Boundedness in a two-dimensional chemotaxis-haptotaxis system

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Abstract
This work studies the chemotaxis-haptotaxis system

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, \ t > 0, \\
v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
w_t &= -vw, \quad x \in \Omega, \ t > 0,
\end{align*}
\]

in a bounded smooth domain \( \Omega \subset \mathbb{R}^2 \) with zero-flux boundary conditions, where the parameters \( \chi, \xi \) and \( \mu \) are assumed to be positive. It is shown that under appropriate regularity assumption on the initial data \( (u_0, v_0, w_0) \), the corresponding initial-boundary problem possesses a unique classical solution which is global in time and bounded. In addition to coupled estimate techniques, a novel ingredient in the proof is to establish a one-sided pointwise estimate, which connects \( \Delta w \) to \( v \) and thereby enables us to derive useful energy-type inequalities that bypass \( w \). However, we note that the approach developed in this paper seems to be confined to the two-dimensional setting.

Key words: chemotaxis, haptotaxis, logistic source, boundedness, coupled estimates
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1 Introduction

1.1 Chemotaxis-haptotaxis model

We consider the chemotaxis-haptotaxis system

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u (1 - u - w), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
    w_t &= -vw, \quad x \in \Omega, \ t > 0,
\end{align*}
\]

in a physical smoothly bounded domain \( \Omega \subset \mathbb{R}^n \), \( n \in \{2, 3\} \), under zero-flux boundary conditions

\[
\frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0
\]

and with prescribed initial data

\[
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega,
\]

where \( \frac{\partial}{\partial \nu} \) denotes differentiation with respect to outward normal on \( \partial \Omega \), and the parameters \( \chi, \xi \) and \( \mu \) are assumed to be positive. This system was initially proposed by Chaplain and Lolas \[3, 4\] to model the process of cancer cell invasion of surrounding tissue. In this context, \( u \) represents the density of cancer cell, \( v \) denotes the concentration of enzyme, and \( w \) stands for the density of extracellular matrix (tissue). In addition to random motion, cancer cells bias their movement towards a gradient of diffusible enzyme as well as a gradient of non-diffusible tissue by detecting matrix molecules such as vitronectin adhered therein. We refer to the aforementioned two directed migrations of cancer cells as chemotaxis and haptotaxis, respectively. The cancer cells are also assumed to undergo birth and death in a logistic manner, competing for space with healthy tissue. The enzyme is produced by cancer cells, and it is supposed to be influenced by diffusion and degradation. The tissue is stiff in the sense that it does not diffuse, but it could be degraded by enzyme upon contact.

1.2 Previous related works on global well-posedness

When \( w \equiv 0 \), the PDE system (1.1) is reduced to the chemotaxis-only system

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u (1 - u), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0.
\end{align*}
\]

This system has been widely studied. In the case \( \mu = 0 \), solutions may blow up in finite time when \( n \geq 2 \) \[7, 13, 24\]; however, it is known that arbitrarily small \( \mu > 0 \) guarantee the global existence and boundedness of solutions when \( n = 2 \) \[14\], and that appropriately large \( \frac{\mu}{\chi} \) preclude blow-up in the case \( n \geq 3 \) \[22\]. Very recently, it is asserted that sufficiently large \( \frac{\mu}{\chi} \) enforce the stability of constant equilibria \[25\].

When \( \chi = 0 \), the PDE system (1.1) becomes the haptotaxis-only system

\[
\begin{align*}
    u_t &= \Delta u - \xi \nabla \cdot (u \nabla w) + \mu u (1 - u - w), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
    w_t &= -vw, \quad x \in \Omega, \ t > 0.
\end{align*}
\]
Global existence theories for this system were explored in [5, 6, 21], whereas the boundedness and asymptotic behavior of solution was studied in [11]. The above results rule out the possibility of blow-up of solutions to this haptotaxis-only system, although the solutions may exhibit some pattern for certain ranges of model parameters and initial data [4].

Compared with the chemotaxis-only system and the haptotaxis-only system, the coupled chemotaxis-haptotaxis system (1.1) is much less understood. As far as we know, when \( n \in \{2, 3\} \), the global boundedness of solutions for this system has remained pending so far, although the global existence was examined in [15, 16]. The purpose of this work is to answer this issue of boundedness when \( n = 2 \).

1.3 Main results

As to the above initial data we suppose that for some \( \alpha \in (0, 1) \) we have

\[
\begin{align*}
  u_0 &\in C^0(\overline{\Omega}) \quad \text{with} \quad u_0 \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u_0 \neq 0, \\
  v_0 &\in W^{1,\infty}(\Omega) \quad \text{with} \quad v_0 \geq 0 \quad \text{in} \quad \Omega, \\
  w_0 &\in C^{2+\alpha}(\overline{\Omega}) \quad \text{with} \quad w_0 > 0 \quad \text{in} \quad \Omega \quad \text{with} \quad \frac{\partial w_0}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]

Under these assumptions, our main result reads as follows.

**Theorem 1.1** Let \( n = 2 \), and suppose that \( \chi > 0, \xi > 0 \) and \( \mu > 0 \). Then for each \((u_0, v_0, w_0)\) fulfilling (1.4), (1.1)-(1.3) possesses a unique classical solution which is global in time and bounded in \( \Omega \times (0, \infty) \).

To the best of our knowledge, this is the first boundedness result addressing the full parabolic-parabolic-ODE chemotaxis-haptotaxis model, despite there exist some boundedness and stabilization results on the simplified parabolic-elliptic-ODE chemotaxis-haptotaxis model in which the spatiotemporal evolution of chemical concentration is described by the elliptic equation \( 0 = \Delta v - v + u \) replacing the original parabolic counterpart [19, 20]. Here we should point out that the full chemotaxis-haptotaxis system is much more mathematically challenging than the aforementioned simplified one.

