About Inverse 3–SAT

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Abstract

The Inverse 3–SAT problem is known to be coNP Complete: Given \( \phi \) a set of models on \( n \) variables, is there a 3–CNF formula such that \( \phi \) is its exact set of models? An immediate candidate formula \( F^3_\phi \) arises, which is the conjunction of all 3–clauses satisfied by all models in \( \phi \). The (co)Inverse 3–SAT problem can then be resumed: Given \( \phi \) a set of models on \( n \) variables, is there a model of \( F^3_\phi \)?

This article uses two important intermediate results: 1- The candidate formula can be easily (i.e. in polynomial time) transformed into an equivalent formula \( F_\phi \) which is 3–closed under resolution. A crucial property of \( F_\phi \) is that the induced formula \( F_\phi \mid I \) by applying any partial assignment \( I \) of the \( n \) variables to \( F_\phi \) is unsatisfiable iff its 3–closure contains the empty clause. 2- A set of partial assignments (of polynomial size) which subsume all assignments \( \phi \) can be easily computed.

The (co)Inverse 3–SAT question is then equivalent to decide whether it exists a partial assignment \( I \) such that the 3–closure of \( F_\phi \mid I \) does not contain the empty clause.

KEYWORDS: Inverse SAT, Closure under Resolution, Partial assignment

1. Introduction

The satisfiability problem has been one of the most studied problems in computational complexity [1, 2, 3, 4, 6, 7, 9]. Kavvadias and Sideri have shown that the Inverse 3–SAT problem is coNP Complete [5]: Given \( \phi \) a set of models on \( n \) variables, is there a 3–CNF formula such that \( \phi \) is its exact set of models? An immediate candidate 3–CNF formula \( F^3_\phi \) arises which is the set of all 3–clauses satisfied by all models in \( \phi \). Since \( F^3_\phi \) is the most restricted 3–CNF formula (in term of its model set) which is satisfied by all models in \( \phi \), the (co)Inverse 3–SAT problem can then be defined: Given \( \phi \) a set of models on \( n \) variables, is there a model of \( F^3_\phi \)? The properties of \( F^3_\phi \) will bring a new interesting way to solve the Inverse 3–SAT problem.

In the next part of the article, all needed notations will be defined. In section 3, the main ideas of the algorithm presented in section 4 will be developed.

2. Preliminaries

3–CNF formula. A CNF propositionnal formula \( F \) is regarded in the standard way as a set of clauses, where each clause is regarded as a set of literals, and each literal as a boolean variable or its negation. Whether \( x \) is a positive or a negative literal, \( \bar{x} \) denotes its complement. The size of a set \( A \) (denoted \( |A| \)) is the number of its elements. A 3–clause is
A clause of size 3. The 3–clause \( c = \{x, y, z\} \) is denoted \((xyz)\). \( c \setminus \{x\} \) is the clause \((yz)\). The empty clause, denoted \((\varnothing)\), is equivalent to \(false\). A 3–CNF formula is a CNF formula containing at least one 3–clause.

**Assignment.** Let \( F \) be a 3–CNF formula on \( n \) variables \( \{x_1, x_2 \ldots x_n\} \). Each variable \( x_i \) can be assigned to the value \( v_i \). A (total) assignment of the \( n \) variables is a set of \( n \) values \( \{v_1, v_2 \ldots v_n\} \), where the value \( v_i \) is assigned to the variable \( x_i \). A value \( v \) is equal to 0 (\(false\)) or 1 (\(true\)), the opposite of the value \( v \), \( \bar{v} = 1 - v \). A clause of \( F \) is satisfied when at least one of its literals is set (assigned) to \(true\). \( F \) is satisfiable if it exists a truth assignment of the \( n \) variables which satisfies all its clauses. Such a truth assignment is called a *model*. A partial assignment on \( k \) variables is the subset of a total assignment restricted to the values of the chosen \( k \) variables \((k \leq n)\).

