Conjugate Flow Action Functionals

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We present a new general method to construct an action functional for a non-potential field theory. The key idea relies on representing the governing equations of the theory relative to a diffeomorphic flow of curvilinear coordinates which is assumed to be functionally dependent on the solution field. Such flow, which will be called the conjugate flow of the theory, evolves in space and time similarly to a physical fluid flow of classical mechanics and it can be selected in order to symmetrize the Gâteaux derivative of the field equations with respect to suitable local bilinear forms. This is equivalent to requiring that the governing equations of the field theory can be derived from a principle of stationary action on a Lie group manifold. By using a general operator framework, we obtain the determining equations of such manifold and the corresponding conjugate flow action functional. In particular, we study scalar and vector field theories governed by second-order nonlinear partial differential equations. The identification of transformation groups leaving the conjugate flow action functional invariant could lead to the discovery of new conservation laws in fluid dynamics and other disciplines.

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I. INTRODUCTION

The problem of finding an action functional whose Euler-Lagrange equations correspond to a prescribed set of partial differential equations (PDEs) is known as the inverse problem of the calculus of variations and it has attracted the attention of researchers for more than one century. Perhaps, one of the main reasons is that the formulation of a field theory in terms of an action functional is very elegant and, more importantly, it allows us to establish an immediate connection between symmetry principles and conservation laws.

For a given system of field equations, the existence conditions of an action functional can be put in a correspondence with the theory of conservative vector fields. Essentially, if the path integral of the nonlinear operator representing the field equations is independent on the trajectory of fields connecting two specified points in a function space, then there exists a scalar field (the action) whose functional derivative yields the governing equations of the theory. The path integral of an operator along a trajectory of fields is defined in terms of a bilinear form which can be selected in a rather arbitrary way. In this sense, the solution to the inverse problem of the calculus of variations is reduced to look for a bilinear form that makes the given nonlinear operator potential, if any. It was shown by Tonti that there exist not just one but an infinite number of such forms. Therefore, an infinite number of action principles can be constructed for a given set of field equations. However, the physical meaning of such generalized principles is often obscured by the bilinear form that has to be adjusted on the given system of field equations. An alternative way to proceed is to select a specific bilinear form, e.g., one having a physical meaning, and then look for ways of modifying the field equations as to obtain a new problem which is potential with respect to the chosen bilinear form. Among known methods devised to do so, we recall the adjoint equation method and the integrating operator method.

The purpose of this paper is to introduce a new general approach to construct an action functional for a non-potential field theory. The key idea relies on representing the governing equations of the theory relative to a curvilinear flow of coordinates that is assumed to be a functional of the solution field. This flow will be called the conjugate flow of the theory and, as we will see, it can be selected in order to satisfy the existence conditions of an action principle for any given set of field equations. Let us briefly describe the main ideas that led us to introduce the conjugate flow and, more importantly, their relevance in the context of
known physical theories. To this end, let us first notice that flows of coordinates depending on solution to a system of field equations arise naturally in many areas of mathematical physics. Perhaps, the most relevant example is in the context of classical fluid mechanics, where the trajectories of fluid elements in space are related to the velocity field that solves, e.g., the Navier-Stokes equations\textsuperscript{1,2,37}. The curvilinear coordinate system advected by such flow is known as \textit{Lagrangian system} and it can be determined by integrating out the definition of the velocity field\textsuperscript{48}. In this sense, physical fluid flow of classical mechanics can be considered as a very particular type of conjugate flow. Another example of conjugate flow is the free-falling coordinate system\textsuperscript{51} in the Einstein’s theory of gravitation. Here the flow appears as a geodesic mapping\textsuperscript{24} in a four-dimensional Riemannian space whose metric is determined by a particular distribution of energy and momentum through the solution of the Einstein’s field equations.

In both examples just discussed, the relation between the solution to the field equations and the conjugate flow is given and it reduces to the definition of the velocity field in the case of Navier-Stokes equations and to the definition of geodesics in the case of Einstein’s theory of gravitation. In a broader framework, however, such functional relation may be left unspecified. This yields an infinite number of functional degrees of freedom (those associated with the conjugate flow) that can be selected, for instance, by requiring that the governing equations of the theory can be derived from a principle of stationary action. This new formulation of the inverse problem of the calculus of variations brings together concepts of differential geometry and nonlinear functional analysis and it results in \textit{new types of action principles} generalizing those ones based on specific functional flows, e.g., the Herivel-Lin principle for perfect fluid\textsuperscript{7,10,20,31}.

This paper is organized as follows. In section \textbf{II} we introduce the theory of the conjugate flow and we characterize the group of infinitesimal perturbations by using methods of nonlinear functional analysis. The representation of arbitrary nonlinear field equations in conjugate flow intrinsic coordinates is discussed in section \textbf{III}. In section \textbf{IV} we determine the existence conditions of conjugate flow action functionals in a rather general operator framework. Determining equations for conjugate flows symmetrizing scalar and vector field theories governed by second-order PDEs are obtained in section \textbf{V}. Finally, the main findings and their implications are summarized in section \textbf{VI}. We also include a brief appendix where we recall some fundamental identities of differential geometry that will be extensively used.
II. THE CONJUGATE FLOW

Conjugate flow is an intuitive physical notion which is represented mathematically by a continuous point transformation of Euclidean or Riemannian space into itself. In order to set up this transformation, let us consider a “particle” labeled by $\sigma^\nu (\nu = 0, 1, ..., n$, where $n$ is the number of spatial dimensions and 0 denotes the temporal component) and represent its trajectory in a fixed space-time Cartesian system as

$$x_\sigma^\mu = \hat{x}_\sigma^\mu (\sigma^\nu; u^j), \quad \mu, \nu = 0, ..., n,$$

(1)

where $u^j (j = 1, ..., N)$ is a vector field that solves a prescribed system of field equations. We adopt the convention that Latin indices $i, j, k$, etc., run over spatial coordinate labels (usually 1, 2, 3) while Greek indices $\mu, \nu, \alpha$, etc., run over space-time coordinate labels (usually 0, 1, 2, 3). Also, repeated indices are summed unless otherwise stated. The trajectory of $\sigma^\nu$ defined by (1) can be of course expressed in arbitrary coordinate systems, although such generalization is not of primary importance in what follows. If we consider three spatial dimensions and we assume that the time variable is not transformed, i.e., $\hat{x}^0 = \sigma^0 = t$, then the system (1) reduces to the more familiar form

$$
\begin{align*}
  x_1^1 &= \hat{x}^1 (\sigma^1, \sigma^2, \sigma^3, t; u^j) \\
  x_2^2 &= \hat{x}^2 (\sigma^1, \sigma^2, \sigma^3, t; u^j) \\
  x_3^3 &= \hat{x}^3 (\sigma^1, \sigma^2, \sigma^3, t; u^j)
\end{align*}
$$

where $\sigma^j = \hat{x}^j (\sigma^1, \sigma^2, \sigma^3, t_0; u^j)$ is the initial position of the particle. The transformation (1) is assumed to be invertible (with differentiable inverse) and to possess continuous derivatives up to a prescribed order, except possibly at certain singular surfaces, curves or points. These requirements make (1) a diffeomorphism, i.e., a time-dependent flow of curvilinear coordinates whose motion in space resembles in toto a physical fluid flow of classical mechanics. Einstein called these coordinate systems “reference-mollusks”. In figure we sketch two realizations of the conjugate flow (1) for two different solution fields corresponding, e.g., to different boundary or initial conditions in an initial/boundary value problem for a field equation.
FIG. 1. Sketch of the transport phenomenon of a volume of particles Σ by two different realizations of the conjugate flow \( \hat{x}^\mu(\sigma'; u') \), namely, \( \hat{x}^\mu(\sigma'; u_1') \) and \( \hat{x}^\mu(\sigma'; u_2') \). Shown are trajectories of two different particles labeled as \( \sigma_1 \) and \( \sigma_2 \).

Coordinate flows that are functionals of the solution to a field equation are obviously not new in the literature. For instance, in the context of symmetry analysis of partial differential equations the so-called non-classical symmetries are remarkable examples of field-dependent transformations. Similarly, in classical Lagrangian fluid dynamics the trajectories of fluid elements in space are obtained as local functionals of the velocity field \( U^j(x^k, t) \) that solves, e.g., the Navier-Stokes equations (\( x^k \) here are fixed Cartesian coordinates)

\[
\frac{\partial U^j}{\partial t} + U^k \frac{\partial U^j}{\partial x^k} = - \frac{\partial P}{\partial x^j} + \frac{1}{Re} \frac{\partial^2 U^j}{\partial x^k \partial x^k}, \quad \frac{\partial U^k}{\partial x^k} = 0, \quad k, j = 1, \ldots, n. \tag{2}
\]

Such functional relation is defined by the solution to the well-known initial value problem

\[
\frac{\partial \hat{X}^j(\sigma^i, t)}{\partial t} = U^j \left( \hat{X}^k(\sigma^i, t), t \right), \quad \hat{X}^j(\sigma^i, t_0) = \sigma^j. \tag{3}
\]

In this sense, the physical fluid flow can be considered as a very particular type of conjugate flow, since \( \hat{X}^j \) is functionally dependent on the solution to Eq. \((2)\) by means of Eq. \((3)\). Another conjugate flow which is different than the physical fluid flow may be defined, e.g., by solving

\[
\frac{\partial \hat{X}^j(\sigma^i, t)}{\partial t} = U^j \left( \hat{X}^k(\sigma^i, t) + B^k_\omega(\sigma^i, t), t \right), \quad \hat{X}^j_\omega(\sigma^i, t_0) = \sigma^j. \tag{4}
\]
where $B^k_\omega(\sigma^i, t)$ is a realization of space-time Brownian motion. A remarkable result by Gomes shows that ensemble averaging - over random $B^k_\omega$ - of diffeomorphisms of type (1) allows to construct a variational principle for the Navier-Stokes equations.

