LECTURES ON CONTROLLED TOPOLOGY: MAPPING CYLINDER NEIGHBORHOODS

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Abstract. The existence theorem for mapping cylinder neighborhoods is discussed as a prototypical example of controlled topology and its applications. The first of a projected series developed from lectures at the Summer School on High-Dimensional Topology, Trieste Italy 2001

1. Introduction

Controlled topology has the hallmarks of a mature mathematical subject: powerful results, sophisticated interactions with, and applications to, other subjects, difficult and unexpectedly beautiful conjectures. It is not very accessible, however. Partly this is because complete results are difficult and there is not a large enough community to sustain interest in partial answers. Another problem is that it bloomed rapidly, so lacks the more-accessible historical development and expositions of most mature subjects. This paper is the first in a projected series to try to address this. Here we outline the setting and applications of the existence theorem for mapping cylinder neighborhoods (originally, “completions” of ends of maps). This illustrates most of the ingredients of the subject: local homotopy theory, local fundamental groups, elaborate algebraic obstructions, interesting applications. The focus is on what all these things mean and how they fit together, and most details are omitted.

This paper is an expansion of the first third of a series of lectures given at the Summer School on High-Dimensional Topology in Trieste, Italy, in the summer of 2001. Other topics were the controlled h-cobordism theorem, illustrating some of the geometric and algebraic techniques; and homology manifolds, illustrating the still-incomplete theory of controlled surgery.

1.1 Locating the subject. In the first half of the 20th century topology had two main branches: point-set topology, concerned with local properties (separation, connectedness, dimension theory etc); and algebraic topology, concerned with definition and detection of global structure (homology, characteristic classes, etc.). In the 50s and 60s the algebraic branch split into homotopy theory and geometric topology. Homotopy theory was still largely descriptive, but in the geometric area the emphasis changed from description to construction. For instance rather than

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1
computing homology of examples of manifolds, the objective was to construct or classify manifolds with given homological structure. This development was mainly restricted to spaces with uniform local structure, i.e. manifolds. Some of the descriptive techniques had extensions to spaces with symmetries (group actions) and stratified spaces such as algebraic varieties. Extensions of constructive methods were very limited due to complexity of interactions between different levels in group actions or stratified sets.

Controlled topology began in the late 1970s and 80s as a way to apply the constructive techniques of geometric topology to local questions more typical of point-set topology. For example, which subspaces of a space have a neighborhood homeomorphic to a mapping cylinder? Mapping cylinders must be constructed rather than simply detected, so although this is a local question it requires constructive techniques.

Controlled topology has had striking successes in elucidating geometric structure. Unexpectedly, it has also had striking applications in algebra. Geometric problems have obstructions related to linear or quadratic algebra ($K$- and $L$-theory). Controlled geometric problems have obstructions in controlled-algebra analogs, essentially homology with coefficients in spectra related to the uncontrolled obstructions. This turns out to be a two-way street: results about ordinary obstructions give information about control and local geometric structure, and conversely direct controlled constructions can give information about ordinary obstructions. For instance the famous “strong Novikov conjecture” asserts that some assembly maps from homology to ordinary obstruction groups are isomorphisms, at least rationally. The homology corresponds to controlled problems, ordinary groups correspond to uncontrolled problems, and the assembly map corresponds to “forgetting control.” If the assembly map is an isomorphism then solvability of an uncontrolled problem determines solvability of a more delicate controlled one. Conversely if there is a geometric construction that “gains control” — produces a controlled solution from an uncontrolled one — then the assembly map must be an isomorphism. Most geometric proofs of cases of the Novikov conjecture rely on this principle, and the most delicate (especially work of Farrell and Jones) use it explicitly.

Controlled topology thus lies at the juncture between geometric and point-set topology, homotopy theory, and stable algebra. Constructions and proofs tend to be elaborate, but outcomes can be deep and powerful.

1.2 Plan. True mastery of a subject requires understanding the details. However to get started, or for those looking for application rather than mastery, an overview can be helpful. The goal here is to give such an overview: definitions and enough explanation for good understanding of the statements of theorems, sketches of proofs in enough detail to show how the hypotheses are used and what the difficult points are. Finally in this paper we focus on the construction and application of mapping cylinders. This illustrates most of the techniques and issues of controlled topology.

The central result in the paper is the existence theorem for mapping cylinder neighborhoods, 3.1. However the hypotheses are quite elaborate, so Section 2 is devoted to developing them. Specifically, 2.1 gives the definition, 2.2 describes the use of “control spaces”, and 2.3 describes the simplest (“uncontrolled”) case of neighborhoods of points. Section 2.4 defines tameness and describes some of the results. Tameness often does not appear in statements of applications because it follows from other hypotheses, but it is central to proofs. Section 2.5 discusses
homotopy links, which provide homotopy models and play an important role in controlling local fundamental groups, as explained in 2.6. Stratified systems of fibrations are introduced in 2.7. These are needed to impose some regularity on local fundamental groups, and appear prominently in the structure of stratified sets. Section 2.8 begins development of the “spectral sheaf homology” used to describe the obstruction groups. A key feature of this theory is the assembly map defined in 2.9. The most elaborate part of the development is in section 2.10, where the controlled K-theory (more precisely, pseudoisotopy) spectrum is discussed. It is in large part the controlled assembly isomorphism theorem for this spectrum that makes the theory accessible and useful.

With the setting and hypotheses explained, the existence theorem for mapping cylinder neighborhoods is stated in 3.1. The proof is outlined in sections 3.2 and 3.3. Some useful refinements are given in 3.4. The first concerns smooth and PL structures, and the second gives a recognition criterion for mapping cylinders.

Applications are given in section 4. The first three are straightforward: the special case of manifolds (4.1) with its corollary the finiteness of compact finite dimensional ANRs (the Borsuk conjecture), and collaring in homology manifolds (4.3). Then we consider mapping cylinders in cases where the tameness and local fundamental group structure are more elaborate, namely stratified spaces (4.4), and a special case where the obstructions can be made relatively explicit, topological actions of finite groups (4.5). The final application (in 4.6) is to define topological regular neighborhoods. These are mapping cylinders in a product with $[0, \infty)$, and generalize the “approximate tubular neighborhoods” in stratified spaces developed by Hughes and others.

2. The setting

To illustrate the basic ideas of control we investigate the existence of mapping cylinder neighborhoods. Fix a space $X$ and a closed subset $Y$. We suppose the complement $X - Y$ is a manifold since we use manifold techniques there. Finally we suppose $X$ and $Y$ are finite-dimensional ANRs (absolute neighborhood retracts), to avoid local point-set problems.

2.1 Definition. A mapping cylinder neighborhood of $Y$ in $X$ is a closed neighborhood $N \supset Y$ with frontier $\partial N$ a submanifold of $X - Y$, a map $f: \partial N \to Y$, and a homeomorphism of the mapping cylinder of $f$ with $N$ which is the identity on $\partial N$ and $Y$.

More explicitly the mapping cylinder is the identification space $\partial N \times I \cup_f Y$, where “$\cup_f$” indicates that points $(x, 0) \in \partial N \times I$ are identified with $f(x) \in Y$, and the homeomorphism $\partial N \times I \cup_f Y \to N$ restricts to the identities on $\partial N \times \{1\} \to \partial N$ and $Y \to Y$.

