Spontaneous Magnetization in Maxwell QED$_{2+1}$

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Abstract

Spontaneous magnetization in the Maxwell QED$_{2+1}$ at finite fermion density is studied. It is shown that at low fermion densities the one-loop free energy has its minimum at some nonvanishing value of the magnetic field. The magnetization is due to the asymmetry of the fermion spectrum of the massive QED$_{2+1}$ in an external magnetic field.

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In (2+1)-dimensional quantum electrodynamics (QED\textsubscript{2+1}) an addition of the Chern-Simons term $\frac{\theta}{4} \varepsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha$ to the bare Lagrangian drastically modifies the theory [1]: the gauge field becomes massive and, due to the presence of the totally antisymmetric tensor $\varepsilon_{\mu\nu\alpha}$, the electric and magnetic components of the modified Maxwell equation are mixed up. As a consequence, in a static uniform magnetic field $B$ the electric charge $j_0^{\text{e}} = \theta B$ associated with the Chern-Simons term is induced. On the other hand, in a uniform magnetic field charge connected to fermions, $j_0^{\text{f}} = \frac{eB}{4\pi}(2N + 1)$ is induced [2, 3] ($N$ is the number of filled Landau levels), thus the electric neutrality condition in Maxwell-Chern-Simons QED\textsubscript{2+1} is $j_0^{\text{f}} + j_0^{\text{cs}} = 0$. Recently Hosotani has shown that in QED\textsubscript{2+1}, with the Chern-Simons term in the bare Lagrangian a (self-consistent) neutral configuration with a uniform magnetic field has the energy lesser than that of the naive vacuum [4]. Since the condition $j_0^{\text{f}} + j_0^{\text{cs}} = 0$ also implies masslessness of the gauge field, the spontaneous magnetization in the neutral Maxwell-Chern-Simons QED\textsubscript{2+1} has Nambu-Goldstone origin [5-8].

In this talk we shall discuss the possibility of the spontaneous magnetization in the Maxwell QED\textsubscript{2+1} with a finite fermion density. We shall consider QED\textsubscript{2+1} with two-component massive fermions with the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\partial^\mu + eA^\mu - m) \psi ,$$

(1)

$\gamma$-matrices are the Pauli matrices, $\gamma^0 = \sigma_3, \gamma^1 = i\sigma_1, \gamma^2 = \sigma_2$. Let us remind that with a uniform external magnetic field $B = \partial_1 A_2 - \partial_2 A_1$ the fermion energy spectrum (Landau levels) in this theory is the following [9, 10]:

$$p_0^{(0)} = -m \text{ sign}(eB), \quad p_0^{(\pm n)} = \pm \sqrt{m^2 + 2\vert eB \vert n}, \quad n = 1, 2, \ldots$$

(2)

(note the asymmetry of the spectrum. The degeneracy of all levels is $\frac{|eB|}{2\pi}$).

Addition of the term $-\mu \psi^\dagger \psi$ to the Lagrangian (1) provides the parameter controlling the fermion density ($\mu$ is the chemical potential, therefore all levels with energies up to $\mu$ are filled). One may calculate the fermion density (induced charge) in QED\textsubscript{2+1} with $B, \mu \neq 0$ using either the corresponding Green function [3] or the spectral properties of the theory [11, 12]. The latter is most straightforward way: here, in case of discrete and equally degenerated levels the density is

$$\rho = \frac{|eB|}{2\pi} \left\{ -\frac{1}{2} \sum_n \text{ sign}(p_0^{(n)}) + \sum_n \left[ \theta(p_0^{(n)}) \theta(\mu - p_0^{(n)}) - \theta(-p_0^{(n)}) \theta(p_0^{(n)} - \mu) \right] \right\} ,$$

(3)

or,

$$\rho(B, \mu) = \frac{eB}{4\pi} + \left\{ \begin{array}{ll}
\frac{|eB|}{2\pi} \left( \frac{\mu^2 - m^2}{2|eB|} \right) + \theta(-eB), & \mu > m; \\
0, & |\mu| < m; \\
-\frac{|eB|}{2\pi} \left( \frac{\mu^2 - m^2}{2|eB|} \right) + \theta(eB), & \mu < -m
\end{array} \right.$$
([... denotes the integral part). The asymmetry of the density as the function of \( \mu \) is the consequence of the asymmetry of the fermion spectrum, \( \left[ \frac{\mu^2 - m^2}{2|eB|} \right] \) or \( \left( \frac{\mu^2 - m^2}{2|eB|} \right) + 1 \) describes the number of filled Landau levels. Unlike QED_{3+1} [13] the magnitude of the magnetic field and the number of filled Landau levels do not define the chemical potential unambiguously – at a fixed \( B \) the fermion density is a step-function of the chemical potential, Fig. 1.

