Splitting Madsen-Tillmann spectra II. The Steinberg idempotents and Whitehead conjecture

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Abstract

We show that, at the prime \( p = 2 \), the spectrum \( \Sigma^{-n}D(n) \) splits off the Madsen-Tillmann spectrum \( MTO(n) = BO(n)^{-\gamma^n} \) which is compatible with the classic splitting of \( M(n) \) off \( BO(n)_+ \). For \( n = 2 \), together with our previous splitting result on Madsen-Tillmann spectra, this shows that \( MTO(2) \) is homotopy equivalent to \( BSO(3)_+ \vee \Sigma^{-2}D(2) \).

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1 Introduction

The Madsen-Tillmann spectrum $MTO(n)$ is defined to be the Thom spectrum of the virtual bundle $-\gamma_n \to BO(n)$ where $\gamma_n$ is the universal $n$-plane bundle. It is known that these spectra filter the spectrum $MO$, i.e. there is a sequence

$$S^0 = MTO(0) \to \Sigma MTO(1) \to \cdots \to \Sigma^{n-1} MTO(n-1) \xrightarrow{\varepsilon_n} \Sigma^n MTO(n) \to \cdots$$

(1)

where $\varepsilon_n$ is induced by the inclusion $O(n-1) \subset O(n)$, with the property $\text{colim} \Sigma^n MTO(n) \cong MO$. Furthermore the cofiber of the successive stages is homotopy equivalent to $BO(n)_+$, that is, we have a cofiber sequence

$$\cdots \to \Sigma^{-1} MTO(n-1) \to MTO(n) \xrightarrow{p_n} BO(n)_+ \to MTO(n-1) \to \cdots$$

(2)

where $p_n$ is the map induced by the “embedding” of $(-1)$ times the canonical bundle into the $0$-dimensional trivial bundle. In other words, the spectrum $MO$ can be built up from pieces $BO(n)_+$.

We have shown that localised away from 2, $MTO(2n) \cong BO(2n)_+$ and $MTO(2n+1) \cong \ast$ for all $n \geq 0$ [4, Theorem 1.1.B]. Thus the study of $MTO(n)$’s become essentially 2-local problems. Therefore we shall work at the prime $p = 2$. For technical reasons, we will rather work with 2-completed spectra instead of 2-local spectra, and in the rest of the paper we identify a spectrum with its 2-completion. Since our main application concerns the mod 2 cohomology of associated infinite loop spaces, by passing to 2-completion, no information will be lost. Throughout the paper homology and cohomology are taken with $\mathbb{Z}/2$ coefficients. As we work essentially with spectra, we identify spaces with its suspension spectra. In the literature, sometimes a space $X$ is identified with the suspension spectra of the space with added basepoint $X_+$, which explains a notational discrepancy the reader may find between the current paper and results we quote.

At the prime 2, Randal-Williams computed $H_*(\Omega^\infty MTO(i))$ for $i = 1$ and 2, [12, Theorems A and B]. Combining the two theorems, we get an exact sequence of Hopf algebras

$$H_*(Q_O BO(2)_+) \to H_*(Q_O BO(1)_+) \to H_*(Q_O BO(0)_+) \to \mathbb{Z}/2$$

(3)

where the (Hopf) kernel of the first two maps are isomorphic to $H_*(\Omega^\infty MTO(i))$ for $i = 2$ and 1 respectively. Thus a natural question to ask was whether if this exact sequence could be extended further to the left with $H_*(\Omega^\infty MTO(i))$ isomorphic to the kernel of each stage. We showed that this was impossible in [4]. So a new question to ask, then, is to what extent we can generalize [12, Theorems A and B].

This question leads to a search for another sequence of spectra with the beginning as in (3). It turns out that there indeed is such a sequence, well-known to stable homotopy theorists, that is:

4 The splitting

4.1 $D(n)$ as a summand of Thom spectrum

4.2 Maps from $\Sigma^{-n} D(n)$ to $MTO(n)$

4.3 Proof of the splitting

4.4 Further refinements

5 Homology of the associated infinite loop spaces

5.1 Exact sequences

5.2 Relations among $\mu$-classes

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These spectra realizes the length filtration of the Steenrod algebra, that is, we have
\[ H^*(D(n)) \cong A/G_n \text{ where } G_n \text{ is the span of } Sq^I, I \text{ admissible, } l(I) > n. \]

We note that \( G_n \) happens to be a left \( \mathcal{A} \)-ideal, so that this isomorphism is as \( \mathcal{A} \) modules.

They originally were defined using the symmetric powers ([9, Proposition 4.3]), and we have \( \text{colim } D(n) = H\mathbb{Z}/2 \), and cofibrations
\[ \rightarrow \Sigma^{-n-1}M(n) \rightarrow D(n-1) \rightarrow D(n) \rightarrow \Sigma^{-n}M(n) \rightarrow \]

The spectrum \( M(n) \) is defined to be the cofibre of the map \( \Sigma^{-n}D(n-1) \rightarrow \Sigma^{-n}D(n) \).

Thus one can say that \( H\mathbb{Z}/2 \) can be built up of \( M(n) \)'s. These building blocks can be described alternatively as follows.

The spectra \( BO(1)_+^\infty \) admit a natural (left) \( Gl_n(\mathbb{Z}/2) \) action. Thus the Steinberg idempotent \( e_n \in \mathbb{Z}/2[Gl_n(\mathbb{Z}/2)] \) gives rise to a splitting of \( BO(1)_+^\infty \) and we have \( M(n) \simeq e_n BO(1)_+^\infty \) [9, Theorem 5.1]. Moreover, through the Becker-Goettlieb transfer map, this splitting gives rise to a splitting of \( M(n) \) off \( BO(n)_+ \) (see Mitchell and Priddy’s paper [9] for more details and for the odd primary splitting results as well).

