ON REGULAR BUT NOT COMPLETELY REGULAR SPACES

PIOTR KALEMBA AND SZYMON PLEWIK

Abstract. We present how to obtain non-comparable regular but not completely regular spaces. We analyze a generalization of Mysior’s example, extracting its underlying purely set-theoretic framework. This enables us to build simple counterexamples, using the Niemytzki plane, the Songefrey plane or Lusin gaps.

1. Introduction

Our discussion focuses around a question: How can a completely regular space be extended by a point to only a regular space? Before A. Mysior’s example, such a construction seemed quite complicated, compare [10] and [3]. R. Engelking included a description of Mysior’s example in the Polish edition of his book [4, p. 55-56]. In [2] there is considered a modification of Mysior’s example which requires no algebraic structure on the space. We present a purely set-theoretic approach which enables us to obtain non-comparable examples, such spaces are $X(\omega, \lambda_1)$ and $X(\lambda_2, \kappa)$, see Section 2. This approach is a step towards a procedure to rearrange some completely regular spaces onto only regular ones. One can find a somewhat similar idea in [6], compare "the Jones’ counterexample machine" in [2, p. 317]. The starting point of our discussion are the cases of completely regular spaces which are not normal. For example, subspaces of the Niemytzki plane are examined in [1] or [11], some $\Psi$-spaces are studied in [5], also the Songefrey plane is commentated in [11]. The key idea of our construction of counterexamples looks roughly as follows. Start from a completely regular space $X$, which is not normal. In fact, we need that $X$ contains countable many pairwise disjoint closed subsets which, even after removal from each of them a small subset, cannot be separated.

2000 Mathematics Subject Classification. Primary: 54D10; Secondary: 54G20.
Key words and phrases. Niemytzki plane; Songefrey plane; Lusin gap.
by open sets. By numbering these closed sets as $\Delta_X(k)$ and assuming that the collections of small sets form proper ideals $I_X(k)$, we should check that the property $(\ast)$ is fulfilled. Copies of $X$ are numbered by integers and then the $k$-th copy is glued along the set $\Delta_X(k)$ to the $(k-1)$-copy, moreover copies of sets $\Delta_X(m)$, for $k \neq n \neq k-1$, are removed from the $k$-th copy. As a result we get the completely regular space $Y_X$, which has a one-point extension to the regular space which is not completely regular.

In fact, given a completely regular space $X$, which we do not know whether it has one-point extension to the space which is only regular, we can build a space $Y_X$ which has such an extension. A somewhat similar method was presented in [6]. For this reason, we look for ways of comparing such spaces. Following the concept of topological ranks, compare [7, p. 112] or [12, p. 24], which was developed in Polish School of Mathematics, we say that spaces $X$ and $Y$ have non-comparable regularity ranks, whenever $X$ and $Y$ are regular but not completely regular and there does not exist a regular but not completely regular space $Z$ such that $Z$ is homeomorphic to a subspace of $X$ and $Z$ is homeomorphic to a subspace of $Y$.

2. On Mysior’s example

We modify the approach carried out in [2], which consists in a generalization of Mysior’s example, compare [10]. Despite the fact that our arguments resemble those used in [2], we believe that this presentation is a bit simpler and enables us to construct some non-homeomorphic examples, for example spaces $X(\lambda, \kappa)$. Let $\kappa$ be an uncountable cardinal and $\{A(k) : k \in \mathbb{Z}\}$ be a countable infinite partition of $\kappa$ into pairwise disjoint subsets of the cardinality $\kappa$, where $\mathbb{Z}$ stands for the integers. Denote the diagonal of the Cartesian product $\kappa^2$ by $\Delta = \{(x, x) : x \in \kappa\}$ and put $\Delta(k) = \Delta \cap A(k)^2$.

