Convergence of the TFDW Energy to the Liquid Drop Model

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Abstract

We consider two nonlocal variational models arising in physical contexts. The first is the Thomas-Fermi-Dirac-von Weizäcker (TFDW) model, introduced in the study of ionization of atoms and molecules, and the second is the liquid drop model with external potential, proposed by Gamow in the context of nuclear structure. It has been observed that the two models exhibit many of the same properties, especially in regard to the existence and nonexistence of minimizers. We show that, under a “sharp interface” scaling of the coefficients and constrained mass, the TFDW energy Gamma-converges to the Liquid Drop model, for a general class of external potentials. Finally, we present some consequences for global minimization of each model.

I. Introduction

The Thomas-Fermi-Dirac-von Weizäcker (TFDW) theory is a variational model for ionization in atoms and molecules. Minimizers $u \in H^1(\mathbb{R}^3)$ of the energy

$$E_{TFDW}(u) = \int_{\mathbb{R}^3} \left( c_T |u|^\frac{10}{3} - c_D |u|^\frac{8}{3} + c_W |\nabla u|^2 - V(x)|u|^2 \right) dx + D(|u|^2, |u|^2)$$

where

$$D(f, g) := \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|} dx dy,$$

subject to an $L^2$ constraint, $\|u\|_{L^2(\mathbb{R}^3)}^2 = M$, model electron density in an atom or molecule whose nuclei act via the electrostatic potential $V$, and total electron charge $M$. The liquid drop model (with potential) is also a variational problem with physical motivations: for sets $\Omega \subset \mathbb{R}^3$ of finite perimeter and given volume $|\Omega| = M$, one minimizes the energy

$$E_{LD}(\Omega) = \text{Per}_{\mathbb{R}^3}(\Omega) - \int_{\Omega} V(x) dx + D(1_{\Omega}, 1_{\Omega}).$$

Here, the first term represents the perimeter of $\partial \Omega$, which may be calculated as the total variation of the measure $|\nabla 1_{\Omega}|$, with $1_{\Omega} \in BV(\mathbb{R}^3; \{0, 1\})$. When $V \equiv 0$, this is Gamow’s problem, a simplified model for the stability of atomic nuclei. The constraint value $M$ represents the number of nucleons bound by the strong nuclear force.
As variational problems, the TFDW and liquid drop models have much in common. Each features a competition between local attractive terms (gradient and potential terms) and a common non-local repulsive term. As such, each problem is characterized by subtle problems of existence and nonexistence due to the translation invariance of the problem “at infinity”: for large values of the “mass” constraint $M$, minimizing sequences may fail to converge due to splitting of mass which escapes to infinity, the “dichotomy” case in the concentration-compactness principle of Lions [22]. (See e.g., [7, 8, 12–14, 17, 18, 23–25, 27].) While this similarity has been often remarked, and one often speaks of the liquid drop models as a sort of “sharp interface” version of TFDW, no direct analytic connection between the two has been made. In this paper we prove that, after an appropriate “sharp interface” scaling and normalization, the TDFW energy converges to the liquid drop model with potential, within the context of Gamma convergence.

In order to establish this connection we select the constants in the TFDW energy so as to set up a sharp interface limit. We note that this choice of scaling is not physically natural for the application to ionization phenomena, but is motivated purely mathematically. We introduce a length-scale parameter $\varepsilon > 0$, and choose constants $c_W = \varepsilon^2$, $c_{TF} = \frac{1}{2\varepsilon}$ and $c_D = \frac{1}{\varepsilon}$. We note that for fixed $\varepsilon$, the qualitative behavior of the minimization problem for TFDW is not affected by the specific choices of the constants $c_W, c_{TF}, c_D$, and the values we select here will lead to the standard choice of constants in the liquid drop model. In addition, we complete the square in the nonlinear potential by adding in a multiple of the constrained $L^2$ norm, which is a constant in the minimization problem and thus has no effect on the existence of minimizers or the Euler-Lagrange equations. That is, the nonlinear potential is rewritten as,

$$
\int_{\mathbb{R}^3} \frac{1}{2\varepsilon} \left( |u|^{\frac{4}{\varepsilon}} - 2|u|^\frac{4}{3} \right) \, dx = \int_{\mathbb{R}^3} \frac{1}{2\varepsilon} |u|^2 \left( |u|^\frac{4}{3} - 1 \right)^2 \, dx - \frac{M}{2\varepsilon},
$$

where $M = \|u\|_{L^2(\mathbb{R}^3)}^2$ according to the constraint. Thus we recognize the triple well potential,

$$W(u) := |u|^2 \left( |u|^\frac{4}{3} - 1 \right)^2,$$

vanishing at $|u| = 0, 1$, and a version of the TFDW energy of the rescaled and normalized form,

$$\mathcal{E}_\varepsilon^V(u) := \int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{2\varepsilon} W(u) - V(x)u^2 \right] \, dx + D(|u|^2, |u|^2), \quad \|u\|_{L^2(\mathbb{R}^3)}^2 = M. \quad (1.2)$$

As $\varepsilon \to 0^+$ we expect that sequences $u_\varepsilon$ of uniformly bounded energy should converge almost everywhere to one of the wells of the potential $W$, that is, in the limit $u(x) \in \{0, \pm 1\}$. As $\mathcal{E}_\varepsilon^V(|u|) = \mathcal{E}_\varepsilon^V(u)$, we expect minimizers of $\mathcal{E}_\varepsilon^V$ to have fixed sign, but families $u_\varepsilon$ with bounded energy might well take both positive and negative values. Hence, we define the limiting liquid drop functional for $u \in BV(\mathbb{R}^3; \{0, \pm 1\})$ as

$$\mathcal{E}_0^V(u) := \frac{1}{8} \int_{\mathbb{R}^3} |\nabla u| - \int_{\mathbb{R}^3} V |u|^2 \, dx + D(|u|^2, |u|^2). \quad (1.3)$$
The first term is the total variation of the measure $|\nabla u|$, and for $u = 1_{\Omega}$ it measures the perimeter of $\partial \Omega$. If $u$ takes both values $\pm 1$, then

$$\int_{\mathbb{R}^3} |\nabla u| = \int_{\mathbb{R}^3} |\nabla u_+| + |\nabla u_-|$$

which measures the perimeter of $\{ x \in \mathbb{R}^3 \mid u(x) = 1 \}$ and that of $\{ x \in \mathbb{R}^3 \mid u(x) = -1 \}$, whereas the other terms yield the same value for $u$ and $|u| = u^2$.

We make the following general hypotheses regarding the potential $V$:

$$V \in L^\frac{5}{2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \quad \text{and} \quad V(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty; \quad (1.4)$$

We define domains for the functionals which incorporate the mass constraint,

$$\mathcal{H}^M := \left\{ u \in H^1(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)}^2 = M \right\},$$

$$\mathcal{X}^M := \left\{ u \in BV(\mathbb{R}^3, \{0, \pm 1\}) : \|u\|_{L^2(\mathbb{R}^3)}^2 = M \right\},$$

and define the infimum values

$$e^V_\varepsilon(M) := \inf \left\{ E^V(u) : u \in \mathcal{H}^M \right\}, \quad e^0_0(M) := \inf \left\{ E^0(u) : u \in \mathcal{X}^M \right\},$$

for the constrained TFDW and liquid drop problems. In recognition of the subtlety of the existence problem for minimizers of both models (see [8], [27], [2], [1], and the excellent review article [9]), the target space and Gamma limit must incorporate the concentration structure of minimizing sequences for the liquid drop model: while minimizing sequences for either TFDW or liquid drop may not converge, they do concentrate at one or more mass centers, and if there is splitting of mass the separate pieces diverge away via translation. We define the energy “at infinity”, $e^0_0(u)$, taking potential $V(x) \equiv 0$, with infimum value $e^0_0(M)$. From this we then define the appropriate Gamma limit as

$$\mathcal{F}^V_0(\{u^i\}_{i=0}^\infty) := \begin{cases} 
E^V_0(u^0) + \sum_{i=1}^\infty E^0_0(u^i), & \{u^i\}_{i=0}^\infty \in \mathcal{H}^M, \\
\infty, & \text{otherwise},
\end{cases} \quad (1.5)$$

on the space of limiting configurations,

$$\mathcal{H}^M := \left\{ \{u^i\}_{i=0}^\infty \subset BV(\mathbb{R}^3, \{0, \pm 1\}) ; \sum_{i=0}^\infty \int_{\mathbb{R}^3} |\nabla u^i| < \infty, \sum_{i=0}^\infty \|u^i\|_{L^2(\mathbb{R}^3)}^2 = M \right\}.$$

We now state our Gamma-convergence result.

**Theorem I.1.** $e^V_\varepsilon$ $\Gamma$–converges to $\mathcal{F}^V_0$, in the sense that:
(i) (Compactness and Lower-bound) For any sequence \( \varepsilon_k \to 0^+ \), if \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \subset \mathcal{H}^M \) and \( \sup_k \mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}) < \infty \), then there exist a subsequence \( \{u_{\varepsilon_k}\} \), a collection \( \{u_i\}_{i=0}^{\infty} \in \mathcal{H}^M \), and translations \( \{x_k^i\}_k \subset \mathbb{R}^3 \), with \( \{x_0^i\} = \{0\} \), so that

\[
\left| u_{\varepsilon_k}(\cdot) - \sum_{i=0}^{\infty} u^i(\cdot-x_k^i) \right| \to 0 \quad \text{in} \quad L^2(\mathbb{R}^3),
\]

\[
|x_k^i - x_k^j| \to \infty, \quad i \neq j,
\]

\[
\mathcal{F}_0^V(\{u_i^{\infty}\}_{i=0}) \leq \liminf_{k \to \infty} \mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}).
\]

(ii) (Upper-bound) Given \( \{u_i\}_{i=0}^{\infty} \in \mathcal{H}^M \) and any sequence \( \varepsilon_k \to 0^+ \), there exist functions \( \{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \subset \mathcal{H}^M \) and translations \( \{x_k^i\}_k \subset \mathbb{R}^3 \), with \( \{x_0^i\} = \{0\} \), such that equations \( (1.6) \) and \( (1.7) \) hold, and

\[
\mathcal{F}_0^V(\{u_i^{\infty}\}_{i=0}) \geq \limsup_{k \to \infty} \mathcal{E}_{\varepsilon_k}(u_{\varepsilon_k}).
\]

The compactness and lower semicontinuity proven in (i) combine two different approaches in the calculus of variations. Local convergence of the singular limits uses BV bounds in the flavor of the Cahn-Hilliard problems, as studied in \([26, 28]\). On the other hand, the lack of global compactness imposes a concentration-compactness structure \([2, 13, 22, 23]\), in order to recover all of the mass escaping to infinity. This form of the Gamma limit, as a sum of disassociated variational problems splitting on different scales is common in droplet breakup for di-block copolymers; see \([4, 8]\). The proof of part (i) is done in section 2.

For the recovery sequence and upper bound (ii), the presence of an infinite number of \( \{u_i\}_{i=0}^{\infty} \) presents some obstacles not normally seen in Cahn-Hilliard-type problems, where the setting is usually a bounded domain or flat torus. Indeed, for (ii) of Theorem 1.1 we must consider \( \{u_i\}_{i=0}^{\infty} \) with infinitely many nontrivial components, and then it is only possible at any fixed \( \varepsilon > 0 \) to construct a trial function approximating \( u^i \) when the scale of its support is large compared to \( \varepsilon \). This construction will be done in section 3.

In the following paragraph, we discuss the implications of our theorem to minimization problems in various settings, and the proofs of those results will be given in section 4.

**On Minimizers**

Here we discuss the implications of the Gamma convergence result on minimizers of TFDW \((1.1)\) and of the liquid drop problem. For minimizers we note that \( \mathcal{E}_{\varepsilon_k}(|u|) = \mathcal{E}_0^V(u) \), \( \mathcal{E}_0^V(|u|) = \mathcal{E}_0^V(u) \), and so we restrict to the cone of positive functions \( \mathcal{H}^+_M, \mathcal{X}^+_M, \mathcal{H}^+_0 \) as the domain for each.

In some sense, one tends to think of a Gamma limit as a framework in which minimizers of the \( \varepsilon \) functionals should converge to minimizers of the limiting energy (see, e.g., \([19]\)), but
given the complexity of the question of the existence of minimizers for each model, this is a subtle point. The notion of generalized minimizers, introduced for the case $V \equiv 0$ in [18, Definition 4.3], provides a useful means of discussing the structure of minimizing sequences which may lose compactness:

**Definition I.2.** Let $V$ satisfy (1.4) and $M > 0$. A generalized minimizer of $e^V_0(M)$ is a finite collection \( \{u^0, u^1, \ldots, u^N\} \), $u^i \in BV(\mathbb{R}^3, \{0, 1\})$, such that:

1. \( \|u^i\|_{L^2(\mathbb{R}^3)}^2 \equiv m^i, i = 0, 1, \ldots, N \), with $\sum_{i=0}^{N} m^i = M$;
2. $u^0$ attains the minimum in $e^V_0(m^0)$ and $u^i$ attains $e^0_0(m^i)$, $i = 1, \ldots, N$;
3. $e^0_0(M) = e^V_0(m^0) + \sum_{i=1}^{N} e^0_0(m^i)$.

