Some fuzzy Korovkin type approximation theorems via power series summability method

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Abstract

This article provides a power series summability-based Korovkin type approximation theorem for any sequence of fuzzy positive linear operators. Using the notion of fuzzy modulus of smoothness, we also derive an associated approximation theorem concerning the fuzzy rate of convergence of these operators. Furthermore, through an example, we illustrate that our summability-based Korovkin type theorem has an advantage over the fuzzy Korovkin type theorem proved in the seminal paper by Anastassiou (Stud. Univ. Babeș-Bolyai Math L(4):3–10, 2005).

Keywords Korovkin theorem · Fuzzy number · Power series summability method · Fuzzy positive linear operators · Fuzzy modulus of continuity

1 Introduction

The concept of fuzziness is a famous key to measure the completeness of a certain assumption or concern. Notably, its applications occur everywhere around us in the areas like stock market, bio-medicines, weather predictions, robotics, and pattern recognitions. It has been observed by the approximation theorists that one of the serious issues in classical approximation theory is to address the limit (or the statistical limit) with the absolute accuracy. To deal with this, fuzzy analogues of many classical approximation theorems have been established (please see Anastassiou 2010), and these efforts have laid the formal foundation of the fuzzy approximation theory.

In memory of a beloved man and an Approx. Theorist Prof. D. V. Pai.

Zadeh (1965) was the first mathematician who systematically introduced and developed the elements of fuzzy set theory, particularly membership function and fuzzy numbers. Goetschel and Voxman (1983) presented a slightly modified definition of fuzzy numbers and also defined a metric for this family of fuzzy sets. Subsequently, researchers defined different classes of sequences of fuzzy numbers and studied them from different perspectives, for instance (cf. Dubois and Prade 1978; Esi and Acikgoz 2011a, b; Savaş 2006; Tripathy and Baruah 2009; Tripathy and Borgogain 2011). Inspired by the work of Gal (2000), Anastassiou (2005) proved a basic fuzzy Korovkin type theorem using Shisha-Mond inequality and established a fuzzy rate of convergence in terms of fuzzy modulus of continuity. Anastassiou (2004) studied some fuzzy-wavelet type operators and fuzzy-neural network type operators. Furthermore, the author established a higher-order point-wise approximation result in fuzzy sense. Using the A-statistical convergence technique, Anastassiou and Duman (2008) proposed some fuzzy Korovkin type results and showed that their results have an advantage over the corresponding results in classical approximation. Recently, Yavuz (2018) established a fuzzy trigonometric Korovkin type approximation theorem by using the power series summability method and also derived another approximation result for fuzzy periodic continuous functions with the help of fuzzy modulus of continuity. Using an example concerning fuzzy Abel–Poisson convolution operators, the author showed that the result established in Anastassiou...
and Gal (2006) (Theorem 1.1) does not work but his result
taken by means of power series method works. In other
words, the author’s result is a non-trivial generalization of the
result given in Anastassiou and Gal (2006). Subsequently,
Yavuz (2021) studied the trigonometric Korovkin type
theorem for fuzzy-valued functions of two variables and also
introduced the double level Fourier series of fuzzy valued
functions to examine the associated approximation by using
Cesáro and Abel methods of summation of infinite series.
For a systematic exposition of the idea of fuzzy approxima-
tion theory, we highly recommend the readers to follow the
classic book by Anastassiou (2010).

Inspired by these studies, we propose to investigate the
fuzzy Korovkin type approximation theorems for fuzzy con-
tinuous functions on a compact support \( J = [a, b] \subset \mathbb{R} \)
by means of power series method. Before proceeding further,
we recall some basic definitions of fuzzy set theory and the
power series summability method in the fuzzy number space.

### 2 Important keywords and power series

**summability**

In fuzzy mathematics, a membership function is a map
\( f : \mathbb{R} \to [0, 1] \) which decides the grade of membership
of any real number in the given fuzzy set \( A \). Subsequently,
any membership function is said to be a fuzzy number if it is
normal, convex, upper semi-continuous, and the closure of
the set \( \{ t \in \mathbb{R} : f(t) > 0 \} \) is compact. We denote
the set of all fuzzy real numbers by the analogous notation \( \mathbb{R}_f \).

