ON A CLASS OF FULLY NONLINEAR CURVATURE FLOWS IN HYPERBOLIC SPACE

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Abstract. In this paper, we study a class of flows of closed, star-shaped hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ with speed $(\sinh r)^{\alpha/\beta} \sigma_k^{1/\beta}$, where $\sigma_k$ is the $k$-th elementary symmetric polynomial of the principal curvatures, $\alpha$, $\beta$ are positive constants and $r$ is the distance from points on the hypersurface to the origin. We obtain convergence results under some assumptions of $k$, $\alpha$ and $\beta$. When $k = 1$, $\alpha > 1 + \beta$, and the initial hypersurface is mean convex, we prove that the mean convex solution to the flow for $k = 1$ exists for all time and converges smoothly to a sphere. When $1 \leq k \leq n$, $\alpha > k + \beta$, and the initial hypersurface is uniformly convex, we prove that the uniformly convex solution to the flow exists for all time and converges smoothly to a sphere. In particular, we generalize Li-Sheng-Wang’s results in [10] from Euclidean space to hyperbolic space.

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1. Introduction

Let $M_0$ be a smooth, closed, and star-shaped hypersurface in $\mathbb{H}^{n+1}$ which encloses the origin, $n \geq 2$. In this paper, we study the following geometric flow,

$$\frac{\partial}{\partial t} X(x, t) = -(\phi(r))^\alpha \beta \sigma_k^{1/\beta}(\kappa) V(x, t) + \gamma V(x, t),$$

where $\alpha, \beta$ are positive constants, $\gamma = \binom{n}{k}$, $\sigma_k$ is the $k$-curvature, given by

$$\sigma_k(\cdot, t) = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k}.$$
and \( \kappa_i = \kappa_i(\cdot, t) \) are the principal curvatures of the hypersurface \( \mathcal{M}_t \), parametrized by \( X(\cdot, t) : S^n \to \mathbb{H}^{n+1} \), and \( \nu(\cdot, t) \) is the unit outer normal of \( \mathcal{M}_t \) at \( X(\cdot, t) \). We denote by \( r \) the distance from the point \( X(x, t) \) to the origin, \( \phi(r) = \sinh r \), and \( V(x, t) = \phi(r) \frac{\partial}{\partial r} \) is a conformal Killing vector field on \( \mathbb{H}^{n+1} \). We shall call \( r \) the radial function of \( \mathcal{M}_t \) in this paper.

The same kind of flow as (1.1) has been studied by Li, Sheng and Wang in the Euclidean space [10]. The purpose of this paper is to generalize their results from the Euclidean space to the hyperbolic space. The motivation of studying the flow (1.1) is its relationship with the prescribed curvature measure problem in hyperbolic space which has been studied recently by Fengrui Yang [13].

**Definition 1.1.**

1. We say that a smooth closed hypersurface \( \mathcal{M} \) is \( k \)-convex, \( k = 1, \cdots, n \), if at every point its principal curvatures satisfy \( \kappa = (\kappa_1, \cdots, \kappa_n) \in \Gamma_k^+ \), where

\[
\Gamma_k^+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sigma_i(x_1, \cdots, x_n) > 0, \text{ for all } 1 \leq i \leq k\},
\]

and \( \sigma_k \) is the \( k \)-th elementary symmetric polynomial. For \( k = 1 \), 1-convex is also called mean convex.

2. We say that a smooth closed hypersurface \( \mathcal{M} \) is uniformly convex if its principal curvatures satisfy \( \kappa_i > 0 \) for all \( i = 1, \cdots, n \).

3. A hypersurface \( \mathcal{M} \) in the hyperbolic space \( \mathbb{H}^{n+1} \) is called star-shaped (with respect to the origin), if its support function \( u = \langle V, \nu \rangle \) is positive everywhere, where \( V = \phi(r) \frac{\partial}{\partial r} \).

**1.1. Main results.** We now state our main results.

**Theorem 1.2.** Let \( \mathcal{M}_0 \) be a smooth, closed, mean convex and star-shaped hypersurface in \( \mathbb{H}^{n+1} \) enclosing the origin. If \( 0 < \beta \leq 1, \alpha > \beta + 1 \), the flow (1.1) with \( k = 1 \) has a unique smooth, mean convex and star-shaped solution \( \mathcal{M}_t \) for all time \( t > 0 \). The flow converges exponentially to a sphere centered at the origin in the \( C^\infty \) topology.

Moreover, when \( \mathcal{M}_0 \) is uniformly convex, we can prove the solution to (1.1) remains uniformly convex for all \( k = 1, \cdots, n \).

**Theorem 1.3.** Let \( \mathcal{M}_0 \) be a smooth, closed, uniformly convex star-shaped hypersurface in \( \mathbb{H}^{n+1} \) enclosing the origin. If \( 0 < \beta \leq 1, \alpha > \beta + k \), the flow (1.1) has a unique smooth and uniformly convex solution \( \mathcal{M}_t \) for all time \( t > 0 \). The flow converges exponentially to a sphere centered at the origin in the \( C^\infty \) topology.

In order to prove our main results in Theorem 1.2 and Theorem 1.3, we shall establish the a priori estimates for the flow (1.1), and show that if \( X(\cdot, t) \) solves (1.1), then the radial function \( r \) converges exponentially to a constant as \( t \to \infty \).

**1.2. Organization of the paper.** This paper is organized as follows. In section 2 we collect some properties of star-shaped hypersurfaces, and show that the flow (1.1) can be reduced to a parabolic equation of the radial function. We will also derive the evolution equations for various geometric quantities in section 2. In section 3 we establish the needed a priori estimates of the radial function \( r \) and its gradient \( \nabla r \) of the solution \( \mathcal{M}_t \). In section 4 we establish the needed a priori estimates of the \( k \)-curvature \( \sigma_k \) of the solution \( \mathcal{M}_t \). In section 5 and section 6 we respectively prove the needed a priori estimates of the principal curvatures for mean convex and uniformly convex solutions. In section 7 we prove the main results of asymptotic convergence.
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2. Preliminaries

In this paper we view $\mathbb{H}^{n+1}$ as the warped product manifold with a coordinate chart home-

omorphic to $\mathbb{R}^{n+1}$ with Riemannian metric $g^\mathbb{H} = dr \otimes dr + (\phi(r))^2 \bar{g}$, where $\bar{g}$ is the standard
metric on $\mathbb{S}^n$ and $\phi(r) = \sinh r$.

2.1. The vector field $V$. We denote the covariant derivative with respect to $g^\mathbb{H}$ in $\mathbb{H}^{n+1}$ by $D$. We have the following basic properties for the vector field

$$V = \phi(r) \partial_r.$$ 

**Lemma 2.1 (see [7]).** For any tangent vector field $X$ on $\mathbb{H}^{n+1}$, we have

$$D_X V = \phi'(r) X.$$ 

**Corollary 2.2.** For any tangent vector field $X$ on $\mathbb{H}^{n+1}$, we have

$$D_X r = \frac{\langle X, V \rangle}{\phi(r)}.$$ 

where $\langle \cdot, \cdot \rangle$ means the inner product of tangent vectors induced by $g^\mathbb{H}$.

**Proof.** Since $V = \phi(r) \partial_r$, $\langle V, V \rangle = (\phi(r))^2$. Thus

$$D_X \langle V, V \rangle = D_X ((\phi(r))^2) = 2\phi(r)\phi'(r)D_X r.$$ 

Meanwhile, by Lemma 2.1, for the left hand side we have

$$D_X \langle V, V \rangle = 2 \langle D_X V, V \rangle = 2 \langle \phi'(r) X, V \rangle = 2\phi'(r) \langle X, V \rangle.$$ 

Hence $D_X r = \frac{\langle X, V \rangle}{\phi(r)}$. \qed

2.2. Elementary symmetric polynomials.

For $n$ variables $x_1, \cdots, x_n$, their $k$ th elementary symmetric polynomial is defined by

$$\sigma_k(x_1, \cdots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k},$$

**Lemma 2.3 (Newton-Maclaurin’s inequalities).** When $(x_1, \cdots, x_n) \in \Gamma_{m-1}$, for $l, m$ such that $1 \leq l < m \leq n$, we have

$$\sigma_m(x_1, \cdots, x_n) \leq \binom{n}{m} \left( \frac{\sigma_l(x_1, \cdots, x_n)}{\binom{m}{l}} \right)^{m/l}. \tag{2.1}$$

Equality holds if and only if $(x_1, \cdots, x_n) = c(1, \cdots, 1)$ for some constant $c > 0$.

**Lemma 2.4 (see [6]).** For a real symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, we write $\sigma_k(A) = \sigma_k(\lambda_1, \ldots, \lambda_n)$, where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $A$. We write $\sigma_k^{ij} = \frac{\partial \sigma_k}{\partial a_{ij}}$. When the eigenvalues of $A$ belong to $\Gamma_k^+$, $k = 1, \ldots, n$, we have that $(\sigma_k^{ij})$ is positive definite.
In the most case of the following paper, by using \( \sigma_k \), we mean the \( k \) th elementary symmetric polynomial of the principal curvatures, that is, locally, eigenvalues of the matrix \( \left( h_i^j \right) \). For consideration of the consistency of superscripts and subscripts, by using the notation \( \sigma_k^{ij} \), we mean \( \sigma_k^{ij} = g^{il} \frac{\partial h^j_l}{\partial t^i} \), and similarly we can define second derivatives by \( \sigma_k^{pq,rs} = g^{il} g^{sm} \frac{\partial^2 h^j_l}{\partial t^i \partial t^m} \).

For general functions on the eigenvalues of a real symmetric matrix, such as \( F = (\sigma_k)^{\frac{1}{\beta}} \), we also write

\[
\dot{F}^{ij} = g^{il} \frac{\partial F}{\partial h^j_l}, \quad \text{and} \quad \ddot{F}^{pq,rs} = g^{il} g^{sm} \frac{\partial^2 F}{\partial h^j_l \partial h^m_r}.
\]

We recall the following result:

**Lemma 2.5 (see [6]).**

\[
\sigma_k^{ij} (h^2)_{ij} = \sigma_1 \sigma_k - (k + 1) \sigma_{k+1}, \quad \sigma_k^{ij} h_{ij} = k \sigma_k, \quad \sigma_k^{ij} g_{ij} = (n - k + 1) \sigma_{k-1},
\]

where \( (h^2)_{ij} = h^{ik} h_{kj} \).

For \( F = (\sigma_k)^{\frac{1}{\beta}} \), we have \( \dot{F}^{ij} = \frac{1}{\beta} (\sigma_k)^{\frac{1}{\beta} - 1} \sigma_k^{ij} \) and

**Corollary 2.6.**

\[
\dot{F}^{ij} (h^2)_{ij} = \frac{1}{\beta} (\sigma_k)^{\frac{1}{\beta} - 1} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}), \quad \dot{F}^{ij} h_{ij} = \frac{k}{\beta} (\sigma_k)^{\frac{1}{\beta}}, \quad \dot{F}^{ij} g_{ij} = \frac{n - k + 1}{\beta} (\sigma_k)^{\frac{1}{\beta} - 1} \sigma_{k-1}.
\]

### 2.3. Radial function and radial graph.

Let \( e_1, \ldots, e_n \) be a smooth local orthonormal frame field on \( S^n \), and let \( \nabla \) be the covariant derivative on \( S^n \). We denote by \( g_{ij}, g^{ij}, \nu, h_{ij} \) the metric on \( M_t \) induced by the metric of \( \mathbb{H}^{n+1} \), the inverse of the metric, the unit outer normal and the second fundamental form of \( M_t \), respectively. Then, in terms of \( r \), we have

\[
u = \frac{\phi^2}{\sqrt{\phi^2 + |\nabla r|^2}}, \quad g_{ij} = \phi^2 \delta_{ij} + r_i r_j, \quad \nu = \frac{1}{\phi^2} \left( \delta^{ij} - \frac{r_i r_j}{\phi^2 + |\nabla r|^2} \right),
\]

\[
h_{ij} = \left( \frac{\phi^2 + |\nabla r|^2}{\phi^2} \right)^{-1} \left( -\phi \nabla_i \nabla_j r + 2 \phi' r_i r_j + \phi^2 \phi' \delta_{ij} \right),
\]

\[
h^i_j = \frac{1}{\phi^2 \sqrt{\phi^2 + |\nabla r|^2}} \left( \delta^{ik} - \frac{r_i r_k}{\phi^2 + |\nabla r|^2} \right) \left( -\phi \nabla_k \nabla_j r + 2 \phi' r_k r_j + \phi^2 \phi' \delta_{kj} \right),
\]

where we denote \( r_i = \nabla_i (r) \). These formulae can be found in a number of papers, see, for example [7].

