Abstract—In this paper we study in detail the so-called Chow-weight homology of Voevodsky motivic complexes and relate it to motivic homology. We generalize earlier results and prove that the vanishing of higher motivic homology groups of a motif $M$ implies similar vanishing for its Chow-weight homology along with effectivity properties of the higher terms of its weight complex $t(M)$ and of higher Deligne weight quotients of its cohomology. Applying this statement to motives with compact support we obtain a similar relation between the vanishing of Chow groups and the cohomology with compact support of varieties. Moreover, we prove that if higher motivic homology groups of a geometric motif or a variety over a universal domain are torsion (in a certain “range”) then the exponents of these groups are uniformly bounded. To prove our main results we study Voevodsky slices of motives. Since the slice functors do not respect the compactness of motives, the results of the previous Chow-weight homology paper are not sufficient for our purposes; this is our main reason to extend them to ($\mathcal{W}_{\text{Chow}}$-bounded below) motivic complexes.

Keywords: motives, triangulated categories, Chow groups, weight structures, Chow-weight homology, Deligne weight filtration, effectivity.

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1. INTRODUCTION

This paper is a continuation of [1] (see also the slightly different [2]). In these papers an extension of the well-known decomposition of the diagonal theory\(^1\) to geometric Voevodsky motives and varieties was proposed; its main tool were the new Chow-weight homology theories. Since the main purpose of these papers was the study of varieties, Chow-weight homology was only defined on the categories of geometric motives.

In the current text we demonstrate that it makes sense to study Chow-weight homology of objects of the bigger category $DM^\text{eff}_R$ of $R$-linear motivic complexes as well. So, we extend the main results of [1] to the class $DM^\text{eff}_{R,\text{Chow}+}$ of motives that are bounded below with respect to the Chow weight structure on the category $DM^\text{eff}_R$, here $R$ is the coefficient ring. This enables us to generalize Corollary 3.4.2 of [1] and prove that the vanishing of higher motivic homology groups of a motif $M$ is equivalent to similar vanishing for Chow-weight homology of $M$ and also to the corresponding effectivity properties of higher terms of the Chow-weight complex $t(M)$ of $M$. The difference with loc. cit. is that we are able to treat motivic homology of positive dimensions (that correspond to complexes of algebraic cycles of dimension $j > 0$) in the corresponding Theorem 3.2.3 below. The proof of that theorem uses Voevodsky slices of motives; thus it is necessary to consider Chow-weight homology and conditions related to it for objects of $DM^\text{eff}_R$ that are not geometric.

Next we apply Theorem 3.2.3 to extend some more results of ibid. We prove that if higher motivic homology groups of a geometric motif $M$ over a universal domain are torsion (in a certain “range”) then the exponents of these groups are uniformly bounded. Moreover, we apply Theorem 3.2.3 to motives with compact support of varieties (cf. Corollary 4.2.3 of ibid.); we obtain that if certain Chow groups of a variety

\(^1\) Recall that this theory originates from [3]. Some of it was recalled and discussed in Propositions 0.4, 4.3.1, and 4.3.4, and Remarks 0.5(1), 3.3.10(1), and 4.3.2(2) of [1].
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over a universal domain $K$ are torsion then they are of bounded exponent. Arguing similarly to Theorem 4.2.1 ibid. we also obtain that this torsion assumption implies certain effectivity conditions for the cohomology of $X$ with compact support; see Theorem 4.2.3.

Now we describe the contents of the paper. More details can be found at the beginnings of sections.

In Section 2 we recall several properties of (smashing) weight structures, Voevodsky’s motivic category $DM^\text{eff}_R$ and its localizations, and Chow weight structures on them.

In Section 3 we define Chow-weight homology on $DM^\text{eff}_R$, and extend its properties (as studied in [1]) to the class of $w_{\text{Chow}}$-bounded below objects of $DM^\text{eff}_R$. This enables us to prove the first main Theorem 3.3.3; it says that the vanishing of higher motivic homology groups of an $M \in DM^\text{eff}_Rw_{\text{Chow}}^+$ (over all finitely generated extensions of the base field $k$) is equivalent to the similar vanishing for Chow-weight homology of $M$. This vanishing is also equivalent to certain effectivity assumptions on the weight complex $\kappa(M)$ (that is, its higher terms should be “big Chow motives” that are “effective enough”); moreover, it has a “coproduct and extension-closure” re-formulation.

In Section 4 we apply Theorem 3.3.3 to geometric motives and combine it with the results and arguments of ibid. Firstly we prove that if higher motivic homology groups of a geometric motif over a universal domain are torsion (in a certain “range”) then the exponents of these groups along with the related Chow-weight homology ones are uniformly bounded. Next, we combine earlier results with the properties of motives with compact support to obtain the aforementioned Theorem 4.2.3.

In Section 5 we prove some properties of motives that are necessary both for the current paper and for [1]. They appear to be well-known even though the authors were not able to find them in the literature.

2. PRELIMINARIES

In Subsection 2.1 we recall some definitions; they are mostly related to (smashing) triangulated categories.

In Subsection 2.2 we recall some basics on ($R$-linear) Voevodsky motives over a perfect field $k$.

In Subsection 2.3 we recall basic definitions and statements on weight structures.

In Subsection 2.4 we discuss purely compactly generated weight structures and the weight structures they induce on (“purely compactly generated”) localizations.

In Subsection 2.5 we recall some of the theory of strong weight complexes, pure (homological) functors and weight spectral sequences.

In Subsection 2.6 we apply the general theory to the category $DM^\text{eff}_R$ and its localizations; this gives certain Chow weight structures whose hearts are “generated” by Chow motives.

2.1. Some Notation and Conventions

- For $a \leq b \in \mathbb{Z}$ we will write $[a,b]$ (resp. $[a,+\infty)$, resp. $[a,+\infty]$) for the set $\{i \in \mathbb{Z} : a \leq i \leq b\}$ (resp. $\{i \in \mathbb{Z} : i \geq a\}$, resp. $[a,+\infty) \cup \{+\infty\} \subset \mathbb{Z} \cup \{+\infty\}$); we will never consider real line segments in this paper. Respectively, when we write $i \geq c$ (for $c \in \mathbb{Z}$) we mean that $i$ is an integer satisfying this inequality.
- Given a category $C$ and $X,Y \in \text{Obj}C$ we will write $C(X,Y)$ for the set of morphisms from $X$ to $Y$ in $C$.
- For categories $C',C$ we write $C' \subset C$ if $C'$ is a full subcategory of $C$.
- Given a category $C$ and $X,Y \in \text{Obj}C$, we say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $Y$.\footnote{Clearly, if $C$ is triangulated or abelian, then $X$ is a retract of $Y$ if and only if $X$ is its direct summand.}
- Let $H$ be a subcategory of an additive category $C$.

Then $H$ is said to be retraction-closed in $C$ if it contains all retracts of its objects in $C$. Moreover, the full subcategory $\text{Kar}_C(H) \subset C$ whose objects are the retracts of objects of a subcategory $H$ (in $C$) will be called the retraction-closure of $H$ in $C$.\footnote{Clearly, if $C$ is triangulated or abelian, then $X$ is a retract of $Y$ if and only if $X$ is its direct summand.}
Let us now list some less common definitions and conventions. Below the symbol $\mathcal{C}$ below will always denote some triangulated category; usually it will be endowed with a \emph{weight structure} $w$ (see Definition 2.3.1 below).

**Definition 2.1.1.** Let $\mathcal{B}$ be an additive category.

1. We call a category $\mathcal{B}/\mathcal{H}$ the factor $\mathcal{B}$ by its full additive subcategory $\mathcal{H}$ if $\text{Obj}(\mathcal{B}/\mathcal{H}) = \text{Obj} \mathcal{B}$ and $$(\mathcal{B}/\mathcal{H})(X,Y) = \mathcal{B}(X,Y)/\left(\sum_{Z \in \text{Obj} \mathcal{H}} \mathcal{B}(Z,Y) \circ \mathcal{B}(X,Z)\right)$$ for any $X,Y \in \text{Obj} \mathcal{B}$.

2. We will write $K(\mathcal{B})$ for the homotopy category of (cohomological) complexes over $\mathcal{B}$. We will write $K^{i}(\mathcal{B})$ if $\mathcal{B}^{i}$ are the terms of the complex $\mathcal{B}$.

3. For any $\mathcal{C}$ we say that $\mathcal{C}$ is an \emph{extension} of $\mathcal{B}$ by $\mathcal{C}$ if there exists a distinguished triangle $A \to C \to B \to A[1]$.

4. A class $\mathcal{D}$ is said to be \emph{extension-closed} if it is closed with respect to extensions and contains $0$. We call the smallest extension-closed subclass of objects of $\mathcal{C}$ that contains a given class $\mathcal{D}$ the \emph{extension-closure} of $\mathcal{D}$.

5. Given a class $\mathcal{D}$ of objects of $\mathcal{C}$ we will write $\mathcal{D}^{\perp}$ or $\mathcal{D}^{\perp}_{\mathcal{C}}$ for the smallest full retraction-closed triangulated subcategory of $\mathcal{C}$ containing $\mathcal{D}$. We call $\mathcal{D}^{\perp}$ the triangulated category \emph{densely generated} by $\mathcal{D}$.

6. For $\mathcal{B}$ we write $\mathcal{B}^{\perp}$ if $\mathcal{B}^{\perp}$ are the terms of the complex $\mathcal{B}$.

7. Assume that $\mathcal{C}$ is connective (in $\mathcal{C}$) if $\mathcal{C} \cap \mathcal{C}^{\perp}$.

8. Given $f \in \mathcal{C}(X,Y)$, where $X,Y \in \text{Obj} \mathcal{C}$, we call the third vertex of (any) distinguished triangle $X \to Y \to Z$ a \emph{cone} of $f$.\footnote{Recall that different choices of cones are connected by non-unique isomorphisms.}

Let us now list some definitions related to smashing triangulated categories.

**Definition 2.1.2.** Assume that $\mathcal{D}$ is a class of objects of $\mathcal{C}$, and $\mathcal{C}$ is smashing, that is, closed with respect to (small) coproducts.

1. We say that a class of objects or a full subcategory of $\mathcal{C}$ is smashing (in $\mathcal{C}$) if it is closed with respect to $\mathcal{C}$-coproducts.

2. We say that a full subcategory $\mathcal{D} \subset \mathcal{C}$ is localizing whenever it is triangulated and smashing. Respectively, we call the smallest localizing subcategory of $\mathcal{C}$ that contains $\mathcal{D}$ the \emph{localizing subcategory of $\mathcal{C}$ generated by $\mathcal{D}$}.

3. If $\mathcal{H}$ is a subcategory of $\mathcal{C}$ then we call the full subcategory of $\mathcal{C}$ whose objects are the retracts of coproducts of objects of $\mathcal{H}$ in $\mathcal{C}$ the \emph{coproductive hull of $\mathcal{H}$} (in $\mathcal{C}$); we will use the notation $\mathcal{H}^{\cop}$ for it.

4. An object $M$ of $\mathcal{C}$ is said to be \emph{compact} if the functor $\mathcal{H}^{M} = \mathcal{C}(M,-) : \mathcal{C} \to \text{Ab}$ respects coproducts.

5. We say that $\mathcal{C}$ is \emph{compactly generated} by $\mathcal{D}$ if $\mathcal{D}$ is an essentially small class of compact objects of $\mathcal{C}$ that generates $\mathcal{C}$ as its own localizing subcategory.

2.2. On Voevodsky Motivic Complexes and Certain Localizations

We start with some preliminaries and notation for motivic complexes.

In this section $k$ will denote a fixed perfect base field of characteristic $p$, and we set $\mathbb{Z}[1/p] = \mathbb{Z}$ if $p = 0$.

The set of smooth projective varieties over $k$ will be denoted by SmPrVar.
• For a (fixed) unital commutative associative $\mathbb{Z}[1/p]$-algebra $R$ we consider the $R$-linear motivic categories $\text{DM}_{gm,R}^{\text{eff}} \subset \text{DM}_R^{\text{eff}} \subset \text{DM}_R$ (see [4], Subsection 3.1). The categories $\text{DM}_R^{\text{eff}}$ and $\text{DM}_R$ are smashing (see Definition 2.1.2), and the embedding $\text{DM}_R^{\text{eff}} \to \text{DM}_R$ respects coproducts. Moreover, $\text{DM}_R^{\text{eff}}$ is compactly generated by its triangulated subcategory $\text{DM}_{gm,R}^{\text{eff}}$ of effective geometric motives.

• There is a functor $M_R$ ($R$-motif) from the category of smooth $k$-varieties into $\text{DM}_{gm,R}^{\text{eff}}$. Actually, $M_R$ extends to the category of all $k$-varieties (see [1, 5]); yet we will mention this extension just a few times. We will write $R$ for the object $M_R$ ($\text{Spec } k$).

