The Bumblebee Model of spontaneous Lorentz symmetry breaking is explored in a cosmological context, considering a single non-zero time component for the vector field. The relevant dynamic equations for the evolution of the Universe are derived and its properties and physical significance studied. We conclude that a late-time de Sitter expansion of the Universe can be replicated, and attempt to constrain the parameter of the potential driving the spontaneous symmetry breaking.

I. INTRODUCTION

Several outstanding questions in cosmology, such as the current accelerated expansion of the Universe [1] or the presence and origin of dark matter [2], have motived many efforts to expand upon the conventional formalism of General Relativity (GR), as the latter cannot provide a direct explanation of such phenomena. A great number of attempts have been made to phenomenologically study putative modifications of GR, ranging from the inclusion of non-trivial curvature terms in the Einstein-Hilbert action [3, 4] to the addition of suitable scalar [5] and vector fields [6–8], among others [9].

Amongst the latter, the Bumblebee Model was initially put forwarded in 1989 [10] during a study of Lorentz symmetry-breaking (LSB) in compactified higher-dimensions, being fully fleshed out by 2010 [11] in an attempt to extend the Standard Model to the gravitational sector. Being an extension of GR, it takes the usual Einstein-Hilbert action and expands it with a vector field and a potential.

The main differences between the Bumblebee and the other, more well-known vector models, the Einstein-Æther models [6, 7], are mainly due to the inclusion of a coupling between the curvature (in the form of the Ricci tensor, \( R_{\mu\nu} \)) and the vector field, meaning the dynamic equations of the system and therefore the perception of how gravity works will be fundamentally different from that of the GR picture. Furthermore, the æther models explicitly break Lorentz symmetry, effectively discarding it as a defining symmetry of the action. The Bumblebee model, on the other hand, incorporates a mechanism of spontaneous breaking of the Lorentz symmetry, which is compatible with general Riemann-Cartan geometries [8, 11–15]. This mechanism is inspired by the Higgs mechanism of the Standard Model of fundamental particles and interactions, (itself an allusion to the Landau theory of phase transitions that first originated in Condensed Matter Physics) and serves to both safeguard Lorentz symmetry as a symmetry of the action and regulate the way in which that symmetry is broken.

II. THE BUMBLEBEE MODEL FOR SPONTANEOUS LORENTZ SYMMETRY BREAKING

As mentioned in the previous section, the Bumblebee model extends the standard formalism of General Relativity by allowing a LSB; this is dynamically driven by...
a suitable potential exhibiting a non-vanishing vacuum expectation value (vev), so that the Bumblebee vector field $B_\mu$ acquires a specific four-dimensional orientation. Thus, the action functional is written as

$$S = \int \sqrt{-g} \left[ \frac{1}{2\kappa} (R + \xi B^\mu B^\nu R_{\mu\nu}) - \frac{1}{4} B^\mu B^\nu B_{\mu\nu} - V(B^\mu B_\mu \pm b^2) + \mathcal{L}_M \right] d^4x,$$  

(1)

where $\kappa = 8\pi G$, $\xi$ is a coupling constant (with dimensions $[\xi] = M^{-2}$), $B_\mu$ is the Bumblebee field (with $|B_\mu| = M$), $B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength tensor, $b^2 \equiv b_\mu b^\mu = \langle B_\mu B^\mu \rangle_0 \neq 0$ is the expectation value for the contracted Bumblebee vector, $V$ is a potential exhibiting a non-vanishing vacuum expectation value ($\xi B^\mu B^\nu \mu\nu \neq 0$) and $\mathcal{L}_M$ is the Lagrangian density for the matter fields.

By varying Eq. (1) with respect to the metric, the modified Einstein equations are obtained,

$$G_{\mu\nu} = \kappa \left[ 2V'B_\mu B_\nu - B_\mu B_\nu - \left( V + \frac{1}{4} B_{\alpha\beta} B^{\alpha\beta} \right) g_{\mu\nu} \right] + \xi \left( \frac{1}{2} B^\alpha B^\beta R_{\alpha\beta} - 2B_\mu B^\alpha R_{\alpha\nu} - B_\nu B^\alpha R_{\alpha\mu} \right)$$

$$+ \frac{1}{2} \nabla_\alpha \nabla_\mu (B^\alpha B_\nu) + \frac{1}{2} \nabla_\alpha \nabla_\nu (B^\alpha B_\mu) - \frac{1}{2} \nabla_\mu \nabla_\nu B_{\alpha\beta} g_{\mu\nu} - \frac{1}{2} \Box \left( \nabla_\mu B_\nu \right) + T_{\mu\nu},$$

(2)

where $V'$ denotes the derivative of the potential $V$ with respect to its argument and $T_{\mu\nu}$ is the stress-energy tensor of matter.