A. Infinitesimal flow perturbations

The components of the vector field $u^j$ appearing in Eq. (1) are, by definition, Cartesian components expressed in terms of conjugate flow intrinsic coordinates $\sigma^\nu$. In other words, if we denote by $U^j(x^\mu)$ the Cartesian components of a vector field that satisfies, e.g., Eq. (2), then $u^j$ are defined as

$$u^j(\sigma^\nu) \overset{\text{def}}{=} U^j(\widehat{x}^\mu(\sigma^\nu; u^k)).$$

(5)

Note that these are *not* the tensorial components of the vector field. Now, let us consider an infinitesimal perturbation of $u^j$ in the form

$$u^j(\sigma^\nu) \rightarrow u^j(\sigma^\nu) + \epsilon \varphi^j(\sigma^\nu),$$

(6)

where $\epsilon$ is a small real parameter. Such perturbation can be obviously expressed relative to arbitrary coordinate systems. For instance, in Cartesian coordinates we have $U^j(x^\mu) \rightarrow U^j(x^\mu) + \epsilon \Phi^j(x^\mu)$. Disregarding the particular choice of the coordinate system, the field perturbation (6) induces the following perturbation in the conjugate flow (1)

$$\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j) \simeq \widehat{x}^\mu(\sigma^\nu; u^j) + \epsilon \delta \widehat{x}^\mu \delta u^j \varphi^j,$$

(7)

where, by definition

$$\frac{\delta \widehat{x}^\mu}{\delta u^j} \varphi^j \overset{\text{def}}{=} \lim_{\epsilon \to 0} \frac{\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j) - \widehat{x}^\mu(\sigma^\nu; u^j)}{\epsilon}.$$

(8)

The quantity $\delta \widehat{x}^\mu / \delta u^j$ is known as Gâteaux derivative of the functional $\widehat{x}^\mu$ with respect to $u^j$ and, under rather weak requirements, it is a continuous linear operator. The perturbed flow $\widehat{x}^\mu(\sigma^\nu; u^j + \epsilon \varphi^j)$ is assumed to have the same regularity properties as the unperturbed one, i.e., invertibility and continuous derivatives up to prescribed order in all variables.

Let us now postulate that the solution field $u^j$ is also functionally connected to the conjugate flow $\widehat{x}^\mu$ and let us denote this functional relation by $u^j(\sigma^\nu; \widehat{x}^\mu)$. This fundamental assumption implies that an infinitesimal flow perturbation $\widehat{x}^\mu + \epsilon \widetilde{\varphi}^\mu$ induces the following variation in the solution field $u^j$

$$u^j(\sigma^\nu; \widehat{x}^\mu + \epsilon \widetilde{\varphi}^\mu) \simeq u^j(\sigma^\nu; \widehat{x}^\mu) + \epsilon \delta u^j \delta \widehat{x}^\mu \widetilde{\varphi}^\mu.$$

(9)
FIG. 2. Conjugate flow perturbation \( \hat{x}^\mu + \epsilon \hat{\phi}^\mu \) induced by a field variation \( u^j + \epsilon \phi^j \) and corresponding deformation of the volume of particles \( \sigma^\nu \) advected by the flow.

where, in analogy with Eq. (8), we have defined the Gâteaux differential as

\[
\frac{\delta u^j}{\delta \hat{x}^\mu} \hat{\phi}^\mu \overset{\text{def}}{=} \lim_{\epsilon \to 0} \frac{u^j(\sigma^\nu; \hat{x}^j + \epsilon \hat{\phi}^\mu) - u^j(\sigma^\nu; \hat{x}^j)}{\epsilon}.
\]  

(10)

In the context of the Navier-Stokes equations, this means that a perturbation in the conjugate flow \( \hat{x}^j \) determines - by assumption - a perturbation in the velocity field \( U^j \) that solves Eq. (2). This ultimately results in a perturbation of the physical fluid flow \( \hat{X}^j \) by means of Eq. (3). In other words, by perturbing the conjugate flow we are actually perturbing the physical fluid flow. At this point it is convenient to set

\[
\hat{\phi}^j = \frac{\delta u^j}{\delta \hat{x}^\mu} \hat{\phi}^\mu, \quad \hat{\phi}^\mu = \frac{\delta \hat{x}^\mu}{\delta u^j} \phi^j
\]  

(11)

(12)

and write Eq. (7) and Eq. (9) as

\[
u^j(\sigma^\mu; \hat{x}^\nu + \epsilon \hat{\phi}^\nu) \simeq u^j(\sigma^\mu; \hat{x}^\nu) + \epsilon \hat{\phi}^j(\sigma^\mu; \hat{x}^\nu),
\]  

(13)

\[
\hat{x}^\mu(\sigma^\nu; u^j + \epsilon \phi^j) \simeq \hat{x}^\mu(\sigma^\nu; u^j) + \epsilon \hat{\phi}^\mu(\sigma^\nu; u^j).
\]  

(14)

Note that in these equations we have \( \hat{\phi}^\mu \neq \hat{\phi}^\mu \) and \( \phi^j \neq \hat{\phi}^j \). In fact, if we arbitrarily perform a simultaneous perturbation of \( u^j \) and \( \hat{x}^\mu \) we cannot obviously expect that, in general, the functional disturbances arising from the Gâteaux differentials (11) and (12) coincide with
the perturbations at the left hand side of Eqs. (13) and (14). This immediately leads us to the question of which variable between \( u^j \) and \( \hat{x}^\mu \) should be chosen as independent when performing perturbations. In the sequel we will be mostly concerned with perturbations induced in the conjugate flow \( \hat{x}^\mu \) through a variation of the solution field \( u^j \), i.e. we will mostly employ Eq. (14), although the other approach, i.e. the one based Eq. (13), can be equivalently considered.

### III. CONJUGATE FLOW REPRESENTATION OF FIELD EQUATIONS

Several field equations of mathematical physics, remarkably the fluid mechanics equations, include naturally the concept of conjugate flow within their formulation. Such flow usually has a direct physical interpretation, e.g., trajectories of fluid elements in space, and it often constitutes the ground work on which dynamical results are constructed\(^{1,2,37}\). Many other field equations, however, do not include explicitly any term having a direct reference to a conjugate flow. This is the case, for example, of the classical heat equation, the Maxwell’s equations of electrodynamics, the laws of elasticity and, undoubtedly, many others. The fundamental question at this point is whether it is possible to formulate a law that include both the field equations and the conjugate flow and it allows to study their interaction, e.g., in the context of the principle of stationary action. The answer is affirmative and the simplest way to achieve this result is to represent the field equations relative to a coordinate system which is advected by the conjugate flow, namely, coordinates \( \sigma^\nu \). In other words, we represent the field equations on a curvilinear manifold\(^8\) which is assumed to be functionally dependent on their solution. As a consequence, the equations look completely different in conjugate flow intrinsic coordinates and, in general, they are highly nonlinear. For example, by using the mathematical tools summarized in appendix\(^{A}\) it can be show that the classical one-dimensional heat equation

\[
\frac{\partial U}{\partial t} - \alpha \frac{\partial^2 U}{\partial x^2} = 0, \tag{15}
\]

where \( U(x,t) \) denotes the temperature field in fixed Cartesian coordinates, can be written in terms of conjugate flow intrinsic coordinates as

\[
\frac{\partial u}{\partial t} - \frac{1}{(\partial \hat{x}/\partial \sigma)^3} \alpha \left( \frac{\partial^2 \hat{x} u}{\partial \sigma^2} - \frac{\partial \hat{x} \partial u}{\partial \sigma} \right) = 0, \tag{16}
\]
where the flow \( \mathbf{x} \) is a functional of \( u \). For illustration purposes, here we have assumed that the time variable is not transformed, i.e. we have set \( \mathbf{x}^0 = \sigma^0 = t \). By examining Eq. (16) under the conjugate flow perspective, we see that a perturbation in the field \( u(\sigma, t) \) induces also a perturbation in the conjugate flow \( \mathbf{x}(\sigma, t; u) \) through Eqs. (14) and (12). Therefore the perturbed equation in conjugate flow intrinsic coordinates includes many terms arising from the perturbations of both \( u \) and \( \mathbf{x} \). Clearly, if the conjugate flow is in rest with respect to the fixed Cartesian coordinate system then Eq. (16) coincides with Eq. (15), although the effects of the aforementioned functional perturbations are still present.

