We will prove that if \( \phi \) belongs to the class \( A^1(\mathbb{R}) \) with constant \( c \geq 1 \) then the decreasing rearrangement of \( \phi \), belongs to the same class with constant not more than \( c \). We also find for such \( \phi \) the exact best possible range of those \( p > 1 \) for which \( \phi \in L^p \). In this way we provide alternative proofs of the results that appear in [1].

1. Introduction

The theory of Muckenhoupt weights has been proved to be an important tool in analysis. One of the most important facts concerning these is their self improving property. A way to express this is through the so-called reverse Hölder or more generally reverse Jensen inequalities (see [2], [3] and [7]).

In this paper we are concerned with such weights and more precisely for those \( \phi \) that belong to the class \( A^1(J) \) where \( J \) is an interval on \( \mathbb{R} \). This is defined as follows:

A function \( \phi : J \rightarrow \mathbb{R}^+ \) belongs to \( A^1(J) \) if there exists a constant \( c \geq 1 \) such that

\[
\frac{1}{|I|} \int_I \phi(x)dx \leq c \cdot \text{ess inf}_I \phi,
\]

for every \( I \) subinterval of \( J \), where \( | \cdot | \) is the Lebesgue measure on \( \mathbb{R} \).

Moreover if the constant \( c \) is the least for which (1.1) is satisfied for any \( I \subseteq J \) we say that the \( A^1 \) constant of \( \phi \) is \( c \) and is denoted by \([\phi]_1\). We say then that \( \phi \) belongs to the \( A^1 \) class of \( J \) with constant \( c \) and we write \( \phi \in A^1(J,c) \).

It is a known fact that if \( \phi \in A^1(J,c) \) then there exists \( p(c) > 1 \) such that \( \phi \in L^p \) for every \( p \in [1,p(c)) \).

Moreover \( \phi \) satisfies a reverse Hölder inequality for every \( p \in [1,p(c)) \). That is for any such \( p \) there exists \( C_p > 1 \) such that

\[
\frac{1}{|I|} \int_I \phi^p(x)dx \leq C_p \left( \frac{1}{|I|} \int_I \phi(x)dx \right)^p,
\]

for every \( I \) subinterval of \( J \) and every \( \phi \in A^1(J,c) \).

The problem of the exact determination of the best possible constant \( p(c) \) has been treated in [1]. More precisely it is shown there the following:

**Theorem A:** If \( \phi \in A^1((0,1), c) \) and \( c \) is greater than 1, then \( \phi \in L^p(0,1) \) for any \( p \).
such that $1 \leq p < \frac{c}{c-1}$. Moreover the following inequality is true

$$
\frac{1}{|I|} \int_I \phi^p(x) dx \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \frac{1}{|I|} \int_I \phi(x) dx \right)^p
$$

for every $I$ subinterval of $(0, 1)$ and for any $p$ in the range $[1, \frac{c}{c-1})$. Additionally, the constant that appears in the right of inequality (1.3) is best possible.

As a consequence of the above theorem we have that the best possible range for the $L^p$-integrability of any $\phi$ with $[\phi]_1 = c$ is $[1, \frac{c}{c-1})$.

The approach for proving the above theorem as is done in [1], is by using the decreasing rearrangement of $\phi$ which is defined by the following equation

$$
\phi^*(t) = \sup_{c \in (0,1), |e| \geq t} \left[ \inf_{x \in e} \phi(x) \right],
$$

for any $t \in (0, 1]$.

Then $\phi^*$ is a function equimeasurable to $\phi$, non-increasing and left continuous.

The immediate step for proving Theorem A, as it appears in [1] is the following:

**Theorem B.** If $\phi \in A_{1}((0, 1), c)$ then $\phi^* \in A_{1}((0, 1), c')$ for some $c'$ such that $1 \leq c' \leq c$.

This is treated in [1] initially for continuous functions $\phi$ and generalized to arbitrary $\phi$ by use of a covering lemma. Then applying several techniques the authors in [1] were able to prove Theorem A firstly for non-increasing functions and secondly for general $\phi$ by use of Theorem B.

