On a more accurate Hardy-Mulholland-type inequality

Bicheng Yang1* and Qiang Chen2

Abstract

By using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard’s inequality, a more accurate Hardy-Mulholland-type inequality with multi-parameters and a best possible constant factor is given. The equivalent forms, the reverses, the operator expressions and some particular cases are considered.

MSC: 26D15; 47A07

Keywords: Mulholland-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0$, and $\|b\|_q > 0$, we have the following Hardy-Hilbert’s inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{1}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q.$$  \hfill (1)

A more accurate inequality of (1) is given as follows (cf. [1], Th. 323 and [2]):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{1}{m+n-\alpha} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q \quad (0 \leq \alpha \leq 1),$$  \hfill (2)

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

Also we have the following Mulholland’s inequality similar to (1) with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3] or [1], Th. 343, replacing $\frac{a_m}{m^p}$, $\frac{b_n}{n^q}$ by $a_m$, $b_n$):

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} a_m b_n \frac{1}{mn} \frac{1}{n^q} \left( \sum_{m=2}^{\infty} \frac{a_m}{m^{1+p}} \right)^{\frac{1}{p}} \left( \sum_{n=2}^{\infty} \frac{b_n}{n^{1+q}} \right)^{\frac{1}{q}}.$$  \hfill (3)

Inequalities (1)–(3) are important in analysis and its applications (cf. [1, 2, 4–18]).
Suppose that $\mu_i, \nu_j > 0 \ (i, j \in \mathbb{N} = \{1, 2, \ldots\})$,

$$U_m = \sum_{i=1}^{m} \mu_i, \quad V_n = \sum_{j=1}^{n} \nu_j \quad (m, n \in \mathbb{N}),$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 321, replacing $\mu^{1/p} a_m$ and $\nu^{1/q} b_n$ by $a_m$ and $b_n)$: If $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} \frac{a_m}{m^{\beta}} < \infty$, $0 < \sum_{n=1}^{\infty} \frac{b_n}{n^{\gamma}} < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} \leq \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} \frac{a_m}{\mu^{(1-\lambda_1)-1}} \right)^{\frac{1}{\lambda}} \left( \sum_{n=1}^{\infty} \frac{b_n}{\nu^{(1-\lambda_2)-1}} \right)^{\frac{1}{\gamma}}.$$  

(5)

For $\mu_i = \nu_j = 1 \ (i, j \in \mathbb{N})$, inequality (5) reduces to (1).

In 2015, Yang [19] gave an extension of (5) as follows: If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are positive and decreasing, with $U_\infty = V_\infty = \infty$, then we have the following inequality with the best possible constant factor $\pi / \sin(\pi/\lambda)$:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} \leq \frac{\pi}{\lambda \sin(\pi/\lambda)} \left[ \sum_{m=1}^{\infty} \frac{U_m^{(1-\lambda_1)-1}}{\mu^{(1-\lambda_1)-1}} \right]^{\frac{1}{\lambda}} \left[ \sum_{n=1}^{\infty} \frac{V_n^{(1-\lambda_2)-1}}{\nu^{(1-\lambda_2)-1}} \right]^{\frac{1}{\gamma}}.$$  

(6)

In this paper, by using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard’s inequality, a new Hardy-Mulholland-type inequality with a best possible constant factor is given as follows: If $\mu_1 = \nu_1 = 1$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are positive and decreasing, with $U_\infty = V_\infty = \infty$, we have the following inequality:

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln \mu_m \ln \nu_n} \leq \frac{\pi}{\sin(\pi/\gamma)} \left[ \sum_{m=2}^{\infty} \frac{U_m^{(1-\lambda_1)-1}}{\mu^{(1-\lambda_1)-1}} \right]^{\frac{1}{\lambda}} \left[ \sum_{n=2}^{\infty} \frac{V_n^{(1-\lambda_2)-1}}{\nu^{(1-\lambda_2)-1}} \right]^{\frac{1}{\gamma}},$$  

(7)

which is an extension of (3). Moreover, the more accurate inequality of (7) and its extension with multi-parameters and the best possible constant factors are obtained. The equivalent forms, the reverses, the operator expressions and some particular cases are considered.

### 2 Some lemmas and an example

In the following, we agree that $p \neq 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $-1 < \gamma \leq 0$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = \lambda$, $\mu_i, \nu_j > 0 \ (i, j \in \mathbb{N})$, with $\mu_1 = \nu_1 = 1$, $U_m$ and $V_n$ are defined by (4),

$$\frac{1}{1 + \frac{\alpha}{2}} \leq \alpha \leq 1, \quad \frac{1}{1 + \frac{\beta}{2}} \leq \beta \leq 1,$$

$$a_m, b_n \geq 0, \quad \|a\|_{p, \Phi_\lambda} := \left( \sum_{m=2}^{\infty} \Phi_\lambda(m) a_m^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|b\|_{q, \Psi_\lambda} := \left( \sum_{n=2}^{\infty} \Psi_\lambda(n) b_n^q \right)^{\frac{1}{q}},$$

where

$$\Phi_\lambda(m) := \left( \frac{U_m}{\mu_{m+1}} \right)^{p-1} \left( \ln \alpha U_m \right)^{(1-\lambda_1)-1},$$

$$\Psi_\lambda(n) := \left( \frac{V_n}{\nu_{n+1}} \right)^{q-1} \left( \ln \beta V_n \right)^{(1-\lambda_2)-1} \quad (m, n \in \mathbb{N} \setminus \{1\}).$$  

(8)
Lemma 1 If \( n \in \mathbb{N} \setminus \{1\} \), \( a \in (n - \frac{1}{2}, n) \), \( f(x) \) is continuous in \( (n - \frac{1}{2}, n + \frac{1}{2}) \), and \( f'(x) \) is strictly increasing in the intervals \((n - \frac{1}{2}, a), (a, n) \) and \((n, n + \frac{1}{2})\) respectively, satisfying
\[
f'(a - 0) \leq f'(a + 0), \quad f'(n - 0) \leq f'(n + 0),
\]
then we have the following Hermite-Hadamard's inequality (cf. [20]).

Proof In view of \( f'(n - 0) \leq f'(n + 0) = \lim_{x \to a^+} f'(x) \) is finite, we set the linear function \( g(x) \) as follows:
\[
g(x) := f'(n - 0)(x - n) + f(n), \quad x \in \left[ n - \frac{1}{2}, n + \frac{1}{2} \right].
\]
Since \( f'(x) \) is strictly increasing in \([n - \frac{1}{2}, a) \) and \((a, n)\), then for \( x \in [n - \frac{1}{2}, a) \),
\[
f'(x) < \lim_{x \to a^-} f'(x) = f'(a - 0) \leq f'(a + 0) < f'(n - 0);
\]
for \( x \in (a, n), f'(x) < \lim_{x \to n^-} f'(x) = f'(n - 0) \). Hence,
\[
(f(x) - g(x))' = f'(x) - f'(n - 0) < 0, \quad x \in \left( n - \frac{1}{2}, a \right) \cup (a, n).
\]
Since \( f(x) - g(x) \) is continuous in \([n - \frac{1}{2}, n]\) with \( f(n) - g(n) = 0 \), it follows that
\[
f(x) - g(x) > 0, \quad x \in \left( n - \frac{1}{2}, n \right).
\]
In the same way, since \( f'(x) \) is strictly increasing in \((n, n + \frac{1}{2})\), then for \( x \in (n, n + \frac{1}{2}) \),
\[
f'(x) > f'(n + 0) \geq f'(n - 0) \). Hence,
\[
(f(x) - g(x))' = f'(x) - f'(n - 0) > 0, \quad x \in \left( n, n + \frac{1}{2} \right).
\]
Since \( f(x) - g(x) \) is continuous in \([n, n + \frac{1}{2}]\) with \( f(n) - g(n) = 0 \), it follows that
\[
f(x) - g(x) > 0, \quad x \in \left( n, n + \frac{1}{2} \right).
\]
Therefore, we have \( f(x) - g(x) > 0, x \in (n - \frac{1}{2}, n + \frac{1}{2}) \setminus \{n\} \). Then we find
\[
\int_{n - \frac{1}{2}}^{n + \frac{1}{2}} f(x) \, dx > \int_{n - \frac{1}{2}}^{n + \frac{1}{2}} g(x) \, dx = f(n),
\]
namely, (9) follows. The lemma is proved.