Therefore we have a construction of \( MO \) with \( BO(n)_+ \)'s as building blocks, a construction of \( H\mathbb{Z}/2 \) with \( M(n) \)'s as building blocks. Furthermore \( H\mathbb{Z}/2 \) and \( M(n) \)'s split off respectively \( MO \) and \( BO(n)_+ \)'s. It is then natural to ask whether one can split intermediate stages. The purpose of this paper is to answer affirmatively to this question, and discuss some consequences, including an answer to the question on generalization of the exact sequence (3).

More detailed statements are given in the next section.

The paper is organized as follows. In section 2 we summarize our results. In section 3, we recall relevant results from [5] and construct a map from \( F_\ast Y \) to \( F_\ast X \). In section 4, we use Takayasu’s results [13] to construct a map going the other way round, and show that we indeed have a splitting. In section 5, we discuss the consequences in homology of infinite loop spaces.

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## 2 Statement of results

To give precise statements, we start with some definitions.

**Definition 2.1.** i). A filtered spectrum is a sequence of spectra \( F_\ast X \)
\[ F_0X \rightarrow F_1X \rightarrow \cdots \rightarrow F_nX \rightarrow F_{n+1}X \rightarrow \cdots \]
with a homotopy equivalence \( \text{hocolim } F_nX \simeq X \).

ii). A map of filtered spectra \( (f_\ast) \) from \( F_\ast X \) to \( F_\ast Y \) is a collection of maps \( f_n: F_nX \rightarrow F_nY \) that makes the squares
\[ F_nX \rightarrow F_{n+1}X \]
\[ F_nY \rightarrow F_{n+1}Y \]
iii). Two filtered spectrum \(F_*X\) and \(F_*Y\) are said to be equivalent if there is a map \(f_*\) from \(F_*X\) to \(F_*Y\) such that \(f_n\)'s are homotopy equivalence for all \(n\).

iv). We say that \(F_*X\) splits off \(F_*Y\) if there are maps \(f_*\) from \(F_*X\) to \(F_*Y\) and \(g_*\) from \(F_*Y\) to \(F_*X\) such that \(H_*(f_* \circ g_*) = \text{id}\).

Our main result then reads as follows.

**Theorem 2.2.** Define filtered spectra \(F_*X\) and \(F_*Y\) by \(F_*X = D(n), F_*Y = \Sigma^n MTO(n)\). Then \(F_*X\) splits off \(F_*Y\).

As our proof doesn’t depend on the decomposition of \(MO\) ([15, Theorem 2.10], [12, Theorem 2]), we obtain a “new” proof of the splitting of \(H\Z/2\) off \(MTO\). However, our method doesn’t allow us to obtain information on other summands.

An immediate consequence is the following.

**Corollary 2.3.** \(H_*(\Omega^\infty \Sigma^{-n} D(n))\) splits off \(H_*(\Omega^\infty \Sigma MTO(n))\) as a Hopf algebra.

Thus the “correct way to extend” the exact sequence ([3]) is just the following standard fact

**Proposition 2.4 ([3]).** The following sequence of Hopf algebras is exact.

\[ \cdots \to H_*(\Omega^\infty M(n)) \to H_*(\Omega^\infty M(n-1)) \to \cdots \to H_*(\Omega^\infty M(2)) \to H_*(Q_0 B\Z/2^\infty) \to H_*(Q_0 S^0) \to \Z/2 \]

Furthermore the image of \(H_*(\Omega^\infty M(n)) \to H_*(\Omega^\infty M(n-1))\) is isomorphic to \(H_*(\Omega^\infty D(n-1))\).

As \(D(0) \cong S^0, \Sigma^{-1} D(1) \cong MTO(1)\) ([3, Proposition 4.4], and \(M(1) \cong BO(1)\), combined with the \(n = 2\) case of Theorem 2.2, we recover Theorems A and B of [12].

Of course, the cohomology being dual of homology, the exact sequences above give some information on certain characteristic classes. More precisely, recall from [12, 4].

**Definition 2.5.** A universally defined characteristic class in \(H^*(\Omega^\infty MTO(n))\) is an element in the subalgebra generated by the image of \(H^*(BO(n)) \to H^*(Q_0 BO(n)) \to H^*(\Omega^\infty MTO(n))\). We denote \(\mu_{i_1, \ldots, i_n} = (\Omega^\infty p_n)^*(\sigma^{i_1^\infty}(\sigma_1^{i_1} \ldots \sigma_n^{i_n}))\) where \(H^*(BO(n)) \cong \Z/2[\sigma_1, \ldots, \sigma_n]\).

In [4], we used the summand \(BSO(2n + 1)\) that split off \(MTO(2n)\) to show that some of these classes remain algebraically independent. Here we use the summand \(D(n)\) that splits off \(MTO(n)\) to show that there are “linear” relations corresponding to elements of \(H^*(M(n))\), and that in the case of dimension 2, these relations together with the ones derived from the action of top Steerond squares are the only relations. More precisely, we will show:

**Theorem 2.6.** i). In \(H^*(\Omega^\infty MTO(n))\), we have relations \((\Omega^\infty p_n)^*(\sigma^{i\infty}(x)) = 0\) for \(x \in H^*(M(n)) \subset H^*(BO(n))\).

ii). For \(n = 2\), the only relations among \(\mu_{i,j}\)'s are the relations above, and \(\mu_{2i,2j} = \mu_{i,j}^2\).

iii). Again for \(n = 2\), the subalgebra of universally defined characteristic classes in \(H^*(\Omega^\infty MTO(2))\) is the polynomial algebra generated by \(\nu_{i,j}\)'s with \(ij\) odd, where \(\nu_{i,j}\) is defined in [4].