Fix an infinite cardinal number $\lambda < \kappa$ and proper $\lambda^+$-complete ideals $I(k, \lambda)$ on the sets $A(k)$. In particular, we assume that singletons are in $I(k, \lambda)$, hence $H \in I(k, \lambda)$ for any $H \subseteq A(k)$ such that $|H| \leq \lambda$. Consider a topology $\mathcal{T}$ on $X = \kappa^2$ generated by the basis consisting of all singletons $\{a\}$, whenever $a \in \kappa^2 \setminus \Delta$, and all the sets

$$\{(x, x)\} \cup \{x\} \times (A(k-1) \setminus G) \cup ((A(k+1) \setminus F) \times \{x\})$$
where \( x \in A(k) \) and \( |G| < \lambda \) and \( F \in I(k+1, \lambda) \). We denote not-singleton basic sets by \( \Gamma(x, G, F) \).

**Lemma 1.** Assume that \( H \subseteq \Delta(k) \cap V \), where \( V \) is an open set in \( X \). If the set \( \{ x \in A(k) : (x, x) \in H \} \) does not belong to the ideal \( I(k, \lambda) \), then the difference \( \Delta(k) \setminus \text{cl}_X(V) \) has the cardinality less than \( \lambda \).

**Proof.** Suppose that a set \( \{(b_\alpha, b_\alpha) : \alpha < \lambda\} \subseteq \Delta(k-1) \) of the cardinality \( \lambda \) is disjoint from \( \text{cl}_X(V) \). For each \( \alpha < \lambda \), fix a basic set \( \Gamma(b_\alpha, G_\alpha, F_\alpha) \) disjoint from \( V \), where \( F_\alpha \in I(k, \lambda) \). The ideal \( I(k, \lambda) \) is \( \lambda^+ \)-complete and the set \( \{ x \in A(k) : (x, x) \in H \} \) does not belong to this ideal. So, there exists a point \( (x, x) \in H \) such that

\[
x \in A(k) \setminus \bigcup\{F_\alpha : \alpha < \lambda\}.
\]

Therefore

\[
(x, b_\alpha) \in (A(k) \setminus F_\alpha) \times \{b_\alpha\} \subseteq \Gamma(b_\alpha, G_\alpha, F_\alpha)
\]

for every \( \alpha < \lambda \). Fix a basic set \( \Gamma(x, G_x, F_x) \subseteq V \) and \( \alpha < \lambda \) such that \( b_\alpha \in A(k-1) \setminus G_x \). We get \( (x, b_\alpha) \in \{x\} \times (A(k-1) \setminus G_x) \subseteq V \), a contradiction. \( \square \)

**Corollary 2.** The space \( X \) is completely regular, but not normal.

**Proof.** The base consists of closed-open sets and one-points subsets of \( X \) are closed. So \( X \), being a zero-dimensional T_1 space, is completely regular. Subsets \( \Delta(k+1) \) and \( \Delta(k) \) are closed and disjoint. By Lemma 1 if a set \( V \subseteq X \) is open and \( \Delta(k+1) \subseteq V \), then \( \text{cl}_X(V) \cap \Delta(k) \neq \emptyset \), which implies that \( X \) is not a normal space. \( \square \)

**Proposition 3.** Assume that the cardinal \( \lambda \) has an uncountable cofinality. If \( f : X \to \mathbb{R} \) is a continuous real valued function, then for any point \( x \in \kappa \) there exists a basic set \( \Gamma(x, G_x, F_x) \) such that the function \( f \) is constant on it.

**Proof.** Without loss of generality, we can assume that \( f(x, x) = 0 \). For each \( n > 0 \), fix a base set \( \Gamma(x, G_n, F_n) \subseteq f^{-1}(((-1/n, 1/n))) \). Then put \( G_x = \bigcup\{G_n : n > 0\} \) and \( F_x = \bigcup\{F_n : n > 0\} \). Since \( \lambda \) has an uncountable cofinality, we get that the set \( \Gamma(x, G_x, F_x) \) belongs to the base. Obviously, if

\[
(a, b) \in \bigcap\{\Gamma(x, G_n, F_n) : n > 0\} = \Gamma(x, G_x, F_x),
\]

then \( f(a, b) = 0 \). \( \square \)
When \( \lambda \) has the countable cofinality, then the above proof also works, but then the set \( G_x = \cup \{ G_n : n > 0 \} \) may have the cardinality \( \lambda \), and therefore \( \Gamma(x, G_x, F_x) \) does not necessarily belong to the base, and also it could be not open. Furthermore, any continuous real valued function must also be constant onto other large subsets of \( X \).

