INNERNESS OF CONTINUOUS DERIVATIONS ON ALGEBRAS OF MEASURABLE OPERATORS AFFILIATED WITH FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. This paper is devoted to derivations on the algebra $S(M)$ of all measurable operators affiliated with a finite von Neumann algebra $M$. We prove that if $M$ is a finite von Neumann algebra with a faithful normal semi-finite trace $\tau$, equipped with the locally measure topology $t$, then every $t$-continuous derivation $D : S(M) \to S(M)$ is inner. A similar result is valid for derivation on the algebra $S(M, \tau)$ of $\tau$-measurable operators equipped with the measure topology $t_\tau$.

1. INTRODUCTION

Given an algebra $A$, a linear operator $D : A \to A$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$ (the Leibniz rule). Each element $a \in A$ implements a derivation $D_a$ on $A$ defined as $D_a(x) = [a, x] = ax - xa, x \in A$. Such derivations $D_a$ are said to be inner derivations. If the element $a$, implementing the derivation $D_a$, belongs to a larger algebra $B$ containing $A$, then $D_a$ is called a spatial derivation on $A$.

One of the main problems in the theory of derivations is to prove the automatic continuity, “innerness” or “spatiality” of derivations, or to show the existence of non-inner and discontinuous derivations on various topological algebras. In particular, it is a general algebraic problem to find algebras which admit only inner derivations. Examples of algebra for which any derivation is inner include:

- finite dimensional simple central algebras (see [9, p. 100]);
- simple unital $C^*$-algebras (see the main theorem of [15]);
- the algebras $B(X)$, where $X$ is a Banach space (see [11, Corollary 3.4]).
- von Neumann algebras (see [14, Theorem 1])

A related problem is:

Given an algebra $A$, is there an algebra $B$ containing $A$ as a subalgebra such that any derivation of $B$ is inner and any derivation of the algebra $A$ is spatial in $B$?

The following are some examples for which the answer is positive:

- $C^*$-algebras (see [10, Theorem 4] or [14, Theorem 2]);
- standard operator algebras on a Banach space $X$, i.e. subalgebras of $B(X)$ containing all finite rank operators (see [11, Corollary 3.4]).

In [11] and [3], derivations on various subalgebras of the algebra $LS(M)$ of locally measurable operators with respect to a von Neumann algebra $M$ has been considered. A complete description of
derivations has been obtained in the case when $M$ is of type I and III. Derivations on algebras of measurable and locally measurable operators, including rather non trivial commutative case, have been studied by many authors [1-8]. A comprehensive survey of recent results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras can be found in [2].

If we consider the algebra $S(M)$ of all measurable operators affiliated with a type III von Neumann algebra $M$, then it is clear that $S(M) = M$. Therefore from the results of [1] it follows that for type $I_\infty$ and type III von Neumann algebras $M$ every derivation on $S(M)$ is automatically inner and, in particular, is continuous in the local measure topology. The problem of description of the structure of derivations in the case of type II algebras has been open so far and seems to be rather difficult.

In this connection several open problems concerning innerness and automatic continuity of derivations on the algebras $S(M)$ and $LS(M)$ for type II von Neumann algebras have been posed in [2]. First positive results in this direction were recently obtained in [6-7], where automatic continuity has been proved for derivations on algebras of $\tau$-measurable and locally measurable operators affiliated with properly infinite von Neumann algebras.

Another problem in [2] Problem 3] asks the following question:

Let $M$ be a type II von Neumann algebra with a faithful normal semi-finite trace $\tau$. Consider the algebra $S(M)$ (respectively $LS(M)$) of all measurable (respectively locally measurable) operators affiliated with $M$ and equipped with the locally measure topology $t$. Is every $t$-continuous derivation $D : S(M) \rightarrow S(M)$ (respectively, $D : LS(M) \rightarrow LS(M)$) necessarily inner?

In the present paper we suggest a solution of this problem for type $I_1$ von Neumann algebras (in this case $LS(M) = S(M)$). Namely, we prove that if $M$ is a finite von Neumann algebra and $D : S(M) \rightarrow S(M)$ is a $t$-continuous derivation then $D$ is inner. A similar result is proved for derivation on the algebra $S(M, \tau)$ of all $\tau$-measurable operators equipped with the measure topology $t_\tau$.

2. ALGEBRAS OF MEASURABLE OPERATORS

Let $B(H)$ be the $*$-algebra of all bounded linear operators on a Hilbert space $H$, and let $1$ be the identity operator on $H$. Consider a von Neumann algebra $M \subset B(H)$ with the operator norm $\| \cdot \|$ and with a faithful normal semi-finite trace $\tau$. Denote by $P(M) = \{ p \in M : p = p^2 = p^* \}$ the lattice of all projections in $M$.

A linear subspace $\mathcal{D}$ in $H$ is said to be affiliated with $M$ (denoted as $\mathcal{D}_\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary $u$ from the commutant

$$M' = \{ y \in B(H) : xy = yx, \forall x \in M \}$$

of the von Neumann algebra $M$.

A linear operator $x : \mathcal{D}(x) \rightarrow H$, where the domain $\mathcal{D}(x)$ of $x$ is a linear subspace of $H$, is said to be affiliated with $M$ (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$.

A linear subspace $\mathcal{D}$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if

1) $\mathcal{D}_\eta M$;
2) there exists a sequence of projections $\{ p_n \}_{n=1}^\infty$ in $P(M)$ such that $p_n \uparrow 1$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$.  

A closed linear operator \( x \) acting in the Hilbert space \( H \) is said to be \textit{measurable} with respect to the von Neumann algebra \( M \), if \( x\eta M \) and \( \mathcal{D}(x) \) is strongly dense in \( H \).

