NON STANDARD SPIN 2 FIELD THEORY.

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It is usually accepted that General Relativity is the only consistent theory which can be obtained starting from the linear Fiertz-Pauli lagrangian. It is the aim of the present paper to study whether, under certain requirements, a different and consistent field theory can be found. These requirements will be the common ones encountered in flat field theory: removal of the non physical degrees of freedom and conservation of the energy and momentum currents determined from Noether’s Theorem. It will be shown that imposing certain constraint (related to the elimination of the undesired components of the reducible representation) on the field manifold, a consistent theory (at least to first order in nonlinearities) is achieved. The theory obtained proceeding this way is characterized, to the lowest non linear order, by certain parameter $\epsilon$. General Relativity’s corresponding term is found to be the limit case of our non standard theory when $\epsilon \to 0$. So, $\epsilon$ measures at this level the size of the breaking of the global symmetry appearing in General Relativity i.e. diff. invariance. It remains as open questions the matters of the new theory’s solvability to all orders and the appearence of it’s quantized version.

1 Introduction.

Since the times of Kraichnan [1], Gupta [2], Feynman [3] etc, the process for re-obtaining Einstein’s General Relativity departing from an (inconsistent) spin 2 linear field theory in the minkowskian space is widely known. We are aware that in the transit to a consistent theory many of the starting model’s essential features are lost: the minkoskian space turns to be a pseudoriemannian one, the interpretational meaning of coordinates changes, the global Lorentzian simmetry gives way to a double fold symmetry (local Lorentz invariance plus global diffeomorphisms), and so on. It is the purpose of the present paper to study the possibility of finding a nonlinear spin 2 consistent theory fulfilling the usual requirements encountered in flat non geometrical field theory, and to establish which are the differences between our non standard construction and Einstein’s geometrical one. We will not accept any arbitrariness on the form of the expressions defining the theory and the determination

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of all the lagrangian terms will be based on consistency or physical grounds.

In section 2 a general review is made of the essential ingredients and requirements met in classical field theory. Special emphasis will be done on gauge reduction, responsible of the non physical degrees of freedom removal, and on the definition of the mathematical object which correctly describes the energy an momentum currents \(i.e.\) the energy momentum tensor, from here called e.m.t.). These two points will be shown to be crucial in our non standard construction. Section 3 is just the application of section 2 to a well known and simple example: electrodynamics. Section 4 reviews the celebrated inconsistency of the linear spin 2 field theory. Mention is made of Deser’s first order formalism outlook [4] and special prominence is given to Feynman’s approach [3] and to the consistency condition therein obtained. We also remind that it is commonly accepted that General Relativity is the only proper theory fulfilling this condition. Section 5 deals with the topic of trying to write a spin 2 field theory satisying section 2 demands. We show that there is certain definition of the field manifold that solves the problems of gauge reduction, Feynman’s consistency condition and is compatible (at least to first order in the nonlinearities) with the conservation criterion stated in section 2.

2 Ingredients and requirements in a relativistic field theory.

We overview the essential ingredients and requirements encountered in classical field theory. Points 2-i) to 2-iv) deal with the ingredients, while 2-v) and 2-vi) do with the requirements.

2-i) Our departure arena is the minkowskian manifold \(M_4\) and the corresponding coordinatizations given by the assembly of inertial observers. As usual, those coordinatizations are related via transformations of the Poincare group.

2-ii) We contemplate the case of \(N\) particles, characterized through their \(M_4\) trajectories \(\{X_i\}_{i=1,...,N} ; X_i : \mathbb{R} \to M_4\). We denote their components in a given inertial system by \(z^\mu_i(\lambda)\), where \(\lambda\) is the line parameter, usually arc length. We also consider several quantities \((\psi^1_i,...,\psi^k_i)\) related to the \(i\)-th particle which are found to be outstanding for a given theory. For instance, when studying electrodynamics we choose \(\psi^1_i = m_i\) \(y\) \(\psi^2_i = q_i\), mass and charge of the \(i\)-th particle. Of course, the generalization of these quantities to the case of finite distributions is immediate introducing the densities \((\psi^1_i(p),...,\psi^k_i(p))\) \(p \in M_4\).

2-iii) We also deal with a manifold \(B\) of fields defined over \(M_4\). The ”points” of \(B\) are the precise values of the fields: \(\{B_j\}_{j=1,...,Q}\) where \(B_j : M_4 \to W_j\) and \(W_j\) is
some representation space. In the theories we shall consider, the fields $B_j$ transform as Lorentz or Poincare representations.

2-iv) The system’s dynamics is determined via the specification of the action functional $S$. It is generally understood that $S$ can be broken into three pieces, the particles’, the field’s and the interaction particles-field’s term:

$$S = S_P + S_F + S_I$$

As usual, the motion an field equations are obtained equating to zero the corresponding ”fixed” functional variations

$$\delta|_{\text{fields fixed}}(S) \equiv \delta S_P + \delta|_{\text{fields fixed}}(S_I) = 0$$

$$\delta|_{\text{traj. fixed}}(S) \equiv \delta S_F + \delta|_{\text{traj. fixed}}(S_I) = 0.$$ 

2-v) Both of the action’s addends ($S_F$ and $S_I$) where the fields $B_j$ appear are written as either line integrals or 4-integrals over $M_4$. So some of the field indices are saturated with the ones belonging to the 4-vector $\dot{z}^\mu$ or with the derivative operator $\partial_\mu$. Therefore it is easily understood that the natural representations fitting the action are tensorial ones. It is well known that these are neither Poincare nor Lorentz irreps. However we usually want to deal with fundamental fields, which are labeled by definite Cassimir values (mass and spin) and are found to be irreducible representations of the Poincare group. Therefore the mathematical objects we handle when writing the action are carriers of superfluous degrees of freedom. This drawback is settled introducing some kind of ”gauge reduction” i.e. certain internal transformation $\{B_j\} \rightarrow \{B'_j\}$ leaving the action invariant: $S(B) = S(B')$. This invariance allows to remove the unwished degrees of freedom. As we shall later see, this is not the only acceptable way to proceed, and the gauge reduction condition can be performed in different manners. Nevertheless we impose the following requirement on our theory.

Requirement 1 - A classical field theory should be such that it enables the removal of the undesired degrees of freedom.

2-vi) The underlying Poincare symmetry determines the existence of two essential objects related to conserved quantities. Associated to the homogeneous part of the group we have the angular momentum tensor, while the e.m.t. is the object which takes account of translational invariance. We shall study how to define the e.m.t. and under what conditions it satisfies the conservation requirement.

For both free particles ($S = S_P$) and free fields ($S = S_F$) use can be done of Noether’s Theorem machinery in order to determine the corresponding conserved current. In first case we obtain the particle’s e.m.t.

$$\tau_{\mu\nu} = m c \delta^{(3)}(\vec{r} - \vec{r}(x_0)) \dot{z}_\mu \dot{z}_\nu \frac{ds}{dx^0}$$  \hspace{1cm} (2.1)
where $\vec{r}(x_0)$ is the 3-trajectory of the mass $m$ particle and $\dot{z}_\mu$ its 4-speed. Generalization to $N$ particles is achieved adding the label $i$ to $\{m, \vec{r}(x_0), z_\mu\}$; $i = 1, \ldots, N$; and to the discrete case introducing a matter distribution function. In the free particle case $\ddot{z}_\mu = 0$ and the conservation condition is automatically fulfilled

$$\tau_{\mu,\nu} = 0.$$  

In the free field context we proceed similarly and, making use of Noether’s Theorem, the following conserved tensor is obtained (see [5])

$$T^{\mu\nu} = L_\alpha^{\mu} B^{\alpha,\nu} - L F \eta^{\mu\nu} - f^{\mu\sigma\nu,\sigma}, \quad (2.2)$$

where $B^\alpha$ are the fields, $L_F$ is the lagrangian field density and

$$L_\alpha^{\mu} \equiv \frac{\partial L_F}{\partial B^{\alpha,\mu}},$$

$$f^{\mu\sigma\nu,\sigma} \equiv \frac{1}{2} (L_\alpha^{\mu} S^{\alpha\sigma\rho} + L_\sigma^{\sigma\rho} - L_\rho^{\sigma\mu}) B^\beta, \quad (2.3)$$

with $S^{\alpha\sigma\rho}$ the quantities related to the infinitesimal Poincare transformations of the fields via

$$B^{\alpha}(x') = \left[ \delta^\alpha_\beta + \frac{1}{2} S^{\alpha\mu\nu} \omega_{\mu\nu} \right] B^\beta(x)$$

and $\omega_{\mu\nu}$ the parameters of the infinitesimal homogeneous transformation.

