EINSTEIN FOUR-MANIFOLDS OF PINCHED SECTIONAL CURVATURE

XIAODONG CAO AND HUNG TRAN

Abstract. In this paper, we obtain classification of four-dimensional Einstein manifolds with positive Ricci curvature and pinched sectional curvature. In particular, the first result concerns with an upper bound of sectional curvature, improving a theorem of E. Costa. The second is a generalization of D. Yang’s result assuming an upper bound on the difference between sectional curvatures.

1. Introduction

A Riemannian manifold \((M, g)\) is called Einstein if it satisfies
\[
\text{Rc} = \lambda g,
\]
where \(\text{Rc}\) is its Ricci curvature and \(\lambda\) is a constant. A fundamental question in differential geometry is to determine whether a smooth manifold admits an Einstein metric or not. In dimension four, the Hitchin-Thorpe inequality gives a topological obstruction to the existence of such a structure. Currently, there is no known topological obstruction in higher dimensions. Thus, a generally crucial problem is to classify Einstein manifolds with suitable curvature assumptions.

Furthermore, it is known that if \(\lambda > 0\) then \(M\) is compact and has finite fundamental group by Myer’s theorem [13]. Again, in dimension four, there are not many examples. The well-known ones (irreducible and reducible symmetric spaces) are the sphere \(S^4\) with round metric \(g_0\), the complex projective space \(\mathbb{CP}^2\) with Fubini-Study metric \(g_{FS}\), the product of two spheres with same curvature \(S^2 \times S^2\), and their quotients. They all have non-negative sectional curvature and are only known examples with that condition. This motivates the following folk conjecture (see, for example, [17]):

Conjecture 1.1. An Einstein four manifold with \(\lambda > 0\) and non-negative sectional curvature must be either \(S^4\), \(\mathbb{CP}^2\), \(S^2 \times S^2\) or quotient.

For convenience, one can normalize the metric such that \(\text{Rc} = g\). M. Berger first obtained classification under the condition that the sectional curvature is \(1/4\)-pinched [1]. In the last decades, there have been many attempts to completely solve this conjecture; for example, see [10, 17, 8, 7] and the references therein. In particular, one interesting approach is to assume a positive lower bound on the sectional curvature, i.e., \(K_{\text{min}} \geq \epsilon > 0\); this immediately implies that \(K_{\text{max}} \leq 1 - 2\epsilon\), so the sectional...
curvature is \((\frac{-\epsilon}{1-\epsilon})\)-pinched. That is, Berger's result implies classification for Einstein manifolds provided \(K_{\min} \geq \frac{1}{6}\). There have been various results in this direction. The lower bound has been improved from \(\frac{1}{120}(\sqrt{1249} - 23)\) by D. Yang [17] to \(\frac{1}{6}(2 - \sqrt{2})\) by E. Costa [8] to \(\frac{1}{12}\) by the first author and P. Wu [7]. There are also classification results under related conditions such as \(K \geq 0\) and positive intersection form [10], non-negative curvature operator [15], nonnegative isotropic curvature [4], and 3-nonnegative curvature operator [7].

It is noted that \(K \geq 0\) implies \(K \leq 1\). So it is also interesting to consider upper bounds on the sectional curvature (say \(K \leq 1\)) instead of lower bound. Costa [8] observed that if \(K \leq \frac{2}{3}\) then the Einstein manifold must be \(S^4\), \(\mathbb{C}P^2\) or their quotients. Note that without assuming non-negativity of sectional curvature, an upper bound on \(K\) gives, due to the algebraic structure, a priori lower bound on \(K\). For instance if \(K \leq 1\) implies \(K \geq -2\). But a careful analysis of the differential structure would give a better bound. In particular, using [7, Proposition 2.4(3)], one can show that the condition \(K < 1\) is equivalent to the Riemannian curvature operator being 4-positive; hence, it follows that

\[K > \frac{1}{28}(7 - \sqrt{105}) \approx -1.1596.\]

That’s our motivation to derive the following improvements.

**Theorem 1.1.** Let \((M, g)\) be a complete smooth 4-manifold such that \(\text{Rc} = g\). Assume the sectional curvature satisfies either one of the following conditions:

a. \(K \leq \frac{14 - \sqrt{19}}{12} \approx .8034\).

b. At each point \(q \in M\), for every orthonormal basis \(\{e_i\}_{i=1}^4 \in T_qM\) which satisfies \(K(e_1, e_2) \geq K(e_1, e_3)\),

\[2K(e_1, e_2) + K(e_1, e_3) \geq \frac{\sqrt{19} - 3}{4} \approx .3397247.\]

Then \((M, g)\) is isometric to either \((S^4, g_0)\), \((\mathbb{R}P^4, g_0)\), or \((\mathbb{C}P^2, g_{FS})\) up to rescaling.

**Remark 1.1.** Condition (a) is an improvement of Costa’s result where the upper bound is \(\frac{2}{3} \approx .6666\). Condition (b) generalizes a result of [17] where the lower bound is \(9/14 \approx .642857\). As explained in Section 4, this is essentially an upper bound on the difference between two highest sectional curvatures; see Berger’s decomposition in Section 2.

Here is a sketch of the proof. The main idea is to make use of elliptic equations, which arise from Ricci flow computation, to study a static metric (see [4, 6, 7, 16] for similar exploitation of this approach). In particular, Brendle first observed a Bochner formula for the Riemannian curvature on an Einstein manifold. Considering this formula at point \(p\), which realizes the minimal sectional sectional curvature, yields an inequality involving only zero order terms. Thus, either condition leads to a lower bound of \(K\), which improves the a priori bound. The rest of the proof is an adaptation of arguments in [17, 7]: integrating a Bochner-Weitzenböck identity of the Weyl
tensor and pinched sectional curvature imply the manifold is half-conformally flat. Then Hitchin’s classification theorem of such manifolds completes our proof.

The organization of the paper is as follows. The next section collects preliminaries including Berger’s curvature decomposition, the inequality at point $p$ realizing the minimal sectional sectional curvature, and classifications by a condition on the Weyl tensor. Section 3 derives estimates from the algebraic structure of the curvature and the inequality resulted from the differential structure. Finally, the last section provides proof of the main theorem.

