W Strings and Cohomology in Parafermionic Theories

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Abstract

By enforcing locality we relate the cohomology found in parafermionic theories to that occurring in W strings. This link provides a systematic method of finding states in the cohomology of $W_{2,s}$ strings.
The states and scattering amplitudes of the $W_3$ string contain Ising model primary fields and correlators respectively [1]. The Ising model that appears in this way is of a specific type, namely the two scalar parafermionic representation [2]. However, the representation of these parafermions was not the standard one [3,5], but it was discovered [4] that there exists a field redefinition which relates the parafermions that occur in the $W_3$ string to the usual ones. In the notation of reference [4] the $W_3$ string is constructed from the fields $\phi_1, \phi_2$ and $x^\mu$; we will use the notation of this paper, with the exception that we replace $\phi_1$ by $\hat{\phi}_1$. Parafermions occur in all critical $W$ strings; for a $W$ string is based on an algebra whose maximal spin is $k+1$, the relevant parafermions are [2,4].

$$\psi_1 = \exp(i\hat{\phi}_1 - \sqrt{\frac{k+2}{k}}\phi_2)$$
$$\psi_{-1} = -\frac{1}{k}\left\{i(k+1)\partial^2\hat{\phi}_1 + (k+1)(\partial\hat{\phi}_1)^2\right\}$$
$$+ i\sqrt{k(k+2)}\partial\hat{\phi}_1\partial\phi_2\exp(-i\hat{\phi}_1 + \sqrt{\frac{k+2}{k}}\phi_2)$$

(1)

The variables $\hat{\phi}_1$ and $\phi_2$ are linearly related to the fields $\varphi$ and $\rho$ that occur in most of the literature on the $W_3$ string by the relation $\hat{\phi}_1 = -(k+1)\rho - i\sqrt{k(k+2)}\varphi$, $\phi_2 = -i\sqrt{k(k+2)}\rho + (k+1)\varphi$. The case $k = 2$ corresponds to the $W_3$ string and $W_N$ strings correspond to taking $N = k + 1$.

The usual parafermions discussed extensively in the literature [3,5] are

$$\psi_1 = -i\partial\phi_1 \exp\left(-i\phi_1 - \sqrt{\frac{k+2}{k}}\phi_2\right)$$
$$\psi_{-1} = -\frac{1}{k}\left\{i(k+1)\partial\phi_1 + \sqrt{k(k+2)}\partial\phi_2\right\}$$
$$\exp(i\phi_1 + \sqrt{\frac{k+2}{k}}\phi_2).$$

(2)

We will refer to $\phi_1$ and $\phi_2$ as the original variables and $\hat{\phi}_1$ and $\phi_2$ as $W$ variables. The relation between the $W$ variables and the original variables being [4]

$$e^{i\phi_1} = i\partial\hat{\phi}_1 e^{-i\hat{\phi}_1}, \quad -i\partial\phi_1 e^{-i\phi_1} = e^{i\hat{\phi}_1}$$

(3)

Parafermions are well known to possess three screening charges [5] which commute with $T(z)$ and $\psi_{\pm 1}(z)$. These are given by

$$Q_i = \oint dz j_i(z) \quad i = 1, 2, 3$$

(4)
where

\[ j_1 = \exp(i(k + 1)\phi_1 + \sqrt{k(k + 2)}\phi_2) \]
\[ j_2 = \exp i\phi_1 \]
\[ j_3 = -i\partial\phi_1 \exp(i\phi_1 - \sqrt{\frac{k}{k + 2}}\phi_2) \]

In terms of \( W_3 \) variables two of the screening charges for \( k = 2 \) are recognisable as charges used in the \( W_3 \) string. The charge \( Q_3 \) is the screening charge \( S \) of reference [6] which was used to construct the infinite number of states that occur in the BRST cohomology of the \( W_3 \) string, while \( Q_1 \) is a nilpotent part of the BRST charge \( Q = Q_0 + Q_1 \) of the \( W_3 \) string. To make this latter identification one must use the formula [4]

\[ e^{in\phi_1} = (-1)^{n+1}e^{-(n+1)\phi_1}\partial^n e^{i\phi_1} \]

The third screening charge \( Q_2 \) becomes a total derivative under the change to \( W \) variables and so, at first sight, appears to play no role.