2.2 The control space. There is a canonical projection of a mapping cylinder to the subspace $Y$. In fact we have assumed $Y$ is an ANR so there is a projection of a neighborhood to $Y$ whether there is a mapping cylinder or not. Denote this by $p: N - Y \to Y$. We refer to $Y$ as the control space and the projection as the control map. To explain the terminology we observe that a mapping cylinder is equivalent to a product structure on the complement $N - Y \simeq \partial N \times (0, 1]$ so that the images of open arcs $p(x) \times (0, 1]$ converge. The map $f: \partial N \to Y$ can be recovered as the limit $f(x) = \lim_{t \to 0} p(x, t)$. Convergence is arranged using the Cauchy
criterion: constructions are done so that images of subintervals $p(x \times [\frac{1}{n+1}, \frac{1}{n}])$ have preassigned small diameter. The crucial issue is control of sizes of images in $Y$, hence the “control” terminology.

2.3 The uncontrolled case. Most controlled theorems have older “uncontrolled” versions in which the control space is implicitly taken to be a point. This version of the mapping cylinder question is: when does a point have a neighborhood homeomorphic to a cone? Recalling that we have assumed the complement is a manifold, this can be reformulated as: when is a noncompact manifold the interior of a compact manifold with boundary? The 1-point compactification then plays the role of $X$, and $Y$ is the point at infinity. Theorems of Browder, Livesay and Levine [BLL] and Siebenmann [S] answer this: there is a necessary homotopy-theoretical “tameness” condition, and then an obstruction in algebraic $K$-theory.

We expand on the $K$-theory part. When the tameness condition is satisfied we can predict the fundamental group of the boundary of the neighborhood: the groups $\pi_1(U - Y)$ indexed by the inverse system of neighborhoods $U$ of $Y$ converges nicely to a finitely presented group $\pi$. In the course of the construction of actual boundaries a finitely presented projective module over the group ring $Z[\pi]$ is encountered. If this projective module is stably free the construction can be continued to give the desired structure. The obstruction is therefore essentially the class of this module in the group of stable equivalence classes of projective modules, $K_0(Z[\pi])$. This is a little too big: $K_0$ records the rank of the module, which is irrelevant to the topology. The actual obstruction group is the reduced group $\tilde{K}_0$, defined to be either the cokernel of the inclusion of the trivial group, $K_0(Z[\{1\}]) \to K_0(Z[\pi])$, or the kernel of the rank homomorphism $K_0(Z[\pi]) \to Z$.

2.4 Tameness. Mapping cylinder neighborhoods have special homotopy properties. The eventual result is that certain of these actually characterize mapping cylinders, modulo $K$-theory problems. We describe these.

The first property is that the neighborhood deformation retracts to $Y$, by pushing toward the 0 end of the mapping cylinder. A key feature of this deformation is that the complement of $Y$ stays in the complement until the last instant when everything collapses into $Y$. We formalize this as:

Definition. An embedding $Y \subset X$ is forward tame if there is a map $f: X \times I \to X$ satisfying

1. $f(x, t) = x$ if $t = 0$ or $x \in Y$;
2. $f^{-1}(Y) = Y \times I \cup U \times \{1\}$, where $U$ is some neighborhood of $Y$.

On the other hand we could pull the complement of $Y$ away from $Y$ by pushing toward the other end of the mapping cylinder. This formalizes to:

Definition. An embedding $Y \subset X$ is backwards tame if there is a map $b: (X - Y) \times I \to X - Y$ satisfying

1. $b(x, t) = x$ if $t = 0$; and
2. for every $t > 0$ the closure in $X$ of $b((X - Y) \times \{t\})$ is disjoint from $Y$.

Putting these together we say:

Definition. An embedding $Y \subset X$ is tame if it is both forward and backward tame.
Quite a bit is known about tameness. For instance if the embedding has finitely presented constant local fundamental groups (see below) then there are homological characterizations, and forward and backward tameness are equivalent because their homological formulations are Poincaré dual [QS, 2.14]. If the embedding has trivial local fundamental groups then it is always tame because the homological conditions are implied by the ANR hypotheses and excision [QS, 2.12]. See [HR] for a treatment in the nonmanifold case.

2.5 Homotopy links. One of the main applications of tameness is to give a comparison of the embedding with a “universal” mapping cylinder constructed using the homotopy link.

Definition. The homotopy link of $Y \subset X$, denoted $\text{holink}(X, Y)$, is a subset of the space of paths in $X$ with the compact-open topology. Specifically it consists of the paths $s: [0, 1] \to X$ with $s^{-1}(Y) = \{0\}$. Evaluation at 0 gives a map $\text{ev}_0: \text{holink}(X, Y) \to Y$. The whole evaluation map is $\text{holink}(X, Y) \times I \to X$, and continuity implies this factors through a map on the mapping cylinder; $\text{ev}: \text{cylinder}(\text{ev}_0) \to X$. This preserves complements and is the identity on $Y$, and in fact is the universal such map from a mapping cylinder.

Now suppose $Y$ has a mapping cylinder neighborhood $N \simeq \text{cylinder}(q)$, with map $q: \partial N \to Y$. Since the homotopy link cylinder is universal, the geometric one factors through it. More explicitly, each point in $\partial N$ determines a cylinder arc in $N$. These arcs are points in the homotopy link so define a map $\partial N \to \text{holink}(X, Y)$. This extends to a map of mapping cylinders. Further, $\partial N \to \text{holink}(X, Y)$ turns out to be a controlled homotopy equivalence over $Y$, so the homotopy link provides a homotopy model for any geometric mapping cylinder neighborhood.

Some of this last construction can be done using forward tameness in place of an actual mapping cylinder. Suppose $f: X \times I \to X$ is a forward-tameness deformation, and $U$ is a neighborhood of $Y$ with $f(U \times \{1\}) \subset Y$. Then the arcs $f: \{x\} \times I \to X$ for $x \in U - Y$ define points in the homotopy link. This defines a map $U - Y \to \text{holink}(X, Y)$. Using this in the first coordinate and distance from $Y$ in the second gives a map to the universal mapping cylinder, $U \to \text{cylinder}(\text{ev}_0)$. When $Y$ is also backwards tame this map is in an appropriate sense a controlled local equivalence near $Y$. Tameness therefore encodes essentially the same local homotopy information as a mapping cylinder neighborhood.

2.6 Controlling local fundamental groups. In standard (uncontrolled) geometric topology the fundamental group plays a central role. Roughly this is because algebraic topology is effective with 1-connected spaces, and general spaces are made 1-connected by taking universal covers. In controlled topology the same principle applies, but fundamental groups cannot be used directly because (among other problems) their definition depends on choices of basepoints. Instead we use comparisons with reference spaces.

The general setting is a reference map $p: E \to Y$, the controlled thing being studied, $W \to Y$, and a map $f: W \to E$ that is required to commute with maps to $Y$ up to some error $\delta$. $f$ is said to be $(\delta, 1)$-connected if given a relative 2-complex
(K, L) and a δ-commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
L & \longrightarrow & W \\
\downarrow \subset & & \downarrow f \\
K & \longrightarrow & E
\end{array}
\end{array}
\]

then there is an extension K → W whose composition into Y is within δ of K → E → Y. When this is satisfied W and E have the same local fundamental group behavior over Y, even if “local fundamental groups” do not make sense.

