Helical fluid and (Hall)-MHD turbulence: a brief review

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Helicity, a measure of the breakage of reflectional symmetry representing the topology of turbulent flows, contributes in a crucial way to their dynamics and to their fundamental statistical properties. We review several of their main features, both new and old, such as the discovery of bi-directional cascades or the role of helical vortices in the enhancement of large-scale magnetic fields in the dynamo problem. The dynamical contribution in magnetohydrodynamic of the cross-correlation between velocity and induction is discussed as well. We consider next how turbulent transport is affected by helical constraints, in particular in the context of magnetic reconnection and fusion plasmas under one- and two-fluid approximations. Central issues on how to construct turbulence models for non-reflectionally symmetric helical flows are reviewed, including in the presence of shear, and we finally briefly mention the possible role of helicity in the development of strongly localized quasi-singular structures at small scale.

This article is part of the theme issue ‘Scaling the turbulence edifice (part 2)’.

1. Introduction

Lorenz knots, by definition, are generated by the closed periodic orbits of the classical Lorenz attractor. One never stops learning. Of course, one should not. Did you know that all torus knots (the simplest of which being the trefoil knot) are Lorenz knots, as discovered recently by Joan Birman and her collaborators [1] (see also [2]), thus
building a bridge between dynamical systems, chaos and topology? Knots being one of the components of helicity, together with twist and writhe, what is then the role of helicity (kinetic, magnetic, cross, generalized) in the dynamics of fluid and (Hall)-MHD turbulence?

The first discussions of helical structures in turbulent flows were likely made in the context of magnetic fields, which are ubiquitous in the universe; they are also often found to be strong when compared with typical velocities, such as for example in the galaxy. This would create an imbalance in the evolution equation—in the simplest case, the magnetohydrodynamic formulation, were it not for the fact that the Lorentz force exerted by such fields can be rendered weak through alignment of the magnetic induction and the current density. Such force-free fields were thus postulated to exist many years ago, although their origin remained somewhat mysterious. This led Woltjer [3] (see also [4]) to realize that magnetic helicity \( H_M \) (definitions and equations are given in §2) was an invariant of the MHD equations in the absence of forcing and of dissipation, as was the cross helicity (or cross-correlation \( H_C \)) between the magnetic field and the velocity.

It was only some years later that a similar conservation law was unraveled in the fluid case for the kinetic helicity \( H_V \) [5–7]. Helicity corresponds to topological properties of the flows and fields, through knots, links, twists and writhes [6,8,9], and their entanglement. Helicity is a pseudo (axial) scalar, since it can change sign upon change of a reference frame from right-handed to left-handed. This corresponds to the (similar) fact that, in a Serret–Frenet frame describing a string in three-dimensional space, the torsion is pseudo scalar, the line being able to exit the plane formed by the tangent and curvature to the string, upward or downward.

Helicity appears to play a role in disparate areas of research beyond fluid dynamics and plasmas, from DNA and bio-chemistry to meta-materials [10]. It is also involved in the problem of entanglements of vortex lines in quantum fluids [11]. Yet another instance is the orientation of the swimming motions of simple biological systems, with possible applications to nanotechnology: chiral behaviour basically allows for more complex three-dimensional motions and thus to follow physical gradients, e.g. of nutrients. Even in an ocean strongly stratified at large scale, at the size of these microorganisms (of the order of 10 \( \mu \text{m} \)), the flow itself indeed is three-dimensional [12].

There is a revival of interest in the properties of helical flows, some of which we discuss here. We first set up the stage in §2 before reviewing briefly in §3 one of the central roles attributed to helicity in its magnetic form, namely the dynamo. Section 4 discusses the recently uncovered properties of helical flows leading to so-called bi-directional cascades and to the link with the existence of sub-invariants, whereas in §5, we deal with the all-important issue of modelling of such flows, in particular in the presence of an imposed large-scale shear flow. We mention applications of these concepts to our close environment, that of the solar wind, and stress the role of the correlation between the velocity and magnetic fields in altering standard results of transport. Finally, §6 presents a brief conclusion. Several documents can be found useful in going over some of the technical details such as the general form of second- and third-rank tensors in the helical case, issues which will be only briefly mentioned here [13–15] (see also [16] for a recent review).

### 2. Equations, definitions and modelling

Let us write the basic equations for the more general case we wish to consider in this paper, that is Hall-MHD, for an incompressible fluid; \( f_{u,b} \) forcing terms are included for both the velocity \( \mathbf{u} \) and the magnetic field \( \mathbf{b} \) expressed in terms of an Alfvén velocity, with \( \mathbf{b} = \mathbf{B}/\sqrt{\mu_0 \rho_0} \)

\[
\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla P + \mathbf{j} \times \mathbf{b} + \nu \nabla^2 \mathbf{u} + f_u, \quad (2.1)
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \eta \nabla \times (\mathbf{j} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b} + f_b \quad (2.2)
\]

and

\[ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0. \quad (2.3) \]
$P$ is the pressure, $\mathbf{j} = \nabla \times \mathbf{b}$ is the current density, $\mathbf{a}$ the magnetic potential, with $\mathbf{b} = \nabla \times \mathbf{a}$; $\rho_0, \mu_0$ are the density assumed constant and the vacuum permeability; $\nu$ and $\eta$ are the kinematic viscosity and magnetic diffusivity. The total energy is the sum of the kinetic and magnetic energy, viz. $K_T = K_V + K_M = \frac{1}{2} \langle |\mathbf{u}|^2 + |\mathbf{b}|^2 \rangle$. The Hall current is controlled by the ion inertial length scale $d_i$; the MHD equations are obtained for $d_i = 0$. Finally, when we also have $\mathbf{b} = 0, f_b = 0$, we recover the Navier–Stokes equations for fluid turbulence.