For any \( x \in \mathbb{R}_f \), we have the following \( \alpha \)-level fuzzy sets:

\[
[x]_\alpha = \begin{cases} 
\{ t \in \mathbb{R} : x(t) \geq \alpha \}, & \text{for } 0 < \alpha \leq 1; \\
\{ t \in \mathbb{R} : x(t) > \alpha \}, & \text{for } \alpha = 0. 
\end{cases}
\]

It is well known Goetschel and Voxman (1986) that the set
\([x]_\alpha\) is compact for each \( \alpha \). Furthermore, for all \( x, y \in \mathbb{R}_f \)
and \( \lambda \in \mathbb{R} \), the algebra on the \( \alpha \)-level sets is defined as follows:

\[
[x \oplus y]_\alpha = [x]_\alpha + [y]_\alpha, \quad [\lambda \odot x]_\alpha = \lambda [x]_\alpha,
\]

where \([z]_\alpha = [z_-^\alpha, z_+^\alpha]\), and \(z_-^\alpha, z_+^\alpha\) are the end points of \([z]_\alpha\)
for each \( \alpha \in [0, 1] \). In addition, we have a partial ordering in
\( \mathbb{R}_f \) by \( x \preceq y \), if and only if \( x^-_\alpha \leq y^-_\alpha \)
and \( x^+_\alpha \leq y^+_\alpha \), \( \forall \alpha \in [0, 1] \). We have a metric \( D : \mathbb{R}_f \times \mathbb{R}_f \to [0, \infty] \) as

\[
D(x, y) = \sup_{\alpha \in [0, 1]} d([x]_\alpha, [y]_\alpha),
\]

where \( d \) is the usual Hausdorff metric given by

\[
d([x]_\alpha, [y]_\alpha) = \max \{|x^-_\alpha - y^-_\alpha|, |x^+_\alpha - y^+_\alpha|\}.
\]

From Xin and Ming (1991), it is known that \((\mathbb{R}_f, D)\) is a
complete metric space.

For the fuzzy number valued functions \( g, h : J \subset \mathbb{R} \to \mathbb{R}_f \),
the distance between \( g \) and \( h \) is defined by

\[
D^*(g, h) = \sup_{t \in J} D(g(t), h(t)).
\]

Let \( C^F(J) \) be the space of all fuzzy continuous functions on
the interval \( J \), then the operator \( L : C^F(J) \to C^F(J) \) is
called a fuzzy linear operator provided

\[
L(a_1 \odot g_1 \oplus a_2 \odot g_2) = a_1 \odot L(g_1) \oplus a_2 \odot L(g_2),
\]

where \( a_1, a_2 \in \mathbb{R} \) and \( g_1, g_2 \in C^F(J) \). Further, the fuzzy
linear operator \( L \) is said to be fuzzy positive linear operator
if for any \( g, h \in C^F(J) \), with \( g(t) \preceq h(t), \forall t \in J \), we have
\( L(g; t) \preceq L(h; t) \).

The power series summation method was introduced by
Borwein (1957), which includes the Abel and Borel summation
methods as special cases. Many researchers have used
the summability method to establish Korovkin type approxima-
tion theorems in the settings of various function spaces,
some recent works can be found in Agrawal et al. (2021),
Agrawal et al. (2022), Demirci et al. (2022), Tas and Atlı-
han (2019), Tas and Yurdakadın (2017), Yavuz and Ó. Talo,
(2019), Yurdakadın and Ta¸s (2022). Recently, Sezer and
Çanak (2017) have applied the power series summability to
the space of fuzzy numbers \( \mathbb{R}_f \) and given some Tauberian
theorems based on the said summability method. We recall
their power series-based convergence definition as follows:

Assume that \( p = (p_n) \) be a sequence of non-negative real
numbers with \( p_1 > 0 \) and such that \( \sum_{n=1}^{\infty} p_n \to \infty \), as
\( k \to \infty \). Further, let the power series \( p(t) = \sum_{n=1}^{\infty} p_n t^{n-1} \)
be associated with \( p \) be convergent for \( 0 < t < 1 \). Then, a
sequence \((u_n)\) of fuzzy numbers is called \( \beta_p \) summable
to some fuzzy number \( \mu \) determined by \( p \) if \( \sum_{n=1}^{\infty} u_n p_n t^{n-1} \)
converges for \( t \in (0, 1) \) and

\[
\lim_{t \to 1^-} \frac{1}{p(t)} \sum_{n=1}^{\infty} u_n p_n t^{n-1} = \mu.
\]

In our further consideration, let \( C(J) \) denote the space of all
continuous functions on \( J \) with the uniform norm \( \| \cdot \| \). The
prime aim of the present paper is to examine a fuzzy Korovkin
type approximation theorem in the algebraic case by using
the $\mathcal{J}_p$-summability method and also prove a related approximation theorem for fuzzy continuous functions on a closed and bounded interval by means of fuzzy modulus of continuity. We also give an example to show that our Korovkin type theorem is a non-trivial generalization of the basic fuzzy Korovkin theorem.

### 3 Fuzzy Korovkin theorem via power series method

For the better understanding of our main results, we shall recall the following standard Korovkin theorem for fuzzy positive linear operators:

**Theorem 1** (Anastassiou 2005, Thm. 4) Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C(J)$ into itself. Assume that there exists a corresponding sequence $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ of positive linear operators into $C(J)$ such that

$$\left\{ T_n(f; x) \right\}_{\alpha}^\pm = \mathcal{T}_n(f_{\alpha}^\pm; x), \quad (3.1)$$

respectively, for all $\alpha \in [0, 1]$, $\forall f \in C(J)$. Further, suppose that

$$\lim_{n \to \infty} \| T_n(e_i) - e_i \| = 0, \quad (3.2)$$

for $i = 0, 1, 2$ with $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$. Then, for all $f \in \mathcal{F}(J)$ we have

$$\lim_{n \to \infty} D^\alpha (T_n(f), f) = 0. \quad (3.3)$$

Let us consider the sequence $\{T_n\}$ of fuzzy positive linear operators from $C(J)$ into itself with the property (3.1). Further, let for each $t \in (0, 1)$

$$\sum_{n=1}^{\infty} \| \mathcal{T}_n(1) \| p_n t^{n-1} < \infty. \quad (3.2)$$

Then for all $f \in \mathcal{F}(J)$, from Sezer and Çanak (2017) it is known that the series $\sum_{n=1}^{\infty} \mathcal{T}_n(f_{\alpha}^\pm) p_n t^{n-1}$ and series $\sum_{n=1}^{\infty} T_n(f) p_n t^{n-1}$ converge for $t \in (0, 1)$.

We now present our first main result, the fuzzy Korovkin theorem via the power series method.

**Theorem 2** Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C(J)$ into itself. Assume that there exists a corresponding sequence $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ of positive linear operators from $C(J)$ into itself with the properties (3.1) and (3.2). Further, let

$$\lim_{i \to \infty} \| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(e_i) - e_i \| = 0, \quad (3.3)$$

for each test function $e_i$, $i = 0, 1, 2$. Then, for all $f \in C(J)$ we have

$$\lim_{i \to \infty} D^\alpha \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(f), f \right) = 0. \quad (3.4)$$

**Proof** Let $f \in C(J)$, $\alpha \in [0, 1]$ and $x \in J$ be fixed. Suppose (3.3) be satisfied. Then from the continuity of $f_{\alpha}^\pm \in C(J)$, it follows that for a given $\epsilon > 0$ there is a number $\delta > 0$ such that $|f_{\alpha}^\pm(z) - f_{\alpha}^\pm(x)| < \epsilon$ holds for every $z \in J$ satisfying $|z - x| < \delta$. Then, for all $z \in J$, we immediately get that

$$|f_{\alpha}^\pm(z) - f_{\alpha}^\pm(x)| \leq \epsilon + \frac{2 M_{\alpha}^\pm (z - x)^2}{\delta^2}, \quad (3.5)$$

