Iterated Brownian motion in bounded domains in $\mathbb{R}^n$

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Abstract
Let $\tau_D(Z)$ is the first exit time of iterated Brownian motion from a domain $D \subset \mathbb{R}^n$ started at $z \in D$ and let $P_z[\tau_D(Z) > t]$ be its distribution. In this paper we establish the exact asymptotics of $P_z[\tau_D(Z) > t]$ over bounded domains as an extension of the result in DeBlassie [14], for $z \in D$

$$P_z[\tau_D(Z) > t] \approx t^{1/2} \exp\left(-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right), \text{ as } t \to \infty.$$  

We also study asymptotics of the life time of Brownian-time Brownian motion (BTBM), $Z_{t}^{1} = z + X(Y(t))$, where $X_t$ and $Y_t$ are independent one-dimensional Brownian motions.

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1 Introduction and statement of main results

Properties of iterated Brownian motion (IBM) analogous to the properties of Brownian motion have been studied extensively by several authors [1, 2, 3, 7, 8, 9, 12, 14, 17, 22, 26, 28]. Several other iterated processes including Brownian-time Brownian motion (BTBM) have also been studied [1, 2, 23]. One of the main differences between these iterated processes and Brownian motion is that they are not Markov processes. However, these processes have connections with the parabolic operator \( \frac{1}{8} \Delta^2 - \frac{\partial}{\partial t} \), as described in [2, 14].

To define the iterated Brownian motion \( Z_t \) started at \( z \in \mathbb{R} \), let \( X_t^+, X_t^- \) and \( Y_t \) be three independent one-dimensional Brownian motions, all started at 0. Two-sided Brownian motion is defined by

\[
X_t = \begin{cases} 
X_t^+, & t \geq 0 \\
X_t^-, & t < 0.
\end{cases}
\]

Then the iterated Brownian motion started at \( z \in \mathbb{R} \) is

\[
Z_t = z + X(Y_t), \quad t \geq 0.
\]

In \( \mathbb{R}^n \), one requires \( X^\pm \) to be independent \( n \)-dimensional Brownian motions. This is the version of the iterated Brownian motion due to Burdzy, see [7].

In what follows we will write \( f \approx g \) and \( f \lesssim g \) to mean that for some positive \( C_1 \) and \( C_2 \), \( C_1 \leq f/g \leq C_2 \) and \( f \leq C_1 g \), respectively. We will also write \( f(t) \sim g(t) \), as \( t \to \infty \), to mean that \( f(t)/g(t) \to 1 \), as \( t \to \infty \).

Let \( \tau_D \) be the first exit time of Brownian motion from a domain \( D \subset \mathbb{R}^n \). The large time behavior of \( P_x[\tau_D > t] \) has been studied for several types of domains, including general cones [5, 13], parabola-shaped domains [4, 25], twisted domains [15] and bounded domains [27]. Our aim in this article is to do the same for the exit time of IBM over bounded domains in \( \mathbb{R}^n \) and for the exit times of BTBM over several domains in \( \mathbb{R}^n \).

In particular, the large time asymptotics of the lifetime of Brownian motion in general cones has been studied by several people including Burkholder [10], DeBlassie [13] and Bañuelos and Smits [5]. Let \( D \) be an open cone with vertex 0 such that \( S^{n-1} \cap D \) is regular for the Laplace-Beltrami operator \( L_{S^{n-1}} \) on the sphere \( S^{n-1} \). Then for some \( p(D) > 0 \) (see [13] and [5])

\[
P_x[\tau_D > t] \sim C(x) t^{-p(D)}, \quad \text{as } t \to \infty.
\]
Now let $D \subset \mathbb{R}^n$. Let $\tau_D(Z) = \inf\{t \geq 0 : Z_t \notin D\}$, be the first exit time of $Z_t$ from $D$. When $D$ is a generalized cone, using the results of Bañuelos and Smits, DeBlassie [14] obtained; for $z \in D$, as $t \to \infty$,

$$P_z[\tau_D(Z) > t] \approx \begin{cases} t^{-p(D)}, & p(D) < 1 \\ t^{-1} \ln t, & p(D) = 1 \\ t^{-(p(D)+1)/2}, & p(D) > 1. \end{cases}$$

