EMBEDDINGS OF $\mathbb{Q}$ INTO SOME FINITELY PRESENTED GROUPS

JAMES BELK, JAMES HYDE, AND FRANCESCO MATUCCI

Abstract. We prove that the lift of Thompson’s group $T$ to the real line contains the additive group $\mathbb{Q}$ of the rational numbers. This gives an explicit, natural example of a finitely presented group that contains $\mathbb{Q}$, answering a Kourovka notebook question of Martin Bridson and Pierre de la Harpe.

We also prove that $\mathbb{Q}$ can be embedded into a finitely presented simple group. Specifically, we describe a simple group $TA$ of homeomorphisms of the circle that contains $\mathbb{Q}$, and we prove that $TA$ is two-generated and has type $F_\infty$.

Our method for proving finiteness properties extends existing techniques to allow for groups of homeomorphisms that are locally more complicated than the standard Thompson groups.

INTRODUCTION

In 1961, Graham Higman proved that any countable group with a computable presentation can be embedded into a finitely presented group [25]. For example, the additive group $\mathbb{Q}$ of rational numbers has computable presentation

$$\langle s_1, s_2, s_3, \ldots \mid s_n^2 = s_{n-1} \text{ for all } n \geq 2 \rangle$$

and can therefore be embedded into some finitely presented group. Though Higman’s proof is constructive, the resulting finite presentations are quite large and unwieldy.

Higman was for many years interested in finding more explicit embeddings of various naturally occurring recursively presented groups such as $\mathbb{Q}$ into finitely presented groups [28]. In 1999, Martin Bridson and Pierre de la Harpe submitted the following question to the Kourovka notebook [32], labelled as a “well-known problem”.

Problem 14.10(a). It is known that any recursively presented group embeds in a finitely presented group. Find an explicit and “natural” finitely presented group $\Gamma$ and an embedding of the additive group of the rationals $\mathbb{Q}$ in $\Gamma$.

The problem then asks the same question for the group $\text{GL}_n(\mathbb{Q})$. Moreover, at the time that the problem was submitted, no explicit and natural examples of finitely generated groups containing $\mathbb{Q}$ were known, so the problem originally included a part (b) asking for such an example. Finitely generated examples were
later supplied by Mikaelian in 2005 [34], but no solution to part (a) of the problem has appeared in the literature.

In this paper we describe two solutions to this problem. The first is the group \( T \) of all piecewise-linear homeomorphisms \( f: \mathbb{R} \to \mathbb{R} \) that satisfy the following conditions:

1. Each linear portion of \( f \) has the form \( f(x) = 2^n x + d \), where \( n \in \mathbb{Z} \) and \( d \) is a dyadic rational.
2. Each breakpoint of \( f \) has dyadic rational coordinates.
3. The homeomorphism \( f \) is periodic in the sense that \( f(x + 1) = f(x) + 1 \) for all \( x \in \mathbb{R} \).

Note that condition (3) implies that \( f \) is either linear or has infinitely many linear segments. We regard such homeomorphisms as “piecewise-linear” as long as the set of breakpoints is discrete.

**Theorem 1.** The group \( T \) is finitely presented and has a subgroup isomorphic to \( \mathbb{Q} \).

This group is closely related to the groups \( F, T, V \) introduced by Richard J. Thompson in the 1960’s, now known collectively as Thompson’s groups [16]. In particular, \( F \) is the group of all piecewise-linear homeomorphisms of the closed interval \([0, 1]\) satisfying conditions (1) and (2) above, \( T \) is the group of all piecewise-linear homeomorphisms of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \) satisfying these same conditions, and \( V \) is a similarly defined group of homeomorphisms of the Cantor set. Thompson proved that all three of these groups are finitely presented and that \( T \) and \( V \) are simple, making them the first known examples of infinite, finitely presented simple groups.

The group \( T \) is precisely the lift of Thompson’s group \( T \) to the real line through the covering map \( \mathbb{R} \to \mathbb{R}/\mathbb{Z} \). As such, it fits into a short exact sequence

\[ \mathbb{Z} \hookrightarrow \overline{T} \twoheadrightarrow T \]

where the kernel is the group of deck transformations, i.e. the cyclic group generated by the map \( z(x) = x + 1 \). Since \( T \) is finitely presented, it follows immediately that \( \overline{T} \) is as well. The kernel is central in \( \overline{T} \) by condition (3) above, and since \( T \) is simple this kernel must be precisely the center of \( \overline{T} \).

The first appearance of \( \overline{T} \) in the literature seems to be the work of Ghys and Sergiescu in 1987 on the cohomology of \( T \) [21]. They proved that \( H^2(T, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \), and that one of the two generators for \( H^2(T, \mathbb{Z}) \) is the Euler class for the extension \( \mathbb{Z} \hookrightarrow \overline{T} \twoheadrightarrow T \). As far as we know it has not previously been observed that \( \overline{T} \) contains \( \mathbb{Q} \), though Bleak, Kassabov, and the third author observed in 2011 that Thompson’s group \( T \) contains \( \mathbb{Q}/\mathbb{Z} \) [5]. We prove in Section 1 that \( \overline{T} \) actually has uncountably many different subgroups isomorphic to \( \mathbb{Q} \), and we give a presentation for \( \overline{T} \) with two generators and four relations. We also prove that \( \overline{T} \) is contained in some other well-known groups:

**Theorem 2.** The following finitely presented groups have subgroups isomorphic to \( \overline{T} \), and hence contain \( \mathbb{Q} \):

1. The automorphism group of Thompson’s group \( F \).
2. The braided Thompson group \( BV \).
Here \( BV \) is the braided version of \( V \) introduced independently by Brin [7] and Dehornoy [17] in 2006. This group has intriguing connections to Teichmüller theory and the study of big mapping class groups [19].

Our second main example is a new group \( TA \) of homeomorphisms of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \), which can be defined as follows. We say that a homeomorphism \( f : S^1 \to S^1 \) is nearly piecewise-linear if either it is piecewise-linear with finitely many pieces or it consists of infinitely many linear pieces whose breakpoints accumulate along a finite set of singular points. Such a homeomorphism has doubling self-similarity at a singular point \( p \in S^1 \) if \( Df(p) \circ f \) agrees with \( f \circ Dp \) in a neighborhood of \( p \), where \( Dp \) and \( Df(p) \) denote the local dilations of \( S^1 \) by a factor of 2 centered at \( p \) and \( f(p) \), respectively. Let \( TA \) be the group of all nearly piecewise-linear homeomorphisms \( f : S^1 \to S^1 \) that satisfy the following conditions:

1. Each linear portion of \( f \) has the form \( f(x) = 2^n x + d \), where \( n \in \mathbb{Z} \) and \( d \) is a dyadic rational.
2. Each breakpoint of \( f \) has dyadic rational coordinates.
3. Each singular point of \( f \) has dyadic rational coordinates as well as doubling self-similarity.

Note that the subgroup of \( TA \) consisting of elements that have no singular points is precisely Thompson’s group \( T \).

**Theorem 3.** The group \( TA \) is two-generated, finitely presented, simple, and contains \( \mathbb{Q} \).

It was not previously known that \( \mathbb{Q} \) could be embedded into a finitely presented simple group. A theorem of Hall asserts that every countable group embeds into a finitely generated simple group [24], and Schupp [40] and Goryushkin [22] proved independently that every countable group embeds into a two-generated simple group.

We actually prove something stronger than finite presentability for \( TA \). Recall that a group \( G \) has type \( F_n \) if there exists a \( K(G, 1) \) complex whose \( n \)-skeleton is finite. For example, a group has type \( F_1 \) if and only if it is finitely generated, and a group has type \( F_2 \) if and only if it is finitely presented. A group has type \( F_\infty \) if it has type \( F_n \) for all \( n \), or equivalently if there exists a \( K(G, 1) \) complex with finitely many cells of each dimension (cf. [20, Proposition 7.2.2]). All of these finiteness properties are commensurability invariants—indeed they are invariant under quasi-isometry [1]. Brown and Geoghegan proved that Thompson’s group \( F \) has type \( F_\infty \) [12], making it the first example of a torsion-free, infinite-dimensional \( F_\infty \) group. Brown later developed general techniques for establishing finiteness properties, and used them to prove that \( T \) and \( V \) have type \( F_\infty \) [11]. An extension of an \( F_\infty \) group by an \( F_\infty \) group has type \( F_\infty \) (cf. [1 Theorem 10]), so it follows from Brown’s result that \( TA \) has type \( F_\infty \). We prove the following.

