The variational complex of a diffeomorphisms group

Marius Buliga

IMB
Bâtiment MA
École Polytechnique Fédérale de Lausanne
CH 1015 Lausanne, Switzerland
Marius.Buliga@epfl.ch

and

Institute of Mathematics, Romanian Academy
P.O. BOX 1-764, RO 70700
București, Romania
Marius.Buliga@imar.ro

This version: 13.01.2000

Abstract

In this paper we propose a variational complex associated to a diffeomorphisms group with first order jet in a Lie group $M$. We study the structure of null lagrangians and we prove some fundamental properties of them, as well as their connection to differential invariants of the group action.

1 Introduction

A variational complex models the Lagrange formalism associated to a given structure. To my knowledge, the Lagrangian formalism has been developed on algebras, which are linear structures. In this paper I construct a variational complex associated to a group of diffeomorphisms. The central notion in this approach is the "null lagrangian" which is basically an integral invariant of the left action of the group on itself.

The classical notion of "null lagrangian" means simply a lagrangian $W = W(x, u, \nabla u)$ with the Euler-Lagrange equation

$$\frac{d}{dy_j} w(u, \nabla u) - \frac{d}{dx_i} \frac{d}{dF_{ij}} w(u, \nabla u) = 0$$

1
satisfied for any \( u \). This is equivalent to the identity (\( \Omega \subset \mathbb{R}^n \) is open, bounded and with smooth boundary):

\[
\int_{\Omega} W(x, u + \phi, \nabla(u + \phi)) \, dx = \int_{\Omega} W(x, u, \nabla u) \, dx
\]

for any \( u \) and for any \( \phi \in C_0^\infty(\Omega) \).

Any homogeneous null lagrangian \( W = W(\nabla u) \) can be written as a linear combination of subdeterminants of \( \nabla u \), cf. Ericksen \[6\], Ball, Currie & Olver \[4\] or Olver \[8\]. This particular structure of (classical) homogeneous null lagrangians leads to the formalization of the calculus of variations in the language of jets. Amongst the contributing papers we cite Tulczyjew \[10\], Anderson & Duchamp \[2\], Olver & Sivaloganathan \[9\]. More recently, the notion of variational bi-complex has been extended to arbitrary graded algebras (see, for example Verbovetsky \[11\] and the references therein) and the theory has found applications in several domains, connected to the existence of variational principles associated to various problems.

In this paper a (homogeneous) \( M \) null lagrangian, where \( M \) is a subgroup of \( GL_n(\mathbb{R}) \), is a function \( W \) with the property:

\[
\int_{\Omega} W(\nabla(u, \phi)) \, dx = \int_{\Omega} W(\nabla u) \, dx
\]

for any \( u, \phi \), diffeomorphisms of \( \Omega \) with compact support in \( \Omega \), such that for any \( x \in \Omega \) we have \( \nabla u(x), \nabla \phi(x) \in M \). By the way, the dot "." means function composition.

It is questionable which is the structure of \( M \) homogeneous null lagrangians and a construction of a variational complex should be done without previous knowledge of the form of null lagrangians.

In this paper we propose a variational complex associated to a diffeomorphisms group with first order jet in a Lie group \( M \). We study the structure of null lagrangians and we prove some fundamental properties of them.

2 Preliminaries

2.1 Notations

\( GL_n(\mathbb{R}) \subset \mathbb{R}^{n \times n} \) is the multiplicative group of all invertible, orientation preserving, matrices, i.e the set of all \( F \) such that \( \det F > 0 \).

2.2 Basic definitions and properties

Definition 2.1 For any Lie subgroup \( M \leq GL_n \) we define the associated local group

\[
[M] = \{ \phi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \mid \forall x \in \mathbb{R}^n \ \nabla \phi(x) \in M \} \tag{2.2.1}
\]

and it’s subgroup of compactly supported diffeomorphisms

\[
[M]_c = \{ \phi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \mid \forall x \in \mathbb{R}^n \nabla \phi(x) \in M , \ supp (\phi - id) \subset \subset \mathbb{R}^n \} \tag{2.2.2}
\]
\( \mathcal{A} \) is the group of affine homothety-translations. Any element of \( \mathcal{A} \) has the form:
\[
\alpha(x_0, y_0, \epsilon)(x) = f(x) = x_1 + \epsilon(x - x_0), \quad x_0, x_1 \in \mathbb{R}^n, \quad \epsilon > 0.
\]
We consider on \( \mathcal{A} \) the punctual convergence of functions defined on \( \mathbb{R}^n \) with values in \( \mathbb{R}^n \).