The importance of using reference maps to control π₁ increases with increased complexity of local π₁ behavior. If the geometric situation is locally 1-connected over Y then no π₁ control is needed. If the local fundamental groups are constant then a locally 1-connected covering space can be used. If local fundamental groups are locally constant over Y then we can use covering spaces of inverse images of open sets in Y. But now the situation starts getting complicated: in geometric constructions we are controlling sizes, so we need to know these open sets are fairly large. In fact we need a priori estimates on these sizes so geometric data can be chosen small in comparison. The simplest way to do this is to control local π₁ using a fixed reference map E → Y. In this way whatever size data we need is determined by the reference map, and doesn’t have to be made explicit to be controlled. In the most general situation local fundamental groups change from place to place. This is easily encoded using reference maps and awful to do with groups.

2.7 Stratified systems of fibrations. In the previous section we described reference maps as a way to avoid the awkwardness of group formulations of local π₁ structure. However geometric constructions do use group formulations. Core steps of proofs are usually done assuming constant local fundamental groups and using locally 1-connected covers. General cases are obtained from this by fitting together locally constant pieces. Thus the general π₁ control apparatus is not intended to feed directly into core proofs, but to formulate general hypotheses that in proofs inductively reduce to constant cases. “Stratified systems of fibrations” [QE2] work well for this.

Definition. Suppose p: E → Y is a map, and Y = Y^n ⊃ Y^{n-1} ⊃ ⋯ ⊃ Y^0 is a filtration by closed subsets. p is a stratified system of fibrations (with filtration Y^*) if

1) the restriction to each of the strata,

\[p^{-1}(Y^i - Y^{i-1}) \cong Y^i - Y^{i-1}\]

is a fibration, and

2) each term in the filtration is a p-NDR. This means there is a neighborhood of Y^i, a deformation of it into Y^i in Y that preserves strata until the very end, and this deformation is covered by a deformation of the inverse image in E.

Lots is known about these. There are many examples, reductions to apparently weaker data, constructions, etc., see [QS, CS, H].

In the mapping cylinder context the tameness hypothesis provides us with a canonical reference map, the homotopy link. Local π₁ hypotheses are formulated in
terms of this. Standard procedure (see the statement in 2.10) is to assume there is a stratified system of fibrations $E \to Y$ and a map $\text{holink}(X, Y) \to E$ that is locally 1-connected over $Y$. In many applications the homotopy link itself is a stratified system of fibrations.

2.8 Homology. The mapping cylinder problem has obstructions lying in locally finite homology with coefficients in a spectral cosheaf. This sounds complicated but is actually good news: nothing simpler could work; it is reasonably accessible to calculation; and the formal properties alone have important applications. In this section we outline the general setup developed in [QE2]. We assume general familiarity with the use of spectra to construct homology theories.

The basic setting is a spectrum-valued functor of maps with locally-compact target. In more detail, the domain of this functor is the category with objects maps $p: E \to B$, with $B$ a locally compact metric space. Morphisms are pairs of maps $(F, f)$ forming a commutative diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B_1 & \xrightarrow{f} & B_2
\end{array}
$$

and so that $f$ is proper (inverse images of compact sets are compact). In the application $B$ is the control space where sizes are measured, and $E$ serves to control local fundamental groups as in Section 2.5.

We explain the local compactness hypothesis. The technical work concerns manifolds mapping to $E$. We work over small open sets in $B$, and the inverse image must have compact closure in the manifold. To get this we assume the map from the manifold to $B$ is proper. But then we have to restrict to proper maps of $B$ to preserve this property. We cannot simply require the manifold to be compact because we need a restriction operation that destroys compactness. If $U \subset B$ is an open set then restriction to inverse images of $U$ gives a map from manifold gadgets over $B$ to ones over $U$. Even if we start with a compact manifold over $B$ the result will usually be noncompact (but proper) over $U$.

The homology of $X$ with coefficients in a spectrum $J$ is the spectrum $X \wedge J$. In more detail, $J$ is a sequence of spaces $J_n$ with various maps. Start with the sequence of spaces $X \times J_n$, divide out $X$ times the basepoint in $J_n$. The maps for $J_n$ then give this sequence of spaces the structure of a suspension spectrum. The associated spectrum is $X \wedge J$. We also denote this by $H(X; J)$. Note this is a spectrum; the homology groups are the homotopy groups of this spectrum

$$H_i(X; J) = \pi_i(H(X; J)).$$

Note also that (unlike ordinary homology) these groups may be nontrivial for $i < 0$.

The locally-compact wrinkle in the theory requires us to work with locally finite homology. This is essentially the relative homology of the 1-point compactification.

The “Atiyah-Hirzebruch” spectral sequence (due originally to G. Whitehead) is a spectral sequence of the form

$$E^2_{i,j} = H_i(X; \pi_j J) \Rightarrow H_{i+j}(X; J).$$
From this one sees, for example, that \( H_j(X; J) \) always vanishes for \( j < j_{\text{min}} \) exactly when \( \pi_j(J) = 0 \) for \( j < j_{\text{min}} \), and that groups near the vanishing line are quite accessible. This turns out to be very useful in applications.

We now return to the context of a spectrum-valued functor \( J(p) \), defined on the category of maps \( p \) with locally compact metric range spaces. In this case we can define a “sheaf” generalization of the homology construction. Suppose \( p : E \to Y \) is a map in the category. We can apply the functor fiberwise to get a spectrum \( J(p^{-1}(y) \to y) \) over each \( y \in Y \). With mild additional assumptions we can fit these together to get a “spectral cosheaf” over \( Y \). This is a sequence of spaces \( J_n(p^{-1}(\#)) \) with maps to \( Y \) and maps to each other making the fibers over \( Y \) into spectra. In the constant-coefficient case \( F \times Y \to Y \) this just gives \( J_n(F \to pt) \times Y \).

By analogy with the constant-coefficient case we define homology with coefficients in this cosheaf by first dividing out the 0-section of each \( J_n(p^{-1}(\#)) \to Y \), then taking the spectrum associated to the resulting suspension spectrum. We use the notation \( H(Y; J(p^{-1}(\#))) \) for this spectrum, and \( H_* \) for its homotopy groups.

Again we actually need locally-finite homology. The spectrum for this is obtained by adding a point at infinity to the spectral cosheaf, over the point at infinity in the 1-point compactification of \( Y \). Then divide out the 0-section and proceed as before. If \( Y \) is already compact this does not change the homology.

There is a generalization of the Atiyah-Hirzebruch spectral sequence to the non-constant coefficient case. Namely in the situation of the previous paragraph we get

\[
E^2_{i,j} = H_{i}(X; \pi_j J(p^{-1}(\#))) \implies H_{i+j}(X; J(p^{-1}(\#))),
\]

where the groups on the right are “ordinary” cosheaf homology groups. Again these are reasonably accessible near the vanishing line for the coefficient spectra.

The is a useful extension of the spectral cosheaf construction. Suppose, as before, that \( E \to Y \) is a map in the domain of the functor, but now assume also that \( f : Y \to Z \) is a proper map. Then we can construct a spectral cosheaf over \( Z \) by applying the functor to inverses under \( f \). More explicitly, over a point \( z \in Z \) we put the spectrum \( J(p^{-1}(f^{-1}(z)) \to f^{-1}(z)) \). As before we can define a homology spectrum by dividing out 0-sections and taking the associated spectrum. The output of this construction is denoted by \( H(Z; J(p^{-1}(f^{-1}(\#)))) \). The notation is a bit tricky. Note we can do the previous construction to the composition \( fp \) and get a spectral cosheaf denoted \( (fp)^{-1}(\#) \). This is different from the cosheaf just constructed, though in some cases they have the same homology.

