Bargmann-Wigner Equations, Fermion-Boson Correspondence and Superradiant Problem in Curved Spacetime

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Abstract

Bargmann-Wigner equations and their solutions are studied in (3+1)-dimensional curved spacetime. Fermion-Boson correspondence for bi-spinor case is studied through the Bargmann-Wigner equations and solutions over curved spacetime. As an application to scattering phenomena of massive Fermions and Bosons on the rotating black holes, the superradiance with negative energy ($\omega < 0$) and positive effective energy in co-rotating coordinate system near horizon ($\omega - m\Omega_H > 0$) is possible to occur as stable physical states in Kerr spacetime.

1 Introduction

It is important to make clear the matter field dynamics in a curved spacetime for the astrophysical observation and theory. As matter fields, Fermions and Bosons have different features in one side and have mutually related features in other side. One of the important theoretical observation for scattering of matter fields is superradiant problem of Fermion and Boson fields in the rotating black hole geometry. The superradiant phenomena were studied not to occur for Fermion fields [1, 2, 3] but to occur for Boson fields [1, 4, 5, 6, 7, 8] in (3+1)-dimensional rotating black hole spacetime. The successive occurrences of superradiant phenomena for Boson fields were studied to cause the serious problem of the instability of black holes especially in Kerr geometry [4, 9, 10, 11]. Therefore the Fermion and Boson relation is one of the important theoretical key to solve the cosmological problems including the superradiant problem.

As a kinematical relation between Fermions and Bosons, the field equations for the direct product of spinor fields are studied to derive those for the higher spin states originally by Bargmann and Wigner in flat Minkowski spacetime, which is known as the Bargmann-Wigner (BW) equations [12, 13]. The BW formulation is expected to analyze the superradiant problem as one of effective applications.

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In this paper, we extend the BW equations in flat Minkowski spacetime to those in general curved spacetime in (3+1)-dimensions. Explicitly we study the BW equations for bi-spinor fields as the direct product of two spinor fields to obtain the Boson field equations for pseudoscalar and vector fields. In this approach, we obtain the consistent solutions for Fermion fields in terms of Boson fields under the torsion free condition.

Applying the result to the superradiant problem, we obtained the result that spinor fields as Fermi particles and pseudoscalar and vector fields as Boson particles should show the corresponding behavior in scattering problem in curved spacetime because they are related by the BW equations. This means that both of them occur superradiant instability or both do not in rotating black hole geometry.

In addition to this result, we will study the type of the superradiance; stable or unstable, considering the co-rotating coordinate frame in Kerr spacetime and obtain the spectrum condition that the effective energy in co-rotating coordinate system near the outer horizon of black hole is positive: \( \omega - m\Omega_H > 0 \). This result is compared with the general superradiant condition \( \omega(\omega - m\Omega_H) < 0 \), the superradiance occur as \( \omega - m\Omega_H > 0 \) with \( \omega < 0 \) for both Fermions and Bosons. We call this as type 2 superradiance. The result is consistent with our previous result in which the type 2 superradiant phenomena are studied as stable physical normalizable states. The negative particle energy (\( \omega < 0 \)) is necessary because of the completeness relation of matter fields and the type 1 superradiance (\( \omega - m\Omega_H > 0 \) with \( \omega < 0 \)) is not normalizable physical states.

We note that in our previous paper, we derived current relations between spinor and scalar fields to discuss the superradiant problem in asymptotic flat spacetime region of Kerr geometry using the BW formulation [14]. In this paper we derive the direct relation of field equations and solutions between Fermi and Bose particles in full spacetime region of general curved geometry. We also note that, in this formulation, the mass of matter fields should be finite because Fermion and Boson fields have different mass dimensions.

It is very interesting to note that, as one of massive pseudoscalar fields, axion fields is important in the superradiant problem and one of the dark matter candidates [15, 16].

The organization of this paper is as follows. In section 2, the BW equations for bi-spinor in flat Minkowski spacetime are extended to include scalar part as well as vector parts in (3+1)-dimensional general curved spacetime. In section 3, Fermion and Boson correspondence is studied using the BW equations and solutions. In section 4, the superradiant problem is studied in the co-rotating coordinate frame in general rotating spacetimes including Kerr geometry. Summary and discussion will be given in the final section.

## 2 Bargmann-Wigner equations in curved spacetime

In this section, we derive the BW equations in general curved spacetime to describe the Boson states consistently with the fundamental spinor states. For this purpose, we start to consider the BW equations for bi-spinor fields in a local Minkowski spacetime attached to each point of the general curved spacetime.
2.1 Bargmann-Wigner equations in local Minkowski spacetime

The B-W equations for the bi-spinor field $\Psi(x)^{(BW)}$ are written [12, 13]:

\[ \begin{align*}
(\gamma^a \partial_a + M)_{\alpha\sigma} \Psi(x)^{(BW)}_{\sigma\beta} &= 0, \\
(\gamma^a \partial_a + M)_{\beta\sigma} \Psi(x)^{(BW)}_{\alpha\sigma} &= 0,
\end{align*} \]  

(2.1)

where $\gamma^a$ and $M$ denote the Dirac gamma matrices in a local Minkowski spacetime and a mass of the bi-spinor field. The index $a = 0, 1, 2, 3$ labels the vector component in local Minkowski coordinate system. We rewrite the BW equations (2.1) omitting spinor indexes $\alpha, \beta$ as

\[ \begin{align*}
(\gamma^a \partial_a + M) \Psi(x)^{(BW)} &= 0, \\
\Psi(x)^{(BW)}(\partial_a \gamma^a + M) &= 0, 
\end{align*} \]  

(2.2)

In order to avoid the transpose operation $T$ in eq. (2.2), we introduce the modified bi-spinor BW field defined as

\[ \Psi(x) = \Psi(x)^{(BW)} C^{-1} , \]  

(2.3)

where $C$ denotes the charge conjugation operator and the relation $C \gamma^a T C^{-1} = -\gamma^a$ is used. Then we obtain the BW equations for the bi-spinor field $\Psi$ as

\[ \begin{align*}
(\gamma^a \partial_a + M) \Psi(x) &= 0, \\
\Psi(x)(\partial_a \gamma^a - M) &= 0. 
\end{align*} \]  

(2.4)

2.2 Relation between local Minkowski and curved spacetimes

In general curved spacetime, the gamma matrices with Roman indexes $(a, b)$ and their algebra are defined in local Minkowski spacetime [12, 17]:

\[ \{ \gamma^a, \gamma^b \} = 2 \eta^{ab}, \eta^{ab} = \text{diag}(-1, 1, 1, 1) . \]  

(2.5)

The gamma matrices with Greek indexes $(\mu, \nu)$ and their algebra in general curved spacetime are obtained through the local Minkowski spacetime:

\[ \begin{align*}
\gamma^\mu &:= e^a_\mu \gamma^a , \\
\{ \gamma^\mu, \gamma^\nu \} &:= 2 g^{\mu\nu}, g^{\mu\nu} = e^a_\mu e^a_\nu ,
\end{align*} \]  

(2.6)

where $e^a_\mu$ and $g^{\mu\nu}$ denote a vierbein (or tetrad) and the metric tensor.