Variation of Eq. (1) with respect to the Bumblebee field yields its equation of motion,

$$\nabla_\mu B_{\mu\nu} = 2 \left( V' B^\nu - \frac{\xi}{2\kappa} B_\mu R_{\mu\nu} \right).$$

(3)

If the l.h.s. of the equation vanishes, the above results in a simple algebraic relation between the Bumblebee, its potential and the geometry of spacetime.

**III. COSMOLOGY**

In most cosmological studies, the Friedmann-Robertson-Walker metric (FRW) metric is assumed, reflecting the Cosmological Principle which posits an homogeneous and isotropic Universe. However, if Lorentz symmetry is spontaneously broken, it is possible that the Bumblebee field acquires a non-vanishing spatial orientation, which would dynamically break the aforementioned isotropy and require a more evolved geometry. This possibility (elaborated in the final section) shall not be pursued in this study: instead, one considers the Ansatz for the Bumblebee field,

$$B_\mu = \left( B(t), 0 \right),$$

(4)

thus upholding the validity of the Cosmological Principle, i.e. maintaining the assumption of large-scale homogeneity and isotropy of the Universe. One thus adopts the flat FRW metric, as given by the line element,

$$ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right],$$

(5)

where $a(t)$ is the scale factor.

From Eq. 4, one sees that the field strength tensor vanishes, $B_{\mu\nu} = 0$, and the only non-trivial component of the Bumblebee Eq. (3) is

$$\left( V' - \frac{3\xi \dot{a}}{2\kappa a} \right) B = 0,$$

(6)

which, for a non-vanishing Bumblebee field, establishes a relation between the dynamics of the potential and the scale factor. This is probably the reason that motivated the authors of Ref. [7] to describe Bumblee-type models as “non-dynamical” — we maintain, however, that the non-minimal coupling of curvature in our case is enough to drive a meaningful process, as shall be studied below.

Using the above expression together with Eq. (4), the $t-t$ component of the modified field Eqs. (2) becomes

$$H^2 (1 - \xi B^2) = \frac{1}{3} \kappa (\rho + V) + H B \dot{B},$$

(7)

while the diagonal $i-i$ components read

$$H^2 + \frac{\dot{a}}{a} (1 - \xi B^2) =$$

$$\kappa V + \xi \left( H^2 B^2 + 4H B \dot{B} + \dot{B}^2 + B \ddot{B} \right),$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Subtracting the two equations above and using the Bumblebee field Eq. (6) yields

$$\kappa \rho = 2 \left( H^2 - \frac{\dot{a}}{a} \right) (1 - \xi B^2) + \xi \left( H B \dot{B} + \dot{B}^2 + B \ddot{B} \right).$$

(9)

Using the Bianchi identities, one also obtains the following modified equation of conservation of energy,

$$\dot{\rho} = -3H \rho - \frac{3\xi}{\kappa a} B(H B + \dot{B}) + 3\frac{\xi}{\kappa a} B^2,$$

(10)

showing that there is an energy exchange between matter and the Bumblebee field.
Due to the complexity of the equations derived in this section, a simple, closed-form solution cannot be obtained by purely analytical means. Given that difficulty, it becomes more enlightening to study instead the full dynamical picture presented by the Friedmann and Raychaudhuri equations under the constraints imposed by the Bumblebee field equation of motion.

A. Generality of a de Sitter solution

Before going into the detailed analysis of the dynamical system of Eqs. (9) and (10) found in the previous section, there is one preliminary result that can be easily obtained, related to the de Sitter solution of said system of equations, i.e. where the scale factor takes an exponential form,

$$a(t) = a_0 e^{H_0 (t-t_0)} .$$

Thus, the Bumblebee field Equation (6) becomes

$$V' B^2 = \frac{3 \xi}{2 \kappa} H_0^2 B^2 .$$

For a non-vanishing Bumblebee field, any potential $V$ which varies non-linearly yields a quantity $V'$ that depend on $B$, and the above implicitly establishes that the Bumblebee field must be constant through $V'(B^2 \pm \xi b^2) = (3\xi/2\kappa)H^2_.$. Of course, if $B = 0$ this relation does not hold, but the Bumblebee remains constant nevertheless. Using $B = B_0 = const.$ on Eqs. (7), (9) and (10) yields

$$H^2_0 = \frac{\kappa V_0}{3(1 - \xi B^2_0)} , \quad \dot{\rho} = -3H_0 \rho = 0 ,$$

where $V_0 \equiv V(B^2_0 \pm \xi b^2)$; this corresponds to a Universe without matter and dominated by the Bumblebee field alone, which acts as a form of dark energy.