For parabola-shaped domains the study of exit time asymptotics for Brownian motion was initiated by Bañuelos, DeBlassie and Smits [4] to answer the question: Are there domains in $\mathbb{R}^n$ for which the distribution of the exit time is sub-exponential? They showed that for the parabola $P = \{(x,y) : x > 0, |y| < A\sqrt{x}\}, A > 0$ there exist positive constants $A_1$ and $A_2$ such that for $z \in P$,

$$-A_1 \leq \liminf_{t \to \infty} t^{-(1/2)} \log P_z[\tau_P > t] \leq \limsup_{t \to \infty} t^{-(1/2)} \log P_z[\tau_P > t] \leq -A_2.$$ 

Subsequently, Lifshits and Shi [25] found that the above limit exists for parabola-shaped domains $P_\alpha = \{(x,Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$, $0 < \alpha < 1$ and $A > 0$ in any dimension; for $z \in P_\alpha$,

$$\lim_{t \to \infty} t^{-(\frac{1-\alpha}{1+\alpha})} \log P_z[\tau_\alpha > t] = -l, \quad (1.1)$$

where

$$l = \left(\frac{1 + \alpha}{\alpha}\right) \left(\frac{\pi J_{(n-3)/2}^{2/\alpha}}{A^{2(3\alpha+1)/(1-\alpha)\alpha}} \frac{\Gamma^2(1-\alpha)}{\Gamma(1/2)} \left[\frac{1}{\Gamma(1/2)}\right]^{\alpha/(1+\alpha)}\right). \quad (1.2)$$

Here $J_{(n-3)/2}$ denotes the smallest positive zero of the Bessel function $J_{(n-3)/2}$ and $\Gamma$ is the Gamma function.

Using the results for Brownian motion in parabola-shaped domains we established in [26] with $l$ given by (1.2), for $z \in P_\alpha$,

$$\lim_{t \to \infty} t^{-(\frac{1-\alpha}{1+\alpha})} \log P_z[\tau_\alpha(Z) > t] = -\left(\frac{3 + \alpha}{2 + 2\alpha}\right) \left(\frac{1 + \alpha}{1 - \alpha}\right) \left(\frac{4 - 2\alpha}{4 + 2\alpha}\right) \left[\frac{1}{\Gamma(1/2)}\right]^{\alpha/(1+\alpha)}.$$

For many bounded domains $D \subset \mathbb{R}^n$ the asymptotics of $P_z[\tau_D > t]$ is well-known (See [27] for a more precise statement of this.) For $z \in D$,

$$\lim_{t \to \infty} e^{\lambda D} P_z[\tau_D > t] = \psi(z) \int_D \psi(y) dy, \quad (1.3)$$
where $\lambda_D$ is the first eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary conditions and $\psi$ is its corresponding eigenfunction.

In [14], DeBlassie proved in the case of iterated Brownian motion in bounded domains for $z \in D$,

$$
\lim_{t \to \infty} t^{-1/3} \log P_z[\tau_D(Z) > t] = -\frac{3}{2} \pi^{2/3} \lambda_D^{2/3}.
$$

(1.4)

The limits (1.3) and (1.4) are very different in that the latter involves taking the logarithm which may kill many unwanted terms in the exponential. It is then natural to ask if it is possible to obtain an analogue of (1.3) for IBM. That is, to remove the log in (1.4). In this paper we prove the following theorem.

**Theorem 1.1.** Let $D \subset \mathbb{R}^n$ be bounded domain for which (1.3) holds point-wise and let $\lambda_D$ and $\psi$ be as above. Then for $z \in D$,

$$2C(z) \leq \liminf_{t \to \infty} t^{-1/2} \exp\left(\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \leq \limsup_{t \to \infty} t^{-1/2} \exp\left(\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \leq \pi C(z),
$$

where $C(z) = \lambda_D \sqrt{2\pi/3} \left(\psi(z) \int_D \psi(y) dy\right)^2$.

We also obtain a version of the above Theorem 1.1 for another closely related process, the so called Brownian-time Brownian motion (BTBM). To define this, let $X_t$ and $Y_t$ be two independent one-dimensional Brownian motions, all started at 0. BTBM is defined to be $Z_1^t = x + X(|Y_t|)$. Properties of this process and its connections to PDE’s have been studied in [1], [2] and [23]. Analogous to Theorem 1.1 we have the following result for this process.

