Simultaneous Lane-Keeping and Obstacle Avoidance by Combining Model Predictive Control and Control Barrier Functions

Sven Brüggemann*, Drew Steeves*, and Miroslav Krstic

Abstract—We combine Model Predictive Control (MPC) and Control Barrier Function (CBF) design methods to create a hierarchical control law for simultaneous lane-keeping (LK) and obstacle avoidance (OA): at the low level, MPC performs LK via trajectory tracking during nominal operation; and at the high level, various CBF-based safety filters that ensure LK and OA are designed and compared across practical scenarios. In particular, we show that Exponential Safety (ESf) and Prescribed-Time Safety (PTSf) filters, which override the MPC control when necessary, result in feasible Quadratic Programs when OA-safety is prioritized. We additionally investigate control designs subject to input constraints by using Input-Constrained-CBFs. Finally, we compare the performance of combinations of ESf, PTSf, and their input-constrained counterparts with respect to the LK and OA goals in two simulation studies for early- and late-detected obstacle scenarios.

I. INTRODUCTION

Achieving stability while satisfying state and input constraints makes MPC [1] tremendously appealing to areas such as autonomous systems [2]–[5]. However, the constrained optimization problem that forms the backbone of MPC can have prohibitively long computing times. This is problematic for fast-moving safety-critical systems since they demand extremely fast system responses when faced with dangerous system “perturbations”. Additionally, such systems require state constraints that are usually time-varying and nonconvex, which also risks recursive feasibility.

CBF-based safety filters on the other hand can directly treat more intricate state constraints while retaining some theoretical guarantees, and prioritizing safety can readily be designed into the safety filter.

In regard to the vehicle lane-keeping (LK), linear MPC is advantageous since, under practical highway-driving assumptions, the (local) vehicle dynamics can be modeled as LTI systems and maximum steering angle inputs are easily handled in the resulting Quadratic Program (QP). For vehicle obstacle avoidance (OA), where (global) nonlinear vehicle dynamics must be considered, CBF-based controllers are desirable since they are computationally cheap, can be designed independent of sampling time, and can natively prioritize safety “violations”.

In this work, we combine these two control design methods to simultaneously perform vehicle LK and OA (for static obstacles with circular planar projections) for highway driving scenarios. We investigate high-performance CBFs which guarantee Prescribed-Time Safety to impose safety only over the finite time for which the obstacle is ahead of the vehicle. For simultaneous LK and OA while constraining the steering angle to practical values, we also investigate the marriage between MPC, where input constraints are natively considered, and Input Constrained Barrier Functions (ICCBFs). Practical aspects such as data acquisition, sensing and computing time are beyond the scope of this work.

Related work

We refer to [6] for a general overview for autonomous highway driving. The predominant implementation of safety filters is QP-based and selects the control input “closest” (in least-squares sense) to a nominal one, subject to linear inequality constraints that enforce safety. The authors of [7] revealed the connection between such Exponential Safety filter (ESf) designs and backstepping through [8]. More recently, the time-varying backstepping methods from [9] were applied in the context of CBF-based safe control design in [10] to generate Prescribed-Time Safety (PTSf), which only invokes safety for as long as it is required.

CBFs have been applied to several practical problems: for example, [11] proposes to design CBFs offline through a combination of sum-of-squares program, and use the related CBF-QP formulation for simultaneous LK and cruise control. Wu et al. [12] design a CBF-CLF hard constraint for the MPC with proven recursive feasibility, producing stability with safety but without computational/feasibility considerations. Using a multi-rate formulation, in [13] a MPC planner is designed in conjunction with a CBF-QP formulation under the assumption that an ICCBF exists. For OA, the authors in [14] propose a continuous-time CBF as an additional state constraint for a discrete-time nonlinear MPC. The work [15] discusses how discrete-time CBF constraints jeopardize recursive feasibility.

II. LATERAL VEHICLE DYNAMICS

We rely on a dynamic model that governs the local vehicle position, heading and associated velocities (all with respect to the road, see Figure 1). We make the following assumption:

Assumption 1. The longitudinal vehicle vel., v1, is constant.