It was also found [2,4] that all the states of the \( W_3 \) string could also be generated by the action of parafermions on four basic physical states. This construction of the physical states and the above identification of \( Q_1 \) leads one to hope that one could exploit the simplicity of \( Q_1 \) in terms of original variables to solve the cohomology of \( W \) string theories. However, the cohomology of \( Q_1 \) is trivial [7]. This is most easily seen by considering the variables

\[ j_1(z) = \eta(z), \quad \xi(z) = \exp\{-i(k + 1)\phi_1 - \sqrt{k(k + 2)}\phi_2\}. \]

It is straightforward to verify that the fields \( \eta \) and \( \xi \) have conformal weights 1 and 0 respectively and operator product expansion \( \eta(z)\xi(w) = \frac{1}{(z-w)} = -\xi(w)\eta(w) \). In other words, they have the same properties as spin 1 and 0 fermionic ghosts. As usual, for such a system [8], we define vacua \( |±> \) obey \( \eta_0 | - > = 0 \) and \( \xi_0 | + > = 0 \) and so \( \eta_0 | + > = | - > \). Since \( Q_1 = \eta_0 \), \( Q_1 \) vanishing on any state means that the state possess only the \( | - > \) vacuum. However, this vacuum can be written as \( Q_1 \) acting on a \( | + > \) vacuum. Introducing the fields \( \eta' = e^{i\phi_1}, \xi' = e^{-i\phi_1} \) we conclude, using the same argument, that the cohomology of \( Q_2 \) is also trivial.

This result is somewhat puzzling when one recalls, that for the cohomology of the \( W_3 \) string, the operator \( Q_1 \) plays an essential role. The key to understanding this apparent paradox is that when calculating a cohomology one must specify not only the BRST charge, but also the set of allowable states or operators. An example of where this is important is the Ramond sector of the superstring where one of the zero modes of the bosonic ghosts is excluded from the space of allowed states. In terms of \( W \) variables a minimum requirement
is that the physical states be single-valued. This means that states which contain terms with $\phi_1$ and $\phi_2$ without derivatives and not in the exponential are excluded. Examining the transformation of equation (3) between the original variables and the $W$ variables one finds that although $e^{i\phi_1}$ and $\partial e^{-i\phi_1}$ lead to well defined operators, many operators, such as $e^{-im\phi_1}, m = 1, 2, 3, \ldots$, do not. As such, when wishing to consider the cohomology of $Q_1$ relevant to the $W$ string we must restrict ourselves to operators which are well defined in terms of $W$ variables and so generated from the operators $e^{i\phi_1}, \partial e^{-i\phi_1}$ in the sense of including all operators contained in the repeated operator product expansions of these operators. This construction includes $e^{in\phi_1}; n = 0, 1, 2, \ldots$, but excludes $e^{-im\phi_1}; m = 0, 1, 2, \ldots$. The charge $Q_2$ is readily found to commute with $e^{i\phi_1}$ and $\partial e^{-i\phi_1}$ and hence all well defined operators. Thus the criterion for operators when written in terms of original variables to be well defined in terms of $W$ variables is that they commute with $Q_2$. Since $\phi_2$ is unchanged by the transition between the two variables, it does not occur in the criterion for being well defined. One can arrive at this result another way, the operator $Q_2$ in terms of $W$ variables takes the form $\oint dz \partial (e^{-i\hat{\phi}_1})$ which is a total derivative. Consequently, acting on any state which is well defined it must vanish.

The above discussion means that when working in terms of original variables we should consider the “well defined” cohomology of $Q_1$ which are states which are well defined in the above sense and in the cohomology of $Q_1$. Such a state, $\psi$ must satisfy

$$Q_1 \psi = Q_2 \psi = 0$$

and we regard two states $\psi'$ and $\psi$ as being equivalent if they differ by $Q_1 \wedge$ where $\wedge$ is a well defined state, ie $Q_2 \wedge = 0$. Since the cohomology of $Q_2$ is trivial this means two equivalent states should differ by $Q_1 Q_2 \wedge$. Another way of stating this result is that the “well defined” cohomology of $Q_1$ is given by

$$\frac{\ker Q_1 \cap \ker Q_2}{\text{Im} Q_1 Q_2} \equiv H_{Q_1}^{WD}$$

We have assumed that $Q_1$ and $Q_2$ have well defined action on all the states we wish to consider. This property is true if we consider Fock spaces based on the exponentials

