THE CATEGORY OF SIMPLE GRAPHS IS COREFLECTIVE IN THE COMMA CATEGORY OF GROUPS UNDER THE FREE GROUP FUNCTOR

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Abstract. We show that the comma category \((F \downarrow \text{Grp})\) of groups under the free group functor \(F : \text{Set} \to \text{Grp}\) contains the category \(\text{Gph}\) of simple graphs as a full coreflective subcategory. More broadly, we generalize the embedding of topological spaces into Steven Vickers' category of topological systems to a simple technique for embedding certain categories into comma categories, then show as a straightforward application that simple graphs are coreflective in \((F \downarrow \text{Grp})\).

1. Introduction

In his 1989 text "Topology Via Logic" [8], Steven Vickers develops a category \(\text{TopSys}\) of topological systems which is noteworthy for containing both the category of topological spaces as a coreflective subcategory and the category of locales as a reflective subcategory. In [1], Adamek and Pedicchio observe that \(\text{TopSys}^{\text{op}}\) is equivalent to the comma category \((\text{Frm} \downarrow \text{CABA})\) of frames over complete atomic boolean algebras, and further show that \(\text{Top}^{\text{op}}\) is equivalent to the regularly epireflective subcategory of monomorphisms.

Rewritten, \(\text{Top}\) is the coreflective subcategory of epimorphisms in

\[(\text{Frm} \downarrow \text{CABA})^{\text{op}} = (\text{CABA}^{\text{op}} \downarrow \text{Frm}^{\text{op}}).\]

As the category of complete atomic boolean algebras is opposite the category of sets via taking the powerset, a topological system can be interpreted as a localic map out of a powerset, ie as an object in the comma category \((\text{P} \downarrow \text{Loc})\) of locales under the powerset locale functor \(\text{P} : \text{Set} \to \text{Loc}.

If 2 is the two-open locale, the hom functor \(\text{hom}_{\text{Loc}}(2, -)\) assigns to each locale its set of locale points; this functor is right adjoint to \(\text{P}\). By adjunction, the category of topological systems is also equivalent to \((\text{Set} \downarrow \text{hom}_{\text{Loc}}(2, -))\).

Note that \(\text{Top}\) fits right in the middle of the adjunction \(\text{P} \dashv \text{hom}_{\text{Loc}}(2, -)\). If \(\text{DT} : \text{Set} \to \text{Top}\) is the discrete space functor, \(\text{U} : \text{Top} \to \text{Set}\) the forgetful functor right adjoint to \(\text{DT}\), and \(\mathcal{L} \circ \mathcal{S}\) is the locale-spectrum adjunction between \(\text{Top}\) and \(\text{Loc}\) (see eg Chapter 2 of Picado and Pultr [6] for details), then \(\mathcal{P} = \mathcal{L} \circ \text{DT}\) and \(\text{hom}_{\text{Loc}}(2, -) \cong \text{U} \circ \mathcal{S}\).

In this language, \(\text{TopSys} \cong (\mathcal{L} \circ \text{DT} \downarrow \text{Loc}) \cong (\text{Set} \downarrow \text{U} \circ \mathcal{S})\). Adjuncting into the category \(\text{Top}\), we have \(\text{TopSys} \cong (\text{DT} \downarrow \mathcal{S})\). Here \(\text{Top}\) embeds naturally as a composition of unit with counit: the map \(\eta^{\mathcal{L} \circ \mathcal{S}} \circ \varepsilon^{\text{DT} \circ \text{U}} : \text{DT} \circ \text{U} \to \text{id} \to \mathcal{S} \circ \mathcal{L}\) gives for each space an object of the comma category \((\text{DT} \downarrow \mathcal{S})\).

In this paper we investigate conditions for which categories in the center of two composable adjoint pairs may be embedded nicely into the appropriate comma categories through this process.
2. The unit-counit composite

Let \( \mathcal{B}, \mathcal{C}, \mathcal{D} \) be complete and cocomplete categories. Let

\[
\begin{align*}
\mathcal{F} : \mathcal{B} & \rightleftarrows \mathcal{C} : \mathcal{G}, \\
\mathcal{L} : \mathcal{C} & \rightleftarrows \mathcal{D} : \mathcal{R},
\end{align*}
\]

with \( \mathcal{F} \dashv \mathcal{G} \) and \( \mathcal{L} \dashv \mathcal{R} \), and define \( \gamma : \mathcal{C} \to (\mathcal{F} \downarrow \mathcal{R}) \) by

\[
\gamma(A) := \eta_{\mathcal{L}} \circ \varepsilon_{\mathcal{F}} \circ \mathcal{G}(A) \circ \varepsilon_{\mathcal{G}},
\]

\[
\gamma(f) := (\mathcal{G}(f) \downarrow \mathcal{L}(f)).
\]

We ask what conditions will ensure that \( \gamma \) embeds \( \mathcal{C} \) as a reflective subcategory of the equivalent comma categories 

\[
(\mathcal{B} \downarrow (\mathcal{G} \circ \mathcal{R})) \cong (\mathcal{F} \downarrow \mathcal{R}) \cong ((\mathcal{L} \circ \mathcal{F}) \downarrow \mathcal{D}).
\]

We can say immediately that \( \gamma \) is faithful as soon as either \( \mathcal{G} \) is or \( \mathcal{L} \) is.