For spinor fields $\psi(x)$, they transform as scalars under general coordinate transformations but as spinors under a local Lorentz transformation $\Lambda(x)$

\[ \psi(x) \rightarrow D(\Lambda(x)) \psi(x) , \]  

(2.7)

where $D(\Lambda)$ is the spinor representation of the homogeneous Lorentz group. A covariant derivative for spinors is introduced as

\[ \mathcal{D}_\mu \psi = (\partial_\mu + \Omega_\mu) \psi , \]  

(2.8)
which is defined to transform under local Lorentz transformations like $\psi$ itself: $\mathcal{D}_\mu \psi(x) \rightarrow D(\Lambda(x))\mathcal{D}_\mu \psi(x)$. The connection matrix $\Omega_\mu$ for spinors is written as

$$\Omega_\mu = \frac{1}{4} \omega^{ab}_{\mu} \Sigma_{ab},$$

(2.9)

where $\omega^{ab}_{\mu}$ is the spin connection and $\Sigma_{ab} = (\gamma_a \gamma_b - \gamma_b \gamma_a)/2$ are the spin matrices representing the generators of homogeneous Lorentz group. Then the covariant Dirac equation in general spacetime is derived as

$$(\gamma^\mu \mathcal{D}_\mu + M)\psi(x) = 0,$$

(2.10)

with the mass parameter $M$.

For vector fields $A_\nu(x)$, they transform as vectors under general coordinate transformations and their covariant derivative is defined as

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\lambda\nu\mu} A_\lambda,$$

(2.11)

where $\Gamma_{\lambda\nu\mu}$ denote the affine connection.

The covariant derivatives for vierbeins with indexes of local Minkowski and general coordinates are expressed as

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^{ab}_{\mu e^b_\nu} - \Gamma_{\mu\nu}^d A_d,$$

(2.12)

The vierbein condition is imposed to determine the geometry:

$$\mathcal{D}_\mu e^a_\nu = 0,$$

(2.13)

which leads the explicit form of the spin connection

$$\omega^{ab}_{\cdot \mu} = g^{\nu\lambda} e^a_\nu \nabla_\mu e^b_\lambda.$$

(2.14)

2.3 Bargmann-Wigner equations for bi-spinor and Boson fields in curved spacetime

Next we consider the bi-spinor BW fields $\Psi(x)$ as possible Boson states. The bi-spinor fields transform under local Lorentz transformations as direct product of spinors from lefthand and righthand sides as

$$\Psi(x) \rightarrow D(\Lambda(x))\Psi(x)D^{-1}(\Lambda(x)),$$

(2.15)

which define the covariant derivatives as

$$\mathcal{D}_\mu \Psi(x) := \partial_\mu \Psi + \Omega_\mu \Psi - \Psi \Omega_\mu,$$

(2.16)

where we take into account transpose symmetry relations: $C\Lambda^T C^{-1} = -\Lambda$ and $C\Sigma_{ab}^T C^{-1} = -\Sigma_{ab}$. Then the covariant field equations for bi-spinors are derived

$$(\gamma^\mu \mathcal{D}_\mu + M)\Psi(x) = 0,$$

(2.17)

$$\Psi(x)(\mathcal{D}_\mu \gamma^\mu - M) = 0.$$

(2.18)
We expand the bi-spinor field in a set of Boson fields as:

$$\Psi(x) = aS(x) + b\gamma_5 P(x) + c\gamma^\mu V_\mu(x) + d\gamma_5 \gamma^\mu A_\mu(x) + \frac{e}{2} \Sigma^{\mu
u} F_{\mu
u}(x),$$  \hspace{1cm} (2.19)

where $S, P, V_\mu, A_\mu$ and $F_{\mu\nu}$ denote scalar, pseudoscalar, vector, axial vector and tensor fields respectively. Coefficients $a, b, c, d, e$ are determined to satisfy the BW equations (2.17) and (2.18).

Subtracting these equations: (2.17) $-$ (2.18), we find the set of relations among Boson fields:

$$aM S(x) = 0,$$  \hspace{1cm} (2.20)

$$bM \gamma_5 P(x) + d\gamma_5 \nabla^\mu A_\mu(x) = 0,$$  \hspace{1cm} (2.21)

$$cM \gamma^\mu V_\mu(x) - e\gamma^\mu \nabla^\nu (F_{\mu\nu}(x) - F_{\nu\mu}(x)) = 0,$$  \hspace{1cm} (2.22)

$$\frac{e}{2} M \Sigma^{\mu
u} F_{\mu\nu}(x) + c \Sigma^{\mu\nu} \nabla_\mu V_\nu(x) = 0.$$  \hspace{1cm} (2.23)

Coefficient parameters are chosen to adjust the field mass dimensions:

$$a = 0, b = c = d = M^2, e = -M,$$  \hspace{1cm} (2.24)

and independent field equations for Boson field relations are obtained from eqs. (2.20)-(2.23):

$$\nabla^\mu \partial_\mu P(x) - M^2 P(x) = 0,$$  \hspace{1cm} (2.25)

$$\nabla^\mu (\nabla_\mu V_\nu(x) - \nabla_\nu V_\mu(x)) - M^2 V_\nu(x) = 0.$$  \hspace{1cm} (2.26)

Other Boson fields are given by independent fields as:

$$S(x) = 0,$$  \hspace{1cm} (2.27)

$$M A_\mu(x) = \partial_\mu P(x),$$  \hspace{1cm} (2.28)

$$F_{\mu\nu}(x) = \nabla_\mu V_\nu(x) - \nabla_\nu V_\mu(x).$$  \hspace{1cm} (2.29)