Making use of the free field’s equations is not hard to see that $T^{\mu\nu}$ identically satisfies

$$T^{\mu\nu},\mu = 0$$

We agree on calling the object (2.2) the true energy momentum tensor of the fields. Due to historical reasons it is known in the literature as Belifante’s e.m.t. It is important to draw attention to the fact that (2.2) is precisely the object obtained when applying Noether’s Theorem to the translational invariant $L_F$. It should also be emphasized that, despite what is quite often asserted, the last addend appearing in (2.2) is not introduced \textit{ad hoc} to symetrize the canonical e.m.t. It naturally appears as a consequence of dealing with Noether’s Theorem’s machinery. As a fact $f^{\mu\sigma\nu,\sigma}$ takes into account the difference $B^{\beta}(x') - B^\beta(x)$ \textit{i.e.} the non scalar nature of the fields.
We know that the quantities \( T^{\mu\nu} \) are not observable. Nevertheless, the ones obtained through the 3-integration

\[
P^\nu = \int d\sigma_\mu T^{\mu\nu}, \tag{2.4}
\]

are by theirselves observable. Of course, \( P^\nu \) is the energy momentum 4-vector of the free field.

(2.4) suggests the following definition. We say that the expressions \( H^{\mu\nu} \) and \( G^{\mu\nu} \), both written in terms of the field \( B^\alpha \), belong to the same 1-orbit if

\[
\int d\sigma_\mu H^{\mu\nu} = \int d\sigma_\mu G^{\mu\nu} + A, \tag{2.5}
\]

where \( A \) is an arbitrary constant and it is understood that use has been made of the free field’s equations in the equality (2.5). Taking into account that the observable object is not \( T^{\mu\nu} \) but \( P^\nu \) we realize that, in the free field case, \( T^{\mu\nu} \) given by (2.2) is not the only acceptable object which describes energy momentum densities, but any other construction belonging to the same 1-orbit. For instance, the canonical e.m.t. \( \tilde{T}^{\mu\nu} \), which is nothing but the part of \( T^{\mu\nu} \) that does not deal with the non scalar nature of the fields i.e.

\[
\tilde{T}^{\mu\nu} = L_\mu B^{\alpha\nu} - L_F \eta^{\mu\nu},
\]

belongs to the same 1-orbit as \( T^{\mu\nu} \) and both are equally acceptable for describing the free field. \( T^{\mu\nu} \) is usually preferred to the canonical e.m.t. because the first one’s construction ensures symmetry under the exchange \( \mu \leftrightarrow \nu \) and also because \( T^{\mu\nu} \) is the object which naturally appears imposing the condition \( \bar{\delta}S = 0 \), where \( \bar{\delta} \) stands for an infinitesimal Poincare transformation. It should be stressed that if no transit is made to the interaction case, it is meaningless to distinguish between different elements in a 1-orbit, as far as (before switching on the interaction!) they provide the same physical predictions. We will use the brackets \( \{ \} \) to denote a given 1-orbit; for instance \( \{ T^{\mu\nu} \}_1 \) is the 1-orbit containing both \( T^{\mu\nu} \) and \( \tilde{T}^{\mu\nu} \).

We now "switch on" the field-particle interaction i.e. we consider a theory described by the action \( S = S_P + S_f + S_I \). It is quite natural to expect from the new situation to satisfy energy-momentum conservation through the equality

\[
(\tau_{\mu\nu} + \bar{T}_\mu)\nu = 0, \tag{2.6}
\]

where, in order to reobtain the free field predictions when the interaction is switched off, \( \bar{T}_\mu \in \{ T^{\mu\nu} \}_1 \).

We will say that \( H^{\mu\nu} \) and \( G^{\mu\nu} \) belong to the same 2-orbit if (2.5) is fulfilled in the new context, i.e. making use of the interaction field equations.

Of course, and similarly to the free field case, if \( \bar{T}_\mu \) and \( \tilde{T}_\mu \) are such that \( \{ \bar{T}_\mu \}_2 = \{ \tilde{T}_\mu \}_2 \), both objects describe the same 4-momentum context once the
interaction is on. The main difference lies in the fact that we now have a definite criterion to discriminate some of the objects in a given 2-orbit, as far as the free field acceptable values are already known. So, we shall request the object appearing in (2.6) to belong to the 1-orbit \( \{ T^{\mu \nu} \}_1 \).

It is not hard to realize that \( T^{\mu \nu} \) and \( \tilde{T}^{\mu \nu} \) do not belong to the same 2-orbit (though they were in the same 1-orbit). As far as the relation between the e.m.t. and the angular momentum tensor suggests symmetry of the first one, we will \textit{a priori} consider as an acceptable candidate to describe the e.m.t. any object \( \tilde{T}^{\mu \nu} \) such that \( \tilde{T}_{[\mu \nu]} = 0 \) and \( \tilde{T}^{\mu \nu} \in \{ T^{\mu \nu} \}_1 \).

Nevertheless the choice of the true e.m.t. \( T^{\mu \nu} \) as the correct object \( \tilde{T}^{\mu \nu} \) appearing in (2.6) lies not only on the satisfaction of the before mentioned conditions but also on the verified fact that it is precisely this object the one which satisfies (2.6) in the well tested electromagnetic case. So we request the theory to fulfill the following Requirement 2. The action describing a realistic field theory should be such that

\[
(\tau_{\mu \nu} + T^{\mu \nu})^{; \mu} = 0 ,
\]

where \( \tau_{\mu \nu} \) and \( T^{\mu \nu} \) are respectively given by (2.1) and (2.2).

Before ending this point we emphasize the following

a) Not all the actions satisfy Req.2. We will say that a theory is truly conservative if it fulfills Req.2. So Req.2 can be restated saying that our field theory should be truly conservative.

b) When dealing with a non truly conservative theory, the common procedure is to seek a symmetrical object \( \Upsilon^{\mu \nu}(B_\alpha) \) which, via the field and motion equations satisfies

\[
(\tau_{\mu \nu} + \Upsilon_{\mu \nu})^{; \mu} = 0 .
\]

It should of course be checked that \( \Upsilon^{\mu \nu} \in \{ T^{\mu \nu} \}_1 \).

Once reviewed the essential features of a flat field theory we apply them to a familiar case.
3 A simple example: spin 1 electrodynamics.

In order to illustrate last section we consider the well known case of electrodynamics. Regarding points 2-i) to 2-iv), the elements involved are

- the minkowskian manifold \( M_4 \);
- a particle of mass \( m \), charge \( e \) and 4-trajectory \( z^{\mu}(s) \);
- the vectorial field \( A^{\mu} \);
- the action functional

\[
S = S_P + S_I + S_F = -mc \int ds - \frac{e}{c} \int A^{\mu} dz^{\mu} - \frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} d^4x ,
\]

(3.1)

and the corresponding motion and field equations

\[
mc \ddot{z}^{\mu} = \frac{e}{c} F^{\mu\nu} \dot{z}_\nu
\]

(3.2)

\[
F^{\mu\nu,\mu} = \frac{1}{c} j^{\nu},
\]

(3.3)

where \( j^{\nu} \) is the 4-vector current \( j^{\nu} \equiv e \frac{dz^{\nu}}{dx_0} / \delta^{(3)}(\vec{r} - \vec{r}(x_0)) \)

Regarding 2-v), \( A^{\mu} \) does not carry an irrep of the transformation group but breaks into direct sum of the \( s = 1 \) irrep and the (undesired) \( s = 0 \) one. According Req.1 we expect the superfluous degree of freedom to be somehow removed. Taking into account the functional dependence of (3.1) it is not hard to see that equations (3.2) and (3.3) are invariant under the internal transformation \( A^{\mu} \rightarrow A^{\mu} + \phi^{\mu} \), with \( \phi \) an arbitrary scalar field with vanishing divergence on the border of the integration domain of \( S_I \). This invariance implies a degeneration of the coupled system (3.2) + (3.3) responsible of the expected gauge reduction.