**Lemma 4.** Let \( k \in \mathbb{Z} \). If \( f : X \to \mathbb{R} \) is a continuous real valued function, then for any \( \varepsilon > 0 \) there exists a real number \( a \) such that \( f[\Delta(k-1)] \subseteq [a, a + 3 \cdot \varepsilon] \) for all but less than \( \lambda \) many points \( (x, x) \in \Delta(k-1) \).

**Proof.** Fix a real number \( b \) and \( \varepsilon > 0 \). The ideal \( I(k, \lambda) \) is \( \lambda^+ \)-complete, so we can choose an integer \( q \in \mathbb{Z} \) such that the subset

\[
\{ x \in A(k) : f(x, x) \in [b + q \cdot \varepsilon, b + (q + 1) \cdot \varepsilon] \}
\]

does not belong to \( I(k, \lambda) \). Use Lemma 1, putting \( a = b + (q - 1) \cdot \varepsilon \) and \( H = f^{-1}([a + \varepsilon, a + 2 \cdot \varepsilon]) \cap \Delta(k) \) and \( V = f^{-1}((a, a + 3 \cdot \varepsilon)) \). Since \( cl_X(V) \subseteq f^{-1}([a, a + 3 \cdot \varepsilon]) \), the proof is completed.

**Corollary 5.** If \( f : X \to \mathbb{R} \) is a continuous real valued function, then for any \( k \in \mathbb{Z} \) there exists a real number \( a_k \) such that \( f(x, x) = a_k \) for all but \( \lambda \) many points \( (x, x) \in \Delta(k) \). Moreover, if \( \lambda \) has uncountable cofinality, then \( f(x, x) = a_k \) for all but less than \( \lambda \) many \( x \in A(k) \).

**Proof.** Apply Lemma 4 substituting consecutively \( \frac{1}{n} \) for \( \varepsilon \), for \( n > 0 \), and \( k + 1 \) for \( k \). \( \square \)

**Theorem 6.** If \( f : X \to \mathbb{R} \) is a continuous real valued function, then there exists \( a \in \mathbb{R} \) such that \( f(x, x) = a \) for all but \( \lambda \) many \( x \in \kappa \). Moreover, when \( \lambda \) has an uncountable cofinality, then \( f(x, x) = a \) for all but less than \( \lambda \) many \( x \in \kappa \).

**Proof.** We shall to prove that the numbers \( a_k \) which appear in Corollary 5 are equal. To do this, suppose that \( a_k \neq a_{k-1} \) for some \( k \in \mathbb{Z} \). Choose disjoint open intervals \( \mathbb{J} \) and \( \mathbb{I} \) such that \( a_k \in \mathbb{J} \) and \( a_{k-1} \in \mathbb{I} \). Apply Lemma 1 taking \( H = \{(x, x) \in \Delta(k) : f(x, x) = a_k \} \) and \( V = f^{-1}(\mathbb{J}) \). Since \( cl_X(V) \cap f^{-1}(\mathbb{I}) = \emptyset \), we get \( f(x, x) \neq a_{k-1} \) for all but less than \( \lambda \) many \( x \in A(k-1) \), a contradiction. \( \square \)