Given a particular potential $V$, Eq. (12) and the above fully determine both $H_0$ and the non-vanishing value of $B_0$. Anticipating the study of the next study, one considers a power-law potential of the form

$$V(B^2 \pm \xi b^2) = M^{4-2n}(B^2 \pm \xi b^2)^n \rightarrow V' = \frac{nV}{B^2 \pm \xi b^2} ,$$

where $M$ has dimensions of mass. Using Eqs. (13) and (12) for $B \neq 0$, one obtains

$$B^2_0 = \frac{2n \mp \xi b^2}{\xi(1 + 2n)} \rightarrow$$

$$V_0 = \left( \frac{2n}{1 + 2n} \frac{1 \pm \xi b^2}{\xi M^2} \right)^n M^4 \rightarrow$$

$$H^2_0 = \frac{2n \kappa}{3 \xi} \left( \frac{2n}{1 + 2n} \frac{1 \pm \xi b^2}{\xi M^2} \right)^n M^2 .$$

If the Bumblebee field vanishes, then one trivially obtains $H^2_0 = \kappa V_0/3$.

Notice that in the above one does not simply find that $B^2 = \mp b^2$ and $V(B^2 \pm \xi b^2) = V(0) = 0$, as could be naively expected if one assumed that the cosmological dynamics should evolve the Bumblebee field until it rests at its non-vanishing vev $< B^2 >= b^2$: indeed, the dynamical effect of the coupling term $\xi B^2 B^\alpha R_{\mu \nu}$ found in the action functional (1) on the field equations can be interpreted as a friction term that arrests the evolution of the Bumblebee field, with the dissipated energy acting to counteract the gravitational attraction of matter and drive an accelerated expansion of the Universe.

One can now look back into the on-shell form of the action (1), which with the prescription (4), $B(t) = B_0$ and the De Sitter solution (13) is reduced to

$$S = \int \frac{1}{2\kappa} \left[ R - 6 \left( 1 - \frac{\xi B^2_0}{2} \right) H_0^2 \right] \sqrt{-\mathbf{g}} d^4 x ,$$

which corresponds to the Hilbert-Einstein action with an added constant term which acts as a cosmological constant $\Lambda$: if the Bumblebee field vanishes, the latter is given by its usual definition $\Lambda = 3H_0^2$, by comparison with the usual Lagrangian density $\mathcal{L} = (R - 2\Lambda)/(2\kappa)$; if one considers $B_0 \neq 0$, then it also includes a contribution from the coupling between the Bumblebee field and the Ricci tensor, $\Lambda^* = 3H_0^2(1 - \xi B^2_0/2)$ (the contribution to a phase of accelerated expansion of the Universe arising from a non-minimal coupling between matter and the Ricci scalar and the ensuing dynamical system analysis can be found in Ref. [19]).

IV. ANALYSIS OF THE DYNAMICAL SYSTEM

In order to explore the possible solutions to Eqs. (7), (9) and (10) and confirm the results obtained in the previous section, one begins by defining the following dimensionless variables (see Refs. [20] for a similar treatment in the context of quintessence),

$$x_1 = \frac{\kappa V}{3H^2} , \quad x_2 = \xi B^2 , \quad x_3 = \frac{\xi B \dot{B}}{H} ,$$

together with the usual relative matter density and deceleration parameter

$$\Omega_M = \frac{\kappa \rho}{3H^2} , \quad q = -\frac{\ddot{a}}{a} .$$

Using Eqs. (6), (7), (9) and (10), one obtains

$$x_1' = 2(1 + \alpha x_3 + q) x_1 ,$$

$$x_2' = 2x_3 ,$$

$$x_3' = (1 - 2q)(1 - x_2) - 3x_1 + x_3(q - 3) .$$
where the prime denotes differentiation with respect to the number of e-folds \( N \equiv \log a \) and one defines

\[
\alpha(x_2) \equiv \frac{V'(x_2)}{V(x_2)} , \quad \beta \equiv \frac{\ddot{a}}{aH^3} .
\] (20)

The modified Friedmann equation (7) reduces to

\[
1 = x_1 + x_2 + x_3 + \Omega_M ,
\] (21)

while the Bumblebee equation of motion becomes

\[
(2\alpha x_1 + q)x_2 = 0 .
\] (22)

The two equations above are algebraic constraints that must be obeyed by any solution to the dynamical system (19).

The form of Eq. (22) indicates two possible branches, corresponding to whether \( x_2 \) is vanishing or not. If the latter is true, this equation collapses to \( q = -2\alpha x_1 \), which can then be used to further simplify the dynamical system (19). The opposite case \( x_2 = 0 \) is more troublesome, as it implies that the dynamical system is no longer autonomous (in the sense that \( x_1' = f(x_1) \) alone); one would have to promote \( q \) and \( \Omega_m \) to dynamical variables and consider the two additional differential equations

\[
\Omega_M' = (2q - 1)\Omega_M + \beta x_2 - 2\alpha x_1 (x_2 + x_3) ,
\] (23)

\[
q' = -\beta + q + 2q^2 .
\]

Furthermore, it can be shown that the determination of the nature of any fixed points would be problematic, as the linearisation of the Jacobian matrix of the system around \( x_2 = 0 \) diverges.

