EQUIVALENCE OF TYPES AND CATLIN BOUNDARY SYSTEMS

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Abstract. We prove the equivalence of the D’Angelo finite type with the Kohn finite ideal type on smooth, pseudoconvex domains in $\mathbb{C}^n$, otherwise known as the Kohn Conjecture. The argument uses Catlin’s notion of a boundary system as well as methods from subanalytic and semialgebraic geometry. When a set contains only two level sets of the Catlin multitype, a lower bound for the subelliptic gain in the $\bar{\partial}$-Neumann problem is obtained in terms of the D’Angelo type, the dimension of the space, and the level of forms.

Dedicated to J.J. Kohn on his 75th birthday

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1. Introduction

In his seminal Acta Mathematica paper of 1979 [17], Joseph J. Kohn gave a sufficient condition for the subellipticity of the $\bar{\partial}$-Neumann problem on a pseudoconvex domain in $\mathbb{C}^n$ by introducing subelliptic multipliers for the $\bar{\partial}$-Neumann problem as well as an algorithm on these multipliers whose termination implies subellipticity. Throughout this paper I shall refer to the latter as the Kohn algorithm. In [17] Kohn defined his subelliptic multipliers as germs of $C^\infty$ functions, and the sufficient condition he obtained for subellipticity applies to $C^\infty$ pseudoconvex domains. Because of the peculiar algebraic properties of the ring of $C^\infty$ functions, however, he restricted his study of the termination of his algorithm to the much better behaved ring of real-analytic functions $C^\omega$. It is for such a pseudoconvex domain $\Omega$ in $\mathbb{C}^n$ with a real-analytic boundary that he proved the equivalence of the following three properties:
(i) subellipticity of the $\bar{\partial}$-Neumann problem for $(p, q)$ forms;
(ii) termination of the Kohn algorithm on $(p, q)$ forms;
(iii) finite order of contact of holomorphic varieties of complex dimension $q$ with the boundary of the domain $\Omega$.

It must be pointed out that the Kohn algorithm involves an increasing chain of ideals, so condition (ii) is called in the literature Kohn finite ideal type.

It already follows from Joseph J. Kohn’s solution to the $\bar{\partial}$-Neumann problem in [14] and [15] for strongly pseudoconvex domains as well as from the weighted estimates for pseudoconvex domains done Hörmander in [13] and Kohn in [16] that the $\bar{\partial}$-Neumann problem is elliptic inside the domain $\Omega$, so the study of subellipticity only needs to be conducted on the boundary of the domain $b\Omega$. By definition, condition (i), subellipticity, is an open condition, namely if it holds at a point $x_0$, then it must hold in a neighborhood of that point $U_{x_0}$. Likewise, the Kohn algorithm terminates when the ideal of multipliers captures a unit, which remains a unit in a neighborhood of $x_0$, making condition (ii) also an open condition. The only condition of the three stated above whose openness Kohn’s paper [17] did not address is (iii).

John D’Angelo picked up precisely this line of inquiry. He showed in his 1982 paper in the Annals of Mathematics [8] that condition (iii) is indeed open for boundaries of $C^\infty$ domains. Remarkably enough, his investigation showed that condition (iii) is fully captured by an algebraic approximation, namely an approximation of the defining function of the domain $\Omega$ by its own Taylor polynomial of a high enough degree, thus establishing that proving openness of condition (iii) for a $C^\infty$ boundary is of the same degree of difficulty as for a $C^\omega$ boundary. Given the significance of D’Angelo’s work on this subject, condition (iii) is known in the literature as D’Angelo finite type.

Following Kohn’s ground-breaking work and D’Angelo’s crucial elucidation of the behavior of condition (iii), David Catlin proved in a series of three very deep papers in the Annals of Mathematics [5], [6], and [7] that for a $C^\infty$ pseudoconvex domain $\Omega$ in $\mathbb{C}^n$, subellipticity of the $\bar{\partial}$-Neumann problem on $\Omega$ is equivalent to finite D’Angelo type, namely condition (iii). This circle of ideas would have been complete if it were not for the fact that Catlin’s work did not investigate the termination of the Kohn algorithm. As stated above, for $C^\infty$ domains Kohn’s construction in [17] shows that the Kohn finite ideal type condition, namely condition (ii), is sufficient for subellipticity. The question is then: Is it also necessary? Joseph J. Kohn conjectured this should indeed be the case. The main result of this paper is then precisely to prove this conjecture due to Kohn:

**Main Theorem 1.1.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with $C^\infty$ boundary. Let $x_0 \in b\Omega$ be any point on the boundary of the domain, and let $U_{x_0}$ be an appropriately small neighborhood around $x_0$. The following three properties are equivalent:

(i) The $\bar{\partial}$-Neumann problem for $(p, q)$ forms is subelliptic on $U_{x_0}$;
(ii) The Kohn algorithm on $(p, q)$ forms terminates at $x_0$ by generating the entire ring $C^\infty(U_{x_0})$;
(iii) The order of contact of holomorphic varieties of complex dimension $q$ with the boundary of the domain $\Omega$ in $U_{x_0}$ is finite.
The main theorem is proven via the following result that gives a precise description of the behavior of the Kohn algorithm under the assumption of finite D’Angelo type:

**Theorem 1.2.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with $C^\infty$ boundary. Let $x_0 \in b\Omega$ be any point on the boundary of the domain, and let the order of contact of holomorphic varieties of complex dimension $q$ with the boundary of $\Omega$ at $x_0$ be finite. If $U_{x_0}$ is an appropriately small neighborhood around $x_0$, then the Kohn algorithm on $(p, q)$ forms terminates at step 1 densely in $U_{x_0} \cap b\Omega$ in the induced topology on $b\Omega$ and by step $N$ otherwise, where $N$ is the number of level sets of the Catlin multitype in $U_{x_0}$.

In case, the neighborhood contains only two level sets of the Catlin multitype, then we obtain an effective lower bound for the subelliptic gain in the $\bar{\partial}$-Neumann problem that is polynomial in the D’Angelo type, and the Kohn algorithm finishes by step 2:

**Theorem 1.3.** Let $\Omega$ in $\mathbb{C}^n$ be a pseudoconvex domain with $C^\infty$ boundary. Let $x_0 \in b\Omega$ be any point on the boundary of the domain, and let the order of contact of holomorphic varieties of complex dimension $q$ with the boundary of $\Omega$ at $x_0$ be finite and equal to $t$. If $U_{x_0}$ is an appropriately small neighborhood around $x_0$ containing at most two level sets of the Catlin multitype, then the Kohn algorithm on $(p, q)$ forms terminates at step 1 densely in $U_{x_0} \cap b\Omega$ and at step 2 otherwise. Furthermore, the subelliptic gain in the $\bar{\partial}$-Neumann problem at $x_0$, which we will denote by $\epsilon$, satisfies

$$\epsilon \geq \frac{1}{4} \max\{([t] - 1)^{n-q}, [t] + 1\}.$$ 

The lower bound depends polynomially on the integer part $[t]$ of the finite D’Angelo type $\Delta_q(b\Omega, x_0) = t \in \mathbb{Q}$.

The termination of the Kohn algorithm at step 1 densely in $b\Omega$ under the assumption of finite D’Angelo type follows from the work of D’Angelo in [8] and of Catlin in [6] and was thus known by the mid 80’s. For all other points though, Kohn’s 1979 result for $C^\omega$ domains in [17] specifies an upper bound of $2n$ for the number of steps until the Kohn algorithm terminates. As for an effective bound for the subelliptic gain $\epsilon$ in the $\bar{\partial}$-Neumann problem in terms of the D’Angelo type and the dimension of the domain, Catlin obtained in [7] a lower bound

$$\epsilon \geq t^{-n^2}e^{n^2}$$

that is exponential in the D’Angelo type $t$ and holds for any pseudoconvex smooth domain. Yum-Tong Siu proved in [19] that a lower bound for $\epsilon$ exists that is polynomial in $t$ and $n$ in the case when the pseudoconvex domain is defined by a function $r(z)$ of the special type

$$r(z) = Re z_n + \sum_{j=1}^{N} |f_j(z_1, \ldots, z_{n-1})|^2,$$

where $N \geq n$ and $f_j$ is holomorphic for all $1 \leq j \leq N$. Theorem 1.3 provides the same answer for the case when $\Omega$ is any smooth domain that is pseudoconvex and such that every point has
a neighborhood that contains at most two level sets of the Catlin multitype. The method of proof of Theorems 1.2 and 1.3 puts in correspondence the smooth pseudoconvex domain with another pseudoconvex domain defined by a polynomial. Given that the behavior of polynomials is easiest to understand, this would suggest an effective estimate of the subelliptic gain for every smooth pseudoconvex domain is within reach.

Theorems 1.2 and 1.3 are proven by giving an algebraic interpretation to the notions of multitype, commutator multitype, and boundary system that Catlin developed in [6] in order to exploit the finite D’Angelo type condition analytically. As it turns out, the functions apart from the defining function of the domain that characterize a boundary system are Tougeron elements, namely they have non-zero gradients but vanish on the variety corresponding to the ideal of multipliers in the Kohn algorithm \( I^q(x_0) \) at some step \( k \) of the Kohn algorithm. To show these elements are in fact multipliers requires a Nullstellensatz. This Nullstellensatz is obtained via a two-step approach. In the first step, it is shown that the Nullstellensatz holds in the polynomial ring, which is the setting of the truncated boundary system defined by Catlin in [6]. The second step consists of showing the same Nullstellensatz with the same exponents holds in the untruncated, original boundary system. This is done via a continuity argument. Finally, a determinant constructed from these Tougeron elements and rows of the Levi form is shown to be a unit. This determinant belongs to the ideal of multipliers at step \( N \) of the Kohn algorithm, where \( N \) is the number of level sets of the Catlin multitype in the neighborhood on which the Kohn algorithm is run. The Kohn algorithm must then terminate by step \( N \).

Joseph J. Kohn showed in [17] that the subelliptic multipliers for the \( \bar{\partial} \)-Neumann problem that he defined form a sheaf of ideals. In the same paper, he proved that for a \( C^\omega \) domain, this sheaf of ideals is defined over the Noetherian ring \( C^\omega \) and is coherent. For a smooth domain, this sheaf of ideals of subelliptic multipliers is defined over the non-Noetherian ring \( C^\infty \). Fortunately, it turns out that this sheaf is quasi-flasque and thus possesses the nice property that the sections that generate the ideal of multipliers for \( (p, q) \) forms \( I^q(x_0) \) at some \( x_0 \in \Omega \) also generate \( I^q(x) \) for \( x \) sufficiently close to \( x_0 \). This property justifies working with the presheaf rather than the multiplier ideal sheaf itself, which greatly simplifies that argument. Its importance thus warrants it being stated as a theorem as follows:

**Theorem 1.4.** Let \( \Omega \) in \( \mathbb{C}^n \) be a domain with \( C^\infty \) boundary. Let \( \mathcal{I}^q \) be the sheaf of subelliptic multipliers for the \( \bar{\partial} \)-Neumann problem on \( (p, q) \) forms. \( \mathcal{I}^q \) is a sheaf over the ring \( C^\infty(\Omega) \) that satisfies the following two properties:

(a) \( \mathcal{I}^q \) is quasi-flasque;

(b) Let \( x_0 \) be any point in \( \overline{\Omega} \). If sections \( s_j \in C^\infty(\Omega) \) generate \( \mathcal{I}_{x_0} \) for \( j \in J \), \( J \) an indexing set, then \( s_j \) also generate \( \mathcal{I}_x \) for \( x \) sufficiently close to \( x_0 \).

Properties (a) and (b) also hold for \( \mathcal{I}_k^q \), the multiplier ideal sheaf given by the Kohn algorithm at step \( k \), for each \( k \geq 1 \).

The quasi-flasque property follows from the fact that \( \mathcal{I}^q \) and each of the sheaves \( \mathcal{I}_k^q \) are closed under the real radical. Joseph J. Kohn proved this closure under the real radical in [17] without using pseudoconvexity, so it is not necessary to impose pseudoconvexity among the
hypotheses of Theorem 1.4. Property (b) is then merely a consequence of the quasi-flasque property, property (a). Its importance resides in the fact that it recovers the chief property of a coherent sheaf in the context where stalks may not be finitely generated as it can certainly be the case for a sheaf over a non-Noetherian ring like \( C^{\infty} \).

This paper is organized as follows: Section 2 is devoted to the Kohn algorithm and other algebraic matters necessary for its successful implementation on \( C^{\infty} \). Section 3 defines quasi-flasque sheaves and proves Theorem 1.4. Section 4 introduces Catlin’s boundary systems as well as his multitype and commutator multitype. Section 5 then defines the notion of a Tougeron element and strengthens a result of Catlin in [6] in order to prove a version of the Diederich-Fornæss theorem for smooth pseudoconvex domains that will imply later on in the argument that the Tougeron elements in the boundary system vanish on \( \mathcal{V}(I^{\infty}_k(x_0)) \), the variety corresponding to the ideal of multipliers at step \( k \) of the Kohn algorithm. The original Diederich-Fornæss theorem in [10] on real-analytic pseudoconvex domains was a crucial ingredient for the equivalence of types in the \( C^{\infty} \) case as proven by Kohn in [17]. Section 6 presents Catlin’s truncation of a boundary system as it appears in [6]. This construction has extraordinary algebraic implications as it allows reducing the proof of the Nullstellensatz to a problem over the polynomial ring. Section 7 then establishes the Nullstellensatz in the polynomial ring using methods from semialgebraic geometry. The result is carried back to the original setting of the Kohn algorithm to prove the Tougeron elements defining the boundary system are subelliptic multipliers, which then establishes Theorems 1.2 and 1.3. The three-way equivalence in the Main Theorem 1.1 follows.

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2. The Kohn algorithm

We start with Joseph J. Kohn’s definition of what it means for the \( \bar{\partial} \)-Neumann problem on \((p, q)\) forms to be subelliptic followed by his definition of a subelliptic multiplier from [17]. We refer the reader to this same paper for details and motivation regarding the setup of the \( \bar{\partial} \)-Neumann problem:

**Definition 2.1.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( x_0 \in \overline{\Omega} \). The \( \bar{\partial} \)-Neumann problem on \( \Omega \) for \((p, q)\) forms is said to be subelliptic at \( x_0 \) if there exist a neighborhood \( U \) of \( x_0 \) and constants \( C, \epsilon > 0 \) such that

\[
\|\varphi\|_\epsilon^2 \leq C (\|\bar{\partial}\varphi\|_0^2 + \|\bar{\partial}^*\varphi\|_0^2 + \|\varphi\|_0^2)
\]  

(2.1)
for all \((p, q)\) forms \(\varphi \in C_0^\infty(U) \cap \text{Dom}(\bar{\partial}^*)\), where \(\| \cdot \|_\epsilon\) is the Sobolev norm of order \(\epsilon\) and \(\| \cdot \|_0\) is the \(L^2\) norm.

**Definition 2.2.** Let \(\Omega\) be a domain in \(\mathbb{C}^n\) and let \(x_0 \in \bar{\Omega}\). A \(C^\infty\) function \(f\) is called a subelliptic multiplier at \(x_0\) for the \(\bar{\partial}\)-Neumann problem on \(\Omega\) if there exist a neighborhood \(U\) of \(x_0\) and constants \(C, \epsilon > 0\) such that

\[
\| f \varphi \|_2^2 \leq C (\| \bar{\partial} \varphi \|_0^2 + \| \bar{\partial}^* \varphi \|_0^2 + \| \varphi \|_0^2)
\]

(2.2) for all \((p, q)\) forms \(\varphi \in C_0^\infty(U) \cap \text{Dom}(\bar{\partial}^*)\). We will denote by \(I^q(x_0)\) the set of all subelliptic multipliers at \(x_0\).

Several remarks about these two definitions are necessary:

1. If there exists a subelliptic multiplier \(f \in I^q(x_0)\) such that \(f(x_0) \neq 0\), then a subelliptic estimate holds at \(x_0\) for the \(\bar{\partial}\)-Neumann problem.
2. As explained on page 2 of the introduction, if \(x_0 \in \Omega\) then automatically estimate (2.2) holds at \(x_0\) with the largest possible \(\epsilon\) allowed by the \(\bar{\partial}\)-Neumann problem, namely \(\epsilon = 1\). This says the problem is elliptic rather than subelliptic inside.
3. The previous remark implies that if \(x_0 \in \partial \Omega\) but \(f = 0\) on \(U \cap \partial \Omega\), then estimate (2.2) again holds for \(\epsilon = 1\). This is the case if we set \(f = r\), where \(r\) is the defining function of the domain \(\Omega\).
4. If \(x_0 \in \partial \Omega\), the highest possible gain in regularity in the \(\bar{\partial}\)-Neumann problem is given by \(\epsilon = \frac{1}{2}\) under the strongest convexity assumption, namely strong pseudoconvexity of \(\Omega\), as proved by Kohn in [14] and [15].
5. This non-ellipticity of the \(\bar{\partial}\)-Neumann problem is coming precisely from the boundary condition given by \(\varphi \in \text{Dom}(\bar{\partial}^*)\).
6. Note that subelliptic multipliers at \(x_0\) for \((p, q)\) forms are denoted by \(I^q(x_0)\) without reference to \(p\), which is the holomorphic part of any such form and which plays no role in the \(\bar{\partial}\)-Neumann problem.

In the paper [17] cited above, Joseph J. Kohn proceeds by considering germs of smooth functions at \(x_0\), which he denotes by \(C^\infty(x_0)\). A germ is an equivalence class of smooth functions determined by the equivalence relation: If \(f_1 \in C^\infty(U_1)\), where \(U_1\) is a neighborhood of \(x_0\), and \(f_2 \in C^\infty(U_2)\), where \(U_2\) is another neighborhood of \(x_0\), then \(f_1 \sim f_2\) if \(f_1 = f_2\) on some neighborhood \(U\) of \(x_0\) such that \(U \subset U_1 \cap U_2\). Given his definition of a subelliptic multiplier, using germs rather than smooth functions makes perfect sense because \(I^q(x_0) \subset C^\infty(x_0)\). For our purposes here, however, it impedes certain algebraic considerations, so we will explain Kohn’s setup using germs, then we will show that we can choose a canonical neighborhood in which to run the Kohn algorithm. Once we do so, we can conduct our argument inside this canonical neighborhood and revert to using smooth functions rather than germs. As explained in the introduction, we will devote the next section, Section 8, to quasi-flasque sheaves and the proof of Theorem 1.4. This material justifies why we can work on a neighborhood with
the presheaf and we do not encounter technical problems of coherence for the multiplier ideal sheaf given by the Kohn algorithm. Beyond providing this justification, however, the material on quasi-flasque sheaves is not used in the rest of the paper and may be skipped by non-algebraically minded readers without any loss in comprehension. It should be noted here that in the real-analytic case for which Kohn proved the equivalence in [17], it makes no difference whether one uses the functions themselves or germs because $\mathbb{C}^\omega$ is a Noetherian ring, so if the neighborhood in which the Kohn algorithm runs is appropriately shrunken, each germ becomes an equivalence class consisting of only one element.

We will proceed now to explain Kohn’s setup of his algorithm. For this we need two more definitions:

**Definition 2.3.** To each $x_0 \in \Omega$ and $q \geq 1$ we associate the module $M^q(x_0)$ defined as the set of $(1,0)$ forms $\sigma$ satisfying that there exist a neighborhood $U$ of $x_0$ and constants $C, \epsilon > 0$ such that
\[
||\text{int}(\bar{\sigma}) \varphi||^2 \leq C(||\bar{\partial} \varphi||^2_0 + ||\bar{\partial}^* \varphi||^2_0 + ||\varphi||^2_0)
\]
for all $(p,q)$ forms $\varphi \in C^\infty_0(U) \cap \text{Dom}(\bar{\partial}^*)$, where $\text{int}(\bar{\sigma}) \varphi$ denotes the interior multiplication of the $(0,1)$ form $\bar{\sigma}$ with the $(p,q)$ form $\varphi$.

The significance of $M^q(x_0)$ is that complex gradients of subelliptic multipliers will be shown to belong to it.

**Definition 2.4.** Let $J \subset C^\infty(x_0)$, then the real radical of $J$ denoted by $\sqrt{J}$ is the set of $g \in C^\infty(x_0)$ such that there exists some $f \in J$ and some positive natural number $m \in \mathbb{N}^*$ such that
\[
|g|^m \leq |f|
\]
on some neighborhood of $x_0$.

The real radical is the correct generalization of the usual radical on the ring of holomorphic functions $\mathcal{O}$ for both $\mathbb{C}$-valued $C^\omega$ functions and $\mathbb{C}$-valued $C^\infty$ functions.

We now have all definitions in place to state Kohn’s Proposition 4.7 from [17] in which he proves the properties characterizing subelliptic multipliers that allow him to put his algorithm together:

**Proposition 2.5.** If $\Omega$ is a smooth pseudoconvex domain and if $x_0 \in \overline{\Omega}$, then $I^q(x_0)$ and $M^q(x_0)$ have the following properties:

(A) $1 \in I^q(x_0)$ and for all $q$, whenever $x_0 \in \Omega$, then $1 \in I^q(x_0)$.
(B) If $x_0 \in b\Omega$, then $r \in I^q(x_0)$.
(C) If $x_0 \in b\Omega$, then $\text{int}(\theta) \bar{\partial}\bar{\partial}r \in M^q(x_0)$ for all smooth $(0,1)$ forms $\theta$ such that $\langle \theta, \bar{\partial}r \rangle = 0$ on $b\Omega$.
(D) $I^q(x_0)$ is an ideal.
(E) If $f \in I^q(x_0)$ and if $g \in C^\infty(x_0)$ with $|g| \leq |f|$ in a neighborhood of $x_0$, then $g \in I^q(x_0)$.
(F) $I^q(x_0) = \sqrt{I^q(x_0)}$. 

(G) $\partial I^q(x_0) \subset M^q(x_0)$, where $\partial I^q(x_0)$ denotes the set of $(1,0)$ forms composed of complex gradients $\partial f$ for $f \in I^q(x_0)$.

(H) $\det_{n-q+1} M^q(x_0) \subset I^q(x_0)$, where $\det_{n-q+1} M^q(x_0)$ is the coefficient of the wedge product of $n - q + 1$ elements of $M^q(x_0)$.