The total kinetic, magnetic, cross and hybrid helicities are defined, respectively, as

$$H_V = \langle \mathbf{u} \cdot \mathbf{\omega} \rangle, \quad H_M = \langle \mathbf{a} \cdot \mathbf{b} \rangle, \quad H_C = \langle \mathbf{b} \cdot \mathbf{u} \rangle, \quad H_G = H_M + 2d_i H_C + d_i^2 H_V,$$

with $\mathbf{\omega} = \nabla \times \mathbf{u}$ the vorticity ($\langle \cdot \rangle$ denotes ensemble average). We also have $H_C = \mathbf{V} \cdot \nabla \times \mathbf{V}$, with $\mathbf{V} = \mathbf{a} + d_i \mathbf{u}$. For $d_i = 0$, the hybrid helicity coincides with the magnetic helicity. Note that $H_V$ is an ideal invariant in the pure fluid case, $H_M, H_C$ in the MHD case and $H_M, H_G$ in Hall MHD.

According to the theorem of Emma Noether [17], invariance properties are due to symmetries of the underlying equations. For helicity, it is the relabelling symmetry of Lagrangian particles. It is at the core of the invariance of potential vorticity in barotropic flows which, for its nonlinear part, is the cross-correlation (and linkage) between vorticity and temperature gradients. It is also responsible for the ideal conservation of cross-helicity in MHD [18] (see also [19,20]). Finally, in Hall-MHD, it is the dual invariance of the magnetic and hybrid helicities that is linked to this relabelling symmetry, for the ion and for the electron fluids [21] (see [22] for extended MHD).

While the fully isotropic fluid case only necessitates one defining function in terms of a field (velocity) correlation tensor, two are needed in the non mirror-symmetric case, corresponding in Fourier space to the energy and helicity spectra. This was re-established recently making use of a helical-vector unit system [23]. It would be of interest to show the equivalence of the second-order Fourier space to the energy and helicity spectra. This was re-established recently making use of the multiple-scale DIA, which is a combination of the original DIA with multiple-scale direct-interaction approximation (DIA), as well as the helical third-order exact law derived in [15,25] (see §4(b) for more details). Thus, we can define the homogeneous isotropic two-mode two-time correlation function of a fluctuating vector field $\chi$ as

$$\langle \chi'(\mathbf{k};t)\chi'(\mathbf{k}';t') \rangle = D_{ij}(\mathbf{k})Q_{\chi}(k; t, t') + \frac{i}{2} \left( \frac{k_i}{k^2} \right) \epsilon_{ijl} H_{\chi}(k; t, t');$$

where $D_{ij} = \delta_{ij} - k_i k_j / k^2$ is the projection operator implementing incompressibility, and $Q_{\chi}, H_{\chi}$ are the energy and helicity spectral functions of the fluctuations of field $\chi$. They will be taken as the lowest-order solutions in a modelling decomposition for dynamos, as described below. Furthermore, note that one can decompose the velocity field on a helical basis $h^{\pm}(\mathbf{k})$ for each Fourier mode $\mathbf{k}$ (see [26,27] and references therein); with time omitted, and with $s = \pm$, we have

$$\mathbf{u}_k = \frac{u^+ h^+_k}{|\mathbf{m}_k|} + \frac{u^- h^-_k}{|\mathbf{m}_k|} + is k \times \frac{\mathbf{m}_k}{|\mathbf{k} \times \mathbf{m}_k|}, \quad \mathbf{m}_k = e_z \times \mathbf{k},$$

where $k^\pm, H^\pm$ will denote their $\pm$ modal energy and helicity.

In the pure fluid case ($\mathbf{b} \equiv 0$), Beltrami solutions with $\mathbf{u} = \pm \phi' \mathbf{\omega}$ have been known to exist for a long time. They are fully helical, but they eventually become unstable and turbulence finally develops and decays at the same rate as for random fields, if later. In MHD, Alfvén waves, with $\mathbf{u} = \pm \phi'' \mathbf{b}$, are well-known as well. Note that these Beltrami (force-free) solutions have been generalized to Hall-MHD as $\nabla \times \mathbf{j} = -\phi \mathbf{b}$ and $\mu \mathbf{b} = 0$ [28]. A related phenomenon is that of Taylor relaxation [29] whereby one can minimize the energy while keeping the magnetic helicity constant. Then, through the action of turbulent eddies leading to reconnection of magnetic field lines, as exemplified in fusion configurations, the flow can evolve towards a fully helical state.

We finish this section by mentioning some of the many modelling techniques that can be used in the context of turbulent fields at high Reynolds number $Re = UL/\nu$, with $U$ the mean field of modulus $U$, and with $L$ a typical large scale. Strongly nonlinear and inhomogeneous turbulence lacking reflectional symmetry can be investigated with the aid of the multiple-scale direct-interaction approximation (DIA), which is a combination of the original DIA with multiple-scale analysis [30,31]. On the basis of this multiple-scale DIA, a propagator (correlation spectrum
Q(k; τ, τ′) and response function G(k; τ, τ′) renormalization perturbation expansion, which is suitable for treating strongly nonlinear turbulence at very high Reynolds number [32], can be performed. In addition, the mean-field inhomogeneity effects, including u′(∂U/∂xj) (with u′ its fluctuation) are taken into account through the derivative expansion following the introduction of multiple scales (with both slow and fast variables). The only assumption in this formulation is that the lowest-order velocity field is homogeneous, isotropic and non-mirror-symmetric, and thus can be written as in eqn. (2.5), with χ′ = u′0, namely

\[
\langle u_0'(k; t)u_0'(k'; t') \rangle = D_{ij}(k)Q_0(k; t, t') + \left( \frac{i}{2} \right) \left( \frac{k_1}{k^2} \right) \epsilon_{ijt} H_0(k; t, t');
\]