We will give more precise description of the inclusion \(H^*(M(n)) \subset H^*(BO(n))\) later.

3 Maps from \(MTO(n)\) to \(\Sigma^{-n} D(n)\)

In this section we use results from [3] to construct maps from \(\Sigma^n MTO(n)'s\) to \(D(n)'s\) that form a map of filtered spectra. First of all, we recall results we will need.
3.1 Exact sequences of spectra and the Whitehead conjecture

We start with a definition.

**Definition 3.1 ([5]).** i). A fibration sequence of spectra $F \to X \xrightarrow{f} Y$ is called exact if there exists a map $g : \Omega^\infty Y \to \Omega^\infty X$ such that $\Omega^\infty f \circ g \simeq \text{id}$.

ii). A sequence of spectra $\cdots \to X_n \to \cdots X_1 \to X_0 \to E_{-1}$ is called exact if for all $n$, $E_n \to X_n \to E_{n-1}$ is exact, where $E_n$ is inductively defined as the fiber of the map $X_n \to E_{n-1}$.

The category of spectra being a triangulated category instead of an abelian category, we have some complication here. The notion of exactness with three terms is more or less a counterpart of a split short exactness in abelian categories. The use of this seemingly too strong condition is motivated by the following fact. By definition, an exact sequence of spectra yields an exact sequence of abelian groups upon applying $[Y, -]$ for a suspension spectrum $Y$, or a spectrum that is a summand of a suspension spectrum. Thus one can regard suspension spectra as free objects, summands of suspension spectra as projective objects, and carry out homological algebra in the category of spectra.

Now, one of the main results of [5], reads as follows.

**Theorem 3.2 ([5] Theorem 1.1]).** Let $d_k$ be defined by the composition

$$d_k : M(k+1) \to BT(k+1)_+ = B(T(1) \times T(k))_+ \xrightarrow{tr} BT(k)_+ \to M(k)$$

where $tr$ is the Becker-Gottlieb transfer, and the first and last map are obtained via the splitting $M(k) \simeq e_k^r BT(k)_+$ with the conjugate Steinberg idempotent $e_k^r$. Then the sequence

$$\cdots d_{k+1} \to M(k+1) \xrightarrow{d_k} M(k) \to \cdots \to M(1) \xrightarrow{d_0} M(0) \xrightarrow{\epsilon} HZ/2$$

is exact.

We note that in [5], the sequence above is shown to be equivalent to another sequence consisting of $M(k+1)$’s, whose exactness is known as the mod 2 Whitehead conjecture (Corollary 1.2 loc.cit.).

3.2 Complexes of spectra

We still need some more definitions.

**Definition 3.3.** i). By a chain complex of spectra $(E_n, d_n)$ we understand a sequence of spectra $E_n$ with maps $d_{n-1} : E_n \to E_{n-1}$ so that the composition $E_{n+1} \to E_n \to E_{n-1}$ is null for all $n$. By a map $f$ of chain complexes of spectra $(E_n, d_E) \to (F_n, d_F)$ we mean a collection of maps $f_n : E_n \to F_n$ such that $f_n d_E = d_F f_{n+1}$. Two complexes are said to be isomorphic if if there are maps $f$ from $E$ to $F$ such that all $f_n$’s are homotopy equivalences.

ii). Let $F_*X$ be a filtered spectrum. Define its associated graded complex $Gr_\bullet(F_*X)$ by $Gr_0(F_*X) = F_0X$, $Gr_i(F_*X) = \Sigma^{-i} \text{cofib}(F_{i-1}X \to F_iX)$, with obvious maps $Gr_iF_*X \to Gr_{i-1}F_*X$.

**Remark 3.4.** It follows from an easy diagram chasing that two filtered spectra are equivalent if and only if their associated graded complexes are isomorphic. Thus $Gr$ provides an embedding of the category of filtered spectra into that of complexes of spectra.
Example 3.5. i). Let $F_n X = D(n)$. Then the associated graded complex $Gr_\ast(F_\ast X)$ is

$$\cdots \to M(n+1) \xrightarrow{\delta_n} M(n) \to \cdots \to M(0)$$

considered in [5 Corollary 1.2]. This complex, with augmentation to $HZ/2$ added, was shown to be equivalent with the complex [7 in [5 section 6].

ii). Let $F_n Y = \Sigma^n MTO(n)$. Then the associated graded complex $Gr_\ast(F_\ast X)$ is given by $(BO(n)_+, tr)$ where $tr$ is the Becker-Gottlieb transfer associated to the inclusion $O(n - 1) \subset O(n)$, as the Becker-Gottlieb transfer $BO(n) \to BO(n - 1)$ factors as $BO(n)_+ \to MTO(n - 1) \to BO(n - 1)_+$. We can also see that $(BO(n)_+, tr)$ is a complex directly by noting that as $O(n) \subset O(n) \times O(2) \subset O(n + 2)$, thus $S^1 \cong \{1\} \times SO(2) \subset O(n) \times O(2)$ normalizes $O(n) \subset O(n + 2)$, so the transfer associated to $O(n) \subset O(n + 2)$ is trivial. Moreover, the complex of free spectra $(BO(n)_+, tr)$ is augmented over $HZ/2$ since the composition $BO(1)_+ \to BO(0)_+ \to HZ/2$ is trivial. This is just another way of saying that the transfer in $\mathbb{Z}/2$-cohomology $H^*(BO(0)_+; \mathbb{Z}/2) \to H^*(BO(1)_+; \mathbb{Z}/2)$ is trivial.

Now we are ready to do homological algebra. We recall the following.

Proposition 3.6. Let $(P_\ast, d_\ast)$ be a cochain complex of projective $R$-modules with an augmentation $P_0 \to A$, and $(A_\ast, d_\ast)$ be a resolution of $A$. Then we get a cochain map from $(P_\ast, d_\ast)$ to $(A_\ast, d_\ast)$.

