A survey on spectral conditions for some extremal graph problems

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Abstract

This survey is two-fold. We first report new progress on the spectral extremal results on the Turán type problems in graph theory. More precisely, we shall summarize the spectral Turán function in terms of the adjacency spectral radius and the signless Laplacian spectral radius for various graphs. For instance, the complete graphs, general graphs with chromatic number at least three, complete bipartite graphs, odd cycles, even cycles, color-critical graphs and intersecting triangles.

The second goal is to conclude some recent results of the spectral conditions on some graphical properties. By a unified method, we present some sufficient conditions based on the adjacency spectral radius and the signless Laplacian spectral radius for a graph to be Hamiltonian, k-Hamiltonian, k-edge-Hamiltonian, traceable, k-path-coverable, k-connected, k-edge-connected, Hamilton-connected, perfect matching and $\beta$-deficient.

Key words: extremal graph theory; spectral radius; Turán theorem; Hamilton cycle; connectivity; matching number.

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Introduction

We only consider simple graphs throughout paper. The notations we used are standard from the monograph written by Bondy and Murty \[24, 25\]. Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ such that $|V(G)| = v(G)$ and $|E(G)| = e(G)$. We usually write $n$ and $m$ for the number of vertices and edges respectively. We use $d(v)$ to denote the degree of the vertex $v$ in $G$. The minimum degree is denoted by $\delta(G)$. We write $K_n$ and $I_n$ for the complete graph and empty graph on $n$ vertices respectively. Let $K_{s,t}$ be the complete bipartite graph with parts of sizes $s$ and $t$. We write $C_t$ for the cycle graph on $t$ vertices. Let $\omega(G)$ denote the clique number, which is defined as the number of vertices in a largest complete subgraph in $G$. The vertex-chromatic number $\chi(G)$ of $G$ is the minimum integer $s \in \mathbb{N}^*$ such that there exists a coloring of $V(G)$ with $s$ colors and the adjacent vertices have different colors. For two vertex-disjoint graphs $G$
and \( H \), \( G \cup H \) denotes the disjoint union of \( G \) and \( H \); \( G \vee H \) denotes the join of \( G \) and \( H \), which is obtained from \( G \cup H \) by adding all possible edges between \( G \) and \( H \). For instance, we have \( K_{s,t} = I_s \vee I_t \) and \( K_n = K_t \vee K_{n-t} \). We use the symbol \( i \sim j \) to denote the vertices \( i \) and \( j \) are adjacent, and \( i \not\sim j \) otherwise.

The Turán number of a graph \( F \) is the maximum number of edges in an \( n \)-vertex graph without a subgraph isomorphic to \( F \), and it is usually denoted by \( \text{ex}(n, F) \). We say that a graph \( G \) is \( F \)-free if it does not contain an isomorphic copy of \( F \) as a subgraph. A graph on \( n \) vertices with no subgraph \( F \) and with \( \text{ex}(n, F) \) edges is called an extremal graph for \( F \) and we denote by \( \text{Ex}(n, F) \) the set of all extremal graphs on \( n \) vertices for \( F \). It is a cornerstone of extremal graph theory to understand \( \text{ex}(n, F) \) and \( \text{Ex}(n, F) \) for various graphs \( F \); see [84, 103, 170] for surveys.

**Question 1.** (Extremal graph problem) **What is the maximum number of edges of a graph \( G \) on \( n \) vertices without a subgraph isomorphic to a given graph \( F \)?**

In 1941, Turán [176] posed the natural question of determining \( \text{ex}(n, K_{r+1}) \) for \( r \geq 2 \). Let \( T_r(n) \) denote the complete \( r \)-partite graph on \( n \) vertices where its part sizes are as equal as possible. In other words, \( T_r(n) = K_{t_1, t_2, \ldots, t_r} \), the complete \( r \)-partite graph on vertex classes with sizes \( t_1, t_2, \ldots, t_r \), where \( \sum_{i=1}^{r} t_i = n \) and \( |t_i - t_j| \leq 1 \) for all \( i \neq j \). Turán [176] (see [15, p. 294]) extended a result of Mantel [141] and obtained that if \( G \) is an \( n \)-vertex graph containing no \( K_{r+1} \), then \( e(G) \leq e(T_r(n)) \), equality holds if and only if \( G = T_r(n) \). There are many extensions and generalizations on Turán’s result. The problem of determining \( \text{ex}(n, F) \) is usually called the Turán-type extremal problem. The most celebrated extension always attributes to a result of Erdős, Stone and Simonovits [60, 66], which states that

\[
\text{ex}(n, F) = \left( 1 - \frac{1}{\chi(F) - 1} + o(1) \right) \frac{n^2}{2},
\]

where \( \chi(F) \) is the vertex-chromatic number of \( F \). This provides good asymptotic estimates for the extremal numbers of non-bipartite graphs. However, for bipartite graphs, where \( \chi(F) = 2 \), it only gives the bound \( \text{ex}(n, F) = o(n^2) \). Although there have been numerous attempts on finding better bounds of \( \text{ex}(n, F) \) for various bipartite graphs \( F \), we know very little in this case. The history of such a case began in 1954 with the theorem of Kővari, Sós and Turán [106], which states that if \( K_{s,t} \) is the complete bipartite graph with vertex classes of size \( s \geq t \), then \( \text{ex}(n, K_{s,t}) = O(n^{2-1/t}) \); see [80, 81] for more details. In particular, we refer the interested reader to the comprehensive survey [84].

Let \( G \) be a simple graph on \( n \) vertices. The **adjacency matrix** of \( G \) is defined as \( A(G) = [a_{ij}]_{n \times n} \) where \( a_{ij} = 1 \) if two vertices \( v_i \) and \( v_j \) are adjacent in \( G \), and \( a_{ij} = 0 \) otherwise. We say that \( G \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \) if these values are eigenvalues of the adjacency matrix \( A(G) \). Let \( \lambda(G) \) be the maximum value in absolute among the eigenvalues of \( G \), which is known as the spectral radius of graph \( G \), that is, \( \lambda(G) = \max\{ |\lambda| : \lambda \text{ is an eigenvalue of } G \} \). By the Perron–Frobenius Theorem [95, p. 534], the spectral radius of a graph \( G \) is actually the largest eigenvalue of \( G \) since the adjacency matrix \( A(G) \) is nonnegative. We usually write \( \lambda_1(G) \) for the spectral radius of \( G \). The spectral radius of a graph sometimes can give some information about the structure of graphs. For example, it
is well-known that the average degree of $G$ is at most $\lambda(G)$, which is at most the maximum degree of $G$. The diagonal matrix of $G$ is $D(G) = [d_{ii}]_{n \times n}$ with diagonal entry $d_{ii} = d(i)$, the degree of vertex $i$. The signless Laplacian matrix is defined as $Q(G) = D(G) + A(G)$. The largest eigenvalue of $Q(G)$, denoted by $q(G)$, is called the $Q$-index or the signless Laplacian spectral radius of $G$. For people that working on spectral graph theory, one of the most well-known problems is the Brualdi–Solheid problem [29], which states that

**Question 2.** (Brualdi–Solheid problem) Given a set $\mathcal{G}$ of graphs, find a tight upper bound for the spectral radius in $\mathcal{G}$ and characterize the extremal graphs.

This problem is well studied in the literature for many classes of graphs, such as graphs with cut vertices [11], given diameter [92], edge chromatic number [73], domination number [173]. For the $Q$-index counterpart of the above problem, Zhang [203] gave the $Q$-index of graphs with given degree sequence, Zhou [206] studied the $Q$-index and Hamiltonicity. Also, from both theoretical and practical viewpoint, the eigenvalues of graphs have been successfully used in many other disciplines, one may refer to [127, 200, 201].

In this paper we shall consider spectral analogues of Turán-type problems for graphs. That is, determining $\text{ex}_\lambda(n, F) = \max\{\lambda(G) : |G| = n, F \not\subseteq G\}$. It is well-known that

$$
\text{ex}(n, F) \leq \frac{n}{2} \text{ex}_\lambda(n, F)
$$

because of the fundamental inequality $2m/n \leq \lambda(G)$. For most graphs, this study is again fairly complete due in large part to a longstanding work of Nikiforov [156]. The Brualdi–Solheid problem asks to determine the maximum spectral radius of a graph $G$ on $n$ vertices belonging to a specified class of graphs. Analogous to the Brualdi–Solheid problem, the following question regarding the adjacency spectral radius is a natural extension.

**Question 3.** (Spectral extremal problem) What is the maximum spectral radius of a graph $G$ on $n$ vertices without a subgraph isomorphic to a given graph $F$?

This problem regarding the adjacency spectral radius was early proposed in [144]. Wilf [183] and Nikiforov [144] obtained spectral strengthening of Turán’s theorem when the forbidden substructure is the complete graph. Soon after, Nikiforov [147] showed that if $G$ is a $K_{r+1}$-free graph on $n$ vertices, then $\lambda(G) \leq \lambda(T_r(n))$, equality holds if and only if $G = T_r(n)$. Moreover, Nikiforov [147] (when $n$ is odd), and Zhai and Wang [196] (when $n$ is even) determined the maximum spectral radius of $K_{2,2}$-free graphs. Furthermore, Nikiforov [154], Babai and Guiduli [7] independently obtained the spectral generalization of the theorem of Kővari, Sós and Turán when the forbidden graph is the complete bipartite graph $K_{s,t}$. Finally, Nikiforov [155] characterized the spectral radius of graphs without paths and cycles of specified length. In addition, Fiedler and Nikiforov [77] obtained tight sufficient conditions for graphs to be Hamiltonian or traceable. For many other spectral analogues of results in extremal graph theory we refer the reader to the survey [156]. It is worth mentioning that a corresponding spectral extension [151] of the theorem of Erdős, Stone and Simonovits [60, 66] states that

$$
\text{ex}_\lambda(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right)n.
$$
In the following sections, we shall survey a large number of extremal graph results and its corresponding spectral extremal results in some details.

1 Extremal spectral problem for Turán type

1.1 Spectral problem for complete graphs

The first theorem of extremal graph theory is commonly regarded as a theorem of Mantel [141], which shows that the extremal triangle-free graphs are balanced complete bipartite graphs. There are many proofs of this result; see, e.g., [15, 168, 204].

**Theorem 1.1** (Mantel [141]). If $G$ is an $n$-vertex graph with no triangle, then

$$e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor,$$

equality holds if and only if $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

A natural extension of Mantel’s theorem is the famous Turán theorem, which extended the forbidden subgraph $K_3$ to large complete subgraph $K_{r+1}$ for every integer $r \geq 2$. The Turán graph $T_r(n)$ is an $n$-vertex complete $r$-partite graph with each part of size $n_i$ such that $|n_i - n_j| \leq 1$ for all $1 \leq i, j \leq r$, that is, $n_i$ equals $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. Sometimes, we call $T_r(n)$ the balanced complete $r$-partite graph and denote the number of its edges by $t_r(n)$.

Clearly, we can verify that

$$e(T_r(n)) = \sum_{0 \leq i < j \leq r-1} \left\lfloor \frac{n+i}{r} \right\rfloor \left\lfloor \frac{n+j}{r} \right\rfloor.$$

Moreover, if we write $n = r \cdot \lfloor \frac{n}{r} \rfloor + s$ for some $0 \leq s < r$, then

$$e(T_r(n)) = \left(1 - \frac{1}{r}\right) \frac{(n^2 - s^2)}{2} + \binom{s}{2}.$$

In particular, we have $T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ and $t_2(n) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$.

**Theorem 1.2** (Turán [176], weak version). If $G$ is a graph on $n$ vertices contains no copy of the complete graph $K_{r+1}$, then

$$e(G) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Furthermore, equality holds if and only if $r$ divides $n$ and $G = T_r(n)$.

By an easy calculation, we can see that $e(T_r(n)) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$, equality holds if and only if $r$ divides $n$. This inequality also can be seen from Theorem 1.2 since $T_r(n)$ does not contain $K_{r+1}$ as a subgraph. More generally, one can prove the strong version of the Turán theorem in a similar way.
Theorem 1.3 (Turán [176], strong version). If $G$ is a graph on $n$ vertices with no copy of $K_{r+1}$, then
\[ e(G) \leq e(T_r(n)), \]
equality holds if and only if $G$ is the Turán graph $T_r(n)$.

Turán’s theorem initiated the rapid development of extremal graph theory and was re-discovered many times with various different proofs; see, e.g., [3, Chapter 40]. In the sequel, we shall present an elegant proof of the weaker version of Turán’s theorem. This proof was proposed by Motzkin and Straus [142] based on applying the optimization method, which is now known as the Lagrangian method; see 174, 175 for more generalizations, and we recommend highly a recent survey written by Yuejian Peng, Hypergraph Lagrangian Function (click). It is worth noting that the theorem of Motzkin and Straus was also applied to the spectral version of Turán’s theorem; see, e.g., [144, 145, 150] for related results.

Definition 1.4. [142] Let $G$ be a graph on vertex set $[n]$. We associate a function
\[ \lambda(G, x) := \sum_{\{i,j\} \in E(G)} x_i x_j, \quad (\forall x \in \mathbb{R}^n). \]
We denote $\Delta := \{x : x_1, \ldots, x_n \geq 0, \sum_{i=1}^n x_i = 1\}$. The Lagrangian of $G$ is defined as the maximum of $\lambda(G, x)$ over all $x \in \Delta$, that is,
\[ \lambda(G) := \max \{ \lambda(G, x) : x \in \Delta \}. \]

We remark that the notation $\lambda(G)$ is frequently denoted by the spectral radius of $G$ beyond this section. Recall that $\omega(G)$ is the number of vertices in a largest complete subgraph of $G$, which is called the clique number of $G$. In 1965, Motzkin and Straus [142] published a new proof of the Turán theorem by proving the following result.

Theorem 1.5 (Motzkin–Straus [142]). For any graph $G$, we have
\[ \lambda(G) = \frac{1}{2} \left( 1 - \frac{1}{\omega(G)} \right). \]

Proof. First of all, we write $\omega(G) = \omega$ for short. Let $K_\omega$ be the largest clique in $G$. We define the vector $w$ by setting $w_i = 1/\omega$ for each $i \in V(K_\omega)$ and $w_i = 0$ for $i \in V \setminus V(K_\omega)$. Clearly, we have $w$ is nonnegative and $\sum_{i=1}^n w_i = 1$. Here, we say that a vector is nonnegative if all of its coordinates are nonnegative. Hence, we can get
\[ \lambda(G) \geq \lambda(G, w) = \sum_{\{i,j\} \in E(K_\omega)} \frac{1}{\omega^2} = \frac{1}{2} \left( 1 - \frac{1}{\omega} \right). \]

There may exist many nonnegative vectors $x$ with $\sum_{i=1}^n x_i = 1$ satisfying $\lambda(G, x) = \lambda(G)$, we further select a vector $x$ among these vectors such that the cardinality of its supporting set $|\text{Supp}(x)| := \# \{i \in [n] : x_i > 0\}$ is minimal. We shall complete the proof by the following two steps.
Step 1. We first show that the vertex set \( \text{Supp}(x) \) forms a clique in \( G \).

If \( \text{Supp}(x) \) does not form a clique in \( G \), we may suppose that \( \{i, j\} \notin E(G) \) for some \( i, j \in \text{Supp}(x) \). Denoting \( s_v := \sum_{u \sim v} x_u \) for each vertex \( v \in V(G) \). We may assume without loss of generality that \( s_i \geq s_j \). Now we move the weight \( x_j \) from vertex \( j \) to vertex \( i \), that is, the new weight of vertex \( i \) is \( x_i + x_j \), while the weight of vertex \( j \) drops to 0. In other words, we construct a new nonnegative vector \( y \) such that \( y_i = x_i + x_j, y_j = 0 \) and \( y_k = x_k \) for \( k \in V \setminus \{i, j\} \). We compute

\[
f_G(y) - f_G(x) = y_is_i + y_js_j - x_is_i - x_js_j = x_js_i - x_js_j \geq 0.
\]

Therefore, \( y \) also satisfies \( \lambda(G, y) = \lambda(G) \) and \( |\text{Supp}(y)| < |\text{Supp}(x)| \), which contradicts the minimality of the supporting set of \( x \).

Step 2. According to Step 1, the clique formed by \( \text{Supp}(x) \) is now denoted by \( K_t \) for some integer \( t \leq \omega \), we next prove that \( x_i = 1/t \) for all \( i \in \text{Supp}(x) \).

Otherwise, if \( x_i > x_j > 0 \) for some \( i, j \in \text{Supp}(x) \), then we choose \( \epsilon \) with \( 0 < \epsilon < x_i - x_j \), and change the weight \( x_i \) to \( x_i - \epsilon \) and \( x_j \) to \( x_j + \epsilon \), that is to say, we define a new nonnegative vector \( z \) with \( z_i = x_i - \epsilon, z_j = x_j + \epsilon \) and \( z_k = x_k \) for \( k \in V \setminus \{i, j\} \). We obtain from Step 1 that

\[
\lambda(G, z) - \lambda(G, x) = \sum_{\{i,j\} \in E(K_t)} z_i z_j - \sum_{\{i,j\} \in E(K_t)} x_i x_j = \epsilon(x_i - x_j) - \epsilon^2 > 0.
\]

This contradicts with the fact \( \lambda(G, x) = \lambda(G) \). Therefore, the values of \( x_i \) are all equal for every \( i \in V(K_t) \), which gives \( x_i = 1/t \) for every \( i \in V(K_t) \). Therefore,

\[
\lambda(G) = \lambda(G, x) = \sum_{\{i,j\} \in E(K_t)} \frac{1}{t^2} = \frac{1}{2} \left( 1 - \frac{1}{t} \right) \leq \frac{1}{2} \left( 1 - \frac{1}{\omega} \right). \tag{2}
\]

Combining (1) and (2), we get \( \lambda(G) = \frac{1}{2} (1 - \frac{1}{t}) \). Moreover, the argument also showed that the nonzero weights of the maximizer \( x \) with minimal supporting set are concentrated on a clique, which is a largest clique of \( G \), and all weights on this clique are equal. \( \square \)

**Corollary 1.6. Motzkin–Straus’ Theorem \( \lambda(G) \Rightarrow \text{Weak Turán’s Theorem} \).**

**Proof.** Let \( G \) be a graph that does not contain \( K_{r+1} \) as a subgraph. Clearly, we have \( \omega(G) \leq r \). We define the vector \( x = (1/n, 1/n, \ldots, 1/n) \), then

\[
\lambda(G, x) = \sum_{\{i,j\} \in E(G)} x_i x_j = \frac{m}{n^2}.
\]

By the Motzkin–Straus theorem, we know that

\[
\lambda(G, x) \leq \lambda(G) = \frac{1}{2} \left( 1 - \frac{1}{\omega} \right) \leq \frac{1}{2} \left( 1 - \frac{1}{r} \right).
\]

Combining the above two results, we can get \( m \leq (1 - \frac{1}{r}) \frac{n^2}{2} \). \( \square \)
For convenience, we usually write the Motzkin–Straus theorem as the following form.

**Theorem 1.7 (Equivalent).** Let $G$ be an $n$-vertex graph. If $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_1, \ldots, x_n \geq 0$ and $x_1 + \cdots + x_n = 1$, then

$$\sum_{\{i,j\} \in E(G)} x_i x_j \leq \frac{\omega(G) - 1}{2\omega(G)}.$$

As promised, we shall compare the extremal results based on the number of edges and the spectral extremal results through the whole survey. In what follows, we shall introduce some spectral version of Mantel’s theorem and Turán’s theorem. Interestingly, the spectral versions of these extremal results are greatly related to the Motzkin–Straus theorem.

**Theorem 1.8 (Nosal, 1970).** Let $G$ be a graph with no triangle. Then

$$\lambda(G) \leq \sqrt{e(G)},$$

equality holds if and only if $G$ is a complete bipartite graph.

This theorem together with Mantel’s Theorem 1.1 yields the following corollary.

**Corollary 1.9.** If $G$ is an $n$-vertex graph without triangles, then

$$\lambda(G) \leq \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \leq n/2.$$

In 1986, Wilf extended this corollary to the following theorem.

**Theorem 1.10 (Wilf [183]).** Let $\lambda(G)$ be the largest eigenvalue of $G$. Then

$$\lambda(G) \leq \left( 1 - \frac{1}{\omega(G)} \right) n.$$

**Corollary 1.11 (Equivalent).** Let $G$ be a $K_{r+1}$-free graph on $n$ vertices. Then

$$\lambda(G) \leq \left( 1 - \frac{1}{r} \right) n.$$

The proof is based on the Motzkin–Straus Theorem 1.5. Recalling that

$$1 - \frac{1}{\omega(G)} = \max_{x \geq 0, \|x\|_1 = 1} x^T A(G)x = \max_{x \geq 0} \frac{x^T A(G)x}{\|x\|_1^2}.$$

**Proof.** We show our proof in three steps.

- By the Perron–Frobenius Theorem, there exists a nonnegative vector $z$ such that

$$A(G)z = \lambda_1 z \Rightarrow z^T A(G)z = \lambda_1 z^T z \Rightarrow \lambda_1 = \frac{z^T A(G)z}{z^T z}.$$
• By the Motzkin–Straus Theorem, we have
\[
\frac{z^T A(G) z}{\left( \sum_{i=1}^{n} z_i \right)^2} \leq 1 - \frac{1}{\omega}.
\]

• By the Cauchy–Schwarz inequality, we get
\[
z^T z = \left( \sum_{i=1}^{n} z_i^2 \right) \geq \frac{1}{n} \left( \sum_{i=1}^{n} z_i \right)^2.
\]

Combining the above three results, we can get the required inequality.

**Remark.** We have the following potential applications. Since the Turán graph \( T_r(n) \) does not contain \( K_{r+1} \) as a subgraph, the Wilf theorem implies that \( \lambda(T_r(n)) \leq (1 - 1/r) n \). Moreover, the Rayleigh theorem implies that \( \lambda(G) = \max_{x \in \mathbb{R}^n} x^T A x / x^T x \geq 2m/n \), hence the Wilf theorem can imply the weak version of the Turán Theorem.

Recall that \( q(G) \) denotes the signless Laplacian spectral radius, i.e., the largest eigenvalue of the signless Laplacian matrix \( Q(G) = D(G) + A(G) \), where \( D(G) = \text{diag}(d_1, \ldots, d_n) \) is the degree diagonal matrix and \( A(G) \) is the adjacency matrix of a graph \( G \). Note that \( Q(G) = D(G) + A(G) = L(G) + 2A(G) \), where \( L(G) = D(G) - A(G) \) is the so-called Laplacian matrix, which is a positive semidefinite matrix, so we can get \( 2\lambda(G) \leq q(G) \).

In 2012, Abreu and Nikiforov [1, 2] enhanced the Wilf theorem in terms of the signless Laplacian spectral radius. They [2] proved that \( q(G) \leq 2 \left( 1 - \frac{1}{\omega(G)} \right) n \), where \( \omega \) is the number of vertices of a largest clique of \( G \). In other words, this result can be equivalently written as that if \( G \) is a \( K_{r+1} \)-free graph on \( n \) vertices, then \( q(G) \leq 2 \left( 1 - \frac{1}{r} \right) n \).

**Theorem 1.12** (Abreu–Nikiforov [2]). Let \( G \) be an \( n \)-vertex graph containing no copy of \( K_{r+1} \). Then
\[
q(G) \leq \left( 1 - \frac{1}{r} \right) 2n.
\]

We remark here that Abreu and Nikiforov did not characterize the extremal graphs attaining the equality. We can show by a careful examination that the equality holds if and only if \( G \) is a complete bipartite for \( r = 2 \), or \( r \) divides \( n \) and \( G = T_r(n) \) for \( r \geq 3 \).

In 2002, Nikiforov [144] generalized the Wilf theorem by given the number of edges. The extremal graphs attaining the equality in Theorem 1.13 was later characterized in [150, Theorem 2]; see [145] for more discussions.

**Theorem 1.13** (Nikiforov [144, 150]). Let \( \lambda(G) \) be the largest eigenvalue of \( G \). Then
\[
\lambda(G) \leq \sqrt{2m \left( 1 - \frac{1}{\omega(G)} \right)}.
\]

Moreover, the equality holds if and only if \( G \) is a complete bipartite graph for \( r = 2 \), or a complete regular \( r \)-partite graph for \( r \geq 3 \) with possibly some isolated vertices.
Corollary 1.14 (Equivalent). Let $G$ be a $K_{r+1}$-free graph with $m$ edges. Then

$$\lambda(G) \leq \sqrt{2m \left(1 - \frac{1}{r}\right)}.$$ 

This result has some similarity with the Wilf theorem. Its proof is also based on the celebrated Motzkin–Straus Theorem, which states that for any $x_1, x_2, \ldots, x_n \geq 0$, we have

$$\frac{\sum_{\{i,j\} \in E(G)} x_i x_j}{\left(\sum_{i=1}^{n} x_i\right)^2} \leq \frac{1}{2} \left(1 - \frac{1}{\omega}\right).$$

Proof. Let $y = (y_1, \ldots, y_n)$ be an (nonnegative) eigenvector of $\lambda(G)$.

• Using the Motzkin–Straus Theorem with respect to $(y_1^2, y_2^2, \ldots, y_n^2)$, we have

$$\frac{\left(\sum_{\{i,j\} \in E(G)} y_i^2 y_j^2\right)}{\left(\sum_{i=1}^{n} y_i^2\right)^2} \leq \frac{1}{2} \left(1 - \frac{1}{\omega}\right).$$

• On the other hand, we can get by Cauchy–Schwarz’s inequality that

$$\lambda_1^2 = \frac{(y^T A(G) y)^2}{(y^T y)^2} = \frac{\left(2 \sum_{\{i,j\} \in E(G)} y_i y_j\right)^2}{\left(\sum_{i=1}^{n} y_i^2\right)^2} \leq \frac{4m \left(\sum_{\{i,j\} \in E(G)} y_i^2 y_j^2\right)}{\left(\sum_{i=1}^{n} y_i^2\right)^2}.$$

This completes the proof by taking square root.

Remark. We have the following interesting applications. Note that the Turán graph $T_r(n)$ does not contain $K_{r+1}$ as a subgraph, Nikiforov’s result tells us that

$$\lambda(T_r(n)) \leq \sqrt{2r(n) \left(1 - \frac{1}{r}\right)} \leq \left(1 - \frac{1}{r}\right)n.$$ 

Moreover, the case $r = 2$ in Nikiforov’s result reduces to the Nosal Theorem 1.8. Additionally, applying the lower bound $\lambda(G) \geq 2m/n$, we can see that Nikiforov’s result implies the weak version of the Turán Theorem 1.2. Furthermore, Nikiforov’s result implies the Wilf Theorem 1.10 by applying the weak version of Turán theorem $m \leq (1 - \frac{1}{r}) \frac{n^2}{2}$.

We make the following conjecture for signless Laplacian spectral radius.

Conjecture 1.15. Let $G$ be a $K_{r+1}$-free graph with $m$ edges. Then

$$q(G) \leq \sqrt{8m \left(1 - \frac{1}{r}\right)}.$$
Remark. The answer of this conjecture is negative. After our manuscript announced on arXiv: 2111.03309, Clive Elphick and Vladimir Nikiforov independently told us that the star graph is a counter-example to Conjecture 1.15. Indeed, by setting \( r = 2 \), the \( n \)-vertex star graph \( K_{1,n-1} \) is clearly \( K_3 \)-free, and \( q(K_{1,n-1}) = n \) but \( \sqrt{8m\left(1 - \frac{1}{r}\right)} = \sqrt{4(n-1)} \). Moreover, the complete bipartite graph \( K_{s,n-s} \) is obviously \( K_{r+1} \)-free for every \( s \geq 1 \) and \( r \geq 2 \), we observe that \( q(K_{s,n-s}) = n \) but \( \sqrt{8m\left(1 - \frac{1}{r}\right)} = \sqrt{s(n-s)(1 - \frac{1}{r})} \).