This caveat, however, may be circumvented if one notices that \( x_2 = 0 \) is not dynamically relevant if \( x_2' \neq 0 \), as the variable just rolls out of its vanishing value and the relation \( q = -2\alpha x_1 \) becomes immediately valid — allowing the aforementioned substitution into Eqs. (19).

Furthermore, since one is not interested in modelling the spontaneous LSB, it is natural to assume that the Bumblebee field has had time to roll from its previous vev (before the LSB) towards the broken phase \( < B^2 >= b^2 \).

Thus, one is left only with the pathological case \( x_2 = x_2' = 0 \), which is physically hard to interpret, as it would imply that the spontaneous LSB has no dynamical effect on the Bumblebee field, which forever rests at its pre-LSB vev.

Fortunately, it can be solved analytically; since the physical implication of this is that the Bumblebee field only imprints an effect on the dynamics through the constant value of the potential \( V_0 \), one expects to recover the usual picture found in GR for a Universe composed of matter and a cosmological constant, where the former becomes more and more diluted and the expansion grows exponentially at late times.

Indeed, taking Eqs. (19) and (23) and replacing \( x_2 = x_2' = 0 \) immediately yields \( x_3 = 0 \), so that

\[
x_1' = 2 (1 + q) x_1 ,
\] (24)

\[
3x_1 = 1 - 2q ,
\]

\[
\Omega_M' = (2q - 1)\Omega_M .
\]

Notice that the last equation is equivalent to the first, as can be seen from the Friedmann equation (21), which reads

\[
\Omega = 1 - x_1 - \Omega_m' = -x_1' .
\]

One obtains the single differential equation

\[
x_1' = 3(1 - x_1) x_1 ,
\] (25)

with solution

\[
x_1(N) = \frac{1}{1 + C e^{-3N}} \rightarrow
\]

\[
\Omega_m(N) = 1 - x_1 = \frac{C}{C + e^{3N}} ,
\]

\[
q(N) = \frac{1 - 3x_1}{2} = -1 + \frac{3}{2C + e^{3N}} ,
\]

where \( C \) is an integration constant. As expected, one finds that \( \Omega_m \) becomes vanishingly small as the Universe expands and approaches a De Sitter phase \( q = -1 \). One concludes that, even if the Bumblebee field becomes locked in the symmetric phase \( < B^2 >= 0 \) so that \( x_2 = x_2' = 0 \), this does not give rise to any unphysical dynamics, and can proceed towards the study of the more relevant scenario \( x_2 \neq 0 \).

As discussed previously, a non-vanishing Bumblebee field implies that \( q = -2\alpha x_1 \); replacing this into the dynamical system (19) yields the closed set

\[
x_1' = 2(1 + \alpha x_3 - 2\alpha x_1) x_1 ,
\] (27)

\[
x_2' = 2x_3 ,
\]

\[
x_3' = (1 + 4\alpha x_1)(1 - x_2) - 3x_1 - x_3(3 + 2\alpha x_1) ,
\]

and the relation

\[
x_1 + x_2 + x_3 + \Omega_M = 1 .
\] (28)

Following the results obtained in the previous section, one adopts a power-law potential of the form (14), here rewritten as

\[
V(x_2) = M_*^4 (x_2 - \xi b^2)^n ,
\] (29)

where the sign is fixed and \( \xi b^2 \) is allowed to have negative values. It is worthy of note that, for this potential, the variable \( \alpha \) becomes:
\[ \alpha = \frac{n}{x_2 - \xi b^2} . \]  

(30)

The presence of this multiplicative term on two of the three dynamical equations \((27)\) renders its behaviour very important in understanding the evolution of the overall system. Since Eq. (21) implies that, for the matter-dominated Universe \(\Omega_m = 1 \rightarrow x_1 + x_2 + x_3 = 0\), if \(x_2 > \xi b^2 > 0, x_1\) will be positive by definition, as seen in Eq. (29). In that case, \(x_3\) will have to be negative, causing \(x_2\) to decrease until it crosses the value of \(\xi b^2\) and \(\alpha\) diverges (and with it, the system’s trajectories in phase space).

Therefore, one must enforce \(x_{2i} < \xi b^2\), where the former is the initial value of the dimensionless variable \(x_2\): notice that, since one is studying the cosmological dynamics after the spontaneous LSB has occurred and the potential \(V\) has settled into the form \((29)\), this initial value is endowed with physical meaning, as it is related to the value acquired by the Bumblebee field after the LSB (which is not modelled directly).

The inequality \(x_{2i} < \xi b^2\) will imply that \(\alpha\) is always negative, in order to avoid the divergence of the system: the closer the value of \(x_2\) is to \(\xi b^2\), the larger the value of \(\alpha\) will assume, and (as will be seen numerically), the more abrupt the evolution in phase-space will be — so it is expectable that some distance must be allowed between the Bumblebee field and the Ricci tensor. Additionally, since the value of \(\xi b^2\) is not expected to be very large (since \(\xi\) is the constant responsible for the Lorentz-violating curvature coupling and no evidence of such effects have yet been observed), the imposed restraint on the initial values \(x_2 < \xi b^2\) becomes very demanding.

