Approximation of Spherical Bodies of Constant Width and Reduced Bodies

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Abstract. We present a spherical version of the theorem of Blaschke that every body of constant width $w < \frac{\pi}{2}$ can be approximated by a body of constant width $w$ whose boundary consists only of pieces of circles of radius $w$ as well as we wish in the sense of the Hausdorff distance. This is a special case of our theorem about approximation of spherical reduced bodies.

Keywords: sphere, spherical convex body, body of constant width, reduced body, Hausdorff distance, approximation

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1 Introduction

A theorem of Blaschke says that for every convex body of constant width $w$ in the Euclidean plane $E^2$ and every $\varepsilon > 0$ there exists a convex body of constant width $w$ whose boundary consists only of pieces of circles of radius $w$ such that the Hausdorff distance between the two bodies is at most $\varepsilon$ (see [1] and also §65 of [2]). A generalization of this fact for normed planes is given in [6], where also approximation of reduced bodies is considered. Corollary at the end of this note presents an analog of this theorem for bodies of constant width on the sphere $S^2$, while Theorem gives a general version for reduced convex bodies on $S^2$.

The method of the proof of Theorem is similar to the proofs from [1] (by Blaschke) and [6]. In order to facilitate the reader a comparison of the present proof with this from [6] for normed planes, in Section 4 we use the notation from [6]. In the present paper we need a number of lemmas and claims for the spherical situation. They are given in Sections 2 and 3.

2 Auxiliary facts from spherical geometry

Let $S^2$ be the unit sphere of the three-dimensional Euclidean space $E^3$. By a great circle we mean the intersection of $S^2$ with any two-dimensional subspace of $E^3$. By a pair of antipodes we mean any pair of points obtained as the intersection of $S^2$ with a one-dimensional subspace of $E^3$. Observe that if two different points $a, b$ are not antipodes, then there is exactly one
great circle containing them. By the *arc of great circle*, or shortly *arc*, \( ab \) connecting them we understand the shorter part of the great circle between \( a \) and \( b \). By the *distance* \( |ab| \), of these points we mean the length of \( ab \).

The set of points of \( S^2 \) whose distance from a point \( c \in S^2 \) equals (is at most) \( r \leq \frac{\pi}{2} \) is called the *circle* (respectively, *disk*) of radius \( r \) and center \( c \). Disks of radius \( \frac{\pi}{2} \) are called *hemispheres*. If \( p \) belongs to a circle, then the set of points which are at distances at most \( \frac{\pi}{2} \) from \( p \) is called a *semicircle*. We say that \( p \) is the *center* of this semicircle.

A subset of \( S^2 \) is called *convex* if it does not contain any pair of antipodes and if together with every two points it contains the arc connecting them. For a set \( A \) contained in the interior of a hemisphere we denote by \( \text{conv}(A) \) the smallest convex set containing \( A \). By a *convex body* we mean a closed convex set with non-empty interior. Let \( p \) be a boundary point of a set \( A \) contained in the interior of a hemisphere. We say that a hemisphere \( K \) containing \( A \) *supports* it if \( A \cap \text{bd}(K) \) is non-empty. If \( p \in A \cap K \), we say that \( K \) supports \( A \) at \( p \).

**Lemma 1.** Let \( A \subset S^2 \) be a closed set with non-empty interior. If at every boundary point the set \( A \) is supported by a hemisphere, then \( A \) is convex.

**Proof.** The proof is analogous to the proof of Theorem 9 of [3]. Namely, suppose that \( A \) is not convex. Then there are points \( x_1, x_2 \in A \) and a point \( y \in x_1x_2 \) such that \( y \notin A \). There exists an interior point \( z \) of \( A \) such that \( z \notin x_1x_2 \). Hence there is \( x_f \in \text{bd}(A) \cap yz \). Every great circle through \( x_f \) separates one of the points \( x_1, x_2, z \) from at least one other of these points. So there is no supporting hemisphere of \( A \) through \( x_f \). A contradiction. \( \square \)

