SCHWARTZ-PICK LEMMA FOR HARMONIC FUNCTIONS

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Abstract. Based on the recently proved Khavinson conjecture, we establish an inequality of Schwarz-Pick type for harmonic functions on the unit ball of $\mathbb{R}^n$.

1. Introduction

This is a sequel to the paper [15]. There we proved the Khavinson conjecture, which says for bounded harmonic functions on the unit ball of $\mathbb{R}^n$, the sharp constants in the estimates for their radial derivatives and for their gradients coincide. In this paper, we further prove the following

Theorem 1. Let $u$ be a real-valued harmonic function on the unit ball $B_n$ of $\mathbb{R}^n$ and $|u| < 1$ on $B_n$.

(i) When $n = 2$ or $n \geq 4$, the following sharp inequality holds:

$$|\nabla u(x)| \leq \frac{2m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{1 - |x|^2}, \quad x \in B_n,$$

where $m_n$ denotes the Lebesgue measure on $\mathbb{R}^n$. Equality holds if and only if $x = 0$ and $u = U \circ T$ for some orthogonal transformation $T$, where $U$ is the Poisson integral of the function that equals 1 on a hemisphere and $-1$ on the remaining hemisphere.

(ii) When $n = 3$ we have

$$|\nabla u(x)| < \frac{8}{3\sqrt{3}} \frac{1}{1 - |x|^2}, \quad x \in B_3.$$  

The constant $\frac{8}{3\sqrt{3}}$ here is the best possible.

Remark 1. Curiously, the inequality (1) fails when $n = 3$. Note that $\frac{8}{3\sqrt{3}} \approx 1.5396$, while the constant $\frac{2m_{n-1}(B_{n-1})}{m_n(B_n)}$ in (1) equals $\frac{3}{2}$ when $n = 3$. According to [11], given $x_0 \in B_3$, there is a $u_0$ harmonic in $B_3$, $|u_0| < 1$, satisfying

$$\left|\frac{\nabla u_0(x)}{|x_0|}\right| = \frac{(9 - |x_0|^2)^2}{3\sqrt{3}(1 - |x_0|^2)(|x_0|^2 + 3)^{3/2} + 3\sqrt{3}(1 - |x_0|^2)}.$$  

Since

$$\lim_{t \to 1^-} \frac{(9 - t^2)^2}{3\sqrt{3}(t^2 + 3)^{3/2} + 3\sqrt{3}(1 - t^2)} = \frac{8}{3\sqrt{3}} > \frac{3}{2},$$

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we see that if \( x_0 \) is sufficiently near the boundary \( S^2 \) then \( |\nabla u_0(x_0)| > \frac{3}{2} \frac{1}{1 - |x_0|^2} \), which contradicts the inequality (1). See Remark 2 in the next section for more explanations.

We refer to Theorem 1 as the Schwarz-Pick lemma for harmonic functions, for (1) it is in analogy to a weaker version of the classical Schwarz-Pick lemma, which states that every holomorphic mapping \( f \) of the unit disk onto itself satisfies the inequality:

\[
|f'(z)| \leq \frac{1}{1 - |z|^2}, \quad |z| < 1;
\]

(2) it is a generalization of the following harmonic Schwarz lemma:

**Theorem A** ([3, Theorem 6.26]). If \( u \) is a real-valued harmonic function on \( B_n \) and \( |u| < 1 \) on \( B_n \), then

\[
|\nabla u(0)| \leq \frac{2m_{n-1}(B_{n-1})}{m_n(B_n)}.
\]

Equality holds if and only if \( u = U \circ T \) for some orthogonal transformation \( T \), where \( U \) is as in Theorem 1.

Some special cases of Theorem 1 are known. When \( n = 2 \), the inequality (1) reads

\[
|\nabla u(z)| \leq \frac{4 \pi}{1 - |z|^2}, \quad |z| < 1;
\]

which is a reformulation of the Lindelöf inequality in the unit disc (see [5, Theorem 3]). Recently, Kalaj [9] established the inequality (1) for \( n = 4 \). When \( n = 3 \), although not explicitly stated, the inequality (2) is an easy consequence of the main result of [20].