**Theorem 1.2.** Let $D \subset \mathbb{R}^n$, $\lambda_D$ and $\psi$ be as in the statement of Theorem 1.1. Let $\tau_D(Z^1)$ be the first exit time of BTBM from $D$. Then for $z \in D$,

$$
\lim_{t \to \infty} t^{-1/6} \exp\left(\frac{3}{2} 2^{-2/3} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z^1) > t] = C(\lambda_D) \psi(z) \int_D \psi(y) dy,
$$

where $C(\lambda_D) = \pi^{-1/6} 2^{13/6} 3^{-1/2} \lambda_D^{1/3}$. This limit is uniform on compact subsets of $D$. 

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Notice that the limits in Theorems 1.1 and 1.2 are different, even at the exponential level.

We obtain the following inequality between distributions of \( \tau_D(Z) \) and \( \tau_D(Z^1) \).

**Theorem 1.3.** Let \( D \subset \mathbb{R}^n \). Then for all \( z \in D \) and all \( t > 0 \),

\[
P_z[\tau_D(Z) > t] \leq 2P_0[\tau_D(Z^1) > t].
\]

**Remark 1.1.** Notice that, from the theorems proved in this paper, the reverse inequality in Theorem 1.3 cannot hold for all large \( t \), in the case of domains \( D \subset \mathbb{R}^n \) considered (i.e. bounded domains with regular boundary, parabola-shaped domains, twisted domains.)

The paper is organized as follows. In §2 we give some preliminary lemmas to be used in the proof of main results. Theorem 1.1 is proved in §3. §4 is devoted to prove Theorem 1.2 and some other results on the exit time asymptotics of BTBM over several domains. In §5, we compare the exit time distributions of IBM and BTBM. In §6, we prove several asymptotic results to be used in the proof of main results.

## 2 Preliminaries

In this section we state some preliminary facts that will be used in the proof of main results.

The main fact is the following Tauberian theorem ([16, Laplace transform method, 1958, Chapter 4]). Laporte [24] also studied this type of integrals. Let \( h \) and \( f \) be continuous functions on \( \mathbb{R} \). Suppose \( f \) is non-positive and has a global max at \( x_0 \), \( f'(x_0) = 0 \), \( f''(x_0) < 0 \) and \( h(x_0) \neq 0 \) and \( \int_{-\infty}^{\infty} h(x) \exp(\lambda f(x)) < \infty \) for all \( \lambda > 0 \). Then as \( \lambda \to \infty \),

\[
\int_0^{\infty} h(x) \exp(\lambda f(x))dx \sim h(x_0) \exp(\lambda f(x_0)) \sqrt{\frac{2\pi}{\lambda |f''(x_0)|}}, \tag{2.1}
\]

It can be easily seen from Laplace transform method that as \( \lambda \to \infty \),

\[
\int_0^{\infty} \exp(-\lambda(x + x^2))dx \sim \exp(-3\lambda 2^{-2/3}) \sqrt{\frac{2^{4/3}\pi}{3\lambda}}. \tag{2.2}
\]
Similarly, as $t \to \infty$,
\[
\int_0^\infty \exp\left(-\frac{at}{u^2} - bu\right)du \sim \sqrt{\frac{\pi}{3}} 2^{2/3} a^{1/6} b^{-2/3} t^{1/6} \exp(-3a^{1/3} b^{2/3} 2^{-2/3} t^{1/3}).
\]

(2.3)

This follows from equation (2.2) and after making the change of variables $u = (atb^{-1})^{1/3} x$.

Finally, we obtain, as $t \to \infty$,
\[
\int_0^\infty u \exp\left(-\frac{at}{u^2} - bu\right)du \sim 2 \sqrt{\frac{\pi}{3}} a^{1/2} b^{-1/2} t^{1/2} \exp(-3a^{1/3} b^{2/3} 2^{-2/3} t^{1/3}).
\]

(2.4)

3 Iterated Brownian motion in bounded domains

If $D \subset \mathbb{R}^n$ is an open set, write
\[
\tau_D^\pm(z) = \inf\{t \geq 0 : X_t^\pm + z \notin D\},
\]
and if $I \subset \mathbb{R}$ is an open interval, write
\[
\eta_I = \eta(I) = \inf\{t \geq 0 : Y_t \notin I\}.
\]

Recall that $\tau_D(Z)$ stands for the first exit time of iterated Brownian motion from $D$. As in DeBlassie [14, §3.], we have by the continuity of the paths for $Z_t = z + X(Y_t)$, if $f$ is the probability density of $\tau_D^\pm(z)$
\[
P_z[\tau_D(Z) > t] = \int_0^\infty \int_0^\infty P_0[\eta(-u,v) > t] f(u) f(v)dvdu.
\]

(3.1)

The proof of Theorem 1.1. The following is well-known
\[
P_0[\eta(-u,v) > t] = \frac{4}{\pi} \sum_{n=0}^\infty \frac{1}{2n+1} \exp\left(-\frac{(2n+1)^2 \pi^2}{2(u+v)^2} t\right) \sin\left(\frac{(2n+1) \pi u}{u+v}\right),
\]

(3.2)

(see Feller [18, pp. 340-342]).