2. Preliminaries

2.1. Curvature decomposition for Einstein four-manifolds. On an oriented four-manifold $(M, g)$, let $R, K, Rc, S, W$ denote the Riemann curvature, sectional curvature, Ricci curvature, scalar curvature and Weyl curvature respectively. Also $\chi$ and $\tau$ denote the Euler characteristic and topological signature. The Hodge star operator induces a natural decomposition of the vector bundle of 2-forms $\Lambda^2 TM$

$$\Lambda^2 TM = \Lambda^+ M \oplus \Lambda^- M,$$

where $\Lambda^\pm M$ are the eigenspaces of $\pm 1$ respectively. Elements of $\Lambda^+ M$ and $\Lambda^- M$ are called self-dual and anti-self-dual. Furthermore, it leads a decomposition for the curvature operator $R: \Lambda^2 TM \to \Lambda^2 TM$:

$$R = \begin{pmatrix} \frac{S}{12} \text{Id} + W^+ & \frac{Rc - \frac{S}{4} \text{Id}}{\frac{S}{12} \text{Id} + W^-} \\ \frac{Rc - \frac{S}{4} \text{Id}}{\frac{S}{12} \text{Id} + W^-} & \frac{\frac{Rc - \frac{S}{4} \text{Id}}{\frac{S}{12} \text{Id} + W^-}}{\frac{R}{S}} \end{pmatrix},$$

for $W^\pm$ the restriction of $W$ to $\Lambda^\pm M$. Here $E = Rc - \frac{S}{4} \text{Id}$ is the traceless Ricci part. If the manifold is closed, then the Gauss-Bonnet-Chern formula for the Euler characteristic and Hirzebruch formulas for the signature (cf. [2] for more details) are,

$$8\pi^2 \chi(M) = \int_M (|W|^2 - \frac{1}{2} |E|^2 + \frac{S^2}{24}) dv;$$

$$12\pi^2 \tau(M) = \int_M (|W^+|^2 - |W^-|^2) dv.$$

Remark 2.1. It follows immediately that if $M$ admits an Einstein metric then $E = 0$,

$$R = \begin{pmatrix} R^+ & 0 \\ 0 & R^- \end{pmatrix} = \begin{pmatrix} \frac{S}{12} \text{Id} + W^+ & 0 \\ 0 & \frac{\frac{Rc - \frac{S}{4} \text{Id}}{\frac{S}{12} \text{Id} + W^-}}{\frac{R}{S}} \end{pmatrix}.$$

Also as a direct consequence of (2.2) and (2.3), we have the Hitchin-Thorpe inequality

$$|\tau(M)| \leq \frac{2}{3} \chi(M).$$

For instance, the curvature of $S^4$ ($\chi = 2, \tau = 0$) with standard metric $g_0$ is:

$$R = \begin{pmatrix} \frac{S}{12} \text{Id} & 0 \\ 0 & \frac{S}{12} \text{Id} \end{pmatrix}.$$
(\mathbb{R}P^4, g_0) is the quotient of (S^4, g_0). The curvature of \mathbb{C}P^2 (\chi = 3, \tau = 1) with Fubini-Study metric \(g_{FS}\) is:

\[ R = \left( \begin{array}{ccc} 0 & 0 & \frac{S}{12} \\ \frac{S}{12} & \text{Id} & 0 \\ 0 & 0 & 0 \end{array} \right). \]

The self-dual part of Weyl tensor \(W^+ = \text{Diag}\{-\frac{S}{12}, -\frac{S}{12}, \frac{S}{6}\}\) and anti-self-dual part \(W^- = 0\).

The curvature of \(S^2 \times S^2 (\chi = 4, \tau = 0)\) with the product metric is

\[ R = \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right) \]

for \(A = \text{Diag}\{0, 0, \frac{S}{4}\}\). The self-dual part and anti-self-dual part of the Weyl tensor are \(W^+ = \text{Diag}\{-\frac{S}{12}, -\frac{S}{12}, \frac{S}{6}\}\).

The duality decomposition also implies that, \(R, R^\pm, W, W^\pm\) of an Einstein four-manifold are all harmonic. Using the harmonicity of \(W^\pm\), A. Derdziński [9] derived the following Weitzenböck formula (also see A. Besse [2, Prop. 16.73]),

Proposition 2.1. Let \((M, g)\) be an Einstein four-manifold, then

\[ \Delta |W^\pm|^2 = 2|\nabla W^\pm|^2 + S|W^\pm|^2 - 36 \det W^\pm. \]

It was observed by Gursky-LeBrun [10] and Yang [17] that \(W^\pm\) satisfies the following refined Kato inequality (proven to be optimal by [3, 5]),

Proposition 2.2. Let \((M, g)\) be an Einstein four-manifold, then

\[ |\nabla W^\pm|^2 \geq \frac{5}{3} |\nabla |W^\pm||^2. \]

On the other hand, Berger [11] has another curvature decomposition for Einstein four-manifolds (also see [14]).

Proposition 2.3. Let \((M, g)\) be an Einstein four-manifold with \(\text{Rc} = \lambda g\). For any \(p \in M\), there exists an orthonormal basis \(\{e_i\}_{1 \leq i \leq 4}\) of \(T_pM\), such that relative to the corresponding basis \(\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}\) of \(\wedge^2 T_pM\), \(R\) takes the form

\[ R = \left( \begin{array}{cc} A & B \\ B & A \end{array} \right), \]

where \(A = \text{Diag}\{a_1, a_2, a_3\}\), \(B = \text{Diag}\{b_1, b_2, b_3\}\). Moreover, we have the followings:

1. \(a_1 = K(e_1, e_2) = K(e_3, e_4) = \min\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}\),
   \(a_3 = K(e_1, e_4) = K(e_2, e_3) = \max\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}\),
   \(a_2 = K(e_1, e_3) = K(e_2, e_4)\), and \(a_1 + a_2 + a_3 = \lambda;\)
2. \(b_1 = R_{1234}, b_2 = R_{1342}, b_3 = R_{1423}\),
3. \(|b_2 - b_1| \leq a_2 - a_1, |b_3 - b_1| \leq a_3 - a_1, |b_3 - b_2| \leq a_3 - a_2\).
One can easily observe that, for Einstein four-manifolds, diagonalization of \((2.10)\) becomes \((2.1)\). As a consequence, eigenvalues of the curvature operator are ordered,

\[
\begin{aligned}
    a_1 + b_1 &\leq a_2 + b_2 \leq a_3 + b_3, \\
    a_1 - b_1 &\leq a_2 - b_2 \leq a_3 - b_3.
\end{aligned}
\]

Here \(a_i + b_i\) (\(1 \leq i \leq 3\)) are eigenvectors of self-dual 2-forms, and \(a_i - b_i\) (\(1 \leq i \leq 3\)) are eigenvectors of anti-self-dual 2-forms. Furthermore, \(W^\pm\) are given by,

\[
\begin{aligned}
    W^+(\omega^+_i, \omega^+_j) &= [(a_i + b_i) - \frac{\alpha}{12}]\delta_{ij}, \\
    W^-(\omega^-_i, \omega^-_j) &= [(a_i - b_i) - \frac{\alpha}{12}]\delta_{ij},
\end{aligned}
\]

where \(\{\omega^\pm_i\}_{1 \leq i \leq 3}\) are the corresponding orthonormal bases of \(\wedge^\pm M\) using Prop \(2.3\).