Denote by \( S(M) \) the set of all linear operators on \( H \), measurable with respect to the von Neumann algebra \( M \). If \( x \in S(M) \), \( \lambda \in \mathbb{C} \), where \( \mathbb{C} \) is the field of complex numbers, then \( \lambda x \in S(M) \) and the operator \( x^* \), adjoint to \( x \), is also measurable with respect to \( M \) (see [16]). Moreover, if \( x, y \in S(M) \), then the operators \( x + y \) and \( xy \) are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators \( x \) and \( y \), and are denoted by \( x + y \) and \( x * y \). It was shown in [16] that \( S(M) \) is a \( * \)-algebra with the identity \( 1 \) over the field \( \mathbb{C} \). Here, \( M \) is a \( * \)-subalgebra of \( S(M) \). In what follows, the strong sum and the strong product of operators \( x \) and \( y \) will be denoted in the same way as the usual operations, by \( x + y \) and \( xy \).

It is clear that if the von Neumann algebra \( M \) is finite then every linear operator affiliated with \( M \) is measurable and, in particular, a self-adjoint operator is measurable with respect to \( M \) if and only if all its spectral projections belong to \( M \).

Let \( \tau \) be a faithful normal semi-finite trace on \( M \). We recall that a closed linear operator \( x \) is said to be \( \tau \)-\textit{measurable} with respect to the von Neumann algebra \( M \), if \( x\eta M \) and \( \mathcal{D}(x) \) is \( \tau \)-dense in \( H \), i.e. \( \mathcal{D}(x)\eta M \) and given \( \varepsilon > 0 \) there exists a projection \( p \in M \) such that \( p(H) \subset \mathcal{D}(x) \) and \( \tau(p^+) < \varepsilon \). Denote by \( S(M, \tau) \) the set of all \( \tau \)-measurable operators affiliated with \( M \).

Note that if the trace \( \tau \) is finite then \( S(M, \tau) = S(M) \).

Consider the topology \( t_\tau \) of convergence in measure or \textit{measure topology} on \( S(M, \tau) \), which is defined by the following neighborhoods of zero:

\[
V(\varepsilon, \delta) = \{ x \in S(M, \tau) : \exists e \in P(M), \tau(e^+) < \delta, x e \in M, \| x e \| < \varepsilon \},
\]

where \( \varepsilon, \delta \) are positive numbers.

It is well-known [13] that \( M \) is \( t_\tau \)-dense in \( S(M, \tau) \) and \( S(M, \tau) \) equipped with the measure topology is a complete metrizable topological \( * \)-algebra.

Let \( M \) be a finite von Neumann algebra with a faithful normal semi-finite trace \( \tau \). Then there exists a family \( \{ z_i \}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee_{i \in I} z_i = 1 \) and such that \( \tau(z_i) < +\infty \) for every \( i \in I \) (such family exists because \( M \) is a finite algebra). Then the algebra \( S(M) \) is \( * \)-isomorphic to the algebra \( \prod_{i \in I} S(z_i M) \) (with the coordinate-wise operations and involution), i.e.

\[
S(M) \cong \prod_{i \in I} S(z_i M)
\]

(\( \cong \) denoting \( * \)-isomorphism of algebras) (see [12]).

This property implies that given any family \( \{ z_i \}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee_{i \in I} z_i = 1 \) and a family of elements \( \{ x_i \}_{i \in I} \) in \( S(M) \) there exists a unique element \( x \in S(M) \) such that \( z_i x = z_i x_i \) for all \( i \in I \).

Let \( t_{\tau_i} \) be the measure topology on \( S(z_i M) = S(z_i M, \tau_i) \), where \( \tau_i = \tau|_{z_i M}, i \in I \). On the algebra \( S(M) \cong \prod_{i \in I} S(z_i M) \) we consider the topology \( t \) which is the Tychonoff product of the topologies \( t_{\tau_i}, i \in I \). This topology coincides with so-called \textit{locally measure topology} on \( S(M) \) (see [7, Remark 2.7]).
It is known \cite{12} that $S(M)$ equipped with the locally measure topology is a topological $*$-algebra. Note that if the trace $\tau$ is finite then $t = t_\tau$.

3. The Main Results

Given a von Neumann algebra $M$ with a faithful normal finite trace $\tau$, $\tau(1) = 1$, we consider the $L_2$-norm

$$\|x\|_2 = \sqrt{\tau(x^*x)}, \ x \in M.$$

Denote by $\mathcal{U}(M)$ and $\mathcal{GN}(M)$ the set of all unitaries in $M$ and the set of all partially isometries in $M$, respectively.

A partial ordering can be defined on the set $\mathcal{GN}(M)$ as follows:

$$u \leq_1 v \iff uu^* \leq vv^*, \ u = uu^*v.$$

It is clear that

$$u \leq_2 v \iff u^*u \leq v^*v, \ u = vu^*u$$

is also defined a partial ordering on the set $\mathcal{GN}(M)$ and

$$u \leq_1 v \iff u^* \leq_2 v^*.$$

Note that $u^*u = r(u)$ is the right support of $u$, and $uu^* = l(u)$ is the left support of $u$.

The $t$-continuity of algebraic operations on $S(M)$ implies that every inner derivation on $S(M)$ is $t$-continuous.

The following main result of the paper shows that the converse implication is also true.

**Theorem 3.1.** Let $M$ be a finite von Neumann algebra with a faithful normal semi-finite trace $\tau$. Then every $t$-continuous derivation $D : S(M) \to S(M)$ is inner.

For the proof of this theorem we need several lemmata.

For $f \in \ker D$ and $x \in S(M)$ we have

$$D(fx) = D(f)x + fD(x) = fD(x),$$

i.e.

$$D(fx) = fD(x).$$

Likewise

$$D(xf) = D(x)f.$$

This simple properties will be frequently used below.