**Definition 2.1.** Given \( F \) a 3–CNF formula on \( n \) variables \( \{x_1, x_2 \ldots x_n\} \); \( c \), a clause in \( F \); \( I \), a partial assignment of \( k \) variables among \( (x_i) \) \((k \leq n)\).

1. Let \( F|I \) be the induced formula by applying \( I \) to \( F \): Any clause that contains a literal which evaluates to \(true\) under \( I \) is deleted from the formula and any literals that evaluate to \(false\) under \( I \) are deleted from all clauses - the clauses that become empty by this deletion remain in the formula as the empty clause.

2. Let \( c|I \) be the induced clause by applying \( I \) to \( c \): If \( c \) contains a literal which evaluates to \(true\) under \( I \) then \( c|I = true; \) If \( c \) contains a subset \( A \) of literals all set to \(false\) under \( I \) then \( c|I = c \setminus A; \) If \( c \) does not contain any literal set by \( I \) then \( c|I = c. \)

**Subsumption.** A clause \( c \) is said to subsume a clause \( d \), and \( d \) is subsumed by \( c \), if the literals of \( c \) are a subset of those of \( d \) (each clause subsumes itself then). A (partial) assignment \( I \) is said to subsume a (partial) assignment \( J \), and \( J \) is subsumed by \( I \), if the values of \( I \) are a subset of those of \( J \).

**Resolution.** Two clauses, \( c_1 = (Ax) \) and \( c_2 = (Bx) \), can be resolved in a third clause \( c = (AB) \), so called resolvent \((c_1 \text{ and } c_2 \text{ are the operands})\), where \( A \) and \( B \) are two subsets of literals. A 3–limited resolution is a resolution in which the resolvent (so called 3–limited resolvent) and the operands have at most 3 literals.

A CNF formula \( F \) is said to be closed under resolution [respectively 3–limited closed under resolution] (or just closed [resp. 3–limited closed]) if no clause of \( F \) is subsumed by a different clause of \( F \), and the resolvent [resp. 3–limited resolvent] of each pair of resolvable clauses is subsumed by some clause of \( F \).

The closure [resp. 3–limited closure] of a CNF formula \( F \) is the CNF formula (denoted \( F^c \) [resp. 3L–\( F^c \)]) that derived from \( F \) by a series of resolutions [resp. 3–limited resolutions] (which add clauses) and subsumptions (which delete clauses), and is closed [resp. 3–limited closed]. Both closure and 3–limited closure are unique [10]. In the same paper [10], the 3–limited closure of a CNF formula has been shown to be computable in polynomial time. \( F^c \) can be separate into 2 disjoint subsets: \( F^c = 3–F^c \cup F^r \), where \( 3–F^c \) is the 3–closure of \( F \), i. e. the subset of \( F^c \) containing only clauses of size 3 or less (each clause of 3L–\( F^c \) is then subsumed by some clause of 3–\( F^c \)), and \( F^r \) contains clauses of size 4 or more.
3. Discussion before the algorithm

Given \( \phi = \{m_1, m_2 \ldots m_{|\phi|}\} \), a set of \(|\phi|\) models on \(n\) variables \((x_i)_{i \leq 1 \leq n}\) (an element of \(\phi\) will be called either assignment or model or simply element according to the context). Let \(F^3_\phi\) the set of all 3–clauses satisfied by all models in \(\phi\).

3.1 The 3–closure of \(F_{\phi\mid I}\) can be computed in polynomial time

Given \(I\), a partial assignment of \(k\) variables among \((x_i)\) \((k \leq n)\).

Proposition 3.1. The 3–closure of \(F^3_\phi\) can be computed in polynomial time.

Proof. Since \(F^3_\phi\) contains all 3–clauses satisfied by all models in \(\phi\), all possible 3–clauses implied by \(F^3_\phi\) are in \(F^3_\phi\). Since any resolvent of size 2 or less results from the resolution of clauses of size 3 or less, the 3–closure under resolution of \(F^3_\phi\) can be computed in polynomial time.

