NILPOTENT GROUPS, O-MINIMAL EULER CHARACTERISTIC, AND LINEAR ALGEBRAIC GROUPS

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Abstract. We establish a surprising correspondence between groups definable in o-minimal structures and linear algebraic groups, in the nilpotent case. It turns out that in the o-minimal context, like for finite groups, nilpotency is equivalent to the normalizer property or to uniqueness of Sylow subgroups. As a consequence, we show algebraic decompositions of o-minimal nilpotent groups, and we prove that a nilpotent Lie group is definable in an o-minimal expansion of the reals if and only if it is a linear algebraic group.

1. Introduction

Groups that are definable in o-minimal structures have been studied by many authors in the past thirty years, often in analogy with Lie groups.

For compact groups, by a conjecture of Pillay in [15], now fully proved, every definable group $G$ has a canonical quotient $G/G^00$ that, endowed with the logic topology, is a compact Lie group [2] with same dimension [9], same homotopy invariants [1], and same first order theory [10].

Strong connections have been found also for groups that are not compact. For instance, every connected abelian real Lie group is the direct product of its maximal torus $T$ by a torsion-free closed subgroup. Similarly, by [3], every o-minimal definably connected abelian group $G$ is the direct product of a maximal abstract torus $T$ (Definition 2.8) and the maximal torsion-free definable subgroup $N(G)$ (Fact 2.9). Therefore every abelian o-minimal group is elementarily equivalent to a linear algebraic group of the same dimension. This is not the case, in general, for solvable groups, as shown by Hrushovski, Peterzil and Pillay in [10]. They give an example of a solvable o-minimal group that is not elementarily equivalent to any definable real Lie group. In this paper we study the intermediate class of nilpotent groups, showing a surprising similarity with the linear algebraic setting, even for finite groups. In Section 2 we prove the following:

**Theorem 1.1.** Let $G$ be a nilpotent group definable in an o-minimal structure. Then

(1) $G$ has maximal abstractly compact subgroups $K$, and

$$G = K \times N(G)$$

where $N(G)$ is the maximal normal definable torsion-free subgroup of $G$.

(2) If $G$ is definably connected then its center $Z(G)$ is definably connected and contains every abstractly compact subgroup of $G$.

As a consequence of decomposition (1) above, in Section 4 we show that linear algebraic groups are the only nilpotent Lie groups that can be defined in an o-minimal expansion of the real field:

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Theorem 1.2. Let $G$ be a nilpotent real Lie group. Then $G$ is definable in an o-minimal structure over the reals if and only if $G$ is Lie isomorphic to a linear algebraic group.

A main tool is the o-minimal Euler characteristic $E$, an invariant under definable bijections that has been used by Strzebonski in [17] to develop a theory of definable $p$-groups and definable $p$-Sylow subgroups, extending classical notions and results for finite groups. In Section 2 and 3 it is used to show the following equivalent characterizations to nilpotency, well-known for finite groups:

Theorem 1.3. Let $G$ be a group definable in an o-minimal structure such that $N(G)$ is nilpotent.

(1) Assume $E(G) \neq 0$. Then the following are equivalent:
   (a) $G$ is nilpotent.
   (b) $G$ has exactly one $p$-Sylow subgroup for each prime $p$ dividing $E(G)$.
   (c) All $p$-Sylow subgroups of $G$ are normal.

(2) Suppose $E(G) = 0$ and $G = G^0$. Then the following are equivalent:
   (a) $G$ is nilpotent.
   (b) $G$ has exactly one 0-Sylow subgroup.
   (c) All 0-Sylow subgroups of $G$ are normal.

(3) Let $G$ be definably connected. Then the following are equivalent:
   (a) $G$ is nilpotent.
   (b) Every proper definable $H < G$ is contained properly in its normalizer.

Finally, Section 4 contains a digression on definable abelian torsion-free groups $G$, for which a decomposition in 1-dimensional definable subgroups is proved, when $\dim \text{Aut}(G) > 0$. This is related to the problem of characterizing definable groups that are elementarily equivalent to a linear algebraic group of the same dimension.