**Remark:** Kohn proved properties (D), (E), and (F) without employing pseudoconvexity. For properties (C) and (H), however, pseudoconvexity is a crucial hypothesis.

By examining the proof of Proposition 2.5 given by Kohn in Section 4 of [17], we get information about the cost in terms of the reduction in the subelliptic gain in the $\partial$-Neumann problem of performing each of the operations specified in parts (C) and (E)-(H). We also have information about the original subelliptic estimate that the defining function $r$ satisfies. All of this data is collected in the following proposition, which will be crucial in Section 7 for obtaining the effective estimate stated in Theorem 1.3.

**Proposition 2.6.** If $\Omega$ is a smooth pseudoconvex domain and if $x_0 \in \overline{\Omega}$, then $I^q(x_0)$ and $M^q(x_0)$ have the following properties:

(i) If $x_0 \in \overline{\Omega}$, then $r \in I^q(x_0)$ satisfies (2.2) with $\epsilon = 1$.

(ii) If $x_0 \in b\Omega$ and $\theta$ is any smooth $(0,1)$ form such that $\langle \theta, \bar{\partial}r \rangle = 0$ on $b\Omega$, then $\text{int}(\theta) \partial \bar{\partial}r \in M^q(x_0)$ satisfies (2.3) with $\epsilon = \frac{1}{2}$.

(iii) If $f \in I^q(x_0)$ satisfies (2.2) with some $\epsilon > 0$ and if $g \in C^\infty(x_0)$ is such that $|g| \leq |f|$ in a neighborhood of $x_0$, then $g \in I^q(x_0)$ satisfies (2.2) with the same $\epsilon$.

(iv) If $f \in I^q(x_0)$ satisfies (2.2) with some $\epsilon > 0$ and if $g \in C^\infty(x_0)$ is such that $|g|^m \leq |f|$ for an integer $m \in \mathbb{N}^*$ in a neighborhood of $x_0$, then $g \in I^q(x_0)$ satisfies (2.2) with $\frac{\epsilon}{m}$.

(v) If $f \in I^q(x_0)$ satisfies (2.2) with some $\epsilon > 0$, then $\partial f \in M^q(x_0)$ satisfies (2.3) with $\frac{\epsilon}{2}$, where $\partial f$ is the complex gradient of $f$.

(vi) If $\sigma_1, \ldots, \sigma_{n+1-q} \in M^q(x_0)$ satisfy (2.3) with $\epsilon_1, \ldots, \epsilon_{n+1-q}$ respectively, then the coefficient of their wedge product $\det_{n-q+1}(\sigma_1, \ldots, \sigma_{n+1-q}) \in I^q(x_0)$ satisfies (2.2) with $\epsilon = \min_{1 \leq j \leq n+1-q} \epsilon_j$.

Kohn gives the following corollary to Proposition 2.5, which is Theorem 1.21 in [17]:

**Corollary 2.7.** If $\Omega$ is a smooth pseudoconvex domain and if $x_0 \in \overline{\Omega}$, then we have:

(a) $I^q(x_0)$ is an ideal.

(b) $I^q(x_0) = \sqrt[+]\mathbb{I}^q(x_0)$.

(c) If $r = 0$ on $b\Omega$, then $r \in I^q(x_0)$ and the coefficients of $\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}$ are in $I^q(x_0)$.

(d) If $f_1, \ldots, f_{n-q} \in I^q(x_0)$, then the coefficients of $\partial r \wedge \bar{\partial} r \wedge \partial f_1 \wedge \cdots \wedge \partial f_j \wedge (\partial \bar{\partial} r)^{n-q-j}$ are in $I^q(x_0)$, for $j \leq n - q$.

**Remark:** Properties (a) and (b) do not require pseudoconvexity whereas properties (c) and (d) cannot be proven in absence of pseudoconvexity.

This corollary precisely motivates how Joseph J. Kohn sets up the algorithm.
The Kohn Algorithm:

Step 1

\[ I_1^q(x_0) = \sqrt{r, \text{coeff}\{\partial r \wedge \bar{\partial}r \wedge (\partial \bar{\partial}r)^{n-q}\}} \]

Step (k+1)

\[ I_{k+1}^q(x_0) = \sqrt{(I_k^q(x_0), A_k^q(x_0))} \]

where

\[ A_k^q(x_0) = \text{coeff}\{\partial f_1 \wedge \cdots \partial f_j \wedge \partial r \wedge \bar{\partial}r \wedge (\partial \bar{\partial}r)^{n-q-j}\} \]

for \( f_1, \ldots, f_j \in I^q(x_0) \) and \( j \leq n - q \). Note that \((\cdot)\) stands for the ideal generated by the functions inside the parentheses and \( \text{coeff}\{\partial r \wedge \bar{\partial}r \wedge (\partial \bar{\partial}r)^{n-q}\} \) is the determinant of the Levi form for \( q = 1 \), namely in the \( \bar{\partial}\)-Neumann problem for \( (0, 1) \) forms. Evidently, \( I_k^q(x_0) \subset I^q(x_0) \) at each step \( k \), and furthermore the algorithm generates an increasing chain of ideals

\[ I_1^q(x_0) \subset I_2^q(x_0) \subset \cdots \]

Note that the two generators of the first ideal of multipliers \( I_1^q(x_0) \) are globally defined objects. Let us then start the algorithm in a given neighborhood \( U \) of \( x_0 \). If it can be shown that none of the moves of the algorithm alters this neighborhood, then we can adopt it as a canonical neighborhood for the algorithm and drop Kohn’s use of germs. A careful analysis of Kohn’s proof of properties (A) through (H) from Proposition 2.5 reveals that no shrinking occurs but property (C) as well as the other parts that involve the result contained in it are proven in a small enough neighborhood of \( x_0 \) that convenient frames of vector fields and dual forms can be defined in which the boundary condition \( \varphi \in \text{Dom}(\bar{\partial}^*) \) has a particularly simple statement. We will adopt this type of neighborhood that Kohn describes in section 2, page 89 of his paper [17] and describe it here for the reader’s convenience. Note that this setup is exactly the standard one used by Joseph J. Kohn and others when working with the \( \bar{\partial}\)-Neumann problem except for the fact that we will exchange indices 1 and \( n \) in order to be consistent with David Catlin’s setup in [6], which we will recall starting with Section 4.

We choose a defining function \( r \) for the domain \( \Omega \) such that \( |\partial r|_x = 1 \) for all \( x \) in a neighborhood of \( b\Omega \). We choose a neighborhood \( U \) of \( x_0 \) small enough so that the previous condition holds on \( U \), and we choose \((1, 0)\) forms \( \omega_1, \ldots, \omega_n \) on \( U \) satisfying that \( \omega_1 = \partial r \) and \( \langle \omega_i, \omega_j \rangle = \delta_{ij} \) for all \( x \in U \). We define by duality \((1, 0)\) vector fields \( L_1, \ldots, L_n \) such that \( \langle \omega_i, L_j \rangle = \delta_{ij} \) for all \( x \in U \). It follows that on \( U \cap b\Omega \),

\[ L_j(r) = \bar{L}_j(r) = \delta_{1j}. \]

We define a vector field \( T \) on \( U \cap b\Omega \) by

\[ T = L_1 - \bar{L}_1. \]
Clearly, the collection of vector fields $L_2, \ldots, L_n, \bar{L}_2, \ldots, \bar{L}_n, T$ gives a local basis for the complexified tangent space $\mathbb{C}T(U \cap b\Omega)$. A $(p, q)$ form $\varphi$ can be expressed in terms of the corresponding local basis of dual forms on $U$ as

$$\varphi = \sum_{|I| = p, |J| = q} \varphi_{IJ} d\omega_I d\bar{\omega}_J,$$

for $I$ and $J$ multi-indices in $\mathbb{N}^n$. As Kohn shows, $\varphi \in \text{Dom}(\bar{\partial}^*)$ means precisely that $\varphi_{IJ}(x) = 0$ when $1 \in J$ and $x \in b\Omega$. The Levi form is likewise computed in this local basis.

The neighborhood $U$ described above is not yet the same as the neighborhood $U_{x_0}$ that appears in the statement of the Main Theorem 1.1 since there are two other conditions we will impose in Section 3 that may shrink $U$ further. In any case, having described $U$, we drop from now on any mention of germs and pass to considering the ring of smooth functions on $U$, $\mathcal{C}^\infty(U)$ without any equivalence relation on it.

Let us now state the main difficulties in working on $\mathcal{C}^\infty(U)$ and explain how each of them will be overcome in the course of the proof:

(i) $\mathcal{C}^\infty(U)$ contains flat functions, which is precisely the reason why it is not Noetherian.
(ii) The Łojasiewicz inequalities do not hold for all elements of $\mathcal{C}^\infty(U)$.
(iii) An ideal $\mathcal{I} \subset \mathcal{C}^\infty(U)$ may not be finitely generated.
(iv) Ascending chains of ideals may not stabilize.

(i) Let us first define what it means for a function to be flat.

**Definition 2.8.** A function $f \in \mathcal{C}^\infty(U)$ is said to be flat at a point $x \in U$ if $f$ vanishes at $x$ along with its derivatives of all orders.

Fortunately, the D’Angelo finite type condition, which will be imposed on the neighborhood $U$, guarantees that the defining function $r$ has to be non-flat in certain directions tangent to the boundary of domain $b\Omega$ besides the non-flatness of $r$ in the normal direction guaranteed by the fact that it defines a manifold. Catlin’s construction of boundary systems in [6] precisely captures this non-flatness of $r$. He differentiates $r$ with respect to certain vector fields and their conjugates in a neighborhood of a point in such a way that he obtains real-valued functions $r_{p+2}, \ldots, r_{n+1-q}$ with non-zero, linearly independent gradients, where $p$ is the rank of the Levi form at the chosen point and $b\Omega$ satisfies finite D’Angelo $q$ type. Thus, Catlin constructs non-flat elements starting from the defining function $r$. This hints at the fact that under the assumption of finite D’Angelo type, the behavior of the ideals in $\mathcal{C}^\infty$ appearing in the Kohn algorithm is much better than that of arbitrary ideals in $\mathcal{C}^\infty$. Catlin’s construction will be explained in detail in Section 4.

(ii) The Łojasiewicz inequalities were discovered by Stanislas Łojasiewicz, when he investigated and solved the Division Problem posed by Laurent Schwarz. These inequalities give analytic restatements of algebraic properties and characterize every function in $\mathcal{O}$ and $\mathcal{C}^\omega$. Furthermore, Edward Bierstone and Pierre Milman showed three years ago in [3] that they also hold for all
functions in Denjoy-Carleman quasianalytic classes, these being local rings of smooth functions
with Taylor expansions that do not necessarily converge but such that they contain all real-
analytic functions and no flat functions. It is very easy to show that a function satisfying
the Łojasiewicz inequalities cannot be flat on any neighborhood in its domain of definition.
It is not known, however, when the converse holds, namely which non-flatness conditions on
a $C^\infty$ function $f$ will imply that $f$ has to satisfy the Łojasiewicz inequalities. The author
has given another argument for the Kohn conjecture that relies upon constructing finitely-
generated subideals of multipliers in the Kohn algorithm such that each generator satisfies a
Łojasiewicz inequality with respect to distance. Such ideals are called Łojasiewicz ideals. In
this paper, however, the approach is to use Catlin’s truncation of a boundary system, which
puts into correspondence the ideals in the Kohn algorithm on the domain $\Omega$ with ideals on
another pseudoconvex domain $\tilde{\Omega}$ where the defining function and all elements in the Kohn
algorithm are polynomials. Since polynomials form a subring of $C^\omega$, each polynomial satisfies
the Łojasiewicz inequalities. Therefore, when interpreted algebraically, this correspondence
that Catlin sets up in [6] precisely provides another argument for the fact that some subideals
of the ideals of multipliers in the Kohn algorithm are Łojasiewicz ideals under the assumption
of finite D’Angelo type.

Let us now state the Łojasiewicz inequality that we will employ in section 6. It should be
noted here that there are three Łojasiewicz inequalities. The interested reader should consult
either the survey article by Stanislas Łojasiewicz [18] or, for an even better presentation, the
article by Bierstone and Milman [3] cited above. Out of these three inequalities, we only need
the one that relates two functions, one of which vanishes on the zero set of the other:

**Definition 2.9.** Let $f \in C^\infty(U)$ and let $X = \{ x \in U \mid f(x) = 0 \}$. Let $g \in C^\infty(U)$ be such that
$g(x) = 0$ for all $x \in X$. $g$ is said to satisfy a Łojasiewicz inequality on $U$ with respect to $f$ if
for any compact subset $K$ of $U$, there exist constants $C > 0$ and $\alpha \geq 0$ with $\alpha \in \mathbb{Q}$ such that
\[ |g(x)|^\alpha \leq C |f(x)| \]
for all $x \in K$.

(iii) The fact that not all ideals $J$ in $C^\infty(U)$ are finitely generated is a consequence of the fact that $C^\infty(U)$ is not Noetherian. Fortunately, for the purposes of this argument, we will only need to work with the two generators of the first ideal in the Kohn algorithm, namely $r$ and $\text{coeff}\{ \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q} \}$ to prove Theorem 1.3. For Theorem 1.2, the correspondence
mentioned earlier between the original domain $\Omega$ and the other defined by a polynomial $\tilde{\Omega}$ puts
to rest all such concerns since polynomials form a subring of $C^\omega$, which is Noetherian. Point
(iii) thus does not affect our argument at all.

(iv) The property that all ascending chains of ideals stabilize characterizes Noetherian rings,
and $C^\infty(U)$ is not Noetherian. The chain of ideals given by the Kohn algorithm is indeed an
ascending chain, so we do not know a priori whether it will stabilize or not. Fortunately, at
any point $x_0 \in b\Omega$, we will be able to construct a unit at some step $k$ in the Kohn algorithm,
where \( k \) is related to the Catlin multitype of \( b\Omega \) at \( x_0 \). Thus, point (iv) does not come into our argument at all.

We shall close this section by recalling from [17] the definition of the Zariski tangent space to an ideal and to a variety, which will allow us to define the holomorphic dimension of a variety. We will then recall from [6] Catlin’s definition of the holomorphic dimension of a variety, which is more restrictive than Kohn’s. These concepts will be significant in Section 5.

Definition 2.10. Let \( \mathcal{I} \) be an ideal in \( C^\infty(U) \) and let \( \mathcal{V}(\mathcal{I}) \) be the variety corresponding to \( \mathcal{I} \). If \( x \in \mathcal{V}(\mathcal{I}) \), then we define \( Z^{1,0}_x(\mathcal{I}) \) the Zariski tangent space of \( \mathcal{I} \) at \( x \) to be

\[
Z^{1,0}_x(\mathcal{I}) = \{ L \in T^{1,0}_x(U) \mid L(f) = 0 \ \forall \ f \in \mathcal{I} \},
\]

where \( T^{1,0}_x(U) \) is the \((1,0)\) tangent space to \( U \subset \mathbb{C}^n \) at \( x \). If \( \mathcal{V} \) is a variety, then we define

\[
Z^{1,0}_x(\mathcal{V}) = Z^{1,0}_x(\mathcal{I}(\mathcal{V})),
\]

where \( \mathcal{I}(\mathcal{V}) \) is the ideal of all functions in \( C^\infty(U) \) vanishing on \( \mathcal{V} \).

The next lemma is Lemma 6.10 of [17] that relates \( Z^{1,0}_x(\mathcal{I}) \) with \( Z^{1,0}_x(\mathcal{V}(\mathcal{I})) \):

Lemma 2.11. Let \( \mathcal{I} \) be an ideal in \( C^\infty(U) \). If \( x \in \mathcal{V}(\mathcal{I}) \), then

\[
Z^{1,0}_x(\mathcal{V}(\mathcal{I})) \subset Z^{1,0}_x(\mathcal{I}). \tag{2.4}
\]

Equality holds in (2.4) if the ideal \( \mathcal{I} \) satisfies the Nullstellensatz, namely \( \mathcal{I} = \mathcal{I}(\mathcal{V}(\mathcal{I})) \).

Let

\[
\mathcal{N}_x = \{ L \in T^{1,0}_x(b\Omega) \mid \langle (\partial \bar{\partial} r)_x, L \wedge \bar{L} \rangle = 0 \}.
\]

\( \mathcal{N}_x \) is precisely the subspace of \( T^{1,0}_x(b\Omega) \) consisting of the directions in which the Levi form vanishes. We end this section by defining the holomorphic dimension of a variety sitting in the boundary of the domain \( \Omega \) first as Kohn defined in [17] and then as Catlin defined it in [6]:

Definition (Kohn) 2.12. Let \( \mathcal{V} \) be a variety in \( U \) corresponding to an ideal \( \mathcal{I} \) in \( C^\infty(U) \) such that \( \mathcal{V} \subset b\Omega \). We define the holomorphic dimension of \( \mathcal{V} \) in the sense of Kohn by

\[
\text{hol. dim} \ (\mathcal{V}) = \min_{x \in \mathcal{V}} \dim Z^{1,0}_x(\mathcal{V}) \cap \mathcal{N}_x.
\]

Definition (Catlin) 2.13. Let \( \mathcal{V} \) be a variety in \( U \) corresponding to an ideal \( \mathcal{I} \) in \( C^\infty(U) \) such that \( \mathcal{V} \subset b\Omega \). We define the holomorphic dimension of \( \mathcal{V} \) in the sense of Catlin by

\[
\text{hol. dim} \ (\mathcal{V}) = \max_{x \in \mathcal{V}} \dim Z^{1,0}_x(\mathcal{V}) \cap \mathcal{N}_x.
\]
3. Quasi-flasque sheaves

The material we will present here comes from Jean-Claude Tougeron’s article [21], which he summarizes in section V.6 of his book [22]. We direct the reader more generally to Robin Hartshorne’s book [12] for elementary material on sheaves. We will use Tougeron’s machinery to prove Theorem 1.4 at the end of this section.

Definition 3.1. Let \( \tilde{U} \) be a given non-empty open set in \( \mathbb{C}^n \), and let \( \mathcal{E} \) be the sheaf of \( C^\infty \) germs on \( \tilde{U} \). A sheaf \( \mathcal{M} \) of \( \mathcal{E} \)-modules is called quasi-flasque if for every open set \( U \subset \tilde{U} \) the canonical homomorphism

\[
\mathcal{M}(\tilde{U}) \otimes_{\mathcal{E}(\tilde{U})} \mathcal{E}(U) \longrightarrow \mathcal{M}(U)
\]

is an isomorphism.

Before we can examine a sheaf of self-radical ideals and prove it is quasi-flasque, we have to recall some algebraic results on the ring \( C^\infty \) that have to do with flatness. If \( U \) and \( \tilde{U} \) are two open sets such that \( U \subset \tilde{U} \), then \( C^\infty(U) \) is a flat module over \( C^\infty(\tilde{U}) \). Here flatness has its algebraic meaning that is different from the concept of flatness defined for functions in Section 2. In [22] Jean-Claude Tougeron obtains this flatness property of the ring of smooth functions over the smaller neighborhood as a module over the ring of smooth functions over the large neighborhood as a corollary to the following technical lemma, which appears as Lemma 6.1 in subsection V.6 on pages 113-4:

Lemma 3.2. Let \( U \subset \tilde{U} \) be open, and let \( \{f_i\}_{i \in \mathbb{N}} \) be a countable family of functions in \( C^\infty(U) \). There exists a function \( \alpha \in C^\infty(\tilde{U}) \) such that

1. \( \alpha \equiv 0 \) on \( \tilde{U} - U \) and \( \alpha(x) \neq 0 \) for all \( x \in U \);
2. The functions \( \alpha \cdot f_i \) can be extended as smooth functions \( f'_i \) in \( C^\infty(\tilde{U}) \) such that

\[
f'_i = \alpha \cdot f_i \equiv 0
\]

on \( \tilde{U} - U \).

Remark: We will be working with open sets \( \tilde{U} \) such that \( \overline{\tilde{U}} \) is compact in \( \mathbb{C}^n \), so without loss of generality, we can in fact assume that \( \alpha \) extends smoothly up to the boundary of \( \tilde{U} \), i.e. that \( \alpha \in C^\infty(\overline{\tilde{U}}) \). Since \( \overline{\tilde{U}} \) is compact in \( \mathbb{C}^n \), there exists an open set \( U' \) such that \( \overline{U} \subset U' \). By applying the lemma twice, once to the pair of sets \( U \) and \( \tilde{U} \) and a second time to the pair \( \tilde{U} \) and \( U' \) with the function \( \alpha \) from the first application replacing the family \( \{f_i\}_{i \in \mathbb{N}} \), it becomes clear we can take \( \alpha \in C^\infty(\overline{\tilde{U}}) \).

Corollary (Flatness Property) 3.3. Let \( U \subset \tilde{U} \) be open, then \( C^\infty(U) \) is a flat module over \( C^\infty(\tilde{U}) \).

Proof:

\[
C^\infty(U) = \{f|_U \mid f \in C^\infty(\tilde{U})\}.
\]
To show $C^\infty(U)$ is a flat module over $C^\infty(\bar{U})$, we need to show that the module of relations among elements of $C^\infty(U)$ is generated by the relations on $C^\infty(\bar{U})$. Let $\phi_1, \ldots, \phi_t \in C^\infty(\bar{U})$. We consider any relation among the restrictions of these elements to $C^\infty(U)$, namely let $f_1, \ldots, f_t \in C^\infty(U)$ be such that

$$f_1 \phi_1 + \cdots + f_t \phi_t = 0. $$

Apply the previous lemma, Lemma 3.2, with $f_1, \ldots, f_t$ instead of the countable family $\{f_i\}_{i \in \mathbb{N}}$. Consider $f'_i \in C^\infty(\bar{U})$ given by $f'_i = \alpha \cdot f_i$. Clearly,

$$f'_1 \phi_1 + \cdots + f'_t \phi_t = 0$$

is a relation on $C^\infty(\bar{U})$ that restricts to $C^\infty(U)$ by restricting each function to $U$. Since $\alpha$ is a unit on $C^\infty(U)$ by Lemma 3.2, set $f_i = \frac{1}{\alpha} f_i$ to obtain that the relation with which we started is indeed generated by a relation on $C^\infty(\bar{U})$. □

Another property related to flatness says that a self-radical ideal stays self-radical if its support is shrunk. We prove this result here also as a corollary to Tougeron’s technical lemma, Lemma 3.2.

