ABSTRACT

We define the Bondi energy for two-dimensional dilatonic gravity theories by generalizing the known expression of the ADM energy. We show that our definition of the Bondi energy is exactly the ADM energy minus the radiation energy at null infinity. An explicit calculation is done for the evaporating black hole in the RST model with the Strominger’s ghost decoupling term. It is shown that the infalling matter energy is completely recovered through the Hawking radiation and the thunderpop.
I. INTRODUCTION

The two-dimensional dilaton gravity theories [1-5] are interesting as toy models for studying many interesting issues in four-dimensional gravity theories. They possess most of the interesting properties of the four-dimensional gravity theories such as the existence of the black hole solutions and Hawking radiations, and at the same time they are more amenable to quantum treatments than their four-dimensional counterparts.

The evaporating black hole solution related to the Hawking radiation [6] is particularly interesting since it is the situation considered by Hawking in the context of the four-dimensional general relativity when he put forward the famous information loss puzzle. His radical proposal for the quantum mechanical evolution associated with the black holes [7] has not yet been solved. However, a recent work on the black-hole evaporation and back reaction of the metric presented by Callan, Giddings, Harvey, and Strominger (CGHS) [1], has shown that the quantum-mechanical gravity puzzles are no more beyond our reach [8,9].

Russo, Susskind, and Thorlacius (RST) obtained the RST model by adding a local covariant counter term to the CGHS model [2,3]. The resulting semiclassical equation is exactly solvable and describes the back reaction of the metric at the one-loop level in the large $N$ limit where $N$ is a number of conformal matter fields. In particular, the model has an exact solution describing the evaporation of black hole via Hawking radiations. It has a mild violation of cosmic censorship hypothesis due to the naked singularity as an isolated event [3].

On the other hand, it is a well known fact that one cannot construct (ordinary) conserved stress-energy-momentum tensor in general relativity except for space-times having particular symmetries [10]. The fact that the stress-energy-momentum tensor for the matter fields alone is not conserved is not surprising since they exchange energies and momenta with the gravitational field. Furthermore, there is no notion corresponding to the conserved stress-energy-momentum tensor because a generally covariant tensor can only satisfy the covariant conservation law. However, one can introduce the concept of stress-energy-momentum for gravity theories if we take the view that the general relativity can be treated as a spin-2 field theory in the Minkowski background [10,16]. Then, stress-energy-momentum will be a pseudotensor in the sense it
is not generally covariant but is Lorentz covariant with respect to the Minkowski background metric. Just as in the case of four-dimensional Einstein gravity, we can show that the pseudotensor corresponding to energy density is a total derivative for the two-dimensional dilaton gravity theories. Therefore, for asymptotically flat space-times, the energy becomes a surface term defined at either spatial or null infinity. The former case is the Arnowitt-Deser-Misner (ADM) energy [11] and the latter is just the Bondi energy [12]. Then, it is obvious that the difference of ADM and Bondi energy is the integral of the current flowing out to null infinity. In the four-dimensional Einstein gravity, this current is interpreted as a radiation energy density. In the case of two-dimensional dilaton gravity theories, the graviton and dilaton fields have no propagating degrees of freedom and only the matter radiation is capable of escaping to null infinity.

In Ref. [1], the Hawking radiation without the back reaction of the metric can be calculated by using the conformal anomaly of the energy-momentum tensor of the matter part by imposing suitable boundary conditions, which is given by

\[
< T^\pm_{\pm}(\sigma^+, \sigma^-) > \bigg|_{\sigma^+ \to \infty} = \frac{\lambda^2}{48} \left[ 1 - \frac{1}{(1 + \frac{m}{\lambda} e^{\lambda(\sigma^- - \sigma_+^+)} )^2} \right]
\]

(1)

where the metric in the tortoise coordinate \(\sigma^\pm\) is asymptotically Minkowskian at the future null infinity. As was emphasized in Ref. [1], the total Hawking radiation is divergent. This result is in contradiction to the energy conservation law, which is not surprising since the effect of the back reaction to the geometry is not taken into account.