(2.7)

here, Q0, H0 are the energy and helicity spectral functions of the lowest-order field. The first (Q0-related) term represents the mirror symmetric part, and the second (H0-related) term represents the non-mirror-symmetric part. We see from (2.5) that the breakage of reflectional symmetry enters through the helicity, as expected. For example, the Reynolds stress is expressed as [33]

\[
\langle u_{i}'u_{j}' \rangle = \langle u_{i0}'u_{j0}' \rangle + \langle u_{i0}'u_{j}' \rangle + \langle u_{i}'u_{j0}' \rangle = \left( \frac{2}{3} \right) K_V \delta_{ij} - \nu_T S_{ij} + [\lambda_{HI} \Omega_{sj} + \lambda_{Hj} \Omega_{si}] D,
\]

(2.8)

where \( K_V \) was previously defined. \( S = \{ S_{ij} \} \) is the rate-of-strain tensor of the mean velocity, \( \Omega_s = \{ \Omega_{si} \} = \Omega + 2\omega_F \) the mean absolute vorticity (with \( \Omega = \nabla \times U \) the mean relative vorticity and \( \omega_F \) the angular velocity). Finally, D denotes the deviatoric part of a tensor. Two transport coefficients emerge from this formulation: a classical turbulent eddy viscosity \( \nu_T \) denoting the effect of small scales on large ones, and \( \lambda_H \) which is related to the helicity-gradient. They are expressed in terms of the spectral functions \( Q(k; \tau, \tau_1), H(k; \tau, \tau_1) \) as well as the response function \( G(k; \tau, \tau_1) \), the exact isotropic Green’s function obtained from the renormalization procedure from the bare Green’s function \( G_j'(k; \tau, \tau_1) \) associated with the lowest-order velocity field \( u_0' \) equation as

\[
\nu_T = \frac{7}{15} \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 G(k; \tau, \tau_1)Q(k; \tau, \tau_1) \quad \text{and} \quad \lambda_H = \frac{1}{30} \int k^{-2} d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 G(k; \tau, \tau_1) \nabla H(k; \tau, \tau_1).
\]

(2.9)

The equation of \( G_j'(k; \tau, \tau_1) \) is given as

\[
\frac{\partial G_j'(k; \tau, \tau_1)}{\partial \tau} + \nu k^2 G_j'(k; \tau, \tau_1)
\]

\[- 2iM_{ijm}(k) \int \delta(k - \mathbf{p} - \mathbf{q}) u_{0'i}(\mathbf{p}; \tau) G_{mj}(\mathbf{q}; \tau, \tau_1) d\mathbf{p} d\mathbf{q} = D_{ij}(k) \delta(\tau - \tau_1),
\]

(2.10)

where \( M_{ijm} = \{ k_mD_{ij}(k) + k_l D_{lij}(k) \}/2 \) and \( \delta(x) \) is the Dirac’s delta function.

Note that (2.8) (together with (2.9)) is not heuristically modelled, but is analytically derived from the fundamental equation with the aid of the multiple-scale renormalized perturbation expansion method (which of course comes with its own set of hypotheses). Since \( G, Q \) and \( H \) are related to the turbulence time and length scales, which are key quantities for a model, (2.8) and (2.9) provide a firm basis for turbulence modelling, that helps as well as guides physical intuition (see also §5). For instance, the expressions in (2.9) provide the information on how the transport coefficients depend on which turbulent statistical quantities are involved, including for the central problem of evaluation of the associated model constants.

3. A brief mention of helical and non-helical dynamos

Magnetic induction, like vorticity, is an axial vector, so it may not be surprising after all that kinetic helicity can be involved in the growth of large-scale magnetic fields, that is in the kinematic dynamo problem for a given velocity field. This phenomenon has been amply reviewed, in terms for example of the so-called alpha-dynamos, in many papers and books (see [34,35], and e.g. [36,37] for references). Similarly, magnetic helicity is involved in the saturation (and possibly the
enhancement) mechanism of the large-scale dynamo when the magnetic field is strong enough to react on the velocity [38].

More specifically, the generation of magnetic fields by turbulent motions was described analytically more than 50 years ago; it did emphasize the central role played by kinetic helicity in the large-scale dynamo. Since helicity can be viewed as a specific combination of velocity shear components, one that remains invariant in the inviscid case, one can remark that helical dynamos in fact also cover the case of shear dynamos. When using a classical turbulence model (one similar to what was described in the previous section, but Markovianized), one observes a substantially different critical magnetic Reynolds number for the onset of dynamo action whether or not the flow is helical [39], a result corroborated by DNS [40]. It has also been realized that, even in the absence of helicity but with helicity fluctuations, a dynamo is still at play [41] although it needs a larger separation of scales to be established.

In fact, the effect on large-scale properties of a turbulent flow in MHD in the helical case was first unraveled with a study of statistical mechanics, taking into account all three invariants \((K_T, H_M, H_C)\) [42]. In that case, there are values of the Lagrange multipliers (corresponding to the invariants) that lead to spectra peaking in the large scales, contrary to the pure hydrodynamic fluid with helicity [43]. It is then not surprising that, in the nonlinear regime, it is now the invariance of magnetic helicity and its subsequent inverse cascade to large scales which ultimately furnishes a saturation mechanism in combination with Alfvén waves [38]. Cross-helicity also plays an important role, contributing an extra term to the turbulent electromotive force [44,45], a role also observed in the solar wind [46] (see also [47]).

For small-scale dynamos in MHD, again Alfvén waves play a central role since they bring into equipartition the (small-scale) kinetic and magnetic energies. Chaos was also seen as a determinant in forging such small-scale magnetic fields [48,49] (see also [36]). Indeed, the enhancement of small-scale magnetic field has been shown to take place because of the presence of chaotic streamlines in fully helical (Beltrami) flows for which \(\mathbf{u} = \pm \phi' \mathbf{\omega}\) [50].