Moreover, $\text{DM}_{gm,R}^{\text{eff}}$ is densely generated (see Subsection 2.1.1) by the $R$-linear motives $M_R$ ($\text{SmPrVar}$) (see Theorem 2.1.2 of ibid.); hence the set $M_R$ (SmPrVar) compactly generates $\text{DM}_R^{\text{eff}}$ as well.

• We will write $\text{Chow}_R^{\text{eff}}$ for the Karoubi-closure in $\text{DM}_R^{\text{eff}}$ of the subcategory whose object class equals $M_R$ (SmPrVar). $\text{Chow}_R^{\text{eff}}$ will be called the category of $R$-linear effective homological Chow motives; see Remark 1.3.2(4) of [4] for a justification of this terminology.

• We also introduce the following notation: $R(\{l\})$ will denote the $R$-linear Lefschetz object (this is $R(1)[2]$ in the notation of [30, pp. 188–238]). For $i \geq 0$ and $M \in \text{Obj} \text{DM}_R^{\text{eff}}$ we will write $M \langle i \rangle$ for the object $M \otimes_{\text{DM}_R^{\text{eff}}} (R(\{l\}))^i$.

Recall that the functor $-\langle i \rangle$ is a full embedding of $\text{DM}_R^{\text{eff}}$ into itself; thus the essential image $\text{DM}_R^{\text{eff}} \langle i \rangle$ of this functor is a full subcategory of $\text{DM}_R^{\text{eff}}$ that is equivalent to $\text{DM}_R^{\text{eff}}$ itself.

Moreover, $-\langle l \rangle$ extends to an exact autoequivalence of $\text{DM}_R$, and the corresponding class $M_R$ (SmVar) $\langle l \rangle$ consists of compact objects for any $i \in \mathbb{Z}$.

• Note that for any $i \geq 0$, $R\langle i \rangle$ is a retract of $M_R((\mathbb{P}^1)^i)$; thus $\text{Chow}_R^{\text{eff}} \langle i \rangle \subset \text{Chow}_R^{\text{eff}}$.

We will also need the following definitions related to motives.

**Definition 2.2.1.** Let $K/k$ be a field extension, and $M$ an object of $\text{DM}_R^{\text{eff}}$.

1. Then $K^{\text{perf}}$ will denote the perfect closure of $K$.
2. We will use the notation $M_K$ for the image of $M$ with respect to the base field change functor $\text{DM}^{\text{eff}}_R \to \text{DM}_R^{\text{eff}} (K^{\text{perf}})$; see Section 5 below for some information on functors of this type.
3. For $j, l \in \mathbb{Z}$ we define $\overline{\text{Chow}}_{\text{Lefschetz}} (M_K, R, l)$ (resp. $\overline{\text{Chow}}_{\text{Lefschetz}} (M_K, R)$) as the group $\text{DM}_R (K^{\text{perf}}) (R(\overline{\{j\}}[l]), M_K)$ (resp. $\text{DM}_R (K^{\text{perf}}) (R(\overline{\{j\}}), M_K)$).
4. For $i \geq -1$ we will write $\text{DM}_R^{\text{eff}} / \text{DM}_R^{\text{eff}} \langle i + 1 \rangle$. $\langle i \rangle$ will denote the corresponding localization functor, and $M_K^{\langle i \rangle} = M^{\langle i \rangle} \circ M_R$.

**Proposition 2.2.2.** Let $j, l \in \mathbb{Z}$, $r \geq 0$, and assume $j - r + l < 0$. Then for any $N \in \text{Obj} \text{Chow}_R^{\text{eff}}$ and any field extension $K/k$ we have $\overline{\text{Chow}}_{\text{Lefschetz}} (N_K \langle r \rangle, R, l) = \{0\}$.

**Proof.** This is an easy consequence of the well-known properties of (Suslin or Bloch) cycle complexes along with Proposition 5.1(1) below; see Proposition 2.3.3(2) of [1].

2.3. Weight Structures: basic Definitions and Statements

Let us recall the definition of the notion that is central for this paper.

**Definition 2.3.1.** I. A pair of subclasses $C_{w \leq 0}, C_{w \geq 0} \subset \text{Obj } C$ will be said to define a weight structure $w$ for a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{w \geq 0}$ and $C_{w \leq 0}$ are retraction-closed in $C$ (i.e., contain all $C$-retracts of their objects).

4 In [1] the group $\text{DM}_R (K^{\text{perf}}) (R(\overline{\{j\}}[l]), M_K)$ is denoted by $h_{2j+r,l} (M_K, R)$.
(ii) Semi-invariance with respect to translations.
\[ C_{w \leq 0} \subset C_{w \leq 0}[1], \quad C_{w \geq 0}[1] \subset C_{w \geq 0}. \]

(iii) Orthogonality.
\[ C_{w \leq 0} \perp C_{w \geq 0}[1]. \]

(iv) Weight decompositions.
For any \( M \) there exists a distinguished triangle
\[ X \to M \to Y \to X[1] \]
such that \( X \in C_{w \leq 0}, \ Y \in C_{w \geq 0}[1]. \)
We will also need the following definitions related to triangulated categories and weight structures.

**Definition 2.3.2.** Let \( i, j \in \mathbb{Z} \); assume that a triangulated category \( C \) is endowed with a weight structure \( w \).

1. The full category \( H^w \subset C \) whose objects class is \( C_{w=0} = C_{w \geq 0} \cap C_{w \leq 0} \) is called the **heart** of \( w \).
2. \( C_{w \geq i} \) (resp. \( C_{w \leq i} \), resp. \( C_{w=0} \)) will denote \( C_{w \geq i}[1] \) (resp. \( C_{w \leq i}[1] \), resp. \( C_{w=0}[1] \)).
3. \( C_{[i,j]} \) denotes \( C_{w \geq i} \cap C_{w \leq j} \); clearly this class equals \( \{0\} \) if \( i > j \).
4. We will call \( C_{w^+} = \cup_{i \in \mathbb{Z}} C_{w \geq i} \) the class of \( w \)-bounded below objects of \( C \).
5. We will say that \( w \) is **smashing** if \( w \) is smashing and the class \( C_{w \geq 0} \) is smashing (in it; see Definition 2.1.2(1) and cf. Proposition 2.3.4(1) below).
6. Assume that a triangulated category \( C \) is endowed with a weight structures \( w \); let \( F : C \to C \) be an exact functor. \( F \) is said to be **weight-exact** (with respect to \( w \)) if it maps \( C_{w \leq 0} \) into \( C_{w \leq 0} \) and sends \( C_{w \geq 0} \) into \( C_{w \geq 0} \).
7. Let \( D \) be a full triangulated subcategory of \( C \).
We will say that \( w \) **restricts to** \( D \) whenever the couple \( (C_{w \leq 0} \cap \text{Obj } D, C_{w \geq 0} \cap \text{Obj } D) \) is a weight structure on \( D \).
8. We will say that \( M \) is left (resp., right) \( w \)-degenerate if \( M \) belongs to \( \cap_{i \in \mathbb{Z}} C_{w \leq i} \) (resp. to \( \cap_{i \in \mathbb{Z}} C_{w \geq i} \)).
Accordingly, \( w \) is left (resp., right) non-degenerate if all left (resp. right) weight-degenerate objects are zero.

**Remark 2.3.3.** 1. A simple (and yet quite useful) example of a weight structure comes from the stupid filtration on \( K(B) \) for an arbitrary additive category \( B \). In this case \( K(B)_{w \leq 0} \) (resp. \( K(B)_{w \geq 0} \)) is the class of complexes that are homotopy equivalent to complexes concentrated in degrees \( \geq 0 \) (resp. \( \leq 0 \)); see ([6], Remark 1.2.3(1)); this weight structure will be denoted by \( w^\circ \).

The heart of this weight structure is the retraction-closure of \( B \) in \( K(B) \) (see Subsection 1.1).
2. A weight decomposition (of any \( M \in \text{Obj } C \)) is (almost) never canonical.
Still for any \( m \in \mathbb{Z} \) the axiom (iv) gives the existence of distinguished triangle
\[ w_{2m}M \to M \to w_{2m+1}M \quad (2.3.1) \]
with some \( w_{2m+1}M \in C_{w \geq 2m+1} \) and \( w_{2m}M \in C_{w \leq 2m} \); we will call it an \( m \)-weight decomposition of \( M \).
We will often use this notation below (even though \( w_{2m+1}M \) and \( w_{2m}M \) are not canonically determined by \( M \)); we will call any possible choice either of \( w_{2m+1}M \) or of \( w_{2m}M \) (for any \( m \in \mathbb{Z} \)) a weight truncation of \( M \). Moreover, when we will write arrows of the type \( w_{2m}M \to M \) or \( M \to w_{2m+1}M \) we will always assume that they come from some \( m \)-weight decompositions.
3. In the current paper we use the “homological convention” for weight structures; it was previously used in [7] and in several papers of the authors, whereas in [8] the “cohomological convention” was used.

In the latter convention the roles of \( C_{w \geq 0} \) and \( C_{w \geq 0} \) are interchanged, i.e., one considers \( C_{w \leq 0} = C_{w \geq 0} \) and \( C_{w \geq 0} \).
We also recall that D. Pauksztello has introduced weight structures independently in [9]; he called them co-\( f \)-structures.
Proposition 2.3.4. Let $\mathcal{C}$ be a triangulated category, $n \geq 0$; we will assume that $w$ is a fixed weight structure on $\mathcal{C}$.

1. $\mathcal{C}_{w\geq 0}$ is closed with respect to all coproducts that exist in $\mathcal{C}$.
2. $\mathcal{C}_{w \geq 2} = (\mathcal{C}_{w \leq -1})^\perp$ and $\mathcal{C}_{w = 0} = \mathcal{C}_{w \geq 1}$.
3. If $M$ belongs to $\mathcal{C}_{w \geq -n}$ then $w_{\leq 0}M$ belongs to $\mathcal{C}_{w \leq -n}$.
4. Assume that $D \subset \mathcal{C}$ is a triangulated subcategory of $\mathcal{C}$ such that $w$ restricts to a weight structure $w_D$ on $D$. Let $M \in \mathcal{C}_{w \geq 0}$, $N \in \mathcal{C}_{w = 0}$, and assume that a morphism $f \in \mathcal{C}(N, M)$ vanishes in the localization $\mathcal{C}/D$.

Then $f$ factors through some object of $Hw_D$.

Proof. Assertions 2–3 were proved in [8] (pay attention to Remark 2.3.3(3)!). Assertion 4 is given by Corollary 1.4.6(2) of [1].

2.4. Some Existence of Weight Structures Statements

Proposition 2.4.1. Let $B$ be a connective additive subcategory of a smashing (triangulated) $\mathcal{C}$, and assume that objects of $B$ are compact in $\mathcal{C}$.

Then there exists a weight structure $w$ on $\mathcal{C}$ such that $\mathcal{C}_{w \geq 0}$ (resp. $\mathcal{C}_{w \geq 0}$) is the smallest subclass of $\text{Obj} \mathcal{C}$ that is closed with respect to coproducts, extensions, and contains $\text{Obj} B[i]$ for $i \leq 0$ (resp. for $i \geq 0$). Moreover, $Hw = B^\oplus$ (see Definition 2.1.2(3)).

In this case we will say that $w$ is purely compactly generated by $B$.

Furthermore, if the objects of $B$ (compactly) generate $\mathcal{C}$ as its own localizing subcategory then $w$ is left non-degenerate.

Proof. The statement easily follows from Corollary 2.3.1 and Lemma 2.3.3 of [10]; cf. Theorem 3.2.2(2,3) of [11].

Now let us discuss certain weight structures in localizations.

Proposition 2.4.2. Assume that $(\mathcal{C}, w, B)$ are as in the previous proposition; in addition, $B$ is essentially small and generates $\mathcal{C}$ as its own localizing subcategory, and $H$ is an additive subcategory of $B$. Denote by $D$ the localizing subcategory of $\mathcal{C}$ generated by $H$. Then the following statements are valid.

1. The Verdier quotient category $\mathcal{C}/D$ exists (i.e., it is a locally small category); the localization functor $\pi : \mathcal{C} \to \mathcal{C}/D$ respects coproducts and converts compact objects into compact ones. Moreover, $\mathcal{C}/D$ is generated by $\pi(\text{Obj} B)$ as its own localizing subcategory, and the corresponding exact functor $(B)_{\mathcal{C}}/(H)_{\mathcal{C}} \to \mathcal{C}/D$ (where $(B)_{\mathcal{C}}/(H)_{\mathcal{C}}$ is the Verdier quotient of the corresponding locally small categories) is a full embedding.