### 2.9 Assembly maps.

We continue with the terminology of the previous section. Suppose \( p : E \to Y \) is an object in the category of “proper maps to locally compact spaces”. Then for each \( y \in Y \) the inclusion

\[
\begin{array}{ccc}
p^{-1}(y) & \longrightarrow & E \\
\downarrow p & & \downarrow p \\
y & \hookrightarrow & Y
\end{array}
\]

is a morphism in the category. Applying \( J \) gives maps from fibers of the spectral cosheaf into \( J(p) \). Under mild continuity hypotheses these fit together to give a map on the total space of the cosheaf. Since the target of this map is a spectrum
the map factors through the associated spectrum of the total space to define a map of spectra

$$H(Y; J(p^{-1})) \to J(p).$$

This is the “general nonsense” description of the assembly map. In special cases there are other descriptions that may give better understanding.

We will make use of the functoriality of assembly maps. Suppose there is a morphism

$$\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_2 \\
\downarrow^{p_1} & & \downarrow^{p_2} \\
Y_1 & \xrightarrow{f} & Y_2
\end{array}$$

in the domain category of $J$. Then following diagram of spectra commutes:

$$\begin{array}{ccc}
H(Y_1; J(p_1^{-1})) & \xrightarrow{H(f)} & H(Y_2; J(p_2^{-1})) \\
\downarrow & & \downarrow \\
J(p_1) & \xrightarrow{} & J(p_2)
\end{array}$$

The top map also factors through the mixed homology $H(Y_2; J(f^{-1}(p_2)))$.

2.10 $K$-theory. The previous section gives the context for homological obstructions. In this section we discuss particular functors used to make contact with the topological problems.

The logical context for the next theorem is that geometric-topological techniques can be used to formulate obstruction groups for controlled problems (see Section 3). This tells us what they are good for, but says very little about their nature. The next theorem provides another description that displays global properties. This is incorporated in the final statement in §3.1.

**Controlled assembly isomorphism theorem.** There is a spectrum-valued functor $S$ defined on maps to locally compact metric spaces, such that if $p: E \to Y$ is a stratified system of fibrations over a locally compact finite-dimensional ANR then

1. $\pi_0S(p)$ is the obstruction group for mapping cylinder neighborhoods of $Y$ with local fundamental groups modeled on $p$, and
2. the assembly map $H^{lf}(Y; S(p^{-1}(#))) \to S(p)$ is an equivalence of spectra.

First we explore the significance of conclusion (1). Uncontrolled work determines some of the homotopy of the “coefficient” spectra ($Y$ a point):

$$\pi_i(S(F \to pt)) = \begin{cases} 
Wh(\pi_1 F) & \text{when } i = 1; \\
K_0(Z[\pi_1 F]) & \text{when } i = 0, \text{ and} \\
K_{-i}(Z[\pi_1 F]) & \text{when } -i < 0.
\end{cases}$$

The $i = 1$ case comes requiring the same spectrum to work for h-cobordisms, $i = 0$ is the uncontrolled end theorem (Siebenmann, see §2.3), and $-i < 0$ comes from seeing Bass’ definition of lower $K$-theory [B] come out of tinkering with Euclidean spaces $Y = \mathbb{R}^i ([PW])$. One can also require a connection to pseudoisotopy and get
The spectral sequence shows the higher homotopy plays no role in the obstructions of interest, so we don’t particularly care what it is.

There are many constructions of spectra encoding lower $K$-theory, and many of these extend to spectrum-valued functors satisfying condition (1) of the theorem. Conclusion (2), which is the source of the real power of the theory, is much more delicate.

The proof of (2) follows the proof of uniqueness of homology, i.e. a morphism of homology theories that induces an isomorphism on homology of a point is an isomorphism on finite-dimensional ANRs. The proof proceeds by induction using exact sequences, first establishing isomorphism for spheres, then finite CW complexes, then (by a trick) ANRs. To prove (2) this way we (i) show the right side $(\mathcal{S}(p))$ satisfies appropriate versions of the axioms of homology in the $Y$ variable; (ii) observe that the map gives isomorphisms over points by definition; and (iii) make minor adjustments to incorporate the “coefficient system” (reference map $p$). The hard part of this is (i). The axioms are homotopy; excision; and a fibration condition for pairs. The fibration hypothesis is the spectrum version of the long exact sequence of homology groups of a pair: the long exact sequence is the homotopy sequence of the fibration. Technically since we are working with locally finite homology the pair axiom is replaced by a condition on restrictions to open sets, but it amounts to the same thing.

Several conclusions can be drawn from this outline. First, we may not care about the higher homotopy of the spectrum but Nature does. The inductive plan of the proof only works if all the groups line up correctly, so we have to get them right whether we want to use them or not. Second, the proof is all-or-nothing. Again because it depends on an induction it either works or fails, and there are no interesting partial results when it fails. Finally, the bottleneck in such arguments is usually the excision axiom. Recall that this is the one that fails for homotopy groups, and thus enables homotopy to be so much more complicated than homology.

The theorem is proved in [QE2] with a spectrum $\mathcal{S}$ constructed using pseudoisotopy of manifolds. This is a version of what is now known as $A$-theory or Waldhausen $K$-theory. For current topological applications one such spectrum is enough. However the proof in [QE2] is complicated and not too clear, so there have been efforts to find other proofs of this key step. Also, a version extending algebraic $K$-theory would have significant applications to algebra. So far this has been not been done: none of the other formulations of Waldhausen $K$-theory and none of the algebraic $K$ constructions (Quillen, Volodin and so on) have been acceptable to the methods of the proof. The author thinks he has a construction for algebraic $K$-theory, but he has thought this before (cf. [QK]) so skepticism is appropriate until details appear.

We offer a philosophical explanation of why the controlled assembly isomorphism theorem is so hard to prove. Frequently a complicated proof is “explained” by the existence of a false similar statement. For example the “reason” Freedman’s topological embedding theorem for 4-manifolds is so hard is that the analogous statement for smooth embeddings is false. The proof must have a topological construction so bizarre that it cannot possibly give a smooth outcome, and must depend on it so essentially that it cannot possibly be avoided. What then is the false thing forcing the isomorphism theorem proof to be so intolerant? The problem is probably in quadratic stable algebra (surgery, $L$-theory). There the geometrically significant lower homotopy groups of the spectrum do have some imprint of strange
behavior in the higher groups. As a result the surgery spectrum constructions now known cannot satisfy the isomorphism theorem. The proof must be delicate enough to reject these impostors. Apparently $K$-theory doesn’t do anything strange enough to deserve the complexity; it is just an innocent victim of problems in surgery.

3. The theorem

Collecting the hypotheses developed in §2, we suppose $X$ is a locally compact finite dimensional metric ANR, $Y \subset X$ is tame, $X - Y$ is a manifold, $p: E \to Y$ is a stratified system of fibrations, and there is a controlled 1-connected map $\text{holink}(X, Y) \to E$.