It will be convenient if we introduce a new variable \( \varphi = \varphi(r) \) satisfying

\[
\frac{d \varphi}{d r} = \frac{1}{\phi(r)}.
\]
Let \( \omega := \sqrt{1 + |\nabla \varphi|^2} \), one can calculate the unit outward normal
\[
\nu = \frac{1}{\omega} \left( 1, -\frac{r_1}{\sigma^2}, \ldots, -\frac{r_n}{\sigma^2} \right)
\]
and the general support function \( u = \langle V, \nu \rangle = \frac{\varphi}{\omega} \). Then we can conclude how the radial function evolves along the flow.

**Lemma 2.7.** The star-shaped solution of the flow
\[
\frac{\partial}{\partial t} X(x, t) = -(\varphi(r)) \hat{\sigma}_k^\frac{1}{n} \nu(x, t) + \gamma V(x, t)
\]
is equivalent to a solution to the scalar parabolic PDE of the radial function \( r \):
\[
\frac{\partial}{\partial t} r = -(\varphi(r)) \hat{\sigma}_k^\frac{1}{n} \omega + \gamma \varphi.
\] (2.4)

**Proof.** Up to a tangential diffeomorphism, the flow (1.1) is equivalent to the following flow
\[
\frac{\partial}{\partial t} X(x, t) = \left( -(\varphi(r)) \hat{\sigma}_k^\frac{1}{n} + \gamma u \right) \nu(x, t).
\]
By [5], It is known that if a closed hypersurface which is a radial graph satisfies
\[
\partial_t X = f \nu,
\]
then the evolution of the scalar function \( r \) satisfies
\[
\partial_t r = f \omega.
\]
Thus
\[
\frac{\partial}{\partial t} r = \left( -(\varphi(r)) \hat{\sigma}_k^\frac{1}{n} + \gamma u \right) \omega = -(\varphi(r)) \hat{\sigma}_k^\frac{1}{n} \omega + \gamma \varphi.
\]

2.4. **Evolution of geometric quantities.** For convenience, we rewrite (1.1) as
\[
\frac{\partial}{\partial t} X(x, t) = -\Phi \nu(x, t) + \gamma V(x, t).
\] (2.5)

where \( \gamma = \binom{0}{k} \) and
\[
\Phi = (\varphi(r)) \hat{\sigma}_k^\frac{1}{n}.
\] (2.6)

We now derive some evolution equations along the flow (2.5). Pick any local coordinate chart \( \{x_i\}_{i=1}^n \) of the hypersurface \( M_t \). We denote \( \nabla \) to be the covariant derivative on \( M_t \) induced by the covariant derivative \( D \) in \( \mathbb{H}^{n+1} \). We denote \( g_{ij} \) and \( \langle \cdot, \cdot \rangle \) to be respectively the metric on \( M_t \) induced by \( g^{\mathbb{H}} \) and the induced inner product of tangent vectors. We denote \( \partial_i = \frac{\partial}{\partial x_i} \), \( X_i = \partial_i X \). Recall the following identities in hyperbolic space, see [2], [4].

\[
D_{X_j}X_j = \Gamma^k_{ij} X_k - h_{ij} \nu,
\] (Gauss formula) (2.7)

\[
\nu_i = h_{ij} g^{ji} X_i,
\] (Weingarten formula) (2.8)

\[
h_{ij,l} = h_{il,j},
\] (Codazzi equation) (2.9)

\[
R_{ijrs} = h_{ir} h_{js} - h_{is} h_{jr} - g_{ir} g_{js} + g_{is} g_{jr},
\] (Gauss equation) (2.10)

where \( \Gamma^k_{ij} \) is the Christoffel symbol of the metric of \( M_t \). Combining the Gauss and Codazzi equations gives the following generalization of Simons’ identity, we have
\[
\nabla_i \nabla_j h_{kl} = \nabla_{(k} \nabla_{l)} h_{ij} + h_{ij}(h^2)_{kl} - h_{kl}(h^2)_{ij} - g_{ij} h_{kl} + g_{kl} h_{ij}
\] (2.11)
where \((h^2)_{ij} = h_k^i h_k^j\) and \(A_{(ab)}\) means the symmetrization of the tensor \(A_{ab}\), that is, \(A_{(ab)} = \frac{1}{2} (A_{ab} + A_{ba})\).

**Lemma 2.8.** Along the flow (2.5), we have the following evolution equations

\[
\frac{\partial}{\partial t} g_{ij} = -2\Phi h_{ij} + 2\gamma\phi' g_{ij}, \quad \frac{\partial}{\partial t} \nu = \nabla \Phi. \tag{2.12}
\]

\[
\frac{\partial}{\partial t} u = -\phi' \Phi + \phi' \gamma u + \langle V, \nabla \Phi \rangle. \tag{2.13}
\]

\[
\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j \Phi - \Phi h_{ij} + \gamma \phi' h_{ij} + (\gamma u - \Phi) g_{ij}. \tag{2.14}
\]

Here \(\nabla\) denotes the covariant derivative on \(\mathcal{M}_t\).

**Proof.** By direct calculations using Lemma 2.1 and the equation (2.8), we have

\[
\frac{\partial}{\partial t} g_{ij} = \partial_t \langle \partial_i X, \partial_j X \rangle
= \langle \nabla_i (\Phi \nu + \gamma V), \partial_j X \rangle + \langle \partial_i X, \nabla_j (\Phi \nu + \gamma V) \rangle
= -\Phi (\langle \partial_i \nu, \partial_j X \rangle + \langle \partial_i X, \partial_j \nu \rangle) + 2\gamma \phi' g_{ij}
= -2\Phi h_{ij} + 2\gamma \phi' g_{ij}.
\]

Since \(\partial_t \nu\) is tangential, we have

\[
\frac{\partial}{\partial t} \nu = \langle \partial_t \nu, \nabla_j X \rangle g^{ij} \partial_i X
= -\langle \nu, \nabla_j (\Phi \nu + \gamma V) \rangle g^{ij} \partial_i X = \nabla_j \Phi g^{ij} \partial_i X = \nabla \Phi.
\]

Now we calculate the evolution of the support function \(u\)

\[
\frac{\partial}{\partial t} u = \partial_t \langle V, \nu \rangle = \langle \partial_t V, \nu \rangle + \langle V, \partial_t \nu \rangle = \langle \phi' \partial_t X, \nu \rangle + \langle V, \nabla \Phi \rangle
= \langle \phi' (-\Phi \nu + \gamma V), \nu \rangle + \langle V, \nabla \Phi \rangle = -\phi' \Phi + \phi' \gamma u + \langle V, \nabla \Phi \rangle.
\]

Now we calculate the evolution of \(h_{ij}\). Note that \(\mathbb{H}^{n+1}\) is of constant sectional curvature of \(K = -1\), thus

\[
\frac{\partial}{\partial t} h_{ij} = -\partial_t \langle DX_i X_j, \nu \rangle
= -\langle DX_i DX_j X_j, \nu \rangle - \langle DX_i X_j, \partial_t \nu \rangle
= -\langle DX_i DX_j X_j + \bar{R} (X_t, X_i) X_j, \nu \rangle - \langle DX_i X_j, \nabla \Phi \rangle
= -\langle DX_i DX_j X_j, \nu \rangle - \langle \bar{R} (X_t, X_i) X_j, \nu \rangle - \langle DX_i X_j, g^{kl} \partial_k \Phi X_k \rangle
= -\langle DX_i DX_j (-\Phi \nu + \gamma V), \nu \rangle + \langle X_t, \nu \rangle \langle X_i, X_j \rangle - \Gamma^k_{ij} \partial_k \Phi
= -\langle DX_i \left(-\partial_j \Phi \nu - \Phi h_{ij} X_t + \gamma \phi' X_j \right), \nu \rangle + g_{ij} (\gamma u - \Phi) - \Gamma^k_{ij} \partial_k \Phi
= \partial_t \partial_j \Phi - \Gamma^k_{ij} \partial_k \Phi - \Phi h_{ij} + \gamma \phi' h_{ij} + g_{ij} (\gamma u - \Phi)
= \nabla_i \nabla_j \Phi - \Phi h_{ij} + \gamma \phi' h_{ij} + g_{ij} (\gamma u - \Phi),
\]

where \(\bar{R}\) denotes the curvature tensor of the metric on \(\mathbb{H}^{n+1}\). \qed
Corollary 2.9. Along the flow (2.5), we have the following evolution equations

\[ \frac{\partial}{\partial t} g^{ij} = 2\Phi h^{ij} - 2\gamma \phi' g^{ij}. \] (2.15)

\[ \frac{\partial}{\partial t} h^i_j = \nabla_i \nabla^j \Phi + \Phi h^k_i h^j_k - \gamma \phi' h^i_j + (\gamma u - \Phi) \delta^i_j. \] (2.16)

Proof. \[ \frac{\partial}{\partial t} g^{ij} = -g^{il} (\partial_t g_{lm}) g^{mj} = 2\Phi h^{ij} - 2\gamma \phi' g^{ij}. \]

Now we calculate \( \nabla_i \nabla_j \Phi \) more precisely. Recall that \( \Phi = \phi \tilde{\sigma} F \), where \( F = \sigma_k^2 \).

Lemma 2.10.

\[ \nabla_i \nabla_j \Phi = (\phi(r)) \frac{\tilde{\sigma}}{\beta} F^{pq} \nabla_p \nabla_q h_{ij} + (\phi(r)) \frac{\tilde{\sigma}}{\beta} F^{pq,rs} h_{rs,i} h_{pq,j} \]

\[ + \left( \frac{\alpha}{\beta} \right)^2 \frac{(\phi(r))^2}{(\phi(r))^2} (\nabla_i r)(\nabla_j r) \Phi - \frac{\alpha}{\beta} \frac{1}{(\phi(r))^2} (\nabla_i r)(\nabla_j r) \Phi - \frac{\alpha}{\beta} \frac{\phi'(r)}{(\phi(r))^2} u h_{ij} \Phi \]

\[ + \frac{\alpha}{\beta} \frac{(\phi(r))^2}{(\phi(r))^2} (g_{ij} - (\nabla_i r)(\nabla_j r)) \Phi + 2 \frac{\alpha}{\beta} \frac{\phi'(r)}{(\phi(r))^2} \phi'(r) \tilde{\sigma} - 1 (\nabla_i r)(\nabla_j r) F \]

\[ + \frac{1}{\beta} (\phi(r)) \frac{\tilde{\sigma}}{\sigma_k^2} (\sigma_k + 1) \phi'(r) \nabla_i h_{ij} - \frac{k}{\beta} \Phi (h^2)_{ij} \]

\[ - \frac{k}{\beta} \Phi g_{ij} + \frac{n - k + 1}{\beta} (\sigma_k + 1) \phi(r) \tilde{\sigma} \sigma_{k-1} h_{ij}. \] (2.17)

Proof. Firstly

\[ \nabla_i \Phi = \nabla_i \left( (\phi(r)) \frac{\tilde{\sigma}}{\beta} F \right) = \frac{\alpha}{\beta} (\phi(r)) \frac{\tilde{\sigma}}{\beta} - 1 \phi'(r) (\nabla_i r) F + (\phi(r)) \frac{\tilde{\sigma}}{\beta} \nabla_i F. \]