Throughout the paper groups are definable with parameters in an o-minimal structure $\mathcal{M}$. We assume $\mathcal{M}$ satisfies the definable choice property (that is, each definable equivalence relation on a definable set has a definable set of representatives) so that, whenever $H < G$ are definable groups, the quotient $G/H$ is a definable set, even if $H$ is not normal.

2. Nilpotency and Euler characteristic

If $\mathcal{P}$ is a cell decomposition of a definable set $X$, the o-minimal Euler characteristic $E(X)$ is defined as the number of even-dimensional cells in $\mathcal{P}$ minus the number of odd-dimensional cells in $\mathcal{P}$, and it does not depend on $\mathcal{P}$ (see [2], Chapter 4). As points are 0-dimensional cells, it follows that for finite sets cardinality and Euler characteristic coincide. Moreover, since for every definable sets $A$, $B$ we have that $E(A \times B) = E(A)E(B)$, the following holds:

Fact 2.1. [17] Let $K < H < G$ be definable groups. Then
   (a) $E(G) = E(H)E(G/H)$
   (b) $E(G/K) = E(G/H)E(H/K)$

Definition 2.2. [17] Let $G$ be a definable group. We say that $G$ is a $p$-group if:
   - $p$ is a prime number and for any proper definable $H < G$,
     $E(G/H) \equiv 0 \mod p$
   - $p = 0$ and for any proper definable subgroup $H < G$,
     $E(G/H) = 0$

A maximal $p$-subgroup of a definable group $G$ is called $p$-Sylow.
Proof. If \( p \) is a prime dividing \( E(G) \), then \( G \) contains an element of order \( p \). In particular, if \( E(G) = 0 \) then \( G \) has elements of each prime order, and

\[
G \text{ is torsion-free } \iff E(G) = \pm 1
\]

Therefore, definable torsion-free groups are definably connected.

(2) Each \( p \)-subgroup is contained in a \( p \)-Sylow, and \( p \)-Sylows are all conjugate.

(3) If \( H \) is a \( p \)-subgroup of \( G \), then

\[
H \text{ is a } p\text{-Sylow } \iff E(G/H) \neq 0 \mod p
\]

(4) If \( E(G) = 0 \), then \( G \) contains a \( 0 \)-subgroup.

(5) Every \( 0 \)-group is abelian and definably connected.

(6) If \( E(G) \neq 0 \), then any \( p \)-subgroup of \( G \) is finite.

(7) Let \( S \subseteq G \) be a subset (definable or not). Then there is a smallest definable subgroup \( H < G \) containing \( S \). We call it the definable subgroup generated by \( S \), and we write \( H = \langle S \rangle \).

Given a definable group \( G \), we denote by \( N(G) \) the maximal normal definable torsion-free subgroup of \( G \) (that exists by Proposition 2.1 in [5]).

We first consider the case where \( E(G) \neq 0 \). If \( G \) is infinite and definably connected, then either \( G \) is torsion-free or \( G \) has elements of each finite order. So if \( E(G) \neq 0 \) then \( G \) is not definably connected and \( G^0 = N(G) \) is torsion-free.

Lemma 2.4. Let \( G \) be a definable group such that \( |E(G)| = p^n \), for some \( p \) prime. Then any \( p \)-Sylow subgroup \( H \) of \( G \) has order \( p^n \), \( H \) is definably isomorphic to \( G/G^0 \) and \( G = G^0 \times H \).

Proof. As \( E(G) = E(G^0)E(G/G^0) \) and \( E(G^0) = \pm 1 \), it follows that \( |E(G)| = E(G/G^0) = |G/G^0| = p^n \), as \( G/G^0 \) is a finite group.

Let \( H \) be a \( p \)-Sylow subgroup of \( G \). By Fact 2.3 we know that \( E(G/H) \neq 0 \mod p \), thus \( E(G/H) = \pm 1 \). So \( E(H) = |H| = |E(G)| = p^n \). Moreover, \( G^0 \) and \( H \) have trivial intersection, as \( G^0 \) is torsion-free and \( H \) is finite. Therefore \( G = G^0 \times H \), as wanted.