Let $D$ be a derivation on $S(M)$. Let us define a mapping $D^* : S(M) \to S(M)$ by setting

$$D^*(x) = (D(x^*))^*, \ x \in S(M).$$

A direct verification shows that $D^*$ is also a derivation on $S(M)$. A derivation $D$ on $S(M)$ is said to be \textit{skew-hermitian}, if $D^* = -D$, i.e. $D(x^*) = -D(x^*)$ for all $x \in S(M)$. Every derivation $D$ on $S(M)$ can be represented in the form $D = D_1 + iD_2$, where

$$D_1 = (D - D^*)/2, \quad D_2 = (-D - D^*)/2i$$

are skew-hermitian derivations on $S(M)$. 
It is clear that a derivation $D$ is inner if and only if the skew-hermitian derivations $D_1$ and $D_2$ are inner.

Therefore further we may assume that $D$ is a skew-hermitian derivation.

**Lemma 3.2.** For every $v \in \mathcal{S}(M)$ the element $vv^*D(v)v^*$ is hermitian.

**Proof.** First note that if $p$ is a projection, then $pD(p)p = 0$. Indeed, $D(p) = D(p^2) = D(p)p + pD(p)$. Multiplying this equality by $p$ from both sides we obtain $pD(p)p = 2pD(p)p$, i.e. $pD(p)p = 0$.

Now take an arbitrary $v \in \mathcal{S}(M)$. Taking into account that $vv^*v = v$ and $D$ is skew-hermitian, we get

\[
(vv^*D(v)v^*)^* = vD(v)^*v = -vD(v^*)v^* =
\]

\[
- vD(v^*v)v^* + vv^*D(v)v^* = -v(v^*vD(v^*v)v^*)v^* +
\]

\[
+ vv^*D(v)v^* = vv^*D(v)v^*,
\]

because $v^*v$ is a projection and therefore $v^*vD(v^*v)v^* = 0$. So

\[
(vv^*D(v)v^*)^* = vv^*D(v)v^*.
\]

The proof is complete.

**Lemma 3.3.** Let $n \in \mathbb{N}$ be a fixed number and let $v \in \mathcal{S}(M)$ be a partially isometry. Then

\[
vv^*D(v)v^* \geq nv^*v
\]

if and only if

\[
v^*vD(v^*)v \leq -nv^*v.
\]

**Proof.** Take an arbitrary $v \in \mathcal{S}(M)$ such that $vv^*D(v)v^* \geq nv^*v$. Multiplying this equality from the left side by $v^*$ and from the right side by $v$, we obtain

\[
v^*vv^*D(v)v^*v \geq nv^*vv^*v,
\]

i.e.

\[
v^*D(v)v^*v \geq nv^*v.
\]

Since

\[
v^*D(v)v^*v = v^*D(vv^*)v - v^*vD(v^*)v = v^*(vv^*D(vv^*)v) -
\]

\[ - v^*vD(v^*)v = -v^*vD(v^*)v,
\]

because $vv^*$ is a projection and therefore $vv^*D(vv^*)vv^* = 0$. So

\[
-v^*vD(v^*)v \geq nv^*v,
\]

i.e.

\[
v^*vD(v^*)v \leq -nv^*v.
\]

In a similar way we can prove the converse implication. The proof is complete.

**Lemma 3.4.** Let $v_1 \in \mathcal{S}(M)$ be a partially isometry and let $v_2 \in \mathcal{S}(pMp)$, where $p = 1 - v_1v_1^* + v_1v_1^* \vee s(iD(v_1v_1^*)) \vee s(iD(v_1v_1^*))$ and $s(x)$ denotes the support of a hermitian element $x$. Then

\[
(v_1 + v_2)(v_1 + v_2)^*D(v_1 + v_2)(v_1 + v_2)^* = v_1v_1^*D(v_1)v_1^* + v_2v_2^*D(v_2)v_2^*.
\]
Proof. Since $v_2 \in \mathcal{S}(pM_p)$ we get

$$v_1 v_2^* = v_2^* v_1 = v_2 v_1^* = v_1^* v_2 = 0,$$
$$v_2^* D(v_1^*) = D(v_1 v_1^*) v_2 = v_2 D(v_1^* v_1) = D(v_1^* v_1) v_2^* = 0.$$

Thus

$$v_1 v_1^* D(v_2) = D(v_1 v_1^*) v_2 - D(v_1 v_1^*) v_2 = 0,$$
$$v_2^* D(v_1) v_2^* = D(v_2^* v_1^*) v_2^* - D(v_2^* v_1^*) v_2 = 0,$$
$$v_1^* D(v_1) v_2^* = D(v_1^* v_1) v_2^* - D(v_1^* v_1) v_2 = 0,$$
$$v_2^* D(v_1) v_1^* = v_2^* D(v_1 v_1^*) - v_2^* v_1 D(v_1^*) = 0,$$
$$D(v_2) v_1^* = D(v_2) v_1^* v_1 = D(v_2 v_1^* v_1) v_1 - D(v_1^* v_1) v_1^* = 0.$$

Taking into account these equalities we get

$$(v_1 + v_2)(v_1 + v_2)^* D(v_1 + v_2)(v_1 + v_2)^* = (v_1 v_1^* + v_2 v_2^*) D(v_1 + v_2)(v_1 + v_2)^* = v_1 v_1^* D(v_1) v_1^* + v_2 v_2^* D(v_2) v_2^* + v_1 v_1^* D(v_2) v_1^* + v_2 v_2^* D(v_1) v_1^* + v_1 v_1^* D(v_2) v_1^* = v_1 v_1^* D(v_1) v_1^* + v_2 v_2^* D(v_2) v_2^*.$$

The proof is complete. \qed

Let $p \in M$ be a projection. It is clear that the mapping

$$pDp : x \to pD(x)p, \ x \in pS(M)p$$

is a derivation on $pS(M)p = S(pM_p)$.

The following lemma is one of the key steps in the proof of the main result.

