹ consisting of the points

$$(\sin 2\pi t, \cos 2\pi t), \quad 0 < s < 1,$$

with the characteristic function μ_U : S¹ → I defined by

$$\mu_U(\sin 2\pi t, \cos 2\pi t) = \sin^2 2\pi t + \cos^2 2\pi t = 1,$$

then the function

$$\phi_1 : S¹ \rightarrow R$$

$$(\sin 2\pi t, \cos 2\pi t) \mapsto t$$

is a bijection onto an open fuzzy set of R and so (U, φ₁) is a fuzzy chart for S¹. If V be another fuzzy set

$$(\sin 2\pi t, \cos 2\pi t), \quad -\frac{1}{2} < t < \frac{1}{2},$$

of S¹ with the characteristic function μ_V : S¹ → I defined by

$$\mu_V(\sin 2\pi t, \cos 2\pi t) = \frac{1}{2},$$

the function

$$\phi_2 : S¹ \rightarrow R$$

$$(\sin 2\pi t, \cos 2\pi t) \mapsto t$$

is another such chart. Each of U and V are fuzzy sets of S¹ such that

$$\sup\{\mu_U(x), \mu_V(x)\} = 1.$$

Since

$$\phi_1 = \phi_2 \text{ if } 0 < \phi_1 < \frac{1}{2} \text{ and } \phi_2 = \phi_1 - 1 \text{ if } \frac{1}{2} < \phi_1 < 1,$$
The mapping $\phi_2 \circ \phi_1^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$
\phi_2 \circ \phi_1^{-1}(t) = \begin{cases} 
t & 0 < t < \frac{1}{2}, \\
t - 1 & \frac{1}{2} < t < 1.
\end{cases}
$$

and is clearly mapping is a $C^1$–fuzzy diffeomorphism, therefor the fuzzy charts $U$ and $V$ form a $C^1$–fuzzy atlas on $S^1$. Now we define another $C^1$–fuzzy atlas on $S^1$. Let $U_1$ be the set of points

$$(z_1, z_2) \in S^1, \quad z_1 > 0.$$ 

We have $U_1$ a fuzzy set of $S^1$ by the characteristic function $\mu_{U_1} : S^1 \rightarrow I$ defined by

$$\mu_{U_1}(z_1, z_2) = \frac{1}{4}.$$ 

Also the function

$$\psi_{U_1} : S^1 \rightarrow \mathbb{R}
\begin{align*}
(z_1, z_2) & \mapsto z_2, \quad (5)
\end{align*}$$

is a bijection onto an open fuzzy set of $\mathbb{R}$. It is therefore a fuzzy chart for $S^1$. Let $U_2, U_3, U_4$ be the sets of points of $S^1$ such that $z_2 > 0, z_1 < 0, z_2 > 0$ respectively. They are fuzzy set by the characteristic functions $\mu_{U_i} : S^1 \rightarrow I$ that $i = 2, 3, 4$ by

$$\mu_{U_2}(z_1, z_2) = \frac{1}{4}, \quad \mu_{U_3}(z_1, z_2) = \frac{1}{4}, \quad \mu_{U_4}(z_1, z_2) = \frac{1}{4}.$$ 

By the following bijection onto an open fuzzy set of $\mathbb{R}$,

$$\psi_{U_2}(z_1, z_2) = z_1, \quad \psi_{U_3}(z_1, z_2) = z_2, \quad \psi_{U_4}(z_1, z_2) = z_1$$

$U_1, U_2, U_3, U_4$ are the fuzzy charts which cover $S^1$ and for any intersections the change of coordinates is a $C^1$ fuzzy diffeomorphism. For instance,

$$\psi_2 = \sqrt{1 - \psi_1^2}, \quad \psi_1 > 0,$$

and so the mapping

$$\psi_2 \circ \psi_1^{-1} : I \rightarrow I
\begin{align*}
t & \mapsto \sqrt{1 - t^2},
\end{align*} \quad (6)$$
is a $C^1$–fuzzy diffeomorphism on the open interval $(0, 1)$. Since, we have