In [2] it is shown that to any minimizing sequence for the liquid drop model with (or without) potential $V$, one may associate a generalized minimizer as above. In this way, up to translation ferrying the components $u^i$ to infinity, the collection of all generalized minimizers of $e^V_0$ with constrained mass $M$ completely characterizes the minimizing sequences of $e^V_0$.

We naturally associate to a generalized minimizer \( \{u^0, u^1, \ldots, u^N\} \) an element \( \{u^i\}_{i=0}^{\infty} \) of $\mathcal{H}^0_0$ by taking \( u^i = 0 \) for all \( i \geq N + 1 \), and then we have \( \mathcal{P}^0_i(\{u^i\}_{i=0}^{\infty}) = e^V_0(M) \). When convenient we abuse notation and denote \( \mathcal{P}^0_i(\{u^i\}_{i=0}^{N}) \) the value of the limiting energy for a generalized minimizer. We may thus address the convergence of minimizers of $e^V_0$ (should they exist) in terms of generalized minimizers of $e^V_0$, using Theorem 1.1.

**Theorem I.3.** Let $M > 0$ and assume that there exists $\varepsilon_n \xrightarrow{n \to \infty} 0^+$ for which $e^V_{\varepsilon_n}(M)$ is attained at $u_n \in \mathcal{H}^M$ for each $n \in \mathbb{N}$. Then, there exists a subsequence and a generalized minimizer \( \{u^0, \ldots, u^N\} \) of $e^V_0$ for which (1.6) and (1.7) hold for $i = 0, \ldots, N$, and

\[
\mathcal{P}^0_i(\{u^i\}_{i=0}^{N}) = e^V_0(M) = \lim_{n \to \infty} e^V_{\varepsilon_n}(M).
\]

A slightly more general version of Theorem I.3 will be proven in Lemma IV.5.

There is a special class of potentials $V$ for which the existence problem $\inf e^V_\varepsilon$ is completely understood for each $\varepsilon$; namely, $V$ of long-range, which are potentials that satisfy

\[
\liminf_{t \to \infty} \left( \inf_{|x| = t} V(x) \right) = \infty. \tag{1.9}
\]

For example, the homogeneous potentials $V^\nu(x) = |x|^{-\nu}$ are of long-range for $0 < \nu < 1$. For $V \in L^{3/2} + L^\infty(\mathbb{R}^3)$ satisfying (1.9) it is known that the global minimum is attained for any $M > 0$, for both the TFDW and liquid drop functionals [3]. For this class of problem, we then obtain the global convergence of minimizers in $L^2$ norm:

**Corollary I.4.** Assume $V$ satisfies (1.4) and (1.9), and for $M > 0$, let $u_\varepsilon \in \mathcal{H}^M$ be a minimizer of $e^V_\varepsilon(M)$. Then, for any sequence $\varepsilon_n \xrightarrow{n \to \infty} 0^+$ there exists a subsequence and a minimizer $u_0 \in BV(\mathbb{R}^3, \{0, 1\})$ of $e^V_0(M)$ with $u_\varepsilon_n \xrightarrow{n \to \infty} u_0$ in $L^2(\mathbb{R}^3)$. 

The most important examples for TFDW are those with atomic or molecular potentials \( V \), as they are related to the Ionization Conjecture \([14, 20, 23, 25, 27]\). We consider the atomic case,

\[
V(x) = V_Z(x) = \frac{Z}{|x|},
\]

with \( Z \geq 0 \) representing a constant nuclear charge. With slight abuse of notation, we denote by \( \varepsilon_Z \), \( \varepsilon_0 \) the energies (1.2) and (1.3), respectively, with the atomic choice \( V = V_Z = Z/|x| \), and

\[
e\varepsilon_Z(M) = \inf \{ \varepsilon_Z(u) : u \in \mathcal{H}^M \}, \quad e_0(M) = \inf \{ \varepsilon_0(u) : u \in \mathcal{H}_+^M \}.
\]

For this choice of potential, and in the liquid drop setting, Lu and Otto \([25]\) proved that there exists \( \mu_0 > 0 \) for which the ball \( B_M = B_{r_M}(0), r_M = \sqrt{\frac{3M}{4\pi}} \), centered at the origin of volume \( M \) is the unique, strict minimizer of \( e_\varepsilon(M) \) for all \( 0 < M < Z + \mu_0 \). The corresponding existence result for TFDW is much weaker: by a result of LeBris \([20]\), for each \( \varepsilon > 0 \) fixed, there exists \( \mu_\varepsilon > 0 \) for which \( e_\varepsilon(M) \) is attained for all \( 0 < M < Z + \mu_\varepsilon \). A natural conjecture is that the intervals of existence converge, that is \( \mu_\varepsilon \longrightarrow \varepsilon \rightarrow 0 + M_0 \). Using Theorem I.1 we are able to prove the following:

**Theorem I.5.** Let \( V(x) = Z/|x|, Z \geq 0 \).

(a) For any \( M \in (0, Z] \), \( e_\varepsilon(M) \) is attained at \( u_\varepsilon \in \mathcal{H}^M \), and \( u_\varepsilon \longrightarrow 1_{B_M} \) in \( L^2 \) norm.

(b) For all \( M \in (Z, Z + \mu_0) \) and \( \varepsilon > 0 \), \( \exists M_\varepsilon \leq M \) with \( M_\varepsilon \longrightarrow M \) such that \( e_\varepsilon(M_\varepsilon) \) attains a minimizer \( u_\varepsilon \in \mathcal{H}^{M_\varepsilon} \). Moreover, \( u_\varepsilon \longrightarrow 1_{B_M} \) in \( L^2 \) norm.

Theorem I.5 is connected to the classical Kohn-Sternberg \([19]\) result on the existence of local minimizers of the \( \varepsilon \)-problem in an \( L^2 \)-neighborhood of an isolated local minimizer of the Gamma-limit. We find minimizers for \( \varepsilon_Z \) which converge to the ball of mass \( M \) as \( \varepsilon \to 0 \) in \( L^2(\mathbb{R}^3) \), which would have the given mass \( M \) except for the possibility of vanishingly small pieces splitting off and diverging to infinity as \( \varepsilon \to 0 \). If it were possible to give a uniform (in \( \varepsilon > 0 \)) lower bound on the quantity of diverging mass in the case of splitting, then we would be able to eliminate this possibility completely and assert that \( M_\varepsilon = M \) in (b), as conjectured above.

**II. Compactness and Lower Bound**

In this section we prove part (i) of Theorem I.1. This involves combining lower bounds on singularly perturbed problems of Cahn-Hilliard type with concentration-compactness methods, to deal with possible loss of compactness via splitting.

Throughout the section we fix a potential \( V \) satisfying (1.4). We begin with some preliminary estimates.
Lemma II.1. Let $\{v_\varepsilon\}_{\varepsilon > 0} \subset H^1(\mathbb{R}^3)$, with $\|v_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq M$ and $\varepsilon^3 V(u_\varepsilon) \leq K_0$. Then there exists a constant $C_0 = C_0(K_0, M, V)$ such that $\forall 0 < \varepsilon < \frac{1}{2}$, we have

$$\int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon} W(v_\varepsilon) \right] dx + D(|v_\varepsilon|^2, |v_\varepsilon|^2) \leq C_0.$$  

Proof. First by (1.4), we write $V = V_{5/2} + V_\infty$, where $V_{5/2} \in L^{\frac{5}{2}}(\mathbb{R}^3)$ and $V_\infty \in L^\infty(\mathbb{R}^3)$, and fix $K > 0$ large enough so that

$$|t| \geq \frac{5}{3} W(t), \quad |t| > K.$$  

Then, by Young’s inequality, for any $u \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} V|u|^2 dx \leq \int_{\mathbb{R}^3} V_{5/2}|u|^2 dx + \|V_\infty\|_{\infty} \int_{\mathbb{R}^3} |u|^2 dx$$

$$\leq \frac{2}{5} \int_{\mathbb{R}^3} V_{5/2}|u|^2 dx + \frac{3}{5} \int_{\mathbb{R}^3} |u|^2 dx + \|V_\infty\|_{\infty} \int_{\mathbb{R}^3} |u|^2 dx$$

$$\leq C \left(1 + \int_{\{u < K\}} u^2 dx\right) + \int_{\{u > K\}} W(u) dx + \|V_\infty\|_{\infty} \int_{\mathbb{R}^3} |u|^2 dx$$

$$\leq C_2 + C_1 \int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} W(u) dx.$$  

Hence, there exist constants $C_1, C_2 > 0$ for which

$$2\varepsilon^3 V(u) + C_1 \int_{\mathbb{R}^3} |u|^2 dx + C_2 \geq \int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{2\varepsilon} W(u) \right] dx + D(|u|^2, |u|^2),$$

and the desired estimate follows. \hfill \Box

Remark II.2. Under the hypotheses of Lemma II.1, $\{v_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^{\frac{5}{2}}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} W(v_\varepsilon) dx \xrightarrow{\varepsilon \to 0^+} 0.$$  

Lemma II.3. Assume $V$ satisfies (1.4), and $\{u_n\}_{n \in \mathbb{N}}$, $\{v_n\}_{n \in \mathbb{N}}$ are sequences which are bounded in $L^2 \cap L^{\frac{10}{7}}(\mathbb{R}^3)$ and such that $(u_n - v_n) \xrightarrow{n \to \infty} 0$ in $L^2_{loc}(\mathbb{R}^3)$. Then,

$$\int_{\mathbb{R}^3} V(x) \left(|u_n|^2 - |v_n|^2\right) dx \xrightarrow{n \to \infty} 0.$$  

Proof. Let $\delta > 0$ be given. By (1.4) we may decompose $V = V_1 + V_2 + V_3$, where:

$$V_1(x) = V(x)(1 - \mathbb{1}_{B_R}(x)), \quad V_2(x) = [V(x) - t] \mathbb{1}_{B_R}(x), \quad V_3(x) = \min\{V(x), t\} \mathbb{1}_{B_R}(x),$$

with $R$ large enough that $\|V_1\|_{\infty} < \delta$; $t$ large enough that $\|V_2\|_{L^2(\mathbb{R}^3)} < \delta$. Note that $V_3$ is compactly supported and uniformly bounded. We then consider each part separately:

$$\int_{\mathbb{R}^3} V_1 |u_n|^2 - |v_n|^2 |dx \leq \delta (\|u_n\|_{L^2(\mathbb{R}^3)}^2 + \|v_n\|_{L^2(\mathbb{R}^3)}^2) \leq c\delta;$$
Dirichlet energy is lost as $\varepsilon$ need for the more stringent $L^\delta > \varepsilon$ the Sobolev embedding. However, given the singularly perturbed nature of $\varepsilon u$, there exists a universal constant $C > 0$ as long as the BV norm is bounded and the $L^\delta$ norm of $u$ is not vanishing.

Next, we prepare the way for the proof of the compactness part of Theorem II.1 by establishing that sequences $u_\varepsilon$ with bounded energy must have centers of concentration, even if they are divergent. The following Lemma will be used to rule out dissipation of $u_\varepsilon$ as long as the BV norm is bounded and the $L^\delta$ norm of $u_\varepsilon$ is not vanishing:

Lemma II.5. There exists a universal constant $C > 0$ such that for all $\psi \in BV(\mathbb{R}^3)$,

\[
\|\psi\|_{BV(\mathbb{R}^3)} \left[ \sup_{a \in \mathbb{R}^3} \int_{B_1(a)} |\psi| \, dx \right]^{\frac{1}{3}} \geq C \int_{\mathbb{R}^3} |\psi|^{\frac{3}{5}} \, dx.
\] (2.1)

Proof. It suffices to prove (2.1) holds for $\psi \in W^{1,1}(\mathbb{R}^3)$, as we can extend it to $\psi \in BV(\mathbb{R}^3)$ by using a density argument [5, Theorem 3.9].

Let $\psi \in W^{1,1}(\mathbb{R}^3)$, and define $\chi_a := \chi(x - a)$, where $\chi \in C_0^\infty(\mathbb{R}^3)$ is any function that is compactly supported in $B_1(0)$.