When \( n = 2 \) or \( n \geq 4 \), Theorem 1 can be restated as

\[
|\nabla u(x)| \leq \frac{2m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{1 - |x|^2} \sup_{y \in B_n} |u(y)|
\]

for bounded harmonic functions \( u \) on \( B_n \). This is obviously related to the following classical estimate (see for instance [21, p.139, (6)]

\[
|\nabla u(x)| \leq \frac{m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{d(x)} \text{osc}_\Omega(u)
\]

for harmonic functions \( u \) in \( \Omega \subset \mathbb{R}^n \), where \( \text{osc}_\Omega(u) \) is the oscillation of \( u \) in \( \Omega \) and \( d(x) \) denotes the distance of \( x \in \Omega \) to the boundary \( \partial \Omega \). In particular, if \( \Omega = B_n \) and \( u \) is a harmonic function on \( B_n \) with \( |u| < 1 \), then (5) reads

\[
|\nabla u(x)| \leq \frac{2m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{1 - |x|}.
\]

Compare this with the inequality (1).

It is also interesting to compare the inequality (1) with the following sharp inequality in [13] (see also [14, p.131]):

\[
|\nabla v(x)| \leq \frac{4(n-1)^{\frac{n+1}{n}}}{n^{\frac{n+1}{2}}} \frac{m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{x_n} \sup_{y \in \mathbb{R}^n_+} |v(y)|.
\]

Here, \( v \) is a bounded harmonic function in the half–space \( \mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\} \).
Recently, several versions of Schwarz lemma for harmonic functions or harmonic mappings were established. See [4, 8, 10, 16, 18, 19]. In particular, Kalaj and Vuorinen [10] obtained the following refinement of the inequality (4):

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad |z| < 1.$$  

This, together with our Theorem 1, suggests the following

Conjecture. Under the hypotheses of Theorem 1(i), we have

$$|\nabla u(x)| \leq \frac{2^{m_n-1} (B_n-1)}{m_n(B_n)} \frac{1 - |u(x)|^2}{1 - |x|^2}, \quad x \in B_n.$$  

We are not able to prove this conjecture and leave it as an open question.

2. OUTLINE OF THE PROOF

Since the cases $n = 2$ and $n = 3$ are known, we shall prove only Theorem 1 for $n \geq 4$.

Recall that we proved in [15] the following

**Theorem B** ([15, Theorem 2]). Let $n \geq 3$ and let $u$ be a real-valued bounded harmonic function on $B_n$. We have the following sharp inequality:

$$|\nabla u(x)| \leq C(x) \sup_{y \in B_n} |u(y)|, \quad x \in B_n,$$

with

$$C(x) := \frac{(n-1)m_{n-1}(B_{n-1})}{m_n(B_n)} \frac{1}{\left(1 - \frac{n-2}{n} |x| \left(1 - t^2 \right)^{\frac{n-3}{2}} \frac{1}{1 - 2t|x| + |x|^2} \right) \frac{1}{1 - |x|^2}}.$$

Thus, in order to prove the inequality (1), it suffices to prove the following

**Proposition 2.** When $n \geq 4$, the function

$$\Phi(\rho) := \frac{1}{\left(1 - \frac{n-2}{n} \rho \left(1 - t^2 \right)^{\frac{n-3}{2}} \frac{1}{1 - 2t\rho + \rho^2} \right) \frac{1}{1 - |x|^2}}$$

is strictly decreasing on $[0, 1]$ and

$$\max_{\rho \in [0,1]} \Phi(\rho) = \Phi(0) = \frac{2}{n-1}.$$  

**Remark 2.** In contrast to the case $n \geq 4$, if $n = 3$ then

$$\Phi(\rho) = \frac{2}{3} \left(1 + \frac{1}{3} \rho^2 \right)^{\frac{1}{2}} - 1 + \rho^2$$

is strictly increasing on $[0, 1]$ and attains its maximum at $\rho = 1$. This explains why the inequality (1) fails when $n = 3$, as well as why, unlike (1), the inequality (2) is sharp but always strict.