With Assumption 1, we obtain the following LTI model that governs the lateral vehicle dynamics [16]:

\[ \dot{x}(t) = Ax(t) + Bu(t) + G\hat{\psi}_{\text{ref}}(t), \]

(1)

where \( x(t) = [e_1 \ \dot{e}_1 \ e_2 \ \dot{e}_2]^T \), where \( u(t) \) is the front wheel steering angle with bounded magnitude \( u_{\text{max}} > 0 \), and
ψ_{ref}(t) ∈ ℜ is the reference yaw rate for t ∈ ℜ≥0. We denote the components of A and B by A = (a_{ij})_{1≤i,j≤4}, B = (b_{ij})_{1≤i≤4}, where b_{i1} = 0 for i = \{1, 3\}, which makes (1) of relative degree two for output functions only considering e_1 and e_2. If the yaw rate ψ_{ref} ≠ 0, it is impossible to stabilize the longitudinal and angular errors to the origin, since G lies outside span(B) in general. However, for a fixed yaw rate ψ_{ref} the steady state and control are\(^1\)

\[ u_s(t) = \dot{\psi}_{ref}(t) \left( \frac{L_l}{v_1} + k_v v_1 \right) = \dot{\psi}_{ref}(t) \bar{u}_s, \]

\[ x_s(t) = \dot{\psi}_{ref}(t) \left[ \begin{array}{c} 0 \\ 0 \\ -\frac{L_r}{v_1} + \alpha_r \end{array} \right] = \dot{\psi}_{ref}(t) \bar{x}_s. \]

### III. High-level Safety Filter

In this section, we assume that a desired nominal control is provided. Our high-level safety filter is designed to perform both of the LK and OA tasks by employing barrier functions which encode these objectives. It is implemented by the common QP-based approach

\[ u_{\text{safe}} = \arg \min_{\bar{u} ∈ ℜ} |\bar{u} - u_{\text{nominal}}|^2 \]

s.t. \[ a_i + b_i \bar{u} ≥ 0, \quad a_k + b_k \bar{u} ≥ 0, \]

where one inequality constraint is used to encode the LK objective, and the other encodes the OA objective. In order to generate safety overriding controllers which can simultaneously satisfy both objectives (and in particular, satisfy the so-called control-sharing property [17], which is a necessary and sufficient condition for (4) to be feasible), we propose a barrier function which smoothly transitions from one of the roadway lane barriers to one encapsulating the obstacle, whereas the other lane barrier either remains the same or expands in a direction away from the obstacle (see Figure 1).

Note that lane modification is not inherent to the MPC design method: in most applications, modifying any existing state constraints in real-time brings into question recursive feasibility.

\(^1\)\(L_l \) (\(L_r \)) is the distance from the center of gravity to the rear (front) tires (resp.), \(k_v \) the under-steer gradient, and \(\alpha_r \), \(\psi_{ref}(t) \) is the slip angle of the rear tires.

### A. Control Barrier Function Design for LK and OA

We begin by designing our barrier functions (BFs) that encode vehicle safety w.r.t. LK and OA. We first define the squared relative distance from vehicle to obstacle,

\[ d(e_1, e_2) := (X_{\text{car}}(e_1, e_2) - X_{\text{obs}})^2 + (Y_{\text{car}}(e_1, e_2) - Y_{\text{obs}})^2 - r_{\text{obs}}^2, \]

where \(r_{\text{obs}} \) denotes the radius of the obstacle-encompassing circle, and \((X_{\text{car}}, Y_{\text{car}}), (X_{\text{obs}}, Y_{\text{obs}})\) denote the global coordinates of the vehicle and the obstacle, resp. We define the smooth step function which depends on the relative distance \(d(e_1, e_2)\) and the relative detection distance \(\delta_2 > 0\):

\[ \Phi(d) = \begin{cases} 0, & d ≥ \delta_2^2, \\ \exp \left( 1 - \frac{d^2}{\delta_2^2 - d} \right), & 0 < d < \delta_2^2, \\ 1, & d ≤ 0. \end{cases} \]