Then, by Hölder’s inequality and Sobolev’s inequality,

\[
\int_{B_1(a)} |\chi_a \psi|^{\frac{3}{5}} \, dx = \int_{B_1(a)} |\chi_a \psi|^{\frac{3}{5}} |\chi_a| \, dx \\
\leq C \left[ \int_{B_1(a)} |\chi_a \psi| \, dx \right]^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |\chi_a \psi|^{\frac{3}{2}} \, dx \right)^{\frac{2}{5}} \\
\leq C \left[ \sup_{a \in \mathbb{R}^3} \int_{B_1(a)} |\psi| \, dx \right]^{\frac{1}{3}} \int_{\mathbb{R}^3} |\nabla (\chi_a \psi)| \, dx \\
\leq C \left[ \sup_{a \in \mathbb{R}^3} \int_{B_1(a)} |\psi| \, dx \right]^{\frac{1}{3}} \int_{\mathbb{R}^3} (\chi_a |\nabla \psi| + |\nabla \chi_a| |\psi|) \, dx.
\]

We conclude the proof of this Lemma by integrating with respect to $a \in \mathbb{R}^3$. 

From this Lemma we may then conclude that noncompactness of sequences with bounded $BV(\mathbb{R}^3)$ norm is due to splitting and translation. The following is an adaptation of [13, Proposition 2.1], which is proven for characteristic functions of finite perimeter sets.
Proposition II.6. Assume $(\psi_n)_{n \in \mathbb{N}}$ is a bounded sequence in $BV(\mathbb{R}^3)$, for which $\liminf_{n \to \infty} \|\psi_n\|_{4/3} > 0$. Then, there exists translations $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^3$, and $0 \neq \psi^0 \in BV(\mathbb{R}^3)$, such that for some subsequence (not relabelled) we have:

(a) $\psi_n(\cdot - a_n) \xrightarrow[n \to \infty]{} \psi^0$ in $L^1_{loc}(\mathbb{R}^3)$,

(b) $\|\psi^0\|_{BV} \leq \liminf_{n \to \infty} \|\psi_n\|_{BV}$.

Proof. By Lemma II.5, we have

$$\sup_{a \in \mathbb{R}^3} \int_{B_1(a)} |\psi_n| dx \geq \left[ C \frac{\int_{\mathbb{R}^3} |\psi_n|^3 dx}{\|\psi_n\|_{BV}} \right]^3 \geq 2c,$$

for some $c > 0$ independent of $n$. Hence, for each $n \in \mathbb{N}$ we may choose $a_n \in \mathbb{R}^3$ for which

$$\int_{B_1(a_n)} |\psi_n| \geq c > 0. \quad (2.2)$$

As $(\psi_n(\cdot - a_n))_{n \in \mathbb{N}}$ is uniformly bounded in $BV(\mathbb{R}^3)$, there exists a subsequence and $\psi^0 \in BV(\mathbb{R}^3)$ for which (a) and (b) hold. By (2.2) and $L^1_{loc}$ convergence, the limit $\psi^0 \neq 0$. \qed

Once we have localized a piece of our $BV(\mathbb{R}^3)$-bounded sequence $\psi_n$ as an $L^1_{loc}$-converging part, we will need to separate the compact piece from the rest, which converges locally to zero but may carry nontrivial $L^1$-mass to infinity. To do this, we first define a smooth cut-off function $\omega : \mathbb{R} \to [0, 1]$, with

$$\omega \mathbb{1}_{(-\infty, 0]} \equiv 1, \quad \omega \mathbb{1}_{[1, \infty)} \equiv 0, \quad \text{and} \quad \|\omega'\|_{\infty} \leq 2,$$

and for any $\rho > 0$,

$$\omega_{\rho}(x) = \omega(|x| - \rho). \quad (2.3)$$

The next Proposition is based on [13, Lemma 2.2.]:

Proposition II.7. Let $(\psi_n)_{n \in \mathbb{N}}$ be bounded in $BV(\mathbb{R}^3)$ with $\psi_n \xrightarrow[n \to \infty]{} \psi^0$ in $L^1_{loc}(\mathbb{R}^3)$ and almost everywhere in $\mathbb{R}^3$, for some function $\psi^0 \in BV(\mathbb{R}^3)$. If $0 < \|\psi^0\|_{L^1(\mathbb{R}^3)} < \liminf_{n \to \infty} \|\psi_n\|_{L^1(\mathbb{R}^3)}$, then there exist radii $(\rho_n)_{n \in \mathbb{N}} \subset (0, \infty)$ such that, up to a subsequence,

$$\int_{\mathbb{R}^3} [\|\nabla \psi_n\| - \|\nabla(\psi_n \omega_{\rho_n})\| - \|\nabla(\psi_n - \psi_n \omega_{\rho_n})\|] \xrightarrow[n \to \infty]{} 0. \quad (2.4)$$

Moreover,

$$\psi_n \omega_{\rho_n} \xrightarrow[n \to \infty]{} \psi^0 \quad \text{in} \quad L^1(\mathbb{R}^3) \quad \text{and} \quad \psi_n(1 - \omega_{\rho_n}) \xrightarrow[n \to \infty]{} 0 \quad \text{in} \quad L^1_{loc}(\mathbb{R}^3), \quad (2.5)$$

with each converging pointwise almost everywhere in $\mathbb{R}^3$. 

Proof. Note that
\[
\int_{\mathbb{R}^3} |\nabla \psi_n| \leq \int_{\mathbb{R}^3} |\nabla (\psi_n \omega_{\rho_n})| + \int_{\mathbb{R}^3} |\nabla [\psi_n (1 - \omega_{\rho_n})]| \\
\leq \int_{\mathbb{R}^3} |\nabla \psi_n| + 2 \int_{\mathbb{R}^3} |\psi_n \nabla (|x| - \rho_n)| \, dx \\
\leq \int_{\mathbb{R}^3} |\nabla \psi_n| + 4 \int_{B_{\rho_n+1}(0) \setminus B_{\rho_n}(0)} |\psi_n| \, dx.
\]
Therefore, (2.4) will hold if we find \(\rho_n\) such that
\[
\int_{B_{\rho_n+1}(0) \setminus B_{\rho_n}(0)} |\psi_n| \, dx \longrightarrow 0. \tag{2.6}
\]

We distinguish between two cases. First, suppose that \(\text{supp} \, \psi_0 \subset B_R(0)\), for some \(R > 0\). In this case, we claim that it suffices to choose \(\rho_n = R\) for all \(n \in \mathbb{N}\). Indeed, by \(L_{1,\text{loc}}(\mathbb{R}^3)\) convergence and the compact support of \(\psi^0\),
\[
\|\psi_n \omega_{\rho_n}\|_{L^1(\mathbb{R}^3)} = \|\psi_n \omega_{\rho_n}\|_{L^1(B_{R+1}(0))} \longrightarrow \|\psi^0\|_{L^1(\mathbb{R}^3)},
\]
and therefore \(\psi_n \omega_{\rho_n} \longrightarrow \psi^0\) in \(L^1(\mathbb{R}^3)\) by the Brezis-Lieb Lemma \[10\]. Also, since \(\psi^0 1_{B_{R+1}(0) \setminus B_R(0)} = 0\) and \(\psi_n \longrightarrow \psi^0\) in \(L^1(B_{R+1}(0))\), we conclude that (2.6) is verified in case \(\text{supp} \, (\psi^0)\) is compact.

In the second case, if \(\text{supp} \, \psi^0\) is unbounded, note that \(\|\psi^0\|_{L^1(\mathbb{R}^3)} < \liminf_{\varepsilon \to 0^+} \|\psi_n\|_{L^1(\mathbb{R}^3)}\) implies that along some subsequence we may choose \(R_n\) such that
\[
\int_{B_{R_n}(0)} |\psi_n| \, dx = \|\psi^0\|_{L^1(\mathbb{R}^3)}. \tag{2.7}
\]
We claim that, chosen this way, \(R_n \longrightarrow \infty\). Indeed, assume that (taking a further subsequence if necessary,) \(R_n \longrightarrow R_0\). Then,
\[
\|\psi^0\|_{L^1(\mathbb{R}^3)} = \liminf_{n \to \infty} \|\psi_n 1_{R_n}\|_{L^1(\mathbb{R}^3)} \leq \liminf_{n \to \infty} \|\psi_n\|_{L^1(B_{R_0})} = \|\psi^0 1_{B_{R_0}}\|_{L^1(\mathbb{R}^3)} < \|\psi^0\|_{L^1(\mathbb{R}^3)},
\]
since we are assuming that \(\text{supp} \, \psi^0\) is essentially unbounded. Thus, \(R_n \longrightarrow \infty\).

Next, fix \(R > 1\) such that
\[
\int_{B_R(0)} |\psi^0| \, dx \geq \frac{1}{2} \|\psi^0\|_{L^1(\mathbb{R}^3)}.
\]
By \(L_{1,\text{loc}}(\mathbb{R}^3)\) convergence, for all sufficiently large \(n\) we have
\[
\int_{B_R(0)} |\psi_n| \, dx \geq \frac{1}{4} \|\psi^0\|_{L^1(\mathbb{R}^3)}. \tag{2.8}
\]
We now claim that there exists $\rho_n \in \left[ \frac{R+R_n}{2}, R_n \right]$ for which
\[
\int_{B_{\rho_n+1}(0) \setminus B_{\rho_n}(0)} |\psi_n| dx \leq \frac{3}{R_n - R} \|\psi^0\|_{L^1(\mathbb{R}^3)}.
\] (2.9)

If so, then (2.6) is satisfied with this choice of $\rho_n \geq r_n := \frac{R+R_n}{2} \to n \to \infty$. To verify the claim, suppose the contrary, and so for every $\rho \in [r_n, R_n]$ we have the opposite inequality to (2.9). For fixed $n$, choose $K \in \mathbb{N}$ with $R_n - 1 \leq r_n + K < R_n$, so there are $K$ intervals of unit length lying in $[r_n, R_n]$. Then, by (2.7), (2.8),
\[
\frac{3}{4} \|\psi^0\|_{L^1(\mathbb{R}^3)} \geq \int_{B_{R_n}(0)} |\psi_n| dx - \int_{B_R(0)} |\psi_n| dx \geq \int_{B_{r_n+K}(0) \setminus B_{r_n}(0)} |\psi_n| dx
\]
\[
> K \cdot \frac{3}{R_n - R} \|\psi^0\|_{L^1(\mathbb{R}^3)} \geq 3 \frac{R_n - r_n - 1}{R_n - R} \|\psi^0\|_{L^1(\mathbb{R}^3)} = \frac{3}{2} \frac{R_n - R - 2}{R_n - R} \|\psi^0\|_{L^1(\mathbb{R}^3)},
\]
for all sufficiently large $n$, a contradiction. This completes the proof of (2.4).

Finally, (2.5) follows from $\psi_n \omega_{\rho_n} \to n \to \infty \psi_0$ almost everywhere in $\mathbb{R}^3$, our definition for $\rho_n$, and the Brezis-Lieb Lemma.

\[\text{Remark II.8. By lower semicontinuity, up to a subsequence,}\]
\[
\int_{\mathbb{R}^3} |\nabla \psi^0| \leq \lim_{n \to \infty} \int_{\mathbb{R}^3} (|\nabla \psi_n| - |\nabla (\psi_n - \psi_n \omega_{\rho_n})|)
\]

We are now ready to prove the compactness and Gamma liminf part of the theorem:

\[\text{Proof of Theorem I.1 (i). Let } \{u_\varepsilon\}_{\varepsilon > 0} \text{ be a family in } \mathcal{H}^M \text{ with } \phi^V_\varepsilon(u_\varepsilon) \leq K_0, \varepsilon > 0.\]

**Step 1: Truncation.**

First, we show that when proving (i) it suffices to restrict to $u_\varepsilon$ satisfying the pointwise bounds $-1 \leq u_\varepsilon \leq 1$ almost everywhere in $\mathbb{R}^3$. Indeed, we define the truncations

\[
u_\varepsilon^* := \begin{cases} 
-1, & u_\varepsilon < -1, \\
u_\varepsilon, & |u_\varepsilon| \leq 1, \\
1, & u_\varepsilon > 1.
\end{cases}
\]

We will show that $\|u_\varepsilon - u_\varepsilon^*\|_{L^2(\mathbb{R}^3)} \to 0$, and

\[
\liminf_{\varepsilon \to 0^+} \phi^V_\varepsilon(u_\varepsilon^*) \leq \liminf_{\varepsilon \to 0^+} \phi^V_\varepsilon(u_\varepsilon). \tag{2.10}
\]

To accomplish this, we first note that by Remark II.2 we have that
\[
0 \leq \int_{\mathbb{R}^3} |u_\varepsilon - u_\varepsilon^*|^2 dx = \int_{\{|u_\varepsilon| > 1\}} (|u_\varepsilon| - 1)^2 dx \leq C \int_{\mathbb{R}^3} W(u_\varepsilon) dx \to 0, \quad \varepsilon \to 0^+.
\]
where $C$ is a constant independent of $\varepsilon$. Also by Remark II.2, $u_\varepsilon$ is bounded in $L^2 \cap L^{10/3}(\mathbb{R}^3)$, and hence its truncation $u_{\varepsilon}^*$ is as well. By Lemma II.3 we conclude that the local potential terms are close,

$$\int_{\mathbb{R}^3} V(|u_\varepsilon|^2 - |u_{\varepsilon}^*|^2) \, dx \rightarrow 0.$$ 

Finally, each of the other terms decreases under truncation,

$$|\nabla u_{\varepsilon}^*| \leq |\nabla u_\varepsilon|, \quad W(u_{\varepsilon}^*) \leq W(u_\varepsilon), \quad D(|u_{\varepsilon}^*|^2, |u_{\varepsilon}^*|^2) \leq D(|u_\varepsilon|^2, |u_\varepsilon|^2),$$

and so (2.10) is verified.

