Convergence properties of 3-point block Adams method with one off-step point for ODEs

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Abstract. Numerous critical and intricate systems from various disciplines of sciences are developed by means of differential equations. Analytical methods are frequently complex or unfeasible to execute for problems due to the difficulty of these systems, therefore, numerical methods are the way out. To integrate differential equations, the most common approaches are single step and multistep techniques. A well-known multistep method Adams formula is one of the appropriate methods for solving non-stiff ordinary differential equation (ODEs). Present research emphases of 3-Point Block Method with one off-step point using Adams Moulton Formula for finding the solution intended for the system of non-stiff first order ODEs. Block method has been derived by considering the Adams Formula. The development of this method will compute the three solution values at \( x_{n+1}, x_{n+2} \) and \( x_{n+3} \) comprising of one off-step point at \( x_{n+2.5} \) using constant step size. The advantage of adding up one off-step point in the implicit Adams method will lead us towards better accuracy. This paper is shaped up with the derivation of the formula followed by its convergence properties by using MATHEMATICA software. As a result, the method is consistent and zero-stable, which implies convergence.

1. Introduction
In this article, we consider the initial value problems (IVPs) method for the first-order ODE system,

\[
y' = f(x, y), \quad y(a) = y_0, \quad a \leq x \leq b
\]  

The first-order ODEs exist among various fields of applied science and engineering represented by mathematical models. Technologically and scientifically, numerical methods for ODEs have immense greatness in the computation for their largely use in real-life problems. For instance, the projectile motion of orbiting bodies, growth of population, and chemical kinetics as defined by [1]. As per [2], certain complications in practice are simulated on first-order ODEs. However, others are developed on second and third ODEs, which are later determined by decreasing them to the first-ordered ODEs system and subsequently were solved by a one-stage process like the Taylor series, Euler’s, and Runge-Kutta Methods. The first order ODEs are therefore critical [3-5]. For solving IVPs consist of first order ODEs, Famurewa et al [6] focused on the development, evaluation, implementation, and comparative analysis of implicit multi-derived linear multistep techniques.

Milne [7] initially suggested implicit block methods. Rosser [8] later applied his concept to the Runge-Kutta method. The one-step implicit block approach can be followed in convergence and stability.
properties [9, 10]. In recent years block methods have been developed to overcome stiff ODEs through backward differentiation formulae and has been studied in [11]. In addition, the Adam-Moulton hybrid blocking approach for the resolution of stiff standard differential equations is seen in two off-step stages [12].

Hamid et al [13], updated second-stage derivative approaches which were constructed to solve ODEs. Sharifi and Seif [14], derived a Hermite interpolation-based, new family of multi-step numerical integration methods. The multistep method is one of the useful techniques with a fast convergence rate and small calculation error, see Lapidus and Seinfeld [15]. This is because the methods of single steps are inefficient as they do not use complete calculating information, see Yaacob and Chang. [16]. The new multi-step methodology has been thoroughly studied by authors such as [17-21].

As, there are several existing methods for solving equation (1) but those methods can only sequentially approximate the numerical solutions at one point. Therefore, we need a faster method that can solve the problem more efficiently. The goal of this paper is to use constant step size to implement the 3-point block method presented in the basic form as the Adams Moulton approach used to solve equation (1). The benefit of the methodology would be apparent as the theoretical work improves when solving the ODE schemes.

The determination of derivation of 3-Point Adams Moulton method with one off-step point is to get the better approximation of the solution. Furthermore, the convergence property of the method has also been discussed in this paper. The following section will briefly describe the formulation of the method.

2. Derivation of 3-point block Adams method with one off-step point

The present part of this paper comprises the derivation of three solution values \( y_{n+1}, y_{n+2} \) and \( y_{n+3} \) having step size \( h \) with one off-step point \( y_{n+2}^{\frac{h}{2}} \) in which half of the step size is shaped up in a block. By utilizing two values \( y_{n-1} \) and \( y_{n} \) in the previous block (Figure 1), these formulae will be computed with a \( h \) step size. Numerous points have been examined for choosing the suitable off-step point. After some observation, it has been noticed that the halved step size helps to acquire the desired stability and optimized point formula [22, 23].