2. $\mathcal{C}/D$ possesses a weight structure $w_{\mathcal{C}/D}$ such that $\pi$ is weight-exact. Moreover, $w_{\mathcal{C}/D}$ is purely compactly generated by its full subcategory corresponding to $B$ (in the sense of Proposition 2.4.1), and the corresponding functor $Hw \to Hw_{\mathcal{C}/D}$ factors as the composition of the obvious functor $Hw \to Hw/H^\oplus$ (see Definition 2.1.1(1)) with a full embedding.

Proof. All these assertions were proved in [12] (see Proposition 4.3.1.3(III) and Theorem 4.3.1.4 of ibid.).

2.5. On Weight Complexes, Pure Functors, and Weight Spectral Sequences

Now we recall the theory of so-called “strong” weight complex functors. Note here that this version of the theory is less general than the “weak” one that was used in [1]. The latter one is sufficient for our purposes (and is somewhat more convenient for them); yet it requires some non-standard definitions.

Proposition 2.5.1. Assume that $\mathcal{C}$ possesses an $\infty$-enhancement (see Subsection 1.1 of [13] for the corresponding references), and satisfies the assumptions of Proposition 2.4.1. Then there exists an exact functor $t^\ast : \mathcal{C} \to K(Hw), M \mapsto (M^i)$, such that the following statements are fulfilled.

1. The composition of the embedding $Hw \to \mathcal{C}$ with $t^\ast$ is isomorphic to the obvious embedding $Hw \to K(Hw)$.
2. Let $C'$ be a triangulated category that possesses an $\infty$-enhancement as well and is endowed with a compactly purely generated weight structure $w'$; let $F : C \to C'$ be a weight-exact functor that lifts to $\infty$-enhancements. Then the composition $t'' \circ F$ is isomorphic to $K(HF) \circ t''$, where $t''$ is the weight complex functor corresponding $w'$, and the functor $K(HF) : K(Hw) \to K(Hw')$ is the obvious $K(-)$-version of the restriction $HF : Hw \to Hw'$ of $F$.

3. Fix a choice of weight truncations $w_i N$ of $N \in \text{Obj} C$ (see Remark 2.3.3(2)) for $i \in \mathbb{Z}$. Then there exist unique morphisms $j_i : w_i N \to w_{i+1} N$ (for $i \in \mathbb{Z}$) that make the corresponding triangles $w_i N \to w_{i+1} N \to N$ commutative. Moreover, the objects $\tilde{N}^{-1-i} = \text{Cone}(j_i)[-1-i]$ belong to $C_{w=0}$ and there exists a complex $\tilde{t} (N)$ whose terms are $\tilde{N}^j$ (set in the corresponding degrees) and $\tilde{t} (N) \cong t'' (N)$ (in $K (Hw)$).

Furthermore, if $0 \leq m \in \mathbb{Z}$ and $w_i N = 0$ then $w_{m+i} N$ belongs to the extension-closure of the set $\{ \tilde{N}^j \mid -m \leq j \leq -l \}$.

4. If $M \in C_{w \leq n}$ (resp. $M \in C_{w \geq n}$) then $t'' (M)$ belongs to $K(Hw)_{w_i \leq n}$ (resp. to $K(Hw)_{w_i \geq n}$; see Remark 2.3.3(1)).

5. Assume that $A$ is an additive covariant functor from $Hw$ into an abelian category $A$. Then the functor $H^A$ that sends $M \in \text{Obj} C$ into the zeroth homology of the complex $A(M')$ is homological.

Moreover, if $A = A' \circ HF$ for some additive functor $A' : Hw' \to A$ in the setting of assertion 2 then $H^A = H^A' \circ F$.

6. If $A$ is an AB4 abelian category then the functor $H^A$ as above is uniquely characterized by the following assumptions: it is homological, respects coproducts, its restriction to the corresponding category $B$ (see Proposition 2.4.1) equals that of $A$, and its restrictions to $B[1]$ for $i \neq 0$ vanish.

**Proof.** Assertions 1 and 2 easily follow from Remark 3.6 of [13], and the first part of assertion 5 is obvious. Next, we recall that the functor $t''$ is “compatible” with the weak weight complex functor as defined in [11]; see Remark 2.5.2(2) below for more detail. Hence one can apply Proposition 1.3.4(4,6) and Lemma 1.3.2(3) of ibid. to obtain assertion 3.

Similarly, (the easy) assertion 4 is given by Proposition 1.3.4(10) of ibid., and the non-trivial (second) part of assertion 5 follows from Proposition 1.3.4(12) of loc. cit. (see also Theorem 2.1.2 of ibid.). Moreover, assertion 6 follows from Proposition 2.3.2(6) of ibid.; to obtain this implication one should note that the functors $Hw \to A$ that respect small coproducts are essentially in one-to-one correspondence with additive functors $B \to A$ (since $Hw = B^\wedge$).

**Remark 2.5.2.** 1. The term “weight complex” originates from [14]; yet the domains of the (“concrete”) weight complex functors considered in that paper were not triangulated.

2. In Proposition 1.3.4 of [11] a certain (canonical) weak weight complex functor was defined as a functor from a category canonically equivalent to $C$ into the “weak” category $K_{\wedge} (Hw)$. Now, our proof above depends on two observations.

Firstly, there exists a canonical additive functor $K(Hw) \to K_{\wedge} (Hw)$, and the weak weight complex functor essentially factors through it; see (Remark 1.3.5(3) of [13] along with Remark 3.6 of [13]).

Secondly, the functors of the type $H^A$ as in part 5 of our proposition (that were called $w$-pure ones in [11]; the terminology was justified in Remark 2.1.3(3) of ibid.) factor through the weak weight complex functor; see Theorem 2.1.2 of [11]. Moreover, the properties of pure functors essentially do not depend on any enhancements (as mentioned in our proposition). In particular, it is easily seen that no $\infty$-lifts for $F$ are necessary for the second part of Proposition 2.5.1(5).

On the other hand, one can probably re-prove some of the statements above via arguments similar to that in Remark 3.6 of [13].

Next we pass to weight spectral sequences. We assume that $C$ possesses an $\infty$-enhancement since we want to cite the previous proposition.
Proposition 2.5.3. Adopt the assumptions of Proposition 2.5.1; assume that $H$ is a homological functor $C \to A$.

Then for any $M \in \text{Obj}_C$ there exists a spectral sequence $T = T_w(H, M)$ with $E_2^{pq}(T) = H_{-q}^{G_p}(M)$, where $G_p$ is the restriction of the functor $H_{-q} = H \circ [q]$ to $H_w$ (and respectively, $H_{-q}^{G_p} = H^{G_p} \circ [p]$; see also Proposition 2.5.1(5)). Moreover, $T_w(H, M)$ is $C$-functorial in $M$ and in $H$ (with respect to composition of $H$ with exact functors of abelian categories), and $T_w(H, M)$ converges to $H_{-q}^{G_p}(M)$ whenever $M$ is bounded below and $H$ kills $C_{wC}$ for $i$ large enough.

Proof. This is (essentially) an easy combination of Theorems 2.3.2 of [8] with our Proposition 2.5.1(5); see also Remark 2.5.2(2).

2.6. Chow Weight Structures on Our Categories

Using the results above we construct and study the Chow weight structures on $DM^\text{eff}_R$ and on $DM'_R$.

Proposition 2.6.1. Assume $r \geq -1$ and $M \in \text{Obj} DM^\text{eff}_R$.

1. Then the categories $DM^\text{eff}_R$ and $DM'_R$ possess $\infty$-enhancements.

2. There exists a left non-degenerate weight structure $w_{Chow}^\text{eff}$ on $DM^\text{eff}_R$ that is purely compactly generated by $Chow^\text{eff}_R$ in the sense of Proposition 2.4.1; thus $DM^{\text{eff}}_{R,w_{Chow}^{\geq 0}}$ (resp. $DM^\text{eff}_{R,w_{Chow}^{< 0}}$) is the smallest subclass of $\text{Obj}_C$ that is closed with respect to coproducts, extensions, and contains Obj $Chow^\text{eff}_R[i]$ for $i \leq 0$ (resp. for $i \geq 0$).

Respectively, $Hw_{Chow} = Chow^{\text{eff}}_R$.

3. The functor $-\langle r+1 \rangle : DM^\text{eff}_R \to DM^\text{eff}_R$ is weight-exact with respect to $w_{Chow}^\text{eff}$.

Moreover, this functor is "strictly weight-exact", i.e., if $M \langle r+1 \rangle$ belongs to $DM^\text{eff}_{R,w_{Chow}^{= 0}}$ (resp. to $DM^\text{eff}_{R,w_{Chow}^{< 0}}$) then $M \in DM^\text{eff}_{R,w_{Chow}^{= 0}}$ (resp. $M \in DM^\text{eff}_{R,w_{Chow}^{< 0}}$) as well.

4. Assume that for some $i \in \mathbb{Z}$ there exist choices of $w_{Chow}^{\leq -i} M$ and $w_{Chow}^{\leq -i+1} M$ that belong to $\text{Obj} DM^\text{eff}_R \langle r+1 \rangle$. Then the corresponding object $\tilde{M}' = \text{Cone}(j_{-i+1})$ as mentioned in Proposition 2.5.1(3) belongs to $DM^\text{eff}_{R,w_{Chow}^{= 0}} \langle r+1 \rangle$.

5. The localization of $DM^\text{eff}_R$ by its subcategory $DM^\text{eff}_R \langle r+1 \rangle$ satisfies the conditions of Proposition 2.4.2 with $H = Chow^\text{eff}_R \langle r+1 \rangle$. Consequently, there exists a purely compactly generated weight structure $w^\text{Chow}_C$ on $DM^\text{eff}_R$ such that the localization functor $\Gamma : DM^\text{eff}_R \to DM^\text{eff}_R$ (see Definition 2.2.1(4)) is weight-exact; moreover, $\Gamma$ respects coproducts and the compactness of objects.

Moreover, if $-1 \leq s \leq r$ then the obvious localization functor $\Gamma^r_s : DM^\text{eff}_R \to DM^\text{eff}_R$ is weight-exact and respects the compactness and coproducts as well.

6. If $K/k$ is a field extension then the base field change functor $-\kappa : DM^\text{eff}_R \to DM^\text{eff}_R$ (see Definition 2.2.1(2) and Proposition 5.1 below) is weight-exact and respects coproducts.

7. If $R$ is not torsion and $k$ is of infinite transcendence degree over its prime subfield, then the weight structure $w_{Chow}$ is right degenerate.

Proof. Assertion 1 is obvious. Now, the subcategory $Chow^\text{eff}_R$ compactly generates $DM^\text{eff}_R$. Consequently, to obtain assertion 2 it suffices to recall that $Chow^\text{eff}_R$ is connective in $DM^\text{eff}_R$ (see Corollary 6.7.3 of [15]) and apply Proposition 2.4.1.

Next, we recall that the functor $-\langle r+1 \rangle$ respects coproducts and sends $Chow^\text{eff}_R$ into itself. Applying the explicit description of $w_{Chow}$ we obtain that $-\langle r+1 \rangle$ is weight-exact. Moreover, if $M \langle r+1 \rangle \in DM^{\text{eff}}_{R,w_{Chow}^{\geq 0}}$ (resp. $M \langle r+1 \rangle \in DM^\text{eff}_{R,w_{Chow}^{< 0}}$) then applying this weight-exactness we obtain that $M \langle r+1 \rangle \perp$
$DM^\text{eff}_R\vert_{w_{\text{Chow}}\geq 1}(r+1)$ (resp. $DM^\text{eff}_R\vert_{w_{\text{Chow}}<1}(r+1)\perp M$). Combining Proposition 2.3.4(2) with the Cancellation Theorem (which says that $-\langle r+1 \rangle$ is fully faithful) we conclude the proof.

4. Proposition 2.5.1(3) says that $\tilde{M}'$ belongs to $DM^\text{eff}_R\vert_{w_{\text{Chow}}=0}$. Since $\tilde{M}'$ also belongs to $\text{Obj } DM^\text{eff}_R\langle r+1 \rangle$, the previous assertion implies that $\tilde{M}' = N(r+1)$, where $N$ belongs to $DM^\text{eff}_R\vert_{w_{\text{Chow}}\leq 0} \cap DM^\text{eff}_R\vert_{w_{\text{Chow}}<0} = DM^\text{eff}_R\vert_{w_{\text{Chow}}=0}$.

Since $\text{Chow}^\text{eff}_R\langle r \rangle \subset \text{Chow}^\text{eff}_R$, we also obtain that the first part of assertion 5 follows from Proposition 2.4.2. Next, to obtain the “moreover” part of the assertion one can apply Proposition 2.4.2 for $\text{chow}^\text{eff}_R$ and $\text{chow}^\text{eff}_R$ consisting of $M^r_S(\text{SmPrVar})$ and $\mathcal{I}^r(M^r_S(\text{SmPrVar})(s+1))$, respectively.

