Finite Size Effects in Vortex Localization

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The equilibrium properties of flux lines pinned by columnar disorder are studied, using the analogy with the time evolution of a diffusing scalar density in a randomly amplifying medium. Near $H_1$, the physical features of the vortices in the localized phase are shown to be determined by the density of states near the band edge. As a result, $H_1$ is inversely proportional to the logarithm of the sample size, and the screening length of the perpendicular magnetic field decreases with temperature. For large tilt the extended ground state turns out to wander in the plane perpendicular to the defects with exponents corresponding to a directed polymer in a random medium, and the energy difference between two competing metastable states in this case is extensive. The divergence of the effective potential associated with strong pinning centers as the tilt approaches its critical value is discussed as well.

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I. INTRODUCTION

The physical properties of vortex lines in the mixed phase of type II superconductors has become subject of intense research in recent years [1]. When an external current density is applied to the bulk of the system, the flux lines may start to move under the action of the Lorentz force. Within a perfectly homogeneous system this driving Lorentz force is balanced only by the friction force opposing the steady state velocity of the flux lines, so that there is a dissipation coupled to the appearance of a finite electric field which results from the flux motion. To avoid this effect, flux lines in the mixed phase should be pinned by inhomogeneities in the underlying crystal structure, such as the point defects associated with vacancies of oxygen atoms in high temperature cuprates [2]. It turns out that the pinning of the vortices is much stronger when these impurities are in the form of correlated disorder, like twin boundaries or columnar defects [3,4], aligned along the direction of the external magnetic field. If the thermal fluctuations are small enough, the flux line may lie along the entire extended defect in the bulk; this is in contrast to short scale disorder pinning, in which the line should accommodate itself to the potential fluctuations and hence increases its elastic energy. However, the correlated defects pinning becomes less effective when the direction of the external magnetic field is tilted with respect to the anisotropy axis, which we take to be along the $\hat{z}$ direction. At some critical tilt, for which the free energy per unit length of the defect is less than the free energy associated with the perpendicular field, a pinning-depinned phase transition occurs and the flux lines are delocalized.

The static and dynamic response of the flux lines in the presence of columnar defects have been considered by Nelson and Vinokur [5]. Using the mapping of flux lines in $d + 1$ dimensional superconductor to the world lines of bosons in a $d$-dimensional quantum system, the authors identified the phase space diagram of the system which contains a high temperature “superfluid” and low-temperature “bose glass” phases, as well as Mott insulator at the matching field, $B_0 = n_{\text{pin}} \phi_0$, for which there is one flux line per defect. At low temperature, this matching field separates the “dilute” region of the bose glass phase, for which the vortex lines are pinned individually by the defects, from the high density region, where interactions are important in determining the localization length and transport properties of the flux lines.

In the low field region, the vortices are localized by the interaction with the correlated defects. Each pin is the analog of a 2D potential well which may be described (up to logarithmic corrections) as a cylindrical square well. The temperature of the superconductor, in turn, corresponds to the Plank constant $\hbar$ of the quantum boson system. For the dilute vortex arrays, where the pinning energy is larger than the interaction energy, there are two regimes. For low temperature the localization length is given approximately by the radius of the defect so that each flux line is localized by one defect. As the temperature increases, the localization length of one defect grows exponentially with $T^2$, and the flux line is then localized by several defects, forming an effective d dimensional potential well in the corresponding boson system.

The depinning of flux line which occurs as a result of external field tilt has been carefully investigated by Hatano and Nelson [6]. The Hamiltonian of the corresponding boson problem is no longer Hermitian; the kinetic term of the Hamiltonian is subject to an imaginary gauge transformation where the gauge field $\mathbf{h}$ is related to the perpendicular magnetic field $\mathbf{H}_\perp$ via $\mathbf{h} = \mathbf{H}_\perp \phi_0/(4\pi)$, where $\phi_0$ is the flux quantum. In the absence of tilt, the probability of finding a point of the vortex at transverse displacement $\mathbf{r}$ relative to the center of the pin is independent of $\hat{z}$, and given by the square of $\phi_{gs}(\mathbf{r})$, the ground-state wavefunction of the Hermitian Hamiltonian. For small tilt, there are left and right ground-state eigenfunctions which correspond to the left and right “tilting” of the Hermitian ground-state are still localized. It turns out that the flux line is described by these left and right eigenfunction at the bottom and the
top of the sample, while deep in the bulk the probability function approaches its Hermitian limit. Typically, the “surface roughness” associated with the tilt extends into the bulk up to some characteristic distance which diverges as the tilting angle approaches the critical angle, for which the flux line delocalizes and the current response becomes linear.

This non-Hermitian delocalization process has been shown to be of importance for other physical systems as well. In general, the time evolution of diffusing scalar field in random environment is determined by the eigenstates of Liouville operator which is the analog of the Hermitian Hamiltonian of a quantum particle in a disordered system; the effect of convection may be modeled by the imaginary gauge transformation. The appearance of extended states as convection is increased beyond a critical value, associated with the complex spectral points of the resulting non-Hermitian operator, is the manifestation of the delocalization transition. The dynamics of strongly driven charge density waves, and the growth of the delocalization transition. The dynamics of the resulting non-Hermitian operator, is the manifestation of the delocalization transition. The dynamics of strongly driven charge density waves, and the growth of the delocalization transition. The dynamics of strongly driven charge density waves, and the growth of the delocalization transition.

This paper is organized as follows. In Section 2, a brief introduction to the boson-vortex analogy is presented, as well as the correspondence between the statistical mechanics of flux lines and the time evolution of a scalar field. Section 3 deals with the finite size effects in the localized phase; both the critical field \( H_{c1} \) and the penetration depth of the perpendicular field are shown to be determined by the statistics of the tail of the density of states. Since this function is strongly related to the size of the system, physical properties of the pinned vortex are determined by the actual size of the bulk. In Section 4 we consider the delocalized phase, in which we recognize that the low lying states are extended and are related to the properties of Burgers’ equation with conservative noise in \( 1+1 \) dimension. Some technical details are given in the appendices.

