LOCAL-TO-GLOBAL EXTENSIONS TO WILDLY RAMIFIED COVERS OF CURVES

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Abstract. Given a Galois cover of curves \( X \to Y \) with Galois group \( G \) which ramified only at \( x \), restriction to the punctured formal neighborhood of \( x \) induces a Galois extension of Laurent series rings \( k(\langle u \rangle)/k(\langle t \rangle) \). If we fix a base curve \( Y \), we can ask when a Galois extension of Laurent series rings comes from a global cover of \( Y \) in this way. Harbater proved that over a separably closed field, every Laurent series extension comes from a global cover for any base curve if \( G \) is a \( p \)-group, and gave a condition for the uniqueness of such an extension. Using a generalization of Artin–Schreier theory to non-abelian \( p \)-groups which gives an explicit description of \( G \)-covers, we fully characterize the curves \( Y \) for which this extension property holds and for which it is unique up to isomorphism, but over a more general ground field.

1. Introduction

Throughout this paper, \( k \) is an arbitrary field of characteristic \( p \), and \( G \) is a finite \( p \)-group.

Let \( Y \) be a smooth proper curve over \( k \) and \( y \in Y(k) \). We define a "\( y \)-ramified \( G \)-cover of \( Y \)" to be a Galois cover of curves \( q : X \to Y \) with Galois group \( G \), which is unramified on \( Y' := Y - \{ y \} \). By the Cohen Structure theorem, we can choose uniformizers \( t, u_i \) such that
\[
\prod_{x_i \in q^{-1}(y)} \mathcal{O}_{X,x_i} \cong \prod_{x_i \in q^{-1}(y)} k[[u_i]] \quad \text{and} \quad \mathcal{O}_{Y,y} \cong k[[t]].
\]
After localization, we obtain a \( G \)-Galois étale algebra \( L := \prod_{x_i \in q^{-1}(y)} k((u_i)) \) over \( k((t)) \). We say that \( L \) arises from the \( G \)-action on \( X \).

Thus, for each curve \( Y \) and point \( y \in Y \) and \( p \)-group \( G \), we obtain a functor
\[
\psi_{Y,y,G} : \left\{ \text{\( y \)-ramified \( G \) covers of \( Y \)} \right\} \to \left\{ \text{Galois étale algebras over \( k((t)) \) with Galois group a \( G \)} \right\}
\]

Understanding the functor \( \psi_{Y,y,G} \) allows us to use the geometry of Galois covers of curves to classify automorphisms of \( k[[t]] \) as in [FBS17]. Conversely, it allows us to use extensions of \( k((t)) \) in order to classify filtrations of ramification groups of Galois covers of curves with Galois group a \( p \)-group as noted in the survey [DHS14].

Questions about \( \psi_{Y,y,G} \) can be approached by turning to étale cohomology. Throughout this paper, for a scheme \( S \) and a (not necessarily abelian) group \( G \), we denote by \( H^1(S,G) \) the Čech cohomology \( H^1_{et}(S,G) \) of the constant sheaf of groups with coefficients in \( G \) with respect to the étale site on \( X \); this cohomology set parameterizes principal \( G \)-bundles on \( X \) [Mil13]. The map from the ring of regular functions on \( Y' \) into the Laurent series field \( \mathcal{O}(Y') \to k((t)) \) coming from the Cohen Structure Theorem induces a map \( \text{Spec } k((t)) \to Y' \) which we can think of as inclusion of the formal deleted neighborhood around \( y \) into \( Y' \). Hence, we obtain a map \( H^1(Y',G) \to H^1(k((t)),G) \) which we denote by \( \Psi_{Y,y,G} \). We note that \( \Psi_{Y,y,G} \) is induced from \( \psi_{Y,y,G} \) by passing to isomorphism classes.

We pose some basic questions about \( \Psi_{Y,y,G} \).

**Question 1.0.1.** When is \( \Psi_{Y,y,G} \) surjective?
This is equivalent to asking when every $G$-Galois extension of $k((t))$ extends to a global, $y$-ramified Galois cover of $Y$. In [Har80], Harbater showed that if the ground field $k$ is algebraically closed, then $\Psi_{Y,y,G}$ is surjective for any $p$-group $G$. In this paper, we provide an answer to Question 1.0.3 over a more general field $k$, not necessarily algebraically or even separably closed, in the following theorem.

Notation: for any ring $R$ of characteristic $p$, let $\varphi: R \to R$ denote the Artin–Schreier map $f \mapsto f^p - f$.

**Theorem 1.0.2.** Let $G$ be a nontrivial finite $p$-group. Then the following are equivalent:

1. The equality $k((t)) = \varphi(k((t))) + \mathcal{O}(Y')$ holds.
2. The map $\Psi_{Y,y,G}$ is surjective.
3. The map $\Psi_{Y,y,G/pZ}$ is surjective.

We can also ask when any lift of an extension of $k((t))$ to a global Galois cover of $Y$ is unique up to isomorphism.

**Question 1.0.3.** When is $\Psi_{Y,y,G}$ injective?

An answer to this over $k$ algebraically closed was given as well by Harbater in [Har80]. In fact, he calculates the size of the fiber of $\Psi_{Y,y,G}$ as $p^r$, where $r$ is the $p$-rank of $Y$. We extend the answer to Question 1.0.3 to a more general field $k$, which may not be algebraically closed.

**Theorem 1.0.4.** Let $G$ be a nontrivial finite $p$-group. Then the following are equivalent:

1. The equality $\varphi(k((t))) + \mathcal{O}(Y') = \varphi(\mathcal{O}(Y'))$ holds.
2. The map $\Psi_{Y,y,G}$ is injective.
3. The map $\Psi_{Y,y,G/pZ}$ is injective.

Combining our answers to Questions 1.0.1 and 1.0.3, without assuming that the base field is algebraically or even separably closed, gives a criterion on $Y$ for $\psi_{Y,y,G}$ to be an equivalence of categories. This generalizes the result of Katz in [Kat86] which states that over any field of characteristic $p$, the functor $\psi_{p^r,\infty}$ is an equivalence of categories. Curves satisfying the criteria of Theorems 1.0.2 and 1.0.4 are particularly useful for relating the geometry of the curve and its covers to properties of $k((t))$ and its extensions. In Section 5 of this paper, we give another explicit example of a class of such curves.