For \(w_1 > 0\) denoting the width of the lane and \(e_{\text{obs}}^f\) denoting the shortest distance of the obstacle-encompassing circle to the centerline (see Figure 1), we define the left BF (ensuring LK or a certain violation of LK)

\[ h_l := \Phi(d) \left( \frac{w_1}{2} - e_1 \cos(e_2) + e_\gamma \right) + (1 - \Phi(d)) \left( \frac{w_1}{2} - e_1 \cos(e_2) \right), \]

where \(e_\gamma \geq 0\) is a design parameter which allows the vehicle to utilize a portion of the highway shoulder or a left-hand lane (assuming no oncoming traffic). By selecting \(e_\gamma = 0\), the left barrier simplifies to 

\[ h_l(e_1, e_2) = \frac{w_1}{2} - e_1, \]

that is, it remains constant regardless of the vehicle’s proximity to the obstacle. We define the right BF (ensuring OA) as

\[ h_r := \Phi(d) \left( e_1 \cos(e_2) - e_{\text{obs}}^f \right) + (1 - \Phi(d)) \left( \frac{w_1}{2} + e_1 \cos(e_2) \right). \]

While the choice \(e_\gamma = 0\) is made throughout Section V with success, to establish theoretical safety guarantees (cf. Lemma 1), we prioritize OA over LK by selecting \(e_\gamma > 0\).

For presentation clarity, we assume w.l.o.g. that the center of the obstacle-encompassing circle is on the right side of the road centerline, i.e., that \(e_{\text{obs}}^f < 0\), and that we seek to pass on the left side of the obstacle. Figure 1 depicts the left and right BFs as red dashed lines. We define the safe sets

\[ S_l := \{ x ∈ ℜ^4 | h_l(e_1, e_2) ≥ 0 \}, \quad S_r := \{ x ∈ ℜ^4 | h_r(e_1, e_2) ≥ 0 \}; \]

goal is to design MPC-oversriding control laws which renders \(S_l ∩ S_r\) positively-invariant.

While the MPC design for lane-keeping is performed using the linear local dynamics (1), the safety filter design, which invokes the vehicle’s global position in (5), involves the vehicle’s nonlinear dynamics: the vehicle coordinates \((X_{\text{car}}, Y_{\text{car}})\) are related to the rel.-deg.-two system (1) by

\[ \frac{dX_{\text{car}}}{dt}(e_1, e_2) = v_1 \cos(e_2 + \psi_r) - (\dot{e_1} - v_1 e_2) \sin(e_2 + \psi_r), \]

\[ \frac{dY_{\text{car}}}{dt}(e_1, e_2) = v_1 \sin(e_2 + \psi_r) + (\dot{e_1} - v_1 e_2) \cos(e_2 + \psi_r). \]

Hence, it follows from the vehicle’s (global) velocities and the dynamics that (6), (7) are rel.-deg.-two CBFs.
B. Exponential Safety by (Time-Invariant) Backstepping

We introduce the time-invariant backstepping method which leads to explicit characterizations for the safety overriding controllers, referred to as ESf control designs, which render either $S_l$ or $S_r$ positively-invariant. This method is a specific case of more general CBF-safety designs, where functions of the CBFs and their derivatives; the backstepping we ensure that $c_{t,j} > 0$ for all times provided that $(10)$ holds true when $h_i$ is the control having no effect on $(10)$. When this does not hold, the barrier constraints in (4) are usually modified to be soft constraints (e.g., $[18–20]$).

For the CBFs (6) and (7), we compute

$$L_g L_f h_{t,i,j} = -b_{21} \cos(e_2) + b_{11} e_1 \sin(e_2) + \frac{2b_{21} \delta_2}{(\delta_2 - d(e_1))^2} \Phi(d(e_1)) e_v \times \left[ (X_{cat} - X_{obs}) \sin(e_2 + \psi_t) - (Y_{cat} - Y_{obs}) \cos(e_2 + \psi_t) \right],$$

which may equate to zero for certain vehicle headings, heading errors, and relative distances between vehicle and obstacle; this is one caveat of using Assumption 1 to obtain (1) which effectively renders our nonholonomic model as underactuated since otherwise, another control term would appear to prevent $L_g L_f h_{t,i,j} = 0$. Preventing a loss of control by controlling the vehicle’s acceleration is a straightforward extension of the work herein.