In 2002, Bollobás and Nikiforov [17] proposed the following conjecture in terms of the first and second largest eigenvalues of \( G \). Let \( \lambda_k(G) \) be the \( k \)-th largest eigenvalue of \( G \).

**Conjecture 1.16** (Bollobás–Nikiforov [17]). If \( G \) does not contain a copy of \( K_{r+1} \), then

\[
\lambda_1^2(G) + \lambda_2^2(G) \leq 2m\left(1 - \frac{1}{r}\right).
\]

In 2021, the special case \( r = 2 \) of this conjecture was proved by Lin, Ning and Wu [130] by using tools from doubly stochastic matrix theory, and also characterize all families of extremal graphs; see Nikiforov [159] for related results. The general case \( r \geq 3 \) remains open. In 2022, Li, Sun and Yu [119] presented the characterization of graphs without short odd cycles. Later, Li and Peng [125] discussed some related spectral problems for graphs with given size containing no short odd cycles.

Recall in Theorem 1.8 that if \( G \) is a \( C_3 \)-free graph on \( n \) vertices with \( m \) edges, then \( \lambda(G) \leq \sqrt{m} \), equality holds if and only if \( G = K_{p,q} \) with \( n - p - q \) isolated vertices. In 2009, Nikiforov [152] proved an interesting result, which gives an upper bound on the spectral radius for \( C_4 \)-free graph with given number of edges. Note that the star graph \( K_{m,1} \) has \( m \) edges and \( m+1 \) vertices, and it contains no copy of \( C_4 \). Let \( S_{m,1} \) be the graph obtained from the star \( K_{m-1,1} \) by adding an edge within its independent set. Namely \( S_{m,1} = K_{m-1,1}^+ \). Note that the graph \( S_{m,1} \) has \( m \) edges and \( m \) vertices. For \( m = 4, 5, 6, 7, 8 \), we can verify that \( \lambda(S_{m,1}) > \sqrt{m} \); for \( m = 9 \), we have \( \lambda(S_{9,1}) = 3 \); for \( m = 10, 11, \ldots \), we can get \( \lambda(S_{m,1}) < \sqrt{m} \). For example, we can calculate that \( \lambda(S_{10,1}) \approx 3.151 < \sqrt{10} \approx 3.162 \).

**Theorem 1.17** (Nikiforov [152]). Let \( m \geq 9 \) and \( G \) be a graph with \( m \) edges. If \( G \) contains no \( C_4 \), then

\[
\lambda(G) \leq \sqrt{m},
\]

equality holds if and only if \( G \) is a star \( K_{1,m} \) or \( S_{9,1} \) with possibly some isolated vertices.

Motivated by Nosal’s Theorem 1.8 and Nikiforov’s Theorems 1.17, Zhai, Lin and Shu [198] proved the following analogue when the forbidden graph is the bipartite graph \( K_{2,r+1} \).

**Theorem 1.18** (Zhai–Lin–Shu [198]). Let \( r \geq 2 \) and \( G \) be a \( K_{2,r+1} \)-free graph of size \( m \geq 16r^2 \). Then

\[
\lambda(G) \leq \sqrt{m},
\]

equality holds if and only if \( G \) is a complete bipartite graph with possibly some isolated vertices.
Moreover, we denote by $B_k$ the book graph with $k$ pages, i.e., the graph obtained from $k$ triangles by sharing a common edge. In [198, Conjecture 5.2], Zhai, Lin and Shu presented a conjecture, which states that if $G$ is a $B_k$-free graph with sufficiently large size $m$, then $\lambda(G) \leq \sqrt{m}$, equality holds if and only if $G$ is a complete bipartite graph with possibly some isolated vertices. Later, Nikiforov [159] confirmed this conjecture by the following stronger theorem. Let $bk(G)$ denote the booksize of $G$, that is, the maximum number of triangles with a common edge in $G$.

**Theorem 1.19** (Nikiforov, 2021). If $G$ is a graph with $m$ edges and $\lambda(G) \geq \sqrt{m}$, then

$$bk(G) > \frac{1}{12} \sqrt{m},$$

unless $G$ is a complete bipartite graph with possibly some isolated vertices.

We end this section with a more general conjecture. Let $B_{r,k}$ be the generalized book, which is a graph by joining every vertex of a clique $K_r$ to every vertex of an independent set $I_k$. In the language of the join of graphs, we have

$$B_{r,k} = K_r \vee I_k.$$  

Clearly, we can see that $B_{2,k} = B_k$ and $B_{r,k}$ contains $k$ copies of $K_{r+1}$. Hence, our conjecture is more general than Corollary 1.14.

**Conjecture 1.20.** Let $G$ be a $B_{r,k}$-free graph with sufficiently large size $m$. Then

$$\lambda(G) \leq \sqrt{\left(1 - \frac{1}{r}\right)2m},$$

equality holds if and only if $G$ is a complete bipartite graph for $r = 2$, and $G$ is a complete regular $r$-partite graph for $r \geq 3$ with possibly some isolated vertices.

Note that the Wilf Theorem [1.10] and Nikiforov Theorem [1.13] can be viewed as the spectral generalization of the weak version of Turán Theorem [1.2]. In what follows, we shall introduce the spectral generalization of the strong version of the Turán Theorem [1.3]. In 2007, Feng, Li and Zhang [71] proved that the Turán graph $T_r(n)$ attains the maximum spectral radius among all $n$-vertex graphs with given chromatic number.

**Theorem 1.21** (Feng–Li–Zhang [71]). Let $G$ be an $n$-vertex complete $r$-partite graph. Then

$$\lambda(G) \leq \lambda(T_r(n)),$$

equality holds if and only if $G$ is the Turán graph $T_r(n)$.

Recall that $Q(G) = D(G) + A(G) = L(G) + 2A(G)$ and $L(G) = D(G) - A(G)$ is positive semidefinite, which implies $q(G) \geq 2\lambda(G)$. The following result on the signless Laplacian spectral radius of a complete $r$-partite graph is also well-known in the literature.
Theorem 1.22 (Cai–Fan [33, 164, 190]). Let $G$ be an $n$-vertex complete $r$-partite graph. Then

$$q(G) \leq q(T_r(n)),$$

equality holds if and only if $r = 2$ and $G = K_{s,n-s}$ for some $s$, or $r \geq 3$ and $G = T_r(n)$.

In the meantime, Nikiforov [147] published the following result.

Theorem 1.23 (Spectral Turán Theorem [147]). Let $G$ be a graph on $n$ vertices not containing $K_{r+1}$ as a subgraph. Then

$$\lambda(G) \leq \lambda(T_r(n)),$$

equality holds if and only if $G$ is the Turán graph $T_r(n)$.

Theorem 1.23 generalized Feng–Li–Zhang’s result since every $r$-partite graph is $K_{r+1}$-free, as well as enhanced Wilf’s Theorem 1.10 since we can verify that $\lambda(T_r(n)) \leq (1 - \frac{1}{r})n$, where the equality holds if and only if $r$ divides $n$. Independently, this result on extremal graph problems was also established by Guiduli in his Ph.D. dissertation [90, pp. 58–61] dating back to 1996 under the guidance of László Babai. In addition, the Wilf Theorem 1.10 can be viewed as the weak version of the following spectral Turán Theorem.

The idea of the proof of Theorem 1.23 in [147] is based on the characteristic polynomial of the Turán graph and two important theorems. The first theorem is also presented early by Nikiforov in [144, Theorem 3.1], which states that if $G$ is a $K_{r+1}$-free graph, then

$$\lambda^r \leq k_2 \lambda^{r-2} + \cdots + (j-1)k_j \lambda^{r-j} + \cdots + (r-1)k_r,$$

where $\lambda = \lambda(G)$ and $k_i = k_i(G)$ are the spectral radius and the number of $i$-cliques of $G$ respectively. In particular, the case $r = 2$ yields the Nosal Theorem 1.8 that is, $\lambda \leq \sqrt{k_2} = \sqrt{e(G)}$ for all $K_3$-free graphs. The second used theorem is an old result of Zykov [213] and independently of Erdős [63], which says that if $G$ is a $K_{r+1}$-free graph, then

$$k_i(G) \leq k_i(T_r(n)).$$

In other words, the Turán graph attains the maximum number of cliques of order $i$ among all $K_{r+1}$-free graphs. Clearly, the case $i = 2$ reduces to the Turán theorem.

We remark here that there is different proof [90] for the spectral Turán theorem; see, e.g., [101]. The proof given in [90] reduces $K_{r+1}$-free graphs in Theorem 1.23 to complete $r$-partite graphs, then one can show that the balanced complete $r$-partite graph attains the maximum value of the spectral radius; see, e.g., [71, Theorem 2.1] or [102, Theorem 2] for more details. The advantage of this proof is that we can avoid the use of the characteristic polynomials of the adjacency matrices of graphs, and this proof allows us to extend the spectral result to the $p$-spectral radius. The $p$-spectral radius of a graph $G$ was introduced by Keevash, Lenz and Mubayi in [104] and was defined as

$$\lambda^{(p)}(G) := \max \{ x^T A(G) x : x \in \mathbb{R}^n, \|x\|_p = 1 \}.$$
Theorem 1.23 was extended by Kang and Nikiforov [101] to the $p$-spectral radius for every $p > 1$. Note that the $p$-spectral radius is just an extremal function on graph, and there is no characteristic polynomial corresponding to the $p$-spectral radius.

In 2022, Li and Peng [124] proved a refinement on the spectral Turán Theorem 1.23. They determined the largest spectral radius for non-$r$-partite $K_{r+1}$-free graphs.

It is easy to see that Theorem 1.24 implies the weak version of Turán’s Theorem. A more natural question one may ask might be the following one.

**Question.** Does Theorem 1.23 imply the strong Turán’s Theorem?

This question was also proposed and answered in [90, 150]. Speaking rather broadly, the spectral bound can imply the edge bound of Turán’s theorem. It is well-known that $e(G) \leq \frac{\sqrt{2}}{2}\lambda(G)$, with equality if and only if $G$ is regular. Although the Turán graph $T_r(n)$ is sometimes not regular, but it is nearly regular. Upon calculation, we can verify that

$$e(T_r(n)) = \left\lfloor \frac{n}{2} \lambda(T_r(n)) \right\rfloor.$$ (3)

Let $n = rs + t$ where $0 \leq t < r$ and $s = \lfloor n/r \rfloor$. Hence the Turán graph $T_r(n)$ has $t$ parts of size $s + 1$ and $r - t$ parts of size $s$, and we can compute that

$$e(T_r(n)) = \frac{1}{2}((n - s - 1)(s + 1)t + (n - s)s(r - t)).$$

and

$$\lambda(T_r(n)) = \frac{1}{2}(n - 2s - 1 + \sqrt{n^2 - 4sn - 2n + 4rs^2 + 4rs + 1}).$$

With the help of the observation in (3), the spectral Turán theorem implies that

$$e(G) \leq \left\lfloor \frac{n}{2} \lambda(G) \right\rfloor \leq \left\lfloor \frac{n}{2} \lambda(T_r(n)) \right\rfloor = e(T_r(n)).$$

Thus the bound of spectral Turán Theorem 1.23 implies the edge bound of the classical Turán Theorem 1.3. To some extent, this interesting implication has shed new lights on the study of spectral extremal graph theory. In addition, the main difficulty in this relation lies in deducing the implication of case of equality, since we cannot get $\lambda(G) = \lambda(T_r(n))$ by using $\lfloor \frac{n}{2}\lambda(G) \rfloor \leq \lfloor \frac{n}{2}\lambda(T_r(n)) \rfloor$. We would like to thank Vladimir Nikiforov for pointing out to us this observation.

Recall that Abreu and Nikiforov (Theorem 1.12) provided a signless Laplacian spectral version of Wilf’s theorem, namely $q(G) \leq 2\left(1 - \frac{1}{r}\right)n$ for every $n$-vertex $K_{r+1}$-free graph $G$. In 2013, He, Jin and Zhang [93, Theorem 1.3] proved further that $q(G) \leq q(T_r(n))$. Their result is regarded as the signless Laplacian spectral version of the strong Turán theorem. Since $q(T_r(n)) \leq 2\left(1 - \frac{1}{r}\right)n$, and if $r$ divides $n$, then $q(T_r(n)) = 2\left(1 - \frac{1}{r}\right)n$, we know that Theorem 1.24 improved the result of Abreu and Nikiforov and the result of Wilf as well. Moreover, they also illustrated that the signless Laplacian spectral version implies the original Turán theorem.

**Theorem 1.24** (He–Jin–Zhang [93]). Let $G$ be a $K_{r+1}$-free graph of order $n$. Then

$$q(G) \leq q(T_r(n)),$$

equality holds if and only if $r = 2$ and $G = K_{s,n-s}$ for some $s$, or $r \geq 3$ and $G = T_r(n)$. 

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As mentioned in Corollary 2.5, this spectral bound of the signless Laplacian radius implies the edge bound in strong version of the Turán Theorem since

\[ e(G) \leq \left\lfloor \frac{n}{4} q(G) \right\rfloor \leq \left\lfloor \frac{n}{4} q(T_r(n)) \right\rfloor = e(T_r(n)). \]

From what has been discussed above, we conclude that the edge version of the Turán theorem can be deduced from the spectral version of either the adjacency matrix or the signless Laplacian matrix. Along with the development of the spectral graph theory, it is commonly recognized that the spectral versions of extremal graph problems are more challenging and interesting.

### 1.2 Spectral problem for general graphs

In this section, we shall consider the extremal number \( \text{ex}(n, F) \) for general graph \( F \). In 1946, Erdős and Stone [60] proved a celebrated theorem, which states that

**Theorem 1.25** (Erdős–Stone [60]). For every \( r \geq 1, t \geq 1 \) and \( \varepsilon > 0 \), there is \( n_0 = n_0(r, t, \varepsilon) \) such that every graph \( G \) on \( n \geq n_0 \) vertices with \( e(G) \geq (1 - \frac{1}{r} + \varepsilon) \frac{n^2}{2} \) contains a copy of a complete \((r + 1)\)-partite graph \( K_{r+1}(t) \), whose every vertex class has \( t \) vertices.

This result is sharp in the following qualitative sense. Let \( T_r(n) \) denote the complete \( r \)-partite graph of order \( n \) whose vertex classes have the number of vertices as equal as possible. Clearly, \( T_r(n) \) contains no complete subgraph of order \( r + 1 \) and \( e(T_r(n)) = (1 - \frac{1}{r}) \frac{n^2}{2} + O(n) \). From this observation, we can get a lower bound on \( \text{ex}(n, K_{r+1}(t)) \).

Hence Theorem 1.25 yields the following two corollaries.

**Corollary 1.26.** For every integers \( r \geq 1 \) and \( t \geq 1 \), then

\[ \text{ex}(n, K_{r+1}(t)) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}, \]

where \( o(1) \to 0 \) whenever \( n \to \infty \).

**Corollary 1.27** (Equivalent). For every integer \( t \geq 1 \), we have

\[ \lim_{n \to \infty} \frac{\text{ex}(n, K_{r+1}(t))}{\binom{n}{2}} = 1 - \frac{1}{r}. \]

A proper vertex-coloring of a graph is a coloring of its vertex set for which each color class forms an independent set. The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the minimum number of colors needed in a proper coloring of \( G \). A graph is \( r \)-partite if it has chromatic number at most \( r \). In other words, the vertex-chromatic number \( \chi(G) \) of \( G \) is the minimum integer \( s \in \mathbb{N}^* \) such that there exists a coloring of \( V(G) \) with \( s \) colors and the adjacent vertices have different colors. For example, we can see that \( \chi(K_{r+1}) = r + 1 \) and \( \chi(K_{s,t}) = 2 \). In fact, a graph \( H \) has \( \chi(H) = r + 1 \) if and only if \( H \) is an \((r + 1)\)-partite graph.
This powerful result of Erdős and Stone has many important applications in extremal
graph theory. Consequently, the Erdős–Stone theorem implies the following theorem since
if $F$ is a graph with chromatic number $\chi(F) = r + 1$, then $F$ is contained in $K_{r+1}(t)$
for an appropriate integer $t$. By applying the monotonicity of the Turán function, we get
$\text{ex}(n, F) \leq \text{ex}(n, K_{r+1}(t))$. This theorem was established by Erdős and Simonovits in [66].

**Theorem 1.28** (Erdős–Simonovits [66]). For every graph $H$, we have
\[
\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} + o(n^2).
\]

Equivalently, we can write the above result as the limitation.

\[
\lim_{n \to \infty} \frac{\text{ex}(n, H)}{\binom{n}{2}} = \frac{1}{\chi(H) - 1}.
\]

Although this result was formally established by Erdős and Simonovits, we always
call it the Erdős–Stone–Simonovits theorem for the sake of their equivalence relation. We
mention here that Erdős, Stone and Simonovits never wrote a paper together. First Erdős
and Stone solved it for $H$ a complete multipartite graph, and then Erdős and Simonovits
proved it for general $H$ in this way.

**Proof.** First of all, we denote $\chi(H) = r + 1$. It is easy to see that the $n$-vertex complete
$r$-partite Turán graph $T_r(n)$ contains no copy of $H$ otherwise $H$ is $r$-partite, so $H$ can be
colored by $r$ colors, a contradiction. Thus, we can get
\[
\text{ex}(n, H) \geq t_r(n) \geq \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - \frac{r}{8} = \left(1 - \frac{1}{r} - o(1)\right) \frac{n^2}{2}.
\]

For the upper bound, since $\chi(H) = r + 1$, there exists an integer $t$ such that $H$ is a
subgraph of $K_{r+1}(t)$. By the Erdős–Stone Theorem 1.25, we can get
\[
\text{ex}(n, H) \leq \text{ex}(n, K_{r+1}(t)) \leq t_r(n) + o(n^2).
\]

Hence, we have proved that for some fixed integer $v$,
\[
t_r(n) \leq \text{ex}(n, H) \leq \text{ex}(n, K_{r+1}(t)) \leq t_r(n) + o(n^2).
\]

This completes the proof. \hfill \qed

The Erdős–Stone–Simonovits theorem implies straightforward the following extremal
number for graphs corresponding to the five regular polyhedrons.

- The regular tetrahedron forms the graph $K_4$, we have $\text{ex}(n, K_4) = e(T_3(n)) = \lfloor n^2/3 \rfloor$.
- Let $Q_8$ be the cube graph on 8 vertices. Then $\chi(Q_8) = 2$ and $\text{ex}(n, Q_8) = o(n^2)$. 

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• Let $O_6$ be the octahedron graph on 6 vertices. It is easy to see that $O_6$ is a complete 3-partite graph with each part of size 2, i.e., $O_6 = K_{2,2,2}$. Then $\chi(O_6) = 3$ and

$$\text{ex}(n, O_6) = \frac{n^2}{4} + o(n^2).$$

• Let $D_{20}$ be the dodecahedron graph on 20 vertices. Then $\chi(D_{20}) = 3$ and

$$\text{ex}(n, D_{20}) = \frac{n^2}{4} + o(n^2).$$

• Let $I_{12}$ be the icosahedron graph on 12 vertices. Then $\chi(I_{12}) = 4$ and

$$\text{ex}(n, I_{12}) = \frac{n^2}{3} + o(n^2).$$

We remark here that although the Erdős–Stone–Simonovits theorem provides good asymptotic estimates for the extremal numbers of non-bipartite graphs. However, for bipartite graphs, where $\chi(F) = 2$, it only gives the bound $\text{ex}(n, F) = o(n^2)$. Although there have been numerous attempts of finding better bounds on $\text{ex}(n, F)$ for various bipartite graphs $F$, we know very little in this case; see [84] for a survey.

In 2009, Nikiforov [151] proved a spectral generalization of the Erdős–Stone–Simonovits theorem, as we know that the Rayleigh formula implies $\lambda(G) \geq 2e(G)/n$.

**Theorem 1.29** (Nikiforov [151]). Let $n \geq r \geq 3$ be integers with $(c/r^r) \ln n \geq 1$. If $G$ is a graph on $n$ vertices with

$$\lambda(G) \geq \left(1 - \frac{1}{r-1} + c\right)n,$$

then $G$ contains a copy of $K_r(s, \ldots, s, t)$ with $s \geq (c/r^r)^r \ln n$ and $t > n^{1-c^{-1}}$.

**Remark.** The main idea of the proof of Nikiforov is based on two elegant results from [17] and [149]. Moreover, Nikiforov also gave an explicit bound on the size of each vertex part of the complete $r$-partite subgraph in terms of the adjacency spectral radius. Furthermore, Theorem 1.29 is very precise and does not use the Regularity Lemma. Its edge version is a lot stronger than the Erdős–Stone and the Erdős–Simonovits theorems. Thus, it can be applied to obtain extremal results for concrete graphs, in particular, graphs whose order is not fixed and grows with $n$.

Analogous to the Turán function, we define the spectral Turán function as $\text{ex}_\lambda(n, F)$, which is the maximum spectral radius in an $n$-vertex graph without subgraph isomorphic to $F$. It is important and meaningful to determine the function $\text{ex}_\lambda(n, F)$ for various graphs $F$. From Theorem 1.29 one can easily get the following weak corollary, which gives the asymptotic spectral extremal number for general graphs. This corollary was also proved by Guiduli in his PH.D. thesis [90] in 1996.
Corollary 1.30. For every integers $r \geq 2$ and $t \geq 1$, we have

$$\text{ex}_\lambda(n, K_r(t)) = \left(1 - \frac{1}{r-1}\right)n + o(n).$$

Consequently, if $F$ is a graph with chromatic number $\chi(F)$, then

$$\text{ex}_\lambda(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right)n + o(n).$$

This result can be rewritten as the following limitation.

$$\lim_{n \to \infty} \frac{\text{ex}_\lambda(n, F)}{n} = 1 - \frac{1}{\chi(F)-1}.$$

We mentioned here that Guiduli [90] proved Corollary 1.30 by using an analogous result proved by Erdős, Frankl and Rödl [67], which states that for any $r, t \geq 1$ and $\varepsilon > 0$, there exists an integer $n_0$ such that if $G$ is a $K_r(t)$-free graph on $n \geq n_0$ vertices, then we can remove at most $\varepsilon n^2$ edges from $G$ to make it being $K_r$-free. This result is a direct consequence of the celebrated Szemerédi Regularity Lemma.

Next, we shall provide an alternative proof of Corollary 1.30 by applying the Graph Removal Lemma, which states that for every graph $H$ on $h$ vertices and every $\varepsilon > 0$, there exists $\delta = \delta_H(\varepsilon) > 0$ such that any graph on $n$ vertices with at most $\delta n^h$ copies of $H$ can be made $H$-free by removing at most $\varepsilon n^2$ edges; see, e.g., [78, 51].

Alternative Proof. Let $F$ be a fixed graph with $\chi(F) = r$ for some $r \geq 2$. We choose $t$ as an integer such that $F \subseteq K_r(t)$. For any $\varepsilon > 0$, if $G$ is an $n$-vertex graph with $\lambda(G) \geq (1 - \frac{1}{r-1} + \varepsilon)n$, the Wilf theorem implies that $G$ contains at least one copy of $K_r$. Next, we shall show that the number of copies of $K_r$ is at least $\delta_{K_r}(\varepsilon/2) \cdot n^r$ where $\delta_H(\varepsilon)$ is the function determined in the Graph Removal Lemma. This is the spectral version of the supersaturation lemma for $K_r$. Indeed, suppose on the contrary that $G$ has less than $\delta_{K_r}(\varepsilon/2) \cdot n^r$ copies of $K_r$, then the Graph Removal Lemma yields that we can remove at most $\frac{\varepsilon}{2} n^2$ edges from $G$ such that the remaining subgraph $G^*$ is $K_r$-free. The Wilf theorem gives that $\lambda(G^*) \leq (1 - \frac{1}{r-1})n$. Hence, the Rayleigh formula yields

$$\lambda(G) \leq \lambda(G^*) + \lambda(G \setminus G^*) < \lambda(G^*) + \sqrt{2e(G \setminus G^*)} \leq \left(1 - \frac{1}{r-1}\right)n + \varepsilon n,$$

a contradiction. Hence $G$ has at least $\delta_{K_r}(\varepsilon/2) \cdot n^r$ copies of $K_r$.

Now, we define an $r$-uniform hypergraph $G^{(r)}$ on the vertex set $V(G)$ where an $r$-element subset $E \subseteq V(G)$ is defined to be a hyperedge if and only if it induces a copy of $K_r$ in the 2-graph $G$. Under this definition, we know that $G^{(r)}$ has at least $\delta_{K_r}(\varepsilon/2) \cdot n^r$ edges. We denote by $K^{(r)}_{t,t,\ldots,t}$ the complete $r$-partite $r$-uniform hypergraph with each vertex part of size $t$. A well-known result of Erdős [64] states that

$$\text{ex}(n, K^{(r)}_{t,t,\ldots,t}) = O(n^{r-(1/t)^{r-1}}).$$
Therefore, for sufficiently large $n$, the hypergraph $G^{(r)}$ contains a copy of $K_{t,t,...,t}^{(r)}$. Note that every copy of $K_{t,t,...,t}^{(r)}$ in $G^{(r)}$ corresponds to a copy of $K_r(t)$ in $G$. Hence $G$ contains a copy of $K_r(t)$ for any $\varepsilon > 0$ and large enough $n$. \hfill \Box

**Conjecture 1.31.** If $G$ is an $F$-free graph with $m$ edges, then

$$
\lambda(G) \leq \sqrt{\left(1 - \frac{1}{\chi(F)} + 1 + o(1)\right) 2m}.
$$

After our manuscript announced, Nikiforov (private communication) told us that he can prove Conjecture 1.31 for 3-chromatic graphs. A natural question is to extend the above-mentioned results on the adjacency spectral radius to that of the signless Laplacian spectral radius. We define $ex_q(n, F)$ to be the largest eigenvalue of the signless Laplacian matrix in an $n$-vertex graph that contains no copy of $F$. That is,

$$
ex_q(n, F) := \max\{q(G) : |G| = n \text{ and } F \not\subseteq G\}.
$$

Note that $Q(G) = D(G) - A(G) + 2A(G)$ and $D(G) - A(G)$ is positive semidefinite. It is known by the Weyl theorem for monotonicity of eigenvalues that $q(G) \geq 2\lambda(G)$. Comparing to Corollary 1.30, one may ask the following question:

**Question.** Let $F$ be a fixed graph. Is it true that $ex_q(n, F) = (1 - \frac{1}{\chi(F) - 1})2n + o(n)$.

The answer of this question is negative. Let $k \geq 2$ be a positive integer and $S_{n,k}$ be the graph consisting of a clique on $k$ vertices and an independent set on $n - k$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. In the language of join of graphs, we have $S_{n,k} = K_k \vee I_{n-k}$. We can observe that $S_{n,k}$ does not contain $C_{2k+2}$ as a subgraph. Furthermore, let $S^+_{n,k}$ be the graph obtained from $S_{n,k}$ by adding an edge to the independent set $I_{n-k}$. Namely, we have $S^+_{n,k} = K_k \vee I^+_{n-k}$. Clearly, we can see that $S^+_{n,k}$ is still $C_{2k+2}$-free and

$$
q(S^+_{n,k}) > q(S_{n,k}) = \frac{n + 2k - 2 + \sqrt{(n + 2k - 2)^2 - 8k^2 + 8k}}{2}.
$$

Hence either the graph $S_{n,k}$ or $S^+_{n,k}$ can yield $ex_q(n, C_{2k+2}) \geq n + o(n)$. However we have $\chi(C_{2k+2}) = 2$ and $(1 - \frac{1}{2})2n + o(n) = o(n)$. In 2015, Nikiforov and Yuan [157] proved that $ex_q(n, C_{2k+2}) = q(S^+_{n,k})$ for every $n \geq 400k^2$. However $\chi(C_{2k+2}) = 2$ and $q(S^+_{n,k}) \neq o(n)$.