1.4 Approaches used in the paper

A main technical difficulty in the proof of Theorem 1.1 emanates from consequences of the strong coupling in (1.1) on the spatial regularity of \( u, v \) and \( w \). Another analytical obstacle stems from the fact that a bound of \( \nabla v \) in \( L^p(\Omega) \) cannot leads to a time-independent bound for \( \nabla w \) in \( L^p(\Omega) \), because

\[
\nabla w(x, t) = \nabla w_0(x) e^{-\int_0^t v(x,s)ds} - w_0(x) e^{-\int_0^t v(x,s)ds} \int_0^t \nabla v(x,s)ds
\]

in which the last term \( \int_0^t \nabla v(x,s)ds \) is non-local in time. It will be crucial to our approach to build a one-sided pointwise estimate which connects \( \Delta w \) to \( v \) (see Lemma 2.2 below). Relying on such a pointwise estimate, we can derive two useful energy-type inequalities that bypass \( w \) (see Lemmata 2.3 and 3.3 below). Using such information along with coupled estimate techniques, we establish estimates on \( \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 \) and \( \int_{\Omega} u^p \) for any \( p > 2 \), which results in the boundedness of \( u \) in \( L^\infty(\Omega) \) by performing the Moser iteration procedure (see Lemmata 3.8, 3.10 and 4.1 below).
Finally, we mention that the methods developed in this paper are restricted to the two-dimensional setting, and thus the boundedness for the corresponding three-dimensional problem largely remains open.

## 2 Local existence and a one-sided pointwise estimate for \( \Delta w \)

With a slight adaption to the proof of [20, Lemma 4.1], we have the following statement on local existence.

**Lemma 2.1** Let \( \chi > 0, \xi > 0 \) and \( \mu > 0 \). Then for any \( u_0, v_0 \) and \( w_0 \) fulfilling (1.1) there exists \( T_{\text{max}} \in (0, \infty) \) with the property that (1.1)-(1.3) possesses a unique classical solution

\[
\begin{align*}
  u &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
v &\in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
w &\in C^{2,1}(\Omega \times [0, T_{\text{max}}]),
\end{align*}
\]

such that

\[
 u \geq 0, \quad v \geq 0 \quad \text{and} \quad 0 < w \leq \|w_0\|_{L^\infty(\Omega)} \quad \text{for all} \quad (x, t) \in \Omega \times [0, T_{\text{max}}) \quad (2.1)
\]

and such that

\[
 \text{either} \quad T_{\text{max}} = \infty, \quad \text{or} \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}. \quad (2.2)
\]

Since the third equation in (1.1) is an ODE, \( w \) can be expressed explicitly in terms of \( v \). This results in the representation formulae

\[
\begin{align*}
w(x, t) &= w_0(x) e^{-\int_0^t v(x, s)ds} \quad \text{and} \\
\nabla w(x, t) &= \nabla w_0(x) e^{-\int_0^t v(x, s)ds} - w_0(x) e^{-\int_0^t v(x, s)ds} \int_0^t \nabla v(x, s)ds \quad (2.3)
\end{align*}
\]

as well as

\[
- \Delta w(x, t) = -\Delta w_0(x) e^{-\int_0^t v(x, s)ds} + 2e^{-\int_0^t v(x, s)ds} \nabla w_0(x) \cdot \int_0^t \nabla v(x, s)ds \\
- w_0(x) e^{-\int_0^t v(x, s)ds} \left| \int_0^t \nabla v(x, s)ds \right|^2 + w_0(x) e^{-\int_0^t v(x, s)ds} \int_0^t \Delta v(x, s)ds \quad (2.5)
\]

for \( (x, t) \in \Omega \times (0, T_{\text{max}}) \).

The following one-sided pointwise estimate for \( -\Delta w \) will serve as a cornerstone for our subsequent analysis (see the proofs of Lemmata 2.3 and 3.3 below).

**Lemma 2.2** Assume that \( \chi > 0, \xi > 0 \) and \( \mu > 0 \), and let \( (u, v, w) \) solve (1.1)-(1.3) in \( \Omega \times (0, T) \) with some some \( (u_0, v_0, w_0) \) satisfying (1.4). Then

\[
- \Delta w(x, t) \leq \|w_0\|_{L^\infty(\Omega)} \cdot v(x, t) + K \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \in (0, T), \quad (2.6)
\]

where

\[
K := \|\Delta w_0\|_{L^\infty(\Omega)} + 4\|\nabla \sqrt{w_0}\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \quad (2.7)
\]
Proof. Start from (2.5), the nonnegativity of \( v \) leads to
\[
- \Delta w_0(x) e^{- \int_0^t v(x,s) ds} \leq \| \Delta w_0 \|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega \text{ and } t \in (0,T)
\]
and a simple but important observation yields
\[
e^{- \int_0^t v(x,s) ds} \left[ 2 \nabla w_0(x) \cdot \int_0^t \nabla v(x,s) ds - w_0(x) \cdot \left| \int_0^t \nabla v(x,s) ds \right|^2 \right] = -w_0(x) e^{- \int_0^t v(x,s) ds} \left| \int_0^t \nabla v(x,s) ds - \nabla w_0(x) \right|^2 + e^{- \int_0^t v(x,s) ds} \frac{|\nabla w_0(x)|^2}{w_0(x)} \leq \frac{|\nabla w_0(x)|^2}{w_0(x)} \leq \frac{4}{\epsilon} \| \nabla \sqrt{w_0} \|_{L^\infty(\Omega)}^2 \quad \text{for all } x \in \Omega \text{ and } t \in (0,T)
\]
thanks to the nonnegativity of \( v \) and the positivity of \( w_0 \).