The growth of large-scale magnetic energy can be seen as arising from a destabilizing mechanism due to small-scale helical properties. On the other hand, for fluids, the turbulent viscosity computed through a Renormalization Group approach has been shown to be third order in an expansion in terms of wavenumbers \((k \to 0)\), and thus sub-dominant to the eddy viscosity [51] (see also [52]). However, breaking the assumption of isotropy leads to the so-called anisotropic kinetic alpha (AKA) effect [53], which is a three-dimensional large-scale instability of the hydrodynamic flow lacking parity invariance. In the basic procedure deriving the AKA effect, the turbulent Reynolds number is assumed to be small, and the mean velocity is assumed to be uniform over the small-scale length and time. In astrophysical and geophysical environments, some range of fluid motions with three-dimensional fluctuations lacking reflectional symmetry may be approximated as independent of the large-scale inhomogeneities. In such a case, the mean velocity evolution can be treated with the AKA large-scale instability (see [54] for an application to vertical hot towers and the onset of tropical cyclones).

4. The role of invariants on the dynamics

(a) Helical invariants

It was realized, early on in the study of turbulent flows, that invariants, such as the total energy, play a key role since interactions between all modes have to sum-up to zero change in the energy in the absence of viscosity and forcing. Thus, the discovery of the invariance of kinetic helicity \(H_V\) [5,6] led to the examination of its contribution to the dynamics of turbulent flows. Written in vector-product form, the Euler equation \(\partial_t \mathbf{u} = \mathbf{u} \times \mathbf{\omega} - \nabla P^*\), with \(P^* = P + |\mathbf{u}|^2 / 2\) the modified pressure, makes the invariance of kinetic helicity particularly straightforward to derive.

Progress in our understanding of helical processes in fluids has been made recently in a variety of ways. Kinetic helicity has been observed in the laboratory [9], making a grid (with a three-dimensional printer) in the form, for example, of a trefoil knot. Such experiments have
opened the way to detailed investigations of helical fluids. Helicity affects both the processes of vortex twisting and of vortex stretching. These processes may differ for flows more complex than fully developed turbulence (FDT) with, for FDT, stronger intermittency and possibly an inverse cascade of energy [55]. Direct numerical simulations of reconnection processes in fluids were performed for a trefoil knot configuration [56] (as well as for anti-parallel vortices), and helicity spectra and reconnection of vortex structures were recently computed using a vortex method [57].

One would thus expect that, upon the presence of non-zero dissipation and helical forcing at a rate $\varepsilon_V$, a cascade of helicity towards the small scales would take place, as postulated by Kraichnan on the basis of statistical mechanics [43]. Dimensional analysis immediately gives the Brissaud–Frisch–Lesieur–Mazure (BFLM) spectra [58] $K_V(k) \sim \varepsilon^{2/3}_V k^{-7/3}$, $H_V(k) \sim \varepsilon^{-1/3}_V k^{-5/3}$ (with the helicity transfer rate: $\dot{\varepsilon}_V = D_t H_V$). These spectra are not observed, and are deemed to be impossible in a study relying on a closure of turbulence [59], which also indicates the inhibition of energy transfer to small scales due to helicity. Rather, helicity behaves as a passive scalar would, with a cascade Fourier spectrum of the form, with $\varepsilon_V = D_t K_V$

$$K_V(k) \sim \varepsilon^{2/3}_V k^{-5/3} \quad \text{and} \quad H_V(k) \sim \varepsilon^{-1/3}_V k^{-5/3}. \quad (4.1)$$

This allows for a slow recovery of full isotropy at small scales, since $\tilde{\sigma}_V(k) \equiv H_V(k)/[kK_V(k)] \sim 1/k$, whereas in the BFLM spectrum, we have $\tilde{\sigma}_V^{(bflm)}(k) = 1, \forall k$ in the inertial range (note that, in the inviscid case, $|\tilde{\sigma}_V(k)| \sim |k|^{-3} \leq 1$ [43], where $\kappa, \xi$ are the Lagrangian multipliers appearing in the distribution function). It was also found recently that the angle between Fourier components of velocity and vorticity or induction decays as $k^{-3/2}$ in inertial Alfvénic or fluid turbulence [60,61]. Helicity can also be defined in quantum turbulence [62], and this dual cascade of energy and helicity to small scales also occurs in that case, as shown for example using DNS of the Gross–Pitaevskii equations [63]. Since energy and helicity are conserved in the inviscid case for any (Fourier) truncation of the system, this includes the minimal set of three modes, or elementary triadic interaction. This strong conservation property is called detailed balance (see [64] for an analysis of interactions with a small number of triads of wavenumbers).

From the $\pm$ helical base written in equation (2.6), it immediately follows that there are three distinguished subsets for which conservation properties hold (together with using the $\pm$ symmetry). For example, the $\{+/++\}$ subset of three interacting $+$ modes (and also for $\{-/−−\}$) is globally conservative and has been shown to lead to an inverse cascade of energy [65] (see [66] for experimental evidence). It was shown in [67] using DNS that strong fluxes of the $+$ species, and separately of the $-$ species were observed to almost cancel each other out as one approached the smallest excited scales of the flow, allowing for a (slow) return to full isotropy at small scale. Furthermore, when restricting the nonlinear interactions of the Navier–Stokes equations to only one-signed helical modes, it was also shown in [68] that the helicity cascaded to the small scales while the energy cascaded to the large scales, and in fact, in that case, regularity could be proven, as in two space dimensions for these bi-directional constant-flux cascades [69].

There are in fact three "separate" constant fluxes in the helicity cascade [27]. To our knowledge, this study has not been performed for MHD, and it would be of interest to do so. It might help unravel the different results concerning cascades of magnetic helicity [70] as well as in the decay case [71]. Indeed, as for kinetic helicity as we mentioned before, it is not the spectrum predicted by dimensional analysis based on the magnetic helicity injection rate that occurs in DNS (although it does so in Markovian closures of MHD [38]), but it is rather spectra that are significantly steeper. Magnetic helicity spectra $H_M(k) \sim k^{-3.3}$ have been observed, and in that case the large scales play a predominant role, with strong non-local interactions between small scales and large scales. More investigations of these spectra must be performed (see [72] for the fluid case). Finally note that new helical invariants have also been analysed in [73].