\([M]_c \) is a set of functions from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), which satisfies the following axioms:

A1/ \( ([M]_c, .) \) is a group with the function composition operation ";.;"

A2/ the following action is well defined:
\[
A : \mathcal{A} \times [M]_c \to [M]_c, \quad A(f, \phi) = f \cdot \phi \cdot f^{-1}.
\]

**Remark 2.2** These axioms are respected also by \([M] \).

**Definition 2.3** For any open set \( E \subset \mathbb{R}^n \) we define
\[
[M](E) = \{ \phi \in [M] : \text{supp} (\phi - \text{id}) \subset \subset E \}.
\]
For any \( x_0 \in \mathbb{R}^n \) the first order jet of \( M \) in \( x_0 \) is:
\[
J^1(x_0, M) = \{ \nabla \phi(x_0) : \phi \in [M] \}.
\]
The first order jet of \( M \) compactly supported diffeomorphisms is:
\[
J^1_c(x_0, M) = \{ \nabla \phi(x_0) : \phi \in [M]_c \}.
\]

**Proposition 2.4** If \([M]_c \) acts transitively on \( \mathbb{R}^n \) then there is a sub semigroup \( J^1_c(M) \) of the multiplicative group \( GL_n(R) \) such that for any \( x_0 \in \mathbb{R}^n \) we have
\[
J^1_c(x_0, M) = J^1_c(M).
\]

**Proof.** We first prove that \( J^1_c(x_0, M) \) is semigroup. We have \( \text{id} \in [M]_c \), hence \( I \), the identity matrix, belongs to \( J^1_c(x_0, M) \). Let us consider \( R, S \in J^1_c(x_0, M) \) and \( \phi, \psi \in [M]_c \) such that \( R = \nabla \phi(x_0), \ S = \nabla \psi(x_0) \). We define the translation \( f \in \mathcal{A} : f(x) = x + \psi(x_0) - x_0 \). From A2/ we have \( \tilde{\phi} = f \cdot \phi \cdot f^{-1} \in [M]_c \), hence from \( \nabla \tilde{\phi}(\psi(x_0)) = \nabla \phi(x_0) \) and A2/ we infer that
\[
RS = \nabla \phi(x_0) \nabla \psi(x_0) = \nabla \tilde{\phi}(\psi(x_0)) \nabla \psi(x_0) = \nabla (\tilde{\phi} \cdot \psi)(x_0) \in J^1_c(x_0, M).
\]
A simple argument based on A2/ shows that \( J^1_c(x_0, M) \) does not depend on \( x_0 \in \mathbb{R}^n \).

For a fixed, arbitrarily chosen, \( x_0 \) we define \( J^1_c(G) = J^1_c(x_0, M) \).

The proof of the fact that if \( F \in J^1_c(G) \) then \( F^{-1} \) exists and \( F^{-1} \in J^1_c(G) \) is similar.

The last equality is trivial because we have \( J(M) \subset M \) and for any \( F \in M \) we have \( F \in [M] \) (as a linear function), therefore \( F \in J(G) \). [Q.E.D.]

**Remark 2.5** We notice that the groups \([M] \) are determined by \( J(M) \), that is: if \( J(M_1) = J(M_2) \) then \([M_1] = [M_2] \). This property justifies the name "local group" for a diffeomorphisms group \([M] \).

**Proposition 2.6** \( J^1_c(M) \) is a normal Lie subgroup of \( M \).
Proof. The fact that \(J_c(M)\) is a Lie group follows from the continuity properties of the composition between smooth diffeomorphisms.

Consider now \(F \in M\). Then, for any \(\phi \in [M]_c\) we have \(F.\phi.F^{-1} \in [M]_c\). We deduce that \(J_c(M)\) is a normal subgroup of \(M\). ■

Example 2.1 We obviously have \([GL_n]_c = Diff^\infty_0\), and (not so obvious) \(J_c(GL_n) = GL_n\). Also, for any \(\Omega \subset R^n\) with lipschitz boundary the group \([GL_n](\Omega)\) acts transitively on \(\Omega\).

Example 2.2 Let us consider \(Diff^\infty_0(dx)\), the subgroup of \(Diff^\infty_0\) containing all volume preserving smooth diffeomorphisms with compact support. We have \(Diff^\infty_0(dx) = [SL_n(R)]_c\) and \(J_c(SL_n) = SL_n\). For any \(\Omega \subset R^n\) with lipschitz boundary the group \([SL_n](\Omega)\) acts transitively on \(\Omega\).

Example 2.3 For any \(u : R^{2n} \rightarrow R^{2n}\) and \(\omega\), the canonical symplectic 2-form on \(R^{2n}\), we denote by \(u^*(\omega)\) the transport of \(\omega\). Let us define \(Diff^\infty_0(\omega)\):

\[
Diff^\infty_0(\omega) = \{ \phi \in Diff^\infty_0 : \phi^*(\omega) = \omega \}.
\]

We have the equalities:

\[
J(Diff^\infty_0(\omega)) = Sp_n(R) = \{ F \in \mathbb{R}^{2n \times 2n} : F\omega F^T = \omega \}
\]

and \(J_c(Sp_n) = Sp_n\). Also, as in the previous examples, \([Sp_n](\Omega)\) acts transitively on \(\Omega\).