3.1 Mapping cylinder existence theorem. Under these conditions there is an invariant $q_0(X, Y) \in H^0_{\lf}(Y; S(p^{-1}(\#)))$. This vanishes if $Y$ has a mapping cylinder neighborhood, and conversely if the invariant vanishes and $\dim X - Y \geq 6$ then there is a mapping cylinder neighborhood.

Dimension 5. This is still true in dimension 5 when the local fundamental groups of $p$ have subexponential growth \([FQ, KQ]\).

We outline the proof only well enough to show the major features.

3.2 Nice neighborhoods and the obstruction. The key objective is to find neighborhoods $N$ with the right controlled homotopy type: closed manifold neighborhoods so that $\partial N \to N - Y$ is an $\epsilon$ homotopy equivalence over $Y$. Tameness is the main ingredient. Choose any small manifold neighborhood $N$, and choose handlebody structures. The tameness deformations provide homotopy data to show how to swap handles to make $\partial N \to N - Y$ highly connected. The final step, which would make it a homotopy equivalence, is obstructed. In the uncontrolled case we see a single nonvanishing relative homology group. If it is stably free over the group ring then we can stabilize and swap handles corresponding to a basis to get a good $N$. This module is a direct summand of a finitely generated free chain group, so is finitely generated projective. The obstruction is the equivalence class of the projective module, modulo stably free modules. In other words, its image in $K_0(Z[\pi_1]).$ We indicate modifications needed in the controlled setting. We can’t use homology because this is a quotient and quotients destroy size estimates. Instead we directly use the projection on controlled chain groups. We define $\tilde{K}_0(Y; p, \epsilon, \delta)$ to be free modules over $Y$ with \(Z[\pi_1(p^{-1}(\#))]\) coefficients (this is clarified in §3), with projections of radius $< \delta,$ modulo ones with basis-preserving projections of radius $< \epsilon.$ Adding estimates to the uncontrolled argument gives an element of this set, and shows that if it is trivial then the argument can be completed to get a nice $N$.

We pause the proof to expand on the obstructions. The main point is that although we have an “obstruction”, and can arrange for the set $K_0(Y; p, \epsilon, \delta)$ in which it lies to be a group, we know nothing about it. This is where the characterization theorem of §9 takes over. This shows:

1. these groups are stable in the sense that for every $\epsilon > 0$ there is $\delta > 0$ so that the map from the inverse limit

$$\tilde{K}_0(Y; p, \epsilon, \delta) \leftarrow \lim_{\leftarrow \epsilon} \tilde{K}_0(Y; p, *, *)$$

is an isomorphism; and

2. the inverse limit is the spectral sheaf homology group $H^0_{\lf}(Y; S(p^{-1}(\#)))$. 


The stability in (1) is subtle and actually harder to prove than the description of the limit in (2). For instance in the controlled algebra used here, sizes grow when morphisms are composed. This means morphisms of fixed size do not form a category, and in place of the homological and categorical techniques of the uncontrolled theory we have to work with chain complexes and constantly estimate sizes. In contrast it is possible to set up the inverse limit theory directly so the work takes place in a category \([P]\). The setup is more elaborate, but no estimates are needed and the group \(\lim_{\leftarrow} K_0(Y; p, *, *)\) appears as ordinary \(K\)-theory of a category. In some applications the stability property is essential (see §5). However for mapping cylinders it is not. We have extracted an invariant from a single sufficiently small neighborhood \(N\). But one could repeat the construction at smaller scales to get a sequence of neighborhoods \(N_i\) with estimates going to 0. From this we could extract a sequence of related algebraic objects with estimates going to 0, or in other words an element of the inverse limit. This approach may yet have significant applications. However so far the benefits (convenience for the categorically sophisticated) do not seem to outweigh the drawbacks (weaker theorems, more elaborate setups).

3.3 Getting mapping cylinders. Returning to the proof, we suppose the obstruction vanishes so we can find nice neighborhoods \(N\). Repeat at smaller scales to get a decreasing sequence \(N_i \supset N_{i+1} \ldots\) which are “nice” with decreasing size estimates. Recall that “nice” meant roughly that the inclusion \(\partial N_i \to N_i - Y\) is a controlled homotopy equivalence. It follows that the regions between these are controlled h-cobordisms. Explicitly, the inclusions of \(\partial N_i\) and \(\partial N_{i+1}\) in \(N_i - \text{interior}(N_{i+1})\) are controlled homotopy equivalences. If these h-cobordisms are all products then we can fit together product structures \(\partial N_i \times [\frac{1}{i+1}, \frac{1}{i}] \approx N_i - \text{interior}(N_{i+1})\) to get a product structure \(\partial N_1 \times (0, 1] \approx N_1\). The control on the size of the product structures shows the images of the arcs converge in \(Y\), so this gives a mapping cylinder.

The intermediate regions originally constructed may not be products, but we can use a “swindle” to make them so. If we factor each \(N_i - \text{interior}(N_{i+1})\) as a composition of h-cobordisms \(U_i \cup V_i\), \(N_1\) becomes an infinite union \((U_1 \cup V_1) \cup (U_2 \cup V_2) \cup \ldots\). Reassociating expresses it as \(U_1 \cup (V_1 \cup U_2) \cup \ldots\). The idea is choose the decompositions so the new pieces, \(V_i \cup U_{i+1}\) are all products. In the uncontrolled setting this is a simple consequence of the invertibility of h-cobordisms. The controlled version is not so simple. We want to inductively choose the decomposition \(U_{i+1} \cup V_{i+1}\) so the union \(V_i \cup U_{i+1}\) is a product. But we must maintain finer control on \(U_{i+1}\) than is available on \(V_i\). The argument thus uses stability of h-cobordism obstructions: we need not only that \(V_i\) has some inverse, but that it has one with arbitrarily finer control. This is a deep fact, so this “swindle” is not just a formal argument. As explained above this can be avoided by working with a sequence \(\{N_i\}\) to formulate the obstruction directly as an element of the inverse limit. When this vanishes the intermediate regions are automatically already products.

This completes the sketch of the proof.

3.4 Refinements. We give two refinements that follow from the proof. The first concerns smooth or PL structures, and the second provides a way to recognize mapping cylinders themselves, not just existence of neighborhoods.

Smooth and PL cylinders. If the manifold in Theorem 3.1 has a smooth or PL structure then the mapping cylinder can be chosen to be smooth or PL, in the sense
that the submanifold $N - Y$ is, and the map $\partial N \times (0, 1] \to N - Y$ is a diffeomorphism or PL isomorphism.

The proof of the Theorem uses handlebody theory, which works in any category of manifolds (dim $\geq 4$ in the topological case). Thus the argument and obstructions are category-independent: if $X - Y$ has a smooth structure we get a smooth $N$, etc.

There is also a smoothing and triangulation theory that shows a topological mapping cylinder in a PL manifold can be made PL, and similarly for smoothing. Using this we could deduce the structure refinement from the topological case. The point of observing it directly from the proof is that eventually it is possible to run the argument backwards and derive the smoothing and triangulation structure theory from controlled theorems. In such ways the controlled theory unifies as well as extends the older work.

The structure refinement is not true in dimension 5, no matter how nice the local fundamental groups are.