Our proofs of these theorems use new and more explicit methods. Proofs in previous work, as in [Kat86], have reduced the problem to the case in which $G$ is abelian. In this case, one can use the vanishing of certain $H^2$ groups or a characterization of abelian $p$-group field extensions using Witt vector theory, as noted in [DHS14]. In this paper, we describe and work with an explicit characterization of $G$-Galois étale algebras for $G$ not necessarily abelian. This characterization, which we will call the Inaba classification, is a generalization of a theorem of Inaba in [Ina61] which extends Artin–Schreier–Witt theory to nonabelian Galois étale algebras.

We introduce some notation: let $U_n(R)$ denote the group of upper triangular $n \times n$ matrices with entries in $R$ such that all diagonal entries are 1, and.

In order to be more explicit about the structure of these extensions, we put an order on the indices of entries of matrices in $U_n(R)$. The indices $(i, j): 1 \leq i < j \leq n$ are ordered by lexicographical order on $(j - i, j)$; that is, they are ordered by going down along consecutive diagonals.

**Theorem 1.0.5.** Let $G$ be a finite $p$-group, and fix an injective homomorphism $\Lambda: G \to U_n(\mathbb{F}_p)$ for some suitable $n$. Let $R$ be a ring of characteristic $p$ such that $\text{Spec } R$ is connected, and let $L/R$ be a Galois étale algebra with Galois group $G$.

1. The $R$-algebra $L$ is generated by elements $a_{i,j} \in L$ for $1 \leq i < j \leq n$ such that the unipotent matrix $A := (a_{i,j})$ satisfies $A^{(p)} = MA$ for some $M \in U_n(R)$. We also have that for $\sigma \in G$, $^\sigma A = A\Lambda(\sigma)$, where $\sigma$ acts entry-wise on $A$. We denote this situation by $L = R[A]$ and say that $L$ is of type $M$. 

(2) Given two algebras $L, L' \in H^1(\text{Spec } R, G)$, if we choose $(A, M)$ for $L$ and $(A', M')$ for $L'$, then $L, L'$ are isomorphic if and only if $M = C^0(M'\lambda^{-1})$ for some $C \in U_n(R)$.

(3) In addition, $L/R$ can be decomposed as a composition of subextensions

\[ (*) \quad L = L_{1,n}/L_{2,n}/L_{1,n-1}/.../L_{2,3}/L_{1,2}/R \]

where $L_{i,j} = R\{a_{lm} : (\ell, m) \leq (i,j)\}$. And let $L_{<i,j}$ denote $R\{a_{lm} : (\ell, m) < (i,j)\}$. Then for each $(i,j)$, either $a_{ij} \in L_{<i,j}$ or $L_{i,j} \cong L_{<i,j}[x]/(x^{p^n} - x - (\sum_{i<k<j} m_{ik}a_{kj}) - m_{ij})$. The latter is the case if and only if there is an element $g \in G$ such that $\Lambda(g)_{i,j} \neq 0$ but $\Lambda(g)_{i,m} = 0$ whenever $(l, m) < (i,j)$.

Note that the above is exactly the Artin–Schreier characterization of $\mathbb{Z}/p\mathbb{Z}$-Galois étale algebras when $G = \mathbb{Z}/p\mathbb{Z} \cong U_2(\mathbb{F}_p)$.

Our proof of the Inaba classification uses modern methods of étale cohomology and hence provides an alternative proof of Inaba’s result while also generalizing the result. Toward this end, we present several useful results in nonabelian étale cohomology in Section 2.

2. Exact Sequences for Nonabelian Čech Cohomology

In order to prove our results in the setting where $G$ is nonabelian, we present some results about exact sequences of Čech cohomology.

If $G$ is a sheaf of abelian groups, we define $\bar{H}^i(X, G)$ to be the $i$th Čech cohomology group with respect to the étale site, and if $X$ is affine, then $\bar{H}^i(X, G)$ is isomorphic to $H^i_{\text{ét}}(X, G)$ by Theorem 10.2 of [Mil13].

But if $G$ is a sheaf of groups which are nonabelian, only $\bar{H}^0(X, G)$ and $\bar{H}^1(X, G)$ are defined, and the latter may not be a group.

2.1. Nonabelian Čech cohomology. First, we recall definitions of these cohomology sets and describe natural maps between them, following [Mil13].

Let $G$ be a sheaf of groups on $X_{\text{ét}}$ and let $U = (U_i \to X)_{i \in I}$ be an étale covering of $X$. Let $U_{i_1...i_k}$ denote $U_{i_1} \times_X ... \times_X U_{i_k}$. A 1-cocycle for $U$ with values in $G$ is a family $(g_{ij})_{(i,j) \in I \times I}$ with $g_{ij} \in G(U_{ij})$ such that

\[(g_{ij}|U_{ijk}) \cdot (g_{jk}|U_{ijk}) = g_{ik}|U_{ijk} \quad \text{for all } i,j,k.\]

We define an equivalence relation $\sim$ as follows; for two cocycles $g = (g_{ij})$ and $g' = (g'_{ij})$, we write $g \sim g'$ if there is a family $(\gamma_i)_{i \in I}$ with $\gamma_i \in G(U_i)$ such that

\[g'_{ij} = (\gamma_i|U_{ij}) \cdot g_{ij} \cdot (\gamma_j|U_{ij})^{-1}\]

for all $i,j$. Then $\bar{H}^1(X_{\text{ét}}, G)$ is defined to be the limit over all étale coverings of $X$ of 1-cocycles modulo $\sim$.

As mentioned before and in [Mil13], $\bar{H}^1(X_{\text{ét}}, G)$ parameterizes principal $G$-bundles (when $G$ is a constant sheaf with group $G$, these are principal $G$-bundles), but it is not necessarily a group. However, $\bar{H}^1(X_{\text{ét}}, G)$ has a distinguished element $g_{ij} = 1$ for all $i,j$. If $\phi$ is a map to $\bar{H}^1(X_{\text{ét}}, G)$, we define $\ker \phi$ to be the preimage of this distinguished element.

2.2. Partial delta functors. In the setting of Čech cohomology, we also have partial delta functors connecting cohomology sets, which we define below.