Note With the assumptions of Lemma 1, if (i) \( a \in (n, n + \frac{1}{2}) \), \( f'(x) \) is strictly increasing in the intervals \((n - \frac{1}{2}, n), (n, a) \) and \((a, n + \frac{1}{2})\), respectively, or (ii) \( a = n \), \( f'(x) \) is strictly increasing in the intervals \((n - \frac{1}{2}, n) \) and \((n, n + \frac{1}{2})\), respectively, then in the same way, we still can obtain (9).
**Example 1** \(\{\mu_n\}_{n=1}^{\infty}\) and \(\{v_n\}_{n=1}^{\infty}\) are decreasing, we set functions \(\mu(t) := \mu_m, t \in (m - 1, m] (m \in \mathbb{N}), v(t) := v_n, t \in (n - 1, n] (n \in \mathbb{N})\), and

\[
U(x) := \int_0^x \mu(t) \, dt \quad (x \geq 0), \quad V(y) := \int_0^y v(t) \, dt \quad (y \geq 0).
\]

Then it follows that \(U(m) = U_m, V(n) = V_n, U(\infty) = U_\infty, V(\infty) = V_\infty\) and

\[
U'(x) = \mu(x) = \mu_m, \quad x \in (m - 1, m),
\]

\[
V'(y) = v(y) = v_n, \quad y \in (n - 1, n) \ (x, y \in \mathbb{N}).
\]

For \(0 < \lambda \leq 1, -1 < \gamma \leq 0\), we set

\[
k_\lambda(x, y) := \frac{1}{x^{\lambda} + y^{\lambda} + \gamma |x^{\lambda} - y^{\lambda}|} \quad (x, y > 0).
\]

We find

\[
0 < K_\gamma(\lambda_1) := \int_0^\infty k_\lambda(1, t)t^{i_2 - 1} \, dt = \int_0^\infty k_\lambda(t, 1)t^{i_1 - 1} \, dt
\]

\[
= \int_0^\infty \frac{t^{i_1 - 1}}{t^\lambda + 1 + \gamma |t^{\lambda} - 1|} \, dt = \int_0^1 \frac{t^{i_1 - 1} + t^{i_2 - 1}}{1 + \gamma + (1 - \gamma)t^\lambda} \, dt
\]

\[
\leq \int_0^1 \frac{t^{i_1 - 1} + t^{i_2 - 1}}{1 + \gamma} \, dt = \frac{1}{1 + \gamma} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \infty,
\]

namely, \(K_\gamma(\lambda_1) \in \mathbb{R}_+\). In the following, we express \(K_\gamma(\lambda_1)\) in other forms.

(i) For \(\gamma = 0\), we obtain

\[
K_0(\lambda_1) = \int_0^\infty \frac{t^{i_1 - 1}}{t^\lambda + 1} \, dt = \frac{1}{\lambda} \int_0^\infty \frac{v^{(\lambda_1 - 1)} - 1}{v + 1} \, dv = \frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})};
\]

(ii) for \(-1 < \gamma < 0, 0 < \frac{1 + \gamma}{1 - \gamma} < 1\), by the Lebesgue term by term integration theorem (cf. [21]), we find

\[
K_\gamma(\lambda_1) = \frac{1}{1 - \gamma} \int_0^1 \frac{t^{\lambda_2 - 1} + t^{\lambda_1 - 1}}{t^\lambda + 1} \, dt
\]

\[
= \frac{1}{\lambda(1 - \gamma)} \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\frac{\lambda_1 - 1}{\lambda}} \int_0^1 \frac{1}{v + 1} \left[ \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} + \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} \right] \, dv
\]

\[
= \frac{1}{\lambda(1 - \gamma)} \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\frac{\lambda_1 - 1}{\lambda}} \int_0^1 \frac{1}{1 + \gamma} \left[ \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} + \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} \right] \, dv
\]

\[
= \frac{1}{\lambda(1 - \gamma)} \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\frac{\lambda_1 - 1}{\lambda}} \frac{\pi}{\sin(\frac{\pi \lambda_2}{\lambda})} + \frac{1}{\lambda(1 - \gamma)} \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\frac{\lambda_1 - 1}{\lambda}} \frac{\pi}{\sin(\frac{\pi \lambda_2}{\lambda})}
\]

\[
= \frac{1}{\lambda(1 - \gamma)} \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\frac{\lambda_1 - 1}{\lambda}} \sum_{k=0}^\infty (-1)^k \lambda \left[ \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} + \left( \frac{1 + \gamma}{1 - \gamma} \right)^{\lambda_2 - 1} v^{\lambda_2 - 1} \right] \, dv
\]
\begin{align}
\frac{1}{\lambda(1-\gamma)} & \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} + \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \right] \frac{\pi}{\sin \left( \frac{\pi}{\lambda} \right)} \\
- \frac{1}{\lambda(1-\gamma)} & \int_{0}^{\frac{1}{\lambda}} \sum_{k=0}^{\infty} (\nu^{2k} - \nu^{2k+1}) \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \nu^{\frac{1}{\lambda} - 1} + \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \nu^{\frac{1}{\lambda} - 1} \right] dv \\
+ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \nu^{\frac{1}{\lambda} - 1} & \right] dv \\
= \frac{1}{1+\gamma} \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} + \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \right] \frac{\pi}{\lambda \sin \left( \frac{\pi}{\lambda} \right)} \\
- \frac{1}{(1+\gamma)} \sum_{k=0}^{\infty} \int_{0}^{\frac{1}{\lambda}} (-1)^{k} \nu^{\frac{1}{\lambda} - 1} \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \nu^{\frac{1}{\lambda} - 1} + \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \nu^{\frac{1}{\lambda} - 1} \right] dv \\
= \frac{1}{1+\gamma} \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} + \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \right] \frac{\pi}{\lambda \sin \left( \frac{\pi}{\lambda} \right)} \\
- \frac{1}{1+\gamma} \sum_{k=0}^{\infty} (-1)^{k} \left( \frac{1+\gamma}{1-\gamma} \right)^{\frac{1}{\lambda} - 1} \left( \frac{1}{\lambda k + \lambda_2} + \frac{1}{\lambda k + \lambda_1} \right).
\end{align}

(iii) for \( \lambda_1 = \lambda_2 = \frac{1}{2}, \) \(-1 < \gamma < 0\), we find

\[ K_{\gamma} \left( \frac{\lambda}{2} \right) = 2 \int_{0}^{1} \frac{t^{(\lambda / 2) - 1}}{1 + \gamma + (1 - \gamma) t^2} \, dt \quad \frac{u^{(1/\lambda + 1/\gamma) - 1}}{1 + (1 + \gamma) t^2} = \frac{4}{\lambda(1+\gamma)} \left( \frac{1+\gamma}{1-\gamma} \right) \frac{1}{\lambda} \int_{0}^{(1/\lambda) - 1/\gamma} \, du \]

For fixed \( m \in \mathbb{N} \setminus \{1\} \), we define the function \( f(y) \) as follows:

\[ f(y) := k_{\gamma} (\ln \alpha U_m, \ln \beta V(y)), \quad y \in \left( n - \frac{1}{2}, n + \frac{1}{2} \right) \quad (n \in \mathbb{N} \setminus \{1\}). \]

Then \( f(y) \) is continuous in \( (n - \frac{1}{2}, n + \frac{1}{2}) \) \((n \in \mathbb{N} \setminus \{1\})\). There exists a unified number \( y_0 > \frac{3}{2} \)

(i) If \( y_0 \in (n - \frac{1}{2}, n + \frac{1}{2}) \), we find

\[ f(y) = \begin{cases} 
\frac{1}{(1+\gamma) \ln \alpha U_m + (1-\gamma) \ln \beta V(y)}, & n - \frac{1}{2} < y < y_0, \\
\frac{1}{(1-\gamma) \ln \alpha U_m + (1+\gamma) \ln \beta V(y)}, & y_0 < y < n + \frac{1}{2}.
\end{cases} \]
For \( y_0 \neq n \), we obtain for \( y \neq n \) that

\[
 f'(y) = \begin{cases} 
\frac{-\lambda(1-\gamma)V'(y)\ln^{n-1}\beta V(y)}{V(y)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(y)\}^2}, & n - \frac{1}{2} < y < y_0, \\
\frac{-\lambda(1-\gamma)ln^{n-1}\beta U_n}{V(y)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(y)\}^2}, & y_0 < y < n + \frac{1}{2}, \\
\end{cases}
\]

for \( y_0 \neq n \), we obtain for \( y = n \) that

\[
 f'(n) = \begin{cases} 
\frac{-\lambda(1-\gamma)ln^{n-1}\beta U_n}{V(n)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(n)\}^2}, & n - \frac{1}{2} < y < y_0, \\
\frac{-\lambda(1-\gamma)ln^{n-1}\beta U_n}{V(n)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(n)\}^2}, & y_0 < y < n + \frac{1}{2}, \\
\end{cases}
\]

Since \( 0 < \lambda \leq 1, -1 < \gamma \leq 0 \), \((1-\gamma)\nu_n \geq (1+\gamma)\nu_{n+1}\), in view of the above results, we find \( f'(n-0) \leq f'(n+0) \) (\( n \neq 0 \)), and \( f'(y) < 0 \) is strictly increasing in \((n - \frac{1}{2}, y_0)\), \((y_0, n)\) and \((n, n + \frac{1}{2})\) for \( y < n \) or in \((n - \frac{1}{2}, n), (n, y_0)\) and \((y_0, n + \frac{1}{2})\) for \( y > n \).