The second result of the section gives a criterion for $X$ itself to be a mapping cylinder over $Y$. As in the discussion of tameness in 2.4 we extract properties of the radial deformation of a mapping cylinder. If $X = \text{cyl}(g)$ for some map $N \to Y$ then the radial deformation to $Y$ is a map $f: X \times I \to X$ satisfying:

1. $f^{-1}(Y) = Y \times I \cup X \times \{1\}$
2. $f(x, t) = x$ if $t = 0$ or $x \in Y$, and
3. $f(f(x, t), 1) = f(x, 1)$

The last condition means that if we use the time-1 retraction $f_1: X \to Y$ as a control map then the deformation $f$ has radius 0 in $Y$. The criterion relaxes this, requiring only that $f$ has radius less than some appropriate $\delta$. Note that to be useful this “appropriate $\delta$” must be known in advance, before $X$ and $f$ are chosen. Note also that if $Y$ is not compact then this sort of control uses a function $\delta: Y \to (0, \infty)$. Typically these functions go to 0, so provide progressively finer control near the ends of $Y$.

**Mapping cylinder recognition.** Suppose $Y$ is a locally compact finite dimensional ANR, $p: E \to Y$ is a stratified system of fibrations, and a dimension $n \geq 6$ is given. Then there is $\delta > 0$ so that if

1. $X \supset Y$ with $X - Y$ a manifold (with boundary) of dimension $n$;
2. there is a map $\text{holink}(X, Y) \to E$ that is $(\delta, 1)$-connected over $Y$;
3. $f: X \times I \to X$ is a proper deformation retraction of $X$ to $Y$ that preserves the complement of $Y$ when $t < 1$, and $f_1 f$ has radius $< \delta$ in $Y$.
4. the inclusion $\partial X \subset X - Y$ is $(\delta, 1)$-connected over $Y$, using $f_1$ as control map.

Then there is a map $g: \partial X \to Y$ and a homeomorphism $\text{cyl}(g) \to X$ that is the identity on the boundary and $Y$.

We can further arrange for the radial deformation in $\text{cyl}(g)$ to be close to the deformation $f$. More specifically choose $\epsilon > 0$ along with $Y$, $p$, and $n$. Then there is
a choice of $\delta$ so we can get $g$ and homeomorphism $h: \text{cyl}(g) \to X$ with the diagram

$$
\begin{array}{c}
\text{cyl}(g) \times I \\
\downarrow \text{cyl. projection} \\
Y
\end{array}
\xrightarrow{h \times \text{id}}
\begin{array}{c}
X \times I \\
\downarrow f_1, f \\
Y
\end{array}
\xrightarrow{\text{id}} Y
$$

commutative within $\epsilon$.

The hypotheses collected in this theorem encode the properties of the “nice neighborhoods” used in the proof of the theorem of 3.1. The part of the proof outlined in 3.2 shows the obstruction vanishes if and only if nice neighborhoods exist. The argument in 3.3 then proves the theorem stated here, that nice neighborhoods are mapping cylinders.

We remark on the role of stability. The argument in 3.3 requires finding a descending sequence of nice neighborhoods with size parameters going to 0. These exist because the obstructions are stable (the same at all sufficiently small scales), and the initial nice neighborhood is chosen to have size in the stable range so it’s existence shows the obstructions are trivial. The abstract existence of mapping cylinder neighborhoods can be formulated directly in terms of the inverse limit, avoiding stability. However the recognition theorem is not accessible to this approach because the existence of a single nice neighborhood does not show the vanishing of the obstruction.

4. Applications

The applications detailed here are briefly described in the introduction.

4.1 Mapping cylinders in manifolds [QE1]. Suppose $Y \subset M$ is a closed embedding with locally 1-connected complement, of an ANR in the interior of a manifold of dimension $\geq 5$. Then $Y$ has a mapping cylinder neighborhood.

To derive this from the main theorem we must show tameness and vanishing of the obstruction. The homological characterization of forward tameness follows from excision and triviality of local fundamental groups. Backwards tameness follows from this. Since the local fundamental groups are trivial we can use the identity $Y \to Y$ as the control map. The main theorem identifies the obstruction as lying in $H^j_0(Y; S(id))$. The spectral sequence for this has $E^2$ terms $H^j_0(Y; \tilde{K}_0(Z))$ and $H^j_{i-1}(Y; K_{-i}(Z))$ for $i > 0$. But $\tilde{K}_0$ and the lower $K$-theory of $Z$ is all trivial, so the obstruction group is trivial. Therefore a mapping cylinder exists.

4.2 Finiteness of ANRs. A compact finite-dimensional ANR is homotopy equivalent to a finite complex.

This statement is the “Borsuk conjecture”, and was proved for all compact ANRs (not necessarily finite-dimensional) by J. West. The finite-dimensional case follows from the previous result as follows: a compact finite-dimensional ANR has an embedding in some Euclidean space. If this is has locally 1-connected complement then there is a mapping cylinder neighborhood. The neighborhood is a manifold (smooth, actually) so is a finite complex. The mapping cylinder projection is a homotopy equivalence. What if the embedding does not have locally 1-connected complement? The inclusion into $R^{n+1}$ has locally 0-connected complement, and if
an embedding has locally 0-connected complement then the inclusion into $\mathbb{R}^{n+1}$ has locally 1-connected complement. Thus increasing dimension by two always makes the complement locally 1-connected.

4.3 Collaring in homology manifolds. Suppose $M$ is an ANR homology manifold of dimension $\geq 5$, the embedding $\partial M \subset M$ has locally 1-connected complement, and $M - \partial M$ is a manifold. Then there is a mapping cylinder neighborhood of $\partial M$. There is a collar (neighborhood homeomorphic to $(\partial M) \times I$) if and only if $(\partial M) \times \mathbb{R}$ is a manifold.

The inclusion of the boundary of a homology manifold is homologically locally infinitely-connected, but local fundamental groups may be nontrivial. For instance closure of the strange component of the complement of the Alexander horned sphere in $S^3$ is a homology manifold with non-locally 1-connected complement of the boundary. With the 1-connected hypothesis the proof of mapping cylinders is the same as the previous theorem with a little modification of the proof of forward tameness.

We give some context for the collaring statement. First, the map in the mapping cylinder must be a resolution (map from a manifold to a homology manifold that is a local homotopy equivalence, see §4). Edwards’ resolution theorem is that when the dimension is $\geq 5$ this map can be approximated by a homeomorphism if and only if the homology manifold has the “disjoint 2-disk property.” If $N$ is a homology manifold then an easy argument shows $N \times \mathbb{R}^2$ has the disjoint 2-disk property so $N$ resolvable implies $N \times \mathbb{R}^2$ is resolvable, and therefore a manifold. It is one of the outstanding conjectures in the area that $N \times \mathbb{R}$ is already a manifold. The theorem shows this conjecture is equivalent to existence of collars of boundaries in certain homology manifolds.

The proof of the collaring statement follows from the fact that the mapping cylinder map is a resolution, so Edwards’ theorem shows the product with $\mathbb{R}$ can be approximated by a homeomorphism if and only if $(\text{partial} M) \times I$ is a manifold.

4.4 Stratified spaces. Many interesting spaces are not manifolds, but are built of manifold pieces. These include algebraic varieties, stratifications coming from singularities, and polyhedra. Generally a “stratified space” has a closed filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ and the strata $X_i - X_{i-1}$ are required to be manifolds. There are several versions that differ in the way the strata fit together. The geometric versions (Whitney, Thom, and PL) have mapping cylinder neighborhoods and complicated relations among them as part of their structure. The most successful topological version, homotopy, or “Quinn” stratified spaces [QS], were identified as an outgrowth of controlled topology and have local homotopy conditions relating the strata. In these the strata may not have mapping cylinder neighborhoods, and the obstruction is exactly the one identified in 2.10.