Let $\epsilon > 0$. From Lemma 6.1 choose $M > 0$ so large that
\[
(1 - \epsilon) \frac{4}{\pi} e^{-\pi^2 t} \sin \pi x \leq P_x[\eta_{(0,1)} > t] \leq (1 + \epsilon) \frac{4}{\pi} e^{-\pi^2 t} \sin \pi x,
\]

(3.3)
for \( t \geq M \), uniformly \( x \in (0, 1) \). Let \( 0 < \delta < 1/2 \), from the Jordan inequality for the sine function in the interval \((0, \pi/2]\),

\[
2x \leq \sin \pi x \leq \pi x, \quad x \in (0, \delta].
\]

(3.4)

For a bounded domain with regular boundary it is well-known (see [27, page 121-127]) that there exists an increasing sequence of eigenvalues, \( \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \), and eigenfunctions \( \psi_k \) corresponding to \( \lambda_k \) such that,

\[
P_z[\tau_D \leq t] = \sum_{k=1}^{\infty} \exp(-\lambda_k t) \psi_k(z) \int_D \psi_k(y)dy.
\]

(3.5)

From the arguments in DeBlassie [14, Lemma A.4]

\[
f(t) = \frac{d}{dt} P_z[\tau_D \leq t] = \sum_{k=1}^{\infty} \lambda_k \exp(-\lambda_k t) \psi_k(z) \int_D \psi_k(y)dy.
\]

(3.6)

Finally choose \( K > 0 \) so large that

\[
A(z)(1 - \epsilon) \exp(-\lambda_D u) \leq f(u) \leq A(z)(1 + \epsilon) \exp(-\lambda_D u)
\]

for all \( u \geq K \), where

\[
A(z) = \lambda_1 \psi_1(z) \int_D \psi_1(y)dy = \lambda_D \psi(z) \int_D \psi(y)dy.
\]

We further assume that \( t \) is so large that \( K < \delta \sqrt{t/M} \). Define \( A \) for \( \delta < 1/2 \), \( K > 0 \) and \( M > 0 \) as

\[
A = \left\{ (u, v) : K \leq u \leq \delta \sqrt{t/M}, \quad \frac{1 - \delta}{\delta} u \leq v \leq \sqrt{t/M} - u \right\}.
\]

On the set \( A \), since \( \delta < 1/2 \), we have \( v \geq (\frac{1}{\delta} - 1)u > u > K \) and \( u + v > \frac{u}{\delta} \), this gives \( \frac{u}{u+v} \leq \delta \).

By equations (3.3) and (3.4), \( P_z[\tau_D(Z) > t] = P[\eta(\tau_D(z), \tau_{\eta}^{\delta}(z)) > t] \) is

\[
\geq C^1 \int_K^{\delta \sqrt{t/M}} \int_{(1-\delta)u/\delta}^{\sqrt{t/M}-u} \frac{u}{(u+v)} \exp\left(-\frac{\pi^2 t}{2(u+v)^2}\right) \exp(-\lambda_D(u + v)) dv du.
\]
where $C^1 = C^1(z) = 4(4/\pi)A(z)^2(1 - \epsilon)^3$. Changing the variables $x = u + v, z = u$ the integral is

$$C^1 \int_{K/\delta}^{\delta/K} \int_{z/\delta}^{\sqrt{t/M}} \frac{z}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dx dz,$$

and reversing the order of integration

$$C^1 \int_{K/\delta}^{\delta/K} \int_{z/\delta}^{\sqrt{t/M}} \frac{z}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dz dx$$

$$= C^1/2 \int_{K/\delta}^{\delta/K} \frac{1}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x)(\delta^2 x^2 - K^2) dx$$

$$\geq \delta^2 C^1/2 \int_{K/\delta}^{\delta/K} x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dx - I,$$

where

$$I = (C^1/2)K^2 \int_0^\infty \frac{1}{x} \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dx.$$

From the Laplace transform method, equation (2.1), there exists $C_0 > 0$ such that as $t \to \infty$,

$$I \sim C_0 t^{-1/6} \exp\left(-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right). \quad (3.7)$$