Notation. Call \(F^3_\phi\) (or \(F\) if it is not confusing) the 3–closure of \(F^3_\phi\).

Remark. (1) Each clause of \(F^3_\phi\) is subsumed by a clause in \(F_\phi\) and \(F_\phi\) is equivalent to \(F^3_\phi\).

(2) As \(F^c_\phi = F_\phi \cup F^c_\phi\) then all clauses of \(F^c_\phi\) result from resolution of clauses of \(F^3_\phi\) or some iterated resolvents of clauses of \(F^3_\phi\).

Example 1. Take \(n = 5\) and 8 models \((m_i)_{1 \leq i \leq 8}\) in \(\phi\).

\(\phi = \{00111, 01011, 10101, 11100, 11111, 10011, 01011, 00100\}\)

By gathering all 3–clauses satisfied by all models of \(\phi\):

\[ F^3_\phi = (x_1 x_2 x_3)(\bar{x}_1 \bar{x}_2 x_3)(x_1 \bar{x}_2 x_5)(\bar{x}_1 x_2 x_5)(x_1 x_3 x_4)(\bar{x}_1 x_3 x_5)(x_1 \bar{x}_4 x_5)(\bar{x}_1 \bar{x}_4 x_5)(\bar{x}_2 x_3 x_4)(x_2 x_3 x_5)(\bar{x}_2 \bar{x}_4 x_5)(x_3 x_4 x_5)(x_3 x_4 \bar{x}_5) \]

Its 3–closure is:

\[ F_\phi = (x_1 x_2 x_3)(\bar{x}_1 \bar{x}_2 x_3)(x_1 \bar{x}_2 x_5)(\bar{x}_1 x_2 x_5)(x_3 x_4 x_5)(x_3 x_4 \bar{x}_5) \]

Proposition 3.2. Given \(I\), a partial assignment of \(k\) variables among \((x_i)\) \((k \leq n)\), the 3–closure of \(F_{\phi\mid I}\) is computable in polynomial time.

Proof. By recurrence.

Let \(R_{\mid I}\) the 3–limited closure of \(F_\phi \cup F_{\phi\mid I}\), i.e. the set of clauses easily reachable from \(F_\phi\) or \(F_{\phi\mid I}\). Given \(c\) a clause implied by \(F_\phi\), it exists at least one subset of \(R_{\mid I}\) whose clauses imply \(c\). Name \(R_c\) such a subset.

Let \(P(k)\) the following property :

\[ P(k) : \text{For all } c \text{ implied by } F_\phi \text{ such that } |c_{\mid I}| \leq 3, \]

\[ \exists R_c \subseteq R_{\mid I} \text{ such that } |R_c| \leq k \Rightarrow c_{\mid I} \text{ is subsumed by some clause } \in 3L-F^c_{\phi\mid I} \]

\(3L-F^c_{\phi\mid I}\) the 3–limited closure of \(F_{\phi\mid I}\)

Here does the recurrence begin.

Given \(c\) implied by \(F_\phi\) such that \(|c_{\mid I}| \leq 3\) i.e. \(c_{\mid I} \in \text{the 3–closure of } F_{\phi\mid I}\).
1. \( k = 1 \). If \( \exists R_c \subseteq R_I | R_c | = 1 \) then \( R_c = \{d\} (d \in F_\phi \cup F_{\phi I} \) subsumes \( c \) and \( c_I \) is subsumed by \( d, I \in F_{\phi I} \) (note that any clause of \( F_{\phi I} \) is subsumed by some clause of \( G_{\phi I} \)). Thus \( P(1) \).