Under this potential, the system exhibits fixed points on

\[
\begin{align*}
\mathbf{x}_A &= (0, 0, 0, 1, \frac{1}{2}) \\
\mathbf{x}_B &= (1, 0, 0, 0, -1) \\
\mathbf{x}_C &= \left(\frac{1 + 3\xi b^2}{1 + 2n}, \frac{2n - 3\xi b^2}{1 + 2n}, 0, 0, -1\right) \\
\mathbf{x}_D &= (0, 1, 0, 0, 0)
\end{align*}
\]

| Table I: Fixed points of the model, Eq. (27). |
|-----------------------------------------------|
| \(\mathbf{x}_A\) & \((0, 0, 0, 1, \frac{1}{2})\) |
| \(\mathbf{x}_B\) & \((1, 0, 0, 0, -1)\) |
| \(\mathbf{x}_C\) & \(\left(\frac{1 + 3\xi b^2}{1 + 2n}, \frac{2n - 3\xi b^2}{1 + 2n}, 0, 0, -1\right)\) |
| \(\mathbf{x}_D\) & \((0, 1, 0, 0, 0)\) |

The fourth fixed point, \(\mathbf{x}_D\), is unphysical, as it corresponds to a Universe without matter and only a constant Bumblebee field, which either expands or contracts linearly or is static (so that \(q = 0\); although it is not mandatory for this fixed point to be repulsive (as it suffices that our Universe undergoes a trajectory in phase space sufficiently far from it), it is a desirable feature.

The analysis of the stability of the system can be achieved from studying the eigenvalues of the Jacobian matrices pertaining to the linearised system in each fixed point, which are:

\[
D_A = \begin{bmatrix}
2 & 0 & -3 - \frac{4n}{\xi b^2} & \frac{4n}{\xi b^2} \\
0 & 0 & -1 & 0 \\
0 & 2 & -3 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

(31)

\[
D_B = \begin{bmatrix}
2 + \frac{8n}{\xi b^2} & 0 & -3 - \frac{4n}{\xi b^2} & 0 \\
0 & -1 + \frac{4n(\xi b^2 - 1)}{\xi b^2} & 0 & 0 \\
- \frac{2n}{\xi b^2} & 2 & -3 + \frac{2n}{\xi b^2} & \frac{2n}{\xi b^2} \\
0 & 0 & 0 & -1 + \frac{4n}{\xi b^2}
\end{bmatrix},
\]

(32)

\[
D_C = \begin{bmatrix}
\frac{1}{n} & 0 & -3 - \frac{1}{n} & 0 \\
0 & 2 & -4 & -\frac{2n + 3n - 2n\xi b^2}{n\xi b^2 - n} \\
1 & 2 & -4 & -3 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

(33)

\[
D_D = \begin{bmatrix}
2 & 0 & -3 & 0 \\
0 & 0 & -1 & 0 \\
0 & 2 & -3 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

(34)

The eigenvalues \(\lambda_i\) for matrices \(D_A\) and \(D_D\) do not depend on \(n\) or \(\xi b^2\):

\[
\lambda_A = (-2, -1, -1, 2) , \quad \lambda_D = (-2, 2, -1, -1) ,
\]

(35)

(36)

and, since at least one of their components has a positive real part, the two associated fixed points are unstable. For the remaining two fixed points, the issue of stability will vary, depending on the values taken by \(n\) or \(\xi b^2\). The specific form of the dependence is quite complex, but it can be understood in a fairly simple way by defining the third degree polynomials (of variable \(\lambda\)):

\[
p_B(\lambda) = \lambda^3 + \xi b^2 \left(\xi b^2 - 10n\right) \lambda^2 + 2 \left(\xi b^2\right)^2 \left[4n + 4n^2 - 17n\xi b^2 - 2 \left(\xi b^2\right)^2\right] \lambda - 4 \left(\xi b^2\right)^3 \left[8n^2 - 2n\xi b^2 - 16n^2\xi b^2 + \left(\xi b^2\right)^3\right],
\]

(37)
\[ p_C(\lambda) = \lambda^3 + 12n^3 (\xi b^2 - 1)^3 + 2n (\xi b^2 - 1) (3\lambda + \xi b^2 - 1) \lambda + 3n^2 (\xi b^2 - 1)^2 (5\lambda + 2\xi b^2 - 2) \]  

(38)

Denoting the three roots of each polynomial \( r_{B,1}, r_{B,2}, r_{B,3} \) and \( r_{C,1}, r_{C,2}, r_{C,3} \) respectively, the eigenvalues for \( D_B \) and \( D_C \) become:

\[ \lambda_B = \left( \frac{4n}{\xi b^2 - 1}, \frac{r_{B,1}}{(\xi b^2 - 1)^2}, \frac{r_{B,2}}{(\xi b^2 - 1)^2}, \frac{r_{B,3}}{(\xi b^2 - 1)^2} \right) \],  

(39)

and

\[ \lambda_C = \left( -3, \frac{r_{C,1}}{n (\xi b^2 - 1)}, \frac{r_{C,2}}{n (\xi b^2 - 1)}, \frac{r_{C,3}}{n (\xi b^2 - 1)} \right) \].  