Assuming Proposition 2 for the moment, we shall prove the second assertion of Theorem 1(i). In view of Theorem B, it follows from the strict monotonicity of $\Phi$ that the equality in (1) takes place if and only if $x = 0$. Then, by Theorem A, $u$ must be of the form $u = U \circ T$, with $T$ an orthogonal transformation and $U$
the Poisson integral of the function that equals 1 on a hemisphere and −1 on the remaining hemisphere.

We now turn to the proof of Proposition 2. An easy computation leads to \( \Phi'(0) = 0 \). Thus, the problem is further reduced to the following

**Theorem 3.** If \( n \geq 4 \) then \( \Phi \) is a strictly concave function on the interval \((0,1)\).

We divide the proof of Theorem 3 into the following two propositions.

**Proposition 4.** We have

\[
\Phi''(\rho) = \frac{2(n-2)}{\rho^2} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \left( 1 - \frac{n-4}{n} \rho^2 \right)^{-\frac{2n-3}{n-4}}
\times \frac{1 - \frac{(n-2)(n-3)}{n(n-1)} \rho^2}{1 - \frac{(n-2)^2}{n^2} \rho^2}
\times \frac{1 - \frac{n-2}{n} \rho^2}{1 - \frac{(n-2)(n-3)}{n^2} \rho^2}
\times \frac{1 - \frac{n-2}{n} \rho^2}{1 - \frac{(n-2)(n-3)}{n^2} \rho^2}
\times \left[ 1 - \frac{1}{n+1} \rho^2 \right]^{n+1} \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k.
\]

Here and throughout the paper, \( \text{hypergeom}[a, b ; c ; z] \) denotes the Gauss hypergeometric function defined by

\[
\text{hypergeom}[a, b ; c ; z] := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k
\]

for \(|z| < 1\), with \((\lambda)_k\) the Pochhammer symbol (or the extended factorial), which is defined by

\[
(\lambda)_0 := 1, \quad (\lambda)_k := \lambda(\lambda + 1) \ldots (\lambda + k - 1) \quad \text{for} \ k \geq 1.
\]

**Proposition 5.** If \( n \geq 4 \) then

\[
\text{hypergeom}[1, \frac{n}{2} ; \frac{n+1}{2} ; t] > \frac{(1 - \frac{n-4}{n} t) \left[ 1 - \frac{(n-2)(n-3)}{n^2} t \right]}{(1 - \frac{n-2}{n} t) \left[ 1 - \frac{(n-2)(n-3)}{n^2} t \right]} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]
\]

holds for all \( t \in [0,1] \).

**Remark 3.** As we will see in the proof, if \( n = 3 \) then the inequality in (8) is reversed.

Propositions 4 and 5 will be proved in the next two sections.

3. **The Proof of Proposition 4**

The proof will be divided into three steps.
Step 1. We express the function \( \Phi \) as follows.

\[
\Phi(\rho) = \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-3}{2}} dt \\
+ (n-2)\rho \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-1}{2}} dt \\
+ \sum_{k=2}^{\infty} \frac{2n(n-2)}{k(k-1)(k+n-2)(k+n-1)} \\
\times \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right]^{\frac{n+2}{2}} C_{k-2}^{\frac{n}{2}} \left( \frac{n-2}{n} \rho \right) \rho^k,
\]

where \( C_k^{\lambda}(x) \) is the Gegenbauer polynomial (also known as the ultraspherical polynomials) of degree \( k \) associated to \( \lambda \), which is defined by the generating relation

\[
(1 - 2xz + z^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k^{\lambda}(x) z^k, \quad -1 < x < 1, \, |z| < 1.
\]

