A simple transport model for the temperature-dependent linear magnetoresistance of high-temperature superconductors

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Taken in conjunction with the temperature ($T$) dependence of the zero-field resistivity, a simple transport model invoking realistic variations in charge-carrier density is shown to be sufficient to explain the linear magnetoresistance and field-temperature resistance scaling recently observed in high-temperature pnictide and cuprate superconductors. Hence, though the $T$-linear zero-field resistance is a definite signature of the “strange metal” state of high-temperature superconductors, their linear magnetoresistance and its scaling need not be. Straightforward experimental tests of these assertions are proposed.

In recent experimental\textsuperscript{1–3} tours de force, the magnetoresistance of pnictide and cuprate superconductors has been measured in magnetic fields $\mu_0 H$ of up to 92 T as a function of temperature $T$. It was found that the transverse magnetoresistance $\rho$ (measured along the length of a bar-like sample with the current $I$ applied along the bar axis, $\perp H$ - see Figure 1) exhibited interesting scaling behaviour with $H$ and $T$. For instance, in Ref.\textsuperscript{1} when $(\rho(H,T)-\rho_0)/T$, where $\rho_0$ is the residual resistivity, is plotted against $H/T$, the data map onto a single curve, which tends to a straight line (i.e. linear magnetoresistance) at larger values of $H/T$. Any observation that gives a “clue towards our... understanding of the strange metal state in high-temperature superconductors\textsuperscript{4}” is very welcome; consequently, the linear magnetoresistance data have stimulated theoretical models such as those in Refs.\textsuperscript{1} and \textsuperscript{5}. For example, Ref.\textsuperscript{5} invokes a disordered strange metal consisting of itinerant electrons interacting via random couplings with naturally formed “quantum dots” containing localized electrons; the $T$-linear zero-field resistivity and some aspects of the magnetoresistance scaling are reproduced\textsuperscript{5}. However, it is legitimate to ask whether the linear magnetoresistance is a defining property of the strange metal state of the high-temperature superconductors, or whether it is merely a consequence of their disordered nature. The present paper uses a simple model to show that expected levels of disorder are sufficient to account for the observed scaling behaviour of the linear magnetoresistance observed in Refs.\textsuperscript{1–5}.

Despite the above-mentioned recent interest, linear magnetoresistance is in fact a decades-old problem (see e.g., Refs.\textsuperscript{6–9} and references therein). A particular issue has been the experimental observation of linear magnetoresistance in “simple metals” such as Al\textsuperscript{10,12}, as is well known from text book\textsuperscript{13} such systems should not do this. However, useful progress towards a resolution of this difficulty was made by George Bruls and coworkers, who explored the idea that variations in Hall voltage $V_{\text{H}}$ along the length of a bar-like sample can result in linear magnetoresistance\textsuperscript{10,12}. (For a simple metal with a spherical Fermi surface, populated by a density $n$ electrons per unit volume, the Hall voltage is\textsuperscript{13}

$$V_{\text{H}} = -\frac{I}{ned},$$

(1)

where $d$ is the sample thickness and $e$ is the magnitude of the charge of the electron.) Bruls et al. proved this idea by measuring linear magnetoresistance in bars of very pure Al that were machined so that $d$ (and hence $V_{\text{H}}$ - see Equation 1) varied along the length of the bar\textsuperscript{10,12}. An equivalent effect is obtained by varying $n$ along the bar\textsuperscript{13}; the current paper shows that this principle can be applied to the cuprates and pnictides to generate linear magnetoresistance with the scaling behaviour mentioned above, without additional recourse to explicitly “strange-metal” physics.

The first part of the derivation in the present paper is similar to that given in Ref.\textsuperscript{14}; however, (i) the system dealt with by the authors of Ref.\textsuperscript{14}is very different from the pnictides and cuprates; (ii) an exploration of linear magnetoresistance is not their primary purpose; and (iii) their expression relevant to the discussion below
is given in terms of a plethora of parameters that are not easily understood. Therefore I give a simplified, but complete form of the derivation (in SI units and using contemporary notation) in the hope that it will encourage others to play with the model and/or develop more sophisticated calculations from a similar starting point.