(40)

The constraints imposed on the stability of these two fixed points by the variations of \( n \) and \( \xi b^2 \) can be found in Fig. 1.

![FIG. 1: Parameter constraints assuming \( x_B \) and \( x_C \) are unstable (white), only \( x_C \) is stable (light grey) and both \( x_B \) and \( x_C \) are stable (dark grey). The points used in the following analysis of phase-space trajectories are highlighted.](image)

It can be seen immediately that only the region pertaining to \(-1/2 < n < 0\) warrants the instability of all of the system’s fixed points, regardless of the value of \( \xi b^2 \). Additionally, if \( n \) and \( \xi b^2 \) have the same sign, then only one stable fixed point is to be found (\( x_C \)), this fact being especially relevant since the simpler and more readily interpretable cases are those for which \( n \) is positive (and small).

If one allows \( \xi b^2 \) to assume negative values while \( n \) remains positive, then the two possible cases of only \( x_C \) being stable or both \( x_B \) and \( x_C \) being stable are separated by a line that can be roughly described by the equation \( \xi b^2 \approx -4.76n \), with the former lying below the line and latter, above. What this means is that to ensure the stability of \( x_B \), \( n \) must increase with \( \xi b^2 \). A similar behaviour can be noted if, on the other hand, \( n \) assumes negative values (lower than \(-1/2\)) and \( \xi b^2 \) is positive, although in this case the equation describing the dividing line appears shifted (closer to \( \xi b^2 \approx -4.76n - 2.38 \)) and the area in which both fixed points are stable only asymptotically approaches the axis at \( \xi b^2 = 0 \). Again, this is worthy of note since, as mentioned before, \( \xi b^2 \) is expected to be small, \( \xi b^2 \ll 1 \).

### V. Trajectory and Evolution Analysis

Following the previous results, one now studies the trajectories of the variables in the phase-space, especially \( x_2 \), \( \Omega_M \) and \( q \), i.e. the Bumblebee field, matter density and deceleration parameter, respectively - which better provide a physical picture of the conditions of the Universe at the given time. One thus numerically integrates the system (27), with initial values corresponding to the departure from the matter dominated era, \( \Omega_M \approx 1 \) and \( q \approx 0.5 \) (the evolution is not affected by choosing initial conditions sufficiently close to these values). A note should be made concerning the initial value of \( x_2 \), or \( x_{2i} \). The default would be to start the numerical simulation at \( x_2 = 0 \), but since the Bumblebee field is in that position only before symmetry is broken, it is reasonable to expect a deviation from the origin, albeit small, for that value.

![FIG. 2: Example of spontaneous symmetry breaking in a quartic potential.](image)

#### A. Example of System Convergence

The case where \( n = 4 \) is depicted in Figs. 3-5, showing that the system collapses entirely to the single attractor, as determined by the constraints displayed in Fig. (1).

These trajectories describe a Universe that transitions from an initial stage of matter domination (\( \Omega_M \approx 1, q \approx 0.5 \)) to a dark energy dominated era (\( \Omega_M \approx 0, q \approx -1 \)). It is interesting to note that the behaviour indicates that instead of evolving smoothly and monotonically towards
FIG. 3: Projection of the phase-space into the $\Omega_M$-$q$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

FIG. 4: Projection of the phase-space into the $x_2$-$q$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

a De Sitter phase, the system overshoots that mark and dwells into $q < -1$ before asymptotically approaching $q = -1$. Since the current value of $\Omega_M \approx 0.7$ is well before that overshoot and cosmographic surveys indicate that $q$ has been steadily declining [21], it is reasonable to assume that the lowest value of $q$ lies still in the future.

Furthermore, because of the discussion surrounding Eq. (30), it can be seen that the restrictions imposed on the initial values of $x_2$ drive this overshoot deeply into negative values, $q \ll -1$: mathematically, this issue could be settled by allowing $x_2 < 0$, which allows for much smoother trajectories (if still overshooting $q = -1$), but results on an added strain on the restriction equation (21), causing $\Omega_M$ to dip into unphysical (negative) values.

FIG. 5: Projection of the phase-space into the $x_2$-$\Omega_M$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

B. Example of system divergence

The second case, for $n = 2$, is depicted in Figs. 6, 7 and 8, and exhibits a divergence of the trajectories. In this case, the system's attractor is insufficient to counteract the repulsive effect of the repulsor $x_A$ close to the system’s initial values at the start of the simulation.