From what has been said, it is clear that the conjugate flow representation of a field equation is much more complicated than a standard formulation in fixed Cartesian coordinates (see table 1). This has been observed, e.g., by Temam\(^{38} \), in the context of the Lagrangian representation of the Navier-Stokes equations. He pointed out that “the Lagrangian representation is not used too often because the Navier-Stokes equations in Lagrangian coordinates are highly nonlinear”. Indeed, by using results of appendix A, it can be shown that these equations can be written in general conjugate flow intrinsic coordinates as

\[
\frac{\partial u^j}{\partial \sigma^\nu} A_0^\nu + u^k \frac{\partial u^j}{\partial \sigma^\nu} A_k^\nu = -\frac{\partial p}{\partial \sigma^\nu} A_j^\nu + \frac{1}{Re} \left( \frac{\partial^2 u^j}{\partial \sigma^\nu \partial \sigma^\lambda} A_k^\nu A_k^\lambda + \frac{\partial u^j}{\partial \sigma^\nu} \frac{\partial A_k^\nu}{\partial \sigma^\lambda} A_k^\lambda \right),
\]

where the quantities \( A_\mu^\nu \), defined in Eq. (A6), are rather complicated functions of \( \mathbf{x}^\mu \). Clearly, when the coordinate system \( \sigma^\nu \) is advected exactly by the physical fluid flow, i.e. when the functional link between \( u^j \) and \( \mathbf{x}^j \) is defined by Eq. (3), then Eq. (17) coincides with the Lagrangian representation of the Navier-Stokes equations.

We remark both the conjugate flow perturbation as well as the perturbation induced in the field equations can be represented relative to arbitrary curvilinear coordinates\(^{1,48,51} \). The choice of a coordinate system advected by the conjugate flow, however, is convenient since the flow map \( \mathbf{x}^\mu(\sigma^\nu; u^j) \) then appears explicitly in the equations of motion, and it can be selected in order to satisfy existence conditions of an action functional (see section IV). On the contrary, if we consider fixed Cartesian coordinates then the conjugate flow perturbation involves the inverse of the flow map \( \mathbf{x}^\mu(\sigma^\nu; u^j) \).

### A. Functional setting

Let us associate with the physical system the linear function space \( \mathcal{U} \), whose elements are the \( N \)-tuples \( u = (u^1, ..., u^N) \). Similarly, let us consider the configuration space \( \mathcal{X} \), whose
\[
\begin{array}{c|c}
\text{Field Equation} & \text{Conjugate Flow Representation} \\
\hline
\nabla \cdot \mathbf{U} = F & \frac{\partial w_j}{\partial \sigma^\nu} A_j^\nu = f \\
\nabla^2 U = F & \left( \frac{\partial^2 u}{\partial \sigma^\lambda \partial \sigma^\rho} A_\rho^\lambda A_k^\nu + \frac{\partial u}{\partial \sigma^\lambda} \frac{\partial A_k^\lambda}{\partial \sigma^\rho} A_\rho^\nu \right) = f \\
\frac{\partial \mathbf{U}}{\partial t} = \alpha \nabla^2 U & \frac{\partial u}{\partial \sigma^\nu} A_0^\nu = \alpha \left( \frac{\partial^2 u}{\partial \sigma^\lambda \partial \sigma^\rho} A_k^\lambda A_k^\nu + \frac{\partial u}{\partial \sigma^\lambda} \frac{\partial A_k^\lambda}{\partial \sigma^\rho} A_\rho^\nu \right) \\
\frac{\partial U^j}{\partial t} + \mathbf{U} \cdot \nabla U^j = -\nabla P + \frac{1}{Re} \nabla^2 U^j & \frac{\partial w^j}{\partial \sigma^\nu} A_\rho^\nu + u^k \frac{\partial w^j}{\partial \sigma^\rho} A_k^\nu = -\frac{\partial p}{\partial \sigma^\nu} A_j^\nu + \frac{1}{Re} \left( \frac{\partial^2 w^j}{\partial \sigma^\lambda \partial \sigma^\rho} A_k^\lambda A_k^\nu + \frac{\partial w^j}{\partial \sigma^\lambda} \frac{\partial A_k^\lambda}{\partial \sigma^\rho} A_\rho^\nu \right)
\end{array}
\]

TABLE I. Conjugate flow representation of well-known PDEs of mathematical physics. The quantities \( A_\rho^\nu \) are rather complicated functions of \( \hat{x}^\mu \) defined in Eq. (A6).

elements, denoted as \( \hat{x} = (\hat{x}^0, \ldots, \hat{x}^n) \), represent \((n + 1)\)-dimensional conjugate flows, \( n \) being the number of spatial dimensions. In general, the configuration space is not a linear space because the summation of two conjugate flows is not a conjugate flow. This is due to the fact that the superimposition of two invertible flows may not be invertible (the summation of two invertible Jacobian matrices is not necessarily invertible). However, the requirement that the perturbed conjugate flow has the same properties of the unperturbed one, i.e. that it is still a diffeomorphism, is equivalent to state that locally, i.e., in the neighborhood of a particular flow \( \hat{x} \), the configuration space \( \mathcal{X} \) is linear or can be linearized. This is equivalent to assume that the flow perturbation \( \hat{\phi} \) is invertible, i.e., that the Gâteaux differential (12) defines an invertible flow map. In this sense we can say that the configuration space \( \mathcal{X} \) is \textit{locally linear}. Given this, an arbitrary field equation (or a system of field equations) written in terms of conjugate flow intrinsic coordinates can be synthesized as

\[
\mathbf{N}_{\hat{x}}(u) = \emptyset_{\mathcal{V}},
\]

where \( \mathbf{N}_{\hat{x}} \) is, in general, a nonlinear operator while \( \emptyset_{\mathcal{V}} \) denotes the null element of a third topological linear space \( \mathcal{V} \). The subscript \( \hat{x} \) in \( \mathbf{N}_{\hat{x}} \) reminds us that the operator is written in terms of conjugate flow intrinsic coordinates \( \sigma^\nu \), i.e. on the manifold defined by \( \hat{x} \). For
example, Eq. (16) can be put in the form (18) by defining \( \hat{N}_x \) as
\[
\hat{N}_x(u) \equiv \frac{\partial u}{\partial t} - \frac{1}{\partial \hat{x}/\partial \sigma \partial \sigma /\partial t} - \alpha \left( \frac{\partial^2 u}{\partial \hat{x}^2 \partial \sigma} - \frac{\partial^2 \hat{x} \partial u}{\partial \sigma^2 \partial \sigma} \right).
\]
(19)

The domain of the nonlinear operator \( \hat{N}_x \) is a suitable space of functions satisfying the initial or the boundary conditions of the problem. In the conjugate flow theory, however, the operator \( \hat{N}_x(u) \) acts on both \( u \) and \( \hat{x} \) and therefore it implicitly identifies two different domains, one within the space of fields \( U \) and the other one within the configuration space \( X \). These two domains will be denoted by \( D_U(\hat{N}_x) \subseteq U \) and \( D_X(\hat{N}_x) \subseteq X \), respectively (see figure 3). The range of the operator \( \hat{N}_x \) will be denoted by \( R(\hat{N}_x) \subseteq V \). The next fundamental step in the functional setting of the conjugate flow theory is to introduce duality pairings between the linear spaces \( U, V \) and the locally linear one \( X \) through non-degenerate local bilinear forms. To this end, let us define
\[
\langle \cdot, \cdot \rangle_u : V \times U \to \mathbb{R},
\]
(20)
\[
\langle \cdot, \cdot \rangle_{\hat{x}} : V \times X \to \mathbb{R}.
\]
(21)

The subscripts \( u \) and \( \hat{x} \) in Eqs. (20) and (21) emphasize the fact that such forms depend also on \( u \) and \( \hat{x} \), in a possibly nonlinear way. An explicit expression of (20) will be given in section IV B. The forms (20) and (21) can be put in a correspondence through the linear transformations defined by Eqs. (11) and (12). In fact, as shown in figure 3 the elements of \( D_U(\hat{N}_x) \) in the neighborhood of a certain \( u \) are in correspondence with the elements of \( D_X(\hat{N}_x) \) in the neighborhood of a certain \( \hat{x} \). In practice, such correspondence can be established locally through the linear operators \( \delta u / \delta \hat{x} \) and \( \delta \hat{x} / \delta u \). For instance, by using Eq. (11) we obtain
\[
\langle v, \tilde{\phi} \rangle_u = \langle v, \frac{\delta u}{\delta \hat{x}} \tilde{\phi} \rangle_u = \langle v, \tilde{\phi} \rangle_{\hat{x}}.
\]
(22)

We shall conclude this section by explaining why we have chosen the definition “conjugate flow” for the transformation (1). To this end, let us recall that the Gâteaux differential of \( \hat{x} \) with respect to \( u \) defines a linear functional from the space \( U \) to the space \( X \equiv U^\dagger \), which is the conjugate space of \( U \). Thus, for every admissible \( u \in U \), the flow \( \hat{x} \) belongs to the conjugate space of \( U \), hence the definition “conjugate flow”. In a broader sense, the adjective “conjugate” simply emphasizes that there exists a functional relation between the flow \( \hat{x} \), the governing equations of the field theory, \( \hat{N}_x(u) = \emptyset_V \), and the solution field \( u \). We also remark that a definition of conjugate flow already appeared in the literature, as
FIG. 3. Sketch of the function spaces employed in the functional setting of the conjugate flow theory. Shown are the domains $D_U(N_{\hat{x}}) \subseteq U$ and $D_X(N_{\hat{x}}) \subseteq X$ of the nonlinear operator $N_{\hat{x}}$ representing the field equations. The range of $N_{\hat{x}}$ is denoted by $R(N_{\hat{x}}) \subseteq V$. We also show the correspondence between field perturbations $u + \epsilon \varphi$, conjugate flow perturbations $\hat{x} + \epsilon \hat{\phi}$ and corresponding perturbations induced in the field equations $N_{\hat{x}+\epsilon \hat{\phi}}(u + \epsilon \varphi)$ relative to a specific representation $(u, \hat{x}, N_{\hat{x}})$. The local bilinear forms that put the various spaces in duality are indicated in between the sets.