where $x \in J$ and $\alpha \in [0, 1]$ where $M_{\alpha}^\pm = \|f_{\alpha}^\pm\|$ (see Korovkin 1960). In view of the linearity and positivity of the operators $\mathcal{T}_n$, we have, for each $n \in \mathbb{N}$, that

$$\left| \frac{1}{p(t)} \sum_{n=1}^{\infty} \mathcal{T}_n(f_{\alpha}^\pm; x) p_n t^{n-1} - f_{\alpha}^\pm(x) \right| \leq \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(f_{\alpha}^\pm(1); x) - 1$$

$$+ M_{\alpha}^\pm \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(1; x) - 1 \right| \leq \frac{\epsilon}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(1; x) + M_{\alpha}^\pm \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n(1; x) - 1 \right|$$

$$+ 2 M_{\alpha}^\pm \left| \frac{1}{\delta^2 p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \mathcal{T}_n((z - x)^2; x) \right|,$$

which yields

$$\left| \frac{1}{p(t)} \sum_{n=1}^{\infty} \mathcal{T}_n(f_{\alpha}^\pm; x) p_n t^{n-1} - f_{\alpha}^\pm(x) \right| \leq \epsilon + \frac{2 M_{\alpha}^\pm (z - x)^2}{\delta^2}, \quad (3.5)$$
\[ \leq \epsilon + (M^\pm_a + \epsilon + \frac{2d^2M^\pm_a}{\delta^2}) \]
\[ \times \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_0; x) - e_0 \right| \]
\[ + \frac{2M^\pm}{\delta^2} \left\{ \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_2; x) - e_2(x) \right| \right\} \]
\[ + \frac{4d^2M^\pm}{\delta^2} \left\{ \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_1; x) - e_1(x) \right| \right\} \]
\[ \leq \epsilon + K^\pm_a \sum_{i=0}^{2} \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(e_i; x) p_n t^{n-1} - e_i(x) \right|, \]
(3.6)

where \( K^\pm_a = \max \{ \frac{2M^\pm}{\delta^2}, \frac{4d^2M^\pm}{\delta^2}, \epsilon + M^\pm_a + \frac{2d^2M^\pm_a}{\delta^2} \} \) and \( d = \max(|a|, |b|) \). Hence by using definition of the metric \( D \) and in view of the property (3.1) we get

\[ D \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(f), f(x) \right) \leq \epsilon + K \sum_{i=0}^{2} \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(e_i; x) p_n t^{n-1} - e_i(x) \right|, \]

where \( K = K(\epsilon) = \sup_{\alpha \in [0,1]} \{ K^\alpha_+, K^\alpha_- \} \). Then taking supremum over \( x \in J \), we have

\[ D^*( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(f), f) \]
\[ \leq \epsilon + K \sum_{i=0}^{2} \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(e_i; x) p_n t^{n-1} - e_i(x) \right|. \]

Finally, by taking limit as \( t \to 1^- \) on both sides and using hypothesis (3.3), we complete the proof. \( \square \)

Our next result is devoted to discuss fuzzy approximation by the operators \( T_n \) for any function \( f \in C(J) \) by means of the fuzzy modulus of continuity based on the \( 3_p \)-summability. From Anastassiou (2005), the fuzzy analogue of the first-order modulus of continuity is defined as:

\[ \omega^F_1(f; \delta) = \sup_{z, x \in J; |z - x| \leq \delta} D(f(z), f(x)), \]

for any \( \delta > 0 \), \( f \in C(J) \).

**Lemma 1** Anastassiou (2004) Let \( f \in C(J) \). Then for any \( \delta > 0 \),

\[ \omega^F_1(f; \delta) = \sup_{\alpha \in [0,1]} \max \{ \omega_1(f^\alpha_-; \delta), \omega_1(f^\alpha_+; \delta) \}. \]

**Theorem 3** Consider a sequence of fuzzy positive linear operators \( T_n : C(J) \to C(J) \). Assume that there exists a corresponding sequence \( \{ T_n \}_{n \in \mathbb{N}} \) from \( C(J) \) into itself with the properties (3.1) and (3.2). Further suppose that the following conditions hold:

1. \( \lim_{t \to 1^-} \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(e_0; x) p_n t^{n-1} - e_0 \right| = 0, \)
2. \( \lim_{t \to 1^-} \omega^F_1(f; \gamma(t)) = 0, \) where

\[ \gamma(t) = \sqrt{\left| \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(\phi) p_n t^{n-1} \right|} \]

with \( \phi(z) = (z - x)^2 \), for each \( x \in J \).