**Corollary 3.4.** Let $\tilde{U}$ be an open set such that $\overline{U}$ is compact in $\mathbb{C}^n$, and let $I \subset C^\infty(\bar{U})$ be an ideal satisfying $I = \sqrt{I}$, where the real radical is taken on $U$. Let $U$ be any open subset of $\tilde{U}$, then

$$I|_U = \sqrt{I}|_U$$

where the real radical is taken on $U$.

**Proof:** Given any $f / \sqrt{I}|_U$ we want to show that $f \in I|_U$. $f \in \sqrt{I}|_U$ means that there exists $g \in I|_U$ and $m \in \mathbb{N}^*$ such that

$$|f|^m \leq |g|$$

on $U$. Consider the function $\alpha$ from Tougeron’s Lemma 3.2

$$|\alpha|^m |f|^m \leq |\alpha|^m |g|$$

on $U$ since $\alpha \neq 0$ on $U$. It follows that

$$|\alpha f|^m \leq |\alpha|^m |g| = |\alpha^m g|$$

(3.1)

on the larger neighborhood $\tilde{U}$. Clearly, $\alpha^m g \in C^\infty(\bar{U})$. By hypothesis and the remark following Lemma 3.2, $\alpha \in C^\infty(\bar{U})$ so there exists some $M > 0$ such that $|\alpha| \leq M$ on $U$. It follows that

$$|\alpha|^m |g| \leq M^m |\tilde{g}|,$$

where $\tilde{g} \in I$ is such that $\tilde{g}|_U = g$. Thus $\alpha^m g \in \sqrt{I} = I$. From (3.1) we conclude that $\alpha f \in I$, thus $(\alpha f)|_U \in I|_U$. By its very definition, however, $\alpha$ is a unit in $C^\infty(U)$, so $f \in I|_U$. □

Let us now consider an ideal $\mathcal{I} \subset C^\infty(\bar{U})$ satisfying that $\mathcal{I} = \sqrt{\mathcal{I}}$. Clearly $\mathcal{I}$ can be viewed as a $\mathcal{E}$-module on $\tilde{U}$. We would like to show $\mathcal{I}$ produces a quasi-flasque sheaf $\mathcal{J}$ on $\tilde{U}$. The
hypothesis $\mathcal{I} = \sqrt[\mathcal{I}]{\mathcal{I}}$ is essential in establishing that $\mathcal{I}$ is quasi-flasque. By Corollary 3.3 for any $U \subset \tilde{U}$ open, $\mathcal{E}(U)$ is a flat $\mathcal{E}(\tilde{U})$-module, which implies

$$\mathcal{I} \otimes_{\mathcal{E}(\tilde{U})} \mathcal{E}(U) \simeq \mathcal{I} \mathcal{E}(U).$$

$I(U) = \mathcal{I} \otimes_{\mathcal{E}(\tilde{U})} \mathcal{E}(U)$ is then a presheaf, and we let $\mathcal{I}$ be the sheaf of $\mathcal{E}$-ideals generated by the presheaf $I(U)$. We denote by $\Gamma(U, \mathcal{I})$ the sections of $\mathcal{I}$ on $U$. The fact that $\mathcal{I}$ is quasi-flasque follows from the next lemma:

**Lemma 3.5.** If an ideal $\mathcal{I} \subset C^\infty(\tilde{U})$ satisfies that $\mathcal{I} = \sqrt[\mathcal{I}]{\mathcal{I}}$ on $\tilde{U}$ and $\tilde{U}$ is compact in $\mathbb{C}^n$, then for any open $U \subset \tilde{U}$

$$\mathcal{I}(U) \subseteq \Gamma(U, \mathcal{I}) \subseteq \sqrt[\mathcal{I}]{\mathcal{I}(U)} = \mathcal{I}(U) \hookrightarrow \mathcal{E}(U).$$

**Remark:** The hypothesis $\mathcal{I} = \sqrt[\mathcal{I}]{\mathcal{I}}$ is essential. The reader may consult page 708 of [1] for an example of a sheaf $\mathcal{I}$, where $\mathcal{I}(U) \subseteq \Gamma(U, \mathcal{I})$.

**Proof:** Since $\mathcal{I}(U)$ is a presheaf, then clearly $\mathcal{I}(U) \subseteq \Gamma(U, \mathcal{I})$. Consider now any section $s \in \Gamma(U, \mathcal{I}) \subset \mathcal{E}(U)$. For all $x \in U$, $s(x) \in \mathcal{I}_x = \mathcal{I}_x \otimes_{\mathcal{E}(\tilde{U})} \mathcal{E}_x = \mathcal{I}_x \mathcal{E}_x$, where $\mathcal{E}_x$ are the germs of smooth functions at $x$ and $\mathcal{I}_x$ are the germs of functions in $\mathcal{I}$ at $x$. This means that for each $x \in U$, there exists an open neighborhood $V_x$ of $x$ in $U$ such that $\tilde{V}_x \subset U$, $\tilde{V}_x$ is compact, and

$$s|_{V_x} = \sum_{i=1}^{n_x} f^i|_{V_x} g^i|_{V_x},$$

where $f^i \in \mathcal{I}$ and $g^i \in \mathcal{E}(\tilde{U})$ for $i = 1, \ldots, n_x$. Since $\tilde{V}_x$ is compact, let $M_i = \sup_{y \in V_x}|g^i(y)|$. We consider the function $\alpha$ given by Lemma 3.2, which satisfies that $\alpha(y) \neq 0$ for all $y \in V_x$ and $\alpha \equiv 0$ on $U - V_x$. Let $M = \sup_{y \in V_x}|\alpha(y)|$. Clearly, $\alpha s$ can be extended to all of $U$. Furthermore,

$$|\alpha s| \leq \sum_{i=1}^{n_x} |f^i| M_i M$$

on $U$, so $\alpha s \in \sqrt[\mathcal{I}]{\mathcal{I}(U)}$. Now restrict $\alpha s$ to $V_x$, where $\alpha$ is a unit and multiply by $\frac{1}{\alpha}$. Clearly $s \in \sqrt[\mathcal{I}]{\mathcal{I}(U)}$. The same proof as given for Lemma 3.4 also shows that $\sqrt[\mathcal{I}]{\mathcal{I}(U)} = \mathcal{I}(U)$. □

Before we can state the next two corollaries, we have to specify on which type of neighborhood we run the Kohn algorithm, so that we obtain reasonable presheaves $I^q(U_{x_0})$ and $I^q_k(U_{x_0})$.

**Properties of the neighborhood $U_{x_0} 3.6.$** The neighborhood $U_{x_0} \ni x_0$ in which we will run the Kohn algorithm satisfies:

1. $U_{x_0} \subset U$, where $U$ is the neighborhood of $x_0$ described on page [17] of Section 2.
2. $b\Omega \cap U_{x_0}$ has D’Angelo finite type with respect to $q$ dimensional complex varieties;
3. The closure $\overline{U}_{x_0}$ is compact in $\mathbb{C}^n$. 

Corollary 3.7. Let $I^q(U_{x_0})$ be the ideal of subelliptic multipliers on a neighborhood $U_{x_0}$ that satisfies all properties in 3.6. The sheaf $\mathcal{I}^q$ generated by $I^q(U_{x_0})$ is quasi-flasque.

Corollary 3.8. Let $I^q_k(U_{x_0})$ be the ideal of multipliers at step $k$ given by the Kohn algorithm on a neighborhood $U_{x_0}$ that satisfies all properties in 3.6. The sheaf $\mathcal{I}^q_k$ generated by $I^q_k(U_{x_0})$ is quasi-flasque.

We conclude this section by stating Tougeron’s Proposition 6.4 from section V.6 of his book [22], which shows that if $I^q_k(U_{x_0})$ is generated by “global” sections, i.e., by sections $s_j$ defined on the entire neighborhood $U_{x_0}$, then the same will be true of the stalks of the sheaf at all neighboring points that are close enough to $x_0$. This result precisely justifies the approach taken in the paper of working on the presheaf $I^q_k(U_{x_0})$ rather than on the sheaf of multiplier ideals $I^q(U_{x_0})$ given by the Kohn algorithm.

Proposition 3.9. Let $\mathcal{M}$ be a quasi-flasque sheaf on $\tilde{U}$ and let $a$ be a point in $\tilde{U}$. If sections $s_j \in \mathcal{M}(\tilde{U})$ for $j \in J$ generate $\mathcal{M}_a$, then these also generate $\mathcal{M}_x$ for $x$ sufficiently close to $a$.

Proof of Theorem 1.4
Part (a) follows from Corollary 3.7 for $\mathcal{I}^q$ and from Corollary 3.8 for $\mathcal{I}^q_k$ for every $k \geq 1$.
Part (b) follows from Proposition 3.9 for both $\mathcal{I}^q$ and $\mathcal{I}^q_k$ for every $k \geq 1$. □

Remarks on Theorem 1.4:
(1) It stands to reason the self-radical property of the multiplier ideals $I^q(x_0)$ and $I^q_k(x_0)$ for all $k \geq 1$ should be the essential hypothesis in this construction since it is the same for the sheaves of real-analytic multipliers considered by Joseph J. Kohn in Proposition 6.5 on page 111 of [17].
(2) Joseph J. Kohn has an example of a domain $\Omega$ that is strongly pseudoconvex everywhere but at one point $x_0$ and satisfies that it is of infinite D’Angelo type and pseudoconvex at $x_0$. In this case, $I^q(x) = I^q_k(x) = C^\infty(x)$ for all $x \in \overline{\Omega} - \{x_0\}$, but $I^q_k(x_0) \subseteq I^q(x_0) \subsetneq C^\infty(x_0)$ for all $k \geq 1$ because the Kohn algorithm does not terminate and the point has infinite order of contact with holomorphic curves. The sections $s_j$ that generate $I^q(x_0)$ all vanish at $x_0$, but Proposition 3.9 guarantees the same sections generate $I^q(x)$ for $x$ close enough to $x_0$, and at least one $s_j(x)$ is non-zero for every such $x$ and some $j \in J$ since $I^q(x)$ is the entire ring.

4. Catlin’s multitype and boundary systems
This section is devoted to recalling the concepts of boundary system, multitype, and commutator multitype from David Catlin’s paper [9]. Let $x_0 \in b\Omega$. A boundary system
\[ \mathfrak{B}_\nu = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu \} \]
of rank $p$ and codimension $n - \nu$ is a collection of $\mathbb{R}$-valued smooth functions with linearly independent gradients and $(1, 0)$ vector fields in a neighborhood of $x_0$ in $\mathbb{C}^n$. The Levi form has
rank $p$ at $x_0$, $r_1 = r$ is the defining function of the domain, and the other functions $r_{p+2}, \ldots, r_\nu$ are obtained from $r$ by differentiation with respect to the vector fields $L_2, \ldots, L_\nu$ and their conjugates in a certain order that will be described in detail shortly. Both the multitype $M(x_0)$ and the commutator multitype $C(x_0)$ are $n$-tuples of positive rational numbers that will be shown to equal each other and satisfy that their first $\nu$ entries measure how many times $r$ has to be differentiated via $L_2, \ldots, L_\nu$ and their conjugates until the functions $r_{p+2}, \ldots, r_\nu$ are obtained. It should be noted that in order for $b\Omega$ to exist and for both the multitype $M(x_0)$ and the commutator multitype $C(x_0)$ to have only finite entries up to their $\nu^{th}$ one, it is necessary for $b\Omega$ to be of finite $q$ D’Angelo type at $x_0$, where $\nu = n + 1 - q$.

Among the concepts of boundary system, multitype, and commutator multitype, the easiest notion to introduce is that of multitype, so we will follow Catlin in [6] in describing it first. Both $M(x_0)$ and $C(x_0)$ are $n$-tuples of rational numbers, so we need to consider all such $n$-tuples that satisfy certain properties. We will call these weights. We will first define these weights and then describe a subset of weights with even better properties, which we will call the set of distinguished weights.

**Definition 4.1.** Let $\Gamma_n$ denote the set of $n$-tuples of rational numbers $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with $1 \leq \lambda_i \leq +\infty$ satisfying the following two properties:

(i) $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(ii) For each $k$ such that $1 \leq k \leq n$, either $\lambda_k = +\infty$ or there exists a set of integers $a_1, \ldots, a_k$ such that $a_j > 0$ for all $1 \leq j \leq k$ and

$$\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1.$$  

The set $\Gamma_n$ is ordered lexicographically, i.e. given $\Lambda', \Lambda'' \in \Gamma_n$ such that $\Lambda' = (\lambda'_1, \ldots, \lambda'_n)$ and $\Lambda'' = (\lambda''_1, \ldots, \lambda''_n)$, then $\Lambda' < \Lambda''$ if there exists $k$ with $1 \leq k \leq n$ such that $\lambda'_j = \lambda''_j$ for all $j < k$ and $\lambda'_k < \lambda''_k$. The set $\Gamma_n$ is called the set of weights.

**Remarks:**

(1) Requiring the sum to equal 1 at each step $k$ in property (ii) is one of Catlin’s most remarkable ideas as it will enable the truncation of the defining function $r$ with respect to a weight $\Lambda = (\lambda_1, \ldots, \lambda_n)$ in a way that will preserve all terms in the Taylor expansion of $r$ that are essential for producing the functions $r_{p+2}, \ldots, r_\nu$ under differentiation. This is exactly the crucial property for passing from a boundary system to a truncation that stays a boundary system, i.e. for reducing the problem from $C^\infty$ to the polynomial ring. This construction will be described in detail in Section [6].

(2) Let $t \in \mathbb{Q}$ be such that $0 < t < \infty$. The set

$$\{ \Lambda \in \Gamma_n \mid \Lambda \leq (t, t, \ldots, t) \}$$

is finite. If the D’Angelo 1 type $\Delta_1(b\Omega, x_0)$ is finite and equal to $t$ at a point $x_0 \in b\Omega$, then this observation will show there are only finitely many possible values for both the multitype $M(x_0)$ and the commutator multitype $C(x_0)$ at $x_0$ since it is a result due to
Catlin in [6] that the entries of either of these are controlled from above by the D’Angelo type. If the D’Angelo q type $\Delta_q(b\Omega, x_0)$ is finite for $q > 1$, then we get the corresponding result for the $\nu$-tuples $\mathcal{M}^\nu(x_0)$ and $\mathcal{C}^\nu(x_0)$ consisting of the first $\nu$ entries of $\mathcal{M}(x_0)$ and $\mathcal{C}(x_0)$ respectively for $\nu = n + 1 - q$.

Let us now define distinguished weights and the multitype $\mathcal{M}(x_0)$:

**Definition 4.2.** Let $\Omega \subset \mathbb{C}^n$ be a smooth domain with defining function $r$. A weight $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n$ is called distinguished if there exist holomorphic coordinates $(z_1, \ldots, z_n)$ around $x_0$ such that

(i) $x_0$ is mapped at the origin;

(ii) If $\sum_{i=1}^n \frac{\alpha_i + \beta_i}{\lambda_i} < 1$, then $D^\alpha D^\beta r(0) = 0$, where $D^\alpha = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$ and $D^\beta = \frac{\partial^{\beta_1}}{\partial z_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial z_n^{\beta_n}}$.

We will denote by $\tilde{\Gamma}_n(x_0)$ the set of distinguished weights at $x_0$.

**Remark:** Property (ii) in Definition 4.1 and property (ii) in Definition 4.2 taken together show that the underlying idea of this setup is to measure the order of vanishing of the defining function $r$ in various directions. It turns out this measure is a weight in $\Gamma_n$ called the multitype. We clearly have to measure the vanishing of $r$ in a way that is independent of the local coordinates chosen. This precisely justifies the wording of the next definition.

**Definition 4.3.** The multitype $\mathcal{M}(x_0)$ is defined to be the smallest weight in lexicographic sense $\mathcal{M}(x_0) = (m_1, \ldots, m_n)$ such that $\mathcal{M}(x_0) \geq \Lambda$ for every distinguished weight $\Lambda \in \tilde{\Gamma}_n(x_0)$.

We will now state the main theorem of Catlin’s paper [6] from page 531 that summarizes the properties of the multitype $\mathcal{M}(x_0)$:

**Theorem 4.4.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary. Let $x_0 \in b\Omega$. The multitype $\mathcal{M}(x_0)$ has the following properties:

1. $\mathcal{M}(x_0)$ is upper semi-continuous with respect to the lexicographic ordering, i.e. there exists a neighborhood $U \ni x_0$ such that for all $x \in U \cap b\Omega$, $\mathcal{M}(x) \leq \mathcal{M}(x_0)$.

2. If $\mathcal{M}(x_0) = (m_1, \ldots, m_n)$ satisfies that $m_{n-q} < \infty$, then there exist a neighborhood $U \ni x_0$ and a submanifold $M$ of $U \cap b\Omega$ of holomorphic dimension at most $q$ in the sense of Catlin such that $x_0 \in M$ and the level set of $\mathcal{M}(x_0)$ satisfies

$$\{x \in U \cap b\Omega \mid \mathcal{M}(x) = \mathcal{M}(x_0)\} \subset M.$$ 

3. If $\mathcal{M}(x_0) = (m_1, \ldots, m_n)$, then there exist coordinates $(z_1, \ldots, z_n)$ around $x_0$ such that $x_0$ is mapped to the origin and if $\sum_{i=1}^n \frac{\alpha_i + \beta_i}{m_i} < 1$, then $D^\alpha D^\beta r(0) = 0$. If one of the entries $m_i = +\infty$ for some $1 \leq i \leq n$, then these coordinates should be interpreted in the sense of formal power series.
If $M(x_0) = (m_1, \ldots, m_n)$, then for each $q = 1, \ldots, n$,

$$m_{n+1-q} \leq \Delta_q(b\Omega, x_0),$$

where $\Delta_q(b\Omega, x_0)$ is the D’Angelo $q$ type of the point $x_0$, i.e. the maximum order of contact of varieties of complex dimension $q$ with the boundary of $\Omega$ at $x_0$.

It is clear that Definition 4.3 does not specify a procedure for computing $M(x_0)$ for a domain $\Omega$ at the boundary point $x_0$. Instead, Catlin defined another weight $C(x_0) \in \Gamma_n$ called the commutator multitype, which he proceeded to compute by differentiating $r$. In the process of computing $C(x_0)$, Catlin came up with the definition of a boundary system. He then showed that $C(x_0) = M(x_0)$. We will now explain his construction of the commutator multitype $C(x_0)$ and of a boundary system $B_\nu(x_0) = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu\}$.

The commutator multitype $C(x_0) = (c_1, \ldots, c_n) \in \Gamma_n$ always satisfies that $c_1 = 1$ because as explained on page 10, $L_1(r) = 1$, which comes from the fact that $r$ describes a manifold and thus its gradient has a non-zero component in the normal direction. This means only one differentiation of $r$ in the direction of $L_1$ suffices to produce a non-vanishing function at $x_0$, hence $c_1 = 1$. We set $r_1 = r$. Next, suppose that the Levi form of $b\Omega$ at $x_0$ has rank equal to $p$. In this case, set $c_i = 2$ for $i = 2, \ldots, p + 1$. Without loss of generality, we can choose in the construction on page 10 the smooth vector fields of type $(1, 0)$ $L_2, \ldots, L_{p+1}$ such that $L_i(r) = \partial r(L_i) \equiv 0$ and the $p \times p$ Hermitian matrix $\partial \bar{\partial} r(L_i, L_j)(x_0)$ for $2 \leq i, j \leq p + 1$ is nonsingular. The reader should note that round parentheses stand for the evaluation of forms on vector fields. If $p + 1 \geq \nu$, we have finished the construction of both the boundary system $B_\nu(x_0)$ and of the commutator multitype $C(x_0)$.

If $p + 1 < \nu$, we need to explain next how the rest of the vector fields $L_{p+2}, \ldots, L_\nu$ and the functions $r_{p+2}, \ldots, r_\nu$ are chosen in the boundary system $B_\nu(x_0)$. Let us consider the $(1, 0)$ smooth vector fields in the kernel of the Levi form at $x_0$. We thus denote by $T_{p+2}^{(1,0)}$ the bundle consisting of $(1, 0)$ vector fields $L$ such that $\partial r(L) = 0$ and $\partial \bar{\partial} r(L, L_j) = 0$ for $j = 2, \ldots, p + 1$. We follow Catlin in passing to the set of germs of sections of $T_{p+2}^{(1,0)}$, which we will denote by $T_{p+2}$. Germs are not technically necessary for defining a boundary system and the commutator multitype, but they become essential later on when Catlin describes truncated boundary systems because one can then pick a representative in the equivalence class of a germ given by a vector field with polynomial coefficients. This construction will be mentioned again in Section 5.

All the directions in which the defining function vanishes up to order 2 have already been identified. It is thus clear we have to consider next lists of vector fields of length at least 3. Let $l \in \mathbb{N}$ be such that $l \geq 3$. Denote by $L$ a list of vector fields $L = \{L^1, \ldots, L^l\}$ such that there is a fixed, non-vanishing vector field $L \in T_{p+2}^{(1,0)}$ and $L^j = L$ or $L^j = \bar{L}$ for all $1 \leq i \leq l$. Let $L\partial r$ be the function

$$L\partial r(x) = L^1 \cdots L^{l-2} \partial r ([L^{l-1}, L^l])(x)$$

for $x \in b\Omega$. The reader should note that if both $L^{l-1}$ and $L^l$ are $L$ or both of them are $\bar{L}$, then the commutator $[L^{l-1}, L^l]$ vanishes. Thus, to have any chance of obtaining a non-vanishing
function as $\mathcal{L}\partial r$, one of $L^{l-1}$ and $L^l$ should be $L$ and the other one $\bar{L}$. Therefore, $\mathcal{L}\partial r(x_0)$ measures the vanishing order of the diagonal entry of the Levi form at $x_0$ corresponding to $L$. We distinguish two cases:

**Case 1:** If $\mathcal{L}\partial r(x_0) = 0$ for every such list $\mathcal{L}$, then set $c_{p+2} = \infty$. Given that weights are increasing $n$-tuples of rational numbers, it follows $c_i = \infty$ for all $i = p + 2, \ldots, n$. We have finished the construction of both the boundary system $\mathcal{B}_v(x_0)$ and of the commutator multitype $\mathcal{C}(x_0)$.