**Theorem 4.** The group \( TA \) has type \( F_\infty \).

We prove this theorem by constructing an action of \( TA \) on a certain simplicial complex \( K \) and then using a combination of Brown’s criterion [11, Theorem 2.2] and Bestvina–Brady Morse theory [3]. Our construction of \( K \) is surprisingly efficient, and can be generalized to prove finiteness properties for a large family of groups of homeomorphisms whose elements have finitely many ‘singular points’. We will discuss this generalization in [2], and show that the resulting family of \( F_\infty \) groups includes many Röver–Nekrashevych groups (obtained by combining a Thompson
group with a self-similar group of tree automorphisms \[36\] [11], as well as a new simple group \(VA\) that contains every countable abelian group.

**Acknowledgments.** The authors would like to thank Collin Bleak for many helpful conversations and suggestions about this work. We would also like to thank Matthew Brin and Matthew Zaremsky for their comments on an early draft of this manuscript and Martin Bridson for comments on the historical perspective.

1. The group \(T\)

In this section we prove that the group \(T\) has uncountably many subgroups isomorphic to \(Q\), and we classify such subgroups. We also give a presentation for \(T\) with 2 generators and 4 relations, and we prove that \(T\) is isomorphic to a subgroup of the braided Thompson group \(BV\).

1.1. Inclusion of \(Q\) into \(T\). Here and in the future, we say that a homeomorphism is Thompson-like if it is piecewise-linear and satisfies conditions (1) and (2) for elements of \(T\) given in the introduction. Let \(PL_2(\mathbb{R})\) denote the group of the homeomorphism \(z(x) = x + 1\).

**Lemma 1.1.** Let \(g\) be an element of \(PL_2(\mathbb{R})\) without fixed points and let \(n \geq 2\). Then there exist infinitely many different \(f \in PL_2(\mathbb{R})\) such that \(f^n = g\).

**Proof.** Without loss of generality, suppose that \(g(0) > 0\). It is well-known that if \([a,b]\) and \([c,d]\) are closed intervals in \(\mathbb{R}\) with dyadic rational endpoints, then there exists at least one Thompson-like homeomorphism \([a,b] \to [c,d]\) (cf. [10 Lemma 4.2]). Choose dyadic rationals \(0 = p_0 < p_1 < \cdots < p_n = g(0)\), and for each \(1 \leq i < n\) choose a Thompson-like homeomorphism \(f_i: [p_{i-1}, p_i] \to [p_i, p_{i+1}].\)

Let \(f_n: [p_{n-1}, p_n] \to [p_n, g(p_1)]\) be the homeomorphism \(gf_1^{-1}f_2^{-1}\cdots f_{n-1}^{-1}\), and let \(f \in PL_2(\mathbb{R})\) be the homeomorphism that agrees with \(f_i\) on each \([p_i, p_{i+1}](1 \leq i \leq n)\) and satisfies

\[f(x) = g^k fg^{-k}(x)\]

for each \(x \in [g^k(0), g^{k+1}(0)]\) with \(k \neq 0\).

To prove that \(f^n = g\), observe that on the interval \([p_{i-1}, p_i]\), the function \(f^n\) restricts to the composition

\[(gf_{i-1}g^{-1}) \cdots (gf_2g^{-1})(gf_1g^{-1})f_n \cdots f_{i+1}f_i\]

Since \(f_n = gf_1^{-1}f_2^{-1}\cdots f_{n-1}^{-1}\), the expression above simplifies to \(g\). Thus \(f^n\) agrees with \(g\) on \([0, g(0)]\), and it follows easily that \(f^n = g\). Moreover, since there are infinitely many possible choices for \(p_1, \ldots, p_{n-1}\) and \(f_1, \ldots, f_{n-1}\), there are infinitely many possibilities for \(f\).

**Remark 1.2.** The proof of Lemma 1.1 is actually an algorithm for constructing all possible \(n\)th roots \(f\) of \(g\). Indeed, given

1. Any \(g \in PL_2(\mathbb{R})\) satisfying \(g(x) > x\) for all \(x \in \mathbb{R}\),
2. Any dyadic rationals \(a \leq b\) in the interval \((0, g(0))\), and
3. Any Thompson-like homeomorphism \(f: [0, b] \to [a, g(0)]\) for which \(f^n(0)\) is defined and equal to \(g(0)\),

there always exists a unique extension of \(f\) to an element of \(PL_2(\mathbb{R})\) satisfying \(f^n = g\), which can be found using the procedure in the above proof.
Lemma 1.3. Let \( m,n \geq 1 \), and let \( g \in T \) so that \( g^m = z \). Then there exist infinitely many different \( f \in T \) so that \( f^n = g \).

Proof. Note that \( g \) cannot have any fixed points, since these would also be fixed points of \( z \). Therefore, by Lemma 1.1, there exist infinitely many \( f \in \text{PL}_2(\mathbb{R}) \) such that \( f^n = g \). Any such homeomorphism commutes with \( z \) since \( f^{mn} = z \), and therefore every such \( f \) lies in \( T \). \( \square \)

Theorem 1.4. There are uncountably many subgroups of \( T \) isomorphic to \( \mathbb{Q} \).

Proof. Observe that \( \mathbb{Q} \) has presentation \( \langle s_1,s_2,\ldots \mid s_n^n = s_{n-1} \text{ for } n \geq 2 \rangle \). To obtain an embedding of \( \mathbb{Q} \) into \( T \) it suffices to find a sequence \( \{s_n\} \) of elements of \( T \) such that \( s_1 \) has infinite order and \( s_n^n = s_{n-1} \) for all \( n \geq 2 \). Such a sequence can be defined recursively by letting \( s_1 = z \) and then repeatedly applying Lemma 1.3 to find, for each \( n \geq 2 \), an element \( s_n \in T \) such that \( s_n^n = s_{n-1} \). Since there are infinitely many choices for \( s_n \) at each stage, this procedure constructs uncountably many different copies of \( \mathbb{Q} \).

Remark 1.5. Note that the proofs of Lemma 1.3 and Theorem 1.4 rely only on the fact that \( T \) is the centralizer of \( z \) in \( \text{PL}_2(\mathbb{R}) \). Indeed, these proofs apply equally well to the centralizer of any element of \( \text{PL}_2(\mathbb{R}) \) that has no fixed points. Any such element is conjugate to \( z \) or \( z^{-1} \) in \( \text{PL}_2(\mathbb{R}) \), so any such centralizer is isomorphic to \( T \).

Remark 1.6. The choice of the elements \( s_n \) in the proof of Theorem 1.4 can be carried out constructively. For example, let \( \{a_n\} \) be the decreasing sequence of dyadics in \([0,1]\) defined recursively by \( a_1 = 1 \) and \( a_n = a_{n-1}/2^{n-1} \). Let \( s_1 = z \), and for each \( n \geq 2 \) let \( s_n \) the the \( n \)th root of \( s_{n-1} \) in \( \text{PL}_2(\mathbb{R}) \) that satisfies

\[
{s_n(x) = \begin{cases} 
  x + a_n & \text{if } 0 \leq x \leq a_n, \\
  2x & \text{if } a_n < x \leq 2a_{n-1}.
\end{cases}}
\]

Note that the given formula for \( s_n \) maps \([0,\frac{1}{2}a_{n-1}]\) to \([a_n,a_{n-1}]\) and satisfies \( s_n^0(0) = a_{n-1} = s_{n-1}(0) \) for each \( n \), so such roots exist and are unique as described in Remark 1.2. By the proofs of Lemma 1.3 and Theorem 1.4, the sequence \( \{s_n\} \) generates a subgroup of \( T \) isomorphic to \( \mathbb{Q} \).

Remark 1.7. The copy of \( \mathbb{Q} \) constructed in Remark 1.6 has the property that the orbit of 0 is dense in \( \mathbb{R} \). For such a copy, the resulting action of \( \mathbb{Q} \) on \( \mathbb{R} \) is conjugate by a homeomorphism of \( \mathbb{R} \) to the usual action of \( \mathbb{Q} \) on \( \mathbb{R} \) by translation. However, there are also “exotic” copies of \( \mathbb{Q} \) in \( T \) for which the orbit of 0 is not dense in \( \mathbb{R} \). For example, by Remark 1.2 we can choose a sequence \( \{s_n\} \) in \( T \) with \( s_1 = z \) and \( s_n^n = s_{n-1} \) (\( n \geq 2 \)) such that \( s_n(0) = \frac{1}{2} + \frac{1}{2^n} \) for all \( n \). In this case, the subgroup \( \langle s_1,s_2,s_3,\ldots \rangle \) is isomorphic to \( \mathbb{Q} \), but the orbit of 0 under the action of this subgroup does not intersect the interval \((0,1/2]\). One consequence of this is that the restricted wreath product \( F \wr \mathbb{Q} \) embeds into \( T \), where the direct sum \( \bigoplus_{\mathbb{Q}} F \) is supported on the complement of the closure of the orbit of 0 under such an exotic action.