We obtain

\[
 f'(y_0 - 0) = \frac{-\lambda(1-\gamma)V'(y_0 - 0)\ln^{n-1}\beta V(y_0)}{V(y_0)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(y_0)\}^2},
\]

\[
 f'(y_0 + 0) = \frac{-\lambda(1+\gamma)V'(y_0 + 0)\ln^{n-1}\beta V(y_0)}{V(y_0)\{(1+\gamma)\ln^2\alpha U_m+(1-\gamma)\ln^2\beta V(y_0)\}^2},
\]

Since for \( y_0 = n \), \( V'(y_0 - 0) = \nu_n \), \( V'(y_0 + 0) = \nu_{n+1} \) and for \( y_0 \neq n \), \( V'(y_0 - 0) = V'(y_0) \), then we have \( \lambda(1-\gamma)V'(y_0 - 0) \geq \lambda(1+\gamma)V'(y_0 + 0) \), namely, \( f'(y_0 - 0) \leq f'(y_0 + 0) \).

(ii) If \( y_0 \notin (n - \frac{1}{2}, n + \frac{1}{2}) \), then it follows that \( f'(y) = \frac{V'(y)}{V(y)^{2}} \frac{d}{dy} k_{\lambda}(\ln\alpha U_m, \ln\beta V(y)) < 0 \), \( y \in (n - \frac{1}{2}, n + \frac{1}{2}) \setminus \{n\} \). We still can find that

\[
 \frac{\nu_n}{V_n} \frac{d}{dy} k_{\lambda}(\ln\alpha U_m, \ln\beta V(y)) \bigg|_{y=n} = f'(n-0)
\]

\[
 \leq f'(n+0) = \frac{\nu_{n+1}}{V_n} \frac{d}{dy} k_{\lambda}(\ln\alpha U_m, \ln\beta V(y)) \bigg|_{y=n},
\]

and \( f'(y) < 0 \) is strictly increasing in \((n - \frac{1}{2}, n)\) and \((n, n + \frac{1}{2})\).

Therefore, \( f(y) \) satisfies the conditions of Lemma 1 with Note. So does \( g(y) = \frac{f(y)}{V(y)^{n+2}\beta V(y)} \). Hence, by (9), we have

\[
 k_{\lambda}(\ln\alpha U_m, \ln\beta V_a) \leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\nu_{n+1}}{V_n} \frac{d}{dy} k_{\lambda}(\ln\alpha U_m, \ln\beta V(y)) \left( \nu_{n+1} \frac{d}{dy} k_{\lambda}(\ln\alpha U_m, \ln\beta V(y)) \right) dy \quad (n \in \mathbb{N}\setminus\{1\}).
\]
Definition 1 Define the following weight coefficients:

\[
\omega(\lambda_2, m) = \sum_{n=2}^{\infty} k_\lambda (\ln \alpha U_m, \ln \beta V_n) \frac{\nu_{n+1} \ln^{1+2} \alpha U_m}{V_n \ln^{1+2} \beta V_n}, \quad m \in \mathbb{N} \setminus \{1\},
\]

\[
\sigma(\lambda_1, n) = \sum_{m=2}^{\infty} k_\lambda (\ln \alpha U_m, \ln \beta V_n) \frac{\mu_{m-1} \ln^{1} \beta V_n}{U_m \ln^{1} \alpha U_m}, \quad n \in \mathbb{N} \setminus \{1\}.
\]

(17)

Lemma 2 If \(\{\mu_m\}_{m=2}^{\infty}\) and \(\{\nu_n\}_{n=4}^{\infty}\) are decreasing and \(U_\infty = V_\infty = \infty\), then for \(m, n \in \mathbb{N} \setminus \{1\}\), we have the following inequalities:

\[
\omega(\lambda_2, m) < K_\gamma (\lambda_1),
\]

(18)

\[
\sigma(\lambda_1, n) < K_\gamma (\lambda_1),
\]

(19)

where \(K_\gamma (\lambda_1)\) is determined by (12).

Proof For \(y \in \left( n - \frac{1}{2}, n + \frac{1}{2} \right) \setminus \{n\}\), \(\nu_{n+1} \leq V'(y)\), by (16), we find

\[
\omega(\lambda_2, m) < \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda (\ln \alpha U_m, \ln \beta V(y)) \frac{\nu_{n+1} \ln^{1+2} \alpha U_m}{V(y) \ln^{1+2} \beta V(y)} \, dy
\]

\[
\leq \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda (\ln \alpha U_m, \ln \beta V(y)) \frac{V'(y) \ln^{1+2} \alpha U_m}{V(y) \ln^{1+2} \beta V(y)} \, dy
\]

\[
= \int_{n}^{n} k_\lambda (\ln \alpha U_m, \ln \beta V(y)) \frac{V'(y) \ln^{1+2} \alpha U_m}{V(y) \ln^{1+2} \beta V(y)} \, dy.
\]

Setting \(t = \int_{n}^{t} k_\lambda (1, t) t^{1+2-1} \, dt = K_\gamma (\lambda_1)\).

Hence, we obtain (18). In the same way, we obtain (19). \(\square\)

Note For example, \(\mu_n = \nu_n = \frac{1}{m^2} (0 \leq \sigma \leq 1)\) satisfies the conditions of Lemma 2.

Lemma 3 With regard to the assumptions of Lemma 2, (i) for \(m, n \in \mathbb{N} \setminus \{1\}\), we have

\[
K_\gamma (\lambda_1)(1 - \theta(\lambda_2, m)) < \omega(\lambda_2, m),
\]

(20)

\[
K_\gamma (\lambda_1)(1 - \theta(\lambda_1, n)) < \sigma(\lambda_1, n),
\]

(21)

where

\[
\theta(\lambda_2, m) = \frac{k_\lambda (1, \frac{\ln \beta (1 + \nu_2)}{\ln \alpha U_m}) \ln^{1+2} \beta (1 + \nu_2)}{\lambda_2 K_\gamma (\lambda_1) \ln^{1+2} \alpha U_m}
\]

\[
= O\left(\frac{1}{\ln^{1+2} \alpha U_m}\right) \in (0, 1) \quad \left(\theta(m) \in \left( \frac{1 - \beta}{\beta \nu_2}, 1 \right) \right),
\]

(22)
\[
\phi(\lambda_1, n) = \frac{k_1(\ln(1 + \mu_2^2)}{\lambda_1 K_\beta(\lambda_1)} \ln^{\alpha} (1 + \mu_2) \\
\ln^{\alpha} (1 + \mu_2)
\]

\[
= O\left(\frac{1}{\ln^{\alpha} \beta V_n}\right) \in (0, 1) \quad \left(\phi(n) \in \left(\frac{1 - \alpha}{\alpha \mu_2}, 1\right)\right); \quad (23)
\]

(ii) for any \( c > 0 \), we have

\[
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+c} \alpha U_m} = \frac{1}{c} \left[\frac{1}{\ln^\alpha (1 + \mu_2)} + cO(1)\right], \quad (24)
\]

\[
\sum_{n=2}^{\infty} \frac{\nu_{n+1}}{V_n \ln^{1+c} \beta V_n} = \frac{1}{c} \left[\frac{1}{\ln^\beta (1 + \nu_2)} + c\tilde{O}(1)\right]. \quad (25)
\]

**Proof** In view of \( \beta \geq 1 \) and \( \beta \geq \frac{1}{1 + \nu_2} > \frac{1}{1 + \nu_2} \), it follows that \( 1 \leq \frac{1 - \beta \nu_2}{1 + \beta \nu_2} + 1 < 2 \). Since, by Examples 1, \( g(y) \) is strictly decreasing in \([m, n+1]\), then for \( m \in \mathbb{N} \setminus \{1\} \), we find

\[
\omega(\lambda_2, m) > \sum_{n=2}^{\infty} \int_y^{y_+1} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{\ln^{\alpha^2} \alpha U_m}{V(y) \ln^{1+c} \beta V(y)} dy
\]

\[
= \int_{2}^{\infty} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{V'(y) \ln^{\alpha^2} \alpha U_m}{V(y) \ln^{1+c} \beta V(y)} dy
\]

\[
= \int_{\frac{1}{1 + \beta \nu_2} + 1}^{\infty} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{V'(y) \ln^{\alpha^2} \alpha U_m}{V(y) \ln^{1+c} \beta V(y)} dy
\]

\[
= \int_{\frac{1}{1 + \beta \nu_2} + 1}^{2} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{V'(y) \ln^{\alpha^2} \alpha U_m}{V(y) \ln^{1+c} \beta V(y)} dy.
\]