2. Suppose \( P(k) \) for \( k \geq 1 \). If \( \exists R_c \subseteq R_I \) such that \( |R_c| \leq k + 1 \) (and no other \( R_c \) of size 1 such that \( c \not\in F_\phi \) and \( |c| > 3 \)) then suppose \( c = (\alpha \beta \gamma L_I) \) where \( \alpha, \beta, \gamma \) are literals not set by \( I \) and \( L_I \) is a subset of literals all evaluate to 0 under \( I(L_I \neq \emptyset) \), i.e \( c_I = (\alpha \beta \gamma) \), with \( \alpha, \beta, \gamma \) not necessarily different.

3. Remove a clause \( d_i \) such that \( |d_i| < |d_i| \leq 3 \), in other words, such that \( d_i \) contains some literal from \( L_I \) (there is at least one such clause in \( R_c \) since \( L_I \neq \emptyset \)) and \( |d_i| \leq 2 \).

4. The size of the remaining set \( R_c \setminus d_i \) is \( \leq k \). If a certain clause \( c' = (\alpha \beta \gamma L_I') \) is implied by \( R_c \setminus d_i \) (where \( L_I' \) is a subset of literals all evaluate to 0 under \( I \)) then \( |c'| = 3 \) and \( \exists R_{c'} = R_c \setminus d_i \subseteq R_I \) such that \( |R_{c'}| \leq k \). By \( P(k) \), \( c'_I = (\alpha \beta \gamma) \) is then subsumed by some clause \( \in 3L_1 - F_{\phi I} \), inducing \( P(k + 1) \) for \( c \).

5. If \( d_i |I| \) contains \( \bar{\alpha} \) or \( \bar{\beta} \) or \( \bar{\gamma} \) then \( d_i |I| \) is useless to imply [some clause subsuming] \( c \). Then \( R_c \setminus d_i \) implies \( c \), inducing \( P(k + 1) \) as shown previously.

6. If \( d_i |I| \in F_{\phi I} \) subsumes \( c_I \) then \( P(k + 1) \) is satisfied for \( c \).

7. If \( d_i |I| \) does not subsume \( c_I \) and does not contain \( \bar{\alpha} \) or \( \bar{\beta} \) or \( \bar{\gamma} \) then either (a) \( d_i |I| = (x) \) or (b) \( d_i |I| = (ax) \) or (c) \( d_i |I| = (xy) \), where \( x \) and \( y \notin \{\alpha \beta \gamma\} \) and are not set by \( I \), and \( a \in \{\alpha \beta \gamma\} \).

   (a) If \( d_i |I| = (x) \) then \( R_c \setminus d_i \) implies \( (\bar{x} \alpha \beta \gamma L_I) \) (recall that implying a certain clause \( C \) means implying a clause which subsumes \( C \)). Since any resolution with \( d_i |I| = (x) \) as operand removes \( \bar{x} \) from the other operand then no clause of \( R_c \setminus d_i \) contains \( \bar{x} \) (for \( R_c \setminus d_i \subseteq R_I \) which is the 3-limited closure of \( F_\phi \cup F_{\phi I} \)). Then \( R_c \setminus d_i \) implies \( (\alpha \beta \gamma L_I) \), inducing \( P(k + 1) \) as shown in Point (4).

   (b) If \( d_i |I| = (ax) \) then \( R_c \setminus d_i \) implies \( (\bar{x} \alpha \beta \gamma L_I) \). Replace \( \bar{x} \) by \( a \) in each possible clause of \( R_c \setminus d_i \) (if the new clause is subsumed by some clause in \( R_I \), keep the subsuming clause instead. Anyway, the replacing clause is in \( R_I \)). Name \( R_{c,d_i} \) the resulting set \( (R_{c,d_i} \subseteq R_I) \). Then \( R_{c,d_i} \) implies \( (\alpha \beta \gamma L_I) \), inducing \( P(k + 1) \) as above.