We now turn to estimate the last term in (2.5). By the second equation in (1.1) and the nonnegativity of \( u, v, v_0 \) and \( w_0 \) we have
\[
w_0(x) e^{- \int_0^t v(x,s) ds} \int_0^t \Delta v(x,s) ds = w_0(x) e^{- \int_0^t v(x,s) ds} \int_0^t (v_t(x,s) + v(x,s) - u(x,s)) ds \\
\leq w_0(x) e^{- \int_0^t v(x,s) ds} \left( v(x,t) - v_0(x) + \int_0^t v(x,s) ds \right) \\
\leq \| w_0 \|_{L^\infty(\Omega)} \cdot v(x,t) + \frac{\| w_0 \|_{L^\infty(\Omega)}}{\epsilon}
\]
for all \( x \in \Omega \) and \( t \in (0,T) \), where we have used the facts that \( z e^{- z} \leq \frac{1}{\epsilon} \) for all \( z \in \mathbb{R} \) and that \( 0 < e^{- \int_0^t v(x,s) ds} \leq 1 \) thanks to \( v \geq 0 \). Finally, collecting (2.8)-(2.10) in conjunction with (2.5) yields (2.6).

Here we stress the fact that the pointwise estimate (2.6) connects \( \Delta w \) to \( v \), which enables us to establish the following useful energy-type inequality that will be used in the proofs of Lemmata 3.5, 3.10 and 4.1 below.

**Lemma 2.3** Let \( n \in \{2,3\}, T \in (0,T_{\max}), \chi > 0, \xi > 0 \text{ and } \mu > 0 \), and assume (1.4). Then the solution of (1.1)-(1.3) satisfies
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u^p + \frac{p-1}{2} \int_\Omega u^{p-2} |\nabla u|^2 \leq \frac{(p-1)\chi^2}{2} \int_\Omega |\nabla v|^2 + \xi \| w_0 \|_{L^\infty(\Omega)} \int_\Omega u^p v + (\mu + \xi K) \int_\Omega u^p - \mu \int_\Omega u^{p+1}
\]
for any \( p > 1 \) and each \( t \in (0,T) \).
Proof. We test the first equation in (1.1) by \( u^{p-1} \), which leads to the identity
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 = \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \xi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \\
+ \mu \int_{\Omega} u^p - \mu \int_{\Omega} u^{p+1} - \mu \int_{\Omega} u^p w
\]
for all \( t \in (0, T) \). Here using the Young inequality we see that
\[
\chi(p-1) \int_{\Omega} u \nabla u \cdot \nabla v \leq \frac{p-1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{(p-1)\chi^2}{2} \int_{\Omega} u^2 |\nabla u|^2 \quad \text{for all } t \in (0, T). \tag{2.13}
\]
Unlike the handling of the above chemotaxis-related integral, we now invoke Lemma 2.2 in conjunction with integration by parts and the Young inequality to estimate the haptotaxis-related integral on the right of (2.12)
\[
\xi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w = -\frac{(p-1)\xi}{p} \int_{\Omega} u^p \Delta w \\
\leq \frac{(p-1)\xi}{p} \| w_0 \|_{L^\infty(\Omega)} \int_{\Omega} u^p v + \frac{(p-1)\xi K}{p} \int_{\Omega} u^p \\
\leq \xi \| w_0 \|_{L^\infty(\Omega)} \int_{\Omega} u^p v + \xi K \int_{\Omega} u^p \quad \text{for all } t \in (0, T) \tag{2.14}
\]
thanks to \( 0 < \frac{p-1}{p} < 1 \). We next observe that
\[
- \mu \int_{\Omega} uw \leq 0 \quad \text{for all } t \in (0, T) \tag{2.15}
\]
because \( w \geq 0 \). Finally, collecting (2.12)-(2.15) yields (2.11) upon a simple rearrangement. □

The following basic property on mass can be easily checked.

**Lemma 2.4** The solution \((u, v, w)\) of (1.1)-(1.3) fulfills
\[
\int_{\Omega} u(x, t) dx \leq m^* := \max \left\{ |\Omega|, \int_{\Omega} u_0(x) dx \right\} \quad \text{for all } t \in (0, T_{\max}). \tag{2.16}
\]

**Proof.** We integrate the first equation in (1.1) with respect to space to obtain
\[
\frac{d}{dt} \int_{\Omega} u = \mu \int_{\Omega} u - \mu \int_{\Omega} u^2 - \mu \int_{\Omega} uw \leq \mu \int_{\Omega} u - \mu \int_{\Omega} u^2 \quad \text{for all } t \in (0, T_{\max}), \tag{2.17}
\]
because \( w > 0 \) by Lemma 2.1. Thanks to the Cauchy-Schwarz inequality, we have \( \int_{\Omega} u^2 \geq \frac{1}{|\Omega|} (\int_{\Omega} u)^2 \), and thus (2.17) entails that \( y(t) := \int_{\Omega} u(x, t) dx, \ t \in [0, T_{\max}), \) satisfies
\[
y'(t) \leq \mu y(t) - \mu \frac{\mu}{|\Omega|} y^2(t) \quad \text{for all } t \in (0, T_{\max}) \tag{2.18}
\]
By an ODE comparison, we therefore obtain that \( y(t) \leq \max \{|\Omega|, y(0)\} \), which precisely results in (2.16). □
3 Energy-type estimates for any $\mu > 0$

In order to establish a bound of $u$ in $L^\infty(\Omega)$, we first build an estimate on $\int_\Omega u \ln u$ as a starting point of our reasoning, which heavily depends on the pointwise estimate (2.6) for $-\Delta w$.

3.1 A coupled estimate on $\int_\Omega u \ln u + \int_\Omega |\nabla v|^2$

We begin with an elementary lemma.