II. FLUX LINE IN COLUMNAR DISORDER

We review here the basic physics of a single flux line in \( d+1 \) dimensional superconductor, as has been discussed in and .

Consider a flux line in a superconducting sample of thickness \( L_0 \) when the sample is pierced by columnar pins, such as long aligned columns of damaged material, as illustrated in Fig. 1.

Neglecting overhangs of the line, we define the flux line by its trajectory \( r(z) \). The free energy of this line may be written as,

\[
F = \frac{\tilde{\epsilon}_1}{2} \int_0^{L_0} \left( \frac{dr(z)}{dz} \right)^2 \, dz + \frac{1}{4\pi} \int_0^{L_0} [a + U(r(z))] \, dz \quad (2.1)
\]

where \( \tilde{\epsilon}_1 \) is the tilt modulus of the flux line, i.e., the energy of the flux line per unit length, and the elastic contribution \( \frac{\tilde{\epsilon}_1}{2} \left( \frac{dr(z)}{dz} \right)^2 \) is the first nontrivial term in the small tipping angle expansion of the line energy of a nearly straight vortex line. \( U(r) \) is the random potential which arises from a \( z \)-independent set of disorder-induced columnar pinning potentials, and \( H_L \) is the perpendicular field. \( \bar{U}(r) \) is taken from some bounded distribution, such as square distribution of width \( \Delta \) around zero; its bias, \( a \), is determined by \( \tilde{\epsilon}_1 \) and the parallel component of the external magnetic field, i.e., \( a = \frac{H_L \bar{\phi}_0}{4\pi} \). Note that we assume \( H_L \) to be much smaller than \( H_c \), hence we neglect terms of order \( H_L^2 \) in (2.1).

Using the analogy between the partition function for this flux line, \( Z = e^{-F/T} \), and the path integral formulation of quantum mechanics, it is easy to see that the conditional probability \( Z(r,z;r_0,z_0) \) obeys the Schrödinger-like equation,
where $h_{\perp} \equiv \frac{H_{\perp}}{4\pi}$ is the dimensionless perpendicular field.

Let us now map this flux line model, which satisfies Eq. (2.2), onto a model for the diffusion of a passive scalar density, $\phi(x)$, suppressed/amplified by a quenched random potential, and driven by an external drift. The time evolution of such a system is described by the equation:

$$\partial_t \phi(r,t) = D \nabla^2 \phi(r,t) - \frac{[b + V(r)]\phi(r,t) + \mathbf{h} \cdot \nabla \phi(r,t)}{T}, \quad \phi \equiv \frac{\phi}{\phi_0},$$

where $V(r)$ is the random amplification/suppression rate, and $D$ is the diffusivity. Clearly, the partition function of the flux line satisfies Eq. (2.3) for $\phi$, where $D$ plays the role of temperature divided by the tilt modulus, $D = T/(2\epsilon_1)$, $\mathbf{h}$ is proportional to the perpendicular magnetic field, $\mathbf{h} = \frac{h}{\epsilon_0}$ and the columnar disorder in the superconductor, as well as its bias, are normalized by the temperature, i.e., $V(r) = U(r)/T$ and $b = a/t$. The correspondence between the various scalar field and magnetic vortex quantities are summarized in table 1.

Table 1

| Scalar Density | Flux Line |
|----------------|-----------|
| $D$            | $T/2\epsilon_1$ |
| $V(r)$         | $U(r)/T$ |
| $\mathbf{h}$  | $h_{\perp}/T$ |
| $b$            | $(\epsilon_1 - \frac{H_{\perp}\phi_0}{2\pi})/T$ |
| $\phi$         | $Z$ |

Eq. (2.3) may be written as $\partial_t \phi = -\mathcal{H}\phi$, where $\mathcal{H}$, the Hamiltonian, is the linear operator which generates the time evolution of $\phi$.

$$\mathcal{H} = -D\nabla^2 + [b + V(r)] - \mathbf{h} \cdot \nabla. \quad (2.4)$$

The dynamics of this system is determined by the eigenvalues and the eigenvectors of the Hamiltonian. For $\mathbf{h} = 0$, the operator, $\mathcal{H} = -D\nabla^2 + [b + V(r)]$, is Hermitian and it is well known that, for strong enough disorder, all its eigenfunctions are real and localized; the localization length is maximal at the center of the energy band, and minimal at the tails $|10|$. When $\mathbf{h} \neq 0$, the above Hamiltonian is no longer Hermitian, and it may be diagonalized using a system of left and right eigenvectors. Since the perpendicular field term may be absorbed into the Laplacian by completing the square, $\nabla \rightarrow \nabla + \frac{\mathbf{h}}{2\pi}$, the right and left eigenfunctions of the new Hamiltonian are related to the eigenfunctions of $\mathcal{H}(\mathbf{h} = 0)$ via an imaginary gauge transformation; if $\phi_{n,0}^R(\mathbf{r})$ is an eigenfunction of the Hermitian problem, then