By selecting $e_v = \left( \frac{u_t}{2} + e_{obs}^t \right)$ in (12), we obtain $L_g L_f h_{t,i,j} = -L_g L_f h_{t,i,j}$, generating the following result.

**Lemma 1.** Suppose a vehicle governed by (1) detects an obstacle and has access to $d(e_1,e_2)$. Under Assumption 1, and if $L_g L_f h_{t,i}(x) \neq 0$ for all $x \in S_r$, $i \in \{t,r\}$, if we permit the left-lane expansion

$$e_v = \left( \frac{u_t}{2} + e_{obs}^t \right) > 0, \quad \text{where} \quad e_{obs}^t < 0,$$

then for $i \in \{t,r\}$ and the overriding controller (11), if we select $c_{t,j} = c_{t,j} > 0$ for $j = 1,2$, then

$$u_{\text{override}} = u_{\text{override}} = (L_g L_f h_{t,i}(-1)^{-1} c_{t,i} e_{2} \overline{u}^i_{\text{obs}}).$$

In other words, the CBFs (6), (7) have the control-sharing property [17], and hence generate the feasible QP

$$u_{\text{safety}} = \arg \min_{u \in \mathbb{R}} |\bar{u} - u_{\text{MPC}}|^2 \quad \text{s.t.} \quad \bar{u} \geq u_{\text{override}},$$

where $L_g L_f h_{t,i} < 0$ (otherwise, the barrier inequalities must be flipped). Moreover, if we additionally select $c_{t,1}$ to satisfy (9), then the safety filter (15) ensures that $S_r \cap S_l \neq \emptyset$ remains positively-invariant for (1) for all times provided that $x(0) \in S_r \cap S_l$, and (1) with (15) is ESf.

The proof of Lemma 1 directly follows from the overriding controllers, the selection (13), and [17, Thm. 1], [7, Cor. 2].

C. Prescribed-Time Safety by Time-Varying Backstepping

Since the obstacle threatens the safe operation of the vehicle only while the vehicle is moving towards it, the OA problem is finite-time in nature, and we would like the safety filter to only perform LK after passing the obstacle.

PTSf uses time-varying gains to balance performance and safety by small overriding inputs, but to enforce safety only for as long as required. PTSf designs retain $h_{i,j}(t) \geq 0$ but drive $h_{i,j}(t) \to 0$ within a finite time that can be a priori prescribed, for $i \in \{t,r\}$ and $j = 1,2$ (see [10] for a more extensive discussion on PTSf).

PTSf designs rely on the following time-varying function

$$\mu_2(t - t_{\text{obs}}) := \frac{1}{(1 - \frac{t - t_{\text{obs}}}{\tau})^2}, \quad t \in [t_{\text{obs}}, t_{\text{obs}} + T],$$

which equals one at the detection time $t_{\text{obs}}$ (d(e_1,e_2) = $\delta_2$ in (5)) but equals infinity at the passing time $t_{\text{obs}} + T$.

Since the longitudinal vehicle velocity $v_t$ is constant, we can estimate the passing time quite accurately for highway driving scenarios by integrating along the flow of the system.
We now exchange the constant gains in (11) with time-varying ones: for $t \in [t_{obs}, t_{obs} + T)$, we define
\[ c_{i,j}(t) = c_{i,j}^0 h_2(t - t_{obs}, T), \quad i \in \{\ell, r\}, \quad j = 1, 2, \]
where $c_{i,j}^0 > 0$ also satisfying (9) are the initial gains at detection time and can be chosen to be as small as possible while retaining a large safe operating envelope. Employing these time-varying gains generates the following result, which is a consequence of Lemma 1 and the treatment in the proof of [10, Thm. 1].

