Bounds on Tracking Error using Closed-Loop Rapidly-Exploring Random Trees

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Motivation

- Autonomous vehicles operating in complex, real-world scenarios
  - Many different navigation scenarios
  - Dynamic and uncertain environment
  - Numerous physical and logical constraints
  - Complex and unstable vehicle dynamics
- Approach: closed-loop rapidly-exploring random trees (CL-RRT) (Kuwata et al. 2009)
  - Maintains advantages of RRT (LaValle 1998)
  - Planning on a closed-loop system
  - Safety guarantees (Frazzoli et al. 2002)

(courtesy DARPA)
Motivation II

- Critical that predicted trajectories are accurately tracked for feasibility

- Successfully demonstrated in DARPA Urban Challenge (Leonard et al. 2008), Agile Robotics for Logistics (Teller et al. 2010)

**Objective:** Characterize theoretical properties of algorithm, particularly tracking
Contributions

1. Under certain assumptions (linear system, bounded disturbance), CL-RRT maintains bounded tracking error for predicted trajectory.
2. Can tighten constraints to guarantee long-term robust feasibility for CL-RRT.
3. Results for linear and nonlinear systems.
Problem Statement

- Uncertain, nonlinear, discrete-time system subject to disturbance
  \( w_t \sim P(W) \)

\[
x_{t+1} = f(x_t, u_t, w_t)
\]

- Constraints acting on system state and input

\[
x_t \in \mathcal{X}_t \\
u_t \in \mathcal{U}_t
\]

Primary Objective

Identify a path (via \( u_t \)) which reaches the goal region \( \mathcal{X}_{goal} \) while satisfying the constraints for all timesteps.
Rapidly-Exploring Random Trees

- System at tree root (R) attempting to reach goal region (G)

- Grow $T$ for some duration: loop through
  1. Sample $x_{samp} \in \mathcal{X}$
  2. Find node $N_{near} \in T$ “nearest” to $x_{samp}$
  3. From $N_{near}$, select inputs $u(t) \in \mathcal{U}$
  4. Simulate trajectory,
     $N_{new} \leftarrow \Phi(N_{near}, u, x_{samp})$
  5. If $N_{new}$ is feasible, add it to $T$

- Determine best path in $T$ according to cost function and execute it
Closed-Loop RRT

- Sample inputs to low-level controller $u = \kappa(r, x)$, then propagate state trajectory (Kuwata et al. 2009)
- Maintain trees for both reference and state
- Resulting tree is still dynamically feasible
- Each sample can generate long-timescale maneuvers $\rightarrow$ very efficient

Key Question:
How do the two algorithms compare in terms of robustness to disturbances/uncertainty?
Open-Loop Model, Open-Loop System

- **Assumptions:** LTI system, additive process noise
- **Approach:** Develop error dynamics of system vs. model

Model:

\[
\hat{u}_t \quad \text{open-loop} \\
\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t
\]

System:

\[
u_t = \hat{u}_t \\
x_{t+1} = Ax_t + Bu_t + w_t
\]

Open-Loop Error Dynamics

\[
e_t = x_t - \hat{x}_t \Rightarrow e_{t+1} = Ae_t + w_t
\]

- $A$ stable $\iff$ Error dynamics stable
- Error propagation unaffected by input sequence
Closed-Loop Model, Closed-Loop System

Model:
\[
\hat{u}_t = K(\hat{x}_t - r_t)
\]
\[
\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t
\]
\[
= (A + BK)\hat{x}_t - BKr_t
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System:
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\]

Closed-Loop Error Dynamics
\[
e_{t+1} = (A + BK)e_t + w_t
\]

- \(A + BK\) stable \(\Leftrightarrow\) Error dynamics stable
- Provides mechanism for shaping error propagation

BIBO Stability of Error Dynamics
\[
A + BK\text{ stable, } w \in \mathcal{W}\text{ bounded} \Rightarrow e_t\text{ bounded} \quad \forall t
\]
Ultimately want robust feasibility: state constraints $X$ and input constraints $U$ satisfied at all timesteps for all possible disturbances $w \in W$

Suppose error bounds known
  - Show error bounds satisfy nominal constraints ... or ...
  - Show nominal path satisfies tightened constraints
Achieving Robust Feasibility