$$\phi_{n,\mathbf{h}}^R = e^{\frac{i}{\epsilon_0}h_{\perp}/D} \phi_{n,0}^R$$

$$\phi_{n,\mathbf{h}}^L = e^{-\frac{i}{\epsilon_0}h_{\perp}/D} \phi_{n,0}^L$$

are the eigenfunctions of the non-Hermitian operator with the same eigenvalue, provided that $\xi_n$, the localization length in the non-driven problem, is less than $D/\epsilon_0$. Thus, for small tilt, there is a spectral rigidity - the eigenvalue spectrum is robust. When $\mathbf{h}$ increases, the eigenfunctions become extended and the boundary conditions of the sample should be taken into account. Since (2.3), in general, does not satisfy these boundary conditions, the new eigenfunctions are no longer related to the $\mathbf{h} = 0$ case by a simple gauge transformation. As a result, the eigenvalues are changed; in case of periodic boundary conditions, complex eigenvalues appear when $D/\epsilon_0$ becomes smaller then $\xi_n$. As $\mathbf{h}$ is increased, these delocalized states appear first at the band center, for which the localization length is maximal, then move outwards, as has been discussed in $[3]$. The mapping of Eq. (2.2) to Eq. (2.3) implies that for thick samples, $L_0 \rightarrow \infty$, the thermally excited flux line relaxes into the spatial configuration equivalent to the ground state eigenstate of $\mathcal{H}$. For small $\mathbf{h}_{\perp}$, this state is localized, so that the flux line is localized around some spatial point at the bulk of the sample. This phenomenon is known as the transverse Meissner effect. As has been shown by $[3]$, the ends of the pinned flux line begin to tear away from the pinning center as the tilt increases. The “penetration depth” of the perpendicular field, associated with the width of the region near the surface in which the transverse Meissner effect breaks down, diverges as the tilt approaches $\mathbf{h}_{\perp}$. Above this value, the eigenstate delocalizes, and the resulting trajectory wanders across the entire sample.

III. MAGNETIC RESPONSE IN THE PINNED PHASE

A. Size dependence of $H_{c1}$

The lower critical field, $H_{c1}$, is defined in a random system as the minimal field in which the first vortex line enters the sample. One could easily recognize from Eqs. (2.1-2.4) that the condition for one flux line in the sample to be energetically favored over the completely diamagnetic phase, is that the Hamiltonian (2.4) admits negative eigenvalues; the spectral points of (2.4) are proportional to the energies per unit length of the corresponding configurations of the vortex. $H_{c1}$ depends, thus, on the ground states of the Hamiltonian; for random potential one should consider the statistical properties of the lower tail of the density of states. Note that we consider here the physical case of bounded disorder, such that for each sample there is a well defined lower critical field. This is in contrast to the discussion given by Larkin and Vinokur, who address the question of magnetic response
For unbounded, Gaussian random potential in an infinite sample. In such a case there is no well defined $H_{c1}$ \[1\].

For a statistically homogeneous sample, any rare fluctuation is realized as one takes the bulk size to infinity. In particular, one may find any large region in which the potential is close to its lower bound, so for infinite sample the minimal value of the potential $V(x)$ sets the magnitude of $H_{c1}$, i.e.,

$$H_{c1}^{\infty} = \frac{4\pi(\epsilon_1 - \Delta)}{\phi_0}. \tag{3.1}$$

For finite sample this is not the case. Typically, as the sample size increases, $H_{c1} \rightarrow H_{c1}^{\infty}$; however, the form of the size dependence of the critical magnetic field is not trivial.

Consider, for example, a hole in the superconductor, i.e., a region of linear size $L$ where the elastic energy of the flux line vanishes, $\epsilon_1 = 0$. The free energy of a vortex line which is trapped in this hole comes entirely from the limits on its transverse motion sets by the boundaries of this hole. Far away from the bulk critical temperature (i.e., when $\epsilon_1$ in the bulk is much bigger than the typical free energy per unit length associated with the trapped flux line) one may assume no vortex displacement along the hole boundaries, and the situation corresponds to a quantum mechanical particle in a box. In such a case the tail of the spectrum of (2.4) looks like

$$\epsilon(k) \sim Dk^2, \tag{3.2}$$

where the minimal value of $k$ is set by the system linear size $L$, $k_{\text{min}} \sim 1/L$, so that,

$$H_{c1} \sim \frac{4\pi T^2}{\epsilon_1 \phi_0 L^2}. \tag{3.3}$$

This result reflects the fact that the entropy of the vortex line decreases if it is confined to finite region, hence its free energy per unit length increases. At zero temperature this finite size correction to $H_{c1}$ disappears; as the temperature approaches its critical value, the potential steps at the hole boundaries become small, and in 2D this contribution to the free energy vanishes exponentially.

Let us consider now the disordered case. In Appendix A we show that for a simple lattice model with square distribution of the disorder the density of states per unit volume near the minimal possible value of the energy (which we set, for convenience, to zero since this value is absorbed in the definition of $H_{c1}^{\infty}$) is

$$g(\epsilon) \sim e^{-\left(\frac{\epsilon}{l_0}\right)^{d/2}} \tag{3.4}$$

where $l_0$ is the lattice constant. One may estimate the energy difference $\Delta \epsilon$ between the ground state and zero for a typical sample of linear size $L$ by the condition

$$\left(\frac{L}{l_0}\right)^d g(\epsilon) \Delta \epsilon \sim 1 \tag{3.5}$$

so for large samples the ground state energy is given by

$$\epsilon_0(L) \sim \frac{D}{l_0^2 \ln^{2/d}(L/l_0)}. \tag{3.6}$$

and

$$H_{c1} \sim H_{c1}^{\infty} + \frac{4\pi T^2}{\epsilon_1 \phi_0 l_0^2 \ln^{2/d}(L/l_0)}. \tag{3.7}$$

In contrast to the hole example, one sees that the critical parallel field approaches $H_{c1}^{\infty}$ much slower, i.e., finite size effects are much stronger in the presence of disorder. For the tilt field below its critical value the situation remains the same, since the spectrum does not change.

Upon taking reasonable parameters for high temperature superconductors, such as $T \sim 700K$, $l_0 \sim 100 - 1000 \AA$ and $\lambda \sim 1000 \AA$, one finds, however, that the entropic corrections to $H_{c1}$ are very small, $\delta H_{c1}/H_{c1} \sim 10^{-5}$. Thus, for almost any realistic sample size one could take the value of $H_{c1}^{\infty}$ as the lower critical field. Since in most cases the radius of the columnar pins is comparable to, or bigger than, the superconducting coherence length, the range of the external field corresponding to the Meissner phase turns out to be very narrow. For other physical situations when the time evolution of some scalar density is given by (2.3), these finite size corrections may be of importance.