In this paper we provide alternative proofs of the Theorems A and B. We first prove Theorem B without any use of covering lemmas. Then we provide a proof of Theorem A for non-increasing functions $\phi$. The proof gives in an immediate way the inequality (1.3). At last we prove Theorem A in it’s general form by using the above mentioned results.

Additionally, we need to say that the dyadic analogue of the above problem is solved in [6] while in [4] and [5] related problems for estimates for the range of $p$ in higher dimensions have been treated.

2. **Rearrangements of $A_{1}$ weights on $(0, 1)$**

We are now ready to state and prove the main theorem in this section.

**Theorem 1:** Let $\phi : (0, 1) \to \mathbb{R}^+$ which satisfies condition (1.1) for any subinterval $I$ of $(0, 1)$, and for a constant $c \geq 1$. Then $\phi^*$ satisfies this condition with the same constant.
**Proof.** It is easy to see that in order to prove our result, we need to prove the following inequality:

\[
\frac{1}{t} \int_0^t \phi^*(u) du \leq c \phi^*(t)
\]

for any \( t \in (0, 1] \), due to the fact that \( \phi^* \) is left continuous and non-increasing.

For any \( \lambda > 0 \) we consider the set \( E_\lambda = \{ x \in (0, 1) : \phi(x) > \lambda \} \). Let now \( \varepsilon > 0 \). Then we can find for any such \( \varepsilon \) an open set \( G_\varepsilon \subseteq (0, 1) \) for which \( G_\varepsilon \supseteq E_\lambda \) and \(|G_\varepsilon \setminus E_\lambda| < \varepsilon\). Then \( G_\varepsilon \) can be decomposed as follows: \( G_\varepsilon = \bigcup_{j=1}^{+\infty} I_{j, \varepsilon} \), where \((I_{j, \varepsilon})\) is a family of non-overlapping open subintervals of \((0, 1)\). If any two of these have a common endpoint we replace them by their union. We apply the above procedure to the new family of intervals and at last we reach to a family \((I'_{j, \varepsilon})_j\) of non-overlapping open intervals such that, if \( G'_{\varepsilon} = \bigcup_{j=1}^{+\infty} I'_{j, \varepsilon} \) we still have that \( G_\varepsilon \supseteq E_\lambda \) and \(|G_\varepsilon \setminus E_\lambda| < \varepsilon\).

Additionally we have that for any \( j \) such that \( I'_{j, \varepsilon} \neq (0, 1) \) there exists an endpoint of it such that if we enlarge this interval in the direction of this point, thus producing the interval \( I'_{j, \varepsilon, \delta} \) with \( \delta \) small enough, we have that \( \text{ess inf}(\phi) \leq \lambda \). This follows by our construction and the definition of \( E_\lambda \). Suppose now that \(|E_\lambda| < 1\). Thus \( I'_{j, \varepsilon, \delta} \neq (0, 1) \) for any \( j, \varepsilon \) and \( \delta \). On each of these intervals we apply (1.1). So we conclude that

\[
\frac{1}{|I'_{j, \varepsilon, \delta}|} \int_{I'_{j, \varepsilon, \delta}} \phi \leq c \cdot \text{ess inf}_{I'_{j, \varepsilon, \delta}}(\phi) \leq c \lambda,
\]

for every \( \varepsilon, \delta > 0 \) and \( j = 1, 2, \ldots \).

Letting \( \delta \to 0^+ \) we reach to the inequality \( \frac{1}{|I'_{j, \varepsilon}|} \int_{I'_{j, \varepsilon}} \phi \leq c \lambda \) for any \( j = 1, 2, \ldots \) and every \( \varepsilon > 0 \).