The sum of the BW eqs. (2.17)+ (2.18) is calculated to be

$$(\gamma^\mu \partial_\mu + M) \Psi(x) + \Psi(x)(\overrightarrow{\partial}_\mu \gamma^\mu - M)$$

$$= 2M^2 (\gamma^\mu \partial_\mu S(x) - \nabla^\mu V_\mu(x) - \gamma_5 \Sigma^{\mu\nu} \nabla_\mu A_\nu(x)) + M \Sigma^{\mu\nu\lambda} \nabla_\mu F_{\nu\lambda}(x),$$

$$= 0,$$  \hspace{1cm} (2.30)

where the third rand anti-symmetric tensor of gamma matrices is denoted: $\Sigma^{\mu\nu\lambda} = (\gamma^{\mu\nu\lambda} + \gamma^{\nu\mu\lambda} + \gamma^{\lambda\mu\nu} - \gamma^{\mu\lambda\nu} - \gamma^{\nu\lambda\mu} - \gamma^{\lambda\nu\mu})/3!$. Here we should check the consistency between the independent terms in the sum BW equations (2.30) and the subtraction BW equations (2.25)-(2.29). Using the subtraction BW equations we can derive the following relations:

$$\gamma^\mu \partial_\mu S(x) = 0,$$  \hspace{1cm} (2.31)

$$M^2 \nabla^\mu V_\mu(x) = [\nabla^\mu, \nabla^\nu] \nabla_\mu V_\nu(x) = C^{\mu\nu\lambda} \nabla_\mu \nabla^\nu V_\lambda(x),$$  \hspace{1cm} (2.32)

$$\Sigma^{\mu\nu\lambda} \nabla_\mu A_\nu(x) = \frac{1}{2} \Sigma^{\mu\nu\lambda} C^{\lambda}_{\mu\nu\lambda} \partial_\lambda P(x),$$  \hspace{1cm} (2.33)

$$\Sigma^{\mu\nu\lambda} \nabla_\mu F_{\nu\lambda}(x) = \Sigma^{\mu\nu\lambda} (-R^{\mu\nu}_{\mu\lambda\nu} V_\mu(x) + C^{\rho\mu}_{\mu\nu} \nabla_\rho V_\lambda(x)), $$  \hspace{1cm} (2.34)
where \( C^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \) and \( R^\rho_{\lambda,\mu\nu} \) denote the torsion tensor and the Riemann curvature tensor. The proof of the consistency equations (2.31)-(2.34) is given in Appendix A. The lower suffix in the curvature tensor are totally antisymmetric because of the factor \( \Sigma^{\rho\nu\lambda} \) and the identity relation holds

\[
R^\rho_{\lambda,\mu\nu} = -\nabla_{[\mu} C^\rho_{\nu],\lambda} - C^\tau_{\nu\mu} C^\rho_{\lambda,\tau},
\]

where the anti-symmetrized suffix notation in square bracket is defined:

\[
A_{[\mu\nu\lambda]} := \frac{1}{3!} (A_{\mu\nu\lambda} + A_{\lambda\mu\nu} + A_{\nu\lambda\mu} - A_{\mu\lambda\nu} - A_{\nu\mu\lambda} - A_{\lambda\nu\mu}).
\]

In order to satisfy the sum of the BW equations, we require the torsion-free condition for the background spacetime geometry:

\[
C^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = 0.
\]

Inserting the independent Boson fields of pseudoscalar and vector fields in the bi-spinor field expression in eq. (2.19), we find

\[
\Psi(x) = M(M - \gamma^\mu \nabla_\mu)(\gamma_5 P(x) + \gamma^\nu V_\nu(x)),
\]

with the supplementary condition, which is from eq. (2.32) with the torsion free condition (2.37),

\[
\nabla^\nu V_\nu(x) = 0.
\]

The equation eq. (2.38) shows the direct relation between bi-spinor and Boson fields.

We note that the transpose operation with the charge conjugation \( C \) for pseudoscalar and vector fields show even and odd properties respectively:

\[
C[(M - \gamma^\mu \nabla_\mu)\gamma_5 P(x)]^T C^{-1} = (M - \gamma^\mu \nabla_\mu)\gamma_5 P(x),
\]

\[
C[(M - \gamma^\mu \nabla_\mu)\gamma^\nu V_\nu(x)]^T C^{-1} = -(M - \gamma^\mu \nabla_\mu)\gamma^\nu V_\nu(x).
\]

This even and odd property corresponds to their spins 0 and 1 for pseudoscalar and vector fields respectively.

## 3 Fermion-Boson correspondence and spin structure

Next we will show to express bi-spinor fields as a direct product of two spinor fields. Then we will derive the direct relations between spinors and Boson fields.

The general curved coordinate system is related to the local inertial coordinate system by the vierbeins \( e^\mu_a \). The BW equations for bi-spinors in the local inertial coordinate system are the Dirac type equations in eqs. (2.4) as

\[
(\gamma^a \partial_a + M)\Psi(x) = 0, \quad \Psi(x)(\partial_a \gamma^a - M) = 0.
\]
Similarly for each bra spinor we obtain from the definition in eq. (3.44) as equation of mass

\[ \Psi_P(x) = M(M - \gamma^a \partial_a)\gamma_5 P(x). \]  

(3.41)

An analytic pseudoscalar field \( P(x) \) can be expanded in the Fourier series with momentum \( k^a \):

\[ P(x) = \sum_k P(k) \quad \text{with} \quad P(k) = c_k \exp(ik^ax_0), \]  

(3.42)

where \( c_k \) is the expansion coefficients. For each plane wave \( P(k) \), the bi-spinor field can be decomposed into sets of ket spinors and bra spinors as

\[ |P(k)^j| = (M/2 - \gamma^a \partial_a) \sqrt{2P(k)}|j>, \]

(3.43)

\[ <P(k)^j| = <j|\sqrt{2P(k)}(M/2 - \partial_a \gamma^a)\gamma_5, \]

(3.44)

where the ket and bra unit spinors are the base of a complete set:

\[ |1> = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |2> = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |3> = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |4> = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

(3.45)

with \( \sum_{j=1}^4 |j><j| = 1 \). The ket and bra spinors defined in eqs. (3.43)-(3.44) satisfy the Dirac equation of mass \( M/2 \):

\[ (\gamma^a \partial_a + M/2)|P(k)^j| = 0 \quad \text{and} \quad <P(k)^j|\partial_a \gamma^a - M/2 = 0. \]

(3.46)