This is an easy exercise using the definition of projectives and exactness, and proof is omitted. We now translate this to our setting.

Proposition 3.7. Let $(P_\ast, d_\ast)$ be a chain complex of projective spectra with an augmentation $P_0 \to A$, and $(A_\ast, d_\ast)$ be a resolution of $A$. Then we get a map of chain complexes from $(P_\ast, d_\ast)$ to $(A_\ast, d_\ast)$.

Proof. First, note that Proposition 3.6 is proved using the fact that if $P$ is projective and $C_0 \to C_1 \to C_2$ is a short exact sequence, then $\text{Hom}(P, C_0) \to \text{Hom}(P, C_1) \to \text{Hom}(P, C_2)$ is short exact. Thus it suffices to show that if $P$ is a suspension spectrum (or a summand of a suspension spectrum, but this doesn’t affect anything), and $C_0 \to C_1 \to C_2$ is a short exact sequence of spectra, then $[P, C_0] \to [P, C_1] \to [P, C_2]$ is a short exact sequence of abelian groups. Write $P = \Sigma^\infty X$. Then we have $[P, C_1] \cong [X, \Omega^\infty C_1]$. By definition of the short exactness, the map $\Omega^\infty C_1 \to \Omega^\infty C_2$ (as well as loop on this) admits a section, thus $[P, C_1] \cong [X, \Omega^\infty C_1] \to [X, \Omega^\infty C_2] \cong [P, C_2]$ is (split-)epi. Since $C_0 \to C_1 \to C_2$ is a cofibration of spectra, it follows that $[P, C_0] \to [P, C_1] \to [P, C_2]$ is a short exact.

3.3 Construction of the maps

By the examples of previous section, and Proposition 3.7, we have a maps of chain complexes of spectra $(BO(n)_+, tr) \to (M(n), d_\ast)$ That is, we have proven the existence of maps $f_n$ such that the following square commutes

$$\begin{array}{ccc}
BO(n)_+ & \xrightarrow{tr} & BO(n - 1)_+ \\
\downarrow f_n & & \downarrow f_{n-1} \\
M(n) & \xrightarrow{d_{n-1}} & M(n - 1)
\end{array}$$
Remark 3.8. The spectrum $M(0)$ is just $S^0$, and $M(1) = BO(1)_+$. The maps $BO(0)_+ \to H\mathbb{Z}/2$ and $\epsilon : M(0) \to H\mathbb{Z}/2$ coincide with the unit of $H\mathbb{Z}/2$, and the maps $f_1$ and $f_0$ can be taken to be the identity.

Now we are ready to prove the following.

Theorem 3.9. Fix maps $f_n : (BO(n)_+, tr) \to (M(n), d_n)$. Then there exists a map $\alpha_n : MTO(n) \to \Sigma^{-n}D(n)$ which makes the following diagram commutative for each $n$

$$
\begin{array}{ccc}
MTO(n) & \longrightarrow & BO(n)_+ \\
\downarrow^{\alpha_n} & & \downarrow^{f_n} \\
\Sigma^{-n}D(n) & \longrightarrow & M(n)
\end{array}
$$

Proof. We proceed by induction on $n$. The case $n = 0$ is trivial. Suppose that we have constructed such $\alpha_{n-1}$. Consider the following diagram.

$$
\begin{array}{ccc}
BO(n)_+ & \longrightarrow & MTO(n-1) \\
\downarrow^{f_n} & & \downarrow^{\alpha_{n-1}} \\
M(n) & \longrightarrow & \Sigma^{1-n}D(n-1)
\end{array}
$$

If we can show that this diagram commutes, then we can define the map $\alpha_n$ using the cofibrations $[\Pi]$ and $[\xi]$, which will conclude the proof. Note that the two horizontal maps induces trivial maps in cohomology, which implies that the two compositions from the top left corner to bottom right corner factor through $E_{n-1}$. Here $E_n$ is the spectrum defined in $[\xi]$, in other words, $E_n$ is the fiber of the map $\Sigma^nD(n) \to \Sigma^nH\mathbb{Z}/2$. Thus we need to show that the two elements in $[BO(n)_+, E_{n-1}]$ agree. However, by $[\xi]$ Theorem 1, $[BO(n)_+, E_{n-1}]$ injects to $[BO(n)_+, M(n-1)]$. Thus it suffices to show that the two maps agree after composition with the map $E_{n-1} \to \Sigma^{1-n}D(n-1) \to M(n-1)$. Now, consider the following diagram.

$$
\begin{array}{ccc}
BO(n)_+ & \longrightarrow & MTO(n-1) & \longrightarrow & BO(n-1)_+ \\
\downarrow^{f_n} & & \downarrow^{\alpha_{n-1}} & & \downarrow^{f_{n-1}} \\
M(n) & \longrightarrow & \Sigma^{1-n}D(n-1) & \longrightarrow & M(n-1)
\end{array}
$$

The right square is commutative by inductive hypothesis. But we chose our maps $f_n$ so that the big square commutes. This finishes the proof.

Note that by construction, the maps $\alpha_n$ form a map of filtered spectra.

Remark 3.10. A reader familiar with $[\xi]$ must have noticed that our arguments are slightly upside-down. If one follows the proofs in $[\xi]$, we get $\alpha_{n-1}$ before $f_n$. As main interests of the authors of loc.cit. were not on $D(n)$’s, some statements that could have been proved in loc.cit and that we could have quoted are not there. We chose to quote the statements that could be easily found, instead of details of proofs.
4 The splitting

In this section, we construct a map of filtered spectra from $D(n)$’s to $\Sigma^n MTO(n)$’s and conclude the proof of Theorem 4.2. The main ingredient here is the description of $D(n)$ as a summand of Thom spectrum, which is implicit in [13].