Remark 1.8. It follows from known results that \( \mathbb{Q} \) does not embed into Thompson’s group \( V \). For example, Higman proved that for every element \( f \in V \) of infinite order there exists only finitely many \( n \in \mathbb{N} \) so that \( f \) has an \( n \)th root \( [26 \text{ Corollary 9.3}] \). Since \( T \) embeds into \( V \), it follows that \( \mathbb{Q} \) does not embed into \( T \). Note, however,
that $\mathbb{Q}/\mathbb{Z}$ embeds into $T$ [5 Theorem 1.6], and indeed every copy of $\mathbb{Q}$ in $T$ maps to a copy $\mathbb{Q}/\mathbb{Z}$ in $T$.

Every copy of $\mathbb{Q}$ obtained from the proof of Theorem 1.4 contains the center $\langle z \rangle$ of $T$. Indeed, using the root-finding algorithm given in Remark 1.2 we can produce every copy of $\mathbb{Q}$ in $\tilde{T}$ that contains the center. The following proposition asserts that these are all of the subgroups of $T$ isomorphic to $\mathbb{Q}$.

**Proposition 1.9.** Every subgroup of $T$ isomorphic to $\mathbb{Q}$ contains the center of $T$.

**Proof.** Let $Q$ be a subgroup of $\tilde{T}$ isomorphic to $\mathbb{Q}$. Since $\mathbb{Q}$ does not embed into $T$ (see Remark 1.8), the projection homomorphism $\tilde{T} \to T$ cannot be injective on $Q$, so $Q$ must intersect the center of $\tilde{T}$ nontrivially. In particular, $Q$ must contain $z^n$ for some $n \geq 1$. Since $\mathbb{Q}$ is isomorphic to $\mathbb{Q}$, there exists an $f \in Q$ so that $f^n = z^n$. Since $f$ and $z$ commute it follows that $(fz)^{-1} = 1$, and since $\tilde{T}$ is torsion-free we conclude that $fz = 1$, and therefore $Q$ contains $z$. □

1.2. **Presentation for $T$.** Here we give a presentation for $T$ with two generators and four relations.

Note first that each element of $T$ is determined by its restriction to the interval $[0, 1]$. Indeed, if $d$ is any dyadic rational and we choose a Thompson-like homeomorphism $f : [0, 1] \to [d, d+1]$, then $f$ extends uniquely to an element of $\tilde{T}$ by the formula

$$f(x) = \lfloor x \rfloor + f(x - \lfloor x \rfloor)$$

for $x \in \mathbb{R} \setminus [0, 1]$.

**Proposition 1.10.** The group $\tilde{T}$ has presentation

$$\langle a, b \mid a^4 = b^3, (ba)^5 = b^3, [bab, a^2baba^2] = [bab, a^2b^2a^2baba^2ba^2] = 1 \rangle.$$ 

where $a, b \in T$ are the elements satisfying

$$a(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{3}{4}, \\ x + \frac{1}{8} & \text{if } \frac{3}{4} < x \leq \frac{7}{8}, \\ 4x - \frac{5}{2} & \text{if } \frac{7}{8} < x \leq 1, \end{cases} \quad b(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{4}, \\ x + \frac{1}{4} & \text{if } \frac{1}{4} < x < \frac{1}{2}, \\ 2x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

**Proof.** Recall that $T$ fits into a short exact sequence $\mathbb{Z} \to \tilde{T} \to T$. In [29], Lochak and Schneps give a presentation for $T$ with two generators $\alpha, \beta \in T$ and the following relations:

$$\alpha^4 = 1, \beta^3 = 1, [\beta \alpha \beta, \alpha^2 \beta \alpha \beta \alpha^2] = 1, [\beta \alpha \beta, \alpha^2 \beta \alpha \beta \alpha^2 \beta \alpha^2 \beta \alpha^2] = 1, (\beta \alpha)^5 = 1.$$

(Note that the version in [29] contains a typo. See [15] Proposition 1.3 for a corrected version.) The elements $a, b \in T$ defined above map to $\alpha$ and $\beta$, respectively, under the epimorphism $\tilde{T} \to T$. Applying the standard algorithm for finding a presentation for a group extension (cf. [27 Proposition 2.55]), it follows that $\tilde{T}$ is generated by $a, b$, and the generator $z(x) = x + 1$ for the kernel of the epimorphism $\tilde{T} \to T$, with relations

$$a^4 = z, \quad b^3 = z, \quad [bab, a^2baba^2] = 1, \quad [bab, a^2b^2a^2baba^2ba^2] = 1, \quad (ba)^5 = z^3$$
as well as $a^{-1}za = z$ and $b^{-1}zb = z$. These last two relations follow from the first two, and the desired presentation now follows by eliminating the generator $z$. □
EMBEDDINGS OF $\mathbb{Q}$ INTO SOME FINITELY PRESENTED GROUPS

Remark 1.11. There is a natural copy of Thompson’s group $F$ inside of $T$ consisting of all elements that fix the integers pointwise, and it is easy to see that $T$ is generated by this copy of $F$ together with the element $\delta(x) = x + \frac{1}{2}$. Using similar reasoning to the proof of Proposition 1.10, one can derive the presentation

$$T = \langle x_0, x_1, \delta \mid x_2 x_1 x_2^{-1} = x_3, x_3 x_2^{-1} = x_4, x_4 \delta = (x_0 \delta x_0)^{-1}, x_1 \delta = y_1, x_2 \delta = y_2 \rangle$$

where $x_0$ and $x_1$ are the usual generators for $F$, with $x_n = x_0^{-n} x_1 x_0^{n-1}$ for $n \geq 1$, $y_1 = x_0^2 x_1 x_0^{-1}$, and $y_2 = x_0^2 x_2 x_1 x_0^{-1}$.

Remark 1.12. It follows from the work of Ghys and Sergiescu [21] that $T$ is perfect, but there is an elementary proof of this fact using the copy of $F$ inside of $T$ described in the previous remark. Specifically, it is well-known that the commutator subgroup of $F$ consists of all elements that are the identity in neighborhoods of 0 and 1 (cf. [16, Theorem 4.1]), and it follows that any element of $T$ which is the identity in a neighborhood of the integers lies in $[T, T]$. If we let $f(x) = x + 1/4$ and $g$ be any element of $T$ that is the identity in a neighborhood of the integers and agrees with $f$ in a neighborhood of 1/4, then $k = f^{-2} g f$ is also the identity in a neighborhood of the integers. Then $g$ and $k$ both lie in $[T, T]$, so $f = g f k^{-1} f^{-1}$ does as well, and hence $z = f^4$ lies in $[T, T]$. Thus the kernel of the epimorphism $T \to T$ lies in the commutator subgroup, and since $T$ is simple and hence perfect it follows that $T$ is perfect.

1.3. Inclusion of $T$ into $BV$. The braided Thompson group $BV$ was introduced independently by Brin [7] and Dehornoy [17]. It fits into a short exact sequence

$$B_{2^n} \hookrightarrow BV \twoheadrightarrow V$$

where $V$ is Thompson’s group and $B_{2^n}$ is a certain ascending union of the braid groups $B_{2^n}$. Brin proved that $BV$ is finitely presented [8], and Bux, Fluch, Marscher, Witzel, and Zaremsky proved that $BV$ has type $\text{F}_\infty$ [15].

Proposition 1.13. The braided Thompson group $BV$ has a subgroup isomorphic to $T$ and hence contains $\mathbb{Q}$.

Sketch of Proof. Let $\beta : T \to BV$ be the homomorphism that maps the three generators $x_0, x_1, \delta$ of $T$ (see Remark 1.11) to the elements of $BV$ shown in Figure 1. It is easy to check that these elements satisfy the necessary relations, so $\beta$ is indeed a well-defined homomorphism. Moreover, observe that $\beta(\delta^n)$ is non-trivial for every $n \geq 1$, and hence $\beta$ is injective on the center of $T$. But since $T$ is simple and $T$ is a perfect central extension of $T$ (see Remark 1.12), every normal subgroup of $T$ is contained in the center of $T$, and therefore the kernel of $\beta$ must be trivial. □
2. THE GROUP TA

Let TA be the group of homeomorphisms of the circle S^1 = \mathbb{R}/\mathbb{Z} defined in the introduction. The goal of this section is to prove that TA is two-generated, simple, and contains \( \mathbb{Q} \) (Theorem 3) and that it has type F∞ (Theorem 4).

2.1. The groups Aut(F) and A. As we will see, the group TA is closely related to the automorphism group of Thompson’s group F. The structure of this automorphism group was analyzed by Brin in 1996 [6] using a theorem of Rubin [38].

Brin’s analysis is based on a well-known action of Thompson’s group F on the real line. For the following proposition, let \( z(x) = x + 1 \), let \( F_\mathbb{R} \) be the group of all \( f \in \text{PL}_2(\mathbb{R}) \) with the property that \( f \) agrees with some \( z^m \) and \( z^n \) \((m, n \in \mathbb{Z})\) on neighborhoods of \(-\infty\) and \(+\infty\), and let \( h : \mathbb{R} \to (0, 1) \) be the Thompson-like homeomorphism shown in Figure 2.