In the following we will therefore assume without loss of generality that $-1 \leq u_\varepsilon \leq 1$, $\varepsilon > 0$, almost everywhere in $\mathbb{R}^3$.

**Step 2: Passing to the first limit.**

Let $\phi_\varepsilon := \Phi(u_\varepsilon)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi(t) := \int_0^t \sqrt{W(\tau)} \, d\tau.$$ 

Then,

$$\phi_\varepsilon = \int_0^{u_\varepsilon} |t|(1 - |t|^{2/3}) \, dt = sign(u_\varepsilon) \left( \frac{1}{2} |u_\varepsilon|^2 - \frac{3}{8} |u_\varepsilon|^{2/3} \right),$$

and since $\|u_\varepsilon\|_{\infty} \leq 1$,

$$\frac{1}{8} |u_\varepsilon|^2 \leq |\phi_\varepsilon| \leq \frac{1}{2} |u_\varepsilon|^2 \quad \text{and} \quad |\phi_\varepsilon| \leq \phi_\varepsilon(1) = \frac{1}{8}. \quad (2.11)$$

In particular, $\|\phi_\varepsilon\|_{L^1(\mathbb{R}^3)} \leq \frac{1}{2} \|u_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{M}{2}$. Furthermore, $\{\phi_\varepsilon\}_{\varepsilon > 0}$ is bounded in $BV(\mathbb{R}^3)$. Indeed, by Young’s inequality and Lemma II.1 with $v_\varepsilon = u_\varepsilon$,

$$\int_{\mathbb{R}^3} |\nabla \phi_\varepsilon| \, dx = \int_{\mathbb{R}^3} \sqrt{W(u_\varepsilon)} |\nabla u_\varepsilon| \, dx \leq \int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} W(u_\varepsilon) \right] \, dx \leq K_1, \quad (2.12)$$

with constant $K_1 = K_1(K_0, M, V)$. Consequently, $\|\phi_\varepsilon\|_{BV(\mathbb{R}^3)}$ is uniformly bounded.

Now let $\varepsilon_k \longrightarrow 0^+$ be any sequence. By the compact embedding of $BV(\mathbb{R}^3)$ in $L^1_{loc}(\mathbb{R}^3)$ there exist a subsequence, which we continue to denote by $\varepsilon_k \longrightarrow 0^+$, and a function $\phi^0 \in BV(\mathbb{R}^3)$ so that $\phi_{\varepsilon_k} \longrightarrow \phi^0$ in $L^1_{loc}(\mathbb{R}^3)$ and almost everywhere in $\mathbb{R}^3$. Moreover, by lower semicontinuity of the total variation,

$$\int_{\mathbb{R}^3} |\nabla \phi^0| \leq \liminf_{k \to \infty} \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon_k}| \, dx. \quad (2.13)$$

Now we can use the invertibility of $\Phi$ and the local uniform continuity of $\Phi^{-1}$ to obtain that $u_{\varepsilon_k} \longrightarrow u^0 := \Phi^{-1}(\phi^0)$ almost everywhere in $\mathbb{R}^3$. Then, by Fatou’s Lemma and Lemma II.1 we have

$$0 \leq \int_{\mathbb{R}^3} W(u^0) \, dx \leq \liminf_{k \to \infty} \int_{\mathbb{R}^3} W(u_{\varepsilon_k}) \, dx = 0.$$
hence \( W(u^0) \equiv 0, u^0(x) \in \{0, \pm 1\} \) almost everywhere, and
\[
\phi^0 = \frac{1}{8} u^0 \text{ almost everywhere in } \mathbb{R}^3.
\] (2.14)

As a result, by Fatou’s Lemma and (2.11), for any compact \( K \subset \mathbb{R}^3 \),
\[
\int_K |\phi^0| dx = \frac{1}{8} \int_K |u^0|^2 dx \leq \frac{1}{8} \liminf_{k \to \infty} \int_K |u_{\varepsilon_k}|^2 dx \leq \lim_{k \to \infty} \int_K |\phi_{\varepsilon_k}| dx = \int_K |\phi^0| dx.
\] (2.15)

Thus, \( u_{\varepsilon_k} \longrightarrow u^0 \) in \( L^2_{loc}(\mathbb{R}^3) \) and pointwise almost everywhere in \( \mathbb{R}^3 \), and \( \|u^0\|^2_2(\mathbb{R}^3) \leq M \).

For the nonlocal term, as \( u_{\varepsilon_k} \longrightarrow u^0 \) locally, along a further subsequence it converges almost everywhere, and hence by Fatou’s Lemma,
\[
D(|u^0|^2, |u^0|^2) \leq \liminf_{k \to \infty} D(|u_{\varepsilon_k}|^2, |u_{\varepsilon_k}|^2),
\] (2.16)
and by Lemma II.3 (2.13), (2.12), and (2.14) we have
\[
\phi^0_{\varepsilon_k}(u^0) \leq \liminf_{k \to \infty} \phi^0_{\varepsilon_k}(u_{\varepsilon_k}).
\]

If \( \phi_{\varepsilon_k} \longrightarrow \phi^0 \) in the \( L^1 \) norm, then by the same argument as (2.15) we may conclude that \( u_{\varepsilon_k} \longrightarrow u^0 \) converges in \( L^2 \) norm, and so \( m^0 := \|u^0\|^2_2(\mathbb{R}^3) = M \), and setting \( u^i \equiv 0 \) for all \( i \geq 1 \), the proof is complete.

**Step 3:** Splitting off the remainder sequence.

To continue we assume that \( m^0 := \|u^0\|^2_2(\mathbb{R}^3) < M \), so the first limit does not capture all of the mass in the sequence \( u_{\varepsilon_k} \). In this case, both \( u_{\varepsilon_k} \) and \( \phi_{\varepsilon_k} \) converge only locally (and not in norm), that is
\[
\|\phi^0\|_{L^1(\mathbb{R}^3)} < \liminf_{k \to \infty} \|\phi_{\varepsilon_k}\|_{L^1(\mathbb{R}^3)},
\]
and similarly for \( u_{\varepsilon_k} \).

Applying Proposition II.7 and Remark II.8 to \( \phi_{\varepsilon_k} \), there exist radii \( \{\rho_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) so that, for
\[
\phi^0_{\varepsilon_k} := \omega_{\rho_k} \phi_{\varepsilon_k}, \quad \phi^1_{\varepsilon_k} := (1 - \omega_{\rho_k}) \phi_{\varepsilon_k},
\]
where \( \omega_{\rho} \) is defined in (2.3), and for a subsequence (which we continue to write as \( \varepsilon_k \longrightarrow 0^+ \)),
\[
\phi^0_{\varepsilon_k} \longrightarrow \phi^0 \text{ in } L^1 \text{ norm, } \phi^1_{\varepsilon_k} \longrightarrow 0 \text{ in } L^1_{loc}(\mathbb{R}^3), \text{ and }
\]
\[
\int_{\mathbb{R}^3} |\nabla \phi^0| + \int_{\mathbb{R}^3} |\nabla \phi^1_{\varepsilon_k}| dx \leq \int_{\mathbb{R}^3} |\nabla \phi_{\varepsilon_k}| dx + o(1).
\] (2.17)
Moreover, the convergence is also pointwise almost everywhere in \( \mathbb{R}^3 \), and from (2.6) and (2.15), the mass contained in the cut-off region is negligible:
\[
\lim_{k \to \infty} \int_{B_{\rho_k+1}(0) \setminus B_{\rho_k}(0)} |\phi_{\varepsilon_k}| dx = 0 = \lim_{k \to \infty} \int_{B_{\rho_k+1}(0) \setminus B_{\rho_k}(0)} |u_{\varepsilon_k}|^2 dx.
\] (2.18)
We also decompose $u_{\varepsilon_k}$ into two pieces,

$$u^0_{\varepsilon_k} = u_{\varepsilon_k} \sqrt{\omega_{\rho_k}}, \quad \text{and} \quad u^1_{\varepsilon_k} = u_{\varepsilon_k} \sqrt{1 - \omega_{\rho_k}}, \quad (2.19)$$

so that $(u_{\varepsilon_k})^2 = (u^0_{\varepsilon_k})^2 + (u^1_{\varepsilon_k})^2$ pointwise. Note that $\phi^1_{\varepsilon_k} = \Phi(u^i_{\varepsilon_k})$ holds in $\mathbb{R}^3 \setminus \{ \rho_k < |x| < \rho_k + 1 \}$, and by (2.18) the region where they are no longer explicitly related carries a negligible amount of the mass of $u_{\varepsilon_k}$.

Equations (2.17), (2.14) and (2.12) give

$$\frac{1}{8} \int_{\mathbb{R}^3} |\nabla u^0|^2 + \lim_{k \to \infty} \int_{\mathbb{R}^3} |\nabla \phi^1_{\varepsilon_k}| dx \leq \liminf_{k \to \infty} \int_{\mathbb{R}^3} \left[ \frac{\varepsilon_k}{2} |\nabla u_{\varepsilon_k}|^2 + \frac{1}{2 \varepsilon_k} W(u_{\varepsilon_k}) \right] dx \leq K_1, \quad (2.20)$$

and hence $\{\phi^1_{\varepsilon_k}\}_{k \in \mathbb{N}}$ is uniformly bounded in $BV(\mathbb{R}^3)$. The nonlocal term also splits in the same way. Indeed, by (2.19), the positivity of $D(f, g)$ for $f, g \geq 0$, and (2.16)

$$\liminf_{k \to \infty} D(|u^0_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2) = \liminf_{k \to \infty} D(|u^0_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2, |u^0_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2)$$

$$\geq \liminf_{k \to \infty} D(|u^0_{\varepsilon_k}|^2, |u^0_{\varepsilon_k}|^2) + D(|u^1_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2) \quad (2.21)$$

Moreover, (2.17), (2.14), and (2.11) give

$$\int_{\mathbb{R}^3} |\phi^0| dx = \frac{1}{8} \int_{\mathbb{R}^3} |u^0|^2 dx \leq \frac{1}{8} \liminf_{k \to \infty} \int_{\mathbb{R}^3} |u^0_{\varepsilon_k}|^2 dx$$

$$\leq \lim_{k \to \infty} \int_{\mathbb{R}^3} |\phi^0_{\varepsilon_k}| dx = \int_{\mathbb{R}^3} |\phi^0| dx,$$

thus $u^0_{\varepsilon_k} \xrightarrow[k \to \infty]{L^2(\mathbb{R}^3)} u^0$ in $L^2(\mathbb{R}^3)$. As a result,

$$M = m^0 + \lim_{k \to \infty} M^1_{\varepsilon_k}, \quad \text{where} \quad M^1_{\varepsilon_k} := \|u^1_{\varepsilon_k}\|_{L^2(\mathbb{R}^3)}^2 = \|u^1_{\varepsilon_k} - u^0\|_{L^2(\mathbb{R}^3)}^2 + o(1). \quad (2.22)$$

Lastly, as $u_{\varepsilon_k} \xrightarrow[k \to \infty]{L^2_{L^2}(\mathbb{R}^3)} u^0$, by Lemma 3.3 we have

$$\int_{\mathbb{R}^3} V |u_{\varepsilon_k}|^2 dx = \int_{\mathbb{R}^3} V |u^0|^2 dx + o(1),$$

and hence we conclude,

$$\mathcal{E}_0^V(u^0) + \liminf_{k \to \infty} \left[ \int_{\mathbb{R}^3} |\nabla \phi^1_{\varepsilon_k}| dx + D \left( |u^0_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2 \right) \right] \leq \liminf_{\varepsilon \to 0^+} \mathcal{E}_\varepsilon^V(u_{\varepsilon}).$$

**Step 4: Concentration in the remainder sequence.**

For any bounded sequence $\{\tilde{\psi}_k\}_{k \in \mathbb{N}}$ in $L^1(\mathbb{R}^3)$ we define

$$\mathcal{M}(\{\tilde{\psi}_k\}) := \sup\{ ||\tilde{\psi}||_{L^1(\mathbb{R}^3)} : \exists x_k \in \mathbb{R}^3, \tilde{\psi}_k(\cdot + x_k) \xrightarrow[k \to \infty]{L^1(\mathbb{R}^3)} \psi \in L^1_{L^1}(\mathbb{R}^3) \},$$
So \(\mathcal{M} ( \{ \psi_k \} )\) identifies the largest possible \(L^1_{loc}\) limiting mass of the sequence, up to translation.