### 1.3 Spectral problem for bipartite graphs

The history of studying the extremal number for bipartite graphs began in 1954 with the Kővari–Sós–Turán theorem [106], which states that if $K_{s,t}$ is the complete bipartite graph with vertex classes of size $s \geq t$, then $ex(n, K_{s,t}) = O(n^{2-1/t})$; see [80] [81] for more details. In particular, we refer the interested reader to the comprehensive survey by Füredi and Simonovits [84] for the history of bipartite extremal graph problems.
Theorem 1.32 (Kővari–Sós–Turán, 1954). For all \( t \geq s \geq 2 \), if \( G \) is an \( n \)-vertex graph and contains no copy of \( K_{s,t} \), then

\[
e(G) \leq \frac{1}{2} (t - 1)^{\frac{1}{s}} n^{2 - \frac{1}{s}} + \frac{1}{2} (s - 1)n.
\]

In other words, we have

\[
ex(n, K_{s,t}) \leq \frac{1}{2} (t - 1)^{\frac{1}{s}} n^{2 - \frac{1}{s}} + \frac{1}{2} (s - 1)n \approx \frac{1}{2} (t - 1)^{\frac{1}{s}} n^{2 - \frac{1}{s}}.
\]

For convenience of readers, we here include a standard proof, which is a well application of the double counting technique.

**Proof.** Let \( G \) be a \( K_{s,t} \)-free \( n \)-vertex graph with \( m \) edges. We denote the number of copies of \( K_{s,1} \) in \( G \) as \( \#K_{s,1} \). The proof establishes an upper bound and a lower bound on \( \#K_{s,1} \), and then gets a bound on \( m \) by combining the upper bound and the lower bound. Since \( K_{s,1} \) is a tree, we can call the side with \( s \) vertices the leaf vertices, and the side with \( 1 \) vertex the root vertex.

On the one hand, we can count \( \#K_{s,1} \) by enumerating the leaf vertices. For any subset \( S \) of \( s \) vertices, the number of \( K_{s,1} \)’s that are leaves is exactly the number of common neighbors of these \( s \) vertices. We write \( d(S) \) for the number of common neighbors of vertices of \( S \), i.e., \( d(S) = | \cap_{v \in S} N(v) | \). Since \( G \) is \( K_{s,t} \)-free, we get \( d(S) \leq t - 1 \). Thus, we establish that

\[
\#K_{s,1} = \sum_{S \in \binom{V}{s}} d(S) \leq \binom{n}{s} (t - 1).
\]

On the other hand, for each vertex \( v \in V(G) \), the number of copies of \( K_{s,1} \) where \( v \) is the root vertex is exactly \( \binom{d(v)}{s} \). Therefore,

\[
\#K_{s,1} = \sum_{v \in V(G)} \binom{d(v)}{s} \geq n \left( \frac{1}{n} \sum_{v \in V(G)} d(v) \right) = n \left( \frac{2m}{n} \right),
\]

where the inequality uses the convexity of \( x \mapsto \binom{x}{s} \). Here we regard \( \binom{x}{s} \) as a degree \( s \) polynomial in \( x \), so it makes sense for \( x \) to be non-integers.

Combining the upper bound and lower bound of \( \#K_{s,1} \), we obtain that \( n \left( \frac{2m}{n} \right) \leq \binom{n}{s} (t - 1) \). For constant \( s \), we can use \( \binom{x}{s} = (1 + o(1)) \frac{x^s}{s!} \) to get \( n \left( \frac{2m}{n} \right)^s \leq (1 + o(1)) n^{s(t - 1)} \). The above inequality simplifies to \( m \leq \left( \frac{1}{2} + o(1) \right) (t - 1)^{1/s} n^{2 - 1/s} \).

**Definition 1.33** (Zarankiewicz). Let \( z(m, n, s, t) \) be the maximum number of edges in a bipartite graph \( G \) with vertex parts \( M \) of size \( m \) and \( N \) of size \( n \) such that \( G \) does not contain a copy of \( K_{s,t} \) with vertex part of size \( s \) in \( M \) and vertex part of size \( t \) in \( N \).

In the language of matrix, the value \( z(m, n, s, t) \) can be defined as the maximum number of ones in a matrix consisting only of zeros and ones of size \( m \times n \) that does not contain an all ones submatrix of size \( s \times t \). The problem of finding this number is known as the Zarankiewicz problem.
Clearly, we can see from the definition that \( z(n, n, s, t) \leq \text{ex}(2n, K_{s,t}) \). We next show that \( 2\text{ex}(n, K_{s,t}) \leq z(n, n, s, t) \). To prove this inequality, we just need to construct a bipartite graph \( G \) that has \( 2\text{ex}(n, K_{s,t}) \) edges and contains no copy of \( K_{s,t} \). Let \( G \) be an extremal graph with \( \text{ex}(n, K_{s,t}) \) edges and contains no copy of \( K_{s,t} \). We define a bipartite graph \( H \) on vertex sets \( V_1 \) and \( V_2 \) where \( V_1 = V_2 = V(G) \) with edge set \( E(H) = \{\{u, v\} : u \in V_1, v \in V_2, \{u, v\} \in E(G)\} \). We can verify that \( H \) does not contain any copy of \( K_{s,t} \) and \( H \) has \( 2\text{ex}(G) \) edges. Thus, we get \( 2\text{ex}(n, K_{s,t}) \leq z(n, n, s, t) \). We now state the above observation as the following proposition.

**Proposition 1.34.** \( 2\text{ex}(n, K_{s,t}) \leq z(n, n, s, t) \leq \text{ex}(2n, K_{s,t}) \).

From this proposition, it turns the problems of finding an upper bound for the Turán number into an upper bound of the Zarankiewicz problems. Using the counting method in the proof of the Kővári–Sós–Turán theorem, we can similarly prove that

\[
z(m, n, s, t) \leq (s - 1)^{\frac{1}{t}} \cdot (n - t + 1)m^{1 - \frac{1}{t}} + (t - 1)m.
\]

In 1996, Füredi \[80\] improved the coefficient of this bound to \((s - t + 1)^{1/t}\).

**Theorem 1.35** (Füredi \[80\]). For all \( s \geq t \geq 1 \), we have

\[
z(m, n, s, t) \leq (s - t + 1)^{\frac{1}{t}}nm^{1 - \frac{1}{t}} + tm^{2 - \frac{2}{t}} + tn. \tag{4}
\]

In 2010, Nikiforov \[154\] proved by an elegant induction that

**Theorem 1.36** (Nikiforov \[154\]). If \( s, t \geq 2 \), then for every \( k = 0, 1, \ldots, s - 2 \),

\[
z(m, n, s, t) \leq (s - k - 1)^{\frac{1}{t}}nm^{1 - \frac{1}{t}} + (t - 1)m^{1 + \frac{k}{t}} + kn.
\]

For \( k = 0 \), the upper bound of Kővári, Sós and Turán is obtained, and, if \( s \geq t \), the author also improved Füredi’s bound, since the case \( k = t - 2 \) leads to \( z(m, n, s, t) \leq (s - t + 1)^{1/t}nm^{1 - 1/t} + (t - 1)m^{2 - 2/t} + (t - 2)n \).

We remark that finding the lower bound of \( \text{ex}(n, K_{s,t}) \) is extremely difficult. By using the Probabilistic Method, one can get a lower bound of \( \text{ex}(n, K_{s,t}) \).

**Theorem 1.37.** For all \( t \geq s \geq 2 \), we have

\[
\text{ex}(n, K_{s,t}) \geq \frac{1}{16}n^{2 - \frac{s+t-2}{s}}.
\]

**Corollary 1.38.** For all positive integer \( s \), we have

\[
c_1(s) \cdot n^{2 - \frac{2}{s+1}} \leq \text{ex}(n, K_{s,s}) \leq c_2(s) \cdot n^{2 - \frac{1}{s}},
\]

where \( c_1(s) \) and \( c_2(s) \) are constants depending only on \( s \).
Proof of Theorem 1.37. We consider the random graph $G(n, p)$, a graph on $n$ vertices where each each is chosen independently from $K_n$ with probability $p$. Let $\#K_{s,t}$ denote the number of copies of $K_{s,t}$ in $G(n, p)$. Then the expected number of edges and copies of $K_{s,t}$ are given by $\mathbb{E}[e(G)] = p\binom{n}{2}$ and $\mathbb{E}[\#K_{s,t}] = \binom{n}{s} \binom{n-s}{t} p^t$ respectively. By the linearity of expectation, we have

$$\mathbb{E}[e(G) - \#K_{s,t}] = \mathbb{E}[e(G)] - \mathbb{E}[\#K_{s,t}] \geq \frac{1}{4}pn^2 - n^{s+t}p^t.$$ 

We now want to pick $p$ so that the right hand side in above is as large as possible. By setting $p = \frac{1}{16}n^{-\frac{s+t-2}{s+t-1}}$, we can get $\mathbb{E}[e(G) - \#K_{s,t}] \geq \frac{1}{8}pn^2 = \frac{1}{16}n^{2-\frac{s+t-2}{s+t-1}}$. Thus there exists a graph $G$ in which $e(G) - \#K_{s,t} \geq \frac{1}{16}n^{2-\frac{s+t-2}{s+t-1}}$, and by removing one edge from each copy of $K_{s,t}$, we can get a $K_{s,t}$-free graph with at least $\frac{1}{16}n^{2-\frac{s+t-2}{s+t-1}}$ edges. \hfill \Box

Conjecture 1.39. For $t \geq s \geq 2$, we have $ex(n, K_{s,t}) = \Theta(n^{2-1/s})$.

We here list the recent progress of this conjecture in the literature. In 1996, Kollár, Rónyai and Szabó \[107\] confirmed this conjecture for the case $t > s$. In 1999, Alon, Rónyai and Szabó \[5\] slightly improved to the case $t > (s-1)!$. In 2013, Blagojević, Bukh and Karasev \[14\] and later Bukh \[30\] used the random algebraic method to give different constructions which yield the same lower bound $ex(n, K_{s,t}) = \Omega(n^{2-1/s})$, provided that $t$ is sufficiently large. In 2021, Bukh made a further improvement \[32\] by applying the random algebraic construction again.

Note that $2e(G)/n \leq \lambda(G)$. Hence finding the upper bounds on the adjacency spectral radius extends the general extremal bounds on number of edges. In 2007, Babai and Guiduli \[7\] published an asymptotic bound of the Kővári–Sós–Turán upper bound on the spectral radius (Their work dates back to 1996 on the Ph.D. thesis of Guiduli \[90\]).

Theorem 1.40 (Babai–Guiduli \[90, 7\]). Let $\lambda(G)$ denote the spectral radius of $G$.

(1) Let $s \geq 2$ be an integer. If $G$ is an $n$-vertex $K_{s,2}$-free graph, then

$$\lambda(G) \leq (s - 1)^{\frac{1}{2}}n^{\frac{1}{2}} + O(n^{\frac{1}{2}}).$$

(2) Let $s \geq t \geq 2$ be integers. If $G$ is an $n$-vertex $K_{s,t}$-free graph, then

$$\lambda(G) \leq \left((s - 1)^{\frac{1}{t}} + o(1)\right)n^{1-\frac{1}{t}}.$$

Nikiforov \[147\] proved the following spectral extremal problem for $K_{2,\ell}$ and the book graph $B_{k+1} = K_2 \vee I_{k+1}$, a graph obtained by joining each vertex of $K_2$ to each vertex of an independent set of size $k+1$.

Theorem 1.41 (Nikiforov \[147\]). Let $\ell \geq k \geq 0$ be integers. If $G$ is a graph on $n$ vertices and contains no copy of $K_{2,\ell+1}$ and $K_2 \vee I_{k+1}$, then

$$\lambda(G) \leq \frac{1}{2}\left(k - \ell + 1 + \sqrt{(k - \ell + 1)^2 + 4\ell(n-1)}\right).$$

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In particular, this theorem implies the following result.

**Corollary 1.42.** Let \( s \geq 2 \). If \( G \) is an \( n \)-vertex \( K_{s,2} \)-free graph, then

\[
\lambda(G) \leq \frac{1}{2} + \sqrt{(s - 1)(n - 1)} + 1/4.
\]

As a consequence of Theorem 1.36, Nikiforov [154] improved the result of Babai and Guiduli concerning the largeness of the spectral radius of a graph of certain order that does not contain a complete bipartite subgraph \( K_{s,t} \).

**Theorem 1.43** (Nikiforov [154]). Let \( s \geq t \geq 3 \) be integers. If \( G \) is an \( n \)-vertex \( K_{s,t} \)-free graph with spectral radius \( \lambda(G) \). Then

\[
\lambda(G) \leq (s - t + 1)^{1/7}n^{1/7} + (t - 1)n^{1/7} + t - 2.
\]

A natural question is that how to characterize the extremal graphs attaining the equality in Theorems 1.36 and 1.43. For the edge version, it is extremely difficult to find the graph attaining the upper bound even for the special graph \( K_{2,2} \). Surprisingly, the extremal graphs of the spectral problem for \( K_{2,2} \) was determined by Nikiforov [147], and Zhai and Wang [196]. We infer the readers to Subsection 1.5 for more details.

### 1.4 Spectral problem for odd cycles

Let \( C_{2k+1} \) denote the cycle on \( 2k + 1 \) vertices. Note that \( C_{2k+1} \) has chromatic number \( \chi(C_{2k+1}) = 3 \). The Erdős–Stone Theorem 1.25 or Erdős–Simonovits Theorem 1.28 implies that \( \text{ex}(n, C_{2k+1}) = n^2/4 + o(n^2) \). In 1971, Bondy [21] and Woodall [184] proved independently a more stronger result, which implies that the extremal number for odd-length cycle is exactly equal to \( n^2/4 \); see [15, p. 150] for more details.

**Theorem 1.44** (Bondy [21]; Woodall [184]). If \( G \) is a graph on \( n \) vertices with \( e(G) > \lfloor n^2/4 \rfloor \), then \( G \) contains a cycle of length \( \ell \) for every \( \ell = 3, 4, \ldots, \lfloor (n + 3)/2 \rfloor \).

From this theorem, we can easily get the following corollary.

**Corollary 1.45.** If \( k \geq 1 \) and \( n \geq 4k \), then

\[
\text{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor.
\]

This result can also be proved by the well-known stability method for \( n \) sufficiently large; see [103, Theorem 5.3] for more details. In 2015, Füredi and Gunderson [85] determined the exact value of \( \text{ex}(n, C_{2k+1}) \) for all \( n \geq 1 \) and \( k \geq 2 \). The case \( k = 1 \) was early characterized by Mantel who showed \( \text{ex}(n, C_3) = \lfloor n^2/4 \rfloor \) for all \( n \).

A wheel \( W_n \) is a graph on \( n \) vertices obtained from \( C_{n-1} \) by adding one vertex \( w \) and making \( w \) adjacent to all vertices of the \( C_{n-1} \). In the language of join graph, we have

\[
W_n = K_1 \vee C_{n-1}.
\]
It is easy to see that the vertex-chromatic number \( \chi(W_{2k}) = 4 \) and \( W_{2k} \) is color-critical, that is, there is an edge \( e \in W_{2k} \) such that \( \chi(W_{2k} \setminus e) = 3 \). The Erdős–Stone–Simonovits Theorem 1.28 implies that \( \text{ex}(n, W_{2k}) = e(T_3(n)) + o(n^2) \). In 2013, Dzido proved the exact Turán number by using Corollary 1.45.

**Theorem 1.46 (Dzido [58]).** For all \( k \geq 3 \) and \( n \geq 6k - 10 \), then

\[
\text{ex}(n, W_{2k}) = e(T_3(n)) = \left\lfloor \frac{n^2}{3} \right\rfloor.
\]

For odd wheels, we know that \( \chi(W_{2k+1}) = 3 \) and the Erdős–Stone–Simonovits Theorem 1.28 implies the asymptotic value \( \text{ex}(n, W_{2k+1}) = n^2/4 + o(n^2) \). We remark here that the exact extremal number of odd wheels was recently determined by Yuan [193] for sufficiently large \( n \) by applying the stability method. The problem for odd wheels is more complicated than that of even wheels since the odd wheels are not color-critical. In 2022, the spectral extremal problem was proved by Cioabă, Desai and Tait [46].

In 2008, Nikiforov [148] proved an analogue of the Bondy–Woo dall Theorem 1.44 in terms of the spectral radius for the existence of cycles with consecutive lengths.

**Theorem 1.47 (Nikiforov [148]).** Let \( G \) be a graph of sufficiently large order \( n \) with \( \lambda(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \). Then \( G \) contains a cycle of length \( t \) for \( t \leq n/320 \).

The result is sharp because the complete bipartite graph \( T_2(n) \) with parts of size \( \lfloor n/2 \rfloor \) and \( \lceil n/2 \rceil \) has no odd cycles, and its largest eigenvalue is \( \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \). Moreover, it is clear that the constant 1/320 in Theorem 1.47 can be increased even with careful calculations in the Nikiforov methods. Thus, the following question may arise:

**Conjecture 1.48.** What is the maximum \( C \) such that for all positive \( \varepsilon < C \) and sufficiently large \( n \), every graph \( G \) of order \( n \) with \( \lambda(G) > \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \) contains a cycle of length \( t \) for every \( t \leq (C - \varepsilon)n \).

It is known from Theorem 1.44 that if \( G \) is a graph of order \( n \) with \( e(G) > \left\lfloor \frac{n^2}{4} \right\rfloor \), then \( G \) contains a cycle of length \( t \) for every \( 3 \leq t \leq \lceil n/2 \rceil \). Therefore, one can conjecture that \( C = 1/2 \). Unfortunately, this is not true by taking the join of a complete graph of order \( k \) and an empty graph of order \( n - k \), we can obtain a graph

\[
S_{n,k} = K_k \lor I_{n-k}.
\]

Setting \( k = \lceil (3 - \sqrt{5})n/4 \rceil \), an easy calculation gives

\[
\lambda(S_{n,k}) = \frac{k-1}{2} + \sqrt{k n - \frac{1}{4}(3k^2 + 2k - 1)} > \frac{n}{2} \geq \left\lfloor \frac{n^2}{4} \right\rfloor.
\]

However, \( S_{n,k} \) does not contain cycles longer than \( 2k \approx 0.382n < n/2 \). Recently, Ning and Peng [161] slightly refined this as \( C = 1/160 \). Zhai and Lin [199] improved these results to \( C \geq 1/7 \), and Li and Ning [118] proved that \( C \geq 1/4 \) by applying a method different from previous ones.

As a consequence of Theorem 1.47 we can get the following corollary.
Corollary 1.49. Let \( k \geq 1 \) and \( G \) be a graph of sufficiently large order \( n \geq 640k + 320 \). If \( G \) does not contain a copy of \( C_{2k+1} \), then \( \lambda(G) \leq \sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \).

For the extremal problems on the signless Laplacian spectral radius, the following results on odd cycles are proved in [87] and [194].

Theorem 1.50 (Freitas–Nikiforov–Patuzzi [87]). If \( G \) is a graph on \( n \geq 6 \) vertices with no copy of \( C_5 \), then either \( q(G) < q(K_2 \vee I_{n-2}) \) or \( G = K_2 \vee I_{n-2} \).

In 2014, Yuan proved the general result for the signless Laplacian radius.

Theorem 1.51 (Yuan [194]). Let \( k \geq 3, n \geq 100k^2 \) and \( G \) be an \( n \)-vertex graph. If \( G \) has no copy of \( C_{2k+1} \), then either \( q(G) < q(K_k \vee I_{n-k}) \) or \( G = K_k \vee I_{n-k} \).

Recall that \( S_{n,k} = K_k \vee I_{n-k} \), it is not hard to show that
\[
q(S_{n,k}) = \frac{n + 2k - 2 + \sqrt{(n + 2k - 2)^2 - 8k(k - 1)}}{2}.
\]

Note that the extremal graphs for spectral problem and signless Laplacian spectral problem are different when we forbid the odd-length cycles, the former is the balanced complete bipartite graph \( T_2(n) \) and the latter is the split graph \( S_{n,k} \).

1.5 Spectral problem for even cycles

In this section, we shall review extremal problems involving the cycles of even length. Note that even cycles are clearly bipartite, the Kővari–Sós–Turán Theorem 1.32 implies that \( \text{ex}(n, C_{2k}) \leq \text{ex}(n, K_{k,k}) = O(n^{2-1/k}) \) for every integer \( k \geq 2 \).

There have abundant references of the study of \( \text{ex}(n, C_4) \) in the literature; see, e.g., [65, 28, 82]. By double counting method, Reiman showed an upper bound that
\[
\text{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n - 3}).
\] (5)

Furthermore, the equality holds if and only if every pair of vertices has exactly one common neighbor. However, The well-known Friendship Theorem (see, e.g., [65] or [3, Chapter 43]) states that if \( G \) is an \( n \)-vertex graph in which every pair of vertices has exactly one common neighbor, then \( n \) is odd and there is a vertex which is adjacent to all other vertices in \( G \), i.e., \( G = K_1 \vee \frac{n-1}{2}K_2 \). This implies that \( e(G) = 3(n - 1)/2 \). Hence, the equality in (5) never holds. Such graph \( G \) is called the friendship graph or the windmill graph.

Using orthogonal polarity graphs constructed from finite projective planes, Erdős, Rényi and Sós [65] and Brown [28] proved a lower bound, which states that
\[
\text{ex}(n, C_4) \geq \frac{1}{2}n^{3/2} + o(n^{3/2}).
\]

This result together with [65] implies the following asymptotic Turán number.

Theorem 1.52 (Erdős–Rényi–Sós, 1966). \( \text{ex}(n, C_4) = \frac{1}{2}n^{3/2} + o(n^{3/2}) \).
We remark here that there are many constructions of graphs attaining this lower bound asymptotically; see, e.g., [24]. We next include a famous construction by using the projective plane over the finite field, which is now known as the Erdős–Rényi projective polarity graph.

Proof. Let \( q \) be a prime power. Then the field of order \( q \) exists and denoted by \( \mathbb{F}_q \). We consider the three-dimensional vector space \( \mathbb{F}_q^3 \) over the field \( \mathbb{F}_q \). Since space \( \mathbb{F}_q^3 \) contains \( q^3 - 1 \) non-zero vectors, \( q^2 + q + 1 \) one-dimensional subspaces and \( q^2 + q + 1 \) two-dimensional subspaces. Moreover, each one-dimensional subspace contains \( q - 1 \) nonzero vectors, every two distinct one-dimensional subspaces intersect only in \((0,0,0)\). Furthermore, each one-dimensional subspace is contained in \( q + 1 \) two-dimensional subspaces, and each two-dimensional subspace contains \( q + 1 \) one-dimensional subspaces. For every nonzero vector \( \vec{a} = (a_1, a_2, a_3) \in \mathbb{F}_q^3 \), let \( A \) denote the one-dimensional subspace spanning by \( \vec{a} \), that is, \( A := \{k(a_1, a_2, a_3) : k \in \mathbb{F}_q \} \).

**Step 1.** We now construct the desired graph \( G \) whose vertices are all one-dimensional subspaces of \( \mathbb{F}_q^3 \), that is, \( V(G) = \{A = \text{Span} \{\vec{a}\} : \vec{a} \in \mathbb{F}_q^3 \} \). Therefore, we get \( |V(G)| = \frac{q^3 - 1}{q - 1} = q^2 + q + 1 \). Let two vertices \( A \) and \( B \) be adjacent if and only if

\[
a_1b_1 + a_2b_2 + a_3b_3 = 0 \quad \text{(over } \mathbb{F}_q)\.
\]

This definition is clearly well-defined, in other words, it does not depend on the choices of representative elements of the one-dimensional subspaces.

Next, we are going to count the number of edges. For each vertex \( A = \text{Span} \{\vec{a}\} \), since \( a_1x_1 + a_2x_2 + a_3x_3 = 0 \) has \( q^2 - 1 \) nonzero solutions in \( \mathbb{F}_q^3 \), which forms exactly \( q + 1 \) one-dimensional subspaces (vertices). Note that the graph \( G \) may have loops (the case \( a_1^2 + a_2^2 + a_3^2 = 0 \) is possible), we remove all loops of \( G \) to get a simple subgraph \( \bar{G} \) on the same vertex set. Thus,

\[
d_{\bar{G}}(A) = \begin{cases} 
q, & \text{if } a_1^2 + a_2^2 + a_3^2 = 0, \\
q + 1, & \text{otherwise}.
\end{cases}
\]

Anyway, we have \( d_{\bar{G}}(A) \geq q \) for every vertex \( A \). Therefore, we get

\[
e(\bar{G}) = \frac{1}{2} \sum_{v \in V} d(v) \geq \frac{1}{2}(q^2 + q + 1)q.
\]

**Step 2.** We are going to prove that \( \bar{G} \) is \( K_{2,2} \)-free.

Let \( V \) and \( W \) be two distinct vertices of \( G \), if \( U = \text{Span} \{\vec{u}\} \) is a common neighbor of \( V \) and \( W \), then \( u_1, u_2, u_3 \) is a solution of the linear equations

\[
\begin{align*}
v_1u_1 + v_2u_2 + v_3u_3 &= 0, \\
w_1u_1 + w_2u_2 + w_3u_3 &= 0.
\end{align*}
\]

Since \( V \) and \( W \) are two distinct one-dimensional subspaces, then the vectors \((v_1, v_2, v_3)\) and \((w_1, w_2, w_3)\) are linearly independent, which means that the space of solutions has
dimension one. Hence the number of common neighbors of \( V \) and \( W \) in \( G \) is exactly one. Note that the unique common neighbor of \( V \) and \( W \) in \( G \) may be \( V \) or \( W \) itself, and we have removed all loops from \( G \) to obtain \( \tilde{G} \) so \( V \) and \( W \) is at most one common neighbor in the loopless graph \( \tilde{G} \). For example, when \( q = 2 \), the vertices \( \text{Span}\{(0,0,1)\} \) and \( \text{Span}\{(1,1,0)\} \) have the common neighbor \( \text{Span}\{(1,1,0)\} \) in \( G \). However, in the loopless graph \( \tilde{G} \), \( \text{Span}\{(0,0,1)\} \) and \( \text{Span}\{(1,1,0)\} \) have no common neighbor.

Combining the above two steps, we conclude that

\[
\text{ex}(q^2 + q + 1, K_{2,2}) \geq e(\tilde{G}) \geq \frac{1}{2}(q^2 + q + 1)q.
\]

By the denseness of the prime numbers, we get \( \text{ex}(n, K_{2,2}) = (\frac{1}{2} + o(1))n^{3/2} \).

We mention here that the graph \( \tilde{G} \) constructed in the above is called the Erős–Renyi projective polarity graph, and denoted by \( ER_q \). Furthermore, we can compute the exact number of edges of \( ER_q \) by applying the argument of the standard linear algebra method. More precisely, the graph \( ER_q \) has \( q+1 \) vertices with degree \( q \), and \( q^2 \) vertices with degree \( q+1 \). Consequently, we get

\[
e(ER_q) = \frac{1}{2}((q+1)q + q^2(q+1)) = \frac{1}{2}(q+1)^2q.
\]

Generally speaking, determining the exact value of \( \text{ex}(n, C_4) \) seems to be an extremely difficult problem and far beyond reach. In 1996, Füredi [82] showed that for \( n \) large enough with the form \( n = q^2 + q + 1 \), the extremal number is attained by the polarity graph \( ER_q \) of a projective plane.

**Theorem 1.53** (Füredi [82]). Let \( q \geq 15 \) be a prime power. Then

\[
\text{ex}(q^2 + q + 1, C_4) = \frac{1}{2}q(q+1)^2.
\]

For the extremal number of \( C_6 \), the best possible is the following.

**Theorem 1.54** (Füredi–Naor–Verstraëte [83]).

\[
0.5338n^{\frac{4}{3}} \leq \text{ex}(n, C_6) \leq 0.6272n^{\frac{4}{3}}.
\]

The asymptotic extremal number for \( C_8 \) remains open.