Fullness is more precarious. For \( A, B \) objects in \( \mathcal{C} \), \( f_B \in \text{hom}_\mathcal{B}(\mathcal{G}(A), \mathcal{G}(B)) \), \( f_D \in \text{hom}_\mathcal{D}(\mathcal{L}(A), \mathcal{L}(B)) \), assume that we have a diagram in \( \mathcal{C} \):

\[
\begin{array}{ccc}
(F \circ \mathcal{G})(A) & \xrightarrow{F(f_B)} & (F \circ \mathcal{G})(B) \\
\downarrow \varepsilon_{\mathcal{F}} & & \downarrow \varepsilon_{\mathcal{F}} \\
A & \xrightarrow{f_B} & B \\
\eta_{\mathcal{L}} & \downarrow & \eta_{\mathcal{L}} \\
(R \circ \mathcal{L})(A) & \xrightarrow{R(f_D)} & (R \circ \mathcal{L})(B)
\end{array}
\]

We are interested in conditions that will guarantee that the pair \((f_B, f_D)\) is in the image of \( \gamma \), that is, conditions under which the commutativity of (1) implies existence of \( f \in \text{hom}_\mathcal{C}(A, B) \) satisfying both \( f_B = \mathcal{G}(f) \) and \( f_D = \mathcal{L}(f) \).

The above diagram expands to:

\[
\begin{array}{ccc}
(F \circ \mathcal{G})(A) & \xrightarrow{F(f_B)} & (F \circ \mathcal{G})(B) \\
\downarrow \varepsilon_{\mathcal{F}} & & \downarrow \varepsilon_{\mathcal{F}} \\
A & \xrightarrow{f'_B} & B \\
\eta_{\mathcal{L}} & \downarrow & \eta_{\mathcal{L}} \\
(R \circ \mathcal{L})(A) & \xrightarrow{R(f'_D)} & (R \circ \mathcal{L})(B)
\end{array}
\]

Suppose there is some \( f' \in \text{hom}_\mathcal{C}(A, B) \) filling in the diagram:

\[
\begin{array}{ccc}
(F \circ \mathcal{G})(A) & \xrightarrow{F(f_B)} & (F \circ \mathcal{G})(B) \\
\downarrow \varepsilon_{\mathcal{F}} & & \downarrow \varepsilon_{\mathcal{F}} \\
A & \xrightarrow{f'_B} & B \\
\eta_{\mathcal{L}} & \downarrow & \eta_{\mathcal{L}} \\
(R \circ \mathcal{L})(A) & \xrightarrow{R(f'_D)} & (R \circ \mathcal{L})(B)
\end{array}
\]

Then as the diagrams:

\[
\begin{array}{ccc}
(F \circ \mathcal{G})(A) & \xrightarrow{F(G(f'))} & (F \circ \mathcal{G})(B) \\
\downarrow \varepsilon_{\mathcal{F}} & & \downarrow \varepsilon_{\mathcal{F}} \\
A & \xrightarrow{f'_B} & B \\
\eta_{\mathcal{L}} & \downarrow & \eta_{\mathcal{L}} \\
(R \circ \mathcal{L})(A) & \xrightarrow{R(L(f'))} & (R \circ \mathcal{L})(B)
\end{array}
\]
each also commute, we must have \( f_B = G(f') \) and \( f_D = L(f') \) by the universality of the counit and unit respectively.

This means that the condition that every diagram of shape \( \Box \) can be filled in as \( \Theta \) is equivalent to \( \gamma \) being full.

One easy way to achieve this is for \( \varepsilon^{F \dashv G} \) to be epic at \( A \), for \( \eta^{L \dashv R} \) to be monic at \( B \), and for one of those two to be strong. The first two conditions would be guaranteed by faithfulness of \( G \) and \( L \) respectively. Let’s make that assumption.

Since \( C \) is complete and cocomplete, Proposition 4.3.7(3) of Borceux book 1 \( [2] \) tells that \( \varepsilon^{F \dashv G} \) will be strong epic as soon as \( G \) is conservative, and that \( \eta^{L \dashv R} \) will be strong monic as soon as \( L \) is conservative.

We conclude:

**Lemma 1.** If \( G \) and \( L \) are both faithful and at least one of the two is conservative, then \( \gamma \) is full and faithful.