In the following, we normalize the metric such that \(Rc = g\). Recall that for Einstein manifolds (see \([\Pi] \) or \([9]\)),

\[
\Delta R(e_i, e_j, e_k, e_l) + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{dljk}) = 2R_{ijkl},
\]

where \(B_{ijkl} = g^{mn}g^{pq}R_{mijnp}R_{klnq}\). Berger’s curvature decomposition yields explicitly that

\[
\Delta R(e_1, e_2, e_1, e_2) + 2(a_1^2 + b_1^2 + 2a_2^2a_3 + 2b_2b_3) = 2a_1.
\]

Let \(p\) be the point that realizes the minimum of the sectional curvature of \((M^4, g)\) by the tangent plane spanned by \(\{e_1, e_2\} \subset T_pM\). For any \(v \in T_pM\) and geodesic \(\gamma(t)\) with \(\gamma(0) = p, \gamma'(0) = v\), let \(\{e_1, e_2, e_3, e_4\}\) be a parallel orthonormal frame along \(\gamma(t)\), then we have

\[
(D^2_{v,v}R)(e_1, e_2, e_1, e_2)(p) = D^2_{v,v}(R(e_1, e_2, e_1, e_2))(p) \geq 0.
\]

Hence it follows that \((\Delta R)(e_1, e_2, e_1, e_2)(p) \geq 0\). Thus, at \(p\), the following holds

\[
a_1^2 + b_1^2 + 2(a_2^2a_3 + b_2b_3) \leq a_1.
\]

2.2. **Classification by estimates on \(W^\pm\).** We also need the following, which is implicit in \([7]\). A proof is provided for completeness.

**Proposition 2.4.** Let \((M, g)\) be an Einstein four-manifold \(Rc = g\) such that,

\[
|W^+| + |W^-| \leq \sqrt{\frac{3}{2}}
\]

Then it is isometric to either \((S^4, g_0)\), \((\mathbb{R}P^4, g_0)\), or \((\mathbb{C}P^2, g_{FS})\) up to rescaling.

**Proof.** If the manifold is half-conformally flat, then, by Hitchin’s classification \([21]\) Theorem 13.30], the result follows. If not, then for some \(\alpha > 0\) to be determined later, and any \(\epsilon > 0\), there exists \(t = t(\alpha, \epsilon) \in \mathbb{R}^+\), such that

\[
\int_M (|W^+|^2 + \epsilon)^{\frac{3}{2}} - t(|W^-|^2 + \epsilon)^{\frac{3}{2}} dv = 0.
\]

Also \(t(\alpha, \epsilon) \to t(\alpha, 0)\) as \(\epsilon \to 0\).
Applying Weitzenböck formula (2.8) and refined Kato inequality (2.9) yields,
\[ \Delta[(|W^+|^2 + \epsilon)^\alpha + t^2(|W^-|^2 + \epsilon)^\alpha] \]
\[= \alpha(|W^+|^2 + \epsilon)^{\alpha-2} [(|W^+|^2 + \epsilon)(2|\nabla W^+|^2 + S|W^+|^2 - 36 \det W^+) + (\alpha - 1)|\nabla|W^+|^2|^2] \]
\[+ t^2 \alpha(|W^-|^2 + \epsilon)^{\alpha-2} [(|W^-|^2 + \epsilon)(2|\nabla W^-|^2 + S|W^-|^2 - 36 \det W^-) + (\alpha - 1)|\nabla|W^-|^2|^2] \]
\[\geq [(4 - \frac{2}{3\alpha})|\nabla((|W^+|^2 + \epsilon)^{\alpha/2}|^2 + \alpha(|W^+|^2 + \epsilon)^{\alpha-1}(S|W^+|^2 - 36 \det W^+)] \]
\[+ t^2 [(4 - \frac{2}{3\alpha})|\nabla((|W^-|^2 + \epsilon)^{\alpha/2}|^2 + \alpha(|W^-|^2 + \epsilon)^{\alpha-1}(S|W^-|^2 - 36 \det W^-)).(\ast) \]

Using the Poincaré inequality, we have
\[ (4 - \frac{2}{3\alpha}) \int_M (|\nabla(|W^+|^2 + \epsilon)^{\alpha/2}|^2 + t^2|\nabla(|W^-|^2 + \epsilon)^{\alpha/2}|^2)dv \]
\[\geq (2 - \frac{1}{3\alpha}) \int_M \left| \nabla[(|W^+|^2 + \epsilon)^{\alpha/2} - t(|W^-|^2 + \epsilon)^{\alpha/2}] \right|^2 dv \]
\[\geq (2 - \frac{1}{3\alpha}) \lambda_1 \int_M \left[ (|W^+|^2 + \epsilon)^{\alpha/2} - t(|W^-|^2 + \epsilon)^{\alpha/2} \right]^2 dv, \]
where \( \lambda_1 \) is the lowest positive eigenvalue of the Laplace operator. In our case that \( \text{Ric} = g \), we have \( \lambda_1 \geq \frac{4}{3} \) (see, for example, [12]). Picking \( \alpha = \frac{1}{3} \), which maximizes the value of \( \frac{1}{\alpha}(2 - \frac{1}{3\alpha}) \), substituting \( S = 4 \), and integrating the above inequality (\( \ast \)), we obtain
\[ 0 \geq \frac{1}{3} \int_M \left( 4 \left[ (|W^+|^2 + \epsilon)^{\alpha/2} - t(|W^-|^2 + \epsilon)^{\alpha/2} \right]^2 + t^2(|W^-|^2 + \epsilon)^{-2/3}(4|W^-|^2 - 36 \det W^-) \right. \]
\[\left. + (|W^+|^2 + \epsilon)^{-2/3}(4|W^+|^2 - 36 \det W^+) \right) dv. \]