Then, for all \( f \in C(J) \), we have

\[ \lim_{t \to 1^-} D^* \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} T_n(f) p_n t^{n-1}, f \right) = 0. \]

**Proof** Let \( f \in C(J) \) and \( x \in J \) be arbitrary but fixed. Since \( f^\pm_a \in C(J) \), we can write, for every \( \epsilon > 0 \), that there exists a number \( \delta > 0 \) such that \( |f^\pm_a(z) - f^\pm_a(x)| < \epsilon \) holds for every \( z \in J \), satisfying \( |z - x| < \delta \). Then for all \( z \in J \), we immediately get

\[ |f^\pm_a(z) - f^\pm_a(x)| \leq \left( 1 + \frac{(z - x)^2}{\delta^2} \right) \omega_1(f^\pm_a; \delta) \]

and hence we obtain

\[ \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n T_n(f^\pm_a(z); x) p_n t^{n-1} - f^\pm_a(x) \right| \]
\[ \leq \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n \left( |f^\pm_a(z) - f^\pm_a(x)| ; x \right) \]
\[ + M^\pm_a \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_0; x) - e_0(x) \right| \]
\[ \leq \omega_1(f^\pm_a; \delta) \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n \left( 1 + \frac{(z - x)^2}{\delta^2} ; x \right) \]
\[ + M^\pm_a \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_0; x) - e_0(x) \right| \]
\[ \leq \omega_1(f^\pm_a; \delta) \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n \left( e_0; x \right) - e_0 \]
\[ + \omega_1(f^\pm_a; \delta) + M^\pm_a \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(e_0; x) - e_0(x) \]
\[ + \omega_1(f^\pm_a; \delta) \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} T_n(\phi(z); x) , \]
where \( M_a^\pm = ||f_a^\pm|| \). Then considering property (3.1), Lemma (1) and definition of the metric \( D \) we obtain

\[
D \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n T_n(f; x) x^{n-1}, f(x) \right)
\]

\[
\leq \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n r^{-1} \mathcal{T}_n(e_0; x) - e_0 \right| \omega(T; f, \gamma(t)) + 2 \omega(T; f, \gamma(t)) + M \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n \mathcal{T}_n(e_0; x) x^{n-1} - e_0 \right|
\]

\[
+ \omega(T; f, \gamma(t)) + \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n \mathcal{T}_n(e_0; x) x^{n-1} - e_0 \right|
\]

where \( M \) is defined as:

\[
M := \sup_{x \in J} \max(M_a^+, M_a^-). \]

Taking supremum over \( x \in J \) and putting \( \delta = \gamma(t) \), we conclude that

\[
D^\star \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n T_n(f; x) x^{n-1}, f(x) \right)
\]

\[
\leq \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n r^{-1} \mathcal{T}_n(e_0; x) - e_0 \right| \omega(T; f, \gamma(t)) + 2 \omega(T; f, \gamma(t)) + M \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n \mathcal{T}_n(e_0; x) x^{n-1} - e_0 \right|
\]

\[
+ \omega(T; f, \gamma(t)) + \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n \mathcal{T}_n(e_0; x) x^{n-1} - e_0 \right|
\]

where \( K = \max[M, 2] \). Finally, taking the limit as \( t \to 1^- \), and considering the hypotheses (i) and (ii) of the theorem, the proof is completed.

\[ \square \]

4 Illustrative example

In the following example, we elaborate that our Theorem 2 is a non-trivial generalization of the classical fuzzy Korovkin type result given in Anastassiou (2005).