FIG. 6: Projection of the phase-space into the $\Omega_M$-$q$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

In this case the analysis is much more straightforward: the fixed point $x_A$ is so repulsive that it causes the trajectories to steer clear of the close vicinity of the basin of
FIG. 7: Projection of the phase-space into the $x_2$-$q$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

C. Overshoot variation

As mentioned before, the analysed stable trajectories present a behaviour for the deceleration parameter $q$ overshooting its final convergence value at the respective fixed point for the chosen potential. The magnitude of this effect, though it may appear to depend on the initial value taken for $x_2$, indeed flattens out after an initial phase (for approximately $-5.5 < x_{2i} < -2$) and responding linearly, eventually stabilising at a final value of $q_{Min} \approx -15.78$, for $n = 4$, as can be seen in Fig. 9.

FIG. 8: Projection of the phase-space into the $x_2$-$\Omega_M$ plane for $n = 4$ (two darker plots) and $n = 6$ (two lighter plots). Fixed points are marked as dots in dark grey ($n = 4$) and light grey ($n = 6$). For both cases $\xi b^2 = 10^{-2}$ and $x_{2i} = 10^{-3}$ (darker), $10^{-12}$ (lighter).

FIG. 9: Variation of the minimum of $q$ with the initial value of $x_2$ for $n = 4$.

This result seems to suggest two zones of effective system evolution: the first one ($-5.5 < \log_{10}(x_{2i}) < \log_{10}(\xi b^2)$) more dynamical and vulnerable to the initial values, and the second one ($\log_{10}(x_{2i}) < -5.5$) more stable and robust to variations of the initial conditions.

As for the other studied case, $n = 6$, presented in Fig. 10, it depicts a similar behaviour, with the differences being a faster stabilization (linear response is seen for $\log_{10}(x_{2i}) < -4.2$, leveling out at $q_{Min} \approx -20.47$), comparatively to when $n = 4$.

FIG. 10: Variation of the minimum of $q$ with the initial value of $x_2$ for $n = 4$.

VI. CONCLUSIONS AND OUTLOOK

In this work, one has considered the Bumblebee model, a vectorial extension of General Relativity under a spontaneous symmetry breaking mechanism. Since the study is made considering that the Bumblebee vector is only non-trivial in the time component, which depends only on time, no additional breaking of Lorentz symmetry occurs, aside from the one derived from adopting the FRW
metric, which is equivalent to a foliation of space-time into spatial pictures labelled by a cosmological time.

It was found that the model predicts four points of equilibrium, in agreement with the preliminary study of the equations; two are unstable regardless of the value of the model’s parameters, corresponding to the static and matter-dominated cases. The remaining two yield a Universe undergoing an accelerated expansion, which can be ascribed to the arise of a quantity akin to a cosmological constant: this can be caused by the effects of the potential alone (for practically all values of the parameters, with the exception of $-\frac{1}{2} < n < 0$), making the identification $\Lambda = 3H_0^2$, or to the effects of a concerted evolution between the potential and the Bumblebee vector’s non-zero component, resulting in $\Lambda^* = -3H_0^2(1 + V_C)/2$, where $V_C = (1 \pm \xi b^2)/(1 + 2n)$ is the value of the potential on the fixed point $x_C$.

It was noted that, when considering a power-law type of potential, even within the parametrical constraints that guarantee the existence of one attractor, a convergent scenario isn’t always found: in particular, this was not attained in the case of odd powers, which, for all tested values, appear to cause the system’s variables to be strongly repelled initially, thus diverging from the De Sitter attractor. This can be explained by the fact that, for odd $n$, the variable $x_1$ will be initially negative if $x_2 < \xi b^2$ and positive if $x_2 > \xi b^2$. But the discussion concerning Eq. 30 guarantees the system will diverge in the latter case, and in the former, the negativity of $x_1$ disturbs the balance of the Friedmann Equation (21), disallowing convergence.

The above does not guarantee that the system does not allow converging trajectories, only that they are not dense in the (continuous) set of all possible trajectories with physically significant starting points, and even if one was to be found, there would be no basis to argue the great fine-tuning needed for its manifestation.

Additionally, the analysis of convergent trajectories in phase-space predict that $q$ always overshoots the final value of $-1$, but since the present-day Universe is still on the descending slope of the trajectory, prior to that overshoot, such behaviour constitutes a prediction of the model, rather than an experimental constraint (albeit one not to be verified in the foreseeable future).

Expectations concerning $\xi b^2$, $x_2$ and convergence of the system may, however, be used in restricting the window of accessible values to the dynamical system’s variables: the major factors in that process can be categorised into three primary groups: (1) regularity of the solutions, (2) elimination of unphysical behaviour and (3) mathematical stability of dynamical system.

1. $-1/2 < n < 0$: This bound is required from a stability analysis of the fixed points of the dynamical system, as there are no converging trajectories for values of the exponent $n$ outside the indicated window. However, it can be argued that there is no guarantee that the evolution of the Universe must be one of stable convergence, or that even in the presence of attractors, a region of converging trajectories can be found for a specific choice of model parameters $n$ and $\xi b^2$ without fine-tuning the initial conditions.