**Conjecture 1.55.** Let \( C_8 \) be the cycle on 8 vertices. Then

\[
\text{ex}(n, C_8) = \Theta(n^{\frac{5}{4}}).
\]

**Theorem 1.56** (Lazebnik–Ustimenko–Woldar [114]).

\[
\text{ex}(n, C_{10}) \geq (4 \cdot 5^{-\frac{5}{3}} + o(1))n^{\frac{6}{5}} \approx (0.5798 + o(1))n^{\frac{6}{5}}.
\]
Theorem 1.57 (Bondy–Simonovits [26]). If $G$ is an $n$-vertex graph with $$e(G) > 100k \cdot n^{1+\frac{1}{k}},$$ then $G$ contains a copy of $C_{2\ell}$ for every integer $\ell \in [k, kn^{1/k}]$.

The Bondy–Simonovits theorem implies that $\text{ex}(n, C_{2k}) \leq 100k n^{1+1/k}$. Next, we shall list some improvements on this upper bound in the literature.

- (Verstraëte [178]) $\text{ex}(n, C_{2k}) \leq 8(k-1)n^{1+1/k}$.
- (Pikhurko [167]) $\text{ex}(n, C_{2k}) \leq (k-1)n^{1+1/k} + 16(k-1)n$.
- (Bukh–Jiang [31]) $\text{ex}(n, C_{2k}) \leq 80\sqrt{k} \log k \cdot n^{1+1/k} + O(n)$.
- (He [94]) $\text{ex}(n, C_{2k}) \leq (16\sqrt{5} \sqrt{k} \log k + o(1)) \cdot n^{1+1/k}$.

Conjecture 1.58 (Erdős–Simonovits, 1982). $$\text{ex}(n, C_{2k}) = \Theta(n^{1+\frac{1}{k}}).$$

In the sequel, we shall review the results on extremal spectral problems for even cycles. Let $n$ be an odd integer and $G = K_1 \lor n-1/2K_2$ be the friendship graph on $n$ vertices. By an easy calculation, we know that $e(G) = 3/2(n-1)$ and $\lambda(G) = 1 + \sqrt{4n-3}$.

Theorem 1.59 (Nikiforov [147]). Let $G$ be a graph on $n$ vertices. If $G$ has no $4$-cycle $C_4$, then $\lambda(G) \leq \frac{1 + \sqrt{4n-3}}{2}$, equality holds if and only if $n$ is odd and $G = K_1 \lor \frac{n-1}{2}K_2$.

This theorem states that if $G$ is $C_4$-free, then

$$\frac{2m}{n} \leq \lambda(G) \leq \frac{1 + \sqrt{4n-3}}{2}. \quad (6)$$

This spectral result implies $\text{ex}(n, C_4) \leq \left(\frac{1}{2} + o(1)\right)n^{3/2}$.

The Friendship Theorem [3, Chapter 43] states that if $G$ is a graph on $n$ vertices such that every two distinct vertices have exactly one common neighbor, then $n$ is odd and $G = K_1 \lor \frac{n-1}{2}K_2$, a graph obtained from $\frac{n-1}{2}$ triangles sharing a single common vertex. Such a graph is usually called the friendship graph. The equality in Theorem 1.59 is best possible only for $n$ odd, and Nikiforov’s result may be improved for even $n$.

Erdős and Renyi showed that if $q$ is a prime power, then the polarity graph $ER_q$ is $C_4$-free on $n = q^2 + q + 1$ vertices with $e(ER_q) = \frac{2(q+1)^2}{2}$. Thus, we have

$$\lambda(ER_q) \geq \frac{2e(ER_q)}{n} = \frac{q(q+1)^2}{q^2+q+1} > q + 1 - \frac{1}{q} = 1 + \frac{\sqrt{4n-3}}{2} - \frac{1}{\sqrt{n-1}}.$$

This lower bound is quite close to the upper bound in (6).

In 2012, Zhai and Wang improved the result of Nikiforov when $n$ is even.
Theorem 1.60 (Zhai–Wang [196]). Let $G$ be a graph of even order $n$. If $G$ has no copy of $C_4$, then $\lambda(G) \leq x_0$, where $x_0$ is the largest root of the equation

$$x^3 - x^2 - (n - 1)x + 1 = 0.$$  

Moreover, the equality holds if and only if $G$ is a graph obtained from the star $K_{1,n-1}$ by adding $\frac{n}{2} - 1$ disjoint edges in its independent set.

Let $K_k \vee I_{n-k}$ be the graph obtained by joining each vertex of $K_k$ to $n - k$ isolated vertices of $I_{n-k}$, and let $K_k \vee I_{n-k}^+$ be the graph obtained by adding one edge within the independent set of $K_k \vee I_{n-k}$. We can see that

- $C_{2k+1} \notin K_k \vee I_{n-k}$ and $C_{2k+2} \notin K_k \vee I_{n-k}$.
- $C_{2k+1} \subseteq K_k \vee I_{n-k}$ and $C_{2k+2} \subseteq K_k \vee I_{n-k}^+$.

Conjecture 1.61 (Nikiforov [155]). Let $k \geq 2$ and $G$ be a graph of sufficiently large order $n$. If $G$ is $C_{2k+2}$-free, then

$$\lambda(G) \leq \lambda(K_k \vee I_{n-k}^+),$$

equality holds if and only if $G = K_k \vee I_{n-k}^+ + 1$.

In 2020, Zhai and Lin [197] proved Conjecture 1.61 for the case $k = 2$ and $n \geq 23$.

Conjecture 1.62 (Nikiforov [155]). Let $k \geq 2$ and $G$ be a graph of sufficiently large order $n$. If $G$ is both $C_{2k+1}$-free and $C_{2k+2}$-free, then

$$\lambda(G) \leq \lambda(K_k \vee I_{n-k}),$$

equality holds if and only if $G = K_k \vee I_{n-k}^+ + 1$.

The case $k = 1$ in Conjecture 1.62 was early proved by Favaron, Mahéo and Saclé [69].

In 2012, Yuan, Wang and Zhai [195] proved Conjecture 1.62 for the case $k = 2$ and $n \geq 6$.

In 2019, Gao and Hou [91] proved Conjectures 1.61 and 1.62 under stronger conditions. More precisely, Conjectures 1.61 holds when we forbid all cycles of length at least $2k + 2$. Conjectures 1.62 holds when we forbid all cycles of length at least $2k + 1$.

Theorem 1.63 (Gao–Hou [91]). Let $k \geq 2$ and $G$ be a graph of order $n \geq 13k^2$. If $G$ does not contain a cycle of length at least $2k + 2$, then $\lambda(G) \leq \lambda(K_k \vee I_{n-k}^+)$, equality holds if and only if $G = K_k \vee I_{n-k}^+ + 1$.

Theorem 1.64 (Gao–Hou [91]). Let $k \geq 2$ and $G$ be a graph of order $n \geq 13k^2$. If $G$ does not contain a cycle of length at least $2k + 1$, then $\lambda(G) \leq \lambda(K_k \vee I_{n-k})$, equality holds if and only if $G = K_k \vee I_{n-k}$.

Remark. In 2022, Cioabă, Desai and Tait [47] confirmed Nikiforov’s conjecture.

In what follows, we shall introduce some extremal graph results in terms of the signless Laplacian spectral radius. For odd $n$, we write $F_n$ for the friendship graph on $n$ vertices, that is, $F_n = K_1 \vee \frac{n-1}{2}K_2$; for even $n$, we write $F_n$ for the graph obtained from $F_{n-1}$.
by adding an extra edge hung to its center. In other words, the $F_n$ can be viewed as a graph obtained from $K_{1,n-1}$ by adding a maximum matching within the independent set $I_{n-1}$. The following result is an analogue of both Theorem 1.59 and Theorem 1.60 for the signless Laplacian spectral radius.

**Theorem 1.65** (Freitas–Nikiforov–Patuzzi [87]). If $G$ is a graph on $n \geq 4$ vertices with no copy of $C_4$, then $q(G) \leq q(F_n)$, equality holds if and only if $G = F_n$.

In addition, Freitas et. al. [87] proposed a conjecture for forbidden even cycles of length at least 6, which was solved by Nikiforov and Yuan [157].

**Theorem 1.66** (Nikiforov–Yuan [157]). Let $k \geq 2$, $n \geq 400k^2$ and $G$ be a graph on $n$ vertices. If $G$ has no copy of $C_{2k+2}$, then $q(G) \leq q(K_k \vee I_{n-k}^+)$, equality holds if and only if $G = K_k \vee I_{n-k}^+$.

### 1.6 Spectral problem for color-critical graphs

In this section, we shall review a special class of graphs, which is now known as the color-critical graphs (also known as edge-color-critical graphs). The definition is stated as below.

**Definition 1.67.** Let $e$ be an edge of graph $F$. We say that $e$ is a color-critical edge if $\chi(F - e) < \chi(F)$. We say that $F$ is color-critical if $F$ contains a color-critical edge.

The following are some examples. Clearly, the complete graph $K_{r+1}$ is color-critical and $\chi(K_{r+1}) = r + 1$. The odd cycle $C_{2k+1}$ is color-critical and $\chi(C_{2k+1}) = 3$.

Let $G$ and $H$ be two graphs. The join of $G$ and $H$ is defined as a graph $G \cup H$ obtained from the disjoint union $G \cup H$ by adding all edges connecting every vertex of $G$ to every vertex of $H$. For example, the wheel graph $W_n = K_1 \vee C_{n-1}$ and the book graph $B_t = K_2 \vee K_t$. In addition, we can also see that $K_{s_1,s_2,...,s_r} = I_{s_1} \vee K_{s_2,...,s_r} = K_{s_1,s_2} \vee K_{s_3,...,s_r}$.

**Example 1.68** (The wheel graph). Let $W_{2k} = K_1 \vee C_{2k-1}$ be the wheel graph, a vertex that connects all vertices of an odd cycle $C_{2k-1}$. Then $W_{2k}$ is color-critical and $\chi(W_{2k}) = 4$. However, we can check that $W_{2k+1} = K_1 \vee C_{2k}$ is not color-critical.

Moreover, for integer $s \geq 2$, we define $W_{s,m} := K_s \vee C_m$ as the generalized wheel graph. Since $s \geq 2$, the generalized wheel graph $W_{s,m}$ is always color-critical. Moreover, we have $\chi(W_{s,2k-1}) = s + 3$ and $\chi(W_{s,2k}) = s + 2$.

**Example 1.69** (The book graph). Let $B_k = K_2 \vee I_{k-2}$ be the book graph, that is, $k$ triangles sharing a common edge. Then $B_k$ is color-critical and $\chi(B_k) = 3$.

To extend the above result, we can consider the extremal number of the generalized book graph $B_{s,t} := K_s \vee K_t$ for some integer $s \geq 3$. This graph can also be viewed as $t$ cliques $K_{s+1}$ sharing a common sub-clique $K_s$. We can see that $B_{s,t}$ is color-critical and $\chi(B_{s,t}) = s + 1$.

On the other hand, the book graph $B_t$ is defined as the $t$ triangles sharing a common edge. Let $SE_{G,t}$ be the graph obtained from $t$ copies of $G$ by sharing a common edge.
We can extend this definition to \( t \) cliques \( K_r \) sharing a common edge. It is easy to see that \( SE_{K_r,t} \) is color-critical and \( \chi(SE_{K_r,t}) = r \). In addition, we also can extend the book graph to \( t \) odd-length cycles \( C_{2k+1} \) sharing a common edge. It is clear that \( SE_{C_{2k+1},t} \) is a color-critical graph and \( \chi(SE_{C_{2k+1},t}) = 3 \).

Let \( K_{1,1,n_3,\ldots,n_r} \) be the complete \( r \)-partite graph with two parts of order 1 and other parts of order \( n_i \) for each \( i \in [3,r] \). Then \( K_{1,1,n_3,\ldots,n_r} \) is color-critical and \( \chi(K_{1,1,n_3,\ldots,n_r}) = r \). Note that \( SE_{K_r,t} \subseteq K_{1,1,t,\ldots,t} \). Let \( K_r(s_1,s_2,\ldots,s_r) \) be the complete \( r \)-partite graph on the vertex sets \( V_1, V_2, \ldots, V_s \), where each part \( V_i \) has \( s_i \) vertices.

**Example 1.70.** Let \( s_1 \geq 2 \) be an integer. We write \( K_r^+(s_1,s_2,\ldots,s_r) \) for the graph obtained from \( K_r(s_1,s_2,\ldots,s_r) \) by adding an edge within the part \( V_1 \). It is easy to see that \( K_r^+(s_1,s_2,\ldots,s_r) \) is color-critical and \( \chi = r + 1 \).

It is important that for any color-critical graph \( G \) with \( \chi(G) = r + 1 \), there exist some integers \( s_1, \ldots, s_r \) such that \( G \) is a subgraph of \( K_r^+(s_1,s_2,\ldots,s_r) \).

The famous Erdős–Stone–Simonovits Theorem 1.28 states that if \( F \) is a graph with the vertex-chromatic number \( \chi(F) = r + 1 \), then
\[
t_r(n) \leq \text{ex}(n,F) = \left(1 - \frac{1}{r}\right) \frac{n^2}{2} + o(n^2) = t_r(n) + o(n^2).
\]

In previous section, we have proved that for some special graphs, we can remove the error term \( o(n^2) \). For example, the Turán Theorem 1.3 states that \( \text{ex}(n,K_{r+1}) = t_r(n) \), the Bondy–Woodall Theorem 1.44 implies that \( \text{ex}(n,C_{2k+1}) = t_2(n) \) holds for \( n \geq 4k + 1 \), and the Dzido Theorem 1.46 states that \( \text{ex}(n,W_{2k}) = t_3(n) \) holds for \( n \geq 6k - 10 \). Moreover, a result of Edwards [59], and Khadžiivanov–Nikiforov [105] independently implies that \( \text{ex}(n,B_k) = t_2(n) \) holds for \( n \geq 6k \); see, e.g., [16] [18] [19] for more details.

In what follows, we present a more general result proved by Simonovits.

**Theorem 1.71 (Simonovits [169]).** Let \( F \) be a graph with \( \chi(F) = r + 1 \) where \( r \geq 2 \). If \( F \) is color-critical, then for sufficiently large \( n \),
\[
\text{ex}(n,F) = t_r(n).
\]

Moreover, the unique extremal graph is Turán graph \( T_r(n) \).

In 2009, Nikiforov [153] extended Theorem 1.71 by showing that there is \( n_0 \) such that if \( G \) has \( n \geq n_0 \) vertices and \( \lambda(G) > \lambda(T_r(n)) \), then \( G \) contains a copy of the complete \( r \)-partite graph with parts of size \( \Omega(\ln n) \) plus an extra edge.

**Theorem 1.72 (Nikiforov [153]).** Let \( r \geq 2 \) and \( 2/\ln n \leq c \leq 1/r^{(2r+9)(r+1)} \). If \( G \) is a graph on \( n \) vertices with \( \lambda(G) > \lambda(T_r(n)) \), then \( G \) contains a copy of
\[
K_r^+([c \ln n], \ldots, [c \ln n], [n^{1-\sqrt{c}}]).
\]

As a consequence of Theorem 1.72 we can easily get the following weak corollary.
Corollary 1.73. If $F$ is color-critical and $\chi(F) = r + 1$ where $r \geq 2$, then

$$\text{ex}_\chi(n, F) = \lambda(T_r(n))$$

holds for sufficiently large $n$, and the unique extremal graph is $T_r(n)$.

Remark. Recall that the $p$-spectral radius \cite{104} of a graph $G$ is defined as $\lambda^{(p)}(G) := \max\{x^T A(G)x : x \in \mathbb{R}^n, \|x\|_p = 1\}$. In 2014, Keevash, Lenz and Mubayi \cite{104}, Corollary 1.5] extended the above spectral version in Corollary 1.73 to the $p$-spectral radius $\lambda^{(p)}(G)$ for every $p > 1$. Furthermore, Kang and Nikiforov \cite{101}, Theorem 6] proved the $p$-spectral version of Theorem 1.72 and hence generalized the result in \cite{104}, since the order in each part of $K^+_r(\lceil c\ln n \rceil)$ grows with $n$, instead of a fixed integer in the forbidden subgraph $K^+_r(t)$. This difference makes the proof longer with more advanced techniques.

1.7 Spectral problem for intersecting triangles

Let $F_k$ denote the $k$-fan graph which is the graph consisting of $k$ triangles that intersect in exactly one common vertex. Note that $F_k$ has $2k + 1$ vertices. This notation is slightly different from that in Section 1.5. This graph is known as the friendship graph because it is the only extremal graph in the well-known Friendship Theorem \cite[Chapter 43]{3]. Since $\chi(F_k) = 3$, the Erdős–Stone–Simonovits Theorem 1.28 implies that $\text{ex}(n, F_k) = n^2/4 + o(n^2)$. In 1995, Erdős, Füredi, Gould and Gunderson \cite{68} proved the following exact result.

Theorem 1.74 (Erdős et al. \cite{68}). For every $k \geq 1$ and $n \geq 50k^2$, we have

$$\text{ex}(n, F_k) = \left\lceil \frac{n^2}{4} \right\rceil + \begin{cases} k^2 - k, & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even}. \end{cases}$$

The extremal graphs of Theorem 1.74 are as follows. For odd $k$ (where $n \geq 4k - 1$), the extremal graphs are constructed by taking $T_2(n)$, the balanced complete bipartite graph, and embedding two vertex disjoint copies of $K_k$ in one side. For even $k$ (where now $n \geq 4k - 3$), the extremal graphs are constructed by taking $T_2(n)$ and embedding a graph with $2k - 1$ vertices, $k^2 - \frac{3}{2}k$ edges with maximum degree $k - 1$ in one side.

The $(k, r)$-fan is the graph consisting of $k$ copies of the clique $K_r$ which intersect on a single vertex, and is denoted by $F_{k,r}$. In particular, we have $F_{1,r} = K_r$ and $F_{k,3} = F_k$. Note that $\chi(F_{k,r}) = r$. Similarly, the Erdős–Stone–Simonovits Theorem 1.28 also implies that $\text{ex}(n, F_{k,r}) = (1 - \frac{1}{r-1})\frac{n^2}{2} + o(n^2) = t_{r-1}(n) + o(n^2)$. In 2003, Chen, Gould, Pfender and Wei \cite{35} proved an exact answer and generalized Theorem 1.74 as follows.

Theorem 1.75 (Chen et al. \cite{35}). For every $k \geq 2$ and $r \geq 3$, if $n \geq 16k^3r^8$, then

$$\text{ex}(n, F_{k,r}) = t_{r-1}(n) + \begin{cases} k^2 - k, & \text{if } k \text{ is odd}, \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even}. \end{cases}$$
We remark here that there is a typo in [35] since the correct condition should be \( r \geq 3 \).

For the \( r = 2 \) case, we observe that \( F_{k,2} \) is a star graph and \( \mathrm{ex}(n,F_{k,2}) = \Theta(n) \).

The extremal graphs of Theorem 1.75 are constructed by taking the \((r - 1)\)-partite Turán graph \( T_{r - 1}(n)\) and embedding a graph \( G_0 \) in one vertex part, denoted by \( G_{n,k,r} \). If \( k \) is odd, \( G_0 \) is isomorphic to two vertex disjoint copies of \( K_k \). If \( k \) is even, \( G_0 \) is isomorphic to the graph with \( 2k - 1 \) vertices, \( k^2 - \frac{3k}{2} \) edges with maximum degree \( k - 1 \).

Let \( C_{k,q} \) be the graph consisting of \( k \) cycles of length \( q \) which intersect exactly in one common vertex. The graph \( C_{k,q} \) for even \( q \) is a star graph and \( \mathrm{ex}(n,C_{k,q}) = n^2/4 + o(n^2) \). In 2016, Hou, Qiu and Liu [98] determined exactly the extremal number for \( C_{k,q} \) with \( k \geq 1 \) and odd integer \( q \geq 5 \).

**Theorem 1.76** (Hou–Qiu–Liu [98]). For an integer \( k \geq 1 \) and an odd integer \( q \geq 5 \), there exists \( n_0(k,q) \) such that for all \( n \geq n_0(k,q) \), we have

\[
\mathrm{ex}(n,C_{k,q}) = \left\lfloor \frac{n^2}{4} \right\rfloor + (k - 1)^2.
\]

The extremal graphs are a Turán graph \( T_2(n) \) with a \( K_{k-1,k-1} \) embedding into one class.

**Remark.** We emphasize here that when \( q \) is even, then \( C_{k,q} \) is a bipartite graph where every vertex in one of its parts has degree at most 2. For such a sparse bipartite graph, a classical result of Füredi [79] or Alon, Krivelevich and Sudakov [6] implies that \( \mathrm{ex}(n,C_{k,q}) = \Theta(n^{3/2}) \). Recently, a breakthrough result of Conlon, Lee and Janzer [52, 53] shows that for even \( q \geq 6 \) and \( k \geq 1 \), we have \( \mathrm{ex}(n,C_{k,q}) = \Theta(n^{3/2-\delta}) \) for some \( \delta = \delta(k,q) > 0 \). It is a challenging problem to determine the value \( \delta(k,q) \). For instance, the special case \( k = 1 \), this problem reduces to determine the extremal number for even cycle.

Next, we introduce a unified extension of both Theorems 1.74 and 1.76. Let \( s \) be a positive integer and \( t_1, \ldots, t_k \geq 5 \) be odd integers. We write \( H_{s,t_1,\ldots,t_k} \) for the graph consisting of \( s \) triangles and \( k \) odd cycles of lengths \( t_1, \ldots, t_k \) in which these triangles and cycles intersect in exactly one common vertex. The graph \( H_{s,t_1,\ldots,t_k} \) is also known as the flower graph with \( s+k \) petals. We remark that the \( k \) odd cycles can have different lengths. Clearly, when \( t_1 = \cdots = t_k = 0 \), then \( H_{s,0,\ldots,0} = F_s \), the \( s \)-fan graph; see Theorem 1.74. In addition, when \( s = 0 \) and \( t_1 = \cdots = t_k = q \), then \( H_{0,q,\ldots,q} = C_{k,q} \); see Theorem 1.76.

In 2018, Hou, Qiu and Liu [99] and Yuan [192] independently determined the extremal number of \( H_{s,k} \) for \( s \geq 0 \) and \( k \geq 1 \). Let \( F_{n,s,k} \) be the family of graphs with each member being a Turán graph \( T_2(n) \) with a graph \( H \) embedded in one partite set, where

\[
H = \begin{cases} 
K_{s+k-1,s+k-1}, & \text{if } (s,k) \neq (3,1), \\
K_{3,3} \text{ or } 3K_3, & \text{if } (s,k) = (3,1), 
\end{cases}
\]

where \( 3K_3 \) is the union of three disjoint triangles.

**Theorem 1.77** (Hou–Qiu–Liu [99]; Yuan [192]). For every graph \( H_{s,t_1,\ldots,t_k} \) with \( s \geq 0 \) and \( k \geq 1 \), there exists \( n_0 \) such that for all \( n \geq n_0 \), we have

\[
\mathrm{ex}(n,H_{s,t_1,\ldots,t_k}) = \left\lfloor \frac{n^2}{4} \right\rfloor + (s+k-1)^2.
\]
Moreover, the only extremal graphs for $H_{s,t_1,...,t_k}$ are members of $F_{n,s,k}$.

Another interesting problem on this topic is to determine the Turán number of $C_k,q$ for even $q$. More general, it is challenging to determine the Turán number of $H_{s,t_1,...,t_k}$ where the cycles have even lengths.

Recall that $\text{Ex}(n,F)$ denotes the set of graphs that contain no copy of $F$ and have maximum number of edges. For example, the Turán theorem gives $\text{Ex}(n,K_{r+1}) = \{T_r(n)\}$ for every $r \geq 2$, and the Simonovits theorem yields that for every fixed color-critical graph $F$ with $\chi(F) = r + 1 \geq 3$, we have $\text{Ex}(n,F) = \{T_r(n)\}$ for sufficiently large $n$. In 2020, Cioabă, Feng, Tait and Zhang [45] proved the spectral version of Theorem 1.74.

**Theorem 1.78** (Cioabă et al. [45]). Let $k \geq 2$ and $G$ be a graph of order $n$ that does not contain a copy of $F_k$. For sufficiently large $n$, if $G$ has the maximal spectral radius, then

$$G \in \text{Ex}(n,F_k).$$

In 2021, it was proved by Li and Peng in [122] that the spectral version of intersecting odd cycles is also similar. Soon after, Desai, Kang, Li, Ni, Tait and Wang [54] proved the analogous result for intersecting cliques.

**Theorem 1.79** (Li–Peng [122]). Let $G$ be a graph of order $n$ that does not contain a copy of $H_{s,t_1,...,t_k}$, where $s \geq 0$ and $k \geq 1$. For sufficiently large $n$, if $G$ has the maximal spectral radius, then

$$G \in \text{Ex}(n,H_{s,t_1,...,t_k}).$$

**Theorem 1.80** (Desai et al. [54]). Let $G$ be a graph of order $n$ that does not contain a copy of $F_{k,r}$, where $k \geq 2$ and $r \geq 3$. For sufficiently large $n$, if $G$ has the maximal spectral radius, then

$$G \in \text{Ex}(n,F_{k,r}).$$

Recently, Cioabă, Desai and Tait [46] investigated the largest spectral radius of an $n$-vertex graph that does not contain the odd-wheel graph $W_{2k+1}$, which is the graph obtained by joining a vertex to a cycle of length $2k$. Moreover, they raised the following more general conjecture.

**Conjecture 1.81.** Let $F$ be any graph such that the graphs in $\text{Ex}(n,F)$ are Turán graphs plus $O(1)$ edges. Then for sufficiently large $n$, a graph attaining the maximum spectral radius among all $F$-free graphs is a member of $\text{Ex}(n,F)$.

**Remark.** This conjecture was recently confirmed by Wang, Kang and Xue [180].

Recall that $F$ is called edge-color-critical if there exists an edge $e$ of $F$ such that $\chi(F-e) < \chi(F)$. Let $F$ be an edge-color-critical graph with $\chi(F) = r + 1$. By a result of Simonovits [169] and a result of Nikiforov [153] or Keevash et al. [104], we know that $\text{Ex}(n,F) = \text{Ex}_F(n,F) = \{T_r(n)\}$ for sufficiently large $n$, this shows that Conjecture 1.81 is true for all edge-color-critical graphs. As we mentioned before, Theorems 1.78, 1.80 and 1.79 say that Conjecture 1.81 holds for the $k$-fan graph $F_k$, the $(k,r)$-fan graph $F_{k,r}$ and
the intersecting odd cycle $H_{s,t_1,...,t_k}$. Note that these graphs $F_k, F_{k,r}$ and $H_{s,t_1,...,t_k}$ are not edge-color-critical.

Recall that $S_{n,k} = K_k \lor I_{n-k}$. Clearly, we can see that $S_{n,k}$ does not contain $F_k$ as a subgraph. Recently, Zhao, Huang and Guo [205] proved that $S_{n,k}$ is the unique graph attaining the maximum signless Laplacian spectral radius among all graphs of large order $n$ containing no copy of $F_k$.

**Theorem 1.82** (Zhao–Huang–Guo [205]). Let $k \geq 2$ and $n \geq 3k^2 - k - 2$. If $G$ is an $n$-vertex graph that does not contain a copy of $F_k$, then $q(G) \leq q(S_{n,k})$, with equality holding if and only if $G = S_{n,k}$.

It is worth mentioning that the extremal graphs in Theorem 1.82 are not the same as those of Theorem 1.78. In addition, for $k = 1$, i.e., $G$ is triangle-free, from Theorem 1.24 we know that $q(G) \leq n$, equality holds if and only if $G$ is a complete bipartite graph.

It is a natural question to consider the maximum signless Laplacian spectral radius among all graphs containing no copy of $C_{k,q}$ or $H_{s,t_1,...,t_k}$. When the paper [122] was announced (arXiv:2106.00587), Chen, Liu and Zhang [41] proved quickly the signless Laplacian spectral version for $C_{k,q}$. More generally, they showed the result for graph $H_{s,t_1,...,t_k}$ for odd integers $t_1, \ldots, t_k$.