**Proposition 1.** Suppose a vehicle governed by (1) detects an obstacle and has access to $d(e_1, e_2)$. Under Assumption 1, and if $L_gL_fh_{i,1}(x) \neq 0$ for all $x \in S_i$, $i \in \{\ell, r\}$, if we select (13), then for $i \in \{\ell, r\}$ and
\[ u_{t,\text{override}} = -\frac{L_f^2 h_{i,1} + (c_{i,1} + c_{i,2}) L_fh_{i,1} + (c_{i,1} + c_{i,1}c_{i,2}) h_{i,1}}{L_gL_fh_{i,1}}, \]
if we select $c_{\ell,j} = c_{r,j} > 0$ for $j = 1, 2$, then
\[ u_{t,\text{override}} = (L_gL_fh_{i,1})^{-1}(\dot{c}_{\ell,1} + c_{i,1}c_{i,2}) w_1. \]
In other words, the CBFs (6), (7) have the control-sharing property, and hence generate the feasible QP
\[ u_{\text{safe}} = \arg \min_{u \in \mathbb{R}} |\bar{u} - u_{\text{MPC}}|^2 \]
\[ \text{s.t.} \quad u_{t,\text{override}} \leq u \leq u_{t,\text{override}}, \quad (16) \]
when $L_gL_fh_{i,1} < 0$ (otherwise, the barrier inequalities must be flipped). If we additionally select $c_{\ell,1}$ to satisfy (9), then the safety filter (16) ensures that $S_\ell \cap S_r \neq \emptyset$ remains positively-invariant for (1) over the interval $[t_{obs}, t_{obs} + T)$ provided that $x(t_{obs}) \in S_\ell \cap S_r$, and (1) with (16) is PTSf. Moreover, the time-varying overriding control laws are uniformly bounded for $t \in [t_{obs}, t_{obs} + T]$.

**D. Input-Constrained CBFs (ICCBFs)**

One incompatibility between MPC- and CBF-based control designs is that the former can handle input constraints, whereas the latter cannot natively while retaining guarantees.

The work [21] introduces ICCBF, whose designs that are similar to CBF designs, except they restrict the safe sets $S_i$ further by iteratively removing states from which system safety can only be achieved by violating the input constraints. In the (time-invariant) backstepping framework, the first iteration of the ICCBF design replaces (8), (10) by
\[ \frac{d}{dt} h_{i,1}(x) = -c_{i,1} h_{i,1}(c_{i,1}, e_2) + h_{i,2}(x), \]
\[ \frac{d}{dt} h_{i,2} \geq -c_{i,2} h_{i,2} - \inf_{|u| \leq u_{max}} \{L_gL_fh_{i,1} u\}, \]
which translates to the BF constraint
\[ b_{i,2}(x) := \frac{L_f^2 h_{i,1} + (c_{i,1} + c_{i,2}) L_fh_{i,1} + (c_{i,1} + c_{i,1}c_{i,2}) h_{i,1}}{L_gL_fh_{i,1}}, \]
the manipulation (17) effectively treats the control term as a disturbance and adds a margin of safety to the dynamics governing $h_{i,2}$ equal to the disturbance’s upper bound. Notice that the relative degree of the ICCBF is no longer two, as is the case for the CBFs in Sections III-B and III-C; indeed, the ICCBF methodology iterates backstepping at least once more by enforcing (similar to (10)) $\frac{d}{dt} b_{i,2} + c_{i,3} b_{i,2} \geq 0$ for $c_{i,3} > 0$, which is equivalent to the CBF constraint
\[ L_fh_{i,2}(x) + L_gh_{i,2}(x) + c_{i,3} b_{i,2}(x) \geq 0. \]
We say that $b_{i,2}$ is an ICCBF if
\[ L_fh_{i,2}(x) + \sup_{|u| \leq u_{max}} \{L_fh_{i,2}(x) u + c_{i,3} b_{i,2}(x) \geq 0 \}
holds only on the set $x \in S_i \cap \{b_{i,2}(x) \geq 0\}$ (see [21, Def. 4] for details).

As for standard CBFs, validating (18) is difficult in practice. However, minimizing the left-hand side of (18) is a test to invalidate candidate ICCBFs; in these cases, the authors of [21] propose to iterate the backstepping procedure $M \in \mathbb{N}$ times to further restrict the safe set (yet existence of $M$ guaranteeing (18) is open). A theoretical study of combining PTSf and ICCBF is not in the scope of this work.