Knowing infinite cardinal numbers \( \lambda < \kappa \) and proper \( \lambda^+ \)-complete ideals \( I(k, \lambda) \) on sets \( A(k) \), one can extend the space \( X \) by one or two points so as to get a regular space which is not completely regular.
This is a standard construction, compare [6], [10] and [2] or [4, Example 1.5.9], so we will describe it briefly. Fix points $+\infty$ and $-\infty$ that do not belong to $X$. On the set $X^* = X \cup \{-\infty, +\infty\}$ we introduce the following topology. Let open sets in $X$ be open in $X^*$, too. But the sets

$$V^+_m = \{+\infty\} \cup \bigcup \{A(n) \times \kappa : n > m\}$$

form a base at the point $+\infty$ and the sets

$$V^-_m = \{-\infty\} \cup \bigcup \{A(n) \times \kappa : n \leq m\} \setminus \Delta(m)$$

form a base at the point $-\infty$. Thus we have

$$\Delta(m) = cl_{X^*}(V^+_m) \cap cl_{X^*}(V^-_m) = cl_X(V^+_m \cap X) \cap cl_X(V^-_m \cap X),$$

which gives that the space $X^*$ is regular and not completely regular. Indeed, consider a closed subset $D \subseteq X^*$ and a point $p \in X^* \setminus D$. When $p \in X$, then $p$ has a closed-open neighborhood in $X^*$ which is disjoint with $D$. When $p = +\infty$, then consider the basic set $V^+_m$ which is disjoint with $D$ and check $cl_{X^*}(V^+_m) \subseteq V^+_m$. Analogously, when $p = -\infty$, then consider the basic set $V^-_m$ which is disjoint with $D$ and check $cl_{X^*}(V^-_m) \subseteq V^-_m$. By Theorem 6, no continuous real valued function separates an arbitrary closed set $\Delta(k)$ from a point $p \in \{+\infty, -\infty\}$. Hence the space $X^*$ is not completely regular. The same holds for subspaces $X^* \setminus \{+\infty\}$ and $X^* \setminus \{-\infty\}$. Moreover, if $f : X^* \to \mathbb{R}$ is a continuous function, then $f(+\infty) = f(-\infty)$.

Now for convenience, the above defined space $X$ is denoted $X(\lambda, \kappa)$, whenever the ideals $I(\lambda, \kappa)$ consist of sets of the cardinality less than $\lambda$. Assuming $\omega < \lambda_1 < \lambda_2 < \kappa$ we get two (non-comparable) non-homeomorphic spaces $X(\omega, \lambda_1)$ and $X(\lambda_2, \kappa)$, since the first one has the cardinality $\lambda_1$. But a subspace of $X(\lambda_2, \kappa)$ of the cardinality $\lambda_1$ is discrete and its closure in $(X(\omega, \lambda_2))^*$, being zero-dimensional, is completely regular. In other words, spaces $(X(\omega, \lambda_1))^*$ and $(X(\lambda_2, \kappa))^*$ have non-comparable regularity ranks.

3. General approach

The analysis conducted above can be generalized using some known counterexamples. We apply such a generalization to the Niemytzki plane, cf. [11, p. 34] or [11, pp. 100 - 102], the Songefrey’s half-open square topology, cf. [11, pp. 103 - 105] and special Isbell-Mrówka spaces (which are also known as $\Psi$-spaces).
Given a space $X$ and a closed and discrete subset $\Delta_X \subseteq X$, assume that $\Delta_X$ can be partitioned onto pairwise disjoint subsets $\Delta_X(k)$. For each $k \in \mathbb{Z}$, let $I_X(k)$ be a proper ideal on $\Delta_X(k)$. Suppose that the following property is fulfilled:

\((\ast)\). If a set $V \subseteq X$ is open and the set $\Delta_X(k) \setminus V$ belongs to $I_X(k)$, then the set $\Delta_X(k-1) \setminus cl_X(V)$ belongs to $I_X(k-1)$.