Then we can read the energy and spin dependence for each ket spinor from the expression in eq. (3.45) as

\[ |P(k)^1| = (M + E)|u_1(k^a)| > \sqrt{P(k)/2}, \]

\[ |P(k)^2| = (M + E)|u_1(k^a)| > \sqrt{P(k)/2}, \]

\[ |P(k)^3| = (M - E)|v_1(-k^a)| > \sqrt{P(k)/2}, \]

\[ |P(k)^4| = -(M - E)|v_1(-k^a)| > \sqrt{P(k)/2}, \]

(3.47)

where spinors or anti-spinors are denoted by \( u \) and \( v \), spins up or down are denoted by \( \uparrow \) or \( \downarrow \) respectively, and four momentum is denoted as \( k^a = (E, k_1, k_2, k_3) \). The charge conjugation relates them as

\[ C : \gamma_2(|u_1(k^a)|)^* = |v_1(k^a)| > \quad \text{and} \quad \gamma_2(|v_1(k^a)|)^* = |u_1(k^a)| >. \]

(3.48)

Similarly for each bra spinor we obtain from the definition in eq. (3.44) as

\[ <P(k)^1| = -(M + E) <v_1(\tilde{k}^a)| \sqrt{P(k)/2}, \]

\[ <P(k)^2| = (M + E) <v_1(\tilde{k}^a)| \sqrt{P(k)/2}, \]

\[ <P(k)^3| = -(M - E) <u_1(-\tilde{k}^a)| \sqrt{P(k)/2}, \]

\[ <P(k)^4| = -(M - E) <u_1(-\tilde{k}^a)| \sqrt{P(k)/2}. \]

(3.49)
The ket and bra spinors are related by the transpose operation as
\[
(|u_\uparrow(k^a)|)^T = <u_\uparrow(k^a)|, \quad (|u_\downarrow(k^a)|)^T = <u_\downarrow(k^a)|,
\]
and similar to the anti-spinors. The modified four momentum in eqs. (3.49) is denoted as \(\tilde{k}^a = (E, -k_1, k_2, -k_3)\), where the sign of momenta of \(k_1\) and \(k_2\) is changed due to the transpose property of the gamma matrices: \(\gamma^1 T = -\gamma^1\) and \(\gamma^2 T = -\gamma^2\) to express bra spinors from ket spinors. The explicit expression for the ket and bra spinors for pseudoscalar fields is in Appendix B.

Using these spinors, the bi-spinor for the pseudoscalar field is recovered:
\[
\sum_{j=1}^{4} |P(k^j)| = \frac{1}{2} \left( (M + E)^2 (|u_\uparrow(k^a)| > v_\uparrow(\tilde{k}^a) > < u_\uparrow(k^a) | v_\downarrow(\tilde{k}^a)) + (M - E)^2 (|v_\downarrow(-k^a)| > u_\downarrow(-\tilde{k}^a) > < v_\downarrow(-k^a) | u_\uparrow(-\tilde{k}^a)) \right) P(k)/2,
\]
where we have used the field equation for the pseudoscalar field (2.25) and the identity relation:
\[
(\gamma^a \partial_a + M)^2 \gamma_5 P(x) = 2M(\gamma^a \partial_a + M)\gamma_5 P(x).
\]

From the spinor decomposition equation (3.51), we can recognize that direct product of spinors and anti-spinors forms a spin 0 pseudoscalar field.

In the parallel way, we can confirm that the direct product of spinors and anti-spinors forms a spin 1 vector field, whose proof is in Appendix C.

Therefore the BW formulation is a kind of dynamical realization of algebraic direct product representation of rotation group:
\[
\frac{1}{2} \otimes \frac{1}{2} = 0^- \oplus 1^-,
\]
where the numbers and the minus sign show the magnitude of spins and their parity.

Based on the BW formulation, one of the important statements on the Fermion-Boson correspondence is the superradiant problem in the black hole background geometry. If the Bose particles behave like superradiant instability, the Fermi particles should behave like superradiant instability too, and vice versa under the Fermion-Boson correspondence.

In the next section, we shall study to make clear the superradiant problem for both of Fermi and Bose particles in the view of particle spectra.

## 4 Spectrum condition in co-rotating coordinate system and superradiant problem

### 4.1 Spectrum condition in co-rotating coordinate system

In this subsection, we try to derive the spectrum condition for Fermi and Bose particles around rotating geometry as radiant phenomena. The (3+1)-dimensional spacetime is assumed to be
stationary and axially symmetric for which the metric tensors are independent of the time $t$ and the azimuthal angle $\phi$. The invariant length is expressed in the polar coordinate as

$$ds^2 = g_{tt}(r,\theta)dt^2 + 2g_{t\phi}(r,\theta)dtd\phi + g_{\phi\phi}(r,\theta)d\phi^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2.$$  \hfill (4.1)

In order to study the co-rotating coordinate system, which is locally non-rotating observer system \[18, 19\], we make to diagonalize the $t-\phi$ part of the invariant length:

$$ds_{t-\phi}^2 = g_{tt}d\bar{t}^2 + g_{t\phi}d\bar{t}d\phi + g_{\phi\phi}d\phi^2 \equiv g_{\bar{t}\bar{t}}d\bar{t}^2 + g_{\bar{\phi}\bar{\phi}}d\bar{\phi}^2,$$  \hfill (4.2)

where $\bar{t}$ and $\bar{\phi}$ denote the time and azimuthal angle in co-rotating coordinate system respectively. The local co-rotation coordinate system and the general rotation spacetime are related by the linear transformation:

$$(d\bar{t}, d\bar{\phi}) = (dt, d\phi) U,$$  \hfill (4.3)

and the metric tensors are related as

$$\bar{G} = \begin{pmatrix} g_{\bar{t}\bar{t}} & 0 \\ 0 & g_{\bar{\phi}\bar{\phi}} \end{pmatrix} = U^{-1}G(U^T)^{-1}, \quad G = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$ \hfill (4.4)

with the transfer matrix:

$$U = \begin{pmatrix} 1 & X(r, \theta) \\ Y(r, \theta) & 1 \end{pmatrix},$$  \hfill (4.5)

where the diagonal element is set to one as our normalization and elements $X$, $Y$ are to be determined taking account of their mass dimensions $\text{dim}[X]=1$, $\text{dim}[Y]=-1$.