6. We recall that if $f : \text{Spec } K \rightarrow \text{Spec } k$ is the corresponding morphism then $-\kappa$ can be defined as the restriction of $DM^\text{eff}_R$ of the functor $f^* : DM_R \rightarrow DM_R(K^\text{perf})$; see Proposition 5.1(1) below. Now, the latter functor respects coproducts since there exists a functor $f_*$ that is right adjoint to it; see Theorem 3.1 of [16], and Definitions 1.1.12 and 1.4.2 of [17]. Hence $-\kappa$ respects coproducts as well.

Next, $f^*$ sends (effective) Chow motives over $k$ into that over $K^\text{perf}$ (see Proposition 5.1(1) below or Proposition 2.3.2 of [29]); thus applying the explicit description of $w_{\text{Chow}}$ once again we obtain that $-\kappa$ is weight-exact indeed.

Lastly, assertion 7 is given by Proposition 3.2.6 of [11].

3. ON CHOW-WEIGHT HOMOLOGY AND ITS RELATION TO MOTIVIC HOMOLOGY

In Subsection 3.1 we define and study Chow-weight homology functors $DM^\text{eff}_R \rightarrow \text{Ab}$; these statements generalize the ones of [1, Subsection 3.1].

In Subsection 3.2 we extend the equivalent criteria for the vanishing of Chow-weight homology from $DM^\text{eff}_R(\text{gm},R)$ (as studied in ibid.) to $DM^\text{eff}_R\vert_{w_{\text{Chow}}^+}$.

In Subsection 3.3 we prove our central Theorem 3.2.3 that relates the vanishing of motivic homology in a staircase range to certain Chow-weight homology vanishing conditions. This is related to the study of slices.

3.1. On Chow-weight Homology of General Motives: basic Properties

**Definition 3.1.1.** 1. Let $i,l,j \in \mathbb{Z}$; let $K$ be a field extension of $k$.

Then we will write $\text{CWH}_i^j(-\kappa,R,l)$ for the functor $\mathcal{H}_{\text{Chow}}^j \circ [i]$, where we apply Proposition 2.5.1(5) to the weight structure $w_{\text{Chow}}$ (see Proposition 2.6.1(2) and $\mathcal{A}$ is the restriction of the functor $N \mapsto \mathcal{H}_{10}^j(N(K,R,l))$ (see Definition 2.2.1(3)) to $\text{Chow}^\text{eff}_R$. We will sometimes omit $R$ in this notation. Moreover, we will often write $\text{CWH}_i^j(M_K,R)$ for $\text{CWH}_i^j(M_K,R,0)$.

2. We will use the notation $DM^\text{eff}_R\vert_{w_{\text{Chow}}^+}$ for the class of $w_{\text{Chow}}$-bounded below motives (see Definition 2.3.2(4)).

3. We will need the following convention (cf. Subsection 2.2): $I^{\text{dim}}_\infty = I^{\text{dim}}_0$ is the identity on $DM^\text{eff}_R$, $I^{\text{dim}}_0 = I^j$, $w^{\text{dim}}_{\text{Chow}} = w_{\text{Chow}}$, $\text{Chow}^\text{eff}_R(\{+\infty\}) = DM^\text{eff}_R(\{+\infty\}) = \{0\}$, etc.

4. We will write $I^R_{\text{dim}}$ for the $R$-linear version of the homotopy $t$-structure of Voevodsky; see Subsection 4.4 of [15] or Example 2.3.13 of [18].

Let us prove some properties of these functors.

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5 The reader can easily verify that this class yields a (full) triangulated subcategory of $DM^\text{eff}_R(\text{gm},R)$; yet we will not need this fact below.
Proposition 3.1.2. Let $i, l, j, K$ be as above and $r \in [0, +\infty]$.

1. $\text{CWH}_j^i(-K, R,l)$ is a homological functor on $\text{DM}_R^{\text{eff}}$ that respects coproducts. Moreover, this functor factors through the base field change functor $\text{DM}_R^{\text{eff}} \to \text{DM}_R^{\text{eff}}(K^{\text{perf}})$.

2. Assume $r \geq j + 1$. Then the functor $\text{CWH}_j^i(-K, R,l)$ kills $\text{DM}_R^{\text{eff}}(r + 1)$; thus it induces a well-defined functor $\text{DM}_R^{\prime} \to \text{Ab}$ (see Definition 2.2.1(4)).

Moreover, this functor is pure with respect to the weight structure $w_{\text{Chow}}^r$ (see Proposition 2.6.1(5)).

3. For any smooth projective connected variety $P/k$ the functors $\text{DM}_R^{\prime}[l'](M_R(P)(j)), -$ and $\text{CWH}_j^i(-K(P), R)$ are canonically isomorphic; note that the latter functor is well-defined according to the previous assertion.

4. Let $N$ belong to $\text{DM}_{R_{\text{Chow}}}^{\prime}(2-n)$. Then $\text{CWH}_j^i(N_K,l) = \{0\}$ whenever either $i > n$ and $j \leq r-l$ or if $l < 0$.

5. Moreover, if $m \in [0, r]$ then the following assumptions on $N \in \text{DM}_{R_{\text{Chow}}}^{\prime}(2-n)$ are equivalent.

   a) $\text{CWH}_j^i(N_K) = \{0\}$ for all $0 \leq j \leq m$ and all function fields (that is, finitely generated extensions) $K/k$.

   b) The object $N_m = i_m^*(N)$ belongs to $\text{DM}_R^{\text{eff}}(m) \cap \text{DM}_{R_{\text{Chow}}}^{\prime}(2-n)$.

   c) There exists a choice of $w_{\text{Chow},\leq n}^r$ that belongs to $l'(\text{Obj } \text{DM}_R(m+1))$.

6. The class $\text{DM}_{R_{\text{Chow}}}^{\text{eff}}(2-n)$ (see Definition 3.1.1(4)) equals the smallest extension-closed subclass of $\text{Obj } \text{DM}_R^{\text{eff}}$ that is closed with respect to coproducts and contains $\text{Obj } \text{Chow}_{\text{eff}}(a)[a \cdot b]$ for all $a, b \geq 0$. Moreover, the functor $\text{CWH}_j^i(N_K,l) = \{0\}$ kills this class whenever $i > j + l$.

Proof. 1. All the statements easily follow from Proposition 2.5.1(5, 6) along with Proposition 2.6.1(6); note here that the functor $\mathcal{C} \mathfrak{h} \mathfrak{o} \mathfrak{u} \mathfrak{v}_j(-K, R,l)$ respects coproducts since the object $R(\langle j \rangle_{K^{\text{perf}}})$ of the category $\text{DM}_R(K^{\text{perf}})$ is perfect (see Subsection 2.2).

2. To prove the first part of the assertion we should verify that $\text{CWH}_j^i(-K, R,l) \circ (r + 1) = 0$. Recall that the functor $-(r + 1) : \text{DM}_R^{\text{eff}} \to \text{DM}_R^{\text{eff}}$ is weight-exact with respect to $w_{\text{Chow}}$ by Proposition 2.6.1(3); hence Proposition 2.5.1(5, 6) reduces the statement to the vanishing of the restriction of $\mathcal{C} \mathfrak{h} \mathfrak{o} \mathfrak{u} \mathfrak{v}_j(M_K^i, R,l)$ to $\text{Chow}_{\text{eff}}^i(R_{K^{\text{perf}}})$. The latter fact is given by Proposition 2.2.2.

To prove the “moreover” statement we invoke Proposition 2.5.1(6) once again. Since the functor $\text{DM}_R^{\text{eff}} \to \text{Ab}$ induced by $\text{CWH}_j^i(-K, R,l)$ respects coproducts, it suffices to note that its restrictions to $M_K^i(\text{SmPrVar})[s]$ for $s \neq 0$ vanish since the functor $\text{CWH}_j^i(-K, R,l)$ is pure with respect to $w_{\text{Chow}}$.

3. Since the object $l'(M_R(P)(j))$ is compact in $\text{DM}_R^{\prime}$ (see Proposition 2.4.2(1)), both of the functors in question respect coproducts. Now, they are also homological, and the functor $\text{CWH}_j^i(-K(P), R)$ is $w_{\text{Chow}}$-pure by definition; thus to obtain the isomorphism in question it suffices to compare their restrictions to the categories $l'(\text{Chow}_{\text{eff}}^i)[i] \subset \text{Obj } \text{DM}_K^i$ for $i \in \mathbb{Z}$.

Next, recall that the localization $\text{DM}_R^{\text{eff}} / \text{DM}_R^{\text{eff}}(j + 1)$ embeds into $\text{DM}_R^{\prime}$ by Proposition 2.4.2(1). Thus Proposition 2.2.6(6) of [1] yields the result easily.

4. Clearly, we can assume $n = 0$.

By Proposition 2.5.1(4) the corresponding weight complex $l^\prime(N)$ is homotopy equivalent to a complex concentrated in non-positive degrees. Next we recall that the functor $\text{CWH}_j^i$ is pure with respect to $w_{\text{Chow}}^r$ (see Definition 3.1.1(1) and assertion 2 of this proposition); applying the relation of pure functors to weight complexes we obtain that $\text{CWH}_j^i(N_K,l) = \{0\}$ for all $i > 0$ (and the corresponding values of $j$) indeed.
Lastly, $\text{CWH}^j(N_K,I) = \{0\}$ if $j < 0$ since the corresponding restriction of the functor $N \mapsto \mathcal{H}_j(H^1_{\text{Chow}}(N_K,R,I))$ to $\text{H}_{\text{Chow}}$ is zero immediately from the connectivity of the category $\text{CWH}(K^{\text{perf}})$.

5. If $j < m$ then the functor $\text{CWH}^j(-K) = \{0\}$ factors through $s_n$ by assertion 2; hence the implication (b) $\Rightarrow$ (a) follows from assertion 4. Next, condition (b) trivially implies condition (c) if $r = +\infty$, and it does so in the case $r < +\infty$ as well by Theorem 3.3.1 of [19]; cf. Remark 3.3.2(1) of loc. cit. Moreover, recall that the functor $s_n$ is weight-exact by Proposition 2.6.1(5); hence condition (c) implies condition (b).

It remains to verify that (a) implies (b). We assume $n = 0$ once again. First assume that $m < +\infty$. It clearly suffices to verify that condition (a) implies the following: if $-1 < s < m$ and $l^s_r(N) \in \text{DM}^s_{\text{eff},\text{Chow}}$ then $N_{s+1}$ belongs to $\text{DM}^{s+1}_{\text{eff},\text{Chow}}$.

We take a weight decomposition $w^{s+1}_{\text{Chow}}(N_{s+1}) \xrightarrow{g_{s+1}} N_{s+1} \xrightarrow{w^{s+1}_{\text{Chow}}(N_{s+1})}$ of $N_{s+1}$, and apply the localization $l^s_r : \text{DM}^{s+1}_{\text{eff},\text{Chow}} \rightarrow \text{DM}^s_{\text{eff},\text{Chow}}$. Since $l^s_r$ is weight-exact, we have $l^s_r(w^{s+1}_{\text{Chow}}(N_{s+1})) \in \text{DM}^s_{\text{eff},\text{Chow}}$. Since $l^s_r(N) = l^s_r(N_{s+1})$, the orthogonality axiom (iii) of Definition 2.3.1 gives $l^s_r(g_{s+1}) = 0$.

Next, Proposition 2.3.4(3) implies $w^{s+1}_{\text{Chow}}(N_{s+1}) \in \text{DM}^s_{\text{eff},\text{Chow}}$. Thus combining Proposition 2.3.4(4) with Proposition 2.6.1(3) we obtain that $g_{s+1}$ factors through an element of $\text{DM}^{s+1}_{\text{eff},\text{Chow}}$. Thus it factors through a coproduct of $M^{s+1}_{\text{eff}}(P_a)(s+1)$ for some (connected) varieties $P_a \in \text{SmPrVar}$.

Applying assertion 3, we obtain that $M^{s+1}_{\text{eff}}(P_a)(s+1) \perp N_{s+1}$ since $\text{CWH}^s(N_{s+1},k(P_a)) = 0$. Thus $g_{s+1} = 0$. It follows that $N_{s+1}$ is a retract of $w^{s+1}_{\text{Chow}}(N_{s+1})$, hence $N_{s+1}$ belongs to $\text{DM}^{s+1}_{\text{eff},\text{Chow}}$ indeed.