Example 1 Let us consider the operators of fuzzy-Bernstein defined as:

\[
\mathcal{B}_n(f; x) = \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} f(x)
\]

where \( f \in C^\infty[0, 1], x \in [0, 1] \) and \( n \in \mathbb{N} \). Using these operators, for \( f \in C^\infty[0, 1] \), let us consider the fuzzy pos-

\[ \mathcal{B}_n(f; x) = (1 + x_n) \circ \mathcal{B}_n(f; x) \]

where \( \{x_n\} = \begin{cases} 1, & \text{if } n = m^3, m \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} \). Then, we have

\[
\mathcal{B}_n(f(\alpha); x) = \mathcal{B}_n(f(\alpha); x)
\]

\[
= (1 + x_n) \sum_{j=0}^{\infty} x_j (1 - x)^{n-j} f(\alpha) (j/n)
\]

It is clear that the sequence \( \{x_n\} \) diverges in the classical sense. Further, we observe that

\[
\mathcal{B}_n(e_0; x) = (1 + x_n);
\]

\[
\mathcal{B}_n(e_1; x) = (1 + x_n)x;
\]

\[
\mathcal{B}_n(e_2; x) = (1 + x_n) \left( x^2 + \frac{x(1-x)}{i} \right).
\]

Now if we take \( p_n = 1 \) for all \( n \in \mathbb{N} \), then we obtain \( p(x) = \sum_{n=1}^{\infty} p_n x^{n-1} = \frac{1}{1-x} \), \( |x| < 1 \) which implies that \( R = 1 \). Further, we obtain

\[
\lim_{x \to 1^-} \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} x_n = \lim_{x \to 1^-} \frac{(1 - x)}{x} \sum_{m=1}^{\infty} x^m^3
\]

From Exercise 35 on page 54 of Polya and Szegö (1972), we have

\[
\lim_{x \to 1^-} \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} x_n = \lim_{x \to 1^-} \frac{(1 - x)}{x} \sum_{m=1}^{\infty} x^m^3
\]

\[
= \lim_{x \to 1^-} \frac{(1 - x)^{2/3}}{x} \Gamma \left( \frac{4}{3} \right) = 0.
\]

Hence, using \( \mathcal{B}_n(e_0; x) = (1 + x_n) \), we conclude that

\[
\lim_{x \to 1^-} \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} ||\mathcal{B}_n(e_0) - e_0|| = 0.
\]

Moreover, from \( \mathcal{B}_n(e_1; x) = (1 + x_n)x \), we have

\[
||\mathcal{B}_n(e_1) - e_1|| \leq x_n, \forall i \in \mathbb{N}.
\]
Hence, in view of equation (4.2) we get
\[
\lim_{x \to 1^-} \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} ||B_n(e_1) - e_1|| = 0.
\]

Finally, we have
\[
||B_n(e_2) - e_2|| \leq \frac{1}{4n} + x_n \left(1 + \frac{1}{4n}\right) = \tau_1 + \tau_2.
\]

Since \(\tau_1 \to 0\), as \(n \to \infty\), it follows that it converges to 0 in terms of the power series method. Further, since \((1 + \frac{1}{4n})\) is a convergent sequence as \(n \to \infty\), it is bounded and therefore in view of (4.2), \(\tau_2\) converges to 0 in terms of the power series method. Consequently, we obtain
\[
\lim_{x \to 1^-} \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n x^{n-1} ||B_n(e_2) - e_2|| = 0.
\]

So, by Theorem 2, we obtain, for all \(f \in C^{(F)}[0, 1]\) that
\[
\lim_{t \to 1^-} D^s \left( \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} B_n^F(f), f \right) = 0. \tag{4.2}
\]

However, it can be said that the fuzzy Korovkin theorem 1 does not work for the operators defined by (4.1), since \((x_n)\) is not convergent to 0 as \(n \to \infty\), (in the usual sense).

5 Conclusions and comments

In the paper, we have made efforts to establish a power series summability analogue of a basic fuzzy Korovkin type approximation theorem. Further with some suitable assumptions, we have proved an approximation theorem based on the first-order fuzzy modulus of continuity. In addition to this, we found an example which shows the significance of our study. Interestingly, one can extend our study in the modular spaces (or in the multivariate settings too). It would be also nice to see the new developments in our results through the other known summability techniques.

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Declarations

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