As to the expressions for the transfer matrix elements, $X$ is chosen to obtain the local co-rotating coordinate system adjusting the angular velocity of rotating spacetime \[18, 19\]:

$$X = -\Omega(r, \theta) = g_{t\phi}(r,\theta)/g_{\phi\phi}(r,\theta),$$ \hfill (4.6)

where $\Omega(r, \theta)$ ($=: d\phi/dt$) denotes the angular velocity of spacetime. From and the $\bar{G}$ and $G$ relation of eq.(4.4) and the co-rotating condition of eq.(4.6), we obtain another matrix element as

$$Y = 0,$$ \hfill (4.7)

and the $t-\phi$ part of the invariant length is

$$ds_{t-\phi}^2 = (g_{tt} - \frac{g_{t\phi}^2}{g_{\phi\phi}})d\bar{t}^2 + g_{t\phi}(d\phi + \frac{g_{t\phi}}{g_{\phi\phi}}d\bar{t})^2$$ \hfill (4.8)

In addition to the coordinate forms, the differential operators are introduced to form an invariant quantity:

$$d\bar{t}\partial/\partial_{\bar{t}} + d\phi\partial/\partial_{\bar{\phi}} = d\bar{t}\partial/\partial_t + d\bar{\phi}\partial/\partial_{\phi},$$  \hfill (4.9)
where the differential operators transform as

\[(\partial/\partial \bar{t}, \partial/\partial \bar{\phi}) = (\partial/\partial t, \partial/\partial \phi)U.\]  

(4.10)

Because metric tensors are independent of the time \(t\) and the azimuthal angle \(\phi\), the \(t-\phi\) factor of matter fields is the exponential form as

\[\Phi \propto \exp(-i\omega t + im\phi),\]  

(4.11)

where the matter field \(\Phi\) stands for bi-spinor fields \(\Psi\), pseudoscalar fields \(P\), vector fields \(V_{\mu}\), spinors, and anti-spinors explained in the previous sections. The notations \(\omega\) and \(m\) denote the frequency as a field (the energy as a particle) and azimuthal angular momentum respectively.

Applying the differential operator to matter fields of the form eq.(4.11), the invariant quantity of eq.(4.9) becomes

\[-\omega dt + m d\phi = -\bar{\omega} d\bar{t} + \bar{m} d\bar{\phi},\]  

(4.12)

where

\[\bar{\omega} = \omega + Xm, \quad \bar{m} = m.\]  

(4.13)

As to the stationary and axially symmetric spacetime, the reflection symmetry on \(t-\phi\) part holds: \((dt, d\phi) \rightarrow -(dt, d\phi)\). And by this symmetry, the reflection symmetry on the spectrum also holds:

\[(\omega, m) \leftrightarrow -(\omega, m), \quad (\bar{\omega}, \bar{m}) \leftrightarrow -\bar{(\omega, m)}\]  

(4.14)

in general rotating geometry and in co-rotating coordinate system respectively. On the frequency, one side of the spectra is physical modes and other is unphysical modes. The physical normal modes are half of the full spectrum and the physical modes in co-rotating coordinate system are chosen as

\[\bar{\omega} > 0,\]  

(4.15)

which naturally connect the non-rotating coordinate systems. We call the spectrum condition for the radiant processes in the rotating spacetime.

For the explicit expression for the metric tensors, we take the Kerr metric, which is

\[g_{tt} = \frac{1}{\rho^2}(-\Delta + a^2 \sin^2 \theta), \quad g_{t\phi} = \frac{a \sin^2 \theta}{\rho^2}(\Delta - (r^2 + a^2))\]  

\[g_{\phi\phi} = \frac{\sin^2 \theta}{\rho^2}(-a^2 \sin^2 \theta \Delta + (r^2 + a^2)^2), \quad g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2\]  

(4.16)

with

\[\Delta = r^2 + a^2 - 2M_{BH}r, \quad \rho = r^2 + a^2 \cos^2 \theta,\]  

(4.17)
where $M_{BH}$ and $a$ denote the mass and angular momentum per unit mass of the black hole. Near the outer horizon $r = r_H := M_{BH} + (M_{BH}^2 - a^2)^{1/2}$, the spectrum condition becomes
\[
\omega - m\Omega_H > 0, \tag{4.19}
\]
where $\Omega_H = a/(r_H^2 + a^2)$ denotes the angular velocity of black hole near the horizon. As another understanding of the spectrum condition (4.19), we can express the radial part of the wave function as
\[
\Phi_{\text{radial}} \approx \exp \left[ -i(\omega - m\Omega_H)r_* \right], \tag{4.24}
\]
where the radial tortoise coordinate \cite{21} is defined $dr_* = (r^2 + a^2)\Delta^{-1}dr$ and the radial momentum is denoted by $\omega - m\Omega_H$ as the positive value is for in-going wave to the black hole. Therefore we analyze the superradiant problem under the consideration of the spectrum condition relating the role in the frequency $\omega$ and the radial momentum $\omega - m\Omega_H$ in the next subsection.

### 4.2 Superradiant problem

The superradiant problem is one of the longstanding problems and is still important nowadays. In this subsection, we study the superradiant problem taking account the Fermion-Boson correspondence and the spectrum condition.

\textsuperscript{1}We note that the geodesic motion of a particle shows the Coriolis force in an asymptotic region:
\[
d\bar{\phi} \rightarrow d\phi - 2aM_{BH}dt/r^3, \quad (a, 2M_{BH} \ll r), \tag{4.18}
\]
which support our choice of co-rotating coordinate system of eq.(4.6).

\textsuperscript{2}It is noted that another interesting choice of co-rotating coordinate system is to put the angular velocity with the horizon condition ($\Delta = 0$) to the matrix element in the Kerr metric:
\[
X' = -\Omega' = g_{\theta\phi}/g_{\phi\phi} \big|_{\Delta=0} = -a/(r^2 + a^2), \tag{4.20}
\]
where the new choice of co-rotating coordinate system is denoted by the prime suffix. In this choice, new coordinate system is obtained as
\[
Y' = (X'g_{\phi\phi} - g_{\theta\phi})/(X'g_{\phi\theta} - g_{\theta\theta}) = -a \sin^2 \theta, \quad dt' = dt - a \sin^2 \theta d\phi, \quad d\phi' = d\phi - a dt/(r^2 + a^2), \tag{4.21}
\]
and matrix tensors are
\[
g'_{\mu} = -\Delta/\rho^2, \quad g'_{\mu\nu} = \sin^2 \theta (r^2 + a^2)^2/\rho^2. \tag{4.22}
\]
This coordinate system becomes the Boyer-Lindquist coordinates \cite{20}:
\[
ds^2_{\text{LQ}} = g'_{tt}dt' + g'_{\phi\phi}d\phi'^2 = -\Delta(dt - a \sin^2 \theta d\phi)^2/\rho^2 + \sin^2 \theta ((r^2 + a^2)d\phi - a dt)^2/\rho^2. \tag{4.23}
\]
The metric condition on the horizon is the same form as in eq.(4.19) in this co-rotating coordinate system.
For the Bose particles, the superradiant condition was derived from the current conservation near the rotating black hole horizon \[1, 4, 5, 6, 7, 8\]:

\[\omega - m\Omega_H < 0 \text{ with } \omega > 0,\]  