$$\exp(in\phi_1 + \frac{q}{\sqrt{k(k+2)}\phi_2})$$

for $n, q \in \mathbb{Z}$ and we restrict our space of states to be of this form. In terms of the $W$ string variables $\rho$ and $\varphi$ this restriction means the states can be written in terms of the ghosts $d = e^{-i\rho}$ and $e = e^{i\rho}$ and have $\varphi$ momenta quantized in units of $\sqrt{k(k+2)}$. It will often prove useful to reparameterize this exponential in terms of $\ell$ and $m \in \mathbb{Z}$, $\ell - m \in 2\mathbb{Z}$.
where \( n = \frac{\ell - m}{2}, q = \frac{\ell - m}{2}k - m \). We refer to space of oscillator states based on such an exponential as \( F^\ell_m \) and the exponential itself are \( \phi^\ell_m \).

The cohomology of equation (9) is precisely that discussed in reference [7] where it was argued that it coincided with the usual cohomology studied in parafermionic theories. The latter is usually formulated in terms of the charge \( Q_3 = S \) and it is known to consist of descendants of the parafermionic primary fields which are not themselves parafermionic highest weight states. It was shown in reference [7] that for the cohomology of equation (9) there existed an isomorphism between the cohomology of \( F^\ell_m \) and \( F^\ell_{m+2nk} \). As such we can restrict our attention to \(| m | \leq k\).

The parafermions \( \psi_{+1} \) and \( \psi_{-1} \) can be written in the form of a commutator with \( Q_2 \) and \( Q_1 \) respectively. Using these results, the operator product expansion \( \psi_1(z)\psi_p(w) = (z - w)^{2\ell}c_{1,p}\psi_{p+1}(w) \), and the analogous result for \( \psi_{-p} \), one finds [15] that

\[
\psi_p \propto [Q_2, \phi^{-(k+2)}_{-k+2p}], \quad \psi_{-p} \propto [Q_1, \phi^{-(k+2)}_{k-2p}]
\]

Evaluating these expressions yields the results

\[
\psi_p \propto \left( e^{-i\phi_1} \frac{\partial^p}{p!} e^{i\phi_1} \right) e^{-ip\phi_1 - p\phi_2 \sqrt{\frac{k+2}{k}}} = (-1)^{p+1} \frac{p+1}{p!} e^{ip\phi_1 - p\phi_2 \sqrt{\frac{k+2}{k}}}
\]

\[
\psi_{-p} \propto \left( e^{-i\phi_1} \frac{\partial^p}{p!} e^{i\phi_1} \right) e^{ip\phi_1 + p\phi_2 \sqrt{\frac{k+2}{k}}}
\]

The fields \( \psi_k \) and \( \psi_{-k} \) have conformal weight zero and for any field \( L(z) \) we obtain a new field \( (\psi_k L)z = \oint d w \psi_k(w) L(z) \) and similarly for \( \psi_{-k} \). Using equation (12) one can show that the operator product expansion \( \psi_k(z) \psi_{-k}(w) \) does not contain any singular terms as \( z \to w \), but does contain the identity. As such we may act with \( \psi_{-k} \) to invert the effect of \( \psi_k \) or visa versa. This implements the isomorphism \( F^\ell_m \to F^\ell_{m+2nk} \) discussed above. In fact, one can find expressions that write \( \psi_p \) and \( \psi_{-p} \) as commutators of \( Q_1 \) and \( Q_2 \) respectively.

To explore which states are in \( H^{W.D}_{Q_1} \) let us consider the field

\[
Y^{k+1}(z) = e^{-i(k+2)\phi_1 - \sqrt{k(k+2)}\phi_2(z)}.
\]

which has a conformal weight \( k + 1 \). Taking its commutation with \( Q_2 \) we find

\[
[Q_2, Y^{k+1}(z)] = \frac{1}{(k+1)!} e^{-i(k+2)\phi_1 - \sqrt{k(k+2)}\phi_2} \phi^{k+1} e^{i\phi_1}
\]
\[ \oint \frac{d\varepsilon}{\varepsilon^{k+2}} \exp\{-i(k+2)\phi_1(z) - \sqrt{k(k+2)}\phi_2(z) + i\phi_1(z + \varepsilon)\} \]  