Setting \( t = \frac{\ln \beta V(y)}{\ln \alpha U_m} \), we have \( \ln \beta V(\frac{1}{1 + \beta \nu_2} + 1) = \ln \beta (1 + \frac{1 - \beta \nu_2}{1 + \beta \nu_2} \nu_2) = 0 \) and

\[
\omega(\lambda_2, m) > \int_0^{\infty} k_\beta(1, t) t^{1+c} dt - \int_{\frac{1}{1 + \beta \nu_2} + 1}^{2} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{V'(y) \ln^{\alpha^2} \alpha U_m}{V(y) \ln^{1+c} \beta V(y)} dy
\]

\[
= K_\beta(\lambda_1) (1 - \theta(\lambda_2, m)),
\]

where

\[
\theta(\lambda_2, m) := \frac{\ln^{\alpha^2} \alpha U_m}{K_\beta(\lambda_1)} \int_{\frac{1}{1 + \beta \nu_2} + 1}^{2} k_\beta(\ln \alpha U_m \ln \beta V(y)) \frac{V'(y)}{V(y) \ln^{1+c} \beta V(y)} dy \in (0, 1).
\]

In view of the integral mid-value theorem, for fixed \( m \in \mathbb{N} \setminus \{1\} \), there exists \( \theta(m) \in (\frac{1}{1 + \beta \nu_2}, 1) \) such that

\[
\theta(\lambda_2, m) = \frac{\ln^{\alpha^2} \alpha U_m}{K_\beta(\lambda_1)} k_\beta(\ln \alpha U_m \ln \beta V(1 + \theta(m))) \int_{\frac{1}{1 + \beta \nu_2} + 1}^{2} \frac{V'(y)}{V(y) \ln^{1+c} \beta V(y)} dy
\]

\[
= \frac{\ln^{\alpha^2} \alpha U_m}{\lambda_2 K_\beta(\lambda_1)} k_\beta(\ln \alpha U_m \ln \beta V(1 + \theta(m))) \ln^{\alpha^2} \beta (1 + \nu_2)
\]

\[
= \frac{1}{\lambda_2 K_\beta(\lambda_1)} k_\beta(1, \ln \beta V(1 + \theta(m))) \ln^{\alpha^2} \beta (1 + \nu_2) \ln^{\alpha^2} \alpha U_m.
\]
Hence, we find

\[ 0 < \theta(\lambda_2, m) \leq \frac{1}{\lambda_2 K_p(\lambda_1)} \frac{\ln^{12} \beta(1 + \nu_2)}{(1 + \gamma) \ln^{12} \alpha U_m}, \]

namely, \( \theta(\lambda_2, m) = O(\frac{1}{\ln^{2} \alpha U_m}) \). Then we obtain (20) and (22). In the same way, we obtain (21) and (23).

For \( c > 0 \), we find

\[
\sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1+\epsilon} \alpha U_m} \leq \sum_{m=2}^{\infty} \frac{\mu_m}{U_m \ln^{1+\epsilon} \alpha U_m} = \frac{\mu_2}{U_2 \ln^{1+\epsilon} \alpha U_2} + \sum_{m=3}^{\infty} \frac{\mu_m}{U_m \ln^{1+\epsilon} \alpha U_m}
\]

\[
= \frac{\mu_2}{U_2 \ln^{1+\epsilon} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x) \ln \alpha U(x)}{U_m \ln^{1+\epsilon} \alpha U_m} dx
\]

\[
= \frac{\mu_2}{U_2 \ln^{1+\epsilon} \alpha U_2} + \sum_{m=3}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U(x) \ln^{1+\epsilon} \alpha U(x)} \ln \alpha U(x) dx
\]

\[
= \frac{\mu_2}{U_2 \ln^{1+\epsilon} \alpha U_2} + \frac{1}{c \ln \alpha (1 + \mu_2)} \cdot \left[ \int_{m-1}^{m} \frac{1}{U_m \ln^{1+\epsilon} \alpha U_m} dx \right]
\]

Hence, we obtain (20). In the same way, we obtain (21).

**Lemma 4** If \(-1 < \gamma \leq 0, 0 < \lambda_1, \lambda_2 < 1, \lambda_1 + \lambda_2 \leq 1, K_p(\lambda_3)\) is determined by (12), then for \(0 < \delta < \min(\lambda_1, \lambda_2),\) we have

\[ K_p(\lambda_1 \pm \delta) = K_p(\lambda_1) + o(1) \quad (\delta \to 0^+). \]  

**Proof** We find, for \(0 < \delta < \min(\lambda_1, \lambda_2),\)

\[
|K_p(\lambda_1 + \delta) - K_p(\lambda_1)| \leq \int_{0}^{\infty} \frac{t^{\lambda_1-1} |t^\delta - 1|}{t^\delta + 1 + \gamma |t^\delta - 1|} dt
\]

\[
= \int_{0}^{1} \frac{t^{\lambda_1-1} (1 - t^\delta)}{1 + \gamma + (1 - \gamma) t^\delta} dt + \int_{1}^{\infty} \frac{t^{\lambda_1-1} (t^\delta - 1)}{1 - \gamma + (1 + \gamma) t^\delta} dt
\]

\[
\leq \frac{1}{1 + \gamma} \left[ \int_{0}^{1} t^{\lambda_1-1} (1 - t^\delta) dt + \int_{1}^{\infty} \frac{t^{\lambda_1-1} (t^\delta - 1)}{t^\delta} dt \right]
\]

\[
= \frac{1}{1 + \gamma} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \delta} + \frac{1}{\lambda_2 - \delta} - \frac{1}{\lambda_2} \right) \to 0 \quad (\delta \to 0^+). \]
In the same way, we find

\[ |K_\nu(\lambda_1 - \delta) - K_\nu(\lambda_1)| \leq \frac{1}{1 + \gamma} \left[ \int_0^1 t^{\nu+1-1} (t^{\delta} - 1) \, dt + \int_1^{\infty} t^{\nu+1-1} (1 - t^{\delta}) \, dt \right] \]

\[ = \frac{1}{1 + \gamma} \left( \frac{1}{\lambda_1 - \delta} - \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_2 + \delta} \right) \to 0 \quad (\delta \to 0^+), \]

and then we have (26).

3 Main results

In the following, we also set

\[ \Phi_\nu(m) := \omega(\lambda_2, m) \left( \frac{U_m}{\mu_{m+1}} \right)^{p-1} (\ln \alpha U_m)^p (\ln \beta V_n)^q \quad (m \in \mathbb{N} \setminus \{1\}), \]

\[ \Psi_\nu(n) := \sigma(\lambda_1, n) \left( \frac{V_n}{\upsilon_{n+1}} \right)^{q-1} (\ln \beta V_n)^p (\ln \beta V_n)^q \quad (n \in \mathbb{N} \setminus \{1\}). \]

Theorem 1

(i) For \( p > 1 \), we have the following equivalent inequalities:

\[ I := \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) a_n b_n \leq \|a\|_{\sum_{p} \Phi_\nu} \|b\|_{q, \Psi_\nu}, \]

\[ J := \left\{ \sum_{n=2}^{\infty} \upsilon_{n+1} \ln(\lambda_2) - 1 \beta V_n \left( \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) a_m \right)^p \right\}^{\frac{1}{p}} \leq \|a\|_{\sum_{p} \Phi_\nu}. \]

(ii) for \( 0 < p < 1 \) (or \( p < 0 \)), we have the equivalent reverse of (28) and (29).