Example 2.4 Let us take \(C_n = \{ \lambda R | \lambda > 0, RR^T = I_n \}\), the group of linear conformal matrices. Then \([C_n]_c\) is the group of conformal diffeomorphisms and \([C_n]_c = \{ id \}\), therefore \(J_c(C_n) = \{ I_n \}\).

For all the transitivity results needed in these examples we refer to Michor & Vizman \[7\].

In the following lemmas we collect some elementary facts connected to the algebraic structure previously introduced.

Lemma 2.7 Let \(A, B\) be non empty open subsets of \(R^n\). If \(A\) is bounded then there exists \(f \in A\) such that the application \(A(f, \cdot) : [M](A) \rightarrow [M](B)\) is well defined, injective and continuous.

Lemma 2.8 If \(A\) and \(B\) are two open disjoint sets then for any \(\phi \in [M](A), \psi \in [M](B)\) we have \(\phi.\psi = \psi.\phi \in [M](A \cup B)\).

Lemma 2.9 For any group \([M]_c\) we have \(F.\phi.F^{-1} \in [M]_c\) if \(F \in M\).
3 Null lagrangians and group invariants

We shall denote by $T[M](\Omega)$ the space of all vector fields $\eta$ over $\Omega$ with the associated one-parameter flow in $[M](\Omega)$.

We associate to any $W \in \Lambda^s(M)$ the integral $I_W$ defined over $[M](\Omega) \times (T[M](\Omega))^s$ by the formula:

$$I_W(\phi, \eta) = \int_{\Omega} W(\phi, \nabla \phi)(\nabla \eta_1, \nabla \eta_2, \ldots, \nabla \eta_s) \, dx$$

$\Lambda(M)$ is the graded algebra of differential forms over $J_c(M)$. The Lie algebra associated to $J_c(M)$ will be denoted by $j_c(M)$. The space of constant differential forms of order $s$ over $j_c(M)$ is $\lambda^s(M)$. Then $\Lambda^s(M) = C^\infty(R^n \times j_c(M), \lambda^s(M))$

**Definition 3.1** A differential form $\alpha \in \Lambda^r(R^n)$ is a M differential invariant if for any $\phi \in \Lambda_c(M)$ the form $\phi^*\alpha - \alpha$ is exact.

One can extend this definition to "G differential invariant" simply by replacing ",$\phi \in \Lambda_c(M)\) with ",$\phi \in G\), where $G$ is a diffeomorphisms group.

**Definition 3.2** A function $W \in \Lambda^0(M)$ is a M null lagrangian if for any $f \in A$, with the notation $(W, f)(x, F) = W(f(x), F)$, we have: $I_{(W, f)}(\cdot; \Omega)$ is constant over $[M](\Omega)$.

**Proposition 3.3** The definition does not depend on the choice of $\Omega$ in the class of smooth open bounded sets.

**Proof.** Let $\Omega'$ smooth, bounded and $\phi' \in \Lambda^i(M)$. There is an element $f \in A$, $f(x) = a + \varepsilon x$, such that $f(\Omega') \subset \Omega$. Then we have $f, \phi', f^{-1} \in \Lambda^i(M)$. $W$ verifies definition therefore:

$$\int_{\Omega} W(x, I) \, dx = \int_{\Omega} W(f(\phi'(f^{-1}(x))), \nabla \phi'(f^{-1}(x))) \, dx$$

We change variables $y = f^{-1}(x)$ and we obtain:

$$\int_{f^{-1}(\Omega)} W(f(x), I) \, dy = \int_{f^{-1}(\Omega)} W(f(\phi'(y)), \nabla \phi'(y)) \, dy$$

Because $\phi'$ has compact support in $\Omega'$, we have:

$$\int_{\Omega'} W(f(y), I) \, dy = \int_{\Omega'} W(f(\phi'(y)), \nabla \phi'(y)) \, dy$$

This resumes the proof. ■

5
The class of null lagrangians which generate null integrals is denoted by

\[ NL^0(M) = \{ W \in \Lambda^0(M) : I_W = 0 \} \]

We have fixed the constant value of \( I_W \) to be 0 for further technical reasons. However, this not causes problems, because we may think instead that a factorization by \( R \) has been performed.

The class of homogeneous null lagrangians is made by all \( W = W(\nabla u) \) which are null lagrangians. This class is denoted by \( nl^0(M) \). No factorization by \( R \) has been done in this case.

To any differential form \( \alpha \in \Lambda^r(R^n) \) and any \((v_1,\ldots,v_r)\in R^{nr}\) we associate a potential \( \alpha^*(v_1,\ldots,v_r) \in \Lambda^0(GL_n) \) in the following way:

\[
\alpha^*(y,F)(v_1,\ldots,v_r) = \alpha(y(Fv_1,\ldots,Fv_r))
\]

In order to shorten the notation we shall denote by \( I_\alpha(u;\Omega) \) the mapping defined over \( X^r_1(M) \) with values in \( R^{nr} \), given by:

\[
I_\alpha(u;\Omega)(v_1,\ldots,v_r) = \int_\Omega \alpha(u(x),\nabla u(x))(v_1,\ldots,v_r) \, dx
\]

**Proposition 3.4** If \( \alpha \) is a \( M \) (or \([M](\Omega)\)) differential invariant then for any \((v_1,\ldots,v_r)\) the mapping \((y,F)\mapsto \alpha^*(y,F)(v_1,\ldots,v_r)\) is a null lagrangian (shortly: \( \alpha^* \) is a vectorial null lagrangian).