$$\psi_2 = -\sqrt{1 - \psi_3^2} , \quad \psi_3 > 0,$$

$$\psi_3 = \sqrt{1 - \psi_1^2} , \quad \psi_1 < 0,$$

$$\psi_4 = \sqrt{1 - \psi_3^2} , \quad \psi_3 < 0,$$

then the other changes coordinates are also $C^1$–fuzzy diffeomorphisms. Therefore the charts $\psi_1, \psi_2, \psi_3, \psi_4$ form a $C^1$–fuzzy atlas. In fact, this $C^1$–fuzzy atlas is compatible with the pervious $C^1$–fuzzy atlas. To prove this, we need to show that, the additional change of coordinates are also $C^1$–fuzzy diffeomorphism. This is the case, since

$$\psi_1 = \begin{cases} 
\cos(2\pi \phi_1) & 0 < \phi_1 < \frac{1}{2}, \\
\cos(2\pi \phi_2) & 0 < \phi_2 < \frac{1}{2},
\end{cases}$$

$$\psi_2 = \begin{cases} 
\cos(2\pi \phi_1) & 0 < \phi_1 < \frac{1}{4} \text{ or } \frac{3}{4} < \phi_1 < 1, \\
\cos(2\pi \phi_2) & -\frac{1}{4} < \phi_2 < \frac{1}{4},
\end{cases}$$

and so on.

**Definition 3.6.** A fuzzy differentiable function $\psi : M' \to M$ is called a fuzzy immersion if its rank is equal to the dimension of $M'$ at each point of its domain. If its domain is the whole of $M'$, $\psi$ is said to be a fuzzy immersion of $M'$ into $M$.

**Definition 3.7.** A $C^1$–fuzzy manifold $M'$ is said to be a $C^1$–fuzzy submanifold of a $C^1$–fuzzy manifold $M$ if

(i) $M'$ is a fuzzy subset of $M$,

(ii) Natural fuzzy injection $j : M' \to M$ is a fuzzy immersion.

**Example 3.8.** Clearly $M(n \times n, \mathbb{R})$, the set of real $n \times n$ matrices, is a $C^1$–fuzzy manifold and $\text{GL}(n, \mathbb{R})$ ia a fuzzy subset of it. If $j : \text{GL}(n, \mathbb{R}) \to M(n \times n, \mathbb{R})$ is the natural fuzzy injection and

$$\det \circ j : \text{GL}(n, \mathbb{R}) \to \mathbb{R}$$
is fuzzy differentiable, then \( j \) is a fuzzy immersion so \( GL(n, \mathbb{R}) \) is a \( C^1 \)–fuzzy submanifold of \( M(n \times n, \mathbb{R}) \).

4 Fuzzy Lie group

**Definition 4.1.** A fuzzy Lie group \( G \) is a \( C^1 \)–fuzzy manifold \( G \) which is also a group, such that the mappings

\[
m : (G \times G, \tau \times \tau) \rightarrow (G, \tau), \quad i : (G, \tau) \rightarrow (G, \tau)
\]

defined in (1) and (2), are fuzzy differentiable.

**Example 4.2.** (i) One of the simplest example of a fuzzy Lie group is \( \mathbb{R}^n \), that is commutative fuzzy Lie group. The group operation is given by vector addition. The identity element is the zero vector, and the inverse of a vector \( x \) is the vector \(-x\). If \( \mathbb{R}^n \) equipped with the ordinary fuzzy topology, it is trivial which the mappings

\[
m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (x, y) \mapsto x + y,
\]

\[
i : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad x \mapsto x^{-1},
\]

are fuzzy differentiable.

(ii) The other example of fuzzy Lie group is the general linear group \( GL(n, \mathbb{R}) \) consisting of all invertible \( n \times n \) real matrices, with matrix multiplication defining the group multiplication, and matrix inversion defining the inverse. In fact, \( GL(n, \mathbb{R}) \) is an \( n^2 \)-dimensional \( C^1 \)–fuzzy manifold such that

\[
m : GL(n, \mathbb{R}^n) \times GL(n, \mathbb{R}^n) \rightarrow GL(n, \mathbb{R}^n) \quad (A, B) \mapsto AB,
\]

\[
i : GL(n, \mathbb{R}^n) \rightarrow GL(n, \mathbb{R}^n) \quad A \mapsto A^{-1},
\]

**Definition 4.3.** A fuzzy transformation group acting on a \( C^1 \)–fuzzy manifold is determined by a fuzzy Lie group \( G \) and fuzzy differentiable map \( \Phi : G \times M \rightarrow M \), which satisfies
i) $\Phi$ is a fuzzy global surjective,

ii) $\Phi(g,\Phi(h,x)) = \Phi(gh,x)$, for any $x \in M$ and $g, h \in G$.