Proof (i) By Hölder’s inequality with weight (cf. [20]) and (17), we have

\[ \left( \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) a_m \right)^p \]

\[ = \left[ \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) \left( \frac{U_m}{\mu_{m+1}} (\frac{\ln \alpha U_m)^{(1-\lambda_1)/q}}{\ln \beta V_n)^{(1-\lambda_2)/p}} \right)^{p-1} \right]^p \]

\[ \times \left( \frac{U_m}{\mu_{m+1}} (\ln \alpha U_m)^{(1-\lambda_1)/q} \upsilon_{m+1}^{\frac{1}{p}} \right)^{\upsilon_{m+1}^{\frac{1}{p}}} \]

\[ \leq \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) \frac{U_m^{p-1} (\ln \alpha U_m)^{(1-\lambda_1)/q} \upsilon_{m+1}^{\frac{1}{p}}} {U_m^{p-1} (\ln \beta V_n)^{(1-\lambda_2)/p} \mu_{m+1}^{\frac{1}{q}}} \chi_{m+1}^{\frac{1}{p}} \]

\[ \times \left[ \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) \frac{(\ln \beta V_n)^{(1-\lambda_2)(q-1)} \mu_{m+1}^{p-1}} {U_m^{(1-\lambda_1)/q} \upsilon_{m+1}^{\frac{1}{p}}} \right]^{p-1} \]

\[ = \frac{(\sigma(\lambda_1, n))^{p-1} V_n}{(\ln \beta V_n)^{p-2}} \sum_{m=2}^{\infty} k_\nu(\ln \alpha U_m, \ln \beta V_n) \frac{U_m^{p-1} (\ln \alpha U_m)^{(1-\lambda_1)/q} \upsilon_{m+1}^{\frac{1}{p}}} {U_m^{p-1} (\ln \beta V_n)^{(1-\lambda_2)/p} \mu_{m+1}^{p-1}} \chi_{m+1}^{\frac{1}{p}}. \]
Then, by (16), we find

$$J \leq \left[ \sum_{m=2}^{\infty} \left( \sum_{n=2}^{\infty} k_n (\ln \alpha U_m, \ln \beta V_n) \frac{\nu_{n+1} U_m^{p-1} (\ln \alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln \beta V_n)^{(1-\lambda_2)} \mu_{m+1}^p} a_m \right)^{\frac{1}{p}} \right]^{\frac{1}{p}}$$

$$= \left[ \sum_{m=2}^{\infty} \left( \sum_{n=2}^{\infty} k_n (\ln \alpha U_m, \ln \beta V_n) \frac{\nu_{n+1} U_m^{p-1} (\ln \alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln \beta V_n)^{(1-\lambda_2)} \mu_{m+1}^p} a_m \right)^{\frac{1}{p}} \right]^{\frac{1}{p}}$$

$$= \left[ \sum_{m=2}^{\infty} o(\lambda_2, m) \left( \frac{U_m}{\mu_{m+1}} \right)^{p-1} \left( \frac{\ln \alpha U_m}{\ln \beta V_n} \right)^{(1-\lambda_1)(p-1)} a_m \right]^{\frac{1}{p}},$$

and then (29) follows.

By Hölder’s inequality (cf. [20]), we have

$$I = \sum_{m=2}^{\infty} \left[ \sum_{n=2}^{\infty} \frac{\nu_{n+1} U_m^{p-1} (\ln \alpha U_m)^{(1-\lambda_1)(p-1)}}{(\ln \beta V_n)^{(1-\lambda_2)} \mu_{m+1}^p} a_m \right]^{\frac{1}{p}} \left( \frac{\sigma (\lambda_1, n)}{(\ln \beta V_n)^{(1-\lambda_2)} \mu_{m+1}} b_n \right)$$

$$\leq J \|b\|_{q, \bar{\psi}_k},$$

(32)

Then, by (29), we have (28).

On the other hand, assuming that (28) is valid, we set

$$b_n := \frac{\nu_{n+1} \ln^{p-1} \beta V_n (\sum_{m=2}^{\infty} k_n (\ln \alpha U_m, \ln \beta V_n) a_m)^{p-1}}{\sigma (\lambda_1, n) \mu_{m+1}^p}, \quad n \in \mathbb{N} \setminus \{1\}.$$ (33)

Then we find $J' = \|b\|_{q, \bar{\psi}_k}^q$. If $J = 0$, then (29) is trivially valid; if $J = \infty$, then by (31), (29) takes the form of equality. Suppose that $0 < J < \infty$. By (28), it follows that

$$\|b\|_{q, \bar{\psi}_k}^q = J' \leq \|a\|_{p, \phi_k} \|b\|_{q, \bar{\psi}_k},$$

(34)

$$\|b\|_{q, \bar{\psi}_k}^{p-1} = J \leq \|a\|_{p, \phi_k},$$

(35)

and then (29) follows, which is equivalent to (28).

(ii) For $0 < p < 1$ (or $p < 0$), by the reverse Hölder’s inequality with weight (cf. [20]) and (13), we obtain the reverse of (30) (or (30)), then we have the reverse of (31), and then the reverse of (29) follows. By Hölder’s inequality (cf. [20]), we have the reverse of (32), and then by the reverse of (29), the reverse of (28) follows.

On the other hand, assuming that the reverse of (28) is valid, we set $b_n$ as (33). Then we find $J' = \|b\|_{q, \bar{\psi}_k}^q$. If $J = \infty$, then the reverse of (29) is trivially valid; if $J = 0$, then by the reverse of (31), (29) takes the form of equality ($= 0$). Suppose that $0 < J < \infty$. By the reverse of (28), it follows that the reverses of (34) and (35) are valid, and then the reverse of (29) follows, which is equivalent to the reverse of (28). \[\square\]

**Theorem 2** If $p > 1$, $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are decreasing, $U_\infty = V_\infty = \infty$, $\|a\|_{p, \phi_k} \in \mathbb{R}_+$, and $\|b\|_{q, \bar{\psi}_k} \in \mathbb{R}_+$, then we have the following equivalent inequalities:

$$\sum_{n=2}^{\infty} k_n (\ln \alpha U_m, \ln \beta V_n) a_m b_n < K_p (\lambda_1) \|a\|_{p, \phi_k} \|b\|_{q, \bar{\psi}_k},$$

(36)
\[
J_1 := \left\{ \sum_{n=2}^{\infty} \frac{V_{n+1}}{V_n} \ln^{p-2} \beta V_n \left( \sum_{m=2}^{\infty} k_2 (\ln \alpha U_m, \ln \beta V_m) a_m \right) \right\}^{\frac{1}{p}} < K_{\gamma}(\lambda_1) \| \tilde{a} \|_{p, \Phi_1}, \tag{37}
\]

where the constant factor \( K_{\gamma}(\lambda_1) \) is the best possible.

**Proof.** Using (18) and (19) in (28) and (29), we obtain the equivalent inequalities (36) and (37).

For \( \varepsilon \in (0, \min(p \lambda_1, p(1-\lambda_2))] \), we set \( \tilde{\lambda}_3 = \lambda_3 - \frac{\varepsilon}{p} (\in (0,1)) \), \( \tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} (\in (0,1)) \), and

\[
\tilde{a}_m := \frac{\mu_{m+1}}{U_m} \ln^{\tilde{\lambda}_3 - 1} \alpha U_m = \frac{\mu_{m+1}}{U_m} \ln^{\tilde{\lambda}_2 - 1} \alpha U_m,
\]

\[
\tilde{b}_n := \frac{V_{n+1}}{V_n} \ln^{\tilde{\lambda}_2 - 1} \beta V_n = \frac{V_{n+1}}{V_n} \ln^{\tilde{\lambda}_2 - 1} \beta V_n.
\tag{38}
\]

Then, by (24), (25) and (21), we have

\[
\| \tilde{a} \|_{p, \Phi_1} \| \tilde{b} \|_{q, \Psi_1} = \left( \sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m} \ln^{p+1+\varepsilon} \alpha U_m \right)^{\frac{1}{p}} \left( \sum_{n=2}^{\infty} \frac{V_{n+1}}{V_n} \ln^{p+1+\varepsilon} \beta V_n \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{\varepsilon} \left[ \ln^\varepsilon \alpha (1 + \mu_2) + \varepsilon O(1) \right]^{\frac{1}{p}} \left[ \ln^\varepsilon \beta (1 + \nu_2) + \varepsilon \tilde{O}(1) \right]^{\frac{1}{q}},
\]

\[
\tilde{I} := \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_2 (\ln \alpha U_m, \ln \beta V_m) \tilde{a}_m \tilde{b}_n
\]

\[
= \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_2 (\ln \alpha U_m, \ln \beta V_m) \frac{\mu_{m+1} \ln^{\lambda_2} \beta V_n}{U_m \ln^{\lambda_3} \alpha U_m} \frac{V_{n+1}}{V_n} \ln^{\lambda_2 - 1} \beta V_n
\]

\[
= \sum_{n=2}^{\infty} \frac{V_{n+1}}{V_n} \ln^{\lambda_2 + \varepsilon} \beta V_n \geq K_{\gamma}(\tilde{\lambda}_1) \sum_{n=2}^{\infty} \left( 1 - O\left( \frac{1}{\ln^{\lambda_1 + \varepsilon} \beta V_n} \right) \right) \frac{V_{n+1}}{V_n} \ln^{\lambda_1 + \varepsilon} \beta V_n
\]

\[
= K_{\gamma}(\tilde{\lambda}_1) \left[ \sum_{n=2}^{\infty} \frac{V_{n+1}}{V_n} \ln^{\lambda_1 + \varepsilon} \beta V_n - \sum_{n=2}^{\infty} O\left( \frac{1}{\ln^{\lambda_1 + \varepsilon} \beta V_n} \right) \frac{V_{n+1}}{V_n} \right]
\]

\[
= \frac{1}{\varepsilon} K_{\gamma}(\tilde{\lambda}_1) \left[ \frac{1}{\ln^\varepsilon \beta (1 + \nu_2)} + \varepsilon \tilde{O}(1) - O(1) \right].
\]