In the presence of helicity, with broken mirror-symmetry, the $±$ degeneracy of the fully isotropic case is lifted and the $K^{±}, H^{±}$ fluxes can be distinguished. An inverse energy cascade, for a subset of all the possible $±$ interactions, is found; it is associated with backscatter and eddy noise although this point needs further investigation. The other two sub-categories of interactions are
Different subsets of these interactions conserve different combinations of $K^2$, $H^2$ [27], leading to different sub-dynamics. Kraichnan [43] had already noted the possibility that the $[+/-+]$ and $[-/-]$ modal subsets could lead to (partial) inverse energy cascades, but he also remarked that the other remaining nonlinear interactions would swamp such effects.

Moreover, it is shown in [27] that these several fluxes are constant separately in the inertial range. In fact, the presence of several invariants, beyond global energy and helicity, has also been discussed by a number of authors. For example, the case of helical symmetry, as a generalization of axial symmetry (the axis being now an helix) is discussed in [23]. Such flows are, again, quasi-two-dimensional; they have an infinite number of invariants, like for the two-dimensional Euler equations, and are proven as well to be integrable [74].

A priori, the same type of argument can be used for magnetic helicity, and one expects as well sub-invariants that are separately conserved. It is not clear whether this would explain the behaviour observed in DNS of MHD for three different flows having the same global invariants well sub-invariants that are separately conserved. It is not clear whether this would explain the ratio of the forcing scale to the ion inertial scale $\delta$ in this case, this leads to the Kolmogorov 4/5th law [81]; with $\delta V(r) = \langle u_L(r) \rangle$. This result can be recovered using a simple physical argument in which the dimensional scaling law for the helicity spectra, $H_M^r(k) \sim k^{-2/3}$, plays a central role [80]. This was verified by a series of (moderate-resolution) numerical simulations. It would certainly be of interest to pursue this study for substantially higher numerical resolution at small scale, leading to higher Reynolds numbers $Re$, in order to investigate the role small scales play in the development of inverse cascades in that case. For example, does the $k^{-2}$ inverse scaling of $H_M$ persist at high $Re$ in Hall-MHD?

(b) Exact laws in the presence of helicity

A conservation law, such as that of kinetic energy, is a global volume-integrated property of non-dissipative physics. These laws are in fact rather constraining in the sense that they are valid, in Fourier space, for each individual (but isolated) triad. They also lead, under a suite of hypotheses (homogeneity, isotropy, stationarity, and non-zero dissipation in the limit of high Reynolds number), to an exact relationship linking third-order structure functions to the kinetic energy dissipation rate $\varepsilon_V$ and the spatial distance $r$ between two points. In the simplest (incompressible) case, this leads to the Kolmogorov 4/5th law [81]; with $\delta u(r) = u(x + r) - u(x)$, and with $u_L$ the longitudinal component of the velocity (along $r$), it is written as $\langle \delta u_L(r)^3 \rangle = -(4/5)\varepsilon_V r$. This law has been generalized to include all components of the velocity, to the dynamics of the passive scalar and to that of MHD [82] and Hall-MHD [83]. In the helical case, some rewriting of second- and third-rank tensors taking into account the fact that vorticity and helicity are pseudo (axial) vectors and scalars, was deemed necessary [15, 84]. Vectorial versions of these exact laws stemming from conservation properties can in fact be derived in a simpler manner [85]. In the fluid helical case, the exact relationship for helicity reads, with $\tilde{\varepsilon}_V$ the dissipation rate of $H_M$

$$\langle \delta u_L(r) \delta u_i(r) \delta \omega_i(r) \rangle - \left( \frac{1}{2} \right) \langle \delta \omega_L(r) \delta u_i(r) \rangle^2 = -\frac{4\tilde{\varepsilon}_V}{3} r, \quad \tilde{\varepsilon}_V \equiv D_I H_V . \quad \text{(4.2)}$$

For incompressible helical MHD, working on the magnetic induction equation leads to, with $\mathcal{E}^{\text{turb}} = u \times b$ the electromotive force (EMF) (see [85] for Hall-MHD):

$$3 \langle \mathcal{E}^{\text{turb}}(x) \times \mathbf{a}(x + r) \rangle = \tilde{\varepsilon}_{HM} r, \quad \tilde{\varepsilon}_{HM} \equiv D_I H_M . \quad \text{(4.3)}$$
Five remarks are in order. (i) The magnetic helicity law is not in terms of structure functions, but of correlations functions, likely because a large-scale magnetic field cannot be eliminated from the dynamics (and is the source of Alfvén waves), whereas Galilean invariance allows us to ignore the mean velocity field. (ii) These exact helical relationships involve cross-correlations, e.g. between velocity and vorticity $\omega$, or the electromotive force and the magnetic potential. (iii) In their vectorial form [85], such laws clearly indicate the relationship between the lack of Beltramisation (non-zero Lamb vector, Lorentz force and Ohm’s Law) and the amount of transfer through scales. (iv) These helical expressions have barely been analysed on data, with the notable exception of the fluid case for the EDQNM (Eddy Damped Quasi-Normal Markovian) closure [86], and for a series of DNS [87]. It would be of interest to see such analyses carried out for MHD, both for EDQNM and for DNS, as well as in the context of the solar wind for which more refined data is now available, including at small (ionic and even electronic) scales through the recently launched MMS (Magnetospheric Multi-Scale) mission and the Parker Solar Probe. And finally, (v): the direction of the cascade is not determined by these exact laws which can be observed to change sign, for example in the solar wind for the total energy [88]. In fact, helical invariants in extended MHD, which covers both the regimes of Hall current and electron inertia, have a more complex behaviour [89,90]. $H_C$, which in that paper is called ion canonical helicity, is shown to undergo either a direct or inverse cascade (in the latter case when the magnetic energy is in excess), a fact that may be linked to its dominant physical dimensionality as the ion inertial scale varies. This may also be associated with the non-locality of energy transfer, for example in Hall-MHD (see [91] and references therein).