   (c) If \( d_i |I| = (xy) \) then \( R_c \setminus d_i \) implies \( (\bar{x} \alpha \beta \gamma L_I) \) and \( (\bar{y} \alpha \beta \gamma L_I) \). Replace \( \bar{x} \) by \( y \) in each possible clause of \( R_c \setminus d_i \) (as above, if the new clause is subsumed by some clause in \( R_I \), keep the subsuming clause instead). Name \( R_{c,d_i} \) the resulting set \( (R_{c,d_i} \subseteq R_I) \). Then \( R_{c,d_i} \) implies \( (\bar{y} \alpha \beta \gamma L_I) \). Since it implies also \( (\bar{y} \alpha \beta \gamma L_I) \) then it implies the resolvent \( (\alpha \beta \gamma L_I) \), inducing \( P(k + 1) \).

By this recurrence, any clause \( \varepsilon \) the 3-closure of \( F_{\phi I} \) is subsumed by some clause \( \in 3L_1 - F_{\phi I} \) (the other way holds as well). Then the 3-limited closure of \( F_{\phi I} \) (computable in polynomial time) corresponds to the 3-closure of \( F_{\phi I} \).

\( \square \)
3.2 $F_{\phi|I}$ is unsatisfiable iff its 3–closure contains the empty clause

Given $I$, a partial assignment of $k$ variables among $(x_i)$ ($k \leq n$).

**Proposition 3.3.** Given $F$, a 3–CNF formula on $n$ variables $(x_i)_{i \leq 1 \leq n}$. $F$ is closed under resolution implies $F|_I$ is closed under resolution.

**Proof.** If $c_1 \ni x_i$ and $c_2 \ni \overline{x_i}$ are in $F|_I$ (in particular, $x_i$ is unset by $I$), pick clauses $d_1, d_2$ in $F$ which restrict to $c_1$ and $c_2$, respectively. Then $x_i \in d_1$ and $\overline{x_i} \in d_2$, hence their resolvent $(d_1 \setminus \{x_i\}) \cup (d_2 \setminus \{x_i\})$ is subsumed by some $d \in F$. If $d$ contains a literal made true under $I$, then so does $d_1$ or $d_2$, contradicting their choice. Thus, $d|_I$ is in $F|_I$, and it subsumes $(c_1 \setminus \{x_i\}) \cup (c_2 \setminus \{\overline{x_i}\})$.

Thanks to Emil Jeřábek (http://cstheory.stackexchange.com/a/16835/6346). \qed

**Proposition 3.4.** $F_{\phi|I}$ is unsatisfiable iff its 3–closure contains the empty clause.

**Proof.** As $F_\phi^c = F_\phi \cup F_\phi^r$ then $F_{\phi|I}^c = F_{\phi|I} \cup F_{\phi|I}^r$. Suppose the 3–closure of $F_{\phi|I}$ is unsatisfiable (the other implication is obvious). Then $F_{\phi|I}^c$ is unsatisfiable and it contains the empty clause (from the previous proposition and the Quine’s theorem [8]: A formula closed under resolution is unsatisfiable iff it contains the empty clause).

1. As $F_\phi^c$ is equivalent to $F_\phi$ then $F_{\phi|I}^c$ is equivalent to $F_{\phi|I}$.
2. Two equivalent formulas have the same 3-closure.
3. If the empty clause is in a formula then it is in its 3-closure (since $|()| = 0$).

Hence $(\emptyset)$ is in the 3–closure of $F_{\phi|I}$.

\qed

3.3 $\bar{\phi}$, a set of partial assignments subsuming all assignments $\not\in \phi$, can be computed in polynomial time

Consider some total order among the $n$ variables, say the lexicographic one.