Since \( G_\varepsilon = \bigcup_{j=1}^{+\infty} I'_{j, \varepsilon} \) is disjoint we must have that:

\[
\frac{1}{|G_\varepsilon|} \int_{G_\varepsilon} \phi \leq \sup \left\{ \frac{1}{|I'_{j, \varepsilon}|} \int_{I'_{j, \varepsilon}} \phi : j = 1, 2, \ldots \right\} \leq c \lambda
\]

for every \( \varepsilon > 0 \) and letting \( \varepsilon \to 0^+ \) we have as a result that

\[
\frac{1}{|E_\lambda|} \int_{E_\lambda} \phi \leq c \lambda \leq c \cdot \text{ess inf}_{E_\lambda}(\phi).
\]

By the definition of \( E_\lambda \) we have that

\[
\frac{1}{|E_\lambda|} \int_{E_\lambda} \phi = \frac{1}{|E_\lambda|} \int_{0,1} \phi^*(u) du
\]

and of course

\[
\text{ess inf}_{E_\lambda}(\phi) = \text{ess inf}_{(0,1)}(\phi^*) = \phi^*(|E_\lambda|).
\]
since $\phi^*$ is left continuous. As a consequence from the above we immediately see that

$$\frac{1}{|E_\lambda|} \int_0^{|E_\lambda|} \phi^*(u) du \leq c\phi^*(|E_\lambda|).$$

The same inequality holds even in the case where $|E_\lambda| = 1$, so $G_\varepsilon = (0, 1)$. Then by relation (1.1) that holds for the interval $(0, 1)$ we conclude (2.5). Thus we have proved that $\frac{1}{t} \int_0^t \phi^*(u) du \leq c\phi^*(t)$, for every $t$ of the form $t = |E_\lambda|$ for some $\lambda > 0$.

Let now $t \in (0, 1]$ and define

$$\phi^*(t) = \lambda_1, \quad t_1 = \min \left\{ s \in (0, 1] : \phi^*(s) = \lambda_1 \right\} \leq t.$$

Additionally $|E_{\lambda_1}| = t_1$. As a result

$$\frac{1}{t} \int_0^t \phi^*(u) du \leq \frac{t_1}{t} \left( \frac{1}{|E_{\lambda_1}|} \int_0^{|E_{\lambda_1}|} \phi^*(u) du \right) + \frac{t - t_1}{t} \lambda_1 \leq \frac{t_1}{t} c\lambda_1 + \frac{t - t_1}{t} \lambda_1 \leq c\lambda_1 = \phi^*(t).$$

where in the second inequality we have used the above results. Theorem 1 is now proved.

We proceed now to the next section.

3. $L^p$ integrability for $A_1$ weights on $(0, 1)$

We shall now prove the following:

**Theorem 2.** Let $\phi \in A_1((0, 1), c)$ where $c$ is greater than 1. Then, for every $p \in [1, c^{-1})$, $\phi \in L^p$ and satisfies the following inequality

$$\frac{1}{|I|} \int_I \phi^p \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \frac{1}{|I|} \int_I \phi \right)^p,$$

for every $I$ subinterval of $(0, 1)$. Moreover, inequality (3.6) is best possible.

We will need first a preliminary lemma which we state as

**Lemma 1.** Let $g : (0, 1] \to \mathbb{R}^+$ be a non-increasing function. Then the following inequality is true for any $p > 1$ and every $\delta \in (0, 1)$

$$\int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^p dt = -\frac{1}{p - 1} \left( \int_0^\delta g \right)^p \frac{1}{\delta^{p-1}} + \frac{p}{p - 1} \int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) dt.$$

**Proof.** By using Fubini’s theorem it is easy to see that

$$\int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^p dt = \int_{\lambda=0}^{+\infty} p\lambda^{p-1} \left\{ t \in (0, \delta) : \frac{1}{t} \int_0^t g \geq \lambda \right\} dt.$$

Let now $\frac{1}{\delta} \int_0^\delta g = f_\delta \geq f = \int_0^1 g$. 