![Figure 1. 3-Points Block with one-off step point](image)

The interval \( a \leq x \leq b \) in equation (1) is split into a sequence of blocks, with each block having the 3-point block method with one off-step point as shown in Figure 1. Utilizing two previous values \( x_{n} \) and \( x_{n-1} \) of the previous block with step size \( h \), the solutions of \( y_{n+1}, y_{n+2}, y_{n+2}^{\frac{h}{2}} \) and \( y_{n+3} \) at the \( x_{n+1}, x_{n+2}, x_{n+2}^{\frac{h}{2}} \) and \( x_{n+3} \) respectively with step size \( h \) were estimated simultaneously in a block. The process would simultaneously measure three points with one off-step using one prior block. The 3-point formulae are fully tacit methods derived using the polynomial of Lagrange interpolation. The four values of \( y_{n+1}, y_{n+2}, y_{n+2}^{\frac{h}{2}} \) and \( y_{n+3} \) can be obtained by integrating (1) over the interval \( (x_{n}, x_{n+1}), (x_{n}, x_{n+2}), (x_{n}, y_{n+2}^{\frac{h}{2}}) \) and \( (x_{n}, x_{n+3}) \), respectively using MATHEMATICA.
To derive the formula at the point \( x_{n+r} \), \( r = 1, 2, 5/2, \) and 3, equation (1) is integrated over the interval \((x_n, x_{n+r})\) to get,

\[
\int_{x_n}^{x_{n+r}} y' dx = \int_{x_n}^{x_{n+r}} f(x, y) dx
\]  
(2)

\[
y(x_{n+r}) = y(x_n) + \int_{x_n}^{x_{n+r}} f(x, y) dx
\]  
(3)

The function \( f(x, y) \) in equation (3) is evaluated by using Lagrange interpolation polynomial and the interpolation points included are \((x_{n-1}, y_{n-1}), (x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+2}, y_{n+2}), (x_{n+3}, y_{n+3})\) and \((x_{n+5}, y_{n+5})\) and \((x_{n+3}, y_{n+3})\). The process is given as,

\[
P_q(x) = \sum_{j=0}^{k} L_{q,j}(x)f(x_{n+3-j})
\]  
(4)

\[
L_{q,j} = \prod_{i=0}^{k-1} \frac{x-x_{n+3-i}}{x_{n+3-j}-x_{n+3-i}} \quad \text{for } j = 0, \frac{1}{2}, 1, 2, \ldots, k
\]

\[
P_3(x) = \frac{(x - x_{n-1})(x - x_{n})(x - x_{n+1})(x - x_{n+2})(x - x_{n+3})(x - x_{n+3})(x - x_{n+3})}{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})} f(x_{n+3})
\]

\[
+ \frac{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})}{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})} f(x_{n+3})
\]

\[
+ \frac{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})}{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})} f(x_{n+3})
\]

\[
+ \frac{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})}{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})} f(x_{n+3})
\]

\[
+ \frac{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})}{(x_{n+3} - x_{n-1})(x_{n+3} - x_n)(x_{n+3} - x_{n+1})(x_{n+3} - x_{n+2})(x_{n+3} - x_{n+3})(x_{n+3} - x_{n+3})} f(x_{n+3})
\]

Let \( s = \frac{x - x_{n+3}}{h} \) whereas \( x = sh + x_{n+3} \). Therefore, equation (5) becomes,
\[ P(x_{n+3} + sh) = \left\{ \begin{array}{c}
\frac{(sh + 3h + h)(sh + 3h)(sh + 2h)(sh + h) \left(sh + \frac{1}{2}h\right)}{(3h + h)(3h)(2h)(h) \left(\frac{h}{2}\right)} f(x_{n+3}) \\
+ \frac{(sh + 3h + h)(sh + 3h)(sh + 2h)(sh + h)(sh)}{(5h + 2h)(3h)(2h)(h) \left(\frac{h}{2}\right)} f(x_{n+2}) \\
+ \frac{(sh + 3h + h)(sh + 3h)(sh + 2h) \left(sh + \frac{1}{2}h\right)(sh)}{(2h + h)(2h)(h) \left(-\frac{h}{2}\right)} f(x_{n+1}) \\
+ \frac{(sh + 3h + h)(sh + 3h)(sh + h) \left(sh + \frac{1}{2}h\right)(sh)}{(h + h)(h)(-h) \left(-\frac{3h}{2}\right)} f(x_n) \\
+ \frac{(sh + 3h + h)(sh + 2h)(sh + h) \left(sh + \frac{1}{2}h\right)(sh)}{(h)(-h)(-2h) \left(-\frac{5h}{2}\right)} f(x_{n-1}) \\
+ \frac{(sh + 3h)(sh + 2h)(sh + h) \left(sh + \frac{1}{2}h\right)(sh)}{(-h)(-h - h)(-h - 2h) \left(-\frac{5h}{2}\right)} f(x_{n-2}) \\
\end{array} \right. \] (6)

\[ P(x_{n+3} + sh) = \frac{f_{n+1}}{12} \left[ (s + 4)(s + 3)(s + 2)(s + 1) \left(s + \frac{1}{2}\right) \right] - \frac{32f_{n+2}}{105} \left[ (s + 4)(s + 3)(s + 2)(s + 1) \right] \\
+ \frac{f_{n+2}}{3} \left[ (s + 4)(s + 3)(s + 2) \left(s + \frac{1}{2}\right)(s) \right] - \frac{f_{n+1}}{6} \left[ (s + 4)(s + 3)(s + 1) \left(s + \frac{1}{2}\right)(s) \right] \\
+ \frac{f_n}{15} \left[ (s + 4)(s + 2)(s + 1) \left(s + \frac{1}{2}\right)(s) \right] - \frac{f_{n-1}}{84} \left[ (s + 3)(s + 2)(s + 1) \left(s + \frac{1}{2}\right)(s) \right] \] (7)

