A KRONECKER LIMIT TYPE FORMULA FOR ELLIPTIC EISENSTEIN SERIES

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Abstract. Let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane \( \mathbb{H} \), and let \( M = \Gamma \backslash \mathbb{H} \) be the associated finite volume hyperbolic Riemann surface. Associated to any cusp of \( M \), there is the classically studied non-holomorphic (parabolic) Eisenstein series. In [KM79], Kudla and Millson studied non-holomorphic (hyperbolic) Eisenstein series associated to any closed geodesic on \( \Gamma \backslash \mathbb{H} \). Finally, in [JK11] and [KvP12], so-called elliptic Eisenstein series associated to any elliptic fixed point of \( M \) were introduced. In the present article, we prove the meromorphic continuation of the elliptic Eisenstein series and we explicitly compute its poles and residues. Further, we derive a Kronecker limit type formula for elliptic Eisenstein series for general \( \Gamma \). Finally, for the full modular group \( \text{PSL}_2(\mathbb{Z}) \), we give an explicit formula for the Kronecker’s limit functions in terms of holomorphic modular forms.

1. Introduction

1.1. Classical Eisenstein series. Classically, in the theory of holomorphic modular forms, the Eisenstein series of weight \( 2k \) \((k \in \mathbb{N}, k \geq 2)\) for the full modular group \( \text{PSL}_2(\mathbb{Z}) \) are defined by

\[
E_{2k}(z) := \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \frac{1}{(cz + d)^{2k}} \quad (z \in \mathbb{H}).
\]

The arithmetic significance of these series is reflected by the fact that their Fourier coefficients are given by certain divisor sums. More generally, in the theory of automorphic functions, the non-holomorphic Eisenstein series associated to the cusp \( \infty \) of \( \text{PSL}_2(\mathbb{Z}) \) is defined by

\[
E_{\text{par}}(z,s) := \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \frac{\text{Im}(z)^s}{(cz + d)^{2s}} \quad (z \in \mathbb{H}; s \in \mathbb{C}, \text{Re}(s) > 1);
\]

where \( \Gamma_{\infty} \) denotes the stabilizer of the cusp \( \infty \) in \( \Gamma \).

More generally, let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a Fuchsian subgroup of the first kind, then, a non-holomorphic Eisenstein series \( E_p(\text{par}) \( z, s \) \) associated to any cusp \( p \) of \( \Gamma \) can be defined for \( z \in M \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \). As a function of \( s \), this so-called parabolic Eisenstein series is a holomorphic function for \( \text{Re}(s) > 1 \) and admits a meromorphic continuation to the whole complex \( s \)-plane. The parabolic Eisenstein series play an important role in number theory, in particular in the spectral theory of automorphic functions on \( M \).

1.2. Hyperbolic and elliptic Eisenstein series. In [KM79], Kudla and Millson first studied a generalized, form-valued Eisenstein series associated to any closed geodesic of \( M \), or equivalently to any primitive, hyperbolic element of \( \Gamma \). A scalar-valued, hyperbolic Eisenstein series was defined in [JKvP10], and the authors proved that the series admits a meromorphic continuation to all \( s \in \mathbb{C} \).

Analogously, associated to any elliptic fixed point \( w \) of \( M \), and, more generally, associated to any point of \( M \), there is an elliptic Eisenstein series, which was first introduced in the unpublished document [JK04] by Jorgenson and Kramer. The motivation to consider these series has arisen from Arakelov geometry. To compare the canonical (or Arakelov) metric \( \mu_{\text{can}} \) and the scaled hyperbolic metric \( \mu_{\text{shyp}} \), one considers the quotient

\[
d_\Gamma := \sup_{z \in \Gamma \backslash \mathbb{H}} \frac{\mu_{\text{can}}(z)}{\mu_{\text{shyp}}(z)}.
\]
The main result proven in [JK11] is the bound \( d_\Gamma = O_{\gamma_0}(1) \), where \( \Gamma \backslash \mathbb{H} \) is a finite degree cover of \( \Gamma_0 \backslash \mathbb{H} \). The proof in [JK11] uses an identity which relates \( \mu_{\text{can}} \) to \( \mu_{\text{hyp}} \) and an integral involving the hyperbolic heat kernel on \( \Gamma \backslash \mathbb{H} \) (see [JK06]). Decomposing this heat kernel into terms involving parabolic, hyperbolic, and elliptic elements of \( \Gamma \) (a method that is also used in the proof of Selberg’s trace formula), each term can be bounded by the special value of a parabolic, hyperbolic, and elliptic Eisenstein series at \( s = 2 \), respectively.

For \( z, w \in M \) with \( z \neq w \), the elliptic Eisenstein series associated to the point \( w \in M \) for the subgroup \( \Gamma \) is defined by

\[
E^{\text{ell}}_w(z, s) = \sum_{\gamma \in \Gamma_w \backslash \Gamma} \sinh(d_{\text{hyp}}(\gamma z))^{-s};
\]

here, \( \Gamma_w \) denotes the stabilizer of \( w \) in \( \Gamma \), and \( d_{\text{hyp}}(w, \gamma z) \) denotes the hyperbolic distance from \( w \) to \( \gamma z \). The parabolic, hyperbolic, and elliptic Eisenstein series fulfill various relations. For example, if one considers a sequence of elliptically degenerating hyperbolic Riemann surfaces, then the elliptic Eisenstein series associated to the degenerating elliptic point converges to the rescaled parabolic Eisenstein series associated to the newly developed cusp of the limit surface [GvP09]. An analogous result was proven in [GJM08] for the hyperbolic Eisenstein series. In [KvP12], the Fourier expansion of the elliptic Eisenstein series for the full modular group \( \text{PSL}_2(\mathbb{Z}) \) was computed and its meromorphic continuation was established from this.

In the present article, the meromorphic continuation of the elliptic Eisenstein series \( E^{\text{ell}}_w(z, s) \) for any Fuchsian subgroup \( \Gamma \) of the first kind to the whole \( s \)-plane is proven using methods from the spectral theory on \( M \). More precisely, we employ a relation between \( E^{\text{ell}}_w(z, s) \) and an elliptic Poincaré series, which is square-integrable on \( M \) and which can be meromorphically continued via its spectral expansion. Furthermore, we determine the possible poles of \( E^{\text{ell}}_w(z, s) \) and we compute its residues.

1.3. Kronecker limit type formula. Classically, for \( \Gamma = \text{PSL}_2(\mathbb{Z}) \), Kronecker’s limit formula evaluates the special value of the parabolic Eisenstein series at \( s = 0 \) in terms of the logarithm of the absolute value of Dedekind’s Delta function. More precisely, at \( s = 0 \), there is a Laurent expansion of the form

\[
E^{\text{par}}_\infty(z, s) = 1 + \log(|\Delta(z)|^{1/6} \text{Im}(z)) \cdot s + O(s^2),
\]

with Dedekind’s Delta function given by

\[
\Delta(z) = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2),
\]

For general Fuchsian subgroups of the first kind, analogues of Kronecker’s limit formula were investigated in [Gol73].

The special value of the hyperbolic Eisenstein series at \( s = 0 \) is a harmonic form which is the Poincaré dual to the considered geodesic [KM79].

In the present paper, we first prove a Kronecker limit type formula for elliptic Eisenstein series for an arbitrary Fuchsian subgroup \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) of the first kind. Having substracted an expression involving all parabolic Eisenstein series for \( \Gamma \), this formula expresses the special value of \( E^{\text{ell}}_w(z, s) \) at \( s = 0 \) in terms of the norm of a holomorphic modular form which vanishes only at the elliptic point \( w \).

In case that \( \Gamma = \text{PSL}_2(\mathbb{Z}) \), we then explicitly determine this holomorphic modular form, and we prove that, at \( s = 0 \), there are Laurent expansions of the form

\[
E^{\ell}_t(z, s) = -\log(|E_6(z)| |\Delta(z)|^{-1/2}) \cdot s + O(s^2),
\]

\[
E^{\ell}_p(z, s) = -\log(|E_4(z)| |\Delta(z)|^{-1/3}) \cdot s + O(s^2)
\]

with \( \rho := \exp(2\pi/3) \). Combining the Kronecker’s limit formulae (1.1), (1.2), and (1.3), one easily concludes for a modular form \( f \) for \( \text{PSL}_2(\mathbb{Z}) \), that \( \log |f| \) can be expressed as the special value of a combination of the Eisenstein series \( \bar{E}^{\text{par}}_\infty(z, s) \), \( E^{\ell}_t(z, s) \), and \( E^{\ell}_p(z, s) \). Further, we note that the leading terms in (1.2) resp. (1.3) are equal to \( -\log((j(i) - j(z))^2) \) and \( -\log((j(i) - j(z))^3) \), respectively. This phenomenon is explained at the end of this article, by proving a relation between the elliptic
Eisenstein series and the automorphic Green’s function on $M$, which is a fundamental object in arithmetic geometry, see, e.g., [GZ86]. In particular, we prove that, for an arbitrary Fuchsian subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$, the elliptic Eisenstein series and the rescaled automorphic Green’s function on $M$ coincide at $s = 0$.

1.4. Outline of the article. The paper is organized as follows. In section 2, we recall and summarize basic notation and definitions used in this article, and we cite relevant results from the literature. In section 3, we define the elliptic Eisenstein series $E_{ell}^\sharp(z, s)$ for a Fuchsian subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$ of the first kind and we prove some of its basic properties. In particular, we show that the series is holomorphic for $\text{Re}(s) > 1$ and an automorphic function for $\Gamma$. In contrast to the parabolic series, the elliptic Eisenstein series fails to be an eigenfunction of $\Delta_{hyp}$; instead it satisfies a differential difference equation. In section 4, we prove the meromorphic continuation of the elliptic Eisenstein series to the whole $s$-plane and we determine its poles and compute its residues. In section 5, we prove a Kronecker limit type formula for the elliptic Eisenstein series for an arbitrary Fuchsian subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$ of the first kind. In section 6, we establish the explicit Kronecker’s limit formulae (1.2) and (1.3) for the hyperbolic line element, resp. the hyperbolic Laplacian take the form

$\Delta_{hyp} = -\frac{\partial^2}{\partial \vartheta^2} - \frac{1}{\tanh(\vartheta)} \frac{\partial}{\partial \vartheta} - \frac{1}{\sinh^2(\vartheta)} \frac{\partial^2}{\partial \vartheta^2}$.

In section 7, we conclude by proving a relation between the elliptic Eisenstein series and the automorphic Green’s function on $M$ for an arbitrary Fuchsian subgroup $\Gamma \subset \text{PSL}_2(\mathbb{R})$ of the first kind.

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2. Background material

2.1. Basic notation. As mentioned in the introduction, we let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} | x, y \in \mathbb{R}; y > 0\}$. We let $M := \Gamma \backslash \mathbb{H}$, which is a finite volume hyperbolic Riemann surface, and denote by $p : \mathbb{H} \rightarrow M$ the natural projection. By $P_{\Gamma}$ and $E_{\Gamma}$ we denote a complete set of $\Gamma$-inequivalent cusps and elliptic fixed points of $\Gamma$, respectively, and we set $p_{\Gamma} := \sharp P_{\Gamma}$, $e_{\Gamma} := \sharp E_{\Gamma}$, that is, we assume that $M$ has $e_{\Gamma}$ elliptic fixed points and $p_{\Gamma}$ cusps. We identify $M$ locally with its universal cover $\mathbb{H}$.

We let $\mu_{hyp}$ denote the hyperbolic metric on $M$, which is compatible with the complex structure of $M$, and has constant negative curvature equal to minus one. The hyperbolic line element $ds_{hyp}^2$, resp. the hyperbolic Laplacian $\Delta_{hyp}$, are given as

$$ds_{hyp}^2 := \frac{dx^2 + dy^2}{y^2}, \quad \Delta_{hyp} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

By $\mathcal{F}_{\Gamma}$ we denote a fundamental domain of $\Gamma$. Since $\Gamma$ is of the first kind, the hyperbolic volume $\text{vol}_{hyp}(M) = \text{vol}_{hyp}(\mathcal{F}_{\Gamma})$ is finite.

By $d_{hyp}(z, w)$ we denote the hyperbolic distance from $z \in \mathbb{H}$ to $w \in \mathbb{H}$, which satisfies the relation

$$\cosh(d_{hyp}(z, w)) = 1 + 2u(z, w)$$

with the point-pair invariant

$$u(z, w) = \frac{|z - w|^2}{4 \text{Im}(z) \text{Im}(w)}.$$ 

For $z = x + iy \in \mathbb{H}$, we define the hyperbolic polar coordinates $\varrho = \varrho(z), \vartheta = \vartheta(z)$ centered at $i \in \mathbb{H}$ by

$$\varrho(z) := d_{hyp}(i, z), \quad \vartheta(z) := \angle(\mathcal{L}, T_z),$$

where $\mathcal{L} := \{z \in \mathbb{H} | x = \text{Re}(z) = 0\}$ denotes the positive $y$-axis and $T_z$ is the euclidean tangent at the unique geodesic passing through $i$ and $z$ at the point $i$. In terms of the hyperbolic polar coordinates, the hyperbolic line element, resp. the hyperbolic Laplacian take the form

$$ds_{hyp}^2 = \sinh^2(\varrho)d\varrho^2 + d\vartheta^2, \quad \Delta_{hyp} = -\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\tanh(\varrho)} \frac{\partial}{\partial \varrho} - \frac{1}{\sinh^2(\varrho)} \frac{\partial^2}{\partial \vartheta^2}.$$
2.2. Parabolic Eisenstein series. For a parabolic fixed point/cusp \( p_j \in P\Gamma \) \((j = 1, \ldots, p\Gamma)\), let \( \Gamma_{p_j} \) denote its stabilizer subgroup \( \Gamma_{p_j} := \text{Stab}_H(p_j) = \langle \gamma_{p_j} \rangle \) generated by a primitive, parabolic element \( \gamma_{p_j} \in \Gamma \). Further, let \( \sigma_{p_j} \in \text{PSL}_2(\mathbb{R}) \) be a (parabolic) scaling matrix, that is a matrix satisfying
\[
\sigma_{p_j} \gamma_{p_j} \sigma_{p_j} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
the scaling matrix \( \sigma_{p_j} \) is unique up to multiplication on the right by elements \( \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \) with \( \xi \in \mathbb{R} \).

For \( z \in \mathbb{H} \) and \( s \in \mathbb{C} \), the parabolic Eisenstein series associated to the parabolic fixed point \( p_j \in P\Gamma \) \((j = 1, \ldots, p\Gamma)\) is defined by
\[
\mathcal{E}^{\text{par}}_{p_j}(z,s) = \sum_{\gamma \in \Gamma_{p_j} \setminus \Gamma} \text{Im}(\sigma_{p_j}^{-1} \gamma z)^s.
\]

Referring to [Hej83], [Iwa02], or [Kub73], where detailed proofs are provided, we recall that the series (2.3) converges absolutely and locally uniformly for any \( z \in \mathbb{H} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), and that it is holomorphic for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \). Moreover, it is invariant with respect to \( \Gamma \), i.e. \( \mathcal{E}^{\text{par}}_{p_j}(z,s) \in \mathcal{A}(\Gamma \setminus \mathbb{H}) \). A straightforward computation shows that the series (2.3) satisfies the differential equation
\[
(\Delta_{\text{hyp}} - s(1 - s)) \mathcal{E}^{\text{par}}_{p_j}(z,s) = 0,
\]
i.e. it is an eigenfunction of \( \Delta_{\text{hyp}} \) and, therefore, it is a real-analytic function (with respect to \( z = x + iy \)).

The parabolic Eisenstein series \( \mathcal{E}^{\text{par}}_{p_j}(z,s) \) \((j = 1, \ldots, p\Gamma)\) admits a meromorphic continuation to the whole \( s \)-plane. For \( \text{Re}(s) \geq 1/2 \), there are only finitely many poles; they are located in the interval \((1/2, 1]\) and they are simple. There is always a pole at \( s = 1 \) with residue
\[
\text{Res}_{s=1} \mathcal{E}^{\text{par}}_{p_j}(z,s) = \frac{1}{\text{vol}_{\text{hyp}}(\mathcal{F}_{\Gamma})}.
\]
Moreover, the following functional equations hold
\[
\mathcal{E}^{\text{par}}_{p_j}(z,s) = \sum_{k=1}^{p\Gamma} \phi_{p_j,p_k}(s) \mathcal{E}^{\text{par}}_{p_k}(z,1-s)
\]
and
\[
\sum_{\ell=1}^{p\Gamma} \phi_{p_j,p_{\ell}}(s) \phi_{p_{\ell},p_k}(1-s) = \delta_{p_j,p_k};
\]
here
\[
\phi_{p_j,p_k}(s) := \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{1}{c^s} \prod_{c \mod d} \frac{1}{(c \overline{d} \in \sigma_{p_j}^{-1} \Gamma \sigma_{p_k})},
\]
are the scattering entries (see, e.g., [Iwa02] or [Kub73]).