Then, what about the energy conservation in the formation and evaporation of the two-dimensional dilatonic black hole when we consider the back reaction of the metric? In this paper, we will consider the energy conservation in the two-dimensional dilaton gravity theories, especially for the RST model. In Sec. II, we define the notion of the Bondi energy by generalizing the known expression for the ADM energy for the dilatonic gravity models. In Sec. III, we will calculate the Hawking radiation rate and the integrated Hawking radiation for the evaporating black hole in the RST model. In order to get the positive definite Hawking flux [14], we include the Strominger’s ghost decoupling term [15]. Then, we obtain a desirable result, the total outgoing radiation due to the Hawking radiation and the classical thunderpop being equal to
the energy of the infalling matter fields. It means that the infalling matter energy is completely recovered in the RST model. In Sec. IV, we calculate the Bondi energy and the Hawking radiation at null infinity, and show that the total energy is conserved. Finally, some discussions are given in Sec. V.

II. ADM, BONDI, AND RADIATION ENERGY IN TWO-DIMENSIONAL DILATON GRAVITIES

In this section, we present the definitions of the ADM and Bondi energy, and their relation. We consider dilaton gravity theories described by the action,

$$S_T = S_{DG} + S_f + S_{qt},$$

$$S_{DG} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla\phi)^2 + 4\lambda^2) \right],$$

$$S_f = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ -\frac{1}{2} \sum_{i=1}^N (\nabla f_i)^2 \right],$$

$$S_{qt} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ -\kappa^2 \phi R - \frac{Q^2}{2} R \frac{\Box}{\Box} \right],$$

where $\kappa = Q^2 = 0$ gives the classical action, $\kappa = 0$, $2Q^2 = \frac{N}{12}$ gives the CGHS model [1], and $\kappa = 2Q^2 = \frac{(N-24)}{12}$ gives the RST model [2]. One then splits the above action as

$$S_T = S_{DG} + S_M$$

where $S_M = S_f + S_{qt}$. Then, the equation of motion for the metric is given by

$$G_{\mu\nu} = T^M_{\mu\nu},$$

$$G_{\mu\nu} = \frac{2\pi}{\sqrt{-g}} \frac{\delta S_{DG}}{\delta g^{\mu\nu}} = 2e^{-2\phi} \left[ \nabla_\mu \nabla_\nu \phi + g_{\mu\nu}((\nabla\phi)^2 - \Box \phi - \lambda^2) \right],$$

$$T^M_{\mu\nu} = -\frac{2\pi}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}},$$

where $T^M_{\mu\nu}$ is the stress-energy-momentum tensor composed of classical and quantum matter parts.

In order to obtain the ordinary conserved quantity instead of the covariant conserved one, we expand the metric and dilaton fields around the linear dilaton vacuum (LDV),

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi} + \psi,$$

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \bar{\phi} = -\lambda x^{\alpha} \eta_{\alpha\beta} e^\beta$$
where $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$ are the LDV configuration, and $\eta_{11} = -\eta_{00} = 1$ and $\epsilon^\alpha$ satisfies $\epsilon^\alpha \eta_{\alpha\beta} \epsilon^\beta = 1$. One then linearizes the equation of motion (7) [16]:

$$G^{(1)}_{\mu\nu} = T^M_{\mu\nu} - G^{(2)}_{\mu\nu}$$  \hspace{1cm} (12)

where $G^{(1)}_{\mu\nu}$ is the linear part of $G_{\mu\nu}$ in $h_{\mu\nu}$ and $\psi$ expansions, and $G^{(2)}_{\mu\nu}$ is the rest. If we take the time and space coordinate $(t, q)$ such that $x^\alpha \eta_{\alpha\beta} \epsilon^\beta = q$, then it is straightforward to show that the left hand side of (12) identically satisfies the conservation law,

$$\partial_\mu G^{(1)\mu0} = 0,$$  \hspace{1cm} (13)

thanks to the linearized Bianchi identity [13]. Note that the total momentum, $\int dq G^{(1)01}$, is not conserved since the translational symmetry in the spatial direction is spontaneously broken by the LDV. Then, the right-hand side of (12) can be thought of as the energy-momentum tensor for the fields, $f$, $\phi$, and $g$. Defining the matter current as

$$J^\mu = T^\mu^0_M - G^{(2)\mu0},$$  \hspace{1cm} (14)

we see that it satisfies the conservation law

$$\partial_\mu J^\mu = 0$$  \hspace{1cm} (15)

due to Eq. (12) and Eq. (13). If we consider a space-time which approaches the LDV at spatial infinity fast enough, then $J^1$ would vanish at $q \to \infty$ and the total energy,

$$E_{ADM}(t) \equiv \int_{-\infty}^{\infty} dq J^0(t, q)$$

$$= \int_{-\infty}^{\infty} dq G^{(1)00}(t, q),$$  \hspace{1cm} (17)

is a conserved quantity, which is also called the ADM energy(mass).