**Case 2:** There exists at least one list $\mathcal{L}$ such that $\mathcal{L}\partial r(x_0) \neq 0$. Among all lists with this property, we choose one list for which the length $l$ is the smallest. Clearly, there might exist more than one list of smallest length, but the entries $c_i$ of the commutator multitype $\mathcal{C}(x_0)$ will turn out to be independent of the choice made here. Set $c_{p+2} = l$, where $l$ is this smallest value of the length of the list. Let $\mathcal{L}_{p+2} = \{L^1, \ldots, L^l\}$ be the list chosen whose length satisfies $l = c_{p+2}$ and let $L \in T_{p+2}$ be the germ of the fixed vector field in $T_{p+2}$ such that $L^i = L$ or $L^i = \bar{L}$ for all $1 \leq i \leq l$. Define functions $f$ and $g$ by

$$f(x) = \text{Re}\{L^2 \cdots L^{l-2} \partial_r ([L^{l-1}, L^i])(x)\}$$

and

$$g(x) = \text{Im}\{L^2 \cdots L^{l-2} \partial_r ([L^{l-1}, L^i])(x)\}.$$ 

Since $l \geq 3$, the definitions of $f$ and $g$ make sense. We define $\mathbb{R}$-valued vector fields $X$ and $Y$ such that $L = X + iY$. Since $l$ was chosen to be minimal, it follows that $f(x_0) = g(x_0) = 0$, but $L^1(f + ig)(x_0) \neq 0$. This implies at least one of $Xf(x_0)$, $Xg(x_0)$, $Yf(x_0)$, and $Yg(x_0)$ does not vanish. Without loss of generality, let $Xf(x_0) \neq 0$. We set $r_{p+2}(x) = f(x)$ and $L_{p+2} = L$, the vector field used in constructing the list $\mathcal{L}_{p+2}$. It follows that $L_{p+2}(r_{p+2})(x_0) \neq 0$. This concludes the second case and thus the construction at step $p + 2$.

We proceed inductively. Assume that for some integer $\nu - 1$ with $p + 2 \leq \nu - 1 < n$ we have already constructed finite positive numbers $c_1, \ldots, c_{\nu-1}$ as well as functions $r_1$, $r_{p+2}, \ldots, r_{\nu-1}$ and vector fields $L_2, \ldots, L_{\nu-1}$. Let $T^{(1,0)}_\nu$ denote the set of $(1, 0)$ smooth vector fields $L$ such that $\partial \partial_r(L, L_j) = 0$ for $j = 2, \ldots, p + 1$ and $L(r_k) = 0$ for $k = 1, p + 2, p + 3, \ldots, \nu - 1$. Let $T_\nu$ be the set of germs of sections of $T^{(1,0)}_\nu$. For each $k = p + 2, \ldots, \nu - 1$, $L_k \in T_k$ and $L_k(r_k)(x_0) \neq 0$, which implies that $T^{(1,0)}_\nu$ is a subbundle of $T^{(1,0)}(b\Omega)$ of dimension $n + 1 - \nu$ because the vector fields $L_2, \ldots, L_{\nu-1}$ were chosen to be linearly independent. We have to describe next the list $\mathcal{L}$ for which we will compute $\mathcal{L}\partial r(x)$. We are allowed to use both vector fields from among $L_{p+2}, \ldots, L_{\nu-1}$ as well as vector fields in $T_\nu$. Thus, we fix some vector field $L$ in $T_\nu$ and consider the list $\mathcal{L} = \{L, \ldots, L_i\}$ such that each $L_i$ is one of the vector fields from the set $\{L_{p+2}, \bar{L}_{p+2}, \ldots, L_{\nu-1}, \bar{L}_{\nu-1}, L, \bar{L}\}$. Let $l_i$ denote the total number of times both $L_i$ and $\bar{L}_i$ occur in $\mathcal{L}$ for $p + 2 \leq i \leq \nu - 1$ and let $l_\nu$ denote the total number of times both $L$ and $\bar{L}$ occur in the list $\mathcal{L}$. We now introduce two definitions that pertain to the list $\mathcal{L}$ and explain their significance:

**Definition 4.5.** A list $\mathcal{L} = \{L^1, \ldots, L^l\}$ is called ordered if

(i) $L^j = L$ or $L^j = \bar{L}$ for $1 \leq j \leq l_\nu$. 

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(ii) $L^j = L_i$ or $L^j = \bar{L}_i$ for $1 + \sum_{k=i+1}^{\nu} l_k \leq j \leq \sum_{k=i}^{\nu} l_k$.

Remarks:

(1) Part (i) says that the differentiation with respect to the extra vector field $L \in T_\nu$ or its conjugate should be done outside of any differentiation with respect to the previously chosen vector fields $L_{p+2}, \ldots, L_{\nu-1}$.

(2) Part (ii) of the definition says that if we look from left to right at the list $L$, we should have differentiation with respect to $L$ or $\bar{L}$, then differentiation with respect to $L_{\nu-1}$ or its conjugate if it takes place at all, then differentiation with respect to $L_{\nu-2}$ or its conjugate if it takes place, and so on. Therefore, we differentiate $r$ with respect to previously chosen vector fields inside and with respect to the more recently chosen vector fields outside. Not allowing vector fields to mix makes vanishing orders of $r$ easier to understand and exploit.

Definition 4.6. A list $\mathcal{L} = \{L^1, \ldots, L^l\}$ is called $\nu$-admissible if

(i) $l_\nu > 0$;

(ii) $\sum_{i=p+2}^{\nu-1} l_i < 1$, where $\mathcal{C}^{\nu-1} = (c_1, \ldots, c_{\nu-1})$ is the $(\nu - 1)^{th}$ commutator multitype.

Remarks:

(1) It is obvious that condition (i) should be imposed because if we do not differentiate with respect to the new vector field $L \in T_\nu$ or its conjugate, then we cannot expect to obtain anything that we did not already have by step $\nu - 1$.

(2) Condition (ii) follows from the minimality of the length of the lists chosen at the previous steps. In other words, if we strip away the first $l_\nu$ vector fields from $\mathcal{L}$ and look at $\mathcal{L}' = \{L^{\nu+1}, \ldots, L^l\}$, then this is a list that appeared at one of the previous steps, so we know $\mathcal{L}' \partial r(x_0) = 0$ if (ii) holds.

(3) It also makes perfect sense that lists with property (ii) should be considered since we are trying to construct a commutator multitype $\mathcal{C}(x_0)$ such that $\mathcal{C}(x_0) = \mathcal{M}(x_0)$, and the multitype $\mathcal{M}(x_0)$ is defined as the weight that dominates all distinguished weights.

We now consider only $\nu$-admissible, ordered lists $\mathcal{L}$ and distinguish two cases:

**Case 1:** For all such lists $\mathcal{L}$, $\mathcal{L} \partial r_1(x_0) = 0$. In this case, we set $c_\nu = \infty$. It follows that $c_i = \infty$ for all $\nu \leq i \leq n$. We have finished the construction of both the boundary system $\mathcal{B}_\nu(x_0)$ and of the commutator multitype $\mathcal{C}(x_0)$. 

Case 2: There exists at least one such list $L$ for which $L\partial r_1(x_0) \neq 0$. We would like to choose the list $L$ with minimal length just as before. Let $c(L)$ denote the solution to the equation

$$\sum_{i=p+2}^{\nu-1} \frac{l_i}{c_i} + \frac{l_{\nu}}{c(L)} = 1.$$ 

Because $L$ is $\nu$-admissible and thus satisfies condition (ii) of Definition 4.6

$$1 - \sum_{i=p+2}^{\nu-1} \frac{l_i}{c_i} > 0,$$

and $1 - \sum_{i=p+2}^{\nu-1} \frac{l_i}{c_i}$ is a rational number since all entries $c_i$ are rational and all numbers $l_i$ are positive integers. $l_{\nu}$ is also a positive integer, so the solution $c(L)$ has to be a positive rational number. Set

$$c_{\nu} = \inf \{ c(L) \mid L \text{ is } \nu \text{-admissible, ordered, and satisfies } L\partial r_1(x_0) \neq 0 \}.$$ 

If there exists more than one such list for which $c(L)$ reaches the infimum, we make an arbitrary choice and denote it by $L_\nu = \{ L^1, \ldots , L^j \}$. Next, we set $L'_\nu = \{ L^2, \ldots , L^l \}$, and define $\mathbb{R}$-valued vector fields $X$ and $Y$ such that $L^1 = X + iY$. Just as before, we let functions $f$ and $g$ be defined by $f(x) = Re\{ L'_\nu \partial r_1(x) \}$ and $g(x) = Im\{ L'_\nu \partial r_1(x) \}$. The minimality of the length of $L_\nu$, as well as of vector fields $X$ and $Y$, follows from the minimality of $c_\nu$. In the case $l_{\nu} > 0$, this follows from the minimality of the previously chosen $c_{\nu+2}, \ldots , c_{\nu-1}$.

(1) If $L = \{ L^1, \ldots , L^j \}$ is any ordered list such that $L^i = L_j$ or $L^i = \bar{L}_j$ for all $1 \leq i \leq l$ and all $p + 2 \leq j \leq \nu$ and if $\sum_{i=p+2}^{\nu} \frac{1}{c_i} < 1$, then $L\partial r_1(x_0) = 0$. In the case $l_{\nu} = 0$, this follows from the minimality of $c_{\nu}$. In the case $l_{\nu} > 0$, this follows from the minimality of the previously chosen $c_{\nu+2}, \ldots , c_{\nu-1}$.

(2) The construction of the boundary system $\mathfrak{B}_\nu(x_0) = \{ r_1, r_{p+2}, \ldots , r_{\nu}; L_2, \ldots , L_\nu \}$ depends on the choices of lists $L_{p+2}, \ldots , L_\nu$ as well as of vector fields $L_{p+2}, \ldots , L_\nu$ in these lists. Fortunately, by minimality of the lengths of the lists chosen, the commutator multitype $\mathcal{C}(x_0) = (c_1, \ldots , c_n)$ is the same regardless of these choices. The algebraic constructions we will embark upon shortly are likewise not affected by these choices.

We shall call a collection

$$\mathfrak{B}_\nu(x_0) = \{ r_1, r_{p+2}, \ldots , r_{\nu}; L_2, \ldots , L_\nu \}$$

of functions and vector fields a boundary system of rank $p$ and codimension $n - \nu$ if it is obtained by the procedure described above. The vector fields $L_2, \ldots , L_\nu$ are called the special vector fields associated to the boundary system $\mathfrak{B}_\nu$. The $\nu^{th}$ commutator multitype of the boundary
system $\mathcal{B}_\nu$ is the $\nu$-tuple $\mathcal{C}^\nu = (c_1, \ldots, c_\nu)$. We summarize in the next theorem two of the most important properties of $\mathcal{C}^\nu$, which are contained in Proposition 2.1 on page 536 and Theorem 2.2 on page 538 of Catlin’s paper [6]:

**Theorem 4.7.** Let the domain $\Omega = \{z \in \mathbb{C}^n \mid r(z) < 0\}$ be pseudoconvex in a neighborhood of a point $x_0 \in \partial \Omega$. The $\nu^{th}$ commutator multitype $\mathcal{C}^\nu = (c_1, \ldots, c_\nu)$ of the boundary system $\mathcal{B}_\nu$ satisfies the following two properties:

(i) $\mathcal{C}^\nu$ is upper semi-continuous with respect to the lexicographic ordering, i.e. there exists a neighborhood $U \ni x_0$ such that for all $x \in U \cap \partial \Omega$, $\mathcal{C}^\nu(x) \leq \mathcal{C}^\nu(x_0)$.

(ii) $\mathcal{C}^\nu(x_0) = \mathcal{M}^\nu(x_0)$, where $\mathcal{M}^\nu = (m_1, \ldots, m_\nu)$ consists of the first $\nu$ entries of the multitype $\mathcal{M} = (m_1, \ldots, m_n)$.

**Remark:** Pseudoconvexity is irrelevant for part (i) but essential for part (ii) of this theorem.

We conclude this section with a lemma that is easy to derive from Catlin’s construction of a boundary system explained above and yet crucial for the proof of both Theorem 1.2 and Theorem 1.3.

**Lemma 4.8.** Let the domain $\Omega = \{z \in \mathbb{C}^n \mid r(z) < 0\}$ be pseudoconvex in a neighborhood of a point $x_0 \in \partial \Omega$ and of finite D’Angelo $q$ type. Let

$$
\mathcal{B}_{n+1-q}(x_0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}
$$

be a boundary system of rank $p$ and codimension $n - (n + 1 - q) = q - 1$ at $x_0$, then

$$
\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}(x_0) \neq 0.
$$

**Remark:** We have not yet shown that $r_{p+2}, \ldots, r_{n+1-q}$ are multipliers in the ideal $I_k^q(x_0)$ at step $k$ of the Kohn algorithm. Once we do so, then

$$
\text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}\} \in I_k^q(x_0)
$$

because $n + 1 - q - (p + 1) + p = n - q$, so coeff\{$\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}$\} will be a unit in the ideal $I_k^q(x_0)$. Thus the Kohn algorithm will terminate at step $k + 1$.

**Proof:** Having finite D’Angelo $q$ type at $x_0$ allows us to conclude that the full set of functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ and of vector fields $L_2, \ldots, L_{n+1-q}$ can be constructed in the boundary system $\mathcal{B}_{n+1-q}(x_0)$, so the $(n + 1 - q)^{th}$ commutator multitype $\mathcal{C}^{n+1-q} = (c_1, \ldots, c_{n+1-q})$ has only finite entries by part (4) of Theorem 4.4 and part (ii) of Theorem 4.7. As explained on page 20 for $i = 1, p + 2, p + 3, \ldots, n + 1 - q$ and $j = 2, \ldots, n + 1 - q$, it holds that

$$
L_j(r_i) \begin{cases} 
= 0 & \text{if } j > i \\
\neq 0 & \text{if } j = i \\
\text{no information} & \text{if } j < i
\end{cases}
$$

The vector fields $L_2, \ldots, L_{n+1-q}$ are linearly independent and belong to $T^{(1,0)}(b\Omega \cap U)$ for $U \supset x_0$ an open set around $x_0$. Let us complete these to a basis of $T^{(1,0)}(b\Omega)$ in accordance with the setup on page 10. Since the imaginary part of $L_1$ is $T$ and its real part is the normal direction
to $b\Omega$ as explained on page \[10\] it follows that at $x_0$ the wedge product $\partial r \wedge \bar{\partial} r \wedge (\bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}$ is given by the determinants of all the $(n-q) \times (n-q)$ minors of the $(n-1) \times (n-q)$ matrix:

$$
\begin{pmatrix}
A_p & 0 \\
* & B
\end{pmatrix},
$$

where $A_p$ is a $p \times p$ nonsingular matrix coming from the fact that the Levi form has rank $p$ at $x_0$, $0$ is a $(n-1-p) \times p$ matrix of all zero entries, $*$ is a $p \times (n-q-p)$ matrix for which we have no information, and $B$ is the following lower triangular $(n-1-p) \times (n-q-p)$ matrix:

$$
\begin{pmatrix}
L_{p+2}(r_{p+2}) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
* & L_{p+3}(r_{p+3}) & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & L_{n-q}(r_{n-q}) & 0 & 0 & \cdots & 0 \\
* & * & \cdots & * & L_{n+1-q}(r_{n+1-q}) & 0 & 0 & 0
\end{pmatrix}
$$

Here $*$ denotes an entry for which we have no information, and the right side block of zero entries occurs only if $q > 1$. Given the location of the zero entries, it is clear that the wedge product

$$(\partial r \wedge \bar{\partial} r \wedge (\bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q})(x_0) = (\det A_p) L_{p+2}(r_{p+2}) \cdots L_{n+1-q}(r_{n+1-q}),$$

all of which are nonzero by construction. \qed

5. Togeran Elements and the Stratification Theorem

The aim of this section is to define Togeran elements, explain the relationship between Togeran elements and Catlin’s functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ in a boundary system $\mathcal{B}_{n+1-q}$, and then prove a stratification theorem. The latter will relate the zero set of the ideal of multipliers at the first step of the Kohn algorithm $\mathcal{V}(I(U))$, out of which we remove certain level sets of the Catlin multitype, to the vanishing set of functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ in the boundary system $\mathcal{B}_{n+1-q}$. Using a result of Catlin in \[6\], this stratification theorem can be reinterpreted as a generalization for smooth pseudoconvex domain of a theorem proven by Klas Diederich and John Erik Fornæss in \[10\] for real-analytic pseudoconvex domains, which Joseph J. Kohn used in a crucial way to complete the proof of the equivalence of types in the $C^\infty$ case in \[17\].

Varieties corresponding to ideals in the ring of smooth functions are notoriously ill-behaved, but in some special circumstances, they can have reasonable properties. The first to prove that a variety corresponding to an ideal of smooth functions has an open and dense set of smooth points, was René Thom in \[20\] under the hypothesis that the ideal $\mathcal{I} \subset C^\infty$ is Lojasiewicz. In his proof, René Thom employed Jacobian extensions, which are very similar to the way the Kohn algorithm proceeds. Jean-Claude Togeran gave a much simpler proof of Thom’s result in his book \[22\], this result being Proposition 4.6 of subsection V.4. Jean-Claude Togeran’s construction is significant because it employs certain functions, which we will call Togeran elements in his honor, that vanish on the variety but have non-zero gradients. These Togeran
elements are the crucial objects needed to understand precisely how the Kohn algorithm works since working with the wedge products of their gradients, which are non-vanishing, is much easier than working with the usual wedge products in the Kohn algorithm whose vanishing orders may be much harder to ascertain.

Incidentally, if a variety corresponding to an ideal in the ring of smooth functions has an open and dense set of smooth points, its behavior from an algebraic standpoint is quite close to that of a variety corresponding to an ideal in \( C^\infty \). It should be noted here that the presence of Lojasiewicz inequalities in an ideal immediately leads to the conclusion that the variety corresponding to that ideal has an open and dense set of smooth points. Thus, all varieties corresponding to ideals in \( \mathcal{O}, C^\infty \), and all Denjoy-Carleman quasianalytic rings considered by Bierstone and Milman in \( [3] \) have this property.

We shall now define Tougeron elements. Since these elements are \( \mathbb{R} \)-valued smooth functions, the reader should note that if \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \) is a multi-index, then by \( D^k \) it will be meant the differentiation \( \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \).

**Definition 5.1.** Consider \( f \in C^\infty(\mathbb{R}^n) \) and \( V = \{ x \in \mathbb{R}^n \mid f(x) = 0 \} = \mathcal{V}(f) \). If \( x_0 \in V \) and there exist an open set \( U \subset \mathbb{R}^n \), \( x_0 \in U \), and a multi-index \( k \in \mathbb{N}^n \) with \( |k| = d \geq 1 \) such that \( D^k f(x_0) \neq 0 \) but \( D^h f(x) = 0 \) for all \( h \in \mathbb{N}^n \) such that \( |h| < |k| = d \) and all \( x \in U \cap V \), then we call \( g(x) = D^k f(x) \) a Tougeron element corresponding to \( f(x) \) provided that the multi-index \( k' \in \mathbb{N}^n \) is such that \( |k'| = d - 1 \) and \( \frac{\partial}{\partial x_j} D^k f(x_0) = D^k f(x_0) \neq 0 \) for some \( 1 \leq j \leq n \).

**Remarks:**

1. By construction, the gradient of the Tougeron element \( g(x) \) satisfies that \( \nabla g(x_0) \neq 0 \), so there exists a perhaps smaller open set \( \bar{U} \subset U \) with \( x_0 \in \bar{U} \) such that \( \mathcal{V}(g) \cap \bar{U} \) is a hypersurface, where \( \mathcal{V}(g) \) is the vanishing set of \( g(x) \).

2. Objects similar to the one defined above may have been used by Whitney and others before Tougeron in \( [22] \), but we make use of this construction here in a manner so similar to Tougeron’s that it is only fair this object should bear his name.

Tougeron elements are \( \mathbb{R} \)-valued smooth functions, whereas Catlin’s setup in \( [6] \) and Kohn’s setup in \( [17] \) involve \( \mathbb{C} \)-valued smooth functions. Furthermore, in both \( [6] \) and \( [17] \), the differentiations are done with respect to \( (1, 0) \) and \( (0, 1) \) vector fields in the tangent space of \( b\Omega \) only, so obviously Definition 5.1 does not apply verbatim. Furthermore, Catlin’s construction of a boundary system \( \mathfrak{B}_\nu(x_0) \) involves differentiating only one function, namely the defining function \( r \), in such a way that the object that counts vanishing orders \( \mathfrak{M}(x_0) = \mathfrak{C}(x_0) \) controls all distinguished weights. A quick comparison of Definition 4.2 with Definition 5.1 shows that the former aims at a behavior in which differentiations are ordered as in Definition 4.5 whereas the latter insists that \( D^h f(x) = 0 \) for all \( h \in \mathbb{N}^n \) such that \( |h| < |k| = d \), i.e. it insists on vanishing of derivatives of all lower orders. Still, Tougeron’s setup was the author’s motivation for looking at Catlin’s boundary system construction in \( [6] \) with the aim of finding distinguished elements that vanished on the varieties corresponding to the ideals \( I^q_k(x_0) \) in the Kohn algorithm and had nonzero gradients. We shall thus call functions \( r_1, r_{p+2}, \ldots, r_{n+1-q} \) in
the boundary system $\mathfrak{B}_{n+1-q}(x_0)$ Tougeron elements for the Kohn algorithm. A consequence of Lemma 1.8 in the previous section is that functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ have linearly independent, nonzero gradients. We now have to show that indeed $r_1, r_{p+2}, \ldots, r_{n+1-q}$ vanish on the variety corresponding to the first ideal of multipliers $I^1(U)$, provided we take out of it certain level sets of the $(n+1-q)^{th}$ commutator multitype $\mathcal{C}^{n+1-q}$ and $U$ is an appropriately small neighborhood around $x_0$. This result is precisely the stratification theorem mentioned at the very beginning of this section.

We start the discussion of the stratification theorem by stating Proposition 2.1 on page 536 of Catlin’s paper [6] strengthened in an obvious manner. The differences between this statement and Catlin’s original statement will be outlined in a remark following the proposition.