Recall a few notions and facts from [7]. We say that \( e \) is an *extreme point* of \( C \) provided the set \( C \setminus \{ e \} \) is convex. If hemispheres \( G \) and \( H \) are different and their centers are not antipodes, then \( L = G \cap H \) is called a *lune*. The semicircles bounding \( L \) and contained respectively in \( \text{bd}(G) \) and \( \text{bd}(H) \) are denoted by \( G/H \) and \( H/G \). The *thickness* \( \Delta(L) \) of \( L \) is defined as the distance of the centers of \( G/H \) and \( H/G \). For every hemisphere \( K \) supporting a convex body \( C \) we find hemispheres \( K^* \) supporting \( C \) such that the lunes \( K \cap K^* \) are of the minimum thickness. At least one such a \( K^* \) exists. By the *width* \( \text{width}_{K}(C) \) of \( C \) determined by \( K \) we mean the thickness of the lune \( K \cap K^* \). It changes continuously, as the position of \( K \) changes. From Part I of Theorem 1 of [4] we know that such \( K^* \) is unique if \( \text{width}_{K}(C) < \frac{\pi}{2} \).

When we go on \( \text{bd}(C) \), then always counterclockwise. Let us recall, but in a slightly different form than in [10], that if hemispheres \( X, Y, Z \) support a convex body \( C \) in this order, then we write \( X \prec Y \prec Z \). Let \( X \) and \( Z \) be hemispheres supporting \( C \) at \( p \) and let \( L = X \cap Y \cap Z \) for every hemisphere \( Y \) supporting \( C \) at \( p \). Then \( X \) is said to be the *right supporting* hemisphere at \( p \) and \( Z \) is said to be the *left supporting* hemisphere at \( p \).

By the *thickness* \( \Delta(C) \) of \( C \) we mean the minimum of \( \text{width}_{K}(C) \) over all supporting hemispheres \( K \) of \( C \).
Assume that a lune \( L \) of thickness \( \Delta C \) contains a convex body \( C \). Then by Claim 2 of [7] both the centers of the semicircles bounding \( L \) belong to \( \text{bd}(C) \). We call such a lune \( L \) a \textit{supporting lune of} \( C \). We say that the lune supports \( C \) \textit{at} the mentioned centers. The arc connecting the centers of the semicircles bounding \( L \) is called a \textit{thickness chord of} \( C \). This notion is an analog of the notion of a thickness chord of a convex body in \( E^d \) considered by many authors under this name or without any name, as for instance in [2].

If for all hemispheres \( K \) supporting \( C \) the numbers \( \text{width}_K(C) \) are equal, we say that \( C \) is \textit{of constant width}. Of course, a convex body \( C \subset S^2 \) with \( \Delta(C) < \frac{\pi}{2} \) is of constant width if and only if for every hemisphere \( K \) supporting \( L \) the lune \( K \cap K^* \) has thickness \( \Delta(C) \).

**Lemma 2.** Consider a non-degenerate spherical triangle \( abd \subset S^2 \) and a point \( e \in ab \) such that \( de \) is orthogonal to \( ab \). The distance between \( d \) and \( ab \) is at most

\[
\arcsin \frac{|ab|}{\tan \frac{1}{2} |adb|}.
\]

**Proof.** Let \( e' \) be the center of \( ab \) and \( d' \) be such that \( |d'e'| = |de| \) with \( d'e' \) orthogonal to \( ab \). Clearly, \( \angle e'd'a = \frac{1}{2} \angle adb \) and \( |ae'| = \frac{1}{2} |ab| \). Since \( ae'd' \) is a right triangle, \( \sin |d'e'| \cdot \tan \frac{1}{2} |adb| = \tan \frac{1}{2} |ab| \) and our thesis for \( d' \) instead of \( d \) is true. By \( |d'e'| = |de| \) we get the thesis for our \( d \). \( \square \)

Here is a spherical version of the classic Blaschke selection theorem.

**Claim 1.** From every sequence of convex bodies on \( S^d \) of thickness at most a fixed constant smaller than \( \pi \) we may select a subsequence which tends to a spherical convex body.

**Proof.** First we select a subsequence which is contained in a certain spherical disk of a radius below \( \pi \). Then from this subsequence we select the final subsequence (see [7] p. 558 and [5]). \( \square \)

## 3 Reduced bodies on sphere

After [7] we say that a spherical convex body \( R \) is \textit{reduced} if \( \Delta(Z) < \Delta(R) \) for every convex body \( Z \subset R \) different from \( R \). For basic properties of reduced bodies on \( S^2 \) see [10]. The class of spherical reduced bodies is larger than the class of bodies of constant width. Earlier many papers considered reduced bodies in the Euclidean and normed \( d \)-dimensional spaces; see the survey articles [8] and [9].

Later tacitly assume that all considered reduced bodies \( R \subset S^2 \) are of thickness below \( \frac{\pi}{2} \).