From equation (2.4) as $t \to \infty$,

$$\int_0^\infty x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dx \sim 2 \sqrt{\frac{\pi}{3}} \frac{\pi^2}{2}^{1/2} \lambda_D^{-1/2} e^{-1/2} \exp\left(-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right). \quad (3.8)$$

Now for some $c_1 > 0$,

$$\int_0^{K/\delta} x \exp\left(-\frac{\pi^2 t}{2x^2} - \lambda_D x\right) dx \leq e^{-\pi^2 \delta^2 t/2K^2} \int_0^{K/\delta} x \exp(-\lambda_D x) dx \lesssim e^{-c_1 t}, \quad (3.9)$$

and

$$\int_{\sqrt{t/M}}^\infty x \exp\left(-\frac{\pi^2 t}{2x^2}\right) \exp(-\lambda_D x) dx \leq \int_{\sqrt{t/M}}^\infty x \exp(-\lambda_D x) dx$$

$$= (\sqrt{t/M} \lambda_D^{-1} + \lambda_D^{-2}) \exp(-\lambda_D \sqrt{t/M}). \quad (3.10)$$
Now from equations (3.7)-(3.10) we get

\[
\liminf_{t \to \infty} t^{-1/2} \exp \left( \frac{3}{2} \pi^{2/3} \lambda_{D}^{2/3} t^{1/3} \right) P_{z}[\tau_{D}(Z) > t] \geq \delta^{2}(C^{1/2})^{2} \sqrt{\frac{\pi}{3}} \left( \frac{\pi^{2}}{2} \right)^{1/2} \lambda_{D}^{1/3}.
\]

(3.11)

For the upper bound for \( P[\tau_{D}(Z) > t] \) from equation (3.10) in [14],

\[
P_{z}[\tau_{D}(Z) > t] = 2 \int_{0}^{\infty} \int_{u}^{\infty} P_{\eta(0,1)} \left[ \eta(0,1) > \frac{t}{(u+v^2)} \right] f(u)f(v)dvdu.
\]

(3.12)

We define the following sets that make up the domain of integration,

\[
A_{1} = \{(u,v) : v \geq u \geq 0, \ u + v \geq \sqrt{t/M}\},
\]

\[
A_{2} = \{(u,v) : u \geq 0, \ v \geq K, \ u \leq v, \ u + v \leq \sqrt{t/M}\},
\]

\[
A_{3} = \{(u,v) : 0 \leq u \leq v \leq K\}.
\]

Over the set \( A_{1} \) we have for some \( c > 0 \),

\[
\int \int_{A_{1}} P_{\eta(0,1)} \left[ \eta(0,1) > \frac{t}{(u+v^2)} \right] f(u)f(v)dvdu \leq \int \int_{A_{1}} f(u)f(v)dvdu \leq \exp(-c\sqrt{t/M}).
\]

(3.13)

The equation (3.13) follows from the distribution of \( \tau_{D} \) from Lemma 2.1 in [26].

Since on \( A_{3}, \ t/(u+v^2) \geq M, \)

\[
\int \int_{A_{3}} P_{\eta(0,1)} \left[ \eta(0,1) > \frac{t}{(u+v^2)} \right] f(u)f(v)dvdu \leq \int_{0}^{K} \int_{0}^{K} \exp(-\frac{\pi^{2}t}{2(u+v^2)^2}) f(u)f(v)dvdu.
\]

\[
\leq \exp(-\frac{\pi^{2}t}{8K^2}) \int_{0}^{K} \int_{0}^{K} f(u)f(v)dvdu \leq \exp(-\frac{\pi^{2}t}{8K^2}).
\]

(3.14)
Let $C_1 = C_1(z) = 2\pi(4/\pi)A(z)^2(1 + \epsilon)^3$. For the integral over $A_2$ we get,

$$\int \int_{A_2} P_{u+v}[\eta(0,1)] \frac{t}{(u+v)^2} f(u) f(v) dvdu$$

$$\leq C_1 \int_0^K \int_{\sqrt{t/M} - u}^{\sqrt{t/M}} f(u) \exp(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D v) dvdu$$

$$+ C_1 \int_K^{1/2\sqrt{t/M}} \int_u^{\sqrt{t/M} - u} \frac{u}{u+v} \exp(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D(u+v)) dvdu$$