We claim that for our remainder sequence, \(\mathcal{M}(\{\phi^1_{\varepsilon_k}\}) > 0\). Indeed, this will follow from Proposition II.6 once we have established the hypotheses. We first note that by (2.20), \(\{\phi^1_{\varepsilon_k}\}_{k \in \mathbb{N}}\) is bounded in \(BV(\mathbb{R}^3)\). Next, we must show that the \(L^\frac{4}{3}\) norm of \(\phi^1_{\varepsilon_k}\) is bounded below. As \(u^1_{\varepsilon_k} = u_{\varepsilon_k}\) almost everywhere in \(\mathbb{R}^3 \setminus B_{\rho_k+1}(0)\), we have

\[
C_0\varepsilon_k \geq \int_{\mathbb{R}^3 \setminus B_{\rho_k+1}(0)} W(u_{\varepsilon_k}) \, dx = \int_{\mathbb{R}^3 \setminus B_{\rho_k+1}(0)} W(u^1_{\varepsilon_k}) \, dx,
\]

and thus, from (2.11), (2.6) and (2.22) we have:

\[
\int_{\mathbb{R}^3 \setminus B_{\rho_k+1}(0)} |\phi^1_{\varepsilon_k}|^\frac{4}{3} \, dx \geq \frac{1}{16} \int_{\mathbb{R}^3 \setminus B_{\rho_k+1}(0)} |u^1_{\varepsilon_k}|^\frac{4}{3} \, dx \\
\geq \frac{1}{32} \int_{\mathbb{R}^3 \setminus B_{\rho_k+1}(0)} \left( |u^1_{\varepsilon_k}|^\frac{10}{3} + |u_{\varepsilon_k}|^2 \right) \, dx - C_0\varepsilon_k \\
> \frac{1}{32} \int_{\mathbb{R}^3 \setminus B_{\rho_k}(0)} |u^1_{\varepsilon_k}|^2 \, dx - o(1) \\
\geq \frac{1}{32} \int_{\mathbb{R}^3} |u^1_{\varepsilon_k}|^2 \, dx + o(1) = \frac{M^1_{\varepsilon_k}}{32} + o(1) = \frac{1}{32} (M - m^0) + o(1) > 0.
\]

Applying Proposition II.6 the claim follows.

By the claim and Proposition II.6 we may choose a subsequence, translations \(\{x^1_k\}_{k \in \mathbb{N}}\), and \(\phi^1 \in BV(\mathbb{R}^3)\) with

\[
\phi^1_{\varepsilon_k} (-x^1_k) \xrightarrow[k \to \infty]{} \phi^1 \text{ in } L^1_{loc}(\mathbb{R}^3), \quad \|\phi^1\|_{L^1(\mathbb{R}^3)} \geq \frac{1}{2} \mathcal{M}(\{\phi^1_{\varepsilon_k}\}).
\]

Note that since \(\phi^1_{\varepsilon_k} \xrightarrow[k \to \infty]{} 0\) in \(L^1_{loc}(\mathbb{R}^3)\), the sequence \(|x^1_k| \xrightarrow[k \to \infty]{} \infty\). By the same arguments as in Step 1 we may conclude that \(u^1_{\varepsilon_k} (-x^1_k) \xrightarrow[k \to \infty]{} u^1 = 8\phi^1\) in \(L^2_{loc}(\mathbb{R}^3)\), with \(W(u^1) \equiv 0\) almost everywhere in \(\mathbb{R}^3\), and hence \(u^1 \in BV(\mathbb{R}^3, \{0, \pm 1\})\) with \(\|u^1\|_{L^2(\mathbb{R}^3)}^2 = m^1 \leq (M - m^0)\).

Finally, the nonlocal term, which splits as in (2.21), passes to the limit using Fatou’s Lemma,

\[
D(|u^0|^2, |u^0|^2) + D(|u^0|^2, |u^1|^2) \leq D(|u^0|^2, |u^0|^2) + \liminf_{k \to \infty} D(|u^1_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2) \\
\leq \liminf_{k \to \infty} D(|u^1_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2).
\]

In conclusion, we have

\[
\varepsilon_0^{\psi}(u^0) + \varepsilon_0^V(u^1) \leq \varepsilon_0^{\psi}(u^0) + \liminf_{k \to \infty} \left[ \int_{\mathbb{R}^3} |\nabla \phi^1_{\varepsilon_k}| \, dx + D(|u^1_{\varepsilon_k}|^2, |u^1_{\varepsilon_k}|^2) \right] \leq \liminf_{\varepsilon \to 0^+} \varepsilon^V(u_\varepsilon),
\]

with \(m^0 + m^1 \leq M\). If \(m^1 = \|u^1\|_{L^2(\mathbb{R}^3)}^2 = M - m^0\), then \(u^1 \equiv 0\) for all \(i \geq 2\).
Step 5: Iterating the argument.

If \( m^0 + m^1 < M \), then as in Step 3, the convergence of \( \phi^1_{\epsilon_k} (\cdot - x_k^1) \xrightarrow{k \to \infty} \phi^1 \) is only local and not in the norm of \( L^1(\mathbb{R}^3) \) (and similarly for \( u^1_{\epsilon_k} (\cdot - x_k^1) \xrightarrow{k \to \infty} u^1 \) in \( L^2(\mathbb{R}^3) \)) and so there is again a remainder part to be separated via Proposition II.7. That is, we may choose radii \( \{\rho_k^1\}_{k \in \mathbb{N}} \) and further decompose \( \phi^1_{\epsilon_k} (\cdot - x_k^1) \),

\[
\phi^1_{\epsilon_k} (\cdot - x_k^1) \omega_{\rho_k^1} \xrightarrow{k \to \infty} \phi^1 \text{ in } L^1 \text{ norm, } \phi^2_{\epsilon_k} := \phi^1_{\epsilon_k} (\cdot - x_k^1) (1 - \omega_{\rho_k^1}) \xrightarrow{k \to \infty} 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^3),
\]

with the same consequences as in Step 4, identifying a mass center for \( \phi^2_{\epsilon_k} \) via Proposition II.6 translating and passing to a local \( L^1 \) limit to find \( \phi^2 = \frac{1}{8} u^2 \), and creating an refined lower bound.

Assuming the procedure has been done for the first \( n \) steps, we would have \( u^{i_0}, \ldots, u^n \in BV(\mathbb{R}^3, \{0, \pm 1\}) \) with masses \( ||u^i||_{L^2(\mathbb{R}^3)} = m^i \), and translations \( \{x_k^i\}_{k \in \mathbb{N}} \) for each \( i = 1, \ldots, n \), such that:

\[
\begin{align*}
\begin{cases}
 u_{\epsilon_k} = u^0 + \sum_{i=1}^n u^i (\cdot - x_k^i) + u_{\epsilon_k}^{n+1}(\cdot - x_k^n), \quad \text{and } u_{\epsilon_k}^{n+1}(\cdot - x_k^n) \xrightarrow{k \to \infty} 0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^3); \\
 m^i = ||u^i||_{L^2(\mathbb{R}^3)}^2, \quad i = 0, \ldots, n; \\
x_k^i \xrightarrow{k \to \infty} \infty, \quad |x_k^i - x_k^j| \xrightarrow{k \to \infty} \infty, \quad 1 \leq i \neq j; \\
 M = \sum_{i=0}^n m^i + \lim_{k \to \infty} ||u_{\epsilon_k}^{n+1}||_{L^2(\mathbb{R}^3)}^2; \\
 \mathcal{E}_0^V (u^0) + \sum_{i=1}^n \mathcal{E}_0^0 (u^i) \leq \lim_{\epsilon \to 0^+} \mathcal{E}_0^V (u_{\epsilon_k}).
\end{cases}
\end{align*}
\]

If for some \( n \in \mathbb{N} \), the remainder \( \phi_{\epsilon_k}^i \to 0 \) in the \( L^1 \) norm, then the iteration terminates at that \( n \), and the proof (i) of Theorem II.4 is completed by choosing \( u^i = 0 \) for all \( i \geq n + 1 \). If the iteration continues indefinitely, we must verify that the entire mass corresponding to \( \{u_{\epsilon_k}\}_{k \in \mathbb{N}} \) is exhausted by the \( \{u^i\}_{i=0}^\infty \). It is here that we use \( \mathcal{M} (\{\phi^i_{\epsilon_k}\}) \). When localizing mass in the remainder term \( \phi_{\epsilon_k}^i \), the translations \( \{x_k^i\} \) and limit \( \phi^i = \frac{1}{8} u^i \) are chosen via Proposition II.6 in such a way that \( ||\phi^i||_{L^1(\mathbb{R}^3)} \geq \frac{1}{2} \mathcal{M} (\{\phi^i_{\epsilon_k}\}), i = 1, \ldots, n \). In this way, the boundedness of the partial sums \( \sum_{i=0}^n m^i \leq M \) implies that, should the process continue indefinitely, the residual mass \( \mathcal{M} (\{\phi^i_{\epsilon_k}\}) \leq 2m^i \xrightarrow{i \to \infty} 0 \). We claim that this implies that

\[
M = \sum_{i=0}^\infty m^i = \sum_{i=0}^\infty ||u^i||_{L^2(\mathbb{R}^3)}^2,
\]

and that the entire mass corresponding to \( \{u_{\epsilon_k}\}_{k \in \mathbb{N}} \) is exhausted by the \( \{u^i\}_{i=0}^\infty \). Indeed, if \( \sum_{i=0}^\infty m^i = M' < M \), then each remainder sequence has \( ||\phi_{\epsilon_k}^i||_{L^1(\mathbb{R}^3)} \geq \frac{1}{8} ||u^i||_{L^2(\mathbb{R}^3)}^2 \geq \frac{M - M'}{8} \).

Returning to Step 4, and calculating as in (2.23), we obtain a lower bound

\[
\int_{\mathbb{R}^3} |\phi_{\epsilon_k}^i|^2 dx \geq C(M - M'),
\]
for a constant $C$ independent of $k, i$. Using Lemma III.5 we then have a uniform lower bound,

$$\mathcal{M}(\{\phi_{e_k}^i\}) \geq \sup_{a \in \mathbb{R}^3} \int_{B_1(a)} |\phi_{e_k}^i| \, dx \geq C'(M - M')^3,$$

for each $i \in \mathbb{N}$, with $C'$ depending on the upper energy bound $K_0$, but independent of $k, i$. This contradicts $\mathcal{M}(\{\phi_{e_k}^i\}) < 2m^i \xrightarrow{i \to \infty} 0$. Hence (2.25) is established, and passing to the limit $n \to \infty$ in (2.24) we conclude the proof of (i) of Theorem I.1.

\[\square\]

### III. Upper Bound

In this section we prove part (ii) of Theorem I.1, the construction of recovery sequences in the Gamma convergence of $\mathcal{E}_\varepsilon^V$. As the space $\mathcal{H}_0^M$ consists of a collection of functions in $BV(\mathbb{R}^3, \{0, \pm 1\})$, we build the recovery sequence by superposition of each, using the following lemma:

**Lemma III.1.** Given $v^0 \in BV(\mathbb{R}^3, \{0, \pm 1\})$ with $\|v^0\|_{L^2(\mathbb{R}^3)} = M$, there exists $\varepsilon_0 = \varepsilon_0(v^0) > 0$ and functions $\{v_\varepsilon\}_{0 < \varepsilon < \varepsilon_0} \subset \mathcal{H}_0^M$ of compact support, such that

$$\|v_\varepsilon - v^0\|_{L^r(\mathbb{R}^3)} \xrightarrow{\varepsilon \to 0^+} 0, \quad \forall 1 \leq r < \infty,$$

and

$$\mathcal{E}_\varepsilon^V(v_\varepsilon) \xrightarrow{\varepsilon \to 0^+} \mathcal{E}_0^V(v^0).$$