We assume a bar-like sample fabricated from a simple metal (Figure 1); the latter has a single, negatively charged carrier type of number density $n$ and with effective mass $m^*$ (it is easy to show that the same result obtains for positively charged holes). If $\mathbf{H}$ is parallel to $z$ (as it is in most of the experiment)\cite{13} it does not matter whether this material has a three-dimensional (spherical) or two-dimensional (cylindrical) Fermi surface, as long as the axis of the latter is also parallel to $z$. The current runs along $x$, parallel to the bar long axis, and the field is applied parallel to $z$. I assume that the sample is not very magnetic, so that the magnetic flux density within the sample and the applied field are roughly the same: i.e., $B \approx \mu_0 H$.

The nature of the Lorentz force ($-e\mathbf{v} \times \mathbf{B}$) means that there is no change to the carrier motion in the $z$ direction\cite{17}, parallel to $\mathbf{B}$. It is therefore sufficient to consider the following elements of the conductivity tensor $\sigma$:

$$
\sigma_{xx} = \sigma_{yy} = \sigma' = \frac{ne^2 \tau}{m^* (1 + \omega_c^2 \tau^2)}
$$

$$
\sigma_{yx} = -\sigma_{xy} = \sigma' \omega_c \tau.
$$

(2)

Here, $\tau$ is the scattering time within the relaxation-time approximation and $\omega_c = \frac{eB}{m^*}$ is the cyclotron frequency\cite{17}.

In the steady state, the continuity equation\cite{17} is

$$
\nabla \cdot \mathbf{j} = 0,
$$

(3)

where $\mathbf{j} = \sigma \mathbf{E}$ is the current density in the sample due to electric field $\mathbf{E}$. Experimentally\cite{1} we measure the voltages $V = V(x, y)$ at various points on the rod; remembering that $\mathbf{E} = -\nabla V$. Equations $[2]$ and $[3]$ combine to yield

$$
\sigma'[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}] + \frac{\partial V}{\partial x} \left[ \frac{\partial \sigma'}{\partial x} + \frac{\partial (\omega_c \tau \sigma')}{\partial y} \right] + \frac{\partial V}{\partial y} \left[ \frac{\partial \sigma'}{\partial y} - \frac{\partial (\omega_c \tau \sigma')}{\partial x} \right] = 0,
$$

(4)

where the possibility that $\sigma'$, $\omega_c$ and/or $\tau$ might vary with $(x, y)$ has been introduced; the disorder and inhomogeneities that might produce such variations are a common feature of many models of linear magnetoresistance in conventional metals and semiconductors\cite{17}.

The pnictides\cite{17} $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ and cuprates $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ used in the linear magnetoresistance studies are alloys, and it is inevitable that some inhomogeneity of composition (and hence carrier density $n$) will occur. In addition, cuprates suffer from slight variations in oxygen stoichiometry, again leading to variable carrier density (see e.g., Ref. [16] and references therein). Finally, the carrier density in both types of high-temperature superconductor is much smaller than that in conventional metals such as Al and Cu\cite{1},\cite{10} potentially leading to much less effective screening of disorder due to impurities and other defects\cite{17}. Even in crystalline organic superconductors, which are relatively clean systems compared to the cuprates, this reduced screening was shown to lead to an inhomogeneous carrier density\cite{1}. The current simple model therefore assumes that only $n$ varies with position in the sample\cite{1} and that other quantities such as $m^*$ and $\tau$ are much less affected by the inhomogeneities.