**Proof.** We remark that generally, if \( \alpha \) is exact and with compact support in \( \Omega \), then \( \alpha = d\beta \), with \( \beta \) with compact support in \( \Omega \). For any \( \phi \in [M](\Omega) \) the differential form \( \phi^*\alpha - \alpha \) has compact support in \( \Omega \).

We apply the definition of \( \alpha^* \), definition 3.1 and integration by parts (Gauss formula) for the integral \( I_\alpha(u;\Omega)(v_1,\ldots,v_r) \) and we conclude the proof. ■

The following theorem will be essential in further proofs.

**Theorem 3.5** If \( W : \Omega \times \Omega \times J_c(M) \to R \) is continuous and

\[
I_W(u) = \int_\Omega W(x,u(x),\nabla u(x)) \, dx
\]

is constant over \([M](\Omega)\) then for any \( x_1 \in \Omega \) and any \( \psi \in [M](\Omega) \) the mapping \( F \in J_c(M) \mapsto W(x_1,\psi(x_1),\nabla \psi(x_1)F) \) is a homogeneous null lagrangian.

**Proof.** Let us consider \( x_1 \in \Omega \) and \( h > 0 \). \( Q_h \) is the cube \( x_1^i < x^i < x^i + 1/h \). We take \( \phi \in [M](Q_1) \) and \( k \in N \). The extension of \( \phi \) by periodicity over \( \mathbb{R}^n \) is denoted by \( \tilde{\phi} \). We define then:

\[
\phi_{h,k}(x) = \begin{cases} (hk)^{-1} \left( \tilde{\phi}(hk(x - x_1) + x_1) - x_1 \right) + x_1 & \text{if } x \in Q_h \\ x & \text{otherwise} \end{cases}
\]
We have \( \phi_{h,k} \in [M](\Omega) \). Any set \( Q_h \) decomposes in \( k^n \) cubes which will be denoted by \( Q_{hk,j}, j = 1, \ldots, k^n \), such that \( Q_{hk,1} = Q_{hk} \). The corner of \( Q_{hk,j} \) with least distance from \( x_1 \) is denoted by \( x_j \).

Let us now consider \( \psi \in [M](\Omega) \). For a sufficiently large \( h \) we have \( Q_h \subset \Omega \), hence \( I(\psi, \phi_{h,k}; \Omega) \) makes sense. We decompose this integral in two parts:

\[
I(\psi, \phi_{h,k}; \Omega) = I(\psi, \phi_{h,k}; Q_h) + I(\psi; \Omega \setminus Q_h) ,
\]

\[
I(\psi, \phi_{h,k}; Q_h) = \sum_{j=1}^{k^n} \int_{Q_{hk,j}} [W(x, \psi, \phi_{h,k}(x), \nabla(\psi, \phi_{h,k})(x))
- W(x_j, \psi, \phi_{h,k}(x_j), \nabla(\psi, \phi_{h,k})(x_j)])
+ \sum_{j=1}^{k^n} \int_{Q_{hk,j}} W(x_j, \psi, \phi_{h,k}(x_j), \nabla(\psi, \phi_{h,k})(x_j)) \, dx .
\]

Notice that \( \phi_{h,k} \) converges weakly to \( \text{id} \). Because \( W \) and \( \nabla \psi \) are continuous and \( \phi_{h,k} \) converges uniformly to \( \text{id} \) with \( k \to \infty \), it follows that the first sum from the right-handed member of the equality (3.0.2) converges to zero.

By the change of variable \( y = hk(x - x_j) + x_1 \) we obtain:

\[
\int_{Q_{hk,j}} W(x_j, \psi, \phi_{h,k}(x_j), \nabla(\psi, \phi_{h,k})(x_j)) \, dx = (hk)^{-n} \int_{Q_1} W(x_j, \psi(x_j), \nabla(\psi)(y)) \, dy .
\]

We deduce from here that the second sum of the right-handed member (3.0.2) is a Cauchy sum. By a passage to the limit as \( k \to \infty \) we get the equality:

\[
\lim_{k \to \infty} I(\psi, \phi_{h,k}; Q_h) = \int_{Q_h} \int_{Q_1} W(x, \psi(x), \nabla(\psi)(y)) \, dy \, dx .
\]