Therefore,

\[ \nabla_i \nabla_j \Phi = \nabla_i \left( \frac{\alpha \phi'(r)}{\beta \phi(r)} \Phi \nabla_j r + (\phi(r)) \frac{\tilde{\sigma}}{\beta} \nabla_j F \right) \]

\[ = - \frac{1}{\beta} \frac{\phi'(r)}{(\phi(r))^2} (\nabla_i r)(\nabla_j r) \Phi + \frac{\phi'(r)}{\beta \phi(r)} (\nabla_i \nabla_j r) \Phi \]

\[ + \frac{\phi'(r)}{\beta \phi(r)} (\phi(r)) \frac{\tilde{\sigma}}{\beta} \nabla_i r \nabla_j F + (\phi(r)) \frac{\tilde{\sigma}}{\beta} \nabla_i \nabla_j F \]

\[ + \frac{\phi'(r)}{\beta \phi(r)} \nabla_j r \left( \frac{\phi'(r)}{\beta \phi(r)} \Phi \nabla_i r + (\phi(r)) \frac{\tilde{\sigma}}{\beta} \nabla_i F \right). \] (2.18)

Note that by Corollary 2.2,

\[ \nabla_i \nabla_j r = \nabla_i \left( \frac{\langle X_j, V \rangle}{\phi(r)} \right) = \frac{\langle X_i, X_j, V \rangle}{\phi(r)} + \frac{\langle X_j, \nabla_i V \rangle}{\phi(r)} - \frac{\phi'(r)}{(\phi(r))^2} (\nabla_i r) \langle X_j, V \rangle. \]
\[ \binom{-h_{ij} \nu}{\phi(r)} + \binom{X_{j}, \phi'(r) X_{i}}{\phi(r)} - \frac{\phi'(r)}{\phi(r)} (\nabla_{i} r)(\nabla_{j} r) = - \frac{u}{\phi(r)} h_{ij} + \frac{\phi'(r)}{\phi(r)} g_{ij} - \frac{\phi'(r)}{\phi(r)} (\nabla_{i} r)(\nabla_{j} r). \] \tag{2.19} \]

We also have

\[ \nabla_{i} \nabla_{j} F = \nabla_{i} \left( \mathcal{E}^{pq} \nabla_{p} \nabla_{q} h_{pq} \right) = \mathcal{E}^{pq,r s} \nabla_{r} \nabla_{s} h_{pq} + \mathcal{E}^{pq} \nabla_{i} \nabla_{j} h_{pq} \] \tag{2.20} \]

Plugging (2.19) and (2.20) into (2.18) and symmetrize the tensor \( \nabla_{i} \nabla_{j} \Phi \), we get

\[ \nabla_{(i} \nabla_{j)} \Phi = \left( \frac{\alpha}{\beta} \right)^{2} \frac{(\phi'(r))^{2}}{(\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi - \frac{1}{\beta (\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi \]

\[ + \frac{\alpha \phi'(r)}{\beta \phi(r)} (\nabla_{i} r) h_{ij} + \frac{\phi'(r)}{\phi(r)} g_{ij} - \frac{\phi'(r)}{\phi(r)} (\nabla_{i} r)(\nabla_{j} r) \Phi \]

\[ + 2 \frac{\alpha \phi'(r)(\phi(r))}{\beta \phi(r)} \frac{\Phi}{\nabla_{i} r)(\nabla_{j} F} + (\phi(r))^{\frac{\gamma}{2}} \nabla_{(i} \nabla_{j)} F \]

\[ = \left( \frac{\alpha}{\beta} \right)^{2} \frac{(\phi'(r))^{2}}{(\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi - \frac{1}{\beta (\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi \]

\[ - \frac{\alpha \phi'(r)}{\beta (\phi(r))^{2}} u h_{ij} \Phi + \frac{\alpha (\phi'(r))^{2}}{\beta (\phi(r))^{2}} (g_{ij} - (\nabla_{i} r)(\nabla_{j} r)) \Phi \]

\[ + 2 \frac{\alpha \phi'(r)(\phi(r))}{\beta \phi(r)} \frac{\Phi}{\nabla_{i} r)(\nabla_{j} F} + (\phi(r))^{\frac{\gamma}{2}} \mathcal{E}^{pq,r s} h_{rs,i} h_{pq,j} + \mathcal{E}^{pq} \nabla_{(i} \nabla_{j)} h_{pq} \].

By using generalisation of Simons’ identity in (2.11), we have that

\[ \nabla_{(i} \nabla_{j)} h_{kl} = \nabla_{(k} \nabla_{l)} h_{ij} + h_{ij} (h^{2})_{kl} - h_{kl}(h^{2})_{ij} - g_{ij} h_{kl} + g_{kl} h_{ij}. \]

Thus

\[ \nabla_{(i} \nabla_{j)} \Phi = \left( \frac{\alpha}{\beta} \right)^{2} \frac{(\phi'(r))^{2}}{(\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi - \frac{1}{\beta (\phi(r))^{2}} (\nabla_{i} r)(\nabla_{j} r) \Phi - \frac{\alpha \phi'(r)}{\beta (\phi(r))^{2}} u h_{ij} \Phi \]

\[ + \frac{\alpha (\phi'(r))^{2}}{\beta (\phi(r))^{2}} (g_{ij} - (\nabla_{i} r)(\nabla_{j} r)) \Phi + (\phi(r))^{\frac{\gamma}{2}} \mathcal{E}^{pq,r s} h_{rs,i} h_{pq,j} \]

\[ + 2 \frac{\alpha \phi'(r)(\phi(r))}{\beta \phi(r)} \frac{\Phi}{\nabla_{i} r)(\nabla_{j} F} + (\phi(r))^{\frac{\gamma}{2}} \mathcal{E}^{pq} \left( \nabla_{(p} \nabla_{q)} h_{ij} + h_{ij} (h^{2})_{pq} - h_{pq}(h^{2})_{ij} - g_{ij} h_{pq} + g_{pq} h_{ij} \right) \].

Plugging Corollary 2.6 into the last line of the above equation, we obtain (2.17).

\[ \square \]

3. \( C^0 \) and \( C^1 \) Estimates

In this section, we establish the needed a priori estimates of the radial function \( r \) and its gradient \( \nabla r \) of the solution \( \mathcal{M}_t \).

**Lemma 3.1.** Let \( r(\cdot, t) \) be a smooth solution to (2.4) on \( S^n \times [0, T) \). If \( \alpha > k + \beta \), there is a positive constant \( C \) depending only on \( k, \alpha, \max_{S^n} r(\cdot, 0) \) and \( \min_{S^n} r(\cdot, 0) \) such that

\[ 1/C \leq r(\cdot, t) \leq C \quad \forall t \in [0, T). \]
Proof. By (2.4), we conclude that
\[ \frac{\partial r}{\partial t} = -\Phi \omega + \gamma \phi. \]
Here by (2.6),
\[ \Phi = (\phi(r))^\alpha \sigma_k(\kappa)^{\frac{1}{\beta}}. \]
For each time \( t \), if \( r(x,t) \) attains its spatial minimum at the point \((x_t,t)\), then we have
\[ \nabla_i r = 0, \quad \text{for all } 1 \leq i \leq n, \]
\[ \nabla_i \nabla_j r \geq 0, \quad \text{as a matrix.} \]
Hence \( \omega = \sqrt{1 + |\nabla \varphi|^2} = \sqrt{1 + \frac{\nabla r^2}{\omega^2}} = 1 \) at \((x_t,t)\). Therefore we have \( u = \frac{\phi}{\omega} = \phi \) and
\[ \frac{\partial r}{\partial t} = -\Phi + \gamma \phi \quad \text{at} \quad (x_t,t). \]
Moreover, by the last equation of (2.2), at \((x_t,t)\),
\[ h^i_j = h_{ik}g^{kj} \leq \phi \phi' \delta_{ik} \cdot \frac{1}{\phi^2} \delta^{kj} = \frac{\phi'}{\phi} \delta^i_j \]
and
\[ \sigma_k(\kappa) = \sigma_k(h^i_j) \leq \sigma_k \left( \frac{\phi'}{\phi} \delta^i_j \right) = \sigma_k \left( \frac{\phi'}{\phi} \right)^k \gamma. \]

We now prove the existence of uniform positive lower bound of \( r \). The proof of the existence of uniform positive upper bounds is similar.

Since \( \phi(r) = \sinh(r) \), \( \Phi = (\sinh(r))^{\alpha} \sigma_k(\kappa)^{\frac{1}{\beta}} \leq (\sinh(r))^{\frac{\alpha-k}{\beta}} (\cosh(r))^\frac{k}{\beta} \gamma \), we have
\[ \frac{\partial r}{\partial t} = -\Phi + \gamma \phi \geq -\gamma (\sinh(r))^{\frac{\alpha-k}{\beta}} (\cosh(r))^\frac{k}{\beta} \gamma + \gamma \sinh(r) \]
\[ = -\gamma \sinh(r) \left( (\sinh(r))^{\frac{\alpha-k-\beta}{\beta}} (\cosh(r))^\frac{k}{\beta} - 1 \right). \]

By \( \alpha > k + \beta \), we know that \( (\sinh(r))^{\frac{\alpha-k-\beta}{\beta}} (\cosh(r))^\frac{k}{\beta} \) is a monotonically increasing function as \( r > 0 \). This function takes value 0 when \( r = 0 \), and tends to \( +\infty \) and \( r \) tends to \( +\infty \). So there exists a unique \( \tilde{r} = \tilde{r}(k,\alpha,\beta) \in (0, +\infty) \), such that \( (\sinh(\tilde{r}))^{\frac{\alpha-k-\beta}{\beta}} (\cosh(\tilde{r}))^{\frac{k}{\beta}} = 1 \).

Hence if \( r(x,t) = r_{\min}(t) \leq \tilde{r} \), we have \( (\sinh(r))^{\frac{\alpha-k-\beta}{\beta}} (\cosh(r))^\frac{k}{\beta} \leq 1 \). It follows that
\[ \frac{d}{dt} r_{\min}(t) = \frac{\partial r}{\partial t} (x_t,t) \geq 0. \]
This implies that \( r_{\min}(t) \geq \min \{ \min_{S^n} r(\cdot,0), \tilde{r}(k,\alpha) \} \). That is,
\[ r(x,t) \geq \min \{ \min_{S^n} r(\cdot,0), \tilde{r}(k,\alpha,\beta) \}. \]
By similar argument, we can prove that
\[ r(x,t) \leq \max \{ \max_{S^n} r(\cdot,0), \tilde{r}(k,\alpha,\beta) \}. \]
\[ \square \]

Lemma 3.2. Let \( r(\cdot,t) \) be a smooth \( k \)-convex solution to (2.4) on \( S^n \times [0,T) \). If \( \alpha > k + \beta \), we have
\[ |\nabla r| \leq C, \]
where \( \nabla \) is the covariant derivative on \( S^n \). \( C \) is positive constant which only depends on \( M_0 \).
Proof. By (2.4), we have
\[ \partial_t r = -(\sinh(r))^{\frac{1}{\beta}} (\sigma_k)^{\frac{1}{\beta}} \omega + \gamma \sinh(r). \]
As in (2.3), take a new function \( \varphi(r) = \log \left( 1 - \frac{2}{\sigma_k + r} \right) \), then \( \varphi \) is monotonically increasing and \( \varphi(r) \in (-\infty, 0) \), for all \( r > 0 \), and it satisfies
\[ \frac{d\varphi}{dr} = \frac{1}{\sinh(r)}. \]
Then by \( \omega = \sqrt{1 + |\nabla \varphi|^2} \), and \( r = \log \left( \frac{2}{1+\varphi} - 1 \right) \), we have
\[ \sinh(r) = \frac{1}{\sinh(-\varphi)} \text{ and } \cosh(r) = \frac{\cosh(-\varphi)}{\sinh(-\varphi)}. \]
So
\[ \partial_t \varphi = -(\sinh(-\varphi))^{\frac{1}{\beta} + 1} \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} + \gamma. \]
For a fixed time \( t \), we assume that \( \theta_t \in \mathbb{S}^n \) is the spatial maximum point of \( \frac{1}{2} |\nabla \varphi|^2 \). At \((\theta_t, t)\), we have
\[ \partial_t \left( \frac{1}{2} |\nabla \varphi|^2 \right) = \varphi_i (\partial_t \varphi)_i = \varphi_i \left[ -(\sinh(-\varphi))^{\frac{1}{\beta} + 1} \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} + \gamma \right]_i \]
\[ = \varphi_i \left[ - \left( \frac{\alpha}{\beta} - 1 \right) (\sinh(-\varphi))^{-\frac{1}{\beta}} \cosh(-\varphi) \varphi_i \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} \right]_i \]
\[ = - \left( \frac{\alpha}{\beta} - 1 \right) (\sinh(-\varphi))^{-\frac{2}{\beta} + 1} \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} \varphi_i \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} \varphi_i \]
\[ = - (\sinh(-\varphi))^{-\frac{2}{\beta} + 1} \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} \varphi_i \sqrt{1 + |\nabla \varphi|^2} \sigma_k^{\frac{1}{\beta}} \varphi_i, \tag{3.1} \]
where we used \( \left( \sqrt{1 + |\nabla \varphi|^2} \right)_i = 0 \) at \((\theta_t, t)\). For the last term, we know
\[ \left( \sigma_k^{\frac{1}{\beta}} \right)_i = \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \frac{\partial \sigma_k}{\partial h_p^i} (h_p^i) = \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \sigma_k^{pq} g_{qs} (h_p^i) \]
\[ = \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \sigma_k^{pq} (g_{qs} h_p^i) - (g_{qs}) h_p^i \]
\[ = \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \sigma_k^{pq} (h_{pq}^i - h_p^s (g_{qs})^i), \]
where we regarded \( h_{pq}, g_{qs} \) as tensors on \( \mathbb{S}^n \).