After some straightforward algebra, the expression (17) reduces to

$$E_{ADM} = 2e^{2\lambda q}(\partial_q \psi + \frac{\lambda h_{11}}{2})|_{q \to 0,}.$$  \hspace{1cm} (18)

The contribution from $q \to -\infty$ is easily seen to be zero for a space-time which asymptotically approaches the LDV. In the conformal gauge, $g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}$, assuming the following asymptotic field configuration as $q \to \infty$,

$$\rho \approx A(t)e^{-2\lambda q}, \quad \psi \approx A(t)e^{-2\lambda q},$$  \hspace{1cm} (19)
we obtain the expression
\[ E_{\text{ADM}} = 2e^{2\lambda q}(\partial_q \psi + \lambda \rho)|_{q \to \infty} \] (20)
by keeping only the linear term in \( \rho \). Note that the expression (20) is valid also for any coordinate choice which approaches the conformal gauge at infinities fast enough. It is just the expression used often in the literatures [17,18].

To define the Bondi energy, we need the boundary conditions at null infinity. We require that
\[ \psi \approx D(y^-)e^{-\lambda y^+}, \quad \rho \approx D(y^-)e^{-\lambda y^+} \] (21)
where we used the light-cone coordinate, \( y^\pm = t \pm q \). For \( y^+ \to -\infty \), it is enough to require that the configuration approaches the LDV. We now define the Bondi energy \( B(y^-) \) as the energy evaluated along the null line,
\[ B(y^-) = \frac{1}{2} \int_{-\infty}^{\infty} dy^+ G^{(1)-0}(y^+, y^-) \]
\[ = 2e^{\lambda(y^+-y^-)}(\partial_+ - \partial_- + \lambda)\psi|_{y^+ \to \infty}. \] (22)
Note that the Bondi energy is defined at the null infinity while the ADM energy is defined at the spatial infinity.

Now we show the difference between \( E_{\text{ADM}} \) and \( B(y^-) \). Obviously, it can be represented by the integral of \( G^{(1)+0} \) along the null line, from the point \((\infty, -\infty)\) to the point \((\infty, y^-)\),
\[ E_{\text{ADM}} - B(y^-) = -\frac{1}{2} \int_{-\infty}^{y^-} dy^- G^{(1)+0}(y^+, y^-)|_{y^+ \to \infty} \]
\[ = -\int_{-\infty}^{y^-} dy^- \partial_- \left(2e^{\lambda(y^+-y^-)}(\partial_+ - \partial_- + \lambda)\psi\right)|_{y^+ \to \infty} \] (23)
which is just an identity. After some calculation, the right-hand side of (23) can be identified by the integral of the radiation flux of matters \( T_{M_-} \) at null infinity by using (12) under the boundary condition (21), i.e.,
\[ E_{\text{ADM}} - B(y^-) = \int_{-\infty}^{y^-} dy^- \left(T_{q-} + T_{q-}^{\text{qt}}\right)|_{y^+ \to \infty}. \] (24)
It is plausible to regard the quantum matter part of the radiation as a Hawking radiation which is explicitly given by
\[ h(y^-) = T_{q-}^{\text{qt}}|_{y^+ \to \infty} \]
\[
\begin{aligned}
&= -2Q^2 \left( (\partial_- \rho)^2 - \partial_+ \rho + t_- (y^-) \right) \bigg|_{y^+ \to \infty} + \frac{\kappa}{4} (4 \partial_- \rho \partial_- \phi - 2 \partial_+^2 \phi) \bigg|_{y^+ \to \infty} \\
&\approx -2Q^2 t_- (y^-),
\end{aligned}
\]

where the function \( t_- (y^-) \) reflects the non-locality of the conformal anomaly term of Eq. (1). As a result, the radiation is composed of the classical (conformal) matter and the Hawking radiation, and the energy conservation relation (24) is written as

\[
E_{\text{ADM}}(t) - B(y^-) = \frac{1}{2} \sum_{i=1}^{N} \int_{-\infty}^{y^-} dy^- (\partial_- f_i)^2 \bigg|_{y^+ \to \infty} - 2Q^2 \int_{-\infty}^{y^-} dy^- t_- (y^-). 
\]

Note that the Bondi energy is just the remaining energy after the classical and quantum Hawking radiation has been emitted from the system. We will explicitly study this energy conservation relation in the RST model.