$$= I + II. \quad (3.15)$$

Changing variables $u + v = z$, $u = w$

$$I = \int_0^K \int_{\sqrt{t/M}}^{\sqrt{t/M} - u} \exp(-\frac{\pi^2 t}{2(u+v)^2}) f(u) \exp(-\lambda_D v) dvdu$$

$$\leq \int_0^K \int_{w+K}^{\sqrt{t/M}} \exp(-\frac{\pi^2 t}{2z^2}) f(w) \exp(-\lambda_D z) \exp(\lambda_D w) dzdw$$

$$\leq \exp(\lambda_D K) \int_0^K f(w) dw \int_0^{\sqrt{t/M}} \exp(-\frac{\pi^2 t}{2z^2}) \exp(-\lambda_D z) dz$$

$$\lesssim t^{1/6} \exp(-3/2 \pi^{2/3} \lambda_D^{2/3} t^{1/3}). \quad (3.16)$$

Equation (3.16) follows from equation (2.3), with $a = \pi^2/2$, $b = \lambda_D$.

Changing variables $u + v = z$, $u = w$

$$II = C_1 \int_K^{1/2\sqrt{t/M}} \int_u^{\sqrt{t/M} - u} \frac{u}{u+v} \exp(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D(u+v)) dvdu$$

$$\leq C_1 \int_{K2}^{\sqrt{t/M}} \int_{2w}^{\sqrt{z/2} - u} \frac{w}{z} \exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z) dzdw$$

$$= C_1 \int_{2K}^{\sqrt{t/M}} \int_{z/2}^{\sqrt{z/2}} \frac{w}{z} \exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z) dzdw$$

$$\lesssim C_1/8 \int_{2K}^{\sqrt{t/M}} z \exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z) dz$$

$$\leq (1 + \epsilon)(C_1/8) \frac{1}{2} \frac{\pi^2}{3} \lambda_D^{1/2} t^{1/2} (-\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}). \quad (3.17)$$
Equation (3.17) follows by changing the order of the integration. And equation (3.18) follows from equation (2.4).

Now from equations (3.13), (3.14), (3.16) and (3.18) we obtain

$$\limsup_{t \to \infty} t^{-1/2} \exp\left(\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \leq (1 + \epsilon)\left(\frac{C_1}{8}\right)^2 \sqrt{\frac{\pi}{3}} \left(\frac{\pi^2}{2}\right)^{1/2} \lambda_D^{-1}.$$  

(3.19)

Finally, from equations (3.11) and (3.19) and letting $$\epsilon \to 0, \delta \to 1/2$$,

$$2C(z) \leq \liminf_{t \to \infty} t^{-1/2} \exp\left(\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \leq \limsup_{t \to \infty} t^{-1/2} \exp\left(\frac{3}{2} \pi^{2/3} \lambda_D^{2/3} t^{1/3}\right) P_z[\tau_D(Z) > t] \leq \pi C(z),$$

where $$C(z) = \lambda_D \sqrt{2\pi/3} (\psi(z) \int_D \psi(y) dy)^2$$.  

4 The process $$Z_t^1$$; Brownian-time Brownian motion

In this section we study Brownian-time Brownian motion (BTBM), $$Z_t^1$$ started at $$z \in \mathbb{R}$$. Let $$X_t$$ and $$Y_t$$ be two independent one-dimensional Brownian motions, all started at 0. BTBM is defined to be $$Z_t^1 = x + X_t^1$$, for $$x \in \mathbb{R}$$. In $$\mathbb{R}^n$$, we require $$X$$ to be independent one dimensional iterated Brownian motions. If $$D \subset \mathbb{R}^n$$ is an open set, write

$$\tau_D(z) = \inf\{t \geq 0 : X_t + z \notin D\},$$

and if $$I \subset \mathbb{R}$$ is an open interval, we write

$$\gamma_I = \inf\{t \geq 0 : |Y_t| \notin I\},$$

and

$$\eta_I = \inf\{t \geq 0 : Y_t \notin I\}.$$ 

Let $$\tau_D(Z^1)$$ stand for the first exit time of BTBM from $$D$$. We have by the continuity of paths

$$P_z[\tau_D(Z^1) > t] = P[\eta(-\tau_D(z),\tau_D(z)) > t].$$  

(4.1)
Theorem 4.1. Let $0 < \beta$. Let $\xi$ be a positive random variable such that

$$- \log P[\xi > t] \sim ct^\beta, \text{ as } t \to \infty.$$  

If $\xi$ is independent of the Brownian motion $Y$, then

$$- \log P[\eta(-\xi, \xi) > t] \sim 2^{-\frac{2\alpha}{2+\alpha}} \frac{2+\beta}{2} c^{\frac{2}{2+\beta}} \pi^{\frac{2-2\alpha}{2+\alpha}} l^\frac{2+2\alpha}{2+\alpha},$$

as $t \to \infty$.