If there exists a positive constant \( K \leq K_{\gamma}(\lambda_1) \) such that (36) is valid when replacing \( K_{\gamma}(\lambda_1) \) by \( K \), then, in particular, we have \( \varepsilon \tilde{I} \leq \varepsilon K \| \tilde{a} \|_{p, \Phi_1} \| \tilde{b} \|_{q, \Psi_1} \), namely,

\[
K_{\gamma} \left( \lambda_1 - \frac{\varepsilon}{p} \right) \left[ \frac{1}{\ln^\varepsilon \beta (1 + \nu_2)} + \varepsilon \left( \tilde{O}(1) - O(1) \right) \right]
\]

\[
< K \left[ \frac{1}{\ln^\varepsilon \alpha (1 + \mu_2)} + \varepsilon O(1) \right]^\frac{1}{p} \left[ \frac{1}{\ln^\varepsilon \beta (1 + \nu_2)} + \varepsilon \tilde{O}(1) \right]^\frac{1}{q}.
\]

In view of (26), it follows that \( K_{\gamma}(\lambda_1) \leq K(\varepsilon \to 0^+) \). Hence, \( K = K_{\gamma}(\lambda_1) \) is the best possible constant factor of (36).

Similarly to (32), we still can find the following inequality:

\[
I \leq J_1 \| \tilde{b} \|_{q, \Psi_1}.
\tag{39}
\]

Hence, we can prove that the constant factor $K_γ(λ_1)$ in (37) is the best possible. Otherwise, we would reach the contradiction by (39) that the constant factor in (36) is not the best possible.

**Remark 1**

(i) For $α = β = 1$ in (36) and (37), setting

$$φ_λ(λ_1):= \left( \frac{U_m}{μ_{m+1}} \right)^{p-1} \left( \ln U_m \right)^{g(1-λ_1)-1},$$

$$ψ_λ(λ_2):= \left( \frac{V_n}{μ_{n+1}} \right)^{q-1} \left( \ln V_n \right)^{g(1-λ_2)-1} \quad (m, n \in \mathbb{N}\backslash\{1\}),$$

we have the following equivalent Mulholland-type inequalities:

$$\sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{a_m b_n}{\ln(αβ U_m V_n) + γ \ln U_m} < K_{1,γ}(λ_1) ∥a∥_{p,φ_λ} ∥b∥_{q,ψ_λ}.$$  \hspace{1em} (40)

$$\left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{a_m}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{p}} \left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{b_n}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{q}} < K_{1,γ}(λ_1) ∥a∥_{p,φ_λ}.$$  \hspace{1em} (41)

(40) is an extension of (7) and the following inequality (for $λ = 1, λ_1 = \frac{1}{q}, λ_2 = \frac{1}{p}, γ = 0$):

$$\left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{a_m b_n}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{p}} \left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{b_n}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{q}} < \frac{π}{\sin(\frac{π}{p})} \left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{U_m}{μ_{m+1}} \right]^{p-1} \left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{V_n}{μ_{n+1}} \right]^{q-1}.$$  \hspace{1em} (42)

(ii) For $λ = 1, λ_1 = \frac{1}{q}, λ_2 = \frac{1}{p}$ in (36) and (37), we have the following equivalent inequalities:

$$\sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{a_m b_n}{\ln(αβ U_m V_n) + γ \ln U_m} < K_{1,γ}(λ_1) ∥a∥_{p,φ_λ} ∥b∥_{q,ψ_λ}.$$  \hspace{1em} (43)

$$\left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{a_m}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{p}} \left[ \sum_{n=2}^{∞} \sum_{m=2}^{∞} \frac{b_n}{\ln(αβ U_m V_n) + γ \ln U_m} \right]^{\frac{1}{q}} < K_{1,γ}(λ_1) ∥a∥_{p,φ_λ}.$$  \hspace{1em} (44)

where

$$K_{1,γ}(λ_1):= \frac{1}{q} \int_0^1 \frac{t^{\frac{1}{p}} \gamma + (1-\gamma) t}{1 + \gamma} \, dt = \frac{1}{1 + \gamma} \left[ \frac{1 + \gamma}{1 - \gamma} \right]^{\frac{1}{p}} \frac{π}{\sin(\frac{π}{p})} \left[ \frac{1}{\gamma} + \frac{1}{1 - \gamma} \right]^{\frac{1}{q}}.$$  \hspace{1em} (45)
(iii) For \( \gamma = 0 \), (43) reduces to the following more accurate Hardy-Mulholland-type inequality (7):

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_mb_n}{\ln(\alpha \beta U_m V_n)} < \pi \sin\left(\frac{\pi}{p}\right) \left[ \sum_{m=2}^{\infty} \left( \frac{U_m}{m^{p-1}} \right) \right]^{1/p} \left[ \sum_{n=2}^{\infty} \left( \frac{V_n}{n^{q-1}} \right) \right]^{1/q}.
\] (46)

In particular, for \( \mu_i = \nu_j = 1 \) (\( i, j \in \mathbb{N} \)), (46) reduces to the following more accurate Mulholland’s inequality (\( \frac{2}{3} \leq \alpha, \beta \leq 1 \)):

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_mb_n}{\ln(\alpha \beta mn)} < \pi \sin\left(\frac{\pi}{p}\right) \left[ \sum_{m=2}^{\infty} m^{p-1} a_m^p \right]^{1/p} \left[ \sum_{n=2}^{\infty} n^{q-1} b_n^q \right]^{1/q}.
\] (47)

For \( p > 1 \), \( \Psi_1^{-p}(n) = \frac{\log(\beta V_n)}{n^{p-1}} \), we define the following normed spaces:

\[
\ell_{p, \Phi_2} := \{ a = (a_m)_{m=2}^{\infty} : \|a\|_{p, \Phi_2} < \infty \},
\]

\[
\ell_{q, \Psi_2} := \{ b = (b_n)_{n=2}^{\infty} : \|b\|_{q, \Psi_2} < \infty \},
\]

\[
\ell_{p, \Psi_1^{-p}} := \{ c = (c_n)_{n=2}^{\infty} : \|c\|_{p, \Psi_1^{-p}} < \infty \}.
\]

Assuming that \( a = (a_m)_{m=2}^{\infty} \in \ell_{p, \Phi_2} \), setting

\[
c = (c_n)_{n=2}^{\infty}, \quad c_n := \sum_{m=2}^{\infty} k_2(\ln \alpha U_m, \ln \beta V_n) a_m, \quad n \in \mathbb{N}\setminus\{1\},
\]

we can rewrite (37) as follows:

\[
\|c\|_{p, \Psi_1^{-p}} < K_\gamma(\lambda_1) \|a\|_{p, \Phi_2} < \infty,
\]

namely, \( c \in \ell_{p, \Psi_1^{-p}} \).

**Definition 2** Define a Hardy-Mulholland-type operator \( T : \ell_{p, \Phi_2} \to \ell_{p, \Psi_1^{-p}} \) as follows: for any \( a = (a_m)_{m=2}^{\infty} \in \ell_{p, \Phi_2} \), there exists a unique representation \( Ta = c \in \ell_{p, \Psi_1^{-p}} \). Define the formal inner product of \( Ta \) and \( b = (b_n)_{n=2}^{\infty} \in \ell_{q, \Psi_2} \) as follows:

\[
(Ta, b) := \sum_{n=2}^{\infty} \left( \sum_{m=2}^{\infty} k_2(\ln \alpha U_m, \ln \beta V_n) a_m \right) b_n.
\] (48)

Then we can rewrite (36) and (37) as follows:

\[
(Ta, b) < K_\gamma(\lambda_1) \|a\|_{p, \Phi_2} \|b\|_{q, \Psi_2},
\] (49)

\[
\|Ta\|_{p, \Psi_1^{-p}} < K_\gamma(\lambda_1) \|a\|_{p, \Phi_2}.
\] (50)

Define the norm of operator \( T \) as follows:

\[
\|T\| := \sup_{a \neq 0 \in \ell_{p, \Phi_2}} \frac{\|Ta\|_{p, \Psi_1^{-p}}}{\|a\|_{p, \Phi_2}}.
\]
Then, by (50), we find \( \| T \| \leq K_y(\lambda_1) \). Since the constant factor in (50) is the best possible, we have

\[
\| T \| = K_y(\lambda_1) = \int_0^1 \frac{t^\gamma - 1 + t^{\gamma - 1}}{1 + \gamma + (1 - \gamma)t} dt.
\]

(51)

4 Some reverses

In the following, we also set

\[
\hat{\Omega}_x(m) := (1 - \theta(\lambda_2, m)) \left( \frac{U_m}{\mu_{m+1}} \right)^{p - 1} \left( \ln \alpha U_m \right)^{2(1 - \lambda_2) - 1},
\]

\[
\hat{\gamma}_x(n) := (1 - \theta(\lambda_1, n)) \left( \frac{V_n}{\nu_{n+1}} \right)^{q - 1} \left( \ln \beta V_n \right)^{2(1 - \lambda_1) - 1} \quad (m, n \in \mathbb{N} \setminus \{1\}).
\]

(52)

For \( 0 < p < 1 \) or \( p < 0 \), we still use the formal symbols \( \| a \|_{p, \psi_2}, \| b \|_{q, \psi_2}, \| a \|_{p, \hat{\Omega}_x} \) and \( \| b \|_{q, \hat{\gamma}_x} \) et al.