It should be remarked that passing from the free field case, characterized by the equations \( F^{\mu\nu,\mu} = 0 \) to the interaction context, where the corresponding equations are (3.2) + (3.3), the symmetry underlying the gauge reduction does not break down. This is due to the fact that the source of the field trivially fulfills the condition \( j^{\mu,\mu} = 0 \). Of course, this equality implies no condition neither on the trajectory \( z^{\mu}(s) \) nor on it’s derivatives. This is one of the essential differences between the spin 1 and the spin 2 cases.

Regarding 2-vi) it is not hard to see that (3.1) describes a truly conservative
theory (*i.e.* it satisfies Req.2). It can be checked that the corresponding true e.m.t. is

\[
T^{\mu\nu} = F^{\mu}_{\alpha} F^{\alpha\nu} + \frac{1}{4} \eta^{\mu\nu} F^{\rho\epsilon} F_{\rho\epsilon},
\]

which is symmetrical and fulfills (2.5).

The relation connecting the true e.m.t. and the canonical one is

\[
T^{\mu\nu} = \overset{\circ}{T}^{\mu\nu} + F^{\mu\sigma} A^{\nu,\sigma}.
\]

As expected \(T^{\mu\nu}\) and \(\overset{\circ}{T}^{\mu\nu}\) belong to the same 1-orbit but they are in different 2-orbits, as far as \(\int F^{\mu\sigma} A^{\nu,\sigma} d\sigma^{\mu}\) vanishes when the fields \(A_{\mu}\) follow the free field equations.

We finish quoting without demonstration the following statement: the form of the spin 1 linear field lagrangian appearing in (3.1) can be determined by either imposing Req.1 or Req.2.

4 The spin 2 case. General Relativity.

It is widely known that the path followed to obtain the commonly accepted spin 2 field theory, *i.e.* Einstein’s General Relativity (from here on G.R.), differs from the one sketched in section 2. The cornerstones for G.R. are covariance, the strong equivalence principle and the requirement on the action of being second order in the fields’ derivatives. These conditions suffice to determine the Hilbert-Einstein’s (from here on H-E) lagrangian and of course no reference needs to be done to the points listed in section 2.

Anyway it is an old issue (see [1],[2],[3],[4]) to try to reobtain G.R. as the limit case of a flat field theory. We shall review at bird sight two of these constructions checking wether they fulfill the requirements stated in section 2.

In this direction we start considering the symmetrical field \(h_{\alpha\beta}\), which is the mathematical carrier of the gravitational interaction (for a justification see [4] or [6]). We call \(H\) the manifold of the second order symmetrical tensors *i.e.* \(H = \{ h_{\alpha\beta} : h_{[\alpha\beta]} = 0 \}\). Following section 2 we start with the minkowskian manifold \(M_{4}\) and a particle of mass \(m\) characterized by its’ 4-trajectory \(z^{\mu}\). We also consider the action functional
\[
S_0 = -mc \int ds - \lambda \int h^{\mu\nu} \tau_{\mu\nu} d^4x + \int L_{F.P.} d^4x,
\]
where the appearance of the particle’s e.m.t. \( \tau_{\mu\nu} \) in the interaction term is due to: i) the tensorial rank of the field, ii) the requirement of locality and iii) the imposition that the highest degree of \( \frac{d^n}{ds^n} z^\mu(s) \) in the equation of motion is given by \( n = 2 \). \( \lambda \) is the coupling constant. Concerning last term in \( S_0 \), \( L_{F.P.} \) is the Fiertz-Pauli lagrangian. It can be obtained imposing Req.1 to the free field case, taking into account that \( h_{\alpha\beta} \) carries undesired degrees of freedom different from the spin 2 ones. They can be removed using the condition \( \left( \frac{\delta L_{F.P.}}{\delta h_{\alpha\beta}} \right)_{\alpha} \equiv 0 \). This reduces in 4 the number of linearly independent free field equations and implies the invariance of these equations under the internal transformation \( h_{\alpha\beta} \to h_{\alpha\beta} + \chi_{\{\alpha,\beta\}} \), with \( \chi_{\alpha} \) an arbitrary vectorial field. Imposing the before mentioned condition it is not hard to see that

\[
L_{F.P.} \equiv \frac{1}{2} h_{\epsilon\gamma,\alpha} h^{\epsilon\gamma,\alpha} - h_{\epsilon\gamma} h^{\epsilon\gamma,\beta} + h_{\epsilon} h^{\epsilon,\beta} h_{\alpha\beta,\alpha} - \frac{1}{2} h_{\epsilon} h^{\epsilon,\alpha} h_{\gamma,\alpha}.
\]

Problems come when the interaction is switched on, as the new field equations

\[
-\lambda \tau_{\mu\nu} + \frac{\delta L_{F.P.}}{\delta h_{\mu\nu}} = 0
\]
leads to the pointless condition

\[
\tau_{\mu\nu,\mu} = 0; \quad (4.1)
\]
senseless as far as \( \ddot{z}^\nu \neq 0 \). We try to recuperate coherence rewriting the action as

\[
S = -mc \int ds - \lambda \int h^{\mu\nu} \tau_{\mu\nu} d^4x + \int (L_{F.P.} + \Delta L) d^4x \quad (4.2)
\]
where \( \Delta L \) are new terms appearing in the field lagrangian; terms of order higher than 2 in \( h_{\alpha\beta} \) and so responsible of the nonlinearity of the theory. Roughly speaking, the object of the before mentioned references ([1],[2],[3],[4]) is to determine, once given suitable consistency conditions, how do \( \Delta L \) looks like. Of course the expected result, i.e. G.R. is always obtained. The difference between the different approaches found in the literature lay on the particular procedure followed and (as denounced by Weinberg in [6]) in the amount of arbitrariness in the election of the corresponding “specific lagrangians”.

We will start reviewing [4]. It was published quite later than the rest of the before cited papers, but it has the advantage of being particularly direct. Deser considered a first order lagrangian \( L_D (\varphi^{\mu\nu}, \Gamma^\alpha_{\beta\epsilon}) \), where \( \varphi^{\mu\nu} \) is a field related to the deviation from flatness of the covariant metric density and \( \Gamma^\alpha_{\beta\epsilon} \) is an (independent of
\( \varphi^{\mu
u} \) symmetric connection. \( L_D \) is such that the "\( h \)" and "\( \Gamma \)" field equations make it equivalent (modulo a field redefinition) to the second order lagrangian \( L_{F.P.} \).

We already know that the inconsistency (4.1) is mended adding source terms \( (j)_{\theta_{\mu\nu}} \), which are increasing order energy momentum contributions due to the self interaction of the field, to the field equations. This is an alternative point of view to the \( \Delta L \) addition in (4.2). Following this line Deser added a source term to the field equations which was somehow related to the e.m.t. of the original lagrangian. The essential point in the construction is that the action giving way (via variational derivation) to the modified equations can be rewritten as the \textit{a la} Palatini version of H-E action. So, proceeding this way we have succeed the program of passing from the linear form to the non linear equations which take into account all the source terms needed to fix inconsistency; and these source precisely reproduce the energy-momentum contribution of the complete field lagrangian. Furtermore the final result is equivalent to G.R. No iteration is needed as far as as in first order formalism G.R. is third order in the chosen variables.

Nevertheless we find several uneasy features in Deser’s construction. First of all the energy momentum conservation condition used as consistency condition seems to fade away when translated to the natural second order formalism: according [4] obtention via the action’s functional derivative of an expression equivalent to Belifante’s e.m.t. is the cornerstone of the construction. Direct calculation shows that this is not the case in second order formalism, as far as the condition

\[
T_{\alpha\beta} = \frac{\delta(L_{F.P.} + \Delta L)}{\delta h_{\alpha\beta}}
\]  

implies that the second order version of \( T^{\mu\nu} \) i.e. \( T^{(2)}_{\mu\nu} \) can be written as \( \frac{\delta L_{(3)}}{\delta h^{\mu\nu}} \) (see (4.4) for the meaning of \( L_{(3)} \)). It is not hard to check that it is not possible to adjust the coefficients of \( L_{(3)} \) in order to satisfy (4.3).