“a flow uniform in the direction of streaming which separately satisfy the hydrodynamical equations”. Such definition is of course very different from ours.

B. Perturbation expansions

By using the general operator framework developed in the previous section, we can easily synthesize in a single operator equation the perturbative form of an arbitrary set of field equations in the presence of a conjugate flow perturbation, i.e. a simultaneous perturbation of both the solution field and the conjugate flow. Indeed, to the first-order in $\epsilon$ we have

$$N_{\hat{x}+\epsilon \hat{\phi}}(u + \epsilon \varphi) = N_{\hat{x}}(u) + \epsilon \left[ \frac{\delta N_{\hat{x}}}{\delta u} \varphi + \frac{\delta N_{\hat{x}}}{\delta \hat{x}} \hat{\phi} \right] + \cdots,$$

(23)
where the Gâteaux differentials appearing in Eq. (23) are defined as

\[
\frac{\delta N_{\hat{x}}}{\delta u} \varphi \defeq \lim_{\epsilon \to 0} \frac{N_{\hat{x}}(u + \epsilon \varphi) - N_{\hat{x}}(u)}{\epsilon},
\]

(24)

\[
\frac{\delta N_{\hat{x}}}{\delta \hat{x}} \phi \defeq \lim_{\epsilon \to 0} \frac{N_{\hat{x}+\epsilon \phi}(u) - N_{\hat{x}}(u)}{\epsilon},
\]

(25)

provided that such limits exist. A function space representation of the conjugate flow perturbation is sketched in figure 3.

IV. CONJUGATE FLOW ACTION FUNCTIONALS

Let us consider a field \( u \in D_{IU}(N_{\hat{x}}) \) and a conjugate flow \( \hat{x} \in D_{IX}(N_{\hat{x}}) \). The couple \((u, \hat{x})\) does not necessarily have to be a solution to the field equation, i.e. \( N_{\hat{x}}(u) \neq \emptyset \). Disregarding the particular form of the operator \( N_{\hat{x}} \), it is useful to consider

\[
v = N_{\hat{x}}(u) \in R(N_{\hat{x}})
\]

(26)
as a definition two vector fields, one in \( D_{IU}(N_{\hat{x}}) \) and the other one in \( D_{IX}(N_{\hat{x}}) \), respectively. This allows us to introduce in a conceptually simple way the notion of a line integral of an operator according to a geometric standpoint which seems originally due to Volterra\textsuperscript{50}. To this end, let us consider a one-parameter family of fields in the domain \( D_{IU}(N_{\hat{x}}) \)

\[
u = u_{\lambda} \quad (0 \leq \lambda \leq 1).
\]

(27)

This can be regarded as a line in the function space \( D_{IU}(N_{\hat{x}}) \). With such line we can associate the number

\[
\ell_{u} = \int_{0}^{1} \langle N_{\hat{x}}(u_{\lambda}), \frac{\partial u_{\lambda}}{\partial \lambda} \rangle u_{\lambda} d\lambda,
\]

(28)
i.e. the path integral of the operator \( N_{\hat{x}} \) along the trajectory of functions \( u_{\lambda} \in D_{IU}(N_{\hat{x}}) \). We recall that \( \langle \cdot, \cdot \rangle_{u} \) in (28) denotes the local bilinear form (20). In the context of the conjugate flow theory, we can also define the path integral of the operator \( N_{\hat{x}} \) along a trajectory of flows \( \hat{x}_{\lambda} \) in the space \( D_{IX}(N_{\hat{x}}) \), i.e.

\[
\ell_{\hat{x}} = \int_{0}^{1} \langle N_{\hat{x}_{\lambda}}(u), \frac{\partial \hat{x}_{\lambda}}{\partial \lambda} \rangle \hat{x}_{\lambda} d\lambda,
\]

(29)

where \( \langle \cdot, \cdot \rangle_{\hat{x}} \) denotes the local bilinear form (21). If the line integrals (28) and (29) are independent of the path of integration then the operator \( N_{\hat{x}} \) is said to be potential with
respect to the chosen local bilinear form. In this case the integral from a prefixed element \( u_0 \) to any element \( u \) in \( D_u(N_{\hat{x}}) \) along an arbitrarily chosen path defines the action functional

\[
A_u[u] = A_u[u_0] + \int_0^1 \langle N_{\hat{x}}(u_\lambda), \frac{\partial u}{\partial \lambda} \rangle_{u_\lambda} d\lambda. \tag{30}
\]

Similarly, the line integral from a prefixed conjugate flow \( \hat{x}_0 \) to another flow \( \hat{x} \) along an arbitrarily chosen line in \( D_{\hat{x}}(N_{\hat{x}}) \) defines another (dual) action functional

\[
A_{\hat{x}}[\hat{x}] = A_{\hat{x}}[\hat{x}_0] + \int_0^1 \langle N_{\hat{x}}(u), \frac{\partial \hat{x} \lambda}{\partial \lambda} \rangle_{\hat{x}_\lambda} d\lambda. \tag{31}
\]

In turn, the operator \( N_{\hat{x}} \) is said to be the gradient of the functionals \( A_u[u] \) or \( A_{\hat{x}}[\hat{x}] \). This definition relies on the fact that if we calculate the infinitesimal variation of (30) and (31) with respect to independent variations of \( u \) and \( \hat{x} \), respectively, then we obtain

\[
\delta A_u[u] = \langle N_{\hat{x}}(u), \delta u \rangle_u, \quad \delta A_{\hat{x}}[\hat{x}] = \langle N_{\hat{x}}(u), \delta \hat{x} \rangle_{\hat{x}}. \tag{32}
\]

These relations show that the equations of motion of the system, i.e. \( N_{\hat{x}}(u) = 0 \), can be obtained as a stationary point of either \( A_u[u] \) or \( A_{\hat{x}}[\hat{x}] \), for arbitrary variations \( \delta u \) and \( \delta \hat{x} \), respectively. Thus, the theory of conjugate flows allows us to look for action functionals associated with field equations in two different ways, depending on which variable between \( u \) or \( \hat{x} \) is assumed as independent. Clearly, if we consider \( u \) as independent then we are looking for the set of conjugate flows such that the field equation is potential. On the contrary, if we consider the conjugate flow \( \hat{x} \) as independent then we are looking for the set of fields \( u \) such that the field equation is potential. In the sequel we will be mostly concerned with conjugate flows action functionals where the field \( u \) is considered as independent variable.

### A. Existence conditions

In order to formulate the existence conditions of conjugate flow action functionals we follow the approach of Magri\(^{27}\). To this end, we consider two infinitesimal trajectories (two infinitesimal straight lines) of the field \( u \) in the function space \( D_u(N_{\hat{x}}) \)

\[
\begin{align*}
I & : u \to u + \epsilon \varphi \\
II & : u \to u + \nu \psi
\end{align*}
\]
Correspondingly, we have the following infinitesimal conjugate flow perturbations

\[ I : \hat{x} \rightarrow \hat{x} + \epsilon \hat{\phi} \]

\[ II : \hat{x} \rightarrow \hat{x} + \nu \hat{\eta} \]

where \( \hat{\phi} \) and \( \hat{\eta} \) are related to the field perturbations \( \phi \) and \( \psi \) by Eq. (12). Due to this fundamental relation, an infinitesimal circulation of the operator \( N_{\hat{x}} \) around the element \( u \in D_{\hat{u}}(N_{\hat{x}}) \) induces an infinitesimal circulation of \( N_{\hat{x}} \) around a flow \( \hat{x} \in D_{\hat{x}}(N_{\hat{x}}) \). The vanishing of these simultaneous circulations with respect to the local bilinear form (20) is synthesized by the condition

\[
\langle N_{\hat{x}}(u), \epsilon \varphi \rangle_u + \langle N_{\hat{x}+\epsilon \hat{\phi}}(u + \epsilon \varphi), \nu \psi \rangle_{u+\epsilon \varphi} = \\
\langle N_{\hat{x}}(u), \nu \psi \rangle_u + \langle N_{\hat{x}+\nu \hat{\eta}}(u + \nu \psi), \epsilon \varphi \rangle_{u+\nu \psi}.
\] (33)

To the second-order in \( \epsilon \) and \( \nu \) we have

\[
\langle N_{\hat{x}+\epsilon \hat{\phi}}(u + \epsilon \varphi), \nu \psi \rangle_{u+\epsilon \varphi} = \langle N_{\hat{x}}(u), \psi \rangle_u + \\
\epsilon \nu \langle \frac{\delta N_{\hat{x}}}{\delta u} \varphi + \frac{\delta N_{\hat{x}}}{\delta \hat{x}} \hat{\phi}, \psi \rangle_u + \epsilon \nu \langle \varphi; N_{\hat{x}}(u), \psi \rangle_u,
\] (34)

where

\[
\langle \varphi; v, \psi \rangle_u \overset{\text{def}}{=} \lim_{\epsilon \to 0} \frac{\langle v, \psi \rangle_{u+\epsilon \varphi} - \langle v, \psi \rangle_u}{\epsilon}
\] (35)

denotes the Gâteaux differential of the local bilinear form (20), considered as a particular type of nonlinear operator on \( u \). A substitution of Eq. (34) into Eq. (33) gives

\[
\langle \frac{\delta N_{\hat{x}}}{\delta u} \varphi + \frac{\delta N_{\hat{x}}}{\delta \hat{x}} \hat{\phi}, \psi \rangle_u + \langle \varphi; N_{\hat{x}}(u), \psi \rangle_u = \\
\langle \frac{\delta N_{\hat{x}}}{\delta u} \psi + \frac{\delta N_{\hat{x}}}{\delta \hat{x}} \hat{\eta}, \varphi \rangle_u + \langle \psi; N_{\hat{x}}(u), \varphi \rangle_u.
\] (36)