It remains to consider the case $m = r = +\infty$. Similarly to the argument above, it suffices to verify that $M_{\text{eff}}(P_a) \perp N$ for any smooth projective $k$-variety $P_a$. Now, assume that $P_a$ is of dimension $d$ and apply the equivalence of our conditions in the case $m = d$ (note that we have just proved it). Condition (b) in this case gives a weight decomposition triangle with $w^{d+1}_{\text{Chow}}(N_{d+1}) \in \text{Obj}(\text{CWH}(K^{d+1}(d+1))$ (see the arguments above). The definition of Chow groups immediately implies that $M_{\text{eff}}(P_a) \perp M_{\text{eff}}(\text{SmPrVar}) \perp N$ as well. Since $M_{\text{eff}}(P_a) \perp w^{d+1}_{\text{Chow}}(N_{d+1})$ by the orthogonality axiom for $w_{\text{Chow}}$, we obtain that $M_{\text{eff}}(P_a) \perp N$ indeed.

6. The first part of the assertion is given by Theorem 2.4.3 along with Example 2.3.13 of [18]; it also can be easily deduced from Theorem 2.2.1(3) of [20] (cf. also Theorem 6.2.1(1) of [21]). Next, the Chow-weight homology functors respect coproducts; hence the vanishing in question follows from this first part combined with Proposition 2.2.2.

**Remark 3.1.3.** Since the weight structure $w_{\text{Chow}}$ is right degenerate (at least, in some cases; see Proposition 2.6.1(7)), and for right weight-degenerate objects $M$ the weight complex $\ell_K(M)$ vanishes, below we will mainly concentrate on $w_{\text{Chow}}$-bounded below motives. Recall here that Lemma 2.4 of [22] gives an interesting example of right $w_{\text{Chow}}$-degenerate motif; see Proposition 3.2.6 of [11]. Note also that this motif is infinitely effective, i.e., belongs to $\cap_{r \geq 0} \text{Obj} DM_{\text{eff}}^r$; cf. Remark 3.2.4 below.

Another evidence for certain problems with applying arguments similar to our ones to objects that are not $w$-bounded below is given by Remark 2.2.6(3) of ibid.

2. Recall that Chow-weight homology for geometric motives was introduced and thoroughly studied in [1]. Our version of these theories is the only pure extension of these homology theories to $DM_{\text{eff}}$ that respects coproducts (see Proposition 2.5.1(6)).
3.2. Some Chow-weight Homology Vanishing Criteria

To formulate our statements in the most general case we recall the following technical definition.

**Definition 3.2.1.** Let $\mathcal{J}$ be a subset of $\mathbb{Z} \times [0, +\infty)$ (see Subsection 2.1).

We will call it a *staircase* set if for any $(i, j) \in \mathcal{J}$ and $(i', j') \in \mathbb{Z} \times [0, +\infty)$ such that $i' \geq i$ and $j' \leq j$ we have $(i', j') \in \mathcal{J}$.

For $i \in \mathbb{Z}$ the minimum of $j \in [0, +\infty]$ such that $(i, j) \notin \mathcal{J}$ will be denoted by $a_{i,j}$.

2. For $m \in \mathbb{Z}$ we will write $d_{,m} \mathcal{DM}_R^{\text{eff}}$ for the localizing subcategory of $\mathcal{DM}_R^{\text{eff}}$ generated by $\{M_R (X)\}$ for $X$ running through smooth $k$-varieties of dimension at most $m$; thus this category is zero if $m < 0$.

Moreover, we will denote by $d_{,m} \mathcal{Chow}^{\text{eff}}_R$ the subcategory of $\mathcal{Chow}^{\text{eff}}_R$ consisting of the retracts of the motives of smooth projective varieties of dimension at most $m$.

**Remark 3.2.2.** Obviously, $\mathcal{J} \subset \mathbb{Z} \times [0, +\infty)$ is a staircase set if and only if it equals the union of the strips

$$\bigcup_{(b_i, b_j) \in \mathcal{J}} \mathcal{J}_{b_i, b_j},$$

where $\mathcal{J}_{b_i, b_j} = \left[ b_i, +\infty \right) \times \left[ b_j, 0 \right]$.

Consequently, the union of any set of staircase sets is a staircase set as well; cf. Theorem 3.2.3(4) below.

Now we study the vanishing of $\mathcal{CWH}^*_\mathcal{J} (M_R)$ in staircase degrees.

**Theorem 3.2.3.** Let $\mathcal{J} \subset \mathbb{Z} \times [0, +\infty)$ be a staircase set, $M \in \mathcal{DM}_R^{\text{eff}} \mathcal{Chow}^+$ (see Definition 3.1.1(2)).

1. Then the following conditions are equivalent.

A. $\mathcal{CWH}^1_i (M_R) = \{0\}$ for all function fields (see Proposition 3.1.2(5a)) $K/k$ and $(i, j) \in \mathcal{J}$.

B. The object $t^i (M)$ belongs to $\mathcal{DM}_R^{b_j, \mathcal{Chow} \geq \infty}$ whenever $(i, j) \in \mathcal{J}$.

C. For any $i \in \mathbb{Z}$ there exists a choice of $w_{\mathcal{Chow} \leq \infty} M$ that belongs to $\operatorname{Obj} \mathcal{DM}_R^{\mathcal{Chow} (a_{i,j})}$.

D. $M$ belongs to the smallest extension-closed class $\mathcal{D}_\mathcal{J}$ of objects of $\mathcal{DM}_R^{\mathcal{Chow} (a_{i,j})}$ that also closed with respect to coproducts and contains $\mathcal{CWH}^1 \mathcal{DM}_R^{\mathcal{Chow} (a_{i,j})} [-i]$.

E. There exists a $\mathcal{Chow}^{\mathcal{Chow} (a_{i,j})}$-complex $t \mathcal{M} \equiv t (M)$ such that its $i$-th term $\tilde{M}^i$ is an object of $\mathcal{CWH}^{\mathcal{Chow} (a_{i,j})} [-i]$.

2. If $M \in \mathcal{DM}_R^{R, n}$ (for some $a \leq b \in \mathbb{Z}$) then $M$ belongs to $\mathcal{D}_\mathcal{J}$ (see Condition 1.D) if and only if $M$ belongs to the extension-closure of $\bigcup_{b_i, b_j \leq \infty} \left( \operatorname{Obj} \mathcal{CWH}^{\mathcal{Chow} (a_{i,j})} \right) [-i]$.

3. If $M$ is of dimension at most $r \geq 0$ then the (equivalent) conditions of assertion 1 are also equivalent to the following ones:

A'. $\mathcal{CWH}^1_i (M_R) = \{0\}$ whenever $(i, j) \in \mathcal{J}$ and $K = k (P)$, where $P$ is smooth projective $k$-variety of dimension at most $r - j$.

C'. For any $i \in \mathbb{Z}$ there exists a choice of $w_{\mathcal{Chow} \leq \infty} M$ that belongs to $\operatorname{Obj} (d_{s_r - a_{j,i}} \mathcal{DM}_R^{\mathcal{Chow} (a_{j,i})} [-i])$.

D'. $M$ belongs to the smallest extension-closed class of objects of $\mathcal{DM}_R^{\mathcal{Chow} (a_{j,i})}$ that also closed with respect to coproducts and contains $\mathcal{CWH}^1 \mathcal{DM}_R^{\mathcal{Chow} (a_{j,i})} [-i]$.

E'. There exists a $\mathcal{Chow}^{\mathcal{Chow} (a_{j,i})}$-complex $t \mathcal{M} \equiv t (M)$ such that its $i$-th term $\tilde{M}^i$ is an object of $\mathcal{CWH}^{\mathcal{Chow} (a_{j,i})} [-i]$.

Moreover, a similar modification can also be made in assertion 2.

4. Assume that $\mathcal{J}$ are staircase sets for $j$ running through some index set $J$, and $\mathcal{J} = \bigcup \mathcal{J}_j$. Then $M$ belongs to $\mathcal{D}_\mathcal{J}$ if and only if it belongs to $\bigcap \mathcal{D}_j$.

**Proof.** Consider the strip $\mathcal{J}_{(b_i, b_j)} = \left[ b_i, +\infty \right) \times \left[ b_j, 0 \right]$. Applying Proposition 3.1.2(5) inductively we easily obtain that the vanishing of $\mathcal{CWH}^1_i (M_R)$ for all $(i, j) \in \mathcal{J}_{(b_i, b_j)}$ is equivalent to $t^i (M) \in \mathcal{DM}_R^{\mathcal{Chow} \geq \infty}$, cf. the proof of (11), Theorem 3.2.1(2)). Using Remark 3.2.2 we get the equivalence $A \Leftrightarrow B$. Similarly, the equivalence $B \Leftrightarrow C$ follows from Proposition 3.1.2(5) as well.
Next, we can take $w_{\leq i}M = 0$ for $i$ small enough. Thus if condition C is fulfilled then combining Proposition 2.6.1(4) with the extension-closure statement in Proposition 2.5.1(3) we obtain that the choices of $w_{\text{Chow} \leq -i}M$ provided by condition C actually belong to the class $D_{\delta}$.

Now, $w_{\text{Chow}}$ is smashing and left non-degenerate by Proposition 2.6.1(2). Hence for (any object of $M$ of $\text{DM}_{\text{eff}}^R$ and) any choices of $w_{\text{Chow} \geq 0}M$ there exists a (countable homotopy colimit) distinguished triangle $\prod_{j \geq 0} w_{\text{Chow} \geq j}M \to \prod_{j \geq 0} w_{\text{Chow} \geq j}M \to M \to \prod_{j \geq 0} w_{\text{Chow} \geq j}M[1]$; see Theorem 4.1.3(1,2), Definition 4.1.1, and Remark 1.2.6(1) of [10]. Thus if we take $w_{\text{Chow} \geq 0}M$ that belong to $D_{\delta}$ then we conclude that $M$ belongs to $D_{\delta}$ as well. Consequently, condition C implies condition D.

Next, we prove the implication $D \Rightarrow E$ similarly to Proposition 2.3.2(9) of [11] (thus one can use weak weight complexes instead of strong ones in our proof). Recall that the functor $i^*\text{eff}$ is exact and respects coproducts. Since the class $C_{\delta}$ of objects of $(\text{Chow}_{\text{eff}}^R)$ that are isomorphic to the ones satisfying our effectivity assumptions on terms is obviously closed with respect to extensions and coproducts, the class $D'_{\delta}$ of those $N \in \text{Obj} \text{DM}_{\text{eff}}^R$ such that $t^{\text{eff}}(N) \subset C_{\delta}$ is closed with respect to extensions and coproducts as well. Since $D'_{\delta}$ obviously contains $\cup_i \text{Obj} \text{Chow}_{\text{eff}}^R(a_{\delta,i})[-i]$, we obtain $D_{\delta} \subset D'_{\delta}$.

For the remaining implication $E \Rightarrow A$ note that if $\tilde{t}(M) = (\tilde{M}^i)$ then $\text{CWH}_j(M_K)$ is a subquotient of $\mathbb{C}\text{holw}_j(\tilde{M}^i_K, R)$, and the latter group vanishes by Proposition 2.2.2.

2. Assume that $M$ satisfies Condition C of the previous assertion and belongs to $\text{DM}_{\text{eff}}^R(a_{\delta,-b})$. Then we can set $w_{\text{Chow} \leq 0}M = 0$. Moreover, condition D in condition 1 implies that $M$ is an object of $\text{DM}_{\text{eff}}^R(a_{\delta,-b})$; thus if we set $w_{\text{Chow} \leq 0}M = M$ and choose $w_{\text{Chow} \geq -i}M$ to belong to $\text{Obj} \text{DM}_{\text{eff}}^R(a_{\delta,i})$ for $-b < i < -a$ then this condition would be fulfilled for our choices of weight truncations for $-b < i < -a$.

Now we combine Proposition 2.6.1(4) with Proposition 2.5.1(3). Similarly to the proof of the implication $C \Rightarrow D$ in the previous assertion, we obtain that $w_{\text{Chow} \leq 0}M = M$ belongs the extension-closure in question.

3. Obviously, condition A of assertion 1 implies our condition $A'$, whereas conditions $C'$, $D'$, and $E'$ imply conditions $1.C$, $1.D$, and $1.E$, respectively. Thus it suffices to verify that these conditions are equivalent, and also imply the “bounded dimension” version of assertion 2.

Next, it can be easily checked that the arguments above carry over to our setting if one invokes the following statements: for any $j \geq 0$ the weight structure $w_{\text{Chow}}$ restricts to the (triangulated) subcategory of $\text{dim}_{\text{eff}} \text{DM}_R$ whose objects are the $j$-effective and $w_{\text{Chow}}$-bounded below (in $\text{DM}_{\text{eff}}^R$ $\supseteq \text{dim}_{\text{eff}} \text{DM}_R$) motives, and the heart of this restriction equals $(\text{dim}_{\text{eff}} \text{Chow}_R(a_{\delta,j}))[j]$. Now, this statement easily follows from Theorem 2.2 of [22] (along with its proof and Proposition 1.7 of ibid. that give the calculation of the heart).