2.3. Spectral expansions. We enumerate the discrete eigenvalues of the hyperbolic Laplacian \( \Delta_{\text{hyp}} \) acting on smooth functions on \( M \) by
\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots;
\]
we write \( \lambda_r = 1/4 + t_r^2 = s_r(1 - s_r) \), i.e., \( s_r = 1/2 + it_r \) with \( t_r > 0 \) or \( t_r \in [-i/2, 0] \). An eigenvalue \( \lambda_r \) with \( 0 < \lambda_r < 1/4 \), that is \( 1/2 < s_r < 1 \), is called exceptional. In case that \( p\Gamma > 0 \), \( \Delta_{\text{hyp}} \) has also a continuous spectrum covering the interval \([1/4, \infty)\) uniformly with multiplicity 1.

The eigenvalues of the continuous spectrum are of the form \( \lambda = 1/4 + t^2 = s(1 - s) \), i.e., \( s = 1/2 + it \) with \( t \in \mathbb{R} \). The corresponding eigenfunctions are given by the parabolic Eisenstein series \( \mathcal{E}^{\text{par}}_{p_k}(z,1/2+it) \) \((k = 1, \ldots, p\Gamma)\).

Under certain hypotheses on a function \( f \) on \( M \), which are defined carefully in numerous references such as [Iwa97], [Hej83], or [Kub73], there is a spectral expansion in terms of the eigenfunctions \( \psi_r \) associated to the discrete eigenvalues \( \lambda_r \) of the hyperbolic Laplacian \( \Delta_{\text{hyp}} \) and the parabolic Eisenstein
series \( \mathcal{E}_{p_k}^{\text{par}} \) associated to the cusps \( p_k \in P_1 \); without loss of generality, we assume that all eigenfunctions of the Laplacian are real-valued.

\[
(2.6) \quad f(z) = \sum_{r=0}^{\infty} \langle f, \psi_r \rangle \psi_r(z) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \langle f, \mathcal{E}_{p_k}^{\text{par}}(\cdot, \frac{1}{2} + it) \rangle \mathcal{E}_{p_k}^{\text{par}}(z, \frac{1}{2} + it) dt.
\]

By \( L^2(M) \) we denote the space of square-integrable functions on \( M \). For any \( f \in L^2(M) \) such that \( f \) and \( \Delta_{\text{hyp}} f \) are smooth and bounded, the series \( (2.6) \) converges absolutely and uniformly for \( z \) ranging over compacta \( K \subseteq \mathbb{H} \).

2.4. Elliptic Poincaré series. For an elliptic fixed point \( e_j \in E_\Gamma \) (\( j = 1, \ldots, e_\Gamma \)), let \( \Gamma_{e_j} \) denote its stabilizer subgroup \( \Gamma_{e_j} := \text{Stab}_\Gamma(e_j) = \langle \gamma_{e_j} \rangle \) generated by a primitive, elliptic element \( \gamma_{e_j} \in \Gamma \). Hence there is an elliptic scaling matrix \( \sigma_{e_j} \in \mathrm{PSL}_2(\mathbb{R}) \) satisfying

\[
\sigma_{e_j}^{-1} \gamma_{e_j} \sigma_{e_j} = \begin{pmatrix} \cos(\pi/n_{e_j}) & \sin(\pi/n_{e_j}) \\
-\sin(\pi/n_{e_j}) & \cos(\pi/n_{e_j}) \end{pmatrix} \begin{pmatrix} k(\pi/n_{e_j}) \end{pmatrix},
\]

where

\[
n_{z} := \text{ord}_\Gamma(z) \quad (n_z \in \mathbb{N})
\]

denotes the order of the point \( z \) in \( \Gamma \); if \( z \) is an elliptic fixed point, then \( n_z \geq 1 \). Therefore, the group \( \sigma_{e_j}^{-1} \Gamma \sigma_{e_j} \subset \mathrm{PSL}_2(\mathbb{R}) \) is a Fuchsian group of the first kind with the elliptic fixed point \( i \); the scaling matrix \( \sigma_{e_j} \) is unique up to multiplication on the right by elements \( k(\theta) \in \mathbb{K}, \) where \( \theta \in [0, 2\pi) \).

For \( z \in \mathbb{H}, s \in \mathbb{C} \), the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) associated to the elliptic fixed point \( e_j \in E_\Gamma \) is defined by

\[
(2.7) \quad P_{e_j}^{\text{ell}}(z, s) = \sum_{\gamma \in \Gamma_{e_j} \setminus \Gamma} \cosh(\theta(\sigma_{e_j}^{-1} \gamma z))^{-s}
\]

For fixed \( z \in \mathbb{H} \), the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) converges absolutely and locally uniformly for \( s \in \mathbb{C} \) with \( \operatorname{Re}(s) > 1 \), and hence defines a holomorphic function. The elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) is invariant under the action of \( \Gamma \), i.e. we have

\[
P_{e_j}^{\text{ell}}(\gamma z, s) = P_{e_j}^{\text{ell}}(z, s)
\]

for any \( \gamma \in \Gamma \). For fixed \( s \in \mathbb{C} \) with \( \operatorname{Re}(s) > 1 \), the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) converges absolutely and uniformly for \( z \) ranging over compacta \( K \subseteq \mathbb{H} \). Moreover, we have \( P_{e_j}^{\text{ell}}(z, s) \in L^2(M) \). For \( z \in \mathbb{H} \) and \( s \in \mathbb{C} \) with \( \operatorname{Re}(s) > 1 \), the series \( P_{e_j}^{\text{ell}}(z, s) \) satisfies the differential equation (\( j = 1, \ldots, e_\Gamma \))

\[
(\Delta_{\text{hyp}} - s(1 - s))P_{e_j}^{\text{ell}}(z, s) = s(s + 1)P_{e_j}^{\text{ell}}(z, s + 2).
\]

2.5. Special functions. For special functions, we refer to the vast literature (see, e.g., [GR07] or [EMOT81]). However, for the convenience of the reader, we recall several useful identities.

The \( \Gamma \)-function satisfies the following duplication formula

\[
(2.8) \quad \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} 2^{1-2s}\Gamma(2s).
\]

By Stirling’s asymptotic formula, we have for fixed \( \sigma \in \mathbb{R} \) and \( |t| \to \infty \) the asymptotics

\[
|\Gamma(\sigma + it)| \sim \sqrt{2\pi |t|}^{\sigma-1/2} \exp\left( -\frac{\pi |t|^2}{2} \right)
\]

with an implied constant depending on \( \sigma \).

For \( m \in \mathbb{N} \), the Pochhammer symbol is given by

\[
(s)_m := \frac{\Gamma(s + m)}{\Gamma(s)};
\]
for \( m \in \mathbb{N} \) with \( m > 0 \), we note the alternative formula \((s)_m = \prod_{j=0}^{m-1}(s+j)\). Furthermore, for \( m \in \mathbb{Z} \), we have
\[
\Gamma(s - m) = \frac{(-1)^m \Gamma(s)}{(1-s)_m}.
\]

There are various addition formulas; we will made use of the formula
\[
(a + b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k}
\]
which holds for \( n \in \mathbb{N} \).

Finally, for \( a, b, c \in \mathbb{C} \), \( c \neq -n \) \((n \in \mathbb{N})\), and \( w \in \mathbb{C} \), we denote Gauss’s hypergeometric function by \( F(a, b; c; w) \); it is defined by the series
\[
F(a, b; c; w) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} w^k,
\]
which converges absolutely for \( w \in \mathbb{C} \) with \(|w| < 1\).

3. Definition and basic properties

In this section, we define the elliptic Eisenstein series and we prove some of its basic properties.

3.1. Definition. For \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), and \( s \in \mathbb{C} \), the elliptic Eisenstein series associated to the elliptic fixed point \( e_j \in E_\Gamma \) \((j = 1, \ldots, r_\Gamma)\) is defined by
\[
\mathcal{E}^{\text{ell}}_{e_j}(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma} \sinh(\varrho(\sigma_{e_j}^{-1}\gamma z))^{-s};
\]
here, are \( \sigma_{e_j} \) and \( \Gamma_{e_j} \) are defined in subsection 2.3.

3.2. Lemma. The following assertions hold

(i) For fixed \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) converges absolutely and locally uniformly for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), and hence defines a holomorphic function.

(ii) The elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) is invariant under the action of \( \Gamma \), i.e. we have
\[
\mathcal{E}^{\text{ell}}_{e_j}(\gamma z, s) = \mathcal{E}^{\text{ell}}_{e_j}(z, s)
\]
for any \( \gamma \in \Gamma \), and fulfills \( n_{e_j} \mathcal{E}^{\text{ell}}_{e_j}(z, s) = n_{e_j} \mathcal{E}^{\text{ell}}_{e_j}(e_j, s) \).

(iii) For fixed \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) converges absolutely and uniformly for \( z \) ranging over compacta \( K \subseteq \mathbb{H} \) not containing any translate \( \gamma e_j \) of \( e_j \) by \( \gamma \in \Gamma \).

Proof. (i) We first assume that \( e_1 = i \) is an elliptic fixed point of \( \Gamma \), and we will prove the assertion for the elliptic Eisenstein series
\[
\mathcal{E}^{\text{ell}}_{e_1}(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma} \sinh(\varrho(\gamma z))^{-s} = \frac{1}{\text{ord}(i)} \sum_{\gamma \in \Gamma} \sinh(\varrho(\gamma z))^{-s}.
\]

To do this, we write \( s = \sigma + it \in \mathbb{C} \) and we assume that \( \sigma = \text{Re}(s) > 1 \). We fix \( z \in \mathbb{H} \) such that \( z \neq \gamma i \) for any \( \gamma \in \Gamma \). Since \( \Gamma \) acts properly discontinuously on \( \mathbb{H} \) and \( z \neq \gamma i \) for any \( \gamma \in \Gamma \), the minimum
\[
R_1(z) := \min_{\gamma \in \Gamma} d_{\text{hyp}}(i, \gamma z) = \min_{\gamma \in \Gamma} \varrho(\gamma z)
\]
exists and is strictly positive. Hence, introducing the quantity
\[
C_1(z) := \frac{1 - \exp(-2R_1(z))}{2} > 0,
\]
we derive the estimate
\[
\sinh(\varrho(\gamma z)) = \exp(\varrho(\gamma z)) \left(1 - \frac{\exp(-2\varrho(\gamma z))}{2}\right) \geq C_1(z) \exp(\varrho(\gamma z))
\]
for all \( \gamma \in \Gamma \). Therefore, we obtain the estimate

\[
(3.1) \quad \sum_{\gamma \in \Gamma \setminus \Gamma} \left| \sinh (\varphi (\gamma z))^{-s} \right| = \frac{1}{\text{ord}(i)} \sum_{\gamma \in \Gamma} \sinh (\varphi (\gamma z))^{-\sigma} \leq C_1(z)^{-\sigma} \sum_{\gamma \in \Gamma} \exp(-\sigma \varphi (\gamma z)).
\]

Since the Poincaré exponent of \( \Gamma \) equals 1, the series on the right-hand side of (3.1) converges locally uniformly for \( \sigma > 1 \). This completes the proof in the case that \( e_i = i \).

Now, let \( e_j \) be an arbitrary elliptic fixed point of \( \Gamma \). Then, \( \sigma_{e_j}^{-1} \Gamma \sigma_{e_j} \) has the elliptic fixed point \( i \) and, for \( \text{Re}(s) > 1 \) and \( w = \sigma_{e_j}^{-1} z \), we have the equality

\[
E^\text{ell}_{\Gamma \sigma_{e_j}} (z, s) = E^\text{ell}_{\Gamma \sigma_{e_j}} (\sigma_{e_j} w, s) = E^\text{ell}_{\Gamma \sigma_{e_j}^{-1} \Gamma \sigma_{e_j} \cdot \delta} (w, s)
\]

Furthermore, \( w \) is equivalent to \( i \) with respect to \( \sigma_{e_j}^{-1} \Gamma \sigma_{e_j} \) if and only if \( z = \sigma_{e_j} w \) is equivalent to \( e_j \) with respect to \( \Gamma \). The absolute and locally uniform convergence of \( E^\text{ell}_{\Gamma \sigma_{e_j}^{-1} \Gamma \sigma_{e_j} \cdot \delta} (w, s) \) \((w \not\equiv i \mod \Gamma)\) for \( \text{Re}(s) > 1 \) for fixed \( w \in \mathbb{H} \) therefore implies the absolute uniform convergence of \( E^\text{ell}_{\Gamma \sigma_{e_j}} (z, s) \) \((z \not\equiv e_j \mod \Gamma)\) for \( \text{Re}(s) > 1 \) for fixed \( z = \sigma_{e_j} w \in \mathbb{H} \). This completes the proof of assertion (i).

(ii) From Definition 3.1 we immediately deduce for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \)

\[
E^\text{ell}_{e_j} (\gamma z, s) = E^\text{ell}_{e_j} (z, s)
\]

for all \( \gamma \in \Gamma \) and \( n_{e_j} \).

(iii) Let finally \( K \subseteq \mathbb{H} \) be a compact subset not containing any translate \( \gamma e_j \) of \( e_j \) by \( \gamma \in \Gamma \). Then, the expression \( C_1(z) \cdot \exp(\varphi (\gamma z)) \) given in the first part of the proof can be bounded uniformly for all \( z \in K \). For fixed \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the series \( E^\text{ell}_{e_j} (z, s) \) therefore converges absolutely and uniformly on \( K \subseteq \mathbb{H} \).

3.3. Lemma. For \( z \in \mathbb{H} \) with \( z \not\equiv \gamma e_j \) for any \( \gamma \in \Gamma \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the elliptic Eisenstein series \( E^\text{ell}_{e_j} (z, s) \) can be written as

\[
E^\text{ell}_{e_j} (z, s) = \sin \left( \frac{\pi}{n_{e_j}} \right) s \sum_{\gamma \in K_{e_j}} u(z, \gamma z)^{-s/2}
\]

with the \( \Gamma \)-conjugacy class \( K_{e_j} = \{ \sigma^{-1} \gamma e_j \sigma \mid \sigma \in \Gamma \} \) of the generator \( \gamma e_j \) of \( \Gamma_{e_j} \); here, \( u(z, w) \) is defined by (2.2).