(c) The role of cross helicity

The cross-helicity $H_C$ is the correlation between the velocity and magnetic fields. It is a conserved volume-averaged correlation directly related to Alfvén waves and, in the compressible case, to magnetosonic waves as well [92]. $H_C$ also plays a role in the estimation of the dissipation and of the magnetic reconnection rate, in particular in shaping the structures within the fluid and at its boundaries [93,94]. It has been shown numerically to be weak in the reconnection region between two large eddies, allowing it to be effective within the fluid [95]. The analysis presented in these papers (see also [96,97]) gives this result a theoretical basis within a wider modelling framework, and it confirms the finding elaborated in [98] (see also [99]) indicating that the development of the cross-helicity spectrum is progressive in time, with positive and negative lobes and finally, at and it confirms the finding elaborated in [98] (see also [99]) indicating that the development of the cross-helicity spectrum is progressive in time, with positive and negative lobes and finally, at

\[
\frac{\partial \mathbf{u}' \cdot \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = \ldots - \mathbf{u}' \cdot \nabla \mathbf{U} + \ldots, \quad \frac{\partial \mathbf{b}' \cdot \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = \ldots + \mathbf{b}' \cdot \nabla \mathbf{U} + \ldots.
\]

It follows that the equation for $\langle \mathbf{u}' \times \mathbf{b}' \rangle$ is subject to the mean velocity shear with the coupling coefficients being the velocity–magnetic-field correlation. This implies that, in the presence of large-scale velocity inhomogeneities, the turbulent cross helicity may contribute to the large-scale magnetic field. In the multiple-scale DIA framework, the turbulent EMF in the mean magnetic induction equation is expressed as (with $\beta$, $\zeta_R$, $\gamma$ and $\alpha_R$ new transport coefficients [45,100,101])

\[
\langle \mathbf{u}' \times \mathbf{b}' \rangle = - (\beta + \zeta_R)\nabla \times \mathbf{B} + \gamma \mathbf{Q} + \alpha_R \mathbf{B} - \nabla (\zeta_R) \times \mathbf{B}. \quad (4.4)
\]

Thus, the large-scale magnetic-field is subject to a turbulent EMF. The different transport coefficients involve turbulent statistical quantities based on the fluctuating fields (turbulent total and residual energies in $\beta$, and $\zeta_R$, as well as turbulent cross-correlation and residual helicity in $\gamma$ and $\alpha_R$), with specifically

\[
\beta \propto \frac{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle}{2}, \quad \zeta_R \propto \frac{\langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle}{2}, \quad \gamma \propto \langle \mathbf{u}' \cdot \mathbf{b}' \rangle, \quad \alpha_R \propto \langle -\mathbf{u}' \cdot \omega' + \mathbf{b}' \cdot \mathbf{j}' \rangle. \quad (4.5)
\]
Detailed expressions in the case of strong compressibility are found in [101]. We see in (4.4) that the turbulent cross helicity enters the EMF as the coupling coefficient for the mean absolute vorticity $\Omega_s$. The full analytical expression for this transport coefficient is

$$\gamma = \int dk \left[ \tau_1 \{ G_b(k; \tau, \tau_1) + G_b(k; \tau, \tau_1) \} Q_w(k; \tau, \tau_1) \right],$$

(4.6)

where $G_u$ and $G_b$ are the response functions for the velocity and the magnetic field, respectively, and $Q_w$ is the cross-helicity spectral function (see equation (2.5)). This contribution to the EMF is called the cross-helicity effect in dynamos. It can be an important and very relevant extension of mean-field theory. When implemented in a simple dynamo model constituted by the toroidal and poloidal magnetic-field equations with the EMF (Parker equations), it reproduces an oscillatory behaviour of the mean magnetic field similar to the solar and stellar magnetic-field cycles, even without resorting to a presumed differential rotation profile [102,103].

How much this effect works in a dynamo process depends on how much cross helicity we have. The relative cross helicity (denoted $\sigma_C$ in the solar-wind community) is defined by $H_C$ normalized by the total turbulent MHD energy: $\sigma_C = H_C/K_T = 2(u' \cdot b')/(u^2 + b^2)$. This quantity gives a dimensionless measure of the importance of cross-helicity effects. If the turbulent magnetic diffusivity is mainly balanced by the cross-helicity effect, it is expected that the alignment between the mean electric-current density $J$ and the mean absolute vorticity $\Omega_s$ occurs ($J \propto \Omega_s$). This is in marked contrast with the case of balancing between the turbulent magnetic-diffusivity effect and the helicity or $\alpha$ effect, where the mean electric-current density $J$ configuration parallel to the mean magnetic field $B$ is realized (force-free state, $J \propto B$).

Another important cross-helicity effect is on the momentum transport. The Reynolds and turbulent Maxwell stresses in the mean momentum equation are expressed as [45]

$$\langle u'_i u'_j - b'_i b'_j \rangle = \left( \frac{2}{3} \right) K_R \delta_{ij} - \nu_T S_{ij} + \nu_M M_{ij} + [\lambda_{Hi} \Omega_{sj} + \lambda_{Hi} \Omega_{sj}] D,$$

(4.7)

where

$$K_R = \frac{(u'^2 - b'^2)}{2}, \quad \nu_T = \left( \frac{7}{5} \right) \beta, \quad \nu_M = \left( \frac{7}{5} \right) \gamma.$$

(4.8)

$K_R$, $\nu_T$, $\nu_M$ are, respectively, the turbulent MHD residual energy, and the transport coefficients (eddy diffusivity $\nu_T$, and cross-helicity-related coefficient $\nu_M$). Furthermore, $M = \{M_{ij}\}$ is the rate-of-strain tensor of the mean magnetic field, and $\lambda_{Hi}$ is the helicity-gradient-related coefficient (see equations (2.9), (4.5) and (4.6)). The $\nu_M$-related term implies that the turbulent cross helicity coupled with the mean magnetic strain affects the momentum transport. This cross-helicity effect in the momentum transport plays a key role in suppressing the turbulent viscosity and/or inducing a global flow in the solar torsional oscillation [104], as well as spontaneous zonal flow generation in fusion plasmas [105]. It also intervenes in enhancing the magnetic reconnection rate by modulating the outflow configuration [97]. The origin of this effect is reckoned as the vortex-motive force ($u' \times \omega'$) due to the fluctuating Lorentz force $J \times b'$.