It follows from Eq.(4) that the equation \( \rho(B, \mu) = \text{const} \) has an infinite number of solutions enumerated by the filled Landau levels number \( [12] \). For instance, at \( \rho = \text{const}, \mu, B > 0 \) the solutions are:

\[
e B = \frac{4\pi \rho}{2N + 1}, \quad N = 0, 1, 2, \ldots
\]  

(5)

In \((B, \mu)\)-plane these solutions are intervals parallel to the \( \mu \)-axis, Fig. 2 (at the vanishing magnetic field the fermion density is \( \rho(\mu) = \frac{1}{4\pi} \left( \mu^3 - m^3 \right) \theta(\mu^2 - m^2) \sign(\mu) \)).

One may minimize the energy of the above-mentioned configurations of the equal fermion density varying magnetic field. The energy density is \( \mathcal{E} = \mu \rho - \mathcal{L}^{\text{eff}}(B, \mu) = \mu \rho + \frac{B^2}{2} - \mathcal{L}^{\text{eff}}(B) - \tilde{\mathcal{L}}^{\text{eff}}(B, \mu) \). The one-loop effective Lagrangian \( \mathcal{L}^{\text{eff}}(B) \) was calculated in Ref. [14]:

\[
\mathcal{L}^{\text{eff}}(B) = \frac{1}{8\pi^{3/2}} \int_0^\infty \frac{ds}{s^{5/2}} e^{-m^2 s} (eBs \coth(eBs) - 1),
\]

(6)

while the \( \mu \)-dependent contribution \( \tilde{\mathcal{L}}^{\text{eff}}(B, \mu) = \int_0^{\mu} \rho(B, \mu') d\mu' \) may be easily obtained from Eq.(4).

We shall start with \( \mu, B > 0 \) case and take \( eB \ll m^2 \). The energy density \( \mathcal{E}_N \) of the configuration with specific \( N \) is:

\[
\mathcal{E}_N = \frac{B^2}{2} \left( 1 - \frac{e^2}{12\pi m} \right) + \frac{eB}{2\pi} \sum_{n=1}^{N} \sqrt{m^2 + 2eBn}
\]

(7)

(the second term in the right-hand side of Eq.(7) is just the sum of the energies of the filled Landau levels \( \times \) degeneracy).

The energy density \( \mathcal{E}_N \) may be rewritten in terms of the fermion density,

\[
\mathcal{E}_N = \frac{8\pi^2 \rho^2}{e^2(2N + 1)^2} \left( 1 - \frac{e^2}{12\pi m} \right) + \frac{2\rho}{2N + 1} \sum_{n=1}^{N} \sqrt{m^2 + \frac{8\pi \rho n}{2N + 1}}
\]

(8)

or, by introducing dimensionless variables \( \tilde{\mathcal{E}}_N = \mathcal{E}_N/m^3, \tilde{\rho} = \rho/m^2 \) and \( \alpha = e^2/8\pi m \),

\[
\tilde{\mathcal{E}}_N = \frac{\tilde{\rho}^2}{\alpha(2N + 1)^2} \left( 1 - \frac{2}{3\alpha} \right) + \frac{2\tilde{\rho}}{2N + 1} \sum_{n=1}^{N} \sqrt{1 + \frac{8\pi \tilde{\rho} n}{2N + 1}}
\]

(9)

To analyze the possibility of the spontaneous magnetization one must compare the energy \( \mathcal{E}_N \) of the configuration with \( N \) filled Landau levels with the energy \( \mathcal{E}_\mu \) of the configuration of the same density but zero magnetic field. If for some \( N \), one has \( \mathcal{E}_N < \mathcal{E}_\mu \), then the spontaneous magnetization may take place. The energy density \( \mathcal{E}_\mu \) is equal to \( \mathcal{E}_\mu = \frac{1}{6\pi} (\mu^3 - m^3) \theta(\mu^2 - m^2) \), or \( \tilde{\mathcal{E}}_\mu = \frac{1}{6\pi} ((1 + 4\pi \tilde{\rho})^{3/2} - 1) \), thus the condition of magnetization is as follows:
\[
\frac{1}{6\pi} \left( (1 + 4\pi\bar{\rho})^{3/2} - 1 \right) > \frac{\pi\bar{\rho}^2}{\alpha(2N + 1)^2} \left( 1 - \frac{2}{3}\alpha \right) + \frac{2\bar{\rho}}{2N+1} \sum_{n=1}^{N} \sqrt{1 + \frac{8\pi\bar{\rho}n}{2N+1}} .
\]  