Then it is possible to give a general scheme of a construction of a completely regular space $Y = Y_X$, which has one-point extension to a regular space which is not completely regular and two-point extension to a regular space such that no continuous real valued function separates the extra points, whereas removing a single point we get a regular space which is not completely regular. To get this we put

$$x_k = \begin{cases} (k, x), & \text{when } x \in X \setminus \Delta_X; \\ \{(k, x), (k + 1, x)\}, & \text{when } x \in \Delta_X(k). \end{cases}$$

And then put $Y_X = \{x_k : x \in X$ and $k \in \mathbb{Z}\}$. Endow $Y_X$ with the topology as follows. If $k \in \mathbb{Z}$ and $V \subseteq X \setminus \Delta_X$ is an open subset of $X$, then the set $\{x_k : x \in V\}$ is open in $Y_X$. Thus we define neighborhoods of the point $x_k$ where $x \notin \Delta_X$. To define neighborhoods of the point $x_k$, where $x \in \Delta_X$, we use the formula: If $k \in \mathbb{Z}$ and $V \subseteq X$ is an open subset, then the set $$\{x_k : x \in V\} \cup \{x_{k+1} : x \in V \setminus \Delta_X\}$$ is open in $Y_X$. To get a version of $\ast$, we put the following: $\Delta_Y(k) = \{x_k : x \in \Delta_X(k)\}; \Delta_Y = \bigcup \{\Delta_X(k) : k \in \mathbb{Z}\};$ Let $I_Y(k)$ be a proper ideal which consists of sets $\{x_k : x \in A\}$ for $A \in I_X(k); Y_k = \{y_k : y \in X \setminus \Delta_X\}$. So, if $k \in \mathbb{Z}$, then

$$\Delta_Y(k) = cl_Y(\{y_k : y \in X \setminus \Delta_X\}) \cap cl_Y(\{y_{k+1} : y \in X \setminus \Delta_X\}).$$

As we can see, the properties of the space $Y_X$ can be automatically rewritten from the relevant properties of $X$, so we leave details to the reader.

**Proposition 7.** Assume that a space $X$ satisfied $\ast$ and the space $Y$ is as above. If a set $V \subseteq Y$ is open and the set $\Delta_Y(k) \setminus V$ belongs to $I_Y(k)$, then the set $\Delta_Y(k-1) \setminus cl_Y(V)$ belongs to $I_Y(k-1)$.

**Proposition 8.** If a space $X$ is completely regular, then the space $Y$ is completely regular, too.
Now, fix points $+\infty$ and $-\infty$ that do not belong to $Y$. On the set $Y^* = Y \cup \{-\infty, +\infty\}$ we introduce the following topology. Let open sets in $Y$ be open in $Y^*$, too. But the sets

$$V_m^+ = \{+\infty\} \cup \bigcup \{Y_n : n \geq m\} \cup \bigcup \{\Delta_Y(n) : n > m\}$$

form a base at the point $+\infty$ and the sets

$$V_m^- = \{-\infty\} \cup \bigcup \{Y_n : n \leq m\} \cup \bigcup \{\Delta_Y(n) : n < m\}$$

form a base at the point $-\infty$. Thus we have

$$\Delta_Y(m) \subseteq cl_{Y^*}(V_m^+) \cap cl_{Y^*}(V_m^-) = \Delta_Y(m) \cup Y_m,$$

which implies the following.

**Theorem 9.** If $f : Y^* \to \mathbb{R}$ is a continuous real valued function, then $f(+\infty) = f(-\infty)$.

**Proof.** Suppose $f : Y^* \to \mathbb{R}$ is a continuous function such that $f(+\infty) = 1$ and $f(-\infty) = 0$. Fix a decreasing sequence $\{\epsilon_n\}$ which converges to $\frac{1}{2}$. Thus

$$f^{-1}((\epsilon_n, 1]) \subseteq cl_{Y^*}(f^{-1}((\epsilon_n, 1])) \subseteq f^{-1}([\epsilon_n, 1]) \subseteq f^{-1}((\epsilon_{n+1}, 1]).$$