Theorem 3 If \( 0 < p < 1, \{ \mu_m \}_{m=1}^\infty \) and \( \{ \nu_n \}_{n=1}^\infty \) are decreasing, \( U_\infty = V_\infty = \infty, \| a \|_{p, \psi_2} \in \mathbb{R}_+, \) and \( \| b \|_{q, \psi_2} \in \mathbb{R}_+ \), then we have the following equivalent inequalities with the best possible constant factor \( K_y(\lambda_1) \):

\[
\sum_{n=2}^\infty \sum_{m=2}^\infty k_3 \left( \ln \alpha U_m, \ln \beta V_n \right) a_m b_n > K_y(\lambda_1) \| a \|_{p, \hat{\Omega}_x} \| b \|_{q, \psi_2},
\]

(53)

\[
\left\{ \sum_{n=2}^\infty \frac{\nu_{n+1}}{V_n} \ln^{p_2-1} \beta V_n \left( \sum_{m=2}^\infty k_3 \left( \ln \alpha U_m, \ln \beta V_n \right) a_m \right)^{\frac{p}{p-1}} \right\}^{\frac{p}{p-1}} > K_y(\lambda_1) \| a \|_{p, \hat{\Omega}_x}.
\]

(54)

Proof Using (20) and (19) in the reverses of (28) and (29), since

\[
\left( \omega(\lambda_2, m) \right)^{\frac{1}{p}} > \left( K_y(\lambda_1) \right)^{\frac{1}{p}} \left( 1 - \theta(\lambda_2, m) \right)^{\frac{1}{p}} \quad (0 < p < 1),
\]

\[
\left( \sigma(\lambda_1, n) \right)^{\frac{1}{q}} > \left( K_y(\lambda_1) \right)^{\frac{1}{q}} \quad (q < 0),
\]

and

\[
\frac{1}{(K_y(\lambda_1))^{\frac{1}{p-1}}} > \frac{1}{(\sigma(\lambda_1, n))^{\frac{1}{p-1}}} \quad (0 < p < 1),
\]

we obtain equivalent inequalities (53) and (54).

For \( \varepsilon \in (0, \min(p\lambda_1, p(1 - \lambda_2))) \), we set \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{a}_m \) and \( \tilde{b}_n \) as (38). Then, by (24), (25) and (19), we find

\[
\| \tilde{a} \|_{p, \hat{\Omega}_x} \| \tilde{b} \|_{q, \psi_2}
\]

\[
= \left( \sum_{m=2}^\infty (1 - \theta(\lambda_2, m)) \mu_{m+1} \right)^{\frac{1}{p}} \left( \sum_{n=2}^\infty \frac{\nu_{n+1}}{V_n} \ln^{1+p_2} \beta V_n \right)^{\frac{1}{q}}
\]

\[
= \left( \sum_{m=2}^\infty \mu_{m+1} \ln^{1+p_2} \alpha U_m \right) - \sum_{m=2}^\infty O \left( \mu_{m+1} \ln^{1+p_2} \alpha U_m \right) \left( \sum_{n=2}^\infty \frac{\nu_{n+1}}{V_n} \ln^{1+p_2} \beta V_n \right)^{\frac{1}{q}}
\]
It is evident that (52) and (53) are extensions of the following equivalent inequalities:

\[ K \frac{1}{\ln^\varepsilon (1 + \mu_2)} \geq K \frac{1}{\ln^\varepsilon (1 + \nu_2)} + \varepsilon O(1), \]

\[ \tilde{I} := \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_0 \ln \alpha \ln \beta V_n a_m b_n. \]

where the constant factor \( K \) is the best possible.

\[ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_0 \ln \alpha \ln \beta V_n a_m b_n > K \varepsilon \|a\|_{p,\phi_0} \|b\|_{q,\psi_1}, \]

\[ \left\{ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_0 \ln \alpha \ln \beta V_n a_m b_n \right\}^\frac{1}{p} > K \varepsilon \|a\|_{p,\phi_0}, \]

where the constant factor \( K \) is the best possible.

**Remark 2** For \( \alpha = \beta = 1 \), set

\[ \tilde{\theta}(\lambda_n, m) := \frac{1}{\ln^{\mu_1 + \nu_2}(1)} \ln^{\mu_1 + \nu_2}(1) = \frac{1}{\ln^{\mu_1 + \nu_2}(1)} \in (0, 1), \quad (\theta(m) \in (0, 1)), \]

\[ \phi_0(m) := (1 - \tilde{\theta}(\lambda_n, m)) \left( \frac{U_m}{\mu_{m+1}} \right)^{p-1} \ln U_m. \]

It is evident that (53) and (54) are extensions of the following equivalent inequalities:

\[ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_0 \ln \alpha \ln \beta V_n a_m b_n > K \varepsilon \|a\|_{p,\phi_0} \|b\|_{q,\psi_1}, \]
constant factor $K_f(\lambda_1)$:

$$
\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_i (\ln \alpha u_m, \ln \beta v_n) a_m b_n > K_f(\lambda_1) \|a\|_{p, \phi_1} \|b\|_{q, \tilde{\phi}_1},
$$

(57)

$$
J_2 := \left\{ \sum_{n=2}^{\infty} \frac{\nu_{n+1} \ln^{p \lambda - 1} \beta v_n}{(1 - \vartheta (\lambda_1, n))^{p - 1} v_n} \left( \sum_{m=2}^{\infty} k_i (\ln \alpha u_m, \ln \beta v_n) a_m \right)^p \right\}^{1/p} > K_f(\lambda_1) \|a\|_{p, \phi_1}.
$$

(58)

**Proof** Using (18) and (21) in the reverses of (28) and (29), since

$$
(o(\lambda_2, m))^{1/p} > (K_f(\lambda_1))^{1/p} \quad (p < 0),
$$

$$
(o(\lambda_1, n))^{1/q} > (K_f(\lambda_1))^{1/q} (1 - \vartheta (\lambda_1, n))^{1/q} \quad (0 < q < 1),
$$

and

$$
\left[ \frac{1}{(K_f(\lambda_1))^{p - 1} (1 - \vartheta (\lambda_1, n))^{p - 1}} \right]^{1/p} > \left[ \frac{1}{(o(\lambda_1, n))^{p - 1}} \right]^{1/p} \quad (p < 0),
$$

we obtain equivalent inequalities (57) and (58).