If

\[ 1 \to G' \to G \to G'' \to 0 \]

is an exact sequence of sheaves of groups, we can define a map

\[ \delta_0 : G''(X) \to \bar{H}^1(X_{\text{ét}}, G') \]
as follows. Let $c$ be an element of $G''(X)$. Then since $\pi : G \to G''$ is locally surjective, there exists an étale cover $U = (U_i)_{i \in I}$ of $X$, and elements $b_i \in G(U_i)$ such that $\pi(b_i) = c|U_i$. We define
\[
\delta_0(c) = (b_i^{-1}b_j)_{ij}.
\]
One can check that $\delta_0(c)$ is in $\check{H}^1(X, G')$ and that it is well-defined. We also note that $\delta_0$ can be defined in terms of the characterization of $\check{H}^1(X, G')$ as principal $G'$-bundles by mapping $c \in G''(X)$ to the subsheaf of $G$ defined on each open as the preimage of $c$.

When $G'$ is a sheaf of abelian groups such that $G'(U)$ is central in $G(U)$ for all $U$ in an étale cover, we can define a map
\[
\delta_1 : \check{H}^1(X, G'') \to \check{H}^2(X, G')
\]
as follows.

Let $\mathcal{U} = (U_i \to X)_{i \in I}$ be an étale covering of $X$, and let $(c_{ij})_{i \in I}$ be an element of $\check{H}^1(X, G'')$, with $c_{ij} \in G''(U_{ij})$. Let $b_{ij} \in G(U_{ij})$ such that $\pi(b_{ij}) = c_{ij}$. Then, from the cocycle condition on $(c_{ij})$, we have that for each triple $i, j, k$, there exists $h_{ijk} \in G'(U_{ij}k)$ such that $h_{ijk} = b_{jk}^{-1}b_{ij}b_{jk}$. We define
\[
\delta_1((c_{ij})_{i,j}) = (h_{ijk})_{i,j,k}.
\]
The map $\delta_1$ takes values in $\check{H}^2(X, G')$ and is well-defined, as can be checked directly and is proved in Section 4.2 of Chapitre III of [Gro71].

2.3. Long exact sequences. We now have maps that we can arrange into a sequence which we shall show to be “exact,” which we define in the usual sense but with kernels of pointed sets. Additionally, we may have group actions on cohomology sets which allow for a classification of the fibers; we describe these group actions below.

Let $b \in G(X)$ and $c \in (G/H)(X)$. So there is an étale cover $\{U_i\}_{i \in I}$ and elements $b_i \in G(U_i)$ such that $c|U_i$ is the image of $b_i$ for all $i$. Then $b \cdot c \in (G/H)(X)$ denote the section whose restriction to $U_i$ is the image of $b \cdot b_i$ for each $i$.

If $H(U)$ is a normal subgroup of $G(U)$ for all étale opens $U$, then we have an action of $(G/H)(X)$ on $H^1(X, H)$ as follows. If $c \in (G/H)(X)$, there is a family $(b_i)_i$ with $b_i \in G(U_i)$ such that the image of $b_i$ in $G(U_i)/H(U_i)$ is $c|U_i$. If $a = (a_{ij})$ is an element of $H^1(X, H)$, then $c \cdot a := (b_i a_{ij} b_i^{-1})_{ij}$.

If $H(U)$ is a central subgroup of $G(U)$ for all étale open sets $U$, then we have an action of the abelian group $H^1(X, H)$ on $H^1(X, G)$ as follows. If $a = (a_{ij})$ is an element of $H^1(X, H)$ and $b = (b_{ij})$ is an element of $H^1(X, G)$, then $a \cdot b := (a_{ij}b_{ij})$. We summarize some observations about fibers of cohomology maps with this in mind.

**Lemma 2.3.1.** Let $X$ be a connected scheme, and let $G$ be a sheaf of (not necessarily abelian) groups, and $H$ a subsheaf of groups.

1. The Čech cohomology sequence
\[
1 \to H(X) \to G(X) \to (G/H)(X) \overset{\delta_0}{\to} H^1(X, H) \overset{\iota}{\to} H^1(X, G) \overset{\pi}{\to} H^1(X, G/H)
\]
is exact, and for elements $c, c' \in (G/H)(X)$, $\delta_0(c) = \delta_0(c')$ if and only if there exists $\beta \in G(X)$ such that $c = \beta c'$.

2. If, in addition, $H(U)$ is a normal subgroup of $G(U)$ for all étale open sets $U$, two elements of $H^1(X, H)$ have the same image in $H^1(X, G)$ if and only if they are in the same $(G/H)(X)$-orbit.

3. If, in addition, $H(U)$ is a central subgroup of $G(U)$ for all étale open sets $U$, then the extended sequence
\[
1 \to H(X) \to G(X) \to (G/H)(X) \to H^1(X, H) \overset{\iota}{\to} H^1(X, G) \overset{\pi}{\to} H^1(X, G/H) \overset{\delta_1}{\to} H^2(X, H)
\]
is exact, and two elements of $H^1(X, G)$ have the same image in $H^1(X, G/H)$ if and only if they are in the same $H^1(X, H)$-orbit.
Proof. We prove part (3); its proof is similar to the proofs of parts (1) and (2), which follow from Propositions 3.3.3 and 3.4.5 of Chapitre III of [Gir71].

Proof of part (3): an element $(c_{ij})_{i,j}$ is in the kernel of the map $H^1(X, G/H) \to H^2(X, H)$ if and only if representatives $b_{ij}$ for $c_{ij}$ in $G(U_{ij})$ satisfy the cocycle condition, which implies exactness.

Now we consider the fibers of $\pi$, where $\pi$ denotes the map $H^1(X, G) \to H^1(X, G/H)$. Let $b, b'$ be elements of $H^1(X, G)$ represented by cocycles $(b_{ij})_{i,j}$ and $(b'_{ij})_{i,j}$ respectively, and let $c_{ij}$ denote the image of $b_{ij}$ in $(G/H)(U_{ij})$ (and similarly for $c'_{ij}$). Suppose that $\pi(b) = \pi(b')$, so there exist $\gamma_i \in (G/H)(U_i)$ such that $c_{ij} = \gamma_i b_{ij} \gamma_j^{-1}$ for all $i, j$. Now let $\beta_i \in G(U_i)$ such that the image of $\beta_i$ in $G/H(U_i)$ is $\gamma_i$. Then $a_{ij} := b_{ij}^{-1} \beta_i b_{ij}^{-1}$ is an element of $H(U_{ij})$. Since $H(U_{ij})$ is central in $G(U_{ij})$, we also have that $a_{ij} = \beta_i b_{ij}^{-1} b^{-1}$ is central in $H(U_{ij})$. Thus $a_{ij}$ is a cocycle, and so $b$ and $b'$ are in the same $H^1(X, H)$-orbit. The converse is straightforward. 