**Example 4.4.** $\text{GL}(n, \mathbb{R}^n)$ acts on $\mathbb{R}^n$ as a fuzzy transformation group with the map

$$
\Phi : \text{GL}(n, \mathbb{R}^n) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad (A,x) \mapsto Ax.
$$

**Definition 4.5.** Let $G$ be a fuzzy Lie group, then $H \subset G$ is called a *fuzzy Lie subgroup* if $H$ is both a subgroup and a $\mathcal{C}^1$—fuzzy submanifold. For instance, $O(n, \mathbb{R})$, that is real orthogonal $n \times n$ matrices, is a fuzzy Lie subgroup of $\text{GL}(n, \mathbb{R})$.

**Proposition 4.6.** If a fuzzy Lie group $G$ acts on $\mathcal{C}^1$—fuzzy manifold $M$ as a fuzzy transformation group then so does any fuzzy Lie subgroup $H$ of $G$.

**Proof:** If $j : H \longrightarrow G$ is the natural fuzzy injection and $id : M \longrightarrow M$ be a fuzzy identity map. There is a suitable global function $\Phi_H : H \times M \longrightarrow M$ such that

$$
\Phi_H = \Phi \circ (j \times i),
$$

where $j \times i : H \times M \longrightarrow G \times M$. Therefore $\Phi_H$ is a fuzzy surjective and fuzzy differentiable function which we have

$$
\Phi_H(h,\Phi_H(h',x)) = (\Phi \circ (j \times i))(h, (\Phi \circ (j \times i))(h',x))
$$

$$
= \Phi(h,\Phi(h',x)) = \Phi(hh',x)
$$

$$
= (\Phi \circ (j \times i))(hh',x) = \Phi_H(hh',x)
$$

for any $x \in M$ and $h, h' \in H$.

**Example 4.7.** In the example 3.4., we see that the $\text{GL}(n, \mathbb{R})$ be a fuzzy Lie group, and $O(n, \mathbb{R})$ is one of its fuzzy Lie subgroups, then $O(n, \mathbb{R})$ also acts on $\mathbb{R}^3$ as a fuzzy transformation group.
Definition 4.8. Let \( \Phi : G \times M \rightarrow M \) be a fuzzy transformation group. A subset \( S \subset M \) is called \textit{fuzzy \( G \)-invariant subset} of \( M \), if \( \Phi(G \times S) \subseteq S \).

Proposition 4.9. Consider \( \Phi : G \times M \rightarrow M \) as a fuzzy transformation group. If a regular \( C^1 \)-fuzzy submanifold \( M' \) of \( C^1 \)-fuzzy manifold \( M \) is \( G \)-invariant, then \( G \) acts naturally on \( M' \) as a fuzzy transformation group.

Proof: Suppose that \( k : G \times M' \rightarrow G \times M \) is the natural fuzzy injection, then the function \( \Phi' : G \times M' \rightarrow M' \) induced by \( \Phi \circ k \) is fuzzy differentiable that is defines the required action of \( G \) on \( M' \).

Proposition 4.10. If \( M/\rho \) is a quotient \( C^1 \)-fuzzy manifold of \( M \) and equivalence relation \( \rho \) is preserved by a fuzzy Lie transformation group \( G \) on \( M \) then \( G \) acts naturally on \( M/\rho \).

Proof: Let \( \alpha : M \rightarrow M/\rho \) be natural fuzzy surjection and \( \Phi : G \times M \rightarrow M \) be fuzzy differentiable map so \( \alpha \circ \Phi : G \times M \rightarrow M/\rho \) is a fuzzy differentiable function. \( G\times(M/\rho) \) is a quotient \( C^1 \)-fuzzy manifold of \( G \times M \) and \( \alpha \circ \Phi \) is an invariant of corresponding equivalence relation on \( G \times M \). It therefore projects to the fuzzy differentiable function

\[
\Phi : G \times M/\rho \rightarrow M/\rho \quad (g, \alpha m) \mapsto \alpha(gm).
\]

This defines the required action \( G \) on \( M/\rho \).

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