Proposition 2.1. The group \( F_\mathbb{R} \) is isomorphic to Thompson’s group F. In particular, the mapping \( f \mapsto h^{-1} fh \) is an isomorphism \( F \to F_\mathbb{R} \). □

It follows from Rubin’s work that the automorphism group of F is precisely the normalizer of \( F_\mathbb{R} \) in the group of all homeomorphisms of \( \mathbb{R} \). Every homeomorphism of \( \mathbb{R} \) is either orientation-preserving or orientation-reversing, and this gives a decomposition of Aut(F) as a semidirect product \( A \rtimes \mathbb{Z}_2 \), where \( A \) is the subgroup of all orientation-preserving automorphisms (denoted \( \text{Aut}_+ (F) \) by Brin). Brin gave a complete description of the elements of \( A \).

Theorem 2.2 (Brin [6]). The group \( A \) consists of all homeomorphisms \( f \in \text{PL}_2(\mathbb{R}) \) that have the following property: there exist \( g, h \in T \) so that \( f \) agrees with \( g \) on a neighborhood of \(-\infty\) and \( f \) agrees with \( h \) on a neighborhood of \(+\infty\). □

Remark 2.3. Theorem 2.2 is equivalent to saying that \( A \) consists of all elements \( f \in \text{PL}_2(\mathbb{R}) \) for which the commutator \([f, z]\) has compact support.

Corollary 2.4. The group A contains T, and therefore both A and Aut(F) contain \( \mathbb{Q} \).

Brin also expressed A as a group extension in two different ways. First, observe that the elements \( g, h \in T \) in Theorem 2.2 are uniquely determined by the given element \( f \in A \). Brin showed that the mapping \( f \mapsto (g, h) \) determines an epimorphism \( A \to T \times T \) whose kernel is precisely the commutator subgroup of the group \( F_\mathbb{R} \). This gives a short exact sequence

\[ [F, F] \hookrightarrow A \twoheadrightarrow T \times T. \]
Next, if we compose the epimorphism \( A \to T \times T \) with the projection \( T \times T \to T \times T \) we obtain an epimorphism \( \varphi: A \to T \times T \) whose kernel is precisely \( F_\mathbb{R} \). This yields a short exact sequence

\[ F \hookrightarrow A \overset{\varphi}{\twoheadrightarrow} T \times T. \]

**Proposition 2.5.** The group \( A \) is perfect and finitely presented. Indeed, it has type \( F_\infty \).

**Proof.** It is well-known that \([F,F]\) is simple (cf. [11, Theorem 4.5]) and hence perfect. Since \( T \) is perfect (see Remark 1.12), the direct product \( T \times T \) is also perfect, and therefore \( A \) is perfect by the first short exact sequence above.

Brown and Geoghegan proved that \( F \) has type \( F_\infty \) [12, Corollary 5.4]. Brown later proved that \( T \) has type \( F_\infty \) [11, Theorem 4.17], so \( T \times T \) has type \( F_\infty \). By [11, Theorem 10], an extension of an \( F_\infty \) group by an \( F_\infty \) group has type \( F_\infty \), so we conclude that \( A \) has type \( F_\infty \) by the second short exact sequence above. \( \square \)

**Remark 2.6.** Burillo and Cleary have derived an explicit presentation for \( A \) with 8 generators and 35 relations [13, Proposition 3.1].

### 2.2 Embeddings of \( A \) into \( TA \)

There is a close relationship between Brin’s group \( A \) and the group \( TA \). For the following proposition, let \( \text{sing}(f) \) denote the set of singular points of a given \( f \in TA \). Note that

\[ \text{sing}(f^{-1}) = f(\text{sing}(f)) \quad \text{and} \quad \text{sing}(fg) \subseteq g^{-1}(\text{sing}(f)) \cup \text{sing}(g) \]

for any \( f, g \in TA \), and that \( \text{sing}(f) = \emptyset \) if and only if \( f \in T \).

**Proposition 2.7.** For any dyadic point \( p \) on the circle, the group

\[ A_p = \{ f \in TA \mid f(p) = p \text{ and } \text{sing}(f) \subseteq \{p\} \} \]

is isomorphic to \( A \). Moreover, \( TA \) is generated by \( A_p \) and Thompson’s group \( T \).

To prove this proposition, recall the Thompson-like homeomorphism \( h: (0,1) \to \mathbb{R} \) shown in Figure 2. Conjugating \( A \) by \( h \) gives a group \( A_{(0,1)} = h^{-1}Ah \) of Thompson-like homeomorphisms of the interval \((0,1)\). Note that such a homeomorphism might have infinitely many linear pieces, with breakpoints accumulating near 0 and 1 (which we allow since 0 and 1 are not in the domain). The following lemma characterizes the elements of \( A_{(0,1)} \).

**Lemma 2.8.** Let \( f: (0,1) \to (0,1) \) be a Thompson-like homeomorphism. Then \( f \in A_{(0,1)} \) if and only if \( f \) has doubling self-similarity at 0 and 1, i.e. if and only if \( D_0 \circ f \) agrees with \( f \circ D_0 \) in a neighborhood of 0 and \( D_1 \circ f \) agrees with \( f \circ D_1 \) in a neighborhood of 1, where \( D_0(x) = 2x \) and \( D_1(x) = 2x - 1 \).

**Proof.** By Brin’s characterization (Theorem 2.2), the elements of \( A \) are all of the Thompson-like homeomorphisms of \( \mathbb{R} \) that commute with \( z(x) = x + 1 \) in neighborhoods of \( -\infty \) and \( +\infty \). But it is easy to check that \( h^{-1}zh \) is a homeomorphism of \((0,1)\) that agrees with \( D_0 \) in a neighborhood of 0 and agrees with \( D_1^{-1} \) in a neighborhood of 1, so the desired statement follows. \( \square \)

For example, Figure 3 shows the graphs of two of the generators \( x_0, \delta \in T \) (see Remark 1.11) acting on the interval \((0,1)\) as elements of \( A_{(0,1)} \). The doubling self-similarity indicated by Lemma 2.8 manifests as geometric self-similarity of the graphs in neighborhoods of 0 and 1.
Proof of Proposition 2.7. Let $p$ be a dyadic point on the circle, and let $A_p$ be the given subgroup of $T\mathcal{A}$. Let $q: (0,1) \to S^1$ be the linear function that wraps the interval $(0,1)$ counterclockwise around $S^1 \setminus \{p\}$. Then by Lemma 2.8 the elements of $A_p$ are precisely the homeomorphisms of $S^1$ that fix $p$ and lift to elements of $A_{(0,1)}$ via $q$, and therefore $A_p$ is isomorphic to $A_{(0,1)}$.

To prove that $T\mathcal{A}$ is generated by $A_p \cup T$, let $f$ be any element of $T\mathcal{A}$. If $f$ has no singular points then $f \in T$ and we are done, so suppose that $f$ has singular points $p_1, \ldots, p_n$, where each $p_i$ and $f(p_i)$ are dyadic. Multiplying $f$ by an element of Thompson’s group $T$ if necessary, we may assume that each $p_i$ is fixed by $f$. Let $U_1, \ldots, U_n$ be pairwise disjoint neighborhoods of $p_1, \ldots, p_n$ respectively. Then for each $i$ we can find an element $a_i \in A_{p_i}$ which is supported on $U_i$ and agrees with $f$ in a neighborhood of $p_i$. Note that each $A_{p_i}$ is conjugate to $A_p$ by an element of $T$, so each $a_i$ lies in the subgroup of $T\mathcal{A}$ generated by $A_p \cup T$. But $a_1^{-1} \cdots a_n^{-1}f$ has no singular points and is therefore an element of $T$, which proves that $f$ lies in the subgroup of $T\mathcal{A}$ generated by $A_p \cup T$.

Corollary 2.9. The group $T\mathcal{A}$ is finitely generated and has subgroups isomorphic to $T$. In particular, $T\mathcal{A}$ has subgroups isomorphic to $Q$.

Remark 2.10. If $[p,q]$ is any arc on the circle with dyadic endpoints, then by Lemma 2.8 the subgroup

$$A_{[p,q]} = \{ f \in T\mathcal{A} \mid f \text{ is supported on } [p,q] \text{ and } \text{sing}(f) \subseteq \{p,q\} \}$$

is also isomorphic to $A$. Indeed, if $[[p_n,q_n]]_{n \in \mathbb{N}}$ are pairwise disjoint arcs on the circle with dyadic endpoints then the subgroup of $T\mathcal{A}$ generated by $\bigcup_{n \in \mathbb{N}} A_{[p_n,q_n]}$ is isomorphic to the infinite direct sum $\bigoplus_{n \in \mathbb{N}} A$.