By (2.2), we know that
\[ h_{ij} = \left( \sqrt{\phi^2 + |\nabla r|^2} \right)^{-1} (-\phi \nabla_i \nabla_j r + 2\phi' r_i r_j + \phi^2 \delta_{ij}). \]
Since \( \nabla_i \varphi = \frac{1}{\phi} \nabla_i r \), we have
\[ \nabla_q \nabla_p \varphi = \frac{1}{\phi} \nabla_q \nabla_p r - \frac{\phi'}{\phi^2} \nabla_q r \nabla_p r. \]
Thus
\[ h_{pq} = \frac{\phi}{\sqrt{1 + \lvert \nabla \varphi \rvert^2}} (-\varphi_{pq} + \phi' \varphi_p \varphi_q + \phi' \delta_{pq}) \]
\[ = \frac{1}{\sinh(-\varphi) \sqrt{1 + \lvert \nabla \varphi \rvert^2}} \left( -\varphi_{pq} + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_p \varphi_q + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \delta_{pq} \right). \]
Hence at \((\theta_t, t)\) we have
\[ h_{pq,i} = \frac{\cosh(-\varphi)}{(\sinh(-\varphi))^2} \frac{1}{\sqrt{1 + \lvert \nabla \varphi \rvert^2}} \varphi_i \left( -\varphi_{pq} + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_p \varphi_q + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \delta_{pq} \right) \]
\[ + \frac{1}{\sinh(-\varphi)} \frac{1}{\sqrt{1 + \lvert \nabla \varphi \rvert^2}} \left( -\varphi_{pqi} + \frac{1}{(\sinh(-\varphi))^2} \varphi_i \varphi_p \varphi_q + \frac{1}{(\sinh(-\varphi))^2} \varphi_i \delta_{pq} \right) \]
\[ + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_{pi} \varphi_q + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_p \varphi_{qi} \right) \]
\[ = \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_i h_{pq} + \frac{1}{\sinh(-\varphi)} \frac{1}{\sqrt{1 + \lvert \nabla \varphi \rvert^2}} \left( -\varphi_{pqi} + \frac{1}{(\sinh(-\varphi))^2} \varphi_i (\varphi_p \varphi_q + \delta_{pq}) \right) \]
\[ + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_{pi} \varphi_q + \frac{\cosh(-\varphi)}{\sinh(-\varphi)} \varphi_p \varphi_{qi} \right) \] (3.2)

Also we know that
\[ g_{ij} = (\phi(r))^2 \delta_{ij} + r_i r_j = (\phi(r))^2 \left( \frac{r_i r_j}{(\phi(r))^2} \right) \]
\[ = (\phi(r))^2 (\delta_{ij} + \varphi_i \varphi_j) = \frac{1}{(\sinh(-\varphi))^2} (\delta_{ij} + \varphi_i \varphi_j). \]
Hence
\[ g_{pq,i} = \frac{2 \cosh(-\varphi)}{(\sinh(-\varphi))^3} \varphi_i (\delta_{pq} + \varphi_p \varphi_q) + \frac{1}{(\sinh(-\varphi))^2} (\varphi_{pi} \varphi_q + \varphi_p \varphi_{qi}) \]
\[ = \frac{2 \cosh(-\varphi)}{\sinh(-\varphi)} \varphi_i g_{pq} + \frac{1}{(\sinh(-\varphi))^2} (\varphi_{pi} \varphi_q + \varphi_p \varphi_{qi}) \] (3.3)

By using (3.2), (3.3) and \( \nabla_p \left( \frac{1}{2} \lvert \nabla \varphi \rvert^2 \right) \varphi_i = \varphi_i \delta_{pi} = 0 \) at \((\theta_t, t)\), we get
\[ \varphi_i \left( \sigma_k^l \right)_i = \frac{1}{\beta} \sigma_k^l \varphi_i \left( h_{pq} - h_p^s (g_{qs})_i \right) \]
\[ = \frac{1}{\beta} \sigma_k^l \varphi_i \left( h_{pq} - \frac{\cosh(-\varphi)}{\sinh(-\varphi)} h_{pq} \lvert \nabla \varphi \rvert^2 \right) \]
\[ = \frac{1}{\beta} \sigma_k^l \left( \frac{\cosh(-\varphi)}{\sinh(-\varphi)} h_{pq} \lvert \nabla \varphi \rvert^2 \right) \]
\[ = \frac{1}{\beta} \sigma_k^l \left( -\varphi_{pq} + \frac{1}{(\sinh(-\varphi))^2} (\varphi_p \varphi_q + \delta_{pq}) \right) \]
\[ + \frac{1}{\beta} \sigma_k^l \left( \frac{1}{\sinh(-\varphi)} \varphi_i \varphi_p \varphi_q + \frac{1}{\sinh(-\varphi)} \varphi_p \varphi_{qi} \right) \] (3.4)

By the Ricci identity
\[ \varphi_{pqi} = \varphi_{pqi} + (\lambda R_{pqi}^q) = \varphi_{pqi} + \varphi_i (\delta_{iq} \delta_{pi} - \delta_{it} \delta_{pq}) = \varphi_{ipq} + \varphi_q \delta_{pi} - \varphi_i \delta_{pq}. \]
Thus
\[
\varphi_i \varphi_{pq} = \varphi_i \varphi_{pq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 = \left( \frac{1}{2} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2.
\] (3.5)

Plugging (3.5) into (3.4), we get
\[
\varphi_i \left[ \frac{1}{2} \nabla |\varphi|^2 \right]_{pq} = -k \frac{1}{\beta} \frac{1}{\alpha} \cosh(-\varphi) \nabla_i |\varphi|^2 + \frac{1}{\beta} \frac{1}{\alpha} \sigma_k \left( \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right).
\]

Substituting (3.6) into (3.1) gives that
\[
\partial_t \left( \frac{1}{2} |\nabla \varphi|^2 \right) = -\left( \frac{\alpha}{\beta} - 1 \right) (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \cosh(-\varphi) \sigma_k \frac{1}{\beta} \nabla |\varphi|^2 + \frac{1}{\beta} \frac{1}{\alpha} \sigma_k \left( \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right)
\]
\[
+ (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \cosh(-\varphi) \sigma_k \frac{1}{\beta} \nabla |\varphi|^2 + \frac{1}{\beta} \frac{1}{\alpha} \sigma_k \left( \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right)
\]
\[
- (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \cosh(-\varphi) \sigma_k \frac{1}{\beta} \nabla |\varphi|^2 + \frac{1}{\beta} \frac{1}{\alpha} \sigma_k \left( \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right)
\]
\[
\times \left[ \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right]
\]
\[
= \frac{1}{\beta} (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \sigma_k \frac{1}{\beta} \sigma_k \left( \frac{1}{2} \left( \frac{1}{\beta} \nabla |\varphi|^2 \right)_{pq} - \varphi_i \varphi_{iq} + \varphi_p \varphi_q - \delta_{pq} |\nabla \varphi|^2 \right)
\]
\[
- \frac{1}{\beta} (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \sigma_k \frac{1}{\beta} \sigma_k \delta_{pq} |\nabla \varphi|^2
\]
\[
- \frac{1}{\beta} (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \sigma_k \frac{1}{\beta} \sigma_k \varphi_p \varphi_{pq} - \frac{1}{\beta} (\sinh(-\varphi))^{-\frac{\alpha}{\beta}} \sigma_k \frac{1}{\beta} \sigma_k \varphi_p \varphi_{pq} \delta_{pq} |\nabla \varphi|^2 - \varphi_p \varphi_q)
\]
\[
\leq 0,
\] (3.7)

where the last inequality is from \((|\nabla \varphi|)^2_{pq} \leq 0\) at \((t, \theta), (\sigma_k^{pq}) \geq 0\) and \((\delta_{pq} |\nabla \varphi|^2 - \varphi_p \varphi_q) \geq 0\).

When \(\alpha > k + \beta\), we get \(\frac{d}{dt} \left( \frac{1}{2} |\nabla \varphi|^2 \right)_{\max}(t) \leq 0\), so \(|\nabla \varphi|\) is bounded from above by a constant. Since \(\nabla_i r = \nabla(r_\varphi) \cdot \nabla_i \varphi\), we have \(|\nabla r| \leq C\) for some constant \(C\).

\textbf{Corollary 3.3.} \textit{Under the same assumptions in Lemma 3.2, along the flow (2.5), the hypersurface }\(M_t\textit{ preserves star-shapedness and the support function }u\textit{ satisfies}
\[
\frac{1}{C} \leq u \leq C
\] 
\textit{for some constant }\(C > 0\).
Proof. We know \( u = \phi \omega = \frac{\phi(r)}{\sqrt{1+|\varphi|^2}} \). Since \( r \) is bounded from both above and below, we only need to prove \( \omega = \sqrt{1+|\varphi|^2} \) is bounded from both above and below. This follows from \( \omega = \sqrt{1+|\varphi|^2} \geq 1 \) and Lemma 3.2 immediately. \( \square \)

4. Estimates for \( k \)-curvature

In the following sections, we will derive the \( C^2 \) estimates of flow (2.5). To simplify the statements of following lemmas, we always assume \( \alpha > k + \beta \) and the initial hypersurface is smooth, closed, star-shaped and \( k \)-convex. Besides, we denote \( F = (\sigma_k)^{\frac{\beta}{\alpha}} \) and we define a parabolic operator

\[
\mathcal{L} = \partial_t - \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\sinh(r))^{\frac{\beta}{2}} \sigma_k^{ij} \nabla_i \nabla_j = \partial_t - (\phi(r))^{\frac{\beta}{2}} \dot{F}^{ij} \nabla_i \nabla_j.
\]

In this section, we prove that the \( k \)-curvature is bounded from both above and below. We first calculate the evolution equations of \( \Phi \). From Corollary 2.9,

\[
\partial_t \Phi = \partial_t \left( (\phi(r))^{\frac{\beta}{\alpha}} \sigma_k^{\frac{1}{\beta}} \right)
= \frac{\alpha}{\beta} \phi(r) \left( \frac{\beta}{\alpha} \phi(r) \right) \Phi^2 + (\phi(r))^{\frac{\beta}{2}} \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \Phi (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1})
+ n - k + 1 \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\beta}{2}} (\gamma u - \Phi) \sigma_{k-1}.
\]

(4.1)

and

\[
\mathcal{L} u = \gamma \phi'(r) u - \frac{k + \beta}{\beta} \phi'(r) \Phi + \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi + \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\beta}{2}} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) u.
\]

(4.2)

Proof. First, we calculate the evolution equation of \( \Phi \). From Corollary 2.9,

\[
\partial_t \Phi = \partial_t \left( (\phi(r))^{\frac{\beta}{\alpha}} \sigma_k^{\frac{1}{\beta}} \right)
= \frac{\alpha}{\beta} \phi(r) \left( \frac{\beta}{\alpha} \phi(r) \right) \Phi^2 + (\phi(r))^{\frac{\beta}{2}} \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} \Phi (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1})
+ n - k + 1 \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\beta}{2}} (\gamma u - \Phi) \sigma_{k-1}.
\]

By using Corollary 2.6, we have

\[
\partial_t \Phi = \frac{\alpha}{\beta} \gamma \phi'(r) \Phi - \frac{\alpha}{\beta} \omega \phi'(r) \Phi^2 + (\phi(r))^{\frac{\beta}{2}} \dot{F}^{ij} \nabla_i \nabla_j \Phi
+ \frac{1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\beta}{2}} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) - \frac{k}{\alpha} \phi'(r) \Phi
+ \frac{n - k + 1}{\beta} \sigma_k^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\beta}{2}} (\gamma u - \Phi) \sigma_{k-1}
\]
Also we have

\[
(\phi(r))^{\frac{\beta}{\alpha}} F^{ij} \nabla_i \nabla_j \Phi + \frac{\alpha - k}{\beta} \gamma \phi'(r) \Phi - \frac{\alpha}{\beta} \omega \frac{\phi'(r)}{\phi(r)} \Phi^2
\]

\[
+ (\phi(r))^{\frac{\beta}{\alpha}} \frac{1}{2} \sigma_k^{\frac{k}{\alpha}} \Phi (\sigma_k \sigma_k - (k + 1) \sigma_k+1)
\]

\[
+ \frac{n - k + 1}{\beta} \sigma_k^{\frac{k}{\alpha} - 1} (\phi(r))^\frac{\beta}{\alpha} (\gamma u - \Phi) \sigma_k-1,
\]

which implies (4.1).