Using the generating relation (10) and noting that 
\( C_{\frac{n-2}{2}}^{\frac{n}{2}}(t) \equiv 1 \) and 
\( C_{\frac{n-2}{2}}^{\frac{n}{2}}(t) = (n-2)t \),

we obtain

\[
\Phi(\rho) = \sum_{k=0}^{\infty} \left\{ \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-3}{2}} C_{k}^{\frac{n}{2}} \left( \frac{n-2}{n} \rho \right) dt \right\} \rho^k
\]

\[
= \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-3}{2}} dt \\
+ (n-2)\rho \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-1}{2}} dt \\
+ \sum_{k=2}^{\infty} \left\{ \int_{-1}^{1} \left| t - \frac{n-2}{n} \rho \right| (1 - t^2)^{\frac{n-3}{2}} C_{k-2}^{\frac{n}{2}} \left( \frac{n-2}{n} \rho \right) dt \right\} \rho^k.
\]

Then (9) follows by an application of Lemma 6 below, with \( \lambda = \frac{n-2}{2} \) and \( s = \frac{n-2}{n} \rho \).

**Lemma 6 ([15 Lemma 5]).** Let \( \lambda > -1/2 \) and \(-1 < s < 1\). Then we have

\[
\int_{-1}^{1} |x-s|(1-x^2)^{\lambda-\frac{1}{2}} C_k^{\lambda}(x) dx
\]

\[
= \frac{8\lambda(\lambda+1)}{k(k-1)(k+2\lambda)(k+2\lambda+1)} (1 - s^2)^{\lambda+\frac{1}{2}} C_{k-2}^{\lambda+2}(s)
\]

for \( k = 2, 3, \ldots \).
**Step 2.** We claim the formula

\[
\Phi''(\rho) = \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} C_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right)^k \rho^k
\]

\[
- \frac{4(n-2)^2}{n(n-1)} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} (n-1)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right)^k \rho^k
\]

\[
+ \frac{2(n-2)}{n+1} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} (n+2)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right)^k \rho^k.
\]

The proof is similar to that of Lemma 8 in [15]. In view of (11), we write

\[
\Phi_1(\rho) := \int_{-1}^{1} \left| \frac{n-2}{n} \rho - x \right| (1-x^2)^{\frac{n-4}{2}} dx,
\]

\[
\Phi_2(\rho) := (n-2) \rho \int_{-1}^{1} \left| \frac{n-2}{n} \rho - x \right| (1-x^2)^{\frac{n-4}{2}} x dx,
\]

\[
\Phi_3(\rho) := \sum_{k=2}^{\infty} \frac{2n(n-2)}{k(k-1)(n+2)(k+n-1)} \times \left\{ \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=2}^{\infty} C_{k-2}^{\frac{n+2}{n}} \left( \frac{n-2}{n} \rho \right) \right\} \rho^k.
\]

Straightforward computations yield

\[
\Phi_1''(\rho) = \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} C_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right)
\]

and

\[
\Phi_2''(\rho) = \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} C_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho
\]

\[
- \frac{4(n-2)^2}{n(n-1)} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=1}^{\infty} (n-1)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho^k
\]

\[
+ \frac{2(n-2)}{n+1} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} (n+2)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho^k.
\]

So, it remains to show that

\[
\Phi_3''(\rho) = \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=2}^{\infty} C_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho^k
\]

\[
- \frac{4(n-2)^2}{n(n-1)} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=1}^{\infty} (n-1)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho^k
\]

\[
+ \frac{2(n-2)}{n+1} \left[ 1 - \frac{(n-2)^2}{n^2} \rho^2 \right] \sum_{k=0}^{\infty} (n+2)_k \frac{n-2}{n} \left( \frac{n-2}{n} \rho \right) \rho^k.
\]

We need the following lemma.
Lemma 7 ([15] Lemma 4). If \( \lambda \neq 1 \) then

\[
\frac{d}{dx} \left\{ (1 - x^2)^{\lambda - \frac{1}{2}} C_k(x) \right\} = -(k + 1)(k + 2\lambda - 1) \frac{1}{2(\lambda - 1)} (1 - x^2)^{\lambda - \frac{3}{2}} C_{k+1}(x).
\]