Recall the algebraic inequalities,
\[ 36 \det(W^\pm) \leq 2\sqrt{6}|W^\pm|^3. \]

Now let \( \epsilon \to 0 \), we get
\[ 0 \geq \int_M \left( t^2|W^-|^{-4/3}(S|W^-|^2 - 36 \det W^-) + 4(|W^+|^{1/3} - t|W^-|^{1/3})^2 \right. \]
\[\left. + |W^+|^{-4/3}(4|W^+|^2 - 36 \det W^+) \right) dv, \]
\[\geq \int_M \left( t^2|W^-|^{-2/3}(4 - 2\sqrt{6}|W^-|) + 4(|W^+|^{1/3} - t|W^-|^{1/3})^2 \right. \]
\[\left. + |W^+|^{-2/3}(4 - 2\sqrt{6}|W^+|) \right) dv \]
\[\geq \int_M \left( t^2|W^-|^{-2/3}(8 - 2\sqrt{6}|W^-|) - 8t|W^+|^{1/3}|W^-|^{1/3} + |W^+|^{2/3}(8 - 2\sqrt{6}|W^+|) \right) dv. \]
The integrand is a quadratic function of $t$, with positive leading coefficient and discriminant

$$D = |W^+|^{2/3}|W^-|^{2/3}(16 - 64 + 16\sqrt{6}(|W^+| + |W^-|) - 24|W^+||W^-| \leq 0.$$ 

So equality must happen at each point and so either $|W^+|$ or $|W^-|$ must vanish at each point. As both are analytic functions, one of them must be vanishing at every point. So the manifold is half-conformally flat, a contradiction. \hfill \Box

3. Estimates

In this section, we derive estimates from the algebraic structure of $W^\pm$ and the differential inequality (2.13).

3.1. Algebraic Estimates. The technical lemma below estimates the norm of $W^\pm$ by pinching of sectional curvature.

**Lemma 3.1.** Using the Berger’s decomposition Prop. 2.3, suppose

$$a_3 - a_2 = \frac{\delta}{2} \geq 0,$$

$$a_3 + a_2 = \alpha > 0.$$ 

Then

$$\delta \leq 6\alpha - 4;$$

$$|W^+| + |W^-| \leq \frac{6\alpha - 4 + \delta}{\sqrt{6}},$$

$$|W^+|^2 + |W^-|^2 \leq \frac{1}{2}(12\alpha^2 - 16\alpha + \frac{16}{3} + \delta^2)$$

$$= 8(a_3^2 - (1 - a_1)(1 - a_2) + \frac{1}{3}).$$

**Proof.** We’ll prove the first estimate while the second one follows from the same method. Let $\{\lambda_i, \mu_i\}_{i=1}^3$ be eigenvalues of $W^\pm$. By (2.12) the assumptions above translate to

$$\lambda_3 + \lambda_2 + \mu_3 + \mu_2 = \alpha_1 = 2(\alpha - \frac{2}{3}),$$

$$\lambda_3 - \lambda_2 + \mu_3 - \mu_2 = \delta,$$

$$-\frac{\lambda_3}{2} \leq \lambda_2 \leq \lambda_3,$$

$$-\frac{\mu_3}{2} \leq \mu_2 \leq \mu_3.$$ 

Then

$$|W^+| + |W^-| = \sqrt{2}(\sqrt{\frac{3}{4}(\lambda_3 + \lambda_2)^2 + \frac{1}{4}((\lambda_3 - \lambda_2)^2 + \sqrt{\frac{3}{4}(\mu_3 + \mu_2)^2 + \frac{1}{4}((\mu_3 - \mu_2)^2)}}.$$

$$= \sqrt{2}(\sqrt{\frac{3}{4}(\lambda_3 + \lambda_2)^2 + \frac{1}{4}((\lambda_3 - \lambda_2)^2 + \sqrt{\frac{3}{4}(\mu_3 + \mu_2)^2 + \frac{1}{4}((\mu_3 - \mu_2)^2)}}.$$


So we consider the following problem. For,
\[ \lambda_3 + \lambda_2 = p, \]
\[ \lambda_3 - \lambda_2 = q, \]
\[ \mu_3 + \mu_2 = m, \]
\[ \mu_3 - \mu_2 = n. \]
The constraints become:
\[ p + m = \alpha_1, \]
\[ q + n = \delta, \]
\[ 0 \leq q \leq 3p, \]
\[ 0 \leq n \leq 3m. \]
The goal is to maximize
\[ f(p, q, m, n) = \sqrt{3p^2 + q^2} + \sqrt{3m^2 + n^2}. \]
First, since the constraint is a closed set, the maximum exists.
Next, we consider the problem with 2 variables \( m, n \). The constraints become,
\[ 0 \leq 3m - n = \ell \leq 3\alpha_1 - \delta, \]
\[ 0 \leq n \leq \delta. \]
The function to maximize is
\[ f(m, n) = \sqrt{3(\alpha_1 - m)^2 + (\delta - n)^2} + \sqrt{3m^2 + n^2}, \]
\[ \sqrt{3} f(m, n) = \sqrt{(3\alpha_1 - \ell - n)^2 + 3(\delta - n)^2} + \sqrt{(\ell + n)^2 + 3n^2}, \]
\[ = g(\ell, n). \]
We consider the following cases.
**Case 1:** \( n = 0 \) then
\[ g(\ell, 0) = \sqrt{(\ell - 3\alpha_1)^2 + 3\delta^2} + \ell. \]
Since the function \( f(x) = \sqrt{x^2 + a^2 + x}, \ a > 0, \) is strictly increasing,
\[ g(\ell, 0) \leq 3\alpha_1 + \delta. \]
**Case 2:** \( \ell = 0 \) then
\[ g(n, 0) = \sqrt{4n^2 - 6(\alpha_1 + \delta)n + 9\alpha_1^2 + 3\delta^2 + 2n}, \]
\[ = \sqrt{(2n - \frac{3}{2}(\alpha_1 + \delta))^2 + (\frac{3\sqrt{3}}{2} \alpha_1 - \frac{\sqrt{3}}{2} \delta)^2 + 2n} \]
\[ \leq 3\alpha_1 + \delta. \]
**Case 3:** \( n = \delta \) then
\[ g(\ell, \delta) = (3\alpha_1 - \delta) - \ell + \sqrt{(\ell + \delta)^2 + 3\delta^2}, \]
\[ \leq 3\alpha_1 + \delta. \]
Case 4: $\ell = 3\alpha_1 - \delta$ then
\[
g(3\alpha_1 - \delta, n) = 2(\delta - n) + \sqrt{\left(2n + \frac{1}{2}(3\alpha_1 - \delta)\right)^2 + \frac{3}{4}(3\alpha_1 - \delta)^2} \\
\leq 3\alpha_1 + \delta.
\]