Remark 2.5. The semidirect product may be not direct. E.g., \( G = \mathbb{R} \rtimes \mathbb{Z}_2 \) (where \( \mathbb{Z}_2 = \{\pm 1\} \) acts on \( \mathbb{R} \) by multiplication) is a centerless group with \( E(G) = -2 \).

But when \( G \) is nilpotent, much more can be said:

Proposition 2.6. Let \( G \) be a nilpotent definable group such that \( E(G) \neq 0 \). Then

1. the center \( Z(G) \) is infinite whenever \( G \) is infinite;
2. for each \( p \) prime dividing \( |E(G)| \), \( G \) has exactly one \( p \)-Sylow subgroup;
3. \( G = F \times N(G) \), where \( F \) is the direct product of the (unique) \( p \)-Sylow subgroups of \( G \).

Proof. If \( G \) is finite, then \( N(G) = \{e\} \), and (2) and (3) are well-known. So let \( G \) be infinite with \( \dim G = n > 0 \) and \( |E(G)| = m = p_1^{a_1} \cdots p_k^{a_k} \). We will prove the three statements by induction on \( n + m \).

Suppose, by a contradiction, that \( Z = Z(G) \) is finite of cardinality \( r \). Then \( G/Z \) is a nilpotent group of dimension \( n \) and Euler characteristic \( m/r < m \). By induction, \( G/Z = F' \times N' \), where \( F' \) is the direct product of its unique \( p \)-Sylow subgroups. Let now \( F \) be the pull-back in \( G \) of \( F' \). This is a finite nilpotent group so it is the direct product of its unique \( p \)-Sylow subgroups and \( G = N(G) \times F \). However this implies that the infinite center of \( N(G) \) is included in the center of \( G \) that was assumed to be finite, contradiction. So \( Z(G) \) is infinite and (1) holds.
Now assume $Z(G)^0 = G^0 = \mathcal{N}(G)$. If $k = 1$ and $|E(G)| = p^a$, then by Lemma 2.8 we know that $G = G^0 \ltimes G/G^0$. But as $Z(G)^0 = G^0$, the product is direct and $G$ has exactly one $p$-Sylow subgroup.

Suppose $k > 1$. As $G/G^0$ is a finite nilpotent group, it is the direct product of its (unique) $p_i$-Sylow subgroups $H_1, \ldots, H_k$. Let $K_1 < G$ be the pull-back of the product of the first $k - 1$ factors, and $K_2$ be the pull-back of $H_k$. By induction $K_1 = G^0 \times F_1 \times \cdots \times F_{k-1}$ and $K_2 = G^0 \times F_k$, where each $F_i$ is the unique $p_i$-Sylow in $G$ (therefore normal) and $F$ in (3) is the product $F_1 \times \cdots \times F_k$.

Finally, assume $Z = Z(G)^0 \subsetneq G^0$ and let $G_1 = G/Z$. As $E(Z) = \pm 1$ (because $G^0 = \mathcal{N}(G)$ is torsion-free), then $|E(G_1)| = |E(G)| = m$ and $\dim G_1 < \dim G$.

By induction $G_1 = F' \ltimes G_1^0$, where $F'$ is the direct product of its (unique) $p_i$-Sylow subgroups $(i = 1, \ldots, k)$. Let now $K$ be the pull-back in $G$ of $F'$. Then, by the previous case, $K = Z(G)^0 \times F$. As $G/K = G^0_1$ is torsion-free, all $p$-subgroups of $G$ are contained in $K$, so (2) and (3) hold. \hfill \Box

Remark 2.7. In the proposition above, $G$ nilpotent is an essential assumption for all three conditions. For conditions (2) and (3), we have already noticed this in Remark 2.3. For condition (1), it is enough to consider a definable centerless torsion-free group, such as $\mathbb{R} \times \mathbb{R}^{>0}$.

We can now show the first part of Theorem 1.3.

Proof of Theorem 1.3(1). Suppose $G$ is a definable group such that $E(G) \neq 0$.

$(a) \Rightarrow (b)$ If $G$ is nilpotent, then by Proposition 2.10, $G$ has exactly one $p$-Sylow subgroup for each $p$-prime dividing $E(G)$.

$(b) \Rightarrow (c)$ Obvious.