### B. Transverse Meissner Effect for Dilute Concentration

Given a system described by the equation

$$\partial_t \phi = -\mathcal{H}\phi \tag{3.8}$$

where $\mathcal{H}$ is, in general, a time independent non-Hermitian operator with a complete set of left and right eigenstates $\phi_{n,L}$, $\phi_{n,R}$ with the corresponding eigenvalues $\{\epsilon_n\}$ (i.e., the complex “energy spectrum” of the possibly non-Hermitian operator), the time evolution of the normalized amplitude with initial condition $\phi(\mathbf{r},0)$ is given by its spectral decomposition:

$$\dot{\phi} = \frac{\phi(\mathbf{r},t)}{\int d\mathbf{r} \phi(\mathbf{r},t)} \int d\mathbf{r} \sum_n \phi_{n,L}(\mathbf{r},0) \phi_{n,R}(\mathbf{r}) \exp(-\epsilon_n t) \tag{3.9}$$

where $\langle .. \rangle$ is defined as the inner product $\langle \psi \phi \rangle \equiv \int d\mathbf{x} \psi^*(\mathbf{x}) \phi(\mathbf{x})$. At long times, the system is dominated by the ground state, i.e., the state for which the real part of $\epsilon_n$, $\Re \epsilon_n$, is minimal. Using the mapping between the partition function of the vortex system and the scalar field $\phi$, one observes that Eq. (3.3) describes the surface roughness of the flux line: although deep in the bulk,
which is the equivalent to the “long time” of (3.9), the line is stuck to the spatial configuration corresponding to the ground state, this is not the case near the surface, where the “short time” limit of (3.9) is relevant.

Let us discuss this phenomenon for the $\hbar = 0$ case, when all the wavefunctions are localized. Spectral decomposition of small, localized, initial conditions near the most rapidly growing state requires projection upon the eigenvectors of the Hamiltonian. These projections, in general, decrease exponentially with the distance between the fluctuation center and the localized eigenvector. For example, if the initial fluctuation is centered around zero, the projection on an eigenstate localized at $x$ is

$$
<\phi_{x=0}(r,0)|\phi_x(r) > \sim \exp^{-\kappa_n|x|},
$$

(3.10)

where $\kappa_n$ is the inverse localization length of the $n$-th eigenstate. The fastest growing states of $\mathcal{H}$ dominate the system at long times; at any finite time, however, there is a competition between the initial conditions and the growth rates, which we explore using the following model.

Assume for simplicity that the localization length is constant for all states of the system, i.e., it is independent of the eigenenergy as well as the spatial position. The probability amplitude $\phi_x$ generated by a localized initial condition $\phi_{x=0}(r,0)$ is proportional to

$$
\frac{\phi_x(t)}{\phi_{x=0}(0,t)} \sim \exp^{-\epsilon_x|x|},
$$

(3.11)

where $\epsilon_x$ is the eigenvalue of a growth mode localized at $x$. We shall study the relaxation of the first moment of $\phi(x, t)$,

$$
\bar{x}(t) = \int dx x \frac{\phi_x(t)}{\phi_{x=0}(0,t)} = \int dx x \frac{\phi(x,t)}{\phi(x,0)}
$$

(3.12)

to the value $x_{gs}$, it assumes at long times,

$$
x_{gs} = \int dx x \phi_{gs}(x)
$$

(3.13)

where $\phi_{gs}(x)$ is the lowest energy eigenmode of the Hamiltonian. Given $\epsilon_x$, one can optimize (3.11) in order to find the location of the eigenstate which dominates the system at finite time. For a finite sample it is clear that as $t \to \infty$ the ground state (lower free energy) mode localized at position $x_{gs}$ will dominate. The long time convergence of the normalized amplitude to the ground state location is also given by (3.11), where we use a normalization with respect to the final location amplitude, i.e.,

$$
\frac{\phi_x(t)}{\phi_{x=x_{gs}}(t)} \sim \exp^{-\epsilon_{x_{gs}}|x|\epsilon_x=\epsilon_{x_{gs}} - \epsilon_x}.
$$

(3.14)

In random samples one may assume that the eigenenergies are described by a statistically homogeneous density $g(\epsilon)$, and a typical spatial distance $R = |x - x_{gs}|$ associated with energy difference $\Delta \epsilon = \epsilon_{x_{gs}} - \epsilon_x$ is given by the analog of Eq. (3.6)

$$
\Delta \epsilon R^d g(\epsilon) = 1.
$$

(3.15)

A similar mapping into variable range hopping in semiconductors has already been exploited in [6] for finite density of vortices in the superconducting medium. The long range repulsive interaction between flux lines has been taken into account, in the strongly localized regime, by forbidden multiple occupancy of localized states, and the “ground state” is defined by filling the energy band up to the “Fermi surface” determined by the chemical potential of the particles. If all the states are localized and the filling factor is such that the Fermi surface is not at the tails of the band, the density of states at the chemical potential, $g($µ$)$, may be taken as a constant independent of the energy. Upon substituting $\Delta \epsilon(R) = 1/(g($µ$)R^d)$ into Eq. (3.14), one can study the long time convergence to the ground state location, in terms of the distance $R^*$ away from $x_{gs}$ for which the normalized amplitude is maximal. One then finds that the approach of $|x - x_{gs}|$ to zero at long times is given by a stretched exponential [3],

$$
|x - x_{gs}| \sim \exp[-(\frac{t}{t^*})^{\tau^*}],
$$

(3.16)

where $\tau^*$ is the characteristic time of the decay, $t^* = g($µ$)/k^d$. Since the time in Eq. (2.3) corresponds to the length in the correlated direction of the flux problem normalized by the temperature, the penetration depth of the perpendicular magnetic field is $T g($µ$)/k^d$.