By Proposition 7 if $K_m \in I_Y(m)$ and $\Delta_Y(m) \setminus K_m \subseteq f^{-1}((\epsilon_n, 1])$, then

$$f^{-1}((\epsilon_{n+1}, 1]) \supseteq \Delta_Y(m-1) \setminus K_{m-1},$$

for some $K_{m-1} \in I_Y(m-1)$. Since there exists $m \in \mathbb{Z}$ such that $+\infty \in V_m^+ \subseteq f^{-1}((\epsilon_0, 1])$, inductively, we get

$$\Delta_Y \setminus \bigcup \{K_n : n \in \mathbb{Z}\} \subseteq f^{-1}(\frac{1}{2}, 1]),$$

which implies that each $V_n^-$ contains a point $y \in Y$ such that $f(y) \geq \frac{1}{2}$. Hence $f(-\infty) \geq \frac{1}{2}$, a contradiction. \qed

### 3.1. Application of the Niemytzki plane.

Recall that the Niemytzki plane $P = \{(a, b) \in \mathbb{R} \times \mathbb{R} : 0 \leq b\}$ is the closed half-plane which is endowed with the topology generated by open discs disjoint with the real axis $\Delta_P = \{(x, 0) : x \in \mathbb{R}\}$ and all sets of the form $\{a\} \cup D$ where $D \subseteq P$ is an open disc which is tangent to $\Delta_P$ at the point $a \in \Delta_P$. Choose pairwise disjoint subsets $\Delta_P(k) \subseteq \Delta_P$, where $k \in \mathbb{Z}$, such that each set $\Delta_P(k)$ meets every dense $G_\delta$ subset of the real axis. To do that is enough to slightly modify the classic construction of a Bernstein set. Namely, fix an enumeration $\{A_\alpha : \alpha < c\}$ of all dense $G_\delta$ subsets of the real axis. Defining inductively at step $\alpha$ choose
a (1-1)-numerated subset \( \{p_k^\alpha : k \in \mathbb{Z}\} \subseteq A_\alpha \setminus \{p_k^\beta : k \in \mathbb{Z} \ and \ \beta < \alpha\} \). Then, for each \( k \in \mathbb{Z} \), put \( \Delta_\varphi(k) = \{p_k^\alpha : \alpha < \epsilon\} \).

Let us assume that if \( F \subseteq \mathbb{R} \times \mathbb{R} \), then the topology on \( F \) induced from the Euclidean topology will be called the natural topology on \( F \). A set, which is a countable union of nowhere dense subsets in the natural topology on \( F \), will be called a set of first category in \( F \). Our proof of the following lemma is a modification of known reasoning justifying that \( \mathbb{P} \) is not a normal space, compare \([11, pp. 101 -102]\).

**Lemma 10.** Let a set \( F \subseteq \Delta_\varphi \) be a dense subset in the natural topology on the real axis \( \Delta_\varphi \). If a set \( V \) is open in \( \mathbb{P} \) and \( F \subseteq V \), then the set \( \Delta_\varphi \setminus \text{cl}_\varphi(V) \) is of first category in \( \Delta_\varphi \).

**Proof.** To each point \( a \in \Delta_\varphi \setminus \text{cl}_\varphi(V) \) there corresponds a disc \( D_a \subseteq \mathbb{P} \setminus \text{cl}_\varphi(V) \) of radius \( r_a \) tangent to \( \Delta_\varphi \) at the point \( a \). Put

\[
S_n = \{a \in \Delta_\varphi \setminus \text{cl}_\varphi(V) : r_a \geq \frac{1}{n}\}
\]

and use density of \( F \) to check that each \( S_n \) is nowhere dense in the natural topology on \( \Delta_\varphi \). So \( \bigcup \{S_n : n > 0\} = \Delta_\varphi \setminus \text{cl}_\varphi(V) \). □

The space \( \mathbf{Y}_\varphi \) is completely regular. The subspaces \( \mathbf{Y}_\varphi \cup \{-\infty\} \), \( \mathbf{Y}_\varphi \cup \{+\infty\} \) and and the space \( \mathbf{Y}_\varphi^* \) are regular. Moreover, if \( f : \mathbf{Y}_\varphi^* \to \mathbb{R} \) is a continuous real valued function, then \( f(+\infty) = f(-\infty) \).