(14)

Taking the commutator with \( Q_1 \), and expanding the bracket with the \( \varepsilon \) inside it one finds that

\[ U^{k+1}(z) = \frac{1}{(k+1)!} \sum_{p=0}^{k+1} (-1)^p \left( \begin{array}{c} k+1 \\ p \end{array} \right)^2 \left[ e^{-i\phi'_1 \partial p} e^{i\phi_1} \right] \left[ e^{-i\phi_1 \partial^{k+1-p}} e^{i\phi_1} \right] \]  

(15)

where \( \phi'_1 = (k+1)\phi_1 - i\sqrt{k(k+2)}\phi_2 \). In fact \( U^{k+1} \) is the spin \( k+1 \) primary field that occurs in \( \psi(z)\psi_1(w) \). This follows from the observation that \( [Q_2, Y^{k+1}(z)] = e^{-i\rho} = d \).

Given any state \( \psi \in H^{W.D.}_{Q_1} \), then \( U^{k+1}_0 \psi = Q_1 Q_2 Y^{k+1}_0 \psi \) and consequently unless \( \psi \), which can be chosen to be an eigenvalue of \( U^{k+1}_0 \), has eigenvalue zero it will be in the trivial equivalence class of \( H^{W.D.}_{Q_1} \). If we restrict our attention to states \( \psi \) which are exponentials \( \phi^\ell_m \) of the form of equation (9) then they are annihilated by \( Q_1 \) and \( Q_2 \) if \( \ell \geq |m| \). Their eigenvalue of \( U^{k+1}_0 \) is found from

\[ U^{k+1}_0 \phi^\ell_m(z) = \oint (w - z)^k U^{k+1}(w) \phi^\ell_m(z) \]

\[ = \sum_p (-1)^p \left( \begin{array}{c} k+1 \\ p \end{array} \right) \oint \frac{d\tau}{\tau^{p+1}} \oint \frac{d\varepsilon}{\varepsilon^{k+2-p}} \oint dw (w - z)^{k-n(k+1)+q-n} \]

\[ (w + \tau - z)^n(k+1) - q(w + \varepsilon - z)^n \exp\left(-i\phi'_1(w) + i\phi'_1(w + \tau)\right) \]

\[ - i\phi_1(w) + i\phi_1(w + \varepsilon) + i\phi_1(z) + \frac{q}{\sqrt{k(k+2)}} \phi_2(z) \]

\[ = \lambda \phi^\ell_m(z) \]  

(16)

where

\[ \lambda = \sum_{p=0}^{k+1} (-1)^p \left( \begin{array}{c} k+1 \\ p \end{array} \right) \left( \begin{array}{c} \ell+m \\ 2p \end{array} \right) \left( \begin{array}{c} \ell-m \\ 2(k+1-p) \end{array} \right) \]  

(17)

The final step is achieved by expanding the brackets containing \( \tau \) and \( \varepsilon \), carrying out the \( w \) integral and observing which terms do not vanish.

Clearly, every term in the above sum vanishes if \( p > \frac{\ell+m}{2} \) and \( (k+1-p) > \frac{\ell-m}{2} \) which imply \( k+1 > \ell \). As such exponentials for which \( \ell \leq k \) and \( |m| \leq \ell \) have eigenvalue zero and can be non-trivial elements of \( H^{W.D.}_{Q_1} \). We have not carried out a systematic search for other solutions, but low level cases suggest that for \( |m| \leq \ell \) these are the only solutions.
We now assume that there are no other solutions. Since $Q_1$ and $Q_2$ commute with the parafermions, an element of $F^\ell_m$ constructed by the action of the parafermionic modes will be trivial if $\phi^\ell_m$ is trivial. As such, cohomologies based on all other exponentials which do not have eigenvalue zero will be trivial. It may be possible to extend this result to the full Fock space. The above argument, when taken with the results of reference [7], allow one to conclude that the cohomology of equation (9) is the same as the more usual cohomology, which is based on $S$, studied in parafermionic theories.