**Lemma 3.1** Let $\mu > 0$ and $A > 0$. Then there exists $L := L(\mu, A) > 0$ such that

$$(1 + \mu)z \ln z + Az^2 - \mu z^2 \ln z \leq L$$

for all $z > 0$. \hspace{1cm} (3.1)

**Proof.** The function $\varphi : [0, \infty) \to \mathbb{R}$ defined by

$$\varphi(z) := \begin{cases} (1 + \mu)z \ln z + Az^2 - \mu z^2 \ln z, & z > 0, \\ 0, & z = 0 \end{cases}$$

satisfies

$$\frac{\varphi(z)}{z^2 \ln z} \to -\mu \quad \text{as} \quad z \to \infty,$$

so that for some $z_0 > 0$ we have $\varphi < 0$ on $(z_0, \infty)$. Since clearly $\varphi$ is continuous on $[0, \infty)$, (3.1) thus holds with $L := \max_{z \in [0, z_0]} \varphi(z)$. \hfill \Box

In the two-dimensional setting, using the properties of the Neumann heat semigroup (23) and the estimate (2.16) on $u$ in $L^1(\Omega)$ provided by Lemma 2.4, we can derive a $L^p$ estimate on $v$.

**Lemma 3.2** Let $n = 2$, and assume (1.4). Then for any $1 \leq p < \infty$ there exists a positive constant $M(p) := M(p, |\Omega|, \|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^\infty(\Omega)}) > 0$ such that the solution of (1.1)-(1.3) satisfies

$$\int_\Omega v^p \leq M(p) \quad \text{for all} \quad t \in (0, T_{\max}).$$

(3.2)

**Proof.** Since the proof was given in [12, Lemma 3.1], we refrain us from repeating it here. \hfill \Box

Strongly depending on the estimate (2.6) for $-\Delta w$ once again, along with the above two preparations, we now can establish two estimates on $\int_\Omega u \ln u$ and $\int_\Omega |\nabla v|^2$ via coupled estimate techniques.

**Lemma 3.3** Let $n = 2$, $T \in (0, T_{\max})$, $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume (1.4). Then there exists $C > 0$ independent of $T$ such that the solution of (1.1)-(1.3) possesses the properties

$$\int_\Omega u \ln u \leq C \quad \text{for all} \quad t \in (0, T) \quad \text{and} \quad$$

$$\int_\Omega |\nabla v|^2 \leq C \quad \text{for all} \quad t \in (0, T).$$

(3.3)\hspace{1cm} (3.4)
Proof. Testing the first equation of (1.1) against \((1 + \ln u)\), we obtain
\[
\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{|\nabla u|^2}{u} = \chi \int_{\Omega} \nabla u \cdot \nabla v + \xi \int_{\Omega} \nabla u \cdot \nabla w + \mu \int_{\Omega} u(1 + \ln u)(1 - u - w) \quad \text{for all } t \in (0, T).
\] (3.5)

Once more integrating by parts, in light of the Young inequality we have
\[
\chi \int_{\Omega} \nabla u \cdot \nabla v = -\chi \int_{\Omega} u \Delta v \leq \frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{\chi^2}{2} \int_{\Omega} u^2 \quad \text{for all } t \in (0, T).
\] (3.6)

Similarly,
\[
\xi \int_{\Omega} \nabla u \cdot \nabla w = -\xi \int_{\Omega} u \Delta w = \xi \int_{\Omega} u(-\Delta w),
\]
which in view of Lemma 2.2 entails that
\[
\xi \int_{\Omega} \nabla u \cdot \nabla w \leq \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u + \xi K \int_{\Omega} u \quad \text{for all } t \in (0, T)
\] (3.7)
because \(u \geq 0\). Here we use the Young inequality and Lemma 3.2 to estimate the first term on the right
\[
\xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u \leq \frac{\xi \|w_0\|_{L^\infty(\Omega)}}{2} \int_{\Omega} u^2 + \frac{\xi \|w_0\|_{L^\infty(\Omega)}}{2} \int_{\Omega} v^2 \leq \frac{\xi \|w_0\|_{L^\infty(\Omega)}}{2} \int_{\Omega} u^2 + c_1 \quad \text{for all } t \in (0, T)
\]
with \(c_1 := \frac{\xi \|w_0\|_{L^\infty(\Omega)} K}{2} M(2)\), whereas we employ Lemma 2.4 to deal with the second term on the right of (3.7)
\[
\xi K \int_{\Omega} u \leq \xi K m^* \quad \text{for all } t \in (0, T).
\] Thus, we find that
\[
\xi \int_{\Omega} \nabla u \cdot \nabla w \leq \frac{\xi \|w_0\|_{L^\infty(\Omega)}}{2} \int_{\Omega} u^2 + c_2 \quad \text{for all } t \in (0, T),
\] (3.8)
where \(c_2 := c_1 + \xi K m^*\). As to the last term in (3.5), by \(u \geq 0, w \geq 0, (2.1)\), Lemma 2.4 and the basic inequality \(\max_{x \geq 0}(-z \ln z) = \frac{1}{e}\) we obtain
\[
\mu \int_{\Omega} u(1 + \ln u)(1 - u - w) = \mu \int_{\Omega} u - \mu \int_{\Omega} u^2 - \mu \int_{\Omega} uw + \mu \int_{\Omega} u \ln u - \mu \int_{\Omega} u^2 \ln u + \mu \int_{\Omega} (-u \ln u)w \leq \mu \int_{\Omega} u + \mu \int_{\Omega} u \ln u - \mu \int_{\Omega} u^2 \ln u + \mu \int_{\Omega} (-u \ln u)w \leq \mu \int_{\Omega} u \ln u - \mu \int_{\Omega} u^2 \ln u + \mu m^* + \frac{\mu}{e} \|w_0\|_{L^\infty(\Omega)} \cdot |\Omega|
\] (3.9)
8
for all \( t \in (0, T) \). Collecting (3.6), (3.8) and (3.9) along with (3.5) leads to

\[
\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{\vert \nabla u \vert^2}{u} \leq \frac{1}{2} \int_{\Omega} \vert \Delta v \vert^2 + \mu \int_{\Omega} u \ln u + c_3 \int_{\Omega} u^2 - \mu \int_{\Omega} u^2 \ln u + c_4
\]

for all \( t \in (0, T) \), where \( c_3 := \frac{x^2 + \xi \|w_0\|_{L^\infty(\Omega)}}{2} \) and \( c_4 := c_2 + \mu m^* + \frac{\xi}{\varepsilon} \|w_0\|_{L^\infty(\Omega)} \cdot |\Omega| \).