### III. HAWKING RADIATION IN THE RST MODEL

In this section, we apply the formal concepts developed in the above section to the RST model by explicit calculations of relevant quantities. From the action (2), the RST model with the Strominger’s ghost decoupling term \( S_{\text{St}} \) [15] in the conformal gauge is given by

\[
S = S_T + S_{\text{St}},
\]

\[
S_T = \frac{1}{\pi} \int d^2 x \left[ e^{-2\phi} (2 \partial_+ \partial_- \rho - 4 \partial_+ \phi \partial_- \phi + \lambda^2 e^\rho) \\
+ \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i - \kappa \partial_+ \rho \partial_- \rho - \kappa \phi \partial_+ \partial_- \rho \right],
\]

\[
S_{\text{St}} = \frac{1}{\pi} \int d^2 x \left[ 2 \partial_+ (\rho - \phi) \partial_- (\rho - \phi) \right].
\]

We introduced \( S_{\text{St}} \) to make the Hawking radiation positive definite for arbitrary \( N \). In the conformal gauge, one must impose two constraint equations corresponding to the vanishing metric components:

\[
T_{\pm \pm} = (e^{-2\phi} + \frac{\kappa}{4}) (4 \partial_+ \rho \partial_- \phi - 2 \partial_+^2 \phi) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i - \kappa (\partial_+ \rho \partial_- \rho - \partial_+^2 \rho) \\
+ 2 (\partial_+ (\rho - \phi) \partial_- (\rho - \phi) - \partial_+^2 (\rho - \phi)) + \kappa t_\pm (x^\pm) = 0
\]
where the functions $t_{\pm}(x^\pm)$ are needed to satisfy asymptotic physical boundary conditions. Following the Bilal and Callan [4] and de Alwis’s method [5], we perform field redefinition to a Liouville theory [14],

\[
\Omega = \frac{\kappa}{2\sqrt{\kappa - 2}} \phi + \frac{e^{-\phi}}{\sqrt{\kappa - 2}},
\]
\[
\chi = \sqrt{\kappa - 2} \rho - \frac{(\kappa - 4)}{2\sqrt{\kappa - 2}} \phi + \frac{e^{-\phi}}{\sqrt{\kappa - 2}}.
\]

(31)

Then, the action (27) and the two constraints (30) in terms of the redefined fields are given by

\[
S = \frac{1}{\pi} \int d^2 x \left[ -\partial_+ \chi \partial_+ \chi + \partial_+ \partial_- \Omega \partial_- \Omega + \kappa^2 e^{2(\chi - \Omega)} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right],
\]

(32)

\[
\kappa t_{\pm} = -\partial_{\pm} \chi \partial_{\pm} \chi + \partial_{\pm} \partial_- \Omega \partial_{\pm} \Omega + \sqrt{\kappa - 2} \partial_{\pm}^2 \chi + \frac{1}{2} \sum_{i=1}^{N} \partial_{\pm} f_i \partial_{\pm} f_i.
\]

(33)

From the constraint equations (33), we can determine the total central charge,

\[
c = c_\chi + c_\Omega + c_M + c_{\text{ghost}}
\]

\[
eq [1 - 12(\kappa - 2)] + 1 + N - 26,
\]

(34)

together with the ghost contribution. To impose the constraints consistently at the quantum level, we fix $\kappa = \frac{N}{12}$ which is positive, and the Hawking radiation becomes positive definite. From the action (32), we obtain the equations of motion

\[
\partial_+ \partial_- \chi + \frac{\lambda^2}{\sqrt{\kappa - 2}} e^{\frac{2}{\kappa - 2}(\chi - \Omega)} = 0,
\]

(35)

\[
-\partial_+ \partial_- \Omega - \frac{\lambda^2}{\sqrt{\kappa - 2}} e^{\frac{2}{\kappa - 2}(\chi - \Omega)} = 0,
\]

(36)

\[
\partial_i \partial_- f_i = 0.
\]

(37)

In the Kruskal gauge, $\rho = \phi$, the static solution is given by

\[
\Omega(x^+, x^-) = \chi(x^+, x^-)
\]

\[
= -\frac{\lambda^2}{\sqrt{\kappa - 2}} x^+ x^- + P \frac{\kappa}{\sqrt{\kappa - 2}} \ln(-\lambda^2 x^+ x^-) + \frac{m}{\lambda \sqrt{\kappa - 2}}
\]