**IV. LOW-LEVEL MPC**

The low-level MPC ensures LK via trajectory tracking and runs at a lower sampling rate than the safety-critical control. Hence, we discretize the system (1) and design the controller
\[ u(t_k) = u_s(t_k) + v(t_k), \]
where $u_s(t_k)$ is sampled from (2). The evolution of the discrete-time error signal $e_x(t_k) = x(t_k) - x_s(t_k)$ is then described by
\[ e_x(t_{k+1}) = A_d e_x(t_k) + B_d v(t_k) - w(t_k), \]
where subscript $d$ denotes the matrices related to the discretized dynamics of (1), using a zero-order-hold, and $w$ denotes the system’s deviation from the steady state (3) due to a change in desired yaw rate over one time step:
\[ w(t_k) = A_d \Delta x_s(t_k) + B_d \Delta u_s(t_k) + G_d \Delta \psi_{ref}(t_k), \]
with $\Delta x_s(t_k) = x_s(t_{k+1}) - x_s(t_k)$, $\Delta u_s(t_k) = u_s(t_{k+1}) - u_s(t_k)$, $\Delta \psi_{ref}(t_k) = \psi_{ref}(t_{k+1}) - \psi_{ref}(t_k)$. Let the MPC-related cost function be
\[ J_N(e_x(t_k), v(t_k), \psi_{ref}(t_k)) = |e_x(t_{N|k})|^2_Q + \sum_{i=0}^{N-1} |e_x(t_{i|k})|^2_Q + |v(t_{i|k})|^2_R, \]
\[ \text{s.t.} \quad e_x(t_{0|k}) = e(t_k), \quad e_x(t_{i+1|k}) = A_d e_x(t_{i|k}) + B_d v(t_{i|k}) - w(t_k), \]
where $\psi(t_k) \equiv \{\psi(t_{0|k}), \ldots, \psi(t_{N-1|k})\}$ and $\psi_{ref}(t_k) \equiv \{\psi_{ref}(t_{0|k}), \ldots, \psi_{ref}(t_{N-1|k})\}$. The matrices $Q$ and $R$ are positive definite, $|x|_Q \leq \sqrt{x^T Q x}$, and for $i \in \{0, \ldots, N - 1\}$ we define the set of admissible control inputs by
\[ \mathcal{V}_{t_k} \equiv \{v(t_k) : |u_s(t_{k+i}) + v(t_{i|k})| \leq u_{max}\}. \]
At every time instance, the MPC solves the optimization problem $P(t_k)$:
\[ \min_{v(t_k) \in V_{t_k}} J_N(e(t_k), v(t_k), \ldots) \]
\[ (e(t_k), v(t_k), \ldots) \text{ in Figure 5 are of large amplitude and are saturated at their maximum for much of the time. Despite being saturated,} \]

Remark 1. Since $e_1$ is the lateral error along the local vehicle axis (see Figure 1), state constraints (incl. cosine terms) would result in a non-convex nonlinear program; hence, only the safety filter incorporates them.

Note that $w(t_k)$ are bounded if $\dot{\psi}_{\text{ref}}$ and $\Delta \dot{\psi}_{\text{ref}}$ are bounded. Hence, assuming such bounds, towards recursive feasibility and convergence, instead of defining a robust maximal invariant constraint admissible set (cf. e.g. [1]), we choose the terminal cost related to $P$ sufficiently large [22].

V. SIMULATIONS

We use typical highway conditions in the U.S., and assume a constant longitudinal vehicle velocity of $v_1 = 20\text{m/s}$ while being controlled by the MPC- and CBF-based safety filter in (15) or (16) on a single-lane road (with $e_v = 0$ in (6)). The desired path (the center line) is reachable given the vehicle dynamics and steering constraints. The related reference yaw rate and acceleration are provided as time-dependent discrete points and interpolated for the continuous-time safety filter. Due to the high velocity, in some simulations, we impose $u_{\text{max}} = 5^\circ$ and saturate the controls accordingly. The dynamics from (1) are used as a plant model for the MPC, which is applied at a frequency of 20Hz; the safety filter is computed continuously. The car width is encoded within the lane width, $w_1$, and obstacle radius $r_{\text{obs}}$. We present two simulation studies: A. comparing ESf and PTSf (while using ESf for LK) during early obstacle detection scenarios, and; B. comparing input-constrained PTSf and PT-ICCBF (while using ICCBF for LK) during late obstacle detection scenarios.