Then
\[
\frac{1}{t} \int_0^t g > f_\delta, \quad \forall \ t \in (0, \delta) \quad \text{while}
\]
\[
\frac{1}{t} \int_0^t g \leq f_\delta, \quad \forall \ t \in [\delta, 1].
\]

Let now \( \lambda \) be such that: \( 0 < \lambda < f_\delta \). Then for every \( t \in (0, \delta) \) we have that \( \frac{1}{t} \int_0^t g \geq \frac{1}{\delta} \int_0^\delta g = f_\delta > \lambda \). Thus
\[
\left| \left\{ t \in (0, \delta] : \frac{1}{t} \int_0^t g \geq \lambda \right\} \right| = |(0, \delta]| = \delta.
\]

Now for every \( \lambda \geq f_\delta \) there exists unique \( a(\lambda) \in (0, \delta] \) such that \( \frac{1}{a(\lambda)} \int_0^{a(\lambda)} g = \lambda \). It’s existence is guaranteed by the fact that \( \lambda > f_\delta \), that \( g \) is non-increasing and that \( g(0^+) = +\infty \) which may without loss of generality be assumed (otherwise we work for the \( \lambda \)'s on the interval \( (0, \|g\|_\infty) \)).

Then
\[
\left\{ t \in (0, \delta] : \frac{1}{t} \int_0^t g \geq \lambda \right\} = (0, a(\lambda]].
\]

Thus from (3.8) we conclude that
\[
\int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^p dt = \int_{\lambda=0}^{f_\delta} p\lambda^{p-1} \cdot \delta \cdot d\lambda + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1} a(\lambda) d\lambda
\]
\[
= \delta (f_\delta)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1} \frac{1}{\lambda} \left( \int_0^{a(\lambda)} g(u) du \right) d\lambda,
\]
by the definition of \( a(\lambda) \).

As a consequence (3.9) becomes
\[
\int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^p dt = \frac{1}{\delta^{p-1}} \left( \int_0^\delta g \right)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2} \left( \int_0^{a(\lambda)} g(u) du \right) d\lambda
\]
\[
= \frac{1}{\delta^{p-1}} \left( \int_0^\delta g \right)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2} \left( \int_{\{u \in (0, \delta] : \|g\|_\infty < \lambda\}} g(u) du \right) d\lambda
\]
\[
= \frac{1}{\delta^{p-1}} \left( \int_0^\delta g \right)^p + \frac{p}{p-1} \int_0^\delta g(t) \left[ \lambda^{p-1} \frac{1}{\lambda} \int_{\{u \in (0, \delta] : \|g\|_\infty < \lambda\}} g(u) du \right]_{\lambda=f_\delta}^{f_\delta} dt
\]
\[
= \frac{1}{\delta^{p-1}} \left( \int_0^\delta g \right)^p + \frac{p}{p-1} \left[ \int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) \left( \int_0^\delta g(t) dt \right) f_\delta^{p-1} \right]
\]
\[
= - \frac{1}{p-1} \frac{1}{\delta^{p-1}} \left( \int_0^\delta g \right)^p + \frac{p}{p-1} \int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) dt,
\]
where in the third equality we have used Fubini’s theorem and the fact that \( \frac{1}{\delta} \int_0^\delta g = f_\delta \).

Lemma 1 is now proved. \( \square \)
Before we prove Theorem 2 we will need the following:

**Lemma 2.** Let \( g : (0, 1] \to \mathbb{R}^+ \) be non-increasing such that

\[
\frac{1}{t} \int_0^t g(u)du \leq cg(t), \quad \text{for every } t \in (0, 1).
\]

Then for every \( \delta \in (0, 1] \) we have the following inequality

\[
\frac{1}{\delta} \int_0^\delta g^p \leq \frac{1}{c^{p-1}(c + p - pc)} \left( \frac{1}{\delta} \int_0^\delta g \right)^p,
\]

for every \( p \) such that \( 1 \leq p < \frac{p}{p-1} \). Moreover the above inequality is sharp.