This has a few consequences. Since $A$ contains $Q$, this proves that $T\mathcal{A}$ contains the infinite direct sum $\bigoplus_{n \in \mathbb{N}} Q$, and therefore $T\mathcal{A}$ contains every countable, torsion-free abelian group. Also, since $A$ has nonabelian free subgroups, $T\mathcal{A}$ contains an infinite direct sum of nonabelian free groups. In particular, $T\mathcal{A}$ contains $F_2 \times F_2$, so by a result of Mihailova it has unsolvable subgroup membership problem [33]. Note that Thompson’s group $T$ does not contain $F_2 \times F_2$, since by [31] Theorem 7.2.3 the centralizer of any infinite-order element of $T$ must virtually embed into the group.
whether of Burillo, the third author, and Ventura [14, Corollary 3.3].

2.3. The functions $\varphi_p$ and $\Phi_p$. Recall that $A$ fits into a short exact sequence

$$F \hookrightarrow A \xrightarrow{\varphi} T \times T.$$ 

For a given $f \in A$, the two components of $\varphi(f)$ depend only on the behavior of $f$ in neighborhoods of $-\infty$ and $+\infty$. Conjugating by the Thompson-like homeomorphism $h: (0,1) \to \mathbb{R}$ shown in Figure 1, we obtain an epimorphism $\varphi_{(0,1)}: A_{(0,1)} \to T \times T$ whose two components depend only on the behavior of an element of $A_{(0,1)}$ in neighborhoods of 0 and 1. The kernel of $\varphi_{(0,1)}$ is precisely the usual Thompson’s group $F$ acting on $(0,1)$.

Moving to the circle, we obtain for each dyadic point $p \in S^1$ an epimorphism $\varphi_p: A_p \to T \times T$ whose two components depend only on the restrictions of a homeomorphism to arcs of the form $[p, p+\epsilon)$ and $(p-\epsilon, p]$, respectively. The kernel of $\varphi_p$ consists of all elements of $A_p$ that do not have a singularity at $p$, and hence belong to Thompson’s group $T$. Indeed, $\ker(\varphi_p)$ is precisely the stabilizer of $p$ in $T$, which again is isomorphic to Thompson’s group $F$.

We can extend the homomorphism $\varphi_p$ to a function $\Phi_p: TA \to T \times T$ defined as follows. Given any $f \in TA$, let $r$ be the rotation of the circle that maps $p$ to $f(p)$. Then there exists a $g \in A_p$ so that $rg$ agrees with $f$ on a neighborhood of $p$, and we define

$$\Phi_p(f) = \varphi_p(g).$$

Note that this is well-defined since $\varphi_p(g) = \varphi_p(g')$ whenever $g, g' \in A_p$ agree on a neighborhood of $p$. Note also that $\Phi_p(f)$ is the identity if and only if $p$ is not a singular point of $f$. If $\Phi_p(f)$ is not the identity, then we can think of its value as determining the “type” of singularity that $f$ has at $p$.

The functions $\Phi_p$ are not homomorphisms, but they do have the following multiplicative property.

**Proposition 2.11.** Let $f_1, f_2 \in TA$, let $p \in S^1$ be a dyadic point, and let $q = f_2(p)$. Then

$$\Phi_p(f_1f_2) = \Phi_q(f_1) \Phi_p(f_2).$$

**Proof.** Let $r_1$ and $r_2$ be rotations such that $r_1(q) = f_1(q)$ and $r_2(p) = f_2(p) = q$, and let $g_1 \in A_q$ and $g_2 \in A_p$ so that $r_1g_1$ agrees with $f_1$ on a neighborhood of $q$ and $r_2g_2$ agrees with $f_2$ on a neighborhood of $p$. Then $r = r_1r_2$ is the rotation that maps $p$ to $f_1f_2(p)$ and $g = r_2^{-1}g_1r_2g_2$ is an element of $A_p$ such that $rg$ agrees with $f_1f_2$ on a neighborhood of $p$, so

$$\Phi_p(f_1f_2) = \varphi_p(g) = \varphi_p(r_2^{-1}g_1r_2) \varphi_p(g_2).$$

It follows from the definitions of $\varphi_p$ and $\varphi_q$ that $\varphi_p(r_2^{-1}g_1r_2) = \varphi_q(g_1)$, and therefore

$$\Phi_p(f_1f_2) = \varphi_q(g_1) \varphi_p(g_2) = \Phi_q(f_1) \Phi_p(f_2).$$
2.4. Further properties of \( TA \). In this subsection we prove that \( TA \) is simple and two-generated.

**Theorem 2.12.** \( TA \) is simple.

**Proof.** Let \( N \) be a nontrivial normal subgroup of \( TA \), and fix a dyadic point \( p \in S^1 \). By Proposition 2.7, it suffices to prove that \( N \) contains both \( T \) and \( A_p \).

To prove that \( N \) contains \( T \), fix any nontrivial element \( n \in N \). Then we can find an open arc \( I \) on the circle such that \( I \) contains no singular points of \( n \) and \( n(I) \) is disjoint from \( I \). If \( t \) is any element of \( T \) that is supported on \( I \), then \( ntn^{-1}t^{-1} \) lies in \( T \) as well and is supported on \( n(I) \), so the commutator \( [n,t] = ntn^{-1}t^{-1} \) is a nontrivial element of \( T \). We conclude that \( N \cap T \) is nontrivial, and since \( T \) is simple it follows that \( T \subseteq N \).

To prove that \( N \) contains \( A_p \), let \( f, g \in T \times T \). Fix an element \( t \in T \) such that \( t(p) \neq p \), and let \( J \) be an open arc containing \( p \) such that \( t(J) \) is disjoint from \( J \). Then we can find elements \( \tilde{f}, \tilde{g} \in A_p \) that are supported on \( J \) such that \( \varphi_p(\tilde{f}) = f \) and \( \varphi_p(\tilde{g}) = g \). Since \( \tilde{f} \) is supported on \( J \) and \( t \tilde{g}t^{-1} \) is supported on \( t(J) \), we know that \( \tilde{f} \) commutes with \( t \tilde{g}t^{-1} \). But if \( t \in N \) by the previous paragraph, so \( \tilde{f} \) commutes with \( \tilde{g} \) modulo \( N \), and therefore \( [\tilde{f}, \tilde{g}] \in N \). Then \( [f, g] = \varphi_p(N \cap A_p) \), so \( \varphi_p(N \cap A_p) \) contains every commutator in \( T \times T \). Since \( T \times T \) is perfect, it follows that \( \varphi_p \) maps \( N \cap A_p \) onto \( T \times T \). Since \( N \cap A_p \) contains the kernel of \( \varphi_p \) by the previous paragraph, we conclude that \( N \cap A_p = A_p \).

**Remark 2.13.** The simplicity of \( TA \) can also be proven using the existing theory of homeomorphism groups. For example, since \( T \) and \( A_p \) are both perfect and they generate \( TA \), we know that \( TA \) is perfect, so it suffices to prove that the commutator subgroup of \( TA \) is simple. This follows from a theorem of Ling [30, Theorem 1.9], which states roughly that any group of homeomorphisms of a paracompact space which acts transitively on small open sets and is generated by elements supported on small open sets must have simple commutator subgroup.

The following theorem is based on ideas from [4] applied to the circle rather than the Cantor set.

**Theorem 2.14.** The group \( TA \) is 2-generated.

**Proof.** Since \( A \) is perfect and finitely generated by Proposition 2.5, there exist finitely many elements \( f_1, \ldots, f_k \in A \) such that the commutators \( [f_i, f_j] \) (\( i \neq j \)) generate \( A \). Choose an \( n \in \mathbb{N} \) and integers \( 0 \leq u_1 < u_2 < \cdots < u_k < n \) so that all of the differences \( u_i - u_j \) (\( i \neq j \)) are distinct modulo \( n \) (e.g. \( u_i = 2^i \) and \( n = 2^{k+1} \)). Fix dyadic points \( p_0, q_0, p_1, q_1, \ldots, p_{n-1}, q_n-1 \) on the circle in counterclockwise order, and let \( t \) be an element of \( T \) so that

\[
 t(p_i) = q_i \quad \text{and} \quad t(q_i) = p_{i+1}
\]

for each \( i \), where the subscripts are modulo \( n \). Let \( r = t^2 \), and note that each power \( r^i \) maps the interval \( [p_0, p_1] \) to \( [p_i, p_{i+1}] \).