In order to cancel the first term on the right of (3.10), we test the second equation of (1.1) by \(-\Delta v\) and use the Young inequality to find

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \vert \nabla v \vert^2 + \int_{\Omega} \vert \nabla v \vert^2 = - \int_{\Omega} \vert \Delta v \vert^2 - \int_{\Omega} u \Delta v
\]

\[
\leq - \frac{1}{2} \int_{\Omega} \vert \Delta v \vert^2 + \frac{1}{2} \int_{\Omega} u^2 \quad \text{for all } t \in (0, T).
\]

Adding this to (3.10) yields

\[
\frac{d}{dt} \left\{ \int_{\Omega} u \ln u + \frac{1}{2} \int_{\Omega} \vert \nabla v \vert^2 \right\} + \int_{\Omega} \frac{\vert \nabla u \vert^2}{u} + \int_{\Omega} \vert \nabla v \vert^2 \leq \mu \int_{\Omega} u \ln u + A \int_{\Omega} u^2 - \mu \int_{\Omega} u^2 \ln u + c_4
\]

for all \( t \in (0, T) \), where \( A := \frac{1}{2} + c_3 \). Adding \( \int_{\Omega} u \ln u \) to both sides of this and dropping the nonnegative term \( \int_{\Omega} \frac{\vert \nabla v \vert^2}{u} \) on the left, we find that \( y(t) := \int_{\Omega} u \ln u + \frac{1}{2} \int_{\Omega} \vert \nabla v \vert^2, t \in (0, T) \), satisfies the differential inequality

\[
y'(t) + y(t) \leq \int_{\Omega} \left[ (1 + \mu)u \ln u + Au^2 - \mu u^2 \ln u \right] + c_4,
\]

which in view of Lemma 3.1 implies

\[
y'(t) + y(t) \leq c_5 \quad \text{for all } t \in (0, T)
\]

with \( c_5 := L \cdot |\Omega| + c_4 \). Upon ODE comparison, this yields

\[
y(t) \leq \max \left\{ c_5, y(0) \right\} \quad \text{for all } t \in (0, T),
\]

which proves (3.3) and (3.4).

### 3.2 A bound for \( \int_{\Omega} u^2 + \int_{\Omega} \vert \nabla v \vert^4 \)

To build a bound for \( \int_{\Omega} u^2 \), we shall need the following generalization of the Gagliardo-Nirenberg inequality for the general case when \( r > 0 \) (cf. [18, Lemma A.5] for a detailed proof), which extends the standard case when \( r \geq 1 \) in [2].

**Lemma 3.4** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary, and let \( p \in (1, \infty) \) and \( r \in (0, p) \).

Then there exists \( C > 0 \) such that for each \( \eta > 0 \) one can pick \( C_\eta > 0 \) with the property that

\[
\left\| u \right\|_{L^p(\Omega)}^p \leq \eta \left\| \nabla u \right\|_{L^2(\Omega)}^{p-r} \left\| u \ln |u| \right\|_{L^r(\Omega)}^r + C \left\| u \right\|_{L^r(\Omega)}^r + C_\eta
\]

holds for all \( u \in W^{1,2}(\Omega) \).
By applying (2.11) to $p = 2$ and using Lemma 3.2 we establish an energy inequality involving $\int_{\Omega} u^2$.

**Lemma 3.5** Let $n = 2$, $T \in (0, T_{\text{max}})$, $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume (1.4). Then there exists $C_1 > 0$ independent of $T$ such that the solution of (1.1)-(1.3) satisfies

$$\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq \chi^2 \int_{\Omega} u^2 |\nabla v|^2 - \mu \int_{\Omega} u^3 + C_1 \quad \text{for all } t \in (0, T).$$

**(3.13)**

**Proof.** We apply (2.11) to $p = 2$ to obtain

$$\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq \chi^2 \int_{\Omega} u^2 |\nabla v|^2 + 2\xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^2 v + 2(\mu + \xi K) \int_{\Omega} u^2 - 2\mu \int_{\Omega} u^3$$

for any $p > 1$ and each $t \in (0, T)$. Here we invoke the Young inequality and Lemma 3.2 to estimate

$$2\xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^2 v \leq \frac{\mu}{2} \int_{\Omega} u^3 + \frac{128}{2\mu^2} \left(\xi \|w_0\|_{L^\infty(\Omega)}\right)^2 \int_{\Omega} v^3$$

$$\leq \frac{\mu}{2} \int_{\Omega} u^3 + c_1 \quad \text{for all } t \in (0, T)$$

(3.15)

with $c_1 := \frac{128}{2\mu^2} (\xi \|w_0\|_{L^\infty(\Omega)})^2 \cdot M(3)$, where $M(3)$ is defined by Lemma 3.2. Similarly, we have

$$2(\mu + \xi K) \int_{\Omega} u^2 \leq \frac{\mu}{2} \int_{\Omega} u^3 + c_2 \quad \text{for all } t \in (0, T)$$

with $c_2 := \frac{128}{2\mu^2} (\mu + \xi K)^3 \cdot |\Omega|$. This in conjunction with (3.14) and (3.15) leads to (3.13). \qed

In order to deal with the first integral term on the right of (3.13), we further derive the following energy inequality for $\int_{\Omega} |\nabla v|^4$.