(38)

where $P$ and $m$ parametrize different solutions. For $P = -\frac{1}{4}$ and $m = 0$, it becomes LDV solution, and for $P = 0$ and $m \neq 0$, it is a thermal equilibrium solution since the metric is independent of time in appropriate coordinates.
Let us now consider an evaporating black-hole solution formed by an incoming shock wave at \( x^+ = x_0^+ \) given by \( T^t_{++} = \frac{m}{\lambda x_0^+} \delta(x^+ - x_0^+) \) [2],

\[
\Omega(x^+, x^-) = \chi(x^+, x^-) = -\frac{\lambda^2}{\sqrt{\kappa - 2}} x^+ x^- - \frac{m}{\lambda x_0^+ \sqrt{\kappa - 2}} (x^+ - x_0^+) \Theta(x^+ - x_0^+) \tag{39}
\]

where the matching condition at \( x^+ = x_0^+ \) are obtained by the ++ constraint equation (33) with the above incoming pulse wave. Then, the singularity can form at \( \phi_c = -\frac{1}{2} \ln \kappa/4 \) where \( \frac{d\Omega(\phi_c)}{d\phi} = 0 \). The singularity occurs at the boundary of the range of \( \Omega \) where \( \Omega(\phi_c) = \kappa/4 \sqrt{\kappa - 2}[1 - \ln \kappa/4] \). From (39), the curve \( \phi(\bar{x}^+, \bar{x}^-) = \phi_c \) is given by

\[
1 - \ln \frac{\kappa}{4} = -\frac{4\lambda^2}{\kappa} \bar{x}^+ \bar{x}^- - \ln(-\lambda^2 \bar{x}^+ \bar{x}^-) - \frac{4m}{\lambda x_0^+ \kappa} (\bar{x}^+ - x_0^+) \Theta(\bar{x}^+ - x_0^+). \tag{40}
\]

This is the same form as in the case without the Strominger term except for the change of the value of \( \kappa \). The location of the singularity is inside an apparent horizon which is given by \( \partial_+ \phi = 0 \). The apparent horizon gives another curve:

\[
\hat{x}^+(\hat{x}^- + \frac{m}{\lambda^3 x_0^+}) + \frac{\kappa}{4\lambda^2} = 0. \tag{41}
\]

Following the suggestion of Hawking [19], RST showed that the singularity and apparent horizon collide in a finite proper time and the singularity is naked after the two have merged [2]. From (40) and (41), the intersection point is given by

\[
x^+_s = \frac{\kappa \lambda x_0^+}{4m} \left(e^{4m/\lambda} - 1\right), \quad x^-_s = -\frac{1}{\lambda^3 x_0^+ \left(1 - e^{-4m/\lambda}\right)}. \tag{42}
\]

As shown by RST, it is possible to match the evaporating solution (39) with a shifted LDV solution at the null line, \( x^- = x^-_s \).

The conformal transformation, \( x^\pm = \pm \frac{1}{\lambda} e^{\pm \lambda \sigma} \), does not give an asymptotically static configuration and in particular the dilaton and graviton fields do not approach the correct form of LDV at infinity, so we introduce a quasi-static coordinate \( y^\pm \) where
the fields approach LDV in spatial and null infinities in both $q \to \pm \infty$ and $y^+ \to \pm \infty$ [4],

\[ x^+ = \frac{1}{\lambda} e^{\lambda y^+}, \quad x^- = -\frac{1}{\lambda} e^{-\lambda y^-} - \frac{m}{\lambda^2 x_0^+} \Theta(y^+ - y_0^+). \] (43)

In this coordinate, $\Omega$ and $\chi$ are static to the leading order. Note that this coordinate transformation is not conformal due to the presence of the $\Theta$ function, however, it does not matter since the coordinate transformation asymptotically goes to conformal gauge at infinities. We denote the intersection point in this coordinate by $(y^+_s, y^-_s)$,

\[ y^+_s = \frac{1}{\lambda} \ln(\lambda x^+_s), \quad y^-_s = -\frac{1}{\lambda} \ln\left(-\lambda x^-_s - \frac{m}{\lambda^2 x^+_s}\right) \] (44)

which will be used in later. The Penrose diagram of the RST model is depicted in Fig. 1.