**Theorem 1.83** (Chen–Liu–Zhang [41]). For integers $k \geq 2, t \geq 1$ and $q = 2t + 1$, there exists an integer $n_0$ such that if $n \geq n_0$ and $G$ is a $C_{k,q}$-free graph on $n$ vertices, then

$$q(G) \leq q(S_{n,kt}),$$

equality holds if and only if $G = S_{n,kt}$.

It is natural to ask the following problem, which is the signless Laplacian spectral version of the extremal problem for intersecting cliques. Clearly, when $r = 3$, our conjecture reduces to the result of Zhao et al. [205].

**Conjecture 1.84.** [54] For integers $k \geq 2$ and $r \geq 3$, there exists an integer $n_0(k,r)$ such that if $n \geq n_0(k,r)$ and $G$ is a $F_{k,r}$-free graph on $n$ vertices, then

$$q(G) \leq q(S_{n,k(r-2)}),$$

equality holds if and only if $G = S_{n,k(r-2)}$.

### 2 Spectral problem for Hamiltonianity

A cycle passing through all vertices of a graph is called a Hamilton cycle. A graph containing a Hamilton cycle is called a Hamiltonian graph. A path passing through all vertices of a graph is called a Hamilton path and a graph containing a Hamilton path is said to be traceable.

Every complete graph on at least three vertices is evidently Hamiltonian. Indeed, the vertices of a Hamilton cycle can be selected one by one in an arbitrary order. But suppose
that our graph has considerably fewer edges. In particular, we may ask how large the
minimum degree must be in order to guarantee the existence of a Hamilton cycle. The
celebrated Dirac theorem [57] answers this question. It states that every graph with \( n \geq 3 \)
vertices and minimum degree at least \( n/2 \) has a Hamilton cycle. The condition is sharp
when we consider the complete bipartite graph with the parts of sizes \( \left\lfloor \frac{n-1}{2} \right\rfloor \) and \( \left\lceil \frac{n+1}{2} \right\rceil \).

The closure operation introduced by Bondy and Chvátal [23] is a powerful tool for
the problems of Hamiltonicity of graphs. Let \( G \) be a graph of order \( n \). The \( s \)-closure of
\( G \), denoted by \( \text{cl}_s(G) \), is the graph obtained from \( G \) by recursively joining pairs of non-
adjacent vertices whose degree sum is at least \( s \) until no such pair remains. It is not
hard to prove that the \( s \)-closure of \( G \) is uniquely determined; see, e.g., [23]. Clearly, if \( G \)
contains a Hamilton cycle, then so does \( \text{cl}_s(G) \) for every \( s \) since \( G \) is a subgraph of \( \text{cl}_s(G) \).
Surprisingly, Ore [165] proved that both \( \text{cl}_n(G) \) and \( G \) keep in line with the existence of
Hamilton cycle.

**Theorem 2.1** (Ore [165]). A graph \( G \) is Hamiltonian if and only if the closure graph
\( \text{cl}_n(G) \) is Hamiltonian. In particular, if \( d(u) + d(v) \geq n \) for all non-edges \( \{u, v\} \), then
\( \text{cl}_n(G) = K_n \) and \( G \) is Hamiltonian.

The proof is simple and similar with that of the Dirac theorem, so we here conclude
the proof briefly. Without loss of generality, we only consider the case \( \text{cl}_n(G) = G + uv \),
where \( d_G(u) + d_G(v) \geq n \). If \( G + uv \) is Hamiltonian, then \( G \) has a Hamiltonian path
\( u = u_1, u_2, \ldots, u_n = v \). We denote by \( S = \{ i : uu_{i+1} \in E(G) \} \) and \( T = \{ i : u_iv \in E(G) \} \).
Clearly, we have \( |S| = d_G(u) \) and \( |T| = d_G(v) \). Note that \( u_n \notin S \cup T \), which implies that
\( |S \cup T| \leq n - 1 \). Since \( d_G(u) + d_G(v) \geq n \), we can get
\( |S \cap T| = |S| + |T| - |S \cup T| \geq d_G(u) + d_G(v) - (n - 1) \geq 1 \). Thus there must be some \( i \) such that \( u \) is adjacent to \( u_{i+1} \) and \( v \) is
adjacent to \( u_i \), which implies that \( G \) has the Hamilton cycle \( u_1u_{i+1}u_{i+2} \cdots u_nu_1 \).

In 1972, Chvátal [43] proved the following theorem, which characterizes the degree
sequence of Hamiltonian graph. Theorem 2.2 becomes a classic conclusion in many compre-
hensive textbooks; see [25] pp. 485–488 or [180] pp. 288–290 or [50] pp. 308–312 for
more details.

**Theorem 2.2** (Chvátal [43]). Let \( G \) be an \( n \)-vertex graph with degree sequence \( d_1 \leq d_2 \leq \cdots \leq d_n \).
If \( G \) is not Hamiltonian, then there exists an integer \( i < n/2 \) such that \( d_i \leq i \)
and \( d_{n-i} \leq n - i - 1 \).

The following result is due to Ore [165] and Bondy [22] independently. It is a direct
consequence of the Chvátal theorem; see [24] p. 60 for more details.

**Theorem 2.3** (Ore [165], Bondy [22]). Let \( G \) be a graph on \( n \geq 3 \) vertices. If
\[
e(G) \geq \left( \frac{n-1}{2} \right) + 1,
\]
then \( G \) has a Hamilton cycle unless \( G = K_1 \lor (K_1 \cup K_{n-2}) \) or \( n = 5 \) and \( G = K_2 \lor I_3 \).

In 1962, Erdős improved the above result for graphs with given minimum degree.
Theorem 2.4 (Erdős [62]). Let \( G \) be a graph on \( n \) vertices. If the minimum degree \( \delta(G) \geq \delta \) and
\[
e(G) > \max\left\{ \left(\frac{n-\delta}{2}\right) + \delta^2, \left(\frac{n - \lfloor \frac{n-1}{2} \rfloor}{2}\right) + \left(\frac{n-1}{2}\right)^2 \right\},
\]
then \( G \) has a Hamilton cycle.

We mention here that \( f(\delta) = \binom{n-\delta}{2} + \delta^2 \) is decreasing with respect to \( \delta \) for fixed \( n \geq 6\delta. \) It is sufficient to prove the theorem under the condition \( \delta(G) = \delta. \) Indeed, if \( G \) satisfies \( \delta(G) = t > \delta \) and \( e(G) > f(\delta), \) then we have \( e(G) > f(t). \) The special case \( \delta(G) = t \) can deduce the general case \( \delta(G) > \delta. \) Hence, the condition can be reduced to \( \delta(G) = \delta. \)

To see the sharpness of the bound in Theorem 2.4, we consider the graph \( H_{n,\delta} \) obtained from a copy of \( K_{n-\delta} \) by adding an independent set of \( \delta \) vertices with degree \( \delta \) each of which is adjacent to the same \( \delta \) vertices in \( K_{n-\delta}. \) In the language of graph join and union, that is,
\[
H_{n,\delta} := K_\delta \vee (K_{n-2\delta} \cup I_\delta).
\]
Clearly, we can verify that \( \delta(H_{n,\delta}) = \delta \) and \( H_{n,\delta} \) does not contain Hamilton cycle. Moreover, we have \( e(H_{n,\delta}) = \binom{n-\delta}{2} + \delta^2 \) and \( \lambda(H_{n,\delta}) > \lambda(K_{n-\delta}) = n - 1 - \delta \) since \( K_{n-\delta} \) is a proper subgraph of \( H_{n,\delta}. \) When \( n \geq 6\delta, \) we have \( e(H_{n,\delta}) \geq e(H_{n,\lfloor (n-1)/2 \rfloor}). \) Thus, we get the following corollary.

Corollary 2.5 (Erdős). Let \( k \geq 1 \) and \( n \geq 6\delta. \) If \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq \delta \) and \( e(G) \geq e(H_{n,\delta}) \), then either \( G \) has a Hamilton cycle or \( G = H_{n,\delta}. \)

2.1 Spectral results for Hamilton cycle

Apparently there are very few sufficient conditions for the existence of a Hamilton cycle in graphs. As it turns out spectral properties of graphs can supply rather powerful sufficient conditions for Hamiltonicity. We denote by \( \mu := \max \{ |\lambda_i| : 2 \leq i \leq n \} \) the second largest eigenvalue in absolute value. A famous such result proved by Krivelevich and Sudakov [109] states that if \( G \) is a \( d \)-regular graph with sufficiently large order \( n \) such that
\[
\mu \leq \frac{(\log \log n)^2}{1000 \log n \log \log n} d,
\]
then \( G \) has a Hamilton cycle. The proof of this result is quite involved technically. We omit the proof details here, referring the reader to [109]. The parameter \( \mu \) in the above formulation is usually called the second eigenvalue of the \( d \)-regular graph \( G \) (the first and the trivial eigenvalue being \( \lambda_1 = d \)). To some extent, this terminology is inaccurate, as in fact \( \mu = \max \{ \lambda_2, -\lambda_n \}. \) We will call a \( d \)-regular graph \( G \) on \( n \) vertices in which all eigenvalues, but the first one, are at most \( \mu \) in their absolute values, an \((n,d,\mu)\)-graph. Krivelevich and Sudakov [109] also conjectured that an even stronger result is true, namely, there exists a positive constant \( C \) such that for large enough \( n, \) any \((n,d,\mu)\)-graph that satisfies \( \mu/d < C \) contains a Hamilton cycle.

In 2010, Fiedler and Nikiforov proved a spectral version of Theorem 2.4.
Theorem 2.6 (Fiedler–Nikiforov [77]). If $G$ is a graph on $n \geq 3$ vertices and 

$$\lambda(G) > n - 2,$$

then either $G$ has a Hamilton cycle or $G = K_1 \lor (K_1 \cup K_{n-2})$.

This spectral result can easily be deduced from Theorem 2.3.

**Proof.** The well-known Stanley inequality asserts that $\lambda(G) \leq -1/2 + \sqrt{2e(G) + 1}/4$, which together with $\lambda(G) > n - 2$, implies that $e(G) > \binom{n-1}{2}$. Hence, for $n \neq 5$, the desired result follows from Theorem 2.3. For $n = 5$, if $G$ is the another possible exception $K_2 \lor I_3$, we can calculate that $\lambda(K_2 \lor I_3) = 3$, which contradicts the condition $\lambda(G) > n - 2$. This completes the proof. \qed

In 2013, Yu and Fan [188] gave the corresponding spectral version for the signless Laplacian radius. Recall that $q(G)$ stands for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix.

**Theorem 2.7 (Yu–Fan [188]).** If $G$ is a graph on $n \geq 3$ vertices and 

$$q(G) > 2(n - 2),$$

then $G$ has a Hamilton cycle or $G = K_1 \lor (K_1 \cup K_{n-2})$, or $n = 5$ and $G = K_2 \lor I_3$.

In [188], the counterexample of $n = 5, G = K_2 \lor I_3$ is missed. This tiny flaw has already been pointed out by Liu et al. [138] and by Li and Ning [116] as well.

**Proof.** An important inequality in [72] states that $q(G) \leq \frac{2e(G)}{n-1} + n - 2$, which together with $q(G) > 2(n - 2)$ yields that $e(G) > \binom{n-1}{2}$. Hence, for $n \neq 5$, the desired result follows from Theorem 2.3. For $n = 5$, if $G$ is the another possible exception $K_2 \lor I_3$, we can calculate that $q(K_2 \lor I_3) = \frac{7 + \sqrt{33}}{2} \approx 6.372$. This completes the proof. \qed

Observing that a graph $G$ with minimum degree $\delta(G) = 1$ does not contain Hamilton cycle. Thus $\delta(G) \geq 2$ is a trivial necessary condition for $G$ to be Hamiltonian. In 2015, Ning and Ge [160] refined the extremal result of Ore and the spectral result of Fiedler and Nikiforov for graphs with minimum degree at least two.

**Theorem 2.8 (Ning–Ge [160]).** Let $G$ be a graph on $n \geq 5$ vertices and $\delta(G) \geq 2$. If 

$$e(G) \geq \binom{n-2}{2} + 4,$$

then $G$ has a Hamilton cycle unless $G \in \{K_2 \lor (K_{n-4} \cup I_2), K_3 \lor I_4, K_2 \lor (K_{1,3} \cup K_1), K_1 \lor K_{2,4}, K_3 \lor (K_2 \cup I_3), K_4 \lor I_5, K_3 \lor (K_{1,4} \cup K_1), K_2 \lor K_{2,5}, K_5 \lor I_6\}$. 

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Theorem 2.9 (Ning–Ge [160]). Let \( n \geq 14 \) be an integer. If \( G \) is a graph on \( n \) vertices with minimum degree \( \delta(G) \geq 2 \) and

\[
\lambda(G) \geq \lambda(K_2 \lor (K_{n-4} \cup I_2)),
\]

then either \( G \) has a Hamilton cycle or \( G = K_2 \lor (K_{n-4} \cup I_2) \).

Upon some computations, we can find the following exceptions: when \( n = 7, \) \( K_3 \lor I_4 \) is an exception that contains no Hamilton cycle since \( \lambda(K_3 \lor I_4) = 1 + \sqrt{13} \approx 4.605 > \lambda(K_2 \lor (K_3 \cup I_2)) \approx 4.404 \); when \( n = 9, \) \( K_4 \lor I_5 \) contains no Hamilton cycle and \( \lambda(K_4 \lor I_5) = \frac{3+\sqrt{73}}{2} \approx 6.217 > \lambda(K_2 \lor (K_5 \cup I_2)) \approx 6.197 \). Moreover, Ning and Ge [160] conjecture that the bound \( n \geq 14 \) in Theorem 2.9 can be sharpened to \( n \geq 10 \). Soon after, Chen, Hou and Qian [36], Theorem 1.6] proved that Theorem 2.9 holds for every integer \( n \geq 10 \).

In addition, Chen et al. [36] also proved the version for the signless Laplacian radius.

Theorem 2.10 (Chen–Hou–Qian [36]). Let \( n \geq 11 \) be an integer. If \( G \) is a graph on \( n \) vertices with minimum degree \( \delta(G) \geq 2 \) and

\[
q(G) \geq q(K_2 \lor (K_{n-4} \cup I_2)),
\]

then \( G \) has a Hamilton cycle unless \( G = K_2 \lor (K_{n-4} \cup I_2) \).

In 2016, Benediktovich [10] improved slightly the result of Ning and Ge.

Theorem 2.11 (Benediktovich [10]). Let \( G \) be a graph on \( n \geq 8 \) vertices with \( \delta \geq 2 \). If

\[
\lambda(G) \geq n-3,
\]

then \( G \) has a Hamilton cycle unless \( G = K_2 \lor (K_{n-4} \cup I_2) \) or \( G = K_1 \lor (K_{n-3} \cup K_2) \), or \( n = 9 \), there are two more exceptions \( G \in \{I_5 \lor K_4, K_3 \lor (K_{1,4} \lor K_1)\} \).

By an easy computation, we know that \( \lambda(I_5 \lor K_4) = \frac{3+\sqrt{73}}{2} \approx 6.217 \) and \( \lambda(K_3 \lor (K_{1,4} \lor K_1)) \approx 6.032 \). These two graphs do not contain Hamilton cycle.

By introducing the minimum degree of a graph as a new parameter, Li and Ning [116] Theorems 1.5, 1.8] obtained the spectral analogue of the result of Erdős (Corollary 2.5).

Theorem 2.12 (Li–Ning [116]). Suppose \( \delta \geq 1 \) and \( n \geq \max\{6\delta + 5, (\delta^2 + 6\delta + 4)/2\} \). If \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq \delta \) and

\[
\lambda(G) \geq \lambda(H_{n,\delta}),
\]

then either \( G \) has a Hamilton cycle or \( G = H_{n,\delta} \).

Theorem 2.13 (Li–Ning [116]). Suppose \( \delta \geq 1 \) and \( n \geq \max\{6\delta + 5, (3\delta^2 + 5\delta + 4)/2\} \). If \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq \delta \) and

\[
q(G) \geq q(H_{n,\delta}),
\]

then either \( G \) has a Hamilton cycle or \( G = H_{n,\delta} \).
Clearly, Theorem 2.12 and Theorem 2.13 extended Theorem 2.9 and Theorem 2.10 for sufficiently large \( n \), respectively. Although Theorem 2.12 and Theorem 2.13 are algebraic, their proofs need some detailed graph structural analysis. The key part of the proof of these theorems attributes to the stability result for Hamilton cycle.

Soon after, Nikiforov [158, Theorem 1.4] further proved the following result, which improved Theorem 2.12. Recall that \( H_{n,\delta} = K_{\delta} \lor (K_{n-2\delta} \cup I_{\delta}) \) is not Hamiltonian and \( \delta(H_{n,\delta}) = \delta \). For notational convenience, we need to introduce a new graph. We denote

\[
L_{n,\delta} := K_1 \lor (K_{n-\delta-1} \lor K_\delta).
\]

Trivially, the case \( \delta = 1 \) yields \( H_{n,1} = L_{n,1} = K_1 \lor (K_{n-2} \lor K_1) \). We can observe that \( L_{n,\delta} \) does not contain Hamilton cycles and \( \delta(L_{n,\delta}) = \delta \). Moreover, we have \( e(L_{n,\delta}) = \binom{n-\delta}{2} + \binom{\delta+1}{2} \) and \( \lambda(L_{n,\delta}) > \lambda(K_{n-\delta}) = n - \delta - 1 \).

**Theorem 2.14** (Nikiforov [158]). Suppose that \( \delta \geq 1 \) and \( n \geq \delta^3 + \delta + 4 \). If \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq \delta \) and

\[
\lambda(G) \geq n - \delta - 1,
\]

then \( G \) has a Hamilton cycle, or \( G = H_{n,\delta} \), or \( G = L_{n,\delta} \).

Note that \( \lambda(H_{n,\delta}) > \lambda(K_{n-\delta}) = n - \delta - 1 \), so the condition in Theorem 2.14 is weaker and succincter than that in Theorem 2.12. Moreover, we can show that \( e(H_{n,\delta}) > e(L_{n,\delta}) \) and \( \lambda(H_{n,\delta}) > \lambda(L_{n,\delta}) \). Theorem 2.14 is a unified improvement on Theorem 2.11 as well as Theorem 2.12. Indeed, we can easily see that \( \lambda(H_{n,\delta}) > n - \delta - 1 \). Furthermore, by applying the Kelmans operation, we get \( \lambda(H_{n,\delta}) > \lambda(L_{n,\delta}) \); see [116, Lemma 6] for more details.

Although Theorem 2.14 is indeed a generalization of Theorem 2.12. Unfortunately, one dissatisfaction in Theorem 2.14 is that the requirement of the order of graph is stricter than that in Theorem 2.12. One open problem is to relax this requirement. In fact, by a tiny modification of the proof of Theorem 2.12 in [116], we can prove that if \( G \) is non-Hamiltonian and \( \lambda(G) \geq n - \delta - 1 \), then \( e(G) > e(H_{n,\delta+1}) \). The stability result (see [116, Lemma 2]) implies that \( G \) is a subgraph of \( H_{n,\delta} \) and \( L_{n,\delta} \). As pointed out by Nikiforov [158], the crucial point of the argument of Theorem 2.14 is based on proving that for large \( n \geq \delta^3 + \delta + 4 \), if \( G \) is a subgraph of \( H_{n,\delta} \) with \( \delta(G) \geq \delta \), i.e., \( G \) is obtained from \( H_{n,\delta} \) by deleting edges from the clique \( K_{n-\delta} \), then one can show that \( \lambda(G) < n - \delta - 1 \), unless \( G = H_{n,\delta} \). The same argument holds for the subgraph of \( L_{n,\delta} \). Generally speaking, for sufficiently large \( n \) with respect to \( \delta \), both \( H_{n,\delta} \) and \( L_{n,\delta} \) consist of a large clique \( K_{n-\delta} \) together with a few number of outgrowth edges. We know that \( \lambda(H_{n,\delta}) \) and \( \lambda(L_{n,\delta}) \) are very close to \( \lambda(K_{n-\delta}) = n - \delta - 1 \). The key idea of Nikiforov exploits the fact that if \( G \) is a subgraph of \( H_{n,\delta} \) or \( L_{n,\delta} \) with \( \delta(G) \geq \delta \), then all the outgrowth edges contribute to \( \lambda(G) \) much less than a single edge within the dense clique \( K_{n-\delta} \), thus \( \lambda(G) < \lambda(K_{n-\delta}) = n - \delta - 1 \); see [158, Theorem 1.6] for more details.

On the other hand, a natural question is that whether the value bound \( q(G) \geq 2(n - \delta - 1) \) corresponding to Theorem 2.13 holds or not. The similar problems under the condition of
By the Rayleigh Formula, we have
\[ G \in H_{n,\delta} \]
We define a vector \( h \)
Note that \( \{ v \in V(H_{n,\delta}) : d(v) = \delta \} \)
and the Kelmans transformation.

Recall that \( H_{n,\delta} = K_n \cup (K_{n-2} \cup I_\delta) \), we denote \( X = \{ v \in V(H_{n,\delta}) : d(v) = \delta \} \), \( Y = \{ v \in V(H_{n,\delta}) : d(v) = n - 1 \} \) and \( Z = \{ v \in V(H_{n,\delta}) : d(v) = n - \delta - 1 \} \). Let \( E_1(H_{n,\delta}) \) be the set of those edges of \( H_{n,\delta} \) whose both endpoints are from \( Y \cup Z \). We define
\[
\mathcal{H}_{n,\delta}^{(1)} = \{ H_{n,\delta} \setminus E' : E' \subseteq E_1(H_{n,\delta}) \text{ with } |E'| \leq \lceil \delta^2/4 \rceil \}.
\]
Here, we denote by \( H_{n,\delta} \setminus E' \) the graph obtained from \( H_{n,\delta} \) by deleting all edges of the edge set \( E' \). Similarly, for the graph \( L_{n,\delta} \), we denote \( X = \{ v \in V(L_{n,\delta}) : d(v) = \delta \} \), \( Y = \{ v \in V(L_{n,\delta}) : d(v) = n - 1 \} \) and \( Z = \{ v \in V(L_{n,\delta}) : d(v) = n - \delta - 1 \} \). The notation is clear although we used the same alphabets to denote the sets of vertices. It is easy to see that \( Y \) contains only one vertex. We use \( E_1(L_{n,\delta}) \) to denote the set of edges of \( L_{n,\delta} \) whose both endpoints are from \( Y \cup Z \). We define
\[
\mathcal{L}_{n,\delta}^{(1)} = \{ L_{n,\delta} \setminus E' : E' \subseteq E_1(L_{n,\delta}) \text{ with } |E'| \leq \lceil \delta/4 \rceil \}.
\]
First of all, we can show that if \( G \in \mathcal{H}_{n,\delta}^{(1)} \cup \mathcal{L}_{n,\delta}^{(1)} \), then \( G \) contains no Hamilton cycle and \( q(G) \geq 2(n - \delta - 1) \). Indeed, recall the subsets \( X, Y \) and \( Z \) defined as above. For each case, we define a vector \( h \) such that \( h_v = 1 \) for every \( v \in Y \cup Z \) and \( h_v = 0 \) for every \( v \in X \).
Note that \( q(K_{n-\delta} \cup I_\delta) = q(K_{n-\delta}) = 2(n - \delta - 1) \) and \( h \) is a corresponding eigenvector. If \( G \in \mathcal{H}_{n,\delta}^{(1)} \), then we get
\[
h^T Q(G) h - h^T Q(K_{n-\delta} \cup I_\delta) h = \delta^2 - 4|E'| \geq 0.
\]
By the Rayleigh Formula, we have
\[
q(G) \geq \frac{h^T Q(G) h}{h^T h} \geq \frac{h^T Q(K_{n-\delta} \cup I_\delta) h}{h^T h} = 2(n - \delta - 1).
\]
Similarly, we can show that \( q(G) \geq 2(n - \delta - 1) \) for every \( G \in \mathcal{L}_{n,\delta}^{(1)} \).

**Theorem 2.15** (Li–Liu–Peng [120]). Assume that \( \delta \geq 1 \) and \( n \geq \delta^4 + \delta^3 + 4\delta^2 + \delta + 6 \).
Let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta(G) \geq \delta \). If
\[
q(G) \geq 2(n - \delta - 1),
\]
then \( G \) has a Hamilton cycle unless \( G \in \mathcal{H}_{n,\delta}^{(1)} \) or \( G \in \mathcal{L}_{n,\delta}^{(1)} \).

In 2021, Zhou, Broersma, Wang and Lu [212] proved a slight improvement on Theorem 2.15. Their proofs are based on the Bondy–Chvátal closure, a degree sequence condition, and the Kelmans transformation.
2.2 Hamilton cycle in balanced bipartite graph

Let $G$ be a bipartite graph with vertex sets $X$ and $Y$. The bipartite graph $G$ is called balanced if $|X| = |Y|$. We remark here that if a bipartite graph has Hamilton cycle, then it must be balanced. So we consider the existence of Hamilton cycle only in balanced bipartite graph.

**Lemma 2.16** (See [25, p. 490]). Let $G$ be a balanced bipartite graph on $2n$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_{2n}$. If $G$ is not Hamiltonian, then there exists an integer $i \leq n$ such that $d_i \leq i$ and $d_n \leq n - i$.

Motivated by the work of Erdős [62] in Theorem 2.4, Moon and Moser [143] provided a corresponding result for the balanced bipartite graphs.

**Theorem 2.17** (Moon–Moser [143]). Let $G$ be a balanced bipartite graph on $2n$ vertices. If the minimum degree $\delta(G) \geq \delta$ for some $1 \leq \delta \leq n/2$ and

$$e(G) > n(n - \delta) + \delta^2,$$

then $G$ has a Hamilton cycle.

The condition $\delta \leq n/2$ is well comprehensible since Moon and Moser [143] also pointed out that if $G$ is a balanced bipartite on $2n$ vertices with $\delta(G) > n/2$, then $G$ must be Hamiltonian. This is a bipartite version of the Dirac theorem.

Let $B_{n,\delta}$ be the bipartite graph obtained from the complete bipartite graph $K_{n,n}$ by deleting all edges in its one subgraph $K_{\delta,n-\delta}$. More precisely, the two vertex parts of $B_{n,\delta}$ are $V = V_1 \cup V_2$ and $U = U_1 \cup U_2$ where $|V_1| = |U_1| = \delta$ and $|V_2| = |U_2| = n - \delta$, we join all edges between $V_1$ and $U_1$, and all edges between $V_2$ and $U$. We denote

$$B_{n,\delta} := K_{n,n} \setminus K_{\delta,n-\delta}.$$

It is easy to see that $e(B_{n,\delta}) = n(n - \delta) + \delta^2$ and $B_{n,\delta}$ contains no Hamilton cycle. This implies that the condition in Moon–Moser’s theorem is best possible.

For a bipartite graph, Lu, Liu and Tian [139] gave a sufficient condition for a balanced bipartite graph being Hamiltonian in terms of the number of edges and the spectral radius of its quasi-complement. Let $K_{n,n-1} + e$ be the bipartite graph obtained from $K_{n,n-1}$ by adding a pendent edge to one of vertices in the part of size $n$.

**Theorem 2.18** (Lu–Liu–Tian [139]). Let $G$ be a balanced bipartite graph on $2n \geq 4$ vertices. If the minimum degree $\delta(G) \geq 1$ and

$$e(G) \geq n(n - 1) + 1,$$

then $G$ is Hamiltonian unless $G = K_{n,n-1} + e$.

Liu, Shi and Xue [138] gave sufficient conditions on the spectral radius for a balanced bipartite graph being Hamiltonian. Their results extended Theorem 3.7. Moreover, they also provided tight sufficient conditions on the signless Laplacian spectral radius for a graph to be Hamiltonian and traceable, which improve the results of Yu and Fan [188]. Let $K_{n,n-2} + 4e$ be a bipartite graph obtained from $K_{n,n-2}$ by adding two vertices which are adjacent to two common vertices with degree $n - 2$ in $K_{n,n-2}$.
**Theorem 2.19** (Li–Shiu–Xue [138]). Let $G$ be a balanced bipartite graph on $2n \geq 8$ vertices. If $\delta(G) \geq 2$ and
$$\lambda(G) \geq \sqrt{n(n-2)} + 4,$$
then $G$ is Hamiltonian unless $G = K_{n,n-2} + 4e$.