2. $x_{21} < \xi b^2$: After the spontaneous LSB has occurred and the potential has settled into its form (29), this upper bound for the value of the Bumblebee field must be obeyed, otherwise the system diverges through the action of $\alpha$.

3. $x_{21} \ll \xi b^2$: The stronger restriction, not contemplated in the Friedmann Equation, is related with the behaviour of the deceleration parameter $q$, and ascertains that the minimum of the solution for this variable (the so-called overshoot) shows a tendency to explode if the absolute initial values of $x_2$ and $\xi b^2$ are too close. Once more, this is due to the dimensionless quantity $\alpha$ diverging and dominating the dynamics of the system, making impossible the smooth transition required in order for there not to be an overshoot.

As a final remark, it can be said that the model possesses a fairly rich structure, with the obtained results allowing for more focused subsequent studies. Going beyond further inquiries into the cosmological dynamics here approached (e.g. by selecting a different potential $V(B^2 \pm b^2)$), one anticipates that interesting results should arise from the consideration of a spatially oriented Bumblebee that breaks the homogeneity and isotropy postulated by the Cosmological Principle: this could bear a possible relation with the putative “axis of evil” reported in the cosmological background radiation [22].

A different approach could also consider several local instances of spontaneous Lorentz symmetry breaking, in analogy with Condensed Matter Physics [23]: one can argue that this is compatible with the above-mentioned possibility of a spontaneous LSB in a spatial direction, by considering that the Bumblebee field acquires a particular (random) orientation only within a region with a well defined coherence length — so that, beyond this distance, another orientation is randomly adopted during the LSB. If the size of the observable Universe is much larger than the coherence length, it is physically plausible that the large scale superposition of several regions with different spatial orientations for the Bumblebee field gives rise to a globally homogeneous and isotropic Universe. The collision of different “bubbles” could also produce interesting physics and hypothetically lead to observable relics of the LSB.

The in-depth study of such a mechanism would most probably require modelling the actual LSB (i.e. the evolution of the shape of the potential, so that the vev of the Bumblebee field becomes non-vanishing) and encompass similar considerations to the well-known Kibble-Zurek mechanism [23], albeit with the added complexity of considering a vector field instead of a scalar. By the same
token, one remarks that the interface between regions with different spatial LSB could give rise to topological defects, which would physically correspond to areas where Lorentz symmetry holds.

However mesmerising, the above scenario clearly requires heavy-duty numerical computations, and is thus incompatible with the stated purpose of this work: to clearly identify the possibility of driving an accelerated expansion of the Universe via a non-minimally coupled Bumblebee vector field.

[1] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116 (1998) 1009.
[2] V. Trimble, Ann. Rev. Astron. Astrophys. 25 (1987) 425.
[3] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010) 3.
[4] O. Bertolami and J. Páramos, Int. J. Geom. Meth. Mod. Phys. 11 (2014) 1460003.
[5] I. Zlatev, L. -M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82 (1999) 896.
[6] A. Tartaglia and N. Radicella, Phys. Rev. D 76 (2007) 083501.
[7] C. Armendariz-Picon and A. Diez-Tejedor, JCAP 0912 (2009) 018.
[8] V. A. Kostelecky, Phys. Rev. D 69 (2004) 105009.
[9] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513 (2012) 1.
[10] V. A. Kostelecky and S. Samuel, In Davis 1989, Proceedings, Differential geometric methods in theoretical physics 715-726 (1989).
[11] V. A. Kostelecky and J. D. Tasson, Phys. Rev. D 83 (2011) 016013.
[12] Q. G. Bailey and V. A. Kostelecky, Phys. Rev. D 74(2006) 045001.
[13] M. D. Seifert, Phys. Rev. D 81 (2010) 065010.
[14] O. Bertolami and J. Faramos, Phys. Rev. D 72 (2005) 044001.
[15] G. Guiomar and J. Páramos, Phys. Rev. D 90 (2014) 082002.
[16] J. B. Jimenez and A. L. Maroto, Phys. Rev. D 78 (2008) 063005.
[17] J. Zuntz, T. G. Zlosnik, F. Bourliot, P. G. Ferreira and G. D. Starkman, Phys. Rev. D 81 (2010) 104015.
[18] T. Koivisto and D. F. Mota, JCAP 0808 (2008) 021.
[19] R. Ribeiro and J. Páramos, Phys. Rev. D 90 (2014) 124065.
[20] S. Tsujikawa, arXiv:1004.1493 [astro-ph.CO].
[21] Y. -G. Gong and A. Wang, Phys. Rev. D 75 (2007) 043520.
[22] A. Bershadskii and K. R. Sreenivasan, Phys. Lett. A 319 (2003) 21.
[23] T. W. B. Kibble, arXiv:cond-mat/0211110.