Let us now consider the Hawking radiation. From the fundamental condition that $T_{\pm \pm}$ must be a true tensor without anomaly, we require the anomalous transformation as

\[ t_\pm(y^\pm) = \left( \frac{\partial y^\pm}{\partial \sigma^\pm} \right)^{-2} \left( t_\pm(\sigma^\pm) - \frac{1}{2} D_{\sigma^\pm}(y^\pm) \right) \] (45)

where $D_{\sigma^\pm}(y^\pm)$ is the Schwarzian derivative. Then, following [4], we obtain the Hawking radiation,

\[ h(y^-) = -\kappa t_-(y^-) = \frac{\kappa \lambda^2}{4} \left[ 1 - \frac{1}{(1 + m e^{\lambda(y^- - y_0^-)})^2} \right] \] (46)

for $y^- < y^-_s$ and $h(y^-) = 0$ for $y^- > y^-_s$. A typical form of the Hawking radiation is illustrated in Fig. 2. Note that the expression for the Hawking radiation (46) is same as (25) by identifying $2Q^2 = \kappa$. As expected, for $y^- \to -\infty$, there is no Hawking radiation. In the limit $y^- \to y^-_s - 0$, the radiation is

\[ h(y^-_s - 0) = \frac{\kappa \lambda^2}{4} (1 - e^{-\frac{8m}{\kappa \lambda}}). \] (47)

For $y^- < y^-_s$, the integrated Hawking flux $H(y^-)$, is calculated as

\[ H(y^-) = \int_{-\infty}^{y^-} dy^- h(y^-) = \frac{\kappa \lambda}{4} \left[ 1 - \frac{1}{(1 + m e^{\lambda(y^- - y_0^-)})^2} + \ln(1 + m e^{\lambda(y^- - y_0^-)}) \right]. \] (48)
For the interesting limit, $y^- \to y_s^- - 0$, we obtain

$$H(y_s^- - 0) = m + \frac{\kappa \lambda}{4} (1 - e^{-\frac{4\kappa}{\lambda^2}})$$  \hspace{1cm} (49)$$

which is greater than the total energy of the infalling matter field. This point is clarified in the next section.

On the other hand, for $y^- > y_s^-$, $H(y^-)$ is given by

$$H(y^-) = \int_{-\infty}^{y_s^- - 0} dy^- h(y^-) + \int_{y_s^- + 0}^{y^-} dy^- h(y^-)$$

$$= H(y_s^- - 0) + 0$$

$$= H(y_s^- + 0).$$  \hspace{1cm} (50)$$

Therefore, the integrated Hawking flux $H(y^-)$ is saturated when the black hole is completely evaporated and the total Hawking flux is a just $H(y_s^-)$.

**IV. ENERGY CONSERVATION IN THE RST MODEL**

In this section, we calculate the Bondi energy, and prove the energy conservation at the arbitrary time in the RST model.

Let us first show the solution (39) satisfies the boundary conditions (19) and (21) in the asymptotically quasi-static coordinates (43). For $y^+ > y_0^+$, the evaporating black-hole solution (39) can be written as the following form,

$$\Omega(y^+, y^-) = \frac{1}{\sqrt{\kappa - 2}} (e^{\lambda (y^+ - y^-)} + m e^{\lambda (y^+ - y_0^+)} - \frac{\kappa}{4 \sqrt{\kappa - 2}} \ln(e^{\lambda (y^+ - y^-)} + m e^{\lambda (y^+ - y_0^+)})$$

$$- \frac{m}{\lambda \sqrt{\kappa - 2}} (e^{\lambda (y^+ - y_0^+)} - 1),$$  \hspace{1cm} (51)$$

$$\chi(y^+, y^-) = \Omega(y^+, y^-) + \frac{\lambda \sqrt{\kappa - 2}}{2} (y^+ - y^-),$$  \hspace{1cm} (52)$$

and the vacuum solution ($\bar{\Omega}$, $\bar{\chi}$) is

$$\bar{\Omega}(y^+, y^-) = \frac{1}{\sqrt{\kappa - 2}} e^{\lambda (y^+ - y^-)} - \frac{\kappa \lambda}{4 \sqrt{\kappa - 2}} (y^+ - y_0^+),$$  \hspace{1cm} (53)$$

$$\bar{\chi}(y^+, y^-) = \bar{\Omega}(y^+, y^-) + \frac{\lambda \sqrt{\kappa - 2}}{2} (y^+ - y^-).$$  \hspace{1cm} (54)$$