Proof. We start by considering the hyperbolic triangle with vertices \( e_j, z, \) and \( \gamma e_j z, \) and corresponding angles \( 2\pi/n_{e_j}, \pi/2, \) and \( \pi/2, \) respectively. Bisecting the angle at \( e_j \) yields a right-angled triangle with angles \( \pi/n_{e_j}, \pi/2, \) and \( \pi/2. \) By a well-known formula for right-angled triangles (see [Bea95], p. 147), we obtain the identity

\[
(3.2) \quad \sinh (d_{\text{hyp}} (e_j, z)) = \sinh \left( \frac{\pi}{n_{e_j}} \right) \sinh \left( \frac{d_{\text{hyp}} (z, \gamma e_j z)}{2} \right).
\]

Using the formula \( \sinh^2 (r/2) = (\cosh (r) - 1)/2 \) and formula (2.1), this leads to the identity

\[
\sinh^2 (d_{\text{hyp}} (e_j, z)) = \sin^2 \left( \frac{\pi}{n_{e_j}} \right) \frac{|z - \gamma e_j z|^2}{4 \text{Im}(z) \text{Im}(\gamma e_j z)} = \sin^2 \left( \frac{\pi}{n_{e_j}} \right) u(z, \gamma e_j z).
\]

Therefore, the elliptic Eisenstein series can be written as

\[
(3.3) \quad E^\text{ell}_{e_j} (z, s) = \sin \left( \frac{\pi}{n_{e_j}} \right) s \sum_{\gamma \in \Gamma \setminus \Gamma} u(z, \gamma^{-1} \gamma e_j z)^{-s/2},
\]

where \( z \in \mathbb{H} \) with \( z \not\equiv \gamma e_j \) for any \( \gamma \in \Gamma \); note that for \( \gamma := \gamma^{-1} \gamma e_j \gamma (\gamma \in \Gamma_{e_j} \setminus \Gamma) \), we have

\[
u(z, \gamma z) = 0 \iff z = \gamma z \iff z = \gamma^{-1} e_j.
\]
in which case the elliptic Eisenstein series $E_{e_j}^\text{ell}(z, s)$ is not defined. Since the map $\phi : \Gamma_{e_j} \setminus \Gamma \to K_{e_j}$, given by $\Gamma_{e_j} \gamma \mapsto \gamma^{-1} \gamma_{e_j} \gamma$, is bijective, (3.3) can finally be rewritten as

$$E_{e_j}^\text{ell}(z, s) = \sin \left( \frac{\pi}{n_{e_j}} \right)^s \sum_{\gamma \in K_{e_j}} u(z, \gamma z) z^{-s/2}.$$ 

This completes the proof of the lemma. \hfill \Box

3.4. Remark. In [Hüb56], the following series associated to a pair of hyperbolic fixed points $h_j$ of $\Gamma$ is studied

$$G(z, s) = \sum_{\gamma \in K_{h_j}} u(z, \gamma z)^{-s}$$

with the $\Gamma$-conjugacy class $K_{h_j} = \{ \sigma^{-1} \gamma_{h_j} \sigma \mid \sigma \in \Gamma \}$ of the generator $\gamma_{h_j}$ of the stabilizer group $\Gamma_{h_j}$ of $h_j$ in $\Gamma$. Hence, the elliptic Eisenstein series can be considered as an elliptic analogue of the series studied in [Hüb56].

3.5. Lemma. For $z \in \mathbb{H}$ with $z \neq e_j$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 0$, the elliptic Eisenstein series $E_{e_j}^\text{ell}(z, s)$ satisfies the differential equation ($j = 1, \ldots, e_{\Gamma}$)

$$\left( \Delta_{\text{hyp}} - s(1 - s) \right) E_{e_j}^\text{ell}(z, s) = -s^2 E_{e_j}^\text{ell}(z, s + 2).$$

Proof. Since the differential operator

$$\Delta_{\text{hyp}} = -\frac{\partial^2}{\partial \ell^2} - \frac{1}{\tanh^2(\ell)} \frac{\partial}{\partial \ell} - \frac{1}{\sinh^2(\ell)} \frac{\partial^2}{\partial \ell^2}$$

is invariant under the action of $\Gamma$, it suffices by Lemma 3.2 to prove the equality

$$\left( \Delta_{\text{hyp}} - s(1 - s) \right) \sinh(\ell)^{-s} = -s^2 \sinh(\ell)^{-(s+2)}.$$ 

This follows immediately by a straight-forward calculation. \hfill \Box

The elliptic Eisenstein series $E_{e_j}^\text{ell}(z, s)$ is not a square-integrable function on $M$. However, there exists an infinite relation to the following square-integrable function.

3.6. Lemma. For $z \in \mathbb{H}$ with $z \neq e_j$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have the relation

$$E_{e_j}^\text{ell}(z, s) = \sum_{k=0}^{\infty} \frac{(\pi i)^k}{k!} P_{e_j}^\text{ell}(z, s + 2k)$$

with $P_{e_j}^\text{ell}(z, s)$ defined by (2.7).

Proof. We first check the absolute and local uniform convergence of the series in the claimed relation for fixed $z \in \mathbb{H}$ with $z \neq e_j$ for any $\gamma \in \Gamma$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Since $\Gamma$ acts properly discontinuously on $\mathbb{H}$ and $z \neq e_j$ for any $\gamma \in \Gamma$, the minimal distance $d_{\text{hyp}}(e_j, \gamma z)$ exists and is strictly positive. Using the estimate

$$\cosh(\rho(\sigma_{e_j}^{-1} \gamma z)) > C,$$

where $C = C(z) > 1$ is a positive constant depending on $z$, but which is independent of $\gamma \in \Gamma$, together with the estimate $|\Gamma'(s')| \leq |\Gamma(\text{Re}(s'))| = (\Gamma(\text{Re}(s')))^2$ for $s' \in \mathbb{C}$ with $\text{Re}(s') > 0$, we obtain the bound

$$\sum_{k=0}^{\infty} \left| \frac{(\pi i)^k}{k!} P_{e_j}^\text{ell}(z, s + 2k) \right| \leq \sum_{k=0}^{\infty} \frac{(\text{Re}(s))^{2k}}{k!} \sum_{\gamma \in \Gamma \setminus \Gamma} \cosh(\rho(\sigma_{e_j}^{-1} \gamma z))^{\text{Re}(s) - 2k}$$

$$\leq \sum_{k=0}^{\infty} \frac{(\text{Re}(s))^{2k}}{k!} C^{-2k} \sum_{\gamma \in \Gamma \setminus \Gamma} \cosh(\rho(\sigma_{e_j}^{-1} \gamma z))^{\text{Re}(s)}$$

$$= (1 - C^{-2})^{-\text{Re}(s)/2} P_{e_j}^\text{ell}(z, \text{Re}(s)).$$
This proves that the series in question converges absolutely and locally uniformly for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \).

Now, the claimed relation can easily be derived by changing the order of summation, namely we compute

\[
\sum_{k=0}^{\infty} \frac{(\frac{2}{s})^k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k) = \sum_{\gamma \in \Gamma \setminus \Gamma} \cosh(\varrho(\sigma_{e_j}^{-1} \gamma z))^{-s} \sum_{k=0}^{\infty} \frac{(\frac{2}{s})^k}{k!} \cosh(\varrho(\sigma_{e_j}^{-1} \gamma z))^{-2k} = \sum_{\gamma \in \Gamma \setminus \Gamma} \cosh(\varrho(\sigma_{e_j}^{-1} \gamma z))^{-s} \left( 1 - \cosh(\varrho(\sigma_{e_j}^{-1} \gamma z))^{-2} \right)^{-s/2} = \mathcal{E}_{e_j}^{\text{ell}}(z, s).
\]

This completes the proof of the lemma. \( \square \)

4. Meromorphic continuation

In this section, we establish the meromorphic continuation of the elliptic Eisenstein series to the whole \( s \)-plane, and we determine its possible poles and residues. The proof relies on the relation of the elliptic Eisenstein series to the elliptic Poincaré series which is given in Lemma 3.6. We first state the spectral expansion of the elliptic Poincaré series which is well-known to the experts.

4.1. Proposition. For \( z \in \mathbb{H} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) associated to the elliptic fixed point \( e_j \in E_{\Gamma} \) admits the spectral expansion

\[
(4.1) \quad P_{e_j}^{\text{ell}}(z, s) = \sum_{r=0}^{\infty} a_{r,e_j}(s) \psi_r(z) + \frac{1}{4\pi} \sum_{k=1}^{p_r} \int_{-\infty}^{\infty} a_{1/2+it,p_k,e_j}(s) \mathcal{E}_{p_k}^{\text{par}}(z, \frac{1}{2} + it) \, dt;
\]

here, the coefficients \( a_{r,e_j}(s) \) and \( a_{1/2+it,p_k,e_j}(s) \) are given by the formulas

\[
a_{r,e_j}(s) = \frac{2^{s-1} \sqrt{\pi}}{n_{e_j} \Gamma(s)} \Gamma\left( \frac{s - s_r}{2} \right) \Gamma\left( \frac{s - 1 + s_r}{2} \right) \psi_r(e_j),
\]

\[
a_{1/2+it,p_k,e_j}(s) = \frac{2^{s-1} \sqrt{\pi}}{n_{e_j} \Gamma(s)} \Gamma\left( \frac{s - \frac{1}{2} - it}{2} \right) \Gamma\left( \frac{s - \frac{1}{2} + it}{2} \right) \mathcal{E}_{p_k}^{\text{par}}(e_j, \frac{1}{2} - it),
\]

respectively. For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the expansion (4.1) converges absolutely and uniformly for \( z \) ranging over compacta \( K \subset \mathbb{H} \).

Proof. The proof can be obtained by an explicit computation of the spectral coefficients (see, e.g., [vP10]). Alternatively, one observes that the Helgason transform \( \cosh(\varrho(z))^{-s}(s_r) \) of the function \( \cosh(\varrho(z))^{-s} \) evaluated at \( s_r \) is given by

\[
\cosh(\varrho(z))^{-s}(s_r) = \frac{2^{s-1} \sqrt{\pi}}{\Gamma(s)} \Gamma\left( \frac{s - s_r}{2} \right) \Gamma\left( \frac{s - 1 + s_r}{2} \right),
\]

hence, the spectral expansion (4.1) can be obtained as a special case of a so-called non-euclidean Poisson summation formula (see, e.g., [Hel84], [Sel56], or [Ter85]). \( \square \)

The explicit knowledge of the spectral coefficients of the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) enables us to prove its meromorphic continuation to the whole \( s \)-plane, to determine its possible poles and to compute its residues.

4.2. Proposition. For \( z \in \mathbb{H} \), the elliptic Poincaré series \( P_{e_j}^{\text{ell}}(z, s) \) admits a meromorphic continuation to the whole \( s \)-plane. The possible poles of the function \( \Gamma(s)\Gamma(s - 1/2)^{-1} P_{e_j}^{\text{ell}}(z, s) \) are located at the points:
Proof. In order to derive the meromorphic continuation of $P_{e_j}^\ell(z,s)$, we use the spectral expansion (4.1). We start by giving the meromorphic continuation for the series in (4.1) arising from the discrete spectrum. The explicit formula

$$a_{r,e_j}(s) = \frac{2^{s-1}}{n_{e_j}} \frac{\Gamma(s - \frac{1}{2}) \Gamma \left( \frac{s - 1 + s_r}{2} \right)}{\Gamma(s) \Gamma \left( \frac{s - s_r}{2} \right)} \psi_r(e_j)$$

in terms of $\Gamma$-functions proves the meromorphic continuation for the coefficients $a_{r,e_j}(s)$ to the whole $s$-plane. Writing

$$\Gamma \left( \frac{s - s_r}{2} \right) \Gamma \left( \frac{s - 1 + s_r}{2} \right) = \Gamma \left( \frac{s - 1}{2} - i \ell_r \right) \Gamma \left( \frac{s - 1}{2} + i \ell_r \right),$$

and applying Stirling’s asymptotic formula, we get

$$\Gamma \left( \frac{s - s_r}{2} \right) \Gamma \left( \frac{s - 1 + s_r}{2} \right) = O \left( e^{\Re(s) - 3/2} e^{-\pi \ell_r/2} \right),$$

as $\ell_r \to \infty$, with an implied constant depending on $s$. Using this bound together with the well-known sup-norm bound

$$\sup_{z \in \Gamma_{\mathrm{H}}} |\psi_r(z)| = O \left( e^{-\sqrt{t_r}} \right),$$

(a) $s = 1/2 \pm it_r - 2n$, where $n \in \mathbb{N}$ and $\lambda_r = s_r(1 - s_r) = 1/4 + t_r^2$ is the eigenvalue of the eigenfunction $\psi_r$, which is a simple pole with residue

$$\text{Res}_{s = 1/2 \pm it_r - 2n} \left[ \frac{\Gamma(s)}{\Gamma(s - \frac{1}{2})} P_{e_j}^\ell(z,s) \right] = \frac{(-1)^n 2^{1/2 \pm it_r - 2n} \sqrt{\pi} \Gamma(\pm it_r - n)}{n! n_{e_j} \Gamma(\pm it_r - 2n)} \sum_{\ell : n_{e_j} = n_r} \psi_\ell(e_j) \psi_\ell(z);$$

in case $t_r = 0$, the factor in front of the sum reduces to $2^{1/2 - 2n} \sqrt{\pi} (2n)!/(n!)^2 n_{e_j}$.

(b) $s = 1 - \rho - 2n$, where $n \in \mathbb{N}$ and $w = \rho$ is a pole of the Eisenstein series $E_{pk}^{\text{par}}(z,w)$ with $\Re(\rho) \in (1/2,1]$, which is a simple pole with residue

$$\text{Res}_{s = 1 - \rho - 2n} \left[ \frac{\Gamma(s)}{\Gamma(s - \frac{1}{2})} P_{e_j}^\ell(z,s) \right] = \frac{(-1)^n 2^{1 - \rho - 2n} \sqrt{\pi} \Gamma(\frac{1}{2} - \rho - n)}{n! n_{e_j} \Gamma(\frac{1}{2} - \rho - 2n)} \times$$

$$\times \sum_{k \geq 1} \left[ \text{Res}_{w = \rho} E_{pk}^{\text{par}}(e_j,w) \cdot \text{CT}_{w = \rho} E_{pk}^{\text{par}}(z,1-w) + \text{CT}_{w = \rho} E_{pk}^{\text{par}}(e_j,w) \cdot \text{Res}_{w = \rho} E_{pk}^{\text{par}}(z,1-w) \right].$$

(c) $s = \rho - 2n$, where $n \in \mathbb{N}$ and $w = \rho$ is a pole of the Eisenstein series $E_{pk}^{\text{par}}(z,w)$ with $\Re(\rho) < 1/2$. If $\rho$ is a simple pole, the residue is given by

$$\text{Res}_{s = \rho - 2n} \left[ \frac{\Gamma(s)}{\Gamma(s - \frac{1}{2})} P_{e_j}^\ell(z,s) \right] = \frac{2^{\rho - 2n} \sqrt{\pi} \Gamma(\frac{1}{2} - \rho - n)}{n! n_{e_j} \Gamma(\frac{1}{2} - \rho - 2n)} \times$$

$$\times \sum_{k \geq 1} \left[ \text{CT}_{w = \rho - 2(n-\ell)} E_{pk}^{\text{par}}(e_j,1-w) \cdot \text{Res}_{w = \rho - 2(n-\ell)} E_{pk}^{\text{par}}(z,w) + \right.$$

$$\left. + \text{Res}_{w = \rho - 2(n-\ell)} E_{pk}^{\text{par}}(e_j,1-w) \cdot \text{CT}_{w = \rho - 2(n-\ell)} E_{pk}^{\text{par}}(z,w) \right];$$

here $m \in \mathbb{N}$ is such that $-3/2 - 2m < \Re(s) \leq 1/2 - 2m$. In case $\Re(s) = 1/2 - 2m$, the summand for $\ell = m$ has to be multiplied by $1/2$.

The poles given in cases (a), (b), (c) might coincide in parts; if this is the case, the corresponding residues have to be added accordingly.
we find for all, but the finitely many $r$ with $t_r \in [-i/2, 0]$ corresponding to eigenvalues $s_r(1 - s_r) = \lambda_r < 1/4$, the bound

$$a_{r, \epsilon_j}(s) \psi_r(z) = O \left( t_r^{\text{Re}(s) - 1/2} e^{-\pi t_r/2} \right),$$

as $t_r \to \infty$, with an implied constant depending on $s$. This proves that the series in (4.1) arising from the discrete spectrum converges absolutely and locally uniformly for all $s \in \mathbb{C}$, and hence defines a holomorphic function away from the poles of $a_{r, \epsilon_j}(s)$.

The location of the poles of the series in (4.1) arising from the discrete spectrum multiplied by the factor $\Gamma(s)\Gamma(s - 1/2)^{-1}$ and the calculation of the residues arising from this part is straightforward referring to the fact that $\Gamma(s'/2)$ is a meromorphic function for all $s' \in \mathbb{C}$, which has a simple pole at $s' = -2n$ ($n \in \mathbb{N}$) with residue equal to $2(-1)^n/n!$.