Finally, by using Eq. (12) we can write the vanishing condition of the infinitesimal circulation entirely in terms of field perturbations \( \psi \) and \( \varphi \) as

\[
\langle G_{\hat{x}} \varphi, \psi \rangle_u + \langle \varphi; N_{\hat{x}}(u), \psi \rangle_u = \langle G_{\hat{x}} \psi, \varphi \rangle_u + \langle \psi; N_{\hat{x}}(u), \varphi \rangle_u,
\] (37)

where the linear operator \( G_{\hat{x}} \) is defined as

\[
G_{\hat{x}} \overset{\text{def}}{=} \frac{\delta N_{\hat{x}}}{\delta u} + \frac{\delta N_{\hat{x}}}{\delta \hat{x}} \frac{\delta \hat{x}}{\delta u}.
\] (38)

Thus, if the circulation vanishes along any infinitesimal closed line in \( D_{\hat{u}}(N_{\hat{x}}) \) then Eq. (37) must hold for every \( \varphi, \psi \) and for all admissible \( u \). This is the necessary condition for
operators to be potential with respect to the local bilinear form \((20)\). If the domain of the operator \(N_\hat{x}\) is simply connected, then this condition is also sufficient. This happens, e.g., when \(D_u(N_\hat{x})\) is defined by linear homogeneous initial or boundary conditions (in this case \(D_u(N_\hat{x})\) is a convex set). We remark that the general symmetry condition \((37)\) includes interesting subcases. For example, if the flow \(\hat{x}\) is not a functional of \(u\) then we have
\[
\langle \frac{\delta N_\hat{x}}{\delta u} \varphi, \psi \rangle_u + \langle \varphi; N_\hat{x}(u), \psi \rangle_u = \langle \frac{\delta N_\hat{x}}{\delta u} \psi, \varphi \rangle_u + \langle \psi; N_\hat{x}(u), \varphi \rangle_u.
\]
(39)

In addition, if the bilinear form \(\langle \cdot, \cdot \rangle_u\) does not depend on \(u\), i.e., if we are dealing with a standard (non-local) bilinear form, then the symmetry condition \((39)\) coincides with the classical one obtained by Vainberg\(^{47}\)
\[
\langle \frac{\delta N_\hat{x}}{\delta u} \varphi, \psi \rangle = \langle \frac{\delta N_\hat{x}}{\delta u} \psi, \varphi \rangle,
\]
(40)

namely, the Gâteaux derivative of the operator \(N_\hat{x}\) must be symmetric with respect to the bilinear form \(\langle \cdot, \cdot \rangle\).

**B. Choice of the local bilinear form**

In classical, relativistic and quantum field theories the action functional has the standard form\(^{23,24,51}\)
\[
A = \int L \sqrt{g} d^4 \eta.
\]
(41)

where \(L\) denotes the Lagrangian density, \(g\) is the determinant of the metric tensor associated with the coordinate system \(\eta\) and \(\sqrt{g} d^4 \eta\) is the invariant space-time volume element (\(d^4 \eta\) being a shorthand notation for \(d\eta^0 d\eta^1 \cdots d\eta^3\)). A comparison between Eq. \((41)\) and Eq. \((30)\) suggests that the local bilinear form to be considered for the conjugate flow formulation of the inverse problem of the calculus of variations is
\[
\langle a, b \rangle_u \overset{\text{def}}{=} \int_{\Sigma} ab J d^4 \sigma, \quad a \in U, \quad b \in V,
\]
(42)

where the Jacobian determinant \(J\) is a rather complicated function of \(\hat{x}\) (see Eq. \((A5)\)). The domain \(\Sigma\) appearing in the integral \((A2)\) is a four-dimensional volume of particles \(\sigma^\nu\) advected by the conjugate flow (see figure \(1\)). Before proceeding further, it is useful to clarify the physical meaning of the bilinear form \((42)\). To this end, with reference to figure \(1\) let \(V(t; u)\) be the volume of particles obtained by advecting \(\Sigma\) with the conjugate flow
\( \hat{x}^\mu(\sigma^\nu; u) \). In particular, in figure 1 we show two different volumes at time \( t_1 \), corresponding to two realizations of the flow, namely, \( \hat{x}^\mu(\sigma^\nu; u_1) \) and \( \hat{x}^\mu(\sigma^\nu; u_2) \). The bilinear form (42) is defined by integration over a volume of particles co-moving with the conjugate flow. In fact,

\[
\int_{V(t;u)} A(x^\mu) B(x^\mu) d^4 x = \int_{\Sigma} a(\sigma^\nu) b(\sigma^\nu) J(\hat{x}^\mu(\sigma^\nu; u)) d^4 \sigma.
\]

(43)

At the left hand side of this equation we are using fixed Cartesian coordinates \( x^\mu \), while at the right hand side we are using conjugate flow intrinsic coordinates \( \sigma^\nu \) (\( J \) is the Jacobian determinant of the transformation from fixed Cartesian to conjugate flow intrinsic coordinates). At each specific time, e.g., at time \( t_1 \) in figure 1 the trajectory of fields \( u_\lambda = (1 - \lambda) u_1 + \lambda u_2 \) (\( 0 \leq \lambda \leq 1 \)) defines a transformation that takes the volume \( V(t; u_1) \) into the volume \( V(t; u_2) \).

The bilinear form (42) generalizes the one appearing in the Herivel-Lin variational principle 7,10,20,36, where the volume of particles is advected precisely by the physical fluid flow. Note also that (42) is symmetric, non-degenerate and non-negative, i.e., it satisfies all the properties of an inner product. By using Eq. (A21) we obtain the following Gâteaux derivative

\[
\langle \varphi; a, b \rangle_u \overset{\text{def}}{=} \frac{d}{d\epsilon} \left[ \langle a, b \rangle_u + \epsilon \varphi \right]_{\epsilon=0} = \int_{\Sigma} ab J \hat{\nabla} \cdot \hat{\varphi} d^4 \sigma
\]

(44)

where, according to Eq. (12), \( \hat{\varphi} \) is a linear functional of \( \varphi \), i.e. (44) is a trilinear form in \( a \), \( b \) and \( \varphi \). A substitution of Eqs. (42) and (44) into the symmetry condition (37) suggests that for incompressible flow perturbations (\( \nabla \cdot \hat{\varphi} = 0 \)) we have

\[
\langle G_\hat{x} \varphi, \psi \rangle_u = \langle G_\hat{x} \psi, \varphi \rangle_u.
\]

(45)

Equation (45) requires the symmetry of the operator \( G_\hat{x} \) defined in (38) relative to the local inner product (42). Thus, the application of the conjugate flow theory to the inverse problem of the calculus of variations is now reduced to look for an incompressible four-dimensional flow symmetrizing the operator \( G_\hat{x} \) with respect to the bilinear form (42).

C. Conservation laws

Once a conjugate flow satisfying the symmetry conditions (37) or (45) has been identified, we can write the action functional (30) entirely in terms of the field \( u \). In particular, if we
consider the local bilinear form \((42)\) we obtain

\[
A[u] = \int_0^1 \int_\Sigma N_{\tilde{x}(u_\lambda)}(u_\lambda) \frac{\partial u_\lambda}{\partial \lambda} J(u_\lambda) d^4 \sigma d\lambda,
\]

where we have set \(A[u_0] = 0\). A comparison between \((46)\) and \((41)\) yields the following Lagrangian density

\[
\mathcal{L} = \int_0^1 N_{\tilde{x}(u_\lambda)}(u_\lambda) J(u_\lambda) \frac{\partial u_\lambda}{\partial \lambda} d\lambda,
\]

which is expressed in conjugate flow intrinsic coordinates \(\sigma^\nu\). Such representation can be of course transformed to arbitrary coordinates, leading us to the general form \((41)\). Before concluding this section, let us briefly recall that the symmetries of the action functional \((46)\) are associated with conservation laws. The classical approach to perform this type of analysis relies on the identification of suitable transformation groups (symmetry groups) leaving the action invariant. Therefore, once the conjugate flow action functional is available, we can use a fairly well developed framework to compute the Noether invariants of the field theory and the associated currents. This could lead to the discovery of new conservation laws in fluid dynamics and other disciplines.