More specifically, suppose $X$ is a homotopy stratified space in the sense of [QS]. Then more-or-less by definition

- the embedding $X_{i-1} \subset X_i$ is tame;
- the projection of the homotopy link $\text{ev}_0 \colon \text{holink}(X_i, X_{i-1}) \to X_{i-1}$ is a stratified system of fibrations; and
- $X_i - X_{i-1}$ is a manifold.

Thus we conclude there is an obstruction in $\mathcal{H}^0_0(\text{X}_{i-1}; S(\text{ev}_0^{-1}(\#)))$ to the existence of a mapping cylinder neighborhood of $X_{i-1}$ in $X_i$. Vanishing of the obstruction
implies existence of such a neighborhood if either $\dim X_i \geq 6$ or $\dim X_i = 4$ and fundamental groups of point-inverses in the homotopy link are “good.”

We note that the fact that existence is obstructed means that mapping cylinders are not the natural local structure in these spaces. A weaker version developed by Hughes and others seems to be correct, see §3.5.

4.5 Topological actions of finite groups. Suppose a finite group $G$ acts on a manifold $M$. We can filter the quotient $M/G$ by orbit types: images of points lie in the same stratum of the quotient if their isotropy subgroups are conjugate. If the action is smooth or PL then the quotient is a smooth or PL stratified space, though the stratification may not be exactly the orbit type stratification. If the action is just topological then really awful point-set things can happen in the quotient. A nice compromise is the class of “homotopically stratified” actions [QS], where the quotient with orbit type filtration is assumed to be homotopically stratified in the sense discussed above. This rules out weird point-set behavior but allows many other things. For instance these can have mapping cylinder problems.

An “equivariant mapping cylinder neighborhood” of a $G$-invariant subset of $M$ is just what it sounds like: a mapping cylinder structure invariant under the action of $G$. The quotient is an ordinary mapping cylinder neighborhood of the quotient subset. Therefore Theorem 3.1 can be applied in the quotient to determine the existence of equivariant mapping cylinder neighborhoods. We discuss the easiest case, neighborhoods of the non-free points.

**Theorem.** Suppose the finite group $G$ acts in a homotopically-stratified way on a compact manifold, let $Y \subset M$ be the points not moved freely by $G$, and suppose the codimension of $Y$ is $\geq 3$. Then

1. there is a stratified system of fibrations $p: B_{G_x} \to Y/G$ whose fiber over $x \in Y$ is the classifying space of the subgroup $G_x \subset G$ fixing $x$;
2. there is an obstruction in $H_0(Y/G; S(p^{-1}(\#)))$ to the existence of an equivariant mapping cylinder neighborhood of $Y$; and
3. if there is an equivariant cylinder neighborhood of $Y \cap \partial M$ in $\partial M$ and $\dim M \geq 5$, then the obstruction vanishes if and only if there is an extension of the boundary structure to an equivariant mapping cylinder neighborhood of all of $Y$.

The hypothesis that $Y$ has codimension is at least 3 implies the embedding $Y \subset M$ has locally 1-connected complement. Therefore local fundamental groups in the quotient come from the the group action, and are modeled by the isotropy groups described in the theorem. The obstructions are often quite accessible:

1. $K_{-i}(Z[H]) = 0$ for finite groups $H$ and $-i \leq -2 |C|$. Therefore the spectral sequence for the obstruction group has $E^2$ terms only $H_0(Y/G; K_0(Z[G_\#]))$ and $H_1(Y/G; K_{-1}(Z[G_\#]))$. (these are group, not spectral, cosheaf homology groups);
2. if $Y/G$ is connected and there is a point fixed by $G$ then the $H_0$ term reduces to $K_0(Z[G])$; and
3. there is an action of a finite group on a disk that is smooth on the boundary and locally linear (in fact can be smoothed in the complement of any fixed point), but the non-free set does not have an equivariant mapping cylinder neighborhood because the $\tilde{K}_0(Z[G])$ part of the obstruction is nontrivial [QE2].
4.6 Topological regular neighborhoods. Mapping cylinders are wonderful, but since they do not always exist they are not satisfactory objects for a topological theory of “regular neighborhoods.” The appropriate notion seems to be a skewed mapping cylinder neighborhood in $X \times [0, \infty)$.

**Definition.** Suppose $X$ is locally compact and $Y \subset X$ is closed. A *topological regular neighborhood* of $Y$ consists of

1. an open neighborhood $U$ of $Y$,
2. a proper map $q: U \to Y \times [0, 1)$ that is the identity $Y \to Y \times \{0\}$ and preserves complements of these sets,
3. a homeomorphism of the relative mapping cylinder $\text{cyl}(q, \text{id}_Y)$ with a neighborhood of $Y \times \{0\}$ in $Y \times [0, \infty)$ that is the identity on $Y \times [0, 1)$ and $U \times \{0\}$.

The relative mapping cylinder is the ordinary mapping cylinder with the cylinder arcs in the subset $Y \times I$ identified to points. The result contains a copy of $Y$, and the complement of this is the mapping cylinder of the restriction of $q$ to $U - Y \to Y \times (0, 1)$. The homeomorphism in (3) takes the cylinder arcs to arcs that start on $U \times \{0\}$ and go diagonally to $Y \times [0, 1)$, see the figure.

The idea is that we may not be able to find mapping cylinder neighborhoods because there is an obstruction to finding an appropriate domain for the map. So we use the neighborhood itself as the domain for a map, and get a mapping cylinder in the next higher dimension.

**Existence of regular neighborhoods.** Suppose $X$ is a locally compact ANR, $Y \subset X$ is tame, and there is a map, controlled 1-connected over $Y$, from the homotopy link of $Y$ in $X$ to a stratified system of fibrations over $Y$. Finally suppose $X - Y$ is a manifold of dimension $\geq 5$. Then there is a topological regular neighborhood of $Y$ in $X$.

As usual this also holds for $X$ of dimension 4, provided the local fundamental groups have subexponential growth.

Before indicating the proof we discuss some of the structure of these neighborhoods. Suppose $B \subset A$ has a mapping cylinder neighborhood with map $q: U \to B$. The homotopy link is in a sense universal for mapping cylinders mapping to $(A, B)$, so the cylinder structure defines a map $U \to \text{holink}(A, B)$. This is an “approximate fiber homotopy equivalence” over $Y$ [QS, 2.7]. In the regular neighborhood situation the homotopy link of $Y \times (0, 1) \subset X \times (0, 1)$ is the pullback of the homotopy link of $Y \subset X$. Thus the regular neighborhood structure gives an approximate fiber homotopy equivalence over $Y \times (0, 1), U \to \text{holink}(X, Y) \times (0, 1)$.
In the important special case where $X$ is a homotopically stratified space as in §4.4 and $Y$ is the union of the lower strata, these regular neighborhoods are the same as the “approximate tubular neighborhoods” developed by Hughes [H], and earlier in special cases by Hughes, Taylor, Weinberger and Williams [HTWW]. In a stratified set $\text{holink}(X, Y) \to Y$ is itself a stratified system of fibrations. Thus $q: U - Y \to Y \times (0, 1)$ is approximately fiber homotopy equivalent to a stratified system of fibrations. This identifies $q$ as a “manifold stratified approximate fibration” over $Y \times (0, 1)$.