**Definition 3.1.** Some additional useful definitions:

1. Let $M_k$ be the set of all $2^k$ partial assignments $(I_k)$ on the first $k$ values of the variables ($1 \leq k \leq n$).
2. Let $\phi_k = \{I_k \in M_k/I_k \in \phi\}$
3. Let $\bar{\phi}_k = \{I_k \in M_k/I_{k-1} \in \phi_{k-1} \text{ and } I_k \not\in \phi_k\}$ ($I_0 = \emptyset$ and $\phi_0$ is the empty set)
4. Let $\bar{\phi} = \bigcup_k \bar{\phi}_k$
5. Let $m_{i,j}$ the restriction of $m_i \in \phi$ to its first $j$ values and $\bar{m}_{i,j}$ the restriction of $m_i \in \phi$ to its first $j - 1$ values ($j \geq 1$) concatenated with the opposite of its $j^{th}$ value (as last value).

**Proposition 3.5.** About $\bar{\phi}_k$
1. The extension to the rest of the $n$ variables of any partial assignment of $\tilde{\phi}_k$ is not in $\phi$.

2. An assignment $I_n$ of the $n$ variables does not belong to $\phi$ iff $\exists k \leq n, I_k \in \tilde{\phi}_k$ where $I_k$ is the partial assignment issued from $I_n$ restricted to the first $k$ values.

3. The computation of $\tilde{\phi}_k$ can be done in polynomial time.

Proof. (1) Since any element of $\tilde{\phi}_k$ is not in $\phi$, neither is any extension of it.
(2) If $I_n \notin \phi$ then obviously $\exists k \leq n, I_k \in \tilde{\phi}_k$. If $\exists k \leq n, I_k \in \tilde{\phi}_k$ where $I_k$ is the partial assignment issued from $I_n$ restricted to the first $k$ values then by (1) any extension of $I_k \notin \phi$ and $I_n \notin \phi$.
(3) $|\phi_k|, |\tilde{\phi}_k| \leq |\phi|$ (and $|\tilde{\phi}| \leq n|\phi|$). The computation of $\phi_k$ can obviously be done in polynomial time. So can be the computation of $\tilde{\phi}_k$: for each model $m_i \in \phi$, compute $\tilde{m}_{i,k}$, put it in $\tilde{\phi}_k$ if it does not belong to $\phi$.

Proposition 3.6. About $\tilde{\phi}$

1. The extension to the rest of the $n$ variables of any partial assignment of $\tilde{\phi}$ is not in $\phi$.

2. $\tilde{\phi}$ is a set of partial assignments subsuming all assignments of the $n$ variables which are not in $\phi$ ($|\tilde{\phi}| \leq n|\phi|$).

3. $\tilde{\phi}$ can be computed in polynomial time.

Proof. Directly from the previous proposition and the definition of $\tilde{\phi}$.

Remark. As we are interested in partial assignments which could be extended to an entire model for the 3–CNF $F$, we can only consider the $\tilde{\phi}_k$ sets for $k > 3$ without changing anything further.

Example 2. Take $n = 5$ and 8 models $(m_i)_{1 \leq i \leq 8}$ in $\phi$.

$\phi = \{00111, 01011, 10101, 11100, 11111, 10011, 01101, 00100\}$ (as Example 1)

The 3–closure of the candidate formula has been established:

$F_{\phi} = (x_1 x_2 x_3) (\bar{x}_1 x_2 x_5) (x_1 \bar{x}_2 x_5) (x_3 x_4) (\bar{x}_3 x_5) (\bar{x}_4 x_5)$

Let build the sets $(\tilde{\phi})_k$ for $4 \leq k \leq n(= 5)$:

- $k = 4$
  $\phi_4 = \{0011, 0101, 1010, 1110, 1111, 1001, 0110, 0010\}$ $\bar{m}_{1,4} = 0010 \in \phi_4$ ($= m_{8,4}$ so $\bar{m}_{8,4} = m_{1,4} \in \phi_4$)
  $\bar{m}_{2,4} = 0100 \notin \phi_3$ ($\notin \tilde{\phi}_4$)
  and so on until $\phi_4 = \{0100, 1011, 1000, 0111\}$

- $k = 5$
  In the same way, $\tilde{\phi}_5 = \{00110, 01010, 10100, 11101, 11110, 10010, 01100, 00101\}$

Hence $\tilde{\phi} = \tilde{\phi}_4 \cup \tilde{\phi}_5$
An equivalent formulation of the (co)Inverse 3–SAT question: Is there a partial assignment $I \in \bar{\phi}$ such that the 3–closure of $F_{\phi|I}$ does not contain the empty clause?