We close this section with a useful lemma about maps of cohomology induced by subgroups of $p$-groups.

**Lemma 2.3.2.** Let $R$ be a ring of characteristic $p$ such that $\text{Spec } R$ is connected. If $G'$ is a $p$-group and $G$ is a subgroup of $G'$, then the induced map $\rho : H^1(\text{Spec } R, G) \to H^1(\text{Spec } R, G')$ is injective.

**Proof.** We first prove the statement for $G$ central in $G'$. In this case, the action of $G'/G$ on $H^1(\text{Spec } R, G)$ is trivial, so by Lemma 2.3.1, $\rho$ is injective.

We now proceed by induction on $|G'|$, doing a diagram chase of pointed sets. Let $G'_1$ be a non-trivial central subgroup of $G'$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (every $p$-group has a non-trivial center, as guaranteed by the class equation), and let $G_1 = G_1' \cap G$. By Lemma 2.3.1 and Lemma 1.4.3 of [Kat86] (the latter stating that $H^2(\text{Spec } R, \mathbb{Z}/p\mathbb{Z}) = 0$), the rows of the following commutative diagram are exact.

\[
\begin{array}{ccc}
1 & \longrightarrow & H^1(\text{Spec } R, G_1) \\
\downarrow \rho_1 & & \downarrow \rho \\
1 & \longrightarrow & H^1(\text{Spec } R, G_1')
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
\downarrow \rho_2 & & \\
& & 1
\end{array}
\]

By induction, $\rho_1$ and $\rho_2$ are injective, and by the case in the beginning of this proof, $\iota$ and $\iota'$ are injective. Now suppose that $b_1, b_2 \in H^1(\text{Spec } R, G)$ and $\rho(b_1) = \rho(b_2)$. Then $\rho_2 \circ \rho_1(b_1) = \rho_2 \circ \rho_1(b_2)$, so $\rho_2(b_1) = \rho_2(b_2)$. By Lemma 2.3.1 there exists $a \in H^1(\text{Spec } R, G_1)$ such that $a \cdot b_1 = b_2$. Then $\rho_1(a) \rho(b_1) = \rho(b_2) = \rho(b_1)$, so $\rho_1(a) = 1$ by injectivity of $\iota'$, and so $a = 1$ by injectivity of $\rho_1$. Thus $b_1 = b_2$ and $\rho$ is injective. 

\[\square\]

3. **Inaba Classification of $p$-group Covers**

We now provide generalizations of Artin–Schreier theory to non-abelian groups. We first establish some notation. For an $\mathbb{F}_p$-algebra $R$, we define $U_n(R)$ to be the group of upper triangular $n \times n$ matrices with coefficients in $R$, such that all the diagonal entries are 1, and the group action is matrix multiplication. We also denote by $X$ the upper triangular matrix of indeterminates, where the $(i,j)$ entry is the indeterminate $x_{ij}$ for $j - i > 0$ and the diagonal entries are 1. For a characteristic $p$ ring $R$ and a matrix $C \in U_n(R)$, we denote by $C^{(p)}$ the matrix obtained by raising each entry of $C$ to the $p$th power (different from matrix multiplication of $C$ with itself $p$ times). For a matrix $M \in U_n(R)$, we denote by $L_M$ the $R$-algebra $R[X]/(X^{(p)} - MX)$, by which we mean the $R$-algebra generated by the indeterminate entries of $X$, modulo the relations coming from the matrix equation $X^{(p)} = MX$. This has a $U_n(\mathbb{F}_p)$-action given by $X \mapsto X \cdot g$ for $g \in U_n(\mathbb{F}_p)$ (with indeterminates mapping to the corresponding entry of the matrix $X \cdot g$). Lastly, we say
Lemma 3.0.1. Let $R$ be a ring of characteristic $p$ such that $\Spec R$ is connected. Then the finite Galois étale algebras over $\Spec R$ with Galois group $U_n(\mathbb{F}_p)$ are the algebras $R[X]/(X^{(p)} = MX)$ where $M$ ranges over all matrices in $U_n(R)$, and the Galois action is given by matrix multiplication $X \mapsto X \cdot g$. Two such Galois algebras defined by matrices $M, M'$ are isomorphic as $R$-algebras with $U_n(\mathbb{F}_p)$-action if and only if $M = C^{(p)} M' C^{-1}$ for some $C \in U_n(R)$.

Proof. Let $U_n$ be the $\mathbb{F}_p$-group scheme representing the functor which sends a ring $A$ to $U_n(A)$, and let $U_n(\mathbb{F}_p)$ be the constant group scheme. We have an exact sequence

$$1 \to U_n(\mathbb{F}_p) \to U_n \xrightarrow{\zeta} U_n \to 1$$

where $\zeta$ is the morphism (which is not a group homomorphism) $B \mapsto B^{(p)} B^{-1}$. By Lang’s theorem [Lan56], $\zeta$ is surjective and identifies $U_n(\mathbb{F}_p)$ with $U_n$. Since $\Spec R$ is connected, we know that $H^0(\Spec R, U_n(\mathbb{F}_p)) = U_n(\mathbb{F}_p)$, so by Lemma 2.3.1 we have an exact sequence of pointed sets

$$1 \to U_n(\mathbb{F}_p) \to U_n(R) \xrightarrow{\delta} H^1(\Spec R, U_n(\mathbb{F}_p)) \to H^1(\Spec R, U_n)$$

where $\delta$ sends a matrix $M \in U_n(R)$ to the principal $U_n(\mathbb{F}_p)$-bundle given by $\mathcal{L}^{-1}(M)$ on each étale open of $\Spec R$. That is, $\delta(M)$ is exactly the étale algebra $R[X]/(X^{(p)} = MX)$.