We now turn to give the meromorphic continuation of the integral

$$
\frac{2^{1 - s} \pi^{-1/2} \Gamma(s) n_{\epsilon_j}}{4\pi} \int_{-\infty}^{\infty} a_{1/2 + i t, p_k, \epsilon_j}(s) E_{p_k}^\text{par} \left( z, \frac{1}{2} + it \right) dt =
$$

(4.2)

$$
\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma \left( s - \frac{1}{2} - i t \right)}{\Gamma \left( s - \frac{1}{2} + i t \right)} \frac{\Gamma \left( s - \frac{1}{2} + i t \right)}{\Gamma \left( s - \frac{1}{2} - i t \right)} E_{p_k}^\text{par} \left( e_j, 1 - t \right) E_{p_k}^\text{par} \left( z, t \right) dt.
$$

arising from the continuous part of the spectral expansion (4.1) for $k = 1, \ldots, p_r$ after multiplication by $2^{1 - s} \pi^{-1/2} \Gamma(s)n_{\epsilon_j}$. Substituting $t \mapsto 1/2 + it$, the integral (4.2) can be rewritten as

(4.3)

$$I_{1/2, p_k}(s) := \frac{1}{4\pi i} \int_{\text{Re}(t)=1/2} \frac{\Gamma \left( s - t \right)}{\Gamma \left( s - 1/2 \right)} E_{p_k}^\text{par} \left( e_j, 1 - t \right) E_{p_k}^\text{par} \left( z, t \right) dt.$$

The function $I_{1/2, p_k}(s)$ is holomorphic for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, in fact, it is holomorphic for $s \in \mathbb{C}$ satisfying $\text{Re}(s) \neq 1/2 - 2n$ ($n \in \mathbb{N}$).

Now, let $\varepsilon > 0$ be sufficiently small such that $E_{p_k}^\text{par} (z, s)$ has no poles in the strip $1/2 - \varepsilon < \text{Re}(s) < 1/2 + \varepsilon$. For $s \in \mathbb{C}$ with $1/2 < \text{Re}(s) < 1/2 + \varepsilon$, we then have by the residue theorem

(4.4)

$$I_{1/2, p_k}(s) = I_{1/2 + \varepsilon, p_k}(s) - \frac{1}{2} \text{Res}_{t=0} \left[ \Gamma \left( \frac{s - t}{2} \right) \Gamma \left( \frac{s - 1 + t}{2} \right) E_{p_k}^\text{par} \left( e_j, 1 - t \right) E_{p_k}^\text{par} \left( z, t \right) \right].$$

The right-hand side of (4.4) is a meromorphic function for $1/2 - \varepsilon < \text{Re}(s) < 1/2 + \varepsilon$ giving the meromorphic continuation $I_{1/2, p_k}^{(1)}(s)$ of the integral $I_{1/2, p_k}(s)$ to the strip $1/2 - \varepsilon < \text{Re}(s) < 1/2 + \varepsilon$.

Now, assuming $1/2 - \varepsilon < \text{Re}(s) < 1/2$, and using the residue theorem once again, we obtain

(4.5)

$$I_{1/2, p_k}^{(1)}(s) = I_{1/2, p_k}(s) + \frac{1}{2} \text{Res}_{t=1-\varepsilon} \left[ \Gamma \left( \frac{s - t}{2} \right) \Gamma \left( \frac{s - 1 + t}{2} \right) E_{p_k}^\text{par} \left( e_j, 1 - t \right) E_{p_k}^\text{par} \left( z, t \right) \right] +$$

$$+ \Gamma \left( s - \frac{1}{2} \right) E_{p_k}^\text{par} \left( e_j, 1 - s \right) E_{p_k}^\text{par} \left( z, s \right)$$

$$= I_{1/2, p_k}(s) + \Gamma \left( s - \frac{1}{2} \right) E_{p_k}^\text{par} \left( e_j, 1 - s \right) E_{p_k}^\text{par} \left( z, 1 - s \right) + \Gamma \left( s - \frac{1}{2} \right) E_{p_k}^\text{par} \left( e_j, 1 - s \right) E_{p_k}^\text{par} \left( z, s \right).$$

The right-hand side of (4.5) is a meromorphic function for $-3/2 < \text{Re}(s) < 1/2$ giving the meromorphic continuation $I_{1/2, p_k}^{(2)}(s)$ of the integral $I_{1/2, p_k}^{(1)}(s)$ to the strip $-3/2 < \text{Re}(s) < 1/2$. Summing up, we find that the formulas (4.4) and (4.5) provide the meromorphic continuation of the integral $I_{1/2, p_k}(s)$ to the strip $-3/2 < \text{Re}(s) \leq 1/2$.

Continuing this two-step process, the meromorphic continuation of the integral $I_{1/2, p_k}(s)$ to the strip
\(-3/2 - 2m < \text{Re}(s) \leq 1/2 - 2m\) \((m \in \mathbb{N})\) is given by means of the formula

\[
I_{1/2, p_k}(s) + \sum_{\ell=0}^{m} \frac{(-1)^{\ell}}{\ell!} \Gamma\left(s - \frac{1}{2} + \ell\right) \epsilon_{p_k}^{\text{par}}(e_j, 1 - s - 2\ell) \epsilon_{p_k}^{\text{par}}(z, s + 2\ell)
\]

\addtocounter{equation}{1}

(4.6)

where, on the line \(\text{Re}(s) = 1/2 - 2m\), the integral \(I_{1/2, p_k}(s)\) has to be replaced by \(I_{1/2+\rho, p_k}(s)\) and the summand for \(\ell = m\) in the second sum has to be omitted. In this way, we obtain the meromorphic continuation of the integral \(I_{1/2, p_k}(s)\) to the whole \(s\)-plane.

Adding up and using the identity

\[
\sum_{k=1}^{p \rho} \epsilon_{p_k}^{\text{par}}(e_j, 1 - s') \epsilon_{p_k}^{\text{par}}(z, s') = \sum_{k=1}^{p \rho} \epsilon_{p_k}^{\text{par}}(e_j, s') \epsilon_{p_k}^{\text{par}}(z, 1 - s'),
\]

which can be derived from the functional equations (2.4) and (2.5), the meromorphic continuation of the continuous part of the spectral expansion (1.1) after multiplication by \(\Gamma(s)\), i.e. of

\[
\frac{\Gamma(s)}{4\pi} \sum_{k=1}^{p \rho} \int_{-\infty}^{\infty} a_{1/2+it, p_k, e_j}(s) \epsilon_{p_k}^{\text{par}}(z, 1/2 + it) dt,
\]

to the strip \(-3/2 - 2m < \text{Re}(s) \leq 1/2 - 2m\) \((m \in \mathbb{N})\) is given by means of the formula

\[
\frac{2^{s-1}}{\sqrt{n}} \frac{\Gamma(s)}{n_{e_j}} \sum_{k=1}^{p \rho} I_{1/2, p_k}(s) \frac{2^{s}}{\sqrt{n}} \sum_{\ell=0}^{m} \frac{(-1)^{\ell}}{\ell!} \Gamma\left(s - \frac{1}{2} + \ell\right) \sum_{k=1}^{p \rho} \epsilon_{p_k}^{\text{par}}(e_j, 1 - s - 2\ell) \epsilon_{p_k}^{\text{par}}(z, s + 2\ell),
\]

(4.8)

where, on the line \(\text{Re}(s) = 1/2 - 2m\), the integral \(I_{1/2, p_k}(s)\) has to be replaced by \(I_{1/2+\rho, p_k}(s)\) and the summand for \(\ell = m\) in the sum over \(\ell\) has to be multiplied by the factor 1/2. In this way, we obtain the meromorphic continuation of the continuous part of the spectral expansion (1.1) multiplied by \(\Gamma(s)\) to the whole \(s\)-plane.

In order to determine the poles arising from the continuous spectrum, we work from formula (1.8), valid in the strip \(-3/2 - 2m < \text{Re}(s) \leq 1/2 - 2m\) \((m \in \mathbb{N})\). After dividing by \(\Gamma(s - 1/2)\), the only poles can arise from the functions \(\epsilon_{p_k}^{\text{par}}(e_j, 1 - s - 2\ell)\) and \(\epsilon_{p_k}^{\text{par}}(z, s + 2\ell)\) \((\ell = 0, \ldots, m)\). The poles arising from \(\epsilon_{p_k}^{\text{par}}(e_j, 1 - s - 2\ell)\) are located at \(s_{\ell} = 1 - \rho - 2\ell\) \((\ell = 0, \ldots, m)\), where \(\rho\) is a pole of \(\epsilon_{p_k}^{\text{par}}(e_j, s)\) with \(1/2 + 2(m - \ell) \leq \text{Re}(\rho) < 1/2 + 2(m - \ell)\); therefore, by the results of subsection 2.2 there is only a simple pole for \(\ell = m\), i.e. at \(s_{m} = 1 - \rho - 2m\), where \(\rho\) is a pole of \(\epsilon_{p_k}^{\text{par}}(e_j, s)\) with \(\rho \in (1/2, 1]\). The poles arising from \(\epsilon_{p_k}^{\text{par}}(z, s + 2\ell)\) are located at \(s_{\ell}' = \rho - 2\ell\) \((\ell = 0, \ldots, m)\), where \(\rho\) is a pole of \(\epsilon_{p_k}^{\text{par}}(z, s)\) with \(-3/2 - 2(m - \ell) < \text{Re}(\rho) \leq 1/2 - 2(m - \ell)\). The residues can now be derived from formula (1.8). This completes the proof of the proposition. \(\square\)

4.3. Theorem. For \(z \in \mathbb{H}\) with \(z \neq \gamma e_j\) for any \(\gamma \in \Gamma\), the elliptic Eisenstein series \(\epsilon_{e_j}^{\text{ell}}(z, s)\) associated to the elliptic fixed point \(e_j \in E\) admits a meromorphic continuation the whole \(s\)-plane. The possible poles of the function \(\Gamma(s - 1/2)^{-1} \epsilon_{e_j}^{\text{ell}}(z, s)\) are located at the points:

(a) \(s = 1/2 \pm it_r - 2n\), where \(n \in \mathbb{N}\) and \(\lambda_r = s_r(1 - s_r) = 1/4 + t_r^2\) is the eigenvalue of the eigenfunction \(\psi_r\), which is a simple pole with residue

\[
\text{Res}_{s=1/2 \pm it_r - 2n} \left[\Gamma(s - \frac{1}{2})^{-1} \epsilon_{e_j}^{\text{ell}}(z, s)\right] = \frac{(-1)^n 2^{1/2 \pm it_r} \sqrt{\pi} \Gamma(\pm it_r)(\frac{3}{4} \pm it_r)^2}{n! n_{e_j} \Gamma(\frac{3}{4} \pm it_r) \Gamma(\pm it_r - 2n)(1 \pm it_r) n} \sum_{\ell, s_{\ell}\;=s_r} \psi_{\ell}(e_j) \psi_{\ell}(z);
\]

in case \(t_r = 0\), the factor in front of the sum reduces to \((-1)^n 2^{3/2}(2n)! (\frac{3}{4})^2/(n!)^2 n_{e_j}\).
(b) \( s = 1 - \rho - 2n, \) where \( n \in \mathbb{N} \) and \( w = \rho \) is a pole of the Eisenstein series \( E_{p_k}^{\text{par}}(z, w) \) with \( \text{Re}(\rho) \in (1/2, 1] \), which is a simple pole with residue

\[
\text{Res}_{s=1-\rho-2n} \left[ \Gamma(s - \frac{1}{2})^{-1} E_{e_j}^{\text{ell}}(z, s) \right] = \frac{(-1)^n 2^{1-\rho} \sqrt{\pi} \Gamma(\frac{1}{2} - \rho)(\frac{1}{2} + \rho)^2}{n! n_{e_j} \Gamma(1 - \rho) \Gamma(\frac{1}{2} - \rho - 2n)(\frac{1}{2} + \rho)_n} \times
\]

\[
\times \sum_{k=1}^{\text{pr}} \left[ \text{Res}_{w=\rho} E_{p_k}^{\text{par}}(e_j, w) \cdot \text{CT}_{w=\rho} E_{p_k}^{\text{par}}(z, 1-w) + \text{CT}_{w=\rho} E_{p_k}^{\text{par}}(e_j, w) \cdot \text{Res}_{w=\rho} E_{p_k}^{\text{par}}(z, 1-w) \right].
\]

(c) \( s = \rho - 2n, \) where \( n \in \mathbb{N} \) and \( w = \rho \) is a pole of the Eisenstein series \( E_{p_k}^{\text{par}}(z, w) \) with \( \text{Re}(\rho) < 1/2. \) If \( \rho \) is a simple pole, the residue is given by

\[
\text{Res}_{s=\rho-2n} \left[ \Gamma(s - \frac{1}{2})^{-1} E_{e_j}^{\text{ell}}(z, s) \right] = \frac{2\pi}{n_{e_j} \Gamma(\frac{e_j}{2} - n)} \sum_{k=0}^{m+2k'} \frac{(-1)^{\ell-k'} (\rho - \frac{1}{2} - 2n)_{k'+\ell}}{k'! (\ell - k')! \Gamma(\frac{e_j}{2} + \frac{1}{2} - n + k')} \times
\]

\[
\times \sum_{k=1}^{\text{pr}} \left[ \text{CT}_{w=\rho-2(n-\ell)} E_{p_k}^{\text{par}}(e_j, 1-w) \cdot \text{Res}_{w=\rho-2(n-\ell)} E_{p_k}^{\text{par}}(z, w) + \right.
\]

\[
\left. + \text{Res}_{w=\rho-2(n-\ell)} E_{p_k}^{\text{par}}(e_j, 1-w) \cdot \text{CT}_{w=\rho-2(n-\ell)} E_{p_k}^{\text{par}}(z, w) \right];
\]

here \( m \in \mathbb{N} \) is such that \(-3/2 - 2m < \text{Re}(s) \leq 1/2 - 2m, \) \( n' = m \) for \(-1 - 2m < \text{Re}(s) \leq 1/2 - 2m, \) and \( n' = m + 1 \) for \(-3/2 - 2m < \text{Re}(s) \leq -1 - 2m. \) In case \( \text{Re}(s) = 1/2 - 2m, \) the summand for \( \ell = m + 2k' \) \( (k' = 0, \ldots, n') \) has to be multiplied by \( 1/2. \)

The poles given in cases (a), (b), (c) might coincide in parts; if this is the case, the corresponding residues have to be added accordingly.

**Proof.** We start by proving that the function \( E_{e_j}^{\text{ell}}(z, s) \) has a meromorphic continuation to the half-plane

\[ \mathcal{H}_n := \{ s \in \mathbb{C} \mid \text{Re}(s) > -1 - 2n \} \]

for any \( n \in \mathbb{N}. \) By Lemma 3.6, which states the relation

\[ E_{e_j}^{\text{ell}}(z, s) = \sum_{k=0}^{\infty} \frac{(\frac{e_j}{2})_k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k), \]

we can write

\[
E_{e_j}^{\text{ell}}(z, s) = \sum_{k=0}^{n} \frac{(\frac{e_j}{2})_k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k) + \sum_{k=n+1}^{\infty} \frac{(\frac{e_j}{2})_k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k).
\]

Now, we prove that the series

\[
\sum_{k=n+1}^{\infty} \frac{(\frac{e_j}{2})_k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k)
\]

is a holomorphic function on the half-plane \( \mathcal{H}_n. \) For this we prove the absolute and locally uniform convergence of this series for fixed \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 - 2n. \) Since \( \Gamma \) acts properly discontinuously on \( \mathbb{H} \) and \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma, \) the minimal distance \( \min_{\gamma \in \Gamma} d_{\text{hyp}}(e_j, \gamma z) \) exists and is strictly positive. Using the estimate

\[ \cosh \left( g(\sigma_{e_j}^{-1} \gamma z) \right) > C, \]
where $C = C(z) > 1$ is a positive constant depending on $z$, but which is independent of $\gamma \in \Gamma$, together with the estimate $|\Gamma(s')| \leq |\Gamma(\text{Re}(s'))| = \Gamma(\text{Re}(s'))$ for $s' \in \mathbb{C}$ with $\text{Re}(s') > 0$, we obtain the bound

$$\sum_{k=n+1}^{\infty} \left| \frac{(\frac{\gamma}{2})_k}{k!} P_{e_j}^\text{ell}(z, s + 2k) \right| = \sum_{k=0}^{\infty} \left| \frac{(\frac{\gamma}{2})_{k+n+1}}{(k+n+1)!} P_{e_j}^\text{ell}(z, s + 2(k+n+1)) \right| \leq P_{e_j}^\text{ell}(z, \text{Re}(s) + 2(n+1)) \sum_{k=0}^{\infty} \left| \frac{(\text{Re}(s))^2}{2} \right| \frac{1}{(k+n+1)!} C^{-2k}.$$  

From this, we derive that the series (4.10) converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\text{Re}(s) > -1 - 2n$, which proves that is a holomorphic function on the half-plane $\mathcal{H}_n$. Since the finite sum

$$\sum_{k=0}^{n} \frac{(\frac{\gamma}{2})_k}{k!} P_{e_j}^\text{ell}(z, s + 2k)$$

in (1.9) is a meromorphic function on the whole $s$-plane by Proposition 4.2, we conclude that $\mathcal{E}_{e_j}^\text{ell}(z, s)$ has a meromorphic continuation to the half-plane $\mathcal{H}_n$. Since $n$ was chosen arbitrarily, this proves the meromorphic continuation of $\mathcal{E}_{e_j}^\text{ell}(z, s)$ to the whole $s$-plane.