$(c) \Rightarrow (a)$ Suppose all $p$-Sylow subgroups of $G$ are normal, and let $H$ be their product.

Clearly $H$ is a normal subgroup of $G$ and $\mathcal{N}(G) \cap H = \{e\}$, since all $p$-subgroups of $G$ are finite by Fact 2.3(4). Therefore $G = H \times \mathcal{N}(G)$.

As finite $p$-groups are nilpotent and we are assuming $\mathcal{N}(G)$ is nilpotent, it follows that $G$ is nilpotent as well. \hfill \Box

We now consider the case where $E(G) = 0$. It is well-known that $G$ may have no maximal definably compact subgroup (for instance, see Example 5.3 in [17]). However, by Theorem 1.5 in [3], if $G$ is definably connected then $G$ always has maximal abstracly compact subgroups, all conjugate (if definable). We will show that when $G$ is nilpotent (definably connected or not), then maximal abstractly compact subgroups of $G$ are a direct complement of $\mathcal{N}(G)$.

Definition 2.8. Let $G$ be a definable group and let $P$ be a property. We say that a subgroup $H \subset G$ is abstractly $P$ if $H$ is a section of a definable subgroup with property $P$ in a definable quotient of $G$.

That is, there is a definable normal subgroup $N$ of $G$ and a definable subgroup $H'$ of $G/N$ with property $P$, whose pull-back in $G$ is $N \rtimes H$. In particular, $H$ is abstractly isomorphic to a definable group $H'$ with property $P$. We call $H$ an abstract torus when $H'$ is a definable torus (that is, abelian, definably connected and definably compact).

Fact 2.9. [3] If $G$ is a definable solvable definably connected group and $A$ is any $0$-Sylow of $G$, then $G = \mathcal{N}(G) \rtimes A = \mathcal{N}(G) \rtimes T$, where $T \cong A/\mathcal{N}(A)$ is an abstract torus (and, therefore, a maximal abstractly compact subgroup of $G$). In particular, if $G$ is abelian, then $G = \mathcal{N}(G) \rtimes T$.

Proposition 2.10. Let $G$ be a nilpotent group such that $E(G) = 0$. Suppose $G$ is definably connected. Then

$(1)$ $G$ has a unique $0$-Sylow subgroup $A$ and it is contained in the center of $G$;

$(2)$ $G$ is a direct product of a normal abelian group and a definably compact group.

$(3)$ $G$ is a direct product of a normal abelian group and a definably connected group.

$(4)$ $G$ is a direct product of a normal abelian group and a definably compact group.

$(5)$ $G$ is a direct product of a normal abelian group and a definably connected group.
(2) Any maximal abstract torus $T$ of $G$ is contained in $A$, and

$$G = \mathcal{N}(G) \times T$$

Proof. By induction on $n = \dim G$. If $n = 1$, then by Fact 2.3(4)(5), $G$ is a 0-group and there is nothing to prove. Suppose $n > 1$. If $G$ is abelian, see Fact 2.9. So let $G$ be non-abelian and set $Z = Z(G)$. Note that both $Z$ and $G/Z$ are infinite, because $G$ is nilpotent and definably connected. Suppose first $G/Z$ is definably compact. Then $\mathcal{N}(G) \subseteq Z$, $G = \mathcal{N}(G) \times T$ (Fact 2.9) and $G$ has a unique 0-Sylow $A$ that is, moreover, central.

So let $G/Z$ be not definably compact. By induction, $G/Z$ has a unique 0-Sylow $A_1$, and $G/Z = N_1 \times T_1$, where $N_1 = \mathcal{N}(G/Z)$ is definable torsion-free, and $T_1 \cong A_1/\mathcal{N}(A_1)$ is a maximal abstract torus of $G/Z$. Note that $A_1$ is the image of any 0-Sylow $A$ of $G$. By Proposition 2.6 in [3], $\mathcal{N}(A)$ is central in $G$, therefore $A_1$ is definably compact and $T_1 = A_1$.