From the hypothesis we have:

\[
\lim_{k \to \infty} I(\psi, \phi_{h,k}; \Omega) = \lim_{k \to \infty} I(\psi, \phi_{h,k}; Q_h) + I(\psi; \Omega \setminus Q_h)
= I(\psi; Q_h) + I(\psi; \Omega \setminus Q_h) ,
\]

therefore (3.0.4) implies that:

\[
\int_{Q_h} \int_{Q_1} W(x, \psi(x), \nabla(\psi)(y)) \, dy \, dx = \int_{Q_h} W(x, \psi(x), \nabla(\psi)(x)) \, dx .
\]

We multiply the relation (3.0.6) with \( h^n \) and pass to the limit as \( h \to \infty \). The result is:

\[
\int_{Q_1} W(x_1, \psi(x_1), \nabla(\psi)(x_1)) \, dy = W(x_1, \psi(x_1), \nabla(\psi)(x_1))
\]

which concludes the proof. ■
4 The variational complex of the diffeomorphism group

We introduce two graded vector spaces:

\[ K^0(M) = NL^0(M), \quad K^p(M) = \{ W \in \Lambda^p(M) : \forall H \in j_c(M), W(\cdot)(H, ...) \in K^{p-1}(M) \} \]

\[ NL^p(M) = \{ W \in K^p(M) : I_W = 0 \} \]

The graded derivation operator \( D \) on \( \Lambda(M) \) is defined further. It is sufficient to define \( D^0, D^1 \).

\( D^0 : \Lambda^0(M) \to \Lambda^1(M) \) is defined by the formula:

\[ Dw(y, F)H = w(y, F) \text{tr}H - \langle \frac{\partial w}{\partial F}(y, F), FH \rangle \]

\( D^1 : \Lambda^1(M) \to \Lambda^2(M) \) is defined by:

\[ Dw(y, F)(H, P) = D(w_{ij}H_{ij})(y, F)P - D(w_{ij}P_{ij})(y, F)H + w(F)[P, H] \]

We have then the obvious proposition:

**Proposition 4.1** The sequence

\[ \Lambda^0(M) \xrightarrow{D^0} \Lambda^1(M) \xrightarrow{D^1} \Lambda^2(M) \]

is semi-exact, that is \( D^2 = 0 \).

**Proof.** We leave to the reader to verify, by direct calculus. ■

Remark that if \( j(M) \subset sl_n \) (the algebra of null trace matrices), then the complex defined above is simply the de Rham complex.

More interesting is the following:

**Proposition 4.2** The sequences

\[ NL^{p-1}(M) \xrightarrow{D} NL^p(M) \xrightarrow{D} NL^{p+1}(M) \]

\[ K^{p-1}(M) \xrightarrow{D} K^p(M) \xrightarrow{D} K^{p+1}(M) \]

are well defined and, of course, semi-exact.

**Proof.** We shall prove only that

\[ K^0(M) \xrightarrow{D} K^1(M) \]

is well defined. The rest follows.

Let \( u \in [M](\Omega) \) and \( \eta \in T[M](\Omega) \). The flow on \( \Omega \) generated by \( \eta \) is \( t \mapsto \phi_t \in [M](\Omega) \).
We consider the function:

\[ I(u, t) = \int_{\Omega} w(u_\phi^{-1}, \nabla(u_\phi^{-1})) \, dx \]

If \( w \in NL^0(M) \) then \( I(u, t) \) is constant with respect to \( t \), therefore the derivative of \( I(u, t) \) relative to \( t \) at \( t = 0 \) is null, for any \( u \). We have the following representation of this derivative:

\[ \frac{\partial I}{\partial t}(u, 0) = \int_{\Omega} \hat{w}(x, u(x), \nabla u(x)) \, dx \]

where the potential \( \hat{w} \) is given by the expression:

\[ \hat{w}(x, y, F) = w(y, F) \text{div}\eta(x) - \left( \frac{\partial w}{\partial F}(y, F), F\nabla\eta(x) \right) \]

We deduce that \( \hat{w} \) is a non-homogeneous null lagrangian. From theorem 3.5 we deduce that \( Dw \), previously defined, indeed belongs to \( K^1(G) \).

**Definition 4.3** The complex \( NL(M) \) introduced in proposition 4.2 is called the complex of \( [M] \) null lagrangians. Analogously, the complex of homogeneous null lagrangians is \( nl(M) \).

In the context of this paper we introduce the following definition of the Euler-Lagrange operator:

**Definition 4.4** The Euler-Lagrange operator associated to \( w \in \Lambda^0(M) \) is the function:

\[ Ew : [M](\Omega) \rightarrow T[M](\Omega)^* \]

defined by

\[ Ew(u)\eta = \int_{\Omega} \left( \frac{d}{dy_j} w(u, \nabla u) - \frac{d}{dx_i} \frac{d}{dF_{ij}} w(u, \nabla u) \right) \nabla u_{jk} \eta_k \, dx \quad (4.0.1) \]

From the proof of the previous proposition we extract the formula:

\[ \frac{d}{dt} \int_{\Omega} w(u_\phi^{-1}, \nabla(u_\phi^{-1})) \, dx = \int_{\Omega} Dw(u, \nabla u)\nabla \eta \, dx \quad (4.0.2) \]