For the Hamiltonicity of balanced bipartite graphs, Li and Ning [116] also proved the spectral version of the Moon–Moser Theorem 2.17.

**Theorem 2.20** (Li–Ning [116]). Suppose that $\delta \geq 1$ and $n \geq (\delta+1)^2$. Let $G$ be a $2n$-vertex balanced bipartite graph. If the minimum degree $\delta(G) \geq \delta$ and
$$\lambda(G) \geq \lambda(B_{n,\delta}),$$
then either $G$ has a Hamilton cycle or $G = B_{n,\delta}$.

**Theorem 2.21** (Li–Ning [116]). Suppose that $\delta \geq 1$ and $n \geq (\delta+1)^2$. Let $G$ be a $2n$-vertex balanced bipartite graph. If the minimum degree $\delta(G) \geq \delta$ and
$$q(G) \geq q(B_{n,\delta}),$$
then either $G$ has a Hamilton cycle or $G = B_{n,\delta}$.

Recall that Theorem 2.14 extended slightly Theorem 2.12. For the case of bipartite graphs, we can see that $\lambda(B_{n,\delta}) > \lambda(K_{n,n-\delta}) = \sqrt{n(n-\delta)}$ since $K_{n,n-\delta}$ is a proper subgraph of $B_{n,\delta}$. Motivated by this observation, Ge and Ning [88] Theorem 1.4] proved the following improvement on Theorem 2.20.

**Theorem 2.22** (Ge–Ning [88]). Suppose that $\delta \geq 1$ and $n \geq \delta^3 + 2\delta + 4$. Let $G$ be a balanced bipartite graph on $2n$ vertices. If the minimum degree $\delta(G) \geq \delta$ and
$$\lambda(G) \geq \sqrt{n(n-\delta)},$$
then either $G$ has a Hamilton cycle or $G = B_{n,\delta}$.

We remark here that Jiang, Yu and Fang [100] Theorem 1.2] proved independently the result in Theorem 2.22 under a weak condition $n \geq \max\left\{\frac{\delta^3 + \delta^2 + 3}{2}, \frac{(\delta + 2)^2}{2}\right\}$ with the same method only by some careful calculations. In 2020, Liu, Wu and Lai [137] unified these several former spectral Hamiltonian results on balanced bipartite graphs and complementary graphs. In addition, Lu [140] extended some spectral conditions for the Hamiltonicity of balanced bipartite graphs.

Correspondingly, Li, Liu and Peng [120] Theorem 4] also gave an improvement on Theorem 2.21 by observing that $q(B_{n,\delta}) > q(K_{n,n-\delta}) = 2n - \delta$. For the graph $B_{n,\delta}$, let $S$ and $T$ be the vertex sets such that the degree of vertices from $T$ is either $n$ or $n - \delta$. Let $X = \{v \in S : d(v) = \delta\}, Y = \{v \in T : d(v) = n\}, W = \{v \in T : d(v) = n - \delta\}$ and $Z = \{v \in S : d(v) = n\}$. We can see from the definition that $S = X \cup Z$ and $T = Y \cup W$. We denote $E_1(B_{n,\delta})$ by those edges of $B_{n,\delta}$ whose both endpoints are from $Y \cup W \cup Z$. Let
$$B_{n,\delta}^{(1)} = \left\{B_{n,\delta} \setminus E' : E' \subseteq E_1(B_{n,\delta}) \text{ with } |E'| \leq \lfloor \delta^2/4 \rfloor \right\}.$$
2.3 Problem for $k$-Hamiltonicity

A graph $G = (V, E)$ is $k$-Hamiltonian if for all $X \subset V$ with $|X| \leq k$, the subgraph induced by $V \setminus X$ is Hamiltonian. Thus 0-Hamiltonian graph is the same as the general Hamiltonian graph. Similarly, a graph $G$ is $k$-edge-Hamiltonian if any collection of vertex-disjoint paths with at most $k$ edges altogether belong to a Hamiltonian cycle in $G$. In [34], it is obtained that for a graph $G$, if $\delta(G) \geq \frac{n+k}{2}$, then $G$ is $k$-Hamiltonian. Clearly, when $k = 0$, it reduces to the Dirac theorem.

**Theorem 2.24** (Chartrand et al. [34], Bondy–Chvátal [23]). A graph $G$ is $k$-Hamiltonian if and only if the closure graph $cl_{n+k}(G)$ is $k$-Hamiltonian. In particular, if $d(u) + d(v) \geq n + k$ for all non-edges $\{u, v\}$, then $cl_{n+k}(G) = K_n$ and $G$ is $k$-Hamiltonian.

In 1972, Chvátal [43] proved the following theorem, which characterizes the sufficient condition of degree sequence of $k$-Hamiltonian graphs.

**Theorem 2.25** (Chvátal [43]). Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $n \geq 3$, and $0 \leq k \leq n-3$. If $G$ is not $k$-Hamiltonian, then there exists $1 \leq i < \frac{n-k}{2}$ such that $d_i \leq i + k$ and $d_{n-i-k} \leq n - i - 1$.

In [74], Feng et al. obtained sufficient conditions for a graph to be $k$-Hamiltonian.

**Theorem 2.26** (Feng et al. [74]). Let $k \geq 1$, and let $G$ be a graph of order $n \geq k + 6$. If $e(G) \geq \binom{n-1}{2} + k + 1$, then either $G$ is $k$-Hamiltonian or $G = K_{k+1} \cup (K_1 \cup K_{n-k-2})$.

In this survey, we shall provide an extension by giving minimum degree. When $\delta = k+1$, the following theorem reduces to Theorem 2.26. For convenience, we denote by

$$H_{n,k,\delta} := K_\delta \cup (K_{n-2\delta+k} \cup I_{\delta-k})$$
and

\[ L_{n,k,\delta} := K_{k+1} \cup (K_{n-\delta-1} \cup K_{\delta-k}). \]

It is easy to see that both \( H_{n,k,\delta} \) and \( L_{n,k,\delta} \) are not \( k \)-Hamiltonian and has minimum degree \( \delta(H_{n,k,\delta}) = \delta(L_{n,k,\delta}) = \delta \). In particular, when \( k = 0 \), we can see that \( H_{n,0,\delta} \) is the same as \( H_{n,\delta} \), and \( L_{n,0,\delta} \) is the same as \( L_{n,\delta} \), which are defined in equations (7) and (8).

**Theorem 2.27.** Let \( \delta \geq k + 1 \geq 1 \) and \( n \geq 6\delta - 5k \). If \( G \) is a graph on \( n \) vertices with minimum degree \( \delta(G) \geq \delta \) and

\[ e(G) \geq e(H_{n,k,\delta}), \]

then either \( G \) is \( k \)-Hamiltonian or \( G = H_{n,k,\delta} \).

**Proof.** Suppose that \( G \) is not \( k \)-Hamiltonian. According to Theorem 2.24, we know that the closure graph \( H := cl_{n+k}(G) \) is also not \( k \)-Hamiltonian. Let \( d_1 \leq d_2 \leq \cdots \leq d_n \) be the degree sequence of \( H \). By Theorem 2.25, there exists an integer \( 1 \leq m \leq \frac{n-k-1}{2} \) such that \( d_m \leq m + k \) and \( d_{n-m-k} \leq n - m - 1 \). Hence, we have

\[ 2e(H) = \sum_{i=1}^{n} d_i \leq m(m+k) + (n-2m-k)(n-m-1) + (m+k)(n-1) \]

\[ = 3m^2 - (2n - 2k - 1)m + n^2 - n. \] (9)

First of all, we denote \( f(m) = 3m^2 - (2n - 2k - 1)m \). Since \( \delta \leq \delta(G) \leq \delta(H) \leq m+k \), we can see that \( \delta - k \leq m \leq \frac{n-k-1}{2} \). The quadratic function \( f(m) \) is minimized at \( m_0 = \frac{2n - 2k - 1}{6} \).

Note that \( n \geq 6\delta - 5k \) implies \( m_0 - (\delta - k) > \frac{n-k-1}{2} - m_0 \). Therefore, we have \( f(m) \) is maximized at \( m = \delta - k \). Then

\[ 2e(G) \leq 2e(H) \leq f(\delta - k) + n^2 - n = 2e(K_{\delta} \cup (K_{n-2\delta+k} \cup I_{\delta-k})). \]

By the condition, we get \( e(G) = e(K_{\delta} \cup (K_{n-2\delta+k} \cup I_{\delta-k})) \). All the inequalities above become equalities, that is \( m = \delta - k \) and \( G = cl_{n+k}(G) \), moreover the degrees of \( G \) are \( d_1 = \cdots = d_{\delta-k} = \delta \), \( d_{\delta-k+1} = \cdots = d_{n-\delta} = n - \delta + k - 1 \) and \( d_{n-\delta+1} = \cdots = d_n = n - 1 \).

Next we shall prove that \( G = K_{\delta} \cup (K_{n-2\delta+k} \cup I_{\delta-k}) \).

Recall that \( d_1,d_2,\ldots,d_n \) are the degrees of \( G \), and the corresponding vertices are \( v_1,v_2,\ldots,v_n \). We next denote by \( X = \{v_1,v_2,\ldots,v_{\delta-k}\} \) and \( Y = \{v_{\delta-k+1},\ldots,v_n\} \). We next show that the induced subgraph \( G[Y] = K_{n-\delta+k} \). Otherwise, we choose two non-adjacent vertices \( v_r \) and \( v_s \) implies that \( d_r + d_s < n + k \). On the other hand, we know from the definition of \( Y \) that \( d_r + d_s \geq 2(n - \delta + k - 1) > n + k \), a contradiction. Thus we get \( G[Y] = K_{n-\delta+k} \).

The previous result on degree sequence implies that there are \( \delta \) vertices in \( Y \) with degree \( n - 1 \), we now denote these vertices by \( F = \{v_{n-\delta+1},\ldots,v_n\} \). Let \( G[X,F] \) be the induced bipartite subgraph in \( G \) between vertex sets \( X \) and \( F \). Finally, we shall show that \( G[X,F] = K_{\delta-k,\delta} \). Note that \( G = cl_{n+k}(G) \), which implies that every two vertices with the degree sum at least \( n + k \) are adjacent. Thus, every vertex in \( X \) is adjacent to every vertex in \( F \). So we can see that \( G[X,F] = K_{\delta-k,\delta} \), and then \( X \) is an independent set in \( G \). Hence we get \( G = K_{\delta} \cup (K_{n-2\delta+k} \cup I_{\delta-k}) \). \( \square \)
Recently, by utilizing the degree sequences and the closure concept, Liu et al. [134] generalized Theorem 2.14 to \(k\)-Hamiltonian graphs. Their results can be considered as the spectral counterpart for the above Dirac-type condition. We mention here that there is a tiny typo at the end of the proof in [134, Theorem 4] since the extremal graph is not the only one. Clearly, the graph \(H_{n,k,\delta} = K_\delta \cup (K_{n-2\delta+k} \cup I_{\delta-k})\) is not \(k\)-Hamiltonian and \(\lambda(H_{n,k,\delta}) > n - \delta + k - 1\). By a careful modification, the correct result should be the following.

**Theorem 2.28** (Liu et al. [134]). Let \(k \geq 0, \delta \geq k + 2\) and \(n \geq n_0(k, \delta)\) where
\[
n_0(k, \delta) = \max\{2\delta^2 - 2k\delta + 2\delta - k + 2, (\delta - k)(k^2 + 2k + 5) + 1\}.
\]

If \(G\) is a connected graph of order \(n\) with minimum degree \(\delta(G) \geq \delta\) and
\[
\lambda(G) \geq n - \delta + k - 1,
\]
then \(G\) is \(k\)-Hamiltonian unless \(G = H_{n,k,\delta}\) or \(G = L_{n,k,\delta}\).

Since \(\lambda(H_{n,k,\delta})\) contains \(K_{n-\delta+k}\) as a subgraph, we have \(\lambda(H_{n,k,\delta}) \geq n - \delta + k - 1\). Note that \(\lambda(H_{n,k,\delta}) > \lambda(L_{n,k,\delta})\). So we can immediately get the spectral version of Theorem 2.28 for graphs with sufficient large order.

**Corollary 2.29.** Let \(k \geq 0, \delta \geq k + 2\) and \(n \geq n_0(k, \delta)\). If \(G\) is a connected graph of order \(n\) with minimum degree \(\delta(G) \geq \delta\) and
\[
\lambda(G) \geq \lambda(H_{n,k,\delta}),
\]
then \(G\) is \(k\)-Hamiltonian unless \(G = H_{n,k,\delta}\).

Recall that \(H_{n,k,\delta} = K_\delta \cup (K_{n-2\delta+k} \cup I_{\delta-k})\). Let \(X\) be the set of \(\delta - k\) vertices forming by the independent set \(I_{\delta-k}\), \(Y\) be the set of \(\delta\) vertices corresponding to the clique \(K_\delta\), and \(Z\) be the set of the remaining \(n - 2\delta + k\) vertices consisting of the clique \(K_{n-2\delta+k}\). This notation is clear although it is not standard. We write \(E_1(H_{n,k,\delta})\) for the set of edges of \(H_{n,k,\delta}\) whose both endpoints are from \(Y \cup Z\). Furthermore, we define the family \(\mathcal{H}^{(1)}_{n,k,\delta}\) of graphs as below.

\[
\mathcal{H}^{(1)}_{n,k,\delta} = \{ H_{n,k,\delta} \setminus E' : E' \subseteq E_1(H_{n,k,\delta}) \text{ with } |E'| \leq \lfloor \delta(\delta-k)/4 \rfloor \}.
\]

Here, we write \(H_{n,k,\delta} \setminus E'\) for the graph obtained from \(H_{n,k,\delta}\) by deleting all edges of the edge set \(E'\). Similarly, for the graph \(L_{n,k,\delta} = K_{k+1} \cup (K_{n-\delta-1} \cup K_{\delta-k})\), we denote \(X\) by the set of vertices corresponding to the clique \(K_{\delta-k}\), \(Y\) by the set of vertices corresponding to the clique \(K_{k+1}\), and \(Z\) by the set of the remaining \(n - \delta - 1\) vertices. We write \(E_1(L_{n,k,\delta})\) for the set of edges of \(L_{n,k,\delta}\) whose both endpoints are from \(Y \cup Z\). Moreover, we define the family \(\mathcal{L}^{(1)}_{n,k,\delta}\) of graphs as follows.

\[
\mathcal{L}^{(1)}_{n,k,\delta} = \{ L_{n,k,\delta} \setminus E' : E' \subseteq E_1(L_{n,k,\delta}) \text{ with } |E'| \leq \lfloor (k+1)(\delta-k)/4 \rfloor \}.
\]

Recently, Li and Peng [121] presented the following signless Laplacian spectral conditions for \(k\)-Hamiltonian graphs with large minimum degree.
Theorem 2.30 (Li–Peng [121]). Let $k \geq 0, \delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq 2(n - \delta + k - 1),$$

then $G$ is $k$-Hamiltonian unless $G \in H_{n,k,\delta}^{(1)}$ or $G \in L_{n,k,\delta}^{(1)}$.

Corollary 2.31. Let $k \geq 0, \delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq q(H_{n,k,\delta}),$$

then $G$ is $k$-Hamiltonian unless $G = H_{n,k,\delta}$.

2.4 Problem for $k$-edge-Hamiltonicity

Recall that a graph $G$ is $k$-edge-Hamiltonian if any collection of vertex-disjoint paths with at most $k$ edges altogether belong to a Hamiltonian cycle in $G$.

Theorem 2.32 (Kronk [111], Bondy–Chvátal [23]). A graph $G$ is $k$-edge-Hamiltonian if and only if the closure graph $cl_{n+k}(G)$ is $k$-edge-Hamiltonian. In particular, if $d(u) + d(v) \geq n + k$ for all non-edges $\{u, v\}$, then $cl_{n+k}(G) = K_n$ and $G$ is $k$-edge-Hamiltonian.

Lemma 2.33 (Kronk [112]). Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $n \geq 3$, and $0 \leq k \leq n - 3$. If $G$ is not $k$-edge-Hamiltonian, then there exists $1 \leq i < \frac{n+k}{2}$ such that $d_{n-k} \leq i$ and $d_{n-i} \leq n-i+k-1$.

Theorem 2.34 (Feng et al. [74]). Let $k \geq 1$ and $G$ be a graph of order $n \geq k + 6$. If

$$e(G) \geq \left(\frac{n-1}{2}\right) + k + 1,$$

then either $G$ is $k$-edge-Hamiltonian or $G = K_{k+1} \cup (K_1 \cup K_{n-k-2})$.

To avoid unnecessary calculations, we do not attempt to get the best bound on the order of graphs in the proof.

Theorem 2.35 (Li–Peng [121]). Let $k \geq 0, \delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq n - \delta + k - 1,$$

then $G$ is $k$-edge-Hamiltonian unless $G = H_{n,k,\delta}$ or $G = L_{n,k,\delta}$.

Since $\lambda(H_{n,k,\delta})$ contains $K_{n-\delta+k}$ as a proper subgraph, we have $\lambda(H_{n,k,\delta}) > n - \delta + k - 1$. Moreover, applying the Kelmans operations on $L_{n,k,\delta}$, we can get a proper subgraph of $H_{n,k,\delta}$, this implies $\lambda(H_{n,k,\delta}) > \lambda(L_{n,k,\delta})$; see, e.g., [116, Theorem 2.12]. With this observation in mind, Theorem 2.35 implies the following corollary, which is an extension on Theorem 2.12.
Corollary 2.36. Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and
\[
\lambda(G) \geq \lambda(H_{n,k,\delta}),
\]
then $G$ is $k$-edge-Hamiltonian unless $G = H_{n,k,\delta}$.

Moreover, Li and Peng [121] also presented the following sufficient conditions on the signless Laplacian spectral radius for $k$-Hamiltonian graphs with large minimum degree.

Theorem 2.37 (Li–Peng [121]). Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and $q(G) \geq 2(n - \delta + k - 1)$,
\[
\text{then } G \text{ is } k\text{-edge-Hamiltonian unless } G \in H_{n,k,\delta}^{(1)} \text{ or } G \in L_{n,k,\delta}^{(1)}.
\]

As a consequence, we get the following corollary.

Corollary 2.38. Let $k \geq 0$, $\delta \geq k + 2$ and $n$ be sufficiently large. If $G$ is an $n$-vertex graph with minimum degree $\delta(G) \geq \delta$ and $q(G) \geq q(H_{n,k,\delta})$,
\[
\text{then } G \text{ is } k\text{-edge-Hamiltonian unless } G = H_{n,k,\delta}.
\]

3 Spectral problem for path coverability

3.1 Problem for Hamilton path

Let $G$ be a simple graph. A path is called a Hamilton path if it contains all vertices of $G$. When $G$ contains a Hamiltonian path, we usually say that $G$ is traceable. It is well-known that $G$ contains a Hamilton path if and only if $G \lor K_1$ contains a Hamilton cycle. With the help of this relation, almost all results involving Hamilton cycle can imply the corresponding results for Hamilton path. Thus, almost all problems involving Hamilton path can be reduced to the problem of Hamilton cycle in this way. For any non-negative integer $q$, a graph $G$ with $n \geq 3$ vertices is called $q$-traceable if any removal of at most $q$ vertices of $G$ results in a traceable graph.

Theorem 3.1 (Dirac [57], Ore [165]). If $G$ is a graph on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n-1}{2}$, then $G$ has a Hamilton path.

Theorem 3.2 (Bondy–Chvátal [23]). A graph $G$ is traceable if and only if the closure graph $\text{cl}_{n-1}(G)$ is traceable. In particular, if $d(u) + d(v) \geq n - 1$ for all non-edges $\{u, v\}$, then $\text{cl}_{n-1}(G) = K_n$ and $G$ is traceable.

Lemma 3.3 (Chvátal [43]). Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $G$ has no Hamilton path, then there exists $i < \frac{n+1}{2}$ such that $d_1 \leq i - 1$ and $d_{n-i+1} \leq n - i - 1$. 48
Theorem 3.4 (Ore [165], Bondy [22]). Let $G$ be a graph on $n \geq 3$ vertices. If
\[ e(G) \geq \left( \frac{n-1}{2} \right), \]
then $G$ has a Hamilton path unless $G = K_1 \cup K_{n-1}$ or $n = 4$ and $G = K_{1,3}$.

Theorem 3.5 (Fiedler–Nikiforov [77]). Let $G$ be a graph on $n$ vertices and
\[ \lambda(G) \geq n - 2, \]
then either $G$ has a Hamilton path or $G = K_{n-1} \cup K_1$.

The proof is short and similar with that in Theorem 2.6.

Proof. The well-known Stanley inequality asserts that
\[ \lambda(G) \leq -1/2 + \sqrt{2e(G) + 1/4}, \]
which together with $\lambda(G) \geq n - 2$, implies that $e(G) \geq \left( \frac{n-1}{2} \right)$. Hence, for $n \neq 4$, the desired result follows from Theorem 3.4. For $n = 4$, if $G$ is the another possible exception $K_{1,3}$, we can calculate that $\lambda(K_{1,3}) = \sqrt{3}$, which contradicts the condition $\lambda(G) \geq n - 2$. This completes the proof. \qed

In 2013, Yu and Fan [188] provided the condition for the existence of Hamiltonian paths in terms of the signless Laplacian spectral radius of the graph.

Theorem 3.6 (Yu–Fan [188]). Let $G$ be a graph on $n$ vertices and
\[ q(G) \geq 2(n-2), \]
then $G$ has a Hamilton path or $G = K_{n-1} \cup K_1$, or $n = 4$ and $G = K_{1,3}$.

Proof. An important inequality in [72] states that $q(G) \leq \frac{2e(G)}{n-1} + n - 2$, which together with $q(G) \geq 2(n-2)$ yields that $e(G) \geq \left( \frac{n-1}{2} \right)$. Hence, for $n \neq 4$, the desired result follows from Theorem 3.4. For $n = 4$, when $G$ is the another possible exception $K_{1,3}$, we can calculate that $q(K_{1,3}) = 4$. This completes the proof. \qed

Note that the extremal graph in Fielder–Nikiforov’s result is $K_{n-1} \cup K_1$, which is not connected. Moreover, if a graph contains a Hamilton path, then it must be connected. In 2012, Lu, Liu and Tian [139] generalized this spectral condition to connected graphs.

Theorem 3.7 (Lu–Liu–Tian [139]). Let $G$ be a graph on $n \geq 7$ vertices. If $G$ is connected and
\[ \lambda(G) \geq \sqrt{(n-3)^2 + 2}, \]
then $G$ contains a Hamilton path.

We remark here that the original statement of Theorem 3.7 (see [139, Theorem 3.4]) requires the restriction $n \geq 5$, this is a typo because for $n = 6$, we can verify that $\lambda(K_2 \vee I_4) = \frac{1+\sqrt{33}}{2} \approx 3.372 > \sqrt{3^2 + 2} \approx 3.316$ and $K_2 \vee I_4$ does not contain Hamilton path. By a careful examination, the restriction should be modified to $n \geq 7$. 49
Theorem 3.8 (Ning–Ge [160]). Let $G$ be a graph on $n \geq 4$ vertices with $\delta(G) \geq 1$. If
\[ \lambda(G) > n - 3, \]
then $G$ has a Hamilton path or $G \in \{ K_1 \cup (K_{n-3} \cup I_2), K_2 \cup I_4, K_1 \cup (K_{1,3} \cup K_1) \}$.

When the minimum degree is involved in contrast with the results in [77], using the adjacency spectral radius, Li and Ning [116] obtained the following result.

Theorem 3.9 (Li–Ning [116]). Let $\delta \geq 0$ and $n \geq \max\{6\delta + 10, (\delta^2 + 7\delta + 8)/2 \}$. If $G$ is a graph of order $n$ with minimum degree $\delta(G) \geq \delta$ and
\[ \lambda(G) \geq \lambda(K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})), \]
then $G$ has a Hamilton path, unless $G = K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})$.

Theorem 3.10 (Li–Ning [116]). Let $\delta \geq 0$ and $n \geq \max\{6\delta + 10, (3\delta^2 + 9\delta + 8)/2 \}$. If $G$ is a graph of order $n$ with minimum degree $\delta(G) \geq \delta$ and
\[ q(G) \geq q(K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})), \]
then $G$ has a Hamilton path, unless $G = K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})$.

Theorem 3.9 was generalized by Nikiforov [158, Theorem 1.5] as below.

Theorem 3.11 (Nikiforov [158]). Let $\delta \geq 1$ and $n \geq \max\{6\delta + 10, (\delta^2 + 7\delta + 8)/2 \}$. If $G$ is a graph on $n$ vertices with minimum degree $\delta(G) \geq \delta$ and
\[ \lambda(G) \geq \lambda(K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})), \]
then $G$ has a Hamilton path unless $G = K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})$ or $G = K_{\delta+1} \lor K_{n-\delta-1}$.

The following version of the signless Laplacian spectral radius is a special case of Theorem 3.19, which will be introduced in next subsection.

Theorem 3.12 (Cheng et al. [42]). Let $\delta \geq 1$ and $n \geq \delta^4 + 9\delta^3 + 24\delta^2 + 23\delta + 15$. If $G$ is a connected graph on $n$ vertices with minimum degree $\delta(G) \geq \delta$ and
\[ q(G) \geq 2(n - \delta - 2), \]
then either $G$ has a Hamilton path or $G$ is a subgraph of $K_\delta \lor (K_{n-2\delta-1} \cup I_{\delta+1})$ by removing at most $\lfloor \delta(\delta + 1)/4 \rfloor$ edges.

Moreover, comparing the Moon–Moser theorem with the Erdős theorem, we can similarly consider the existence of Hamilton path in bipartite graphs. Note that if a bipartite graph has a Hamilton path, then it must be balanced and nearly balanced, i.e., the sizes of two vertex parts differs at most one.

In 2017, Li and Ning [117] established several spectral analogues of Moon and Moser’s theorem on Hamilton paths in balanced bipartite graphs and nearly balanced bipartite graphs. We remark that one main ingredient of their proofs is a structural stability result.
involving Hamilton paths in balanced bipartite graphs with given minimum degree and number of edges. To some extent, the line of proof is similar with that in [116].

In 2020, Wei and You [182, Theorem 1.5] further extended Niki forov’s result (Theorem 3.11) by improving the bound of order $n$ by almost a half. In addition, the authors [182, Theorem 1.8] also obtained spectral sufficient conditions for a graph to be traceable among graphs or balanced bipartite graphs with large minimum degree, which extended a result of Li and Ning for balanced bipartite graph being traceable. Furthermore, Liu, Wu and Lai [137] unified these several former spectral Hamiltonian results on balanced bipartite graphs and complementary graphs.

3.2 Problem for path coverability

In this section, we consider the $k$-path-coverable problem. A graph $G$ is $k$-path-coverable if $V(G)$ can be covered by $k$ or fewer vertex-disjoint paths. In particular, 1-path-coverable is the same as traceable. So the concept of $k$-path coverability is a natural extension of the Hamilton path. The disjoint path cover problem is strongly related to the well-known hamiltonian problem (one may refer to [126] for a survey), which is among the most fundamental ones in graph theory, and attracts much attention in theoretical computer science. However, this problem is NP-complete [171], therefore, finding their guaranteed sufficient conditions becomes an interesting work. In [128], such Ore-type condition is obtained.

**Theorem 3.13** (Bondy–Chvátal [23]). A graph $G$ is $k$-path coverable if and only if the closure graph $\text{cl}_{n-k}(G)$ is $k$-path coverable. In particular, if $d(u) + d(v) \geq n - k$ for all non-edges $\{u, v\}$, then $\text{cl}_{n-k}(G) = K_n$ and $G$ is $k$-path coverable.