**Proposition 5.2.** Let $\mathfrak{B}_\nu$ for $p + 2 \leq \nu \leq n$ be a boundary system of rank $p$ and codimension $n - \nu$ in a neighborhood of a given boundary point $x_0$. There exists a neighborhood $U$ of $x_0$ such that all the following conditions are satisfied on its closure $\overline{U}$:

(i) For all $x \in \overline{U} \cap b\Omega$, $\mathcal{C}^{\nu}(x) \leq \mathcal{C}^{\nu}(x_0)$, where $\mathcal{C}^{\nu} = (c_1, \ldots, c_\nu)$ is the $\nu^{th}$ commutator multitype;

(ii) $M^{\nu} = \{x \in \overline{U} \cap b\Omega \mid r_j(x) = 0, \ j = 1, p + 2, \ldots, \nu\}$ is a submanifold of $\overline{U} \cap b\Omega$ of holomorphic dimension $n - \nu$ in the sense of Catlin;

(iii) The level set of the commutator multitype at $x_0$ satisfies that

$$\{x \in \overline{U} \cap b\Omega \mid \mathcal{C}^{\nu}(x) = \mathcal{C}^{\nu}(x_0)\} \subset M^{\nu};$$

(iv) For all $x \in \overline{U} \cap b\Omega$, the Levi form has rank at least $p$ at $x$;

(v) For all $x \in \overline{U} \cap b\Omega$, $\mathcal{L}_j \partial r_1(x) \neq 0$ for all $j = p + 2, \ldots, \nu$, where $\mathcal{L}_{p+2}, \ldots, \mathcal{L}_{\nu}$ are the $\nu$-admissible, ordered lists used in defining the boundary system $\mathfrak{B}_\nu$.

**Remark:** The difference between this statement and Catlin’s original Proposition 2.1 in [6] is in shrinking $U$ such that all properties hold on the closure of $U$ in $b\Omega$, $\overline{U}$. Parts (i)-(iii) hold for a given neighborhood as shown by Catlin, so they will clearly hold on any smaller neighborhood of $x_0$. Catlin proved that properties in (iv) and (v) hold at $x_0$. These are open conditions, however, and there are only finitely many of them, so it is obvious the neighborhood $U$ can be shrunk, if necessary, so that they hold on the closure $\overline{U}$ of the shrunken neighborhood. Furthermore, note that condition (v) implies that the gradients of the functions $r_1, r_{p+2}, \ldots, r_{\nu}$ are nonzero on $\overline{U}$ and linearly independent, which makes $M^{\nu}$ a manifold as stated in (ii).

The study of level sets of the $(n+1-q)^{th}$ commutator multitype $\mathcal{C}^{n+1-q}$ yields very interesting conclusions in the neighborhood of a point $x_0$ of D’Angelo finite type $q$. As D’Angelo proved in [8], the finiteness of the D’Angelo type is an open condition, so we can shrink the neighborhood $\overline{U}$ from Proposition 5.2 to a neighborhood $\tilde{U}$, where $x_0 \in \tilde{U}$ and $\tilde{U} \subset U$, so that the D’Angelo type is finite at all $x \in \tilde{U}$. By remark 1.2 on page 532 of Catlin’s paper [6], which is also the content of remark 2 following Definition 4.1, the $(n + 1 - q)^{th}$ commutator multitype
\( \mathcal{C}^{n+1-q} \) can assume only finitely many values \( \mathcal{C}_1^{n+1-q}, \ldots, \mathcal{C}_N^{n+1-q} \) at all points in \( \tilde{U} \cap b\Omega \), where \( \mathcal{C}_1^{n+1-q} < \mathcal{C}_2^{n+1-q} < \cdots < \mathcal{C}_N^{n+1-q} \) and \( N \) is some positive natural number, \( N \geq 1 \). Let 
\[
S_j = \{x \in \tilde{U} \cap b\Omega \mid \mathcal{C}_j^{n+1-q}(x) = \mathcal{C}_j^{n+1-q}\}
\]
be the level sets of the \( (n+1-q) \)th commutator multitype for \( 1 \leq j \leq N \). Note that unlike in Proposition 5.2 we are working here with the open set \( \tilde{U} \) and not its closure. Clearly,
\[
\tilde{U} \cap b\Omega = \bigcup_{j=1}^{N} S_j.
\]
For pseudoconvex smooth domains \( \Omega \), we seek to show that \( S_1 \), the level set of the lowest \( (n+1-q) \)th commutator multitype, is an open set in the induced topology on \( b\Omega \).

**Lemma 5.3.** Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex smooth domain, and let \( x_0 \in b\Omega \) be a boundary point of finite \( D'\text{Angelo} \) \( q \) type such that the \( (n+1-q) \)th commutator multitype \( \mathcal{C}_n^{n+1-q}(x_0) = (1, 2, \ldots, 2) \), then there exists a neighborhood \( U' \) of \( x_0 \) such that for all \( x \in U' \cap b\Omega \), \( \mathcal{C}_n^{n+1-q}(x) = \mathcal{C}_n^{n+1-q}(x_0) \).

**Remark:** We will provide two proofs for this lemma.

**Proof 1:** \( \mathcal{C}_n^{n+1-q}(x_0) = (1, 2, \ldots, 2) \) means that the boundary system at \( x_0 \) is given by
\[
\mathfrak{B}_{n+1-q}(x_0) = \{r_1; L_2, \ldots, L_{n+1-q}\}
\]
since the Levi form has rank at least \( n + 1 - q - 1 = n - q \) at \( x_0 \). In the language given by the Kohn algorithm, this condition can be restated as \( \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}(x_0) \neq 0 \), which is an open condition, hence it holds on some neighborhood \( U' \) of \( x_0 \), i.e. on \( U' \cap b\Omega \).

**Proof 2:** The \( (n+1-q) \)th commutator multitype \( \mathcal{C}_n^{n+1-q} \) is upper semi-continuous with respect to the lexicographic ordering, i.e. there exists a neighborhood \( U' \ni x_0 \) such that for all \( x \in U' \cap b\Omega \), \( \mathcal{C}_n^{n+1-q}(x) \leq \mathcal{C}_n^{n+1-q}(x_0) \), but \( \mathcal{C}_n^{n+1-q} \) is the lowest possible \( (n+1-q) \)th commutator multitype. This implies \( \mathcal{C}_n^{n+1-q}(x) = \mathcal{C}_n^{n+1-q}(x_0) \) for all \( x \in U' \cap b\Omega \).

**Corollary 5.4.** Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex smooth domain, and let \( x_0 \in b\Omega \) be a boundary point of finite \( D'\text{Angelo} \) \( q \) type. There exists a neighborhood \( U' \) of \( x_0 \) such that the Kohn algorithm terminates at step 1 densely in \( U' \cap b\Omega \) in the induced topology of \( b\Omega \).

**Proof:** By the first proof given for Lemma 5.3, \( \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q} \neq 0 \) densely in \( U' \cap b\Omega \), but \( \text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\} \in I_1^q \), the first ideal of multipliers in the Kohn algorithm. Thus \( I_1^q(x) = \mathcal{C}^\infty \) for a dense set in \( U' \cap b\Omega \), i.e. the Kohn algorithm terminates at step 1 at each of the points in this dense set.

We are finally ready to state and prove the stratification theorem that is the object of this section:
Stratification Theorem 5.5. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $x_0 \in \partial \Omega$ be a boundary point of finite D’Angelo $q$ type. Let

$$\mathcal{B}_{n+1-q}(x_0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$

be the boundary system at $x_0$. There exists a neighborhood $U'$ of $x_0$ such that

$$r_1, r_{p+2}, \ldots, r_{n+1-q} \in \mathcal{I}\left(\mathcal{V}(I_1^q(U')) - \bigcup_{j=2}^{N-1} S_j\right),$$

i.e. the functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ vanish on the zero set of the first ideal of multipliers $I_1^q(U')$ of the Kohn algorithm on the neighborhood $U'$ after we remove from the zero set all the level sets of the commutator multitype except for the lowest one and the highest one, which is the one at the point $x_0$ itself.

Remarks:

1. As it will be apparent in the proof of this theorem,

$$\mathcal{V}(I_1^q(U')) - \bigcup_{j=2}^{N-1} S_j = S_N,$$

which is the level set of the top commutator multitype in the lexicographic ordering in the open set $U'$. Because of the upper semi-continuity of the commutator multitype, any sequence of points in $S_N$ has to accumulate at a point in $S_N$, so $S_N$ is a closed set. When working over $C^\infty(U')$, any closed set is a variety, which implies that

$$\mathcal{V}(I_1^q(U')) - \bigcup_{j=2}^{N-1} S_j$$

is a variety as well, and the notation

$$r_1, r_{p+2}, \ldots, r_{n+1-q} \in \mathcal{I}\left(\mathcal{V}(I_1^q(U')) - \bigcup_{j=2}^{N-1} S_j\right)$$

makes sense.

2. It is entirely possible that the functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ vanish on some other level sets of the commutator multitype, but the statement of this theorem reflects the information we have so far about their behavior.

3. The way we will prove Theorem 1.2 in Section 7 will be precisely by showing that the functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ vanish on $\mathcal{V}(I_k^q(U')) - \bigcup_{j=k+1}^{N-1} S_j$ for $1 \leq k \leq N - 1$. Once we get to step $N - 1$ in the Kohn algorithm, we will have that

$$r_1, r_{p+2}, \ldots, r_{n+1-q} \in \mathcal{I}(\mathcal{V}(I_{N-1}^q(U'))).$$

From here, an argument involving the truncation of the boundary system will allow us to conclude that the functions $r_1, r_{p+2}, \ldots, r_{n+1-q}$ are in fact multipliers in the ideal $I_{N-1}^q(U')$, so by Lemma 4.8 we will obtain a unit in $I_N^q(U')$. This will show the Kohn
algorithm terminates by step $N$. Note that the number of level sets of the commutator multitype acts then as the counter in the Kohn algorithm.

**Proof:** We distinguish two cases:

**Case 1:** The Levi form has rank $n - q$ at $x_0$. In this case, the commutator multitype $C_{n+1-q}(x_0) = (1, 2, \ldots, 2)$ and the boundary system has the form

$$\mathfrak{B}_{n+1-q}(x_0) = \{r_1; L_2, \ldots, L_{n+1-q}\}.$$  

We choose $U'$ to be precisely the neighborhood guaranteed by the conclusion of Lemma 5.3. Since $r_1 = r$, by the very definition of $I^q_1$, $r_1 \in I^q_1(U')$. No level sets of $C_{n+1-q}$ need to be removed because by the upper semi-continuity of $C_{n+1-q}$, it stays the same on all of $U'$, and it is the lowest one.

**Case 2:** The Levi form has rank $p$ with $p < n - q$ at $x_0$. This means $C_{n+1-q}(x_0) > (1, 2, \ldots, 2)$. Let $U'$ be a neighborhood of $x_0$, where Proposition 5.2 holds and the D'Angelo $q$ type is finite for all $x \in U' \cap b\Omega$.

$$U' \cap b\Omega = \bigcup_{j=1}^{N} S_j,$$

where $\mathfrak{c}_1^{n+1-q} < \mathfrak{c}_2^{n+1-q} < \cdots < \mathfrak{c}_N^{n+1-q}$ and

$$S_j = \{x \in U' \cap b\Omega \mid \mathfrak{c}_{n+1-q}(x) = \mathfrak{c}_j^{n+1-q}\}$$

are the level sets of the $(n+1-q)^{th}$ commutator multitype for $1 \leq j \leq N$. By part (i) of Proposition 5.2, $x_0 \in S_N$. By part (iii) of Proposition 5.2 it follows that $S_N \subset M^{n+1-q}$, where

$$M^{n+1-q} = \{x \in U' \cap b\Omega \mid r_j(x) = 0, j = 1, p + 2, \ldots, n + 1 - q\}.$$  

For all $x \in S_1$, coeff$\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\} \neq 0$, so $S_1 \cap \mathcal{V}(I^q_1(U')) = \emptyset$. For all $2 \leq j \leq N$ and all $x \in S_j$, coeff$\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\}(x) = 0$. Therefore,

$$\mathcal{V}(I^q_1(U')) = \bigcup_{j=2}^{N} S_j,$$

which is equivalent to

$$\mathcal{V}(I^q_1(U')) - \bigcup_{j=2}^{N-1} S_j = S_N \subset M^{n+1-q}$$

since $S_i \cap S_j = \emptyset$ if $i \neq j$. Given the definition of $M^{n+1-q}$, this means precisely that

$$r_1, r_{p+2}, \ldots, r_{n+1-q} \in \left(\mathcal{V}(I^q_1(U')) - \bigcup_{j=2}^{N-1} S_j\right).$$

$\square$
Corollary 5.6. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $x_0 \in \partial \Omega$ be a boundary point of finite D’Angelo $q$ type. There exist a neighborhood $U'$ of $x_0$ such that for all $k \geq 1$ the variety corresponding to the ideal of multipliers at step $k$ of the Kohn algorithm $\mathcal{V}(I_0^k(U'))$ satisfies that it has holomorphic dimension at most $q - 1$ in the sense of Catlin at every point.

Proof: Apply Theorem 5.5 to obtain that there exists a neighborhood $U' \ni x_0$ such that

$$\mathcal{V}(I_0^k(U')) = \bigcup_{j=2}^{N} S_j,$$

where $\mathcal{C}_{1}^{n+1-q} = (1, 2, \ldots, 2) < \mathcal{C}_{2}^{n+1-q} < \cdots < \mathcal{C}_{N}^{n+1-q}$ and

$$S_j = \{x \in U' \cap \partial \Omega \mid \mathcal{C}_{j}^{n+1}(x) = \mathcal{C}_{j}^{n+1-q}\}$$

are the level sets of the $(n + 1 - q)^{th}$ commutator multitype for $1 \leq j \leq N$. $S_1$ is dense in $U' \cap \partial \Omega$ by Lemma 5.3. The level sets $S_j$ are mutually disjoint, i.e. $S_i \cap S_j = \emptyset$ if $i \neq j$.

We consider now any point $x \in \mathcal{V}(I_0^k(U'))$. Clearly, $x \in S_j$ for some $2 \leq j \leq N$. The boundary system at $x$ is given by

$$\mathcal{B}_{n+1-q}(x) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\},$$

which contains at least one other function besides $r_1$ precisely because $x$ does not belong to the lowest level set of the commutator multitype $S_1$. By part (ii) of Proposition 5.2 there exists a neighborhood $U_x \ni x$ such that $S_j \subset M^{n+1-q}$, where

$$M^{n+1-q} = \{x \in U' \cap \partial \Omega \mid r_j(x) = 0, j = 1, p + 2, \ldots, n + 1 - q\}$$

has holomorphic dimension $n - (n + 1 - q) = q - 1$ in the sense of Catlin. There are finitely many disjoint sets $S_1, \ldots, S_N$, so the result follows for $k = 1$. The general case is a consequence of the observation that $I_0^q(U') \subset I_1^q(U') \subset \cdots$ implies $\mathcal{V}(I_0^q(U')) \supset \mathcal{V}(I_1^q(U')) \supset \cdots$. \hfill $\Box$

Remark: The contrapositive to Corollary 5.6 is a generalization of the following theorem proven by Klas Diederich and John Erik Fornæss, which appears as Theorem 3 on page 374 of [10]:

Theorem 5.7. Let $S$ be a pseudoconvex $C^\omega$ hypersurface in $\mathbb{C}^n$. Suppose that $M \subset S$ is a not necessarily closed $C^\omega$ subvariety with holomorphic dimension $q$. Let $z_0 \in M$ be an arbitrary point and $U = U(z_0)$ an open neighborhood of $z_0$. Then there is a complex submanifold $V \subset U \cap S$ of dimension at least $q$. The manifold $V$ can always be chosen in such a way that $M \cap V \neq \emptyset$ and in fact the holomorphic dimension of $M \cap V$ precisely equals $q$.

In the terminology employed in this paper, holomorphic dimension in the statement of Theorem 5.7 is meant in the sense of Kohn. It should be noted that Catlin mentioned in the introduction of his paper [6] that the construction involving the manifold $M''$ associated to a boundary system $\mathcal{B}_\nu$, leads to a contrapositive of the Diederich-Fornæss Theorem, Theorem 5.7. The additional information obtained here merely relates this setup to the Kohn algorithm, which leads to two very useful conclusions, namely Theorem 5.6 and Corollary 5.6. A quick
comparison of Corollary 5.6 with Theorem 5.7 shows that Corollary 5.6 pertains to the more inclusive class of smooth domains rather than just $C^\omega$ ones. We should note here that the proof of Theorem 5.7 in [10] is executed with methods that do not extend to the smooth case. In spite of this gain in generality, it is also clear that the original Diederich-Fornaess Theorem pertains to any variety of holomorphic dimension $q$ inside a pseudoconvex $C^\omega$ boundary of a domain, whereas Corollary 5.6 only yields information regarding the varieties corresponding to the ideals of multipliers in the Kohn algorithm.

6. Truncated boundary systems

Truncation of a boundary system will take place with respect to a weight $(\lambda_1, \ldots, \lambda_n) \in \Gamma_n$. We thus start by defining two classes of functions that have a special relationship to a certain weight.

**Definition 6.1.** Let the weight $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n$ be given. We will denote by $\mathcal{M}(t; \Lambda)$ the set of germs of smooth functions $f$ defined near the origin such that
\[
D^\alpha \bar{D}^\beta f(0) = 0 \quad \text{whenever} \quad \sum_{i=1}^n \frac{\alpha_i + \beta_i}{\lambda_i} < t.
\]

**Definition 6.2.** Let the weight $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n$ be given. We will denote by $\mathcal{H}(t; \Lambda)$ the set of polynomials $f$ defined near the origin such that
\[
D^\alpha \bar{D}^\beta f(0) = 0 \quad \text{whenever} \quad \sum_{i=1}^n \frac{\alpha_i + \beta_i}{\lambda_i} \neq t.
\]

**Remarks:**

1. A quick glance at Definition 6.2 will show that Definition 6.1 and Definition 6.2 have as their aim setting up the stage for understanding distinguished weights.
2. The reader should note that Definition 6.2 intends to set up a certain type of homogeneity with respect to the weight $\Lambda$, thus differentiation with respect to multi-indices that lie both above and below $\Lambda$ should yield 0.

We will now state Catlin’s Proposition 3.6 from page 542 of [6]. Given a boundary system at $x_0$,
\[
\mathcal{B}_\nu(x_0) = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu\},
\]
this result shows in what sets $\mathcal{M}(t; \Lambda)$ we can place the additional functions $r_{p+2}, \ldots, r_\nu$ obtained by differentiating $r$ and relates this information to the $\nu^{th}$ commutator multitype $\mathcal{C}^\nu(x_0)$.

**Proposition 6.3.** Let $\mathcal{B}_\nu = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu\}$ be a boundary system of rank $p$ and codimension $n - \nu$ about the origin in $\mathbb{C}^n$. Suppose that the $\nu^{th}$ commutator multitype of $\mathcal{B}_\nu$ at the origin $\mathcal{C}^\nu(0) = (\lambda_1, \ldots, \lambda_\nu)$ and let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a weight in $\Gamma_n$ that agrees with $\mathcal{C}^\nu(0)$ up to the $\nu^{th}$ entry and furthermore satisfies that $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_{p+1} = 2$, and $\lambda_j \geq 3$
for $j \geq p + 2$. If $r_1 \in \mathcal{M}(1; \Lambda)$, then $r_k \in \mathcal{M}(\frac{1}{\lambda_k}; \Lambda)$ for all $k = p + 2, \ldots, \nu$. Moreover, if $c_{\nu + 1}$ denotes the $(\nu + 1)^{th}$ entry of $\mathcal{C}^{\nu + 1}(0)$, then $c_{\nu + 1} \geq \lambda_{\nu + 1}$.

**Remark:** The reader should note that we have translated the point $x_0$ to the origin already in order to satisfy part (i) of Definition 4.2, the definition of a distinguished weight.

As Proposition 6.3 shows, to a boundary system $\mathfrak{B}_\nu$ is associated the $\nu$-tuple $\mathcal{C}(0)$, but spaces $\mathcal{M}(t; \Lambda)$ and $\mathcal{H}(t; \Lambda)$ require a weight $\Lambda \in \Gamma_n$, which is an $n$-tuple. We thus need to manufacture a weight starting with an $\nu$-tuple. The most natural way to do so is precisely the content of the next definition:

**Definition 6.4.** Let $\Gamma_{n,\nu + 1}$ be the set of weights $(\lambda_1, \ldots, \lambda_n)$ in $\Gamma_n$ such that $\lambda_{\nu + 1} = \cdots = \lambda_n$, i.e. the last $n - \nu$ entries coincide.

Having dealt with all the preliminary considerations, we are now poised to explain Catlin’s truncation a boundary system $\mathfrak{B}_\nu$ with respect to a given weight $\Lambda$ as it appears in Section 4 of [6]. Starting with a boundary system

$$\mathfrak{B}_\nu = \{r_1, r_{p + 2}, \ldots, r_{\nu}; L_2, \ldots, L_\nu\},$$

we would like to construct a corresponding boundary system

$$\tilde{\mathfrak{B}}_\nu = \{\tilde{r}_1, \tilde{r}_{p + 2}, \ldots, \tilde{r}_{\nu}; \tilde{L}_2, \ldots, \tilde{L}_\nu\}$$

such that the functions $\tilde{r}_k$ are polynomials, the vector fields $\tilde{L}_k$ have polynomial coefficients, and both are homogeneous with respect to $\Lambda$. We shall do so by considering those terms in the Taylor expansions of the functions $r_k$ and the coefficients of the vector fields $L_k$ that are of lowest order relative to $\Lambda$.