We integrate by parts, taking into account that \( \text{supp } \eta \subset \subset \Omega \), and we obtain:

\[ \frac{d}{dt} \int_{\Omega} w(u_\phi^{-1}, \nabla(u_\phi^{-1})) \, dx = \int_{\Omega} \mathcal{E}w(u, \nabla u)\nabla u_{jk} \eta_k \, dx \quad (4.0.3) \]

where \( \mathcal{E}w \) is the classical Euler-Lagrange operator defined by:

\[ \mathcal{E}_j w(u, \nabla u) = \frac{d}{dy_j} w(u, \nabla u) - \frac{d}{dx_i} \frac{d}{dF_{ij}} w(u, \nabla u) \quad (4.0.4) \]

We construct next the variational complex associated to \([M]\). The following proposition is straightforward, due to proposition 4.2.
Proposition 4.5 The following sequence is semi-exact:

\[
\Lambda^p(M) \xrightarrow{D} \Lambda^{p+1}(M) \quad , \quad D(W + NL^p(M)) = DW + NL^{p+1}(M) \quad (4.0.5)
\]

Definition 4.6 With the notation

\[
V^p(M) = \frac{\Lambda^p(M)}{NL^p(M)}
\]

the complex \( V(M) \) is called the variational complex associated to \( M \). For any \( w \in \Lambda^p(M) \), we shall denote by \( NLw \) the equivalence class:

\[
NLw = w + NL^p(M) \in V^p(M)
\]

The name "variational complex" given to \( V(M) \) is justified by the following proposition, which tells that the Euler-Lagrange operator \( Ew \) can be identified with \( DNLw \), that is the derivation \( D : V^0 \rightarrow V^1 \) "is" the Euler-Lagrange operator:

Proposition 4.7 For any \( w \in \Lambda^0(M), Ew = 0 \in T[M](\Omega)^* \) if and only if \( DNLw = 0 \in V^1(M) \).

Proof. Recall the formula (4.0.2) (using also definition 4.4):

\[
\int_{\Omega} Dw(u, \nabla u) \nabla \eta \, dx = Ew(u)\eta \quad (4.0.6)
\]

If \( Ew = 0 \in T[M](\Omega)^* \) then \( w \) is constant along any one parameter flow in \( [M](\Omega) \), hence \( w \in NL^0 \), which is the same as \( DNLw = 0 \in V^1(M) \). Conversely, suppose that \( w \in NL^0 \). Then we have immediately \( Ew(u)\eta = 0 \), for any \( u \in [M](\Omega) \) and \( \eta \in T[M](\Omega) \). □

Let us introduce another complex, made by variational integrals:

Definition 4.8 The graded vector space of variational integrals \( VI(M) \) is:

\[
VI^p(M) = \{I_W : W \in \Lambda^p(M)\} \quad (4.0.7)
\]

We consider the derivation operator \( DI_W = IDW \).

We have therefore, the following easy consequence of proposition 4.5:

Proposition 4.9 The sequence

\[
VI^{p-1}(M) \xrightarrow{D} VI^p(M) \xrightarrow{D} VI^{p+1}(M)
\]

is semi-exact. Moreover, the mapping

\[
NL \, w \in V^p(M) \mapsto I_W \in VI^p(M)
\]

is an isomorphism from the variational complex \( V(M) \) to the variational integrals complex \( VI(M) \).
Proof. Recall that we have demanded that \( w \in NL^p(M) \) if and only if \( I_w = 0 \). Therefore we have \( w - w' \in NL^p(M) \) if and only if \( I_w = I_{w'} \).

The second part of the proposition is straightforward from definition 4.8 and proposition 4.2. ■

As a conclusion, we get a more precise image of the variational complex if we look at the complex of variational integrals \( VI(M) \). It is useful to consider also the relation, coming from (4.0.2),

\[
DI_w(u, \eta) = \frac{d}{dt} I_w(u, \phi_t) |_{t=0}
\]

where \( w \in \Lambda^0(M) \) and \( \phi_t \) is the one-parameter flow generated by \( \eta \).

5 Examples

It is visible now that a central object in the construction presented in this paper is the space of homogeneous null lagrangians \( nL^0(M) \). We shall look to this space in the followings.

We derive first a necessary condition for \( W \) to be a homogeneous \( M \) null lagrangian.

**Theorem 5.1** Let \( W = W(F) \) be a homogeneous \( M \) null lagrangian. Then for any \( F \in J_c(G) \) and for any \( \eta \in T[M](\Omega) \) we have the inequality:

\[
\int_{\Omega} D(DW(F)\nabla\eta(x)) \nabla\eta(x) \, dx = 0 .
\]  
(5.0.1)

**Proof.** Consider the function

\[
I(t) = \int_{\Omega} W(F\nabla\phi_{-t}(x)) \, dx .
\]

This is a \( C^2 \) function which is constant, according to hypothesis upon \( W \) and theorem 3.5. This fact implies that

\[
\frac{\partial I}{\partial t}(0) = 0 , \quad \frac{\partial^2 I}{\partial t^2}(0) = 0 .
\]

The first relation is trivially satisfied.