**Theorem 3.14.** [23, 115] Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $k \geq 1$. If $G$ is not $k$-path-coverable, then there exists $1 \leq i < \frac{n-k}{2}$ such that $d_{i+k} \leq i$ and $d_{n-i} \leq n - i - k - 1$.

**Theorem 3.15** (Feng et al. [74]). Let $k \geq 2$, and let $G$ be a graph of order $n \geq 5k + 6$. If
\[
e(G) \geq \left(\frac{n - k - 1}{2}\right) + k + 1,
\]
then either $G$ is $k$-path-coverable or $G = K_1 \vee (K_{n-k-2} \cup I_{k+1})$.

For every integers $k \geq 1$ and $\delta \geq 1$, we denote
\[
B_{n,k,\delta} := K_\delta \vee (K_{n-k-2} \cup I_{\delta+k}).
\]
Clearly, $B_{n,k,\delta}$ is not $k$-path-coverable and $e(B_{n,k,\delta}) = \left(\frac{n-\delta-k}{2}\right) + (\delta + k)\delta$. We next extend Theorem 3.15 by introducing the minimum degree. Clearly, when we set $\delta = 1$, the following theorem reduces to Theorem 3.15.

**Theorem 3.16.** Let $k \geq 1$ and $G$ be a graph on $n \geq 5k + 6\delta$ vertices. If the minimum degree $\delta(G) \geq \delta$ and
\[
e(G) \geq e(B_{n,k,\delta}),
\]
then $G$ is $k$-path-coverable or $G = B_{n,k,\delta}$.
Proof. Assume that \( G \) is not \( k \)-path-coverable. By Theorem 3.13 we know that the closure graph \( H := c_{n-k}(G) \) is also not \( k \)-path-coverable. We denote by \( d_1 \leq d_2 \leq \cdots \leq d_n \) the degree sequence of \( H \). By Theorem 3.14 there exists an integer \( 1 \leq m \leq \frac{n-k-1}{2} \) such that \( d_{m+k} \leq m \) and \( d_{n-m} \leq n - m - k - 1 \). Thus we get

\[
2e(H) = \sum_{i=1}^{n} d_i \leq (m+k)m + (n-2m-k)(n-m-k-1) + m(n-1) = 3m^2 - (2n-4k-1)m + (n-k)(n-k-1). \tag{10}
\]

We denote by \( f(m) := 3m^2 - (2n-4k-1)m \). Since \( \delta \leq \delta(G) \leq \delta(H) \leq m \leq \frac{n-k-1}{2} \). The function \( f(m) \) is minimized at \( m_0 = \frac{2n-6k-1}{6} \). By the condition \( n \geq 6\delta + 5k \), we have \( m_0 - \delta > \frac{n-k-1}{2} - m_0 \). Hence, the function \( f(m) \) is maximized at \( m = \delta \), which yields

\[
2e(G) \leq 2e(H) \leq f(\delta) + (n-k)(n-k-1) = 2e(K_\delta \cup (K_{n-2\delta-k} \cup I_{\delta+k})).
\]

Combining the condition, we get \( e(G) = e(K_\delta \cup (K_{n-2\delta-k} \cup I_{\delta+k})) \), and all the above inequalities in (10) become equalities. Hence, we have \( m = \delta \) and \( G = c_{n-k}(G) \), the degrees of \( G \) are \( d_1 = \cdots = d_{\delta+k} = \delta, d_{\delta+k+1} = \cdots = d_{n-\delta} = n-\delta-k-1 \) and \( d_{n-\delta+1} = \cdots = d_n = n-1 \). Next, we shall prove that \( G = K_\delta \cup (K_{n-2\delta-k} \cup I_{\delta+k}) \).

Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( G \) corresponding to the degrees \( d_1, d_2, \ldots, d_n \). We denote by \( X := \{v_1, v_2, \ldots, v_{\delta+k}\} \) and \( Y := \{v_{\delta+k+1}, \ldots, v_n\} \). First of all, we show that the induced subgraph \( G[Y] \) forms a clique. Otherwise, let \( u \) and \( v \) be two non-adjacent vertices in \( Y \). From the definition of the \( (n-k) \)-closure graph, we know that \( d_G(u) + d_G(v) < n-k \). On the other hand, we have \( d_G(u) + d_G(v) \geq 2(n-\delta-k-1) > n-k \), a contradiction. So we get \( G[Y] = K_{n-\delta-k} \). Let \( F \) be the set of vertices in \( Y \) with degree \( n-1 \), that is, \( F = \{v_{n-\delta+1}, \ldots, v_n\} \). We denote by \( G[X,F] \) the induced bipartite subgraph of \( G \) between vertex sets \( X \) and \( F \). Finally, we shall prove \( G[X,F] = K_{\delta+k} \). Since each vertex of \( F \) has degree \( \delta \), so it adjacent to every vertex of \( X \). Note that the vertex of \( X \) has degree \( \delta = |F| \), so \( X \) is an independent set of \( G \). Then \( G[X,F] = K_{\delta+k} \). Hence, we get \( G = K_\delta \cup (K_{n-2\delta-k} \cup I_{\delta+k}) \). \( \square \)

In 2020, by generalizing the results in Theorem 3.9 and Theorem 3.11 Liu et al. 134 obtained the following sufficient conditions by using the adjacency spectral radius.

**Theorem 3.17** (Liu et al. 134). Let \( k \geq 1, \delta \geq 2 \) and \( n \geq n_0(\delta, k) \) where

\[
n_0(\delta, k) = \max\{\delta^2(\delta + k) + \delta + k + 5k + 6\delta + 6\}.
\]

If \( G \) is a connected graph on \( n \) vertices with minimum degree \( \delta(G) \geq \delta \) and

\[
\lambda(G) \geq n - \delta - k - 1,
\]

then \( G \) is \( k \)-path-coverable unless \( G = B_{n,k,\delta} \).

Since \( K_{n-\delta-k} \) is a proper subgraph of \( B_{n,k,\delta} \), we have

\[
\lambda(B_{n,k,\delta}) > \lambda(K_{n-\delta-k}) = n - \delta - k - 1.
\]

Hence, we get immediately the following corollary.
Corollary 3.18 (Liu et al. [134]). Let $k \geq 1$, $\delta \geq 2$ and $n \geq n_0(\delta, k)$. If $G$ is a connected graph of order $n$ with minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq \lambda(B_{n,k,\delta}),$$

then $G$ is $k$-path-coverable unless $G = B_{n,k,\delta}$.

In the sequel, we will consider the $Q$-index version of Theorem 3.17. Recall that $B_{n,k,\delta} = K_\delta \lor (K_{n-2\delta-k} \cup I_{\delta+k})$. For the graph $B_{n,k,\delta}$, let $X = \{v \in V(B_{n,k,\delta}) : d(v) = k\}$, $Y = \{v \in V(B_{n,k,\delta}) : d(v) = n - 1\}$, and $Z = \{v \in V(B_{n,k,\delta}) : d(v) = n - k - \delta - 1\}$. We denote by $E_1(B_{n,k,\delta})$ the edge set of $B_{n,k,\delta}$ whose endpoints are both from $Y \cup Z$. Moreover, we define

$$B_{n,k,\delta}^{(1)} = \{B_{n,k,\delta} \setminus E' : E' \subseteq E_1(B_{n,k,\delta}) \text{ with } |E'| \leq \lfloor (\delta + k)/4\rfloor\}.$$

Here, the symbol $B_{n,k,\delta} \setminus E'$ stands for the subgraph of $B_{n,k,\delta}$ by deleting all edges from the edge set $E'$. Recently, Cheng, Feng, Li and Liu [42] proved the following extension on the sufficient conditions of the existence of a Hamilton path.

Theorem 3.19 (Cheng et al. [42]). Let $k \geq 1$, $\delta \geq 2$ and $n \geq n_1(\delta, k)$ where

$$n_1(\delta, k) = (\delta^2 + k\delta + 7\delta + 6k + 9)(\delta^2 + k\delta + 1)$$

If $G$ is a connected graph on $n$ vertices and minimum degree $\delta(G) \geq \delta$ such that

$$q(G) \geq 2(n - \delta - k - 1),$$

then either $G$ is $k$-path-coverable or $G \in B_{n,k,\delta}^{(1)}$.

Corollary 3.20. Let $k \geq 1$, $\delta \geq 2$ and $n \geq n_1(\delta, k)$. If $G$ is a connected graph of order $n$ with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq q(B_{n,k,\delta}),$$

then $G$ is $k$-path-coverable unless $G = B_{n,k,\delta}$.

4 Spectral problem for connectivity

4.1 Problem for $k$-connectivity

A connected graph $G$ is said to be $k$-connected (or $k$-vertex-connected) if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are deleted. In particular, 1-connected graphs are the general connected graphs. The connectivity of $G$, written as $\kappa(G)$, is defined as the maximum integer $k$ such that $G$ is $k$-connected. In other words, the connectivity of a graph $G$ is the minimum number of vertices that we need to delete to make $G$ disconnected. From the definition, we can see that $\kappa(G) \geq k$ is equivalent to say that $G$ is $k$-connected.
When we talk about the connectivity and the eigenvalues, perhaps the most famous one is due to Fiedler [76] which states that the second smallest Laplacian eigenvalue is at most the connectivity for any non-complete graph. For adjacency eigenvalues, the relation between the connectivity, edge-connectivity and the eigenvalues was reported in [50, 89, 208, 211].

Let $G$ be a simple graph of order $n \geq k + 1$. It is known [15, Page 4] that if $\delta(G) \geq \frac{1}{2}(n + k - 2)$, then $G$ is $k$-connected. In 1969, Bondy [20] provided a sufficient condition on the degree sequence.

**Theorem 4.1** (Bondy–Chvátal [23]). A graph $G$ is $k$-connected if and only if the closure graph $\text{cl}_{n+k-2}(G)$ is $k$-connected. In particular, if $d(u) + d(v) \geq n + k - 2$ for all non-edges $\{u, v\}$, then $\text{cl}_{n+k-2}(G) = K_n$ and $G$ is $k$-connected.

**Lemma 4.2** (Bondy [20]). Suppose that $n \geq k + 1$ and $G$ is an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $G$ is not $k$-connected, then there exists $1 \leq i \leq \frac{n-k+1}{2}$ such that $d_i \leq i + k - 2$ and $d_{n-k+1} \leq n - i - 1$.

Sometimes we can characterize the vertex-connectivity using the information about co-degrees of vertices in our graph. Such result was used in [61] to determine the vertex-connectivity of dense random $d$-regular graphs.

**Theorem 4.3** (Krivelevich et al. [108]). Let $G = (V, E)$ be a $d$-regular graph on $n$ vertices such that $\sqrt{n} \log n < d \leq 3n/4$ and the number of common neighbors for every two distinct vertices in $G$ is $(1 + o(1))d^2/n$. Then the graph $G$ is $d$-vertex-connected.

**Theorem 4.4** (Wu–Zhang–Feng [185]). Let $G$ be a connected graph on $n \geq 5$ vertices. If the minimum degree $\delta(G) \geq 2$ and

$$e(G) \geq \left(\frac{n-2}{2}\right) + 3,$$

then $G$ is 2-connected unless $G = K_1 \lor (K_{n-3} \cup K_2)$.

**Theorem 4.5** (Feng et al. [74]). Let $G$ be a graph of order $n \geq k + 1 \geq 2$. If

$$e(G) \geq \left(\frac{n-1}{2}\right) + k - 1,$$

then either $G$ is $k$-connected or $G = K_{k-1} \lor (K_1 \cup K_{n-k})$.

The above result was extended by Feng et al. [75, Theorem 3.2]. Clearly, when we set $\delta = k - 1$, then Theorem 4.4 reduces to Theorem 4.5. For convenience, we denote

$$A_{n,k,\delta} := K_{k-1} \lor (K_{\delta-k+2} \cup K_{n-\delta-1}).$$

It is easy to see that $A_{n,k,\delta}$ is not $k$-connected and $e(A_{n,k,\delta}) = \frac{1}{2}n^2 - (\delta - k + \frac{5}{2})n + (\delta + 1)(\delta - k + 2)$. In 2017, Feng et al. [75, Theorem 3.2] proved that $A_{n,k,\delta}$ is the unique graph that is not $k$-connected and has the maximum number of edges.
Theorem 4.6 (Feng et al. [75]). Let \( G \) be a connected graph on \( n \) vertices with minimum degree \( \delta(G) \geq \delta \). If \( e(G) \geq e(A_{n,k,\delta}) \), then either \( G \) is \( k \)-connected or \( G = A_{n,k,\delta} \).

From the spectral point of view, we naturally ask the following question: can one find a sufficient spectral condition for a connected graph to be \( k \)-connected? For pseudo-random graphs, Krivelevich and Sudakov [110, Theorem 4.1] provided a sufficient condition on the second largest eigenvalue in absolute value for regular graphs.

Theorem 4.7. [110] Let \( G \) be a \( d \)-regular graph on \( n \) vertices with \( d \leq n/2 \). We denote \( \mu = \max\{|\lambda_i| : i \in [n], \lambda_i \neq d\} \). Then the connectivity of \( G \) satisfies \( \kappa(G) \geq d - 36\mu^2/d \).

In 2019, Wu, Zhang and Feng [185] presented the sufficient spectral condition of a connected graph to be 2-connected with small minimum degree.

Theorem 4.8 (Wu–Zhang–Feng [185]). Let \( G \) be a connected graph on \( n \geq 5 \) vertices. If the minimum degree \( \delta(G) \geq 2 \) and \( \lambda(G) \geq \sqrt{(n-3)^2 + 4} \), then \( G \) is 2-connected.

Theorem 4.9 (Feng et al. [75]). Let \( \delta \geq k \geq 3 \) and \( n \geq n_0(k, \delta) \) where
\[
n_0(k, \delta) = (\delta - k + 2)(k^2 - 2k + 4) + 3.
\]
If \( G \) is a connected graph of order \( n \) with minimum degree \( \delta(G) \geq \delta \) and
\[
\lambda(G) \geq n - \delta + k - 3,
\]
then \( G \) is \( k \)-connected unless \( G = A_{n,k,\delta} \).

Since \( \lambda(A_{n,k,\delta}) \geq n - \delta + k - 3 \), we immediately have

Corollary 4.10 (Feng et al. [75]). Let \( \delta \geq k \geq 3 \) and \( n \geq n_0(k, \delta) \). Let \( G \) be a connected graph of order \( n \) and minimum degree \( \delta(G) \geq \delta \). If
\[
\lambda(G) \geq \lambda(A_{n,k,\delta}),
\]
then \( G \) is \( k \)-connected unless \( G = A_{n,k,\delta} \).

Recall that \( A_{n,k,\delta} = K_{k-1} \lor (K_{\delta-k+2} \lor K_{n-\delta-1}) \). For the graph \( A_{n,k,\delta} \), let \( X = \{v \in V(A_{n,k,\delta}) : d(v) = \delta\} \), \( Y = \{v \in V(A_{n,k,\delta}) : d(v) = n-1\} \) and \( Z = \{v \in V(A_{n,k,\delta}) : d(v) = n - \delta + k - 3\} \). Let \( E_1(A_{n,k,\delta}) \) denote the edge set of \( A_{n,k,\delta} \) whose endpoints are both from \( Y \cup Z \). We define
\[
A_{n,k,\delta}^{(1)} = \{A_{n,k,\delta} \setminus E' : E' \subseteq E_1(A_{n,k,\delta}) \text{ with } |E'| \leq \lfloor (\delta - k + 2)(k - 1)/4 \rfloor \}.
\]
Here, we denote by \( A_{n,k,\delta} \setminus E' \) the graph obtained from \( A_{n,k,\delta} \) by deleting all edges of the edge set \( E' \). Recently, it was proved in [202] that
Theorem 4.11 (Zhang et al. [202]). Let $\delta \geq k \geq 3$ and $n \geq n_1(k, \delta)$ where
\[ n_1(k, \delta) = (k^2 + 2k - 3)\delta^2 - (2k^3 - k^2 - 17k + 8)\delta + (k^4 - 3k^3 - 8k^2 + 23k + 4). \]
If $G$ is a connected graph of order $n$ with minimum degree $\delta(G) \geq \delta$ and
\[ q(G) \geq 2(n - \delta + k - 3), \]
then $G$ is $k$-connected unless $G \in A^{(1)}_{n,k,\delta}$.

Corollary 4.12. Let $\delta \geq k \geq 3$ and $n \geq n_1(k, \delta)$. Let $G$ be a connected graph of order $n$ and minimum degree $\delta(G) \geq \delta$. If
\[ q(G) \geq q(A_{n,k,\delta}), \]
then $G$ is $k$-connected unless $G = A_{n,k,\delta}$.

4.2 Problem for $k$-edge-connectivity

Similarly to vertex-connectivity, we now define the edge-connectivity of a graph. A simple graph $G$ is $k$-edge-connected if it has at least two vertices and remains connected whenever fewer than $k$ edges are deleted. The edge-connectivity of $G$, written as $\kappa'(G)$, is defined as the maximum $k$ such that $G$ is $k$-edge-connected. In particular, 1-edge-connected graphs are the general connected graphs. Clearly, it is well-known that $\kappa(G) \leq \kappa'(G)$. Moreover, the edge-connectivity is always at most the minimum degree of a graph.

Theorem 4.13 (Bondy–Chvátal [23]). A graph $G$ is $k$-edge-connected if and only if the closure graph $\text{cl}_{n+k-2}(G)$ is $k$-edge-connected. In particular, if $d(u) + d(v) \geq n + k - 2$ for all non-edges $\{u, v\}$, then $\text{cl}_{n+k-2}(G) = K_n$ and $G$ is $k$-edge-connected.

Lemma 4.14 (Bauer et al. [9]). Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $n \geq k+1$, and $d_1 \geq k \geq 1$. If $G$ is not $k$-edge-connected, then there exists $k + 1 \leq i \leq \lfloor n/2 \rfloor$ such that $d_{i-k+1} \leq i - 1$, $d_i \leq i + k - 2$ and $d_n \leq n - i + k - 2$. In particular, if $k \geq \lfloor n/2 \rfloor$, then $G$ is $k$-edge-connected.

For adjacency eigenvalues, the relation between the connectivity, edge-connectivity and the eigenvalues was reported in [44, 50, 89, 135, 163].

First of all, we introduce some results that only deal with regular graphs.

Theorem 4.15 (Krivelevich–Sudakov [110]). Let $G$ be a $d$-regular graph on $n$ vertices. If $\lambda_2(G) \leq d - 2$, then $\kappa'(G) = d$.

Extending and improving the the previous results, Cioabă [44] obtained that

Theorem 4.16 (Cioabă [44]). Let $d \geq k \geq 2$. If the second largest eigenvalue $\lambda_2$ of a $d$-regular graph satisfies
\[ \lambda_2 < d - \frac{(k - 1)n}{(d + 1)(n - d - 1)}, \]
then the edge-connectivity of $G$ is at least $k$. 56
Now, we consider the $k$-edge-connected property. Since every $k$-connected graph is also $k$-edge-connected. So Theorem 4.5 also provided a sufficient condition for a graph to be $k$-edge-connected. Next, we shall present a sufficient condition with a smaller lower bound, which is slightly different from Theorem 4.5.

**Theorem 4.17** (Feng et al. [74]). Let $G$ be a graph of order $n \geq k + 1$. If $\frac{1}{2}(n^2 - (k + 4)n + 2k^2 + 2k + 4)$, then $G$ is $k$-edge-connected.

**Proof.** Suppose that $G$ is not $k$-edge-connected. Then from Lemma 4.14, there exists an integer $k + 1 \leq i \leq \frac{n}{2}$ such that $d_{i-k+1} \leq i - 1$, $d_i \leq i + k - 2$ and $d_n \leq n - i + k - 2$. In particular, note that $n \geq 2k + 2$. We have

$$2e(G) \leq (i - k + 1)(i - 1) + (k - 1)(i + k - 2) + (n - i)(n - i + k - 2)$$

$$= n(n - 1) + (k - 1)(n + k - 1) + 2i^2 - (2n + k - 1)i.$$

Suppose $f(x) = 2x^2 - (2n + k - 1)x$ with $k + 1 \leq x \leq \frac{n}{2}$. Then it is easy to check that $f(k + 1) - f\left(\frac{5}{2}\right) = \frac{(n-k-3)(n-2k-2)}{2} \geq 0$, with equality if and only if $n = 2k + 2$. It follows that $f_{\max}(x) = f(k + 1) = -(k + 1)(2n - k - 3)$, and we have

$$2e(G) \leq n(n - 1) + (k - 1)(n + k - 1) - (k + 1)(2n - k - 3)$$

$$= n^2 - (k + 4)n + 2k^2 + 2k + 4.$$

Thus $e(G) = \frac{1}{2}(n^2 - (k + 4)n + 2k^2 + 2k + 4)$, and we have $i = k + 1$, and the degree sequence is given as $d_1 = d_2 = k$, $d_3 = \cdots = d_{k+1} = 2k - 1$ and $d_{k+2} = \cdots = d_n = n - 3$.

Next, we show that if $G$ has this degree sequence, then it must be $k$-edge-connected, which will be a contradiction. It suffices to show that for any partition $V(G) = A \cup B$, where $|A| + |B| = n$ and $1 \leq |A| \leq |B|$, we have $e(A, B) \geq k$.

Since $G$ has minimum degree $\delta(G) = k$, if $|A| = 1$, then $e(A, B) \geq k$, and if $|A| = 2$, then $e(A, B) \geq 2(k - 1) \geq k$.

Now, let $|A| \geq 3$. Note that $n \geq 2k + 2$ and $|A| \leq |B|$ imply $|B| \geq k + 1$. Firstly, if $|B| = k + 1$, then we also have $|A| = k + 1$. Thus by the symmetry of $A$ and $B$, we may assume that $A$ has $k$ vertices with degree at least $2k - 1$ in $G$. Then, each such vertex in $A$ has at least $(2k - 1) - k \geq 1$ neighbour in $B$, and hence $e(A, B) \geq k$.

Secondly, suppose that $|B| \geq k + 2$. If $A$ contains a vertex $v$ with $d_G(v) = n - 3$, then $v$ has at least $|B| - 2 \geq k$ neighbours in $B$, and $e(A, B) \geq k$. Otherwise, all vertices with degree $n - 3$ in $G$ lie in $B$. Since $|A| \geq 3$, each such vertex in $B$ has at least $|A| - 2 \geq 1$ neighbour in $A$. Since there are $n - (k + 1) \geq (2k + 2) - (k + 1) > k$ such vertices, it follows again that $e(A, B) \geq k$. This completes the proof.}

### 4.3 Problem for Hamilton-connectivity

A graph is Hamilton-connected if for every pair of vertices $u, v$, there is a Hamilton path from $u$ to $v$. It is well-known [25] p. 474 that a graph $G$ is traceable from every vertex if and only if $G \lor K_1$ is Hamilton-connected.
Theorem 4.18 (Dirac [57], Ore [165]). If $G$ is a graph on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n+1}{2}$, then $G$ is Hamilton-connected.

Theorem 4.19 (Bondy–Chvatal [23]). Let $G$ be a simple graph on $n$ vertices. Then $G$ is Hamilton-connected if and only if the closure graph $cl_{n+1}(G)$ is Hamilton-connected. In particular, if $d(u) + d(v) \geq n + 1$ for all non-edges $\{u, v\}$, then $cl_{n+1}(G) = K_n$ and $G$ is Hamilton-connected.

Theorem 4.20 (Berge [13]). Let $G$ be an $n$-vertex graph with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If $G$ is not Hamilton-connected, then there exists an integer $2 \leq i \leq n/2$ such that $d_i - 1 \leq i$ and $d_n - i \leq n - i$.

The following sufficient condition on the number of edges is due to Ore [166].

Theorem 4.21 (Ore [166]). Let $G$ be a graph on $n$ vertices. If $e(G) \geq \left(\frac{n}{2}\right)^2 + 3$, then $G$ is Hamilton-connected.

In 2017, Zhou and Wang [207] gave some sufficient conditions on the number of edges, the spectral radius and the signless Laplacian spectral radius for a graph to be Hamilton-connected. And they also studied sufficient conditions on the number of edges, the spectral radius and the signless Laplacian spectral radius for a graph to be traceable from every vertex.

Theorem 4.22 (Zhou–Wang [207]). Let $G$ be a connected graph on $n \geq 6$ vertices with minimum degree $\delta(G) \geq 3$. If $e(G) \geq \left(\frac{n-2}{2}\right) + 6$, then $G$ is Hamilton-connected unless $G = K_3 \lor (K_{n-5} \cup K_2)$ and the following more exceptions: when $n = 8$ and $G \in \{K_4 \lor I_4, K_3 \lor (K_{1,3} \cup K_1), K_3 \lor (K_{1,2} \cup K_2), K_2 \lor K_{2,4}\}$; when $n = 9$ and $G = K_4 \lor (K_2 \cup I_3)$; when $n = 10$ and $G \in \{K_5 \lor I_5, K_4 \lor (K_{1,4} \cup K_1), K_3 \lor K_{2,5}\}$; when $n = 11$ and $G = K_4 \lor (K_{1,3} \cup K_2)$; when $n = 12$ and $G = K_6 \lor I_6$.

Theorem 4.23 (Zhou–Wang [207]). Let $G$ be a connected graph on $n \geq 6$ vertices with minimum degree $\delta(G) \geq 3$. If $\lambda(G) \geq \sqrt{(n-3)^2 + 10}$, then $G$ is Hamilton-connected.

Theorem 4.24 (Zhou–Wang [207]). Let $G$ be a connected graph on $n \geq 6$ vertices with minimum degree $\delta(G) \geq 3$. If $q(G) \geq 2(n-3) + \frac{14}{n-1}$, then $G$ is Hamilton-connected unless $n = 8$ and $G = K_4 \lor I_4$. 

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In 2020, Zhou, Wang and Lu [209] provided a further improvement on Theorems 4.22, 4.23 and 4.24. More precisely, they proved that if \( n \geq 11 \) and \( G \) satisfies \( \delta(G) \geq 3 \) and \( e(G) \geq (n-3)^2 + 13 \), then \( G \) is Hamilton-connected unless \( G \) belongs to some exceptions.

**Theorem 4.25 (Zhou–Wang–Lu [209])**. Let \( G \) be a connected graph on \( n \geq 14 \) vertices with minimum degree \( \delta \geq 3 \). If

\[
\lambda(G) > n - 3,
\]

then \( G \) is Hamilton-connected unless \( G = K_3 \lor (K_{n-5} \cup I_2) \) or \( G = K_2 \lor (K_{n-4} \cup K_2) \).

**Theorem 4.26 (Zhou–Wang–Lu [209])**. Let \( G \) be a connected graph on \( n \geq 13 \) vertices with minimum degree \( \delta \geq 3 \). If

\[
q(G) > 2(n - 3) + \frac{6}{n - 1},
\]

then \( G \) is Hamilton-connected unless \( G = K_3 \lor (K_{n-5} \cup I_2) \).

For notational convenience, we denote

\[
S_{n,\delta} := K_\delta \lor (K_{n-2\delta+1} \cup I_{\delta-1})
\]

and

\[
T_{n,\delta} := K_2 \lor (K_{n-\delta-1} \cup K_{\delta-1}).
\]

Clearly, we can verify that both \( S_{n,\delta} \) and \( T_{n,\delta} \) are not Hamilton-connected and \( \delta(S_{n,\delta}) = \delta(T_{n,\delta}) = \delta \). Moreover, we have \( e(S_{n,\delta}) = \binom{n-\delta+1}{2} + \delta(\delta - 1) \) and \( e(T_{n,\delta}) = \binom{n-\delta+1}{2} + \binom{\delta-1}{2} + 2(\delta - 1) \). Trivially, we have \( S_{n,2} = T_{n,2} \) and \( e(S_{n,\delta}) > e(T_{n,\delta}) \) for every \( \delta \geq 3 \).