For a very dilute concentration of flux lines, the ground state (to which the system eventually evolves), is in the lower tail of the density of states and is always localized, even for weak disorder in high dimensions. The nature of the surface roughness is determined by the other low energy states at the tail of the density of states function. Since the ground state is now near the bottom of the band, the assumption that the density of states in this region is energy independent is no longer valid. One may estimate the time needed for the decay of a state in the tail to the ground state by optimizing (3.14) with respect to $R$, using (3.13) and (3.4). Taking the logarithm of both sides of (3.15), we get, up to logarithmic corrections,

$$
\Delta \epsilon \sim \frac{D}{(\ln R)^{2/d}}.
$$

(3.17)

From this we obtain the semiclassical expression for the distance away from the ground state at time $t$:

$$
|x - x_{gs}| \sim \exp[-\frac{t/t^*}{\log (t/t^*)^{1+2/d}}]
$$

(3.18)

where now $t^* = l_0^2/D$, or in the vortex language, the width of the surface roughness is proportional to $\tilde{\epsilon}_1 l_0^2/T$. 


Note that the effect of temperature (or the diffusion constant) in this expression is to decrease the surface roughness rather than increasing it as in the former case; the reason for this counter-intuitive result is that the temperature dictates the energy spacing at the tail. Lower temperature implies bigger spacing, so that the effect of neighboring sites on a pinned flux line becomes smaller. Of course, the width of the tail itself is proportional to the temperature, so that the result (3.18) becomes meaningless at the limit $T \to 0$.

This result implies that at small concentrations, i.e., close to $H_{c1}$, the localization of flux lines is more effective than in the case considered by [6]. Samples which are not thick enough to localize the vortices at high magnetic field may do so near the lower critical field, when the effects of tail statistics are of importance.

For a finite but small tilt $h$, the low-lying states are still localized, and the situation is essentially the same. At short times, there can be transition effects due to extended states which overlap with the initial condition, but at long times the normalized amplitude is localized into the ground state. For strongly localized states near the band edge the tilt can be absorbed into the gauge transformation (2.6), the only change introduced in the consideration above is that the inverse localization length $\kappa$ becomes anisotropic,

$$\kappa(h) = \kappa(h = 0) - \tilde{h}. \quad (3.19)$$

As a result, one should optimize (3.11, 3.14) with respect to $\theta$, the angle between $\tilde{h}$ and $x$. The result turns out to be equivalent to replacing $h$ in (3.19) by its absolute value $\tilde{h}$.

### IV. EXTENDED STATES AT LARGE TILT

#### A. Delocalized flux line as directed polymer

Let us consider now the limit of very large tilt, in the sense that even the ground state of our system is delocalized, i.e., $h \gg D/\xi_0$. As $h \to \infty$, the external magnetic field is perpendicular to the columnar defects; the flux line is now free in the $\tilde{z}$ direction, while in the $xy$ plane it sees the cross section of these defects, i.e., point disorder. One might then guess that the flux line freely diffuses in the $\tilde{z}$ direction, while in the $xy$ plane it looks like a directed polymer in a $1 + 1$ dimensional random medium, that is, the projection of the flux line on the $xy$ plane is a pinned string characterized by the wandering exponent $1/z = 2/3$. It is separated from competing metastable states by energy barriers which (up to logarithmic corrections) scale like $L^\beta$, where $\beta = 1/3$ and $L$ is the linear system size [11, 12].

The same features characterize the most rapidly growing eigenmode of the Hamiltonian (2.4). The low-lying eigenstates of the non-Hermitian Hamiltonian are extended in the tilt direction while wandering in the transverse direction with the same exponents $1/z$ and $\beta$. Suppose $L$ is the length of the sample in the $\tilde{h}$ direction, and $W$ is the width in the perpendicular direction, measured in units of some short length cutoff. For $W > L^{2/3}$ there are many, spatially uncorrelated, competing metastable states in the sample; the decay of such a state into the ground state involves activation above a potential barrier of order $L^{1/3}$ [13], so that the time needed for such a process is of order $t_0 \sim \exp(L^{1/3})$. On the other hand, as $W < L^{2/3}$, there is only one effective low energy state: since the energy barriers of the directed polymer are equivalent to the surface roughness in KPZ model, the potential barriers scale as $W^{1/2}$ in this case. In general, one sees that, in $2 + 1$ dimensions, when the low-lying states are extended, the time needed for the relaxation to the ground state is macroscopic and proportional to the (longitudinal or transverse) size of the system.

In order to quantify these considerations, we may map Eq. (2.3) into a nonlinear problem with additive disorder using Cole-Hopf transformation, i.e.,

$$\phi(x,t) = \exp\left(\frac{\rho(x,t)}{\gamma}\right). \quad (4.1)$$

Separating the result into longitudinal terms, i.e., parallel to the drift $\tilde{h}$, and the transverse terms, perpendicular to $\tilde{h}$, one gets,

$$\partial_t \rho(x, r \perp, t) = D_x \partial_x^2 \rho(x, r \perp, t) + D_\perp \nabla_\perp^2 \rho(x, r \perp, t) \quad (4.2)$$

$$+ \frac{D}{\gamma} [\nabla_\perp \rho(x, r \perp, t)]^2 + \frac{D}{\gamma} [\partial_x \rho(x, r \perp, t)]^2 - \tilde{h} \cdot \partial_x \rho(x, r \perp, t) + \gamma (b + V(x, r_\parallel, r_\perp)).$$

Where for concreteness we take $\tilde{h}$ along the $x$ direction and denote the remaining $d - 1$ directions by $r \perp$.

At the Burgers’s fixed point, i.e., for large $\tilde{h}$, $x$ scales as $r_\parallel^{\alpha}$ and $\rho$ scales as $r_\perp^{\gamma}$, where $\alpha = 1/2$ and $\gamma = 3/2$ for the $2 + 1$ dimensional realization of the model [11]. From Eq. (4.2) one sees that the terms proportional to $D_x$ and $(\partial_x \rho)^2$ scale like $r_\parallel^{\alpha-2\gamma}$ and $r_\perp^{2\gamma-2\gamma}$ respectively, which implies that they are irrelevant on large length scales. Thus, the eigenstates of (4.2) satisfy the 1+1 Burgers type equation, where the first order derivative with respect to time is multiplied by $\tilde{h}$. For more details see [8].