Let \( \psi: A \to A_{[p_0, p_1]} \) be an isomorphism, where \( A_{[p, q]} \) denotes the copy of \( A \) supported on an arc \( [p, q] \) (see Remark 2.10), and let

\[
 f = \left( r^{u_1} \psi(f_1) \right) \left( r^{u_2} \psi(f_2) \right) \cdots \left( r^{u_k} \psi(f_k) \right).
\]

Note then that \( f \) is supported on \( [p_{u_1}, p_{u_1+1}] \cup \cdots \cup [p_{u_k}, p_{u_k+1}] \), and agrees with \( r^{u_i} \psi(f_i) r^{-u_i} \) on each \( [p_{u_i}, p_{u_i+1}] \). We claim that \( t \) and \( f \) generate \( TA \).
Observe that, since all of the differences \( u_i - u_j \) \((i \neq j)\) are distinct modulo \( n \), all of the intervals of support of \( r^{-u_i} f_r^{u_i} \) and \( r^{-u_j} f_r^{u_j} \) for \( i \neq j \) are different except for \([p_0, p_1]\). It follows that

\[
[r^{-u_i} f_r^{u_i}, r^{-u_j} f_r^{u_j}] = \psi([f_i, f_j])
\]

for each \( i \neq j \). These commutators generate \( A_{[p_0, p_1]} \), and therefore \( A_{[p_0, p_1]} \) is contained in the subgroup generated by \( t \) and \( f \). Conjugating by powers of \( t \), we deduce that each \( A_{[p_1, p_{i+1}]} \) as well as each \( A_{[q_i, q_{i+1}]} \) lies in the subgroup generated by \( t \) and \( f \).

Now, Thompson’s group \( T \) is generated by the elements of \( T \) supported on each \([p_i, p_{i+1}]\) and each \([q_i, q_{i+1}]\) (since the interiors of these intervals cover the circle), so \( T \) lies in the subgroup generated by \( t \) and \( f \). But \( A_{p_1} \) is generated by the elements of \( T \) that fix \( p_1 \) together with the elements of \( A_{[p_0, p_1]} \cap A_{p_1} \) and \( A_{[p_1, p_2]} \cap A_{p_1} \), and therefore \( A_{p_1} \) also lies in the subgroup generated by \( t \) and \( f \). By Proposition 2.7, we conclude that \( t \) and \( f \) generate \( TA \). \( \square \)

2.5. **Finiteness Properties.** In this section we prove that \( TA \) has type \( F_\infty \), which proves in particular that \( TA \) is finitely presented.

Our strategy is to use a combination of Brown’s criterion [11] Theorem 2.2 and Bestvina–Brady Morse theory [9]. For the following theorem, recall that a **Morse function** on a simplicial complex \( K \) (with vertex set \( K^0 \)) is a function \( \mu: K^0 \to \mathbb{Z} \) that satisfies \( \mu(v) \neq \mu(w) \) for every edge \( \{v, w\} \) of \( K \). Given such a Morse function and any \( n \in \mathbb{Z} \), the corresponding **sublevel complex** \( K_{\leq n} \) is the subcomplex of \( K \) induced by all the vertices in \( \mu^{-1}((-\infty, n]) \). The **descending link** \( \text{lk}_v \) of a vertex \( v \in K^0 \) is the link of \( v \) in the sublevel complex \( K_{\leq \mu(v)} \). That is, \( \text{lk}_v \) is the subcomplex of \( K \) induced by all vertices \( w \in K^0 \) that are adjacent to \( v \) and satisfy \( \mu(w) < \mu(v) \).

**Theorem 2.15** (Brown–Bestvina–Brady). Let \( G \) be a group acting simplicially on a contractible simplicial complex \( K \), and let \( \mu: K^0 \to \mathbb{Z} \) be a \( G \)-invariant Morse function on \( K \). Suppose that:

1. Each sublevel complex \( K_{\leq n} \) has finitely many orbits of simplices under the action of \( G \).
2. The stabilizer in \( G \) of each simplex of \( K \) has type \( F_\infty \), and
3. For each \( m \in \mathbb{N} \) there exists an \( n \in \mathbb{Z} \) so that the descending link of each vertex in \( \mu^{-1}([n, \infty)) \) is \( m \)-connected.

Then \( G \) has type \( F_\infty \). \( \square \)

The remainder of this section is devoted to defining a contractible simplicial complex \( K \) on which \( TA \) acts, defining a \( TA \)-invariant Morse function \( \mu \) on \( K \), and then verifying conditions (1), (2), and (3) above.

Let \( D \) be the set of dyadic points on the circle. Given any finite set \( M \subset D \) (possibly empty), put an equivalence relation \( \sim_M \) on \( TA \) by

\[
g \sim_M h \iff \Phi_p(g) = \Phi_p(h) \text{ for all } p \in D \setminus M.
\]

That is, \( g \sim_M h \) if \( g \) and \( h \) have the same “singular structure” on the complement of \( M \). Let \([g]_M \) denote the equivalence class of a given \( g \in TA \) under \( \sim_M \), and let

\[
K^0 = \{ [g]_M \mid g \in TA \text{ and } M \text{ a finite subset of } D \}.
\]

Note that we allow \( M \) to be empty, in which case \([g]_M \) is precisely the coset \( Tg \).
Define a relation $\leq$ on $K^0$ by $[g]_M \leq [h]_{M'}$ if and only if $M \subseteq M'$ and $g \sim_{M'} h$. Note that $g \sim_M h$ implies $g \sim_{M'} h$ whenever $M \subseteq M'$, and therefore $\leq$ is a partial order on $K^0$. Let $K$ be the resulting flag complex, i.e. the simplicial complex with vertex set $K^0$ and one simplex for each finite chain in $K^0$.

**Proposition 2.16.** $K^0$ forms a directed set under $\leq$, and therefore $K$ is contractible.

**Proof.** Let $[g_1]_{M_1}, [g_2]_{M_2} \in K^0$, and let

$$M = M_1 \cup M_2 \cup \text{sing}(g_1) \cup \text{sing}(g_2).$$

Note that $M$ is finite since $M_1$, $M_2$, $\text{sing}(g_1)$, and $\text{sing}(g_2)$ are finite sets. Then $g_1$ and $g_2$ have no singular points on $S^1 \setminus M$, so $g_1 \sim_M g_2 \sim_M e$, where $e$ is the identity homeomorphism of $S^1$. Then $[g_1]_{M_1} \leq [e]_M$ and $[g_2]_{M_2} \leq [e]_M$, so $[g_1]_{M_1}$ and $[g_2]_{M_2}$ have an upper bound in $K^0$. This proves that $K^0$ is a directed set, and it follows that $K$ is contractible (cf. [20 Proposition 9.3.14]). \hfill \Box

Now, define a right action of $T\mathcal{A}$ on $K^0$ by

$$[f]_M g = [fg]_{g^{-1}(M)}$$

for all $f, g \in T\mathcal{A}$ and all finite $M \subseteq D$. Note that $\Phi_{g^{-1}(p)}(fg) = \Phi_{g(p)}(\Phi_{g^{-1}(p)}(g))$ for all $p \in M$ by Proposition 2.11, and therefore this action is well-defined. It is also clearly order-preserving, so it extends to a right action of $T\mathcal{A}$ on $K$.

For the following lemma, define a **basic simplex** in $K$ to be any simplex that corresponds to a chain in $K^0$ of the form

$$[e]_{M_0} < [e]_{M_1} < \cdots < [e]_{M_k},$$

where $M_0 \subset M_1 \subset \cdots \subset M_k$ and $e$ denotes the identity element of $T\mathcal{A}$.

**Lemma 2.17.** Every simplex in $K$ lies in the orbit of a basic simplex under $T\mathcal{A}$.

**Proof.** Let $\Delta$ be any simplex in $K$, corresponding to a chain of the form

$$[g_0]_{M_0} < [g_1]_{M_1} < \cdots < [g_k]_{M_k}$$

for some $g_0, g_1, \ldots, g_k \in T\mathcal{A}$ and some finite sets $M_0 \subset M_1 \subset \cdots \subset M_k \subseteq D$. Since $[g_0]_{M_0} \leq [g_i]_{M_i}$ for each $i$, we know that $g_0 \sim_{M_i} g_i$, and hence $[g_i]_{M_i} = [g_0]_{M_i}$. Thus we can rewrite $\Delta$ as

$$[g_0]_{M_0} < [g_0]_{M_1} < \cdots < [g_0]_{M_k}.$$

Then $\Delta g_0^{-1}$ is the basic simplex

$$[e]_{g_0(M_0)} < [e]_{g_0(M_1)} < \cdots < [e]_{g_0(M_k)}.$$

Now define a function $\mu: K^0 \to \mathbb{Z}$ by

$$\mu([g]_M) = |M|$$

for all $[g]_M \in K^0$. Then clearly $\mu([g]_M) < \mu([h]_{M'})$ whenever $[g]_M < [h]_{M'}$, so $\mu$ is a Morse function on $K$. Moreover, since

$$\mu([g]_M h) = \mu([gh]_{h^{-1}(M)}) = |h^{-1}(M)| = |M| = \mu([g]_M)$$

for all $[g]_M \in K^0$ and $h \in T\mathcal{A}$, the function $\mu$ is $G$-invariant.