Theorem 3.1 of [10] presents the boundary structure of a reduced body \( R \subset S^2 \). Namely, assume that \( M_1, M_2 \) are supporting hemispheres of \( R \) with \( \text{width}_{M_1}(R) = \Delta(R) = \text{width}_{M_2}(R) \) such that \( \text{width}_{M_1}(R) > \Delta(R) \) for every \( M \) fulfilling \( \prec M_1MM_2 \). Consider the lunes \( L_1 = M_1 \cap M_1^* \) and \( L_2 = M_2 \cap M_2^* \). Then the arcs \( a_1a_2 \) and \( b_1b_2 \) are in \( \text{bd}(R) \), where \( a_i \) is the center of \( M_i/M_i^* \) and \( b_i \) is the center of \( M_i^*/M_i \) for \( i = 1, 2 \).
Denote by $c$ the intersection point of the arcs $a_1a_2$ and $b_1b_2$. The union of the triangles $a_1a_2c$ and $b_1b_2c$ is called a butterfly, while $a_1a_2$ and $b_1b_2$ its arms.

Consider any maximum piece $\widehat{gh}$ of the boundary of $R$ which does not contain any arc Let $K$ be any hemisphere supporting $R$ at a point of $\widehat{gh}$. But at $g$ we agree only for the right, and at $h$ we agree only for the left. By Part I of Theorem 1 of [7] for every $K$ there exists exactly one hemisphere $K^*$ supporting $R$ such that the lune $K \cap K^*$ has thickness $\text{width}_K(R)$. From this theorem we also conclude that $K^*$ touches $R$ at a unique point. In particular, for the right hemisphere supporting $R$ at $g$ denote this unique point by $g'$, and for the left hemisphere supporting $R$ at $h$ by $h'$.

**Claim 2.** For any hemisphere $K$ supporting $R$ at a point of $\widehat{gh}$ we have $\text{width}_K(R) = \Delta(R)$.

**Proof.** Clearly, $\text{width}_K(R)$ cannot be smaller than $\Delta(R)$, because then we get a contradiction with the definition of the thickness of $R$.

Suppose that $\text{width}_K(R) > \Delta(R)$. Then by continuity arguments (see Theorem 2 of [7]) we obtain that if $K$ supports $R$ at a point different from $g$ and $h$, then there are supporting hemispheres $K_1$ and $K_2$ with $\angle K_1 KK_2$ such that for every supporting hemisphere $H$ fulfilling $\angle K_1 H K_2$ we have $\Delta(H) > \Delta(R)$. By the just recalled Theorem 3.1 of [10], the piece $\widehat{gh}$ of the boundary of $R$ contains an arc. A contradiction with the assumption on $\widehat{gh}$ from the paragraph preceding our claim. If $K$ supports $R$ at $g$ or $h$, then similarly we show that $\widehat{gh}$ contains an arc whose one end-point is $g$ or $h$, which again gives a contradiction with the choice of $\widehat{gh}$.

By the two preceding paragraphs we get $\text{width}_K(R) = \Delta(R)$.

From this claim and Lemma 2.2 of [10] we see that all the points at which all our hemispheres $K^*$ touch $R$ form a curve $g'h'$ being a piece of $\text{bd}(R)$. We call it the curve opposite to the curve $\widehat{gh}$. Vice-versa, from this lemma we obtain that $g'h'$ determines $\widehat{gh}$. So we say that $\widehat{gh}$ and $g'h'$ is a pair of opposite curves of constant width $\Delta(R)$. Let us summarize the above consideration as the following claim analogous to Corollary 11 of [4] on the situation in a normed plane.

**Claim 3.** The boundary of a reduced body $R \subset S^2$ consists of countably many pairs of arms of butterflies and of countably many pairs of opposite pairs of curves of constant width $\Delta(R)$.

Clearly any particular pair of opposite pieces of curves of constant width $\Delta(R)$ consists of end-points of thickness chords; they are the centers of the semicircles bounding the lunes of thickness $\Delta(R)$ supporting $R$ at the points of these pieces of curves of constant width.

In particular, when $R$ from Claim 3 is a body of constant width, then its boundary is the union of one pair of curves of constant width $\Delta(R)$. We may present $\text{bd}(R)$ on infinitely many ways as such an union. Each time the curves end at the end-points of a thickness chord of $R$. 