Furthermore Deser’s construction is not exclusive in the sense that it does not rule out the possibility of (using second order formalism) finding a non linear theory with linear part given by the Fiertz-Pauli piece but, when going to higher orders, different from G.R. Another uneasy feature of [4] is that it does not provide a definite criterion (like the one given in eq. (6.2.3) of [3]) to “discriminate” whether a tentative theory is acceptable or not. Besides, it is quite hard to determine if the procedure described in [4] satisfies the program sketched in section 2. As a fact, the use of first order variables darkens the study of Req.1 and (as already mentioned) Req.2.

The approach followed by Feynman [3] is quite tiresome, for use is made of second order formalism which is the natural language of the problem, but requires to take into account the contributions of an infinite series. Lengthy calculations become more and more horrible when going to higher orders. The nice side of the approach is that it provides a necessary condition on \( \Delta L \) based on the simultaneous fulfillment of the field and motion equations.

Starting from the general action (4.2), we break \( \Delta L \) into the sum
\[ \Delta L = \sum_{j=3}^{\infty} L(j). \] (4.4)

In order to simplify the notation we schematically write
\[
L(j) = \sum_{n=1}^{q} a^2_n \left( h^{-2} \partial h \partial h \right)_n ,
\] (4.5)

where \( \left\{ \left( h^{-2} \partial h \partial h \right)_n \right\}_{n=1}^{q} \) stands for the set of \( q \) linearly independent (modulo an additive 4-divergence) terms of order \( (j-2) \) in the non derivative \( h^s \) and order 2 in the derivative ones. \( \{a^2_n\} \) are certain constants. To clarify the notation we consider the Fiertz-Pauli lagrangian \( (i.e. \ the \ L(2) \ case) \) and define

\[
\begin{align*}
a_1^2 &\equiv \frac{1}{2} , \quad (\partial h \partial h)_1 \equiv h_{\epsilon \gamma,\alpha} h^{\epsilon \gamma,\alpha} \\
a_2^2 &\equiv -1 , \quad (\partial h \partial h)_2 \equiv h_{\epsilon \gamma} h^{\gamma,\beta} \\
a_3^2 &\equiv 1 , \quad (\partial h \partial h)_3 \equiv h_{\epsilon \beta} h_{\alpha \beta}^{\alpha} \\
a_4^2 &\equiv -\frac{1}{2} , \quad (\partial h \partial h)_4 \equiv h_{\epsilon} h^{\gamma,\alpha} h^{\gamma,\alpha} .
\end{align*}
\]

Returning to (4.2) it is straightforward to realize that corresponding motion and field equations are

\[
(\eta^{\mu\nu} + 2\lambda h^{\mu\nu}) \ddot{z}_\mu + 2\lambda h^{\mu\nu,\rho} \dot{z}^\rho \dot{z}_\mu - \lambda h^{\mu\tau,\nu} \dot{z}_\mu \dot{z}_\tau = 0 \] (4.6)

\[
-\lambda \tau_{\mu\nu} + \frac{\delta L_F}{\delta h^{\mu\nu}} = 0 ,
\] (4.7)

with \( L_F \equiv L_{F,P} + \Delta L \). It is not hard to check that (4.6) leads to

\[
g_{\mu\lambda} \tau^{\rho\mu,\rho} = -\left[ \mu\rho, \lambda \right] \tau^{\mu\rho} ,
\] (4.8)

where used has been made of the following notation

\[
g_{\mu\nu} \equiv \eta_{\mu\nu} + 2\lambda h_{\mu\nu} ,
\] (4.9)

\[
[\mu\rho, \nu] \equiv \lambda \left[ h_{\mu\nu,\rho} + h_{\rho\nu,\mu} - h_{\mu\rho,\nu} \right].
\]

Using (4.8) and the field equation (4.7) we infer the following consistency condition:

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**Condition A.** \( L_F \) appearing in the action (4.2) of consistent spin 2 field theory necessarily fulfills the condition
\[ g_{\mu\lambda} \left( \frac{\delta L_F}{\delta h_{\rho\mu}} \right)_{,\rho} = -[\mu, \lambda] \frac{\delta L_F}{\delta h_{\rho\mu}} ; \quad h_{\mu\rho} \in H. \] (4.10)

where \( L_F \) is the complete field lagrangian

From last expression it is possible to recursively infer the form of \( L_{(j)} \) using \( L_{(k)} \), \( k < j \). Starting from \( L_{F,P} \), we use the second order version of (4.10) to determine \( L_{(3)} \) through

\[ \eta_{\mu\lambda} \left( \frac{\delta L_3}{\delta h_{\rho\mu}} \right)_{,\rho} = -[\mu, \lambda] \frac{\delta L_{F,P}}{\delta h_{\rho\mu}}. \] (4.11)

Once obtained \( L_{(3)} \) we can use the third order version of (4.10) to determine \( L_{(4)} \), and so on.

Proceeding this way we obtain (see Appendix A)

\[ L_{(3)} = \lambda \left( -2 \, [1] - 2 \, [2] + 2 \, [4] + 4 \, [5] - [6] + \frac{1}{2} \, [7] - \\
- [8] - [9] + 2 \, [10] - 3 \, [11] + [12] + [13] - \frac{1}{2} \, [14] \right), \] (4.12)

where the value of \([j] ; j = 1, ..., 14\) is given in Appendix A.

We can make use of (4.9); identify \( g_{\mu\nu} \) with a metric tensor and write H-E lagrangian as a function of \( h_{\mu\nu} \). Working this way we check that, not only the second order term obtained reproduces \(-2 \lambda^2 \) times \( L_{F,P} \), but the third order one is precisely \(-2 \lambda^2 L_{(3)} \). Instead of going to higher and higher orders in (4.10), Feynman observed that condition A can be rewritten as an invariance condition of \( L_F \) under infinitesimal transformations of \( h_{\alpha\beta} \). This is precisely the invariance underlying H-E lagrangian and so G.R. is a solution of (4.10) to all orders. So, imposing Condition A to a standard field theory the machinery of pseudoriemannian manifolds and differential geometry is recovered.

Several remarks should be done:

a) Not only G.R. fulfills Condition A but, as far as we know, it is the only theory of the form (4.5) + (4.10) which follows it.

b) In the "intergated to all orders" theory, i.e. G.R., the gauge reduction given by invariance under \( h_{\alpha\beta} \rightarrow h_{\alpha\beta} + \chi_{(\alpha,\beta)} \), with finite \( \chi_{\beta} \), is no longer valid. Instead,
an infinitesimal version underlies the new context. It is well known that these transformations reproduce infinitesimal general diffeomorphisms. Initially Einstein elucidated this symmetry as the one due to the change in the observer’s state of motion. As it will be demonstrated in [7], this interpretation is not attainable. Anyway, the passing from the finite to the infinitesimal symmetry is closely related to the passage from flat to curve scenario.

c) The before described reobtention of G.R. has been achieved without necessity of invoking any restriction on the form of the field e.m.t. Only a posteriori and once determined the aspect of \( \Delta L \), we tackle the problem of studying the corresponding e.m.t. Anyway it is clear that the true e.m.t. is not an acceptable object to describe the energy-momentum density, as far as it would imply

\[
T_{\mu\nu}^{\mu} + \frac{1}{\lambda} \left( \frac{\delta L_F}{\delta h_{\mu\nu}} \right)_{,\mu} = 0 ,
\]

which is a condition over \( \Delta L \) that G.R. does not satisfy. Actually it is an unattainable condition over any \( \Delta L \), (see Appendix B). Looking for coherence with section 2 we might seek a symmetrical object \( \Theta^{\mu\nu} \), which belongs to \( \{T^{\mu\nu}\}_1 \) and fulfills

\[
\Theta_{\mu\nu}^{\mu} + \frac{1}{\lambda} \left( \frac{\delta L_F}{\delta h_{\mu\nu}} \right)_{,\mu} = 0 .
\]

Nevertheless this is not the common procedure. It is well known that G.R. bears a new frame where there is no room for section 2 requirements. As a fact, in the geometrical scenario use is customarily done of Weyl’s e.m.t.:

\[
T^{\mu\nu}_W = \gamma \left( \frac{\delta L_G}{\delta g_{\mu\nu}} \right) ,
\]

where \( L_G \) is the geometrical field lagrangian (usually H-E) and \( \gamma \) is a factor dependent of the lagrangian normalization and chosen in such a way that \( T^{\mu\nu}_W \) identically fulfills (2.5). Anyway, the subject of the e.m.t. in G.R. is complicated, as far as there are several candidates depending, for instance, if a field theoretical approach to the problem or a geometric one is used. There is a wide literature on the subject (see [8]).