Lemma 3.5. Let $M$ be a von Neumann algebra with a faithful normal finite trace $\tau$, $\tau(1) = 1$. There exists a sequence of projections $\{p_n\}$ in $M$ with $\tau(1 - p_n) \to 0$ such that the derivation $p_n Dp_n$ maps $p_n M p_n$ into itself for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ consider the set

$$\mathcal{F}_n = \{v \in \mathcal{S}(M) : vv^* D(v)v^* \geq nn v v^*\}.$$

Note that $0 \in \mathcal{F}_n$, so $\mathcal{F}_n$ is not empty. Let us show that the set $\mathcal{F}_n$ has a maximal element with respect to the order $\leq_1$.

Let $\{v_\alpha\} \subset \mathcal{F}_n$ be a totally ordered net. We will show that $v_\alpha \xrightarrow{\tau} v$ for some $v \in \mathcal{F}_n$. For $\alpha \leq \beta$ we have

$$\|v_\beta - v_\alpha\|_2 = \|l(v_\beta) v_\beta - l(v_\alpha) v_\beta\|_2 = \|l(v_\beta) - l(v_\alpha)\|_2 \|v_\beta\| = \sqrt{\tau(l(v_\beta) - l(v_\alpha))} \to 0,$$
because \( \{l(v_\alpha)\} \) is an increasing net of projections. Thus \( \{v_\alpha\} \) is a \( \| \cdot \|_2 \)-fundamental, and hence there exists an element \( v \) in the unit ball \( M \) such that \( v_\alpha \xrightarrow{\| \cdot \|_2} v \). Therefore \( v_\alpha \xrightarrow{t_\tau} v \), and thus we have
\[
 v_\alpha v_\alpha^* \xrightarrow{t_\tau} vv^*, \quad v_\alpha^* v_\alpha \xrightarrow{t_\tau} v^*v. 
\]

Therefore
\[
 vv^*, \quad v^*v \in P(M). 
\]

Thus \( v \in \mathcal{S}N(M) \).

Since \( \{v_\alpha v_\alpha^*\} \) is an increasing net of projections it follows that \( v_\alpha v_\alpha^* \uparrow vv^* \). Also, \( v_\alpha = v_\alpha v_\alpha^* v_{\beta} \) for all \( \beta \geq \alpha \) implies that \( v_\alpha = v_\alpha v_\alpha^* v \). So \( v_\alpha \leq_1 v \) for all \( \alpha \). Since \( v_\alpha \xrightarrow{t_\tau} v \) by \( t_\tau \)-continuity of \( D \) we have that \( D(v_\alpha) \xrightarrow{t_\tau} D(v) \). Taking into account that \( v_\alpha v_\alpha^* D(v_\alpha) v_\alpha^* \geq n v_\alpha v_\alpha^* \) we obtain \( vv^* D(v) v^* \geq n v v^* \), i.e. \( v \in \mathcal{F}_n \).

So, any totally ordered net in \( \mathcal{F}_n \) has the least upper bound. By Zorn’s Lemma \( \mathcal{F}_n \) has a maximal element, say \( v_n \).

Put
\[
 p_n = 1 - v_n v_n^* \lor v_n^* v_n \lor s(iD(v_n v_n^*)) \lor s(iD(v_n^* v_n)).
\]

Let us prove that
\[
 \|vv^* D(v) v^*\| \leq n
\]
for all \( v \in \mathcal{U}(p_n M p_n) \).

The case \( p_n = 0 \) is trivial.

Let us consider the case \( p_n \neq 0 \). Take \( v \in \mathcal{U}(p_n M p_n) \). Let \( vv^* D(v) v^* = \int_{-\infty}^{+\infty} \lambda \, d e_\lambda \) be the spectral resolution of \( vv^* D(v) v^* \). Assume that \( p = e_n^{\perp} \neq 0 \). Then
\[
 pvv^* D(v) v^* p \geq np.
\]

Denote \( u = pv \). Then since \( p \leq p_n = vv^* \), we have
\[
 uu^* D(u) u^* = pvv^* D(pv) v^* p =
\]
\[
 = pvv^* D(p) vv^* p + pvv^* p D(v) v^* p =
\]
\[
 = pvv^* D(p) pvv^* + pvv^* D(v) v^* p =
\]
\[
 = 0 + pvv^* D(v) v^* p \geq np,
\]
 i.e.
\[
 uu^* D(u) u^* \geq np.
\]
Since \( uu^*, u^* u \leq p_n = 1 - v_n v_n^* \lor v_n^* v_n \lor s(iD(v_n v_n^*)) \lor s(iD(v_n^* v_n)) \) it follows that \( u \) is orthogonal to \( v_n \), i.e. \( uv_n^* = v_n^* u = 0 \). Therefore \( w = v_n + u \in \mathcal{S}N(M) \). Using Lemma 3.4 we have
\[
 ww^* D(w) w^* = v_n v_n^* D(v_n) v_n^* + uu^* D(u) u^* \geq n(v_n v_n^* + p) = nww^*,
\]
because
\[
 ww^* = (v_n + u)(v_n + u)^* = v_n v_n^* + uu^* =
\]
\[
 = v_n v_n^* + pvv^* p = v_n v_n^* + p.
\]
So
\[ w w^* D(w) w^* \geq n w w^*. \]
This is contradiction with maximality \( v_n^* \). From this contradiction it follows that \( e_n^+ = 0 \). This means that
\[ v v^* D(v) v^* \leq n v v^* \]
for all \( v \in U(p_n M p_n) \).

Set
\[ S_n = \{ v \in \mathcal{G}(M) : v v^* D(v) v^* \leq -n v v^* \}. \]
By Lemma 3.3 it follows that \( v \in \mathcal{F}_n \) is a maximal element of \( \mathcal{F}_n \) with respect to the order \( \leq_1 \) if and only if \( v^* \) is a maximal element of \( S_n \) with respect to the order \( \leq_2 \).