Let $\Lambda$ denote the weight $(\lambda_1, \ldots, \lambda_n)$. For $t$ such that $0 < t < 1$, we define $\Lambda_t : \mathbb{C}^n \to \mathbb{C}^n$ as the map given by

$$\Lambda_t(z) = (t^{\frac{1}{\lambda_1}}z_1, \ldots, t^{\frac{1}{\lambda_n}}z_n).$$

Proposition 6.3 applied with a weight $\Lambda \in \Gamma_{n,\nu + 1}$ that matches the commutator multitype $\mathcal{C}(0)$ in the first $\nu$ entries yields that

$$r_k \in \mathcal{M}\left(\frac{1}{\lambda_k}; \Lambda\right), \quad k = 1, p + 2, \ldots, \nu.$$

Define $\tilde{r}_k$ for $k = 1, p + 2, \ldots, \nu$ by

$$\tilde{r}_k = \lim_{t \to 0} t^{-\frac{1}{\lambda_k}} \Lambda_t^* r_k$$

where $\Lambda_t^*$ is the pullback under the map $\Lambda_t$. This guarantees that $\tilde{r}_k \in \mathfrak{H}\left(\frac{1}{\lambda_k}; \Lambda\right)$ as the terms in the Taylor expansion of $r_k$ around $0$ that are not of lowest order with respect to $\Lambda$ are sent to $0$ in the limit in the $C^\infty$ topology. The polynomials $\tilde{r}_k$ are therefore homogeneous with respect to $\Lambda$. 
Similarly, define $\tilde{L}_k$ for $k = 2, \ldots, \nu$ by

$$\tilde{L}_k = \lim_{t \to 0} t^{1/k} (d\Lambda_t^{-1}) L_k.$$  

We will call the vector fields $\tilde{L}_k$ homogeneous vector fields. The interested reader should consult Section 4 of Catlin’s paper [6] for more information on the behavior of the coefficients of vector fields $L_k$ and $\tilde{L}_k$ as well as the behavior under the action of $\Lambda_t$ of lists $L_k$ used in defining $\mathfrak{B}_\nu$. In particular, Section 4 of [6] contains a careful discussion of which kind of spaces $\mathfrak{M}(t; \Lambda)$ and $\mathfrak{F}(t; \Lambda)$ the coefficients of $L_k$ and $\tilde{L}_k$ belong respectively. Obtaining optimal properties for the coefficients of $\tilde{L}_k$ comes about precisely as a result of having considered germs rather than vector fields on a neighborhood in the construction of the boundary system. This point was touched upon on page 19. We are only interested in the algebraic behavior of the functions $\tilde{r}_k$ versus the functions $r_k$, so we will not dwell here on matters connected to vector fields.

We would like to show that the homogeneous (truncated)

$$\tilde{\mathfrak{B}}_\nu = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_\nu; \tilde{L}_2, \ldots, \tilde{L}_\nu\}$$

is a boundary system of rank $p$ and codimension $n - \nu$ around the origin in $\mathbb{C}^n$. This is the content of Catlin’s Proposition 4.4 from page 546 of [6], which we recall here:

**Proposition 6.5.** Let $\mathfrak{B}_\nu = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu\}$ be a boundary system of rank $p$ and codimension $n - \nu$ about the origin in $\mathbb{C}^n$. Suppose that the $\nu$th commutator multitype of $\mathfrak{B}_\nu$ at the origin $\mathfrak{C}_\nu(0) = (\lambda_1, \ldots, \lambda_\nu)$ and let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ be a weight in $\Gamma_{n,\nu+1}$ that agrees with $\mathfrak{C}_\nu(0)$ up to the $\nu$th entry and furthermore satisfies that $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_{p+1} = 2$, and $\lambda_j \geq 3$ for $j \geq p + 2$. If $r_k \in \mathfrak{M}\left(\frac{1}{\lambda_k}; \Lambda\right)$ for all $k = 1, p + 2, \ldots, \nu$, then the following assertions hold:

(i) $\tilde{\mathfrak{B}}_\nu = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_\nu; \tilde{L}_2, \ldots, \tilde{L}_\nu\}$ is a homogeneous boundary system of rank $p$ and codimension $n - \nu$ around the origin in $\mathbb{C}^n$.

(ii) Let $\mathfrak{C}_{\nu+1}(0) = (\tilde{c}_1, \ldots, \tilde{c}_{\nu+1})$ denote the $(\nu + 1)$th commutator multitype of $\tilde{\mathfrak{B}}_\nu$ at the origin. Then $\tilde{c}_k = \lambda_k$ for all $k = 1, \ldots, \nu$.

(iii) If $c_{\nu+1} \leq \lambda_{\nu+1}$, where $c_{\nu+1}$ denotes the $(\nu + 1)$th entry of $\mathfrak{C}_{\nu+1}(0)$, then $\tilde{c}_{\nu+1} = \lambda_{\nu+1}$.

(iv) If $c_{\nu+1} > \lambda_{\nu+1}$, then $\mathcal{L}' \partial \tilde{r}_1(0) = 0$ for every homogeneous, $(\nu + 1)$-admissible, ordered list $\mathcal{L}'$.

(v) If $r_1$ is the defining function of a pseudoconvex domain, then so is $\tilde{r}_1$. In fact, $\tilde{r}_1$ is plurisubharmonic.

**Remark:** Part (v) of Proposition 6.5 holds because if $\Omega$ is pseudoconvex, then $\tilde{r}_1$ equals the limit in the $C^\infty$ topology of $t^{-1} \partial_t r_1$, which is for each $t$ in $(0, 1)$ the defining function of a pseudoconvex domain. This means the domain $\tilde{\Omega}$ defined by $\tilde{r}_1$ is pseudoconvex. Part (v) of Proposition 6.5 will have far reaching algebraic implications below.
Now that we have finished explaining Catlin's truncation of a boundary system, we must discuss its significance and relate it to other such constructions. First of all, there is a remarkable connection between this truncation with respect to the weight $\Lambda$ and Hironaka's use of the Weierstrass Division Theorem in his proof of desingularization in the analytic case. Given an ideal $\mathcal{I} \subset C^\omega$, Hironaka constructed a standard basis for the ideal $\mathcal{I}$ by identifying the elements $f \in \mathcal{I}$ with the lowest weight when measured against a linear form $L: \mathbb{N}^n \to \mathbb{R}$ given by

$$L(k_1, \ldots, k_n) = \sum_{j=1}^{n} l_j k_j,$$

where the coefficients $l_j \geq 0$ for all $1 \leq j \leq n$. $L$ is evaluated on multi-indices $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ corresponding to terms $x^k$ with non-zero coefficients in the convergent Taylor expansion of $f$. Let $f_1, \ldots, f_N$ be the finitely many generators of the ideal $\mathcal{I}$ that form the standard basis. Given any $f \in \mathcal{I}$, there exist functions $h_1, \ldots, h_N \in C^\omega$ such that

$$f = f_1 h_1 + \cdots + f_N h_N.$$

This decomposition is unique for all $f \in \mathcal{I}$ provided that the basis $f_1, \ldots, f_N$ is constructed by using a linear form $L$ whose coefficients $l_1, \ldots, l_n$ are linearly independent over $\mathbb{Z}$. The interested reader should consult either the exposition in [2] by Aroca, Hironaka, and Vincente or for an even more comprehensive treatment the paper [4] by Jöel Briançon and the paper [11] by André Galligo. If we compare Hironaka's construction to Catlin's, we realize that the linear form used by Catlin is

$$L(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \sum_{j=1}^{n} \frac{1}{\lambda_j} (\alpha_j + \beta_j),$$

where $L$ is evaluated on multi-indices $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ corresponding to terms $z^\alpha \bar{z}^\beta$ in the Taylor expansion around 0 of elements in $C^\omega$. As it can be easily seen, the coefficients in the linear form cannot be altered since they come from the commutator multitype $\mathfrak{C}(0)$. Still, it will be shown that the truncated boundary system $\tilde{B}_\nu(0)$, which is the boundary system of the domain $\tilde{\Omega}$, carries within it the algebra of the original boundary system $B_\nu(0)$ of the domain $\Omega$. Therefore, heuristically at least, Catlin's construction is not very far from Hironaka's.

The second term of comparison comes from the truncation of the defining function of the domain done by John D’Angelo in [8] in order to show that the D’Angelo finite type condition is an open condition. The success of this construction prompted D’Angelo to inquire whether the Kohn Conjecture could be attacked by truncating the $C^\omega$ defining function of a pseudoconvex domain $\Omega$ appropriately. The most natural way of doing so would be to go far out enough into the Taylor expansion at a point to guarantee that the truncated $\tilde{r}$ retains the same D’Angelo type as the original defining function $r$. The problematic part of this attempt comes from the fact that pseudoconvexity is not a finitely determined condition and may be lost under such a truncation. If pseudoconvexity is lost, then it is not clear whether the crucial element $\text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\}$ is still a subelliptic multiplier, so the Kohn algorithm no longer proceeds as defined. The interested reader should consult D’Angelo’s book [9] or Siu’s paper [19] for a more comprehensive discussion of these matters. Instead, Catlin’s approach is to
truncation the defining function in such a way that $\tilde{r}$ stays plurisubharmonic, so the domain $\tilde{\Omega}$ is pseudoconvex. It is not evident a priori what happens to the D’Angelo $q$ type under this truncation, but as parts (ii) and (iii) of Proposition 6.5 show, one has at least control over the commutator multitype $\tilde{C}^{n+1-q}(0)$. Furthermore, we will show next that the truncation of the boundary system cannot increase the number of level sets of the commutator multitype in a well-chosen neighborhood of the origin. We start by defining exactly what we mean here by a well-chosen neighborhood:

**Definition 6.6.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $x_0 \in b\Omega$ be a boundary point of finite D’Angelo $q$ type. Let

$$\mathcal{B}_{n+1-q}(x_0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$

be the boundary system at $x_0$. A neighborhood $U$ of $x_0$ is called optimal for the commutator multitype if there does not exist a smaller neighborhood $U' \subset U$ such that $x_0 \in U'$ and $U'$ contains a strictly smaller number of level sets of the commutator multitype than $U$.

**Remark:** Since the number of level sets of the commutator multitype is finite in the neighborhood of a point of finite D’Angelo type, it is clear that this definition makes sense, and we have to shrink the neighborhood $U$ at most finitely many times. In fact, even if there are sequences on each of the upper level sets $S_2, \ldots, S_{N-1}$ of the commutator multitype that accumulate at $x_0$, which belongs to $S_N$, it is clear that we can choose an optimal neighborhood for the commutator multitype, which will in this case retain all of the level sets $S_1, \ldots, S_N$, where $S_1$ in known from Lemma 5.3 to be dense in $b\Omega$.

**Proposition 6.7.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $0 \in b\Omega$ be a boundary point of finite D’Angelo $q$ type. Let

$$\mathcal{B}_{n+1-q}(0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$

be the boundary system at 0, and let

$$\mathcal{B}_{n+1-q}(0) = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}; \tilde{L}_2, \ldots, \tilde{L}_{n+1-q}\}$$

be its truncation. If $U \ni 0$ is an optimal neighborhood for the commutator type for the untruncated domain $\Omega$ containing $N$ level sets of the commutator multitype, then there exists a neighborhood $U' \ni 0$ for the truncated domain $\tilde{\Omega}$ such that $U'$ contains $\tilde{N}$ level sets of the commutator multitype $\tilde{C}^{n+1-q}$, where $\tilde{N} \leq N$. The neighborhood $U'$ is optimal for the commutator multitype $\tilde{C}^{n+1-q}$.

**Proof:** By part (ii) of Proposition 6.5 it is clear that the top commutator multitype for both $\Omega$ at 0 and $\tilde{\Omega}$ at 0 is the same. By Lemma 5.3 the level set of the lowest commutator multitype will be present in both $U$ and $U'$ regardless of how we choose these open sets since that level set is dense in $b\Omega$ under the assumption of finite D’Angelo type. The question is whether the truncated defining function $\tilde{r}$ can generate more level sets of the commutator multitype than the original defining function $r$. The answer turns out to be no. The intermediate commutator
multitypes of the level sets $S_2, \ldots, S_{N-1}$ in the optimal set $U$ around the origin come about because $\mathcal{L} \partial r_1(x) \neq 0$ for $x \in U$ and ordered, admissible lists $\mathcal{L}$ of shorter length than those that define $\mathcal{C}^{n+1-q}(0)$. Since we truncate with respect to the weight that agrees with $\mathcal{C}^{n+1-q}(0)$ up to the $(n + 1 - q)^{th}$ entry, this means we are truncating with respect to the weight that allows for the largest number of terms in the Taylor expansion since $\mathcal{C}^{n+1-q}(0)$ is the top commutator multitype in the lexicographic ordering in all of $U$. We conclude that terms in the Taylor expansion of $r(x)$ for $x \in U$ that enable the conclusion $\mathcal{L} \partial r_1(x) \neq 0$ either are

(a) retained in $\tilde{r}$, in which case that particular commutator multitype $\mathcal{C}^{n+1-q}(x)$ will be represented in $U'$ or

(b) dropped from $\tilde{r}$, in which case that particular commutator multitype $\mathcal{C}^{n+1-q}(x)$ will not be represented in $U'$.

Clearly, no new commutator multitypes appear in $U'$ in $\partial \Omega$. To make this construction even easier to trace, we note that all terms in $\tilde{r}$ come from terms of the type $\frac{1}{|\alpha| |\beta|} D^\alpha \bar{D}^\beta r(0) z^\alpha z^\beta$ such that $D^\alpha \bar{D}^\beta r(0) \neq 0$, which is an open condition. The number of terms in $\tilde{r}$ is finite. We shrink $U$ such that $D^\alpha \bar{D}^\beta r(x) \neq 0$, for all $x \in U$ and all terms in $\tilde{r}$. The original neighborhood $U$ was optimal with respect to the commutator type, so shrinking it will not change the number of level sets of the commutator type by its very definition. It is clear $U'$ can be chosen to be optimal for the commutator multitype $\tilde{\mathcal{C}}^{n+1-q}$.

We will show next that the previous proposition and the control over the commutator multitype $\tilde{\mathcal{C}}^{n+1-q}(0)$ given by parts (ii) and (iii) of Proposition 6.5 suffice for proving a version of the Stratification Theorem, Theorem 5.5, as well as of a corollary to it, which sets the stage for Theorem 1.3. Here the finite D’Angelo $q$ type condition at 0 is replaced by the finite commutator multitype condition guaranteed by part (ii) of Proposition 6.5.

**Proposition 6.8.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $0 \in \partial \Omega$ be a boundary point of finite D’Angelo $q$ type such that the neighborhood $\bar{U}$ of 0 is optimal with respect to the commutator multitype $\mathcal{C}^{n+1-q}$ and contains exactly $N$ level sets of the commutator multitype. Let

$$\mathcal{B}_{n+1-q}(0) = \{ r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q} \}$$

be the boundary system at 0, and let

$$\tilde{\mathcal{B}}_{n+1-q}(0) = \{ \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}; \tilde{L}_2, \ldots, \tilde{L}_{n+1-q} \}$$

be its truncation. There exists a neighborhood $U'$ of 0 that is optimal for $\tilde{\mathcal{C}}^{n+1-q}$ such that

$$\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \mathcal{I} \left( \mathcal{V}(\tilde{I}_1^q(U')) - \bigcup_{j=2}^{\tilde{N}-1} \tilde{S}_j \right),$$

i.e. the functions $\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}$ vanish on the zero set of the first ideal of multipliers $\tilde{I}_1^q(U')$ of the Kohn algorithm on the neighborhood $U'$ of the domain $\bar{\Omega}$ that corresponds to $\Omega$ under truncation after we remove from the zero set all the level sets $\tilde{S}_2, \ldots, \tilde{S}_{\tilde{N}-1}$ of the commutator.
The finiteness of the D’Angelo \( q \) type at the origin and pseudoconvexity of \( \Omega \) imply by part (4) of Theorem 4.4 along with part (ii) of Theorem 4.7 that the \( (n + 1 - q)^{th} \) commutator multitype at the origin \( \mathcal{C}^{n+1-q}(0) = (c_1, \ldots, c_{n+1-q}) \) has only finite entries. By part (ii) of Proposition 6.5 it follows that the commutator multitype of the truncation at the origin \( \mathcal{C}^{n+1-q}(0) = \mathcal{C}^{n+1-q}(0) \), i.e. \( \mathcal{C}^{n+1-q}(0) \) only has finite entries. By the upper semi-continuity of the commutator multitype, part (i) of Theorem 4.7 it follows that there exists a neighborhood \( U' \) of the origin such that for all \( x \in U' \cap b\tilde{\Omega} \), \( \mathcal{C}^{n+1-q}(x) \leq \mathcal{C}^{n+1-q}(0) \). Therefore, there exist only finitely many level sets of the commutator multitype \( \mathcal{C}^{n+1-q}(x) \) in \( U' \), where \( U' \) can be chosen to be optimal for the commutator multitype \( \mathcal{C}^{n+1-q} \). As a result, Lemma 5.3 goes through verbatim for the truncation. This means that the rest of the proof of the Stratification Theorem 5.3 goes through verbatim as well. \( \square \)

**Corollary 6.9.** Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex smooth domain, and let \( 0 \in b\Omega \) be a boundary point of finite D’Angelo \( q \) type. Let \( U \) be an optimal neighborhood of the origin with respect to the commutator multitype \( \mathcal{C}^{n+1-q} \) that contains only two level sets of the commutator multitype. Let

\[
\mathcal{B}_{n+1-q}(0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}
\]

be the boundary system at 0, and let

\[
\tilde{\mathcal{B}}_{n+1-q}(0) = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}; \tilde{L}_2, \ldots, \tilde{L}_{n+1-q}\}
\]

be its truncation. There exists a neighborhood \( U' \) of 0 such that

\[
\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_1(U'),
\]

i.e. the functions \( \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \) are subelliptic multipliers in the first ideal \( \tilde{I}_1(U') \) of the Kohn algorithm on the neighborhood \( U' \) of the domain \( \tilde{\Omega} \) that corresponds to \( \Omega \) under truncation. The Kohn algorithm terminates at step 1 densely in \( U' \cap b\tilde{\Omega} \) in the induced topology on \( b\tilde{\Omega} \) and at step 2 otherwise.

**Proof:** By part (v) of Proposition 6.5 the domain \( \tilde{\Omega} \) is pseudoconvex, so the Kohn algorithm is defined on it in the same way as on the original \( \Omega \). As mentioned in the proof of Proposition 6.8 Corollary 5.4 applies to \( \tilde{\Omega} \) because the entries of the commutator multitype \( \tilde{\mathcal{C}}^{n+1-q}(0) \) are finite. Therefore, we know there exists a neighborhood \( U' \) of the origin such that the Kohn algorithm terminates at step 1 densely in \( U' \cap b\tilde{\Omega} \) in the induced topology on \( b\tilde{\Omega} \). We choose the neighborhood \( U' \) so that it is optimal for the commutator multitype \( \tilde{\mathcal{C}}^{n+1-q} \). To see what happens at the points that are left over, we apply Proposition 6.7 to conclude that there are at most two level sets of \( \tilde{\mathcal{C}}^{n+1-q} \) in \( U' \). Since there is more than one function in \( \tilde{\mathcal{B}}_{n+1-q}(0) \), we conclude that the origin in the boundary \( \tilde{\Omega} \) cannot have the lowest commutator multitype \( (1, 2, \ldots, 2) \). This means there are exactly two level sets of the commutator multitype \( \tilde{\mathcal{C}}^{n+1-q} \).
in $U'$. To finish off the proof, we apply Proposition 6.8 to conclude that
\[ \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \mathcal{I}(\mathcal{V}(\tilde{I}_1^q(U'))), \]
where $U'$ may be shrunk a little, if necessary. Clearly, there are no intermediate level sets of the commutator type to be taken out in this case. Since $\tilde{r}_1$ is a polynomial, all multipliers in $\tilde{I}_k^q(U')$ at every step $k \geq 1$ of the Kohn algorithm are $C^\omega$ functions. Thus, the Lojasiewicz inequalities apply on $\tilde{I}_k^q(U')$ as well as the Lojasiewicz Nullstellensatz, which says that
\[ \mathcal{I}(\mathcal{V}(\tilde{I}_k^q(U'))) = \sqrt[k]{\tilde{I}_k^q(U')} = \tilde{I}_k^q(U'), \]
because $\tilde{I}_k^q(U')$ is self-radical. It follows that
\[ \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_1^q(U'). \]
Since by part (i) of Proposition 6.5, $\mathcal{B}_{n+1-q}$ is still a boundary system and the full set of functions $\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}$ have been constructed, Lemma 4.8 goes through. We conclude
\[ \partial \bar{r}_1 \wedge \partial \bar{r}_1 \wedge (\partial \bar{r}_1)^p \wedge \partial \bar{r}_{p+2} \wedge \cdots \wedge \partial \bar{r}_{n+1-q}(x) \neq 0 \]
for all $x \in \mathcal{V}(\tilde{I}_1^q(U'))$ by the construction of $U'$. The function
\[ \text{coeff}\{\partial \bar{r}_1 \wedge \partial \bar{r}_1 \wedge (\partial \bar{r}_1)^p \wedge \partial \bar{r}_{p+2} \wedge \cdots \wedge \partial \bar{r}_{n+1-q}\} \in \tilde{I}_2^q(U'), \]
so the Kohn algorithm terminates at step 2 at every point $x \in \mathcal{V}(\tilde{I}_1^q(U'))$. \hfill \Box

In case there are only two levels of the commutator mutitype present in an optimal neighborhood of 0 in the boundary of the original domain $\Omega$, we thus have obtained by the previous corollary that
\[ \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_1^q(U'). \]
We would like to relay this conclusion back to the domain $\Omega$, i.e. we would like to obtain $r_1, r_{p+2}, \ldots, r_{n+1-q} \in I_1^q(U'')$ with control over the order of the roots that need to be taken in the Nullstellensatz, where $U''$ is some neighborhood of 0 in the boundary of $\Omega$ that is potentially smaller than the optimal neighborhood with which we started. More generally, we will show that
\[ \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_1^q(\Omega), \]
where $N$ is the number of level sets of the commutator multitype in an optimal neighborhood around 0 in the boundary of the original domain $\Omega$, and we would like to show this implies that
\[ r_1, r_{p+2}, \ldots, r_{n+1-q} \in I_N^q(\Omega'') \]
on a potentially smaller neighborhood around 0 in the original domain $\Omega$. Both of these assertions will be obtained via a continuity argument:

Define $r_{k,t}$ for $k = 1, p + 2, \ldots, \nu$ by
\[ r_{k,t} = t^{\frac{1}{n_k}} \Lambda_t^* r_k \]
where $\Lambda_t^*$ is the pullback under the map $\Lambda_t(z) = (t^{\frac{1}{n_1}} z_1, \ldots, t^{\frac{1}{n_\nu}} z_\nu)$. 