We leave the detail for this argument to the reader, since we will not apply this assertion below. We only note that we propose to take the subcategories of dimension at most $r$ in the localizations of the type $\text{DM}_R$ in it and not to consider the corresponding localizations of $\text{dim}_{\text{eff}} \text{DM}_R$ (even though the latter can probably be used as well; cf. Proposition 2.2.6(7) of [1]).

4. Obvious; see condition A in assertion 1.

Remark 3.2.4. Taking $\mathcal{I} = \mathbb{Z} \times [0, +\infty)$ in our theorem we immediately obtain that any infinitely effective motif in $\text{DM}_{\text{eff}}^R_{w_{\text{Chow}}}$ is zero; cf. Remark 3.1.3(1).

Now we relate our theorem to higher Chow-weight homology.

Proposition 3.2.5. Let $\mathcal{I} \subset \mathbb{Z} \times [0, +\infty)$ and $M \in \text{DM}_{\text{eff}}^R_{w_{\text{Chow}}}$. Consider the following assumptions on $M$.

1. $\text{CWH}_j^i(M_K, R) = \{0\}$ for all $(i, j) \in \mathcal{I}$ and all function fields $K/k$.

2. For all rational extensions $K/k$ and $(i, j) \in \mathcal{I}$ we have $\text{CWH}_j^i(M_K, 1) = \{0\}$.

3. $\text{CWH}_0^i(M_K, j) = \{0\}$ for all $(i, j) \in \mathcal{I}$ and all function fields $K/k$.

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4. \(\text{CWH}_a^i(M_K, j-a) = \{0\}\) for all \((i, j) \in \mathcal{I}, a \in \mathbb{Z},\) and all function fields \(K/k.\)

Then the following statements are valid.

1. Condition 4 implies conditions 3 and 2, and either of the latter two conditions implies condition 1.

2. Let \(\mathcal{J}\) be a staircase set. Then our conditions 1–4 are equivalent.

**Proof.** 1. Obviously, condition 4 implies all other ones. The proofs of the remaining implications are similar to that in Proposition 3.4.1 of [1].

2. It remains to check that condition 1 implies condition 4. For this purpose we combine Theorem 3.2.3(1) (see Conditions A and D in it) with Proposition 3.1.2(1, 4).

**Remark 3.2.6.** Note that the implication 1 \(\Rightarrow\) 4 from Proposition 3.2.5 may be false if \(\mathcal{J}\) is not a staircase set. For example, take \(\mathcal{J} = [2, +\infty) \times [0, +\infty) \cup \{0\} \times [0, 5].\) Then the motif \(\mathbb{Q}(l)[-1]\) obviously satisfies the vanishing in condition 1; yet \(\text{CWH}_1^0(\mathbb{Q}(l)[-1], 1) \cong \mathbb{Q}.\)

### 3.3. On the Relation to Motivic Homology

Now we define certain “steep” staircase sets and birational motives.

**Definition 3.3.1.** 1. Let \((i_0, j_0) \in \mathbb{Z} \times [0, +\infty).\) Then we define \(S_{(i_0, j_0)} \subset \mathbb{Z} \times [0, +\infty)\) as the set \(\{(i, j) : i \geq i_0, 0 \leq j \leq j_0 + (i - i_0)\};\) we illustrate this definition by marking in grey the points of the set \(S_{(2,2)}\) on the following picture:

2. Let \(\mathcal{J}\) be a subset of \(\mathbb{Z} \times [0, +\infty)\) (see Subsection 2.1).

We will call it a *superstaircase* set if for any \((i_0, j_0) \in \mathcal{J}\) we have \(S_{(i_0, j_0)} \subset \mathcal{J}.\)

3. We will say that an object \(N\) of \(\text{DM}_{RM}^{eff}\) is *birational* if \(\text{Obj} \text{DM}_{RM}^{eff}(\mathcal{I}) \perp N.\)

Let us make some observations related to superstaircase sets and slices.

**Remark 3.3.2.** 1. Obviously, any superstaircase is a staircase one.

Moreover, if \((i, j) \in S_{(i_0, j_0)}\) then \(S_{(i, j)} \subset S_{(i_0, j_0)};\) it clearly follows that a subset of \(\mathbb{Z} \times [0, +\infty)\) is superstaircase if (and only if) it can be presented as the union of “sectors” \(S_{(i_0, j_0)}\) for some \((i_0, j_0) \in S_{(i_0, j_0)};\)

2. Consider the embedding \(\text{DM}_{RM}^{eff}(\mathcal{I}) \to \text{DM}_{MR}^{eff}\); the composition \(v^{21}\) of this inclusion with its right adjoint can clearly be described as \(\text{Hom}(R(\mathcal{I}), -)(\mathcal{I});\) see Proposition 4.6.2 of [24]. Moreover, for any object \(M\) of \(\text{DM}_{RM}^{eff}\) Proposition 4.6.2 of [26] gives the following *slice filtration* triangle

\[
v^{21}(M) \to M \to v^0(M) \to v^{21}(M)[l].
\] (3.3.1)

Recall also that the object \(v^0(M)\) is birational in the sense of Definition 3.3.1(3); see Lemma 4.5.4 of [24].

Now we relate the Chow-weight homology to higher motivic homology.
Theorem 3.3.3. Let \( \mathcal{J} = \cup S_{(t, r)} \) for some \( t, r \in \mathbb{Z} \), \( n_r \geq 0 \). Then the following assumptions on \( M \in DM_R^{\text{eff}} \) are equivalent:

(a) \( \text{CWH}_j^u(M_K) = \{0\} \) for all \( (i, j) \in \mathcal{J} \) and all function fields \( K/k \);
(b) \( \text{CWH}_j^w_r(M_K, R, -c) = \{0\} \) for all these \( K \), \( 0 \leq r \leq n_r \), and \( c \geq t \);
(c) \( \text{CWH}_j^w_r(M_K, R, -c) = \{0\} \) for all these \( K \) and \( (r, c) \in \mathcal{J} \).

Proof. Clearly, Condition (c) implies Condition (b).

Next, assume that Condition (a) is fulfilled and \( (i, j) \in \mathcal{J} \). Since \( \mathcal{J} \) is a superstaircase set, \( (u, j + u - i) \in \mathcal{J} \) whenever \( u \geq i \). Hence \( \text{CWH}_j^u(M_K, u - i) = \{0\} \) for all \( u \in \mathbb{Z} \); see conditions 4 and 1 in Proposition 3.2.5 and Proposition 3.1.2(4).

Next, the functor \( H = \text{CWH}_j^w_r(-K, R, u - i) \) kills \( DM_R^{\text{eff}} \) since \( w_{\text{Chow}} \) is a weight structure. Thus we have a converging Chow-weight spectral sequence

\[
T(H, M) : E_2^{a,b} = \text{CWH}_j^w(M_K, R, -q) \Rightarrow E_\infty^{a,b} = \text{CWH}_j^w(M_K, R, -u - q),
\]

that corresponds to \( w_{\text{Chow}} \); see Proposition 2.5.3. Applying the aforementioned vanishing of the groups \( \text{CWH}_j^u(M_K, u - i) = \{0\} \) we obtain \( \text{CWH}_j^w_r(M_K, R, -i) = \{0\} \); thus Condition (a) implies Condition (c).

It remains to verify that (b) implies (a). Since \( \mathcal{J} = \cup S_{(t, r)} \), it suffices to prove that this implication is valid for \( \mathcal{J} = S_{(t, r)} \), where \( t \in \mathbb{Z} \) and \( n \geq 0 \). Moreover, we can clearly assume that \( t = 0 \). Hence it remains to prove the following lemma.

Lemma 3.3.4. Assume that \( n \geq 0 \), \( M \in \text{Obj} DM_R^{\text{eff}} \), and \( \text{CWH}_j^w_r(M_K, R, -c) = \{0\} \) if \( c \geq 0 \), \( 0 \leq r \leq n \), and \( K/k \) is a function field.

Then \( \text{CWH}_j^u(M_K) = \{0\} \) for all \( (i, j) \in S_{(0, n)} \) (and all function fields \( K/k \)).

Proof. Let us prove the statement by induction on \( n \geq 0 \).

In the case \( n = 0 \) the statement is a simple generalization of Corollary 3.4.2 of [1]; we essentially repeat the argument here. By Proposition 5.1(3) below, our assumption implies that \( M \) belongs to \( DM_R^{\text{eff}, 1} \) (see Definition 3.1.1(4)). Thus it remains to apply Proposition 3.1.2(6). Now we assume that the statement in question is valid if \( n \leq m \) for some \( m \geq 0 \). We should verify it for \( n = m + 1 \). Since we have just proved it in the case \( n = 0 \), it suffices to verify that \( \text{CWH}_j^u(M_K) = \{0\} \) for \( (i, j - 1) \in S_{(0, n - 1)} \).

We take the slice filtration distinguished triangle (3.3.1) and denote \( \nu^0(M) \) and \( \nu^{21}(M) \) by \( M^0 \) and \( M^1 \), respectively. Then for any \( i \in \mathbb{Z} \), \( j \geq 0 \), and function field \( K \) (3.3.1) gives a long exact-sequence \( \cdots \rightarrow \text{CWH}_j^u(M_K) \rightarrow \text{CWH}_j^u(M_K) \rightarrow \text{CWH}_j^u(M_K^0) \rightarrow \cdots \) for Chow-weight homology. Thus it suffices to verify that \( \text{CWH}_j^u(M_K^1) = \text{CWH}_j^u(M_K^1) = \{0\} \) whenever \( (i, j - 1) \in S_{(0, n - 1)} \). Now, it is easily seen that for any \( i \in \mathbb{Z} \) and \( j \geq 1 \) we have \( \text{CWH}_j^u(M^1_K) = \text{CWH}_j^u(M^1_K) = \{0\} \) if \( (i, j - 1) \) belongs to \( S_{(0, n - 1)} \). Thus \( M^2 = \text{Hom}(R(1), M) \) (see Remark 3.3.2(2)) we have \( \text{CWH}_j^u(M^2_K, R, -i) = \{0\} \) whenever \( i \geq 0 \) and \( j - 1 \leq n - 1 \). Applying the inductive assumption we obtain \( \text{CWH}_j^u(M^2_K) = \{0\} \) if \( (i, j - 1) \in S_{(0, n - 1)} \). Now, Proposition 2.5.1(5) easily implies that \( \text{CWH}_j^u(-K) \circ (1) \cong \text{CWH}_j^u(-K) \); hence \( \text{CWH}_j^u(M^2_K) = \{0\} \) if \( (i, j - 1) \in S_{(0, n - 1)} \) (recall that \( M^1 \cong M^2(1) \)). Moreover, \( M^1 \) belongs to \( DM_R^{\text{eff}, \text{Chow}<1} \) since \( M \) does; see Corollary 3.3.7(2) (along with Theorem 3.3.1) of [18] or Proposition 5.1(3) below. It easily follows that \( M^0 \) belongs to \( DM_R^{\text{eff}, \text{Chow}<1} \) as well. Since \( M^0 \) is birational, it belongs to \( DM_R^{\text{eff}} \) by Lemma 3.3.5(2) below. Thus \( \text{CWH}_j^u(M^0_K) = \{0\} \) whenever \( i \geq 0 \) (see Proposition 3.1.2(4)) and we can conclude the proof.
Lemma 3.3.5. Let $N \in DM^{{eff}}_{R, w_{\text{chow}} \leq 0}$.

1. Then for any $j \geq 0$ there exists a choice of $w_{\text{Chow}} \leq j \ N$ that belongs to $\text{Obj} \ DM^{{eff}}_{R, j}$.

2. Assume in addition that $N$ is birational in the sense of Definition 3.3.1(3). Then $M \in DM^{{eff}}_{R, w_{\text{chow}} > 0}$.

Proof. 1. If $N \in DM^{{eff}}_{R, w_{\text{chow}} ^+}$ then the statement is an easy combination of the case $n = 0$ of Lemma 3.3.4 (note that this case of that lemma does not depend on our one) with Theorem 3.2.3(1) (see conditions A and C in it).

The argument for the general case is similar to the proof of ([11], Proposition 2.3.2(10)). Let us fix $j \geq 0$. Since the weight structure $w_{\text{Chow}}$ is smashing, the class $C$ of those $M \in \text{Obj} \ DM^{{eff}}_{R, j}$ such that there exists $w_{\text{Chow}} \leq j M$ that belongs to $\text{Obj} \ DM^{{eff}}_{R, j}$ is smashing (see Definition 2.1.2(1)) in $\mathcal{C}$; see Proposition 2.3.2(3) of loc. cit. Moreover, $C$ is extension-closed by Proposition 1.2.4(12) ibid. Recalling Proposition 3.1.2(6) once again we obtain that there exists $w_{\text{Chow}} \leq j M$ whenever $M \in \text{Obj} \ DM^{{eff}}_{R, j}$; thus $N$ belongs to $\mathcal{C}$ itself indeed.