From the viewpoint of transport enhancement and suppression, the role of cross helicity is to suppress the enhanced transports (turbulent viscosity $\nu_T$ and turbulent magnetic diffusivity $\eta_T (= \beta + \xi_R)$) arising from the turbulent energies $K_V$, $K_M$ and $K_T$. In inhomogeneous turbulence where the levels of turbulent energies and helicities vary depending on the production mechanisms due to the mean-field inhomogeneities, the local balance between the transport enhancement and suppression alters in space and time. In the flow domain where the relative cross helicity is large, the effective transport is suppressed. On the other hand, when $\sigma_C$ is small, the enhanced diffusivity shows a local dominance. This is one of the reasons why the magnetic reconnection rate is expected to be locally drastically enhanced when and where the turbulent cross helicity vanishes, as observed in two-dimensional DNS and models of MHD [95,106].
5. Modelling helical turbulence for fluids and MHD

It is known that the standard or simplest model with a gradient-transport approximation works poorly in turbulence with cross-flow configurations. For instance, an eddy viscosity applied to a turbulent swirling pipe flow completely fails to reproduce the axial mean velocity profile near the centreline, as experimentally observed. This is because the eddy viscosity effect is so strong that an inhomogeneous flow structure is rapidly smeared out. The expression of the velocity correlation (2.5) implies that the turbulent kinetic helicity and energy should be statistical quantities for the modelling of non-reflectional symmetric turbulence. Of course, one can introduce expansions of the eddy viscosity model that takes into account the role of helicity. For example, in [107], it was shown that a helical model performs better than when not taking into account helicity correction terms, also allowing us to display the role of rotation in partitioning the large-scale energy cascade.

In (2.8), the eddy-viscosity $-\nu_T S$ is a gradient-diffusion model for momentum transport. On the other hand, the helicity—or $\lambda_H$-related term—represents a deviation from the gradient-diffusion model. In non-mirror-symmetric turbulence, helicity being a structural property of turbulence, should be included in the model expression, in addition to kinetic energy. One can model the transport coefficients $\nu_T$, $\lambda_H$ (2.9), on the basis of the theoretical results, in terms of one-point turbulent statistical quantities ($K_V$, $\varepsilon_V$ and $H_V$) as

$$\nu_T = C_v \tau K_V = \frac{C_v K_V^2}{\varepsilon_V}, \quad \lambda_H = C_\eta \tau \ell^2 \varepsilon_V = C_\eta \left( \frac{K_V}{\varepsilon_V} \right) \left( \frac{K_V^2}{\varepsilon_V} \right) \nabla H_V.$$

Such a turbulence helicity model with coherent structure effects incorporated through $H_V$ was successfully applied to a turbulent swirling pipe flow [33]. It thus allowed us to unravel the role of helicity on coherent structures, namely that the presence of turbulent helicity leads to a suppression of turbulent transport, and contributes to a persistent presence of coherent vortical structures against the turbulent mixing due to eddy viscosity. Its systematic extension to subgrid-scale (SGS) helicity models for large-eddy simulations (LES) has been proposed [108] (also see [109] for a recent attempt to formulate a statistical mechanics framework for large-scale structures of turbulent von Kármán flows). This helicity effect works as well for global structure formation. In a series of DNS in which non-uniform turbulent helicity is injected, a global flow is induced from the initial no-mean flow turbulence in coupling with a system rotation [110]. By examining the budget of the Reynolds-stress equation, it was pointed out that the pressure–diffusion correlation as well as the Coriolis-force correlation both play a key role for producing the effect counter-balancing the eddy viscosity [111]. Finally, note that the helicity cascade to small scales has been found to be slightly less local in scale than its energetic counterpart [112]. This may lead again to the need of modifying the modelling of helical flows.

In geophysical and astrophysical turbulence with large-scale inhomogeneities, without resorting to any external forcing, helicities are self-generated by these inhomogeneities in combination with rotation or magnetic fields. For helicity invariants, such as $H_V$ for neutral fluids and $H_C$ in MHD, the evolution equations of helicity can be written in a simplified form for $F = (H_V, H_C)$, with $P_F$, $\varepsilon_F$ the production and dissipation rates and $T_F$ the transport flux

$$\partial_t F + (U \cdot \nabla) F = P_F - \varepsilon_F + \nabla \cdot T_F. \quad (5.1)$$

The production rate $P_F$ stems from the mean-field helicity cascade, and is constituted by the inhomogeneity of the mean field coupled with the turbulent flux. For instance, $P_{H_V}$ for the turbulent kinetic helicity is expressed as

$$P_{H_V} = -\left< u_i' u_j' \right> \left( \frac{\partial \Omega_j}{\partial x_i} \right) + \Omega_j \left( \frac{\partial}{\partial x_j} \right) \left< u_i' u_j' \right> .$$

The dissipation rate $\varepsilon_F$ in equation (5.1) stems from the viscosity and/or diffusivity, representing the helicity decay rate. An elaborate evaluation of $\varepsilon_F$ requires modelling of the $\varepsilon_F$ equation itself, considering the theoretical derivations of the equations of the kinetic helicity dissipation rate $\dot{\varepsilon}_V$.
and the cross-helicity dissipation rate $\tilde{e}_{HC}$ (see [93,113]). The transport rate $\nabla \cdot T_F$ in equation (5.1), written in the divergent form, represents the flux through the boundary. Among the several terms in $\nabla \cdot T_F$, the inhomogeneity of turbulence along the angular velocity vector or the mean magnetic field, provides helicity generation through transport. For $H_V$, we have $(2\omega F \cdot \nabla)\langle u^2 \rangle/2$, which means that the turbulence inhomogeneity along the rotation direction generates a turbulent kinetic helicity. This is related to an important helicity generation mechanism in rotating stratified turbulence [78,79]. These production and transport rates, as well as the dissipation rate due to viscosity and magnetic resistivity, constitute the evolution equation of the turbulent helicity.