V. SYMMETRIZING FLOWS

A field equation is said to be formally symmetric when the operator symmetry condition, e.g. Eq. \((45)\), is satisfied disregarding the particular form of the boundary or the initial conditions associated with the problem. Clearly, when the domain of the operator \(N_{\tilde{x}}\) is formed by a set of functions satisfying local homogeneous boundary and initial conditions then formal symmetry is a necessary condition for symmetry. Such a condition, however, is not sufficient even in the case of homogeneous boundaries. In any case, it is useful to establish formal symmetry conditions for particular classes of field equations. This has been done, e.g., by Tonti by using classical inner products in fixed coordinate systems. In this section we obtain similar conditions for incompressible conjugate flow variations. In particular, we study scalar and vector field theories governed by second-order nonlinear partial differential equations. These theories include many well-known equations of mathematical physics, e.g., the Navier-Stokes equations, the Maxwell’s equations in potential formula-
tion, the laws of elasticity, the advection-reaction-diffusion equations and the Schrödinger equation.

A. Vector field theories

The conjugate flow representation of second-order vector field equations can be written in the general form

$$N \hat{x}(u) = f_k \left( u^j; w_{j,\mu}; w_{j,\mu\nu}; \hat{x}_\\mu; \hat{x}_{\mu\nu} \right) = 0, \quad (48)$$

where the comma denotes partial differentiation with respect to $\sigma^\mu$ ($\mu = 0, ..., 3$), i.e., $u^j_\mu \equiv \partial u^j / \partial \sigma^\mu$. An example of (48) is the intrinsic form of the Navier-Stokes equations (17), together with the divergence-free condition of the velocity field. The Gâteaux differential of (48), for arbitrary variations of $u^j$ and $\hat{x}_\mu$, reads

$$\delta N \hat{x} + \delta \hat{x} \hat{\phi} = \frac{\partial f_k}{\partial u^j_\mu} \phi^j_\mu + \frac{\partial f_k}{\partial u^j_{\mu\nu}} \varphi^j_{\mu\nu} + \frac{\partial f_k}{\partial \hat{x}_\mu} \hat{\phi}_\mu + \frac{\partial f_k}{\partial \hat{x}_{\mu\nu}} \hat{\phi}_{\mu\nu}. \quad (49)$$

At this point we recall that the conjugate flow perturbation $\hat{\phi}_\mu$ is related to the field perturbation $\phi^j$ by Eq. (12). In general, such relation involves both derivatives and integrals. For example, it could be in the form

$$\hat{\phi}_\mu = \int_\Sigma K^\mu_j \left( \sigma^\nu; u^k \right) \phi^j \, d^4 \sigma + W^\mu_j \left( \sigma^\nu; u^k \right) \phi^j + M^\mu_{j,\lambda} \left( \sigma^\nu; u^k \right) \phi^j, \quad (50)$$

where $\Sigma$ is a four-dimensional volume of particles advected by the conjugate flow and $K^\mu_j$, $W^\mu_j$, $M^\mu_{j,\lambda}$ are suitable functions.

If we proceed in the most general case, i.e., by using Eq. (12) without any further specification, then the equations for the conjugate flow satisfying the symmetry condition (45) will be functional differential equations. In practical application, however, it might be convenient to restrict the type of functional dependence to specific forms, for example (50). In these cases, the type of functional dependence between $\hat{\phi}_\mu$ and $\phi^j$ is somehow imposed and it defines a class of conjugate flows. If such class is large enough, then the symmetry condition (45) will extract one or more flows for which the vector field theory is potential. Let us illustrate this procedure with reference to incompressible conjugate flows that are local functionals of the solution field $u^j$. In particular, we shall consider

$$x^\mu = \hat{x}_\mu \left( \sigma^\nu, u^j \right), \quad (51)$$
where \( \hat{x}^\mu \) (to be determined) are algebraic functions of \( u^j \) which do not involve integrals in space-time, e.g., \( \hat{x}^\mu \) are polynomials of \( u^j \). Note that the assumption that \( \hat{x}^\mu \) are local functionals of \( u^j \) could lead to determining equations for the conjugate flow with no solutions. In other words, it might be possible that the class of algebraic conjugate flows is not large enough to accommodate the symmetry requirement (45) for a specific set of PDEs. In any case, it is interesting to analyze such class and derive evolution equations for the symmetrizing flow. To this end, let us first set

\[
ad^\mu_j \overset{\text{def}}{=} \frac{\partial \hat{x}^\mu}{\partial u^j},
\]

(52)

\[
bd^\mu_j\nu \overset{\text{def}}{=} ad^\mu_j\nu + \frac{\partial a^\mu_j}{\partial u^k} u^k_{\nu},
\]

(53)

\[
c^\mu_j\nu\rho \overset{\text{def}}{=} bd^\mu_j\nu\rho + \frac{\partial b^\mu_j\nu}{\partial u^i} u^i_{\nu\rho}.
\]

(54)

This allows us to write the conjugate flow perturbation \( \hat{\phi}^\mu \) and its partial derivatives as

\[
\hat{\phi}^\mu = a^\mu_j \varphi^j,
\]

(55)

\[
\hat{\phi}^\mu_j\nu = b^\mu_j\nu \varphi^j + a^\mu_j \varphi^j_{\nu},
\]

(56)

\[
\hat{\phi}^\mu_j\nu\rho = c^\mu_j\nu\rho \varphi^j + b^\mu_j\nu \varphi^j_{\rho} + b^\mu_j\rho \varphi^j_{\nu} + a^\mu_j \varphi^j_{\nu\rho}.
\]

(57)

Remarkably, the highest derivative order in \( \hat{\phi}^\mu \) and \( \varphi^j \) is the same. In fact, the \( k \)th-order derivative of \( u^j \) with respect to \( \sigma^\nu \) involves the \( k \)th-order derivative of \( \hat{x}^\mu \) (see Eqs. (A14) and (A15)). This is why we have included the second-order derivative of the conjugate flow in the second-order vector field equation (48). A substitution of (56)-(57) into (49) yields the following operator \( G_{\hat{x}} \) (see Eq. (38))

\[
G_{\hat{x}} \varphi = H_{kj} \varphi^j + B^\nu_{kj} \varphi^j_{\nu} + F^\mu\nu_{kj} \varphi^j_{\mu\nu},
\]

(58)

where

\[
H_{kj} \overset{\text{def}}{=} \frac{\partial f_k}{\partial u^j} + \frac{\partial f_k}{\partial \hat{x}^\mu \nu} b^\mu_j + \frac{\partial f_k}{\partial \hat{x}^\mu \nu \rho} c^\mu_j \nu\rho,
\]

(59)

\[
B^\nu_{kj} \overset{\text{def}}{=} \frac{\partial f_k}{\partial u^j_{\nu}} + \frac{\partial f_k}{\partial \hat{x}^\mu \nu} a^\mu_j + \left( \frac{\partial f_k}{\partial \hat{x}^\mu \nu \rho} + \frac{\partial f_k}{\partial \hat{x}^\mu \nu} \right) b^\mu_j
\]

(60)

\[
F^\mu\nu_{kj} \overset{\text{def}}{=} \frac{\partial f_k}{\partial u^j_{\nu\mu}} + \frac{\partial f_k}{\partial \hat{x}^\mu \nu} a^\rho_j.
\]

(61)

Note that \( F^\mu\nu_{kj} \) is symmetric with respect to \( \mu \) and \( \nu \) by construction. At this point, we use
the local inner product (42) and write explicitly the symmetry condition (45) as

\[ \int \Sigma \psi^k H_{kj} \varphi^j J^d \sigma + \int \Sigma \psi^k B^\mu_{kj} \varphi^i J^d \sigma + \int \Sigma \psi^k F^\mu^\nu_{kj} \varphi^j J^d \sigma = \]
\[ \int \Sigma \varphi^j H_{jk} \psi^k J^d \sigma + \int \Sigma \varphi^j B^\mu_{jk} \psi^i J^d \sigma + \int \Sigma \varphi^j F^\mu^\nu_{jk} \psi^k J^d \sigma. \] (62)

The next step is to reduce the derivative order of the field \( \psi^k \) at the right hand side by using integration by parts. Since we are looking for a formal symmetry condition, we neglect all the boundary terms. This yields, for instance

\[ \int \Sigma \varphi^j B^\mu_{jk} \psi^k J^d \sigma = - \int \Sigma \left( \varphi^j B^\mu_{jk} J \right)_{,\mu} \psi^k J^d \sigma \]
\[ = - \int \Sigma \psi^k B^\mu_{jk} \varphi^j J^d \sigma - \int \Sigma \psi^k B^\mu_{jk} \varphi^j J^d \sigma - \int \Sigma \psi^k B^\mu_{jk} \Gamma^\lambda_{\mu \nu} \varphi^j J^d \sigma, \] (63)

where the derivative of the Jacobian \( J \) has been represented in terms of the affine connection (A18) by using Eq. (A22). Similarly, the integral involving \( \psi^k_{,\mu}\) in (62) can be rewritten as

\[ \int \Sigma \varphi^j F^\mu^\nu_{jk} \psi^k_{,\mu} J^d \sigma = \int \Sigma \left( \varphi^j F^\mu^\nu_{jk} J \right)_{,\mu} \psi^k J^d \sigma \]
\[ = \int \Sigma \left[ \varphi^j F^\mu^\nu_{jk} + \varphi^j F^\nu_{jk,\mu} + \varphi^j F^\mu_{jk} \Gamma^\lambda_{\lambda \nu} + \varphi^j F^\mu_{jk} \Gamma^\lambda_{\lambda \mu} + \varphi^j F^\mu_{jk} \Gamma^\lambda_{\lambda \nu} \right] \psi^k J^d \sigma. \] (64)