Next we calculate the evolution of \( u \). Note that the covariant derivatives of \( u \) satisfy (see Lemma 2.6 of [6])

\[
\nabla_i u = h^k_i (V, X_k)
\]

and

\[
\nabla_i \nabla_j u = \nabla^k h_{ij} (V, X_k) + \phi' h_{ij} - h_{ik} h^k_j u.
\]

By (2.8), we have

\[
\frac{\partial}{\partial t} u = \phi'(r)(\gamma u - \Phi) + \langle V, \nabla \Phi \rangle.
\]

Also we have

\[
\nabla_i \Phi = \nabla_i \left( (\phi(r))^{\frac{\beta}{\alpha}} \frac{1}{2} \sigma_k \right)
\]

\[
= \frac{\alpha}{\beta} (\phi(r))^{\frac{\beta}{\alpha} - 1} \phi'(r) (\nabla_i r) \sigma_k^{\frac{k}{\alpha}} + (\phi(r))^{\frac{\beta}{\alpha}} \nabla_i \left( \frac{1}{2} \sigma_k \right)
\]

\[
= \frac{\alpha}{\beta} \phi'(r) \Phi \nabla_i r + (\phi(r))^{\frac{\beta}{\alpha}} F^{pq} \nabla_i h_{pq}.
\]

By using \( \nabla_j r = \frac{(X_i, V)}{\phi(r)} \), we have

\[
\langle V, \nabla \Phi \rangle = g^{ij} (\nabla_i \Phi) (V, X_j)
\]

\[
= \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi + (\phi(r))^{\frac{\beta}{\alpha}} g^{ij} F^{pq} \nabla_i h_{pq} \langle V, X_j \rangle
\]

\[
= \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi + (\phi(r))^{\frac{\beta}{\alpha}} F^{pq} \nabla^k h_{pq} \langle V, X_k \rangle.
\]

Combining (4.3), (4.4) and (4.5), and by using Corollary 2.6, we get

\[
\mathcal{L} u = \partial_t u - (\phi(r))^{\frac{\beta}{\alpha}} F^{pq} \nabla_p \nabla_q u
\]

\[
= \phi'(r)(\gamma u - \Phi) + \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi - \phi'(r)(\phi(r))^{\frac{\beta}{\alpha}} F^{pq} h_{pq} + (\phi(r))^{\frac{\beta}{\alpha}} F^{pq} (h^2)_{pq} u
\]

\[
= \phi'(r)(\gamma u - \Phi) - \frac{k}{\beta} \phi'(r) \Phi + \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi + \frac{1}{\beta} \sigma_k^{\frac{k}{\alpha} - 1} (\phi(r))^{\frac{\beta}{\alpha}} (\sigma_k \sigma_k - (k + 1) \sigma_k+1) u
\]

\[
= \gamma \phi'(r) u - \frac{k + \beta}{\beta} \phi'(r) \Phi + \frac{\alpha}{\beta} \phi'(r) |\nabla r|^2 \Phi + \frac{1}{\beta} \sigma_k^{\frac{k}{\alpha} - 1} (\phi(r))^{\frac{\beta}{\alpha}} (\sigma_k \sigma_k - (k + 1) \sigma_k+1) u.
\]

\[
\square
\]

**Proposition 4.2.** Along the flow (2.5), if the initial hypersurface \( \mathcal{M}_0 \) is smooth, closed and \( k \)-convex, then there exists a constant \( C > 0 \) such that

\[
\sigma_k \geq C
\]

on \( \mathcal{M}_t \) for \( t \in [0, T) \).
Proof. By Lemma 2.3, we know that \( \sigma_1 \sigma_k - (k + 1) \sigma_{k+1} \geq C_1(\sigma_k)^{1+\frac{k}{2}} > 0 \) for some constant \( C_1 = C_1(n, k) > 0 \), and \( \sigma_{k-1} \geq C_2(\sigma_k)\frac{1}{\sigma_k} > 0 \) for some constant \( C_2 = C_2(n, k) > 0 \).

We apply maximum principle to the evolution equation (4.1) of \( \Phi \). At the spatial minimum point of \( \Phi \) on \( M_t \), we have \( (\nabla_i \nabla_j \Phi) \geq 0 \), hence \( \sigma_k \nabla_i \nabla_j \Phi \geq 0 \), then \( \tilde{F}^{ij} \nabla_i \nabla_j \Phi \geq 0 \). If \( \Phi \) is small enough such that \( \Phi \leq \gamma \min_{S^n \times [0, T]} u \), we have

\[
\partial_t \Phi \geq \frac{\alpha - k}{\beta} \gamma \phi'(r) \Phi - \frac{\alpha}{\beta} \frac{\phi'(r)}{\phi(r)} \Phi^2 = \frac{\alpha - k}{\beta} \gamma \phi'(r) \Phi - \frac{\alpha}{\beta} \phi'(r) \Phi^2
\]

at the spatial minimum point of \( \Phi \). By Lemma 3.1 and Lemma 3.2, let

\[
C_1 = \frac{\alpha}{(\alpha - k) \gamma u_{\text{min}}},
\]

we have

\[
\partial_t \Phi \geq \frac{\alpha - k}{\beta} \gamma \phi'(r) \left( \Phi - C_1 \Phi^2 \right).
\]

Therefore, if \( 0 \leq \Phi_{\text{min}}(t) \leq \min \left\{ \frac{1}{C_1}, \gamma \min_{S^n \times [0, T]} u \right\} \), then \( \frac{d}{dt} \Phi_{\text{min}}(t) \geq 0 \). In all, we conclude that \( \Phi \) is bounded from below by a positive constant. Since \( \sigma_k^{\frac{1}{\beta}} = \frac{1}{(\phi(r))^{\frac{1}{\beta}}} \Phi \), we know that \( \sigma_k \) is bounded from below by a positive constant \( C \). \( \square \)

**Proposition 4.3.** Along the flow (2.5), if the initial hypersurface \( M_0 \) is smooth, closed and \( k \)-convex, then there exists a constant \( C > 0 \), such that

\[
\sigma_k \leq C
\]

on \( M_t \) for \( t \in [0, T) \).

Proof. To prove the upper bound of \( \sigma_k \), we apply the technique of Tso [3]. We consider the auxiliary function \( Q = \log \Phi - \log(u - a) \), where \( a = \frac{1}{2} \inf_{S^n \times [0, T]} u \). Using the equations (4.1) and (4.2), at the maximum point of \( Q \) on \( M_t \), we have

\[
\mathcal{L} Q = \mathcal{L} \Phi \leq \frac{\mathcal{L} u}{u - a} - \frac{\mathcal{L} u}{u - a}
\]

\[
= \frac{\alpha - k}{\beta} \gamma \phi'(r) \Phi + \frac{\alpha}{\beta} \frac{\phi'(r)}{\phi(r)} \Phi - \frac{n - k + 1}{\beta} \frac{1}{\sigma_k^{\frac{1}{\beta}}} (\gamma u - \Phi) \frac{1}{\Phi} \sigma_k - (k + 1) \sigma_{k+1}
\]

\[
+ \frac{n - k + 1}{\beta} \frac{1}{\sigma_k} (\phi(r))^{\frac{1}{\beta}} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) \frac{1}{u - a} + \frac{(a + k)}{\beta} \phi'(r) \Phi
\]

\[
\leq - \frac{a}{u - a} \Phi^{\frac{1}{\beta}} (\phi(r))^{\frac{1}{\beta}} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) + C \Phi + C
\]

\[
= - \frac{a}{\beta(u - a)} \sigma_1 \sigma_k - (k + 1) \sigma_{k+1} \Phi + C \Phi + C
\]

for some \( C > 0 \) by Lemma 3.1 and Lemma 3.2 and Corollary 3.3, upon assuming that \( \Phi \) is large enough such that \( \Phi \geq \gamma \max_{S^n \times [0, T]} u \). By Lemma 2.3, we can estimate

\[
\frac{\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}}{\sigma_k} \geq \frac{k}{(n-1)\beta} (\sigma_k)^{\frac{1}{\beta}}.
\]
Thus

\[ \mathcal{L}Q \leq -C_1 \Phi^{1+\frac{\beta}{n}} + C \Phi + C. \]  

(4.6)

for some constant \( C, C_1 > 0 \). Thus, there exists a constant \( C > 0 \) that only depend on \( \mathcal{M}_0 \), so that whenever \( \Phi > C \), we have \( \frac{\Phi}{Q_{\max}}(t) < 0 \). From Lemma 3.1 and Lemma 3.2, \( r \) and \( u \) is bounded. Hence \( \Phi \) goes to infinity when \( Q \) goes to infinity. Therefore \( Q \) is bounded from above by a constant, which gives an upper bound of \( \Phi \). \( \square \)

5. \( C^2 \) ESTIMATE FOR \( k = 1 \) AND MEAN CONVEX SOLUTIONS

In this section, we consider the case \( k = 1 \) of the flow (2.5). Since \( 0 < \beta \leq 1 \), the flow is fully nonlinear except \( \beta = 1 \), we still need to get \( C^2 \) estimate. We will prove the boundness of principal curvatures when the initial hypersurface is mean convex. We first calculate the evolution equation of \( |A|^2 \), where \( |A| \) denotes the tensor \( (h^i_j) \) and \( |A|^2 = h^i_i h^j_j \).