(4.25)

which we call the type 1 superradiance. Especially for the small rotating black hole and the positive incident particle energy, the superradiant instability are studied to occur under the condition of (4.25) \[4, 9, 10, 11\]. As we pointed out in the previous subsection in eq.(4.14), the particles have the reflection symmetry on the spectrum \((\omega, m) \rightarrow -(\omega, m)\), there is another superradiant possibility:

\[\omega - m\Omega_H > 0 \text{ with } \omega < 0,\]  

(4.26)

which we call the type 2 superradiance. The type 2 superradiant condition is consistent with our spectrum condition in eq.(4.19).

On the other hand, for Fermi particles, the type 1 superradiance has been studied not to occur \[1, 2, 3\]. One of the reasons is the spectrum region of type 1 is occupied by the Fermion particles and cannot realize in this spectrum region because of the Fermi statistics. We note that there is another possible superradiant spectrum region of type 2, because this spectrum region is not occupied and possible to occur.

We also note that the type 2 superradiance is possible for both Bose and Fermi particles and the result is consistent our previous study on Fermion-Boson correspondence and the spectrum condition. The type 2 superradiance is the necessary and stable spectrum modes because of the reflection symmetry with respect to \(\omega\) and \(m\). As black holes would not recognize particles as elementary or composite, the similar radiative behavior for the Fermi and Bose particles to black holes is desirable based on the Fermion-Boson correspondence.

## 5 Summary and discussion

We have studied the interacting relations of matter fields with the curved spacetime. We obtained the Bargmann-Wigner formulation on general curved spacetime, which describe the bi-spinors as pseudoscalar and vector fields consistently as in eq.(2.38) under the torsion-free condition in eq.(2.37) for the background spacetime geometry. We also derived the direct relations between Fermi (spinor and anti-spinors) and Boson (pseudoscalar and vector) fields via bi-spinor fields. The spin structures show that the BW formulation is a kind of dynamical realization of the representation under rotation group as quark and anti-quark system forms a \(\pi\)- and \(\rho\)- mesons as in eq.(3.53). Following this Fermion-Boson correspondence, we investigated the field spectrum around the rotation black hole and obtained the spectrum condition \(\omega - m\Omega_H > 0\) in eq.(4.19) for both Boson and Fermion fields in the co-rotating coordinate system as locally non-rotating observer \[4, 6\]. We also applied these results to the superradiant problem in the Kerr spacetime geometry and obtained the result that the type 2 superradiance is possible to occur as \(\omega - m\Omega_H > 0\) with \(\omega < 0\) in eq.(4.26).

These results are consistent with our previous research works based on the BW formulation \[14\] and the quantum field theory \[22\] in (3+1)-dimensional spacetime and the BTZ spacetime in (2+1)-dimension \[23, 24\].
The type 2 superradiance is the positive radial momentum ($\omega - m\Omega_H > 0$) with negative energy ($\omega < 0$) modes. We can interpret this phenomenon of the annihilation of anti-particle with negative energy into a black hole as the creation of particle with positive energy from the black hole. This process can be expressed by the Bogoliubov transformation for Fermi particles:

$$U_\theta = \exp \left[ \theta \sum_{\alpha+\beta=0} (a_\alpha b_\beta - b_\beta^\dagger a_\alpha^\dagger) \right],$$

(5.1)

where $a_\alpha, b_\beta$ denote the particle, anti-particle annihilation operators with positive or negative quantum number $\alpha = -\beta = (\omega, m, \ldots)$ respectively under the spectrum condition (4.19). The mixing parameter $\theta$ is proportional to the angular velocity of the black hole $\Omega_H$. The dressed anti-particle operator is obtained by the pair creation effect as $b_\beta := U_\theta b_\beta U_\theta^\dagger = \cos \theta b_\beta + \sin \theta a_\alpha^\dagger$. The annihilation of the anti-particle into black hole shows that the creation of particle with positive energy from the black hole as

$$H\hat{b}_\beta|0> = \sin \theta \omega a_\alpha^\dagger |0> , \quad (\alpha + \beta = 0),$$

(5.2)

where the Hamiltonian operator is $H = \sum \omega (a_\alpha^\dagger a_\alpha - b_\beta^\dagger b_\beta)$ and the vacuum represents the black hole state as $b_\beta|0> = 0$. Similarly the annihilation of particle with negative energy into a black hole can be interpreted as the creation of anti-particle with positive energy from the black hole.

For Boson particles, the mixing angle should be replaced: $\sin \theta \rightarrow \sinh \theta$ in eq.(5.2). By this mechanism, the energy can be transferred to Fermi and/or Bose particles from rotating black holes. This is known as the Penrose mechanism to extract the black hole energy in the ergo region [25].

Combination the type 2 superradiant modes ($\omega - m\Omega_H > 0$ with $\omega < 0$) and standard normal modes ($\omega - m\Omega_H > 0$ with $\omega > 0$) forms a complete set of physical normal modes both for Fermion and Boson independent of elementary or composite particles [22].

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Appendix

A  Proof of the consistency equations (2.31)-(2.34)

We shall show the consistency equations step by step in the following.