A. ESf and PTSf during early obstacle detection

We assume the obstacle is detected 40m ahead and use the ESf design in Section III-B for LK. We compare the closed-loop performance of MPC with ESf OA to that of MPC with PTSf OA. In Figure 2, we observe that both designs successfully avoid the obstacle while staying on the road, even though the desired trajectory would lead to unsafe operation. For the ESf OA design, we select $c_{i,i} = 15$ for $i = \{1,2\}$ satisfying (9), which allows the vehicle to approach the obstacle very closely (and seemingly match the performance of PTSf); this performance is innate to the PTSf OA design and is desirable because it allows less intervention with the MPC controller, which tracks the desired path well.

We now investigate the control effort required for this OA task. It is clear from Figure 3 that the nominal MPC control input, which seeks to track the centerline, is overwritten by the safety filters to avoid the obstacle. The filters mainly differ early on when safety constraints become active. We observe a large control input generated by the ESf OA design which is undesirable (see [10] for a further discussion). We can alleviate this by lowering the gains $c_{i,i}$ at the cost of increased conservatism, which can become problematic when the lane width is limited, potentially violating the control sharing property. The PTSf OA controller has a significantly smaller peak but allows the vehicle to approach the barrier equally to the ESf OA design. After passing the obstacle, control authority is gradually ceded to the MPC using a smooth step function similar to (5); see [10] for details.

Throughout the maneuver, the control sharing property among the LK and OA controllers was verified.

B. PTSf and PT-ICCBF during late obstacle detection

Now suppose that the obstacle is detected when only 15m ahead of the vehicle. We saturate the inputs of the MPC with ESf for LK and PTSf for OA control designs to $|u| \leq u_{\text{max}}$ and compare the results to MPC with the input-constrained equivalent designs following the methodology in Section III-D. Figure 4 illustrates the non-constrained designs steering the vehicle away from the obstacle, but due to the input saturation, violate safety since the vehicle contour intersects with the obstacle. In comparison, the ICCBF LK with PT-ICCBF OA design successfully avoids the obstacle while steering the vehicle to the barrier. Due to the late obstacle detection, the control inputs for both safety filters in Figure 5 are of large amplitude and are saturated at their maximum for much of the time. Despite being saturated,

\[ \text{Fig. 2: The OA designs override the MPC control to enforce safety. The ESf filter is tuned to approximately match the performance of the PTSf one. Both designs pass the obstacle at the boundary of the safe set.} \]

\[ \text{Fig. 3: For similar closed-loop vehicle trajectories, the PTSf filter overrides the MPC control less aggressively than the ESf filter. The discrete-time MPC sends the minimum steering command as the vehicle deviates from the desired trajectory.} \]
Fig. 4: The vehicle leaves the desired trajectory to avoid the obstacle. The ICPTSf renders the closed loop safe. On the contrary, the zoom reveals that the PTSf violate safety due to saturated control inputs.

Fig. 5: For a similar trajectory the PT safety filter shows a smoother interjection with less jerk.

the ICCBF for LK with PT-ICCBF for OA controller ensures safe vehicle operation, whereas this same input saturation renders the controller absent of the input-constrained design consideration unsafe.

VI. CONCLUSION AND ACKNOWLEDGEMENTS

Our multi-layer MPC and CBF-Safety design exploits the advantages (numerical cost, nonlinear-model-based control fidelity, and ability to perform swift interventions) of both control strategies while allowing the encoding of safety prioritization of OA over LK. This prioritization allows us to establish CBF-QP feasibility and hence OA-safety guarantees. Additional to ESf filter designs, we explore PTSf and input-constrained safety designs, which bring the advantages of retaining safety while balancing performance, and practicality. Our ongoing research aims to provide some theoretical guarantees for the combinations of ESfs, PTSfs and their input-constrained counterparts for simultaneous LK and OA.

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