4 Approximation of reduced bodies

Theorem. Let \( R \subset S^2 \) be a reduced body of thickness below \( \frac{\pi}{2} \). For any \( \varepsilon > 0 \) there exists a reduced body \( R_\varepsilon \subset S^2 \) with \( \Delta(R_\varepsilon) = \Delta(R) \) whose boundary consists only of arms of butterflies and arcs of circles of radius \( \Delta(R) \) such that the Hausdorff distance between \( R_\varepsilon \) and \( R \) is at most \( \varepsilon \).

Proof. We omit the trivial case when the boundary of \( R \) consists only of arms of butterflies.

Take any positive \( \varepsilon < \frac{1}{2}\pi \). Put \( \rho_\varepsilon = 2 \cdot \arctan(\sin \varepsilon) \).

Consider any pair \( F, G \) of opposite curves of constant width \( \Delta(R) \) in the boundary of \( R \). Exceptionally, when \( R \) is a body of constant width, we divide \( \text{bd}(R) \) into a pair of curves \( F \) and \( G \) by an arbitrary thickness chord. Denote the endpoints of \( F \) by \( f' \) and \( f'' \), and the endpoints of \( G \) by \( g' \) and \( g'' \), in both cases according to the positive orientation.

Part A. The aim of this part is to construct the set \( R_\varepsilon \).

From Claim 3 we know that \( \text{bd}(R) \) contains countably many pairs of opposite pieces of curves of constant width \( \Delta(R) \). For each such a pair \( F, G \) we provide a number of different thickness chords \( f_1g_1, \ldots, f_ng_n \) of \( R \) such that \( f_1, \ldots, f_n \in F \) (with \( f_1 = f' \) and \( f_n = f'' \)), and \( g_1, \ldots, g_n \in G \) (with \( g_1 = g' \) and \( g_n = g'' \)), taking care that \( |f_i f_{i+1}| \) and \( |g_i g_{i+1}| \) be below \( \varepsilon \) for \( i = 1, \ldots, n - 1 \), and that the positively oriented angle between every two successive chords is below \( \frac{\pi}{2} \). Clearly, some of points \( f_1, \ldots, f_n \) (some of \( g_1, \ldots, g_n \)) may coincide. Let \( c_i \in \Gamma_i \) be a point of intersection of circles of

\[
\begin{align*}
\text{Figure: Illustration to the proof of Theorem}
\end{align*}
\]
radius \( \Delta(R) \) with centers \( f_i \) and \( f_{i+1} \) (so \( c_i \) is in equal distances from \( f_i \) and \( f_{i+1} \)). Such \( c_i \) exists since, thanks to Claim 2, we have \( |f_ig_{i+1}| \leq \Delta(R) \) and \( |f_{i+1}g_i| \leq \Delta(R) \). Moreover, by \( c_0 \) we mean \( g' \), and by \( c_n \) we mean \( g'' \).

For every \( i \in \{1, \ldots, n - 1 \} \) take the arc \( F_i \) of the circle \( F_i^c \) of radius \( \Delta(R) \) with center \( c_i \) and endpoints \( f_i \) and \( f_{i+1} \) which is in \( \Phi_i \). Moreover, for every \( i \in \{1, \ldots, n \} \) take the arc \( G_i \) of the circle \( G_i^c \) of radius \( \Delta(R) \) with center \( f_i \) which begins at \( c_{i-1} \) and ends at \( c_i \). Created arcs are marked by broken lines in Figure. Clearly, \( G_1 \subset \Gamma_1 \), \( G_i \subset \Gamma_{i-1} \cup \Gamma_i \) for \( i = 2, \ldots, n - 1 \), and \( G_n \subset \Gamma_{n-1} \). We have constructed the pair of curves \( F^* = F_1 \cup \ldots \cup F_{n-1} \) and \( G^* = G_1 \cup \ldots \cup G_n \).

Denote by \( U_\varepsilon \) the closure of the union of all arms of the butterflies of \( R \) and of all pairs of curves of the form \( F^* \) and \( G^* \). We see that \( U_\varepsilon \) is obtained from \( \text{bd}(R) \) by exchanging all pairs of opposite curves \( F \) and \( G \) by the constructed pairs of curves \( F^* \) and \( G^* \).

We define \( R_\varepsilon \) as the set bounded by \( U_\varepsilon \).

Part B. We define a sequence of sets \( R^j \subset S^2 \) and show that they are convex bodies.

We define \( R_0, R_1, \ldots \) by induction.

Put \( R^0 = R \). Clearly it is a convex body.