4.1 $D(n)$ as a summand of Thom spectrum

Consider the reduced regular representation $\rho_n$ of $T(n) = O(1)^{x_n}$ and let $M(n)_k$ be the stable summand of the Thom spectrum $BT(n)^{k\rho_n}$ corresponding to the Steinberg idempotent $e_n$. In particular, $M(n)_0 = M(n)$ defined in previous section. According to Takayasu [13] there is a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \rightarrow M(n)_k \rightarrow M(n)_{k+1}$$

where the mapping $M(n)_k \rightarrow M(n)_{k+1}$ is induced by the bundle map $k\rho_n \rightarrow (k+1)\rho_n$. We are interested in the case $k = -1$ where Takayasu’s cofibration looks like

$$\Sigma^{-1}M(n-1)_{-1} \xrightarrow{i'_{-1}} M(n)_{-1} \xrightarrow{j_{-1}} M(n)_0.$$  \hspace{1cm} (9)

Furthermore, the map $i'_{-1}$ satisfies the property $i'_{-1} \circ j_{-2} = i_{-1}$, where $j_{-2} : \Sigma^{-1}M(n-1)_{-2} \rightarrow M(n)_{-1}$ is induced by the inclusion $T(n-1) \subset T(n)$ [13, Theorem A]. We record the following observation which is implicit in [13].

**Lemma 4.1.** There is a homotopy equivalence $M(n)_{-1} \rightarrow \Sigma^{-n}D(n)$, i.e. $\Sigma^{-n}D(n)$ is a stable summand of $BT(n)^{-\rho_n}$ corresponding to the Steinberg idempotent.

**Proof.** By [13, Proposition 4.1.6], we have $H^*(M(n)_{-1}) \cong H^*(D(n))$. By Corollary 4.2.3 loc.cit., $j_{-1}^{x}$ is monomorphism, so by the long exact sequence for the cofibration [9] $i'_{-1}$ is epimorphism, and the filtered spectrum

$$M(0)_{-1} \rightarrow \cdots \rightarrow \Sigma^{-n}M(n-1)_{-1} \rightarrow \Sigma^n M(n)_{-1} \rightarrow \cdots$$

realizes the length filtration of the Steenrod algebra. Thus by [13 Corollary 1.4.1] is equivalent to

$$D(0) \rightarrow \cdots \rightarrow D(n) \rightarrow D(n+1) \rightarrow \cdots.$$  

\hspace{1cm} \Box

**Remark 4.2.** The above can also be proved by direct cohomology calculation using [13, Proposition 4.1.6] and [9, Theorem 5.8].

4.2 Maps from $\Sigma^{-n}D(n)$ to $MTO(n)$

Denote by $\beta_n$ the composition

$$\Sigma^{-n}D(n) \rightarrow BT(n)^{-\rho_n} \rightarrow BT(n)^{(-\gamma)x_n} \rightarrow BO(n)^{-\gamma_n} \cong MTO(n)$$

where the map at the middle is induced by the embedding $\gamma_1^{x_n} \subset \rho_n$ (or the embedding of virtual vector bundles $-\rho_n \subset (-\gamma)^{x_n}$), and the last map is induced by the inclusion $T(n) \rightarrow O(n)$.

Unlike the maps $\alpha_n$ that are constructed as maps between cofibers, the compatibility between different $\beta_n$’s are not immediate from the definition. Thus our first task is to show that they form a map of filtered spectra.
**Lemma 4.3.** We have following commutative diagram. Thus the maps $\beta_n$’s form a map of filtered spectra.

![Diagram](image.png)

Proof. First consider the square on the left. Since the target of the two compositions is $(−1)$-connected, and the fiber of the map $j_2: \Sigma^{-1}M(n−1)−2 \to \Sigma^{-1}M(n−1)−1 = D(n−1)$ has no cell in positive dimension, we see that it suffices to show that they become homotopic after the composition with $j_2$. Now, consider the diagram

![Diagram](image.png)

Everything except on the left commutes by naturality of Thom spectra construction. The top left square commutes because the map on the right is $GL_{n−1}(\mathbb{Z}/2)$-equivariant. The bottom left square commutes because of the equality $e_{n−1}e_n = e_n$ ([5 Corollary 2.6 (2)]), this also follows easily from ([9 Proposition 2.5]) and the fact that the map on the right is $GL_{n−1}(\mathbb{Z}/2)$-equivariant. By [13 Theorem A], the map $i_−1 : D(n−1) \to D(n)$ satisfies the property $i_−1 \circ j_2 = i_−1$, so the conclusion follows. A similar but easier argument applies to the map $D(n) \to \Sigma^n M(n)$.

4.3 Proof of the splitting

So far, we have defined the maps $\alpha_n$ and $\beta_n$ so that the following diagram commutes.

![Diagram](image.png)

Now, consider $H^∗(\alpha_n \circ \beta_n)$. As $H^∗(D(n))$ is monogenic over the Steenrod algebra ([9 Proposition 4.3]), this map is determined by its image on the bottom class. However, the bottom
cell of \( D(n) \) is just the image of \( D(0) \), so the bottom class is detected by the pull-back to \( H^0(D(0)) \). The commutativity of the above diagram then implies that \( H^0(\alpha_n \circ \beta_n) \) restricted to \( H^0(D(0)) \) is identity. Thus \( H^*(\alpha_n \circ \beta_n) \) is identity, which concludes the proof of Theorem 2.2.

### 4.4 Further refinements

We have shown in [4] that \( BSO(2n+1)_+ \) splits off \( MTO(2n) \). One may ask how this splitting interacts with the splitting of current paper. We show that they are complementary.