Thus, we have completed the proof of Theorem 3.3.3.

Remark 3.3.6. 1. Below we will mostly apply Theorem 3.3.3 to geometric motives. Note however that our argument relies on slices; thus one cannot “apply it inside $DM^{{eff}}_{gm, R}$” (see [25]).

Moreover, the proofs of Theorems 3.2.3 and 3.3.3 hint that it can make sense to study conditions of these theorems for subcategories of $DM^{{eff}}_{R}$ that are bigger than $DM^{{eff}}_{R, w_{\text{chow}} ^+}$. In particular, one may treat the Voevodsky category $DM^{{eff}}_{R, \subset} \supset DM^{{eff}}_{R, w_{\text{chow}} ^+}$; cf. Subsection 2.3 of [23].

2. Since the slice functors are exact and respect coproducts, for any staircase set $\mathcal{J}$ an object $M$ of $DM^{{eff}}_{R}$ belongs to the class $D_\mathcal{J}$ (see condition D in Theorem 3.2.3(1)) if and only if $v_0^j (M)$ and $v_\geq 1^j (M)$ do. Moreover, similar implications hold for other “slices” of $M$. We leave the detail for these statements to the reader.

3. Obviously, there are plenty of staircase subsets of $\mathbb{Z} \times [0, +\infty)$ that are not superstaircase ones. However, the only “concrete” staircase of this sort that were considered in [1] are sets of the form $\mathbb{Z} \times [0, c - 1]$ (for $c > 0$). They correspond to the $c$-effectivity of motives; cf. Remark 3.3.2(2) of ibid.

4. Now we demonstrate that Theorem 3.3.3 does not extend to the case where $\mathcal{J}$ is an arbitrary subset of $\mathbb{Z} \times [0, +\infty)$. Let $R = \mathbb{Q}$ and $M = \mathbb{Q}(\{1\} [-1]$. Then $\text{CWH}^\mathcal{J} (M) = \{0\}$ for $(i, j) \neq (1, 1)$.

Next, assume that $k$ is not a union of finite fields, and $\mathcal{J} = [0, +\infty) \times [0, +\infty) \setminus \{(1, 1)\}$. Then $\text{CWH}^0_0 (N, Q) = k^\times \otimes Q \setminus \{0\}$; yet $(0, 0) \in \mathcal{J}$.

5. Assume $M \in DM^{{eff}}_{R, w_{\text{chow}} ^+}$, $t \geq 0$, and for any $t \geq 0$ the $E_2^{*, *}$ terms of the Chow-weight spectral sequence $T_{w_{\text{chow}}} (H, M)$ for the homological functor $DM^{{eff}}_{R, t} (R(t), -)$ are concentrated in the first quadrant (in particular, this is the case if $M \in DM^{{eff}}_{R, w_{\text{chow}} ^+}$). Thus we have the so-called five-term exact sequence:

$$0 \to \text{CWH}^0 (M) \to \text{CWH}_{w_{\text{chow}}}^0 (M, -1) \to \text{CWH}^1 (M, -1) \to \text{CWH}^2 (M) \to \text{CWH}_{w_{\text{chow}}}^0 (M, -2).$$

Clearly one may obtain some homology vanishing statements from this sequence.
4. APPLICATIONS TO GEOMETRIC MOTIVES

In this section we apply Theorem 3.3.3 to obtain some new statements on geometric motives. We argue similarly to [1].

In Subsection 4.1 we combine our Theorem 3.2.3 with the results of [1, Subsection 3.6]; this roughly gives the finiteness of exponents of higher Chow-weight homology and lower motivic homology groups provided that they are torsion.

In Subsection 4.2 we apply our results to the motif with compact support of a variety \(X\). It follows that if certain Chow homology groups of \(X\) (over a universal domain \(K\) containing \(k\)) are torsion then they are of finite exponent, and also estimates the effectivity of the corresponding Deligne weight factors of singular and étale cohomology of \(X\) with compact support.

4.1. On Chow-weight and Motivic Homology of Bounded Exponent

Now we apply our results to geometric motives.

**Theorem 4.1.1.** Let \(M \in \text{Obj} \ DM_{gm,\mathbb{Z}}^\text{eff}(\mathbb{Q})\), \(K\) be a universal domain (that is, \(K\) is an algebraically closed field that is of infinite transcendence degree over its prime subfield) containing \(k\), and \(\mathcal{F} = \cup S_{(t,n)}\) (see Definition 3.3.1(1)).

The following conditions are equivalent.

1. \(\mathcal{C} \text{H}o\mathcal{I}o\mathcal{V}_r((M \otimes \mathbb{Q})_K, \mathbb{Q}, -c) = \{0\}\) for \(0 \leq r \leq n_s\) and \(c \geq n_s\); here \(M \otimes \mathbb{Q}\) is the result of the application to \(M\) of the extension of scalars functor \(- \otimes \mathbb{Q} = \otimes_{\mathbb{Z}[1/p]} \mathbb{Q}: DM_{gm,\mathbb{Z}[1/p]}^\text{eff} \to DM_{gm,\mathbb{Q}}^\text{eff}\) provided by Proposition 3.6.2(I.1) of [1].

2. There exists \(E_M > 0\) such that \(E_M \mathcal{C} \text{H}o\mathcal{I}o\mathcal{V}_r(M_k', -c, \mathbb{Z}[1/p]) = \{0\}\) for any \((r,c) \in \mathcal{F}\) and any field extension \(k'/k\).

3. \(\mathcal{C} \text{WH}^j((M \otimes \mathbb{Q})_K, \mathbb{Q}) = \{0\}\) whenever \((i,j) \in \mathcal{F}\).

4. There exists \(E_M' > 0\) such that \(E_M' \mathcal{C} \text{WH}^{j-a}(M_k', a) = \{0\}\) for all \(a \in \mathbb{Z}, (i,j) \in \mathcal{F}\), and all field extensions \(k'/k\).

**Proof.** Condition 2 obviously implies condition 1.

Next, Proposition 2.3.4(II) of [1] implies that condition 1 (resp. 3) is fulfilled if and only if we have similar vanishing over any function field \(k'/k\). Hence these conditions are equivalent according to Theorem 3.3.3 applied to the motif \(M \otimes \mathbb{Q} \in \text{Obj} \ DM_{\mathbb{Q}}^\text{eff} \).

Moreover, conditions 3 and 4 are equivalent according to Theorem 3.6.4 of ibid. (see condition II.B in it).

Lastly we argue similarly to the proof of Corollary 3.6.5(II) of ibid. As we have already noted in the proof of Theorem 3.3.3, we have a convergent Chow-weight spectral sequence \(T(H, M)\) as follows:

\[E_2^{u,q}T(H, M) = \mathcal{C} \text{WH}^{u+q}(M_k, \mathbb{Z}[1/p], -q) \Rightarrow E_\infty^{u+q} = \mathcal{C} \text{H}o\mathcal{I}o\mathcal{V}_r(M_k, \mathbb{Z}[1/p], -u - q).\]

Next, the complex \(i^*(M)\) is isomorphic to a bounded one (see Definition 3.1.1(1) and Proposition 2.2.1(1) of [1]); hence the simple index computation made above yields that for \((r,c) \in \mathcal{F}\) the group \(\mathcal{C} \text{H}o\mathcal{I}o\mathcal{V}_r(M_k, -c, \mathbb{Z}[1/p])\) possesses a filtration of a uniformly bounded length whose factors are killed by the multiplication by \(E_M'\). Thus we can take \(E_M\) to be a high enough power of \(E_M'\).

**Remark 4.1.2.** It is quite remarkable that certain Chow-weight homology has finite exponents. Note that (in general) Chow-weight homology groups and motivic homology of geometric motives can have really “weird” torsion.
4.2. Applications to Motives with Compact Support

Let us apply our results to motives with compact support. Let us recall some basics on these motives.

**Proposition 4.2.1.** 1. There exists a functor $M_{\text{gm}}^{c,R}$ of the motif with compact support from the category $\text{SchPr}$ of $k$-varieties with morphisms being proper ones into $DM_{\text{gm},R}^{\text{eff}}$.

2. For any $j, l \in \mathbb{Z}$, $X \in \text{Var}$, $M = M_{\text{gm}}^{c,R}(X)$, and any field extension $k'/k$ the group $\hat{\mathcal{H}}_{\text{eff}}(M_{k'}, R/l)$ (see Definition 2.2.1(3)) is naturally isomorphic to the higher Chow group $CH^j_{\text{eff}}(X_{k'}, l, R)$ (cf. Theorem 5.3.14 of [5] for the $R = \mathbb{Z}[1/p]$-version of this notation).

**Proof.** These statements easily follow from their $\mathbb{Z}[1/p]$-linear versions provided by Subsection 5.3 of [5] along with (Proposition 1.3.3 of [4] and) Proposition 5.1(2) below; see Proposition 4.1.2 of [1].

**Remark 4.2.2.** Recall that actually the functor $M_R$ is defined on the category of all $k$-varieties, and we have $M_R(X) = M_{\text{gm}}^{c,R}(X)$ whenever $X$ is proper (see Proposition 5.3.5 of [5]). In particular, $M_R(X) = M_{\text{gm}}^{c,R}(X)$ if $X$ is smooth projective.

Now we combine Theorem 3.3.3 with certain results of [1] to obtain an extension of Theorem 4.2.1 of ibid. Note that this statement does not mention Chow-weight homology.

**Theorem 4.2.3.** Assume that $X \in \text{Var}$, $K$ is a universal domain containing $k$, and for some set of $\{ (t_i, n_i) \} \subset \mathbb{Z} \times [0, +\infty)$ we have $CH_r(X_{K}, \mathcal{S} \in \mathfrak{F}) = \{ 0 \}$ whenever there exists $s$ such that $0 \leq r \leq n_i$ and $c \geq t_i$.

1. Then there exists $X^\mathcal{F}_0 > 0$ such that $X^\mathcal{F}_0 CH_r(X_{K}, \mathcal{S} \in \mathfrak{F}, \mathbb{Z}[1/p]) = \{ 0 \}$ for all $r < \infty$, and any field extension $k'/k$, where $\mathcal{F} = \bigcup_{(t_i, n_i)}$ (see Definition 3.3.1(1)).

2. If $k$ is a subfield of $\mathbb{C}$ and $l, m \in \mathbb{Z}$ then the $m + 1$-th (Deligne) weight factor of $H^m_c(X_k)$ of the (\mathbb{Q}\text{-linear}) singular cohomology of $X_k$ with compact support is $d_{j,k}$-effective as a pure Hodge structure; see Definition 3.2.1(1) above and (1), Definition 3.5.3, Theorem 3.5.4(2)) for the corresponding definitions.

Moreover, the same effectivity properties hold for Deligne weight factors of étale cohomology $H^j_{\text{et}}(X_{\text{et}}, \mathbb{Q}_l)$ if $k$ is the perfect closure of a field that is of finite transcendence degree over its prime subfield.

**Proof.** 1. Immediate from Proposition 4.2.1(2) combined with Theorem 4.1.1 (see conditions 1 and 2 in it).

2. For $M = M_{\text{gm}}^{c,Q}(X)$ we have $\text{CWH}^j_{\text{eff}}(M_{k}) = \{ 0 \}$ if $(i, j) \in \mathfrak{F}$; see condition 4 of that theorem (or Theorem 3.3.3). Given this statement, one can argue similarly to Theorem 4.2.1(2) of [1].

**Remark 4.2.4.** 1. Moreover, for $X$ as above, $M = M_{\text{gm}}^{c,Q}(X)$, and any cohomological functor $H$ from $DM_{\text{gm},Q}^{\text{eff}}$ into an abelian category $\mathcal{A}$ one can combine the Chow-weight homology vanishing mentioned in the proof with Proposition 3.5.1(1) of ibid. to obtain the following: for any $l, m \in \mathbb{Z}$ both $E_2^{-l,m} H^l(M)$ and $(G_{n-l} H_m(M))$ are subquotients of $H^m(M_{l}(P)(a_{j,l}))$ for some $P \in \text{SmPrVar}$ whenever $a_{j,l} < +\infty$, and these two objects vanish if $a_{j,l} = +\infty$; see Definition 1.4.4(2) and Proposition 1.4.5(2) of ibid. for the corresponding notation.

2. The combination of two or more or less “standard” motivic conjectures yields that the Hodge effectivity condition in Theorem 4.2.3(2) is actually equivalent to our assumptions on Chow groups of $X$. This statement is an easy implication of Proposition 3.5.6 of [1] (combined with our Theorem 3.3.3).