Assume that \( R^{j-1} \) is a convex body, where \( j \geq 1 \). We get the boundary of \( R^j \) by exchanging a pair of opposite curves \( F, G \), if any remains, from \( \text{bd}(R^{j-1}) \) into a pair \( F^*, G^* \) as in Part A.

At every boundary point \( p \), the set \( R^j \) is supported by a hemisphere. If \( p \in F^* \cup G^* \) this follows from the construction of \( F^* \) and \( G^* \). If \( p \in \text{bd}(R^j) \) does not belong to any curves \( F^* \) or \( G^* \), then in the part of the supporting hemisphere take this supporting \( R^{j-1} \). From Lemma 1 we conclude that \( R^j \) is convex. Clearly, \( R^j \) is a convex body.

Part C. We show that \( R_\varepsilon \) is a convex body and \( \Delta(R_\varepsilon) = \Delta(R) \).

If after a finite number of steps in Part B we get \( R_\varepsilon \), we see that it is a convex body.

In the opposite case, \( R_\varepsilon = \lim_{j \to \infty} R^j \). By Claim 1 we see that \( R_\varepsilon \) is also a convex body.

From the construction of \( R^j \) we see that for every supporting hemisphere \( K \) of \( R^j \) at every point of \( F^* \cup G^* \) we have \( \text{width}_K = \Delta(R) \). The remaining part of \( \text{bd}(R^j) \) does not differ from \( \text{bd}(R^{j-1}) \). So by induction we get \( \Delta(R^j) = \Delta(R) \). Thus if after a finite number of steps we obtain \( R_\varepsilon \), it has thickness \( \Delta(R) \). By Lemma 4 of [7], if \( R_\varepsilon = \lim_{j \to \infty} R^j \), then the same is true.

Part D. Let us prove that \( R_\varepsilon \) is a reduced body.

We should show that for any convex body \( Z \subset R_\varepsilon \) different from \( R_\varepsilon \) the inequality \( \Delta(Z) < \Delta(R_\varepsilon) \) holds true. The body \( Z \) does not contain an extreme point \( e \) of \( R_\varepsilon \) as it follows from the spherical analog of the Krein-Millman theorem formulated on p. 565 of [7]. Consequently, \( Z \) is disjoint with an open disk \( D \) centered at \( e \). Now exactly as in Part 4 of [6] we show that \( \Delta(Z) < \Delta(R_\varepsilon) \), so that \( R_\varepsilon \) is a reduced body.

Part E. We show that every \( F^* \) and every \( G^* \) are in the union of some triangles.
Take a pair of curves $F^*$ and $G^*$ constructed in Part A. The bounding semicircle of the first lune supporting $R_\varepsilon$ at $f_i$ is denoted by $K_i$, and that at $g_i$ by $L_i$ (again see Figure).

For $i \in \{1, \ldots, n-1\}$, by $k_i$ denote the point of intersection of $K_i$ with $K_{i+1}$ (if they are subsets of a great circle, take $k_i$ as the center of $f_if_{i+1}$). By $l_i$ denote the point of intersection of $L_i$ with $L_{i+1}$ (if they are subsets of a great circle, take $l_i$ as the center of $g_ig_{i+1}$).

For every $c \in G_i$ take the lune supporting $R_\varepsilon$ such that $cf_i$ is the thickness chord and denote by $T(c)$ the bounding semicircle of this lune through $f_i$. In particular, $T(g_i) = K_i$. When we move $c \in G_i$ counterclockwise from $g_i$ to $c_i$, the lune and thus also $T(c)$ “rotate” counterclockwise. So since the distance between $c_i$ and any point of $T(c_i)$ is at least $\Delta(R)$ we see that the distance from $c_i$ to every point of the arc $f_ik_i$ is at least $\Delta(R)$. Analogously, the distance from $c_i$ to any point of $f_{i+1}k_i$ is at least $\Delta(R)$. So since every point of $F_i$ is at the distance $\Delta(R)$ from $c_i$, we get $F_i \subset f_ik_if_{i+1}$. Thus $F^*$ is contained in the union of triangles $f_ik_if_{i+1}$, where $i = 1, \ldots, n$. Similarly, $G^*$ is in the union of triangles $g_ig_ig_{i+1}$, where $i = 1, \ldots, n$.

Part F. We show that the Hausdorff distance between $R$ and $R_\varepsilon$ is at most $\varepsilon$.