**Lemma 3.6** Let $n \in \{2, 3\}$, $T \in (0, T_{\text{max}})$, $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume (1.4). Then there exists $C_2 > 0$ independent of $T$ such that the solution of (1.1)-(1.3) fulfills

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^4 + \frac{1}{2} \int_{\Omega} |\nabla u|^4 + \int_{\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial t} \leq 2 \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial t} + (n + 4) \int_{\Omega} u^2 |\nabla v|^2$$

(3.16)

for all $t \in (0, T)$.

**Proof.** The proof is based on straightforward computations using the second equation in (1.1), and it was actually proved in [17, the proof of Lemma 3.3; see (3.12)-(3.13) therein]. Thus, we prevent us from repeating the details here. \qed

**Corollary 3.7** Let $n = 2$, $T \in (0, T_{\text{max}})$, $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume (1.4). Then there exists $C_2 > 0$ independent of $T$ such that the solution of (1.1)-(1.3) carries the property

$$\frac{d}{dt} \left\{ \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 \right\} + \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla v|^2 \right\} \leq \frac{2}{\chi^2} \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial t} + (\chi^2 + 6) \int_{\Omega} u^2 |\nabla v|^2 + C_2 \quad \text{for all } t \in (0, T).$$

(3.17)
Proof. Adding (3.16) to (3.13) yields
\[
\frac{d}{dt} \left\{ \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 \right\} + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla |\nabla v|^2|^2 \leq 2 \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} + (\chi^2 + 6) \int_{\Omega} u^2 |\nabla v|^2 - \mu \int_{\Omega} u^3 + C_1
\]
for all \( t \in (0, T) \), where \( C_1 \) is provided by Lemma 3.5. Adding \( \int_{\Omega} u^2 \) to both sides of this and using the inequality
\[
\int_{\Omega} u^2 \leq \mu \int_{\Omega} u^3 + \frac{4}{2\mu^2} \cdot |\Omega|
\]
thanks to the Young inequality, we obtain (3.17) with \( C_2 := C_1 + \frac{4}{2\mu^2} \cdot |\Omega| \). \( \square \)

In the two-dimensional setting, we shall show that the two integrals on the right of (3.17) can be cancelled by \( \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla |\nabla v|^2|^2 \) on the left, which thereby results in a bound for \( \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 \).

Lemma 3.8 Let \( n = 2 \), \( T \in (0, T_{\text{max}}) \), \( \chi > 0 \), \( \xi > 0 \), and \( \mu > 0 \), and assume (1.4). Then there exists \( C > 0 \) independent of \( T \) such that the solution of (1.1)-(1.3) enjoys the property
\[
\int_{\Omega} u^2 \leq C \quad \text{for all } t \in (0, T) \quad \text{and}
\int_{\Omega} |\nabla v|^4 \leq C \quad \text{for all } t \in (0, T).
\]

Proof. Starting from (3.17), we first estimate \( \int_{\partial \Omega} |\nabla v|^2 \left| \frac{\partial |\nabla v|^2}{\partial \nu} \right| \). This boundary-related integral has been mainly studied in [2, (3.15)], and accordingly we have
\[
2 \int_{\partial \Omega} |\nabla v|^2 \left| \frac{\partial |\nabla v|^2}{\partial \nu} \right| \leq \frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + c_1 \quad \text{for all } t \in (0, T).
\]

We next deal with \( \int_{\Omega} u^2 |\nabla v|^2 \). For any \( \eta > 0 \), Young’s inequality yields
\[
(\chi^2 + 6) \int_{\Omega} u^2 |\nabla v|^2 \leq \eta \int_{\Omega} |\nabla v|^6 + \frac{(\chi^2 + 6)^2}{2\eta} \int_{\Omega} u^3 \quad \text{for all } t \in (0, T).
\]

Here we use the Gagliardo-Nirenberg inequality and (3.4) to estimate
\[
\eta \int_{\Omega} |\nabla v|^6 = \eta \| |\nabla v|^2 \|_{L^3(\Omega)}^3 \leq \eta c_2 \| |\nabla v|^2 \|_{L^4(\Omega)}^4 \| |\nabla v|^2 \|_{L^1(\Omega)} + c_2 \| |\nabla v|^2 \|_{L^4(\Omega)}^3 \leq \eta c_3 \| |\nabla v|^2 \|_{L^2(\Omega)}^2 + c_3 \quad \text{for all } t \in (0, T)
\]
and invoke Lemma 3.4 along with (3.3) and Lemma 2.4 to handle
\[
\frac{(\chi^2 + 6)^{\frac{3}{2}}}{\sqrt{\eta}} \int_{\Omega} u^3 = \frac{(\chi^2 + 6)^{\frac{3}{2}}}{\sqrt{\eta}} \|u\|^3_{L^3(\Omega)} \\
\leq \frac{(\chi^2 + 6)^{\frac{3}{2}}}{\sqrt{\eta}} \left[ \eta \|\nabla u\|^2_{L^2(\Omega)} \cdot \|u \ln u\|_{L^1(\Omega)} + c_4 \|u\|^3_{L^3(\Omega)} + c_5(\eta) \right] \\
\leq c_6 \sqrt{\eta} \|\nabla u\|^2_{L^2(\Omega)} + c_7(\eta) \text{ for all } t \in (0, T). \tag{3.23}
\]
Taking \(\eta > 0\) sufficiently small fulfilling \(\eta \leq \min\{\frac{1}{2c_3}, \frac{1}{c_6}\}\), from (3.21)-(3.23) we infer that
\[
(\chi^2 + 6) \int_{\Omega} u^2 |\nabla z|^2 \leq \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} \left| \nabla |\nabla v|^2 \right|^2 + c_8 \text{ for all } t \in (0, T). \tag{3.24}
\]
Thus, from (3.17), (3.20) and (3.24) we obtain that \(y(t) := \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4, t \in (0, T)\), satisfies the differential inequality
\[
y'(t) + y(t) \leq c_9 \tag{3.25}
\]
with \(c_9 := c_1 + c_8 + C_2\). Upon an ODE comparison, this entails
\[
y(t) \leq \max \left\{ y(0), c_9 \right\}, \text{ for all } t \in (0, T),
\]
which implies (3.18) and (3.19). \(\square\)

Lemma 3.8 results in the following useful corollary that will be used in the proof of Lemma 3.10 below.