**Proposition 3.7.** The (co)Inverse 3–SAT question "Is there a model of $F^3_{\phi} \notin \phi"$ is equivalent to the question "Is there a partial assignment $I \in \bar{\phi}$ such that the 3–closure of $F_{\phi|I}$ does not contain the empty clause?"

**Proof.** If it exists a partial assignment $I \in \bar{\phi}$ such that the 3–closure of $F_{\phi|I}$ does not contain the empty clause then:
1) All extensions of $I$ on the rest of the $n$ variables are not in $\phi$ (from Prop. 3.4).
2) $F_{\phi|I}$ is satisfiable (from Prop. 3.2).

Then $I$ extended (concatenated) with a model of $F_{\phi|I}$ is a model of $F^3_{\phi} \notin \phi$.

If it exists $m$, a model of $F^3_{\phi} \notin \phi$ ($m$ is also a model of $F_{\phi}$) then it exists a partial assignment $I_m \in \bar{\phi}$ which subsumes $m$ (since $\bar{\phi}$ is a set of partial assignments which subsume all assignment $\notin \phi$). Then $F_{\phi|I_m}$ is satisfiable (if not, no extension of $I_m$ can satisfy neither $F_{\phi}$ nor $F^3_{\phi}$: contradiction) and its 3–closure does not contain the empty clause. \[\square\]

**4. The algorithm**

**Input:** $\phi$, a set of models over $n$ variables.

**Step 1:** Compute $F_{\phi}$, the 3–closure of the candidate formula.

**Step 2:** Compute $\bar{\phi}$, a set of partial assignments subsuming all assignments $\notin \phi$.

**Step 3:** For each partial assignment $I \in \bar{\phi}$, compute the 3–closure of $F_{\phi|I}$ and check whether it contains the empty clause.

**Output:** Yes or No, answering the question: Is there a partial assignment $I \in \bar{\phi}$ such that the 3–closure of $F_{\phi|I}$ does not contain the empty clause?

**Proposition 4.1.** This algorithm lets solve the (co)Inverse 3–SAT problem. Each step can be computed in polynomial time.

**Proof.** This algorithm obviously finishes. It outputs the answer to the question: Is there a partial assignment $I \in \bar{\phi}$ such that the 3–closure of $F_{\phi|I}$ does not contain the empty clause? which is equivalent to the classical (co)Inverse 3–SAT question. Its polynomial-time computation comes directly from the previous results of the article (since $|\bar{\phi}| \leq n|\phi|$, there is no exponential increase in size). \[\square\]

**Example 3.** Take $n = 5$ and 8 models $(m_i)_{1 \leq i \leq 8}$ in $\phi$.

$\phi = \{00111, 01011, 10101, 11100, 11111, 10011, 01101, 00100\}$ (as Example 1 and 2)

$F_{\phi}$ and $\bar{\phi}$ have been found:

$F_{\phi} = (x_1x_2x_3)(\bar{x}_1\bar{x}_2x_3)(x_1\bar{x}_2x_5)(\bar{x}_1x_2x_5)(x_3x_4)(x_3x_5)(\bar{x}_4x_5)$

$\phi = \{0100, 1011, 1000, 0111, 00110, 01010, 10100, 11101, 11110, 10010, 01100, 00101\}$

$F_{|0100} = (\emptyset)$ but $F_{|0111} = (x_3)$ so the candidate formula has at least one model $m \notin \phi$ ($m = 10111$).
5. Conclusion

The (co)Inverse 3–SAT problem can be solved in polynomial time.

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