Case 5: At a critical point,
\[
0 = \partial_\ell g = -\frac{3\alpha_1 - \ell - n}{\sqrt{(3\alpha_1 - \ell - n)^2 + 3(\delta - n)^2}} + \frac{\ell + n}{\sqrt{(\ell + n)^2 + 3n^2}},
\]
\[
0 = \partial_n g = -\frac{3\alpha_1 + 3\delta - \ell - 4n}{\sqrt{(3\alpha_1 - \ell - n)^2 + 3(\delta - n)^2}} + \frac{4n + \ell}{\sqrt{(\ell + n)^2 + 3n^2}}.
\]
Therefore, at that point,
\[
\frac{\ell + n}{(3\alpha_1 - \delta - \ell) + (\delta - n)} = \frac{4n + \ell}{(3\alpha_1 - \delta - \ell) + 4(\delta - n)} = \frac{n}{\delta - n} = \frac{\ell}{3\alpha_1 - \delta - \ell} = \frac{1}{x}.
\]
Then,
\[
g(\ell, n) = (x + 1)\sqrt{\left(\frac{3\alpha_1 - \delta}{x + 1} + \frac{\delta}{x + 1}\right)^2 + \left(\frac{\delta}{x + 1}\right)^2} \\
= \sqrt{9\alpha_1^2 + \delta^2} \leq 3\alpha_1 + \delta.
\]

\[\square\]

Remark 3.1. The first estimate generalizes and unifies [17, Lemma 4.1] which treats the case $\alpha \leq 1 - \epsilon$, $\delta \leq 2(1 - 3\epsilon)$ in part (a.) and $\alpha \leq 1$, $\delta \leq 1 - 6\epsilon$ in part (b.).

Remark 3.2. The condition that $6\alpha + \delta - 4 \leq 3$ implies 3-non-negative curvature.

Corollary 3.1. Let $(M, g)$ be an Einstein four-manifold $Rc = g$ such that at each point, $\alpha \leq a_1 \leq a_3 \leq \beta$, then
\[
8\pi^2 \chi(M) \leq \left(8(\beta^2 - (1 - \alpha)(\alpha + \beta) + \frac{10}{3}\right)Vol(M).
\]

Proof. Recall,
\[
8\pi^2 \chi(M) = \int_M (|W^+|^2 + |W^-|^2 + \frac{S^2}{24})d\mu.
\]
By Lemma 3.1 at each point,
\[
|W^+|^2 + |W^-|^2 \leq 8(a_3^2 - (1 - a_1)(1 - a_2) + \frac{1}{3})
\]
So it remains to maximize, given the pinching condition on $a_1$, $a_3$,
\[
f(a_1, a_3) = a_3^2 - (1 - a_1)(a_1 + a_3).
\]
By the algebraic properties of $a_1, a_3$, the domain here is a quadrilateral determined by lines $x = \alpha, y = \beta, 2x + y = 1, 2y + x = 1$ (which is already within the half-plane $y \geq x$). So standard technique yields,

$$|W^+|^2 + |W^-|^2 \leq 8(\beta^2 - (1 - \alpha)(\alpha + \beta) + \frac{1}{3}).$$

The result then follows. \qed

**Remark 3.3.** In [10], the authors show that if $W^+ \neq 0$ then,

$$\int_M |W^+|^2 dv \geq \frac{2}{3} \text{Vol}(M).$$

That is, if the Einstein 4-manifold is not half-conformally flat then,

$$8\pi^2 \chi(M) \geq 2 \text{Vol}(M).$$

They also observe that if $a_1 \geq 0$ then,

$$8\pi^2 \chi(M) \leq \frac{10}{3} \text{Vol}(M).$$

As a consequence, it follows that

$$9 \geq \chi(M) > \frac{15}{4} |\tau(M)|.$$  

Therefore, there are only finitely many homeomorphism types for an Einstein 4-manifold with non-negative sectional curvature and not half-conformally flat. Corollary 3.1 then gives a more precise description of the relation between the topology type and bound on the sectional curvature. For instance, if $a_1 > \alpha = \frac{2 - \sqrt{3}}{6} \approx .04466$ then we could choose $\beta = 1 - 2\alpha$ and then $(|\tau|, \chi)$ could only be one of the following choices (1, 5), (1, 7), (0, 2), (0, 4), (0, 6).

### 3.2. Differential Estimates

Here we derive several consequences of (2.13). First we observe the following.

**Lemma 3.2.** Let $xy \leq 0$ and assume

$$|2x + y| \leq a,$$

$$|x - y| \leq b.$$

If $2a < b$ then

$$4xy + x^2 + y^2 \geq \frac{1}{3}(2a^2 - 2ab - b^2).$$

If $2a \geq b$ then

$$4xy + x^2 + y^2 \geq -\frac{1}{2}b^2.$$  

**Proof.** Let $m = 2x + y$ and $n = x - y$ then

$$9(4xy + x^2 + y^2) = (m + n)^2 + (m - 2n)^2 + 4(m + n)(m - 2n)$$

$$= 6m^2 - 3n^2 - 6mn = f(m, n).$$
Consider the region $D = \{-a \leq m \leq a; -b \leq n \leq b\}$. The only critical point of $f(m, n)$ is $(0, 0)$ and $f(0, 0) = 0$. So we consider the function along the boundary of $D$ and the result follows.

Recall that $p$ is the point that realizes the minimum of the sectional curvature of $(M^4, g)$ by the tangent plane spanned by $\{e_1, e_2\}$. At point $p$, we get:

$$a_1^2 + b_1^2 + 2(a_2a_3 + b_2b_3) \leq a_1.$$ 

Also by Prop 2.3 we observe,

$$|b_2 - b_3| \leq a_3 - a_2 = b,$$

$$|b_2 - b_1| = |2b_2 + b_3| \leq a_2 - a_1 = a,$$

$$b_1^2 + 2b_2b_3 = b_2^2 + b_3^2 + 4b_2b_3.$$

So Lemma 3.2 implies the followings:

- If $2a < b$ or $a_1 + 1 \geq 4a_2$ then

  $$b_1^2 + 2b_2b_3 \leq (a_2 - a_1)^2 - \frac{1}{3}(a_3 - a_1)^2.$$ 

- If $2a \geq b$ or $a_1 + 1 \leq 4a_2$ then

  $$b_1^2 + 2b_2b_3 \leq -\frac{1}{2}(a_3 - a_2)^2.$$ 

Consequently, it is possible to obtain a lower bound for sectional curvature given an upper bound.