To derive the formulae for \(y_{n+1}, y_{n+2}, y_{n+5/2}\) and \(y_{n+3}\), integrating the above equation (7) with respect to \(s\), \(s = \frac{x-x_{n+3}}{h}\) and replacing \(dx = h ds\) and altering the limits of integration from -3 to -2, -3 to -1, -3 to \(\frac{1}{2}\) and -3 to 0 in equation (7). The final set of formulae are given as,

\[ y_{n+1} = y_n + h \left[ \frac{-11f_{n+3}}{180} + \frac{98f_{n+2}}{315} - \frac{49f_{n+1}}{120} + \frac{283f_n}{120} + \frac{15f_{n-1}}{360} - \frac{13f_{n-2}}{840} \right] \]
\[ y_{n+2} = y_n + h \left[ \frac{-11f_{n+3}}{90} + (0)f_{n+2} + \frac{57f_{n+1}}{45} - \frac{17f_n}{45} + \frac{17f_{n-1}}{90} \right] \]
\[ y_{n+5/2} = y_n + h \left[ \frac{-125f_{n+3}}{4608} + \frac{65f_{n+2}}{252} + \frac{125f_{n+1}}{192} - \frac{2875f_n}{1204} + \frac{55f_{n-1}}{144} - \frac{125f_{n-2}}{10752} \right] \]
\[ y_{n+3} = y_n + h \left[ \frac{3f_{n+3}}{20} + \frac{24f_{n+2}}{35} + \frac{21f_{n+1}}{40} + \frac{51f_n}{40} + \frac{3f_{n-1}}{8} - \frac{3f_{n-2}}{280} \right] \] (8)

In the next section convergence properties of the 3-Point Adams method is discussed.
3. The convergence of the method

The linear multistep method (LMM) is acceptable if the method’s solution converges, to a theoretical way of solving as the step-length \( h \) reaches zero. If the term “convergent” is to be applied to the equation

\[
\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \quad (9)
\]

Then the convergence property must hold for all initial value problems equation (1). Where \( \alpha_j \) and \( \beta_j \) are the coefficients that would be taken from equation (8).

**Theorem 3.1**

The method’s necessary conditions for convergence are given by [17] as,

1. The approach needs to be consistent.
2. The method must have stability (zero stable).

[17] proved this theorem with the statement that:
The equation (9) has order \( p \) if \( C_0 = C_1 = \cdots = C_p = 0 \) and \( C_{p+1} \neq 0 \).

By following this theorem, the consistency of LMM in equation (9) is endorsed by the following definitions.

**Definition 3.1. Consistent**
The LMM equation (9) is said to be consistent if it possess the order \( p \geq 1 \).

**Definition 3.2. Zero stable:** The equation (9) is zero stable if first characteristics of the polynomial possess no root with modulus greater than one, also each root with modulus one is simple [25].

**Definition 3.2. A-stable:** The method is A-stable if for all \( \lambda \) in the left-half plane \( Re(\lambda) \leq 0 \). In which the whole negative left-half plane is covered to display the stability region.

3.1. Consistency of the 3-point Adams Method with one off-step point

In this section, we are formalizing the Taylor series method, for finding out the order by using LMM. With equation (9), we co-relate the linear difference operator as

\[
L[y(x_n); h] = \sum_{j=0}^{k+\frac{5}{2}+1} \alpha_{j-(k-1),T} y(x_n + Th) - h\beta_{T,T} y'(x_n + Th) \quad (10)
\]

where an arbitrary constant \( y \), is a continuous and differentiable function [5, 26]. \( k \) is total number of back values and \( s \) represents future values. Whereas \( T \) pointed out the number of equations at point \( 1, 2, \frac{5}{2}, 3 \).

The general linear multistep method for the 3-point Adams method with one off-step point is defined by:

\[
\sum_{j=0}^{k+\frac{5}{2}+1} \alpha_{j-(k-1),T} y_{n+j-(k-1)} = h\beta_{T,T} f_{n+T} \quad (11)
\]

The function \( y(x + Th) \) is expanded with its derivative \( y'(x + Th) \), and gathering terms in equation (8) will get,

\[
L = [y(x); h] = C_0y(x) + C_1y'(x) + \ldots + C_q h^q y^{(q)}(x) + \ldots \quad (12)
\]
whereas, in equation (12) $C_q$ are the constants and varied for each $q$.