Denote by $P$ the closure of the convex hull of all points $f_i$ and $g_i$ and of endpoints of all arms of butterflies of $R$. Denote by $Q$ the union of $P$ and all triangles $f_ik_if_{i+1}$ and $g_ig_ig_{i+1}$.

Part E implies inclusions $P \subset R \subset Q$ and $P \subset R_\varepsilon \subset Q$. So in order to estimate the Hausdorff distance between $R$ and $R_\varepsilon$ by $\varepsilon$ it is sufficient to estimate the Hausdorff distance between $P$ and $Q$ by $\varepsilon$. Since $P \subset Q$, the Hausdorff distance between them is $\sup_{q \in Q} \inf_{p \in P} |pq|$. Hence it is sufficient to show that all (i.e., for all pairs $F$, $G$ and all $i$) distances between $k_i$ and $f_ik_if_{i+1}$, and also between $l_i$ and $g_ig_ig_{i+1}$, are at most $\varepsilon$.

Since the sum of angles of any quadrangle $a_1k_ia_if_{i+1}$ is over $2\pi$, from the assumption in Part A on the angle between two successive chords below $2\pi$, we get $\angle f_ia_1f_{i+1} < \frac{\pi}{2}$. So $\angle f_ik_if_{i+1} > 2\pi - 3\cdot \frac{\pi}{2} = \frac{\pi}{2}$. Hence $\frac{1}{2} \angle f_ik_if_{i+1} > 1$. By Lemma 2 the distance from $k_i$ to $f_ik_if_{i+1}$ is at most $\arcsin \frac{1}{2} |f_ik_if_{i+1}| < \arcsin \frac{1}{2} |f_ik_if_{i+1}|$. By $|f_i f_{i+1}| \leq \rho_\varepsilon$ (see Part A) and $\rho_\varepsilon = 2\arctan(\sin \varepsilon)$ we get that it is at most $\arcsin \frac{1}{2} \rho_\varepsilon = \arcsin \frac{2\arctan(\sin \varepsilon)}{2} = \varepsilon$. Analogously, the distance between $l_i$ and $g_ig_ig_{i+1}$ is at most $\varepsilon$. We see that the Hausdorff distance between $P$ and $Q$, and thus between $R$ and $R_\varepsilon$, is at most $\varepsilon$.

If $R$ is a body of constant width, from Parts A and B of this proof we see that $R_\varepsilon$ (so $R^1$ in this case) is also a body of constant width. So we get the following corollary.

**Corollary.** For every body $W \subset S^2$ of constant width and for arbitrary $\varepsilon > 0$ there exists a body $W_\varepsilon \subset S^2$ of constant width $\Delta(W_\varepsilon) = \Delta(W)$ whose boundary consists only of arcs of circles of radius $\Delta(W)$, such that the Hausdorff distance between $W$ and $W_\varepsilon$ is at most $\varepsilon$.

Applications for Barbier’s theorem in Part 4 of [5] do not hold true here for the sphere.

Let us correct misprints in this Part 4 on p. 873: three times $2\pi$ should be changed into $\pi$. 

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**References:**

[5] Barbier’s theorem.
References

[1] W. Blaschke: Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts, Math. Ann, 76 (1915) 504–513.

[2] T. Bonnesen, W. Fenchel: Theorie der konvexen Körper, Springer, Berlin et al., 1934, (Engl. transl. Theory of Convex Bodies, BCS Associated, Moscow, Idaho USA, 1987).

[3] H. G. Eggleston: Convexity, Cambridge University Press, 1958.

[4] E. Fabińska, M. Lassak: Reduced bodies in normed spaces, Isr. J. Math. 161 (2007) 75–88.

[5] N. N. Hai, P. T. Ann: A generalization of Blaschke’s converegence theorem in metric space, J. Convex Anal. 20 1013–1024.

[6] M. Lassak: Approximation of bodies of constant width and reduced bodies in a normed plane, J. Convex Anal. 19 (2012) 865–874.

[7] M. Lassak: Width of spherical convex bodies, Aeq. Math. 89 (2015), 555–567.

[8] M. Lassak, H. Martini: Reduced convex bodies in Euclidean space - a survey, Expo. Math. 29 (2011) 204–21.

[9] M. Lassak, H. Martini: Reduced convex bodies in finite-dimensional normed spaces – a survey, Results Math. 66 (2014) 405–426.

[10] M. Lassak, M. Musielak: Reduced spherical convex bodies, Bull. Pol. Acad. Sci., Math. 66 (2018) 87–97.

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