**Lemma 3.3.** At point $p$, suppose $a_3 = \alpha \leq 1$ then we have:

$$4a_2 \leq a_1 + 1,$$

$$a_1 \geq \frac{1}{28}(15 - 8\alpha - \sqrt{3}\sqrt{96\alpha^2 - 80\alpha + 19}).$$

**Proof.** Let $\delta = a_1 = \min K$ at point $p$ then we have

$$a_3 = \alpha,$$

$$a_1 = \delta,$$

$$a_2 = 1 - \alpha - \delta,$$

$$a_2 + a_3 = 1 - \delta,$$

$$a_2a_3 = (1 - \alpha - \delta)\alpha.$$ 

If $b_2b_3 \geq 0$ then equation (2.13) becomes

$$\delta = a_1 \geq a_1^2 + 2a_2a_3 \geq \delta^2 + 2\delta(1 - 2\delta) = 2\delta - 3\delta^2.$$ 

If $a_1 < 0$ then $a_2 < 0$ and, consequently $a_3 > 1$, a contradiction. So either $a_1 = a_2 = 0$ or $a_1 = a_2 = \frac{1}{3}$.

If $b_2b_3 < 0$ then we consider two cases.
Case 1: \( 4a_2 > 1 + a_1 \) and, by the discussion following Lemma 3.2, we have
\[
\delta = a_1 \geq a_1^2 + 2a_2a_3 + b_2^2 + b_3^2 + 4b_2b_3
\]
\[
\delta \geq \delta^2 + 2(1 - \alpha - \delta)\alpha - \frac{(2\alpha - 1 + \delta)^2}{2};
\]
\[
0 \geq \delta^2 - 8\alpha\delta + 8\alpha - 8\alpha^2 - 1
\]
\[
0 \geq \delta^2 - 8\alpha\delta + \alpha_1.
\]
As a consequence,
\[
4\alpha - \sqrt{16\alpha^2 - \alpha_1} \geq \delta \geq 4\alpha - \sqrt{16\alpha^2 - \alpha_1}.
\]
So,
\[
1 = a_1 + a_2 + a_3
\]
\[
> a_1 + \frac{1}{4}(1 + a_1) + \alpha,
\]
\[
\frac{3}{4} > \frac{5}{4}(4\alpha - \sqrt{24\alpha^2 + 1 - 8\alpha} + \alpha
\]
\[
> 6\alpha - \frac{5}{4}\sqrt{24\alpha^2 + 1 - 8\alpha}.
\]
But that is a contradiction for \( \frac{1}{3} \leq \alpha \leq 1 \). Thus this case does not hold.

Case 2: \( 4a_2 \leq 1 + a_1 \) and, by the discussion following Lemma 3.2, we have
\[
\delta = a_1 \geq a_1^2 + 2a_2a_3 + b_2^2 + b_3^2 + 4b_2b_3
\]
\[
\delta \geq \delta^2 + 2(1 - \alpha - \delta)\alpha + (1 - \alpha - 2\delta)^2 - \frac{(\alpha - \delta)^2}{3};
\]
\[
0 \geq 14\delta^2 + (8\alpha - 15)\delta + 3 - 4\alpha^2.
\]
As a consequence,
\[
\frac{1}{28}(15 - 8\alpha + \sqrt{3\sqrt{96\alpha^2 - 80\alpha + 19} \geq \delta \geq \frac{1}{28}(15 - 8\alpha - \sqrt{3\sqrt{96\alpha^2 - 80\alpha + 19}}.
\]

\[\square\]

Corollary 3.2. At \( p \), if \( a_3 \leq \frac{\sqrt{7}}{2} \approx .866025 \) then \( K \geq 0 \).

Proof. Let \( a_3 = \alpha \) at point \( p \). Then by Lemma 3.3,
\[
a_1 \geq 0
\]
\[
15 - 8\alpha \geq \sqrt{3\sqrt{96\alpha^2 - 80\alpha + 19}}
\]
\[
\leftrightarrow -\frac{\sqrt{3}}{2} \leq \alpha \leq \frac{\sqrt{3}}{2}.
\]

\[\square\]

Remark 3.4. Z. Zhang [18] obtains a similar result but we fail to follow the proof.
Lemma 3.4. At point $p$ suppose $0 \leq a_3 - a_2 = \alpha < 2$ then

\[ 4a_2 \leq a_1 + 1, \]
\[ a_1 \geq \beta = \frac{1}{6} (3 - 2\alpha - \sqrt{1 + 8\alpha^2 - 4\alpha}). \]

Proof. Let $\delta = a_1 = \min K$ at point $p$ then we have

\[ a_2 + a_3 = 1 - \delta, \]
\[ a_3 - a_2 = \alpha, \]
\[ a_2 - a_1 = \frac{1 - \alpha - 3\delta}{2}, \]
\[ 4a_2a_3 = (a_2 + a_3)^2 - (a_2 - a_3)^2 = (1 - \delta)^2 - \alpha^2. \]

If $b_2b_3 \geq 0$ then equation (2.13) yields

\[ \delta = a_1 \geq a_1^2 + 2a_2a_3 \geq \delta^2 + \frac{1}{2}((1 - \delta)^2 - \alpha^2); \]
\[ 0 \geq \frac{3}{2}\delta^2 - 2\delta + \frac{1}{2}(1 + \alpha^2) \]

Therefore if $\alpha > \frac{1}{\sqrt{3}}$ there is a contradiction. For $0 \leq \alpha \leq \frac{1}{\sqrt{3}}$, we have,

\[ \frac{1}{3} \leq \frac{1}{3} (2 - \sqrt{1 - 3\alpha^2}) \leq \delta \leq \frac{1}{3} (2 + \sqrt{1 - 3\alpha^2}). \]

Therefore, $a_1 = a_2 = a_3 = \frac{1}{3}$.

If $b_2b_3 < 0$ then we consider two cases.