**Proposition 2.18.** Each sublevel complex $K_{\leq n}$ has finitely many orbits of simplices under the action of $T\mathcal{A}$. 
Proof. Note that a basic simplex $\Delta$ corresponding to a chain 

$$[e]_{M_0} < [e]_{M_1} < \cdots < [e]_{M_k}$$

lies in $K_{\leq n}$ if and only if $|M_k| \leq n$. Note also that if $t \in T$ then $\Delta^{-1}$ corresponds to the chain 

$$[e]_{t(M_0)} < [e]_{t(M_1)} < \cdots < [e]_{t(M_k)}.$$ 

Since $T$ acts transitively on $k$-element subsets of $D$ for every $k \geq 1$ (cf. [16, Lemma 4.2]), there are only finitely many $T$-orbits of chains $M_0 \subset M_1 \subset \cdots \subset M_k$ in $D$ with $|M_k| \leq n$, and therefore there are only finitely many orbits of basic simplices in $K_{\leq n}$ under the action of $T$. By Lemma [2.17] we conclude that there are only finitely many orbits of simplices in $K_{\leq n}$ under the action of $TA$. □

**Proposition 2.19.** The stabilizer of any simplex in $K$ has type $F_\infty$.

**Proof.** By Lemma [2.17] it suffices to prove this for a basic simplex. The stabilizer of a basic simplex 

$$[e]_{M_0} < [e]_{M_1} < \cdots < [e]_{M_k}$$

consists of all $g \in TA$ so that $\text{sing}(g) \subseteq M_0$ and $g(M_i) = M_i$ for each $i$. Let $S$ be the subgroup of the stabilizer consisting of elements that fix $M_k$ pointwise. Then $S$ has finite index in the stabilizer, and since finiteness properties are quasi-isometry invariants [1] it suffices to prove that $S$ has type $F_\infty$.

Let $p_1, \ldots, p_n$ be the points of $M_k$ in counterclockwise order around the circle. Then 

$$S \cong S_1 \times \cdots \times S_n$$

where $S_i$ is the subgroup of $S$ consisting of elements that are supported on the arc $[p_i, p_{i+1}]$ (where $p_{n+1} = p_1$). Each $S_i$ has one of the following three isomorphism types:

1. If both $p_i$ and $p_{i+1}$ lie in $M_0$, then $S_i$ is isomorphic to Thompson’s group $F$.
   
   This has type $F_\infty$ by a result of Brown and Geoghegan [12].
2. If neither $p_i$ nor $p_{i+1}$ lies in $M_0$, then $S_i$ is isomorphic to $A$, so it has 
   type $F_\infty$ by Proposition 2.5.
3. If exactly one of $p_i$ and $p_{i+1}$ lies in $M_0$, then $S_i$ is isomorphic to the subgroup 
   $A^+ = \varphi^{-1}(T \times \{1\})$ of $A$, which fits into a short exact sequence 
   $$F \hookrightarrow A^+ \twoheadrightarrow T.$$ 

But $F$ has type $F_\infty$ [12] and $T$ has type $F_\infty$ [11, Theorem 4.17], and it is well-known that an extension of an $F_\infty$ group by an $F_\infty$ group has type $F_\infty$ (cf. [11, Theorem 10]), so it follows that $A^+$ has type $F_\infty$.

Since a direct product of finitely many groups of type $F_\infty$ has type $F_\infty$, it follows that $S$ has type $F_\infty$. □

**Lemma 2.20.** Let $p_1, \ldots, p_n$ be distinct points in $D$ and let $f_1, \ldots, f_n \in T \times T$. Then there exists a $g \in TA$ so that $\text{sing}(g) \subseteq \{p_1, \ldots, p_n\}$ and $\Phi_{p_i}(g) = f_i$ for each $i$.

**Proof.** Let $U_1, \ldots, U_n$ be pairwise disjoint open neighborhoods of $p_1, \ldots, p_n$, respectively. Then for each $i$ we can find an element $g_i \in A_{p_i}$ which is supported on $U_i$ and satisfies $\varphi_{p_i}(g_i) = f_i$, in which case the product $g = g_1 \cdots g_n$ has the desired properties. □
Let \( \mu \) be the identity function \( \mu : \mathcal{V} \to \mathcal{V} \), and \( \xi \) be the identity function \( \xi : \mathcal{V} \to \mathcal{V} \).

The descending link of each \( [g]_M \in K^0 \) is \((|M| - 2)\)-connected.

Proof. It suffices to prove this for the vertices of the form \([e]_M\), where \( e \) is the identity in \( TA \) and \( M \) is a finite set of dyadic points. Given such an \([e]_M\), let \( X \) be the simplicial complex with vertex set \( M \times (\bigtimes T) \), where vertices \((p_1, f_1), \ldots, (p_n, f_n)\) form a simplex in \( X \) if and only if their first coordinates \( p_1, \ldots, p_n \in M \) are distinct. Then \( X \) is precisely the join \( *_{p \in M} X_p \), where \( X_p \) is the discrete subcomplex of \( X \) consisting of all vertices in \( \{p\} \times (\bigtimes T) \). By a theorem of Milnor \[35\] Lemma 2.3, the join of any \( n \) nonempty simplicial complexes is always \((n - 2)\)-connected, and therefore \( X \) is \((|M| - 2)\)-connected. We claim that the descending link of \([e]_M\) is isomorphic to the barycentric subdivision of \( X \).

Observe that the vertices of the descending link \( \text{lk}_{A}([e]_M) \) are the set

\[
L_M = \{ [g]_{M'} \mid M' \text{ is a proper subset of } M \text{ and } \Phi_p(g) = 1 \text{ for } p \notin M \}.
\]

Two such vertices \([g_1]_{M'} \) and \([g_2]_{M'} \) are the same if and only if \( \Phi_p(g_1) = \Phi_p(g_2) \) for all \( p \in M \setminus M' \). Then each \([g]_{M'} \in L_M \) has an associated simplex \( \sigma([g]_{M'}) \) in \( X \), namely the simplex with vertices \((p, \Phi_p(g)) \) for \( p \in M \setminus M' \), and this defines an injection \( \sigma \) from \( L_M \) to the collection of simplices of \( X \). By Lemma 2.20, this function \( \sigma \) is also surjective, and therefore a bijection. It is easy to check that two vertices \( v, w \in L_M \) satisfy \( v \leq w \) if and only if \( \sigma(w) \) is a face of \( \sigma(v) \), so it follows that the subcomplex of \( K \) induced by \( L_M \) is isomorphic to the barycentric subdivision of \( X \).

This completes the proof that \( TA \) has type \( F_{\infty} \).

2.6. Presentation for \( TA \). We can use the machinery developed in the previous section to derive a presentation for \( TA \). For the following proposition, recall that a group \( G \) is the amalgamated sum of the subgroups \( H_1, \ldots, H_n \) if \( G \) can be obtained from the free product \( H_1 * \cdots * H_n \) by identifying the copies of \( H_i \cap H_j \) in \( H_i \) and \( H_j \) for each \( i \neq j \). Note that an amalgamated sum of three or more subgroups cannot necessarily be described as a graph of groups if all of the intersections \( H_i \cap H_j \) are nontrivial.

Proposition 2.22. Let \( p \) and \( q \) be distinct dyadic points on the circle. Then \( TA \) is the amalgamated sum of Thompson’s group \( T \), the subgroup \( A_p \), and the subgroup

\[
H = \{ f \in TA \mid f(\{p, q\}) = \{p, q\} \text{ and } \text{sing}(f) \subseteq \{p, q\} \}
\]

which is isomorphic to \((A \times A) \rtimes \mathbb{Z}_2\).

Proof. It follows from Proposition 2.21 that the descending link \( \text{lk}_{A}(v) \) for a vertex \( v \in K^0 \) is simply connected as long as \( \mu(v) \geq 3 \). Since \( K \) is contractible by Proposition 2.16 and hence simply connected, it follows from a result of Bestvina and Brady \[3\] Corollary 2.6 that the sublevel complex \( K_{\leq 2} \) is simply connected.

Now observe that every vertex in \( K_{\leq 2} \) is in the orbit of exactly one of the vertices \([e]_g \), \([e]_{(p)} \), and \([e]_{(p, q)} \). Indeed, the 2-simplex in \( K_{\leq 2} \) spanned by these three vertices is a fundamental domain for the action of \( TA \) on \( K_{\leq 2} \). By a theorem of Brown \[9\] Theorem 3), it follows that \( TA \) is the amalgamated sum of the stabilizers of these three vertices. But the stabilizer of \([e]_g \) is Thompson’s group \( T \), the stabilizer of \([e]_{(p)} \) is the group \( A_p \), and the stabilizer of \([e]_{(p, q)} \) is the group \( H \) defined above.
This proposition can be used to derive an explicit presentation for $T\mathcal{A}$. Note that the three intersections

$$T \cap \mathcal{A}_p, \quad T \cap H, \quad \mathcal{A}_p \cap H$$

are isomorphic to Thompson’s group $F$, the group $(F \times F) \rtimes \mathbb{Z}_2$, and the group $\mathcal{A}^+ \times \mathcal{A}^+$, respectively, all of which are finitely generated. (Here $\mathcal{A}^+$ denotes the subgroup $\varphi^{-1}(T \times \{1\})$ of $\mathcal{A}$, which fits into a short exact sequence $F \to \mathcal{A}^+ \to T$.) Then we can obtain a presentation for $T\mathcal{A}$ by starting with a presentation for $T \ast \mathcal{A}_p \ast H$ and then adding one relation for each generator of each of the three intersections above that specifies that the two images of the generator in $T \ast \mathcal{A}_p \ast H$ are equal.