3. One can certainly consider Chow-weight spectral sequences for non-geometric objects of $DM_{Q}^{\text{eff}}$ (or $DM_{R}^{\text{eff}}$ for any $R$). In particular, one may extend to $DM_{Q}^{\text{eff}}$ singular and étale homology functors similar to the ones mentioned in Theorem 4.2.3(2). Note here that there exist homological functors of this sort that take values in the corresponding ind-completed categories and respect coproducts; see Lemma 2.2 of [26].

An important observation here is that these functors convert objects of $\text{Chow}_{Q}^{\text{eff}}$ into (ind-pure) objects of weight 0 in the corresponding mixed categories; hence these weight spectral sequences degenerate at $E_2$ (cf. Theorem 3.5.4 of [1]).
5. SOME MOTIVIC STATEMENTS

Let us prove some statements that were used both in [1] and in the current paper. The authors were not able to find these formulations (over fields) in the literature; yet certainly no originality is claimed.

We will not introduce any notation or definitions that will be used below; most of it can be found in [16] (cf. also Theorem 1.2.1 of [29]).

**Proposition 5.1.** Let $K/k$ be an extension of perfect fields, $f : \text{Spec} K \to \text{Spec} k$ is the corresponding morphism, and $X$ is a $k$-variety.

Then the following statements are valid.

1. The functor $- \otimes_k$ in Definition 2.2.1(2) is essentially the restriction to $\text{DM}^\text{eff}_R$ of the functor $f^* : \text{DM}^\text{eff}_R \to \text{DM}^\text{eff}_{R(K)}$, and we have $f^*(M_R(X)) \cong M_R(X_k)$.

2. Moreover, $f^*(M_R^c(X)) \cong M^c_R(X_K)$.

3. For an object $N$ of $\text{DM}^\text{eff}_R$ we have $\text{DM}^\text{eff}_R \otimes_{\text{DM}^\text{eff}_{R(K)}} = 0$ if and only if $\langle \text{Chow}_0(N_k); E, l \rangle = [0]$ for all $l < 0$ and all function fields $k'/k$.

Moreover, these conditions are equivalent to the vanishing of $\langle \text{Chow}_0(N_k); E, l \rangle = [0]$ for all $l < 0$, $r \geq 0$, and all function fields $k'/k$.

**Proof.** 1. We pass to the “stable” motivic category $\text{DM}^\text{eff}_R \cong \text{DM}_{cdh}(\text{Spec} K, R) \supseteq \text{DM}^\text{eff}_{R(K)}$ (see Remark Subsection 1.2, Definition 1.5, and Proposition 8.1(c) of [16]). Then the second part of the assertion implies that we can define $- \otimes_k$ as the restriction to $\text{DM}^\text{eff}_R$ of the functor $f^*$ indeed.

To obtain the isomorphism in question we recall that $M_R(X)$ can be computed as $\chi^X(R)$, where $x : X \to \text{Spec} k$ is the structure morphism; see the formula (8.7.1) (and Subsection 1.6) of ibid. We take the corresponding Cartesian square

\[
\begin{array}{ccc}
X_k & \xrightarrow{f_X} & X \\
\downarrow x_k & & \downarrow x \\
\text{Spec} K & \xrightarrow{f} & \text{Spec} k
\end{array}
\]

and recall that the categories $\text{DM}_{cdh}(-, R)$ give a motivic category over the category of noetherian $k$-schemes of finite dimension that is continuous with respect to the twists ($n$) for $n \in \mathbb{Z}$; see Proposition 4.3 and Theorem 5.1 of ibid. for this statement, and Definitions 2.4.45 and 4.3.2 of [17] for the corresponding definitions. Thus we can apply the base change isomorphism (see Theorem 2.4.50(4) of ibid.) to obtain $f^*X_k \cong f_k^* x_k^*$.

Next, the motivic category $\text{DM}_{cdh}(-, R)$ is generated by the aforementioned twists (see Definition 2.5 and Proposition 4.3 of [16]) and the morphism $f$ is regular immediately from the Popescu–Spivakovsky theorem (see Section 4.1.5 of [17]). Thus we can apply Propositions 4.3.12 of ibid. to obtain $f^*_k x_k \cong x_k^* f^*$. It remains to recall that $f^*R_{\text{Spec} k} \cong R_{\text{Spec} k}$ (see Subsection 1.1.1 of ibid.); hence $f^*(M_R(X)) \cong x_k^* f^*_k R_{\text{Spec} k} \cong M_R(X_K)$ indeed.

2. The proof differs just a little from the previous one. Proposition 8.10 of [16] says that $M^*_{\text{gen}}(X)$ can be computed as $\chi^X(R)$; here we use the notation of [17] (and write $x_k^*$ instead of $R_{\text{Spec} k}$). Next, the observations made above yield that we can apply Proposition 4.3.15 of ibid. to obtain $f^* x_k \cong x_k f^*_k$. Combining this statement with the isomorphisms mentioned above we obtain $f^*(M^*_{\text{gen}}(X)) \cong x_k^* f_k^* x_k^* R_{\text{Spec} k} \cong M^*_{\text{gen}}(X_K)$ indeed.

3. $N$ belongs to $\text{DM}^\text{eff}_R \otimes_{\text{DM}^\text{eff}_{R(K)}} = 0$ if and only if for any function field $k'/k$, any presentation $\text{Spec} k' = \lim X_i$ for $X_i \in \text{SmVar}$ (recall that a presentation of this type exists since $k$-varieties are generically smooth), $r \geq 0$, and $l < 0$ we have $\text{DM}^\text{eff}_R (M_{R(X)}(r)[l-r], N) = [0]$; see Corollary 5.2, Subsection 1.18, and Theorem 3.7 of [27].

---

6 Alternatively, one may apply Corollary 2.3.12 and Theorem 3.3.1 of [18]; see also Definitions 1.3.10 and 3.2.3, and the formula (2.3.4(a)) of ibid.
Next we recall that the motives $M_R(X_j)$ can also be computed as $x_j^*: R_{X_j} \rightarrow \text{Spec } k$ are the structure morphisms and $x_{j*}$ are left adjoint to $x^{\#}_j$; see the formula (8.5.3) of [16] (yet we omit $L$’s in the notation of loc. cit.). Thus $DM^\text{eff}(M_R(X_j) \langle r \rangle |l - r], N) \cong DM_{\text{cdh}}(X_j, R) \langle R_{X_j} \langle \langle r \rangle |l - r], x_N^* R)$. We can pass to the limit in this isomorphism using the aforementioned continuity property (see Definition 2.5 of ibid.) to obtain

$$\lim_{\text{lim}} DM_{\text{cdh}}(X_j, R) \langle R_{X_j} \langle \langle r \rangle |l - r], x_N^* R) \cong DM_{\text{cdh}}(\text{Spec } k, R) \langle R_{\text{Spec } k} \langle \langle r \rangle |l - r], f^* R_N).$$

The latter group is isomorphic to $\mathbb{C} \int_{\text{mot}}(N_{k^r}, R, l - r)$ since we can replace $k$ by its perfect closure; see Proposition 8.1 of [16].

It suffices to verify the vanishing in question for $r = 0$. This statement easily follows from Proposition 5.2.6(8) and Remark 5.2.7(7) of [28]. Moreover, it can be easily deduced from the well-known Theorem 4.19 of [30, pp. 87–137] along with the fact that $R(n)[-n]$ is a retract of $M_R(\mathbb{G}^n_m)$ (where $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$).

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REFERENCES

1. M. V. Bondarko and V. A. Sosnilo, “On Chow-weight homology of geometric motives,” Preprint (2020). https://www.researchgate.net/publication/340849991_On_Chow-weight_homology_of_geometric_motives. Accessed September 7, 2020.

2. M. V. Bondarko and V. A. Sosnilo, “Detecting the c-effectivity of motives, their weights, and dimension via Chow-weight (co)homology: A "mixed motivic decomposition of the diagonal,“ Preprint (2014). https://arxiv.org/abs/1411.6354.

3. S. Bloch, Lectures on Algebraic Cycles (Duke Univ., Durham, N. C., 1980), in Ser.: Duke University Mathematics Series, Vol. 4.

4. M. V. Bondarko and D. Z. Kumallagov, “On Chow weight structures without projectivity and resolution of singularities,” St. Petersbourg Math. J. 30, 803–819 (2019).

5. S. Kelly, “Voevodsky motives and ldh-descent,” Asterisque 391, 1–134 (2017).

6. M. V. Bondarko and V. A. Sosnilo, “On constructing weight structures and extending them to idempotent extensions,” Homol., Homotopy Appl. 20 (1), 37–57 (2018).

7. J. Wildeshaus, “Chow motives without projectivity,” Compos. Math. 145, 1196–1226 (2009).

8. M. V. Bondarko, “Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general),” J. K-Theory 6, 387–504 (2010). https://arxiv.org/abs/0704.4003.

9. D. Pauksztello, “Compact cochain objects in triangulated categories and co-t-structures,” Cent. Eur. J. Math. 6, 25–42 (2008).

10. M. V. Bondarko and V. A. Sosnilo, “On purely generated $\alpha$-smashing weight structures and weightexact localizations,” J. Algebra 535, 407–455 (2019).

11. M. V. Bondarko, “On weight complexes, pure functors, and detecting weights,” Preprint (2018). https://arxiv.org/abs/1812.11952.

12. M. V. Bondarko and V. A. Sosnilo, “Non-commutative localizations of additive categories and weight structures; applications to birational motives,” J. Inst. Math. Jussieu 17, 785–821 (2018).

13. V. A. Sosnilo, “Theorem of the heart in negative $K$-theory for weight structures,” Doc. Math. 24, 2137–2158 (2019).

14. H. Gillet and C. Soulé, “Descent, motives and $K$-theory,” J. Reine Angew. Math. 478, 127–176 (1996).

15. A. Beilinson and V. Vologodsky, “A DG guide to Voevodsky motives,” Geom. Funct. Anal. 17, 1709–1787 (2008).

16. D.-C. Cisinski and F. Déglise, “Integral mixed motives in equal characteristic,” Doc. Math. Extra Volume, 145–194 (2015).

17. D.-C. Cisinski and F. Déglise, Triangulated Categories of Mixed Motives (Springer-Verlag, Cham, 2019), in Ser.: Springer Monographs in Mathematics.

18. M. V. Bondarko and F. Déglise, “Dimensional homotopy t-structures in motivic homotopy theory,” Adv. Math. 311, 91–189 (2017).
19. M. V. Bondarko and V. A. Sosnilo, “On the weight lifting property for localizations of triangulated categories,” Lobachevskii J. Math. 39, 970–984 (2018).
20. M. V. Bondarko, “$\mathbb{Z}[1/p]$-motivic resolution of singularities,” Compos. Math. 147, 1434–1446 (2011).
21. M. V. Bondarko, “Differential graded motives: Weight complex, weight filtrations and spectral sequences for realizations: Voevodsky vs. Hanamura,” J. Inst. Math. Jussieu 8, 39–97 (2009). https://arxiv.org/abs/math.AG/0601713.
22. J. Ayoub, “Motives and algebraic cycles: A selection of conjectures and open questions,” in Hodge Theory and $L^2$-Analysis, Ed. by L. Ji (International, Somerville, Mass., 2017), in Ser.: Advanced Lectures in Mathematics, vol. 39, pp. 87–125.
23. M. V. Bondarko, “Intersecting the dimension and slice filtrations for motives,” Homol., Homotopy Appl. 20 (1), 259–274 (2018).
24. B. Kahn and R. Sujatha, “Birational motives, II: Triangulated birational motives,” Int. Math. Res. Not. 2017, 6778–6831 (2017).
25. J. Ayoub, “The slice filtration on $DM(k)$ does not preserve geometric motives. Appendix to A. Huber’s ”Slice filtration on motives and the Hodge conjecture,” Math. Nachr. 281, 1764–1776 (2008).
26. H. Krause, “Smashing subcategories and the telescope conjecture — An algebraic approach,” Invent. Math. 139, 99–133 (2000).
27. F. Déglise, “Modules homotopiques (Homotopy modules),” Doc. Math. 16, 411–455 (2011).
28. M. V. Bondarko, “Gersten weight structures for motivic homotopy categories; retracts of cohomology of function fields, motivic dimensions, and coniveau spectral sequences,” Preprint (2018). https://arxiv.org/abs/1803.01432.
29. M. V. Bondarko and M. A. Ivanov, “On Chow weight structures for cdh-motives with integral coefficients,” St. Petersburg Math. J. 27 (6), 869–888 (2016).
30. V. Voevodsky, A. Suslin, and E. Friedlander, Cycles, Transfers and Motivic Homology Theories, Annals of Mathematical studies, vol. 143 (Princeton Univ. Press, Princeton, NJ, 2000).