This picture is essentially valid in higher spatial dimensions. Eq. (4.2) is not specific to any dimension, so it gives the correct scaling behavior in terms of $\alpha$ and $\gamma$. The fact that $\alpha + \gamma = 2$ in any dimension, where $\alpha$ decreases with the dimensionality of the system, implies that the irrelevant terms become even more irrelevant near the Burgers’s fixed point for more than two spatial dimensions. Thus, one should expect that the delocalized ground state in $d + 1$ spatial dimensions has the scaling properties of a directed polymer in $d$ dimensions while extended in the direction of the tilt. For $d > 3$ this implies that for weak disorder the system may be in
the weak coupling regime, so that the nonlinear terms in \( L_2 \) are both irrelevant, and the system flows into the Gaussian (Edwards-Wilkinson) fixed point. As a result, the wandering exponent of the low-lying states becomes \( 1/2 \), like a classic lattice random walk, and the energy barriers between these eigenstates do not scale with the size of the system. However, above a critical threshold value of the disorder strength, the system renormalizes into the strong coupling regime, and a scenario similar to the \( 2 + 1 \) dimension case should apply.

B. Energy statistics and finite size effects in the extended phase

In this subsection we address the interpolation between the localized and the Burgers, directed polymer like, limit. As the tilt \( h \) becomes strong, there is a linear response of the bulk magnetization to the tilt; the average perpendicular component of the magnetization \( b_{\perp} = \frac{B_1 \phi_0}{4\pi} \) is related to the external field as \( b_{\perp} = \sigma(h_{\perp}) b_{\perp} \) where \( \sigma(h_{\perp}) \rightarrow 0 \) as \( h_{\perp} \rightarrow h_c \), and \( \sigma(h_{\perp}) \rightarrow 1 \) as \( h_{\perp} \rightarrow \infty \).

For \( h_{\perp} > h_{\perp}^c \), the vortex line proliferates superkinks of the same sign \( \perp \). Thus, the energy associated with the existence of the flux line at point \( r \) is not \( U(r) \), but rather \( \tilde{U}(r) = U(r) \tau(r, U(r)) \), where \( \tau(r, U(r)) \) is the longitudinal length, the flux line spent on the point \( r \), which is itself a function of the potential energy at the point. Since the flux line tends to spend more time on strongly attractive points, \( \tilde{U}(r) \) is a nonlinear function of \( U(r) \), and the energy distribution function \( P(\tilde{U}) \) is no longer symmetric; instead, it tends to emphasize the attractive regions while deemphasizing the repulsive ones.

Calculation of \( P(\tilde{U}) \) in some limits of a simple model are presented in Appendix B. It is shown that the effective disorder strength diverges as \( (h-h_c)^{-1/2} \) as \( h \) approaches the critical field from above.

Since actual systems are of finite size, it may lead to completely different response to the external drift. In the directed polymer phase, the disorder strength and the tilt modulus lead to a length scale \( L_c \), above which the polymer adapts itself to the pinning potential. Below \( L_c \), the line remains stiff, since the fluctuations of the pinning energy grows only sublinearly with length \( L \). Thus, in the Burgers’ phase, the low lying eigenstates may be fluctuating, depending on the actual length of the sample and the strength of the disorder. As \( h \rightarrow h_c \) from above, the effective disorder strength diverges, such that the ground state is always fluctuating in this limit. The various crossover regions for different sized samples are illustrated in Fig. 2.

It is interesting to note that the critical length \( L_c \) may be estimated by two different ways \( L_2 \). First, one may extract the relevant part of the Eq. \( L_2 \) (where we omit the bias \( b \), since it does not effect the results)

![FIG. 2. “Phase diagram” for the different states of the flux line. In order to see the localized phase, the longitudinal dimension of the sample should be longer than the localization length, which diverges like \( (h_c - h)^{-1} \) as \( h \rightarrow h_c \) from above. Above the transition, the potential barriers between two competing metastable states scales with the length of the sample only if \( L_\parallel > D h / \Delta^2 l_0^2 \). As \( h \rightarrow \infty \), \( L_\parallel \) saturates to its value for point disorder in the \( xy \) plane. When the convection is smaller, the effective disorder becomes stronger; as a result, \( L_\parallel \sim (h-h_c)^{3/2} \) near the transition.](image-url)

\[
\frac{\partial \phi}{\partial x} = D_\perp \nabla_\perp^2 \phi + \frac{D}{\gamma} (\nabla_\perp \phi)^2 + \gamma V(\mathbf{r}_\perp, \mathbf{x}). \tag{4.3}
\]

Division by \( h \) and rescaling of the length units by some short length cutoff \( l_0 \), i.e., \( y = x/l_0 \), one gets the dimensionless form,

\[
\frac{\partial \phi}{\partial y_\parallel} = D_\perp l_0 \nabla_\perp^2 \phi + \frac{D}{\gamma h l_0} (\nabla_\perp \phi)^2 + \frac{\gamma l_0}{h} V(y_\perp, y_\parallel). \tag{4.4}
\]

The Fourier transform of \( \phi(y_\perp, y_\parallel) \), \( \phi(k, \omega) \) satisfy the integral equation,

\[
\phi(k, \omega) = G_0(k, \omega) V(k, \omega) + \frac{D}{\gamma h l_0} \int \frac{d\Omega}{(2\pi)^2} \int d\mathbf{q} \cdot (k - \mathbf{q}) \phi(\mathbf{q}, \Omega) \phi(k - \mathbf{q}, \omega - \Omega) \]

where

\[
G_0(k, \omega) = \frac{1}{i\omega - D h l_0 k^2}, \tag{4.6}
\]

and \( V(k, \omega) \) is the Fourier transform of the growth rate. A loop renormalization group correction to the dimensionless diffusion constant \( D = \frac{D}{h l_0} \) comes from the self energy diagram and is of order \( g^2 \int d\mathbf{q} q^{\delta-3} \) where the dimensionless coupling constant \( g^2 = \frac{\Delta^2}{D h} \). In the \( 2 + 1 \)
dimensional realization of our system, Eq. (4.4) is reduced to $1 + 1$ dimensions, and this correction diverges like $g^2/q$ as the incoming momentum vanishes; this implies that the transverse dimension of the system, $L_\perp$, should satisfy $L_\perp > \frac{D \hbar}{\Delta \varepsilon_0^3}$, in order to see the effect of the disorder, i.e., the transition from stiff ground state to a wandering one. A trivial scaling argument then shows that the longitudinal dimension of the system is given by $L_{c||} = (D \hbar / \Delta^2 \varepsilon_0^3)^{3/2}$.