Following formulae are used to find out the constant $C_q$ in terms of $\alpha_j$ and $\beta_j$.

\begin{align*}
C_0 &= \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k, \\
C_1 &= \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k), \\
\vdots \\
C_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \ldots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k).
\end{align*}

By using equation (13) for all four formulae, we got $C_1 = \ldots = C_6 = 0$ and $C_7 \neq 0$. It shows that the order of 3-Point Adams Moulton method with one off-step point is 6 and the error constant is $C_7 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
311 \\
120960 \\
756 \\
100 \\
896
\end{bmatrix}$.

Hence, we can conclude that the 3-point Adams Method is consistent of orders 6. From Definition 3.1, since the block method with off-step point have order $p \geq 1$ therefore is it consistent.

3.2 Stability of the 3-Point Block Adams method with one off-step point

In the test equation, the stability of the 3-point block Adams system with one off-step point is applied to the problem of first order linear equation, $y' = f = \lambda y$.  

The equation (8) is then inscribed into the matrix form to get the matrix as follows

\begin{equation}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{n+1} \\
y_{n+2} \\
y_{n+3}
\end{bmatrix}
+ h
\begin{bmatrix}
283\lambda & -49\lambda & 88\lambda \\
360 & 120 & 315 & 180 \\
19\lambda & 17\lambda & 0 & -1\lambda \\
15 & 45 & 90 & 90 \\
2875\lambda & 125\lambda & 65\lambda & -125\lambda \\
2304 & 182 & 252 & 4608 \\
51\lambda & 21\lambda & 24\lambda & 3\lambda \\
40 & 40 & 35 & 20
\end{bmatrix}
\begin{bmatrix}
f_{n+1} \\
f_{n+2} \\
f_{n+3} + h
\end{bmatrix}
+ h
\begin{bmatrix}
0 & -13\lambda & 0 \\
0 & 17\lambda & 0 \\
0 & 90 & 45 \\
0 & -125\lambda & 55\lambda \\
0 & 10752 & 144 \\
0 & -3\lambda & 3\lambda \\
0 & 280 & 0 & 8
\end{bmatrix}
\begin{bmatrix}
f_{n-2} \\
f_{n-1} \\
f_{n} \\
f_{n+1} \\
f_{n+2} \\
f_{n+3}
\end{bmatrix}
\end{equation}

The stability polynomial for equation (8) of the block system is,
\( R(t,H) = t^4 \left( -\frac{H}{2520} + \frac{11H^2}{60480} + \frac{H^3}{4320} + \frac{H^4}{20160} \right) \\
+ t^3 \left( -1 - \frac{257H}{180} - \frac{6557H^2}{7560} - \frac{583H^3}{2160} - \frac{29H^4}{672} \right) \\
+ t^4 \left( 1 - \frac{3961H}{2520} - \frac{13081H^2}{12096} - \frac{431H^3}{1080} + \frac{283H^4}{4032} \right) \)  

(15)

where \( H = h\lambda \) and \( H = 0 \), we have 

\[
R(t,H) = -t^3 + t^4 = 0
\]

Solving equation (16) for \( t \), gives \( \{ t \to 0 \} \), \( \{ t \to 0 \} \), \( \{ t \to 0 \} \), and \( \{ t \to 1 \} \). Hence, referring to Definition 3.2, the method is zero stable where all of the major roots lie in the unit circle or on it. If the size of step for the method is constant, the stability region is investigated that is shown in Figure 2.

![Stability region of the 3-Point Adams Method (6) with off-step point](image)

**Figure 2.** Stability region of the 3-Point Adams Method (6) with off-step point

4. Discussion

In section 3.1, the proposed method is proven to be consistent as the order is 6. Hence Definition 3.1, demonstrates that the method satisfies the consistency condition. Correspondingly, in section 3.2, the zero stability of the developed method has been analysed which also assures the condition emphasised in Definition 3.2. Figure 2 shows the stability region of the method corresponds to the 3-point Adams method which elaborates that the region lies outside the blue line is stable region. Hence on the basis of Theorem 3.1, the 3-Point Adams Moulton method with one off-step points is convergent therefore it is appropriate solver for non-stiff ODEs.

5. Conclusion

For solving non-stiff ODEs with constant step size, this paper includes a 3-Point Adams Moulton technique with a single off-step point. The order and stability properties of the 3-Point Adams Moulton Method are also determined. The analysis shows that the method is consistent and zero stable, representing that it is convergent. As a result, on the basis of convergence property it is concluded that the method can be used to solve non-stiff ODEs.

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