Let $H$ be the pull-back of $A_1$ in $G$. As $A_1$ is normal, $H$ is normal as well. The quotient $G/H = N_1$ is torsion-free, so $H$ contains all 0-subgroups of $G$. By induction (since $\dim N_1 > 0$, as $G/Z$ is not definably compact), $H$ has a unique 0-Sylow, so $G$ has a unique 0-Sylow $A = \mathcal{N}(A) \times T$.

Note that since $A$ is the only 0-Sylow of $G$, it contains all $k$-torsion elements $G[k]$ of $G$, for each $k \in \mathbb{N}$. Each $G[k]$ is a finite normal subgroup of $G$, therefore central. Let $S$ be the union of all $G[k]$. That is, $S$ is the torsion subgroup of $G$. We claim that $A$ is the definable subgroup generated by $S$. If not, let $(S) = K \subset A$. Note that, by minimality, $K$ is definably connected. Since $A$ is a 0-group, it follows that $E(A/K) = 0$. So by Fact 2.9(1)(5), the abelian group $A/K$ contains a 0-subgroup. But this is impossible, because $K$ contains all torsion elements of $A$ and it is a direct factor of $A$ (as both $A$ and $K$ are abelian and divisible), so $A/K$ is torsion-free. Therefore $(S) = A$. Since $S$ is central, $A$ is central as well. Finally, notice that every maximal abstract torus $T$ of $G$ contains $S$, therefore $T \subset A$. □

Remark 2.11. The nilpotency assumption in Proposition 2.10 cannot be extended to solvability, not even for linear groups. For instance, the group $G = \mathbb{R}^2 \rtimes \text{SO}_2(\mathbb{R})$, where $\text{SO}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication, is a centerless solvable linear group with several 0-Sylows.

We now show the second part of Theorem 1.3.

Proof of Theorem 1.3(2). Let $G$ be a definably connected group with $E(G) = 0$.

(a) ⇒ (b) If $G$ is nilpotent, then by Proposition 2.10 $G$ has exactly one 0-Sylow.

(b) ⇒ (c) Obvious.

(c) ⇒ (a) By Theorem 1.5 in [3], $G = PH$ where $P$ is a union of conjugates of a 0-Sylow $A$ and $H$ is definable torsion-free. Since $A$ is normal in $G$ by assumption, then $P = A$ and $G$ is solvable. Whenever $G$ is solvable and definably connected, then $G/\mathcal{N}(G)$ is definably compact and therefore abelian by [13]. As we are assuming $\mathcal{N}(G)$ nilpotent, then $G$ is nilpotent as well. □

We conclude the section with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $G$ be a nilpotent definable group.

(1) We want to show that $G$ has maximal abstractly compact subgroups $K$, and any such $K$ is a direct complement of $\mathcal{N}(G)$. If $E(G) \neq 0$, then $K = F$ from Proposition 2.9. If $G = G^0$, then $K = T$ in Proposition 2.10. If $E(G) = 0$ and $G \neq G^0$, then $K = F \cdot T$, where $F$ is a finite normal subgroup of $G$ such that $G = F \cdot G^0$ [8 Theo 6.10], and $T$ is a maximal abstract torus of $G^0$. 
(2) If $G$ is torsion-free, there is nothing to prove. Set $N = \mathcal{N}(G) \subseteq G$. As $G$ is definably connected, then $E(G) = 0$. By Proposition 2.10
\[ Z(G) = Z(N) \times T \]
for every maximal abstract torus $T$ of $G$. Therefore $Z(G)$ is definably connected and contains every abstractly compact subgroup of $G$. \qed

3. Nilpotency and normalizers

It is well-known that a finite group $G$ is nilpotent if and only if $G$ has the normalizer property (also called normalizers grow). That is, every proper subgroup $H$ of $G$ is contained properly in its normalizer $N_G(H) = \{g \in G : H^g = H\}$.

For infinite groups one implication still holds: every nilpotent group has the normalizer property. However, there are infinite groups with this property that are not even solvable. We show below that for groups definable in o-minimal structures nilpotency is equivalent to the normalizer property, even when restricted to definable subgroups, assuming $\mathcal{N}(G)$ is nilpotent:

**Proposition 3.1.** Let $G$ be a definably connected group such that $\mathcal{N}(G)$ is nilpotent. Then $G$ is nilpotent if and only if $H \subseteq N_G(H)$, for every proper definable $H < G$.