References

[1] J. Alonso, Finiteness conditions on groups and quasi-isometries. *Journal of Pure and Applied Algebra* **95.2** (1994): 121–129. [doi:10.1016/0022-4049(94)90069-8]

[2] J. Belk, J. Hyde, and F. Matucci, Finite germ extensions. In preparation.

[3] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups. *Inventiones mathematicae* **129.3** (1997): 445–470. [doi:10.1007/s002220050168]

[4] C. Bleak, L. Elliott, and J. Hyde, Sufficient conditions for a group of homeomorphisms of the Cantor set to be two generated. In preparation.

[5] C. Bleak, M. Kassabov, and F. Matucci, Structure theorems for groups of homeomorphisms of the circle. *International Journal of Algebra and Computation* **21.06** (2011): 1007–1036. [doi:10.1142/S0218196711006571]

[6] M. Brin, The chameleon groups of Richard J. Thompson: Automorphisms and dynamics. *Publications Mathématiques de l’IHÉS* **84** (1996): 5–33. [doi:10.1007/BF02698834]

[7] M. Brin, The algebra of strand splitting. I. A braided version of Thompson’s group V. *Journal of Group Theory* **10.6** (2007): 757–788. [doi:10.1515/JGT.2007.055]

[8] M. Brin, The algebra of strand splitting. II. A presentation for the braid group on one strand. *International Journal of Algebra and Computation* **16.1** (2006): 203–219. [doi:10.1142/S021819670600286X]

[9] K. Brown, Presentations for groups acting on simply-connected complexes. *Journal of Pure and Applied Algebra* **32.1** (1984): 1–10. [doi:10.1016/0022-4049(84)90009-4]

[10] M. Brin and C. Squier, Groups of piecewise linear homeomorphisms of the real line. *Inventiones mathematicae* **79.3** (1985), 485–498. [doi:10.1007/BF01388519]

[11] K. Brown, Finiteness properties of groups. *Journal of Pure and Applied Algebra* **44.1–3** (1987): 45–73. [doi:10.1016/0022-4049(87)90015-6]

[12] K. Brown and R. Geoghegan, An infinite-dimensional torsion-free $F_\infty$ group. *Inventiones mathematicae* **77.2** (1984): 367–381. [doi:10.1007/BF01388451]

[13] J. Burillo and S. Cleary, The automorphism group of Thompson’s group $F$: subgroups and metric properties. *Revista Matemática Iberoamericana* **29.3** (2013): 809–828. [doi:10.4171/RMI/741]

[14] J. Burillo, F. Matucci, E. Ventura, The conjugacy problem in extensions of Thompson’s group $F$. *Israel J. Math.* **216** (2016), no. 1, 15–59. [doi:10.1007/s11856-016-1403-9]

[15] K. Bux, M. Fluch, M. Marschler, M. Witzel, and M. Zaremsky, The braided Thompson’s groups are of type $F_\infty$. *Journal für die reine und angewandte Mathematik (Crelles Journal)* **718** (2016): 59–101. [doi:10.1515/crelle-2014-0030]

[16] J. Cannon, W. Floyd, and W. Parry, Introductory notes on Richard Thompson’s groups. *Enseignement Mathématique* **42** (1996): 215–256. [doi:10.5169/seals-87877]

[17] P. Dehornoy, The group of parenthesized braids. *Advances in Mathematics* **205.2** (2006): 354–409. [doi:10.1016/j.aim.2005.07.012]
18 JAMES BELK, JAMES HYDE, AND FRANCESCO MATUCCI

[18] L. Funar and C. Kapoudjian, The braided Ptolemy-Thompson group is finitely presented. Geom. Topol. 12.1 (2008), 475–530. doi:10.2140/gt.2008.12.475

[19] L. Funar, C. Kapoudjian, and V. Sergiescu, Asymptotically rigid mapping class groups and Thompson groups. Handbook of Teichmüller Theory, Volume III (2012): 595–664. doi:10.4171/103-1/11

[20] R. Geoghegan, Topological Methods in Group Theory. Graduate Texts in Mathematics, vol 243. Springer, New York, NY, 2007. doi:10.1007/978-0-387-74614-2

[21] É. Ghys and V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle. Commentarii Mathematici Helvetici 62.1 (1987): 185–239. doi:10.1007/BF02564445

[22] A.P. Goryushkin, Imbedding of countable groups in 2-generated simple groups. Mathematical Notes of the Academy of Sciences of the USSR 16.2 (1974): 725–727. doi:10.1007/BF01105577

[23] V. Guba and M. Sapir, Diagram Groups. Memoirs of the American Mathematical Society 130, no. 620, American Mathematical Society, 1997. doi:10.1090/memo/0620

[24] P. Hall, On the embedding of a group in a join of given groups. Journal of the Australian Mathematical Society 17.4 (1974): 434–495. doi:10.1017/S1446788700018073

[25] G. Higman, Subgroups of finitely presented groups. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 262.1311 (1961): 455–475. doi:10.1098/rspa.1961.0132

[26] G. Higman, Finitely presented infinite simple groups. Notes on Pure Mathematics 8 (1974), Australian National University, Canberra.

[27] D. Holt and B. Eick and E. A. O’Brien, Handbook of computational group theory. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005. xvi+514 pp

[28] D. L. Johnson, Embedding some recursively presented groups. Groups St. Andrews 1997 in Bath, II, 410–416, London Math. Soc. Lecture Note Ser., 261, Cambridge Univ. Press, Cambridge, 1999.

[29] P. Lochak and L. Schneps, The universal Ptolemy-Teichmüller groupoid, Geometric Galois actions, 2, 325–347, London Math. Soc. Lecture Note Ser., 243, Cambridge Univ. Press, Cambridge, 1997.

[30] F. Matucci, Algorithms and classification in groups of piecewise-linear homeomorphisms, Ph.D. Thesis, Cornell University, 2008. arXiv:0807.2871

[31] V. Mazurov and E. Khukhro, Unsolved problems in group theory: The Kourovka notebook, No. 14. Sobolev Institute of Mathematics, 1999. arXiv:1401.0300

[32] K. A. Mihailova, The occurrence problem for free products of groups, Mat. Sb. (N.S.) 75(117) 1968 199–210. http://mi.mathnet.ru/eng/msb3978

[33] V. Mikaelian, On a problem on explicit embeddings of the group $\mathbb{Q}$. International Journal of Mathematics and Mathematical Sciences 2005.13 (2005): 2119–2123. doi:10.1155/IJMMS.2005.2119

[34] J. Milnor, Construction of universal bundles, II. Annals of Mathematics (1956): 430–436. doi:10.2307/1970012

[35] V. Nekrashevych, Cuntz–Pimsner algebras of group actions. Journal of Operator Theory (2004): 223–249. https://www.jstor.org/stable/24715627

[36] M. Rubin, Locally moving groups and reconstruction problems. In: Holland W.C. (eds) Ordered Groups and Infinite Permutation Groups. Mathematics and Its Applications, vol. 354. Springer, Boston, MA, 1996, 121–157. doi:10.1007/978-1-4613-3433-9_5

[37] O. Salazar-Díaz, Thompson’s group $V$ from a dynamical viewpoint. International Journal of Algebra and Computation 20.01 (2010): 39–70. doi:10.1142/S0218196710005534

[38] P. Schupp, Embeddings into simple groups. Journal of the London Mathematical Society 2.1 (1976): 90–94. doi:10.1112/jlms/s2-13.1.90
[41] R. Skipper and M. Zaremsky, Almost-automorphisms of trees, cloning systems and finiteness properties. *Journal of Topology and Analysis* (2019): 1–46. doi:10.1142/S1793525320500077

School of Mathematics & Statistics, University of St Andrews, St Andrews, KY16 9SS, Scotland.

E-mail address: [jmb42@st-andrews.ac.uk](mailto:jmb42@st-andrews.ac.uk)

Department of Mathematics, Cornell University, Ithaca, New York 14853.

E-mail address: [jth263@cornell.edu](mailto:jth263@cornell.edu)

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Milan 20125, Italy.

E-mail address: [francesco.matucci@unimib.it](mailto:francesco.matucci@unimib.it)