By substituting (63)-(64) into (62) and assuming that the field variations \( \varphi^j \) and \( \psi^k \) are arbitrary, we obtain that the formal symmetry condition is satisfied if and only if

\[ F^\mu^\nu_{kj} = F^\mu^\nu_{jk}, \] (65)
\[ B^\nu_{kj} = -B^\nu_{jk} + 2 \left( F^\mu^\nu_{jk,\mu} + F^\mu^\nu_{jk} \Gamma^\lambda_{\lambda \mu} \right), \] (66)
\[ H_{kj} = H_{jk} + F^\mu^\nu_{jk,\mu} + 2 F^\mu^\nu_{jk,\mu} \Gamma^\lambda_{\lambda \mu} + F^\mu^\nu_{jk} \Gamma^\lambda_{\lambda \mu}, \] (67)

This is a system of nonlinear PDEs that, in principle, allows us to identify the functional relation between the conjugate flow \( \tilde{x}^\mu \) and the field \( u^j \), i.e., the functional manifold on which the vector field theory is potential. We shall call Eqs. (65)-(67) the determining equations of the conjugate flow. It is interesting to note that if we remove the functional link between \( \tilde{x}^\mu \) and \( u^j \), and we consider fixed Cartesian coordinates then the conditions (65)-(67) consistently reduce to those of Tonti [42].
B. Scalar field theories

Let us consider a scalar field theory governed by the nonlinear second-order PDE

$$N(x) = f(u, u_{,\mu}; u_{,\mu\nu}; \hat{x}^\mu, \hat{x}^\nu, \hat{x}^\nu_{,\mu}) = 0. \quad (68)$$

A simple example of (68) is the intrinsic form of the diffusion equation (16). The operator $G$ in this case is obtained as

$$G(x) = Q + Z_{,\mu} + R_{\mu\nu}, \quad (69)$$

where

$$Q(x) \equiv \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \hat{x}^\mu} b^\mu_{,\nu} + \frac{\partial f}{\partial \hat{x}^\mu_{,\nu}} c^\mu_{,\nu}, \quad (70)$$

$$Z_{,\mu} \equiv \frac{\partial f}{\partial u_{,\mu}} + \frac{\partial f}{\partial \hat{x}^\mu} a^\mu + \left( \frac{\partial f}{\partial \hat{x}^\mu_{,\nu}} + \frac{\partial f}{\partial \hat{x}^\nu_{,\mu}} \right) b^\mu_{,\nu}, \quad (71)$$

$$R_{\mu\nu} \equiv \frac{\partial f}{\partial u_{,\nu\rho}} + \frac{\partial f}{\partial \hat{x}^\mu_{,\nu\rho}} b^\mu_{,\nu}, \quad (72)$$

By using the local inner product (42) we write explicitly the symmetry condition (45) as

$$\int_{\Sigma} (\psi Q + \psi Z_{,\mu} + \psi R_{\mu\nu}) Jd^4\sigma = \int_{\Sigma} (Q\psi + Z_{,\mu} + R_{\mu\nu}) Jd^4\sigma. \quad (74)$$

At this point, we integrate by parts the terms at the right hand side and neglect all the boundary contributions. This yields, for arbitrary field variations $\varphi$ and $\psi$

$$Z_{,\mu} = R_{,\mu} + R_{\mu\nu} \Gamma_{,\lambda}^\nu, \quad (75)$$

$$Z_{,\mu} = R_{,\mu} + 2R_{\mu\nu} \Gamma_{,\lambda}^\nu + R_{\mu\nu} \Gamma_{,\lambda}^\nu \Gamma_{,\lambda}^\rho + R_{\mu\nu} \Gamma_{,\lambda\mu}^\rho - Z_{,\mu} \Gamma_{,\lambda}^\nu. \quad (76)$$

A differentiation of Eq. (75) with respect to $\sigma^\mu$ and subsequent substitution into Eq. (76) gives the single relation

$$\left( R_{,\mu} + R_{\mu\nu} \Gamma_{,\lambda}^\nu - Z_{,\mu} \right) \Gamma_{\mu\nu} = 0, \quad (77)$$

which is equivalent to the following system of determining equations for the conjugate flow

$$R_{,\mu} + R_{\mu\nu} \Gamma_{,\lambda}^\nu = Z_{,\mu}. \quad (78)$$
### Field Equation

\[ f \left( u; u_{,\mu}; u_{,\mu\nu}; \hat{x}_{,\mu}^{\mu}; \hat{x}_{,\mu\nu}^{\mu\nu} \right) = 0 \]

### Determining Equations for the Conjugate Flow

\[ R_{i,\nu}^{\mu\nu} + R_{i,\nu}^{\mu\nu} \Gamma_{\lambda\nu}^{\lambda} = Z^{\nu} \]

\[ f_k \left( u^j; u_{,\mu}^j; u_{,\mu\nu}^j; \hat{x}_{,\mu}^{\mu}; \hat{x}_{,\mu\nu}^{\mu\nu} \right) = 0 \]

\[ F_{\mu\nu}^{jk} = F_{\mu\nu}^{jk} \]

\[ B_{\nu}^{\mu} = -B_{\nu}^{\mu} + 2 \left( F_{j,\mu}^{\mu\nu} + F_{j,\mu}^{\mu\nu} \Gamma_{\lambda\mu}^{\lambda} \right), \]

\[ H_{jk} = H_{jk} + F_{j,\mu}^{\mu\nu} + 2F_{j,\mu}^{\mu\nu} \Gamma_{\lambda\nu}^{\lambda} + F_{j,\mu}^{\mu\nu} \Gamma_{\lambda\nu}^{\lambda}, \]

\[ \Gamma_{\lambda\mu}^{\lambda}, \Gamma_{\rho\nu}^{\rho} - B_{\rho\nu}^{\mu}, -B_{\rho\nu}^{\mu} \Gamma_{\lambda\nu}^{\lambda}. \]

**TABLE II.** Determining equations for the conjugate flow that symmetrizes second-order scalar and vector field theories. Once the conjugate flow is available, the Lagrangian density and the action functional can be determined by calculating the integral (47) along an arbitrary trajectory of fields.

This system defines the functional relation between the field \( u \) and the flow \( \hat{x}^{\mu} \) for which the scalar field theory is potential. Note that if we remove the functional link between \( \hat{x}^{\mu} \) and \( u \), then the conditions (78) consistently reduce to those of Tonti \( 41,42 \). In order to see this, we simply set \( a^{\mu}, b^{\mu}_{\nu} \) equal to zero in Eq. (71) and Eq. (72) and then substitute them into Eq. (78). The result in fixed Cartesian coordinates \( (\Gamma_{\beta\rho}^{3} = 0) \) is

\[
\frac{\partial}{\partial x^{\mu}} \left( \frac{\partial f}{\partial u_{,\mu\nu}} \right) - \frac{\partial f}{\partial u_{,\mu\nu}} = 0. \quad (79)
\]

This is the classical condition arising from the symmetry requirement of a second-order nonlinear scalar field equation \( 13,15 \).

### VI. SUMMARY

We developed a new general approach to construct an action functional for a non-potential field theory. The key idea relies on representing the governing equations of the theory relative to a functional flow of curvilinear coordinates (the conjugate flow) which is assumed to be a dependent on the solution field. We have shown that such flow can be selected in order to symmetrize the Gâteaux derivative of the field equations with respect to suitable
local bilinear forms. This is equivalent to requiring that the field equations of the theory can be derived from a principle of stationary action on a Lie group manifold. By using a general operator framework, we obtained the determining equations of such manifold for second-order scalar and vector field theories and shown that they are consistent with classical results in fixed coordinates. Once the symmetrizing conjugate flow is available, the action functional of the theory can be constructed explicitly by using path integration. In particular, the duality principle between the conjugate flow and the solution field allows us to perform integrations either in terms of flows or in terms of fields, yielding two different types of actions.

The proposed new methodology can be applied to scalar, vector and tensor field theories. In particular, it can be applied to the Navier-Stokes equations, for which a great research effort has focused in obtaining a physically meaningful principle of stationary action. Recent results of Gomes and Eyink indeed have shown that an action principle can be constructed on random diffeomorphisms. These random flows are usually defined in terms of stochastic perturbations of Lagrangian base flows. By using the methods of the present paper, the variational principle for the Navier-Stokes equations may be constructed on a deterministic functional diffeomorphism, i.e., on a conjugate flow satisfying the set of determining equations summarized in table 2. In fact, the representation of the Navier-Stokes equations in terms of a symmetrizing conjugate flow yields a potential field theory which admits a Lagrangian density and an action functional. These quantities can be explicitly determined by using the methods of section IV. In particular, once the conjugate flow is available, the Lagrangian density can be determined by computing the integral along a trajectory of velocity fields. The identification of groups of transformations leaving the conjugate flow action functional invariant could lead to the discovery of new conservation laws in fluid dynamics and other disciplines.