**Corollary 3.9** Let \(n = 2, T \in (0, T_{\text{max}})\), \(\chi > 0\), \(\xi > 0\) and \(\mu > 0\), and assume (1.4). Then there exists \(C_3 > 0\) independent of \(T\) such that the solution of (1.1)-(1.3) possesses the property
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 \text{ for all } t \in (0, T). \tag{3.26}
\]
Moreover, for any \(4 \leq q < \infty\) there exists \(M_1(q) > 0\) such that the solution of (1.1) satisfies
\[
\int_{\Omega} |\nabla v|^q \leq M_1(q) \text{ for all } t \in (0, T). \tag{3.27}
\]

**Proof.** (3.19) in conjunction with Lemma 3.2 leads to
\[
\|v(\cdot, t)\|_{W^{1,4}(\Omega)} \leq c_1 \text{ for all } t \in (0, T).
\]
This, along with the Sobolev embedding \(W^{1,4}(\Omega) \hookrightarrow C^0(\bar{\Omega})\) thanks to \(4 > n = 2\), yields (3.26). As to (3.27), it immediately follows from (3.18) and the standard parabolic regularity theory (cf. [8 Lemma 4.1] or [10 Lemma 1]). \(\square\)
3.3 A bound of \(u\) in \(L^p(\Omega)\)

**Lemma 3.10** Let \(n = 2\), \(T \in (0, T_{\text{max}})\), \(\chi > 0\), \(\xi > 0\) and \(\mu > 0\), and assume (1.4). Then for any \(p > 2\) there exists \(C(p) > 0\) independent of \(T\) such that the solution of (1.1)-(1.3) fulfills

\[
\int_{\Omega} u^p \leq C(p) \quad \text{for all } t \in (0, T).
\]  

**Proof.** Starting from (2.11) once again and neglecting the nonnegative term \(\frac{p-1}{2} \int_{\Omega} u^{p-2} \vert \nabla u \vert^2\) on the left, we arrive at

\[
\frac{d}{dt} \int_{\Omega} u^p \leq \frac{p(p-1)}{2} \int_{\Omega} u^p \vert \nabla v \vert^2 + p \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^p v + p(\mu + \xi K) \int_{\Omega} u^p - p \mu \int_{\Omega} u^{p+1}
\]

for any \(p > 2\) and each \(t \in (0, T)\). Adding \(\int_{\Omega} u^p\) to both sides of this and using (3.26) we see that

\[
\frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p \leq \frac{p(p-1)}{2} \int_{\Omega} u^p \vert \nabla v \vert^2 + c_1 \int_{\Omega} u^p - p \mu \int_{\Omega} u^{p+1}
\]

(3.29)

for any \(p > 2\) and each \(t \in (0, T)\), where \(c_1 := 1 + p(\xi \|w_0\|_{L^\infty(\Omega)} \cdot C_3 + \mu + \xi K)\). Here we invoke the Young inequality and (3.27) to estimate

\[
\frac{p(p-1)}{2} \int_{\Omega} u^p \vert \nabla v \vert^2 \leq \frac{p \mu}{2} \int_{\Omega} u^{p+1} + c_2 \int_{\Omega} \vert \nabla v \vert^{2(p+1)}
\]

\[
\leq \frac{p \mu}{2} \int_{\Omega} u^{p+1} + c_2 \cdot M_1(2p + 2)
\]

(3.30)

with some \(c_2 > 0\) and \(M_1(\cdot)\) defined by Corollary 3.9. Similarly, we have

\[
c_1 \int_{\Omega} u^p \leq \frac{p \mu}{2} \int_{\Omega} u^{p+1} + c_3
\]

(3.31)

with some \(c_3 > 0\). Collecting (3.29)-(3.31) yields \(y(t) := \int_{\Omega} u^p, t \in (0, T)\), satisfies the differential inequality

\[
y(t) + y(t) \leq c_4
\]

where \(c_4 := c_2 \cdot M_1(2p + 2) + c_3\). Upon an ODE comparison, this yields

\[
y(t) \leq \max \left\{ y(0), c_4 \right\} \quad \text{for all } t \in (0, T),
\]

which leads to (3.30). \(\square\)

**Corollary 3.11** Let \(n = 2\), \(T \in (0, T_{\text{max}})\), \(\chi > 0\), \(\xi > 0\) and \(\mu > 0\), and assume (1.4). Then there exists \(C > 0\) independent of \(T\) such that the solution of (1.1)-(1.3) possesses the property

\[
\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \quad \text{for all } t \in (0, T).
\]

**Proof.** (3.32) is a direct consequence of (3.28) for a fixed \(p > 2\) and the standard parabolic regularity theory (cf. [8] Lemma 4.1 or [10] Lemma 1). \(\square\)
4 Boundedness. Proof of Theorem 1.1

Although (3.32) shows that $\nabla v(\cdot, t)$ is bounded in $L^\infty(\Omega)$, $\nabla w(\cdot, t)$ might become unbounded in $L^\infty(\Omega)$ in light of (2.4). Therefore, we cannot directly apply the result of the well-known Moser-Alikakos iteration \cite{Alikakos1985} to the first equation in (1.1) to gain the boundedness of $u(\cdot, t)$ in $L^\infty(\Omega)$. To bypass $w$, our strategy is to use (2.11) as a starting point for our proof.