**Corollary 4.4.** \( \Sigma^{-2n}D(2n) \vee BSO(2n + 1)_+ \) splits off \( MTO(2n) \). When \( n = 1 \), we have homotopy equivalence \( MTO(2) \cong \Sigma^{-2}D(2) \vee BSO(3)_+ \).

**Proof.** Denote by \( f_{2n} \) the inclusion \( O(2n) \subset SO(2n + 1) \) given by \( A \mapsto \det(A)(A \oplus 1) \), and by \( Tr_{f_{2n}} \) the associated Miller-Mann-Mann transfer \( BSO(2n + 1)_+ \to MTO(2n) \). Consider the composition

\[
(\alpha_{2n} \lor Tr_{2n} \circ p_{2n}^* \circ (\beta_{2n} \lor Tr_{f_{2n}})^* : H^*(BSO(2n+1)) \oplus H^*(\Sigma^{-2n}D(2n)) \to H^*(BSO(2n+1)) \oplus H^*(\Sigma^{-2n}D(2n)).
\]

The components \( H^*(BSO(2n+1)) \to H^*(BSO(2n+1)) \) and \( H^*(\Sigma^{-2n}D(2n)) \to H^*(\Sigma^{-2n}D(2n)) \) are automorphisms. Consider now the component \( H^*(\Sigma^{-2n}D(2n)) \to H^*(BSO(2n+1)) \).

This is trivial since the source is generated over the Steenrod algebra by negative-degree elements, and the target is concentrated in non-negative degrees. Thus the map \( (\alpha_{2n} \lor Bf_{2n} \circ p_{2n})^* \circ (\beta_{2n} \lor Tr_{f_{2n}})^* \) is an automorphism. This proves the splitting for general \( n \). When \( n = 1 \), it suffices to compare the cohomology of both sides, or alternatively, by comparing the fibrations \( MTO(2) \to BO(2)_+ \to MTO(1) \) and \( \Sigma^{-2}D(2) \to M(2) \to D(1) \), noting that \( BO(2)_+ \cong M(2) \vee BSO(3)_+ \) (c.f. [4] Theorem C), we see that \( (\alpha_{2n} \lor Bf_{2n} \circ p_{2n})^* \) induces mod 2 homology equivalence. Since everything in sight is of finite type, this implies that we have a 2-local homotopy equivalence.

### 5 Homology of the associated infinite loop spaces

In this section, we discuss the consequences of our splitting theorem to homology of associated infinite loop spaces.

#### 5.1 Exact sequences

We start with generalities on summands of suspension spectra.

**Lemma 5.1.** Let \( M \) be a spectrum such that \( M \) splits of \( \Sigma^\infty X \) where \( X \) is a space. Denote \( B \) a basis of \( \tilde{H}_s(M) \). Then there are elements \( s_x \in H_s(\Omega^\infty M) \) such that

\[
H_s(\Omega^\infty M) = \mathbb{Z}/2Q^I(s_x); \text{ excess}(I) > |x|, I \text{ allowable}, \sigma^s_*(s_x) = x,
\]

where \( Q^I \) denotes the Dyer-Lashof operation ([10]). Suppose further that the splitting of \( M \) is obtained by an idempotent of the form \( f = \Sigma_i f_i \), where \( f_i \)'s are self-maps of the space \( X \). Then we can choose \( s_x \) in such a way that in \( H_s(QX) \) we have \( s_x = x \) modulo decomposables in \( H_s(QX) \), where we identify \( H_s(X) \) with its image in \( H_s(QX) \) via the canonical map \( X \to QX \), and \( H_s(\Omega^\infty X) \) with its image in \( H_s(QX) \) via the splitting map.
Proof. The first statement is straightforward, so its proof is omitted. The second statement follows as we have
\[ H_\ast(\Omega^\infty(f)) = H_\ast(\Omega^\infty(\Sigma_i f_i)) \equiv \Sigma_i H_\ast(\Omega^\infty(f_i)) \]
modulo decomposables.

Thus \( H_\ast(\Omega^\infty M(n)) \)'s are polynomial Hopf algebra. As a matter of fact they are bipo-
nomial, but we will not need this. A nice feature of such Hopf algebras is that a short
exact sequence involving only these Hopf algebras always splits as algebras, thus inducing a
short exact sequence of indecomposables. Besides, the map showing up in such a short exact
sequence can be effectively studied by studying the induced map in the indecomposables,
with little loss of information. All these considerations lead to the following refinement of
Proposition 2.4.

**Proposition 5.2.** The following sequence of Hopf algebra is exact. It gives rise to an exact
sequence of graded vector spaces after taking the module of indecomposables.

\[ \cdots \to H_\ast(\Omega^\infty M(n)) \to H_\ast(\Omega^\infty M(n-1)) \to \cdots \to H_\ast(\Omega^\infty M(2)) \to H_\ast(\Omega^\infty_{0} BZ/2_+) \to H_\ast(Q_{0} S^{0}) \to \mathbb{Z}/2 \]

Furthermore the image of \( H_\ast(\Omega^\infty M(n)) \to H_\ast(\Omega^\infty M(n-1)) \) is isomorphic to \( H_\ast(\Omega^\infty_{0} D(n-1)) \).