The result of the above discussion is that to find non-trivial elements of $H^W_D$ we must consider Fock spaces based on $\phi^\ell_m = \exp\left(\frac{i}{2}(\ell - m)\phi_1 + \frac{\phi_2}{2\sqrt{k(k+2)}}(\ell k - m(k+2))\right)$ (18)

$\ell = 0,1,...,k$ ; $m = -\ell,-\ell+2,...,\ell$ These are the parafermionic primary fields which have conformal dimensions $\Delta^\ell_m = \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k}$. It is also useful to consider the fields $\hat{\phi}^\ell_m = A_{m+1}^+ A_{m+2}^+ ... A_{m+k}^+ \phi^\ell_m$, $m = \ell,\ell+2,...,2k-\ell$ which have conformal dimensions $\Delta^{k-\ell}_m = \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k} + \frac{m-\ell}{2}$. In particular, one finds that $\hat{\phi}_2^0 \propto \psi_p$. In terms of $W$ string variables, the $\phi^\ell_m$ fields become

$\phi^\ell_m = (-1)^{\frac{\ell-m}{2}} \exp\left(-i\frac{\ell-m}{2} + 1\right) \hat{\phi}^{\ell-m}_m \exp(i\hat{\phi}_1) \exp\left(\frac{\phi_2}{2\sqrt{k(k+2)}}(\ell k - m(k+2))\right)$ (19)

upon use of equation (3).

It was observed in reference [18] that the unitary minimal series for the $W_n$ algebra contained as its first member a theory with central charge $c = \frac{2(n-1)}{n+2}$ which coincided with that for the parafermionic theory with $k = n$ and that the former theory possesses highest weight fields, with respect to the $W_n$ algebra, with dimensions equal to the spin and thermal operators of the parafermionic theory. The parafermions are known [23] to generate in their operator product expansion $\psi_1(z)\psi_{-1}(w)$ a $W$ algebra which is made up from a $W_k$ algebra and an infinite number of higher spin, but null generators. Since the fields $\phi^\ell_m$ are annihilated by $A_{m+p}^+$, $p \geq 1$ and $A_{m-p}^+$, $p \geq 1$ they will be annihilated by all the positive modes of the $W$ algebra. As such the $\phi^\ell_m$ will be highest weight fields with respect to the $W_n$ algebra. The same applies to parafermionic descendants which are themselves parafermionic primary fields. As such, the states of the parafermionic theory carry a representation of the $W_k$ algebra. The uniqueness of the $W$ unitary series then allows one to conclude that the parafermionic theory contains the highest weight states and descendants of the first member of the $W_k$ minimal series. Checking the dimensions of the operators that appear in the two theories for low values of $k$ one indeed finds that there
is a one to one match of the conformal dimensions of the fields of the two theories. The operator product suggests that all bilinear parafermionic excitations $\psi_1 \psi_{-1}$ can be written in terms $W$ descendants and the null nature of the higher spin generators suggests that the irreducible representation contains only the descendants of the $W_k$ algebra. Thus, it seems likely that the parafermionic states can be identified with those of the first member of the minimal series.

Since the $W$ generators are found in the operator product of the parafermions it follows that they commute with $Q_1$ and $Q_2$. Consequently, we can act with them on any element of $H^{W,D}_{Q_1}$ and create another element. From the arguments given above it is clear that states which can be written in terms of the spin $k + 1$ $W$ generator are cohomologically trivial and the case of the parafermions with $k = 2$ suggests that the same applies to all the infinite number of higher spin $W$ generators. However, this statement does not apply, in general, to states which can be written in terms of the generators of the $W_k$ algebra. Thus it would seem likely that the cohomology of equation (9), is none other than the states of the first member of the minimal series, in agreement with the statement at the end of the above paragraph.

Let us consider a BRST charge $Q$ of the form

$$Q = Q_0 + Q_1$$

where

$$Q_0 = \oint dz c \{ T_{\phi_1,\phi_2} + T^x - \frac{1}{2} T^{b,c} \}$$

For the case of $k = 2$ this is the BRST charge of the $W_3$ string constructed by Thiery Mieg [10] and written in the above form in reference [4]. In the above system of fields we have two types of ghost numbers corresponding to the assignments $(0, -1), (0, 1), (-1, 0)$ and $(1,0)$ for the ghosts $b,c,d$ and $e$ respectively. With this choice, $Q_1$ and $Q_0$ have ghost numbers $(1,0)$ and $(0,1)$ respectively and consequently, $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. For $k \geq 3$ the charge of equation (20) is the BRST charge for the so called $W_{2,s}$ strings discussed in reference [16]. While for $k = 2$ it is the BRST charge for the $W_3$ string.