Proof. The proof follows similar to the proof of Theorem 3.1 in Nane [26], by integration by parts,

$$P[\eta(-\xi, \xi) > t] = \int_0^\infty \frac{du}{P_0(\eta(u, u) > t) P[\xi > u]} du. \tag{4.2}$$

We use the distribution of $\eta(u, u)$ given in (3.2). We use the asymptotics from equation (6.1) on the set $A = \{u > 0 : K \leq u \leq \sqrt{t/M}\}$. For the lower bound we use Lemma 2.4 in [26], but for the upper bound we use deBruijn Tauberian Theorem as in [26, Lemma 2.2].

From Theorem 4.1 we obtain similar results for the asymptotic distribution of the first exit time of $Z^1$ from the interior of several open sets $D \subset \mathbb{R}^n$.

Corollary 4.1. Let $0 < \alpha < 1$. Let $P_\alpha = \{(x, Y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x > 0, |Y| < Ax^\alpha\}$. Then for $z \in P_\alpha$,

$$\lim_{t \to \infty} t^{-\frac{1}{\frac{3}{2}+\alpha}} \log P_z[\tau_\alpha(Z^1) > t] = -2^{\frac{2\alpha-2}{2+2\alpha}} \frac{3 + \alpha}{2+2\alpha} \frac{1 + \alpha}{1-\alpha} \pi^{\frac{2-2\alpha}{2+2\alpha}} l^\frac{2+2\alpha}{2+2\alpha},$$

where $l$ is the limit given by (1.2).

Corollary 4.2. Let $D \subset \mathbb{R}^2$ be a twisted domain with growth radius $\gamma r^p$, $\gamma > 0$, $0 < p < 1$. Then for $z \in D$,

$$\lim_{t \to \infty} t^{-\frac{1}{\frac{3}{2}+p}} \log P_z[\tau_D(Z^1) > t] = -2^{\frac{2p-2}{2+2p}} \frac{3 + p}{2+2p} \frac{1 + p}{1-p} \pi^{\frac{2-2p}{2+2p}} l_1^\frac{2+2p}{2+2p},$$

where $l_1$ is the limit given by the limit in [15, Theorem 1.1].

Remark 4.1. Notice that there is only a constant difference in the limit of the asymptotic distribution of $\tau_D(Z)$ and that of $\tau_D(Z^1)$ (compare with the results in Nane [26].)
Proof of Theorem 1.2. From equations (3.2), (3.6) and (4.1)

\[ P_z[\tau_D(Z^1) \leq t] = \int_0^\infty P_0[\eta(-u,u) > t] f(u)du \]  

(4.3)

\[ = \frac{4}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \lambda_k \psi_k(z) \int_D \psi_k(y) dy \int_0^\infty \exp(-\frac{(2n+1)^2\pi^2 t}{8u^2} - \lambda_k u) du. \]

From equation (2.3), for each \( n, k \) we have with \( a = \frac{(2n+1)^2\pi^2}{8} \) and \( b = \lambda_k \)

\[ \int_0^\infty \exp(-\frac{(2n+1)^2\pi^2 t}{8u^2} - \lambda_k u) du \]

\[ \sim \pi^{5/6} 2^{1/3} (2n + 1)^{1/3} \lambda_k^{-2/3} t^{1/6} \exp\left(-\frac{3}{2} 2^{-2/3} (2n + 1)^{2/3} \pi^{2/3} \lambda_k^{2/3} t^{1/3}\right). \]

With this, equation (4.3) becomes

\[ \int_0^\infty P_0[\eta(-v,v) > t] f(v)dv \]  

(4.4)

\[ \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \lambda_k \psi_k(z) \int_D \psi_k(y) dy \]

\[ \times \pi^{5/6} 2^{1/3} (2n + 1)^{1/3} \lambda_k^{-2/3} t^{1/6} \exp\left(-\frac{3}{2} 2^{-2/3} (2n + 1)^{2/3} \pi^{2/3} \lambda_k^{2/3} t^{1/3}\right). \]