5 Non standard nonlinear spin 2 theory.

With all we have seen and taking into account Condition A, it might seem nonsensical to seek a spin 2 theory based on different grounds, as far as (4.10) is a necessary condition on any consistent theory. Anyway we will temporary ignore it
and deal with the question of determining $\Delta L$ appearing in (4.2) following the path sketched in section 2.

We already mentioned that both Req.1 or Req.2 suffice to find the form of the spin 1 field lagrangian. We know study wether something similar happens in a non linear spin 2 context. Unfortunately the gauge reduction condition Req.1 has little to say about the non linear action. It was found to be enough to obtain the free lagrangian $L_{F.P.}$, but apparently it has no influence on $\Delta L$. If we try to build a parallel condition to the one used to determine $L_{F.P.}$ we observe that 

$$\left( \frac{\delta \Delta L}{\delta h_{\alpha \beta}^\alpha} \right)_{, \alpha} = 0$$

is unattainable. Besides the before mentioned condition would not fix the inconsistency (4.1). Furthermore the new nonlinear action does not seem to be invariant under the old finite transformation $h_{\alpha \beta} \rightarrow h_{\alpha \beta} + \chi_{\{\alpha, \beta\}}$. So, in the search of a non linear theory, it is by no means evident to find a clue from gauge reduction.

We next consider the consequences of imposing Req.2. Together with the field equation (4.7), Req.2 suggests the identification

$$T_{\mu \nu} \overset{?}{=} -\frac{1}{\lambda} \left( \frac{\delta L_{F.P.}}{\delta h_{\mu \nu}} + \frac{\delta \Delta L}{\delta h_{\mu \nu}} \right)$$

(5.1)

where the symbol $\overset{?}{=} \overset{?}{=}$ means that in $L.H.S. \overset{?}{=} R.H.S.$, it is necessary to check the existence of certain value of the coefficients $\{a_{i}^\alpha\}$ such that the equality $L.H.S. = R.H.S.$ holds.

We rewrite (5.1) to second order as

$$T_{\mu \nu}^{(2)} \overset{?}{=} -\frac{1}{\lambda} \left( \frac{\delta L_{3}}{\delta h_{\mu \nu}} \right)$$

(5.2)

where $T_{\mu \nu}^{(2)}$ is built using the the construction described in 2-vi), introducing the value of $L_{F.P.}$. It is easy to check that the coefficients in $L_{3}$ cannot be adjusted in such a way that (5.2) holds.

Condition (5.1) is too restrictive. In order to ensure the true conservativity of the theory it is enough to demand the less stringent condition

$$(T_{\mu \nu})_{, \mu} \overset{?}{=} -\frac{1}{\lambda} \left( \frac{\delta \Delta L}{\delta h_{\mu \nu}} \right)_{, \mu}$$

(5.3)

which can be written to lowest order as

$$-\lambda (T_{\mu \nu}^{(2)})^{, \mu} \overset{?}{=} \left( \frac{\delta L_{3}}{\delta h_{\mu \nu}} \right)^{, \mu}$$

(5.4)
(4.11) and (5.4) have similar structures. Both reproduce a system of linear equations with many more equations than unknowns. The main difference lies in the fact that (4.11) is solvable while (5.4) is not (see Appendix B). So we arrive to the conclusion that the structure (5.3) is incompatible.

Nevertheless it is possible to find a consistent way to handle the problem. In this direction we consider the gauge reduction requirement. Instead of starting with an action invariant under certain 4-parameter internal transformation we shall assume that the field manifold $H$ is somehow reduced to a smaller manifold $H^*$ which is nothing but the restriction of $H$ to the (non algebraic) condition $\varphi_\nu(h_{\alpha\beta}) = 0$; i.e.

$$H^* = \{h_{\alpha\beta} \in H : \varphi_\nu(h_{\alpha\beta}) = 0 ; \nu = 0, 1, 2, 3\}.$$

This is an alternative and admissible way to impose the removal of the undesired degrees of freedom. We shall implement this condition imposing the requirement that the fields appearing in the dynamical equations are restricted to belong to $H^*$.

In this direction we suppose that $\varphi_\nu(h_{\alpha\beta})$ has the following form

$$\varphi_\nu(h_{\alpha\beta}) \equiv (\eta_{\mu\nu} + 2\lambda h_{\mu\nu}) \left( \frac{\delta L_F}{\delta h_{\rho\mu}} \right)_{,\rho} + [\mu, \nu] \frac{\delta L_F}{\delta h_{\rho\mu}}, \quad (5.5)$$

and hence $H^*$ is the manifold of the symmetrical tensors $h_{\alpha\beta}$ fulfilling $\varphi_\nu(h_{\alpha\beta}) = 0$, where $\varphi_\nu(h_{\alpha\beta})$ is given by (5.5).

It is important to emphasize the differences between (4.10) and our condition $\varphi_\nu(h_{\alpha\beta}) = 0$. (4.10) was used to determine the value of the coefficients $\{a^n_j\}$ satisfying

$$\text{Actually the condition } \varphi_\nu(h_{\alpha\beta}) = 0 \text{ on } H \text{ should be imposed prior to the determination of the dynamical equations given by the functional variation } \delta. \text{ Nevertheless the suggested procedure demonstrates to be "good enough" when the reduction of the manifold is too complicated. To simplify we consider an easy example. Imagine we start from a kynematical manifold } [\omega_1, \omega_2], \text{ where } \omega_1 \text{ and } \omega_2 \text{ are scalar fields, which is reduced, via the condition } \varphi(\omega_1, \omega_2) = 0. \text{ We shall assume that we know how to invert the before mentioned condition and write } \omega_2 = \omega_2(\omega_1). \text{ So we introduce the reduced manifold } [\omega_1] = [\omega_1, \omega_2(\omega_1)]. \text{ We define the new action using the old one as } S(\omega_1) \equiv S(\omega_1, \omega_2(\omega_1)). \text{ Our real dynamical problem is to determine the solutions of } \frac{\delta S(\omega_1)}{\delta \omega_1} = 0. \text{ If we have problems to deal with this equation, } i.e. \text{ if it is not trivial to invert } \varphi(\omega_1, \omega_2) = 0 \text{ we observe that}

$$
\frac{\delta S(\omega_1)}{\delta \omega_1} = \frac{\delta S(\omega_1, \omega_2)}{\delta \omega_1} \bigg|_{\omega_2=\omega_2(\omega_1)} + \frac{\delta S(\omega_1, \omega_2)}{\delta \omega_2} \bigg|_{\omega_2=\omega_2(\omega_1)} \frac{\partial \omega_2}{\partial \omega_1},
$$

so, the simultaneous imposition of $\frac{\delta S(\omega_1, \omega_2)}{\delta \omega_1} \bigg|_{\omega_2=\omega_2(\omega_1)} = 0$ and $\frac{\delta S(\omega_1, \omega_2)}{\delta \omega_2} \bigg|_{\omega_2=\omega_2(\omega_1)} = 0$ is a sufficient (though not necessary!) condition for a solution of $\frac{\delta S(\omega_1)}{\delta \omega_1} = 0$. 15
this requirement. In this way we arrived to the value of the parameters describing
Hilbert-Einstein’s lagrangian. This procedure implies no previous condition over the
fields $h_{\alpha\beta}$.