In order to determine the poles of $\mathcal{E}_{e_j}^\text{ell}(z, s)$, we calculate its poles in the strip

$$S_n := \{s \in \mathbb{C} \mid -1 - 2n < \text{Re}(s) \leq -1 - 2n\}$$

for any $n \in \mathbb{N}$. By considering $\mathcal{E}_{e_j}^\text{ell}(z, s)$ with its decomposition (4.9) in the strip $S_n$, we see that the possible poles arise from the finite sum

$$F_n(z, s) := \sum_{k=0}^{n} \frac{(\frac{\gamma}{2})_k}{k!} P_{e_j}^\text{ell}(z, s + 2k).$$

Therefore, by Proposition 4.2, the possible poles of the function $F_n(z, s)$ in the strip $S_n$ are located at the points $s = 1/2 \pm it - 2n$, where $\lambda_r = 1/4 + t^2_r$ is the eigenvalue of the eigenfunction $\psi_r$, at the points $s = 1/2 - 2n$, where $w = \rho$ is a pole of the Eisenstein series $\mathcal{E}_{\mathcal{P}, b}^\text{par}(z, w)$ with $\text{Re}(\rho) \in (1/2, 1]$, and at the points $s = 1/2 - 2n'$, where $w = \rho$ is a pole of the Eisenstein series $\mathcal{E}_{\mathcal{P}, b}^\text{par}(z, w)$ with $\text{Re}(\rho) < 1/2$ and where $n' \in \mathbb{N}$ satisfying $-1 - 2n < \text{Re}(\rho) - 2n' \leq -1 - 2n$.

We now turn to compute the residues of the function $\Gamma(s - 1/2)^{-1} \mathcal{E}_{e_j}^\text{ell}(z, s)$, at the possible poles in the strip $S_n$ for any $n \in \mathbb{N}$. To do this, we write

$$\frac{F_n(z, s)}{\Gamma(s - \frac{1}{2})} = \frac{2\sqrt{\pi}}{\Gamma(\frac{\gamma}{2})} \sum_{k=0}^{n} \frac{2^{-(s+2k)} \Gamma(s - \frac{1}{2} + 2k)}{\Gamma(\frac{\gamma}{2} + \frac{1}{2} + k)} \Gamma(s + 2k) \frac{1}{\Gamma(\frac{\gamma}{2} + \frac{1}{2} + k)} P_{e_j}^\text{ell}(z, s + 2k),$$

where we used the equality

$$\Gamma(\frac{\gamma}{2} + k) = \frac{2^{1-(s+2k)} \sqrt{\pi} \Gamma(s + 2k)}{\Gamma(\frac{\gamma}{2} + \frac{1}{2} + k)},$$

which follows by means of the duplication formula (2.8). The explicit formula for the residues of the function $\Gamma(s)\Gamma(s - 1/2)^{-1} \mathcal{E}_{e_j}^\text{ell}(z, s)$ given in Proposition 4.2 (a) now leads to the following residue of the function $\Gamma(s - 1/2)^{-1} F_n(z, s)$ at $s = s_r - 2n = 1/2 + it - 2n$ ($t_r \neq 0$)

$$\text{Res}_{s = s_r - 2n} \left[ \frac{F_n(z, s)}{\Gamma(s - \frac{1}{2})} \right] = \frac{(-1)^n 2\pi}{n_{e_j}} \Gamma(\frac{\gamma}{2} - n) \Gamma(s_r - \frac{1}{2} - 2n) \sum_{k=0}^{n} \frac{(-1)^k \Gamma(s_r - \frac{1}{2} - n + k)}{k!(n-k)! \Gamma(\frac{\gamma}{2} + \frac{1}{2} + n - k)} \sum_{t \equiv s \equiv s_r} \psi_t(e_j) \psi(t).$$

Now, applying formula

$$(a + b)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (-b)_k (a + k)_n$$

with $a := s_r/2 + 1/2 - n$ and $b := -s_r + 1/2 + n$, adding up, and using the equality

$$\Gamma(s_r - \frac{1}{2} - n) = \frac{2^{s_r - 1/2} \Gamma(s_r - \frac{1}{2})(1 - \frac{\gamma}{2})_n}{\sqrt{\pi} \Gamma(s_r)(\frac{\gamma}{2} - s_r)_n},$$

we get
which can be derived by using twice formula (2.9) and then duplication formula (2.8), we obtain
\[ \text{Res}_{s=s_r-2n} \left[ \frac{F_n(z, s)}{\Gamma(s - \frac{1}{2})} \right] = \frac{(-1)^n 2^{2r} \sqrt{\pi} \Gamma(s_r - \frac{1}{2}) (1 - \frac{s_r}{2})_{2n}}{n! n_{e_j} \Gamma(s_r) \Gamma(s_r - \frac{1}{2} - 2n)(\frac{3}{2} - s_r)_{n}} \sum_{\ell : \ell s = s_r} \psi_{\ell}(e_{j}) \psi_{\ell}(z). \]

The residue of the function \( \Gamma(s - 1/2)^{-1} F_n(z, s) \) at \( s = 1/2 - it_r - 2n \) \((t_r \neq 0)\) is computed similarly replacing \( s_r = 1/2 + it_r \) by \( 1 - s_r = 1/2 - it_r \). Summing up, the residue of the function \( \Gamma(s - 1/2)^{-1} F_n(z, s) \) at \( s = 1/2 \pm it_r - 2n \) \((t_r \neq 0)\) is given by
\[ \text{Res}_{s=1/2 \pm it_r-2n} \left[ \frac{F_n(z, s)}{\Gamma(s - \frac{1}{2})} \right] = \frac{(-1)^n 2^{1/2 \pm it_r} \sqrt{\pi} \Gamma(\pm it_r) (\frac{1}{2} \pm it_r)_n^2}{n! n_{e_j} \Gamma(\frac{3}{2} \pm it_r) \Gamma(\pm it_r - 2n)(1 \mp it_r)_n} \sum_{\ell : \ell s = s_r} \psi_{\ell}(e_{j}) \psi_{\ell}(z). \]

The residues in the remaining cases are computed similarly. This completes the proof of the theorem.

An immediate consequence of Theorem 4.3 is the following corollary.

4.4. Corollary. For \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) associated to the elliptic fixed point \( e_j \in E_\Gamma \) admits a simple pole at \( s = 1 \) with residue
\[ \text{Res}_{s=1} \mathcal{E}^{\text{ell}}_{e_j}(z, s) = \frac{2\pi}{n_{e_j} \text{vol}_{\text{hyp}}(\mathcal{F}_{\Gamma})}. \]

5. A Kronecker limit type formula

In this section, we study the behaviour of the elliptic Eisenstein series at \( s = 0 \) for an arbitrary Fuchsian subgroup \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) of the first kind.

5.1. Proposition. For \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) associated to the elliptic fixed point \( e_j \in E_\Gamma \) admits a Laurent expansion at \( s = 0 \) of the form
\[ \mathcal{E}^{\text{ell}}_{e_j}(z, s) = 2^{s} \sqrt{\pi} \Gamma(s - \frac{1}{2}) \sum_{k=1}^{\text{pr}} \mathcal{E}^{\text{par}}_{p_k}(e_j, 1 - s) \mathcal{E}^{\text{par}}_{p_k}(z, s) = \frac{2\pi}{n_{e_j} \text{vol}_{\text{hyp}}(\mathcal{F}_{\Gamma})} + \mathcal{K}_{e_j}(z) \cdot s + O(s^2), \]
where \( \mathcal{K}_{e_j}(z) \) is a real-valued function, which fulfills \( n_{e_j} \mathcal{K}_{e_j}(z) = n_{e_j} \mathcal{K}_{e_j}(e_j) \) and which is invariant with respect to \( \Gamma \). Moreover, for any \( \gamma \in \Gamma \), we have the estimate
\[ \mathcal{K}_{e_j}(z) = - \log |z - \gamma e_j| + O(1) \]
as \( z \to \gamma e_j \).

Proof. For \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 \), by the proof of Theorem 4.3, the elliptic Eisenstein series \( \mathcal{E}^{\text{ell}}_{e_j}(z, s) \) is given by decomposition (4.9) with \( n = 0 \), i.e., we have
\[ \mathcal{E}^{\text{ell}}_{e_j}(z, s) = F_{e_j}(z, s) + \sum_{k=1}^{\infty} \frac{(\frac{3}{2})^k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k). \]

Hence, introducing the notation
\[ R_{e_j}(z, s) := 2^{s} \sqrt{\pi} \Gamma(s - \frac{1}{2}) \sum_{k=1}^{\text{pr}} \mathcal{E}^{\text{par}}_{p_k}(e_j, 1 - s) \mathcal{E}^{\text{par}}_{p_k}(z, s), \]
we have for \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 \) the identity
\[ \mathcal{E}^{\text{ell}}_{e_j}(z, s) - R_{e_j}(z, s) = P_{e_j}^{\text{ell}}(z, s) - R_{e_j}(z, s) + \sum_{k=1}^{\infty} \frac{(\frac{3}{2})^k}{k!} P_{e_j}^{\text{ell}}(z, s + 2k). \]
In the first step, we determine the Laurent expansion of the function $P^\text{rell}_{\epsilon_j}(z, s) - R_{\epsilon_j}(z, s)$ at $s = 0$. By the proof of Proposition 4.2, we derive for $s \in \mathbb{C}$ with $-3/2 < \text{Re}(s) < 1/2$ the identity

$$(5.4) \quad P^\text{rell}_{\epsilon_j}(z, s) - R_{\epsilon_j}(z, s) = \sum_{r=0}^{\infty} a_{r, \epsilon_j}(s) \psi_r(z) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} a_{1/2+it, \epsilon_j}(s) \mathcal{E}^\text{par}_{pk}(z, \frac{1}{2} + it) dt,$$

where the coefficients $a_{r, \epsilon_j}(s)$ resp. $a_{1/2+it, \epsilon_j}(s)$ are explicitly given in Proposition 4.2. For $r = 0$, i.e., $t_r = -i/2$ and $s_r = 1$, the function

$$a_{0, \epsilon_j}(s) \psi_0(z) = \frac{2^{s-1}\sqrt{\pi}}{n_{\epsilon_j} \text{vol}_\text{hyp}(\mathcal{F}_{\Gamma})} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right)$$

in the series arising from the discrete spectrum in (5.4) admits a Laurent expansion at $s = 0$ of the form

$$a_{0, \epsilon_j}(s) \psi_0(z) = -\frac{2\pi}{n_{\epsilon_j} \text{vol}_\text{hyp}(\mathcal{F}_{\Gamma})} \cdot s + O(s^2).$$

For $r > 0$, the function $a_{r, \epsilon_j}(s) \psi_r(z)$ admits a Laurent expansion at $s = 0$ of the form

$$a_{r, \epsilon_j}(s) \psi_r(z) = \frac{\sqrt{\pi}}{2n_{\epsilon_j}} \Gamma\left(-\frac{s_r}{2}\right) \Gamma\left(\frac{s_r - 1}{2}\right) \psi_r(\epsilon_j) \psi_r(z) \cdot s + O(s^2).$$

Furthermore, for $s_r = 1/2 + it_r$, with $t_r \in (-i/2, 0]$, the term $\Gamma(-s_r/2)\Gamma((s_r - 1)/2)$ is real-valued, and, for $s_r = 1/2 + it_r$ with $t_r > 0$, we have

$$\Gamma\left(-\frac{s_r}{2}\right) \Gamma\left(\frac{s_r - 1}{2}\right) = \left|\Gamma\left(-\frac{1}{2} + it_r\right)\right|^2,$$

which again is real-valued. Next, the function $a_{1/2+it, \epsilon_j}(s)$ for $k = 1, \ldots, pr$ in the integral arising from the continuous spectrum in (5.4) admits a Laurent expansion at $s = 0$ of the form

$$a_{1/2+it, \epsilon_j}(s) = \frac{\sqrt{\pi}}{2n_{\epsilon_j}} \left|\Gamma\left(-\frac{1}{2} + it\right)\right|^2 \mathcal{E}^\text{par}_{pk}\left(\epsilon_j, \frac{1}{2} - it\right) \cdot s + O(s^2).$$

Furthermore, using identity (1.7) together with $\mathcal{E}^\text{par}_{pk}(\cdot, 1/2 + it) = \mathcal{E}^\text{par}_{pk}(\cdot, 1/2 - it)$ for $t \in \mathbb{R}$, we deduce that

$$S_{\epsilon_j}(z) := \sum_{k=1}^{pr} \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{2n_{\epsilon_j}} \left|\Gamma\left(-\frac{1}{2} + it\right)\right|^2 \mathcal{E}^\text{par}_{pk}\left(\epsilon_j, \frac{1}{2} - it\right) \mathcal{E}^\text{par}_{pk}\left(z, \frac{1}{2} + it\right) dt$$

$$= \sum_{k=1}^{pr} \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{2n_{\epsilon_j}} \left|\Gamma\left(-\frac{1}{2} + it\right)\right|^2 \mathcal{E}^\text{par}_{pk}\left(\epsilon_j, \frac{1}{2} - it\right) \mathcal{E}^\text{par}_{pk}\left(z, \frac{1}{2} + it\right) dt = S_{\epsilon_j}(z),$$

hence, $S_{\epsilon_j}(z)$ is also real-valued. All in all, we obtain a Laurent expansion at $s = 0$ of the form

$$P^\text{rell}_{\epsilon_j}(z, s) - R_{\epsilon_j}(z, s) = -\frac{2\pi}{n_{\epsilon_j} \text{vol}_\text{hyp}(\mathcal{F}_{\Gamma})} + F_{\epsilon_j}(z) \cdot s + O(s^2),$$

where

$$F_{\epsilon_j}(z) := -\frac{2\pi}{n_{\epsilon_j} \text{vol}_\text{hyp}(\mathcal{F}_{\Gamma})} + \frac{\sqrt{\pi}}{2n_{\epsilon_j}} \sum_{r>0}^{\infty} \Gamma\left(-\frac{s_r}{2}\right) \Gamma\left(\frac{s_r - 1}{2}\right) \psi_r(\epsilon_j) \psi_r(z) +$$

$$+ \frac{1}{8\sqrt{\pi}n_{\epsilon_j}} \sum_{k=1}^{pr} \int_{-\infty}^{\infty} \left|\Gamma\left(-\frac{1}{2} + it\right)\right|^2 \mathcal{E}^\text{par}_{pk}\left(\epsilon_j, \frac{1}{2} - it\right) \mathcal{E}^\text{par}_{pk}\left(z, \frac{1}{2} + it\right) dt$$

is a real-valued function, which clearly fulfills $n_{\epsilon_j}F_{\epsilon_j}(z) = n_z F_z(\epsilon_j)$ and which is invariant with respect to $\Gamma$. 

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In the second step, we determine the Laurent expansion of the series

$$\sum_{k=1}^{\infty} \frac{1}{k!} P_{e_j}^\text{ell}(z, s + 2k)$$

in (5.3) at $s = 0$. By the proof of Theorem 4.3, this series is a holomorphic function for $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$, and $s \in \mathbb{C}$ with $\text{Re}(s) > -1$. Since the functions $P_{e_j}^\text{ell}(z, s + 2k)$ ($k \in \mathbb{N}, k \geq 1$) are non-vanishing and the Pochhammer symbol admits a Laurent expansion at $s = 0$ of the form

$$\left( \frac{s}{2} \right)_k = \frac{(k-1)!}{2} \cdot s + O(s^2),$$

we have the following Laurent expansion at $s = 0$

$$\sum_{k=1}^{\infty} \frac{1}{k!} P_{e_j}^\text{ell}(z, s + 2k) = G_{e_j}(z) \cdot s + O(s^2)$$

with

$$G_{e_j}(z) := \sum_{k=1}^{\infty} \frac{1}{2k} P_{e_j}^\text{ell}(z, 2k).$$

For $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$, the function $G_{e_j}(z)$ is a real-valued function, which fulfills $n_{e_j} G_{e_j}(z) = n_z G_z(e_j)$ and which is invariant with respect to $\Gamma$, since the function $P_{e_j}^\text{ell}(z, 2k)$ ($k \in \mathbb{N}, k \geq 1$) is real-valued, fulfills $n_{e_j} P_{e_j}^\text{ell}(z, 2k) = n_z P_{z}^\text{ell}(e_j, 2k)$, and is invariant with respect to $\Gamma$, by the very definition of the series.