For $\varepsilon \in (0, \min(q_\lambda, q_1(1 - \lambda_1)))$, we set $\tilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q} \in (0, 1)$, $\tilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \in (0, 1)$, and

$$
\tilde{a}_m := \frac{\mu_{m+1}}{U_m} \ln^{1 - \varepsilon} \alpha u_m = \frac{\mu_{m+1}}{U_m} \ln \lambda_1^{1 - \frac{\varepsilon}{q}} \alpha u_m,
$$

$$
\tilde{b}_n := \frac{\nu_{n+1} \ln^{1 - \varepsilon} \beta V_n}{V_n} = \frac{\nu_{n+1}}{V_n} \ln \lambda_2^{1 - \frac{\varepsilon}{q}} \beta V_n.
$$

Then, by (24), (25) and (18), we have

$$
\|\tilde{a}\|_{p, \phi_1} \|\tilde{b}\|_{q, \tilde{\phi}_1}
$$

$$
= \left( \sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m} \ln^{1 - \varepsilon} \alpha u_m \right)^{1/p} \left( \sum_{n=2}^{\infty} \frac{(1 - \vartheta (\lambda_1, n))\nu_{n+1}}{V_n \ln^{1 - \varepsilon} \beta V_n} \right)^{1/q}
$$

$$
= \left( \sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m} \ln^{1 - \varepsilon} \alpha u_m \right)^{1/p} \left( \sum_{n=2}^{\infty} \frac{\nu_{n+1}}{V_n \ln^{1 - \varepsilon} \beta V_n} - \sum_{n=2}^{\infty} O \left( \frac{\nu_{n+1}}{V_n \ln^{1 - \varepsilon} \beta V_n} \right) \right)^{1/q}
$$

$$
= \frac{1}{\varepsilon} \left[ \frac{1}{\ln \alpha (1 + \vartheta_2)} + \varepsilon O(1) \right]^{1/p} \left[ \frac{1}{\ln \beta (1 + v_2)} + \varepsilon (\tilde{O}(1) - O(1)) \right]^{1/q},
$$

$$
\tilde{I} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_i (\ln \alpha u_m, \ln \beta v_n) \tilde{a}_m \tilde{b}_n
$$

$$
= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} k_i (\ln \alpha u_m, \ln \beta v_n) \frac{\nu_{n+1} \ln^{1 - \varepsilon} \alpha u_m}{V_n \ln^{1 - \varepsilon} \beta V_n} \frac{\mu_{m+1}}{U_m \ln^{1 - \varepsilon} \alpha u_m}
$$

$$
= \sum_{m=2}^{\infty} \frac{\mu_{m+1} \omega(\tilde{\lambda}_2, m)}{U_m \ln^{1 - \varepsilon} \alpha u_m} \leq K_f(\tilde{\lambda}_1) \sum_{m=2}^{\infty} \frac{\mu_{m+1}}{U_m \ln^{1 - \varepsilon} \alpha u_m}
$$

$$
= \frac{\varepsilon}{\varepsilon} K_f(\tilde{\lambda}_1) \left[ \frac{1}{\ln \beta (1 + v_2)} + \varepsilon O(1) \right].
$$
If there exists a positive constant $K \geq K_{\gamma}(\lambda_1)$ such that (57) is valid when replacing $K_{\gamma}(\lambda_1)$ by $K$, then, in particular, we have $\varepsilon \tilde{I} > \varepsilon K \|\tilde{a}_{\alpha, \varphi \lambda} \|_{q, \tilde{\gamma}_2}$, namely,

$$K_{\gamma} \left( \frac{\varepsilon}{q} + \varepsilon \frac{1}{\ln^2(1 + \mu_2)} + \varepsilon O(1) \right)$$

$$> K \left[ \frac{1}{\ln^2(1 + \mu_2)} + \varepsilon O(1) \right] \frac{1}{\ln^2(1 + \nu_2)} + \varepsilon \left( \tilde{O}(1) - O(1) \right) \right].$$

It follows that $K_{\gamma}(\lambda_1) \geq K (\varepsilon \to 0^+)$. Hence, $K = K_{\gamma}(\lambda_1)$ is the best possible constant factor of (57).

Similarly to the reverse of (32), we still can find that

$$I \geq f_2 \|b\|_{q, \tilde{\gamma}_2}. \quad (59)$$

Hence, the constant factor $K_{\gamma}(\lambda_1)$ in (58) is still the best possible. Otherwise, we would reach the contradiction by (59) that the constant factor in (57) is not the best possible. □

**Remark 3** For $\alpha = \beta = 1$, set

$$\tilde{\vartheta}(\lambda_1, n) = k_{\lambda_1} \frac{\ln(1 + \vartheta(n))}{\ln \lambda_1 \lambda_{\gamma}(\lambda_1)} \ln^2(1 + \mu_2) \ln^2 \beta V_n = O \left( \frac{1}{\ln^2 U_n} \right) \in (0, 1) \quad (\vartheta(n) \in (0, 1)),$$

$$\tilde{\psi}_n(n) := (1 - \tilde{\vartheta}(\lambda_1, n)) \left( \frac{V_n}{V_{n+1}} \right)^{q-1} (\ln V_n)^{q(1-\lambda_2)-1}.$$

It is evident that (57) and (58) are extensions of the following equivalent inequalities:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_{\lambda_1} (\ln U_m, \ln V_n) a_m b_m > K_{\gamma}(\lambda_1) \|a\|_{p, \varphi \lambda} \|b\|_{q, \tilde{\gamma}_2},$$

$$\left\{ \sum_{n=2}^{\infty} \frac{V_{n+1}}{V_n} \ln(1 - \tilde{\vartheta}(\lambda_1, n)) \left( \sum_{m=2}^{\infty} k_{\lambda_1} (\ln U_m, \ln V_n) a_m \right)^p \right\}^{\frac{1}{p}} \left( \sum_{m=2}^{\infty} k_{\lambda_1} (\ln U_m, \ln V_n) a_m \right)^{\frac{q}{q}} > K_{\gamma}(\lambda_1) \|a\|_{p, \varphi \lambda},$$

where the constant factor $K_{\gamma}(\lambda_1)$ is the best possible.

**5 Conclusions**

In this paper, by using the way of weight coefficients, the technique of real analysis, and Hermite-Hadamard’s inequality, a more accurate Hardy-Mulholland-type inequality with multi-parameters and a best possible constant factor is given by Theorems 1, 2, and the equivalent forms are considered. The equivalent reverses with the best possible constant factor are obtained by Theorems 3, 4. Moreover, the operator expressions and some particular cases are considered. The method of weight coefficients is very important, which helps us to prove the main inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type of inequalities.

**Acknowledgements**

This work is supported by the National Natural Science Foundation (No. 61370186, No. 61640222), and Appropriative Researching Fund for Professors and Doctors, Guangdong University of Education (No. 2015ARF25). We are grateful for this help.
Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

Author details
1Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, PR. China.
2Department of Computer Science, Guangdong University of Education, Guangzhou, Guangdong 510303, PR. China.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 March 2017 Accepted: 16 June 2017 Published online: 12 July 2017

References
1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
2. Yang, BC: Discrete Hilbert-Type Inequalities. Bentham Science Publishers Ltd, The United Arab Emirates (2011)
3. Mulholland, HP: Some theorems on Dirichlet series with positive coefficients and related integrals. Proc. Lond. Math. Soc. 29(2), 281-292 (1929)
4. Mitrinovi ´c, DS, Pe ˇcari´c, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Boston (1991)
5. Yang, BC: Hilbert-Type Integral Inequalities. Bentham Science Publishers Ltd, The United Arab Emirates (2009)
6. Yang, BC: On Hilbert’s integral inequality. J. Math. Anal. Appl. 220, 778-785 (1998)
7. Yang, BC: An extension of Mulholland’s inequality. Jordan J. Math. Stat. 3(3), 151-157 (2010)
8. Yang, BC: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
9. Rassias, M, Yang, BC: On half-discrete Hilbert’s inequality. Appl. Comput. Math. 220, 75-93 (2013)
10. Rassias, M, Yang, BC: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Comput. Math. 225, 263-277 (2013)
11. Huang, QL, Yang, BC: A more accurate half-discrete Hilbert inequality with a non-homogeneous kernel. J. Funct. Spaces Appl. 2013, 628250 (2013)
12. Huang, QL, Wang, AZ, Yang, BC: A more accurate half-discrete Hilbert-type inequality with a general non-homogeneous kernel and operator expressions. Math. Inequal. Appl. 17(1), 367-388 (2014)
13. Liu, T, Yang, BC: On a half-discrete reverse Mulholland-type inequality and extension. J. Inequal. Appl. 2014, 103 (2014)
14. Rassias, M, Yang, BC: On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function. Appl. Comput. Math. 242, 800-813 (2014)
15. Huang, QL, Wu, SH, Yang, BC: Parameterized Hilbert-type integral inequalities in the whole plane. Sci. World J. 2014, 169061 (2014)
16. Chen, Q, Yang, BC: On a more accurate multidimensional Mulholland-type inequality. J. Inequal. Appl. 2014, 322 (2014)
17. Rassias, M, Yang, BC: On a multidimensional Hilbert-type integral inequality associated to the gamma function. Appl. Comput. Math. 249, 408-418 (2014)
18. Rassias, M, Yang, BC: A Hilbert-type integral inequality in the whole plane related to the hyper geometric function and the beta function. J. Math. Anal. Appl. 428(2), 1286-1308 (2015)
19. Yang, BC: An extension of a Hardy-Hilbert-type inequality. J. Guangdong Univ. Educ. 35(3), 1-7 (2015)
20. Kuang, JC: Applied Inequalities. Shandong Science Technic Press, Jinan (2004)
21. Kuang, JC: Real and Functional Analysis (Continuation), vol. 2. Higher Education Press, Beijing (2015)