Proof. Denote by \( D'(n) \) the fiber of the map \( \Sigma^{-n} D(n) \to \Sigma^{-n} HZ/2 \) corresponding to the
bottom class. Then by theorem 3.2 we see that the fibration \( D'(n) \to M(n) \to D'(n-1) \) is

exact. Note that \( \Omega^\infty D'(n) \cong \Omega^\infty \Sigma^{-n} D(n) \) for \( n \geq 1 \), and \( \Omega^\infty D'(0) \cong Q_{0} S^{0} \). By definition

a three-term exact sequence of spectra leads to a short exact sequence of Hopf algebras by

applying \( H_\ast(\Omega^\infty (-); \mathbb{Z}/2) \). Splicing together the short exact sequence thus obtained, we get

an exact sequence as in the statement of Proposition, except the last entries which are

\[ \cdots \to H_\ast(\Omega^\infty BZ/2_+) \to H_\ast(Q_{0} S^{0}) \to \mathbb{Z}/2[\mathbb{Z}/2] \to \mathbb{Z}/2. \]

Noting that we have \( H_\ast(\Omega^\infty X_+) \cong H_\ast(\Omega^\infty_{0} X_+) \otimes \mathbb{Z}/2[\mathbb{Z}] \) for connected \( X \), in particular

\( X = BZ/2 \) and \( X = pt \), and that \( \mathbb{Z}/2[\mathbb{Z}] \to \mathbb{Z}/2[\mathbb{Z}] \to \mathbb{Z}/2[\mathbb{Z}/2] \) is exact, we see that the

sequence in the Proposition is exact. As everything insight is polynomial, they remain exact

after passing to indecomposables.

**Remark 5.3.** The \( \text{Gl}_{n}(\mathbb{Z}/2) \) action on \( BT(n)_+ \) extends that on \( BT(n) \). Thus it is easy
to see from the definition of \( e'_n \) (2.2) that we have \( e'_n BT(n) = e'_n BT(n)_+ \) for \( n > 1 \). Thus
\( \Omega^\infty_{0} M(n) = \Omega^\infty M(n) \) for \( n > 1 \), and the above exact sequence can be expressed entirely in
terms of \( \Omega^\infty_{0} M(n) \)'s.

An immediate consequence is

**Corollary 5.4.** \( H^\ast(\Omega^\infty_{0} MTO(2)) \) is a polynomial algebra.

Proof. By Corollary 2.4 we have \( \Omega^\infty_{0} MTO(2) \cong Q_{0} BSO(3)_+ \times \Omega^\infty D'(2) \), noting that \( \pi_{0}(D'(2)) = 0 \) since it is a direct factor of \( \pi_{0}(M(2)) \). The short exactsequence above implies that \( H^\ast(\Omega^\infty D'(2)) \)

injects to \( H^\ast(\Omega^\infty M(3)) \). Since \( M(3) \) is a stable summand of \( BO(3) \), we see that \( H^\ast(\Omega^\infty D'(2)) \)

injects to \( H^\ast(Q_{0} BO(3)) \) which is polynomial (18 Theorem 3.11). Since \( H^\ast(\Omega^\infty D'(2)) \) is

a conned Hopf algebra, by the structure theorem of Hopf algebras over \( \mathbb{Z}/2 \) (11 Theorem
6.1] or [7 Theorem 7.11]), this implies that \( H^\ast(\Omega^\infty D'(2)) \) itself is a polynomial algebra. Now

the Corollary follows as the other factor \( H^\ast(Q_{0} BO(3)) \) is polynomial again by [16 Theorem
3.11].
5.2 Relations among $\mu$-classes

As an application of the above, we now prove Theorem 2.6.

**Definition 5.5.** Define the weight on elements of $H_*(QX)$ by $w(Q^i x) = 2^i$ for $x \in \text{Im}(H_*(X) \to H_*(QX))$, and extend it by $w(xy) = w(x) + w(y)$. Denote by $W_i(H_*(QX))$ the set of weight $i$ elements. The module of indecomposables $QH_*(QX)$ inherits the weights from $H_*(QX)$, and we have $QH_*(QX) \cong \oplus_i W_i QH_*(QX)$, one can define similarly the weights on elements of $QH_*(\Omega^\infty M(n))$ via the inclusion $QH_*(\Omega^\infty M(n)) \subset QH_*(QX)$. By the second statement of Lemma 5.7, $QH_*(\Omega^\infty M(n))$ also admits a direct sum decomposition $QH_*(\Omega^\infty M(n)) \cong \oplus_i W_i QH_*(\Omega^\infty M(n))$. In each case, we say that the elements of $W_i(\cdot)$ are homogeneous of weight $i$.

With this definition, we can state:

**Lemma 5.6.** The map $QH_*(\Omega^\infty d_k) : QH_*(\Omega^\infty \mathcal{M}(k)) \to QH_*(\Omega^\infty \mathcal{M}(k))$ sends homogeneous elements to homogeneous elements, and multiplies the weight by 2.

**Proof.** This is immediate from [5] Proposition 3.6.

Define a decreasing filtration $F_i$ on $QH_*(\Omega^\infty \mathcal{M}(n))$ by $F_i QH_*(\Omega^\infty \mathcal{M}(n)) = \oplus j \geq i W_j QH_*(\Omega^\infty \mathcal{M}(n))$. Then it satisfies the following conditions.

i). $F_i(H_*(\Omega^\infty \mathcal{M}(n)) = H_*(\Omega^\infty \mathcal{M}(n))$

ii). if $x \in F_i$ then $Q^i(x) \in F_{2i}$

Thus by naturality of the Dyer-Lashof operations, we see that any map of infinite loop spaces between $\Omega^\infty \mathcal{M}(n)$’s induce filtration preserving map in homology, even though most of the time they don’t send homogeneous elements to homogeneous elements. We also note that the homology suspension $\sigma^\infty_x$ maps isomorphically $F_1/F_2$ to $H_*(\mathcal{M}(n))$. We will show the following.