**Lemma 4.1** Let $n = 2$, $T \in (0, T_{\text{max}})$, $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume (1.4). Then there exists $C > 0$ independent of $T$ such that the solution of (1.1)-(1.3) satisfies

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T).$$

**(Proof.** We begin with (2.11)

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{p-1}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 \leq \frac{(p-1)\chi^2}{2} \int_{\Omega} u^p |\nabla v|^2 + \xi \|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^p v$$

$$+ (\mu + \xi K) \int_{\Omega} u^p - \mu \int_{\Omega} u^{p+1}$$

for any $p > 1$ and each $t \in (0, T)$. Adding $\int_{\Omega} u^p$ to both sides of this and invoking (3.32), we obtain

$$\frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p + \int_{\Omega} |\nabla u^\frac{p}{2}|^2 \leq c_1 p^2 \int_{\Omega} u^p$$

(4.2)

for any $p \geq 2$ and each $t \in (0, T)$, where $c_1 > 0$, as all subsequently appearing constants $c_2, c_3, \cdots$ are independent of $T$ as well as of $p \geq 2$. We now use the Gagliardo-Nirenberg inequality to deal with the last integral

$$\int_{\Omega} u^p = \|u^\frac{p}{2}\|_{L^2(\Omega)}^2 \leq c_2 \|\nabla u^\frac{p}{2}\|_{L^2(\Omega)} \cdot \|u^\frac{p}{2}\|_{L^1(\Omega)} + \|u^\frac{p}{2}\|_{L^1(\Omega)}^2$$

for any $p \geq 2$ and each $t \in (0, T)$. By Young’s inequality, this yields

$$c_1 p^2 \int_{\Omega} u^p \leq \int_{\Omega} |\nabla u^\frac{p}{2}|^2 + c_3 p^4 \|u^\frac{p}{2}\|_{L^1(\Omega)}^2$$

$$= \int_{\Omega} |\nabla u^\frac{p}{2}|^2 + c_3 p^4 \left( \int_{\Omega} u^\frac{p}{2} \right)^2 \quad \text{for all } t \in (0, T).$$

Hence, (4.2) entails that

$$\frac{d}{dt} \int_{\Omega} u^p + \int_{\Omega} u^p \leq c_3 p^4 \left( \int_{\Omega} u^\frac{p}{2} \right)^2 \quad \text{for all } t \in (0, T).$$

Upon integration, this shows that

$$\int_{\Omega} u^p \leq \int_{\Omega} u^p_0 + c_3 p^4 \int_0^t e^{-(t-\tau)} \left( \int_{\Omega} u^\frac{p}{2}(\cdot, \tau) \right)^2 d\tau.$$
Writing \( p_k := 2^k \) and

\[
B_k := \max_{t \in (0,T)} \int_\Omega u^{p_k}(\cdot, t)
\]

for \( k \in \{1, 2, \cdots\} \), we see that

\[
B_k \leq |\Omega| \cdot \|u_0\|_{L^\infty(\Omega)}^{p_k} + c_3 p_k^4 B_{k-1}^2 \quad \text{for all } k \geq 1,
\]

where we have used the simple fact that \( \int_0^t e^s ds \leq 1 \). Now if \( p_k^4 B_{k-1}^2 \leq \|u_0\|_{L^\infty(\Omega)}^{p_k} \) for infinitely many \( k \geq 1 \), we have

\[
\sup_{t \in (0,T)} \left( \int_\Omega u^{p_k-1}(\cdot, t) \right)^{\frac{1}{p_k-1}} \leq \left( \frac{\|u_0\|_{L^\infty(\Omega)}^{p_k}}{p_k^4} \right)^{\frac{1}{p_k-1}} = \frac{\|u_0\|_{L^\infty(\Omega)}}{p_k^4} \rightarrow \|u_0\|_{L^\infty(\Omega)} \quad \text{as } k \to \infty,
\]

which implies that

\[
\sup_{t \in (0,T)} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}
\]

and thereby proves the lemma in this case.

Conversely, if \( p_k^4 B_{k-1}^2 > \|u_0\|_{L^\infty(\Omega)}^{p_k} \) for all sufficiently large \( k \), then (4.3) yields some \( c_4 > 0 \) such that

\[
B_k \leq c_4 p_k^4 B_{k-1}^2 \quad \text{for all } k \geq 1.
\]

In view of the definition of \( p_k \), this implies

\[
B_k \leq c_4 (16)^k B_{k-1}^2 \leq a^k B_{k-1}^2 \quad \text{for all } k \geq 1
\]

with \( a := (\max\{c_4, 16\})^2 \). Thus, by induction we obtain

\[
B_k \leq a^{k+\sum_{j=1}^{k-1} 2^j(k-j)} \cdot B_0^2 \quad \text{for all } k \geq 1.
\]

(4.4)

Here we observe that

\[
k + \sum_{j=1}^{k-1} 2^j(k-j) = 2 + 2^2 + \cdots + 2^k - k \leq 2^{k+1} \quad \text{for all } k \geq 1.
\]

From this and (4.4) we infer

\[
\frac{1}{B_k^{p_k}} \leq a B_0 \cdot a^{p_k} \quad \text{for all } k \geq 1,
\]

which after taking \( k \to \infty \) implies that

\[
\sup_{t \in (0,T)} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq a B_0
\]

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and thereby yields the assertion in this case. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The statement of global classical solvability and boundedness is a straightforward consequence of Lemma 2.1 and Lemma 4.1. □

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