**Proof.** Assume $H \subseteq N_G(H)$ for every proper definable $H < G$. We will show that $G$ is nilpotent. If $G$ is not solvable, let $R$ be the solvable radical of $G$. Then the quotient of $G/R$ by its finite center is a centerless semisimple group $\check{G}$.

Suppose $G$ is definably compact and let $H$ be the normalizer of a maximal definable torus $T$ of $G$. We claim that $H$ is self-normalizing. Suppose $g \in G$ normalizes $H$. Then $T^g$ is a maximal definable torus of $H$. Therefore $T^g = T^x$ for some $x \in H$, and $g \in H$ as well. Now the pull-back of $H$ in $G$ is a proper definable subgroup equal to its normalizer, contradiction.

If $G$ is not definably compact, then by [3], $\bar{G} = \check{K}\check{H}$, where $\check{K}$ is definably compact and $\check{H}$ is torsion-free. By [12], $G$ is elementarily equivalent to a connected centerless semisimple Lie group, for which maximal compact subgroups are self-normalizing subgroups. Therefore the pre-image of $\check{K}$ in $G$ is a proper definable subgroup equal to its normalizer, contradiction.

Hence $G$ must be solvable. If $G$ is not torsion-free, let $A$ be a 0-Sylow of $G$. Then $G = \mathcal{N}(G) \cdot A$ by Fact 2.3. Let $H = N_G(A)$. If $H = G$, then $A$ is normal in $G$. By Theorem 1.3(2), then $G$ is nilpotent, and we are done. Assume that $H$ is a proper subgroup of $G$. By Theorem 1.3(2), this is equivalent to say that $G$ is not nilpotent. We claim that $N_G(H) = H$. Since $A$ is normal in $H$, then by Theorem 1.3(2), $H$ is nilpotent. Let now $g \in G$ be such that $H^g = H$. As $H$ is nilpotent, by Proposition 2.10 $A$ is the only 0-Sylow of $G$ and $A^g = A$. Therefore $g \in N_G(A) = H$. So $H$ is a proper definable subgroup of $G$ equal to its normalizer, contradiction.

Thus we have shown that every time $G$ is not nilpotent, there is a definable subgroup $H < G$ such that $N_G(H) = H$. \qed

Proposition 3.1 finishes the proof of Theorem 1.3

4. Nilpotent groups and linear algebraic groups

Connected solvable Lie groups that are definable in an o-minimal expansion of the reals are completely characterized in [4]. Some of them, for instance the group in [18] pg. 327, are not Lie isomorphic to any linear algebraic group. However, if we restrict to nilpotent groups, the only definable Lie groups are linear algebraic:
Proof of Theorem 1.2. Clearly linear algebraic groups over the reals are definable in the real field. Conversely, let $G$ be a nilpotent real Lie group definable in an o-minimal structure.

First assume $G$ is connected. By Proposition 2.10, $G$ has a closed, simply-connected normal subgroup $N = N(G)$ and a central connected compact subgroup $T$ such that $G = N \times T$. By Theorem 4.5 in [4], $N$ is a triangular group, so is isomorphic to a closed connected subgroup of $UT_m(\mathbb{R})$, the group of unipotent upper triangular matrices, for some $m \in \mathbb{N}$. All such groups are algebraic, as the exponential map is polynomial for nilpotent Lie algebras. If $\dim T = k$, then the subgroup $T$ is Lie isomorphic to the algebraic group $SO_2^k(\mathbb{R})$.

If $G$ is not definably connected, then by [8], $G = F \cdot G^0$, for some finite normal subgroup $F$. By the connected case, $G^0$ is linear algebraic. Since finite groups are linear algebraic, so is $G$. Therefore a definable real nilpotent $G$ is Lie isomorphic to $U \times K$, where $U$ is a closed connected subgroup of some $UT_m(\mathbb{R})$, and $K$, the maximal compact subgroup of $G$, is isomorphic to $F \cdot SO_2^k(\mathbb{R})$, for some finite nilpotent $F$.

By Theorem 1.1 and results of Hrushovski, Peterzil and Pillay [9, 10] on compact groups, the problem of determining whether a definable nilpotent group is elementarily equivalent to a linear algebraic group reduces to the torsion-free case.