Nevertheless the condition which defines $H^*$ i.e. $\varphi_\nu(h_{\alpha\beta}) = 0$ is not a condition
over the lagrangian coefficients but over the own kinematically acceptable fields $h_{\alpha\beta}$.
It is important to realize that the coefficients which appear in the H-E lagrangian are
not acceptable for our gauge reduction condition, as far as they identically cancel
(5.5) and so imply no condition over the fields $h_{\alpha\beta}$.

This approach simultaneously gives solution to the superfluous degrees of freedom
removal and to the consistency problem before stated. Nevertheless if we want to
complete the program outlined in Section 2 we still have to deal with Req.2. In this
direction we impose the

\textbf{Condition B.} A theory consistent with the requirements developed in section
2 describing a spin 2 field is determined by the condition

$$-\lambda \left( T^{\mu\nu} \right)_{,\mu} = \left( \frac{\delta \Delta L}{\delta h_{\mu\nu}} \right)_{,\mu};\; h_{\alpha\beta} \in H^*, \quad (5.6)$$

where $T^{\mu\nu}$ depends on $\Delta L$ through (2.2) and $H^*$ is defined via (5.5).

We know study the consistency of Condition B. In this direction we consider the
lowest order version of (5.6)

$$-\lambda \left( T^{(2)}_{\mu\nu} \right)_{,\mu} = \left( \frac{\delta L^{N}_{(3)}}{\delta h_{\mu\nu}} \right)_{,\mu};\; h_{\alpha\beta} \in H^*, \quad (5.7)$$

where we have introduced the label $N$ (of nonstandard) to distinguish the lagrangian
fulfilling (4.10) from the one given by (5.6).

We have already seen that (5.4) is incompatible. Nevertheless, taking into ac-
count the condition $h_{\alpha\beta} \in H^*$, one equation can be eliminated from the total system.
In other words, a new system of equations is obtained writing one of the vectorial
terms appearing in $\varphi_\nu(h_{\alpha\beta}) = 0$ as afuction of the rest and substituting it next
in (5.4). Surprisingly, the system obtained proceeding this way is compatible, (see
Appendix B). As a fact there is a degeneracy of order 1 in it’s solution. It is not
hard to check that our third order solution can be parametrized as

$$L^N_{(3)} = \lambda \left( (-2 + \epsilon) [1] + (-2 + \epsilon) [2] + (2 - \epsilon) [4] + 4 [5] + \left( -1 - \frac{1}{2} \epsilon \right) [6] +$$
\[
+ \left( \frac{1}{2} + \frac{1}{4} \epsilon \right) [7] + \left( -1 - \frac{1}{2} \epsilon \right) [8] + \left( -1 + \frac{1}{2} \epsilon \right) [9] + (2 - \epsilon) [10] + \\
+ \left( -3 - \frac{1}{2} \epsilon \right) [11] + \left( 1 + \frac{1}{2} \epsilon \right) [12] + \left( 1 + \frac{1}{2} \epsilon \right) [13] + \left( -\frac{1}{2} - \frac{1}{4} \epsilon \right) [14],
\]

(5.8)

where \( \epsilon \) is a real number.

The resolubility of the new system of equations means that Condition B can be implemented at least to first order in the nonlinear contribution \( \Delta L \). It opens the horizon for the construction of a spin 2 theory written following the path indicated in section 2.

We now remark several important points.

a) (5.8) tends to the G.R. term (4.12) when \( \epsilon \to 0 \). It should be emphasized that G.R. can by no means be a particular solution of (5.6), as far as in Einstein’s theory \( \varphi_\nu(h_{\alpha\beta}) \) identically vanishes and there is no reduction from \( H \) to \( H^* \). So, solution (5.8) is valid \( \forall \epsilon \in \{(-\infty, 0) \cup (0, \infty)\} \)

b) A look at (5.8) shows that, to the given order, \( \epsilon \) breaks G.R.’s global symmetry i.e. diff. invariance. The absolute value of \( \epsilon \) measures the size of this breaking.

c) The fact that G.R. gives the correct predictions for the classical tests suggests that the absolute value of \( \epsilon \) is small.

d) If we write (5.6) to fourth order and find the system to be solvable (which for the time being is only a hypothesis) we expect to find something like

\[
L_{(4)}^N = L_{(4)}^N \left( \{a^4_n\}_{n=1}^q, \epsilon, \{\epsilon'_p\}_{p=1}^r \right)
\]

where \( a^4_n \in \mathbb{R} \) are \( q \) fixed constants, \( \epsilon \) is the indeterminacy appearing in \( L_{(3)}^N \) and \( \epsilon'_p \) are \( r \) free parameters responsible of the \( r \)-indeterminacy present in the fourth order version of (5.6). Of course \( 0 \leq r \leq q \).

e) According to the footnote in page 15, the dynamical problem can be treated equating to zero the functional derivative of the lograngian fulfilling Condition B with respect to an arbitrary \( h_{\alpha\beta} \) and imposing as a simultaneous equation \( \varphi_\nu(h_{\alpha\beta}) = 0 \).

f) It remains unknown wether the infinite number of systems involved in (5.6) are solvable. Going to higher and higher order does not ensure the integrability to all orders. It would be necessary to find a qualitative argumento to determine it.

g) At this point the sensible reader might be wondering what is the necessity for
"it all". We have already got G.R., a familiar theory which fulfills condition A to all orders and provides correct prediction for the classical tests. Then, why complicate life with a tentative theory of dubious behavior when going to higher orders?. We find several good reasons. First of all the new theory might introduce new features for quantization. Furthermore it shall be demonstrated in [7] that intrinsical diff. invariance entails unavoidable complications. A general solution fulfilling Condition B is not diff. invariant, but for small values of $\epsilon$, gives good predictions. So it keeps away from the undesired difficulties and maintains agreement with experiments. Besides, if condition B is integrable, we have at our disposal a consistent spin 2 theory following the usual field theory requirements; characterized by a well defined value of the e.m.t. (actually by the true e.m.t. introduced in section 2).

6 Conclusions.

We have seen that, starting from the second order linear Fiertz-Pauli lagrangian, there seems to be two roads which, going to a non linear field self interaction, avoid inconsistency. The first one is the well known Condition A, giving way to Einstein's G.R. The second is condition B which we have proved to be integrable to third order in the field lagrangian. The corresponding term depends on certain free parameter which shows the deviation from the diff. invariance solution. It remains as an open question wether a theory fulfilling condition B is feasible to all orders or not, and which are the quantum implications of the (hypothetical) complete field lagrangian.

7 Appendices.

A. Some calculations concerning Condition A.

Following (4.4) we write

$$L_{(3)} = \sum_{n=1}^{q} a_n^3 \left( h \partial h \partial h \right)_n .$$

It is easy to check that in this case $n = 14$, i.e. we can choose $14 \left( h \partial h \partial h \right)$ linearly independent terms (modulo an additive 4-divergence) in such a away that the rest
of the terms are written as linear combination (modulo an additive 4-divergence) of the former 14. In order to simplify the notation we write

\[ x_n \equiv a^3_n \quad [n] \equiv (h \partial h \partial h)_n \quad n = 1, \ldots, 14, \]

i.e.

\[ L^{(3)} = \sum_{n=1}^{14} x_n [n]. \]

We can choose

\[ 1 \equiv h^\beta_\alpha h^\alpha_\gamma h^\delta_\rho; \quad 2 \equiv b^\alpha h^\alpha_\gamma h^\alpha_\gamma; \quad 3 \equiv b^\alpha b^\beta h^\alpha_\beta; \quad 4 \equiv h_{\alpha\beta,\gamma} h^\alpha_\epsilon h^\epsilon_\alpha; \]
\[ 5 \equiv h_{\alpha\gamma,\beta} h^\alpha_\gamma h^\beta_\epsilon; \quad 6 \equiv h_{\alpha\gamma,\beta} h^\alpha_\epsilon h^\beta_\epsilon; \quad 7 \equiv h h_{\alpha,\beta} h^\alpha_\beta; \quad 8 \equiv h b^\beta b^\epsilon; \]
\[ 9 \equiv c^\alpha h_{\alpha,\beta} h^\beta_\epsilon; \quad 10 \equiv c^\alpha h_{\alpha,\beta} h^\beta_\epsilon; \quad 11 \equiv h_{\alpha,\beta} c^\alpha b^\epsilon; \quad 12 \equiv h c_\alpha b^\epsilon; \]
\[ 13 \equiv h_{\alpha,\beta} c^\alpha c^\beta; \quad 14 \equiv h c_\alpha c^\alpha, \]

where

\[ b_\alpha \equiv h^\epsilon_\alpha; \quad c_\alpha \equiv h_\alpha; \quad h \equiv h_\epsilon. \]