An alternative way to see the same result is the following: consider a 2D isotropic homogeneous tight binding model, which is a discretization of Eq. (2.3). The hopping term, thus, is $D/\varepsilon_0^2$, where $\varepsilon_0$ the lattice constant. The energy spectrum of this model is simply $E(k_x, k_y) = \frac{D}{\varepsilon_0^2} [\cos(k_x \varepsilon_0) + \cos(k_y \varepsilon_0)]$, where $\frac{\pi}{\varepsilon_0} < k_x, y < \pi$. Thus, the minimal separation between nondegenerate energy levels is $\Delta E \sim \frac{D}{\varepsilon_0^2}$. Introducing the tilt $\tilde{h}/D$ pointing in the $x$ direction, leads to a simple shift of $k_x \rightarrow k_x + i \frac{\tilde{h}}{D}$, and the corresponding states are tilted in the $x$ direction, without any wandering in the direction transverse to $x$. As a result, the eigenstates are now complex, and the minimal imaginary distance between energy levels is $\frac{D \hbar}{L_{c\perp} \varepsilon_0^3}$. Now let us introduce disorder into the system. Using first order perturbation theory one finds that the leading order correction to the energy levels, is proportional to $\Delta^2$; thus, we have the same result, i.e., that non trivial effects of the disorder may be seen if $L_{\perp} > L_{c\perp} \sim (D \hbar / \Delta^2 \varepsilon_0^3)$.

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**APPENDIX A: TAIL STATES STATISTICS**

In this appendix we calculate the density of states (DOS) at the tails of the band, i.e., near the ground state. We then use the results to study the decay of these states to the ground state. The model we use is a discretization of Eq. (2.3), where we neglect the constant bias of the growth rate $b$, since it does not effect the statistics of the DOS. The discussion below assumes localization of the wavefunction of the relevant part of the eigenvalue spectrum, hence, the tilt term $\tilde{h}$ does not change the spectrum; it effects the decay rate via the modified localization length, as discussed in Section 3b.

We consider, thus, a model of a $d$ dimensional configuration space tilted by square blocks of length $\varepsilon_0$, where the potential energy in each of these blocks is a constant number taken from a square distribution between 0 and $\Delta$ (the choice of these limits is only for the sake of convenience, since the results are independent of the potential bias $b$, on the other hand, other probability measures may give different results. In particular, for any unbounded distribution, such as Gaussian distributions, the spectrum of $\tilde{H}$ is unbounded from above, and there is no delocalization transition in the sense we discuss here). The dimensionless Hamiltonian is:

$$H = -D \nabla^2 + V(x).$$  \hspace{1cm} (A1)

It is easy to see that the DOS function $\rho(\epsilon)$ in this model is bounded from below by zero, and $\rho(\epsilon) \to 0$ as $\epsilon \to 0$. The tail of the density of states is determined by the range of parameters in which rare events, such as large spatial regions with low potential energy, determine the DOS.

Let us estimate these fluctuations in the following way: the probability to find a hypersphere of radius $R$ which contains only blocks of potential energy less then $V_0$ is

$$P(R, V < V_0) \sim \exp ([R/\varepsilon_0]^d \ln[V_0/\Delta]).$$  \hspace{1cm} (A2)

The ground state energy of these fluctuations is given approximately by

$$\epsilon_0 \sim V_0 + \frac{D}{R^{2d/3}}$$  \hspace{1cm} (A3)

so that the probability to get the energy between $\epsilon$ and $\epsilon + d\epsilon$ using a sphere of radius $R$ is,

$$p(R, \epsilon) \sim \exp \left( \frac{R}{\varepsilon_0} \right)^d \left( \ln \epsilon - \frac{D}{R^2} \right)$$  \hspace{1cm} (A4)

This expression is well defined for $\sqrt{D}/\epsilon \leq R < \infty$. Optimizing $p(R, \epsilon)$ with respect to $R$ gives, up to logarithmic corrections, a maximum at the lower limit $R^* = \sqrt{D}/\epsilon$, so that as $\epsilon \to 0$ from below,

$$P(\epsilon) \sim \exp (-D/(-\epsilon)^{d/2}).$$  \hspace{1cm} (A5)

We assume that the DOS is proportional to $P(\epsilon)$, i.e., at the tail of the distribution we have $g(\epsilon) \sim g_0 P(\epsilon)$, where $g_0$ is some normalization proportional to the DOS at the middle of the band.

Let us consider now the tight binding analog of the above model. The on site potential is now $w_i$, taken from a square distribution between zero and $\Delta$. The Hamiltonian is

$$H = \frac{t}{2} \sum_x \sum_{\nu=1}^d (a_{x\nu}^\dagger \epsilon_\nu a_x + h.c.) - \sum_i v_i a_{x}^\dagger a_x$$  \hspace{1cm} (A6)

where $a_{x}^\dagger, a_x$ are boson creation and annihilation operators, $\epsilon_\nu$ are unit lattice vectors, and the hopping element $t$ is normalized by $D$.

The eigenenergies of this Hamiltonian are bounded, $-\tau < \epsilon_n < \Delta + \tau$. The states at the tails correspond to a
rare spatial fluctuations of $v_i$’s; in general, the chance to find such fluctuations (e.g., a region of radius $R$ in which the on site potential is less then $V$) is the same as in the previous model. The energy spectrum of such fluctuation is given approximately by

$$\epsilon = V + t \sum_{\nu=1}^{d} \cos(k_{\nu})$$  \hspace{1cm} (A7)

where $k_{\text{min}} \sim 1/R$, thus, the tail states obey the relation

$$\epsilon \sim V - t + t/R^2$$  \hspace{1cm} (A8)

so that the result (A5) is applicable here also, with the energy measured from the lower bound of the band.