Summing up, and letting

$$K_{e_j}(z) := F_{e_j}(z) + G_{e_j}(z)$$

for $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$, this proves the asserted Laurent expansion at $s = 0$.

We are left to prove the estimate (5.1). From (5.5) we immediately derive the bound $F_{e_j}(z) = O(1)$ as $z \to \gamma e_j$ for any $\gamma \in \Gamma$. Therefore, by (5.7), it remains to prove the estimate

$$G_{e_j}(z) = - \log |z - \gamma e_j| + O(1)$$

as $z \to \gamma e_j$ for any $\gamma \in \Gamma$. To do this we fix some $\gamma \in \Gamma/\Gamma_{e_j}$. We choose $\varepsilon = \varepsilon(e_j, \Gamma) \in \mathbb{R}_{>0}$ sufficiently small such that

$$\{ \gamma' \in \Gamma \mid \gamma' B_{2\varepsilon}(\gamma e_j) \cap B_{2\varepsilon}(\gamma e_j) \neq \emptyset \} = \Gamma_{e_j},$$

where $B_{2\varepsilon}(\gamma e_j) := \{ w \in \mathbb{H} \mid \rho(\sigma_{e_j}^{-1}\gamma^{-1}w) < 2\varepsilon \}$ denotes the open hyperbolic disc of radius $2\varepsilon$ centered at $\gamma e_j$. Without loss of generality we may assume that $z \in \mathbb{H}$, $z \neq \gamma e_j$, is sufficiently close to $\gamma e_j$, say

$$d_{\text{hyp}}(\gamma e_j, z) = \rho(\sigma_{e_j}^{-1}\gamma^{-1}z) < \varepsilon.$$

We note that for any $\gamma' \in \Gamma/\Gamma_{e_j}$ with $\gamma' \neq \gamma$, we have $d_{\text{hyp}}(\gamma' e_j, \gamma e_j) > 2\varepsilon$, which, by the triangle equality $d_{\text{hyp}}(\gamma' e_j, z) + d_{\text{hyp}}(\gamma e_j, z) \geq d_{\text{hyp}}(\gamma' e_j, \gamma e_j)$, gives the bound

$$d_{\text{hyp}}(\gamma' e_j, z) = \rho(\sigma_{e_j}^{-1}\gamma'^{-1}z) > 2\varepsilon - \varepsilon = \varepsilon > \rho(\sigma_{e_j}^{-1}\gamma^{-1}z).$$

Introducing the notation $C := \cosh(\rho(\sigma_{e_j}^{-1}\gamma^{-1}z))$, we write

$$G_{e_j}(z) = \sum_{k=0}^{\infty} \frac{1}{2k+2} \sum_{\gamma' \in \Gamma/\Gamma_{e_j}} \cosh(\rho(\sigma_{e_j}^{-1}\gamma'^{-1}z))^{-(2k+2)}$$

$$= -\frac{1}{2} \log (1 - C^{-2}) + \sum_{k=0}^{\infty} \frac{1}{2k+2} \sum_{\gamma' \neq \gamma \in \Gamma/\Gamma_{e_j}} \cosh(\rho(\sigma_{e_j}^{-1}\gamma'^{-1}z))^{-(2k+2)},$$

where for the last equality we used the identity

$$\sum_{k=0}^{\infty} \frac{C^{-2k+2}}{2k+2} = -\frac{1}{2} \log (1 - C^{-2}),$$

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keeping in mind that $C > 1$. Now, using formula (2.1), namely
\[
\cosh(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z)) = \cosh(d_{\text{hyp}}(\gamma e_j, z)) = 1 + \frac{|z - \gamma e_j|^2}{2 \text{Im}(z) \text{Im}(\gamma e_j)},
\]
we derive the identity
\[
-\frac{1}{2} \log \left( \sinh(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z))^2 \right) = -\frac{1}{2} \log \left( \frac{|z - \gamma e_j|^2}{\text{Im}(z) \text{Im}(\gamma e_j)} \right) - \frac{1}{2} \log \left( 1 + \frac{|z - \gamma e_j|^2}{4 \text{Im}(z) \text{Im}(\gamma e_j)} \right).
\]
This leads to the estimate
\[
-\frac{1}{2} \log(1 - C^{-2}) = -\frac{1}{2} \log \left( \sinh(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z))^2 \right) + \log \left( \cosh(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z)) \right)
\]
\[
= - \log |z - \gamma e_j| + O(1)
\]
as $z \to \gamma e_j$. Finally, using the bound (5.8) and the identity (5.9) with $C = \cosh(\varepsilon)$, we get
\[
\left| \sum_{k=0}^{\infty} \frac{1}{2k + 2} \sum_{\gamma' \in \Gamma/\Gamma_{e_j}} \text{cosh}(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z))^{-(2k+2)} \right| \leq \cosh(\varepsilon)^2 \log(\tanh(\varepsilon)^{-1}) \sum_{\gamma' \neq \gamma} \text{cosh}(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z))^{-2}
\]
\[
\leq \cosh(\varepsilon)^2 \log(\tanh(\varepsilon)^{-1}) P_{\text{ell}}(\gamma e_j, z, 2).
\]
Therefore, we get the bound
\[
\sum_{k=0}^{\infty} \frac{1}{2k + 2} \sum_{\gamma' \in \Gamma/\Gamma_{e_j}} \text{cosh}(\rho(\sigma_{e_j}^{-1} \gamma^{-1} z))^{-(2k+2)} = O(1)
\]
as $z \to \gamma e_j$. Hence, adding up, we have proven the asserted estimate
\[
G_{e_j}(z) = - \log |z - \gamma e_j| + O(1)
\]
as $z \to \gamma e_j$. This completes the proof of the proposition. \(\square\)

By standard arguments (see, e.g., [Fav77]), we can now establish the following Kronecker limit type formula for elliptic Eisenstein series.

5.2. Theorem. For $z \in \mathbb{H}$ with $z \neq \gamma e_j$ for any $\gamma \in \Gamma$, the elliptic Eisenstein series $E_{\text{ell}}(z, s)$ associated to $e_j \in E_\Gamma$ admits a Laurent expansion at $s = 0$ of the form
\[
E_{\text{ell}}(z, s) = \frac{2^s \sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)} \sum_{k=1}^{pr} c_{p_k}(e_j, 1 - s) E_{p_k}^{\text{par}}(z, s) = - C_{e_j} - \log(|H_{e_j}(z)| \text{Im}(z)^{C_{e_j}} \text{Im}(e_j)^{C_{e_j}}) \cdot s + O(s^2),
\]
where we have set (for $w \in \mathbb{H}$)
\[
C_w := \frac{2\pi}{n_w \text{vol}_{\text{hyp}}(F_\Gamma)};
\]
further, $H_{e_j}(z)$ is a holomorphic function, unique up to multiplication with a complex constant of absolute value 1, which vanishes if and only if $z = \gamma e_j$ for any $\gamma \in \Gamma$, which fulfills $|H_{e_j}(z)|^{n_{e_j}} = |H_w(e_j)|^{n_{w}}$ and which satisfies
\[
H_{e_j}(\gamma z) = \varepsilon_{e_j}(\gamma)(cz + d)^{2C_{e_j}} H_{e_j}(z)
\]
for any $\gamma = (a_{b \ c \ d}) \in \Gamma$. Here, $\varepsilon_{e_j}(\gamma) \in \mathbb{C}$ is a constant of absolute value 1 depending on $e_j$ and $\gamma$ but which is independent of $z$.

Proof. Using the notation (5.2), i.e.,
\[
E_{\text{ell}}(z, s) := \frac{2^s \sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)} \sum_{k=1}^{pr} c_{p_k}(e_j, 1 - s) E_{p_k}^{\text{par}}(z, s),
\]
we recall that Proposition 5.1 provides the following Laurent expansion at \( s = 0 \)

\[
(5.11) \quad \mathcal{E}_{e_j}^{\text{ell}}(z, s) - R_{e_j}(z, s) = \sum_{r=0}^{\infty} a_{r,e_j}(z) \cdot s^r
\]

with \( a_{r,e_j}(z) \in \mathcal{A}(\Gamma \setminus \mathbb{H}) \) for \( r \in \mathbb{N} \), and with \( a_{0,e_j}(z) = -2\pi/n_e \text{ vol}_{\text{hyp}}(\mathcal{F}_r) \) and \( a_{1,e_j}(z) = K_{e_j}(z) \).

Further, for \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the function \( \mathcal{E}_{e_j}^{\text{ell}}(z, s + 2) \) is holomorphic at \( s = 0 \) by lemma 3.2 and non-vanishing by the very definition of the series. Therefore, we have a Laurent expansions at \( s = 0 \) of the form

\[
(5.12) \quad \mathcal{E}_{e_j}^{\text{ell}}(z, s + 2) = \sum_{r=0}^{\infty} b_{r,e_j}(z) \cdot s^r
\]

with \( b_{r,e_j}(z) \in \mathcal{A}(\Gamma \setminus \mathbb{H}) \) for \( r \in \mathbb{N} \). Substituting the expansions (5.11) and (5.12) into the differential equation

\[
(\Delta_{\text{hyp}} - s(1 - s)) (\mathcal{E}_{e_j}^{\text{ell}}(z, s) - R_{e_j}(z, s)) = (\Delta_{\text{hyp}} - s(1 - s)) \mathcal{E}_{e_j}^{\text{ell}}(z, s) = -s^2 \mathcal{E}_{e_j}^{\text{ell}}(z, s + 2),
\]

we derive the identity

\[
\sum_{r=0}^{\infty} \Delta_{\text{hyp}} a_{r,e_j}(z) \cdot s^r = \sum_{r=1}^{\infty} a_{r-1,e_j}(z) \cdot s^r - \sum_{r=2}^{\infty} (a_{r-2,e_j}(z) + b_{r-2,e_j}(z)) \cdot s^r.
\]

Comparison of the coefficients now leads to the following recurrence formula

\[
\Delta_{\text{hyp}} a_{r,e_j}(z) = a_{r-1,e_j}(z) - a_{r-2,e_j}(z) - b_{r-2,e_j}(z),
\]

where \( a_{r,e_j}(z) = b_{r,e_j}(z) = 0 \) for \( r < 0 \). In particular, for \( r = 0 \), we recover

\[
\Delta_{\text{hyp}} a_{0,e_j}(z) = -\Delta_{\text{hyp}} C_{e_j} = 0.
\]

Further, for \( r = 1 \), we get

\[
(5.13) \quad \Delta_{\text{hyp}} a_{1,e_j}(z) = \Delta_{\text{hyp}} K_{e_j}(z) = a_{0,e_j}(z) = -C_{e_j}.
\]

Since \( \Delta_{\text{hyp}} \log(y) = 1 \), this implies the identity

\[
\Delta_{\text{hyp}} (K_{e_j}(z) + C_{e_j} \log(y) + C_{e_j} \log(\text{Im}(e_j))) = -C_{e_j} + C_{e_j} = 0.
\]

Hence, for \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the function

\[
h_{e_j}(z) := K_{e_j}(z) + C_{e_j} \log(\text{Im}(z)) + C_{e_j} \log(\text{Im}(e_j))
\]

is a non-constant, real-valued harmonic function, which fulfills \( n_e h_{e_j}(z) = n_z h_{e_j}(e_j) \). Now, for fixed \( w \in \mathbb{H} \) and for a constant \( A \in \mathbb{C} \) of absolute value 1, we consider the function

\[
H_{e_j}(z) := A \exp(-h_{e_j}(w)) \exp\left( \int_z^w 2 \frac{\partial}{\partial \bar{z}} h_{e_j}(\bar{z}) d\bar{z} \right).
\]

Since \( h_{e_j}(z) = K_{e_j}(z) + C_{e_j} \log(\text{Im}(z)) + C_{e_j} \log(\text{Im}(e_j)) \) is harmonic, the function \( H_{e_j}(z) \) is independent of the path from \( z \) to \( w \) and is analytic in \( z \). In particular, for any path from \( z \) to \( w \), we have the identity

\[
\text{Re}\left( \int_z^w 2 \frac{\partial}{\partial \bar{z}} h_{e_j}(\bar{z}) d\bar{z} \right) = h_{e_j}(w) - h_{e_j}(z).
\]

Therefore, we conclude that

\[
|H_{e_j}(z)| = \exp(-\text{Re}(h_{e_j}(w))) \exp(h_{e_j}(w) - h_{e_j}(z)) = \exp(-h_{e_j}(z)).
\]

Further, \( H_{e_j}(z) \) vanishes if and only if \( z = \gamma e_j \) for some \( \gamma \in \Gamma \), and by the estimate (5.1) we derive the following Laurent expansion at \( z = \gamma e_j \ (\gamma \in \Gamma) \)

\[
(5.14) \quad H_{e_j}(z) = \sum_{r=1}^{\infty} a_{r,\gamma e_j} \cdot (z - \gamma e_j)^r
\]
with \(a_{r,\gamma e_j} \in \mathbb{C}\) for \(r \geq 1\) and \(a_{1,\gamma e_j} \neq 0\). Adding up, we have

\[
h_{e_j}(z) = K_{e_j}(z) + C_{e_j} \log(\Im(z)) + C_{e_j} \log(\Im(e_j)) = -\log|H_{e_j}(z)| \iff K_{e_j}(z) = -\log|H_{e_j}(z)\Im(z)^{C_{e_j}}\Im(e_j)^{C_{e_j}}|.
\]

Therefore, the \(\Gamma\)-invariance of the function \(K_{e_j}(z)\) yields the \(\Gamma\)-invariance of \(\log|H_{e_j}(z)\Im(z)^{C_{e_j}}\Im(e_j)^{C_{e_j}}|\).

This leads to the identity

\[
|H_{e_j}(\gamma z)| \Im(\gamma z)^{C_{e_j}} \Im(e_j)^{C_{e_j}} = |H_{e_j}(z)| \Im(z)^{C_{e_j}} \Im(e_j)^{C_{e_j}} \iff |H_{e_j}(\gamma z)| = |H_{e_j}(z)| \cdot |cz+d|^{2C_{e_j}} \iff |H_{e_j}(\gamma z)| = |H_{e_j}(z)\cdot (cz+d)^{2C_{e_j}}|
\]

for any \(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\). From this we deduce that the function

\[
f(z) := \frac{H_{e_j}(\gamma z)}{H_{e_j}(z)(cz+d)^{2C_{e_j}}}
\]

has absolute value 1. Moreover, if \(z \in \mathbb{H}\) tends to \(\gamma' e_j\) for some \(\gamma' \in \Gamma\), the translate \(\gamma z \in \mathbb{H}\) tends to \(\gamma' e_j\). Hence, from (5.14) we get the following Laurent expansions at \(z = \gamma' e_j\) (\(\gamma' \in \Gamma\))

\[
H_{e_j}(z) = \sum_{r=1}^{\infty} a_{r,\gamma' e_j} \cdot (z - \gamma' e_j)^r,
\]

\[
H_{e_j}(\gamma z) = \sum_{r=1}^{\infty} a_{r,\gamma\gamma' e_j} \cdot (\gamma z - \gamma' e_j)^r = \sum_{r=1}^{\infty} \frac{a_{r,\gamma\gamma' e_j}}{(c\gamma' e_j + d)^r} \cdot \frac{(z - \gamma' e_j)^r}{(cz + d)^r}
\]

with \(a_{r,\gamma' e_j}, a_{r,\gamma\gamma' e_j} \in \mathbb{C}\) for \(r \geq 1\), \(a_{1,\gamma' e_j} \neq 0\), and \(a_{1,\gamma\gamma' e_j} \neq 0\); here, we used that for \(z, z' \in \mathbb{H}\) we have the identity

\[
\gamma z - \gamma' z' = \frac{z - z'}{(cz + d)(cz' + d)}.
\]

Since \((cz + d)^{2C_{e_j}}\) never vanishes for \(z \in \mathbb{H}\), this implies that the function \(f(z)\) is regular for \(z \in \mathbb{H}\).