**Proof.** The basic construction is familiar, based on that of Sternberg [28]. Proof of inequalities (1.12) and (1.13), so we highlight the modifications necessary for our case. The first step is to regularize $v^0$. As compactly supported functions are dense in the $BV(\mathbb{R}^3)$ norm, we may assume that $\text{supp} \ v^0$ is bounded. Next, define a smooth mollifier, using $\varphi \in C_0^\infty(B_1(0))$, $\varphi(x) \geq 0$, $\int_{B_1(0)} \varphi \, dx = 1$ to generate $\varphi_n(x) = n^3 \varphi(nx) \in C_0^\infty(B_{1/3}(0))$. Following the proof of regularization of BV functions (see [28, Theorem 3.42,]), we create a sequence $w_n = \varphi_n * v^0$ which is smooth and supported in a $\frac{1}{n}$-neighborhood of the support of $v^0$. As in [28], the regularization is obtained as a level surface of $w_n$. Here, we have two components, corresponding to the regularizations of $v_+^0$ and $v_-^0$, in case $v^0$ takes on both values $\pm 1$. By Sard’s Theorem [28, 3.4.3,], there exist values $t_+ \in (0, 1)$ and $t_- \in (-1, 0)$ for which the boundaries of the sets

$$F_n^+ := \{x \in \mathbb{R}^3 \mid w_n(x) > t_+ > 0\}, \quad F_n^- := \{x \in \mathbb{R}^3 \mid w_n(x) < t_- < 0\}$$

are smooth for each $n \in \mathbb{N}$, $v_n^\pm := 1_{F_n^\pm} \xrightarrow{n \to \infty} v_0^\pm$ converge in $L^1(\mathbb{R}^3)$, and

$$\int_{\mathbb{R}^3} |\nabla v_n^\pm| \xrightarrow{n \to \infty} \int_{\mathbb{R}^3} |\nabla v_0^\pm|.$$

Note by this construction that the sets $F_n^\pm$ are smooth and disjoint for each $n$. Hence, the construction in [28] may be done separately for the components $F_n^\pm$, for any $0 < \varepsilon < \eta_n$, with $\eta_n > 0$ being chosen so that the neighborhoods of radius $\sqrt{\varepsilon}$ of the boundaries $F_n^\pm$ are
defines a function in \( v \). There exists \( \tilde{v}_{n, \varepsilon} \) with \( \tilde{v}_{n, \varepsilon}^+, \tilde{v}_{n, \varepsilon}^- \) disjointly supported, \( 0 \leq \tilde{v}_{n, \varepsilon}^\pm \leq 1 \), and

\[
\| \tilde{v}_{n, \varepsilon}^\pm - v_n^\pm \|_{L^1(\mathbb{R}^3)} \xrightarrow{\varepsilon \to 0^+} 0, \quad \text{and} \quad \int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |H \nabla \tilde{v}_{n, \varepsilon}^\pm|^2 + \frac{1}{2\varepsilon} W(\tilde{v}_{n, \varepsilon}^\pm) \right] \xrightarrow{\varepsilon \to 0^+} \frac{1}{8} \int_{\mathbb{R}^3} |\nabla v_n^\pm|.
\]

(3.1)

Writing \( \tilde{v}_{n, \varepsilon} = \tilde{v}_{n, \varepsilon}^+ - \tilde{v}_{n, \varepsilon}^- \) (again, a disjoint sum for all \( 0 < \varepsilon < \eta_n \)) the same properties (3.1) hold for \( \tilde{v}_{n, \varepsilon} \) and \( v_0 = v_n^+ - v_n^- \).

Next, we adjust the \( \tilde{v}_{n, \varepsilon} \) so that for each \( n, \varepsilon \), each has \( L^2 \) norm equal to \( M \), and hence defines a function in \( H^M \). For this we use dilation: let \( \lambda_\varepsilon := (\| \tilde{v}_{n, \varepsilon} \|^2_{L^2(\mathbb{R}^3)}/M)^{\frac{1}{2}} \xrightarrow{\varepsilon \to 0^+} 1 \).

We define the rescaled functions \( \hat{v}_{n, \varepsilon} : \mathbb{R}^3 \to \mathbb{R} \) by:

\[
\hat{v}_{n, \varepsilon}(x) := \tilde{v}_{n, \varepsilon}(\lambda_\varepsilon x), \quad \text{and} \quad \hat{v}_n^\pm(x) := v_n^\pm(\lambda_\varepsilon x).
\]

First, by rescaling we have \( \| \hat{v}_{n, \varepsilon} \|^2_{L^2(\mathbb{R}^3)} = M \), and so \( \hat{v}_{n, \varepsilon} \in H^M \) for all \( n, \varepsilon \). Next, we observe that, since the supports \( F_n^\pm \) of the components of \( v_0 \) are smooth, for \( |\lambda_\varepsilon - 1| \) sufficiently small, we may estimate

\[
\| \hat{v}_n^0 - v_0^\pm \|_{L^1(\mathbb{R}^3)} \leq c |\lambda_\varepsilon^{\frac{1}{2}} - 1| \int_{\mathbb{R}^3} |\nabla v_0^\pm|.
\]

Hence, we have convergence in the \( L^1 \) norm,

\[
0 \leq \| \hat{v}_{n, \varepsilon} - v_n^0 \|_{L^1(\mathbb{R}^3)} \leq \| \hat{v}_{n, \varepsilon} - \hat{v}_n^0 \|_{L^1(\mathbb{R}^3)} + \| \hat{v}_n^0 - v_n^0 \|_{L^1(\mathbb{R}^3)} \leq \lambda_\varepsilon^{-\frac{1}{2}} \| \hat{v}_{n, \varepsilon} - v_n^0 \|_{L^1(\mathbb{R}^3)} + c |\lambda_\varepsilon^{\frac{1}{2}} - 1| \int_{\mathbb{R}^3} |\nabla v_n^0| \xrightarrow{\varepsilon \to 0^+} 0.
\]

As each of \( |\hat{v}_{n, \varepsilon}| \leq 1 \) almost everywhere in \( \mathbb{R}^3 \), and for fixed \( n \) each is of uniformly bounded support, the convergence extends to any \( L^r(\mathbb{R}^3), r \geq 1 \). Moreover,

\[
\int_{\mathbb{R}^3} \left[ \frac{\varepsilon}{2} |H \nabla \hat{v}_{n, \varepsilon}^\pm|^2 + \frac{1}{2\varepsilon} W(\hat{v}_{n, \varepsilon}^\pm) \right] dx
\]

\[
= \left[ \lambda_\varepsilon^{-\frac{1}{2}} \int_{\mathbb{R}^3} \frac{\varepsilon}{2} |H \nabla \hat{v}_{n, \varepsilon}^\pm|^2 dx + \lambda_\varepsilon^{-1} \int_{\mathbb{R}^3} \frac{1}{2\varepsilon} W(\hat{v}_{n, \varepsilon}^\pm) dx \right] \xrightarrow{\varepsilon \to 0^+} \frac{1}{8} \int_{\mathbb{R}^3} |\nabla v_n^0|,
\]

which holds for each \( n \in \mathbb{N} \). As in \cite{28}, by a diagonal argument, there exists \( \varepsilon_0 = \varepsilon_0(v^0) > 0 \) so that for any sequence \( \varepsilon_k \xrightarrow{k \to \infty} 0^+ \) with \( \varepsilon_k < \varepsilon_0 \), we obtain a sequence \( v_{\varepsilon_k} \) with

\[
\| v_{\varepsilon_k} - v^0 \|_{r \to \infty} \xrightarrow{r \geq 1} 0, \quad \text{and} \quad \int_{\mathbb{R}^3} \left[ \frac{\varepsilon_k}{2} |\nabla v_{\varepsilon_k}|^2 + \frac{1}{2\varepsilon_k} W(v_{\varepsilon_k}) \right] dx \xrightarrow{k \to \infty} \frac{1}{8} \int_{\mathbb{R}^3} |\nabla v^0|.
\]

The local potential terms also converge by Lemma \cite{11.3}. Furthermore, by the Hardy-Littlewood-Sobolev inequality \cite{21, Theorem 4.3} (with \( p = 6/5 = r \)),

\[
0 \leq |D(|v_{\varepsilon_k}|^2, |v_{\varepsilon_k}|^2) - D(|v^0|^2, |v^0|^2)|
\]

\footnote{We note that the potential in \cite{28} has two wells at \( u = \pm 1 \), whereas our transitions connect \( v = 0 \) to \( v = \pm 1 \), and so our \( \tilde{v}_{n, \varepsilon}^\pm = \frac{1}{2}(\rho_\varepsilon + 1) \) for \( \rho_\varepsilon \) as constructed in \cite{28}.}
\[
\begin{align*}
|D(|v_{\varepsilon k}|^2 - |v^0|^2, |v_{\varepsilon k}|^2 + |v^0|^2)| \\
\leq \|v_{\varepsilon k}|^2 - |v^0|^2\|_{L^2(\mathbb{R}^3)} \|v_{\varepsilon k}|^2 + |v^0|^2\|_{L^2(\mathbb{R}^3)} \xrightarrow{k \to \infty} 0.
\end{align*}
\]

This completes the proof of Lemma III.1.

Proof of (ii) of Theorem III.2. If \( \{u^i\} \) is a finite collection with \( N \) nontrivial components, this follows easily from Lemma III.1. Indeed, for any sequence \( \varepsilon_k \xrightarrow{k \to \infty} 0^+ \) with \( 0 < \varepsilon_k < \min_{i=0,\ldots,N} \{\varepsilon_0(u^i)\} \), we apply the lemma to find \( u^i_\varepsilon \to u^i \), and form the disjoint sum,

\[
u_{\varepsilon k}(x) = u^0_{\varepsilon k} + \sum_{i=1}^N u^i_{\varepsilon k}(x - x^i_k),\]

by choosing translations \( x^i_k \) which tend to infinity and far from each other quickly enough in \( k \).

If \( \{u^i\}_{i=0}^\infty \) has an infinite number of nontrivial elements, we must be more careful. In particular, as we go down the list of the \( \{u^i\}_{i=0}^\infty \), the characteristic length scale of each \( u^i \) gets smaller, and for any particular \( \varepsilon > 0 \) there can only be a finite number of \( i \) with \( 0 < \varepsilon < \varepsilon_0(u^i) \), for which the trial functions \( u^i_\varepsilon \) can be constructed via Lemma III.1. Take any sequence \( \varepsilon_k \xrightarrow{k \to \infty} 0^+ \). By Lemma III.1 for each \( i = 0, 1, 2, \ldots \) there exists \( \varepsilon^i = \varepsilon_0(u^i) > 0 \) and a sequence \( u^i_{\varepsilon_k} \), for which

\[
\begin{align*}
&\left|\varepsilon^V_{\varepsilon k}(u^0_{\varepsilon_k}) - \varepsilon^V_0(u^0)\right| < \frac{\varepsilon^V_0(u^0)}{10} \quad \text{and} \quad \|u^0_{\varepsilon_k} - u^0\|_{L^2(\mathbb{R}^3)} < \frac{m^0}{10}, \\&\left|\varepsilon^V_{\varepsilon k}(u^i_{\varepsilon_k}) - \varepsilon^V_0(u^i)\right| < \frac{\varepsilon^V_0(u^i)}{10} \quad \text{and} \quad \|u^i_{\varepsilon_k} - u^i\|_{L^2(\mathbb{R}^3)} < \frac{m^i}{10}, \quad 0 < \varepsilon_k < \varepsilon^0, \quad 0 < \varepsilon_k < \varepsilon^i, \quad i = 1, 2, 3, \ldots
\end{align*}
\]

(3.2)

By taking \( \varepsilon^i \) smaller if necessary we may assume \( 0 < \varepsilon^i < \varepsilon^{i-1} \). We now construct \( U_{\varepsilon k} \) as follows: for each \( k \in \mathbb{N} \), choose \( n_k \in \mathbb{N} \) such that \( 0 < \varepsilon_k < \varepsilon^i \) for all \( i \leq n_k \). As the \( u^i_{\varepsilon_k} \) are all compactly supported, we may choose points \( x^i_{\varepsilon_k} \in \mathbb{R}^3, i = 1, \ldots, n_k \), so that the functions \( u^i_{\varepsilon_k}(x - x^i_{\varepsilon_k}) \) are disjointly supported, and

\[
|x^i_{\varepsilon_k} - x^j_{\varepsilon_k}| > 2^k.
\]

Then, we set

\[
U_{\varepsilon k}(x) := u^0_{\varepsilon k}(x) + \sum_{i=1}^{n_k} u^i_{\varepsilon k}(x - x^i_{\varepsilon_k}).
\]

Note that by the choice of the \( x^i_{\varepsilon_k} \), the nonlocal interactions are negligible,

\[
\varepsilon^V_{\varepsilon k}(U_{\varepsilon k}) = \varepsilon^V_{\varepsilon k}(u^0_{\varepsilon_k}) + \sum_{i=1}^{n_k} \varepsilon^V_{\varepsilon k}(u^i_{\varepsilon_k}) + cM2^{-k}.
\]