The charge $Q_0$ can be regarded as that for a critical string constructed from $\phi_1, \phi_2, x^\mu$ with the usual ghosts $b,c$. Although three of the former fields possess background charges we can, by a linear transformation, rotate the background charge into one direction without affecting the cohomology. The physical states for such a system were found [12] to be constructed from D.D.F. like operators in the same way as for the bosonic string without a background charge. It would then seem most likely that one could apply the arguments of reference [13] to conclude that the cohomology of $Q$ contained operators of the form

$$cR(\phi_1, \phi_2, x^\mu)$$
Given $\chi = \sum_{p+q=r} \chi^{(p',q')}$, the equation $Q\chi = 0$ can be separated into a number of equations, one for each ghost number pair $(p',q')$. For the highest $q$ ghost number in $\chi$, $Q_0\chi^{(p,q)} = 0$ and the lower ghost number equations are $Q_1 \chi^{(p,q)} + Q_0 \chi^{(p+1,q-1)} = 0$ etc. This situation is just that required to apply the theory of the spectral sequence of a double complex to deduce the cohomology [14]. In this method one constructs a series of spaces $E_r^{(p',q')}$ and operators $d_r : E_r^{(p',q')} \to E_r^{(p'+r,q'-r+1)}$ which eventually terminate in the sense that $d_s$ vanishes or $E_{s+1}^{(p',q')} = E_s^{(p',q')}$. The cohomology of $Q$ is then $H^t_Q = \sum_{p'+q' = t} E_s^{(p',q')}$. Applying this method to our case, we have $E_1 = H_{Q_0}$ and $E_2 = H_{Q_1}(H_{Q_0})$ with $d_1 = Q_1$.

Carrying out the first step of the spectral sequence, we consider a non-trivial cohomology class of $Q_0$ as our highest ghost number state. When $q = 1$, this state is of the form of equation (22), we have $\chi = cR + U$ terms of lower $q$ ghost number, where $R$ and $U$ are conformal fields of weight 1 and 0 respectively. Applying $Q\chi = 0$ we find $Q_1 cR = -Q_0 U$ which implies that

$$Q_1 R = \partial U$$  \hspace{1cm} (23)

This equation in turn implies that $Q_1 U = 0$. In principle, even though $U$ has $q$ ghost number zero it could contain a $bc$ term. However, an examination of this possibility shows that such a term must have zero coefficient and we believe this applies to all terms with more than one $b$. Since $Q_1 U = 0$ the sequence of equations terminates.

Applying a similar analysis to the case where the highest ghost number in $\chi$ has $q = 1$ i.e $\partial ccR$ one finds that $Q_1 R = 0$. The identity state and its conjugate are automatically annihilated by $Q_0$ and $Q_1$ and so are also members of the cohomology of $H_Q$ by themselves. In terms of the spectral sequence method the above implies that the operator $d_2$ which
maps $\chi^{p,q}$ to $Q_1\chi^{p+1,q-2}$ vanishes and so the cohomology of $Q$ is given by $H_Q = H_{Q_1}H_{Q_0}$.

Clearly, we can choose $U$ to be the identity operator; $\phi_0^0 = 1$ in which case $Q_1 R = 0$. In this case the physical states are annihilated by $Q_1$ and $Q_0$ and are a product of an operator function of $\phi_1$ and $\phi_2$ and one of $x^\mu$. States of this type are of the form

$$c\phi^\ell_m \sqrt{x^{-\Delta^\ell_m}}$$

where $\sqrt{x^{-\Delta^\ell_m}}$ is a conformal field constructed from $x^\mu$ of weight $1 - \Delta^\ell_m$. Clearly, for every non trivial element of $H_{Q_1}^{W,D}$ of fixed conformal weight we can find a physical state. As discussed above, we will, in general, find $W_k$ or parafermionic descendants of $\phi^\ell_m$ which are elements of $H_{Q_1}^{W,D}$ and so they can be used to construct physical states. All these states have an $x^\mu$ momentum that is not fixed and are sometimes called continuous momentum states. For the cases of $k = 3, 4$ and 5 we can recover the low level states found in reference [16] using mathematica.