1  The proof of equation (2.31)

The equation $\gamma^\mu \partial_\mu S(X) = 0$ is simply by the equation $S(x) = 0$ in eq. (2.27).
2 The proof of equation (2.32)
For the general tensor field $T^{\rho\sigma}(x)$, we can show the relation:

$$[\nabla_\nu, \nabla_\mu]T^{\rho\sigma}(x) = -R_{\lambda,\mu\nu}^{\rho}T^{\lambda\sigma}(x) - R_{\lambda,\rho\nu}^{\sigma}T^{\lambda\mu}(x) + C_{\lambda\rho\sigma} A^{\lambda}T^{\rho\sigma}(x),$$

(A.1)

where $C_{\lambda\rho\sigma}$ is the torsion tensor. Setting the index parameters as $\rho = \mu$ and $\sigma = \nu$, we obtain

$$[\nabla_\nu, \nabla_\mu]T^{\mu\nu}(x) = C_{\lambda\mu\nu} A^{\lambda}T^{\mu\nu}(x).$$

(A.2)

Identifying $T^{\mu\nu} = \nabla_\mu V^\nu(x)$ in this formula

$$M^2 \nabla_\mu V^\nu(x) = C_{\lambda\mu\nu} A^{\lambda}V^\nu(x),$$

(A.3)

the equation (2.32) is proved.

3 The proof of equation (2.33)
From the expression for the axial vector $A_\mu(x)$ in eq. (2.28), we obtain

$$2\Sigma^{\mu\nu} \nabla_\mu A_\nu(x) = [\nabla_\mu, \nabla_\nu]P(x) = \Sigma^{\mu\nu}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) \partial_\lambda P(x).$$

(A.4)

Using the relation $\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = C^{\lambda}_{\mu\nu}$, we have proved the equation (2.33).

4 The proof of the equation (2.34)
From the expression for tensor field in eq. (2.29), we obtain

$$\Sigma^{\mu\lambda} \nabla_\mu F_{\lambda\tau}(x) = \Sigma^{\mu\lambda}[\nabla_\mu, \nabla_\lambda]V_\tau(x).$$

(A.5)

Using the formula $[\nabla_\mu, \nabla_\lambda]V_\tau(x) = -R_{\tau,\mu\lambda}^\rho V_\rho(x) + C_{\mu\rho\lambda} \nabla_\lambda V_\sigma(x)$, we get

$$\Sigma^{\mu\lambda} \nabla_\mu F_{\lambda\tau}(x) = \Sigma^{\mu\lambda}(-R_{\tau,\mu\lambda}^\rho V_\rho(x) + C_{\mu\rho\lambda} \nabla_\lambda V_\sigma(x)).$$

(A.6)

Because of the factor $\Sigma^{\mu\lambda}$ the lower suffix $\tau, \mu, \lambda$ in $R_{\tau,\mu\lambda}^\rho$ can be totally antisymmetrized, and the equation (2.34) has been proved.

B The ket and bra spinors for pseudoscalar fields
Gamma matrices in local inertial coordinate system (flat Minkowski spacetime) is represented:

$$\gamma^0 = -i \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\sigma^\mu & 0 \end{pmatrix},$$

(B.1)

where $\sigma^\mu$ stands for Pauli matrices.

For ket spinors defined in eq. (3.43), the explicit matrix expression is obtained as

$$(M/2 - \gamma^\mu \partial_\mu) \sqrt{2P(k)} |j> = (M - i\gamma^\mu k_\mu)|j> \sqrt{P(k)/2},$$

$$= \begin{pmatrix} M + E & -\sigma \cdot \vec{k} \\ \sigma \cdot \vec{k} & M - E \end{pmatrix} |j> \sqrt{P(k)/2},$$

(B.2)
where $P(k) = c_k \exp (ik^a x_a)$ in eq. (3.42). From this expression we can read each component of the ket spinors and anti-spinors in eqs. (3.47) as

\[
|u_1(k^a)\rangle = \begin{pmatrix} 1 \\ 0 \\ k_3/(M + E) \\ (k_1 + k_2)/(M + E) \end{pmatrix}, \quad |u_1(-k^a)\rangle = \begin{pmatrix} 0 \\ 1 \\ (k_1 - ik_2)/(M + E) \\ -k_3/(M + E) \end{pmatrix},
\]

\[
|v_1(-k^a)\rangle = \begin{pmatrix} -k_3/(M - E) \\ -(k_1 + k_2)/(M - E) \\ 1 \\ 0 \end{pmatrix}, \quad |u_1(-k^a)\rangle = \begin{pmatrix} -(k_1 - ik_2)/(M - E) \\ k_3/(M - E) \\ 0 \\ 1 \end{pmatrix},
\]

where spinors $u$ and anti-spinors $v$ are charge conjugation pair each other defined in eqs. (3.48).

For bra spinors defined in eq. (3.44), the explicit matrix expression is obtained as

\[
\sqrt{2}P(k) < j/(M/2 - \overset{\rightarrow}{\gamma} \gamma^a)\gamma_5 = < j/(M - iy^a k_a)\gamma_5 \sqrt{P(k)/2},
\]

\[
= -< j \begin{pmatrix} -\overset{\rightarrow}{\gamma} \cdot k \\ M + E \\ M - E \end{pmatrix} \frac{\gamma^a}{\partial \cdot k} \sqrt{P(k)/2}, \quad (B.4)
\]

Here we know that the ket and bra relation is $u_1(k^a) \rightarrow -v_1(\tilde{k}^a)$ and $u_2(k^a) \rightarrow v_1(\tilde{k}^a)$ by the effect of the operation $\gamma_5$. The modified four momentum $\tilde{k}^a$ is explained below eq. (3.49). Then we obtain the expression for bra spinors in eq. (3.49).

### C  The ket and bra spinors for vector fields

For the vector fields in the curved spacetime, we can set up a local inertial coordinate system similar to the pseudoscalar case. The BW solution for the vector fields is from eq. (2.38) as

\[
\Psi_V(x) = M(M - \gamma^a \partial_a)\gamma^a V_a(x).
\]

The analytic solution for the vector fields can be expanded in Fourier series,

\[
V_a(x) = \sum_{k, I} \epsilon_a^{(I)}(k) V(k)^{(I)} \text{ with } V(k)^{(I)} = \delta_k^{(I)} \exp (ik^a x_a),
\]

where $\epsilon_a^{(I)}(k)$ and $\delta_k^{(I)}$ with $(I = 1, 2, 3)$ denote three independent polarization vectors and expansion constants respectively. And for more simple treatment we choose a co-moving coordinate frame, where fields are at rest, that is $k^a = (M, 0, 0, 0)$. The supplementary condition eq. (2.39) is now $k^a \epsilon_a^{(I)}(k) = 0$ and three independent solutions are simply obtained:

\[
\epsilon_a^{(1)} = (0, 1, 0, 0), \quad \epsilon_a^{(2)} = (0, 0, 1, 0), \quad \epsilon_a^{(3)} = (0, 0, 0, 01), \quad (C.3)
\]