Note that the mass \( \|U_{\varepsilon k}\|_{L^2(\mathbb{R}^3)}^2 = \sum_{i=0}^{n_k} m^i := M^k < M \), but it will approach \( M \) as \( k \to \infty \), as more components are added to the sum.
We next show that $\mathcal{E}_ε^V(U_{ε_k}) \xrightarrow{k \to \infty} \mathcal{F}_0^V(\{u^i\}_{i=0}^\infty)$. Let $δ > 0$ be given, and choose $N \in \mathbb{N}$ for which both
\begin{align*}
\sum_{i=N+1}^{\infty} m^i < \frac{δ}{5}, \quad \sum_{i=N+1}^{\infty} \mathcal{E}_0^0(u^i) < \frac{δ}{5}.
\end{align*}
(3.3)
From Lemma [III.1], there exists $K \in \mathbb{N}$ such that for all $k \geq K$,
\begin{align*}
\sum_{i=0}^{N} \|u^i_{ε_k} - u^i\|^2_{L^2(\mathbb{R}^3)} + \|\mathcal{E}_ε^V(u_{ε_k}) - \mathcal{E}_0^V(u^0)\| + \sum_{i=1}^{N} \|\mathcal{E}_ε^V(u_{ε_k}) - \mathcal{E}_0^V(u^0)\| < \frac{δ}{5}.
\end{align*}
(3.4)
Then, for all $k \geq K$, using (3.4), (3.2), and (3.3), we estimate
\begin{align*}
\|U_{ε_k} - \left(u_0 + \sum_{i=0}^{\infty} u^i(x - x^i_k)\right)\|_{L^2(\mathbb{R}^3)} &\leq \sum_{i=0}^{N} \|u^i_{ε_k} - u^i\|_{L^2(\mathbb{R}^3)} + \sum_{i=N+1}^{\infty} \|u^i_{ε_k} - u^i\|_{L^2(\mathbb{R}^3)} \\
&\quad + \sum_{i=n_k+1}^{\infty} \|u^i\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{δ}{5} + \sum_{i=N+1}^{\infty} \frac{m^i}{10} + \frac{δ}{5} < δ,
\end{align*}
and
\begin{align*}
\left|\mathcal{E}_ε^V(U_{ε_k}) - \mathcal{E}_0^V(u^0) - \sum_{i=0}^{N} \mathcal{E}_0^0(u^i)\right| \\
&\leq \left|\mathcal{E}_ε^V(U_{ε_k}) - \mathcal{E}_0^V(u^0)\right| + \sum_{i=0}^{N} \left|\mathcal{E}_ε^V(u^i_{ε_k}) - \mathcal{E}_0^0(u^i)\right| + \sum_{i=N+1}^{n_k} \mathcal{E}_0^0(u^i_{ε_k}) + cM2^{-k} \\
&\leq \frac{δ}{5} + \frac{11}{10} \sum_{i=N+1}^{n_k} \mathcal{E}_0^0(u^i) + cM2^{-k} < \frac{δ}{2}.
\end{align*}
Then, by (3.3), for all $k \geq K$ we have
\begin{align*}
\left|\mathcal{E}_ε^V(U_{ε_k}) - \mathcal{F}_0^V(\{u^i\}_{i=0}^\infty)\right| \leq \left|\mathcal{E}_ε^V(U_{ε_k}) - \mathcal{E}_0^V(u^0) - \sum_{i=0}^{N} \mathcal{E}_0^0(u^i)\right| + \sum_{i=N+1}^{\infty} \mathcal{E}_0^0(u^i) < δ.
\end{align*}
It remains to correct the mass of $U_{ε_k}$, so that each $\|U_{ε_k}\|^2_{L^2(\mathbb{R}^3)} = M$. This is done as in Lemma [III.1] dilating each component $u^i_{ε_k}$ by the scaling factor $λ_k = (M^k/M)^{\frac{1}{2}} \xrightarrow{k \to \infty} 1$, that is, by setting
\begin{align*}
u_{ε_k}(x) = u^0_{ε_k}(λ_k x) + \sum_{i=1}^{n_k} u^i_{ε_k}(λ_k(x - x_k)).
\end{align*}
Then $\|u_{ε_k}\|^2_{L^2(\mathbb{R}^3)} = M$, $k \in \mathbb{N}$, $\|u_{ε_k} - U_{ε_k}\|^2_{L^2(\mathbb{R}^3)} \xrightarrow{k \to \infty} 0$, and $|\mathcal{E}_ε^V(u_{ε_k}) - \mathcal{E}_ε^V(U_{ε_k})| \xrightarrow{k \to \infty} 0$, since $λ_k \to 1$. This concludes the proof of Theorem [I.1] □
IV. Minimizers

In this section we examine the connection between minimizers of the Liquid Drop and TDFW functionals. The compactness of minimizing sequences being a delicate issue which is shared by the two models,

First, whether the minimum in $e^V_\varepsilon(M)$ is attained or not, the infimum values converge as $\varepsilon \to 0^+$:

**Lemma IV.1.** Assume $V$ satisfies (1.4). Then, for all $M > 0$, $e^V_\varepsilon(M) \xrightarrow{\varepsilon \to 0^+} e^V_0(M)$.

**Proof.** The proof is standard. First, $\forall \varepsilon > 0$, $\exists u_\varepsilon \in \mathcal{H}^M$ with $\|u_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 = M$ and $e^V_\varepsilon(u_\varepsilon) \leq e^V_\varepsilon(M) + \varepsilon$. By Theorem I.1 (i), $\exists \{u^i\}_{i=0}^\infty \in \mathcal{H}^M$ and a subsequence $\varepsilon_n \xrightarrow{n \to \infty} 0^+$ with

$$e^V_0(M) \leq \mathcal{F}_0^V(\{u^i\}_{i=0}^\infty) \leq \liminf_{\varepsilon \to 0^+} e^V_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0^+} e^V_\varepsilon(M).$$

For the complementary inequality, for any $\delta > 0$, $\exists \{v^i\}_{i=0}^\infty \in \mathcal{H}^M$ with $\mathcal{F}_0^V(\{v^i\}_{i=0}^\infty) < e^V_0(M) + \delta$. Then, by (ii) in Theorem I.1, for any $\varepsilon > 0$, $\exists v_\varepsilon \in \mathcal{H}^M$ with

$$e^V_\varepsilon(M) + \delta > \mathcal{F}_0^V(\{v^i\}_{i=0}^\infty) \geq \limsup_{\varepsilon \to 0^+} e^V_\varepsilon(u_\varepsilon) \geq \limsup_{\varepsilon \to 0^+} e^V_\varepsilon(M).$$

Putting the above inequalities together, and letting $\delta \to 0^+$, we obtain the desired conclusion. \hfill \Box

**Proof of Corollary I.4.** In [3] it is proven that for $V$ satisfying (1.9), the minimum for both $e^V_\varepsilon$ and $e^V_0$ are attained. Indeed, the proof of these results in [3] actually yields the stronger conclusion that all minimizing sequences for either the TDFW or Liquid Drop functionals are convergent. Thus, $\forall \varepsilon > 0$, $\exists u_\varepsilon \in \mathcal{H}^M$ which attains the minimum, $e^V_\varepsilon(M) = e^V_\varepsilon(u_\varepsilon)$. By Lemma IV.1, $e^V_\varepsilon(u_\varepsilon) \xrightarrow{\varepsilon \to 0^+} e^V_0(M)$, so for any sequence $\varepsilon_n \xrightarrow{n \to \infty} 0^+$, by Theorem I.1 (i), $\exists \{u^i\}_{i=0}^\infty \in \mathcal{H}^M$ with

$$\mathcal{F}_0^V(\{u^i\}_{i=0}^\infty) \leq \liminf_{n \to \infty} e^V_\varepsilon(u_\varepsilon) = e^V_0(M).$$

Defining $m^i = \|u^i\|_{L^2(\mathbb{R}^3)}^2$, we have

$$e^V_0(M) = e^V_0(m^0) + \sum_{i=1}^\infty e^V_0(m^i), \quad (4.1)$$

We then obtain a contradiction by using Step 6 in the proof of Theorem 1 of [3]. Indeed, by choosing compactly supported $v^0, v^1 \in \mathcal{H}^M$ whose energy is close to the infimum $e^V_0(m^0), e^V_0(m^1)$ as in Step 6, we obtain the strict subadditivity condition,

$$e^V_0(M) < e^V_0(m^0) + e^V_0(m^1) + e^V_0(M - m^0 - m^1) \leq e^V_0(m^0) + \sum_{i=1}^\infty e^V_0(m^i),$$

and the desired contradiction to (4.1). \hfill \Box
Analyzing the possible loss of compactness in minimizing sequences for $e^Z_\varepsilon(M)$, $\varepsilon \geq 0$ and $Z \geq 0$, requires the use of concentration-compactness methods \cite{22}. The following are standard results for problems where loss of compactness entails splitting of mass to infinity:

**Lemma IV.2.** Assume $V$ satisfies (1.4). Then, for any $\varepsilon \geq 0$ and $M > 0$,

(i) If $\forall m^0 \in (0, M)$,

\[ e^V_\varepsilon(M) < e^V_\varepsilon(m^0) + e^0_\varepsilon(M - m^0), \]  

then all minimizing sequences for $e^V_\varepsilon(M)$ are precompact.

(ii) If there exist noncompact minimizing sequences for $e^V_\varepsilon(M)$, then $\exists m^0 \in (0, M)$ such that $e^V_\varepsilon(m^0)$ attains a minimizer and $e^V_\varepsilon(M) = e^V_\varepsilon(m^0) + e^0_\varepsilon(M - m^0)$.

Statement (ii) is a useful precision of the contrapositive of (i). The proof for the TFDW functional was done in \cite{23}, and for Liquid Drop models it may be derived from the more detailed concentration lemma in \cite{2}; although it is stated there for $V$ of a special form, in fact it is true for a much larger class including those satisfying (1.4).

Next, we specialize to the atomic case, $V(x) = \frac{Z}{|x|}$, and present the following refinement of the existence result of \cite{25} for the Liquid Drop model with atomic potential:

**Proposition IV.3.** There exists a constant $\mu_0 > 0$ such that for all $Z \geq 0$ and for all $M \in (0, Z + \mu_0)$:

(i) All minimizing sequences for $e^Z_0(M)$ are precompact.

(ii) The unique minimizer of $e^Z_0(M)$ is the ball $B_M(0)$ of radius $r_M = \left(\frac{3M}{\pi}\right)^{1/3}$.

**Proof.** Statement (ii) is proven in Theorem 2 of \cite{25}, using Theorem 2.1 in \cite{16}. The case $Z = 0$ was proven in \cite{17}. We sketch the proof of (i), since we will need certain definitions and estimates for (ii). As in Julin \cite{16}, we define an asymmetry function corresponding to a fixed set $\Omega$ of finite perimeter,

\[ \gamma(\Omega) := \min_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1_B(x) - 1_\Omega(x + y)}{|x|} dx, \]

where $B = B_M(0)$ is the ball of mass $M$ centered at the origin. The quantitative isoperimetric inequality (see (2.3) of \cite{16} or \cite{15}) then asserts the existence of a universal constant $\mu_0 > 0$, such that

\[ \int_{\mathbb{R}^3} |\nabla 1_\Omega| - \int_{\mathbb{R}^3} |\nabla 1_B| \geq \mu_0 \gamma(\Omega), \]
with equality if and only if $\Omega$ is a translate of $B$. Then, as in the proof of Theorem 1.1 of [10] in the three-dimensional case, we may estimate the difference in the nonlocal terms by the asymmetry,

$$D(1_B, 1_B) - D(1_\Omega, 1_\Omega) \leq |B| \gamma(\Omega).$$

The optimality of the ball $B = B_M$ follows easily from this: assume $\Omega$ is of finite perimeter, with $|\Omega| = M$. Then, provided $\Omega$ is not a translate of the ball $B = B_M$,

$$\mathcal{E}^Z_0(1_\Omega) - \mathcal{E}^Z_0(1_B) > (\mu_0 - M) \gamma(\Omega) + Z \left( \int_{\mathbb{R}^3} \frac{1_B(x) - 1_\Omega(x)}{|x|} dx \right)$$

$$\geq (Z + \mu_0 - M) \gamma(\Omega) > 0, \quad (4.3)$$

for all $M < Z + \mu_0$.

To obtain (i), the precompactness of all minimizing sequences, we use the above to establish strict subadditivity of $e^Z_0(M)$, as in Lions [22]. Let $M = m^0 + m^1$ with $m^0, m^1 > 0$; we will show that (1.2) holds, and then by Lemma IV.2 all minimizing sequences for $e^Z_0(M)$ are precompact.