(10)

Below we shall suppose \( \bar{\rho} \) to be small, \( \bar{\rho} \ll 1 \). After making an expansion in powers of \( \bar{\rho} \) in the inequality (10) one has

\[
\frac{1}{6\pi} \left( \frac{3}{2} 4\pi\bar{\rho} + \frac{3}{8}(4\pi\bar{\rho})^2 \right) > \frac{\pi\bar{\rho}^2}{\alpha(2N + 1)^2} \left( 1 - \frac{2}{3}\alpha \right) + \frac{2\bar{\rho}}{2N+1} \sum_{n=1}^{N} \left( 1 + \frac{4\pi\bar{\rho}n}{2N+1} \right) + O(\bar{\rho}^3) .
\]  

(11)

By collecting similar terms, we finally obtain:

\[
\frac{\bar{\rho}}{2N+1} > \frac{\pi\bar{\rho}^2}{\alpha(2N + 1)^2} - \frac{5}{3} \frac{\pi\bar{\rho}^2}{(2N + 1)^2} + O(\bar{\rho}^3) ,
\]  

(12)

i.e. inequality (10) does have solutions for \( \bar{\rho} < \frac{\alpha}{\pi}(2N + 1) \).

Thus we have shown that the spontaneous magnetization may take place in the Maxwell QED\(_{2+1}\) at low fermion density: at \( \bar{\rho} < \frac{\alpha}{\pi}(2N + 1) \ll 1 \) the energy is minimum when \( \left[ \frac{\pi\rho}{\alpha} \right] + 1 \) Landau levels are filled. For example, at \( \rho < \frac{e^2m}{16\pi} \) the induced magnetic field is

\[
B = \frac{4\pi\bar{\rho}}{e} .
\]

Let us check now whether magnetization is possible with \( \mu > 0, B < 0 \). In this case the fermion density is \( \rho = \frac{|eB|}{4\pi}(2N + 1), N = 1, 2, \ldots, \)

\[
\mathcal{E}_N = \frac{8\pi^2\rho^2}{e^2(2N + 1)^2} \left( 1 - \frac{e^2}{12\pi m} \right) + \frac{2\rho}{2N+1} \sum_{n=0}^{N} \sqrt{m^2 + \frac{8\pi\rho n}{2N+1}}
\]  

(14)

and the magnetization condition \( \mathcal{E}_\mu > \mathcal{E}_N \) is equivalent to

\[
\bar{\rho} + \pi\bar{\rho}^2 > \frac{\pi\bar{\rho}^2}{\alpha(2N + 1)^2} + \frac{2N + 2}{2N+1}\bar{\rho} + \frac{4(N + 1)(N + 2)}{(2N + 1)^2} \pi\bar{\rho}^2 + O(\bar{\rho}^3) ,
\]  

(15)

which has no solutions.

Therefore, for \( \mu > 0 \), i.e. for finite particle density the spontaneous magnetization arises with a definite sign of magnetic field \( (B > 0) \). For \( \mu < 0 \), i.e. for finite antiparticle density the field has the opposite sign \( (B < 0) \).

To better understand the mechanism of the magnetization it is instructive to consider (2+1)-dimensional field theory possessing symmetrical fermion spectrum in a uniform magnetic field, e.g. QED\(_{2+1}\) with four-component spinors. Calculating \( \mathcal{E}_\mu \) and \( \mathcal{E}_N \) in this model one may see that the energy of magnetized system is always greater than that with \( B = 0 \) (as well as in QED\(_{3+1}\)).

The examples described above indicate that spontaneous magnetization in the finite fermion density QED\(_{2+1}\) is due to the asymmetry of the fermion spectrum in a magnetic field. The latter is of topological nature (see Ref. \[1\] and references therein). Therefore, the above arguments for magnetization based on the one-loop calculations will hold true in all orders of the perturbation theory.
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Figure 1: Fermion density as a function of chemical potential, $B=\text{const.}$
Figure 2: Solutions of the equation $\rho(B, \mu) = \rho$ in $(B, \mu^2 - m^2)$–plane, $eB_0 = 4\pi\rho$