By [12], every linearizable abelian torsion-free definable group can be decomposed into the product of definable 1-dimensional subgroups. This definable splitting has been proved also in [14] for groups definable in several o-minimal structures, and by an induction argument it reduces to the 2-dimensional case:

Conjecture 4.1. Every abelian 2-dimensional torsion-free group definable in an o-minimal structure $\mathcal{M}$ is the product of two definable 1-dimensional subgroups.

It is unknown whether Conjecture 1.1 holds in an arbitrary o-minimal structure. We give below a positive answer for groups with an infinite definable family of definable automorphisms:

Proposition 4.2. Let $(G, +)$ be an abelian 2-dimensional torsion-free group definable in an o-minimal structure $\mathcal{M}$, and let $\text{Aut}(G)$ be the group of $\mathcal{M}$-definable automorphisms of $G$. If $\dim \text{Aut}(G) > 0$, then $G$ can be decomposed as a direct product of definable 1-dimensional subgroups.

Proof. We know by [15] that $G$ has a 1-dimensional definable subgroup $H$. Suppose $A$ is a different 1-dimensional definable subgroup of $G$. Then $A$ is a definable complement of $H$, and we are done. This is because $A \cap H = \{0\}$, as both $A$ and $H$ have no proper non-trivial definable subgroups, and $H + A = G$, because $H + A$ is a definable subgroup of full dimension, and $G$ is definably connected.

So assume for a contradiction that $H$ is the only non-trivial definable subgroup of $G$, and set $G = G/H$. Thus $H$ is definably characteristic and for each $x \in G$, $x \notin H$, $G = \langle x \rangle$ and each definable homomorphism from $G$ is determined by its value on $x$. Therefore no definable automorphism of $G$ can send in $H$ an element that is not in $H$ already.

Lemma 4.3. Let $\varphi_1, \varphi_2 \in \text{Aut}(G)$, and let $\bar{\varphi}_1, \bar{\varphi}_2 \in \text{Aut}(\bar{G})$ be the induced maps on the quotient $\bar{G}$. Then

$$\bar{\varphi}_1 = \bar{\varphi}_2 \implies \varphi_1 = \varphi_2$$

Therefore $\text{Aut}(G) \xhookrightarrow{} \text{Aut}(\bar{G})$.

Proof. Let $x \in G \setminus H$, so that $G = \langle x \rangle$. Then

$$\varphi_1(x) = \varphi_2(x) + h, \text{ for some } h \in H$$
because \( \varphi_1 = \varphi_2 \). Consider now the kernel of the homomorphism \( \varphi_1 - \varphi_2 \):

\[
K = \ker(\varphi_1 - \varphi_2) = \{ g \in G : \varphi_1(g) = \varphi_2(g) \}.
\]

If \( K = \{0\} \) then \( \varphi_1 - \varphi_2 \in \text{Aut}(G) \) and \( (\varphi_1 - \varphi_2)(x) = h \in H \), impossible. Then \( K \) is a non-trivial definable subgroup of \( G \), so \( H \subset K \), and \( \varphi_1 = \varphi_2 \). \( \square \)

As \( \dim \text{Aut}(G) > 0 \), there is an infinite definable family \( F \) in \( \text{Aut}(G) \). Let \( F \subset \text{Aut}(\bar{G}) \) be the induced definable family on the quotient \( \bar{G} \). By Lemma 4.3, we know that \( F \) is infinite as well. By [13], there is a definable product \( \cdot \) on \( \bar{G} \), such that \( (\bar{G}, +, \cdot) \) is a definable field. We show below that \( \text{Aut}(G) \) is a 1-dimensional definable group, and it is definably isomorphic to the multiplicative group of \( G \), \( G^* = \bar{G} \setminus \{0\} \):