Since $0 < m^0 < M < Z + \mu_0$, both $e^Z_0(M)$, $e^Z_0(m^0)$ are attained by balls $B = B_M(0)$, $B^0 = B_{r_{m^0}}(0)$. For any $\delta > 0$ (to be chosen later,) we may choose a bounded open set $\omega$ with $0 \in \omega$, $|\omega| = m^1$, and $e_0(1_\omega) < e_0^0(m^1) + \delta$. Note that if $m^0 \geq Z$, then $0 < m^1 < \mu_0$ and we may choose $\omega = B^1 = B_{r_{m^1}}$, which attains $e^0(m^1)$.

Define $\omega_\xi = \omega + \xi$, and $\Omega = \Omega_\xi = B^0 \cup \omega_\xi$, with $|\xi|$ sufficiently large that the union is disjoint. We first claim that $\exists R > 1$ such that $\gamma(\Omega_\xi) \geq \gamma(\Omega) > 0$ is bounded away from zero for all $\xi$ with $|\xi| > R$, with constant $C = C(m^0, m^1)$. Indeed, for $y \in \mathbb{R}^3$ define

$$v = v^0 + v^1, \quad v^0(y) = \int_{B^0} \frac{dx}{|x - y|}, \quad v^1(y) = \int_{\omega + \xi} \frac{dx}{|x - y|},$$

so that

$$\gamma(\Omega_\xi) = \int_B \frac{dx}{|x|} - \max_{y \in \mathbb{R}^3} v(y).$$

Hence, to bound $\gamma(\Omega_\xi)$ from below we must bound $v(y)$ uniformly from above. As $-\Delta v = 4\pi(1_{B^0}(y) + 1_{\omega + \xi}(y))$ in $\mathbb{R}^3$, it attains its maximum at $y \in \Omega_\xi = B^0 \cup \omega_\xi$. Thus, there are two possibilities: if the maximum occurs at $y \in B^0$, then $v(y) = v^0(y) + O(|\xi|^{-1})$. Since $v^0$ is maximized at $y = 0$, there exists $C_0 = C_0(M, m^0)$ and $R > 1$ with

$$\gamma(\Omega_\xi) \geq \int_{B \setminus B^0} \frac{dx}{|x|} - O(|\xi|^{-1}) \geq C_0 > 0,$$

for all $|\xi| > R$.

In case the maximum of $v$ occurs at $y \in \omega + \xi$, then $v(y) = v^1(y) + O(|\xi|^{-1})$. For any domain $D$ with $|D| = m^1$ we have

$$\int \frac{dx}{|x|} \leq \int_{B^1} \frac{dx}{|x|}.$$
where \( B^1 = B_{r_m}(0) \) is the ball with mass \( m^1 \). It follows that
\[
v^1(y) = \int_{\omega + \xi} \frac{dx}{|x - y|} \leq \int_{B^1} \frac{dx}{|x|}.
\]
Therefore, as in the previous case, there exist \( C_1 = C_1(M, m^1) \) and \( R > 1 \) with \( \gamma(\Omega_{\xi}) \geq C_1 > 0 \), for all \( |\xi| > R \), and the claim is established, with \( C = \min\{C_0, C_1\} \).

To conclude, we choose \( 0 < \delta < \frac{1}{2} (Z + \mu_0 - M) \) and using (4.3),
\[
e^Z_0(M) = e^Z_0(1_B) < e^Z_0(1_{\Omega_{\xi}}) - (Z + \mu_0 - M) \gamma(\Omega_{\xi})
\]
\[
\leq e^Z_0(1_B^0) + e^Z_0(1_{\omega + \xi}) - (Z + \mu_0 - M) \gamma(\Omega_{\xi}) + 2 \int_{B^0} \int_{\omega + \xi} \frac{dx \, dy}{|x - y|}
\]
\[
\leq e^Z_0(1_B^0) + e_0(1_{\omega}) - (Z + \mu_0 - M) \gamma(\Omega_{\xi}) + O(|\xi|^{-1})
\]
\[
\leq e^Z_0(m^0) + e^Z_0(m^1) + \varepsilon - (Z + \mu_0 - M) \gamma(\Omega_{\xi}) + O(|\xi|^{-1}).
\]

Taking \( |\xi| \) sufficiently large, (4.2) holds for all \( M \in (0, Z + \mu_0) \).

**Remark IV.4.** Thanks to Proposition [IV.3], we may conclude that for the liquid drop model with \( V(x) = Z/|x| \) with \( 0 < M < Z + \mu_0 \), the unique generalized minimizer (see Definition (1.2)) is the singleton \( \{u^0 = 1_{B_M}\} \). Indeed, this will be true for any functional which satisfies the strict subadditivity condition (1.2).

Next, we prove Theorem [1.3] In fact, we prove the following slightly more general version, which will also be a step towards the proof of Theorem [1.5].

**Lemma IV.5.** Let \( M > 0 \) and \( \delta_n, \varepsilon_n \to 0 \). Assume \( u_n \in \mathcal{H}^M \) for which \( e^V_\varepsilon(u_n) \leq e^V_\varepsilon(M) + \delta_n \) for each \( n \in \mathbb{N} \). Then, there exists a subsequence and a generalized minimizer \( \{u^0, \ldots, u^N\} \) of \( e^V_\varepsilon \) for which (1.6) and (1.7) hold for \( i = 0, \ldots, N \), and
\[
\mathcal{F}^V_0(\{u^i\}_{i=0}^N) = e^V_0(M) = \lim_{n \to \infty} e^V_{\varepsilon_n}(M).
\]

**Proof.** By (i) of Theorem [1.1] there exists a subsequence along which \( u_n \) decomposes as in (1.6), with \( \{u^i\}_{i=0}^\infty \in \mathcal{H}^M \) satisfying (1.8). By Lemma IV.1 we have
\[
\mathcal{F}^V_0(\{u^i\}_{i=0}^\infty) = \lim_{n \to \infty} e^V_\varepsilon(u_n) = \lim_{n \to \infty} e^V_{\varepsilon_n}(M) = e^V_0(M).
\]
Let \( m^i = \|u^i\|_{L^2(\mathbb{R}^3)}^2 \). It suffices to show that \( u^0 \) minimizes \( e^V_0(m^0) \) and \( u^i \) minimizes \( e^0_0(m^i) \), for each \( i \), and that all but a finite number of the \( u^i \equiv 0 \). First, by (1.5) we have

\[
e^V_0(m^0) + \sum_{i=1}^\infty e^0_0(m^i)
\]
\[
\leq e_0(u^0) + \sum_{i=1}^\infty e_0^0(u^i) = \mathcal{F}^V_0(\{u^i\}_{i=0}^\infty) = e^V_0(M) \leq e^V_0(m^0) + \sum_{i=1}^\infty e^0_0(m^i),
\]
the last step by the Binding Inequality (subadditivity) of $e_0$ see e.g. [2]. As each term is non-negative, equality holds in each relation. Furthermore, as $e_0^V(m^0) \leq e_0^V(u^0)$ and each $e_0^Z(m^i) \leq e_0^Z(u^i)$, we must have equality in these as well. This proves that each $u^i$, $i \geq 0$, is minimizing.

Finally, suppose infinitely many $u^i \neq 0$. Then, by the convergence of the series, $0 < m^i < \mu_0$ for all but finitely many $i$; assume $0 < m^i, m^{i+1} < \mu_0$. Then by the strict subadditivity, provin in Proposition [IV.3] $e_0^Z(m^i) + e_0^Z(m^{i+1}) > e_0^Z(m^i + m^{i+1})$. But then,

$$e_0^V(M) = e_0^V(m^0) + \sum_{i=1}^{\infty} e_0^Z(m^i) > e_0^V(m^0) + \sum_{i \neq j, j+1} e_0^Z(m^i) + e_0^Z(m^j + m^{j+1}) \geq e_0^V(M),$$

a contradiction. \hfill \square

We finish with the proof of Theorem [I.5].

**Proof.** Recall that we assume $V(x) = \mathbb{Z}/|x|$, $\mathbb{Z} > 0$. For (a), $0 < M \leq \mathbb{Z}$, the compactness of all minimizing sequences for $e_0^Z(M)$ was proven by Lions [23]. Let $\varepsilon \in \mathcal{H}^M$ with $e_0^Z(\varepsilon) = e_0^Z(M)$. By Lemma [IV.5] there exists a generalized minimizer of $e_0^Z(M)$, $\{u^i\}_{i=0}^N$, such that [1.6] and [1.7] hold for $\varepsilon_0^Z(\mathbb{Z}) = \lim_{n \to \infty} e_0^Z(\varepsilon_0^Z(\mathbb{Z}) = \lim_{n \to \infty} e_0^Z(M)$. By Remark [IV.4] $N = 0$ and $\varepsilon \xrightarrow{n \to \infty} u^0$, which attains the minimum in $e_0^Z(M)$.

For (b), first note that if there is a sequence $\varepsilon_n \xrightarrow{n \to \infty} 0^+$ for which $e_0^Z(\varepsilon_n)$ attains its minimum at $u_n \in \mathcal{H}^M$, then by the same argument as for (a) we obtain the conclusion of the Theorem with $M_{e_n} = M$. It therefore suffices to consider sequences $\varepsilon_n \xrightarrow{n \to \infty} 0^+$ for which the minimum in $e_0^Z(\varepsilon_n)$ is not attained. By Lemma [IV.2] for each $n$ there exist $m^0_n \in (0, M)$ such that

$$e_0^Z(\varepsilon_n) = e_0^Z(m^0_n) + e_0^Z(M - m^0_n),$$

and there exists $u_n \in H^1(\mathbb{R}^3)$ with $\|u_n\|_{L^2(\mathbb{R}^3)}^2 = m^0_n$ and $e_0^Z(u_n) = e_0^Z(m^0_n)$. For each $n$, we may choose $u_n \in H^1(\mathbb{R}^3)$ with compact support and $\|u_n\|_{L^2(\mathbb{R}^3)}^2 = M - m^0_n$ and for which $e_0^Z < e_0^Z(M - m^0_n) + \varepsilon_n$. Next, choose radii $\rho_n$ in the smooth cut-off $\omega_{\rho_n}$ defined in (2.3), such that $\tilde{u}_n = u_n\omega_{\rho_n}$ satisfies both $\|\tilde{u}_n - u_n\|_{L^2(\mathbb{R}^3)} \xrightarrow{n \to \infty} 0$ and $|e_0^Z(\tilde{u}_n) - e_0^Z(u_n)| \xrightarrow{n \to \infty} 0$. We also choose $\varepsilon_n$ such that $\tilde{u}_n \in \mathcal{H}^3$ such that $\tilde{u}_n$ and $v_n(\cdot + \varepsilon_n)$ have disjoint supports for each $n$, and $|\varepsilon_n| \xrightarrow{n \to \infty} \infty$. Set $U_n(x) = \tilde{u}_n(x) + v_n(\cdot + \varepsilon_n), \varepsilon_n >>>> 0$.

$$\|U_n\|_{L^2(\mathbb{R}^3)} = \|\tilde{u}_n\|_{L^2(\mathbb{R}^3)}^2 + \|v_n\|_{L^2(\mathbb{R}^3)}^2 \xrightarrow{n \to \infty} M, \varepsilon_n >>>> 0.$$

By Lemma [IV.1] $e_0^Z(U_n) \xrightarrow{n \to \infty} e_0^Z(M)$, so applying (i) of Theorem [I.1] there exists $\{u^i\}_{i=0}^\infty \in \mathcal{H}^M$ for which (1.6) and (1.7) hold, and $\mathcal{F}_0^Z(\{u^i\}_{i=0}^\infty) = e_0^Z(M)$. By Remark [IV.3] $u^i \equiv 0$ for all $i \geq 1$ and $u^0 = 1_{\mathbb{R}_m}$ minimizes $e_0^V(M)$. From (1.6) we conclude that $U_n = \tilde{u}_n + v_n(\cdot + \varepsilon_n) \xrightarrow{n \to \infty} u^0$ in $L^2(\mathbb{R}^3)$. Since for every fixed compact set $K \subset \mathbb{R}^3$ we have $U_n = u_n$ almost
everywhere in $K$ and for all sufficiently large $n$, it follows that $u_n \xrightarrow{n \to \infty} u^0$ in $L^2_{loc}(\mathbb{R}^3)$ and pointwise almost everywhere. Consequently, we have $v_n \xrightarrow{n \to \infty} 0$ and $u_n \xrightarrow{n \to \infty} u^0$ globally in $L^2(\mathbb{R}^3)$. In conclusion, taking $M_{\varepsilon_n} = m_{n}, e_{\varepsilon_n}^Z(M_{\varepsilon_n})$ is attained at $u_{\varepsilon_n} = u_n; M_{\varepsilon_n} \xrightarrow{n \to \infty} M$, and $u_n \xrightarrow{n \to \infty} u^0 = \mathbb{1}_{\mathbb{R}^3}$ in $L^2(\mathbb{R}^3)$.

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