Since $H^1(X, \mathcal{O}_X) = 1$ for $X$ affine and $U_n$ has a composition series whose factors are $\mathbb{G}_a$, we see by induction that $H^1(\Spec R, U_n) = 1$, so the map $U_n(R) \to H^1(\Spec R, U_n(\mathbb{F}_p))$ is surjective, and every element of $H^1(\Spec R, U_n(\mathbb{F}_p))$ can be represented by a $U_n(\mathbb{F}_p)$-algebra of the form $L_M$. By Lemma 2.3.1 $L_M \cong L_M^*$ if and only if there exists a matrix $C \in U_n(R)$ such that $M' = C * M$, where $*$ is the left action of $U_n(R)$ on $U_n/U_n(R)$. Let $N$ be an element of $U_n(S)$ for $S$ an étale $R$-algebra such that $\mathcal{L}(N) = M$. Then $C * M = \mathcal{L}(CN) = C^{(p)} M C^{-1}$. This yields the result. \qed

Now we look at $G$ a general $p$-group and fix an embedding $\Lambda : G \to U_n(\mathbb{F}_p)$ (such an embedding is guaranteed by Proposition 2.4.12 of [Spr98]).

3.1. Proof of Theorem 1.0.5. Part (1): First, we note that the inclusion $\Lambda : G \to U_n(\mathbb{F}_p)$ induces a map

$$H^1(\Spec R, G) \to H^1(\Spec R, U_n(\mathbb{F}_p))$$

sending

$$L \mapsto \prod_{G \setminus U_n(\mathbb{F}_p)} L =: \tilde{L}$$

with the following left $U_n(\mathbb{F}_p)$-action. Let $u_1, ..., u_r$ be coset representatives for $G \setminus U_n(\mathbb{F}_p)$, with $u_1 = e$ being the identity element. Then we can write any element of $\prod_{G \setminus U_n(\mathbb{F}_p)} L$ as $(\ell_1)_{i=1}^{r}$ with $\ell_i \in L$. For each $u_1 \in U_n(\mathbb{F}_p)$, there exist $g_i \in G$ such that $u_1 u = g_i u_j$, where $j(i)$ is the index of the coset $u_i u$. Then $u \cdot (\ell_i)_{i=1}^{r} = (g_i^{-1} \ell_i^{-1})_{i=1}^{r}$.

Now consider the map $\pi : \tilde{L} \to L$ which is projection onto the first component, and note that for $g \in G$, the first coordinate of $g \cdot (\ell_i)_{i=1}^{r} = g \ell_i$, so $\pi$ is a map of étale $G$-algebras. But by Lemma 3.0.1 $\tilde{L} \cong R[X]/(X^{(p)} = MX)$ as $G$-algebras, so the surjection $\pi$ expresses $L$ as $R[A]$, where $A$ is the matrix with $ij$-coordinate $\pi(x_{ij})$. And since $\pi$ is compatible with the action of $G$, the original $G$-action on $L$ agrees with the action coming from matrix multiplication by $\Lambda(G)$.

Part (2): following [Ina61], we show that the matrix $M$ is unique up to $p$-equivalence. Suppose that $L = R[\{a_{ij}\}]$ and $A := (a_{ij})$ satisfies $A^{(p)} = MA$ for some $M \in U_n(R)$ such that the action of $G$ is given by $\sigma A = A \cdot \sigma$. Suppose that $L$ can similarly be generated by the entries of $B$ with $B^{(p)} = MB$ for some $M \in U_n(R)$. Then $BA^{-1} \in U_n(R)$ since $BA^{-1}$ is fixed by $G$, so $B = CA$ for some $C \in U_n(R)$, and thus $M = C^{(p)} M C^{-1}$. 


Conversely, suppose $L, L'$ are $G$-Galois $R$-algebras of type $M$ and $M'$, respectively, and that $M$ is $p$-equivalent to $M'$. By the above arguments, $L$ maps to $L_M$ in $H^1(\text{Spec } R, U_n(\mathbb{F}_p))$ and $L'$ maps to $L_{M'}$, and by Lemma 3.0.1, $L_M$ and $L_{M'}$ are isomorphic. Since the map $H^1(\text{Spec } R, G) \to H^1(\text{Spec } R, U_n(\mathbb{F}_p))$ is injective by Lemma 2.3.2, $L$ must be isomorphic to $L'$.

Part (3): First note that, since $L/R$ has degree a power of $p$, each of the subextensions in $(*)$ must have degree a power of $p$. Let $1 \leq i < j \leq n$. The equation $A^{(p)} = MA$ at the $(i, j)$ coordinate gives $a_{ij}^{(p)} = a_{ij} + \sum_{i < k < j} m_{ik}a_{kj} + m_{ij}$. Since $L_{i,j} = L_{<i,j}[a_{ij}]$, the extension $L_{i,j}/L_{<i,j}$ must have degree 1 or $p$, giving the first claim.

For the second claim, note the Galois group of $L$ over $L_{i,j}$, where $N_{i,j}$ is the subgroup of matrices in $U_n(\mathbb{F}_p)$ with $(\ell, m)$-entry equal to zero for $(\ell, m) \leq (i, j)$ (for entries above the main diagonal). We similarly have that $G \cap N_{<i,j} (\text{matrices with } (\ell, m)\text{-entry equal to zero for } (\ell, m) < (i, j))$ is the Galois group of $L$ over $L_{<i,j}$. Hence $(G \cap N_{<i,j})/(G \cap N_{i,j})$ is the Galois group of $L_{i,j}/L_{<i,j}$, giving the second claim.

4. Main Proofs

We now show that properties of the map $\Psi_{Y,y,G}$ can be checked for $G = \mathbb{Z}/p\mathbb{Z}$, or equivalently that it suffices to know the behavior of $\varphi$ on $k((t))$ and $O(Y')$. We begin with a lemma about the structure of $U_n(R)$.

**Lemma 4.0.1.** Let $R$ be a ring of characteristic $p$. Suppose $M = (m_{ij}), M' = (m'_{ij}) \in U_n(R)$ are $p$-equivalent, so $M = B^{(p)}M'B^{-1}$ for some $B = (b_{ij}) \in U_n(R)$. Then for each pair $i, j$, there exists an element $c$ of the $\mathbb{Z}$-algebra generated by $\{m_{ij}, b_{ij}, m'_{ij}, |i' - j' < i - j\}$ such that $m_{ij} = \varphi(b_{ij}) + m'_{ij} + c$. That is, $m_{ij} = \varphi(b_{ij}) + m'_{ij}$ modulo the elements on the lower diagonals.