Using the notation

\[ (1)_\beta \equiv h_{\epsilon,\alpha,\beta} \square h^\alpha_\epsilon; \quad (2)_\beta \equiv h^\alpha_\epsilon \square h_{\beta,\epsilon,\alpha}; \quad (3)_\beta \equiv h^\alpha_\epsilon \square h_{\alpha,\epsilon,\beta}; \quad (4)_\beta \equiv h_{\epsilon,\alpha,\beta} \square h^\alpha_\epsilon; \]
\[ (5)_\beta \equiv h_{\beta,\epsilon,\alpha,\beta} \square h^\delta_\alpha; \quad (6)_\beta \equiv h_{\alpha,\beta,\gamma} h^\delta_\alpha; \quad (7)_\beta \equiv h_{\alpha,\beta,\gamma} h^\delta_\alpha; \quad (8)_\beta \equiv h_{\beta,\epsilon,\alpha} h^\gamma_\delta; \]
\[ (9)_\beta \equiv h_{\epsilon,\beta} \square b^\epsilon; \quad (10)_\beta \equiv b^\epsilon \square h_{\epsilon,\beta}; \quad (11)_\beta \equiv b_{\beta,\gamma} h^\gamma_\delta; \quad (12)_\beta \equiv b_{\delta,\gamma,\beta} h^\gamma_\delta; \]
\[ (13)_\beta \equiv b_{\gamma,\alpha} h_{\alpha,\beta}; \quad (14)_\beta \equiv b_{\alpha,\beta} h_{\beta,\gamma}; \quad (15)_\beta \equiv b^\alpha h_{\beta,\gamma,\alpha}; \quad (16)_\beta \equiv b^\alpha h_{\beta,\gamma,\alpha}; \]
\[ (17)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (18)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (19)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (20)_\beta \equiv c^\alpha h_{\alpha,\beta}; \]
\[ (21)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (22)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (23)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (24)_\beta \equiv c^\alpha h_{\alpha,\beta}; \]
\[ (25)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (26)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (27)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (28)_\beta \equiv c^\alpha h_{\alpha,\beta}; \]
\[ (29)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (30)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (31)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (32)_\beta \equiv c^\alpha h_{\alpha,\beta}; \]
\[ (33)_\beta \equiv c^\alpha h_{\alpha,\beta}; \quad (34)_\beta \equiv c^\alpha h_{\alpha,\beta}; \]

it can be demonstrated that

\[ ^3 \text{There are some algebraic errors in ref. [3] probably due to a mistranscription of Feyman’s lectures (though the conclusions are absolutely correct!). First of all the minimum number of elements of the form } h \partial h \partial h \text{ needed to write } L^{(3)} \text{ is not 18 but 14. There is also an error in eq.}(6.1.13). \text{ In order to check it, suffices to realize that } (6.1.13) \text{ does not reproduce the corresponding term in the H-E lagrangian.} \]
\[-[\mu_\beta, \beta] \frac{\delta L_{F,P}}{\delta h_{\rho\nu}} = \lambda \left[ 2(1)_\beta - (4)_\beta - 2(14)_\beta - 2(15)_\beta + 2(16)_\beta + 2(19)_\beta - (20)_\beta + + 2(22)_\beta - 2(28)_\beta - (29)_\beta + (31)_\beta \right], \tag{A.1} \]

In a similar way, some lengthy algebra leads to

\[
\eta_{\mu\beta} \left( \frac{\delta L_3}{\delta h_{\rho\nu}} \right)_{\phi} = \left[ -x_1 - x_4 + \frac{1}{2}x_5 \right]_{(1)_\beta} + \left[ -x_1 - \frac{1}{2}x_5 \right]_{(2)_\beta} + + \left[ -x_2 - x_{10} \right]_{(3)_\beta} + \left[ x_6 - x_2 - x_{10} \right]_{(4)_\beta} + \left[ x_6 - 2x_2 + 2x_7 - 2x_{10} \right]_{(5)_\beta} + + \left[ -2x_6 - x_5 + x_4 \right]_{(6)_\beta} + \left[ -x_4 - 2x_9 \right]_{(7)_\beta} + \left[ -x_4 - x_1 \right]_{(8)_\beta} + + \left[ -x_1 - x_4 - x_3 \right]_{(9)_\beta} + \left[ -x_1 - x_3 - \frac{1}{2}x_5 \right]_{(10)_\beta} + \left[ -2x_6 - \frac{1}{2}x_5 \right]_{(11)_\beta} + + \left[ -x_4 - x_3 - \frac{1}{2}x_5 - x_9 - x_{11} \right]_{(12)_\beta} + \left[ -x_2 - \frac{1}{2}x_5 \right]_{(13)_\beta} + \left[ -x_1 - 2x_3 - x_4 - \frac{1}{2}x_5 \right]_{(14)_\beta} + + \left[ x_4 - \frac{3}{2}x_5 - 2x_6 \right]_{(15)_\beta} + \left[ -x_3 - x_4 - x_9 - x_{11} \right]_{(16)_\beta} + \left[ -\frac{1}{2}x_9 - x_{10} - \frac{1}{2}x_{11} \right]_{(17)_\beta} + + \left[ -\frac{1}{2}x_{11} + \frac{1}{2}x_9 - 2x_7 \right]_{(18)_\beta} + \left[ -2x_7 - x_{11} \right]_{(19)_\beta} + \left[ -2x_{13} - \frac{1}{2}x_{11} + \frac{1}{2}x_9 \right]_{(20)_\beta} + + \left[ -\frac{1}{2}x_9 - \frac{1}{2}x_{11} - 2x_{13} \right]_{(21)_\beta} + \left[ -x_2 + x_3 \right]_{(22)_\beta} + \left[ -x_4 - 2x_3 - \frac{1}{2}x_5 + 2x_8 - 2x_{11} \right]_{(23)_\beta} + + \left[ -2x_6 - \frac{1}{2}x_5 + x_3 \right]_{(24)_\beta} + \left[ -2x_7 - 2x_8 + \frac{1}{2}x_{11} - \frac{1}{2}x_9 \right]_{(25)_\beta} + + \left[ -\frac{1}{2}x_{11} + \frac{1}{2}x_9 - x_8 - 2x_{13} \right]_{(26)_\beta} + \left[ -x_9 - x_8 - 2x_{13} \right]_{(27)_\beta} + + \left[ -x_8 - \frac{1}{2}x_9 + \frac{1}{2}x_{11} - x_{10} \right]_{(28)_\beta} + \left[ -x_8 - \frac{1}{2}x_9 + \frac{1}{2}x_{11} - x_{12} \right]_{(29)_\beta} + + \left[ x_{13} - 2x_{12} - 2x_{14} \right]_{(30)_\beta} + \left[ x_{13} - x_{12} - 2x_{14} \right]_{(31)_\beta} + \left[ -x_8 - 2x_7 \right]_{(32)_\beta} + + \left[ -x_8 - x_{12} \right]_{(33)_\beta} + \left[ -x_{12} - 2x_{14} \right]_{(34)_\beta} . \tag{A.2} \]

Equating terms in (A.1) and (A.2) we obtain a system of 34 equations and 14 unknowns:

\[-x_1 - x_4 + \frac{1}{2}x_5 = 2\lambda \]
\[-x_1 - \frac{1}{2} x_5 = 0 \]
\[-x_2 - x_{10} = 0 \]
\[... = ...\]
\[-x_8 - x_{12} = 0 \]
\[-x_{12} - 2x_{14} = 0. \]

(A.3)

Despite of the much bigger number of equation than unknowns, system (A.3) is compatible, being the coefficients appearing in (4.12) its solution:

\[x_1 = -2 \lambda, \quad x_2 = -2 \lambda, \quad x_3 = 0 \quad x_4 = 2 \lambda,\]
\[x_5 = 4 \lambda, \quad x_6 = -\lambda, \quad x_7 = \frac{1}{2} \lambda, \quad x_8 = -\lambda,\]
\[x_9 = -\lambda, \quad x_{10} = 2 \lambda, \quad x_{11} = -3 \lambda, \quad x_{12} = \lambda,\]
\[x_{13} = \lambda, \quad x_{14} = -\frac{1}{2} \lambda.\]

**B. Some calculations concerning Condition B.**

We now set out the resolubility of (5.6); beginning with it’s lowest order version (5.7). The first step is to determine \((T^{(2)}_{\alpha\beta})^\alpha\) using the machinery developed in section 2.