**Lemma 5.7.** The map $H_*(\Omega^\infty d_{n-1})$ induces an injection

$$H_*(\mathcal{M}(n)) \cong F_1/F_2(QH_*(\Omega^\infty \mathcal{M}(n))) \to F_2/F_4(QH_*(\Omega^\infty \mathcal{M}(n-1))).$$

**Proof.** By Lemma 5.6 for $k = n - 1$ we see that $QH_*(\Omega^\infty d_{n-1})$ induces a map from $F_1/F_2$ to $F_2/F_4$. By applying Lemma 5.6 to the case $k = n$, we see that the image of $QH_*(\Omega^\infty d_n)$ is included in $F_2$. Alternatively, one can see this by noting that the map $H_*(d_n) = 0$. Thus by the exactness of Proposition 2.4, $F_1/F_2(QH_*(\Omega^\infty \mathcal{M}(n)))$ injects to $F_2(QH_*(\Omega^\infty \mathcal{M}(n-1)))$. But as we have $F_1/F_2 \cong W_1$ and by Lemma 5.6 $W_1$ maps to $W_2$, we get the desired result.

Now we are ready to prove Theorem 2.6. The inclusion $H^*(\mathcal{M}(n)) \subset H^*(BO(n))$ is given by $H^*(f_n)$, and this is determined uniquely by its compatibility with $H^*(\alpha_n)$, which in turn is determined uniquely by the fact that $H^{-n}(MTO(n))$ contains only one non-trivial element, and the fact that $H^*(D(n))$ is generated by the bottom class as a module over the Steenrod algebra. The cofibration sequence (2) imply that such a class vanishes if its preimage in $H^*(QBO(n))$ belongs to the image of $H^*(\Omega^\infty MTO(n-1))$. Now, Theorem 3.9 implies that we have a commutative diagram

\[
\begin{array}{ccc}
BO(n) & \xrightarrow{f_n} & M(n) \\
\downarrow & & \downarrow \\
MTO(n-1) & \xrightarrow{\Sigma^{1-n} D(n-1)} & M(n-1)
\end{array}
\]
Thus we get
\[
\begin{align*}
H_\ast(Q_0BO(n)_+) & \to H_\ast(\Omega^\infty M(n)) \\
H_\ast(\Omega^\infty MTO(n-1)) & \to H_\ast(\Omega^\infty (\Sigma^{1-n}D(n-1))) \\
& \to H_\ast(\Omega^\infty M(n-1))
\end{align*}
\]

Dualizing Lemma above, we see that the space of functionals on \(QH_\ast(\Omega^\infty M(n-1))\) vanishing on \(F_4\) surjects to the space of functionals on \(QH_\ast(\Omega^\infty M(n))\) vanishing on \(F_2\), which is precisely the image of \(\sigma^{\infty\ast}\). Thus by the commutativity of the diagram above, we see that the image of the composition \(PH_\ast(M(n)) \to PH_\ast(\Omega^\infty M(n)) \to PH_\ast(Q_0BO(n))\) is contained in the image of \(PH_\ast(\Omega^\infty MTO(n-1))\). This concludes the proof of i). The other statements follow from [1] Theorem 1.8 and Corollary 4.4.

**Remark 5.8.** The formula in [11 Corollary 4.4] involves a map that sends homogeneous elements of weight 1 to a sum of homogeneous elements of weight 2 and 4. The proof of Proposition 4.5’ loc. cit. shows that the terms of weight 4 can be ignored. In the above argument, we show that actually one can use another map which are homogeneous.

To conclude, we give some explicit examples of those relations. First of all, we have [?, Corollary 3.11]

**Proposition 5.9.** The image of \(H_\ast(M(n))\) in \(H_\ast(BO(n))\) is the a free-module over \(H_\ast(BT(n))^{Gl_n(\mathbb{Z}/2)}\) generated by a basis of \(A(n-2)Sq^{2n-1}, \ldots, 2^1 (x_1^{-1} \cdots x_n^{-1})\) where \(A(k)\) is the subalgebra of the Steenrod algebra generated by \(Sq^1, Sq^2, \ldots, Sq^{2^k}\). Here we identify \(H_\ast(BO(n))\) with its image in
\[
H_\ast(BT(n)) \subseteq H_\ast(BT(n))^{- \gamma_0} \cong (x_1 \cdots x_n)^{-1} H_\ast(BT(n))
\]
via \(Bi^\ast\) where \(i : T(n) \subseteq O(n)\).

When \(n = 2\), \(A(0)\) is just the exterior algebra generated by \(Sq^1\), and \(Sq^{2,1}(x_1^{-1}x_2^{-1}) = x_1 + x_2 = \sigma_1, Sq^1(\sigma^1) = x_1^2 + x_2^2 = \sigma_2^2\), whereas the Dickson invariants \(H_\ast(BT(n))^{Gl_n(\mathbb{Z}/2)}\) is generated by \(w_2 = x_1^2 + x_1 x_2 + x_2^2 = \sigma_1^2 + \sigma_2, w_3 = x_1 x_2 (x_1 + x_2) = \sigma_1 \sigma_2\), we derive

**Corollary 5.10.** The set
\[
\{(\sigma_1^j + \sigma_2)^i(\sigma_1 \sigma_j)^j \sigma_1^i; i \geq 0, j \geq 0, \epsilon \in \{1, 2\}\}
\]
forms a basis of the image of \(H_\ast(M(2))\) in \(H_\ast(BO(2))\).

Below is a table of these relations in low dimensions.

| Relation | Value |
|----------|-------|
| \(\mu_{1,0}\) | 0 |
| \(\mu_{3,0} + \mu_{1,1}\) | 0 |
| \(\mu_{2,1}\) | 0 |
| \(\mu_{5,0} + \mu_{3,1} + \mu_{1,2}\) | 0 |
| \(\mu_{3,1}\) | 0 |
| \(\mu_{4,1} + \mu_{2,2}\) | 0 |

Here we have omitted the relations that follow from lower degree relations and the general relation \(\mu_{2i,2j} = \mu_{i,j}^2\).
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