Lemma 5.1. Along the flow (2.5),

\[
\mathcal{L}(|A|^2) = -2 \frac{\alpha}{\beta} \frac{\phi}{\beta} H \frac{n}{\beta} - 1 |\nabla A|^2 + 2(\phi(r)) \frac{n}{\beta} (1 - \beta) H \frac{n}{\beta} - 1 h^i_i \nabla_i F \nabla_j F
\]

\[
+ 2 \frac{\alpha}{\beta} \left( \frac{\alpha}{\beta} - 1 \right) \left( \frac{\phi'(r)}{\beta} \right)^2 \Phi h^i_i (\nabla_r, \nabla_j r) - 2 \frac{\alpha}{\beta} \Phi (\phi(r))^2 \Phi h^i_i (\nabla_r, \nabla_j r)
\]

\[
- 2 \frac{\alpha}{\beta} \Phi (\phi(r))^2 u |A|^2 + 2 \frac{\alpha}{\beta} \Phi (\phi(r))^2 \Phi H + 4 \frac{\alpha}{\beta} \Phi (\phi(r))^2 \Phi H + 2 \Phi (\phi(r))^2 \Phi H + 2 (1 - \beta) \Phi h^i_i h^j_j.
\]

(5.1)

\[
-4 \gamma \phi'(r) |A|^2 + 2(\gamma u - \Phi) H + 2(\gamma u - \Phi) H + 2(1 - \beta) \Phi h^i_i h^j_j.
\]

Proof. By Corollary 2.9, we have

\[
\frac{\partial}{\partial t} |A|^2 = 2 h^i_i \frac{\partial}{\partial t} h^j_j = 2 h^i_i \left( \nabla_i \nabla_j \Phi + \Phi h^k_k h^i_i - \gamma \phi' h^i_i + (\gamma u - \Phi) \delta^i_i \right)
\]

\[
= 2 h^i_i \nabla_i \nabla_j \Phi + 2 \Phi h^i_i h^j_j - 4 \gamma \phi' |A|^2 + 2(\gamma u - \Phi) H.
\]

(5.2)

Since \( k = 1, \sigma_k = H, F = H + \frac{1}{\beta} \), \( \mathcal{L} = \partial_t - \frac{1}{\beta} \phi \frac{\phi}{\beta} H \frac{n}{\beta} - 1 g_{ij} \nabla_i \nabla_j \). We observe that

\[
\hat{F}_{pq} = \frac{1}{\beta} \frac{H + \frac{1}{\beta} - 1}{\beta} g_{pq}, \quad \hat{F}_{pq,rs} = \frac{1}{\beta^2} \frac{H + \frac{1}{\beta} - 2}{\beta} g_{pq} g_{rs}.
\]

Thus

\[
\hat{F}_{pq,rs} \nabla_i h_{pq} \nabla_j h_{rs} = (1 - \beta) H \frac{n}{\beta} - 1 \nabla_i F \nabla_j F.
\]

By (2.17), we have

\[
\nabla_i (\nabla_j \Phi) = \frac{1}{\beta} (\phi(r)) \frac{n}{\beta} H \frac{n}{\beta} - 1 g_{pq} \nabla_p \nabla_q h_{ij} + (\phi(r)) \frac{n}{\beta} (1 - \beta) H \frac{n}{\beta} - 1 i \nabla_j F F
\]

\[
+ \frac{\alpha}{\beta} \frac{\phi'(r)}{\beta} \left( \frac{\phi'(r)}{\beta} \right)^2 (\nabla_i r) (\nabla_j r) - \frac{\alpha}{\beta} \Phi (\phi(r))^2 (\nabla_i r) (\nabla_j r) - \frac{\alpha}{\beta} \Phi (\phi(r))^2 u h_{ij}
\]

\[
+ \frac{\alpha}{\beta} (\phi(r))^2 (g_{ij} - (\nabla_i r) (\nabla_j r)) \Phi + 2 \frac{\alpha}{\beta} \phi'(r) (\phi(r))^2 - 1 (\nabla_i r) (\nabla_j r) F
\]

\[
+ \frac{1}{\beta} (\phi(r))^2 H \frac{n}{\beta} - 1 |A|^2 h_{ij} - \frac{1}{\beta} \Phi (H^2)_{ij} - \frac{1}{\beta} \Phi g_{ij} + \frac{n}{\beta} H \frac{n}{\beta} - 1 \phi(r) h_{ij}.
\]

(5.3)
Plugging (5.3) into (5.2), we get
\[
\frac{\partial}{\partial t}|A|^2 = \frac{2}{\beta}(\phi(r))^\frac{\alpha}{\beta} H^{\frac{1}{\beta} - 1}g_i^j \nabla_p \nabla_q h_{ij} + 2(\phi(r))^{\frac{\alpha}{\beta}}(1 - \beta)H^{-\frac{\alpha}{\beta}} h_i^j \nabla_i F \nabla_j F
\]
\[+ \frac{2\alpha}{\beta} \left( \frac{\alpha}{\beta} - 1 \right) \left( \phi'(r) \right)^2 \Phi h_i^j (\nabla_i r)(\nabla_j r) - \frac{2\alpha}{\beta} \phi(r)^2 \Phi h_i^j (\nabla_i r)(\nabla_j r)
\]
\[+ \frac{2\alpha}{\beta} \phi'(r)^2 \nabla_i \Phi |A|^2 + 2\alpha \phi'(r) \Phi H + 4\alpha \phi'(r)(\Phi(r))^\frac{\alpha}{\beta} H^{-\frac{\alpha}{\beta}} h_i^j (\nabla_i r)(\nabla_j F)
\]
\[+ \frac{2\alpha}{\beta} \Phi H + \frac{2n}{\beta} H^{\frac{\alpha}{\beta} - 1} \phi(r)^2 |A|^2
\]
\[+ 2(2\alpha - 1) \Phi h_i^j h_i^j.
\]

Also,
\[g_i^j \nabla_p \nabla_q |A|^2 = g_i^j \nabla_p \left( 2h_i^j \nabla q h_i^j \right) = 2g_i^j \nabla_p \left( h_i^j \nabla q h_i^j \right) + 2g_i^j h_i^j \nabla_p \nabla_q h_i^j
\]
\[= 2g_i^j \left( h_i^j \nabla_q h_i^j \right) + 2g_i^j h_i^j h_{i,pq} = 2|\nabla A|^2 + 2g_i^j h_i^j h_{i,pq}.
\]

Combining the above two equations, we get (5.1). \( \square \)

**Proposition 5.2.** Under the flow (2.5) for \( k = 1 \), when the initial hypersurface is mean convex, if \( \alpha > 1 + \beta \), the principal curvatures of the mean convex solution have a uniform bound, i.e.
\[|\kappa_i| \leq C \quad i = 1, \ldots, n.
\]

**Proof.** Plugging \( k = 1 \), \( \Phi = (\phi(r))^\frac{\alpha}{\beta} H \) into (4.1), we get
\[\mathcal{L}\Phi = \frac{\alpha - 1}{\beta} \gamma \phi'(r) \Phi - \frac{\alpha}{\beta} \phi'(r) \Phi^2 + \frac{1}{\beta} (\phi(r))^\frac{\alpha}{\beta} H^{\frac{1}{\beta} - 1} \Phi |A|^2
\]
\[+ \frac{n}{\beta} H^{\frac{1}{\beta} - 1} (\phi(r))^\frac{\alpha}{\beta} (\gamma u - \Phi).
\]

Inspired by the work [9] of Li, Xu and Zhang, we define an auxiliary function
\[Q = \log |A|^2 - 2B \log (\Phi - a),
\]
where \( B = 1 - \frac{\alpha}{2c}, \ a = \frac{1}{2} \inf_{S^* \times [0,T]} \Phi, \) and \( c = \sup_{S^* \times [0,T]} \Phi. \) Thus
\[\mathcal{L}Q = \mathcal{L} \left( \log (|A|^2) - 2B \log (\Phi - a) \right)
\]
\[= \mathcal{L} \left( \log (|A|^2) - 2B \log (\Phi - a) \right)
\]
\[= \mathcal{L} \left( |A|^2 \right) \frac{1}{|A|^2} + \frac{1}{\beta} \phi(\Phi)^{\frac{\alpha}{\beta}} H^{\frac{1}{\beta} - 1} g_i^j \nabla_i |A|^2 \nabla_j |A|^2 - 2B \frac{\mathcal{L}\Phi}{\Phi - a}
\]
\[\quad - 2B \frac{1}{\beta} \phi(\Phi)^{\frac{\alpha}{\beta}} H^{\frac{1}{\beta} - 1} g_i^j \nabla_i \Phi \nabla_j \Phi \Phi - a \Phi - a .
\]

At the spatial maximum point of \( Q \), we have
\[\frac{\nabla_i |A|^2}{|A|^2} = 2B \frac{\nabla_i \Phi}{\Phi - a}.
\]

Plugging (5.1), (5.4) and (5.6) into (5.5), we get
\[\mathcal{L}Q
\]
\[= \mathcal{L} \left( |A|^2 \right) \frac{1}{|A|^2} + 4B^2 \frac{1}{\beta} \phi(\Phi)^{\frac{\alpha}{\beta}} H^{\frac{1}{\beta} - 1} g_i^j \nabla_i \Phi \nabla_j \Phi \Phi - a \Phi - a - 2B \frac{\mathcal{L}\Phi}{\Phi - a} - 2B \frac{1}{\beta} \phi(\Phi)^{\frac{\alpha}{\beta}} H^{\frac{1}{\beta} - 1} g_i^j \nabla_i \Phi \nabla_j \Phi \Phi - a \Phi - a.
\]
Without loss of generality, we may assume the local coordinate have thus

\[ \frac{\phi(r)}{\beta} \frac{\phi'(r)}{\beta} H_{ij} \left( \nabla_i r \right) \left( \nabla_j r \right) - 2 \frac{\phi'(r)}{\beta} \left( \phi(r) \right) H_{ij} \left( \nabla_i r \right) \left( \nabla_j r \right) \]

\[ - 2 \frac{\phi'(r)}{\beta} \left( \phi(r) \right)^2 H_{ij} \left( \nabla_i r \right) \left( \nabla_j r \right) + 4 \frac{\phi'(r)}{\beta} \left( \phi(r) \right) \frac{\phi'}{\beta} H_{ij} \left( \nabla_i r \right) \left( \nabla_j r \right) \]

\[ + \frac{2}{\beta} H^{\frac{1}{\beta} - 1} \left( \phi(r) \right)^{\frac{2}{\beta} - 1} |A|^2 - 2 \frac{\phi'}{\beta} \left( \phi(r) \right) \Phi H_{ij} \left( \nabla_i r \right) \left( \nabla_j r \right) \]

\[ - 2 \frac{2B}{\beta} (\phi(r))^{\frac{2}{\beta} - 1} \frac{\phi}{\Phi - a} + \frac{2B}{\beta} \phi(r) \frac{\phi}{\Phi - a} \]

\[ + \frac{4B^2 - 2B}{\beta} \phi^{\frac{2}{\beta} - 1} \frac{\Phi}{\Phi - a} \]

\[ = I_1 + \cdots + I_7, \quad (5.7) \]

where \( I_i \) denotes the \( i \)th line on the right hand side of (5.7).

Since the eigenvalues of \( (h_i^j) \) are \( \kappa_1, \cdots, \kappa_n \), and \( |A| = \sqrt{(\kappa_1)^2 + \cdots + (\kappa_n)^2} \geq |\kappa_i|, \forall i \), we have

\[ -|A| g^{ij} \leq h^{ij} \leq |A| g^{ij}, \]

Without loss of generality, we may assume the local coordinate \( \{x_1, \cdots, x_n\} \) is orthonormal at the spatial maximum point of \( R \), thus

\[ \nabla_i r = \frac{\langle X_i, V \rangle}{\phi} \leq 1, \]

which implies that

\[ |\nabla r|^2 \leq n. \]

So we have

\[ \left| \frac{h^{ij}}{|A|} \left( \nabla_i r \right) \left( \nabla_j r \right) \right| \leq |\nabla r|^2 \leq n, \quad (5.8) \]

\[ \left| \frac{h^{ij}}{|A|} \left( \nabla_i F \right) \left( \nabla_j F \right) \right| \leq |\nabla F|^2, \quad (5.9) \]

\[ \left| \frac{h^{ij}}{|A|} \left( \nabla_i r \right) \left( \nabla_j r \right) \right| \leq |\nabla r||\nabla F| \leq \sqrt{n} |\nabla F|. \quad (5.10) \]

Using the Cauchy inequality, it is easy to see

\[ \frac{\langle \nabla |A|^2, \nabla \Phi \rangle}{(\Phi - a)^2} = 2 \nabla_i h^{ij} \left( h^{ij} \frac{\nabla_i \Phi}{\Phi - a} \right) \leq |\nabla A|^2 + |A|^2 \frac{|\nabla \Phi|^2}{(\Phi - a)^2}. \]

By (5.6),

\[ \frac{\nabla_i |A|^2}{|A|^2} = 2B \frac{\nabla_i \Phi}{\Phi - a}, \]

thus

\[ \frac{|\nabla A|^2}{|A|^2} + \frac{|\nabla \Phi|^2}{(\Phi - a)^2} \geq \frac{\langle \nabla |A|^2, \nabla \Phi \rangle}{|A|^2(\Phi - a)^2} = 2B \frac{|\nabla \Phi|^2}{(\Phi - a)^2}. \]
That is,
\[
\frac{|\nabla A|^2}{|A|^2} \geq (2B - 1) \frac{|\nabla \Phi|^2}{(\Phi - a)^2}.
\]  
(5.11)

By (5.9) and (5.11), we estimate the first line of (5.7) as
\[
I_1 \leq - \frac{4B - 2}{\beta} \phi^a H^\frac{1}{\beta} - 1 \frac{|\nabla \Phi|^2}{(\Phi - a)^2} + 2(1 - \beta) \phi^a H^\frac{1}{\beta} \frac{1}{|A|} |\nabla F|^2.
\]  
(5.12)

By (5.8) and (5.10), we estimate the second and the third line of (5.7) as
\[
I_2 + I_3 \leq C + \frac{C}{|A|} + C |\nabla F|.
\]  
(5.13)

for some constant $C > 0$, by the boundness of $r$, $u$, and $H$.