- Ultimately want robust feasibility: state constraints $\mathcal{X}$ and input constraints $\mathcal{U}$ satisfied at all timesteps for all possible disturbances $w \in \mathcal{W}$
- Suppose error bounds known
  - Show error bounds satisfy nominal constraints . . . or . . .
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Achieving Robust Feasibility

- Ultimately want **robust feasibility**: state constraints $\mathcal{X}$ and input constraints $\mathcal{U}$ satisfied at all timesteps for all possible disturbances $w \in \mathcal{W}$
- Suppose error bounds known
  - Show error bounds satisfy nominal constraints ... or ...
  - Show **nominal** path satisfies tightened constraints
**Tube MPC**

- **Approach:** Leverage robust model predictive control (MPC) (Mayne et al. 2000) to show CL-RRT satisfies tightened constraints
  - RRT paths ⇔ feasible MPC solutions
- **Example:** Identify tube of nominally feasible constraints (Langson et al. 2004, Mayne et al. 2005)
  - **Centerline:** disturbance-free trajectory
  - **Cross-section:** robust positively invariant set $Z$

\[(A + BK)x + w \in Z \quad \forall \; x \in Z, \; \forall \; w \in \mathcal{W}\]
**Approach:** Leverage robust model predictive control (MPC) (Mayne et al. 2000) to show CL-RRT satisfies tightened constraints
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**Tube MPC**

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Robust Feasibility

**Theorem (Closed-loop RRT with Robust Feasibility)**

Given an LTI system and linear feedback $K$ such that $A + BK$ is stable, tighten the state constraints $\mathcal{X}$ and input constraints $\mathcal{U}$ according to

$$\mathcal{X}^- = \mathcal{X} \ominus Z, \quad \mathcal{U}^- = \mathcal{U} \ominus KZ.$$  

Then any path followed using CL-RRT with feedback $K$ satisfies all constraints for all $w \in \mathcal{W}$.

- Proof in paper (same error dynamics)
- Leads to **Tube-RRT**: “tree of tubes”
- Tube cross-section is fixed off-line $\Rightarrow$ negligible complexity increase
- Other approaches for tightening constraints (CT-RRT)
Linear Example

- Double integrator (quadrotor) navigating 2D obstacle field in windy env. (30% of input)
- 20 trials

| Algorithm       | % Feas. | Avg Error, m | Max Error, m | Time per Node, ms |
|-----------------|---------|--------------|--------------|-------------------|
| RRT (←)         | 10      | 0.341        | 0.997        | 7.04              |
| CL-RRT (→)      | 100     | 0.025        | 0.057        | 6.77              |

Luders et al. (LIDS, MIT) Bounds on Tracking Error using CL-RRT
Nonlinear Example

- Badly-modelled skid-steering vehicle operating in uneven terrain
  - Terrain disturbance $\leq 20\%$ of speed
  - Steering map bias (10\% of speed)
  - Steered using pure pursuit (Park et al. 2007, Kuwata et al. 2008)

- Open-loop: heading drift $\Rightarrow$ certain infeasibility

- Closed-loop RRT: identifies feasible path in 50\% of trials
  - Converts poor mapping into bounded offset
Conclusions

- CL-RRT can be used to accurately track a trajectory with **known error bounds** and **robust feasibility guarantees**, without replanning.
- Accurate trajectory tracking $\Rightarrow$ improved likelihood of long-term feasibility.
- Introduced Tube-RRT: augments CL-RRT with robust feasibility by tightening constraints.
- Demonstrated robustness in many domains:
  - Linear vs. nonlinear
  - Simulation vs. hardware
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### Tube Error Propagation

**Model:** same as CL-RRT

\[
\begin{align*}
\hat{u}_t &= K(\hat{x}_t - r_t) \\
\hat{x}_{t+1} &= A\hat{x}_t + B\hat{u}_t \\
           &= (A + BK)\hat{x}_t - BKr_t
\end{align*}
\]

**System:** use same \( K \)

\[
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x_{t+1} &= Ax_t + Bu_t + w_t \\
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**Closed-Loop Error Dynamics**

\[
e_{t+1} = (A + BK)e_t + w_t \iff \text{same as CL-RRT}
\]

- Same model trajectory, error dynamics \(\Rightarrow\) same system trajectory
- Tube MPC performs error propagation in same manner as closed-loop RRT
- Only necessary to tighten constraints to achieve robust feasibility

Luders et al. (LIDS, MIT)
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