**Theorem 4.27.** Let \( \delta \geq 2 \) and \( G \) be a graph on \( n \geq 6\delta - 5 \) vertices. If \( \delta(G) \geq \delta \) and

\[
e(G) \geq e(S_{n,\delta}),
\]

then either \( G \) is Hamilton-connected or \( G = S_{n,\delta} \).

**Proof.** Assume that \( G \) is not Hamilton-connected. By Theorem 4.19, we know that the closure graph \( H := cl_{n+1}(G) \) is also not Hamilton-connected. Let \( d_1' \leq d_2' \leq \cdots \leq d_n' \) be the degree sequence of \( H \). By Theorem 4.20, there is an integer \( 2 \leq m \leq \frac{n}{2} \) such that \( d_{m-1} \leq m \) and \( d_{n-m} \leq n - m \). Hence, we can get

\[
2e(H) = \sum_{i=1}^{n} d_i' \leq (n - 1)m + (n - 2m + 1)(n - m) + m(n - 1) = 3m^2 - (2n + 3)m + n^2 + n. \tag{11}
\]

Let \( f(m) = 3m^2 - (2n + 3)m \) be a function on \( m \). Note that \( \delta \leq \delta(G) \leq \delta(H) \leq m \leq \frac{n}{2} \), and the symmetric line of \( f(m) \) is \( m_0 = \frac{2n+3}{6} \). The condition \( n \geq 6\delta - 5 \) implies \( \frac{n}{2} - \frac{2n+3}{6} < \frac{2n+3}{6} - \delta \). So the maximum value of \( f(m) \) is attained at \( m = \delta \), therefore

\[
e(G) \leq 2e(H) \leq f(\delta) + n^2 + n = 2e(S_{n,\delta}),
\]

as desired.
where $S_{n,\delta} = K_\delta \vee (I_{\delta-1} \cup K_{n-2\delta+1})$. The condition in the theorem implies that $e(G) = e(H) = e(S_{n,\delta})$, and all inequalities in (11) must be equalities, that is, $m = \delta$ and $G = \text{cl}_{n+1}(G)$, and hence the degree sequence of $G$ is $d_1 = \cdots = d_{\delta-1} = \delta$, $d_\delta = \cdots = d_{n-\delta} = n - \delta$, and $d_{n-\delta+1} = \cdots = d_n = n - 1$. Next we shall show that $G = S_{n,\delta}$.

Let $v_1, v_2, \ldots, v_n$ be vertices of $G$ corresponding to the degrees $d_1, d_2, \ldots, d_n$. We denote by $X = \{v_1, v_2, \ldots, v_{\delta-1}\}$ and $Y = \{v_\delta, v_{\delta+1}, \ldots, v_n\}$. First of all, we shall prove that the induced subgraph $G[Y] = K_{n-\delta+1}$. Otherwise, there are two non-adjacent vertices $v_r, v_s \in G[Y]$. By the definition of the $(n+1)$-closure graph, we know that

$$d_G(v_r) + d_G(v_s) = d_H(v_r) + d_H(v_s) < n + 1.$$ 

On the other hand, we have $d_G(v_r) + d_G(v_s) \geq 2(n - \delta) > n + 1$, a contradiction. Thus, the induced subgraph $G[Y] = K_{n-\delta+1}$. Note that $Y$ contains $\delta$ vertices with degree $n - 1$, we denote these vertices by $F = \{v_{n-\delta+1}, \ldots, v_n\}$. Every vertex of $F$ is adjacent to the vertex of $X$. Since the vertex of $X$ has degree $\delta$, we can get that $X$ is an independent set in $G$, thus we have $G[X, F] = K_{\delta-1, \delta}$. The above discussion reveals that $G = K_{\delta} \vee (I_{\delta-1} \cup K_{n-2\delta+1})$. This completes the proof.

Recently, Chen et al. [37], Yu et al. [189], and Wei et al. [181] independently showed the spectral version of Hamilton-connectedness for graphs with large minimum degree.

**Theorem 4.28** (Wei et al. [181]). Let $\delta \geq 3$ and $n \geq n_0(\delta)$ where

$$n_0(\delta) = \max\{6\delta, (\delta^3 - \delta^2 + 2\delta + 8)/2\}.$$ 

If $G$ is a graph on $n$ vertices with the minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq n - k,$$

then $G$ is Hamilton-connected unless $G = S_{n,\delta}$ or $G = T_{n,\delta}$.

**Corollary 4.29.** Let $\delta \geq 3$ and $n \geq n_0(\delta)$. If $G$ is a graph on $n$ vertices with the minimum degree $\delta(G) \geq \delta$ and

$$\lambda(G) \geq \lambda(S_{n,\delta}),$$

then $G$ is Hamilton-connected unless $G = S_{n,\delta}$.

In 2020, Zhou, Wang and Lu [210] presented the signless Laplacian spectral conditions for Hamilton-connected graphs with large minimum degree.

For the graph $S_{n,\delta}$, let $X = \{v \in V(S_{n,\delta}) : d(v) = \delta\}, Y = \{v \in V(S_{n,\delta}) : d(v) = n - 1\}$ and $Z = \{v \in V(S_{n,\delta}) : d(v) = n - \delta\}$. Let $E_1(S_{n,\delta})$ be the subset of edges of $S_{n,\delta}$ whose both endpoints are from $Y \cup Z$. We define the family of graphs as below

$$S_{n,\delta}^{(1)} = \{S_{n,\delta} \setminus E' : E' \subseteq E_1(S_{n,\delta}) \text{ and } |E'| \leq |\delta(\delta - 1)/4|\}.$$ 

Here, we denote by $S_{n,\delta} \setminus E'$ the graph obtained from $S_{n,\delta}$ by deleting all edges of the edge set $E'$. Similarly, for the graph $T_{n,\delta}$, let $X = \{v \in V(T_{n,\delta}) : d(v) = \delta\}, Y = \{v \in V(T_{n,\delta}) :$
\(d(v) = n - 1\) and \(Z = \{v \in V(T_{n, \delta}) : d(v) = n - \delta\}\). Let \(E_1(T_{n, \delta})\) be the subset of \(E(T_{n, \delta})\) containing the edges whose both endpoints are from \(Y \cup Z\). We denote
\[
\mathcal{T}^{(1)}_{n, \delta} = \left\{ T_{n, \delta} \setminus E' : E' \subseteq E_1(T_{n, \delta}) \text{ and } |E'| \leq \lfloor (\delta - 1)/2 \rfloor \right\}.
\]
When \(G\) is a subgraph of \(S_{n, \delta}\) or \(T_{n, \delta}\), we can easily see that \(G\) is not Hamilton-connected. Moreover, we can verify that \(\delta(G) = \delta\) and \(q(G) \geq 2(n - \delta)\) for every \(G \in S^{(1)}_{n, \delta} \cup \mathcal{T}^{(1)}_{n, \delta}\).

**Theorem 4.30** (Zhou–Wang–Lu [210]). Let \(\delta \geq 2\) and \(n \geq n_1(\delta)\) where
\[
n_1(\delta) = \delta^4 + 5\delta^3 + 2\delta^2 + 8\delta + 12.
\]
If \(G\) is a graph of order \(n\) with minimum degree \(\delta(G) \geq \delta\) and
\[
q(G) \geq 2(n - \delta),
\]
then \(G\) is Hamilton-connected unless \(G \in S^{(1)}_{n, \delta}\) or \(G \in \mathcal{T}^{(1)}_{n, \delta}\).

**Corollary 4.31.** Let \(\delta \geq 3\) and \(n \geq n_1(\delta)\). If \(G\) is a graph on \(n\) vertices with minimum degree \(\delta(G) \geq \delta\) and
\[
q(G) \geq q(S_{n, \delta}),
\]
then \(G\) is Hamilton-connected unless \(G = S_{n, \delta}\).

Recently, Xu, Zhai and Wang [187] present some sufficient conditions for a graph to be Hamilton-connected in terms of size, spectral radius and signless Laplacian spectral radius, respectively, which extend some corresponding results.

## 5 Spectral problem for deficiency

### 5.1 Problem for given matching number

A matching in a graph is a set of disjoint edges, the matching number of \(G\), denoted by \(\alpha'(G)\), is the maximum size of a matching in \(G\). A perfect matching in \(G\) is a matching covering all vertices.

In 1959, Erdős and Gallai [61] proved the following extremal problem for \(M_{k+1}\), a matching of \(k + 1\) edges; see [15] p. 58 and [4, Theorem 2] for alternative proofs.

**Theorem 5.1** (Erdős–Gallai [61]). For any \(n \geq 2k + 1\), if \(G\) is an \(n\)-vertex graph without copy of \(M_{k+1}\), then
\[
e(G) \leq \max \left\{ \left(\frac{2k + 1}{2}\right), \left(\frac{k}{2}\right) + (n-k)k \right\}.
\]
Moreover, the extremal are the following.

1. If \(2k + 1 \leq n < \frac{5k + 3}{2}\), then \(K_{2k+1} \cup I_{n-2k-1}\) is the unique extremal graph.
2. If \(n = \frac{5k + 3}{2}\), then there are two extremal graphs \(K_{2k+1} \cup I_{n-2k-1}\) and \(K_k \cup I_{n-k}\).
3. If \(n > \frac{5k + 3}{2}\), then \(K_k \cup I_{n-k}\) is the unique extremal graph.
Recently, Wang [179] proved that the two extremal graphs in Theorem 5.1 not only have the maximum number of $s$-cliques for every $s \geq 2$, but also attain the maximum number of $K_k \vee I_t$, which is the graph obtained from $K_k \cup I_t$ by replacing the part of size $s$ by a clique of the same size. We infer the interested reader to [179,55] for recent progress.

In 2007, Feng, Yu and Zhang [70] proved the spectral version of Erdős–Gallai’s theorem and determined the largest spectral radius with given matching number. First of all, we can see that $K_k \vee I_{n-k}$ does not contain a matching of size $k+1$ and

$$\lambda(K_k \vee I_{n-k}) = \frac{k - 1 + \sqrt{(k - 1)^2 + 4k(n-k)}}{2}.$$ 

**Theorem 5.2** (Feng et el. [70]). Let $G$ be a graph on $n$ vertices with matching number $k$.
(1) If $n = 2k$ or $2k + 1$, then $\lambda(G) \leq \lambda(K_n)$ with equality if and only if $G = K_n$.
(2) If $2k + 2 \leq n < 3k + 2$, then $\lambda(G) \leq 2k$ with equality if and only if $G = K_{2k+1} \cup I_{n-2k-1}$.
(3) If $n = 3k + 2$, then $\lambda(G) \leq 2k$ with equality if and only if $G = K_k \vee I_{n-k}$ or $G = K_{2k+1} \cup I_{n-2k-1}$.
(4) If $n > 3k + 2$, then $\lambda(G) \leq \lambda(K_k \vee I_{n-k})$ with equality if and only if $G = K_k \vee I_{n-k}$.

In 2008, Yu [191] presented the spectral version for the signless Laplacian radius. To some extent, the results and the extremal graphs are similar with that of the adjacency spectral radius. Some tedious calculation yields

$$q(K_k \vee I_{n-k}) = \frac{n + 2k - 2 + \sqrt{(n + 2k - 2)^2 - 8k^2 + 8k}}{2}.$$ 

**Theorem 5.3** (Yu [191]). Let $G$ be a graph on $n$ vertices with matching number $k$.
(1) If $n = 2k$ or $2k + 1$, then $q(G) \leq q(K_k)$, with equality if and only if $G = K_n$.
(2) If $2k + 2 \leq n < \frac{5k+3}{2}$, then $q(G) \leq 4k$, with equality if and only if $G = K_{2k+1} \cup I_{n-2k-1}$.
(3) If $n = \frac{5k+3}{2}$, then $q(G) \leq 4k$, with equality if and only if $G = K_k \vee I_{n-k}$ or $G = K_{2k+1} \cup I_{n-2k-1}$.
(4) If $n > \frac{5k+3}{2}$, then $q(G) \leq q(K_k \vee I_{n-k})$ with equality if and only if $G = K_k \vee I_{n-k}$.

### 5.2 Problem for perfect matching

The study of the relation between the eigenvalues and the matching number was initiated by Brouwer and Haemers [27]. For regular graphs, they obtained

**Theorem 5.4**. [27] Let $G$ be a connected $k$-regular graph on $n$ vertices with adjacency eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. If $n$ is even and

$$\lambda_3 \leq \begin{cases} k - 1 + \frac{3}{k+1}, & \text{if } k \text{ is even}, \\ k - 1 + \frac{3}{k+2}, & \text{if } k \text{ is odd}, \end{cases}$$

then $G$ has a perfect matching.

Subsequently, Cioabă, Gregory and Haemers [48,49] refined and generalized the above result to obtain the following result for the $(r + 1)$-th largest eigenvalue.

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Theorem 5.5. [48] Let $G$ be a connected $k$-regular graph on $n$ vertices with eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. Assume $r > 0$ is an integer such that $n \equiv r \pmod{2}$. If

$$k - \lambda_{r+1} > \begin{cases} 
0.1475, & \text{if } k = 3, \\
1 - \frac{3}{k+1} - \frac{1}{(k+1)(k+2)}, & \text{if } k \text{ is even}, \\
1 - \frac{3}{k+2} - \frac{1}{(k+2)^2}, & \text{if } k \geq 5 \text{ is odd},
\end{cases}$$

then $G$ has a matching of size at least $\frac{r}{2} + 1$.

Further, they found an explicit expression $\rho(k)$ in [49], and proved that if $\lambda_3(G) < \rho(k)$, then $G$ has a matching of size $\lfloor \frac{r}{2} \rfloor$. Other related results can be found in [135, 163].

In what follows, we shall consider the sufficient conditions on the existence of perfect matching in terms of the number of edges. First of all, we define

$$R_n := K_1 \lor (K_{n-3} \lor I_2).$$

It is easy to verify that $R_n$ has no perfect matching and $e(R_n) = \left(\frac{n-2}{2}\right) + 2$. In 2021, Suil O [162] proved that if an $n$-vertex ($n \geq 10$) graph has more edges than $R_n$, then it has a perfect matching. More precisely, they showed the following theorem even for graphs with small order.

Theorem 5.6 (Suil O [162]). Let $n \geq 10$ be an even integer or $n = 4$. If $G$ is a connected graph of order $n$ with $e(G) > \left(\frac{n-2}{2}\right) + 2$, then $G$ has a perfect matching. For $n = 6$ or $n = 8$, if $e(G) > 9$ or $e(G) > 18$, respectively, then $G$ has a perfect matching.

We mention here that the extremal graphs in Theorem 5.6 can be characterized easily. By a careful examination, we get that if $e(G) \geq \left(\frac{n-2}{2}\right) + 2$, then $G$ has a perfect matching unless $n \geq 12$ and $G = R_n$, or $n = 6$ and $G \in \{R_6, K_2 \lor I_4, K_{2,4}, K_1 \lor (K_1 \lor K_{1,3})\}$, or $n = 8$ and $G \in \{R_8, K_1 \lor K_{2,5}, K_2 \lor (K_1 \lor K_{1,4}), K_2 \lor (K_2 \lor K_{1,3}), K_3 \lor I_5\}$, or $n = 10$ and $G \in \{R_{10}, K_4 \lor I_6\}$. The proof is a standard case analysis, so we omit the details. Moreover, Suil O [162] proved the following lower bound for spectral radius in an $n$-vertex graph to guarantee the existence of a perfect matching.

Theorem 5.7 (Suil O [162]). Let $n$ be an even integer and $G$ be an $n$-vertex graph.

1. If $n = 4$ or $n \geq 8$ and $\lambda(G) > \lambda(R_n)$, then $G$ has a perfect matching.
2. If $n = 6$ and $\lambda(G) > 1 + \sqrt{33} \over 2$, then $G$ has a perfect matching.

The condition is best possible since both $R_n$ and $K_2 \lor I_4$ have no perfect matching. Some calculations yield that $\lambda(K_2 \lor I_4) = 1 + \sqrt{33} \approx 3.372$ and $\lambda(R_6) \approx 3.177$. Moreover, it is not hard to see that $\lambda(R_n)$ is the largest root of

$$x^3 - (n - 4)x^2 - (n - 1)x + 2(n - 4) = 0.$$

We here make a brief proof. Let $x = (x_1, \ldots, x_n)$ be the eigenvector corresponding to $\lambda = \lambda(R_n)$. By the symmetry, we know that the entries of eigenvector corresponding to the two vertices of degree 1 are equal, we may denote by $x_1 = x_2 = x$. Similarly, the
entries of eigenvector corresponding to the $n - 3$ vertices with degree $n - 3$ are denoted by $x_3 = \cdots = x_{n-1} = y$, and the entry of eigenvector corresponding to the vertex with degree $n - 1$ is denoted by $x_n = z$. The eigen-equation $Ax = \lambda x$ implies that

$$\begin{cases}
\lambda x = z, \\
\lambda y = (n - 4)y + z, \\
\lambda z = 2x + (n - 3)y.
\end{cases}$$

We can obtain that $x = \frac{1}{\lambda}z$ and $y = \frac{1}{\lambda-4}z$, and thus

$$\lambda^3 - (n - 4)\lambda^2 - (n - 1)\lambda + 2(n - 4) = 0$$

Next, we shall show that

$$n - 3 < \lambda(R_n) < \sqrt{(n - 3)^2 + 1}.$$ 

Since $K_{n-2}$ is a proper subgraph of $K_1 \lor (K_{n-3} \lor I_2)$, the second inequality holds immediately. Let $f(x) = x^3 - (n - 4)x^2 - (n - 1)x + 2(n - 4)$. We can compute the derivation $f'(x) = 3x^2 - 2(n - 4)x - (n - 1)$, which has two distinct roots $x_1 = \frac{1}{3}(n - 4 - \sqrt{n^2 - 5n + 13})$ and $x_2 = \frac{1}{3}(n - 4 + \sqrt{n^2 - 5n + 13})$. Note that $f(x_2) < 0$. An easy calculation reveals that $x_2 < \sqrt{(n - 3)^2 + 1}$ and $f(\sqrt{(n - 3)^2 + 1}) > 0$ for $n \geq 8$, hence $\sqrt{(n - 3)^2 + 1}$ is greater than the largest root of $f(x)$. By the above discussion, we know that the largest root of $f(x)$ is approximately equal to $n - 3$. Given a graph $G$, it is not straightforward to verify the condition $\lambda(G) > \lambda(R_n)$. Hence, it is possible and meaningful to relax the condition to $\lambda(G) \geq n - 3$ and characterize more extremal graphs.

In 2020, Liu, Pan and Li [131] presented the result about the relationship between the signless Laplacian spectral radius and prefect matching in graphs. Recall that $R_n = K_1 \lor (K_{n-3} \lor I_2)$ and $R_n$ has no perfect matching.

**Theorem 5.8 (Liu–Pan–Li [131]).** Let $n$ be even and $G$ be an $n$-vertex connected graph.

1. If $n = 4$ or $n \geq 10$ and $q(G) > q(R_n)$, then $G$ has a perfect matching.
2. If $n = 6$ and $q(G) > 4 + 2\sqrt{3}$, then $G$ has a perfect matching.
3. If $n = 8$ and $q(G) > 6 + 2\sqrt{6}$, then $G$ has a perfect matching.

First of all, we can show that $q(R_n)$ is the largest root of the equation

$$x^3 - (3n - 7)x^2 + n(2n - 7)x - 2(n^2 - 7n + 12) = 0.$$ 

Moreover, we can see that $K_2 \lor I_4$ and $K_3 \lor I_5$ have no perfect matching. By some calculations, we can get that $q(K_2 \lor I_4) = 4 + 2\sqrt{3} \approx 7.464$ and $q(R_6) \approx 6.909$. For $n = 8$, we have $q(K_3 \lor I_5) = 6 + 2\sqrt{6} \approx 10.899$ and $q(R_8) \approx 10.513$.

### 5.3 Problem for graph deficiency

The *deficiency* of a graph $G$, denoted by $\text{def}(G)$, is the number of unmatched vertices under a maximum matching in $G$. In particular, $G$ has a perfect matching if and only if $\text{def}(G) = 0$. And $G$ has a nearly perfect matching if and only if $\text{def}(G) = 1$. 

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Let $G = (V, E)$ be a simple graph on $n$ vertices and $\alpha'(G)$ denote the number of edges in a maximum matching of $G$. To generalize the result of Tutte [177], Berge [12] obtained the well-known Berge–Tutte formula.

**Theorem 5.9** (Berge–Tutte). Let $S \subseteq V$ be a set of vertices. We denote by $\text{odd}(G - S)$ the number of components with odd vertices in the induced subgraph $G - S$. Then

$$\alpha'(G) = \frac{1}{2} \left( n - \max_{S \subseteq V} \{ \text{odd}(G - S) - |S| \} \right).$$

With the help of the Berge–Tutte formula, we can get

$$\text{def}(G) = \max_{S \subseteq V} \{ \text{odd}(G - S) - |S| \}.$$  

We say that $G$ is $\beta$-deficient if $\text{def}(G) \leq \beta$. In particular, $G$ is 0-deficient if and only if $G$ has a perfect matching. And $G$ is 1-deficient if and only if $G$ has a perfect matching or a nearly perfect matching.

**Theorem 5.10** (Bondy–Chvatal [23]). Let $G$ be a simple graph on $n$ vertices. Then $G$ is $\beta$-deficient if and only if the closure graph $\text{cl}_{n-\beta-1}(G)$ is $\beta$-deficient. In particular, if $d(u) + d(v) \geq n - \beta - 1$ for all non-edges $\{u, v\}$, then $\text{cl}_{n-\beta-1}(G) = K_n$ and $G$ is $\beta$-deficient.

**Lemma 5.11**. [8, 113] Let $G$ be a connected graph on $n$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Suppose $0 \leq \beta \leq n$ and $n \equiv \beta \pmod{2}$. If $G$ is not $\beta$-deficient, then there exists $1 \leq i \leq \frac{n+\beta-2}{2}$ such that $d_{i+1} \leq i - \beta$ and $d_{n+i-1} \leq n - i - 2$.

**Lemma 5.12** (Feng et al. [74]). Let $G$ be a graph of order $n \geq 11$. Let $0 \leq \beta \leq n$ with $n \equiv \beta \pmod{2}$. If $e(G) \geq \frac{1}{2} \left( n^2 - 5n - 4\beta + 10 \right)$, then $G$ is $\beta$-deficient, unless $\beta = 0$ and $G = K_1 \cup (I_2 \cup K_{n-3})$, or $\beta = 1$ and $G = I_2 \cup K_{n-2}$.

For every integers $\beta \geq 0$ and $\delta \geq 1$, we denote

$$R_{n,\delta,\beta} := K_\delta \cup (K_{n-2\delta-\beta-1} \cup I_{\delta+\beta+1}).$$

It is easy to see that $R_{n,\delta,\beta}$ is a graph with minimum degree $\delta$ and deficiency $\beta + 1$, so it is not $\beta$-deficient. Furthermore, we can easily prove the following theorem.

**Theorem 5.13.** For every $\beta \geq 0$ and $\delta \geq 1$, there exists an $n_0(\delta, \beta)$ such that for every $n \geq n_0(\delta, \beta)$ with $n \equiv \beta \pmod{2}$, if $G$ is an $n$-vertex connected graph with minimum degree $\delta(G) \geq \delta$ and

$$e(G) \geq e(R_{n,\delta,\beta}),$$

then $G$ is $\beta$-deficient unless $G = R_{n,\delta,\beta}$.

In 2018, Liu et al. [132] proved the following spectral extremal problem for $\beta$-deficient.
Theorem 5.14 (Liu et al. [132]). Let $\beta \geq 0$ and $\delta \geq 1$ and $n \equiv \beta (\text{mod } 2)$ with $n \geq \tilde{n}_0(\delta, \beta)$ where

$$
\tilde{n}_0(\delta, \beta) = \max \{ \delta^3 + \delta^2 (\beta + 1) + \delta + 6, 2\delta^2 + 2\delta \beta + 7\delta + 3\beta + 7, (\delta + \beta + 2)(\delta + 1)\delta \}. 
$$

If $G$ is an $n$-vertex connected graph with minimum degree $\delta(G) \geq \delta$ and

$$
\lambda(G) \geq n - \delta - \beta - 2, 
$$

then $G$ is $\beta$-deficient unless $G = R_{n, \delta, \beta}$.

Corollary 5.15 (Liu et al. [132]). Let $\beta \geq 0$ and $\delta \geq 1$ and $n \equiv \beta (\text{mod } 2)$ with $n \geq n_0(\delta, \beta)$ where

$$
n_0(\delta, \beta) := \max \{ 7 + 7\delta + 2\delta^2 + 2\delta \beta + 3\beta, (\delta + \beta + 2)(\delta + 1)\delta \}. 
$$

If $G$ is an $n$-vertex connected graph with minimum degree $\delta(G) \geq \delta$ and

$$
\lambda(G) \geq \lambda(R_{n, \delta, \beta}), 
$$

then $G$ is $\beta$-deficient unless $G = R_{n, \delta, \beta}$.

For some special cases, one can relax the bound on order of graphs. For example, when $\delta = 1$ and $\beta = 0$, i.e., the existence of perfect matching, the desired result holds for all $n \geq 8$; see the previous Theorem 5.7.

Furthermore, Liu et al. [133] obtained a tight $Q$-spectral condition for a graph to be $\beta$-deficient. In order to state their main result, we need to fix some notation. Recall that $R_{n, \delta, \beta} = K_{\delta} \lor \left( K_{n - 2\delta - 1 - \beta} \cup I_{\delta + \beta + 1} \right)$. In the graph $R_{n, \delta, \beta}$, we denote by $X = \{ v \in V(R_{n, \delta, \beta}) : d(v) = \delta \}$, $Y = \{ v \in V(R_{n, \delta, \beta}) : d(v) = n - 1 \}$, and $Z = \{ v \in V(R_{n, \delta, \beta}) : d(v) = n - \beta - 2 \}$. Let $E_1(R_{n, \delta, \beta})$ denote the set of edges of $R_{n, \delta, \beta}$ whose endpoints are both in $Y \cup Z$. We further define the following family of graphs.

$$
\mathcal{R}^{(1)}_{n, \delta, \beta} = \left\{ R_{n, \delta, \beta} \setminus E' : E' \subseteq E_1(R_{n, \delta, \beta}) \text{ with } |E'| \leq \lfloor (\delta + \beta + 1)\delta/4 \rfloor \right\}. 
$$

Here, we denote by $R_{n, \delta, \beta} \setminus E'$ the graph obtained from $R_{n, \delta, \beta}$ by deleting all edges of the edge set $E'$. As a $Q$-spectral counterpart, we obtain the following analogue of signless Laplacian spectral radius.

Theorem 5.16 (Liu et al. [133]). Let $\beta \geq 0$ and $\delta \geq 1$ and $n \equiv \beta (\text{mod } 2)$ with $n \geq n_1(\delta, \beta)$ where

$$
n_1(\delta, \beta) = \delta^4 + (2\beta + 7)\delta^3 + (\beta^2 + 11\beta + 15)\delta^2 + (4\beta^2 + 13\beta + 20)\delta + 10\beta + 21. 
$$

If $G$ is a connected graph on $n$ vertices with the minimum degree $\delta(G) \geq \delta$ and

$$
q(G) \geq 2(n - \delta - \beta - 2), 
$$

then either $G$ is $\beta$-deficient or $G \in \mathcal{R}^{(1)}_{n, \delta, \beta}$.  

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Corollary 5.17. Let $\beta \geq 0$ and $\delta \geq 1$ and $n \equiv \beta(\text{mod } 2)$ with $n \geq n_1(\delta, \beta)$. If $G$ is an $n$-vertex connected graph with minimum degree $\delta(G) \geq \delta$ and

$$q(G) \geq q(R_{n, \delta, \beta}),$$

then $G$ is $\beta$-deficient unless $G = R_{n, \delta, \beta}$.

For some special cases, one can relax the bound on order of graphs. For example, when $\delta = 1$ and $\beta = 0$, i.e., the existence of perfect matching, the desired result holds for all $n \geq 10$; see the previous Theorem 5.8.

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