For the rest four lines of (5.7), we have
\[
I_4 + I_5 + I_6 + I_7 \leq \frac{2}{\beta} \left(1 - B \frac{\Phi}{\Phi - a}\right) \phi^a H^\frac{1}{\beta} - 1 |A|^2 + \frac{4B^2 - 2B}{\beta} \phi^a H^\frac{1}{\beta} - 1 \frac{|\nabla \Phi|^2}{(\Phi - a)^2} + C |A| + C |\nabla F| + C
\]  
(5.14)

for some constant $C > 0$.

Combining (5.12), (5.13) and (5.14), and assuming that $|A|$ is large enough, we get
\[
\mathcal{L} Q \leq \frac{2}{\beta} \left(1 - B \frac{\Phi}{\Phi - a}\right) \phi^a H^\frac{1}{\beta} - 1 |A|^2 + \frac{2}{\beta} (2B^2 - 3B + 1) \phi^a H^\frac{1}{\beta} - 1 \frac{|\nabla \Phi|^2}{(\Phi - a)^2} + C |A| + C |\nabla F| + C,
\]  
(5.15)

for some constant $C > 0$ at the spatial maximum point of $Q$.

For the first term in (5.15), since
\[
1 - B \frac{\Phi}{\Phi - a} = 1 - \frac{1 - \frac{a}{2c}}{1 - \frac{a}{\Phi}} \leq 1 - \frac{1 - \frac{a}{2c}}{1 - \frac{a}{c}} = - \frac{a}{2c} \frac{1 - \frac{a}{c}}{1 - \frac{a}{c}},
\]
and $(\phi(r))^\frac{a}{\beta} H^\frac{1}{\beta} - 1 \geq c'$ for some positive constant $c'$, we have
\[
\frac{2}{\beta} \left(1 - B \frac{\Phi}{\Phi - a}\right) (\phi(r))^\frac{a}{\beta} H^\frac{1}{\beta} - 1 |A|^2 \leq - c_1 |A|^2,
\]  
(5.16)

for some constant $c_1 > 0$.

For the second term in (5.15),
\[
2B^2 - 3B + 1 = (2B - 1)(B - 1) = - \frac{a}{2c} \left(1 - \frac{a}{c}\right),
\]
which is a negative constant, and
\[
|\nabla \Phi|^2 = \left|\nabla \left((\phi(r))^\frac{a}{\beta} F\right)\right|^2 = \left|\frac{a}{\beta} (\phi(r))^\frac{a}{\beta} - 1 \phi'(r) F \nabla r + (\phi(r))^\frac{a}{\beta} \nabla F\right|^2
\]
\[
= \left(\frac{a}{\beta}\right)^2 (\phi(r))^2 (\frac{a}{\beta} - 1) (\phi'(r))^2 F^2 |\nabla r|^2 + 2 \frac{a}{\beta} (\phi(r))^\frac{2a}{\beta} - 1 \phi'(r) F \langle \nabla r, \nabla F\rangle + (\phi(r))^\frac{a}{\beta} |\nabla F|^2
\]
\[
\geq c' |\nabla F|^2 - C |\nabla F|,
\]
for some positive constant \( c', C \). Thus
\[
\frac{2}{\beta} (2B^2 - 3B + 1) \phi \Phi H^\frac{1}{2} - 1 \frac{|\nabla \Phi|^2}{(\Phi - a)^2} \leq -c_2 |\nabla F|^2 + C |\nabla F|, \tag{5.17}
\]
for some positive constant \( c_2, C \).

Plugging (5.16) and (5.17) into (5.15), we get
\[
\mathcal{L}Q \leq -c_1 |A|^2 - c_2 |\nabla F|^2 + C |A| + C |\nabla F| + C \leq -c_1 |A|^2 + C |A| + C. \tag{5.18}
\]
Since \( |A| \) tends to \(+\infty\) as \( Q \) tends to \(+\infty\), if \( Q \) is large enough, \( \mathcal{L}Q \leq 0 \), thus we prove the boundness of \( Q \), which implies the boundness of \( |A| \). So the principal curvatures \( \kappa_i (i = 1, \cdots, n) \) are bounded.

6. \( C^2 \) estimate for uniformly convex solutions

In this section, we consider uniformly convex solution of the flow (2.5). We will prove that if the initial hypersurface is uniformly convex, then there exists a positive uniform lower bound of principal curvatures.

To estimate the lower bound of the principal curvatures, we need the following lemma.

**Lemma 6.1** (see [10], [12]). Let \( \tilde{h}^{ij} \) be the inverse of \( \{h_{ij}\} \). Then \( \{\tilde{h}^{ij}\} \) represents the inverse of Weingarten map. If \( \{\tilde{h}^{ij}\} > 0 \), we have
\[
(G^{pq,lm} + 2G^{pqm}\tilde{h}^l_q) \eta^n_p \eta^m_l \geq 2G^{-1} (G^{pq} \eta^n_p)^2.
\]
for any tensor \( \{\eta^n_p\} \), where \( G = \sigma_k^\frac{1}{2} (\tilde{h}^{ij}) \).

Using Lemma 6.1 now we can prove

**Proposition 6.2.** Let \( X(\cdot, t) \) be a smooth, closed and uniformly convex solution to the flow (2.5) for \( t \in [0, T) \), which enclosed the origin. If \( \alpha > k + \beta \), there exists a positive constant \( C \) depending only on \( \alpha \) and \( M_0 \), such that the principal curvatures of \( X(\cdot, t) \) satisfy
\[
\frac{1}{C} \leq \kappa_i (\cdot, t) \leq C, \quad \forall t \in [0, T) \text{ and } i = 1, 2, \cdots, n.
\]

**Proof.** As we have already proved the upper bound of \( \sigma_k \), to derive the uniform bounds on the principal curvatures, it suffices to prove the uniform positive lower bound on the principal curvatures
\[
\kappa_i (x, t) \geq \frac{1}{C} > 0.
\]
Let \( \lambda (x, t) \) denote the maximal principal radii (that is, the reciprocal of the minimal principal curvature) at \( X(x, t) \). In the following, we will prove the upper bound of \( \lambda (x, t) \).

Let \( (x_0, t_0) \) denote the point where \( \lambda (x, t) \) attains its spatial maximum at time \( t = t_0 \). Without loss of generality, we choose a local normal coordinate \( \{x_1, \cdots, x_n\} \) near \( P_0 = (x_0, t_0) \) such that the second fundamental form \( h_{ij} \) is diagonal at \( P_0 \). For now, the equation about evolution of geometric quantities, that is, Lemma 2.8 and Corollary 2.9 still holds.

Furthermore, we can assume \( \partial_t |_{(x_0, t_0)} \) is an eigenvector with respect to \( \lambda (x_0, t_0) \), i.e., \( \lambda (x_0, t_0) = \tilde{h}^1_1 (x_0, t_0) \). Using this coordinate system, we can calculate the evolution of \( \tilde{h}^1_1 \) at \( (x_0, t_0) \).
From
\[ \frac{\partial}{\partial t} \hat{h}_1 = - \left( \hat{h}_1 \right)^2 \partial_t \hat{h}_1, \]
\[ \nabla_i \hat{h}_1 = \frac{\partial \hat{h}_1}{\partial h_p^i} \nabla_j h_p^q = - \hat{h}_1^p \hat{h}_1^q \nabla_j h_p^q = - \left( \hat{h}_1 \right)^2 \nabla_j h_{11}, \]
and
\[ \nabla_j \nabla_i \hat{h}_1 = \nabla_j \left( - \hat{h}_1^p \hat{h}_1^q \nabla_i h_p^q \right) \]
\[ = - \nabla_j \hat{h}_1^p \hat{h}_1^q \nabla_i h_p^q - \hat{h}_1^p \nabla_j \hat{h}_1^q \nabla_i h_p^q - \hat{h}_1^p \nabla_i \hat{h}_1^q \nabla_j h_p^q \]
\[ = \hat{h}_1^p \hat{h}_1^q \nabla_j \hat{h}_1^q \nabla_i h_p^q + \hat{h}_1^p \hat{h}_1^q \nabla_j \hat{h}_1^q \nabla_i h_p^q - \hat{h}_1^p \hat{h}_1^q \nabla_j \nabla_i h_p^q \]
\[ = - \left( \hat{h}_1 \right)^2 \nabla_j \nabla_i h_{11} + 2 \left( \hat{h}_1 \right)^2 \hat{h}_1^{pq} \nabla_i h_{1p} \nabla_j h_{1q}, \]
we get
\[ \mathcal{L} \hat{h}_1 = \frac{\partial \hat{h}_1}{\partial t} - \left( \phi(r) \right)^{\frac{\alpha}{\beta}} \hat{F}^{ij} \nabla_i \nabla_j \hat{h}_1 \]
\[ = - \left( \hat{h}_1 \right)^2 \partial_t \hat{h}_1 + \left( \phi(r) \right)^{\frac{\alpha}{\beta}} \hat{F}^{ij} \left( \hat{h}_1 \right)^2 \nabla_i \nabla_j \hat{h}_1 \]
\[ - 2\left( \phi(r) \right)^{\frac{\alpha}{\beta}} \hat{F}^{ij} \left( \hat{h}_1 \right)^2 \hat{h}_1^{pq} \nabla_i h_{1p} \nabla_j h_{1q}. \]
(6.1)

By Corollary 2.9,
\[ \partial_t \hat{h}_1 = \nabla_1 \nabla^1 \Phi + \Phi \left( \hat{h}_1 \right)^2 - \gamma \phi'(r) \hat{h}_1 + (\gamma u - \Phi). \]
(6.2)

By (2.17),
\[ \nabla_1 \nabla^1 \Phi = \left( \phi(r) \right)^{\frac{\alpha}{\beta}} \hat{F}^{pq} \nabla_p \nabla_q \phi^1 + \left( \phi(r) \right)^{\frac{\alpha}{\beta}} \hat{F}^{pq, rs} \nabla_{pq, 1} h_{rs, 1} \]
\[ + \left( \frac{\alpha}{\beta} \right)^2 \frac{\left( \phi'(r) \right)^2}{\left( \phi(r) \right)^2} (\nabla_1 r)^2 \Phi - \frac{\alpha}{\beta} \frac{1}{\left( \phi(r) \right)^2} (\nabla_1 r)^2 \Phi - \frac{\alpha}{\beta} \frac{\phi'(r)}{\phi(r)^2} u \hat{h}_1 \Phi \]
\[ + \frac{\alpha}{\beta} \frac{(\phi'(r))^2}{\left( \phi(r) \right)^2} \left( 1 - (\nabla_1 r)^2 \right) \Phi + 2 \frac{\alpha}{\beta} \frac{\phi'(r)}{\phi(r)^2} \hat{F}^{pq, rs} (\nabla_1 r) (\nabla_1 F) \]
\[ + \frac{1}{\beta} \left( \phi(r) \right)^{\frac{\alpha}{\beta}} \left( \sigma_k \right)^{\frac{1}{\beta} - 1} (\sigma_k \sigma_k - (k + 1) \sigma_{k+1}) \hat{h}_1 \]
\[ - k \Phi \frac{n - k + 1}{\beta} \left( \sigma_k \right)^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\alpha}{\beta}} \sigma_{k-1} \hat{h}_1. \]
(6.3)