To get the desired result we must prove that the following series converge absolutely, which implies that the first term in the series in (4.4) is the dominant term,

\[ \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} (2n + 1)^{-2/3} \lambda_k^{1/3} \exp\left(-\frac{3}{2} 2^{-2/3} (2n + 1)^{2/3} \pi^{2/3} \lambda_k^{2/3} \delta/2\right) < \infty. \]

The series in \( n \) for \( k \) fixed

\[ \sum_{n=0}^{\infty} (2n + 1)^{-2/3} \exp\left(-\frac{3}{2} 2^{-2/3} (2n + 1)^{2/3} \pi^{2/3} \lambda_k^{2/3} \delta/2\right) \]

\[ \leq \exp\left(-\frac{3}{2} 2^{-2/3} \pi^{2/3} \lambda_k^{2/3} \delta/2\right) \]

\[ \leq \frac{\exp\left(-\frac{3}{2} 2^{-2/3} \pi^{2/3} \lambda_k^{2/3} \delta/2\right)}{1 - \exp\left(-\frac{3}{2} \pi^{2/3} \lambda_k^{2/3} \delta/2\right)}. \]
Since for $\delta > 0$, 
\[ \sum_{k=1}^{\infty} \exp\left(-\frac{3}{2}2^{-2/3}\pi^{2/3}\lambda_k^{2/3}\delta/3\right) \leq \infty, \]
we are done. This follows from the Weyl’s asymptotic formula for the eigenvalues $\lambda_k$, $\lambda_k \geq C_{n,D}k^{n/2}$, see I. Chavel [11], where $C_{n,D}$ depends only the dimension $n$, and the domain $D$, independent of $k$. From above equation (4.4) the constant $C(\lambda_D) = \pi^{-1/6}2^{13/6}3^{-1/2}\lambda_1^{1/3}$, where $\lambda_D = \lambda_1$ is the first eigenvalue of the Dirichlet Laplacian in $D$. 

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\textbf{Proof of Theorem 1.3.} From equation (3.10) in [14] we get

\[ P_z[\tau_D(Z) > t] = 2 \int_{\infty}^{\infty} \int_{0}^{\infty} P_0[\eta(-u,v) > t]f(u)f(v)dvdu \]

\[ \leq 2 \int_{0}^{\infty} \int_{u}^{\infty} P_0[\eta(-v,v) > t]f(u)f(v)dvdu \]  

(5.1)

\[ \leq 2 \int_{0}^{\infty} \int_{0}^{\infty} P_0[\eta(-v,v) > t]f(u)f(v)dvdu \]

\[ = 2 \int_{0}^{\infty} P_0[\eta(-v,v) > t]f(v)dv \]

\[ = 2P_z[\tau_D(Z^1) > t]. \]  

(5.2)

The inequality (5.1) follows from the fact that $(-u, v) \subset (-v, v)$. The equality (5.2) follows from equation (4.1).

Let $\phi$ be an increasing function. If we multiply the inequality in the Theorem 1.3 by the derivative of $\phi$ and integrate in time we get

\[ E_z(\phi(\tau_D(Z))) \leq 2E_z(\phi(\tau_D(Z^1))). \]

In particular, for $p \geq 1$,

\[ E_z((\tau_D(Z))^p) \leq 2E_z((\tau_D(Z^1))^p). \]
6 Asymptotics

In this Section we will prove some lemmas that were used in section 3 and section 4. The following lemma is proved in [14, Lemma A1] (it also follows from more general results on “intrinsic ultracontractivity”). We include it for completeness.

Lemma 6.1. As $t \to \infty$,

$$P_x[\eta(0,1) > t] \sim \frac{4}{\pi} e^{-\frac{\pi^2}{4} t} \sin \pi x, \text{ uniformly for } x \in (0, 1).$$

We will next prove the similar results to Nane [26, Lemma 4.2] that will be used for the process $Z^1$.

Lemma 6.2. Let $B = \{u > 0 : t/u^2 > M\}$ for $M$ large. Then on $B$,

$$\frac{d}{du} P_0[\eta(-u,u) > t] \sim \exp\left(-\frac{\pi^2 t}{8 u^2}\right) \frac{\pi t}{u^3}. \quad (6.1)$$

Proof. If we differentiate $P_0[\eta(-u,u) > t]$ which is given in (3.2) we get

$$\frac{d}{du} P_0[\eta(-u,u) > t] = \frac{\pi t}{u^3} \sum_{n=0}^{\infty} (2n+1)(-1)^n \exp\left(-\frac{(2n+1)^2 \pi^2}{8 u^2} t\right).$$

The result follows from this. $\square$

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