Taking into account this observation in a similar way we can show that
\[ v v^* D(v) v^* \geq -n v v^* \]
for all \( v \in U(p_n M p_n) \). So
\[ -n v v^* \leq v v^* D(v) v^* \leq n v v^*. \]
This implies that \( v v^* D(v) v^* \in M \) and
\[ \| v v^* D(v) v^* \| \leq n \]
for all \( v \in U(p_n M p_n) \).

Let us show that the derivation \( p_n D p_n \) maps \( p_n M p_n \) into itself. Take \( v \in U(p_n M p_n) \). Then \( v v^* = v^* v = p_n \) and hence
\[ (p_n D p_n)(v) = p_n D(p_n v p_n) p_n = v v^* D(v) v^* v \in p_n M p_n. \]
Since any element from \( p_n M p_n \) is a finite linear combination of unitaries from \( U(p_n M p_n) \) it follows that
\[ p_n D(x) p_n \in p_n M p_n \]
for all \( x \in p_n M p_n \), i.e. the derivation \( p_n D p_n \) maps \( p_n M p_n \) into itself.

Let us show that \( \tau(v_n v_n^*) \to 0 \). Let us suppose the opposite, e.g. there exist a number \( \varepsilon > 0 \) and a sequence \( n_1 < n_2 < \ldots < n_k < \ldots \) such that
\[ \tau(v_{n_k} v_{n_k}^*) \geq \varepsilon \]
for all \( k \geq 1 \). Since \( v_{n_k} \in \mathcal{F}_{n_k} \) we have
\[ v_{n_k} v_{n_k}^* D(v_{n_k}) v_{n_k}^* \geq n_k v_{n_k} v_{n_k}^* \]
for all \( k \geq 1 \).

Now take an arbitrary number \( c > 0 \) and let \( n_k \) be a number such that \( n_k > c \delta \), where \( \delta = \frac{\varepsilon}{2} \). Suppose that
\[ v_{n_k} v_{n_k}^* D(v_{n_k}) v_{n_k}^* \in c V (\delta, \delta) = V (c \delta, \delta). \]
Then there exists a projection \( p \in M \) such that
\[ \| v_{n_k} v_{n_k}^* D(v_{n_k}) v_{n_k}^* p \| < c \delta, \tau(p^+) < \delta. \]
Let \( v_{n_k}^* D(v_{n_k}) v_{n_k}^* \) be the spectral resolution of \( v_{n_k}^* D(v_{n_k}) v_{n_k}^* \). From (3.3) using [12], Lemma 2.2.4] we obtain that \( e_n^\perp \leq p^\perp \). Taking into account (3.2) we have that \( v_{n_k}^* v_{n_k}^* \leq e_n^\perp \). Since \( n_k > \epsilon \delta \) it follows that \( e_{n_k}^\perp \leq e_{\epsilon \delta}^\perp \).

Thus

\[
\epsilon \leq \tau(v_{n_k}^* v_{n_k}^*) \leq \tau(p^\perp) < \delta = \frac{\varepsilon}{2}.
\]

This contradiction implies that

\[
v_{n_k}^* D(v_{n_k}) v_{n_k}^* \not\in cV(\delta, \delta)
\]

for all \( n_k > \epsilon \delta \). Since \( c > 0 \) is arbitrary it follows that the sequence \( \{v_{n_k}^* D(v_{n_k}) v_{n_k}^*\}_{k \geq 1} \) is unbounded in the measure topology. Therefore the set \( \{v v^* D(v) v^* : v \in \mathcal{M}(M)\} \) is also unbounded in the measure topology.

On the other hand, the continuity of the derivation \( D \) implies that the set \( \{xx^* D(x) x^* : ||x|| \leq 1\} \) is bounded in the measure topology. In particular, the set \( \{uu^* D(u) u^* : u \in \mathcal{M}(M)\} \) is also bounded in the measure topology. This contradiction implies that \( \tau(v_n v_n^*) \to 0 \).

Finally let us show that

\[
\tau(1 - p_n) \to 0.
\]

It is clear that

\[
l(iD(v_n v_n^*) v_n v_n^*) \leq v_n v_n^* \]
\[
r(v_n^* iD(v_n v_n^*)) \leq v_n v_n^*.
\]

Since

\[
D(v_n v_n^*) = D(v_n^* v_n) v_n^* v_n + v_n^* D(v_n v_n^*)
\]

we have

\[
\tau(s(iD(v_n v_n^*))) = \tau(s(iD(v_n v_n^*)) v_n v_n^* + v_n^* iD(v_n v_n^*)) \leq \tau(s(v_n v_n^*) \vee l(iD(v_n v_n^*) v_n v_n^*)) \vee r(v_n^* iD(v_n v_n^*))) \leq \tau(v_n v_n^*) + \tau(v_n v_n^*) + \tau(v_n v_n^*) = 3\tau(v_n v_n^*),
\]

i.e.

\[
\tau(s(iD(v_n v_n^*))) \leq 3\tau(v_n v_n^*).
\]

Similarly

\[
\tau(s(iD(v_n v_n^*))) \leq 3\tau(v_n^* v_n).
\]

Now taking into account that

\[
v_n v_n^* = v_n^* v_n
\]

we obtain

\[
\tau(1 - p_n) = \tau(v_n v_n^* \vee v_n^* v_n \vee s(iD(v_n v_n^*)) \vee s(iD(v_n v_n^*))) \leq \tau(v_n v_n^*) + \tau(v_n^* v_n) + 3\tau(v_n^* v_n) + 3\tau(v_n v_n) = 8\tau(v_n v_n^*) \to 0,
\]
i.e.
\[ \tau(1 - p_n) \to 0. \]
The proof is complete. \hfill \Box

Let \( c \in S(M) \) be a central element. It is clear that the mapping
\[ cD : x \to cD(x), \quad x \in S(M) \]
is a derivation on \( S(M) \).