**Lemma 4.4.** \( \text{Aut}(G) \cong (G^*, \cdot) \).

**Proof.** First let us see that \( \text{Aut}(\bar{G}) \) is a definable group definably isomorphic to \( (G^*, \cdot) \). Let \( f \in \text{Aut}(\bar{G}) \) and let \( f(1) = a \in G^* \). The set \( \{ x \in \bar{G} : f(x) = a \cdot x \} \) is a definable subgroup of \( (\bar{G}, +) \) containing 0 and 1; but \( (\bar{G}, +) \) does not have any proper definable subgroups, so \( f(x) = a \cdot x \) for every \( x \in \bar{G} \). On the other hand, every definable function \( \bar{G} \to \bar{G} \) of the form \( f(x) = a \cdot x \), with \( a \in G^* \), is a definable automorphism of \( (\bar{G}, +) \), so \( \text{Aut}(\bar{G}) \cong (G^*, \cdot) \).

By Lemma 4.3, \( \dim \text{Aut}(G) = 1 \) as well, and \( \text{Aut}(G)^0 \cong \bar{G}^0 \). Moreover (for instance) \( -\text{id}_G \mapsto -1 \in (\bar{G}^*, \cdot) \), so \( \text{Aut}(G) \cong (G^*, \cdot) \). \( \square \)

Fix now \( x \in G \), \( x \notin H \), and consider the set

\[
X = \{ \varphi(x) : \varphi \in \text{Aut}(G) \}.
\]

Clearly \( X \) is a definable set, and \( \dim X = \dim \text{Aut}(G) = 1 \), because \( x \) is a generator of \( G \). Moreover \( X \cap H = \emptyset \), because no element in \( H \) is a generator. We claim that \( K = X \cup \{0\} \) is a subgroup:

- \( a \in K \Rightarrow -a \in K \), because if \( \varphi \in \text{Aut}(G) \), then \( -\varphi \in \text{Aut}(G) \).

- \( a, b \in K \Rightarrow a + b \in K \):
  
  (i) If \( b = -a \), then \( a + b = 0 \), and there is nothing to prove.

  (ii) Let \( b \neq -a \), with \( \varphi(x) = a \) and \( \psi(x) = b \). We claim that \( \varphi + \psi \in \text{Aut}(G) \). Otherwise
  
  \[
  F = \ker(\varphi + \psi) = \{ g \in G : \varphi(g) = -\psi(g) \}
  \]

  would be a proper (because \( \varphi(x) \neq -\psi(x) \)) non-trivial definable subgroup of \( G \), so \( H = F \). Therefore \( f = (-\psi)^{-1} \circ \varphi \) would be a definable automorphism of \( G \) that is the identity on \( H \), and is not the identity on \( G \). So consider the set of all such automorphisms of \( G \):
  
  \[
  Y = \{ \varphi \in \text{Aut}(G) : \varphi|_H = \text{id}_H \}
  \]

  Now \( Y \) would be an infinite (because it contains \( f \) and all its powers) definable subgroup of \( \text{Aut}(G) \). By dimension reasons \( Y^0 = \text{Aut}(G)^0 \), which is impossible, because \( \text{Aut}(G)^0 \) contains all multiplications by positive rational numbers, none of which is the identity on \( H \).

Therefore \( \varphi + \psi \in \text{Aut}(G) \), and \( (\varphi + \psi)(x) = a + b \).
So we have proved that if \( \dim \text{Aut}(G) > 0 \), then the definable 1-dimensional subgroup \( H \) has a definable complement in \( G \), as wanted. \( \square \)

**Question 4.5.** What if \( \dim \text{Aut}(G) = 0 \)?

We conclude with a remark and a question about the general case:

**Remark 4.6.** Let \( G \) be a definably connected group in an o-minimal structure. Assume \( G \) is elementarily equivalent to a real algebraic group. Then \( N(G) \) is nilpotent and \( G \) has a definable Levi decomposition.

**Proof.** In real algebraic groups any normal closed connected simply-connected subgroup is nilpotent, so \( N(G) \) must be nilpotent. Moreover, in real algebraic groups the intersection between the solvable radical and any Levi subgroup is finite, therefore Levi subgroups of \( G \) from \( \square \) must be definable.

**Question 4.7.** Let \( G \) be a definably connected group in an o-minimal structure such that \( N(G) \) is nilpotent and \( G \) has a definable Levi decomposition. Is \( G \) elementarily equivalent to a real algebraic group?

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