Applying Lemma 7 successively, it follows that

\[
\frac{d}{d\rho} \left\{ \left[ 1 - \left( \frac{n - 2)^2}{n^2} \rho^2 \right)^{\frac{n+1}{2}} \right] C_k^{\frac{n+2}{2}} \left( \frac{n-2}{n} \rho \right) \right\} = \frac{n-2}{n} \frac{k(k+n-2)(k+n-1)}{n(n-2)} \left[ 1 - \left( \frac{n - 2)^2}{n^2} \rho^2 \right)^{\frac{n-3}{2}} C_k^{\frac{n+2}{2}} \left( \frac{n-2}{n} \rho \right) \right],
\]

and hence

\[
\Phi_3''(\rho) = \sum_{k=2}^{\infty} \frac{2n(n-2)}{k(k-1)(k+n-2)(k+n-1)} \rho^k
\]

\[
\times \left( \frac{d^2}{d\rho^2} \left\{ \left[ 1 - \left( \frac{n - 2)^2}{n^2} \rho^2 \right)^{\frac{n+1}{2}} \right] C_k^{\frac{n+2}{2}} \left( \frac{n-2}{n} \rho \right) \right\} \rho^k + 2 \frac{d}{d\rho} \left\{ \left[ 1 - \left( \frac{n - 2)^2}{n^2} \rho^2 \right)^{\frac{n+1}{2}} \right] C_k^{\frac{n+2}{2}} \left( \frac{n-2}{n} \rho \right) \right\} \frac{d}{d\rho} (\rho^k) + \left[ 1 - \left( \frac{n - 2)^2}{n^2} \rho^2 \right)^{\frac{n+1}{2}} C_k^{\frac{n+2}{2}} \left( \frac{n-2}{n} \rho \right) \right] \frac{d^2}{d\rho^2} (\rho^k) \right).
\]

This is precisely the desire identity ([13], in view of that

\[
\frac{(n-1)_k}{(n)_k} = \frac{n-1}{n+k-1} \quad \text{and} \quad \frac{(n)_k}{(n+2)_k} = \frac{n(n+1)}{(n+k)(n+k+1)}.
\]
Step 3. We are now ready to conclude the proof of the Proposition 4.

Denote the three terms in the right hand side of (12) by $I$, $II$ and $III$ respectively. By the generating relation (10), we have

\begin{align}
I &= \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left[ 1 - 2 \left( \frac{n-2}{n} \rho + \rho^2 \right)^{-\frac{n+1}{2}} \right. \\
&= \frac{2(n-2)^2}{n^2} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-4}{n} \rho^2 \right)^{\frac{n+1}{2}}.
\end{align}

For $II$, we shall make use of the following

Lemma 8 ([22] p.279, (8)). Suppose that $\nu, \lambda \in \mathbb{R}$ and $2\lambda \neq 0, -1, -2, \ldots$. Then the identity

\begin{equation}
(1 - xz)^{-\nu} \frac{\Gamma(n+\nu+1)}{\Gamma(\lambda+\frac{1}{2})} \frac{z^2(x^2-1)}{(1-xz)^2} = \sum_{k=0}^{\infty} (\nu)_k C_k^{\lambda}(x) z^k
\end{equation}

holds, whenever both sides make sense.

We apply (15) with $\nu = n-1$ and $\lambda = \frac{\rho}{2}$ to obtain

\begin{align}
II &= -\frac{4(n-2)^2}{n(n-1)} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-2}{n} \rho^2 \right)^{-n+1} \\
&\quad \times \, _2F_1 \left[ \frac{n-1}{2}, \frac{n+1}{2}; \frac{n}{2} \frac{(n-2)^2}{n^2 \rho^2} - 1 \right].
\end{align}

On the other hand,

\begin{equation}
\frac{2}{n-2} \frac{\rho^2 \left( (a-2)^2 \rho^2 - 1 \right)}{(1 - \frac{n-2}{n} \rho^2)^2}
\end{equation}

which follows from Pfaff's transformation formula ([6] p.105, (4)))

\[ _2F_1 \left[ \frac{1}{2}, \frac{n+1}{2}; \frac{n}{2} \frac{1 - (n-2)^2 \rho^2}{1 - \frac{n-2}{n} \rho^2} \right]. \]