**Lemma 3.6.** Let \( M \) be a von Neumann algebra with a faithful normal finite trace \( \tau \), \( \tau(1) = 1 \). There exist an invertible central element \( c \in S(M) \) and a faithful projection \( p \in M \) such that the derivation \( cpDp \) maps \( pMp \) into itself.

**Proof.** By Lemma [3.5] there exists a sequence of projections \( \{p_n\} \subset M \) with \( \tau(1 - p_n) \to 0 \) such that the derivation \( p_nDp_n \) maps \( p_nMp_n \) into itself for all \( n \in \mathbb{N} \). By (3.1) we have

(3.4) \[ \|v_nv_n^*D(v_n)v_n^*\| \leq n \]
for all \( v_n \in \mathcal{U}(pMp) \).

Let \( z_n = c(p_n) \) be the central support of \( p_n \), \( n \in \mathbb{N} \). Since \( p_n \leq z_n \) and \( \tau(1 - p_n) \to 0 \) we get \( \tau(1 - z_n) \to 0 \). Thus \( \bigvee_{n \geq 1} z_n = 1 \). Set

\[ f_1 = z_1, \quad f_n = z_n \land \left( \bigvee_{k=1}^{n-1} f_k \right)^\perp, \quad n > 1. \]

Then \( \{f_n\} \) is a sequence of mutually orthogonal central projections with \( \bigvee_{n \geq 1} f_n = 1 \).

Set

\[ c = \sum_{n=1}^{\infty} n^{-1} f_n \]
and

\[ p = \sum_{n=1}^{\infty} f_np_n, \]

where convergence of series means the convergence in the strong operator topology. Then \( c \) is an invertible central element in \( S(M) \) and \( p \) is a faithful projection in \( M \).

Let us show that

\[ \|vv^*cD(v)v^*\| \leq 1 \]
for all \( v \in \mathcal{U}(pMp) \).

Take \( v \in \mathcal{U}(pMp) \) and put \( v_n = f_nv, \ n \in \mathbb{N} \). Since \( f_n \leq z_n \) it follows that \( v_n \in \mathcal{U}(p_nMp_n) \). Taking into account that \( f_nc = n^{-1}f_n \) from the inequality (3.4) we have

\[ \|v_nv_n^*cD(v_n)v_n^*\| \leq 1. \]

Notice that

\[ f_nv_nv^*cD(v)v_n^* = (f_nv)(f_nv)^*cD(f_nv)(f_nv)^* = v_nv_n^*cD(v_n)v_n^*. \]

Since \( \{f_n\} \) is a sequence of mutually orthogonal central projections we obtain that

\[ \|vv^*cD(v)v^*\| = \sup_{n \geq 1} \|v_nv_n^*cD(v_n)v_n^*\| \leq 1. \]
Thus as in the proof of Lemma 3.5 it follows that the derivation \( cpDP \) maps \( pMp \) into itself. The proof is complete.

In the following Lemmata 3.7-3.10 we do not assume the continuity of derivations.

**Lemma 3.7.** Let \( M \) be an arbitrary von Neumann algebra with mutually equivalent orthogonal projections \( e, f \) such that \( e + f = 1 \). If \( D : S(M) \to S(M) \) is a derivation such that \( D|_{fS(M)f} \equiv 0 \) then

\[
D(x) = ax - xa
\]

for all \( x \in S(M) \), where \( a = D(u^*)u \) and \( u \) is a partial isometry in \( M \) such that \( u^*u = e, \ uu^* = f \).

**Proof.** Since \( D(f) = 0 \) we have

\[
D(e) = D(1 - f) = D(1) - D(f) = 0.
\]

Thus

\[
D(ex) = eD(x), \quad D(xe) = D(x)e, \\
D(fx) = fD(x), \quad D(xf) = D(x)f
\]

for all \( x \in S(M) \).

Now take the partially isometry \( u \in M \) such that

\[
u^*u = e, \ uu^* = f.
\]

Set \( a = D(u^*)u \). Since \( eu^* = u^*, \ ue = u \) we have

\[
a = D(u^*)u = D(eu^*)ue = eD(u^*)ue,
\]

i.e. \( a \in eS(M)e = S(eMe) \).

We shall show that

\[
D(x) = ax - xa
\]

for all \( x \in S(M) \).

Consider the following cases.

Case 1. \( x = exe \). Note that

\[
x = exe = u^*uxu^*u
\]

and

\[
uxu^* = f(uxu^*)f \in fS(M)f.
\]

Therefore \( D(uxu^*) = 0 \). Further

\[
D(x) = D(u^*uxu^*u) = \\
= D(u^*)uxu^*u + u^*D(uxu^*)u + u^*uxu^*D(u) = \\
= D(u^*)ux + xu^*D(u),
\]

i.e.

\[
D(x) = D(u^*)ux + xu^*D(u).
\]

Taking into account

\[
u^*D(u) = D(u^*) - D(u^*)u = D(e) - D(u^*)u = -a
\]
we obtain 
\[ D(x) = ax - xa. \]

**Case 2.** \( x = exf. \) Then 
\[ D(x) = D(exf) = D(u^*uxu^*) = \]
\[ = D(u^*)uxu^* + u^*D(uxu^*) = ax, \]
because \( u^*uxu^* \in fS(M)f \) and \( D(uxu^*) = 0. \) Thus \( D(x) = ax. \) Since \( a \in eS(M)e \) we have 
\[ xa = efxae = 0. \]
Therefore 
\[ D(x) = ax - xa. \]

**Case 3.** \( x = fxe. \) Then 
\[ D(x) = D(fxe) = D(uu^*xu^*) = \]
\[ = D(uu^*xu^*)u + uu^*xD(u) = xu^*D(u). \]
Since \( u^*D(u) = -a \) we get 
\[ D(x) = -xa. \] Since \( a \in eS(M)e \) have 
\[ ax = eae = 0. \]
Therefore 
\[ D(x) = ax - xa. \]