**Proof.** Consider two matrices $W = (w_{ij}), Z = (z_{ij}) \in U_n(R)$. We compute that the $(i, j)$ entry of $WZ$ is

$$\sum_{k=1}^{n} w_{ik}z_{kj} = z_{ij} + w_{ij} + \sum_{i < k < j} w_{ik}z_{kj}.$$ 

Applying this to both sides of the equation $MB = B^{(p)}M'$ yields the result. \qed

4.1. **Proof of Theorem 1.0.4.** We first show that (1) implies (2), so suppose $\varphi(k((t))) \cap O(Y') = \varphi(O(Y'))$. Let $\text{Spec } L, \text{Spec } L'$ be two étale $G$-covers of $Y'$. By Theorem 1.0.3 $L = R[A]$ and $L' = R[A']$ with $A^{(p)} = MA$ and $A'^{(p)} = M'A'$ for some $M, M' \in U_n(\mathbb{F}_p)$, and since they are isomorphic over $k((t))$, we have that $M = B^{(p)}M'B^{-1}$ for some $B \in U_n(k((t)))$. Suppose for contradiction that there exists an entry $b_{ij}$ of $B$ not in $O(Y')$, chosen such that all entries on lower diagonals are in $O(Y')$. Then by Lemma 4.0.1, $\varphi(b_{ij}) = m_{ij} - m'_{ij} + c$ where $c$ is a polynomial in the entries of lower diagonals of $B, M,$ and $M'$. Then $\varphi(b_{ij}) \in O(Y')$, and by (1), $b_{ij} \in O(Y')$, a contradiction.

Next we show that (2) implies (3), so assume that $\Psi_{Y,y,G}$ is injective. Since $G$ is a nontrivial $p$-group, it has a nontrivial subgroup $H$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, so by Lemma 2.3.2, we have a commutative square

$$\begin{array}{ccc}
H^1(Y', \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\Psi_{Y,y,\mathbb{Z}/p\mathbb{Z}}} & H^1(Y', G) \\
\downarrow{\Psi_{Y,y,\mathbb{Z}/p\mathbb{Z}}} & & \downarrow{\Psi_{Y,y,G}} \\
H^1(k((t)), \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\Psi_{Y,y,\mathbb{Z}/p\mathbb{Z}}} & H^1(k((t)), G)
\end{array}$$

and since the top and right arrows are injective, we know that $\Psi_{Y,y,\mathbb{Z}/p\mathbb{Z}}$ is injective.

Next we show that (3) implies (1), so assume that $\Psi_{Y,y,\mathbb{Z}/p\mathbb{Z}}$ is injective. Let $b$ be an element of $k((t))$ such that $\varphi(b) \in O(Y')$. Then, by Lemma 3.0.1, we have that $k((t))[x]/(x^p - x)$ is isomorphic
Theorem 1.0.5 tells us that generally we show by induction that for any \((k, \ell)\) and \( \ell \in L \) we can assume that \( \Psi_{Y, y} \) is surjective. The surjectivity of \( \Psi_{Y, y} \) induces the following commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & H^1(Y', \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\psi_{Y', G}} & H^1(Y', G) & \xrightarrow{\phi_{Y', H}} & H^1(Y', G/H) & \longrightarrow & 1 \\
\downarrow{\psi_{Y, y}} & & \downarrow{\phi_{Y, y}} & & \downarrow{\phi_{Y, y}} & & \downarrow{\phi_{Y, y}} & & \downarrow{\phi_{Y, y}} \\
1 & \longrightarrow & H^1(k((t)), \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\iota_{k(t)}} & H^1(k((t)), G) & \xrightarrow{\phi_{k(t)}} & H^1(k((t)), G/H) & \longrightarrow & 1
\end{array}
\]

whose rows are exact. Also by Lemma 2.3.1, two elements of \( H^1(Y', G) \) have the same image in \( H^1(Y', G/H) \) if and only if they are in the same \( H^1(Y', \mathbb{Z}/p\mathbb{Z}) \)-orbit (and similarly for \( H^1(k((t)), G) \)). The surjectivity of \( \Psi_{Y, y, G} \) is proved via the following diagram chase.

Let \( \tilde{\beta} \) be an element of \( H^1(k((t)), G) \), and let \( \tilde{\gamma} := \phi_{k(t)}(\tilde{\beta}) \). By the inductive hypothesis, there exists \( \gamma \in H^1(Y', G/H) \) such that \( \Psi_{Y, y, G/H}(\gamma) = \tilde{\gamma} \). Let \( \beta \) be an element of \( H^1(Y', G) \) mapping to \( \gamma \). Then there exists \( \alpha \in H^1(k((t)), \mathbb{Z}/p\mathbb{Z}) \) such that \( \iota_{k(t)}(\alpha) \cdot \Psi_{Y, y, G}(\beta) = \tilde{\beta} \), and by the inductive hypothesis there is an element \( \alpha \in H^1(Y', \mathbb{Z}/p\mathbb{Z}) \) mapping to \( \alpha \). Then \( \Psi_{Y, y, G}(\alpha \cdot \beta) = \tilde{\beta} \), so \( \Psi_{Y, y, G} \) is surjective.

Next, we show that (2) implies (1), so suppose \( \Psi_{Y, y, G} \) is surjective for some finite \( p \)-group \( G \), and again let \( L \) be a nontrivial central subgroup of \( G \) isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). The diagram above shows that \( \Psi_{Y, y, G/H} \) is surjective; iterating this process shows that \( \Psi_{Y, y, \mathbb{Z}/p\mathbb{Z}} \) is surjective. Let \( f \in k((t)) \), so \( k((t))[x]/(x^p - x - f) \) is a \( \mathbb{Z}/p\mathbb{Z} \)-Galois étale algebra over \( k((t)) \). Since \( \Psi_{Y, y, \mathbb{Z}/p\mathbb{Z}} \) is surjective, Theorem 1.0.5 tells us that \( f \) is \( p \)-equivalent to an element \( g \) of \( O(Y') \), so \( f = \phi(b) + g \) for some \( b \in k((t)) \). Thus the equality in (1) holds.