Other physical states can be found by choosing other elements $U$ in $H_{Q_1}^{W,D}$ which have conformal weight 0. Examining the weights of $\phi^\ell_m$ and $\hat{\phi}^\ell_m$, we find the only possible solutions are $\phi^k_k$ and $\hat{\phi}^0_{2k} = \psi_k$. We also consider $\psi_{-k}$. A solution to $Q_1 R = L_{-1} U$ is found in each case by considering $R$ to be proportional to $d_{-1}\phi^k_k$ and $d_{-1}\psi_{\pm k}$. In the former case, working in terms of $W$ variables we find that

$$d_{-1}\phi^k_k = e^{i(k+1)\hat{\phi}_1 - \sqrt{k+2}(k+3)\phi_2}$$

and

$$Q_1(d_{-1}\phi^k_k)(z) = \oint \frac{d\varepsilon}{\varepsilon^{k+2}} \oint_{z} dw (1 + \frac{\varepsilon}{w - z})^{k+1} (w - z)^{k+1} \exp\{-i\hat{\phi}_1(w) + i\hat{\phi}_1(w + \varepsilon) - i(k+1)\hat{\phi}_1(w) + \sqrt{k(k+2)}\phi_2(w) + i(k+1)\hat{\phi}_1(z) - \sqrt{\frac{k}{k+3}}(k+3)\phi_2(z)\}$$

$$= -(k+2)\partial\phi^k_k$$

Similarly, one finds that

$$Q_1(d_{-1}\psi_k)(z) = -\binom{2k+1}{k} \partial\psi_k, \quad Q_1(d_{-1}\psi_{-k})(z) = -\binom{2k+1}{k+1} \partial\psi_{-k}.$$  

In the above cases, we set $R = d_{-1} U$, whereupon equation (23) becomes $W_{-1} U \propto L_{-1} U$ which we recognise as a highest weight descendant of the $W_k$ algebra.
The above states are those in the cohomology of $Q$ subject to the restriction $\ell \geq |m|$. As discussed earlier, we may construct, using the isomorphism between $F^\ell_m$ and $F^\ell_{m+2k}$, an infinite number of non-trivial elements of the cohomology of $Q_1$ and so also $Q$. Although $\psi_{\pm 1}$ commute with $Q_1$, they do not commute with $Q$. However, we can, using equation (27), extend them to the physical states $\Psi_k \equiv cd_{-1}\psi_k + \left(\frac{2k+1}{k}\right)\psi_k$ and $\Psi_{-k} \equiv cd_{-1}\psi_{-k} + \left(\frac{2k+1}{k+1}\right)\psi_{-k}$ which do commute with $Q$. Thus, given any state $\phi$ in the cohomology of $Q$ we can generate an infinite number of states given $\Psi_{\pm k}\phi$. For the case of $k = 2$, evidence based on mathematica calculations, was given for the above states. One can also use the parafermions $\psi_{\pm 1}$ or the screening charge $S$ in conjunction with an appropriate picture changing operator $P$ to generate an infinite number of physical states. In the latter two methods one need work with basic states that are only a subset of the states included in the above restriction. These latter constructions are discussed in more detail below.

We have taken care to impose that the physical states when written in terms of $W$ variables are local. However, it could happen that some of these physical states which are local in terms of $W$ variables are non-local in terms of original variables. The criterion that a state when written in terms of $W$ variables is local in terms of original variables is that it is annihilated by $Q_w = \oint dze^{-i\phi} = \oint dze^{(-i(k+1)\rho+\sqrt{k(k+2)}\varphi)}$. Since the parafermions and the $\phi_m$ are well defined in terms of $W$ variables, they are annihilated by $Q_w$. Consequently, all the above states and those found using the isomorphism are annihilated by $Q_w$. The conjugates of these states are constructed using not only the parafermions, but also the picture changing operator. However, the picture changing operator is not well defined in terms of original variables and so neither are the conjugate states.