For an arbitrary element \( x \in S(M) \) we consider its representation of the form \( x = exe + exf + fxe + fxf \) and taking into account the above cases we obtain 
\[ D(x) = ax - xa. \]

The proof is complete. \( \square \)

**Lemma 3.8.** Let \( M \) be a von Neumann algebra and let \( e, f \in M \) be projections such that \( 0 \neq f \sim e \leq f^\perp. \) If \( D : S(M) \to S(M) \) is a derivation with \( D|_{fS(M)f} \equiv 0 \) then there exists an element \( a \in S(M) \) such that 
\[ D|_{pS(M)p} \equiv Da|_{pS(M)p}, \]
where \( p = e + f. \)

**Proof.** Denote \( b = D(e)e - eD(e). \) Since \( e \) is a projection, one has \( eD(e)e = 0. \) Thus 
\[ Db(e) = be - eb = \]
\[ = (D(e)e - eD(e))e = (D(e)e - eD(e)) = \]
\[ = D(e)e + eD(e) = D(e^2) = D(e), \]
i.e. \( D(e) = Db(e). \)

Now let \( x = fxf. \) Taking into account that \( ef = 0 \) we obtain 
\[ Db(x) = bx - ba = \]
\[ = (D(e)e - eD(e))fxf - fxf(D(e)e - eD(e)) = \]
\[ = -eD(e)fxf - eD(e)fxf = -eD(e)fxf - fxD(f)e \]
\[ = 0, \]
i.e. $D_b|_{pS(M)p} \equiv 0$. Consider the derivation $\Delta = D - D_b$. We have
\[
\Delta(p) = (D - D_b)(e + f) = D(e) - D_b(e) = 0,
\]
i.e. $\Delta(p) = 0$. Thus
\[
\Delta(pxp) = p\Delta(x)p
\]
for all $x \in S(M)$. This means that $\Delta$ maps $pS(M)p = S(pMp)$ into itself. So the restriction $\Delta|_{pS(M)p}$ of $\Delta$ on $pS(M)p$ is a derivation. Moreover
\[
\Delta|_{pS(M)p} = (D - D_b)|_{pS(M)p} \equiv 0.
\]
By Lemma 3.7 there exists $c \in pS(M)p$ such that
\[
\Delta|_{pS(M)p} \equiv D_c|_{pS(M)p}.
\]
Then
\[
D|_{pS(M)p} = (\Delta + D_b)|_{pS(M)p} = D_c|_{pS(M)p} + D_b|_{pS(M)p} = D_{b+c}|_{pS(M)p}.
\]
So
\[
D|_{pS(M)p} = D_{b+c}|_{pS(M)p}.
\]
The proof is complete.

Lemma 3.9. Let $M$ be a von Neumann algebra of type $II_1$ with faithful normal center-valued trace $\Phi$ and let $f$ be a projection such that $\Phi(f) \geq \varepsilon 1$, where $0 < \varepsilon < 1$. If $D : S(M) \to S(M)$ is a derivation such that $D|_{fS(M)}f \equiv 0$ then $D$ is inner.

Proof. Without loss of generality we may assume that $\Phi(1) = 1$. Choose a number $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Since $M$ is of type $II_1$, there exists a projection $f_1 \leq f$ such that $\Phi(f_1) = 2^{-n}1$. Since $f_1 \leq f$ we have $D|_{f_1S(M)f_1} \equiv 0$. Therefore replacing, if necessary, $f$ by $f_1$, we may assume that $\Phi(f) = 2^{-n}1$.

Consider the following cases.

Case 1. $n = 1$. Then $f \sim f^\perp$. By Lemma 3.7 $D$ is inner.

Case 2. $n > 1$. Take a projection $e \leq f^\perp$ with $e \sim f$. Denote $p = e + f$. Applying Lemma 3.8 we can find an element $a_p \in S(M)$ such that
\[
D|_{pS(M)p} \equiv D_{a_p}|_{pS(M)p}.
\]
Set $\Delta := D - D_{a_p}$. Then $\Phi(p) = 2^{1-n}1$ and
\[
\Delta|_{pS(M)p} \equiv 0.
\]
Similarly, applying Lemma 3.8 $(n - 1)$ times, we can find an element $a \in S(M)$ such that $D = D_a$. The proof is complete.

Lemma 3.10. Let $M$ be a von Neumann algebra of type $II_1$ and let $f$ be a faithful projection. If $D : S(M) \to S(M)$ is a derivation such that $D|_{fS(M)}f \equiv 0$ then $D$ is inner.

Proof. Since $f$ is a faithful, we see that $c(\Phi(f)) = 1$, where $c(x) = \inf\{z \in P(Z(M)) : zx = x\}$ is the central support of the element $x \in S(M)$. There exist a family $\{z_n\}_{n \in F}$, $F \subseteq \mathbb{N}$, of central projections from $M$ with $\bigvee_{n \in F} z_n = 1$ and a sequence $\{\varepsilon_n\}_{n \in F}$ with $\varepsilon_n > 0$ such that
\[
z_n \Phi(f) \geq \varepsilon_n z_n.
\]
for all $n \in F$. Since $z_n$ is a central projection, we have $D(z_n) = 0$. Thus
$$D(z_n x) = z_n D(x)$$
for all $x \in S(M)$. This means that $D$ maps $z_n S(M) = S(z_n M)$ into itself. So $z_n D|_{S(z_n M)}$ is a derivation on $S(z_n M)$. Moreover
$$z_n D|_{S(z_n M)} \equiv 0$$
and
$$z_n \Phi(f) \geq \varepsilon_n z_n.$$
By Lemma 3.9 there exists $a_n = z_n a_n \in S(z_n M)$ such that
$$z_n D|_{S(z_n M)} \equiv D a_n|_{S(z_n M)}$$
for all $n \in F$. There exists a unique element $a \in S(M)$ such that $z_n a = z_n a_n$ for all $n \in F$. It is clear that $D = D_a$. The proof is complete. \hfill $\Box$

Proof of Theorem 3.7. For finite type I von Neumann algebras the assertion has been proved in [1, Corollary 4.5]. Therefore it is sufficient to consider the case of type II$_1$ von Neumann algebras.