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Similarly, define $L_{k,t}$ for $k = 2, \ldots, \nu$ by

$$L_{k,t} = t^{\frac{1}{N}} (d\Lambda_{t}^{-1}) L_{k},$$

In this notation,

$$r_{k,1} = r_{k}$$

for $k = 1, p + 2, \ldots, \nu$, and

$$L_{k,1} = L_{k}$$

for $k = 2, \ldots, \nu$, whereas

$$r_{k,0} = \tilde{r}_{k}$$

for $k = 1, p + 2, \ldots, \nu$, and

$$L_{k,0} = \tilde{L}_{k}$$

for $k = 2, \ldots, \nu$. We are thus running the continuity argument on $t \in [0, 1]$. It is clear from Catlin’s setup that parts (i)-(iii) and (v) of Proposition 6.5 apply to $B_{\nu,t} = \{r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t}; L_{2,t}, \ldots, L_{\nu,t}\}$ for $0 < t < 1$ with the obvious modifications. We state this result here:

**Proposition 6.10.** Let $B_{\nu} = \{r_{1}, r_{p+2}, \ldots, r_{\nu}; L_{2}, \ldots, L_{\nu}\}$ be a boundary system of rank $p$ and codimension $n - \nu$ about the origin in $\mathbb{C}^{n}$. Suppose that the $\nu^{th}$ commutator multitype of $B_{\nu}$ at the origin $C_{\nu}(0) = (\lambda_{1}, \ldots, \lambda_{\nu})$ and let $\Lambda = (\lambda_{1}, \ldots, \lambda_{n})$ be a weight in $\Gamma_{n,\nu+1}$ that agrees with $C_{\nu}(0)$ up to the $\nu^{th}$ entry and furthermore satisfies that $\lambda_{1} = 1$, $\lambda_{2} = \cdots = \lambda_{p+1} = 2$, and $\lambda_{j} \geq 3$ for $j \geq p + 2$. If $r_{k} \in M_{\nu} \left( \frac{1}{\lambda_{k}}; \Lambda \right)$ for all $k = 1, p + 2, \ldots, \nu$, then for every $t$ such that $0 < t < 1$ the following assertions hold:

(i)

$$B_{\nu,t} = \{r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t}; L_{2,t}, \ldots, L_{\nu,t}\}$$

is a boundary system of rank $p$ and codimension $n - \nu$ around the origin in $\mathbb{C}^{n}$.

(ii) Let $C_{\nu+1,t}(0) = (c_{1,t}, \ldots, c_{\nu+1,t})$ denote the $(\nu + 1)^{th}$ commutator multitype of $B_{\nu,t}$ at the origin. Then $c_{k,t} = \lambda_{k}$ for all $k = 1, \ldots, \nu$.

(iii) If $c_{\nu+1} = \lambda_{\nu+1}$, where $c_{\nu+1}$ denotes the $(\nu + 1)^{th}$ entry of $C_{\nu+1}(0)$, then $c_{\nu+1,t} = \lambda_{\nu+1}$.

(iv) If $r_{1}$ is the defining function of a pseudoconvex domain, then so is $r_{1,t}$. In fact, $r_{1,t}$ is plurisubharmonic.

Before we can state the stratification theorem that holds for every $t \in (0, 1)$, we have to state the equivalent result to Proposition 6.7. The proof is very similar to the one of Proposition 6.7, so we omit it:

**Proposition 6.11.** Let $\Omega \subset \mathbb{C}^{n}$ be a pseudoconvex smooth domain, and let $0 \in b\Omega$ be a boundary point of finite D’Angelo $q$ type. Let

$$B_{n+1-q} = \{r_{1}, r_{p+2}, \ldots, r_{n+1-q}; L_{2}, \ldots, L_{n+1-q}\}$$
be the boundary system at 0, and let

$$\mathcal{B}_{n+1-q,t}(0) = \{ r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t}; L_{2,t}, \ldots, L_{\nu,t} \}$$

be its transformation under the action of the map $$\Lambda_t$$ for $$0 < t < 1$$. If $$U \ni 0$$ is an optimal neighborhood for the commutator type for the original domain $$\Omega$$ containing $$N$$ level sets of the commutator multitype, then there exists a neighborhood $$U'_t \ni 0$$ of the domain $$\Omega_t$$ that corresponds to $$\Omega$$ under the action of the map $$\Lambda_t$$ such that $$U'_t$$ contains $$N_t$$ level sets of the commutator multitype $$\mathcal{C}^{n+1-q,t}$$, where $$N_t \leq N$$. For each $$t$$, the neighborhood $$U'_t$$ is chosen so that it is optimal with respect to $$\mathcal{C}^{n+1-q,t}$$.

A quick glance at the proof of the stratification theorem for $$\mathcal{B}_{n+1-q}(0)$$, Proposition 6.8, will show that the same proof goes through verbatim for $$\mathcal{B}_{n+1-q,t}(0)$$ since Proposition 6.11 holds. Let us then state this result below:

**Proposition 6.12.** Let $$\Omega \subset \mathbb{C}^n$$ be a pseudoconvex smooth domain, and let $$0 \in b\Omega$$ be a boundary point of finite D’Angelo $$q$$ type such that the neighborhood $$U$$ of 0 is optimal with respect to the commutator multitype $$\mathcal{C}^{n+1-q}$$ and contains exactly $$N$$ level sets of the commutator multitype. Let

$$\mathcal{B}_{n+1-q}(0) = \{ r_1, r_{p+2}, \ldots, r_{n+1-q}, L_2, \ldots, L_{n+1-q} \}$$

be the boundary system at 0, and let

$$\mathcal{B}_{n+1-q,t}(0) = \{ r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t}; L_{2,t}, \ldots, L_{\nu,t} \}$$

be its transformation under the action of the map $$\Lambda_t$$, for $$0 < t < 1$$. There exists a neighborhood $$U'_t$$ of 0 that is optimal for $$\mathcal{C}^{n+1-q,t}$$ such that

$$r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t} \in \mathcal{I} \left( \mathcal{V}(I^{q}_{1,t}(U'_t)) - \bigcup_{j=2}^{N_t-1} S_{j,t} \right),$$

i.e. the functions $$r_{1,t}, r_{p+2,t}, \ldots, r_{\nu,t}$$ vanish on the zero set of the first ideal of multipliers $$I^{q}_{1,t}(U'_t)$$ of the Kohn algorithm on the neighborhood $$U'_t$$ of the domain $$\Omega_t$$ that corresponds to $$\Omega$$ under the action of the map $$\Lambda_t$$ after we remove from the zero set all the level sets $$S_{2,t}, \ldots, S_{N_t-1,t}$$ of the commutator multitype except for the lowest one and the highest one, which is the one at the origin. $$N_t$$ denotes the number of level sets of $$\mathcal{C}^{n+1-q,t}$$ in $$U'_t$$ and satisfies $$N_t \leq N$$.

We need to state and prove two elementary lemmas dealing with the Lojasiewicz inequality in Definition 2.9. The first one gives an equivalent statement, whereas the second shows how the Lojasiewicz inequality behaves under approximation. Both of these lemmas are essential for the continuity argument.

**Lemma 6.13.** Let $$f \in C^\infty(U)$$ and let $$X = \{ x \in U \mid f(x) = 0 \}$$. Let $$g \in C^\infty(U)$$ be such that $$g(x) = 0$$ for all $$x \in X$$. Let $$U$$ be an open set and let $$K \subset U$$ be any compact subset of $$U$$. The following two conditions are equivalent:
(a) There exist constants $C > 0$ and $\alpha \geq 0$ with $\alpha \in \mathbb{Q}$ such that

$$|g(x)|^\alpha \leq C |f(x)|$$

for all $x \in K$.

(b) There exist constants $C > 0$ and $\alpha \geq 0$ with $\alpha \in \mathbb{Q}$ such that

$$|g(x)|^\alpha < C |f(x)|$$

for all $x \in K - X$, i.e. away from the zero set of $f$.

Remark: Only the constant $C$ might change between (a) and (b). The power $\alpha \in \mathbb{Q}$ stays the same.

Proof: (a) $\implies$ (b) Let us look at the points $x \in K - X$. Since $|g(x)|^\alpha \leq C |f(x)|$ for all $x \in K$, it is possible there exists a point $x \in K - X$ for which $|g(x)|^\alpha = C |f(x)|$ Let us take $C' = 2C$. Clearly, $|g(x)|^\alpha < 2C |f(x)|$ at all $x \in K - X$ since $C > 0$ and $|f(x)| > 0$.

(b) $\implies$ (a) At the points $x \in K \cap X$, $f(x) = g(x) = 0$, so clearly $|g(x)|^\alpha \leq C |f(x)|$ at all $x \in K$ with the same $C > 0$ and the same $\alpha \in \mathbb{Q}$.

Lemma 6.14. Let $f, g, \tilde{f}, \tilde{g} \in C^\infty(U)$. Let $\tilde{X} = \{x \in U \mid \tilde{f}(x) = 0\}$. Let $\tilde{g}$ be such that $\tilde{g}(x) = 0$ for all $x \in \tilde{X}$. If there exist constant $C > 0$ and positive integer $m \in \mathbb{N}^*$ such that

$$|\tilde{g}(x)|^m \leq C |\tilde{f}(x)|$$

for all $x \in U$, then

$$|g(x)|^m \leq 2^m M |g(x) - \tilde{g}(x)|^m + 2^m M C |f(x) - \tilde{f}(x)| + 2^m M C |f(x)|, \quad (6.1)$$

for some constant $M > 0$ and for all $x \in U$. If for some $x$,

$$|\tilde{g}(x)|^m < C |\tilde{f}(x)|,$$

then

$$|g(x)|^m < 2^m M |g(x) - \tilde{g}(x)|^m + 2^m M C |f(x) - \tilde{f}(x)| + 2^m M C |f(x)|. \quad (6.2)$$

Proof:

$$|g(x)|^m \leq (|g(x) - \tilde{g}(x)| + |\tilde{g}(x)|)^m \leq 2^m M |g(x) - \tilde{g}(x)|^m + 2^m M |\tilde{g}(x)|^m,$$

where $M$ is the maximum over all positive coefficients appearing in the binomial expansion.

We now use that $|\tilde{g}(x)|^m \leq C |\tilde{f}(x)|$ for all $x \in U$ to conclude that

$$|g(x)|^m \leq 2^m M |g(x) - \tilde{g}(x)|^m + 2^m M C |\tilde{f}(x)|$$

$$\leq 2^m M |g(x) - \tilde{g}(x)|^m + 2^m M C |f(x) - \tilde{f}(x)| + 2^m M C |f(x)|,$$

for all $x \in U$. The strict version of the estimate, (6.2) follows trivially. 

Since the preliminaries of the continuity argument have been given, we can now state precisely what the Łojasiewicz Nullstellensatz in Corollary 6.9 means, i.e. the Łojasiewicz inequality that each function $\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}$ has to satisfy on some neighborhood around the origin. We
do not need to include $\tilde{r}_1$ in this discussion since it is one of the two generators of the ideal $\tilde{I}_q(U')$.

From the setup of the Kohn algorithm, we know that

$$\tilde{I}_q(x_0) = \sqrt[\sqrt[\sqrt[\cdot \cdot \cdot \sqrt{r (\tilde{r}, \text{coeff}\{\partial \tilde{r} \wedge \bar{\partial} \tilde{r} \wedge (\partial \bar{\partial} \tilde{r})^{n-q}\})}]}.$$  

Let $\tilde{f} = \text{coeff}\{\partial \tilde{r} \wedge \bar{\partial} \tilde{r} \wedge (\partial \bar{\partial} \tilde{r})^{n-q}\}$. Corollary [6.9] says that

$$\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \sqrt[\sqrt[\sqrt[\cdot \cdot \cdot \sqrt{r (\tilde{r}, \text{coeff}\{\partial \tilde{r} \wedge \bar{\partial} \tilde{r} \wedge (\partial \bar{\partial} \tilde{r})^{n-q}\})}]}.$$  

so for each $j$ such that $p + 2 \leq j \leq n + 1 - q$, there exist a positive integer $m_j \in \mathbb{N}^*$ as well as a constant $C_j > 0$ such that

$$|\tilde{r}_j(x)|^{m_j} \leq C_j \left( |\tilde{r}(x)| + |\tilde{f}(x)| \right)$$

for all $x \in U'$, $U'$ being the neighborhood guaranteed by Corollary [6.9]. This is the Lojasiewicz inequality on which the proof of Theorem [1.2] will rest and clearly applies only in the case when there are only two level sets of the commutator multitype in the optimal neighborhood around a point $x_0$ in the boundary of $\Omega$. In the general case of Theorem [1.2] when there are $N$ level sets of the commutator multitype in the optimal neighborhood $U$, we will show that

$$r_{p+2}, \ldots, r_{n+1-q} \in \mathcal{I}(\mathcal{V}(I_{N-1}^q(U))),$$

which in the truncation of the boundary system will amount to

$$\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \mathcal{I}(\mathcal{V}(\tilde{I}_{N-1}^q(U'))).$$

Polynomials form a subring of $C^\omega$, which is Noetherian, so by the same algebraic reasoning as in the proof of Corollary [6.9] we conclude that

$$\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_{N-1}^q(U'),$$

and there exist finitely many generators $\tilde{f}_1, \ldots, \tilde{f}_p$ for the ideal $\tilde{I}_{N-1}^q(U')$, i.e.

$$\tilde{I}_{N-1}^q(U') = (\tilde{f}_1, \ldots, \tilde{f}_p).$$

All functions in $C^\omega$ satisfy the three Lojasiewicz inequalities, so we know there exist a constant $C_j > 0$ and a positive integer $m_j \in \mathbb{N}^*$ for every $p + 2 \leq j \leq n + 1 - q$ such that for all $x \in U'$,

$$|\tilde{r}_j(x)|^{m_j} \leq C_j \left( |\tilde{f}_1(x)| + \cdots + |\tilde{f}_p(x)| \right).$$

We would like the next proposition to apply to both the case $N = 2$ and $N > 2$. For the case when $N = 2$, we change the notation by letting $\tilde{r}_1 = \tilde{f}_1$ and $\tilde{f} = \tilde{f}_2$. Now the notation for the two cases is consistent, and we can state the proposition that gives the continuity argument for the Lojasiewicz inequality:

**Proposition 6.15.** Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex smooth domain, and let $0 \in b\Omega$ be a boundary point of finite D’Angelo $q$ type. Let $U$ be an optimal neighborhood of $0$ with respect to the commutator multitype $\mathcal{C}^{n+1-q}$. Let

$$\mathfrak{B}_{n+1-q}(0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$
be the boundary system at 0, and let
\[ \mathfrak{B}_{n+1-q,t}(0) = \{ r_{1,t}, r_{p+2,t}, \ldots, r_{n+1-q,t}; L_{2,t}, \ldots, L_{n+1-q,t} \} \]
be its transformation under the action of the map \( \Lambda_t \), for \( t \in [0, 1] \). Assume the following two additional hypotheses are satisfied:

(a) \( r_{p+2,t}, \ldots, r_{n+1-q,t} \in \mathcal{I}(V(I_{k,t}^q(U_t))) \),
for the same step \( k \) of the Kohn algorithm on some neighborhood \( U_t \) of the origin and every \( t \in [0, 1] \);

(b) There exist finitely many \( \tilde{f}_1, \ldots, \tilde{f}_p \in I_{k,t}^q(U_0) \) such that for every \( j \) satisfying \( p + 2 \leq j \leq n + 1 - q \),
\[ |r_{j,0}(x)|^{m_j} \leq C_j (|\tilde{f}_1(x)| + \cdots + |\tilde{f}_p(x)|) \] (6.3)
for a positive integer \( m_j \in \mathbb{N}^* \) and a constant \( C_j > 0 \) on the open set \( U_0 \).

Then there exist a perhaps small neighborhood \( U'_t \) of 0 satisfying \( U'_t \subseteq U_t \) as well as functions \( f_{1,t}, \ldots, f_{p,t} \in I_{k,t}^q(U_t) \) for every \( t \in (0, 1] \) such that
\[ \lim_{t \to 0} f_{1,t} = \tilde{f}_1, \ldots, \lim_{t \to 0} f_{p,t} = \tilde{f}_p \]
uniformly in the \( C^\infty \) topology and
\[ |r_{j,0}(x)|^{m_j} \leq C_{j,t} (|f_{1,t}(x)| + \cdots + |f_{p,t}(x)|) \] (6.4)
for every \( 0 \leq t \leq 1 \) and every \( x \in U'_t \), where \( C_{j,t} > 0 \) might be different from \( C_j \). In other words, the Lojasiewicz inequality is satisfied with the same power \( m_j \) for each of the functions \( r_{j,t} \) on \( U_t \). Here \( f_{1,0} = \tilde{f}_1, \ldots, f_{p,0} = \tilde{f}_p \). We conclude that
\[ r_{p+2,t}, \ldots, r_{n+1-q,t} \in I_{k,t}^q(U'_t) \] (6.5)
i.e. the functions \( r_{1,t}, r_{p+2,t}, \ldots, r_{n+1-q,t} \) are subelliptic multipliers in the \( k^{th} \) ideal \( I_{k,t}^q(U'_t) \) of the Kohn algorithm on the neighborhood \( U'_t \) of the domain \( \Omega_t \) that corresponds to \( \Omega \) under the action of the map \( \Lambda_t \).

**Proof:** Let
\[ G = \{ t \in [0, 1] \mid \text{condition (6.4) holds for } t \}. \]
By hypothesis (b), \( 0 \in G \), so \( G \neq \emptyset \). To show that \( G = [0, 1] \), we have to establish that there is a neighborhood around 0 in \( G \) of radius \( \delta > 0 \) and then show that from this conclusion it follows all points in \([0, 1]\) are in \( G \). We start by running the Kohn algorithm for each \( t \in [0, 1] \) on \( U_t \) around the origin, which is on the boundary of the domain \( \Omega_t \) with defining function \( r_{1,t} \). From the definition of the Kohn algorithm is clear that the defining function determines all subelliptic multipliers in the various steps of the Kohn algorithm. Since the moves of the algorithm are the same regardless of \( t \), and passing from \( \Omega_1 = \Omega \) to \( \Omega_0 = \widehat{\Omega} \) happens along a family of defining functions \( r_{1,t} \) that converges uniformly in the \( C^\infty \) topology, it is clear that
for any given \( j \) with \( 1 \leq j \leq P \), there exist families of functions \( \{ f_{1,t} \}_{t \in (0,1]}, \ldots, \{ f_{P,t} \}_{t \in (0,1]} \) such that
\[
\lim_{t \to 0} f_{1,t} = \tilde{f}_1, \quad \ldots, \quad \lim_{t \to 0} f_{P,t} = \tilde{f}_P
\]
and \( f_{1,t}, \ldots, f_{P,t} \in I_{k,t}^q(U_t) \) for every \( t \in (0,1] \). There might in fact be more than one such family for a certain \( j \) such that \( 1 \leq j \leq P \) that converges to \( \tilde{f}_j \) in the \( C^\infty \) topology since the multiplier ideals on the original domain \( \Omega \) are composed of smooth functions and may contain flat functions as well. Note that the addition of a flat function may change the family but not its limit in the \( C^\infty \) topology. We thus choose one appropriate such family for each \( j \) in the first step of the proof.

**Step 1:** \( \exists \delta > 0 \) such that \( B_\delta(0) \cap [0,1] \subset G \)

Since \( 0 \in G \), we restate (6.3) as in part (b) of Lemma 6.14, i.e.
\[
|r_{j,0}(x)|^{m_j} < C_j \left( |f_{1,0}(x)| + \cdots + |f_{P,0}(x)| \right)
\]
on \( U_0 - V(I_{k,0}^q(U_0)) \) and apply (6.2) of Lemma 6.14:
\[
|r_{j,t}(x)|^{m_j} < 2^{m_j} M |r_{j,t}(x) - r_{j,0}(x)|^{m_j} + 2^{m_j} M C_{j,t_0} |f_{1,t}(x) - f_{1,0}(x)| + 2^{m_j} M C_{j,t_0} |f_{P,t}(x)| + 2^{m_j} M C_{j,t_0} |f_{P,t}(x)|
\]

For \( t \) close enough to \( 0 \), \( f_{1,t}, \ldots, f_{P,t} \) do not all vanish at \( x \in U_0 - V(I_{k,0}^q(U_0)) \) and on some \( U_t \) such that \( 0 \in U_t \) and \( U_t \subset U_0 \), the differences \( |f_{1,t}(x) - f_{1,0}(x)|, \ldots, |f_{P,t}(x) - f_{P,0}(x)| \) are controlled by the sup norm on \( U_t \) and can be made very small for appropriately chosen families \( \{ f_{1,t} \}_{t \in (0,1]}, \ldots, \{ f_{P,t} \}_{t \in (0,1]} \) such that
\[
\lim_{t \to 0} f_{1,t} = \tilde{f}_1, \quad \ldots, \quad \lim_{t \to 0} f_{P,t} = \tilde{f}_P
\]
and \( f_{1,t}, \ldots, f_{P,t} \in I_{k,t}^q(U_t) \) for every \( t \in (0,1] \). We conclude all such small values of \( t \) are in \( G \).

**Step 2:** \( G = [0,1] \)

Let \( t' \in B_\delta(0) \) be such that \( t' > 0 \). By step 1, \( t' \in G \). Look at the image of the set \( U_{t'} \) under the inverse of the map \( \Lambda_{t'} \), which is a bijective holomorphic map for neighborhoods of the origin, and choose \( U_{t'}' \subset U \cap (\Lambda_{t'})^{-1}(U_{t'}) \). Clearly, \( 0 \in U_{t'}' \). Furthermore, (6.4) follows for all \( t' < t < 1 \).

The algebraic restatement of the Łojasiewicz inequality given by (6.4) is that
\[
r_p+2,t, \ldots, r_{n+1-q,t} \in \sqrt{p} I_{k,t}^q(U_t') = I_{k,t}^q(U_t').
\]
This is precisely (6.5). \( \square \)

In the next section, we will obtain an effective Nullstellensatz by computing upper bounds for the powers \( m_{p+2}, \ldots, m_{n+1-q} \) explicitly in case there are only two levels of the commutator multitype present in an optimal neighborhood of \( 0 \) in the boundary of the original domain \( \Omega \). Proposition 6.15 will give a non-effective Nullstellensatz in the general case.
7. An effective Nullstellensatz

We will start with effective computations that set the stage for proving Theorem 1.3. We will then give the proofs first for Theorem 1.3, which is the simplest case, and then for Theorem 1.2, which is the general case. We will end with the three-way equivalence that establishes the main theorem, Theorem 1.1.

Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex smooth domain, and let \( x_0 \in b\Omega \) be a boundary point of finite D’Angelo \( q \) type \( \Delta_q(b\Omega, x_0) = t \). By its very definition, \( t \in \mathbb{Q} \) and \( t \geq 2 \). If the D’Angelo \( q \) type achieves its minimum at \( x_0 \), i.e. \( t = 2 \), then the commutator type \( \mathcal{C}^{n+1-q}(x_0) = (1, 2, \ldots, 2) \) because its entries for \( j \) such that \( 2 \leq j \leq n+1-q \) satisfy that \( 2 \leq c_j \leq t \). Therefore, the Levi form has rank at least \( n - q \) at \( x_0 \), and \( \partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}(x_0) \neq 0 \), which means that \( I^q(x_0) = C^\infty(x_0) \), so the Kohn algorithm ends at step 1. Furthermore, by parts (ii) and (vi) of Proposition 2.6, \( \text{coeff} \{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q} \} \) satisfies (2.2) with \( \epsilon = \frac{1}{2} \).

Let us assume that the D’Angelo \( q \) type is finite and satisfies \( \Delta_q(b\Omega, x_0) = t > 2 \) at \( x_0 \). In that case, the fact that the D’Angelo \( q \) type is finite implies that

\[
\mathcal{B}_{n+1-q}(x_0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}
\]

is a boundary system of rank \( p \) and codimension \( q - 1 \). Let us now translate the point \( x_0 \) to the origin and truncate \( \mathcal{B}_{n+1-q}(0) \) with respect to the weight

\[
\Lambda = (1, \lambda_2, \ldots, \lambda_{n+1-q}, \lambda_{n+1-q}, \ldots, \lambda_{n+1-q}),
\]

where

\[
\mathfrak{M}^{n+1-q}(0) = \mathcal{C}^{n+1-q}(0) = (1, \lambda_2, \ldots, \lambda_{n+1-q}).
\]

By part (4) of Theorem 4.4, \( \lambda_j \leq t \) for all \( 2 \leq j \leq n+1-q \), so this same inequality thus holds for all entries of the weight \( \Lambda \), since we defined \( \Lambda \) in such a way that entries beyond the \( (n+1-q) \)th one equal the highest known finite entry \( \lambda_{n+1-q} \leq t \). The truncated boundary system

\[
\mathcal{B}_{n+1-q}(0) = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}; \tilde{L}_2, \ldots, \tilde{L}_{n+1-q}\}
\]

consists of functions \( \tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \) that are all polynomials of degrees that we can estimate. First of all, let us estimate the degree of \( \tilde{r}_1 \). We know that \( \tilde{r}_1 \in \mathfrak{S}(1; \Lambda) \), so the terms from the Taylor expansion of \( r_1 = r \) at 0 that survive are the ones corresponding to \( z^\alpha \bar{z}^\beta \) such that \( \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} = 1 \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \). Since \( \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq t \), it follows that we retain terms of degree at most \( (|\alpha| + |\beta|) \frac{1}{|t|+1} = 1 \), where \( |t| \) is the integer part of \( t \in \mathbb{Q} \), i.e. the highest integer \( d \in \mathbb{Z} \) such that \( d \leq t \). In other words, \( |t| + 1 \) is the lowest integer greater than or equal to \( t \). We have thus obtained that \( \tilde{r}_1 \) is a polynomial of degree at most \( |t| + 1 \).

\[
\tilde{r}_k \in \mathfrak{S} \left( \frac{1}{x_k}; \Lambda \right) \text{ for } p + 2 \leq k \leq n + 1 - q.
\]

The terms from the Taylor expansion of \( r_k \) at 0 that survive the truncation are the ones corresponding to \( z^\alpha \bar{z}^\beta \) such that \( \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} = \frac{1}{x_k} \leq \frac{1}{3} \), since the Levi form has rank \( p \) at 0, so \( \lambda_j \geq 3 \) for \( j \geq p + 2 \). Therefore, the terms \( z^\alpha \bar{z}^\beta \) of \( \tilde{r}_k \) have degrees that satisfy \( (|\alpha| + |\beta|) \frac{1}{|t|+1} \leq \frac{1}{3} \), so \( \tilde{r}_k \) has degree at most \( \left\lfloor \frac{|t|+1}{3} \right\rfloor + 1 \).
Finally, we have to compute the degree of \( \tilde{f} = \text{coeff}\{\partial \tilde{r} \wedge \bar{\partial} \tilde{r} \wedge (\partial \bar{\partial} \tilde{r})^{n-q}\} \). Since \( \tilde{r} \) is a polynomial of degree at most \([t] + 1\), each coefficient of the Levi form on the domain \( \tilde{\Omega} \) defined by \( \tilde{r} \) will be a polynomial of degree at most \([t] - 1\), since these coefficients are obtained by differentiating \( \tilde{r} \) twice. \( \tilde{f} \) involves the determinants of the \( n - q \) minors of the Levi form, so it can have degree at most \(([t] - 1)^{n-q}\).

By Corollary \ref{corollary:degree_bound} we know that for each \( k \) such that \( p + 2 \leq k \leq n + 1 - q \), there exist a positive integer \( m_k \in \mathbb{N}^* \) as well as a constant \( C_k > 0 \) such that

\[
|\tilde{r}_k(x)|^{m_k} \leq C_j (|\tilde{r}(x)| + |\tilde{f}(x)|)
\]

for all \( x \in U', U' \) being the neighborhood guaranteed by Corollary \ref{corollary:degree_bound}. We would like to give an upper bound for the integer \( m_k \) in terms of the degrees of \( \tilde{r}_k, \tilde{r}_1, \) and \( \tilde{f} \). We can rewrite the Lojasiewicz inequality as

\[
|\tilde{r}_k(x)^{m_k}| \leq C_j (|\tilde{r}(x)| + |\tilde{f}(x)|).
\]

Now, by construction we know that \( \tilde{r}_k \) just like \( r_k \) has at least one term of degree 1, since it has a non-zero gradient. The most generous bound for \( m_k \) is obtained by noticing that the lowest terms of \( \tilde{r}_k(x)^{m_k} \) should have degrees that equal the highest degrees of the terms of the right hand side of the inequality. The lowest degree terms on the left hand side have degree \( m_k \) and the highest degree terms on the right hand side have degree \( \max\{([t] - 1)^{n-q}, [t] + 1\} \), which is the maximum of the degree of \( \tilde{f} \) and the degree of \( \tilde{r} \). Therefore, we conclude that

\[
m_k \leq \max\{([t] - 1)^{n-q}, [t] + 1\} \text{ for all } k = p + 2, \ldots, n + 1 - q.
\]

We can now prove Theorem \ref{theorem:algorithm_finishes}

**Proof of Theorem \ref{theorem:algorithm_finishes}**

The assertion that the Kohn algorithm finishes at step 1 densely in \( b\Omega \) follows from Corollary \ref{corollary:algorithm_finishes}. If the Levi form does not have rank at least \( n - q \) at \( x_0 \), we translate \( x_0 \) to the origin and truncate the boundary system. By Proposition \ref{proposition:neighborhood_exists}, there exists a neighborhood \( U_t \) of the origin such that

\[
r_{p+2,t}, \ldots, r_{n+1-q,t} \in I_{1,t}(U_t)
\]

for \( t = 1 \), which is equivalent to

\[
r_{p+2}, \ldots, r_{n+1-q} \in I_1(U_1).
\]

By Lemma \ref{lemma:higher_order_terms}

\[
\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}(0) \neq 0.
\]

This continues to hold on some neighborhood \( U' \) such that \( 0 \in U' \subset U_1 \). Clearly,

\[
\text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}\} \in I_2(U'),
\]

so the algorithm finishes at step 2.

We can easily compute the lower bound for the subelliptic gain. The ideal \( I_1(U') \) has two generators, \( r \) and \( f = \text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^{n-q}\} \). By part (i) of Proposition \ref{proposition:gain_properties} \( r \) satisfies \((\ref{equation:gain_1})\) with \( \epsilon = 1 \). By parts (ii) and (vi) of Proposition \ref{proposition:gain_properties} \( f \) satisfies \((\ref{equation:gain_1})\) with \( \epsilon = \frac{1}{2} \). By
By part (iv) of Proposition 2.6. The element $\partial r_k$ satisfies (2.3) with $\epsilon \geq \left(\frac{1}{2}\right)^2 \frac{1}{\max\{[t]-1, [t]+1\}}$ by part (v) of Proposition 2.6. Finally, by part (vi) of Proposition 2.6, the unit

$$\text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}\}$$

satisfies (2.2) with $\epsilon \geq \min\left\{\frac{1}{2}, \frac{1}{4} \frac{1}{\max\{[t]-1, [t]+1\}}\right\} = \frac{1}{4} \frac{1}{\max\{[t]-1, [t]+1\}}$. \hfill \Box

**Remarks:**

1. The term $([t]-1)^{n-q}$ is very close to $2^{q-1-n} t^{n-q}$, which is the maximal jump of the D'Angelo type in a neighborhood of $x_0$. John D'Angelo calculated this jump in [8] by truncating the defining function $r$ in a way that is very similar to Catlin's truncation of a boundary system as we already remarked in Section 6. It is very possible this similarity in bounds results from an interesting algebraic phenomenon having to do with truncation.

2. It is known that the subelliptic gain $\epsilon$ has to satisfy the upper bound $\epsilon \leq \frac{1}{t}$, for $t$ the finite D'Angelo type. Thus, $\epsilon \geq \frac{1}{4} \frac{1}{\max\{[t]-1, [t]+1\}}$ is a polynomial lower bound in $t$ that is not so far in order of magnitude from the known optimal upper one. It should be remarked here that there are examples in the literature of domains where the optimal bound $\frac{1}{t}$ is clearly not reached.

3. For $q$ not so large compared to $n$, the term $([t]-1)^{n-q}$ dominates in

$$\max\{([t]-1)^{n-q}, [t]+1\}.$$ 

Here is now the general case, Theorem 1.2.

**Proof of Theorem 1.2:** Just as in the proof of Theorem 1.3, the assertion that the Kohn algorithm finishes at step 1 densely in $b\Omega$ follows from Corollary 5.4. If the Levi form does not have rank at least $n-q$ at $x_0$, let

$$\mathcal{B}_{n+1-q}(x_0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$

be the boundary system at $x_0$. We apply the Stratification Theorem, Theorem 5.5, to conclude that there exists a neighborhood $U'$ of $x_0$ optimal for the commutator multitype $\mathcal{C}^{n+1-q}$ such that

$$r_1, r_{p+2}, \ldots, r_{n+1-q} \in \mathcal{T}\left(\mathcal{V}(I_1^q(U')) - \bigcup_{j=2}^{N-1} S_j\right),$$

where $S_1, \ldots, S_N$ are the level sets for $\mathcal{C}^{n+1-q}$ in $U'$ corresponding to increasing values of $\mathcal{C}^{n+1-q}$ in the lexicographic ordering. Given how Theorem 5.5 is proven, it is clear that all points of $U'$ satisfy the finite D'Angelo $q$ type condition. Let us look at all the points $x \in S_2$. We choose for each of these a neighborhood $U_x \subset U'$ such that $x \in U_x$ and $U_x$ is the neighborhood guaranteed
by part (i) of Proposition 5.2. Therefore, \( U_x \) contains only two level sets of the commutator multitype \( \mathfrak{C}^{n+1-q} \). Theorem 5.3 applies to show that at each \( x \in S_2 \), the Kohn algorithm finishes at step 2. In particular this implies that \( \mathcal{V}(I^q_2(U')) \subset \bigcup_{j=3}^{N} S_j \). We no longer have strict equality as in the proof of Theorem 5.5 because termination of the Kohn algorithm at a point is an open condition, so step 2 will remove entire neighborhoods of each of the points \( x \in S_2 \) and not just the points themselves.

Next we look at the points \( x \in S_3 \). We choose for each of these a neighborhood \( U_x \subset U' \) such that \( x \in U_x \) and \( U_x \) is the neighborhood guaranteed by part (i) of Proposition 5.2. Therefore, \( U_x \) contains only three level sets of the commutator multitype \( \mathfrak{C}^{n+1-q} \). We now look at the boundary system
\[
\mathfrak{B}_{n+1-q}(x) = \{ r'_1, r'_{p+2}, \ldots, r'_{n+1-q}; L'_2, \ldots, L'_{n+1-q} \}
\]
at \( x \) defined such that on the neighborhood \( U_x \) all parts of Proposition 5.2 are satisfied. Note the change in notation compared to \( \mathfrak{B}_{n+1-q}(x_0) \). As mentioned above, the D’Angelo q type is finite at \( x \). Since there are only three level sets of the commutator multitype in \( U_x \), it follows that Theorem 5.5 applied to \( U_x \) yields that
\[
r'_1, r'_{p+2}, \ldots, r'_{n+1-q} \in \mathcal{I}(\mathcal{V}(I^q_2(U_x))) - S_2),
\]
where \( S_1, \ldots, S_3 \) are the level sets for \( \mathfrak{C}^{n+1-q} \) in \( U_x \). We may have to shrink \( U_x \) a little for the previous assertion to hold, but we will not change notation as not to complicate this write-up. By the same procedure as we applied at step 2 of the Kohn algorithm on \( U' \) but applied instead to \( U_x \), clearly
\[
r'_1, r'_{p+2}, \ldots, r'_{n+1-q} \in \mathcal{I}(\mathcal{V}(I^q_2(U_x))).
\]
Let us now translate the point \( x \) to the origin and truncate the boundary system to some
\[
\bar{\mathfrak{B}}_{n+1-q}(0) = \{ \bar{r}'_1, \bar{r}'_{p+2}, \ldots, \bar{r}'_{n+1-q}; \bar{L}'_2, \ldots, \bar{L}'_{n+1-q} \}.
\]
By Proposition 6.7, the commutator multitype \( \bar{\mathfrak{C}}^{n+1-q} \) has at most three level sets in an optimal neighborhood \( \bar{U} \). We now apply the equivalent statement to the Stratification Theorem to \( \bar{U} \), namely Proposition 6.8 and then the same procedure as we applied at step 2 of the Kohn algorithm to \( U' \) to remove all points corresponding to the middle level set of the commutator multitype \( \bar{S}_2 \). We conclude that
\[
\bar{r}'_1, \bar{r}'_{p+2}, \ldots, \bar{r}'_{n+1-q} \in \mathcal{I}(\mathcal{V}(I^q_2(\bar{U}))),
\]
where \( I^q_2(\bar{U}) \) is the ideal of multipliers corresponding to step 2 of the Kohn algorithm applied on \( \bar{U} \). Note that it is possible that the functions \( \bar{r}'_1, \bar{r}'_{p+2}, \ldots, \bar{r}'_{n+1-q} \) already belong to \( \mathcal{I}(\mathcal{V}(I^q_1(\bar{U}))) \) in case \( \bar{U} \) contains only two level sets of the commutator multitype \( \bar{\mathfrak{C}}^{n+1-q} \). This does not matter since by definition the ideals in the Kohn algorithm contain all previous multipliers, so
\[
\bar{r}'_1, \bar{r}'_{p+2}, \ldots, \bar{r}'_{n+1-q} \in \mathcal{I}(\mathcal{V}(I^q_2(\bar{U})))
\]
clearly holds in this case as well. Note also that by Proposition 6.11 and Proposition 6.12, the same reasoning holds for all intermediate boundary systems \( \mathfrak{B}_{n+1-q,t}(0) \) on some neighborhood.
Let us now choose some generators $\tilde{f}_1, \ldots, \tilde{f}_P$ for the ideal $\tilde{I}_2^n(\tilde{U})$, which is an ideal in a Noetherian ring, hence it is finitely generated. All subelliptic multipliers for the truncated domain are polynomials and satisfy the Lojasiewicz inequalities, so we know there exist a constant $C_j > 0$ and a positive integer $m_j \in \mathbb{N}^*$ for every $p + 2 \leq j \leq n + 1 - q$ such that for all $x \in \tilde{U}$,

$$|r_j'(x)|^{m_j} \leq C_j \left( |\tilde{f}_1(x)| + \cdots + |\tilde{f}_P(x)| \right).$$

Given how the truncation of a boundary system was done, it is clear that there exist

$$f_1, \ldots, f_P \in I_2^q(U_x)$$

such that $f_j$ corresponds to $\tilde{f}_j$ under the truncation by the weight $\Lambda$ for all $1 \leq j \leq P$ since we apply the Kohn algorithm in the same way on both the original domain and the truncated one. A priori, the smooth functions $f_1, \ldots, f_P$ that satisfy this property may not be unique, but that does not matter as some such functions get chosen in the course of the proof of Proposition 6.15. We thus apply Proposition 6.15 to conclude a Lojasiewicz inequality on potentially some shrinking of $U_x$ holds, namely

$$|r_j'(x)|^{m_j} \leq C_j \left( |f_1(x)| + \cdots + |f_P(x)| \right)$$

for each $p + 2 \leq j \leq n + 1 - q$, for the same integer $m_j \in \mathbb{N}^*$, and some potentially bigger constant $C_j'$. We have already translated the origin back to $x$ in the previous statement. We conclude that

$$r_{p+2}', \ldots, r_{n+1-q}' \in I_2^q(U_x).$$

By Lemma 4.8

$$\partial r \wedge \overline{\partial} r \wedge (\partial \overline{\partial} r)^p \wedge \partial r_{p+2}' \wedge \cdots \wedge \partial r_{n+1-q}'(x) \neq 0.$$  

This continues to hold on some neighborhood $U''$ such that $x \in U'' \subset U_x$. Clearly,

$$\text{coeff}\{\partial r \wedge \overline{\partial} r \wedge (\partial \overline{\partial} r)^p \wedge \partial r_{p+2}' \wedge \cdots \wedge \partial r_{n+1-q}'\} \in I_2^q(U''),$$

so the algorithm finishes at step 3.

We thus remove at each step $j$ of the Kohn algorithm on the neighborhood $U'$ at least all the points in $S_j$. Inductively, we conclude that

$$r_1', r_{p+2}', \ldots, r_{n+1-q}' \in \mathcal{I}(\mathcal{V}(I_{n-1}^q(U'))).$$

Let us now translate the point $x_0$ to the origin and truncate the boundary system

$$\mathfrak{B}_{n+1-q}(0) = \{r_1, r_{p+2}, \ldots, r_{n+1-q}; L_2, \ldots, L_{n+1-q}\}$$

to some

$$\tilde{\mathfrak{B}}_{n+1-q}(0) = \{\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q}; \tilde{L}_2, \ldots, \tilde{L}_{n+1-q}\}.$$

By Proposition 6.7, the commutator multitype $\mathcal{E}_{n+1-q}$ has at most $N$ level sets in an optimal neighborhood $\tilde{U}'$ of the origin. We apply the same procedure in the neighborhood $\tilde{U}'$ as we
applied in the neighborhood $\tilde{U}$ at step 3 but inductively up to step $N-1$ of the Kohn algorithm. We conclude

$$\tilde{r}_1, \tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \mathcal{I}(\mathcal{V}(\tilde{I}_{N-1}^q(\tilde{U}')))$$

As mentioned above, this may be true already at a previous step if the number of level sets of the commutator multitype $\mathfrak{C}^{n+1-q}$ in $\tilde{U}'$ is strictly smaller than $N$. Just as before, on the truncated domain we can immediately conclude that

$$\tilde{r}_{p+2}, \ldots, \tilde{r}_{n+1-q} \in \tilde{I}_{N-1}^q(\tilde{U}')$$

By Proposition 6.11 and Proposition 6.12, for all intermediate boundary systems $\tilde{B}_{n+1-q,t}(0)$ on some neighborhood $U_t$ of the origin, i.e. for every $t \in (0, 1)$,

$$r_{1,t}, r_{p+2,t}, \ldots, r_{n+1-q,t} \in \mathcal{I}(\mathcal{V}(I_{N-1,t}^q(U_t)))$$

We use the same argument involving Proposition 6.15 to bring the Lojasiewicz inequalities corresponding to the previous statement back to the original domain. We translate the origin back to $x_0$. We thus obtain that for some potentially smaller $U'' \subset U'$ containing $x_0$,

$$r_{p+2}, \ldots, r_{n+1-q} \in I_{N-1}^q(U'')$$

By Lemma 4.8,

$$\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}(x_0) \neq 0$$

This continues to hold on some neighborhood $U'''$ such that $x_0 \in U''' \subset U'' \subset U'$. Clearly,

$$\text{coeff}\{\partial r \wedge \bar{\partial} r \wedge (\partial \bar{\partial} r)^p \wedge \partial r_{p+2} \wedge \cdots \wedge \partial r_{n+1-q}\} \in I_N^q(U''')$$

so the algorithm finishes by step $N$ on all of $U'''$ including at $x_0$. We apply the same procedure at all $x \in S_N$. The Kohn algorithm finishes by step $N$ everywhere. □

**Remark:** Note that we have used Theorem 1.4 in an essential way since we can apply the Kohn algorithm at will on smaller or larger neighborhoods as necessary to remove various points. This is true precisely because all multiplier ideal sheaves $I_{q,k}$, at all steps $k \geq 1$ of the Kohn algorithm satisfy parts (a) and (b) of Theorem 1.4.

We finally have all the necessary results for proving the equivalence of types for smooth, pseudoconvex domains in $\mathbb{C}^n$:

**Proof of Theorem 1.1**

(ii) $\implies$ (i) This implication was already proven by Joseph J. Kohn in [17] for the $C^\infty$ case. Definitions of subellipticity of the $\bar{\partial}$-Neumann problem and of subelliptic multipliers, Definition 2.1 and Definition 2.2 respectively, make it clear why this is the case.

(i) $\implies$ (iii) This implication is the contrapositive of Catlin’s Theorem 1 from [5], which says that if a holomorphic variety of dimension $q$ has order of contact $\eta$ with $b\Omega$ at $x_0$, then the $\bar{\partial}$-Neumann problem for $(p,q)$ forms gains at most $\epsilon \leq 1/\eta$. Thus if (iii) is false, and there exists a holomorphic variety $\mathcal{V}$ of complex dimension $q$ passing through $x_0$ that has infinite order of contact with $b\Omega$, as $\eta \to \infty$, $\epsilon \to 0$, so subellipticity cannot hold.
(iii) ⇒ (ii) This is merely the qualitative statement that follows from the more precise Theorem 1.2.

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