Case 1: If $4a_2 > a_1 + 1$ then, by the discussion following Lemma 3.2 equation (2.13) becomes

\[ \delta = a_1 \geq a_1^2 + 2a_2a_3 + b_2^2 + b_3^2 + 4b_2b_3 \]
\[ \delta \geq \delta^2 + \frac{1}{2}((1 - \delta)^2 - \alpha^2) - \frac{\alpha^2}{2}; \]
\[ 0 \geq 3\delta^2 - 4\delta + 1 - 2\alpha^2 \]
\[ \geq 3\delta^2 - 4\delta + \alpha_1. \]

As a consequence,

\[ \frac{2 + \sqrt{4 - 3\alpha_1}}{3} \geq \delta \geq \frac{2 - \sqrt{4 - 3\alpha_1}}{3}. \]

So,

\[ 1 = a_1 + a_2 + a_3, \]
\[ = a_1 + 2a_2 + \alpha \geq \frac{1}{2} + \frac{3}{2}a_1 + \alpha; \]
\[ \frac{1}{2} \geq \alpha + 1 - \frac{1}{2}\sqrt{1 + 6\alpha^2}. \]
For $0 \leq \alpha < 2$, the inequality above is possible only if $\alpha = 0$. That implies $\delta \geq \frac{1}{3}$. So $a_1 = a_2 = a_3 = \frac{1}{3}$, a contradiction to $4a_2 > a_1 + 1$.

**Case 2:** If $4a_2 \leq a_1 + 1$ then, by the discussion following Lemma 3.2, equation (2.13) becomes

\[
\delta = a_1 \geq a_1^2 + 2a_2a_3 + b_2^2 + b_3^2 + 4b_2b_3
\]
\[
\delta \geq \delta^3 + \frac{1}{2}((1 - \delta)^2 - \alpha^2) + \left(\frac{1 - \alpha - 3\delta}{2}\right)^2 - \frac{1}{3}\left(\frac{1 + \alpha - 3\delta}{2}\right)^2;
\]
\[
0 \geq 3\delta^2 - \delta(3 - 2\alpha) + \frac{2}{3} - \frac{2\alpha}{3} - \frac{\alpha^2}{3}.
\]

As a consequence,

\[
\frac{1}{6}(3 - 2\alpha + \sqrt{1 + 8\alpha^2 - 4\alpha}) \geq \delta \geq \frac{1}{6}(3 - 2\alpha - \sqrt{1 + 8\alpha^2 - 4\alpha}).
\]

**Corollary 3.3.** At $p$, if $a_3 - a_2 \leq \sqrt{3} - 1$ then $K \geq 0$.

**Proof.** Let $a_3 - a_2 = \alpha$ at point $p$. Then by Lemma 3.3

\[
a_1 \geq 0
\]
\[
\leftrightarrow 3 - 2\alpha \geq \sqrt{1 + 8\alpha^2 - 4\alpha},
\]
\[
\leftrightarrow -1 - \sqrt{3} \leq \alpha \leq \sqrt{3} - 1.
\]

As $\alpha \geq 0$ the result follows. \qed

Lemmas 3.3 and 3.4 imply that, at $p$, $a_2$ and $a_1$ are close to each other if there is an upper bound on $a_3$ or $a_3 - a_2$. The next result estimates $a_2 - a_1$ by $a_1$.

**Lemma 3.5.** Suppose at point $p$,

\[
a_1 = \delta \geq 0,
\]
\[
a_2 = x + \delta \leq \frac{1}{4} + \frac{\delta}{4}.
\]

Then

\[
0 \leq x \leq 1 - 3\delta - \frac{1}{2}\sqrt{3 + 18\delta^2 - 15\delta}.
\]

**Proof.** We have

\[
a_3 = \alpha = 1 - 2\delta - x.
\]

If $b_2b_3 \geq 0$ then equation (2.13) yields

\[
\delta = a_1 \geq a_1^2 + 2a_2a_3 \geq \delta^2 + 2(x + \delta)(1 - 2\delta - x);
\]
\[
0 \leq 2x^2 + x(6\delta - 2) + 3\delta^2 - \delta.
\]

Therefore,

\[
\delta \geq \frac{1}{6}(1 - 6x + \sqrt{12x^2 + 12x + 1}).
\]
So, for $0 \leq x \leq 1$,

$$a_1 + a_2 = 2a_1 + x \geq x + \frac{1}{3}(1 - 6x + \sqrt{12x^2 + 12x + 1}) \geq \frac{2}{3}.$$ 

Thus, $x = 0$.

We consider $b_2b_3 < 0$. Then equation (2.13) becomes

$$\delta = a_1 \geq \delta^2 + 2(x + \delta)\alpha + (b_2^2 + b_3^2 + 4b_2b_3).$$

Since $4a_2 \leq 1 + a_1$, applying the discussion after Lemma 3.2 into equation (2.13) yields

$$\delta = a_1 \geq \delta^2 + 2(x + \delta)\alpha + \frac{1}{3}(2x^2 - 2x(\alpha - x - \delta) - (\alpha - x - \delta)^2),$$

$$\geq \delta^2 + 2(x + \delta)\alpha + x^2 - \frac{1}{3}(\alpha - \delta)^2;$$

$$0 \geq x^2 + 2x\alpha + 2\delta\alpha + \delta^2 - \delta - \frac{1}{3}(\alpha - \delta)^2.$$ 

Substituting $\alpha = 1 - 2\delta - x$ then yields,

$$0 \leq 4x^2 - 8x(1 - 3\delta) + (1 - 9\delta + 18\delta^2).$$

This quadratic has the discriminant

$$D = 48(1 - 5\delta + 6\delta^2) = 48(1 - 2\delta)(1 - 3\delta) \geq 0.$$ 

As a consequence,

$$x \leq 1 - 3\delta - \frac{1}{2}\sqrt{3 + 18\delta^2 - 15\delta}.$$ 

Then the result follows. \qed

**Remark 3.5.** For $0 \leq \delta \leq \frac{1}{3}$,

$$1 - 3\delta - \frac{1}{2}\sqrt{3 + 18\delta^2 - 15\delta} < \frac{1}{4}(1 - 3\delta).$$

4. **Classification**

This section proves the main result.