Case 1. The trace $\tau$ is finite (we may suppose without loss of generality that $\tau(1) = 1$). By Lemma 3.6 there exist an invertible central element $c \in S(M)$ and a faithful projection $p \in M$ such that the derivation $cpDp$ on $S(pMp) = pS(M)p$ maps $pMp$ into itself. By Sakai’s Theorem [14, Theorem 1] there is an element $a_p \in pMp$ such that $cpD(x)p = a_p x - xa_p$ for all $x \in pMp$. Since $cD$ is $\tau$-continuous it follows that
$$cpD(x)p = a_p x - xa_p$$
for all $x \in S(pMp)$. So
$$pD(x)p = (c^{-1}a_p)x - x(c^{-1}a_p)$$
for all $x \in S(pMp)$.

As in the proof of Lemma 3.8 denote $b = D(p)p - pD(p)$. Then $D(p) = D_b(p)$.

Consider the derivation $\Delta$ on $S(M)$ defined by
$$\Delta = D - D_{c^{-1}a_p} - D_b.$$
Then
$$\Delta(p) = D(p) - D_{c^{-1}a_p}(p) - D_b(p) = 0,$$
because $D(p) = D_b(p)$ and $c^{-1}a_p \in pMp$.

Let $x \in S(pMp)$. Taking into account that $\Delta(p) = 0$ we have
$$\Delta(x) = \Delta(pxp) = p\Delta(pxp)p =$$
$$= pD(pxp)p - pD_{c^{-1}a_p}(p)p - pD_b(pxp)p = 0,$$
because $pD(pxp)p = pD_{c^{-1}a_p}(p)p$ and $pD(p)p = 0$. So
$$\Delta|_{S(pMp)} \equiv 0.$$
Since $p$ is a faithful projection in $M$, by Lemma 3.10 $\Delta = D - D_{c^{-1}a_p} - D_b$ is an inner derivation. This means that there exists an element $h \in S(M)$ such that
$$D = D_h + D_{c^{-1}a_p} + D_b = D_{h+c^{-1}a_p+b}.$$
orthogonal central projections in family exists because

Case 2. Let \( \tau \) be an arbitrary faithful normal semi-finite trace on \( M \). Take a family \( \{ z_i \}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee z_i = 1 \) and such that \( \tau(z_i) < +\infty \) for every \( i \in I \) (such family exists because \( M \) is a finite algebra). The map \( D_i : S(z_i M) \to S(z_i M) \) defined by

\[ D_i(x) = z_i D(z_i x), \quad x \in S(z_i M) \]

is a derivation on \( S(z_i M) \). By the case 1 for each \( i \in I \) there exists \( a_i \in S(z_i M) \) such that \( D_i = D_{a_i} \). Further there is a unique element \( a \in S(M) \) such that \( z_i a = z_i a_i \) for all \( i \in I \). Now it is clear that \( D = D_a \). The proof is complete.

Recall that a \( * \)-subalgebra \( A \) of \( S(M) \) is called absolutely solid if from \( x \in S(M) \), \( y \in A \), and \( |x| \leq |y| \) it follows that \( x \in A \). Note that \( S(M, \tau) \) is an absolutely solid \( * \)-subalgebra in \( S(M) \).

The following theorem gives a solution of the mentioned problem [2] Problem 3] for the algebra \( S(M, \tau) \) of all \( \tau \)-measurable operators affiliated with \( M \).

**Theorem 3.11.** Let \( M \) be a finite von Neumann algebra with a faithful normal semi-finite trace \( \tau \). Then every \( t_\tau \)-continuous derivation \( D : S(M, \tau) \to S(M, \tau) \) is inner.

**Proof.** As above take a family \( \{ z_i \}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee z_i = 1 \) and such that \( \tau(z_i) < +\infty \) for every \( i \in I \). The map \( D_i : S(z_i M, \tau_i) \to S(z_i M, \tau_i) \) defined by

\[ D_i(x) = z_i D(z_i x), \quad x \in S(z_i M, \tau_i) \]

is a derivation on \( S(z_i M, \tau_i) = S(z_i M) \), where \( \tau_i = \tau|_{z_i M}, \ i \in I \). Note that the restriction of the topology \( t_\tau \) on \( S(z_i M, \tau_i) \) coincides with the topology \( t_{\tau_i} \). Since \( \tau(z_i) < +\infty \) we have that the measure topology \( t_{\tau_i} \) on \( S(z_i M, \tau_i) \) coincides with the locally measure topology. Therefore the derivation \( D_i \) is continuous in the locally measure topology. By Theorem 3.11 for each \( i \in I \) there exists \( a_i \in S(z_i M) \) such that \( D_i = D_{a_i} \). Now if we take the unique element \( a \in S(M) \) such that \( z_i a = z_i a_i \) for all \( i \in I \), then we obtain that

\[ z_i D(x) = D(z_i x) = D_i(z_i x) = a_i(z_i x) - (z_i x)a_i = z_i(ax - xa), \]

i.e.

\[ D(x) = ax - xa \]

for all \( x \in S(M, \tau) \), i.e the derivation \( D \) is implemented by the element \( a \in S(M) \). Since \( S(M, \tau) \) is an absolutely solid \( * \)-subalgebra in \( S(M) \), applying [5] Proposition 5.17] we may choose the element \( a \), implementing \( D \), from the algebra \( S(M, \tau) \) itself. So \( D \) is an inner derivation on \( S(M, \tau) \). The proof is complete.

\( \square \)

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