When \( \Psi_{Y, y, G} \) is an injection, we now use the Inaba classification to give a concrete description of the unique cover mapping under \( \Psi_{Y, y, G} \) to a given \( G \)-Galois cover of \( k((t)) \) which is in the image of \( \Psi_{Y, y, G} \). We use the notation and index-ordering described in Theorem 1.0.5.

Lemma 4.2.1. Suppose that \( \Psi_{Y, y, G} \) is an injection, and that \( L \) is a \( G \)-Galois étale \( k((t)) \)-algebra which is isomorphic to \( k((t))[A] \) with \( A^{(p)} = MA \) and \( M \in U_n(O(Y')) \). Let \( E \) denote the subring of \( L \) generated by \( O(Y') \) and the entries \( a_{ij} \) of \( A \), and let \( E_{i,j} = O(Y')[a_{kl} : (k, l) < (i, j)] \). Then \( L_{i,j}/L_{<i,j} \) is a nontrivial extension if and only if \( E_{i,j}/E_{<i,j} \) is a nontrivial extension, and \( \Psi_{Y,g, G}(E) = L \).

Proof. Since \( M \in U_n(O(Y')) \), we have that \( E_{i,j} = E_{<i,j}[a_{ij}] \) where \( \phi(a_{ij}) \in E_{<ij} \).

If \( E_{i,j}/E_{<i,j} \) is a trivial extension, then \( a_{ij} \in E_{<<ij} \subseteq L_{<i,j} \), so \( L_{i,j}/L_{<i,j} \) is a trivial extension.

We next show that if \( L_{i,j}/L_{<i,j} \) is a trivial extension, then \( E_{i,j}/E_{<i,j} \) is trivial. So suppose that \( a_{ij} \in L_{<ij} \). We know that \( \phi(a_{ij}) \in E_{<ij} \) and we want to show that \( a_{ij} \) is also in \( E_{<i,j} \).

More generally we show by induction that for any \( (k, f) \) and \( a \in L_{k,f} \), if \( \phi(a) \in E_{k,f} \), then \( a \in E_{k,f} \). The base case is that \( \phi(k((t))) \cap O(Y') = \phi(O(Y')) \); this follows from Theorem 1.0.4. So suppose
\[ L_{k,t} \text{ strictly contains } k((t)). \] We may assume \( L_{k,t} \) is isomorphic to \( L_{<k,t}[x]/(x^p - x - f) \) for an indeterminate \( x \) and \( f \in L_{<k,t} \), and so \( a \) can be expressed uniquely as a sum

\[ a = \sum_{m=0}^{p-1} c_m x^m \]

where \( c_m \in L_{<k,t} \). The \( p - 1 \)st coefficient of \( \varphi(a) \) is then \( \varphi(c_{p-1}) \), which is in \( E_{<k,t} \), and hence by induction \( c_{p-1} \) is in \( E_{<k,t} \). Subtracting off the top-degree term of \( a \) and iterating this procedure, we can conclude that all coefficients of \( a \) are in \( E_{<k,t} \), and since \( x \in E_{k,t} \), we conclude that \( a \in E_{k,t} \), which is what we wanted to show.

Since \( E \) is described as an iterative extension in a completely analogous way to \( L \) as in Theorem 1.0.5, it follows that \( E \) is an étale \( G \)-Galois \( O(Y') \)-algebra with \( L = \Psi_{Y,Y,G}(E) \).

**Theorem 4.2.2.** For a nontrivial \( p \)-group \( G \), the map \( \Psi_{Y,Y,G} \) is a bijection if and only if \( \psi_{Y,Y,G} \) is an equivalence of categories.

**Proof.** First suppose that \( \psi_{Y,Y,G} \) is an equivalence of categories. Then since \( \psi_{Y,Y,G} \) is essentially surjective, \( \Psi_{Y,Y,G} \) is surjective. And if \( \phi : L_1 \to L_2 \) is an isomorphism of \( G \)-Galois \( k((t)) \)-algebras and \( \Psi_{Y,Y,G}(E_i) = L_i \) for \( i = 1, 2 \), then since \( \psi_{Y,Y,G} \) is fully faithful, \( \phi \) lifts to an isomorphism \( E_1 \to E_2 \), so \( \Psi_{Y,Y,G} \) is injective.

For the converse, suppose that \( \Psi_{Y,Y,G} \) is a bijection. We construct an inverse functor to \( \psi_{Y,Y,G} \). Let \( L \) be a \( G \)-Galois étale \( k((t)) \)-algebra, so \( L \) is of the form \( k((t))[A] \) where \( A^{(p)} = MA \) for some \( M \in U_n(O(Y')) \) as argued in the proof of Lemma 4.2.1. We define \( \psi^{-1}_{Y,Y,G}(L) \) to be \( E \) as defined in Lemma 4.2.1. Now let \( L = k((t))[A] \) and \( L' = k((t))[A'] \) be two \( G \)-Galois étale \( k((t)) \)-algebras, with \( A^{(p)} = MA \) and \( A'^{(p)} = M'A' \), for \( M, M' \in U_n(O(Y')) \). Suppose \( \phi \) is a morphism \( L \to L' \); we will show that \( \phi \) restricts to a homomorphism \( E \to E' \). We induct on \( (i,j) \), showing that \( \phi(E_{i,j}) \subset E' \) for all \( (i,j) \). The base case follows from \( \phi \) being a map of \( k((t)) \)-algebras and hence a map of \( O(Y') \)-algebras. So suppose that \( \phi(E_{<i,j}) \subset E' \). We have that \( E_{i,j} = E_{<i,j}[a_{ij}] \), so we just need to show that \( \phi(a_{ij}) \in E' \). We know that \( \varphi(a_{ij}) \) is in \( E_{<i,j} \), and so \( \varphi(\varphi(a_{ij})) = \phi(\varphi(a_{ij})) \) is in \( E_{<i,j} \) by induction. But then \( \phi(a_{ij}) \in E_{<i,j} \) by an argument as in the proof of Lemma 4.2.1. So we define \( \psi^{-1}_{Y,Y,G}(\phi) \) to be this restriction. Then it is straightforward to see that \( \psi^{-1}_{Y,Y,G} \) and \( \psi_{Y,Y,G} \) are inverses.