Plugging (6.2) and (6.3) into (6.1), we have
\[ \mathcal{L} \hat{h}_1 \]
\[ = - \left( \hat{h}_1 \right)^2 \left[ \left( \phi^1 \right)^{\frac{\alpha}{\beta}} \hat{F}^{pq, rs} \nabla_{pq, 1} h_{rs, 1} + \left( \frac{\alpha}{\beta} \right)^2 \frac{\left( \phi'(r) \right)^2}{\phi^2} (\nabla_1 r)^2 \Phi - \frac{\alpha}{\beta} \frac{1}{\phi^2} (\nabla_1 r)^2 \Phi \right] \]
\[ - \frac{\alpha}{\beta} \frac{\phi'}{\phi^2} u \hat{h}_1 \Phi + \frac{\alpha}{\beta} \frac{(\phi'(r))^2}{\phi^2} \left( 1 - (\nabla_1 r)^2 \right) \Phi + 2 \frac{\alpha}{\beta} \phi' \phi^{\frac{\alpha}{\beta} - 1} (\nabla_1 r) (\nabla_1 F) \]
\[ + \frac{1}{\beta} \phi^{\frac{\alpha}{\beta}} \left( \sigma_k \right)^{\frac{1}{\beta} - 1} (\sigma_k \sigma_k - (k + 1) \sigma_{k+1}) \hat{h}_1 \]
\[ - k \Phi \frac{n - k + 1}{\beta} \left( \sigma_k \right)^{\frac{1}{\beta} - 1} (\phi(r))^{\frac{\alpha}{\beta}} \sigma_{k-1} \hat{h}_1 \]
\[ + 2 \phi^{\frac{\alpha}{\beta}} \hat{F}^{ij} \left( \hat{h}_1 \right)^2 h_{1p} \nabla_i h_{1q} \nabla_j h_{1q}, \]
Then we denote \( G = \sigma_k^\frac{1}{\beta} \), thus \( F = G^\frac{k}{\beta} \). We have

\[
\tilde{F}^{pq} = k G^\frac{k-1}{\beta} G^{pq}
\]

Thus for the first term in the third line of (6.4), we have

\[
\tilde{F}^{pq,rs} h_{rs,1} h_{pq,1} + 2 \tilde{F}^{ij} h^{pq} h_{1p,i} h_{1q,j} = k G^\frac{k-1}{\beta} \left( G^{pq,rs} h_{rs,1} h_{pq,1} + 2 G^{ij} h^{pq} h_{1p,i} h_{1q,j} \right) + \frac{k}{\beta} \left( \frac{k}{\beta} - 1 \right) G^\frac{k-2}{\beta} \tilde{G}^{pq} \tilde{G}^{rs} h_{pq,1} h_{rs,1}
\]

By Lemma 6.1 and Codazzi equation, we have

\[
G^{pq,rs} h_{rs,1} h_{pq,1} + 2 G^{ij} h^{pq} h_{1p,i} h_{1q,j} \geq 2 G^{-1} (G^{pq} h_{pq,1})^2 = 2 G^{-1} (\nabla_1 G)^2.
\]

Combining (6.6) and (6.7), we get

\[
\left( \tilde{h}_1 \right)^2 \tilde{F}^{pq,rs} h_{rs,1} h_{pq,1} + 2 \tilde{F}^{ij} h^{pq} h_{1p,i} h_{1q,j} \geq k \left( \frac{k}{\beta} + 1 \right) \left( \tilde{h}_1 \right)^2 \frac{k}{\beta} G^\frac{k}{\beta} G^{-2} (\nabla_1 G)^2
\]

Substituting (6.8) into (6.4) and observing that \( \phi^2 - \phi^2 = 1 \), we have

\[
\mathcal{L} \tilde{h}_1 \leq - \left( \tilde{h}_1 \right)^2 \left[ \left( \frac{\alpha}{\beta} \right)^2 \frac{(\phi')^2}{\phi^2} (\nabla_1 r)^2 \Phi - \frac{\alpha}{\beta} \frac{(\phi')^2}{\phi^2} (\nabla_1 r)^2 \Phi + \frac{\alpha}{\beta} (\nabla_1 r)^2 \Phi \right]
\]

\[
+ \frac{\alpha (\phi')^2}{\phi^2} \left( 1 - (\nabla_1 r)^2 \right) \Phi + 2 \frac{\alpha}{\beta} \phi' \Phi G^{-1} (\nabla_1 r)(\nabla_1 G) \Phi \right]
\]

\[
- \frac{k}{\beta} \left( \frac{k}{\beta} + 1 \right) \left( \tilde{h}_1 \right)^2 G^{-2} (\nabla_1 G)^2 \Phi - \left( \tilde{h}_1 \right)^2 \left( \gamma u - \frac{k + \beta}{\beta} \Phi \right)
\]

\[
- \frac{n - k + 1}{\beta} \phi \left( \sigma_k \right)^{\frac{1}{\beta} - 1} \sigma_{k-1} \tilde{h}_1 - \frac{1}{\beta} \phi \left( \sigma_k \right)^{\frac{1}{\beta} - 1} (\sigma_1 \sigma_k - (k + 1) \sigma_{k+1}) \tilde{h}_1
\]

\[
+ \left( \frac{k}{\beta} - 1 \right) \Phi + \gamma \phi \tilde{h}_1 + \frac{\alpha}{\beta} \frac{\phi'}{\phi^2} u \tilde{h}_1 \Phi
\]
\[
= - \left( \frac{k\alpha}{\beta}\left(\frac{k}{\beta} + 1\right) - \frac{(\alpha - 1)}{k + \beta}\right) - \left( \frac{\alpha}{\beta}(\gamma u - \frac{k + \beta}{\beta} \Phi) - \left( \frac{k}{\beta} + 1\right) G^{-2}(\nabla_1 G)^2 \right]
\]

\[
= - \left( \frac{\alpha}{\beta}(\gamma u - \frac{k + \beta}{\beta} \Phi) - \left( \frac{k}{\beta} + 1\right) G^{-2}(\nabla_1 G)^2 \right]
\]

For these coefficients, we have
\[
\frac{k\alpha}{\beta^2\left(\frac{k}{\beta} + 1\right)} - \frac{(\alpha - 1)}{k + \beta} = - \frac{\alpha - k - \beta}{k + \beta} < 0,
\]
and
\[
\frac{\alpha^2}{\beta^2\left(\frac{k}{\beta} + 1\right)} \frac{(\gamma u - \frac{k + \beta}{\beta} \Phi)^2}{\phi^2} G^{-2}(\nabla_1 G)^2 \geq 0.
\]

Plugging (6.9) and (6.10) into the inequality of \( \mathcal{L} \tilde{h}_1 \) above, and use the boundness of \( r, u \) and \( \Phi \), we have
\[
\mathcal{L} \tilde{h}_1 \leq \left( \frac{\alpha}{\beta}(\gamma u - \frac{k + \beta}{\beta} \Phi) - \left( \frac{k}{\beta} + 1\right) G^{-2}(\nabla_1 G)^2 \right)
\]

for some constant \( C > 0 \).

Since
\[
\nabla_1 r = \frac{\langle X_1, V \rangle}{\phi} \leq \frac{|X_1||V|}{|V|} = 1,
\]
we can estimate the coefficient of \( \left( \tilde{h}_1 \right)^2 \) in (6.11) as following
\[
- \frac{\alpha}{\beta}(\gamma u - \frac{k + \beta}{\beta} \Phi) - \left( \frac{k}{\beta} + 1\right) G^{-2}(\nabla_1 G)^2 \geq 0.
\]
for a constant $\delta > 0$, by the lower bounds of $u$ and $\Phi$.

Plugging (6.12) into (6.11), we get

$$L \tilde{h}_1^1 \leq -\delta (\tilde{h}_1^1, \tilde{h}_1^1) + C \tilde{h}_1^1 + C, \quad (6.13)$$

for some positive constants $\delta$, and $C$. Thus when $\tilde{h}_1^1$ is large enough, $L \tilde{h}_1^1 \leq 0$, so $\tilde{h}_1^1$ is bounded from above. That is, the principal radii is bounded from above, which implies the principal curvature is bounded from below. Moreover, by Proposition 4.3, $\sigma_k$ is bounded from above, thus the principal curvature is bounded from both above and below. □

7. **Long time existence and convergence**

Now we have obtained the a priori estimates of flow (2.5), with the initial hypersurfaces mentioned in Theorem 1.2 and Theorem 1.3. From a priori estimates, these flows have short time existence. Using the $C^2$ estimates given in Proposition 5.2, Proposition 6.2, we can get the $C^{2,\lambda}$ estimate of the scalar equation (2.4) by using Theorem 6 in Andrews [1], where $\lambda \in (0,1)$. Note that as $0 < \beta \leq 1$ the equation (2.4) is in general not concave with respect to the second spatial derivatives and the result of Krylov [8] can not be applied directly. Then the parabolic Schauder theory [11] implies the $C^{k,\lambda}$ estimates for all $k \geq 2$. Hence, we get the long time existence of these flows.

**Proposition 7.1.** The smooth solution of the flow (2.5) with the initial hypersurface mentioned in Theorem 1.2 and Theorem 1.3 exists for all time $t \in [0, +\infty)$.

We now complete the proofs of Theorem 1.2 and Theorem 1.3.

Proof. By (3.7) in the proof of Lemma 3.2,

$$\partial_t \left( \frac{1}{2} \nabla \varphi^2 \right) \leq -\frac{\alpha - k - \beta}{\beta} (\sinh(-\varphi))^{-2} \cosh(-\varphi) \sigma_k \frac{1}{\sqrt{1 + |\nabla \varphi|^2 |\nabla \varphi|^2}}. \quad (7.1)$$

From Lemma 3.1 and Proposition 4.2 we know there exists a positive constant $c$, such that

$$\partial_t (|\nabla \varphi|^2) \leq -c |\nabla \varphi|^2. \quad (7.2)$$

thus $\max_{S^n} |\nabla \varphi|$ converges to 0 exponentially fast, which implies $\max_{S^n} |\nabla r|$ converges to 0 exponentially fast.

The long time existence of the flow (2.5) has been shown in Proposition 7.1. Now we prove that the hypersurfaces converge to a sphere along the flow (2.5). As $\max_{S^n} |\nabla r|$ converges to 0 exponentially fast when $\alpha > \beta + k$ and $n \geq 2$, we know $\max_{S^n} |\nabla^l r|$ decays exponentially to 0 for any $l \geq 1$ from the interpolation inequality and the a priori estimates we have made. When $\alpha > \beta + k$, we claim that the radial function $r$ converges to $\hat{r}$. Since $M_0$ is closed and star-shaped, it can be bounded by two spheres centred at the origin, we denote them as $S^n (a_1)$
and $\mathbb{S}^n(a_2), a_1 < a_2$. In the proof of Lemma 3.1, we know the radial function of $M_t$ can be bounded by $a_1(t)$ and $a_2(t)$ by the well-known comparison principle, where

$$\frac{da_i(t)}{dt} = -\gamma \sinh(a_i(t)) \left( \eta(a_i(t)) - \eta(\hat{r}) \right), \quad i = 1, 2,$$

(7.3)

where $\eta$ is the monotonically increasing function $\eta(R) = \sinh(R) \left( \frac{\alpha_1}{\beta} \cosh(R) - \frac{\alpha_2}{\beta} \right)$, by estimation of such ODEs, we see $a_1(t)$ and $a_2(t)$ tends to $\hat{r}$ as $t \to +\infty$, which implies that the radial function of $M_t$ converges to 1 in $C^0$ norm. Combining the estimates of $|\bar{\nabla}r|$, we know $M_t$ converges to $\mathbb{S}^n(\hat{r})$ exponentially fast.

Finally, we conclude the paper with the following remark and question.

Remark 7.2. By the same argument, the curvature bounds in Section 6 can also be obtained if we replace $\sigma_k^{1/k}$ by a general 1-homogeneous symmetric curvature function $G(\kappa)$ which is inverse-concave and with dual $G^*$, vanishing on the boundary of the positive cone $\Gamma_n^+$, where $G_*(z_1, \ldots, z_n) = G(z_1^{-1}, \ldots, z_n^{-1})^{-1}$. We focus $G = \sigma_k^{1/k}$ in this paper because of its relationship with the prescribed curvature measure problems in hyperbolic space.

Question 7.3. Is it possible to obtain the curvature estimate for the $k$-convex solution of the flow (1.1) with $k = 2, \ldots, n$?

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