**APPENDIX B: MAGNETIC RESPONSE AT LARGE TILT**

In this Appendix we study the response of the bulk magnetization $B_\perp$ to the tilt of the external field. In the localized phase there is no linear response, i.e., no bulk tilt of the vortex as a result of the external perpendicular magnetic field. Thus, our discussion here is relevant only in the extended phase.

It is important to note that for an *unbounded* distribution of defects strength, such as a Gaussian distribution, one flux line is always localized. For a given tilt, there is always a strong enough defect such that the localization length is small enough and the tilted state is localized. A delocalization of one flux line may occur only if the distribution of the defects is bounded. We will take the quenched random potential from a distribution of width $\Delta$, so that the minimal localization length is of order $1/\sqrt{\Delta}$. The tilt is assumed to be strong such that all the states are localized.

The typical situation, though, is that the flux line “lives”, for a while, on defects which are quite strong with respect to their neighbors, i.e., there are no stronger defects in the vicinity of the strong one. After a while, however, the flux line is going to jump to another defect which is weaker, in order to satisfy the tilt term in the free energy. This process may be described by the optimization of the free energy of the “local jump”

$$\delta F = -\frac{\Delta \tau}{(1 + \Delta g(\mu)R^d)} - \int H \cdot B(z) dz,$$  \hspace{1cm} (B1)

where $\tau$ is the length that the flux line spend on the defect. The first term in (B1) indicates that there is a decrease of the free energy due to the fact that the flux line lives on the defect. This decrease is proportional to the potential energy difference between the current site and the site the flux line jumps to; it is bounded from above by $\Delta$ while it decreases to zero when the jump distance, $R$, is much bigger than the density of states per unit area, since then the flux line may find another strong defect. The denominator of the first term in (B1) reflects the simplest interpolation between these two limits. The term $H \cdot B$ indicates the free energy per unit length decrease since the flux line average direction $B$ tends to be in the direction of the external magnetic field $H$. In terms of the “local jump”, this term takes the form

$$H \cdot B = \frac{H \phi_0}{4\pi} (r^2 + R^2)^{1/2} \cos\left(\frac{H \phi_0}{H} - \frac{r}{2}\right).$$  \hspace{1cm} (B2)

Now let us calculate the effective tilt of the flux line, i.e., the response of the internal magnetic field $B$ to the external field $H$. For $H$ in the direction of the defects, $B$ is parallel to $H$; this is also true if $H$ is perpendicular to the defects. Small tilt of $H$ from the defects direction is not enough to tilt the flux line, so that there is no response of the internal magnetic field (except near the surface, see (B3)). Above the critical tilt there is linear response of the flux line, i.e., $B_\perp = \sigma(H_\perp)H_\perp$, where $\sigma(H_\perp) \to 0$ as $H_\perp \to H_\perp^c$ and $\sigma(H_\perp) \to 1$ as $H_\perp \to \infty$. Using our local jump model, near $H_\perp^c$ i.e., where $H_\perp << H ||$ and $R << \tau$; near the critical tilt $R \to 0$, so that we may assume that $R^d << \Delta g$. $\delta F$ then takes the form

$$\delta F = -\Delta \tau + \frac{H \phi_0}{8\pi} \frac{|H_\perp|}{H ||} - \frac{R}{\tau}^2.$$  \hspace{1cm} (B3)

Optimizing with respect to $\tau$ one gets

$$\frac{H \phi_0}{8\pi} \left(\frac{|H_\perp|}{H ||}^2 - \frac{R^2}{\tau^2}\right) \approx \Delta.$$  \hspace{1cm} (B4)

Since, in this limit $B_\perp \sim R/\tau$ and $H_\perp \sim H_\perp/H ||$, and using the definition $\tilde{c}_1 = H \phi_0/4\pi$, we have

$$B_\perp^2 = H_\parallel^2 - \frac{\Delta}{\tilde{c}_1}$$  \hspace{1cm} (B5)

i.e., below $H_\perp^c = \sqrt{\Delta/\tilde{c}_1}$ there is no response to the external tilt while above $H_\perp^c B_\perp \sim (H_\perp - H_\perp^c)^{\beta}$ where $\beta = 1/2$. This analysis is valid for the region in which both $B_\perp$ and $H_\perp$ are much smaller than one, and the resulting $R$ is small compared to $(g\Delta)^{1/d}$.

In the opposite limit, where the external magnetic field is perpendicular to the defects, $R$ is much bigger than both $\tau$ and $(g\Delta)^{1/d}$, thus the first term is negligible and the flux line is parallel to the external field.

The above treatment was for one jump, i.e., it neglects global effects of the defects. For the actual $B_\perp$, one should take the typical defect instead of the extreme one. However, we may use the same analysis in order to get the “time”, $\tau$, spent by the flux line on a particular defect, by dividing the radius of the defect by the “local velocity”, i.e., the associated perpendicular part of the internal magnetic field, associated with it. The result is
\[ \tau(U) \sim \frac{R_0}{B_\perp} \sim \frac{R_0}{\sqrt{H_\perp^2 - \left(\frac{U}{\tau}\right)^2}} \]  
\tag{B6}

where \( R_0 \) is the radius of the defect. Thus, the effective energy associated with a defect of potential energy \( U \) is tilt dependent; as one approaches the critical tilt for this defect, \( H_\perp \to H_\perp^* + \delta \), the effective energy \( E_{\text{eff}} = U\tau(U) \) diverges as \( 1/\sqrt{\delta} \). (This is in agreement with the result for \( h_c \) from the localized limit, i.e., by equating the typical inverse localization length \( \sqrt{\Delta/D} \) to the \( h/D \) term in the gauge transformation \(^{[15]}\).)

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