As we mentioned in §4(c), the cross-helicity plays a significant role in the alteration of turbulent flow dynamics. In (4.4) and (4.7), the $\gamma$- and $\nu_M$-related terms arise from the turbulent cross helicity. They may counter-balance the terms of the turbulent magnetic diffusivity $\nu_T (= \beta + \zeta R)$ and the turbulent viscosity $\nu_T$. These cross-helicity effects are expected to contribute to the suppression of the turbulent transport in improved-confinement modes in fusion plasmas, and to the localization of the effective magnetic diffusivity into a small region where the cross helicity vanishes, leading to a fast reconnection in a fully turbulent medium (e.g. [95] in two dimensions in MHD). Another interesting case where the turbulent cross helicity plays an important role is the solar-wind (SW). The SW and its magnetic field is a dynamically evolving, inhomogeneous and anisotropic turbulent fluid. Since Alfvén waves propagate predominantly outwardly from the solar surface (the basis of the solar corona), a positive (negative) cross helicity is observed in the inward (outward) heliospheric magnetic sectors. SW turbulence exhibits strong Alfvénicity: the alignment and equipartition between the velocity and magnetic-field fluctuations in the inner heliosphere and the Alfvénicity both decay as the heliocentric distance increases.

Turbulence models for the large-scale evolution of the solar wind have been developed [47,114]. In these models, the evolution of turbulence using the kinetic and magnetic energies $K_V, K_M$ (or equivalently, the total and residual energies $K_T = K_V + K_M, K_R = K_V - K_M$), as well as the turbulent cross helicity $H_C$, has been investigated. Characteristics of the large-scale behaviour of the Alfvénicity represented by the normalized cross helicity $\sigma_C$ were obtained with the aid of spacecraft observations [115]. They were successfully reproduced by a turbulence model based on the multiple-scale DIA analysis of MHD turbulence [46,116,117].

Indeed, high-resolution spacecraft observations of solar-wind turbulence provide important data for constructing models of energy cascades in Hall-MHD and beyond, such as for instance statistical theories of extended MHD where both the Hall drift and the electron inertia are included to treat length and time scales comparable to the ion cyclotron frequency and/or the electron skin depth [90,118] (see also [119]). These formulations are expected to be useful for modelling inertial confinement by high-intensity lasers in the laboratory, as well as turbulence-plasma waves interactions. For example, a recent comparison of high-resolution MMS spacecraft data with the analysis of two-dimensional hybrid-kinetic numerical simulations confirmed that kinetic contributions to inter-scale energy transfer can remain important at scales larger than the ion-inertial length [120], possibly because of the inverse (magnetic and generalized) helicity cascades, and for intermittency constraints as well.

Finally, it will be important in the future to incorporate the mechanisms behind bi-directional cascades in statistical theories because of their role in the mixing and dissipation properties of turbulent flows (see [121] in the oceanographic context for a model taking into account the slowing-down of turbulence transfer in the presence of waves combined with the dominance of rotation at large scale).

6. Conclusion

Helicity is an integrand part of fluid dynamics, including in the possibility of the development of singular structures, a central tenet of turbulence research leading, among other, to intermittency and to anomalous (order unity) dissipation for the high Reynolds number flows encountered in geo- and astrophysics. Indeed, it was shown for fluids that, if a singularity (in the sense of a finite-time blow-up of vorticity) occurs, then the curvature of vortex lines blows up as well [122];
but what of torsion, the third element in the Serret–Frenet frame, which can be associated with helicity? This may well be an open problem, and similarly for MHD. In fact, the role of helicity in such singular structures was emphasized again recently [123] in the framework of the Clebsch variable formalism. Similarly, one could ask what role, if any, helicity might have for turbulence in non-integer dimension, studied in [124] using a fractal Fourier decimation, possibly e.g. with the help of fractional derivatives [125].

Moreover shear, an integral part of the definition of helicity, has been shown recently, for rotating and/or stratified fluids as well as for (Hall)-MHD turbulence, to have a central role in the formation of strong small-scale localized structures, an interpretation of which can be performed for example in the framework of critical phenomena [126–128]. The observation of such shear instabilities in complex turbulent flows affects our evaluation of mixing and dissipation in the presence of rotation and stratification [129,130], with local values comparable to that of FDT. It also leads for example to high (intermittent) vertical drafts in the mesosphere [131] (see [132] for oceanic observations). This is occurring as well in the presence of magnetic fields, in which case cross helicity plays a central role [133], as already elaborated in this review. This in fact can be related to an interpretation in terms of classical avalanches [134,135], as developed earlier to understand the distribution spectrum of the energy of solar flares [136,137]. However, the role of the breaking of reflectional symmetry and of helical structures in general in the establishment of so-called self-organized criticality for such cases has not been entirely elucidated to this date.

Finally, detailed observations of magnetic helicity in solar flares have already allowed for the prediction of strong solar eruptions [138]. In addition to various physics-based models including the helicity-dependent or non-parity-invariance effect on turbulent transport [33,53,107], using highly resolved datasets from detailed laboratory experiments, high-performance computing, high-resolution in situ satellite observations, and analysing and combining them with a variety of modelling and of machine learning techniques [139,140], is all expected to lead to enhanced physical insights in turbulence and nonlinear phenomena in general.

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