Equation [4] could be attacked using a variety of techniques. However, the primary interest of this paper is the transverse magnetoresistance, measured using the voltage drop in the $x$ direction. I therefore consider a variation of $n$ only in the $x$ direction (the $\sim 10$ nm to $\sim 1$ $\mu$m length scales over which this happens will be discussed below). Finally, in a descriptive work, it is useful to be able to apply separation of variables to Equation [4] to produce an analytical solution. We hence choose

$$
n = n_0 e^{kx}
$$

(5)

with $n_0$ and $k \neq 0$ being constants. Insertion of this into Equation [4] produces

$$
\left[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right] + k \left[ \frac{\partial V}{\partial x} - \omega_c \tau \frac{\partial V}{\partial y} \right] = 0.
$$

(6)

This has the solution $V = V_0 e^{-kz} e^{\omega_c \tau y} + C$, where $V_0$ and $C$ are constants; as the data in Refs. [1] deal only with voltage differences measured between contacts, $C$ can be set equal to 0. [Owing to the way in which the variation of $n$ with $x$ was chosen, this solution automatically satisfies the boundary condition that no current should flow out of the sample surfaces at $y = \pm \frac{w}{2}$ (Figure 1).]

The fact that a current $I$ flows along the sample bar (Figure 1) is used to find $V_0$. We have

$$
I = d \int_{-\frac{w}{2}}^{+\frac{w}{2}} j_x dy
$$

$$
= \sigma' \kappa V_0 e^{-kz} (1 + \omega_c^2 \tau^2) \int_{-\frac{w}{2}}^{+\frac{w}{2}} e^{\omega_c \tau y} dy.
$$

(7)

some rearrangement gives $V_0$, yielding

$$
V = \frac{I}{\sigma_0} \frac{1}{\kappa v_0 d} \left[ \frac{\kappa v_0 \tau}{2} e^{\frac{\kappa v_0 \tau y}{2}} \right] e^{-kz} e^{\omega_c \tau y},
$$

(8)

where $\sigma_0 = n_0 e^2 \tau/m^*$. Note that the central bracket tends to the value 1 in a well-behaved fashion as $\omega_c \to 0$.

Equation [5] is quite general and can be used to predict the voltage at any position on the bar. However, to focus on understanding the results in Refs. [1] I consider the special case of voltages $V_1$ and $V_2$ at two contacts on the same face ($\perp y$; Figure 1) of the bar at positions
FIG. 2. (a) Equation 10 plotted for $\kappa w = 10$ (thin blue curve) and $\kappa w = -10$ (thin red curve). The average of the red and blue curves forms the complete expression for the magnetoresistance (Equation 11); this is shown as the thick green curve (bold curves in Figure 2(a,b)). Note that the magnetoresistance, now symmetric about $B = 0$, is a function of only $\kappa w$ and $\omega_c^2$; it will therefore follow exactly the same temperature dependence as $R(0)$ does. The shape is determined by the dimensionless quantities; increasing $\kappa w$ moves the crossover from curved to linear magnetoresistance to lower values of $\omega_c^2$ [compare Figures 2(a) and (b)].

In a real sample, it is likely that the composition variations will be more random than this simple case. One could deal with this by evaluating Equation 11 for many different values of $\kappa$ and averaging the results. How-
It is customary to subtract \( \rho_0 \) from the data and treat the \( H \)– and \( T \)–dependence of what remains. I shall do the same here. For a fixed value of \( \kappa w \), Equation \([11]\) takes the form \( R(B,T) = R(0,T)f(\omega_c \tau) \), i.e., the zero-field resistance multiplied by a function of only \( \omega_c \tau \). If \( R(0,T) = AT \), then

\[
R(B,T) = ATf(\omega_c \tau).
\] (13)

Therefore, the magnetoresistance scales linearly with \( T \), as observed\([8]\). Moreover, if one plots \( R(B,T)/T = Af(\omega_c \tau) \), the data should all collapse onto a single curve that is a function of only \( \omega_c \tau \).

In the absence of temperature-dependent variations in carrier density (not expected in a metallic system\([2]\)), the resistance will be proportional to a temperature-dependent scattering rate. Put simply, \( R(0,T) \propto 1/T \); since \( R(0,T) = AT \), this implies that \( \tau \propto 1/T \). Therefore, if \( B \) is divided by \( T \), one obtains a quantity that is proportional to \( \omega_c \tau \). This is likely to be the reason why the magnetoresistance data from Refs. \([1-3]\) divided by \( T \) collapse onto a single curve when plotted versus \( B/T \) [e.g., Figure \( 2(c) \)].