Substituting (17) into (16) yields

\begin{align}
II &= -\frac{4(n-2)^2}{n(n-1)} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-2}{n} \rho^2 \right) \\
&\quad \times \left( 1 - \frac{n-4}{n} \rho^2 \right)^{-\frac{n}{2}} \, _2F_1 \left[ \frac{n+1}{2}, \frac{n}{2} \frac{1 - (n-2)^2 \rho^2}{1 - \frac{n-4}{n} \rho^2} \right].
\end{align}
In the same way,

\begin{equation}
III = \frac{2(n-2)}{n+1} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-2}{n \rho^2} \right)^{-n} \\
\times \; _2F\!\!\!_1 \left[ \begin{array}{c}
\frac{n}{2}, \frac{n+1}{2} \\
\frac{n+3}{2}
\end{array}; \frac{\rho^2 \left( \frac{(n-2)^2}{n^2 \rho^2} - 1 \right)}{(1 - \frac{n-2}{n \rho^2})^2}\right]
\end{equation}

Applying Gauss’ contiguous relation \([\text{[6, p.103, (38)]})

\begin{equation}
(c-b)z \; _2F\!\!\!_1 \left[ \begin{array}{c}
a, b \\
c+1; z\end{array} \right] = c \; _2F\!\!\!_1 \left[ \begin{array}{c}
a-1, b \\
c; z\end{array} \right] - c(1-z) \; _2F\!\!\!_1 \left[ \begin{array}{c}
a, b \\
c; z\end{array} \right],
\end{equation}

with \(a=1, b=\frac{n}{2}, c=\frac{n+1}{2}\) and

\begin{equation}
z = \frac{\rho^2 \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]}{1 - \frac{n-4}{n \rho^2}},
\end{equation}

we have

\begin{equation}
\; _2F\!\!\!_1 \left[ \begin{array}{c}
\frac{n}{2} \\
\frac{n+3}{2}\end{array}; \frac{\rho^2 \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]}{1 - \frac{n-4}{n \rho^2}}\right]
= \frac{n+1}{\rho^2} \left[ 1 - \frac{n-4}{n \rho^2} \right]^{-\frac{n+1}{2}} \\
- \frac{n+1}{\rho^2} \left[ 1 - \frac{n-2}{n \rho^2} \right] \; _2F\!\!\!_1 \left[ \begin{array}{c}
\frac{n}{2} \\
\frac{n+3}{2}\end{array}; \frac{\rho^2 \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]}{1 - \frac{n-4}{n \rho^2}}\right].
\end{equation}

Substituting this into \((19)\), we obtain

\begin{equation}
III = \frac{2(n-2)}{\rho^2} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-4}{n \rho^2} \right)^{-\frac{n+1}{2}} \\
- \frac{2(n-2)}{\rho^2} \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]^{\frac{n+1}{2}} \left( 1 - \frac{n-4}{n \rho^2} \right)^{-\frac{n+1}{2}} \\
\times \left[ 1 - \frac{n-2}{n \rho^2} \right]^2 \; _2F\!\!\!_1 \left[ \begin{array}{c}
\frac{\nu}{2} \\
\frac{n+1}{2}, \frac{n+3}{2}\end{array}; \frac{\rho^2 \left[ 1 - \frac{(n-2)^2}{n^2 \rho^2} \right]}{1 - \frac{n-4}{n \rho^2}}\right].
\end{equation}

Summing up \((13)\), \((18)\) and \((20)\) leads to the desired equality \((7)\).
4. Proof of Proposition 5

The proof of Proposition 5 is rather lengthy, we only sketch it. Write

\[ \varphi(t) := \frac{t \left[ 1 - \frac{(n-2)^2}{n^2} t \right]}{1 - \frac{n-4}{n} t}, \quad t \in [0, 1]. \]

It is easy to check that the inequality (8) is equivalent to

\[ \varphi(t) \left[ 1 - \varphi(t) \right] \frac{1}{2} \binom{n-3}{n-2} \binom{1-\varphi(t)}{n-1} > \frac{t \left[ 1 - \frac{(n-2)^2}{n^2} t \right]^{n-3} \left[ 1 - \frac{(n-2)(n-3)}{n^2} t \right]}{\left(1 - \frac{n-4}{n} t\right)^{n-4} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]}. \]

So we define

\[ \Psi(t) := \varphi(t) \left[ 1 - \varphi(t) \right] \frac{1}{2} \binom{n-3}{n-2} \binom{1-\varphi(t)}{n-1} \]

and claim that \( \Psi(t) > 0 \) for all \( t \in (0, 1) \). Since \( \Psi(0) = 0 \), it suffices to show that \( \Psi'(t) > 0 \) for all \( t \in (0, 1) \).