In this direction we write the infinitesimal Poincare variation of the field

\[\delta h^{\alpha\beta}_\epsilon (x') = \frac{1}{2} \omega_{\mu\nu} (S^{\mu\nu})^{\alpha\beta}_{\epsilon\tau} h^{\epsilon\tau} (x),\]

where

\[(S^{\mu\nu})^{\alpha\beta}_{\epsilon\tau} = \frac{1}{2} \left( \delta_\epsilon^{\alpha} \eta^{\mu\beta} \delta_\tau - \delta_\epsilon^{\alpha} \eta^{\nu\beta} \delta_\tau + \delta_\epsilon^{\alpha} \eta^{\mu\alpha} \delta_\beta - \delta_\epsilon^{\alpha} \eta^{\nu\alpha} \delta_\beta + \eta^{\mu\alpha} \delta_\epsilon^{\beta} \delta_\beta - \eta^{\nu\alpha} \delta_\epsilon^{\beta} \delta_\beta - \eta^{\mu\beta} \delta_\epsilon^{\alpha} \delta_\beta - \eta^{\nu\beta} \delta_\epsilon^{\alpha} \delta_\beta \right),\]

and according (2.3)

\[f^{\mu\nu} = \frac{1}{2} \left( L_{\alpha\beta}^{\mu} (S^{\rho\nu})^{\alpha\beta}_{\epsilon\tau} + L_{\alpha\beta}^\rho (S^{\nu\rho})^{\alpha\beta}_{\epsilon\tau} - L_{\alpha\beta}^\nu (S^{\mu\nu})^{\alpha\beta}_{\epsilon\tau} \right) h^{\epsilon\tau}\]

\[L_{\alpha\beta}^\mu = \frac{\partial L_F}{\partial h^{\alpha\beta}_{\mu\tau}}.\]
Substituting these values in (2.2) it is not hard to see that

\[-\lambda \left( T_{\alpha\beta}^{(2)} \right)^{\alpha} = -\lambda \left[ -2 (1)_{\beta} + 2 (2)_{\beta} + 2 (4)_{\beta} + 2 (10)_{\beta} - 2 (11)_{\beta} - 2 (12)_{\beta} + 2 (13)_{\beta} + 2 (14)_{\beta} + 2 (15)_{\beta} - 4 (16)_{\beta} - 2 (17)_{\beta} - 2 (19)_{\beta} + 2 (20)_{\beta} + 2 (21)_{\beta} - 2 (23)_{\beta} - 2 (24)_{\beta} + 2 (27)_{\beta} + 2 (29)_{\beta} - 2 (31)_{\beta} \right].\]

(B.1)

Equating now terms in (A.7) and (B.1) we obtain the new system of equations

\[-x_1 - x_4 + \frac{1}{2} x_5 = 2\lambda\]
\[-x_1 - \frac{1}{2} x_5 = -2\lambda\]
\[-x_2 - x_{10} = 0\]
... = ...
\[-x_8 - x_{12} = 0\]
\[-x_{12} - 2x_{14} = 0.\]

Which is incompatible. Anyway we have not considered yet the the implications of \(h_{\alpha\beta} \in H^*.\)

We introduce the notation

\[-[\mu\rho, \beta] \frac{\delta L_{F.P}}{\delta h^\mu} \equiv \sum_{i=1}^{34} \Omega_i (i)_{\beta} + \vartheta \geq 3 \]

(B.2)

\[\eta_{\mu\beta} \left( \frac{\delta L_3}{\delta h^\mu} \right)_{,\rho} \equiv \sum_{i=1}^{34} z_i (i)_{\beta} + \vartheta \geq 3 , \]

(\text{where } \vartheta(p) \text{ stands for the terms of order } p \text{ in the } h's) \text{ and reexpress the second order version of the (5.5) cancelation as}

\[\sum_{i=1}^{34} z_i (i)_{\beta} = \sum_{i=1}^{34} \Omega_i (i)_{\beta} . \]

(Calling
we rewrite the incompatible condition (5.4) as

\[ \sum_{i=1}^{34} z_i (i)_{\beta} = \sum_{i=1}^{34} \Theta_i (i)_{\beta} \]  

(B.3)

Dealing with the resolubility of Condition B we still have to impose the condition $h_{\alpha\beta} \in H^*$, i.e. the $34 (i)_{\beta}$ are not longer independent. We can use (B.2) to write $(j)_{\beta}$; $1 \leq j \leq 34$ as a linear combination of the 33 remainig terms, substitute in (B.3) and obtain a new system of equations. In this direction, we say that a set of values $\{\bar{x}_i\}_{i=1}^{14}$ is a solution to our problem if, finding the expression (B.2) for $\{x_i\}_{i=1}^{14} = \{\bar{x}_i\}_{i=1}^{14}$ and replacing any of the $(k)_{\beta}$ there appearing in (B.3), it identically vanishes. In other words, we have to deal with 34 different systems of equations, depending on the $(j)_{\beta}$ replaced. We shall distinguish two cases

i) If the replaced $(j)_{\beta}$ is such that $\Theta_j \neq \Omega_j$, when identifying the coefficients of $(i)_{\beta}, i \neq j$; a new system (33 equations, 14 unknowns) is obtained.

It is not hard to check that the new system of equations is given by

\[ z_i (x_1, x_2, ..., x_{14}) = \frac{\Omega_i - \Theta_i}{\Omega_j - \Theta_j} z_j (x_1, x_2, ..., x_{14}) + \frac{\Omega_j \Theta_i - \Theta_j \Omega_i}{\Omega_j - \Theta_j} \]

\[ j \text{ fixed; } i \neq j, \]

and, independently of the $(j)_{\beta}$ replaced, the resolution of the system gives a solution which can be parametrized as

\[
\begin{align*}
    x_1 &= -2 \lambda + \gamma, \quad x_2 = -2 \lambda + \gamma, \quad x_3 = 0, \quad x_4 = 2 \lambda - \gamma, \\
    x_5 &= 4 \lambda, \quad x_6 = -\lambda - \frac{1}{2} \gamma, \quad x_7 = \frac{1}{2} \lambda + \frac{1}{4} \gamma, \quad x_8 = -\lambda - \frac{1}{2} \gamma, \\
    x_9 &= -\lambda + \frac{1}{2} \gamma, \quad x_{10} = 2 \lambda - \gamma, \quad x_{11} = -3 \lambda - \frac{1}{2} \gamma, \quad x_{12} = \lambda + \frac{1}{2} \gamma, \\
    x_{13} &= \lambda + \frac{1}{2} \gamma, \quad x_{14} = -\frac{1}{2} \lambda - \frac{1}{4} \gamma.
\end{align*}
\]

(B.4)

Redefining the parameter $\gamma$ as $\gamma \equiv \epsilon \lambda$ we identify the coefficients appearing in (5.8). It should be remarked that if we substitute the values (B.4) in (B.2), all the
(\(j_\beta\)) such that \(\Theta_j = \Omega_j\) decouple from the expression. So, we conclude that (B.4) is a solution to our problem.

ii) If \((j_\beta)\) is such that \(\Theta_j = \Omega_j\), when replacing it all the contributions in \(x_1, ..., x_{14}\) cancel and what is obtained is nothing but a linear condition on the \((i_\beta)\) which is precisely (B.2) using (B.4). We conclude that (B.4) is not only a solution, but the only solution to our problem.

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