For the $W_3$ string, $k = 2$, it was shown in reference [6] that all the physical states could be constructed from four basic states by applying the screening charge $S$ and the picture changing operator $P$ in an appropriate manner. The continuous momentum operators could be found from the basic operators $a(h, 0)$ for $h = 0, \frac{1}{2}$ and $\frac{1}{16}$ in the notation of that reference, while for the discrete operators one used the basic operator $D(0)$ of reference [4]. An alternative method [4] of finding these operators was found by repeatedly applying the parafermion $\psi_{-1}$ to the operators $a(0,0)$, $\bar{a}(0,0)$ and $a(\frac{1}{16}, 0)$. In addition to these operators, the cohomology also contains the conjugates of the above operators that can be generated by the action of screening charges or the parafermion $\psi_1$ and picture changing operator on the basic states[4]. These states were recovered [19] by an alternative, but subsequent method, which is related to the use of screening charges of reference [6]. A number of arguments [12,20,6,4,19] have been given to suggest that these are all the physical states in the $W_3$ string.

The parafermionic primary operators $\phi_m^l$ for $k = 2$ are given by $\phi_0^0$, $\phi_1^1$, $\phi_{-1}^2$, $\phi_2^3$, and $\phi_{-2}^2$. These are none other than the operators $a(0, 1)$, $a(\frac{1}{16}, 1)$, $a(\frac{1}{16}, 2)$, $\bar{a}(0,0)$, $\bar{a}(\frac{1}{16}, 0)$.
and \( \bar{a}(0,1) \) respectively. While taking \( U \) to be \( \psi_2 \) and \( \phi_2^2 \) leads to the discrete states \( D(0) \) and \( \overline{D}(0) = SPD(0) \) respectively. Thus for the \( W_3 \) string, the states of equations (24) to (27), and those generated from them by using the isomorphism relating \( F_m^l \) to \( F_{m+2k}^l \), and their conjugates account for all states in the cohomology of \( Q \).

One can repeat the above pattern of the \( W_3 \) string for the \( W_{2,k+1} \) strings. Rather than use the isomorphism generated by \( \Psi_{\pm k} \) we can use the method of the screening charges. One finds that the action of \( S^n\phi_m^\ell \) is well defined if \( n\left(\ell-\frac{n+1}{2}\right)-\frac{nm}{2} \in \mathbb{Z} \). It is straightforward to find the solutions to this equation. For example, one finds that if \( k = 3 \) then \( S^5\phi_m^\ell \) and \( S^{5q+\ell+1}\phi_m^\ell \), \( q \in \mathbb{Z} \) are the allowed states; while if \( k = 4 \) then \( S^{3q}\phi_m^\ell \) for \( q \in \mathbb{Z} \) and \( S^{3q+\ell+1}\phi_m^\ell \), where \( q \in \mathbb{Z} \) if \( \ell \) is even and \( q \in 2\mathbb{Z} \) if \( \ell \) is odd. In this construction one must insert appropriate picture changing operators to obtain a non-zero result. One can also use the action of the parafermions \( \psi_{\pm 1} \) to create new states. Although conceptually simpler, it is more difficult to find explicit expressions for the states using this method. Unlike the \( W_3 \) string, we do not expect to get all states in this construction unless, as discussed, above, we must take account the \( W_k \) or parafermionic descendants. One could also generate states using the screening charge \( Q_w \) and the picture changing operator which was used for the case of \( k = 2 \) in reference [16].

It should be possible to extend the techniques used in this paper to find not only all the physical states in the \( W_{2,s} \) strings, but also those for the \( W \) strings. This could be achieved by exploiting the knowledge of \( H^{W,D}_Q \) to solve the full cohomology of \( Q \). This is the strategy advocated in reference [24] for the case of \( W_4 \) strings.

Under the change of variables \( \phi_1 \rightarrow \phi_1', \phi_2 \rightarrow \phi_2' = -i\sqrt{k(k+2)}\phi_1 - (k+1)\phi_2 \) one finds that the energy momentum tensor maintains its form and so \( Q_0 \rightarrow Q_0', \psi_1 \rightarrow k\psi_1', \psi_{-1} \rightarrow \frac{1}{k}\psi_1 \) and \( Q_1 \leftrightarrow Q_2 \). Since the cohomology relevant to the \( W \) string is given by equation (9) this change of variables is a symmetry of the physical states. It would interesting to see if this extends to the scattering amplitudes and what is the significance of this symmetry.

In recent series of papers it has been shown that the ordinary bosonic string can be embedded in the \( N = 1,2 \) superstring [21], and the \( W_3 \) string [22]. However, in order to achieve these results, in particular the matching of the cohomologies, non-local field redefinitions similar to the one of equation (3) are required. As such, this paper should shed some light on the relations between these various theories.

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