We now give a concise reformulation of the criterion for when \( \Psi_{Y,Y,G} \) is a bijection. We denote by \( F \) the Frobenius map \( \mathcal{O}_Y \to \mathcal{O}_Y \) sending \( f \to f^p \). This induces a map \( F^*: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \); we also let \( \phi^*: = F^* - Id \) be the map on cohomology induced by Artin–Schreier. Then our reformulation is as follows.

**Corollary 4.2.3.** The map \( \psi_{Y,Y,G} \) is an equivalence of categories if and only if \( \phi^*: H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \) is a bijection.

**Proof.** The natural open immersion \( i : Y' \to Y \) gives the following exact sequence of sheaves on \( Y \):

\[ 0 \to \mathcal{O}_Y \to i_* \mathcal{O}_{Y'} \to \text{skysc}_y \left( \frac{k((t))}{k[[t]]} \right) \to 0 \]

where \( \text{skysc}_y \left( \frac{k((t))}{k[[t]]} \right) \) denotes the skyscraper sheaf at \( y \), with value group \( k((t))/k[[t]] \) where the group structure is given by the additive structure on \( k((t)) \). We see that \( H^1(Y, i_* \mathcal{O}_{Y'}) \cong H^1(Y', \mathcal{O}_{Y'}) = 0 \) since \( Y' \) is affine, so the induced long exact sequence in Zariski cohomology gives us an isomorphism

\[ \frac{H^0(Y, \text{skysc}_y \left( \frac{k((t))}{k[[t]]} \right))}{H^0(Y, i_* \mathcal{O}_{Y'})} \cong \frac{k((t))}{\mathcal{O}(Y') + k[[t]]} \cong H^1(Y, \mathcal{O}_Y). \]
We also see that the map
\[ F^*: \frac{k((t))}{\mathcal{O}(Y') + k[[t]]} \to \frac{k((t))}{\mathcal{O}(Y') + k[[t]]} \]
maps \( f \mapsto f^0 \) for \( f \in k((t)) \).

Now suppose \( \varphi^* \) is surjective. Then for any \( f \in k((t)) \) there exist \( g \in k((t)), h \in \mathcal{O}(Y'), l \in k[[t]] \) such that \( f = g^p - g + h + l \). Let \( a_0 \) be the constant term of \( l \); by setting \( h' := h + a_0 \) and \( l' := l - a_0 \) we can assume \( l \) has no constant term. So \( l = \sum_{i=1}^{\infty} a_i t^i \). We define a power series \( \tilde{l} := \sum_{i=1}^{\infty} b_i t^i \) where \( b_i = -a_i \) for \( i \) not divisible by \( p \) and \( b_{np} = b_0^p - a_{np} \). So \( f = \varphi(g + \tilde{l}) + h \) and \( \Psi_{Y,y,G} \) is surjective for all nontrivial \( p \)-groups \( G \) by Theorem 1.0.2. The converse, that surjectivity of the maps \( \Psi_{Y,y,G} \) implies surjectivity of \( \varphi^* \), is straightforward.

Next, suppose \( \varphi^* \) is injective, and consider \( f \) such that \( \varphi(f) \in \mathcal{O}(Y') \). Then \( f \in k[[t]] + \mathcal{O}(Y') \), so there exist \( g \in \mathcal{O}(Y'), l \in k[[t]] \) such that \( f = g + l \). Again, we can assume that \( l \) has no constant term, so \( l \in tk[[t]] \). But then \( \varphi(g) + \varphi(l) \in \mathcal{O}(Y') \), which implies \( \varphi(l) \in \mathcal{O}(Y') \). But since a nonzero \( l \in tk[[t]] \) would have a zero at \( y \), it could not come from a regular function on \( Y' \), so we must have \( l = 0 \) and so in fact \( \varphi(f) \in \mathcal{O}(Y') \) and \( \Psi_{Y,y,G} \) is injective for all nontrivial \( p \)-groups \( G \) by Theorem 1.0.4. Conversely, suppose \( \Psi_{Y,y,G} \) is injective for all nontrivial \( p \)-groups \( G \), and consider \( f \in k((t)) \) such that \( \varphi(f) = g + l \) for some \( g \in \mathcal{O}(Y'), l \in tk[[t]] \). We can write \( l = \varphi(\tilde{l}) \) for \( \tilde{l} \) as above, so \( \varphi(f - \tilde{l}) \in \mathcal{O}(Y') \), which implies \( f - \tilde{l} \in \mathcal{O}(Y') \) by the hypothesis. Then \( \tilde{f} = \tilde{f} \) in \( k((t))/(k[[t]] + \mathcal{O}(Y')) \), which is what we wanted to show. \( \square \)

5. Examples

In this section, we apply the previous results to study \( \psi_{E,O} \) for \( E \) an elliptic curve over \( \mathbb{F}_p \) and \( O \) the point at infinity. The operator \( F^* \) on \( H^1(E, \mathcal{O}_E) \) acts as multiplication by some \( a \in \mathbb{F}_p \), and in fact \( \#E(\mathbb{F}_p) = p + 1 - a \). We say that \( E \) is anomalous if \( \#E(\mathbb{F}_p) = p \).

**Theorem 5.0.1.** For an elliptic curve \( E \) over \( \mathbb{F}_p \), the following are equivalent:
1. \( E \) is not anomalous.
2. The map \( \Psi_{E,O,G} \) is injective for every nontrivial \( p \)-group \( G \).
3. The map \( \Psi_{E,O,G} \) is surjective for every nontrivial \( p \)-group \( G \).
4. The map \( \psi_{E,O} \) gives an equivalence of categories.

**Proof.** We let \( e \) denote a generator of \( H^1(E, \mathcal{O}_E) \) as a \( \mathbb{F}_p \)-vector space, so \( F^* \) maps \( e \mapsto a \cdot e \). Then for \( x \in \mathbb{F}_p \), we have \( xe \mapsto x^ae \). But since \( x \in \mathbb{F}_p \), the map is \( xe \mapsto xae \), so \( \varphi(xe) = x(a - 1)e \). Then the map \( \varphi^*: H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E) \) is surjective if and only if \( a \neq 1 \) and it’s injective if and only if \( a \neq 1 \). Applying Corollary 1.2.3 gives the result. \( \square \)

As indicated in [Ol876] anomalous curves are, as their name suggests, uncommon, and so Theorem 5.0.1 provides us with a broad class of curves whose \( p \)-group Galois covers correspond nicely to \( p \)-group \( k((t)) \)-extensions.

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