Therefore, by the maximum principle, we obtain that \(f(z) = \varepsilon\) for a constant \(\varepsilon = \varepsilon_{e_j}(\gamma) \in \mathbb{C}\) of absolute value 1 depending on \(e_j\) and \(\gamma\), but which is independent of \(z\). Hence, we get

\[
H_{e_j}(\gamma z) = \varepsilon_{e_j}(\gamma)(cz+d)^{2C_{e_j}} H_{e_j}(z),
\]

as asserted. This completes the proof of the theorem. \(\square\)

6. The case of the full modular group

In this section, we consider the special case that \(\Gamma = \text{PSL}_2(\mathbb{Z})\). Hence, we have \(e_\Gamma = 2\) and \(p_\Gamma = 1\), and we can choose \(E_\Gamma = \{e_1 = i, e_2 = \rho = \exp(2\pi i/3)\}\) and \(P_\Gamma = \{p_1 = \infty\}\). The point \(i \in \mathbb{H}\) is an elliptic fixed point of order \(n_i = 2\) with scaling matrix \(\sigma_i = \text{id}\), and the point \(\rho \in \mathbb{H}\) is an elliptic fixed point of order \(n_\rho = 3\) with scaling matrix

\[
\sigma_\rho = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}/2 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{pmatrix}.
\]

Moreover, \(\infty\) is a cusp of width 1 and scaling matrix \(\sigma_\infty = \text{id}\). The hyperbolic volume is given by

\[
\text{vol}_{\text{hyp}}(\mathcal{F}_\Gamma) = \frac{\pi}{3}
\]

and, therefore, we have \(C_1 = 6/n_i = 3\) and \(C_\rho = 6/n_\rho = 2\).

For \(k = 4, 6\), let

\[
E_k(z) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \Gamma} (cz+d)^{-k} = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} (cz+d)^{-k}.
\]

(6.1)
denote the holomorphic Eisenstein series of weight $k$, which is a modular form satisfying

$$E_k(\gamma z) = (cz + d)^k E_k(z)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The function $E_k(z)$ ($k = 4, 6$) is normalized such that we have the Fourier expansions

$$E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)e(mz),$$

$$E_6(z) = 1 + 504 \sum_{m=1}^{\infty} \sigma_5(m)e(mz),$$

respectively, where $\sigma_k(m)$ ($k = 4, 6$) denotes the divisor function. By $\Delta(z)$, we denote the Dedekind’s Delta function

$$\Delta(z) = \frac{1}{1728} (E_4(z)^3 - E_6(z)^2),$$

which is a cusp form of weight 12 satisfying

$$\Delta(\gamma z) = (cz + d)^{12} \Delta(z)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Further, let $E_{\infty}(z, s)$ denote the parabolic Eisenstein series

$$E_{\infty}(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \text{Im}(\gamma z)^s = \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}} \text{Im}(cz+d)^s,$$

Its parabolic Fourier expansion (with respect to the cusp $\infty$) is given by

$$E_{\infty}(z, s) = y^s + \varphi(s)y^{1-s} + \sum_{m\in\mathbb{Z}} \varphi_m(s) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|m|y)e(mx),$$

where $K_{\nu}(\cdot)$ denotes the modified Bessel function of the second kind and where

$$\varphi(s) = \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s - 1)}{\Gamma(s)} = \frac{\Lambda(2s - 1)}{\Lambda(2s)},$$

$$\varphi_m(s) = \frac{2m^s}{\Gamma(s)} \zeta(2s + 1) \sum_{d|m} d^{-2s+1} = \frac{2}{\Lambda(2s)} \sum_{ab = |m|} \left( \frac{a}{b} \right)^{s-\frac{1}{2}};$$

here, we have set $\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$ with the Riemann zeta function $\zeta(s)$.

We first recall the classical Kronecker limit formula for the parabolic Eisenstein series $E_{\infty}(z, s)$ (see, e.g., [Sie80] or [Zag92]).

6.1. Proposition. For $z \in \mathbb{H}$, the parabolic Eisenstein series $E_{\infty}(z, s)$ admits a Laurent expansion at $s = 1$ of the form

$$E_{\infty}(z, s) = \frac{\text{vol}_{\text{hyp}}(\mathcal{F}_\Gamma)^{-1}}{s - 1} - \frac{1}{2\pi} \log(\text{Im}(z)^6) + C + O(s - 1)$$

with $C = (6 - 72 \zeta(-1) - 6 \log(4\pi))/\pi$. At $s = 0$, it admits a Laurent expansion of the form

$$E_{\infty}(z, s) = 1 + \log(\text{Im}(z)) \cdot s + O(s^2).$$

An analogous result holds for the elliptic Eisenstein series $E_{\infty}(z, s)$ ($j = 1, 2$). To prove it, we first recall the parabolic Fourier expansion of the elliptic Eisenstein series (see [KuPT2] for the special case $\Gamma = \text{PSL}(\mathbb{Z})$, or [VaP] for an arbitrary Fuchsian subgroup $\Gamma$ of the first kind). For $z \in \mathbb{H}$ with
Im(z) > Im(\gamma e_j) for any \gamma \in \Gamma, the elliptic Eisenstein series \( \mathcal{E}_{e_j}^{\text{ell}}(z, s) \) admits the parabolic Fourier expansion (with respect to the cusp \( \infty \))

\[
(6.6) \quad \mathcal{E}_{e_j}^{\text{ell}}(z, s) = \sum_{m \in \mathbb{Z}} a_{m; \infty, e_j}(y, s) e(mx)
\]

with coefficients given by

\[
a_{0; \infty, e_j}(y, s) = 2^s \sqrt{\pi} \Gamma(s - \frac{1}{2}) \sum_{k=0}^{\infty} \frac{(s - \frac{1}{2})k(\frac{s}{2})}{k! (\frac{s}{2} + \frac{1}{2})^k y^{1-s-2k}} \mathcal{E}_{e_j}^{\text{par}}(e_j, s + 2k),
\]

\[
a_{m; \infty, e_j}(y, s) = 2^s y^s \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(\frac{s}{2})k_1 (\frac{s}{2})k_2}{k_1! k_2!} I_m(y, s; k_1, k_2) V_{\infty, m}^{\text{par}}(e_j, s + 2k_1 + 2k_2) \quad (m \neq 0),
\]

with

\[
I_m(y, s; k_1, k_2) = \int_{-\infty}^{\infty} (y + it)^{-s-2k_1} (y - it)^{-s-2k_2} e(-mt) \, dt,
\]

\[
V_{\infty, m}^{\text{par}}(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \text{Im}(\gamma z)^s e\left(-m \text{Re}(\gamma z)\right).
\]

6.2. **Proposition.** For \( z \in \mathbb{H} \) with \( z \neq \gamma i \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}_{i}^{\text{ell}}(z, s) \) admits a Laurent expansion at \( s = 0 \) of the form

\[
\mathcal{E}_{i}^{\text{ell}}(z, s) - \frac{2^{s-1} \sqrt{\pi} \Gamma(s - \frac{1}{2})}{3 \Gamma(s)} \mathcal{E}_{\infty}^{\text{par}}(i, 1-s) \mathcal{E}_{\infty}^{\text{par}}(z, s) =
\]

\[
-3 + \left(-\log(|E_6(z)| \text{Im}(z)^3) + B_i\right) s + O(s^2)
\]

with \( B_i = -72 \zeta'(-1) + 3 \log(2\pi) - 12 \log(\Gamma(\frac{1}{3})) \).

Further, for \( z \in \mathbb{H} \) with \( z \neq \gamma \rho \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}_{\rho}^{\text{ell}}(z, s) \) admits a Laurent expansion at \( s = 0 \) of the form

\[
\mathcal{E}_{\rho}^{\text{ell}}(z, s) - \frac{2^{s-1} \sqrt{\pi} \Gamma(s - \frac{1}{2})}{3 \Gamma(s)} \mathcal{E}_{\infty}^{\text{par}}(\rho, 1-s) \mathcal{E}_{\infty}^{\text{par}}(z, s) =
\]

\[
-2 + \left(-\log(|E_4(z)| \text{Im}(z)^2 \text{Im}(\rho)^2) + B_\rho\right) s + O(s^2)
\]

with \( B_\rho = -48 \zeta'(-1) + 4 \log(\frac{2\pi}{\sqrt{3}}) - 12 \log(\Gamma(\frac{1}{3})) \).

**Proof.** By Theorem 5.2 for \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \) \((j = 1, 2)\), we have a Laurent expansion at \( s = 0 \) of the form

\[
\mathcal{E}_{e_j}^{\text{ell}}(z, s) - \frac{2^s \sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)} \mathcal{E}_{\infty}^{\text{par}}(e_j, 1-s) \mathcal{E}_{\infty}^{\text{par}}(z, s) = -C_{e_j} + \mathcal{K}_{e_j}(z) s + O(s^2)
\]

with

\[
(6.7) \quad \mathcal{K}_{e_j}(z) = -\log(|H_{e_j}(z)| \text{Im}(z)^{C_{e_j}} \text{Im}(e_j)^{C_{e_j}}),
\]

for a holomorphic function \( H_{e_j}(z) \) with properties given in Theorem 5.2.

To explicitly determine the function \( H_{e_j}(z) \) \((j = 1, 2)\), we first determine its behaviour as \( y \to \infty \). To do this, we let \( z \in \mathbb{H} \) be such that \( \text{Im}(z) > \text{Im}(\gamma e_j) \) for any \( \gamma \in \Gamma \), and we consider the parabolic Fourier expansion of \( \mathcal{K}_{e_j}(z) \), which is of the form

\[
\mathcal{K}_{e_j}(z) = \sum_{m \in \mathbb{Z}} b_{m; e_j}(y) e(mx)
\]

with coefficients given by

\[
b_{m; e_j}(y) = \int_{0}^{\frac{1}{y}} \mathcal{K}_{e_j}(z) e(-mx). \]
Since $K_{e_j}(z)$ is real-valued, we have $b_{-m:e_j}(y) = \overline{b_{m:e_j}(y)}$. Further, from the differential equation (see (5.13))

$$\Delta_{\text{hyp}} K_{e_j}(z) = -C_{e_j} = -\frac{2\pi}{n_{e_j} \text{vol}_{\text{hyp}}(F_T)},$$

we obtain for $m = 0$ the identity $\Delta_{\text{hyp}} b_{0:e_j}(y) = -C_{e_j}$, and, for $m \neq 0$, the differential equation $b''_{m:e_j}(y) = (2\pi m)^2 b_{m:e_j}(y)$. From this we derive that

$$b_{0:e_j}(y) = -C_{e_j} \log(y) + A_{e_j} y + B_{e_j},$$

$$b_{m:e_j}(y) = A_{m:e_j} \exp(-2\pi my) + A'_{m:e_j} \exp(2\pi my) \quad (m \neq 0)$$

with constants $A_{e_j}, B_{e_j} \in \mathbb{R}$, and with constants $A_{m:e_j}, A'_{m:e_j} \in \mathbb{C}$ satisfying $A'_{m:e_j} = \overline{A_{m:e_j}}$. However, from the parabolic Fourier expansion of the function

$$e_{\text{ell}}^j(z, s) = \frac{2s\sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)} e_{\infty}(e_j, 1 - s) e_{\infty}(z, s),$$

which is obtained by combining (6.6) with (6.5), we conclude that $A'_{m:e_j} = 0$ for $m > 0$ and $A_{m:e_j} = 0$ for $m < 0$. Hence, we can write

$$(6.8) \quad K_{e_j}(z) = -C_{e_j} \log(y) + A_{e_j} y + B_{e_j} + \sum_{m=1}^{\infty} A_{m:e_j} e(mz) + \sum_{m=1}^{\infty} \overline{A_{m:e_j}} e(-m\overline{z}).$$

To determine the constants $A_{e_j}, B_{e_j} \in \mathbb{R}$, we introduce the notation

$$h(s) := \frac{2s\sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)},$$

and we consider the constant term $\tilde{a}_{0:\infty, e_j}(y, s)$ of the parabolic Fourier expansion of the function

$$e_{\text{ell}}^j(z, s) - h(s) e_{\infty}(e_j, 1 - s) e_{\infty}(z, s) = e_{\text{ell}}^j(z, s) - h(s) e_{\infty}(e_j, s) e_{\infty}(z, 1 - s).$$

The constant term $\tilde{a}_{0:\infty, e_j}(y, s)$ is given by

$$\tilde{a}_{0:\infty, e_j}(y, s) = a_{0:\infty, e_j}(y, s) - h(s) e_{\infty}(e_j, s) \left( y^{1 - s} + \varphi(1 - s)y^s \right)$$

$$= F_{e_j}(y, s) - h(s) \varphi(1 - s)y^s e_{\infty}(e_j, s).$$

with

$$F_{e_j}(y, s) := h(s) \sum_{k=1}^{\infty} \frac{(s - \frac{1}{2})k(\frac{s}{2})k}{k!(\frac{s}{2} + \frac{1}{2}k)} y^{1-s-2k} e_{\infty}(e_j, s + 2k).$$

Since the function $e_{\infty}(e_j, s + 2k)$ ($k \in \mathbb{N}, k > 0$) is holomorphic and non-vanishing for $s \in \mathbb{C}$ with $\text{Re}(s) > -1$, and using the Laurent expansion at $s = 0$

$$h(s) \frac{(s - \frac{1}{2})k(\frac{s}{2})k}{k!(\frac{s}{2} + \frac{1}{2}k)} y^{1-s-2k} = -\frac{\pi y^{1-2k}}{n_{e_j} (k - 2k^2)} \cdot s^2 + O(s^3),$$

we derive that $F_{e_j}(y, s) = O(s^2)$ at $s = 0$. Further, we have at $s = 0$ the following Laurent expansions

$$-h(s) \varphi(1 - s)y^s = -C_{e_j} - C_{e_j} \left( 24 \zeta'(-1) + \log(8\pi^2) + \log(y) \right) \cdot s + O(s^2),$$

$$e_{\infty}(e_j, s) = 1 + \log(|\Delta(e_j)|^{1/6} \text{Im}(e_j)) \cdot s + O(s^2),$$

where the last expansion follows from proposition [6.3]. These expansions lead to the following Laurent expansion at $s = 0$

$$\tilde{a}_{0:\infty, e_j}(y, s) = -C_{e_j} - C_{e_j} \left( 24 \zeta'(-1) + \log(8\pi^2) + \log(y) + \log(|\Delta(e_j)|^{1/6} \text{Im}(e_j)) \right) \cdot s + O(s^2).$$