**Proof.** Fix \( \delta \in (0, 1] \) and \( p \in \left[ \frac{c}{c-1}, \frac{c}{c-1} \right) \). Then by Lemma 1

\[
\int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^p dt = -\frac{1}{p-1} \left( \int_0^\delta g \right)^p + \frac{p}{p-1} \int_0^\delta \left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) dt
\]

\[\Rightarrow \int_0^\delta \left[ \left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) - \frac{p-1}{p} \left( \frac{1}{t} \int_0^t g \right)^p \right] dt \leq \frac{1}{p} \int_0^\delta \left( \frac{1}{\delta} \int_0^\delta g \right)^p.
\]

We now define the following function \( h_y \), of the variable \( x \) for any fixed constant \( y > 0 \)

\[
h_y(x) = x^{p-1}y - \frac{p-1}{p}x^p, \quad \text{for } x \in [y, cy].
\]

Then

\[
h_y'(x) = (p-1)x^{p-2}y - (p-1)x^{p-1} = (p-1)x^{p-2}(y-x) \leq 0, \quad \forall \ x \geq y.
\]

Thus, \( h_y \) is decreasing on the interval \([y, cy]\). We conclude that for any \( x \) such that \( y \leq x \leq cy \) we have \( h_y(x) \geq h_y(cy) \).

Applying the above conclusion in the case where \( x = \frac{1}{t} \int_0^t g, \ y = g(t) \) (noting that \( y \leq x \leq cy \), for any fixed \( t \)) we reach to the inequality:

\[
\left( \frac{1}{t} \int_0^t g \right)^{p-1} g(t) - \frac{p-1}{p} \left( \frac{1}{t} \int_0^t g \right)^p \geq c^{p-1}g^p(t) - \frac{p-1}{p} c^p g^p(t)
\]

\[\Rightarrow c^{p-1} \left[ 1 - \frac{p-1}{p} c \right] g^p(t), \quad \forall \ t \in (0, 1].
\]

Applying \((3.11)\) in \((3.10)\) we have as a result that

\[
c^{p-1} \left[ 1 - \frac{p-1}{p} c \right] \int_0^\delta g^p(t) dt \leq \frac{1}{p} \left( \frac{1}{\delta} \int_0^\delta g \right)^p
\]

\[\Rightarrow \int_0^\delta g^p \leq \frac{1}{c^{p-1}[p + c - pc]} \left( \frac{1}{\delta} \int_0^\delta g \right)^p,
\]

which is the inequality that is stated above.

Additionally \((3.12)\) is sharp as can be seen by using the function \( g(t) = \frac{1}{c} t^{\frac{1}{c}-1}, t \in (0, 1], \) for \( c > 1 \), and \( g = \text{const} \) for \( c = 1 \).

Lemma 2 is now proved.
We are now ready for the

**Proof of Theorem 2.** Let $I \subseteq (0, 1)$ be an interval.

We set $\phi_I : I \to \mathbb{R}^+$ by $\phi_I(x) = \phi(x), \ x \in I$.

Then $\phi_I$ satisfies on $I$ the condition (1.1) with constant $c$. That is $\phi \in A_1(I)$ with $A_1$-constant less or equal then $c$. Then by the results of Section 2 and a dilation argument we conclude that

$$\phi_I^* = g_I : (0, |I|] \to \mathbb{R}^+ \text{ satisfies}$$

$$\frac{1}{t} \int_0^t g_I \leq cg_I(t), \text{ for any } t \in (0, |I|].$$

Then by Lemma 2 and considering the results of this Section we have the inequality:

$$\frac{1}{t} \int_0^t g_I^p(u)du \leq \frac{1}{c^{p-1}(c+p-pc)} \left( \frac{1}{t} \int_0^t g_I(u)du \right)^p$$

for any $t \in (0, |I|]$.

By the fact now that $g_I = (\phi/I)^*$ and (3.13) we see immediately: (for $t = |I|$) that

$$\frac{1}{|I|} \int_I \phi^p \leq \frac{1}{c^{p-1}(c+p-pc)} \left( \frac{1}{|I|} \int_I \phi \right)^p.$$

At last, we mention that the result is best possible since Lemma 2 is proved to be sharp.

Theorem 2 is now proved.

□

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