We apply twice an integration by parts argument to the second equality in order to obtain the desired inequality. ■

In order to see what (5.0.1) means let us take \( M = GL_n \). In this case we have

\[
T[M](\Omega) = C_0^\infty(\Omega, \mathbb{R}^n)
\]
hence for any $\eta \in T[M](\Omega)$ and $F \in J_c(M) = GL_n(R)$, the vector field $F\eta$ belongs to $T[M](\Omega)$. Therefore the relation (5.0.1) can be written as:

$$\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F) \int_{\Omega} \eta_i \eta_j \eta_k \eta_l \, dx = 0 \quad (5.0.2)$$

for any $\eta \in C_0^\infty$. An argument from Ball [3], proof of Theorem 3.4, allow us to consider piecewise affine vector fields $\eta$. It can be shown that (5.0.2) implies the Legendre-Hadamard equality:

$$\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F) a_i a_k b_j b_l = 0 \quad (5.0.3)$$

for any vectors $a, b \in \mathbb{R}^n$.

Therefore any $W \in nl^0(GL_n)$ is a classical null lagrangian.

This means that $W$ can be extended over $R^{n \times n}$ such that for any $F \in R^{n \times n}$ and for any $\eta \in C_0^\infty(\Omega, R^n)$ we have the inequality:

$$\int_{\Omega} W(F + \nabla \eta) \, dx = \int_{\Omega} W(F) \, dx$$

The class of classical null lagrangians is known (see Ball, Currie & Olver [4] or Olver [3]); any homogeneous null lagrangian $W$ is a linear combination of minors of $F$. From the definition of $NL^0(GL_n)$ we see that any $GL_n$ null lagrangian is a classical null lagrangian. The particular structure of null lagrangians leads to the introduction of a variational bi-complex in the language of jets.

For the choice $M = SL_n$, the relation (5.0.1) becomes

$$D(DW(F)a \otimes b)a \otimes b = 0 \quad (5.0.4)$$

for any orthogonal $a, b \in R^n$ (that is $a \cdot b = 0$).

**Proposition 5.2** If $W \in nl^0(SL_n)$ then for any $F \in SL_n$, for any $a, b \in R^n$, $a \cdot b = 0$ and for any $t \in R$ we have $F(I_n + ta \otimes b) \in SL_n$ and the function

$$t \mapsto f(F, a \otimes b, t) = W(F(I_n + ta \otimes b))$$

is linear.

**Proof.** Let us denote, for $H \in j_c(M)$, by $\exp_t H$ the solution of the problem $\dot{H}_t = H F_t$, $F_0 = H$. It is straightforward that if $HH = 0_n$ then $\exp_t H = I_n + tH$. For any $a, b \in R^n$, $a \cdot b = 0$, we take $H = a \otimes b$ and obtain the first part of the proposition.

In order to resume the proof, because of theorem [5.5] it is sufficient to prove that the second derivative of $f(F, a \otimes b, t)$ with respect to $t$ equals 0 for $t = 0$. We use again that if $H = a \otimes b$ and $a \cdot b = 0$ then $HH = 0_n$, and we are led to the equality:

$$\frac{d^2}{dt^2} f(F, a \otimes b, t)_{t=0} = D(DW(F)a \otimes b)a \otimes b$$

12
This resumes the proof, because of the hypothesis [5.0.4]. ■

In the case $M = SL_2$ we proved the following theorem.

**Theorem 5.3** If $W \in n(0)(SL_2)$ then $W(F) = a_{ij}F_{ij} + b$.

**Proof.** We shall use local coordinates of $SL_2$ and apply proposition [5.2]. It is sufficient to consider the coordinates:

\[
F = \begin{pmatrix} X & Y \\ Z & \frac{1+YZ}{X} \end{pmatrix}, \quad F' = \begin{pmatrix} \frac{1+Y'Z'}{X'} & Y' \\ Z' & X' \end{pmatrix}
\]

Take $a = (a_1, a_2)$ and $b = (-a_2, a_1)$. Then the function $f(F, a \otimes b, t)$, expressed in the coordinates $(X, Y, Z)$ or $(X', Y', Z')$, is linear in $t$, as shown in proposition [5.2].