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The asymptotic behaviors of the solution (51) at the spatial and null infinity are,
\[
\begin{align*}
(e^{-2\phi} + \frac{K}{2}\phi)\bigg|_{q \to \infty} &= e^{2\lambda q} - \frac{K\lambda}{2} q + \frac{m}{\lambda} + O(e^{-\lambda q}), \\
(e^{-2\phi} + \frac{K}{2}\phi)\bigg|_{y^+ \to \infty} &= e^{\lambda(y^+ - y^-)} - \frac{K\lambda}{4}(y^+ - y^-) - \frac{K}{4} \ln \left(1 + \frac{m}{\lambda} e^{\lambda(y^- - y_0^+)}\right)e^{-\frac{4m}{\lambda}}.
\end{align*}
\]
(55) (56)

From (55) and (56), the asymptotic forms \( \phi \) are obtained:
\[
\begin{align*}
\phi|_{q \to \infty} &= (\bar{\phi} + \psi)|_{q \to \infty} = -\lambda q + A(t)e^{-2\lambda q} + \cdots, \\
\phi|_{y^+ \to \infty} &= (\bar{\phi} + \psi)|_{y^+ \to \infty} = -\lambda (y^+ - y^-) + D(y^-)e^{-\lambda y^-} + \cdots
\end{align*}
\]
(57) (58)

where \( A(t) = -\frac{m}{2\lambda} \) and \( D(y^-) = \frac{\xi}{\lambda} e^{\lambda y^-} \ln \left(1 + \frac{m}{\lambda} e^{\lambda(y^- - y_0^+)}\right)e^{-\frac{4m}{\lambda}}\). Therefore the solution (39) satisfies the boundary conditions (19) and (21) in the quasi-static coordinate.

Then, it is convenient to write the Bondi energy as
\[
B(y^-) = \sqrt{\kappa - 2(\lambda + \partial_- - \partial_+)}\delta\Omega(y^+, y^-)|_{y^+ \to +\infty}.
\]
(59)

For solution satisfying the boundary condition (21), it is straightforward to show that the Bondi energy (59) reduces to the previous form (22) since
\[
\sqrt{\kappa - 2(\lambda + \partial_- - \partial_+)}\delta\Omega(y^+, y^-)|_{y^+ \to +\infty} = (\lambda + \partial_- - \partial_+) \left[ \frac{K}{2} \psi + e^{\lambda(y^+ - y^-)}(e^{-2\psi} - 1) \right]|_{y^+ \to +\infty} = (\lambda + \partial_- - \partial_+) \left[ e^{\lambda(y^+ - y^-)}(-2\psi + 2\psi^2) \right]|_{y^+ \to +\infty} = 2e^{\lambda(y^+ - y^-)}(\partial_+ - \partial_- + \lambda)(\psi - \psi^2)|_{y^+ \to +\infty} = 2e^{\lambda(y^+ - y^-)}(\partial_+ - \partial_- + \lambda)\psi|_{y^+ \to +\infty}.
\]
(60)

where \( \delta\Omega = \Omega - \bar{\Omega} \). Similarly, the ADM energy can be written as
\[
E_{\text{ADM}}(t) = \sqrt{\kappa - 2(\lambda - \partial_q)} \delta\Omega|_{q \to +\infty}.
\]
(61)

By putting Eq. (51) and Eq. (53) into (61) and (59), we obtain ADM energy and Bondi energy respectively,
\[
\begin{align*}
E_{\text{ADM}}(t) &= m, \\
B(y^-) &= m - \frac{K\lambda}{4} \left[ \frac{m}{\lambda} + e^{-\lambda(y^- - y_0^+)} + \ln(1 + \frac{m}{\lambda} e^{\lambda(y^- - y_0^+)}) \right].
\end{align*}
\]
(62) (63)
Note that at the point \( y_s^− = 0 \), the Bondi energy is given by

\[
B(y_s^− - 0) = -\frac{\kappa \lambda}{4} (1 - e^{-\frac{4m}{\kappa \lambda}})
\]

which is negative. The behavior of the negative Bondi energy is shown in Fig. 3 as an illustration.