The bi-spinor $\Psi_V(x)$ is decomposed into the ket and bra spinors, which are defined:

\[
|V(k)^{(I, j)}\rangle \equiv < \overset{\rightarrow}{V}(k)^{(I, j)} \rangle, \quad (C.4)
\]

\[
<V(k)^{(I, j)}| \equiv < j \sqrt{2}V(k)^{(I)}(M/2 - \overset{\rightarrow}{\gamma} \gamma^a)\gamma^a \epsilon_a^{(I)} \rangle. \quad (C.5)
\]
If we use the explicit form of unit spinors (3.45), we obtain non-vanishing components of ket spinors common for $I = 1, 2, 3$ as
\[
|V^{(I,j=1)}| = M \sqrt{2V(k)^{(I)}} |\hat{u}_1^\dagger|, \quad |V^{(I,j=2)}| = M \sqrt{2V(k)^{(I)}} |\hat{u}_2^\dagger|,
\]
where the unit ket spinors are:
\[
|\hat{u}_1^\dagger| = (1, 0, 0, 0)^T, \quad |\hat{u}_2^\dagger| = (0, 1, 0, 0)^T,
\]
where $T$ denotes the transpose operation. And we obtain non-vanishing components of bra spinors as
\[
< V^{(I=1,j=1)}| = iM \sqrt{2V(k)^{(I=1)}} < \hat{v}_1|, \quad < V^{(I=1,j=2)}| = -iM \sqrt{2V(k)^{(I=1)}} < \hat{v}_1|,
\]
for $\epsilon_a^{(I=1)}$ and
\[
< V^{(I=2,j=1)}| = M \sqrt{2V(k)^{(I=2)}} < \hat{v}_1|, \quad < V^{(I=2,j=2)}| = M \sqrt{2V(k)^{(I=2)}} < \hat{v}_1|,
\]
for $\epsilon_a^{(I=2)}$ and
\[
< V^{(I=3,j=1)}| = -iM \sqrt{2V(k)^{(I=3)}} < \hat{v}_1|, \quad < V^{(I=3,j=2)}| = -iM \sqrt{2V(k)^{(I=3)}} < \hat{v}_1|,
\]
for $\epsilon_a^{(I=3)}$, where the unit bra spinors are:
\[
< \hat{v}_1| \equiv (0, 0, 0, 1), \quad < \hat{v}_1| \equiv (0, 0, 1, 0).
\]
Using these ket and bra spinors, the bi-spinor for the vector fields is reconstructed as
\[
\sum_{j=1}^{4} |V(k)^{(I=1,j)}| < V(k)^{(I=1,j)}| = i2M^2(|\hat{u}_1^\dagger| < \hat{v}_1| - |\hat{u}_2^\dagger| < \hat{v}_1|), \quad M(M - \gamma^a \partial_a) \gamma^b \epsilon_b^{(I=1)} V(k)^{(I=1)}
\]
for $\epsilon_a^{(I=1)}$ and
\[
\sum_{j=1}^{4} |V(k)^{(I=2,j)}| < V(k)^{(I=2,j)}| = 2M^2(|\hat{u}_1^\dagger| < \hat{v}_1| + |\hat{u}_2^\dagger| < \hat{v}_1|), \quad M(M - \gamma^a \partial_a) \gamma^b \epsilon_b^{(I=2)} V(k)^{(I=2)}
\]
for $\epsilon_a^{(I=2)}$ and
\[
\sum_{j=1}^{4} |V(k)^{(I=3,j)}| < V(k)^{(I=3,j)}| = -i2M^2(|\hat{u}_1^\dagger| < \hat{v}_1| + |\hat{u}_2^\dagger| < \hat{v}_1|), \quad M(M - \gamma^a \partial_a) \gamma^b \epsilon_b^{(I=3)} V(k)^{(I=3)}
\]
for $\epsilon_a^{(I=3)}$. These expressions of spin structures in the product of spin 1/2 spinors show that they form three components of spin one vector fields.
References

[1] W. G. Unruh, Phys. Rev. D10, 3194 (1974).
[2] K. Maeda, Prog. Theor. Phys. 55, 1677 (1976).
[3] S. M. Wagh and N. Dadhich, Phys. Rev. D32, 1863 (1985).
[4] W. H. Press and S. A. Teukolsky, Nature 238, 211 (1972).
[5] C. W. Misner, Phys. Rev. Lett. 28, 993 (1972).
[6] S. Detweiler, Phys. Rev. D22, 2323 (1980).
[7] S. Mano and E. Takasugi, Progress of Theoretical Physics, 97, 213 (1997).
[8] S. Mukohyama, Phys. Rev. D61, 124021 (2000).
[9] S. Chandrasekhar, The Mathematical Theory of Black Holes, Claredon Press (1983).
[10] V. Cardoso, O. J. C. Dias, J. P. S. Lemos and S. Yoshida, Phys. Rev. D70, 044039 (2004).
[11] H. Kodama, Prog. Theor. Phys. Supplement 172, 11 (2008).
[12] V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. (USA) 34, 211 (1948).
[13] As a text book, see for example, D. Lurie, Particles and Fields, Interscience Publications (1968).
[14] M. Kenmoku and YM. Cho, Int. J. Mod. Phys. Lett. 30, 1550052 (2015).
[15] H. Yoshino and H. Kodama, Prog. Theor. Phys. 128, 153 (2012).
[16] H. Yoshino and H. Kodama, Prog. Theor. Exp. Phys. 2015, 061E01 (2015).
[17] As a text book, see for example, S. Weinberg, The Quantum Theory of Fields vol. III, Cambridge Univ. Press (2000).
[18] J. M. Bardeen, Astrophys. J. 162, 71 (1970).
[19] C. W. Misner, K. S. Thone and J. A. Wheeler, see section 33.4 in Gravitation, W. H. Freeman and Company (1970).
[20] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).
[21] T. Regge and J. A. Weeler, Phys. Rev. 108, 1063 (1957).
[22] M. Kenmoku, YM. Cho, K. Shigemoto and J. H. Yoon, Class. Quantum Grav. 34, 215009 (2017).
[23] M. Kuwata, M. Kenmoku and K. Shigemoto, Class. Quantum Grav. 25, 145016 (2008).
[24] M. Kuwata, M. Kenmoku and K. Shigemoto, Prog. Theor. Phys. 119, 939 (2008).
[25] R. Penrose, Re. Nuovo Cimento, 1, 252 (1969).