**Theorem 4.1.** Let $(M, g)$ be an Einstein four-manifold $Rc = g$ such that at each point,

$$K \leq \beta = \frac{14 - \sqrt{19}}{12} \approx .8034.$$ 

Then it is isometric to either $(S^4, g_0)$, $(\mathbb{RP}^4, g_0)$, or $(\mathbb{CP}^2, g_{FS})$ up to rescaling.
Proof. We consider Berger’s decomposition Prop \ref{prop:2.3} at point \( p \) realizing the minimum of sectional curvature. Let \( \alpha = a_3 \). By Lemma \ref{lem:3.3},

\[
a_1 \geq \alpha_1 = \frac{1}{28} (15 - 8\alpha - \sqrt{3} \sqrt{96\alpha^2 - 80\alpha + 19}).
\]

Since this function of \( \alpha \) is decreasing for \( \alpha \geq \frac{1}{8} \), we conclude that,

\[
a_1 \geq \beta_1 = \frac{1}{28} (15 - 8\beta - \sqrt{3} \sqrt{96\beta^2 - 80\beta + 19}).
\]

By the choice of \( p \), the sectional curvature is at least \( \beta_1 \) at each point. By Lemma \ref{lem:3.1}, then at each point

\[
|W^+| + |W^-| \leq \frac{6(K_3 + K_2) + 2(K_3 - K_2) - 4}{\sqrt{6}},
\]

\[
\leq \frac{8K_3 + 4K_2 - 4}{\sqrt{6}} = \frac{4K_3 - 4K_1}{\sqrt{6}},
\]

\[
\leq \frac{4\beta - 4\beta_1}{\sqrt{6}}.
\]

Substituting \( \beta = \frac{14 - \sqrt{10}}{12} \) yields,

\[
|W^+| + |W^-| \leq \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}.
\]

The theorem then follows from Prop \ref{prop:2.4}. \( \square \)

Similarly, we have the following.

**Theorem 4.2.** Let \((M, g)\) be an Einstein four-manifold \( Rc = g \) such that at each point,

\[
a_3 - a_2 \leq \beta = \frac{7 - \sqrt{19}}{4} \approx .660275.
\]

Then it is isometric to either \((S^4, g_0)\), \((\mathbb{RP}^4, g_0)\), or \((\mathbb{CP}^2, g_{FS})\) up to rescaling.

Proof. We consider Berger’s decomposition Prop \ref{prop:2.3} at point \( p \) realizing the minimum of sectional curvature. Let \( \alpha = a_3 - a_2 \). By Lemma \ref{lem:3.4},

\[
a_1 \geq \alpha_1 = \frac{1}{6} (3 - 2\alpha - \sqrt{1 + 8\alpha^2 - 4\alpha}).
\]

Since this function of \( \alpha \) is decreasing for \( \alpha \geq 0 \), we conclude that,

\[
a_1 \geq \beta_1 = \frac{1}{6} (3 - 2\beta - \sqrt{1 + 8\beta^2 - 4\beta}).
\]

By the choice of \( p \), the sectional curvature is at least \( \beta_1 \) at each point.
By Lemma 3.1 then at each point
\[ |W^+| + |W^-| \leq \frac{6(a_3 + a_2) + 2(a_3 - a_2) - 4}{\sqrt{6}}, \]
\[ \leq \frac{2 - 6a_1 + 2(a_3 - a_2)}{\sqrt{6}}, \]
\[ \leq \frac{2 + 2\beta - 6\beta_1}{\sqrt{6}}. \]

Substituting \( \beta = \frac{7 - \sqrt{19}}{4} \) yields,
\[ |W^+| + |W^-| \leq \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}. \]

The theorem then follows from Prop 2.4. \( \Box \)

Now the proof of Theorem 1.1 follows immediately.

**Proof.** Condition (a.) is considered in Theorem 4.1
Condition (b.) implies that
\[ 2a_2 + a_1 \geq \frac{\sqrt{19} - 3}{4}, \]
\[ a_2 - a_3 \geq \frac{\sqrt{19} - 7}{4}. \]
This is exactly the condition of Theorem 4.2 so the result follows. \( \Box \)

**References**

[1] Marcel Berger. Sur quelques variétés d’Einstein compactes. *Ann. Mat. Pura Appl. (4)*, 53:89–95, 1961.
[2] Arthur L. Besse. *Einstein manifolds*, volume 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1987.
[3] T. Branson. Kato constants in Riemannian geometry. *Math. Res. Lett.*, 7(2-3):245–261, 2000.
[4] Simon Brendle. Einstein manifolds with nonnegative isotropic curvature are locally symmetric. *Duke Math. J.*, 151(1):1–21, 2010.
[5] David M. J. Calderbank, Paul Gauduchon, and Marc Herzlich. Refined Kato inequalities and conformal weights in Riemannian geometry. *J. Funct. Anal.*, 173(1):214–255, 2000.
[6] Xiaodong Cao and Hung Tran. The Weyl tensor of gradient Ricci solitons. *Geom. Topol.*, 20(1):389–436, 2016.
[7] Xiaodong Cao and Peng Wu. Einstein four-manifolds of three-nonnegative curvature operator. *Unpublished*, 2014.
[8] Ézio de Araujo Costa. On Einstein four-manifolds. *J. Geom. Phys.*, 51(2):244–255, 2004.
[9] Andrzej Derdziński. Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.*, 49(3):405–433, 1983.
[10] Matthew J. Gursky and Claude LeBrun. On Einstein manifolds of positive sectional curvature. *Ann. Global Anal. Geom.*, 17(4):315–328, 1999.
[11] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.
[12] André Lichnerowicz. Géométrie des groupes de transformations. Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958.

[13] S. B. Myers. Riemannian manifolds with positive mean curvature. Duke Math. J., 8:401–404, 1941.

[14] I. M. Singer and J. A. Thorpe. The curvature of 4-dimensional Einstein spaces. In Global Analysis (Papers in Honor of K. Kodaira), pages 355–365. Univ. Tokyo Press, Tokyo, 1969.

[15] Shun-ichi Tachibana. A theorem of Riemannian manifolds of positive curvature operator. Proc. Japan Acad., 50:301–302, 1974.

[16] Hung Tran. On closed manifolds with harmonic weyl. arXiv preprint arXiv:1602.01429, 2016.

[17] DaGang Yang. Rigidity of Einstein 4-manifolds with positive curvature. Invent. Math., 142(2):435–450, 2000.

[18] Zhuhong Zhang. Four-dimensional einstein manifolds with sectional curvature bounded from above. arXiv preprint arXiv:1606.01157, 2016.

Department of Mathematics, Cornell University, Ithaca, NY 14853-4201
E-mail address: cao@math.cornell.edu

Department of Mathematics, University of California at Irvine, Irvine, CA 92697
E-mail address: hungtt1@uci.edu