3.2. **Application of the Songefrey plane**, i.e. application of the Songenfrey’s half-open square topology.

Recall that the Songefrey plane \( \mathbb{S} = \{(a,b) : a \in \mathbb{R} \ and \ b \in \mathbb{R}\} \) is the plane endowed with the topology generated by rectangles of the form \([a,b) \times [c,d)\). Let \( \Delta_\mathbb{S} = \{(x,-x) : x \in \mathbb{R}\} \). Since \( \Delta_\mathbb{S} \) with the topology induced from the Euclidean topology is homeomorphic with the real line, we can choose pairwise disjoint subsets \( \Delta_\mathbb{S}(k) \subseteq \Delta_\mathbb{S} \) such that each set \( \Delta_\mathbb{S}(k) \) meets every dense \( G_\delta \) subset of \( \Delta_\mathbb{S} \). The following lemma can be proved be the second category argument used previously in the proof Lemma [10] so we omit it, compare also \([11, pp. 103 -104]\).

**Lemma 11.** Let a set \( F \subseteq \Delta_\mathbb{S} \) be a dense subset in the topology on \( \Delta_\mathbb{S} \) which is inherited from the Euclidean topology. If a set \( V \) is open in \( \mathbb{S} \) and \( F \subseteq V \), then the set \( \Delta_\mathbb{S} \setminus \text{cl}_\mathbb{S}(V) \) is of first category in \( \Delta_\mathbb{S} \). □
Again, the space $Y_S$ is completely regular. The subspaces $Y_S \cup \{-\infty\}$, $Y_S \cup \{+\infty\}$ and and the space $Y_S^*$ are regular. Moreover, if $f : Y_S^* \to \mathbb{R}$ is a continuous real valued function, then $f(+\infty) = f(-\infty)$.

### 3.3. Applications of some $\Psi$-spaces

Let us recall some notions needed to define a Lusin gap, compare [8]. A family of sets is called almost disjoint, whenever any two members of it have the finite intersection. A set $C$ separates two families, whenever each member of the first family is almost contained in $C$, i.e. $B \setminus C$ is finite for any $B \in Q$, and each member of the other family is almost disjoint with $C$. An uncountable family $\mathcal{L}$, which consists of almost disjoint and infinite subsets of $\omega$, is called Lusin-gap, whenever no two its uncountable and disjoint subfamilies can be separated by a subset of $\omega$. Adapting concepts discussed in [9] or [5], to a Lusin-gap $\mathcal{L}$, let $\Psi(\mathcal{L}) = \mathcal{L} \cup \omega$. A topology on $\Psi(\mathcal{L})$ is generated as follows. Any subset of $\omega$ is open, also for each point $A \in \mathcal{L}$ the sets $\{A\} \cup A \setminus F$, where $F$ is finite, are open.

**Proposition 12.** If $\mathcal{L}$ is a Lusin-gap and $\bigcup \{\Delta_{\mathcal{L}}(k) : k \in \mathbb{Z}\} = \mathcal{L}$, then the space $\Psi(\mathcal{L})$ satisfies the property $(*)$, whenever sets $\Delta_{\mathcal{L}}(k)$ are uncountable and pairwise disjoint and each ideal $I_{\mathcal{L}}(k)$ consists of all countable subsets of $\Delta_{\mathcal{L}}(k)$.