Appendix A: Representation of field equations in conjugate flow intrinsic coordinates

In this appendix we recall some fundamental identities of differential geometry that allow us to write the dynamic equations of a physical system in terms of conjugate flow intrinsic coordinates. To this end, let us first consider the Jacobian of the conjugate flow
transformation (1)

\[ J_\nu^\mu \overset{\text{def}}{=} \frac{\partial \tilde{x}^\mu}{\partial \sigma^\nu}, \quad \mu, \nu = 0, \ldots, n, \tag{A1} \]

where \( n \) is the number of spatial dimensions and 0 denotes the temporal component. It is easy to verify that the transpose of the algebraic complement of \( J_\nu^\mu \) has tensorial expression (repeated indices are summed)

\[ C_\rho^\lambda \overset{\text{def}}{=} \frac{1}{n!} \epsilon^{\lambda \nu \alpha \cdots} \epsilon_{\rho \mu \lambda \cdots} \frac{\partial \tilde{x}^\mu}{\partial \sigma^\nu} \frac{\partial \tilde{x}^\lambda}{\partial \sigma^\alpha} \cdots, \tag{A2} \]

where \( \epsilon^{\lambda \nu \alpha \cdots} \) and \( \epsilon_{\rho \mu \lambda \cdots} \) are multi-dimensional permutation symbols, i.e. Levi-Civita tensorial densities. For example, if we consider one spatial and one temporal dimension then we obtain the simple expression (all indices are from 0 to 1)

\[ C_\rho^\lambda = \epsilon^{\lambda \nu} \epsilon_{\rho \mu} \frac{\partial \tilde{x}^\mu}{\partial \sigma^\nu}. \tag{A3} \]

Similarly, in 1 + 3 dimensions, i.e. one temporal and three spatial dimensions (all indices are from 0 to 3)

\[ C_\rho^\lambda = \frac{1}{6} \epsilon^{\lambda \nu \alpha \beta} \epsilon_{\rho \mu \lambda \delta} \frac{\partial \tilde{x}^\mu}{\partial \sigma^\nu} \frac{\partial \tilde{x}^\lambda}{\partial \sigma^\alpha} \frac{\partial \tilde{x}^\delta}{\partial \sigma^\beta}. \tag{A4} \]

By using Eqs. (A1) and (A2) we obtain the Jacobian determinant

\[ J \overset{\text{def}}{=} \det (J_\nu^\mu) = \frac{1}{n + 1} J_\nu^\mu C_\mu^\nu. \tag{A5} \]

This allows us to write the the inverse of the Jacobian matrix (A1) as

\[ A_\rho^\lambda \overset{\text{def}}{=} \frac{C_\rho^\lambda}{J}. \tag{A6} \]

At this point, let us denote by

\[ \sigma^\nu = \tilde{\sigma}^\nu (x^\mu; u^j), \quad \mu, \nu = 0, \ldots, n \tag{A7} \]

the inverse transformation of (1). Such inverse transformation exists and it is differentiable by definition of conjugate flow. From the well known identity

\[ \frac{\partial \tilde{\sigma}^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial \sigma^\lambda} = \delta_\lambda^\nu \tag{A8} \]

we obtain the following fundamental expression of the partial derivatives \( \partial \tilde{\sigma}^\nu / \partial x^\mu \) as a function of \( \sigma^\lambda \)

\[ \frac{\partial \tilde{\sigma}^\nu}{\partial x^\mu} = A_\mu^\nu (\sigma^\lambda; u^j), \tag{A9} \]
where $A_{\mu}^{\nu}$ is defined in Eq. (A6). It is useful to write down $A_{\mu}^{\nu}$ explicitly in the two-dimensional case

$$
\begin{bmatrix}
A_0^0 & A_0^1 \\
A_1^0 & A_1^1
\end{bmatrix} = \frac{1}{J} \begin{bmatrix}
\frac{\partial \hat{x}^1}{\partial \sigma^1} - \frac{\partial \hat{x}^0}{\partial \sigma^1} \\
-\frac{\partial \hat{x}^1}{\partial \sigma^0} & \frac{\partial \hat{x}^0}{\partial \sigma^0}
\end{bmatrix},
$$

(A10)

where

$$J = \frac{\partial \hat{x}^1}{\partial \sigma^1} \frac{\partial \hat{x}^0}{\partial \sigma^0} - \frac{\partial \hat{x}^0}{\partial \sigma^1} \frac{\partial \hat{x}^1}{\partial \sigma^0}.
$$

(A11)

If time is not transformed, i.e. if $x^0 = \sigma^0 = t$, then Eq. (A10) reduces to

$$
\begin{bmatrix}
A_0^0 & A_0^1 \\
A_1^0 & A_1^1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-(\partial \hat{x}/\partial t)/(\partial \hat{x}/\partial \sigma) & 1/(\partial \hat{x}/\partial \sigma)
\end{bmatrix},
$$

(A12)

where we have denoted by $\sigma \equiv \sigma^1$ and $x \equiv x^1$.

**Partial differentiation in intrinsic coordinates**

Let us consider a vector field $U^j (x^\mu)$ in fixed Cartesian coordinates $x^\mu$. Such field can be equivalently expressed relative to conjugate flow intrinsic coordinates by using the transformation (1). This yields the following equivalent representations

$$U^j (x^\mu) = U^j (\hat{x}^\mu (\sigma^\nu)) = u^j (\sigma^\nu) = u^j (\hat{\sigma}^\nu (x^\mu)).
$$

(A13)

The transformation law for partial derivatives of $U^j$ is easily obtained by differentiating Eq. (A13)

$$\frac{\partial U^j}{\partial x^\mu} = \frac{\partial u^j}{\partial \sigma^\nu} \frac{\partial \hat{x}^\nu}{\partial x^\mu} = \frac{\partial u^j}{\partial \sigma^\nu} A_{\mu}^{\nu},
$$

(A14)

where the quantities $\partial \hat{x}^\nu/\partial x^\mu$ are expressed in coordinates $\sigma^\nu$ through Eq. (A9). Let us now evaluate the second derivative with respect to $x^\nu$ and express the result in conjugate flow intrinsic coordinates. To this end let us perform an additional differentiation of (A14) with respect to $x^\nu$. This yields

$$\frac{\partial^2 U^j}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 u^j}{\partial \sigma^\lambda \partial \sigma^\nu} A_{\mu}^{\lambda} + \frac{\partial u^j}{\partial \sigma^\nu} \frac{\partial A_{\mu}^{\rho}}{\partial \sigma^\nu} A_{\nu}^{\rho}.
$$

(A15)

By using the expressions of $J$ and $C_{\mu}^{\nu}$ obtained in (A5) and (A2) it is possible to manipulate Eq. (A15) further. However, it is more convenient to obtain first $A_{\mu}^{\nu}$ explicitly as a function of $\sigma^\mu$ and then perform the differentiation appearing in (A15).
Perturbations of the metric tensor, affine connection and Jacobian determinant

When the conjugate flow (1) undergoes an infinitesimal disturbance of type (14) then all the quantities related to its intrinsic geometry are subject to small variations. For instance, the metric tensor

$$g_{\mu\nu} = \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu}. \quad (A16)$$

becomes, to the first-order in $\epsilon$, $g_{\mu\nu} + \epsilon h_{\mu\nu}$ where

$$h_{\mu\nu} = \frac{\partial \hat{x}^\beta}{\partial \sigma^\mu} \frac{\partial \hat{x}^\alpha}{\partial \sigma^\nu} + \frac{\partial \hat{x}^\alpha}{\partial \sigma^\mu} \frac{\partial \hat{x}^\beta}{\partial \sigma^\nu}. \quad (A17)$$

The corresponding perturbation in the affine connection (Christoffel symbols of the second kind)

$$\Gamma^\alpha_{\mu\nu} = \frac{\partial \sigma^\alpha}{\partial x^\rho} \frac{\partial^2 \hat{x}^\rho}{\partial \sigma^\mu \partial \sigma^\nu}. \quad (A18)$$

can be obtained by using the well-known representation in terms of metric tensor, i.e.,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\rho} \left( \frac{\partial g_{\rho\nu}}{\partial \sigma^\mu} + \frac{\partial g_{\rho\mu}}{\partial \sigma^\nu} - \frac{\partial g_{\mu\nu}}{\partial \sigma^\rho} \right). \quad (A19)$$

This yields

$$\delta \Gamma^\alpha_{\mu\nu} = -\epsilon g^{\alpha\rho} h_{\rho\beta} \Gamma^\beta_{\mu\nu} + \frac{\epsilon}{2} g^{\alpha\rho} \left( \frac{\partial h_{\rho\nu}}{\partial \sigma^\mu} + \frac{\partial h_{\rho\mu}}{\partial \sigma^\nu} - \frac{\partial h_{\mu\nu}}{\partial \sigma^\rho} \right) \quad (A20)$$

the covariant derivatives ";" being of course constructed by using the unperturbed affine connection $\Gamma^\alpha_{\mu\nu}$. These results allows us to compute the conjugate flow perturbation of other fundamental geometric quantities such as the Riemann-Christoffel curvature tensor and the Jacobian determinant (A5) of the conjugate flow transformation. To this end, we substitute Eq. (14) into Eq. (A5) and we keep only the linear terms in $\epsilon$ to obtain

$$J + \epsilon C^\mu_{\nu} \frac{\partial \hat{x}^\mu}{\partial \sigma^\nu} = J \left( 1 + \epsilon \frac{\partial \hat{x}^\mu}{\partial x^\mu} \right). \quad (A21)$$

The quantity $\partial \hat{x}^\mu/\partial x^\mu$ is the divergence of the flow perturbation (remember that $\hat{x}^\mu$ are Cartesian components). Another useful formula involving the Jacobian determinant is

$$\frac{\partial J}{\partial \sigma^\mu} = J \Gamma^\nu_{\nu \mu}. \quad (A22)$$
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