We derive twice with respect to time the function $f(F, a \otimes b, t)$ at $t = 0$ and we equal the result to 0. After some elementary computation we obtain the following minimal system of equations for $g(X, Y, Z) = f(F(X, Y, Z)):

\[
\begin{align*}
g_{,xx}X^2 &= 2g_{,yz}(1 + YZ) \\
g_{,zz}X &= -g_{,yz}Y \\
g_{,xy}X &= -g_{,yz}Z \\
g_{,yy} &= 0 \\
g_{,zz} &= 0
\end{align*}
\]

From [5.0.8], [5.0.9] we find that:

\[
g(X, Y, Z) = A(X)YZ + B(X)Y + C(X)Z + D(X)
\]

We put the expression of $g$ in [5.0.6] and we obtain the equation:

\[
XC'(X) + XYA'(X) = -A(X)Y
\]

From here we derive that $C(X) = c$ and $A(X) = k/X$. We introduce in [5.0.10] what we have found and use this in [5.0.7]. We find that $B(X) = b$. Finally, we update the form of $g$ and use it in [5.0.5]. It follows that $D''(X) = 2k/X^3$ therefore $D(X) = (k/X) + eX + f$. We collect all the information gained and we arrive at the following expression of $g$:

\[
g(X, Y, Z) = k\frac{1+YZ}{X} + bY + cZ + eX + f
\]

which proves the theorem. ■

A straightforward consequence of the previous theorem is the following one:
Theorem 5.4 Let $W \in C^2(SL_2, R)$ and $\Omega \subset R^2$, bounded, open, with smooth boundary. If for any volume preserving diffeomorphism $\phi : \Omega \to \Omega$, with compact support we have

$$\int_{\Omega} W(\nabla \phi) \, dx = ct.$$ 

then $W(F) = a_{ij} F_{ij} + b$.

Moreover, let $W \in C^2(GL_2, R)$. If for any diffeomorphism $u : \Omega \to \Omega$ with compact support and for any volume preserving diffeomorphism $\phi : \Omega \to \Omega$, with compact support we have

$$\int_{\Omega} W(\nabla (u \cdot \phi)) \, dx = \int_{\Omega} W(\nabla u) \, dx$$

then $W(F) = a_{ij}(\det F) F_{ij} + b(\det F)$.

These results suggest the following conjecture:

Conjecture 5.5 For any $M$ and any $W \in nl^0(M)$ there is a classical null lagrangian $\tilde{W}$ such that $W(F) = \tilde{W}(F)$ for any $F \in J_c(M)$.

Even if homogeneous $M$ null lagrangians are classical null lagrangians, the set $NL^1(M)$ can depend non-trivially on $M$. Indeed, consider the group of symplectic matrices $M = Sp_m$, where $n = 2m$. Then $[M](\Omega)$ is simply the group of all symplectomorphisms with compact support in $\Omega$. Take the one-form $\alpha(x, y) = y \, dx$. Then $\alpha$ is a $Sp_m$ differential invariant (the Calabi invariant), which give raise to a vectorial null lagrangian $\alpha^*$, according to proposition 3.1. We leave the reader to check that (any component of) $\alpha^*$ belongs to $NL^1(Sp_m)$, but not to $NL^1(GL_n)$.

6 Final remarks

It would be very interesting to construct independently the complex of variational integrals $VI(M)$. We think that this can be done in the language of currents, using a similar approach as Ambrosio & Kirchheim [1].

It is to be mentioned that behind the algebraic construction performed in this paper are hidden facts related to continuity of variational integrals defined over groups of diffeomorphisms, as in the case of classical null lagrangians is shown in Ball, Currie & Olver [4]. In the context of diffeomorphisms groups we cite the lecture paper Buliga [5]. A paper concerning necessary and sufficient conditions for lower semicontinuity of variational integrals on diffeomorphisms groups is in preparation.

Aknowledgements. Some results from this paper have been communicated in a talk given at S.I.S.S.A. Trieste, in September 1999. The author wishes to thank A. Braides and G. Dal Maso for the opportunity to give this talk and for interesting discussions during the visit.
References

[1] L. Ambrosio, B. Kirchheim, Currents in metric spaces, preprint (1999)

[2] I. M. Anderson, T. Duchamp, On the existence of global variational principles, Amer. J. of Math., 102, 5, (1980), 781—868

[3] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal., vol. 63, (1977), 337 — 403

[4] J.M. Ball, J.C. Currie, P.J. Olver, Null Lagrangians, weak continuity and variational problems of arbitrary order, J. Funct. Anal., 41, (1981), 135 — 174

[5] M. Buliga, Quasiconvexity versus group invariance, lecture held on Feb. 22 at the Mathematical Institute, Oxford, Applied Analysis and Mechanics Seminars, Hilary Term 1999

[6] L. Ericksen, Nilpotent energies in liquid crystal theory, Arch. Rational Mech. Anal., vol. 10, 3, (1962), 189 — 196

[7] P. W. Michor, C. Vizman, n-transitivity of certain diffeomorphism groups, Acta Math. Univ. Comenianae, 63, 2, (1994)

[8] P. J. Olver, Conservation laws and null divergences, Math. Proc. Camb. Phil. Soc. 94, (1983), 529—540

[9] P.J. Olver, J. Sivaloganathan, The structure of null Lagrangians, Nonlinearity, vol. 1, no. 2, (1988), 389—398

[10] W. M. Tuckey, The Lagrange complex, Bull. Soc. Math. France, 105, (1977), 419—431

[11] A. Verbovetsky, Lagrangian formalism over graded algebras, preprint SISSA 93/94/FM