**Proof.** Consider uncountable and disjoint families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$. Suppose $\mathcal{A} \subseteq V$ and $\mathcal{B} \subseteq W$, where open sets $V$ and $W$ are disjoint. Let

$$C = \bigcup \{A \subseteq \omega : \{A\} \cup A \text{ is almost contained in } V\}.$$  

The set $C$ separates families $\mathcal{A}$ and $\mathcal{B}$, which contradicts that $\mathcal{L}$ is a Lusin-gap. Setting $\mathcal{A} = \Delta_{\mathcal{L}}(k)$ and $\mathcal{B} = \Delta_{\mathcal{L}}(k-1)$, we are done. $\blacksquare$

The space $Y_{\mathcal{L}}$ is completely regular. Again by Theorem 9, we get the following. The subspaces $Y_{\mathcal{L}} \cup \{-\infty\}$, $Y_{\mathcal{L}} \cup \{+\infty\}$ and and the space $Y_{\mathcal{L}}^*$ are regular. Moreover, if $f : Y_{\mathcal{L}}^* \to \mathbb{R}$ is a continuous real valued function, then $f(+\infty) = f(-\infty)$.

### 4. Comment

In [6], F. B. Jones formulated the following problem: Does a non-completely regular space always contain a substructure similar to that possessed by $Y$? Jones’ space $Y$ is constructed by gluing (sewing) countably many disjoint copies of a suitable space $X$. This method fixes two
subsets of $X$ and consists in sewing alternately copies of either of them. On the other hand, our method consists in gluing different sets at each step. The problem of Jones may be understood as an incentive to study the structural diversity of regular spaces, which are not completely regular. Even though the meaning of "a substructure similar to that possessed by $Y$" seems vague, we think that an appropriate criterion for the aforementioned diversity is a slightly modified concept of a topological rank, compare [7] or [12]. We have introduced regularity ranks, but our counterexamples are only a preliminary step to the study of diversity of regular spaces.

References

[1] D. Chodounský, Non-normality and relative normality of Niemytzki plane. Acta Univ. Carolin. Math. Phys. 48 (2007), no. 2, 37-41.
[2] K. Ch. Ciesielski and J. Wojciechowski, Cardinality of regular spaces admitting only constant continuous functions. Topology Proc. 47 (2016), 313-329.
[3] R. Engelking, General topology. Mathematical Monographs, Vol. 60, PWN—Polish Scientific Publishers, Warsaw, (1977).
[4] R. Engelking, Topologia ogólna I, Państwowe Wydawnictwo Naukowe, Warszawa (1989).
[5] F. Hernández-Hernández and M. Hrušák, $Q$-sets and normality of $\Psi$-spaces, Spring Topology and Dynamical Systems Conference. Topology Proc. 29 (2005), no. 1, 155-165.
[6] F. B. Jones, Hereditarily separable, non-completely regular spaces, Proceedings of the Blacksburg Virginia Topological Conference, March (1973).
[7] K. Kuratowski, Topology-Volume I. Transl. by J. Jaworowski, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe Polish Scientific Publishers, Warsaw (1966).
[8] N. Luzin, On subsets of the series of natural numbers, Isv. Akad. Nauk. SSSR Ser. Mat. 11 (1947), 403-411.
[9] S. Mrówka, On completely regular spaces, Fund. Math. 41 (1954), 105-106.
[10] A. Mysior, A regular space which is not completely regular. Proc. Amer. Math. Soc. 81 (1981), no. 4, 652-653.
[11] L.A. Steen and J.A. jun. Seebach, Counterexamples in topology. New York etc.: Holt, Rinehart and Winston, Inc., XIII, (1970).
[12] W. Sierpiński, Introduction to General Topology. Lectures in Mathematics at the University of Toronto. The University of Toronto Press (1934).
PIOTR KALEMBA, INSTITUTE OF MATHEMATICS, UNIVERSITY OF SILESIA, UL.
BANKOWA 14, 40-007 KATOWICE

E-mail address: piotr.kalemba@us.edu.pl

SZYMON PLEWIK, INSTITUTE OF MATHEMATICS, UNIVERSITY OF SILESIA, UL.
BANKOWA 14, 40-007 KATOWICE

E-mail address: plewik@math.us.edu.pl