Applying the elementary relation ([6, p.102, (23)])

\[ \frac{d}{dz} \left\{ \binom{z-a}{c-a} (1-z)^{a+b-c} \binom{a+b-c}{c} \right\} = (c-a) \binom{z-a-1}{c-a} (1-z)^{a+b-c-1} \binom{a-1}{c} \]

and noting that

\[ \varphi'(t) = \frac{(1 - \frac{n-2}{n} t) \left[ 1 - \frac{(n-2)(n-4)}{n^2} t \right]}{(1 - \frac{n-4}{n} t)^2}, \]
we obtain
\[
\frac{d}{dt} \left\{ \varphi(t) \frac{n-1}{2} \left[ 1 - \varphi(t) \right]^{\frac{1}{2}} \frac{d}{dt} \left[ 1, \frac{n}{n+1} ; \varphi(t) \right] \right\}
\]
\[
= \frac{n-1}{2} \varphi(t) \frac{n-3}{n} \left[ 1 - \varphi(t) \right]^{-\frac{1}{2}} \varphi'(t)
\]
\[
= \frac{n-1}{2} t^{\frac{n-3}{n}} \left[ 1 - \frac{n-4}{n} t \right]^{-\frac{2}{n}} \left[ 1 - \frac{(n-2)^2}{n^2} t \right]^{\frac{n-3}{n-1}} \left[ 1 - \frac{(n-2)(n-4)}{n^2} t \right]^{\frac{n-3}{n-3}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{\frac{n-3}{n-3}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{-2}
\]
\[
\times \left\{ \frac{n-1}{2} - \frac{12 - 46n + 56n^2 - 26n^3 + 4n^4}{2n^2(n-1)} \right\}
\]

Also, a direct calculation yields
\[
\frac{d}{dt} \left\{ t^{\frac{n-1}{2}} \left[ 1 - \frac{(n-2)^2}{n^2} t \right]^{\frac{n-3}{n}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{\frac{n-3}{n-3}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{-2}
\]
\[
\times \left\{ \frac{n-1}{2} - \frac{10 - 49n + 57n^2 - 26n^3 + 4n^4}{2n^2(n-1)} \right\}
\]

It follows that
\[
\Psi'(t) = \frac{t^{\frac{n-1}{2}}}{2n^5(n-1)} \left( 1 - \frac{n-4}{n} t \right)^{-\frac{2}{n}} \left[ 1 - \frac{(n-2)^2}{n^2} t \right]^{\frac{n-3}{n}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{\frac{n-3}{n-3}} \left[ 1 - \frac{(n-2)(n-3)}{n(n-1)} t \right]^{-2}
\]
\[
\times \left\{ \frac{n-1}{2} - \frac{10 - 49n + 57n^2 - 26n^3 + 4n^4}{2n^2(n-1)} \right\}
\]
\[
\times \left\{ n^3(n^2 - 3n - 2) - 2n(n-2)(n-4)(n^2 - 3n + 1) t
\]
\[
+ (n-2)^2(n-3)(n-4)^2 t^2 \right\}.
\]
It is easily seen that, when $n \geq 4$, the quadratic polynomial
\[ n^3(n^2 - 3n - 2) - 2n(n - 2)(n - 4)(n^2 - 3n + 1) t + (n - 2)^2(n - 3)(n - 4)^2 t^2 \]
is always positive on the interval $[0,1]$. Consequently $\Psi'(t) > 0$ for all $t \in [0,1]$. This completes the proof.

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