We easily see that for \( y^- < y_s^- \), the total (ADM) energy of the evaporating black hole system is the sum of the integrated Hawking flux (48) and the Bondi energy (63):

\[
E_{\text{ADM}} = B(y^-) + H(y^-)
= m.
\]

In this region, there is no classical contribution to the radiation.

For \( y^- > y_s^- \), we note that the classical negative energy thunderpop should be taken into consideration. Indeed, integrating the energy density carried out by the thunderpop, we obtain [2]

\[
E_{\text{thunderpop}} = \int_{-\infty}^{y^-} dy^- T_{-\sigma}^\ell(y^-)
= -\frac{\kappa \lambda}{4} (1 - e^{-\frac{4m}{\kappa \lambda}}),
\]

while for \( y^- < y_s^- \), there is no thunderpop contribution.

There is no Bondi energy for \( y^- > y_s^- \) because of \( \delta \Omega = 0 \), (i.e., we now have the LDV and the Bondi energy of the LDV is zero) and we have

\[
E_{\text{ADM}}(t) = B(y^-) + H(y^-) + E_{\text{thunderpop}}
= 0 + H(y_s^- + 0) + E_{\text{thunderpop}}
= m
\]

by using \( H(y^-) \) in (50). Therefore we see that for arbitrary \( y^- \) the energy conservation relation is valid:

\[
E_{\text{ADM}}(t) = B(y^-) + H(y^-) + \int_{-\infty}^{y^-} dy^- T_{-\sigma}^\ell(y^-).
\]

Therefore, we proved that the total energy of the infalling classical shock wave is preserved throughout the formation and subsequent evaporation of the black hole.
V. DISCUSSIONS

In this paper, we expressed the Bondi energy which is consistent with the usual definition of the Bondi energy as being the energy left in the system after the radiation has been occurred [10]. Another attempt to construct the Bondi energy was done by Bilal [20]. However, we are puzzled by the fact it violates the energy conservation. He applied his definition of the Bondi energy to the example of evaporating black hole in the RST model, which we also considered. His Bondi energy is positive definite up to the point where the thunderpop is emitted. However, if the Bondi energy is defined as the energy left in the system after the radiation, then it should be negative just before the emission of the negative thunderpop energy in the RST model. On the other hand, our Bondi energy is not necessarily positive definite. We believe our definition of the Bondi energy is more reasonable since it satisfies the usual requirements for the Bondi energy and the energy conservation. Also, the change of Bondi energy could not be followed exactly up to the end point of the Hawking radiation and the calculation was done only to the leading order in $m$ in Ref. [20]. In this paper, we could do the calculations exactly up to the endpoint.

Finally, we comment on the Hawking radiation $h(y_s^- - 0) = \frac{\kappa\lambda^2}{4}[1 - e^{-\frac{2m}{\kappa\lambda}}]$ just before the end point of the black hole evaporation. It is definitely positive due to the decoupling of ghost contribution and depends on the total energy $m$. For $m \to \infty$, it recovers the well-known two-dimensional result without back reaction of the metric, i.e., $h(y_s^- - 0) \to \frac{\kappa\lambda^2}{4}$. This result is natural in that the infinitely large mass of matter fields effectively generates the static solution since the large mass of black hole may radiate eternally as far as $m$ is infinite. Another limit one can consider is for $\kappa\lambda >> m$, where one may properly neglect the subleading quantum effect in $\frac{1}{N}$ expansions among the one-loop graphs. In this case, the Hawking radiation depends on the linear power of mass, $h(y_s^- - 0) \to 2\lambda m$.

ACKNOWLEDGEMENTS

We are very grateful to Choonkyu Lee for helpful discussions. We were supported by the Korea Science and Engineering Foundation through the Center for Theoretical
Physics (1995).

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FIGURE CAPTIONS

FIG. 1 Penrose diagram of the evaporating black hole in the RST model. An incoming shock wave at $y^+ = y_0^+$ produces the black hole and the negative energy thunderpop goes out at $y^- = y_s^-$. The zigzag line denotes the singularity.

FIG. 2 A plot of the Hawking radiation $h(y^-)$ up to $y_s^- \simeq 4$ for the case $m/\lambda = \lambda x_0^+ = 1$. $h(y^-) = 0$ for $y^- > y_s^-$. 

FIG. 3 A plot of the Bondi energy $B(y^-)$ up to $y_s^- \simeq 4$ for the case $m/\lambda = \lambda x_0^+ = 1$. $B(y^-) = 0$ for $y^- > y_s^-$. 

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