Note that this is not a necessary condition. In the case of topological systems, \( L \) is not faithful, as can be seen by looking at maps from a nonempty space to an indiscrete space with more than one element.

We settle here and turn our attention to the existence of a left adjoint to \( \gamma \).

Following the proof of Theorem 5.2.3 in Rydeheard and Burstall \( [7] \), \( (F \downarrow R) \) is complete because \( R \) preserves limits and cocomplete because \( F \) preserves colimits. Those limits are constructed from limits in \( B \) and \( D \), so \( \gamma \) preserves limits as soon as both \( G \) and \( L \) preserve limits themselves.

\( G \) is a right adjoint, so this is only a restriction on \( L \). We mention that in the (dual) case of topological systems, the forgetful functor \( U : \text{Top} \to \text{Set} \) has both a left adjoint and a right adjoint.

By Lemma 1 and the special adjoint functor theorem, we conclude the following.

**Theorem 2.** Let \( B, C, D \) be complete and cocomplete categories, where \( C \) admits a cogenerating set and is well-powered. Let

\[
F : B \rightleftarrows C : G, \\
L : C \rightleftarrows D : S \dashv R,
\]

with \( F \dashv G \) and \( S \dashv L \dashv R \). Assume further that both \( G \) and \( L \) are faithful, and that at least one of the two is conservative. Then \( C \) is a reflective subcategory of \( (B \downarrow (G \circ R)) \cong (F \downarrow R) \cong ((L \circ F) \downarrow D) \) embedded by:

\[
\gamma : C \to (F \downarrow R), \\
\gamma(A) := \eta^{L \dashv R}_A \circ \varepsilon^{F \dashv G}_A, \\
\gamma(f) := (G(f) \downarrow L(f)),
\]

for \( A \) and \( f \) respectively an object and an arrow of \( C \).

Note that our hypotheses on \( C \) imply that \( L \) preserving limits is equivalent to \( L \) admitting a left adjoint \( S \).

3. **Simple graphs**

The theorem requires an adjoint triple \( S \dashv L \dashv R \). One such triple is \( \mathcal{D} \dashv \mathcal{V} \dashv \mathcal{I} \) between \( \text{Set} \) and the category \( \text{Gph} \) of simple graphs, where \( V : \text{Gph} \to \text{Set} \) is the functor taking a simple graph to its set of vertices, and \( D, I : \text{Set} \to \text{Gph} \) are the discrete and indiscrete graph functors respectively.

Note that the category \( \text{Gph} \) here is the category of reflexive graphs with no loops, where graph homomorphisms are permitted to collapse adjacent vertices to a single vertex. This is following the nlab article \( [3] \). \( \text{Gph} \) is a Grothendieck
quasitopos, which implies that it is locally presentable, which in turn (along with
cocompleteness) implies that it is co-well-powered, see [5].

Given a simple graph $G$, the right angled Artin group of $G$ is the quotient of the
free group on the vertices of $G$ by the commutators of adjacent vertex pairs. This
gives a functor $A : \text{Gph} \rightarrow \text{Grp}$.

It is known (referenced in [3]) that $A$ has a right adjoint $C : \text{Grp} \rightarrow \text{Gph}$, which
takes a group $H$ to its commutation graph $\mathcal{C}(H)$. The vertices of $\mathcal{C}(H)$ are the
elements of $H$, where two vertices are defined to be adjacent iff they commute in
$H$.

**Corollary 3.** \text{Gph} embeds as a full coreflective subcategory of $(\mathcal{F} \downarrow \text{Grp})$, where
$\mathcal{F} : \text{Set} \rightarrow \text{Grp}$ is the free group functor.

**Proof.** The situation looks like:

$$
\mathcal{D}, \mathcal{I} : \text{Set} \rightleftharpoons \text{Gph} : V, \\
A : \text{Gph} \rightleftharpoons \text{Grp} : C,
$$

with $\mathcal{D} \dashv V \dashv \mathcal{I}$ and $A \dashv C$. Here $V$ and $A$ are both faithful, and $A$ is conservative.
The single-vertex graph is a generator of $\text{Gph}$, and as noted earlier, the category
is co-well-powered.

Because $\mathcal{F} = A \circ \mathcal{D}$, the result now follows directly from the dual to Theorem 2.

The embedding takes a graph to the quotient map from the free group on the
vertices to the right angled Artin group. A homomorphism out of a free group is
isomorphic to a simple graph when it is surjective and its kernel is generated by
commutators of generators.

Our comma category $(\mathcal{F} \downarrow \text{Grp})$ also contains the category $\text{Grp}$ as a reflective
subcategory, embedded by taking a group to its counit under the free-forgetful
adjunction. The left adjoint to this embedding is simply the codomain projection.

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