Having given a plausible explanation for the scaling behaviour of the data in Refs. \([1-3]\), it remains to discuss reasonable values for the dimensionless quantities in the model.

Values of \( \kappa w \): Based on general properties of fairly disordered alloys\([10]\), one might expect variations of \( n \) of a few tenths of a percent to a few percent over the lengthscale of typical microstructure, which can range in size from \( \sim 10 \) nm to \( \sim 1 \) \( \mu m \). Therefore, \( \kappa \) may be \( \sim 10^4 \) – \( 10^6 \) m\(^{-1}\). Typical samples used for pulsed-field transport measurements tend to have a width \( w \approx 50 \) – \( 500 \) \( \mu m \) (author’s observation). Taking values somewhere in the middle of these ranges, one obtains \( \kappa w \sim 10 \), as used in Figure \( 2(a) \) to give a reasonable simulation of data from Ref. \([1]\) [Figure \( 2(f) \)].

Values of \( \omega_c \tau \) in the cuprates and pnictides: Even good-quality cuprate superconductors that exhibit Shubnikov-de Haas and de Haas-van Alphen oscillations in high magnetic fields have relatively large scattering rates; a typical sample of this sort\([20]\) has \( \omega_c \tau \approx 1 \) at 20 T. The samples used in Refs. \([1-3]\) do not exhibit magnetic quantum oscillations; they are chosen to be close to the quantum-critical point and possess heavy masses and enhanced scattering. I have guessed \( \omega_c \tau \approx 2.5 \) at \( \mu_B H \approx 90 \) T [representing the upper limit of the \( x \)-axis of Figure \( 2(c) \)]. However, note that lower values of \( \omega_c \tau \) can be compensated for by higher values of \( \kappa w \).

One further consequence of very high scattering rates is that the fine details of the Fermi-surface topology are unlikely to be important in determining the transport properties; hence a model that assumes a simple Fermi-surface topology, such as the one presented here, may be adequate to derive general qualitative features of the resistivity.

Finally, note that the predictions of this model are sensitive to the position of contacts on the bar and the sample width (see Equation \([5]\)). A simple test involving several bars of different widths cut from the same crystal would be sufficient to check the importance (or not) of the effects described here\([20]\). The same geometrical nature of the effect also explains the angle dependence of the scaling noted in Ref. \([2]\). If one rotates the bar about the \( x \)-axis in the magnetic field, the magnetoresistance will only depend on the component of \( \mathbf{H} \) parallel to \( z \), as seen in the experiments. In the model discussed here, this is nothing to do with the special nature of the \( c \) axis\([2]\); it is merely a geometrical effect of the relative position/orientation of the current paths between the contacts that measure the voltage and the magnetic field.

In summary, a transport model invoking realistic-sized variations in the charge-carrier density is sufficient to explain the linear magnetoresistance and scaling behaviour reported in Refs. \([1-3]\). The model shows that this simple form of disorder causes the temperature dependence of the magnetoresistance to scale with the \( T \)-linear zero-field resistivity that is a feature of pnictides and cuprates\([10]\). Therefore, though the latter \( T \)-linear zero-field resistance is indeed a signature of the “strange metal” state of high-temperature superconductors, the linear magnetoresistance need not be. Nevertheless, cuprates and pnictides are complex systems, and it is conceivable that other effects may be at play in the experimental data. With this in mind, I propose experiments that involve varying the sample size and geometry to test whether or not simple effects of disorder are important.

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20 In some cases, experiments are carried out using contacts made using wires connected to a smear of silver paint or silver epoxy across the whole width of the top surface (⊥ $z$-see Figure 1) of a bar-shaped sample (author’s observation). Some exploration using Equation 8 shows that this is not a “smoking gun” configuration for investigating the geometrical effects described in the current paper. Such contacts frequently have a resistance $\sim 10 – 100$ times the four-contact resistance of the bar and hence do not represent an equipotential region on the sample. Instead, they will detect an average of the voltages across the bar; the resulting voltage difference between two such contacts will tend to be dominated by the large linear magnetoresistance terms produced as one approaches the sample edges at $y = \pm w/2$. 