Finally we outline how the theorem follows from a version of the mapping cylinder recognition theorem of 3.4. Let $h: X \times [0, 1] \to X$ be a forward-tame deformation, and let $V$ be a neighborhood of $Y$ such that $h(V \times \{1\}) \subset Y$. Define $f: X \times [0, \infty) \times [0, 1] \to X \times [0, \infty)$ by

$$f(x, s, t) = (h(x, t), s + td(x, Y))$$

where $d(x, Y)$ is the distance from the point to the subspace $Y$. Then properties of $h$ imply

1. $f$ is a homotopy from the identity at $t = 0$ to a map at $t = 1$ that takes $V \times [0, \infty)$ into $Y \times [0, \infty)$;
2. when $t < 1$ $f(\#, t)$ takes the complement of $Y \times [0, \infty)$ into itself;
3. $f(\#, t)$ is the identity on $Y \times [0, \infty)$;
4. $f(\#, t)$ takes the complement of $Y \times \{0\}$ into itself for all $t$.

Delete $Y \times \{0\}$, then this is a homotopy of $(X - Y) \times \{0\} \cup X \times (0, \infty)$. We would like to arrange it to satisfy the conditions of the mapping cylinder recognition theorem, to get a mapping cylinder over $Y \times (0, \infty)$. This cannot be done completely: the end near $Y \times \{0\}$ is ok, but none of the conditions hold near $\infty$. Instead we use a relative version: if the conditions hold over $Y \times (0, 1)$ then some neighborhood of $Y \times \{0\}$ is a mapping cylinder over $Y \times (0, 1 - \epsilon)$. Such relative versions are standard parts of controlled theory (see e.g. the remarks before Theorem 1.3 in [Q2]). They are often not stated explicitly because the statements are so complicated, but follow from the proofs. The actual goal is thus to arrange the conditions of the recognition theorem to hold over $Y \times (0, 1)$.

Recall that we want to use $f_1$ as the control map. The first problem is that $f_1$ does not even map all of the space into $Y \times (0, \infty)$. However this does work near 0. Suppose the neighborhood $V$ taken into $Y$ by $h$ contains the points within $\epsilon$ of $Y$. Then $f$ does deform all of $f_1^{-1}(Y \times [0, \epsilon))$ into $Y \times (0, \infty)$. Restrict to this, in the sense that we consider the space $f_1^{-1}(Y \times [0, \epsilon))$ with control map $f_1$ over $Y \times [0, \epsilon)$. Reparameterize $[0, \epsilon)$ as $[0, \infty)$. The situation is now that $f_1$ can serve as a control map. It is also proper. We have lost something: since the original deformation did not preserve $f_1^{-1}(Y \times [0, \epsilon)$, the restriction does not define a deformation of the space. However since $Y \times \{0\}$ is left fixed, by continuity $f$ keeps some $f_1^{-1}(Y \times [0, \tau)$ inside the new space. Reparameterize the interval again to arrange the deformation to be defined on $f_1^{-1}(Y \times [0, 2))$. We now have the control map and deformation defined over $Y \times [0, 2)$.

The last step is to arrange arbitrarily good size control, at least over $Y \times (0, 1)$. The deformation $f$ is the identity on $Y \times \{0\}$, or in other words the composition $f_1 f$ has radius 0 as a homotopy of $Y \times \{0\}$ in itself. It follows that $f_1 f$ has very small radius over $Y \times [0, \epsilon)$, for small $\epsilon$. By reparameterizing $[0, \epsilon)$ to $[0, 2)$ we can arrange that $f_1 f$ has arbitrarily small radius in the $Y$ coordinate over $Y \times [0, 2)$. 


We need to do a little better. We are controlling over the 0 end of $Y \times (0, \infty)$, which is non-compact even if $Y$ is compact. Assume $Y$ is compact to simplify the argument, then the control objective is a continuous function $\delta : (0, \infty) \to (0, \infty)$, not a constant $\delta > 0$. Elaborate the previous argument: since $f_1 f$ has radius 0 over $Y \times \{0\}$, there is a continuous increasing function $\epsilon : [0, \infty) \to [0, \infty)$ taking 0 to 0, so that $f_1 f$ has radius $< \epsilon$ in the $Y$ coordinate, over $Y \times [0, 2)$. Now reparameterize by a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ so that $\epsilon \theta < \delta$. The result is $\delta$ controlled in the $Y$ coordinate. It remains to get control in the $[0, \infty)$ coordinate. This is again a standard argument using continuity and reparameterization.

The outcome of all this is a neighborhood of $Y \times \{0\}$ in $X \times [0, \infty)$, a control map to $Y \times [0, \infty)$, and a deformation defined over $Y \times [0, 2)$ and satisfying the control needed for the Recognition Theorem over $Y \times [0, 1)$. The theorem then asserts that there is a mapping cylinder structure provided the dimension of $X \times [0, \infty)$ is at least 6, or in other words if $X$ has dimension at least 5, or $X$ has dimension 4 and the local fundamental groups are small.

References

[B] H. Bass, *Algebraic K-theory*, W. A. Benjamin, 1968.
[BLL] W. Browder, J. Levine, G. R. Livesay, *Finding a boundary for an open manifold*, Amer. J. Math. 87 (1965), 1017–1028.
[CS] S. Cappell and J. Shaneson, *The mapping cone and mapping cylinder of a stratified map*, Annals of Math. Studies 138 (1995), 58–66.
[C] D. Carter, *Lower K-theory of finite groups*, Comm. Algebra 8 (1980), 1927–1937.
[FQ] Michael Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton University Press, 1990.
[H] Bruce Hughes, *The approximate tubular neighborhood theorem*, Ann. Math. (to appear).
[HR] B. Hughes and A. Ranicki, *Cambridge Tracts in Math. 123* (1996).
[HTWW] B. Hughes, L. Taylor, B. Williams, S. Weinberger, *Neighborhoods in stratified spaces with two strata*, Topology 39 (2000), 873–919.
[KQ] V. S. Krushkal and F. Quinn, *Subexponential groups in 4-manifold topology*, Geom. Topol. 4 (2000), 407–430.
[P] E. Pedersen, *Bounded and continuous control*, London Math. Soc. Lecture Notes 227 (1995), Cambridge Univ. Press, Cambridge, 277–284.
[PW] E. Pedersen and C. Weibel, *K-theory homology of spaces*, Springer Lecture Notes in Math. 1370 (1989), 346–361.
[QE1] Frank Quinn, *Ends of maps, I*, Ann. Math 110 (1979), 275–331.
[QE2] --------, *Ends of maps, II*, Invent. Math. 68 (1982), 353–424.
[QE4] --------, *Ends of maps, IV: Controlled pseudoisotopy*, American J. Math. 108 (1986), 1139–1162.
[QS] --------, *Homotopically stratified sets*, J. Am. Math. Society 1 (1988), 441–499.
[QR] --------, *Algebraic K-theory of poly-(finite or cyclic) groups*, Bull. American Math Soc. 12 (1985), 221–226.
[S] Lawrence Siebenmann, *Thesis*, Princeton University (1965).