From this we derive that

$$b_{0:e_j}(y) = -C_{e_j} \log(y) - C_{e_j} \left( 24 \zeta'(-1) + \log(8\pi^2) + \log(|\Delta(e_j)|^{1/6} \text{Im}(e_j)) \right),$$
and, therefore, we get
\[ A_{e_j} = 0, \]
(6.9)
\[ B_{e_j} = -C_{e_j}(24\zeta'(-1) + \log(8\pi^2) + \log(|\Delta(e_j)|^{1/6}) - C_{e_j}\log(\Im(e_j)). \]
Introducing the notation
\[ f_{e_j}(z) = \exp \left( -2 \sum_{m=1}^{\infty} A_{m,e_j}e(mz) \right), \]
we derive from (6.8) the equality
\[ K_{e_j}(z) = -C_{e_j}\log(y) + B_{e_j} - \log(f_{e_j}(z)^{1/2}) - \log(\overline{f_{e_j}(z)^{1/2}}) \]
\[ = B_{e_j} - \log(|f_{e_j}(z)|\Im(z)^{C_{e_j}}). \]
From this and (6.7), we derive the identity
\[ \log(|H_{e_j}(z)|\Im(z)^{C_{e_j}} \Im(e_j)^{C_{e_j}}) \]
\[ = -B_{e_j} + \log(|f_{e_j}(z)|\Im(z)^{C_{e_j}}), \quad \text{i.e.} \quad |H_{e_j}(z)| = \Im(e_j)^{C_{e_j}} \exp(-B_{e_j})|f_{e_j}(z)|. \]
Since \(|f_{e_j}(z)| \to 1\) as \(y \to \infty\), we have \(|H_{e_j}(z)| \to \Im(e_j)^{-C_{e_j}} \exp(-B_{e_j})\) as \(y \to \infty\), which implies
(6.10)
\[ H_{e_j}(z) = A \Im(e_j)^{-C_{e_j}} \exp(-B_{e_j}) + O(\exp(-2\pi y)) \]
as \(y \to \infty\) with a constant \(A \in \mathbb{C}\) (depending on \(e_j\)) of absolute value 1. We now consider the function
\[ f_{e_j}(z) = \frac{H_{e_j}(z)}{E_{2C_{e_j}}(z)} \]
with the Eisenstein series \(E_{2C_{e_j}}(z)\) of weight \(2C_{e_j}\). The Eisenstein series \(E_{2C_{e_j}}(z)\) vanishes if and only if \(z = \gamma e_j\) for any \(\gamma \in \Gamma\), and it admits a simple zero at \(z = \gamma e_j\) for any \(\gamma \in \Gamma\). Similarly, the function \(H_{e_j}(z)\) vanishes if and only if \(z = \gamma e_j\) for any \(\gamma \in \Gamma\), and, by (5.14), it admits a simple zero at \(z = \gamma e_j\) for any \(\gamma \in \Gamma\). Therefore, the function \(f_{e_j}(z)\) is a regular function on \(\mathbb{H}\). Moreover, from the asymptotics
(6.11)
\[ E_{2C_{e_j}}(z) = 1 + O(\exp(-2\pi y)) \]
as \(y \to \infty\), which can be deduced from the Fourier expansion (6.2), (6.3), respectively, and from the bound (6.10), we deduce that \(f_{e_j}(z)\) is bounded as \(y \to \infty\). Hence, the function \(f_{e_j}(z)\) is a modular function with a finite character. Therefore, there exists \(m_{e_j} \in \mathbb{N}\), \(m_{e_j} \geq 1\), such that \(f_{e_j}(z)^{m_{e_j}}\) is a modular function with trivial character. Hence, we get \(f_{e_j}(z)^{m_{e_j}} = c_{e_j}\) for a constant \(c_{e_j} \in \mathbb{C}\) and, therefore, we get the equality
(6.12)
\[ H_{e_j}(z)^{m_{e_j}} = c_{e_j}E_{2C_{e_j}}(z)^{m_{e_j}}. \]
From the asymptotics (6.10), we derive
\[ H_{e_j}(z)^{m_{e_j}} = A^{m_{e_j}} \Im(e_j)^{-m_{e_j}C_{e_j}} \exp(-m_{e_j}B_{e_j}) + O(\exp(-2\pi y)) \]
as \(y \to \infty\), which, together with the bound (6.11), leads to the equality
(6.13)
\[ c_{e_j} = A^{m_{e_j}} \Im(e_j)^{-m_{e_j}C_{e_j}} \exp(-m_{e_j}B_{e_j}). \]
Substituting (6.13) into (6.12), we obtain the identity
\[ \log(|H_{e_j}(z)^{m_{e_j}}|) = \log(\Im(e_j)^{-m_{e_j}C_{e_j}} \exp(-m_{e_j}B_{e_j})|E_{2C_{e_j}}(z)^{m_{e_j}}|), \]
from which we deduce
\[ \log(|H_{e_j}(z)|) = -B_{e_j} + \log(\Im(e_j)^{-C_{e_j}}|E_{2C_{e_j}}(z)|). \]
Therefore, we get
\[ K_{e_j}(z) = -\log(|H_{e_j}(z)|\Im(z)^{C_{e_j}} \Im(e_j)^{C_{e_j}}) = -\log(|E_{2C_{e_j}}(z)|\Im(z)^{C_{e_j}}) + B_{e_j}. \]
Finally, from the well-known formulas (see, e.g., [DS05, p. 7])

\[ E_4(i) = \frac{3\Gamma(\frac{1}{4})^8}{(2\pi)^6} \text{ resp. } E_6(\rho) = \frac{2^33^3\Gamma(\frac{1}{4})^{18}}{(2\pi)^{12}}, \]

we derive

\[ |\Delta(i)|^{1/6} = \frac{E_4(i)^{1/2}}{1728^{1/6}} = \frac{\Gamma(\frac{1}{4})^4}{2(2\pi)^3}, \quad |\Delta(\rho)|^{1/6} = \frac{E_6(\rho)^{1/3}}{1728^{1/6}} = \frac{3^{1/2}\Gamma(\frac{1}{4})^6}{(2\pi)^4}, \]

respectively. Substituting these identities for \( j = 1, 2 \) into (6.9),

\[ B_{e_j} = -C_{e_j}(24\zeta'(-1) + \log(8\pi^2) + \log(|\Delta(e_j)|^{1/6})) - C_{e_j} \log(\text{Im}(e_j)) \]

we obtain the formulas

\[ B_i = -3\left(24\zeta'(-1) - \log(2\pi) + 4\log(\Gamma(\frac{1}{4}))\right), \]

\[ B_\rho = -2\left(24\zeta(-1) - 2\log(2\pi) + 2\log(\sqrt{3}) + 6\log(\Gamma(\frac{1}{4}))\right), \]

observing that \( \text{Im}(\rho) = \sqrt{3}/2 \). This completes the proof of the proposition. \( \square \)

Combining the classical Kronecker limit formula given in Proposition 6.1 with the Kronecker limit type formula for the elliptic Eisenstein series given in Proposition 6.2 we deduce the following Kronecker limit formula for the elliptic Eisenstein series for \( \text{PSL}_2(\mathbb{Z}) \).

6.3. Corollary. For \( z \in \mathbb{H} \) with \( z \neq \gamma i \) for any \( \gamma \in \Gamma \), there is a Laurent expansion at \( s = 0 \) of the form

\[ \mathcal{E}^\text{ell}_i(z, s) = -\log(|E_4(z)| |\Delta(z)|^{-1/2}) \cdot s + O(s^2). \]

Further, for \( z \in \mathbb{H} \) with \( z \neq \gamma \rho \) for any \( \gamma \in \Gamma \), we have a Laurent expansion at \( s = 0 \) of the form

\[ \mathcal{E}^\text{ell}_\rho(z, s) = -\log(|E_4(z)| |\Delta(z)|^{-1/3}) \cdot s + O(s^2). \]

Proof. Using the notation

\[ h(s) := \frac{2^s\sqrt{\pi} \Gamma(s - \frac{1}{2})}{n_{e_j} \Gamma(s)}, \]

Proposition 6.1 yields the following Laurent expansions at \( s = 0 \)

\[ h(s)\mathcal{E}^\text{par}_\infty(z, 1 - s) = C_{e_j} + C_{e_j}(C' + \log(|\Delta(z)|^{1/6} \text{Im}(z))) \cdot s + O(s^2), \]

\[ \mathcal{E}^\text{par}_\infty(e_j, s) = 1 + \log(|\Delta(e_j)|^{1/6} \text{Im}(e_j)) \cdot s + O(s^2) \]

with \( C' = 24\zeta'(-1) + 8\pi^2 \). Therefore, we obtain a Laurent expansion at \( s = 0 \) of the form

\[ h(s)\mathcal{E}^\text{par}_\infty(e_j, 1 - s)\mathcal{E}^\text{par}_\infty(z, s) = h(s)\mathcal{E}^\text{par}_\infty(e_j, s)\mathcal{E}^\text{par}_\infty(z, 1 - s) = C_{e_j} + C_{e_j}(C' + \log(|\Delta(z)|^{1/6} \text{Im}(z) \text{Im}(e_j))) \cdot s + O(s^2). \]

Substituting this Laurent expansion into the Laurent expansion given by Proposition 6.2 using formula (5.9), and adding up, we conclude that for \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), the elliptic Eisenstein series \( \mathcal{E}^\text{ell}_i(z, s) \) \( (j = 1, 2) \) admits a Laurent expansion at \( s = 0 \) of the form

\[ \mathcal{E}^\text{ell}_j(z, s) = \left(-\log(|E_{2\gamma e_j}(z)| \text{Im}(z)^{C_{e_j}}) + C_{e_j} \log(|\Delta(z)|^{1/6} \text{Im}(z))\right) \cdot s + O(s^2) \]

\[ = -\log(|E_{2\gamma e_j}(z)| |\Delta(z)|^{-C_{e_j}/6}) \cdot s + O(s^2). \]

This completes the proof of the corollary. \( \square \)

6.4. Corollary. For \( z \in \mathbb{H} \) with \( z \neq \gamma i \) for any \( \gamma \in \Gamma \), there is a Laurent expansion at \( s = 0 \) of the form

\[ \mathcal{E}^\text{ell}_i(z, s) = -\log(|j(i) - j(z)|^2) \cdot s + O(s^2). \]

Further, for \( z \in \mathbb{H} \) with \( z \neq \gamma \rho \) for any \( \gamma \in \Gamma \), we have a Laurent expansion at \( s = 0 \) of the form

\[ \mathcal{E}^\text{ell}_\rho(z, s) = -\log(|j(\rho) - j(z)|^2) \cdot s + O(s^2). \]
Proof. By means of the identities
\[ |j(i) - j(z)|^2 = |E_6(z)| |\Delta(z)|^{-1/2}, \quad |j(\rho) - j(z)|^3 = |E_4(z)| |\Delta(z)|^{-1/3}, \]
Corollary 6.4 is an immediate consequence of Corollary 6.3. \qed

6.5. Remark. The proof of Proposition 5.1 provides an expression of \(|j(e_j) - j(z)|\) in terms of spectral data, more precisely, in terms of the functions \(F_{e_j}(z)\) resp. \(G_{e_j}(z)\), given by (5.5), (5.6), respectively.

6.6. Remark. Let \(\Gamma\) be a Fuchsian subgroup satisfying \(p_\Gamma = 1\). Then, at \(s = 1\), we have
\[ \mathcal{E}_{p_1}(z, s) = \frac{\text{vol}_{\text{hyp}}(F_1)^{-1}}{s - 1} + \kappa_{p_1}(s) + O(s - 1), \]
with the Kronecker limit function \(\kappa_{p_1}(s)\) (see [JO05]) and the scattering constant \(\kappa_{p_1,p_1}\). Since \(p_\Gamma = 1\), the functional equations imply that, at \(s = 0\), we have
\[ \mathcal{E}_{p_1}(z, s) = 1 - \text{vol}_{\text{hyp}}(F_1)\mathcal{K}_{p_1}(z) - \kappa_{p_1,p_1} + O(s^2). \]
From this, it seems to be possible to determine explicitly determine the Laurent expansion of the function \(R_{e_j}(z, s)\) given by (5.2), and then to establish a generalization of Corollary 6.4. In case that \(p_\Gamma > 1\), the study of the function \(R_{e_j}(z, s)\) is more difficult, since the behaviour of \(\mathcal{E}_{p_1}(z, s)\) at \(s = 0\) is not known in general.

7. Relation to the Automorphic Green’s Function

For \(z, w \in M\) with \(z \neq w\) and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1\), the automorphic Green’s function on \(M\) is defined as
\[ G_s(z, w) = \sum_{\gamma \in \Gamma} g_s(z, \gamma w), \]
where \(g_s(z, w)\) is the Green’s function on \(\mathbb{H}\) given by
\[ g_s(z, w) = \frac{1}{4\pi \Gamma(2s)} u(z, w)^{-s} F\left(s, s; 2s; -\frac{1}{u(z, w)}\right) \]
with \(u(z, w)\) defined by (2.2) and with the hypergeometric function \(F(s, s; 2s; Z)\) recalled in subsection 2.5. In the literature, there are different normalizations of the functional equations imply that, at \(z, w \in M\) with \(z \neq w\), and that it is holomorphic for \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1\). Moreover, it is invariant with respect to \(\Gamma\). Furthermore, \(G_{e_j}^\text{ell}(z, s)\) admits a meromorphic continuation to the whole \(s\)-plane, assuming \(z, w \in M\) with \(z \neq w\).

To compare the automorphic Green’s function with the elliptic Eisenstein series, we first prove the following infinite relation.

7.1. Lemma. For \(z \in \mathbb{H}\) with \(z \neq e_j\) for any \(\gamma \in \Gamma\), and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1\), we have the relation
\[ G_{e_j}^\text{ell}(z, s) = \frac{2^s \Gamma(s)^2}{4\pi \Gamma(2s)} \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right) k \left(\frac{s}{2} + \frac{1}{2}\right) k}{k!(s + \frac{1}{2})^k} P_{e_j}(z, s + 2k) \]
with \(P_{e_j}(z, s)\) defined by (2.7).
Proof. The absolute and local uniform convergence of the series in the claimed relation for fixed \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), can be proven along the lines of the proof given in Lemma 3.6.

In the next step, for \( z \neq w \), we express the function

\[
g_s(z, w) = \frac{1}{4\pi} \frac{\Gamma(s)^2}{\Gamma(2s)} u(z, w)^{-s} F\left( s, s; 2s; -\frac{1}{u(z, w)} \right);
\]

in terms of \( \cosh(d_{\text{hyp}}(z, w)) \), by applying formula 9.134.1 of [GR07], namely

\[
F(\alpha, \beta; 2\beta; Z) = \left(1 - \frac{Z}{2}\right)^{-\alpha} F\left( \frac{\alpha}{2}, \frac{\alpha + 1}{2}; \beta + \frac{1}{2}; \frac{Z^2}{(2 - Z)^2} \right).
\]

Letting \( \alpha := s \), \( \beta := s \), and \( Z := -1/u(z, w) \), we obtain

\[
u(z, w)^{-s} F\left( s, s; 2s; -\frac{1}{u(z, w)} \right) = 2^s \cosh(d_{\text{hyp}}(z, w))^{-s} F\left( \frac{s + 1}{2}; s + \frac{1}{2} \right)
\]

\[
\quad = 2^s \sum_{k=0}^{\infty} \frac{(\frac{s}{2})_k (\frac{s + 1}{2})_k}{k!(s + \frac{1}{2})_k} \cosh(d_{\text{hyp}}(z, w))^{-s-2k}.
\]

Hence, for \( z \neq w \), we get the desired expression

\[
g_s(z, w) = \frac{2^s \Gamma(s)^2}{4\pi \Gamma(2s)} \sum_{k=0}^{\infty} \frac{(\frac{s}{2})_k (\frac{s + 1}{2})_k}{k!(s + \frac{1}{2})_k} \cosh(d_{\text{hyp}}(z, w))^{-s-2k}.
\]

Applying Definition (2.7) and Definition (7.1), the claimed relation can now be derived by changing the order of summation. This completes the proof of the lemma. \(\square\)

7.2. Remark. Using the relation given in Lemma 7.1 one can establish a new proof of the meromorphic continuation of the automorphic Green’s function, along the lines of the proof of Theorem 4.3.

7.3. Proposition. For \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), we have the relation

\[
\mathcal{E}_{e_j}^{\text{ell}}(z, s) = \frac{2\sqrt{\pi} \Gamma(s + \frac{1}{2})}{\Gamma(s)} G^{\text{ell}}_{e_j}(z, s) = \sum_{k=1}^{\infty} \frac{(\frac{s}{2})_k a_k(s)}{k!} P_{e_j}(z, s + 2k)
\]

with \( a_k(s) := 1 - (\frac{s}{2} + \frac{1}{2})_k/(s + \frac{1}{2})_k \).

Proof. By the duplication formula [2.8] we deduce

\[
\frac{2^{s+1} \sqrt{\pi} \Gamma(s + \frac{1}{2})}{\Gamma(s)} = \frac{4\pi \Gamma(2s)}{\Gamma(s)^2}.
\]

Therefore, the assertion can easily be proven by combining Lemma 4.6 with Lemma 7.1. \(\square\)

7.4. Corollary. For \( z \in \mathbb{H} \) with \( z \neq \gamma e_j \) for any \( \gamma \in \Gamma \), at \( s = 0 \), we have the Laurent expansion

\[
\mathcal{E}_{e_j}^{\text{ell}}(z, s) - \frac{2^{s+1} \sqrt{\pi} \Gamma(s + \frac{1}{2})}{\Gamma(s)} G^{\text{ell}}_{e_j}(z, s) = O(s^2).
\]

Proof. Starting from Proposition 7.3 one can meromorphically continue the relation (7.3) to all \( s \in \mathbb{C} \) with \(-1 < \text{Re}(s) < 1\). From this relation, the claimed Laurent expansion can easily be proven. \(\square\)

7.5. Remark. The question arises what kind of applications can be deduced from the above relation between the elliptic Eisenstein series and the automorphic Green’s function. We leave this for future studies.
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